Spatial dynamics approaches to hydroelastic surface waves

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Abstract

In this dissertation we present existence theories for three distinct types of travelling hydroelastic surface waves: *doubly periodic waves* which are periodic in two distinct horizontal directions, two-dimensional *line solitary waves* which are localised in the horizontal direction and *periodically modulated solitary waves* which are localised in one horizontal direction and periodic in the other. All our existence results are based on formulations of the hydrodynamic problem as a spatial Hamiltonian system. For the existence theory of doubly periodic waves we use a resonant version of the Lyapunov centre theorem which is proved in this thesis. The existence of line solitary waves is proved using a centre-manifold reduction to reduce the problem to a locally equivalent finite-dimensional Hamiltonian system which has homoclinic orbits. Finally we prove existence of periodically modulated solitary waves which emanate from the line solitary waves in a *dimension-breaking* bifurcation using well-developed techniques from existence theory for water waves.

Zusammenfassung

In dieser Dissertation präsentieren wir Existenztheorien für drei verschiedene Typen permanenter hydroelastischer Oberflächenwellen: *doppelt periodische Wellen* die periodisch in zwei verschiedene horizontale Richtungen sind, *zweidimensionale solitäre Wellen*, welche entlang der horizontalen Achse lokalisiert sind und *periodisch modulierte solitäre Wellen*, welche entlang einer horizontalen Richtung lokalisiert und entlang der senkrechten periodisch sind. All unsere Existenzbeweise basieren auf einer Formulierung des hydrodynamischen Problems als räumliches hamiltonsches System. Für die Existenztheorie doppelt periodischer Wellen nutzen wir eine resonante Version des Lyapunovschen Zentrumssatzes, welche in dieser Arbeit bewiesen wird. Die Existenz zweidimensionaler solitärer Wellen wird mithilfe einer Zentrumsmannigfaltigkeitsreduktion bewiesen, die das Problem auf ein endlichdimensionales hamiltonsches System reduziert, welches homokline Lösungen besitzt. Im letzten Teil beweisen wir die Existenz periodisch modulierter solitärer Wellen, welche aus einer zweidimensionalen solitären Welle in einer *dimensionsbrechenden* Bifurkation enstehen, indem wir gut erforschte Techniken aus der Existenztheorie von Wasserwellen verwenden.

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1 Introduction

1.1 Background

In this dissertation we consider two- and three-dimensional irrotational, incompressible flow of a perfect fluid of unit density under an elastic sheet modelled using the Cosserat theory of hyperelastic shells (Plotnikov & Toland [49]). The fluid is bounded below by a rigid horizontal bottom $\{y = -h\}$ and above by a free surface $\{y = \eta(x_1, x_3, t)\}$, where h is the depth of the water in its undisturbed state. In the two-dimensional case η does not depend upon x_3. The equations of motion, as given by Guyenne & Parau [29] are Laplace's equation

$$\phi_{x_1x_1} + \phi_{yy} + \phi_{x_3x_3} = 0, \qquad -h < y < \eta \tag{1.1}$$

with boundary conditions

$$\phi_y = 0, \qquad y = -h, \tag{1.2}$$

$$\phi_y = \eta_t + \phi_{x_1}\eta_{x_1} + \phi_{x_3}\eta_{x_3}, \qquad y = \eta, \tag{1.3}$$

$$\phi_t + \frac{1}{2}(\phi_{x_1}^2 + \phi_y^2 + \phi_{x_3}^2) + g\eta + DU(\eta) = 0, \qquad y = \eta, \tag{1.4}$$

where

$$\begin{aligned} U(\eta) = 2\Bigg\{ \frac{1}{\sqrt{Q}}\Bigg[\partial_{x_1}\left(\frac{1+\eta_{x_3}^2}{\sqrt{Q}}P_{x_1}\right) - \partial_{x_1}\left(\frac{\eta_{x_1}\eta_{x_3}}{\sqrt{Q}}P_{x_3}\right) - \partial_{x_3}\left(\frac{\eta_{x_1}\eta_{x_3}}{\sqrt{Q}}P_{x_1}\right) + \partial_{x_3}\left(\frac{1+\eta_{x_1}^2}{\sqrt{Q}}P_{x_3}\right)\Bigg] \\ + 2P^3 - 2KP\Bigg\}, \end{aligned}$$

and

$$\begin{aligned} Q &= 1 + \eta_{x_1}^2 + \eta_{x_3}^2, \\ P &= \frac{1}{2Q^{\frac{3}{2}}}\Big[(1+\eta_{x_3}^2)\eta_{x_1x_1} - 2\eta_{x_1x_3}\eta_{x_1}\eta_{x_3} + (1+\eta_{x_1}^2)\eta_{x_3x_3}\Big], \\ K &= \frac{1}{Q^2}(\eta_{x_1x_1}\eta_{x_3x_3} - \eta_{x_1x_3}^2). \end{aligned}$$

Here ϕ is an Eulerian velocity potential, g is the gravitational constant and D is the flexural rigidity of the elastic sheet; deep water ($h = \infty$) is included as a special case. For a discussion of the derivation of this and alternative models we refer to Toland [53, 54] and Plotnikov & Toland [49, 50, 51]. *Travelling waves* are solutions of (1.1)–(1.4) of the form

$$\eta(x_1,t)=\eta(x_1-ct), \qquad \phi(x_1,y,x_3,t)=\phi(x_1-ct,y,x_3);$$

they correspond to waves moving in the x_1-direction with constant speed c and without change of shape. The equations for travelling waves are obtained by replacing ∂_t with $-c\partial_{x_1}$ in equations (1.1)–(1.4). In this dissertation we present existence theories for three distinct types of travelling waves: *doubly periodic waves* which are periodic in two distinct horizontal directions (Chapter 2); two-dimensional *line solitary waves* which are localised in the x_1-direction (Chapter 3); and *periodically modulated solitary waves* which are localised in the x_1-direction and periodic in the x_3-direction (Chapter 4). The material in Chapters 2 and 3 has been published by Ahmad, Groves & Nilsson [3] and Ahmad & Groves [2] respectively (see also Ahmad [1] for a preliminary version of Chapter 3).

The starting point of all three existence theories is the fact that the travelling-wave version of equations (1.1)–(1.4) follows from the formal variational principle

$$\delta\int\left\{\int_{-h}^{\eta}\left[-c\phi_{x_1}+\tfrac{1}{2}(\phi_{x_1}^2+\phi_y^2+\phi_{x_3}^2)\right]\mathrm{d}y+\tfrac{1}{2}g\eta^2+2DP^2\sqrt{Q}\right\}\mathrm{d}x_1\,\mathrm{d}x_3=0, \qquad (1.5)$$

in which the variations are taken over η and ϕ (a modified version of the classical variational principle for water waves introduced by Luke [41]); this observation is confirmed by the calculation

$$\begin{aligned}
&\delta\int\left\{\int_{-h}^{\eta}\left[-c\phi_{x_1}+\tfrac{1}{2}(\phi_{x_1}^2+\phi_y^2+\phi_{x_3}^2)\right]\mathrm{d}y+\tfrac{1}{2}g\eta^2+2DP^2\sqrt{Q}\right\}\mathrm{d}x_1\,\mathrm{d}x_3\\
&=\int\bigg\{\int_{-h}^{\eta}-c\dot{\phi}_{x_1}\,\mathrm{d}y-c\phi_{x_1}\Big|_{y=\eta}\dot{\eta}+\int_{-h}^{\eta}\left[\phi_{x_1}\dot{\phi}_{x_1}+\phi_y\dot{\phi}_y+\phi_{x_3}\dot{\phi}_{x_3}\right]\mathrm{d}y\\
&\qquad+\tfrac{1}{2}[\phi_{x_1}+\phi_y+\phi_{x_3}]\Big|_{y=\eta}\dot{\eta}+g\eta\dot{\eta}+2D\frac{\partial}{\partial\eta_{x_3}}\left[P^2\sqrt{Q}\right]\dot{\eta}_{x_3}+2D\frac{\partial}{\partial\eta_{x_1}}\left[P^2\sqrt{Q}\right]\dot{\eta}_{x_1}\\
&\qquad+2D\frac{\partial}{\partial\eta_{x_3x_3}}\left[P^2\sqrt{Q}\right]\dot{\eta}_{x_3x_3}+2D\frac{\partial}{\partial\eta_{x_1x_1}}\left[P^2\sqrt{Q}\right]\dot{\eta}_{x_1x_1}\bigg\}\mathrm{d}x_1\,\mathrm{d}x_3\\
&=\int\bigg\{\int_{-h}^{\eta}\left[-c\dot{\phi}_{x_1}-(\phi_{x_1x_1}+\phi_{yy}+\phi_{x_3x_3})\dot{\phi}\right]\mathrm{d}y+\phi_n(1+\eta_{x_1}^2+\eta_{x_3}^2)^{1/2}\dot{\phi}\Big|_{y=\eta}-\phi_y\dot{\phi}\Big|_{y=-h}\\
&\qquad+\bigg(-2D\left[\frac{\partial}{\partial\eta_{x_3}}\left(P^2\sqrt{Q}\right)\right]_{x_3}+2D\left[\frac{\partial}{\partial\eta_{x_3x_3}}\left(P^2\sqrt{Q}\right)\right]_{x_3x_3}\\
&\qquad\quad+\left[-c\phi_{x_1}+\tfrac{1}{2}(\phi_{x_1}^2+\phi_y^2+\phi_{x_3}^2)\right]\Big|_{y=\eta}+g\eta-2D\left[\frac{\partial}{\partial\eta_{x_1}}\left(P^2\sqrt{Q}\right)\right]_{x_1}\\
&\qquad\quad+2D\left[\frac{\partial}{\partial\eta_{x_1x_1}}\left(P^2\sqrt{Q}\right)\right]_{x_1x_1}\bigg)\dot{\eta}\bigg\}\mathrm{d}x_1\,\mathrm{d}x_3
\end{aligned}$$

$$
\begin{aligned}
&= \int \Bigg\{ \int_{-h}^{\eta} \Big[-c\dot{\phi}_{x_1} - (\phi_{x_1x_1} + \phi_{yy} + \phi_{x_3x_3})\dot{\phi}\Big]\,\mathrm{d}y + [-\eta_{x_1}\phi_{x_1} + \phi_y - \eta_{x_3}\phi_{x_3}]\dot{\phi}\Big|_{y=\eta} \\
&\qquad - \phi_y\dot{\phi}\Big|_{y=-h} + \Bigg(\Big[-c\phi_{x_1} + \tfrac{1}{2}(\phi_{x_1}^2 + \phi_y^2 + \phi_{x_3}^2)\Big]\Big|_{y=\eta} + g\eta \\
&\qquad\qquad - 2D\left[\frac{\partial}{\partial \eta_{x_3}}\big(P^2\sqrt{Q}\big)\right]_{x_3} + 2D\left[\frac{\partial}{\partial \eta_{x_3x_3}}\big(P^2\sqrt{Q}\big)\right]_{x_3x_3} \\
&\qquad\qquad - 2D\left[\frac{\partial}{\partial \eta_{x_1}}\big(P^2\sqrt{Q}\big)\right]_{x_1} + 2D\left[\frac{\partial}{\partial \eta_{x_1x_1}}\big(P^2\sqrt{Q}\big)\right]_{x_1x_1}\Bigg)\dot{\eta}\Bigg\}\mathrm{d}x_1\mathrm{d}x_3 \\
&= \int \Bigg\{ -\int_{-h}^{\eta} [\phi_{x_1x_1} + \phi_{yy} + \phi_{x_3x_3}]\dot{\phi}\,\mathrm{d}y - \phi_y\dot{\phi}\Big|_{y=-h} + [-c\eta_{x_1} - \eta_{x_1}\phi_{x_1} + \phi_y - \eta_{x_3}\phi_{x_3}]\dot{\phi}\Big|_{y=\eta} \\
&\qquad + \Big(\Big[-c\phi_{x_1} + \tfrac{1}{2}(\phi_{x_1}^2 + \phi_y^2 + \phi_{x_3}^2)\Big]\Big|_{y=\eta} + g\eta + DU(\eta)\Big)\dot{\eta}\Bigg\}\,\mathrm{d}x_1\,\mathrm{d}x_3,
\end{aligned}
$$

where the formal first variations of η and ϕ are denoted by respectively $\dot{\eta}$ and $\dot{\phi}$ and we have used integration by parts and Green's integral formula.

We proceed by interpreting the variational functional in (1.5) as an action functional and using an appropriate Legendre transform, which is actually of higher order due to the presence of second-order derivatives in the density (see Lanczos [40, Appendix I]), to formulate the hydrodynamic problem as an (infinite-dimensional) Hamiltonian system in which a distinguished horizontal coordinate plays the role of time. Methods from Hamiltonian-systems theory are then applied to find interesting solutions. This idea is known as *spatial dynamics* and has been successfully employed in the classical gravity-capillary water-wave problem (see Groves & Haragus [20] and Groves [19] for a review). In this thesis we apply it in the hydrodynamic setting in three different ways.

Line solitary waves have been computed numerically by Guyenne & Parau [29] and Milewski, Vanden-Broeck & Wang [46], together with more exotic non-symmetric and multi-pulse line solitary waves (Gao, Vanden-Broeck & Wang [17]). These papers all treat deep water ($h = \infty$); the finite-depth case was studied by Gao, Vanden-Broeck & Wang [16]. At the time of writing there are no numerical computations of doubly periodic or periodically modulated solitary hydroelastic waves, but *fully localised three-dimensional solitary waves* (which are localised in all horizontal directions) have been thoroughly investigated numerically (see Parau & Vanden-Broeck [48] for a survey of these results).

We now give a summary of the nomenclature for Hamiltonian systems used in this thesis followed by an outline of our existence theories for the three types of hydroelastic waves.

Hamiltonian formalism

Let X, Z be real Hilbert spaces, where X is continuously and densely embedded in Z, and Z is equipped with an additional continuous inner product $\langle \cdot\,,\cdot\rangle$ which does not necessarily induce its strong topology (a typical example would be $X = H^2(\mathbb{R})$, $Z = H^1(\mathbb{R})$ and $\langle \cdot\,,\cdot\rangle$ is the $L^2(\mathbb{R})$-inner product). Let $\Lambda_1 \times U$ be a neighbourhood of the origin in $\mathbb{R} \times X$. We regard U as a manifold domain of Z by extending elements of the tangent space $TX|_v \cong X^* \cong X$ of X at the point $v \in U$ to elements of the tangent space $TZ|_v \cong Z^* \cong Z$ of Z at this point. The derivative $\mathrm{d}F^{\mu_1}[v] \in X^*$ of a smooth real-valued function $F^{\mu_1}(v)$ of $(\mu_1, v) \in \Lambda_1 \times U$ has a unique extension $\widetilde{\mathrm{d}F^{\mu_1}}[v] \in Z^*$. We use the same notation for smooth functions $F^{\mu_1}(v)$ of (μ_1, v) with values in X, so that $\widetilde{\mathrm{d}F^{\mu_1}}[v] \in \mathcal{L}(Z)$. Because the inner product $\langle \cdot\,,\cdot\rangle$ does not necessarily induce the strong topology for Z, the adjoint of $\widetilde{\mathrm{d}F^{\mu_1}}[v]$ with respect to $\langle \cdot\,,\cdot\rangle$ does not automatically exist; we therefore assume, whenever needed, that there exists an adjoint operator $\widetilde{\mathrm{d}F^{\mu_1}}[v]^* \in \mathcal{L}(Z)$ such that

$$\langle \widetilde{\mathrm{d}F^{\mu_1}}[v]^*(v_1), v_2\rangle = \langle v_1, \widetilde{\mathrm{d}F^{\mu_1}}[v](v_2)\rangle$$

for all $(\mu_1, v) \in \Lambda_1 \times U$ and $v_1,\ v_2 \in Z$. Both $\widetilde{\mathrm{d}F^{\mu_1}}[v]$ and $\widetilde{\mathrm{d}F^{\mu_1}}[v]^*$ are assumed to depend smoothly upon $(\mu_1, v) \in \Lambda_1 \times U$.

Using this framework we make the following definitions and assumptions.

(i) *A (parameter-dependent) k-form on U* is an alternating, bounded, k-linear mapping $Z^k \to \mathbb{R}$ which depends smoothly upon $(\mu_1, v) \in \Lambda_1 \times U$. An *exact symplectic 2-form* $\Omega^{\mu_1}|_v$ on U is a 2-form given by

$$\Omega^{\mu_1}|_v(v_1, v_2) = \langle J^{\mu_1}(v)v_1, v_2\rangle,$$

where $J^{\mu_1}(v)$ is an invertible, skew-symmetric linear mapping in $\mathcal{L}(Z)$ which depends smoothly upon $(\mu_1, v) \in \Lambda_1 \times U$.

We assume that $\Omega^{\mu_1}|_v$ is the exterior derivative of a 1-form $\omega^{\mu_1}|_v$ given by

$$\omega^{\mu_1}|_v(w) = \langle \alpha^{\mu_1}(v), w\rangle,$$

where $\alpha^{\mu_1}(v)$ is an element of Z which depends smoothly upon $(\mu_1, v) \in \Lambda_1 \times U$; in other words

$$\Omega^{\mu_1}|_v(v_1, v_2) = \langle \widetilde{\mathrm{d}\alpha^{\mu_1}}[v](v_1), v_2\rangle - \langle v_1, \widetilde{\mathrm{d}\alpha^{\mu_1}}[v](v_2)\rangle,$$

so that

$$J^{\mu_1}(v) = \widetilde{\mathrm{d}\alpha^{\mu_1}}[v] - \widetilde{\mathrm{d}\alpha^{\mu_1}}[v]^*.$$

This extra assumption is due to the fact that we allow for a non-constant symplectic structure. (In the special case that J^{μ_1} is constant one can always set $\alpha^{\mu_1} = \frac{1}{2}J^{\mu_1}$.)

(ii) *A (parameter-dependent) Hamiltonian on* U is a smooth real-valued function $H^{\mu_1}(v)$ of $(\mu_1, v) \in \Lambda_1 \times U$ which satisfies $H^{\mu_1}(0) = 0$ and $\mathrm{d}H^{\mu_1}[0] = 0$ for all $\mu_1 \in \Lambda_1$. We assume that its gradient, that is the element $\nabla H^{\mu_1}(v)$ of Z with

$$\widetilde{\mathrm{d}H}^{\mu_1}[v](w) = \langle \nabla H^{\mu_1}(v), w \rangle$$

for all $w \in Z$, exists for all v in a dense subset $\mathcal{D}_\mathrm{H}$ of $\Lambda_1 \times U$ and extends to a smooth function of $(\mu_1, v) \in \Lambda_1 \times U$.

(iii) *The Hamiltonian vector field* $v_\mathrm{H}^{\mu_1}$ of a *Hamiltonian system* $(Z, \Omega^{\mu_1}, H^{\mu_1})$, where Ω^{μ_1} is an exact symplectic 2-form and H^{μ_1} is a Hamiltonian on U, is given by

$$v_\mathrm{H}^{\mu_1}(v) = J^{\mu_1}(v)^{-1} \nabla H^{\mu_1}(v), \qquad (\mu_1, v) \in \mathcal{D}_\mathrm{H};$$

it yields the unique element $v_\mathrm{H}^{\mu_1}(v)$ of Z such that

$$\Omega^{\mu_1}|_v(v_\mathrm{H}^{\mu_1}(v), w) = \widetilde{\mathrm{d}H}^{\mu_1}[v](w)$$

for all $w \in Z$. *Hamilton's equations*

$$v_t = v_\mathrm{H}^{\mu_1}(v), \qquad (\mu_1, v) \in \mathcal{D}_\mathrm{H},$$

determine the orbits generated by the Hamiltonian vector field.

1.2 A resonant Lyapunov centre theorem and doubly periodic travelling hydroelastic waves

Resonant Hamiltonian systems

A linear dynamical system

$$v_t = Lv$$

for which $L \in \mathbb{R}^{2n \times 2n}$ has a pair of simple, purely imaginary eigenvalues $\pm \mathrm{i}\omega$ has a periodic orbit with frequency ω. The classical *Lyapunov centre theorem* asserts that a (smooth) nonlinear perturbation

$$v_t = Lv + N(v) \tag{1.6}$$

of this dynamical system has a family of small-amplitude periodic solutions with frequency near ω provided that (i) it is Hamiltonian or reversible, and (ii) the non-resonance condition that $\mathrm{i}n\omega$ is not an eigenvalue for any integer $n \neq \pm 1$ is satisfied (see Kielhöfer [36, §§I.11.1–I.11.2]). The theorem can be extended to infinite-dimensional, quasilinear systems under additional hypotheses on the decay rate of the resolvent of L along the imaginary axis (Devaney [13]), and furthermore the condition that $0 \notin \sigma(L)$ can also be replaced by a compatibility condition on the range of L (see Iooss [33]).

In Chapter 2 we consider a parameter-dependent evolutionary equation of the form

$$v_t = L^{\mu_1} v + N^{\mu_1}(v)$$

which is both reversible and Hamiltonian; it may be finite- or infinite-dimensional (and quasilinear). The linear operator L^{μ_1} is supposed to have two pairs $\pm \mathrm{i}\kappa_1^{\mu_1}$, $\pm \mathrm{i}\kappa_2^{\mu_1}$ of simple purely imaginary eigenvalues which depend smoothly upon μ_1 and collide with 'non-zero speed' at $\mu_1 = 0$, that is

$$\kappa_1^0 = \kappa_2^0, \qquad \frac{\mathrm{d}}{\mathrm{d}\mu_1}(\kappa_1^{\mu_1} - \kappa_2^{\mu_1})\Big|_{\mu_1=0} \neq 0.$$

The collision is semisimple, that is at criticality the eigenvalues $\pm \mathrm{i}\kappa$, where $\kappa = \kappa_1^0 = \kappa_2^0$, are geometrically and algebraically double. We assume the non-resonance condition $\mathrm{i}n\kappa \notin \sigma(L^0)$ for $n = \pm 2, \pm 3, \ldots$ but allow the origin to be a 'trivial' eigenvalue arising from a translational symmetry or, in an infinite-dimensional setting, to lie in the continuous spectrum of L provided a compatibility condition on its range is satisfied. The result is a two-parameter family $\{v(t_1,t_2), \mu_1(t_1,t_2)\}_{0 \leq t_1, t_2 < \varepsilon}$ of $2\pi/(\kappa + \mu_2(t_1,t_2))$-periodic reversible solutions, where $\mu_1(t_1,t_2), \mu_2(t_1,t_2) \to 0$ along with the amplitude of the solutions as $(t_1,t_2) \to (0,0)$.

A classical Hamiltonian system with n degrees of freedom has the form

$$\dot{v} = J\nabla H(v), \qquad J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix},$$

where the Hamiltonian H is a smooth real-valued function of $v \in \mathbb{R}^{2n}$ and $0, I$ denote the $n \times n$ zero and identity matrices; in the above notation $L = J\nabla H_2$, where $H_2(v) = \frac{1}{2}\mathrm{d}^2 H[0](v,v)$. Examining a two-degree-of-freedom Hamiltonian system whose linearisation has two semisimple eigenvalues $\pm\mathrm{i}\kappa$ one finds that $H_2(q_1,q_2,p_1,p_2) = \frac{1}{2}\kappa(q_1^2 + p_1^2) \pm \frac{1}{2}\kappa(q_2^2 + p_2^2)$, the two cases of which are referred to as a $1:1$ and $1:-1$ resonance respectively. This terminology is also applied to higher-order Hamiltonian systems with no other eigenvalue resonances by restricting to the eigenspaces corresponding to $\pm\mathrm{i}\kappa$. Periodic solutions of two-degree-of-freedom Hamiltonian systems with semisimple $1:1$ and $1:-1$ resonances were studied by Kummer [38, 39], while the corresponding non-semisimple 'Hamiltonian-Hopf' resonance (in which $\pm\mathrm{i}\kappa$ are geometrically simple and algebraically double eigenvalues) was studied by van der Meer [55].

The dynamical system (1.6) is *reversible* if there exists an involution R (the 'reverser') which anticommutes with L and N. A reversible system has the property that Ru is a solution whenever u is a solution; a solution u is termed *reversible* or *symmetric* if it is invariant under R. Reversible Hamiltonian systems have the property that either $H(Rv) = H(v)$ or $H(Rv) = -H(v)$; these cases are referred to as 'antisymplectic' and 'symplectic' respectively. Small-amplitude periodic solutions of symplectically reversible, n-degree-of-freedom Hamiltonian systems which exhibit a semisimple $1 : -1$ resonance were studied by Alomair & Montaldi [4] (see also Buzzi & Lamb [12]) using Lyapunov-Schmidt reduction.

In Chapter 2 we consider parameter-dependent, antisymplectically reversible Hamiltonian systems of the form

$$v_t = J^{\mu_1}(v)\nabla H^{\mu_1}(v) \tag{1.7}$$

which exhibit a semisimple $1 : 1$ or $1 : -1$ resonance in the eigenvalues $\pm\mathrm{i}\kappa$ when $\mu_1 = 0$. The system may be finite- or infinite-dimensional, and $J^{\mu_1}(v)$ is an invertible linear operator which is skew-symmetric with respect to a suitable inner product $\langle\cdot\,,\cdot\rangle$ and, as the notation indicates, is not necessarily constant (see below for a precise statement). We look for periodic solutions of (1.7) with frequency near κ by writing

$$v(t) = u(\tau), \qquad \tau = (\kappa + \mu_2)t$$

and seeking 2π-periodic solutions of the transformed equation. For this purpose we use variational Lyapunov-Schmidt reduction, seeking critical points of the functional

$$S(u, \mu_1, \mu_2) = \frac{1}{2\pi}\int_0^{2\pi} \Big\{ -(\kappa + \mu_2)\langle\alpha^{\mu_1}(u), u_\tau\rangle - H^{\mu_1}(u)\Big\}\,\mathrm{d}\tau, \tag{1.8}$$

where $\alpha^{\mu_1}(v)$ is an anti-derivative of $J^{\mu_1}(v)$ (see above), in a suitable 'loop space'. In the special case that J^{μ_1} is constant one can set $\alpha^{\mu_1} = \frac{1}{2}J^{\mu_1}$ and S reduces to the simpler function used by Buzzi & Lamb [12, 10, eqn. (12)]. A precise statement of our theorem is given below; the proof is presented in Section 2.1.

A resonant Lyapunov centre theorem

Let X, Z be real Hilbert spaces, where X is continuously and densely embedded in Z, and consider the autonomous evolutionary equation

$$v_t = L^{\mu_1}v + N^{\mu_1}(v), \tag{1.9}$$

where $(\mu_1, v) \mapsto N^{\mu_1}(v)$ is a smooth mapping from a neighbourhood $\Lambda_1 \times U$ of the origin in $\mathbb{R}\times X$ into Z which satisfies $N^{\mu_1}(0) = 0$ and $\mathrm{d}N^{\mu_1}[0] = 0$ for all $\mu_1 \in \Lambda_1$, and $L^{\mu_1} : X \subseteq Z \to Z$ is a closed linear operator which depends smoothly upon μ_1 in the topology of $\mathcal{L}(X, Z)$. We study equation (1.9) under the following hypotheses.

(H1) Equation (1.9) represents Hamilton's equations

$$v_t = v_{\mathrm{H}}^{\mu_1}(v)$$

for a Hamiltonian system $(Z, \Omega^{\mu_1}, H^{\mu_1})$ with $\mathcal{D}_{\mathrm{H}} = \Lambda_1 \times U$ (in the nomenclature in Section 1.1), so that

$$L^{\mu_1}(v) + N^{\mu_1}(v) = J^{\mu_1}(v)^{-1}\nabla H^{\mu_1}(v)$$

and in particular

$$L^{\mu_1}v = J^{\mu_1}(0)^{-1}\nabla H_2^{\mu_1}(v),$$

where $H_2^{\mu_1}(v) = \frac{1}{2}\mathrm{d}^2 H^{\mu_1}[0](v,v)$ is the part of H^{μ_1} which is homogeneous of degree 2 in v.

(H2) Equation (1.9) is reversible: both L^{μ_1} and N^{μ_1} anticommute with an involution $R \in \mathcal{L}(X) \cap \mathcal{L}(Z)$. This reverser R satisfies

$$H^{\mu_1}(Rv) = H^{\mu_1}(v), \quad R^*\alpha^{\mu_1}(Rv) = -\alpha^{\mu_1}(v), \quad R^*J^{\mu_1}(Rv)R = -J^{\mu_1}(v)$$

for all $(\mu_1, v) \in \Lambda_1 \times U$.

There are also spectral hypotheses on L^{μ_1}.

(H3) The linear operator L^{μ_1} has two pairs $\pm\mathrm{i}\kappa_1^{\mu_1}$, $\pm\mathrm{i}\kappa_2^{\mu_1}$ of purely imaginary eigenvalues with linearly independent eigenvectors $e_1^{\mu_1}$, $\bar{e}_1^{\mu_1}$, $e_2^{\mu_1}$, $\bar{e}_2^{\mu_1}$, all of which depend smoothly upon μ_1. Furthermore $\bar{e}_1^0 = Re_1^0$, $\bar{e}_2^0 = Re_2^0$ and

$$\kappa_1^0 = \kappa_2^0, \qquad \frac{\mathrm{d}}{\mathrm{d}\mu_1}(\kappa_1^{\mu_1} - \kappa_2^{\mu_1})\Big|_{\mu_1=0} \neq 0.$$

For notational simplicity we henceforth abbreviate L^0, e_1^0 and e_2^0 to respectively L, e_1 and e_2, and define $\kappa := \kappa_1^0 = \kappa_2^0$.

(H4) The origin is one of

(i) a point of the resolvent set of L,

(ii) a point of the continuous spectrum of L,

(iii) a geometrically simple, algebraically double eigenvalue of L with eigenvector f_1 and generalized eigenvector f_2 (possibly embedded in continuous spectrum), where $Rf_1 = -f_1$ and $Rf_2 = f_2$.

Here (ii) and (iii) entail further spectral hypotheses (see (H7) and (H8) below).

(H5) The imaginary number $\mathrm{i}k\kappa$ belongs to the resolvent set of L for each $k \in \mathbb{Z}\backslash\{0,-1,1\}$.

(H6) The linear operator L satisfies

$$\left\|(\mathrm{i}k\kappa I - L)^{-1}\right\|_{Z\to Z} \lesssim \frac{1}{|k|}, \qquad \left\|(\mathrm{i}k\kappa I - L)^{-1}\right\|_{Z\to X} \lesssim 1$$

for all $k \in \mathbb{Z}\backslash\{0,-1,1\}$.

(H7) The zero eigenvalue (if present) is 'trivial' in the following sense: writing $u \in U$ as $u = qf_1 + w$ with $w \in \{f_1\}^\perp$, one finds that $J^{\mu_1}(u)$ and $H^{\mu_1}(u)$ do not depend upon q.

(H8) Let
$$\mathcal{U} = \{u \in H^1_{\mathrm{per}}(\mathbb{R}, Z) \cap L^2_{\mathrm{per}}(\mathbb{R}, X) : u(\tau) \in U \text{ for all } \tau \in \mathbb{R}\}$$
and
$$N^\star(u, \mu_1, \mu_2) = \frac{1}{\sqrt{2\pi}} \int_0^{2\pi} \Big(J^{\mu_1}(u)\Big((\kappa + \mu_2) J^{\mu_1}(u) u_\tau - L^{\mu_1} u - N^{\mu_1}(u)\Big) + J^0(0) L u \Big) \,\mathrm{d}\tau.$$
Suppose that the equation
$$L u^\dagger = J^0(0)^{-1}(I - \Pi_0) N^\star(u, \mu_1, \mu_2)$$
has a unique solution $u^\dagger \in (I - \Pi_0)X$ which depends smoothly upon $u \in \mathcal{U}$, $\mu_1 \in \Lambda_1$ and $\mu_2 \in \Lambda_2$, where Π_0 is the orthogonal projection of X onto $\mathrm{span}\{f_1, f_2\}$.

Remark 1.1.

(i) Hypothesis (H8) is meaningful only if the origin lies in the continuous spectrum of L or is an eigenvalue embedded in continuous spectrum, since it is automatically satisfied if the origin lies in the resolvent set of L or is an isolated eigenvalue of L.

(ii) If $J^{\mu_1}(u)$ is constant and $\Pi_0 = 0$, then
$$N^\star(u, \mu_1, \mu_2) = -\frac{1}{\sqrt{2\pi}} \int_0^{2\pi} N^{\mu_1}(u) \,\mathrm{d}\tau,$$
so that hypothesis (H8) reduces to the condition used by Iooss [33].

Theorem 1.2. *Under hypotheses (H1)–(H8) there exist $\varepsilon > 0$ and a smooth, two-parameter branch $\{v(\chi_1, \chi_2), \mu_1(\chi_1, \chi_2)\}_{0 \leq \chi_1, \chi_2 < \varepsilon}$ of $2\pi/(\kappa + \mu_2(\chi_1, \chi_2))$-periodic reversible solutions to equation (1.9) in $H^1_{\mathrm{loc}}(\mathbb{R}, Z) \cap L^2_{\mathrm{loc}}(\mathbb{R}, X)$. The rescaled function $u(\tau) = v(t)$, $\tau = (\kappa + \mu_2)t$ satisfies $\|u(\chi_1, \chi_2)\|_{L^2_{\mathrm{per}}(\mathbb{R}, Z)} \to 0$, while $\mu_1(\chi_1, \chi_2), \mu_2(\chi_1, \chi_2) \to 0$ as $(\chi_1, \chi_2) \to (0, 0)$.*

Outline proof

Here we sketch the proof of Theorem 1.2 under the hypotheses (H1)–(H8); a complete proof is given in Section 2.1. Our goal is to prove the existence of periodic solutions, with frequency near κ, of the equation
$$v_t = L^{\mu_1} v + N^{\mu_1}(v).$$
We proceed as in the proof of the classical Lyapunov-centre theorem by rescaling in time, substituting $v(t) = u(\tau)$, where $\tau = (\kappa + \mu_2)t$. The task is then to find 2π-periodic solutions
$$u(\tau) = \frac{1}{\sqrt{2\pi}} \sum_{k \in \mathbb{Z}} u_k \mathrm{e}^{\mathrm{i}k\tau}, \qquad u_k \in Z,$$

of

$$(\kappa+\mu_2)u_\tau = L^{\mu_1}u + N^{\mu_1}(u).$$

According to hypothesis (H1) this equation can be reformulated as

$$F(u,\mu_1,\mu_2)=0, \tag{1.10}$$

where

$$F(u,\mu_1,\mu_2) = (\kappa+\mu_2)J^{\mu_1}(u)u_\tau - \nabla H^{\mu_1}(u).$$

In addition to having a Hamiltonian structure, equation (1.10) can also be viewed as the Euler-Lagrange equation of an action integral S: one finds that

$$\mathrm{d}_1 S[u,\mu_1,\mu_2](v) = \frac{1}{2\pi}\int_0^{2\pi} \langle F(u,\mu_1,\mu_2), v\rangle \,\mathrm{d}\tau,$$

where

$$S(u,\mu_1,\mu_2) = \frac{1}{2\pi}\int_0^{2\pi} \Big\{ -(\kappa+\mu_2)\langle \alpha^{\mu_1}(u), u_\tau\rangle - H^{\mu_1}(u)\Big\}\,\mathrm{d}\tau.$$

The next step is to perform a Lyapunov-Schmidt reduction. This procedure is not entirely straightforward due to the several different possibilities for the spectrum of L at the origin given in hypothesis (H4). The idea is to find an orthogonal decomposition $L^2_{\mathrm{per}}(\mathbb{R},Z) = \mathcal{W}_1 \oplus \mathcal{W}_2$ and write equation (1.10) as

$$\widetilde{\Pi}_{\mathcal{W}_1} F(u,\mu_1,\mu_2) = 0, \tag{1.11}$$

$$(I-\widetilde{\Pi}_{\mathcal{W}_1}) F(u,\mu_1,\mu_2) = 0, \tag{1.12}$$

where $\widetilde{\Pi}_{\mathcal{W}_1}$ is the orthogonal projection onto $\mathcal{W}_1$ along $\mathcal{W}_2$, in such a way that (1.12) can be solved for $u_{\mathcal{W}_2} := (I-\widetilde{\Pi}_{\mathcal{W}_1})u$ in terms of $u_{\mathcal{W}_1} := \widetilde{\Pi}_{\mathcal{W}_1}u$, μ_1 and μ_2. A rather natural initial choice for $\mathcal{W}_1$ is

$$\mathcal{W}_1 = \{qf_1 + pf_2 + Ae_1\mathrm{e}^{\mathrm{i}\tau} + Be_2\mathrm{e}^{\mathrm{i}\tau} + \bar{A}\bar{e}_1\mathrm{e}^{-\mathrm{i}\tau} + \bar{B}\bar{e}_2\mathrm{e}^{-\mathrm{i}\tau},\ q,p\in\mathbb{R},\ A,B\in\mathbb{C}\},$$

where $p=q=0$ in the case when zero is not an eigenvalue. However, one finds that with this choice equation (1.12) cannot be solved directly with the implicit-function theorem due to the possibility that zero is a point of the continuous spectrum of L, or is an eigenvalue of L which is embedded in the continuous spectrum. We therefore opt to perform the reduction in two steps.

By letting

$$\mathcal{W}_1 = \{u_0 + Ae_1\mathrm{e}^{\mathrm{i}\tau} + Be_2\mathrm{e}^{\mathrm{i}\tau} + \bar{A}\bar{e}_1\mathrm{e}^{-\mathrm{i}\tau} + \bar{B}\bar{e}_2\mathrm{e}^{-\mathrm{i}\tau},\ A,B\in\mathbb{C},\ u_0\in Z\},$$

we find that we can solve (1.12) for $u_{\mathcal{W}_2}$ as a function of $u_{\mathcal{W}_1}$, μ_1 and μ_2 using hypotheses (H5), (H6). This procedure yields the reduced equation

$$\widetilde{\Pi}_{\mathcal{W}_1} F(u_{\mathcal{W}_1} + u_{\mathcal{W}_2}(u_{\mathcal{W}_1},\mu_1,\mu_2),\mu_1,\mu_2) = 0 \tag{1.13}$$

for $u_{\mathcal{W}_1}$, which is still infinite-dimensional because $Z \subset \mathcal{W}_1$. Equation (1.13) can now be reduced further by introducing the orthogonal decomposition $\mathcal{W}_1 = \mathcal{W}_{1,1} \oplus \mathcal{W}_{1,2}$, where

$$\mathcal{W}_{1,1} = \{qf_1 + pf_2 + Ae_1\mathrm{e}^{\mathrm{i}\tau} + Be_2\mathrm{e}^{\mathrm{i}\tau} + \bar{A}\bar{e}_1\mathrm{e}^{-\mathrm{i}\tau} + \bar{B}\bar{e}_2\mathrm{e}^{-\mathrm{i}\tau},\ q,p \in \mathbb{R},\ A,B \in \mathbb{C}\},$$

and writing it as

$$\widetilde{\Pi}F(u_1 + u_2 + u_{\mathcal{W}_2}(u_1 + u_2, \mu_1, \mu_2), \mu_1, \mu_2) = 0, \tag{1.14}$$

$$(I - \widetilde{\Pi})F(u_1 + u_2 + u_{\mathcal{W}_2}(u_1 + u_2, \mu_1, \mu_2), \mu_1, \mu_2) = 0, \tag{1.15}$$

where $\tilde{\Pi}$ is the orthogonal projection onto $\mathcal{W}_{1,1}$ along $\mathcal{W}_{1,2}$ and $u_1 = \widetilde{\Pi}u_{\mathcal{W}_1}$, $u_2 = (I - \widetilde{\Pi})u_{\mathcal{W}_1}$. Of course here we encounter the original issue, namely that (1.15) cannot be solved directly with the implicit-function theorem due to the possibility that zero is a point of the continuous spectrum of L, or is an eigenvalue of L which is embedded in the continuous spectrum. However, equation (1.15) can be rewritten as

$$Lu_2 = J^0(0)^{-1}(I - \Pi_0)N^\star(u_1 + u_2 + u_{\mathcal{W}_2}(u_1 + u_2, \mu_1, \mu_2), \mu_1, \mu_2),$$

and hypothesis (H8) allows us to solve it for u_2 as a function of u_1, μ_1 and μ_2. Substituting $u_2 = u_2(u_1, \mu_1, \mu_2)$ into equation (1.14) yields the final reduced equation for u_1, which we write as

$$f(u_1, \mu_1, \mu_2) = 0. \tag{1.16}$$

Equation (1.16) is a finite-dimensional problem, with variables $q, p, A, B, \bar{A}, \bar{B}$ and parameters μ_1, μ_2. The number of variables can be reduced further by exploiting the additional structure of the original equation (2.1). We carry out the Lyapunov-Schmidt reduction outlined above at a variational level, showing that (1.16) is the Euler-Lagrange equation of the reduced functional $s(u_1, \mu_1, \mu_2)$ obtained by substituting

$$u = u_1 + u_2(u_1, \mu_1, \mu_2) + u_{\mathcal{W}_2}(u_1 + u_2(u_1, \mu_1, \mu_2), \mu_1, \mu_2)$$

into $S(u, \mu_2, \mu_2)$. Furthermore, it follows from hypothesis (H2) and the autonomy of (2.1) that s depends upon A, B only through the quantities $|A|^2$, $|B|^2$, and by hypothesis (H7) that s does not depend upon q, so that $s(A, B, \bar{A}, \bar{B}, p, q, \mu_1, \mu_2) = \tilde{s}(|A|^2, |B|^2, p, \mu_1, \mu_2)$. The task of solving equation (1.10) is thus reduced to that of finding critical points of $\tilde{s}$, that is solving the equations

$$\partial_1\tilde{s} = 0, \tag{1.17}$$

$$\partial_2\tilde{s} = 0, \tag{1.18}$$

$$\partial_3\tilde{s} = 0. \tag{1.19}$$

Equations (1.17) and (1.18) can be solved for μ_1, μ_2 as functions of $|A|^2$, $|B|^2$ and p using the implicit-function theorem because

$$\det\begin{pmatrix}\partial_1\partial_4\tilde{s}(0,0,0,0,0) & \partial_1\partial_5\tilde{s}(0,0,0,0,0)\\ \partial_2\partial_4\tilde{s}(0,0,0,0,0) & \partial_2\partial_5\tilde{s}(0,0,0,0,0)\end{pmatrix} \simeq \frac{\mathrm{d}}{\mathrm{d}\mu_1}(\kappa_1^{\mu_1}-\kappa_2^{\mu_1})\Big|_{\mu_1=0} \neq 0$$

due to hypothesis (H3). The proof of Theorem 2.1 is completed by inserting $\mu_1 = \mu_1(|A|^2,|B|^2,p)$, $\mu_2 = \mu_1(|A|^2,|B|^2,p)$ into (1.19) and solving it for p as a function of $|A|^2$, $|B|^2$ with a final application of the implicit-function theorem.

Application to hydroelastic waves

Introducing dimensionless variables with unit length $(D/\rho g)^{1/4}$ and unit speed $(\rho/Dg^3)^{1/8}$, so that

$$(x_1,x_2,x_3) \to \left(\frac{D}{\rho g}\right)^{1/4}(x_1,x_2,x_3), \quad \eta \to \left(\frac{D}{\rho g}\right)^{1/4}\eta, \quad \phi \to \left(\frac{D^3 g}{\rho^3}\right)\phi,$$

we write equations (1.1)–(1.4) as

$$\phi_{x_1x_1}+\phi_{x_2x_2}+\phi_{x_3x_3}=0, \qquad -\tfrac{1}{\beta}<x_2<\eta(x_1,x_3), \tag{1.20}$$

$$\phi_{x_2}=0, \qquad x_2=-\tfrac{1}{\beta}, \tag{1.21}$$

$$\phi_{x_2}+\gamma\eta_{x_1}-\phi_{x_1}\eta_{x_1}-\phi_{x_3}\eta_{x_3}=0, \qquad x_2=\eta(x_1,x_3), \tag{1.22}$$

$$-\gamma\phi_{x_1}+\tfrac{1}{2}(\phi_{x_1}^2+\phi_{x_2}^2+\phi_{x_3}^2)+\eta+U(\eta)=0, \qquad x_2=\eta(x_1,x_3), \tag{1.23}$$

where

$$\beta=\left(\frac{D}{\rho g h^4}\right)^{1/4}\geq 0, \qquad \gamma=\left(\frac{c^8\rho}{Dg^3}\right)^{1/8}>0.$$

The dimensionless fluid domain is $\{-\frac{1}{\beta}<x_2<\eta(x_1,x_2)\}$, so that the limit $\beta\to 0$ corresponds to 'infinite depth'.

In Chapter 2 we consider waves which are periodic with periods p_1 and p_2 in two arbitrary horizontal directions x and z which form different angles θ_1, $\theta_2 \in [0,\pi)$ with the x_1-axis respectively, so that

$$x=\csc(\theta_2-\theta_1)(x_1\sin\theta_2-x_3\cos\theta_2), \qquad z=\csc(\theta_1-\theta_2)(x_1\sin\theta_1-x_3\cos\theta_1)$$

(see Figure 1.1). To this end we seek solutions of the governing equations of the form

$$\eta(x_1,x_3)=\tilde{\eta}(\tilde{x},\tilde{z}), \quad \phi(x_1,x_2,x_3)=\tilde{\phi}(\tilde{x},x_2,\tilde{z}),$$

where

$$\tilde{x}=x_1\sin\theta_2-x_3\cos\theta_2, \qquad \tilde{z}=\nu(x_1\sin\theta_1-x_3\cos\theta_1)$$

with $\nu = 2\pi/p_2$ and $\tilde{\eta}$ is 2π-periodic in $\tilde{z}$. We proceed by formulating the hydroelastic problem as a reversible Hamiltonian system in which the horizontal spatial direction $\tilde{x}$ plays the role of the time-like variable ('spatial dynamics'), working in a phase space of functions which are 2π-periodic in $\tilde{z}$. By construction a periodic solution in 'time' of the evolutionary system corresponds to a surface wave which is periodic in both $\tilde{x}$ and $\tilde{z}$ and is found using a Lyapunov centre theorem (although care is required in interpreting such solutions; see below).

Note that the equations for $\tilde{\eta}$ and $\tilde{\phi}$ follow from the formal variational principle

$$\delta \int \int_0^{2\pi} \left\{ \int_{-\frac{1}{\beta}}^{\tilde{\eta}} \frac{1}{2} \left(\tilde{\phi}_{\tilde{x}}^2 + \tilde{\phi}_{x_2}^2 + \nu^2 \tilde{\phi}_{\tilde{z}}^2 + 2\nu \cos(\theta_1 - \theta_2) \tilde{\phi}_{\tilde{x}} \tilde{\phi}_{\tilde{z}} \right) \mathrm{d}x_2 \right.$$
$$\left. + 2\sqrt{Q(\tilde{\eta})} P(\tilde{\eta})^2 + \tfrac{1}{2}\tilde{\eta}^2 + \gamma(\tilde{\eta}_{\tilde{x}} \sin\theta_2 + \nu \tilde{\eta}_{\tilde{z}} \sin\theta_1) \tilde{\phi}|_{x_2=\tilde{\eta}} \right\} \mathrm{d}\tilde{z}\, \mathrm{d}\tilde{x} = 0. \tag{1.24}$$

To formulate the hydroelastic equations as an evolutionary system, we treat the variational functional in equation (1.24) as an action functional in which a density is integrated over the $\tilde{x}$ direction and perform a Legendre transform to derive a formulation of the hydrodynamic problem as an infinite-dimensional, reversible Hamiltonian system in which $\tilde{x}$ is the time-like variable; full details are given in Section 2.2. Finally, we introduce a bifurcation parameter μ_1 by writing $\nu = \nu_0 + \mu_1$, where ν_0 is a reference value for ν (see below) and use a change of variable to linearise a nonlinear boundary condition emerging from the Legrendre transform. The result is a system of the form

$$\hat{v}_{\tilde{x}} = L^{\mu_1} \hat{v} + N^{\mu_1}(\hat{v}) \tag{1.25}$$

which is amenable to Theorem 1.2, satisfying (H1) and (H2) by construction.

A purely imaginary eigenvalue is of $L := L^0$ with corresponding eigenvector in the kth Fourier mode (where $(k, s) \neq (0, 0)$) corresponds to a linear hydroelastic wave of the form $\eta(\tilde{x}, \tilde{z}) = \eta_{s,k} \mathrm{e}^{\mathrm{i}\ell_1 \tilde{x} + \mathrm{i}\ell_2 \tilde{z}}$ with

$$\ell_1 = s \sin\theta_2 + \nu_0 k \sin\theta_1, \tag{1.26}$$
$$\ell_2 = -s \cos\theta_2 - \nu_0 k \cos\theta_1, \tag{1.27}$$

and a solution of this kind exists if and only if ℓ_1 and ℓ_2 satisfy the dispersion relation

$$\mathcal{D}(\ell_1, \ell_2) := (1 + (\ell_1^2 + \ell_2^2)^2) \sqrt{\ell_1^2 + \ell_2^2} \tanh\left(\beta^{-1} \sqrt{\ell_1^2 + \ell_2^2} \right) - \gamma^2 \ell_1^2 = 0.$$

A mode k purely imaginary eigenvalue is therefore corresponds to an intersection in the (ℓ_1, ℓ_2)-plane of the dispersion curve

$$C_{\mathrm{dr}} = \{(\ell_1, \ell_2) \in \mathbb{R}^2 \setminus \{(0, 0)\} : \mathcal{D}(\ell_1, \ell_2) = 0\}$$

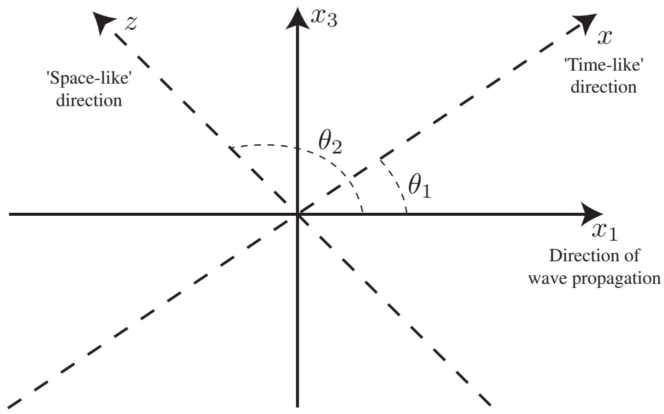

Figure 1.1: In the spatial dynamics formulation of the hydrodynamic problem the x and z directions are treated as respectively 'time-like' and 'space like'.

with the straight line S_k defined by equations (1.26), (1.27). (The solution $(\ell_1, \ell_2) = (0, 0)$ of $\mathcal{D}(\ell_1, \ell_2) = 0$ is excluded since it corresponds to $(k, s) = (0, 0)$.) The dispersion curve is sketched in Figure 1.2(a). The (β, γ)-parameter plane is divided into three regions in which C_{dr} has respectively zero, one, and two nontrivial bounded branches; the delineating curves D_1 and D_2 are given explicitly in Section 2.3.1. The central region is in fact each divided into two subregions, at the mutual boundary of which the qualitative shape of the branches changes, namely, from convex to nonconvex.

A point of intersection of S_k and C_{dr} corresponds to a purely imaginary mode k eigenvalue is; its imaginary part is the value of the parameter s at the point of intersection, that is, the value of S_0 in the (S_0, T)-coordinate system at the intersection, where

$$T = \{(\ell_1, \ell_2) \in \mathbb{R}^2 \ : \ \ell_1 = \sin\theta_1\, a, \ \ell_2 = -\cos\theta_1\, a, \ a \in \mathbb{R}\}$$

(see Figure 1.2(b)). The geometric multiplicity of the eigenvalue is is given by the number of distinct lines in the family $\{S_k\}$ which intersect C_{dr} at this parameter value, and a tangent intersection between S_k and C_{dr} indicates that each eigenvector in mode k has an associated Jordan chain of length at least 2. Notice that the sets $S_k \cap C_{\mathrm{dr}}$ and $S_{-k} \cap C_{\mathrm{dr}}$ have the same cardinality: the purely imaginary number is is a mode k eigenvalue if and only if the purely imaginary number $-$is is a mode $-k$ eigenvalue.

Let us now consider how to apply a Lyapunov centre theorem to equation (1.25). A first attempt might be to choose β, γ and θ_2 such that S_0 does not intersect C_{dr} and θ_1 such that S_1 and S_{-1} intersect C_{dr} in points with coordinates (s, ν_0), (t, ν_0) and $(-s, \nu_0)$, $(-t, \nu_0)$ in the (S_0, T)-coordinate system respectively, while S_k does not intersect C_{dr} for $k = \pm 2$, ± 3, ... (see Figure 1.2(b)). In this configuration L has simple mode 1 eigenvalues is, it with eigenvectors $\hat{v}_{1,s}\mathrm{e}^{\mathrm{i}z}$, $\hat{v}_{1,t}\mathrm{e}^{\mathrm{i}z}$ and mode -1 eigenvalues $-$is, $-$it with eigenvectors $\hat{v}_{-1,s}\mathrm{e}^{-\mathrm{i}z}$, $\hat{v}_{-1,t}\mathrm{e}^{-\mathrm{i}z}$. Assuming that $0 < t < s$ and neglecting any spectrum at the origin for the moment,

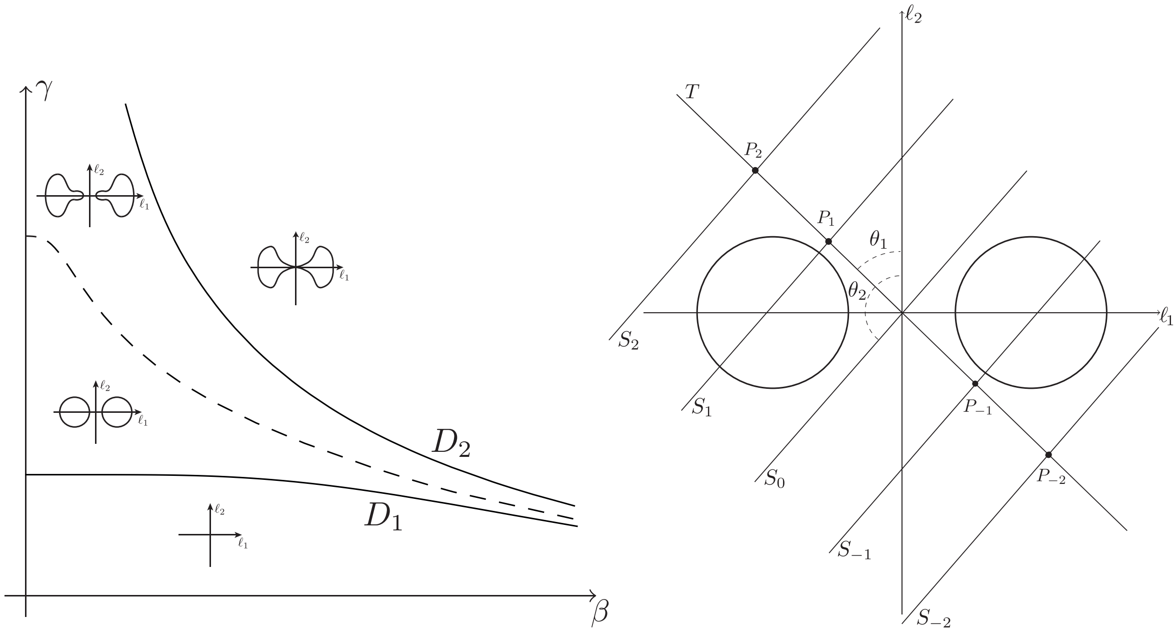

Figure 1.2: (a) Shape of the dispersion curve C_{dr} in the (β, γ)-parameter plane. (b) Position of the lines S_k and the dispersion curve C_{dr} in the (ℓ_1, ℓ_2)-plane; the lines S_0 and T form angles θ_2 and θ_1 respectively with the positive ℓ_2-axis, and the line S_k intersects the line T at the point P_k.

one finds that the purely imaginary eigenvalues $\pm\mathrm{i}s$ satisfy the non-resonance condition in a standard Lyapunov centre theorem, an infinite-dimensional version of which therefore gives a periodic solution of (1.25) with period near $2\pi/s$. However this approach does not yield a genuinely three-dimensional wave: at the linear level the solution takes the form $\hat{v} = \mathrm{Re}(\hat{v}_{1,s}\mathrm{e}^{\mathrm{i}sx}\mathrm{e}^{\mathrm{i}z}) = \mathrm{Re}(\hat{v}_{1,s}\mathrm{e}^{\mathrm{i}(sx+z)})$, which depends upon the single spatial direction $sx + z$. Note that Groves & Haragus [20] and Bagri & Groves [6], while correctly elucidating the use of spatial dynamics and Lyapunov centre theory to construct doubly periodic water waves, actually only detect waves of this kind, often referred to as '$2\frac{1}{2}$-dimensional waves'.

A more promising approach is to choose β, γ and θ_2 such that S_0 does not intersect C_{dr} and ν_0 and θ_1 such that S_1 and S_{-1} each intersect C_{dr} in points with coordinates $(\pm s, \nu_0)$ and $(\pm s, -\nu_0)$ in the (S_0, T)-coordinate system, while S_k does not intersect C_{dr} for $k = \pm 2$, ± 3, ... (see Figure 1.3). In this configuration L exhibits a $1:1$ or $1:-1$ resonance: it has two mode 1 eigenvalues $\pm\mathrm{i}s$ two mode -1 eigenvalues $\pm\mathrm{i}s$ so that $\pm\mathrm{i}s$ are geometrically and algebraically double eigenvalues of L with eigenvectors $\hat{v}_{1,s}\mathrm{e}^{\mathrm{i}z}$, $\hat{v}_{-1,s}\mathrm{e}^{-\mathrm{i}z}$ and $\hat{v}_{1,-s}\mathrm{e}^{\mathrm{i}z}$, $\hat{v}_{-1,-s}\mathrm{e}^{-\mathrm{i}z}$. Under the assumption that the other hypotheses are satisfied, Theorem 1.2 gives a periodic solution of (1.25) with period near $2\pi/s$ which yields a genuinely three-dimensional wave: at the linear level the solution takes the form $\hat{v}_{1,s}\mathrm{e}^{\mathrm{i}sx}\mathrm{e}^{\mathrm{i}z} + \hat{v}_{-1,s}\mathrm{e}^{\mathrm{i}sx}\mathrm{e}^{-\mathrm{i}z} + \hat{v}_{-1,-s}\mathrm{e}^{-\mathrm{i}sx}\mathrm{e}^{-\mathrm{i}z} + \hat{v}_{1,-s}\mathrm{e}^{-\mathrm{i}sx}\mathrm{e}^{\mathrm{i}z}$, which cannot be reduced to a function of a single spatial direction. (One can of course apply this idea in any mode, assuming that $\pm \mathrm{i}s$ are both mode k and mode $-k$ eigenvalues which do not resonate with any other purely imaginary eigenvalue.)

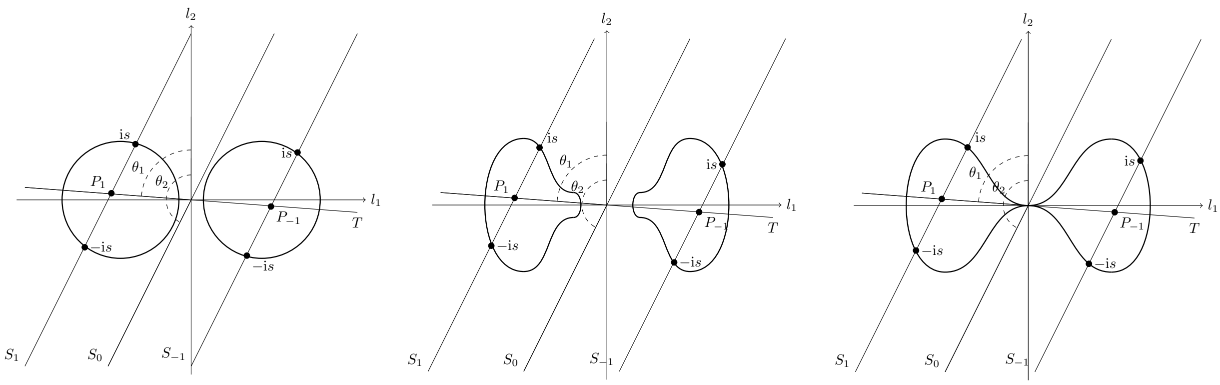

Figure 1.3: Scenarios of interest.

In Chapter 2.3 we apply Theorem 1.2 to the spatial dynamics formulation (1.25) of the hydroelastic problem in the eigenvalue scenarios shown in Figure 1.3, studying the purely imaginary spectrum of L in detail in Section 2.3.1. Hypotheses (H3), (H5) and (H6) are readily verified, while (H4)(iiii) (the origin is a double eigenvalue of L) and (H4)(ii) (the origin is a point of the continuous spectrum of L) arise in the cases $\beta > 0$ and $\beta = 0$ respectively. In the former case we find that the zero eigenvalue is trivial in the sense of (H7) and an isolated spectral point of L, so that (H8) is also satisfied. The additional verification of (H8) in the case $\beta = 0$ is undertaken in Section 2.3.3. Altogether we establish the following result.

Theorem 1.3. *Choose β, γ and θ_2 such that S_0 does not intersect C_{dr} and ν_0 and θ_1 such that S_1 and S_{-1} each intersect C_{dr} in points with coordinates $(\pm s, \nu_0)$ and $(\pm s, -\nu_0)$ in the (S_0, T)-coordinate system, while S_k does not intersect C_{dr} for $k = \pm 2, \pm 3, \ldots$ (see Figure 1.3). There exist $\varepsilon > 0$ and a two-parameter branch $\{(\phi, \eta)(\chi_1, \chi_2)\}_{0 \leq \chi_1, \chi_2 < \varepsilon}$ of doubly periodic solutions of* (1.20)–(1.23) *with periods $2\pi/(s+\mu_2(\chi_1, \chi_2))$, $2\pi/(\nu_0+\mu_1(\chi_1, \chi_2))$ in the variables $x_1 \sin\theta_2 - x_3 \cos\theta_2$ and $x_1 \sin\theta_1 - x_3 \cos\theta_1$ respectively.*

We also derive the following 'inverse' result which shows that under a certain non-resonance condition one can find a family of doubly periodic solutions which are small perturbations of any given periodic cell; these solutions have a fixed dimensionless wave speed γ and fixed periodic directions.

Theorem 1.4. *Choose β, s, ν_0 and $\theta_2 - \theta_1$ arbitrarily. There exist θ_1 and γ such that S_1 and S_{-1} each intersect C_{dr} in points with coordinates $(\pm s, \nu_0)$ and $(\pm s, -\nu_0)$ in the (S_0, T)-coordinate system. Assume that S_k does not intersect C_{dr} for $k \neq \pm 1$. There exist $\varepsilon > 0$ and a two-parameter branch $\{(\phi, \eta)(\chi_1, \chi_2)\}_{0 \leq \chi_1, \chi_2 < \varepsilon}$ of doubly periodic solutions of* (1.20)–(1.23) *with periods* $2\pi/(s+\mu_2(\chi_1,\chi_2))$, $2\pi/(\nu_0+\mu_1(\chi_1,\chi_2))$ *in the variables* $x_1\sin\theta_2 - x_3\cos\theta_2$ *and* $x_1\sin\theta_1 - x_3\cos\theta_1$ *respectively.*

1.3 Centre-manifold reduction and solitary hydroelastic surface waves

In Chapter 3 we consider line solitary waves, replacing x_1 by x for notational simplicity and working in dimensionless variables with unit length h and unit speed c, so that

$$(x,y) \to h(x,y), \quad \eta \to h\eta, \quad \phi \mapsto hc\phi.$$

The governing equations become

$$\phi_{xx} + \phi_{yy} = 0, \qquad\qquad -1 < y < \eta \tag{1.28}$$

with boundary conditions

$$\phi_y = 0, \qquad y = -1, \tag{1.29}$$

$$\phi_y - \eta_x\phi_x + \eta_x = 0, \qquad y = \eta, \tag{1.30}$$

$$\begin{aligned}&-\phi_x + \tfrac{1}{2}(\phi_x^2+\phi_y^2) + \alpha\eta \\ &\quad + \beta\left(\frac{1}{(1+\eta_x^2)^{1/2}}\left[\frac{1}{(1+\eta_x^2)^{1/2}}\left(\frac{\eta_{xx}}{(1+\eta_x^2)^{3/2}}\right)_x\right]_x + \frac{1}{2}\left(\frac{\eta_{xx}}{(1+\eta_x^2)^{3/2}}\right)^3\right) = 0, \quad y = \eta\end{aligned} \tag{1.31}$$

and asymptotic conditions $\eta \to 0$, $(\phi_x,\phi_y) \to 0$ as $x \to \pm\infty$, where

$$\alpha = \frac{gh}{c^2}, \qquad \beta = \frac{D}{\rho h^3 c^2}.$$

Equations (1.28)–(1.31) follow from the formal variational principle

$$\delta \int \left\{ \int_{-1}^{\eta} \Big(-\phi_x + \tfrac{1}{2}(\phi_x^2+\phi_y^2)\Big)\, \mathrm{d}y + \tfrac{1}{2}\alpha\eta^2 + \tfrac{1}{2}\beta\frac{\eta_{xx}^2}{(1+\eta_x^2)^{5/2}} \right\} \mathrm{d}x = 0. \tag{1.32}$$

Weakly nonlinear theory

Let us briefly review the (formal) classical weakly nonlinear theory as it applies to this problem. Figure 1.4 shows the linear dispersion relation

$$\alpha + \beta\mu^4 = \mu\coth(\mu)$$

for a sinusoidal wave train with wave number μ. For each fixed value β_0 of β the dispersion curve has a unique minimum at $(\mu, \alpha^{-1}) = (\mu_0, \alpha_0^{-1})$; the relationship between β_0, α_0 and μ_0 can be expressed in the form

$$\beta_0(\mu_0) = \frac{1}{4\mu_0^3}\coth(\mu_0) - \frac{1}{4\mu_0^2}\operatorname{cosech}^2(\mu_0), \qquad \alpha_0(\mu_0) = \frac{3\mu_0}{4}\coth(\mu_0) + \frac{\mu_0^2}{4}\operatorname{cosech}^2(\mu_0), \tag{1.33}$$

which defines a curve C in the (β, α)-plane parametrised by $\mu_0 \in (0, \infty)$. Setting $\alpha = \alpha_0 + \varepsilon^2$, $\beta = \beta_0$, and substituting the modulational *Ansatz*

$$\eta(x) = \varepsilon\Big(\zeta(\varepsilon x)\mathrm{e}^{\mathrm{i}\mu_0 x} + \bar{\zeta}(\varepsilon x)\mathrm{e}^{-\mathrm{i}\mu_0 x}\Big) + \varepsilon^2\Big(\zeta_2(\varepsilon x)\mathrm{e}^{2\mathrm{i}\mu_0 x} + \bar{\zeta}_2(\varepsilon x)\mathrm{e}^{-2\mathrm{i}\mu_0 x} + \zeta_0(\varepsilon x)\Big) + \cdots$$

into equations (3.2)–(3.6), one finds that to leading order ζ satisfies the nonlinear Schrödinger equation

$$\zeta - b_1\zeta_{XX} - b_2|\zeta|^2\zeta = 0, \tag{1.34}$$

where $X = \varepsilon x$ (see Appendix 3.5.1 for details of the derivation and formulae for the coefficients b_1 and b_2). One finds that b_1 is positive for all values of μ_0, and there exists a critical value $\mu_0^\star$ (numerically $\mu_0^\star \approx 177.33$) such that $b_2 > 0$ for $\mu_0 < \mu_0^\star$ (*'focussing'*) and $b_2 < 0$ for $\mu_0 > \mu_0^\star$ (*'defocussing'*).

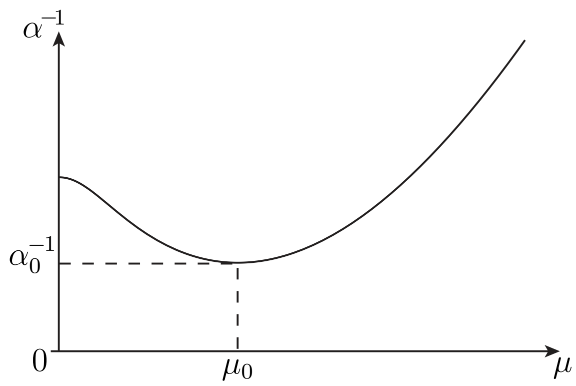

Figure 1.4: The linear dispersion relation for a fixed β_0.

Suppose that $b_2 > 0$, that is, choose μ_0 sufficiently small, or equivalently β_0 sufficiently large (corresponding to sufficiently shallow water in physical variables). Equation (1.34) admits the family

$$\zeta(X) = \left(\frac{2}{b_2}\right)^{1/2}\operatorname{sech}\left(\frac{X}{b_1^{1/2}}\right)\mathrm{e}^{\mathrm{i}\theta}, \qquad \theta \in [0, 2\pi)$$

of homoclinic solutions (solutions which decay to zero as $x \to \pm\infty$), which correspond to the solitary waves

$$\eta(x) = 2\varepsilon \left(\frac{2}{b_2}\right)^{1/2} \operatorname{sech}\left(\frac{\varepsilon x}{b_1^{1/2}}\right) \cos(\mu_0 x + \theta) + O(\varepsilon^2).$$

These waves take the form of periodic wave trains modulated by exponentially decaying envelopes; the wave with $\theta = 0$ is a symmetric wave of elevation while the wave with $\theta = \pi$ is a symmetric wave of depression (see Figure 1.5). In Chapter 3 we confirm the predictions of the weakly nonlinear theory and prove the following theorem.

Theorem 1.5. *Choose $\mu_0 \in (0, \mu_0^\star)$ and let (β_0, α_0) denote the point on the curve C with this parameter value. For each sufficiently small value of $\varepsilon > 0$ and $\nu \in (0,1)$ the hydroelastic problem* (1.28)–(1.31) *with $\beta = \beta_0$ and $\alpha = \alpha_0 + \varepsilon^2$ admits two geometrically distinct, symmetric solitary-wave solutions $(\eta_\varepsilon^\pm, \phi_\varepsilon^\pm)$ which satisfy the estimate*

$$\eta_\varepsilon^\pm(x) = \pm 2\varepsilon \left(\frac{2}{b_2}\right)^{1/2} \operatorname{sech}\left(\frac{\varepsilon x}{b_1^{1/2}}\right) \cos(\mu_0 x) + O(\varepsilon^2 \mathrm{e}^{-\nu b_1^{-1/2}\varepsilon|x|})$$

uniformly over $x \in \mathbb{R}$.

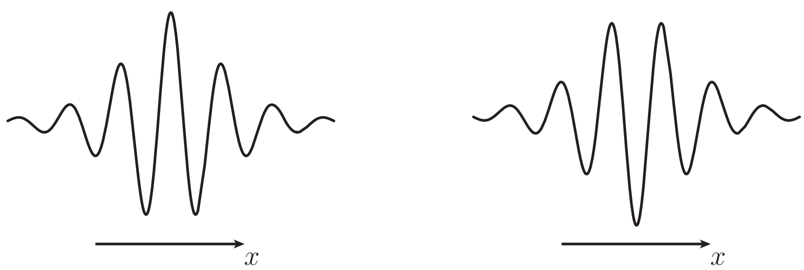

Figure 1.5: Symmetric envelope solitary waves (with scaled amplitudes and wavelengths).

Kirchgässner reduction

We prove Theorem 1.5 using the *Kirchgässner reduction*: the hydrodynamic problem is formulated as a spatial Hamiltonian system and reduced to a locally equivalent Hamiltonian system with finitely many degrees of freedom; homoclinic solutions of the reduced system correspond to solitary waves. The method was introduced by Kirchgässner [37] and has been used for many problems in fluid mechanics, in particular for water waves (see Dias & Iooss [14] for a review), and more recently for water waves with vorticity (Groves & Wahlén [22]) and ferrofluids (Groves, Lloyd & Stylianou [26], Groves & Nilsson [21]). In Chapter 3 we review the method as it applies to hydroelastic solitary waves, presenting various refinements and new features.

To formulate the hydroelastic equations as a Hamiltonian system we treat the variational functional in equation (1.32) as an action functional in which a density is integrated over the x-direction and perform a Legendre transform. Full details of this step and the several necessary changes of variables, which are performed explicitly, are also given; the result is a quasilinear evolution equation of the form

$$u_x = Lu + N^\delta(u) \tag{1.35}$$

for the variable $u = (\eta, \rho, \omega, \xi, \bar{\Gamma}, \bar{\Psi})$ in the phase space

$$Z = \left\{(\eta, \rho, \omega, \xi, \bar{\Gamma}, \bar{\Psi}) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \bar{H}^1(0,1) \times \bar{L}^2(0,1)\right\},$$

where the overline denotes the subspace of functions with zero mean value; the domain of the linear operator L is

$$\mathcal{D}(L) = \left\{(\eta, \rho, \omega, \xi, \bar{\Gamma}, \bar{\Psi}) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \bar{H}^2(0,1) \times \bar{H}^1(0,1) : \bar{\Gamma}_y\Big|_{y=0,1} = 0\right\}$$

and the nonlinear term on the right-hand side of (1.35), which satisfies $N^\delta(u) = O(\|(\delta, u)\|\|u\|)$, maps a neighbourhood of the origin in $\mathbb{R}^2 \times \mathcal{D}(L)$ analytically into Z. Here $\rho, \omega, \xi, \bar{\Gamma}, \bar{\Psi}$ are auxiliary variables related to η and ϕ, we have written $\alpha = \alpha_0 + \delta_1$ and $\beta = \beta_0 + \delta_2$, where α_0 and β_0 are fixed, and the superscript δ denotes the dependence upon this parameter.

In Section 3.2 we show that the spectrum of L is discrete. By reducing the spectral problem to a non self-adjoint Sturm-Liouville problem, we show that a complex number λ is an eigenvalue of L if and only if

$$\alpha_0 + \lambda^4\beta_0 = \lambda \cot(\lambda) \tag{1.36}$$

and deduce that $\sigma(L)$ consists of

(a) a countably infinite family $\{\lambda_k\}_{k\in\mathbb{Z}\setminus\{0\}}$ of simple real eigenvalues, where $\{\lambda_k\}_{k=1}^\infty$ are the positive real solutions of equation (1.36), so that $\lambda_k \in (k\pi, (k+1)\pi)$ for $k = 1, 2, \ldots$ and
$$\lambda_k^2 = k^2\pi^2 + \frac{2}{\beta_0} + o\left(\frac{1}{k}\right)$$
for large k, and $\lambda_{-k} = -\lambda_k$ for $k = 1, 2, \ldots$,

(b) four additional eigenvalues (counted according to multiplicity) which are shown in Figure 1.6. Note in particular that a *Hamiltonian-Hopf bifurcation* occurs at each point $(\beta_0(\mu_0), \alpha_0(\mu_0))$ of the curve C: two pairs of purely imaginary eigenvalues become complex by colliding at the points $\pm \mathrm{i}\mu_0$ on the imaginary axis.

Remarkably, we can treat (1.35) as a dynamical system with countably infinitely many coordinates by showing that L is a *Riesz spectral operator*, that is its generalised eigenvectors form a Riesz basis for Z (a Schauder basis obtained by an isomorphism from an othonormal basis). In particular, at a point $(\beta_0(\mu_0), \alpha_0(\mu_0))$ of the curve C (a 'Hamiltonian-Hopf point') we can write

$$Z = \left\{ u = Ae + Bf + \bar{A}\bar{e} + \bar{B}\bar{f} + \sum_{k \in \mathbb{Z} \setminus \{0\}} \beta_k e_{\lambda_k} : A, B \in \mathbb{C},\ \{\beta_k\} \in \ell^2 \right\},$$

where e, f and e_{λ_k} are suitably normalised generalised eigenvectors with $(L - \mathrm{i}\mu_0 I)e = 0$, $(L - \mathrm{i}\mu_0 I)f = e$ and $(L - \lambda_k I)e_{\lambda_k} = 0$. In the above notation

$$Lu = (\mathrm{i}\mu_0 A + B)e + \mathrm{i}\mu_0 Bf + (-\mathrm{i}\mu_0 \bar{A} + \bar{B})\bar{e} - \mathrm{i}\mu_0 \bar{B}\bar{f} + \sum_{k \in \mathbb{Z} \setminus \{0\}} \lambda_k \beta_k e_{\lambda_k}$$

and $u \in \mathcal{D}(L)$ whenever $\{\lambda_k \beta_k\} \in \ell^2$.

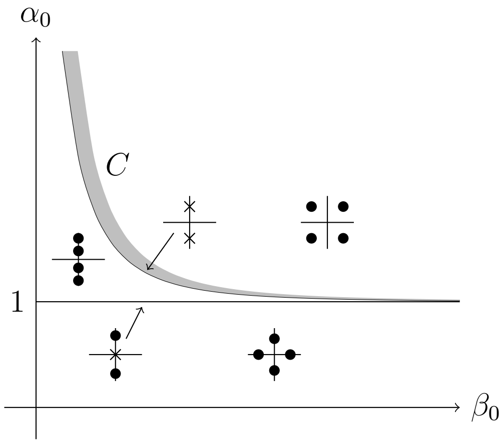

Figure 1.6: The shaded region indicates the parameter regime in which homoclinic bifurcation is detected; dots and crosses denote respectively simple and algebraically double, geometrically simple eigenvalues.

Homoclinic solutions of the Hamiltonian system are of particular interest since they correspond to solitary waves. We detect them using centre-manifold reduction (see Mielke [44, 45] for the version of the reduction theorem used here). Denoting the central and hyperbolic subspaces of Z at a Hamiltonian-Hopf point by

$$Z_1 = \{u_1 = Ae + Bf + \bar{A}\bar{e} + \bar{B}\bar{f} : A, B \in \mathbb{C}\}, \qquad Z_2 = \left\{ u_2 = \sum_{k \in \mathbb{Z} \setminus \{0\}} \beta_k e_{\lambda_k} : \{\beta_k\} \in \ell^2 \right\},$$

one finds that all small, globally bounded solutions to (1.35) lie on a *centre manifold* of the form $\{u_2 = r(u_1; \delta)\}$, where the *reduction function* $r : Z_1 \to \mathcal{D}(L)$ is $O(\|(\delta, u)\| \|u\|)$. The flow on the centre manifold is governed by the reduced system

$$u_{1x} = Lu_1 + N^\delta(u_1 + r(u_1;\delta)), \tag{1.37}$$

which is itself a reversible Hamiltonian system (with two degrees of freedom). One of the key requirements in Mielke's theorem is that the operator $L_2 = L|_{Z_2}$ has *L^p-maximal regularity* in the sense that the differential equation

$$\partial_x u_2 = L_2 u_2 + h$$

admits a unique solution $u_2 \in W^{1,p}(\mathbb{R}, Z_2) \cap L^p(\mathbb{R}, \mathcal{D}(L_2))$ for each $h \in L^p(\mathbb{R}, Z_2)$ and $p > 1$. In fact L^p-maximal regularity for some $p > 1$ implies L^p-maximal regularity for all $p > 1$ (see Mielke [43]), and an operator has L^2-maximal regularity if and only if it satisfies a resolvent esimate (see Arendt & Duelli [5, Theorem 2.4]); the theorem is usually stated with the resolvent estimate as a hypothesis. (Mielke's theorem actually requires maximal regularity in exponentially weighted spaces, a property which is implied by L^p-maximal regularity; see Mielke [43, Lemma 2.3].) In Section 3.3 we however demonstrate directly that a Riesz spectral operator with no imaginary eigenvalues has L^2-maximal regularity, and stipulate L^2-maximal regularity as a hypothesis in Mielke's theorem.

With a slight abuse of notation we write $(\delta_1, \delta_2) = (\delta, 0)$, so that positive values of δ correspond to points on the 'complex' side of C (the shaded region in Figure 1.6), one finds after a Darboux and normal-form transformation that the reduced equation (1.37) can be formulated as the Hamiltonian system

$$A_x = \frac{\partial \tilde{H}^\delta}{\partial \bar{B}}, \qquad B_x = -\frac{\partial \tilde{H}^\delta}{\partial \bar{A}}, \tag{1.38}$$

$$\tilde{H}^\delta(A, B, \bar{A}, \bar{B}) = \mathrm{i}s(A\bar{B} - \bar{A}B) + |B|^2 + \tilde{H}^\delta_{\mathrm{NF}}(|A|^2, \mathrm{i}(A\bar{B} - \bar{A}B)) + O(|(A,B)|^2|(\delta, A, B)|^{n_0}),$$

where $\tilde{H}^\delta_{\mathrm{NF}}(A, B, \bar{A}, \bar{B})$ is a real polynomial function of its arguments which satisfies

$$\tilde{H}^\delta_{\mathrm{NF}}(|A|^2, \mathrm{i}(A\bar{B} - \bar{A}B), \delta) = O(|(A,B)|^2|(\delta, A, B)|);$$

it contains the terms of order $3, \ldots, n_0 + 1$ in the Taylor expansion of $\tilde{H}^\delta(A, B, \tilde{A}, \tilde{B})$. Equations (1.38) inherit the reversibility of the Hamiltonian system: they are invariant under the transformation $(A, B)(x) \mapsto (\bar{A}, -\bar{B})(-x)$. Neglecting the remainder term in the Hamiltonian and introducing the scaled variables

$$A(x) = \varepsilon \mathrm{e}^{\mathrm{i}sx}\tilde{A}(X), \qquad B(x) = \varepsilon^2 \mathrm{e}^{\mathrm{i}sx}\tilde{B}(X), \qquad X = \varepsilon x,$$

where $\varepsilon^2 = \delta$, confirms that the system is at leading order equivalent to the nonlinear Schrödinger equation

$$\tilde{A}_{XX} = -c_1\tilde{A} - d_1\tilde{A}|\tilde{A}|^2,$$

where c_1 and d_1 are the coefficients of respectively $\delta|A|^2$ and $|A|^4$ in the Taylor expansion of $\tilde{H}^{\delta}_{\mathrm{NF}}$. We compute these coefficients explicitly in Appendix 3.5.2 and find that

$$c_1 = -\frac{1}{b_1}, \qquad d_1 = \frac{\sinh^2(\mu_0)}{2\tau_1}\,\frac{b_2}{b_1},$$

where b_1, b_2 are the coefficients in equation (1.34) and $\tau_1 > 0$ is defined in equation (3.39).

A rigorous analysis of (1.38) is given in Section 3.4. Returning to real coordinates $q, p \in \mathbb{R}^2$ given by $A = \frac{1}{\sqrt{2}}(q_1 + \mathrm{i}q_2)$, $B = \frac{1}{\sqrt{2}}(p_1 + \mathrm{i}p_2)$, eliminating p and introducing the scaled variables

$$q(x) = \varepsilon R_{sx} Q(X), \qquad X = \varepsilon x,$$

where $\varepsilon^2 = -c_1\delta$ and R_θ is the matrix representing a rotation through the angle θ, transforms (1.38) into

$$Q_{XX} = Q - CQ|Q|^2 + T_1^\varepsilon(Q, Q_X) + R_{-sX/\varepsilon} T_2^\varepsilon(R_{sX/\varepsilon}Q, R_{sX/\varepsilon}Q_X, R_{sX/\varepsilon}Q_{XX}) \tag{1.39}$$

where $C = -d_1/c_1$ and

$$T_1^\varepsilon(Q, Q_X) = O(\varepsilon|(Q, Q_X)|), \qquad T_2^\varepsilon(Q, Q_X, Q_{XX}) = O(\varepsilon^{n_0-2}|(Q, Q_X, Q_{XX})|).$$

Equation (1.39) is invariant under $X \mapsto -X$, $(Q_1(X), Q_2(X)) \mapsto Q_1(-X), -Q_2(-X))$ and in the limit $\varepsilon = 0$ has the explicit solution

$$Q(X) = \begin{pmatrix} h(X) \\ 0 \end{pmatrix}, \qquad h(X) = \left(\frac{2}{C}\right)^{1/2} \operatorname{sech}(X)$$

which is nondegenerate in the class of symmetric functions (see Section 3.4 for a precise statement of this result). This fact allows one to prove the following theorem with an implicit-function theorem argument.

Theorem 1.6. *For each $\nu \in (0,1)$ and each sufficiently small value of $\varepsilon > 0$ equation (1.39) has two homoclinic solutions $Q^{\varepsilon\pm}$ which are symmetric, that is invariant under the transformation $(Q_1(X), Q_2(X)) \mapsto (Q_1(-X), -Q_2(-X))$, and satisfy the estimate*

$$Q^{\varepsilon\pm}(X) = \pm \begin{pmatrix} h(X) \\ 0 \end{pmatrix} + O(\varepsilon \mathrm{e}^{-\nu|X|})$$

for all $X \in \mathbb{R}$.

Finally, let us briefly mention some related work in the literature. Buffoni & Groves [10] show that (1.38) has an infinite number of geometrically distinct homoclinic solutions which generically resemble multiple copies of one of the ‘primary’ homoclinic solutions found here.

In the present context this result yields the existence of an infinite family of 'multi-pulse' hydroelastic solitary waves. A variational existence theory for hydroelastic solitary waves in the present parameter regime has been given by Groves, Hewer & Wahlén [25], while the Kirchgässner reduction (without the Hamiltonian framework) has also been applied to an alternative model in which the ice sheet is modelled as a Kirchhoff-Love elastic plate with nonzero thickness and inertial effects are taken into account (Ilichev [31] and Ilichev & Tomashpolskii [32]).

1.4 A dimension-breaking phenomenon for solitary hydroelastic surface waves

In Chapter 4 we consider periodically modulated solitary waves, replacing x_1, x_3 by x, z for notational consistency with previous work on water waves (see below) and working in dimensionless variables with unit length h and unit speed c, so that

$$(x, y, z) \to h(x, y, z), \quad \eta \to h\eta, \quad \phi \to hc\phi.$$

The governing equations become

$$\phi_{xx} + \phi_{yy} + \phi_{zz} = 0, \qquad -1 < y < \eta \tag{1.40}$$

with boundary conditions

$$\phi_y = 0, \qquad y = -1, \tag{1.41}$$

$$\phi_y = \phi_x\eta_x + \phi_z\eta_z - \eta_x, \qquad y = \eta, \tag{1.42}$$

$$-\phi_x + \frac{1}{2}(\phi_x^2 + \phi_y^2 + \phi_z^2) + \alpha\eta + \beta U(\eta) = 0, \qquad y = \eta, \tag{1.43}$$

and asymptotic conditions $\eta \to 0$, $(\phi_x, \phi_y, \phi_z) \to 0$ as $x \to \pm\infty$, where

$$\alpha = \frac{gh}{c^2}, \qquad \beta = \frac{D}{c^2h^3}.$$

Equations (1.40)–(1.43) follow from the formal variational principle

$$\delta \int \left\{ \int_{-1}^{\eta} \left[-\phi_x + \tfrac{1}{2}(\phi_x^2 + \phi_y^2 + \phi_z^2) \right] \mathrm{d}y + \tfrac{1}{2}\alpha\eta^2 + 2\beta P^2\sqrt{Q} \right\} \mathrm{d}x \, \mathrm{d}z = 0. \tag{1.44}$$

Weakly nonlinear theory

Let us briefly review the (formal) classical weakly nonlinear theory as it applies to this problem. Recall that, for each fixed value β_0 of β, the linear dispersion curve for a two-dimensional sinusoidal wave train with wave number μ has a unique minimum at $(\mu, \alpha^{-1}) = (\mu_0, \alpha_0^{-1})$ (see Figure 1.4 and equation (1.33)). Substituting

$$\begin{aligned}\alpha &= \alpha_0 + \varepsilon^2, \qquad \beta = \beta_0,\\ \eta(x) &= \varepsilon\Big(\zeta(\varepsilon x, \varepsilon z)\mathrm{e}^{\mathrm{i}\mu_0 x} + \bar{\zeta}(\varepsilon x, \varepsilon z)\mathrm{e}^{-\mathrm{i}\mu_0 x}\Big)\\ &\qquad + \varepsilon^2\Big(\zeta_2(\varepsilon x, \varepsilon z)\mathrm{e}^{2\mathrm{i}\mu_0 x} + \bar{\zeta}_2(\varepsilon x, \varepsilon z)\mathrm{e}^{-2\mathrm{i}\mu_0 x} + \zeta_0(\varepsilon x, \varepsilon z)\Big) + \cdots\end{aligned}$$

into equations (1.40)–(1.43), one finds that to leading order ζ satisfies the Davey-Stewartson system

$$\zeta - A_1\zeta_{XX} - A_2\zeta_{ZZ} - A_3|\zeta|^2\zeta + A_4\zeta\psi_X = 0, \tag{1.45}$$
$$(1-\alpha_0^{-1})\psi_{XX} + \psi_{ZZ} + A_4\partial_X(|\zeta|^2) = 0, \tag{1.46}$$

where $X = \varepsilon x$, $Z = \varepsilon z$ (see Appendix 4.4.2); formulae for the coefficients A_1, A_3 and A_4 are given in Appendix 4.4.1, while $A_2 = \mu_0^{-1}\coth(\mu_0)$. The system (1.45), (1.46) is of *elliptic-elliptic* type since $A_1, A_2 > 0$ and $\alpha_0 > 1$. In the special case of z-independent waves these equations reduce to the cubic nonlinear Schrödinger equation

$$\zeta - A_1\zeta_{XX} - A_5|\zeta|^2\zeta = 0, \tag{1.47}$$

where $A_5 = A_3 + (1-\alpha_0^{-1})^{-1}A_4^2$. (Note that the coefficients A_1, A_5 in equation (1.47) are the same as the coefficients b_1, b_2 in equation (1.34).)

The nonlinear Schrödinger equation (1.47) is focussing if $A_5 > 0$. In this case it has symmetric solitary-wave solutions

$$\zeta^\star(X) = \pm\left(\frac{2}{A_5}\right)^{1/2}\operatorname{sech}\left(\frac{X}{A_1^{1/2}}\right);$$

as Z-independent solutions of (1.45), (1.46) they are again referred to as *line solitary waves*. It was shown by Groves, Sun & Wahlén [28] that Davey-Stewartson systems of elliptic-elliptic focussing type have solitary-wave solutions of the form

$$\zeta(X,Z) = \zeta^\star(X) + \tilde{\zeta}(X,Z),$$

where $\tilde{\zeta}$ is evanescent in its first argument and periodic in its second. These solutions are *periodically modulated solitary waves*, which emerge from line solitary waves in a *dimension-breaking bifurcation*. Similar dimension-breaking bifurcation phenomena for other model equations have been studied by Haragus-Courcelle & Pego [30] and Bridges & Derks [9].

Spatial dynamics

In Chapter 3 we show that the hydroelastic-wave problem (1.40)–(1.43) also has two families of z-independent, small-amplitude, symmetric solitary-wave solutions $(\eta_\varepsilon^\pm, \phi_\varepsilon^\pm)$, either of which we now denote by $(\eta^\star, \phi^\star)$. These line solitary waves are solutions of (1.40)–(1.43) with

$$\alpha = \alpha_0 + \varepsilon^2, \quad \beta = \beta_0, \quad \eta^\star = \varepsilon\zeta^\star(\varepsilon x)\cos(\mu_0 x) + \mathcal{O}(\varepsilon^2),$$

where β_0, α_0 are given by (1.33) and $\mu_0 < \mu_0^\star \approx 177.33$ (see above). The line solitary waves are wave packets consisting of a decaying envelope which modulates an underlying periodic wave train with wave number μ_0 (see Figure 1.7). In Chapter 4 we prove the existence of three-dimensional periodically modulated solitary waves which emanate from the line solitary waves in a dimension-breaking bifurcation; they are sketched in Figure 1.7.

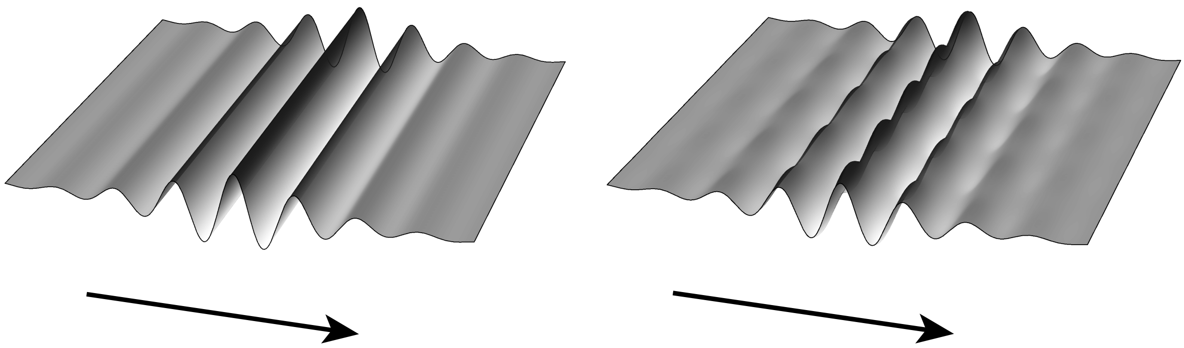

Figure 1.7: The periodically modulated solitary wave on the right emerges from the line solitary wave on the left in a dimension breaking bifurcation.

We use the method by using the variational principle (1.44) in the usual fashion to formulate the hydroelastic problem for travelling waves as a reversible evolutionary equation

$$u_z = f(u), \tag{1.48}$$

where u belongs to a space $\mathcal{X}$ of functions which converge to zero as $x \to \pm\infty$ and $f : \mathcal{D}(f) \subseteq \mathcal{X} \to \mathcal{X}$ is an unbounded, densely defined vector field (see Sections 4.1.1 and 4.1.2). Observe that the nonzero equilibria of equation (1.48) are precisely the line solitary-wave solutions of the hydroelastic-wave problem; of particular interest in this respect are the equilibria $u^\star$ corresponding to the small amplitude line solitary waves. A solution to equation (1.48) of the form

$$u(x,z) = u^\star(x) + u'(x,z),$$

where u' has a small amplitude and is periodic in z, corresponds to a periodically modulated solitary wave which is generated by a dimension-breaking bifurcation from a line solitary wave.

Accordingly, we construct a family of small-amplitude periodic solutions to the equation

$$u'_z = f(u^\star + u') \tag{1.49}$$

for each sufficiently small value of ε. Explicit formulae for equations (1.48) and (1.49) (together with the definitions of suitable function spaces in which to study them) are presented in Section 4.1.1 below.

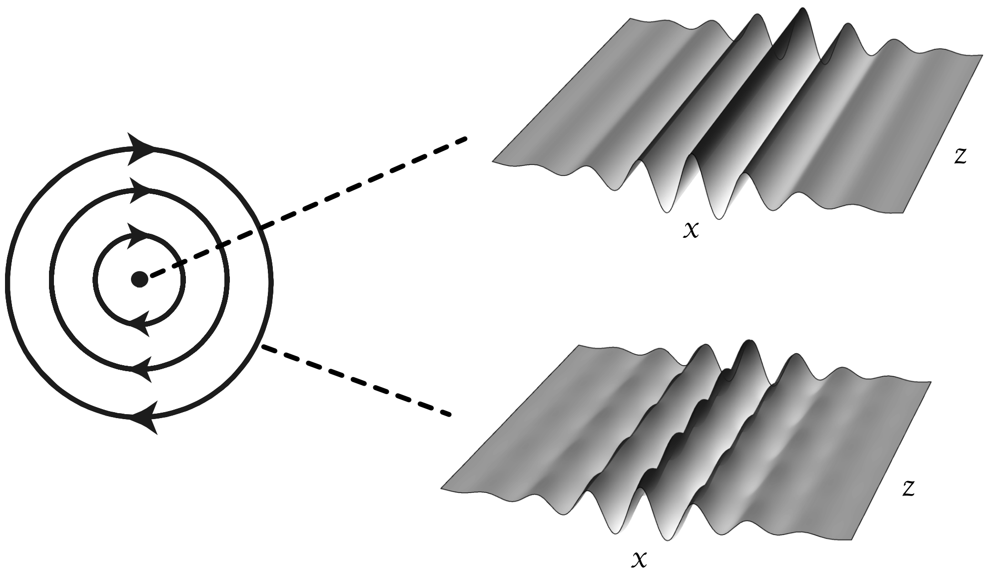

Figure 1.8: A family of periodic orbits surrounding the origin in the phase space $\mathcal{X}$ of (1.49). Each point in $\mathcal{X}$ is a function of x which vanishes as $x \to \pm\infty$, while z is the time-like variable.

The corresponding result for water waves with weak surface tension was established by Groves, Sun & Wahlén [27] (see Groves, Haragus & Sun [23] for water waves with strong surface tension); in Chapter 4 we use their work as a blueprint, adhering closely to their strategy.

The Lyapunov-Iooss theorem

As we have already seen, small-amplitude periodic solutions of reversible evolutionary equations are classically found using the Lyapunov centre theorem, which asserts the existence of a family of these solutions which surround the zero equilibrium and have frequency near ω_0 provided that $\pm\mathrm{i}\omega_0$ are non-resonant imaginary eigenvalues of the corresponding linearised system. Figure 1.8 shows an interpretation of this result in the present setting. In our problem the central subspace is however infinite-dimensional due to the presence of essential spectrum at the origin (a feature typical of spatial dynamics formulations for problems in unbounded domains). We therefore turn to Iooss's generalisation of the reversible Lyapunov centre theorem mentioned in Section 1.2, which we refer to as the Lyapunov-Iooss theorem. Its proof is similar, but more straightforward, than that of Theorem 1.2.

Theorem 1.7. *Consider the differential equation*

$$v_\tau = Lv + N(v), \tag{1.50}$$

in which $v(\tau)$ belongs to a Banach space $\mathcal{X}$. Suppose that $\mathcal{Y}, \mathcal{Z}$ are further Banach spaces with the properties that

(i) $\mathcal{Z}$ *is continuously embedded in* $\mathcal{Y}$ *and continuously and densely embedded in* $\mathcal{X}$*,*

(ii) $L : \mathcal{Z} \subseteq \mathcal{X} \to \mathcal{X}$ *is a closed linear operator,*

(iii) there is a neighbourhood U of the origin in $\mathcal{Y}$ such that $L \in \mathcal{L}(\mathcal{Y}, \mathcal{X})$ and N maps $\mathcal{U}$ (and hence $\mathcal{U} \cap \mathcal{Z}$) smoothly into $\mathcal{X}$ with $N(0) = 0$, $\mathrm{d}N[0] = 0$.

Suppose further that

(iv) equation (1.50) *is reversible: there exists an involution $S \in \mathcal{L}(\mathcal{X}, \mathcal{X})$ with $SLv = -LSv$ and $SN(v) = -N(Sv)$ for all $v \in \mathcal{U}$,*

and the following spectral hypotheses are satisfied:

(v) $\pm \mathrm{i}\omega_0$ *are nonzero simple eigenvalues of* L*;*

(vi) $\mathrm{i}n\omega_0 \in \rho(L)$ *for* $n \in \mathbb{Z} \setminus \{-1, 0, 1\}$*;*

(vii) $\|(L - \mathrm{i}n\omega_0 I)^{-1}\|_{\mathcal{X}\to\mathcal{X}} = o(1)$ *and* $\|(L - \mathrm{i}n\omega_0 I)^{-1}\|_{\mathcal{X}\to\mathcal{Z}} = O(1)$ *as* $n \to \infty$*;*

(viii) For each $v^\dagger \in \mathcal{U}$ the equation

$$Lv = -N(v^\dagger)$$

has a unique solution $v \in \mathcal{Y}$ and $v^\dagger \mapsto v$ is a smooth mapping $\mathcal{U} \to \mathcal{Y}$.

Under these hypotheses there exists a neighbourhood $\mathcal{N}$ of the origin in $\mathbb{R}$ and a continuously differentiable branch $\{(v(s), \omega(s))\}_{s\in S}$ of $2\pi/\omega(s)$-periodic solutions in $C_{\mathrm{per}}(\mathbb{R}, \mathcal{Y}\oplus\mathcal{Z})$ to (1.50) *with amplitude $O(|s|)$. Here the direct sum refers to the decomposition of a function into its mode* 0 *and higher-order Fourier components, the subscript 'per' indicates a $2\pi/\omega(s)$-periodic function and $\omega(s) = \omega_0 + O(|s|^2)$.*

Remark 1.8. *Hypothesis (viii) ('the Iooss condition at the origin') is stronger than, but implies Iooss's original condition (see Remark 1.1(ii)). We impose this stronger condition here since it is more easily verified in applications.*

Our task is thus to study the purely imaginary spectrum of the linearisation L of the vector field on the right-hand side of (1.49) and verify the hypotheses in the Lyapunov-Iooss theorem. This existence theory is given in Section 4.1.3 with proofs of the techincal results in Sections 4.2 and 4.3. Motivated by the weakly nonlinear scaling for modulated wave packets, we use the decomposition $\eta = \eta_1 + \eta_2$, where the supports of the Fourier transforms of η_1 and η_2 lie respectively in $S = [-\mu_0 - \delta, -\mu_0 + \delta] \cup [\mu_0 - \delta, \mu_0 + \delta]$ and $\mathbb{R} \setminus S$, determine η_2 and all other variables as functions of η_1, and write

$$\eta_1(x) = \frac{1}{2}\varepsilon\zeta(\varepsilon x)\mathrm{e}^{\mathrm{i}\mu_0 x} + \frac{1}{2}\varepsilon\overline{\zeta(\varepsilon x)}\mathrm{e}^{-\mathrm{i}\mu_0 x}.$$

We find, after a lengthy calculation, that the pair $\pm\varepsilon k, k > 0$ are eigenvalues of L if and only if the operator

$$\mathcal{B}_{\varepsilon,k}(\zeta,\psi) := \begin{pmatrix} A_2^{-1}\Big(\zeta - A_1\zeta_{xx} - (A_3+A_5)(\zeta^\star)^2\zeta - A_3(\zeta^\star)^2\overline{\zeta} + A_4\zeta^\star\vartheta_x\Big) + k^2\zeta - \mathcal{R}_{\varepsilon,k}(\zeta) \\ -(1-\alpha_0^{-1})\psi_{xx} - \frac{A_4}{2}[\zeta^\star\zeta_1]_x + k^2\psi \end{pmatrix}$$

has a zero eigenvalue, where $\mathcal{R}_{\varepsilon,k}(\zeta)$ is a remainder term which is $O(\varepsilon)$ in an appropriate sense; note that $\mathcal{B}_{0,k}(\zeta,\psi)$ is precisely the operator obtained by linearising the Davey-Stewartson system at its line solitary wave and taking the Fourier transform with respect to z. A comparison with well-studied operators in mathematical physics shows that $\mathcal{B}_{0,0} = \mathcal{B}_{0,k} - k^2 I$ has precisely one simple negative eigenvalue $-k_0^2$, so that $\mathcal{B}_{0,k}$ has a simple zero precisely when $k = k_0$, and we deduce from a spectral perturbation argument that $\mathcal{B}_{\varepsilon,k}$ is invertible for $k \in [k_{\min}, \infty) \setminus \{k_\varepsilon\}$, where $k_{\min}$ is a fixed small, positive number and $|k_0 - k_\varepsilon| = O(\varepsilon)$, while $\mathcal{B}_{\varepsilon,k_\varepsilon}$ has a simple zero eigenvalue; the corresponding eigenvector is invariant under the symmetry $\tilde{S} : (\zeta(x), \psi(x)) \mapsto (\overline{\zeta(-x)}, -\psi(-x))$, thus corresponding to a linear water wave whose surface profile is symmetric in x. We also show that $\mathcal{B}_{\varepsilon,k}|_{\operatorname{Fix}\tilde{S}}$ does not have a zero eigenvalue in the interval $(0, k_{\min})$.

A similar argument is used to verify the Iooss condition at the origin. Applying the above reduction to the equation

$$Lv = -N(v^\dagger),$$

where N is the nonlinear part of the vector field on the right-hand side of (1.49), we obtain the reduced equation

$$\mathcal{C}_\varepsilon(\zeta) = \zeta^\dagger,$$

where $\mathcal{C}_0$ is the operator obtained by linearising the cubic nonlinear Schrödinger equation at its solitary wave and $\mathcal{C}_\varepsilon - \mathcal{C}_0$ is $O(\varepsilon)$ in an appropriate sense. The operator $\mathcal{C}_0$ is invertible in the class of functions which are invariant unter the symmetry $\zeta(x) \mapsto \overline{\zeta(-x)}$, and it follows that the reduced equation is solvable in this class. Working in function spaces corresponding to symmetric surface waves, we therefore obtain the following existence result for periodically modulated solitary waves from Theorem 1.7.

Theorem 1.9. *There exist a neighbourhood $\mathcal{N}_\varepsilon$ of the origin in $\mathbb{R}$ and a family of periodically modulated solitary waves $\{(\eta_\chi(x,z), \phi_\chi(x,y,z))\}_{\chi \in \mathcal{N}_\varepsilon}$, which emerges from the line solitary wave in a dimension-breaking bifurcation. Here*

$$\eta_\chi(x,z) = \eta_\varepsilon^\star(x) + \eta_\chi^\star(x,z),$$

in which $\eta_\chi^\star(\cdot,\cdot)$ has amplitude $O(|\chi|)$ and is even in both arguments and periodic in its second with frequency $\varepsilon k_\varepsilon + O(|\chi|^2)$; the positive number k_ε satisfies $|k_\varepsilon - k_0| = O(\varepsilon)$.

2 A resonant Lyapunov centre theorem and doubly periodic travelling hydroelastic waves

2.1 A resonant Lyapunov centre theorem

In this section we prove the following theorem, working in the framework for Hamiltonian systems set out in Section 1.1.

Theorem 2.1. *Let X, Z be real Hilbert spaces, where X is continuously and densely embedded in Z, and consider the autonomous evolutionary equation*

$$v_t = L^{\mu_1} v + N^{\mu_1}(v), \tag{2.1}$$

where $(\mu_1, v) \mapsto N^{\mu_1}(v)$ is a smooth mapping from a neighbourhood $\Lambda_1 \times U$ of the origin in $\mathbb{R} \times X$ into Z which satisfies $N^{\mu_1}(0) = 0$ and $\mathrm{d}N^{\mu_1}[0] = 0$ for all $\mu_1 \in \Lambda_1$, and $L^{\mu_1} : X \subseteq Z \to Z$ is a closed linear operator which depends smoothly upon μ_1 in the topology of $\mathcal{L}(X, Z)$. Suppose that

(H1) Equation (1.9) *represents Hamilton's equations*

$$v_t = v_{\mathrm{H}}^{\mu_1}(v)$$

for a Hamiltonian system $(Z, \Omega^{\mu_1}, H^{\mu_1})$ with $\mathcal{D}_{\mathrm{H}} = \Lambda_1 \times U$ (in the nomenclature in Section 1.1), so that

$$L^{\mu_1}(v) + N^{\mu_1}(v) = J^{\mu_1}(v)^{-1} \nabla H^{\mu_1}(v)$$

and in particular

$$L^{\mu_1} v = J^{\mu_1}(0)^{-1} \nabla H_2^{\mu_1}(v),$$

where $H_2^{\mu_1}(v) = \frac{1}{2}\mathrm{d}^2 H^{\mu_1}[0](v, v)$ is the part of H^{μ_1} which is homogeneous of degree 2 in v.

(H2) Equation (1.9) *is reversible: both* L^{μ_1} *and* N^{μ_1} *anticommute with an involution* $R \in \mathcal{L}(X) \cap \mathcal{L}(Z)$. *This* reverser R *satisfies*

$$H^{\mu_1}(Rv) = H^{\mu_1}(v), \quad R^*\alpha^{\mu_1}(Rv) = -\alpha^{\mu_1}(v), \quad R^*J^{\mu_1}(Rv)R = -J^{\mu_1}(v)$$

for all $(\mu_1, v) \in \Lambda_1 \times U$.

(H3) The linear operator L^{μ_1} *has two pairs* $\pm \mathrm{i}\kappa_1^{\mu_1}$, $\pm \mathrm{i}\kappa_2^{\mu_1}$ *of purely imaginary eigenvalues with linearly independent eigenvectors* $e_1^{\mu_1}$, $\bar{e}_1^{\mu_1}$, $e_2^{\mu_1}$, $\bar{e}_2^{\mu_1}$, *all of which depend smoothly upon* μ_1. *Furthermore* $\bar{e}_1^0 = Re_1^0$, $\bar{e}_2^0 = Re_2^0$ *and*

$$\kappa_1^0 = \kappa_2^0, \qquad \frac{\mathrm{d}}{\mathrm{d}\mu_1}(\kappa_1^{\mu_1} - \kappa_2^{\mu_1})\Big|_{\mu_1=0} \neq 0.$$

For notational simplicity we henceforth abbreviate L^0, e_1^0 *and* e_2^0 *to respectively* L, e_1 *and* e_2, *and define* $\kappa := \kappa_1^0 = \kappa_2^0$.

(H4) The origin is one of

(i) a point of the resolvent set of L,

(ii) a point of the continuous spectrum of L,

(iii) a geometrically simple, algebraically double eigenvalue of L *with eigenvector* f_1 *and generalized eigenvector* f_2 *(possibly embedded in continuous spectrum), where* $Rf_1 = -f_1$ *and* $Rf_2 = f_2$.

Here (ii) and (iii) entail further spectral hypotheses (see (H7) and (H8) below).

(H5) The imaginary number $\mathrm{i}k\kappa$ *belongs to the resolvent set of* L *for each* $k \in \mathbb{Z}\backslash\{0,-1,1\}$.

(H6) The linear operator L *satisfies*

$$\left\|(\mathrm{i}k\kappa I - L)^{-1}\right\|_{Z\to Z} \lesssim \frac{1}{|k|}, \qquad \left\|(\mathrm{i}k\kappa I - L)^{-1}\right\|_{Z\to X} \lesssim 1$$

for all $k \in \mathbb{Z}\backslash\{0,-1,1\}$.

(H7) The zero eigenvalue (if present) is 'trivial' in the following sense: writing $u \in U$ *as* $u = qf_1 + w$ *with* $w \in \{f_1\}^\perp$, *one finds that* $J^{\mu_1}(u)$ *and* $H^{\mu_1}(u)$ *do not depend upon* q.

(H8) Let

$$\mathcal{U} = \{u \in H^1_{\mathrm{per}}(\mathbb{R}, Z) \cap L^2_{\mathrm{per}}(\mathbb{R}, X) : u(\tau) \in U \text{ for all } \tau \in \mathbb{R}\}$$

and

$$N^\star(u,\mu_1,\mu_2) = \frac{1}{\sqrt{2\pi}}\int_0^{2\pi}\Big(J^{\mu_1}(u)\Big((\kappa+\mu_2)J^{\mu_1}(u)u_\tau - L^{\mu_1}u - N^{\mu_1}(u)\Big) + J^0(0)Lu\Big)\,\mathrm{d}\tau.$$

Suppose that the equation

$$Lu^\dagger = J^0(0)^{-1}(I-\Pi_0)N^\star(u,\mu_1,\mu_2)$$

has a unique solution $u^\dagger \in (I-\Pi_0)X$ *which depends smoothly upon* $u \in \mathcal{U}$, $\mu_1 \in \Lambda_1$ *and* $\mu_2 \in \Lambda_2$, *where* Π_0 *is the orthogonal projection of* X *onto* $\operatorname{span}\{f_1, f_2\}$.

Under these hypotheses there exist $\varepsilon > 0$ *and a smooth, two-parameter branch* $\{v(\chi_1,\chi_2), \mu_1(\chi_1,\chi_2)\}_{0\leq\chi_1,\chi_2<\varepsilon}$ *of* $2\pi/(\kappa+\mu_2(\chi_1,\chi_2))$*-periodic reversible solutions to equation* (2.1) *in* $H^1_{\mathrm{loc}}(\mathbb{R},Z)\cap L^2_{\mathrm{loc}}(\mathbb{R},X)$. *The rescaled function* $u(\tau)=v(t)$, $\tau=(\kappa+\mu_2)t$ *satisfies* $\|u(\chi_1,\chi_2)\|_{L^2_{\mathrm{per}}(\mathbb{R},Z)} \to 0$, *while* $\mu_1(\chi_1,\chi_2), \mu_2(\chi_1,\chi_2) \to 0$ *as* $(\chi_1,\chi_2)\to(0,0)$.

We look for periodic solutions of (2.1) with frequency near κ by writing

$$v(t) = u(\tau), \qquad \tau = (\kappa+\mu_2)t,$$

where μ_2 lies in a neighbourhood Λ_2 of the origin in $\mathbb{R}$, and seeking 2π-periodic solutions of the transformed equation

$$(\kappa+\mu_2)u_\tau = L^{\mu_1}u + N^{\mu_1}(u). \tag{2.2}$$

To this end we introduce the function spaces

$$\begin{aligned}\mathcal{X} &:= H^1_{\mathrm{per}}(\mathbb{R},Z)\cap L^2_{\mathrm{per}}(\mathbb{R},X),\\ \mathcal{Z} &:= L^2_{\mathrm{per}}(\mathbb{R},Z),\end{aligned}$$

equipping $\mathcal{Z}$ with the continuous scalar product

$$(\cdot\,,\cdot) = \frac{1}{2\pi}\int_0^{2\pi}\langle\cdot\,,\cdot\rangle$$

and noting that elements $w \in \mathcal{Z}$ can be expanded in Fourier series

$$w(\tau) = \frac{1}{\sqrt{2\pi}}\sum_{k\in\mathbb{Z}}[w]_k \mathrm{e}^{\mathrm{i}k\tau}, \qquad [w]_k = \frac{1}{\sqrt{2\pi}}\int_0^{2\pi} w(\tau)\mathrm{e}^{-\mathrm{i}k\tau}\,\mathrm{d}\tau \in Z.$$

We seek 2π-periodic solutions of (2.2) by studying the function $F:\mathcal{U}\times\Lambda_1\times\Lambda_2 \mapsto \mathcal{Z}$ defined by

$$F(u,\mu_1,\mu_2) = (\kappa+\mu_2)J^{\mu_1}(u)u_\tau - \nabla H^{\mu_1}(u),$$

where

$$\mathcal{U} = \{u\in\mathcal{X} : u(\tau)\in U \text{ for all } \tau\in\mathbb{R}\}.$$

The equation

$$F(u,\mu_1,\mu_2) = 0, \tag{2.3}$$

has a variational characterisation.

Proposition 2.2. *Equation* (2.3) *is the Euler-Lagrange equation for the action functional* $S : \mathcal{U} \times \Lambda_1 \times \Lambda_2 \to \mathbb{R}$ *given by*

$$S(u, \mu_1, \mu_2) = \frac{1}{2\pi} \int_0^{2\pi} \Big\{ -(\kappa + \mu_2)\langle \alpha^{\mu_1}(u), u_\tau \rangle - H^{\mu_1}(u) \Big\} \, \mathrm{d}\tau.$$

Furthermore S *is invariant with respect to the translation* $T_\theta : u(\tau) \mapsto u(\tau + \theta)$ *and reversing operation* $T : u(\tau) \mapsto (Ru)(-\tau)$.

Proof. The first assertion follows from the calculation

$$\begin{aligned}
&\mathrm{d}_1 S[u, \mu_1, \mu_2](v) \\
&= \frac{1}{2\pi} \int_0^{2\pi} \Big\{ -(\kappa + \mu_2)\Big(\langle \widetilde{\mathrm{d}\alpha}^{\mu_1}[u](v), u_\tau \rangle + \langle \alpha^{\mu_1}(u), v_\tau \rangle\Big) - \langle \nabla H^{\mu_1}(u), v \rangle \Big\} \, \mathrm{d}\tau \\
&= \frac{1}{2\pi} \int_0^{2\pi} \Big\{ -(\kappa + \mu_2)\Big(\langle \widetilde{\mathrm{d}\alpha}^{\mu_1}[u](v), u_\tau \rangle - \langle v, \widetilde{\mathrm{d}\alpha}^{\mu_1}[u](u_\tau) \rangle\Big) - \langle \nabla H^{\mu_1}(u), v \rangle \Big\} \, \mathrm{d}\tau \\
&= \frac{1}{2\pi} \int_0^{2\pi} \langle (\kappa + \mu_2) J^{\mu_1}(u) u_\tau - \nabla H^{\mu_1}(u), v \rangle \, \mathrm{d}\tau \\
&= (F(u, \mu_1, \mu_2), v)
\end{aligned}$$

for $v \in \mathcal{X}$, while the second is a consequence of the periodicity of u and hypothesis (H2). ■

We proceed by solving (2.3) using a Lyapunov-Schmidt reduction. For this purpose it is necessary to examine the solvability conditions for the equations

$$(\pm \mathrm{i}\kappa I - L)u = J^0(0)^{-1} f \tag{2.4}$$

and

$$Lu = J^0(0)^{-1} f, \tag{2.5}$$

where f is a given function in Z. Normalise $e_1^{\mu_1}$, $e_2^{\mu_1}$ such that

$$\Omega^{\mu_1}|_0(e_1^{\mu_1}, \bar{e}_1^{\mu_1}) = \pm\mathrm{i}, \quad \Omega^{\mu_1}|_0(e_2^{\mu_1}, \bar{e}_2^{\mu_1}) = \pm\mathrm{i}, \quad \Omega^{\mu_1}|_0(e_1^{\mu_1}, e_2^{\mu_1}) = 0, \quad \Omega^{\mu_1}|_0(e_1^{\mu_1}, \bar{e}_2^{\mu_1}) = 0$$

and f_1, f_2 such that

$$\Omega^0|_0(f_1, f_2) = 1,$$

where $\Omega^{\mu_1}|_0$ is extended *bilinearly* to the complexification of Z. Observing that L is the Hamiltonian vector field for the linear Hamiltonian system $(Z, \Omega^0|_0, H_2^0)$, we find that the spectral projections $P_{\pm\mathrm{i}\kappa}$ and P_0 onto the eigenspaces $E_{\mathrm{i}\kappa} = \mathrm{span}\{e_1, e_2\}$, $E_{-\mathrm{i}\kappa} = \mathrm{span}\{\bar{e}_1, \bar{e}_2\}$ and generalised eigenspace $E_0 = \mathrm{span}\{f_1, f_2\}$ are given by

$$P_{\mathrm{i}\kappa} u = \sum_{i=1}^{2} s_i \Omega^0|_0(u, \bar{e}_i) e_i = \sum_{i=1}^{2} s_i \langle J^0(0)(u), e_i \rangle e_i,$$

$$P_{-\mathrm{i}\kappa}u = -\sum_{i=1}^{2} s_i \Omega^0|_0(u,e_i)\bar{e}_i = -\sum_{i=1}^{2} s_i \langle J^0(0)(u),\bar{e}_i\rangle \bar{e}_i,$$

where $s_i = -\Omega^{\mu_1}|_0(e_i^{\mu_1},\bar{e}_i^{\mu_1})$, and

$$\begin{aligned} P_0 u &= \Omega^0|_0(u,f_2)f_1 - \Omega^0|_0(u,f_1)f_2, \\ &= \langle J^0(0)(u),f_2\rangle f_1 - \langle J^0(0)(u),f_1\rangle f_2 \end{aligned}$$

(see Mielke [45, §3.1]); here $\langle\cdot\,,\cdot\rangle$ is extended *sesquilinearly* to the complexification of Z. This observation shows in particular that the (necessary and sufficient) solvability condition for (2.4), namely that the spectral projection of its right-hand side onto $E_{\pm\mathrm{i}\kappa}$ vanishes, is equivalent to the requirement that the orthogonal projection $\Pi_{\pm\mathrm{i}\kappa}f$ of f onto $E_{\pm\mathrm{i}\kappa}$ vanishes. In this case it has a unique solution in the orthogonal complement of $E_{\pm\mathrm{i}\kappa}$ in X which depends continuously upon f. Similarly, equation (2.5) is solvable if the orthogonal projection $\Pi_0 f$ of f onto E_0 vanishes (note that this is merely a sufficient condition), and in this case has a unique solution in the orthogonal complement of E_0 in X which depends continuously upon f. In the following analysis we use the convention that $f_1=f_2=0$ and hence $P_0=\Pi_0=0$ if 0 is not an eigenvalue of L.

Define

$$\begin{aligned} \mathcal{W}_1 &= \{u_0 + Ae_1\mathrm{e}^{\mathrm{i}\tau} + Be_2\mathrm{e}^{\mathrm{i}\tau} + \bar{A}\bar{e}_1\mathrm{e}^{-\mathrm{i}\tau} + \bar{B}\bar{e}_2\mathrm{e}^{-\mathrm{i}\tau},\ A,B\in\mathbb{C},\ u_0\in Z\}, \\ \mathcal{W}_2 &= \{u\in\mathcal{Z}\colon [u]_0=0,\ \Pi_{\mathrm{i}\kappa}[u]_1 = \Pi_{-\mathrm{i}\kappa}[u]_{-1}=0\}, \end{aligned}$$

where $\Pi_{\pm\mathrm{i}\kappa}$ are the orthogonal projections onto the eigenspaces $E_{\mathrm{i}\kappa}=\operatorname{span}\{e_1,e_2\}$ and $E_{-\mathrm{i}\kappa}=\operatorname{span}\{\bar{e}_1,\bar{e}_2\}$, so that

$$\begin{aligned} \mathcal{Z} &= \mathcal{W}_1 \oplus \mathcal{W}_2, \\ \mathcal{X} &= (\mathcal{W}_1\cap\mathcal{X}) \oplus (\mathcal{W}_2\cap\mathcal{X}) \end{aligned}$$

and the decompositions are orthogonal. Let $\widetilde{\Pi}_{\mathcal{W}_1}$ be the projection of $\mathcal{Z}$ onto $\mathcal{W}_1$ along $\mathcal{W}_2$, write $u\in\mathcal{U}$ as

$$u = \underbrace{\widetilde{\Pi}_{\mathcal{W}_1}u}_{=:\,u_{\mathcal{W}_1}} + \underbrace{(I-\widetilde{\Pi}_{\mathcal{W}_1})u}_{=:\,u_{\mathcal{W}_2}}$$

and equation (2.3) as

$$\widetilde{\Pi}_{\mathcal{W}_1}F(u_{\mathcal{W}_1}+u_{\mathcal{W}_2},\mu_1,\mu_2)=0, \tag{2.6}$$

$$(I-\widetilde{\Pi}_{\mathcal{W}_1})F(u_{\mathcal{W}_1}+u_{\mathcal{W}_2},\mu_1,\mu_2)=0. \tag{2.7}$$

Proposition 2.3. *The linear operator*

$$(I-\widetilde{\Pi}_{\mathcal{W}_1})\mathrm{d}_1F[0,0,0]\colon (\mathcal{W}_2\cap\mathcal{X})\to\mathcal{W}_2$$

is an isomorphism.

Proof. The equation

$$(I - \widetilde{\Pi}_{\mathcal{W}_1})\mathrm{d}_1 F[0,0,0](v) = w \tag{2.8}$$

with $w \in \mathcal{W}_2$ is equivalent to

$$(\mathrm{i}\kappa k I - L)[v]_k = J^0(0)^{-1}[w]_k, \qquad k \in \mathbb{Z} \setminus \{0\},$$

with $[w]_k \in Z$, $k \notin \{0,-1,1\}$ and $[w]_1 \in E_{\mathrm{i}\kappa}^{\perp}$, $[w]_{-1} \in E_{-\mathrm{i}\kappa}^{\perp}$ (in X). By assumption (H5) the operator $\mathrm{i}\kappa k I - L\colon X \to Z$ is an isomorphism for $k \notin \{0,-1,1\}$ and we have established that the equations

$$\begin{aligned}(\mathrm{i}\kappa I - L)[v]_1 &= J^0(0)^{-1}[w]_1,\\ (-\mathrm{i}\kappa I - L)[v]_{-1} &= J^0(0)^{-1}[w]_{-1}\end{aligned}$$

have unique solutions $[v]_1 \in E_{\mathrm{i}\kappa}^{\perp}$, $[v]_{-1} \in E_{-\mathrm{i}\kappa}^{\perp}$ (in X) which depend continuously upon $[w]_1$, $[w]_{-1}$. It follows that

$$\begin{aligned}\|v\|_{L^2_{\mathrm{per}}(\mathbb{R},X)}^2 &= \sum_{k\in\mathbb{Z}\setminus\{0,-1,1\}} \|[v]_k\|_X^2 + \|[v]_1\|_X^2 + \|[v]_{-1}\|_X^2\\ &= \sum_{k\in\mathbb{Z}\setminus\{0,-1,1\}} \left\|(\mathrm{i}\kappa k I - L)^{-1} J^0(0)^{-1}[w]_k\right\|_X^2 + \|[v]_1\|_X^2 + \|[v]_{-1}\|_X^2\\ &\lesssim \sum_{k\in\mathbb{Z}\setminus\{0,-1,1\}} \|[w]_k\|_Z^2 + \|[w]_1\|_Z^2 + \|[w]_{-1}\|_Z^2\\ &\leq \|w\|_{\mathcal{Z}}^2\end{aligned}$$

and similarly

$$\|v\|_{H^1_{\mathrm{per}}(\mathbb{R},Z)} \lesssim \|w\|_{\mathcal{Z}}$$

by assumption (H6), so that v lies in $\mathcal{X}$. We conclude that equation (2.8) has a unique solution $v \in \mathcal{W}_2 \cap \mathcal{X}$ which depends continuously upon $w \in \mathcal{W}_2$. ∎

Lemma 2.4. *There exist neighbourhoods $\mathcal{U}_1$ and $\mathcal{U}_2$ of the origin in respectively $\mathcal{W}_1$ and $\mathcal{W}_2$ and a reduction function $u_{\mathcal{W}_2} : \mathcal{U}_1 \times \Lambda_1 \times \Lambda_2 \to \mathcal{U}_2$ such that equation (2.7) admits the unique solution $(u_{\mathcal{W}_1}, u_{\mathcal{W}_2}(u_{\mathcal{W}_1}, \mu_1, \mu_2))$ in $\mathcal{U}_1 \times \mathcal{U}_2$. Furthermore $u_{\mathcal{W}_1}(0,0,0) = 0$ and $\mathrm{d}u_{\mathcal{W}_1}[0,0,0] = 0$.*

Proof. This result follows from the implicit-function theorem and Proposition 2.3. ∎

The next step is to further decompose the reduced equation

$$\widetilde{\Pi}_{\mathcal{W}_1} F(u_{\mathcal{W}_1} + u_{\mathcal{W}_2}(u_{\mathcal{W}_1}, \mu_1, \mu_2), \mu_1, \mu_2) = 0 \tag{2.9}$$

by introducing the orthogonal projection $\widetilde{\Pi}$ of $\mathcal{Z}$ onto

$$\mathcal{W}_{1,1} = \{qf_1 + pf_2 + Ae_1\mathrm{e}^{\mathrm{i}\tau} + Be_2\mathrm{e}^{\mathrm{i}\tau} + \bar{A}\bar{e}_1\mathrm{e}^{-\mathrm{i}\tau} + \bar{B}\bar{e}_2\mathrm{e}^{-\mathrm{i}\tau},\ q,p \in \mathbb{R},\ A,B \in \mathbb{C}\},$$

which is given by

$$\widetilde{\Pi}u = \Pi_0[u]_0 + \mathrm{e}^{\mathrm{i}\tau}\Pi_{\mathrm{i}\kappa}[u]_1 + \mathrm{e}^{-\mathrm{i}\tau}\Pi_{-\mathrm{i}\kappa}[u]_{-1},$$

so that

$$\mathcal{W}_1 = \mathcal{W}_{1,1} \oplus \mathcal{W}_{1,2},$$

where $\mathcal{W}_{1,2} = (I - \widetilde{\Pi})\mathcal{W}_1$. Writing $u_{\mathcal{W}_1} \in \mathcal{U}_1$ as

$$u_{\mathcal{W}_1} = \underbrace{\widetilde{\Pi}u_{\mathcal{W}_1}}_{=:\,u_1} + \underbrace{(I - \widetilde{\Pi})u_{\mathcal{W}_1}}_{=:\,u_2},$$

we find that (2.9) is equivalent to

$$\widetilde{\Pi}G(u_1, u_2, \mu_1, \mu_2) = 0, \tag{2.10}$$

$$\underbrace{(I - \widetilde{\Pi})G(u_1, u_2, \mu_1, \mu_2)}_{= (I - \Pi_0)[G(u_1, u_2, \mu_1, \mu_2)]_0} = 0, \tag{2.11}$$

where

$$G(u_1, u_2, \mu_1, \mu_2) = F(u_1 + u_2 + u_{\mathcal{W}_2}(u_1 + u_2, \mu_1, \mu_2), \mu_1, \mu_2).$$

We proceed with a further reduction of Lyapunov-Schmidt type. Solving equation (2.11) for u_2 in terms of u_1, μ_1 and μ_2 requires hypothesis (H8) if the origin lies in the continuous spectrum of L or is an eigenvalue embedded in the continuous spectrum.

Lemma 2.5. *There exist neighbourhoods $\mathcal{U}_{1,1}$ and $\mathcal{U}_{1,2}$ of the origin in respectively $\mathcal{W}_{1,1}$ and $\mathcal{W}_{1,2}$ and a reduction function $u_2 : \mathcal{U}_{1,1} \times \Lambda_1 \times \Lambda_2 \to \mathcal{U}_{1,2}$ such that equation (2.11) admits the unique solution $(u_1, u_2(u_1, \mu_1, \mu_2))$ in $\mathcal{U}_{1,1} \times \mathcal{U}_{1,2}$. Furthermore $u_1(0,0,0) = 0$ and $\mathrm{d}u_1[0,0,0] = 0$.*

Proof. Equation (2.11) is equivalent to

$$Lu_2 = J^0(0)^{-1}(I - \Pi_0)N^\star(u_1 + u_2 + u_{\mathcal{W}_2}(u_1 + u_2, \mu_1, \mu_2), \mu_1, \mu_2).$$

Let $v(u_1, u_2, \mu_1, \mu_2)$ be the unique solution of the equation

$$Lv = J^0(0)^{-1}(I - \Pi_0)N^\star(u_1 + u_2 + u_{\mathcal{W}_2}(u_1 + u_2, \mu_1, \mu_2), \mu_1, \mu_2)$$

and define $\Upsilon : (\mathcal{U}_1 \cap \mathcal{W}_{1,1}) \times (\mathcal{U}_1 \cap \mathcal{W}_{1,2}) \times \Lambda_1 \times \Lambda_2 \to \mathcal{W}_{1,2}$ by

$$\Upsilon(u_1, u_2, \mu_1, \mu_2) = u_2 - v(u_1, u_2, \mu_1, \mu_2).$$

Observing that $\Upsilon(0,0,0,0) = 0$, $\mathrm{d}_2\Upsilon[0,0,0,0] = I$, one therefore obtains the result from the implicit-function theorem. ■

The reduced equation

$$\tilde{\Pi}G(u_1, u_2(u_1,\mu_1,\mu_2),\mu_1,\mu_2) = 0$$

is conveniently written as

$$f(u_1,\mu_1,\mu_2) = 0, \tag{2.12}$$

where

$$f(u_1,\mu_1,\mu_2) = \tilde{\Pi}F(u_1 + h(u_1,\mu_1,\mu_2),\mu_1,\mu_2)$$

and the new reduction function $h : \mathcal{U}_{1,1} \times \Lambda_1 \times \Lambda_2 \to (I - \tilde{\Pi})\mathcal{X}$ is given by

$$h(u_1,\mu_1,\mu_2) = u_2(u_1,\mu_1,\mu_2) + u_{\mathcal{W}_2}(u_1 + u_2(u_1,\mu_1,\mu_2),\mu_1,\mu_2).$$

Note again that $h(0,0,0) = 0$ and $\mathrm{d}_1 h[0,0,0] = 0$. Equation (2.12) inherits the variational structure of (2.3).

Proposition 2.6. *Equation* (2.12) *is the Euler-Lagrange equation for the reduced action functional* $s : \mathcal{U}_{1,1} \times \Lambda_1 \times \Lambda_2 \to \mathbb{R}$ *given by*

$$s(u_1,\mu_1,\mu_2) = S(u_1 + h(u_1,\mu_1,\mu_2),\mu_1,\mu_2),$$

that is

$$\mathrm{d}_1 s[u_1,\mu_1,\mu_2](v_1) = (f(u_1,\mu_1,\mu_2), v_1) \tag{2.13}$$

for all $v_1 \in \tilde{\Pi}\mathcal{X}$.

Proof. This result follows from the calculation

$$\begin{aligned}
\mathrm{d}_1 s[u_1,\mu_1,\mu_2](v_1) &= \mathrm{d}_1 S[u_1 + h(u_1,\mu_1,\mu_2),\mu_1,\mu_2](v_1 + \mathrm{d}_1 h[u_1,\mu_1,\mu_2](v_1)) \\
&= (F(u_1 + h(u_1,\mu_1,\mu_2),\mu_1,\mu_2),\mu_1,\mu_2), \mathrm{d}_1 h[u_1,\mu_1,\mu_2](v_1) + v_1) \\
&= (\tilde{\Pi}F(u_1 + h(u_1,\mu_1,\mu_2),\mu_1,\mu_2), \mathrm{d}_1 h[u_1,\mu_1,\mu_2](v_1) + v_1) \\
&= (\tilde{\Pi}F(u_1 + h(u_1,\mu_1,\mu_2),\mu_1,\mu_2), v_1) \\
&= (f(u_1,\mu_1,\mu_2), v_1),
\end{aligned}$$

where the second line follows from the first by Proposition 2.2, the third follows from the second because

$$(I - \tilde{\Pi})F(u_1 + h(u_1,\mu_1,\mu_2),\mu_1,\mu_2) = 0$$

by construction, and the fourth follows from the third because $h(u_1,\mu_1,\mu_2)$ (and hence all its derivatives) lies in $(I - \tilde{\Pi})\tilde{X}$. ■

Introducing coordinates

$$u_1 = qf_1 + pf_2 + Ae_1\mathrm{e}^{\mathrm{i}\tau} + Be_2\mathrm{e}^{\mathrm{i}\tau} + \bar{A}\bar{e}_1\mathrm{e}^{-\mathrm{i}\tau} + \bar{B}\bar{e}_2\mathrm{e}^{-\mathrm{i}\tau},$$

one finds that the reduced equation (2.12) is given by

$$\partial_{\bar{A}} s = 0, \tag{2.14}$$
$$\partial_{\bar{B}} s = 0, \tag{2.15}$$
$$\partial_p s = 0 \tag{2.16}$$

(recall that s does not depend upon q by hypothesis (H7)). The reduced action functional s remains invariant under the symmetries T_θ and T, whose actions on $\mathcal{W}_{1,1}$ are given by

$$T_\theta(A, B, \bar{A}, \bar{B}, q, p) = (A\mathrm{e}^{\mathrm{i}\theta}, B\mathrm{e}^{\mathrm{i}\theta}, \bar{A}\mathrm{e}^{-\mathrm{i}\theta}, \bar{B}\mathrm{e}^{-\mathrm{i}\theta}, q, p),$$
$$T(A, B, \bar{A}, \bar{B}, q, p) = (\bar{A}, \bar{B}, A, B, -q, p).$$

It follows that s is a real-valued function of the real quantities $|A|^2$, $|B|^2$, $\frac{\mathrm{i}}{2}(\bar{A}B - A\bar{B})$, $\frac{1}{2}(A\bar{B} + \bar{A}B)$, p, μ_1 and μ_2 which is even with respect to $\frac{\mathrm{i}}{2}(\bar{A}B - A\bar{B})$. Restricting to $A = r_1$, $B = \mathrm{i}r_2$, where r_1 and r_2 are real (so that $\frac{\mathrm{i}}{2}(\bar{A}B - A\bar{B}) = r_1 r_2$, $\frac{1}{2}(A\bar{B} + \bar{A}B) = 0$), we find that

$$s(A, B, \bar{A}, \bar{B}, p, \mu_1, \mu_2) = \tilde{s}(r_1^2, r_2^2, r_1 r_2, p, \mu_1, \mu_2),$$

where the right-hand side is even in its third argument, so that in fact, with a slight abuse of notation,

$$s(A, B, \bar{A}, \bar{B}, q, \mu_1, \mu_2) = \tilde{s}(r_1^2, r_2^2, p, \mu_1, \mu_2).$$

Equations (2.14)–(2.16) therefore reduce to

$$r_1 \partial_1 \tilde{s}(r_1^2, r_2^2, p, \mu_1, \mu_2) = 0,$$
$$r_2 \partial_2 \tilde{s}(r_1^2, r_2^2, p, \mu_1, \mu_2) = 0,$$
$$\partial_3 \tilde{s}(r_1^2, r_2^2, p, \mu_1, \mu_2) = 0,$$

and further to

$$\partial_1 \tilde{s}(r_1^2, r_2^2, p, \mu_1, \mu_2) = 0, \tag{2.17}$$
$$\partial_2 \tilde{s}(r_1^2, r_2^2, p, \mu_1, \mu_2) = 0, \tag{2.18}$$
$$\partial_3 \tilde{s}(r_1^2, r_2^2, p, \mu_1, \mu_2) = 0 \tag{2.19}$$

for solutions with non-zero r_1 and r_2 components.

Lemma 2.7. *Denoting by $\tilde{s}_k^{ij}(r_1^2, r_2^2, p)^m \mu_1^{n_1}\mu_2^{n_2}$ the part of $\tilde{s}$ which is homogeneous of order m in (r_1^2, r_2^2, p) and of order n_1, n_2 in μ_1 and μ_2, we have that*

$$\begin{aligned}
\tilde{s}_2^{00} &= \tilde{s}_{002}^{00} p^2,\\
\tilde{s}_2^{10} &= \tilde{s}_{200}^{10}\mu_1 r_1^2 + \tilde{s}_{020}^{10}\mu_1 r_2^2 + \tilde{s}_{002}^{10}\mu_1 p^2,\\
\tilde{s}_2^{01} &= -s_1\mu_2 r_1^2 - s_2\mu_2 r_2^2,
\end{aligned}$$

where

$$\begin{aligned}
\tilde{s}_{002}^{00} &= -\tfrac{1}{2},\\
\tilde{s}_{200}^{10} &= \Omega_0^0(\partial_{\mu_1} L^0 e_1, \bar{e}_1),\\
\tilde{s}_{020}^{10} &= \Omega_0^0(\partial_{\mu_1} L^0 e_2, \bar{e}_2),\\
\tilde{s}_{002}^{10} &= \tfrac{1}{2}\Omega_0^0(\partial_{\mu_1} L^0 f_2, f_2).
\end{aligned}$$

Proof. We begin by recording the formulae

$$\begin{aligned}
\mathrm{d}_1^2 S[0,0,0](v_1, v_2) &= (J^0(0)(\kappa v_{1\tau} - Lv_1), v_2),\\
\mathrm{d}_1^2\mathrm{d}_2 S[0,0,0](v_1, v_2, 1) &= (\partial_{\mu_1} J^0(0)(\kappa v_{1\tau} - Lv_1) + J^0(0)\partial_{\mu_1} L^0 v_1, v_2),\\
\mathrm{d}_1^2\mathrm{d}_3 S[0,0,0](v_1, v_2, 1) &= (J^0(0) v_{1\tau}, v_2)
\end{aligned}$$

for v_1, $v_2 \in \mathcal{X}$, which are obtained by differentiating the identity

$$\mathrm{d}_1 S[u, \mu_1, \mu_2](v) = \Big(J^{\mu_1}(u)\Big((\kappa + \mu_2)u_\tau - L^{\mu_1}u - N^{\mu_1}(u)\Big), v\Big)$$

for $(u, \mu_1, \mu_2) \in \mathcal{U} \times \Lambda_1 \times \Lambda_2$ and $v \in \mathcal{X}$ (see Proposition 2.2).

These formulae show that

$$\tilde{s}_2^{00} = \tilde{s}_{200}^{00} r_1^2 + \tilde{s}_{020}^{00} r_2^2 + \tilde{s}_{002}^{00} p^2,$$

where

$$\begin{aligned}
\tilde{s}_{200}^{00} &= \mathrm{d}_1^2 S[0,0,0](\mathrm{e}^{\mathrm{i}\tau} e_1, \mathrm{e}^{-\mathrm{i}\tau}\bar{e}_1) = (J^0(0)(\mathrm{i}\kappa I - L)\mathrm{e}^{\mathrm{i}\tau} e_1, \mathrm{e}^{\mathrm{i}\tau} e_1) = 0,\\
\tilde{s}_{020}^{00} &= \mathrm{d}_1^2 S[0,0,0](\mathrm{e}^{\mathrm{i}\tau} e_2, \mathrm{e}^{-\mathrm{i}\tau}\bar{e}_2) = (J^0(0)(\mathrm{i}\kappa I - L)\mathrm{e}^{\mathrm{i}\tau} e_2, \mathrm{e}^{\mathrm{i}\tau} e_2) = 0,\\
\tilde{s}_{002}^{00} &= \tfrac{1}{2}\mathrm{d}_1^2 S[0,0,0](f_2, f_2) = -\tfrac{1}{2}(J^0(0)Lf_2, f_2) = -\tfrac{1}{2}\Omega_0^0(f_1, f_2) = -\tfrac{1}{2}.
\end{aligned}$$

Similarly, denoting the part of $h(u_1, \mu_1, \mu_2)$ which is homogeneous of degree i, j, k, ℓ, n_1 and n_2 in respectively A, B, $\bar{A}$, $\bar{B}$, μ_1 and μ_2 by $h_{ijk\ell}^{n_1 n_2} A^i B^j \bar{A}^k \bar{B}^\ell \mu_1^{n_1}\mu_2^{n_2}$, we find that

$$\begin{aligned}
\tilde{s}_2^{10} &= \tilde{s}_{200}^{10}\mu_1 r_1^2 + \tilde{s}_{020}^{10}\mu_1 r_2^2 + \tilde{s}_{002}^{10}\mu_1 p^2,\\
\tilde{s}_2^{01} &= \tilde{s}_{200}^{01}\mu_2 r_1^2 + \tilde{s}_{020}^{01}\mu_2 r_2^2 + \tilde{s}_{002}^{01}\mu_2 p^2,
\end{aligned}$$

where

$$
\begin{aligned}
\tilde{s}^{10}_{200} &= \mathrm{d}_1^2\mathrm{d}_2 S[0,0,0](\mathrm{e}^{\mathrm{i}\tau}e_1, \mathrm{e}^{-\mathrm{i}\tau}\bar{e}_1, 1) + \mathrm{d}_1^2 S[0,0,0](\mathrm{e}^{\mathrm{i}\tau}e_1, h^{10}_{00100}) + \mathrm{d}_1^2 S[0,0,0](\mathrm{e}^{-\mathrm{i}\tau}\bar{e}_1, h^{10}_{10000})\\
&= (J^0(0)\partial_{\mu_1} L^0(\mathrm{e}^{\mathrm{i}\tau}e_1), \mathrm{e}^{\mathrm{i}\tau}e_1)\\
&= \Omega^0_0(\partial_{\mu_1} L^0(e_1), \bar{e}_1),\\
\tilde{s}^{10}_{020} &= \mathrm{d}_1^2\mathrm{d}_2 S[0,0,0](\mathrm{e}^{\mathrm{i}\tau}e_2, \mathrm{e}^{-\mathrm{i}\tau}\bar{e}_2, 1) + \mathrm{d}_1^2 S[0,0,0](\mathrm{e}^{\mathrm{i}\tau}e_2, h^{10}_{00010}) + \mathrm{d}_1^2 S[0,0,0](\mathrm{e}^{-\mathrm{i}\tau}\bar{e}_1, h^{10}_{01000})\\
&= (J^0(0)\partial_{\mu_1} L^0(\mathrm{e}^{\mathrm{i}\tau}e_2), \mathrm{e}^{\mathrm{i}\tau}e_2)\\
&= \Omega^0_0(\partial_{\mu_1} L^0(e_2), \bar{e}_2),\\
\tilde{s}^{10}_{002} &= \tfrac{1}{2}\mathrm{d}_1^2\mathrm{d}_2 S[0,0,0](f_2, f_2, 1) + \mathrm{d}_1^2 S[0,0,0](f_2, h^{10}_{00001})\\
&= \tfrac{1}{2}(J^0(0)\partial_{\mu_1} L^0 f_2, f_2)\\
&= \tfrac{1}{2}\Omega^0_0(\partial_{\mu_1} L^0 f_2, f_2)
\end{aligned}
$$

and

$$
\begin{aligned}
\tilde{s}^{01}_{200} &= \mathrm{d}_1^2\mathrm{d}_3 S[0,0,0](\mathrm{e}^{\mathrm{i}\tau}e_1, \mathrm{e}^{-\mathrm{i}\tau}\bar{e}_1, 1) + \mathrm{d}_1^2 S[0,0,0](\mathrm{e}^{\mathrm{i}\tau}e_1, h^{01}_{00100}) + \mathrm{d}_1^2 S[0,0,0](\mathrm{e}^{-\mathrm{i}\tau}\bar{e}_1, h^{01}_{10000})\\
&= \mathrm{i}(J^0(0)\mathrm{e}^{\mathrm{i}\tau}e_1, \mathrm{e}^{\mathrm{i}\tau}e_1)\\
&= \mathrm{i}\Omega^0_0(e_1, \bar{e}_1)\\
&= -s_1,\\
\tilde{s}^{01}_{020} &= \mathrm{d}_1^2\mathrm{d}_3 S[0,0,0](\mathrm{e}^{\mathrm{i}\tau}e_2, \mathrm{e}^{-\mathrm{i}\tau}\bar{e}_2, 1) + \mathrm{d}_1^2 S[0,0,0](\mathrm{e}^{\mathrm{i}\tau}e_2, h^{01}_{00100}) + \mathrm{d}_1^2 S[0,0,0](\mathrm{e}^{-\mathrm{i}\tau}\bar{e}_2, h^{01}_{01000})\\
&= \mathrm{i}(J^0(0)\mathrm{e}^{\mathrm{i}\tau}e_2, \mathrm{e}^{\mathrm{i}\tau}e_2)\\
&= \mathrm{i}\Omega^0_0(e_2, \bar{e}_2)\\
&= -s_2,\\
\tilde{s}^{01}_{002} &= \tfrac{1}{2}\mathrm{d}_1^2\mathrm{d}_3 S[0,0,0](f_2, f_2, 1) + \mathrm{d}_1^2 S[0,0,0](f_2, h^{10}_{00001})\\
&= 0.
\end{aligned}
$$

Note that the second derivatives are extended *bilinearly* to the complexification of $\mathcal{Z}$ while $(\cdot\,,\cdot)$ is extended *sesquilinearly*. ■

Corollary 2.8. *One has the formulae*

$$\tilde{s}^{10}_{200} = \partial_{\mu_1}\kappa^0_1, \qquad \tilde{s}^{01}_{200} = \partial_{\mu_1}\kappa^0_2.$$

Proof. Observe that

$$\Omega^{\mu_1}_0(L^{\mu_1}e^{\mu_1}_i, \bar{e}^{\mu_1}_i) = \mathrm{i}\kappa^{\mu_1}_i\Omega^{\mu_1}_0(e^{\mu_1}_i, \bar{e}^{\mu_1}_i) = -s_i\kappa^{\mu_1}_i.$$

Differentiating this formula with respect to μ_1 and evaluating the result at $\mu_1 = 0$ yields

$$\begin{aligned}
-s_i\partial_{\mu_1}\kappa_i^0\Big|_{\mu_1=0} &= \Omega_0^0(\partial_{\mu_1}L^0 e_i,\bar{e}_i)+\Omega_0^1(Le_i,\bar{e}_i)+\Omega_0^0(L\partial_{\mu_1}e_i^{\mu_1},\bar{e}_i)+\Omega_0^0(Le_i,\partial_{\mu_1}\bar{e}_i^{\mu_1})\Big|_{\mu_1=0}\\
&= \Omega_0^0(\partial_{\mu_1}L^0 e_i,\bar{e}_i)+\Omega_0^1(Le_i,\bar{e}_i)-\Omega_0^0(\partial_{\mu_1}e_i^{\mu_1},L\bar{e}_i)+\Omega_0^0(Le_i,\partial_{\mu_1}\bar{e}_i^{\mu_1})\Big|_{\mu_1=0}\\
&= \Omega_0^0(\partial_{\mu_1}L^0 e_i,\bar{e}_i)+\mathrm{i}\kappa\Big(\Omega_0^1(e_i,\bar{e}_i)+\Omega_0^0(\partial_{\mu_1}e_i^{\mu_1},\bar{e}_i)+\Omega_0^0(e_i,\partial_{\mu_1}\bar{e}_i^{\mu_1})\Big)\Big|_{\mu_1=0}\\
&= \Omega_0^0(\partial_{\mu_1}L^0 e_i,\bar{e}_i)+\mathrm{i}\kappa\partial_{\mu_1}\underbrace{\Omega_0^{\mu_1}(e_i^{\mu_1},\partial_{\mu_1}\bar{e}_i^{\mu_1})}_{=s_i}\Big|_{\mu_1=0}\\
&= \Omega_0^0(\partial_{\mu_1}L^0 e_i,\bar{e}_i).
\end{aligned}$$

■

Finally, we solve equations (2.17)–(2.19) using the information given by Lemma 2.7 and Corollary 2.8, thus completing the proof of Theorem 2.1.

Lemma 2.9. *There exist* $\varepsilon > 0$ *and functions* $p^\star : B_\varepsilon(0) \to \mathbb{R}$, $\mu_1^\star : B_\varepsilon(0) \to \mathbb{R}$, $\mu_2^\star : B_\varepsilon(0) \to \mathbb{R}$ *such that the solution set of* (2.17)–(2.19) *in* $\mathcal{U}_{1,1} \times \Lambda_1 \times \Lambda_2$ *coincides with*

$$\{(r_1^2, r_2^2, p^\star(r_1^2,r_2^2), \mu_1^\star(r_1^2,r_2^2), \mu_2^\star(r_1^2,r_2^2) : |(r_1^2,r_2^2)| < \varepsilon\}.$$

Proof. Since

$$\begin{pmatrix}\partial_1\tilde{s}(0,0,0,0,0)\\ \partial_2\tilde{s}(0,0,0,0,0)\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}$$

and

$$\begin{aligned}
\det\begin{pmatrix}\partial_1\partial_4\tilde{s}(0,0,0,0,0) & \partial_1\partial_5\tilde{s}(0,0,0,0,0)\\ \partial_2\partial_4\tilde{s}(0,0,0,0,0) & \partial_2\partial_5\tilde{s}(0,0,0,0,0)\end{pmatrix} &= -s_2\tilde{s}_{200}^{10}+s_1\tilde{s}_{020}^{01}\\
&= -s_1s_2\partial_{\mu_1}(\kappa_1^0-\kappa_2^0)\\
&\neq 0,
\end{aligned}$$

we can solve equations (2.17), (2.18) locally for $\mu_1 = \mu_1(r_1^2, r_2^2, p)$, $\mu_2 = \mu_2(r_1^2, r_2^2, p)$ using the implicit-function theorem. Inserting this solution into (2.19) yields

$$\tilde{t}(r_1^2, r_2^2, p) = 0, \tag{2.20}$$

where

$$\tilde{t}(r_1^2,r_2^2,p) = \partial_3\tilde{s}(r_1^2,r_2^2,p,\mu_1(r_1^2,r_2^2,p),\mu_2(r_1^2,r_2^2,p)).$$

Furthermore

$$\tilde{t}(0,0,0) = \partial_3\tilde{s}(0,0,0,\mu_1(0,0,0),\mu_2(0,0,0)) = 0$$

and similarly

$$\begin{aligned}\partial_3\tilde{t}(0,0,0) &= \partial_3^2\tilde{s}(0,0,0,0,0) + \partial_3\mu_1(0,0,0)\partial_3\partial_4\tilde{s}(0,0,0,0,0) + \partial_3\mu_2(0,0,0)\partial_3\partial_5\tilde{s}(0,0,0,0,0)\\ &= 2\tilde{s}^{00}_{002}\\ &= -1.\end{aligned}$$

We can therefore solve equation (2.19) locally for $p = p(r_1^2, r_2^2)$ using the implicit-function theorem.

The assertion follows by setting $p^\star(r_1^2, r_2^2) = p(r_1^2, r_2^2)$, $\mu_1^\star(r_1^2, r_2^2) = \mu_1(r_1^2, r_2^2, p(r_1^2, r_2^2))$ and $\mu_2^\star(r_1^2, r_2^2) = \mu_2(r_1^2, r_2^2, p(r_1^2, r_2^2))$. ■

2.2 Doubly periodic hydroelastic waves

We consider waves which are periodic with periods p_1 and p_2 in two arbitrary horizontal directions x and z which form (different) angles θ_1, $\theta_2 \in [0, \pi)$ with the x_1-axis respectively, so that

$$x = \csc(\theta_2 - \theta_1)(x_1 \sin\theta_2 - x_3\cos\theta_2), \qquad z = \csc(\theta_1 - \theta_2)(x_1\sin\theta_1 - x_3\cos\theta_1)$$

(see Figure 1.1). To this end we seek solutions of the governing equations (1.20)–(1.23) of the form

$$\eta(x_1, x_3) = \tilde{\eta}(\tilde{x}, \tilde{z}), \qquad \phi(x_1, x_2, x_3) = \tilde{\phi}(\tilde{x}, x_2, \tilde{z}),$$

where

$$\tilde{x} = x_1\sin\theta_2 - x_3\cos\theta_2, \qquad \tilde{z} = \frac{2\pi}{p_2}(x_1\sin\theta_1 - x_3\cos\theta_1)$$

and $\tilde{\eta}$, $\tilde{\phi}$ are 2π-periodic in $\tilde{z}$ (the requirement that they are also periodic in $\tilde{x}$ is applied later). Dropping the tildes for notational simplicity, one finds that the governing equations become

$$\phi_{xx} + \phi_{x_2x_2} + \nu^2\phi_{zz} + 2\nu\cos(\theta_1 - \theta_2)\phi_{xz} = 0, \qquad -\tfrac{1}{\beta} < x_2 < \eta, \quad (2.21)$$

$$\phi_{x_2} = 0, \qquad x_2 = -\tfrac{1}{\beta}, \quad (2.22)$$

$$\begin{aligned}\phi_{x_2} = &-\gamma(\sin\theta_2\eta_x + \nu\sin\theta_1\eta_z) + \eta_x\phi_x + \nu^2\eta_z\phi_z\\ &+ \nu\cos(\theta_1-\theta_2)(\eta_x\phi_z + \eta_z\phi_x), \qquad x_2 = \eta, \quad (2.23)\end{aligned}$$

$$\begin{aligned}&-\gamma(\sin\theta_2\phi_x + \nu\sin\theta_1\phi_z)\\ &\quad + \tfrac{1}{2}(\phi_x^2 + \phi_{x_2}^2 + \nu^2\phi_z^2 + 2\nu\cos(\theta_1-\theta_2)\phi_x\phi_z) + \gamma\eta + U(\eta) = 0, \qquad x_2 = \eta, \quad (2.24)\end{aligned}$$

where

$$\begin{aligned}
U(\eta) = 2\Bigg(& \frac{1}{\sqrt{Q}}\Bigg[\partial_x\Bigg(\frac{1+\nu^2\sin^2(\theta_1-\theta_2)\eta_z^2}{\sqrt{Q}}P_x\Bigg) + \nu^2\partial_z\Bigg(\frac{1+\sin^2(\theta_1-\theta_2)\eta_x^2}{\sqrt{Q}}P_z\Bigg) \\
&\qquad + \nu\cos(\theta_1-\theta_2)\Bigg(\partial_x\Bigg(\frac{P_z}{\sqrt{Q}}\Bigg) + \partial_z\Bigg(\frac{P_x}{\sqrt{Q}}\Bigg)\Bigg) \\
&\qquad - \nu^2\sin^2(\theta_1-\theta_2)\Bigg(\partial_x\Bigg(\frac{\eta_x\eta_z}{\sqrt{Q}}P_z\Bigg) + \partial_z\Bigg(\frac{\eta_x\eta_z}{\sqrt{Q}}P_x\Bigg)\Bigg]\\
&+ 2P^3 - 2KP\Bigg), \\
Q = {} & 1 + \eta_x^2 + \nu^2\eta_z^2 + 2\nu\cos(\theta_1-\theta_2)\eta_x\eta_z, \\
P = {} & \frac{1}{2Q^{\frac{3}{2}}}\Big[\eta_{xx} + \nu^2\eta_{zz} + 2\nu\cos(\theta_1-\theta_2)\eta_{xz} \\
&\qquad + \nu^2\sin^2(\theta_1-\theta_2)(\eta_{xx}\eta_z^2 - 2\eta_{xz}\eta_x\eta_z + \eta_{zz}\eta_x^2)\Big], \\
K = {} & \frac{1}{Q^2}\nu^2\sin^2(\theta_1-\theta_2)(\eta_{xx}\eta_{zz} - \eta_{xz}^2)
\end{aligned}$$

and $\nu = 2\pi/p_2$.

We proceed by formulating equations (2.21)–(2.24) as a Hamiltonian system in which the horizontal spatial direction x plays the role of the time-like variable ('spatial dynamics'). Our starting point is the observation that these equations follow from the formal variational principle

$$\begin{aligned}
\delta\int\int_0^{2\pi}\Bigg\{&\int_{-\frac{1}{\beta}}^{\eta}\frac{1}{2}\Big(\phi_x^2 + \phi_{x_2}^2 + \nu^2\phi_z^2 + 2\nu\cos(\theta_1-\theta_2)\phi_x\phi_z\Big)\,\mathrm{d}x_2 \\
&+ 2\sqrt{Q}P^2 + \tfrac{1}{2}\eta^2 + \gamma(\eta_x\sin\theta_2 + \nu\eta_z\sin\theta_1)\phi|_{x_2=\eta}\Bigg\}\,\mathrm{d}z\,\mathrm{d}x = 0, \qquad (2.25)
\end{aligned}$$

in which the variations are taken over η and ϕ. Because of the difficulty in performing analysis on a variable domain, we use the change of variable

$$\phi(x, x_2, z) = \Phi(x, y, z), \qquad x_2 = \begin{cases} y + (1+\beta y)\eta, & \beta > 0, \\ y + \mathrm{e}^y\eta, & \beta = 0, \end{cases}$$

to map the variable fluid domain $\{-\frac{1}{\beta} < x_2 < \eta(x,z)\}$ to the fixed domain $\{-\frac{1}{\beta} < y < 0\}$. The variational principle (2.25) is transformed into

$$\delta\mathcal{L} = 0, \qquad \mathcal{L} = \int L(\eta, \eta_x, \eta_{xx}, \Phi, \Phi_x)\,\mathrm{d}x,$$

in which

$$
\begin{aligned}
&L(\eta,\eta_x,\eta_{xx},\Phi,\Phi_x)\\
&\quad=\int_0^{2\pi}\left\{\int_{-\frac{1}{\beta}}^0 \frac{1}{2K_2(\eta)}\Big((\Phi_x-K_1(\eta)\eta_x\Phi_y)^2+K_2(\eta)^2\Phi_y^2+\nu^2(\Phi_z-K_1(\eta)\eta_z\Phi_y)\right.\\
&\qquad\qquad\qquad\qquad\qquad +2\nu\cos(\theta_1-\theta_2)(\Phi_x-K_1(\eta)\eta_x\Phi_y)(\Phi_z-K_1(\eta)\eta_z\Phi_y)\Big)\,\mathrm{d}y\\
&\qquad\qquad\left.+2\sqrt{Q}P^2+\tfrac{1}{2}\eta^2+\gamma(\eta_x\sin\theta_2+\nu\eta_z\sin\theta_1)\Phi|_{y=0}\right\}\mathrm{d}z
\end{aligned}
$$

and

$$
K_1(\eta)=\begin{cases}\dfrac{1+\beta y}{1+\beta\eta}, & \beta>0,\\[2mm] \dfrac{\mathrm{e}^y}{1+\mathrm{e}^y\eta}, & \beta=0,\end{cases}\qquad
K_2(\eta)=\begin{cases}\dfrac{1}{1+\beta\eta}, & \beta>0,\\[2mm] \dfrac{1}{1+\mathrm{e}^y\eta}, & \beta=0.\end{cases}
$$

The next step is to perform a formal Legendre transformation by introducing the new coordinate

$$\rho=\eta_x$$

and momenta

$$
\begin{aligned}
\zeta&=\frac{\delta L}{\delta\eta_x}-\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\delta L}{\delta\eta_{xx}}\right)\\
&=-\int_{-\frac{1}{\beta}}^0\frac{K_1(\eta)}{K_2(\eta)}\Big((\Phi_x-K_1(\eta)\eta_x\Phi_y)\Phi_y+\nu\cos(\theta_1-\theta_2)(\Phi_z-K_1(\eta)\eta_z\Phi_y)\Phi_y\Big)\,\mathrm{d}y\\
&\quad-10\frac{P^2}{\sqrt{Q}}(\eta_x+\nu\cos(\theta_1-\theta_2)\eta_z)\\
&\quad+\frac{4P}{Q}\nu^2\sin^2(\theta_1-\theta_2)(-\eta_{xz}\eta_z+\eta_{zz}\eta_x)+\gamma\sin\theta_2\Phi|_{y=0}\\
&\quad-\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{2P}{Q}(1+\nu^2\sin^2(\theta_1-\theta_2)\eta_z^2)\right),\\
\xi&=\frac{\delta L}{\delta\eta_{xx}}\\
&=\frac{2P}{Q}(1+\nu^2\sin^2(\theta_1-\theta_2)\eta_z^2),\\
\Psi&=\frac{\delta L}{\delta\Phi_x}\\
&=\frac{1}{K_2(\eta)}(\Phi_x-K_1(\eta)\eta_x\Phi_y)+\frac{1}{K_2(\eta)}\nu\cos(\theta_1-\theta_2)(\Phi_z-K_1(\eta)\eta_z\Phi_y),
\end{aligned}
$$

and defining the Hamiltonian by

$$
\begin{aligned}
&H(\eta,\rho,\Phi,\zeta,\xi,\Psi)\\
&\quad=\int_S \zeta\eta_x\,\mathrm{d}z+\int_S \xi\eta_{xx}\,\mathrm{d}z+\int_\Sigma \Psi\Phi_x\,\mathrm{d}y\,\mathrm{d}z-L(\eta,\eta_x,\eta_{xx},\Phi,\Phi_x)\\
&\quad=\int_\Sigma \biggl\{\frac{K_2(\eta)}{2}(\Psi^2-\Phi_y^2)-\frac{1}{2K_2(\eta)}\nu^2\sin^2(\theta_1-\theta_2)(\Phi_z-K_1(\eta)\eta_z\Phi_y)^2\\
&\qquad\qquad+K_1(\eta)\rho\Phi_y\Psi-\nu\cos(\theta_1-\theta_2)(\Phi_z-K_1(\eta)\eta_z\Phi_y)\Psi\biggr\}\,\mathrm{d}y\,\mathrm{d}z\\
&\qquad+\int_S\biggl\{\zeta\rho-\frac{1}{2}\eta^2-\gamma(\rho\sin\theta_2+\nu\eta_z\sin\theta_1)\Phi|_{y=0}+\frac{Q(\eta,\rho)^{5/2}\xi^2}{2(1+\nu^2\sin^2(\theta_1-\theta_2)\eta_z^2)^2}\\
&\qquad\quad+\frac{\xi}{1+\nu^2\sin^2(\theta_1-\theta_2)\eta_z^2}\Bigl(-(1+\sin^2(\theta_1-\theta_2)\rho^2)\nu^2\eta_{zz}\\
&\qquad\qquad\qquad+2\nu^2\sin^2(\theta_1-\theta_2)\rho\rho_z\eta_z-2\nu\cos(\theta_1-\theta_2)\rho_z\Bigr)\biggr\}\,\mathrm{d}z,
\end{aligned}
\tag{2.26}
$$

where $S=(0,2\pi)$, $\Sigma=(-\frac{1}{\beta},0)\times(0,2\pi)$ and

$$Q(\eta,\rho)=1+\rho^2+\nu^2\eta_z^2+2\nu\cos(\theta_1-\theta_2)\rho\eta_z.$$

Hamilton's equations are

$$
\begin{aligned}
\eta_x&=\frac{\delta H}{\delta\zeta}\\
&=\rho,
\end{aligned}
\tag{2.27}
$$

$$
\begin{aligned}
\rho_x&=\frac{\delta H}{\delta\xi}\\
&=\frac{Q(\eta,\rho)^{5/2}\xi}{(1+\nu^2\sin^2(\theta_1-\theta_2)\eta_z^2)^2}-\frac{(1+\sin^2(\theta_1-\theta_2)\rho^2)\nu^2\eta_{zz}}{1+\nu^2\sin^2(\theta_1-\theta_2)\eta_z^2}\\
&\qquad-\frac{2\nu\cos(\theta_1-\theta_2)\rho_z}{1+\nu^2\sin^2(\theta_1-\theta_2)\eta_z^2}+\frac{2\nu^2\sin^2(\theta_1-\theta_2)\rho\rho_z\eta_z}{1+\nu^2\sin^2(\theta_1-\theta_2)\eta_z^2},
\end{aligned}
\tag{2.28}
$$

$$
\begin{aligned}
\Phi_x&=\frac{\delta H}{\delta\Psi}\\
&=K_2(\eta)\Psi+K_1(\eta)\rho\Phi_y-\nu\cos(\theta_1-\theta_2)(\Phi_z-K_1(\eta)\eta_z\Phi_y),
\end{aligned}
\tag{2.29}
$$

$$
\begin{aligned}
-\zeta_x &= \frac{\delta H}{\delta \eta} \\
&= \int_{-\frac{1}{\beta}}^{0} \Bigg\{ -\frac{K_2(\eta)^2 K_3}{2}(\Psi^2 - \Phi_y^2) - \frac{K_3}{2}\nu^2 \sin^2(\theta_1-\theta_2)(\Phi_z - K_1(\eta)\eta_z\Phi_y)^2 \\
&\qquad - K_1(\eta)K_2(\eta)K_3\rho\Phi_y\Psi - \nu^2\sin^2(\theta_1-\theta_2)K_1(\eta)K_3\eta_z\Phi_y(\Phi_z - K_1(\eta)\eta_z\Phi_y) \\
&\qquad - \nu\cos(\theta_1-\theta_2)K_1(\eta)K_2(\eta)K_3\eta_z\Phi_y\Psi - [\nu\cos(\theta_1-\theta_2)K_1(\eta)\Phi_y\Psi]_z \\
&\qquad - \left[\frac{K_1(\eta)}{K_2(\eta)}\nu^2\sin^2(\theta_1-\theta_2)(\Phi_z - K_1(\eta)\eta_z\Phi_y)\Phi_y\right]_z \Bigg\}\,\mathrm{d}y \\
&\quad + \left[\frac{2\nu^2\sin^2(\theta_1-\theta_2)Q(\eta,\rho)^{5/2}\xi^2\eta_z}{(1+\nu^2\sin^2(\theta_1-\theta_2)\eta_z^2)^3}\right]_z \\
&\quad - \left[\frac{5\nu^2 Q(\eta,\rho)^{3/2}\xi^2\eta_z}{2(1+\nu^2\sin^2(\theta_1-\theta_2)\eta_z^2)^2}\right]_z - \left[\frac{5\nu\cos(\theta_1-\theta_2)Q(\eta,\rho)^{3/2}\xi^2\rho}{2(1+\nu^2\sin^2(\theta_1-\theta_2)\eta_z^2)^2}\right]_z \\
&\quad + \Bigg[\frac{2\nu\sin^2(\theta_1-\theta_2)\xi\eta_z}{(1+\nu^2\sin^2(\theta_1-\theta_2)\eta_z^2)^2}\Big(-(1+\sin^2(\theta_1-\theta_2)\rho^2)\nu^2\eta_{zz} \\
&\qquad\qquad + 2\nu^2\sin^2(\theta_1-\theta_2)\rho\rho_z\eta_z - 2\nu\cos(\theta_1-\theta_2)\rho_z\Big)\Bigg]_z \\
&\quad - \left[\frac{(1+\sin^2(\theta_1-\theta_2)\rho^2)\nu^2\xi}{1+\nu^2\sin^2(\theta_1-\theta_2)\eta_z^2}\right]_{zz} - \left[\frac{2\nu^2\sin^2(\theta_1-\theta_2)\rho\rho_z\xi}{1+\nu^2\sin^2(\theta_1-\theta_2)\eta_z^2}\right]_z - \eta + \gamma\nu\sin\theta_1\Phi_z|_{y=0},
\end{aligned}
\tag{2.30}
$$

$$
\begin{aligned}
-\xi_x &= \frac{\delta H}{\delta \rho} \\
&= \zeta - \gamma\sin\theta_2\Phi|_{y=0} + \int_{-\frac{1}{\beta}}^{0} K_1(\eta)\Psi\Phi_y\,\mathrm{d}y + \frac{5Q(\eta,\rho)^{3/2}\xi^2\rho}{2(1+\nu^2\sin^2(\theta_1-\theta_2)\eta_z^2)^2} \\
&\quad + \frac{5\nu\cos(\theta_1-\theta_2)Q(\eta,\rho)^{3/2}\xi^2\eta_z}{2(1+\nu^2\sin^2(\theta_1-\theta_2)\eta_z^2)^2} - \frac{2\nu^2\sin^2(\theta_1-\theta_2)\xi\rho\eta_{zz}}{1+\nu^2\sin^2(\theta_1-\theta_2)\eta_z^2} \\
&\quad - \left[\frac{2\nu^2\sin^2(\theta_1-\theta_2)\xi\eta_z}{1+\nu^2\sin^2(\theta_1-\theta_2)\eta_z^2}\right]_z \rho + \left[\frac{2\nu\cos(\theta_1-\theta_2)\xi}{1+\nu^2\sin^2(\theta_1-\theta_2)\eta_z^2}\right]_z,
\end{aligned}
\tag{2.31}
$$

$$
\begin{aligned}
-\Psi_x &= \frac{\delta H}{\delta \Phi} \\
&= (K_2(\eta)\Phi_y)_y - (K_1(\eta)\Psi\rho)_y - \left(\frac{K_1(\eta)}{K_2(\eta)}\nu^2\sin^2(\theta_1-\theta_2)(\Phi_z - K_1(\eta)\eta_z\Phi_y)\eta_z\right)_y \\
&\quad + \left[\frac{1}{K_2(\eta)}\nu^2\sin^2(\theta_1-\theta_2)(\Phi_z - K_1(\eta)\eta_z\Phi_y)\right]_z \\
&\quad + \nu\cos(\theta_1-\theta_2)\Psi_z - \nu\cos(\theta_1-\theta_2)(K_1(\eta)\eta_z\Psi)_y,
\end{aligned}
\tag{2.32}
$$

where

$$K_3 = \begin{cases} \beta, & \beta > 0, \\ \mathrm{e}^y, & \beta = 0, \end{cases}$$

with boundary conditions

$$-K_2(\eta)\Phi_y = 0, \qquad y = -\tfrac{1}{\beta}, \qquad (2.33)$$

$$\begin{aligned} &-K_2(\eta)\Phi_y + K_1(\eta)\Psi\rho + \frac{K_1(\eta)}{K_2(\eta)}\nu^2 \sin^2(\theta_1 - \theta_2)(\Phi_z - K_1(\eta)\eta_z\Phi_y)\eta_z \\ &\qquad + \nu\cos(\theta_1 - \theta_2)K_1(\eta)\eta_z\Psi - \gamma(\rho\sin\theta_2 + \nu\eta_z\sin\theta_1) = 0, \qquad y = 0. \qquad (2.34) \end{aligned}$$

We also introduce a bifurcation parameter μ_1 by writing $\nu = \nu_0 + \mu_1$, where ν_0 is a reference value for ν to be chosen later.

To place equations (2.27)–(2.34) on a rigorous footing, we introduce the spaces

$$\begin{aligned} H^m_{\mathrm{per}}(S) &= \{w \in H^m_{\mathrm{loc}}(\mathbb{R}) : w(z + 2\pi) = w(z) \text{ for all } z \in \mathbb{R}\}, \\ H^m_{\mathrm{per}}(\Sigma) &= \{w \in H^m_{\mathrm{loc}}((-\tfrac{1}{\beta}, 0) \times \mathbb{R}) : w(y, z + 2\pi) = w(y, z) \text{ for all } (y, z) \in (-\tfrac{1}{\beta}, 0) \times \mathbb{R}\} \end{aligned}$$

for $m \in \mathbb{N}_0$; recall that $H^1_{\mathrm{per}}(S)$ and $H^2_{\mathrm{per}}(\Sigma)$ are Banach algebras, while the formulae $w \mapsto w|_{y=0}$ and $w \mapsto w|_{y=-\frac{1}{\beta}}$ (for $\beta > 0$) define bounded linear mappings $H^m(\Sigma) \to H^{m-1}(S)$ for $m \in \mathbb{N}$. The following proposition relates to mappings appearing in the above equations (see Bagri & Groves [6, Proposition 2.1] for parts (i), (ii) and Buffoni & Toland [11] for parts (iii), (iv)).

Proposition 2.10.

(i) The formula $(w_1, w_2) \mapsto w_1 w_2$ defines bounded bilinear mappings $L^2_{\mathrm{per}}(\Sigma) \times H^1_{\mathrm{per}}(S) \to L^2_{\mathrm{per}}(\Sigma)$, $H^1_{\mathrm{per}}(\Sigma) \times L^2_{\mathrm{per}}(S) \to L^2_{\mathrm{per}}(\Sigma)$ and $H^1_{\mathrm{per}}(\Sigma) \times H^1_{\mathrm{per}}(S) \to H^1_{\mathrm{per}}(\Sigma)$.

(ii) The formula

$$(w_1, w_2) \mapsto \int_{-\frac{1}{\beta}}^{0} w_1(\cdot, y) w_2(\cdot, y)\,\mathrm{d}y$$

defines bounded bilinear mappings $L^2_{\mathrm{per}}(\Sigma) \times H^1_{\mathrm{per}}(\Sigma) \to L^2_{\mathrm{per}}(S)$, $H^1_{\mathrm{per}}(\Sigma) \times L^2_{\mathrm{per}}(\Sigma) \to L^2_{\mathrm{per}}(S)$ and $H^1_{\mathrm{per}}(\Sigma) \times H^1_{\mathrm{per}}(\Sigma) \to H^1_{\mathrm{per}}(S)$.

(iii) The formulae $\eta \mapsto (1 + \beta\eta)^{-1} - 1$ (for $\beta > 0$) and $\eta \mapsto (1 + \mathrm{e}^y\eta)^{-1} - 1$ (for $\beta = 0$) yield mappings $H^3_{\mathrm{per}}(S) \to H^3_{\mathrm{per}}(S)$ and $H^3_{\mathrm{per}}(S) \to H^3_{\mathrm{per}}(\Sigma)$ respectively which are defined and analytic in a neighbourhood of the origin.

(iv) For each $n \in \mathbb{N}$ the formula $\eta \mapsto (1 + (\nu_0 + \mu)^2 \sin^2(\theta_1 - \theta_2)\eta_z^2)^{-n} - 1$ yields a mapping $\mathbb{R} \times H^3_{\mathrm{per}}(S) \to H^2_{\mathrm{per}}(S)$ which is defined and analytic in a neighbourhood of the origin.

(v) For each $n \in \mathbb{N}$ the formula $(\eta,\rho) \mapsto Q(\eta,\rho)^{\frac{n}{2}}$ yields a mapping $H^3_{\mathrm{per}}(S) \times H^2_{\mathrm{per}}(S) \to H^2_{\mathrm{per}}(S)$ which is defined and analytic in a neighbourhood of the origin.

Let us now define

$$X=\{v=(\eta,\rho,\Phi,\zeta,\xi,\Psi)\in H^3_{\mathrm{per}}(S)\times H^2_{\mathrm{per}}(S)\times H^2_{\mathrm{per}}(\Sigma)\times H^1_{\mathrm{per}}(S)\times H^2_{\mathrm{per}}(S)\times H^1_{\mathrm{per}}(\Sigma)\},$$
$$Z=\{v=(\eta,\rho,\Phi,\zeta,\xi,\Psi)\in H^2_{\mathrm{per}}(S)\times H^1_{\mathrm{per}}(S)\times H^1_{\mathrm{per}}(\Sigma)\times L^2_{\mathrm{per}}(S)\times H^1_{\mathrm{per}}(S)\times L^2_{\mathrm{per}}(\Sigma)\}.$$

The following lemma, which is a consequence of the previous proposition, shows that the right-hand sides of equations (2.27)–(2.32) define an analytic mapping $v^{\mu_1}_{\mathrm{H}} : \Lambda_1 \times U \to Z$, where U is a neighbourhood of the origin of X. In the notation of the lemma, these equations define a quasilinear evolutionary system

$$v_x = v^{\mu_1}_{\mathrm{H}}(v) \tag{2.35}$$

with nonlinear boundary conditions given by

$$\Phi_y = F^{\mu_1}(\eta,\rho,\Phi,\zeta,\xi,\Psi), \qquad y=-\tfrac{1}{\beta}, \tag{2.36}$$
$$\Phi_y + \gamma(\rho\sin\theta_2 + \nu_0\eta_z\sin\theta_1) = F^{\mu_1}(\eta,\rho,\Phi,\zeta,\xi,\Psi), \qquad y=0, \tag{2.37}$$

where

$$\begin{aligned}
&F^{\mu_1}(\eta,\rho,\Phi,\zeta,\xi,\Psi)\\
&\quad= K_2(\eta)K_3\eta\Phi_y + K_1(\eta)\Psi\rho + (\nu_0+\mu_1)\cos(\theta_1-\theta_2)K_1(\eta)\eta_z\Psi\\
&\qquad+\frac{K_1(\eta)}{K_2(\eta)}(\nu_0+\mu_1)^2\sin^2(\theta_1-\theta_2)(\Phi_z - K_1(\eta)\eta_z\Phi_y)\eta_z - \frac{K_1(\eta)}{K_2(\eta)}\gamma\mu_1\eta_z\sin\theta_1,
\end{aligned}$$

and we note for later use that

$$\begin{aligned}
&\mathrm{d}F^{\mu_1}[\eta,\rho,\Phi,\zeta,\xi,\Psi](\tilde\eta,\tilde\rho,\tilde\Phi,\tilde\zeta,\tilde\xi,\tilde\Psi)\\
&\quad= -K_2(\eta)^2K_3^2\eta\Phi_y\tilde\eta + K_2(\eta)K_3(\Phi_y\tilde\eta + \eta\tilde\Phi_y) - K_1(\eta)K_2(\eta)K_3\Psi\rho\tilde\eta + K_1(\eta)(\Psi\tilde\rho+\rho\tilde\Psi)\\
&\qquad+(\nu_0+\mu_1)\cos(\theta_1-\theta_2)\Big(-K_1(\eta)K_2(\eta)K_3\eta_z\Psi\tilde\eta + K_1(\eta)(\Psi\tilde\eta_z + \eta_z\tilde\Psi)\Big)\\
&\qquad+\frac{K_1(\eta)}{K_2(\eta)}(\nu_0+\mu_1)^2\sin^2(\theta_1-\theta_2)\\
&\qquad\qquad\times\Big((\Phi_z - K_1(\eta)\eta_z\Phi_y)\tilde\eta_z + \eta_z\Big(\tilde\Phi_z + K_1(\eta)K_2(\eta)K_3\eta_z\Phi_y\tilde\eta - K_1(\eta)(\Phi_y\tilde\eta_z+\eta_z\tilde\Phi_y)\Big)\Big)\\
&\qquad-\frac{K_1(\eta)}{K_2(\eta)}\gamma\mu_1\sin\theta_1\tilde\eta_z.
\end{aligned}$$

This system is reversible; the reverser is given by

$$\begin{aligned}
&R\Big(\eta(z),\rho(z),\Phi(y,z),\zeta(z),\xi(z),\Psi(y,z)\Big)\\
&\qquad= \Big(\eta(-z),-\rho(-z),-\Phi(y,-z),-\zeta(-z),\xi(-z),\Psi(y,-z)\Big).
\end{aligned}$$

Lemma 2.11. *There exist neighbourhoods U and Λ_1 of the origin in respectively X and $\mathbb{R}$ with the following properties.*

(i) The formula $(\mu_1, v) \mapsto v_{\mathrm{H}}^{\mu_1}(v)$, where $v_{\mathrm{H}}^{\mu_1}(v)$ is defined by the right-hand sides of (2.27)–(2.32) *(with $\nu = \nu_0 + \mu_1$), defines an analytic mapping $\Lambda_1 \times U \to Z$.*

(ii) The formula $(\mu_1, v) \mapsto F^{\mu_1}(v)$ defines an analytic mapping $\Lambda_1 \times U \to H^1(\Sigma)$.

(iii) The derivative $\mathrm{d}F^{\mu_1}[v] \in \mathcal{L}(X, H^1(\Sigma))$ has a unique extension $\widetilde{\mathrm{d}F^{\mu_1}}[v] \in \mathcal{L}(Z, L^2(\Sigma))$ which depends analytically upon $(\mu_1, v) \in \Lambda_1 \times U$.

It remains to confirm that (2.35) has a Hamiltonian structure; for this purpose we use the following lemma, which is proved by direct calculations and Proposition 2.10.

Lemma 2.12.

(i) The formula $(\mu_1, v) \mapsto H^{\mu_1}(v)$, where $H^{\mu_1}(v)$ is defined by the right-hand side of (2.26) *(with $\nu = \nu_0 + \mu_1$), defines an analytic mapping $\Lambda_1 \times U \to \mathbb{R}$.*

(ii) The derivative $\mathrm{d}H^{\mu_1}[v] \in X^*$ has a unique extension $\widetilde{\mathrm{d}H^{\mu_1}}[v] \in Z^*$ which depends analytically upon $(\mu_1, v) \in \Lambda_1 \times U$.

(iii) The formula

$$\widetilde{\mathrm{d}H^{\mu_1}}[v](w) = \langle J v_{\mathrm{H}}^{\mu_1}(v), w \rangle,$$

where $\langle \cdot\,, \cdot \rangle$ denotes the $(L^2_{\mathrm{per}}(S))^2 \times L^2_{\mathrm{per}}(\Sigma) \times (L^2_{\mathrm{per}}(S))^2 \times L^2_{\mathrm{per}}(\Sigma)$ inner product and

$$J(\eta, \rho, \Phi, \zeta, \xi, \Psi) = (-\zeta, -\xi, -\Psi, \eta, \rho, \Phi),$$

holds for all $(\mu_1, v) \in \mathcal{D}_{\mathrm{H}}$ and $w \in Z$, where

$$\mathcal{D}_{\mathrm{H}} = \{(\mu_1, v) \in \Lambda_1 \times U : (2.36),\ (2.37)\ \textit{are satisfied}\},$$

so that the gradient $\nabla H^{\mu_1}(v)$ exists (and equals $J v_{\mathrm{H}}^{\mu_1}(v)$) for all $(\mu_1, v) \in \mathcal{D}_{\mathrm{H}}$ and extends to an analytic function of $(\mu_1, v) \in \Lambda_1 \times U$.

Altogether we conclude that $v_{\mathrm{H}}^{\mu_1}(v) = J^{-1} \nabla H^{\mu_1}(v)$ for $(\mu_1, v) \in \mathcal{D}_{\mathrm{H}}$ defines the Hamiltonian vector field for the Hamiltonian system (Z, Ω, H^{μ_1}), where $\Omega : Z^2 \to \mathbb{R}$ is the constant symplectic 2-form

$$\begin{aligned}
&\Omega\Big((\eta_1, \rho_1, \Phi_1, \zeta_1, \xi_1, \Psi_1), (\eta_2, \rho_2, \Phi_2, \zeta_2, \xi_2, \Psi_2)\Big) \\
&\quad = \langle J(\eta_1, \rho_1, \Phi_1, \zeta_1, \xi_1, \Psi_1), (\eta_2, \rho_2, \Phi_2, \zeta_2, \xi_2, \Psi_2) \rangle \\
&\quad = \int_S (\zeta_2 \eta_1 - \eta_2 \zeta_1 + \xi_2 \rho_1 - \rho_2 \xi_1)\,\mathrm{d}z + \int_\Sigma (\Psi_2 \Phi_1 - \Phi_2 \Psi_1)\,\mathrm{d}y\,\mathrm{d}z.
\end{aligned}$$

Note that Ω is the exterior derivative of the parameter-independent 1-form $\omega|_v$ given by

$$\begin{aligned}\omega|_{(\eta,\rho,\Phi,\zeta,\xi,\Psi)}(\tilde\eta,\tilde\rho,\tilde\Phi,\tilde\zeta,\tilde\xi,\tilde\Psi) &= \int_S (\eta\tilde\zeta+\rho\tilde\xi)\,\mathrm{d}z + \int_\Sigma \Phi\tilde\Psi\,\mathrm{d}y\,\mathrm{d}z\\ &= \langle \alpha(\eta,\rho,\Phi,\zeta,\xi,\Psi),(\tilde\eta,\tilde\rho,\tilde\Phi,\tilde\zeta,\tilde\xi,\tilde\Psi)\rangle,\end{aligned}$$

where

$$\alpha(\eta,\rho,\Phi,\zeta,\xi,\Psi) = (0,0,0,\eta,\rho,\Phi).$$

The system (2.35)–(2.37) is unsuitable for analysis due to its nonlinear boundary conditions. We proceed by replacing Φ with the new variable

$$\Gamma = \Phi - \partial_y\Delta^{-1}F^{\mu_1}(\eta,\rho,\Phi,\zeta,\xi,\Psi),$$

where $\Delta^{-1}\Upsilon$ is the unique solution of the boundary value problem

$$\begin{aligned} \Delta f &= \Upsilon, && -\tfrac{1}{\beta}<y<0,\\ f&=0, && y=0,\\ f&=0, && y=-\tfrac{1}{\beta}.\end{aligned}$$

In the notation of the following lemma we find that for $(\eta,\rho,\zeta,\xi,\Psi)\in W$ and $\mu_1\in\Lambda_1$ the variable $\Phi\in V_1$ satisfies (2.36), (2.37) if and only if the variable $\Gamma\in V_2$ satisfies

$$\Gamma_y = 0, \qquad y=-\tfrac{1}{\beta}, \tag{2.38}$$

$$\Gamma_y + \gamma(\rho\sin\theta_2+\nu_0\eta_z\sin\theta_1) = 0, \qquad y=0, \tag{2.39}$$

because

$$\begin{aligned}\Gamma_y &= \Phi_y - \partial_{yy}\Delta^{-1}F^{\mu_1}(\eta,\rho,\Phi,\zeta,\xi,\Psi)\\ &= \Phi_y - \partial_{yy}\Delta^{-1}F^{\mu_1}(\eta,\rho,\Phi,\zeta,\xi,\Psi) - \underbrace{\partial_{zz}\Delta^{-1}F^{\mu_1}(\eta,\rho,\Phi,\zeta,\xi,\Psi)}_{=0}\\ &= \Phi_y - F^{\mu_1}(\eta,\rho,\Phi,\zeta,\xi,\Psi)\end{aligned}$$

for $y=0$ and $y=-\frac{1}{\beta}$.

Lemma 2.13.

(i) There exist neighbourhoods V_1, V_2 of the origin in $H^2_{\mathrm{per}}(\mathbb{R})$ and W of the origin in

$$X_0 = \{(\eta,\rho,\zeta,\xi,\Psi)\in H^3_{\mathrm{per}}(S)\times H^2_{\mathrm{per}}(S)\times H^1_{\mathrm{per}}(S)\times H^2_{\mathrm{per}}(S)\times H^1_{\mathrm{per}}(\Sigma)\}$$

such that $\Phi\mapsto\Gamma(\Phi,(\eta,\rho,\zeta,\xi,\Psi),\mu_1)$ is an analytic diffeomorphism $V_1\to V_2$ which, together with its inverse, depends analytically upon $(\eta,\rho,\zeta,\xi,\Psi)\in W$ and $\mu_1\in\Lambda_1$.

(ii) The derivative $\mathrm{d}_1\Gamma[\Phi,(\eta,\rho,\zeta,\xi,\Psi),\mu_1] \in \mathcal{L}(H^2_{\mathrm{per}}(\Sigma))$ *extends to an isomorphism in* $\mathcal{L}(H^1_{\mathrm{per}}(\Sigma))$ *which, together with its inverse, depends analytically upon* $\Phi \in V_1$, $(\eta,\rho,\zeta,\xi,\Psi) \in W$ *and* $\mu_1 \in \Lambda_1$.

Proof. (i) This result follows by applying the implicit-function theorem to the equation

$$g(\Phi,\Gamma,(\eta,\rho,\zeta,\xi,\Psi),\mu_1) := \Gamma - \Gamma(\Phi,(\eta,\rho,\zeta,\xi,\Psi),\mu_1) = 0.$$

Here we note that g maps (a neighbourhood of the origin in) $H^2_{\mathrm{per}}(\Sigma) \times H^2_{\mathrm{per}}(\Sigma) \times X_0 \times \mathbb{R}$ into $H^2_{\mathrm{per}}(\Sigma)$ (by Lemma 2.11(ii) and the fact that Δ^{-1} belongs to $\mathcal{L}(H^1_{\mathrm{per}}(\Sigma), H^3_{\mathrm{per}}(\Sigma)))$, and that $g(0,0,0,0) = 0$ and $\mathrm{d}_1 g[0,0,0,0] = -I$.

(ii) It follows from Lemma 2.11(iii) and the fact that Δ^{-1} belongs to $\mathcal{L}(L^2_{\mathrm{per}}(\Sigma), H^2_{\mathrm{per}}(\Sigma))$ that $\mathrm{d}_1\Gamma[\Phi,(\eta,\rho,\zeta,\xi,\Psi),\mu_1] \in \mathcal{L}(H^2_{\mathrm{per}}(\Sigma))$ extends to an element $\widetilde{\mathrm{d}_1\Gamma}[\Phi,(\eta,\rho,\zeta,\xi,\Psi),\mu_1]$ of $\mathcal{L}(H^1_{\mathrm{per}}(\Sigma))$ which depends analytically upon $\Phi \in V_1$, $(\eta,\rho,\zeta,\xi,\Psi) \in W$ and $\mu_1 \in \Lambda_1$. Obviously $\widetilde{\mathrm{d}_1\Gamma}[0,0,0,0] = I$ is an isomorphism, which is an open property. The analyticity of $\widetilde{\mathrm{d}_1\Gamma}[\cdot]$ implies the analyticity of its inverse. ■

It follows from the above lemma that the formula

$$G^{\mu_1}(\eta,\rho,\Phi,\zeta,\xi,\Psi) = (\eta,\rho,\Gamma,\zeta,\xi,\Psi)$$

defines a valid change of variable: it is an analytic diffeomorphism from U to a neighbourhood $\hat{U}$ of the origin in X, the operator $\mathrm{d}G^{\mu_1}[u] \in \mathcal{L}(X)$ extends to an isomorphism $\widetilde{\mathrm{d}G^{\mu_1}}[u] \in \mathcal{L}(Z)$, and G^{μ_1}, $\widetilde{\mathrm{d}G^{\mu_1}}[u]$ and their inverses depend analytically upon $(u,\mu_1) \in U \times \Lambda_1$. The system (2.35)–(2.37) is transformed into

$$\hat{v}_x = \hat{v}^{\mu_1}_{\mathrm{H}}(\hat{v}), \tag{2.40}$$

where

$$\hat{v}^{\mu_1}_{\mathrm{H}}(\hat{v}) = \widetilde{\mathrm{d}G^{\mu_1}}[(G^{\mu_1})^{-1}(\hat{v})](v^{\mu_1}_{\mathrm{H}}((G^{\mu_1})^{-1}(\hat{v})))$$

with linear boundary conditions (2.38), (2.39). Note also that G^{μ_1} and $(G^{\mu_1})^{-1}$ both commute with the reverser R, so that (2.40) inherits the reversibility of equation (2.35).

Writing $(G^{\mu_1})^{-1}$ as K^{μ_1}, one finds that the change of variable transforms (Z,Ω,H^{μ_1}) into the new Hamiltonian system $(Z,\hat{\Omega}^{\mu},\hat{H}^{\mu})$, where

$$\hat{\Omega}^{\mu_1}|_{\hat{v}}(\hat{v}_1,\hat{v}_2) = \langle \hat{J}^{\mu_1}(\hat{v})\hat{v}_1,\hat{v}_2\rangle \tag{2.41}$$

with

$$\hat{J}^{\mu_1}(\hat{v}) = \widetilde{\mathrm{d}K^{\mu_1}}[\hat{v}]^* J \widetilde{\mathrm{d}K^{\mu_1}}[\hat{v}]$$

and

$$\hat{H}^{\mu_1}(\hat{v}) = H^{\mu_1}(K^{\mu_1}(\hat{v})) \tag{2.42}$$

for $(\mu_1, \hat{v}) \in \Lambda_1 \times \hat{U}$. In particular

$$\hat{v}_{\mathrm{H}}^{\mu_1}(\hat{v}) = \hat{J}^{\mu_1}(\hat{v})\nabla \hat{H}^{\mu_1}(\hat{v})$$

for $(\mu_1, \hat{v}) \in \hat{\mathcal{D}}_{\mathrm{H}}$, where

$$\hat{\mathcal{D}}_{\mathrm{H}} = \{(\mu_1, \hat{v}) \in \Lambda_1 \times \hat{U} : (2.38),\ (2.39) \text{ are satisfied}\}.$$

Note further that $\hat{\Omega}^{\mu_1}|_{\hat{v}}$ is the exterior derivative of the 1-form $\hat{\omega}^{\mu_1}|_{\hat{v}}$ given by

$$\hat{\omega}^{\mu_1}|_{\hat{v}}(\hat{w}) = \langle \hat{\alpha}^{\mu_1}(\hat{v}), \hat{w}\rangle$$

with

$$\hat{\alpha}^{\mu_1}(\hat{v}) = \widetilde{\mathrm{d}K^{\mu_1}}[\hat{v}]^*(\alpha(K^{\mu_1}(\hat{v}))).$$

The validity of these calculations relies upon the existence of the adjoint operator $\widetilde{\mathrm{d}K^{\mu_1}}[\hat{v}]^*$, and this assumption is verified in the following result.

Proposition 2.14. *The adjoint operators $\widetilde{\mathrm{d}G^{\mu_1}}[v]^*$, $\widetilde{\mathrm{d}K^{\mu_1}}[\hat{v}]^* \in \mathcal{L}(Z)$ exist and depend analytically upon $(\mu_1, v) \in \Lambda_1 \times U$ and $(\mu_1, \hat{v}) \in \Lambda_1 \times \hat{U}$.*

Proof. The existence of the adjoint $\widetilde{\mathrm{d}G^{\mu_1}}[v]^* \in \mathcal{L}(Z)$ follows by a direct calculation; its components are given by

$$\begin{aligned}
(\widetilde{\mathrm{d}G^{\mu_1}}[v]^*(\tilde{v}))_\eta = \tilde{\eta} + \int_{-\frac{1}{\beta}}^0 \Bigg\{ \Bigg(& - K_2(\eta)^2 K_3^2 \eta \Phi_y + K_2(\eta) K_3 \Phi_y \\
& - K_1(\eta) K_2(\eta) K_3 \Psi \rho - (\nu_0 + \mu_1)\cos(\theta_1 - \theta_2) K_1(\eta) K_2(\eta) K_3 \eta_z \Psi \\
& + \frac{K_1(\eta)}{K_2(\eta)} (\nu_0 + \mu_1)^2 \sin^2(\theta_1 - \theta_2) \eta_z K_1(\eta) K_2(\eta) K_3 \eta_z \Phi_y \Bigg) \Delta^{-1}(\tilde{\Gamma}_y) \\
- \Bigg(\Bigg(& (\nu_0 + \mu_1)\cos(\theta_1 - \theta_2) K_1(\eta) \Psi \\
& + \frac{K_1(\eta)}{K_2(\eta)} (\nu_0 + \mu_1)^2 \sin^2(\theta_1 - \theta_2)(\Phi_z - K_1(\eta)\eta_z \Phi_y) \\
& - \frac{K_1^2(\eta)}{K_2(\eta)} (\nu_0 + \mu_1)^2 \sin^2(\theta_1 - \theta_2) \eta_z \Phi_y \\
& - \frac{K_1(\eta)}{K_2(\eta)} \gamma \mu_1 \sin\theta_1 \Bigg) \Delta^{-1}(\tilde{\Gamma}_y) \Bigg)_z \Bigg\} \,\mathrm{d}y, \\
(\widetilde{\mathrm{d}G^{\mu_1}}[v]^*(\tilde{v}))_\rho = \tilde{\rho} + \int_{-\frac{1}{\beta}}^0 & K_1(\eta) \Psi \Delta^{-1}(\tilde{\Gamma}_y) \,\mathrm{d}y,
\end{aligned}$$

$$
\begin{aligned}
(\widetilde{\mathrm{d}G}^{\mu_1}[v]^*(\tilde{v}))_\Gamma &= \tilde{\Gamma} - \left(\left(K_2(\eta)K_3\eta - \frac{K_1^2(\eta)}{K_2(\eta)}(\nu_0+\mu_1)^2\sin^2(\theta_1-\theta_2)\eta_z^2\right)\Delta^{-1}(\tilde{\Gamma}_y)\right)_y \\
&\qquad - \left(\frac{K_1(\eta)}{K_2(\eta)}(\nu_0+\mu_1)^2\sin^2(\theta_1-\theta_2)\eta_z\Delta^{-1}(\tilde{\Gamma}_y)\right)_z, \\
(\widetilde{\mathrm{d}G}^{\mu_1}[v]^*(\tilde{v}))_\zeta &= \tilde{\zeta}, \\
(\widetilde{\mathrm{d}G}^{\mu_1}[v]^*(\tilde{v}))_\xi &= \tilde{\xi}, \\
(\widetilde{\mathrm{d}G}^{\mu_1}[v]^*(\tilde{v}))_\Psi &= \tilde{\Psi} + \left(K_1(\eta)\rho + (\nu_0+\mu_1)\cos(\theta_1-\theta_2)K_1(\eta)\eta_z\right)\Delta^{-1}(\tilde{\Gamma}_y),
\end{aligned}
$$

from which the analyticity of $\widetilde{\mathrm{d}G}^{\mu_1}[v]^*$ also follows by Proposition 2.10. (Note that the formula $(\Delta^{-1})^* = \Delta^{-1}$ has been used in this calculation.)

Observe that $\widetilde{\mathrm{d}G}^{\mu_1}[v]^*$ is an isomorphism because $\widetilde{\mathrm{d}G}^0[0]^* = I$ is obviously an isomorphism, which is an open property. Furthermore the analytic dependence of $\widetilde{\mathrm{d}G}^{\mu_1}[v]^*$ upon $(\mu_1, v) \in \Lambda_1 \times U$ implies the same of its inverse, and the calculation

$$
\begin{aligned}
\langle \widetilde{\mathrm{d}G}^{\mu_1}[v]^{-1}(v_1), v_2\rangle &= \langle \widetilde{\mathrm{d}G}^{\mu_1}[v]^{-1}v_1, \widetilde{\mathrm{d}G}^{\mu_1}[v]^*(\widetilde{\mathrm{d}G}^{\mu_1}[v]^*)^{-1}(v_2)\rangle \\
&= \langle \widetilde{\mathrm{d}G}^{\mu_1}[v]\widetilde{\mathrm{d}G}^{\mu_1}[v]^{-1}(v_1), (\widetilde{\mathrm{d}G}^{\mu_1}[v]^*)^{-1}(v_2)\rangle \\
&= \langle v_1, (\widetilde{\mathrm{d}G}^{\mu_1}[v]^*)^{-1}(v_2)\rangle
\end{aligned}
$$

shows that $(\widetilde{\mathrm{d}G}^{\mu_1}[v]^{-1})^*$ exists and equals $(\widetilde{\mathrm{d}G}^{\mu_1}[v]^*)^{-1}$. The proof is completed by noting that $\widetilde{\mathrm{d}K}^{\mu_1}[\hat{v}] = \widetilde{\mathrm{d}G}^{\mu_1}[v]^{-1}$ with $v = (G^{\mu_1})^{-1}(\hat{v})$. ■

2.3 Application of Lyapunov centre theory to hydroelastic waves

In this section we apply Theorem 2.1 to the spatial dynamics formulation

$$\hat{v}_x = \hat{v}_{\mathrm{H}}^{\mu_1}(\hat{v})$$

for hydroelastic waves derived in Section 2.2. For this purpose we define

$$
\begin{aligned}
X = \{\hat{v} = (\eta,\rho,\Gamma,\zeta,\xi,\Psi) \in{}& H^3_{\mathrm{per}}(S)\times H^2_{\mathrm{per}}(S)\times H^2_{\mathrm{per}}(\Sigma)\times H^1_{\mathrm{per}}(S)\times H^2_{\mathrm{per}}(S)\times H^1_{\mathrm{per}}(\Sigma): \\
&\Gamma_y\Big|_{y=-\frac{1}{\beta}} = 0,\ \Gamma_y\Big|_{y=0} + \gamma(\rho\sin\theta_2 + \nu_0\eta_z\sin\theta_1) = 0\}, \\
Z = \{v = (\eta,\rho,\Gamma,\zeta,\xi,\Psi) \in{}& H^2_{\mathrm{per}}(S)\times H^1_{\mathrm{per}}(S)\times H^1_{\mathrm{per}}(\Sigma)\times L^2_{\mathrm{per}}(S)\times H^1_{\mathrm{per}}(S)\times L^2_{\mathrm{per}}(\Sigma)\}
\end{aligned}
$$

(note the modification to the space X) and consider $\hat{v}_{\mathrm{H}}^{\mu_1}$ as the Hamiltonian vector field for the Hamiltonian system $(Z, \hat{\Omega}^{\mu_1}, \hat{H}^{\mu_1})$, where $\hat{\Omega}^{\mu_1}$ and $\hat{H}^{\mu_1}$ are defined in equations (2.41) and

(2.42) and $\mathcal{D}_H = \Lambda_1 \times \hat{U}$ is a neighbourhood of the origin in $\mathbb{R} \times X$. Defining $L^{\mu_1} = \mathrm{d}\hat{v}_{\mathrm{H}}^{\mu_1}[0]$ and $N^{\mu_1}(\hat{v}) = \hat{v}_{\mathrm{H}}^{\mu_1}(\hat{v}) - L^{\mu_1}\hat{v}$, one can write Hamilton's equations as

$$\hat{v}_x = L^{\mu_1}\hat{v} + N^{\mu_1}(\hat{v}), \tag{2.43}$$

where in particular $L := L^0$ is given by the explicit formula

$$L\begin{pmatrix}\eta\\ \rho\\ \Gamma\\ \zeta\\ \xi\\ \Psi\end{pmatrix} = \begin{pmatrix}\rho\\ \xi - \nu_0^2\eta_{zz} - 2\nu_0\cos(\theta_1-\theta_2)\rho_z\\ \Psi - \nu_0\cos(\theta_1-\theta_2)\Gamma_z\\ \nu_0^2\xi_{zz} + \eta - \gamma\nu_0\sin\theta_1\Gamma_z|_{y=0}\\ -\zeta + \gamma\sin\theta_2\Gamma|_{y=0} - 2\nu_0\cos(\theta_1-\theta_2)\xi_z\\ -\Gamma_{yy} - \nu_0^2\sin^2(\theta_1-\theta_2)\Gamma_{zz} - \nu_0\cos(\theta_1-\theta_2)\Psi_z\end{pmatrix},$$

which is readily calculated from (2.27)–(2.32) since the change of variable used to linearise the boundary condition is near-identity. Equation (2.43) evidently satisfies hypothesis (H1).

Furthermore, the reverser R clearly satisfies $R^* = R$ and

$$H^{\mu_1}(Rv) = H^{\mu_1}(v), \quad R^*\alpha^{\mu_1}(Rv) = -\alpha^{\mu_1}(v), \quad R^*J^{\mu_1}(Rv)R = -J^{\mu_1}(v)$$

for all $(\mu_1, v) \in \Lambda_1 \times U$; since R commutes with G^{μ_1} and K^{μ_1} we conclude that

$$\hat{H}^{\mu_1}(R\hat{v}) = \hat{H}^{\mu_1}(\hat{v}), \quad R^*\hat{\alpha}^{\mu_1}(R\hat{v}) = -\hat{\alpha}^{\mu_1}(\hat{v}), \quad R^*\hat{J}^{\mu_1}(R\hat{v})R = -\hat{J}^{\mu_1}(\hat{v})$$

for all $(\mu_1, \hat{v}) \in \Lambda_1 \times \hat{U}$. Hypothesis (H2) is therefore also satisfied.

2.3.1 Purely imaginary spectrum

The next step is to examine the purely imaginary spectrum of the linear operator L. This task is readily accomplished by using Fourier-series representations

$$\hat{v}(y,z) = \sum_{k\in\mathbb{Z}} \hat{v}_k(y)\mathrm{e}^{\mathrm{i}kz}, \qquad \hat{v}^\star(y,z) = \sum_{k\in\mathbb{Z}} \hat{v}_k^\star(y)\mathrm{e}^{\mathrm{i}kz}$$

for $\hat{v} \in X$ and $\hat{v}^\star \in Z$ and examining the resulting decoupled spectral problems for each Fourier mode. We begin with the following lemma, which is proved by well-established methods for spatial dynamics problems (see Groves & Haragus [20, Lemma 3], Bagri & Groves [6, Lemmata 3.1, 3.2] and the references therein).

Lemma 2.15.

(i) *Suppose that* $s \in \mathbb{R}$ *and* $k \in \mathbb{Z}$ *are not both zero. The imaginary number* $\mathrm{i}s$ *is a mode* k *eigenvalue of* L *if and only if*

$$(1+\sigma_k^4)\sigma_k - \frac{\gamma^2}{t_k}(k\nu_0 \sin\theta_1 + s\sin\theta_2)^2 = 0, \tag{2.44}$$

and the corresponding eigenvector is $\hat{v}_{k,s}\mathrm{e}^{\mathrm{i}kz}$. *This eigenvalue has a Jordan chain of length at least* 2 *with generalised eigenvector* $\hat{w}_{k,s}\mathrm{e}^{\mathrm{i}kz}$ *if either*

(a) $\beta > 0$, $s = a\beta$, $\nu_0 = \tilde{\nu}_0\beta$, $(5+\tilde{c}_k)\tilde{a}_k\tilde{b}_k - 2\sin\theta_2\tilde{\sigma}_k^2 \neq 0$ *and* (β,γ) *lies on a point of the curve*

$$C_k = \{(\beta_k(a), \gamma_k(a)) : a \in (0,\infty)\},$$

where

$$\beta_k^4(a) = \frac{1}{\tilde{\sigma}_k^4}\cdot\frac{2\sin\theta_2\tilde{\sigma}_k^2 - (1+\tilde{c}_k)\tilde{a}_k\tilde{b}_k}{(5+\tilde{c}_k)\tilde{a}_k\tilde{b}_k - 2\sin\theta_2\tilde{\sigma}_k^2},$$
$$\gamma_k^2(a) = \frac{(1+\beta_k^4(a)\tilde{\sigma}_k^4)\tilde{\sigma}_k\tanh(\tilde{\sigma}_k)}{\beta_k(a)\tilde{b}_k^2};$$

(b) $\beta > 0$, $s = a\beta$, $\nu_0 = \tilde{\nu}_0\beta$ *and* $(5+\tilde{c}_k)\tilde{a}_k\tilde{b}_k - 2\sin\theta_2\tilde{\sigma}_k^2 = 0$, *which implies that* $\theta_2 = 0$ *and* $a = -k\tilde{\nu}_0\cos(\theta_1)$*;*

(c) $\beta = 0$ *and*

$$2\sin\theta_2 - \frac{4\sigma_k^2 a_k b_k}{1+\sigma_k^4} = \frac{a_k b_k}{\sigma_k^2}.$$

Expressions for the quantities a_k, b_k, c_k, $\tilde{a}_k$, $\tilde{b}_k$, $\tilde{c}_k$, t_k, $\tilde{t}_k$, σ_k *and* $\tilde{\sigma}_k$ *as functions of the parameters* s, k, ν_0, θ_1, θ_2 *are given in Appendix 2.4.*

(ii) *Suppose* $\beta > 0$. *Zero is a mode* 0 *eigenvalue of* L *with a Jordan chain of length* 2 *if* $\gamma^{-2} \neq \beta\sin^2\theta_2$ *and length* 4 *if* $\gamma^{-2} = \beta\sin^2\theta_2$*; the generalised eigenvectors are* $\hat{f}_1, \hat{f}_2, \hat{f}_3, \hat{f}_4$ *where* $L\hat{f}_1 = 0$, $L\hat{f}_2 = \hat{f}_1$ *and* $L\hat{f}_3 = \hat{f}_2$, $L\hat{f}_4 = \hat{f}_3$ *if* $\gamma^{-2} = \beta\sin^2\theta_2$.

(iii) *Suppose* $\beta = 0$ *and that* $s = 0$ *does not solve* (2.44) *for any* $k \in \mathbb{Z}\setminus\{0\}$ *(so that zero is not a mode* k *eigenvalue for any* $k \in \mathbb{Z}\setminus\{0\}$*). Zero is not a mode* 0 *eigenvalue of* L*, which instead has essential spectrum at the origin. More precisely, the equation* $L\hat{v} = \hat{v}^\star$ *has a unique solution for each* $\hat{v}^\star \in Z$ *which satisfies the regularity requirement*

$$\int_{-\infty}^{y}\int_{-\infty}^{t}\Psi_0^\star(s)\,\mathrm{d}s\,\mathrm{d}t,\ \int_{-\infty}^{y}\Psi_0^\star(t)\,\mathrm{d}t \in L^2(-\infty,0)$$

and compatibility condition

$$\int_{-\infty}^{0}\Psi_0^\star(t)\,\mathrm{d}t - \gamma\sin\theta_2\eta_0^\star = 0.$$

This solution satisfies the estimate

$$\|\hat{v} - [\![\hat{v}]\!]_0\|_X \lesssim \|\hat{v}^\star\|_Z$$

and its 0*th Fourier component* $[\![\hat{v}]\!]_0$ *is given by the formula*

$$\begin{pmatrix} \hat{\eta}_0 \\ \hat{\rho}_0 \\ \hat{\Gamma}_0 \\ \hat{\zeta}_0 \\ \hat{\xi}_0 \\ \hat{\Psi}_0 \end{pmatrix} = \begin{pmatrix} \zeta_0^\star \\ \eta_0^\star \\ -\int_{-\infty}^{y} \int_{-\infty}^{t} \Psi_0^\star(s)\, \mathrm{d}s\, \mathrm{d}t \\ -\xi_0^\star - \gamma \sin\theta_2 \int_{-\infty}^{0} \int_{-\infty}^{t} \Psi_0^\star(s)\, \mathrm{d}s\, \mathrm{d}t \\ \rho_0^\star \\ \Gamma_0^\star \end{pmatrix}.$$

(iv) Suppose that $s \in \mathbb{R} \setminus \{0\}$ *does not satisfy* (2.44) *for any* $k \in \mathbb{Z}$*. The imaginary number* is *belongs to the resolvent set of* L*.*

(v) The resolvent estimates

$$\|(\mathrm{i}sI - L)^{-1}\|_{Z \to X} \lesssim 1, \qquad \|(\mathrm{i}sI - L)^{-1}\|_{Z \to Z} \lesssim \frac{1}{|s|}$$

hold uniformly over all sufficiently large values of $|s|$*.*

Formulae for the generalised eigenvectors $\hat{v}_{k,s}$*,* $\hat{w}_{k,s}$ *and* $\hat{f}_1$*,* $\hat{f}_2$*,* $\hat{f}_3$*,* $\hat{f}_4$ *are given in Appendix 2.4.*

We proceed by interpreting equation (2.44) geometrically. Let

$$\ell_1 = s\sin\theta_2 + \nu_0 k \sin\theta_1, \tag{2.45}$$

$$\ell_2 = -s\cos\theta_2 - \nu_0 k \cos\theta_1, \tag{2.46}$$

and note that $\ell_1^2 + \ell_2^2 = \sigma_k^2$, so that (2.44) can be written as

$$\mathcal{D}(\ell_1, \ell_2) := (1 + (\ell_1^2 + \ell_2^2)^2)\sqrt{\ell_1^2 + \ell_2^2} \tanh\left(\beta^{-1}\sqrt{\ell_1^2 + \ell_2^2}\right) - \gamma^2 \ell_1^2 = 0. \tag{2.47}$$

A mode k purely imaginary eigenvalue is (with $(k, s) \neq (0, 0)$) therefore corresponds to an intersection in the (ℓ_1, ℓ_2)-plane of the dispersion curve

$$C_{\mathrm{dr}} = \{(\ell_1, \ell_2) \in \mathbb{R}^2 \setminus \{(0, 0)\} : \mathcal{D}(\ell_1, \ell_2) = 0\}$$

with the straight line S_k defined by equations (2.45), (2.46). (The solution $(\ell_1, \ell_2) = (0, 0)$ of $\mathcal{D}(\ell_1, \ell_2) = 0$ is excluded since it corresponds to $(k, s) = (0, 0)$.)

The dispersion curve C_{dr} is described parametrically by

$$C_{\mathrm{dr}} := \Big\{(\ell_1,\ell_2)\in\mathbb{R}^2 : \ell_1^2 = \frac{(1+a^4)a}{\gamma^2}\tanh(\beta^{-1}a),\ \ell_2^2 = a^2 - \frac{(1+a^4)a}{\gamma^2}\tanh(\beta^{-1}a),\ a>0\Big\};$$

its shape is shown in Figure 1.2 (a, insets) in the indicated regions of the (β,γ)-parameter plane. The delimiting curves are

$$D_1 = \Big\{(\beta_0(a),\gamma_0(a))\Big|_{k=0,\,\theta_2=\frac{\pi}{2}} : a\in(0,\infty)\Big\},$$

at each point of which the equation $\mathcal{D}(a\beta,0)=0$ has double roots $\pm a$, and

$$D_2 = \{(\beta,\beta^{-1/2}) : \beta\geq 0\},$$

at each point of which the equation $\mathcal{D}(\ell_1,0)=0$ has a double zero root. We find that $C_{\mathrm{dr}}=\emptyset$ in the region below the curve D_1. In the region between the curves D_1 and D_2 the equation $\mathcal{D}(\ell_1,0)=0$ has two pairs of simple nonzero roots $\pm\ell_1^{(1)}$, $\pm\ell_1^{(2)}$; the branches of C_{dr} intersect the ℓ_1 axis vertically at the points $(\pm\ell_1^{(1)},0)$ and $(\pm\ell_1^{(2)},0)$. Notice the qualitative difference in the shape of C_{dr} in the two subregions. In the upper subregion the portion C_{dr}^{+} of C_{dr} in the positive quadrant has two points of inflection (it is concave to the left of the first and to the right of the second and convex in between); passing into the lower subregion, one finds that the two points of inflection merge and disappear, so that C_{dr}^{+} becomes concave. The points $(\pm\ell_1^{(1)},0)$ approach the origin as one passes through D_2 from below to above, as does the left point of inflection on C_{dr}^{+}. In the region above D_2 the branches of C_{dr} intersect the ℓ_1 axis vertically at the points $(\pm\ell_1^{(2)},0)$, while in a neighbourhood of $(0,0)$ they have the limiting behaviour

$$\ell_2^2 \sim \left(\gamma^2\beta-1\right)\ell_1^2$$

as $\ell_1\to 0$ and therefore make angles $\pm\arctan\sqrt{\gamma^2\beta-1}$ with the ℓ_1 axis at the origin. The subcurve C_{dr}^{+} has a single point of inflection, to the left and right of which it is respectively convex and concave.

For given ν, θ_1 and θ_2 the lines S_k in the (ℓ_1,ℓ_2)-plane are parallel, equidistant and form an angle θ_2 with the positive ℓ_2-axis. They intersect the line

$$T = \{(\ell_1,\ell_2)\in\mathbb{R}^2\ :\ \ell_1=\sin\theta_1\,a,\ \ell_2=-\cos\theta_1\,a,\ a\in\mathbb{R}\}$$

(which passes through the origin and makes an angle θ_1 with the positive ℓ-axis) at the points $P_k = (\sin\theta_1\,k\nu_0, -\cos\theta_1\,k\nu_0)$, $k\in\mathbb{Z}$ (see Figure 1.2(b)). The number of points in the set $S_0\cap C_{\mathrm{dr}}$ depends only upon (β,γ), which determines the shape of C_{dr}, and θ_2, which determines the slope of each line S_k. Furthermore, for fixed β, γ and θ_2 the number of points in the sets $S_k\cap C_{\mathrm{dr}}$, $k=\pm1,\pm2,\dots$ depends only upon ν_0, which determines the distance between the lines S_k. At each fixed point of the (β,γ)-parameter plane the number of purely imaginary eigenvalues of the linear operator L therefore depends upon the two parameters θ_2 and ν_0; the third parameter θ_1, which specifies the slope of the line T, influences only the values of these eigenvalues and their relative positions on the imaginary axis: the imaginary part of a purely imaginary eigenvalue corresponding to an intersection of S_k and C_{dr} is the value of S_0 in the (S_0, T)-coordinate system at the intersection (the signed distance between the intersection and the point P_n). The geometric multiplicity of the eigenvalue, is, is given by the number of distinct lines in the family S_k that intersect C_{dr} at this parameter value, and a tangent intersection between S_k and C_{dr} indicates that each eigenvector in mode k has an associated Jordan chain of length at least 2. Finally, notice that the sets $S_k \cap C_{\mathrm{dr}}$ and $S_{-k} \cap C_{\mathrm{dr}}$ have the same cardinality: the purely imaginary number is is a mode k eigenvalue if and only if the purely imaginary number $-$is is a mode $-k$ eigenvalue.

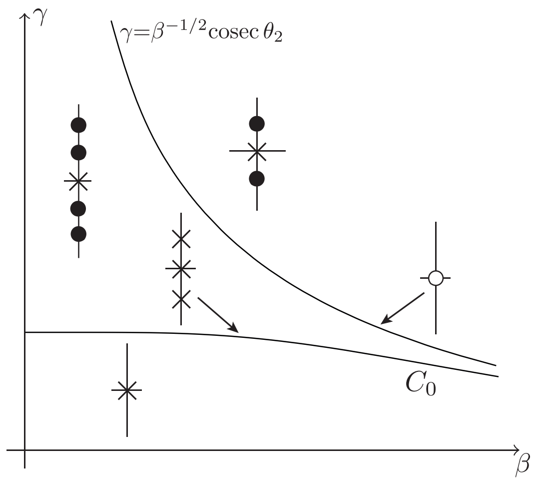

Figure 2.1: Mode 0 eigenvalues in the (β, γ)-parameter plane. Solid dots, crosses and hollow dots represent eigenvalues with Jordan chains of length 1, 2 and 4; the zero eigenvalue becomes essential spectrum at $\beta = 0$.

Figure 2.1 illustrates how the purely imaginary mode 0 eigenvalues depend upon (β, γ), nonzero pairs $\pm$is of which satisfy

$$(1 + s^4)|s| - \frac{\gamma^2 \sin^2 \theta_2 s^2}{\tanh(\beta^{-1}|s|)} = 0.$$

The delimiting curves are C_0 and $\{(\beta, \beta^{-1/2} \operatorname{cosec} \theta_2) : \beta > 0\}$, points of which are associated with respectively non-zero eigenvalues $\pm \mathrm{i} a \beta_0(a)$ with a Jordan chain of length 2 and a zero eigenvalue with a Jordan chain of length 4.

2.3.2 Parameter selection

We now choose β, γ and θ_2 such that S_0 does not intersect C_{dr} (so that (β,γ) lies in the region above the curve C_0 in Figure 2.1), and ν_0 and θ_1 such that S_1 and S_{-1} each intersect C_{dr} in points with coordinates $(\pm s,\nu_0)$ and $(\pm s,-\nu_0)$ in the (S_0,T)-coordinate system, while S_k does not intersect C_{dr} for $k=\pm 2,\ \pm 3,\ \ldots$ (see Figure 1.3). In this configuration L has two mode 1 eigenvalues $\pm\mathrm{i}s$ two mode -1 eigenvalues $\pm\mathrm{i}s$ so that $\pm\mathrm{i}s$ are geometrically and algebraically double eigenvalues of L with eigenvectors $\hat{v}_{1,s}\mathrm{e}^{\mathrm{i}z}$, $\hat{v}_{-1,s}\mathrm{e}^{-\mathrm{i}z}$ and $\hat{v}_{1,-s}\mathrm{e}^{\mathrm{i}z}$, $\hat{v}_{-1,-s}\mathrm{e}^{-\mathrm{i}z}$, and, for $\beta>0$, a mode 0 zero eigenvalue with a Jordan chain $\hat{f}_1$, $\hat{f}_2$ of length 2. Note that L^{μ_1} has two mode 1 eigenvalues $\mathrm{i}s_1^{\mu_1}$, $-\mathrm{i}s_{-1}^{\mu_1}$ and two mode -1 eigenvalues $\mathrm{i}s_{-1}^{\mu_1}$, $-\mathrm{i}s_1^{\mu_1}$ which satisfy

$$g(s_{\pm 1}^{\mu_1},\nu_0+\mu_1,\pm 1)=0, \tag{2.48}$$

where

$$\begin{aligned}g(t,\nu,\pm 1)&=(1+(\sigma_{\pm 1})^4)\sigma_{\pm 1}-\frac{\gamma^2(\pm\nu^2\sin\theta_1+t\sin\theta_2)^2}{\tanh(\beta^{-1}\sigma_{\pm 1})},\\ (\sigma_{\pm 1})^2&=t^2\pm 2\nu t\cos(\theta_1-\theta_2)+\nu^2\end{aligned}$$

and $s_{\pm 1}^0=s$ (the corresponding eigenvectors are given by $\mathrm{d}G[0](v_{1,\pm s_\pm^{\mu_1}})$ and $\mathrm{d}G[0](v_{-1,\pm s_\pm^{\mu_1}})$, where $v_{1,\pm s_\pm^{\mu_1}}$, $v_{-1,\pm s_\pm^{\mu_1}}$ are defined by the right-hand side of equation (2.59) with ν_0 replaced by $\nu_0+\mu_1$). Differentiating (2.48) with respect to μ_1, we obtain the formulae

$$\frac{\mathrm{d}}{\mathrm{d}\mu_1}s_{\pm 1}^{\mu_1}=-\frac{g_\nu(s_{\pm 1}^{\mu},\nu_0+\mu,\pm 1)}{g_t(s_{\pm 1}^{\mu},\nu_0+\mu,\pm 1)}$$

which imply that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\mu}(s_1^{\mu_1}-s_{-1}^{\mu_1})|_{\mu_1=0}&=-\frac{g_\nu(s,\nu_0,1)}{g_t(s,\nu_0,1)}+\frac{g_\nu(s,\nu_0,-1)}{g_t(s,\nu_0,-1)}\\ &=\frac{1}{g_t(s,\nu_0,1)g_t(s,\nu_0,-1)}\begin{vmatrix} g_t(s,\nu_0,1) & g_t(s,\nu_0,-1)\\ g_\nu(s,\nu_0,1) & g_\nu(s,\nu_0,-1)\end{vmatrix}.\end{aligned}$$

It follows that $\frac{\mathrm{d}}{\mathrm{d}\mu}(s_1^\mu-s_{-1}^\mu)|_{\mu=0}=0$ if and only if $\nabla g(s,\nu_0,1),\ \nabla g(s,\nu_0,-1)$ are parallel, in other words if and only if the solution curves of

$$\begin{aligned}g(t,\nu,1)&=0,\\ g(t,\nu,-1)&=0\end{aligned}$$

intersect tangentially at the point (s,ν_0) in the (t,ν)-plane. This observation indicates that generically $\frac{\mathrm{d}}{\mathrm{d}\mu}(s_1^\mu-s_{-1}^\mu)|_{\mu=0}\neq 0$.

Noting that

$$\hat{v}_{1,-s}\mathrm{e}^{\mathrm{i}z}=\overline{\hat{v}_{-1,s}\mathrm{e}^{-\mathrm{i}z}}=R(\hat{v}_{-1,s}\mathrm{e}^{-\mathrm{i}z}),\qquad \hat{v}_{-1,-s}\mathrm{e}^{-\mathrm{i}z}=\overline{\hat{v}_{1,s}\mathrm{e}^{\mathrm{i}z}}=R(\hat{v}_{1,s}\mathrm{e}^{\mathrm{i}z})$$

and $R\hat{f}_1 = -\hat{f}_1$, $R\hat{f}_2 = \hat{f}_2$, we find that there exists parameters such that hypotheses (H3)–(H6) are satisfied (with (H4)(ii) for $\beta = 0$ and (H4)(iii) for $\beta > 0$). Hypothesis (H7) follows from the observations that $(G^{\mu_1})^{-1}(\hat{f}_1) = \hat{f}_1$ and that $\hat{J}^{\mu_1}(\hat{v})$, $\hat{H}^{\mu_1}(\hat{v})$ depend upon Γ and ζ only through Γ_y and $\zeta - \gamma \sin\theta_1 \Gamma|_{y=0}$, while hypothesis (H8) is verified in Section 2.3.3 below.

Applying Theorem 2.1 thus yields a family of doubly periodic waves whose periodic cells are small perturbations of the basic periodic cell defined by the periods $2\pi/\nu_0$, $2\pi/s$ and angle $\theta_2 - \theta_1$ between the periodic directions. The following lemma shows that it is in fact possible to choose the basic periodic cell (and value of β) arbitrarily and adjust the value of γ and the angle θ_1 to ensure that Theorem 2.1 applies in this configuration (under the additional hypothesis that S_k does not intersect C_{dr} for $k \neq \pm 1$) .

Lemma 2.16. *Choose β, s, ν_0 and $\theta_2 - \theta_1$. There exist θ_1 and γ such that S_1 and S_{-1} each intersect C_{dr} in points with coordinates $(\pm s, \nu_0)$ and $(\pm s, -\nu_0)$ in the (S_0, T)-coordinate system.*

Proof. The lines S_1 and S_{-1} intersect C_{dr} at respectively (s, ν_0) and $(s, -\nu_0)$ if and only if

$$\gamma^2 b_1^2 = \tanh(\beta^{-1}\sigma_1)(1 + \sigma_1^4)\sigma_1,$$
$$\gamma^2 b_{-1}^2 = \tanh(\beta^{-1}\sigma_{-1})(1 + \sigma_{-1}^4)\sigma_{-1},$$

and in this case they also intersect C_{dr} at respectively $(-s, -\nu_0)$ and $(-s, \nu_0)$ (see the remarks at the end of Section 2.3.1). Let e_x, e_z and i be unit vectors in the x, z and x_1 directions (see Figure 1.1) and set

$$\ell_1 = s e_z + \nu_0 e_x, \qquad \ell_2 = -s e_z + \nu_0 e_x, \tag{2.49}$$

so that $\sigma_1^2 = |\ell_1|^2$, $\sigma_{-1} = |\ell_2|^2$, $i \cdot \ell_1 = b_1$, $i \cdot \ell_2 = b_{-1}$; in this notation our task is to find a solution of the equations

$$(v \cdot \ell_1)^2 = \tanh(\beta^{-1}|\ell_1|)(1 + |\ell_1|^4)|\ell_1|, \tag{2.50}$$
$$(v \cdot \ell_2)^2 = \tanh(\beta^{-1}|\ell_2|)(1 + |\ell_2|^4)|\ell_2| \tag{2.51}$$

of the form $v = \gamma i$. Let ψ denote the angle between ℓ_1 and ℓ_2; equation (2.49) shows that rotating e_x and e_z through the same angle θ, that is changing θ_1 and θ_2 by θ, causes ℓ_1 and ℓ_2 to rotate through θ and thus does not change ψ.

Observe that the equations

$$v \cdot n_1 = \pm\alpha_1,$$
$$v \cdot n_2 = \pm\alpha_2,$$

where n_1, $n_2 \in \mathbb{R}^2$ are linearly independent and α_1, $\alpha_2 > 0$, represent two pairs of parallel lines which intersect in four points. Each of these points v satisfies

$$|v| = \frac{|\alpha_2 n_1 \pm \alpha_1 n_2|}{|n_1||n_2|},$$

so that $|v|$ depends only upon α_1, α_2, $|n_1|$, $|n_2|$ and the angle between n_1 and n_2. Using this result we find that the solution set to equations (2.50), (2.51) consists of four points. Let v be one of these points, and note that $|v|$ depends only upon $|\ell_1|$, $|\ell_2|$ and ψ (since the right-hand sides of (2.50), (2.51) depend only upon $|\ell_1|$ and $|\ell_2|$.) By rotating ℓ_1 and ℓ_2 through a suitably chosen angle θ, that is changing θ_1 and θ_2 by θ, we can arrange that $v = |v|i$ and hence $v = \gamma \mathrm{i}$ by setting $\gamma = |v|$.

■

2.3.3 Verification of hypothesis (H8)

It remains to verify that hypothesis (H8) is satisfied when $\beta = 0$. The condition is that the equation

$$Lv^\dagger = \hat{J}^0(0)^{-1}[\hat{J}^{\mu_1}(\hat{v})\Big((\kappa_0 + \mu_2)\hat{v}_\tau - \hat{v}_{\mathrm{H}}^{\mu_1}(\hat{v})\Big) + \hat{J}^0(0)L\hat{v}]_0, \tag{2.52}$$

has a unique solution $v^\dagger \in X$ which depends smoothly upon $(\hat{v}, \mu_1, \mu_2) \in \mathcal{U} \times \Lambda_1 \times \Lambda_2$. Since

$$\hat{J}^0(0)^{-1}[\hat{J}^0(0)L\hat{v}]_0 = L[\hat{v}]_0$$

one can rewrite equation (2.52) as

$$L\Big(v^\dagger - [\hat{v}]_0\Big) = \hat{J}^0(0)^{-1}[\hat{J}^{\mu_1}(\hat{v})\Big((\kappa_0 + \mu_2)\hat{v}_\tau - \hat{v}_{\mathrm{H}}^{\mu_1}(\hat{v})\Big)]_0. \tag{2.53}$$

In view of equation (2.53) and Lemma 2.15(iii) our task is to show that

$$(\hat{v}, \mu_1, \mu_2) \mapsto \int_{-\infty}^{y} [\![[\hat{w}_\Gamma]_0]\!]_0, \qquad (\hat{v}, \mu_1, \mu_2) \mapsto \int_{-\infty}^{y}\int_{-\infty}^{t} [\![[\hat{w}_\Gamma]_0]\!]_0 \,\mathrm{d}s\,\mathrm{d}t, \tag{2.54}$$

where

$$\hat{w} = \hat{J}^{\mu_1}(\hat{v})\Big((\kappa_0 + \mu_2)\hat{v}_\tau - \hat{v}_{\mathrm{H}}^{\mu_1}(\hat{v})\Big),$$

are smooth mappings $\mathcal{U} \times \Lambda_1 \times \Lambda_2 \to L^2(-\infty, 0)$ and that

$$\int_{-\infty}^{0} [\![[\hat{w}_\Gamma]_0]\!]_0 \;\mathrm{d}y = -[\![[\hat{w}_\zeta]_0]\!]_0 \gamma \sin\theta_2 \tag{2.55}$$

for each $(\hat{v}, \mu_1, \mu_2) \in \mathcal{U} \times \Lambda_1 \times \Lambda_2$. Here we use the symbols $[\cdot]_0$ and $[\![\cdot]\!]_0$ to denote the projections onto the 0th Fourier modes in the τ and z variables respectively. This task is accomplished in Lemma 2.18 below with the help of the following auxiliary result.

Proposition 2.17. *The formulae*

$$(v, \mu_1) \mapsto \int_{-\infty}^{y} [\![[(v_{\mathrm{H}}^{\mu_1}(v))_\Psi]_0]\!]_0 \,\mathrm{d}t, \qquad (v, \mu_1) \mapsto \int_{-\infty}^{y}\int_{-\infty}^{t} [\![[(v_{\mathrm{H}}^{\mu_1}(v))_\Psi]_0]\!]_0 \,\mathrm{d}s\,\mathrm{d}t$$

define analytic mapppings $\mathcal{U} \times \Lambda_1 \to L^2(-\infty,0)$ *and*

$$\int_{-\infty}^{0} [\![[(v_{\mathrm{H}}^{\mu_1}(v))_\Psi]_0]\!]_0 \,\mathrm{d}y = [\![[(v_{\mathrm{H}}^{\mu_1}(v))_\eta]_0]\!]_0 \gamma \sin\theta_2$$

for each $(v,\mu_1) \in \mathcal{U} \times \Lambda_1$.

Proof. It follows from (2.32) that

$$[\![[(v_{\mathrm{H}}^{\mu_1}(v))_\Psi]_0]\!]_0 = [\![[(g_0^{\mu_1}(v))_y]_0]\!]_0,$$

where

$$\begin{aligned} g_0^{\mu_1}(v) &= \Phi_y - K_1(\eta)\eta\Phi_y - K_1(\eta)\Psi\rho - \frac{K_1(\eta)}{K_2(\eta)}(\nu_0+\mu_1)^2\sin^2(\theta_1-\theta_2)(\Phi_z - K_1(\eta)\eta_z\Phi_y)\eta_z \\ &\qquad - (\nu_0+\mu_1)\cos(\theta_1-\theta_2)K_1(\eta)\eta_z\Psi \\ &= \Phi_y - \Big(K_2(\eta)\eta\Phi_y - K_2(\eta)\Psi\rho - (\nu_0+\mu_1)^2\sin^2(\theta_1-\theta_2)(\Phi_z - K_1(\eta)\eta_z\Phi_y)\eta_z \\ &\qquad\qquad - (\nu_0+\mu_1)\cos(\theta_1-\theta_2)K_2(\eta)\eta_z\Psi\Big)\mathrm{e}^y \end{aligned}$$

and $v = (\eta,\rho,\Phi,\zeta,\xi,\Psi)$. Observing that $(v,\mu_1) \mapsto g_0^{\mu_1}(v)$ maps $U \times \Lambda_1$ analytically into $L^2(-\infty,0)$ (see Proposition 2.10) and the same is true of $(v,\mu_1) \mapsto \int_{-\infty}^{y} g_0^{\mu_1}(v)\,\mathrm{d}t$ because $u \mapsto \int_{-\infty}^{y} u(t)\mathrm{e}^t\,\mathrm{d}t$ belongs to $\mathcal{L}(L^2(-\infty,0))$, we conclude that $(v,\mu_1) \mapsto \int_{-\infty}^{y} [\![[(v_{\mathrm{H}}^{\mu_1}(v))_\Psi]_0]\!]_0\,\mathrm{d}t$ and $(v,\mu_1) \mapsto \int_{-\infty}^{y}\int_{-\infty}^{t} [\![[(v_{\mathrm{H}}^{\mu_1}(v))_\Psi]_0]\!]_0\,\mathrm{d}s\,\mathrm{d}t$ map $\mathcal{U} \times \Lambda_1$ analytically into $L^2(-\infty,0)$. Finally

$$\int_{-\infty}^{0} [\![[(v_{\mathrm{H}}^{\mu_1}(v))_\Psi]_0]\!]_0 \,\mathrm{d}y = [\![[\rho]_0]\!]_0\gamma\sin\theta_2 = [\![[(v_{\mathrm{H}}^{\mu_1}(v))_\eta]_0]\!]_0 \gamma \sin\theta_2$$

because of (2.33), (2.34) and (2.27). ■

Lemma 2.18. *The formulae* (2.54) *define analytic mappings* $\mathcal{U} \times \Lambda_1 \times \Lambda_2 \to L^2(-\infty,0)$ *and the formula* (2.55) *is satisfied for each* $(\hat{v},\mu_1,\mu_2) \in \mathcal{U} \times \Lambda_1 \times \Lambda_2$.

Proof. We first note that

$$\widetilde{\mathrm{d}G^{\mu_1}}[v]^*(\hat{w}) = J\Big((\kappa_0+\mu_2)v_\tau - v_{\mathrm{H}}^{\mu_1}(v)\Big), \tag{2.56}$$

where $v = K^{\mu_1}(\hat{v})$, and using the explicit formulae for $\widetilde{\mathrm{d}G^{\mu_1}}[v]^*$ appearing in the proof of Proposition 2.14, we find that the ζ- and Γ-components of equation (2.56) are

$$\hat{w}_\zeta = (\kappa_0 + \mu_2)\frac{\mathrm{d}}{\mathrm{d}\tau}v_\eta - (v_{\mathrm{H}}^{\mu_1}(w))_\eta\,, \tag{2.57}$$

$$\hat{w}_\Gamma - (g_1^{\mu_1}(v,\hat{w}))_y - (g_2(v,\hat{w}))_z = -(\kappa_0 + \mu_2)\frac{\mathrm{d}}{\mathrm{d}\tau}v_\Psi + (v_{\mathrm{H}}^{\mu_1}(v))_\Psi\,, \tag{2.58}$$

where

$$\begin{aligned} g_1^{\mu_1}(v,\hat{w}) &= \left(K_2(v_\eta)K_3 v_\eta - \frac{K_1^2(v_\eta)}{K_2(v_\eta)}(\nu_0+\mu_1)^2\sin^2(\theta_1-\theta_2)(v_\eta)_z^2\right)\Delta^{-1}((\hat{w}_\Gamma)_y) \\ &= \left(K_2(v_\eta)v_\eta - K_1(v_\eta)(\nu_0+\mu_1)^2\sin^2(\theta_1-\theta_2)(v_\eta)_z^2\right)\mathrm{e}^y\Delta^{-1}((\hat{w}_\Gamma)_y), \\ g_2(v,\hat{w}) &= \left(\frac{K_1(v_\eta)}{K_2(v_\eta)}(\nu_0+\mu_1)^2\sin^2(\theta_1-\theta_2)(v_\eta)_z\right)\Delta^{-1}((\hat{w}_\Gamma)_y); \end{aligned}$$

note in particular that $(\hat{v},\mu_1)\mapsto g_1^{\mu_1}(v,\hat{w})$ maps $\hat{U}\times\Lambda_1$ analytically into $L^2(-\infty,0)$ (see Proposition 2.10) and the same is true of $(\hat{v},\mu_1)\mapsto \int_{-\infty}^y g_1^{\mu_1}(v,\hat{w})\,\mathrm{d}t$ because $u\mapsto\int_{-\infty}^y u(t)\mathrm{e}^t\,\mathrm{d}t$ belongs to $\mathcal{L}(L^2(-\infty,0))$.

Equation (2.58) implies that

$$[\![[\hat{w}_\Gamma]_0]\!]_0 = [\![[(g_1^{\mu_1}(v,\hat{w}))_y]_0]\!]_0 + [\![[(v_{\mathrm{H}}^{\mu_1}(v))_\Psi]_0]\!]_0,$$

and it follows from this identity and Proposition 2.17 that $(\hat{v},\mu_1,\mu_2)\mapsto\int_{-\infty}^y[\![[\hat{w}_\Gamma]_0]\!]_0\,\mathrm{d}t$ and $(\hat{v},\mu_1,\mu_2)\mapsto\int_{-\infty}^y\int_{-\infty}^t[\![[\hat{w}_\Gamma]_0]\!]_0\,\mathrm{d}s\,\mathrm{d}t$ map $\mathcal{U}\times\Lambda_1\times\Lambda_2$ analytically into $L^2(-\infty,0)$. Furthermore, the calculation

$$\begin{aligned} \int_{-\infty}^0[\![[\hat{w}_\Gamma]_0]\!]_0\,\mathrm{d}y &= \underbrace{\Big[[\![[g_1^{\mu_1}(w,\hat{w})]_0]\!]_0\Big]_{-\infty}^0}_{=0} + \int_{-\infty}^0[\![[(v_{\mathrm{H}}^{\mu_1}(w))_\Psi]_0]\!]_0\,\mathrm{d}y \\ &= [\![[(v_{\mathrm{H}}^{\mu_1}(w))_\eta]_0]\!]_0\gamma\sin\theta_2 \\ &= -[\![[\hat{w}_\zeta]_0]\!]_0\gamma\sin\theta_2, \end{aligned}$$

shows that (2.55) is also satisfied; here we have used Proposition 2.17 and the facts that

$$[\![[\hat{w}_\zeta]_0]\!]_0 = -[\![[(v_{\mathrm{H}}^{\mu_1}(w))_\eta]_0]\!]_0$$

(see equation (2.57)) and $\Delta^{-1}((\hat{w}_\Gamma)_y)\Big|_{y=0}=0.$ ■

2.4 Appendix: Eigenvectors

The quantities introduced in Lemma 2.15 are

$$\sigma_k^2 = s^2 + 2k\nu_0 s\cos(\theta_1-\theta_2) + k^2\nu_0^2,$$

$$
\begin{aligned}
t_k &= \tanh(\beta^{-1}\sigma_k),\\
a_k &= s + k\nu_0\cos(\theta_1-\theta_2),\\
b_k &= k\nu_0\sin\theta_1 + s\sin\theta_2,\\
c_k &= 2\beta^{-1}\sigma_k\,\mathrm{cosech}(2\beta^{-1}\sigma_k)
\end{aligned}
$$

and

$$
\begin{aligned}
\tilde{\sigma}_k^2 &= a^2 + 2k\tilde{\nu}_0 a\cos(\theta_1-\theta_2) + k^2\tilde{\nu}_0^2,\\
\tilde{a}_k &= a + k\tilde{\nu}_0\cos(\theta_1-\theta_2),\\
\tilde{b}_k &= a\sin\theta_2 + k\tilde{\nu}_0\sin\theta_1,\\
\tilde{c}_k &= 2\tilde{\sigma}_k\,\mathrm{cosech}(2\tilde{\sigma}_k),
\end{aligned}
$$

and the generalised eigenvectors are given by

$$
\hat{v}_{k,s} = \begin{pmatrix}
\dfrac{\mathrm{i}\gamma b_k}{1+\sigma_k^4}\\[2ex]
-\dfrac{s\gamma b_k}{1+\sigma_k^4}\\[2ex]
\frac{1}{2}\mathrm{e}^{-\sigma_k y}(1-t_k) + \frac{1}{2}\mathrm{e}^{\sigma_k y}(1+t_k)\\[2ex]
\gamma\sin\theta_2 - \dfrac{\gamma\sigma_k^2 a_{2k} b_k}{1+\sigma_k^4}\\[2ex]
-\dfrac{\mathrm{i}\gamma\sigma_k^2 b_k}{1+\sigma_k^4}\\[2ex]
\frac{1}{2}\mathrm{i}a_k\big(\mathrm{e}^{-\sigma_k y}(1-t_k) + \mathrm{e}^{\sigma_k y}(1+t_k)\big)
\end{pmatrix}, \tag{2.59}
$$

$$
\hat{w}_{k,s} = \begin{pmatrix}
\dfrac{\gamma\sin\theta_2}{1+\sigma_k^4} - 2\gamma a_k b_k\left(\dfrac{2\sigma_k^2}{(1+\sigma_k^4)^2} + \dfrac{c_k}{2\sigma_k^2(1+\sigma_k^4)}\right)\\[2ex]
\dfrac{\mathrm{i}\gamma b_k}{1+\sigma_k^4} + \dfrac{\mathrm{i}s\gamma\sin\theta_2}{1+\sigma_k^4} - 2\mathrm{i}\gamma s a_k b_k\left(\dfrac{2\sigma_k^2}{(1+\sigma_k^4)^2} + \dfrac{c_k}{2\sigma_k^2(1+\sigma_k^4)}\right)\\[2ex]
\dfrac{\mathrm{i}a_k}{2\sigma_k}\big((1-t_k)y\mathrm{e}^{-\sigma_k y} - (1+t_k)y\mathrm{e}^{\sigma_k y}\big) + \dfrac{\mathrm{i}c_k a_k}{2\sigma_k^2}(\mathrm{e}^{-\sigma_k y} + \mathrm{e}^{\sigma_k y})\\[2ex]
\mathrm{i}\left(\dfrac{c_k a_k}{\sigma_k^2} + \dfrac{\sigma_k^2 a_{2k}}{1+\sigma_k^4}\right)\gamma\sin\theta_2 + \dfrac{\mathrm{i}\gamma\sigma_k^2 b_k}{1+\sigma_k^4} - 2\mathrm{i}\gamma a_k a_{2k} b_k\left(\dfrac{2\sigma_k^4}{(1+\sigma_k^4)^2} + \dfrac{c_k-2}{2(1+\sigma_k^4)}\right)\\[2ex]
-\dfrac{\sigma_k^2}{1+\sigma_k^4}\gamma\sin\theta_2 + 2\gamma a_k b_k\left(\dfrac{2\sigma_k^4}{(1+\sigma_k^4)^2} + \dfrac{c_k-2}{2(1+\sigma_k^4)}\right)\\[2ex]
\left(\frac{1}{2}(1-t_k) - \dfrac{c_k a_k^2}{2\sigma_k^2} - \dfrac{(1-t_k)a_k^2}{2\sigma_k}y\right)\mathrm{e}^{-\sigma_k y} + \left(\frac{1}{2}(1+t_k) - \dfrac{c_k a_k^2}{2\sigma_k^2} + \dfrac{(1+t_k)a_k^2 y}{2\sigma_k}\right)\mathrm{e}^{\sigma_k y}
\end{pmatrix},
$$

$$\hat{f}_1 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ \gamma \sin\theta_2 \\ 0 \\ 0 \end{pmatrix}, \ \hat{f}_2 = \begin{pmatrix} \gamma \sin\theta_2 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \ \hat{f}_3 = \begin{pmatrix} 0 \\ \gamma \sin\theta_2 \\ -\frac{1}{2}y^2 - \beta^{-1}y \\ 0 \\ 0 \\ 0 \end{pmatrix}, \ \hat{f}_4 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \gamma \sin\theta_2 \\ -\frac{1}{2}y^2 - \beta^{-1}y \end{pmatrix}.$$

3 Centre-manifold reduction and solitary hydroelastic surface waves

3.1 Formulation as a spatial Hamiltonian system

In this section we formulate the two-dimensional hydrodynamic problem (1.28)–(1.31) which follows from the variational principle (1.32), as a spatial Hamiltonian system. This formulation is unfavourable because of the variable fluid domain. It is therefore convenient to introduce the change of variable

$$\tilde{y} = \frac{1+y}{1+\eta(x)}, \qquad \Phi(x,\tilde{y}) = \phi(x,y), \tag{3.1}$$

which maps the variable fluid domain $\{(x,y) : x \in \mathbb{R},\, y \in (-1,\eta(x))\}$ to the fixed strip $\mathbb{R}\times(0,1)$. Dropping the tildes for notational simplicity, one obtains the transformed equations

$$\Phi_{xx} + \Phi_{yy}\frac{1+y^2\eta_x^2}{(1+\eta)^2} - 2\Phi_{xy}\frac{y\eta_x}{1+\eta} - \Phi_y\frac{y\eta_{xx}}{1+\eta} + 2\Phi_y\frac{y\eta_x^2}{(1+\eta)^2} = 0, \qquad 0<y<1 \tag{3.2}$$

$$\Phi_y\Big|_{y=0} = 0, \tag{3.3}$$

$$\frac{\Phi_y}{1+\eta} + \eta_x - \eta_x\left(\Phi_x - \Phi_y\frac{y\eta_x}{1+\eta}\right)\Bigg|_{y=1} = 0, \tag{3.4}$$

$$-\left(\Phi_x - \Phi_y\frac{y\eta_x}{1+\eta}\right) + \frac{1}{2}\left(\left(\Phi_x - \Phi_y\frac{y\eta_x}{1+\eta}\right)^2 + \left(\frac{\Phi_y}{1+\eta}\right)^2\right) + \alpha\eta \tag{3.5}$$

$$+\beta\left(\frac{1}{(1+\eta_x^2)^{1/2}}\left[\frac{1}{(1+\eta_x^2)^{1/2}}\left(\frac{\eta_{xx}}{(1+\eta_x^2)^{3/2}}\right)_x\right]_x + \frac{1}{2}\left(\frac{\eta_{xx}}{(1+\eta_x^2)^{3/2}}\right)^3\right)\Bigg|_{y=1} = 0 \tag{3.6}$$

with asymptotic conditions $\eta \to 0$, $(\Phi_x,\Phi_y) \to 0$ as $x \to \pm\infty$.

The ‘flattened’ hydrodynamic problem follows from the variational principle

$$\delta \int L(\eta,\eta_x,\eta_{xx},\Phi,\Phi_x)\,\mathrm{d}x = 0$$

with Lagrangian

$$L(\eta,\eta_x,\eta_{xx},\Phi,\Phi_x) =$$
$$\int_0^1\!\left(-\left[\Phi_x-\Phi_y\frac{y\eta_x}{1+\eta}\right]+\frac{1}{2}\left[\Phi_x-\Phi_y\frac{y\eta_x}{1+\eta}\right]^2+\frac{1}{2}\frac{\Phi_y^2}{(1+\eta)^2}\right)(1+\eta)\,\mathrm{d}y+\tfrac{1}{2}\alpha\eta^2+\tfrac{1}{2}\beta\frac{\eta_{xx}^2}{(1+\eta_x^2)^{5/2}}.$$

We perform a formal Legendre transformation by defining

$$\begin{aligned}
\rho &= \eta_x,\\
\omega &= \frac{\delta L}{\delta\eta_x}-\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\delta L}{\delta\eta_{xx}}\right)\\
&= \int_0^1\left(y\Phi_y-\left[\Phi_x-\Phi_y\frac{y\eta_x}{1+\eta}\right]y\Phi_y\right)\mathrm{d}y+\tfrac{5}{2}\beta\frac{\eta_x\eta_{xx}^2}{(1+\eta_x^2)^{7/2}}-\beta\frac{\eta_{xxx}}{(1+\eta_x^2)^{5/2}},\\
\xi &= \frac{\delta L}{\delta\eta_{xx}} = \beta\frac{\eta_{xx}}{(1+\eta_x^2)^{5/2}},\\
\Psi &= \frac{\delta L}{\delta\Phi_x} = -(1+\eta)+\left(\Phi_x-\Phi_y\frac{y\eta_x}{1+\eta}\right)(1+\eta)
\end{aligned}$$

and defining the Hamiltonian function by

$$\begin{aligned}
H(\eta,\rho,\omega,\xi,\Phi,\Psi) &= \omega\eta_x+\xi\eta_{xx}+\int_0^1\Psi\Phi_x\,\mathrm{d}y-L(\eta,\rho,\omega,\xi,\Phi,\Psi)\\
&= \omega\rho-\tfrac{1}{2}\alpha\eta^2+\frac{\xi^2}{2\beta}(1+\rho^2)^{5/2}+\tfrac{1}{2}(1+\eta)\\
&\qquad+\int_0^1\left(\frac{1}{2(1+\eta)}(\Psi^2-\Phi_y^2)+\Psi+\frac{\rho y\Phi_y\Psi}{1+\eta}\right)\mathrm{d}y.
\end{aligned}$$

Writing $\alpha=\alpha_0+\delta_1$ and $\beta=\beta_0+\delta_2$, where α_0 and β_0 are fixed, we find that Hamilton's equations are given explicitly by

$$\eta_x = \frac{\delta H^\delta}{\delta\omega} = \rho, \tag{3.7}$$

$$\rho_x = \frac{\delta H^\delta}{\delta\xi} = \frac{(1+\rho^2)^{5/2}}{\beta_0+\delta_2}\xi, \tag{3.8}$$

$$\omega_x = -\frac{\delta H^\delta}{\delta\eta} = \frac{1}{(1+\eta)^2}\int_0^1\left(\tfrac{1}{2}(\Psi^2-\Phi_y^2)+\rho y\Phi_y\Psi\right)\mathrm{d}y+(\alpha_0+\delta_1)\eta-\tfrac{1}{2}, \tag{3.9}$$

$$\xi_x = -\frac{\delta H^\delta}{\delta\rho} = -\omega-\frac{5}{2}\frac{\rho}{\beta_0+\delta_2}\xi^2(1+\rho^2)^{3/2}-\frac{1}{1+\eta}\int_0^1 y\Phi_y\Psi\,\mathrm{d}y, \tag{3.10}$$

$$\Phi_x = \frac{\delta H^\delta}{\delta\Psi} = \frac{\Psi+\eta}{1+\eta}+\frac{\rho y\Phi_y}{1+\eta}, \tag{3.11}$$

$$\Psi_x = -\frac{\delta H^\delta}{\delta\Phi} = \frac{1}{1+\eta}(-\Phi_y+\rho y\Psi)_y, \tag{3.12}$$

where the superscript denotes the dependence upon $\delta = (\delta_1, \delta_2)$, with boundary conditions

$$-\Phi_y + y\rho\Psi\Big|_{y=0,1} = 0,$$

which emerge from the integration by parts used to compute (3.12). A straightforward calculation shows that the η- and Φ-components of any solution to these equations satisfy (3.2)–(3.6).

Note that equations (3.7)–(3.12) are reversible, that is invariant under the transformation $(\eta, \omega, \rho, \xi, \Phi, \Psi)(x) \mapsto R(\eta, \omega, \rho, \xi, \Phi, \Psi)(-x)$, where the reverser is defined by

$$R(\eta, \omega, \rho, \xi, \Phi, \Psi) = (\eta, -\omega, -\rho, \xi, -\Phi, \Psi).$$

They are also invariant under the transformation $\Phi \mapsto \Phi + c$ for any constant c. To eliminate this symmetry it is convenient to replace (Φ, Ψ) with new variables $(\bar{\Phi}, \Phi_0, \bar{\Psi}, \Psi_0)$, where $\bar{\Phi} = \Phi - \Phi_0$, $\bar{\Psi} = \Psi - \Psi_0$ and

$$\Phi_0 = \int_0^1 \Phi \, \mathrm{d}y, \qquad \Psi_0 = \int_0^1 \Psi \, \mathrm{d}y.$$

This transformation leads to a new canonical Hamiltonian system with Hamiltonian

$$\bar{H}(\eta, \omega, \rho, \xi, \bar{\Phi}, \bar{\Psi}, \Phi_0, \Psi_0) = H(\eta, \omega, \rho, \xi, \bar{\Phi} + \Phi_0, \bar{\Psi} + \Psi_0) = H(\eta, \omega, \rho, \xi, \bar{\Phi}, \bar{\Psi} + \Psi_0)$$

and additional constraints

$$\int_0^1 \bar{\Phi} \, \mathrm{d}y = 0, \qquad \int_0^1 \bar{\Psi} \, \mathrm{d}y = 0.$$

Observe that Φ_0 is a cyclic variable whose conjugate Ψ_0 is a conserved quantity; we proceed in standard fashion by setting $\Psi_0 = -1$, considering the equations for $(\eta, \omega, \rho, \xi, \bar{\Phi}, \bar{\Psi})$ and recovering Φ_0 by quadrature. Dropping the bars for notational simplicity, one finds that Hamilton's equations for the reduced system are

$$\eta_x = \rho, \tag{3.13}$$

$$\rho_x = \frac{(1+\rho^2)^{5/2}}{\beta_0 + \delta_2}\xi, \tag{3.14}$$

$$\omega_x = \frac{1}{(1+\eta)^2} \int_0^1 \Big(\tfrac{1}{2}((\Psi - 1)^2 - \Phi_y^2) + \rho y \Phi_y (\Psi - 1)\Big) \, \mathrm{d}y - \tfrac{1}{2} + (\alpha_0 + \delta_1)\eta, \tag{3.15}$$

$$\xi_x = -\omega - \frac{5}{2}\frac{\rho}{\beta_0 + \delta_2}\xi^2(1+\rho^2)^{3/2} - \frac{1}{1+\eta}\int_0^1 y\Phi_y(\Psi - 1)\, \mathrm{d}y, \tag{3.16}$$

$$\Phi_x = \frac{\Psi}{1+\eta} + \frac{\rho}{1+\eta}\left(y\Phi_y - \int_0^1 y\Phi_y\right) \mathrm{d}y, \tag{3.17}$$

$$\Psi_x = \frac{1}{1+\eta}(-\Phi_y + \rho y(\Psi - 1))_y, \tag{3.18}$$

with constraints

$$\int_0^1 \Phi \, \mathrm{d}y = 0, \qquad \int_0^1 \Psi \, \mathrm{d}y = 0 \tag{3.19}$$

and boundary conditions

$$-\Phi_y + y\rho(\Psi - 1)\Big|_{y=0,1} = 0. \tag{3.20}$$

Let

$$Z = \Big\{(\eta, \rho, \omega, \xi, \Phi, \Psi) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \bar{H}^1(0,1) \times \bar{L}^2(0,1)\Big\},$$

where the overline denotes the subspace of functions with zero mean value, and note that the right-hand side of equations (3.13)–(3.18) define an analytic mapping $v_{\mathrm{H}}^\delta : \Lambda \times U \to Z$, where Λ and U are neighbourhoods of the origin in $\mathbb{R}^2$ and

$$X = \{(\eta, \rho, \omega, \xi, \Phi, \Psi) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \bar{H}^2(0,1) \times \bar{H}^1(0,1)\}$$

respectively. Equations (3.13)–(3.18) therefore define a quasilinear evolutionary system

$$v_x = v_{\mathrm{H}}^\delta(v)$$

with nonlinear boundary conditions given by

$$\Phi_y - y\rho(\Psi - 1)\Big|_{y=0,1} = 0.$$

To confirm its Hamiltonian structure, note that $(\delta, u) \mapsto H^\delta(u)$ defines mappings $\Lambda \times Z \to \mathbb{R}$ and $\Lambda \times X \to \mathbb{R}$ which are analytic at the origin, and the formula

$$\mathrm{d}H^\delta[u](w) = \langle J v_{\mathrm{H}}^\delta(u), w \rangle,$$

where $\langle \cdot, \cdot \rangle$ denotes the $\mathbb{R}^4 \times (L^2(0,1))^2$ inner product and

$$J(\eta, \rho, \omega, \xi, \Phi, \Psi) = (-\omega, -\xi, \eta, \rho, -\Psi, \Phi),$$

holds for all $(\delta, u) \in \mathcal{D}_{\mathrm{H}}$ and $w \in Z$, where

$$\mathcal{D}_{\mathrm{H}} = \Big\{(\delta, u) \in \Lambda \times U : \Phi_y - y\rho(\Psi - 1)\Big|_{y=0,1} = 0\Big\}.$$

Altogether we conclude that v_{H}^δ is the Hamiltonian vector field for the Hamiltonian system (Z, Ω, H^δ), where $\Omega : Z^2 \to \mathbb{R}$ is the constant symplectic form

$$\Omega\Big((\eta_1,\rho_1,\omega_1,\xi_1,\Phi_1,\Psi_1),(\eta_2,\rho_2,\omega_2,\xi_2,\Phi_2,\Psi_1)\Big)$$
$$= \int_0^1 (\Psi_2\Phi_1 - \Phi_2\Psi_1)\,\mathrm{d}y + \omega_2\eta_1 + \xi_2\rho_1 - \eta_2\omega_1 - \rho_2\xi_1.$$

One cannot work directly with (3.13)–(3.18) because of the nonlinear boundary condition at $y = 1$ in the domain of the Hamiltonian vector field v_{H}^{δ}. We overcome this difficulty using the change of variable $(\eta,\rho,\omega,\xi,\Phi,\Psi) \mapsto (\eta,\rho,\omega,\xi,\Gamma,\Psi)$, where

$$\Gamma = \Phi - \rho\int_0^y s(\Psi(s)-1)\,\mathrm{d}s + \rho\int_0^1\int_0^y s(\Psi(s)-1)\,\mathrm{d}s\,\mathrm{d}y,$$

which is a smooth diffeomorphism $Z \to Z$ with inverse

$$\Phi = \Gamma + \rho\int_0^y s(\Psi(s)-1)\,\mathrm{d}s - \rho\int_0^1\int_0^y s(\Psi(s)-1)\,\mathrm{d}s\,\mathrm{d}y.$$

This change of variable transforms equations (3.13)–(3.18) into

$$\eta_x = \rho, \tag{3.21}$$

$$\rho_x = \frac{(1+\rho^2)^{5/2}}{\beta_0+\delta_2}\xi, \tag{3.22}$$

$$\begin{aligned}\omega_x = \frac{1}{(1+\eta)^2}\int_0^1 \bigg\{&\frac{1}{2}(\Psi-1)^2 - \frac{1}{2}\Big(\Gamma_y + \rho y(\Psi-1)\Big)^2\Big) \\ &+ \rho y\Gamma_y(\Psi-1) + \rho^2y^2(\Psi-1)^2\bigg\}\,\mathrm{d}y - \tfrac{1}{2} + (\alpha_0+\delta_1)\eta,\end{aligned} \tag{3.23}$$

$$\xi_x = -\omega - \frac{5}{2}\frac{\rho}{\beta_0+\delta_2}\xi^2(1+\rho^2)^{3/2} - \frac{1}{1+\eta}\int_0^1 y(\Gamma_y + \rho y(\Psi-1))(\Psi-1)\,\mathrm{d}y, \tag{3.24}$$

$$\begin{aligned}\Gamma_x = {}&\frac{\Psi}{1+\eta} + \frac{\rho y(\Gamma_y + \rho y(\Psi-1))}{1+\eta} - \frac{\rho}{1+\eta}\int_0^1 y(\Gamma_y + \rho y(\Psi-1))\,\mathrm{d}y \\ &- \frac{(1+\rho^2)^{5/2}}{\beta_0+\delta_2}\xi\int_0^y s(\Psi(s)-1)\,\mathrm{d}s + \frac{(1+\rho^2)^{5/2}}{\beta_0+\delta_2}\xi\int_0^1\int_0^y s(\Psi(s)-1)\,\mathrm{d}s\,\mathrm{d}y \\ &+ \rho\int_0^y \frac{s}{1+\eta}\Gamma_{yy}\,\mathrm{d}s - \rho\int_0^1\int_0^y \frac{s}{1+\eta}\Gamma_{yy}\,\mathrm{d}s\,\mathrm{d}y,\end{aligned} \tag{3.25}$$

$$\Psi_x = -\frac{1}{1+\eta}\Gamma_{yy} \tag{3.26}$$

and the boundary conditions (3.20) into

$$\Gamma_y\Big|_{y=0,1} = 0. \tag{3.27}$$

Equations (3.21)–(3.26) are Hamilton's equations for the Hamiltonian system $(Z,\Upsilon,\hat{H}^{\delta})$, where

$$\begin{aligned}
&\hat{H}^\delta(\eta,\rho,\omega,\xi,\Gamma,\Psi) = \\
&\quad \omega\rho - \frac{1}{2}(\alpha_0+\delta_1)\eta^2 + \frac{\xi^2}{2\beta_0+\delta_2}(1+\rho^2)^{5/2} + \tfrac{1}{2}(\eta-1) \\
&\qquad + \int_0^1 \left\{ \frac{1}{2(1+\eta)}\Big((\Psi-1)^2 - (\Gamma_y+\rho y(\Psi-1))^2\Big) + \frac{1}{1+\eta}\Big(\rho y\Gamma_y(\Psi-1) + \rho^2y^2(\Psi-1)^2\Big)\right\} \mathrm{d}y
\end{aligned}$$

and

$$\begin{aligned}
\Upsilon|_{(\eta,\rho,\omega,\xi,\Gamma,\Psi)}&\Big((\tilde{\eta}_1,\tilde{\rho}_1,\tilde{\omega}_1,\tilde{\xi}_1,\tilde{\Gamma}_1,\tilde{\Psi}_1),(\tilde{\eta}_2,\tilde{\rho}_2,\tilde{\omega}_2,\tilde{\xi}_2,\tilde{\Gamma}_2,\tilde{\Psi}_2)\Big) \\
&= \int_0^1 \Bigg\{ \tilde{\Psi}_2\left(\tilde{\Gamma}_1 + \tilde{\rho}_1\int_0^y s\Psi(s)\,\mathrm{d}s + \rho\int_0^y s\tilde{\Psi}_1(s)\,\mathrm{d}s\right) - \tfrac{1}{2}\tilde{\rho}_1y^2\tilde{\Psi}_2 \\
&\qquad\qquad - \tilde{\Psi}_1\left(\tilde{\Gamma}_2 + \tilde{\rho}_2\int_0^y s\Psi(s)\,\mathrm{d}s + \rho\int_0^y s\tilde{\Psi}_2(s)\,\mathrm{d}s\right) + \tfrac{1}{2}\tilde{\rho}_2y^2\tilde{\Psi}_1\Bigg\}\,\mathrm{d}y \\
&\qquad + \tilde{\omega}_2\tilde{\eta}_1 + \tilde{\xi}_2\tilde{\rho}_1 - \tilde{\eta}_2\tilde{\omega}_1 - \tilde{\rho}_2\tilde{\xi}_1;
\end{aligned}$$

furthermore

$$\mathcal{D}(\hat{v}^\delta_{\mathrm{H}}) = \{(\eta,\rho,\omega,\xi,\Gamma,\Psi)\in Z : \Gamma_y\big|_{y=0,1} = 0\}.$$

We write (3.21)–(3.26) as

$$u_x = Lu + N^\delta(u),$$

in which $L = \mathrm{d}\hat{v}^0_{\mathrm{H}}[0]$, so that

$$L\begin{pmatrix}\eta\\ \rho\\ \omega\\ \xi\\ \Gamma\\ \Psi\end{pmatrix} = \begin{pmatrix}\rho\\ \frac{1}{\beta_0}\xi\\ (\alpha_0-1)\eta\\ -\omega-\frac{1}{3}\rho+\int_0^1 y\Gamma_y\,\mathrm{d}y\\ \frac{1}{2\beta_0}(y^2-\frac{1}{3})\xi+\Psi\\ -\Gamma_{yy}\end{pmatrix}$$

with

$$\mathcal{D}(L) = \Big\{(\eta,\rho,\omega,\xi,\Gamma,\Psi)\in\mathbb{R}\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}\times\bar{H}^2(0,1)\times\bar{H}^1(0,1) : \Gamma_y\big|_{y=0,1} = 0\Big\}.$$

3.2 Spectral analysis

In this section we examine the spectrum of the linear operator $L : \mathcal{D}(L)\subseteq Z\to Z$ in detail. Our first result is obtained by a straightforward calculation.

Proposition 3.1. *A complex number λ is an eigenvalue of L if and only if*

$$\alpha_0 + \lambda^4 \beta_0 = \lambda \cot(\lambda); \tag{3.28}$$

its eigenspace is one-dimensional and spanned by respectively

$$e_\lambda = \begin{pmatrix} \frac{1}{\lambda}\sin(\lambda) \\ \sin(\lambda) \\ \frac{1}{\lambda^2}(\alpha_0 - 1)\sin(\lambda) \\ \frac{1}{\lambda^2}\cos(\lambda) - \frac{1}{\lambda^3}\alpha_0 \sin(\lambda) \\ \frac{1}{\lambda}\cos(\lambda y) - \frac{1}{\lambda^2}\sin(\lambda) + \frac{1}{2}(y^2 - \frac{1}{3})\sin(\lambda) \\ \cos(\lambda y) - \frac{1}{\lambda}\sin(\lambda) \end{pmatrix}, \qquad e_0 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

for $\lambda \neq 0$ and $\lambda = 0$ (which arises only for $\alpha_0 = 1$). All eigenvalues are also algebraically simple, with the exception of the zero eigenvalue at $\alpha_0 = 1$ and the purely imaginary eigenvalues $\pm \mathrm{i}\mu_0$ at the point $(\beta_0(\mu_0), \alpha_0(\mu_0))$ of the curve

$$C = \Bigg\{ (\beta_0(\mu_0), \alpha_0(\mu_0)) = \left(\frac{1}{4\mu_0^3}\coth(\mu_0) - \frac{1}{4\mu_0^2 \sinh^2(\mu_0)}, \frac{3\mu_0}{4}\coth(\mu_0) + \frac{\mu_0^2}{4\sinh^2(\mu_0)} \right) : \\ \mu_0 \in (0, \infty) \Bigg\}$$

in the parameter plane which are algebraically double.

The following lemma gives more precise information on the point spectrum of L.

Lemma 3.2. *Choose $(\beta_0, \alpha_0) \in C$. The point spectrum of L consists of a countably infinite family $\{\lambda_k\}_{k \in \mathbb{Z}\setminus\{0\}}$ of simple real eigenvalues, where $\{\lambda_k\}_{k=1}^{\infty}$ are the positive real solutions of equation (3.28) and $\lambda_{-k} = -\lambda_k$ for $k = 1, 2, \ldots$ together with*

(a) two plus-minus pairs of simple purely imaginary eigenvalues if $\alpha_0 > 1$ and (β_0, α_0) lies to the left of the curve C in the parameter plane,

(b) a plus-minus pair of algebraically double purely imaginary eigenvalues $\pm \mathrm{i}\mu_0$ if (β_0, α_0) is the point with parameter value μ_0 on the curve C,

(c) a plus-minus quartet of genuinely complex eigenvalues if $\alpha_0 > 1$ and (β_0, α_0) lies to the right of the curve C in the parameter plane,

(d) a plus-minus pair of simple purely imaginary eigenvalues and an algebraically double zero eigenvalue if $\alpha_0 = 1$,

(e) an additional plus-minus pair of simple real eigenvalues and a plus-minus pair of simple purely imaginary eigenvalues if $\alpha_0 < 1$.

Furthermore $\lambda_k \in (k\pi, (k+1)\pi)$ *for* $k = 1, 2, \dots$ *and*

$$\lambda_k^2 = k^2\pi^2 + \frac{2}{\beta_0} + o\left(\frac{1}{k}\right)$$

for large k.

Proof. Observe that λ solves (3.28) if and only if $\nu = \lambda^2$ is an eigenvalue of the non-selfadjoint Sturm-Liouville problem

$$\begin{aligned} -v_{yy} &= \nu v, &(3.29)\\ \frac{v_y(1)}{v(1)} &= \alpha_0 + \beta_0\nu^2, &(3.30)\\ v(0) &= 0. &(3.31) \end{aligned}$$

This problem has a countable number of (not necessarily real) eigenvalues $\{\nu_n\}_{n\in\mathbb{N}_0}$, which repeated according to algebraic multiplicity and listed according in increasing absolute value, are given asymptotically for large n by

$$\nu_n = (n-1)^2\pi^2 + \frac{2}{\beta_0} + o\left(\frac{1}{n}\right) \tag{3.32}$$

(see Binding, Browne & Watson [8, Theorem 2.2]). The real eigenvalues of the spectral problem (3.29)–(3.31) correspond to the intersections in the (ν, μ_0) plane of the parabola $\mu_0 = \alpha_0 + \beta_0\nu^2$ and the curve $\mu_0 = B(\nu)$, where $B(\nu) = \sqrt{\nu}\cot\sqrt{\nu}$. The function $B(\nu)$ has poles exactly at the *Dirichlet eigenvalues*

$$\nu_n^{\mathrm{D}} = (n+1)^2\pi^2, \qquad n \in \mathbb{N}_0 \tag{3.33}$$

of the self-adjoint problem in which (3.30) is replaced by $v(1) = 0$; it is strictly decreasing from $+\infty$ to $-\infty$ in each interval $(-\infty, \nu_0^{\mathrm{D}})$ and $(\nu_n^{\mathrm{D}}, \nu_{n+1}^{\mathrm{D}})$, $n \in \mathbb{N}_0$. It follows that (3.29)–(3.31) has at least one real eigenvalue in each interval $(\nu_n^{\mathrm{D}}, \nu_{n+1}^{\mathrm{D}})$, $n \in \mathbb{N}_0$ (see Figure 3.1).

Comparing (3.32) with (3.33) and using the above geometrical characterisation of the real eigenvalues, one concludes that

(1) each interval $(\nu_n^{\mathrm{D}}, \nu_{n+1}^{\mathrm{D}})$, $n \in \mathbb{N}$ contains a simple real eigenvalue;

(2) there are precisely two additional eigenvalues (counted according to algebraic multiplicity) in the form of either

  (a) a complex-conjugate pair (with non-vanishing imaginary part) whose absolute value is less than ν_0^{D} (Figure 3.1 (a)),

  (b) one negative, algebraically double eigenvalue (Figure 3.1 (b)),

  (c) two simple real eigenvalues to the left of ν_0^D, at least one of which is negative (Figure 3.1 (c)–(e)).

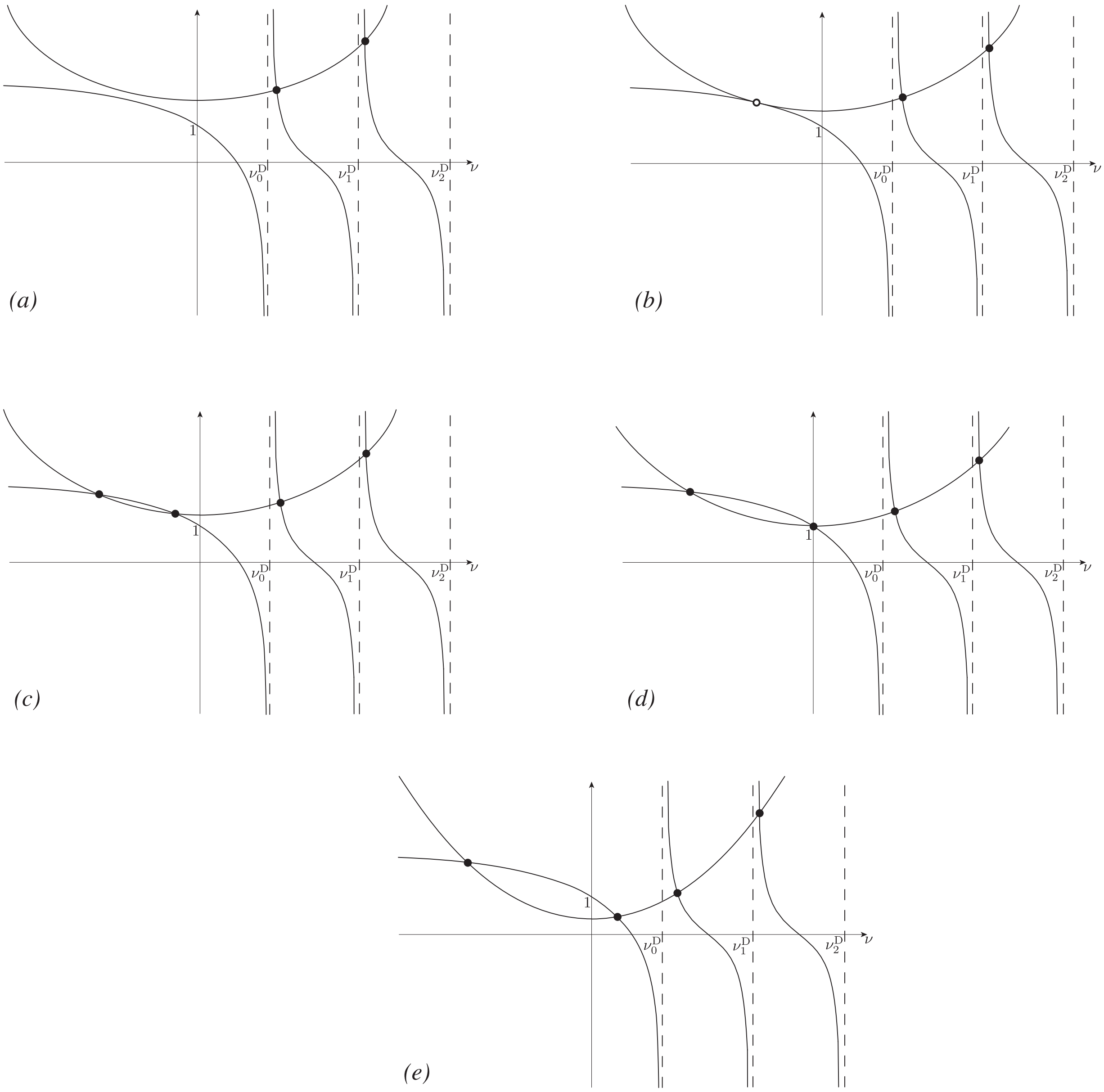

Figure 3.1: Geometric characterisation of the eigenvalues ν_n as the points of intersection of the curve $\mu_0 = B(\nu)$ with the parabola $\mu_0 = \alpha_0 + \beta_0\nu^2$; one real eigenvalue lies in each interval $(\nu_n^{\mathrm{D}}, \nu_{n+1}^{\mathrm{D}})$, $n \in \mathbb{N}_0$. (a) Two additional complex eigenvalues; (b) one additional algebraically double negative eigenvalue; (c)–(e) two additional real eigenvalues.

The solutions λ of (3.28) are recovered from the above analysis by the formula $\nu = \lambda^2$, so that in particular they occur in plus-minus pairs. Clearly (3.28) has a real solution in each interval $((\nu_n^{\mathrm{D}})^{1/2}, (\nu_{n+1}^{\mathrm{D}})^{1/2})$ and $(-(\nu_{n+1}^{\mathrm{D}})^{1/2}, -(\nu_n^{\mathrm{D}})^{1/2})$, $n \in \mathbb{N}_0$ (see point (1) above), and it follows from point (2) that there are four additional solutions (counted according to multiplicity). The results in Proposition 3.1 and the fact that $B(0) = 1$ show that these four solutions are described by precisely one of the statements (a)–(e) (according to which of the scenarios in Figure 3.1 occurs).

The asymptotic formula for λ_k follows by writing $k = n + 1$. ■

According to this lemma the purely imaginary eigenvalues of L appear in pairs $\pm\mathrm{i}\mu_0$ satisfying the dispersion relation

$$\alpha_0 + \mu_0^4\beta_0 = \mu_0 \coth(\mu_0). \tag{3.34}$$

Figure 1.6 shows the dependence of these eigenvalues upon β_0 and α_0. At each point of $\{\alpha_0 = 1\}$ two real eigenvalues become purely imaginary by colliding at the origin, while at each point of the curve C two pairs of purely imaginary eigenvalues become complex by colliding at non-zero points $\pm\mathrm{i}\mu_0$ on the imaginary axis. For later reference we record the formulae

$$e = \begin{pmatrix} \sinh(\mu_0) \\ \mathrm{i}\mu_0 \sinh(\mu_0) \\ -\mathrm{i}\cosh(\mu_0) + \frac{\mathrm{i}}{\mu_0}\sinh(\mu_0) + \mathrm{i}\beta_0\mu_0^3\sinh(\mu_0) \\ -\beta_0\mu_0^2\sinh(\mu_0) \\ -\mathrm{i}\cosh(\mu_0 y) + \frac{\mathrm{i}}{\mu_0}\sinh(\mu_0) - \frac{1}{2}\mathrm{i}\mu_0(y^2 - \frac{1}{3})\sinh(\mu_0) \\ \mu_0\cosh(\mu_0 y) - \sinh(\mu_0) \end{pmatrix},$$

$$f = \begin{pmatrix} -\mathrm{i}\cosh(\mu_0) \\ \sinh(\mu_0) + \mu_0\cosh(\mu_0) \\ -\beta_0\mu_0^2\sinh(\mu_0) - \frac{1}{\mu_0^2}\sinh(\mu_0) - \frac{\alpha_0}{\mu_0}\cosh(\mu_0) + \frac{2}{\mu_0}\cosh(\mu_0) \\ \mathrm{i}\beta_0\mu_0^2\cosh(\mu_0) + 2\mathrm{i}\beta_0\mu_0\sinh(\mu_0) \\ -y\sinh(\mu_0 y) - \frac{1}{\mu_0^2}\sinh(\mu_0) + \frac{1}{\mu_0}\cosh(\mu_0) + \frac{1}{2}(y^2 - \frac{1}{3})(\mu_0\cosh(\mu_0) + \sinh(\mu_0)) \\ -\mathrm{i}\mu_0 y\sinh(\mu_0 y) - \mathrm{i}\cosh(\mu_0 y) + \mathrm{i}\cosh(\mu_0) \end{pmatrix}$$

for an eigenvector e and generalised eigenvector f with eigenvalue $\mathrm{i}\mu_0$ when $(\beta_0, \alpha_0) \in C$ (the corresponding formulae for the zero eigenvalue at $\alpha_0 = 1$ are $e_0 = (1, 0, 0, 0, 0, 0)^{\mathrm{T}}$, $f_0 = (0, 1, -\frac{1}{3}, 0, 0, 0)^{\mathrm{T}}$).

Lemma 3.3. *The operator L is regular, that is its spectrum consists entirely of isolated eigenvalues of finite algebraic multiplicity.*

Proof. Since $\mathcal{D}(L)$ is compactly embedded in X it suffices to show that $\rho(L)$ is non-empty, so that L has compact resolvent (Kato [35, Theorem III.6.29]). In the case $\alpha_0 \neq 1$ a direct calculation shows that L is invertible with

$$L^{-1}\begin{pmatrix}\eta\\ \rho\\ \omega\\ \xi\\ \Gamma\\ \Psi\end{pmatrix}=\begin{pmatrix}\dfrac{1}{\alpha_0-1}\omega\\ \eta\\ -\frac{1}{3}\eta-\xi-\displaystyle\int_0^1 y\int_0^y\Psi(t)\,\mathrm{d}t\,\mathrm{d}y\\ \beta_0\rho\\ -\displaystyle\int_0^y\int_0^s\Psi(t)\,\mathrm{d}t\,\mathrm{d}s+\int_0^1\int_0^y\int_0^s\Psi(t)\,\mathrm{d}t\,\mathrm{d}s\,\mathrm{d}y\\ \Gamma-\frac{1}{2}(y^2-\frac{1}{3})\rho\end{pmatrix}.$$

To deal with the case $\alpha_0=1$ note that $L|_{\alpha_0=1}$ is a compact perturbation of $L|_{\alpha_0=\frac{1}{2}}$, so that the essential spectrum of these two operators (the set of λ for which $(\lambda I-L)$ is not Fredholm with index zero) is identical (see Schechter [52]). It follows that the spectrum of $L|_{\alpha_0=1}$ consists of the solution set of (3.28); in particular its resolvent set is non-empty. ■

Finally, we show that the set of generalised eigenvectors of L form a Schauder basis for Z. In particular, we show that this set is a *Riesz basis*, that is a basis obtained from an orthonormal basis by an isomorphism (see Gohberg & Krein [18, §VI.2]); note that we use the Dirichlet norm for the space $\bar{H}^1(0,1)$.

Proposition 3.4. *The set*

$$\mathcal{A}=\left\{\begin{pmatrix}(k\pi)^{-1}\cos(k\pi y)\\ \cos(k\pi y)\end{pmatrix}\right\}_{k\in\mathbb{Z}\setminus\{0\}}$$

is an orthonormal basis for $\bar{H}^1(0,1)\times\bar{L}^2(0,1)$.

Proof. Note that $\left\{\sqrt{2}\cos(k\pi y)\right\}_{k=1}^{\infty}$, $\left\{\sqrt{2}(k\pi)^{-1}\cos(k\pi y)\right\}_{k=1}^{\infty}$ are orthonormal bases for respectively $\bar{L}^2(0,1)$ and $\bar{H}^1(0,1)$. It therefore follows from

$$\mathrm{sp}\left\{\begin{pmatrix}(k\pi)^{-1}\cos(k\pi y)\\ \cos(k\pi y)\end{pmatrix}\right\}_{k\in\mathbb{Z}\setminus\{0\}}=\mathrm{sp}\left\{\begin{pmatrix}\sqrt{2}(k\pi)^{-1}\cos(k\pi y)\\ 0\end{pmatrix},\begin{pmatrix}0\\ \sqrt{2}\cos(k\pi y)\end{pmatrix}\right\}_{k=1}^{\infty}$$

in $\bar{H}^1(0,1)\times\bar{L}^2(0,1)$ that $\mathcal{A}$ is complete, and it is evidently orthonormal. ■

Corollary 3.5. *Let P be the spectral projection onto the four-dimensional subspace of Z corresponding to the eigenvalues shown in Figure 1.6, and let $\{e_1,e_2,e_3,e_4\}$ be a basis for*

$P[Z]$ consisting of generalised eigenvectors of L. The set

$$\{e_1, e_2, e_3, e_4\} \cup \{f_k\}_{k\in\mathbb{Z}\setminus\{0\}}, \qquad f_k = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ (k\pi)^{-1}\cos(k\pi y) \\ \cos(k\pi y) \end{pmatrix},$$

is a Riesz basis for Z.

Proof. Let $\{g_1, g_2, g_3, g_4\}$ denote the usual basis for the subset $\mathbb{R}^4 \times \{(0,0)\}$ of Z, and note that $\{g_1, g_2, g_3, g_4\} \cup \{f_k\}_{k\in\mathbb{Z}\setminus\{0\}}$ is an orthonormal basis for Z. Let $\pi : Z \to \mathbb{R}^4$ denote the projection $(\eta, \rho, \omega, \xi, \Phi, \Psi) \mapsto (\eta, \rho, \omega, \xi)$, and note that $\{\pi e_1, \pi e_2, \pi e_3, \pi e_4\}$ spans $\mathbb{R}^4$.

The formula $S(\eta, \rho, \omega, \xi, \Phi, \Psi) = (T(\eta, \rho, \omega, \xi), (\Phi, \Psi))$, where $T(\eta, \rho, \omega, \xi)$ is the coordinate vector of $(\eta, \rho, \omega, \xi)$ with respect to the basis $\{\pi e_1, \pi e_2, \pi e_3, \pi e_4\}$ for $\mathbb{R}^4$, defines an isomorphism $X \to X$ with $S[\{g_1, g_2, g_3, g_4\} \cup \{f_k\}_{k\in\mathbb{Z}\setminus\{0\}}] = \{e_1, e_2, e_3, e_4\} \cup \{f_k\}_{k\in\mathbb{Z}\setminus\{0\}}$. It follows that $\{e_1, e_2, e_3, e_4\} \cup \{f_k\}_{k\in\mathbb{Z}\setminus\{0\}}$ is a Riesz basis for Z. ■

Theorem 3.6. *The set $\{e_1, e_2, e_3, e_4\} \cup \{e_{\lambda_k}\}_{k\in\mathbb{Z}\setminus\{0\}}$ is a Riesz basis for Z.*

Proof. We first note that the set $\{e_1, e_2, e_3, e_4\} \cup \{e_{\lambda_k}\}_{k\in\mathbb{Z}\setminus\{0\}}$ is ω-linearly independent since it is the union of bases for the generalised eigenspaces of a regular operator (see Gohberg & Krein [18, p. 329]).

Choose $s^\star \in (0, \lambda_1)$. The function $h : (0, \infty) \to Z$ defined by

$$h(s) = \begin{pmatrix} \frac{1}{s}\sin(s) \\ \sin(s) \\ \frac{1}{s^2}(\alpha_0 - 1)\sin(s) \\ \frac{1}{s^2}\cos(s) - \frac{1}{s^3}\alpha_0\sin(s) \\ \frac{1}{s}\cos(sy) - \frac{1}{s^2}\sin(s) + \frac{1}{2}(y^2 - \frac{1}{3})\sin(s) \\ \cos(sy) - \frac{1}{s}\sin(s) \end{pmatrix}$$

satisfies

$$\|h(s_1) - h(s_2)\| \leq \sup_{s\in[s^\star,\infty)} \|h'(s)\||s_1 - s_2| \lesssim |s_1 - s_2|$$

for all s_1, $s_2 \in (s^\star, \infty)$. With $s_1 = \lambda_k$ and $s_2 = k\pi$ this calculation shows in particular that

$$\left\| e_{\lambda_k} - \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ (k\pi)^{-1}\cos(k\pi y) \\ \cos(k\pi y) \end{pmatrix} \right\| = \left\| e_{\lambda_k} - \begin{pmatrix} 0 \\ 0 \\ 0 \\ (k\pi)^{-2}\cos(k\pi) \\ (k\pi)^{-1}\cos(k\pi y) \\ \cos(k\pi y) \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 0 \\ (k\pi)^{-2}\cos(k\pi) \\ 0 \\ 0 \end{pmatrix} \right\|$$

$$\begin{aligned}
&\leq \left\| e_{\lambda_k} - \begin{pmatrix} 0 \\ 0 \\ 0 \\ (k\pi)^{-2}\cos(k\pi) \\ (k\pi)^{-1}\cos(k\pi y) \\ \cos(k\pi y) \end{pmatrix} \right\| + \frac{1}{k^2\pi^2} \\
&\lesssim |\lambda_k - k\pi| + \frac{1}{k^2\pi^2} \\
&= O(\tfrac{1}{k})
\end{aligned}$$

as $k \to \infty$, and similarly

$$\left\| e_{-\lambda_k} - \begin{pmatrix} 0 \\ 0 \\ 0 \\ (k\pi)^{-2}\cos(k\pi) \\ -(k\pi)^{-1}\cos(k\pi y) \\ \cos(k\pi y) \end{pmatrix} \right\| = \left\| e_{\lambda_k} - \begin{pmatrix} 0 \\ 0 \\ 0 \\ (k\pi)^{-2}\cos(k\pi) \\ (k\pi)^{-1}\cos(k\pi y) \\ \cos(k\pi y) \end{pmatrix} \right\| = O(\tfrac{1}{k})$$

as $k \to \infty$. Hence

$$\sum_{j=1}^{4} \|e_j - e_j\|^2 + \sum_{k\in\mathbb{Z}\setminus\{0\}} \left\| e_{\lambda_k} - \begin{pmatrix} 0 \\ 0 \\ 0 \\ (k\pi)^{-2}\cos(k\pi) \\ (k\pi)^{-1}\cos(k\pi y) \\ \cos(k\pi y) \end{pmatrix} \right\|^2 < \infty$$

and the conclusion now follows by Bari's theorem (Gohberg & Krein [18, Theorem VI.2.3]). ■

Let $\{e^1, e^2, e^3, e^4\} \cup \{e^{\lambda_k}\}_{k\in\mathbb{Z}\setminus\{0\}}$ be the dual Riesz basis to $\{e_1, e_2, e_3, e_4\} \cup \{e_{\lambda_k}\}_{k\in\mathbb{Z}\setminus\{0\}}$ (see Gohberg & Krein [18, §VI.1-2]), so that

$$P = \sum_{i=1}^{4} \langle \cdot, e^i \rangle e_i, \qquad (I-P) = \sum_{k\in\mathbb{Z}\setminus\{0\}} \langle \cdot, e^{\lambda_k} \rangle e_{\lambda_k},$$

and define $Z_2 = (I-P)Z$, $L_2 = (I-P)L$ (with $\mathcal{D}(L_2) = (I-P)\mathcal{D}(L)$). Note that

$$Z_2 = \left\{ \sum_{k\in\mathbb{Z}\setminus\{0\}} \beta_k e_{\lambda_k} : \{\beta_k\} \in \ell^2 \right\}, \qquad \mathcal{D}(L_2) = \left\{ \sum_{k\in\mathbb{Z}\setminus\{0\}} \beta_k e_{\lambda_k} : \{\beta_k\}, \{\lambda_k\beta_k\} \in \ell^2 \right\}.$$

We conclude this section with a maximal regularity result for $\tilde{L}$ which is used in Section 3.3 below.

Lemma 3.7. *The operator $L_2 : \mathcal{D}(L_2) \subseteq Z_2 \to Z_2$ has L^2-maximal regularity in the sense that the differential equation*

$$\dot{w} = L_2 w + h$$

admits a unique solution $w \in H^1(\mathbb{R}, Z_2) \cap L^2(\mathbb{R}, \mathcal{D}(L_2))$ for each $h \in L^2(\mathbb{R}, Z_2)$.

Proof. Writing

$$w = \sum_{k \in \mathbb{Z}\setminus\{0\}} w_k e_{\lambda_k}, \qquad h = \sum_{k \in \mathbb{Z}\setminus\{0\}} h_k e_{\lambda_k}$$

(where $w_k = \langle w, e^{\lambda_k} \rangle$, $h_k = \langle h, e^{\lambda_k} \rangle$), we find that

$$\dot{w}_k = \lambda_k w_k + h_k, \tag{3.35}$$

which is solved by

$$w_k(t) = \begin{cases} \int_{-\infty}^{t} h_k(s) \mathrm{e}^{\lambda_k(t-s)}\,\mathrm{d}s, & k < 0, \\ -\int_{t}^{\infty} h_k(s) \mathrm{e}^{\lambda_k(t-s)}\,\mathrm{d}s, & k > 0. \end{cases}$$

Note that

$$\|w_k\|_{L^2(\mathbb{R},\mathbb{R})} \leq \frac{1}{\lambda_k} \|h_k\|_{L^2(\mathbb{R},\mathbb{R})}$$

because

$$\begin{aligned} \|w_k\|_{L^2(\mathbb{R},\mathbb{R})}^2 &= \int_{-\infty}^{\infty} \left| \int_{t}^{\infty} h_k(s) \mathrm{e}^{\lambda_k(t-s)}\,\mathrm{d}s \right|^2 \mathrm{d}t \\ &\leq \int_{-\infty}^{\infty} \int_{t}^{\infty} \mathrm{e}^{\lambda_k(t-s)}\,\mathrm{d}s \int_{t}^{\infty} \mathrm{e}^{\lambda_k(t-s)} |h_k(s)|^2\,\mathrm{d}s\,\mathrm{d}t \\ &= \frac{1}{\lambda_k} \int_{-\infty}^{\infty} \int_{t}^{\infty} \mathrm{e}^{\lambda_k(t-s)} |h_k(s)|^2\,\mathrm{d}s\,\mathrm{d}t \\ &= \frac{1}{\lambda_k} \int_{-\infty}^{\infty} \int_{-\infty}^{s} \mathrm{e}^{\lambda_k(t-s)}\,\mathrm{d}t\,|h_k(s)|^2\,\mathrm{d}s \\ &= \frac{1}{\lambda_k^2} \|h_k\|_{L^2(\mathbb{R},\mathbb{R})}^2 \end{aligned}$$

for $k > 0$ with a similar calculation for $k < 0$. It follows that

$$\|w\|_{L^2(\mathbb{R},Z_2)}^2 = \int_{-\infty}^{\infty} \sum_{k \in \mathbb{Z}\setminus\{0\}} |w_k(t)|^2\,\mathrm{d}t = \sum_{k \in \mathbb{Z}\setminus\{0\}} \|w_k\|_{L^2(\mathbb{R},\mathbb{R})}^2 \leq \sum_{k \in \mathbb{Z}\setminus\{0\}} \|h_k\|_{L^2(\mathbb{R},\mathbb{R})}^2 = \|h\|_{L^2(\mathbb{R},Z_2)}^2$$

and similarly

$$\|L_2 w\|_{L^2(\mathbb{R},Z_2)}^2 = \int_{-\infty}^{\infty} \sum_{k \in \mathbb{Z}\setminus\{0\}} \lambda_k^2 |w_k(t)|^2\,\mathrm{d}t = \sum_{k \in \mathbb{Z}\setminus\{0\}} \lambda_k^2 \|w_k\|_{L^2(\mathbb{R},\mathbb{R})}^2 \leq \sum_{k \in \mathbb{Z}\setminus\{0\}} \|h_k\|_{L^2(\mathbb{R},\mathbb{R})}^2 = \|h\|_{L^2(\mathbb{R},Z_2)}^2,$$

so that w, $L_2 w \in L^2(\mathbb{R}, X)$. Equation (3.35) shows that w is differentiable, satisfies $\dot{w} \in L^2(\mathbb{R}, X)$ and solves the given differential equation.

The uniqueness of the solution follows by noting that equation (3.35) has no nontrivial solution in $L^2(\mathbb{R}, \mathbb{R})$ when $h_k = 0$. ■

3.3 Centre-manifold reduction

Our strategy in finding solutions to Hamilton's equations (3.21)–(3.26) for (Z, Υ, H^δ) consists in applying a reduction principle which asserts that it is locally equivalent to a finite-dimensional Hamiltonian system. The key result is the following theorem due to Mielke [44, 45].

Theorem 3.8. *Consider the differential equation*

$$\dot{u} = \mathcal{L}u + \mathcal{N}(u; \lambda), \tag{3.36}$$

which represents Hamilton's equations for the reversible Hamiltonian system $(M, \Omega^\lambda, H^\lambda)$. *Here* u *belongs to a Hilbert space* $\mathcal{Z}$, $\lambda \in \mathbb{R}^l$ *is a parameter and* $\mathcal{L} : \mathcal{D}(\mathcal{L}) \subset \mathcal{Z} \to \mathcal{Z}$ *is a densely defined, closed linear operator. Regarding* $\mathcal{D}(\mathcal{L})$ *as a Hilbert space equipped with the graph norm, suppose that* 0 *is an equilibrium point of* (3.36) *when* $\lambda = 0$ *and that*

(H1) The part of the spectrum $\sigma(\mathcal{L})$ *of* $\mathcal{L}$ *which lies on the imaginary axis consists of a finite number of eigenvalues of finite multiplicity and is separated from the rest of* $\sigma(\mathcal{L})$ *in the sense of Kato, so that* $\mathcal{Z}$ *admits the decomposition* $\mathcal{Z} = \mathcal{Z}_1 \oplus \mathcal{Z}_2$, *where* $\mathcal{Z}_1 = \mathcal{P}(\mathcal{Z})$, $\mathcal{Z}_2 = (I - \mathcal{P})(\mathcal{Z})$ *are the centre and hyperbolic subspaces of* $\mathcal{L}$ *defined by the spectral projection* $\mathcal{P}$ *corresponding the purely imaginary part of* $\sigma(\mathcal{L})$.

(H2) The operator $\mathcal{L}_2 = \mathcal{L}|_{\mathcal{Z}_2}$ *has* L^2*-maximal regularity in the sense that the differential equation*

$$\dot{u}_2 = \mathcal{L}_2 u_2 + h$$

admits a unique solution $u_2 \in H^1(\mathbb{R}, \mathcal{Z}_2) \cap L^2(\mathbb{R}, \mathcal{D}(\mathcal{L}_2))$ *for each* $h \in L^2(\mathbb{R}, \mathcal{Z}_2)$.

(H3) There exist a natural number k *and neighbourhoods* $\Lambda \subset \mathbb{R}^l$ *of* 0 *and* $U \subset \mathcal{D}(\mathcal{L})$ *of* 0 *such that* $\mathcal{N}$ *is* $(k+1)$ *times continuously differentiable on* $U \times \Lambda$, *its derivatives are bounded and uniformly continuous on* $U \times \Lambda$ *and* $\mathcal{N}(0, 0) = 0, \mathrm{d}_1\mathcal{N}[0, 0] = 0$.

Under these hypotheses there exist neighbourhoods $\tilde{\Lambda} \subset \Lambda$ *of* 0 *and* $\tilde{U}_1 \subset U \cap \mathcal{Z}_1$, $\tilde{U}_2 \subset U \cap \mathcal{Z}_2$ *of* 0 *and a reduction function* $r : \tilde{U}_1 \times \tilde{\Lambda} \to \tilde{U}_2$ *with the following properties. The reduction function* r *is* k *times continuously differentiable on* $\tilde{U}_1 \times \tilde{\Lambda}$ *and* $r(0; 0) = 0$, $\mathrm{d}_1 r[0; 0] = 0$. *The graph* $\tilde{M}^\lambda = \{u_1 + r(u_1; \lambda) \in \mathcal{Z}_1 \oplus \mathcal{Z}_2 : u_1 \in \tilde{U}_1\}$ *is a Hamiltonian centre manifold for* (3.36), *so that* *(i)* $\tilde{M}^\lambda$ *is a locally invariant manifold of* (3.36)*: through every point in* $\tilde{M}^\lambda$ *there passes a unique solution of* (3.36) *that remains on* $\tilde{M}^\lambda$ *as long as it remains in* $\tilde{U}_1 \times \tilde{U}_2$.

(ii) Every small bounded solution $u(x)$, $x \in \mathbb{R}$ *of* (3.36) *that satisfies* $(u_1(x), u_2(x)) \in \tilde{U}_1 \times \tilde{U}_2$ *lies completely in* $\tilde{M}^\lambda$.

(iii) Every solution $u_1 : (x_1, x_2) \to \tilde{U}_1$ *of the reduced equation*

$$\dot{u}_1 = \mathcal{L}u_1 + \tilde{\mathcal{N}}^\lambda(u_1), \tag{3.37}$$

where $\tilde{\mathcal{N}}^\lambda(u_1) = \mathcal{P}\mathcal{N}(u_1 + r(u_1; \lambda); \lambda)$, *generates a solution*

$$u(x) = u_1(x) + r(u_1(x); \lambda)$$

of the full equation (3.36).

(iv) $\tilde{M}^\lambda$ *is a symplectic submanifold of* M *and the flow determined by the Hamiltonian system* $(\tilde{M}^\lambda, \tilde{\Omega}^\lambda, \tilde{H}^\lambda)$, *where the tilde denotes restriction to* $\tilde{M}^\lambda$, *coincides with the flow on* $\tilde{M}^\lambda$ *determined by* $(M, \Omega^\lambda, H^\lambda)$. *The reduced equation* (3.37) *is reversible and represents Hamilton's equations for* $(\tilde{M}^\lambda, \tilde{\Omega}^\lambda, \tilde{H}^\lambda)$.

Remark 3.9.

(i) We find that

$$\begin{aligned}\tilde{H}^\lambda(u_1) &= H^\lambda(u_1 + r(u_1; \lambda)),\\ \tilde{\Omega}^\lambda|_{u_1}(v_1, v_2) &= \Omega^0|_0(v_1 + \mathrm{d}_1 r[u_1; \lambda](v_1), v_2 + \mathrm{d}r[u_1; \lambda](v_2))\\ &= \Omega^0|_0(v_1, v_2) + O(|(\lambda, u_1)|)\end{aligned} \tag{3.38}$$

as $(\lambda, u_1) \to 0$. *Using a parameter-dependent version of Darboux's theorem (e.g. see Buffoni & Groves [10]), we may assume that the remainder term in* (3.38) *vanishes identically.*

(ii) Substituting $u = u_1 + r(u_1; \lambda)$ *into* (3.36) *and eliminating* $\dot{u}_1$ *using* (3.37) *leads to the equation*

$$\mathcal{L}r(u_1; \lambda) - \mathrm{d}_1 r[u_1; \lambda](\mathcal{L}u_1) = \tilde{\mathcal{N}}^\lambda(u_1) + \mathrm{d}_1 r[u_1; \lambda](\tilde{\mathcal{N}}^\lambda(u_1)) - \mathcal{N}(u_1 + r(u_1; \lambda); \lambda),$$

which can be used to recursively determine the terms in the Taylor series of $r(u_1; \lambda)$ *and* $\mathcal{N}^\lambda(u_1)$.

We proceed by choosing $(\beta_0(\mu_0), \alpha_0(\mu_0)) \in C$, setting $(\delta_1, \delta_2) = (\delta, 0)$, and applying Theorem 3.8 to (Z, Υ, H^δ). Hypothesis (H3) is clearly satisfied for any natural number k, and we henceforth refer to functions which are continuously differentiable an arbitrary, but fixed number of times as 'smooth'. The spectral theory in Section 3.2 shows that (H1), (H2) are also satisfied; indeed, the (complexified) four-dimensional centre subspace of L is spanned by the generalised eigenvectors

$$E = \tau_1^{-1/2}\, e, \quad \bar{E} = \tau_1^{-1/2}\, \bar{e}, \quad F = \tau_1^{-1/2}\Big(f - \frac{\mathrm{i}\tau_2}{2\tau_1}\, e\Big), \quad \bar{F} = \tau_1^{-1/2}\Big(\bar{f} + \frac{\mathrm{i}\tau_2}{2\tau_1}\, \bar{e}\Big),$$

where

$$\begin{aligned} \tau_1 &= -\mu_0 \coth(\mu_0) + \frac{3\sinh(\mu_0)\cosh(\mu_0)}{2\mu_0} - \frac{1}{2} > 0, \\ \tau_2 &= -\frac{\sinh(2\mu_0)}{2\mu_0^2} + \frac{4\mu_0}{3} - \frac{1}{2\mu_0} - \frac{3\cosh(2\mu_0)}{2\mu_0} + \frac{\mu_0 + \sinh(2\mu_0)}{\sinh^2(\mu_0)}, \end{aligned} \tag{3.39}$$

so that the centre and hyperbolic subspaces of L are respectively

$$Z_1 = \{AE + BF + \bar{A}E + \bar{B}F : A, B \in \mathbb{C}\}, \qquad Z_2 = \Big\{ \sum_{k \in \mathbb{Z}\setminus\{0\}} \beta_k f_k : \{\beta_k\} \in \ell^2 \Big\}.$$

The vectors are normalised such that $(L - \mathrm{i}\mu_0 I)E = 0$, $(L - \mathrm{i}\mu_0 I)F = E$ with $SE = \bar{E}$, $SF = -\bar{F}$, and

$$\Upsilon|_0(E, \bar{F}) = \Upsilon|_0(\bar{E}, F) = 1, \qquad \Upsilon|_0(\bar{F}, E) = \Upsilon|_0(F, \bar{E}) = -1$$

and the symplectic product of any other combination of the vectors $E, F, \bar{E}, \bar{F}$ is zero (so that $\{E, F, \bar{E}, \bar{F}\}$ is a symplectic basis for the centre subspace of L). Writing

$$u_1 = AE + BF + \bar{A}\bar{E} + \bar{B}\bar{F},$$

we therefore find that A, B are canonical coordinates for the reduced Hamiltonian system (see Remark 3.9(i)), which can therefore be written as

$$A_x = \frac{\partial \tilde{H}^\delta}{\partial \bar{B}}, \qquad B_x = -\frac{\partial \tilde{H}^\delta}{\partial \bar{A}}$$

(with a slight abuse of notation we abbreviate $\tilde{H}^\delta|_{(\delta_1,\delta_2)=(\delta,0)}$ to $\tilde{H}^\delta$); this system is reversible with reverser $S : (A, B) \to (\bar{A}, -\bar{B})$. Note that the quadratic, parameter-independent part of the Hamiltonian is

$$H_2^0(A, B, \bar{A}, \bar{B}) = \mathrm{i}\mu_0(A\bar{B} - \bar{A}B) + |B|^2,$$

so that in coordinates

$$L \begin{pmatrix} A \\ B \\ \bar{A} \\ \bar{B} \end{pmatrix} = \begin{pmatrix} \mathrm{i}\mu_0 & 1 & 0 & 0 \\ 0 & \mathrm{i}\mu_0 & 0 & 0 \\ 0 & 0 & -\mathrm{i}\mu_0 & 1 \\ 0 & 0 & 0 & -\mathrm{i}\mu_0 \end{pmatrix} \begin{pmatrix} A \\ B \\ \bar{A} \\ \bar{B} \end{pmatrix}.$$

The next step is to use a normal-form transform to simplify the Hamiltonian. For this purpose we use the following result due to Elphick [15].

Lemma 3.10. *Let $n_0 \geq 2$. There exists a near-identity, canonical change of variables which transforms the Hamiltonian to*

$$\mathrm{i}\mu_0(A\bar{B} - \bar{A}B) + |B|^2 + H^{\delta}_{\mathrm{NF}}(A, B, \bar{A}, \bar{B}) + O(|(A, B)|^2|(\delta, A, B)|^{n_0}),$$

where the complexification of H^{δ}_{NF} lies in $\ker \mathcal{L}_{L^*}$, and $\mathcal{L}_{M^*} : \mathbb{C}[Z] \to \mathbb{C}[Z]$ is defined by

$$(\mathcal{L}_{M^*}p)(Z) = M^*Z \cdot \nabla p(Z)$$

for $M \in \mathbb{C}^{4\times 4}$, where the coefficients of the polynomials in the complex polynomial rings depend upon δ and the gradient is taken with respect to $Z = (A, B, \bar{A}, \bar{B})$.

We proceed by characterising $\ker \mathcal{L}_{L^*}$ using the following lemma, the statements in which are obtained from results by Murdock [47, Lemma 3.4.8], Malonza [42, Lemma 4, Theorem 9] and Billera, Cushman and Sanders [7, Section 4] respectively. Corollary 3.12 takes into account that H^{μ}_{NF} is real-valued.

Lemma 3.11. *Let $S = \mathrm{diag}(\mathrm{i}\mu_0, \mathrm{i}\mu_0, -\mathrm{i}\mu_0, -\mathrm{i}\mu_0)$ and $N = L - S$.*

(i) The kernel of $\mathcal{L}_{L^*} : \mathbb{C}[Z] \to \mathbb{C}[Z]$ is given by $\ker \mathcal{L}_{L^*} = \ker \mathcal{L}_{N^*} \cap \ker \mathcal{L}_{S^*}$.

(ii) The kernel of $\mathcal{L}_{N^*}$ is given by $\ker \mathcal{L}_{N^*} = \mathbb{C}[A, \bar{A}, A\bar{B} - \bar{A}B]$.

(iii) The kernel of $\mathcal{L}_{S^*}$ is given by $\ker \mathcal{L}_{S^*} = \mathbb{C}[A\bar{A}, A\bar{B}, B\bar{A}, B\bar{B}]$.

Corollary 3.12. *The kernel of $\mathcal{L}_{L^*} : \mathbb{C}[Z] \to \mathbb{C}[Z]$ is given by $\mathbb{C}[|A|^2, \mathrm{i}(A\bar{B} - \bar{A}B)]$ and $H^{\delta}_{\mathrm{NF}} \in \mathbb{R}[|A|^2, \mathrm{i}(A\bar{B} - \bar{A}B)]$.*

Writing the transformed reduced system as

$$u_{1x} = Lu_1 + P^{\delta}(u_1),$$

where

$$\begin{aligned} u_1 &= AE + BF + \bar{A}\bar{E} + \bar{B}\bar{F}, \\ P^{\delta}(u_1) &= \partial_{\bar{B}}\tilde{H}^{\delta}(A, B, \bar{A}, \bar{B})E - \partial_{\bar{A}}\tilde{H}^{\delta}(A, B, \bar{A}, \bar{B})F \\ &\qquad + \partial_{B}\tilde{H}^{\delta}(A, B, \bar{A}, \bar{B})\bar{E} - \partial_{A}\tilde{H}^{\delta}(A, B, \bar{A}, \bar{B})\bar{F}, \end{aligned}$$

we can compute the Taylor series of $r(u_1; \delta)$ and $\mathcal{N}^{\delta}(u_1)$, and hence $H^{\delta}(A, B, \bar{A}, \bar{B})$, recursively using the equation

$$Lr(u_1; \delta) - \mathrm{d}_1 r[u_1; \delta](Lu_1) = P^{\delta}(u_1) + \mathrm{d}_1 r[u_1; \delta](P^{\delta}(u_1)) - N^{\delta}(u_1 + r(u_1; \delta)) \qquad (3.40)$$

(see Remark 3.9(ii)), where with a slight abuse of notation we have applied the near-identity normal-form transformation to the reduction function. Corollary 3.12 states that there are real constants c_1, c_2, d_1, d_2, d_3 such that

$$\begin{aligned}
&\tilde{H}_2^1(A,B,\bar{A},\bar{B},0) = c_1|A|^2 + c_2\mathrm{i}(A\bar{B}-\bar{A}B),\\
&\tilde{H}_3^0(A,B,\bar{A},\bar{B},0) = 0,\\
&\tilde{H}_4^0(A,B,\bar{A},\bar{B},0) = d_1|A|^4 + d_2\mathrm{i}(A\bar{B}-\bar{A}B)|A|^2 - d_3(A\bar{B}-\bar{A}B)^2,
\end{aligned}$$

where $\delta^j\tilde{H}_k^j(A,B,\bar{A},\bar{B})$ denotes the part of the Taylor expansion of $\tilde{H}^\delta(A,B,\bar{A},\bar{B})$ which is homogeneous of order j in δ and k in $(A,B,\bar{A},\bar{B})$. The coefficients c_1 and d_1, whose values are required in Section 3.4 below, are computed in Appendix 3.5.2; we find that $c_1<0$ and there exists a critical value $\mu_0^\star$ of μ_0 such that $d_1>0$ for $\mu_0<\mu_0^\star$, which we now assume.

3.4 Homoclinic solutions

In this section we examine the reduced Hamiltonian system

$$\begin{aligned}
A_x &= \partial_{\bar{B}}\tilde{H}^\delta(A,B,\bar{A},\bar{B})\\
&= \mathrm{i}\mu_0 A + B + \partial_{\bar{B}}\tilde{H}^\delta_{\mathrm{NF}}(|A|^2,\mathrm{i}(A\bar{B}-\bar{A}B,\delta) + \underline{O}(|(A,B)||(\delta,A,B)|^{n_0}), \qquad (3.41)\\
B_x &= -\partial_{\bar{A}}\tilde{H}^\delta(A,B,\bar{A},\bar{B})\\
&= \mathrm{i}\mu_0 B - \partial_{\bar{A}}\tilde{H}^\delta_{\mathrm{NF}}(|A|^2,\mathrm{i}(A\bar{B}-\bar{A}B),\delta) + \underline{O}(|(A,B)||(\delta,A,B)|^{n_0}), \qquad (3.42)
\end{aligned}$$

where the underscore indicates that the order-of-magnitude estimate remains valid when formally differentiated with respect to (A,B). The truncated system without the remainder terms was examined in detail by Iooss & Pérouème [34], who also studied the 'persistence' of certain solutions as solutions to the full system. Here we present an alternative, functional-analytic proof of the existence of two reversible homoclinic solutions to (3.41), (3.42).

We begin by returning to real coordinates $q=(q_1,q_2)^{\mathrm{T}}$, $p=(p_1,p_2)^{\mathrm{T}}$ given by

$$A = \frac{1}{\sqrt{2}}(q_1+\mathrm{i}q_2), \qquad B = \frac{1}{\sqrt{2}}(p_1+\mathrm{i}p_2)$$

and hence obtaining the real Hamiltonian system

$$q_x = \frac{\partial\tilde{H}^\delta}{\partial p} = p + \mu_0 R_{\frac{\pi}{2}}q + \overbrace{\partial_2\tilde{H}^\delta_{\mathrm{NF}}(\tfrac{1}{2}|q|^2, p\cdot R_{\frac{\pi}{2}}q)R_{\frac{\pi}{2}}q}^{:=P_1^\delta(q,p)} + R_1^\delta(q,p), \qquad (3.43)$$

$$p_x = -\frac{\partial\tilde{H}^\delta}{\partial p} = \mu_0 R_{\frac{\pi}{2}}p \underbrace{-\partial_1\tilde{H}^\delta_{\mathrm{NF}}(\tfrac{1}{2}|q|^2, p\cdot R_{\frac{\pi}{2}}q)q + \partial_2\tilde{H}^\delta_{\mathrm{NF}}(\tfrac{1}{2}|q|^2, p\cdot R_{\frac{\pi}{2}}q)R_{\frac{\pi}{2}}p}_{:=P_2^\delta(q,p)} + R_2^\delta(q,p), \quad (3.44)$$

in which

$$\tilde{H}^\delta(q,p) = \frac{1}{2}|p|^2 + \mu_0 p\cdot R_{\frac{\pi}{2}}q + \tilde{H}^\delta_{\mathrm{NF}}(\tfrac{1}{2}|q|^2, p\cdot R_{\frac{\pi}{2}}q, \delta) + O(|(q,p)|^2|(\delta,q,p)|^{n_0}),$$

so that $P_1^\delta(q,p)$, $P_2^\delta(q,p)$ are polynomials in δ, q and p and

$$R_1^\delta(q,p),\ R_2^\delta(q,p) = \underline{O}(|(q,p)|(\delta,q,p)|^{n_0}).$$

Note that this system is reversible with reverser $S:(q_1,p_1,q_2,p_2)\mapsto(q_1,-p_1,-q_2,p_2)$ and that

$$R_\theta P_1^\delta(q,p) = P_1^\delta(R_\theta q, R_\theta p), \qquad R_\theta P_2^\delta(q,p) = P_2^\delta(R_\theta q, R_\theta p)$$

for all $\theta\in[0,2\pi)$, where R_θ is the matrix representing a rotation through the angle θ.

The next step is to recast equations (3.43), (3.44) as a single second-order equation. Writing

$$p = q_x - \mu_0 R_{\frac{\pi}{2}}q + v,$$

we find from equation (3.43) that

$$v + P_1^\delta(q, q_x - \mu_0 R_{\frac{\pi}{2}}q + v) + R_1^\delta(q, q_x - \mu_0 R_{\frac{\pi}{2}}q + v) = 0, \tag{3.45}$$

and using the implicit-function theorem we now construct a solution of (3.45) of the form

$$v = v_1^\delta(q, q_x - \mu_0 R_{\frac{\pi}{2}}q) + v_2^\delta(q, q_x - \mu_0 R_{\frac{\pi}{2}}q),$$

where v_1^δ solves the truncated equation with $R_1^\delta = 0$ and takes the particular form

$$v_1^\delta(q, q_x - \mu_0 R_{\frac{\pi}{2}}q) = w_1^\delta(|q|^2, R_{\frac{\pi}{2}}q\cdot(q_x - \mu_0 R_{\frac{\pi}{2}}q))R_{\frac{\pi}{2}}q. \tag{3.46}$$

Note that w_1^δ necessarily solves

$$w_1 + \partial_2\tilde{H}^\delta_{\mathrm{NF}}(\tfrac{1}{2}|q|^2, w_1|q|^2 + R_{\frac{\pi}{2}}q\cdot(q_x - \mu_0 R_{\frac{\pi}{2}}q)) = 0, \tag{3.47}$$

while v_2^δ necessarily solves

$$\begin{aligned} v_2 &+ P_1^\delta(q, q_x - \mu_0 R_{\frac{\pi}{2}}q + v_1^\delta(q, q_x - \mu_0 R_{\frac{\pi}{2}}q) + v_2) - P_1^\delta(q, q_x - \mu_0 R_{\frac{\pi}{2}}q + v_1^\delta(q, q_x - \mu_0 R_{\frac{\pi}{2}}q)) \\ &+ R_1^\delta(q, q_x - \mu_0 R_{\frac{\pi}{2}}q + v_1^\delta(q, q_x - \mu_0 R_{\frac{\pi}{2}}q) + v_2) = 0. \end{aligned} \tag{3.48}$$

Proposition 3.13.

(i) Equation (3.47) *has a unique solution $w_1 = w_1^\delta(|q|^2, R_{\frac{\pi}{2}}q\cdot(q_x - \mu_0 R_{\frac{\pi}{2}}q))$ which depends analytically upon δ, $|q|^2$ and $R_{\frac{\pi}{2}}q\cdot(q_x - \mu_0 R_{\frac{\pi}{2}}q)$ and satisfies $w_1^0(0,0)=0$. The function v_1^δ defined by* (3.46) *satisfies*

$$v_1^\delta + P_1^\delta(q, q_x - \mu_0 R_{\frac{\pi}{2}}q + v_1^\delta) = 0$$

and

$$R_\theta v_1^\delta(q, q_x - \mu_0 R_{\frac{\pi}{2}}q) = v_1^\delta(R_\theta q, R_\theta(q_x - \mu_0 R_{\frac{\pi}{2}}q))$$

for all $\theta\in[0,2\pi)$.

(ii) Equation (3.48) *has a unique solution* $v_2 = v_2^\delta(q, q_x - \mu_0 R_{\frac{\pi}{2}} q)$ *which depends smoothly upon* δ, q *and* $q_x - \mu_0 R_{\frac{\pi}{2}} q$ *and satisfies*

$$v_2^\delta(q, q_x - \mu_0 R_{\frac{\pi}{2}} q) = \underline{O}(|(q, q_x - \mu_0 R_{\frac{\pi}{2}} q)||(\delta, q, q_x - \mu_0 R_{\frac{\pi}{2}} q)|^{n_0}).$$

Substituting

$$p = q_x - \mu_0 R_{\frac{\pi}{2}} q + v_1^\delta + v_2^\delta$$

into equation (3.44), where we have omitted the arguments of v_1^δ, v_2^δ for notational simplicity, shows that

$$(\partial_x - \mu_0 R_{\frac{\pi}{2}})^2 q = -(\partial_x - \mu_0 R_{\frac{\pi}{2}})(v_1^\delta + v_2^\delta) + \tilde{P}^\delta(q, q_x - \mu_0 R_{\frac{\pi}{2}} q) + \tilde{R}^\delta(q, q_x - \mu_0 R_{\frac{\pi}{2}} q),$$

in which

$$\begin{aligned}
\tilde{P}^\delta(q, q_x - \mu_0 R_{\frac{\pi}{2}} q) &= P_2^\delta(q, q_x - \mu_0 R_{\frac{\pi}{2}} q + v_1^\delta),\\
\tilde{R}^\delta(q, q_x - \mu_0 R_{\frac{\pi}{2}} q) &= P_2^\delta(q, q_x - \mu_0 R_{\frac{\pi}{2}} q + v_1^\delta + v_2^\delta)\\
&\qquad - P_2^\delta(q, q_x - \mu_0 R_{\frac{\pi}{2}} q + v_1^\delta) + R_2^\delta(q, q_x - \mu_0 R_{\frac{\pi}{2}} q + v_1^\delta + v_2^\delta).
\end{aligned}$$

It follows that

$$\begin{aligned}
(\partial_x - \mu_0 R_{\frac{\pi}{2}})^2 q = {}& -\partial_1 v_1^\delta(q_x - \mu_0 R_{\frac{\pi}{2}} q) - \partial_2 v_1^\delta(\partial_x - \mu_0 R_{\frac{\pi}{2}})^2 q + \tilde{P}^\delta(q, q_x - \mu_0 R_{\frac{\pi}{2}} q)\\
& - \partial_1 v_2^\delta(q_x - \mu_0 R_{\frac{\pi}{2}} q) - \partial_2 v_2^\delta(\partial_x - \mu_0 R_{\frac{\pi}{2}})^2 q\\
& - \partial_1 v_2^\delta \mu_0 R_{\frac{\pi}{2}} q - \partial_2 v_2^\delta \mu_0 R_{\frac{\pi}{2}}(q_x - \mu_0 R_{\frac{\pi}{2}} q) + \mu_0 R_{\frac{\pi}{2}} v_2^\delta + \tilde{R}^\delta(q, q_x - \mu_0 R_{\frac{\pi}{2}} q),
\end{aligned}$$

where $\partial_j v_k^\delta$ is the matrix $\mathrm{d}_j v_k^\delta[q, q_x - \mu_0 R_{\frac{\pi}{2}} q]$ and we have used the calculation

$$\begin{aligned}
&(\partial_x - \mu_0 R_{\frac{\pi}{2}}) v_1^\delta(q, q_x - s R_{\frac{\pi}{2}} q)\\
&\quad = (\partial_x - \mu_0 R_{\frac{\pi}{2}}) R_{sx} v_1^\delta(R_{-\mu_0 x} q, R_{-\mu_0 x}(q_x - \mu_0 R_{\frac{\pi}{2}} q))\\
&\quad = R_{\mu_0 x} \partial_x v_1^\delta(R_{-\mu_0 x} q, R_{-\mu_0 x}(q_x - \mu_0 R_{\frac{\pi}{2}} q))\\
&\quad = R_{\mu_0 x} \partial_1 v_1^\delta(R_{-\mu_0 x} q, R_{-\mu_0 x}(q_x - \mu_0 R_{\frac{\pi}{2}} q)) \partial_x(R_{-\mu_0 x} q)\\
&\qquad\quad + R_{\mu_0 x} \partial_2 v_1^\delta(R_{-\mu_0 x} q, R_{-\mu_0 x}(q_x - \mu_0 R_{\frac{\pi}{2}} q)) \partial_x(R_{-\mu_0 x}(q_x - \mu_0 R_{\frac{\pi}{2}} q))\\
&\quad = R_{\mu_0 x} \partial_1 v_1^\delta(R_{-\mu_0 x} q, (q_x - \mu_0 R_{\frac{\pi}{2}} q)) R_{-\mu_0 x}(q_x - \mu_0 R_{\frac{\pi}{2}} q)\\
&\qquad\quad + R_{\mu_0 x} \partial_2 v_1^\delta(R_{-\mu_0 x} q, (q_x - \mu_0 R_{\frac{\pi}{2}} q)) R_{-\mu_0 x}(\partial_x - \mu_0 R_{\frac{\pi}{2}})^2 q\\
&\quad = \partial_1 v_1^\delta(q, q_x - \mu_0 R_{\frac{\pi}{2}} q)(q_x - \mu_0 R_{\frac{\pi}{2}} q) + \partial_2 v_1^\delta(q, q_x - \mu_0 R_{\frac{\pi}{2}} q)(\partial_x - \mu_0 R_{\frac{\pi}{2}})^2 q.
\end{aligned}$$

Introducing the scaled variables

$$q(x) = \varepsilon R_{\mu_0 x} Q(X), \qquad X = \varepsilon x,$$

where $\varepsilon^2 = -c_1\delta$, so that

$$q_x - \mu_0 R_{\frac{\pi}{2}} q = \varepsilon^2 R_{\mu_0 x} Q_X(X), \qquad (\partial_x - \mu_0 R_{\frac{\pi}{2}})^2 q = \varepsilon^3 R_{\mu_0 x} Q_{XX}(X),$$

transforms the resulting equation into

$$Q_{XX} = Q - CQ|Q|^2 + T_1^\varepsilon(Q, Q_X) + R_{-\mu_0 X/\varepsilon} T_2^\varepsilon(R_{\mu_0 X/\varepsilon} Q, R_{\mu_0 X/\varepsilon} Q_X, R_{\mu_0 X/\varepsilon} Q_{XX}) \tag{3.49}$$

where $C = -d_1/c_1$ and

$$T_1^\varepsilon(Q, Q_X) = O(\varepsilon|(Q, Q_X)|), \qquad T_2^\varepsilon(Q, Q_X, Q_{XX}) = O(\varepsilon^{n_0-2}|(Q, Q_X, Q_{XX})|).$$

Remark 3.14. *The various changes of variable preserve the reversibility symmetry, so that equation* (3.49) *is invariant under the transformation* $X \mapsto -X$, $(Q_1, Q_2) \mapsto (Q_1, -Q_2)$.

Before proving the existence of homoclinic solutions to (3.49) we define the function spaces with which we work and refer to some functional-analytic results which are used in the proof (see Kirchgässner [37, Proposition 5.1]).

Definition 3.15. *Suppose that* $k \in \mathbb{N}_0$ *and* $\nu \geq 0$. *Define*

$$C_\nu^k(\mathbb{R}) = \{f \in C^k(\mathbb{R})\colon \|f\|_{k,\nu} < \infty\}, \qquad \|f\|_{k,\nu} := \sup_{t\in\mathbb{R}} \sum_{j=0}^{k} |f^{(j)}(t)|\mathrm{e}^{\nu|t|}$$

and their subspaces

$$C_{\nu,\mathrm{e}}^k = \{f \in C_\nu^k(\mathbb{R})\colon f(-t) = f(t),\ t \in \mathbb{R}\}, \quad C_{\nu,\mathrm{o}}^k = \{f \in C_\nu^k(\mathbb{R})\colon f(-t) = -f(t),\ t \in \mathbb{R}\}.$$

In the case $k = 0$ *we just write* $C_\nu(\mathbb{R})$, $C_{\nu,\mathrm{e}}(\mathbb{R})$ *and* $C_{\nu,\mathrm{o}}(\mathbb{R})$.

Proposition 3.16.

(i) The formula

$$K\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} z_{1XX} - z_1 \\ z_{2XX} - z_2 \end{pmatrix}$$

defines a bounded linear operator $C_\nu^2(\mathbb{R})^2 \to C_\nu(\mathbb{R})^2$ *and* $C_{\nu,\mathrm{e}}^2(\mathbb{R}) \times C_{\nu,\mathrm{o}}^2(\mathbb{R}) \to C_{\nu,\mathrm{e}}(\mathbb{R}) \times C_{\nu,\mathrm{o}}(\mathbb{R})$ *for each* $\nu \geq 0$.

(ii) For $0 \leq \nu < 1$ *the operator* $K\colon C_\nu^2(\mathbb{R})^2 \to C_\nu(\mathbb{R})^2$ *is invertible with bounded inverse given by*

$$(K^{-1}f)(t) = -\frac{1}{2}\int_{-\infty}^{\infty} \mathrm{e}^{-|t-s|} f(s)\,\mathrm{d}s,$$

where the integration is taken componentwise.

(iii) Suppose that $C > 0$, $h \in C_1(\mathbb{R})$ and $0 \le \nu < 1$. The formula

$$K_h z = K^{-1} \begin{pmatrix} -3Ch^2 z_1 \\ -Ch^2 z_2 \end{pmatrix}$$

defines a bounded linear operator $C_0(\mathbb{R})^2 \to C_\nu^2(\mathbb{R})^2$ and a compact operator $C_\nu(\mathbb{R})^2 \to C_\nu(\mathbb{R})^2$.

Theorem 3.17. *For each $\nu \in (0,1)$ and each sufficiently small value of $\varepsilon > 0$ equation* (3.49) *has two homoclinic solutions $Q^{\varepsilon\pm}$ which are symmetric, that is invariant under the transformation $(Q_1(X), Q_2(X)) \mapsto (Q_1(-X), -Q_2(-X))$, and satisfy the estimate*

$$Q^{\varepsilon\pm}(X) = \pm \begin{pmatrix} h(X) \\ 0 \end{pmatrix} + O(\varepsilon \mathrm{e}^{-\nu|X|})$$

for all $X \in \mathbb{R}$.

Proof. For $\varepsilon = 0$ equation (3.49) has the family

$$\{(Q_1, Q_2)^{\mathrm{T}} = R_\theta(h(X_0 + \cdot), 0)^{\mathrm{T}} \colon \theta \in [0, 2\pi),\ X_0 \in \mathbb{R}\}$$

of homoclinic solutions, where

$$h(X) = \left(\frac{2}{C}\right)^{1/2} \operatorname{sech}(X).$$

Two of these solutions, namely those with $(\theta, X_0) = (0,0)$ and $(\theta, X_0) = (\pi, 0)$, which we denote by respectively Q^+ and Q^-, are symmetric. We seek a solution of (3.49) in the form of a perturbation of Q^+ by writing

$$Q_1 = h + z_1, \quad Q_2 = z_2,$$

so that $z = (z_1, z_2)^{\mathrm{T}}$ satisfies

$$z_{1XX} - z_1 = -3Ch^2 z_1 + r_1^\varepsilon(z_1, z_2, z_{1X}, z_{2X}, z_{1XX}, z_{2XX}, X), \tag{3.50}$$
$$z_{2XX} - z_2 = -Ch^2 z_2 + r_2^\varepsilon(z_1, z_2, z_{1X}, z_{2X}, z_{1XX}, z_{2XX}, X) \tag{3.51}$$

with the obvious definitions of r_1^ε and r_2^ε. We study the system (3.50), (3.51) in the space $C_\nu^2(\mathbb{R})^2$ with fixed $\nu \in (0,1)$ and, with a slight abuse of notation, consider the nonlinearity $r^\varepsilon = (r_1^\varepsilon, r_2^\varepsilon)^{\mathrm{T}}$ as a mapping $C_\nu^2(\mathbb{R})^2 \to C_\nu(\mathbb{R})^2$ and $C_{\nu,\mathrm{e}}^2(\mathbb{R}) \times C_{\nu,\mathrm{o}}^2(\mathbb{R}) \to C_{\nu,\mathrm{e}}(\mathbb{R}) \times C_{\nu,\mathrm{o}}(\mathbb{R})$ with

$$\|r^\varepsilon(z_1, z_2)\|_{0,\nu} = O(\varepsilon) + \underline{O}_1(\|(z_1, z_2)\|_{2,\nu}^2).$$

In terms of the operators K and K_h defined in Proposition 3.16 equations (3.50), (3.51) can thus be written as

$$z = K_h z + K^{-1} r^{\varepsilon}(z). \tag{3.52}$$

The eigenvalue problem

$$K_h z = z$$

is equivalent to the decoupled system

$$z_{1XX} = z_1 - 3Ch^2 z_1, \tag{3.53}$$
$$z_{2XX} = z_2 - Ch^2 z_2 \tag{3.54}$$

of ordinary differential equations. Let

$$\begin{aligned} z_1^1(X) &= \operatorname{sech}(X)\tanh(X), & z_2^1(X) &= \operatorname{sech}(X), \\ z_1^2(X) &= \operatorname{sech}(X)(-3+\cosh^2(X)+3X\tanh(X)), & z_2^2(X) &= \operatorname{sech}(X)(2X+\sinh(2X)), \end{aligned}$$

so that $\{z_1^1, z_1^2\}$ and $\{z_2^1, z_2^2\}$ are fundamental solution sets for respectively (3.53) and (3.54). Since z_1^1, z_2^1 are bounded while z_1^2, z_2^2 are unbounded, we conclude that all bounded solutions of equation (3.53) are multiples of $z_1^1 = -h_X$ and all bounded solutions of equation (3.54) are multiples of $z_2^1 = (2/C)^{-1/2}h$. The eigenspace of $K_h\colon C_\nu(\mathbb{R})^2 \to C_\nu(\mathbb{R})^2$ corresponding to the eigenvalue 1 is therefore

$$\operatorname{sp}\left\{\begin{pmatrix} h_X \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ h \end{pmatrix}\right\},$$

which lies in $C_{\nu,\mathrm{o}}(\mathbb{R}) \times C_{\nu,\mathrm{e}}(\mathbb{R})$. This calculation shows that 1 is not an eigenvalue of $K_h|_{C_{\nu,\mathrm{e}}(\mathbb{R})\times C_{\nu,\mathrm{o}}(\mathbb{R})}$ and since K_h is a compact operator $C_\nu(\mathbb{R})^2 \to C_\nu(\mathbb{R})^2$, one concludes that the spectrum of $K_h|_{C_{\nu,\mathrm{e}}(\mathbb{R})\times C_{\nu,\mathrm{o}}(\mathbb{R})}$ consists only of eigenvalues, so that 1 lies in the resolvent set of $K_h|_{C_{\nu,\mathrm{e}}(\mathbb{R})\times C_{\nu,\mathrm{o}}(\mathbb{R})}$. It follows that

$$I - K_h\colon C_{\nu,\mathrm{e}}(\mathbb{R}) \times C_{\nu,\mathrm{o}}(\mathbb{R}) \to C^2_{\nu,\mathrm{e}}(\mathbb{R}) \times C^2_{\nu,\mathrm{o}}(\mathbb{R})$$

is invertible. We can therefore solve equation (3.52) for sufficiently small values of $\varepsilon > 0$ using the implicit function theorem; the solution $z_\star^\varepsilon$ satisfies $\|z_\star^\varepsilon\|_{2,\nu} = O(\varepsilon)$.

Returning to equation (3.49), we have found a symmetric solution $Q^{\varepsilon+} = Q^+ + z_\star^\varepsilon$ which satisfies the stated estimate. The second homoclinic solution $Q^{\varepsilon-}$ is obtained from Q^- by the same procedure. ■

3.5 Appendix

3.5.1 Formal derivation of the nonlinear Schrödinger equation

Writing $\beta = \beta_0$, $\alpha = \alpha_0 + \varepsilon^2$ and substituting the formal asymptotic expansions

$$\eta(x) = \varepsilon\eta_1(x, X) + \varepsilon^2\eta_2(x, X) + \varepsilon^3\eta_3(x, X) + \dots,$$
$$\Psi(x, y) = \varepsilon\Psi_1(x, X, y) + \varepsilon^2\Psi_2(x, X, y) + \varepsilon^3\Psi_3(x, X, y) + \dots,$$

where $X = \varepsilon x$, into equations (3.2)–(3.6) yields the boundary-value problems

$$\Psi_{1xx} + \Psi_{1yy} = 0, \qquad 0 < y < 1, \tag{3.55}$$
$$\Psi_{1y}\Big|_{y=0} = 0, \tag{3.56}$$
$$\Psi_{1y} + \eta_{1x}\Big|_{y=1} = 0, \tag{3.57}$$
$$\alpha_0\eta_1 - \Psi_{1x} + \beta_0\eta_{1xxxx}\Big|_{y=1} = 0 \tag{3.58}$$

for Ψ_1,

$$\Psi_{2xx} + \Psi_{2yy} + 2\Psi_{1xX} + 2\eta_1\Psi_{1xx} - 2y\eta_{1x}\Psi_{1xy} - y\Psi_{1y}\eta_{1xx} = 0, \qquad 0 < y < 1, \tag{3.59}$$
$$\Psi_{2y}\Big|_{y=0} = 0, \tag{3.60}$$
$$\Psi_{2y} + \eta_{1X} + \eta_{2x} - \eta_{1x}\Psi_{1x} + \eta_{1x}\eta_1\Big|_{y=1} = 0, \tag{3.61}$$
$$-\Psi_{2x} - \Psi_{1X} + \alpha_0\eta_2 + 4\beta_0\eta_{1xxxX} + \beta_0\eta_{2xxxx} + \Psi_{1y}\eta_{1x} + \tfrac{1}{2}\Psi_{1x}^2 + \tfrac{1}{2}\Psi_{1y}^2\Big|_{y=1} = 0 \tag{3.62}$$

for Ψ_2 and

$$\begin{aligned}&\Psi_{3xx} + \Psi_{3yy} + 2\Psi_{2xX} + 4\eta_1\Psi_{1xX} + 2\eta_1\Psi_{2xx} + 2\eta_2\Psi_{1xx}\\ &\quad - 2y\eta_{1x}\Psi_{1Xy} + \Psi_{1XX} - 2y\eta_{1x}\Psi_{2xy} - 2y\eta_{1X}\Psi_{1xy}\\ &\quad - 2y\eta_{2x}\Psi_{1xy} - 2y\eta_{1xX}\Psi_{1y} - y\eta_{2xx}\Psi_{1y} - y\eta_{1xx}\Psi_{2y}\\ &\quad + \eta_1^2\Psi_{1xx} + y^2\eta_{1x}^2\Psi_{1yy} - 2y\eta_1\eta_{1x}\Psi_{1xy} - y\eta_1\eta_{1xx}\Psi_{1y}\\ &\quad + 2y\eta_{1x}^2\Psi_{1y} = 0, \qquad\qquad 0 < y < 1,\end{aligned} \tag{3.63}$$
$$\Psi_{3y}\Big|_{y=0} = 0, \tag{3.64}$$
$$\begin{aligned}&\Psi_{3y} + \eta_{2X} + \eta_{3x} - \eta_{1x}\Psi_{1X} - \eta_{1x}\Psi_{2x} - \eta_{2x}\Psi_{1x} - \eta_{1X}\Psi_{1x}\\ &\quad + \eta_1\eta_{1X} + \eta_{1x}\eta_2 + \eta_1\eta_{2x} - \eta_1\eta_{1x}\Psi_{1x} + y\eta_{1x}^2\Psi_{1y}\Big|_{y=1} = 0,\end{aligned} \tag{3.65}$$
$$\begin{aligned}&-\Psi_{3x} - \Psi_{2X} + \alpha_0\eta_3 + 6\beta_0\eta_{1xxXX} + 4\beta_0\eta_{2xxxX} + \beta_0\eta_{3xxxx}\\ &\quad + \eta_{1x}\Psi_{2y} + \eta_{1X}\Psi_{1y} + \eta_{2x}\Psi_{1y} + \Psi_{1x}\Psi_{1X} + \Psi_{1x}\Psi_{2x}\\ &\quad + \Psi_{1y}\Psi_{2y} - \tfrac{5}{2}\beta_0\eta_{1x}^2\eta_{1xxxx} - \eta_1\eta_{1x}\Psi_{1y} - \eta_1\Psi_{1y}^2\\ &\quad - \eta_{1x}\Psi_{1x}\Psi_{1y} + \eta_1 - 10\beta_0\eta_{1x}\eta_{1xx}\eta_{1xxx} - \tfrac{5}{2}\beta_0\eta_{1xx}^3\Big|_{y=1} = 0\end{aligned} \tag{3.66}$$

for Ψ_3. We proceed by making the modulational *Ansatz*

$$\begin{aligned}
\eta_1(x,X) &= \zeta_1(X)\mathrm{e}^{\mathrm{i}\mu_0 x} + \text{c.c.}\,,\\
\eta_2(x,X) &= \zeta_2(X)\mathrm{e}^{2\mathrm{i}\mu_0 x} + \text{c.c.} + \zeta_0(X),\\
\eta_3(x,X) &= \zeta_3(X)\mathrm{e}^{3\mathrm{i}\mu_0 x} + \zeta_4(X)\mathrm{e}^{2\mathrm{i}\mu_0 x} + \zeta_5(X)\mathrm{e}^{\mathrm{i}\mu_0 x} + \text{c.c.} + \zeta_6(X).
\end{aligned}$$

From (3.55)–(3.57) it follows that

$$\begin{aligned}
\Psi_{1xx} + \Psi_{1yy} &= 0, && 0 < y < 1,\\
\Psi_{1y}\Big|_{y=0} &= 0, &&\\
\Psi_{1y}\Big|_{y=1} &= -\mathrm{i}\mu_0\zeta_1\mathrm{e}^{\mathrm{i}\mu_0 x} + \text{c.c.}\,, &&
\end{aligned}$$

the solution to which is

$$\Psi_1(x,X,y) = -\frac{\mathrm{i}\cosh(\mu_0 y)}{\sinh(\mu_0)}\zeta_1\mathrm{e}^{\mathrm{i}\mu_0 x} + \text{c.c.} + g_1(X),$$

where g_1 is an arbitrary function of a single variable. The equation

$$(\alpha_0 + \beta_0\mu_0^4)\zeta_1 - \Psi_{1x}\Big|_{y=1} = 0,$$

which follows from (3.58), then recovers the dispersion relation (3.34).

From (3.59)–(3.61) it follows that

$$\begin{aligned}
\Psi_{2xx} + \Psi_{2yy} = & -2\mu_0\frac{\cosh(\mu_0 y)}{\sinh(\mu_0)}\zeta_{1X}\mathrm{e}^{\mathrm{i}\mu_0 x}\\
& - \mathrm{i}s(3\mu_0 y\sinh(\mu_0 y) - 2\cosh(\mu_0 y))\zeta_1^{2}\mathrm{e}^{2\mathrm{i}\mu_0 x} + \text{c.c.}, \qquad 0 < y < 1,\\
\Psi_{2y}\Big|_{y=0} = & \; 0,\\
\Psi_{2y}\Big|_{y=1} = & -\zeta_{1X}\mathrm{e}^{\mathrm{i}\mu_0 x} + \mathrm{i}\mu_0\left(\frac{\mu_0\cosh(\mu_0 y)}{\sinh(\mu_0)}\zeta_1^{2} - \zeta_1^2 - 2\zeta_2\right)\mathrm{e}^{2\mathrm{i}\mu_0 x},
\end{aligned}$$

the solution to which is

$$\begin{aligned}
\Psi_2(x,X,y) = & \left(\frac{\coth(\mu_0)}{\sinh(\mu_0)}\cosh(\mu_0 y) - y\frac{\sinh(\mu_0 y)}{\sinh(\mu_0)}\right)\zeta_{1X}\mathrm{e}^{\mathrm{i}\mu_0 x}\\
& + \left(\mathrm{i}\mu_0\left(\frac{\coth(\mu_0)\cosh(2\mu_0 y)}{\sinh(2\mu_0)} - y\frac{\sinh(\mu_0 y)}{\sinh(\mu_0)}\right)\zeta_1^2 + \frac{\mathrm{i}\cosh(2\mu_0 y)}{\sinh(2\mu_0)}\zeta_2\right)\mathrm{e}^{2\mathrm{i}\mu_0 x}\\
& + \text{c.c.} + g_2(X),
\end{aligned}$$

where g_2 is an arbitrary function of a single variable. Substituting the formulae for Ψ_1, Ψ_2 and the modulational *Ansatz* into (3.62), and equating the coefficients of $\mathrm{e}^{0\mathrm{i}\mu_0 x}, \mathrm{e}^{\mathrm{i}\mu_0 x}, \mathrm{e}^{2\mathrm{i}\mu_0 x}$, we then find that

$$\begin{aligned} g_{1X} &= \frac{\mu_0^2}{\sinh^2(\mu_0)}|\zeta_1|^2 + \alpha_0\zeta_0, \\ \zeta_2 &= \frac{1}{2}\,\frac{(1-3\coth^2(\mu_0))\mu_0^2}{\alpha_0+16\mu_0^4\beta_0-\mu_0(\coth(\mu_0)+(\coth(\mu_0))^{-1})}\,\zeta_1^2, \\ \beta_0 &= \frac{1}{4\mu_0^3}\coth(\mu_0) - \frac{1}{4\mu_0^2}\operatorname{cosech}^2(\mu_0). \end{aligned} \tag{3.67}$$

Using the dispersion relation and the above formula for β_0, we find that

$$\alpha_0 = \frac{3\mu_0}{4}\coth(\mu_0) + \frac{\mu_0^2}{4}\operatorname{cosech}^2(\mu_0).$$

Similarly, (3.63)–(3.65) yield a Poisson equation for Ψ_3 with boundary conditions at $y=0$ and $y=1$, the solution to which is

$$\begin{aligned} &\Psi_3(x,X,y) \\ &=\Bigg(\bigg(\frac{3}{2}\,\frac{\mathrm{i}\mu_0^2\cosh(\mu_0 y)}{\sinh(\mu_0)} - \frac{2\mathrm{i}\mu_0^2\coth(2\mu_0)\cosh(\mu_0)\cosh(\mu_0 y)}{\sinh^2(\mu_0)} - \frac{1}{2}\frac{\mathrm{i}\mu_0^2 y^2\cosh(\mu_0 y)}{\sinh(\mu_0)} \\ &\qquad + \mathrm{i}\mu_0^2 y\frac{\sinh(2\mu_0 y)}{\sinh^2(\mu_0)}\bigg)\bar{\zeta}_1\zeta_1^2 \\ &\qquad + \bigg(\frac{(2\mathrm{i}\mu_0\coth(\mu_0)+\mathrm{i}\mu_0\tanh(\mu_0))\cosh(\mu_0 y)}{\sinh(\mu_0)} \\ &\qquad\qquad + \frac{\mathrm{i}\mu_0 y\sinh(\mu_0 y)}{\sinh(\mu_0)} - \frac{2\mathrm{i}\mu_0 y\sinh(2\mu_0 y)}{\sinh(2\mu_0)}\bigg)\bar{\zeta}_1\zeta_2 \\ &\qquad + \bigg(\frac{\mathrm{i}\mu_0\coth(\mu_0)\cosh(\mu_0 y)}{\sinh(\mu_0)} - \frac{\mathrm{i}\mu_0 y\sinh(\mu_0 y)}{\sinh(\mu_0)}\bigg)\zeta_0\zeta_1 \\ &\qquad + \bigg(\frac{\mathrm{i}y^2\cosh(\mu_0 y)}{2\sinh(\mu_0)} + \frac{\mathrm{i}(2\coth^2(\mu_0)-1)\cosh(\mu_0 y)}{2\sinh(\mu_0)} - \frac{\mathrm{i}y\coth(\mu_0)\sinh(\mu_0 y)}{\sinh(\mu_0)}\bigg)\zeta_{1XX} \\ &\qquad - \frac{\mathrm{i}\cosh(\mu_0 y)}{\sinh(\mu_0)}\zeta_5 + \frac{\mathrm{i}\cosh(\mu_0 y)}{\sinh(\mu_0)}\zeta_1 g_{1X}\Bigg)\mathrm{e}^{\mathrm{i}\mu_0 x} + (\ldots)\mathrm{e}^{2\mathrm{i}\mu_0 x} + (\ldots)\mathrm{e}^{3\mathrm{i}\mu_0 x} + \mathrm{c.c} \\ &\qquad + \frac{\sinh(\mu_0 y)(\mu_0 y\coth(\mu_0)+\coth(\mu_0)-y) - \cosh(\mu_0 y)(\mu_0 y^2+\coth(\mu_0))}{\sinh(\mu_0)}\frac{\mathrm{d}}{\mathrm{d}X}|\zeta_1|^2 \\ &\qquad - \tfrac{1}{2}y^2 - \frac{y^2}{2}g_{1XX} \end{aligned}$$

with

$$g_{1XX} - \zeta_{0X} + 2\mu_0 \coth(\mu_0)\frac{\mathrm{d}}{\mathrm{d}X}|\zeta_1|^2 = 0.$$

By integrating this equation and substituting it into (3.67) we find that

$$\zeta_0 = \Big(\frac{\mu_0^2}{\alpha_0 - 1}(1 - \coth^2(\mu_0)) - \frac{2\mu_0 \coth(\mu_0)}{\alpha_0 - 1}\Big)|\zeta_1|^2,$$

so that

$$g_{1X} = \Big(\frac{\mu_0^2 \alpha_0}{\alpha_0 - 1}(1 - \coth^2(\mu_0)) - \frac{2\mu_0 \alpha_0}{\alpha_0 - 1}\coth(\mu_0) - \mu_0^2(1 - \coth^2(\mu_0))\Big)|\zeta_1|^2.$$

Substituting the formulae for ζ_0, ζ_2, g_{1X}, Ψ_1, Ψ_2, Ψ_3 and the modulational Ansatz into (3.66), and equating coefficients of $\mathrm{e}^{\mathrm{i}\mu_0 x}$, finally yields the nonlinear Schrödinger equation

$$\begin{aligned}\zeta_1 &- (6\beta_0\mu_0^2 - (1-\sigma^2)(1-\mu_0\sigma))\zeta_{1XX} \\ &+ \Big(\frac{-\mu_0^4(1-3\sigma^2)^2}{2(\alpha_0 + 16\beta_0\mu_0^4 - \mu_0(\sigma + \sigma^{-1}))} + \mu_0^3(-5\mu_0^3\beta_0 + 4\sigma - 2\sigma^3) \\ &\qquad - \frac{\mu_0^4(1-\sigma^2)^2}{\alpha_0 - 1} + \frac{4\mu_0^3\sigma(1-\sigma^2)}{\alpha_0 - 1} - \frac{4\alpha_0\mu_0^2\sigma^2}{\alpha_0 - 1}\Big)|\zeta_1|^2\zeta_1 = 0,\end{aligned}$$

where $\sigma = \coth(\mu_0)$.

3.5.2 Computation of the normal-form coefficients

For this purpose we make use of the calculation

$$\Upsilon|_0(Lu, v) = H_2^0(u, v) = H_2^0(v, u) = \Upsilon|_0(Lv, u),$$

denote the parts of $H^\delta(w)$, $g^\delta(w)$ which are homogeneous of order m in δ and n in w by $\delta^m H_n^m(w)$, $\delta^m N_n^m(w)$ and the part of $r(u_1;\delta)$ which is homogeneous of order m in δ and n in u_1 by $r_n^m(u_1;\delta)$. With a slight abuse of notation we use the same symbols for the multilinear operators associated with these quantities.

Write

$$r_n^m(u_1;\delta) = \sum_{i+j+k+\ell=m} r_{ijk\ell}^n \delta^m A^i B^j \bar{A}^k \bar{B}^\ell$$

and consider the δA-component of (3.40), namely

$$(L - \mathrm{i}\mu_0 I)r_{1000}^1 = c_2\mathrm{i}E - c_1 F - N_1^1(E).$$

Taking the symplectic product of this equation with $\bar{E}$, we find that

$$c_1 = -\Upsilon|_0(r^1_{1000}, \underbrace{(L + \mathrm{i}\mu_0 I)\bar{E}}_{=0}) + \Upsilon|_0(N^1_1(E), \bar{E}) = 2H^1_2(E, \bar{E}) = -\frac{\sinh^2(\mu_0)}{\tau_1}.$$

To compute d_1 we consider the $A^2\bar{A}$-component of (3.40), namely

$$(L - \mathrm{i}\mu_0 I)r^0_{2010} = \mathrm{i}d_2 E - 2d_1 F - 3N^0_3(E, E, \bar{E}) - 2N^0_2(\bar{E}, r^0_{200000}) - 2N^0_2(E, r^0_{101000}),$$

and again take the symplectic product with $\bar{E}$, so that

$$\begin{aligned} 2d_1 = {} & -\Upsilon|_0(r^0_{2010}, \underbrace{(L + \mathrm{i}\mu_0 I)\bar{E}}_{=0}) + 3\Upsilon|_0(N^0_3(E, E, \bar{E}), \bar{E}) \\ & + 2\Upsilon|_0(N^0_2(\bar{E}, r^0_{2000}), \bar{E}) + 2\Upsilon|_0(N^0_2(E, r_{1010}), \bar{E}). \end{aligned}$$

The functions r^0_{2000} and r^0_{1010} are obtained from the A^2- and $A\bar{A}$-components of (3.40), which are respectively

$$\begin{aligned} (K - 2\mathrm{i}\mu_0 I)r^0_{2000} &= -N^0_2(E, E), \\ K r^0_{1010} &= -2N^0_2(E, \bar{E}) \end{aligned}$$

(note that r^0_{101000} is determined up to addition of a multiple of F). Altogether we find that

$$\begin{aligned} d_1 = \frac{\sinh^4(\mu_0)}{2\tau_1^2} \Bigg(& \frac{\mu_0^4(1 - 3\sigma^2)^2}{2(\alpha_0 + 16\beta_0\mu_0^4 - \mu_0(\sigma + \sigma^{-1})} - \mu_0^3(-5\mu_0^3\beta_0 + 4\sigma - 2\sigma^3) \\ & + \frac{\mu_0^4(1 - \sigma^2)^2}{\alpha_0 - 1} - \frac{4\mu_0^3\sigma(1 - \sigma^2)}{\alpha_0 - 1} + \frac{4\alpha_0\mu_0^2\sigma^2}{\alpha_0 - 1} \Bigg), \end{aligned}$$

where $\sigma = \coth(\mu_0)$.

For completeness, we record the formulae for $\tilde{r}^0_{101000}$ and $\tilde{r}^0_{200000}$, namely

$$\tilde{r}^0_{1010} = \begin{pmatrix} -\mu_0(\alpha_0 - 1)^{-1}(\mu_0 + \sinh(2\mu_0)) \\ 0 \\ 0 \\ 0 \\ 0 \\ -\mu_0 \sinh(2\mu_0) + 2\mu_0 \sinh(\mu_0)(\mu_0 y \sinh(\mu_0 y) + \cosh(\mu_0 y)) \end{pmatrix},$$

$$\tilde{r}^0_{2000} = \tilde{a}^0_{2000} \begin{pmatrix} \mathrm{i}\sinh(2\mu_0) \\ -2\mu_0\sinh(2\mu_0) \\ \frac{1}{2\mu_0}(\alpha_0 - 1)\sinh(2\mu_0) \\ -4\mathrm{i}\mu_0^2\beta_0\sinh(2\mu_0) \\ -(\frac{1}{2\mu_0} + (y^2 - \frac{1}{3})\mu_0)\sinh(2\mu_0) + \cosh(2\mu_0 y) \\ 2\mathrm{i}\mu_0\cosh(2\mu_0 y) - \mathrm{i}\sinh(2\mu_0) \end{pmatrix}$$
$$+ \begin{pmatrix} \frac{1}{2}\mu_0\sinh(2\mu_0) \\ \mathrm{i}\mu_0^2\sinh(2\mu_0) \\ -\frac{\mathrm{i}}{8}(2\alpha_0 - 3)\sinh(2\mu_0) - \frac{\mathrm{i}}{\mu_0}\sinh^2(\mu_0) \\ -2\beta_0\mu_0^3\sinh(2\mu_0) \\ \mathrm{i}\sinh(\mu_0)(-2\mu_0 y\sinh(\mu_0 y) + \cosh(\mu_0 y)) \\ \mathrm{i}(1 + \frac{1}{2}\mu_0^2(y^2 - \frac{1}{3}))\sinh(2\mu_0) + \mathrm{i}(-\frac{3}{\mu_0} + \frac{1}{2}\mu_0(y^2 - \frac{1}{3}))\sinh^2(\mu_0) \\ \mu_0\sinh(\mu_0)(\mu_0 y\sinh(\mu_0 y) + \cosh(\mu_0 y)) - \frac{1}{2}\mu_0\sinh(2\mu_0) \end{pmatrix},$$

where

$$\tilde{a}^0_{2000} = \frac{1}{2}\mathrm{i}\Big(\frac{\mu_0^2(\cosh(2\mu_0) + 2)}{\sinh(2\mu_0)(\alpha_0 + 16\beta_0\mu_0^4) - 2\mu_0\cosh(2\mu_0)} + \mu_0\Big).$$

4 A dimension-breaking phenomenon for solitary hydroelastic surface waves

4.1 Dimension-breaking phenomena

4.1.1 Spatial dynamics

In this section we formulate equations (1.40)–(1.43), which follow from the formal variational principle (1.32), as an evolutionary equation in which the unbounded horizontal direction z plays the role of time. The first step is to use the change of variable

$$\Phi(x, y', z) = \phi(x, y, z), \qquad y' = \frac{1+y}{1+\eta(x,z)}$$

to map the variable fluid domain $\{(x, y, z) : x, z \in \mathbb{R}, y \in (-1, \eta(x, z))\}$ into the fixed strip $\mathbb{R} \times (0, 1) \times \mathbb{R}$; we drop the primes for notational convenience. The transformed equations are

$$\begin{aligned}
&\Phi_{xx} - 2\frac{\eta_x}{1+\eta}y\Phi_{xy} - \frac{\eta_{xx}}{1+\eta}y\Phi_y + 2\frac{\eta_x^2}{(1+\eta)^2}y\Phi_y + \frac{\eta_x^2}{1+\eta}y^2\Phi_{yy} + \frac{\Phi_{yy}}{(1+\eta)^2} \\
&\quad + \Phi_{zz} - \frac{\eta_z}{1+\eta}y\Phi_{yz} - \frac{\eta_{zz}}{1+\eta}y\Phi_{yz} + 2\frac{\eta_z^2}{(1+\eta)^2}y\Phi_y + \frac{\eta_z^2}{1+\eta}y^2\Phi_{yy} = 0, && 0 < y < 1, \\
&\Phi_y = 0, && y = 0, \\
&\frac{\Phi_y}{1+\eta} = -\eta_x + \Phi_x\eta_x - \frac{\eta_x^2}{1+\eta}y\Phi_y + \Phi_z\eta_z - \frac{\eta_z^2}{1+\eta}y\Phi_y, && y = 1, \\
&\Phi_t - \frac{\eta_t}{1+\eta}y\Phi_y - \Phi_x - \frac{\eta_x}{1+\eta}y\Phi_y + \frac{1}{2}\Big(\Phi_x - \frac{\eta_x}{1+\eta}y\Phi_y\Big)^2 \\
&\quad + \frac{\Phi_y^2}{2(1+\eta)^2} + \frac{1}{2}\Big(\Phi_z - \frac{\eta_z}{1+\eta}y\Phi_y\Big)^2 + \alpha\eta + \beta U(\eta) = 0, && y = 1,
\end{aligned}$$

which follow from the transformed formal variational principle $\delta\mathcal{L} = 0$, where

$$\mathcal{L}(\eta, \eta_z, \eta_{zz}, \Phi, \Phi_z) = \int_{\mathbb{R}} L(\eta, \eta_z, \eta_{zz}, \Phi, \Phi_z)\,\mathrm{d}z$$

and

$$L(\eta,\eta_z,\eta_{zz},\Phi,\Phi_z)=\int_{\mathbb{R}}\left\{\int_0^1\left[-\Phi_x+\frac{\eta_x}{1+\eta}+\frac{1}{2}\left(\Phi_x-\frac{\eta_x}{1+\eta}y\Phi_y\right)^2\right.\right.$$
$$\left.+\frac{\Phi_y^2}{2(1+\eta)^2}+\frac{1}{2}\left(\Phi_z-\frac{\eta_z}{1+\eta}y\Phi_y\right)^2\right](1+\eta)\,\mathrm{d}y$$
$$\left.+\frac{1}{2}\alpha(\eta-1)^2+2\beta P^2\sqrt{Q}\right\}\mathrm{d}x.$$

To write the problem as an evolutionary equation in which z plays the role of time, we interpret $\mathcal{L}$ as an action functional and formulate the equations as a Hamiltonian system. We therefore introduce new variables

$$\begin{aligned}
\rho&=\eta_z,\\
\xi&=\frac{\partial L}{\partial\eta_{zz}}=2\beta\frac{(1+\eta_x^2)P}{Q},\\
\omega&=\frac{\partial L}{\partial\eta_z}-\partial_z\left(\frac{\partial L}{\partial\eta_{zz}}\right),\\
&=\int_0^1-\left(\Phi_z-\frac{\eta_z}{1+\eta}y\Phi_y\right)y\Phi_y\,\mathrm{d}y+4\beta\frac{P}{Q}(\eta_z\eta_{xx}-\eta_{xz}\eta_x)-10\beta\frac{P^2}{\sqrt{Q}}\eta_z-\xi_z,\\
\Psi&=\frac{\partial L}{\partial\Phi_z}=\left(\Phi_z-\frac{\eta_z}{1+\eta}y\Phi_y\right)(1+\eta).
\end{aligned}$$

The Hamiltonian is given by

$$\begin{aligned}
H(\eta,\rho,\Phi,\omega,\xi,\Psi)&=\int_{\mathbb{R}}\omega\rho\,\mathrm{d}x+\int_{\mathbb{R}}\xi\eta_{zz}\,\mathrm{d}x+\int_{\mathbb{R}}\int_0^1\Psi\Phi_z\,\mathrm{d}y\,\mathrm{d}x-L\\
&=\int_{\mathbb{R}}\left\{\int_0^1\left[\left(\frac{\Psi}{(1+\eta)^2}+\frac{\rho}{(1+\eta)^2}y\Phi_y\right)\Psi-\frac{1}{2}\frac{\Phi_y^2}{(1+\eta)^2}-\frac{1}{2}\frac{\Psi^2}{(1+\eta)^2}\right.\right.\\
&\qquad\left.+\left(\Phi_x-\frac{\eta_x}{1+\eta}y\Phi_y\right)-\frac{1}{2}\left(\Phi_x-\frac{\eta_x}{1+\eta}y\Phi_y\right)^2\right](1+\eta)\,\mathrm{d}y\\
&\qquad+\omega\rho+\frac{Q(\eta,\rho)^{\frac{5}{2}}}{2\beta(1+\eta_x^2)^2}-\frac{\xi}{1+\eta_x^2}\left[(1+\rho^2)\eta_{xx}-2\rho_x\rho\eta_x\right]\\
&\qquad\left.-\frac{1}{2}\alpha\eta^2\right\}\mathrm{d}x,
\end{aligned}$$

where

$$Q(\eta,\rho)=1+\eta_x^2+\rho^2$$

and we have used the identity

$$\eta_{zz} = \frac{Q(\eta,\rho)^{\frac{5}{2}}}{\beta(1+\eta_x^2)^2}\xi - \frac{1}{1+\eta_x^2}\Big[(1+\rho^2)\eta_{xx} - 2\rho_x\rho\eta_x\Big],$$

which follows from the definition of ξ. Hamilton's equations are given by

$$\eta_z = \frac{\delta H}{\delta \omega} = \rho, \tag{4.1}$$

$$\rho_z = \frac{\delta H}{\delta \xi} = \frac{Q(\eta,\rho)^{\frac{5}{2}}}{\beta(1+\eta_x^2)^2}\xi - \frac{1}{1+\eta_x^2}\Big[(1+\rho^2)\eta_{xx} - 2\rho_x\rho\eta_x\Big], \tag{4.2}$$

$$\Phi_z = \frac{\delta H}{\delta \Psi} = \frac{\Psi}{1+\eta} + \frac{\rho}{1+\eta}y\Phi_y, \tag{4.3}$$

$$\begin{aligned}\omega_z = -\frac{\delta H}{\delta \eta} = -\int_0^1 \Bigg\{ &-\frac{1}{2}\frac{\Psi^2}{(1+\eta)^2} - \frac{\rho}{(1+\eta)^2}y\Phi_y\Psi + \Phi_x - \frac{1}{2}\Phi_x^2 + \frac{1}{2}\frac{\eta_x^2}{(1+\eta)^2}y^2\Phi_y^2 \\ &+ \frac{1}{2}\frac{\Phi_y^2}{(1+\eta)^2} + y\Phi_{xy} - y(\Phi_x\Phi_y)_x + \left[\frac{\eta_x}{1+\eta}y^2\Phi_y^2\right]_x\Bigg\}\,\mathrm{d}y \\ &- \alpha\eta + \left[\frac{\xi^2 Q(\eta,\rho)^{\frac{3}{2}}(5\eta_x(1+\eta_x^2)^2 - 4Q(\eta,\rho)(1+\eta_x^2)\eta_x)}{2\beta(1+\eta_x^2)^4}\right]_x \\ &+ \left[2\xi\frac{\rho_x\rho(1+\eta_x^2) + \eta_x(\eta_{xx} + \rho^2\eta_{xx} - 2\rho_x\rho\eta_x)}{(1+\eta_x^2)^2}\right]_x + \left[\frac{\xi(1+\rho^2)}{1+\eta_x^2}\right]_{xx},\end{aligned} \tag{4.4}$$

$$\xi_z = -\frac{\delta H}{\delta \rho} = -\int_0^1 \frac{y\Phi_y\Psi}{1+\eta}\,\mathrm{d}y - \omega - \frac{5}{2}\frac{Q(\eta,\rho)^{\frac{3}{2}}\xi^2\rho}{\beta(1+\eta_x^2)^2} + \frac{\xi}{1+\eta_x^2}(2\rho\eta_{xx} - 2\rho_x\eta_x) + \left[\frac{2\xi\rho\eta_x}{1+\eta_x^2}\right]_x, \tag{4.5}$$

$$\Psi_z = -\frac{\delta H}{\delta \Phi} = \frac{\rho}{1+\eta}\Big[y\Psi\Big]_y + \eta_x\left[\left(\Phi_x - \frac{\eta_x}{1+\eta}y\Phi_y\right)y\right]_y - \frac{\Phi_{yy}}{1+\eta} - \left[\left(\Phi_x - \frac{\eta_x}{1+\eta}y\Phi_y\right)y\right]_x \tag{4.6}$$

with boundary conditions

$$\Phi_y = 0, \qquad y = 0, \tag{4.7}$$

$$\frac{\rho}{1+\eta}\Psi - \eta_x + \left(\Phi_x - \frac{\eta_x}{1+\eta}\Phi_y\right)\eta_x - \frac{\Phi_y}{1+\eta} = 0, \qquad y = 1. \tag{4.8}$$

Equations (1.40)–(1.43) can therefore be formulated as

$$u_z = f(u) \tag{4.9}$$

for $u = (\eta, \rho, \Phi, \omega, \xi, \Psi)$ with boundary conditions

$$\Phi_y = B(u) \quad \text{ on } y = 0, 1, \tag{4.10}$$

where $f(u)$ is given by the right hand side of equations (4.1)–(4.6) and

$$B(\eta, \rho, \Phi, \omega, \xi, \Psi) = -y\eta_x + y\Phi_x\eta_x + \frac{y(\rho\Psi - y\eta_x^2\Phi_y)}{1+\eta} + \frac{\eta}{1+\eta}\Phi_y;$$

note the helpful identity

$$f_6(u) = -\Big[(1+\eta)\Phi_x - y\eta_x\Phi_y\Big]_x + \Big[-\Phi_y + y\eta_x + B(\eta, \rho, \Phi, \omega, \xi, \Psi)\Big]_y, \tag{4.11}$$

where f_6 is the sixth component of f. Working with the parameter regime in which a line solitary wave exists, we henceforth set

$$\alpha = \alpha_0 + \varepsilon^2, \qquad \beta = \beta_0,$$

where β_0, α_0 are given by equation (1.33) and $\mu_0 < \mu_0^\star$.

Define

$$X^s = H^{s+2}(\mathbb{R}) \times H^{s+1}(\mathbb{R}) \times H^{s+1}(\Sigma) \times H^s(\mathbb{R}) \times H^{s+1}(\mathbb{R}) \times H^s(\Sigma), \quad s \geq 0,$$

where Σ is the strip $\mathbb{R} \times (0, 1)$, and let M be a neighbourhood of the origin in X^1 contained in the set

$$\Big\{u \in X^1 : \eta(x) > -1 \text{ for all } x \in \mathbb{R}\Big\}$$

(note that M is a manifold domain of X^0). The following result (see Bagri and Groves [6, Proposition 2.1] and Propostition 2.10) shows that $B, f : M \to H^1(\Sigma)$ are analytic mappings.

Proposition 4.1.

(i) *The formula* $w \mapsto w|_{y=0}$ *defines continuous mappings* $H^n(\Sigma) \to H^{n-1}(\mathbb{R})$ *for each* $n \in \mathbb{N}$.

(ii) *The formula*

$$(w_1, w_2) \mapsto w_1 w_2$$

defines continuous mappings $L^2(\Sigma) \times H^1(\mathbb{R}) \to L^2(\mathbb{R})$, $H^1(\Sigma) \times L^2(\mathbb{R}) \to L^2(\Sigma)$ *and* $H^1(\Sigma) \times H^1(\mathbb{R}) \to H^1(\Sigma)$.

(iii) *The formula*

$$(w_1, w_2) \mapsto \int_{-\infty}^{0} w_1(\cdot, y) w_2(\cdot, y) \, \mathrm{d}y$$

defines continuous mappings $L^2(\Sigma) \times H^1(\Sigma) \to L^2(\mathbb{R})$, $H^1(\Sigma) \times L^2(\Sigma) \to L^2(\mathbb{R})$ *and* $H^1(\Sigma) \times H^1(\Sigma) \to H^1(\mathbb{R})$.

(iv) *The formula*

$$\eta \mapsto (1 + \eta)^{-1}$$

defines a mapping $H^3(\mathbb{R}) \to H^3(\mathbb{R})$ *which is analytic in a neighbourhood of the origin.*

(v) *For each* $n \in \mathbb{N}$ *the formula*

$$\eta \mapsto (1 + \eta_x^2)^{-n}$$

defines a mapping $H^3(\mathbb{R}) \to H^2(\mathbb{R})$ *which is analytic in a neighbourhood of the origin.*

(vi) *For each* $n \in \mathbb{N}$ *the formula*

$$(\eta, \rho) \mapsto Q(\eta, \rho)^{n/2}$$

defines a mapping $H^3(\mathbb{R}) \times H^2(\mathbb{R}) \to H^2(\mathbb{R})$ *which is analytic in a neighbourhood of the origin.*

Equations (4.9) and (4.10) therefore constitute an evolutionary equation in the infinite-dimensional phase space X^0 with

$$\mathcal{D}(f) = \Big\{ u \in M : \Phi_y = B(u) \text{ on } y = 0, 1 \Big\}.$$

These equations are reversible, i.e. invariant under the transformation

$$z \mapsto -z, \quad u \mapsto R(u),$$

where the reverser $R : X^s \to X^s$ is defined by

$$R(\eta, \rho, \Phi, \omega, \xi, \Psi) = (\eta, -\rho, -\Phi, -\omega, \xi, \Psi).$$

Equations (4.9) and (4.10) are also invariant under the reflection $S : X^s \to X^s$ given by

$$\begin{aligned} &S\Big(\eta(x), \rho(x), \Phi(x,y), \omega(x), \xi(x), \Psi(x,y)\Big) \\ &\quad = \Big(\eta(-x), \rho(-x), -\Phi(-x,y), \omega(-x), \xi(-x), -\Psi(x,y)\Big), \end{aligned}$$

and one may seek solutions which are invariant under this symmetry by replacing X^s by

$$\begin{aligned} X_\mathrm{r}^s &:= X^2 \cap \operatorname{Fix} S \\ &= H_\mathrm{e}^{s+2}(\mathbb{R}) \times H_\mathrm{e}^{s+1}(\mathbb{R}) \times H_\mathrm{o}^{s+1}(\Sigma) \times H_\mathrm{e}^{s}(\mathbb{R}) \times H_\mathrm{e}^{s+1}(\mathbb{R}) \times H_\mathrm{o}^{s}(\Sigma), \end{aligned}$$

where

$$\begin{aligned} H_\mathrm{e}^s(\mathbb{R}) &= \Big\{w \in H^2(\mathbb{R}) : w(x) = w(-x) \text{ for all } x \in \mathbb{R}\Big\}, \\ H_\mathrm{o}^s(\Sigma) &= \Big\{w \in H^2(\Sigma) : w(x,y) = -w(-x,y) \text{ for all } (x,y) \in \Sigma\Big\}, \end{aligned}$$

(with corresponding definitions for $H_\mathrm{o}^s(\mathbb{R})$ and $H_\mathrm{e}^s(\Sigma)$) and M by $M_\mathrm{r} := M \cap \operatorname{Fix} S$. It is also possible to extend M_r by replacing X_r^1 in its definition with the extended function space $X_\star^1$, where

$$X_\star^1 = H_\mathrm{e}^{s+2}(\mathbb{R}) \times H_\mathrm{e}^{s+1}(\mathbb{R}) \times H_{\star,\mathrm{o}}^{s+1}(\Sigma) \times H_\mathrm{e}^{s}(\mathbb{R}) \times H_\mathrm{e}^{s+1}(\mathbb{R}) \times H_{\star,\mathrm{o}}^{s}(\Sigma),$$

and

$$H_{\star,\mathrm{o}}^{s+1}(\Sigma) = \Big\{w \in L^2_\mathrm{loc}(\Sigma) : w_x, w_y \in H^s(\Sigma), w(x,y) = -w(-x,y) \text{ for all } (x,y) \in \Sigma\Big\}$$

(a Banach space with norm $\|u\|_{\star,s+1} := (\|u_x\|_s^2 + \|u_y\|_s^2)^{\frac{1}{2}}$); note the relationship $M_\mathrm{r} = M_\star \cap X_\mathrm{r}^1$ between M_r and its extension $M_\star$. This feature allows us to consider solutions to (4.9), (4.10) whose Φ-component is unbounded; in particular line solitary waves fall into this category.

Proposition 4.1 shows that $H : M \to \mathbb{R}$ is analytic, and the calculation

$$
\begin{aligned}
&\mathrm{d}H[\eta,\rho,\Phi,\omega,\xi,\Psi](\eta_1,\rho_1,\Phi_1,\omega_1,\xi_1,\Psi_1)\\
&\quad=\int_0^1\bigg\{\int_0^1\bigg[-\frac{\Psi^2}{(1+\eta)^2}\eta_1-\frac{\rho y\Phi_y\Psi}{(1+\eta)^2}\eta_1+\frac{1}{2}\frac{\Phi_y^2}{(1+\eta)^2}\eta_1+\frac{1}{2}\frac{\Psi^2}{(1+\eta)^2}\eta_1+\Phi_x\eta_1\\
&\qquad\qquad\qquad-y\Phi_y\eta_{1x}-\frac{\eta_x y\Phi_x\Phi_y}{1+\eta}\eta_1+\frac{\eta_x^2y^2\Phi_y^2}{(1+\eta)^2}\eta_1-\frac{1}{2}\Big(\Phi_x-\frac{\eta_x}{1+\eta}y\Phi_y\Big)^2\eta_1\\
&\qquad\qquad\qquad+\Big(\Phi_x-\frac{\eta_x}{1+\eta}y\Phi_y\Big)y\Phi_y\eta_{1x}+\frac{y\Phi_y\Psi}{1+\eta}\rho_1+\frac{\rho y\Psi}{1+\eta}\Phi_{1y}-\frac{\Phi_y}{1+\eta}\Phi_{1y}\\
&\qquad\qquad\qquad+(1+\eta)\Phi_{1x}-\eta_x y\Phi_{1y}-(1+\eta)\Phi_x\Phi_{1x}+\eta_x y\Phi_y\Phi_{1x}\\
&\qquad\qquad\qquad+\Big(\Phi_x-\frac{\eta_x}{1+\eta}y\Phi_y\Big)\eta_x y\Phi_{1y}+\frac{2\Psi}{1+\eta}\Psi_1+\frac{\rho y\Phi_y}{1+\eta}\Psi_1\bigg]\,\mathrm{d}y\\
&\qquad\quad+\frac{5Q(\eta,\rho)^{3/2}\eta_x}{2\beta(1+\eta_x^2)^2}\eta_{1x}-\frac{2Q(\eta,\rho)^{5/2}\eta_x}{\beta(1+\eta_x^2)^3}\eta_{1x}+\frac{2\xi(1+\rho^2)\eta_{xx}}{(1+\eta_x^2)^2}\eta_{1x}\\
&\qquad\quad+\frac{2\xi\rho_x\rho}{1+\eta_x^2}\Big(1-\frac{2\eta_x^2}{1+\eta_x^2}\Big)\eta_{1x}-\frac{\xi(1+\rho^2)}{1+\eta_x^2}\eta_{1xx}-\alpha\eta\eta_1\\
&\qquad\quad+\omega\rho_1+\frac{5Q(\eta,\rho)^{3/2}\rho}{2\beta(1+\eta_x^2)^2}\rho_1-\frac{\xi}{1+\eta_x^2}(2\rho\eta_{xx}-2\rho_x\eta_x)\rho_1+\frac{2\xi\rho\eta_x}{1+\eta_x^2}\rho_{1x}\\
&\qquad\quad+\rho\omega_1-\frac{1}{1+\eta_x^2}\Big[(1+\rho^2)\eta_{xx}-2\rho_x\rho\eta\Big]\xi_1\bigg\}\,\mathrm{d}x
\end{aligned}
$$

shows that $\mathrm{d}H[u]\in\mathcal{L}(X^1,\mathbb{R})$ extends to an operator $\widetilde{\mathrm{d}H}[u]\in\mathcal{L}(X^0,\mathbb{R})$ which depends analytically upon $u\in M$. Using this fact we find that f is the Hamiltonian vector field for the Hamiltonian system (X^0,Ω,H), i.e.

$$\Omega(f(u),v)=\widetilde{\mathrm{d}H}[u](v)$$

for all $u\in\mathcal{D}(f)$ and $v\in X^0$, where the position-independent symplectic form Ω on X^0 is given by

$$\Omega(v_1,v_2)=\int_{\mathbb{R}}(\eta_1\omega_2-\eta_2\omega_1+\rho_1\xi_2-\rho_2\xi_1)\,\mathrm{d}x+\int_\Sigma\Big(\Phi_1\Psi_2-\Phi_2\Psi_1\Big)\,\mathrm{d}y\,\mathrm{d}x.$$

The line solitary waves found in Chapter 3 are equilibrium solutions of equations (4.9), (4.10) of the form $u^\star=(\eta^\star,0,\Phi^\star,0,\xi^\star,0)$ and we seek solutions of these equations of the form

$$u=u^\star+u',\qquad u'\in M',$$

where M' is a neighbourhood of the origin in X^1 chosen small enough so that $u^\star+M'\subset M$; note that we can replace M with M_r or $M_\star$. Substituting this *Ansatz* into (4.9), (4.10) and again dropping the primes for notational convenience, we find that

$$u_z=f(u^\star+u)\tag{4.12}$$

with boundary conditions

$$\Phi_y = B_{\mathrm{l}}(\eta, \Phi) + B_{\mathrm{nl}}(\eta, \rho, \Phi, \omega, \xi, \Psi) \quad \text{on } y = 0, 1, \tag{4.13}$$

where

$$\begin{aligned} B_{\mathrm{l}}(\eta, \Phi) &= \mathrm{d}B[\eta^\star, 0, \Phi^\star, 0, \xi^\star, 0](\eta, \rho, \Phi, \omega, \xi, \Psi) \\ &= y(-\eta_x + \eta^\star_x \Phi_x + \Phi^\star_x \eta_x) + \frac{\eta^\star \Phi_y}{1+\eta^\star} + \frac{\Phi^\star_y \eta}{(1+\eta^\star)^2} \\ &\qquad + \frac{y^2(\eta^\star_x)^2 \Phi^\star_y \eta}{(1+\eta^\star)^2} - \frac{y^2(\eta^\star_x)^2 \Phi_y}{1+\eta^\star} - \frac{2y^2 \eta^\star_x \Phi^\star_y \eta_x}{1+\eta^\star} \end{aligned}$$

and

$$B_{\mathrm{nl}}(\eta, \rho, \Phi, \omega, \xi, \Psi) = B(\eta^\star + \eta, \rho, \Phi^\star + \Phi, \omega, \xi^\star + \xi, \Psi) - B(\eta^\star, 0, \Phi^\star, 0, \xi^\star, 0) - B_{\mathrm{l}}(\eta, \Phi).$$

A small-amplitude periodic solution of (4.9), (4.10) corresponds to a periodically modulated solitary wave in the form of a three-dimensional perturbation of a line solitary wave.

4.1.2 Boundary conditions

Equations (4.9), (4.10) cannot be studied using standard methods for evolutionary equations because of the nonlinear boundary conditions. This difficulty is handled using a change of variable which leads to an equivalent problem in a linear space. For $(\eta, \rho, \Phi, \omega, \xi, \Psi)$ in M, M_{r} or $M_\star$ define

$$\mathcal{Q}(\eta, \rho, \Phi, \omega, \xi, \Psi) = (\eta, \rho, \Gamma, \omega, \xi, \Psi),$$

where

$$\Gamma = \Phi + \Theta_y$$

and $\Theta \in H^3(\Sigma)$ is the unique solution of the boundary-value problem

$$\begin{aligned} -\Delta\Theta + B_{\mathrm{l}}(0, \Theta_y) &= B_{\mathrm{nl}}(\eta, \rho, \Phi, \omega, \xi, \Psi) \qquad && \text{in } \Sigma, \\ \Theta &= 0 && \text{on } y = 0, 1 \end{aligned}$$

(the linear operator on the left-hand side of this boundary-value problem is uniformly strongly elliptic with smooth coefficients); note in particular that

$$-\Phi_y + B_{\mathrm{l}}(\eta, \Phi) + B_{\mathrm{nl}}(\eta, \rho, \Phi, \omega, \xi, \Psi) = -\Gamma_y + B_{\mathrm{l}}(\eta, \Phi) - \Theta_{xx}. \tag{4.14}$$

Lemma 4.2.

(i) The mapping $\mathcal{Q}$ is a near-identity, analytic diffeomorphism from a neighbourhood M of the origin in X^1 onto a neighbourhood $\tilde{M}$ of the origin in X^1.

(ii) For each $u \in M$ the operator $\mathrm{d}\mathcal{Q}[u]$ extends to an isomorphism $\widetilde{\mathrm{d}\mathcal{Q}}[u] : X^0 \to X^0$, and the operators $\widetilde{\mathrm{d}\mathcal{Q}}[u], \widetilde{\mathrm{d}\mathcal{Q}}[u]^{-1} \in \mathcal{L}(X^0, X^0)$ depend analytically on $u \in M$.

(iii) Statements (i) and (ii) also hold when $M, \tilde{M}$ are replaced by $M_{\mathrm{r}}, \tilde{M}_{\mathrm{r}}$ or $M_\star, \tilde{M}_\star$, where $M_{\mathrm{r}} = M \cap \operatorname{Fix} S = M_\star \cap X^1_{\mathrm{r}}$ and $\tilde{M}_{\mathrm{r}} = \tilde{M} \cap \operatorname{Fix} S = \tilde{M}_\star \cap X^1_{\mathrm{r}}$.

Proof. The mapping $\mathcal{Q} : M \to X^1$ is clearly near identity and analytic since Θ is a linear function of the analytic map $B_{\mathrm{nl}} : M \to X^1$. Replacing M with a smaller neighbourhood of the origin in X^1 if necessary, we obtain the result stated in the first part of the lemma using the inverse function theorem.

For the second assertion we observe that

$$\begin{aligned} B_{\mathrm{nl}}(\eta, \rho, \Phi, \omega, \xi, \Psi) = y\Phi_x\eta_x &- \frac{y^2(\eta^\star_x + \eta_x)^2(\Phi^\star_y + \Phi_y)}{1+\eta^\star+\eta} + \frac{\eta^\star+\eta}{1+\eta^\star+\eta}(\Phi^\star_y + \Phi_y) \\ &+ \frac{y\rho\Psi}{1+\eta^\star+\eta} + \frac{y^2(\eta^\star_x)^2(\Phi^\star_y+\Phi_y)}{(1+\eta^\star)} - \frac{\eta^\star}{1+\eta^\star}(\Phi^\star_y+\Phi_y) \\ &- \frac{\Phi^\star_y\eta}{(1+\eta^\star)^2} - \frac{y^2(\eta^\star_x)^2\Phi^\star_y\eta}{(1+\eta^\star)^2} + \frac{2y^2\eta^\star_x\Phi^\star_y}{1+\eta^\star}\eta_x \end{aligned}$$

and therefore

$$\begin{aligned} &\mathrm{d}B_{\mathrm{nl}}[\eta, \rho, \Phi, \omega, \xi, \Psi](\eta_1, \rho_1, \Phi_1, \omega_1, \xi_1, \Psi_1) = \\ &\quad y\Phi_x\eta_{1x} + y\eta_x\Phi_{1x} - \frac{y^2(\eta^\star_x+\eta_x)^2}{1+\eta^\star+\eta}\Phi_{1y} + \frac{(1+2\eta^\star+2\eta)}{(1+\eta^\star+\eta)^2}(\Phi^\star_y+\Phi_y)\eta_1 + \frac{\eta^\star+\eta}{1+\eta^\star+\eta}\Phi_{1y} \\ &\quad - \frac{2y^2(\eta^\star_x+\eta_x)(\Phi^\star_y+\Phi_y)(1+\eta^\star+\eta)\eta_{1x} + y^2(\eta^\star_x+\eta_x)^2(\Phi^\star_y+\Phi_y)\eta_1}{(1+\eta^\star+\eta)^2} \\ &\quad + \frac{y\Psi}{1+\eta^\star+\eta}\rho_1 + \frac{y\rho}{1+\eta^\star+\eta}\Psi_1 - \frac{y\rho\Psi}{(1+\eta^\star+\eta)^2}\eta_1 + \frac{y^2(\eta^\star_x)^2}{1+\eta^\star}\Phi_{1y} - \frac{\eta^\star}{1+\eta^\star}(\Phi_{1y}) \\ &\quad + \frac{\Phi^\star_y}{(1+\eta^\star)^2}\eta_1 - \frac{y^2(\eta^\star_x)^2\Phi^\star_y}{(1+\eta^\star)^2}\eta_1 + \frac{2y^2\eta^\star_x\Phi^\star_y}{1+\eta^\star}\eta_{1x}. \end{aligned} \tag{4.15}$$

Each of the terms of the right-hand side of (4.15) consists of several functions in $H^1(\Sigma)$ multiplied by a single function in $L^2(\Sigma)$. Using Proposition 4.1, one concludes that each such term respresents a bounded linear mapping from X_0 into $L^2(\Sigma)$ which depends smoothly upon $(\eta, \rho, \Phi, \omega, \xi, \Psi) \in \tilde{M}$. It follows that $\mathrm{d}B_{\mathrm{nl}}[\eta, \rho, \Phi, \omega, \xi, \Psi] : X^1 \to H^1$ extends to a bounded linear operator $\widetilde{\mathrm{d}B_{\mathrm{nl}}}[\eta, \rho, \Phi, \omega, \xi, \Psi] : X^0 \to L^2(\Sigma)$ that depends smoothly upon

$(\eta,\rho,\Phi,\omega,\xi,\Psi)\in\tilde{M}$. The corresponding result holds for $\mathrm{d}\mathcal{Q}$ since $\mathcal{Q}$ depends linearly upon B_{nl}; the extension $\widetilde{\mathrm{d}\mathcal{Q}}$ is unique since X^1 is a dense subspace of X^0. The inverse function theorem now yields the second assertion since $\widetilde{\mathrm{d}\mathcal{Q}}$ is near identity. ∎

The above change of variable transforms (4.12) into the equation

$$v_z = Lv + N(v) \tag{4.16}$$

with linear boundary conditions

$$\Gamma_y = B_{\mathrm{l}}(\eta,\Gamma) \quad \text{on } y=0,1 \tag{4.17}$$

for the variable $v=\mathcal{Q}(u)$, in which $L:=\mathrm{d}\tilde{f}[0]=\mathrm{d}f[u^\star]$ and $N:=\tilde{f}-L$ are the linear and nonlinear parts of the transformed vector field

$$\tilde{f}(v) := \widetilde{\mathrm{d}\mathcal{Q}}[\mathcal{Q}^{-1}(v)](f(u^\star+\mathcal{Q}^{-1}(v))).$$

Here

$$\mathcal{D}(L)=Y^1 := \Big\{(\eta,\rho,\Gamma,\omega,\xi,\Psi)\in X^1 : \Gamma_y = B_{\mathrm{l}}(\eta,\Gamma) \text{ on } y=0,1\Big\}$$

and $\mathcal{D}(N)$ is the neighbourhood $U:=\tilde{M}\cap Y^1$ of the origin in Y^1, and we may replace $\tilde{M},Y^1,U$ by $\tilde{M}_{\mathrm{r}},Y^1_{\mathrm{r}},U_{\mathrm{r}}$ or $\tilde{M}_\star,Y^1_\star,U_\star$ in the usual fashion. The linear operator is given by the explicit formula

$$L\begin{pmatrix}\eta\\ \rho\\ \Gamma\\ \omega\\ \xi\\ \Psi\end{pmatrix} = \begin{pmatrix}\rho\\ \frac{\xi}{\beta_0}-\eta_{xx}+h_1(\eta,\xi)\\ \Psi+H_1(\rho,\Psi)\\ \alpha_0\eta-\Gamma_x|_{y=1}+\xi_{xx}+h_2(\eta,\Gamma,\xi)\\ -\omega+h_3(\rho,\Psi)\\ -\Gamma_{xx}-\Gamma_{yy}+H_2(\eta,\Gamma)\end{pmatrix},$$

where

$$h_1(\eta,\xi)=\frac{Q^{\star\frac{5}{2}}}{\beta_0(1+\eta_x^{\star 2})^2}\xi+\frac{\eta_x^\star\xi^\star}{\beta_0(1+\eta_x^{\star 2})}\eta_x-\frac{1}{1+\eta_x^{\star 2}}\eta_{xx}+\frac{2\eta_{xx}^\star\eta_x^\star}{(1+\eta_x^{\star 2})^2}\eta_x-\frac{\xi}{\beta_0}+\eta_{xx},$$

$$
\begin{aligned}
h_2(\eta,\Gamma,\xi) = \int_0^1 \Bigg\{ & \Phi_x^\star\Gamma_x - \frac{y^2\eta_x^\star\Phi_y^{\star 2}}{(1+\eta^\star)^2}\eta_x + \frac{y^2\eta_x^{\star 2}\Phi_y^{\star 2}}{(1+\eta^\star)^3}\eta - \frac{y^2\eta_x^{\star 2}\Phi_y^\star}{(1+\eta^\star)^2}\Gamma_y \\
& - \frac{\Phi_y^\star}{(1+\eta^\star)^2}\Gamma_y + \frac{\Phi_y^{\star 2}}{(1+\eta^\star)^3}\eta + y\Big[\Phi_y^\star\Gamma_x + \Phi_x^\star\Gamma_y\Big]_x \\
& - \left[\frac{y^2\Phi_y^\star}{1+\eta^\star}\eta_x - \frac{y^2\eta_x^\star\Phi_y^2}{(1+\eta^\star)^2}\eta + \frac{2y^2\eta_x^\star\Phi_y^\star}{1+\eta^\star}\Gamma_y\right]_x \Bigg\}\,\mathrm{d}y \\
& + \left[\frac{\xi^\star Q^{\star\frac{3}{2}}\Big(5\eta_x^\star(1+\eta_x^{\star 2}) - 4Q^\star\eta_x^\star\Big)}{\beta_0(1+\eta_x^{\star 2})^3}\xi\right]_x + \left[\frac{\xi^{\star 2}}{2(\eta_x^{\star 2}+1)^{\frac{3}{2}}\beta_0}\eta_x\right]_x \\
& + \left[\frac{2\eta_x^\star\eta_{xx}^\star}{(1+\eta_x^{\star 2})^2}\xi\right]_x + \left[\frac{2\xi^\star\eta_x^\star}{(1+\eta_x^{\star 2})^2}\eta_{xx}\right]_x + \left[\frac{2\xi^\star(1-3\eta_x^{\star 2})\eta_{xx}^\star}{(1+\eta_x^{\star 2})^3}\eta_x\right]_x \\
& + \left[\frac{1}{1+\eta_x^{\star 2}}\xi\right]_{xx} + \left[\frac{2\eta_x^\star\xi^\star}{(1+\eta_x^{\star 2})^2}\eta_x\right]_{xx} - \xi_{xx}, \\
h_3(\rho,\Psi) = {} & -\int_0^1 \frac{y\Phi_y^\star}{1+\eta^\star}\Psi\,\mathrm{d}y - \frac{5Q^{\star\frac{3}{2}}\xi^{\star 2}}{2\beta_0(1+\eta_x^{\star 2})^2}\rho + \frac{\xi^\star}{1+\eta_x^{\star 2}}(2\eta_{xx}^\star\rho\text{-}2\eta_x^\star\rho_x) + \left[\frac{2\xi^\star\eta_x^\star}{1+\eta_x^{\star 2}}\rho\right]_x, \\
H_1(\rho,\Psi) = {} & \frac{1}{1+\eta^\star}\Psi + \frac{y\Phi_y^\star}{1+\eta^\star}\rho - \Psi, \\
H_2(\eta,\Gamma) = {} & \Big[F_1(\eta,\Gamma)\Big]_x + \Big[F_3(\eta,\Gamma)\Big]_y
\end{aligned}
$$

and

$$
\begin{aligned}
F_1(\eta,\Gamma) &= -\eta^\star\Gamma_x - \Phi_x^\star\eta + y\Phi_y^\star\eta_x + y\eta_x^\star\Gamma_y, \\
F_3(\eta,\Gamma) &= y\eta_x + B_1(\eta,\Gamma).
\end{aligned}
$$

Furthermore, the change of variable preserves the reversibility of the evolutionary equation.

4.1.3 Existence results

In this section we give an outline of our existence proof for periodically modulated solitary waves; the technical details of the reduction are given in Sections 4.2 and 4.3. We construct periodically modulated solitary waves by applying the Lyapunov-Iooss theorem (Theorem 1.7) to (4.16), (4.17), taking $\mathcal{X} = X_\mathrm{r}^0$, $\mathcal{Y} = Y_\star^1$, $\mathcal{Z} = Y_\mathrm{r}^1$ and $\mathcal{U} = U_\star$ with $\tau = z$ and $R(\eta,\rho,\Phi,\omega,\xi,\Psi) = (\eta,-\rho,-\Phi,-\omega,\xi,\Psi)$. The spectral hypotheses are verified by studying the resolvent equations

$$
(L - \mathrm{i}\lambda I)u = u^\dagger, \qquad \lambda \in \mathbb{R}, \tag{4.18}
$$

for L, where $u \in Y^1$ and $u^\dagger \in X_\mathrm{r}^0$, i.e.

$$\rho = \mathrm{i}\lambda\eta + \eta^\dagger, \tag{4.19}$$

$$\frac{\xi}{\beta_0} - \eta_{xx} + h_1(\eta,\xi) = \mathrm{i}\lambda\rho + \rho^\dagger, \tag{4.20}$$

$$\Psi + H_1(\rho,\Psi) = \mathrm{i}\lambda\Gamma + \Gamma^\dagger, \tag{4.21}$$

$$\alpha_0\eta - \Gamma_x|_{y=1} + \xi_{xx} + h_2(\eta,\Gamma,\xi) = \mathrm{i}\lambda\omega + \omega^\dagger, \tag{4.22}$$

$$-\omega + h_3(\rho,\Psi) = \mathrm{i}\lambda\xi + \xi^\dagger, \tag{4.23}$$

$$-\Gamma_{xx} - \Gamma_{yy} + H_2(\eta,\Gamma) = \mathrm{i}\lambda\Psi + \Psi^\dagger \tag{4.24}$$

with

$$\Gamma_y = 0 \qquad \text{on } y = 0, \tag{4.25}$$

$$\Gamma_y = -\eta_x + F_3(\eta,\Gamma) \qquad \text{on } y = 1. \tag{4.26}$$

Since L is real and anticommutes with the reverser R it suffices to examine non-negative values λ, real values of $\eta, \Gamma, \xi, \rho^\dagger, \omega^\dagger, \Psi^\dagger$ and purely imaginary values of $\rho, \omega, \Psi, \eta^\dagger, \Gamma^\dagger, \xi^\dagger$.

The arguments given in Section 1.4 suggest that the support of the Fourier transform $\hat{\eta} = \mathcal{F}[\eta]$ of η is concentrated near wavenumbers $\mu = \pm\mu_0$; we therefore decompose it into the sum of a function $\hat{\eta}_1$ with spectrum near $\mu = \pm\mu_0$ and a function $\hat{\eta}_2$ whose support is bounded away from these points. To this end choose $\delta \in (0, \mu_0/3)$, let χ_0 be the characteristic function of the interval $[-\delta, \delta]$ and define

$$\eta_1 = \chi(D)\eta := \mathcal{F}^{-1}[\chi\hat{\eta}], \qquad \eta_2 = (1-\chi(D))\eta := \mathcal{F}^{-1}[(1-\chi)\hat{\eta}],$$

where $\chi(\mu) = \chi_0(\mu - \mu_0) + \chi_0(\mu + \mu_0)$. In Section 4.2 we reduce (4.19)–(4.26) to a linear equation of Davey-Stewartson type by determining $\eta_2, \rho, \Gamma, \omega, \xi$ and Ψ as functions of η_1 and writing

$$\eta_1(x) = \frac{1}{2}\varepsilon\zeta(\varepsilon x)\mathrm{e}^{\mathrm{i}\mu_0 x} + \frac{1}{2}\varepsilon\overline{\zeta(\varepsilon x)}\mathrm{e}^{-\mathrm{i}\mu_0 x}$$

with $\zeta \in \chi_0(\varepsilon D)L^2(\mathbb{R})$, so that $\eta_1 \in \chi(D)L^2(\mathbb{R})$. The results of these calculations are summarised in the following theorem.

Theorem 4.3.

(i) There exists a constant $k_{\max} > 0$ *such that equations* (4.19)–(4.26) *have a unique solution* $u \in Y^1$ *for each* $\lambda > \varepsilon k_{\max}$ *and each* $u^\dagger \in X^0$.

(ii) Suppose that $\lambda = \varepsilon k$, *where* $0 < k \leq k_{\max}$. *There exist* $\zeta^\dagger_{\varepsilon,k} \in \mathcal{L}(X^0, \chi_0(\varepsilon D)L^2(\mathbb{R}))$ *and an injection* $\check{u}_{\varepsilon,k} \in \mathcal{L}(\chi_0(\varepsilon D)L^2(\mathbb{R}) \times X^0, Y^1)$ *such that* $u \in Y^1$ *solves the resolvent equations*

$$(L - \mathrm{i}\lambda I)u = u^\dagger$$

if and only if there exists a solution (ζ,ψ) *of the reduced equation*

$$\mathcal{B}_{\varepsilon,k}(\zeta,\psi)=(\zeta^{\dagger}_{\varepsilon,k}(u^{\dagger}),0)$$

with $u=\check{u}_{\varepsilon,k}(\zeta,u^{\dagger})$*. The operator* $\mathcal{B}_{\varepsilon,k}:\mathcal{D}_{\mathcal{B}}\subset W\to W$ *is given by*

$$\mathcal{B}_{\varepsilon,k}(\zeta,\psi)=\\ \begin{pmatrix} A_2^{-1}\Big(\zeta-A_1\zeta_{xx}-(A_3+A_5)(\zeta^{\star})^2\zeta-A_3(\zeta^{\star})^2\overline{\zeta}-A_4\zeta^{\star}\psi_x\Big)+k^2\zeta-\mathcal{R}_{\varepsilon,k}(\zeta)\\ -(1-\alpha_0^{-1})\psi_{xx}+k^2\psi-\frac{A_4}{2}\mathrm{Re}(\zeta^{\star}\zeta)_x\end{pmatrix},$$

where $W=L^2(\mathbb{R})\times L^2(\mathbb{R})$*,* $\mathcal{D}_{\mathcal{B}}=H^2(\mathbb{R})\times H^2(\mathbb{R})$ *and* $\mathcal{R}_{\varepsilon,k}\in\mathcal{L}(H^2(\mathbb{R}),L^2(\mathbb{R}))$ *satisfies the estimates*

$$\|\mathcal{R}_{\varepsilon,k}(\zeta)\|_0\lesssim\varepsilon\|\zeta\|_1;$$

each solution (ζ,ψ) *of the reduced equation satisfies* $\zeta\in\chi_0(\varepsilon D)H^2(\mathbb{R})$*. In particular,* $L-\mathrm{i}\varepsilon kL$ *is (semi-)Fredholm if* $\mathcal{B}_{\varepsilon,k}$ *is (semi-)Fredholm, and the kernels of* $L-\mathrm{i}\varepsilon kI$ *and* $\mathcal{B}_{\varepsilon,k}$ *have the same dimension. Furthermore the reduction preserves the invariance of the resolvent equations under the reflection* S*; its action on* W *is given by* $\tilde{S}:(\zeta(x),\psi(x))\mapsto(\overline{\zeta(-x)},-\psi(-x))$*.*

(iii) Define $Z^0_{\mathrm{r}}=H^2_{\mathrm{e}}(\mathbb{R})\times H^1_{\mathrm{e}}(\mathbb{R})\times H^1_{\mathrm{o}}(\Sigma)\times L^2_{\mathrm{e}}(\mathbb{R})\times H^1_{\mathrm{e}}(\mathbb{R})\times H^1_{\mathrm{e}}(\Sigma)\times\mathring{H}^1_{\mathrm{o}}(\Sigma)$*, where*

$$\mathring{H}^1_{\mathrm{o}}(\mathbb{R})=\Big\{u\in H^1_{\mathrm{o}}(\Sigma):u|_{y=0}=u|_{y=1}=0\Big\},$$

and

$$H^s_{\mathrm{c}}(\mathbb{R})=\Big\{u\in H^s(\mathbb{R}):u(-x)=\overline{u(x)}\text{ for all }x\in\mathbb{R}\Big\},\quad s\geq 0.$$

There exist $\zeta^{\dagger}_{\varepsilon,0}\in\mathcal{L}\Big(Z^0_{\mathrm{r}},\chi_0(\varepsilon D)L^2_{\mathrm{c}}(\mathbb{R})\Big)$ *and an injection* $\check{u}_{\varepsilon,0}\in\mathcal{L}\Big(\chi_0(\varepsilon D)L^2_{\mathrm{c}}(\mathbb{R})\times Z^0_{\mathrm{r}},Y^1_{\star}\Big)$ *such that* $u\in Y^1_{\star}$ *solves*

$$Lu=u^{\dagger},$$

where $\Psi^{\dagger}=-(\Psi^{\dagger}_1)_x-(\Psi^{\dagger}_3)_y$ *if and only if there exists a solution* ζ *of the reduced equation*

$$\mathcal{C}_{\varepsilon}\zeta=\zeta^{\dagger}_{\varepsilon,0}(\eta^{\dagger},\rho^{\dagger},\Phi^{\dagger},\omega^{\dagger},\xi^{\dagger},\Psi^{\dagger}_1,\Psi^{\dagger}_3)$$

with $u=\check{u}_{\varepsilon,0}(\zeta,\eta^{\dagger},\rho^{\dagger},\Phi^{\dagger},\omega^{\dagger},\xi^{\dagger},\Psi^{\dagger}_1,\Psi^{\dagger}_3)$*. The operator* $\mathcal{C}_{\varepsilon}:H^2_{\mathrm{c}}(\mathbb{R})\subset L^2_{\mathrm{c}}(\mathbb{R})\to L^2_{\mathrm{c}}(\mathbb{R})$ *is given by*

$$\mathcal{C}_{\varepsilon}\zeta=A_2^{-1}\zeta-A_2^{-1}A_1\zeta_{xx}-\frac{1}{2}A_2^{-1}A_5(\zeta^{\star})^2\zeta-\frac{1}{4}A_2^{-1}A_5(\zeta^{\star})^2\overline{\zeta}-\mathcal{R}_{\varepsilon,0}(\zeta)$$

and $\mathcal{R}_{\varepsilon,0}\in\mathcal{L}\Big(H^2_{\mathrm{c}}(\mathbb{R}),L^2_{\mathrm{c}}(\mathbb{R})\Big)$ *satisfies the estimate*

$$\|\mathcal{R}_{\varepsilon,0}(\zeta)\|_0\lesssim\varepsilon\|\zeta\|_1;$$

each solution of the reduced equation lies in $\chi_0(\varepsilon D)H^2_{\mathrm{c}}(\mathbb{R})$*.*

Part (i) of Theorem 4.3 implies that $\mathrm{i}\lambda \in \rho(L)$ for $\lambda > \varepsilon k_{\max}$, while parts (ii) and (iii) show in particular that $\mathrm{i}\varepsilon k \in \rho(L)$ if $\mathcal{B}_{\varepsilon,k}$ (for $k > 0$) or $\mathcal{C}_\varepsilon$ (for $k = 0$) is invertible. We study the invertibility of these operators in Section 4.3, establishing the following results, whose corollaries relate them to the operator L.

Lemma 4.4. *For each sufficiently small $k_{\min} > 0$ there exists a unique number $k_\varepsilon \in [k_{\min}, k_{\max}]$ with the following properties.*

(i) The operator $\mathcal{B}_{\varepsilon,k} : \mathcal{D}_{\mathcal{B}} \subseteq W \to W$ is an isomorphism for each $k \in [k_{\min}, k_{\max}] \setminus \{k_\varepsilon\}$.

(ii) The operator $\mathcal{B}_{\varepsilon,k_\varepsilon} : \mathcal{D}_{\mathcal{B}} \subseteq W \to W$ is Fredholm with index 0 and has a one-dimensional kernel which lies in $\operatorname{Fix} \tilde{S}$.

Corollary 4.5. *The imaginary number $\mathrm{i}\varepsilon k$ belongs to $\rho(L)$ for each $k \in [k_{\min}, \infty) \setminus \{k_\varepsilon\}$, while $\mathrm{i}\varepsilon k_\varepsilon$ is a simple eigenvalue of L whose eigenspace lies in $\operatorname{Fix} S$.*

Proof. It remains only to show that the eigenvalue $\mathrm{i}\varepsilon k_\varepsilon$ is algebraically simple. Observe that $\Omega(Lu_1, u_2) = -\Omega(u_1, Lu_2)$ and in particular that $\Omega((L - \mathrm{i}\varepsilon k_\varepsilon)u_1, u_2) = -\Omega(u_1, (L + \mathrm{i}\varepsilon k_\varepsilon)u_2)$ for $u_1, u_2 \in \mathcal{D}(L)$. It follows that $\Omega(f, \bar{u}_\varepsilon) = 0$ is a necessary condition for $f \in X_{\mathrm{r}}^0$ to lie in the range of $L - \mathrm{i}\varepsilon k_\varepsilon I$, where u_ε is an eigenvector corresponding to the eigenvalue k_ε. Using the formulae

$$\begin{aligned}
\xi &= \beta_0 \frac{(1+\eta_x^{\star 2})^2}{Q^{\star\frac{5}{2}}} \left(-\frac{\eta_x^\star \xi^\star}{\beta_0(1+\eta_x^{\star 2})}\eta_{\varepsilon x} + \frac{1}{1+\eta_x^{\star 2}}\eta_{\varepsilon xxx} - \frac{2\eta_x^\star \eta_{xx}^\star}{(1+\eta_x^{\star 2})^2}\eta_\varepsilon - \varepsilon^2 k_\varepsilon^2 \eta_\varepsilon \right),\\
\Psi &= (1+\eta^\star)\mathrm{i}\varepsilon k_\varepsilon \Gamma_\varepsilon - y\eta_{\varepsilon x}\Phi_y^\star,\\
\omega &= -\int_0^1 \left\{ y\Phi_y^\star \left(\mathrm{i}\varepsilon k_\varepsilon \Gamma_\varepsilon - \frac{y\Phi_y^\star}{1+\eta^\star}\mathrm{i}\varepsilon k_\varepsilon \eta_\varepsilon \right) \right\} \mathrm{d}y - \frac{2\eta_x^\star \xi^\star}{1+\eta_x^{\star 2}}\mathrm{i}\varepsilon k_\varepsilon \eta_{\varepsilon x}\\
&\qquad + \left(\frac{2\xi^\star \eta_{xx}^\star}{1+\eta_x^{\star 2}} - \frac{5Q^{\star\frac{3}{2}}\xi^{\star 2}}{2\beta_0(1+\eta_x^{\star 2})^2} \right)\mathrm{i}\varepsilon k_\varepsilon \eta_\varepsilon + \left[\frac{2\xi^\star \eta_x^\star}{1+\eta_x^{\star 2}}\mathrm{i}\varepsilon k_\varepsilon \eta_\varepsilon \right]_x\\
&\qquad - \mathrm{i}\varepsilon k_\varepsilon \frac{\beta_0(1+\eta_x^{\star 2})^2}{Q^{\star\frac{5}{2}}} \left(\frac{1}{1+\eta_x^{\star 2}}\eta_{\varepsilon xxx} - \frac{\eta_x^\star \xi^\star}{\beta_0(1+\eta_x^{\star 2})}\eta_{\varepsilon x} - \frac{2\eta_x^\star \eta_{xx}^\star}{(1+\eta_x^{\star 2})^2}\eta_{\varepsilon x} - \varepsilon^2 k_\varepsilon^2 \eta_\varepsilon \right)
\end{aligned}$$

(see equations (4.35), (4.36) and (4.37)), we find that

$$\begin{aligned}
\Omega(u_\varepsilon, \bar{u}_\varepsilon) &= -2\mathrm{i}\varepsilon k_\varepsilon \beta_0 \int_{\mathbb{R}} |\eta_{\varepsilon x}|^2 \,\mathrm{d}x - 2\mathrm{i}\varepsilon k_\varepsilon \int_\Sigma |\Gamma_\varepsilon|^2 \,\mathrm{d}y\,\mathrm{d}y\\
&\qquad + O\Big(\varepsilon^2(\|\eta_\varepsilon\|_2^2 + \|\Gamma_\varepsilon\|_0^2)\Big)\\
&\neq 0,
\end{aligned}$$

so that u_ε does not lie in the range of $L - \mathrm{i}\varepsilon k_\varepsilon I$. ■

Lemma 4.6.

(i) The operator $\mathcal{C}_\varepsilon : H^2_{\mathrm{c}}(\mathbb{R}) \subseteq L^2_{\mathrm{c}}(\mathbb{R}) \to L_{\mathrm{c}}(\mathbb{R})$ *is an isomorphism.*

(ii) There exists $k_{\min} > 0$ *such that* $\mathcal{B}_{\varepsilon,k}\big|_{\mathrm{Fix}\,\tilde{S}}$ *is semi-Fredholm and injective for* $k \in (0, k_{\min})$.

Corollary 4.7. *The operator* $L - \mathrm{i}\varepsilon k I : Y^1_{\mathrm{r}} \to X^0_{\mathrm{r}}$ *is an isomorphism for each* $k \in (0, k_{\min})$.

Proof. Because $k \mapsto \mathcal{B}_{\varepsilon,k}$ is a continuous mapping $(0, k_{\max}) \mapsto \mathcal{L}(\mathcal{D}_{\mathcal{B}} \cap \mathrm{Fix}\,\tilde{S}, \mathrm{Fix}\,\tilde{S})$ (see Section 4.3.3) and the Fredholm index of $\mathcal{B}_{\varepsilon,k}\big|_{\mathrm{Fix}\,\tilde{S}}$ is 0 for $k \in [k_{\min}, k_{\max}]$, the same is true for $k \in (0, k_{\min})$. Using Lemma 4.6, we find that $\mathcal{B}_{\varepsilon,k}\big|_{\mathrm{Fix}\,\tilde{S}}$ is invertible for $k \in (0, k_{\min})$, and the stated result follows from Theorem 4.3 (ii). ■

The resolvent decay estimate is a consequence of Lemma 4.9 below, which relies upon the following proposition (see Appendix 4.4.3 for its proof).

Proposition 4.8. *The solution* $(\eta, \rho, \Phi, \omega, \xi, \Psi) \in X^1$ *of the constant-coefficient spectral problem*

$$\rho - \mathrm{i}\lambda\eta = h^\dagger_0, \tag{4.27}$$

$$\frac{\xi}{\beta_0} - \eta_{xx} - \mathrm{i}\lambda\rho = h^\dagger_1, \tag{4.28}$$

$$\Psi - \mathrm{i}\lambda\Gamma = H^\dagger_1, \tag{4.29}$$

$$\alpha_0\eta - \Gamma_x|_{y=1} + \xi_{xx} - \mathrm{i}\lambda\omega = h^\dagger_2, \tag{4.30}$$

$$-\omega - \mathrm{i}\lambda\xi = h^\dagger_3, \tag{4.31}$$

$$-\Gamma_{xx} - \Gamma_{yy} - \mathrm{i}\Psi = H^\dagger_2, \tag{4.32}$$

with

$$\Gamma_y = 0 \qquad \textit{on } y = 0, \tag{4.33}$$

$$\Gamma_y = -\eta_x + F^\dagger_3 \qquad \textit{on } y = 1, \tag{4.34}$$

where $\lambda > 0$, $(h^\dagger_0, h^\dagger_1, H^\dagger_1, h^\dagger_2, h^\dagger_3, H^\dagger_2) \in X^0_{\mathrm{r}}$ *and* $F^\dagger_3 \in H^1(\Sigma)$ *satisfies the estimate*

$$\begin{aligned}
&\|\eta\|_3 + \|\rho\|_2 + \|\Gamma\|_2 + \|\omega\|_1 + \|\xi\|_2 + \|\Psi\|_1 \\
&\quad \lesssim \|h^\dagger_0\|_2 + \|h^\dagger_1\|_1 + \|H^\dagger_1\|_1 + \|h^\dagger_2\|_0 + \|h^\dagger_3\|_1 + \|H^\dagger_2\|_0 + \|F^\dagger_3\big|_{y=1}\|_{1/2} + \lambda\|F^\dagger_3\big|_{y=1}\|_0, \\
&\|\eta\|_2 + \|\rho\|_1 + \|\Gamma\|_1 + \|\omega\|_0 + \|\xi\|_1 + \|\Psi\|_0 \\
&\quad \lesssim |\lambda|^{-1}\Big(\|h^\dagger_0\|_2 + \|h^\dagger_1\|_1 + \|H^\dagger_1\|_1 + \|h^\dagger_2\|_0 + \|h^\dagger_3\|_1 + \|H^\dagger_2\|_0 + \|F^\dagger_3\big|_{y=1}\|_{1/2} + \lambda\|F^\dagger_3\big|_{y=1}\|_0\Big)
\end{aligned}$$

for all sufficiently large values of λ.

Lemma 4.9. *There exists a constant $\lambda^\star > 0$ such that the solution $u \in Y^1$ of equations* (4.19)–(4.26) *satisfies the estimate*

$$\|u\|_{X^1}^2 + \lambda^2 \|u\|_{X^0}^2 \lesssim \|u^\dagger\|_{X^0}^2$$

for each $\lambda > \lambda^\star$ and each $u^\dagger \in X_{\mathrm{r}}^0$.

Proof. Consider the constant-coefficient spectral problem (4.27)–(4.34) and observe that the resolvent equations (4.18) can be written in this form with

$$\begin{aligned} h_0^\dagger &= \eta^\dagger, & h_1^\dagger &= \rho^\dagger - h_1(\eta,\xi), \\ H_1^\dagger &= \Phi^\dagger - H_1(\rho,\Psi), & h_2^\dagger &= \omega^\dagger - h_2(\eta,\Gamma,\xi), \\ h_3^\dagger &= \xi^\dagger - h_3(\rho,\Psi), & H_2^\dagger &= \Psi^\dagger - H_2(\eta,\Gamma) \end{aligned}$$

and $F_3^\dagger = F_3(\eta,\Gamma)$. These quantities satisfy the estimates

$$\begin{aligned} \|h_0^\dagger\|_1 &\lesssim \|\eta^\dagger\|_1, \\ \|h_1^\dagger\|_1 &\lesssim \|\rho^\dagger\|_1 + \varepsilon^{5/2}\|\xi\|_1 + \varepsilon^2\|\eta_x\|_1 + \varepsilon^2\|\eta_{xx}\|_1, \\ \|H_1^\dagger\|_1 &\lesssim \|\Gamma^\dagger\|_1 + \varepsilon\|\Psi\|_1 + \varepsilon\|\rho\|_1, \\ \|h_2^\dagger\|_0 &\lesssim \varepsilon\|\Gamma_x\|_0 + \varepsilon\|\Gamma_y\|_0 + \varepsilon\|\Gamma_{xy}\|_0 + \varepsilon^2\|\eta\|_0 + \varepsilon\|\eta_x\|_0 + \varepsilon^2\|\xi_x\|_0 + + \varepsilon^2\|\xi_{xx}\|_0, \\ \|h_3^\dagger\|_1 &\lesssim \varepsilon\|\Psi\|_1 + \varepsilon^2\|\rho\|_1 + \varepsilon^2\|\rho_x\|_1, \\ \|H_2^\dagger\|_0 &\lesssim \varepsilon\|\Gamma_x\|_0 + \varepsilon\|\Gamma_y\|_0 + \varepsilon\|\Gamma_{xx}\|_0 + \varepsilon\|\Gamma_{yy}\|_0 + + \varepsilon\|\Gamma_{xy}\|_0 + \varepsilon\|\eta\|_0 + \varepsilon\|\eta_x\|_0, \end{aligned}$$

and combining them with Proposition 4.8, one finds after some elementary manipulation that

$$\|u\|_{X^1}^2 + \lambda^2 \|u\|_{X^0}^2 \lesssim \|u^\dagger\|_{X^0}^2$$

for sufficiently small ε and sufficiently large λ. ■

It remains to establish the Iooss condition at the origin. For this purpose we first record the following consequence of Theorem 4.3 (iii) and Lemma 4.6 (i).

Proposition 4.10. *For each $(\eta^\dagger, \rho^\dagger, \Gamma^\dagger, \omega^\dagger, \xi^\dagger, \Psi_1^\dagger, \Psi_3^\dagger) \in Z_{\mathrm{r}}^0$, where $\Psi^\dagger = -(\Psi_1^\dagger)_x - (\Psi_3^\dagger)_y$, the equation*

$$Lu = u^\dagger$$

has a unique solution $u \in Y_\star^1$ which satisfies the estimate

$$\|u\|_{Y_\star^1} \lesssim \|\eta^\dagger\|_2 + \|\rho^\dagger\|_1 + \|\Gamma^\dagger\|_1 + \|\omega^\dagger\|_0 + \|\xi^\dagger\|_1 + \|\Psi_1^\dagger\|_1 + \|\Psi_3^\dagger\|_1.$$

Lemma 4.11. *For each $v^\dagger \in U_\star$ the equation*

$$Lv = -N(v^\dagger)$$

has a unique solution $v \in Y^1_\star$ which satisfies the estimate

$$\|v\|_{Y^1_\star} \lesssim \|v^\dagger\|_{Y^1_\star}.$$

Proof. Choose $v^\dagger \in U_\star$ and write $u^\dagger = \mathcal{Q}^{-1}(v^\dagger)$. Recall that

$$g(v) = \widehat{\mathrm{d}\mathcal{Q}}[u](f(u^\dagger + u)), \qquad u = \mathcal{Q}^{-1}(v)$$

and that $\mathcal{Q}, \widehat{\mathrm{d}\mathcal{Q}}[u](f(u^\star + u))$ do not alter the sixth component of their arguments. The sixth component of $g(u^\dagger)$ is therefore given by

$$\begin{aligned}
g_6(v^\dagger) &= f_6(u^\star + u^\dagger) \\
&= -\Big[(1+\eta^\star+\eta^\dagger)(\Phi^\star_x + \Gamma^\dagger_x) - y(\eta^\star_x + \eta^\dagger_x)(\Phi^\star_y + \Gamma^\dagger_y)\Big]_x \\
&\qquad + \Big[-\Phi^\star_y - \Gamma^\dagger_y + B(\eta^\star + \eta^\dagger, \rho^\dagger, \Phi^\star + \Gamma^\dagger, \omega^\dagger, \xi^\star + \xi^\dagger, \Psi^\dagger) + y(\eta^\star_x + \eta^\dagger_x)\Big]_y \\
&= -\Big[(1+\eta^\star+\eta^\dagger)\Gamma^\dagger_x + \Phi^\star_x\eta^\dagger - y(\eta^\star_x + \eta^\dagger_x)\Gamma^\dagger_y - y\Phi^\star_y\eta^\dagger_x\Big]_x \\
&\qquad + \Big[-\Gamma^\dagger_y + B_\mathrm{l}(\eta^\dagger, \Gamma^\dagger) + B_\mathrm{nl}(\eta^\dagger, \rho^\dagger, \Gamma^\dagger, \omega^\dagger, \xi^\dagger, \Psi^\dagger) + y\eta^\dagger_x\Big]_y \\
&= -\Big[(1+\eta^\star+\eta^\dagger)\Gamma^\dagger_x + \Phi^\star_x\eta^\dagger - y(\eta^\star_x + \eta^\dagger_x)\Gamma^\dagger_y - y\Phi^\star_y\eta^\dagger_x\Big]_x \\
&\qquad + \Big[-\Gamma^\dagger_y - \Theta^\dagger_{xx} + B_\mathrm{l}(\eta^\dagger, \Gamma^\dagger) + y\eta^\dagger_x\Big] \\
&= -\Big[\Gamma^\dagger_x + (\eta^\star+\eta^\dagger)\Gamma^\dagger_x + \Phi^\star_x\eta^\dagger - y(\eta^\star_x + \eta^\dagger_x)\Gamma^\dagger_y - y\Phi^\star_y\eta^\dagger_x\Big]_x \\
&\qquad + \Big[-\Gamma^\dagger_y + B_\mathrm{l}(\eta^\dagger, \Gamma^\dagger) + y\eta^\dagger_x\Big]_y
\end{aligned}$$

in which the first line follows from (4.11), the second from the fact that $f_6(u^\star) = 0$ and the definitions of $B_\mathrm{l}, B_\mathrm{nl}$, the third from (4.14) and the fourth from the identity $\Gamma^\dagger_{xx} = \Phi^\dagger_{xx} + \Theta^\dagger_{xxy}$. According to the definition $N(v) = g(v) - Lv$ the sixth component of $N(v^\dagger)$ is

$$\begin{aligned}
N_6(v^\dagger) &= f_6(u^\star + u^\dagger) - L_6 v^\dagger \\
&= -\Big[F_1(\eta^\dagger, \Gamma^\dagger) + (\eta^\star + \eta^\star)\Gamma^\dagger_x + \Phi^\star_x\eta^\dagger - y(\eta^\star_x + \eta^\dagger_x)\Gamma^\dagger_y - y\Phi^\star_y\eta^\dagger_x\Big]_x \\
&\qquad + \Big[-F_3(\eta^\dagger, \Gamma^\dagger) + B_\mathrm{l}(\eta^\dagger, \Gamma^\dagger) + y\eta^\dagger_x\Big]_y \\
&= -\Big[F_1(\eta^\dagger, \Gamma^\dagger) + (\eta^\star + \eta^\dagger)\Gamma^\dagger_x + \Phi^\star_x\eta^\dagger - y(\eta^\star_x + \eta^\dagger_x)\Gamma^\dagger_y - y\Phi^\star_y\eta^\dagger_x\Big]_x,
\end{aligned}$$

Applying Proposition 4.10 with

$$\Psi^\dagger_1 = -\Big(F_1(\eta^\dagger, \Gamma^\dagger) + (\eta^\star + \eta^\dagger)\Gamma^\dagger_x + \Phi^\star_x\eta^\dagger - y(\eta^\star_x + \eta^\dagger_x)\Gamma^\dagger_y - y\Phi^\star_y\eta^\dagger_x\Big), \qquad \Psi^\dagger_3 = 0,$$

one finds that the equation

$$Lv = -N(v^\dagger)$$

has a unique solution $v \in Y_\star^1$ which satisfies

$$\|v\|_{Y_\star^1} \lesssim \|N_1(v^\dagger)\|_2 + \|N_2(v^\dagger)\|_1 + \|N_3(v^\dagger)\|_1 + \|N_4(v^\dagger)\|_0 + \|N_5(v^\dagger)\|_1 + \|\Psi_1^\dagger\|_1$$

The assertion follows by combining this estimate with

$$\|\Psi_1^\dagger\|_1 \lesssim \|\eta^\dagger\|_2 + \|\nabla\Gamma^\dagger\|_1 + \|\Gamma^\dagger\|_1 \lesssim \|v^\dagger\|_{Y_\star^1}$$

and

$$\|N(v^\dagger)\|_{X^0} \lesssim \|v^\dagger\|_{Y_\star^1}.$$

■

4.2 Derivation of the reduced equation

4.2.1 Reduction to a single pseudodifferential equation

We begin by eliminating $\rho, \Gamma, \omega, \xi$ and Ψ from (4.19)–(4.26) for $\lambda \geq 0$, writing $\eta = \eta_1 + \eta_2$, where

$$\eta_1 = \chi(D)\eta, \qquad \eta_2 = (1 - \chi(D))\eta$$

and deriving a single, equivalent equation for η_1 (equation (4.47)).

Equations (4.20) and (4.21) yield the explicit formulae

$$\xi = \beta_0 \frac{(1+\eta_x^{\star 2})^2}{Q^{\star\frac{5}{2}}}\left(-\frac{\eta_x^\star \xi^\star}{\beta_0(1+\eta_x^{\star 2})}\eta_x + \frac{1}{1+\eta_x^{\star 2}}\eta_{xx} - \frac{2\eta_x^\star\eta_{xx}^\star}{(1+\eta_x^{\star 2})^2}\eta_x - \lambda^2\eta + \mathrm{i}\lambda\eta^\dagger + \rho^\dagger\right), \quad (4.35)$$

$$\Psi = (1+\eta^\star)(\mathrm{i}\lambda\Gamma + \Gamma^\dagger) - y\rho\Phi_y^\star \quad (4.36)$$

for ξ and Ψ. From these formulae and (4.23) we obtain

$$\begin{aligned}\omega = & -\int_0^1 \left\{ y\Phi_y^\star\left(\mathrm{i}\lambda\Gamma + \Gamma^\dagger - \frac{y\Phi_y^\star}{1+\eta^\star}(\mathrm{i}\lambda\eta + \eta^\dagger)\right)\right\}\mathrm{d}y - \frac{2\eta_x^\star\xi^\star}{1+\eta_x^{\star 2}}(\mathrm{i}\lambda\eta_x + \eta_x^\dagger) \\ & + \left(\frac{2\xi^\star\eta_{xx}^\star}{1+\eta_x^{\star 2}} - \frac{5Q^{\star\frac{3}{2}}\xi^{\star 2}}{2\beta_0(1+\eta_x^{\star 2})^2}\right)(\mathrm{i}\lambda\eta + \eta^\dagger) + \left[\frac{2\xi^\star\eta_x^\star}{1+\eta_x^{\star 2}}(\mathrm{i}\lambda\eta + \eta^\dagger)\right]_x - \xi^\dagger \\ & - \mathrm{i}\lambda\frac{\beta_0(1+\eta_x^{\star 2})^2}{Q^{\star\frac{5}{2}}}\left(\frac{1}{1+\eta_x^{\star 2}}\eta_{xx} - \frac{\eta_x^\star\xi^\star}{\beta_0(1+\eta_x^{\star 2})}\eta_x - \frac{2\eta_x^\star\eta_{xx}^\star}{(1+\eta_x^{\star 2})^2}\eta_x - \lambda^2\eta + \mathrm{i}\lambda\eta^\dagger + \rho^\dagger\right).\end{aligned} \quad (4.37)$$

Substituting these formulae into (4.24)–(4.26) and taking the Fourier transform with respect to x, we obtain the boundary-value problem

$$-\hat{\Gamma}_{yy} + q^2\hat{\Gamma} = \hat{F}^\dagger(u^\dagger) - \mathrm{i}\mu\hat{F}_1(\eta,\Gamma) - \mathrm{i}\lambda\hat{F}_2(\eta,\Gamma) - \Big[\hat{F}_3(\eta,\Gamma)\Big]_y, \qquad 0<y<1, \tag{4.38}$$

$$\hat{\Gamma}_y = 0, \qquad y=0, \tag{4.39}$$

$$\hat{\Gamma}_y = -\mathrm{i}\mu\hat{\eta} + \hat{F}_3(\eta,\Gamma), \qquad y=1 \tag{4.40}$$

for the Fourier transform $\hat{\Gamma}(\mu,y)$ of $\Gamma(x,y)$, where $q=\sqrt{\mu^2+\lambda^2}$,

$$\begin{aligned} F_2(\eta,\Gamma) &= -\mathrm{i}\lambda\eta^\star\Gamma + \mathrm{i}\lambda y\Phi^\star_y\eta, \\ F^\dagger(u^\dagger) &= \Psi^\dagger + \mathrm{i}\lambda(1+\eta^\star)\Gamma^\dagger - \mathrm{i}\lambda y\Phi^\star_y\eta^\dagger \end{aligned}$$

and $\Psi^\dagger = -\Psi^\dagger_{1x} - \Psi^\dagger_{3y}$ for $\lambda=0$. On the other hand, substituting (4.35), (4.36) and (4.37) into (4.22) yields

$$g_\varepsilon(D,\lambda)\eta = \mathcal{N}(\eta,\Gamma,u^\dagger), \tag{4.41}$$

where

$$g_\varepsilon(\mu,\lambda) = \alpha_0 + \varepsilon^2 + \beta_0 q^4 - \frac{\mu^2}{q^2}q\coth(q)$$

and

$$\begin{aligned}
\mathcal{N}(\eta,\Gamma,u^\dagger) = \omega^\dagger - \mathrm{i}\lambda\int_0^1 &\left\{y\Phi^\star_y\Big(\mathrm{i}\lambda\Gamma + \Gamma^\dagger - \frac{y\Phi^\star_y}{1+\eta^\star}(\mathrm{i}\lambda\eta+\eta^\dagger)\Big)\right\}\mathrm{d}y - \frac{2\mathrm{i}\lambda\eta^\star_x\xi^\star}{1+\eta^{\star 2}_x}(\mathrm{i}\lambda\eta_x+\eta^\dagger_x) \\
&+ \mathrm{i}\lambda\left(\frac{2\xi^\star\eta^\star_{xx}}{1+\eta^{\star 2}_x} - \frac{5Q^{\star\frac{3}{2}}\xi^{\star 2}}{2\beta_0(1+\eta^{\star 2}_x)^2}\right)(\mathrm{i}\lambda\eta+\eta^\dagger) + \mathrm{i}\lambda\left[\frac{2\xi^\star\eta^\star_x}{1+\eta^{\star 2}_x}(\mathrm{i}\lambda\eta+\eta^\dagger)\right]_x \\
&+ \lambda^2\frac{\beta_0(1+\eta^{\star 2}_x)^2}{Q^{\star\frac{5}{2}}}\left(\frac{1}{1+\eta^{\star 2}_x}\eta_{xx} - \frac{\eta^\star_x\xi^\star}{\beta_0(1+\eta^{\star 2}_x)}\eta_x - \frac{2\eta^\star_x\eta^\star_{xx}}{(1+\eta^{\star 2}_x)^2}\eta_x \right. \\
&\qquad\qquad \left. - \lambda^2\eta + \mathrm{i}\lambda\eta^\dagger + \rho^\dagger\right) \\
&- \mathrm{i}\lambda\xi^\dagger + \Gamma_x|_{y=1} - 2\beta_0\lambda^2\eta_{xx} + \beta_0\lambda^4\eta + \beta_0\eta_{xxxx} - \mathcal{F}^{-1}\left[\frac{\mu^2}{q^2}q\coth(q)\hat{\eta}\right] \\
&- \int_0^1\left\{\Phi^\star_x\Gamma_x - \frac{y^2\eta^\star_x\Phi^{\star 2}_y}{(1+\eta^\star)^2}\eta_x + \frac{y^2\eta^{\star 2}_x\Phi^{\star 2}_y}{(1+\eta^\star)^3}\eta - \frac{y^2\eta^{\star 2}_x\Phi^\star_y}{(1+\eta^\star)^2}\Gamma_y \right. \\
&\qquad - \frac{\Phi^\star_y}{(1+\eta^\star)^2}\Gamma_y + \frac{\Phi^{\star 2}_y}{(1+\eta^\star)^3}\eta + y\Big[\Phi^\star_y\Gamma_x + \Phi^\star_x\Gamma_y\Big]_x \\
&\qquad \left. - \left[\frac{y^2\Phi^\star_y}{1+\eta^\star}\eta_x - \frac{y^2\eta^\star_x\Phi^2_y}{(1+\eta^\star)^2}\eta + \frac{2y^2\eta^\star_x\Phi^\star_y}{1+\eta^\star}\Gamma_y\right]_x\right\}\mathrm{d}y
\end{aligned}$$

$$
\begin{aligned}
&-\left[\frac{\xi^\star Q^{\star\frac{3}{2}}\left(5\eta_x^\star(1+\eta_x^{\star 2})-4Q^\star\eta_x^\star\right)}{\beta_0(1+\eta_x^{\star 2})^3}\xi\right]_x-\left[\frac{\xi^{\star 2}}{2(\eta_x^{\star 2}+1)^{\frac{3}{2}}\beta_0}\eta_x\right]_x\\
&-\left[\frac{2\eta_x^\star\eta_{xx}^\star}{(1+\eta_x^{\star 2})^2}\xi\right]_x-\left[\frac{2\xi^\star\eta_x^\star}{(1+\eta_x^{\star 2})^2}\eta_{xx}\right]_x-\left[\frac{2\xi^\star(1-3\eta_x^{\star 2})\eta_{xx}^\star}{(1+\eta_x^{\star 2})^3}\eta_x\right]_x\\
&-\left[\frac{1}{1+\eta_x^{\star 2}}\xi\right]_{xx}-\left[\frac{2\eta_x^\star\xi^\star}{(1+\eta_x^{\star 2})^2}\eta_x\right]_{xx}+\xi_{xx}.
\end{aligned}
$$

The next step is to express Γ as a function of η. To this end we formulate the boundary-value problem (4.38)–(4.40) as integral equation

$$
\Gamma=\mathcal{G}_1\Big(F_1(\eta,\Gamma),F_2(\eta,\Gamma),F_3(\eta,\Gamma),\eta\Big)+\mathcal{G}_2(F^\dagger(u^\dagger)), \tag{4.42}
$$

where

$$
\begin{aligned}
\mathcal{G}_1(P_1,P_2,P_3,p)&=\mathcal{F}^{-1}\left[\int_0^1\left\{G(y,\tilde{y})(-\mathrm{i}\mu\hat{P}_1-\mathrm{i}\lambda\hat{P}_2)+G_{\tilde{y}}(y,\tilde{y})\hat{P}_3\right\}\mathrm{d}y-\mathrm{i}\mu G(y,1)\hat{p}\right],\\
\mathcal{G}_2(P)&=\mathcal{F}^{-1}\left[\int_0^1 G(y,\tilde{y})\hat{P}\,\mathrm{d}\tilde{y}\right]
\end{aligned}
$$

and

$$
G(y,\tilde{y})=\begin{cases}\dfrac{\cosh(qy)\cosh(q(1-\tilde{y}))}{q\sinh(q)}, & 0\le y\le\tilde{y}\le 1,\\[2mm] \dfrac{\cosh(q\tilde{y})\cosh(q(1-y))}{q\sinh(q)}, & 0\le\tilde{y}\le y\le 1.\end{cases}
$$

The following proposition records the mapping properties of the integral operators appearing on the right hand side of (4.42) (see Groves, Sun and Wahlén [27, Proposition 3.1]).

Proposition 4.12.

(i) The mapping $(P_1,P_2,P_3,p)\mapsto\mathcal{G}_1(P_1,P_2,P_3,p)$ defines a linear function

$$
\begin{cases}H^1(\Sigma)\times H^1(\Sigma)\times H^1(\Sigma)\times H^2(\mathbb{R})\to H^2(\Sigma), & \lambda>0,\\ H^1_{\mathrm{e}}(\Sigma)\times H^1_{\mathrm{o}}(\Sigma)\times H^1_{\mathrm{o}}(\Sigma)\times H^2_{\mathrm{e}}(\mathbb{R})\to H^2_{\star,\mathrm{o}}(\Sigma), & \lambda=0,\end{cases}
$$

which satisfies the estimate

$$
\begin{aligned}
\|\nabla\mathcal{G}_1\|_1+\lambda\|\mathcal{G}_1\|_1+\lambda^2\|\mathcal{G}_1\|_0\lesssim{}&\|P_1\|_1+\|P_2\|_1+\|P_3\|_1+\lambda(\|P_1\|_0+\|P_2\|_0+\|P_3\|_0)\\
&+(1+\lambda^2)\|p\|_0+\|p_{xx}\|_0.
\end{aligned}
$$

(ii) *Suppose that $\lambda > 0$. The mapping $P \mapsto \mathcal{G}_2(P)$ defines a linear function $L^2(\Sigma) \to H^2(\Sigma)$ which satisfies the estimate*

$$\|\nabla \mathcal{G}_2\|_1 + \lambda\|\mathcal{G}_2\|_1 + \lambda^2\|\mathcal{G}_2\|_0 \lesssim (1+\lambda^{-1})\|P\|_0.$$

(iii) *Suppose that $\lambda = 0$ and $P = -P_{1x} - P_{3y}$, where $P_1 \in H^1_{\mathrm{e}}(\Sigma)$ and $P_3 \in \mathring{H}^1_{\mathrm{o}}(\Sigma)$. The mapping $(P_1, P_3) \mapsto \mathcal{G}_2(P)$ defines a linear function $H^1_{\mathrm{e}}(\Sigma) \times H^1_{\mathrm{o}}(\Sigma) \to H^2_{\star,\mathrm{o}}(\Sigma)$ which satisfies the estimate*

$$\|\nabla \mathcal{G}_2\|_1 \lesssim \Big\||P_1\|_1 + \|P_3\|_1.$$

Theorem 4.13.

(i) *Suppose that $\lambda > 0$. For each sufficiently small value of ε and each $\eta \in H^2(\Sigma)$, $F^\dagger \in L^2(\Sigma)$, equation (4.42) has a unique solution $\tilde{\Gamma} = \tilde{\Gamma}(\eta, u^\dagger) \in H^2(\Sigma)$ which satisfies the estimate*

$$\|\nabla\tilde{\Gamma}\|_1 + \lambda\|\tilde{\Gamma}\|_1 + \lambda^2\|\tilde{\Gamma}\|_0 \lesssim (1+\lambda^2)\|\eta\|_0 + \|\eta_{xx}\|_0 + (1+\lambda^{-1})\|F^\dagger\|_0.$$

(ii) *Suppose that $\lambda = 0$. For each sufficiently small value of ε and each $\eta \in H^2_{\mathrm{e}}(\mathbb{R})$, $\Psi_1^\dagger \in H^1_{\mathrm{e}}(\Sigma)$, $\Psi_3^\dagger \in \mathring{H}^1_{\mathrm{o}}(\Sigma)$, equation (4.42) has a unique solution $\tilde{\Gamma} = \tilde{\Gamma}(\eta, \Psi_1^\dagger, \Psi_3^\dagger) \in H^2_{\star,0}(\Sigma)$ which satisfies the estimate*

$$\|\nabla\tilde{\Gamma}\|_1 \lesssim \|\eta\|_2 + \|\Psi_1^\dagger\|_1 + \|\Psi_3^\dagger\|_1.$$

Proof. It follows from the estimates

$$\begin{aligned}
\|F_1\|_1, \|F_3\|_1 &\lesssim \varepsilon\Big(\|\eta\|_1 + \|\eta_x\|_1 + \|\nabla\Gamma\|_1\Big)\\
&\lesssim \varepsilon\Big(\|\eta\|_0 + \|\eta_{xx}\|_0 + \|\nabla\Gamma\|_1\Big),\\
\lambda\|F_1\|_0, \lambda\|F_3\|_0 &\lesssim \varepsilon\Big(\lambda\|\eta\|_0 + \lambda\|\eta_x\|_0 + \lambda\|\nabla\Gamma\|_0\Big)\\
&\lesssim \varepsilon\Big((1+\lambda^2)\|\eta\|_0 + \|\eta_{xx}\|_0 + \lambda\|\nabla\Gamma\|_0\Big),\\
\|F_2\|_1 &\lesssim \varepsilon\Big(\lambda\|\eta\|_1 + \lambda\|\Gamma\|_1\Big)\\
&\lesssim \varepsilon\Big((1+\lambda^2)\|\eta\|_0 + \|\eta_{xx}\|_0 + \lambda\|\Gamma\|_1\Big),\\
\lambda\|F_2\|_0 &\lesssim \varepsilon\Big(\lambda^2\|\eta\|_0 + \lambda^2\|\Gamma\|_0\Big)
\end{aligned}$$

and Proposition 4.12(i) that

$$\begin{aligned}
&\|\nabla\mathcal{G}_1(\eta,\Gamma)\|_1 + \lambda\|\mathcal{G}_1(\eta,\Gamma)\|_1 + \lambda^2\|\mathcal{G}_1(\eta,\Gamma)\|_0\\
&\qquad \lesssim \varepsilon(\|\nabla\Gamma\|_1 + \lambda\|\Gamma\|_1 + \lambda^2\|\Gamma\|_0) + (1+\lambda^2)\|\eta\|_0 + \|\eta_{xx}\|_0,
\end{aligned}$$

in which $\mathcal{G}_1(\eta,\Gamma)$ is used as an abbreviation for $\mathcal{G}_1(F_1(\eta,\Gamma),F_2(\eta,\Gamma),F_3(\eta,\Gamma),\eta)$. Using this estimate together with Proposition 4.12(ii) or (iii) as appropriate and equipping $H^2(\Sigma)$ with the norm $\Gamma \mapsto \|\nabla\Gamma\|_1 + \lambda\|\Gamma\|_1 + \lambda^2\|\Gamma\|_0$, one finds that (4.42) admits a unique solution which satisfies the stated estimate. ■

Substituting $\Gamma = \tilde{\Gamma}(\eta, u^\dagger)$ into (4.41), we obtain a single equation for η, namely

$$g_\varepsilon(D,\lambda)\eta = \tilde{\mathcal{N}}(\eta, u^\dagger), \tag{4.43}$$

where

$$\tilde{\mathcal{N}}(\eta, u^\dagger) = \mathcal{N}\Big(\eta, \tilde{\Gamma}(\eta, u^\dagger), u^\dagger\Big). \tag{4.44}$$

Finally, we record an estimate for $g_\varepsilon(\mu,\lambda)$ which is obtained by elementary calculus and use it to solve (4.43) for large values of λ, thus proving Theorem 4.3(i).

Proposition 4.14. *For each fixed $\lambda^\star > 0$ the quantity $g_\varepsilon(\mu,\lambda)$, which is even in both arguments, satisfies the estimates*

$$\lambda^4 + (\mu-\mu_0)^2 + \varepsilon^2 \lesssim g_\varepsilon(\mu,\lambda) \lesssim \lambda^4 + (\mu-\mu_0)^2 + \varepsilon^2, \quad (\mu,\lambda) \in S$$

and

$$1 + q^4 + \varepsilon^2 \lesssim g_\varepsilon(\mu,\lambda) \lesssim 1 + q^4 + \varepsilon^2, \quad (\mu,\lambda) \in [0,\infty)^2 \setminus S,$$

where $S = [\mu_0-\delta, \mu_0+\delta] \times [0,\lambda^\star]$.

Lemma 4.15. *Choose $\lambda^\star > 0$. For each sufficiently small value of ε, each $\lambda > \lambda^\star$ and each $u^\dagger \in X^0$, equation* (4.43) *has a unique solution $\eta = \eta(u^\dagger) \in H^4(\mathbb{R})$ which satisfies the estimate*

$$(1+\lambda^4)\|\eta\|_0 + \|\eta_{xxxx}\|_0 \le c_\lambda \|u^\dagger\|_{X^0}.$$

Proof. Write (4.43) as

$$g_\varepsilon(D,\lambda)\eta - \tilde{\mathcal{N}}(\eta,0) = \tilde{\mathcal{N}}(0,u^\dagger)$$

and recall the estimates $g_\varepsilon(\mu,\lambda) \gtrsim 1 + (\mu^2+\lambda^2)^2$ (see Proposition 4.14) and

$$\|\tilde{\mathcal{N}}(\eta,0)\|_0 \lesssim \varepsilon\Big((1+\lambda^4)\|\eta\|_0 + \|\eta_{xxxx}\|_0\Big).$$

Equipping $H^4(\mathbb{R})$ with the norm $\eta \mapsto (1+\lambda^4)\|\eta\|_0 + \|\eta_{xxxx}\|_0$, one finds that the operator

$$g_\varepsilon(D,\lambda) - \tilde{\mathcal{N}}(\cdot,0) : H^4_\mathrm{e}(\mathbb{R}) \to L^2_\mathrm{e}(\mathbb{R})$$

is invertible for each sufficiently small value of ε, and the result follows from this fact and the estimate

$$\|\tilde{\mathcal{N}}(0,u^\dagger)\|_0 \le c_\lambda \|u^\dagger\|_{X^0},$$

which follows from (4.44). ■

In view of the previous lemma we now fix $\lambda^\star > 0$ and suppose that $\lambda \leq \lambda^\star$. We write (4.43) as

$$g_\varepsilon(\mu,\lambda)\hat{\eta}_1 = \chi\mathcal{F}\Big[\tilde{\mathcal{N}}(\eta_1+\eta_2,u^\dagger)\Big] \tag{4.45}$$

and

$$g_\varepsilon(\mu,\lambda)\hat{\eta}_2 = (1-\chi)\mathcal{F}\Big[\tilde{\mathcal{N}}(\eta_1+\eta_2,u^\dagger)\Big], \tag{4.46}$$

where $\hat{\eta}_1 = \chi\hat{\eta}$ and $\hat{\eta}_2 = (1-\chi)\hat{\eta}$.

Lemma 4.16. *The linear operator*

$$g_\varepsilon(D,\lambda) - (1-\chi(D))\tilde{\mathcal{N}}(\cdot,0) : (1-\chi(D))H^4(\mathbb{R}) \to (1-\chi(D))L^2(\mathbb{R})$$

is invertible for each sufficiently small value of ε.

Proof. This result follows from the estimates $g_\varepsilon(\mu,\lambda) \gtrsim 1+q^4$ for $|\mu-\mu_0| > \delta$ (see Proposition 4.14) and

$$\|\tilde{\mathcal{N}}(\eta,0)\|_0 \lesssim \varepsilon\|\eta\|_4,$$

which again follows from (4.44). ∎

Corollary 4.17.

(i) Suppose that $\lambda > 0$. For each sufficiently small value of ε and each $\eta_1 \in \chi(D)L^2(\mathbb{R})$, $u^\dagger \in X^0$, equation (4.46) has a unique solution $\eta_2 = \check{\eta}_2(\eta_1,u^\dagger) \in H^4(\mathbb{R})$ which satisfies the estimate

$$\|\check{\eta}_2\|_4 \lesssim \varepsilon\|\eta_1\|_0 + c_\lambda\|u^\dagger\|_{X^0}.$$

(ii) Suppose that $\lambda = 0$. For each sufficiently small value of ε and each $\eta_1 \in \chi(D)L^2_\mathrm{e}(\mathbb{R})$, $u^\dagger \in X^0_\mathrm{r}$, equation (4.46) has a unique solution $\eta_2 = \check{\eta}_2(\eta_1,u^\dagger) \in H^4_\mathrm{e}(\mathbb{R})$ which satisfies the estimate

$$\|\check{\eta}_2\|_4 \lesssim \varepsilon\|\eta_1\|_0 + \|\omega^\dagger\|_0 + \|\Psi_1^\dagger\|_1 + \|\Psi_3^\dagger\|_1.$$

Proof. Write (4.46) as

$$g_\varepsilon(D,\lambda)(\eta_2) - (1-\chi(D))\tilde{\mathcal{N}}(\eta_2,0) = (1-\chi(D))\Big(\tilde{\mathcal{N}}(\eta_1,0) + \tilde{\mathcal{N}}(0,u^\dagger)\Big)$$

and use Lemma 4.16 and the estimates

$$\begin{aligned}
&\|\tilde{\mathcal{N}}(\eta_1,0)\|_0 \lesssim \varepsilon\|\eta_1\|_0, && \\
&\|\tilde{\mathcal{N}}(0,u^\dagger)\|_0 \leq c_\lambda\|u^\dagger\|_{X^0}, && \lambda > 0, \\
&\|\tilde{\mathcal{N}}(0,u^\dagger)\|_0 \lesssim \|\omega^\dagger\|_0 + \|\Psi_1^\dagger\|_1 + \|\Psi_3^\dagger\|_1, && \lambda = 0.
\end{aligned}$$

∎

Substituting $\check{\eta}_2 = \check{\eta}_2(\eta_1, u^\dagger)$ into (4.45), we obtain a single equation for η_1, namely

$$g_\varepsilon(\mu, \lambda)\hat{\eta}_1 = \chi \mathcal{F}\Big[\check{\mathcal{N}}(\eta_1, u^\dagger)\Big], \tag{4.47}$$

where

$$\begin{aligned}\check{\mathcal{N}}(\eta_1, u^\dagger) &= \tilde{\mathcal{N}}\Big(\eta_1 + \check{\eta}_2(\eta_1, u^\dagger), u^\dagger\Big)\\ &= \mathcal{N}\Big(\eta_1 + \check{\eta}_2(\eta_1, u^\dagger), \check{\Gamma}(\eta_1, u^\dagger), u^\dagger\Big)\end{aligned}$$

and

$$\check{\Gamma}(\eta_1, u^\dagger) = \tilde{\Gamma}\Big(\eta_1 + \check{\eta}_2(\eta_1, u^\dagger), u^\dagger\Big).$$

4.2.2 Intermediate spectral values

The next step is to derive more precise estimates for the functions introduced in the previous section. We fix $\lambda^\star > 0$, suppose that $\lambda \le \lambda^\star$ and expand $\eta^\star$, $\xi^\star$ and $\Phi^\star$ as

$$\begin{aligned}\eta^\star(x) &= \eta_1^\star(x) + \eta_2^\star(x) + \eta_{\mathrm{r}}^\star(x),\\ \xi^\star(x) &= \xi_1^\star(x) + \xi_2^\star(x) + \xi_{\mathrm{r}}^\star(x),\\ \Phi^\star(x, y) &= \Phi_1^\star(x, y) + \Phi_2^\star(x, y) + \Phi_{\mathrm{r}}^\star(x, y),\end{aligned}$$

and $\tilde{\Gamma} = \tilde{\Gamma}(\eta, u^\dagger)$ as

$$\tilde{\Gamma}(\eta, u^\dagger) = \tilde{\Gamma}^{(1)}(\eta) + \tilde{\Gamma}^{(2)}(\eta) + \tilde{\Gamma}^{(3)}(\eta) + \tilde{\Gamma}^{\mathrm{r}}(\eta) + \tilde{\Gamma}^{\dagger}(u^\dagger).$$

Here $\eta_1^\star, \eta_2^\star, \eta_{\mathrm{r}}^\star$ and $\Phi_1^\star, \Phi_2^\star, \Phi_{\mathrm{r}}^\star$ are given in Appendix 4.4.1 and $\xi_1^\star = \beta_0 \eta_{1xx}^\star$, $\xi_2^\star = \beta_0 \eta_{2xx}^\star$ due to

$$\xi^\star = \frac{\beta_0}{(1 + \eta_x^{\star 2})^{3/2}} \eta_{xx}^\star,$$

which follows from the fact, that $(\eta^\star, 0, \Phi^\star, 0, \xi^\star, 0)$ solves (4.1)–(4.6). Furthermore

$$\begin{aligned}\tilde{\Gamma}^{(1)}(\eta) &= \mathcal{G}_1(0, 0, 0, \eta),\\ \tilde{\Gamma}^{(2)}(\eta) &= \mathcal{G}_1(F_1^{(2)}(\eta), F_2^{(2)}(\eta), F_3^{(2)}(\eta), 0),\\ \tilde{\Gamma}^{(3)}(\eta) &= \mathcal{G}_1(F_1^{(3)}(\eta), F_2^{(3)}(\eta), F_3^{(3)}(\eta), 0)\end{aligned}$$

with

$$\begin{aligned}F_1^{(2)}(\eta) &= -\eta_1^\star \tilde{\Gamma}_x^{(1)}(\eta) - \Phi_{1x}^\star \eta + y \Phi_{1y}^\star \eta_x + y \eta_{1x}^\star \tilde{\Gamma}_y^{(1)}(\eta),\\ F_2^{(2)}(\eta) &= -\mathrm{i}\lambda \eta_1^\star \tilde{\Gamma}^{(1)}(\eta) + \mathrm{i}\lambda y \Phi_{1y}^\star \eta,\end{aligned}$$

$$
\begin{aligned}
F_3^{(2)}(\eta) &= y\eta_{1x}^\star \tilde{\Gamma}_x^{(1)}(\eta) + y\Phi_{1x}^\star \eta_x + \eta_1^\star \tilde{\Gamma}_y^{(1)}(\eta) + \Phi_{1y}^\star \eta, \\
F_1^{(3)}(\eta) &= -\eta_1^\star \tilde{\Gamma}_x^{(2)}(\eta) - \eta_2^\star \tilde{\Gamma}_x^{(1)}(\eta) - \Phi_{2x}^\star \eta + y\Phi_{2y}^\star \eta_x + y\eta_{1x}^\star \tilde{\Gamma}_y^{(2)} + y\eta_{2x}^\star \tilde{\Gamma}_y^{(1)}, \\
F_2^{(3)}(\eta) &= -\mathrm{i}\lambda\eta_1^\star \tilde{\Gamma}^{(2)}(\eta) - \mathrm{i}\lambda\eta_2^\star \tilde{\Gamma}^{(1)}(\eta) + \mathrm{i}\lambda y\Phi_{2y}^\star \eta, \\
F_3^{(3)}(\eta) &= y\eta_{1x}^\star \tilde{\Gamma}_x^{(2)} + y\eta_{2x}^\star \tilde{\Gamma}_x^{(1)} + y\Phi_{2x}^\star \eta_x + \eta_1^\star \tilde{\Gamma}_y^{(2)}(\eta) + \eta_2^\star \tilde{\Gamma}_y^{(1)}(\eta) + \Phi_{2y}^\star \eta \\
&\qquad - (\eta_1^\star)^2 \tilde{\Gamma}_y^{(1)}(\eta) - 2\eta_1^\star \Phi_{1y}^\star \eta - y^2(\eta_{1x}^\star)^2 \tilde{\Gamma}_y^{(1)}(\eta) - 2y^2\eta_{1x}^\star \Phi_{1y}^\star \eta_x
\end{aligned}
$$

and $\tilde{\Gamma}^{\mathrm{r}}(\eta), \tilde{\Gamma}^\dagger(u^\dagger)$ are determined implicitly by

$$\tilde{\Gamma}^{\mathrm{r}}(\eta) = \mathcal{G}_1\Big(F_1^{\mathrm{r}}(\eta), F_2^{\mathrm{r}}(\eta), F_3^{\mathrm{r}}(\eta), 0\Big)$$

with

$$F_j^{\mathrm{r}}(\eta) = F_j\Big(\eta, \tilde{\Gamma}^{(1)}(\eta) + \tilde{\Gamma}^{(2)}(\eta) + \tilde{\Gamma}^{(3)}(\eta) + \tilde{\Gamma}^{\mathrm{r}}(\eta)\Big) - F_j^{(2)}(\eta) - F_j^{(3)}(\eta), \quad j = 1, 2, 3,$$

and

$$\tilde{\Gamma}^\dagger = \mathcal{G}_1\Big(F_1(0, \tilde{\Gamma}^\dagger), F_2(0, \tilde{\Gamma}^\dagger), F_3(0, \tilde{\Gamma}^\dagger), 0\Big) + \mathcal{G}_2(F^\dagger(u^\dagger)).$$

The following lemma is proved using the estimates in Theorem 4.13.

Lemma 4.18. *The functions $\tilde{\Gamma}^{(1)}, \tilde{\Gamma}^{(2)}, \tilde{\Gamma}^{(3)}, \tilde{\Gamma}^{\mathrm{r}}$ and $\tilde{\Gamma}^\dagger$ satisfy the estimates*

$$
\begin{aligned}
&||\nabla\tilde{\Gamma}^{(j)}(\eta)||_1 + \lambda||\tilde{\Gamma}^{(j)}(\eta)||_1 + \lambda^2||\tilde{\Gamma}^{(j)}(\eta)||_0 \lesssim \varepsilon^{j-1}||\eta||_2, \quad j = 1, 2, 3, \\
&||\nabla\tilde{\Gamma}^{\mathrm{r}}(\eta)||_1 + \lambda||\tilde{\Gamma}^{\mathrm{r}}(\eta)||_1 + \lambda^2||\tilde{\Gamma}^{\mathrm{r}}(\eta)||_0 \lesssim \varepsilon^3||\eta||_2, \\
&||\nabla\tilde{\Gamma}^\dagger(\eta)||_1 + \lambda||\tilde{\Gamma}^\dagger(\eta)||_1 + \lambda^2||\tilde{\Gamma}^\dagger(\eta)||_0 \le c_\lambda||u^\dagger||_{X^0}, \qquad \lambda > 0, \\
&||\nabla\tilde{\Gamma}^\dagger(\eta)||_1 + \lambda||\tilde{\Gamma}^\dagger(\eta)||_1 + \lambda^2||\tilde{\Gamma}^\dagger(\eta)||_0 \lesssim ||\omega^\dagger||_0 + ||\Psi_1^\dagger||_1 + ||\Psi_3^\dagger||_1, \qquad \lambda = 0.
\end{aligned}
$$

We correspondigly write

$$\tilde{\mathcal{N}}(\eta, u^\dagger) = \tilde{\mathcal{N}}^{(2)}(\eta) + \tilde{\mathcal{N}}^{(3)}(\eta) + \tilde{\mathcal{N}}^{(\mathrm{r})}(\eta) + \tilde{\mathcal{N}}^\dagger(u^\dagger),$$

where

$$
\begin{aligned}
\tilde{\mathcal{N}}^{(2)}(\eta) = \tilde{\Gamma}_x^{(2)}(\eta) &- \mathrm{i}\lambda\int_0^1 \Phi_{1y}^\star \mathrm{i}\lambda\tilde{\Gamma}^{(1)}(\eta)\,\mathrm{d}y \\
&- \int_0^1 \Big\{\Phi_{1x}^\star \tilde{\Gamma}_x^{(1)}(\eta) - \Phi_{1y}^\star \tilde{\Gamma}_y^{(1)}(\eta) + \Big[y\Phi_{1y}^\star \tilde{\Gamma}_x^{(1)}(\eta) + y\Phi_{1y}^\star \tilde{\Gamma}_y^{(1)}(\eta)\Big]_x\Big\}\,\mathrm{d}y, \qquad (4.48)
\end{aligned}
$$

$$\begin{aligned}
\tilde{\mathcal{N}}^{(3)}(\eta) = \tilde{\Gamma}_x^{(3)}(\eta)\Big|_{y=1} &+ \frac{5\lambda^2}{2\beta_0}\xi_1^{\star 2}\eta - 2\lambda^2\xi_1^\star\eta_{1xx}^\star\eta + 2\lambda^2\eta_{1x}^\star\xi_1^\star\eta_x - 2\lambda^2\Big[\xi_1^\star\eta_{1x}^\star\eta\Big]_x \\
&- \lambda^2\eta_{1x}^\star\xi_1^\star\eta_x - \frac{3\lambda^2\beta_0}{2}\eta_{1x}^{\star 2}\eta_{xx} - 2\lambda^2\beta_0\eta_{1x}^\star\eta_{1xx}^\star\eta_x + \frac{\lambda^4\beta_0}{2}\eta_{1x}^{\star 2}\eta \\
&- \Big[\xi_1^\star\eta_{1x}^\star\eta_{xx} - \lambda^2\xi_1^\star\eta_{1x}^\star\eta + 2\beta_0\eta_{1x}^\star\eta_{1xx}^\star\eta_{xx} - 2\beta_0\lambda^2\eta_{1x}^\star\eta_{1xx}^\star\eta - \frac{1}{2\beta_0}\xi_1^{\star 2}\eta_x\Big]_x \\
&- \Big[2\xi_1^\star\eta_{1x}^\star\eta_{xx} + 2\xi_1^\star\eta_{1xx}^\star\eta_x\Big]_x + \Big[\lambda^2\beta_0\eta_{1x}^{\star 2}\eta - \frac{\lambda^2\beta_0}{2}\eta_{1x}^{\star 2}\eta\Big]_{xx} \\
&+ \Big[\eta_{1x}^\star\xi_1^\star\eta_x + \frac{3}{2}\beta_0\eta_{1x}^{\star 2}\eta_{xx} + 2\beta_0\eta_{1x}^\star\eta_{1xx}^\star\eta_x + 2\eta_{1x}^\star\xi_1^\star\eta_x - \beta_0\eta_{1x}^{\star 2}\eta_{xx}\Big]_{xx} \\
&- \mathrm{i}\lambda\int_0^1\Big\{\mathrm{i}\lambda y\Phi_{1y}^\star\tilde{\Gamma}^{(2)}(\eta) + \mathrm{i}\lambda y\Phi_{2y}^\star\tilde{\Gamma}^{(1)}(\eta) - \mathrm{i}\lambda y^2\Phi_{1y}^{\star 2}\eta\Big\}\,\mathrm{d}y \\
&- \int_0^1\Big\{\Phi_{1x}^\star\tilde{\Gamma}_x^{(2)}(\eta) + \Phi_{2x}^\star\tilde{\Gamma}_x^{(1)}(\eta) - \Phi_{1y}^\star\tilde{\Gamma}_y^{(2)}(\eta) - \Phi_{2y}^\star\tilde{\Gamma}_y^{(1)}(\eta)\Big\}\,\mathrm{d}y \\
&+ \int_0^1\Big\{y\Big[\Phi_{1y}^\star\tilde{\Gamma}_x^{(2)}(\eta) + \Phi_{2y}^\star\tilde{\Gamma}_x^{(1)}(\eta) + \Phi_{1x}^\star\tilde{\Gamma}_y^{(2)}(\eta) + \Phi_{2x}^\star\tilde{\Gamma}_y^{(1)}(\eta)\Big]_x\Big\}\,\mathrm{d}y \\
&+ \int_0^1\Big\{2\eta_1^\star\Phi_{1y}^\star\tilde{\Gamma}_y^{(1)}(\eta) + \Phi_{1y}^{\star 2}\eta - y^2\Big[\Phi_{1y}^{\star 2}\eta_x + 2\eta_{1x}^\star\Phi_{1y}^\star\tilde{\Gamma}_y^{(1)}\Big]_x\Big\}\,\mathrm{d}y, \qquad (4.49) \\
\tilde{\mathcal{N}}(\eta) &= \tilde{\mathcal{N}}(\eta,0) - \tilde{\mathcal{N}}^{(2)}(\eta) - \tilde{\mathcal{N}}^{(3)}(\eta), \\
\tilde{\mathcal{N}}^\dagger(u^\dagger) &= \tilde{\mathcal{N}}(0,u^\dagger).
\end{aligned}$$

Lemma 4.19. *The functions $\tilde{\mathcal{N}}^{(2)}, \tilde{\mathcal{N}}^{(3)}, \tilde{\mathcal{N}}^{\mathrm{r}}$ and $\tilde{\mathcal{N}}^\dagger$ satisfy the estimates*

$$\begin{aligned}
\|\tilde{\mathcal{N}}^{(j)}(\eta)\|_0 &\lesssim \varepsilon^{j-1}\|\eta\|_4, \quad j=2,3, \\
\|\tilde{\mathcal{N}}^{\mathrm{r}}(\eta)\|_0 &\lesssim \varepsilon^3\|\eta\|_4, \\
\|\tilde{\mathcal{N}}^\dagger(u^\dagger)\|_0 &\leq c_\lambda\|u^\dagger\|_{X^0}, && \lambda>0, \\
\|\tilde{\mathcal{N}}^\dagger(u^\dagger)\|_0 &\lesssim \|\omega^\dagger\|_0 + \|\Psi_1^\dagger\|_1 + \|\Psi_3^\dagger\|_1, && \lambda=0.
\end{aligned}$$

Similarly we expand $\breve{\eta}_2 = \breve{\eta}_2(\eta_1, u^\dagger)$ as

$$\breve{\eta}_2(\eta_1,u^\dagger) = \breve{\eta}_2^{(2)}(\eta_1) + \breve{\eta}_2^{\mathrm{r}}(\eta_1) + \breve{\eta}_2^\dagger(u^\dagger),$$

where

$$\mathcal{F}\Big[\breve{\eta}_2^{(2)}\Big] = g_\varepsilon(\mu,\lambda)^{-1}(1-\chi)\mathcal{F}\Big[\tilde{\mathcal{N}}^{(2)}(\eta_1)\Big]$$

and $\breve{\eta}_2^{\mathrm{r}}, \breve{\eta}_2^\dagger$ are determined implicitly by

$$\mathcal{F}\Big[\breve{\eta}_2^{\mathrm{r}}\Big] = g_\varepsilon(\mu,\lambda)^{-1}(1-\chi)\mathcal{F}\Big[\tilde{\mathcal{N}}^{(3)}(\eta_1) + \tilde{\mathcal{N}}^{\mathrm{r}}(\eta_1) + \tilde{\mathcal{N}}(\breve{\eta}_2^{(2)} + \breve{\eta}_2^{\mathrm{r}}, 0)\Big],$$

$$\mathcal{F}\Big[\check{\eta}_2^\dagger\Big] = g_\varepsilon(\mu,\lambda)^{-1}(1-\chi)\mathcal{F}\Big[\tilde{\mathcal{N}}(\check{\eta}_2^\dagger, u^\dagger)\Big],$$

and $\check{\Gamma} = \check{\Gamma}(\eta_1, u^\dagger)$ as

$$\check{\Gamma}(\eta_1, u^\dagger) = \check{\Gamma}^{(1)}(\eta_1) + \check{\Gamma}^{(2)}(\eta_1) + \check{\Gamma}^{(3)}(\eta_1) + \check{\Gamma}^{\mathrm{r}}(\eta_1) + \check{\Gamma}^\dagger(u^\dagger),$$

where

$$\begin{aligned}
\check{\Gamma}^{(1)} &= \tilde{\Gamma}^{(1)}(\eta_1),\\
\check{\Gamma}^{(2)} &= \tilde{\Gamma}^{(2)}(\eta_1) + \tilde{\Gamma}^{(1)}(\check{\eta}_2^{(2)}(\eta_1)),\\
\check{\Gamma}^{(3)} &= \tilde{\Gamma}^{(3)}(\eta_1) + \tilde{\Gamma}^{(2)}(\check{\eta}_2^{(2)}(\eta_1)) + \tilde{\Gamma}^{(1)}(\check{\eta}_2^{\mathrm{r}}(\eta_1)),\\
\check{\Gamma}^{\mathrm{r}} &= \tilde{\Gamma}^{\mathrm{r}}\Big(\eta_1 + \check{\eta}_2^{(2)}(\eta_1) + \check{\eta}_2^{\mathrm{r}}(\eta_1)\Big) + \tilde{\Gamma}^{(3)}\Big(\check{\eta}_2^{(2)}(\eta_1) + \check{\eta}_2^{\mathrm{r}}(\eta_1)\Big) + \tilde{\Gamma}^{(2)}(\check{\eta}_2^{\mathrm{r}}(\eta_1)),\\
\check{\Gamma}^\dagger &= \tilde{\Gamma}(\check{\eta}_2^\dagger(u^\dagger), u^\dagger).
\end{aligned}$$

Proposition 4.20.

(i) The functions $\check{\eta}_2^{(2)}, \check{\eta}_2^{\mathrm{r}}$ and $\check{\eta}_2^\dagger$ satisfy the estimates

$$\begin{aligned}
&\|\check{\eta}_2^{(2)}\|_2 \lesssim \varepsilon\|\eta_1\|_0, \quad \|\check{\eta}_2^{\mathrm{r}}\|_2 \lesssim \varepsilon^2\|\eta_1\|_0,\\
&\|\check{\eta}_2^\dagger\|_2 \leq c_\lambda\|u^\dagger\|_{X^0}, && \lambda > 0,\\
&\|\check{\eta}_2^\dagger\|_2 \lesssim \|\omega^\dagger\|_0 + \|\Psi_1^\dagger\|_1 + \|\Psi_3^\dagger\|_1, && \lambda = 0.
\end{aligned}$$

(ii) The functions $\check{\Gamma}^{(1)}, \check{\Gamma}^{(2)}, \check{\Gamma}^{(3)}, \check{\Gamma}^{\mathrm{r}}$ and $\check{\Gamma}^\dagger$ satisfy the estimates

$$\begin{aligned}
&\|\nabla\check{\Gamma}^{(j)}\|_1 + \lambda\|\check{\Gamma}^{(j)}\|_0 \lesssim \varepsilon^{j-1}\|\eta_1\|_0, \quad j = 1,2,3,\\
&\|\nabla\check{\Gamma}^{\mathrm{r}}\|_1 + \lambda\|\check{\Gamma}^{\mathrm{r}}\|_0 \lesssim \varepsilon^3\|\eta_1\|_0,\\
&\|\nabla\check{\Gamma}^\dagger\|_1 + \lambda\|\check{\Gamma}^\dagger\|_0 \leq c_\lambda\|u^\dagger\|_{X^0}, && \lambda > 0,\\
&\|\nabla\check{\Gamma}^\dagger\|_1 + \lambda\|\check{\Gamma}^\dagger\|_0 \lesssim \|\omega^\dagger\|_0 + \|\Psi_1^\dagger\|_1 + \|\Psi_3^\dagger\|_1, && \lambda = 0.
\end{aligned}$$

Finally we write

$$\check{\mathcal{N}}(\eta_1, u^\dagger) = \check{\mathcal{N}}^{(2)}(\eta_1) + \check{\mathcal{N}}^{(3)}(\eta_1) + \check{\mathcal{N}}^{\mathrm{r}}(\eta_1) + \check{\mathcal{N}}^\dagger(u^\dagger),$$

where

$$\begin{aligned}
\check{\mathcal{N}}^{(2)}(\eta_1) &= \tilde{\mathcal{N}}^{(2)}(\eta_1),\\
\check{\mathcal{N}}^{(3)}(\eta_1) &= \tilde{\mathcal{N}}^{(3)}(\eta_1) + \tilde{\mathcal{N}}^{(2)}(\check{\eta}_2^{(2)}(\eta_1)),\\
\check{\mathcal{N}}^{\mathrm{r}}(\eta_1) &= \tilde{\mathcal{N}}^{\mathrm{r}}\Big(\eta_1 + \check{\eta}_2^{(2)}(\eta_1) + \check{\eta}_2^{\mathrm{r}}(\eta_1)\Big) + \tilde{\mathcal{N}}^{(3)}\Big(\check{\eta}_2^{(2)}(\eta_1) + \check{\eta}_2^{\mathrm{r}}(\eta_1)\Big) + \tilde{\mathcal{N}}^{(2)}(\check{\eta}_2^{\mathrm{r}}(\eta_1)),\\
\check{\mathcal{N}}^\dagger(u^\dagger) &= \tilde{\mathcal{N}}\Big(\check{\eta}_2^\dagger(u^\dagger), u^\dagger\Big).
\end{aligned}$$

Proposition 4.21. *The functions* $\check{\mathcal{N}}^{(2)}, \check{\mathcal{N}}^{(3)}, \check{\mathcal{N}}^{(\mathrm{r})}$ *and* $\check{\mathcal{N}}^{\dagger}$ *satisfy the estimates*

$$\begin{aligned} \|\check{\mathcal{N}}^{(j)}(\eta_1)\|_0 &\lesssim \varepsilon^{j-1}\|\eta_1\|_0, \quad j=2,3, \\ \|\check{\mathcal{N}}^{\mathrm{r}}(\eta_1)\|_0 &\lesssim \varepsilon^3\|\eta_1\|_0, \\ \|\check{\mathcal{N}}^{\dagger}(u^\dagger)\|_0 &\leq c_\lambda \|u^\dagger\|_{X^0}, && \lambda > 0, \\ \|\check{\mathcal{N}}^{\dagger}(u^\dagger)\|_0 &\lesssim \|\omega^\dagger\|_0 + \|\Psi_1^\dagger\|_1 + \|\Psi_3^\dagger\|_1, && \lambda = 0. \end{aligned}$$

Observing that $\chi(D)\check{\mathcal{N}}^{(2)}(\eta_1) = 0$ for all $\eta_1 \in \chi(D)L^2(\mathbb{R})$, we find that the reduced equation (4.47) may be written as

$$g_\varepsilon(\mu,\lambda)\hat{\eta}_1 = \chi\mathcal{F}\Big[\check{\mathcal{N}}^{(3)}(\eta_1) + \check{\mathcal{N}}^{\mathrm{r}}(\eta_1) + \check{\mathcal{N}}^{\dagger}(u^\dagger)\Big]. \tag{4.50}$$

Lemma 4.22. *There exsists a constant* $k_{\max} > 0$ *such that the linear operator*

$$g_\varepsilon(D,\lambda) - \chi(D)\Big(\check{\mathcal{N}}^{(3)}(\cdot) + \check{\mathcal{N}}^{\mathrm{r}}(\cdot)\Big) : \chi(D)L^2(\mathbb{R}) \to \chi(D)L^2(\mathbb{R})$$

is invertible for each $\lambda \in [\varepsilon k_{\max}, \lambda^\star]$.

Proof. Write $\lambda = \varepsilon k$ and observe that

$$g_\varepsilon(\mu,\lambda) \gtrsim \varepsilon^2(1+\varepsilon^2 k_{\max}^4)$$

(see Proposition 4.14), while

$$\|\check{\mathcal{N}}^{(3)}(\eta_1) + \check{\mathcal{N}}^{\mathrm{r}}(\eta_1)\|_0 \lesssim \varepsilon^2\|\eta_1\|_0.$$

It follows that the given operator is invertible for sufficiently large values of k. ■

Corollary 4.23. *For each* $\lambda \in [\varepsilon k_{\max}, \lambda^\star]$ *and each* $u \in X^0$, *equation* (4.50) *has a unique solution* $\eta_1 \in \chi(D)L^2(\mathbb{R})$ *which satisfies the estimate*

$$\|\eta_1\|_0 \leq c_\lambda \|u^\dagger\|_{X^0}.$$

4.2.3 Small spectral values

In this section we calculate the leading-order terms in the quantity $\check{\mathcal{N}}^{(3)}(\eta_1)$ appearing in the reduced equation (4.50), writing

$$\eta_1(x) = \frac{1}{2}\varepsilon\zeta(\varepsilon x)\mathrm{e}^{\mathrm{i}\mu_0 x} + \frac{1}{2}\varepsilon\overline{\zeta(\varepsilon x)}\mathrm{e}^{-\mathrm{i}\mu_0 x},$$

where $\zeta \in \chi_0(\varepsilon D)L^2(\mathbb{R})$. In view of Corollary 4.23 we write $\lambda = \varepsilon k$ and henceforth suppose that $k \leq k_{\max}$. For this purpose we first perform the corresponding calculation for $\check{\mathcal{N}}^{(2)}(\eta_1)$ using the formula

$$\begin{aligned}\check{\mathcal{N}}^{(2)}(\eta_1) = \tilde{\Gamma}_x^{(2)}(\eta_1)\Big|_{y=1} + k\varepsilon \int_0^1 y\Phi_{1y}^\star k\varepsilon\tilde{\Gamma}^{(1)}\,\mathrm{d}y \\ - \int_0^1 \Big\{\Phi_{1x}^\star\tilde{\Gamma}_x^{(1)} - \Phi_{1y}^\star\tilde{\Gamma}_y^{(1)} + \Big[y\Phi_{1y}^\star\tilde{\Gamma}_x^{(1)} + y\Phi_{1x}^\star\tilde{\Gamma}_y^{(1)}\Big]_x\Big\}\,\mathrm{d}y\end{aligned} \tag{4.51}$$

(see equation (4.48)) together with

$$\check{\Gamma}^{(1)} = \mathcal{F}^{-1}\Big[-\mathrm{i}\mu G(y,1)\hat{\eta}_1\Big] \tag{4.52}$$

and

$$\tilde{\Gamma}^{(2)}(\eta_1) = \mathcal{F}^{-1}\left[\int_0^1 \Big\{G(y,\tilde{y})\Big(-\mathrm{i}\mu\mathcal{F}[\check{F}_1^{(2)}] - \mathrm{i}k\varepsilon\mathcal{F}[\check{F}_2^{(2)}]\Big) + G_{\tilde{y}}(y,\tilde{y})\mathcal{F}[\check{F}_3^{(2)}]\Big\}\mathrm{d}\tilde{y}\right], \tag{4.53}$$

where

$$\check{F}_1^{(2)} = -\eta_1^\star\check{\Gamma}_x^{(1)} - \Phi_{1x}^\star\eta_1 + y\Phi_{1y}^\star\eta_{1x} + y\eta_{1x}^\star\check{\Gamma}_y^{(1)}, \tag{4.54}$$

$$\check{F}_2^{(2)} = -\mathrm{i}k\varepsilon\eta_1^\star\check{\Gamma}^{(1)} + \mathrm{i}k\varepsilon\Phi_{1y}^\star\eta_1, \tag{4.55}$$

$$\check{F}_3^{(2)} = y\eta_{1x}^\star\check{\Gamma}_x^{(1)} + y\Phi_{1x}^\star\eta_{1x} + \eta_1^\star\check{\Gamma}_y^{(1)} + \Phi_{1y}^\star\eta_1. \tag{4.56}$$

At various stages of the calculation terms of the form $G(y,1)\hat{p}$ and $G(y,\tilde{y})\hat{P}$ arise, where $\hat{p}$ and $\hat{P}$ have compact support. We decompose such terms into a leading-order part and a higher-order remainder term using the following five results by Groves, Sun and Wahlén [27, p. 31–33].

Lemma 4.24.

(i) Suppose that $\underline{\mu} > 0$. The Green's function G admits the decomposition

$$G(y,\tilde{y}) = G(y,\tilde{y};\underline{\mu}) + R(y,\tilde{y}),$$

where

$$G(y,\tilde{y};\underline{\mu}) = \begin{cases} \dfrac{\cosh(\underline{\mu}y)\cosh(\underline{\mu}(1-\tilde{y}))}{\underline{\mu}\sinh(\underline{\mu})}, & 0 \leq y \leq \tilde{y} \leq 1, \\ \dfrac{\cosh(\underline{\mu}\tilde{y})\cosh(\underline{\mu}(1-y))}{\underline{\mu}\sinh(\underline{\mu})}, & 0 \leq \tilde{y} \leq y \leq 1 \end{cases}$$

and R satisfies the estimates

$$|\partial_y^{m_1}\partial_{\tilde{y}}^{m_2}R(y,\tilde{y})| \lesssim |\mu - \underline{\mu}| + \varepsilon^2k^2, \quad m_1, m_2 = 0,1,2,\ldots$$

uniformly over $y, \tilde{y}$ in $[0,1]$ and q in each fixed interval $[q_{\min}, q_{\max}]$ containing $\underline{\mu}$.

(ii) The Green's function G admits the expansion

$$G(y,\tilde{y}) = G(y,\tilde{y};0) + R(y,\tilde{y}),$$

where

$$G(y,\tilde{y};0) = \frac{1}{q^2}$$

and R satisfies the estimate

$$|R(y,\tilde{y})| \lesssim 1$$

uniformly over $y, \tilde{y}$ in $[0,1]$ and q in each fixed interval $[0, q_{\max}]$.

Proposition 4.25. *The function $\check{\Gamma}^{(1)}$ is given by the formula*

$$\check{\Gamma}^{(1)} = -\frac{\mathrm{i}\cosh(\mu_0 y)}{2\sinh(\mu_0)}\Big(\varepsilon\zeta(\varepsilon x)\mathrm{e}^{\mathrm{i}\mu_0 x} - \varepsilon\overline{\zeta(\varepsilon x)}\mathrm{e}^{-\mathrm{i}\mu_0 x}\Big) + R_2,$$

where the symbol R_j denotes a quantity which satisfies the estimate

$$\|R_j\|_j \lesssim \varepsilon^{3/2}\|\zeta\|_1.$$

Substituting this formula into equations (4.54)–(4.56) and approximating $\zeta^\star$ with

$$\zeta_\delta^\star := \chi_0(\varepsilon D)\zeta^\star$$

by means of the estimate

$$\begin{aligned}\varepsilon\|\zeta^\star(\varepsilon\cdot) - \zeta_\delta^\star(\varepsilon\cdot)\|_\infty &\leq \int_{-\infty}^{\infty}\left|\hat{\zeta}^\star\Big(\frac{\mu}{\varepsilon}\Big) - \hat{\zeta}_\delta^\star\Big(\frac{\mu}{\varepsilon}\Big)\right|\mathrm{d}\mu\\ &= \int_{|\mu|\geq\delta}\left|\hat{\zeta}^\star\Big(\frac{\mu}{\varepsilon}\Big)\right|\mathrm{d}\mu\\ &= \mathcal{O}(\varepsilon^n), \quad n > 1\end{aligned}$$

($\zeta^\star$ is a Schwartz-class function) yields the following result.

Corollary 4.26. *The functions $\check{F}_1^{(2)}, \check{F}_2^{(2)}$ and $\check{F}_3^{(2)}$ are given by the formulae*

$$\begin{aligned}\check{F}_1^{(2)} &= \varepsilon^2\frac{\mu_0(y\mu_0\sinh(\mu_0) - \cosh(\mu_0 y))}{\sinh(\mu_0)}\zeta_\delta^\star(\varepsilon x)\Big(\frac{1}{2}\zeta(\varepsilon x)\mathrm{e}^{2\mathrm{i}\mu_0 x} + \frac{1}{2}\overline{\zeta(\varepsilon x)}\mathrm{e}^{-2\mathrm{i}\mu_0 x}\Big)\\ &\quad - \varepsilon^2\frac{\mu_0(y\mu_0\sinh(\mu_0 y) + \cosh(\mu_0 y))}{\sinh(\mu_0)}\zeta_\delta^\star(\varepsilon x)\Big(\frac{1}{2}\zeta(\varepsilon) + \frac{1}{2}\overline{\zeta(\varepsilon x)}\Big) + \varepsilon R_1,\end{aligned}$$

$$\begin{aligned}\check{F}_2^{(2)} &= k\varepsilon^3 \frac{\mu_0(y\mu_0\sinh(\mu_0) - \cosh(\mu_0 y))}{\sinh(\mu_0)}\zeta_\delta^\star(\varepsilon x)\Big(\frac{1}{4}\zeta(\varepsilon x)\mathrm{e}^{2\mathrm{i}\mu_0 x} + \frac{1}{4}\overline{\zeta(\varepsilon x)}\mathrm{e}^{-2\mathrm{i}\mu_0 x}\Big)\\ &\qquad - k\varepsilon^3\frac{\mu_0(y\mu_0\sinh(\mu_0 y) + \cosh(\mu_0 y))}{\sinh(\mu_0)}\zeta_\delta^\star(\varepsilon x)\Big(\frac{1}{4}\zeta(\varepsilon) + \frac{1}{4}\overline{\zeta(\varepsilon x)}\Big) + k\varepsilon^2 R_1,\\ \check{F}_3^{(2)} &= -\mathrm{i}\varepsilon^2\frac{\mu_0(\sinh(\mu_0 y) - y\mu_0\cosh(\mu_0 y))}{\sinh(\mu_0)}\zeta_\delta^\star(\varepsilon x)\Big(\frac{1}{2}\zeta(\varepsilon x)\mathrm{e}^{2\mathrm{i}\mu_0 x} - \frac{1}{2}\overline{\zeta(\varepsilon x)}\mathrm{e}^{-2\mathrm{i}\mu_0 x}\Big) + \varepsilon R_1.\end{aligned}$$

Proposition 4.27. *The function $\tilde{\Gamma}^{(2)}(\eta_1)$ is given by the formula*

$$\begin{aligned}\tilde{\Gamma}^{(2)}(\eta_1) = &-2\mathrm{i}\varepsilon^2\zeta_\delta^\star(\varepsilon x)\Big(\frac{1}{2}\zeta(\varepsilon)\mathrm{e}^{2\mathrm{i}\mu_0 x} - \frac{1}{2}\overline{\zeta(\varepsilon x)}\mathrm{e}^{-2\mathrm{i}\mu_0 x}\Big)W(y)\\ &+\varepsilon\frac{\mu_0\cosh(\mu_0)}{\sinh(\mu_0)}\mathcal{F}^{-1}\left[\frac{\mathrm{i}\mu}{m^2+k^2}\mathcal{F}\Big[\mathrm{Re}(\zeta_\delta^\star\zeta)\Big]\right](\varepsilon x) + \varepsilon S,\end{aligned}$$

where

$$W(y) = -\frac{\mu_0\cosh(2\mu_0 y)}{4\sinh^2(\mu_0)} + \frac{\mu_0 y\sinh(\mu_0 y)}{2\sinh(\mu_0)}$$

and the symbol S denotes a quantity which satisfies the estimate

$$\|\nabla S\|_1 + k\varepsilon\|S\|_0 \lesssim \varepsilon^{3/2}\|\zeta\|_1.$$

Finally, we combine equation (4.51) with the representations of $\check{\Gamma}^{(1)}$ and $\tilde{\Gamma}^{(2)}(\eta_1)$ in Propositions 4.25 and 4.27.

Lemma 4.28. *The quantity $\check{\mathcal{N}}^{(2)}(\eta_1)$ is given by the formula*

$$\begin{aligned}\check{\mathcal{N}}^{(2)}(\eta_1) = &-\varepsilon^2 g_0(2\mu_0,0)B_1\zeta_\delta^\star(\varepsilon x)\Big(\frac{1}{2}\zeta(\varepsilon x)\mathrm{e}^{2\mathrm{i}\mu_0 x} + \frac{1}{2}\overline{\zeta(\varepsilon x)}\mathrm{e}^{-2\mathrm{i}\mu_0 x}\Big)\\ &-\varepsilon^2\frac{\mu_0\cosh(\mu_0)}{\sinh(\mu_0)}\mathcal{F}^{-1}\left[\frac{\mu^2}{\mu^2+k^2}\mathcal{F}[\mathrm{Re}(\zeta_\delta^\star\zeta)]\right](\varepsilon x)\\ &-\varepsilon^2\frac{\mu_0^2}{2\sinh^2(\mu_0)}\mathrm{Re}(\zeta_\delta^\star(\varepsilon x)\zeta(\varepsilon x)) + \varepsilon R_0,\end{aligned}$$

where

$$B_1 = \frac{\mu_0^2(\cosh(2\mu_0)+2)}{2g_0(2\mu_0,0)\sinh^2(\mu_0)}.$$

We now compute the leading-order terms in the quantity

$$
\begin{aligned}
\check{\mathcal{N}}^{(3)}(\eta_1) = {} & \check{\Gamma}_x^{(3)}(\eta_1)\Big|_{y=1} + \tilde{\Gamma}_x^{(2)}(\check{\eta}_2^{(2)})\Big|_{y=1} + \frac{5}{2\beta_0}k^2\varepsilon^2\xi_1^{\star 2}\eta_1 - 2k^2\varepsilon^2\xi_1^\star\eta_{1xx}^\star\eta_1 + 2k^2\varepsilon^2\eta_{1x}^\star\xi_1^\star\eta_{1x} \\
& - 2k^2\varepsilon^2\Big[\xi_1^\star\eta_{1x}^\star\eta_1\Big]_x - k^2\varepsilon^2\eta_{1x}^\star\xi_1^\star\eta_{1x} - \frac{3\beta_0}{2}k^2\varepsilon^2\eta_{1x}^{\star 2}\eta_{1xx} - 2\beta_0 k^2\varepsilon^2\eta_{1x}^\star\eta_{1xx}^\star\eta_{1x} \\
& + \frac{\beta_0}{2}k^4\varepsilon^4\eta_{1x}^{\star 2}\eta_1 + \Big[\eta_{1x}^\star\xi_1^\star\eta_{1x} + \frac{3\beta_0}{2}\eta_{1x}^{\star 2}\eta_{1xx} + 2\beta_0\eta_{1x}^\star\eta_{1xx}^\star\eta_{1x} - \frac{\beta_0}{2}k^2\varepsilon^2\eta_{1x}^{\star 2}\eta_1\Big]_{xx} \\
& + \Big[\xi_1^\star\eta_{1x}^\star\eta_{1xx} - k^2\varepsilon^2\xi_1^\star\eta_{1x}^\star\eta_1 + 2\beta_0\eta_{1x}^\star\eta_{1xx}^\star\eta_{1xx} - 2k^2\varepsilon^2\beta_0\eta_{1x}^\star\eta_{1xx}^\star\eta_1\Big]_x \\
& + \Big[\frac{1}{2\beta_0}\xi_1^{\star 2}\eta_{1x} + 2\xi_1^\star\eta_{1x}^\star\eta_{1xx} + 2\xi_1\eta_{1xx}^\star\eta_{1x}\Big]_x \\
& + \Big[\beta_0\eta_{1x}^{\star 2}\eta_{1xx} - 2\eta_{1x}^\star\xi_1^\star\eta_{1x} - k^2\varepsilon^2\eta_{1x}^{\star 2}\eta_{xx}\Big]_{xx} \\
& - \mathrm{i}k\varepsilon\int_0^1 \Big\{ - \mathrm{i}k\varepsilon y^2\Phi_{1y}^{\star 2}\eta_1 + \mathrm{i}k\varepsilon y\Phi_{1y}^\star\check{\Gamma}^{(2)} + \mathrm{i}k\varepsilon\Phi_{2y}^\star\check{\Gamma}^{(1)}\Big\}\,\mathrm{d}y \\
& - \int_0^1 \Big\{\Phi_{1x}^\star\check{\Gamma}_x^{(2)} - \Phi_{1y}^\star\check{\Gamma}_y^{(2)} + 2\eta_1^\star\Phi_{1y}^\star\check{\Gamma}_y^{(1)} + \Phi_{1y}^{\star 2}\eta_1\Big\}\,\mathrm{d}y \\
& - \int_0^1 \Big\{y\Big[\Phi_{1y}^\star\check{\Gamma}_x^{(2)} + \Phi_{1x}^\star\check{\Gamma}_y^{(2)} - 2y\eta_{1x}^\star\Phi_{1y}^\star\check{\Gamma}_y^{(1)} - y\Phi_{1y}^{\star 2}\eta_{1x}\Big]_x\Big\}\,\mathrm{d}y \\
& - \int_0^1 \Big\{\Phi_{2x}^\star\check{\Gamma}_x^{(1)} - \Phi_{2y}^\star\check{\Gamma}_y^{(1)} + y\Big[\Phi_{2y}^\star\check{\Gamma}_x^{(1)} + \Phi_{2x}^\star\check{\Gamma}_y^{(1)}\Big]_x\Big\}\,\mathrm{d}y,
\end{aligned}
\tag{4.57}
$$

which formula follows from equations (4.48), (4.49) and the relations

$$
\check{\mathcal{N}}^{(3)}(\eta_1) = \tilde{\mathcal{N}}^{(3)}(\eta_1) + \tilde{\mathcal{N}}^{(2)}(\check{\eta}_2^{(2)}(\eta_1)), \quad \check{\Gamma}^{(2)} = \tilde{\Gamma}^{(2)}(\eta_1) + \tilde{\Gamma}^{(1)}(\check{\eta}_2^{(2)}).
$$

For this purpose we also use the formulae

$$
\begin{aligned}
\tilde{\Gamma}^{(1)}(\check{\eta}_2^{(2)}) &= \mathcal{F}^{-1}\Big[- \mathrm{i}\mu G(y,1)\mathcal{F}[\check{\eta}_2^{(2)}]\Big], \\
\check{\eta}_2^{(2)} &= g_\varepsilon(D,\lambda)^{-1}(1-\chi(D))\tilde{\mathcal{N}}^{(2)}(\eta_1),
\end{aligned}
$$

and

$$
\begin{aligned}
& \tilde{\Gamma}^{(3)}(\eta_1) + \tilde{\Gamma}^{(2)}(\check{\eta}_2^{(2)}) \\
& \quad = \mathcal{F}^{-1}\Bigg[\int_0^1 \Big\{G(y,\tilde{y})\Big(- \mathrm{i}\mu\mathcal{F}\Big[\check{F}_1^{(3)}\Big] - \mathrm{i}\lambda\mathcal{F}\Big[\check{F}_2^{(3)}\Big]\Big) + G_{\tilde{y}}(y,\tilde{y})\mathcal{F}\Big[\check{F}_3^{(3)}\Big]\Big\}\,\mathrm{d}\tilde{y}\Bigg],
\end{aligned}
\tag{4.58}
$$

where

$$
\begin{aligned}
\check{F}_1^{(3)} &= -\eta_1^\star\check{\Gamma}_x^{(2)} - \eta_2^\star\check{\Gamma}_x^{(1)} - \Phi_{1x}^\star\check{\eta}_2^{(2)} - \Phi_{2x}^\star\eta_1 + y\Phi_{1y}^\star\check{\eta}_{2x}^{(2)} + y\Phi_{2y}^\star\eta_{1x} + y\eta_{1x}^\star\check{\Gamma}_y^{(2)} + y\eta_{2x}^\star\check{\Gamma}_y^{(1)}, \\
\check{F}_2^{(3)} &= -\mathrm{i}k\varepsilon\eta_1^\star\check{\Gamma}_x^{(2)} - \mathrm{i}k\varepsilon\eta_2^\star\check{\Gamma}^{(1)} + \mathrm{i}k\varepsilon y\Phi_{2y}^\star\eta_1 + \mathrm{i}k\varepsilon y\Phi_{1y}^\star\check{\eta}_2^{(2)},
\end{aligned}
$$

$$\begin{aligned}\check{F}_3^{(3)} &= y\eta_{1x}^\star\check{\Gamma}_x^{(2)} + y\eta_{2x}^\star\check{\Gamma}_x^{(1)} + y\Phi_{1x}^\star\check{\eta}_{2x}^{(2)} + y\Phi_{2x}^\star\eta_{1x} + \eta_1^\star\check{\Gamma}_y^{(2)} + \eta_2^\star\check{\Gamma}_y^{(1)} + \Phi_{1y}^\star\check{\eta}_2^{(2)} + \Phi_{2y}^\star\eta_1\\ &\quad - \eta_1^{\star 2}\check{\Gamma}_y^{(1)} - 2\eta_1^\star\Phi_{1y}^\star\eta_1 - y^2\eta_{1x}^{\star 2}\check{\Gamma}_y^{(1)} - 2y^2\eta_{1x}^\star\Phi_{1y}^\star\eta_{1x}.\end{aligned}$$

Proposition 4.29. *The quantity $\check{\eta}_2^{(2)}$ satisfies the estimate*

$$\begin{aligned}\check{\eta}_2^{(2)} = &-\varepsilon^2 B_1\zeta_\delta^\star(\varepsilon x)\Big(\frac{1}{2}\zeta(\varepsilon x)\mathrm{e}^{2\mathrm{i}\mu_0 x} + \frac{1}{2}\overline{\zeta(\varepsilon x)}\mathrm{e}^{-2\mathrm{i}\mu_0 x}\Big) - \varepsilon^2 B_2\mathrm{Re}\big(\zeta_\delta^\star\big(\varepsilon x)\zeta(\varepsilon x)\big)\\ &-\varepsilon^2 B_3\mathcal{F}^{-1}\bigg[\frac{\mu^2}{(1-\alpha_0^{-1})\mu^2+k^2}\mathcal{F}\Big[2\mathrm{Re}(\zeta_\delta^\star\zeta)\Big](\mu)\bigg](\varepsilon x) + \varepsilon R_2,\end{aligned}$$

where

$$B_2 = \frac{\mu_0^2}{2\alpha_0\sinh^2(\mu_0)}, \qquad B_3 = \frac{\mu_0(\alpha_0\sinh(2\mu_0)+\mu_0)}{4\alpha_0^2\sinh^2(\mu_0)}.$$

Proof. We apply the operator $g_\varepsilon(D,\lambda)^{-1}(1-\chi(D))$ to the expression for $\check{\mathcal{N}}^{(2)}(\eta_1)$ given in Lemma 4.28. The first term on the right-hand side of this expression is handled using the approximation

$$g_\varepsilon(\mu,k\varepsilon)^{-1} = g_0(2\mu_0,0)^{-1} + T_1(\mu,\varepsilon),$$

where

$$|T_1(\mu,\varepsilon)| \lesssim |\mu-2\mu_0| + \varepsilon^2, \qquad |\mu-2\mu_0| \le 2\delta,$$

while the second and third are treated with the approximation

$$g_\varepsilon(\mu,k\varepsilon)^{-1} = \frac{q^2}{(\alpha_0-1)\mu^2+\alpha_0 k^2\varepsilon^2} + T_2(\mu,\varepsilon),$$

where

$$|T_2(q,\varepsilon)| \lesssim \mu^2 + \varepsilon^2, \qquad |\mu| \le 2\delta.$$

Turning to the fourth term, note that

$$g_\varepsilon(D,\lambda)^{-1}(1-\chi(D))R_0 = R_4$$

because $g_\varepsilon(\mu,k\varepsilon) \gtrsim 1+q^4$ for $\mu \in \mathrm{supp}\,(1-\chi)$ (see Proposition 4.14). ∎

The function $\tilde{\Gamma}^{(1)}(\check{\eta}_2^{(2)})$ may now be computed using Lemma 4.24; combining the result with Proposition 4.12 yields the following proposition.

Proposition 4.30. *The function $\check{\Gamma}^{(2)}$ is given by the formula*

$$\check{\Gamma}^{(2)}(\eta_1) = -\mathrm{i}\varepsilon^2\zeta_\delta^\star(\varepsilon x)\Big(\frac{1}{2}\zeta(\varepsilon x)\mathrm{e}^{2\mathrm{i}\mu_0 x} - \frac{1}{2}\overline{\zeta(\varepsilon x)}\mathrm{e}^{-2\mathrm{i}\mu_0 x}\Big)M_1(y) + \varepsilon\vartheta_\delta(\varepsilon x) + \varepsilon S,$$

where

$$\begin{aligned}
\vartheta_\delta(x) &= \frac{M_2}{\alpha_0}\mathcal{F}^{-1}\bigg[\frac{\mathrm{i}\mu}{(1-\alpha_0^{-1})\mu^2+k^2}\mathcal{F}\Big[2\mathrm{Re}(\zeta_\delta^\star\zeta)\Big]\bigg](x),\\
M_1(y) &= M_{11}\frac{\cosh(2\mu_0 y)}{\sinh(2\mu_0)} + \frac{\mu_0 y\sinh(\mu_0 y)}{\sinh(\mu_0)},\\
M_{11} &= -\frac{\mu_0^2(\cosh(2\mu_0)+2) + \mu_0 g_0(2\mu_0,0)\sinh(2\mu_0,0)}{2\sinh^2(\mu_0)g_0(2\mu_0,0)},\\
M_2 &= \frac{\mu_0(\alpha_0\sinh(2\mu_0)+\mu_0)}{4\sinh^2(\mu_0)}.
\end{aligned}$$

Corollary 4.31. *The functions $\check{F}_1^{(3)}, \check{F}_2^{(3)}$ and $\check{F}_3^{(2)}$ are given by the formulae*

$$\begin{aligned}
\check{F}_1^{(3)} &= \varepsilon^3(\zeta_\delta^\star(\varepsilon x))^2\bigg(\frac{1}{2}\Big(\zeta(\varepsilon x)U_{11}(y) + \overline{\zeta(\varepsilon x)}U_{12}(y)\Big)\mathrm{e}^{\mathrm{i}\mu_0 x}\\
&\qquad + \frac{1}{2}\Big(\zeta(\varepsilon x)U_{12}(y) + \overline{\zeta(\varepsilon x)}U_{11}(y)\Big)\mathrm{e}^{-\mathrm{i}\mu_0 x}\bigg)\\
&\quad + \varepsilon^3\zeta_\delta^\star(\varepsilon x)\vartheta_{\delta x}(y)U_2(y)\Big(\frac{1}{2}\mathrm{e}^{\mathrm{i}\mu_0 x} + \frac{1}{2}\mathrm{e}^{-\mathrm{i}\mu_0 x}\Big) + \varepsilon^2 T,\\
\check{F}_2^{(3)} &= \varepsilon T,\\
\check{F}_3^{(3)} &= -\mathrm{i}\varepsilon^3(\zeta_\delta^\star(\varepsilon x))^2\bigg(\frac{1}{2}\Big(\zeta(\varepsilon x)U_{31}(y) + \overline{\zeta(\varepsilon x)}U_{32}(y)\Big)\mathrm{e}^{\mathrm{i}\mu_0 x}\\
&\qquad - \frac{1}{2}\Big(\zeta(\varepsilon x)U_{32}(y) + \overline{\zeta(\varepsilon x)}U_{31}(y)\Big)\mathrm{e}^{-\mathrm{i}\mu_0 x}\bigg)\\
&\quad - \varepsilon^3\zeta_\delta^\star(\varepsilon x)\vartheta_{\delta x}(y)U_4(y)\Big(\frac{1}{2}\mathrm{e}^{\mathrm{i}\mu_0 x} - \frac{1}{2}\mathrm{e}^{-\mathrm{i}\mu_0 x}\Big) + \varepsilon^2 T,
\end{aligned}$$

where

$$\begin{aligned}
U_{11}(y) &= \frac{\mu_0(\cosh(\mu_0 y) + 2\mu_0 y\sinh(\mu_0 y))B_1}{2\sinh(\mu_0)} + \frac{\mu_0\cosh(\mu_0 y)\big(C_2 + \frac{1}{2}B_2\big)}{\sinh(\mu_0)}\\
&\quad - C_4 - \frac{\mu_0 y M_1'(y)}{2} - \mu_0 M_1(y),\\
U_{12}(y) &= \frac{\mu_0(\cosh(\mu_0 y) + 2\mu_0 y\sinh(\mu_0 y))B_1}{2\sinh(\mu_0)} + \frac{\mu_0\cosh(\mu_0 y)B_2}{2\sinh(\mu_0)}\\
&\quad - \mu_0 D_2(\cosh(2\mu_0 y) + \mu_0 y\sinh(2\mu_0 y))\\
&\quad - \frac{\mu_0^2 y(\mu_0 y\cosh(\mu_0 y) + 3\sinh(\mu_0 y))}{4\sinh(\mu_0)},
\end{aligned}$$

$$
\begin{aligned}
U_2(y) &= -1 - \frac{\mu_0 \cosh(\mu_0 y)}{\alpha_0 \sinh(\mu_0)},\\
U_{31}(y) &= \frac{\mu_0(\sinh(\mu_0 y) + 2\mu_0 y \cosh(\mu_0 y))B_1}{2\sinh(\mu_0)} - \frac{\mu_0 \sinh(\mu_0 y)\left(C_2 + \frac{1}{2}B_2\right)}{\sinh(\mu_0)}\\
&\qquad - C_4\mu_0 y + \frac{M_1'}{2} + \mu_0^2 y M_1(y) - \frac{\mu_0 \sinh(\mu_0 y)(1+3\mu_0^2 y^2)}{2\sinh(\mu_0)},\\
U_{32}(y) &= \frac{\mu_0(\sinh(\mu_0 y) + 2\mu_0 y \cosh(\mu_0 y))C_1}{2\sinh(\mu_0)} - \frac{\mu_0 \sinh(\mu_0 y) B_2}{2\sinh(\mu_0)}\\
&\qquad + \mu_0 D_2(\mu_0 y \cosh(2\mu_0 y) + \sinh(2\mu_0 y))\\
&\qquad + \frac{y\mu_0^2(\cosh(\mu_0 y) - \mu_0 y \sinh(\mu_0 y))}{4\sinh(\mu_0)},\\
U_4(y) &= -\mu_0 y + \frac{\mu_0 \sinh(\mu_0 y)}{\alpha_0 \sinh(\mu_0)}
\end{aligned}
$$

and the symbol T denotes a quantity which satisfies the estimate

$$\|\chi(D)T\|_0 \lesssim \varepsilon^{3/2}\|\zeta\|_1.$$

The next result is obtained from equation (4.58) using Lemma 4.24.

Proposition 4.32. *The quantity $\tilde{\Gamma}_x^{(3)}(\eta_1) + \tilde{\Gamma}_x^{(2)}(\check{\eta}_2^{(2)})\Big|_{y=1}$ is given by the formula*

$$
\begin{aligned}
\tilde{\Gamma}_x^{(3)} + \tilde{\Gamma}_x^{(2)}(\check{\eta}_2^{(2)})\Big|_{y=1} &= \mu_0 \varepsilon^2 (\zeta_\delta(\varepsilon x))^2 \bigg(\frac{1}{2}\Big(\zeta(\varepsilon x) N_1 + \overline{\zeta(\varepsilon x)} N_2\Big)\mathrm{e}^{\mathrm{i}\mu_0 x}\\
&\qquad\qquad + \frac{1}{2}\Big(\zeta(\varepsilon x) N_2 + \overline{\zeta(\varepsilon x)} N_1\Big)\mathrm{e}^{-\mathrm{i}\mu_0 x}\bigg)\\
&\quad + \mu_0 \varepsilon^3 \zeta_\delta^\star(\varepsilon x)\vartheta_{\delta x}(\varepsilon x) N_3 \left(\frac{1}{2}\mathrm{e}^{\mathrm{i}\mu_0 x} + \frac{1}{2}\mathrm{e}^{-\mathrm{i}\mu_0 x}\right) + \varepsilon^2 T,
\end{aligned}
$$

where

$$
\begin{aligned}
N_1 &= \frac{\mu_0 \cosh(2\mu_0) B_1}{2\sinh^2(\mu_0)} + \frac{\mu_0\left(C_2 + \frac{1}{2}B_2\right)}{\sinh^2(\mu_0)} - \frac{\cosh(\mu_0) C_4}{\sinh(\mu_0)} - \frac{M_{11}\mu_0}{2\sinh^2(\mu_0)} - \frac{\mu_0^2 \cosh(\mu_0)}{2\sinh(\mu_0)},\\
N_2 &= \frac{\mu_0 \cosh(2\mu_0) C_1}{2\sinh^2(\mu_0)} + \frac{\mu_0 B_2}{2\sinh^2(\mu_0)} - \frac{\mu_0 \sinh(2\mu_0) D_2}{2\sinh^2(\mu_0)} - \frac{\mu_0^2 \cosh(\mu_0)}{4\sinh(\mu_0)},\\
N_3 &= \frac{-2\mu_0 \alpha_0^{-1} - \sinh(2\mu_0)}{2\sinh^2(\mu_0)}.
\end{aligned}
$$

Finally, combining equation (4.57) and the results of Propositions 4.30 and 4.32 yields the following formula.

Lemma 4.33. *The quantity $\check{\mathcal{N}}^{(3)}(\eta_1)$ is given by the formula*

$$\begin{aligned}\check{\mathcal{N}}^{(3)}(\eta_1) &= \varepsilon^3(\zeta_\delta^\star(\varepsilon x))^2\Big((A_3+A_5)\zeta(\varepsilon x)+A_3\overline{\zeta(\varepsilon x)}\Big)\frac{\mathrm{e}^{\mathrm{i}\mu_0 x}}{2}\\ &\quad+\varepsilon^3(\zeta_\delta^\star(\varepsilon x))^2\Big(A_3\zeta(\varepsilon x)+(A_3+A_5)\overline{\zeta(\varepsilon x)}\Big)\frac{\mathrm{e}^{-\mathrm{i}\mu_0 x}}{2}\\ &\quad-\varepsilon^3 A_4\zeta_\delta^\star(\varepsilon x)\vartheta_{\delta x}(\varepsilon x)\Big(\frac{1}{2}\mathrm{e}^{\mathrm{i}\mu_0 x}+\frac{1}{2}\mathrm{e}^{-\mathrm{i}\mu_0 x}\Big)+\varepsilon^2 T.\end{aligned}$$

4.2.4 Derivation of the reduced equation in its final form

In this section we derive the reduced equation in its final form, setting $\lambda=\varepsilon k$ and $k\le k_{\max}$, and thus complete the proof of Theorem 4.3(ii) and (iii). For $k>0$ we define $\check{u}_{\varepsilon,k}:\chi(\varepsilon D)L^2(\mathbb{R})\times X^0\to Y^1$ by the formula

$$\check{u}_{\varepsilon,k}(\zeta,u^\dagger)=\Big(\eta_1+\check{\eta}_2(\eta_1,u^\dagger),\check{\rho}(\eta_1,u^\dagger),\check{\Gamma}(\eta_1,u^\dagger),\check{\omega}(\eta_1,u^\dagger),\check{\xi}(\eta_1,u^\dagger),\check{\Psi}(\eta_1,u^\dagger)\Big),$$

while $\check{u}_{\varepsilon,0}:\chi_0(\varepsilon D)L_c^2(\mathbb{R})\times Z_\mathrm{r}^0\to Y_\star^1$ is defined by

$$\begin{aligned}&\check{u}_{\varepsilon,0}(\zeta,\eta^\dagger,\rho^\dagger,\Phi^\dagger,\omega^\dagger,\xi^\dagger,\Psi_1^\dagger,\Psi_3^\dagger)\\ &\quad=\Big(\eta_1+\check{\eta}_2(\eta_1,u^\dagger),\check{\rho}(\eta_1,u^\dagger),\check{\Gamma}(\eta_1,u^\dagger),\check{\omega}(\eta_1,u^\dagger),\check{\xi}(\eta_1,u^\dagger),\check{\Psi}(\eta_1,u^\dagger)\Big)\end{aligned}$$

with $\Psi^\dagger=-(\Psi_1^\dagger)_x-(\Psi_3^\dagger)_y$; here

$$\eta_1=\varepsilon\zeta(\varepsilon x)\frac{\mathrm{e}^{\mathrm{i}\mu_0 x}}{2}+\varepsilon\overline{\zeta(\varepsilon x)}\frac{\mathrm{e}^{-\mathrm{i}\mu_0 x}}{2}$$

and $\check{\rho},\check{\xi},\check{\Psi},\check{\omega}$ are given by equations (4.19), (4.35), (4.36), (4.37) with $\Gamma=\check{\Gamma}(\eta_1+\check{\eta}_2(\eta_1,u^\dagger))$ and $\eta=\eta_1+\check{\eta}_2(\eta_1,u^\dagger)$. The theory in the sections above shows that

$$\eta_1^+(x)=\varepsilon\zeta(\varepsilon x)\frac{\mathrm{e}^{\mathrm{i}\mu_0 x}}{2},\qquad \zeta\in\begin{cases}\chi_0(\varepsilon D)L^2(\mathbb{R}), & k>0,\\ \chi_0(\varepsilon D)L_\mathrm{c}^2(\mathbb{R}), & k=0,\end{cases}$$

solves the reduced equation

$$g_\varepsilon(\mu,k\varepsilon)\hat{\eta}_1^+=\chi_0(\mu-\mu_0)\mathcal{F}\Big[\check{\mathcal{N}}(\eta_1,u^\dagger)\Big],$$

where $\eta_1=\eta_1^++\overline{\eta_1^+}$, if and only if $u=\check{u}_{\varepsilon,k}$ solves the resolvent equations (4.19)–(4.26). It follows that

$$\eta_1^+(x)=\varepsilon\zeta(\varepsilon x)\frac{\mathrm{e}^{\mathrm{i}\mu_0 x}}{2},\qquad \zeta\in\begin{cases}L^2(\mathbb{R}), & k>0,\\ L_\mathrm{c}^2(\mathbb{R}), & k=0\end{cases}$$

solves the equation

$$g_\varepsilon(\mu, k\varepsilon)\hat{\eta}_1^+ = \chi_0(\mu-\mu_0)\mathcal{F}\Big[\check{\mathcal{N}}(\chi(D)\eta_1, u^\dagger)\Big], \tag{4.59}$$

where $\eta_1 = \eta_1^+ + \overline{\eta_1^+}$, if and only if $\zeta \in \chi_0(\varepsilon D)L^2(\mathbb{R})$ (for $k > 0$) or $\zeta \in \chi_0(\varepsilon D)L^2_{\mathrm{c}}(\mathbb{R})$ (for $k = 0$) and $u = \check{u}_{\varepsilon,k}$ solves the resolvent equations (4.19)–(4.26).

Since all solutions of equation (4.59) have support in the interval $[\mu_0-\delta, \mu_0+\delta]$, its solution set in $L^2(\mathbb{R})$ coincides with its solution set in $H^s(\mathbb{R})$ for any $s \geq 0$, and we henceforth work in $H^4(\mathbb{R})$ by writing it as a second-order pseudodifferential equation. Let

$$\tilde{g}_\varepsilon(\mu, \lambda) = \varepsilon^2 + A_1(\mu-\mu_0)^2 + A_2\lambda^2$$

be the second-order Taylor polynomial of g_ε at the point $(\mu_0, 0)$. We note that

$$\chi_0(\varepsilon\tilde{\mu})\mathcal{F}\Big[\check{\mathcal{N}}^{(2)}(\chi(D)\eta_1)\Big](\mu_0+\varepsilon\tilde{\mu}) = 0$$

since

$$\operatorname{supp}\mathcal{F}\Big[\check{\mathcal{N}}^{(2)}(\chi(D)\eta_1)\Big] \cap [\mu_0-\delta, \mu_0+\delta] = \emptyset,$$

so that we can write (4.59) as

$$\tilde{g}_\varepsilon(\mu, \varepsilon k)\hat{\eta}_1^+ = \frac{\tilde{g}_\varepsilon(\mu, \varepsilon k)}{g_\varepsilon(\mu, \varepsilon k)}\chi_0(\mu-\mu_0)\mathcal{F}\Big[\check{\mathcal{N}}^{(3)}(\chi(D)\eta_1) + \check{\mathcal{N}}^{r}(\chi(D)\eta_1) + \check{\mathcal{N}}^{\dagger}(u^\dagger)\Big]$$

or equivalently as

$$\begin{aligned}
&\tilde{g}_\varepsilon(\mu_0+\varepsilon\tilde{\mu}, \varepsilon k)\hat{\zeta}(\tilde{\mu}) \\
&\quad = 2\frac{\tilde{g}_\varepsilon(\mu_0+\varepsilon\tilde{\mu}, \varepsilon k)}{g_\varepsilon(\mu_0+\varepsilon\tilde{\mu}, \varepsilon k)}\chi_0(\varepsilon\tilde{\mu})\mathcal{F}\Big[\check{\mathcal{N}}^{(3)}(\chi(D)\eta_1) + \check{\mathcal{N}}^{r}(\chi(D)\eta_1) + \check{\mathcal{N}}^{\dagger}(u^\dagger)\Big](\mu_0+\varepsilon\tilde{\mu}).
\end{aligned}$$

Taking the inverse Fourier transform with respect to $\tilde{\mu}$, we find that

$$\begin{aligned}
&A_2^{-1}\zeta - A_2^{-1}A_1\zeta_{xx} + k^2\zeta \\
&\quad = \mathcal{F}^{-1}\Bigg[2\frac{\tilde{g}_\varepsilon(\mu_0+\varepsilon\tilde{\mu}, \varepsilon k)}{g_\varepsilon(\mu_0+\varepsilon\tilde{\mu}, \varepsilon k)}\chi_0(\varepsilon\tilde{\mu})\varepsilon^{-2}A_2^{-1} \\
&\qquad\qquad \times \mathcal{F}\Big[\check{\mathcal{N}}^{(3)}(\chi(D)\eta_1) + \check{\mathcal{N}}^{r}(\chi(D)\eta_1)\Big](\mu_0+\varepsilon\tilde{\mu}) + \hat{\zeta}^{\dagger}_{\varepsilon,k}(u^\dagger)(\tilde{\mu})\Bigg],
\end{aligned} \tag{4.60}$$

where

$$\hat{\zeta}^{\dagger}_{\varepsilon,k}(u^\dagger)(\tilde{\mu}) = 2A_2^{-1}\frac{\tilde{g}_\varepsilon(\mu_0+\varepsilon\tilde{\mu}, \varepsilon k)}{g_\varepsilon(\mu_0+\varepsilon\tilde{\mu}, \varepsilon k)}\chi_0(\varepsilon\tilde{\mu})\varepsilon^{-2}\mathcal{F}\Big[\check{\mathcal{N}}^{\dagger}(u^\dagger)\Big](\mu_0+\varepsilon\tilde{\mu}).$$

We observe that

$$\begin{aligned}\check{\mathcal{N}}^{(3)}(\chi(D)\eta_1) &= \varepsilon^3(\zeta_\delta^\star(\varepsilon x))^2\Big((A_3+A_5)(\chi_0(\varepsilon D)\zeta)(\varepsilon x) + A_3\overline{(\chi_0(\varepsilon D)\zeta)(\varepsilon x)}\Big)\frac{\mathrm{e}^{\mathrm{i}\mu_0 x}}{2}\\ &\quad + \varepsilon^3(\zeta_\delta^\star(\varepsilon x))^2\Big(A_3(\chi_0(\varepsilon D)\zeta)(\varepsilon x) + (A_3+A_5)\overline{(\chi_0(\varepsilon D)\zeta)(\varepsilon x)}\Big)\frac{\mathrm{e}^{-\mathrm{i}\mu_0 x}}{2}\\ &\quad - \varepsilon^3 A_4\zeta_\delta^\star(\varepsilon x)(\chi_0(\varepsilon D)\vartheta_\delta)_x(\varepsilon x)\Big(\frac{1}{2}\mathrm{e}^{\mathrm{i}\mu_0 x} + \frac{1}{2}\mathrm{e}^{-\mathrm{i}\mu_0 x}\Big) + \varepsilon^2 T,\end{aligned}$$

and using the estimate

$$\begin{aligned}||(\chi_0(\varepsilon\tilde{\mu})-1)\hat{f}(\tilde{\mu})||_0^2 &= \int_{|\tilde{\mu}|\geq\delta/\varepsilon} |\hat{f}(\tilde{\mu})|^2\,\mathrm{d}\tilde{\mu}\\ &\leq \delta^{-2}\varepsilon^2\int_{|\tilde{\mu}|\geq\delta/\varepsilon}|\tilde{\mu}|^2|\hat{f}(\tilde{\mu})|^2\,\mathrm{d}\tilde{\mu}\\ &\leq \delta^{-2}\varepsilon^2||f||_1^2, \end{aligned}\tag{4.61}$$

we find that

$$\|((\chi_0(\varepsilon D)-1)\zeta)(\varepsilon x)\mathrm{e}^{\mathrm{i}\mu_0 x}\|_0^2 = \varepsilon^{-1}\|(\chi_0(\varepsilon\tilde{\mu})-1)\hat{\zeta}(\tilde{\mu})\|_0^2 \leq \delta^{-2}\varepsilon\|\zeta\|_1^2$$

and

$$\begin{aligned}\|((\chi_0(\varepsilon D)-1)\vartheta_\delta)_x(\varepsilon x)\mathrm{e}^{\mathrm{i}\mu_0 x}\|_0^2 &= \varepsilon^{-1}\alpha_0^{-1}M_2\left\|\frac{(\chi_0(\varepsilon\tilde{\mu})-1)\tilde{\mu}^2}{(1-\alpha_0^{-1})\tilde{\mu}^2+k^2}\mathcal{F}\Big[2\mathrm{Re}(\zeta_\delta^\star\zeta)\Big](\tilde{\mu})\right\|_0\\ &\lesssim \varepsilon^{-1}\Big\|(\chi_0(\varepsilon\tilde{\mu})-1)\mathcal{F}\Big[2\mathrm{Re}(\zeta_\delta^\star\zeta)\Big](\tilde{\mu})\Big\|_0\\ &\lesssim \varepsilon\|\zeta\|_1,\end{aligned}$$

so that

$$\check{\mathcal{N}}^{(3)}(\chi(D)\eta_1) = \frac{\varepsilon^3}{2}E(\varepsilon x)\mathrm{e}^{\mathrm{i}\mu_0 x} + \frac{\varepsilon^3}{2}F(\varepsilon x)\mathrm{e}^{-\mathrm{i}\mu_0 x} + \varepsilon^2 T, \tag{4.62}$$

where

$$\begin{aligned}E &= (A_3+A_5)(\zeta_\delta^\star)^2\zeta + A_3(\zeta_\delta^\star)^2\zeta - A_4\zeta_\delta^\star\vartheta_{\delta x},\\ F &= A_3(\zeta_\delta^\star)^2\chi_0(\varepsilon D)\zeta + (A_3+A_5)(\zeta_\delta^\star)^2\chi_0(\varepsilon D)\overline{\zeta} - A_4\zeta_\delta^\star(\chi_0(\varepsilon D)\vartheta_\delta)_x.\end{aligned}$$

Furthermore

$$\chi_0(\varepsilon\tilde{\mu})\mathcal{F}\Big[\varepsilon F(\varepsilon x)\mathrm{e}^{-\mathrm{i}\mu_0 x}\Big](\mu_0+\varepsilon\tilde{\mu}) = \chi_0(\varepsilon\tilde{\mu})\hat{F}(\tilde{\mu}+2\varepsilon^{-1}\mu_0) = 0$$

because $\operatorname{supp}\hat{F} \in [-3\varepsilon^{-1}\delta, 3\varepsilon^{-1}\delta]$.

Using the estimate

$$\left|\frac{\tilde{g}_\varepsilon(\mu,\lambda)}{g_\varepsilon(\mu,\lambda)}-1\right| \lesssim |(\mu-\mu_0,\lambda)|^2, \qquad |\mu-\mu_0|\leq\delta,\ \lambda<\lambda^\star, \tag{4.63}$$

we find in particular that

$$\left|\frac{\tilde{g}_\varepsilon(\mu_0+\varepsilon\tilde{\mu},\varepsilon k)}{g_\varepsilon(\mu_0+\varepsilon\tilde{\mu},\varepsilon k)}\chi_0(\varepsilon\tilde{\mu})\right| \lesssim 1.$$

It follows that

$$\begin{aligned}
&\left\|\mathcal{F}^{-1}\left[\frac{\tilde{g}_\varepsilon(\mu_0+\varepsilon\tilde{\mu},\varepsilon k)}{g_\varepsilon(\mu_0+\varepsilon\tilde{\mu},\varepsilon k)}\chi_0(\varepsilon\tilde{\mu})\varepsilon^{-2}\mathcal{F}\Big[\check{\mathcal{N}}^{\mathrm{r}}(\chi(D)\eta_1)\Big](\mu_0+\varepsilon\tilde{\mu})\right]\right\|_0\\
&\quad\lesssim \varepsilon^{-5/2}\Big\|\check{\mathcal{N}}^{\mathrm{r}}(\chi(D)\eta_1)\Big\|_0\\
&\quad\lesssim \varepsilon^{1/2}\Big\|\chi(D)\eta_1\Big\|_0\\
&\quad\lesssim \varepsilon||\zeta||_0,
\end{aligned}$$

where we have used Proposition 4.21, the identity

$$\int_{-\infty}^{\infty}\left|\hat{f}(\mu_0+\varepsilon\tilde{\mu})\right|^2\mathrm{d}\tilde{\mu}=\varepsilon^{-1}\Big\|\hat{f}\Big\|_0^2,$$

and

$$\left\|\mathcal{F}^{-1}\left[\frac{\tilde{g}_\varepsilon(\mu_0+\varepsilon\tilde{\mu},\varepsilon k)}{g_\varepsilon(\mu_0+\varepsilon\tilde{\mu},\varepsilon k)}\chi_0(\varepsilon\tilde{\mu})\varepsilon^{-2}\mathcal{F}[T](\mu_0+\varepsilon\tilde{\mu})\right]\right\|_0 \lesssim \varepsilon^{-1/2}\Big\|\chi_+(D)T\Big\|_0 \lesssim \varepsilon|\zeta|_1.$$

Finally we write

$$\begin{aligned}
&\mathcal{F}^{-1}\left[2\frac{\tilde{g}_\varepsilon(\mu_0+\varepsilon\tilde{\mu},\varepsilon k)}{g_\varepsilon(\mu_0+\varepsilon\tilde{\mu},\varepsilon k)}\chi_0(\varepsilon\tilde{\mu})\varepsilon^{-2}\mathcal{F}\left[\frac{\varepsilon^3}{2}E(\varepsilon x)\mathrm{e}^{\mathrm{i}\mu_0 x}\right](\mu_0+\varepsilon\tilde{\mu})\right]\\
&\quad=\mathcal{F}^{-1}\left[\left(\frac{\tilde{g}_\varepsilon(\mu_0+\varepsilon\tilde{\mu},\varepsilon k)}{g_\varepsilon(\mu_0+\varepsilon\tilde{\mu},\varepsilon k)}-1\right)\chi_0(\varepsilon\tilde{\mu})\hat{E}(\tilde{\mu})\right]\\
&\qquad\quad+\mathcal{F}^{-1}\Big[(\chi_0(\varepsilon\tilde{\mu})-1)\hat{E}(\tilde{\mu})\Big]+E(x)
\end{aligned}$$

and note that

$$\begin{aligned}
\left\|\mathcal{F}^{-1}\left[\left(\frac{\tilde{g}_\varepsilon(\mu_0+\varepsilon\tilde{\mu},\varepsilon k)}{g_\varepsilon(\mu_0+\varepsilon\tilde{\mu},\varepsilon k)}-1\right)\chi_0(\varepsilon\tilde{\mu})\hat{E}(\tilde{\mu})\right]\right\|_0 &\lesssim \varepsilon\delta\|E'\|+k^2\varepsilon^2\|E\|_0\\
&\lesssim \varepsilon\|\zeta\|_1
\end{aligned}$$

(see estimate (4.63)), and

$$\left\|\mathcal{F}^{-1}\left[(\chi_0(\varepsilon\tilde{\mu})-1)\hat{E}(\tilde{\mu})\right]\right\|_0 \lesssim \varepsilon\|E\|_1 \lesssim \varepsilon\|\zeta\|_1$$

(see estimate (4.61)).

According to the above calculations equation (4.60) can be written as

$$\begin{aligned}A_2^{-1}\zeta - A_1A_2^{-1}\zeta_{xx} + k^2\zeta &= A_2^{-1}(A_3+A_5)(\zeta_\delta^\star)^2\zeta + A_2^{-1}A_3(\zeta_\delta^\star)^2\overline{\zeta}\\ &\qquad - A_2^{-1}A_4\zeta_\delta^\star\vartheta_{\delta x} + \tilde{\mathcal{R}}_{\varepsilon,k}(\zeta) + \zeta_{\varepsilon,k}^\dagger(u^\dagger)\\ &= A_2^{-1}(A_3+A_5)(\zeta^\star)^2\zeta + A_2^{-1}A_3(\zeta^\star)^2\overline{\zeta}\\ &\qquad - A_2^{-1}A_4\zeta^\star\vartheta_x + \mathcal{R}_{\varepsilon,k}(\zeta) + \zeta_{\varepsilon,k}^\dagger(u^\dagger),\end{aligned}$$

where

$$\vartheta = \frac{A_4}{4}\mathcal{F}^{-1}\left[\frac{\mathrm{i}\mu}{(1-\alpha_0^{-1})\mu^2+k^2}\mathcal{F}\left[2\mathrm{Re}(\zeta^\star\zeta)\right]\right]$$

and

$$\left\|\tilde{\mathcal{R}}_{\varepsilon,k}(\zeta)\right\|_0, \left\|\mathcal{R}_{\varepsilon,k}(\zeta)\right\|_0 \lesssim \varepsilon\|\zeta\|_1.$$

4.3 Spectral theory for the reduced equation

We now turn our attention to the solvability of the reduced equation

$$\begin{aligned}&A_2^{-1}\zeta - A_2^{-1}A_1\zeta_{xx} - A_2^{-1}(A_3+A_5)(\zeta^\star)^2\zeta\\ &\quad - A_2^{-1}A_3(\zeta^\star)^2\overline{\zeta} + A_2^{-1}A_4\zeta^\star\vartheta_x + k^2\zeta - \mathcal{R}_{\varepsilon,k}(\zeta) = \zeta^\dagger\end{aligned}\tag{4.64}$$

for arbitrary $\zeta^\dagger \in L^2(\mathbb{R})$ (for $k>0$) or $\zeta^\dagger \in L^2_c(\mathbb{R})$ (for $k=0$). Recall that $\lambda \in \rho(L)$ for $\lambda > \varepsilon k_{\max}$ and the invertibility of $L-\varepsilon kI$ for $k<k_{\max}$ is determined by the solvability of (4.64). For $k>0$ we also note that each step in the reduction procedure preserves the invariance of the resolvent equations under the reflection S; in the present coordinates the action of this symmetry is given by $\zeta(x)\mapsto\overline{\zeta(-x)}$, so that (4.64) is invariant under this transformation.

The results in this section are quoted from Section 4.1 of Groves, Sun and Wahlén [27].

4.3.1 Invertibility of the reduced operator

In this section we reformulate the reduced equation (4.64) once more, reducing the question of its solvability to determining the location of an eigenvalue of an unbounded linear operator.

We first present an auxiliary spectral result, one consequence of which is the solvability of (4.64) for $k = 0$.

Proposition 4.34. *The formulae*

$$\mathcal{C}_{0,1}(\zeta_1) = A_2^{-1}\Big(\zeta_1 - A_1\zeta_{1xx} - 6\operatorname{sech}^2(A_1^{-1/2}x)\zeta_1\Big),$$
$$\mathcal{C}_{0,2}(\zeta_2) = A_2^{-1}\Big(\zeta_2 - A_1\zeta_{2xx} - 2\operatorname{sech}^2(A_1^{-1/2}x)\zeta_2\Big)$$

define

(i) self-adjoint operators $\mathcal{C}_{0,1}, \mathcal{C}_{0,2} : H^2(\mathbb{R}) \subseteq L^2(\mathbb{R}) \to L^2(\mathbb{R})$ whose spectrum consists of essential spectrum $[A_2^{-1}, \infty)$ and respectively two simple eigenvalues at $-3A_2^{-1}, 0$ and a simple eigenvalue at 0,

(ii) self-adjoint operators $\mathcal{C}_{0,1} : H^2_{\mathrm{e}}(\mathbb{R}) \subseteq L^2_{\mathrm{e}}(\mathbb{R}) \to L^2_{\mathrm{e}}(\mathbb{R}), \mathcal{C}_{0,2} : H_{\mathrm{o}}(\mathbb{R}) \subseteq L^2_{\mathrm{o}}(\mathbb{R}) \to L^2_{\mathrm{e}}(\mathbb{R})$ whose spectrum consists of essential spectrum $[A_2^{-1}, \infty)$ and, in the case of $\mathcal{C}_{0,1}$, a simple eigenvalue at $-3A_2^{-1}$.

Lemma 4.35. *Suppose that $k = 0$. The reduced equation* (4.64) *has a unique solution $\zeta \in H^2_{\mathrm{c}}(\mathbb{R})$ for each $\zeta^\dagger \in L^2_{\mathrm{c}}(\mathbb{R})$.*

Let W be the real Hilbert space $L^2(\mathbb{R}) \times L^2(\mathbb{R}) \times L^2(\mathbb{R})$ equipped with the inner product

$$\langle\!\langle (\zeta_1, \zeta_2, \psi), (\tilde{\zeta}_1, \tilde{\zeta}_2, \tilde{\psi}) \rangle\!\rangle = \langle \zeta_1, \tilde{\zeta}_1 \rangle_0 + \langle \zeta_2, \tilde{\zeta}_2 \rangle_0 + \frac{\alpha_0}{2A_2}\langle \psi, \tilde{\psi} \rangle_0$$

and define the unbounded operators $\mathcal{B}_{\varepsilon,k}, \tilde{\mathcal{B}} : \mathcal{D}_{\mathcal{B}} \subseteq W \to W$ by the formulae

$$\tilde{\mathcal{B}}_{\varepsilon,k}\begin{pmatrix}\zeta_1\\ \zeta_2\\ \psi\end{pmatrix}$$
$$= \begin{pmatrix} A_2^{-1}\zeta_1 - A_2^{-1}A_1\zeta_{1xx} - A_2^{-1}(2A_3 + A_5)(\zeta^\star)^2\zeta_1 + A_2^{-1}A_4\zeta^\star\psi_x - \operatorname{Re}\mathcal{R}_{\varepsilon,k}(\zeta) \\ A_2^{-1}\zeta_2 - A_2^{-1}A_1\zeta_{2xx} - A_2^{-1}A_5(\zeta^\star)^2\zeta_2 - \operatorname{Im}\mathcal{R}_{\varepsilon,k}(\zeta) \\ -(1-\alpha_0^{-1})\psi_{xx} - \frac{A_4}{2}[\zeta^\star\zeta_1]_x \end{pmatrix}$$

and

$$\mathcal{B}_{\varepsilon,k} = \tilde{\mathcal{B}}_{\varepsilon,k} + k^2 I,$$

where $\mathcal{D}_{\mathcal{B}} := H^2(\mathbb{R}) \times H^2(\mathbb{R}) \times H^2(\mathbb{R})$. In this framework equation (4.64) may be written as

$$\mathcal{B}_{\varepsilon,k}(\zeta_1, \zeta_2, \vartheta) = \Big(\operatorname{Re}\zeta^\dagger, \operatorname{Im}\zeta^\dagger, 0\Big),$$

where $\zeta_1 = \operatorname{Re}\zeta, \zeta_2 = \operatorname{Im}\zeta$; recall that this equation is invariant under the transformation $\tilde{S} : (\zeta_1(x), \zeta_2(x), \psi(x)) = (\zeta_1(-x), -\zeta_2(-x), -\psi(-x))$ (the action of the reflection S).

The spectra of the self-adjoint operators $\mathcal{B}_{0,k}$ and $\tilde{\mathcal{B}}_{0,k}$ (the second of which does not depend upon k) can be determined precisely.

Lemma 4.36. *The spectrum of the operator* $\tilde{\mathcal{B}}_{0,k}$ *consists of essential spectrum* $[0,\infty)$ *and a simple negative eigenvalue* $-k_0^2$ *whose eigenspace lies in* $\operatorname{Fix}\tilde{S} = L^2_{\mathrm{e}}(\mathbb{R}) \times L^2_{\mathrm{o}}(\mathbb{R}) \times L^2_{\mathrm{o}}(\mathbb{R})$.

Lemma 4.37. *Let* m *and* M *be positive real numbers with* $m < k_0^2 < M$ *and* γ *be an ellipse in the complex plane with major axis* $[-M,-m]$. *For each* $k \in [0, k_{\max}]$ *the portion of the spectrum of* $\tilde{\mathcal{B}}_{\varepsilon,k}$ *within* γ *consists of precisely one simple real eigenvalue* $\kappa_{\varepsilon,k}$ *with* $\kappa_{0,k} = -k_0^2$; *its eigenspace lies in* $\operatorname{Fix}\tilde{S}$. *In particular,* $\tilde{\mathcal{B}}_{\varepsilon,k}$ *is closed and* $\tilde{\mathcal{B}}_{\varepsilon,k} - \kappa I : \mathcal{D}_{\mathcal{B}} \to W$ *is Fredholm with index* 0 *for* $k \in [0, k_{\max}]$ *and* $\kappa \in [-M,-m]$.

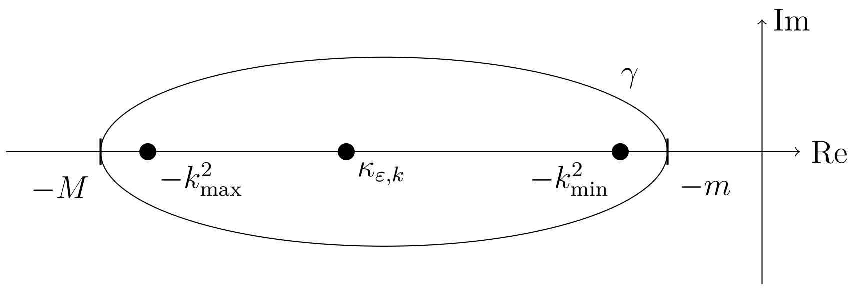

Figure 4.1: The portion of $\sigma(\tilde{\mathcal{B}}_{\varepsilon,k})$ within γ consists of precisely one simple real eigenvalue $\kappa_{\varepsilon,k} \in [-k_{\max}^2, -k_{\min}^2]$ with $\kappa_{0,k} = -k_0^2$.

Lemma 4.38. *The spectral projection* $P_{\varepsilon,k} \in \mathcal{L}(W, \mathcal{D}_{\mathcal{B}})$ *defined by*

$$P_{\varepsilon,k} w = \frac{1}{2\pi \mathrm{i}} \int_\gamma (\kappa I - \tilde{\mathcal{B}}_{\varepsilon,k})^{-1} w \, \mathrm{d}\kappa$$

satisfies the estimate

$$\|P_{\varepsilon,k} - P_{0,k}\|_{\mathcal{L}(W,\mathcal{D}_{\mathcal{B}})} \lesssim \varepsilon$$

uniformly in $k \in [0, k_{\max}]$.

Finally, choose $k_{\max} > k_0$ and $k_{\min} \in (0, k_0)$ and apply Lemma 4.37, choosing m, M such that $M > k_{\max}^2$ and $m < k_{\min}^2$, so that the point $-k^2$ lies in the interior of γ for all $k \in [k_{\min}, k_{\max}]$ (see Figure 4.1). The following lemma follows from the relation $\mathcal{B}_{\varepsilon,k} = \tilde{\mathcal{B}}_{\varepsilon,k} + k^2 I$.

Lemma 4.39. *Fix* $k \in [k_{\min}, k_{\max}]$.

(i) The operator $\mathcal{B}_{\varepsilon,k} : \mathcal{D}_{\mathcal{B}} \subseteq W \to W$ *is invertible provided that* $\kappa_{\varepsilon,k} + k^2 \neq 0$.

(ii) The operator $\mathcal{B}_{\varepsilon,k} : \mathcal{D}_{\mathcal{B}} \subseteq W \to W$ *has a simple zero eigenvalue whenever* $\kappa_{\varepsilon,k} + k^2 = 0$.

In Section 4.3.3 we show that the equation $\kappa_{\varepsilon,k} + k^2 = 0$ has precisely one solution k_ε in the interval $[k_{\min}, k_{\max}]$ (see Figure 4.1).

4.3.2 Small spectral values

In this section we prove that $\mathcal{B}_{\varepsilon,k}\big|_{\mathrm{Fix}\,\tilde{S}}$ is semi-Fredholm and injective for $k \in (0, k_{\min})$. Because this operator is closed (see Lemma 4.37) it suffices to establish the *a priori* estimate

$$\|w\|_W \lesssim k^{-2}\|\mathcal{B}_{\varepsilon,k}w\|_W, \qquad k \in (0, k_{\min}),$$

for sufficiently small values of $k_{\min}$. For notational simplicity we abbreviate $\mathrm{Fix}\,\tilde{S}$ and $\mathcal{D}_\mathcal{B} \cap \mathrm{Fix}\,\tilde{S}$ to respectively W and $\mathcal{D}_\mathcal{B}$, so that $W = L^2_\mathrm{e}(\mathbb{R}) \times L^2_\mathrm{o}(\mathbb{R}) \times L^2_\mathrm{o}(\mathbb{R})$ and $\mathcal{D}_\mathcal{B} = H^2_\mathrm{e}(\mathbb{R}) \times H^2_\mathrm{o}(\mathbb{R}) \times H^2_\mathrm{o}(\mathbb{R})$.

The first step is to choose an eigenvector corresponding to the eigenvalue $\kappa_{\varepsilon,k}$ of $\mathcal{B}_{\varepsilon,k}$ in a perturbative fashion.

Lemma 4.40. *There exists an eigenvector $w_{\varepsilon,k} = (\zeta_{\varepsilon,k}, \psi_{\varepsilon,k})$ of $\tilde{\mathcal{B}}_{\varepsilon,k}$ corresponding to the eigenvalue $\kappa_{\varepsilon,k}$, such that $\|w_{\varepsilon,k}\|_W = 1$ and $\|w_{\varepsilon,k} - w_{0,k}\|_{\mathcal{D}_\mathcal{B}} \lesssim \varepsilon$. Furthermore, there exists a primitive $\Psi_{\varepsilon,k} \in H^3_\mathrm{e}(\mathbb{R})$ of $\psi_{\varepsilon,k}$ with $\|\Psi_{\varepsilon,k} - \Psi_{0,k}\|_0 \lesssim \varepsilon$.*

We now define the auxiliary operator $\mathcal{A}_{\varepsilon,k} : \mathcal{D}_\mathcal{A} \subseteq V \to V$ by the formula

$$\mathcal{A}_{\varepsilon,k}\begin{pmatrix}\zeta_1\\ \zeta_2\\ \phi\end{pmatrix} = \begin{pmatrix} A_2^{-1}\zeta_1 - A_2^{-1}A_1\zeta_{1xx} - A_2^{-1}(2A_3 + A_5)(\zeta^\star)^2\zeta_1 + A_2^{-1}A_4\zeta^\star\phi - \operatorname{Re}\mathcal{R}_{\varepsilon,k}(\zeta) \\ A_2^{-1}\zeta_2 - A_2^{-1}A_1\zeta_{2xx} - A_2^{-1}A_5(\zeta^\star)^2\zeta_2 - \operatorname{Im}\mathcal{R}_{\varepsilon,k}(\zeta) \\ (1-\alpha_0^{-1})\phi + \frac{A_4}{2}\zeta^\star\zeta_1 \end{pmatrix},$$

where $V = L^2_\mathrm{e}(\mathbb{R}) \times L^2_\mathrm{o}(\mathbb{R}) \times L^2_\mathrm{e}(\mathbb{R})$ and $\mathcal{D}_\mathcal{A} = H^2_\mathrm{e}(\mathbb{R}) \times H^2_\mathrm{o}(\mathbb{R}) \times L^2_\mathrm{e}(\mathbb{R})$. Observe that $\mathcal{A}_{0,k}$ is self-adjoint with respect to the inner product $\langle\!\langle \cdot, \cdot \rangle\!\rangle$ for V and its spectrum consists of essential spectrum $\{1 - \alpha_0^{-1}\} \cup [A_2^{-1}, \infty)$ and a simple negative eigenvalue $-\omega_1^2$ (the point $1 - \alpha_0^{-1}$ is an eigenvalue with infinite-dimensional eigenspace $\{(0,0)\} \times L^2_\mathrm{e}(\mathbb{R})$) (see Groves, Sun and Wahlén [27, p. 44]). Below we write $v = (\zeta, \psi_x) \in \mathcal{D}_\mathcal{A}$ for $w = (\zeta, \psi) \in \mathcal{D}_\mathcal{B}$ and in particular set $v_{\varepsilon,k} = (\zeta_{\varepsilon,k}, \psi_{\varepsilon,k,x})$.

Lemma 4.41. *The operator $\mathcal{A}_{0,k} : \mathcal{D}_\mathcal{A} \subseteq V \to V$ satisfies*

$$\langle\!\langle \mathcal{A}_{0,k}v_{0,k}, v_{0,k} \rangle\!\rangle = -k_0^2, \quad \langle\!\langle \mathcal{A}_{0,k}v_{0,k}, v \rangle\!\rangle = 0, \quad \langle\!\langle \mathcal{A}_{0,k}v, v \rangle\!\rangle \gtrsim \|v\|_V^2$$

for each $w \in \mathcal{D}_\mathcal{B}$ with $\langle\!\langle w, w_{0,k} \rangle\!\rangle = 0$.

The *a priori* estimate announced at the start of this section (Theorem 4.43) is a corollary of the following lemma.

Lemma 4.42. *The operator $\mathcal{B}_{\varepsilon,k} : \mathcal{D}_{\mathcal{B}} \subseteq W \to W$ satisfies the estimate*

$$\langle\!\langle \mathcal{B}_{\varepsilon,k} w \rangle\!\rangle \gtrsim k^2 \|w\|_1^2, \quad k \leq k_{\min}$$

for each $w \in \mathcal{D}_{\mathcal{B}}$ such that $\langle\!\langle w, w_{\varepsilon,k} \rangle\!\rangle = 0$ and each sufficiently small value of $k_{\min}$.

Theorem 4.43. *Choose $k \in (0, k_{\min}]$ and $w^\dagger \in W$. Any solution $w \in \mathcal{D}_{\mathcal{B}}$ of the equation*

$$\mathcal{B}_{\varepsilon,k} w = w^\dagger$$

satisfies the estimate

$$\|w\|_W \lesssim k^{-2} \|w^\dagger\|_W.$$

4.3.3 Solution of the eigenvalue equation

In this section we determine the solution set of the equation

$$\kappa_{\varepsilon,k} + k^2 = 0 \tag{4.65}$$

in the interval $[k_{\min}, k_{\max}]$. The solution set is known explicitly for $\varepsilon = 0$, and we approach the problem for $\varepsilon > 0$ using a perturbation argument. The estimate

$$\|\tilde{\mathcal{B}}_{\varepsilon,k} - \tilde{\mathcal{B}}_{0,k}\|_{\mathcal{L}(\mathcal{D}_{\mathcal{B}}, W)} \lesssim \varepsilon \tag{4.66}$$

yields the necessary estimate for $\kappa_{\varepsilon,k}$, and to obtain the corresponding estimate for $\kappa_{\varepsilon,k}'$ we show that the linear mapping $\mathcal{R}_{\varepsilon,k} : H^2(\mathbb{R}) \to L^2(\mathbb{R})$ is a continuously differentiable function of $k > 0$ which satisfies the estimate

$$\|\mathcal{R}_{\varepsilon,k}(\zeta)'\|_0 \lesssim \varepsilon$$

uniformly over $k \in [k_{\min}, k_{\max}]$ (which implies the corresponding result for $\tilde{\mathcal{B}}_{\varepsilon,k}$). This property of $\mathcal{R}_{\varepsilon,k}$ is established by systematically examining each step in its derivation in Section 4.2 and checking the continuous differentiability of the operators appearing there.

Proposition 4.44. *The linear mappings $\tilde{\Gamma}(\cdot, 0), \tilde{\Gamma}^{(j)}, j = 1, 2, 3,$ and $\tilde{\Gamma}^{\mathrm{r}} : H^2(\mathbb{R}) \to H^2(\Sigma)$ are continuously differentiable functions of $k > 0$.*

Corollary 4.45. *The linear mappings $\tilde{\mathcal{N}}(\cdot, 0), \tilde{\mathcal{N}}^{(2)}, \tilde{\mathcal{N}}^{(3)}$ and $\tilde{\mathcal{N}}^{r} : H^4(\mathbb{R}) \to L^2(\mathbb{R})$ are continuously differentiable.*

Lemma 4.46. *The linear mappings $\check{\eta}_2(\cdot, 0), \check{\eta}_2^{(2)}$ and $\check{\eta}_2^{\mathrm{r}} : \chi(D) L^2(\mathbb{R}) \to L^2(\mathbb{R})$ are continuously differentiable functions of $k > 0$.*

Proof. Recall that

$$\check{\eta}_2(\eta_1,0)=\Big(g_\varepsilon(D,\varepsilon k)-(1-\chi(D))\tilde{\mathcal{N}}(\cdot)\Big)^{-1}(1-\chi(D))\tilde{\mathcal{N}}(\eta_1,0).$$

The mapping $(1-\chi(D))H^4(\mathbb{R})\to(1-\chi(D))L^2(\mathbb{R})$ defined by

$$k\mapsto g_\varepsilon(D,\varepsilon k)-(1-\chi(D))\tilde{\mathcal{N}}(\cdot)$$

is continuously differentiable and invertible (Lemma 4.16) . It follows that its inverse and hence $\check{\eta}_2(\eta_1)$ is continuously differentiable with respect to k. The continuous differentiability of $k\mapsto\check{\eta}_2^{(2)}(\cdot)$ follows from that of $k\mapsto g_\varepsilon(D,\varepsilon k)$, and the continuous differentiability of $k\mapsto\check{\eta}_2^{\mathrm{r}}(\cdot)$ is a consequence of the formula $\check{\eta}_2^{\mathrm{r}}(\cdot)(\eta_1)=\check{\eta}_2(\eta_1,0)-\check{\eta}_2^{(2)}(\eta_1)$. ∎

Corollary 4.47.

(i) The linear mappings $\check{\Gamma}^{(j)}, j=1,2,3$ and $\check{\Gamma}^{\mathrm{r}}:\chi(D)L^2(\mathbb{R})\to H^2(\Sigma)$ are continuously differentiable functions of $k>0$.

(ii) The linear mappings $\check{\mathcal{N}}^{(2)},\check{\mathcal{N}}^{(3)}$ and $\check{\mathcal{N}}^{\mathrm{r}}:\chi(D)L^2(\mathbb{R})\to L^2(\mathbb{R})$ are continuously differentiable functions of $k>0$.

Corollary 4.48. *The linear mappings $\tilde{\mathcal{R}}_{\varepsilon,k},\mathcal{R}_{\varepsilon,k}:H^2(\mathbb{R})\to L^2(\mathbb{R})$ are continuously differentiable functions of $k>0$.*

Having established the continuous differentiability of $\tilde{\mathcal{R}}_{\varepsilon,k}(\zeta)$ and $\mathcal{R}_{\varepsilon,k}(\zeta)$, we now turn to the task of estimating their derivatives, again by systematically examining each step in their derivation. The following result, which is established in the same way as Proposition 4.12, is used in the first step.

Proposition 4.49. *The mapping $(P_1,P_2,P_3,P_4,P_5)\mapsto\mathcal{H}(P_1,P_2,P_3,P_4,P_5)$, where*

$$\begin{aligned}&\mathcal{H}(P_1,P_2,P_3,P_4,P_5)\\&\quad=\mathcal{F}^{-1}\left[\int_0^1\Big\{G(y,\tilde{y})\Big(-2\varepsilon^2k\hat{P}_1-\mathrm{i}\mu\hat{P}_2-\mathrm{i}\varepsilon\hat{P}_3-\mathrm{i}k\varepsilon\hat{P}_4\Big)+G_{\tilde{y}}(y,\tilde{y})\hat{P}_5\Big\}\,\mathrm{d}\tilde{y}\right]\end{aligned}$$

defines a linear function $H^1(\Sigma)\times H^1(\Sigma)\times H^1(\Sigma)\times H^1(\Sigma)\times H^1(\Sigma)\to H^2(\Sigma)$ which satisfies the estimate

$$\|\nabla\mathcal{H}\|_1+\varepsilon\|\mathcal{H}\|_1+\varepsilon^2\|\mathcal{H}\|_0\lesssim\varepsilon\|P_1\|_1+\sum_{j=2}^5\|P_j\|_1+\varepsilon^2\|P_1\|_0+\varepsilon\sum_{j=2}^5\|P_j\|_0.$$

Lemma 4.50. *The linear mappings $\tilde{\Gamma}(\cdot,0), \tilde{\Gamma}^{(j)}, j=1,2,3,$ and $\tilde{\Gamma}^{\mathrm{r}}: H^2(\mathbb{R}) \to H^2(\Sigma)$ satisfy the estimates*

$$\begin{aligned}\|\nabla\tilde{\Gamma}(\eta,0)'\|_1 + \varepsilon\|\tilde{\Gamma}(\eta,0)'\|_0 &\lesssim \|\eta\|_2,\\ \|\nabla\tilde{\Gamma}^{(j)}(\eta)'\|_1 + \varepsilon\|\tilde{\Gamma}^{(j)}(\eta)'\|_0 &\lesssim \varepsilon^{j-1}\|\eta\|_2, \quad j=1,2,3,\\ \|\nabla\tilde{\Gamma}^{\mathrm{r}}(\eta)'\|_1 + \varepsilon\|\tilde{\Gamma}^{\mathrm{r}}(\eta)'\|_0 &\lesssim \varepsilon^3\|\eta\|_2\end{aligned}$$

uniformly over $k \in [k_{\min}, k_{\max}]$.

Corollary 4.51. *The linear mappings $\tilde{\mathcal{N}}(\cdot,0), \tilde{\mathcal{N}}^{(2)}, \tilde{\mathcal{N}}^{(3)}$ and $\tilde{\mathcal{N}}^{\mathrm{r}}: H^4(\mathbb{R}) \to L^2(\mathbb{R})$ satisfy the estimates*

$$\begin{aligned}\|\tilde{\mathcal{N}}(\eta,0)'\|_0 &\lesssim \varepsilon\|\eta\|_4,\\ \|\tilde{\mathcal{N}}^{(j)}(\eta)'\|_0 &\lesssim \varepsilon^{j-1}\|\eta\|_4, \quad j=2,3,\\ \|\tilde{\mathcal{N}}^{\mathrm{r}}(\eta)'\|_0 &\lesssim \varepsilon^3\|\eta\|_4\end{aligned}$$

uniformly over $k \in [k_{\min}, k_{\max}]$.

The next estimates are obtained by differentiating the formulae defining $\check{\eta}_2(\eta_1,0), \check{\eta}_2^{(2)}$ and $\check{\eta}_2^{\mathrm{r}}$ with respect to k and using Lemma 4.19, Proposition 4.20(i) and Corollary 4.51 and the fact that $|\partial_k g_\varepsilon(\mu,\varepsilon,k)| \leq c$ for $k \in [k_{\min}, k_{\max}]$.

Lemma 4.52. *The linear mappings $\check{\eta}_2(\cdot,0), \check{\eta}_2^{(2)}$ and $\check{\eta}_2^{\mathrm{r}}: \chi(D)L^2(\mathbb{R}) \to L^2(\mathbb{R})$ satisfy the estimates*

$$\begin{aligned}\|\check{\eta}_2(\eta_1,0)'\|_4 &\lesssim \varepsilon\|\eta_1\|_0,\\ \|\check{\eta}_2^{(2)}(\eta_1)'\|_4 &\lesssim \varepsilon\|\eta_1\|_0,\\ \|\check{\eta}_2^{\mathrm{r}}(\eta_1)'\|_4 &\lesssim \varepsilon^2\|\eta_1\|_0\end{aligned}$$

uniformly over $k \in [k_{\min}, k_{\max}]$.

Corollary 4.53.

(i) The linear mappings $\check{\Gamma}^{(j)}, j=1,2,3$ and $\check{\Gamma}^{\mathrm{r}}: \chi(D)L^2(\mathbb{R}) \to H^2(\Sigma)$ satisfy the estimates

$$\begin{aligned}\|\nabla\check{\Gamma}^{(j)}(\eta_1)'\|_1 + \varepsilon\|\check{\Gamma}^{(j)}(\eta_1)'\|_0 &\lesssim \varepsilon^{j-1}\|\eta_1\|_0, \qquad j=1,2,3,\\ \|\nabla\check{\Gamma}^{\mathrm{r}}(\eta_1)'\|_1 + \varepsilon\|\check{\Gamma}^{\mathrm{r}}(\eta_1)'\|_0 &\lesssim \varepsilon^3\|\eta_1\|_0\end{aligned}$$

uniformly over $k \in [k_{\min}, k_{\max}]$.

(ii) The linear mappings $\check{\mathcal{N}}^{(2)}, \check{\mathcal{N}}^{(3)}$ *and* $\tilde{\mathcal{N}}^{\mathrm{r}} : \chi(D)L^2(\mathbb{R}) \to L^2(\mathbb{R})$ *satisfy the estimates*

$$\begin{aligned}\|\check{\mathcal{N}}^{(j)}(\eta_1)'\|_0 &\lesssim \varepsilon^{j-1}\|\eta_1\|_0, \qquad j=2,3,\\ \|\check{\mathcal{N}}^{\mathrm{r}}(\eta_1)'\|_0 &\lesssim \varepsilon^3\|\eta_1\|_0\end{aligned}$$

uniformly over $k \in [k_{\min}, k_{\max}]$.

An estimate for $\check{\mathcal{N}}^{(2)}(\eta_1)'$ as a function of ζ is given in Lemma 4.55. This result is established by differentiating (4.51) and using Propositions 4.25, 4.27 and 4.54 below, which is derived from the formulae

$$\check{\Gamma}^{(1)\prime} = \mathcal{F}^{-1}\left[\int_0^1 G(y,\tilde{y})(-2\varepsilon^2 k\mathcal{F}[\check{\Gamma}^{(1)}])\,\mathrm{d}\tilde{y}\right]$$

and

$$\begin{aligned}\tilde{\Gamma}^{(2)}(\eta_1)' = \mathcal{F}^{-1}\Bigg[\int_0^1 \Big\{ G(y,\tilde{y})\Big(-2\varepsilon^2 k\mathcal{F}[\tilde{\Gamma}^{(2)}(\eta_1)] - \mathrm{i}\mu\mathcal{F}[\check{F}_1^{(2)\prime}] - \mathrm{i}\varepsilon\mathcal{F}[\check{F}_2^{(2)}] - \mathrm{i}k\varepsilon\mathcal{F}[\check{F}_2^{(2)\prime}]\Big)\\ + G_{\tilde{y}}(y,\tilde{y})\mathcal{F}[\check{F}_3^{(2)\prime}]\Big\}\,\mathrm{d}\tilde{y}\Bigg]\end{aligned}$$

by the methods used in Section 4.2.3.

Proposition 4.54. *The quantities* $\check{\Gamma}^{(1)\prime}$ *and* $\tilde{\Gamma}^{(2)}(\eta_1)'$ *satisfy the estimates*

$$\begin{aligned}\check{\Gamma}^{(1)\prime} &= S,\\ \tilde{\Gamma}^{(2)}(\eta_1)' &= -2\varepsilon\mu_0\coth(\mu_0)\mathcal{F}^{-1}\left[\frac{\mathrm{i}\mu k}{(\mu^2+k^2)^2}\mathcal{F}[\mathrm{Re}(\zeta^\star\zeta)](\mu)\right](\varepsilon x) + \varepsilon S\end{aligned}$$

uniformly over $k \in [k_{\min}, k_{\max}]$.

Lemma 4.55. *The quantity* $\check{\mathcal{N}}^{(2)}(\eta_1)'$ *satisfies the estimate*

$$\check{\mathcal{N}}^{(2)}(\eta_1)' = 2\varepsilon^2\mu_0\coth(\mu_0)\mathcal{F}^{-1}\left[\frac{\mu^2 k}{(\mu^2+k^2)^2}\mathcal{F}[\mathrm{Re}(\zeta_\delta^\star\zeta)](\mu)\right](\varepsilon x) + \varepsilon R_0$$

uniformly over $k \in [k_{\min}, k_{\max}]$.

Lemma 4.56. *The quantity* $\check{\eta}_2^{(2)\prime}$ *satisfies the estimate*

$$\check{\eta}_2^{(2)\prime} = 2\varepsilon^2 B_3\mathcal{F}^{-1}\left[\frac{\mu^2 k}{((1-\alpha_0^{-1})\mu^2+k^2)^2}\mathcal{F}[2\mathrm{Re}(\zeta_\delta^\star\zeta)](\mu)\right](\varepsilon x) + \varepsilon R_4$$

uniformly over $k \in [k_{\min}, k_{\max}]$.

Proof. Differentiating the formula

$$g_\varepsilon(\mu,\varepsilon k)\mathcal{F}[\check{\eta}^{(2)}] = (1-\chi)\mathcal{F}[\check{\mathcal{N}}^{(2)}(\eta_1)] = \mathcal{F}[\check{\mathcal{N}}^{(2)}(\eta_1)]$$

with respect to k yields

$$\begin{aligned} g_\varepsilon(\mu,\varepsilon k)\mathcal{F}[\check{\eta}_2^{(2)\prime}] &= \mathcal{F}[\check{\mathcal{N}}^{(2)}(\eta_1)'] - \partial_k g_\varepsilon(\mu,\varepsilon k)\mathcal{F}[\check{\eta}^{(2)}(\eta_1)] \\ &= 2\varepsilon\frac{(\varepsilon^{-1}\mu)^2 k}{((\varepsilon^{-1}\mu)^2+k^2)^2}\mu_0\coth(\mu_0)\mathcal{F}[\mathrm{Re}(\zeta_\delta^\star\zeta)](\varepsilon^{-1}\mu) \\ &\quad + \varepsilon\partial_k g_\varepsilon(\mu,\varepsilon k)\bigg(\frac{1}{2}B_1\mathcal{F}[\zeta_\delta^\star\zeta](\varepsilon^{-1}(\mu-2\mu_0)) \\ &\qquad\qquad + \frac{1}{2}B_1\mathcal{F}[\zeta_\delta^\star\overline{\zeta}](\varepsilon^{-1}(\mu+2\mu_0))\bigg) \\ &\quad + \varepsilon\partial_k g_\varepsilon(\mu,\varepsilon k)\left(B_2 + B_3\frac{2(\varepsilon^{-1}\mu)^2}{(1-\alpha_0^{-1})(\varepsilon^{-1}\mu)^2+k^2}\right)\mathcal{F}[\mathrm{Re}(\zeta_\delta^\star\zeta)](\varepsilon^{-1}\mu) \\ &\quad + \varepsilon(1-\chi)\hat{R}_0, \end{aligned}$$

in which we have used Proposition 4.29 and Lemma 4.55. Noting that

$$\partial_k g_\varepsilon(\mu,\varepsilon k) = \frac{2k(\varepsilon^{-1}\mu)^2}{((\varepsilon^{-1}\mu)^2+k^2)^2} + O(\varepsilon)$$

uniformly over $\{|\mu|\in[0,\mu_{\max}]\}$ for each fixed $\mu_{\max}>0$ (in particular for $\mu_{\max}=2\delta$), whence $\partial_k g_\varepsilon(\mu,\varepsilon k) = O(\varepsilon)$ uniformly over $\{|\mu|\in[\mu_{\min},\mu_{\max}]\}$ for each fixed $\mu_{\min},\mu_{\max}>0$ (in particular for $\mu_{\min}=2\mu_0-2\delta$, $\mu_{\max}=2\mu_0+2\delta$), we find that

$$\begin{aligned} g_\varepsilon(\mu,\varepsilon)\mathcal{F}[\check{\eta}_2^{(2)\prime}] = 2\varepsilon\frac{(\varepsilon^{-1}\mu)^2 k}{((\varepsilon^{-1}\mu)^2+k^2)^2}&\left(\mu_0\coth(\mu_0)+B_2+B_3\frac{2(\varepsilon^{-1}\mu)^2}{(1-\alpha_0^{-1})(\varepsilon^{-1}\mu)^2+k^2}\right) \\ &\times\mathcal{F}[\mathrm{Re}(\zeta_\delta^\star\zeta)](\varepsilon^{-1}\mu) + \varepsilon(1-\chi)\hat{R}_0. \end{aligned}$$

Finally, we use the estimate

$$g_\varepsilon(\mu,k\varepsilon)^{-1} = \frac{(\varepsilon^{-1}\mu)^2+k^2}{(\alpha_0-1)(\varepsilon^{-1}\mu)^2+\alpha_0 k^2} + T_2(\mu,\varepsilon),$$

where

$$|T_2(\mu,\varepsilon)| \lesssim \mu^2+\varepsilon^2, \qquad |\mu|<2\delta,$$

and the fact that

$$g_\varepsilon(D,\lambda)^{-1}(1-\chi(D))R_0 = R_4$$

because $g_\varepsilon(\mu, k\varepsilon) \gtrsim 1 + q^4$ for $\mu \in \operatorname{supp}(1-\chi)$ (see Proposition 4.14). The calculation

$$\begin{aligned}
&2\varepsilon\frac{(\varepsilon^{-1}\mu)^2 k}{((\varepsilon^{-1}\mu)^2+k)^2}\,\frac{(\varepsilon^{-1}\mu)^2+k^2}{(\alpha_0-1)(\varepsilon^{-1}\mu)^2+\alpha_0 k^2}\left(\mu_0\coth(\mu_0)+B_2+B_3\frac{2(\varepsilon^{-1}\mu)^2}{(1-\alpha_0^{-1})(\varepsilon^{-1}\mu)^2+k^2}\right)\\
&\quad = 4B_3\alpha_0\varepsilon\frac{(\varepsilon^{-1}\mu)^2 k}{((1-\alpha_0^{-1})(\varepsilon^{-1}\mu)^2+k^2)^2}
\end{aligned}$$

yields the stated estimate. ■

Finally, we compute estimates for $\check{\Gamma}^{(2)\prime}$ and $\tilde{\Gamma}^{(3)}(\eta_1)'+\tilde{\Gamma}^{(2)}$ using the formulae

$$\begin{aligned}
\check{\Gamma}^{(2)\prime} &= \tilde{\Gamma}^{(2)}(\eta_1)'+\tilde{\Gamma}^{(1)}(\check{\eta}_2^{(2)})',\\
\tilde{\Gamma}^{(1)}(\check{\eta}_2^{(2)})' &= \mathcal{F}^{-1}\left[\int_0^1 G(y,\tilde{y})\Big(-2\varepsilon^2 k\mathcal{F}[\tilde{\Gamma}^{(1)}(\check{\eta}_2^{(2)})]\Big)\,\mathrm{d}\tilde{y}\right]
\end{aligned}$$

and

$$\begin{aligned}
\tilde{\Gamma}^{(3)}(\eta_1)'+\tilde{\Gamma}^{(2)}(\check{\eta}_2^{(2)})' = \mathcal{F}^{-1}\Bigg[\int_0^1 \Big\{&G(y,\tilde{y})\Big(-2\varepsilon^2 k\mathcal{F}\Big[\tilde{\Gamma}^{(3)}(\eta_1)+\tilde{\Gamma}^{(2)}(\check{\eta}_2^{(2)})\Big]-\mathrm{i}\mu\mathcal{F}[\check{F}_1^{(3)\prime}]\\
&\qquad -\mathrm{i}\varepsilon\mathcal{F}[\check{F}_2^{(3)}]-\mathrm{i}k\varepsilon\mathcal{F}[\check{F}_2^{(3)\prime}]\Big)\\
&+G_{\tilde{y}}(y,\tilde{y})\mathcal{F}[F_3^{(3)\prime}]\Big\}\,\mathrm{d}\tilde{y}\Bigg]
\end{aligned}$$

and deduce an estimate for $\check{\mathcal{N}}(\eta_1)'$ by differentiating equation (4.57).

Proposition 4.57.

(i) The quantity $\check{\Gamma}^{(2)}(\eta_1)'$ *satisfies the estimate*

$$\check{\Gamma}^{(2)}(\eta_1)' = -\varepsilon\vartheta_\delta(\varepsilon x)'M_2+\varepsilon S$$

uniformly over $k\in[k_{\min},k_{\max}]$, *where*

$$\vartheta_\delta(x)' = -\frac{A_4}{2}\mathcal{F}^{-1}\left[\frac{\mathrm{i}\mu k}{((1-\alpha_0^{-1})\mu^2+k^2)^2}\mathcal{F}[2\mathrm{Re}(\zeta_\delta^\star\zeta)](\mu)\right](x).$$

(ii) The quantity $\tilde{\Gamma}_x^{(3)}(\eta_1)'+\tilde{\Gamma}_x^{(2)}(\check{\eta}_2^{(2)})'\Big|_{y=1}$ *satisfies the estimate*

$$\tilde{\Gamma}_x^{(3)}(\eta_1)'+\tilde{\Gamma}_x^{(2)}(\check{\eta}_2^{(2)})'\Big|_{y=1} = \mu_0\varepsilon^3\zeta_\delta^\star(\varepsilon x)\vartheta_{\delta x}(\varepsilon x)'N_3\left(\frac{1}{2}\mathrm{e}^{\mathrm{i}\mu_0 x}+\frac{1}{2}\mathrm{e}^{-\mathrm{i}\mu_0 x}\right)+\varepsilon^2 T$$

uniformly over $k\in[k_{\min},k_{\max}]$.

Lemma 4.58. *The quantity $\check{\mathcal{N}}^{(3)}(\eta_1)'$ is given by the formula*

$$\check{\mathcal{N}}^{(3)}(\eta_1)' = -\varepsilon^3 A_4 \zeta_\delta^\star(\varepsilon x)\vartheta_{\delta x}(\varepsilon x)'\left(\frac{1}{2}\mathrm{e}^{\mathrm{i}\mu_0 x} + \frac{1}{2}\mathrm{e}^{-\mathrm{i}\mu_0 x}\right) + \varepsilon^2 T$$

uniformly over $k \in [k_{\min}, k_{\max}]$.

We now prove our main result using Lemmata 4.28, 4.33, 4.55 and 4.58.

Theorem 4.59. *The linear mappings $\tilde{\mathcal{R}}_{\varepsilon,k}, \mathcal{R}_{\varepsilon,k} : H^2(\mathbb{R}) \to L^2(\mathbb{R})$ satisfy the estimate*

$$\|\tilde{\mathcal{R}}_{\varepsilon,k}(\zeta)'\|_0, \|\mathcal{R}_{\varepsilon,k}(\zeta)'\|_0 \lesssim \varepsilon\|\zeta\|_1$$

uniformly over $k \in [k_{\min}, k_{\max}]$.

Proof. Recall that

$$\begin{aligned}\tilde{\mathcal{R}}_{\varepsilon,k}(\zeta) = A_2^{-1}\mathcal{F}^{-1}\Bigg[&2\frac{\tilde{g}_\varepsilon(\mu_0+\varepsilon\tilde{\mu},\varepsilon k)}{g_\varepsilon(\mu_0+\varepsilon\tilde{\mu},\varepsilon k)}\chi_0(\varepsilon\tilde{\mu})\varepsilon^{-2} \\ &\times \mathcal{F}\Big[\check{\mathcal{N}}^{(3)}(\chi(D)\eta_1) + \check{\mathcal{N}}^{\mathrm{r}}(\chi(D)\eta_1)\Big](\mu_0+\varepsilon\tilde{\mu})\Bigg] \\ &- A_2^{-1}E(x),\end{aligned}$$

so that

$$\begin{aligned}\tilde{\mathcal{R}}_{\varepsilon,k}(\zeta)' = A_2^{-1}\mathcal{F}^{-1}\Bigg[&2\left(\frac{\tilde{g}_\varepsilon(\mu_0+\varepsilon\tilde{\mu},\varepsilon k)}{g_\varepsilon(\mu_0+\varepsilon\tilde{\mu},\varepsilon k)}\right)'\chi_0(\varepsilon\tilde{\mu})\varepsilon^{-2} \\ &\times \mathcal{F}\Big[\check{\mathcal{N}}^{(3)}(\chi(D)\eta_1) + \check{\mathcal{N}}^{\mathrm{r}}(\chi(D)\eta_1)\Big](\mu_0+\varepsilon\tilde{\mu})\Bigg] \\ &+ A_2^{-1}\mathcal{F}^{-1}\Bigg[2\frac{\tilde{g}_\varepsilon(\mu_0+\varepsilon\tilde{\mu},\varepsilon k)}{g_\varepsilon(\mu_0+\varepsilon\tilde{\mu},\varepsilon k)}\chi_0(\varepsilon\tilde{\mu})\varepsilon^{-2} \\ &\times \mathcal{F}\Big[\check{\mathcal{N}}^{(3)}(\chi(D)\eta_1)' + \check{\mathcal{N}}^{\mathrm{r}}(\chi(D)\eta_1)'\Big](\mu_0+\varepsilon\tilde{\mu})\Bigg] \\ &- A_2^{-1}E(x)'.\end{aligned}$$

Noting that

$$\check{\mathcal{N}}^{(3)}(\chi(D)\eta_1)' = \frac{\varepsilon^3}{2}E(\varepsilon x)'\mathrm{e}^{\mathrm{i}\mu_0 x} + \frac{\varepsilon^3}{2}F(\varepsilon x)'\mathrm{e}^{-\mathrm{i}\mu_0 x} + \varepsilon^2 T,$$

one finds by repeating the arguments below equation (4.62) that

$$\begin{aligned}&\bigg\| A_2^{-1}\mathcal{F}^{-1}\bigg[2\frac{\tilde{g}_\varepsilon(\mu_0+\varepsilon\tilde{\mu},\varepsilon k)}{g_\varepsilon(\mu_0+\varepsilon\tilde{\mu},\varepsilon k)}\chi_0(\varepsilon\tilde{\mu})\varepsilon^{-2}\\ &\qquad\qquad \times \mathcal{F}\Big[\check{\mathcal{N}}^{(3)}(\chi(D)\eta_1)'+\check{\mathcal{N}}^{\mathrm{r}}(\chi(D)\eta_1)'\Big](\mu_0+\varepsilon\tilde{\mu})\bigg] - A_2^{-1}E(x)'\bigg\|_0\\ &\quad\lesssim \|\zeta\|_1.\end{aligned}$$

Moreover, the calculation

$$\left(\frac{\tilde{g}_\varepsilon(\mu,\varepsilon k)}{g_\varepsilon(\mu,\varepsilon k)}\right)' = \frac{1}{g_\varepsilon(\mu,\varepsilon k)}\left(\left(1-\frac{\tilde{g}_\varepsilon(\mu,\varepsilon k)}{g_\varepsilon(\mu,\varepsilon k)}\right)\partial_k g_\varepsilon(\mu,\varepsilon k)+\partial_k g_\varepsilon(\mu,\varepsilon k)-\partial_k\tilde{g}_\varepsilon(\mu,\varepsilon k)\right)$$

and the estimates

$$\begin{aligned}|(\mu-\mu_0,\varepsilon k)|^4 &\lesssim g_\varepsilon(\mu,\varepsilon k),\\ \left|\frac{\tilde{g}_\varepsilon(\mu,\varepsilon k)}{g_\varepsilon(\mu,\varepsilon k)}-1\right| &\lesssim |(\mu-\mu_0,\varepsilon k)|^2,\\ |\partial_k g(\mu,\varepsilon k)| &\lesssim \varepsilon,\\ |\partial_k g_\varepsilon(\mu,\varepsilon k)-\partial_k\tilde{g}(\mu,\varepsilon k)| &\lesssim \varepsilon|(\mu-\mu_0,\varepsilon k)|^5\end{aligned}$$

for $|\mu-\mu_0|<\delta$ imply that

$$\left|\left(\frac{\tilde{g}_\varepsilon(\mu_0+\varepsilon\tilde{\mu},\varepsilon k)}{g_\varepsilon(\mu_0+\varepsilon\tilde{\mu},\varepsilon k)}\right)'\chi_0(\varepsilon\tilde{\mu})\right| \lesssim \varepsilon,$$

whence

$$\begin{aligned}&\bigg\|\mathcal{F}^{-1}\bigg[2\bigg(\frac{\tilde{g}_\varepsilon(\mu_0+\varepsilon\tilde{\mu},\varepsilon k)}{g_\varepsilon(\mu_0+\varepsilon\tilde{\mu},\varepsilon k)}\bigg)'\chi_0(\varepsilon\tilde{\mu})\varepsilon^{-2}\mathcal{F}\Big[\check{\mathcal{N}}^{(3)}(\eta_1)+\check{\mathcal{N}}^{\mathrm{r}}(\eta_1)\Big](\mu_0+\varepsilon\tilde{\mu})\bigg]\bigg\|_0\\ &\quad\lesssim \varepsilon^{-3/2}\Big(\|\check{\mathcal{N}}^{(3)}(\chi(D)\eta_1)\|_0+\|\check{\mathcal{N}}^{\mathrm{r}}(\chi(D)\eta_1)\|_0\Big)\\ &\quad\lesssim \varepsilon^{1/2}\|\eta_1\|_1\\ &\quad\lesssim \varepsilon\|\zeta\|_1.\end{aligned}$$

According to the above calculation we have

$$\|\tilde{\mathcal{R}}_{\varepsilon,k}(\zeta)'\|_0 \lesssim \varepsilon\|\zeta\|_1,$$

and the observation

$$\mathcal{R}_{\varepsilon,k}(\zeta)'-\tilde{\mathcal{R}}_{\varepsilon,k}(\zeta)' = A_4(\zeta_\delta^\star\vartheta_{\delta x}'-\zeta\vartheta_x'),$$

shows that

$$\|\mathcal{R}_{\varepsilon,k}(\zeta)' - \tilde{\mathcal{R}}_{\varepsilon,k}(\zeta)'\|_0 \lesssim \varepsilon \|\zeta\|_1.$$

■

Corollary 4.60. *The linear mapping* $\tilde{\mathcal{B}}_{\varepsilon,k} : \mathcal{D}_{\mathcal{B}} \to V$ *satisfies the estimate*

$$\|\tilde{\mathcal{B}}'_{\varepsilon,k}\|_{\mathcal{L}(\mathcal{D}_{\mathcal{B}},V)} \lesssim \varepsilon$$

uniformly over $k \in [k_{\min}, k_{\max}]$.

The results in the next lemma are deduced from the estimates

$$\|\tilde{\mathcal{B}}_{\varepsilon,k} - \tilde{\mathcal{B}}_{0,k}\|_{\mathcal{L}(\mathcal{D}_{\mathcal{B}},V)} \lesssim \varepsilon, \qquad \|\tilde{\mathcal{B}}'_{\varepsilon,k}\|_{\mathcal{L}(\mathcal{D}_{\mathcal{B}},V)} \lesssim \varepsilon$$

using the method given by Groves, Haragus and Sun [24, Lemma 3.15].

Lemma 4.61. *The eigenvalue* $\kappa_{\varepsilon,k}$ *of* $\tilde{\mathcal{B}}_{\varepsilon,k} : \mathcal{D}_{\mathcal{B}} \subset V \to V$ *is a differentiable function of* k *and satisfies the estimates*

$$|\kappa_{\varepsilon,k} - \kappa_{0,k}| \leq c^\star \varepsilon, \qquad |\kappa'_{\varepsilon,k}| \leq c^\star \varepsilon$$

uniformly in $k \in [k_{\min}, k_{\max}]$.

Finally the solution set of (4.65) in the interval $[k_{\min}, k_{\max}]$ can be determined from the estimates presented in Lemma 4.61 using a perturbation argument and the contraction-mapping principle (cf. Groves, Haragus and Sun [24, Theorem 3.16]).

Theorem 4.62. *Equation* (4.65) *has precisely one solution* k_ε *in the interval* $[k_{\min}, k_{\max}]$. *This solution lies in the set*

$$\mathcal{S} = \left\{ k : |k - k_0| \leq \frac{c^\star \varepsilon}{k_0} \right\}.$$

4.4 Appendix

4.4.1 The line solitary waves

An asymptotic expansion of the line solitary waves found in Chapter 3 in powers of ε may be computed in the framework of their existence theory, using a suitable *Ansatz* in the truncated reduced Hamiltonian system (see equations (3.41) and (3.42)). One finds that

$$\eta^\star(x) = \eta_1^\star(x) + \eta_2^\star(x) + \eta_{\mathrm{r}}^\star(x), \qquad \Phi^\star(x,y) = \Phi_1^\star(x,y) + \Phi_2^\star(x,y) + \Phi_{\mathrm{r}}^\star(x,y),$$

where

$$
\begin{aligned}
\eta_1^\star(x) &= \varepsilon\zeta^\star(\varepsilon x)\cos(\mu_0 x),\\
\eta_2^\star(x) &= C_0\varepsilon^2\xi^\star(\varepsilon x)\sin(\mu_0 x) - C_1\varepsilon^2(\zeta^{\star 2})(\varepsilon x)\cos(2\mu_0 x)\\
&\qquad - C_2\varepsilon^2(\zeta^{\star 2})(\varepsilon x) - C_3\varepsilon^2(\zeta^\star)'(\varepsilon x)\sin(\mu_0 x),\\
\Phi_1^\star(x,y) &= \varepsilon\zeta^\star(\varepsilon x)\sinh(\mu_0 x)\frac{\cosh(\mu_0 y)}{\sinh(\mu_0)},\\
\Phi_2^\star(x,y) &= \ - C_0\varepsilon^2\xi^\star(\varepsilon x)\cosh(\mu_0 x)\frac{\cosh(\mu_0 y)}{\sinh(\mu_0)} + \varepsilon^2(\zeta^{\star 2})(\varepsilon x)\sin(2\mu_0 x)\frac{\mu_0 y\sinh(\mu_0 y)}{2\sinh(\mu_0)}\\
&\qquad + C_4\varepsilon\partial_x^{-1}(\zeta^{\star 2})(\varepsilon x) + C_5\varepsilon^2(\zeta^{\star 2})(\varepsilon x)\sinh(2\mu_0 x)\cosh(2\mu_0 y),
\end{aligned}
$$

and $\eta_{\mathrm{r}}^\star, \nabla\Gamma_{\mathrm{r}}^\star = \underline{O}(\varepsilon^3)$, the symbol $\underline{O}(\varepsilon^\alpha)$ denoting a smooth quantity whose derivatives are all $O(\varepsilon^\alpha \mathrm{e}^{-\rho\varepsilon|x|})$ for some $\rho > 0$ (uniformly over $y \in [0,1]$). The functions $\zeta^\star$ and $\xi^\star$ are given by the formulae

$$
\zeta^\star(x) = \pm\left(\frac{2}{A_5}\right)^{1/2}\operatorname{sech}\left(\frac{x}{A_1^{1/2}}\right), \qquad \xi^\star(x) = x\zeta^\star(x)
$$

with positive constants

$$
A_1 = 6\mu_0^2\beta_0 + (1-\mu_0\coth(\mu_0))\operatorname{cosech}^2(\mu_0), \qquad A_5 = A_3 + (1-\alpha_0^{-1})^{-1}A_4^2
$$

where

$$
\begin{aligned}
A_3 &= \frac{\mu_0^4(1-3\sigma^2)^2}{2(\alpha + 16\beta_0\mu_0^4 - \mu_0(\sigma+\sigma^{-1}))} + \mu_0^3(5\mu_0^3\beta_0 - 4\sigma + 2\sigma^2) + \frac{\mu_0^4}{\alpha_0}(1-\sigma^2)^2,\\
A_4 &= \frac{\mu_0(\alpha_0\sinh(2\mu_0)+\mu_0)}{\alpha_0\sinh^2(\mu_0)}
\end{aligned}
$$

and

$$
\sigma = \coth(\mu_0) = \frac{\alpha + \beta_0\mu_0^4}{\mu_0}.
$$

Finally, the positive coefficients C_1, C_2, C_4 and C_5 are given by

$$
\begin{aligned}
C_1 &= \frac{\mu_0^2(\cosh(2\mu_0)+2)}{4\sinh^2(\mu_0)g_0(2\mu_0,0)}, & C_2 &= \frac{\mu_0(\sinh(2\mu_0)+\mu_0)}{4\sinh^2(\mu_0)(\alpha_0-1)},\\
C_4 &= \frac{\mu_0}{4\sinh^2(\mu_0)}\Big(\mu_0 - C_2\mu_0 - \frac{3}{2}C_2\sinh(2\mu_0)\Big), & C_5 &= -\frac{2C_1 + \mu_0\coth(\mu_0)}{2\sinh(2\mu_0)},
\end{aligned}
$$

while the values (and signs) of the coefficients C_0 and C_3 are unimportant.

4.4.2 Formal derivation of the Davey-Stewartson equation

Substituting $\beta = \beta_0$, $\alpha = \alpha_0 + \varepsilon^2$ and

$$\rho = \varepsilon\eta, \qquad \phi = \varepsilon\Psi$$

into (1.40)–(1.43) and expanding the boundary conditions using the formulae

$$\begin{aligned}
\Psi_x(x, 1+\varepsilon\eta, z) &= \Psi_x(x,1,z) + \varepsilon\eta\Psi_{xy}(x,1,z) + \frac{1}{2}(\varepsilon\eta)^2\Psi_{xyy}(x,1,z) + \dots,\\
\Psi_y(x, 1+\varepsilon\eta, z) &= \Psi_y(x,1,z) + \varepsilon\eta\Psi_{yy}(x,1,z) + \frac{1}{2}(\varepsilon\eta)^2\Psi_{yyy}(x,1,z) + \dots,\\
\Psi_z(x, 1+\varepsilon\eta, z) &= \Psi_z(x,1,z) + \varepsilon\eta\Psi_{yz}(x,1,z) + \frac{1}{2}(\varepsilon\eta)^2\Psi_{yyz}(x,1,z) + \dots
\end{aligned}$$

we find that

$$\begin{aligned}
&\Psi_{xx} + \Psi_{yy} + \Psi_{zz} = 0, && 0 < y < 1,\\
&\Psi_y = 0, && y = 0,\\
&-\Psi_y - \varepsilon\eta\Psi_{yy} - \frac{1}{2}\varepsilon^2\eta^2\Psi_{yyy}\\
&\quad + \varepsilon\eta_x(\Psi_x + \varepsilon\eta\Psi_{xy}) + \varepsilon\eta_z(\Psi_z + \varepsilon\eta\Psi_{yz}) - \eta_x + \dots = 0, && y = 1,\\
&-\Psi_x - \varepsilon\eta\Psi_{xy} - \frac{1}{2}\varepsilon^2\eta^2\Psi_{xyy}\\
&\quad + \frac{\varepsilon}{2}(\Psi_x^2 + \Psi_y^2 + \Psi_z^2 + 2\varepsilon\eta\Psi_x\Psi_{xy} + 2\varepsilon\eta\Psi_y\Psi_{yy} + 2\varepsilon\eta\Psi_z\Psi_{yz})\\
&\quad + (\alpha_0 + \varepsilon^2)\eta + \beta_0(\eta_{xxxx} + 2\eta_{xxzz} + \eta_{zzzz})\\
&\quad + 2\beta_0\varepsilon^2\bigg(-\frac{1}{2}[\eta_x\eta_{xx}\eta_{zz}]_x - \frac{3}{2}[\eta_z\eta_{xz}\eta_{zz}]_x - [\eta_{xxz}\eta_x\eta_z]_x - [\eta_x^2\eta_{xxx}]_x - [\eta_z^2\eta_{xxx}]_x\\
&\qquad - [\eta_x^2\eta_{xzz}]_x - [\eta_z^2\eta_{xzz}]_x - \frac{3}{2}[\eta_z\eta_{xx}\eta_{xz}]_x + \frac{1}{2}[\eta_{xzz}\eta_x^2]_x + \frac{1}{2}[\eta_{xzz}\eta_z^2] - \frac{3}{2}[\eta_x\eta_{xx}^2]_x\\
&\qquad + [\eta_{xxx}\eta_z^2]_x - [\eta_{xz}^2\eta_x]_x - \frac{1}{2}[\eta_x\eta_z\eta_{xxz}]_x - \frac{1}{2}[\eta_x\eta_z\eta_{xxx}]_x - \frac{1}{4}\eta_x^2\eta_{xxxx} - \frac{1}{4}\eta_z^2\eta_{xxxx}\\
&\qquad - \frac{1}{2}[\eta_x\eta_z\eta_{xxx}]_z - \frac{1}{2}[\eta_x\eta_z\eta_{xzz}]_z - \frac{1}{2}[\eta_z\eta_{xx}\eta_{zz}]_z - [\eta_x^2\eta_{xxz}]_z - [\eta_z^2\eta_{xxz}]_z\\
&\qquad - [\eta_x^2\eta_{zzz}]_z - [\eta_z^2\eta_{zzz}]_z - [\eta_{xzz}\eta_x\eta_z]_z - \frac{3}{2}[\eta_x\eta_{xx}\eta_{xz}]_z - \frac{3}{2}[\eta_x\eta_{xz}\eta_{zz}]_z\\
&\qquad + \frac{1}{2}[\eta_x^2\eta_{xxz}]_z - \frac{3}{2}[\eta_z\eta_{zz}^2]_z + \frac{1}{2}[\eta_z^2\eta_{xxz}]_z - [\eta_z\eta_{xz}^2]_z + [\eta_x^2\eta_{zzz}]_z\\
&\qquad + \frac{7}{4}\eta_{xx}\eta_{zz}^2 + \frac{7}{4}\eta_{xx}^2\eta_{zz} + \frac{1}{4}\eta_{xx}^3 + \frac{1}{4}\eta_{zz}^3 - \eta_{xx}\eta_{xz}^2 - \eta_{xz^2}\eta_{zz}\bigg) + \dots = 0, && y = 1.
\end{aligned}$$

Substituting the asymptotic expansions

$$\eta = \eta_1(x, X, Z) + \varepsilon \eta_2(x, X, Z) + \varepsilon^2 \eta_3(x, X, Z) + \dots,$$
$$\Psi = \Psi_1(x, X, y, Z) + \varepsilon \Psi_2(x, X, y, Z) + \varepsilon^2 \Psi_3(x, X, y, Z) + \dots,$$

where $X = \varepsilon x, Z = \varepsilon z$, into the above equations yields the boundary value problems

$$\Psi_{1xx} + \Psi_{1yy} = 0, \qquad 0 < y < 1, \tag{4.67}$$
$$\Psi_{1y}\Big|_{y=0} = 0, \tag{4.68}$$
$$\Psi_{1y} + \eta_{1x}\Big|_{y=1} = 0, \tag{4.69}$$
$$\alpha_0 \eta_1 + \beta_0 \eta_{1xxxx} - \Psi_{1x}\Big|_{y=1} = 0 \tag{4.70}$$

for Ψ_1,

$$\Psi_{2xx} + \Psi_{2yy} + 2\Psi_{1xX} = 0, \qquad 0 < y < 1, \tag{4.71}$$
$$\Psi_{2y}\Big|_{y=0} = 0, \tag{4.72}$$
$$\Psi_{2y} + \eta_{1X} + \eta_{2x} + \eta_1 \Psi_{1yy} - \eta_{1x}\Psi_{1x}\Big|_{y=1} = 0, \tag{4.73}$$
$$\alpha_0 \eta_2 + \beta_0 \eta_{2xxxx} + 4\beta_0 \eta_{1xxxX} - \Psi_{1X} - \Psi_{2x} + \frac{1}{2}\Psi_{1x}^2 + \frac{1}{2}\Psi_{1y}^2 - \eta_1 \Psi_{1xy}\Big|_{y=1} = 0, \tag{4.74}$$

for Ψ_2 and

$$\Psi_{3xx} + \Psi_{3yy} + \Psi_{1XX} + 2\Psi_{2xX} + \Psi_{1ZZ} = 0, \qquad 0 < y < 1, \tag{4.75}$$
$$\Psi_{3y}\Big|_{y=0} = 0, \tag{4.76}$$
$$\begin{aligned}\Psi_{3y} &+ \eta_{2X} + \eta_{3x} + \eta_2 \Psi_{1yy} + \eta_1 \Psi_{2yy} - \eta_{1x}\Psi_{1X} \\ &- \eta_{1X}\Psi_{1x} - \eta_{2x}\Psi_{1x} - \eta_{1x}\Psi_{2x} + \frac{1}{2}\eta_1^2 \Psi_{1yyy} - \eta_1 \eta_{1x}\Psi_{1xy}\Big|_{y=1} = 0,\end{aligned} \tag{4.77}$$
$$\begin{aligned}\alpha_0 \eta_3 &+ \beta_0 \eta_{3xxxx} + 4\beta_0 \eta_{2xxxX} + 6\beta_0 \eta_{1xxXX} + 2\beta_0 \eta_{xxZZ} \\ &+ \eta_1 - 10\beta_0 \eta_{1x}\eta_{1xx}\eta_{1xxx} - \frac{5}{2}\beta_0 \eta_{1x}^2 \eta_{1xxxx} - \frac{5}{2}\beta_0 \eta_{1xx}^3 \\ &- \Psi_{2X} - \Psi_{3x} + \Psi_{1y}\Psi_{2y} - \eta_1 \Psi_{1Xy} + \Psi_{1x}\Psi_{1X} + \Psi_{1x}\Psi_{2x} \\ &- \eta_2 \Psi_{1xy} - \eta_1 \Psi_{2xy} + \eta_1 \Psi_{1y}\Psi_{1yy} + \eta_1 \Psi_{1x}\Psi_{1xy} - \frac{1}{2}\eta_1^2 \Psi_{1xyy}\Big|_{y=1} = 0\end{aligned} \tag{4.78}$$

for Ψ_3.

The next step is the *Ansatz*

$$\eta_1 = \zeta(X, Z)\mathrm{e}^{\mathrm{i}\mu_0 x} + \text{c.c.},$$

$$\eta_2 = \zeta_2(X,Z)\mathrm{e}^{2\mathrm{i}\mu_0 x} + \text{c.c.} + \zeta_0(X,Z),$$
$$\eta_3 = \zeta_3(X,Z)\mathrm{e}^{3\mathrm{i}\mu_0 x} + \zeta_4(X,Z)\mathrm{e}^{2\mathrm{i}\mu_0 x} + \zeta_5(X,Z)\mathrm{e}^{\mathrm{i}\mu_0 x} + \text{c.c.} + \zeta_6(X,Z).$$

From (4.67)–(4.69) it follows that

$$\begin{aligned}\Psi_{1xx} + \Psi_{1yy} &= 0, \qquad 0 < y < 1,\\ \Psi_{1y}\Big|_{y=0} &= 0,\\ \Psi_{1y}\Big|_{y=1} &= -\mathrm{i}\mu_0\zeta\mathrm{e}^{\mathrm{i}\mu_0 x} + \text{c.c.}\end{aligned}$$

and the solution to this boundary value problem is given by

$$\Psi_1 = -\frac{\cosh(\mu_0 y)}{\sinh(\mu_0)}\zeta\mathrm{e}^{\mathrm{i}\mu_0 x} + \text{c.c.} + g_1(X,Z),$$

where g_1 is an arbitrary function of two variables. From the boundary condition (4.70) it follows that

$$(\alpha_0 + \beta_0\mu_0^4 - \mu_0\coth(\mu_0))\zeta\mathrm{e}^{\mathrm{i}\mu_0 x} + \text{c.c.} = 0,$$

which recovers the dispersion relation

$$\alpha_0 + \beta_0\mu_0^4 = \mu_0\coth(\mu_0).$$

The formula for Ψ_1 and (4.71)–(4.73) yield the boundary-value problem

$$\begin{aligned}&\Psi_{2xx} + \Psi_{2yy} = -2\mu_0\frac{\cosh(\mu_0 y)}{\sinh(\mu_0)}\zeta_X\mathrm{e}^{\mathrm{i}\mu_0 x} + \text{c.c.}, \qquad 0 < y < 1,\\ &\Psi_{2y}\Big|_{y=0} = 0,\\ &\Psi_{2y}\Big|_{y=1} = -\zeta_X\mathrm{e}^{\mathrm{i}\mu_0 x} - 2\mathrm{i}\mu_0\zeta_2\mathrm{e}^{2\mathrm{i}\mu_0 x} + 2\mathrm{i}\mu_0^2\coth(\mu_0)\zeta^2\mathrm{e}^{2\mathrm{i}\mu_0 x} + \text{c.c.},\end{aligned}$$

the solution to which is given by

$$\begin{aligned}\Psi_2 = &\left(\frac{\coth(\mu_0)}{\sinh(\mu_0)}\cosh(\mu_0 y) - \frac{y\sinh(\mu_0 y)}{\sinh(\mu_0)}\right)\zeta_X\mathrm{e}^{\mathrm{i}\mu_0 x}\\ &-\frac{\mathrm{i}\cosh(2\mu_0 y)}{\sinh(2\mu_0)}\zeta_2\mathrm{e}^{2\mathrm{i}\mu_0 x} + \frac{\mathrm{i}\mu_0\coth(\mu_0)}{\sinh(2\mu_0)}\cosh(2\mu_0 y)\zeta^2\mathrm{e}^{2\mathrm{i}\mu_0 x} + \text{c.c.} + g_2(X,Z),\end{aligned}$$

where g_2 is an arbitrary function of two variables. Substituting into the boundary condition (4.74) yields

$$\begin{aligned}
&\alpha_0\zeta_0+\frac{\mu_0^2}{\sinh^2(\mu_0)}|\zeta|^2-g_{1X}(X,Z)\\
&-\frac{\mathrm{i}}{2\sinh^2(\mu_0)}\Big(-2\mu_0(-1+2\beta_0\mu_0^2)+4\mu_0^3\beta_0\cosh(2\mu_0)-\sinh(2\mu_0)\Big)\zeta_X\mathrm{e}^{\mathrm{i}\mu_0x}\\
&+\left(\Big(\alpha_0+16\beta_0\mu_0^4-2\mu_0\coth(2\mu_0)\Big)\zeta_2+\frac{\mu_0^2(2+\cosh(2\mu_0))}{2\sinh^2(\mu_0)}\zeta^2\right)\mathrm{e}^{2\mathrm{i}\mu_0x}+\text{c.c.}=0.
\end{aligned}$$

By equating the coefficients of $\mathrm{e}^{0\mathrm{i}\mu_0x},\mathrm{e}^{\mathrm{i}\mu_0x},\mathrm{e}^{2\mathrm{i}\mu_0x}$ with zero, we obtain

$$g_{1\tilde{X}}=\alpha_0\zeta_0+\frac{\mu_0^2}{\sinh^2(\mu_0)}|\zeta|^2, \tag{4.79}$$

$$\beta_0=\frac{1}{4\mu_0^3}\coth(\mu_0)-\frac{1}{4\mu_0^2\sinh^2(\mu_0)},$$

$$\zeta_2=-\frac{\mu_0^2(2+\cosh(2\mu_0))}{2\sinh^2(\mu_0)(\alpha_0+16\beta_0\mu_0^4-2\mu_0\coth(2\mu_0))}\zeta^2 \tag{4.80}$$

and with this formula for β_0 and the dispersion relation we find that

$$\alpha_0=\frac{3}{4}\mu_0\coth(\mu_0)+\frac{\mu_0^2}{4\sinh^2(\mu_0)}.$$

The formulae for Ψ_1,Ψ_2 and (4.75)–(4.77) yield

$$\begin{aligned}
\Psi_{3xx}+\Psi_{3yy}&=\left(\frac{\mathrm{i}(1-2\mu_0\coth(\mu_0))}{\sinh(\mu_0)}\cosh(\mu_0y)+\frac{2\mathrm{i}\mu_0y\sinh(\mu_0y)}{\sinh(\mu_0)}\right)\zeta_{XX}\mathrm{e}^{\mathrm{i}\mu_0x}\\
&\quad+\frac{\mathrm{i}\cosh(\mu_0y)}{\sinh(\mu_0)}\zeta_{ZZ}\mathrm{e}^{\mathrm{i}\mu_0x}+\text{c.c.}-g_{1XX}(X,Z)-g_{1ZZ}(X,Z),\quad 0<y<1,\\
\Psi_{3y}\Big|_{y=0}&=0,\\
\Psi_{3y}\Big|_{y=1}&=\Big(-\mathrm{i}\mu_0\zeta_5+\frac{1}{2}\mathrm{i}\mu_0^3(1-2\coth^2(\mu_0))\zeta^2\overline{\zeta}+\mathrm{i}\mu_0^2\coth(\mu_0)\zeta_0\overline{\zeta}\\
&\qquad+\mathrm{i}\mu_0^2(2\coth(\mu_0)+\tanh(\mu_0))\zeta_2\overline{\zeta}+\mathrm{i}\mu_0\zeta g_{1X}(X,Z)\Big)\mathrm{e}^{\mathrm{i}\mu_0x}+\text{c.c.}\\
&\quad-\zeta_{0X}+2\mu_0\coth(\mu_0)\partial_X(|\zeta|^2)+\ldots.
\end{aligned}$$

The solution to this boundary value problem is given by

$$\begin{aligned}\Psi_3 = \bigg[&- \frac{\mathrm{i}\cosh(\mu_0 y)}{\sinh(\mu_0)}\zeta_5 + \frac{\mathrm{i}\cosh(\mu_0 y)}{\sinh(\mu_0)}\zeta g_{1X}(X,Z) \\ &+ \bigg(\frac{\mathrm{i}}{2\sinh(\mu_0)} y^2 \cosh(\mu_0 y) \\ &\qquad + \frac{\mathrm{i}(2\coth^2(\mu_0)-1)}{2\sinh(\mu_0)}\cosh(\mu_0 y) - \frac{\mathrm{i}\coth(\mu_0)}{\sinh(\mu_0)} y\sinh(\mu_0 y)\bigg)\zeta_{XX} \\ &+ \bigg(\frac{\mathrm{i}}{2\sinh(\mu_0)} y \sinh(\mu_0 y) - \frac{\mathrm{i}(1+\mu_0\coth(\mu_0))}{2\mu_0^2\sinh(\mu_0)}\cosh(\mu_0 y)\bigg)\zeta_{ZZ} \\ &+ \frac{\mathrm{i}\mu_0\coth(\mu_0)}{\sinh(\mu_0)}\cosh(\mu_0 y)\zeta_0\zeta + \frac{1}{2}\mathrm{i}\mu_0^2(1-2\coth^2(\mu_0))\frac{\cosh(\mu_0 y)}{\sinh(\mu_0)}\zeta^2\overline{\zeta} \\ &+ \frac{\mathrm{i}\mu_0}{\sinh(\mu_0)}(2\coth(\mu_0)+\tanh(\mu_0))\cosh(\mu_0 y)\zeta_2\overline{\zeta}\bigg]\mathrm{e}^{\mathrm{i}\mu_0 x} + \text{c.c.} \\ &- \frac{1}{2}\Big(g_{1XX}(X,Z) + g_{1ZZ}(X,Z)\Big)y^2 + g_3(X,Z) + \dots\end{aligned}$$

where g_3 is an arbitrary function of two variables and

$$g_{1XX}(X,Z) + g_{1ZZ}(X,Z) - \zeta_{0X} + 2\mu_0\coth(\mu_0)\partial_X(|\zeta|^2) = 0, \tag{4.81}$$

which is obtained from (4.77). Equation (4.78) thus shows that

$$\begin{aligned}\bigg[&\underbrace{(\alpha_0 + \beta_0\mu_0^4 - \mu_0\coth(\mu_0))}_{=0}\zeta_5 + \mu_0^2(3\coth^2(\mu_0)-1)\zeta_2\overline{\zeta} \\ &- \frac{1}{2}\mu_0^3\Big(10\mu_0^3\beta_0 - 8\coth(\mu_0) + 4\coth^3(\mu_0)\Big)\zeta^2\overline{\zeta} \\ &+ \Big(-6\mu_0^2\beta_0 + (\mu_0\coth(\mu_0)-1)(\coth^2(\mu_0)-1)\Big)\zeta_{XX} \\ &- \bigg(2\mu_0^2\beta_0 - \frac{\mu_0}{2} + \frac{\coth(\mu_0)}{2\mu_0} + \frac{\coth^2(\mu_0)}{2}\bigg)\zeta_{ZZ} \\ &+ \Big(1 - \mu_0^2(1-\coth^2(\mu_0))\zeta_0 + 2\mu_0\coth(\mu_0)g_{1X}\Big)\zeta\bigg]\mathrm{e}^{\mathrm{i}\mu_0 x} + \text{c.c.} \\ + \alpha_0\zeta_6 &+ \mathrm{i}\mu_0^2\frac{\coth(\mu_0)}{\sinh^2(\mu_0)}(\zeta_X\overline{\zeta} - \zeta\overline{\zeta}_X) - \frac{\mathrm{i}\mu_0}{\sinh^2(\mu_0)}(\zeta_X\overline{\zeta} - \zeta\overline{\zeta}_X) - g_{2X} + \dots = 0.\end{aligned} \tag{4.82}$$

Finally, we write

$$\zeta_2 = \frac{1}{2}\frac{(1-3\coth^2(\mu_0))\mu_0^2}{\alpha_0 + 16\beta_0\mu_0^4 - \mu_0(\coth(\mu_0)+\tanh(\mu_0))}$$

(see (4.80)) and

$$\zeta_0 = \frac{1}{\alpha_0} g_{1X} - \frac{\mu_0^2}{\alpha_0 \sinh^2(\mu_0)} |\zeta|^2$$

$$(1-\alpha_0^{-1}) g_{1XX} + g_{1ZZ} + \left(2\mu_0 \coth(\mu_0) + \frac{\mu_0^2}{\alpha_0 \sinh^2(\mu_0)}\right) \partial_X(|\zeta|^2)$$

(see (4.79) and (4.81)). From (4.82) we obtain the Davey-Stewartson system

$$\zeta - A_1 \zeta_{XX} - A_2 \zeta_{ZZ} - A_3 |\zeta|^2 \zeta + A_4 \zeta \psi_X = 0 \tag{4.83}$$

with

$$(1-\alpha_0^{-1}) \psi_{XX} + \psi_{ZZ} + A_4 \partial_X(|\zeta|^2) = 0, \tag{4.84}$$

where we have written $\psi = g_1$ and used the formula for β_0.

4.4.3 Resolvent estimate

In this section we present the proof of Proposition 4.8. To prove this result we first solve the constant-coefficient spectral problem (4.27)–(4.34) explicitly.

Proposition 4.63. *Suppose that*

$$(\alpha_0 + \beta_0 q^4) q \sinh(q) + \mu^2 \cosh(q) \neq 0$$

for all $\mu \in \mathbb{R}$, where $q = \sqrt{\mu^2 + \lambda^2}$. The constant-coefficient spectral problem (4.27)–(4.34)*, where $(h_0^\dagger, h_1^\dagger, H_1^\dagger, h_2^\dagger, h_3^\dagger, H_2^\dagger) \in X_\mathrm{r}^0$ and $F_3^\dagger \in H^1(\Sigma)$, has a unique solution $(\eta, \rho, \Phi, \omega, \xi, \Psi) \in X^1$ whose Fourier transform $(\hat{\eta}, \hat{\rho}, \hat{\Phi}, \hat{\omega}, \hat{\xi}, \hat{\Psi})$ is given by*

$$\hat{\eta} = \frac{1}{\alpha_0 + \beta_0 q^4} \Big(\mathrm{i}\beta_0 q^2 \lambda \widehat{h_0^\dagger} + \beta_0 q^2 \widehat{h_1^\dagger} + \widehat{h_2^\dagger} + \mathrm{i}\mu \hat{\Gamma}\Big|_{y=1} - \mathrm{i}\lambda \widehat{h_3^\dagger} \Big), \tag{4.85}$$

$$\hat{\rho} = \widehat{h_0^\dagger} + \frac{\mathrm{i}\lambda}{\alpha_0 + \beta_0 q^4} \Big(\mathrm{i}\beta_0 q^2 \lambda \widehat{h_0^\dagger} + \beta_0 q^2 \widehat{h_1^\dagger} + \widehat{h_2^\dagger} + \mathrm{i}\mu \hat{\Gamma}\Big|_{y=1} - \mathrm{i}\lambda \widehat{h_3^\dagger} \Big), \tag{4.86}$$

$$\begin{aligned} \hat{\Gamma}(y) = & \int_0^1 \tilde{G}(y, \tilde{y}) \Big(\widehat{H_2^\dagger} + \mathrm{i}\lambda \widehat{H_1^\dagger} \Big) \, \mathrm{d}y \\ & + \tilde{G}(y, 1) \Big(\widehat{F_3^\dagger}\Big|_{y=1} - \frac{\mathrm{i}\mu}{\alpha_0 + \beta_0 q^4} \Big(\mathrm{i}\beta_0 q^2 \lambda \widehat{h_0^\dagger} + \beta_0 q^2 \widehat{h_1^\dagger} + \widehat{h_2^\dagger} - \mathrm{i}\lambda \widehat{h_3^\dagger} \Big) \Big), \end{aligned} \tag{4.87}$$

$$\hat{\omega} = -\widehat{h_3^\dagger} - \mathrm{i}\lambda \beta_0 \widehat{h_1^\dagger} + \lambda^2 \beta_0 \widehat{h_0^\dagger} + \frac{\mathrm{i}\lambda \beta_0 q^2}{\alpha_0 + \beta_0 q^4} \Big(\mathrm{i}\beta_0 q^2 \lambda \widehat{h_0^\dagger} + \beta_0 q^2 \widehat{h_1^\dagger} + \widehat{h_2^\dagger} + \mathrm{i}\mu \hat{\Gamma}\Big|_{y=1} - \mathrm{i}\lambda \widehat{h_3^\dagger} \Big), \tag{4.88}$$

$$\hat{\xi} = \beta_0 \widehat{h_1^\dagger} + \mathrm{i}\lambda\beta_0 \widehat{h_0^\dagger} - \frac{\beta_0 q^2}{\alpha_0 + \beta_0 q^4}\Big(\mathrm{i}\beta_0 q^2 \lambda \widehat{h_0^\dagger} + \beta_0 q^2 \widehat{h_1^\dagger} + \widehat{h_2^\dagger} + \mathrm{i}\mu\hat{\Gamma}\Big|_{y=1} - \mathrm{i}\lambda \widehat{h_3^\dagger}\Big), \tag{4.89}$$

$$\hat{\Psi}(y) = \mathrm{i}\lambda\hat{\Gamma} + \widehat{H_2^\dagger}, \tag{4.90}$$

where

$$\tilde{G}(y,\tilde{y}) = \begin{cases} -\dfrac{\cosh(qy)(A\cosh(q\tilde{y}) - \sinh(q\tilde{y})}{q}, & 0 \le y \le \tilde{y} \le 1, \\ -\dfrac{\cosh(q\tilde{y})(A\cosh(qy) - \sinh(qy)}{q}, & 0 \le \tilde{y} \le y \le 1 \end{cases}$$

and

$$A = \frac{q\cosh(q) - \sinh(q)\frac{\mu^2}{\alpha_0 + \beta_0 q^4}}{-q\sinh(q) + \cosh(q)\frac{\mu^2}{\alpha_0 + \beta_0 q^4}}.$$

To prove the stated estimates in Proposition 4.8 we need the identities

$$\int_0^1 \tilde{G}_y(y,\tilde{y}) f(\tilde{y})\,\mathrm{d}\tilde{y} = \int_0^1 \tilde{H}_{\tilde{y}}(y,\tilde{y}) f(\tilde{y})\,\mathrm{d}\tilde{y},$$
$$\int_0^1 \tilde{G}(y,\tilde{y}) f(\tilde{y})\,\mathrm{d}\tilde{y} = \int_0^1 \tilde{F}_y(y,\tilde{y}) f(\tilde{y})\,\mathrm{d}\tilde{y},$$

where

$$\tilde{H}(y,\tilde{y}) = \begin{cases} -\dfrac{\sinh(qy)(A\sinh(q\tilde{y}) + \cosh(q\tilde{y})}{q}, & 0 \le y \le \tilde{y} \le 1, \\ -\dfrac{\sinh(q\tilde{y})(A\sinh(qy) + \cosh(qy)}{q}, & 0 \le \tilde{y} \le y \le 1 \end{cases}$$

and

$$\tilde{F}(y,\tilde{y}) = \begin{cases} -\dfrac{\cosh(qy)(A\sinh(q\tilde{y}) + \cosh(q\tilde{y})}{q^2}, & 0 \le y \le \tilde{y} \le 1, \\ -\dfrac{\sinh(q\tilde{y})(A\cosh(qy) + \sinh(qy)}{q^2}, & 0 \le \tilde{y} \le y \le 1 \end{cases}$$

and the estimates

$$|\tilde{G}(y,\tilde{y})| \lesssim \frac{\mathrm{e}^{-|q||y-\tilde{y}|}}{|q|}, \qquad |\tilde{G}_y(y,\tilde{y})| \lesssim \mathrm{e}^{-|q||y-\tilde{y}|},$$
$$|\tilde{H}(y,\tilde{y})| \lesssim \frac{\mathrm{e}^{-|q||y-\tilde{y}|}}{|q|}, \qquad |\tilde{F}(y,\tilde{y})| \lesssim \frac{\mathrm{e}^{-|q||y-\tilde{y}|}}{q^2}$$

for sufficiently large q, which are obtained by elementary calculations.

Proposition 4.8. *The solution* $(\eta, \rho, \Phi, \omega, \xi, \Psi) \in X^1$ *of the constant-coefficient spectral problem*

$$\rho - \mathrm{i}\lambda\eta = h_0^\dagger, \tag{4.27}$$

$$\frac{\xi}{\beta_0} - \eta_{xx} - \mathrm{i}\lambda\rho = h_1^\dagger, \tag{4.28}$$

$$\Psi - \mathrm{i}\lambda\Gamma = H_1^\dagger, \tag{4.29}$$

$$\alpha_0\eta - \Gamma_x|_{y=1} + \xi_{xx} - \mathrm{i}\lambda\omega = h_2^\dagger, \tag{4.30}$$

$$-\omega - \mathrm{i}\lambda\xi = h_3^\dagger, \tag{4.31}$$

$$-\Gamma_{xx} - \Gamma_{yy} - \mathrm{i}\Psi = H_2^\dagger, \tag{4.32}$$

with

$$\Gamma_y = 0 \qquad \text{on } y = 0, \tag{4.33}$$

$$\Gamma_y = -\eta_x + F_3^\dagger \qquad \text{on } y = 1, \tag{4.34}$$

where $\lambda > 0$, $(h_0^\dagger, h_1^\dagger, H_1^\dagger, h_2^\dagger, h_3^\dagger, H_2^\dagger) \in X_\mathrm{r}^0$ *and* $F_3^\dagger \in H^1(\Sigma)$ *satisfies the estimate*

$$\begin{aligned}
&\|\eta\|_3 + \|\rho\|_2 + \|\Gamma\|_2 + \|\omega\|_1 + \|\xi\|_2 + \|\Psi\|_1 \\
&\quad\lesssim \|h_0^\dagger\|_2 + \|h_1^\dagger\|_1 + \|H_1^\dagger\|_1 + \|h_2^\dagger\|_0 + \|h_3^\dagger\|_1 + \|H_2^\dagger\|_0 + \|F_3^\dagger\big|_{y=1}\|_{1/2} + \lambda\|F_3^\dagger\big|_{y=1}\|_0, \\
&\|\eta\|_2 + \|\rho\|_1 + \|\Gamma\|_1 + \|\omega\|_0 + \|\xi\|_1 + \|\Psi\|_0 \\
&\quad\lesssim |\lambda|^{-1}\Big(\|h_0^\dagger\|_2 + \|h_1^\dagger\|_1 + \|H_1^\dagger\|_1 + \|h_2^\dagger\|_0 + \|h_3^\dagger\|_1 + \|H_2^\dagger\|_0 + \|F_3^\dagger\big|_{y=1}\|_{1/2} + \lambda\|F_3^\dagger\big|_{y=1}\|_0\Big)
\end{aligned}$$

for all sufficiently large values of λ.

Proof. Using formula (4.87), we find that

$$\begin{aligned}
\|\hat{\Gamma}(y)\|_0^2 \lesssim &\int_0^1 \left| \int_0^1 \tilde{G}(y,\tilde{y})\Big(\widehat{H_2^\dagger}(\tilde{y}) + \mathrm{i}\lambda\widehat{H_1^\dagger}(\tilde{y})\Big)\,\mathrm{d}\tilde{y}\right|^2 \mathrm{d}y \\
&+ \int_0^1 \left|\tilde{G}(y,1)\bigg(\widehat{F_3^\dagger}\Big|_{y=1} - \frac{\mathrm{i}\mu}{\alpha_0 + \beta_0 q^4}\Big(\mathrm{i}\beta_0 q^2\lambda\widehat{h_0^\dagger} + \beta_0 q^2\widehat{h_1^\dagger} + \widehat{h_2^\dagger} - \mathrm{i}\lambda\widehat{h_3^\dagger}\Big)\bigg)\right|^2 \mathrm{d}y
\end{aligned}$$

and we can estimate the first summand by

$$\begin{aligned}
&\int_0^1 \left| \int_0^1 \tilde{G}(y,\tilde{y})\Big(\widehat{H_2^\dagger}(\tilde{y}) + \mathrm{i}\lambda\widehat{H_1^\dagger}(\tilde{y})\Big)\,\mathrm{d}\tilde{y}\right|^2 \mathrm{d}y \\
&\quad\lesssim \int_0^1 \left(\int_0^1 \frac{\mathrm{e}^{-|q||y-\tilde{y}|}}{|q|}\Big|\widehat{H_2^\dagger}(\tilde{y}) + \mathrm{i}\lambda\widehat{H_1^\dagger}(\tilde{y})\Big|\,\mathrm{d}\tilde{y}\right)^2 \mathrm{d}y
\end{aligned}$$

$$\begin{aligned}
&= \int_0^1 \left(\int_0^1 \frac{\left(\mathrm{e}^{-\frac{|q|}{2}|y-\tilde{y}|}\right)^2}{|q|} \left|\widehat{H_2^\dagger}(\tilde{y}) + \mathrm{i}\lambda \widehat{H_1^\dagger}(\tilde{y})\right| \mathrm{d}\tilde{y} \right)^2 \mathrm{d}y \\
&\lesssim \int_0^1 \left(\int_0^1 \mathrm{e}^{-|q||y-\tilde{y}|}\,\mathrm{d}\tilde{y} \int_0^1 \left|\widehat{H_2^\dagger}(\tilde{y}) + \mathrm{i}\lambda \widehat{H_1^\dagger}(\tilde{y})\right|^2 \frac{\mathrm{e}^{-|q||y-\tilde{y}|}}{|q|^2}\,\mathrm{d}\tilde{y} \right) \mathrm{d}y \\
&\lesssim \frac{1}{|q|} \int_0^1 \left(\int_0^1 \left|\widehat{H_2^\dagger}(\tilde{y}) + \mathrm{i}\lambda \widehat{H_1^\dagger}(\tilde{y})\right|^2 \frac{\mathrm{e}^{-|q||y-\tilde{y}|}}{|q|^2}\,\mathrm{d}\tilde{y} \right) \mathrm{d}y \\
&\lesssim \frac{1}{|q|} \int_0^1 \left(\int_0^1 \left|\widehat{H_2^\dagger}(\tilde{y}) + \mathrm{i}\lambda \widehat{H_1^\dagger}(\tilde{y})\right|^2 \frac{\mathrm{e}^{-|q||y-\tilde{y}|}}{|q|^2}\,\mathrm{d}y \right) \mathrm{d}\tilde{y} \\
&\lesssim \frac{1}{|q^3|} \int_0^1 \left(\int_0^1 \mathrm{e}^{-|q||y-\tilde{y}|}\,\mathrm{d}y \left|\widehat{H_2^\dagger}(\tilde{y}) + \mathrm{i}\lambda \widehat{H_1^\dagger}(\tilde{y})\right|^2 \right) \mathrm{d}\tilde{y} \\
&\lesssim \frac{1}{q^4} \int_0^1 \left|\widehat{H_2^\dagger}(\tilde{y}) + \mathrm{i}\lambda \widehat{H_1^\dagger}(\tilde{y})\right|^2 \mathrm{d}\tilde{y}
\end{aligned}$$

and the second summand by

$$\begin{aligned}
&\int_0^1 \left|\frac{\tilde{G}(y,1)}{\alpha_0+\beta_0 q^4}\right|^2 \left|(\alpha_0+\beta_0 q^4)\widehat{F_3^\dagger}\Big|_{y=1} - \mathrm{i}\mu\Big(\mathrm{i}\beta_0 q^2 \lambda \widehat{h_0^\dagger} + \beta_0 q^2 \widehat{h_1^\dagger} + \widehat{h_2^\dagger} - \mathrm{i}\lambda \widehat{h_3^\dagger}\Big)\right|^2 \mathrm{d}y \\
&\quad\lesssim \frac{1}{(\alpha_0+\beta_0 q^4)^2} \\
&\qquad\times \int_0^1 \frac{\mathrm{e}^{-2|q||y-1|}}{q^2}\,\mathrm{d}y \left|(\alpha_0+\beta_0 q^4)\widehat{F_3^\dagger}\Big|_{y=1} - \mathrm{i}\mu\Big(\mathrm{i}\beta_0 q^2 \lambda \widehat{h_0^\dagger} + \beta_0 q^2 \widehat{h_1^\dagger} + \widehat{h_2^\dagger} - \mathrm{i}\lambda \widehat{h_3^\dagger}\Big)\right|^2 \\
&\quad\lesssim \frac{1}{(\alpha_0+\beta_0 q^4)^2 |q|^3} \left|(\alpha_0+\beta_0 q^4)\widehat{F_3^\dagger}\Big|_{y=1} - \mathrm{i}\mu\Big(\mathrm{i}\beta_0 q^2 \lambda \widehat{h_0^\dagger} + \beta_0 q^2 \widehat{h_1^\dagger} + \widehat{h_2^\dagger} - \mathrm{i}\lambda \widehat{h_3^\dagger}\Big)\right|^2.
\end{aligned}$$

In the same fashion we obtain the estimate

$$\begin{aligned}
\|\hat{\Gamma}_y(y)\|_0^2 \lesssim\; & \int_0^1 \left| \int_0^1 \left(\tilde{G}_y(y,\tilde{y})\widehat{H_2^\dagger}(\tilde{y}) - \mathrm{i}\lambda \tilde{H}(y,\tilde{y})\widehat{H_{1\tilde{y}}^\dagger}(\tilde{y}) \right) \mathrm{d}\tilde{y} + \mathrm{i}\lambda \tilde{H}(y,1)\widehat{H_1^\dagger}(1) \right|^2 \mathrm{d}y \\
&+ \int_0^1 \left| \frac{\tilde{G}_y(y,1)}{\alpha_0+\beta_0 q^4} \right. \\
&\qquad \left. \times \left((\alpha_0+\beta_0 q^4)\widehat{F_3^\dagger}\Big|_{y=1} - \mathrm{i}\mu\Big(\mathrm{i}\beta_0 q^2 \lambda \widehat{h_0^\dagger} + \beta_0 q^2 \widehat{h_1^\dagger} + \widehat{h_2^\dagger} - \mathrm{i}\lambda \widehat{h_3^\dagger}\Big) \right) \right|^2 \mathrm{d}y.
\end{aligned}$$

Note that

$$|\widehat{H_1^\dagger}(1)| = \left| \int_0^1 y \widehat{H_{1\tilde{y}}^\dagger}(y)\,\mathrm{d}y + \int_0^1 \widehat{H_1^\dagger}(y)\,\mathrm{d}y \right| \lesssim \|H_1^\dagger\|_1.$$

From these estimates we obtain

$$\|\hat{\Gamma}\|_0 \lesssim \frac{1}{\lambda}\Big(\|h_0^\dagger\|_0 + \|h_1^\dagger\|_0 + \|H_1^\dagger\|_1 + \|h_2^\dagger\|_0 + \|h_3^\dagger\|_0 + \|H_2^\dagger\|_0 + \|F_3^\dagger\big|_{y=1}\|_0 \Big),$$

$$\|\hat{\Gamma}_y\|_0 \lesssim \frac{1}{\lambda}\Big(\|h_0^\dagger\|_0 + \|h_1^\dagger\|_0 + \|H_1^\dagger\|_1 + \|h_2^\dagger\|_0 + \|h_3^\dagger\|_0 + \|H_2^\dagger\|_0 + \|F_3^\dagger\big|_{y=1}\|_0\Big),$$
$$\|\mu\hat{\Gamma}\|_0 \lesssim \frac{1}{\lambda}\Big(\|h_0^\dagger\|_0 + \|h_1^\dagger\|_0 + \|H_1^\dagger\|_1 + \|h_2^\dagger\|_0 + \|h_3^\dagger\|_0 + \|H_2^\dagger\|_0 + \|F_3^\dagger\big|_{y=1}\|_0\Big)$$

and hence

$$\begin{aligned}\|\Gamma\|_1^2 &= \|(1+\mu^2)\hat{\Gamma}\|_0^2 + ||\hat{\Gamma}_y\|_0^2 \\ &\lesssim \frac{1}{\lambda^2}\Big(\|h_0^\dagger\|_2^2 + \|h_1^\dagger\|_1^2 + \|H_1^\dagger\|_1^2 + \|h_2^\dagger\|_0^2 + \|h_3^\dagger\|_1^2 + \|H_2^\dagger\|_0^2 + \|F_3^\dagger\big|_{y=1}\|_0^2\Big).\end{aligned}$$

From (4.85) we obtain the estimates

$$\|\hat{\eta}\|_0 \lesssim \frac{1}{\lambda}\Big(\|h_0^\dagger\|_0 + \|h_1^\dagger\|_0 + \|H_1^\dagger\|_1 + \|h_2^\dagger\|_0 + \|h_3^\dagger\|_0 + \|H_2^\dagger\|_0 + \|F_3^\dagger\big|_{y=1}\|_0\Big),$$
$$\|\mu\hat{\eta}\|_0 \lesssim \frac{1}{\lambda}\Big(\|h_0^\dagger\|_1 + \|h_1^\dagger\|_0 + \|H_1^\dagger\|_1 + \|h_2^\dagger\|_0 + \|h_3^\dagger\|_0 + \|H_2^\dagger\|_0 + \|F_3^\dagger\big|_{y=1}\|_0\Big)$$
$$\|\mu^2\hat{\eta}\|_0 \lesssim \frac{1}{\lambda}\Big(\|h_0^\dagger\|_2 + \|h_1^\dagger\|_1 + \|H_1^\dagger\|_1 + \|h_2^\dagger\|_0 + \|h_3^\dagger\|_0 + \|H_2^\dagger\|_0 + \|F_3^\dagger\big|_{y=1}\|_{1/2}\Big)$$

and hence

$$\begin{aligned}\|\eta\|_2^2 &= \|(1+\mu^2+\mu^4)\hat{\eta}\|_0^2 \\ &\lesssim \frac{1}{\lambda}\Big(\|h_0^\dagger\|_2^2 + \|h_1^\dagger\|_1^2 + \|H_1^\dagger\|_1^2 + \|h_2^\dagger\|_0^2 + \|h_3^\dagger\|_0^2 + \|H_2^\dagger\|_0^2 + \|F_3^\dagger\big|_{y=1}\|_{1/2}^2\Big).\end{aligned}$$

Using the formulae (4.86)–(4.90) we therefore obtain the estimates

$$\|\rho\|_1^2 \lesssim \frac{1}{\lambda^2}\Big(\|h_0^\dagger\|_2 + \|h_1^\dagger\|_1 + \|H_1^\dagger\|_1 + \|h_2^\dagger\|_0 + \|h_3^\dagger\|_1 + \|H_2^\dagger\|_0 + \|F_3^\dagger\big|_{y=1}\|_{1/2}^2\Big),$$
$$\|\omega\|_0^2 \lesssim \frac{1}{\lambda^2}\Big(\|h_0^\dagger\|_2 + \|h_1^\dagger\|_1 + \|H_1^\dagger\|_1 + \|h_2^\dagger\|_0 + \|h_3^\dagger\|_1 + \|H_2^\dagger\|_0 + \|F_3^\dagger\big|_{y=1}\|_{1/2}^2\Big),$$
$$\|\xi\|_1^2 \lesssim \frac{1}{\lambda^2}\Big(\|h_0^\dagger\|_2 + \|h_1^\dagger\|_1 + \|H_1^\dagger\|_1 + \|h_2^\dagger\|_0 + \|h_3^\dagger\|_1 + \|H_2^\dagger\|_0 + \|F_3^\dagger\big|_{y=1}\|_{1/2}^2\Big),$$
$$\|\Psi\|_0^2 \lesssim \frac{1}{\lambda^2}\Big(\|h_0^\dagger\|_2 + \|h_1^\dagger\|_1 + \|H_1^\dagger\|_1 + \|h_2^\dagger\|_0 + \|h_3^\dagger\|_1 + \|H_2^\dagger\|_0 + \|F_3^\dagger\big|_{y=1}\|_{1/2}^2\Big).$$

The estimate

$$\begin{aligned}&\|\eta\|_3^2 + \|\rho\|_2^2 + \|\Gamma\|_2^2 + \|\omega\|_1^2 + \|\xi\|_2^2 + \|\Psi\|_1^2 \\ &\quad\lesssim \|h_0^\dagger\|_2 + \|h_1^\dagger\|_1 + \|H_1^\dagger\|_1 + \|h_2^\dagger\|_0 + \|h_3^\dagger\|_1 + \|H_2^\dagger\|_0 + \|F_3^\dagger\big|_{y=1}\|_{1/2}^2 + \lambda\|F_3^\dagger\big|_{y=1}\|_0^2\end{aligned}$$

is proven in the same way. ■

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