Kontsevich–Zorich monodromy of Origamis from the aspect of Arithmeticity

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Abstract

A translation surface is a compact Riemann surface obtained as a collection of finitely many polygons in the plane with glued parallel edges of the same length by translations. There is a natural action of the group of invertible real 2x2-matrices with positive determinant on the moduli space of translation surfaces by applying matrices to the polygons and regluing them by the same combinatorial relation. Under rare circumstances an orbit of this action is closed in the moduli space of translation surfaces. One class of translation surfaces where this happens are origamis, which consist of finitely many unit squares. The orbit of an origami parametrizes a family of compact Riemann surfaces. We consider the family of first singular cohomology groups with complex coefficients defined by this family of compact Riemann surfaces. It can be given the structure of a flat holomorphic vector bundle. For certain classes of origamis we will study the monodromy group of this vector bundle from the aspect of arithmeticity.

Zusammenfassung

Eine Translationsfläche ist eine kompakte Riemannsche Fläche, die man dadurch erhält, dass man endlich viele Polygone in der Ebene an parallelen Kanten gleicher Länge durch Translationen verklebt. Es gibt eine natürliche Operation der Gruppe der invertierbaren reellen 2x2-Matrizen mit positiver Determinante auf dem Modulraum der Translationsflächen, indem wir die Matrizen auf die Polygone anwenden und dann erneut nach den gleichen kombinatorischen Relationen verkleben. In seltenen Fällen ist eine Bahn dieser Aktion abgeschlossen im Modulraum der Translationsflächen. Eine Klasse von Translationsflächen, für die dies eintritt, sind Origamis. Diese bestehen aus endlich vielen verklebten Einheitsquadraten. Die Bahn eines Origamis parametrisiert eine Familie kompakter Riemannscher Flächen. Wir betrachten die Familie der ersten singulären Kohomologien mit komplexen Koeffizienten, die durch diese Familie kompakter Riemannscher Flächen definiert wird. Ihr kann die Struktur eines flachen holomorphen Vektorbündels gegeben werden. Wir werden für bestimmte Klassen von Origamis die Monodromiegruppe dieses Vektorbündels unter dem Gesichtspunkt der Arithmetizität studieren.

Acknowledgment

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Preface

Griffiths-Schmid arithmeticity conjecture. Let S, $\mathcal{X}$ be complex algebraic manifolds such that $\mathcal{X}$ is bimeromorphic to a Kähler manifold and let $f\colon \mathcal{X} \to S$ be a flat proper holomorphic map such that every fiber $X_s = f^{-1}(s)$ is a connected algebraic manifold. We can consider $f\colon \mathcal{X} \to S$ as a family of algebraic manifolds parametrized by S. Furthermore, let ω be a global section of $R^2 f_* \mathbb{Z}_{\mathcal{X}}$ such that for every $s \in S$ the restriction of ω to the stalk $(R^2 f_* \mathbb{Z}_{\mathcal{X}})_s$ is an integral Kähler class $\omega_s \in H^{1,1}(X_s) \cap H^2(X_s, \mathbb{Z})$. By the Koidara embedding theorem every fiber X_s is hence a projective algebraic variety and we call such an $f\colon \mathcal{X} \to S$ a *family of polarized algebraic manifolds*. The family $f\colon \mathcal{X} \to S$ naturally carries such an ω if $\mathcal{X}$ itself is projective. Then for any embedding $\mathcal{X} \to \mathbb{P}^N$ we can pull back the Kähler class associated to the Fubini-Study metric on $\mathbb{P}^N$ to $\mathcal{X}$ and this defines in a natural way a global section of $R^2 f_* \mathbb{Z}_{\mathcal{X}}$ as we want it. It is a really hard task to understand such families geometrically, i.e. how are the topological and geometric features of the fibers related to each other?

One of the tools to understand the topology of the fibers X_s of $f\colon \mathcal{X} \to S$ is to consider their cohomology groups $H^i(X_s, \mathbb{Q})$ ($i \in \mathbb{N}$). We want to describe the family $f\colon \mathcal{X} \to S$ with the help of the cohomology of the fibers in the following way. We fix a base point $s_0 \in S$ and measure how elements of the cohomology group $H^i(X_{s_0}, \mathbb{Q})$ change if we transport them along closed paths in S in a "continuous way". This idea leads to the concept of "monodromy" which we will explain in detail in Chapter 4.

With the theorem of Ehresmann one can show that for every natural number $i \in \mathbb{N}$ the cohomology groups $H^i(X_s, \mathbb{Q})$ glue together to a locally constant sheaf $R^i f_* \mathbb{Q}_{\mathcal{X}}$ over S[1]. If we fix a base point $s_0 \in S$ we can consider the associated monodromy representation

$$\rho_i\colon \pi_1(S, s_0) \longrightarrow \mathrm{GL}(H^i(X_{s_0}, \mathbb{Q})).$$

Our goal is thus to understand the representation ρ_i respectively the image $G_i = \mathrm{Im}(\rho_i)$ of ρ_i. The groups G_i are often called the *algebraic monodromy groups of the family* $f\colon \mathcal{X} \to S$. In general it is really hard to describe the group G_i directly and it is way easier to determine the Zariski closure $\overline{G_i}$ of G_i in the linear group $\mathrm{GL}(H^i(X_{s_0}, \mathbb{Q}))$. By a result of Deligne [20] we know for example that $\overline{G_i}$ is a semi-simple algebraic group. Write $\overline{G_i}(\mathbb{Z})$ for the subgroup of $\overline{G_i}(\mathbb{Q})$ which stabilizes the lattice $H^i(X_{s_0}, \mathbb{Z})$/torsion. One can show that G_i is a subgroup of $\overline{G_i}(\mathbb{Z})$. Hence a natural question is to ask whether the monodromy group G_i is of finite index or of infinite index in $\overline{G_i}(\mathbb{Z})$. In the first case we call G_i *arithmetic* and *thin* otherwise. Phillip Griffiths developed a Hodge

[1] We will do this for a family of compact Riemann surfaces in Section 2.2.3

theoretical approach which brings us a little bit closer to the answer of this question. This approach even led him and Wilfried Schmid to the conjecture which we formulate in the next paragraph.

Each fiber X_s of $f\colon \mathcal{X} \to S$ $(s \in S)$ is a Kähler manifold on which we have the Hodge decomposition

$$H^i(X_s, \mathbb{C}) = \bigoplus_{p+q=i} H_s^{p,q},$$

where $H_s^{p,q}$ is the subspace of classes of closed (p,q)-forms on X_s. In Chapter 4 we will sketch the proof that the subspaces $H_s^{p,q}$ are the fibers of C^∞-subbundles $\mathbf{H}^{p,q}$ of the holomorphic vector bundle $\mathbf{H}^i$ associated to $R^i f_* \mathbb{Z}_{\mathcal{X}}$. All these information can be summarized in the notion of a *mixed variation of Hodge structure* on S (c.f. Definition 4.4.9). Results from [48] on mixed variation of Hodge structures led Griffiths and Schmidt in 1973 to conjecture that all monodromy groups that arise from families of algebraic manifolds as above could be arithmetic [49, Appendix (d)]. This conjecture was the starting point of the work on the question whether monodromy groups of families of algebraic manifolds are arithmetic or thin. A part of the wonderful survey [108] of Peter Sarnak is dedicated to this topic where he explains some history and progress relating this question. The first examples of families with thin monodromy were given by Deligne and Mostow in 1986 [23]. Meanwhile other examples of families with thin monodromy could be found. Brav and Thomas for example found new families with thin monodromy among families of Calabi-Yau three-folds which arise as the set of solutions of a differential equation on $\mathbb{P}^1(\mathbb{C})\backslash\{0,1,\infty\}$ [11]. Their proofs rely on the Ping-Pong Lemma. This approach was later extended for example in [36] and [4]. In the last article the authors constructed Ping-Pong tables with the help of a computer.

Families of linear deformations of translation surfaces. In this thesis we will focus on Griffiths' and Schmid's arithmeticity conjecture in the context of families of compact Riemann surfaces that come from linear deformations of origamis. Before we can present results, we have to describe the families we are interested in.

A *translation surface* is a compact connected Riemann surface constructed by gluing finitely many polygons P_i $(i = 1, \ldots, n)$ in the Euclidean plane $\mathbb{R}^2$ along their edges by translations. This leads to a flat metric on the Riemann surface where we remove finitely many points which correspond to vertices of the polygons P_i. We denote translation surfaces by pairs (X, ω) where X is a connected compact Riemann surface and ω is a non-trivial holomorphic one-form which provides the surface with a flat metric outside the finite set of zeros of ω. The moduli space of genus g translation surfaces $\Omega\mathcal{M}_g$ is in a natural way a bundle over the moduli space of compact connected Riemann surfaces $\mathcal{M}_g$ (see Section 3.2.2). The zeros of ω give a partition $\underline{\kappa}$ of $2g-2$ by the Theorem of Gauss-Bonnet and in this way we get a stratification $\Omega\mathcal{M}_g = \bigsqcup_{\underline{\kappa}} \Omega\mathcal{M}_g(\underline{\kappa})$ of the moduli space of translastion surfaces (c.f. Section 3.2.3). Applying matrices in $\mathrm{GL}_2^+(\mathbb{R})$ to the polygons $P_i \subset \mathbb{R}^2$ of a translation surface and regluing them leads to an action of $\mathrm{GL}_2^+(\mathbb{R})$ on the moduli space of genus g translation surfaces $\Omega\mathcal{M}_g$ (see Figure 1 for an example). If $A \in \mathrm{GL}_2^+(\mathbb{R})$ is a matrix and $(X, \omega) \in \Omega\mathcal{M}_g$ is a translation surface then we write $A.(X, \omega)$ for the translation surface obtained by the construction above.

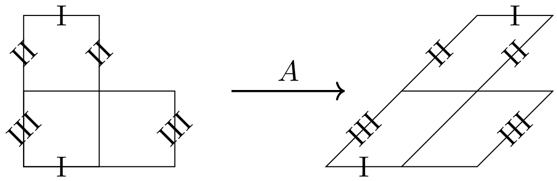

Figure 1.: Deformation of an L-shaped translation surface by the matrix $A = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$.

We want to relate geometric an topological features of the linear deformation $A.(X, \omega)$ of a translation surface (X, ω) by a matrix $A \in \mathrm{GL}_2^+(\mathbb{R})$ to the corresponding features of (X, ω). This is possible in many ways and for many features. See Section 4.5 of the survey [33] for an overview. We want to restrict ourselves to translation surfaces (X, ω) and matrices $A \in \mathrm{GL}_2^+(\mathbb{R})$ such that all the $A.(X, \omega)$ fit into a family in terms of complex geometry respectively algebraic geometry. We explain now how the last sentence is meant.

For a translation surface (X, ω) linear deformation by homothety and rotation lead to translation surfaces in the same fiber of the projection map $\Omega\mathcal{M}_g \to \mathcal{M}_g$. Thus the projection of the $\mathrm{GL}_2^+(\mathbb{R})$-orbit of (X, ω) to $\mathcal{M}_g$ lead to a mapping $\mathbb{H} \to \mathcal{M}_g$, which can be shown to be holomorphic. Under rare circumstances such an $\mathrm{GL}_2^+(\mathbb{R})$-orbit is closed in $\Omega\mathcal{M}_g$ or equivalent [67] there is a lattice $\mathrm{SL}(X, \omega) \leqslant \mathrm{SL}_2(\mathbb{R})$ stabilizing (X, ω). The lattice $\mathrm{SL}(X, \omega)$ is called *Veech group* of the translation surface (X, ω). The projection $\mathbb{H} \to \mathcal{M}_g$ factors through a conjugate $\mathrm{SL}^*(X, \omega)$ of $\mathrm{SL}(X, \omega)$ called *mirror Veech group* and the image of $\mathbb{H}/\mathrm{SL}^*(X, \omega) \to \mathcal{M}_g$ is an algebraic but non-complete curve in the moduli space of compact Riemann surfaces $\mathcal{M}_g$. We call a translation surface (X, ω) with these properties a *Veech surface*. A certain class of Veech surfaces are *origamis* or *square-tiled*-surfaces which are built out of finitely many Euclidean unit squares.

We now consider a Veech surface (X, ω) of genus $g \geqslant 2$. Recall that the points of $\mathbb{H}/\mathrm{SL}^*(X, \omega) \to \mathcal{M}_g$ parametrize compact Riemann surfaces. After passing to a certain torsion free finite index subgroup Γ of the mirror Veech group $\mathrm{SL}^*(X, \omega)$, we can put together the compact Riemann surfaces parametrized by $\mathbb{H}/\mathrm{SL}^*(X, \omega)$ to a family of compact Riemann surfaces $f\colon \mathcal{X} \to \mathbb{H}/\Gamma$ (c.f. Section 4.6.1). If we choose $c \in \mathbb{H}/\Gamma$ with fiber $f^{-1}(\{c\}) = X$, then the local system $R^1 f_* \mathbb{Q}_{\mathcal{X}}$ has a monodromy representation

$$\pi_1(\mathbb{H}/\Gamma, c) \longrightarrow \mathrm{GL}(H^1(X, \mathbb{Q})).$$

We can describe the monodromy representation as follows. We have chosen the finite index subgroup $\Gamma \leqslant \mathrm{SL}(X, \omega)$ torsion free and hence an element $[\gamma] \in \pi_1(\mathbb{H}/\Gamma, c)$ corresponds to an element $A_\gamma \in \Gamma$. We will later show that the class $[\gamma]$ acts on $H^1(X, \mathbb{Q})$ as the pullback by a homeomorphism φ_{A_γ} on X (see Proposition4.6.5). Furthermore the homeomorphism φ_{A_γ} preserves the set of zeros $Z(\omega) \subset X$ of ω and φ_{A_γ} acts on the translation charts of $X \backslash Z(\omega)$ as an affine map with linear part $A_\gamma \in \Gamma$.

We write in the following $\mathrm{Aff}^+(X,\omega)$ for the group of orientation preserving homeomorphisms of X, which respect $Z(\omega)$ and which are affine on the translation charts of (X,ω). Since $\Gamma \leqslant \mathrm{SL}(X,\omega)$ is torsion free, we can identify the group Γ from above with a torsion free subgroup of $\mathrm{Aff}^+(X,\omega)$. The action of $\mathrm{Aff}^+(X,\omega)$ on $H^1(X,\mathbb{Z})$ by pullback respects the symplectic intersection pairing on $H^1(X,\mathbb{Q})$ and thus the monodromy representation of $R^1f_*\mathbb{Q}_{\mathfrak{X}}$ is given by the action

$$\rho\colon \Gamma \longrightarrow \mathrm{Sp}(H^1(X,\mathbb{Q})).$$

of Γ on $H^1(X,\mathbb{Q})$ by pullback.

We can describe the monodromy of $R^1f_*\mathbb{Q}_{\mathfrak{X}}$ even further if we use Teichmüller theory. Let $K(X,\omega) = \mathbb{Q}(\mathrm{tr}(\mathrm{SL}(X,\omega))$ be the trace field of the Veech group $\mathrm{SL}(X,\omega)$. It is a totally real algebraic extension of $\mathbb{Q}$ of degree at most g [87, Theorem 5.1]. Write F for the Galois closure of $K(x,\omega)$. The subspace $H^1_{st}(X) = \mathrm{span}_{\mathbb{R}}(\{\mathrm{Re}(\omega),\mathrm{Im}(\omega)\})$ of $H^1(X,\mathbb{R})$ is invariant under the action of $\mathrm{Aff}^+(X,\omega)$ on $H^1(X,\mathbb{R})$ by pull back. Furthermore, one can show that an element $\varphi \in \mathrm{Aff}^+(X,\omega)$ acts on $H^1_{st}(X)$ by its linear part $D(\varphi) \in \mathrm{SL}(X,\omega)$ with respect to the basis $(\mathrm{Re}(\omega),\mathrm{Im}(\omega))$. We consider $H^1_{st}(X)$ as a subspace of $H^1(X,\mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{R}$. The subspace $H^1_{st}(X)$ is defined over the trace field $K(X,\omega)$ [94, Coroallary 2.10] and so we can consider the Galois conjugate representations of $\mathrm{Aff}^+(X,\omega)$ on it with respect to $\mathrm{Gal}(F|\mathbb{Q})$. By [94] for every $\sigma \in \mathrm{Gal}(F|\mathbb{Q})$ the Galois conjugate representation of $\mathrm{Aff}^+(X,\omega)$ on $H^1_{st}(X)^\sigma$ is isomorphic to that of $\mathrm{Aff}^+(X,\omega)$ on $H^1_{st}(X)$ if and only if σ fixes $K(X,\omega)$. Let σ_i $(i = 1,\dots,r)$ be a system of representatives of $\mathrm{Gal}(F,\mathbb{Q})/\mathrm{Gal}(F,K(X,\omega))$, where $\sigma_1 = \mathrm{id}$. Then the subspace $\oplus_{i=1}^r H^1_{st}(X)^{\sigma_i}$ is invariant under $\mathrm{Gal}(F,\mathbb{Q})$ and is thus defined over $\mathbb{Q}$. Hence we get the following decomposition into $\mathrm{Aff}^+(X,\omega)$-subrepresentations

$$H^1(X,\mathbb{Q}) = W \oplus H^1_{(0)} \quad \text{with} \quad W \otimes_{\mathbb{Q}} \mathbb{R} = \bigoplus_{i=1}^{r} H^1_{st}(X)^{\sigma_i}, \tag{0.0.0.1}$$

where $H^1_{(0)}$ is the orthogonal complement of W with respect to the intersection pairing. We already pointed out that we perfectly understand the action of $\mathrm{Aff}^+(X,\omega)$ on the subspace $W \otimes_{\mathbb{Q}} \mathbb{R}$ of $H^1(X,\mathbb{R})$. This is in general not the case for the remaining part $H^1_{(0)} \subset H^1(X,\mathbb{Q})$. Hence we will focus on the action of Γ respectively of $\mathrm{Aff}^+(X,\omega)$ on $H^1_{(0)}$. We call the associated subrepresentations

$$\rho^{(X,\omega)}_{\mathrm{KoZo}}\colon \mathrm{Aff}^+(X,\omega) \longrightarrow \mathrm{Sp}(H^1_{(0)}) \quad \text{and} \quad \rho^{\Gamma}_{\mathrm{KoZo}}\colon \Gamma \longrightarrow \mathrm{Sp}(H^1_{(0)}),$$

the *Kontsevich–Zorich (monodromy) representation of the translation surface* (X,ω), respectively the *Kontsevich–Zorich (monodromy) representation of the family* $f\colon \mathfrak{X} \to \mathbb{H}/\Gamma$. Furthermore, we write $G_{(X,\omega)}$ for the image of the representation $\rho^{(X,\omega)}_{\mathrm{KoZo}}$, respectively G_Γ for the image of $\rho^{\Gamma}_{\mathrm{KoZo}}$. We call $G_{(X,\omega)}$ the *Kontsevich–Zorich monodromy.* The group G_Γ is a finite index subgroup of $G_{(X,\omega)}$ and since we are interested in arithmeticity, we will mainly focus on $G_{(X,\omega)}$. A first question is again how does the Zariski closure of the Kontsevich–Zorich monodromy looks like? A very good answer was given by Filip in [34]. He showed that the Zariski closure of the monodromy group of any direct summand of $\rho_{\mathrm{KoZo}}^{(X,\omega)}\colon \mathrm{Aff}^+(X,\omega) \to \mathrm{Sp}(H^1_{(0)})$ is up to finite index and compact factors one of the following groups and representations:

- $\mathrm{Sp}(2d,\mathbb{R})$ in the standard representation.
- $\mathrm{SU}_{\mathbb{C}}(p,q)$ in the standard representation.
- $\mathrm{SU}_{\mathbb{C}}(p,1)$ in an exterior power representation.
- $\mathrm{SO}^*(2n)$ in the standard representation.
- $\mathrm{SO}_{\mathbb{R}}(n,2)$ in a spin representation.

Hence the question which Zariski closures can occur has a satisfying answer and the next step is then the study of arithmeticity for the Kontsevich–Zorich monodromy. One extreme case of the splitting 0.0.0.1 is realized by translation surfaces which are constructed by gluing finitely many unit squares. These special translation surfaces are called *origamis*. Their Veech groups are finite index subgroups of $\mathrm{SL}_2(\mathbb{Z})$. Thus the trace field of their Veech groups is always given by the rational numbers $\mathbb{Q}$. Hence for an origami $\mathcal{O} = (X,\omega)$ the two-dimensional subspace $H^1_{st}(X)$ is defined over $\mathbb{Q}$ and with a slight abuse of notation we get the splitting

$$H^1(X,\mathbb{Q}) = H^1_{st}(X) \oplus H^1_{(0)}(X).$$

The action of the affine group $\mathrm{Aff}^+(\mathcal{O})$ of an origami $\mathcal{O} = (X,\omega)$ on the cohomology $H^1(X,\mathbb{Q})$ is also way easier to understand then for general translation surfaces. For example for origamis without automorphisms we can describe the action of the affine group on the (relative) homology combinatorially [85, Section 3]. Hence we will restrict ourselves in the proceeding investigation on the monodromy arising from origamis.

Simion Filip proved in [33] that in each stratum $\Omega\mathcal{M}_g(\underline{\kappa})$ ($g \geqslant 2$) the Zariski closure of the Kontsevich–Zorich monodromy of all origamis outside of a finite set of $\mathrm{GL}_2(\mathbb{R})$-suborbit closures is isomorphic to $\mathrm{Sp}_{2g-2}(\mathbb{R})$ and thus as big as possible. The question of arithmeticity for an origami with Kontsevich–Zorich monodromy with largest possible Zariski closure was first answered in genus three by Hubert and Matheus [63]. They answered it positively although they intended to get a different result. The investigation in genus three was continued in [8], where the authors found seven infinite families in the stratum $\Omega\mathcal{M}_3(4)$ with arithmetic Kontsevich–Zorich monodromy. Furthermore, we found in [8] a finite family of genus four origamis in the stratum $\Omega\mathcal{M}_4(6)$ with arithmetic Kontsevich–Zorich monodromy and full Zariski closures. In [71] Carlos Matheus and the author of this thesis extended the results of [8] on infinite families in genus four, five and six with arithmetic Kontsevich–Zorich monodromies and Zariski closures the whole symplectic group. As far as the author of this text knows there are no such examples for higher genera then genus six. A special role in this context is played by origamis of genus two. Martin Möller observed that the Kontsevich–Zorich monodromy of all origamis of genus two is arithmetic. We explained and elaborated his ideas which are based on

Hodge theory in Appendix B of [8]. A natural further question for genus two origamis is therefore to ask about the index of the Kontsevich–Zorich monodromy in $\mathrm{PSL}_2(\mathbb{Z})$. Computer experiments for origamis in the stratum $\Omega\mathcal{M}_2(2)$ and primitive origamis in $\Omega\mathcal{M}_2(1,1)$ showed that the index in $\mathrm{PSL}_2(\mathbb{Z})$ is expected to be either one or three (c.f. [8, Section 6]). This conjecture was partially proved in [76].

Results and structure of this thesis. In this thesis we present our contributions to the investigation of the Kontsevich–Zorich monodromy of origamis. In the second part of this thesis we present our results which divide into two blocks according to the very different tools we use. More precisely, there is one block which relies on Hodge theory and one block where we use concepts from the theory of algebraic groups. In the first part of this text respectively the first five chapters we provide the necessary theoretical background for this thesis. My goal, in the first place, is to develop the full theory to understand the results. In particular we provide references which in our opinion are useful to learn the different aspects of mathematics which play a role in this thesis and more generally in the field of Teichmüller dynamics. In this sense we hope that this text can help newcomers to get a first idea of the fascinating area of Teichmüller dynamics.

Now, we come to a more detailed outline of this text. The idea of Chapter 1, on the one hand, is to give a short general introduction to the theory of algebraic groups. On the other hand we will sketch the proof of [100, Theorem 9.10], a beautiful result of Prasad and Rapinchuk, which is together with [111, Theorem 1.2] of Singh and Venkataramana, one of the main ingredients of the arithmeticity results for our families from [71].

In Chapter 2 and Chapter 3 we recall some important facts about Teichmüller theory. Hereby we provide information for both aspects, the analytic theory of Teichmüller spaces and Grothendieck's complex geometric approach to Teichmüller theory. Furthermore, Chapter 3 provides the mathematical preliminaries for translation surfaces in particular the moduli space of translation surfaces and Teichmüller curves. We will end Chapter 3 with Section 3.3, where we study Dehn (multi-)twists about the core curves of cylinder decompositions of translation surfaces. This will be a very important tool for computing explicit elements of the Kontsevich–Zorich monodromy.

After that we will explain in the first part of Chapter 4 the different notions and concepts of monodromy and parallel transport which are relevant for this text. We will give a lot of proofs in this part because it was either hard to find references which fit to our needs or they were missing relevant details. In the second part of Chapter 4 we will provide a short introduction to Hodge theory and period mappings. In particular, we collect all the necessary concepts and tools to state Griffiths' and Schmid's conjecture about the algebraic monodromy of families of curves from the beginning of the introduction (c.f. 4.4.18). But we do not only state their conjecture but we will also state the theorem which led to this conjecture (c.f. 4.5.7). Finally, we will properly define Kontsevich–Zorich monodromy in the last section of this chapter.

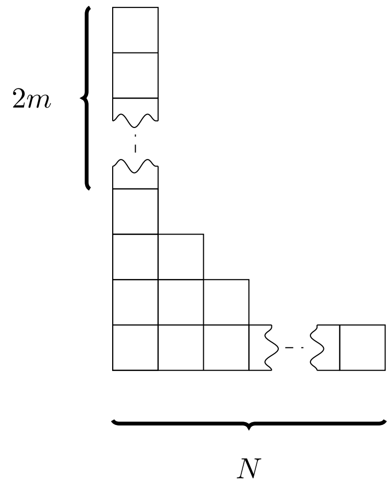

Figure 2.: The origami $\mathcal{O}^{(4)}_{N,M}$.

The last chapter of the preliminary part is dedicated to Kontsevich–Zorich cocycle for Teichmüller curves. There is a deep connection between Hodge theory and the Kontsevich–Zorich cocycle, in particular its Lyapunov exponents. We will state in this chapter the results in this context which we will need later in this thesis.

Section 6.1 of Chapter 6 primarily aims at proving the observation of Martin Möller which was published in appendix B of [8]. The observation is formulated in the following theorem:

A Theorem (see Theorem 6.1.5)**.** For every genus two origami $\mathcal{O} = (X, \omega)$ the Kontsevich–Zorich monodromy $G_{\mathcal{O}}$ has Zariski closure isomorphic to $\mathrm{SL}_2(\mathbb{R})$ and it is an arithmetic group.

The proof of this theorem is based on ideas of Martin Möller and arose in collaboration with Carlos Matheus. As we already mentioned, the techniques we use to prove Theorem A rely heavily on Hodge theory and its applications to Teichmüller dynamics. In particular Theorem 4.5.7 of Griffiths, which led to Griffiths' and Schmid's artihmeticity conjecture 4.4.18, is an important ingredient of the proof. Note that Griffiths used in [48, Appendix D] the same ideas as we did for the proof of Theorem A to show that a family of K3 surfaces has an arithmetic algebraic monodromy group. In Section 6.4 we use the tools which we developed in Section 6.1 to gain some insights for the Kontsevich–Zorich monodromy of certain coverings of L-shaped translation surfaces.

All the remaining arithmeticity results of this thesis rely on a different approach which is based on algebraic groups and geometric group theory. Let us now state these remaining results. Let $N \geqslant 4$ and $M = 4 + 2m$ with $m \geqslant 0$. We consider the stairs origami $\mathcal{O}^{(4)}_{N,M}$ of degree $N + M + 2$ which can be seen in Figure 2. We showed in Chapter 8 for a finite family of origamis $\mathcal{O}^{(4)}_{N,M}$ that the monodromy groups $G_{\mathcal{O}^{(4)}_{N,M}}$ are arithmetic. More precisely we obtain the following theorem:

B Theorem (see Theorem 8.1.3)**.** For all $N = 3m+4$ and $M = 2m+4$ with $4 \leqslant N \leqslant 50$ and $m \in \{0, \ldots, 50\}$ the monodromy groups $G_{\mathcal{O}^{(4)}_{N,M}}$ have full Zariski closure and they are arithmetic groups.

For the proof of Theorem B we used a theorem of Singh and Venkataramana (see Theorem 1.6.1). It requires that the Kontsevich–Zorich monodromy has full Zariski closure which turned out to be the hardest part. In the case of Theorem B we used a computational approach to obtain full Zariski closures and this is why we could only prove it for a finite family. In [71] Carlos Matheus and the author of this thesis overcame this problem with a theorem of Prasad and Rapinchuk (see Theorem 1.5.8) and Galois theory. This idea was already used in [8] for genus three origamis. We extended these results to infinite families of origamis in genus four, five and six. The origamis that we considered in genus five and six are very similar to the origamis $\mathcal{O}^{(4)}_{N,M}$, we only added stairs to get higher genus. The results from [71] are summarized in the following theorem.

C Theorem (see Theorem 9.2.3, Theorem 9.3.5 and Theorem 9.4.5)**.** In genus four, five and six we have infinite families of stairs origamis, such that the Kontsevich–Zorich monodromy of these origamis have full Zariski closure and furthermore they are arithmetic subgroups.

The last theorem is part of joined work with Pascal Kattler and Gabriela Weitze-Schmithüsen which we started quite recently. We showed that there are indeed origamis which do have thin Kontsevich–Zorich monodromy.

D Theorem (see Theorem 7.2.3)**.** In genus three there is an origami whose Kontsevich monodromy has Zariski closure $\mathbf{SL}_2(\mathbb{R}) \times \mathbf{SL}_2(\mathbb{R})$ but the monodromy group is a thin subgroup of the closure.

Due to computer experiments by Pascal Kattler we can assume that there are more examples like this. But the question whether there are thin Kontsevich–Zorich monodromy groups with largest possible Zarsiki closure remains open.

Previously published content. Parts of this thesis have been published in the article [8] which is accepted for publication in Transactions of the AMS and the article [71] which was published in Mathematische Nachrichten.

Chapter 6 is based on the tools developed in Appendix B of [8]. But we present the results in a slightly different way so that we can use them more easily for Section 6.4 and Section 7.1.

The calculations in the first part of 7.2 are taken from Appendix A of [8] which is due to Etienne Bonnafoux and Carlos Matheus.

In Chapter 8 we integrated the work of Section 5 of [8]. Chapter 9 is taken from [71] with some minor changes such that it better fits to the preliminary part of this thesis.

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Part I.

Preliminaries

1. Algebraic groups

In Section 1.1– Section 1.4 of this chapter we will provide the background in the theory of algebraic groups and Lie algebras which we need for this thesis. An algebraic group is classically a group defined by polynomial equations over a field k. We should be able to consider the solutions of the equations in any k-algebra R. To take this idea into account we have to use the language of group schemes. Since we are mainly working with classical groups, we are only interested in closed points of schemes. Luckily the books [91] and [92] of James Milne are perfectly suitable for this. For beginners a book that uses the classical approach of algebraic groups over algebraically closed fields is still useful and here I can recommend the book [84] of Gunter Malle, which is for me the best introduction to this field. In the last sections of this chapter we want to explain results of Prasad, Rapinchuk [100] and Singh, Venkataramana [111] which we used in the proof of Theorem B and Theorem C.

1.1 Definitions of algebraic groups

As we already mentioned we are only working with classical groups and are only interested in closed points of them. Let k be a field and let A be a k-algebra. Instead of considering the topological space $\mathrm{Spec}(A)$ which consists of the set of all prime ideals on A we consider the subset $\mathrm{Spm}(A) \subset \mathrm{Spec}(A)$ which consists of all maximal ideals of A. In the same way as for Spec we can define the Zariski topology on $\mathrm{Spm}(A)$ and furthermore we can associate to $\mathrm{Spm}(A)$ a sheaf of k-algebras $\mathcal{O}_{\mathrm{Spm}(A)}$ (see Appendix A of [92]). An *affine algebraic scheme* over k is per definition a locally k-ringed space which is isomorphic to the pair $(\mathrm{Spm}(A), \mathcal{O}_{\mathrm{Spm}(A)})$ for some finitely generated k-algebra A. A *morphism of affine algebraic schemes over k* is a morphism of locally k-ringed spaces. Affine algebraic schemes over k with morphisms of locally k-ringed spaces forms a category which we will denote in the following by $\underline{k\text{−AffSchm}}$. Furthermore, we write $\underline{k\text{−AffSchm}}^0$ for the category of affine schemes of finite type over a field k, i.e. the category of schemes over k which have an open affine cover with obvious morphisms. By a little abuse of notation we often write $\mathrm{Spm}(A)$ if we mean the topological space equipped with the sheaf of k-algebras. The functor $A \mapsto (\mathrm{Spm}(A), \mathcal{O}_{\mathrm{Spm}(A)})$ is a contravariant equivalence between the category of k-algebras and the category of affine k-schemes (see Appendix A of [92] for more details).

1.1.1 Definition (Algebraic groups as affine group schemes)**.** An *algebraic group G (defined) over a field k* is an affine algebraic scheme $(\mathrm{Spm}(A), \mathcal{O}_{\mathrm{Spm}(A)})$ for a finitely generated k-algebra A with morphisms of locally ringed spaces $m\colon G\times G\to G$, $\mathrm{inv}\colon G\to G$ and $e\colon \mathrm{Spm}(k)\to G$ such that the following diagrams commute:

$$\begin{tikzcd} G\times G\times G \arrow[r, "\mathrm{id}\times m"] \arrow[d, "m\times \mathrm{id}"] & G\times G \arrow[d, "m"] \\ G\times G \arrow[r, "m"] & G \end{tikzcd} \qquad \begin{tikzcd} \mathrm{Spm}(k)\times G \arrow[r, "(e{,}\mathrm{id})"] \arrow[dr, "\cong"'] & G\times G \arrow[d, "m"] & G\times \mathrm{Spm}(k) \arrow[l, "(\mathrm{id}{,}e)"'] \arrow[dl, "\cong"] \\ & G & \end{tikzcd}$$

$$\begin{tikzcd} G \arrow[r, "(\mathrm{inv}{,}\mathrm{id})"] \arrow[d] & G\times G \arrow[d, "m"] & G \arrow[l, "(\mathrm{id}{,}\mathrm{inv})"'] \arrow[d] \\ \mathrm{Spm}(k) \arrow[r, "e"] & G & \mathrm{Spm}(k) \arrow[l, "e"'] \end{tikzcd}$$

The morphism $G\to\mathrm{Spm}(k)$ exists since G is of finite type over k. The isomorphism of $\mathrm{Spm}(k)\times G\cong G$ in the diagram on the right above is just the natural isomorphism $\mathrm{Spm}(k)\times G=\mathrm{Spm}(A\otimes_k k)\cong G$. We say that A is a *coordinate ring of* G and we write $\mathcal{O}(G)=A$. A *group variety* is an algebraic group such that the coordinate ring $\mathcal{O}(G)=A$ of G is reduced, i.e. $A\otimes_k\overline{k}$ has no non-zero nilpotent elements for an algebraic closure $\overline{k}$ of k.

A *morphism of algebraic groups* $\varphi\colon (G,m_G)\to(H,m_H)$ is a morphism of locally ringed spaces $\varphi\colon G\to H$ such that $\varphi\circ m_G=m_H\circ(\varphi\times\varphi)$.

Consider the functor between the category of affine schemes of finite type over a field k and the category of functors between the category of affine schemes of finite type and the category of sets

$$\underline{k\mathrm{-AffSchm}^0}\longrightarrow \mathrm{Func}(\underline{k\mathrm{-AffSchm}^0},\underline{Sets}),$$

which sends an affine scheme X of finite type to the Hom-functor $h_X=\mathrm{Mor}(\cdot,X)$ and a morphisms $X\to Y$ between affine schemes of finite type to the natural transformation $h_X\to h_Y$. The Lemma of Yoneda says that this functor is fully faithful. We will use this fact to define algebraic groups as a functor.

Let (G,m) be an algebraic group with coordinate ring $\mathcal{O}(G)=A$. For a k-algebra R we write $G(R)$ for the set of points of G with coordinates in R, i.e.

$$G(R)=h_G(\mathrm{Spm}(R))=\mathrm{Mor}(\mathrm{Spm}(R),G)\cong\mathrm{Hom}_{k-\mathrm{Alg}}(A,R).$$

Furthermore the group multiplication $m\colon G\times G\to G$ induces a natural transformation $h_{G\times G}\to h_G$ and for a k-algebra R we write $m(R)$ for the map

$$m(R)\colon h_{G\times G}(\mathrm{Spm}(R))\longrightarrow h_G(\mathrm{Spm}(R)).$$

Hence the map $R \longmapsto (G(R), m(R))$ defines a functor from the category of k-algebras to the category of groups and we gave sense to what we mean by the R-points of an algebraic group defined over k. These considerations also give rise to the second definition of algebraic groups.

1.1.2 Definition (Algebraic groups as functors)**.** An *algebraic group defined over k* is a functor

$$F_G \colon \underline{k\text{−Alg}} \longrightarrow \underline{\text{Grps}}$$

from the category of finitely generated k-algebras to the category of groups that is representable, i.e. there is a finitely generated k-algebra A such that F_G is naturally equivalent to the functor h^A.

In the setting of Definition 1.1.2 a homomorphism of algebraic groups $\alpha \colon F_G \to F_H$ is a natural transformation such that $\alpha_R \colon F_G(R) \to F_H(R)$ is a group homomorphism for every k-algebra R.

Sketch of equivalence of Definition 1.1.1 and Definition 1.1.2. If we have a functor F_G as above with a finitely generated k-algebra A such that $h^A(R) = \mathrm{Hom}_{k-\mathrm{Alg}}(A, R)$ for every k-algebra R, then we define $G = \mathrm{Spm}(A)$. Thus

$$F_G(R) \cong \mathrm{Mor}(\mathrm{Spm}(R), G)$$

for every k-algebra R. In particular for every finitely generated k-algebra R we have a group multiplication $m_R \colon h^A(R) \times h^A(R) \to h^A(R)$, a map that builds the inverse group elements $\mathrm{inv}_R \colon h^A(R) \to h^A(R)$ and a neutral element $e \in h^A(K)$. By the Lemma of Yoneda this implies that there exist morphisms of schemes $m \colon G \times G \to G$, $\mathrm{inv} \colon G \to G$ and $e \colon \mathrm{Spm}(k) \to G$ which deliver commutative diagrams as in 1.1.1. □

One can use Definition 1.1.2 to define normal algebraic subgroups.

1.1.3 Definition. Let G be an algebraic group over k and H an algebraic subgroup of G. We say that H is *normal* in G if $H(R)$ is a normal subgroup of $G(R)$ for every finitely generated k-algebra R.

We will give some important examples.

1.1.4 Example. Let k be a field.

(i) The *additive group* $\mathbb{G}_a$ is the algebraic group with coordinate ring $k[\mathbb{G}_a] = k[X]$. It can be characterized by a functor that maps a k-algebra R to its additive group $(R, +)$.

(ii) The *multiplicative group* $\mathbb{G}_m$ is the algebraic group with coordinate ring $k[\mathbb{G}_m] = k[X, Y]/(XY - 1)$. It corresponds to the functor that maps a k-algebra R to its multiplicative group $(R^x, \cdot)$.

(iii) An algebraic group T over k is a *torus*, if after extending the base field k by a finite separable field F, it becomes isomorphic to a product of copies of $(\mathbb{G}_m)_F$ as locally ringed spaces, i.e.

$$T_F \cong (\mathbb{G}_m)_F \times \cdots \times (\mathbb{G}_m)_F.$$

We say that the torus T *splits over the field* F.

(iv) Let G be an algebraic group over k. We write G^0 for the connected component of G which contains the neutral element. Then G^0 is a normal algebraic subgroup of G [92, Proposition 1.34 and Proposition 1.52]

1.1.5 Remark (Subspace topology of k-points)**.** Let G be an algebraic group over a field k with coordinate ring $\mathcal{O}(G) = A$, where A is a finitely generated k-algebra. Again a point $P \in G(k)$ is a homomorphism of k-algebras $P\colon A \to k$. Thus the kernel $m_P = \ker(P)$ is a maximal ideal in A and hence a point in $\mathrm{Spm}(A)$. In this way we get an inclusion $G(k) \hookrightarrow \mathrm{Spm}(A)$. In the future we will equip $G(k)$ with the subspace topology of the Zariski topology on $\mathrm{Spm}(A)$.

Let G be an algebraic group over a field k and let $S \leqslant G(k)$ be an arbitrary subgroup. By [92, Lemma 1.40] the closure $\overline{S}$ of S in $G(k)$ equipped with the subspace topology is again a subgroup of $G(k)$ and by [92, Theorem 1.45] there is a unique algebraic subgroup H of G with k-points $H(k) = \overline{S}$.

1.1.6 Definition (Zariski closure)**.** Let G be an algebraic group over k and S a subgroup of $G(k)$. The unique algebraic subgroup H of G such that $H(k)$ is the closure of S in $G(k)$ is called the *Zariski closure* of S in G.

1.2 Representations and arithmeticity

Let k be a field and V a finite dimensional k-vector space. For R a k-algebra we write $V_R = V \otimes_k R$. We denote by GL_V the functor

$$\mathrm{GL}_V\colon \underline{k\text{−Alg}} \longrightarrow \underline{\mathrm{Grps}},$$

where $\mathrm{GL}_V(R) = \mathrm{Aut}(V_R)$ is the group of R-linear automorphisms of V_R. If V is a vector space of dimension n, then a choice of a k-basis of V determines an isomorphism of functors $\mathrm{GL}_V \to \mathrm{GL}_n$ where GL_n is the algebraic group with coordinate ring

$$k[\mathrm{GL}_n] = k[T_{11}, T_{12}, \ldots, T_{nn}, T]/(\det(T_{ij})_{ij}T - 1).$$

This shows that GL_V is also an algebraic group over k.

1.2.1 Definition. Let k be a field and let G be an algebraic group. A *linear representation* of G on a finite-dimensional k-vector space V is a morphism of algebraic groups $\rho\colon G \to \mathrm{GL}_V$. We say that ρ is *faithful* if $\rho(R)\colon G(R) \to \mathrm{GL}_V(R)$ is injective for every k-algebra R.

1.2.2 Definition. Let G be an affine algebraic group over $\mathbb{Q}$ and let $\rho\colon G \to \mathrm{GL}_V$ be a faithful linear representation on a finite dimensional $\mathbb{Q}$-vector space V. For a lattice $L \subset V$ we define

$$G(\rho, L) = \mathrm{Stab}_{G(\mathbb{Q})}(L) = \{g \in G(\mathbb{Q}) \mid g \cdot L = L\}.$$

An *arithmetic subgroup* of $G(\mathbb{Q})$ is any subgroup commensurable with $G(\rho, L)$, i.e. a subgroup S of $G(\mathbb{Q})$ such that $S \cap G(\rho, L)$ has finite index in S and finite index in $G(\rho, L)$.

The definition of arithmeticity in 1.2.2 is independent of the choice of a faithful representation $\rho\colon G \to \mathrm{GL}_V$ and a lattice $L \subset V$ [90, Proposition 28.8].

1.3 Roots and one-parameter subgroups

1.3.1 Functor Lie. Lie algebras are one of the most important tools when it comes to work with algebraic groups, especially in characteristic zero. We want to give some basic constructions on Lie algebras associated to algebraic groups.

Let G be an algebraic group over a field k with structure sheaf $\mathcal{O}_G$ and coordinate ring $\mathcal{O}(G)$. The neutral element is by definition an element $e \in G(k)$ or equivalent a morphism $\mathrm{Spm}(k) \to G$ or a morphism $\mathcal{O}(G) \to k$. We will write $\mathrm{Lie}(G)$ or $\mathfrak{g}$ for the tangent space of G at the neutral element $e \in G(k)$. In the following we want to equip $\mathrm{Lie}(G)$ with Lie brackets but we will start with more details on $\mathrm{Lie}(G)$ as a vector space since this will help us to understand this object.

For every finitely generated k-algebra R we define the algebra $R[\epsilon] = R[X]/(X^2)$, so $\epsilon^2 = 0$ in $R[\epsilon]$. We denote by $\iota_R\colon R \to R[\epsilon]$ the inclusion and by $\pi_R\colon R[\epsilon] \to R$ the homomorphism with

$$\pi_R(a + b \cdot \epsilon) = a \quad (a, b \in R).$$

We write $\mathfrak{g}(k)$ for the kernel $\mathrm{Ker}(G(k[\epsilon]) \xrightarrow{\pi_k} G(k))$. This means that an element in $\mathfrak{g}(k)$ is a homomorphism $\varphi\colon \mathcal{O}(G) \to k[\epsilon]$ such that the composition $\pi_k \circ \varphi$ is the homomorphism $\mathcal{O}(G) \to k$ corresponding to the neutral element e. The kernel of the homomorphism $e\colon \mathcal{O}(G) \to k$ of the neutral element is the maximal ideal m_e. Since $\epsilon^2 = 0$ the homomorphism $\varphi\colon \mathcal{O}(G) \to k[\epsilon]$ from above factors through $\mathcal{O}(G)]/m_e^2$ and by [92, Lemma 3.22] we have $\mathcal{O}(G)/m_e^2 \cong k \oplus m_e/m_e^2$. Hence the homomorphism φ is of the form

$$k \oplus m_e/m_e^2 \longrightarrow k[\epsilon], \quad (a, b) \longmapsto a + D_\varphi(b) \cdot \epsilon$$

with a linear map $D_\varphi\colon m_e/m_e^2 \to k$, uniquely determined by φ. We showed

$$\mathrm{Lie}(G) = \mathrm{Hom}(m_e/m_e^2\,,\,k) \cong \mathrm{Ker}(G(k[\epsilon]) \xrightarrow{\pi_k} G(k)) = \mathfrak{g}(k).$$

Now we have defined the map on objects for our functor Lie. Next we want to make clear what it does with morphisms. Let H be a second algebraic group and $\varphi\colon G \to H$ a morphism of algebraic groups. Since φ is a natural transformation the following diagram commutes.

$$\begin{tikzcd}
G(k[\epsilon]) \arrow[r, "\varphi(k[\epsilon])"] \arrow[d, "\pi_k"] & H(k[\epsilon]) \arrow[d, "\pi_k"] \\
G(k) \arrow[r, "\varphi(k)"] & H(k)
\end{tikzcd}$$

By the diagram above we get a homomorphism $\mathfrak{g}(k) \to \mathfrak{h}(k)$, where $\mathfrak{h}(k)$ is the kernel $\mathrm{Ker}(H(k[\epsilon]) \to H(k))$. Thus the morphism of algebraic groups $\varphi\colon G \to H$ defines a k-vector space homomorphism $\mathrm{Lie}(G) \to \mathrm{Lie}(H)$. We showed that associating the vector space $\mathrm{Lie}(G)$ to an algebraic group is functorial.

1.3.2 Definition. Let k be a field. We call the functor

$$\mathrm{Lie}\colon \underline{k\text{--AlgGr}} \longrightarrow \underline{k\text{--Vec}}$$

from the category of algebraic groups over k to the category of vector spaces over k, which maps an algebraic group G to the vector space $\mathrm{Lie}(G)$, the *Lie functor*.

By definition $\mathrm{Lie}(G) = \mathfrak{g}$ is a vector space. If we want to call it Lie algebra, then we have to define Lie brackets for $\mathrm{Lie}(G)$. This is what we want to do next. Again by [92, Lemma 3.22] we have a split exact sequence

$$0 \longrightarrow m_e \longrightarrow \mathcal{O}(G) \xrightarrow{\;e\;} k \longrightarrow 0.$$

Tensoring it with a finitely generated k-algebra R, we get a short exact sequence of R-modules

$$0 \longrightarrow (m_e)_R \longrightarrow \mathcal{O}(G)_R \xrightarrow{\;e_R\;} R \longrightarrow 0. \tag{1.3.2.1}$$

We define $\mathfrak{g}(R) = \mathrm{Ker}(G(R[\epsilon]) \to G(R))$. An element in $\mathfrak{g}(R)$ is hence a homomorphism $\psi\colon \mathcal{O}(G)_R \to R[\epsilon]$ with $\pi_R \circ \psi = e_R$, where e_R is the extension of the homomorphism $e\colon \mathcal{O}(G) \to k$. From the short exact sequence (1.3.2.1) we get with similar arguments as above

$$\mathfrak{g}(R) \cong \mathrm{Hom}((m_e)_R/(m_e)_R^2\,,\,R) \cong \mathfrak{g}(k) \otimes_k R.$$

Using the identification $\mathfrak{g}(R) = \mathfrak{g} \otimes_k R$ we define the algebraic group $\mathrm{GL}_{\mathfrak{g}}$ as a functor

$$\mathrm{GL}_{\mathfrak{g}}\colon \underline{k\text{--Alg}} \longrightarrow \underline{\mathrm{Grps}},$$

which sends a k-algebra R to the group $\mathrm{Aut}_{R-\mathrm{lin}}(\mathfrak{g}(R))$ of R-linear automorphisms of $\mathfrak{g}(R)$. The group $G(R)$ is a subgroup of $G(R[\epsilon])$ by the inclusion map $\iota_R\colon R \to R[\epsilon]$ and both groups act on $\mathfrak{g}(R)$ by inner automorphisms. Hence we can define a group homomorphism $\mathrm{Ad}(R)\colon G(R) \to \mathrm{GL}_{\mathfrak{g}}(R)$ by

$$\mathrm{Ad}(R)(g)\, x = g \cdot x \cdot g^{-1}, \quad x \in \mathfrak{g}(R),\ g \in G(R)$$

This clearly defines a natural transformation $\mathrm{Ad}\colon G \to \mathrm{GL}_{\mathfrak{g}}$ or in other words a representation of the algebraic group G. By [92, p. 10.7] we have $\mathrm{Lie}(\mathrm{GL}_{\mathfrak{g}})) \cong \mathrm{End}(\mathfrak{g})$. If we apply the functor Lie on the morphism of algebraic groups Ad, we get a homomorphism of k-vector spaces

$$\mathrm{ad}\colon \mathrm{Lie}(G) \longrightarrow \mathrm{Lie}(\mathrm{GL}_{\mathfrak{g}})) \cong \mathrm{End}(\mathfrak{g}).$$

For $A, X \in \mathrm{Lie}(G)$ we define the Lie brackets as $[A, X] = \mathrm{ad}(A)(X)$. With these brackets $\mathrm{Lie}(G)$ becomes a Lie algebra and we call it the *Lie algebra associated to* G. Of course we still have to show that for every morphism of algebraic groups $\varphi\colon G \to H$ the vector space homomorphism $\mathrm{Lie}(\varphi)\colon \mathrm{Lie}(G) \to \mathrm{Lie}(H)$ preserves the Lie brackets we just defined but we omit this part here and refer to [92, pp. 10.19–10.22].

1.3.3 Remark. The Lie functor is the unique functor in the sense that the Lie algebra of the linear group GL_n is given by $\mathrm{Lie}(\mathrm{GL}_n) = k^{n\times n}$ with Lie brackets $[A, B] = A\,B - B\,A$ [92, Theorem 10.23].

The Lie functor is far away from being one to one even for characteristic zero. But it is still very useful, especially if we apply it to algebraic subgroups of an algebraic group. We want to collect some important properties of the functor Lie in this direction from [92, 10.15–10.17, section 10.k] respectively chapter II section 2 and especially Theorem 2.11 of [91]:

1.3.4 Theorem. Let G be an algebraic group over a field k. In the situation of algebraic groups we have the special situation that G is smooth if and only if $\dim(G) = \dim(\mathrm{Lie}(G))$ [92, Propsition 1.37]. The functor Lie has the following properties.

(i) Let H_1, H_2 be smooth connected (irreducible) subgroups of G such that $\mathrm{Lie}(H_1) = \mathrm{Lie}(H_2)$. If $H_1 \cap H_2$ is smooth then $H_1 = H_2$

(ii) Let $H_1, \dots, H_n$ be smooth algebraic subgroups of a connected (irreducible) algebraic group G. If the Lie algebras of the H_i generate the Lie algebra $\mathrm{Lie}(G)$, then the subgroups H_i generate G as an algebraic group.

For the following property it is really important that the characteristic of the field k of the algebraic group G is zero.

(iii) For a connected algebraic group G over a field k of characteristic zero the connected algebraic subgroups of G are in natural one-to-one correspondence with the Lie subalgebras of $\mathrm{Lie}(G)$.

1.3.5 Reductive and semisimple groups. Our next goal is to present the structure theorems of reductive and semisimple algebraic groups. Before we can do this, we have to define these objects of course. Let G be an algebraic group over a field k and $\overline{k}$ an algebraic closure of k. We say that G is *solvable* if there is a finite sequence $(G_i \mid i = 0, \ldots, s)$ of algebraic subgroups of G such that $G_0 = G$, $G_s = \{e\}$ and G_i is a normal subgroup of G_{i-1} with commutative quotient G_{i-1}/G_i for every $i = 1, \ldots, s$.

Let G be a connected group variety over k. By [92, Proposition 6.42] the group variety G contains a largest connected solvable normal subgroup variety. This is called the *radical* $R(G)$ of G. The connected group variety G is called *semisimple* if after extension of the base field k to its closure $\overline{k}$, the radical $R(G_{\overline{k}})$ of $G_{\overline{k}}$ is trivial.

An algebraic group U over k is called *unipotent* if every nonzero representation $\rho\colon U \to \mathrm{GL}_V$ has a nonzero fixed vector $v \in V$, i.e. for every k-algebra R the representation $\rho(R)$ fixes $v \in V \otimes_k R$. By [92, subsection 6.45] every connected group variety G over k has a largest connected normal unipotent subgroup variety. This is called the *unipotent radical* $R_u(G)$ of G. We call G *reductive*, if after extension of the base field k to $\overline{k}$, the unipotent radical $R_u(G_{\overline{k}})$ of $G_{\overline{k}}$ is trivial.

1.3.6 Root space decomposition for Lie algebras of reductive groups. Let G be a reductive linear algebraic group defined over a field k and let $T \leqslant G$ be a torus which splits over k, in which case we refer to the pair (G, T) as a *split reductive group.* We write $X(T) = \mathrm{Hom}(T, \mathbb{G}_m)$ for the commutative group of characters of T. We can consider $X(T)$ as a free $\mathbb{Z}$-module [92, chapters 12.b and 12.c]. The rank of $X(T)$ equals the dimension of T. We have a morphism of algebraic groups

$$\mathrm{Ad}|_T \colon T \to \mathrm{GL}(\mathrm{Lie}(G))$$

induced by the adjoint representation $\mathrm{Ad}\colon G \to \mathrm{GL}(\mathrm{Lie}(G))$. Since T is a torus the representation $(\mathrm{Lie}(G), \mathrm{Ad}|T)$ also induces a decomposition

$$\mathrm{Lie}(G) = \bigoplus_{\chi \in X(T)} \mathrm{Lie}(G)_\chi$$

by [92, Theorem 4.25 and Theorem 12.12]. Here $\mathrm{Lie}(G)_\chi$ are character eigenspaces, on which $Ad|_T$ acts through the character $\chi \in X(T)$, i.e. $\mathrm{Lie}(G)_\chi$ is the greatest subspace among the subvector spaces $V \subset \mathrm{Lie}(G)$ such that $\mathrm{Ad}(t)\, v = \chi(t)\, v$ for all $t \in T(R)$ and $v \in V_R$ with R a k-algebra. We write $\Phi(G, T)$ for the non-trivial characters $\chi \in X(T)$ with non-trivial character eigenspaces $\mathrm{Lie}(G)_\chi$ and call the elements of $\Phi(G, T)$ the *roots* of (G, T).

1.3.7 Remark. A reference for this remark is Chapter III of [91]. We want to introduce an alternative way to obtain the root space decomposition in 1.3.6 but for this approach we have to restrict ourselves to a semisimple linear algebraic group G over a field k of **characteristic zero**. Let T be a maximal torus in G. We consider the pair $(\mathrm{Lie}(T), \mathrm{Lie}(G))$ and the homomorphism of k-vector spaces

$$\mathrm{ad}\colon \mathrm{Lie}(G) \longrightarrow \mathrm{End}(\mathrm{Lie}(G))$$

from Subsection 1.3.1 and Definition 1.3.2. The Lie algebra $\mathrm{Lie}(T)$ is an abelian Lie algebra and for every $A \in \mathrm{Lie}(T)$ the elements $\mathrm{ad}(A)$ of the endomorphism algebra $\mathrm{End}(\mathrm{Lie}(G))$ are semisimple [91, Proposition 2.12, Example 2.14, Lemma 3.15]. Hence for every $A \in \mathrm{Lie}(T)$ all the endomorphisms $\mathrm{ad}(A)$ are simultaneously diagonalizable. If $X \in \mathrm{Lie}(G)$ is a common eigenvector for all $\mathrm{ad}(A)$ with $A \in \mathrm{Lie}(T)$ then $\mathrm{ad}(A)(X) = [A, X] = \alpha(A)X$ for an element $\alpha(A) \in k$. We obtain linear maps $\alpha\colon \mathrm{Lie}(T) \to k$ and write

$$\mathfrak{g}_\alpha = \{X \in \mathrm{Lie}(G) \mid \mathrm{ad}(A)(X) = [A, X] = \alpha(A)X \ \forall\, A \in \mathrm{Lie}(T)\}$$

for the eigenspace of the linear map α. We can decompose $\mathrm{Lie}(G)$ in eigenspaces

$$\mathrm{Lie}(G) = \mathrm{Lie}(T) \oplus \bigoplus_{\alpha\in\Phi} \mathfrak{g}_\alpha, \tag{1.3.7.1}$$

where $\mathrm{Lie}(T) = \mathfrak{g}_0$ is the root space with respect to the trivial linear form. The nonzero linear forms α such that $\mathfrak{g}_\alpha \neq \{0\}$ are called *roots* of the tuple $(\mathrm{Lie}(T), \mathrm{Lie}(G))$ and we denote them by $\Phi = \Phi(\mathrm{Lie}(T), \mathrm{Lie}(G))$.

By identifying $X(T) \otimes_{\mathbb{Z}} k$ with $\mathrm{Hom}_k(\mathrm{Lie}(T), k)$ via the Lie algebra functor, the nonzero $\chi \in X(T)$ with $\mathrm{Lie}(G)_\chi \neq \{0\}$ are in one-to-one correspondence with the roots Φ of $(\mathrm{Lie}(T), \mathrm{Lie}(G))$. Thus we will also consider Φ as the set $\Phi(G, T)$ of non-trivial characters $\chi \in X(T)$ with non-trivial character eigenspaces $\mathrm{Lie}(G)_\chi$.

1.3.8 Example (Lie algebra of the symplectic group)**.** We want to do some calculations for the symplectic group

$$\mathrm{Sp}_{2n}(k) = \{A \in \mathrm{GL}_{2n}(k) \mid {}^tA\, J\, A = J\} \quad \text{with} \quad J = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}.$$

For a reference see Example (C_n) in section 21 of [92]. A nice reference in the case $n = 2$ is [83]. One can easily extend the calculations in there to arbitrary $n \geqslant 2$.

A maximal torus of $\mathrm{Sp}_{2n}(k)$ is given by the algebraic subgroup $T(k) \leqslant \mathrm{Sp}_{2n}(k)$ consisting of diagonal matrices of the form

$$\mathrm{diag}(a_1, \ldots, a_n, a_1^{-1}, \ldots, a_n^{-1}).$$

The Lie algebra $\mathfrak{sp}_{2n}(k) \subset \mathfrak{gl}_{2n}(k) = k^{2n\times 2n}$ has an explicit description as

$$\begin{aligned}\mathfrak{sp}_{2n}(k) &= \{X \in k^{2n\times 2n} \mid {}^tXJ + JX = 0\} \\ &= \left\{ \begin{pmatrix} A & B \\ C & -{}^tA \end{pmatrix} \mid A, B, C \in k^{n\times n},\ {}^tB = B,\ {}^tC = C \right\}\end{aligned}$$

with Lie brackets $[X,Y] = XY - YX$ inherited from $\mathfrak{gl}_{2n}(k)$. If $E_{ij} \in k^{n\times n}$ denotes the matrix with entries $(E_{ij})_{kl} = 1$ if $i = k$, $j = l$ and $(E_{ij})_{kl} = 0$ if $i \neq k$ or $j \neq l$ then a basis of $\mathfrak{sp}_{2n}(k)$ is given by $\{X_{ij}, Y_{ij}, {}^tY_{ij} \mid i,j = 1,\dots,n\}$ with matrices

$$X_{ij} = \begin{pmatrix} E_{ij} & 0 \\ 0 & -E_{ji} \end{pmatrix} \quad \text{and} \quad Y_{ij} = \begin{pmatrix} 0 & E_{ij} + Eji \\ 0 & 0 \end{pmatrix}$$

We write $\mathfrak{h}$ for the Lie algebra of the maximal torus $T(k)$. It consists of diagonal matrices of the form $H = \operatorname{diag}(\lambda_1,\dots,\lambda_n,-\lambda_1,\dots,-\lambda_n)$ with $\lambda_i \in k$. For $H \in \mathfrak{h}$ as before and $i,j \in \{1,\dots n\}$ with $i \neq j$ we have

$$\begin{aligned}
[H, X_{ij}] &= (\lambda_i - \lambda_j)\cdot X_{ij}, \\
[H, Y_{ij}] &= (\lambda_i + \lambda_j)\cdot Y_{ij} \quad \text{and} \quad [H, {}^tY_{ij}] = -(\lambda_i + \lambda_j)\cdot {}^tY_{ij}, \\
[H, Y_{ii}] &= 2\lambda_i \cdot Y_{ii} \quad \text{and} \quad [H, {}^tY_{ii}] = -2\lambda_i \cdot {}^tY_{ii}.
\end{aligned}$$

For every $i = 1,\dots,n$ we can define a linear map $e_i\colon \mathfrak{h} \to k$ on the set of diagonal matrices $\mathfrak{h} \leqslant \mathfrak{sp}_{2n}(k)$ by

$$e_i\left(\operatorname{diag}(\lambda_1,\dots,\lambda_n,-\lambda_1,\dots,-\lambda_n)\right) = \lambda_i.$$

This leads to a decomposition $\mathfrak{sp}_{2n}(k) = \mathfrak{h} \oplus \bigoplus_{\alpha\in\Phi} \mathfrak{g}_\alpha$, where the set of roots Φ is given by

$$\Phi = \Phi(\mathfrak{sp}_{2n}(k), \mathfrak{h}) = \{(e_i - e_j),\ \pm(e_i + e_j),\ \pm 2e_i \mid i \neq j\}.$$

We want to write down the root spaces explicitly. For $i,j \in \{1,\dots n\}$ with $i \neq j$ we have

$$\begin{aligned}
\mathfrak{g}_{e_i - e_j} &= \operatorname{span}_k(X_{ij}), \\
\mathfrak{g}_{e_i + e_j} &= \operatorname{span}_k(Y_{ij}) \quad \text{and} \quad \mathfrak{g}_{-(e_i+e_j)} = \operatorname{span}_k({}^tY_{ij}), \\
\mathfrak{g}_{2e_i} &= \operatorname{span}_k(Y_{ii}) \quad \text{and} \quad \mathfrak{g}_{-2e_i} = \operatorname{span}_k({}^tY_{ii}).
\end{aligned}$$

1.3.9 Weyl group of an algebraic group. Let (G,T) be a split reductive group. The normalizer $N_G(T)$ of the torus T in G and the centralizer $C_G(T)$ of the torus T in G are the algebraic groups such that for any k-algebra R,

$$\begin{aligned}
N_G(T)(R) &= G(R) \cap \bigcap \left(N_{G(R')}(T(R')) \mid R' \text{ is } R\text{-algebra}\right) \\
C_G(T)(R) &= G(R) \cap \bigcap \left(C_{G(R')}(T(R')) \mid R' \text{ is } R\text{-algebra}\right)
\end{aligned}$$

For a proof that $N_G(T)$ and $C_G(T)$ are algebraic groups see [92, 1.j and 1.k]. We have the identity $C_G(T) = N_G(T)^0$ [92, Corollary 17.39] and since T is maximal we have the identity $C_G(T) = T$ [92, Corllary 17.84]. We call the quotient $N_G(T)/C_G(T)$ the *Weyl group* of the tuple (G,T) and denote it by $W(G,T)$. We want to collect some information about the Weyl group in the next proposition. For a proof of it see [92, Proposition 21.1].

Proposition. The Weyl group $W(G,T)$ of a split reductive group (G,T) is a finite algebraic group and for every field F containing k we have the identities

$$W(G,T)(k) = W(G,T)(F) = N_G(T)(F)/T(F).$$

Since the group $W(G,T)(F)$ is the same for every field F containing k, we often abuse notation and write $W(G,T)$ if we mean $W(G,T)(F)$.

Let R be a k-algebra. Every element $g \in N_G(T)(R)$ induces an action on $T(R)$ by conjugation, $T(R) \to T(R)$, $t \mapsto g^{-1}\,t\,g$. This action on T leads to an action of the Weyl group $W(G,T)$ on the characters $X(T)$ which preserves the set of roots $\Phi(G,T)$. The group $N(G,T)(k)$ acts on the Lie algebra $\mathrm{Lie}(G) = \mathfrak{g}$ via the adjoint representation $\mathrm{Ad}\colon G \to \mathrm{GL}_{\mathfrak{g}}$. These two actions on the Lie algebra fit together very nicely as we will see in the next proposition (for a proof see [92, Proposition 21.2]).

Proposition. Let (G,T) be a split reductive group. Consider the decomposition

$$\mathrm{Lie}(G) = \mathrm{Lie}(T) \oplus \bigoplus_{\alpha\in\Phi} \mathfrak{g}_\alpha$$

from (1.3.7.1). Let $s \in W(G,T)(k)$ be represented by an element $n \in N_G(T)(k)$ and let $\alpha \in \Phi(G,T)$ be a root of (G,T). Then $s\alpha$ is also a root of (G,T) and $\mathfrak{g}_{s\alpha} = \mathrm{Ad}(n)\mathfrak{g}_\alpha$

1.3.10 Structure of reductive and semisimple groups. We want to collect some important properties about the structure of split reductive groups and split semisimple groups. For a reference of the first five items about reductive groups in the following list see [92, Theorem 21.11]. For the statement about split semisimple groups see [92, Proposition 21.49].

So let G be a reductive group and let $T \leqslant G$ be a maximal torus that splits over k. We denote by $\Phi = \Phi(G,T)$ again the roots of (G,T) . Then for a root $\alpha \in \Phi$ we have:

(i) In the decomposition $\mathrm{Lie}(G) = \mathrm{Lie}(T) \oplus \bigoplus_{\alpha\in\Phi} \mathfrak{g}_\alpha$ we have $\dim(\mathfrak{g}_\alpha) = 1$ for every $\alpha \in \Phi$.

(ii) There is a unique unipotent algebraic subgroup $U_\alpha \leqslant G$[1] (called *root subgroup*) with the following properties. It is isomorphic to $\mathbb{G}_a$, U_α is normalized by T and for every isomorphism $u_\alpha\colon \mathbb{G}_a \to U_\alpha$ and every k-algebra R the relation

$$t \cdot u_\alpha(b) \cdot t^{-1} = u_\alpha(\alpha(t) \cdot b)$$

holds for arbitrary $t \in T(R)$ and $b \in \mathbb{G}_a(R)$.

(iii) The subgroup $U_\alpha \leqslant G$ has Lie algbra $\mathrm{Lie}(U_\alpha) = \mathfrak{g}_\alpha$. Furthermore an algebraic subgroup of G contains U_α if and only if its Lie algebra contains $\mathfrak{g}_\alpha$.

[1]For a definition of the subgroup $U_\alpha \leqslant G$ see [92, Propsition 13.29, Definition 21.10]

(iv) For any element $w \in W(G,T)(k)$ in the Weyl group of (G,T) represented by $n \in N_G(T)(k)$ we have $n\,U_\alpha n^{-1} = U_{w.\alpha}$.

(v) The reductive group G is generated by T and the unipotent subgroups $U_\alpha \leqslant G$ where $\alpha \in \Phi$.

(vi) If G is semisimple, then G is already generated by the unipotent subgroups $U_\alpha \leqslant G$ where $\alpha \in \Phi$.

1.4 Root systems

In this section let always V be a finite dimensional vector space over a field k of characteristic zero.

1.4.1 Definition (Reflection)**.** Let $\alpha \in V\backslash\{0\}$. A *reflection with vector* α is a vector space endomorphism $s \in \mathrm{End}(V)$ such that $s(\alpha) = -\alpha$ and the set of vectors fixed by s is a hyperplane $H \subset V$.

1.4.2 Remark. Let V be a vector space over a field k of characteristic zero. We denote $V^\vee = \mathrm{Hom}(V,k)$ and write $\langle \cdot,\cdot\rangle$ for the natural pairing $V \times V^\vee \to k$.

(i) Let $\alpha \in V\backslash\{0\}$. If $\alpha^\vee \in V^\vee$ is an element of $V^\vee$ such that $\langle \alpha, \alpha^\vee\rangle = 2$, then a reflection s with vector α is given by

$$s\colon V \longrightarrow V, \quad v \longmapsto v - \langle v, \alpha^\vee\rangle \cdot \alpha.$$

Furthermore every reflection with vector α is of this form for a unique $\alpha^\vee \in V^\vee$ see [92, Lemma C.1].

(ii) Let $R \subset V$ be a finite set spanning V. There exists at most one reflection s with vector α such that $s\,R \subset R$ [92, Lemma C.3].

1.4.3 Definition (Root system of vector spaces)**.** Let V be a finite dimensional vector space over a field k of characteristic zero. A subset R of V is a *root system* in V if the following conditions hold:

(i) The set R is finite, $0 \notin R$ and $\mathrm{Span}_k(R) = V$.

(ii) For each $\alpha \in R$, there exists a unique reflection s_α with vector α such that $s_\alpha(R) \subset R$.

(iii) For all $\alpha, \beta \in R$, the vector $s_\alpha(\beta) - \beta$ is an integer multiple of α.

The *Weyl group* $W(R) \leqslant \mathrm{End}(V)$ of a root system (V,R) is the subgroup generated by the reflections s_α with $\alpha \in R$.

1.4.4 Inner product on a root system. Let R be a root system in a vector space V over a subfield k of $\mathbb{R}$ and let $W(R) \leqslant \mathrm{End}(V)$ be the Weyl group of the root system R. The Weyl group $W(R)$ is finite because it acts faithfully on R. Let $(\cdot,\cdot)_0 \colon V \times V \to k$ be an inner product. Then the bilinear form

$$(\cdot,\cdot)\colon V \times V \longrightarrow k, \quad (x,y) = \sum_{w \in W} (w\,x, w\,y)_0 \tag{1.4.4.1}$$

is again an inner product on V and by definition the elements of $W(R)$ are orthogonal maps with respect to the inner product in (1.4.4.1) see [92, p. C19] or [91, Proposition III.1.9].

Let now $(\cdot,\cdot)$ be an arbitrary inner product on V. We write again $\langle\cdot,\cdot\rangle$ for the natural pairing $V \times V^\vee \to k$. For each nonzero $\beta \in V$ we can define $\beta^\vee = 2(\cdot,\beta)/(\beta,\beta) \in V^\vee$ and obtain $\langle\beta,\beta^\vee\rangle$=2. Thus a reflection s_β with vector β is given by

$$s_\beta \colon V \longrightarrow V, \quad v \longmapsto v - \langle v, \beta^\vee\rangle \cdot \beta, \tag{1.4.4.2}$$

The endomorphism s_β fixes the hyperplane $H_\beta = \mathrm{span}_k(\{\beta\})^\perp$.

We consider a root system $R \subset V$ and an inner product $(\cdot,\cdot)$ such that the elements of $W(R)$ are orthogonal with respect to $(\cdot,\cdot)$. Let $\alpha \in R$ and let $s \in W(R)$ be the unique reflection with vector α such that $s(R) \subset R$. Let H be the hyperplane fixed by s. Then for every $w \in H$ we have

$$(w,\alpha) = (s(w), s(\alpha)) = (w, -\alpha)$$

This implies $(w,\alpha) = 0$ for every $w \in H$. Thus the linear map

$$V \longrightarrow V, \quad v \longmapsto v - 2\frac{(v,\alpha)}{(\alpha,\alpha)} \cdot \alpha \tag{1.4.4.3}$$

agrees with s on H, it agrees with s on the subspace generated by α and thus it agrees with s on the whole space V. We showed that s is given as in 1.4.4.3. By 1.4.2 the reflection s with vector $\alpha \in R$ is defined by $s(v) = v - \langle v, \alpha^\vee\rangle\alpha$ with a unique $\alpha^\vee \in V^\vee$ such that $\langle\alpha,\alpha^\vee\rangle = 2$. We conclude

$$(v,\alpha)/(\alpha,\alpha) = 1/2\langle v, \alpha^\vee\rangle.$$

for every $v \in V$. Hence the ratio $(v,\alpha)/(\alpha,\alpha)$ is independent of the choice of an inner product such that all the elements of $W(R)$ are orthogonal with respect to it. This shows that two roots α,β are orthogonal with respect to $(\cdot,\cdot)$ if and only if the elements α,β are orthogonal for all inner products for which the elements of $W(R)$ are orthogonal. Hence the following definition makes sense.

1.4.5 Definition. Let (V,R) be a root system equipped with an inner product $(\cdot,\cdot)$ such that the elements of $W(R)$ are orthogonal maps. We say that (V,R) is *indecomposable or irreducible* if R can not be written as the disjoint union of two proper subsets $\varnothing \neq R_1, R_2 \subset R$ such that R_1 and R_2 are orthogonal for the inner product $(\cdot,\cdot)$.

1.4.6 Proposition (Lengths of roots in indecomposable root systems)**.** Let (V, R) be an irreducible root system. Then there is a partition $R^{>} \sqcup R^{<} = R$ of the set of roots such that for every inner product $(\cdot, \cdot)$ for which the elements of $W(R)$ are orthogonal maps, the following properties hold:

(i) All the elements in $R^{>}$ and $R^{<}$ have the same length with respect to $(\cdot, \cdot)$ and $(r, r) > (s, s)$ for every $r \in R^{>}$ and $s \in R^{<}$.

(ii) All roots of a given length are conjugate under the action of the Weyl group $W(R)$ of (V, R).

We call the elements in $R^{>}$ the *long roots* and the elements in $R^{<}$ the *short roots* of the root system (V, R).

Proof. [64, §10.4, Lemma C] □

1.4.7 Theorem (Root system of a reductive algebraic group)**.** Let G be a reductive algebraic group and $T \leqslant G$ a maximal torus that splits over k, a field of characteristic zero. Write again $\Phi(G, T) \subset X(T)$ for the subset of characters from the splitting in Section 1.3.6 and write $\Phi(\mathrm{Lie}(G), \mathrm{Lie}(T))$ for the subset of $\mathrm{Lie}(T)^\vee = \mathrm{Hom}(\mathrm{Lie}(T), k)$ defined in Section 1.3.6. Then the pair

$$(X(T) \otimes_{\mathbb{Z}} k, \Phi(G, T)) \quad \text{respectively} \quad (\mathrm{Lie}(T)^\vee, \Phi(\mathrm{Lie}(G), \mathrm{Lie}(T)))$$

is a root system over k in the sense of Definition 1.4.3 and the group

$$W(G, T)(k) = N_G(T)(k)/C_G(T)(k)$$

from Section 1.3.9 can be identified with the Weyl group of this root system.

Proof. See [92, Corollary 21.12] and [92, Proposition C.29] for the results on the root systems. See [92, Corollary 21.38] for the statement on the Weyl group. □

1.4.8 Remark. We note here again the very important fact that the root system of a split reductive group does not change under extension of the base field [92, Note 21.17].

1.5 A criterion for Zariski density

1.5.1 Generic elements. We will first give some basic definitions and statements about generic elements. We can recommend the article [70] which gives a nice introduction to this theory and which was the main source for this subsection.

Let k be a field of characteristic zero and let $\overline{k}$ be an algebraic closure of k. Furthermore let G be a reductive group over k with coordinate ring $\mathcal{O}(G)$. We have a natural action of $\mathrm{Gal}(\overline{k}|k)$ on the $\overline{k}$-points

$$G(\overline{k}) = \mathrm{Hom}_k(\mathcal{O}(G), \overline{k})$$

by post-composition. Fix a torus $T \leqslant G$ of dimension n. We denote by X_T the group of characters

$$\chi\colon T(\overline{k}) \longrightarrow \mathbb{G}_m(\overline{k}).$$

We have an action of the Galois group $\mathrm{Gal}(\overline{k}|k)$ on the characters X_T given by

$$\mathrm{Gal}(\overline{k}|k) \times X_T \longrightarrow X_T, \quad (\sigma, \chi) \longmapsto {}^{\sigma}\chi,$$

where ${}^{\sigma}\chi \in X_T$ is defined as ${}^{\sigma}\chi(t) = \sigma(\chi(\sigma^{-1}(t)))$ with $t \in T(\overline{k})$. Thus the action is giving us a representation

$$\varphi_T\colon \mathrm{Gal}(\overline{k}|k) \to \mathrm{Aut}_{\mathbb{Z}}(X_T)$$

By a little abuse of notation we write $W(G,T)$ for the group $W(G,T)(\overline{k})$. This abuse is not too bad if we remember the proposition in 1.3.9. Furthermore we have a natural inclusion of the finite Weyl group $W(G,T)$ into $\mathrm{Aut}_{\mathbb{Z}}(X_T)$ induced by the faithful action of the $\overline{k}$-points $N_G(T)(\overline{k})$ on the torus $T(\overline{k})$ by conjugation [84, Exercise 10.30].

For an arbitrary torus $D \leqslant G$ we write k_D for the minimal field extension of k such that the subgroup $\mathrm{Gal}(\overline{k}|k_D)$ of $\mathrm{Gal}(\overline{k}|k)$ acts trivially on the characters X_D. In this case the representation φ_D descends to an injective representation

$$\varphi_D\colon \mathrm{Gal}(k_D|k) \longrightarrow \mathrm{Aut}_{\mathbb{Z}}(X_T).$$

The field k_D is a Galois extension and it is minimal among the field extensions K of k such that D_K is split, i.e. $D_K \cong \mathbb{G}_{m,K}^n$.

The group $\varphi_T(\mathrm{Gal}(\overline{k}|k))$ and the Weyl group $W(G,T)$ considered as a subgroup of $\mathrm{Aut}(X_T)$, have the following relationship (see [70, Lemma 2.2]):

Proposition. If k_G denotes the field extension $k_G = \bigcap_T k_T$ of k where the intersection is over all maximal tori $T \leqslant G$, then for every maximal torus T we have

$$\varphi_T(\mathrm{Gal}(\overline{k}|k_G)) \leqslant W(G,T)$$

Especially if G is split then $\varphi_T(\mathrm{Gal}(\overline{k}|k)) \leqslant W(G,T)$ for every maximal torus $T \leqslant G$.

Let G be an algebraic group over k. We have a group action $\mu\colon G \times G \to G$ on G by conjugation, i.e. μ is a natural transformation such that for every k-algebra R we have that $\mu(R)$ is the group action of $G(R)$ on itself by conjugation (see [92, section 1.f. and section 7] for more details about actions of algebraic groups). For every $g \in G(k)$ the orbit map $\mu_g\colon G \to G$ is defined to be the restriction of μ to $G \times \{g\}$. We write $Z_G(g)$

for the fiber of the orbit map $\mu_g \colon G \to G$ over g. Then $Z_G(g)$ is an algebraic subgroup of G and for an arbitrary k-algebra R we have

$$Z_G(g)(R) = \{h \in G(R) \mid h \cdot g_R = g_R \cdot h\},$$

where g_R is the image of g under the natural inclusion of $G(k)$ into $G(R)$. We call a semisimple element $g \in G(k)$ *regular* if the identity component $Z_G(g)^0$ of $Z_G(g)$ is a maximal torus in G. See the article [113] of Robert Steinberg for more properties of regular elements.

1.5.2 Definition (Generic tori and generic elements)**.** Let G be a reductive group.

- We say that a maximal torus T of G is *generic* if the equality

  $$\varphi_T(\mathrm{Gal}(\overline{k}|k)) = \varphi_T(\mathrm{Gal}(k_T|k)) = W(G,T)$$

  holds.
- We call a regular element $g \in G(k)$ *generic* if the associated maximal torus $Z_G(g)^0$ is generic.

1.5.3 Example (Regular and generic elements of the symplectic group)**.** Consider the symplectic group Sp_{2n} defined over $\mathbb{Q}$. Furthermore let $g \in \mathrm{Sp}_{2n}(\mathbb{Q})$ be semisimple with eigenvalues in $\mathbb{R}\backslash\mathbb{Q}$. The characteristic polynomial $P_g(t) \in \mathbb{Q}[t]$ is a reciprocal polynomial, i.e. the coefficients a_i of the characteristic polynomial $P_g(t)$ satisfy $a_i = a_{2n-i}$ for every $i = 0, 1, \ldots, 2n$. The $2n$ distinct eigenvalues of g come in pairs $\lambda_i, \lambda_i^{-1}$ $(i = 1, \ldots, n)$. Let k_g be the field extension of $\mathbb{Q}$ which we obtain by adjoining the eigenvalues

$$\{\lambda_i, \lambda_i^{-1} \mid i = 1, \ldots, n\} \subset \mathbb{R}$$

of g to $\mathbb{Q}$. We consider $Z_{\mathrm{Sp}_{2n}}(g)^0 \leqslant \mathrm{Sp}_{2n}$ and write $(Z_{\mathrm{Sp}_{2n}}(g)^0)_{k_g}$ for the extension of the base field to k_g. Since g is diagonalizable over k_g and all eigenvalues are pairwise different, the group $(Z_{\mathrm{Sp}_{2n}}(g)^0)_{k_g}(R)$ is isomorphic to $T_{k_g}(R)$ for every k_g-algebra R. Here $T_{k_g}(R)$ is the standard torus of $\mathrm{Sp}_{2n}(R)$. This shows that $Z_{\mathrm{Sp}_{2n}}(g)^0$ is a torus and that $g \in \mathrm{Sp}_{2n}(\mathbb{Q})$ is regular.

In Chapter 9 we will see in detail that we can identify the Galois group $\mathrm{Gal}(k_g|\mathbb{Q})$ with a subgroup of the hyperoctahedral group $\mathbb{Z}_2^n \rtimes S_n$. Furthermore the Weyl group $W(G,T)$ of Sp_{2n} can also be identified with the hyperoctahedral group and thus g is generic if and only if $\mathrm{Gal}(k_g|\mathbb{Q})$ has order $2^n \cdot n!$.

Let Ω be a symplectic form on $\mathbb{Q}^{2n}$ taking integral values on the lattice $\mathbb{Z}^{2n}$. We write $\mathrm{Sp}_\Omega(\mathbb{Z})$ for the elements in $\mathbb{Z}^{2n\times 2n}$ which respect the form Ω. In Chapter 9 generic elements play an important role for the proof of Theorem C. We will always consider special generic elements with the same properties as the element $g \in \mathrm{Sp}_{2n}(\mathbb{Q})$ from the last example. We will sum up the properties of the element $g \in \mathrm{Sp}_{2g}(\mathbb{Q})$ in Definition

1.5.4 below and will call these elements *Galois pinching*. As in Example 1.5.3 it is easy to see that Galois pinching elements in the group $\mathrm{Sp}_\Omega(\mathbb{Z})$ are generic elements of the algebraic group Sp_Ω. Galois pinching elements were first introduced in [86, Section 4].

1.5.4 Definition. A matrix $A \in \mathrm{Sp}_\Omega(\mathbb{Z})$ is called *Galois pinching* if A has the following properties.

(i) The characteristic polynomial $P_A(t) \in \mathbb{Z}[t]$ of the matrix $A \in \mathrm{Sp}_\Omega(\mathbb{Z})$ is irreducible over $\mathbb{Q}$.

(ii) All the roots of the matrix A are real numbers.

(iii) The Galois group of the characteristic polynomial $P_A(t) \in \mathbb{Z}[t]$ of A is as large as possible and thus isomorphic to the hyperoctahedral group $\mathbb{Z}_2^n \rtimes S_n$.

The next theorem by Prasad and Rapinchuk [100, Theorem 9.10] is one of the key ingredients to prove Theorem C. It enables us to describe the Zariski closure of a subgroup $\Gamma \leqslant G(k)$ generated by a generic element $g \in G(k)$ of infinite order and an element $x \in G(k)$ of infinite order not commuting with g. We need the following two Lemmas 1.5.5 and 1.5.7 before we can prove it.

Therefore let G be a split reductive group over k with split torus T. For every root $\alpha \in \Phi(G,T)$ we write again U_α for the root subgroup from 1.3.10. There is an isomorphism $u\colon \mathbb{G}_a(\overline{k}) \to U_\alpha(\overline{k})$ such that

$$t \cdot u(c) \cdot t^{-1} = u(\alpha(t) \cdot c)$$

for every $t \in T(\overline{k})$ and $c \in \mathbb{G}_a(\overline{k})$. Write ${}^\sigma u$ for the homomorphism $\mathbb{G}_a(\overline{k}) \to G(\overline{k})$ defined by ${}^\sigma u(c) = \sigma \circ u(\sigma^{-1} \circ c)$ for every $c \in \mathbb{G}_a(\overline{k})$.

1.5.5 Lemma. Let $\alpha, \beta \in \Phi(G,T)$ and $\sigma \in \mathrm{Gal}(\overline{k}|k)$ with ${}^\sigma\alpha = \beta$. Then the image of ${}^\sigma u$ is the root subgroup $U_\beta(\overline{k})$ of $G(\overline{k})$ and $\sigma(U_\alpha(\overline{k})) = U_\beta(\overline{k})$.

Proof. For every $t \in T(\overline{k})$ and every $c \in \mathbb{G}_a(\overline{k})$ we have

$$\begin{aligned} t \cdot {}^\sigma u(c) \cdot t^{-1} =& t \cdot \sigma \circ u(\sigma^{-1} \circ c) \cdot t^{-1} \\ =& \sigma \left((\sigma^{-1} \circ t) \cdot u(\sigma^{-1} \circ c) \cdot (\sigma^{-1} \circ t)^{-1}\right) \\ =& \sigma \circ u \left(\alpha(\sigma^{-1} \circ t) \cdot (\sigma^{-1} \circ c)\right) \\ =& \sigma \circ u \left(\sigma^{-1}({}^\sigma\alpha)(t) \ \cdot (\sigma^{-1} \circ c)\right) \\ =& {}^\sigma u \left({}^\sigma\alpha(t) \cdot c\right) \\ =& {}^\sigma u(\beta(t) \cdot c). \end{aligned}$$

We showed that the image U of ${}^\sigma u\colon \mathbb{G}_a(\overline{k}) \to G(\overline{k})$ is normalized by $T(\overline{k})$ and that $T(\overline{k})$ acts on U as the character ${}^\sigma\alpha$. Furthermore $\mathrm{Lie}(U) = \mathfrak{g}_{{}^\sigma\alpha}(\overline{k})$ by [84, Theorem 8.17 (c)]. From [92, Proposition 21.11] respectively [84, Theorem 8.17 (d)] we conclude that the image U of ${}^{\sigma}u$ is a root subgroup of $G(\overline{k})$ with $U = U_{{}^{\sigma}\alpha}(\overline{k}) = U_{\beta}(\overline{k})$. Since $\sigma(\overline{k}) = \overline{k}$ we have

$$\sigma(U_{\alpha}(\overline{k})) = \sigma(u(\overline{k})) = {}^{\sigma}u(\sigma(\overline{k})) = U_{\beta}(\overline{k}).$$

□

1.5.6 Remark. Let G be a split reductive group G with maximal torus T that is split over k. We denote by $\Phi^{>}(G,T)$ the long roots and by $\Phi^{<}(G,T)$ the short roots of the root system $\Phi(G,T)$, compare Proposition 1.4.6. The long roots $\Phi^{>}(G,T)$ are a *closed* and *symmetric* subset of $\Phi(G,T)$ [84, Proposition B.14, Example B.15 (3)]. This means the following:

- For $\alpha, \beta \in \Phi^{>}(G,T)$ with $\alpha + \beta \in \Phi(G,T)$ we have $\alpha + \beta \in \Phi^{>}(G,T)$.
- If $\alpha \in \Phi^{>}(G,T)$ then $\mathbb{Z}\alpha \cap \Phi^{>}(G,T) = \{-\alpha, +\alpha\}$.

From [92, Remark 21.94] we conclude that the algebraic subgroup

$$G(\Phi^{>}(G,T)) = \langle T, U_{\alpha} \mid \alpha \in \Phi^{>}(G,T)\rangle$$

is a split reductive group with root system $(X(T), \Phi^{>}(G,T))$.

Furthermore if G is semisimple then $\mathbb{Z}\Phi(G,T)$ is of finite index in $X(T)$ [92, C.34 and Proposition 21.48] and the same holds for the sublattice $\mathbb{Z}\Phi^{>}(G,T)$. By [92, Proposition 21.48] the reductive group $G(\Phi^{>}(G,T))$ is also semisimple and by [92, Proposition 21.49] we have

$$G(\Phi^{>}(G,T)) = \langle U_{\alpha} \mid \alpha \in \Phi^{>}(G,T)\rangle.$$

We call G *almost simple* if it is semisimple, noncommutative and every proper normal algebraic subgroup is finite. A very important property of an almost simple group is, that the root system $\Phi(G,T)$ of G is indecomposable [92, Proposition 24.1] or equivalent the Weyl group acts irreducibly on $X_T \otimes_{\mathbb{Z}} \mathbb{Q}$ [84, Proposition A.16].

1.5.7 Lemma. Let G be an almost simple group over k and let T be a generic torus of G. Then the following holds.

(i) Every $t \in T(k)$ of infinite order generates a subgroup of $T(k)$ with Zariski closure the torus T.

(ii) If $n \in G(k)$ is an element of infinite order then $n \in N_G(T)(k)$ implies $n \in T(k)$.

Proof. Part (i) is [101, Proposition 1] and part (ii) can be found in subsection 9.5 of [100]. □

1.5.8 Theorem. Let G be an almost simple algebraic group over k and $g \in G(k)$ a generic element of infinite order with maximal torus $T = Z_G(g)^\circ$. Furthermore let $x \in G(k)$ be an element of infinite order with $x \notin T(k)$. Let $\Gamma = \langle g, x\rangle \leqslant G(k)$ be the group generated by g and x. Let $H \leqslant G$ be the identity component of the Zariski closure of Γ. Then either

$$H = G \quad \text{or} \quad H = \langle U_\alpha \mid \alpha \in \Phi^>(G,T)\rangle,$$

where U_α is the root subgroup from 1.3.10 corresponding to the root $\alpha \in \Phi^>(G,T)$.

Proof. By assumption the root system $\Phi(G,T)$ is indecomposable or equivalent the Weyl group acts irreducibly on $X_T \otimes_{\mathbb{Z}} \mathbb{Q}$. The torus T is generic. This implies $\varphi_T(\mathrm{Gal}(\overline{k}|k)) = W(G,T)$ and hence the Galois group $\mathrm{Gal}(\overline{k}|k)$ acts irreducibly on $X_T \otimes_{\mathbb{Z}} \mathbb{Q}$ as well. For the generic element g we conclude from Lemma 1.5.7 part (i) that the Zariski closure of $\langle g\rangle \leqslant G(k)$ equals T and the Zariski closure of the group generated by xgx^{-1} equals xTx^{-1}. Thus H contains the maximal torus T and xTx^{-1}. Since $x \notin T(k)$ it can not normalize T by Lemma 1.5.7 part (ii). Hence $\mathrm{Lie}(H)$ contains at least one one-dimensional $\mathfrak{g}_\alpha$, where $\alpha \in \Phi(\mathrm{Lie}(G), \mathrm{Lie}(T))$. Since the Lie-functor is injective on subgroups of G we conclude $U_\alpha \leqslant H$ for the unique root subgroup U_α with $\mathrm{Lie}(U_\alpha) = \mathfrak{g}_\alpha$.

We consider the natural action of $\mathrm{Gal}(\overline{k}|k)$ on $G(\overline{k})$ and the action of $\varphi_T(\mathrm{Gal}(\overline{k}|k))$ on $\Phi(G,T)$ from Section 1.5.1. Since H is defined over k we conclude $\sigma.H(\overline{k}) = H(\overline{k})$ for every $\sigma \in \mathrm{Gal}(\overline{k}|k)$. On the other hand $\sigma(U_\alpha(\overline{k})) = U_{{}^\sigma\alpha}(\overline{k})$ by Lemma 1.5.5 and thus $U_\beta(\overline{k}) \subset H(\overline{K})$ for every $\beta \in \Phi(G,T)$ with $\beta = {}^\sigma\alpha$ for a $\sigma \in \mathrm{Gal}(\overline{k}|k)$. From [92, Corollary 1.44] we conclude that U_β is an algebraic subgroup of H for every $\beta \in \Phi(G,T)$ such that there exists an element $\sigma \in \mathrm{Gal}(\overline{k}|k)$ with $\beta = {}^\sigma\alpha$.

Since T is generic we have that $\varphi_T(\mathrm{Gal}(\overline{k}|k))$ equals the Weyl group $W(G,T)$ of $\Phi(G,T)$. The algebraic group G is almost simple and hence the root system $\Phi(G,T)$ is indecomposable. Thus the Weyl group acts transitively on the roots of same length by Proposition 1.4.6. This implies

$$\langle T, U_\beta \mid \beta \in \Phi^<(G,T)\rangle \leqslant H \quad \text{or} \quad \langle T, U_\gamma \mid \gamma \in \Phi^>(G,T)\rangle \leqslant H.$$

Now the statement follows from Remark 1.5.6 since G is almost simple and in particular semisimple. □

1.5.9 Example. We now come back to the symplectic group Sp_{2n} defined over $\mathbb{Q}$. It is an almost simple group [92, p. 459]. Again the calculations from [83] can easily be generalized to arbitrary $n \geqslant 2$. Remember from Example 1.3.8 that the set of roots of $\mathfrak{sp}_{2n}$ with respect to the diagonal matrices $\mathfrak{h} \leqslant \mathfrak{sp}_{2n}$ is given by

$$\Phi = \Phi(\mathfrak{sp}_{2n}(k), \mathfrak{h}) = \{e_i - e_j,\ \pm(e_i + e_j),\ \pm 2e_i \mid i \neq j\}.$$

The set of long roots $\Phi^>$ is given by all the elements $\pm 2e_i$ for $i = 1, \ldots, n$. The set of short roots $\Phi^<$ consists of the elements $e_i - e_j$ and $\pm(e_i + e_j)$ with $i \neq j$. See Figure 1.1 for the case $n = 2$. Remember that we had for every $i = 1, \ldots, n$ the identities

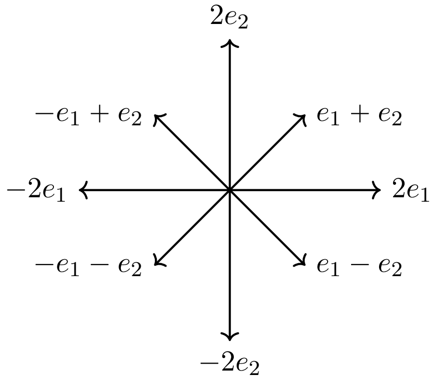


Figure 1.1.: Root system of $\mathfrak{sp}_4$.

$$\mathfrak{g}_{2e_i} = \operatorname{span}_{\mathbb{Q}}\left(\begin{pmatrix} 0 & E_{ii} \\ 0 & 0 \end{pmatrix}\right) \quad \text{and} \quad \mathfrak{g}_{-2e_i} = \operatorname{span}_{\mathbb{Q}}\left(\begin{pmatrix} 0 & 0 \\ E_{ii} & 0 \end{pmatrix}\right)$$

Furthermore the commutator group $\mathfrak{h}_\mathfrak{i} = [\mathfrak{g}_{2e_i}, \mathfrak{g}_{-2e_i}]$ is given by

$$\mathfrak{h}_i = \operatorname{Span}_k\left(\begin{pmatrix} E_{ii} & 0 \\ 0 & -E_{ii} \end{pmatrix}\right).$$

We conclude that $\mathfrak{h} = \bigoplus_{i=1}^n \mathfrak{h}_i$ and $\mathfrak{g}_{2e_i} \oplus \mathfrak{g}_{-2e_i} \oplus \mathfrak{h}_i \cong \mathfrak{sl}_2$ for every $i = 1, \ldots, n$. Finally we get

$$\mathfrak{h} \oplus \bigoplus_{\alpha \in \Phi^>} \mathfrak{g}_\alpha \cong \bigoplus_{i=1}^n \mathfrak{sl}_2.$$

This implies that the subgroup H of Sp_{2n} which is generated by all the U_α with $\alpha \in \Phi^>$ is isomorphic to $\prod_{i=1}^n \mathrm{SL}_2$.

Let again Ω be a symplectic form on $\mathbb{Q}^{2n}$ which takes integral values on $\mathbb{Z}^n$. With the help of Theorem 1.5.8 we will now formulate a criterion when two matrices $A, B \in \mathrm{Sp}_\Omega(\mathbb{Z})$ generate a Zariski dense subgroup in $\mathrm{Sp}_{\Omega,\mathbb{R}}$. Note that a *Lagrangian subspace with respect to* Ω is a subspace $W \subset \mathbb{R}^{2n}$ such that W coincides with its orthogonal symplectic complement, i.e. $W = W^\perp$.

1.5.10 Criterion. Let $A, B \in \mathrm{Sp}_\Omega(\mathbb{Z}) \leqslant \mathbb{Z}^{2n \times 2n}$ be two matrices.

(i) If A is Galois pinching and $B \neq \mathrm{id}$ is unipotent such that $(B - \mathrm{id})(\mathbb{R}^{2n})$ is not a Lagrangian subspace with respect to Ω, then the subgroup generated by A and B is Zariski dense in Sp_Ω.

(ii) If A is Galois pinching and $B \neq \mathrm{id}$ is unipotent such that the dimension of the subspace $(B - \mathrm{id})(\mathbb{R}^{2n})$ is not n, then the subgroup generated by A and B is Zariski dense in Sp_Ω.

Proof. Note that part (ii) of Criterion 1.5.10 directly follows from (i) since Lagrangian subspaces of Ω in $\mathbb{R}^{2n}$ have dimension n. This follows from $\dim(W) + \dim(W^{\perp}) = 2n$ for every subspace $W \subset \mathbb{R}^{2n}$.

We will continue with the proof of part (i). Since the matrix B is unipotent, it has infinite order. If A and B would commute, then the eigenspace $\mathrm{Eig}(B, 1)$ would be a common proper invariant subspace for both A and B. But this would imply by [86, Proposition 4.3] that $(B - \mathrm{id})(\mathbb{R}^{2n})$ is a Lagrangian subspace with respect to Ω, a contradiction to our assumption. This shows that A and B do not commute. The algebraic group Sp_{Ω} is almost simple [92, p. 459, Theorem 24.47]. Since A is Galois-pinching and in particular a generic element, we can apply Theorem 1.5.8 and conclude that the Zariski closure of the group generated by A and B is either Sp_{Ω} or isomorphic to $\prod_{i=1}^{n} \mathrm{SL}_2$ by Example 1.5.9. If the Zariski closure of A and B would be isomorphic to $\prod_{i=1}^{n} \mathrm{SL}_2$, then the two matrices would have a common proper invariant subspace, what is again impossible by [86, Proposition 4.3]. This shows that the Zariski closure of the group generated by A and B is Sp_{Ω}. ☐

1.6 A criterion for arithmeticity

Let G be a smooth algebraic group over a field k. There exists a largest smooth connected normal unipotent subgroup $R_u(G)$ of G called the unipotent radical of G (see Section 1.3.5).

We need the following theorem of Singh and Venkataramana [111, Theorem 1.2].

1.6.1 Theorem. Suppose that Ω is a non-degenerate symplectic form on the rational vector space $\mathbb{Q}^n$. Consider the lattice $\mathbb{Z}^n \subset \mathbb{Q}^n$ and write again $\mathrm{Sp}_{\Omega}(\mathbb{Z})$ for the stabilizer

$$\mathrm{Stab}_{\mathrm{Sp}_{\Omega}(\mathbb{Q})}(\mathbb{Z}^n) = \{g \in \mathrm{Sp}_{\Omega}(\mathbb{Q}) \mid g \cdot \mathbb{Z}^n = \mathbb{Z}^n\}.$$

Suppose $\Gamma \leqslant \mathrm{Sp}_{\Omega}(\mathbb{Z})$ is a dense subgroup which contains three transvections C_1, C_2, $C_3 \in \Gamma$. Let $w_1, w_2, w_3 \in \mathbb{Z}^n$ with $(C_i - I_n)(\mathbb{Z}^n) = \mathbb{Z}\, w_i$. If there is $i, j \in \{1, 2, 3\}$ with $\Omega(w_i, w_j) \neq 0$ and $W = \mathrm{Span}_{\mathbb{Q}}(w_1, w_2, w_3)$ is three dimensional such that the group generated by $C_1|_W$, $C_2|_W$, $C_3|_W$ contains a non-trivial element of the unipotent radical of $\mathrm{Sp}_{\Omega|W}(\mathbb{Q})$, then Γ has finite index in $\mathrm{Sp}_{\Omega}(\mathbb{Z})$.

For this reason we will describe the unipotent radical of the group $\mathrm{Sp}_{\Omega|W}(\mathbb{Q})$ a bit further. Since W has dimension three the form $\Omega|W$ is degenerate and there is a one-dimensional null subspace $E = \mathbb{Q}e$ with $e \in W\backslash\{0\}$ such that $\Omega(e, w) = 0$ for every $w \in W$. Since every element in $\mathrm{Sp}_{\Omega|W}(\mathbb{Q})$ preserves the degenerate form $\Omega|W$, we conclude that $\mathrm{Sp}_{\Omega|W}(\mathbb{Q})$ preserves the null space $E = \mathbb{Q}e$. Hence the kernel of the projection map $\mathrm{Sp}_{\Omega|W}(\mathbb{Q}) \to \mathrm{Sp}(W/E)$ is given by

$$\mathrm{Hom}(W/E, E) \times \mathrm{End}(E) \cong \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ x & y & z \end{pmatrix} \in \mathrm{GL}(3, \mathbb{Q}) \mid z \neq 0 \right\} \cong \mathbb{Q}^2 \times \mathbb{Q}^\times. \quad (1.6.1.1)$$

The group in 1.6.1.1 is solvable and a normal subgroup of $\mathrm{Sp}_{\Omega|W}(\mathbb{Q})$. Since $\mathrm{Sp}(W/E) \cong \mathrm{SL}(2, \mathbb{Q})$, we can write $\mathrm{Sp}_{\Omega|W}(\mathbb{Q})$ as the semidirect product

$$\begin{aligned} \mathrm{Sp}_{\Omega|W}(\mathbb{Q}) &= \mathrm{SL}(2, \mathbb{Q}) \ltimes (\mathbb{Q}^2 \times \mathbb{Q}^\times) \\ &= \left\{ \begin{pmatrix} a & b & 0 \\ c & d & 0 \\ x & y & z \end{pmatrix} \in \mathrm{GL}(3, \mathbb{Q}) \mid z \neq 0,\ a\,d - b\,c = 1 \right\} \end{aligned}$$

The group $\mathrm{SL}_2(\mathbb{Q})$ is simple and thus the subgroup in (1.6.1.1) is the largest normal solvable subgroup of $\mathrm{SP}_{\Omega|W}(\mathbb{Q})$ or in other words the radical $R(\mathrm{Sp}_{\Omega|W}(\mathbb{Q}))$. Thus the unipotent radical $R_u(\mathrm{Sp}_{\Omega|W}(\mathbb{Q}))$ of $\mathrm{Sp}_{\Omega|W}(\mathbb{Q})$ is given by

$$R_u(\mathrm{Sp}_{\Omega|W}(\mathbb{Q})) = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ x & y & 1 \end{pmatrix} \in \mathrm{GL}(3, \mathbb{Q}) \right\}. \quad (1.6.1.2)$$

2. Families of Riemann surfaces

2.1 Families of (stable) Riemann surfaces

First of all we want to collect some important definitions and results from Grothendieck's talks in Cartan's seminar from 1960 and 1961 recorded in [50], [52]. Hereby we can warmly recommend the survey paper [2] on Grothendieck's contributions to Teichmüller theory and the survey paper [1] on the original work of Oswald Teichmüller. Furthermore, the paper [29] of Michael Engber can be very helpful since the author repeats a lot of ideas of Grothendieck and generalizes it in the context of Riemann surfaces of finite type. We can also recommend the lecture notes of Johannes Schmitt [109] which is a nice introduction to the general theory of moduli spaces and for moduli spaces of compact Riemann surfaces and stable Riemann surfaces. Before we define what we understand by a stable Riemann surface, we will repeat the definition of a complex space.

Let $D \subset \mathbb{C}^n$ be a domain and let $\mathcal{J}$ be an ideal sheaf in $\mathcal{O}_D$ which is of finite type on D, i.e. for every point $z \in D$ there exists an open neighbourhood $U \subset D$ of z and functions $f_1, \ldots, f_k \in \mathcal{O}_D(U)$ such that

$$\mathcal{J}(U) = \mathcal{O}_D(U)\, f_1 + \ldots \mathcal{O}_D(U)\, f_k.$$

We consider the support V of the quotient sheaf $\mathcal{O}_D/\mathcal{J}$ which is defined as

$$V = \operatorname{supp}(\mathcal{O}_D/\mathcal{J}) = \{z \in D \mid (\mathcal{O}_D/\mathcal{J})_z \neq 0\}.$$

We have that $V \cap U = N(f_1, \ldots, f_k)$ and thus V is an analytic subvariety of D. If $\iota \colon V \to D$ is the inclusion and ι^{-1} the inverse image functor, then we write $\mathcal{O}_V$ for the restriction $(\mathcal{O}_D/\mathcal{J})|_V = \iota^{-1}(\mathcal{O}_D/\mathcal{J})$ and call the pair $(V, \mathcal{O}_V)$ a *complex model space*. A $\mathbb{C}$-ringed space $(X, \mathcal{O}_X)$ is called a *complex space* if X is Hausdorff and if every $x \in X$ has a neighbourhood U such that $(U, \mathcal{O}_{X|U})$ is isomorphic to a complex model space $(V, \mathcal{O}_V)$. A *holomorphic map* between complex spaces $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ is a morphism of $\mathbb{C}$-ringed spaces. We write $\underline{\text{CoSp}}$ for the category of complex spaces.

Now a one-dimensional connected compact complex space X is called a *stable Riemann surface* if the following holds:

- All singular points are ordinary double points, i.e. have a neighbourhood isomorphic to the neighbourhood of the origin of the locus $\{(z, w) \in \mathbb{C}^2 \mid z \cdot w = 0\}$ and

- Every irreducible component L of X that is isomorphic to the projective line $\mathbb{P}^1(\mathbb{C})$ intersects $\overline{X\backslash L}$ in at least three points.

2.1.1 Definition. Let C be a fixed complex space.

(i) A *complex space over* C is a pair $(\mathcal{X}, f)$, where $\mathcal{X}$ is a complex space and $f\colon \mathcal{X} \to C$ is a holomorphic map, i.e. a morphism of complex spaces. We use the notation $\mathcal{X}/C$ for a complex space $\mathcal{X}$ over C.

A *morphism* between two complex spaces $(\mathcal{X}, f)$ and $(\mathcal{X}', f')$ over C is a holomorphic map $\varphi\colon \mathcal{X} \to \mathcal{X}'$ such that $f' \circ \varphi = f$. In other words, we have the following commutative diagram:

$$\begin{tikzcd} \mathcal{X} \arrow[rr, "\varphi"] \arrow[dr, "f"'] & & \mathcal{X}' \arrow[dl, "f'"] \\ & C & \end{tikzcd}$$

The set of complex spaces over C with morphisms as above defines a category which we denote by $\underline{\mathrm{A}_C}$.

(ii) A *family of compact (stable) Riemann surfaces over* C is a complex space $f\colon \mathcal{X} \to C$ over C with a surjective, proper and flat morphism f, such that all the fibers are compact (stable) Riemann surfaces embedded in $\mathcal{X}$.

In the case that all fibers of a (stable) family of Riemann surfaces $\mathcal{X}/C$ have the same genus $g \geqslant 0$, we call g the *genus of the family* $\mathcal{X}/C$.

We denote the category of families of compact Riemann surfaces of genus g over C by $\underline{\mathrm{FR}^{C,g}}$ and the category of stable families of Riemann surfaces of genus g over C by $\underline{\mathrm{FR}^{C,g}_{\mathrm{stable}}}$.

2.1.2 Proposition (Family in the sense of Kodaira–Spencer)**.** Let $B, \mathcal{X}$ be holomorphic manifolds and let B be connected. A holomorphic map $f\colon \mathcal{X} \to B$ between the holomorphic manifolds B and $\mathcal{X}$ is a family of compact Riemann surfaces if and only if f is a proper submersion and every fiber is a compact Riemann surface embedded in $\mathcal{X}$.

Proof. Combine the implicit function theorem as on page 47 of [5] with [51, Theorem 3.1]. □

2.1.3 Remark. Let $B, \mathcal{X}$ be complex spaces and $f\colon \mathcal{X} \to B$ be a morphism.

(i) Kodaira and Spencer used the following definition for family of compact Riemann surfaces. If B and $\mathcal{X}$ are holomorphic manifolds and if $f\colon \mathcal{X} \to B$ is a proper submersion as in Proposition 2.1.2 then they called $f\colon \mathcal{X} \to B$ a family of compact Riemann surfaces (c.f. [5, Definition 2.4.1.1]). By Proposition 2.1.2 our definition can be understood as generalisation to non-smooth complex spaces.

(ii) Another consequence of [51, Theorem 3.1] is that the complex space $\mathcal{X}$ is smooth if B is smooth or in other words $\mathcal{X}$ is a manifold if B is a manifold.

For every holomorphic map $h\colon C' \to C$ between complex spaces we can define a functor

$$F_h\colon \underline{\mathrm{A}_C} \longrightarrow \underline{\mathrm{A}_{C'}} \tag{2.1.3.1}$$

by sending a complex space $(\mathcal{X}, f)$ over C to the complex space $\mathcal{X} \times_C C' \to C'$, which is the fiber product of $\mathcal{X}$ and C' over the maps $h\colon C' \to C$ and $f\colon \mathcal{X} \to C$[1].

$$\begin{tikzcd}
\mathcal{X} \times_C C' \arrow[r] \arrow[d] & \mathcal{X} \arrow[d, "f"] \\
C' \arrow[r, "h"] & C
\end{tikzcd}$$

Furthermore, we define that F_h sends a morphism $\varphi\colon \mathcal{X}_1 \to \mathcal{X}_2$ between $\mathcal{X}_1/C$ and $\mathcal{X}_2/C$ to the morphism $F_h(\varphi)\colon \mathcal{X}_1 \times_C C' \to \mathcal{X}_2 \times_C C'$ by the universal property of fiber product. This defines a fibered category in the sense of [16, Definition 1.1], see [16, Example 1].

We now fix a genus $g \geqslant 2$ for our families of compact (stable) Riemann surfaces. Consider the functors

$$F^g\colon \underline{\mathrm{CoSp}} \longrightarrow \underline{\mathrm{Sets}}$$

and

$$F^g_{\mathrm{stable}}\colon \underline{\mathrm{CoSp}} \longrightarrow \underline{\mathrm{Sets}}$$

from the category of complex spaces $\underline{\mathrm{CoSp}}$ to the category of sets $\underline{\mathrm{Sets}}$, which associate to a complex space C the set of families of (stable) Riemann surfaces over C up to isomorphisms and which associates to a holomorphic map $h\colon C' \to C$ the functor F_h restricted to families of (stable) Riemann surfaces modulo isomorphisms. In other words for a complex space C the set $F^g(C)$ consists of the objects in $\underline{\mathrm{FR}^{C,g}}$ up to isomorphism and the objects in F^g_{stable} are the objects of $\underline{\mathrm{FR}^{C,g}_{\mathrm{stable}}}$ up to isomorphism.

The functors F^g and F^g_{stable} are not representable (see for example [56, Ex.23.2] in the algebraic geometry setting). But there are coarse moduli spaces $\mathcal{M}_g$ and $\overline{\mathcal{M}}_g$ for them, which we will introduce in Section 3.1 about Teichmüller theory. The latter space $\overline{\mathcal{M}}_g$ is compact and was introduced by Deligne and Mumford [24]. There is a natural inclusion $\mathcal{M}_g \subset \overline{\mathcal{M}}_g$ as dense open complex subspaces and this is why $\overline{\mathcal{M}}_g$ is called the *Deligne-Mumford compactification* of $\mathcal{M}_g$. The complement $\partial\mathcal{M}_g = \overline{\mathcal{M}}_g \backslash \mathcal{M}_g$ is a Weil divisor (see [24] or [77] for proofs in the algebraic geometry setting).

[1]For a proof of existence in the category of analytic spaces see [37, Corollary 0.32].

2.2 Local system associated to a family of compact Riemann surfaces

Before we continue with solutions for the problem that the functor F^g is not representable, we will explain how we can associate a local system to a family of compact Riemann surfaces. This sheaf will turn out to be very helpful for many aspects in this thesis.

2.2.1 Families of compact Riemann surfaces and Ehresmann's theorem. Let $B, \mathcal{X}$ be holomorphic manifolds and let B be connected. Furthermore let $f\colon \mathcal{X} \to B$ be a surjective proper submersion. By the theorem of Ehresmann [28] for every $b \in B$ there is an open neighbourhood $U \subset B$ of b and a diffeomorphism $F_b\colon f^{-1}(U) \to \mathcal{X}_b \times U$ with $F_b \circ pr_2 = f|_{f^{-1}(U)}$ for every $x \in f^{-1}(U)$, where $\mathcal{X}_b = f^{-1}(\{b\})$. From the construction of the trivializations F_b with the help of an Ehresmann connection, we can achieve that for all charts (U_1, F_{b_1}) and (U_2, F_{b_2}) with $U_1 \cap U_2 \neq \varnothing$, the image of the cocycle map

$$U_1 \cap U_2 \to \operatorname{Diff}(\mathcal{X}_{b_2}, \mathcal{X}_{b_1}), \quad x \longmapsto F_{b_1,x} \circ F_{b_2,x}^{-1} \tag{2.2.1.1}$$

is in the subgroup of orientation preserving diffeomorphisms $\operatorname{Diff}(\mathcal{X}_{b_2}, \mathcal{X}_{b_1})^+$ between $\mathcal{X}_{b_2}$ and $\mathcal{X}_{b_1}$. In Subsection 4.1.4 we will come back to Ehresmann's theorem.

Let now $g\colon \mathcal{Y} \to C$ be a family of compact Riemann surfaces of genus g. In [51, Proposition 1.8] Grothendieck showed that for every $c \in C$ there is a neighbourhood U of c in C, and a family of genus g compact Riemann surfaces $f\colon \mathcal{X} \to B$ where $\mathcal{X}$ and B are smooth, as well as a morphism $\varphi\colon U \to B$ such that we have an isomorphism between the complex space $g^{-1}(U)$ and $\mathcal{X} \times_B U$ over U. i.e. we get the following commutative diagram:

$$\begin{tikzcd}
g^{-1}(U) \arrow[rr, "\cong"] \arrow[dr] & & \mathcal{X} \times_B U \arrow[dl] \\
& U &
\end{tikzcd}$$

Since $f\colon \mathcal{X} \to B$ is locally trivializable, we made plausible that the following Lemma of Grothendieck [50, Lemma 2.1] holds, c.f [29, Lemma 2.3].

2.2.2 Lemma. Every family of compact genus g Riemann surfaces $f\colon \mathcal{X} \to C$ is locally topological trivial.

2.2.3 Constructing the local system. Let C and $\mathcal{X}$ be complex spaces and let $f\colon \mathcal{X} \to C$ be a family of compact Riemann surfaces over C. The direct image functor $f_*\colon \underline{\mathrm{Ab}}(\mathcal{X}) \to \underline{\mathrm{Ab}}(C)$ from the category of sheaves of abelian groups on $\mathcal{X}$ to the category of sheaves of abelian groups on C is left exact. Since the category $\underline{\mathrm{Ab}}(\mathcal{X})$ has enough injectives, we can consider for every natural number $k \in \mathbb{N}$ the right derived functor

$$R^k f_*\colon \underline{\mathrm{Ab}}(\mathcal{X}) \to \underline{\mathrm{Ab}}(C).$$

Consider $\mathbb{Z}_{\mathcal{X}}$, the constant sheaf of stalk $\mathbb{Z}$ on $\mathcal{X}$. Then $R^k f_* \mathbb{Z}_{\mathcal{X}}$ is the sheaf associated to the presheaf $U \mapsto H^k(f^{-1}(U), \mathbb{Z}_{\mathcal{X}}|_{f^{-1}(U)})$ on C [55, Proposition III.8.1].

We know that $f\colon \mathcal{X} \to C$ is topologically locally trivial, so for every $c \in C$ there is an open neighbourhood $U \subset C$ and a homeomorphism

$$\varphi\colon f^{-1}(U) \longrightarrow \mathcal{X}_c \times U.$$

If U is contractible we further have

$$H^k(f^{-1}(U), \mathbb{Z}_{\mathcal{X}}|_{f^{-1}(U)}) \cong H^k(\mathcal{X}_c, \mathbb{Z}),$$

where the isomorphism follows from [117, Theorem 4.47] and invariance of cohomology under homotopy (c.f. [118, section 3]). Since every analytic variety is locally contractible [43] and C is locally isomorphic to a model space, we showed that $R^k f_* \mathbb{Z}_{\mathcal{X}}$ is a locally constant sheaf.

2.3 Rigidifying families of compact Riemann surfaces

To eliminate the non-trivial automorphisms between families of Riemann surfaces over a complex space C, we have to rigidify our families of Riemann surfaces by adding extra structure. We will sketch some ideas of Grothendieck to rigidify the functor F^g [50].

2.3.1 Definition. Let Γ be a discrete group and $g \geqslant 2$ a natural number. A *rigidifying functor of group* Γ is a functor

$$RI\colon \underline{\mathrm{FR}^g} \longrightarrow \underline{\mathrm{Prip}^\Gamma}$$

from the category $\underline{\mathrm{FR}^g}$ of genus g families of compact Riemann surfaces to the category $\underline{\mathrm{Prip}^\Gamma}$ of principal bundles of group Γ with the following properties:

(i) For every complex space C the functor RI restricts to a functor

$$\underline{\mathrm{FR}^{C,g}} \longrightarrow \underline{\mathrm{Prip}^{C,\Gamma}}$$

from the category of families of genus g Riemann-surfaces over C to the category of principal Γ-bundles over C.

(ii) There are compatibility isomorphisms in the sense of [16, Definition 1.4] for the pull back functors F_h as in 2.1.3.1, where $h\colon C' \to C$ is a holomorphic map.

(iii) Every automorphism of an object $\mathcal{X}/C$ in $\underline{\mathrm{FR}^{C,g}}$, which induces the identity on $RI(\mathcal{X}/C)$ is the identity.

An *RI-structure* on a family of Riemann surfaces $\mathcal{X}/C$ is by definition a global section of the bundle $RI(\mathcal{X}/C)$.

2.3.2 Remark. Let C be a complex space and $\mathcal{X}/C$ a family of genus $g \geqslant 2$ Riemann surfaces over C. There are two important things we have to note for a rigidifying functor

$$RI\colon \underline{\mathrm{FR}^g} \longrightarrow \underline{\mathrm{Prip}^\Gamma}$$

as above: First of all, if there exists a global section of the principal Γ-bundle $RI(\mathcal{X}/C)$ then it is trivial [96, Section 9.4.3]. And the most important property is that any automorphism ϕ of the family $\mathcal{X}/C$ such that $RI(\phi)$ preserves a section of the principal Γ-bundle $RI(\mathcal{X}/C)$, is the identity by condition (iii) of 2.3.1. This follows since $RI(\phi)$ is the identity on the bundle $RI(\mathcal{X}/C)$ [29, page 219] and is one of the main ingredients in the proof of [50, Theorem 3.1]. We want to state this important theorem of Grothendieck right after this remark.

2.3.3 Theorem (Theorem 3.1, [50])**.** Assume for a discrete group Γ and a natural number $g \geqslant 2$ we are given a rigidifying functor of group Γ:

$$RI\colon \underline{\mathrm{FR}^g} \longrightarrow \underline{\mathrm{Prip}^\Gamma}.$$

Then the functor

$$F^g_{RI}\colon \underline{\mathrm{CoSp}} \longrightarrow \underline{\mathrm{Sets}},$$

which associates to a complex space C the set of isomorphism classes of genus g families of Riemann surfaces over C equipped with an RI-structure, is representable.

Next we want to explain the two most famous examples of rigidifying functors.

2.3.4 Teichmüller-functor. We will first construct the main tool for rigidifying our families of Riemann surfaces, the so called *Teichmüller functor*, a rigidifying functor which is especially important since every other rigidifying functor can be constructed from it [50, §4]. We will hereby mainly follow [29] and the survey [2].

Let C be a connected complex space and let $f\colon \mathcal{X} \to C$ be a family of compact Riemann surfaces of genus $g \geqslant 2$. We fix a base point $c \in C$ and write $\mathcal{X}_c$ for the fiber $f^{-1}(\{c\})$ over c. For a point $t \in C$ and the fiber $\mathcal{X}_t$, we denote by $I(X_c, X_t)$ the space of homotopy classes of homeomorphisms between the two Riemann surfaces $\mathcal{X}_c$ and $\mathcal{X}_t$. Furthermore we write G for the group $I(\mathcal{X}_c, \mathcal{X}_c)$. Since $f\colon \mathcal{X} \to C$ is locally topological trivial we can equip the union

$$\mathcal{R}(\mathcal{X}/C) = \bigcup (I(\mathcal{X}_c, \mathcal{X}_t) \mid t \in C)$$

with the structure of a principal bundle of group G over C [29, Proposition 2.4]. Let $\Gamma(\mathcal{X}_c)$ be the index two subgroup of G consisting of homotopy classes of orientation preserving diffeomorphisms of C. Remember how we constructed the local trivializations of $f\colon \mathcal{X} \to B$ by the Ehresmann connection and that we could choose the cocycle maps in (2.2.1.1) orientation preserving. This shows that we could construct in the same way as above a principal $\Gamma(\mathcal{X}_c)$-bundle $\mathcal{P}(\mathcal{X}/C)$ of C, which we obtain from $\mathcal{R}(\mathcal{X}/C)$ by reducing the structure group G of $\mathcal{R}(\mathcal{X}/C)$ to $\Gamma(\mathcal{X}_c)$, i.e. there is an isomorphism

$$\mathcal{P}(\mathcal{X}/C) \times_{\Gamma(\mathcal{X}_c)} G = (\mathcal{P}(\mathcal{X}/C) \times G)/\Gamma(\mathcal{X}_c) \longrightarrow \mathcal{R}(\mathcal{X}/C).$$

Since $\Gamma(\mathcal{X}_c)$ is discrete, we can equip $\mathcal{P}(\mathcal{X}/C)$ with an analytic structure via the local projections of $\mathcal{P}(\mathcal{X}/C)$ to C (c.f. section 4 of [2]). The functor

$$RI^{C,g}_{\text{Teich}}\colon \underline{\mathrm{FR}^{C,g}} \longrightarrow \underline{\mathrm{Prip}^{C,\Gamma}},$$

which associates to a family $\mathcal{X}/C$ of genus g Riemann surfaces the principal $\Gamma(\mathcal{X}_c)$-bundle $\mathcal{P}(\mathcal{X}/C) \to C$ from above leads to the so called *Teichmüller-functor*

$$RI^{g}_{\text{Teich}}\colon \underline{\mathrm{FR}^{g}} \longrightarrow \underline{\mathrm{Prip}^{\Gamma}},$$

which is indeed a rigidifying functor.

2.3.5 Definition. A *Teichmüller-marking* or *Teichmüller-structure* on a family $\mathcal{X}/C$ of genus g Riemann surfaces is a global section of the bundle $\mathcal{P}(\mathcal{X}/C)$ over C.

2.3.6 Jacobi-functor of level n and level n-structures. Grothendieck presented in [50] a second rigidifying functor which plays a very important role in algebraic geometry.

Let C be again a complex space and $\mathcal{X}/C$ a family of genus $g \geqslant 2$ Riemann surfaces over it. We fix a base point $c \in C$ and write $\Gamma = \Gamma(\mathcal{X}_c)$ for the group of homotopy classes of orientation preserving diffeomorphisms of $\mathcal{X}_c$. The group Γ acts on the first cohomology group $H^1(\mathcal{X}_c, \mathbb{Z})$ and respects the intersection form on it. Thus by choosing a symplectic basis of $H^1(\mathcal{X}_c, \mathbb{Z})$ we obtain representations

$$\Gamma \longrightarrow \mathrm{Sp}(2g, \mathbb{Z}) \quad \text{and} \quad \Gamma \longrightarrow \mathrm{Sp}(2g, \mathbb{Z}_n)\ (n \in \mathbb{N}).$$

For a family of genus g Riemann surfaces $f\colon \mathcal{X} \to C$ we thus obtain two associated principal bundles

$$\begin{aligned} \mathcal{P}^{\mathrm{Tor}}(\mathcal{X}/C) &:= \mathcal{P}(\mathcal{X}/C) \times_\Gamma \mathrm{Sp}(2g, \mathbb{Z}) \qquad \text{and} \\ \mathcal{P}^{[n]}(\mathcal{X}/C) &:= \mathcal{P}(\mathcal{X}/C) \times_\Gamma \mathrm{Sp}(2g, \mathbb{Z}_n) \quad (n \in \mathbb{N}) \end{aligned}$$

Now for every $n \geqslant 3$ the functor

$$RI^g_{[n]} \colon \underline{\mathrm{FR}^g} \longrightarrow \underline{\mathrm{Prip}}^{\mathrm{Sp}(2g,\mathbb{Z}_n)},$$

which associates to a family $\mathcal{X}/C$ of genus g Riemann surfaces the bundle $\mathcal{P}^{[n]}(\mathcal{X}/C)$, is another example of a rigidifying functor. A section of $\mathcal{P}^{[n]}(\mathcal{X}/C)$ is called a *level-n-structure of the family $\mathcal{X}/C$*. Grothendieck used in his proof of the statement that the Teichmüller functor RI^g_{Teich} is a rigidifying functor, that there exists an interger $n > 0$ such that $RI^g_{[n]}$ is a rigidfiying functor.

3. Teichmüller theory and translation surfaces

3.1 Teichmüller theory

In this section we want to explain several ways how to analytically construct the fine moduli space $\mathcal{T}_g$ for the functor that associates to a complex space the set of isomorphism classes of genus $g \geqslant 2$ families of compact Riemann surfaces that have a Teichmüller marking (c.f. Theorem 2.3.3 and Section 2.3.4).

3.1.1 First Definition of Teichmüller Space. Let S be a compact connected orientable surface of genus $g \geqslant 2$. We want to define the Teichmüller space $\mathcal{T}_g(S)$ as the set of complex structures on S up to isotopy. Formally this can be done as follows. Let X be a compact Riemann surface and $\varphi\colon S \to X$ an orientation preserving diffeomorphism. Then we call the pair (X, φ) a *complex structure on* S. Write $\mathrm{Diff}^+(S)$ for the group of orientation preserving diffeomorphisms and $\mathrm{Diff}^0(S)$ for the normal subgroup of orientation preserving diffeomorphisms homotopic to the identity. Let (X_1, φ_1) and (X_2, φ_2) be two complex structures on S. We say that the structures (X_1, φ_1) and (X_2, φ_2) are equivalent if there exists an element $h \in \mathrm{Diff}^0(S)$ and a biholomorphic map $b\colon X_1 \to X_2$ such that the following diagramm commutes.

$$\begin{tikzcd} S \arrow[r, "\varphi_1"] \arrow[d, "h"] & X_1 \arrow[d, "b"] \\ S \arrow[r, "\varphi_2"] & X_2 \end{tikzcd}$$

Now we define the *Teichmüller space* as the set of pairs $\mathcal{T}_g(S) = \{(X, \varphi)\}/\sim$ modulo the equivalence from above.

3.1.2 Second Definition of Teichmüller Space. Next we want to give a second definition of the Teichmüller space $\mathcal{T}_g(S)$. We need this characterisation of Teichmüller space later for constructing Teichmüller disks in Section 3.2.5. In the following let $g \geqslant 2$ and in contrast to the first defintion let $S = S_g$ be a hyperbolic compact Riemann surface of genus g.

Denote by $\mathrm{QC}(S)$ the set of quasiconformal homeomorphisms of S onto itself and we write $\mathrm{QC}_0(S)$ for the normal subgroup of elements $f \in \mathrm{QC}(S)$ homotopic to the identity.

Let X_1, X_2 be two Riemann surfaces and let $\varphi_i \colon S \to X_i$ be two quasiconformal mappings for $i = 1, 2$. We say that the pairs (X_1, φ_1) and (X_2, φ_2) are *Teichmüller equivalent* if there is an element $f \in \mathrm{QC}_0(S)$ such that $\varphi_2 \circ f \circ \varphi_1^{-1}$ is a biholomorphic map. In other words there is a biholomorphic map $b \colon X_1 \to X_2$ and an element $f \in \mathrm{QC}_0(S)$ such that the following diagram commutes.

$$\begin{array}{ccc} S & \xrightarrow{\varphi_1} & X_1 \\ \downarrow f & & \downarrow b \\ S & \xrightarrow{\varphi_2} & X_2 \end{array}$$

The *Teichmüller space* $\mathcal{T}_g(S)$ is now the set of Teichmüller equivalence classes of pairs (X, φ), where X is a Riemann surface and $\varphi \colon S \to X$ is a quasiconformal mapping. The map $\varphi \colon S \to X$ is called a *(Teichmüller) marking* (c.f. Definition 2.3.5). For (X_1, φ_1) and (X_2, φ_2) two marked Riemann surfaces, we define the Teichmüller metric

$$d((X_1, \varphi_1), (X_2, \varphi_2)) = \inf_f \log K(f),$$

where f runs over all quasiconformal maps $f \colon X_1 \to X_2$ such that f is homotopic to $\varphi_2 \circ \varphi_1^{-1}$ on S. Here $K(f)$ denotes the quasiconformal constant of f (see [61, Definition 4.1.2]). This makes $\mathcal{T}_g$ a complete metric space. We can also equip $\mathcal{T}_g(S)$ with the structure of a holomorphic manifold. Due to Bers $\mathcal{T}_g(S)$ is homeomorphic to a bounded domain in $\mathbb{C}^{3g-3}$, see [Chapter 6][66] for more details.

3.1.3 Third Definition of Teichmüller Space. For the sake of completeness we also want to discuss the third definition of Teichmüller space via Beltrami forms. Let S be a compact connected orientable surface of genus $g \geqslant 2$. Denote by $B(S)$ the space of Beltrami forms on S, which consists of differential forms μ of type $(-1, 1)$ such that the restriction of μ to a domain D of any local coordinate z has the form $f\, d\bar{z}/dz$, where f is a bounded measurable function on U. We have a norm $\|\cdot\|_\infty$ on $B(S)$, where $\|\mu\|_\infty$ is given by the supremum of the L^∞ norms of all the functions f which describe μ locally. Write $B(S)_1$ for the Beltrami forms of norm less than one and consider two Beltrami forms to be the same if they coincide outside a subset of measure zero.

Every quasiconformal mapping $\varphi \colon S \to X$ gives us a Beltrami form $\mu_\varphi \in B(S)_1$ that is locally given by $\bar\partial(u \circ \varphi)/\partial(u \circ \varphi)$ whenever u is a coordinate of X. On the other hand every $\mu \in B(S)_1$ defines a Riemann surface structure S_μ whose underlying topological space is the same as the topological space of S and whose atlas is given by homeomorphisms u of open subsets of S to $\mathbb{C}$ such that $\mu = \bar\partial u/\partial u$ [61, Proposition and Definition 4.8.12]. The identity map $\mathrm{id} \colon S \to S_\mu$ is a quasiconformal mapping and the Beltrami form μ_{id}

that is given by the identity map from above equals $\mu \in B(S)_1$. Every quasiconformal map $f \in \mathrm{QC}(S)$ induces isometric automorphisms $\rho_f\colon \mathcal{T}_g(S) \to \mathcal{T}_g(S)$ and $\sigma_f\colon B(S)_1 \to B(S)_1$ by

$$\rho_f([\varphi]) = [\varphi \circ f^{-1}] \quad \text{and} \quad \sigma_f(\mu_\varphi) = \mu_{\varphi\circ f^{-1}},$$

where $\varphi\colon S \to X$ is a quasiconformal map. For the definition of σ_f remember from the last paragraph that every element $\mu \in B(S)_1$ comes from a quasiconformal mapping.

For the map $\Phi\colon B(S)_1 \to \mathcal{T}_g(S)$ defined by $\Phi(\mu) = [\mathrm{id}\colon S \to S_\mu]$ and every quasiconformal homeomorphism $f \in \mathrm{QC}(S)$ we have the identity $\Phi \circ \sigma_f = \rho_f \circ \Phi$. Hence Φ factors through the action of the normal subgroup $\mathrm{QC}_0(S)$ on $B(S)_1$. The map Φ induces an identification $\mathcal{T}_g(S) = B(S)_1/\mathrm{QC}_0(S)$ (see [61, Proposition 6.4.11, 6.4.12] and [66, Theorem 1.6] for more details). This is why we can take $B(S)_1/\mathrm{QC}_0(S)$ as a third definition for Teichmüller space.

3.1.4 Definition (Mapping class group)**.** Let again $S = S_g$ be a compact hyperbolic Riemann surface of genus g. We gave several definitions of Teichmüller space. Thus we also give two definitions of *mapping class group* or *Teichmüller modular group*.

(i) In the situation of the first definition of Teichmüller space from Subsection 3.1.1 we define the mapping class group as the quotient

$$\Gamma_g(S) = \mathrm{Diff}^+(S)/\mathrm{Diff}^0(S).$$

(ii) We already mentioned that the subgroup $\mathrm{QC}_0(S)$ of quasiconformal homeomorphisms on S homotpic to the identity is normal in $\mathrm{QC}(S)$. We define in the situation of the second and third definition of the Teichmüller space (Section 3.1.2 and Section 3.1.3) the mapping class group as the quotient group

$$\Gamma_g(S) = \mathrm{QC}(S)/\mathrm{QC}_0(S).$$

The homomorphism $\mathrm{QC}(S) \to \mathrm{Aut}(\mathcal{T}_g(S))$ defined by $f \mapsto \rho_f$ factors through $\mathrm{QC}_0(S)$ and produces a homomorphism $\Gamma_g(S) \to \mathrm{Aut}(\mathcal{T}_g(S))$ to the group of holomorphic automorphisms $\mathrm{Aut}(\mathcal{T}_g(S)))$ of $\mathcal{T}_g(S)$.

3.1.5 Remark (Teichmüller universal curve)**.** We can also analytically construct a universal curve respectively universal family for our moduli-problem by taking the quotient

$$\mathcal{U}_g = (S \times B(S)_1)/\mathrm{QC}_0(S),$$

where $\mathrm{QC}_0(S)$ acts on $S \times B(S)_1$ by $f.(s,\mu) = (f^{-1}(s), \sigma_f(\mu))$ [61, Theorem 6.8.3]. The quotient $\mathcal{U}_g$ is called *Teichmüller universal curve*. It is topologically a trivial fiber bundle $S \times \mathcal{T}_g$ [61, Theorem 6.8.4]. Furthermore we have a natural action of the mapping class group $\Gamma_g(S)$ on $\mathcal{U}_g$ which turn out to be automorphisms of the holomorphic family $\mathcal{U}_g \to \mathcal{T}_g$ (c.f. [29] and [27]).

3.1.6 Moduli space of compact Riemann surfaces and level-structures. References for this part are [56, Section 27], [54] and [61].

The action of the mapping class group Γ_g on the Teichmüller space $\mathcal{T}_g$ is holomorphic and properly discontinuous. The *moduli space of compact Riemann surfaces* is now defined as the quotient $\mathcal{M}_g = \mathcal{T}_g/\Gamma_g$. By a theorem of Cartan [14] $\mathcal{M}_g$ is a complex space and it is a coarse moduli space for the moduli functor

$$F^g\colon \underline{\mathrm{CoSp}} \longrightarrow \underline{\mathrm{Sets}}$$

from Section 2.1 which associates to a complex space C the set of families of compact Riemann surfaces of genus g over C up to isomorphism. A coarse moduli space for the moduli functor

$$F^g_{\text{stable}}\colon \underline{\mathrm{CoSp}} \longrightarrow \underline{\mathrm{Sets}}$$

is for example given by the quotient $\widehat{\mathcal{T}}_g/\Gamma_g$ where $\widehat{\mathcal{T}}_g$ is Abikoff's augmented Teichmüller space. See [57] for more details.

Let us now come again to the rigidification of the moduli problem with level-n-structures ($n \in \mathbb{N}$). We said in Section 2.3.6 that

$$RI^g_{[n]}\colon \underline{\mathrm{FR}^g} \longrightarrow \underline{\mathrm{Prip}}^{\mathrm{Sp}(2g,\mathbb{Z}_n)},$$

is a rigidifying functor for our moduli problem if $n \geqslant 3$. Hence for $RI = RI^g_{[n]}$ the functor

$$F^g_{RI}\colon \underline{\mathrm{CoSp}} \longrightarrow \underline{\mathrm{Sets}},$$

which associates to a complex space C the set of isomorphism classes of genus g families of compact Riemann surfaces over C equipped with a non-trivial level-n-structure, is representable by Theorem 2.3.3. We can represent the functor by the quotient $\mathcal{M}_g^{[n]} = \mathcal{T}_g(S)/\Gamma_g^{[n]}$, where $\Gamma_g^{[n]}$ is the kernel of the action of the mapping class group on the cohomology group $H^1(S, \mathbb{Z}/n\mathbb{Z})$. The group $\Gamma_g^{[n]}$ is torsion free for all $n \geqslant 3$ by a result of Serre what makes $\mathcal{M}_g^{[n]}$ a complex manifold for these natural numbers.

3.2 Translation surfaces

3.2.1 Definition of translation surfaces and origamis. A *translation surface* is a pair (X, ω) consisting of a compact Riemann surface X and a non-zero holomorphic 1-form $\omega \in \Omega^1_X$. We write $Z(\omega)$ for the set of zeros of ω. By the theorem of Gauss-Bonnet the total numbers of zeros (count in multiplicity) is $2g-2$, where $g \in \mathbb{N}$ is the genus of the Riemann surface X. The name translation surface comes from the fact that the pair (X, ω) comes with an atlas of charts on $X \backslash Z(\omega)$. The new coordinates in the old coordinate patches $\{(U, \eta)\}$ of $X \backslash Z(\omega)$ are given by the integrals

$$z(p) = \int_{p_0}^{p} \omega \quad (p \in U),$$

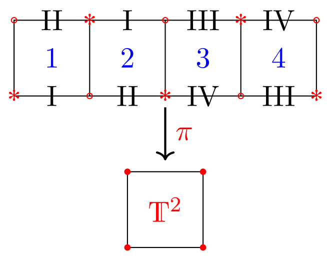


Figure 3.1.: X-origami

where $U \subset X\backslash Z(\omega)$ is simply connected and $p_0 \in U$ is a basepoint. In these new coordinates we have $\omega = dz$. An *origami* or *square tiled surface* is a pair (X, ω) where X is a Riemann surface obtained as a finite cover

$$\pi\colon X \longrightarrow \mathbb{C}/(\mathbb{Z} \oplus \sqrt{-1}\,\mathbb{Z}) =: \mathbb{T}$$

of the torus branched only at the origin $0 \in \mathbb{T}$ and where ω is defined as $\omega = \pi^*(dz)$. For example the X-origami in Figure 3.1.

3.2.2 Hodge bundle. In this section we follow the construction in [80]. See also the PhD-Thesis of Fabian Ruoff [106] for a construction of the Hodge-Bundle in the language of stacks. Let $\pi\colon \mathcal{U}_g \to \mathcal{T}_g$ be the universal Teichmüller curve. The direct image $\pi_*\,\Omega^1_{\mathcal{U}_g|\mathcal{T}_g}$ by π of the sheaf $\Omega^1_{\mathcal{U}_g|\mathcal{T}_g}$ of relative Kähler-differentials is locally free of rank g. Define $\Omega\mathcal{T}_g$ to be the associated holomorphic vector-bundle minus its zero-section. For any point $p \in \mathcal{T}_g$ and fiber $X_p = \pi^{-1}(p)$ the identity

$$\left(\pi_*\,\Omega^1_{\mathcal{U}_g|\mathcal{T}_g}\right)\Big|_p = \Gamma(X_p, \Omega^1_{X_p})$$

holds, what makes $\Omega\mathcal{T}_g$ a holomorphic fiber-bundle with fiber $\mathbb{C}^g\backslash\{0\}$. Altogether $\Omega\mathcal{T}_g$ consists of equivalence classes of triples (X, φ, ω), where X is a Riemann surface of genus g with marking $\varphi\colon S \to X$ and where ω is a holomorphic 1-form on X with a relation that says that two triples are equivalent if they only differ by a quasiconformal homeomorphism in $QC_0(S)$ (c.f. Equation (3.2.2.1)).

There is an action of the mapping class group $\Gamma_g(S)$ on $\Omega\mathcal{T}_g$ in the following way. For a mapping class $[f]$ represented by $f \in \mathrm{QC}(S)$ and a triple (X, φ, ω) in $\Omega\mathcal{T}_g$ define

$$[f]\cdot(X, \varphi, \omega) = (X, \varphi \circ f^{-1}, \omega). \tag{3.2.2.1}$$

This action is continuous, properly discontinuous and the map $\Omega\mathcal{T}_g \to \mathcal{T}_g(S)$ is equivariant with respect to the two actions of the mapping class group $\Gamma_g(S)$. If we factor out the action of the modular group $\Gamma_g(S)$, we obtain $\Omega\mathcal{M}_g = \Omega\mathcal{T}_g/\Gamma_g(S)$, the *moduli space of abelian differentials* and the map $\Omega\mathcal{T}_g \to \mathcal{T}_g(S)$ descends to a map $\Omega\mathcal{M}_g \to \mathcal{M}_g$ called *Hodge-bundle*. Over a point $[X] \in \mathcal{M}_g$ represented by a compact Riemann-surface X, the Hodge bundle $\Omega\mathcal{M}_g$ has fiber $\Gamma(X, \Omega^1_X)\backslash\{0\}$ (modulo an action of $\mathrm{Aut}(X)$).

3.2.3 Stratification. A good reference for this subsection is [39]. Let X be a compact Riemann surface of genus $g \geqslant 2$ and ω a non-zero holomorphic 1-form on X. By the theorem of Gauss-Bonnet ω has a finite set of zeros $Z(\omega) = \{z_1, \ldots, z_n\}$ of multiplicity $\mathrm{mult}(z_i) = k_i$ such that $\sum_{i=1}^n k_i = 2g-2$. Since the action of the mapping class group $\Gamma_g(S)$ respects the multiplicities of the zeros we obtain subsets $\Omega\mathcal{T}_g(\underline{\kappa}) \subset \Omega\mathcal{T}_g$ and $\Omega\mathcal{M}_g(\underline{\kappa}) \subset \Omega\mathcal{M}_g$, consisting of equivalence classes (X,φ,ω) respectively (X,ω), where the multiplicities of the zeros of ω form the partition $\underline{\kappa}$ of $2g-2$. This leads to natural stratifications

$$\Omega\mathcal{T}_g = \bigsqcup_{\underline{\kappa}} \Omega\mathcal{T}_g(\underline{\kappa}) \quad \text{and} \quad \Omega\mathcal{M}_g = \bigsqcup_{\underline{\kappa}} \Omega\mathcal{M}_g(\underline{\kappa}),$$

where $\underline{\kappa}$ runs over all partitions of $2g-2$.

3.2.4 $\mathbf{GL}^+(\mathbf{2},\mathbb{R})$-action on Strata. In this subsection we want to explain the action of the orientation preserving invertible matrices $\mathrm{GL}^+(2,\mathbb{R})$ on the bundle $\Omega\mathcal{T}_g$ and $\Omega\mathcal{M}_g$ which gives rise to affine invariant manifolds in strata of $\Omega\mathcal{M}_g$ and Teichmüller curves in $\mathcal{M}_g$. We will hereby follow [87] and [33].

Let (X,φ,ω) be a representant of an element in $\Omega\mathcal{T}_g$ and $A \in \mathrm{GL}^+(2,\mathbb{R})$. We define a harmonic 1-form ω_A on X by

$$\omega_A = \begin{pmatrix} 1 & \sqrt{-1} \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \mathrm{Re}\,\omega \\ \mathrm{Im}\,\omega \end{pmatrix}.$$

There is a unique complex structure on the topological space underlying X such that ω_A is again a holomorphic 1-form. We denote the corresponding Riemann surface by X_A. The identity map $\mathrm{id}\colon X \to X_A$ defines an affine map with constant partial derivatives and hence a quasiconformal map. We define the marking $\varphi_A\colon S \to X_A$ as the composition $\mathrm{id} \circ \varphi$. The element $A.(X,\varphi,\omega) \in \Omega\mathcal{T}_g$ is by definition given as the equivalence class of the triple (X_A,φ_A,ω_A). The right-action of the mapping class group $\Gamma_g(S)$ on $\Omega\mathcal{T}_g$ commutes with the left-action of $\mathrm{GL}^+(2,\mathbb{R})$ on $\Omega\mathcal{T}_g$ from above and hence descends to an $\mathrm{GL}^+(2,\mathbb{R})$-action on the Hodge-bundle $\Omega\mathcal{M}_g$.

A very important class of objects in the theory of Teichmüller dynamics are so called *Veech surfaces*, i.e. points $(X,\omega) \in \Omega\mathcal{M}_g$ whose $\mathrm{GL}^+(2,\mathbb{R})$-orbits are closed in $\Omega\mathcal{M}_g$. We want to investigate these objects a little bit further in the next subsection.

3.2.5 Teichmüller disks. The following can be found for example in the work of Earle and Gardiner[26] or in the survey paper [80]. Fix a point $(X,\varphi,\omega) \in \Omega\mathcal{T}_g$ and consider the map

$$\mathrm{SL}_2(\mathbb{R}) \to \Omega\mathcal{T}_g \to \mathcal{T}_g, \quad A \mapsto \pi_g\Big(A.(X,\varphi,\omega)\Big) = (X_A,\varphi_A).$$

For a rotation matrix $R \in \mathrm{SO}(2)$ one can show that the identity $\mathrm{id}\colon X \to X_R$ defines a biholomorphic map. This means that (X,φ) and (X_R,φ_R) are the same point in $\mathcal{T}_g$. By identifying $\mathbb{D} \cong \mathrm{SL}(2,\mathbb{R})/\mathrm{SO}(2)$ in the standard way via Möbius transformations, the map from above descends to a map $g_\omega\colon \mathbb{D} \to \mathcal{T}_g$ which is an isometry of the unit disk $\mathbb{D}$ into the Teichmüller space with respect to the Poincaré metric on $\mathbb{D}$ and the Teichmüller metric on $\mathcal{T}_g$. The image $g_\omega(\mathbb{D}) \subset \mathcal{T}_g$ is therefore called *Teichmüller disk*.

We want to give a second equivalent definition of the map $g_\omega\colon \mathbb{D} \to \mathcal{T}_g$ from which the isometry can be seen more easily. The quotient $\overline{\omega}/\omega$ defines a Beltrami form of norm one and hence every $t \in \mathbb{D}$ defines an element $t\,\overline{\omega}/\omega$ in $B(X)_1$, the set of Beltrami-forms on X with norm less than one. We can now define $g_\omega\colon \mathbb{D} \to \mathcal{T}_g$ by

$$\mathbb{D} \longrightarrow B(X)_1 \longrightarrow \mathcal{T}_g, \quad t \longmapsto [(X_{t\,\overline{\omega}/\omega},\ \mathrm{id} \circ \varphi)],$$

where $\mathrm{id}\colon X \to X_{t\,\overline{\omega}/\omega}$ is the quasiconformal map which comes from the identity (see also [80, Proposition 4.4.1] for a nice and very detailed proof).

Let (X, ω) be a tuple consisting of a compact Riemann surface X and a holomorphic 1-form ω on it. We denote by $\mathrm{Aff}^+(X, \omega)$ the group of quasiconformal homeomorphisms f of X onto itself with the following properties. The quasiconformal homeomorphism f maps $X \backslash Z(\omega)$ onto itself and it is given by affine maps $z \mapsto Az + c$ ($A \in \mathrm{GL}^+(2, \mathbb{R})$, $c \in \mathbb{R}^2$) on the charts of the translation atlas given by ω. Since X is connected the matrix $A \in \mathrm{GL}^+(2, \mathbb{R})$ is the same for every translation chart of (X, ω). Furthermore since X has finite volume, we conclude $A \in \mathrm{SL}(2, \mathbb{R})$. Thus there is a natural group homomorphism

$$D\colon \mathrm{Aff}^+(X, \omega) \longrightarrow \mathrm{SL}(2, \mathbb{R}), \quad f \longmapsto D(f) = A,$$

which maps an affine homeomorphism f to the linear part A appearing in the description of f on the translation charts. The group homomorphism D is called the *derivative* and its image the *Veech group* of (X, ω). In the following we denote the Veech group of (X, ω) by $\mathrm{SL}(X, \omega)$. The kernel of the derivative D is a finite group denoted by $\mathrm{Aut}(X, \omega)$. Hence we get the following short exact sequence:

$$1 \longrightarrow \mathrm{Aut}(X, \omega) \longrightarrow \mathrm{Aff}^+(X, \omega) \xrightarrow{D} \mathrm{SL}(X, \omega) \longrightarrow 1$$

The intersection of $\mathrm{Aff}^+(X, \omega)$ with the group $\mathrm{QC}_0(X)$ contains only the identity [26, Lemma 5.2] and thus we can identify $\mathrm{Aff}^+(X, \omega)$ with a subgroup of the mapping class group $\Gamma_g(X) = \mathrm{QC}(X)/\mathrm{QC}_0(X)$.

Let $g_\omega\colon \mathbb{D} \to \mathcal{T}_g$ be a Teichmüller disk. The global stabilizer of the image $\Delta = g_\omega(\mathbb{D})$ with respect to the action of the mapping class group Γ_g on the Teichmüller space $\mathcal{T}_g$ is the affine group $\mathrm{Aff}^+(X, \omega)$ [26, Theorem 1]. Furthermore, the group $D^{-1}(\pm I) = \mathrm{Aut}(X, \omega) \cdot \{\pm I\}$ stabilizes every point in Δ [80, Proposition 4.3.5].

3.2.6 Teichmüller curves. By composing $g_\omega\colon \mathbb{D} \to \mathcal{T}_g$ with the projection map $\pi_g\colon \mathcal{T}_g \to \mathcal{M}_g$, we get a holomorphic map

$$f_\omega\colon \mathbb{D} \longrightarrow \mathcal{M}_g.$$

We already mentioned that the global stabilizer of the action of Γ_g on the Teichmüller disk $\Delta := g_\omega(\mathbb{D}) \subset \mathcal{T}_g$ is the group $\mathrm{Aff}^+(X,\omega)$. We write $H(X,\omega) = D^{-1}(\pm I)$ for the subgroup of $\mathrm{Aff}^+(X,\omega)$ which stabilizes every point in Δ. Let $R = \mathrm{diag}(1,-1)$ and

$$\mathrm{Stab}(f_\omega) = \{A \in \mathrm{Aut}(\mathbb{D}) \mid f_\omega(A\,t) = f_\omega(t)\ \forall t \in \mathbb{D}\}.$$

By [87, Prop. 3.2] we have that $R\,\mathrm{Stab}(f_\omega)\,R$ coincides with the projection of the Veech group $\mathrm{SL}(X,\omega)$ of (X,ω) to $\mathrm{PSL}_2(\mathbb{R})$. Thus we can identify the quotient $\mathrm{Aff}^+(X,\omega)/H(X,\omega)$ with $\mathrm{Stab}(f_\omega)$. Since g_ω is injective we obtain an isomorphism $\mathbb{D}/\mathrm{Stab}(f_\omega) \cong \Delta/\mathrm{Aff}^+(X,\omega)$. The map $f_\omega\colon \mathbb{D} \to \mathcal{M}_g$ clearly factors through its stabilizer and we call

$$j_\omega\colon \mathbb{D}/\mathrm{Stab}(f_\omega) \to \mathcal{M}_g$$

or the quotient $C_1 := \Delta/\mathrm{Aff}^+(X,\omega)$ a *Teichmüller curve* if one of the following equivalent statements is true:

(i) The stabilizer group $\mathrm{Stab}(f_\omega) \subset \mathrm{Aut}(\mathbb{D})$ is a lattice.

(ii) The manifold $\Delta/\mathrm{Aff}^+(X,\omega)$ has finite volume or equivalently has finitely many cusps and no funnel or flaring ends.

3.2.7 Proposition. If $j_\omega\colon \mathbb{D}/\mathrm{Stab}(f_\omega) \to \mathcal{M}_g$ is a Teichmüller curve then the image j_ω is an algebraic curve in $\mathcal{M}_g$, whose normalization is $C_1 = \mathbb{D}/\mathrm{Stab}(f_\omega)$.

Proof. The mapping class group Γ_g acts holomorphically and properly discontinuously on $\mathcal{T}_g$ and thus we get a holomorphic proper map between complex spaces $j_\omega\colon C_1 \to \mathcal{M}_g$. We already know from the theorem of Remmert ([103] or [45]) that the image of j_ω is an analytic set in $\mathcal{M}_g$. In the proof of [44, Theorem 1] the authors showed that $j_\omega(C_1) \subset \mathcal{M}_g$ has finite fibers and that j_ω is injective outside the preimage of the critical locus. By [45, Theorem 9.3.3] the holomorphic map $j_\omega\colon C_1 \to j_\omega(C_1)$ is a one-sheeted analytic covering. Since C_1 is a manifold and hence normal, the holomorphic map $j_\omega\colon C_1 \to j_\omega(C_1)$ is the normalization of $j_\omega(C_1)$ in the sense of [45]. Let $\overline{\mathcal{M}_g}$ denote the Deligne-Mumford compactification and write $\overline{\mathcal{T}_g}$ for the augmented Teichmüller space. Furthermore let $c_1, \dots, c_n$ denote the cusps of $\mathrm{Stab}(f_\omega)$. By [57, Proposition 4.13] the map $g_\omega\colon \mathbb{D} \to \mathcal{T}_g$ extends continuously to a map

$$\overline{g_\omega}\colon \mathbb{D} \cup \{c_1, \dots, c_n\} \to \overline{\mathcal{T}}_g$$

and thus the map j_ω extends continuously to a map

$$\overline{j_\omega}\colon \mathbb{D} \cup \{c_1, \dots, c_n\}/\mathrm{Stab}(f_\omega) \to \overline{\mathcal{M}}_g.$$

Hence for every holomorphic map $f\colon \overline{\mathcal{M}_g} \to \mathbb{C}$ the composition $f \circ \overline{j_\omega}$ is holomorphic by the Riemann extension Theorem [45, Theorem 7.4.2]. This shows that the map $\overline{j_\omega}$ is holomorphic and the image of $\overline{j}_\omega$ is an analytic subset of the projective algebraic variety $\overline{\mathcal{M}}_g$. We conclude that the image of $\overline{j}_\omega$ is an algebraic curve in $\overline{\mathcal{M}}_g$ by Chow's Theorem[1]. $\square$

[1]See [45, Corollary 9.5.1] for a proof of Chow's Theorem.

$$\begin{tikzcd}
\mathbb{D} \arrow[r, "g_\omega"] \arrow[d] & \Delta \arrow[r] \arrow[d] & \mathcal{T}_g \arrow[d, "\pi_g"] \\
\mathbb{D}/\mathrm{Stab}(f_\omega) \arrow[r, "\cong"] & C_1 = \Delta/\mathrm{Aff}^+(X,\omega) \arrow[r, "j_\omega"] & \mathcal{M}_g
\end{tikzcd}$$

Figure 3.2.: Diagram of the construction of Teichmüller curves.

If the curve C_1 was constructed from a pair $(X,\omega) \in \Omega\mathcal{M}_g$, we say that (X,ω) *generates the Teichmüller curve* C_1. The construction made above is visualized in Figure 3.2.

3.2.8 Remark. If $(X,\omega) = \mathcal{O}$ defines an origami, the group $\mathrm{SL}(X,\omega)$ is a finite index subgroup of $\mathrm{SL}(2,\mathbb{Z})$ (see [53]). This implies that $\mathrm{Stab}(f_\omega)$ is a lattice in $\mathrm{Aut}(\mathbb{D})$ and hence every origami defines a Teichmüller curve. We will call a Teichmüller curve, which comes from an origami, an *origami-curve*.

3.3 Dehn twists and cylinder decomposition

References for this section are the book [31] the article [119] and the lecture notes [102]. In this section let (X,ω) always be a translation surface of genus $g \geqslant 2$.

3.3.1 Definition (Cylinder of a translation surface)**.** A *cylinder* in (X,ω) is an open subset of $X\backslash Z(\omega)$ which is isometric to an euclidean cylinder of the form $\mathbb{R}/k\mathbb{Z} \times (0,a)$ with respect to the flat metric on $X\backslash Z(\omega)$, where $k, a > 0$ are positive numbers. We call k the *width* and a the *height* of the cylinder. The *modulus* μ of a cylinder is the ratio of height to width, i.e. $\mu = a/k$.

3.3.2 Remark. A more topological definition is if we say that a cylinder is a connected set of homotopic simple closed geodesics. Every geodesic is an image of $\mathbb{R}/k\mathbb{Z} \times b$ for $b \in (0,a)$. We have a partial order on the set of cylinders by subset relation. Thus it makes sense to speak of *maximal* cylinders. We will show that every maximal cylinder of a genus $g \geqslant 2$ translation surface (X,ω) is bounded by a concatenation of saddle connections.

3.3.3 Definition. The *direction* of a cylinder is the direction of one of the closed geodesics which build the cylinder. A *cylinder decomposition* of (X,ω) is a set of maximal, pairwise disjoint cylinders such that X is the union of the closures of the cylinders. The *direction of the cylinder decomposition* is the direction of each of the cylinders.

Let now $\phi\colon \mathbb{R}/k\mathbb{Z} \times (0,a) \to X\backslash Z(\omega)$ be an isometry. Then the twist

$$T\colon \mathbb{R}/k\mathbb{Z} \times (0,a) \longrightarrow \mathbb{R}/k\mathbb{Z} \times (0,a), \quad (r,t) \longmapsto (r + k/a \cdot t,\, t)$$

defines a homeomorphism of X as follows. We write Z for the image of the isometry ϕ. We define $T_Z\colon X \to X$ by

$$T_Z(x) = \begin{cases} \phi \circ T \circ \phi^{-1}(x) & \text{if } x \in Z \\ x & \text{if } x \in X\backslash Z \end{cases}. \tag{3.3.3.1}$$

If γ is the core curve of the cylinder Z, then the map T_Z is called *(Dehn) twist about the cylinder Z* or *(Dehn) twist about the core curve γ*. The equivalence class of the map T_Z is a well defined element in the mapping class group $\Gamma_g(X) = \mathrm{Diff}^+(X)/\mathrm{Diff}^0(X)$ of X. For us it will be important how a twist T_Z acts on the homology $H_1(X,\mathbb{Z})$ by push forward. Therefore we state the following proposition which can be found in the book of Farb and Margalit [31, Proposition 6.3.1] in a sligthly more general way.

3.3.4 Proposition. Let Z be a cylinder of a translation surface (X,ω) with simple core curve γ. If c denotes the corresponding class of γ in $H_1(X,\mathbb{Z})$ and if Q denotes the intersection pairing on $H_1(X,\mathbb{Z})$ then the multiple twist T_Z^k ($k \in \mathbb{N}$) acts by push forward on the singular homology as

$$(T_Z^k)_*(v) = v + k \cdot Q(v,c)\, c.$$

for every $v \in H_1(X,\mathbb{Z})$.

In this thesis we want to study the action of $\mathrm{Aff}^+(X,\omega)$ on $H_1(X,\mathbb{R})$ by push forward and Dehn twists will play an important role to do so. Thus given a cylinder decomposition of the translation surface (X,ω) we want to show next how we can construct an element in $\mathrm{Aff}^+(X,\omega)$ by a composition of multiple twists on the cylinders belonging to the cylinder decomposition. On the other hand parabolic elements of the Veech group $\mathrm{SL}(X,\omega)$ lead to cylinder decompositions of the translation surface (X,ω). More concretely in part (i) of Proposition 3.3.5 we show a theorem of W. A. Veech [116, Proposition 2.4] (see also [119, Lemma 3.8]). The proof of it shows how we can explicitly construct a reducible element in $\mathrm{Aff}^+(X,\omega) \leqslant \Gamma_g(X)$ from a given cylinder decomposition of a translation surface if the cylinders have commensurable moduli. Conversely in part (ii) of 3.3.5 we construct a parabolic element in $\mathrm{SL}(X,\omega)$ from a cylinder decomposition with commensurable moduli. You can find the main idea of the proof of (ii) in [119, Lemma 3.9]. Anja Randecker worked out all the missing steps in her lecture [102], which I used as a main source for the following parts.

3.3.5 Proposition (Cylinder decomposition and parabolic elements)**.** Let (X,ω) be a finite translation surface of genus $g \geqslant 2$.

(i) Assume there is a cylinder decomposition on (X, ω), such that all the moduli are integer multiples of a fixed $\alpha \in \mathbb{R}\backslash\{0\}$. Then there is a parabolic element $M \in \mathrm{SL}(X, \omega)$ and a concatenation φ of multiple twists on the cylinders such that $\varphi \in \mathrm{Aff}^+(X, \omega)$ with derivative $D\varphi = M$.

(ii) For every parabolic element in $\mathrm{SL}(X, \omega)\backslash\{\pm I\}$ there is a cylinder decomposition on (X, ω) in the direction of an eigenvector of the parabolic element.

Before we can prove this result, we have to show the following statement which is worth being formulated as a proposition itself.

3.3.6 Proposition. Let (X, ω) be a finite translation surface of genus $g \in \mathbb{N}$. If the Veech group of (X, ω) contains a parabolic element of the form

$$M_h = \begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix} \in \mathrm{SL}(X, \omega)$$

for an element $\alpha \in \mathbb{R}\backslash\{0\}$, then every horizontal trajectory is either closed or a saddle connection.

Proof. If the translation surface (X, ω) has genus $g \geqslant 2$, then it has singularities and horizontal saddle connections.

(1) Let $\varphi \in \mathrm{Aff}^+(X, \omega)$ be an affine diffeomorphism with derivative

$$D\varphi = \begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix} \in \mathrm{SL}(X, \omega).$$

Let $Z(\omega)$ be the set of zeros of ω with $s = \#Z(\omega) \in \mathbb{N}$ the number of singularities of the translation surface (X, ω). And let $H = \{h_1, \ldots, h_k\}$ be the set of horizontal trajectories in X which start in a singularity, i.e. the set of horizontal separatrices and saddle connections. Both sets $Z(\omega)$ and H are nonempty and finite. Furthermore, the homeomorphism φ induces a permutation of $Z(\omega)$ and H. Let $n = s! \cdot k!$ and $\psi = \varphi^n$. Then we have $\psi|_{Z(\omega)} = \mathrm{id}$ and $\psi(h_i) = h_i$ for all $i \in \{1, \ldots, k\}$.

The diffeomorphism φ preserves the horizontal direction and hence ψ preserves the horizontal direction. We conclude that ψ is a translation on every $h_i \in H$ for $i \in \{1, \ldots, k\}$. Let the singularity $s_i \in Z(\omega)$ be the starting point of the trajectory h_i. We have $\psi|_{h_i}(s_i) = s_i$ and hence $\psi|_{h_i} = \mathrm{id}$, respectively $\psi|_{h_1 \cup \cdots \cup h_k} = \mathrm{id}$.

(2) The next step is to show that every element in H is a saddle connection. So assume that there is an $i \in \{1, \ldots, k\}$ such that $h_i \in H$ is not a saddle connection. The trajectory h_i is a separatrix and hence not bounded in one direction.

(2.1) We show that there is an open set $U \subset X$ such that $h_i \cap U$ is dense in U. So let $p \in h_i$ be a point with $p \notin Z(\omega)$ and let v be a vertical geodesic that starts in p. Since there are only finitely many singularities there are only finitely many points r in v such that the horizontal geodesic which starts in r runs into a singularity s before it returns to the segment v. So let $v' \subset v$ be a vertical segment (that starts in $p \in v$ and ends in a point $p' \in v$), such that no horizontal geodesic, which starts in a point of $v' \backslash \{p\}$, ends in a singularity $s \in Z(\omega)$ before it returns to v.

Now shift v' along h_i and consider the rectangle that is generated by the shifting process (see the yellow rectangle in Figure 3.3). The volume of X is bounded and the volume of the rectangle increases monotonously. This implies that there is a point of time when a shifted copy of v' and the original v' intersect. We conclude that either h_i and v' have an intersection point or the horizontal trajectory starting in p' and v' have an intersection point. In the second case we can find a point $p'' \in v'$ and a horizontal trajectory h_i' that starts in $p'' \in v'$ and ends in $p \in v'$. Both horizontal trajectories h_i and h_i' are trajectories through p and thus $h_i = h_i'$. In both cases the horizontal trajectory h_i has an intersection point p'' with the vertical geodesic v.

We write $\overline{pp''}$ for the vertical geodesic from p to p''. We assume now that there is a point q in the interior $\operatorname{int}(\overline{pp''})$ of $\overline{pp''}$ which does not belong to the closure $\overline{h}_i$ of h_i. Since $\overline{h}_i$ is closed, we can find and choose an open geodesic segment $I \subset v$ of maximal size such that $q \in I$ and $I \cap \overline{h}_i = \varnothing$. Let $q' \in \partial I$ be the endpoint of I that is closer to p. Since I was maximally chosen, we get $q' \in \overline{h}_i$ and all the points of the forward, horizontal trajectory h_i'' that starts in q' also belong to $\overline{h}_i$. With the same arguments as above we conclude that either h_i'' is closed or we can find a returning point of h_i'' in I, i.e. $h_i'' \cap I \neq \varnothing$. The first case is impossible since X is a translation surface and q' lies between the two points p, $p'' \in h_i$. This would imply that h_i is closed as well. The second case is impossible because otherwise we have $\overline{h}_i \cap I \neq \varnothing$.

By shifting the vertical segment $\overline{pp''}$ in horizontal direction we can now choose an open set $U \subset X$ in the covered rectangle such that U contains the vertical segment $\overline{pp''}$ and such that $h_i \cap U$ is dense in U.

(2.2) We have $\psi|_{h_i} = \mathrm{id}$ from (1) and thus $\psi|_U = \mathrm{id}$. Let (V, z) be a chart of X with $U \cap V \neq \varnothing$. We have $\psi|_{U \cap V} = \mathrm{id}$ and thus $\psi = \mathrm{id}$. But since $\alpha \neq 0$ this is a contradiction to

$$D\psi = \begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix}^n = \begin{pmatrix} 1 & n \cdot \alpha \\ 0 & 1 \end{pmatrix} \neq \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

where $D\psi \in \mathrm{SL}(X, \omega)$ is the derivative of ψ. This shows that H only contains horizontal saddle connections.

(3) It is left to show that every horizontal trajectory which is not a saddle connection is closed. Again, we will show this by contradiction. So let $h \subset X$ be a horizontal trajectory which is not a saddle connection. We assume that h is not closed and hence h is infinitely long.

(3.1) We assume that h is not dense in X. Thus there exists a point $p \in \partial\overline{h}$ the boundary of $\overline{h}$. Consider a horizontal trajectory l through p and a vertical geodesic v through p

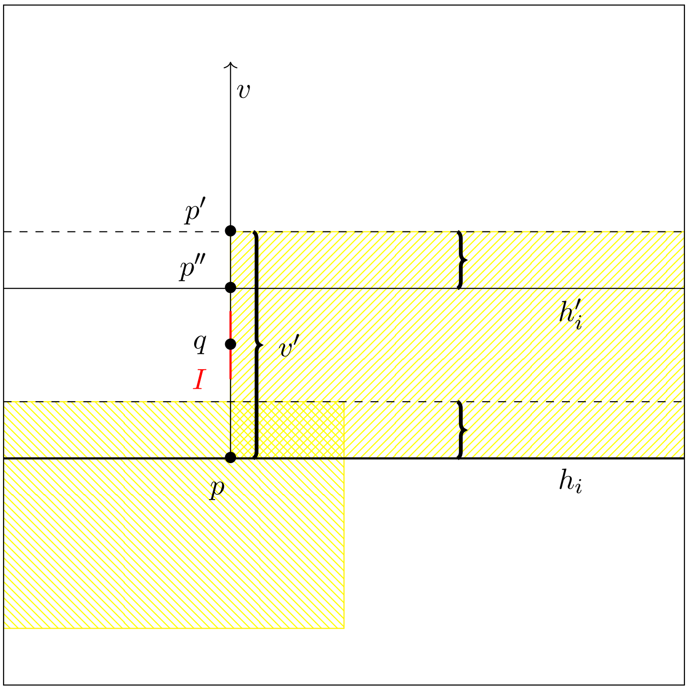


Figure 3.3.: Horizontal trajectories and geodesic vertical segments

which starts in p. Let v' be the reflection of the vertical segment v at the horizontal trajectory l. Then v' is also a vertical segment which has p as an endpoint. From $p \in \overline{h}$ we conclude $l \subset \overline{h}$. If l is infinitely long, we can show as above that the trajectory l intersects the vertical segment v in a point p' and the vertical segment v' in a point p''. And again as above we can show that every point on the vertical segment between p' and p'' belongs to the closure $\overline{l}$ and hence belongs to the closure $\overline{h}$. Thus we can find a neighborhood U of p which belongs to $\overline{h}$. But this shows $p \notin \partial\overline{h}$ the boundary of $\overline{h}$. We conclude that l is not infinitely long and l is closed or a saddle connection.

(3.2) If h is dense in X then of course all horizontal saddle connections lie in $\overline{h}$.

(3.3) In both cases (3.1) and (3.2) we found a horizontal trajectory l which is either a closed geodesic or a saddle connection and which has an intersection point with $\overline{h}$. But h is parallel to l and thus has to be closed or a saddle connection as well. □

Proof of Proposition 3.3.5. (i) Let $k \in \mathbb{N}$ such that $k \cdot \alpha$ is the least common multiple of all the moduli of the cylinder.

At first we assume that we have a cylinder decomposition in horizontal direction. We define the matrix $M \in \mathrm{SL}(2, \mathbb{R})$ by

$$M = \begin{pmatrix} 1 & k \cdot \alpha \\ 0 & 1 \end{pmatrix}.$$

The matrix M defines a (possibly multiple) twist on every cylinder which fixes the edges of every cylinder pointwise. If we now combine the single actions on every cylinder to a homeomorphism φ as in (3.3.3.1), then we get an affine map with derivative $D\varphi = M \in \mathrm{SL}(X, \omega)$.

If the direction of the cylinder decomposition is not horizontal then choose a rotation matrix $R \in \mathrm{GL}(2, \mathbb{R})$ that maps the direction of the cylinder decomposition to the horizontal direction. Since R is a rotation we hereby do not change the moduli of the cylinders. With the arguments from above we have $M \in \mathrm{SL}(R.(X, \omega))$ and thus $R^{-1}MR$ is an element of $\mathrm{SL}(X, \omega)$. For the trace of the matrix M and $R^{-1}MR$ we get $\mathrm{trace}(M) = \mathrm{trace}(R^{-1}MR) = 2$ and hence both matrices are parabolic.

(ii) Let $M \in \mathrm{SL}(X, \omega)$ be a parabolic element. First of all we assume that M is of the form

$$M \in \left\{ \begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & \alpha \\ 0 & -1 \end{pmatrix} \right\}$$

for $\alpha \in \mathbb{R}\backslash\{0\}$.

As in Proposition 3.3.6 we can show that every horizontal trajectory is either a saddle connection or closed. If we have a closed horizontal trajectory there is an ε-neighborhood in which we also have only closed horizontal trajectories. In this way we can define cylinders on (X, ω). If we now choose maximal cylinders we obtain a cylinder decomposition for (X, ω).

If we do not have a parabolic element as above then we bring it to the above form by conjugation as we will now explain. Every parabolic element has trace ± 2 and determinant 1. Thus the characteristic polynomial is of the form $t^2 \pm 2t + 1 \in \mathbb{R}[t]$. This implies that every parabolic element has eigenvalue 1 or -1. Let $v \in \mathbb{R}^2$ be an eigenvector for the parabolic element and choose a rotation matrix $R \in \mathrm{GL}(2, \mathbb{R})$ that maps v to the horizontal direction. We rotate the whole translation surface (X, ω) by R such that the eigenvector v gets mapped in the horizontal direction. We receive the new Veech group $\mathrm{SL}(R.(X, \omega)) = R\,\mathrm{SL}(X, \omega)\,R^{-1}$. The parabolic element $R\,M\,R^{-1}$ of $\mathrm{SL}(R.(X, \omega))$ has $(1, 0)$ as an eigenvector and thus the first column of the matrix $R\,M\,R^{-1}$ is of the form $(\pm 1, 0)$. The trace of $R\,M\,R^{-1}$ is again ± 2 and thus the second column is of the form $(\alpha, \pm 1)$. We conclude that the element $R\,M\,R^{-1}$ is of the form above. □

3.3.7 Remark (Parabolic elements and moduli of cylinder)**.** Let (X, ω) be a finite translation surface of genus $g \geqslant 2$ such that its Veech group $\mathrm{SL}(X, \omega)$ contains a parabolic element. Then the moduli of the cylinders of the corresponding cylinder decomposition are commensurable.

Proof. Let $\varphi \in \mathrm{Aff}^+(X, \omega)$ such that the derivative $D\varphi \in \mathrm{SL}(X, \omega)$ is a parabolic element. Without loss of generality we assume that $D\varphi$ induces a horizontal cylinder decomposition. Let $\psi = \varphi^n$ be defined as in Proposition 3.3.6 such that ψ is the identity on every horizontal saddle connection. The horizontal saddle connections form the boundaries of the maximal cylinders and thus ψ acts like a twist or multiple twist on every cylinder. We fix a cylinder and let $\mu \in \mathbb{R}$ be the modulus of the fixed cylinder. If ψ acts as a k-twist on the fixed cylinder, then we get the identity

$$D\psi = \begin{pmatrix} 1 & n \cdot \alpha \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & k \cdot \mu \\ 0 & 1 \end{pmatrix}.$$

Hence $\mu = n/k \cdot \alpha$ and the moduli of the cylinders are all rational multiples of an element $\alpha \in \mathbb{R}$. $\square$

4. Mondromy representations and Hodge theory

4.1 Fibrations

Main references for this section are the book of Spanier [112] and the paper [97] of Palmer and Tillman. If not stated otherwise we consider in this section always the category of topological spaces with continuous maps. If we write homotopy of paths we always mean homotopy relative to the endpoints of the interval.

We write I for the closed interval $[0,1]$. Let X be a topological space. The *fundamental groupoid* $\Pi_1(X)$ of the topological space X is the category where the objects $\mathrm{Obj}(\Pi_1(X)) = X$ are the elements of X and where the morphisms between two elements $x_0, x_1 \in X$ is the set $\mathrm{Mor}_{\Pi_1(X)}(x_0, x_1)$ of homotopy classes of paths relative to $\{0,1\}$, which start in x_0 and end in x_1. Composition of morphisms is the product of homotopy classes.

Choose a base point $x_0 \in X$. The *fundamental group with base point $x_0 \in X$* can then be identified with the subcategory $\pi_1(X, x_0)$ of $\Pi_1(X)$ whose single object is the element $x_0 \in X$.

4.1.1 Definition. (i) A continuous map $p\colon E \to B$ between topological spaces E and B is said to have the *homotopy lifting property with respect to a topological space S* if the following holds. Given two continuous maps $f\colon S \to E$ and $F\colon S \times I \to B$ such that $F(s,0) = p \circ f(s)$ for every $s \in S$, there exists a map $H\colon S \times I \to E$ that makes the diagram

$$\begin{tikzcd} S \times \{0\} \arrow[r, "f"] \arrow[d, hook] & E \arrow[d, "p"] \\ S \times I \arrow[r, "F"'] \arrow[ur, dashed, "H"] & B \end{tikzcd}$$

commute.

(ii) A map $p\colon E \to B$ is called *(Hurewicz) fibration* if it has the homotopy lifting property with respect to every topological space S.

(iii) A map $p\colon E \to B$ has *the unique path lifting property* if given two paths $\alpha, \gamma\colon I \to E$ with $\gamma(0) = \alpha(0)$ and $p \circ \gamma = p \circ \alpha$ then $\alpha = \gamma$.

Let $p\colon E \to B$ be a fibration and let $\gamma\colon I \to B$ with $\gamma(0) = b_0$ and $\gamma(1) = b_1 \in B$. Let $\iota\colon p^{-1}(\{b_0\}) \hookrightarrow E$ be the inclusion. It follows from the homotopy lifting property of the fibration $p\colon E \to B$ (applied to the fiber $p^{-1}(\{b_o\})$) that there exists a map $H_\gamma\colon p^{-1}(\{b_o\}) \times I \to E$ such that the following diagram commutes:

$$\begin{tikzcd}
p^{-1}(\{b_o\}) \times \{0\} \arrow[rr, "\iota \circ pr_1"] \arrow[d, hook] & & E \arrow[d, "p"] \\
p^{-1}(\{b_o\}) \times I \arrow[r, "pr_2"'] \arrow[urr, dashed, "H_\gamma"] & I \arrow[r, "\gamma"] & B
\end{tikzcd}$$

We define a map $f_{[\gamma]}$ between the fibers $p^{-1}(b_0)$ and $p^{-1}(b_1)$ by

$$f_{[\gamma]}\colon p^{-1}(b_0) \longrightarrow p^{-1}(b_1), \quad f_{[\gamma]}(e) = H_\gamma(e, 1),$$

which is well defined by the commutativity of the diagram from above.

4.1.2 Theorem. Let $p\colon E \to B$ be a fibration.

(i) There is a functor
$$F_h\colon \Pi_1(B) \longrightarrow \underline{h\mathrm{Top}}$$
from the fundamental groupoid $\Pi_1(B)$ of B to the homotopy category $\underline{h\mathrm{Top}}$ of topological spaces, which assigns to $b \in B$ the fiber $p^{-1}(\{b\})$ over b and to $[\gamma]$ the homotopy class $[f_{[\gamma]}]$ of the map $f_{[\gamma]}$ from above.

(ii) If furthermore the fibration $p\colon E \to B$ has the unique path lifting property then there is a functor
$$F\colon \Pi_1(B) \longrightarrow \underline{\mathrm{Top}}$$
from the fundamental groupoid $\Pi_1(B)$ of B to the category of topological spaces which assigns to $b \in B$ the fiber $p^{-1}(\{b\})$ over b and to $[\gamma]$ the map $f_{[\gamma]}$ from above.

It is really hard to find proper proofs for the statements of the preceding theorem in the literature. Hence we will give the proofs here.

Proof. **(1) The functors are well defined:**

Let $b_0, b_1 \in B$ and $\gamma_1, \gamma_2\colon I \to B$ two paths in B with $\gamma_1(0) = \gamma_2(0) = b_0$ and $\gamma_1(1) = \gamma_2(1) = b_1$ such that $\gamma_1 \simeq \gamma_2$ i.e., both paths are homotopic relative to $\{0, 1\}$. As before we write $\iota\colon p^{-1}(\{b_0\}) \hookrightarrow E$ for the inclusion. Let $H_1, H_2\colon p^{-1}(\{b_o\}) \times I \to E$ be two continuous maps which make the two diagrams commute:

$$\begin{tikzcd}
p^{-1}(\{b_o\}) \times \{0\} \arrow[rr, "\iota \circ pr_1"] \arrow[d, hook] & & E \arrow[d, "p"] \\
p^{-1}(\{b_o\}) \times I \arrow[r, "pr_2"'] \arrow[urr, "H_1"] & I \arrow[r, "\gamma_1"] & B
\end{tikzcd}
\qquad
\begin{tikzcd}
p^{-1}(\{b_o\}) \times \{0\} \arrow[rr, "\iota \circ pr_1"] \arrow[d, hook] & & E \arrow[d, "p"] \\
p^{-1}(\{b_o\}) \times I \arrow[r, "pr_2"'] \arrow[urr, "H_2"] & I \arrow[r, "\gamma_2"] & B
\end{tikzcd}$$

(i) We have to show that the maps $f_{[\gamma_1]} = H_1(-,1)$ and $f_{[\gamma_2]} = H_2(-,1)$ are homotopic. For this we use the homotopy lifting property: Let $h\colon I \times I \to B$ be a homotopy $\gamma_1 \simeq \gamma_2$. We define $f\colon p^{-1}(\{b_o\}) \times (\{0\} \times I \cup I \times \{0,1\}) \to E$ by

$$f(e,t,0) = H_1(e,t), \quad f(e,t,1) = H_2(e,t), \quad f(e,0,t) = e,$$

where $e \in p^{-1}(\{b_o\}$ and $t \in I$.

Since there is a homeomorphism from $I \times I$ to itself that maps $I \times \{0\}$ to $\{0\} \times I \cup I \times \{0,1\}$, we can apply the homotopy lifting property of the fibration on $p^{-1}(\{b_o\}) \times I$ to get a continuous map $K\colon p^{-1}(\{b_o\}) \times I \times I \to E$ that makes the following diagram commute:

$$\begin{tikzcd}
p^{-1}(\{b_o\}) \times (\{0\} \times I \cup I \times \{0,1\}) \arrow[rr, "f"] \arrow[d, hook] & & E \arrow[d, "p"] \\
p^{-1}(\{b_o\}) \times I \times I \arrow[r, "pr_{2,3}"'] \arrow[urr, dashed, "K"] & I \times I \arrow[r, "h"] & B
\end{tikzcd}$$

Now the map $p^{-1}(\{b_o\}) \times I \to E$ with $(e,t) \mapsto K(e,1,t)$ is a homotopy between $f_{[\gamma_1]}$ and $f_{[\gamma_2]}$ because for every $e \in p^{-1}(\{b_o\})$ we have the equalities

$$\begin{aligned} K(e,1,0) &= H_1(e,1) = f_{[\gamma_1]}(e) \quad \text{and} \\ K(e,1,1) &= H_2(e,1) = f_{[\gamma_2]}(e). \end{aligned}$$

(ii) In the second case we have to show that the two maps $f_1 = H_1(-,1)$ and $f_2 = H_2(-,1)$ are equal. This is a consequence of unique path lifting in the following way. For every $e \in p^{-1}(\{b_o\})$ we have two paths $H_1(e,-)\colon I \to E$ and $H_2(e,-)\colon I \to E$ such that

$$p \circ H_1(e,-) = \gamma_1 \simeq \gamma_2 = p \circ H_2(e,-).$$

Hence by [112, Lemma 2.3.3] for every $e \in p^{-1}(\{b_o\})$ we have the homotopy of paths $H_1(e,-) \simeq H_2(e,-)$ and in particular $H_1(e,1) = H_2(e,1)$.

(2) The functors preserve composition of morphisms: Let $\gamma_1\colon [0,1] \to B$ and $\gamma_2\colon [0,1] \to B$ be two paths that are composable i.e., $b_1 = \gamma_1(1) = \gamma_2(0)$. Let $b_0 = \gamma(0)$. By the homotopy path lifting property of the fibration $p\colon E \to B$ we get again two continuous maps $H_1\colon p^{-1}(\{b_o\}) \times [0,1] \to E$ and $H_2\colon p^{-1}(\{b_1\}) \times [1,2]$ such that the following two diagrams commute:

$$\begin{tikzcd}
p^{-1}(\{b_o\}) \times \{0\} \arrow[rr, "pr_1"] \arrow[d, hook] & & E \arrow[d, "p"] \\
p^{-1}(\{b_o\}) \times [0,1] \arrow[r, "pr_2"'] \arrow[urr, dashed, "H_1"] & {[0,1]} \arrow[r, "\gamma_1"] & B
\end{tikzcd}
\qquad
\begin{tikzcd}
p^{-1}(\{b_1\}) \times \{0\} \arrow[rr, "pr_1"] \arrow[d, hook] & & E \arrow[d, "p"] \\
p^{-1}(\{b_1\}) \times [0,1] \arrow[r, "pr_2"'] \arrow[urr, dashed, "H_2"] & {[0,1]} \arrow[r, "\gamma_2"] & B
\end{tikzcd}$$

We define a continuous map $L\colon p^{-1}(\{b_o\}) \times [0,1] \to E$ by

$$L(e,t) = \begin{cases} H_1(e,\, 2t), & t \in [0, \frac{1}{2}] \\ H_2(H_1(e,1),\, 2t-1), & t \in [\frac{1}{2}, 1] \end{cases}$$

The map L makes the following diagram commute:

$$\begin{tikzcd} p^{-1}(\{b_0\}) \times \{0\} \arrow[rrr, "\iota \circ pr_1"] \arrow[d, hook] & & & E \arrow[d, "p"] \\ p^{-1}(\{b_0\}) \times [0,1] \arrow[urrr, "L"] \arrow[r, "pr_2"'] & {[0,1]} \arrow[rr, "\gamma_1 * \gamma_2"] & & B \end{tikzcd}$$

By definition we have $f_{[\gamma_1 * \gamma_2]} = L(-,1)$ and we get the relation (in (i) up to homotopy)

$$f_{[\gamma_1 * \gamma_2]} = L(-,1) = H_2(H_1(-,1),1) = H_2(-,1) \circ H_1(-,1) = f_{[\gamma_2]} \circ f_{[\gamma_1]}.$$

□

4.1.3 Remark. Let $p\colon E \to B$ be a fibration. Fix a base point $b_0 \in B$. If we restrict the fundamental groupoid functor in Theorem 4.1.2 (i) to the subcategory $\pi_1(B, b_0)$ of $\Pi(B)$, then the mapping on the morphisms induces a group homomorphism

$$\pi_1(B, b_0) \longrightarrow \pi_0 \mathrm{Aut}(h\, p^{-1}(\{b_0\})), \tag{4.1.3.1}$$

where $\pi_0 \mathrm{Aut}(h\, p^{-1}(\{b_0\}))$ is the group of homotopy classes of homotopy equivalences of $p^{-1}(\{b_0\})$. We call the group homomorphism (4.1.3.1) the *monodromy of the fibration $p\colon E \to B$ in the point $b_0 \in B$.*

There are many examples of fibrations. The most common class are covering maps between topological spaces $\pi\colon Y \to X$, which are known to have unique path lifting [112, Theorem 2.2.3]. A different important class of fibrations are fiber bundles $f\colon E \to B$ with B a paracompact Hausdorff space [112, Corollary 2.7.14]. Differentiable manifolds are paracompact because they are Hausdorff spaces, second countable and locally compact [82, Theorem 1.15]. In the next example we want to mention another class of fibrations, which are important in this text.

4.1.4 Proper submersions as fibrations. In this subsection we will mainly follow section 4.1 of [13] and the paper [60] of Matias del Hoyo. Let B and $\mathcal{X}$ be holomorphic manifolds and let $f\colon \mathcal{X} \to B$ be a proper surjective submersion. We write df for the differential $df\colon T\mathcal{X} \to TB$. If we equip $\mathcal{X}$ with a Riemannian metric g_E we can define a subbundle T^h complementary to the kernel $\ker(df)$ of df by taking orthogonal complements on every fiber of $T\mathcal{X}$. We call T^h a *horizontal subbundle of* $T\mathcal{X}$. For every smooth curve $\gamma\colon I \to B$ with velocity $\gamma'\colon I \to TB$ and every $x \in \mathcal{X}_{\gamma(0)}$ there is a unique lift $\zeta_{\gamma,x}\colon I \to \mathcal{X}$ (i.e. $f \circ \zeta_{\gamma,x} = \gamma$) with the following properties:

- The element $\zeta'_{\gamma,x}(t)$ is an element of the horizontal tangent space $T^h_{\zeta_{\gamma,x}(t)}\mathcal{X}$ for every $t \in I$ and furthermore $df\zeta'_{\gamma,x}(t) = \gamma'(t)$ for every $t \in I$.
- The starting point of $\zeta_{\gamma,x}$ is $\zeta_{\gamma,x}(0) = x$.

We hence get a notion of parallel transport on $f\colon \mathcal{X} \to B$ and the corresponding connection is called *Ehresmann connection*. For every smooth curve $\gamma\colon I \to B$ we can find a lift $H\colon I \times \mathcal{X}_{\gamma(0)} \to \mathcal{X}$ by defining $H(t,x) = \zeta_{\gamma,x}(t)$. With this connection we can locally smoothly trivialize the family $f\colon \mathcal{X} \to B$. For example in a neighbourhood of B which is isomorphic to a ball in $\mathbb{R}^n$ we can trivialize by parallel transport along radial geodesic segments. This result is known as the theorem of Ehresmann. The family $f\colon \mathcal{X} \to B$ is a fibration since B is paracompact as a smooth manifold. Moreover for every $t \in I$ the diffeomorphisms

$$H(t,\cdot)\colon \mathcal{X}_{\gamma(0)} \longrightarrow \mathcal{X}_{\gamma(t)}$$

are orientation preserving by the following argument. The map $H(0,\cdot)$ is the identity by construction. Moreover the determinant is continuous and thus the determinant of the differential $dH(t,\cdot)$ is greater than zero on charts because it does not vanish. This shows that for every $b \in B$ we get a monodromy representation

$$\mathrm{mon}_f\colon \pi_1(B,b) \longrightarrow \pi_0\mathrm{Homeo}^+(\mathcal{X}_b),$$

where $\pi_0\mathrm{Homeo}^+(\mathcal{X}_b)$ is the group of orientation preserving homeomorphisms of $\mathcal{X}_b$ up to homotopy.

4.1.5 An equivariant action on families of Riemann surfaces. In this section we will explain how the fundamental group $\pi_1(B)$ of the base space B of a holomorphic family of Riemann surfaces $f\colon \mathcal{X} \to B$ acts on the family $f\colon \mathcal{X} \to B$ as holomorphic automorphisms of families. This can for example be found in the article [27] of Clifford Earle and Patricia Sipe.

We start with a family $f\colon \mathcal{X} \to B$ of Riemann surfaces of genus $g \geqslant 2$ with $\mathcal{X}$ and B manifolds. Let $\pi\colon \widetilde{B} \to B$ be the holomorphic universal covering of B and $\widetilde{\mathcal{X}} = \mathcal{X} \times_B \widetilde{B}$ the pull back of the family $f\colon \mathcal{X} \to B$, where

$$\mathcal{X} \times_B \widetilde{B} = \{(x,t) \in \mathcal{X} \times \tilde{B} \mid f(x) = \pi(t)\}$$

as a set.

Then $\widetilde{f}\colon \widetilde{\mathcal{X}} \to \widetilde{B}$ is again a holomorphic family of compact Riemann surfaces and thus a proper surjective submersion. We identify $\pi_1(B,b)$ with the group of Deck transformations of the cover $\pi\colon \tilde{B} \to B$. Choose a point $\tilde{b} \in \pi^{-1}(\{b\})$. Each Deck transformation $\gamma \in \pi_1(B,b)$ defines a bundle morphism A_γ of the family $\widetilde{f}\colon \widetilde{\mathcal{X}} \to \widetilde{B}$ by the universal property of the pullback. On the level of sets we have

$$A_\gamma(x,t) = (x,\gamma(t)) \quad \text{for } (x,t) \in \mathcal{X} \times_B \widetilde{B}.$$

The universal cover $\widetilde{B}$ is simply connected and the fiber $\widetilde{f}^{-1}(\{\tilde{b}\})$ is biholomorphic to the compact Riemann surface $S = f^{-1}(\{b\})$. By parallel transport with respect to the Ehresmann connection we can define for every $t \in \widetilde{B}$ an orientation preserving diffeomorphism

$$S \longrightarrow \widetilde{f}^{-1}(t)$$

and since $\widetilde{B}$ is simply connected this defines a well defined global section of the principal bundle $\mathcal{P}(\widetilde{\mathcal{X}}/\widetilde{B})$. Hence the family $\widetilde{f}\colon \widetilde{\mathcal{X}} \to \widetilde{B}$ is a marked family. We write $\mathcal{U}_g \to \mathcal{T}_g(S)$ for the universal family of the Teichmüller space $\mathcal{T}_g(S)$. By the universal property of $\mathcal{U}_g \to \mathcal{T}_g(S)$ we get a unique holomorphic map $h\colon \widetilde{B} \to \mathcal{T}_g(S)$ such that we can write the family $\widetilde{\mathcal{X}} \to \widetilde{B}$ as the pullback of $\mathcal{U}_g \to \mathcal{T}_g$ via the morphism $h\colon \widetilde{B} \to \mathcal{T}_g(S)$. By the definition of the pullback we also get a natural projection map $H\colon \widetilde{\mathcal{X}} \to \mathcal{U}_g$. The mapping class group $\Gamma_g(S)$ naturally lies in the automorphism group of the family $\mathcal{U}_g \to \mathcal{T}_g(S)$. Furthermore, the map H is equivariant with respect to the action of the fundamental group $\pi_1(B,b)$ on $\widetilde{f}\colon \widetilde{\mathcal{X}} \to \widetilde{B}$ as above and the action of $\pi_1(B,b)$ on $\mathcal{U}_g$ via the monodromy representation $\mathrm{mon}_f\colon \pi_1(B,b) \to \Gamma_g(S)$, i.e.

$$H \circ A_\gamma = \mathrm{mon}_f(\gamma) \circ H \tag{4.1.5.1}$$

for all $\gamma \in \pi_1(B,b)$. See [27] for a proof.

4.1.6 Remark. By a result of Grothendieck [50, Theorem 4.1] it is possible to reconstruct the family $f\colon \mathcal{X} \to B$ with the help of formula (4.1.5.1) from the morphism $h\colon \widetilde{B} \to \mathcal{T}_g(S)$ and the monodromy representation $\mathrm{mon}_f\colon \pi_1(B,b) \to \Gamma_g(S)$.

This is a really interesting result because we want to consider families for covers of Teichmüller curves later in this text. More concretely: Let $g \geqslant 2$, then a Teichmüller disk is by definition a holomorphic isometric embedding $\mathbb{D} \to \mathcal{T}_g$ into the Teichmüller space of genus g compact Riemann surfaces which is an isometry for the Poincaré metric on $\mathbb{D}$ and the Teichmüller metric on $\mathcal{T}_g$. Under rare circumstances the image Δ of $\mathbb{D} \to \mathcal{T}_g$ projects onto an algebraic curve in $\mathcal{M}_g$. Let $\Gamma \leqslant \Gamma_g$ be the global stabilizer of Δ under the action of the mapping class group. Then the projection $\Delta \to \mathcal{M}_g$ factor through Γ and we obtain a holomorphic map $\Delta/\Gamma \to \mathcal{M}_g$. One can show that $C = \Delta/\Gamma$ is the normalization of the image of Δ in $\mathcal{M}_g$ (see Proposition 3.2.7). Remark 4.1.6 and especially formula (4.1.5.1) can help us to understand families of compact Riemann surfaces over covers of $C = \Delta/\Gamma$.

4.2 Monodromy of locally constant sheaves

Another important class where the notion of monodromy appears, are sheaves, which are locally isomorphic to a constant sheaf of R-modules for a ring R. Such sheaves are also called *local systems*. For every locally constant sheaf of R-modules $\mathbb{L}$ on a path connected topological space X one can construct monodromy maps

$$\pi_1(X, x_0) \longrightarrow \mathrm{Aut}(\mathbb{L}_{x_0})$$

as in section 4.1 about fibrations. (We denote by $\mathbb{L}_{x_0}$ the stalk at the point $x_0 \in X$). We want to explain in the following why the construction in Claire Voisin's book [118] is more or less similar to that of fibrations from the previous section.

Essential for the construction of the monodromy functor was the homotopy lifting property of fibrations. The equivalent statement for locally constant sheaves is provided by the next lemma.

4.2.1 Lemma. Let M be a R-module and let S be a locally connected topological space and let $\mathcal{F}$ be a local system with stalk M on $S \times [0,1]$. A global section σ^0 on $\mathcal{F}|_{S\times\{0\}} = \iota^{-1}\mathcal{F}$ for the inclusion $\iota\colon S \times \{0\} \hookrightarrow S \times [0,1]$ extends uniquely to a global section σ of $\mathcal{F}$.

Proof. Since $\mathcal{F}$ is locally constant and $[0,1]$ is compact we can find open connected subsets $U_i \subset S$ ($i \in I$) and for every $i \in I$ real numbers $0 \leqslant r_{i,k} < r_{i,k+1} \leqslant 1$ ($1 \leqslant k \leqslant N_i$) sucht that $r_{i,0} = 0$ and $r_{i,N_i} = 1$ for every $i \in I$ and such that the restriction of $\mathcal{F}$ to $U_i \times [r_{i,k}, r_{i,k+1}]$ is a constant sheaf for every $i \in I$ and every $1 \leqslant k < N_i$. Since U_i is connected we get for every $1 < k < N_i$ the identities

$$\Gamma(U_i \times [r_{i,k-1}, r_{i,k}], \mathcal{F}) = \Gamma(U_i, \mathcal{F}|_{U_i\times\{r_{i,k}\}}) = \Gamma(U_i \times [r_{i,k}, r_{i,k+1}], \mathcal{F})$$

Thus $\sigma^0|_{U_i\times\{0\}}$ extends to a section $\sigma_i \in \Gamma(U_i \times [0,1], \mathcal{F})$ for every $i \in I$. Again, since $\mathcal{F}$ is locally constant one can show that sections σ_i and σ_j coincide on the intersection $U_i \times [0,1] \cap U_j \times [0,1]$. Hence the sections σ_i ($i \in I$) glue to a global section σ of $\mathcal{F}$. $\square$

From this Lemma one can deduce the following.

4.2.2 Lemma. Let $\mathcal{F}$ be a local system with stalk M on $S \times [0,1]$, where S is again a locally connected topological space. We write $\mathrm{pr}_1\colon S \times [0,1] \to S$ for the projection to the first component and we write $\mathcal{G}$ for the sheaf $\mathrm{pr}_1^{-1}(\mathcal{F}|_{S\times\{0\}})$. Then there is a canonical isomorphism of sheaves

$$\mathcal{F} \xrightarrow{\cong} \mathcal{G}.$$

Proof. We write $\mathcal{H}om(\mathcal{F}, \mathcal{G})$ for the Hom-sheaf between $\mathcal{F}$ and $\mathcal{G}$. For an open set $U \subset S \times [0,1]$ the elements in $\Gamma(U, \mathcal{H}om(\mathcal{F}, \mathcal{G}))$ are the morphisms of sheaves between $\mathcal{F}|U$ and $\mathcal{G}|U$. Since $\mathcal{F}$ and $\mathcal{G}$ are local systems the sheaves $\mathcal{H}om(\mathcal{F}|_{S\times\{0\}}, \mathcal{G}|_{S\times\{0\}})$ and $\mathcal{H}om(\mathcal{F}, \mathcal{G})|_{S\times\{0\}}$ are isomorphic. The restrictions $\mathcal{F}|_{S\times\{0\}}$ and $\mathcal{G}|_{S\times\{0\}}$ are canonically isomorphic by construction. Let

$$\sigma \in \Gamma(S \times \{0\}, \mathcal{H}om(\mathcal{F}|_{S\times\{0\}}, \mathcal{G}|_{S\times\{0\}}))$$

be the corresponding canonical section. By [74, Proposition 2.3.10] the sheaf $\mathcal{H}om(\mathcal{F}, \mathcal{G})$ is locally constant and thus by the previous Lemma 4.2.1 the section σ can be extended to a morphism of sheaves between $\mathcal{F}$ and $\mathcal{G}$. It turns out that this extension is an isomorphism (see Lemma 3.8 in [118]). $\square$

4.2.3 Theorem. Let X be a topological space and let $\mathbb{L}$ be a locally constant sheaf on X. The construction in the following proof induces a well defined functor

$$F \colon \Pi_1(X) \longrightarrow \underline{R\text{–Mod}}$$

from the fundamental groupoid $\Pi_1(X)$ of X to the category of R-modules which maps a point $x \in X$ to the stalk $\mathbb{L}_x$ and which maps a class $[\gamma]$ to an isomorphism of R-modules from $\mathbb{L}_{\gamma(0)}$ to $\mathbb{L}_{\gamma(1)}$.

Proof. Let $\gamma \colon [0,1] \to X$ be a path from x to y. Consider the inverse image $\mathcal{G}^\gamma = \gamma^{-1}\mathbb{L}$ which is a locally constant sheaf on $[0,1]$. We have that $\mathcal{G}^\gamma|_{\{0\}}$ is the constant sheaf with stalk $\mathbb{L}_x$ and $\mathcal{G}^\gamma|_{\{1\}}$ is the constant sheaf with stalk $\mathbb{L}_y$. If we apply the previous Lemma 4.2.2 twice (with $S = \{\mathrm{pt}\}$) we get for $i = 0, 1$ canonical isomorphisms

$$\mathcal{G}^\gamma \longrightarrow \mathrm{pr}_1^{-1}(\mathcal{G}^\gamma|_{\{i\}}) \tag{4.2.3.1}$$

For $i = 0, 1$ the sheaf $\mathrm{pr}_1^{-1}(\mathcal{G}^\gamma|_{\{i\}})$ is the constant sheaf with stalk $\mathbb{L}_x$ respectively $\mathbb{L}_y$ on $[0,1]$ since the inverse image of the constant sheaf $\mathcal{G}|_{\{i\}}$ is constant. The set $[0,1]$ is connected and hence for $i = 0, 1$ every stalk of the constant sheaf $\mathrm{pr}_1^{-1}(\mathcal{G}^\gamma|_{\{i\}})$ is naturally isomorphic to the module of global sections. This is especially true for the stalks $\mathbb{L}_x = (\mathrm{pr}_1^{-1}\mathcal{G}^\gamma|_{\{0\}})_0$ and $\mathbb{L}_y = (\mathrm{pr}_1^{-1}\mathcal{G}^\gamma|_{\{1\}})_1$. Hence the morphisms of sheaves from (4.2.3.1) lead to isomorphisms

$$\varphi_{x,\gamma} \colon \Gamma([0,1], \mathcal{G}^\gamma) \longrightarrow \Gamma([0,1], \mathrm{pr}_1^{-1}(\mathcal{G}^\gamma|_{\{0\}})) \longrightarrow \mathbb{L}_x$$

and

$$\varphi_{y,\gamma} \colon \Gamma([0,1], \mathcal{G}^\gamma) \longrightarrow \Gamma([0,1], \mathrm{pr}_1^{-1}(\mathcal{G}^\gamma|_{\{1\}})) \longrightarrow \mathbb{L}_y.$$

We define $A_\gamma = \varphi_{y,\gamma} \circ \varphi_{x,\gamma}^{-1} \in \mathrm{Iso}(\mathbb{L}_x, \mathbb{L}_y)$.

Next we have to show that our map is well defined, i.e. that $A_\gamma = A_\eta$ for paths $\gamma, \eta \colon [0,1] \to X$ which are homotopic. Let $h \colon [0,1] \times [0,1] \to X$ be a homotopy of paths with

$$h(s,0) = \gamma(s), \quad h(s,1) = \eta(s), \quad h(0,t) = x \quad h(1,t) = y$$

for $s, t \in [0,1]$. We consider the inverse image sheaf $h^{-1}\mathbb{L}$. It is easy to see that

$$(h^{-1}\mathbb{L})|_{I \times \{0\}} = \gamma^{-1}\mathbb{L} \quad \text{and} \quad (h^{-1}\mathbb{L})|_{I \times \{1\}} = \eta^{-1}\mathbb{L}.$$

If we apply Lemma 4.2.2 to $(h^{-1}\mathbb{L})|_{I\times\{0\}}$ and to $(h^{-1}\mathbb{L})|_{I\times\{1\}}$ we get for γ and η isomorphisms that identify $h^{-1}\mathbb{L}$ with the constant sheaf with stalk $\mathbb{L}_x$ on $[0,1] \times [0,1]$ and isomorphisms that identify $h^{-1}\mathbb{L}$ with the constant sheaf with stalk $\mathbb{L}_y$ on $[0,1] \times [0,1]$. By Lemma 4.2.1 (and the proof of the Lemma) these isomorphisms coincide pairwise since γ and η start and end in the same point. Thus the maps $\varphi_{x,\gamma}$, $\varphi_{x,\eta}$ coincide and the maps $\varphi_{y,\gamma}$, $\varphi_{y,\eta}$ coincide. We conclude $A_\gamma = A_\eta$.

Now it is left to show that $[\gamma * \eta] \mapsto A_\eta \circ A_\gamma$ for paths $\gamma, \eta \colon [0,1] \to X$ with $\gamma(1) = \eta(0)$. We omit this part since it uses basically the same arguments as before. $\square$

From this theorem we get two corollaries.

4.2.4 Corollary. Let X be a path connected and simply connected topological space. Then every local system $\mathcal{G}$ with stalk M on X is isomorphic to the constant sheaf M_X with stalk M on X.

Idea of the proof. Let $x \in X$ then for every $y \in X$ there is a path $\gamma\colon [0,1] \to X$ with $\gamma(0) = x$ and $\gamma(1) = y$ and an isomorphism A_γ between the stalks $\mathcal{G}_x$ and $\mathcal{G}_y$. In the proof of Proposition 3.9 in the book [118] Voisin describes how the A_γ glue together to a global section of the sheaf $\mathcal{H}om(M_X, \mathcal{G})$ of locally constant homomorphisms between M_X and $\mathcal{G}$. She argues that this section is an isomorphism since M_X and $\mathcal{G}$ are isomorphic in a neighbourhood of x by assumption. $\square$

4.2.5 Corollary. Let M be a R-module and let $\mathbb{L}$ be a locally constant sheaf of stalk M on a path connected, locally simply connected topological space X. Fix a point $x_0 \in X$ and an isomorphism $\varphi\colon \mathbb{L}_{x_0} \to M$. Then the construction in the proof of this corollary is a well defined group homomorphism

$$\pi_1(X, x_0) \longrightarrow \mathrm{Aut}(M).$$

There are two ways to obtain the group homomorphism. One can either use Theorem 4.2.3 directly or one can do an intermediate step with the universal cover of X and Corollary 4.2.4. Both approaches appear in the literature and especially the second one is important later in Subsection 4.5.4 about period mappings. This is why we want to explain both ways and why they lead to the same group homomorphism.

Proof of Corollary 4.2.5. Approach 1: Let $c \in \pi_1(X, x_0)$ and let $\gamma\colon [0,1] \to X$ be a closed path which represents c. By Theorem 4.2.3 we get an isomorphism $A_\gamma\colon \mathbb{L}_{x_0} \to \mathbb{L}_{x_0}$ which is independent of the representantive of the class c. Define

$$\pi_1(X, x_0) \longrightarrow \mathrm{Aut}(M), \quad [\gamma] \longmapsto \varphi \circ A_\gamma \circ \varphi^{-1}.$$

Approach 2: If $\pi\colon \widetilde{X} \to X$ is an universal cover of X then $\pi^{-1}(\mathbb{L})$ is isomorphic to the constant sheaf with stalk M on $\widetilde{X}$ by Corollary 4.2.4. Write

$$\beta\colon \pi^{-1}\mathbb{L} \longrightarrow M_{\widetilde{X}}$$

for the isomorphism of sheaves. Then β is uniquely determined by the chosen isomorphism $\varphi\colon \mathbb{L}_{x_0} \cong M$.

Fix a point $y_0 \in \pi^{-1}(\{x_0\})$, then for every $c \in \pi_1(X, x_0)$ the isomorphism β induces an isomorphisms $\beta_c\colon \pi^{-1}(\mathbb{L})_{c\cdot y_0} \to M$ between the stalk at $c \cdot y_0$ and M. Furthermore $(\pi^{-1}\mathbb{L})_{c\cdot y_0}$ and $\mathbb{L}_{x_0}$ are naturally isomorphic by the definition of the inverse image $\pi^{-1}\mathbb{L}$. Write $\alpha_c\colon (\pi^{-1}\mathbb{L})_{c\cdot y_0} \to \mathbb{L}_{x_0}$ for this isomorphism. Then define the group homomorphism

$$\pi_1(X, x_0) \longrightarrow \mathrm{Aut}(M), \quad c \longmapsto \varphi \circ \alpha_c \circ \beta_c^{-1}$$

Compare Approach 1 and Approach 2: We explain now why the construction in Approach 2 leads to the same group homomorphism as the construction in Approach 1 with the monodromy functor from Theorem 4.2.3. We identify $\pi_1(X, x_0)$ with the Deck group of $\pi\colon \widetilde{X} \to X$. Let $c \in \pi_1(X, x_0)$ and $\widetilde{\gamma}\colon [0,1] \to \widetilde{X}$ a continuous path from y_0 to $c \cdot y_0$ such that $\gamma = \pi(\widetilde{\gamma})$ is an element of the class $c \in \pi_1(X, x_0)$. By Theorem 4.2.3 the paths $\widetilde{\gamma}$ and $\gamma = \pi(\widetilde{\gamma})$ induce isomorphisms of stalks

$$A_{\widetilde{\gamma}}\colon \left(\pi^{-1}\mathbb{L}\right)_{y_0} \longrightarrow \left(\pi^{-1}\mathbb{L}\right)_{c\cdot y_0} \quad \text{and} \quad A_\gamma\colon \mathbb{L}_{x_0} \longrightarrow \mathbb{L}_{x_0}.$$

By definition of the inverse image sheaf $\pi^{-1}\mathbb{L}$ we get again isomorphisms

$$\alpha_1\colon (\pi^{-1}\mathbb{L})_{y_0} \longrightarrow \mathbb{L}_{x_0} \quad \text{and} \quad \alpha_c\colon (\pi^{-1}\mathbb{L})_{c\cdot y_0} \longrightarrow \mathbb{L}_{x_0}.$$

Now we have the identity $A_\gamma = \alpha_c \circ A_{\widetilde{\gamma}} \circ \alpha_1^{-1}$. This is because we can cover the compact set $\gamma([0,1])$ by open sets U_i such that $\pi|_{U_i}$ is a homeomorphism onto its image and in this case $\pi^{-1}(\mathbb{L})(U_i) = \mathbb{L}(\pi(U_i))$.

We recall the construction of the isomorphism $\beta\colon \pi^{-1}\mathbb{L} \to M_{\widetilde{X}}$ in 4.2.4. The main idea was to identify the stalk $(\pi^{-1}\mathbb{L})_{y_0}$ with any other stalk $(\pi^{-1}\mathbb{L})_z$ via the isomorphism A_η from Theorem 4.2.3, where $\eta\colon [0,1] \to \widetilde{X}$ is a continuous path which starts in y_0 and ends in z. This immediately implies the identity

$$\beta_c = \varphi \circ \alpha_1 \circ A_{\widetilde{\gamma}}^{-1}.$$

for $c \in \pi_1(X, x_0)$ and $\widetilde{\gamma}$ a path from y_0 to $c \cdot y_0$. We showed that both approaches lead to the same group homomorphism. □

4.2.6 Definition. Let $\mathbb{L}$ be a locally constant sheaf of R-modules and X a path connected locally simply connected topological space X as in the previous corollary. Fix again a point $x_0 \in X$ and write M for the stalk $\mathbb{L}_{x_0}$. Then the group homomorphism

$$\pi_1(X, x_0) \longrightarrow \operatorname{Aut}(M)$$

is called the *monodromy representation of the local system* $\mathbb{L}$ *in the point* $x_0 \in X$.

If our base space X is path-connected and locally simply connected we even have that the monodromy maps from above define an equivalence of categories between the category of local systems on X and the category of abelian groups equipped with a $\pi_1(X, x_0)$-action. Given an abelian group A equipped with a $\pi_1(X)$-action we can naturally associate a local system $\mathbb{L}$ to it as we want to explain now. If $\pi\colon \widetilde{X} \to X$ denotes the universal cover of X and $A_{\widetilde{X}}$ is the constant sheaf with stalk A on $\widetilde{X}$, then we define the sections of $\mathbb{L}$ over an open set $U \subset X$ as follows. The elements of $\Gamma(U, \mathbb{L})$ are the sections σ of $\Gamma(\pi^{-1}(U), A_{\widetilde{X}})$ with the property

$$\sigma(\gamma \cdot x) = \gamma \cdot \sigma(x)$$

for all $x \in \widetilde{X}$ and for all $\gamma \in \pi_1(X)$. For the whole proof see [118] Corollary 3.10.

4.2.7 Proposition. Let X be a topological space, Let M be a R-module and let $\mathbb{L}$ be a locally constant sheaf of stalk M on X. Let W be a second R-module and let

$$Q\colon \mathbb{L} \otimes_R \mathbb{L} \to W_X$$

be a bilinear map. Let $x, y \in X$ and $\gamma\colon [0,1] \to X$ be a path from x to y. Write $A_\gamma\colon \mathbb{L}_x \to \mathbb{L}_y$ for the isomorphism by parallel transport from Theorem 4.2.3. Then $Q_y \circ (A_\gamma \otimes A_\gamma) = Q_x$. Or in other words parallel transport preserves the bilinear map Q.

Proof. Let $\gamma\colon [0,1] \to X$ be a path from x to y. We denote again $\mathcal{G}^\gamma := \gamma^{-1}\mathbb{L}$. Then there are the isomorphisms of sheaves

$$\psi_x\colon \mathcal{G}^\gamma \longrightarrow (\mathbb{L}_x)_{[0,1]} \quad \text{and} \quad \psi_y\colon \mathcal{G}^\gamma \longrightarrow (\mathbb{L}_y)_{[0,1]}.$$

from the proof of Theorem 4.2.3. Let $\mathcal{S}$ be the presheaf $U \mapsto \mathcal{G}^\gamma(U) \otimes_R \mathcal{G}^\gamma(U)$, where $U \subset [0,1]$ is open. Since W_X is constant, the morphism of sheaves Q induces a morphism of presheaves between $\mathcal{S}$ and $\gamma^{-1}W_X = W_{[0,1]}$. Thus the universal property of tensor products on sheaves induces a morphism of sheaves

$$Q_\gamma\colon \mathcal{G}^\gamma \otimes_R \mathcal{G}^\gamma \longrightarrow W_{[0,1]}.$$

By Lemma 4.2.1 the bilinear maps $Q_x\colon \mathbb{L}_x \otimes_R \mathbb{L}_x \to W$ and $Q_y\colon \mathbb{L}_y \otimes_R \mathbb{L}_y \to W$ extend to sheaf morphisms

$$\widetilde{Q}^{(x)}\colon (\mathbb{L}_x)_{[0,1]} \otimes_R (\mathbb{L}_x)_{[0,1]} \to W_{[0,1]} \quad \text{and} \quad \widetilde{Q}^{(y)}\colon (\mathbb{L}_y)_{[0,1]} \otimes_R (\mathbb{L}_y)_{[0,1]} \to W_{[0,1]}$$

The universal property of tensor-products on sheaves leads to sheaf morphisms

$$\widetilde{\psi}_x\colon \mathcal{G}^\gamma \otimes_R \mathcal{G}^\gamma \to (\mathbb{L}_x \otimes \mathbb{L}_x)_{[0,1]} \quad \text{and} \quad \widetilde{\psi}_y\colon \mathcal{G}^\gamma \otimes_R \mathcal{G}^\gamma \to (\mathbb{L}_y \otimes \mathbb{L}_y)_{[0,1]}$$

induced by $\psi_x \times \psi_x$ respectively $\psi_y \times \psi_y$. The construction of trivialisations in Lemma 4.2.1 shows that the following diagrams commute:

$$\begin{tikzcd}
\mathcal{G}^\gamma \otimes_R \mathcal{G}^\gamma \arrow[r, "Q_\gamma"] \arrow[d, "\widetilde{\psi}_x"] & W_{[0,1]} \arrow[d, "\mathrm{id}"] \\
(\mathbb{L}_x)_{[0,1]} \otimes (\mathbb{L}_x)_{[0,1]} \arrow[r, "\widetilde{Q}^{(x)}"'] & W_{[0,1]}
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{G}^\gamma \otimes_R \mathcal{G}^\gamma \arrow[r, "Q_\gamma"] \arrow[d, "\widetilde{\psi}_y"] & W_{[0,1]} \arrow[d, "\mathrm{id}"] \\
(\mathbb{L}_y)_{[0,1]} \otimes_R (\mathbb{L}_y)_{[0,1]} \arrow[r, "\widetilde{Q}^{(y)}"'] & W_{[0,1]}
\end{tikzcd}$$

The natural isomorphism from the global sections $\Gamma([0,1], W_{[0,1]}) = W$ to the stalks $(W_{[0,1]})_0 = W$ and $(W_{[0,1]})_1 = W$ is also given by the identity map. We thus get the following commutative diagrams:

$$\begin{tikzcd}
\mathbb{L}_x \otimes_R \mathbb{L}_x \arrow[d, "Q_x"] & \Gamma([0,1], (\mathbb{L}_x)_{[0,1]} \otimes_R (\mathbb{L}_x)_{[0,1]}) \arrow[l, "\varphi_0"'] \arrow[d, "\Gamma\widetilde{Q}^{(x)}"] & \Gamma([0,1], \mathcal{G}^\gamma \otimes_R \mathcal{G}^\gamma) \arrow[l, "\Gamma\widetilde{\psi}_x"'] \arrow[d, "\Gamma Q_\gamma"] \\
W & W = \Gamma([0,1], W_{[0,1]}) \arrow[l, "\mathrm{id}"] & W = \Gamma([0,1], W_{[0,1]}) \arrow[l, "\mathrm{id}"]
\end{tikzcd}$$

and

$$\begin{tikzcd}
\Gamma([0,1],\mathcal{G}^\gamma\otimes_R\mathcal{G}^\gamma) \arrow[r, "\Gamma\widetilde{\psi}_y"] \arrow[d, "\Gamma Q_\gamma"] & \Gamma([0,1],(\mathbb{L}_y)_{[0,1]}\otimes_R(\mathbb{L}_y)_{[0,1]}) \arrow[r, "\varphi_1"] \arrow[d, "\Gamma\widetilde{Q}^{(y)}"] & \mathbb{L}_y\otimes_R\mathbb{L}_y \arrow[d, "Q_y"] \\
W=\Gamma([0,1],W_{[0,1]}) \arrow[r, "\mathrm{id}"] & W=\Gamma([0,1],W_{[0,1]}) \arrow[r, "\mathrm{id}"] & W
\end{tikzcd}$$

Here the homomorphisms φ_0 and φ_1 are the natural isomorphisms of the global sections of $(\mathbb{L}_x)_{[0,1]}\otimes_R(\mathbb{L}_x)_{[0,1]}$ respectively $(\mathbb{L}_y)_{[0,1]}\otimes_R(\mathbb{L}_y)_{[0,1]}$ to the stalks in the point $0\in[0,1]$ respectively $1\in[0,1]$. The concatenation of the isomorphisms in the first two rows of the diagrams is the homomorphism $A_\gamma\otimes A_\gamma\colon\mathbb{L}_x\otimes_R\mathbb{L}_x\to\mathbb{L}_y\otimes_R\mathbb{L}_y$. The commutativity of the diagrams shows $Q_y\circ A_\gamma\otimes A_\gamma=Q_x$. □

4.2.8 Local systems associated to proper submersions. Let $f\colon\mathcal{X}\to B$ be a proper submersion between holomorphic manifolds B and $\mathcal{X}$. We saw in Section 4.1.4 that for every $b\in B$ we get a monodromy representation

$$\mathrm{mon}_f\colon\pi_1(B,b)\longrightarrow\pi_0\mathrm{Homeo}^+(\mathcal{X}_b),$$

where $\pi_0\mathrm{Homeo}^+(\mathcal{X}_b)$ are the orientation preserving homeomorphisms of $\mathcal{X}_b$ up to homotopy. Since homotopic maps induce the same homomorphisms on cohomology, for any $k\geqslant0$ we get a homomorphism

$$\psi_k\colon\pi_0\mathrm{Homeo}^+(\mathcal{X}_b)\longrightarrow\mathrm{GL}(H^k(\mathcal{X}_b,\mathbb{Q})),$$

induced by the action of $\pi_0\mathrm{Homeo}^+(\mathcal{X}_b)$ on the cohomology groups $H^k(\mathcal{X}_b,\mathbb{Q})$ by pull back. This leads to a homomorphism $\rho_k=\psi_k\circ\mathrm{mon}_f$ from $\pi_1(B,b)$ to the group $\mathrm{GL}(H^k(\mathcal{X}_b,\mathbb{Q}))$. Since orientation preserving homeomorphisms preserve the intersection pairing Q_k on $H^k(\mathcal{X}_b,\mathbb{Q})$, we get for every $k\in\mathbb{N}$ a homomorphism

$$\rho_k\colon\pi_1(B,b)\longrightarrow\mathrm{Aut}(H^k(\mathcal{X}_b,\mathbb{Q}),Q_k).\tag{4.2.8.1}$$

As in Remark 2.2.3 we can show with the Theorem of Ehresmann that for every natural number $k\in\mathbb{N}$ the sheaf $R^kf_*\mathbb{Z}_\mathcal{X}$ is locally constant. Furthermore if we glue local trivialisations of $f\colon\mathcal{X}\to B$ we can easily describe the monodromy representation associated to the local system $R^kf_*\mathbb{Z}_\mathcal{X}$. We will formulate this in the next proposition.

4.2.9 Proposition. Let $f\colon\mathcal{X}\to B$ be a proper submersion between holomorphic manifolds and let $b\in B$. Then for every $k\in\mathbb{N}$ the monodromy representation of the local system $R^kf_*\mathbb{Z}_\mathcal{X}$ is given by the group homomorphism

$$\rho_k\colon\pi_1(B,b)\longrightarrow\mathrm{Aut}(H^k(\mathcal{X}_b,\mathbb{Q}),Q_k).$$

from (4.2.8.1).

Proof. [118, Section 3.1.2] or [6, Proposition 1.2.6]) □

We want to state an important result from Alan Landman [81] for the algebraic representation.

4.2.10 Theorem (Monodromy Theorem)**.** Now let $\mathcal{X} \to \mathbb{D}^*$ be a family of compact Riemann surfaces over the punctured unit disk $\mathbb{D}^* = \{t \in \mathbb{C} \mid 0 < |t| < 1\}$ and let $\gamma \in \pi_1(\mathbb{D}^*)$ be a generator. Denote by $\rho_k(\gamma) = \gamma_s \gamma_u$ the Jordan decomposition of $\rho_k(\gamma)$ into its semistable and unipotent part. Then there is a natural number $m \in \mathbb{N}$ with $(\gamma_u - \mathrm{id})^m = 0$ and γ_s has finite order.

Let $f\colon \mathcal{X} \to \mathbb{D}^*$ be a family of compact Riemann surfaces. If we replace $\mathbb{D}^*$ by a finite cyclic cover we can assume that the family has unipotent algebraic monodromy by Theorem 4.2.10. This is important because it allows us to extend the local system $R^1 f_* \mathbb{Z}_{\mathcal{X}}$. This is the content of the next section.

4.3 Flat vector bundles

4.3.1 Local systems and flat vector bundles. There is another important characterization of local systems as vector bundles with flat connections. A good reference is [117, section 9.2.1].

Let B be a complex manifold and denote by $\mathcal{O}_B$ the sheaf of holomorphic functions on B. If $\mathbb{V}$ is a local system of $\mathbb{C}$-vector spaces then we can associate to $\mathbb{V}$ a locally free $\mathcal{O}_B$-module $\mathbf{V}$, respectively a vector bundle by

$$\mathbf{V} = \mathbb{V} \otimes_{\mathbb{C}} \mathcal{O}_B.$$

This vector bundle naturally comes with a flat connection $\nabla_{\mathbb{V}}\colon \mathbf{V} \to \Omega_B \otimes_{\mathcal{O}_B} \mathbf{V}$ in the following way. Let s be a section of the vector bundle $\mathbf{V}$ and let $U \subset B$ be an open subset, such that $\Gamma(U, \mathbb{V})$ is isomorphic to a $\mathbb{C}$-vector space with basis $s_1, \ldots, s_n$. Then $s|_U = \sum_{i=1}^n f_i\, s_i$ with $f_i \in \mathcal{O}_B(U)$ and we define

$$\nabla_{\mathbb{V}}\, s|_U = \sum_{i=1}^n df_i \otimes s_i = \sum_{i,j=1}^n \frac{\partial f_i}{\partial z_j}\, dz_j \otimes s_i. \tag{4.3.1.1}$$

This definition is independent of the chosen basis and glues together to a map of sheaves. A proof of the following proposition can be found in [117, Proposition 9.11].

4.3.2 Proposition. Let B be a complex manifold. The functor

$$\mathbb{V} \longmapsto (\mathbb{V} \otimes_{\mathbb{C}} \mathcal{O}_B, \nabla_{\mathbb{V}}),$$

defines an equivalence of the category of local systems over B and the category of complex vector bundles over B equipped with a flat connection.

4.3.3 Deligne canonical extension. We write $\mathbb{D}$ for the unit disk and denote $\mathbb{D}^* = \mathbb{D}\backslash\{0\}$. Let $n, k \in \mathbb{N}$ with $k \leqslant n$ and let $B = (\mathbb{D}^*)^k \times \mathbb{D}^{n-k}$ as well as $X = \mathbb{D}^n$. Then $D = X\backslash B$ is a divisor with normal crossing singularity since D is given by the equation $\prod_{i=1}^k t_i = 0$ for local coordinates $t_1, \ldots, t_n$ of the polydisk X.

Let $\mathbb{V}$ be a local system of $\mathbb{C}$-vector spaces over B and let $\mathbf{V} = \mathbb{V} \otimes_{\mathbb{C}} \mathcal{O}_B$ be the associated holomorphic vector bundle with flat connection

$$\nabla_{\mathbb{V}} \colon \mathbf{V} \longrightarrow \Omega_B \otimes_{\mathcal{O}_B} \mathbf{V}.$$

Choose an isomorphism of the fundamental group $\pi_1(B, b)$ of B with base point $b \in B$ and the group $\mathbb{Z}^k$. We write $\rho \colon \pi_1(B, b) \to \mathrm{GL}(\mathbb{V}_b)$ for the monodromy representation of the local system $\mathbb{V}$. Choose generators $\gamma_1, \ldots, \gamma_k \in \pi_1(B, b)$ corresponding to the standard generators $e_1, \ldots, e_k$ of $\mathbb{Z}^k$ under our identification and assume that the monodromy operators $T_i := \rho(\gamma_i) \in \mathrm{GL}(\mathbb{V}_b)$ are unipotent elements for every $i = 1, \ldots, k$.

We will sketch the construction of Deligne's canonical extension of $(\mathbf{V}, \nabla)$ to a vector bundle over X with a meromorphic connection $(\mathbf{V}_{\mathrm{ext}}, \nabla_{\mathrm{ext}})$ [19, Proposition II.5.2]. Without loss of generality we assume $k = n$ and hence $B = (\mathbb{D}^*)^n$. Let $\mathbf{V}$ be a vector bundle of rank r. For every $i = 1, \ldots, n$ we define a nilpotent matrix

$$N_i = \frac{-1}{2\pi\sqrt{-1}} \log T_i = \frac{1}{2\pi\sqrt{-1}} \sum_{j=1}^{\infty} \frac{1}{j}(I_r - T_i)^j,$$

where the sum on the right hand side is finite since the matrices $T_i \in \mathbb{C}^{r \times r}$ are unipotent by assumption. A universal cover $\pi_{\mathrm{univ}} \colon \mathbb{H}^n \to B$ is given by

$$\pi_{\mathrm{univ}}(z_1, \ldots, z_n) = (\exp(2\pi\sqrt{-1}\, z_1), \ldots, \exp(2\pi\sqrt{-1}\, z_n)).$$

The fundamental group $\mathbb{Z}^n$ of B with standard generators $e_1, \ldots, e_n$ acts on $\mathbb{H}^n$ by Deck transformations by the rule

$$e_i.(z_1, \ldots, z_i, \ldots, z_n) = (z_1, \ldots, z_i - 1, \ldots, z_n).$$

The pull back $\pi_{\mathrm{univ}}^* \mathbf{V}$ of the bundle $\mathbf{V}$ to the universal cover $\mathbb{H}^n$ can be globally trivialized and hence the sections of $\mathbf{V}$ over B correspond to holomorphic maps $s \colon \mathbb{H}^n \to \mathbb{C}^r$ which are equivariant with respect to the action of the fundamental group $\mathbb{Z}^n$ on $\mathbb{H}^n$ and the action of the group of monodromy operators on $\mathbb{C}^r$. This means that for every $z \in \mathbb{H}^n$ and every $i = 1, \ldots, n$ the equality $s(e_i.z) = T_i\, s(z)$ is required. For any vector $v \in \mathbb{C}^r$ we define a holomorphic map

$$\widetilde{s}_v \colon \mathbb{H}^n \longrightarrow \mathbb{C}^r, \quad \widetilde{s}_v((z_1, \ldots, z_n)) = \exp\left(2\pi\sqrt{-1} \sum_{j=1}^{n} z_j N_j\right) v.$$

We fix a vector $v \in \mathbb{C}^r$ and the holomorphic map $\widetilde{s}_v \colon \mathbb{H}^n \to \mathbb{C}^r$. Then for every $i = 1, \ldots, n$ and every $z \in \mathbb{H}^n$ we have the property

$$\begin{aligned}\widetilde{s}_v(e_i.z) &= \exp\left(2\pi\sqrt{-1}\left(\sum_{j=1}^{n} z_j N_j - N_i\right)\right) \cdot v \\ &= \exp(-2\pi\sqrt{-1}\, N_i) \cdot \widetilde{s}_v(z) = T_i \cdot \widetilde{s}_v(z).\end{aligned}$$

This shows $\widetilde{s}_v(e_i.z) = T_i \cdot \widetilde{s}_v(z)$. In other words, the holomorphic map $\widetilde{s}_v$ defines a holomorphic section s_v of $\mathbf{V} \to B$. By construction of the universal cover $\pi_{\text{univ}} \colon \mathbb{H}^n \to B$ and by construction of the section $s_v \in \Gamma(B, \mathbf{V})$, we see that s_v is meromorphic on X and holomorphic on $B = X \backslash D$ with growth $O(\log\|t\|^k)$ close to D, where $\|t\| = 1/d(t, D)$. Let $\iota \colon B \to X$ be the inclusion. For every $v \in \mathbb{C}^r$ we have $s_v \in \Gamma(X, \iota_* \mathbf{V})$ and we define $\mathbf{V}_{\text{ext}}$ as

$$\langle s_v \mid v \in \mathbb{C}^r \rangle \otimes_{\mathbb{C}} \mathcal{O}_X \subset \iota_* \mathbf{V}.$$

All the matrices N_j commute with each other and hence we obtain for all $(z_1, \ldots, z_n) \in \mathbb{H}^n$ the equality

$$\exp\left(2\pi\sqrt{-1} \sum_{j=1}^{n} z_j N_j\right) = \prod_{j=1}^{n} \exp\left(2\pi\sqrt{-1}\, z_j N_j\right).$$

For $i = 1, \ldots, n$, we conclude for the partial derivatives

$$\partial/\partial z_i \left(\exp\left(2\pi\sqrt{-1} \sum_{j=1}^{n} z_j N_j\right)\right) = 2\pi\sqrt{-1} \cdot N_i \cdot \exp\left(2\pi\sqrt{-1} \sum_{j=1}^{n} z_j N_j\right).$$

On the pull back $\pi_{\text{univ}}^{-1}\mathbb{V}$ of the local system $\mathbb{V}$ to $\mathbb{H}^n$ we have a natural connection ∇ defined as in 4.3.1.1. If $v \in \mathbb{C}^r$ and $\widetilde{s}_v \colon \mathbb{H}^n \to \mathbb{C}^r$ is defined as above, then

$$\nabla \widetilde{s}_v = 2\pi\sqrt{-1} \sum_{j=1}^{n} dz_j \otimes N_j\, \widetilde{s}_v.$$

For the coordinate $t_k = \exp(2\pi\sqrt{-1} z_k)$ of B we get $dt_k/t_k = 2\pi\sqrt{-1}\, dz_k$ and hence the connection $\nabla_{\text{ext}} \colon \mathbf{V}_{\text{ext}} \to \Omega_X^1 \otimes_{\mathcal{O}_X} \mathbf{V}_{\text{ext}}$ that fits to our construction is given by

$$\nabla_{\text{ext}}(s_v) := \sum_{j=1}^{n} \frac{dt_j}{t_j} \otimes N_j\, s_v.$$

The 1-forms dt_j/t_j are so called logarithmic 1-forms[1]. The tuple $(\mathbf{V}_{\text{ext}}, \nabla_{\text{ext}})$ is a regular connection in the sense of [19, Definition II.1.11]. The construction of the Deligne canonical extension is the main part of the following theorem (see [107, Chapter 6, Section 6.1] or [98, Theorem 11.7] for more details).

[1] The sheaf of logarithmic one forms on X which are holomorphic on $B = X \backslash D$ is often written in the literature as $\Omega_X^1(\log D)$.

4.3.4 Theorem (Deligne's Riemann Hilbert correspondance)**.** The construction delivers a functor

$$\mathbb{V} \longmapsto ((\mathbb{V} \otimes_{\mathbb{C}} \mathcal{O}_B)_{\mathrm{ext}}, (\nabla_{\mathbb{V}})_{\mathrm{ext}}),$$

which includes the category of locally constant sheaves on $B = X \backslash D$ into the category of holomorphic vector bundles on X with a regular meromorphic connection.

4.4 Mixed Hodge structures

The proof of Theorem A relies on Hodge theory. In this section we want to collect all the tools we need. We can warmly recommend the book [98], which we used as a main reference in this section, and the book [117]. But also chapter 3 of [65] summarizes parts of the theory very nicely.

4.4.1 Definition (Hodge structure and Hodge decomposition for modules)**.** Let $V_{\mathbb{Z}}$ be a $\mathbb{Z}$-module of finite rank and let $k \in \mathbb{Z}$.

(i) A *Hodge structure of weight k on $V_{\mathbb{Z}}$* is by definition a decreasing filtration $\{F^p V_{\mathbb{C}}\}_{p \in \mathbb{Z}}$ of the associated complex vector space $V_{\mathbb{C}} = V_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{C}$ by complex subspaces such that for every $p, q \in \mathbb{Z}$ with $p + q = k + 1$ the condition

$$V_{\mathbb{C}} = F^p V \oplus \overline{F^q V}$$

holds.

(ii) A *Hodge decomposition of weight k on $V_{\mathbb{Z}}$* is a decomposition of the associated complex vector space

$$V_{\mathbb{C}} = \bigoplus_{p+q=k} V^{p,q}$$

such that $V^{p,q} = \overline{V^{q,p}}$ for the complex vector spaces $V^{p,q} \subset V_{\mathbb{C}}$ $(p + q = k)$. The numbers $h^{p,q} := \dim(V^{p,q})$ are called *Hodge numbers* of the Hodge structure.

(iii) Let $V_{\mathbb{Z}}$ and $W_{\mathbb{Z}}$ be two $\mathbb{Z}$-modules equipped with Hodge decompositions of weight k. We say that a morphism $f : V_{\mathbb{Z}} \to W_{\mathbb{Z}}$ is a *morphism of Hodge decompositions* if $f_{\mathbb{C}}(V^{p,q}) \subset W^{p,q}$ for every $p, q \in \mathbb{Z}$, where $f_{\mathbb{C}}$ is the complexification $f_{\mathbb{C}}$ of f.

4.4.2 Remark. Let $V_{\mathbb{Z}}$ be a $\mathbb{Z}$-module of finite rank and $k \in \mathbb{Z}$. The definition of Hodge structure and Hodge decomposition in Definition 4.4.1 describe the same structure on $V_{\mathbb{C}}$ by the following arguments. Given a decreasing Hodge filtration $\{F^p V_{\mathbb{C}}\}_{p \in \mathbb{Z}}$ on $V_{\mathbb{C}}$, the complex spaces

$$V^{p,q} := F^p V_{\mathbb{C}} \cap \overline{F^q V_{\mathbb{C}}} \quad (p, q \in \mathbb{Z},\ p + q = k)$$

define a Hodge decomposition. The other way around, if we have a decomposition $V_{\mathbb{C}} = \oplus_{p+q=k} V^{p,q}$ as in Definition 4.4.1 (ii) then the subspaces

$$F^p V_{\mathbb{C}} := \bigoplus_{i \geqslant p} V^{i,k-i} \quad (p \in \mathbb{Z})$$

define a Hodge filtration $\{F^p V_{\mathbb{C}}\}_{p \in \mathbb{Z}}$ on $V_{\mathbb{C}}$.

4.4.3 Example (Hodge structure on integral cohomology)**.** Let X be a compact Kähler manifold then the cohomology group $H^k(X, \mathbb{Z})$ has a Hodge decomposition of weight k. We identify $H^k(X, \mathbb{C})$ by the isomorphism of De Rham with the k-th De Rham cohomology, which we want to denote in the following by $H^k_{\mathrm{DR}}(X, \mathbb{C})$. The k-th De Rham cohomology admits the well known decomposition

$$H^k(X, \mathbb{C}) = H^k_{\mathrm{DR}}(X, \mathbb{C}) = \bigoplus_{p+q=k} H^{p,q}(X),$$

where $H^{p,q}(X)$ is the space of harmonic (p,q)-forms on X (see [98, Theorem 1.8]).

Furthermore if Y is a second compact Kähler manifold and $f \colon X \to Y$ is a holomorphic map, then the pull back map $f^* \colon H^k(Y, \mathbb{C}) \to H^k(X, \mathbb{C})$ maps $H^{p,q}(Y)$ to $H^{p,q}(X)$ for $p + q = k$ and hence defines a morphism of Hodge decompositions.

4.4.4 Example (Hodge structure of Tate)**.** Let $m \in \mathbb{Z}$. An important and basic example of Hodge decomposition is the *Hodge decomposition of Tate* on the module $\mathbb{Z}(m) := (2\pi\sqrt{-1})^m \mathbb{Z}$, which consists of the single vector space

$$H^{-m,-m} := \mathbb{Z}(m) \otimes_{\mathbb{Z}} \mathbb{C} = \mathbb{C}.$$

It has weight $-2m$.

4.4.5 Constructions. We want to introduce the most relevant constructions on Hodge structures. Let $V_{\mathbb{Z}}$, $W_{\mathbb{Z}}$ be $\mathbb{Z}$-modules of finite rank with Hodge decompositions

$$V_{\mathbb{C}} = \bigoplus_{p+q=k} V^{p,q} \quad \text{and} \quad W_{\mathbb{C}} = \bigoplus_{r+s=m} W^{r,s}$$

of weight k and weight m.

(i) *Morphisms:* If $\mathrm{Hom}_{\mathbb{Z}}(V_{\mathbb{Z}}, W_{\mathbb{Z}})$ is the module of all $\mathbb{Z}$-module homomorphisms between $V_{\mathbb{Z}}$ and $W_{\mathbb{Z}}$, then we can identify $\mathrm{Hom}_{\mathbb{Z}}(V_{\mathbb{Z}}, W_{\mathbb{Z}}) \otimes_{\mathbb{Z}} \mathbb{C}$ with the finite vector space $\mathrm{Hom}_{\mathbb{C}}(V_{\mathbb{C}}, W_{\mathbb{C}})$ and there is a Hodge decomposition of weight $m - k$ on $\mathrm{Hom}_{\mathbb{Z}}(V_{\mathbb{Z}}, W_{\mathbb{Z}})$ given by the subspaces

$$\mathrm{Hom}(V, W)^{u,z} := \{\varphi \colon V_{\mathbb{C}} \to W_{\mathbb{C}} \mid \varphi(V^{p,q}) \subset W^{p+u,\, q+z} \, \forall \; p, q\}$$

of the complex vector space $\mathrm{Hom}_{\mathbb{C}}(V_{\mathbb{C}}, W_{\mathbb{C}})$.

(ii) *Tensor product:* We can identify $(V_{\mathbb{Z}} \otimes_{\mathbb{Z}} W_{\mathbb{Z}}) \otimes_{\mathbb{Z}} \mathbb{C}$ with the complex tensor product $V_{\mathbb{C}} \otimes_{\mathbb{C}} W_{\mathbb{C}}$ and get the natural subspaces

$$(V_{\mathbb{C}} \otimes W_{\mathbb{C}})^{u,z} := \bigoplus_{p+r=u,\, q+s=z} V^{p,q} \otimes_{\mathbb{C}} W^{r,s},$$

which define a Hodge decomposition of weight $k+m$ on $V_{\mathbb{Z}} \otimes_{\mathbb{Z}} W_{\mathbb{Z}}$.

4.4.6 Definition. An *integral polarization* of a Hodge structure of weight k on a $\mathbb{Z}$-module $V_{\mathbb{Z}}$ is a morphism of Hodge decompositions

$$Q\colon V_{\mathbb{Z}} \otimes_{\mathbb{Z}} V_{\mathbb{Z}} \longrightarrow \mathbb{Z}(-k),$$

which is $(-1)^k$-symmetric, i.e. $Q(v \otimes w) = (-1)^k Q(w \otimes v)$ for all $v, w \in V_{\mathbb{Z}}$ and such that the hermitian form

$$H(v,w) := (2\pi\sqrt{-1})^k Q_{\mathbb{C}}(C\, v \otimes \overline{w})$$

is positive-definite on $V_{\mathbb{C}}$. Here C denotes the Weil operator, which is defined by $C|V^{p,q} = \sqrt{-1}^{p-q}$ id and hence respects the Hodge decomposition.

4.4.7 Remark. Let $V_{\mathbb{Z}}$ be a $\mathbb{Z}$-module equipped with a Hodge structure with Hodge decomposition $V_{\mathbb{C}} = \oplus_{p+q=k} V^{p,q}$ respectively a decreasing filtration $\{F^p V_{\mathbb{C}}\}_{p\in\mathbb{Z}}$. Furthermore let $Q\colon V_{\mathbb{Z}} \otimes_{\mathbb{Z}} V_{\mathbb{Z}} \to \mathbb{Z}(-k)$ be a polarization for the Hodge structure in the sense of Definition 4.4.6. The only relevant part of the Hodge decomposition of $\mathbb{Z}(-k)$ is given by $H^{-k,-k} = \mathbb{C}$. Hence we directly deduce from the definition of polarization the following relations:

$$\begin{aligned} Q_{\mathbb{C}}(V^{p,q}, V^{r,s}) &= 0 \quad \text{unless } p = s \text{ and } q = r \\ \sqrt{-1}^{p-q} H(v,v) &> 0 \quad \text{for } p+q = k \text{ and } 0 \neq v \in V^{p,q} \end{aligned}$$

The first relations together with an argument on dimensions shows that $F^{k-m+1}V_{\mathbb{C}}$ is the orthogonal complement of $F^m V_{\mathbb{C}}$ with respect to $Q_{\mathbb{C}}$ for every $m \in \mathbb{Z}$. This is indeed an equivalent formulation of the first relation. The two relations above are called *Riemann relations.* The first relation also ensures that H is an hermitian form on the whole space $V_{\mathbb{C}}$. The second one that H is positive-definite. Alternatively we could have defined a polarization as a bilinear form on $V_{\mathbb{Z}}$ which is symmetric for even k and alternating for odd k such that the Riemann relations are fulfilled. See the discussion in [98] Section 2.1. for more details.

4.4.8 Hodge-Riemann pairing. Let X be a compact complex manifold and let $\omega \in H^2(X,\mathbb{Z})$ be an integral Kähler class. We call such a pair (X,ω) a *polarized manifold.* The manifold X is projective by the Koidara embedding theorem, i.e. we have an embedding $X \to \mathbb{P}^n$ for some $n > 0$. Let $d = \dim_{\mathbb{C}}(X)$ the dimension of X and let $k \leqslant d$. We write L for the cup-product

$$L\colon H^k(X,\mathbb{Z}) \longrightarrow H^{k+2}(X,\mathbb{Z}), \quad \alpha \longmapsto \omega \wedge \alpha.$$

We define the *intersection pairing* or *Hodge-Riemann* pairing $Q_k\colon H^k(X,\mathbb{Z}) \times H^k(X,\mathbb{Z}) \to \mathbb{Z}(-k)$ by

$$(\alpha,\beta) \longmapsto \frac{(-1)^{k(k-1)/2}}{(2\pi\sqrt{-1})^k}\int_X \alpha \wedge \beta \wedge \omega^{d-k}.$$

The primitive cohomology $H^k(X,\mathbb{Z})_{\mathrm{prim}} = \ker(L^{d-k+1})$ is naturally equipped with a Hodge decomposition by the spaces

$$H^{p,q}_{\mathrm{prim}} := H^k(X,\mathbb{C})_{\mathrm{prim}} \cap H^{p,q}(X) \quad (p+q=k).$$

The pairing Q_k on $H^k(X,\mathbb{Z})$ now defines an integral polarization on the primitive part $H^k(X,\mathbb{Z})_{\mathrm{prim}}$. See section 7.1.2 of [117] for more details.

4.4.9 Definition (Mixed variation of Hodge structures and Hodge decomposition)**.** Let B be a complex manifold and let $k \in \mathbb{Z}$.

(i) A *(mixed) variation of Hodge structure of weight k on B* (short: VHS of weight k) consists of a local system $\mathbb{V}_{\mathbb{Z}}$ of finitely generated abelian groups on B and a finite decreasing filtration $\{F^p\,\mathbf{V}\}_{p\in\mathbb{Z}}$ of the holomorphic vector bundle $\mathbf{V} := \mathbb{V}_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathcal{O}_B$ by holomorphic subbundles with the following properties:

- For each $b \in B$ the filtration $\{F^p\mathbf{V}_b\}$ defines a Hodge structure of weight k on the finite complex vector space $\mathbb{V}_{\mathbb{Z},b} \otimes_{\mathbb{Z}} \mathbb{C}$.
- The flat connection $\nabla_{\mathbb{V}_{\mathbb{Z}}}\colon \mathbf{V} \to \Omega^1_B \otimes_{\mathcal{O}_B} \mathbf{V}$ associated to the local system $\mathbb{V}_{\mathbb{Z}}$ satisfies *Griffith's transversality condition*

$$\nabla_{\mathbb{V}_{\mathbb{Z}}}(F^p\mathbf{V}) \subset \Omega^1_B \otimes_{\mathcal{O}_B} F^{p-1}\mathbf{V}.$$

(ii) A *morphism of variation of Hodge structures of degree r* is a morphism of the underlying local systems $\varphi\colon \mathbb{V}_{\mathbb{Z}} \to \mathbb{W}_{\mathbb{Z}}$ such that the extension of φ satisfies $\varphi(F^p\mathbf{V}) \subset F^{p+r}\mathbf{W}$.

4.4.10 Remark. Given two VHS $(\mathbb{V}_Z, \{F^p\mathbf{V}\}_{p\in\mathbb{Z}})$ and $(\mathbb{W}_{\mathbb{Z}}, \{F^q\mathbf{W}\}_{q\in\mathbb{Z}})$ of weight k and m on a complex manifold B, then there is a VHS on $\mathbb{V}_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{W}_{\mathbb{Z}}$ of degree $k+m$ and a VHS on $\mathrm{Hom}(\mathbb{V}_{\mathbb{Z}}, \mathbb{W}_{\mathbb{Z}})$ of degree $m-k$. In other words the category $\underline{\mathrm{VHS}(B)}$ of variations of Hodge structure on a complex manifold B is equipped with the operations tensor product and homomorphisms.

4.4.11 Remark. Let $(\mathbb{V}_Z, \{F^p\mathbf{V}\})$ be a VHS of weight k on the complex manifold B as in Definition 4.4.9. We can consider the holomorphic subbundles $F^p\mathbf{V}$ ($p \in \mathbb{Z}$) also as real C^∞-bundles on B and thus their complex conjugates are anti-holomorphic. We define

$$\mathbf{V}^{p,q} := F^p\mathbf{V} \cap \overline{F^q\mathbf{V}} \quad (p, q \in \mathbb{Z}).$$

From this we get a decomposition of $\mathbf{V}$ in C^∞-subbundles

$$\mathbf{V} = \bigoplus_{p+q=k} \mathbf{V}^{p,q}. \tag{4.4.11.1}$$

The decomposition in (4.4.11.1) defines a Hodge decomposition on every fiber $\mathbf{V}_b$ ($b \in B$) of $\mathbf{V}$ and hence we want to call it *Hodge decomposition* of the VHS.

On the other hand, if there is a decomposition of the flat vector bundle $\mathbf{V} = \mathbb{V}_\mathbb{Z} \otimes_\mathbb{Z} \mathcal{O}_B$ as in equation (4.4.11.1) such that for an integer k the bundles $\bigoplus_{i \geqslant p} \mathbf{V}^{i,k-i}$ are holomorphic and moreover fulfill Griffith's transversality condition, then we obtain in this way a variation of Hodge structure of weight k.

4.4.12 Variation of Hodge structures on a family of algebraic manifolds. We want to discuss an example from [46], where Griffiths defined a variation of Hodge structure on a family of algebraic manifolds. So let B and $\mathcal{X}$ be complex manifolds and let $f\colon \mathcal{X} \to B$ be a family of algebraic manifolds. Furthermore we assume that $\mathcal{X}$ is bimeromorphic to a Kähler manifold. The complex space $\mathcal{X}$ is for example bimeromorphic to a Kähler manifold if it is an algebraic manifold.

Consider the local system $R^q f_* \mathbb{Z}_\mathcal{X}$, where $R^q f_*$ is the q-th right derived functor of the direct image functor. We associate to it the vector bundle

$$H^q_{\mathrm{DR}}(\mathcal{X}/B) := R^q f_* \mathbb{Z}_\mathcal{X} \otimes_\mathbb{Z} \mathcal{O}_B.$$

We write $df\colon T\mathcal{X} \to f^*TB$ for the differential of $f\colon \mathcal{X} \to B$. If $T(\mathcal{X}/B)$ denotes the kernel of df and $N(\mathcal{X}/B)$ the image of df then we have an exact sequence

$$0 \longrightarrow T(\mathcal{X}/B) \longrightarrow T\mathcal{X} \longrightarrow N(\mathcal{X}/B) \longrightarrow 0.$$

If $Df\colon f^*\Omega^1_B \to \Omega^1_\mathcal{X}$ is the dual morphism of the differential df with image $Df(f^*\Omega^1_B) \subset \Omega^1_\mathcal{X}$ then we get the dual exact sequence

$$0 \longrightarrow Df(f^*\Omega^1_B) \longrightarrow \Omega^1_\mathcal{X} \longrightarrow \Omega^1_\mathcal{X}/Df(f^*\Omega^1_B) \longrightarrow 0.$$

In the following we write $\Omega^1_{\mathcal{X}/B}$ for $\Omega^1_\mathcal{X}/Df(f^*\Omega^1_B)$. The locally free sheaves $\Omega^p_{\mathcal{X}/B} := \bigwedge^p_{\mathcal{O}_\mathcal{X}} \Omega^1_{\mathcal{X}/B}$ form a complex

$$\Omega^\bullet_{\mathcal{X}/B} = (\mathcal{O}_\mathcal{X} \longrightarrow \Omega^1_{\mathcal{X}/B} \longrightarrow \Omega^2_{\mathcal{X}/B} \longrightarrow \dots),$$

the *relative de Rham complex*. Let $\Omega^\bullet_{\mathcal{X}/B} \to \mathcal{I}^\bullet$ be an injective resolution. Inductively we can built a simultaneous injective resolution $\mathcal{I}^\bullet \to \mathcal{J}^{\bullet,\bullet}$ such that $\mathcal{I}^p \to \mathcal{J}^{p,\bullet}$ is an injective resolution for every $p \geqslant 0$. This is called a Cartan-Eilenberg resolution. Read more about Cartan-Eilenberg resolutions in [115, p. 23.3.7]. We write $\mathcal{J}^\bullet$ for the total complex of $\mathcal{J}^{\bullet,\bullet}$. One can show that $\mathcal{I}^\bullet$ and $\mathcal{J}^\bullet$ are quasi-isomorphic. We apply the functor f_* to $\mathcal{I}$ and $\mathcal{J}^{\bullet,\bullet}$ and consider the spectral sequence $E_r^{p,q} = E_r^{p,q}(f_*\mathcal{J}^{\bullet,\bullet})$ with **upward orientation** associated to $f_*\mathcal{J}^{\bullet,\bullet}$. The spectral sequence $E_r^{p,q}$ degenerates at E_1 [98, Proposition 10.29] and the $E_1^{p,q}$ converge to

$$H^{p+q}(f_*\mathcal{J}^\bullet) \cong H^{p+q}(f_*\mathcal{I}^\bullet) \cong R^{p+q}f_*(\Omega^\bullet_{\mathcal{X}/B}).$$

By [98, Theorem 10.26, Corollary 10.27] we have $H^k_{\mathrm{DR}}(\mathcal{X}/B) \cong R^k f_*(\Omega^\bullet_{\mathcal{X}/B})$ and the above translates to

$$E_1^{p,q} = R^q f_* \Omega^p_{\mathcal{X}/B} \underset{p}{\Longrightarrow} H^{p+q}_{\mathrm{DR}}(\mathcal{X}/B) \cong R^{p+q} f_*(\Omega^\bullet_{\mathcal{X}/B}).$$

Hence we get for every $k \in \mathbb{N}$ a filtration

$$E_1^{k,0} = F^k \subset F^{k-1} \subset \cdots \subset F^1 \subset F^0 = H^k_{\mathrm{DR}}(\mathcal{X}/B)$$

with $E_1^{k-i,i} \cong F_{k-i}/F_{k-i+1}$ for every $i = 0, \ldots, k$.

4.4.13 Theorem. Let $f\colon \mathcal{X} \to B$ be a family of algebraic manifolds such that $\mathcal{X}$ and B are smooth and such that $\mathcal{X}$ is bimeromorphic to a Kähler manifold. Then for every $k \in \mathbb{N}$ the local system $R^k f_* \mathbb{Z}_{\mathcal{X}}$ with the filtration $\{F^p\}$ on $H^k_{\mathrm{DR}}(\mathcal{X}/B)$ from above is a variation of Hodge structure with respect to the Gauss-Manin connection.

Proof. [21, Theorem 3.6] or [98, Corollary 10.31] □

4.4.14 Remark. In the following text we consider most of the time $f\colon \mathcal{X} \to B$ a family of compact Riemann surfaces and the local system $R^1 f_* \mathbb{Z}_{\mathcal{X}}$. For $k = 1$ we get $E_1^{0,1} = f_*\Omega^1_{\mathcal{X}/B}$ and $E_1^{1,0} = R^1 f_* \mathcal{O}_{\mathcal{X}}$. The filtration on $H^1_{\mathrm{DR}}(\mathcal{X}/B)$ is then given by

$$0 = F^2 \subset F^1 = f_*\Omega^1_{\mathcal{X}/B} \subset F^0 = H^1_{\mathrm{DR}}(\mathcal{X}/B). \tag{4.4.14.1}$$

4.4.15 Variation of Hodge structure for the Leray primitive cohomology sheaf. Let $f\colon \mathcal{X} \to B$ be a family of algebraic manifolds as in section 4.4.12. Furthermore let ω be a global section of $R^2 f_* \mathbb{Z}_{\mathcal{X}}$ such that the restriction ω_b of ω to the stalks $H^2(X_b, \mathbb{Z})$ ($b \in B$) are integral Kähler classes. We call a family $f\colon \mathcal{X} \to B$ together with a global section ω as above a *polarized family of algebraic manifolds*. Recall that for every $b \in B$ the pair $(\mathcal{X}_b, \omega_b)$ is a polarized manifold in the sense of 4.4.8.

Let $d = \dim_{\mathbb{C}}(\mathcal{X}_b)$ ($b \in B$) be the dimension of the fibers of $f\colon \mathcal{X} \to B$ and let $k \leqslant d$. Cup-product with the Kähler class ω_b leads to a map

$$H^k(\mathcal{X}_b, \mathbb{Z}) \longrightarrow H^{k+2}(\mathcal{X}_b, \mathbb{Z}), \quad \alpha \longmapsto \omega_b \wedge \alpha$$

between the stalks $H^k(\mathcal{X}_b,\mathbb{Z})$ of $R^kf_*\mathbb{Z}_{\mathcal{X}}$ and $H^{k+2}(\mathcal{X}_b,\mathbb{Z})$ of $R^{k+2}f_*\mathbb{Z}_{\mathcal{X}}$. The morphisms on the stalks glue together to a morphism of sheaves

$$L\colon R^kf_*\mathbb{Z}_{\mathcal{X}} \to R^{k+2}f_*\mathbb{Z}_{\mathcal{X}}.$$

Now we define the *Leray primitive cohomology sheaf* $(R^kf_*\mathbb{Z}_{\mathcal{X}})_{\mathrm{prim}}$ as the kernel of L^{d-k+1}. We write $\mathbf{V}_{\mathrm{prim}}$ for the associated holomorphic bundle of $(R^kf_*\mathbb{Z}_{\mathcal{X}})_{\mathrm{prim}}$ which is a subbundle of $H^k_{\mathrm{DR}}(\mathcal{X}/B)$. If $\{\mathbf{V}^{p,q} \mid p,q \in \mathbb{Z}\}$ is a Hodge decomposition of $H^k_{\mathrm{DR}}(\mathcal{X}/B)$ then the intersections

$$\mathbf{V}_{\mathrm{prim}} \cap \mathbf{V}^{p,q} \quad (p,q \in \mathbb{Z})$$

are again C^∞-bundles and define a Hodge decomposition on $(R^kf_*\mathbb{Z}_{\mathcal{X}})_{\mathrm{prim}}$ (c.f. (3.2) in [49]).

4.4.16 Definition. Let B be a complex manifold and let $(\mathbb{V}_{\mathbb{Z}}, \{F^p\mathbf{V}\}_{p\in\mathbb{Z}})$ be a VHS of weight k on B. We write $\mathbb{Z}(-k)_B$ for the constant sheaf of stalk $\mathbb{Z}(-k) = (2\pi\sqrt{-1})^{-k}\mathbb{Z}$ on B. The constant sheaf $\mathbb{Z}(-k)_B$ is naturally equipped with an abstract variation of Hodge structures of weight $2k$ which fits together with the one on $\mathbb{Z}(-k)$. A *polarization* of the VHS $(\mathbb{V}_{\mathbb{Z}}, \{F^p\mathbf{V}\}_{p\in\mathbb{Z}})$ is a morphism of Hodge structures of degree zero

$$Q\colon \mathbb{V}_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{V}_{\mathbb{Z}} \longrightarrow \mathbb{Z}(-k)_B,$$

such that the induced morphisms of Hodge structure $\mathbb{V}_{\mathbb{Z},b} \otimes_{\mathbb{Z}} \mathbb{V}_{\mathbb{Z},b} \longrightarrow \mathbb{Z}(-k)$ are polarizations for every $b \in B$ in the sense of Definition 4.4.6.

4.4.17 Example (Polarization for a family of polarized algebraic manifolds)**.** Let $f\colon \mathcal{X} \to B$ be a family of algebraic manifolds such that $\mathcal{X}$ and B are manifolds and such that $\mathcal{X}$ is bimeromorphic to a Kähler manifold.

(i) We assume that $f\colon \mathcal{X} \to B$ is a family of polarized algebraic manifolds with global section $\omega \in \Gamma(B, R^2f_*\mathbb{Z}_{\mathcal{X}})$. By definition the restriction of the global section ω to the stalk $H^2(\mathcal{X}_b,\mathbb{Z})$ leads to an integral Kähler class $\omega_b \in H^2(\mathcal{X}_b,\mathbb{Z})$ for every $b \in B$. Thus on every stalk $H^k(\mathcal{X}_b,\mathbb{Z})$ we have the Hodge-Riemann pairing

$$Q^b_k\colon H^k(\mathcal{X}_b,\mathbb{Z}) \otimes_{\mathbb{Z}} H^k(\mathcal{X}_b,\mathbb{Z}) \longrightarrow \mathbb{Z}(-k)$$

from section 4.4.8. The Hodge-Riemann pairing defines an integral polarization on the primitive part $H^k(\mathcal{X}_b,\mathbb{Z})_{\mathrm{prim}}$. The polarizations Q^b_k restricted to $H^k(\mathcal{X}_b,\mathbb{Z})_{\mathrm{prim}}$ glue to a polarization Q_k on the Leray primitive cohomology sheaf $(R^kf_*\mathbb{Z}_{\mathcal{X}})_{\mathrm{prim}}$ equipped with the VHS which comes from the VHS on $R^kf_*\mathbb{Z}_{\mathcal{X}}$.

(ii) If $f\colon \mathcal{X} \to B$ is a family of compact Riemann surfaces then it is automatically a polarized family for the following reason. The complex structure of $\mathcal{X}$ induces the complex structures on every compact Riemann surface $\mathcal{X}_b$ ($b \in B$). Hence if ω is an element of $H^2(\mathcal{X},\mathbb{Z}) \cap H^{1,1}(\mathcal{X})$ it restricts to an element $\omega_b \in H^2(\mathcal{X}_b,\mathbb{Z}) \cap H^{1,1}(\mathcal{X}_b)$ for every fiber $\mathcal{X}_b$ ($b \in B$). Every $\mathcal{X}_b$ has complex dimension one and thus $d\omega_b = 0$.

In other words ω_b is an integral Kähler class for $\mathcal{X}_b$. Recall that the Ehresmann connection induces canonical isomorphisms $H^2(f^{-1}(U), \mathbb{Z}) \cong H^2(\mathcal{X}_u, \mathbb{Z})$ for every contractible $U \subset B$ and $u \in U$. Hence we can consider ω as a global section of $R^2 f_* \mathbb{Z}_{\mathcal{X}}$. In contrast to the general case, the Hodge-Riemann pairing

$$Q_1^b \colon H^1(\mathcal{X}_b, \mathbb{Z}) \otimes_{\mathbb{Z}} H^1(\mathcal{X}_b, \mathbb{Z}) \longrightarrow \mathbb{Z}(-1)$$

is a polarization on the whole cohomology $H^1(\mathcal{X}_b, \mathbb{Z})$ for every $b \in B$ and hence we get an integral polarization for the VHS $(R^1 f_* \mathbb{Z}_{\mathcal{X}}, f_* \Omega^1_{\mathcal{X}/B})$ on B.

4.4.18 Question. Let $f \colon \mathcal{X} \to B$ be a family of polarized algebraic manifolds with $\mathcal{X}, B$ manifolds and $\mathcal{X}$ bimeromorphic to a Kähler manifold. Then we constructed an integral polarized VHS on the Leray primitive cohomology sheaf $(R^k f_* \mathbb{Z}_{\mathcal{X}})_{\mathrm{prim}}$. Let $b \in B$ and let

$$\rho_k \colon \pi_1(B, b) \longrightarrow \mathrm{Aut}_{\mathbb{Z}}(H^k(\mathcal{X}_b, \mathbb{Z})_{\mathrm{prim}})$$

be the monodromy representation of the local system $(R^k f_* \mathbb{Z}_{\mathcal{X}})_{\mathrm{prim}}$. In [49] Griffiths and Schmid asked the question whether the image of ρ_k is always an arithmetic group?

Griffiths and Schmid assumed that the answer to Question 4.4.18 is yes, which turned out to be wrong. The first counterexamples were given by Deligne and Mostow [23]. Nevertheless we want to state in Theorem 4.5.7 the theorem of Griffiths, which has led to the conjecture. This theorem also plays an important role in the proof of Theorem 6.1.5. Before we can state Griffith's theorem, we have to introduce period mappings. We will do this in the next section.

4.5 Period domains and Period mappings

As a reference of the following subsections we used the books [13], [117] and the papers [48] and [49].

4.5.1 Construction of period domains. Motivated by the definition of Hodge structures and Hodge decompositions in Definition 4.4.1 and the definition of integral polarizations of Hodge structures in Definition 4.4.6 we want to fix the following data:

(i) Let $V_{\mathbb{Z}}$ be a finitely generated lattice in a $\mathbb{Q}$-vector space. For a field $\mathbb{F} \in \{\mathbb{Q}, \mathbb{R}, \mathbb{C}\}$ we denote by $V_{\mathbb{F}} = V_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{F}$.

(ii) Let $k \in \mathbb{Z}$ be an integer and for every $p, q \in \mathbb{Z}$ with $p + q = k$ let $h^{p,q} \in \mathbb{N} \cup \{0\}$ such that $h^{p,q} = h^{q,p}$ and $\sum_{p+q=k} h^{p,q} = \dim_{\mathbb{C}}(V_{\mathbb{C}})$. We denote by

$$f^m = \sum_{i \geqslant m} h^{i,k-i}$$

(iii) Furthermore let Q be a bilinear form on $V_{\mathbb{Z}}$ which shall be symmetric if k is even and skew-symmetric if k is odd.

Note especially for item (ii) how we constructed a Hodge filtration out of a Hodge decomposition in Remark 4.4.2: If $V_{\mathbb{Z}}$ is a $\mathbb{Z}$- module of rank k and $V_{\mathbb{C}} = \oplus_{p+q=k} V^{p,q}$ a Hodge decomposition with Hodge numbers $h^{p,q} = \dim(V^{p,q})$ then the subspaces $F^m V_{\mathbb{C}} = \bigoplus_{i \geqslant m} V^{i,k-i}$ define a Hodge filtration on $V_{\mathbb{C}}$ and we get the obvious relation $\dim(F^m V_{\mathbb{C}}) = \sum_{i \geqslant m} h^{i,k-i}$.

Definition. We write $D = D(V_{\mathbb{Z}}, Q, k, \{h^{p,q}\})$ for the set consisting of all Hodge structures of weight k on $V_{\mathbb{Z}}$ with Hodge numbers $\{h^{p,q}\}$ and which are integrally polarized by Q. We call D the *classifying space of Hodge structures* with these data.

In the following we will give D the structure of a complex space. First of all we write $\check{D}$ for the subspace of the finite product of Grassmannians

$$\prod_{f^m \neq 0} \mathrm{Grass}(f^m, V_{\mathbb{C}})$$

consisting of filtrations $\{F^m \mid f^m \neq 0\}$ of $\mathbb{V}_{\mathbb{C}}$ with $\dim(F^m) = f^m$, satisfying the following two conditions :

$$\{0\} \subset \cdots \subset F^{m+1} \subset F^m \subset F^{m-1} \subset \cdots \subset V_{\mathbb{C}} \tag{4.5.1.1}$$

and

$$Q(F^m, F^{k-m+1}) = 0 \tag{4.5.1.2}$$

Then $\check{D}$ is per definition a subvariety of a product of Grassmann varieties and D is the open subset of all those points in $\check{D}$ with the property

$$Q(Cv, \overline{v}) > 0 \quad \text{for all } 0 \neq v \in F^p \cap \overline{F^q} \text{ with } p+q=k, \tag{4.5.1.3}$$

where C is the Weil operator from Definition 4.4.6. The definition of $\check{D}$ and D are absolutely intuitive if we compare the equations 4.5.1.2 and 4.5.1.3 with Definition 4.4.6 respectively the Riemann bilinear relations in Remark 4.4.7.

The space $\check{D}$ and the classifying space of Hodge structures D has also a description as homogeneous space. Write $G_{\mathbb{C}} = \mathrm{Aut}(V_{\mathbb{C}}, Q)$ for the subgroup of $\mathrm{GL}(V_{\mathbb{C}})$, which consists of elements which preserve the extension of Q to $V_{\mathbb{C}}$. One can show that $G_{\mathbb{C}}$ acts transitively on $\check{D}$. Furthermore, if $B_{\mathbb{C}} = \mathrm{Stab}_{G_{\mathbb{C}}}(F_0)$ is the stabilizer of a fixed flag $F_0 = \{F_0^p\} \in \check{D}$, then $\check{D} = G_{\mathbb{C}}/B_{\mathbb{C}}$.

Now we denote by $G_{\mathbb{R}} \leqslant G_{\mathbb{C}}$ the subgroup of elements $T \in G_{\mathbb{C}}$ such that $T V_{\mathbb{R}} \subset V_{\mathbb{R}}$. It is possible to show that the action of $G_{\mathbb{R}}$ on $\check{D}$ preserves D and that $G_{\mathbb{R}}$ acts transitively on D with isotropy group $K = B_{\mathbb{C}} \cap G_{\mathbb{R}}$ at F_0. Since the elements of K commute with complex conjugation they do not only preserve the flag F_0 but also the Hodge decomposition on $V_{\mathbb{C}}$ induced by F_0. We conclude that K preserves the Hermitian form

$$H(v, w) = Q(C\,v, \overline{w})$$

on $V_{\mathbb{C}}$ and thus K is a compact group. We will collect important properties of $\check{D}$ and D in the next proposition.

4.5.2 Proposition. The space $\check{D}$ is a smooth projective complete algebraic variety which is a homogeneous space $\check{D} = G_{\mathbb{C}}/B_{\mathbb{C}}$, where $B_{\mathbb{C}}$ is a parabolic group. The set D of Hodge structures of weight k on $V_{\mathbb{Z}}$, which are polarized by Q is an open complex submanifold of $\check{D}$ and it is homogeneous $D = G_{\mathbb{R}}/K$ with a compact group K.

Proof. Proposition 8.2 and Proposition 8.12 in [48]. □

4.5.3 Remark. We can equip D with a $G_{\mathbb{R}}$-invariant Hermitian metric ds^2_D induced by the Cartan-Killing form on the Lie algebra of $G_{\mathbb{R}}$, see [48] and the references in there.

4.5.4 Construction of period mappings. Let B be a connected holomorphic manifold and let $(\mathbb{V}_{\mathbb{Z}}, \{F^p\mathbf{V}\}, Q)$ be an integral polarized VHS of weight k on B. Let again $\mathbf{V} = \mathbb{V}_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathcal{O}_B$. We fix a base point $b \in B$ and we write W for the fiber $\mathbf{V}_b$ of the holomorphic bundle $\mathbf{V}$ at the point $b \in B$. We denote $f^p = \dim(F^p\mathbf{V}_b)$ and write

$$D \subset \prod_p \mathrm{Grass}(f^p, \mathbf{V}_b),$$

for the classifying space of all Hodge structures on $\mathbf{V}_b$ polarized by Q_b.

Denote by

$$\pi\colon \widetilde{B} \longrightarrow B$$

the universal cover of B. The pullback $\pi^{-1}\mathbb{V}_{\mathbb{Z}}$ of the local system $\mathbb{V}_{\mathbb{Z}}$ is again locally constant or equivalently the associated holomorphic vector bundle $\widetilde{\mathbf{V}} := \pi^{-1}\mathbb{V}_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathcal{O}_{\widetilde{B}}$ is equipped with a flat connection. Furthermore the polarization Q pulls back to a polarization on $\pi^{-1}\mathbb{V}_{\mathbb{C}}$.

Let $\tilde{b} \in \widetilde{B}$ be a point with $\pi(\tilde{b}) = b$. The stalk $(\pi^{-1}\mathbb{V}_{\mathbb{C}})_{\tilde{b}}$ is naturally isomorphic to the stalk $W = (\mathbb{V}_{\mathbb{C}})_b$ and the sheaf $\pi^{-1}\mathbb{V}_{\mathbb{C}}$ is isomorphic to the constant sheaf $W_{\widetilde{B}}$ of stalk W on $\widetilde{B}$ by Corollary 4.2.4.

The isomorphism of sheaves $\beta\colon \pi^{-1}\mathbb{V}_{\mathbb{C}} \cong W_{\widetilde{B}}$ is uniquely determined by the isomorphism $(\pi^{-1}\mathbb{V})_{\tilde{b}} \cong W$ and it induces for every $t \in \widetilde{B}$ an isomorphism of vector spaces

$$\beta_t\colon \left(\pi^{-1}\mathbb{V}_{\mathbb{C}}\right)_t \longrightarrow W.$$

The fiber $\widetilde{\mathbf{V}}_{\tilde{b}}$ is naturally isomorphic to the fiber $\mathbf{V}_b$. On the level of holomorphic vector bundles with flat connection, we could interprete β as the global trivialization

$\widetilde{\mathbf{V}} \to \widetilde{B} \times \mathbf{V}_b$ induced by the flat connection. Every holomorphic subbundle $F^p\mathbf{V}$ of $\mathbf{V}$ pulls back to a holomorphic subbundle $F^p\widetilde{\mathbf{V}}$ of $\widetilde{\mathbf{V}}$. For every $t \in \widetilde{B}$ and every $p \in \mathbb{Z}$ with $f^p \neq 0$ we thus get a subspace $F^p\widetilde{\mathbf{V}}_t \subset \widetilde{\mathbf{V}}_t$ and obviously

$$\beta_t(F^p\widetilde{\mathbf{V}}_t) \in \mathrm{Grass}(f^p, \mathbf{V}_b).$$

Indeed since the polarization Q can be viewed as a natural transformation between locally constant sheaves the second Riemann relation is preserved under the isomorphisms β_t and hence $(\beta_t\, F^p\widetilde{\mathbf{V}}_t)_p \in D$. We get the so called *period mapping*

$$P\colon \widetilde{B} \longrightarrow D, \quad t \longmapsto (\beta_t\, F^p\widetilde{\mathbf{V}}_t)_p$$

We will collect two very important properties of the period mapping in the following proposition.

4.5.5 Proposition. Let B be a connected holomorphic manifold with universal cover $\widetilde{B}$ and let $(\mathbb{V}_{\mathbb{Z}}, \{F^p\mathbf{V}\}, Q)$ be an integral polarized VHS of weight k on B. Furthermore let $b \in B$ be a base point and

$$\rho\colon \pi_1(B, b) \longrightarrow \mathrm{GL}(\mathbb{V}_{\mathbb{C},b})$$

be the monodromy representation associated to $\mathbb{V}_{\mathbb{C}}$. Then the period mapping $P\colon \widetilde{B} \to D$ constructed above has the following properties:

(i) The period mapping $P\colon \widetilde{B} \to D$ is holomorphic

(ii) The image Γ of the monodromy representation ρ is a subgroup of $\mathrm{Aut}(\mathbb{V}_{\mathbb{Z},b}, Q_b)$. Furthermore the period mapping P is equivariant with respect to the action of $[\gamma] \in \pi_1(B, b)$ on $\tilde{B}$ by deck transformations and the action of $\rho([\gamma])^{-1} \in \mathrm{Aut}(\mathbb{V}_{\mathbb{Z},b}, Q_b)$ on D, i.e.
$$P([\gamma] \cdot t) = \rho([\gamma])^{-1} \cdot P(t) \quad (t \in \widetilde{B}).$$

Proof. Part(i) was first shown by Griffiths [47, Theorem 1.27]. You can also find a proof of part (i) in [13, Theorem 4.5.6].

Since Q is a bilinear morphism of sheaves between locally constant sheaves, the polarization is invariant under monodromy and thus the image Γ of the monodromy representation ρ lies in $\mathrm{Aut}((\mathbb{V}_{\mathbb{Z},b}, Q_b)$ (c.f. Proposition 4.2.7).

As above we write $\pi\colon \widetilde{B} \to B$ for the universal cover of B. Fix a base point $b \in B$ and a point $\widetilde{b} \in \widetilde{B}$ with $\pi(\widetilde{b}) = b$. The relation $P(\gamma \cdot t) = \rho(\gamma)^{-1} \cdot P(t) \quad (t \in \widetilde{B})$ follows from the construction of the monodromy map and the construction of the period map as we will explain now. Fix an element $t \in \widetilde{B}$ and let $c = [\gamma] \in \pi_1(B, b)$ be represented by a path $\gamma\colon [0,1] \to B$ with $\gamma(0) = b = \gamma(1)$. Let $\widetilde{\gamma}\colon [0,1] \to \widetilde{B}$ be a lift of γ with $\widetilde{\gamma}(0) = \widetilde{b}$ and let $\widetilde{\eta}\colon [0,1] \to \widetilde{B}$ be a path with $\widetilde{\eta}(0) = \widetilde{b}$ and $\widetilde{\eta}(1) = t$. We denote $\eta = \pi \circ \widetilde{\eta}$. Let $\widetilde{\gamma * \eta}\colon [0,1] \to \widetilde{B}$ be the unique lift of $\gamma * \eta$ with $\widetilde{\gamma * \eta}(0) = \widetilde{b}$. Then by definition of the action of $\pi_1(B, b)$ on $\widetilde{B}$ we have $c.t = \widetilde{\gamma * \eta}(1)$. Let $c.\widetilde{\eta}\colon [0,1] \to \widetilde{B}$ be defined by

$c.\widetilde{\eta}(s) = c.s$. The unique path lifting property of the universal cover $\pi\colon \widetilde{B} \to B$ implies that $\widetilde{\gamma} * c.\widetilde{\eta} = \widetilde{\gamma * \eta}$. Let

$$A_{\widetilde{\gamma}}\colon (\pi^{-1}\mathbb{V}_{\mathbb{C}})_{\widetilde{b}} \longrightarrow (\pi^{-1}\mathbb{V}_{\mathbb{C}})_{c.\widetilde{b}} \quad \text{and} \quad A_{c.\widetilde{\eta}}\colon (\pi^{-1}\mathbb{V}_{\mathbb{C}})_{c.\widetilde{b}} \longrightarrow (\pi^{-1}\mathbb{V}_{\mathbb{C}})_{c.t}$$

be the isomorphism induced by parallel transport from Theorem 4.2.3. Furthermore let $\alpha_{\widetilde{b}}\colon (\pi^{-1}\mathbb{V}_{\mathbb{C}})_{\widetilde{b}} \to \mathbb{V}_{\mathbb{C},b}$ and $\alpha_{c.\widetilde{b}}\colon (\pi^{-1}\mathbb{V}_{\mathbb{C}})_{c.\widetilde{b}} \to \mathbb{V}_{\mathbb{C},b}$ be the natural isomorphism. Then we get the identity

$$\begin{aligned}\beta_{ct}(-) &= \alpha_{\widetilde{b}} \circ A_{\widetilde{\gamma}}^{-1} \circ \alpha_{c.\widetilde{b}}^{-1} \circ \alpha_{c.\widetilde{b}} \circ A_{c.\widetilde{\eta}}^{-1}(-) \\ &= \rho([\gamma])^{-1}\left(\alpha_{c.\widetilde{b}} \circ A_{c.\widetilde{\eta}}^{-1}(-)\right).\end{aligned}$$

By definition of the inverse image functor π^{-1} and the properties of π as a covering we get for every p with $f^p \neq 0$ the identity

$$\alpha_{c.\widetilde{b}} \circ A_{c.\widetilde{\eta}}^{-1}(F^p\widetilde{\mathbf{V}}_{c.t}) = \alpha_{\widetilde{b}} \circ A_{\widetilde{\eta}}^{-1}(F^p\widetilde{\mathbf{V}}_t) = \beta_t(F^p\widetilde{\mathbf{V}}_t).$$

This shows $P(c.t) = \rho(c)^{-1} \cdot P(t)$ as claimed. □

4.5.6 Remark. The period mapping $P\colon \widetilde{B} \to D$ from Proposition 4.5.5 descends to a holomorphic map $\phi\colon B \to D/\Gamma$ which satisfies a technical condition, that is in the literature known as *Griffith's infinitesimal period relation*, see (9.1) respectively (9.2) in [48] for a definition.

Indeed giving a polarized VHS on B with monodromy group Γ is equivalent to giving a holomorphic map $\phi\colon B \to D/\Gamma$ with the following two properties:

(i) The map $\phi\colon B \to D/\Gamma$ is locally liftable i.e. for every $b \in B$ there exists a neighbourhood $U \subset B$ of b and a holomorphic map $\widetilde{\phi}\colon U \to D$ such that the diagram

$$\begin{tikzcd} U \arrow[r, "\widetilde{\phi}"] \arrow[dr, "\phi"'] & D \arrow[d] \\ & D/\Gamma \end{tikzcd}$$

commutes.

(ii) The map $\phi\colon B \to D/\Gamma$ satisfies Griffiths infinitesimal period relation.

For a proof of this result see [48, Proposition 9.3] or [13, Lemma-Definition 4.6.3].

Let B be a complex algebraic manifold and let $\overline{B}$ be a smooth, complete, projective variety which contains B as a Zariski open set. We assume that $S = \overline{B}\backslash B$ is locally given by

$$b_1 \cdot \dots \cdot b_k = 0,$$

where $b_1, \dots, b_k$ are part of a holomorphic coordinate system $b_1, \dots, b_d$ of $\overline{B}$. Let $(\mathbb{V}_{\mathbb{Z}}, \{F^p\mathbf{V}\}, Q)$ be an integral polarized VHS of weight k on B. Fix a basepoint $b \in B$ and write again Γ for the image of the monodromy map

$$\pi_1(B, b) \longrightarrow \mathrm{Aut}(\mathbb{V}_{\mathbb{Z},b}, Q_b)$$

The period mapping of the polarized VHS descends to a holomorphic map $\phi\colon B \to D/\Gamma$. Assume that $s \in S$ is a regular point of the divisor $S = \overline{B}\backslash B$. By the local description of the divisor S we know that there is a neighbourhood U of $s \in \overline{S}$ such that $U \cap S$ is isomorphic to $\mathbb{D}^* \times \mathbb{D}^{d-1}$. The fundamental group $\pi_1(U)$ is generated by a path γ around s. Let $T = \rho([\gamma]) \in \Gamma$ be the corresponding element in the monodromy group Γ. If T is of finite order then we can extend ϕ to a holomorphic map $B \cup \{s\} \to D/\Gamma$ by a result of Griffiths [48, Theorem 9.5]. Write B^* for the union of B with all of these regular points of S for which the mapping ϕ has a holomorphic extension as above. Denote by $\phi^*\colon B^* \to D/\Gamma$ the resulting holomorphic extension of ϕ. We are now able to state Griffith's Theorem, which led to Question 4.4.18.

4.5.7 Theorem (Griffiths, see Theorem 9.9 [48])**.** The image $\phi^*(B^*)$ is a closed analytic subvariety of D/Γ and has finite volume with respect to the metric ds_D^2 on D from Remark 4.5.3.

We will end this section with how the period domain and period mapping looks like for our running example, a family of polarized compact Riemann surfaces over a hyperbolic Riemann surface.

4.5.8 Example. We will follow in this example [46, Section 1(e)] and [47, Section II.1]. Let $f\colon \mathcal{X} \to B$ be a family of compact Riemann surfaces such that $\mathcal{X}$ and B are holomorphic manifolds and such that $\mathcal{X}$ is bimeromorphic to a Kähler manifold. We assume that B is a hyperbolic Riemann surface, i.e. $B = \mathbb{H}/\Gamma$ for a Fuchsian group $\Gamma \leqslant \mathrm{SL}_2(\mathbb{R})$. By 4.4.14 and Example 4.4.17 we can associate the polarized VHS $(R^1f_*\mathbb{Z}_{\mathcal{X}}, f_*\Omega^1_{\mathcal{X}/B}, Q_1)$ of weight one to $f\colon \mathcal{X} \to B$. Let $g \geqslant 2$ be the genus of the fibers $f^{-1}(\{b\})$ $(b \in B)$, then $H^{1,0}(\mathcal{X}) = \Omega^1_{\mathcal{X}_b}$ has complex dimension g and thus the period domain D of the polarized VHS is contained in $\mathrm{Grass}(g, 2g)$. Note that for every $b \in B$ the pairing Q_1^b on $H^1(\mathcal{X}_b, \mathbb{Q})$ induced by Q_1 is a symplectic pairing. Indeed one can find for every $b \in B$ a basis $\gamma_1^b, \dots \gamma_{2g}^b$ of $H^1(\mathcal{X}_b, \mathbb{Z})$ such that the matrix $M_b = (m_{i,j})_{i,j}$ with entries $m_{i,j} = \int_{\mathcal{X}_b} \gamma_i^b \wedge \gamma_j^b$ is given by

$$M = \begin{pmatrix} \underline{0} & I_g \\ -I_g & \underline{0} \end{pmatrix}.$$

This shows that $G_{\mathbb{R}}$ is isomorphic to the symplectic group $\mathrm{Sp}(2g, \mathbb{R})$ and the image of the monodromy representation $\pi_1(B, b) \to \mathrm{Aut}_{\mathbb{Z}}(H^1(\mathcal{X}_b, \mathbb{Z}))$ is contained in $\mathrm{Sp}(2g, \mathbb{R})$.

Let $\pi\colon \mathbb{H} \to B = \mathbb{H}/\Gamma$ be a universal cover of B and denote $\mathbb{V}_{\mathbb{Z}} = R^1f_*\mathbb{Z}_{\mathcal{X}}$ as well as $F^1\mathbf{V} = f_*\Omega^1_{\mathcal{X}/B}$. From [40, Theorem 30.1 and Theorem 30.4] we know that we can find global section $\omega_1, \ldots, \omega_g$ for the pull back $F^1\widetilde{\mathbf{V}}$ of $f_*\Omega^1_{\mathcal{X}/B}$ to the universal cover $\mathbb{H}$ of $B = \mathbb{H}/\Gamma$ such that for every $t \in \mathbb{H}$ the elements $\omega_1(t), \ldots, \omega_g(t)$ are a basis of $F^1\widetilde{\mathbf{V}}_t$. Consider an element $t \in \mathbb{H}$ and let $\alpha_1(t), \beta_1(t), \ldots, \alpha_g(t), \beta_g(t)$ be a symplectic basis of $(\pi^{-1}\mathbb{V}_Z)_t^\vee$. Consider the matrix

$$\Omega_t = (F_t, E_t) \in \mathbb{C}^{g \times 2g}$$

with $F_t = \left(\int_{\beta_i(t)} \omega_j(t)\right)_{i,j}$ and $E_t = \left(\int_{\alpha_i(t)} \omega_j(t)\right)_{i,j}$.

The Plücker coordinates of $F^1\widetilde{\mathbf{V}} \in \mathrm{Grass}(g, 2g)$ are now given by the rows of the matrix Ω_t. One can show that the Riemann relations for Q_1 imply that $Z_t := F_t^{-1}E_t \in \mathbb{C}^{g\times g}$ is positive definite and that Z_t coincides with its transpose [46, Section 1] for more details). This implies Z_t is contained in $\mathbb{H}_g$, the Siegel space of degree g and thus we get a map $P \colon \mathbb{H} \to \mathbb{H}_g$ defined by $P(t) = Z_t$.

4.6 Algebraic monodromy representations for Veech fibrations

We will end this chapter with a section, where we want to explain how one can use Hodge theory to describe Teichmüller curves. We can warmly recommend Simion Filip's survey [33] which gives insights in the Hodge theoretical approach to Teichmüller theory in a by far more elaborate way.

4.6.1 Veech fibrations. We are interested in the algebraic monodromy of families of algebraic manifolds. So we start this section by recalling the construction of families of curves coming from Teichmüller curves as in section 1.4 of [94] or section 3.1 of [93].

Let $g \geqslant 2$ and let $j_\omega \colon C_1 \to \mathcal{M}_g$ be a Teichmüller curve, which comes from a Veech surface $(X, \omega) \in \Omega\mathcal{M}_g$ as in Section 3.2.6. Let $n \geqslant 3$ and $\mathcal{M}_g^{[n]} = \mathcal{T}_g(S)/\Gamma_g^{[n]}$ be the moduli space of curves with level-n structure. Here $\Gamma_g^{[n]}$ is the kernel of the action of the mapping class group $\Gamma_g(S)$ on $H^1(S, \mathbb{Z}/n\mathbb{Z})$. We have that $\Gamma_g^{[n]} \leqslant \Gamma_g$ is a torsion free finite index subgroup. Furthermore $\mathcal{M}_g^{[n]}$ is a fine moduli space. Hence there is a universal family $f^{[n]} \colon \mathcal{X}_{\mathrm{univ}}^{[n]} \to \mathcal{M}_g^{[n]}$ over $\mathcal{M}_g^{[n]}$.

Let $g_\omega \colon \mathbb{D} \to \mathcal{T}_g(S)$ be the map from Subsection 3.2.5 with image $\Delta = g_\omega(\mathbb{D})$ the Teichmüller disk associated to (X, ω). We write Γ_1 for the global stabilizer of Δ with respect to the action of $\Gamma_g^{[n]}$ on $\mathcal{T}_g(S)$ and define $C_1^{[n]}$ as the quotient $C_1^{[n]} = \Delta/\Gamma_1$. The inclusion $\Delta \hookrightarrow \mathcal{T}_g(S)$ induces a map $C_1^{[n]} \to \mathcal{M}_g^{[n]}$ on the quotients. The moduli space $\mathcal{M}_g^{[n]}$ admits a universal family $f^{[n]} \colon \mathcal{X}_{\mathrm{univ}}^{[n]} \to \mathcal{M}_g^{[n]}$, which we can pull back via $C_1^{[n]} \to \mathcal{M}_g^{[n]}$ to get a family of curves $\mathcal{X}_{C_1}^{[n]} \to C_1^{[n]}$.

By the stable reduction theorem we can now pass to a finite index subgroup $\Gamma \leqslant \Gamma_1$, such that the pull back of the universal family via the map $C := \Delta/\Gamma \to \mathcal{M}_g^{[n]}$ delivers a family of curves $f\colon \mathcal{X} \to C$, which can be completed to a stable family $\overline{f}\colon \overline{\mathcal{X}} \to \overline{C}$ over the smooth completion (smooth compactification) $\overline{C} = \overline{\Delta/\Gamma}$ of C.

This implies that monodromies around the cusps $\partial C = \overline{C}\backslash C$ are unipotent [98, Corollary 11.19]. The whole situation is visualized in Figure 4.1.

$$\begin{tikzcd}
\overline{\mathcal{X}} \arrow[d, "\overline{f}"] & \mathcal{X} \arrow[l] \arrow[r] \arrow[d, "f"] & \mathcal{X}_{C_1}^{[n]} \arrow[r] \arrow[dd, "f^{[n]}"] & \mathcal{X}_{\mathrm{univ}}^{[n]} \arrow[dd, "f^{[n]}"] \\
\overline{C} & C = \Delta/\Gamma \arrow[l] \arrow[dr] & & \\
& & C_1^{[n]} = \Delta/\Gamma_1 \arrow[r] \arrow[d] & \mathcal{M}_g^{[n]} \arrow[d] \\
& & C_1 = \Delta/\mathrm{Aff}^+(X,\omega) \arrow[r] & \mathcal{M}_g
\end{tikzcd}$$

Figure 4.1.: Diagram visualizing the construction of the family of curves associated to a Teichmüller curve.

In his PhD thesis, André Kappes collects the most relevant properties of the subgroup $\Gamma \leqslant \mathrm{Aff}^+(X,\omega)$ and the properties of the family of curves $f\colon \mathcal{X} \to \Delta/\Gamma$ from above in a definition [73, Remark 5.4 and Definition 5.5]. We want to do this in the same way because we want to be able to refer to it later in this text.

4.6.2 Definition. We say that a subgroup $\Gamma \leqslant \mathrm{Aff}^+(X,\omega)$ has condition $(\star)$ if the following holds:

(i) The group $\Gamma \leqslant \mathrm{Aff}^+(X,\omega)$ is torsion free and has finite index.

(ii) There is a natural number $n \geqslant 3$ such that the holomorphic map $\Delta/\Gamma \to \mathcal{M}_g$ factors over $\mathcal{M}_g^{[n]}$.

(iii) The pullback $f\colon \mathcal{X} \to \Delta/\Gamma$ of the universal curve over $\mathcal{M}_g^{[n]}$ with n as in (ii) can be completed to a stable family $\overline{f}\colon \overline{\mathcal{X}} \to \overline{\Delta/\Gamma}$ over the smooth compactification $\overline{\Delta/\Gamma}$ of Δ/Γ.

We call the family $f\colon \mathcal{X} \to \Delta/\Gamma$, respectively the family $\overline{f}\colon \overline{\mathcal{X}} \to \overline{\Delta/\Gamma}$ the *Veech fibration associated to* Γ.

4.6.3 Proposition. The construction in Subsection 4.6.1 shows that for every $g \geqslant 2$ and every Veech surface $(X,\omega) \in \Omega\mathcal{M}_g$ we can find a subgroup $\Gamma \leqslant \mathrm{Aff}^+(X,\omega)$ which has condition $(\star)$ and a stable family $\overline{f}\colon \overline{\mathcal{X}} \to \overline{\Delta/\Gamma}$ associated to Γ.

4.6.4 Remark. In the literature a family $f\colon \mathcal{X} \to \Delta/\Gamma$ as in Section 4.6.1 is often called "family coming from the Teichmüler curve $j_\omega\colon C_1 \to \mathcal{M}_g$" associated to (X,ω). I prefer to call them "Veech fibrations" as Freedman and Lucas did in [41]. Inspired by results on elliptic fibrations, Freedman and Lucas were the first who studied families $\overline{f}\colon \overline{\mathcal{X}} \to \overline{\Delta/\Gamma}$ as interesting spaces in their own right and I share their enthusiasm for this lovely construction which helps to study Teichmüller curves in so many ways.

4.6.5 A monodromy representation for Teichmüller curves. Let $(X,\omega) \in \Omega\mathcal{M}_g$ be a Veech surface which defines a Teichmüller curve $j_\omega\colon C_1 \to \mathcal{M}_g$. The affine homeomorphisms $\mathrm{Aff}^+(X,\omega)$ acts on the integral homology group $H_1(X,\mathbb{Z})$ by push forward and on the integral cohomology group $H^1(X,\mathbb{Z})$ by pull back. The actions preserve the intersection form on $H_1(X,\mathbb{Z})$ and the Hodge-Riemann pairing on $H^1(X,\mathbb{Z})$, respectively. This leads to representations

$$\begin{aligned} &\mathrm{Aff}^+(X,\omega) \longrightarrow \mathrm{Sp}(H_1(X,\mathbb{Z})) \quad \text{and} \\ &\mathrm{Aff}^+(X,\omega) \longrightarrow \mathrm{Sp}(H^1(X,\mathbb{Z})). \end{aligned} \tag{4.6.5.1}$$

In the following we describe this action and explain a geometric meaning. As we already mentioned in the preface , the representations above can be considered as the monodromy representation of a deformation of the Veech surface (X,ω) by matrices in $\mathrm{SL}_2(\mathbb{R})$. We will do this mathematically precise now. Let Γ be a subgroup of $\mathrm{Aff}^+(X,\omega)$ which has condition $(\star)$ and let $f\colon \mathcal{X} \to C = \Delta/\Gamma$ be the Veech fibration associated to Γ. Here $\Delta = \Delta(X,\omega) \subset \mathcal{T}_g$ is again the Teichmüller disk associated to the triple (X,id,ω). We continue as in Section 4.1.5. We consider the Teichmüller disk Δ as the universal cover of $C = \Delta/\Gamma$. Let $f_\Delta\colon \mathcal{H} \to \Delta$ be the pullback of the family $f\colon \mathcal{X} \to C$ via the projection $\Delta \to C = \Delta/\Gamma$. Then $f_\Delta\colon \mathcal{H} \to \Delta$ is naturally a marked family. Choose a point $\tilde{c} \in \Delta$ with fiber $f_\Delta^{-1}(\tilde{c}) = X$ which is marked by the identity $\mathrm{id}\colon X \to X$. Let $c \in C$ be the image of $\tilde{c}$ under the projection $\Delta \to C$, then $f^{-1}(\{c\}) = X$ as well.

First we want to describe the monodromy representation $\mathrm{mon}_f\colon \pi_1(C,c) \to \Gamma_g(X)$ of the family of curves $f\colon \mathcal{X} \to C$ as in Subsection 4.1.4. We identify the deck group of the universal cover $\Delta \to C$ with the subgroup $\Gamma \leqslant \mathrm{Aff}^+(X,\omega)$. For every $a \in \pi_1(C,c)$ let $\varphi_a \in \mathrm{Aff}^+(X,\omega)$ be the corresponding element in the group of affine diffeomorphisms. Then Equation 4.1.5.1 in combination with Remark 3.1.5 says that for every fiber Y of the bundle $f_\Delta\colon \mathcal{H} \to \Delta$ with marking $m\colon X \to Y$, we have the equality

$$(Y, m \circ \varphi_a) = \mathrm{mon}_f(a) \cdot (Y,m).$$

This implies $\mathrm{mon}_f(a) = \varphi_a^{-1} \in \Gamma_g(X)$. By Poincaré duality we can also interpret the stalk of the local system $R^1(f_\Delta)_*\mathbb{Z}_\Delta$ at the point $(X,\mathrm{id}) \in \Delta$ as the singular homology $H_1(X,\mathbb{Z})$. By the description of the monodromy map mon_f of the family $f\colon \mathcal{X} \to C$ from above and the way we lifted paths in Proposition 4.2.3, it is immediately clear that the monodromy representation of $R^1f_*\mathbb{Z}_{\mathcal{X}}$ is as follows:

Proposition. We identify the stalk of $R^1 f_* \mathbb{Z}_{\mathcal{X}}$ at $c \in C$ with the singular homology $H_1(X, \mathbb{Z})$ respectively by the singular cohomology $H^1(X, \mathbb{Z})$. Then the monodromy operation of a class $a \in \pi_1(C, c)$ is given by pushforward on $H_1(X, \mathbb{Z})$ respectively by pullback on $H^1(X, \mathbb{Z})$ of the inverse φ_a^{-1} of the unique affine homeomorphism $\varphi_a \in \mathrm{Aff}^+(X, \omega)$ corresponding to $a \in \pi_1(C, c)$.

You can find two different approaches for proofs of the previous proposition in [6] and [80]. We showed that the restriction of the representations in (4.6.5.1) to the subgroup $\Gamma \leqslant \mathrm{Aff}^+(X, \omega)$ correspond to local systems which we identify in the following with $R^1 f_* \mathbb{Z}_{\mathcal{X}}$.

4.6.6 Splitting the monodromy representation. Let again $(X, \omega) \in \Omega\mathcal{M}_g$ be a Veech surface of genus $g \geqslant 2$ and let $\Gamma \leqslant \mathrm{Aff}^+(X, \omega)$ be a subgroup which has condition $(\star)$ with associated Veech fibration $f : \mathcal{X} \to C$. We identify the fundamental group $\pi_1(C)$ with the subgroup Γ. In 4.6.5 we saw that the monodromy representation of the local system $R^1 f_* \mathbb{Z}_{\mathcal{X}}$ is isomorphic to the representation

$$\rho^\Gamma : \Gamma \longrightarrow \mathrm{Sp}(H^1(X, \mathbb{Z})).$$

induced by the action of $\Gamma \leqslant \mathrm{Aff}^+(X, \omega)$ on $H^1(X, \mathbb{Z})$ via pullback. We want to describe the representation a little bit further. Let $K(X, \omega) = \mathbb{Q}(\mathrm{tr}(\mathrm{SL}(X, \omega))$ be the trace field of the Veech group $\mathrm{SL}(X, \omega)$, i.e. the field extension of $\mathbb{Q}$ obtained by adjoining the traces of the matrices in $\mathrm{SL}(X, \omega) \leqslant \mathrm{SL}_2(\mathbb{R})$. It is a totally real algebraic extension of $\mathbb{Q}$ of degree at most g, see [87, Theorem 5.1, Theorem 5.2] and [94, Proposition 2.6]. The subspace $H^1_{st}(X) = \mathrm{span}_{\mathbb{R}}(\{\mathrm{Re}(\omega), \mathrm{Im}(\omega)\})$ of $H^1(X, \mathbb{R})$ is invariant under the action of $\mathrm{Aff}^+(X, \omega)$ on $H^1(X, \mathbb{R})$ via pullback. More precisely an element $\varphi \in \mathrm{Aff}^+(X, \omega)$ acts on $H^1_{st}(X)$ by

$$\varphi^*(\mathrm{Re}(\omega), \mathrm{Im}(\omega)) = D\varphi \cdot (\mathrm{Re}(\omega), \mathrm{Im}(\omega)), \tag{4.6.6.1}$$

where $D\varphi \in \mathrm{SL}(X, \omega)$ is the derivative of $\varphi \in \mathrm{Aff}^+(X, \omega)$ (see [80, Proposition 5.4.2]).

Write F for the Galois closure of $K(x, \omega)$. We consider $H^1_{st}(X)$ as a subspace of $H^1(X, \mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{R}$. The subspace $H^1_{st}(X)$ is defined over a finite field extension of the trace field $K(X, \omega)$ and so we can consider the Galois conjugate representations of $\mathrm{Aff}^+(X, \omega)$ on it with respect to the Galois group $\mathrm{Gal}(F|\mathbb{Q})$. By [94] for every $\sigma \in \mathrm{Gal}(F|\mathbb{Q})$ the Galois conjugate representation of $\mathrm{Aff}^+(X, \omega)$ on $H^1_{st}(X)^\sigma$ is isomorphic to that of $\mathrm{Aff}^+(X, \omega)$ on $H^1_{st}(X)$ if and only if σ fixes $K(X, \omega)$. For $i = 1, \ldots, r$ let σ_i be a system of representatives of the quotient $\mathrm{Gal}(F, \mathbb{Q}) / \mathrm{Gal}(F, K(X, \omega))$, where $\sigma_1 = \mathrm{id}$. Then the subspace $\oplus_{i=1}^r H^1_{st}(X)^{\sigma_i}$ is invariant under $\mathrm{Gal}(F, \mathbb{Q})$ and is thus defined over $\mathbb{Q}$. Let $W \subset H^1(X, \mathbb{Q})$ such that $W \otimes_{\mathbb{Q}} \mathbb{R} = \bigoplus_{i=1}^r H^1_{st}(X)^{\sigma_i}$. Hence we get the following decomposition into $\mathrm{Aff}^+(X, \omega)$-invariant subspaces

$$H^1(X, \mathbb{Q}) = W \oplus H^1_{(0)}, \tag{4.6.6.2}$$

where $H^1_{(0)}$ is the orthogonal complement of W with respect to the symplectic Hodge-Riemann pairing on $H^1(X, \mathbb{Q})$.

4.6.7 Definition (Kontsevich–Zorich monodromy)**.** Let (X, ω) be a Veech surface and let again $\Gamma \leqslant \mathrm{Aff}^+(X, \omega)$ be a subgroup which has condition $(\star)$. Consider the action of $\mathrm{Aff}^+(X, \omega)$ and Γ on $H^1_{(0)}$. We call the corresponding representations

$$\rho^{(X,\omega)}_{\mathrm{KoZo}} \colon \mathrm{Aff}^+(X, \omega) \longrightarrow \mathrm{Sp}(H^1_{(0)}) \quad \text{and} \quad \rho^{\Gamma}_{\mathrm{KoZo}} \colon \Gamma \longrightarrow \mathrm{Sp}(H^1_{(0)}),$$

the *Kontsevich–Zorich (monodromy) representation of the translation surface* (X, ω), respectively the *Kontsevich–Zorich monodromy representation of the Veech fibration* $f \colon \mathcal{X} \to \Delta(X, \omega)/\Gamma$. Furthermore, we write $G_{(X,\omega)} = G^{\mathrm{KoZo}}_{(X,\omega)}$ for the image of the representation $\rho^{(X,\omega)}_{\mathrm{KoZo}}$ respectively G_Γ for the image of $\rho^{\Gamma}_{\mathrm{KoZo}}$ and call the groups the *Kontsevich–Zorich monodromy of the translation surface* (X, ω), respectively the *Kontsevich–Zorich monodromy of the Veech fibration* $f \colon \mathcal{X} \to \Delta(X, \omega)/\Gamma$.

4.6.8 Splitting the Variation of Hodge structure associated to Teichmüller curves. We consider the local system $\mathbb{V}_\mathbb{Z} = R^1 f_* \mathbb{Z}_\mathcal{X}$ and the flat vector bundle

$$H^1_{\mathrm{DR}}(\mathcal{X}/C) = R^1 f_* \mathbb{Z}_\mathcal{X} \otimes_\mathbb{Z} \mathcal{O}_C$$

associated to a Veech fibration $f \colon \mathcal{X} \to C$ of a Veech surface $(X, \omega) \in \Omega\mathcal{M}_g$. From Theorem 4.4.13 we know that there is a VHS of weight one on C, where the only relevant part of the filtration is given by the subbundle

$$f_* \Omega^1_{\mathcal{X}/C} \subset H^1_{\mathrm{DR}}(\mathcal{X}/C).$$

(see Equation (4.4.14.1)). The VHS $(\mathbb{V}_\mathbb{Z}, f_* \Omega^1_{\mathcal{X}/C})$ has a natural polarization which induces the Hodge-Riemann pairing on each fiber of $H^1_{\mathrm{DR}}(\mathcal{X}/C)$ (c.f. Example 4.4.17).

The subspace $H^1_{st}(X) = \mathrm{Span}_\mathbb{R}(\{\mathrm{Re}(\omega), \mathrm{Im}(\omega)\}) \subset H^1(X, \mathbb{R})$ is invariant under the action of $\mathrm{Aff}^+(X, \omega)$ and hence corresponds to a rank two linear subsystem

$$\mathbb{L}_\mathbb{R} \subset \mathbb{V}_\mathbb{R} = (R^1 f_* \mathbb{Z}_\mathcal{X}) \otimes_\mathbb{Z} \mathbb{R},$$

such that the stalk of $\mathbb{L}_\mathbb{R}$ at the base point $c \in C$ is without loss of generality given by the subspace $H^1_{st}(X)$.

Again by [94, Lemma 2.2] the local system $\mathbb{L}_\mathbb{R}$ is defined over a field K_1 which has degree at most two over the trace field $K(X, \omega)$ of (X, ω), that means that there is a local system $\mathbb{L}$ defined over the field K_1 with $\mathbb{L}_\mathbb{R} = \mathbb{L} \otimes_{K_1} \mathbb{R}$. We write F for the Galois closure of the trace field $K(X, \omega)$ and we write $\mathbb{L}^\sigma$ for the Galois conjugate local systems of $\mathbb{L}$. Then Möller deduced from Deligne's semisimplicity theorem [22, Proposition 1.13] the following theorem.

Theorem (Möller, [94] Prop. 2.4)**.** Let $K = K(X, \omega)$ be the trace field of $(X, \omega) \in \Omega\mathcal{M}_g$. The polarized VHS $(\mathbb{V}_\mathbb{Z}, f_* \Omega^1_{\mathcal{X}/C}, Q)$ associated to the family of curves $f \colon \mathcal{X} \to C$ splits over $\mathbb{Q}$ into two subsystems

$$\mathbb{V}_\mathbb{Q} = \mathbb{W}_\mathbb{Q} \oplus \mathbb{M}_\mathbb{Q}$$

where $\mathbb{M}_{\mathbb{Q}}$ carries a polarized $\mathbb{Q}$-VHS of weight one and the local system $\mathbb{W}_{\mathbb{Q}}$ splits over the Galois closure F of the trace field K as

$$\mathbb{W}_F = \bigoplus_{\sigma} \mathbb{L}^{\sigma}, \qquad \sigma \in \mathrm{Gal}(F|\mathbb{Q})/\mathrm{Gal}(F|K(X,\omega))$$

such that each of the Galois-conjugate rank two subsystems $\mathbb{L}^{\sigma}$ carries a polarized F-VHS of weight one. The sum of these VHS gives back the original VHS on $\mathbb{V}_{\mathbb{C}}$.

4.6.9 Remark. Assume we are in the situation of Theorem 4.6.8. We want to explain two direct corollaries of this theorem.

(i) Let $(X,\omega) \in \Omega\mathcal{M}_g$ be a Veech surface with group of affine homeomorphisms $\mathrm{Aff}^+(X,\omega)$. We deduce from (4.6.6.1) that the representation matrix of the action of an element $\varphi \in \mathrm{Aff}^+(X,\omega)$ on $H^1_{st}(X)$ with respect to the basis $(\mathrm{Re}(\omega), \mathrm{Im}(\omega))$ is given by ${}^tD\varphi$, the transpose of the derivative of φ. Hence φ acts on the Galois conjugates L^{σ} of L by $({}^tD\varphi)^{\sigma}$. Let $K = K(X,\omega)$ be the trace field of the Veech group $\mathrm{SL}(X,\omega)$. Let $\mathcal{O}_K$ be the ring of integers in K. Let $\Gamma \leqslant \mathrm{Aff}^+(X,\omega)$ be a subgroup which has condition $(\star)$ and let $f\colon \mathcal{X} \to C = \Delta/\Gamma$ be the associated Veech fibration. Let $c \in C$. We identify again the fundamental group $\pi_1(C,c)$ with the group Γ. The group Γ can be considered as a subgroup of $\mathrm{SL}_2(\mathcal{O}_K)$ by Corollary 2.11 in [94]. From Theorem 4.6.8 we conclude that the monodromy representation $\pi_1(C,c) \to \mathrm{Sp}(H^1(X,\mathbb{R}))$ of $\mathbb{V}_{\mathbb{R}}$ has a subrepresentation

$$\alpha\colon \pi_1(C,c) \longrightarrow \prod_{\sigma} \mathrm{SL}_2(\mathbb{R}), \qquad \sigma \in \mathrm{Gal}(F|\mathbb{Q})/\mathrm{Gal}(F|K(X,\omega)) \tag{4.6.9.1}$$

which can be described as follows. We have a group homomorphism

$$\rho\colon \mathrm{SL}_2(\mathcal{O}_K) \longrightarrow \prod_{\sigma} \mathrm{SL}_2(\mathbb{R}), \quad \rho(\gamma) = \prod_{\sigma} {}^t\gamma^{\sigma}$$

where σ runs over $\mathrm{Gal}(F|\mathbb{Q})/\mathrm{Gal}(F|K)$. We identify $\pi_1(C,c)$ with Γ and $H^1_{st}(X)$ with $\mathbb{R}^2$ via the basis $(\mathrm{Re}(\omega), \mathrm{Im}(\omega))$. Under the identifications from above the monodromy representation in (4.6.9.1) is given by the restriction of the group homomorphism ρ to Γ. By [95, Proposition 5.5.8] it is possible to find a lattice $L \subset \bigoplus_{\sigma} \mathbb{R}^2$ such that $G = \prod_{\sigma} \mathrm{SL}_2(\mathbb{R})$ is defined over $\mathbb{Q}$ with respect to $L \otimes_{\mathbb{Z}} \mathbb{Q}$ and such that the image $\rho(\mathrm{SL}_2(\mathcal{O}_K))$ corresponds to the elements $g \in G(\mathbb{Q})$ with $g \cdot L = L$. Thus $\rho(\mathrm{SL}_2(\mathcal{O}_K))$ is an arithmetic group in G but this is not necessarily the case for $\rho(\Gamma)$ as we will see now. The group Γ is a lattice and thus Zariski dense in $\mathrm{SL}_2(\mathbb{R})$ by the density Theorem of Borel. By a result of Gutkin and Judge (for a proof see [62]) the Veech group of (X,ω) is conjugate to an arithmetic subgroup of $\mathrm{SL}_2(\mathbb{R})$ if and only if the Teichmüller curve $C \to \mathcal{M}_g$ is an origami curve. Thus the projection of the monodromy group $\rho(\Gamma)$ on the copy of $\mathrm{SL}_2(\mathbb{R})$ with $\sigma = \mathrm{id}$ is thin except for the case that $C \to \mathcal{M}_g$ is an origami curve.

(ii) We can associate to the Veech fibration $f\colon \mathcal{X} \to C$ equipped with the polarized VHS $(\mathbb{V}_{\mathbb{Z}}, f_*\Omega^1_{\mathcal{X}/C}, Q)$ a family of Jacobians $\mathrm{Jac}(\mathcal{X}/C)$ as follows. Note that

$$H^1_{\mathrm{DR}}(\mathcal{X}/C)/f_*\Omega^1_{\mathcal{X}/C} \cong R^1 f_*\mathcal{O}_{\mathcal{X}}.$$

We glue the Jacobians $\Omega^1_{X_t}(X_t)^\vee/H^1(X_t,\mathbb{Z})$ of the fibers $X_t = f^{-1}(t)$ $(t \in C)$ and obtain the intermediate Jacobian $\mathrm{Jac}(\mathcal{X}/C) = R^1 f_*\mathcal{O}_{\mathcal{X}}/R^1 f_*\mathbb{Z}_{\mathcal{X}}$ (compare [48, Section 7]). The Torelli map $\mathcal{M}_g \to \mathcal{A}_g$ sends a point in $\mathcal{M}_g$ to its Jacobian. From Theorem 4.6.8 Möller concludes in [94, Theorem 2.7] that the image of the composition

$$C \longrightarrow \mathcal{M}_g \longrightarrow \mathcal{A}_g$$

lies in the locus of $\mathcal{A}_g$ where the abelian varieties split up to isogeny into $A_1 \times_{\mathbb{C}} A_2$ with A_1 has dimension $r = \dim_{\mathbb{Q}}(K(X,\omega))$ and real multiplication by the trace field $K(X,\omega)$ [94, Theorem 2.7].

4.6.10 Monodromy for origami curves. One extreme case of the splitting (4.6.6.2) is given for origamis. The Veech group of an origami $\mathcal{O} = (X,\omega)$ is a finite index subgroup of $\mathrm{SL}_2(\mathbb{Z})$ and thus the trace field of $\mathcal{O}$ is $K(\mathcal{O}) = \mathbb{Q}$. By [89, Theorem 9.5] the elements $\mathrm{Re}(\omega)$ and $\mathrm{Im}(\omega)$ are contained in $H^1(X,\mathbb{Q})$. Thus $H^1_{st}(X)$ is naturally defined over $\mathbb{Q}$. Write $H^1_{st}(X,\mathbb{Q})$ for $\mathrm{span}_{\mathbb{Q}}(\{\mathrm{Re}(\omega),\mathrm{Im}(\omega)\})$. If $W \subset H^1(X,\mathbb{Q})$ is defined as in Subsection 4.6.6 such that $W \otimes_{\mathbb{Q}} \mathbb{R}$ is given by the sum over all Galois conjugates of $H^1_{st}(X)$, then $W = H^1_{st}(X,\mathbb{Q})$ since $K(X,\omega) = \mathbb{Q}$ and

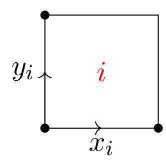
$$H^1(X,\mathbb{Q}) = H^1_{st}(X,\mathbb{Q}) \oplus H^1_{(0)}(X). \tag{4.6.10.1}$$

We can find an analogue of the splitting (4.6.10.1) for the homology $H_1(X,\mathbb{Q})$ as well, as we want to explain next. First of all we want to define the subspaces of $H_1(X,\mathbb{Q})$ which correspond to $H^1_{st}(X,\mathbb{Q})$ and $H^1_{(0)}(X)$. In a second step we want to explain why the chosen subspaces translate to $H^1_{st}(X,\mathbb{Q})$ and $H^1_{(0)}(X)$ by Poincaré duality.

Let $\mathcal{O} = (X,\omega)$ be an origami with $n \in \mathbb{N}$ squares with a branched covering

$$\pi\colon X \to \mathbb{T} = \mathbb{C}/(\mathbb{Z} + \sqrt{-1}\,\mathbb{Z}).$$

For each square $\mathrm{Sq}(i) \subset X\backslash Z$, $i \in \{1,\dots n\}$ let us denote by x_i, y_i the cycles of the relative homology $H_1(X,Z,\mathbb{R})$ as in the picture, where we write Z for the set of corners of the squares of $\mathcal{O}$.



The cycles x_i, y_i ($i = 1, \ldots, n$) generate the relative homology $H_1(X, Z, \mathbb{Z})$ and the absolute homology $H_1(X, \mathbb{Z})$ is the kernel of the connecting map

$$\partial \colon H_1(X, Z, \mathbb{Z}) \longrightarrow H_0(Z, \mathbb{Z}), \tag{4.6.10.2}$$

which sends a cycle representing a homology class in $H_1(X, Z, \mathbb{Z})$ to its boundary. We define the submodule $H_1^{(0)}(X, \mathbb{Z})$ of $H_1(X, \mathbb{Z})$ as the kernel of the pushforward

$$\pi_* \colon H_1(X, \mathbb{Z}) \longrightarrow H_1(\mathbb{T}^2, \mathbb{Z}).$$

We call $H_1^{(0)}(X, \mathbb{Z})$ the *non-tautological part of the homology* $H_1(X, Z, \mathbb{Z})$. The orthogonal complement of $H_1^{(0)}(X, \mathbb{Q})$ with respect to the intersection form is the subspace $H_1^{st}(X, \mathbb{Q}) \leqslant H_1(X, \mathbb{Q})$ generated by the cycles

$$x = \sum_{i=1}^{n} x_i \quad \text{and} \quad y = \sum_{i=1}^{n} y_i.$$

Indeed if $c = \sum_{i=1}^{n} k_i\, x_i + m_i y_i \in H_1(X, \mathbb{Z})$ is in the kernel of π_* with $k_i, m_i \in \mathbb{Q}$, then $\sum_{i=1}^{n} k_i = 0$ and $\sum_{i=1}^{n} m_i = 0$. If $(\cdot, \cdot)$ is the intersection form on $H_1(X, \mathbb{Q})$, then

$$(c, x) = \sum_{i=1}^{n} m_i = 0 \quad \text{and} \quad (c, y) = \sum_{i=1}^{n} k_i = 0.$$

This shows that x and y are in the orthogonal complement of $H_1^{(0)}(X, \mathbb{Q})$. For dimension reasons we must have that $H_1^{st}(X, \mathbb{Q})$ is the orthogonal complement of $H_1^{(0)}(X, \mathbb{Q})$. We call the submodule $H_1^{st}(X, \mathbb{Z}) = \langle x, y \rangle_{\mathbb{Z}}$ respectively the subspace $H_1^{st}(X, \mathbb{Q})$ the *tautological part of the homology*. We get the splitting $H_1(X, \mathbb{Q}) = H_1^{st}(X, \mathbb{Q}) \oplus H_1^{(0)}(\mathbb{Q})$.

For every $c \in H_1(X, \mathbb{Z})$ we have $\int_c \mathrm{Re}(\omega) \in \mathbb{Z}$ and $\int_c \mathrm{Im}(\omega) \in \mathbb{Z}$ since $H_1(X, \mathbb{Z}) = \ker(\partial)$ is the kernel of the connecting map in (4.6.10.2) and

$$\int_{x_i} \mathrm{Re}(\omega) = 1, \quad \int_{y_i} \mathrm{Re}(\omega) = 0, \quad \int_{x_i} \mathrm{Im}(\omega) = 0, \quad \int_{y_i} \mathrm{Im}(\omega) = 1$$

for every $i = 1, \ldots, n$. Hence we can consider $\mathrm{Re}(\omega)$ and $\mathrm{Im}(\omega)$ from $H^1(X, \mathbb{R})$ as elements of $H^1(X, \mathbb{Z})$. By definition $\int_c \mathrm{Re}(\omega) = 0$ and $\int_c \mathrm{Im}(\omega) = 0$ for every cycle $c \in H_1^{(0)}(X, \mathbb{Q})$ and

$$\int_{x} \mathrm{Re}(\omega) = n, \quad \int_{y} \mathrm{Re}(\omega) = 0, \quad \int_{x} \mathrm{Im}(\omega) = 0, \quad \int_{y} \mathrm{Im}(\omega) = n.$$

The Hodge-Riemann pairing is Poincaré dual to the intersection paring on $H_1(X, \mathbb{Q})$ and thus the splitting $H_1(X, \mathbb{Q}) = H_1^{st}(X, \mathbb{Q}) \oplus H_1^{(0)}(\mathbb{Q})$ translates to the splitting in (4.6.10.1) by Poincaré duality.

The intersection form on $H_1(X,\mathbb{Z})$ is invariant under the action of the affine diffeomorphisms $\mathrm{Aff}^+(X,\omega)$. Thus the action of $\mathrm{Aff}^+(X,\omega)$ also preserves the decomposition

$$H_1(X,\mathbb{Z}) = H_1^{\mathrm{st}}(X,\mathbb{Z}) \oplus H_1^{(0)}(X,\mathbb{Z}).$$

We conclude that the representation $\mathrm{Aff}^+(X,\omega) \to \mathrm{Sp}(H_1(X,\mathbb{Z}))$ also restricts to a subrepresentation

$$\mathrm{Aff}^+(X,\omega) \longrightarrow \mathrm{Sp}(H_1^{(0)}(X,\mathbb{Q})).$$

In the following we will also call this representation the Kontsevich–Zorich monodromy of the origami $\mathcal{O} = (X,\omega)$.

5. Ergodic theory and dynamics

5.1 Linear cocycles and Oseledets' multiplicative ergodic theorem

In the following section let Ω be a separable, second-countable metric space, $\mathcal{B}$ its Borel σ-algebra and $\mu\colon \mathcal{B} \to [0,1]$ a probability measure. Let $T\colon \Omega \to \Omega$ be a measurable map. We say that the measure μ is *T-invariant* if for every $A \in \mathcal{B}$ the equality

$$T_*\mu(A) := \mu\left(T^{-1}(A)\right) = \mu(A)$$

holds.

We denote by $\mathrm{MPT}(\Omega, \mathcal{B}, \mu)$ the group of all measure preserving invertible maps. Given a locally compact second-countable group G which acts measurably on Ω, a *measure preserving system* is a group homomorphism

$$T\colon G \longrightarrow \mathrm{MPT}(\Omega, \mathcal{B}, \mu), \quad g \longmapsto T_g.$$

We denote the measure preserving systems by triples (Ω, μ, G). We say that a measurable action $G \times \Omega \to \Omega$ is *ergodic* if for every G-invariant set $S \in \mathcal{B}$ either $\mu(S) = 0$ or $\mu(\Omega \backslash S) = 0$ holds.

5.1.1 Definition (Linear Cocycles)**.** Let (Ω, μ, G) be a measure preserving system with group homomorphism $T\colon G \to \mathrm{MPT}(\Omega, \mathcal{B}, \mu)$ and let $\mathbf{V} \to \Omega$ be a real or complex vector bundle of rank n over Ω. We say that an action $C\colon G \times \mathbf{V} \to \mathbf{V}$ of the group G on $\mathbf{V}$ is a *linear cocylce for* (Ω, μ, G) if it lifts the action of G on (Ω, μ) to the bundle $\mathbf{V} \to \Omega$ by linear transformations i.e.,

(i) For every $g \in G$ the equality $T_g \circ \pi = \pi \circ C(g, -)$ holds. In other words the following diagram commutes

$$\begin{tikzcd} \mathbf{V} \arrow[r, "C(g,-)"] \arrow[d, "\pi"] & \mathbf{V} \arrow[d, "\pi"] \\ \Omega \arrow[r, "T_g"'] & \Omega \end{tikzcd}$$

(ii) The cocycle C restricts for every $g \in G$ and every $\omega \in \Omega$ to linear invertible maps

$$C_g(\omega) := C(g, -)|_{\mathbf{V}_\omega} \colon \mathbf{V}_\omega \longrightarrow \mathbf{V}_{T_g(\omega)}$$

between the fibers.

(iii) The map $C\colon G \times \mathbf{V} \to \mathbf{V}$ is a measurable action with respect to the Borel σ-algebra.

5.1.2 Remark (Cocycles as in Zimmer's book)**.** Let $\mathbb{F}$ be either $\mathbb{R}$ or $\mathbb{C}$. Furthermore let $\mathbf{V}$ be a vector bundle and let $C\colon G \times \mathbf{V} \to \mathbf{V}$ be a linear cocycle as in Definition 5.1.1. We will show that our definition of linear cocycle is also a cocycle in the sense of Zimmer [122, Definition 4.2.1] if the bundle $\mathbf{V} \to \Omega$ can be trivialized by a measurable map on a set of full measure.

We assume that we have a measurable trivialization $\Omega \times \mathbb{F}^n \to \mathbf{V}$ of the vector bundle $\mathbf{V} \to \Omega$. Via this trivialization we identify the linear maps $C_g(\omega)\colon \mathbf{V}_\omega \to \mathbf{V}_{T_g(\omega)}$ with elements in $\mathrm{GL}(n, \mathbb{F})$. Then condition (iii) says that the maps $C_g(\omega)$ vary measurably in ω and g, that is

$$\alpha\colon G \times \Omega \to \mathrm{GL}(n, \mathbb{F}), \quad (g, \omega) \longmapsto C_g(\omega)$$

is a measurable map, where we consider $\mathrm{GL}(n, \mathbb{F}) \subset \mathbb{F}^{n\times n}$ as a Lie group. Since the cocycle $C\colon G \times \mathbf{V} \to \mathbf{V}$ is itself a group action we obtain the compatibility condition

$$\alpha(g\,h, \omega) = C_{gh}(\omega) = C_g(T_h(\omega)) \circ C_h(\omega) = \alpha(g, T_h(\omega)) \cdot \alpha(h, \omega)$$

with maps

$$C_{gh}(\omega)\colon \mathbf{V}_\omega \to \mathbf{V}_{T_{gh}(\omega)}, \quad C_h(\omega)\colon \mathbf{V}_\omega \to \mathbf{V}_{T_h(\omega)}, \quad C_g(T_h(\omega))\colon \mathbf{V}_{T_h(\omega)} \to \mathbf{V}_{T_{gh}(\omega)}.$$

Thus a linear cocycle is also a cocycle in the sense of Zimmer [122, Definition 4.2.1].

For a proof of the following theorem of Oseledets see for example [32] or [105].

5.1.3 Theorem (Oseledets' Theorem for linear cocycles)**.** Assume we are given a measure preserving ergodic system $(\Omega, \mu, \mathbb{Z})$ respectively $(\Omega, \mu, \mathbb{R})$ and $\mathbf{V} \to \Omega$ a real vector bundle over Ω. Assume furthermore there is a linear cocycle

$$C^d\colon \mathbb{Z} \times \mathbf{V} \to \mathbf{V} \quad \text{respectively} \quad C^c\colon \mathbb{R} \times \mathbf{V} \to \mathbf{V}$$

and that the bundle $\mathbf{V}$ is equipped with a norm $\| - \|$ (on each fiber) such that

$$\log^+ \|C^d_{-1}(\omega)\|_{\mathrm{op}} \in L^1(\Omega, \mu) \quad \text{and} \quad \log^+ \|C^d_1(\omega)\|_{\mathrm{op}} \in L^1(\Omega, \mu), \tag{5.1.3.1}$$

respectively

$$\sup_{-1 \leqslant t \leqslant 1} \log^+ \|C^c_t(\omega)\|_{\mathrm{op}} \in L^1(\Omega, \mu). \tag{5.1.3.2}$$

Here $\log^+(x) := \max(0, \log x)$ and $\| - \|_{\mathrm{op}}$ denotes the operator norm of linear maps between vector spaces.

Then there exist real numbers $\lambda_1 > \cdots > \lambda_k$ (with perhaps $\lambda_k = -\infty$) and $\mathbb{Z}$-invariant respectively $\mathbb{R}$-invariant subbundles $\mathbf{V}^{\lambda_i}$ $(i = 1, \dots, k)$ defined on a set of full measure of Ω with $\mathbf{V} = \mathbf{V}^{\lambda_1} \oplus \cdots \oplus \mathbf{V}^{\lambda_k}$ such that

$$\lim_{N \to \pm\infty} \frac{1}{N} \log \|C^d_N(\omega) v\| = \lambda_i \tag{5.1.3.3}$$

and in the second case

$$\lim_{t \to \pm\infty} \frac{1}{t} \log \|C^c_t(\omega) v\| = \lambda_i \tag{5.1.3.4}$$

for every $\omega \in \Omega$ where the splitting exists and every $v \in \mathbf{V}^{\lambda_i}_\omega \backslash \{0\}$ $(i = 1, \dots, k)$. The numbers λ_i $(i = 1, \dots, k)$ from (5.1.3.3) and (5.1.3.4) are called the *Lyapunov exponents* of the cocycle C^d respectively C^c.

5.1.4 Remark. The $\mathbb{Z}$-invariance respectively $\mathbb{R}$-invariance of the subbundles $\mathbf{V}^{\lambda_i}$ in the previous Theorem 5.1.3 means that the cocycles C^d respectively C^c take the fiber $\mathbf{V}^{\lambda_i}_\omega$ to $\mathbf{V}^{\lambda_i}_{T(\omega)}$ $(i = 1, \dots, k)$ for $\omega \in \Omega$, where the splitting exists.

5.1.5 Remark. Note that in the discrete case $(\Omega, \mu, \mathbb{Z})$ as well as the continuous case $(\Omega, \mu, \mathbb{R})$ the numbers λ_i and subbundles $\mathbf{V}^{\lambda_i}$ of $\mathbf{V}$ are unchanged if we replace the support of the measure μ by a finite unramified covering of Ω with a lift of the cocycle C^d respectively C^c on the pullback of $\mathbf{V}$ to the covering.

5.2 Teichmüller dynamics

5.2.1 Kontsevich–Zorich cocycle for Teichmüller curves. Let $\Omega\mathcal{M}_g$ denote again the bundle of non-zero holomorphic 1-forms over the moduli space of curves $\mathcal{M}_g$. We write in the following $\Omega_1\mathcal{M}_g$ respectively $\Omega_1\mathcal{T}_g$ for the subset of translation surfaces and marked translation surfaces with area one. Note that we have a natural area changing diffeomorphism

$$\Omega_1\mathcal{M}_g \times \mathbb{R}_{>0} \longrightarrow \Omega\mathcal{M}_g.$$

The $\mathrm{GL}^+(2, \mathbb{R})$-action on the bundles $\Omega\mathcal{M}_g$ and $\Omega\mathcal{T}_g$ explained in Section 3.2.4 induces an $\mathrm{SL}(2, \mathbb{R})$-action on $\Omega_1\mathcal{M}_g$ and $\Omega_1\mathcal{T}_g$.

We consider a closed $\mathrm{SL}(2, \mathbb{R})$-orbit

$$M = \mathrm{SL}(2, \mathbb{R}).(X, \omega) \subset \Omega_1\mathcal{M}_g,$$

where $(X, \omega) \in \Omega_1\mathcal{M}_g$ is a Veech surface of area one. This means that M corresponds to a Teichmüller curve $C \to \mathcal{M}_g$, where C is the normalization of the image of M under the projection to $\mathcal{M}_g$. To obtain a linear cocycle we have to add level structures once again. Consider the orbit $\mathrm{SL}(2,\mathbb{R}).(X, \mathrm{id}, \omega) \subset \Omega_1\mathcal{T}_g$, which is a lift of M to $\Omega_1\mathcal{T}_g$. Denote its projection to $\Omega_1\mathcal{T}_g/\Gamma_g^{[3]}$ by $M^{[3]}$. We have

$$M^{[3]} \longrightarrow \Omega_1\mathcal{T}_g/\Gamma_g^{[3]} \longrightarrow \mathcal{T}_g/\Gamma_g^{[3]} = \mathcal{M}_g^{[3]}$$

and pull back the universal family $\mathcal{X}_{\mathrm{univ}}^{[3]} \to \mathcal{M}_g^{[3]}$ over the fine moduli space $\mathcal{M}_g^{[3]}$ to a family $f\colon \mathcal{X} \to M^{[3]}$. The local system $\mathbb{V}_{\mathbb{Z}} = R^1 f_* \mathbb{Z}_{\mathcal{X}}$ has stalk $H^1(X,\mathbb{Z})$ at (X,ω). We denote by $\mathbf{V} = \mathbb{V}_{\mathbb{Z}} \otimes_{\mathbb{Z}} C^\infty_{M^{[3]}}$ the corresponding real smooth vector bundle. The geodesic flow $\{g_t\}_{t\in\mathbb{R}}$ on $M^{[3]}$ is the restriction of the $\mathrm{SL}(2,\mathbb{R})$-action to the subgroup $\mathrm{diag}(e^t, e^{-t})$ $(t \in \mathbb{R})$.

We equip $M^{[3]}$ with the finite Borel measure λ induced from the Haar measure on $\mathrm{SL}(2,\mathbb{R})$. We can lift the flow $\{g_t\}_{t\in\mathbb{R}}$ on $M^{[3]}$ to the bundle $\mathbf{V}$ by parallel transport with respect to the flat connection on $\mathbf{V}$. This lifted action varies measurably on the fibers of $\mathbf{V}$ in $\{g_t\}_{t\in\mathbb{R}}$ and $M^{[3]}$ since our connnection on $\mathbf{V}$ is flat and parallel transport is smooth. We define the Kontsevich–Zorich cocycle for the Teichmüller curve generated by (X,ω) to be this lift of the geodesic flow

$$G^{KoZo}\colon \mathbb{R} \times \mathbf{V} \longrightarrow \mathbf{V}, \quad (t,v) \longmapsto g_t.v.$$

5.2.2 Lyapunov exponents for the Kontsevich–Zorich cocycle. Let $(X,\omega) \in \Omega_1\mathcal{M}_g$ be again a Veech surface of genus g with area one which generates a Teichmüller curve $C \to \mathcal{M}_g$ and let

$$G^{KoZo}\colon \mathbb{R} \times \mathbf{V} \to \mathbf{V}$$

be the Kontsevich–Zorich cocycle for the Teichmüller curve from Section 5.2.1 with a real flat bundle $\mathbf{V}$ coming from the local system $\mathbb{V}_{\mathbb{Z}}$.

We saw in Example 4.4.17 and Theorem 4.4.13 that the local system $\mathbb{V}_{\mathbb{Z}}$ has a weight one VHS and a polarization Q on $\mathbb{V}_{\mathbb{Z}}$, which admits a polarization and a Hermitian positive definite form H on every fiber of $\mathbb{V}_{\mathbb{C}}$. On every stalk of $\mathbb{V}_{\mathbb{C}}$ we have the Hodge decomposition

$$H^1(X,\mathbb{C}) = H^{1,0}(X) \oplus H^{0,1}(X) = \Omega^1_X(X) \oplus \overline{\Omega^1_X(X)},$$

compare Example 4.4.3. Using the isomorphism $\Omega^1_X(X) \cong H^1(X,\mathbb{R})$ which sends a holomorphic 1-form η to $\mathrm{Re}(\eta)$, we can define the *Hodge norm* $\|\cdot\|$ on $H^1(X,\mathbb{R})$ as follows. For $v \in H^1(X,\mathbb{R})$ there is a 1-form $\eta_v \in \Omega^1_X(X)$ with $v = \mathrm{Re}(\eta_v)$. Now define $\|v\| = H(\eta_v,\eta_v)^{1/2}$. In this way we can define a norm on every fiber of the smooth bundle $\mathbf{V} = \mathbb{V}_{\mathbb{Z}} \otimes_{\mathbb{Z}} C^\infty_{M^{[3]}}$.

In Section 5.2.1 we equipped the projection of the orbit $SL(2,\mathbb{R}).(X,\omega) \subset \Omega_1\mathcal{T}_g$ to the quotient $\Omega_1\mathcal{T}_g/\Gamma_g^{[3]}$ with a measure λ inherited from the Haar measure on $\mathrm{SL}(2,\mathbb{R})$. The geodesic flow $\{g_t\}_{t\in\mathbb{R}}$ is ergodic with respect to λ [17, Theorem 4.4.1]. Furthermore, the integrability conditions of Oseledets' theorem are fulfilled for the Hodge norm on the Kontsevich–Zorich cocycle $G^{KoZo}\colon \mathbb{R} \times \mathbf{V} \longrightarrow \mathbf{V}$ [93, Lemma 6.10] and thus we can speak of Lyapunov exponents for G^{KoZo}. Since our cocycle respects the symplectic intersection form on $H^1(X, \mathbb{R})$, the Lyapunov exponents are symmetric to the origin. Indeed the Lyapunov spectrum of the Kontsevich–Zorich cocycle is of the form

$$1 = \lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_g \geqslant 0 \geqslant -\lambda_g \geqslant \cdots \geqslant -\lambda_2 \geqslant -\lambda_1 = -1$$

and there are many authors and articles where it is studied, see for example [79] or [38] just to mention two of them.

5.2.3 Lyapunov exponents and Hodge theory. Let $g \geqslant 2$ and $(X, \omega) \in \Omega\mathcal{M}_g$ a Veech surface. Furthermore, let $\Gamma \leqslant \mathrm{Aff}^+(X, \omega)$ be a subgroup which has property $(\star)$ and let $f\colon \mathcal{X} \to C = \Delta(X, \omega)/\Gamma$ be the associated Veech fibration and let $\overline{f}\colon \overline{\mathcal{X}} \to \overline{C}$ be the stable family over the smooth completion $\overline{C}$ of C. Write $S = \overline{C} \backslash C$. The flat vector bundle $\mathbf{V} = R^1 f_* \mathbb{Z}_{\mathcal{X}} \otimes_{\mathbb{Z}} \mathcal{O}_C$ has due to Deligne (see Section 4.3.3) an extension $\mathbf{V}_{\mathrm{ext}}$ to a vector bundle on $\overline{C}$. The $(1,0)$-parts on the fibers of $\mathbf{V}$ form a holomorphic bundle $f_* \Omega_{\mathcal{X}/C}$ and this bundle extends to the subbundle $\overline{f}_* \Omega_{\overline{\mathcal{X}}/\overline{C}} \subset \mathbf{V}_{\mathrm{ext}}$ over $\overline{C}$. In [79] Kontsevich and Zorich already discovered the connection between Hodge theory and the Lyapunov exponents of the Kontsevich–Zorich cocycle G^{KoZo}. For example by[79, Section 9] the identity

$$\lambda_1 + \cdots + \lambda_g = \frac{\deg(f_* \Omega_{\mathcal{X}/C})}{2g - 2 + |S|}$$

holds. We have more results of this type, which will be relevant for us in the next chapter. We write $K(X, \omega)$ for the trace field of the Veech group of (X, ω). Let $\sigma\colon K(X, \omega) \to \mathbb{R}$ be an embedding of the trace field into the real numbers and consider the rank two subsystems $\mathbb{L}^\sigma$ in the decomposition of $R^1 f_* \mathbb{R}_{\mathcal{X}}$ from Theorem 4.6.8 or consider a rank two subsystem $\mathbb{F}$ of $\mathbb{M}_{\mathbb{R}}$ as in the splitting of Theorem 4.6.8. Here $\mathbb{L}^{\mathrm{id}}$ is the local subsystem which has stalk $\mathrm{Span}_{\mathbb{R}}(\{\mathrm{Re}(\omega), \mathrm{Im}(\omega)\}) \subset H^1(X, \mathbb{R})$ over the base point $c \in C$. We write $\mathbf{F}^{(1,0)}_{\mathrm{ext}}$ for the $(1,0)$-part of the Hodge filtration on the Deligne extension $(\mathbb{F} \otimes_{\mathbb{R}} \mathcal{O}_C)_{\mathrm{ext}}$ and write $\deg(\mathbf{F}^{(1,0)}_{\mathrm{ext}})$ for the degree of the line bundle $\mathbf{F}^{(1,0)}_{\mathrm{ext}}$. Write d_σ for the degree of the $(1,0)$-part of the Hodge filtration on the Deligne extension $(\mathbb{L}^\sigma \otimes_{\mathbb{R}} \mathcal{O}_C)_{\mathrm{ext}}$.

We can apply Oseledets' Theorem to each of the summands $\mathbb{L}^\sigma$ or the rank two summand $\mathbb{F}$ of $\mathbb{M}_{\mathbb{R}}$ individually and obtain for each summand a Lyapunov exponent in the spectrum of the Kontsevich–Zorich cocycle G^{KoZo} corresponding to it. In [10] and [9] Bouw and Möller gave results how to write the Lyapunov exponents of these summands in terms of degrees of the line bundles d_σ respectively $\deg(\mathbf{F}^{(1,0)}_{\mathrm{ext}})$ from above.

5.2.4 Theorem (Bouw, Möller, [10], Theorem 8.2, Proposition 8.5)**.** Let λ be the finite $\mathrm{SL}(2, \mathbb{R})$-invariant measure on $M^{[3]}$ as in Section 5.2.1. Then the following holds for the Lyapunov spectrum of the Kontsevich–Zorich cocycle G^{KoZo}:

(i) For each embedding $\sigma\colon K(X,\omega) \to \mathbb{C}$, the Lyapunov exponent associated to the summand $\mathbb{L}^{\sigma}$ is given by the quotient

$$d_{\sigma}/d_{\mathrm{id}}.$$

(ii) For every rank two summand $\mathbb{F}$ of $\mathbb{M}_{\mathbb{R}}$ the Lyapunov exponent is given by the quotient

$$\deg(\mathbf{F}_{\mathrm{ext}}^{(1,0)})/d_{\mathrm{id}}$$

We know that in genus $g = 2$ the VHS over a Teichmüller curve splits over $\mathbb{R}$ into two direct summands of rank two. The full set of Lyapunov exponents is the union of the Lyapunov exponents of the two summands. In [9] Bouw and Möller used Theorem 5.2.4 to compute these two Lyapunov exponents. We want to state their result which was also proven by Bainbridge in [3], in the following Corollary.

5.2.5 Corollary (Bouw, Möller, [9], Corollary 2.4)**.** Let $C \to \mathcal{M}_2$ be a Teichmüller curve in genus $g = 2$ generated by the translation surface (X,ω). The positive Lyapunov exponents are

$$(\lambda_1, \lambda_2) = \begin{cases} (1, 1/3) & \text{if } (X,\omega) \in \Omega\mathcal{M}_2(2), \\ (1, 1/2) & \text{if } (X,\omega) \in \Omega\mathcal{M}_2(1,1) \end{cases}.$$

Part II.

Discussion of results

6. Arithmeticity and period mappings

6.1 Arithmeticity and rank two summands

We consider a genus $g \geqslant 2$ Veech surface $(X, \omega) \in \Omega\mathcal{M}_g$. By Proposition 4.6.3 there exists a subgroup $\Gamma \leqslant \mathrm{Aff}^+(X, \omega)$ which has condition $(\star)$. Let $f\colon \mathcal{X} \to C = \Delta/\Gamma$ be the associated Veech fibration which completes to a stable family $\overline{f}\colon \overline{\mathcal{X}} \to \overline{C} = \overline{\Delta/\Gamma}$ over the smooth compactification $\overline{C}$ of C. In the following we write $\mathbb{V}_{\mathbb{Z}}$ for the local system $R^1 f_* \mathbb{Z}_{\mathcal{X}}$.

For the rest of this section we assume there is a rank two submodule L of $H^1(X, \mathbb{Z})$ which is invariant under the action of $\Gamma \leqslant \mathrm{Aff}^+(X, \omega)$ on $H^1(X, \mathbb{Z})$. We equip L with the restriction of the intersection pairing on $H^1(X, \mathbb{Z})$. That means we get a subrepresentation

$$\rho\colon \Gamma \longrightarrow \mathrm{Sp}(L) \cong \mathrm{SL}_2(\mathbb{Z}).$$

of the monodromy representation $\Gamma \to \mathrm{Sp}(H^1(X, \mathbb{Z}))$. We can associate to ρ a local subsystem $\mathbb{L}_{\mathbb{Z}}$ of $\mathbb{V}_{\mathbb{Z}}$ and it carries a polarized sub-VHS of $(\mathbb{V}_{\mathbb{Z}}, f_*\Omega^1_{\mathcal{X}/C}, Q)$ by [22, Proposition 1.13].

The following proposition was communicated by Martin Möller to us. It was part of his idea how to prove that the the monodromy groups of the Kontsevich–Zorich monodromy for all origamis of genus two are arithmetic. We used it in Appendix B of [8].

6.1.1 Proposition. Suppose the local subsystem $\mathbb{L}_{\mathbb{Z}}$ associated to $\rho\colon \Gamma \to \mathrm{SL}_2(\mathbb{Z})$ carries a non-trivial polarized sub-VHS of $(\mathbb{V}_{\mathbb{Z}}, f_*\Omega^1_{\mathcal{X}/C}, Q)$, i.e. the $(1,0)$-part of the Hodge decomposition of $\mathbb{L}_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathcal{O}_C$ is a non-trivial line bundle. Then the image $\rho(\Gamma)$ is of finite index in $\mathrm{Sp}(L) \cong \mathrm{SL}(2, \mathbb{Z})$.

Proof. Write $\mathbf{L} = \mathbb{L}_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathcal{O}_C$ for the flat vector bundle associated to the local system $\mathbb{L}_{\mathbb{Z}}$ and let $(\mathbb{L}_{\mathbb{Z}}, \mathbf{L}^{(1,0)}, Q_{\mathbb{L}_{\mathbb{Z}}})$ be the non-trivial polarized sub-VHS of $(\mathbb{V}_{\mathbb{Z}}, f_*\Omega^1_{\mathcal{X}/C}, Q)$. Thus there exists a period mapping for the weight one polarized VHS $(\mathbb{L}_{\mathbb{Z}}, \mathbf{L}^{(1,0)}, Q_{\mathbb{L}_{\mathbb{Z}}})$ which is in our situation given as follows (compare Example 4.5.8) or [93, Section 4]). By [40, Theorem 30.3] there exists a global section ω of the $(1,0)$-part of the pullback of $\mathbf{L}$ to the universal cover $\pi\colon \mathbb{H} \to C$ of C. Furthermore, there are sections a, b of $\pi^{-1}\mathbb{L}_{\mathbb{Z}}$ that are locally a symplectic basis adapted to ω, i.e. for every $t \in \mathbb{H}$ we have

$$\int_{a(t)} \omega(t) \in \mathbb{H} \quad \text{and} \quad \int_{b(t)} \omega(t) = 1.$$

The period domain is in our situation analytically isomorphic to the Siegel upper half-plane $\mathbb{H}$ by [46, Proposition 1.24]. The period mapping is given by

$$P\colon \mathbb{H} \longrightarrow \mathbb{H}, \quad \tau \longmapsto \int_{a(t)} \omega(t).$$

By Remark 4.5.6 the period mapping $P\colon \mathbb{H} \to \mathbb{H}$ descends to a holomorphic map $\phi\colon C \to \mathbb{H}/\rho(\Gamma)$, whereby $\rho(\Gamma)$ acts holomorphically, properly and discontinuously on $\mathbb{H}$ (see [7, Proposition 8.2.2 and Proposition 8.2.5]). Also by Remark 4.5.6 we know that the mapping ϕ must be non-constant since the polarized VHS $(\mathbb{L}_{\mathbb{Z}}, \mathbf{L}^{(1,0)}, Q_{\mathbb{L}_{\mathbb{Z}}})$ is by assumption non-trivial. This shows that ϕ is open because it is holomorphic and non-constant .

Next we want to show that $\mathbb{H}/\rho(\Gamma)$ has finite hyperbolic volume. If we follow the discussion which preceeds Theorem 4.5.7, then we see that it is possible to extend the period mapping ϕ holomorphically to a map

$$\widetilde{\phi}\colon C \cup S \longrightarrow \mathbb{H}/\rho(\Gamma),$$

by Theorem 9.5 in [48], where $S \subset \partial C$ denotes the set of cusps for which the monodromy representation ρ maps the corresponding parabolic elements in Γ to elements of finite order in $\mathrm{Sp}(L) \cong SL(2,\mathbb{Z})$. From Proposition 9.11 and the proof of Theorem 9.6 in [48] we conclude that the mapping $\widetilde{\phi}$ is proper and since $\widetilde{\phi}\colon C \cup S \to \mathbb{H}/\rho(\Gamma)$ is holomorphic and non-constant, it is surjective. Furthermore Theorem 4.5.7 implies that $\mathbb{H}/\rho(\Gamma) = \widetilde{\phi}(C \cup S)$ has finite hyperbolic volume or equivalently, the group $\rho(\Gamma)$ has finite index in $\mathrm{SL}(2,\mathbb{Z})$ (see [110, Prop. 1.31]). □

6.1.2 Remark. We know that $\mathrm{SL}_2(\mathbb{Z})$ is a lattice in $\mathrm{SL}_2(\mathbb{R})$ [95, Theorem 7.0.1] and that every subgroup of $\mathrm{SL}_2(\mathbb{R})$ which is commensurable to a lattice, is a lattice as well [95, Example 4.2.2]. By Borel Density Theorem (see [95, Corollary 4.5.6] for a proof) any finite index subgroup of $\mathrm{SL}_2(\mathbb{Z})$ is Zariski dense in $\mathrm{SL}_2(\mathbb{R})$. This shows that in the situation of Proposition 6.1.1 the image of the representation

$$\rho\colon \Gamma \longrightarrow \mathrm{Sp}(L) \cong \mathrm{SL}_2(\mathbb{Z}).$$

has Zariski closure isomorphic to $\mathrm{SL}_2(\mathbb{R})$.

6.1.3 Remark (Lyapunov exponents, period mappings and monodromy representation)**.** Assume we are in the situation of Proposition 6.1.1, respectively the proof of it. Again we assume that we have a rank two submodule L of $H^1(X,\mathbb{Z})$ which is invariant under the action of a finite index subgroup $\Gamma \leqslant \mathrm{Aff}^+(X,\omega)$ which has condition $(\star)$. In Proposition 6.1.1 we showed that the monodromy group $\rho(\Gamma)$ has finite index in $\mathrm{SL}_2(\mathbb{Z})$ if the local system $\mathbb{L}_{\mathbb{Z}}$ associated to ρ carries a non-trivial polarized VHS of weight one. In this case it is possible to compute the non-negative Lyapunov exponent λ_L associated to $\mathbb{L}_{\mathbb{Z}}$ in terms of the monodromy representation ρ, respectively the holomorphic mapping $\phi\colon \mathbb{H}/\Gamma \to \mathbb{H}/\rho(\Gamma)$ induced by the period mapping $P\colon \mathbb{H} \to \mathbb{H}$ defined by $(\mathbb{L}_{\mathbb{Z}}, \mathbf{L}^{(1,0)}, Q_{\mathbb{L}_{\mathbb{Z}}})$. By results of André Kappes [72, Theorem 1.1] or [73, Theorem 9.18, Proposition 9.19] the Lyapunov exponent λ_L is given by the quotient

$$\lambda_L = \frac{\deg(\phi) \cdot \mathrm{vol}(\mathbb{H}/\rho(\Gamma))}{\mathrm{vol}(\mathbb{H}/\Gamma)}. \tag{6.1.3.1}$$

In the next lemma we want to give an easy criterion under which circumstances it is possible to apply Proposition 6.1.1 to the Γ-invariant submodule L of $H^1(X, \mathbb{Z})$. Again, let $\mathbb{L}$ be the local subsystem of $\mathbb{V}_{\mathbb{Z}}$ associated to $\rho\colon \Gamma \to \mathrm{SL}_2(\mathbb{Z})$. In Subsection 5.2.3 we saw that we can apply Oseledets' theorem to the summand $\mathbb{L}$ individually and in this way we obtain a non-negative Lyapunov exponent λ_L. Of course, the Lyapunov exponent λ_L lies in the Lyapunov spectrum of the Kontsevich–Zorich cocycle G^{KoZo}.

6.1.4 Lemma. If the Lyapunov exponent λ_L associated to $\mathbb{L}$ is strictly positive, i.e. $\lambda_L > 0$, then the $(1,0)$-part of the Hodge decomposition of $\mathbb{L} \otimes_{\mathbb{Z}} \mathcal{O}_C$ is a non-trivial line bundle.

Proof. We write $\mathbf{L}^{(1,0)}_{\mathrm{ext}}$ for the $(1,0)$-part of the Hodge filtration on the Deligne extension $(\mathbb{L}_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathcal{O}_C)_{\mathrm{ext}}$ and write $\deg(\mathbf{L}^{(1,0)}_{\mathrm{ext}})$ for the degree of the line bundle $\mathbf{L}^{(1,0)}_{\mathrm{ext}}$. By Theorem 5.2.4 the Lyapunov exponent λ_L is given by the quotient

$$\lambda_L = \deg(\mathbf{L}^{(1,0)}_{\mathrm{ext}})/d_{\mathrm{id}}.$$

This shows $\deg(\mathbf{L}^{(1,0)}_{\mathrm{ext}}) \neq 0$. Thus the $(1,0)$-part of the Hodge decomposition of $\mathbb{L}_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathcal{O}_C$ is a non-trivial line bundle □

From this lemma we can now easily conclude the following theorem, which was first observed by Möller. It is Theorem 38 in our article [8].

6.1.5 Theorem. Every origami $\mathcal{O} = (X, \omega) \in \Omega\mathcal{M}_2$ of genus two has arithmetic Kontsevich–Zorich-monodromy.

Proof. The rank two submodule $H_1^{(0)}(X, \mathbb{Z})$ of $H_1(X, \mathbb{Z})$ is invariant under the action of the affine group $\mathrm{Aff}^+(X, \omega)$. The associated non-negative Lyapunov exponent is either $1/3$ or $1/2$ by Corollary 5.2.5. Now the Theorem follows from Lemma 6.1.4 and Proposition 6.1.1. □

6.2 Period data in rank two

We will come back to the general situation of families of complex algebraic manifolds as in the introduction of this text. Let S and $\mathcal{X}$ be complex connected algebraic manifolds with $\mathcal{X}$ bimeromorphic to a Kähler manifold and let $f\colon \mathcal{X} \to S$ be a family of polarized algebraic manifolds. We fix a natural number $i \in \mathbb{N}$ and consider the local system $R^i f_* \mathbb{Q}_{\mathcal{X}}$. If we fix a base point $s_0 \in S$, then we can consider the associated algebraic monodromy representation

$$\pi_1(S, s_0) \longrightarrow \mathrm{GL}(H^i(X_{s_0}, \mathbb{Q})).$$

Deligne proved in [22] the following theorem for local systems, respectively monodromy representations, which come from families of algebraic manifolds as above.

6.2.1 Theorem (see Theorem 0.1, Variant 0.2 in [22])**.** Let S be a complex connected algebraic manifold with $s_0 \in S$ a base point and let $n \in \mathbb{N}$ be a natural number. Let $\mathbb{L}$ be a local system of $\mathbb{Q}$-vector spaces of dimension n over S and let

$$\rho_{\mathbb{L}}\colon \pi_1(S, s_0) \longrightarrow \mathrm{GL}((\mathbb{L}_{s_0})$$

be the associated monodromy representation. If the local system $\mathbb{L}$ comes from a family of polarized algebraic manifolds parametrized by S as above, then there are up to isomorphism only finitely many different irreducible direct summands which can occur in the splitting of the local system $\mathbb{L}$, respectively the splitting of the representation $\rho_{\mathbb{L}}$.

We want to stress out the importance of Hodge theory for this theorem. Indeed, Deligne deduced Theorem 6.2.1 from the following theorem.

6.2.2 Theorem (see Theorem 0.5 in [22])**.** Let S be a complex connected algebraic manifold and let $n \in \mathbb{N}$ be a natural number. For a local systems $\mathbb{L}$ of $\mathbb{Q}$-vector spaces of dimension n over S there are up to isomorphism only finitely many different irreducible direct summands underlying a polarized VHS which can appear in a splitting of $\mathbb{L}$.

We are interested in the algebraic monodromy of Veech fibrations. Having Theorem 6.2.2 in mind, it would be interesting to know which isomorphism classes of irreducible polarized sub-VHS can occur in the polarized VHS of a Veech fibration. Partial answers in this direction are Theorem 4.6.8 and [94, Corollary 2.11] of Möller. But both results do not help us with the Kontsevich–Zorich monodromy of a Veech fibration. Thus we want to introduce work of André Kappes which can be used to gain information for rank two summands. The work of André Kappes from [72] and [73] is a natural way to gain information because it also relies on Hodge theory. More concretely, it relies on period mappings. After a quick summary of some concepts developed in [72], we will continue our studies with two examples.

Note that we saw in Remark 4.5.6 that a polarized variation of Hodge structure can be equivalently described by its period mapping. Remember that the period domain of weight one polarized Hodge structures on rank two modules is given by the upper half plane $\mathbb{H}$. This motivates the following definition:

6.2.3 Definition. A *modular embedding* or *period datum* (of rank two and weight one) is a triple (P, Γ, ρ) consisting of the following objects:

(i) A group $\Gamma \leqslant \mathrm{SL}_2(\mathbb{R})$ which is a cofinite Fuchsian group, so Γ is a lattice.

(ii) A group homomorphism $\rho\colon \Gamma \to \mathrm{SL}_2(\mathbb{Z})$ such that the image of ρ has finite index in $\mathrm{SL}_2(\mathbb{Z})$.

(iii) A holomorphic map $P\colon \mathbb{H} \to \mathbb{H}$ which is equivariant with respect to the group homomorphism ρ.

6.2.4 Proposition (Proposition 5.4 [72])**.** Assume $\Gamma \leqslant \mathrm{SL}_2(\mathbb{R})$ is a cofinite Fuchsian group and $\rho\colon \Gamma \to \mathrm{SL}_2(\mathbb{Z})$ is a non-trivial group homomorphism. Then there is at most one holomorphic map $P\colon \mathbb{H} \to \mathbb{H}$, which makes (P, Γ, ρ) a modular embedding.

6.2.5 Remark. We consider a Veech surface $(X, \omega) \in \mathcal{M}_g$ of genus $g \geqslant 2$ and a subgroup $\Gamma \leqslant \mathrm{Aff}^+(X, \omega)$ which has condition $(\star)$. Remember that Γ is by assumption torsion free and hence can be identified with a finite index subgroup of the Veech group $\mathrm{SL}(X, \omega) \leqslant \mathrm{SL}_2(\mathbb{R})$. Assume we have a rank two submodule $L \subset H^1(X, \mathbb{Z})$ which is invariant under the action of Γ. Equip L with the restriction of the intersection form on $H^1(X, \mathbb{Z})$ and consider the corresponding representation

$$\rho\colon \Gamma \longrightarrow \mathrm{Sp}(L) \cong \mathrm{SL}_2(\mathbb{Z}).$$

If the corresponding non-negative Lyapunov spectrum λ_L of the Kontsevich–Zorich cocycle is positive then we obtain by Proposition 6.1.1 and 6.1.4 a modular embedding. By Proposition 6.2.4 the modular embedding is uniquely determined by ρ and Γ.

There is a left action of $\mathrm{SL}_2(\mathbb{Z}) \times \mathrm{SL}_2(\mathbb{R})$ on modular embeddings. For $(g, h) \in \mathrm{SL}_2(\mathbb{Z}) \times \mathrm{SL}_2(\mathbb{R})$ and a modular embedding (P, Γ, ρ) define

$$(g, h).(P, \Gamma, \rho) = (g \circ P \circ h^{-1},\ h \cdot \Gamma \cdot h^{-1},\ c_g \circ \rho \circ c_{h^{-1}}),$$

where c_g is the action of g on $\mathrm{SL}_2(\mathbb{Z})$ by conjugation and $c_{h^{-1}}$ is the action of h^{-1} on $\mathrm{SL}_2(\mathbb{R})$ by conjugation.

The next definition gives expression to the fact that we always consider families of compact Riemann surfaces on coverings of Teichmüller curves. Thus we want to be able to replace a subgroup $\Gamma \leqslant \mathrm{Aff}^+(X, \omega)$ which has condition $(\star)$ by a different finite index subgroup of $\mathrm{Aff}^+(X, \omega)$ which also has condition $(\star)$ and still speak about "the same" direct summand of the algebraic monodromy. It also gives credit to the fact that we only want to characterize direct summands of algebraic monodromy representations up to isomorphism.

6.2.6 Definition (Commensurability and weak commensurability)**.** Let (P_1, Γ_1, ρ_1) and (P_2, Γ_2, ρ_2) be two modular embeddings of rank two and weight one.

(i) We say that (P_1, Γ_1, ρ_1) and (P_2, Γ_2, ρ_2) are *commensurable* if there exists a subgroup $\Gamma \leqslant \mathrm{SL}_2(\mathbb{R})$ which is a finite index subgroup of Γ_1 and Γ_2 and such that $\rho_1|_\Gamma = \rho_2|_\Gamma$.

(ii) We say that (P_1, Γ_1, ρ_1) and (P_2, Γ_2, ρ_2) are *weakly commensurable* if there is an element $(g, h) \in \mathrm{SL}_2(\mathbb{Z}) \times \mathrm{SL}_2(\mathbb{R})$ such that (P_1, Γ_1, ρ_1) is commensurable to $(g, h).(P_2, \Gamma_2, \rho_2)$.

For modular embeddings the concept of Lyapunov exponents will be an important tool as well. Remark 6.1.3 justifies the following definition.

6.2.7 Definition. Let (P, Γ, ρ) be a modular embedding of rank two and weight one. The map $P\colon \mathbb{H} \to \mathbb{H}$ descends to holomorphic map $\phi\colon \mathbb{H}/\Gamma \to \mathbb{H}/\rho(\Gamma)$. We define the Lyapunov exponent of the modular embedding by

$$\lambda_{(P,\Gamma,\rho)} = \frac{\deg(\phi) \cdot \mathrm{vol}(\mathbb{H}/\rho(\Gamma))}{\mathrm{vol}(\mathbb{H}/\Gamma)}. \tag{6.2.7.1}$$

Lyapunov exponents will help us with the study of modular embeddings because of the following proposition.

6.2.8 Proposition (Proposition 5.12 [72])**.** The Lyapunov exponents $\lambda_{(P_i,\Gamma_i,\rho_i)}$ of two weakly commensurable modular embeddings (P_i, Γ_i, ρ_i) $(i = 1, 2)$ coincide, i.e

$$\lambda_{(P_1,\Gamma_1,\rho_1)} = \lambda_{(P_2,\Gamma_2,\rho_2)}.$$

6.3 Algebraic monodromy groups and coverings

Intuitively a translation covering of a translation surface (X, ω) arises as follows. Cut up the translation surface (X, ω) such that it becomes a simply connected polygon P. Then each edge in P has an associated parallel edge. Now take copies $P_1, \dots P_d$ of the polygon P and glue each edge of P_i to its associated edge in some P_j $(i, j = 1, \dots, d)$, here i may or may not be equal to j. Do this in such a way that the gluing leads to a connected surface, a covering surface of X. We will see in this section how coverings can help to characterize direct summands of the algebraic monodromy of Veech fibrations.

6.3.1 Definition (Translation covering and Veech covering)**.** Let (X, ω), (Y, η) be translation surfaces. We call a holomorphic covering $\pi\colon Y \to X$ a *translation covering* if $\pi^*(\omega) = \eta$. A translation covering $\pi\colon Y \to X$ is called *Veech covering* if (X, ω) and (Y, η) are Veech surfaces and if the branch points of π have finite $\mathrm{Aff}^+(X, \omega)$-orbits.

6.3.2 Remark. Let (X, ω) and (Y, η) be translation surfaces and let $\pi\colon Y \to X$ be a translation covering. By [53, Theorem 4.9] the translation surfaces (X, ω) and (Y, η) have commensurable Veech groups $\mathrm{SL}(X, \omega)$ and $\mathrm{SL}(Y, \eta)$. Thus if $\mathrm{SL}(X, \omega) \leqslant \mathrm{SL}_2(\mathbb{R})$ is a lattice then $\mathrm{SL}(Y, \eta) \leqslant \mathrm{SL}_2(\mathbb{R})$ is a lattice as well. In other words a translation covering of a Veech surface is a Veech surface.

We consider Veech coverings in the next section because they induce subrepresentations of the algebraic monodromy representation associated to a Veech fibration. We want to make this precise. Let (X, ω) and (Y, η) be Veech surfaces and $\pi\colon Y \to X$ a Veech covering. Write $\mathrm{Aff}^+(Y, \eta)_\pi$ for the elments in $\mathrm{Aff}^+(Y, \eta)$ which descend to (X, ω) via π, i.e. an element $\varphi \in \mathrm{Aff}^+(Y, \eta)$ is contained in $\mathrm{Aff}^+(Y, \eta)_\pi$ if and only if there is an element $\varphi^- \in \mathrm{Aff}^+(X, \omega)$ with $\pi^* \circ \varphi = \varphi^- \circ \pi^*$. Let

$$\Phi_\pi\colon \mathrm{Aff}^+(Y, \eta)_\pi \longrightarrow \mathrm{Aff}^+(X, \omega), \quad \varphi \longmapsto \varphi^-.$$

By [53, Theorem 4.8] the group $\mathrm{Aff}^+(Y, \eta)_\pi$ is a finite index subgroup of $\mathrm{Aff}^+(Y, \eta)$. Again by [53, Theorem 4.8] the image $\mathrm{Aff}^+(X, \omega)^\pi$ of Φ_π is the finite index subgroup of $\mathrm{Aff}^+(X, \omega)$ of elements which lift to (Y, η). This implies the following Lemma which can be found in [72, Proposition 4.2].

6.3.3 Lemma. Let $\pi\colon Y \to X$ be a Veech covering between Veech surfaces (X, ω) and (Y, η) and consider the natural actions of $\mathrm{Aff}^+(Y, \eta)$ on $H^1(Y, \mathbb{Z}))$ and $\mathrm{Aff}^+(X, \omega)$ on $H^1(X, \mathbb{Z}))$ which preserve the intersection forms. Let U be the image of $H^1(X, \mathbb{Z})$ under the group homomorphism

$$\pi^*\colon H^1(X, \mathbb{Z}) \longrightarrow H^1(Y, \mathbb{Z}).$$

Write Q_Y for the intersection form on $H^1(Y, \mathbb{Z})$. Then U is an $\mathrm{Aff}^+(Y, \eta)_\pi$-invariant submodule of $H^1(Y, \mathbb{Z})$ polarized by the restriction of $\deg(\pi) \cdot Q_Y$ on U. Furthermore the maps π^* is equivariant for the action of $\mathrm{Aff}^+(Y, \eta)_\pi$ on U and the action of $\mathrm{Aff}^+(X, \omega)^\pi$ on $H^1(X, \mathbb{Z})$.

Corollary 12 of [99] implies the following Proposition, which can be found in the PhD-Thesis of André Kappes [73, Theorem 9.3].

6.3.4 Proposition (Veech coverings and local systems)**.** We consider Veech surfaces (X, ω) and (Y, η) of genus g and genus h respectively together with a Veech covering $p\colon Y \to X$. Then the following holds:

(i) There is a subgroup $\Gamma \leqslant \mathrm{SL}(X, \omega) \cap \mathrm{SL}(Y, \eta)$ which is isomorphic to a torsion free subgroup of $\mathrm{Aff}^+(X, \omega)$ and $\mathrm{Aff}^+(Y, \eta)$ such that condition $(\star)$ is simultaneously fulfilled for the covers

$$\mathbb{H}/\Gamma \to \Delta(X, \omega)/\mathrm{Aff}^+(X, \omega) \to \mathcal{M}_g \quad \text{and} \quad \mathbb{H}/\Gamma \to \Delta(Y, \eta)/\mathrm{Aff}^+(Y, \eta) \to \mathcal{M}_h$$

(ii) If $f_X\colon \mathcal{X} \to \mathbb{H}/\Gamma$ and $f_Y\colon \mathcal{Y} \to \mathbb{H}/\Gamma$ are the associated Veech fibrations, then we have an inclusion of local systems $R^1(f_X)_*\mathbb{Z}_{\mathcal{X}} \to R^1(f_Y)_*\mathbb{Z}_{\mathcal{Y}}$ which is also a morphism of the corresponding polarized VHS.

6.4 Coverings of L-shaped translation surfaces

Let $0 < a, b < 1$ be parameters. The Veech surfaces in the stratum $\Omega\mathcal{M}_2(2)$ are well understood. An important role is played by the translation surfaces $L(a, b)$ with parameters $0 < a, b < 1$ as shown in Figure 6.1. In this section we want to study the algebraic monodromy of Veech fibrations associated to translation surfaces $K(a, b)$ and $X(a, b)$ in $\Omega\mathcal{M}_5(2^4)$ which we will obtain as coverings of a L-shaped translation surface $L(a, b) \in \Omega\mathcal{M}_2(2)$. Both covers will have Deck-group the Klein four-group $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. The following theorem of Calta and McMullen clearifies for which $0 < a, b < 1$ the translation surface $L(a, b)$ is a Veech surface.

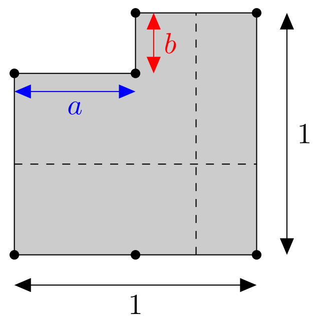


Figure 6.1.: The surface $L(a, b)$. Opposite sides are glued.

6.4.1 Theorem (Calta [12], McMullen [87], [88])**.** The surface $L(a, b)$ is a Veech surface if and only if we are for $0 < a, b < 1$ in one of the following situations.

(i) Both a, and b are rational.

(ii) There are $x, y \in \mathbb{Q}$ and $D > 1$ a square-free integer such that

$$1/(1-a) = x + y\sqrt{D} \quad \text{and} \quad 1/(1-b) = (1-x) + y\sqrt{D}.$$

The trace field of $L(a, b)$ is in this case $K(L(a, b)) = \mathbb{Q}(\sqrt{D})$.

6.4.2 Remark. Let $0 < a, b < 1$ be parameters such that condition (i) or (ii) of Theorem 6.4.1 are fulfilled. Let $\Gamma \leqslant \mathrm{Aff}^+(L(a, b))$ be a finite index subgroup which has condition $(\star)$ and let $f\colon \mathcal{X} \to \Delta/\Gamma$ be the associated Veech fibration. We denote $\mathbb{L}_{a,b} = R^1 f_* \mathbb{Z}_{\mathcal{X}}$. Depending on whether the parameters a, b are rational or not, we get two different splittings of the local system $\mathbb{L}_{a,b}$ due to Theorem 4.6.8:

(i) If we are in situation (i) of Theorem 6.4.1 and a, b are rational then the trace field $K(L(a, b))$ of $L(a, b)$ is $\mathbb{Q}$ and the local system $\mathbb{L}_{a,b}$ decomposes over $\mathbb{Q}$ in rank two summands

$$(\mathbb{L}_{a,b})_{\mathbb{Q}} = \mathbb{L}_{st} \oplus \mathbb{L}_{(0)}$$

(ii) If we are in situation (ii) of Theorem 6.4.1 then the trace field $K(L(a,b))$ is given by $F := \mathbb{Q}(\sqrt{D})$. Write id and σ for the two embeddings of $F = \mathbb{Q}(\sqrt{D})$ into $\mathbb{R}$. Then $\mathbb{L}_{a,b}$ decomposes over F in rank two summands

$$(\mathbb{L}_{a,b})_F = \mathbb{L}^{\mathrm{id}} \oplus \mathbb{L}^{\sigma},$$

where $\mathbb{L}^{\sigma}$ is the Galois-conjugate of $\mathbb{L}^{\mathrm{id}}$.

The local system $\mathbb{L}^{st}$ in case (i) and the local system $\mathbb{L}^{\mathrm{id}}$ in (ii) corresponds to the standard-action of $\Gamma \leqslant \mathrm{Aff}^+(L(a,b)$ on the tautological part

$$H^1_{st}(L(a,b),\mathbb{R}) = \langle \mathrm{Re}(\omega), \mathrm{Im}(\omega) \rangle,$$

where ω is the holomorphic one-form which induces the translation structure on $L(a,b)$.

We will now fix parameters $0 < a,b < 1$ such that (i) or (ii) in 6.4.1 is fulfilled. With the help of Proposition 6.3.4 we can find a finite index subgroup $\Gamma \leqslant \mathrm{SL}(L(a,b))$ which also has finite index in $\mathrm{SL}(K(a,b))$ and $\mathrm{SL}(X(a,b))$ such that condition $(\star)$ is fulfilled for the covers

$$\mathbb{H}/\Gamma \to \Delta(K(a,b))/\mathrm{Aff}^+(K(a,b)) \to \mathcal{M}_5$$

and

$$\mathbb{H}/\Gamma \to \Delta(X(a,b))/\mathrm{Aff}^+(X(a,b)) \to \mathcal{M}_5.$$

This means that $\mathbb{H}/\Gamma$ is a cover for both Teichmüller curves, the Teichmüller curve coming from $K(a,b)$ as well as the Teichmüller curve coming from $X(a,b)$. We obtain two Veech fibrations $f_K \colon \mathcal{X}_K \to \mathbb{H}/\Gamma$ and $f_W \colon \mathcal{X}_W \to \mathbb{H}/\Gamma$ and two local systems $\mathbb{V}^{(K)} := R^1(f_K)_*\mathbb{Q}_{\mathcal{X}_K}$ and $\mathbb{V}^{(W)} := R^1(f_W)_*\mathbb{Q}_{\mathcal{X}_W}$ over $\mathbb{H}/\Gamma$. Since $K(a,b)$ and $X(a,b)$ have genus five, the local systems $\mathbb{V}^{(K)}$ and $\mathbb{V}^{(W)}$ are local systems of rational vector spaces of dimension ten. Following the theme of Theorem 6.2.1 and Theorem 6.2.2 we will know try to gain information about the polarized sub-VHS of the summands respectively the modular embeddings associated to rank two summands.

In the following subsections we will make use of the following lemma several times.

6.4.3 Lemma. Let $(X,\omega) \in \Omega\mathcal{M}_g$ be a translation surface of genus $g \geqslant 2$ equipped with an involution $\sigma \in \mathrm{Aff}^+(X,\omega)$, i.e. $\sigma^2 = \mathrm{id}$. Let X/σ be a half translation surface of genus $h = g(X/\sigma)$. The involution defines a linear mapping $\sigma^* \colon H^1(X,\mathbb{Z}) \to H^1(X,\mathbb{Z})$, which induces a splitting

$$H^1(X,\mathbb{Q}) = \mathrm{Eig}(\sigma^*,1) \oplus \mathrm{Eig}(\sigma^*,-1) \tag{6.4.3.1}$$

into eigenspaces over the rational numbers such that $\dim(\mathrm{Eig}(\sigma^*,1)) = 2h$. Furthermore $\mathrm{Eig}(\sigma^*,1)$ and $\mathrm{Eig}(\sigma^*,-1)$ are orthogonal to each other with respect to the intersection form on $H^1(X,\mathbb{Q})$ and the affine group $\mathrm{Aff}^+(X,\omega)$ respects the splitting of $H^1(X,\mathbb{Q})$ in (6.4.3.1).

Proof. The involution $\sigma \in \mathrm{Aff}^+(X,\omega)$ induces linear maps σ_* on $H_1(X,\mathbb{Z})$ and σ^* on $H^1(X,\mathbb{Z})$. We write $\pi_\sigma\colon X \to X/\sigma$ for the double covering induced by σ. The covering $\pi_\sigma\colon X \to X/\sigma$ induces two linear maps

$$\pi_{\sigma,*}\colon H_1(X,\mathbb{Z}) \longrightarrow H_1(X/\sigma,\mathbb{Z}) \quad \text{and} \quad \pi_\sigma^*\colon H^1(X/\sigma,\mathbb{Z}) \longrightarrow H^1(X,\mathbb{Z}).$$

Since $\sigma^2 = \mathrm{id}$, we obviously get the following splittings over the complex numbers:

$$H_1(X,\mathbb{C}) = \mathrm{Eig}(\sigma_*,1) \oplus \mathrm{Eig}(\sigma_*,-1)$$

and

$$H^1(X,\mathbb{C}) = \mathrm{Eig}(\sigma^*,1) \oplus \mathrm{Eig}(\sigma^*,-1).$$

The covering is invariant under the involution, i.e. $\pi_\sigma \circ \sigma = \pi_\sigma$. This implies

$$\pi_{\sigma,*} \circ \sigma_* = \pi_{\sigma,*} \quad \text{and} \quad \sigma^* \circ \pi_\sigma^* = \pi_\sigma^*.$$

This shows $\mathrm{Eig}(\sigma_*,-1) \subset \ker(\pi_{\sigma,*})$ and $\mathrm{im}(\pi_\sigma^*) \subset \mathrm{Eig}(\sigma^*,1)$.

Remember that we have a positive definite Hermitian form H on $H^1(X,\mathbb{C})$ induced by the intersection form. By the Lemma of Riesz for every element $c \in \ker(\pi_{\sigma,*})$ there is an element $\alpha_c \in H^1(X,\mathbb{C})$ with $\int_c \beta = H(\alpha_c,\beta)$ for all $\beta \in H^1(X,\mathbb{C})$. Let $W = \langle \alpha_c \mid c \in \ker(\pi_{\sigma,*})\rangle \subset H^1(X,\mathbb{C})$ be the subspace generated by all the $\alpha_c \in H^1(X,\mathbb{C})$. Let $\gamma \in \mathrm{im}(\pi_\sigma^*)$ and $c \in \ker(\pi_{\sigma,*})$. Thus we have an element $\gamma_0 \in H^1(X/\sigma,\mathbb{C})$ with $\gamma = \pi_\sigma^*(\gamma_0)$ and

$$H(\alpha_c,\gamma) = \int_c \pi_\sigma^*(\gamma_0) = \int_{\pi_{\sigma,*}(c)} \gamma_0 = 0.$$

This shows that W and $\mathrm{im}(\pi_\sigma^*)$ are orthogonal with respect to H and thus $H^1(X,\mathbb{C}) = W \oplus \mathrm{im}(\pi_\sigma^*)$ for dimension reasons.

We show next that $\mathrm{Eig}(\sigma^*,1) = \mathrm{im}(\pi_\sigma^*)$ holds. Let $c_1,\dots,c_{2h}$ be a basis of $H_1(X/\sigma,\mathbb{Z})$. There are linearly independent $d_1,\dots,d_{2h} \in H_1(X,\mathbb{Z})$ with $\pi_{\sigma,*}(d_i) = c_i$ for every $i = 1,\dots,2h$. Then all the elements $d_i + \sigma_*(d_i) \in \mathrm{Eig}(\sigma_*,1)$ are linearly independent since $\pi_{\sigma,*}(d_i + \sigma_*(d_i)) = 2c_i$ for every $i = 1,\dots,2h$. This shows $\dim(\mathrm{Eig}(\sigma^*,1)) \leqslant \dim(\mathrm{im}(\pi_\sigma^*))$. From $\mathrm{im}(\pi_\sigma^*) \subset \mathrm{Eig}(\sigma^*,1)$ we conclude $\mathrm{Eig}(\sigma^*,1) = \mathrm{im}(\pi_\sigma^*)$.

We write Q for the intersection form on $H^1(X,\mathbb{Q})$. Note that $\sigma \in \mathrm{Aff}^+(X,\omega)$ respects the intersection form. If now $c \in \mathrm{Eig}(\sigma^*,1)$ and $d \in \mathrm{Eig}(\sigma^*,-1)$, then

$$Q(c,d) = Q(\sigma^*(c),\sigma^*(d)) = Q(c,-d) = -Q(c,d).$$

This shows that $\mathrm{Eig}(\sigma^*,1)$ and $\mathrm{Eig}(\sigma^*,-1)$ are orthogonal to each other with respect to the intersection form on $H^1(X,\mathbb{Q})$ and that $\mathrm{Aff}^+(X,\omega)$ respects the decomposition of $H^1(X,\mathbb{Q})$ into eigenspaces as in (6.4.3.1). $\square$

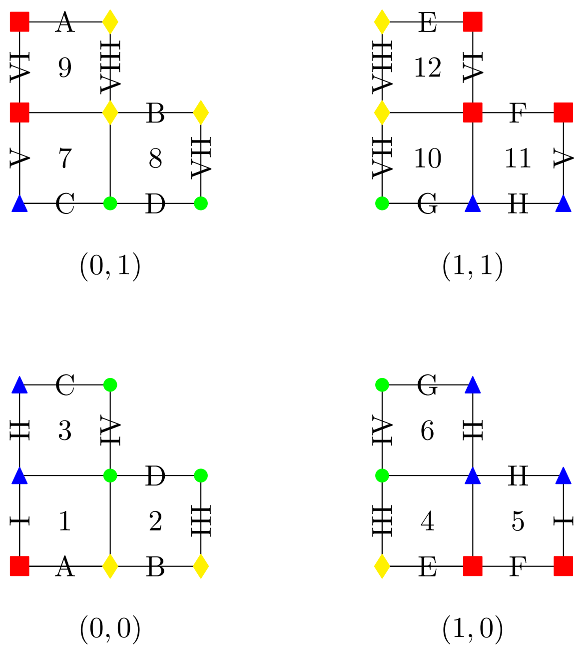


Figure 6.2.: The translation surface $K(1/2, 1/2)$.

The translation surface $K(a, b)$

Again we consider parameters $0 < a, b < 1$ such that (i) or (ii) in 6.4.1 is fulfilled. In this part of the section we consider the translation surface $K(a, b)$ which we obtain as covering $\pi\colon K(a, b) \to L(a, b)$ of the Veech surface $L(a, b)$. If we rescale the area of $K(1/2, 1/2)$ the we can obtain an origami K. The origami K can be found in the article [35] of Filip, Forni and Matheus. For each element in the Klein four-group $V = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ take a copy $L(a, b)_g$ of $L(a, b)$. We obtain $K(a, b)$ by the following construction. For every $g \in V$ glue the two rightmost vertical sides of $L(a, b)_g$ to the two leftmost vertical sides of $L(a, b)_{g+(1,0)}$ and glue the two topmost horizontal sides of $L(a, b)_g$ to the two bottom most horizontal sides of $L(a, b)_{g+(0,1)}$. The translation surface $K(a, b)$ lies in the stratum $\Omega\mathcal{M}_5(2^4)$. It can be seen in Figure 6.2 for the parameters $a = 1/2$ and $b = 1/2$.

6.4.4 Description of automorphisms and singular homology of $K(a, b)$. Every g in the Klein four-group V gives us an element in the automorphism group $\mathrm{Aut}(K(a, b))$ of the translation surface by mapping the copy $L(a, b)_h$ to the copy $L(a, b)_{g+h}$. The automorphism group $\mathrm{Aut}(K(a, b))$ of the translation surface $K(a, b)$ is indeed isomorphic to the Klein four-group V. Note that $K(a, b)$ is a Veech surface as well since it is obtained as a translation covering $\pi\colon K(a, b) \to L(a, b)$ of $L(a, b)$. For every $g \in V \backslash \{(0, 0)\}$ we get a translation surface $K(a, b)/\langle g \rangle \in \Omega\mathcal{M}(2, 2)$ which is an unramified double cover of the surface $L(a,b) \in \Omega\mathcal{M}(2)$. For later use we write

$$\pi_g \colon K(a,b) \longrightarrow K(a,b)/\langle g\rangle$$

for the covering of $K(a,b)/\langle g\rangle$ by $K(a,b)$. Note that the coverings π_g are Veech coverings. In Figure 6.3b you can see the translation surface $K(a,b)/\langle(1,0)\rangle$ and in Figure 6.3a you can see the surface $K(a,b)/\langle(1,1)\rangle$. In both figures we have parameters $a = 1/2$ and $b = 1/2$. Every $g \in V$ induces a linear automorphism g^* on the singular cohomology

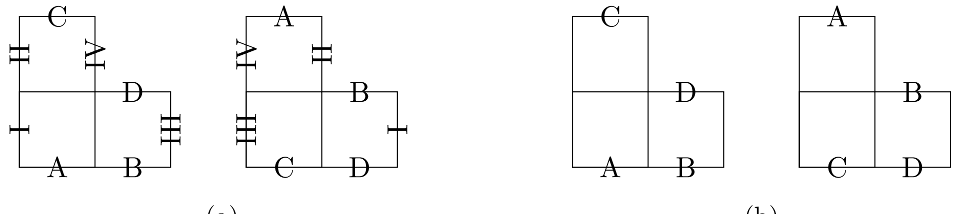


Figure 6.3.: In Figure 6.3b you can see the origami $K(1/2,1/2)/\langle(1,0)\rangle$ and in Figure 6.3a the origami $K(1/2,1/2)/\langle(1,1)\rangle$ can be seen.

$H^1(K(a,b),\mathbb{Q})$. Furthermore the coverings $\pi_g \colon K(a,b) \to K(a,b)/\langle g\rangle$ and the covering $\pi\colon K(a,b) \to L(a,b)$ induce linear maps

$$\pi_g^* \colon H^1(K(a,b)/\langle g\rangle,\mathbb{Q}) \longrightarrow H^1(K(a,b),\mathbb{Q})$$

and

$$\pi^* \colon H^1(L(a,b),\mathbb{Q})) \longrightarrow H^1(K(a,b),\mathbb{Q})$$

by pull back. Every covering π_g is invariant under the automotphism $g \in V \cong \operatorname{Aut}(K(a,b))$ and with the help of 6.4.3 we obtain splittings

$$H^1(K(a,b),\mathbb{Q}) = \operatorname{Eig}(g^*,1) \oplus \operatorname{Eig}(g^*,-1),$$

as well as

$$H^1(K(a,b),\mathbb{Q}) = \operatorname{im}(\pi^*) \oplus \sum_{g\neq(0,0)} \operatorname{Eig}(g^*,-1).$$

As in the proof of Lemma 6.4.3 we can indentify $H^1(K(a,b)/\langle g\rangle,\mathbb{Q})$ with the subspace of $H^1(K(a,b),\mathbb{Q})$ which is invariant under g^*. Furthermore we can identify the singular cohomology $H^1(L(a,b),\mathbb{Q})$ of the Veech surface $L(a,b)$ with the subspace of $H^1(K(a,b),\mathbb{Q})$ which is invariant under all the g^* with $g \in V\backslash\{(0,0)\}$. This shows that we can find a decomposition

$$H^1(K(a,b),\mathbb{Q}) = M \oplus \bigoplus_{g\neq(0,0)} M_g \tag{6.4.4.1}$$

into summands with the following properties:

$$H^1(L(a,b),\mathbb{Q}) \cong M \tag{6.4.4.2}$$

and

$$H^1(K(a,b)/\langle g\rangle,\mathbb{Q}) \cong M \oplus M_g \qquad (g \in V\backslash\{(0,0)\}) \tag{6.4.4.3}$$

Note that by Lemma 6.4.3 all the summands in the decomposition (6.4.4.1) are orthogonal to each other with respect to the intersection form on $H^1(K(a,b),\mathbb{Q})$.

6.4.5 Veech fibration and algebraic monodromy of $K(a,b)$. We apply 6.3.3 respectively Proposition 6.3.4 to $\pi\colon K(a,b) \to L(a,b)$ and each of the $\pi_g\colon K(a,b) \to K(a,b)/\langle g\rangle$ and find a torsion free finite index subgroup $\Gamma_{a,b}^{(K)} \leqslant \Gamma_0$ such that condition $(\star)$ is simultaneously fulfilled for each of the following holomorphic coverings

$$\mathbb{H}/\Gamma_{a,b}^{(K)} \longrightarrow \Delta(K(a,b))/\mathrm{Aff}^+(K(a,b)) \longrightarrow \mathcal{M}_5 \tag{6.4.5.1}$$

$$\mathbb{H}/\Gamma_{a,b}^{(K)} \longrightarrow \Delta(K(a,b)/\langle g\rangle)/\mathrm{Aff}^+(K(a,b)/\langle g\rangle) \longrightarrow \mathcal{M}_3 \qquad (g \neq (0,0)) \tag{6.4.5.2}$$

$$\mathbb{H}/\Gamma_{a,b}^{(K)} \longrightarrow \Delta(L(a,b))/\mathrm{Aff}^+(L(a,b)) \longrightarrow \mathcal{M}_2. \tag{6.4.5.3}$$

We write $f_K\colon \mathcal{X}_K \to \mathbb{H}/\Gamma_{a,b}^{(K)}$ for the Veech fibration associated to the cover (6.4.5.1), respectively $f_g\colon \mathcal{Y}_g \to \mathbb{H}/\Gamma_{a,b}^{(K)}$ and $f_V\colon \mathcal{Y}_V \to \mathbb{H}/\Gamma_{a,b}^{(K)}$ for the Veech fibrations associated to (6.4.5.2) and (6.4.5.3). Furthermore, Proposition 6.3.4 says that we have inclusions of local systems

$$R^1(f_V)_*\mathbb{Z}_{\mathcal{Y}_V} \longrightarrow R^1(f_K)_*\mathbb{Z}_{\mathcal{X}} \quad \text{and} \quad R^1(f_g)_*\mathbb{Z}_{\mathcal{Y}_g} \longrightarrow R^1(f_K)_*\mathbb{Z}_{\mathcal{X}} \quad (g \neq (0,0)),$$

which are also morphisms of polarized VHS. All of these inclusions correspond to pairwise different local subsystems of $\mathbb{V}^{(K)} = R^1(f_K)_*\mathbb{Z}_{\mathcal{X}}$ as they correspond to pairwise different subrepresentations of $H^1(K,\mathbb{Q})$.

The inclusion of local systems from above together with Deligne's semisimplicity result (see [22] or [121, Theorem 7.25]) and the splitting theorem of Möller 4.6.8 shows that $\mathbb{V}_{\mathbb{Q}}^{(K)}$ splits into a direct sum of local subsystems

$$(\mathbb{V}^{(K)})_{\mathbb{Q}} = \mathbb{M} \oplus \bigoplus_{g\neq(0,0)} \mathbb{M}_g \tag{6.4.5.4}$$

such that each of the subsystems $\mathbb{M}_g$ $(g \in V\backslash\{(0,0)\})$ carries a polarized $\mathbb{Q}$-VHS of weight one.

In the next Lemma we will determine the Lyapunov exponents for the Kontsevich-Zorich cocycle over $\mathbb{H}/\Gamma_{a,b}^{(K)}$. We will hereby mimic the proof in Section 5.4 of [35] respectively the proof of [18, Theorem 7].

6.4.6 Lemma. The non-negative Lyapunov exponents associated to the summand $\mathbb{M}$ are 1 and 1/3. The Lyapunov exponents λ_g of the summands $\mathbb{M}_g$ of $\mathbb{V}_{\mathbb{Q}}^{(K)}$ are all given by $\lambda_i = 1/3$ $(g \neq (0,0))$.

Proof. We identify the smooth bundle associated to the local subsystem $\mathbb{M}$ of $\mathbb{V}_{\mathbb{Q}}^{(K)}$ with the smooth bundle $R^1(f_V)_*\mathbb{Z}_{\mathcal{Y}_V} \otimes_{\mathbb{Z}} C^{\infty}_{\mathbb{H}/\Gamma_{a,b}^{(K)}}$ underlying the Kontsevich–Zorich cocycle of the Veech surface $L(a,b) \in \Omega\mathcal{M}_2(2)$. In this stratum we have Laypunov exponents $(1, 1/3)$ by Corollary 5.2.5. As we explained above, we can identify for every $g \neq (0,0)$ the local subsystems

$$\mathbb{M} \oplus \mathbb{M}_g \cong R^1(f_g)_*\mathbb{Q}_{\mathcal{Y}_g}.$$

Every Veech surface $K(a,b)/\langle g\rangle$ is contained in the component $\Omega\mathcal{M}_3^{\mathrm{odd}}(2,2)$ and we know that the sum of Lyapunov exponents is non-varying in this component by results of Chen and Möller [15]. Indeed the sum of Lyapunov exponents is given by $5/3$. With the argument above the sum of Lyapunov exponents of the Kontsevich–Zorich cocycle associated to $R^1(f_g)_*\mathbb{Q}_{\mathcal{Y}_g} \otimes_{\mathbb{Q}} C^{\infty}_{\mathbb{H}/\Gamma_{a,b}^{(K)}}$ must satisfy the relation

$$1 + 1/3 + \lambda_g = 5/3.$$

This shows $\lambda_g = 1/3$ for every $g \neq (0,0)$. $\square$

Let $\Gamma_{a,b}^{(K)} \leqslant \mathrm{Aff}^+(K(a,b)$ be as in Subsection 6.4.5. Recall that all the summands in the splitting

$$H^1(K(a,b),\mathbb{Q}) = M \oplus \bigoplus_{g\neq(0,0)} M_g \qquad (6.4.6.1)$$

from (6.4.4.1) are orthogonal to each other and hence they are invariant under the action of $\Gamma_{a,b}^{(K)}$ on $H^1(K(a,b),\mathbb{Q})$. Depending on $a,b \in \mathbb{Q}$ or not we get a further splitting of the summand as follows (c.f Remark 6.4.2): If a,b are as in part (i) of Theorem 6.4.1, then the summand M splits over $\mathbb{Q}$ as

$$M = M_{st} \oplus M_{(0)},$$

if a,b are as in part (ii) of Theorem 6.4.1 then the summand M splits over the $F := K(L(a,b))$ as

$$M \otimes_{\mathbb{Q}} F = M_{st}^{\mathrm{id}} \oplus M_{st}^{\sigma},$$

where id and σ are the different embeddings of $F = K(L(a,b))$ into $\mathbb{R}$. Note that for $a,b \in \mathbb{Q}$ the summands M_{st} and $M_{(0)}$ of $H^1(K(a,b),\mathbb{Q})$ are orthogonal to all the other summands M_g $(g \neq (0,0))$ with respect to the intersection form. We can restrict $\Gamma_{a,b}^{(K)}$ to a finite index subgroup Γ which still has condition $(\star)$ such that Γ respects a lattice in $M_g \cap H^1(K(a,b),\mathbb{Z})$ with $g \neq (0,0)$ respectively a lattice in $M_{(0)} \cap H^1(K(a,b),\mathbb{Z})$. We can now apply Proposition 6.1.1 and Lemma 6.1.4 on Γ and the rank two summand M_g $(g \neq (0,0))$ respectively $M_{(0)}$ if $a,b \in \mathbb{Q}$. In this way we obtain together with the density theorem of Borel (c.f. Remark 6.1.2) the following proposition.

6.4.7 Proposition. For all parameters $0 < a, b < 1$ as in Theorem 6.4.1 the Kontsevich–Zorich monodromy representation

$$\Gamma_{a,b}^{(K)} \longrightarrow \mathrm{Sp}(H^1_{(0)}(K(a,b),\mathbb{Q}))$$

splits into rank two summands such that the monodromy group projects for each of the summands to an arithmetic subgroup of $\mathrm{SL}_2(\mathbb{R})$.

Wind tree model

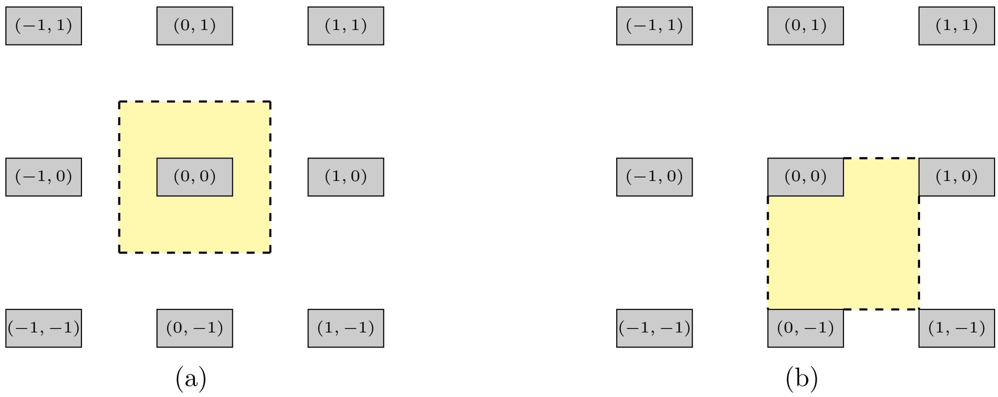


Figure 6.4.: Two fundamental regions for the billiard table of the wind tree model.

In this section we want to study the monodromy groups of local systems associated to Teichmüller curves coming from a famous family of translation surfaces. These translation surfaces were constructed to study a polygonal billiard called *wind tree model* (see Figure 6.4 and Figure 6.5).

We will now explain to the construction of the wind tree model. The Katok-Zemliakov unfolding procedure of the billiard in the yellow-colored fundamental domains shown in Figure 6.4 leads to a translation surface $X(a, b)$ which is made of four reflected copies of the fundamental domain see Figure 6.6. We will see that for the cases of $0 < a, b < 1$

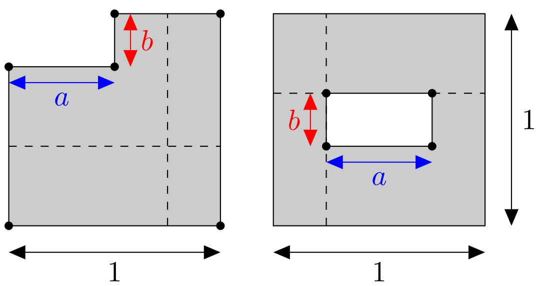


Figure 6.5.: Cutting and pasting the fundamental regions of the wind tree model.

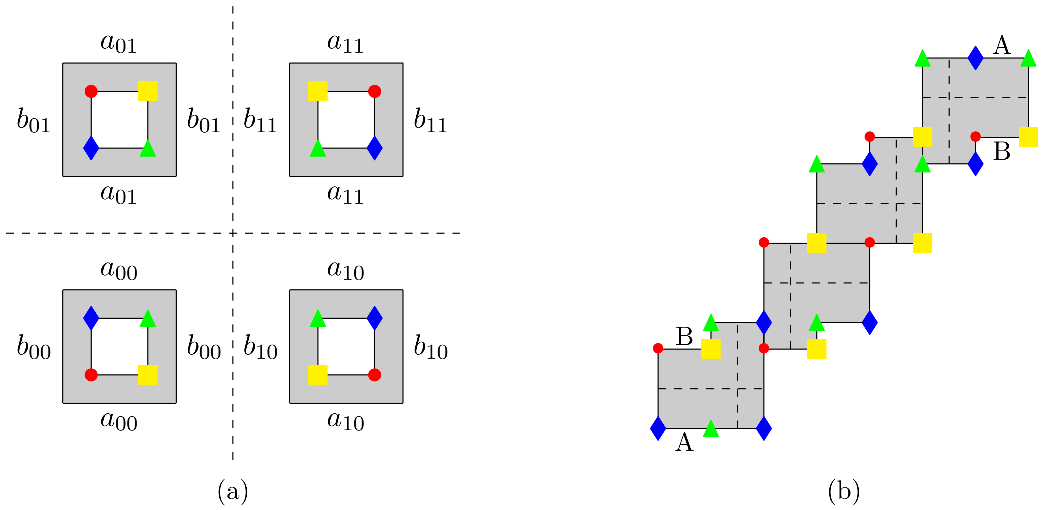


Figure 6.6.: Two different presentations of the translation surface $X(a, b)$.

as in Theorem 6.4.1 the translation surface $X(a, b)$ is also a Veech surface and in the following we want to study the monodromy group of the local system associated to the Teichmüller curve coming from $X(a, b)$.

6.4.8 Lemma (Delecroix, Hubert, Lelièvre [18])**.** The following holds:

(i) The surface $X(a, b)$ is a genus five surface in $\Omega\mathcal{M}_5(2^4)$. It is a normal unramified translation covering of $L(a, b)$ with Deck group G isomorphic to the Klein four-group $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. The Deck group G of the covering is generated by an element $\tau_h \in G$ which acts by translation in direction $(0, 1)$ on the copies in Figure 6.6a and by an element $\tau_v \in G$ which acts by translation in direction $(1, 0)$ on the copies in Figure 6.6a.

(ii) Now consider the translation surfaces from Figure 6.7. The translation surfaces $X(a, b)/\langle\tau_v\rangle$ and $X(a, b)/\langle\tau_h\rangle$ belong to the hyperelliptic component $\Omega\mathcal{M}_3^{\mathrm{hyp}}(2, 2)$ while the translation surface $X(a, b)/\langle\tau_h\tau_v\rangle$ belongs to the component $\Omega\mathcal{M}_3^{\mathrm{odd}}(2, 2)$.

We write again G for the Deck group of the covering $\pi\colon X(a, b) \to L(a, b)$. We will now sum up the main parts of [18, Section 3.3] and [18, Lemma 4]. The Deck group G acts on the cohomology $H^1(X(a, b), \mathbb{Q})$ by pull back. Every element $g \neq 1_G$ of G acts on $H^1(X(a, b), \mathbb{Q})$ by a linear transformation g^* which has only the eigenvalues -1 and 1. Furthermore for every $g \neq 1_G$ there is a decomposition in eigenspaces

$$H^1(X(a, b), \mathbb{Q}) = \mathrm{Eig}(g, 1) \oplus \mathrm{Eig}(g, -1).$$

By Lemma 6.4.3. This leads to a decomposition

$$H^1(X(a, b), \mathbb{Q}) = E \oplus \bigoplus_{g \neq 1_G} E_g, \tag{6.4.8.1}$$

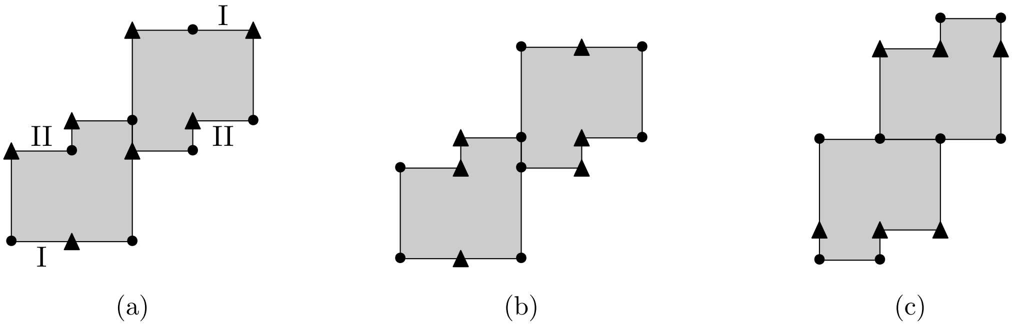


Figure 6.7.: In 6.7a we can see the translation surface $X(a,b)/\langle\tau_h\tau_v\rangle$, in 6.7b we can see $X(a,b)/\langle\tau_h\rangle$ and in 6.7c we can see the translation surface $X(a,b)/\langle\tau_v\rangle$.

such that $g^* \cdot w = w$ for every $w \in E \oplus E_g$ and $g^* \cdot w = -w$ for every $w \in E_h \oplus E_k$ with $h, k \neq g$. For every $g \neq 1_G$ the translation covering $\pi_g \colon X(a,b) \to X(a,b)/\langle g\rangle$ is obviously a Veech covering. As in the proof of Lemma 6.4.3 we obtain isomorphisms of vector spaces

$$H^1(X(a,b)/\langle g\rangle, \mathbb{Q}) \longrightarrow E \oplus E_g \quad \text{and} \quad H^1(L(a,b), \mathbb{Q}) \longrightarrow E \tag{6.4.8.2}$$

We apply Proposition 6.3.4 to the Veech cover $\pi \colon X(a,b) \to L(a,b)$ and each of the coverings $\pi_g \colon X(a,b) \to X(a,b)/\langle g\rangle$ with $g \neq 1_G$. We conclude that there is a subgroup $\Gamma_{a,b}^{(W)} \leqslant \mathrm{SL}_2(\mathbb{R})$, which is contained in the intersection of the Veech group $\mathrm{SL}(X(a,b))$ with the Veech group $\mathrm{SL}(L(a,b))$ and in the intersection of $\mathrm{SL}(X(a,b))$ with all of the groups $\mathrm{SL}(X(a,b)/\langle g\rangle)$ $(g \neq 1_G)$. Furthermore we can assume that there are subgroups $G_0 \leqslant \mathrm{Aff}^+(X(a,b))$, $G_1 \leqslant \mathrm{Aff}^+(L(a,b))$and $G_g \leqslant \mathrm{Aff}^+(X(a,b)/\langle g\rangle)$ $(i = 1,2,3)$ which map isomorphically to $\Gamma_{a,b}^{(W)}$ under the derivative and such that condition $(\star)$ is fulfilled for each of the following covers

$$\mathbb{H}/\Gamma_{a,b}^{(W)} \longrightarrow \Delta(X(a,b))/\mathrm{Aff}^+(X(a,b)) \longrightarrow \mathcal{M}_5 \tag{6.4.8.3}$$

$$\mathbb{H}/\Gamma_{a,b}^{(W)} \longrightarrow \Delta(X(a,b)/\langle g\rangle)/\mathrm{Aff}^+(X(a,b)/\langle g\rangle) \longrightarrow \mathcal{M}_3 \qquad (g \neq 1_G) \tag{6.4.8.4}$$

$$\mathbb{H}/\Gamma_{a,b}^{(W)} \longrightarrow \Delta(L(a,b))/\mathrm{Aff}^+(X(a,b)) \longrightarrow \mathcal{M}_2 \tag{6.4.8.5}$$

We write $f_W \colon \mathcal{X}_W \to \mathbb{H}/\Gamma_{a,b}^{(W)}$ for the Veech fibration associated to the covering in (6.4.8.3), respectively $f_g \colon \mathcal{Y}_g \to \mathbb{H}/\Gamma_{a,b}^{(W)}$ for the Veech fibrations associated to (6.4.8.4) and $f_G \colon \mathcal{Y}_G \to \mathbb{H}/\Gamma_{a,b}^{(W)}$ for the Veech fibration associated to the covering (6.4.8.5). By Proposition 6.3.4 we get inclusions of local systems

$$R^1(f_G)_*\mathbb{Z}_{\mathcal{Y}_G} \longrightarrow R^1(f_W)_*\mathbb{Z}_{\mathcal{X}_W} \tag{6.4.8.6}$$

and

$$R^1(f_g)_*\mathbb{Z}_{\mathcal{Y}_g} \longrightarrow R^1(f_W)_*\mathbb{Z}_{\mathcal{X}_W} \quad (g \neq 1_G), \tag{6.4.8.7}$$

which are furthermore morphisms of polarized VHS. All of these inclusions correspond to pairwise different local subsystems of $\mathbb{V}^{(W)} = R^1(f_W)_*\mathbb{Z}_{\mathcal{X}_W}$ as they correspond to pairwise different subrepresentations of $H^1(X(a,b),\mathbb{Q})$ as indicated in (6.4.8.2).

The inclusion of local systems from (6.4.8.6) and (6.4.8.7) together with Deligne's semisimplicity [22] result and the splitting theorem of Möller 4.6.8 shows that $\mathbb{V}^{(W)}$ splits over $\mathbb{Q}$ as stated below:

$$\mathbb{V}_{\mathbb{Q}}^{(W)} = \mathbb{E} \oplus \bigoplus_{g \neq 1_G} \mathbb{E}_g, \tag{6.4.8.8}$$

where $\mathbb{E} \oplus \mathbb{E}_g \cong R^1(f_g)_*\mathbb{Q}_{\mathcal{Y}_g}$ for $g \neq 1_G$ and where $\mathbb{E} \cong R^1(f_G)_*\mathbb{Q}_{\mathcal{Y}_G}$ is the local system corresponding to the action of $\Gamma_{a,b}^{(W)}$ on $H^1(L(a,b),\mathbb{Q})$. Furthermore each rank two local subsystem $\mathbb{E}_g$ $(g \neq 1_G)$ carries a polarized $\mathbb{Q}$-VHS of weight one.

6.4.9 Lemma. The non-negative Lyapunov exponents associated to the local subsystem $\mathbb{E}$ are $(1, 1/3)$ and the Lyapunov exponents of the subsystems $\mathbb{E}_g$ of $(\mathbb{V}^{(W)})_{\mathbb{Q}}$ are given by $\lambda_g = 2/3$ for the element $g = \tau_h$ and $g = \tau_v$. In the case $g = \tau_h\tau_v$ it is given by $\lambda_{\tau_h\tau_v} = 1/3$ for the local subsystem $\mathbb{E}_{\tau_h\tau_v}$.

Proof. We just mimic the proof of [18, Theorem 7] in our simplified situation. By Proposition 6.3.4 the Kontsevich–Zorich cocycle on the smooth bundle $\mathbb{E} \otimes_{\mathbb{Q}} C^{\infty}_{\mathbb{H}/\Gamma_{a,b}^{(W)}}$ associated to the local subsystem $\mathbb{E}$ of $\mathbb{V}_{\mathbb{Q}}^{(W)}$ can be identified with the Kontsevich–Zorich cocycle for $L(a,b) \in \Omega\mathcal{M}_2(2)$. In the stratum $\Omega\mathcal{M}_2(2)$ we have Lyapunov exponents $(1, 1/3)$ by Corollary 5.2.5. As we explained before we identify for every $g \neq 1_G$ the local subsystems

$$\mathbb{E} \oplus \mathbb{E}_g \cong R^1(f_g)_*\mathbb{Q}_{\mathcal{Y}_g}.$$

In the case of $g = \tau_h$ and $g = \tau_v$ the local system $R^1(f_g)_*\mathbb{Q}_{\mathcal{Y}_g}$ comes from the Veech surface $X(a,b)/\langle g\rangle \in \Omega\mathcal{M}_3^{\mathrm{hyp}}(2,2)$ while the translation surface $X(a,b)/\langle \tau_h\tau_v\rangle$ belongs to $\Omega\mathcal{M}_3^{\mathrm{odd}}(2,2)$. We konw that the sum of Lyapunov exponents is non-varying in these components by results of Chen and Möller [15]. Furthermore, the sum of the Lyapunov exponents is given by 2 in the component $\mathcal{M}_3^{\mathrm{hyp}}(2,2)$ and it is given by 5/3 in the component $\Omega\mathcal{M}_3^{\mathrm{odd}}(2,2)$. This implies:

$$1 + 1/3 + \lambda_g = 2 \quad \text{for } g = \tau_h \text{ and } g = \tau_v$$

$$1 + 1/3 + \lambda_g = 5/3 \quad \text{for } g = \tau_h\tau_v.$$

This shows $\lambda_g = 2/3$ for $g = \tau_h$ and $g = \tau_v$ as well as $\lambda_{\tau_h\tau_v} = 1/3$. $\square$

We can now argue as in Proposition 6.4.7 to obtain the following result.

6.4.10 Proposition. For all parameters $0 < a, b < 1$ as in Theorem 6.4.1 the Kontsevich–Zorich monodromy representation

$$\Gamma_{a,b}^{(W)} \longrightarrow \mathrm{Sp}(H^1_{(0)}(K(a,b),\mathbb{Q}))$$

splits into rank two summands such that the monodromy group projects for each of the summands to an arithmetic subgroup of $\mathrm{SL}_2(\mathbb{R})$.

Summary

We will now sum up some obvious insigths for the period data which come from the rank two summands in the splittings (6.4.5.4) and (6.4.8.8). The group $\Gamma_{a,b}^{(K)}$ as well as the group $\Gamma_{a,b}^{(W)}$ are isomorphic to finite index subgroups of the Veech group $\mathrm{SL}(L(a,b))$. We got a Veech fibration $f_K\colon \mathcal{X}_K \to \mathbb{H}/\Gamma_K$ for the cover $\mathbb{H}/\Gamma_K$ of the Teichmüller curve associated to K and a Veech fibration $f_W\colon \mathcal{X}_W \to \mathbb{H}/\Gamma_W$ for the cover $\mathbb{H}/\Gamma_W$ of the Teichmüller curve associated to W. We found two splittings over the rational numbers of the following local systems

$$R^1(f_K)_*\mathbb{Q}_{\mathcal{X}_K} = \mathbb{M} \oplus \bigoplus_{g \neq (0,0)} \mathbb{M}_g$$

and

$$R^1(f_W)_*\mathbb{Q}_{\mathcal{X}_W} = \mathbb{E} \oplus \bigoplus_{g \neq 1_G} \mathbb{E}_g.$$

All of the direct summands in both decompositions are of rank two and carry polarized sub-VHS. The period data of $\mathbb{E}_{\tau_h}$ and $\mathbb{E}_{\tau_g}$ are not weakly commensurable to any other direct summand of rank two since their Lyapunov exponent is $2/3$ instead of $1/3$. It would be interesting to find out more about the period data which can appear for Veech coverings of the translation surface $L(a,b)$. Maybe one can proceed as in [72, Example 5.10] to show that the period data of the summands $\mathbb{M}_g$ $(g \neq (0,0))$ or the period datum of $\mathbb{E}_{\tau_h\tau_v}$ is not induced by a Veech covering.

7. Prym origamis

In this chapter we will investigate a so called *Prym origami* in the stratum $\mathbf{\Omega\mathcal{M}_3(4)}$. In Section 7.1 we will shortly elaborate some general properties of their Kontsevich–Zorich monodromy and Kontsevich–Zorich representation. After that we will investigate a special Prym origami with thin Kontsevich–Zorich monodromy in Section 7.2. This is part of joined work with Pascal Kattler and Gabriela Weitze-Schmithüsen which we started a few weeks ago.

7.1 The Prym locus of $\mathbf{\Omega\mathcal{M}_3(4)}$

By results of Kontsevich and Zorich [78, Theorem 2] the stratum $\Omega\mathcal{M}_3(4)$ has two connected components

$$\Omega\mathcal{M}_3^{\mathrm{odd}}(4) \quad \text{and} \quad \Omega\mathcal{M}_3^{\mathrm{hyp}}(4).$$

We say that an origami $\mathcal{O} = (X, \omega) \in \Omega\mathcal{M}_3^{\mathrm{odd}}(4)$ belongs to the *Prym-locus* if there is a holomorphic involution $\sigma\colon X \to X$, which has four fixed points such that $\sigma^*\omega = -\omega$. The last condition implies that σ is an affine homeomorphism of $\mathcal{O}$ with derivative $D(\sigma) = -I_2$. By the Riemann-Hurwitz formula we see that the genus of X/σ is $g(X/\sigma) = 1$.

We want to describe the general features of the Kontsevich–Zorich monodromy representation of origamis in the Prym locus a little bit before we come to the main result of this section. The main feature is that the Kontsevich–Zorich representation splits into two rank two summands and that the Kontsevich–Zorich monodromy projects to arithmetic groups with Zariski closure isomorphic to $\mathrm{SL}_2(\mathbb{R})$ (see Proposition 7.1.3). We start our investigation with the following lemma.

7.1.1 Lemma. Let $\mathcal{O} = (X, \omega)$ be an origami in the Prym-locus of $\Omega\mathcal{M}_3^{\mathrm{odd}}(4)$ with associated involution $\sigma\colon X \to X$. Consider the action of σ on the non-tautological part $H_1^{(0)}(X, \mathbb{Q})$ by push forward. Then $H_1^{(0)}(X, \mathbb{Q})$ decomposes into eigenspaces H_+ and H_- for the eigenvalues 1 and -1. Both eigenspaces H_+ and H_- are two-dimensional and orthogonal to each other with respect to the intersection form Q on $H_1(X, \mathbb{Q})$. Furthermore every element in the affine group $\mathrm{Aff}^+(\mathcal{O})$ respects the splitting

$$H_1^{(0)}(X, \mathbb{Q}) = H_+ \oplus H_-.$$

Proof. Since $\mathcal{O} = (X, \omega)$ is an origami, the tautological part $H_1^{st}(X, \mathbb{Q})$ of $H_1(X, \mathbb{Q})$ is defined over $\mathbb{Q}$. Furthermore the involution $\sigma\colon X \to X$ has by definition the property $\sigma^*(\omega) = -\omega$. This implies that $H_1^{st}(X, \mathbb{Q})$ is a subspace of $\mathrm{Eig}(\sigma_*, -1)$ by duality (c.f. Section 4.6.10).

The genus of X is $g(X) = 3$ and the genus of X/σ is $g(X/\sigma) = 1$. By Lemma 6.4.3 and duality $H^1(X, \mathbb{Q}) \to H_1(X, \mathbb{Q})$ the homology $H_1(X, \mathbb{Q})$ splits into eigenspaces

$$H_1(X, \mathbb{Q}) = \mathrm{Eig}(\sigma_*, 1) \oplus \mathrm{Eig}(\sigma_*, -1), \tag{7.1.1.1}$$

of σ_*, which are orthogonal to each other with respect to the intersection form. Furthermore, $\mathrm{Eig}(\sigma_*, 1)$ has dimension two and the eigenspace $\mathrm{Eig}(\sigma_*, -1)$ has dimension four. This shows that the non-tautological part $H_1^{(0)}(X, \mathbb{Q})$ splits into eigenspaces H_+ and H_- of dimension two. $\square$

Let again $\mathcal{O} = (X, \omega)$ be an origami in the Prym-locus of $\Omega\mathcal{M}_3^{\mathrm{odd}}$. Let $\epsilon \in \{+, -\}$ and write U_ϵ for the lattice $H_1(X, \mathbb{Z}) \cap H_\epsilon$. For every $\epsilon \in \{+, -\}$, the lattice U_ϵ is invariant under the action of $\mathrm{Aff}^+(X, \omega)$ and U_ϵ is naturally polarized by the restriction of the intersection form on $H_1(X, \mathbb{Z})$.Thus the action of $\mathrm{Aff}^+(X, \omega)$ on U_ϵ induces a representation

$$\rho_\epsilon\colon \mathrm{Aff}^+(X, \omega) \to \mathrm{Sp}(U_\epsilon). \tag{7.1.1.2}$$

We choose a subgroup $\Gamma \leqslant \mathrm{Aff}^+(X, \omega)$ which has condition $(\star)$. Let $f\colon \mathcal{X} \to C = \Delta(X, \omega)/\Gamma$ be the associated Veech fibration. For every $\epsilon \in \{+, -\}$, we can associate a local system $\mathbb{U}_\epsilon$ to the representation $\rho_\epsilon|_\Gamma$. We denote $\mathbb{V}_\mathbb{Z} = R^1 f_* \mathbb{Z}_\mathcal{X}$ as before. Lemma 7.1.1 induces a splitting of local systems (c.f. Theorem 4.6.8)

$$\mathbb{V}_\mathbb{Q} = \mathbb{W} \oplus \mathbb{E}_+ \oplus \mathbb{E}_-, \tag{7.1.1.3}$$

where $\mathbb{E}_\epsilon \cong \mathbb{U}_\epsilon \otimes_\mathbb{Z} \mathbb{Q}$ by duality for each $\epsilon \in \{+, -\}$. On $\mathbb{V}_\mathbb{Z}$ we have the polarized VHS $(\mathbb{V}_\mathbb{Z}, f_*\Omega^1_{\mathcal{X}/C}, Q)$ of weight one. Thus from Deligne's semisimplicity result and the splitting of the local system $\mathbb{V}_\mathbb{Z}$ in (7.1.1.3), we obtain a polarized VHS of weight one on $\mathbb{U}_\epsilon$ for both of the $\epsilon \in \{+, -\}$ or equivalently a holomorphic mapping

$$\phi_\epsilon\colon C \to \mathbb{H}/\left(\rho_\epsilon|_\Gamma(U_\epsilon)\right),$$

which is locally liftable and has Griffith's infinitesimal period relation.

We denote by $\mathcal{U}_\epsilon^{1,0}$ the $(1,0)$-part of the Hodge filtration of the Deligne extension of $\mathbb{U}_\epsilon \otimes_\mathbb{Z} \mathcal{O}_C$ to the smooth compactification $\overline{C}$ of C. Write $\deg(\mathcal{U}_\epsilon^{1,0})$ for the degree of the line bundle $\mathcal{U}_\epsilon^{1,0}$. The next lemma together with 5.2.4 implies that $\deg(\mathcal{U}_\epsilon^{1,0}) \neq 0$ for every $\epsilon \in \{+, -\}$ or equivalently that ϕ_ϵ is non-constant and hence surjective. This lemma can also be found in Section 6.6 of [86].

7.1.2 Lemma. The Kontsevich–Zorich cocycle on $\mathbb{V}_\mathbb{Z} \otimes_\mathbb{Z} C^\infty_C$ restricts to cocycles on each of the smooth subbundles $\mathbb{U}_+ \otimes_\mathbb{Z} C^\infty_C$ and $\mathbb{U}_- \otimes_\mathbb{Z} C^\infty_C$ with positive Lyapunov exponents $\lambda_+ = 2/5$ and $\lambda_- = 1/5$.

Proof. The involution $\sigma\colon X \to X$ has four fixed points and induces a covering $\pi\colon X \to X/\sigma$ with X/σ an elliptic curve. The holomorphic quadratic differential ω^2 descends to a quadratic differential q on X/σ, i.e. $\pi^*q = \omega^2$. The four ramification points of π correspond to zeroes of odd orders or simple poles of q. By (2.5) in [30] we know that the single zero of order four of ω gives rise to a zero of order three of q. Since X/σ has genus one, we conclude that q has three simple poles. By [30, Theorem 1, Theorem 2] and [15, Section 5.1] we conclude

$$1 + \lambda_- + \lambda_+ = 8/5 \tag{7.1.2.1}$$

and

$$1 + \lambda_- - \lambda_+ = 1/4 \cdot (1 + 1 + 1 + 1/5) = 4/5 \tag{7.1.2.2}$$

This shows $\lambda_+ = 2/5$ and $\lambda_- = 1/5$. □

From Proposition 6.1.1, Lemma 6.1.4 and Borel Density Theorem (c.f. Remark 6.1.2) we obtain the following proposition which was already pointed out in Appendix A of [8], which is due to Etienne Bonnafoux and Carlos Matheus.

7.1.3 Proposition. For each $\epsilon \in \{+,-\}$, the image $\mathrm{im}(\rho_\epsilon)$ of the representation

$$\rho_\epsilon\colon \mathrm{Aff}^+(X,\omega) \to \mathrm{Sp}(U_\epsilon).$$

from (7.1.1.2) has Zariski closure isomorphic to $\mathrm{SL}_2(\mathbb{R})$ and it is an arithmetic group. In other words the monodromy group of the Kontsevich–Zorich representation projects to Zariski dense and arithmetic groups in $\mathrm{SL}_2(\mathbb{R})$.

7.2 A Prym-origami with thin Kontsevich–Zorich monodromy

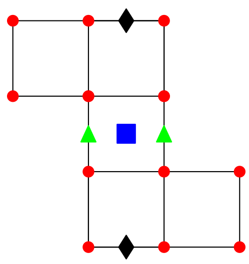


Figure 7.1.: The origami $\mathcal{O}$ with its four fixed points under the involution.

We want to consider the origami $\mathcal{O} \in \Omega\mathcal{M}_3^{\mathrm{odd}}(4)$ from Figure 7.1 associated to the pair of permutations

$$\sigma_h = (1,2)(4,5) \quad \text{and} \quad \sigma_v = (2,3,4).$$

It has degree five and lies in the Prym-locus of $\Omega\mathcal{M}_3^{\mathrm{odd}}(4)$. Etienne Bonnafoux and Carlos Matheus computed the Kontsevich-Zorich monodromy of $\mathcal{O}$ in Appendix A of [8]. We will now repeat their result. The origami $\mathcal{O}$ has three horizontal cylinders with waist curves σ_1, σ_0 and σ_2 with holonomies $(2,0)$, $(1,0)$ and $(2,0)$. Furthermore, $\mathcal{O}$ has three vertical cylinders with waist curves ζ_1, ζ_0 and ζ_2 with holonomy vectors $(0,1)$, $(0,3)$ and $(0,1)$. A basis of the non-tautological part $H_1^{(0)}(\mathcal{O},\mathbb{Q})$ is given by $\Sigma_i = \sigma_i - 2\sigma_0$ $(i = 1,2)$ and $Z_i = 3\zeta_i - \zeta_0$ $(i = 1,2)$. Again the involution splits the non-tautological part in eigenspaces $H_1^{(0)}(\mathcal{O},\mathbb{Q}) = H_1^+ \oplus H_1^-$. The eigenspace H_1^+ of the eigenvalue 1 is generated by the elements $\Sigma^+ = \Sigma_1 - \Sigma_2$ and $Z^+ = Z_1 - Z_2$. The eigenspace H_1^- of the eigenvalue -1 is generated by $\Sigma^- = \Sigma_1 + \Sigma_2$ and $Z^- = Z_1 + Z_2$. The Veech group $\mathrm{SL}(\mathcal{O})$ of $\mathcal{O}$ is generated by the following matrices in $\mathrm{SL}_2(\mathbb{Z})$:

$$m_1 = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, \quad m_2 = \begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix}$$

and

$$m_3 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \quad m_4 = \begin{pmatrix} 6 & -5 \\ 5 & -4 \end{pmatrix}$$

The elements m_1 and m_2 act on the basis $(\Sigma^+, Z^+, \Sigma^-, Z^-)$ of eigenvectors of $H_1^{(0)}(\mathcal{O},\mathbb{Q})$ by the matrices

$$A = \begin{pmatrix} 1 & 3 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

The element $m_3 \in \mathrm{SL}(\mathcal{O})$ comes from the involution on $\mathcal{O}$ and m_4 acts by a Dehn multi-twist in direction $(1,1)$ on $\mathcal{O}$ and thus acts trivially. In the next section we will study the Kontsevich–Zorich monodromy of $\mathcal{O}$.

Thin Kontsevich–Zorich monodromy. In this subsection we finally show parts of the work which Pascal Kattler, Gabriela Weitze Schmithüsen and myself started quite recently. We denote in the following

$$T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad L = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}.$$

We write A, B for the elements $A = (T^3, T)$ and $B = (L, L)$ of $\mathrm{SL}_2(\mathbb{Z}) \times \mathrm{SL}_2(\mathbb{Z})$ and we write $U = \langle A, B \rangle$ for the subgroup generated by A and B. Let

$$P \colon \mathrm{SL}_2(\mathbb{Z}) \longrightarrow \mathrm{PSL}_2(\mathbb{Z})$$

be the projection map and write $\overline{T} = P(T)$ and $\overline{L} = P(L)$ for the images of T and L under P. In the following we write PU for the subgroup of $\mathrm{PSL}_2(\mathbb{Z}) \times \mathrm{PSL}_2(\mathbb{Z})$ generated by the elements $\overline{A} := (\overline{T}^3, \overline{T})$ and $\overline{B} := (\overline{L}, \overline{L})$. For $i = 1, 2$ let $\mathrm{pr}_i\colon \mathrm{PSL}_2(\mathbb{Z}) \times \mathrm{PSL}_2(\mathbb{Z}) \to \mathrm{PSL}_2(\mathbb{Z})$ be the projection onto the first, respectively second component. The image $\mathrm{pr}_1(PU) \leqslant \mathrm{PSL}_2(\mathbb{Z})$ of PU under the projection onto the first component is given by the subgroup of $\mathrm{PSL}_2(\mathbb{Z})$ generated by the elements $\overline{T}^3$ and $\overline{L}$ and we have the identity

$$P\Gamma_1(3) = \langle \overline{T}^3, \overline{L} \rangle = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{PSL}_2(\mathbb{Z}) \mid a \equiv \pm 1 \equiv d \pmod 3,\ b \equiv 0 \pmod 3 \right\}.$$

Consider the following commutative diagram:

$$\begin{tikzcd} PSL_2(\mathbb{Z}) \times \{1\} \arrow[r, hook] & PSL_2(\mathbb{Z}) \times PSL_2(\mathbb{Z}) \arrow[r, "\mathrm{pr}_2"] & PSL_2(\mathbb{Z}) \\ PU \cap (PSL_2(\mathbb{Z}) \times \{1\}) \arrow[r, hook] \arrow[u, hook] & PU \arrow[r, "\mathrm{pr}_2"] \arrow[u, hook] & PSL_2(\mathbb{Z}) \arrow[u, "\mathrm{id}"] \end{tikzcd}$$

We want to show in this section that the group PU has infinite index in $\mathrm{PSL}_2(\mathbb{Z}) \times \mathrm{PSL}_2(\mathbb{Z})$. The projection of PU onto the second component is the full group $\mathrm{PSL}_2(\mathbb{Z})$. Hence the two rows of the commutative diagram above are exact. We conclude that it suffices to show that the group

$$H := PU \cap (PSL_2(\mathbb{Z}) \times \{1\}) = PU \cap \ker(\mathrm{pr}_2)$$

has infinite index in $PSL_2(\mathbb{Z}) \times \{1\}$, respectively that $G = \mathrm{pr}_1(H)$ has infinite index in $PSL_2(\mathbb{Z})$. The group $P\Gamma_1(3)$ has finite index in $\mathrm{PSL}_2(\mathbb{Z})$ and we know from above that G is a subgroup of $P\Gamma_1(3)$. Thus if we can show that $G = \mathrm{pr}_1(H) \leqslant P\Gamma_1(3)$ has infinite index in $P\Gamma_1(3)$, then we are done.

A group presentation of $\mathrm{PSL}_2(\mathbb{Z})$ is given by

$$G_{\mathrm{PSL}_2(\mathbb{Z})} = \langle x, y \mid (y^2 x^{-1})^2,\ (x\, y^{-1})^3 \rangle. \tag{7.2.0.1}$$

The group homomorphism $G_{\mathrm{PSL}_2(\mathbb{Z})} \to \mathrm{PSL}_2(\mathbb{Z})$ induced by $x \mapsto \overline{T}$, $y \mapsto \overline{L}$ is an isomorphism of groups. Consider the element $R \in P\Gamma_1(3)$ given by

$$R = (\overline{L}^2 \overline{T}^{-3})^2 = \pm \begin{pmatrix} -5 & 12 \\ -8 & 19 \end{pmatrix}. \tag{7.2.0.2}$$

7.2.1 Lemma. The group G is given by the normal hull of the element R in the group $\Gamma_1(3)$, i.e $G = \ll R \gg_{P\Gamma_1(3)}$.

Proof. Let $F_2 = F_{x,y}$ be the free group in two generators x and y. Consider the group homomorphism

$$\varphi\colon F_{x,y} \longrightarrow \mathrm{PSL}_2(\mathbb{Z}) \times \mathrm{PSL}_2(\mathbb{Z}), \quad \varphi(x) = \overline{A},\ \varphi(y) = \overline{B}.$$

By definition of the group homomorphism φ, we have the identity $\mathrm{im}(\varphi) = PU$. We get the relation $H = \mathrm{im}(\varphi) \cap \ker(\mathrm{pr}_2)$. Furthermore, the identity

$$\mathrm{im}(\varphi) \cap \ker(\mathrm{pr}_2) = \varphi(\ker(\mathrm{pr}_2 \circ \varphi)) \tag{7.2.1.1}$$

holds. Consider the elements $S_1 = (y^2x^{-1})^2$ and $S_2 = (x\,y^{-1})^3$ of $F_{x,y}$. If we recall the group presentation of $\mathrm{PSL}_2(\mathbb{Z})$ in (7.2.0.1), then it is clear that the kernel of the concatenation $\mathrm{pr}_2 \circ \varphi$ is given by the normall hull $\ll S_1, S_2 \gg_{F_{x,y}}$ in $F_{x,y}$ generated as a normal subgroup by S_1 and S_2. This shows

$$H = \varphi(\ker(\mathrm{pr}_2 \circ \varphi)) = \ll \varphi(S_1), \varphi(S_2) \gg_{PU} .$$

We have the identities

$$\mathrm{pr}_1 \circ \varphi(S_2) = (\overline{T}^3\, \overline{L}^{-1})^3 = I_2 \quad \text{and} \quad \mathrm{pr}_1 \circ \varphi(S_1) = (\overline{L}^2 \overline{T}^{-3})^2 = R.$$

We conclude that G is given by the normall hull of the element R in $P\Gamma_1(3)$ since $\mathrm{pr}_1(PU) = P\Gamma_1(3)$, i.e. $G = \ll R \gg_{P\Gamma_1(3)}$. □

The element $R \in P\Gamma_1(3)$ from (7.2.0.2) is obviously an element of the congruence subgroup $P\Gamma(2) = \langle \overline{T}^2, \overline{L}^2 \rangle$ of $\mathrm{PSL}_2(\mathbb{Z})$ as well. From the description of G from Lemma 7.2.1 we conclude that $G \leqslant \Gamma(2)$ hence $G \leqslant P\Gamma_1(3) \cap P\Gamma(2)$. We write in the following ISC for the intersection $P\Gamma_1(3) \cap P\Gamma(2)$.

7.2.2 Lemma. Let ISC $= P\Gamma_1(3) \cap P\Gamma(2)$ as above. The following holds:

(i) The group ISC has index four in $P\Gamma(2)$. Furthermore, the group ISC is a free group in the set of generators $s_1, \dots, s_5 \in P\Gamma(2)$, where

$$\begin{aligned}
s_1 &= \overline{L}^2 = \pm \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}, \\
s_2 &= \overline{T}^2 \cdot \overline{L}^2 \cdot \overline{T}^{-4} = \pm \begin{pmatrix} 5 & -18 \\ 2 & -7 \end{pmatrix} \\
s_3 &= \overline{T}^4 \cdot \overline{L}^2 \cdot \overline{T}^2 \cdot \overline{L}^{-2} \cdot \overline{T}^{-4} = \pm \begin{pmatrix} -35 & 162 \\ -8 & 37 \end{pmatrix} \\
s_4 &= \overline{T}^4 \cdot \overline{L}^4 \cdot \overline{T}^{-2} = \pm \begin{pmatrix} 17 & -30 \\ 4 & -7 \end{pmatrix}, \\
s_5 &= \overline{T}^6 = \pm \begin{pmatrix} 1 & 6 \\ 0 & 1 \end{pmatrix}.
\end{aligned}$$

(ii) The group ISC has index six in $P\Gamma_1(3)$ and a set of right cosets is given by the elements $A_1, \dots, A_6 \in P\Gamma_1(3)$, where

$$\begin{aligned}
&A_1 = \pm I_2, && A_2 = \overline{T}^3, && A_3 = \overline{L} \\
&A_4 = \overline{T}^3 \cdot \overline{L}, && A_5 = \overline{L} \cdot \overline{T}^3, && A_6 = \overline{T^3} \cdot \overline{L} \cdot \overline{T^3}.
\end{aligned}$$

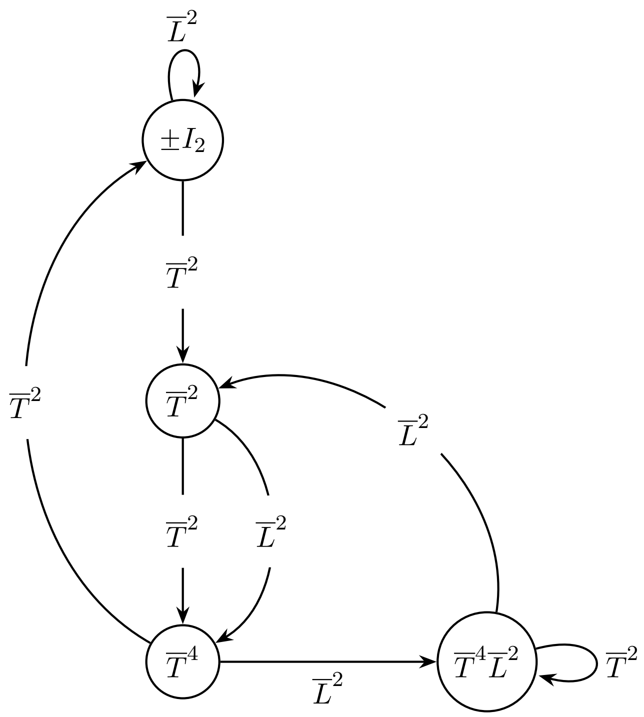


Figure 7.2.: The right coset graph of the group ISC in $P\Gamma(2)$.

Proof. The right coset graph of ISC in $P\Gamma(2)$ is given by the graph in Figure 7.2. A standard argument with the Ping-Pong Lemma shows that $P\Gamma(2)$ can be written as the free product $P\Gamma(2) = \langle \overline{T}^2 \rangle * \langle \overline{L}^2 \rangle$ and hence $P\Gamma(2)$ is a free group in the two generators $\overline{T}^2$ and $\overline{L}^2$. Thus ISC is a free group in five generators by the rank-index-formula. The fundamental group with base point $\pm I_2$ of the right coset graph in Figure 7.2 is naturally isomorphic to the group ISC. Thus we get $S = \{s_1, \ldots, s_5\} \leqslant P\Gamma(2)$ as above as a set of generators of ISC. The right coset graph of ISC in $P\Gamma_1(3)$ is given by the graph in Figure 7.3. $\square$

With Lemma 7.2.1 and Lemma 7.2.2 we can now continue our investigations on the group G. We know that the elements $A_1, \ldots, A_6$ as in Lemma 7.2.2 build a set of right cosets of ISC in $P\Gamma_1(3)$ and $G = \ll R \gg_{P\Gamma_1(3)}$. All the elements $R_i := A_i \, R \, A_i^{-1}$ $(i = 1, \ldots, 6)$ are elements of ISC. All together we conclude that the group $G \leqslant$ ISC is given by the normall hull

$$G = \ll R \gg_{P\Gamma_1(3)} = \ll R_i = A_i \, R \, A_i^{-1} \mid i = 1, \ldots, 6 \gg_{\mathrm{ISC}},$$

In terms of the generators $s_1, \ldots, s_5$ of ISC from Lemma 7.2.2 we have the following

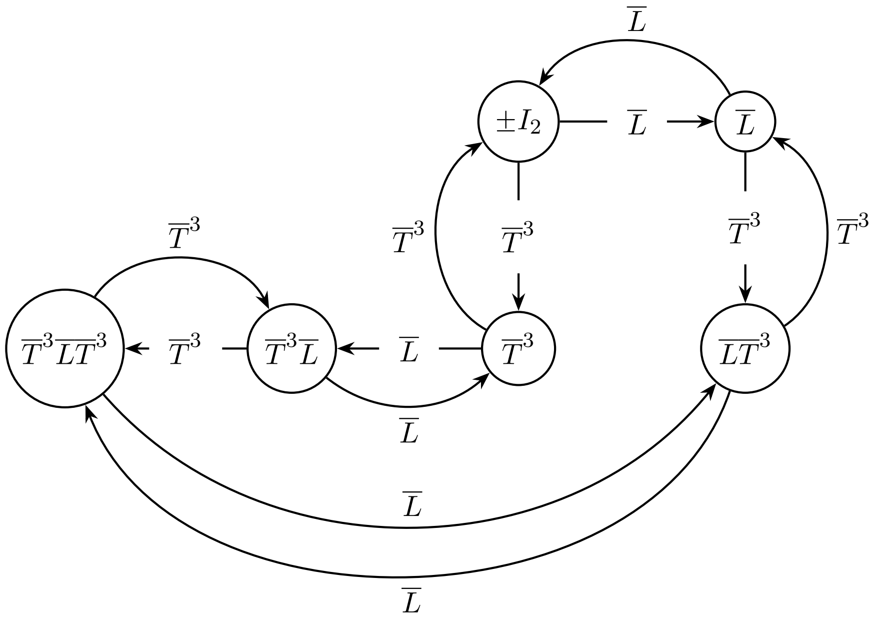


Figure 7.3.: The right coset graph of the group ISC in $P\Gamma_1(3)$

identities for the elements $R_i = A_i\, R\, A_i^{-1}$ $(i = 1, \dots, 6)$ from above:

$$\begin{aligned}
R_1 &= s_1 \cdot s_5^{-1} \cdot s_2^{-1} = \pm \begin{pmatrix} -5 & 12 \\ -8 & 19 \end{pmatrix}, \\
R_2 &= s_2^{-1} \cdot s_1 \cdot s_5^{-1} = \pm \begin{pmatrix} -29 & 156 \\ -8 & 43 \end{pmatrix}, \\
R_3 &= s_1 \cdot s_4^{-1} \cdot s_5 \cdot s_1^{-1} = \pm \begin{pmatrix} -17 & 12 \\ -44 & 31 \end{pmatrix}, \\
R_4 &= s_2^{-1} \cdot s_3^{-1} \cdot s_4 \cdot s_2 = \pm \begin{pmatrix} -149 & 552 \\ -44 & 163 \end{pmatrix} \\
R_5 &= s_1 \cdot s_5^{-1} \cdot s_3^{-1} \cdot s_5 \cdot s_4^{-1} \cdot s_3 \cdot s_5 \cdot s_1^{-1} = \pm \begin{pmatrix} -185 & 156 \\ -236 & 199 \end{pmatrix} \\
R_6 &= s_2^{-1} \cdot s_4^{-1} \cdot s_3^{-1} \cdot s_4^2 \cdot s_2 = \pm \begin{pmatrix} -893 & 3432 \\ -236 & 907 \end{pmatrix}
\end{aligned}$$

We consider the group homomorphism of ISC onto its abelization

$$\phi_{\mathrm{ab}} \colon \mathrm{ISC} \to \mathrm{ISC}/[\mathrm{ISC}, \mathrm{ISC}].$$

It is easy to see that the image $\phi_{\mathrm{ab}}(G)$ is a subgroup of infinite index in $\mathrm{ISC}/[\mathrm{ISC}, \mathrm{ISC}]$ and this shows that G has also infinite index in ISC. Furthermore an easy computation in GAP shows that the matrices $\pm I_2 \cdot R_i$ $(i = 1, \dots, 6)$ project surjectively onto $\mathrm{SL}_2(\mathbb{Z}/5\mathbb{Z})$. By a theorem of Weigel ([120],[104, Theorem 2.2]) we conclude that the Zariski closure of

G is $\mathrm{PSL}_2(\mathbb{R})$ respectively that the Zariski closure of $PU \cap \mathrm{PSL}_2(\mathbb{R}) \times \{\pm I_2\}$ is isomorphic to $\mathrm{SL}_2(\mathbb{R})$. Since $\mathrm{pr}_2(PU) = \mathrm{PSL}_2(\mathbb{Z})$, this shows that the subgroup $PU \leqslant \mathrm{PSL}_2(\mathbb{R}) \times \mathrm{PSL}_2(\mathbb{R})$ is Zariski dense. We will sum up the main result of this section in the next Theorem.

7.2.3 Theorem. The Kontsevich–Zorich monodromy of the prym-origami $\mathcal{O} \in \Omega\mathcal{M}_3^{\mathrm{odd}}(4)$ associated to the permutations $\sigma_h = (1,2)(4,5)$ and $\sigma_v = (2,3,4)$ is thin.

8. Arithmeticity for a finite family of stairs origamis in genus four

In this chapter we integrate the content of [8, Section 5]. We will show for a finite family in the stratum $\Omega\mathcal{M}_4(6)$, that the monodromy group of the Kontsevich–Zorich monodromy representation is an arithmetic group. In contrast to the results from the previous chapter which are based on Hodge theory, we will hereby use the theory of algebraic groups. We laid the foundation of these techniques in Chapter 1. The main difficulty for the sequences of origamis in this chapter and the next chapter is to prove that the monodromy group of the Kontsevich–Zorich representation is Zariski-dense. In this chapter the proof is computer-aided (c.f. Theorem 8.3.1) and hence results in a finite family. In chapter 9 we obtain Zariski-density by a Galois-theoretical approach. Nevertheless we need the explicit description of several elements in the Kontsevich–Zorich monodromy in both cases which we develop in Section 8.2. In Section 8.3 we prove the density of the Kontsevich-Zorich monodromy of the origamis considered in Theorem 8.1.3. Finally, in Section 8.4 we show arithmeticity with the help of 1.6.1.

8.1 Stairs origamis in genus four

Before we start with the construction of our families, we want to explain what we mean by *length or combinatorial length* of a curve on an origami.

8.1.1 Definition. For an origami $\pi\colon \mathcal{O} \to \mathbb{R}^2/\mathbb{Z}^2$ and a closed curve $\gamma\colon [0,1] \to \mathcal{O}$ we mean by *(combinatorial) length* the number of $t \in (0,1]$ such that $\pi(\gamma(t)) = \pi(\gamma(0))$.

First of all we want to introduce the members of the finite family which we want to consider in this section.

8.1.2 Definition. Let $N \geqslant 4$ and $M = 4+2m$ with $m \geqslant 0$. Let $\mathcal{O}^{(4)}_{N,M}$ be the origami of degree $N+M+2$ associated to the pair of permutations $h, v \in \mathrm{Sym}(\{1,\dots,N+M+2\})$, where

$$\begin{aligned} h =& (1,2,3\dots,N)(N+1,N+2,N+3)(N+4,N+5)(N+6)\dots(N+M+2)\\ v =& (1,\ N+1,\ N+4,\ N+6,\dots,N+M+2)(2,\ N+2,\ N+5)(3,\ N+3)(4)\dots(N). \end{aligned}$$

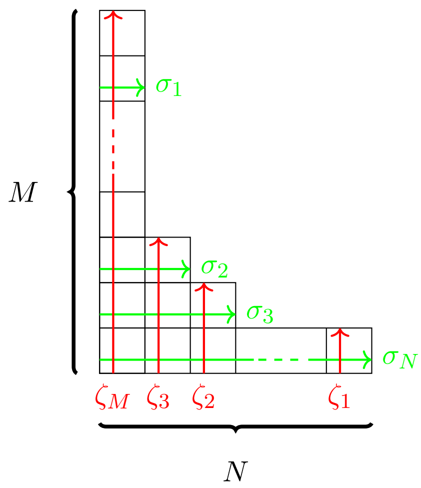


Figure 8.1.: Origami $\mathcal{O}_{N,M}^{(4)}$ with horizontal waist curves $\sigma_1, \sigma_2, \sigma_3, \sigma_N$ and vertical waist curves $\zeta_1, \zeta_2, \zeta_3, \zeta_M$

The origamis from Definition 8.1.2 are visualized in Figure 8.1. The main result of this chapter is the following. This is Theorem 2 respectively Theorem 22 in our article [8]. Note that in Chapter 9 we will obtain an infinite family with arithmetic Kontsevich–Zorich monodromy among the origamis $\mathcal{O}_{N,M}^{(4)}$ but we have more constraints on the parameters $N, M \in \mathbb{N}$.

8.1.3 Theorem. For all $N = 3m + 4$ with $4 \leqslant N \leqslant 50$ and $M = 2m + 4$ with $m \in \{0, \ldots, 50\}$ the Kontsevich–Zorich monodromy of $\mathcal{O}_{N,M}^{(4)}$ is arithmetic.

8.2 Our favorite Dehn twists

In this subsection we compute explicit elements in the Kontsevich–Zorich monodromy. We use cylinder decompositions of the origami $\mathcal{O}_{N,M}^{(4)}$ in several directions to construct Dehn multitwists. Rember that we defined $H_1^{(0)}(\mathcal{O}_{N,M}^{(4)}, \mathbb{Z})$ as the kernel of the push-forward π_* induced by the covering $\pi\colon \mathcal{O}_{N,M}^{(4)} \to \mathbb{T}^2$ of the torus (c.f Subsection 4.6.10). We will compute the transformation matrices of the Dehn-multitwist with respect to a basis $B^{(0)}$ of $H_1^{(0)}(\mathcal{O}_{N,M}^{(4)}, \mathbb{Z})$. This will be the cornerstone for the arguments in Section 8.3 and Section 8.4.

The waist curves $\sigma_1, \sigma_2, \sigma_3, \sigma_N$ of the four maximal horizontal cylinders and the four waist curves $\zeta_1, \zeta_2, \zeta_3, \zeta_M$ of the four maximal vertical cylinders form a basis B of $H_1(\mathcal{O}_{N,M}^{(4)}, \mathbb{Q})$

(see Figure 8.1). We have the following holonomy vectors for these waist curves:

$$\begin{aligned}
&\mathrm{hol}(\sigma_1) = (1,0), \quad \mathrm{hol}(\sigma_2) = (2,0), \quad \mathrm{hol}(\sigma_3) = (3,0), \quad \mathrm{hol}(\sigma_N) = (N,0)\\
&\mathrm{hol}(\zeta_1) = (0,1), \quad \mathrm{hol}(\zeta_2) = (0,2), \quad \mathrm{hol}(\zeta_3) = (0,3), \quad \mathrm{hol}(\zeta_M) = (0,M).
\end{aligned}$$

We have that $B^{(0)} = \{\Sigma_1, \Sigma_2, \Sigma_N, Z_1, Z_2, Z_M\}$ is a basis of the non-tautological part $H_1^{(0)}(\mathcal{O}_{N,M}^{(4)}, \mathbb{Q})$, where

$$\begin{aligned}
\Sigma_1 &= \sigma_2 - 2\sigma_1, \quad \Sigma_2 = \sigma_3 - 3\sigma_1, \quad \Sigma_N = \sigma_N - N\sigma_1,\\
Z_1 &= \zeta_2 - 2\zeta_1, \quad Z_2 = \zeta_3 - 3\zeta_1, \quad Z_M = \zeta_M - M\zeta_1.
\end{aligned}$$

That $B^{(0)}$ is indeed a basis follows for example from the fact that the fundamental matrix $\tilde{G}$ of the intersection form Ω with respect to $B^{(0)}$ (see 8.2) is regular:

$$\tilde{G} = \begin{pmatrix}
0 & 0 & 0 & 0 & 1 & -1\\
0 & 0 & 0 & 1 & 1 & -2\\
0 & 0 & 0 & -1 & -2 & -N-M+1\\
0 & -1 & 1 & 0 & 0 & 0\\
-1 & -1 & 2 & 0 & 0 & 0\\
1 & 2 & N+M-1 & 0 & 0 & 0
\end{pmatrix}$$

We will use the Dehn multitwists along the waist curves of the cylinders in the directions $(1,1)$, $(1,-1)$, $(1,2)$, $(1,-2)$ as well as the horizontal and vertical direction. In the following we compute their actions on $H_1^{(0)}(\mathcal{O}_{N,M}^{(4)}, \mathbb{Q})$.

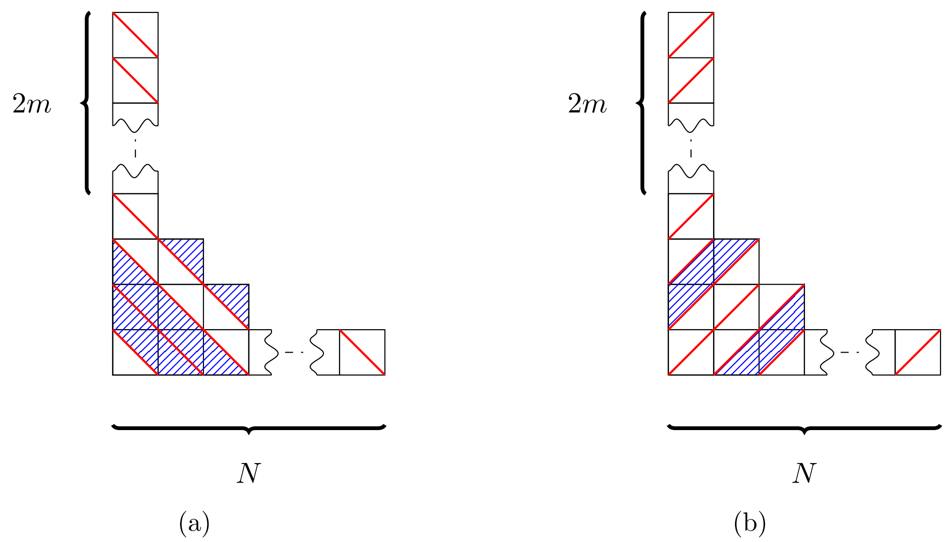


Figure 8.2.: Cylinder decomposition in direction $(1,-1)$ and $(1,1)$ of the origami $\mathcal{O}_{N,M}^{(4)}$. Here χ_1 and δ_1 are the waist curves of the blue cylinders.

For the direction $(1,1)$ we have a decomposition into two maximal cylinders of equal height with waist curves δ_1 of combinatorial length 3 and δ_2 of combinatorial length

$N+M-1$ (see Figure 8.2b). In direction $(1,-1)$ we also have a decomposition into two maximal cylinders (see Figure 8.2a) again of equal height. We denote the waist curve of combinatorial length 5 by χ_1 and the waist curve of combinatorial length $N+M-3$ by χ_2. The associated Dehn multitwists along the waist curves of these maximal cylinders then act as linear maps D_δ and D_χ on $H_1^{(0)}(\mathcal{O}_{N,M}^{(4)}, \mathbb{Q})$ given by (c.f. Proposition 3.3.4 and Formula (2.4) [8]):

$$D_\delta\colon v \longmapsto v + (N+M-1)\ \Omega(v,\delta_1)\,\delta_1 + 3\ \Omega(v,\delta_2)\,\delta_2 \tag{8.2.0.1}$$

$$D_\chi\colon v \longmapsto v + (N+M-3)\ \Omega(v,\chi_1)\,\chi_1 + 5\ \Omega(v,\chi_2)\,\chi_2 \tag{8.2.0.2}$$

We will now count the intersection points of the curves $\sigma_i, \sigma_N, \zeta_i, \zeta_M$ $(i=1,2,3)$ with the waist curves of the cylinders to finally compute matrix representations for D_δ and D_χ:

Ω	δ_1	δ_2	χ_1	χ_2
σ_1	0	1	0	-1
σ_2	1	1	-1	-1
σ_3	1	2	-2	-1
σ_N	1	$N-1$	-2	$-(N-2)$
ζ_1	0	-1	0	-1
ζ_2	-1	-1	-1	-1
ζ_3	-1	-2	-2	-1
ζ_M	-1	$-(M-1)$	-2	$-(M-2)$

Table 8.1.: Number of intersection points between the waist curves δ_1, , δ_2 of the cylinders in direction $(1,1)$ and the waist curves χ_1, χ_2 of the cylinders in direction $(1,-1)$ with the elements $\sigma_i, \sigma_N, \zeta_i, \zeta_M$ $(i=1,2,3)$.

As next step we compute the matrix representations $M_\delta^{(0)}$ and $M_\chi^{(0)}$ of the linear maps $D_\delta, D_\chi \in \mathrm{Sp}_\Omega(H_1^{(0)}(\mathcal{O}_{N,M}^{(4)}, \mathbb{Q}))$ with respect to $B^{(0)}$. The following two elements of the homology will be essential for this:

$$\Delta = (N+M-1)\cdot\delta_1 - 3\cdot\delta_2 \in H_1^{(0)}(\mathcal{O}_{N,M}^{(4)}, \mathbb{Z})$$

$$X = (N+M-3)\cdot\chi_1 - 5\cdot\chi_2 \in H_1^{(0)}(\mathcal{O}_{N,M}^{(4)}, \mathbb{Z})$$

They both have zero holonomy and are thus elements of the non-tautological part $H_1^{(0)}(\mathcal{O}_{N,M}^{(4)}, \mathbb{Z})$. The intersection numbers obtained above allow us by a simple but longish computation to determine the coefficients of Δ and X with respect to the basis B of $H_1(\mathcal{O}_{N,M}^{(4)}, \mathbb{Q})$. We then convert them to coefficients with respect to the basis $B^{(0)}$ of $H_1^{(0)}(\mathcal{O}_{N,M}^{(4)}, \mathbb{Q})$ and obtain:

$$\begin{aligned}
\Delta = {} & -3\,\Sigma_1 + (N+M-1)\,\Sigma_2 - 3\,\Sigma_N \\
& -3\,Z_1 + (N+M-1)\,Z_2 - 3\,Z_M, \\
X = {} & \ \ (N+M-3)\,\Sigma_1 + (N+M-3)\,\Sigma_2 - 5\,\Sigma_N \\
& -(N+M-3)\,Z_1 - (N+M-3)\,Z_2 + 5\,Z_M
\end{aligned} \tag{8.2.0.3}$$

Now, using (8.2.0.1) and (8.2.0.2) we obtain:

$$\begin{aligned}
D_\delta(\Sigma_1) &= \Sigma_1 + \Delta, & D_\delta(\Sigma_2) &= \Sigma_2 + \Delta, & D_\delta(\Sigma_N) &= \Sigma_N + \Delta, \\
D_\delta(Z_1) &= Z_1 - \Delta, & D_\delta(Z_2) &= Z_2 - \Delta, & D_\delta(Z_M) &= Z_M - \Delta, \\
D_\chi(\Sigma_1) &= \Sigma_1 - X, & D_\chi(\Sigma_2) &= \Sigma_2 - 2X, & D_\chi(\Sigma_N) &= \Sigma_N - 2X, \\
D_\chi(Z_1) &= Z_1 - X, & D_\chi(Z_2) &= Z_2 - 2X, & D_\chi(Z_M) &= Z_M - 2X
\end{aligned} \tag{8.2.0.4}$$

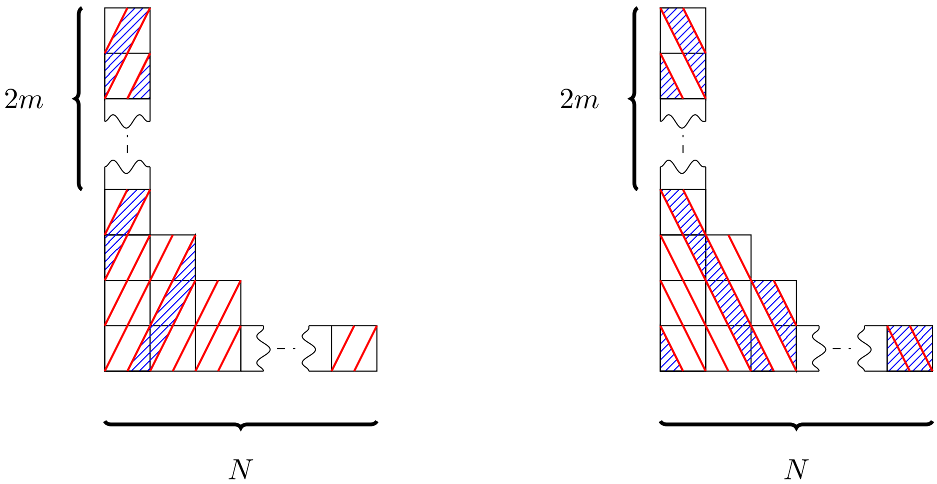
From (8.2.0.4) we see that the linear maps D_δ and D_χ are transvections, i.e. $D_\delta - \mathrm{id}$ and $D_\chi - \mathrm{id}$ have one-dimensional images. This finally leads to the matrix representations $M_\delta^{(0)},\ M_\chi^{(0)} \in \mathbb{R}^{6\times 6}$, where

$$M_\delta^{(0)} = \begin{pmatrix}
-2 & -3 & -3 & 3 & 3 & 3 \\
N+M-1 & N+M & N+M-1 & -N-M+1 & -N-M+1 & -N-M+1 \\
-3 & -3 & -2 & 3 & 3 & 3 \\
-3 & -3 & -3 & 4 & 3 & 3 \\
N+M-1 & N+M-1 & N+M-1 & -N-M+1 & -N-M+2 & -N-M+1 \\
-3 & -3 & -3 & 3 & 3 & 4
\end{pmatrix}$$

and

$$M_\chi^{(0)} = \begin{pmatrix}
-M-N+4 & -2M-2N+6 & -2M-2N+6 & -M-N+3 & -2M-2N+6 & -2M-2N+6 \\
-M-N+3 & -2M-2N+7 & -2M-2N+6 & -M-N+3 & -2M-2N+6 & -2M-2N+6 \\
5 & 10 & 11 & 5 & 10 & 10 \\
M+N-3 & 2M+2N-6 & 2M+2N-6 & M+N-2 & 2M+2N-6 & 2M+2N-6 \\
M+N-3 & 2M+2N-6 & 2M+2N-6 & M+N-3 & 2M+2N-5 & 2M+2N-6 \\
-5 & -10 & -10 & -5 & -10 & -9
\end{pmatrix}.$$

For the directions $(1,2)$ and $(1,-2)$ we assume that $M = 2m+4$ ($m \in \mathbb{N}_{\not{0}}$). We then



(a) (b)

Figure 8.3.: Cylinder decomposition in direction $(1,2)$ and $(1,-2)$ of the origami $\mathcal{O}_{N,M}^{(4)}$. Here γ_1 and α_1 are the waist curves of the blue cylinders.

again have decompositions into two cylinders of equal height (see Figure 8.3a and Figure 8.3b). For direction $(1,2)$ the waist curves γ_1, γ_2 have combinatorial length $\frac{M}{2}$ and

$\frac{2N+M+4}{2}$. For direction $(1,-2)$ the waist curves α_1, α_2 have combinatorial length $N+m$ and $m+6$. We get the following transvections D_γ and D_α:

$$D_\gamma\colon v \longmapsto v + (2N+M+4)\ \Omega(v,\gamma_1)\gamma_1 + M\ \Omega(v,\gamma_2)\gamma_2$$
$$D_\alpha\colon v \longmapsto v + (m+6)\ \Omega(v,\alpha_1)\alpha_1 + (N+m)\ \Omega(v,\alpha_2)\alpha_2$$

For the intersection points of the waist curve γ_1, γ_2 and α_1, α_2 with $\sigma_i, \sigma_N, \zeta_i, \zeta_M$ we counted:

Ω	γ_1	γ_2	α_1	α_2
σ_1	1	1	-1	-1
σ_2	1	3	-1	-3
σ_3	1	5	-2	-4
σ_N	1	$2N-1$	$4-2N$	-4
ζ_1	0	-1	-1	0
ζ_2	0	-2	-1	-1
ζ_3	-1	-2	-1	-2
ζ_M	$-(1+m)$	$-(3+m)$	$-(1+m)$	$-(3+m)$

Table 8.2.: Number of intersection points between the waist curves γ_1, γ_2 of the cylinders in direction $(1,2)$ and the waist curves α_1, α_2 of the cylinders in direction $(1,-2)$ with the elements $\sigma_i, \sigma_N, \zeta_i, \zeta_M$ $(i=1,2,3)$.

With these data we compute similarly as above the representation matrices $M_\gamma^{(0)}$ and $M_\alpha^{(0)}$ of the maps D_γ, D_α with respect to the basis $B^{(0)}$. The crucial elements of the non tautological part of homology are in this case:

$$\Gamma = (2N+M+4)\gamma_1 - M\gamma_2 \in H_1^{(0)}(\mathcal{O}^{(4)}_{N,M}, \mathbb{Z})$$
$$A = (m+6)\alpha_1 - (N+m)\alpha_2 \in H_1^{(0)}(\mathcal{O}^{(4)}_{N,M}, \mathbb{Z})$$

We get for the coefficients of Γ and A in the basis $B^{(0)} = \{\Sigma_1, \Sigma_2, \Sigma_n, Z_1, Z_2, Z_M\}$:

$$\begin{aligned} \Gamma = \ & (2N+M+4)\,\Sigma_1 - M\,\Sigma_2 - M\,\Sigma_N \\ & -2M\,Z_1 - 2M\,Z_2 + (2N+4)\,Z_M, \\ A = & -(N+m)\,\Sigma_1 - (N+m)\Sigma_2 + (m+6)\Sigma_n \\ & +(N-6)Z_1 + 2(N+m)Z_2 + (N-6)Z_M \end{aligned} \tag{8.2.0.5}$$

Furthermore we compute

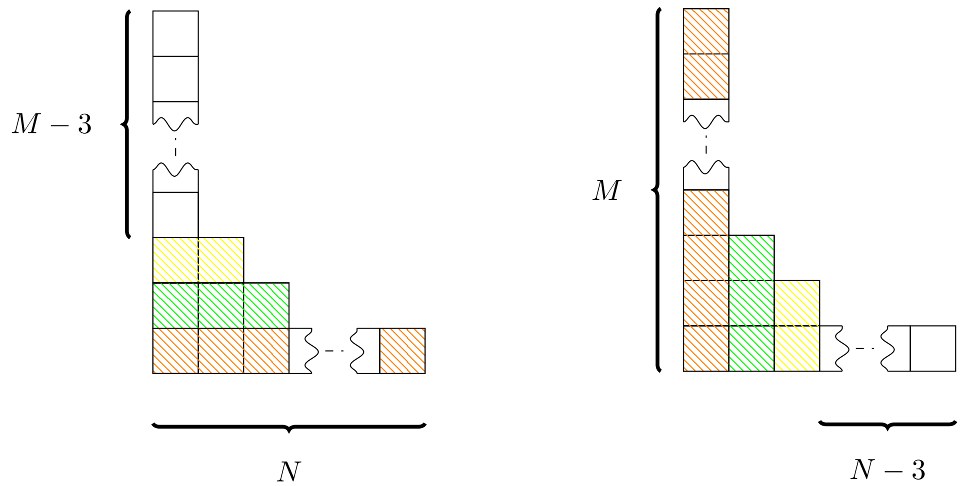
$$\begin{aligned} &D_\gamma(\Sigma_1) = \Sigma_1 - \Gamma, && D_\gamma(\Sigma_2) = \Sigma_2 - 2\Gamma, && D_\gamma(\Sigma_N) = \Sigma_N - (N-1)\Gamma, \\ &D_\gamma(Z_1) = Z_1, && D_\gamma(Z_2) = Z_2 - \Gamma, && D_\gamma(Z_M) = Z_M - (m+1)\Gamma, \\ &D_\alpha(\Sigma_1) = \Sigma_1 + A, && D_\alpha(\Sigma_2) = \Sigma_2 + A, && D_\alpha(\Sigma_N) = \Sigma_N - (N-4)A, \\ &D_\alpha(Z_1) = Z_1 + A, && D_\alpha(Z_2) = Z_2 + 2A, && D_\alpha(Z_M) = Z_M + (3+m)A \end{aligned} \tag{8.2.0.6}$$

We conclude that the maps $D_\gamma,\ D_\alpha$ have the matrix representations $M_\gamma^{(0)}$ and $M_\alpha^{(0)} \in \mathbb{R}^{6\times 6}$ on $H_1^{(0)}(\mathcal{O}_{N,M}^{(4)}, \mathbb{Q})$ with respect to the basis $B^{(0)}$ which are given as follows:

$$M_\gamma^{(0)} = \begin{pmatrix} -M-2N-3 & -2M-4N-8 & -(N-1)(M+2N+4) & 0 & -M-2N-4 & -(m+1)(M+2N+4) \\ M & 2M+1 & (N-1)M & 0 & M & (m+1)M \\ M & 2M & (N-1)M+1 & 0 & M & (m+1)M \\ 2M & 4M & 2(N-1)M & 1 & 2M & 2(m+1)M \\ 2M & 4M & 2(N-1)M & 0 & 2M+1 & 2(m+1)M \\ -2N-4 & -4N-8 & -(N-1)(2N+4) & 0 & -2N-4 & -(m+1)(2N+4)+1 \end{pmatrix}$$

and

$$M_\alpha^{(0)} = \begin{pmatrix} -(N+m)+1 & -(N+m) & (N-4)(N+m) & -(N+m) & -2(N+m) & -(3+m)(N+m) \\ -(N+m) & -(N+m)+1 & (N-4)(N+m) & -(N+m) & -2(N+m) & -(3+m)(N+m) \\ m+6 & m+6 & -(N-4)(m+6)+1 & m+6 & 2(m+6) & (3+m)(m+6) \\ N-6 & N-6 & -(N-4)(N-6) & N-5 & 2(N-6) & (3+m)(N-6) \\ 2(N+m) & 2(N+m) & -2(N-4)(N+m) & 2(N+m) & 4(N+m)+1 & 2(3+m)(N+m) \\ N-6 & N-6 & -(N-4)(N-6) & N-6 & 2(N-6) & (3+m)(N-6)+1 \end{pmatrix}.$$



(a) (b)

Figure 8.4.: Cylinder decomposition in direction $(1,0)$ and $(0,1)$ of the origami $\mathcal{O}_{N,M}^{(4)}$.

In the cylinder decomposition of the origami $\mathcal{O}_{N,M}^{(4)}$ in horizontal and vertical direction we have in both cases four maximal cylinders with moduli $M-3,\ 1/2,\ 1/3,\ 1/N$ for the horizontal direction and moduli $N-3,\ 1/2,\ 1/3,\ 1/M$ for the vertical direction (see Figure 8.4a and Figure 8.4b). Hence we obtain two corresponding multitwists which act on $H_1^{(0)}(\mathcal{O}_{N,M}^{(4)}, \mathbb{Q})$ by:

$$D_h\colon w \mapsto w + 6(M-3)N\ \Omega(w,\sigma_1)\sigma_1 + 3N\ \Omega(w,\sigma_2)\sigma_2 + 2N\ \Omega(w,\sigma_3)\sigma_3 + 6\ \Omega(w,\sigma_N)\sigma_N,$$
$$D_v\colon w \mapsto w + 6(N-3)M\ \Omega(w,\zeta_1)\zeta_1 + 3M\ \Omega(w,\zeta_2)\zeta_2 + 2M\ \Omega(w,\zeta_3)\zeta_3 + 6\ \Omega(w,\zeta_M)\zeta_M.$$

We read the intersection numbers of the σ_i and ζ_j from Figure 8.1. A straight forward computation now gives the representation matrices $M_h^{(0)}$ and $M_v^{(0)}$ for the action of the horizontal and vertical twist on $H_1^{(0)}(\mathcal{O}_{N,M}^{(4)},\mathbb{Q})$ with respect to $B^{(0)}$:

$$M_h^{(0)} = \begin{pmatrix} 1 & 0 & 0 & 0 & -3N & -3N \\ 0 & 1 & 0 & -2N & -2N & -2N \\ 0 & 0 & 1 & 6 & 12 & 6(M-1) \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

and

$$M_v^{(0)} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 3M & 3M & 1 & 0 & 0 \\ 2M & 2M & 2M & 0 & 1 & 0 \\ -6 & -12 & -6(N-1) & 0 & 0 & 1 \end{pmatrix}.$$

8.3 Zariski density in genus four

For the proof of the Zariski-density of the Kontsevich-Zorich monodromy of the $\mathcal{O}_{N,M}^{(4)}$ we will follow an approach which differs from the one in the previous sections. Let us first describe the idea before we go into the details:

The key ingredient of our arguments is the following theorem of Detinko, Flannery and Hulpke.

8.3.1 Theorem (Detinko, Flannery, Hulpke, [25] Prop. 3.7)**.** Suppose that a subgroup $H \leqslant \mathrm{Sp}(2n,\mathbb{Z})$ contains a transvection $t \in H$, i.e. $\mathrm{rank}(t-\mathrm{id}) = 1$. Then H is Zariski dense if and only if the normal closure $\langle t\rangle^H$ of t in H is absolutely irreducible, i.e. $\langle t\rangle^H$ is irreducible for arbitrary extensions of scalars by field extensions of the rational numbers.

In order to use this theorem we firstly have to overcome the issue that the Kontsevich monodromy $\Gamma^{(0)}$ by definition lives in $\mathrm{Sp}_\Omega(H_1^{(0)}(\mathcal{O}_{N,M}^{(4)},\mathbb{Z}))$ and thus per se is not a subgroup of $\mathrm{Sp}(2n,\mathbb{Z})$. We solve this problem by passing to a finite index subgroup. More precisely, we choose in the beginning a $\mathbb{Z}$-submodule $\Gamma_{N,M}$ of $H_1^{(0)}(\mathcal{O}_{N,M}^{(4)},\mathbb{Z})$ such that with a suitable choice of a base of $\Gamma_{N,M}$ the intersection form Ω restricted to $\Gamma_{N,M}$ is a multiple of the standard symplectic form on $\mathbb{Z}^6$. Thus the elements of $\mathrm{Sp}_\Omega(H_1^{(0)}(\mathcal{O}_{N,M}^{(4)},\mathbb{Z}))$ which stabilize $\Gamma_{N,M}$ can be identified with elements of the standard symplectic group $\mathrm{Sp}(6,\mathbb{Z})$. Therefore our goal is to find a transvection t in the image G of the action

$$\mathrm{Aff}^+(\mathcal{O}_{N,M}^{(4)}) \longrightarrow \mathrm{Sp}_\Omega(H_1^{(0)}(\mathcal{O}_{N,M}^{(4)},\mathbb{Z})),$$

which stabilizes the lattice $\Gamma_{N,M}$ and to show that the normal closure $\langle t\rangle^H$ is absolutely irreducible, where we identify

$$H = G \cap \mathrm{Stab}_{\mathrm{Sp}_\Omega(H_1^{(0)}(\mathcal{O}_{N,M}^{(4)},\mathbb{Z}))}(\Gamma_{N,M})$$

with a subgroup of $\mathrm{Sp}(6,\mathbb{Z})$. With this approach we will show Zariski-density for finitely many $N, M \in \mathbb{N}$ using a computer aided proof for the irreducibility of $\langle t\rangle^H$.

We now start going into details. Consider the $\mathbb{Z}$-submodule $\Gamma_{N,M}$ of $H_1^{(0)}(\mathcal{O}_{N,M}^{(4)},\mathbb{Z})$ generated by the following elements:

$$\begin{aligned}
c_1 &= (N+M+2)\Sigma_1, & c_2 &= (N+M+2)(-2\Sigma_1 - \Sigma_N),\\
c_3 &= (-1-N-M)Z_1 + Z_2 + Z_M, & c_4 &= Z_2,\\
c_5 &= Z_1, & c_6 &= \Sigma_1 + \Sigma_2 + \Sigma_N
\end{aligned}$$

The submodule $\Gamma_{N,M}$ has finite index in $H_1^{(0)}(\mathcal{O}_{N,M}^{(4)},\mathbb{Z})$ and if we restrict the symplectic intersection-form Ω to $\Gamma_{N,M}$ we get the following matrix representation in $I_\Omega^C \in \mathbb{Q}^{6\times 6}$ with respect to the basis $C = \{c_1, c_2, c_3, c_4, c_5, c_6\}$:

$$I_\Omega^C = (N+M+2)\cdot \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0 & 0 & 1\\ -1 & 0 & 0 & 0 & 0 & 0\\ 0 & -1 & 0 & 0 & 0 & 0\\ 0 & 0 & -1 & 0 & 0 & 0 \end{pmatrix}$$

Let G be the image of the action $\mathrm{Aff}^+(\mathcal{O}_{N,M}^{(4)}) \to \mathrm{Sp}_\Omega(H_1^{(0)}(\mathcal{O}_{N,M}^{(4)},\mathbb{Z}))$. We conclude that choosing the basis C of $\Gamma_{N,M}$ identifies the elements $\phi \in G$ which stabilize the lattice $\Gamma_{N,M}$ with elements of the standard symplectic group $\mathrm{Sp}(6,\mathbb{Z})$ i.e.,

$$H = G \cap \mathrm{Stab}_{\mathrm{Sp}_\Omega(6,\mathbb{Z})}(\Gamma_{N,M}) \leqslant \mathrm{Sp}(6,\mathbb{Z}).$$

In the following we describe the elements of $\mathrm{Sp}_\Omega(H_1^{(0)}(\mathcal{O}_{N,M}^{(4)},\mathbb{Z}))$ which stabilize the lattice $\Gamma_{N,M}$. More precisely, we find conditions for their matrix representations to do so. Denote by $C \in \mathbb{Q}^{6\times 6}$ the matrix, which has as columns the coefficients of the vectors c_i $(i = 1, \ldots, 6)$ written as a linear combination of elements in $B^{(0)}$. Furthermore let $C^{-1} \in \mathbb{Q}^{6\times 6}$ be the inverse of C i.e.,

$$C = \begin{pmatrix} N+M+2 & -2(N+M+2) & 0 & 0 & 0 & 1\\ 0 & 0 & 0 & 0 & 0 & 1\\ 0 & -(N+M+2) & 0 & 0 & 0 & 1\\ 0 & 0 & -1-N-M & 0 & 1 & 0\\ 0 & 0 & 1 & 1 & 0 & 0\\ 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}$$

and

$$C^{-1} = \begin{pmatrix} \frac{1}{M+N+2} & \frac{1}{M+N+2} & \frac{-2}{M+N+2} & 0 & 0 & 0 \\ 0 & \frac{1}{M+N+2} & \frac{-1}{M+N+2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 & 0 & M+N+1 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

An element $\phi \in \mathrm{Sp}_\Omega(H_1^{(0)}(\mathcal{O}_{N,M}^{(4)}, \mathbb{Z}))$ stabilizes the lattice $\Gamma_{N,M}$ if and only if $\phi(c_i)$ is an element of $\mathrm{Span}_\mathbb{Z}(\{c_1, \ldots, c_6\})$ for each $i \in \{1, \ldots, 6\}$ or equivalent

$$C^{-1} \cdot D_{B^{(0)}}(\phi(c_i))) \in \mathbb{Z}^6$$

for every $i \in \{1, \ldots, 6\}$. Here we write $D_{B^{(0)}}(\phi(c_i)) \in \mathbb{R}^6$ for the coefficients of $\phi(c_i)$ with respect to the basis $B^{(0)}$.

Thus ϕ stabilizes $\Gamma_{N,M}$ if and only if for each $i \in \{1, \ldots, 6\}$ the element $D_{B^{(0)}}(\phi(c_i))$ is in both the kernels of the following two maps

$$\begin{aligned} g_1 \colon \mathbb{Z}^6 &\longrightarrow \mathbb{Z}/(N+M+2)\,\mathbb{Z}, \quad (v_1, \ldots, v_6) \longmapsto \overline{v_1 + v_2 - 2v_3} \\ g_2 \colon \mathbb{Z}^6 &\longrightarrow \mathbb{Z}/(N+M+2)\,\mathbb{Z}, \quad (v_1, \ldots, v_6) \longmapsto \overline{v_2 - v_3}. \end{aligned}$$

Easy but boring calculations show that for every $i \in \{1, \ldots, 6\}$ and every matrix M in the set $\{M_\delta^{(0)},\ M_\chi^{(0)},\ M_\gamma^{(0)},\ M_\alpha^{(0)},\ M_v^{(0)}\}$, we have

$$M \cdot D_{B^{(0)}}(c_i) \in \ker(g_1) \cap \ker(g_2).$$

Hence $D_\delta, D_\chi, D_\gamma, D_\alpha, D_v \in \mathrm{Stab}_{\Gamma_{N,M}}(\mathrm{Sp}_\Omega(H_1^{(0)}(\mathcal{O}_{N,M}^{(4)}, \mathbb{Z})))$. Furthermore we computed with GAP that

$$(M_h^{(0)})^{N+M+2} \cdot D_{B^{(0)}}(c_i) \in \ker(g_1) \cap \ker(g_2)$$

for all $N \in \{4, 5, ..., 50\}$ and $M = 2m + 4$ with $m \in \{0, 1, ..., 50\}$. Consider the algebra $A_{N,M}$ generated by the transvection $t = M_\delta^{(0)}$ and the elements $M^{-1}tM \in \langle t \rangle^H$ where

$$M \in \{M_\chi^{(0)}, M_\gamma^{(0)}, M_\alpha^{(0)}, M_v^{(0)}, (M_h^{(0)})^{N+M+2}\}.$$

Here $\langle t \rangle^H$ denotes the normal closure of the transvection t in H. For $N \in \{4, 5, ..., 50\}$ and $M = 2m + 4$ with $m \in \{0, 1, ..., 50\}$ we calculated with GAP $\dim_\mathbb{Q}(A_{N,M}) = 36$ for the vector space dimension of the algebra $A_{N,M}$. The code for this can be found in [75]. This shows that $\langle t \rangle^H$ is an absolutely irreducible group [114, IV Theorem 2.10]) We obtain now with Theorem 8.3.1 the following Proposition

8.3.2 Proposition. The group H is Zariski dense in $\mathrm{Sp}_\Omega(H_1^{(0)}(\mathcal{O}_{N,M}^{(4)}, \mathbb{R}))$ for every $N \in \{4, 5, ..., 50\}$ and $M = 2m + 4$ with $m \in \{0, 1, ..., 50\}$.

8.4 Arithmeticity for a finite family in genus four

Recall from Section 8.2 that for each of the cylinder decompositions in direction $(1,1)$, $(1,-1)$ and $(1,2)$ we get two maximal cylinders. Their waist curves are $\delta_1, \delta_2 \in H_1(\mathcal{O}_{N,M}^{(4)}, \mathbb{Z})$ for direction $(1,1)$, $\chi_1, \chi_2 \in H_1(\mathcal{O}_{N,M}^{(4)}, \mathbb{Z})$ for the direction $(1,-1)$ and γ_1, γ_2 for the direction $(1,2) \in H_1(\mathcal{O}_{N,M}^{(4)}, \mathbb{Z})$. Furthermore we introduced in 8.2.0.3 and 8.2.0.5 the following elements in the non-tautological part $H_1^{(0)}(\mathcal{O}_{N,M}^{(4)}, \mathbb{Z})$:

$$\begin{aligned}
\Delta = & -3\,\Sigma_1 + (N+M-1)\Sigma_2 - 3\,\Sigma_N \\
& -3\,Z_1 + (N+M-1)\,Z_2 - 3\,Z_M, \\
X = & \quad (N+M-3)\,\Sigma_1 + (N+M-3)\,\Sigma_2 - 5\,\Sigma_N \\
& -(N+M-3)\,Z_1 - (N+M-3)\,Z_2 + 5\,Z_M, \\
\Gamma = & \quad (2N+M+4)\,\Sigma_1 - M\,\Sigma_2 - M\,\Sigma_N \\
& -2M\,Z_1 - 2M\,Z_2 + (2N+4)\,Z_M
\end{aligned}$$

Set $W = \mathrm{Span}_{\mathbb{Q}}(\Delta,\ X,\ \Gamma)$. The vector space W has dimension $\dim_{\mathbb{Q}}(W) = 3$. We set $A = -22 + 4N + 4M$, $B = -6 - 3m$ and $C = -12 + 3N - 9m$. Using (8.2.0.4) and (8.2.0.6) we obtain that the restrictions of the transvections D_δ, D_χ and D_γ to W have the following matrix representations with respect to the basis $\{\Delta,\ X,\ \Gamma\}$:

$$\begin{pmatrix} 1 & A & -2B \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 \\ -A & 1 & -2C \\ 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ B & C & 1 \end{pmatrix}$$

The vector $e = -2C\Delta + 2BX + A\Gamma$ is fixed by all the three elements D_δ, D_χ and D_γ. Furthermore $\Omega(e, w) = 0$ for all $w \in W$. With respect to the new basis $\{\Delta,\ X,\ e\}$ we get the following matrix representations for D_δ, D_χ and D_γ:

$$\begin{pmatrix} 1 & A & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 \\ -A & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 2\frac{BC}{A} + 1 & 2\frac{C^2}{A} & 0 \\ -2\frac{B^2}{A} & -2\frac{BC}{A} + 1 & 0 \\ \frac{B}{A} & \frac{C}{A} & 1 \end{pmatrix}$$

If we choose $C = 0$ or equivalent $N = 3m + 4$, we have

$$\Omega(\Delta, X) = -50\,(2m^2 + 5m + 2) < 0$$

for all $m > 0$ and the group generated by $D_\delta|_W$, $D_\chi|_W$, $D_\gamma|_W$ contains a non-trivial element of the unipotent radical of the symplectic group on W, namely $(D_\chi|_W)^{-2B^2} \circ (D_\gamma|_W)^{A^2}$ is represented by

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ BA & 0 & 1 \end{pmatrix}$$

with respect to the basis $\{\Delta,\ X,\ e\}$. With Theorem 1.6.1 and Proposition 8.3.2 we conclude that the Kontsevich–Zorich monodromy of the origami $\mathcal{O}^{(4)}_{N,M}$ is a finite index subgroup of $\mathrm{Sp}_{\Omega}(H_1^{(0)}(\mathcal{O}^{(4)}_{N,M}, \mathbb{Z}))$ for every $4 \leqslant N \leqslant 50$ with $N = 3m + 4$ and every $M = 2m + 4$ with $0 \leqslant m \leqslant 50$. This ends the proof of Theorem 8.1.3.

9. Arithmeticity for infinite families of stairs origamis in genus four, five and six

In Chapter 8 we were only able to show arithmeticity of the Kontsevich–Zorich monodromy for a finite family of origamis. The reason for that was our computational approach to show Zariski density of the monodromy groups. In this chapter we use Theorem 1.5.8 respectively Criterion 1.5.10 and Galois theory to overcome this problem. Showing Zariski density for the Kontsevich–Zorich monodromy translates with Criterion 1.5.10 to finding Galois pinching elements in the monodromy group. If we show Zariski density then we can apply Theorem 1.6.1 as in Chapter 8. Everything which is written in this chapter is joined work with Carlos Matheus. It is published in the article [71].

9.1 Preliminaries from Galois theory

9.1.1 Galois groups as permutation groups. Consider a monic irreducible polynomial $P(X) \in \mathbb{Z}[X]$ of degree n with the set of complex roots $S = \{\lambda_1, \dots, \lambda_n\}$. Let $Z(P) = \mathbb{Q}(\lambda_1, \dots, \lambda_n)$ be the splitting field of the polynomial $P(X) \in \mathbb{Z}[X]$. We consider the standard embedding of $\mathrm{Gal}(P) = \mathrm{Aut}_{\mathbb{Q}}(Z(P))$ in the permutation group $\mathrm{Sym}(S)$ via

$$\mathrm{Gal}(P) \longrightarrow \mathrm{Sym}(S), \quad \sigma \longmapsto \sigma|_S.$$

The theorem of Dedekind is a useful tool to study the Galois group of a polynomial $P(X) \in \mathbb{Z}[X]$ as above:

Theorem (Dedekind)**.** Let $P(X) \in \mathbb{Z}[X]$ be monic irreducible of degree n. For an arbitrary prime number p not dividing the discriminant of $P(X) \in \mathbb{Z}[X]$, let the monic irreducible factorization of $P(X) \in \mathbb{Z}[X]$ modulo p be

$$P(X) \equiv \pi_1(X) \cdot \ldots \cdot \pi_k(X) \bmod p$$

with $\pi_i(X)$ pairwise distinct and set $d_i := \deg \pi_i(X)$, so $d_1 + \cdots + d_k = n$. Then the Galois group $\mathrm{Gal}(P)$ of $P(X) \in \mathbb{Z}[X]$ viewed as a subgroup of $\mathrm{Sym}(S)$ contains an element that permutes the roots S of $P(X)$ with cycle type $(d_1, \dots, d_k)$.

9.1.2 Galois groups of polynomials of degree four and five. For a reference of the following see [69]. We consider in this subsection a monic irreducible polynomial

$$Q(X) = X^k + \sum_{i=0}^{k-1} b_i X^i \in \mathbb{Q}[X]$$

of degree four or five with set of roots $S = \{\mu_1, \ldots, \mu_k\}$ ($k = 4$ or $k = 5$).

Let first $k = 4$. We define the *cubic resolvent* $CR_Q(Y) \in \mathbb{Q}[Y]$ of the polynomial $Q(X)$ as

$$CR_Q(Y) = (Y - (\mu_1\mu_2 + \mu_3\mu_4))\,(Y - (\mu_1\mu_3 + \mu_2\mu_4))\,(Y - (\mu_1\mu_4 + \mu_2\mu_3)).$$

Direct calculations show

$$CR_Q(Y) = Y^3 - b_2\,Y^2 + (b_1\,b_3 - 4b_0)\,Y - (b_0\,b_3^2 - 4\,b_0\,b_2 + b_1^2).$$

Furthermore a computation reveals the equality

$$\mathrm{Disc}(Q(X)) = \mathrm{Disc}(CR_Q(Y))$$

between the discriminant $\mathrm{Disc}(Q(X))$ of $Q(X) \in \mathbb{Q}[X]$ and the discriminant $\mathrm{Disc}(CR_Q)$ of $CR_Q(Y) \in \mathbb{Q}[Y]$. The five transitive subgroups of the permutation group S_4 are the Klein-four group V_4, the cyclic group C_4, the dihedral group D_4, A_4 and S_4 itself. The next theorem will help to determine the Galois group of the polynomial $Q(X) \in \mathbb{Q}[X]$. A proof can be found in [69, Theorem 2.2.2].

9.1.3 Theorem. Let $Z(CR_Q)$ be the splitting field of the cubic resolvent $CR_Q(Y) \in \mathbb{Q}[Y]$ from above and let $m = [Z(CR_Q) : \mathbb{Q}]$ be the degree of the field extension over the rational numbers. Then we have for the Galois group $\mathrm{Gal}(Q) \leqslant S_4$ of the irreducible polynomial $Q(X) \in \mathbb{Q}[X]$ from above:

$$\mathrm{Gal}(Q) = \begin{cases} S_4 & \text{if } m = 6 \\ A_4 & \text{if } m = 3 \\ D_4 \text{ or } C_4 & \text{if } m = 2 \\ V_4 & \text{if } m = 1 \end{cases}$$

9.1.4 Remark. Since the cubic resolvent $CR_Q(Y) \in \mathbb{Q}(Y)$ of $Q(X) \in \mathbb{Q}[X]$ is a degree three polynomial it is sufficient for the splitting field $Z(CR_Q)$ to be a degree six field extension over the rational numbers, that $CR_Q(Y) \in \mathbb{Q}(Y)$ is irreducible and that the discriminant $\mathrm{Disc}(CR_Q) = \mathrm{Disc}(Q)$ of $CR_Q(Y)$ respectively $Q(X)$ is not a square of a rational number.

Now let $k = 5$. The *Weber sextic resolvent* $SWR_Q(Y) \in \mathbb{Q}[Y]$ defined as in Definition 2.3.2 of [69] is a degree six polynomial which helps to determine the Galois group of the quintic polynomial $Q(X) \in \mathbb{Q}[X]$ as we will explain in the following Theorem and Remark (see [69, Theorem 2.3.3] for a proof of the theorem):

9.1.5 Theorem. The Galois group $\mathrm{Gal}(Q) \leqslant S_5$ of the irreducible monic polynomial $Q(X) \in \mathbb{Q}[X]$ is solvable if and only if the sextic Weber resolvent $SWR_Q(Y) \in \mathbb{Q}[Y]$ has a root in the rational numbers $\mathbb{Q}$.

9.1.6 Remark. The only transitive subgroups of S_5 are C_5, the dihedral group D_5, the affine group $F_{20} \cong \mathbb{Z}/5\mathbb{Z} \rtimes (\mathbb{Z}/5\mathbb{Z})^\times$, A_5 and S_5 itself. Hence the only transitive subgroups that are non-solvable are A_5 and S_5. Thus it is easy to ensure that $\mathrm{Gal}(Q) = S_5$ with the help of the previous theorem and the fact that $\mathrm{Gal}(Q) \leqslant A_5$ if and only if the discriminant $\mathrm{Disc}(Q)$ is a square of a rational number.

9.1.7 Galois group of certain reciprocal polynomials. Consider from now on an irreducible monic polynomial

$$P(X) = \sum_{i=0}^{n} c_i\, X^i \in \mathbb{Z}[X]$$

of degree $n = 2k$ which is *reciprocal*, i.e. $c_n = c_0 = 1$ and $c_i = c_{n-i}$ for $i = 1, \ldots, k$.

The Galois group of such a $P(X) \in \mathbb{Z}[X]$ can be naturally seen as a subgroup of the hyperoctahedral group G_k as we will see in the next paragraph. But first we want to explain the hyperoctahedral group as we want to use it in this text:

For $k \geqslant 1$ we define the hyperoctahedral group as the semidirect product $G_k = \mathbb{Z}_2^k \rtimes S_k$, where S_k is the permutation group on a set with k elements. The group S_k acts on $\mathbb{Z}_2^k$ by $\tau(\epsilon_1, \ldots, \epsilon_k) = (\epsilon_{\tau(1)}, \ldots, \epsilon_{\tau(k)})$ ($\epsilon_i \in \{\pm 1\}$) and the multiplication on the hyperoctahedral group G_k is defined by

$$(\epsilon, \tau) \cdot (\tilde{\epsilon}, \tilde{\tau}) = (\tilde{\tau}(\epsilon) \cdot \tilde{\epsilon},\ \tau \circ \tilde{\tau}). \tag{9.1.7.1}$$

Compare section two in [68].

Since $P(X) \in \mathbb{Z}[X]$ is reciprocal and its splitting field is of zero characteristic and hence perfect, the polynomial $P(X)$ has $n = 2k$ distinct roots which come in pairs $\{\lambda_i, \lambda_i^{-1}\}$ for $i \in \{1, \ldots, k\}$. Denote by $Z(P)$ the splitting field of the polynomial $P(X) \in \mathbb{Z}[X]$ and by $\mathrm{Gal}(P) = \mathrm{Aut}_{\mathbb{Q}}(Z(P))$ the Galois group of $P(X) \in \mathbb{Z}[X]$. An automorphism $\sigma \in \mathrm{Gal}(P)$ necessarily permutes the k pairs of roots $\{\lambda_i, \lambda_i^{-1}\}$ of $P(X) \in \mathbb{Z}[X]$ and this leads to a group homomorphism

$$\phi\colon \mathrm{Gal}(P) \longrightarrow S_k, \quad \sigma \longmapsto \tau_\sigma,$$

where we set $\tau_\sigma(i) = j$ if $\sigma(\{\lambda_i, \lambda_i^{-1}\}) = \{\lambda_j, \lambda_j^{-1}\}$ ($i, j \in \{1, \ldots, k\}$). The kernel N of ϕ is given by the automorphisms $\sigma \in \mathrm{Gal}(P)$ such that $\sigma(\{\lambda_i, \lambda_i^{-1}\}) = \{\lambda_i, \lambda_i^{-1}\}$ for every $i \in \{1, \ldots, k\}$. Hence, we can identify N with a subgroup of $\mathbb{Z}_2^k$ via the following homomorphism

$$\iota\colon N \longrightarrow \mathbb{Z}_2^k, \quad \sigma \longmapsto \epsilon^\sigma = (\epsilon_1^\sigma, \ldots, \epsilon_k^\sigma),$$

where $\epsilon_i^\sigma = 1$ if $\sigma(\lambda_i) = \lambda_i$ and $\epsilon_i^\sigma = -1$ if $\sigma(\lambda_i) = \lambda_i^{-1}$. Since N is the kernel of ϕ, it is normal and together with the map ι, we can represent $\mathrm{Gal}(P) = N \rtimes \mathrm{Im}(\phi)$ as a subgroup of the hyperoctahedral group $G_k = \mathbb{Z}_2^k \rtimes S_k$. The whole situation is visualized in the following commutative diagram:

$$\begin{tikzcd}
N \arrow[r, "\iota"] \arrow[d] & \mathrm{Gal}(P) \arrow[r, "\phi"] \arrow[d] & S_k \arrow[d] \\
\mathbb{Z}_2^k \arrow[r, "i_k"] & G_k = \mathbb{Z}_2^k \rtimes S_k \arrow[r, "\pi_k"] & S_k
\end{tikzcd}$$

Here $i_k\colon \mathbb{Z}_2^k \to G_k$ is the inclusion map and $\pi_k\colon G_k \to S_k$ is the projection map, what makes of course $\mathbb{Z}_2^k \xrightarrow{i_k} G_k \xrightarrow{\pi_k} S_k$ a split exact sequence.

9.1.8 Action of the hyperoctahedral group on the splitting field. In the following we want to consider the action of the hyperoctahedral group G_k on the splitting field $Z(P)$ of $P(X) \in \mathbb{Z}[X]$ that fits together with the action of $\mathrm{Gal}(P)$ from above. Hence we define the action of G_k on $Z(P)$ via a permutation of the roots $\{\lambda_i, \lambda_i^{-1} \mid i = 1, \ldots, k\}$ in the following way: For $\epsilon = (\epsilon_1, \ldots, \epsilon_k) \in \{(\pm 1, \ldots, \pm 1)\}$, every $\tau \in S_k$ and $i = 1, \ldots, k$ we define

$$(\epsilon, \tau) \,.\, \lambda_i = \lambda_{\tau(i)}^{\epsilon_i} \quad \text{and} \quad (\epsilon, \tau) \,.\, \lambda_i^{-1} = \lambda_{\tau(i)}^{-\epsilon_i}. \tag{9.1.8.1}$$

9.1.9 Lemma. The equalities in (9.1.8.1) indeed define an action of the group G_k on the set of roots $\{\lambda_i, \lambda_i^{-1} \mid i = 1, \ldots, k\}$.

Proof. We have to show the compatibility with the multiplication on G_k defined in (9.1.7.1). Let $\delta \in \{\pm 1\}$ and $i \in \{1, \ldots, k\}$. For $(\epsilon, \tau), (\tilde{\epsilon}, \tilde{\tau}) \in G_k$ we have

$$(\epsilon, \tau) . \Big((\tilde{\epsilon}, \tilde{\tau}) \,.\, \lambda_i^\delta \Big) = (\epsilon, \tau) \,.\, \lambda_{\tilde{\tau}(i)}^{\delta \cdot \tilde{\epsilon}_i} = \lambda_{\tau \circ \tilde{\tau}(i)}^{(\delta \cdot \tilde{\epsilon}_i) \cdot \epsilon_{\tilde{\tau}(i)}} = (\tilde{\tau}(\epsilon) \cdot \tilde{\epsilon},\ \tau \circ \tilde{\tau}) \,.\, \lambda_i^\delta,$$

what ends the proof since $(\epsilon, \tau) \cdot (\tilde{\epsilon}, \tilde{\tau}) = (\tilde{\tau}(\epsilon) \cdot \tilde{\epsilon},\ \tau \circ \tilde{\tau})$ by definition of the product on G_k. □

In our subsequent discussion, we need to find reciprocal polynomials such that the map $\phi\colon \mathrm{Gal}(P) \to S_k$ is onto. The next proposition will help us with that.

9.1.10 Proposition. Let $k \geqslant 2$ and $P(X) \in \mathbb{Z}[X]$ be an irreducible reciprocal polynomial of degree $2k$ with $\phi(\mathrm{Gal}(P)) = S_k$. Then $\mathrm{Gal}(P)$ is isomorphic to one of the following subgroups of $G_k = \mathbb{Z}_2^k \rtimes S_k$:

Either $\mathrm{Gal}(P) \cong S_k$, $\mathrm{Gal}(P) \cong G_k$ or $\mathrm{Gal}(P)$ is isomorphic to one of the following three subgroups $H_{k,1}$, $H_{k,2}$, $H_{k,3} \leqslant G_k$, where

$$\begin{aligned} H_{k,1} &:= \{((\epsilon_1,\dots,\epsilon_k),\tau) \mid \prod_{i=1}^{k} \epsilon_i = 1\}, \\ H_{k,2} &:= \{((\epsilon_1,\dots,\epsilon_k),\tau) \mid \operatorname{sign}(\tau)\prod_{i=1}^{k} \epsilon_i = 1\}, \\ \text{and} \quad H_{k,3} &:= \{(+1,\dots,+1), (-1,\dots,-1)\} \times S_k. \end{aligned}$$

Proof. For $k = 2$ this is an easy exercise since the only possible subgroups of $G_2 = \mathbb{Z}_2^2 \rtimes S_2$ which surject onto S_2 and which are not the full group or the trivial group, are groups of order four. In this case

$$H_{2,1} = H_{2,3} = \langle\ (\lambda_1,\lambda_2)(\lambda_1^{-1},\lambda_2^{-1}), (\lambda_1,\lambda_1^{-1})(\lambda_2,\lambda_2^{-1})\ \rangle \cong V_4$$

and

$$H_{2,2} = \langle\ (\lambda_1\lambda_2^{-1}\lambda_1^{-1}\lambda_2)\ \rangle \cong C_4,$$

where we used the same identification of G_2 with a subgroup of the permutation group $\mathrm{Sym}(\{\lambda_1,\lambda_1^{-1},\lambda_2,\lambda_2^{-1}\})$ as in Section 9.1.8.

For $k = 3, 4$ and all $k \geqslant 5$ the result follows as in Proposition 4 in [68] since A_3 is a simple group as well as all A_k with $k \geqslant 5$ and the only non-trivial normal subgroup of A_4 is the Klein four-group V_4 which has index three in A_4. □

Together with the fundamental theorem of Galois theory we will use the next lemma to ensure that the Galois group of certain reciprocal polynomials $P(X) \in \mathbb{Z}[X]$ equals the whole hyperoctahedral group, i.e. $\mathrm{Gal}(P) = G_k$.

9.1.11 Lemma. Let the hyperoctahedral group $G_k = \mathbb{Z}_2^k \rtimes S_k$ act on the splitting field $Z(P) = \mathbb{Q}(\{\lambda_i,\, \lambda_i^{-1} \mid i = 1,\dots,k\})$ of the polynomial $P(X) \in \mathbb{Z}[X]$ as explained in Section 9.1.8. Consider the two subgroups $H_{k,1}$ and $H_{k,2}$ of G_k defined in Proposition 9.1.10. Then:

(i) The expression $\delta_{k,1} := \prod_i \left(\lambda_i - \lambda_i^{-1}\right)$ is invariant under the action of $H_{k,1}$ but not G_k.

(ii) The expression $\delta_{k,2} := \prod_{i<j}\left(\lambda_i + \lambda_i^{-1} - \lambda_j - \lambda_j^{-1}\right)\prod_i\left(\lambda_i - \lambda_i^{-1}\right)$ is invariant under the action of $H_{k,2}$ but not G_k.

Proof. For every $\tau \in S_k$ and every $i \in \{1,\dots,k\}$ we have

$$((+1,\dots,+1),\tau)\,.\,(\lambda_i - \lambda_i^{-1}) = \lambda_{\tau(i)} - \lambda_{\tau(i)}^{-1}.$$

This shows that $\delta_{k,1} \in Z(P)$ is invariant under the action of S_k.

Now consider $(\epsilon, \mathrm{id}).(\lambda_i - \lambda_i^{-1})$ for $\epsilon = (\epsilon_1, \ldots, \epsilon_k) \in \mathbb{Z}_2^k$ and $i \in \{1, \ldots, k\}$. We have

$$\begin{aligned}(\epsilon, \mathrm{id}) \,.\, (\lambda_i - \lambda_i^{-1}) &= \lambda_i - \lambda_i^{-1} = \epsilon_i \, (\lambda_i - \lambda_i^{-1}) \quad \text{if } \epsilon_i = 1, \\ (\epsilon, \mathrm{id}) \,.\, (\lambda_i - \lambda_i^{-1}) &= \lambda_i^{-1} - \lambda_i = \epsilon_i \, (\lambda_i - \lambda_i^{-1}) \quad \text{if } \epsilon_i = -1.\end{aligned}$$

Furthermore $(\epsilon, \mathrm{id}).(\lambda_i + \lambda_i^{-1}) = \lambda_i + \lambda_i^{-1}$ for every $\epsilon \in \mathbb{Z}_2^k$.

We know $\mathrm{sgn}(\tau) = (-1)^{\mathrm{inv}(\tau)}$ for every element $\tau \in S_k$ with $\mathrm{inv}(\tau) \in \mathbb{N}_0$ is the number of elements $(i, j) \in \{1, \ldots, k\} \times \{1, \ldots, k\}$ with $i < j$ but $\tau(i) > \tau(j)$ and thus

$$\begin{aligned}&((+1, \ldots, +1), \tau) \,.\, \Big(\prod_{i<j} (\lambda_i + \lambda_i^{-1} - \lambda_j - \lambda_j^{-1})\Big) \\ &= \mathrm{sgn}(\tau) \Big(\prod_{i<j} (\lambda_i + \lambda_i^{-1} - \lambda_j - \lambda_j^{-1})\Big).\end{aligned}$$

Putting this together with the arguments from above we conclude for every $(\epsilon, \tau) \in G_k = \mathbb{Z}_2^k \rtimes S_k$ where $\epsilon = (\epsilon_1, \ldots, \epsilon_k) \in \{(\pm 1, \ldots, \pm 1)\}$:

$$(\epsilon, \tau) \,.\, \delta_{k,1} = \left(\prod_{i=1}^{k} \epsilon_i\right) \delta_{k,1} \quad \text{and} \quad (\epsilon, \tau) \,.\, \delta_{k,2} = \mathrm{sgn}(\tau) \left(\prod_{i=1}^{k} \epsilon_i\right) \delta_{k,2}$$

This proves (i) and (ii). □

9.1.12 Remark. For a reciprocal monic polynomial $P(X) \in \mathbb{Z}[X]$ of degree $n = 2k$ with set of roots $\{\lambda_i, \lambda_i^{-1} \mid i = 1, \ldots, k\} \subset \mathbb{C}$ we can always find a polynomial $Q(Y) \in \mathbb{Z}[Y]$ such that $1/X^k \cdot P(X) = Q(X + 1/X + 2)$. The polynomial $Q(Y) \in \mathbb{Z}[Y]$ has distinct roots μ_i $(i = 1, \ldots, k)$ such that without loss of generality $\mu_i = \lambda_i + \lambda_i^{-1} + 2$ for all $i = 1, \ldots, k$. We write in the following $\Delta_{k,1} := \delta_{k,1}^2$ and $\Delta_{k,2} := \delta_{k,2}^2$ for the squares of the expressions $\delta_{k,1}$ and $\delta_{k,2}$ from Lemma 9.1.11. We have

(i) $\Delta_{k,1} = \delta_{k,1}^2 = \prod_{i=1}^k (\lambda_i - \lambda_i^{-1})^2 = \prod_{i=1}^k \mu_i(\mu_i - 4) = Q(0)\, Q(4)$

and

(ii) $\Delta_{k,2} = \delta_{k,2}^2 = \Big(\prod_{i<j} (\mu_i - \mu_j)\Big)^2 \Big(\prod_i (\lambda_i - \lambda_i^{-1})\Big)^2 = \mathrm{Disc}(Q)\, \Delta_{k,1}$.

This shows $\Delta_{k,1},\ \Delta_{k,2} \in \mathbb{Q}$ and delivers an easy way how to write $\Delta_{k,1}$ and $\Delta_{k,2}$ in terms of coefficients of $Q(Y) \in \mathbb{Q}[Y]$.

9.1.13 Real roots of cubic, quartic and quintic polynomials. For a cubic polynomial $Q(X) \in \mathbb{R}[X]$ with discriminant $\mathrm{Disc}(Q) \neq 0$ it is well known that the number of real roots can be read off from the discriminant as follows. If

$$\begin{aligned} &\mathrm{Disc}(Q) > 0, \quad \text{then } Q(X) \text{ has three real roots,} \\ &\mathrm{Disc}(Q) < 0, \quad \text{then } Q(X) \text{ has one real root and two non-real roots.} \end{aligned} \tag{9.1.13.1}$$

Real roots for quartic polynomials. We have a slightly more complicated statement of this form for quartic polynomials as well, so let

$$Q(X) = X^4 + a\,X^3 + b\,X^2 + c\,X + d \in \mathbb{R}[X]$$

be a real monic polynomial of degree four. By substituting $X = Y - a/4$, we get the depressed quartic polynomial

$$DQ(Y) = Y^4 + q\,Y^2 + rY + s \in \mathbb{R}[Y], \tag{9.1.13.2}$$

with coefficients

$$\begin{aligned} &q = b - (3/8)\,a^2, \quad r = c - (1/2)\,a\,(b - (1/4)\,a^2) \quad \text{and} \\ &s = d - (3/256)\,a^4 + (1/16)\,a^2\,b - (1/4)\,a\,c. \end{aligned}$$

For a quartic polynomial in the depressed form as in (9.1.13.2), there is an easy criterion whether the polynomial has four real roots or no real roots (see [42]). We want to state the result and denote by $\mathrm{Disc}(DQ)$ the discriminant of the polynomial in 9.1.13.2 and by $F(DQ)$ the expression $F(DQ) := q^2 - 4s$. If we have

$$\begin{aligned} \mathrm{Disc}(DQ) > 0,\ q \geqslant 0, &\quad \text{then (9.1.13.2) has no real roots,} \\ \mathrm{Disc}(DQ) > 0,\ F(DQ) \leqslant 0, &\quad \text{then (9.1.13.2) has no real roots,} \\ \mathrm{Disc}(DQ) > 0,\ q < 0,\ F(DQ) > 0, &\quad \text{then (9.1.13.2) has four real roots.} \end{aligned} \tag{9.1.13.3}$$

Real roots for quintic polynomials. We want to end this section with a criterion from [59] with which we can find out whether a quintic polynomial in $\mathbb{R}[X]$ has simple real roots. Let

$$Q(X) = X^5 + a\,X^4 + b\,X^3 + c\,X^2 + d\,X + e \in \mathbb{R}[X].$$

By substituting $X = Y - a/5$ we get a depressed polynomial

$$DQ(Y) = Y^5 + p\,Y^3 + q\,Y^2 + r\,Y + s \in \mathbb{R}[X]. \tag{9.1.13.4}$$

With the help of the following four discriminants we can find out wether $DQ(Y)$ has simple real roots. We define

$$\begin{aligned}
F_1(DQ) &= -p,\\
F_2(DQ) &= 40\,r\,p - 12\,p^3 - 45\,q^2,\\
F_3(DQ) &= 12\,p^4\,r - 4\,p^3\,q^2 + 117\,p\,r\,q^2 - 88\,r^2\,p^2 - 40\,q\,p^2\,s\\
&\quad +125\,p\,s^2 - 27\,q^4 - 300\,q\,r\,s + 160\,r^3,\\
F_4(DQ) &= -1600\,q\,s\,r^3 - 3750\,p\,s^3\,q + 2000\,p\,s^2\,r^2 - 4\,p^3\,q^2\,r^2\\
&\quad +16\,p^3\,q^3\,s - 900\,r\,s^2\,p^3 + 825\,q^2\,p^2\,s^2 + 144\,p\,q^2\,r^3\\
&\quad +2250\,q^2\,r\,s^2 + 16\,p^4\,r^3 + 108\,p^5\,s^2 - 128\,r^4\,p^2 - 27\,q^4\,r^2\\
&\quad +108\,q^5\,s + 256\,r^5 + 3125\,s^4 - 72\,p^4\,r\,s\,q + 560\,r^2\,p^2\,s\,q\\
&\quad -630\,p\,r\,q^3\,s.
\end{aligned} \tag{9.1.13.5}$$

In [59] they classified the number of real roots and their multiplicity of a depressed quintic polynomial as in (9.1.13.4) using six discriminants among which are the four discriminants from Equation (9.1.13.5). We only state the for us relevant case, namely if the four discriminants $F_1(DQ)$, $F_2(DQ)$, $F_3(DQ)$ and $F_4(DQ)$ are positive then the polynomial $DQ(Y)$ in (9.1.13.4) has five simple real roots.

9.2 Stairs origamis in genus four revisited

Let $N \geqslant 4$ and $M = 4 + 2m$ with $m \geqslant 0$. We consider again the origami $\mathcal{O}^{(4)}_{N,M} \in \Omega\mathcal{M}_4(6)$ from Chapter 8. Recall that it is associated to the pair of permutations $h, v \in \mathrm{Sym}(\{1, \ldots, N + M + 2\})$, where

$$\begin{aligned}
h =&(1,2,3\ldots,N)(N+1,N+2,N+3)(N+4,N+5)(N+6)\ldots(M)\\
v =&(1,\ N+1,\ N+4,\ N+6,\ldots,N+M)(2,\ N+2,\ N+5)(3,\ N+3)\\
&(4)\ldots(N).
\end{aligned}$$

The Kontsevich–Zorich monodromy of $\mathcal{O}^{(4)}_{N,M}$ was already studied in Section 8.1 via the analysis of Dehn twists in several rational directions. You can find all the details how we constructed the following matrices in Section 8.2. But note that we consider in this Chapter Dehn twists in reverse direction of the waist curves. We will now extend our investigations on an infinite subfamily of the stairs origamis $\mathcal{O}^{(4)}_{N,M}$.

For the purpose of finding a Galois pinching element in $Sp(H_1^{(0)}(\mathcal{O}^{(4)}_{N,M}, \mathbb{Z}))$, we consider the horizontal and vertical directions which lead to Dehn twists acting on the basis $B^{(0)}$ of $H_1^{(0)}(\mathcal{O}^{(4)}_{N,M}, \mathbb{Q})$ from Section 8.2 via the matrices (c.f. the matrices $M_h^{(0)}$ and $M_v^{(0)}$ at the end of Section 8.2.

$$M_{-h}^{(0)} = \begin{pmatrix} 1 & 0 & 0 & 0 & 3N & 3N \\ 0 & 1 & 0 & 2N & 2N & 2N \\ 0 & 0 & 1 & -6 & -12 & -6(M-1) \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

and

$$M_{-v}^{(0)} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & -3M & -3M & 1 & 0 & 0 \\ -2M & -2M & -2M & 0 & 1 & 0 \\ 6 & 12 & 6(N-1) & 0 & 0 & 1 \end{pmatrix}.$$

9.2.1 Zariski density and arithmeticity for a genus four family. At this point, we are ready to establish the arithmeticity of the Kontsevich–Zorich monodromy of $\mathcal{O}_{N,M}^{(4)}$ for many choices of N, M such that $N = 4 + 3m$ and $M = 4 + 2m$ ($m \in \mathbb{N}$). We start to show the Zariski density with Criterion 1.5.10. More precisely, consider the matrix

$$A_4(N, M) := M_{-h}^{(0)} \cdot M_{-v}^{(0)} \in \mathbb{R}^{6\times 6}.$$

By a straight forward computation of the characteristic polynomial of $A_4(N, M)$ one obtains a reciprocal, sextic polynomial

$$P(X) = \chi_A(X) = X^6 + a_1 X^5 + a_2 X^4 + a_3 X^3 + a_2 X^2 + a_1 X + 1 \in \mathbb{Z}[X]$$

with coefficients $a_1, a_2, a_3 \in \mathbb{R}$ given by

$$\begin{aligned} a_1 &= 312\, m^2 + 650\, m + 238 \\ a_2 &= 22032\, m^4 + 98280\, m^3 + 146568\, m^2 + 84520\, m + 15743 \\ a_3 &= 279936\, m^6 + 2099520\, m^5 + 6161184\, m^4 + 8927280\, m^3 \\ &\quad + 6611328\, m^2 + 2317980\, m + 299812 \end{aligned}$$

We have $1/X^3 \cdot P(X) = Q(X + 1/X + 2)$ for the cubic polynomial

$$Q(Y) = Y^3 + (a_1 - 6)\, Y^2 + (-4\, a_1 + a_2 + 9)\, Y + 2a_1 - 2a_2 + a_3 - 2 \tag{9.2.1.1}$$

(c.f. Remark 9.1.12). For $m \equiv 1$ modulo 13, the polynomials $P(X)$ and $Q(Y)$ can be written by irreducible factors modulo 13 as

$$P(X) \equiv X^6 + 4\, X^5 + 10\, X^4 + 6\, X^3 + 10\, X^2 + 4\, X + 1 \text{ modulo } 13 \quad \text{and}$$
$$Q(Y) \equiv Y^3 + 11\, Y^2 + 3\, Y + 5 \text{ modulo } 13.$$

In the sequel, we will assume $m \equiv 1$ modulo 13 and thus the polynomials $P(X)$ and $Q(Y)$ are irreducible over the rational numbers $\mathbb{Q}$.

As in Section 9.1.7 we will identify the Galois group $\mathrm{Gal}(P)$ of the reciprocal degree six polynomial $P(X) \in \mathbb{Z}[X]$ with a subgroup of the hyperoctahedral group G_3 as well as with a subgroup of the permutation group S_6 (see Section 9.1.1). The discriminant $\mathrm{Disc}(Q)$ of the polynomial $Q(Y) \in \mathbb{Z}[Y]$ has an irreducible factorization in terms of m as

$$\mathrm{Disc}(Q) = c_9 \left(\sum_{i=0}^{8} c_i\, m^i\right) (m+2)^2\,(3\,m+4)^2,$$

with coefficients

$$\begin{array}{lll} c_9 = 186624, & c_8 = 1778112, & c_7{=}7832160, \\ c_6 = 14307444, & c_5 = 13909500, & c_4{=}8133701, \\ c_3 = 2980770, & c_2 = 676093, & c_1{=}87020, \\ c_0 = 4900. & & \end{array}$$

By [86, Proposition 6.17] we can apply Siegel's theorem on integral points of algebraic curves[1], we have that the discriminant of $Q(Y)$ is a square of a rational number only for finitely many choices of m. This implies that $\mathrm{Gal}(Q) = S_3$ for all but finitely many $m \in \mathbb{N}$ with $m \equiv 1$ modulo 13. Furthermore $\mathrm{Gal}(P)$ the Galois-group of $P(X) \in \mathbb{Z}[X]$ is a subgroup of the hyperoctahedral group $G_3 = \mathbb{Z}_2^3 \rtimes S_3$ such that $\mathrm{Gal}(P)$ projects surjectively onto S_3.

The only non-trivial subgroups of $G_3 = \mathbb{Z}_2^3 \rtimes S_3$ which project surjectively onto S_3 are the groups $H_{3,1}, H_{3,2}$ and $H_{3,3}$ defined in Lemma 9.1.10.

Next we want to factorize the expressions $\Delta_{3,1} = \delta_{3,1}^2$ and $\Delta_{3,2} = \delta_{3,2}^2$ with $\delta_{3,1}$ and $\delta_{3,2}$ from Lemma 9.1.11 in terms of m. With the formulas from Remark 9.1.12 we get

$$\Delta_{3,1} = \delta_{3,1}^2 = c_7 \left(\sum_{i=0}^{6} c_i\, m^i\right) (2\,m+1)(3\,m+1)(m+2)^2(3\,m+4)^2$$

with coefficients

$$\begin{array}{llll} c_7 = 165888, & c_6 = 8748, & c_5 = 65610, & c_4 = 191160, \\ c_3 = 272835, & c_2 = 197463, & c_1 = 67195, & c_0 = 8400. \end{array}$$

and $\Delta_{3,2} = \delta_{3,2}^2 = \Delta_{3,1} \cdot \mathrm{Disc}(Q)$.

By applying Siegel's theorem again, we see that the expressions $\delta_{3,1}$ and $\delta_{3,2}$ are not rational numbers for all but finitely many $m \in \mathbb{N}$ with $m \equiv 1$ modulo 13. In particular, the Galois group $\mathrm{Gal}(P)$ of $P(X)$ is not contained in the subgroups $H_{3,1}$ or $H_{3,2}$ of the

[1] For example [58] is a reference for Siegel's theorem.

hyperoctahedral group G_3 by the fundamental theorem of Galois theory and Lemma 9.1.11.

Furthermore, if we have $m \equiv 1$ modulo 11, the discriminant $\mathrm{Disc}(P)$ of $P(X) \in \mathbb{Z}[X]$ is not divisible by 11 since $\mathrm{Disc}(P) = 9$ modulo 11 and the polynomial $P(X) \in \mathbb{Z}[X]$ can be written by irreducible factors as

$$P(X) \equiv (X^2 + 10\,X + 1)(X^4 + 2\,X^3 + 8\,X^2 + 2\,X + 1) \text{ modulo } 11.$$

If we view $\mathrm{Gal}(P)$ as a subgroup of $\mathrm{Sym}(\{\lambda_i, \lambda_i^{-1} \mid i = 1, 2, 3\})$, then Dedekind's theorem (cf. Section 9.1.1) says that $\mathrm{Gal}(P)$ contains a permutation of type $(2, 4)$ for $m \equiv 1$ modulo 11. The groups S_3 and $H_{3,3} \leqslant G_3$ on the other hand contain only non-trivial permutations of cycle type $(6), (3, 3), (2, 2, 2)$ or $(1, 1, 2, 2)$ (see Appendix A.1). Hence $\mathrm{Gal}(P)$ is not contained in one of the groups S_3, or $H_{3,3}$ of G_3 for $m \equiv 1$ modulo 11.

In summary, we showed the main part of the following proposition:

9.2.2 Proposition. For all but finitely many choices of $m \in \mathbb{N}$ such that $m \equiv 1$ modulo p, where $p \in \{11,\ 13\}$, we have that $A_4(N, M) = M_h^{(0)} \cdot M_v^{(0)} \in \mathbb{R}^{6\times 6}$ is a Galois pinching matrix.

Proof. The discriminant $\mathrm{Disc}(Q)$ of the cubic polynomial $Q(Y) \in \mathbb{Z}[Y]$ from (9.2.1.1) which we computed in 9.2.1, converges to infinity for growing m. This shows that $Q(Y)$ has three distinct real roots for $m \equiv 1$ modulo 13 big enough. Furthermore for m big enough all the coefficients of the polynomial $Q(Y)$ are positive and hence by Décarte's rule of signs the three roots μ_1, μ_2 and μ_3 of $Q(Y)$ are negative. By the choice of $Q(Y)$ we have

$$\mu_i = \lambda_i + \lambda_i^{-1} + 2 \quad \text{for } i = 1, 2, 3,$$

for the six roots $\{\lambda_i,\ \lambda_i^{-1} \mid i = 1, 2, 3\}$ of $P(X)$. With $\lambda_i^{-1} = \overline{\lambda_i}/|\lambda_i|^2$, we conclude for the imaginary part $\mathrm{Im}(\mu_i)$ of μ_i for every $i = 1, 2, 3$:

$$0 = \mathrm{Im}(\mu_i) = \mathrm{Im}(\lambda_i)(1 - 1/|\lambda_i|^2)$$

This shows $|\lambda_i| = 1$ or $\mathrm{Im}(\lambda_i) = 0$ for every $i = 1, 2, 3$. Assume that $\mathrm{Im}(\lambda_i) \neq 0$ for some $i \in \{1, 2, 3\}$. Then $|\lambda_i| = 1$ and

$$0 > \mathrm{Re}(\mu_i) = \mathrm{Re}(\lambda_i + \lambda_i^{-1} + 2) = \mathrm{Re}(\lambda_i) + \mathrm{Re}(\overline{\lambda_i}) + 2.$$

This would imply $\mathrm{Re}(\lambda_i) < -1$ a contradiction to $|\lambda_i| = 1$.

This shows that all roots of $P(X)$ are real for $m \equiv 1$ modulo 13 big enough. Together with the calculations on $\mathrm{Gal}(P)$ from this section we conclude that $A_4(N, M) \in \mathbb{R}^{6\times 6}$ is a Galois pinching matrix for every natural number m big enough such that $m \equiv 1$ modulo p, where $p \in \{11,\ 13\}$. □

From this statement, it is not hard to show with Theorem 1.6.1 that:

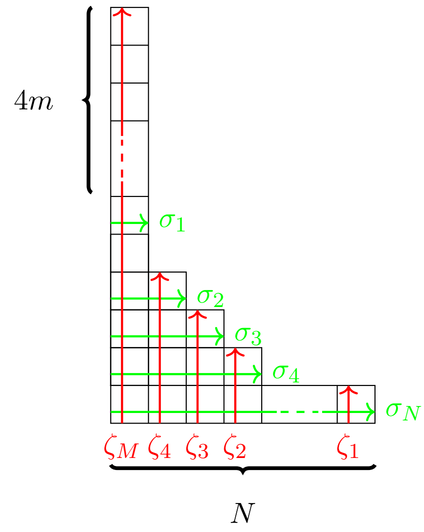


Figure 9.1.: Origami $\mathcal{O}^{(5)}_{N,M}$ with horizontal waist curves $\sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_N$ and vertical waist curves $\zeta_1, \zeta_2, \zeta_3, \zeta_4, \zeta_M$.

9.2.3 Theorem. The Kontsevich-Zorich monodromies of the genus four origamis $\mathcal{O}^{(4)}_{N,M} \in \Omega\mathcal{M}_4(6)$ with $M = 2m + 4$ and $N = 3m + 4$ are finite index subgroups of the symplectic group $\mathrm{Sp}(H_1^{(0)}(\mathcal{O}^{(4)}_{N,M}, \mathbb{Z}))$ for all but perhaps finitely many $m \in \mathbb{N}$ such that $m \equiv 1$ modulo p, where $p \in \{11, 13\}$.

Proof. One can check that the matrix $B \neq \mathrm{Id}$ associated to an appropriate Dehn twist in the direction $(1, 1)$ is a unipotent matrix such that the image $(B - \mathrm{Id})(\mathbb{R}^6)$ is one-dimensional and hence not a Lagrangian subspace (cf. the relevant matrix B is called $M^{(0)}_\delta$ in Section 8.2. Since the matrix $A_4(N, M)$ is Galois pinching for all but finitely many choices of $m \in \mathbb{N}$ with $m \equiv 1$ modulo p, where $p \in \{11, 13\}$ as in Proposition 9.2.2. Zariski density follows now from Criterion 1.5 and arithmeticity follows from Singh–Venkataramana's criterion (c.f Theorem 1.6.1). $\square$

9.3 Stairs origamis in genus five

Now we construct infinite families in genus five with similar methods as in Section 9.2. For $N, M \in \mathbb{N}$ with $M = 6 + 4m$ ($m \in \mathbb{N}$), we consider the origami $\mathcal{O}^{(5)}_{N,M} \in \Omega\mathcal{M}_5(8)$ that is given by the following horizontal and vertical permutation $h, v \in \mathrm{Sym}(\{1, 2, \ldots, N +$

$M + 5\}$):

$$\begin{aligned} h =& (1, \dots, N)(N+1, \dots, N+4)(N+5, \dots, N+7) \\ & (N+8, N+9)(N+10) \dots (N+M+5) \\ v =& (1, N+1, N+5, N+8, N+10, \dots, N+M+5) \\ & (2, N+2, N+6, N+9)(3, N+3, N+7)(4, N+4)(5) \dots (N) \end{aligned}$$

The five waist curves $\sigma_1, \dots, \sigma_4, \sigma_N$ of the maximal horizontal cylinders together with the waist curves $\zeta_1, \dots, \zeta_5, \zeta_M$ of the maximal vertical cylinders form a basis B of the absolute homology $H_1(\mathcal{O}^{(5)}_{N,M}, \mathbb{Q})$ of the origami $\mathcal{O}^{(5)}_{N,M}$ see Figure 9.1. With respect to the basis B the symplectic intersection form Ω on $H_1(\mathcal{O}^{(5)}_{N,M}, \mathbb{Q})$ has a matrix representation $M\Omega = (\Omega(\sigma_i, \zeta_j)_{i,j}$given by

$$M\Omega = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & -1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 \\ -1 & -1 & -1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

If we compare the length of the waist curves of the five maximal horizontal and vertical cylinders of $\mathcal{O}^{(5)}_{N,M}$, we see that

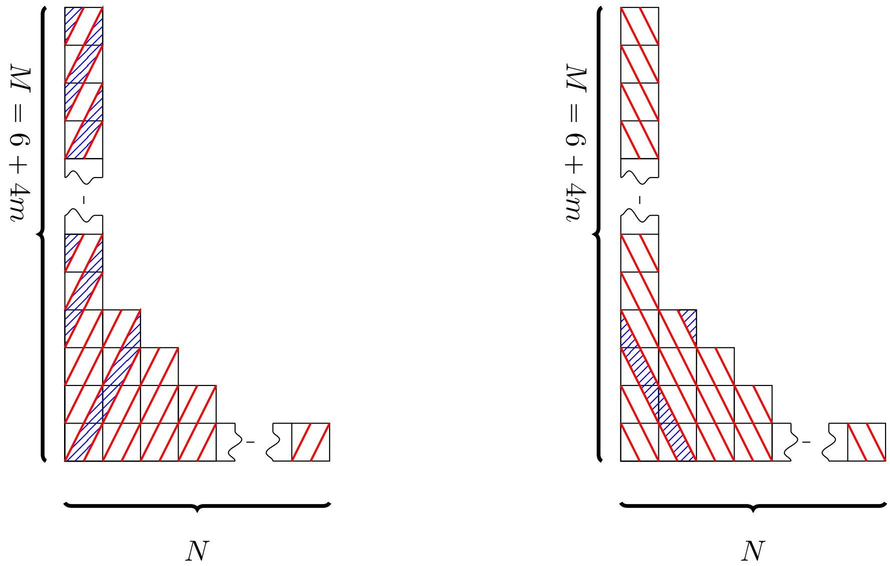
$$B^{(0)} = \{\Sigma_1,\, \Sigma_2,\, \Sigma_3,\, \Sigma_N,\, Z_1,\, Z_2,\, Z_3,\, Z_M\}$$

is a basis of the non-tautological part $H_1^{(0)}(\mathcal{O}^{(5)}_{N,M}, \mathbb{Q})$, where

$$\begin{aligned} &\Sigma_1 := \sigma_2 - 2\,\sigma_1, \quad \Sigma_2 := \sigma_3 - 3\,\sigma_1, \quad \Sigma_3 := \sigma_4 - 4\,\sigma_1, \quad \Sigma_N := \sigma_N - N\,\sigma_1, \\ &Z_1 := \zeta_2 - 2\,\zeta_1, \quad Z_2 := \zeta_3 - 3\,\zeta_1, \quad Z_3 := \zeta_4 - 4\,\zeta_1, \quad Z_M := \zeta_M - M\,\zeta_1. \end{aligned}$$

If we restrict the intersection form Ω to the subspace $H_1^{(0)}(\mathcal{O}^{(5)}_{N,M}, \mathbb{Q})$ of the absolute homology then it can be represented by the following matrix $M\Omega^{(0)} = (\Omega|H_1^{(0)}(\Sigma_i, Z_j))_{i,j}$ with respect to the basis $B^{(0)}$ from above:

$$M\Omega^{(0)} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & -2 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & -3 \\ 0 & 0 & 0 & 0 & -1 & -2 & -3 & 1-N-M \\ 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & -1 & 2 & 0 & 0 & 0 & 0 \\ -1 & -1 & -1 & 3 & 0 & 0 & 0 & 0 \\ 1 & 2 & 3 & M+N-1 & 0 & 0 & 0 & 0 \end{pmatrix}$$



(a) (b)

Figure 9.2.: Cylinder decomposition in direction $(1, 2)$ and direction $(1, -2)$ of the origami $\mathcal{O}^{(5)}_{N,M}$. Here γ_1 is the waist curve of the blue cylinder in direction $(1, 2)$ and α_1 is the waist curve of the blue cylinder in direction $(1, -2)$.

9.3.1 Dehn twists in genus five. Now we can start with our calculations. Recall that $M = 6 + 4m$. In this case, the cylinder decompositions of $\mathcal{O}^{(5)}_{N,M}$ in the directions $(1, 2)$, $(1, -2)$ and $(1, 4)$ have the following structure.

In the direction $(1, 2)$, we find a waist curve γ_1 of length $3 + 2m$ and a waist curve γ_2 of length $8 + N + 2m$. Thus, $\Gamma := (8 + N + 2m)\gamma_1 - (3 + 2m)\gamma_2 \in H_1^{(0)}(\mathcal{O}^{(5)}_{N,M}, \mathbb{Z})$ and this direction yields a transvection[2].

$$D_\gamma \colon v \longmapsto v + (8 + N + 2m)\,\Omega(\gamma_1, v)\,\gamma_1 + (3 + 2m)\,\Omega(\gamma_2, v)\,\gamma_2.$$

For later reference, let us observe that:

$$\begin{aligned}
&\Omega(\gamma_1, \sigma_1) = -1, && \Omega(\gamma_1, \sigma_2) = -1, && \Omega(\gamma_1, \sigma_3) = -1,\\
&\Omega(\gamma_1, \sigma_4) = -1, && \Omega(\gamma_1, \sigma_N) = -1, && \\
&\Omega(\gamma_1, \zeta_1) = 0, && \Omega(\gamma_1, \zeta_2) = 0, && \Omega(\gamma_1, \zeta_3) = 0,\\
&\Omega(\gamma_1, \zeta_4) = 1, && \Omega(\gamma_1, \zeta_M) = 2 + 2m &&
\end{aligned}$$

[2]That D_γ is indeed a transvection was shown for general multitwists along the waist curves of a two-cylinder decomposition in [8, Lemma 12]

and

$$\begin{aligned}
&\Omega(\gamma_2,\sigma_1) = -1, && \Omega(\gamma_2,\sigma_2) = -3, && \Omega(\gamma_2,\sigma_3) = -5,\\
&\Omega(\gamma_2,\sigma_4) = -7, && \Omega(\gamma_2,\sigma_N) = 1-2N, && \\
&\Omega(\gamma_2,\zeta_1) = 1, && \Omega(\gamma_2,\zeta_2) = 2, && \Omega(\gamma_2,\zeta_3) = 3,\\
&\Omega(\gamma_2,\zeta_4) = 3, && \Omega(\gamma_2,\zeta_M) = 4+2m &&
\end{aligned}$$

We can write $\Gamma \in H_1^{(0)}(\mathcal{O}_{N,M},\mathbb{Z})$ as a linear combination of elements of $B^{(0)}$ in the following way:

$$\begin{aligned}
\Gamma = &-(N+2m+8)\,\Sigma_1 + (2m+3)\,\Sigma_2 + (2m+3)\,\Sigma_3 + (2m+3)\,\Sigma_N\\
&+ M\,Z_1 + M\,Z_2 + M\,Z_3 - (N+5)\,Z_M
\end{aligned}$$

Furthermore we calculate

$$\begin{aligned}
&D_\gamma(\Sigma_1) = \Sigma_1 + \Gamma, && D_\gamma(\Sigma_2) = \Sigma_2 + 2\Gamma,\\
&D_\gamma(\Sigma_3) = \Sigma_3 + 3\Gamma, && D_\gamma(\Sigma_N) = \Sigma_N + (N-1)\Gamma,\\
&D_\gamma(Z_1) = Z_1, && D_\gamma(Z_2) = Z_2,\\
&D_\gamma(Z_3) = Z_3 + \Gamma, && D_\gamma(Z_M) = Z_M + (2+2m)\Gamma.
\end{aligned} \tag{9.3.1.1}$$

In the direction $(1,-2)$, we have a waist curve α_1 of length 2 and a waist curve α_2 of length $9+N+4m$. This yields a transvection

$$D_\alpha\colon v \longmapsto v + (9+N+4m)\,\Omega(\alpha_1,v)\,\alpha_1 + 2\,\Omega(\alpha_2,v)\,\alpha_2.$$

Again, for later reference, we note that:

$$\begin{aligned}
&\Omega(\alpha_1,\sigma_1) = 0, && \Omega(\alpha_1,\sigma_2) = 1, && \Omega(\alpha_1,\sigma_3) = 1,\\
&\Omega(\alpha_1,\sigma_4) = 1, && \Omega(\alpha_1,\sigma_N) = 1, && \\
&\Omega(\alpha_1,\zeta_1) = 0, && \Omega(\alpha_1,\zeta_2) = 0, && \Omega(\alpha_1,\zeta_3) = 0,\\
&\Omega(\alpha_1,\zeta_4) = 1, && \Omega(\alpha_1,\zeta_M) = 1, &&
\end{aligned}$$

and

$$\begin{aligned}
&\Omega(\alpha_2,\sigma_1) = 2, && \Omega(\alpha_2,\sigma_2) = 3, && \Omega(\alpha_2,\sigma_3) = 5,\\
&\Omega(\alpha_2,\sigma_4) = 7, && \Omega(\alpha_2,\sigma_N) = 2N-1, && \\
&\Omega(\alpha_2,\zeta_1) = 1, && \Omega(\alpha_2,\zeta_2) = 2, && \Omega(\alpha_2,\zeta_3) = 3,\\
&\Omega(\alpha_2,\zeta_4) = 3, && \Omega(\alpha_2,\zeta_M) = 5+4m, &&
\end{aligned}$$

We can write $A = (9+N+4M)\alpha_1 - 2\alpha_2 \in H_1^{(0)}(\mathcal{O}_{N,M}^{(5)},\mathbb{Z})$ as a linear combination of elements of $B^{(0)}$ as

$$A = -(9+N+4m)\Sigma_1 + 2\,\Sigma_2 + 2\,\Sigma_3 + 2\Sigma_N - 4\,Z_1 - 4\,Z_2 + (7+N+4m)\,Z_3 - 4\,Z_M.$$

We calculate

$$\begin{aligned} D_\alpha(\Sigma_1) &= \Sigma_1 + A, & D_\alpha(\Sigma_2) &= \Sigma_2 + A, \\ D_\alpha(\Sigma_3) &= \Sigma_3 + A, & D_\alpha(\Sigma_N) &= \Sigma_N + A, \\ D_\alpha(Z_1) &= Z_1, & D_\alpha(Z_2) &= Z_2, \\ D_\alpha(Z_3) &= Z_3 + A, & D_\alpha(Z_M) &= Z_M + A. \end{aligned} \tag{9.3.1.2}$$

This leads to a matrix representation $M_\alpha^{(0)}$ of D_α on $H_1^{(0)}(\mathcal{O}_{N,M}^{(5)}, \mathbb{Q})$ with respect to the basis $B^{(0)}$.

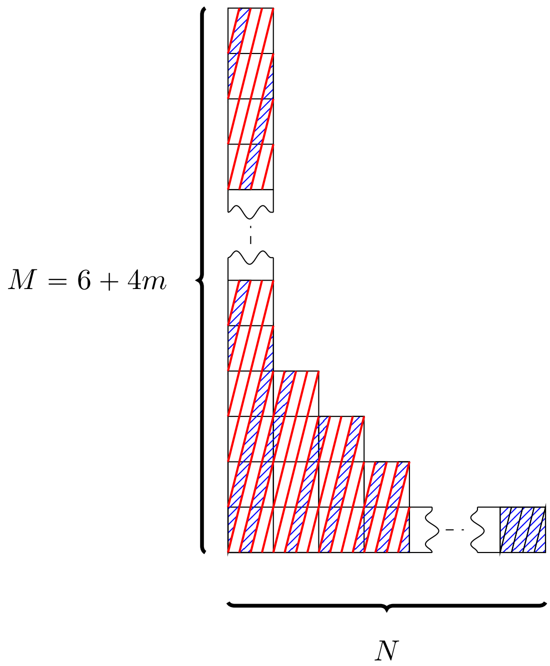


Figure 9.3.: Origami $\mathcal{O}_{N,M}^{(5)}$ with cylinder decomposition in direction $(1,4)$. Here χ_1 is the waist curve of the blue cylinder.

Finally, in the direction $(1,4)$, we find a waist curve χ_1 of length $1 + N + m$ and a waist curve χ_2 of length $10 + 3m$. The transvection associated to this direction is:

$$D_\chi \colon v \longmapsto v + (10 + 3m)\,\Omega(\chi_1, v)\,\chi_1 + (1 + N + m)\,\Omega(\chi_2, v)\,\chi_2$$

Also, let us remark that:

$$\begin{aligned} &\Omega(\chi_1, \sigma_1) = -1, && \Omega(\chi_1, \sigma_2) = -2, && \Omega(\chi_1, \sigma_3) = -4, \\ &\Omega(\chi_1, \sigma_4) = -6, && \Omega(\chi_1, \sigma_N) = -6 - 4(N - 4), && \\ &\Omega(\chi_1, \zeta_1) = 1, && \Omega(\chi_1, \zeta_2) = 1, && \Omega(\chi_1, \zeta_3) = 1, \\ &\Omega(\chi_1, \zeta_4) = 1, && \Omega(\chi_1, \zeta_M) = 2 + m, && \end{aligned} \tag{9.3.1.3}$$

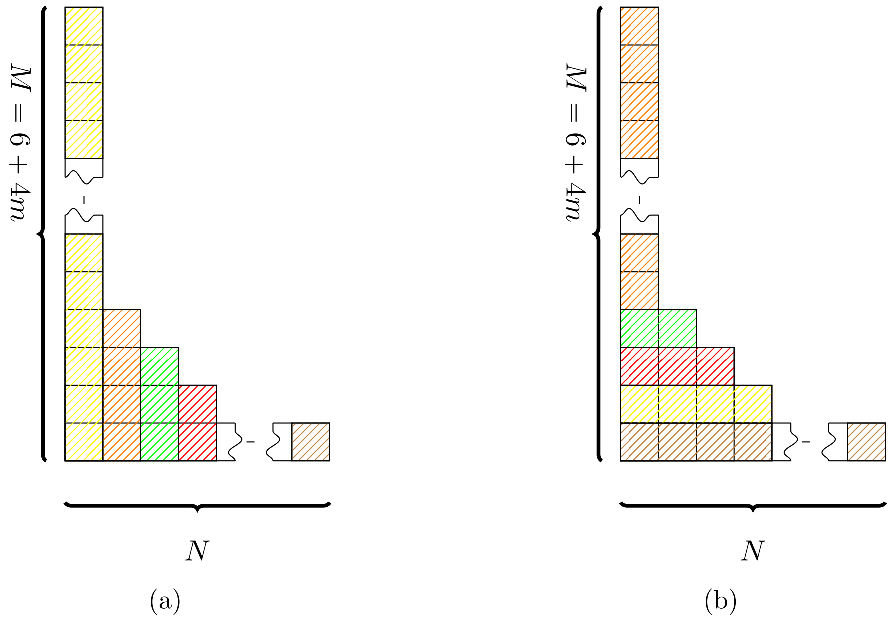


Figure 9.4.: Origami $\mathcal{O}^{(5)}_{N,M}$ with cylinder decomposition in vertical and horizontal direction.

as well as

$$\begin{aligned} &\Omega(\chi_2,\sigma_1) = -3, && \Omega(\chi_2,\sigma_2) = -6, && \Omega(\chi_2,\sigma_3) = -8,\\ &\Omega(\chi_2,\sigma_4) = -10, && \Omega(\chi_2,\sigma_N) = -10, && \\ &\Omega(\chi_2,\zeta_1) = 0, && \Omega(\chi_2,\zeta_2) = 1, && \Omega(\chi_2,\zeta_3) = 2,\\ &\Omega(\chi_2,\zeta_4) = 3, && \Omega(\chi_2,\zeta_M) = 4+3m. && \end{aligned} \tag{9.3.1.4}$$

We can write $X = (3m+10)\,\chi_1 - (N+m+1)\,\chi_2 \in H_1^{(0)}(\mathcal{O}^{(5)}_{N,M},\mathbb{Z})$ as a linear combination of elements of $B^{(0)}$ as

$$\begin{aligned} X = \;& (N+m+1)\,\Sigma_1 + (N+m+1)\,\Sigma_2 + (N+m+1)\Sigma_3 - (3m+10)\Sigma_N \\ &+ (2N-4m-18)\,Z_1 + (2N-4m-18)\,Z_2 + (3N-7)\,Z_3 + (3N-7)\,Z_M. \end{aligned}$$

We calculate for the image of $B^{(0)}$ under D_χ:

$$\begin{aligned} &D_\chi(\Sigma_1) = \Sigma_1, && D_\chi(\Sigma_2) = \Sigma_2 - X,\\ &D_\chi(\Sigma_3) = \Sigma_3 - 2X, && D_\chi(\Sigma_N) = \Sigma_N - (3N-10)X,\\ &D_\chi(Z_1) = Z_1 - X, && D_\chi(Z_2) = Z_2 - 2X,\\ &D_\chi(Z_3) = Z_3 - 3X, && D_\chi(Z_M) = Z_M - (4+3m)X. \end{aligned} \tag{9.3.1.5}$$

For the cylinder decompositions of the origami $\mathcal{O}^{(5)}_{N,M}$ in horizontal, resp. vertical direction we get in both cases five maximal cylinders with moduli $M-4,\ 1/2,\ 1/3, 1/4,\ 1/N$

for the horizontal direction and moduli $N-4,\ 1/2,\ 1/3,\ 1/4,\ 1/M$ for the vertical direction (see Figure 9.4a and Figure 9.4b). We get two Dehn twists which act on $H_1^{(0)}(\mathcal{O}_{N,M}^{(5)},\mathbb{Q})$ by the following mapping rules:

$$\begin{aligned} D_h\colon w \longmapsto w &+ 12(M-4)N\ \Omega(\sigma_1,w)\,\sigma_1 + 6N\,\Omega(\sigma_2,w)\,\sigma_2 \\ &+4N\ \Omega(\sigma_3,w)\,\sigma_3 + 3N\,\Omega(\sigma_4,w)\,\sigma_4 + 12\ \Omega(\sigma_N,w)\,\sigma_N, \end{aligned}$$

$$\begin{aligned} D_v\colon w \longmapsto w &+ 12(N-4)M\ \Omega(\zeta_1,w)\,\zeta_1 + 6M\ \Omega(\zeta_2,w)\,\zeta_2 \\ &+4M\ \Omega(\zeta_3,w)\,\zeta_3 + 3N\,\Omega(\zeta_4,w)\,\zeta_4 + 12\ \Omega(\zeta_M,w)\,\zeta_M. \end{aligned}$$

It is now easy to calculate representation matrices $M_h^{(0)}$ and $M_v^{(0)}$ for the action of the horizontal and vertical twist on $H_1^{(0)}(\mathcal{O}_{N,M}^{(5)},\mathbb{Q})$ with respect to the basis $B^{(0)}$:

$$M_h^{(0)} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 6\,N & 6\,N \\ 0 & 1 & 0 & 0 & 0 & 4\,N & 4\,N & 4\,N \\ 0 & 0 & 1 & 0 & 3\,N & 3\,N & 3\,N & 3\,N \\ 0 & 0 & 0 & 1 & -12 & -24 & -36 & 12-12\,M \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

and

$$M_v^{(0)} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -6\,M & -6\,M & 1 & 0 & 0 & 0 \\ 0 & -4\,M & -4\,M & -4\,M & 0 & 1 & 0 & 0 \\ -3\,M & -3\,M & -3\,M & -3\,M & 0 & 0 & 1 & 0 \\ 12 & 24 & 36 & -12+12\,N & 0 & 0 & 0 & 1 \end{pmatrix}$$

9.3.2 Finding a family of candidates in genus five. Recall that we found in the previous section the following elements of the non-tautological part $H_1^{(0)}(\mathcal{O}_{N,M}^{(5)}, \mathbb{Z})$:

$$\begin{aligned}
X &= (3m+10)\,\chi_1 - (N+m+1)\,\chi_2 \\
&= (N+m+1)\,\Sigma_1 + (N+m+1)\,\Sigma_2 \\
&\quad +(N+m+1)\Sigma_3 - (3m+10)\Sigma_N \\
&\quad +(2N-4m-18)\,Z_1 + (2N-4m-18)\,Z_2 \\
&\quad +(3N-7)\,Z_3 + (3N-7)\,Z_M, \\
A &= (9+N+4M)\alpha_1 - 2\alpha_2 \\
&= -(9+N+4m)\Sigma_1 + 2\,\Sigma_2 + 2\,\Sigma_3 + 2\Sigma_N \\
&\quad -4\,Z_1 - 4\,Z_2 + (7+N+4m)\,Z_3 - 4\,Z_M, \\
\Gamma &= (8+N+2m)\gamma_1 - (3+2m)\gamma_2 \\
&= -(N+2m+8)\,\Sigma_1 + (2m+3)\,\Sigma_2 + (2m+3)\,\Sigma_3 + (2m+3)\,\Sigma_N \\
&\quad +M\ Z_1 + M\,Z_2 + M\,Z_3 - (N+5)\,Z_M.
\end{aligned}$$

Consider the transvections D_χ, D_α and D_γ from the previous section and recall that the image of $D_\chi - \mathrm{id}$, $D_\alpha - \mathrm{id}$ respectively $D_\gamma - \mathrm{id}$ was generated by X, A respectively Γ. We consider the subspace $W = \mathrm{Span}_{\mathbb{Q}}(\{X, A, \Gamma\})$ of $H_1^{(0)}(\mathcal{O}_{N,M}^{(5)}, \mathbb{Q})$. With respect to the basis $\{X, A, \Gamma\}$, we can represent the restrictions of the transvections D_χ, D_α and D_γ to W by the following three matrices (compare Equations 9.3.1.1, 9.3.1.2 and 9.3.1.5):

$$\begin{pmatrix} 1 & b & a \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad \begin{pmatrix} 1 & 0 & 0 \\ -b & 1 & c \\ 0 & 0 & 1 \end{pmatrix}, \qquad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -a & -c & 1 \end{pmatrix},$$

where $a = -5N - 3Nm + 5m + 5$, $b = -9N + 21$ and $c = -2N + 8m + 2$. The element $e := -c\,X + a\,A - b\,\Gamma \in W$ is invariant under $(D_\chi)|_W$, $(D_\alpha)|_W$ respectively $(D_\gamma)|_W$. The two waist curves γ_1, γ_2 are linearly independent in $H_1(\mathcal{O}_{N,M}^{(5)}, \mathbb{Q})$ and the same holds for the waist curves α_1, α_2 and χ_1, χ_2. From the definition of the transvections D_χ, D_α and D_γ and the fact that the element e is invariant under them, we can directly see $\Omega(e, w) = 0$ for all $w \in W$. With respect to the new basis $\{X, A, e\}$ of W we have the following matrix representations for $(D_\chi)|_W$, $(D_\alpha)|_W$ and $(D_\gamma)|_W$:

$$\begin{pmatrix} 1 & b & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad \begin{pmatrix} 1 & 0 & 0 \\ -b & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad \begin{pmatrix} \frac{ac}{b} + 1 & \frac{c^2}{b} & 0 \\ -\frac{a^2}{b} & -\frac{ac}{b} + 1 & 0 \\ \frac{a}{b} & \frac{c}{b} & 1 \end{pmatrix}$$

where $b = -9N + 21 \neq 0$ for all $N \in \mathbb{N}$. If we now choose $c = 0$ or equivalently $N = 1 + 4m$ one can easily see that $a, b \neq 0$ for all N, $m \in \mathbb{N}$. We can see that the subgroup of $\mathrm{Sp}_\Omega(W)$ generated by $(D_\chi)|_W$, $(D_\alpha)|_W$ and $(D_\gamma)|_W$ contains a non-trivial element of the unipotent radical, namely $(D_\alpha)|_W^{-a^2} \circ (D_\gamma)|_W^{b^2}$. We would like to apply

Theorem 1.6.1 to obtain arithmeticity but first we have to show Zariski-density for the Kontsevich–Zorich monodromy of a decent subfamily among the $\mathcal{O}^{(5)}_{N,M}$. To find such a subfamily is our task in the next subsection.

9.3.3 Zariski density and arithmeticity for a genus five family. In this subsection we fix $M = 6 + 4m$ and $N = 1 + 4m$ ($m \in \mathbb{N}$). The characteristic polynomial of the matrix $A := A_5(N, M) := M_h^{(0)} \cdot M_v^{(0)} \in \mathbb{R}^{8\times 8}$ is given by a reciprocal polynomial

$$P(X) = \chi_A(X) = \sum_{i=0}^{8} a_i\, X^i \in \mathbb{Z}[X]$$

with $a_0 = a_8 = 1$ and $a_i = a_{8-i}$ for $i = 1, \ldots, 4$.

We have $1/X^4 \cdot P(X) = Q(X + 1/X + 2)$ for the quartic polynomial

$$Q(Y) = Y^4 + \sum_{i=0}^{3} b_i\, Y^i \in \mathbb{Z}[Y] \tag{9.3.3.1}$$

with coefficients

$$\begin{aligned} b_3 &= a_1 - 8, & b_2 &= a_2 - 6\,a_1 + 20,\\ b_1 &= a_3 - 4\,a_2 + 9\,a_1 - 16, & b_0 &= a_4 - 2\,a_3 + 2\,a_2 - 2\,a_1 + 2. \end{aligned}$$

Let now μ_1, μ_2, μ_3, $\mu_4 \in \mathbb{C}$ be the roots of the polynomial $Q(Y) \in \mathbb{Z}[Y]$ and $CR_Q(Y) \in \mathbb{Q}[Y]$ its cubic resolvent (cf. Section 9.1.2).

For $m \equiv 1$ modulo 31 one can compute with a computer algebra system like MATLAB that the polynomials $P(X) \in \mathbb{Z}[X]$ and $Q(Y) \in \mathbb{Z}[Y]$ are irreducible modulo 31 and hence irreducible over the rational numbers $\mathbb{Q}$. Furthermore the cubic resolvent $CR_Q(Y) \in \mathbb{Z}[Y]$ is irreducible modulo 11 if $m \equiv 1$ modulo 11 and in this case also irreducible over the rational numbers. The discriminant $\mathrm{Disc}(Q)$ of the polynomial $Q(Y)$ can be written by irreducible factors in terms of m as

$$\mathrm{Disc}(Q) = c \cdot f(m) \cdot (2\,m + 3)^6 \cdot (4\,m + 1)^6$$

for a positive integer c and a monic polynomial $f(m)$ of degree 12. With Siegel's theorem of integral points we conclude that $\mathrm{Disc}(Q)$ can only be a square of a rational number for finitely many $m \in \mathbb{N}$. As in Section 9.2.1 we conclude that the Galois group $\mathrm{Gal}(Q)$ of $Q(Y)$ can be identified with the full symmetric group $\mathrm{Sym}(\{\mu_1, \ldots, \mu_4\})$ for all but perhaps finitely many $m \in \mathbb{N}$ with $m \equiv 1$ modulo $p \in \{11, 31\}$. In these cases the Galois group $\mathrm{Gal}(P) \leqslant \mathbb{Z}_2^4 \rtimes S_4$ of our reciprocal polynomial $P(X) = \mathrm{char}_A(X)$ projects surjectively on S_4 and hence $\mathrm{Gal}(P)$ can be identified with S_4, with one of the groups $H_{4,i}$ ($i = 1, 2, 3$) or with the full hyperoctahedral group $G_4 = \mathbb{Z}_2^4 \rtimes S_4$ (see Proposition 9.1.10).

With the help of MATLAB we see that we can write $\Delta_{4,1} = \delta_{4,1}^2$ from Lemma 9.1.11 by irreducible factors in terms of m as

$$\Delta_{4,1} = c \cdot g(m) \cdot (2\,m+1)(4\,m-3)(2\,m+3)^3(4\,m+1)^3$$

for an integer c and a monic polynomial $g(m)$ of degree 8. Furthermore $\Delta_{4,2} = \mathrm{Disc}(Q) \cdot \Delta_{4,1}$.

With [86, Proposition 6, 17] respectively Siegel's theorem on integral points we conclude that $\delta_{4,1}$ and $\delta_{4,2}$ are rational numbers only for finitely many $m \in \mathbb{N}$. Since $H_{4,3}$ is a subgroup of $H_{4,1}$ the fundamental theorem of Galois theory together with Lemma 9.1.11 shows that $\mathrm{Gal}(P) \neq H_{4,i}$ for $i = 1, 2, 3$.

We showed the hardest part of the following proposition:

9.3.4 Proposition. The matrix $A = A_5(N, M) \in \mathbb{R}^{8\times 8}$ is Galois pinching for all but finitely many $m \in \mathbb{N}$ with $m \equiv 1$ modulo p, where $p \in \{11,\ 31\}$.

Proof. The only thing that is left to show, is that for $m \in \mathbb{N}$ as in the statement and big enough, the matrix $A_5(N, M)$ has only real eigenvalues. For this reason we analyse the roots of the polynomial $Q(Y) \in \mathbb{Z}$ from Equation (9.3.3.1) with the help of (9.1.13.3) in Section 9.1.13. First we bring $Q(Y)$ in the depressed form $DQ(t)$ by substituting $t = Y - b_3/4$. If we determine the roots of the depressed form $DQ(t)$, we can determine them directly for $Q(Y)$ as well. By a boring but not to complicated analysis of the expressions $\mathrm{Disc}(DQ)$, $F(DQ)$ and q for $DQ(Y)$ as in 9.1.13.3 (or by using a computer algebra system), we see that

$$\mathrm{Disc}(DQ) > 0, \quad F(DQ) > 0 \quad \textit{and} \quad q < 0$$

for m big enough.

Hence in that case all the roots of $DQ(t)$ and thus all the roots of $Q(Y)$ are real. As in the proof of 9.2.2 we conclude that $A_5(N, M)$ has only real eigenvalues for $m \in \mathbb{N}$ with $m \equiv 1$ modulo p, where $p \in \{11,\ 31\}$ and m big enough. $\square$

Since $B_5(N, M) := M_\alpha^{(0)}$ is an unipotent matrix such that the image $(B_5(N, M) - \mathrm{Id})(\mathbb{R}^8)$ is not a Lagrangian subspace, the matrices $A_5(N, M)$ and $B_5(N, M)$ generate a Zariski-dense subgroup of $\mathrm{Sp}(H_1^{(0)}(\mathcal{O}_{N,M}^{(5)}, \mathbb{Z}))$ by Criterion 1.5.10 for all $m \in \mathbb{N}$ as in Proposition 9.3.4. Together with the theorem of Singh-Venkataramana 1.6.1 and the result in Subsection 9.3.2, we conclude:

9.3.5 Theorem. The genus five origamis $\mathcal{O}_{N,M}^{(5)} \in \Omega\mathcal{M}_5(8)$ with $N = 1 + 4\,m$ and $M = 6 + 4\,m$ have Kontsevich–Zorich monodromies with finite index in $\mathrm{Sp}(H_1^{(0)}(\mathcal{O}_{N,M}^{(5)}, \mathbb{Z}))$ for all but finitely many $m \in \mathbb{N}$ such that $m \equiv 1$ modulo p, where $p \in \{11,\ 31\}$.

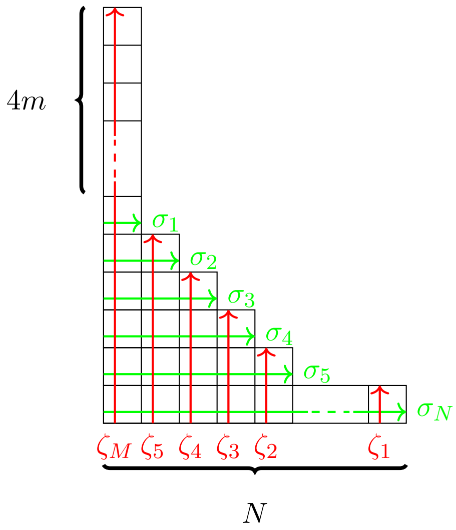


Figure 9.5.: Origami $\mathcal{O}^{(6)}_{N,M}$ with horizontal waist curves $\sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5, \sigma_N$ and vertical waist curves $\zeta_1, \zeta_2, \zeta_3, \zeta_4, \zeta_5, \zeta_M$.

9.4 Stairs origamis in genus six

Lastly we will expand out investigation in genus six. We will see that the Galois theory is even more complicated then for the genus four and five families. Thus we did not try to find families of origamis with arithmetic Kontsevich–Zorich monodromy in higher genus. For $N, M \in \mathbb{N}$ with $M = 6 + 4m$ ($m \in \mathbb{N}$), we consider the genus six origami $\mathcal{O}^{(6)}_{N,M} \in \Omega\mathcal{M}_6(10)$ that is given by the following horizontal and vertical permutation $h, v \in \mathrm{Sym}(\{1, 2, \ldots, N + M + 9\})$:

$$\begin{aligned} h =&(1, \ldots, N)(N + 1, \ldots, N + 5)(N + 6, \ldots, N + 9) \\ &(N + 10, N + 11, N + 12) \\ &(N + 13, N + 14)(N + 15) \ldots (N + M + 9) \end{aligned}$$

$$\begin{aligned} v =&(1, N + 1, N + 6, N + 10, N + 13, N + 15, \ldots, N + M + 9) \\ &(2, N + 2, N + 7, N + 11, N + 14)(3, N + 3, N + 8, N + 12) \\ &(4, N + 4, N + 9)(5, N + 5)(6) \ldots (N) \end{aligned}$$

The six waist curves $\sigma_1, \ldots, \sigma_5,\ \sigma_N$ of the maximal horizontal cylinders together with the waist curves $\zeta_1, \ldots, \zeta_5, \zeta_M$ of the maximal vertical cylinders form again a basis of the absolute homology $H_1(\mathcal{O}^{(6)}_{N,M}, \mathbb{Z})$ of the origami $\mathcal{O}^{(6)}_{N,M}$ see Figure 9.5.

It is easy to see that $B^{(0)} = \{\Sigma_i,\ \Sigma_N,\ Z_i,\ Z_M \mid i = 1, \ldots, 4\}$ is a basis of the non-tautological part $H_1^{(0)}(\mathcal{O}_{N,M}^{(6)}, \mathbb{Q})$, where

$$\Sigma_i := \sigma_{i+1} - (i+1)\,\sigma_1 \text{ for } i = 1,\dots,4, \quad \Sigma_N := \sigma_N - N\,\sigma_1,$$
$$Z_i := \zeta_{i+1} - (i+1)\,\zeta_1 \text{ for } i = 1,\dots,4, \quad Z_N := \zeta_N - N\,\zeta_1.$$

We can represent the restriction of the intersection form Ω to the non-tautological part $H_1^{(0)}(\mathcal{O}_{N,M}^{(6)}, \mathbb{Q})$ of the absolute homology by the following matrix with respect to the basis $B^{(0)}$ from above:

$$\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & -2 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & -3 \\
0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & -4 \\
0 & 0 & 0 & 0 & 0 & -1 & -2 & -3 & -4 & 1-N-M \\
0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & -1 & 2 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & -1 & -1 & 3 & 0 & 0 & 0 & 0 & 0 \\
-1 & -1 & -1 & -1 & 4 & 0 & 0 & 0 & 0 & 0 \\
1 & 2 & 3 & 4 & M+N-1 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}$$

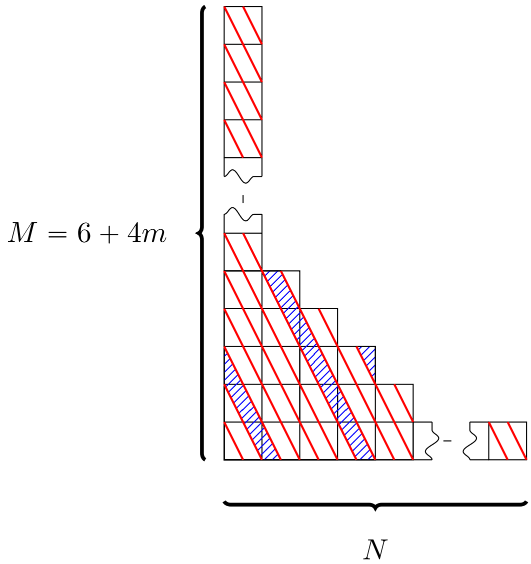


Figure 9.6.: Origami $\mathcal{O}_{N,M}^{(6)}$ with cylinder decomposition in direction $(1, -2)$. Here γ_1 is the waist curve of the blue cylinder.

9.4.1 Dehn twists in genus six. In direction $(1, -2)$ there are two maximal cylinders with one waist curve γ_1 of combinatorial length 4 and one waist curve γ_2 of combinatorial length $N+4m+11$. We count intersection points of γ_1 and γ_2 with the elements of $B^{(0)}$ and get

$$\begin{aligned}
&\Omega(\gamma_1,\sigma_1)=0, && \Omega(\gamma_1,\sigma_2)=1, && \Omega(\gamma_1,\sigma_3)=1,\\
&\Omega(\gamma_1,\sigma_4)=2, && \Omega(\gamma_1,\sigma_5)=2, && \Omega(\gamma_1,\sigma_N)=2,
\end{aligned}$$

$$\begin{aligned}
&\Omega(\gamma_1,\zeta_1)=0, && \Omega(\gamma_1,\zeta_2)=0, && \Omega(\gamma_1,\zeta_3)=1,\\
&\Omega(\gamma_1,\zeta_4)=1, && \Omega(\gamma_1,\zeta_5)=1, && \Omega(\gamma_1,\zeta_M)=1,
\end{aligned}$$

$$\begin{aligned}
&\Omega(\gamma_2,\sigma_1)=2, && \Omega(\gamma_2,\sigma_2)=3, && \Omega(\gamma_2,\sigma_3)=5,\\
&\Omega(\gamma_2,\sigma_4)=6, && \Omega(\gamma_2,\sigma_5)=8, && \Omega(\gamma_2,\sigma_N)=2(N-1),
\end{aligned}$$

$$\begin{aligned}
&\Omega(\gamma_2,\zeta_1)=1, && \Omega(\gamma_2,\zeta_2)=2, && \Omega(\gamma_2,\zeta_3)=2,\\
&\Omega(\gamma_2,\zeta_4)=3, && \Omega(\gamma_2,\zeta_5)=4, && \Omega(\gamma_2,\zeta_M)=5+4m.
\end{aligned}$$

We can write $\Gamma := (N+4m+11)\gamma_1 - 4\gamma_2 \in H_1^{(0)}(\mathcal{O}_{N,M}^{(6)},\mathbb{Z})$ as a linear combination of elements of $B^{(0)}$ in the following way:

$$\begin{aligned}
\Gamma = \;& 4\,\Sigma_1 + 4\,\Sigma_2 - (N+4m+11)\,\Sigma_3 + 4\,\Sigma_4 + 4\,\Sigma_N\\
&- 8\,Z_1 + (N+4m+7)\,Z_2 - 8\,Z_3 + (N+4m+7)\,Z_4 - 8\,Z_M.
\end{aligned}$$

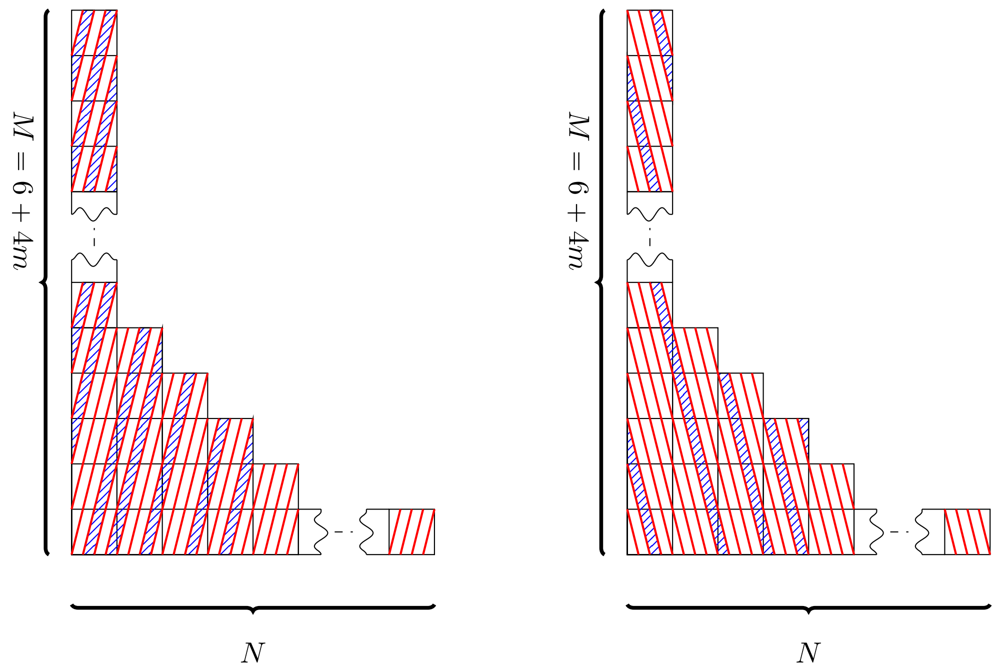
The Dehn twist along the waist curves γ_1 and γ_2 acts on $H_1^{(0)}(\mathcal{O}_{N,M}^{(6)},\mathbb{Q})$ via the mapping

$$D_\gamma\colon v \longmapsto v + (N+4m+11)\,\Omega(\gamma_1,v)\,\gamma_1 + 4\,\Omega(\gamma_2,v)\,\gamma_2$$

and for the images of the elements in $B^{(0)}$ under D_γ we get

$$\begin{aligned}
&D_\gamma(\Sigma_1)=\Sigma_1+\Gamma, && D_\gamma(\Sigma_2)=\Sigma_2+\Gamma, && D_\gamma(\Sigma_3)=\Sigma_3+2\,\Gamma,\\
&D_\gamma(\Sigma_4)=\Sigma_4+2\,\Gamma, && D_\gamma(\Sigma_N)=\Sigma_N+2\,\Gamma, && \\
&D_\gamma(Z_1)=Z_1, && D_\gamma(Z_2)=Z_2+\Gamma, && D_\gamma(Z_3)=Z_3+\Gamma\\
&D_\gamma(Z_4)=Z_4+\Gamma, && D_\gamma(Z_M)=Z_M+\Gamma. &&
\end{aligned}$$

For direction $(1,4)$ there are two maximal cylinders with waist curve δ_1 of length $2m+6$



(a) (b)

Figure 9.7.: Origami $\mathcal{O}^{(6)}_{N,M}$ with cylinder decomposition in direction $(1, 4)$ and $(1, -4)$. Here δ_1 and α_1 are the waist curves of the blue cylinders.

and waist curve δ_2 of length $N+2m+9$. We have

$$\begin{aligned}
&\Omega(\delta_1,\sigma_1) = -2, && \Omega(\delta_1,\sigma_2) = -3, && \Omega(\delta_1,\sigma_3) = -4,\\
&\Omega(\delta_1,\sigma_4) = -5, && \Omega(\delta_1,\sigma_5) = -5, && \Omega(\delta_1,\sigma_N) = -5,\\
&\Omega(\delta_1,\zeta_1) = 0, && \Omega(\delta_1,\zeta_2) = 0, && \Omega(\delta_1,\zeta_3) = 1,\\
&\Omega(\delta_1,\zeta_4) = 1, && \Omega(\delta_1,\zeta_5) = 2, && \Omega(\delta_1,\zeta_M) = 2+2m,\\
&\Omega(\delta_2,\sigma_1) = -2, && \Omega(\delta_2,\sigma_2) = -5, && \Omega(\delta_2,\sigma_3) = -8,\\
&\Omega(\delta_2,\sigma_4) = -11, && \Omega(\delta_2,\sigma_5) = -15, && \Omega(\delta_2,\sigma_N) = -4N+5,\\
&\Omega(\delta_2,\zeta_1) = 1, && \Omega(\delta_2,\zeta_2) = 2, && \Omega(\delta_2,\zeta_3) = 2,\\
&\Omega(\delta_2,\zeta_4) = 3, && \Omega(\delta_2,\zeta_5) = 3, && \Omega(\delta_2,\zeta_M) = 4+2m.
\end{aligned}$$

The element $\Delta := (N+2m+9)\delta_1 - (2m+6)\delta_2 \in H_1^{(0)}(\mathcal{O}^{(6)}_{N,M},\mathbb{Z})$ can be written as a linear combination of elements of $B^{(0)}$ as

$$\begin{aligned}
\Delta = &-(N+2m+9)\,\Sigma_1 + (2m+6)\,\Sigma_2 - (N+2m+9)\,\Sigma_3\\
&+ (2m+6)\,\Sigma_4 + (2m+6)\,\Sigma_N\\
&+ (8m+24)\,Z_1 + (4m-N+9)\,Z_2 + (4m-N+9)\,Z_3\\
&+ (4m-N+9)\,Z_4 - (2N+6)\,Z_M.
\end{aligned}$$

The Dehn twist along the waist curves δ_1 and δ_2 of the maximal cylinders acts on the non-tautological part of the absolute homology via the mapping

$$D_\delta\colon v \longmapsto v + (N+2m+9)\,\Omega(\delta_1,v)\,\delta_1 + (2m+6)\,\Omega(\delta_2,v)\,\delta_2.$$

If we evaluate the elements of the basis $B^{(0)}$ of $H_1^{(0)}(\mathcal{O}^{(6)}_{N,M},\mathbb{Q})$, then we get

$$\begin{aligned}
&D_\delta(\Sigma_1) = \Sigma_1 + \Delta, && D_\delta(\Sigma_2) = \Sigma_2 + 2\,\Delta, && D_\delta(\Sigma_3) = \Sigma_3 + 3\,\Delta,\\
&D_\delta(\Sigma_4) = \Sigma_4 + 5\,\Delta, && D_\delta(\Sigma_N) = \Sigma_n + (2N-5)\,\Delta, &&\\
&D_\delta(Z_1) = Z_1, && D_\delta(Z_2) = Z_2 + \Delta, && D_\delta(Z_3) = Z_3 + \Delta,\\
&D_\delta(Z_4) = Z_4 + 2\,\Delta, && D_\delta(Z_M) = Z_M + (2+2m)\,\Delta. &&
\end{aligned}$$

We have two maximal cylinders in direction $(1,-4)$ with waist curve α_1 of combinatorial length $4+m$ and waist curve α_2 of combinatorial length $N+3m+11$. We calculate the following intersection points with the elements of the basis $B^{(0)}$:

$$\begin{array}{lll}
\Omega(\alpha_1,\sigma_1) = 1, & \Omega(\alpha_1,\sigma_2) = 1, & \Omega(\alpha_1,\sigma_3) = 2,\\
\Omega(\alpha_1,\sigma_4) = 4, & \Omega(\alpha_1,\sigma_5) = 4, & \Omega(\alpha_1,\sigma_N) = 4,\\
\Omega(\alpha_1,\zeta_1) = 0, & \Omega(\alpha_1,\zeta_2) = 0, & \Omega(\alpha_1,\zeta_3) = 1,\\
\Omega(\alpha_1,\zeta_4) = 1, & \Omega(\alpha_1,\zeta_5) = 1, & \Omega(\alpha_1,\zeta_M) = 1+m,\\
\Omega(\alpha_2,\sigma_1) = 3, & \Omega(\alpha_2,\sigma_2) = 7, & \Omega(\alpha_2,\sigma_3) = 10,\\
\Omega(\alpha_2,\sigma_4) = 12, & \Omega(\alpha_2,\sigma_5) = 16, & \Omega(\alpha_2,\sigma_N) = N-4,\\
\Omega(\alpha_2,\zeta_1) = 1, & \Omega(\alpha_2,\zeta_2) = 2, & \Omega(\alpha_2,\zeta_3) = 2,\\
\Omega(\alpha_2,\zeta_4) = 3, & \Omega(\alpha_2,\zeta_5) = 4, & \Omega(\alpha_2,\zeta_M) = 5+3m.
\end{array}$$

With this information we can write the element $A := (N+3m+11)\alpha_1 - (4+m)\alpha_2$ in the basis $B^{(0)}$:

$$\begin{aligned}
A = \;& (m+4)\,\Sigma_1 + (m+4)\,\Sigma_2 - (N+3m+11)\,\Sigma_3\\
&+ (m+4)\,\Sigma_4 + (m+4)\,\Sigma_N\\
&- (4m+16)\,Z_1 + (2N+4m+14)\,Z_2 + (N-1)\,Z_3\\
&- (4m+16)\,Z_4 + (N-1)\,Z_M.
\end{aligned}$$

The map

$$D_\alpha\colon v \longmapsto v + (N+3m+11)\,\Omega(\alpha_1,v)\,\alpha_1 + (m+4)\,\Omega(\alpha_2,v)\,\alpha_2$$

has images

$$\begin{array}{lll}
D_\alpha(\Sigma_1) = \Sigma_1 - A, & D_\alpha(\Sigma_2) = \Sigma_2 - A & D_\alpha(\Sigma_3) = \Sigma_3,\\
D_\alpha(\Sigma_4) = \Sigma_4 - A, & D_\alpha(\Sigma_N) = \Sigma_N - (N-4)\,A, & \\
D_\alpha(Z_1) = Z_1, & D_\alpha(Z_2) = Z_2 + A, & D_\alpha(Z_3) = Z_3 + A,\\
D_\alpha(Z_4) = Z_4 + A, & D_\alpha(Z_M) = Z_M + (m+1)\,A. &
\end{array}$$

In horizontal direction we have six maximal cylinders with moduli $M-5$, $1/2$, $1/3$, $1/4$, $1/5$ and $1/N$ respectively in vertical direction there are six maximal cylinders with moduli $N-5$, $1/2$, $1/3$, $1/4$, $1/5$ and $1/N$. As in the sections before we can calculate representation matrices for the action of the associated Dehn twists on $H_1^{(0)}(\mathcal{O}_{N,M}^{(6)},\mathbb{Q})$.

In horizontal direction we have:

$$M_h^{(0)} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 30N & 30N \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 20N & 20N & 20N \\ 0 & 0 & 1 & 0 & 0 & 0 & 15N & 15N & 15N & 15N \\ 0 & 0 & 0 & 1 & 0 & 12N & 12N & 12N & 12N & 12N \\ 0 & 0 & 0 & 0 & 1 & -60 & -120 & -180 & -240 & -60(M-1) \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

and in vertical direction we get:

$$M_v^{(0)} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -30M & -30M & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -20M & -20M & -20M & 0 & 1 & 0 & 0 & 0 \\ 0 & -15M & -15M & -15M & -15M & 0 & 0 & 1 & 0 & 0 \\ -12M & -12M & -12M & -12M & -12M & 0 & 0 & 0 & 1 & 0 \\ 60 & 120 & 180 & 240 & 60(N-1) & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

9.4.2 Finding a family of candidates in genus six. Recall that we obtained elements $\Gamma,\ \Delta,\ A$ of the non-tautological part $H_1^{(0)}(\mathcal{O}_{N,M}^{(6)}, \mathbb{Z})$ of the absolute homology by comparing the waist curves of the maximal cylinders in direction $(1,-2)$, $(1,4)$ and $(1,-4)$. We wrote them as a linear combination of elements of $B^{(0)}$ in the following way:

$$\begin{aligned}
A = \; & (m+4)\,\Sigma_1 + (m+4)\,\Sigma_2 - (N+3m+11)\,\Sigma_3 \\
& + (m+4)\,\Sigma_4 + (m+4)\,\Sigma_N \\
& - (4m+16)\,Z_1 + (2N+4m+14)\,Z_2 + (N-1)\,Z_3 \\
& - (4m+16)\,Z_4 + (N-1)\,Z_M \\
\Gamma = \; & 4\,\Sigma_1 + 4\,\Sigma_2 - (N+4m+11)\,\Sigma_3 \\
& + 4\,\Sigma_4 + 4\,\Sigma_N \\
& - 8\,Z_1 + (N+4m+7)\,Z_2 - 8\,Z_3 + (N+4m+7)\,Z_4 - 8\,Z_M \\
\Delta = & -(N+2m+9)\,\Sigma_1 + (2m+6)\,\Sigma_2 - (N+2m+9)\,\Sigma_3 \\
& + (2m+6)\,\Sigma_4 + (2m+6)\,\Sigma_N \\
& + (8m+24)\,Z_1 + (4m-N+9)\,Z_2 + (4m-N+9)\,Z_3 \\
& + (4m-N+9)\,Z_4 - (2N+6)\,Z_M.
\end{aligned}$$

Let $W = \mathrm{Span}_{\mathbb{Q}}(A,\,\Gamma,\,\Delta\}$ the $\mathbb{Q}$-linear subspace of $H_1^{(0)}(\mathcal{O}_{N,M}^{(6)},\mathbb{Q})$ spanned by $A,\ \Gamma$ and Δ. The three maps $D_\alpha,\ D_\gamma$ and D_δ are transvections on $H_1^{(0)}(\mathcal{O}_{N,M}^{(6)},\mathbb{Q})$. The images $D_\alpha - \mathrm{id},\ D_\gamma - \mathrm{id}$ respectively $D_\delta - \mathrm{id}$ are generated by the elements A, Γ respectively Δ of $H_1^{(0)}(\mathcal{O}_{N,M}^{(6)},\mathbb{Z})$. If we restrict $D_\alpha,\ D_\gamma$ and D_δ to the subspace W, then we obtain the following three matrix representations with respect to $\{A,\,\Gamma,\,\Delta\}$:

$$\begin{pmatrix} 1 & b & a \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad \begin{pmatrix} 1 & 0 & 0 \\ -b & 1 & c \\ 0 & 0 & 1 \end{pmatrix}, \qquad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -a & -c & 1 \end{pmatrix},$$

where $b = 2 - 2N$, $a = -10\,N + 12\,m - 4\,mN + 42$ and $c = 16\,m + 24 - 8\,N$. The element $e := c\,A - a\,\Gamma + b\,\Delta$ is invariant under the elements $(D_\alpha)|_W,\ (D_\gamma)|_W,\ (D_\delta)|_W \in \mathrm{Sp}_\Omega(W)$ and an element of the nullspace W^Ω. The restrictions of $D_\alpha,\ D_\gamma$ and D_δ to the subspace W have the following matrix representations with respect to the basis $\{A,\,\Gamma,\,e\}$:

$$\begin{pmatrix} 1 & b & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad \begin{pmatrix} 1 & 0 & 0 \\ -b & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad \begin{pmatrix} \frac{ac}{b}+1 & \frac{c^2}{b} & 0 \\ \frac{a^2}{b} & \frac{ac}{b}+1 & 0 \\ -\frac{a}{b} & -\frac{c}{b} & 1 \end{pmatrix}.$$

If we choose $c = 0$ or $N = 3 + 2m$, then we can easily find an element of the unipotent radical of $\mathrm{Sp}_\Omega(W)$ in the subgroup generated by the three transvections $(D_\alpha)|_W,\ (D_\gamma)|_W$ and $(D_\delta)|_W$, for example $(D_\gamma)|_W^{a^2} \circ (D_\delta)|_W^{b}$ lies in the unipotent radical.

9.4.3 Zariski density and arithmeticity for a genus six family. We consider in this subsection the family of origamis $\mathcal{O}_{N,M}^{(6)}$, where $M = 6 + 4m$, $N = 3 + 2m$ and $m \in \mathbb{N}$. As before in the genus four and five section, we try first to determine an infinite family of natural numbers $m \in \mathbb{N}$ for which the matrix $A = A_6(N, M) = M_h^{(0)} \cdot M_v^{(0)} \in \mathbb{R}^{10\times 10}$ is Galois pinching. The characteristic polynomial

$$P(X) := \chi_A(X) = \sum_{i=0}^{10} a_i\, X^i \in \mathbb{Z}[X]$$

of the matrix $A = A_6(N, M) \in \mathbb{R}^{10\times 10}$ is monic and reciprocal, i.e. $a_{10} = a_0 = 1$ and $a_i = a_{10-i}$ for $i = 1, \ldots, 5$. Hence there is a cubic polynomial

$$Q(Y) = Y^5 + \sum_{i=0}^{4} b_i\, Y^i \in \mathbb{Q}[Y] \tag{9.4.3.1}$$

such that $1/X^5 \cdot P(X) = Q(X + 1/X + 2)$. The coefficients of $Q(Y) \in \mathbb{Q}[Y]$ are

$$\begin{aligned} &b_4 = a_1 - 10, \quad b_3 = \; a_2 - 8\, a_1 + 35, \quad b_2 = \; a_3 - 6\, a_2 + 20\, a_1 - 50, \\ &b_1 = a_4 - 4\, a_3 + 9\, a_2 - 16\, a_1 + 25, \quad b_0 = \; a_5 - 2\, a_4 + 2\, a_3 - 2\, a_2 + 2\, a_1 - 2. \end{aligned}$$

Denote the sextic Weber resolvent of $Q(Y) \in \mathbb{Q}[Y]$ again by $SWR_Q(Y) \in \mathbb{Q}[Y]$. For $m \equiv 2$ modulo 89 we computed with MATLAB that $P(X) = \chi_A(X)$ can be irreducibly written modulo 89 as

$$\begin{aligned} P(X) \equiv X^{10} &+ 4\, X^9 + 63\, X^8 + 33\, X^7 + 39\, X^6 + 71\, X^5 \\ &+39\, X^4 + 33\, X^3 + 63\, X^2 + 4\, X + 1 \text{ modulo } 89. \end{aligned}$$

Furthermore for $m \equiv 2$ modulo 17, respectively $m \equiv 2$ modulo 19 we computed that $Q(Y)$ respectively $SWR_Q(Y)$ factorizes as

$$\begin{aligned} Q(Y) &\equiv Y^5 + 4\, Y^4 + Y^2 + 6\, Y + 16 \text{ modulo } 17 \qquad \text{and} \\ SWR_Q(Y) &\equiv Y^6 + 13\, Y^5 + 7\, Y^4 + 3\, Y^3 + 15\, Y^2 + 17\, Y + 16 \text{ modulo } 19. \end{aligned}$$

Hence $Q(Y)$ is irreducible modulo 17 and $SWR_Q(Y)$ is irreducible modulo 19. Since $P(X) \in \mathbb{Q}[X]$ is irreducible for $m \equiv 2$ modulo 89, we identify its Galois group $\mathrm{Gal}(P)$ of $P(X) \in \mathbb{Z}[X]$ again with a subgroup of the hyperoctahedral group $G_5 = \mathbb{Z}_2^5 \rtimes S_5$.

To ensure that $A = A_6(N, M)$ is Galois pinching we need that $\mathrm{Gal}(P)$ projects surjectively onto S_5 or equivalently that $\mathrm{Gal}(Q) = S_5$. With Theorem 9.1.5 and Remark 9.1.6 it suffices to find $m \in \mathbb{N}$ such that the discriminant $\mathrm{Disc}(SWR_Q) = \mathrm{Disc}(Q)$ is not a rational square and that the sextic Weber resolvent $SWR_Q(Y) \in \mathbb{Q}[Y]$ does not have a rational root. We computed

$$\mathrm{Disc}(Q) = c\, f(m)\, (2\, m + 3)^{24},$$

where $c > 0$ and with an irreducible polynomial $f(m) \in \mathbb{Z}[m]$ of degree 16. With Siegel's theorem of integral points and the equations above we conclude that $\mathrm{Gal}(Q) = S_5$ for all but finitely many $m \in \mathbb{N}$ with $m \equiv 2$ modulo $p \in \{17, 19\}$.

Next we want to restrict $m \in \mathbb{N}$ further such that $\mathrm{Gal}(P) \neq S_5$ and $\mathrm{Gal}(P) \neq H_{5,i}$ for $i = 1, 2, 3$. For the expressions $\Delta_{5,1} = \delta_{5,1}^2$ and $\Delta_{5,2} = \delta_{5,2}^2$ from Lemma 9.1.11 and Remark 9.1.12 we computed

$$\begin{aligned}\Delta_{5,1} &= Q(0)\,Q(4) = c \cdot g(m) \cdot (m-1)(4m+1)(2m+3)^8 \quad \text{and}\\ \Delta_{5,2} &= \mathrm{Disc}(Q) \cdot \Delta_{5,1},\end{aligned}$$

where $c \in \mathbb{Z}$ and $g(m) \in \mathbb{Z}[m]$ is an irreducible polynomial of degree 10. With [86, Proposition 6.17] we conclude that $\mathrm{Gal}(P) \neq H_{5,i}$ for $i = 1, 2$ and almost all $m \in \mathbb{N}$ with $m \equiv 2$ modulo p, where $p \in \{17, 19, 89\}$. Furthermore if $m \equiv 2$ modulo 29 then $P(X)$ can be written in irreducible factors modulo 29 as

$$\begin{aligned}P(X) \equiv (X+15)(X+2)(X^2+7X+7)(X^2+X+25)\\ (X^4+22X^3+21X^2+22X+1) \text{ modulo } 29.\end{aligned}$$

One can compute that 29 does not divide the discriminant $\mathrm{Disc}(P)$ of $P(X) \in \mathbb{Q}[X]$ for $m \equiv 2$ modulo 29, we conclude with the theorem of Dedekind that in this case $\mathrm{Gal}(P)$ contains a permutation of cycle type $(4, 2, 2, 1, 1)$. But $H_{5,3}$ does not contain a permutation of this cycle type as we showed in Appendix A.1.

The discussion from above almost showed:

9.4.4 Proposition. The matrix $A_6(N, M) \in \mathbb{R}^{10\times 10}$ is Galois pinching for all but perhaps finitely many $m \in \mathbb{N}$ with $m \equiv 2$ modulo p, where $p \in \{17, 19, 29, 89\}$.

Proof. Let $m \in \mathbb{N}$ such that the quintic polynomial $Q(Y) \in \mathbb{Q}[Y]$ from Equation (9.4.3.1) is irreducible. By substituting $t = Y - b_4/5$ we can bring $Q(Y)$ in depressed form

$$DQ(t) = t^5 + p\,t^3 + q\,t^2 + r\,t + s \in \mathbb{Q}[t].$$

By an analysis of the four discriminants $F_i(DQ)$ $(i = 1, 2, 3, 4)$ from (9.1.13.5) for the polynomial $DQ(t)$ from above, we can see that for all $i = 1, 2, 3, 4$ and m big enough the inequality $F_i(DQ) > 0$ holds. From [59] we know that in this case the depressed polynomial $DQ(t)$ and hence $Q(Y)$ has five real roots. Denote the roots of $Q(Y)$ by μ_i $(i = 1, \ldots, 5)$. We have the equality

$$\mu_i = \lambda_i + \lambda_i^{-1} + 2$$

for all $i = 1, \ldots, 5$, where λ_i and λ_i^{-1} are roots of the reciprocal characteristic polynomial $P(X)$ of $A_6(N, M)$. Furthermore for m big enough all the coefficients b_i $(i = 1, \ldots, 5)$ of $Q(Y)$ are positive. With Décarte's rule of signs we conclude that in this situation all the roots μ_i $(i = 1, \ldots, 5)$ are negative real numbers. As before in 9.2.2 we now know that the roots $\{\lambda_i, \lambda_i^{-1} \mid i = 1, \ldots, 5\}$ of $P(X)$ are real. Putting this argument together with the arguments we did before for $\mathrm{Gal}(P)$, we see that $A_6(N, M)$ is Galois pinching for $m \in \mathbb{N}$ big enough such that $m \equiv 2$ modulo p, where $p \in \{17, 19, 29, 89\}$. $\square$

Denote by $B_6(N, M) = M_\gamma^{(0)} \in \mathbb{R}^{10\times 10}$ the representation matrix with respect to the basis $B^{(0)}$ of the map D_γ acting on the non-tautological $H_1^{(0)}(\mathcal{O}_{N,M}^{(6)}, \mathbb{R})$ from Section 9.4.1. Then $B_6(N, M)$ is unipotent and the subspace

$$(B_6(N, M) - \mathrm{Id})(\mathbb{R}^{10})$$

is one-dimensional and hence not a Lagrangian subspace with respect to Ω. Furthermore $A_6(N, M)$ and $B_6(N, M)$ do not commute but perhaps for finitely many $m \in \mathbb{N}$. Putting hhis together with the previous Proposition about the matrix $A_6(N, M)$, Criterion 1.5.10 implies:

9.4.5 Theorem. The genus six Origamis $\mathcal{O}_{N,M}^{(6)} \in \Omega\mathcal{M}_6(10)$ with $N = 3 + 2m$ and $M = 6{+}4m$ have Kontsevich-Zorich monodromies with finite index in $\mathrm{Sp}(H_1^{(0)}(\mathcal{O}_{N,M}^{(6)}, \mathbb{Z}))$ for all but finitely many $m \in \mathbb{N}$ such that $m \equiv 2$ modulo p, where $p \in \{17, 19, 29, 89\}$.

Appendix A.

A.1 Permutation types

We call φ_k the map that identifies the group $H_{k,3} \leqslant G_k$ from Proposition 9.1.10 with a subgroup of the permutation group $\mathrm{Sym}(\{\lambda_i, \lambda_i^{-1} \mid i = 1, \ldots, k\})$.

We first consider $k = 3$ and $\varphi_3\colon H_{3,3} \to \mathrm{Sym}(\{\lambda_i, \lambda_i^{-1} \mid i = 1, 2, 3\})$. We have

$$\begin{aligned}
\varphi_3((-1,-1,-1),(123)) =& (\lambda_1\lambda_2^{-1}\lambda_3\lambda_1^{-1}\lambda_2\lambda_3^{-1}),\\
\varphi_3((-1,-1,-1),(1,2)) =& (\lambda_1\lambda_2^{-1})(\lambda_2\lambda_1^{-1})(\lambda_3\lambda_3^{-1}),\\
\varphi_3((+1,+1,+1),(123)) =& (\lambda_1\lambda_2\lambda_3)(\lambda_1^{-1}\lambda_2^{-1}\lambda_3^{-1})\\
\varphi_3((+1,+1,+1),(1,2)) =& (\lambda_1\lambda_2)(\lambda_1^{-1}\lambda_2^{-1})(\lambda_3)(\lambda_3^{-1}).
\end{aligned}$$

The element $(123) \in S_3$ is of permutation type (3) and $(12) \in S_3$ is of type $(2,1)$. These two types are the only non-trivial types that can appear in S_3. Hence the calculations from above show that the only non-trivial permutation types that occur in $\varphi_3(H_{3,3}) \leqslant \mathrm{Sym}(\{\lambda_i, \lambda_i^{-1} \mid i = 1, 2, 3\})$ are $(6), (3,3), (2,2,2)$ and $(2,2,1,1)$.

For $k = 5$ the permutation group S_5 has permutations of type (5), $(4,1)$, $(3,2)$, $(3,1,1)$, $(2,2,1)$, $(2,1,1,1)$ and $(1,1,1,1,1)$. If we want to determine the permutation types of the elements of $H_{5,3}$ as a subgroup of $\mathrm{Sym}(\{\lambda_i,\ \lambda_i^{-1} \mid i = 1, \ldots, 5\})$ then because of symmetry reasons it is sufficient to determine the permutations $\varphi_5((-1, \ldots, -1), \sigma))$ and $\varphi_5((+1, \ldots, +1), \sigma))$, where $\sigma \in S_5$ is a represent of a permutation type of S_5 as above. We have

$$\begin{aligned}
\varphi_5((-1,\ldots,-1),(12345)) =& (\lambda_1\lambda_2^{-1}\lambda_3\lambda_4^{-1}\lambda_5\lambda_1^{-1}\lambda_2\lambda_3^{-1}\lambda_4\lambda_5^{-1})\\
\varphi_5((-1,\ldots,-1),(1234)) =& (\lambda_1\lambda_2^{-1}\lambda_3\lambda_4^{-1})(\lambda_2\lambda_3^{-1}\lambda_4\lambda_1^{-1})(\lambda_5\lambda_5^{-1})\\
\varphi_5((-1,\ldots,-1),(123)(45)) =& (\lambda_1\lambda_2^{-1}\lambda_3\lambda_1^{-1}\lambda_2\lambda_3^{-1})(\lambda_4\lambda_5^{-1})(\lambda_5\lambda_4^{-1})\\
\varphi_5((-1,\ldots,-1),(123)) =& (\lambda_1\lambda_2^{-1}\lambda_3\lambda_1^{-1}\lambda_2\lambda_3^{-1})(\lambda_4\lambda_4^{-1})(\lambda_5\lambda_5^{-1})\\
\varphi_5((-1,\ldots,-1),(12)(34)) =& (\lambda_1\lambda_2^{-1})(\lambda_2\lambda_1^{-1})(\lambda_3\lambda_4^{-1})(\lambda_4\lambda_3^{-1})(\lambda_5\lambda_5^{-1})\\
\varphi_5((-1,\ldots,-1),(12)) =& (\lambda_1\lambda_2^{-1})(\lambda_2\lambda_1^{-1})(\lambda_3\lambda_3^{-1})(\lambda_4\lambda_4^{-1})(\lambda_5\lambda_5^{-1}).
\end{aligned}$$

We conclude that for the subgroup $\varphi_k(H_{k,3}) \leqslant \mathrm{Sym}(\{\lambda_i,\ \lambda_i^{-1} \mid i = 1, \ldots, 5\})$ only non-trivial permutations of type (10), $(4,4,2)$, $(6,2,2)$, $(2,2,2,2,2)$ and of type $(5,5)$, $(4,4,1,1)$, $(3,3,2,2)$, $(3,3,1,1,1,1)$, $(2,2,2,2,1,1)$, $(2,2,1,1,1,1,1,1)$ can occur.

A.2 Representation matrix for Dehn twist

The following matrix is the representation matrix $M_\alpha^{(0)}$ for the restriction of the Dehn twist D_α in direction $(1,-2)$ to $H_1^{(0)}(\mathcal{O}_{N,M}^{(5)},\mathbb{Q})$ with respect to the basis $B^{(0)}$ from Section 9.3.1.

$$\begin{pmatrix}
-8-N-4m & -9-N-4m & 0 & -9-N-4m & -9-N-4m & -9-N-4m & 0 & -9-N-4m \\
2 & 3 & 0 & 2 & 2 & 2 & 0 & 2 \\
2 & 2 & 1 & 2 & 2 & 2 & 0 & 2 \\
2 & 2 & 0 & 3 & 2 & 2 & 0 & 2 \\
-4 & -4 & 0 & -4 & -3 & -4 & 0 & -4 \\
-4 & -4 & 0 & -4 & -4 & -3 & 0 & -4 \\
7+N+4m & 7+N+4m & 0 & 7+N+4m & 7+N+4m & 7+N+4m & 1 & 7+N+4m \\
-4 & -4 & 0 & -4 & -4 & -4 & 0 & -3
\end{pmatrix}$$

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