International Journal of Game Theory (2025) 54:27
https://doi.org/10.1007/s00182-025-00949-z

ORIGINAL PAPER

Blend-in fairness and equal split

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Accepted: 10 June 2025

Abstract
Blending in with others is a possible self-serving motivation when people participate in cooperative situations. We use this motivation to formulate a corresponding fairness principle, combine it with rather weak standard axioms from cooperative game theory, and show that it leads to equal split of coalitional gains. The same normative principles characterize this solution when only cohesive games (where it is optimal for the coalition of all players to form) are considered.

Keywords Blend-in fairness · Cohesive games · Cooperative games · Equal division solution

JEL Classification: A13 · C71 · D63 · D91

1 Introduction

A key question in coalitional game theory is how the total payoff a coalition can achieve through cooperation should be divided among its members. One of the most influential answers to this question within the context of cooperative transferable utility games (TU-games) has been provided by the *Shapley value* (cf. Shapley 1953). According to this single-valued solution, each player should be assigned a share of

We are grateful to Sylvain Béal, Rene van den Brink, Youngsub Chun, Yves Sprumont, two anonymous referees, and various seminar audiences for valuable suggestions.

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Published online: 26 June 2025

the payoff that is proportional to his marginal contributions in the corresponding game. From an axiomatic perspective, the Shapley value is the unique solution which is *efficient* (the entire gain when all players cooperate (i.e., the *worth* of the 'grand' coalition) is distributed among the players), *symmetric* (players receive equal payoffs if they are exchangeable in generating coalitional gains), satisfies the *null player property* (no contribution to any coalition results in zero payoff), and is *additive* over games.

The experimental validity of the above principles was recently tested in de Clippel and Rozen (2022). The data analysis of these authors provides strong evidence for the symmetry and additivity axioms with the efficiency axiom being trivially satisfied due to the experimental design. However, no such evidence was found for the null player property; that is, even null players were assigned to payoffs which are significantly different from zero.

The lack of experimental evidence for the null player property naturally draws one's attention to another famous single-valued solution for TU-games, the *equal division solution*. For each game, the latter distributes the worth of the grand coalition equally among all players and it violates the null player property while satisfying the other three mentioned principles. As originally shown in van den Brink (2007), the equal division solution can be characterized by replacing, in the above axiomatic system, the null player property by a *nullifying player* requirement (each player is assigned zero payoff if the worth of each coalition containing that player is zero). We refer the reader to Alonso-Meijide et al. (2019) for an excellent and detailed survey of the corresponding strand of the literature, to Hernandez-Lamoneda et al. (2008) for a characterization of the class of all additive, symmetric, efficient and continuous solutions, as well as to Chapter 4 in Branzei et al. (2008) for an overview of other egalitarianism-based solution concepts.

In this paper we follow a normative perspective on cooperative games (cf. Moulin 2003) and provide an axiomatic support for the idea that the equal division solution springs out of a fairness principle which is rather "... strongly shaped by cultural values" (cf. Young 1994, p. xii). That is, our work fits into the strand of literature emphasizing the context-dependence of fairness and equity "... not because of the lack of general principles of justice, but due to its effect on the interpretation of those principles" (cf. Konow 2001, p. 139). For instance, Alesina and Angeletos (2005) study the extent to which the difference in social perceptions regarding the fairness of market outcomes and the underlying sources of income inequality (Americans vs Europeans) are consistent with equilibrium behavior. Almås et al. (2020) provide experimental support for the differences between Americans and Scandinavians with respect to what kind of inequalities they consider fair and to the importance assigned to fairness relative to efficiency. Cappelen et al. (2007) consider a dictator game in which the distribution phase is preceded by a production phase and show how one may simultaneously estimate the prevalence of different fairness ideals and the weight people attach to fairness considerations. Finally, Gelfand et al. (2002) stress the differences when it comes to self-serving motivations in individualistic or in collectivistic cultures. These authors argue that, in the former, the self is served by enhancing one's positive attributes to "stand out" and be better than others, while in collectivistic cultures the focus is rather on how individuals “blend in” and maintain interdependence with others.

Since the Shapley value averages players’ contributions to every coalition they join, it can be axiomatically rooted via a corresponding marginality principle (cf. Young 1985) in rather individualistic societies. In the present paper we follow a possible self-serving motivation for *collectivistic cultures*, introduce a corresponding fairness principle (blend-in fairness), and show that it crucially shapes the characterization of the equal division solution.

We start in Sect. 2 by providing the formal definitions of the mentioned standard axioms and first weaken efficiency and symmetry to a *symmetric efficiency* principle. The latter imposes on a single-valued solution the equal split of the worth of the grand coalition, provided that *all players* are symmetric in the corresponding game. A *weak null player property* additionally postulates null players to be assigned zero payoffs in games, where the total gain when all players cooperate is zero. These two properties, together with additivity, are clearly satisfied by both the Shapley value and the equal division solution.

In Sect. 3 we develop a notion of *cooperation equivalence of games for players*. To fix ideas, consider two superadditive games (that is, games where the formation of larger coalitions is worthy) and a particular player. Imagine now that what the player realizes when comparing these two games is that there is only a renaming of the *other* players but no change in the corresponding coalitional worths. That is, in terms of the offered ‘blending-in with others’ possibilities, the two games are cooperation equivalent for that particular player. The *blend-in fairness* principle stated in Sect. 4 then requires a player to be assigned the same payoff in two superadditive games which are cooperation equivalent for him. This principle turns out to shape the characterization of the equal division solution both on the entire set of games (Theorem 1) and on the subset of cohesive games, where it is optimal that the grand coalition forms (Theorem 2). We conclude in Sect. 5 with some final remarks.

The proofs of our characterization results are relegated to the Appendix, where we first introduce the class of partition games and show that they form a basis of the entire set of games (Theorem 0). Propositions 1–5 are then crucial for the proofs of Theorem 1 and Theorem 2 as they explain how additivity, symmetric efficiency, the weak null player property, and blend-in fairness generate the equal division solution on partition games. The Appendix also contains examples showing the independence of the utilized axioms.

2 Normative principles and solutions

Let N be a set of n individuals. A *coalition* is any subset of N. A *cooperative transferable utility game* (TU-game) is a pair (N, v), where v is the *characteristic function* of the game assigning a *worth* $v(S)$ to each coalition $S \subseteq N$ such that $v(\emptyset) = 0$. The amount $v(S)$ represents how much the members of S can share should they cooperate. In what follows, we fix the player set N and therefore denote the game (N, v) by its characteristic function v. The notation $\mathcal{G}^N$ stands for the set of all TU-games on the player set N.

Two players $i, j \in N$ are called *symmetric* in a game $v \in \mathcal{G}^N$, if $v(S \cup \{i\}) = v(S \cup \{j\})$ holds for every $S \subseteq N \setminus \{i, j\}$. A player $i \in N$ is a *null player* in $v \in \mathcal{G}^N$ if $v(S \cup \{i\}) = v(S)$ is valid for every $S \subseteq N \setminus \{i\}$. The *sum* of two games $v, w \in \mathcal{G}^N$ is defined by $(v + w)(S) = v(S) + w(S)$ for each $S \subseteq N$.

It is an implicit assumption in cooperative game theory models that the grand coalition forms (all players cooperate) and the basic question is then how should the corresponding proceeds be distributed among all individuals. Thus, a (single-valued) *solution* $f : \mathcal{G}^N \to \mathbb{R}^n$ assigns a payoff vector to each game $v \in \mathcal{G}^N$. The next four normative principles that could be imposed on a solution *f* are standard in the literature.

Efficiency For all $v \in \mathcal{G}^N$: $\Sigma_{i \in N} f_i(v) = v(N)$.
Symmetry For all $v \in \mathcal{G}^N$: If $i, j \in N$ are symmetric in *v*, then $f_i(v) = f_j(v)$.
Null player property For all $v \in \mathcal{G}^N$: If $i \in N$ is a null player in *v*, then $f_i(v) = 0$.
Additivity For all $v, w \in \mathcal{G}^N$: $f(v + w) = f(v) + f(w)$.

The *Shapley value* is the unique solution which is efficient, symmetric, additive, and satisfies the null player property. This solution concept was introduced in Shapley (1953) and it assigns to each player the average of his marginal contributions in the corresponding game. Given a game $v \in \mathcal{G}^N$, the *marginal contribution* of player *i* to a coalition $S \subseteq N \setminus \{i\}$ is just the difference $\Delta_i(S) = v(S \cup \{i\}) - v(S)$. The Shapley value *Sh* is then defined by the condition

$$Sh_i(v) = \frac{1}{|N|!} \sum_{R \in \mathcal{R}} \Delta_i(S_i(R)) \text{ for each } i \in N,$$

where $\mathcal{R}$ is the set of all $|N|!$ orderings of *N* and $S_i(R)$ is the set of players preceding *i* in the ordering *R*.

From the above four axioms, it is only the null player property that is violated by another famous procedure for sharing coalitional gains, the *equal division solution*. For each game $v \in \mathcal{G}^N$ it is defined by

$$ED_i(v) = \frac{v(N)}{n} \text{ for each } i \in N.$$

That is, this solution distributes the worth of the grand coalition equally among all players.

In the next section we present our axiomatic characterization of the equal division solution where all utilized axioms but one are also satisfied by the Shapley value. Besides additivity, these include the weak null player property and symmetric efficiency.

Weak null player property For all $v \in \mathcal{G}^N$: If $i \in N$ is a null player in *v* and $v(N) = 0$, then $f_i(v) = 0$.

Symmetric efficiency For all $v \in \mathcal{G}^N$: If all players are symmetric in *v*, then $f_i(v) = \frac{v(N)}{n}$ for each $i \in N$.

The weak null player property is clearly implied by the null player property and it is stronger than the null player in a null environment property introduced in Casajus and Huettner (2013); the latter imposes that a null player gets a non-negative payoff in a game in which the grand coalition has zero worth. On the other hand, the combination of efficiency and symmetry (but none of these properties taken separately) implies the symmetric efficiency axiom. Symmetric efficiency is stronger than the triviality axiom introduced in Chun (1989) and the weak symmetry axiom used in van den Brink (2007).

3 Cooperation equivalence of games

Imagine a player who evaluates his participation in two different cooperative situations and claims equal payoffs in the correspondingly generated games. As already argued above, one possible way of rationalizing such a claim is to look at the ways in which the particular player maintains interdependence with others and blends in when working with them in the respective situations. The notion of cooperation equivalence of games we introduce below formalizes the idea that, *from the viewpoint of a given player*, two games offer the same possibilities when it comes to fitting in with other players. We use then this notion to formally state in Sect. 4 a corresponding blend-in fairness principle.

In order to equip the reader with a suitable intuition for the notion of cooperation equivalence, let us consider two games in which three players cooperate on their investment decisions.[1] Players 1 and 2 are domestic investors lacking any experience and suitable investment technologies, while player 3 is an experienced foreign investor who possesses 100 units of capital and is able to double any fraction of it he invests. There is no government restriction on such investments when made in cooperation with at least one domestic investor. However, the foreign investor when acting on his own is allowed to invest only 40 units (in the game *v*) or 30 units (in the game *w*).

	$\{1\}$	$\{2\}$	$\{3\}$	$\{1,2\}$	$\{1,3\}$	$\{2,3\}$	N
v	10	20	$2 \times 40 + 60$	30	220	240	260
w	20	10	$2 \times 30 + 70$	30	240	220	260

In the game *v*, players 1 and 2 would like to invest 10 and 20 units of capital, respectively. However, due to their lack of investment technology, one has $v(\{1\}) = 10$, $v(\{2\}) = 20$, and $v(\{1,2\}) = 30$. In a coalition containing player 3, the foreign investor is allowed to use the entire 100 units of his capital and each domestic coalitional member has access to the corresponding investment technology doubling the sum of individual capitals. The interpretation of the coalitional worths in the game *w* is analogous.

Let us now compare the two games *v* and *w from the viewpoint of the foreign investor*. In both games he intends to invest 100 units of capital and this is exactly the amount he puts on the table when joining any non-empty coalition of domestic

[1] Each of these games can be seen as an appropriate modification of the landlord game in Shapley and Shubik (1967).

players. Moreover, when looking at the domestic investors in the two games, he realizes that there is only a renaming of these players (player 1 becomes player 2 and vice versa) but no change in the worth of any coalition containing some of these players. Notice additionally that each of these games is *superadditive* as, exemplified with respect to the game *v*, the inequality $v(S \cup T) \geq v(S) + v(T)$ holds for all $S, T \subseteq N$ with $S \cap T = \emptyset$. In other words, the domestic investors and the foreign investor are "incentivized" in both games to form larger coalitions; hence, it seems natural for a player to focus rather on larger coalitions when evaluating his participation in such situations. Correspondingly, we declare the two superadditive games as being cooperation equivalent for the foreign investor.

Let us denote by $\mathcal{G}_{sa}^N, \mathcal{G}_{sa}^N \subset \mathcal{G}^N$, the set of all superadditive games on the player set *N*. Consider two games $v, w \in \mathcal{G}_{sa}^N$, and *fix* a player $i \in N$*whose viewpoint* we would like to describe. Suppose further that there exists a *bijection*$\sigma : N \to N$*with*$\sigma(i) = i$ such that $v(S) = w(\sigma(S))$ holds for all $S \subseteq N$ with $j \in S$ for some $j \in N \setminus \{i\}$. In other words, when evaluating the two superadditive games, player *i* focusses on the coalitions *each of the other players* might belong to and realizes only a permutation of the players' names but no changes at all in the corresponding coalitional worths. We call the games *v* and *w cooperation equivalent for playeri.*

4 Blend-in fairness leads to equal split

The blend-in fairness principle we introduce below relies on the notion of cooperation equivalence of games for players. More precisely, it requires from a solution to assign the same payoff to a player in superadditive games which are cooperation equivalent for him.

Blend-in fairness For all $v, w \in \mathcal{G}_{sa}^N$ and $i \in N$: If *v* and *w* are cooperation equivalent for player *i*, then $f_i(v) = f_i(w)$.

We would like to mention the fact that (1) it is a specific bijection $\sigma($ satisfying $\sigma(i) = i$ for the corresponding player $i \in N)$ we use in the formulation of cooperation equivalence of games and (2) the implication refers only to the payoff of player *i* in the two games. The reader might then wonder whether there is a logical relation (on the class of superadditive games) between the above axiom and the anonymity property of single-valued solutions requiring that a solution should not discriminate between the players solely on the basis of their "names". Notice that, in the definition of blend-in fairness, it might happen that $v(\{i\}) \neq w(\{\sigma(i)\})$ for player *i* whose viewpoint the bijection σ describes. Clearly then, the Shapley value violates the newly introduced requirement, while satisfying anonymity. On the other hand, one can define a dictatorship solution $d^k : \mathcal{G}_{sa}^N \to \mathbb{R}^n$ with respect to a pre-specified player $k \in N$, where for $v \in \mathcal{G}_{sa}^N$ one has

$$d_j^k(v) = \begin{cases} v(N) & \text{if } j = k, \\ 0 & \text{otherwise.} \end{cases}$$

As it can be easily seen, d^k satisfies blend-in fairness and violates anonymity. Hence, these two requirements are independent.

Our main characterization result is stated in Theorem 1 below and it basically says that, along with additivity and the weak versions of standard requirements, it is the blend-in fairness that leads to equal split of coalitional gains.

Theorem 1 *A solution $f : \mathcal{G}^N \to \mathbb{R}^n$ satisfies additivity, the weak null player property, symmetric efficiency, and blend-in fairness if and only if it is the equal division solution.*

It is worth mentioning that the equal division solution also satisfies a stronger version of the blend-in fairness axiom requiring that for all pairs of (not necessarily superadditive) cooperation equivalent games for some particular player, this player receives the same payoff in both games. As we show in the proof of Theorem 1, there is no need to impose this stronger version on a solution since its weaker version stated only with respect to superadditive games suffices for providing the characterization of the equal division solution on the entire set of games.

Observe additionally that two of the other axioms we utilize, symmetric efficiency and the weak null player property, rather indicate the formation of the grand coalition, the reader might ask whether a corresponding characterization could be reached when only cohesive games are considered. In a *cohesive game* $v \in \mathcal{G}^N$ it is optimal for the grand coalition to form since by definition $v(N) \geq \Sigma_{k=1}^{K} v(S_k)$ holds for every partition $\{S_1, \ldots, S_K\}$ of N (cf. Osborne and Rubinstein (1994), p. 258). Theorem 2 shows that there are the same four normative principles[2] which characterize the equal division solution on the mentioned subclass (denoted by $\mathcal{G}^N_{coh}$) of games.

Theorem 2 *A solution $f : \mathcal{G}^N_{coh} \to \mathbb{R}^n$ satisfies additivity, the weak null player property, symmetric efficiency, and blend-in fairness if and only if it is the equal division solution.*

As we show in the Appendix, the proof of Theorem 2 relies on the fact that all games used in the proof of Theorem 1 are cohesive (with the partition games serving as a basis of the entire set of games being even superadditive (cf. Remark 0 in Sect. 6.1).

5 Concluding remarks

The notion of blend-in fairness introduced in this paper relies on the very basic idea that individuals in cooperative situations are usually construed as being fundamentally connected to others. Such a fundamental connection in superadditive games is mirrored by the formation of larger coalitions; in other words, players' "incentives" to look at individual worths when comparing two such games are rather moderate. The latter interpretation naturally leads to the developed notion of cooperation equiv-

[2] Additivity, the weak null player property, and symmetric efficiency should then be stated with respect to cohesive games. There is no need to change the formulation of blend-in fairness since it is anyway stated with respect to superadditive games and superadditive games are also cohesive.

alence of games with blend-in fairness requiring a player to be assigned the same payoff in two superadditive games that are cooperation equivalent for that player.

As already elaborated in Sect. 4, the standard anonymity axiom (satisfied by the Shapley value) and blend-in fairness (violated by the Shapley value) are independent requirements. Nevertheless, blend-in fairness can alternatively be seen as a local version of anonymity in the sense that (1) anonymity in the corresponding superadditive games holds with the possible exception for the player under consideration and (2) the payoff equivalence is required only for that particular player (whereas it holds for all players in the classical axiom of anonymity). In other words, an axiom in the spirit of anonymity (blend-in fairness) combined with (small modifications of) classical axioms satisfied by the Shapley value turns out to have rather dramatic consequences.

It seems therefore reasonable to further study and axiomatically characterize solutions that satisfy the blend-in fairness axiom and are as close as possible to the formal definition of the Shapley value. Notice that the Shapley value violates blend-in fairness due to the fact that it particularly accounts for players' marginal contributions to the empty set and to singleton coalitions. Hence, any solution respecting only the marginal contributions to coalitions of size at least two would satisfy our fairness axiom. One particular example of a symmetrically efficient and additive solution satisfying blend-in fairness is provided in Sect. 6.3.

Appendix

In Sect. 6.1 we introduce the collection of partition games and show that they form a basis of the space of all games. This fact is then used in Sect. 6.2 to show how blend-in fairness shapes the characterization of the equal division solution (Theorem 1 and Theorem 2). Section 6.3 contains examples for the independence of the utilized axioms.

Partition games

Let $T \subseteq N$ be a nonempty coalition. The *partition game over T* is the game u_T defined as follows.

(1) If $T \neq N$, then for each coalition $S \subseteq N$,

$$u_T(S) := \begin{cases} 0 & \text{if (1) } S \in \{\emptyset, T, N\} \text{ or (2) } |S| > |T| \text{ with } (S,T) \text{ partitioning } N, \\ -1 & \text{otherwise.} \end{cases}$$

(2) If $T = N$, then for each coalition $S \subseteq N$,

$$u_N(S) := \begin{cases} 0 & \text{if } S \neq N, \\ 1 & \text{otherwise.} \end{cases}$$

Remark 0 Notice that for each nonempty $T \subseteq N$ the inequality $u_T(P \cup Q) \geq u_T(P) + u_T(Q)$ holds for all $P, Q \subseteq N$ with $P \cap Q = \emptyset$, i.e., partition games are *superadditive*.[3]

Let us now show that the set $\left\{u_T : T \in 2^N \setminus \{\emptyset\}\right\}$ formed by partition games is a basis of the linear (vector) space $\mathcal{G}^N$.

Theorem 0 *Every game $v \in \mathcal{G}^N$ is a linear combination of partition games.*

Proof Observe that the number of partition games equals the number $2^n - 1$ of nonempty coalitions in *N*. Since the dimension of the space $\mathcal{G}^N$ is also $2^n - 1$, it suffices to show that the partition games are linearly independent over $\mathbb{R}^{2^n-1}$.

Suppose, by contradiction, that the partition games are linearly dependent. Then there exist real numbers $(a_T)_{\{T \subseteq N, T \neq \emptyset\}}$, not all zero, such that

$$\sum_{\{T \subseteq N, T \neq \emptyset\}} a_T u_T(S) = 0, \quad \forall S \subseteq N. \tag{1}$$

By proving the next six consecutive claims, we show that the above system of equalities requires $a_T = 0$ to hold for each nonempty $T \subseteq N$ in contradiction to the fact that the partition games are linearly dependent. □

Claim 1 $a_N = 0$.

Proof of Claim 1 Given the definition of a partition game, we have $u_N(N) = 1$ and $u_T(N) = 0$ holding for each nonempty $T \subset N$. For $S = N$ we have from (1) that

$$\sum_{\{T \subseteq N, T \neq \emptyset\}} a_T u_T(N) = \sum_{\{T \subset N, T \neq \emptyset\}} a_T u_T(N) + a_N u_N(N) = a_N u_N(N) = a_N = 0$$

follows. □

Claim 2 $a_T = 0$ for each $T \subset N$ with $|T| > |N|/2$.

Proof of Claim 2 Let $S \subset N$ be such that $0 < |S| < |N|/2$. By the definition of a partition game,

$$u_T(S) = \begin{cases} 0 & \text{if } T \in \{S, N\}, \\ -1 & \text{otherwise.} \end{cases}$$

For the selected coalition *S*, we have from (1) that

[3] Note additionally that there are no nullifying players in partition games and that each $i \in N$ is a null player in exactly one partition game ($u_{\{i\}}$).

$$\sum_{\{T\subseteq N,T\neq\emptyset\}} a_T u_T(S) = \sum_{\{T\subset N,T\neq\emptyset,T\neq S\}} -a_T = 0 \tag{2}$$

holds. On the other hand, for the coalition $N \setminus S$(with $|N| > |N\backslash S| > |S|$) we have by definition

$$u_T(N \setminus S) = \begin{cases} 0 & \text{if } T \in \{S, N \setminus S, N\}, \\ -1 & \text{otherwise.} \end{cases}$$

For the coalition $N \setminus S$, it follows from (1) that

$$\sum_{\{T\subseteq N,T\neq\emptyset\}} a_T u_T(N \setminus S) = \sum_{\{T\subset N,T\neq\emptyset,T\neq S,T\neq N\setminus S\}} -a_T = 0 \tag{3}$$

should hold. Substituting (3) in (2) results in $a_{N\setminus S} = 0$. Repeating this step as many times as necessary yields $a_T = 0$ for each $T \subset N$ with $|T| > |N|/2$. □

Claim 3 $a_{\{i\}} = a_{\{j\}} := a$ for all $i, j \in N$.

Proof of Claim 3 Take $i, j \in N$ and notice that, by applying the definition of a partition game and in view of (1),

$$\sum_{\{T\subseteq N,T\neq\emptyset\}} a_T u_T(\{i\}) - \sum_{\{T\subseteq N,T\neq\emptyset\}} a_T u_T(\{j\}) = -a_{\{j\}} + a_{\{i\}} = 0$$

holds. Thus, we have $a_{\{i\}} = a_{\{j\}}$ for all $i, j \in N$. Set $a := a_{\{i\}}$. □

Claim 4 $a_T = a$ for each $T \subset N$ with $1 < |T| < |N|/2$.

Proof of Claim 4 Let $i \in N$ and $S \subset N$ be such that $2 \leq |S| < |N|/2$ holds. By Claims 1-3 and in view of (1), we have

$$\sum_{\{T\subseteq N,T\neq\emptyset\}} a_T u_T(\{i\}) = -(n-1)a - \sum_{\{T\subseteq N,T\neq\emptyset,1<|T|\leq|N|/2\}} a_T = 0 \tag{4}$$

and

$$\sum_{\{T\subseteq N,T\neq\emptyset\}} a_T u_T(S) = -na - \sum_{\{T\subseteq N,T\neq\emptyset,T\neq S,1<|T|\leq|N|/2\}} a_T = 0. \tag{5}$$

Subtracting (4) from (5) gives us $a_S = a$. Repeating this step as many times as necessary yields $a_T = a$ for each $T \subset N$ with $1 < |T| < |N|/2$. □

Claim 5 $a_T = a$ for each $T \subset N$ with $1 < |T| = |N|/2$.

Proof of Claim 5 Let $i \in N$ and $S \subset N$ be such that $|S| = |N|/2$(which is possible only if $|N|$ is even). By Claims 1-4 and in view of (1), we have

$$\sum_{\{T \subseteq N, T \neq \emptyset\}} a_T u_T(\{i\}) = -(n-1)a - |\{T : T \subseteq N, T \neq \emptyset, 1 < |T| < |N|/2\}|\, a \\ - \sum_{\{T \subseteq N, T \neq \emptyset, 1 < |T| = |N|/2\}} a_T = 0 \tag{6}$$

and

$$\sum_{\{T \subseteq N, T \neq \emptyset\}} a_T u_T(S) = -na - |\{T : T \subseteq N, T \neq \emptyset, 1 < |T| < |N|/2\}|\, a \\ - \sum_{\{T \subseteq N, T \neq \emptyset, T \neq S, 1 < |T| = |N|/2\}} a_T = 0. \tag{7}$$

Subtracting (6) from (7) gives us $a_S = a$. Repeating this step as many times as necessary yields $a_T = a$ for each $T \subset N$ with $1 < |T| = |N|/2$. □

Claim 6 $a_T = 0$ for each nonempty $T \subseteq N$.

Proof of Claim 6 Due to the above claims it suffices to show that $a = 0$. Take $i \in N$ and notice that (4) transforms into

$$\sum_{\{T \subseteq N, T \neq \emptyset\}} a_T u_T(\{i\}) = -\left[n - 1 + |\{T : T \subseteq N, T \neq \emptyset, 1 < |T| \leq |N|/2\}|\right] a = 0.$$

We conclude that $a = 0$ should hold.

The assertion in Theorem 0 then follows from the contradiction between Claim 6 and the partition games being linearly dependent. □

Proofs of Theorem 1 and Theorem 2

The structure of the proof of Theorem 1 is as follows. We start by deriving the players' payoffs in partition games and show first that, when there are two players in a game, only three axioms are needed (Proposition 1). When the player set contains at least three players, we first show that each player $i \in N$ gets zero payoff in the games $u_{\{i\}}$ and $u_{N\setminus\{i\}}$(Lemma 1) and continue showing that i's payoff is the same in any two games u_T and $u_{T'}$, provided that the coalitions T and T' are of the same size and player i belongs to either $T \cap T'$ or $N \setminus (T \cup T')$(Lemma 2). By using these two lemmas, we prove that all players get zero payoffs in the partition games defined either over singleton coalitions (Proposition 2) or over coalitions of size $|N| - 1$(Proposition 3).

Lemma 3 is then crucial for the rest of the proof as it describes the transfer of arguments with respect to the payoff of a player $k \in T \subset N$, $|T| \geq 2$, from the partition game over T to the partition game over $T \setminus \{k\}$. Thanks to Proposition 2 and Lemma

3, we show in Proposition 4 that, in fact, each player gets zero payoff in every partition game over a coalition of size less than $|N|$, while Proposition 5 delivers the corresponding (short) argument for the equal distribution of $u_N(N)$ in the partition game u_N. Finally, we use the set of partition games as basis of $\mathcal{G}^N$(cf. Theorem 0) and the solution's additivity to complete the characterization.

Proposition 1 *If a solution $f : \mathcal{G}^{\{1,2\}} \to \mathbb{R}^2$ satisfies additivity, symmetric efficiency, and blend-in fairness, then*

(1) $f_i(u_{\{k\}}) = 0$ for each $i, k \in \{1, 2\}$;
(2) $f_i(u_{\{1,2\}}) = \frac{1}{2}$ for each $i \in \{1, 2\}$.

Proof Let z be the game on the player set $\{1, 2\}$ that is identically zero. Since all players are symmetric in z, we have by symmetric efficiency that $f_1(z) = f_2(z) = 0$.

(1) Consider the games z and $u_{\{1\}}$(with $u_{\{1\}}(\{1\}) = u_{\{1\}}(\{1, 2\}) = 0$ and $u_{\{1\}}(\{2\}) = -1$) as well as the bijection $\sigma(1) = 1$, $\sigma(2) = 2$. With respect to the two coalitions containing player 1 we have

$$z(\{1\}) = u_{\{1\}}(\{\sigma(1)\}) = u_{\{1\}}(\{1\}) = 0$$

and

$$z(\{1, 2\}) = u_{\{1\}}(\{\sigma(1), \sigma(2)\}) = u_{\{1\}}(\{1, 2\}) = 0.$$

By blend-in fairness, $f_2(u_{\{1\}}) = f_2(z) = 0$. By an analogous argument applied with respect to the games z and $u_{\{2\}}$(with $u_{\{2\}}(\{2\}) = u_{\{2\}}(\{1, 2\}) = 0$ and $u_{\{2\}}(\{1\}) = 1$), we get $f_1(u_{\{2\}}) = f_1(z) = 0$. Notice finally that all players are symmetric in the game $u_{\{1\}} + u_{\{2\}}$ since we have $\left(u_{\{1\}} + u_{\{2\}}\right)(\{1\}) = \left(u_{\{1\}} + u_{\{2\}}\right)(\{2\}) = -1$ and $\left(u_{\{1\}} + u_{\{2\}}\right)(\{1, 2\}) = 0$. Thus, by symmetric efficiency, $f_i(u_{\{1\}} + u_{\{2\}}) = 0$ holds for each $i \in \{1, 2\}$. By additivity, $f_1(u_{\{1\}}) = f_2(u_{\{2\}}) = 0$.

(2) Since all players are symmetric in $u_{\{1,2\}}$(due to $u_{\{1,2\}}(\{1\}) = u_{\{1,2\}}(\{2\}) = 0$ and $u_{\{1,2\}}(\{1, 2\}) = 1$), the assertion follows by symmetric efficiency. □

Assume now that the player set N contains at least three players.

Lemma 1 *If a solution $f : \mathcal{G}^N \to \mathbb{R}^n$ satisfies the weak null player property and blend-in fairness, then $f_i(u_{\{i\}}) = f_i(u_{N\setminus\{i\}}) = 0$ for each $i \in N$.*

Proof Fix $i \in N$ and consider first the partition game $u_{\{i\}}$. Recall that $u_{\{i\}}(S) = 0$ if $S \in \{\emptyset, \{i\}, N\setminus\{i\}, N\}$ and $u_{\{i\}}(S) = -1$, otherwise. We have then

$$u_{\{i\}}(S \cup \{i\}) - u_{\{i\}}(S) = \begin{cases} 0 - 0 = 0 & \text{if } S \in \{\emptyset, N \setminus \{i\}\}, \\ -1 + 1 = 0 & \text{otherwise,} \end{cases}$$

i.e., player i is a null player in $u_{\{i\}}$. By $u_{\{i\}}(N) = 0$ and the weak null player property, $f_i(u_{\{i\}}) = 0$ follows.

Consider next the partition game $u_{N\setminus\{i\}}$ and recall that $u_{N\setminus\{i\}}(S) = 0$ if $S \in \{\emptyset, N\setminus\{i\}, N\}$ and $u_{N\setminus\{i\}}(S) = -1$, otherwise. Let the bijection $\sigma : N \to N$ by defined by $\sigma(k) = k$ for each $k \in N$. Fix then $j \in N\setminus\{i\}$ and consider $S \subseteq N$ with $j \in S$. We have

$$u_{\{i\}}(S) = u_{N\setminus\{i\}}(\sigma(S)) = u_{N\setminus\{i\}}(S) = \begin{cases} 0 & \text{if } S \in \{\emptyset, N \setminus \{i\}, N\}, \\ -1 & \text{otherwise.} \end{cases}$$

By blend-in fairness, $f_i(u_{N\setminus\{i\}}) = f_i(u_{\{i\}})$ follows. □

Lemma 2 *Let $T, T' \subset N$ be two different coalitions with $|T| = |T'|$. If a solution $f : \mathcal{G}^N \to \mathbb{R}^n$ satisfies blend-in fairness, then $f_i(u_T) = f_i(u_{T'})$ for each $i \in (T \cap T') \cup (N\setminus(T \cup T'))$.*

Proof Recall that $u_T(N) = u_{T'}(N) = 0$ holds and let us consider the following two possible cases.

Case 1 ($|T| < |N|/2$): For the two partition games we have in this case $u_T(S) = 0$ if $S \in \{\emptyset, T, N\setminus T, N\}$ and $u_T(S) = -1$, otherwise; $u_{T'}(S) = 0$ if $S \in \{\emptyset, T', N\setminus T', N\}$ and $u_{T'}(S) = -1$, otherwise. Notice further that, due to $|T| = |T'|$, we have $|T\setminus T'| = |T'\setminus T|$. Define then the bijection (of order two) $\sigma : N \to N$ by $\sigma(k) = k$ for each $k \in (T \cap T') \cup (N\setminus(T \cup T'))$ and $\sigma(\ell) = m$ for $\ell \in T\setminus T'$ and $m \in T'\setminus T$.

Fix then $i \in (T \cap T') \cup (N\setminus(T \cup T'))$ and consider any $S \subset N$ with $j \in S$ for some $j \in N \setminus \{i\}$. Notice first that $j \in T$ makes $S = N \setminus T$ impossible and thus, $u_T(S) = u_{T'}(\sigma(S)) = 0$ holds only if $S \in \{\emptyset, T, N\}$(with $\sigma(T) = T'$ and $\sigma(N) = N$) while for all other relevant coalitions S we have $u_T(S) = u_{T'}(\sigma(S)) = -1$. On the other hand, $j \in N\setminus T$ would make $S = T$ impossible and thus, $u_T(S) = u_{T'}(\sigma(S)) = 0$ holds only if $S \in \{\emptyset, N\setminus T, N\}$(with $\sigma(N\setminus T) = N\setminus T'$ and $\sigma(N) = N$) while for all other relevant coalitions S we have $u_T(S) = u_{T'}(\sigma(S)) = -1$. We conclude by blend-in fairness that $f_i(u_T) = f_i(u_{T'})$ follows.

Case 2 ($|T| \geq |N|/2$): For the two partition games we have in this case $u_T(S) = 0$ if $S \in \{\emptyset, T, N\}$ and $u_T(S) = -1$, otherwise; $u_{T'}(S) = 0$ if $S \in \{\emptyset, T', N\}$ and $u_{T'}(S) = -1$, otherwise. Consider then the bijection σ, players i and j, and coalition S as defined in Case 1. Notice then that we have $u_T(S) = u_{T'}(\sigma(S)) = 0$ only if $S \in \{\emptyset, T, N\}$, while for all other relevant coalitions S we have $u_T(S) = u_{T'}(\sigma(S)) = -1$. Hence, $f_i(u_T) = f_i(u_{T'})$ follows by blend-in fairness. □

Proposition 2 *If a solution $f : \mathcal{G}^N \to \mathbb{R}^n$ satisfies additivity, the weak null player property, symmetric efficiency, and blend-in fairness, then $f_i(u_{\{k\}}) = 0$ for each $i, k \in N$.*

Proof Notice that $f_k(u_{\{k\}}) = 0$ for each $k \in N$ follows from Lemma 1. Moreover, by Lemma 2, $f_m\left(u_{\{i\}}\right) = f_m\left(u_{\{k\}}\right) := x_m$ holds for each $m \in N \setminus \{i, k\}$. Fix next $m \in N\setminus \{i, k\}$ and $q \in N\setminus \{i, k, m\}$, and consider the games $u_{\{i\}}$ and $u_{\{q\}}$. Again by Lemma 2, $f_m\left(u_{\{q\}}\right) = f_m\left(u_{\{i\}}\right) = x_m$ follows. Repeating this argument as many times as necessary results in $f_i(u_{\{k\}}) = x_i$ for each $i, k \in N$ with $i \neq k$.

Consider now the game $u = \Sigma_{\ell \in N} u_{\{\ell\}}$ and notice that

$$u(S) = \begin{cases} 0 & \text{if } S \in \{\emptyset, N\}, \\ -|N| + 1 & \text{if } |S| \in \{1, |N| - 1\}, \\ -|N| & \text{otherwise,} \end{cases}$$

and thus, all players are symmetric in *u*. By symmetric efficiency, $f_i(u) = 0$ holds for each $i \in N$. Applying additivity requires $(-|N| + 1)\, x_i = 0$ to hold for each $i \in N$ and thus, $f_i(u_{\{k\}}) = 0$ for each $i, k \in N$ with $i \neq k$ follows. □

Remark 1 Notice that the game *u* constructed in the proof of Proposition 2 is superadditive and thus, cohesive.

Proposition 3 *If a solution $f : \mathcal{G}^N \to \mathbb{R}^n$ satisfies additivity, the weak null player property, symmetric efficiency, and blend-in fairness, then $f_i(u_{N\setminus\{k\}}) = 0$ for each $i, k \in N$.*

Proof Notice first that, for $i = k$, the assertion follows from Lemma 1. Denote by $\mathcal{T}_{|N|-1}$ the set of all coalitions of size $|N| - 1$, fix $i \in N\setminus \{k\}$ and let $\mathcal{T}_{|N|-1}(i)$ stand for the set of all coalitions in $\mathcal{T}_{|N|-1}$ containing player *i*; notice that $\mathcal{T}_{|N|-1} = \mathcal{T}_{|N|-1}(i) \cup (N\setminus \{i\})$. In view of Lemma 2, $f_i(u_T) = f_i(u_{T'}) := x_i$ holds for each $T, T' \in \mathcal{T}_{|N|-1}(i)$.

Consider now the game $u = \Sigma_{k \in N} u_{N\setminus\{k\}}$ and notice that

$$u(S) = \begin{cases} 0 & \text{if } S \in \{\emptyset, N\}, \\ -\left|\mathcal{T}_{|N|-1}\right| + 1 & \text{if } |S| = |N| - 1, \\ -\left|\mathcal{T}_{|N|-1}\right| & \text{otherwise,} \end{cases}$$

and thus, all players are symmetric in *u*. By symmetric efficiency, $f_i(u) = 0$. By $f_i(u_{N\setminus\{i\}}) = 0$ and additivity, $\left|\mathcal{T}_{|N|-1}(i)\right| x_i = 0$ should hold. We have then $x_i = 0$ and thus, $f_i(u_{N\setminus\{k\}}) = 0$ follows. □

Remark 2 Notice that the game *u* constructed in the proof of Proposition 3 is superadditive and thus, cohesive.

Lemma 3 *Fix $T \subset N$, $k \in T$ with $|T| \geq 2$, and let $f : \mathcal{G}^N \to \mathbb{R}^n$ satisfy additivity, the weak null player property, and symmetric efficiency. Then,*

(1) $[|T| \neq |N|/2$ and $f_k(u_T) = 0] \Rightarrow f_k(u_{T\setminus\{k\}}) = 0$;
(2) $[|T| = |N|/2$ and $f_k(u_T) = f_k(u_{N\setminus T}) = 0] \Rightarrow f_k(u_{T\setminus\{k\}}) = 0$.

Proof Fix $T \subset N$ with $|T| \geq 2$ and $k \in T$. Clearly then, the coalitions T and $T \setminus \{k\}$ are nonempty. Recall further that $u_T(N) = u_{N\setminus T}(N) = 0$ holds.

(1) Notice that $|T| \neq |N|/2$ implies either ($|T| > |N|/2$ and $|T\setminus\{k\}| \geq |N|/2$) or ($|T| < |N|/2$ and $|T\setminus\{k\}| < |N|/2$). We consider these two possibilities separately.

(1.1) ($|T| > |N|/2$ and $|T\setminus\{k\}| \geq |N|/2$): For the two partition games u_T and $u_{T\setminus\{k\}}$ we have: $u_T(S) = 0$ for $S \in \{\emptyset, T, N\}$ and $u_T(S) = -1$, otherwise; $u_{T\setminus\{k\}}(S) = 0$ for $S \in \{\emptyset, T\setminus\{k\}, N\}$ and $u_{T\setminus\{k\}}(S) = -1$, otherwise. Define the games c_T and $c_{T\setminus\{k\}}$ by

$$c_T(S) = \begin{cases} -1 & \text{if } S = T, \\ 0 & \text{otherwise,} \end{cases}$$

and

$$c_{T\setminus\{k\}}(S) = \begin{cases} -1 & \text{if } S = T \setminus \{k\}, \\ 0 & \text{otherwise,} \end{cases}$$

and proceed as follows.

For the game $u_T + c_T$ we have $(u_T + c_T)(S) = 0$ for $S \in \{\emptyset, N\}$ and $(u_T + c_T)(S) = -1$, otherwise. That is, all players are symmetric in $u_T + c_T$ and thus, by symmetric efficiency, $f_k(u_T + c_T) = 0$ in particular holds. By assumption, $f_k(u_T) = 0$ holds as well and thus, by additivity,

$$f_k(c_T) = 0 \tag{8}$$

follows. Considering now the game $c_T + c_{T\setminus\{k\}}$ with $\left(c_T + c_{T\setminus\{k\}}\right)(S) = -1$ for $S \in \{T, T\setminus\{k\}\}$ and $\left(c_T + c_{T\setminus\{k\}}\right)(S) = 0$, otherwise, we have that player k is a null player in the game. By $\left(c_T + c_{T\setminus\{k\}}\right)(N) = 0$ and the weak null player property,

$$f_k\left(c_T + c_{T\setminus\{k\}}\right) = 0 \tag{9}$$

follows. We have then from (8), (9), and additivity that

$$f_k\left(c_{T\setminus\{k\}}\right) = 0 \tag{10}$$

should hold. Notice finally that for the game $u_{T\setminus\{k\}} + c_{T\setminus\{k\}}$ we have $\left(u_{T\setminus\{k\}} + c_{T\setminus\{k\}}\right)(S) = 0$ for $S \in \{\emptyset, N\}$ and $\left(u_{T\setminus\{k\}} + c_{T\setminus\{k\}}\right)(S) = -1$, otherwise. In other words, all players are symmetric in $u_{T\setminus\{k\}} + c_{T\setminus\{k\}}$ and thus, by symmetric efficiency, $f_k\left(u_{T\setminus\{k\}}+c_{T\setminus\{k\}}\right)=0$ in particular holds. By (10) and additivity, $f_k\left(u_{T\setminus\{k\}}\right)=0$ follows.

(1.2) ($|T|<|N|/2$ and $|T\setminus\{k\}|<|N|/2$): Recall that the coalitions T and $T\setminus\{k\}$ are nonempty and observe further that the coalitions $N\setminus T$ and $N\setminus(T\setminus\{k\})$ are nonempty as well. For the two partition games u_T and $u_{T\setminus\{k\}}$ we have: $u_T(S)=0$ for $S\in\{\emptyset,T,N\setminus T,N\}$ and $u_T(S)=-1$, otherwise; $u_{T\setminus\{k\}}(S)=0$ for $S\in\{\emptyset,T\setminus\{k\},N\setminus(T\setminus\{k\}),N\}$ and $u_{T\setminus\{k\}}(S)=-1$, otherwise. Define the games c_T and $c_{T\setminus\{k\}}$ by

$$c_T(S)=\begin{cases}-1 & \text{if } S\in\{T,N\setminus T\},\\ 0 & \text{otherwise,}\end{cases}$$

and

$$c_{T\setminus\{k\}}(S)=\begin{cases}-1 & \text{if } S\in\{T\setminus\{k\},N\setminus(T\setminus\{k\})\},\\ 0 & \text{otherwise,}\end{cases}$$

and proceed as follows.

For the game u_T+c_T we have $(u_T+c_T)(S)=0$ for $S\in\{\emptyset,N\}$ and $(u_T+c_T)(S)=-1$, otherwise. That is, all players are symmetric in u_T+c_T and thus, by symmetric efficiency, $f_k(u_T+c_T)=0$ in particular holds. By assumption, $f_k(u_T)=0$ holds as well and thus, by additivity,

$$f_k(c_T)=0 \tag{11}$$

follows. Consider now the game $c_T+c_{T\setminus\{k\}}$ with $\left(c_T+c_{T\setminus\{k\}}\right)(S)=-1$ for $S\in\{T\setminus\{k\},T,N\setminus T,N\setminus(T\setminus\{k\})\}$ and $\left(c_T+c_{T\setminus\{k\}}\right)(S)=0$, otherwise. Clearly, player *k* is a null player in this game. By $\left(c_T+c_{T\setminus\{k\}}\right)(N)=0$ and the weak null player property,

$$f_k\left(c_T+c_{T\setminus\{k\}}\right)=0 \tag{12}$$

follows. We have then from (11), (12), and additivity that

$$f_k\left(c_{T\setminus\{k\}}\right)=0 \tag{13}$$

should hold. Notice finally that for the game $u_{T\setminus\{k\}}+c_{T\setminus\{k\}}$ we have $\left(u_{T\setminus\{k\}}+c_{T\setminus\{k\}}\right)(S)=0$ for $S\in\{\emptyset,N\}$ and $\left(u_{T\setminus\{k\}}+c_{T\setminus\{k\}}\right)(S)=-1$, otherwise. In other words, all players are symmetric in $u_{T\setminus\{k\}}+c_{T\setminus\{k\}}$ and thus, by symmetric efficiency, $f_k\left(u_{T\setminus\{k\}}+c_{T\setminus\{k\}}\right)=0$ in particular holds. By (13) and additivity, $f_k\left(u_{T\setminus\{k\}}\right)=0$ follows.

(2) Notice that $|T|=|N|/2$ implies $|T|=|N\setminus T|=|N|/2$. Consider then the partition games u_T, $u_{N\setminus T}$, and $u_{T\setminus\{k\}}$ recalling that they are defined as follows: $u_T(S)=0$ for $S\in\{\emptyset,T,N\}$ and $u_T(S)=-1$, otherwise; $u_{N\setminus T}(S)=0$ for $S\in\{\emptyset,N\setminus T,N\}$ and $u_T(S)=-1$, otherwise; $u_{T\setminus\{k\}}(S)=0$ for

$S \in \{\emptyset, T \backslash \{k\}, N \backslash (T \backslash \{k\}), N\}$ and $u_{T \backslash \{k\}}(S) = -1$, otherwise. Define the games c_T, $c_{N \backslash T}$, and $c_{T \backslash \{k\}}$ by

$$c_T(S) = \begin{cases} -1 & \text{if } S = T, \\ 0 & \text{otherwise,} \end{cases}$$

$$c_{N \backslash T}(S) = \begin{cases} -1 & \text{if } S = N \setminus T, \\ 0 & \text{otherwise,} \end{cases}$$

and

$$c_{T \backslash \{k\}}(S) = \begin{cases} -1 & \text{if } S \in \{T \setminus \{k\}, N \setminus (T \setminus \{k\})\}, \\ 0 & \text{otherwise,} \end{cases}$$

and proceed as follows.

For the game $u_T + c_T$ we have $(u_T + c_T)(S) = 0$ for $S \in \{\emptyset, N\}$ and $(u_T + c_T)(S) = -1$, otherwise. That is, all players are symmetric in $u_T + c_T$ and thus, by symmetric efficiency, $f_k(u_T + c_T) = 0$ in particular holds. By assumption, $f_k(u_T) = 0$ holds as well and thus, by additivity,

$$f_k(c_T) = 0 \tag{14}$$

follows. Applying the same argument with respect to the game $u_{N \backslash T} + c_{N \backslash T}$ and recalling $f_k(u_{N \backslash T}) = 0$ holds by assumption, we get

$$f_k(c_{N \backslash T}) = 0. \tag{15}$$

Consider now the game $c_T + c_{N \backslash T} + c_{T \backslash \{k\}}$ with $(c_T + c_{N \backslash T} + c_{T \backslash \{k\}})(S) = -1$ for $S \in \{T \backslash \{k\}, T, N \backslash T, N \backslash (T \backslash \{k\})\}$ and $(c_T + c_{N \backslash T} + c_{T \backslash \{k\}})(S) = 0$, otherwise. Clearly, player k is a null player in this game. By $(c_T + c_{N \backslash T} + c_{T \backslash \{k\}})(N) = 0$ and the weak null player property,

$$f_k(c_T + c_{N \backslash T} + c_{T \backslash \{k\}}) = 0 \tag{16}$$

follows. We have then from (14), (15), (16), and additivity that

$$f_k(c_{T \backslash \{k\}}) = 0 \tag{17}$$

should hold. Notice finally that for the game $u_{T \backslash \{k\}} + c_{T \backslash \{k\}}$ we have $(u_{T \backslash \{k\}} + c_{T \backslash \{k\}})(S) = 0$ for $S \in \{\emptyset, N\}$ and $(u_{T \backslash \{k\}} + c_{T \backslash \{k\}})(S) = -1$, otherwise. In other words, all players are symmetric in $u_{T \backslash \{k\}} + c_{T \backslash \{k\}}$ and thus, by symmetric efficiency, $f_k(u_{T \backslash \{k\}} + c_{T \backslash \{k\}}) = 0$ in particular holds. By (17) and additivity, $f_k(u_{T \backslash \{k\}}) = 0$ follows.

Remark 3 Notice that the games c_T, $c_{T\setminus\{k\}}$, and $c_{N\setminus T}$ constructed in the proof of Lemma 3 are not superadditive but cohesive.

Proposition 4 *If a solution $f:\mathcal{G}^N\to\mathbb{R}^n$ satisfies additivity, the weak null player property, symmetric efficiency, and blend-in fairness, then $f_i(u_T)=0$ for each $i\in N$ and $T\subset N$.*

Proof We proceed by induction on the cardinality of $T\subset N$.

Initialization: For $|T|=|N|-1$, $f_i(u_T)=0$ for each $i\in N$ follows from Proposition 3.

Induction Hypothesis: Suppose that $f_i(u_T)=0$ holds for each $i\in N$ and each $T\subset N$ with $|T|>t$.

Take $T\subset N$ with $|T|=t$ and $i\in N\setminus T$. Notice that if $t=1$, then the assertion directly follows from Proposition 2. Suppose now that $t\geq 2$ holds. Since $|T\cup\{i\}|=t+1>t$, we get $f_i(u_{T\cup\{i\}})=0$ by the induction hypothesis. By the same reasoning and for the case when $t+1=|N|/2$, we additionally get $f_i(u_{N\setminus(T\cup\{i\})})=0$ due to $|N\setminus(T\cup\{i\})|=t+1$. Applying Lemma 3 results then in $f_i(u_T)=0$. Since player $i\in N\setminus T$ was arbitrary chosen, we conclude that $f_i(u_T)=0$ holds for each $i\in N\setminus T$.

Hence, it remains to be shown that $f_i(u_T)=0$ holds for each $i\in T$ as well. For this, fix $i\in T$, denote by $\mathcal{T}_{|T|}$ the set of all coalitions of size $|T|$ and by $\mathcal{T}_{|T|}(i)$ the set of all coalitions in $\mathcal{T}_{|T|}$ containing player *i*. Notice that we have $i\notin T'$ for each $T'\in\mathcal{T}_{|T|}\setminus\mathcal{T}_{|T|}(i)$. Hence, as already shown above, $f_i(u_{T'})=0$ holds for each $T'\in\mathcal{T}_{|T|}\setminus\mathcal{T}_{|T|}(i)$.

Consider now the game $u=\Sigma_{T'\in\mathcal{T}_{|T|}}u_{T'}$ and notice that, when $|T|\geq|N|/2$ holds, we have

$$u(S)=\begin{cases}0 & \text{if } S\in\{\emptyset,N\},\\ -\left|\mathcal{T}_{|T|}\right|+1 & \text{if } |S|=|T|,\\ -\left|\mathcal{T}_{|T|}\right| & \text{otherwise,}\end{cases}$$

while, when $|T|<|N|/2$ is the case, we have

$$u(S)=\begin{cases}0 & \text{if } S\in\{\emptyset,N\},\\ -\left|\mathcal{T}_{|T|}\right|+1 & \text{if } |S|\in\{|T|,|N\setminus T|\},\\ -\left|\mathcal{T}_{|T|}\right| & \text{otherwise.}\end{cases}$$

Observe that, in either of these cases, all players are symmetric in *u*. By symmetric efficiency, $f_i(u)=0$. By $f_i(u_{T'})=0$ for each $T'\in\mathcal{T}_{|T|}\setminus\mathcal{T}_{|T|}(i)$ and additivity, $\left|\mathcal{T}_{|T|}(i)\right|x_i=0$ should hold. We have then $x_i=0$ and thus, $f_i(u_T)=0$ follows.

Remark 4 Notice that the game *u* constructed in the proof of Proposition 4 is superadditive and thus, cohesive.

Proposition 5 *If a solution* $f : \mathcal{G}^N \rightarrow \mathbb{R}^n$ *satisfies symmetric efficiency, then* $f_i(u_N) = \frac{1}{n}$ *for each* $i \in N$.

Proof Recall that $u_N(N) = 1$ and $u_N(S) = 0$ for each $S \neq N$ holds. Clearly then, all players are symmetric in $u_N(N)$ and thus, by symmetric efficiency, $f_i(u_N) = \frac{1}{n}$ for each $i \in N$ follows. □

Propositions 1–5 show that, thanks to additivity, the weak null player property, symmetric efficiency, and blend-in fairness, one has $f_i(u_T) = ED_i(u_T)$ for each partition game $u_T(T \subseteq N, T \neq \emptyset)$ and each $i \in N$.

Proof of Theorem 1 It can be easily verified that the equal division solution satisfies the four axioms. As for the reverse implication, let *T* be a non-empty coalition, *b* a real number, and define the game u_T^b as follows:

(1) If $T \neq N$, then for each coalition $S \subseteq N$,

$$u_T^b(S) := \begin{cases} 0 & \text{if (1) } S \in \{\emptyset, T, N\} \text{ or (2) } |S| > |T| \text{ with } (S,T) \text{ partitioning } N, \\ -1 & \text{otherwise.} \end{cases}$$

(2) If $T = N$, then for each coalition $S \subseteq N$,

$$u_N^b(S) := \begin{cases} 0 & \text{if } S \neq N, \\ b & \text{otherwise.} \end{cases}$$

In view of Propositions 1–5, we conclude for each $i \in N$ that

$$f_i(u_T^b) = \begin{cases} 0 & \text{if } T \subset N, \\ \frac{b}{n} & \text{if } T = N. \end{cases}$$

Theorem 0 further implies the existence of real numbers $(a_T)_{\{T \subseteq N, T \neq \emptyset\}}$ such that, for $v \in \mathcal{G}^N$, one has $v = \Sigma_{T \subseteq N, T \neq \emptyset} a_T u_T^b$. By the additivity of *f*,

$$f_i(v) = \Sigma_{T \subseteq N, T \neq \emptyset} f_i(a_T u_T^b) \tag{18}$$

follows. We further make use of the following result. *Claim*$\Sigma_{T \subset N, T \neq \emptyset} f_i(a_T u_T^b) = 0$. □

Proof of the Claim Notice first that Proposition 1–5 and the additivity of *f* give us $f_i(a u_T^b) = 0$ for each $i \in N$, each non-empty $T \subset N$, and any $a \geq 0$; in particular, we have $f_i(a_T u_T^b) = 0$ for each $i \in N$ and each non-empty $T \subset N$ whenever $a_T \geq 0$ holds. Suppose now that we have $a_T < 0$ for some non-empty $T \subset N$. Notice then that, with *z* being the zero game, $a_T u_T^b + (-a_T) u_T^b = z$ holds and thus, due to the additivity of *f* and $f(z) = 0$ following from symmetric efficiency, we get $f_i(a_T u_T^b) = 0$ for each $i \in N$. Hence, the assertion follows.

The combination of (18) with the above Claim gives us $f_i(v) = f_i(a_N u_N^b) = f_i(u_N^{ba_N}) = f_i(u_N^{v(N)}) = \frac{v(N)}{n}$ for each $i \in N$. □

Proof of Theorem 2 Recall that partition games are superadditive (Remark 0) and thus, cohesive, and that all games constructed in the corresponding proofs of Propositions 1–5 are cohesive as well (Remarks 1–4). Since the zero game is also superadditive, the proof of Theorem 2 is fully analogous to the proof of Theorem 1. □

Independence of the axioms

In what follows, we assume that the player set contains at least three players and construct four examples where the correspondingly defined solution satisfies all axioms but the mentioned one. Observe that all solutions utilized to show axioms' logical independence on $\mathcal{G}^N$ can also be used to show the corresponding independence on $\mathcal{G}^N_{coh}$ as well. This is due to the fact that each non-superadditive game constructed in these examples is cohesive.

Additivity The solution f^A, given by $f^A(v) = ED(v)$ if $v \in \mathcal{G}^N_{sa}$ and $f^A(v) = Sh(v)$ if $v \in \mathcal{G}^N \backslash \mathcal{G}^N_{sa}$, satisfies all axioms but additivity. As to see the latter fact, let $N = \{1, 2, 3\}$ and consider the games *v* and *w* defined as follows: $v(S) = 1$ if $S \in \{\{1, 2\}, N\}$, and $v(S) = 0$, otherwise; $w(S) = 1$ if $S \in \{\{3\}, N\}$, and $w(S) = 0$, otherwise. The game *v* is superadditive but *w* is not and thus, we have $f^A(v) = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$ and $f^A(w) = \left(\frac{1}{6}, \frac{1}{6}, \frac{4}{6}\right)$. For the game $(v + w)$ we have $(v + w)(N) = 2$, $(v + w)(S) = 1$ if $S \in \{\{3\}, \{1, 2\}\}$, and $(v + w)(S) = 0$, otherwise. Since $(v + w)$ is not a superadditive game, we get $f^A(v + w) = \left(\frac{4}{6}, \frac{4}{6}, \frac{4}{6}\right) \neq f^A(v) + f^A(w)$.

Weak null player property Denote by $\mathcal{R}^3_i$ the set of all permutations of the player set *N* at which player $i \in N$ is at least the third player in the corresponding order. Notice that $\left|\mathcal{R}^3_i\right| = \left|\mathcal{R}^3_j\right|$ holds for all $i, j \in N$ and set $r := \left|\mathcal{R}^3_i\right|$. For $i \in N$, let

$$g_i(v) = \frac{1}{r} \sum_{R \in \mathcal{R}^3_i} \Delta_i(S_i(R)).$$

Define then the solution f^{WNP} by

$$f^{WNP}_i(v) = g_i(v) + \frac{v(N) - \Sigma_{i \in N} g_i(v)}{n} \text{ for each } i \in N.$$

In other words, the payoff $f^{WNP}_i(v)$ is just the sum of the average of player *i*'s marginal contributions to coalitions of size at least two ($g_i(v)$) and the equal division of the excess $v(N) - \Sigma_{i \in N} g_i(v)$. This solution clearly satisfies additivity and symmetric efficiency. In order to see that it satisfies blend-in fairness as well, recall that $\sigma(i) = i$ holds for the permutation σ guiding the cooperation equivalence of two superadditive games *v* and *w* for player $i \in N$. Given that $v(S) = w(\sigma(S))$ holds for all $S \subseteq N$ with $j \in S$ for some $j \in N \setminus \{i\}$, $f^{WNP}_i(v) = f^{WNP}_i(w)$ follows. Notice finally that f^{WNP} violates the weak null player property. As to see it, take $N = \{1, 2, 3\}$ and consider the superadditive game *v* defined by $v(S) = 0$ if $S \in \{\{1\}, \{2, 3\}, N\}$, and $v(S) = -1$, otherwise. Notice that player 1 is a null player in *v* and $v(N) = 0$. We get $g(v) = (0, 1, 1)$ and

$f^{WNP}(v) = \left(-\frac{2}{3}, \frac{1}{3}, \frac{1}{3}\right)$ in violation of the weak null player property requiring player 1 to get zero payoff in the game v.

Symmetric efficiency The solution f^{SE}, defined by $f^{SE}(v) = 0$ for each $v \in \mathcal{G}^N$, satisfies all axioms but symmetric efficiency.

Blend-in fairness The Shapley value violates blend-in fairness while satisfying all other axioms.

Funding Open Access funding enabled and organized by Projekt DEAL.

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