Compact quantum groups based on combinatorial structures

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Abstract

This thesis contributes to the theory of compact quantum groups, more specifically to the theory of quantum symmetries of graphs as well as quantum groups defined by tensor categories of partitions.

In the first part, we study quantum symmetries of hypergraphs by introducing a corresponding quantum automorphism group. We show that this quantum group generalizes the quantum automorphism group of Bichon for classical simple and directed graphs, as well as a quantum automorphism group of Goswami and Hossain for multigraphs. Additionally, we prove that our quantum automorphism group of hypergraphs acts maximally on hypergraph C^*-algebras recently introduced by Trieb, Weber and Zenner. This generalizes a result by Schmidt and Weber, where the quantum automorphism group of a graph acts maximally on the corresponding graph C^*-algebra.

In the second part, we study categories of spatial partitions and their associated spatial partition quantum groups in the sense of Cébron and Weber. We show that natural problems about categories of partitions are algorithmically undecidable and generalize categories of spatial partitions by introducing new base partitions. These new base partitions allow us to construct additional examples of free orthogonal quantum groups, but turn out to yield the same class of spatial partition quantum groups as before. Moreover, using our new base partitions, we are able to show that the class of spatial partition quantum groups is closed under taking projective versions, which allows us to give explicit descriptions of the projective versions of certain easy quantum groups in the sense of Banica and Speicher.

This thesis contains results from the articles [33, 32, 37] by the author.

Zusammenfassung

Diese Dissertation leistet einen Beitrag zur Theorie der kompakten Quantengruppen, genauer zur Theorie der Quantensymmetrien von Graphen und zu Quantengruppen, die durch Tensorkategorien von Partitionen definiert sind.

Im ersten Teil untersuchen wir Quantensymmetrien von Hypergraphen, indem wir eine entsprechende Quantenautomorphismengruppe definieren. Wir zeigen, dass diese Quantengruppe die Quantenautomorphismengruppe von Bichon für klassische einfache und gerichtete Graphen, sowie die Quantenautomorphismengruppe von Goswami und Hossain für Multigraphen verallgemeinert. Darüber hinaus beweisen wir, dass unsere Quantenautomorphismengruppe maximal auf den kürzlich von Trieb, Weber und Zenner eingeführten Hypergraph-C^*-Algebren wirkt. Dies verallgemeinert ein Ergebnis von Schmidt und Weber, wonach die Quantenautomorphismengruppe eines Graphen maximal auf der entsprechenden Graph-C^*-Algebra wirkt.

Im zweiten Teil untersuchen wir Kategorien von räumlichen Partitionen und die dadurch definierten Quantengruppen im Sinne von Cébron und Weber. Wir zeigen, dass natürliche Probleme in Kategorien von Partitionen algorithmisch unentscheidbar sind und verallgemeinern Kategorien von räumlichen Partitionen, indem wir neue Basispartitionen einführen. Diese Basispartitionen ermöglichen die Konstruktion freier orthogonaler Quantengruppen, liefern jedoch dieselbe Klasse von Quantengruppen wie zuvor. Darüber hinaus zeigen wir, dass die Klasse der auf räumlichen Partitionen basierenden Quantengruppen unter der Bildung projektiver Versionen abgeschlossen ist. Dies ermöglicht uns die explizite Beschreibung projektiver Versionen bestimmter einfacher Quantengruppen im Sinne von Banica und Speicher.

Diese Dissertation enthält Ergebnisse aus den Artikeln [33, 32, 37] des Autors.

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1. Introduction

In the following, we briefly introduce quantum permutations and motivate compact quantum groups, before we present our main results. These results fall into the two areas of quantum symmetries of graphs and quantum groups obtained from categories of partitions. Finally, we provide an overview of the structure of the thesis.

This thesis contains results from the articles [33, 32, 37] by the author. The additional work [34, 44, 38, 35, 36] and the Bachelor's thesis [87] supervised by the author are mentioned in the margin.

Motivation: Quantum permutation groups

Compact quantum groups were introduced by Woronowicz [95, 97] as a generalization of classical compact groups and provide a framework for describing symmetries in the setting of operator algebras and noncommutative geometry. A main example is the quantum permutation group S_n^+, which was introduced by Wang [89] as the quantum symmetry group of $\mathbb{C}^n$. It can be defined by the universal unital C^*-algebra

$$C(S_n^+) := C^*(u^i_j \mid u \text{ is a quantum permutation}),$$

where a matrix $u := (u^i_j)_{i,j=1}^n$ is called quantum permutation if

$$(u^i_j)^2 = (u^i_j)^* = u^i_j, \quad \sum_{k=1}^n u^i_k = \sum_{k=1}^n u^k_j = 1 \quad (1 \leq i, j \leq n).$$

Note that quantum permutations with entries in $\mathbb{C}$ are precisely classical permutations matrices, which justifies the name quantum permutation group.

Although the quantum permutation group S_n^+ is not a group in the usual sense, we can think of the algebra $C(S_n^+)$ as the algebra of continuous functions on some underlying matrix group, where the group multiplication is encoded by the unital $*$-homomorphism

$$\Delta\colon C(S_n^+) \to C(S_n^+) \otimes C(S_n^+), \quad u^i_j \mapsto \sum_{k=1}^n u^i_k \otimes u^k_j \quad (1 \leq i, j \leq n).$$

For $n \leq 3$, the C^*-algebra $C(S_n^+)$ is commutative, and the Gelfand-Naimark theorem implies that it is isomorphic to the algebra of functions on the classical permutation group S_n. However, if $n \geq 4$, then the algebra $C(S_n^+)$ becomes noncommutative, and there exists no underlying classical group. Still, it is useful to consider $C(S_n^+)$ as algebra of "noncommutative functions" on a virtual quantum group, and we can recover $C(S_n)$ as the abelianization of $C(S_n^+)$.

In addition to describing symmetries in the context of noncommutative geometry [89], quantum permutations appear in the formulation of noncommutative de Finetti theorems in free probability [56] and provide a characterization of perfect quantum strategies in non-local games in quantum information theory [64]. Moreover, so-called easy quantum groups naturally extend classical Schur-Weyl duality to more general partition algebras [10]. Finally, compact quantum groups generalize classical Pontryagin duality to arbitrary discrete groups, leading to the theory of locally compact quantum groups [55].

We refer to [93] for a more detailed motivation of compact quantum groups and to Chapter 2 for their definition and further examples. See also the recent survey [94] for more information on quantum permutations and the quantum permutation group S_n^+ in an operator-algebraic framework.

1.1. Quantum symmetries of graphs

Building on Wang's work, Bichon [13] and Banica [5] introduced two versions of a quantum automorphism group of a finite graph. These quantum groups generalize the classical automorphism group of a graph and are obtained by imposing the additional relation $Au = uA$ on a quantum permutation u, see Definition 4.1.1 and Definition 4.1.4. Here, A denotes the adjacency matrix of the graph, which requires the magic unitary u to respect the graph's structure. Quantum automorphism groups of graphs provide a large class of examples of compact quantum groups and have for example been studied in [21, 75, 57, 28]. In particular, these quantum automorphism groups have been further generalized to different structures such as multigraphs [45], Hadamard matrices [49] and quantum graphs [16, 17].

Quantum automorphism groups of hypergraphs

In this thesis, we present a definition for the quantum automorphism group of a hypergraph. Hypergraphs generalize classical graphs by allowing an edge to connect not only two but an arbitrary number of vertices. This makes hypergraphs quite general and gives them many applications in discrete mathematics and computer science, see [12, 2, 43] for further details. In the following, we focus on directed hypergraphs defined as follows.

Definition 1 (Definition 3.1.6)**.** A *(directed) hypergraph* $\Gamma := (V, E)$ consists of a finite set of vertices V, a finite set of edges E, and two maps $s\colon E \to 2^V$ and $r\colon E \to 2^V$, where 2^V denotes the power set of V.

An edge $e \in E$ can be depicted by an arrow from the set of source vertices $s(e)$ to the set of range vertices $r(e)$. Thus, classical directed edges correspond exactly to hyperedges with $|s(e)| = |r(e)| = 1$. In this context, our quantum automorphism group $\mathrm{Aut}^+(\Gamma)$ of a hypergraph Γ is given by the following compact matrix quantum group.

Definition 2 (Definition 3.2.5)**.** Let $\Gamma := (V, E)$ be a hypergraph and $\mathcal{A}$ be the universal unital C^*-algebra generated by elements u_w^v for all $v, w \in V$ and u_f^e for all $e, f \in E$ such that

1. $u_V := (u^v_w)_{v,w\in V}$ and $u_E := (u^e_f)_{e,f\in E}$ are quantum permutations,

2. $A_s u_E = u_V A_s$ and $A_r u_E = u_V A_r$, where $A_r, A_s \in B(\mathbb{C}^E, \mathbb{C}^V)$ are defined by

$$(A_s)^v_e = \begin{cases} 1 & \text{if } v \in s(e), \\ 0 & \text{otherwise}, \end{cases} \qquad (A_r)^v_e = \begin{cases} 1 & \text{if } v \in r(e), \\ 0 & \text{otherwise}, \end{cases}$$

for all $v \in V$ and $e \in E$.

Then $\mathrm{Aut}^+(\Gamma) := (\mathcal{A}, u_V \oplus u_E)$ is the *quantum automorphism group* of Γ.

Intuitively, $\mathrm{Aut}^+(\Gamma)$ consists of a quantum permutation u_V on the vertices and a quantum permutation u_E on the edges, which are compatible by intertwining the incidence matrices A_s and A_r. If u_V and u_E are classical permutation matrices, this definition recovers the classical automorphism group of a hypergraph, see Section 3.2.

In contrast to the quantum automorphism groups of Bichon and Banica, our definition includes a second magic unitary for the edges. This is necessary to capture quantum symmetries between multiple edges, which are allowed in our definition of a hypergraph. See Section 3.3 for an example of the quantum symmetries of a concrete family of hypergraphs with multiple edges. However, if a hypergraph Γ or its dual Γ^* contains no multiple edges, our definition reduces to only a single quantum permutation.

Theorem 3 (Definition 3.5.4, Definition 3.5.5)**.** *Let $\Gamma := (V, E)$ be a hypergraph.*

1. *If Γ has no multiple edges, then $\mathrm{Aut}^+(\Gamma) \subseteq S^+_V$.*

2. *If Γ^* has no multiple edges, then $\mathrm{Aut}^+(\Gamma) \subseteq S^+_E$.*

Link to quantum symmetries of classical graphs

Since hypergraphs generalize classical graphs and multigraphs, it is natural to ask how our quantum automorphism group relates to the quantum automorphism groups of Bichon [13], Banica [5], and Goswami and Hossain [45]. Denote by $\mathrm{Aut}^+_{\mathrm{Bic}}(\Gamma)$ the quantum automorphism group of Bichon, and denote by $\mathrm{Aut}^+_{\mathrm{GH,Bic}}(\Gamma)$ the quantum automorphism group of Goswami and Hossain in the sense of Bichon. Then the following theorem shows that we obtain both quantum groups as special case when representing classical graphs and multigraphs as hypergraphs.

Theorem 4 (Definition 4.2.3, Definition 4.3.4, Definition 4.4.4)**.**

1. *Let $\Gamma := (V, E)$ be a directed graph as in Definition 3.1.2. Define source and range maps by*
$$s(v, w) = \{v\}, \quad r(v, w) = \{w\} \qquad \forall (v, w) \in E.$$
Then Γ is a hypergraph with $\mathrm{Aut}^+(\Gamma) = \mathrm{Aut}^+_{\mathrm{Bic}}(\Gamma)$.

2. *Let* $\Gamma := (V, E)$ *be a simple graph as in Definition 3.1.1. Define source and range maps by*
$$s(\{v,w\}) = \{v,w\}, \quad r(\{v,w\}) = \{v,w\} \qquad \forall \{v,w\} \in E.$$
Then Γ *is a hypergraph with* $\mathrm{Aut}^+(\Gamma) = \mathrm{Aut}^+_{\mathrm{Bic}}(\Gamma)$.

3. *Let* $\Gamma := (V, E)$ *be a multigraph as in Definition 3.1.4 with source map* s' *and range map* r'. *Define new source and range maps by*
$$s(e) = \{s'(e)\}, \quad r(e) = \{r'(e)\} \qquad \forall e \in E.$$
Then Γ *is a hypergraph with* $\mathrm{Aut}^+(\Gamma) = \mathrm{Aut}^+_{\mathrm{GH,Bic}}(\Gamma)$.

Note that the previous theorem can be used to construct many concrete examples of quantum automorphism groups of hypergraphs.

Actions on hypergraph C^*-algebras

Related to quantum automorphism groups of classical graphs is the study of quantum symmetries of graph C^*-algebras in [77, 52]. These C^*-algebras are defined in terms of an underlying graph and have been studied since the 1980s. They include many concrete examples such as matrix algebras, the algebra of continuous functions on the circle and the Cuntz algebras, see [70] for further details. Recently, Trieb, Weber and Zenner [84] introduced hypergraph C^*-algebras, a generalization of graph C^*-algebras to the setting hypergraphs. This new class includes all graph C^*-algebras, as well as additional examples of non-nuclear C^*-algebras. See the work of Schäfer and Weber [78] for a characterization of the nuclearity of hypergraph C^*-algebras in terms of minors of the underlying hypergraph.

In [77], Schmidt and Weber showed that Banica's quantum automorphism group of a graph acts maximally on the corresponding graph C^*-algebra. In this thesis, we generalize this result to hypergraphs by showing that our quantum automorphism group $\mathrm{Aut}^+(\Gamma)$ acts on the corresponding hypergraph C^*-algebra $C^*(\Gamma)$. Similar to graph C^*-algebras, hypergraph C^*-algebras are generated by a family of projections $\{p_v\}_{v\in V}$ and a family of partial isometries $\{s_e\}_{e\in E}$, see Definition 5.1.2. In this context, our action is defined by permuting these generators using the magic unitaries u_V and u_E, i.e.
$$\alpha(p_v) = \sum_{w\in V} p_w \otimes u_v^w, \quad \alpha(s_e) = \sum_{f\in E} s_f \otimes u_e^f \qquad \forall v \in V,\, e \in E.$$

This action coincides with the action of Schmidt and Weber in the case of classical graphs, but our quantum automorphism group is no longer maximal with respect to it. However, maximality is obtained under the additional assumption that $\mathrm{Aut}^+(\Gamma)$ also acts on $C^*(\Gamma')$, where Γ' is obtained by reversing all edge directions and interchanging the vertices and edges of Γ.

Theorem 5 (Definition 5.2.3, Definition 5.3.4). *Let* $\Gamma := (V, E)$ *be a hypergraph and define* $\Gamma' := (\Gamma^*)^{\mathrm{op}}$. *Then* $\mathrm{Aut}^+(\Gamma)$ *is the largest compact matrix quantum group that acts* *faithfully on both* $C^*(\Gamma)$ *and* $C^*(\Gamma')$ *via the actions*

$$\alpha_1 \colon C^*(\Gamma) \to C^*(\Gamma) \otimes C(\mathrm{Aut}^+(\Gamma)),$$
$$\alpha_2 \colon C^*(\Gamma') \to C^*(\Gamma') \otimes C(\mathrm{Aut}^+(\Gamma))$$

defined by

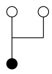
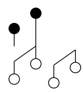
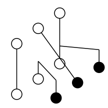
$$\alpha_1(p_v) = \sum_{w \in V} p_w \otimes u_v^w, \quad \alpha_1(s_e) = \sum_{f \in E} s_f \otimes u_e^f \qquad \forall v \in V,\, e \in E,$$
$$\alpha_2(p_e) = \sum_{f \in E} p_f \otimes u_e^f, \quad \alpha_2(s_v) = \sum_{w \in V} s_w \otimes u_v^w \qquad \forall v \in V,\, e \in E.$$

1.2. Quantum groups based on partitions

In addition to quantum automorphism groups, compact quantum groups can be obtained from tensor categories via Woronowicz's Tannaka-Krein duality [96]. A special class of such quantum groups are easy quantum groups [10, 80], where the corresponding tensor categories are defined by colored set partitions. Since easy quantum groups are based on partitions, they form a concrete class of quantum groups that can be studied and classified using combinatorial methods [8, 90, 71, 42, 50, 41]. In the case of orthogonal easy quantum groups, the classification has been completed in [73], whereas the classification in the unitary case is still ongoing. See for example [81, 46, 39] and the more recent work by Mang [60, 61, 62, 63].

Spatial partition quantum groups

In [25], Cébron and Weber introduced spatial partition quantum groups, which generalize easy quantum groups by replacing two-dimensional partitions with three-dimensional spatial partitions. A spatial partition on m levels $p \in \mathcal{P}^{(m)}$ consists of $k \cdot m$ upper points and $\ell \cdot m$ lower points that are partitioned into disjoint subsets by strings. Additionally, we allow both upper and lower points to be uniformly colored along the levels. For example, we have

 $\in \mathcal{P}^{(1)}$,  $\in \mathcal{P}^{(2)}$,  $\in \mathcal{P}^{(3)}$.

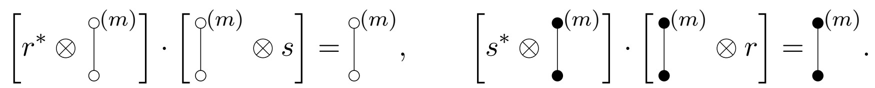
Given spatial partitions on the same number of levels, we can construct new spatial partitions by taking their tensor product, involution and composition. A category of spatial partitions in the sense of Cébron and Weber is a set of spatial partitions closed under these operations and containing the base partitions $^{(m)}$, $^{(m)}$, $^{(m)}$ and $^{(m)}$. Here, $p^{(m)} \in \mathcal{P}^{(m)}$ denotes the spatial partition obtained by placing m copies of the partition p along each level.

By representing spatial partitions as linear operators, categories of spatial partitions give rise to concrete C^*-tensor categories, which then correspond to spatial partition quantum groups via Woronowicz's Tannaka-Krein duality [96]. In the case of partitions on one level, these quantum groups are precisely unitary easy quantum groups [10, 80] and include for example the free orthogonal and free unitary quantum groups [88], the quantum permutation group [89] and the hyperoctahedral quantum group [7].

Spatial partition quantum groups have been studied in [25, 31]. In [25], Cébron and Weber showed that these quantum groups are closed under glued products, implying that the class of spatial partition quantum groups is strictly larger than the class of easy quantum groups. Furthermore, they provided a partial classification of categories of spatial pair partitions on two levels and discuss links to the quantum symmetries of finite-dimensional C^*-algebras. In [31], the author showed that the category $P_2^{(2)}$ of all spatial pair partitions on two levels gives rise to the classical projective orthogonal group PO_n, yielding a simpler example of a non-easy spatial partition quantum group. Additionally, an explicit description of the category of spatial partitions corresponding to the quantum symmetry group of $M_n(\mathbb{C}) \otimes \mathbb{C}^m$ is given.

Generalizing spatial base partitions

In [25], Cébron and Weber raised the question of generalizing the base partitions $^{(m)}$ and $^{(m)}$ while still allowing the construction of compact matrix quantum groups from categories of spatial partitions. In this thesis, we answer this question by showing that the previous two base partitions can be replaced with any pairs of spatial partitions r and s satisfying the conjugate equations



In the case of one level, the partitions $r =$ and $s =$ are the unique solutions to these equations. However, for $m \geq 2$, we obtain not only the previous base partitions $r = {}^{(m)}$ and $s = {}^{(m)}$ but also twisted versions such as

See Definition 6.3.2 for a characterization of all possible solutions to these conjugate equations in the context of spatial partitions.

Using our new base partitions, we can construct additional examples of free orthogonal quantum groups in the sense of Van Daele and Wang [85]. Following the notation in [3], these quantum groups are denoted by $O^+(F_\sigma)$ and with parameters

$$F_\sigma \colon (\mathbb{C}^n)^{\otimes m} \to (\mathbb{C}^n)^{\otimes m}, \quad F_\sigma(e_{i_1} \otimes \cdots \otimes e_{i_m}) = e_{i_{\sigma^{-1}(1)}} \otimes \cdots \otimes e_{i_{\sigma^{-1}(m)}}$$

for all $n \in \mathbb{N}$ and $\sigma \in S_m$. See Definition 7.2.5 for further details and the precise statement in full generality.

Although we can construct new examples of spatial partition quantum groups, our new base partitions yield the same class of quantum groups as defined by Cébron and Weber.

Theorem 6 (Definition 7.3.3)**.** *Let G be a spatial partition quantum group defined by any pair of spatial base partitions satisfying the conjugate equations. Then G is equivalent to a spatial partition quantum group in the sense of Cébron and Weber defined by the base partitions $\sqcap_{\circ\bullet}^{(m)}$ and $\sqcap_{\bullet\circ}^{(m)}$.*

More generally, we show in Section 7.3 that permuting the points of any category of spatial partitions along the levels leaves the corresponding quantum group invariant, which allows us to reduce any base partitions to the case of $\sqcap_{\circ\bullet}^{(m)}$ and $\sqcap_{\bullet\circ}^{(m)}$. Still, our new base partitions are useful from a combinatorial perspective and allow us to show that the class of spatial partition quantum groups is closed under taking projective versions.

Projective spatial partition quantum groups

Consider a compact matrix quantum group G with fundamental representation u, and assume that $\overline{u}$ is unitary. Then its projective version PG is the compact matrix quantum group defined by the representation $u \mathbin{\textcircled{\scriptsize T}} \overline{u}$. If G is a classical group, then PG corresponds exactly to the quotient

$$PG = G/(G \cap \{\lambda I \mid \lambda \in \mathbb{C}\}).$$

Furthermore, in the case quantum groups, projective versions have been studied for example in [4, 8, 11, 48]. In this context, our main result can be formulated as follows.

Theorem 7 (Definition 8.1.9)**.** *Let G be a spatial partition quantum group. Then PG is a spatial partition quantum group. Its category of spatial partitions is given by $\mathrm{Flat}_{m,\circ\bullet}^{-1}(\mathcal{C})$, where $\mathcal{C} \subseteq \mathcal{P}^{(m)}$ is the category of spatial partitions corresponding to G, and $\mathrm{Flat}_{m,\circ\bullet}$ is the functor defined in Section 6.4.*

Note that this result applies not only to projective versions PG, but also to any compact matrix quantum group defined by a tensor product of the fundamental representation u and a unitary conjugate representation $u^\bullet$. See Section 8.1 for further details.

As an application, we consider the category $\mathcal{P}_2^{(m)}$ of all spatial pair partitions on m levels. In [10, 31], it is shown that $\mathcal{P}_2^{(1)}$ corresponds to the classical orthogonal group O_n and that $\mathcal{P}_2^{(2)}$ corresponds to its projective version PO_n. Using the previous theorem, we generalize these results to all $m \in \mathbb{N}$ and obtain in Section 8.2 that

$$\mathcal{P}_2^{(m)} \quad \longleftrightarrow \quad \begin{cases} PO_n & \text{if } m \text{ is even,} \\ O_n & \text{if } m \text{ is odd.} \end{cases}$$

Finally, we consider projective versions of easy quantum groups. Because easy quantum groups are a subclass of spatial partition quantum groups, our main result implies that their projective versions are also spatial partition quantum groups. Using a result of Gromada [48], we derive sets of spatial partitions that generate the categories of the projective versions of orthogonal easy quantum groups with degree of reflection two. This allows us to explicitly describe these quantum groups as universal C^*-algebras defined by a finite set of relations, see Section 8.3.

Algorithmic problems in categories of partitions

In addition to spatial partition quantum groups, we also study general algorithmic problems arising from partitions. First, we present efficient data structures and algorithms for partitions and their basic operations, which have been developed in [87] and subsequently implemented by Volz and the author in the computer algebra system [69].

Second, we study algorithmic problems arising from categories of partitions such as determining membership of a given partition $p \in \mathcal{P}$. While enumerating the elements of a category $\mathcal{C}$ eventually yields the partition p, it is not immediately clear how to determine whether $p \notin \mathcal{C}$. We answer this question by showing that it is generally not possible to decide if $p \notin \mathcal{C}$ and to solve the related problem of counting partitions of a given size.

Theorem 8 (Definition 6.6.9, Definition 6.6.10)**.** *There exists a recursively enumerable category of partitions $\mathcal{C}$ such that the following problems are algorithmically undecidable:*

1. *Decide if $p \in \mathcal{C}$ for a given partition $p \in \mathcal{P}$.*
2. *Determine the number of partitions $|\mathcal{C}(k, \ell)|$ for given $k, \ell \in \mathbb{N}$.*

Here, a category of partitions is recursively enumerable if its elements can be enumerated by a Turing machine, and a problem is algorithmically undecidable if no Turing machine can solve the problem for all possible input values in finite time.

1.3. Structure of the thesis

We start in Chapter 2 with preliminaries about compact matrix quantum groups. This includes the necessary background on C^*-algebras, actions of quantum groups on C^*-algebras and Woronowicz's Tannaka-Krein duality.

In Chapter 3, we introduce our quantum automorphism groups of hypergraphs and study their basic properties. We provide first examples of such quantum groups and show in Chapter 4 that our definition generalizes the quantum automorphism groups in these sense of Bichon for classical graphs and multigraphs. Furthermore, in Chapter 5, we construct an action of our quantum automorphism group on hypergraph C^*-algebras, generalizing a result of Schmidt and Weber for graph C^*-algebras.

The second part of the thesis begins in Chapter 6, where we consider the combinatorics of spatial partitions. This includes the definition of our new base partitions and the study of algorithmic problems in the context of partitions. In Chapter 7, we construct quantum groups using our new base partitions, which turn out to coincide with the class of spatial partition quantum groups in the sense of Cébron and Weber. This allows us to show in Chapter 8 that the class of spatial partition quantum groups is closed under taking projective versions. In particular, we can give concrete descriptions of the projective versions of certain easy quantum groups.

Finally, Chapter 9 contains remaining open questions about quantum automorphism groups of hypergraphs and quantum groups based on partitions.

2. Preliminaries

In this chapter, we present the necessary preliminaries on the theory of compact quantum groups that are used throughout the thesis. We begin by introducing C^*-algebras, before we define compact matrix quantum groups, including first examples and various basic notions. Moreover, we discuss actions of quantum groups on C^*-algebras and formulate Woronowicz's Tannaka-Krein duality, which are used in Chapter 5 and Chapter 7 respectively.

2.1. C^*-algebras

C^*-algebras are a class of operator algebras on Hilbert spaces that appear for example in the mathematical formulation of quantum mechanics [20], in Connes' noncommutative geometry [22] and in the context of compact quantum groups [95]. In the following, we present only basic definitions and constructions that are used throughout this thesis. For more details on the theory of C^*-algebras, we refer to [65, 26, 15, 19].

We begin with the axiomatic definition of a C^*-algebra, before we come to concrete examples and the characterization of C^*-algebras as algebras of bounded operators on Hilbert spaces.

Definition 2.1.1. A *C^*-algebra* $\mathcal{A}$ is a complex associative algebra equipped with a norm $\|\cdot\|$ and an involution $*\colon \mathcal{A} \to \mathcal{A}$ satisfying the following properties:

1. $\mathcal{A}$ is complete with respect to the norm, and $\|xy\| \leq \|x\| \cdot \|y\|$ for all $x, y \in \mathcal{A}$.

2. The involution is anti-linear and anti-multiplicative, i.e.

$$(\alpha x + \beta y)^* = \overline{\alpha} x^* + \overline{\beta} y^*, \quad (x \cdot y)^* = y^* \cdot x^* \qquad \forall x, y \in \mathcal{A},\, \alpha, \beta \in \mathbb{C}.$$

3. The C^*-identity $\|x\|^2 = \|x^* x\|$ holds for all $x \in \mathcal{A}$.

Additionally, $\mathcal{A}$ is *unital* if it contains a unit element $1 \in \mathcal{A}$ making it a unital algebra.

Let $\mathcal{A}$ and $\mathcal{B}$ be C^*-algebras. Then an algebra homomorphism $\phi\colon \mathcal{A} \to \mathcal{B}$ is called *$*$-homomorphism* if $\phi(a^*) = \phi(a)^*$ for all $a \in \mathcal{A}$. If ϕ is bijective, it is a *$*$-isomorphism*. Note that $*$-homomorphisms between C^*-algebras are automatically continuous. Similarly, $\mathcal{B} \subseteq \mathcal{A}$ is a *C^*-subalgebra* if $\mathcal{B} \subseteq \mathcal{A}$ is subalgebra that is closed under the involution and with respect to the norm. A $*$-homomorphism or C^*-subalgebra is called *unital* if its underlying algebra homomorphism or subalgebra is unital.

Next, we present several examples of C^*-algebras.

Example 2.1.2. Let X be a compact Hausdorff space. Then the set

$$C(X) := \{f \colon X \to \mathbb{C} \mid f \text{ is continuous}\}$$

with point-wise addition, point-wise multiplication, and norm and involution defined by

$$\|f\| := \sup_{x \in X} |f(x)|, \qquad f^*(x) := \overline{f(x)} \quad \forall x \in X$$

is a unital commutative C^*-algebra.

Example 2.1.3. Let X be a finite set. If we equip X with the discrete topology, the previous example shows that the set $\mathbb{C}^X := C(X)$ of all $\mathbb{C}$-valued functions on X is a C^*-algebra. In the special case of $X = \{1, \ldots, n\}$, we identify $\mathbb{C}^X$ with $\mathbb{C}^n$.

Example 2.1.4. Let V be a Hilbert space. Then the set $B(V)$ of all bounded linear operators on V is a unital C^*-algebra with respect to point-wise addition, composition, and operator norm and adjoint defined by

$$\|T\| := \sup_{\|v\|=1} \|Tv\|, \qquad \langle Tv, w\rangle = \langle v, T^*w\rangle \quad \forall v, w \in V.$$

Furthermore, the set $K(V)$ of compact operators on V is a non-unital C^*-subalgebra of $B(V)$.

If we fix an orthonormal basis $(v_i)_{i \in I}$ of V, we can identify $B(V)$ with the set of matrices with rows and columns indexed by I. In this case, the multiplication and adjoint are explicitly defined by

$$(a \cdot b)^i_j = \sum_{k \in I} a^i_k \cdot b^k_j, \quad (a^*)^i_j = \overline{(a^j_i)} \qquad \forall i, j \in I,$$

for all matrices $a := (a^i_j)_{i,j \in I}$, $b := (b^i_j)_{i,j \in I} \in B(V)$.

Motivated by the previous example of bounded operators on a Hilbert space, we define the following types of elements in arbitrary C^*-algebras.

Definition 2.1.5. Let $\mathcal{A}$ be a (unital) C^*-algebra and $a \in \mathcal{A}$.

- If $a^* = a$, then a is called *self-adjoint*.
- If $a^2 = a^* = a$, then a is called a *projection*.
- If $aa^*a = a$, then a is called a *partial isometry*.
- If $a^*a = aa^* = 1$, then a is called a *unitary*.

In addition to the previous elements, there exists also a notion of positive elements in general C^*-algebras.

Definition 2.1.6. Let $\mathcal{A}$ be a C^*-algebra. An element $a \in \mathcal{A}$ is called *positive*, and we write $a \geq 0$, if there exists an element $b \in \mathcal{A}$ such that $a = b^*b$. Positive elements induce a partial order $\leq$ on $\mathcal{A}$, where $a \leq b$ if and only if $b - a \geq 0$.

Note that positive elements are necessarily self-adjoint. Furthermore, it can be shown that positive elements in $C(X)$ are precisely real-valued functions with $f(x) \geq 0$ for all $x \in X$ and that positive operators in $B(H)$ are precisely self-adjoint operators with spectrum $\sigma(a) \subseteq [0, \infty)$.

In Definition 2.1.2, we introduced the C^*-algebras $C(X)$ as examples of unital commutative C^*-algebras. The following theorem shows that every commutative unital C^*-algebra is of this form. We refer to [65, Chapter 2] for further details and a proof of the statement.

Theorem 2.1.7 (Gelfand-Naimark)**.** *Let $\mathcal{A}$ be a commutative unital C^*-algebra. Then $\mathcal{A}$ is $*$-isomorphic to $C(X)$ for a compact Hausdorff space X. Moreover, X is explicitly given by*

$$\operatorname{Spec} \mathcal{A} := \{\phi \colon \mathcal{A} \to \mathbb{C} \mid \phi \textit{ is a } *\textit{-homomorphism}\}$$

with the topology of point-wise convergence.

Remark 2.1.8. The previous theorem can be extended by showing that unital $*$-homomorphisms between commutative C^*-algebras correspond precisely to continuous mappings between their underlying spaces. This establishes a duality between the category of compact Hausdorff spaces with continuous maps and the category of commutative unital C^*-algebras with unital $*$-homomorphisms. In analogy to this duality, it is often useful to view general noncommutative C^*-algebras as algebras of "noncommutative functions" on some underlying space, although such a space does technically not exist. This point of view also underlies the theory of compact quantum groups and will be used throughout the following sections.

In addition to the previous theorem, every C^*-algebras can be represented as a concrete algebra of operators on a Hilbert space. Therefore, C^*-algebras can alternatively be defined as norm-closed $*$-subalgebras of $B(H)$ for some Hilbert space H.

Theorem 2.1.9 (GNS representation)**.** *Let $\mathcal{A}$ be a C^*-algebra. Then there exists a Hilbert space H and an injective $*$-homomorphism $\pi \colon \mathcal{A} \to B(H)$.*

Next, we introduce several constructions that will be used throughout this thesis and yield additional examples of C^*-algebras. We begin with universal C^*-algebra, which allow us to define C^*-algebras in terms of generators and relations.

Let $X = \{x_1, \ldots, x_n\}$ be a set of variables. Then we denote by $\mathbb{C}\langle X, X^* \rangle$ the set of noncommutative $*$-polynomials in X. In the following, we identify elements $r \in \mathbb{C}\langle X, X^* \rangle$ with the relations $r = 0$, and we write $r(a_1, \ldots, a_n) \in \mathcal{A}$ for the element obtained by substituting each x_i with $a_i \in \mathcal{A}$ in a $*$-algebra $\mathcal{A}$. Moreover, if $R \subseteq \mathbb{C}\langle X, X^* \rangle$ is a set of relations, then (R) denotes the two-sided $*$-ideal generated by R. Finally, a seminorm p on a $*$-algebra is a C^*-seminorm if it satisfies all norm axioms in Definition 2.1.1.

Definition 2.1.10. Let $X := \{x_1, \dots, x_n\}$ be a set of variables and $R \subseteq \mathbb{C}\langle X, X^*\rangle$ be a set of relations. If

$$\sup\big\{p(x) \mid p \text{ is a } C^*\text{-seminorm on } \mathbb{C}\langle X, X^*\rangle/(R)\big\} < \infty \qquad \forall x \in X,$$

then the *universal unital C^*-algebra* $C^*(X \mid R)$ with generators X and relations R exists. It is uniquely determined by the following universal property: If $\mathcal{A}$ is a unital C^*-algebra with elements $a_1, \dots, a_n \in \mathcal{A}$ satisfying $r(a_1, \dots, a_n) = 0$ for all $r \in R$, then there exists a unital $*$-homomorphism

$$\phi\colon C^*(X \mid R) \to \mathcal{A}, \quad x_i \mapsto a_i \quad (1 \le i \le n).$$

Remark 2.1.11. Note that the precondition in the previous definition is necessary. For example, the universal unital C^*-algebra $C^*(a \mid a^* = a)$ generated by a single self-adjoint element does not exist, since bounded self-adjoint operators on a Hilbert space can have arbitrarily large operator norms. However, throughout this thesis, we will only consider universal C^*-algebras generated by projections, partial isometries or entries of unitary matrices. In these cases, the C^*-norm will always be bounded.

Example 2.1.12. Let X be a finite set. Then the C^*-algebra $\mathbb{C}^X$ form Definition 2.1.3 is $*$-isomorphic to the universal unital C^*-algebra

$$C^*\big(e_i \mid e_i^2 = e_i^* = e_i, \sum_{i \in X} e_i = 1\big),$$

where the generators $(e_i)_{i \in X}$ correspond to the indicator functions defined by

$$e_i(j) = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{otherwise,} \end{cases} \qquad \forall j \in X.$$

Next, we introduce two product constructions that allow the construction of additional examples of C^*-algebras from existing ones. We begin with the minimal tensor product of C^*-algebras, which can be defined using the GNS representation in Definition 2.1.9.

Definition 2.1.13. Let $\mathcal{A}$ and $\mathcal{B}$ be C^*-algebras with GNS representations

$$\pi_1\colon \mathcal{A} \to B(H_1), \quad \pi_2\colon \mathcal{B} \to B(H_2),$$

and denote by π the $*$-homomorphism

$$\pi\colon \mathcal{A} \otimes_{alg} \mathcal{B} \to B(H_1 \otimes H_2), \quad a \otimes b \mapsto \pi_1(a) \otimes \pi_2(b) \qquad \forall a \in \mathcal{A},\, b \in \mathcal{B}.$$

Then the *(minimal) tensor product* $\mathcal{A} \otimes \mathcal{B}$ is the completion of the algebraic tensor product $\mathcal{A} \otimes_{alg} \mathcal{B}$ with respect to the C^*-norm $\|\cdot\| := \|\pi(\cdot)\|$.

Note that multiple C^*-completions on the algebraic tensor product $\mathcal{A} \otimes_{alg} \mathcal{B}$ may exist. However, the minimal tensor product is uniquely determined and satisfies $\|a \cdot b\| = \|a\| \cdot \|b\|$ for all $a \in \mathcal{A}$ and $b \in \mathcal{B}$. Furthermore, in the unital case, both $\mathcal{A}$ and $\mathcal{B}$ are contained in $\mathcal{A} \otimes \mathcal{B}$ as the C^*-subalgebras $\mathcal{A} \otimes 1$ and $1 \otimes \mathcal{B}$ respectively. Next, we give two examples of C^*-algebras constructed from tensor products.

Example 2.1.14. Let $\mathcal{A}$ be a unital C^*-algebra and V be a finite dimensional Hilbert space with orthonormal basis $(v_i)_{i\in I}$. Then every element $a \in \mathcal{A} \otimes B(V)$ can be uniquely written as

$$a = \sum_{i,j\in I} a^i_j \otimes e^i_j,$$

for some $a^i_j \in \mathcal{A}$, where e^i_j are the standard matrix units defined by $(e^i_j)^k_\ell := \delta_{ik}\delta_{j\ell}$ for all $i, j, k, \ell \in I$. Thus, we can identify the element $a \in \mathcal{A} \otimes B(V)$ with the $\mathcal{A}$-valued matrix $a = (a^i_j)_{i,j\in I}$.

Example 2.1.15. Let X, Y be compact Hausdorff spaces. Then the map

$$f \otimes g \in C(X) \otimes C(Y) \ \longmapsto\ \big((x, y) \mapsto f(x)g(y)\big) \in C(X \times Y)$$

extends to a $*$-isomorphism between the tensor product $C(X) \otimes C(Y)$ and $C(X \times Y)$.

In addition to the previous examples, we will need the following proposition about positive elements in tensor products in Chapter 5.

Proposition 2.1.16. *Let $\mathcal{A}$ be a C^*-algebra and $x, y, z \in \mathcal{A}$. If $x \le y$ and $z \ge 0$, then $x \otimes z \le y \otimes z$.*

Proof. Let $x, y, z \in \mathcal{A}$ with $x \le y$ and $z \ge 0$. Then there exist $a, b \in \mathcal{A}$ such that $y - x = a^*a$ and $z = b^*b$. Thus,

$$y \otimes z - x \otimes z = (y - x) \otimes z = a^*a \otimes b^*b = (a \otimes b)^*(a \otimes b) \ge 0,$$

which yields $x \otimes z \le y \otimes z$. $\square$

Finally, we introduce the unital free product $\mathcal{A} * \mathcal{B}$ of two unital C^*-algebras $\mathcal{A}$ and $\mathcal{B}$, which is a noncommutative analogue of the tensor product of C^*-algebras. Intuitively, the *unital free product* is generated by two noncommuting copies of $\mathcal{A}$ and $\mathcal{B}$ that are embedded into $\mathcal{A} * \mathcal{B}$ via canonical inclusions

$$\iota_1 \colon \mathcal{A} \hookrightarrow \mathcal{A} * \mathcal{B}, \quad \iota_2 \colon \mathcal{B} \hookrightarrow \mathcal{A} * \mathcal{B},$$

where the units of $\mathcal{A}$ and $\mathcal{B}$ are identified via $\iota_1(1) = \iota_2(1)$. In the following, we characterize the unital free product via its universal property. See [86, Chapter 1] for more details on its concrete construction.

Proposition 2.1.17. *Let $\mathcal{A}$ and $\mathcal{B}$ be unital C^*-algebras. Then the unital free product $\mathcal{A} * \mathcal{B}$ satisfies the following universal property: For every unital C^*-algebra $\mathcal{C}$ and unital $*$-homomorphisms*

$$\phi_1 \colon \mathcal{A} \to \mathcal{C}, \quad \phi_2 \colon \mathcal{B} \to \mathcal{C},$$

there exists a unique unital $*$-homomorphism

$$\phi \colon \mathcal{A} * \mathcal{B} \to \mathcal{C}, \quad a \mapsto \phi_1(a), \quad b \mapsto \phi_2(b), \qquad \forall a \in \mathcal{A},\, b \in \mathcal{B}$$

where $\mathcal{A}, \mathcal{B} \subseteq \mathcal{A} * \mathcal{B}$ via the canonical inclusions.

2.2. Compact matrix quantum groups

Compact quantum groups were first introduced by Woronowicz in [95, 97] and are a generalization of classical compact groups used to describe symmetries in the setting of C^*-algebras. In the following, we introduce compact matrix quantum groups, a subclass of compact quantum groups analogous to classical matrix groups. We refer to [92, 40] for more information on compact matrix quantum groups and to [83, 67] for the general theory of compact quantum groups.

Before we can define compact matrix quantum groups, we first must introduce two operations on matrices with entries in a C^*-algebra.

Definition 2.2.1. Let $\mathcal{A}$ be a unital C^*-algebra and V be a Hilbert space. The *conjugate map* is the anti-linear map $\overline{\cdot}\colon \mathcal{A}\otimes B(V) \to \mathcal{A}\otimes B(\overline{V})$ defined on pure tensors by

$$\overline{(a\otimes T)} = a^*\otimes \overline{T} \qquad \forall a\in A,\, T\in B(V),$$

where $\overline{V}$ denotes the conjugate Hilbert space of V, and $\overline{T}\in B(\overline{V})$ is defined by $\overline{T}(\overline{v}) = \overline{T(v)}$ for all $v\in V$.

Definition 2.2.2. Let $\mathcal{A}$ be a unital C^*-algebra and V be a Hilbert space. The *Woronowicz tensor product* ⓣ is the bilinear map

$$ⓣ\colon \big(\mathcal{A}\otimes B(V)\big)\times\big(\mathcal{A}\otimes B(V)\big)\to A\otimes A\otimes B(V)$$

defined on pure tensors by

$$(a\otimes T)ⓣ(b\otimes S) = a\otimes b\otimes TS$$

for all $a,b\in\mathcal{A}$ and $S,T\in B(V)$.

Using these operations, we can now define compact matrix quantum groups as follows.

Definition 2.2.3. Let V be a finite-dimensional Hilbert space, $\mathcal{A}$ be a unital C^*-algebra and $u\in\mathcal{A}\otimes B(V)$. Then $G := (\mathcal{A},u)$ is a *compact matrix quantum group* if the following conditions hold:

1. The C^*-algebra $\mathcal{A}$ is generated by the matrix coefficients

   $$\{(\mathrm{id}_{\mathcal{A}}\otimes\varphi)(u)\mid \varphi\in V^*\},$$

   where $V^* := B(V,\mathbb{C})$ denotes the linear dual of V.

2. The element u is unitary, and its conjugate $\overline{u}$ is invertible,

3. There exists a $*$-homomorphism $\Delta\colon\mathcal{A}\to\mathcal{A}\otimes\mathcal{A}$ satisfying

   $$\big(\Delta\otimes\mathrm{id}_{B(V)}\big)(u) = uⓣu.$$

The element u is called the *fundamental representation* of G. Furthermore, we denote the C^*-algebra $\mathcal{A}$ by $C(G)$ and the dense $*$-algebra generated by the matrix coefficients by $\mathcal{O}(G)$.

Remark 2.2.4. Following the philosophy of Definition 2.1.8 by viewing C^*-algebras as algebras of "noncommutative functions", the C^*-algebra $\mathcal{A}$ in the previous definition is denoted by $C(G)$. However, in general, no underlying classical space G exists. Still, we will show below that for a compact matrix quantum group the C^*-algebra $\mathcal{A}$ is commutative if and only if it is of the form $C(G)$ for some classical compact matrix groups G.

Remark 2.2.5. We have chosen a basis-independent definition of compact matrix quantum groups, which allows us to keep track of any tensor product structure of the underlying Hilbert space V. If we choose an orthonormal basis $(v_i)_{i\in I}$ of V, we can identify the fundamental representation u with a matrix $(u^i_j)_{i,j\in I}$, and the axioms of a compact matrix quantum groups can be expressed in coordinates as follows:

1. The C^*-algebra $\mathcal{A}$ is generated by the matrix coefficients u^i_j for all $i, j \in I$.
2. The matrix u is unitary, and its conjugate $\overline{u} = \big((u^i_j)^*\big)_{i,j\in I}$ is invertible.
3. There exists a unital $*$-homomorphism $\Delta\colon \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$ with
$$\Delta(u^i_j) = \sum_{k\in I} u^i_k \otimes u^k_j \qquad \forall i, j \in I.$$

Next, we present several examples of compact matrix quantum groups, starting with the special case of classical compact matrix groups.

Example 2.2.6. Let $G \subseteq U_n$ be a closed subgroup of the group unitary matrices. Define $\mathcal{A} := C(G)$ as the C^*-algebra of continuous functions on G and $u := (u^i_j)^n_{i,j=1}$ as the matrix with entries $u^i_j \in \mathcal{A}$ defined by $(u^i_j)(g) = g^i_j$ for all $1 \leq i, j \leq n$ and $g \in G$. Then $(\mathcal{A}, u)$ is a compact matrix quantum group.

First, the matrix entries u^i_j separate the points of G, i.e. for any distinct $g, h \in G$ there exists an entry u^i_j such that $u^i_j(g) \neq u^i_j(h)$. Thus, the elements u^i_j generate the C^*-algebra $C(G)$ by the Stone-Weierstrass theorem. Furthermore, since $G \subseteq U_n$, the matrix u is unitary and its conjugate $\overline{u}$ is invertible with inverse given by $(\overline{u}^{-1})^i_j = u^j_i$. Finally, the multiplication on G defines a unital $*$-homomorphism
$$\Delta\colon C(G) \to C(G \times G), \quad \big[\Delta(f)\big](g, h) = f(g \cdot h) \qquad \forall g, h \in G.$$
For the matrix entries u^i_j, we have
$$\big[\Delta(u^i_j)\big](g, h) = u^i_j(g \cdot h) = \sum_{k=1}^{n} u^i_k(g) \cdot u^k_j(h).$$
Thus, using the isomorphism $C(G \times G) = C(G) \otimes C(G)$ from Definition 2.1.15, we obtain
$$\Delta(u^i_j) = \sum_{k=1}^{n} u^i_k \otimes u^k_j.$$

The previous example shows that compact matrix quantum groups generalize classical groups of unitary matrices. Moreover, we can use Definition 2.1.7 to characterize all compact matrix quantum groups that arise from classical matrix groups.

Proposition 2.2.7. *Let $(\mathcal{A}, u)$ be a compact matrix quantum group with fundamental representation on $\mathbb{C}^n$. Then $\mathcal{A}$ is commutative if and only if $\mathcal{A}$ is $*$-isomorphic to $C(G)$ for some classical closed subgroup $G \subseteq U_n$. In this case, the group G is explicitly given by*

$$G = \big\{(\phi \otimes \mathrm{id}_{\mathbb{C}^n})(u) \mid \phi \in \operatorname{Spec} \mathcal{A}\big\}.$$

Proof. See [83, Proposition 6.1.11] □

Examples of non-classical compact matrix quantum groups include Wang's free orthogonal and free unitary quantum groups [88] and their deformations in the sense of Van Daele and Wang [85]. Using the notation of Banica [3], these quantum groups are defined as follows.

Definition 2.2.8. Let $n \in \mathbb{N}$, $F \in B(\mathbb{C}^n)$ be invertible, and denote by $\iota\colon \overline{\mathbb{C}^n} \to \mathbb{C}^n$ the linear isomorphism defined by $\iota(\overline{e_i}) = e_i$ for all $1 \leq i \leq n$. Define the universal unital C^*-algebras

$$\begin{aligned} A_o(F) &:= C^*\big(u^i_j \mid u \text{ is unitary and } u = (F\iota)\,\overline{u}\,(F\iota)^{-1}\big),\\ A_u(F) &:= C^*\big(u^i_j \mid u \text{ and } (F\iota)\,\overline{u}\,(F\iota)^{-1} \text{ are unitary}\big) \end{aligned}$$

generated by the entries of a matrix $u := (u^i_j)^n_{i,j=1}$. Then

$$O^+(F) := (A_o(F), u), \quad U^+(F) := (A_u(F), u)$$

are the *free orthogonal* and the *free unitary* quantum groups with parameter F.

Remark 2.2.9. Note that both quantum groups $O^+(F)$ and $U^+(F)$ in the previous definition are well-defined. By definition, their corresponding C^*-algebras are generated by the matrix coefficients u^i_j, and the matrices u and $\overline{u}$ are unitary and invertible respectively. Furthermore, the unital $*$-homomorphism Δ can directly be constructed using the universal property of their C^*-algebras.

Example 2.2.10. Let $n \in \mathbb{N}$. For $F := \mathrm{id}_{\mathbb{C}^n}$, we define

$$O^+_n := O^+(\mathrm{id}_{\mathbb{C}^n}), \quad U^+_n := U^+(\mathrm{id}_{\mathbb{C}^n}).$$

In this case, the relations making both u and $\overline{u}$ unitary can be written in coordinates as

$$\sum_{k=1}^n u^i_k (u^j_k)^* = 1, \quad \sum_{k=1}^n (u^k_i)^* u^k_j = 1 \qquad (1 \leq i, j \leq n),$$

$$\sum_{k=1}^n (u^i_k)^* u^j_k = 1, \quad \sum_{k=1}^n u^k_i (u^k_j)^* = 1 \qquad (1 \leq i, j \leq n).$$

Moreover, the relation $u = \iota\overline{u}\iota^{-1}$ reduces to $u^i_j = (u^i_j)^*$.

Next, we introduce magic unitaries before defining the quantum permutation group S_n^+. It was first introduced by Wang [89] as the quantum automorphism group of the finite set $X := \{1, \dots, n\}$. However, we will consider arbitrary finite sets X and denote the corresponding quantum permutation group by S_X^+.

Definition 2.2.11. Let X be a finite set and $\mathcal{A}$ be a unital C^*-algebra. An element $u := (u_j^i)_{i,j \in X} \in \mathcal{A} \otimes B(\mathbb{C}^X)$ is a *magic unitary* if

$$(u_j^i)^2 = (u_j^i)^* = u_j^i, \quad \sum_{k \in I} u_k^i = \sum_{k \in I} u_i^k = 1 \qquad \forall i, j \in X.$$

Magic unitaries with entries in $\mathbb{C}$ are precisely classical permutation matrices, which gives magic unitaries also the name quantum permutation matrices. These matrices have applications in quantum information in the context non-local games [58], and concrete magic unitaries have been recently studied for example in [34, 66]. For further details and open problems related to magic unitaries, see [94].

The quantum permutation group S_X^+ is the compact matrix quantum group with a universal magic unitary as its fundamental representation.

Definition 2.2.12. Let X be a finite set and $\mathcal{A}$ be the universal unital C^*-algebra with generators u_j^i for all $i, j \in X$ such that $u := (u_j^i)_{i,j \in X}$ is a magic unitary. Then $S_X^+ := (\mathcal{A}, u)$ is the *quantum permutation group* on X.

As in the case of the free orthogonal quantum group O_n^+ and the free unitary quantum group U_n^+, on can directly verify that the quantum permutation group S_n^+ is well-defined, i.e. all axioms of a compact matrix quantum group are satisfied.

Additional examples of compact matrix quantum groups include the hyperoctahedral quantum group H_n^+ [7], and more generally (unitary) easy quantum groups [10, 80] and spatial partition quantum groups [25]. We introduce spatial partition quantum groups in Chapter 6 and Chapter 7, which includes easy quantum groups as a special case. For a more detailed introduction to easy quantum groups, see [91, 92, 40].

Moreover, one can generalize S_n^+ as quantum automorphism group of n points and obtain quantum automorphism groups of various discrete structures such as graphs [5, 13], multigraphs [45], Hadamard matrices [49] and quantum graphs [16, 17]. For further information, see Chapter 3 and Chapter 4, as well as [76].

2.3. Basic notions for quantum groups

As in the case of classical compact groups, various notions like homomorphisms, subgroups and quotients can be generalized to the setting of compact matrix quantum groups.

Definition 2.3.1. Let G and H be compact matrix quantum groups.

1. A *morphism of compact quantum groups* is a unital $*$-homomorphism $\phi\colon C(G) \to C(H)$ satisfying
$$(\phi \otimes \phi) \circ \Delta_G = \Delta_H \circ \phi.$$

2. G and H are *isomorphic* if there exists a morphism of compact quantum groups $\phi\colon C(G) \to C(H)$ that is a $*$-isomorphism. In this case, ϕ^{-1} is also a morphism of compact quantum groups.

3. H is a *subgroup* of G, and we write $H \subseteq G$, if there exists a surjective morphism of compact quantum groups $\phi\colon C(G) \to C(H)$.

4. H is a *quotient* of G if there exists an injective morphism of compact quantum groups $\phi\colon C(H) \to C(G)$. In this case, $C(H)$ can be identified with a C^*-subalgebra of $C(G)$.

Example 2.3.2. Consider the free orthogonal quantum groups O_n^+ and O_{n+1}^+ with fundamental representations u and v respectively. Then $O_n^+ \subseteq O_{n+1}^+$ via the unital $*$-homomorphism

$$\phi\colon C(O_{n+1}^+) \to C(O_n^+), \quad v \mapsto \begin{pmatrix} u & 0 \\ 0 & 1 \end{pmatrix}.$$

Note that the notion of an isomorphism between compact quantum groups is quite general and does not necessarily preserve the fundamental representations of a compact matrix quantum group. In particular, it allows us to relate quantum groups with fundamental representations of different dimensions. However, we will also use the following stronger notions of isomorphism and subgroup between compact matrix quantum groups that preserve their fundamental representations.

Definition 2.3.3. Let G and H be compact matrix quantum groups with fundamental representations u and w on Hilbert spaces V and W respectively.

1. H is a *subgroup (as a compact matrix quantum group)* of G if there exists a unitary $Q\colon W \to V$ and a unital $*$-homomorphism $\phi\colon C(G) \to C(H)$ such that

$$\phi(u) = QwQ^{-1}.$$

   In this case, ϕ is automatically a morphism of compact quantum groups.

2. G and H are *isomorphic (as compact matrix quantum) groups* if the previous $*$-homomorphism ϕ is a $*$-isomorphism.

Example 2.3.4. Consider the free orthogonal quantum group O_n^+ with fundamental representations u and the quantum permutation group S_n^+ with fundamental representations v. Since v is orthogonal, there exists a unital $*$-homomorphism

$$\phi\colon C(O_n^+) \to C(S_n^+), \quad u \mapsto v.$$

Thus, S_n^+ is a subgroup of O_n^+ as compact matrix quantum groups.

In addition to the previous notions, various products exist for compact matrix quantum groups. In the following, we introduce the free product for compact matrix quantum groups [88], which will be used in Chapter 3 in the context of quantum symmetries of hypergraphs.

Definition 2.3.5. Let $G := (\mathcal{A}, u)$ and $H := (\mathcal{B}, v)$ be compact matrix quantum groups. Then their *free product* $G * H$ is the compact matrix quantum group

$$G * H := (\mathcal{A} * \mathcal{B}, u \oplus v),$$

where $\mathcal{A} * \mathcal{B}$ denotes the unital free product of the C^*-algebras $\mathcal{A}$ and $\mathcal{B}$, and the fundamental representations u and v are identified with $\mathcal{A} * \mathcal{B}$-valued matrices via the canonical inclusions $\mathcal{A}, \mathcal{B} \hookrightarrow \mathcal{A} * \mathcal{B}$.

Example 2.3.6. Consider finite sets X and Y with corresponding quantum permutation groups S_X^+ and S_Y^+. Their free product $S_X^+ * S_Y^+$ is given by the universal unital C^*-algebra

$$C(S_X^+ * S_Y^+) = C^*(u_j^i, v_\ell^k \mid u \text{ and } v \text{ are magic unitaries})$$

generated by the entries of the matrices $u := (u_j^i)_{i,j \in X}$ and $v := (v_\ell^k)_{k,\ell \in Y}$. Furthermore, the comultiplication is given by

$$\Delta(u_j^i) = \sum_{k \in X} u_k^i \otimes u_j^k \qquad \forall i, j \in X,$$

$$\Delta(v_j^i) = \sum_{k \in Y} v_k^i \otimes v_j^k \qquad \forall i, j \in Y.$$

2.4. Actions on C^*-algebras

As classical group actions describe the symmetries of classical objects, we can define actions of compact matrix quantum groups on C^*-algebras to describe their quantum symmetries.

In the following, we adopt an algebraic approach to quantum group actions and consider the $*$-algebra $\mathcal{O}(G)$ generated by the matrix coefficients u_j^i of a fundamental representation u indexed by $i, j \in I$. While the map

$$\Delta\colon C(G) \to C(G) \otimes C(G), \quad u_j^i \mapsto \sum_{k \in I} u_k^i \otimes u_j^k \qquad i, j \in I$$

generalizes the multiplication of a classical group, there exists further structure on the $*$-algebra $\mathcal{O}(G)$ that corresponds to the neutral element and the inverse mapping in the classical case. This structure is given by a unital $*$-homomorphism $\varepsilon\colon \mathcal{O}(G) \to \mathbb{C}$ called the *counit* and a unital anti-homomorphism $S\colon \mathcal{O}(G) \to \mathcal{O}(G)$ called the *antipode*, which are defined by

$$\varepsilon(u_j^i) = \delta_{ij}, \quad S(u_j^i) = (u_i^j)^* \qquad \forall i, j \in I.$$

These additional homomorphisms make $\mathcal{O}(G)$ a Hopf $*$-algebra, see [83] for further details.

Using the Hopf $*$-algebra structure on $\mathcal{O}(G)$, we can define actions of compact matrix quantum groups as in [89] and [13].

Definition 2.4.1. Let G be a compact matrix quantum group and $\mathcal{A}$ be a unital C^*-algebra. An *action* of G on $\mathcal{A}$ is a unital $*$-homomorphism $\alpha\colon \mathcal{A} \to \mathcal{A} \otimes C(G)$ such that

1. $(\alpha \otimes \mathrm{id}) \circ \alpha = (\mathrm{id} \otimes \Delta) \circ \alpha$,
2. there exists a dense $*$-subalgebra $\mathcal{B} \subseteq \mathcal{A}$ with $\alpha(\mathcal{B}) \subseteq \mathcal{B} \otimes \mathcal{O}(G)$,
3. $(\mathrm{id} \otimes \varepsilon) \circ \alpha|_{\mathcal{B}} = \mathrm{id}$.

Alternatively, the second and third conditions can be replaced by the more analytic condition that $(1 \otimes C(G))\alpha(\mathcal{A})$ is linearly dense in $\mathcal{A} \otimes C(G)$, see for example [77]. However, Definition 2.4.1 will be easier to verify in our case.

Definition 2.4.2. Let α be an action of a compact matrix quantum groups G on a C^*-algebra $\mathcal{A}$. Then α is *faithful* if, for any quotient H of G such that $\alpha|_{C(H)}$ is an action on $\mathcal{A}$, we have $C(H) = C(G)$.

In [89], Wang showed that the quantum permutation group S_X^+ acts on $\mathbb{C}^X$ in the following sense.

Proposition 2.4.3. *Let X be a finite set. Then S_X^+ acts faithfully on $\mathbb{C}^X$ via*

$$\alpha\colon \mathbb{C}^X \to \mathbb{C}^X \otimes C(S_X^+), \quad \alpha(e_i) = \sum_{j \in X} e_j \otimes u_i^j \qquad \forall i \in X,$$

where u denotes the fundamental representation of S_X^+ and $(e_i)_{i \in X}$ is the standard basis of $\mathbb{C}^X$.

Moreover, Wang showed that this action is *maximal* in the following sense, making S_X^+ the largest compact matrix quantum group acting in $\mathbb{C}^X$ as before. Thus, we can consider S_X^+ as the quantum automorphism group of $\mathbb{C}^X$ or equivalently of the underlying set X.

Proposition 2.4.4. *Let X be a finite set and G be a compact matrix quantum group acting faithfully on $\mathbb{C}^X$ via $\alpha\colon \mathbb{C}^X \to \mathbb{C}^X \otimes C(G)$ such that*

$$\alpha(e_i) = \sum_{j \in X} e_j \otimes u_i^j \qquad \forall i \in X$$

for some $u_j^i \in C(G)$. Then G is a subgroup of S_X^+ via the map

$$\phi\colon C(S_X^+) \to C(G), \quad v_j^i \mapsto u_j^i \qquad \forall i, j \in X,$$

where v denotes the fundamental representation of S_X^+.

Additional examples of actions of compact matrix quantum groups include the action of the free orthogonal quantum group on the free sphere [9] and the action of the quantum automorphism group of a graph on its corresponding graph C^*-algebra [77]. See Chapter 5 for further information on the second case.

2.5. Representation categories and Tannaka-Krein duality

Finally, we introduce representation categories and formulate Woronowicz's Tannaka-Krein duality for compact matrix quantum groups, which allows us to construction of quantum groups from categories of partitions in Chapter 7. We begin with the definition of representations of compact matrix quantum groups.

Definition 2.5.1. Let G be a compact matrix quantum group. A *representation* of G on a finite-dimensional Hilbert space V is an invertible element $v \in C(G) \otimes B(V)$ satisfying

$$(\Delta \otimes \mathrm{id}_{B(V)})(v) = v \oplus v.$$

Additionally, a representation is called *unitary* if the element v is unitary.

Remark 2.5.2. Let V be a finite-dimensional Hilbert space with an orthonormal basis indexed by I. Then a representation of a compact matrix quantum group G on V is given by an invertible matrix $v := (v^i_j)_{i,j \in I} \in C(G) \otimes B(V)$ satisfying

$$\Delta(v^i_j) = \sum_{k \in I} v^i_k \otimes v^k_j \qquad \forall i, j \in I.$$

Next, we present several examples of representations of compact matrix quantum groups.

Example 2.5.3. Let $\rho\colon G \to B(V)$ be a continuous unitary representation of a classical compact matrix group G on a finite-dimensional Hilbert space V. Then $\rho \in C(G, B(V))$ and ρ corresponds precisely to a representation of G in the compact matrix quantum group sense after identifying $C(G, B(V))$ with $C(G) \otimes B(V)$. Thus, Definition 2.5.1 generalizes classical representations of compact groups to the quantum group setting.

Example 2.5.4. Let G be a compact matrix quantum group with fundamental representation u.

1. The *trivial representation* $1 \in C(G) \otimes B(\mathbb{C})$ is a representation of G.
2. The *fundamental representation* u is a representation of G.
3. If $v \in C(G) \otimes B(V)$ is a representation of G, then its *conjugate* $\overline{v} \in C(G) \otimes B(\overline{V})$ is a representation of G.

In addition to the previous examples, it is always possible to form direct sums and tensor products of representations. In the following, we focus mainly on tensor products of representations. For further details on direct sums and the decomposition of representations into irreducible representations, see [83, 67].

Definition 2.5.5. Let G be a compact matrix quantum group. Assume v is a representation of G on a Hilbert space V and w is a representation of G on a Hilbert space W. Then their *tensor product* $v \textcircled{\scriptsize T} w$ is defined by the bilinear operator

$$\textcircled{\scriptsize T}\colon \big(C(G) \otimes B(V)\big) \times \big(C(G) \otimes B(W)\big) \to C(G) \otimes B(V \otimes W)$$

given on pure tensors by

$$(a \otimes S) \textcircled{\scriptsize T} (b \otimes T) = ab \otimes (S \otimes T)$$

for all $a, b \in C(G)$, $S \in B(V)$ and $T \in B(W)$.

Remark 2.5.6. Let v and w be representations of a compact matrix quantum group as in the previous definition. Fix an orthonormal bases of V indexed by I and an orthonormal basis of W indexed by J. Then $v \textcircled{\scriptsize T} w$ is given by

$$(v \textcircled{\scriptsize T} w)^{i_1 j_1}_{i_2 j_2} = v^{i_1}_{i_2} \cdot w^{j_1}_{j_2} \qquad \forall i_1, i_2 \in I,\ j_1, j_2 \in J,$$

where the coordinates are with respect to the induced basis of $V \otimes W$ indexed by $I \times J$. In particular, the tensor product for representations of compact matrix quantum groups generalizes the tensor product for representations of classical groups.

Next, we show that the representations of a compact matrix quantum group form a tensor category, which allows us to formulate Woronowicz's Tannaka-Krein duality. We refer to [96, 83, 67, 47] for more details on the representation theory of compact matrix quantum groups and to [30] for tensor categories in general.

We begin by introducing intertwiners between representations.

Definition 2.5.7. Let G be a compact matrix quantum group with representations v and w on Hilbert spaces V and W respectively. Then the space of *intertwiners* between v and w is defined by

$$\mathrm{Hom}(v, w) := \{T \in B(V, W) \mid Tv = wT\}.$$

Let G be a compact matrix quantum group. Then we can construct a category $\mathrm{Rep}(G)$, where the objects are the finite-dimensional unitary representations of G and the morphisms $\mathrm{Hom}(v, w)$ are intertwiners between representations v and w. Furthermore, using the $\textcircled{\scriptsize T}$-operator and the adjoint T^* of intertwiners, we can define a monoidal and a $*$-structure on this category. The following proposition summarizes its most important properties.

Proposition 2.5.8. *The finite-dimensional unitary representations of a compact matrix quantum group form a concrete monoidal $*$-category in the following sense:*

1. $\mathrm{id}_V \in \mathrm{Hom}(v, v)$ *for every representation v on a Hilbert space V.*
2. $\mathrm{Hom}(v, w)$ *is a linear subspace of $B(V, W)$ for all representations v and w on Hilbert spaces V and W respectively.*
3. *If $S \in \mathrm{Hom}(w, x)$ and $T \in \mathrm{Hom}(v, w)$, then $ST \in \mathrm{Hom}(v, x)$.*
4. *If $T \in \mathrm{Hom}(v, w)$, then $T^* \in \mathrm{Hom}(w, v)$.*
5. *If $S \in \mathrm{Hom}(v, w)$ and $T \in \mathrm{Hom}(x, y)$, then $S \otimes T \in \mathrm{Hom}(v \textcircled{\scriptsize T} x, w \textcircled{\scriptsize T} y)$.*

In addition to the previous properties, the representation category of a compact matrix quantum group is rigid in the sense that every unitary representation has a conjugate representation. However, before we define conjugate representations, we first introduce colors.

Definition 2.5.9. Let $\circ$ and $\bullet$ be two colors. Then

$$\{\circ, \bullet\}^* := \{1, \circ, \bullet, \circ\circ, \circ\bullet, \bullet\circ, \bullet\bullet, \circ\circ\circ, \ldots\}$$

denotes the free monoid generated by $\{\circ, \bullet\}$. It is the set of all finite words over $\{\circ, \bullet\}$ with concatenation as multiplication and the empty word 1 as the identity element. For $x \in \{\circ, \bullet\}^*$, we denote by $|x|$ the length of the word $x \in \{\circ, \bullet\}^*$ and by $x_i \in \{\circ, \bullet\}$ its i-th color. Furthermore, let $\overline{\cdot}\colon \{\circ, \bullet\}^* \to \{\circ, \bullet\}^*$ be the anti-homomorphism defined by

$$\overline{\circ} = \bullet, \quad \overline{\bullet} = \circ, \quad \overline{x \cdot y} = \overline{y} \cdot \overline{x} \qquad \forall x, y \in \{\circ, \bullet\}^*.$$

Throughout this thesis, we will use the color $\bullet$ to denote the conjugate of an object with the color $\circ$. In particular, we introduce the following notation for Hilbert spaces.

Definition 2.5.10. Let V be a Hilbert space. Then we define $V^\circ := V$ and $V^\bullet := \overline{V}$, as well as $v^\circ := v$ and $v^\bullet := \overline{v}$ for all $v \in V$, where $\overline{V}$ denote the conjugate Hilbert space of V and $\overline{v}$ the conjugate of a vector v. Furthermore, we extend this notation to tensor products by defining $V^1 := \mathbb{C}$ and

$$V^{\otimes x} := V^{x_1} \otimes \cdots \otimes V^{x_k} \qquad \forall x \in \{\circ, \bullet\}^*,\ k := |x| > 1.$$

If $T\colon V \to W$ is a linear operator, we additionally write $T^1 := \mathrm{id}_{\mathbb{C}}$, $T^\circ := T$, $T^\bullet := \overline{T}$ and

$$T^{\otimes x} := T^{x_1} \otimes \cdots \otimes T^{x_k} \qquad \forall x \in \{\circ, \bullet\}^*,\ k := |x| > 1,$$

where the conjugate operator $\overline{T}\colon \overline{V} \to \overline{W}$ is defined by $\overline{T}(\overline{v}) = \overline{T(v)}$ for all $v \in V$.

Using the previous colors, we can also define conjugate representations.

Definition 2.5.11. Let u° and $u^\bullet$ be representations of a compact matrix quantum group on Hilbert spaces V and W respectively. Then $u^\bullet$ is *conjugate* to u° if there exist intertwiners $R \in \mathrm{Hom}(1, u^\circ \mathbin{ⓣ} u^\bullet)$ and $S \in \mathrm{Hom}(1, u^\bullet \mathbin{ⓣ} u^\circ)$ satisfying the conjugate equations

$$(R^* \otimes \mathrm{id}_V) \cdot (\mathrm{id}_V \otimes S) = \mathrm{id}_V, \qquad (S^* \otimes \mathrm{id}_W) \cdot (\mathrm{id}_W \otimes R) = \mathrm{id}_W.$$

Note that the representation $u^\bullet$ is also called a dual of u° and the intertwiners R and S are called duality morphisms. We refer to [30] and [67] for the definition of rigidity in a more general context.

In [96], it is shown that the representation category of a compact matrix quantum groups is generated by tensor powers of its fundamental representation u and a corresponding conjugate representation. Therefore, it suffices to consider only the fundamental representation $u^\circ := u$ and its conjugate $u^\bullet$ in the following. Furthermore, we introduce the notation $u^x := u^{x_1} \mathbin{ⓣ} \dots \mathbin{ⓣ} u^{x_k}$ for all $x \in \{\circ, \bullet\}^*$ with $k := |x| > 0$. In the case $x = 1$, we define $u^1 := 1$ as the trivial representation. In particular, if u is defined on a Hilbert space V, then u^x is defined on V^x.

Note that the conjugate of a representation is not unique but is determined only up to unitary equivalence, i.e. conjugation by a unitary. Furthermore, the representation $\overline{u^\circ}$ is not conjugate to u° in general because it is not necessarily unitary. However, in the following, we show that any conjugate representation $u^\bullet$ is equivalent to $\overline{u^\circ}$ and that this equivalence is defined by the solutions R and S of the conjugate equations.

Proposition 2.5.12. *Consider a compact matrix quantum group with fundamental representation u° on a Hilbert space V. Let $F \in B(\overline{V})$ be an invertible operator such that $u^\bullet := F\overline{u^\circ}F^{-1}$ is unitary. Then u° and $u^\bullet$ are conjugate with duality morphisms R and S defined by*

$$R^{ij} = F_i^j, \quad S^{ij} = (\overline{F}^{-1})_i^j \qquad \forall i, j \in I,$$

where the coordinates are with respect to any orthonormal basis of V index by I and its conjugate basis of $\overline{V}$.

Proof. A detailed proof is given in [47]. One verifies directly that

$$u^\circ (u^\circ)^* = 1 \iff R \in \mathrm{Hom}(1, u^\circ \mathbin{ⓣ} u^\bullet),$$

$$u^\bullet (u^\bullet)^* = 1 \iff S \in \mathrm{Hom}(1, u^\bullet \mathbin{ⓣ} u^\circ).$$

Hence, R and S are intertwiners. Furthermore, they satisfy the conjugate equations because

$$(R^* \otimes \mathrm{id}_V) \cdot (\mathrm{id}_V \otimes S) = \overline{F}^{-1}\overline{F}, \qquad (S^* \otimes \mathrm{id}_{\overline{V}}) \cdot (\mathrm{id}_{\overline{V}} \otimes R) = FF^{-1}.$$

□

Proposition 2.5.13. *Let G be a compact matrix quantum group with unitary representation u° on a Hilbert space V. Assume $u^\bullet$ is a unitary representation on $\overline{V}$ that is conjugate to u° via duality morphisms R and S. Then $u^\bullet = F\overline{u^\circ}F^{-1}$, where $F \in B(\overline{V})$ is defined by $F_j^i = R^{ji}$ for all $i, j \in I$ with respect to any orthonormal basis of V indexed by I and its conjugate basis of $\overline{V}$.*

Proof. The representation $\overline{u^\circ}$ is equivalent to a unitary representation that is conjugate to u°, see [67, Example 2.2.3]. Furthermore, conjugates are unique up to equivalence by [67, Proposition 2.2.4]. Hence, there exists an invertible $\widetilde{F} \in B(\overline{V})$ such that $u^\bullet = \widetilde{F}\overline{u^\circ}\widetilde{F}^{-1}$. The previous proposition implies that

$$\widetilde{R} \in \mathrm{Hom}(1, u^\circ \mathbin{ⓣ} u^\bullet), \quad \widetilde{S} \in \mathrm{Hom}(1, u^\bullet \mathbin{ⓣ} u^\circ),$$

where $\widetilde{R}$ and $\widetilde{S}$ are the solutions to the conjugate equations defined by $\widetilde{F}$. As in [46, Theorem 3.4.6], we compute

$$(\widetilde{S}^* \otimes \mathrm{id}_{\overline{V}}) \cdot (\mathrm{id}_{\overline{V}} \otimes R) = F\widetilde{F}^{-1} \in \mathrm{Hom}(u^\bullet, u^\bullet),$$

which yields

$$u^\bullet = F\widetilde{F}^{-1}u^\bullet\widetilde{F}F^{-1} = F\overline{u^\circ}F^{-1}.$$

□

In the case of classical groups, Tannaka-Krein duality [29] allows the reconstruction of a compact group from an abstract category of representations. A similar result for compact matrix quantum groups was first proven by Woronowicz in [96].

There exist multiple proofs of Woronowicz's Tannaka-Krein duality at various levels of abstraction, see for example [96, 67, 59, 47]. In the following, we choose the approach of Gromada [47], which is most suitable for the setting of categories defined by partitions. Thus, we begin by introducing abstract two-colored representation categories that capture the properties of the representation category of a compact matrix quantum group discussed earlier.

Definition 2.5.14. Let V be a finite-dimensional Hilbert space. A *two-colored representation category* $\mathcal{C}$ is a collection of linear subspaces $\mathcal{C}(x,y) \subseteq B(V^{\otimes x}, V^{\otimes y})$ for all $x, y \in \{\circ, \bullet\}^*$ satisfying the following properties:

1. $\mathrm{id}_{V^{\otimes x}} \in \mathcal{C}(x,x)$ for all $x \in \{\circ, \bullet\}^*$.
2. If $S \in \mathcal{C}(y,z)$ and $T \in \mathcal{C}(x,y)$, then $ST \in \mathcal{C}(x,z)$.
3. If $T \in \mathcal{C}(x,y)$, then $T^* \in \mathcal{C}(y,x)$.
4. If $S \in \mathcal{C}(w,x)$ and $T \in \mathcal{C}(y,z)$, then $S \otimes T \in \mathcal{C}(wy,xz)$.
5. There exist $R \in \mathcal{C}(1, \circ\bullet)$ and $S \in \mathcal{C}(1, \bullet\circ)$ satisfying the conjugate equations

$$(R^* \otimes \mathrm{id}_V) \cdot (\mathrm{id}_V \otimes S) = \mathrm{id}_V, \qquad (S^* \otimes \mathrm{id}_{\overline{V}}) \cdot (\mathrm{id}_{\overline{V}} \otimes R) = \mathrm{id}_{\overline{V}}.$$

Woronowicz's Tannaka-Krein duality states that given any two-colored representation category $\mathcal{C}$, there exists a unique compact matrix quantum group whose representation category $\mathrm{Rep}(G)$ is given by $\mathcal{C}$.

Theorem 2.5.15 (Woronowicz's Tannaka-Krein duality)**.** *Let $\mathcal{C}$ be a two-colored representation category on a Hilbert space V. Then there exists a unique compact matrix quantum group G with fundamental representation u° on V and unitary representation $u^\bullet$ on $\overline{V}$, such that*

$$\mathrm{Hom}(u^x, u^y) = \mathcal{C}(x,y) \qquad \forall x, y \in \{\circ, \bullet\}^*.$$

First, let us comment on the uniqueness of the unitary representation $u^\bullet$ in the previous theorem. Since $\mathcal{C}$ contains a pair of morphisms R and S satisfying the conjugate equations, the representations u° and $u^\bullet$ are conjugate. Definition 2.5.13 then implies that $u^\bullet = F\overline{u^\circ}F^{-1}$ with $F^i_j = R^{ji}$. Therefore, the representation $u^\bullet$ is uniquely determined by the category $\mathcal{C}$.

Next, we consider the uniqueness of the quantum group G. The proof of Woronowicz's Tannaka-Krein duality in [47] shows that all relations of the dense $*$-algebra $\mathcal{O}(G)$ are spanned by the intertwiner relations

$$Tu^x = u^y T \qquad \forall x, y \in \{\circ, \bullet\}^*,\ T \in \mathcal{C}(x, y).$$

Hence, $\mathcal{O}(G)$ is uniquely determined by the category $\mathcal{C}$ up to $*$-isomorphism. In contrast, the C^*-algebra $C(G)$ is not unique. However, it is always possible to choose $C(G)$ as the maximal C^*-completion of $\mathcal{O}(G)$. In this case, G is uniquely defined by the following universal property.

Proposition 2.5.16. *Consider the compact matrix quantum group G defined in Definition 2.5.15. Let H be a compact matrix quantum group with fundamental representation w° on a Hilbert space W and unitary representation $w^\bullet$ on $\overline{W}$. If there exists a unitary $Q\colon V \to W$ such that*

$$\mathrm{Hom}(u^x, u^y) \subseteq (Q^{-1})^{\otimes y} \cdot \mathrm{Hom}(w^x, w^y) \cdot Q^{\otimes x} \qquad \forall x, y \in \{\circ, \bullet\}^*,$$

then H is a subgroup G via the map $u \mapsto QwQ^{-1}$.

Motivated by Woronowicz's Tannaka-Krein duality and the previous proposition, we introduce the following notion of equivalence between compact matrix quantum groups.

Definition 2.5.17. Let G be a compact matrix quantum group with fundamental representation u° on a Hilbert space V and conjugate representation $u^\bullet$ on $\overline{V}$. A compact matrix quantum group H with fundamental representation w° on a Hilbert space W and unitary representation $w^\bullet$ on $\overline{W}$ is *equivalent* to G if there exists a unitary $Q\colon V \to W$ such that

$$\mathrm{Hom}(u^x, u^y) = (Q^{-1})^{\otimes y} \cdot \mathrm{Hom}(w^x, w^y) \cdot Q^{\otimes x} \qquad \forall x, y \in \{\circ, \bullet\}^*.$$

One verifies that the previous definition indeed defines an equivalence relation on compact matrix quantum groups. Furthermore, Woronowicz's Tannaka-Krein duality and the previous discussion show that two compact matrix quantum groups G and H are equivalent if and only if $\mathcal{O}(G)$ and $\mathcal{O}(H)$ are $*$-isomorphic. Moreover, this $*$-isomorphism is given by $u = Q^{-1}wQ$, where u and w are the fundamental representations of G and H respectively.

Part I.

Quantum groups from hypergraphs

3. Quantum symmetries of hypergraphs

As classical groups often arise as symmetries of objects, many compact quantum groups can be constructed as quantum automorphism groups of classical structures. In the simplest case of n points, Wang [89] introduced the quantum permutation group S_n^+ as a generalization of the classical symmetric group S_n. Building on this work, Bichon [13] and Banica [5] defined two versions of quantum automorphism groups of finite graphs, yielding a large class of examples of compact matrix quantum groups. Recently, these quantum automorphism groups have been further generalized to different structures such as multigraphs [45], Hadamard matrices [49] and quantum graphs [16, 17].

In this chapter, we introduce a quantum automorphism group for hypergraphs and study its basic properties. In particular, we construct a family of hypergraphs with maximal quantum symmetries and compute the quantum symmetries of opposite and dual hypergraphs. Moreover, we consider hypergraphs without multiple edges and show that in this case quantum automorphisms reduce to a single quantum permutation on the vertices of the underlying hypergraph.

For more information on the quantum automorphism groups of classical graphs and a comparison of those quantum groups to our quantum automorphism group of hypergraphs, see Chapter 4.

3.1. Graphs and hypergraphs

Before introducing quantum automorphism groups of hypergraphs, we first recall the definition of various types of graphs and hypergraphs. These are combinatorial objects consisting of a set of vertices that are connected by edges. We begin with the definition of simple and directed graphs before proceeding to multigraphs and directed hypergraphs. See [27, 12, 2] for further details on graphs and hypergraphs. In the following, we denote the power set of a set X by 2^X.

Definition 3.1.1. A *simple graph* $\Gamma := (V, E)$ consists of a finite set of vertices V and a finite set of edges $E \subseteq 2^V$ satisfying $|e| = 2$ for all $e \in E$.

Definition 3.1.2. A *directed graph* $\Gamma := (V, E)$ consists of a finite set of vertices V and a finite set of edges $E \subseteq V \times V$.

As shown in Figure 3.1, an edge $\{v, w\}$ in a simple graph can be visualized by an undirected line from v to w, whereas an edge (v, w) in a directed graph can be visualized by an arrow from v to w. Furthermore, directed graphs may contain self-loops (v, v), which are excluded in our definition of simple graphs.

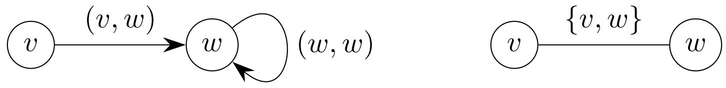


Figure 3.1.: Examples of directed and undirected edges.

Although the previous definitions model edges differently, we can uniformly describe the edge structure of simple and directed graphs using an adjacency matrix.

Definition 3.1.3. Let $\Gamma := (V, E)$ be a simple or directed graph. Then two vertices $v, w \in V$ are *adjacent*, denoted by $v \sim w$, if $\{v, w\} \in E$ or $(v, w) \in E$ respectively. Furthermore, we define the *adjacency matrix* $A \in B(\mathbb{C}^V)$ by

$$A^v_w = \begin{cases} 1 & \text{if } v \sim w, \\ 0 & \text{otherwise,} \end{cases} \qquad \forall v, w \in V.$$

By allowing multiple edges between each pair of vertices, we can generalize directed graphs to multigraphs.

Definition 3.1.4. A *(directed) multigraph* $\Gamma := (V, E)$ consists of a finite set of vertices V, a finite set of edges E, and two maps $s\colon E \to V$ and $r\colon E \to V$.

As shown in Figure 3.2, an edge $e \in E$ in a multigraph can be depicted by a directed arrow from its source vertex $s(e)$ to its range vertex $r(e)$ similar to the case of directed graphs. Furthermore, in the context of multigraphs, we will be interested in vertices with only incoming or outgoing edges.

Definition 3.1.5. Let $\Gamma := (V, E)$ be a multigraph. Then a vertex $v \in V$ is

1. a *source* if $v \neq r(e)$ for all $e \in E$,
2. a *sink* if $v \neq s(e)$ for all $e \in E$,
3. *isolated* if it is a source and a sink.

By replacing the vertices $s(e)$ and $r(e)$ in the definition of a multigraph with arbitrary subsets of vertices, we arrive at the definition of a hypergraph.

Definition 3.1.6. A *(directed) hypergraph* $\Gamma := (V, E)$ consists of a finite set of vertices V, a finite set of edges E, and two maps $s\colon E \to 2^V$ and $r\colon E \to 2^V$.

As shown in Figure 3.2, an edge $e \in E$ can be depicted by an arrow from the set of source vertices $s(e)$ to the set of range vertices $r(e)$. Thus, classical directed edges correspond to hyperedges with $|s(e)| = |r(e)| = 1$, see also Definition 3.1.11 and Definition 3.1.12 below. Note that we allow both $s(e)$ and $r(e)$ to be empty sets.

Similar to the adjacency matrix of classical graphs, the edge structure of a hypergraph can be described by incidence matrices.

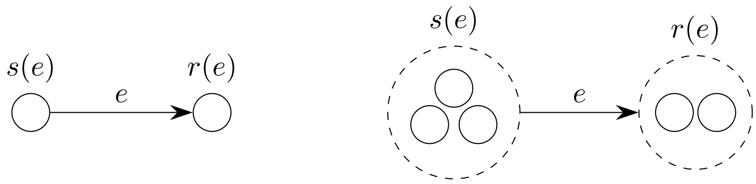


Figure 3.2.: Examples of directed edges and hyperedges.

Definition 3.1.7. Let $\Gamma := (V, E)$ be a hypergraph. The *incidence matrices* $A_s, A_r \in B(\mathbb{C}^E, \mathbb{C}^V)$ are defined by

$$(A_s)^v_e = \begin{cases} 1 & \text{if } v \in s(e), \\ 0 & \text{otherwise,} \end{cases} \quad (A_r)^v_e = \begin{cases} 1 & \text{if } v \in r(e), \\ 0 & \text{otherwise,} \end{cases} \qquad \forall v \in V,\ e \in E.$$

Additionally, the notations of sources and sinks can be generalized from multigraphs to hypergraphs.

Definition 3.1.8. Let $\Gamma := (V, E)$ be a hypergraph. Then a vertex $v \in V$ is

1. a *source* if $v \notin r(e)$ for all $e \in E$,
2. a *sink* if $v \notin s(e)$ for all $e \in E$,
3. *isolated* if it is a source and a sink.

Furthermore, the following properties can be used to describe hypergraphs with special structure.

Definition 3.1.9. Let $\Gamma := (V, E)$ be a hypergraph. Then

1. Γ has *no multiple edges* if $s(e_1) = s(e_2)$ and $r(e_1) = r(e_2)$ implies $e_1 = e_2$ for all $e_1, e_2 \in E$,
2. Γ is *undirected* if $s(e) = r(e)$ for all $e \in E$,
3. Γ is *k-uniform* if $|s(e)| = |r(e)| = k$ for all $e \in E$.

Given a simple graph, a directed graph or a multi-graph, it can naturally be regarded as a hypergraph. Moreover, the resulting hypergraphs can be characterized using the previous properties.

Example 3.1.10. Let $\Gamma := (V, E)$ be a simple graph. Then Γ can be regarded as a hypergraph with source and range maps defined by

$$s(\{v, w\}) = \{v, w\}, \quad r(\{v, w\}) = \{v, w\} \qquad \forall \{v, w\} \in E.$$

Conversely, 2-uniform undirected hypergraphs without multiple edges correspond exactly to simple graphs in this way.

Example 3.1.11. Let $\Gamma := (V, E)$ be a directed graph. Then Γ can be regarded as a hypergraph with source and range maps defined by

$$s(v, w) = \{v\}, \quad r(v, w) = \{w\} \qquad \forall (v, w) \in E.$$

Conversely, 1-uniform hypergraphs without multiple edges correspond exactly to directed graphs in this way.

Example 3.1.12. Let $\Gamma := (V, E)$ be a multigraph with source map $s' \colon E \to V$ and range map $r' \colon E \to V$. Then Γ can be regarded as a hypergraph with new source and range maps defined by

$$s(e) = \{s'(e)\}, \quad r(e) = \{r'(e)\} \qquad \forall e \in E.$$

Conversely, 1-uniform hypergraphs correspond exactly to multigraphs in this way.

Given a hypergraph, it is always possible to obtain a new hypergraph by interchanging the source and range map or by interchanging the vertices and edges.

Definition 3.1.13. Let $\Gamma = (V, E)$ be a hypergraph. The *opposite hypergraph* Γ^{op} is the hypergraph (V, E) with source and range maps s^{op} and r^{op} defined by

$$s^{\mathrm{op}}(e) = r(e), \quad r^{\mathrm{op}}(e) = s(e) \qquad \forall e \in E.$$

Definition 3.1.14. Let $\Gamma = (V, E)$ be a hypergraph. The *dual hypergraph* Γ^* is the hypergraph (E, V) with source and range maps s^* and r^* defined by

$$s^*(v) = \{e \in E \mid v \in s(e)\}, \quad r^*(v) = \{e \in E \mid v \in r(e)\} \qquad \forall e \in E.$$

Note that the dual source and range maps s^* and r^* can be expressed in terms of the source and range maps s and r as follows:

$$s(e) = \{v \in V \mid e \in s^*(v)\}, \quad r(e) = \{v \in V \mid e \in r^*(v)\} \quad \forall e \in E.$$

Therefore, we have $(\Gamma^*)^* = \Gamma$.

3.2. Quantum automorphism groups of hypergraphs

We begin by recalling the definition of the classical automorphism group of a hypergraph and characterize it in terms of permutation matrices, before we introduce the quantum automorphism group of a hypergraph. Let X be a finite set. Then S_X denotes the classical permutation group of X consisting of all bijections from X to itself.

Definition 3.2.1. Let $\Gamma := (V, E)$ be a hypergraph. Then its *automorphism group* $\mathrm{Aut}(\Gamma)$ consists of all pairs of permutations $(\sigma, \tau) \in S_V \times S_E$ satisfying

$$\sigma(s(e)) = s(\tau(e)), \qquad \sigma(r(e)) = r(\tau(e)) \qquad \forall e \in E,$$

where $\sigma \in S_V$ acts on subsets $\{v_1, \dots v_k\} \subseteq V$ by

$$\sigma(\{v_1, \dots v_k\}) = \{\sigma(v_1), \dots, \sigma(v_k)\}.$$

Thus, a hypergraph automorphism is given by a permutation of the vertices and a permutation of the edges, which are compatible by respecting the source and range maps. Next, we show how these compatibility conditions can be reformulated when both permutations are represented as permutation matrices.

Definition 3.2.2. Let X be a finite set. The *permutation representation* of the symmetric group S_X is given by the map $S_X \to B(\mathbb{C}^X)$, $\sigma \mapsto P_\sigma$, where

$$(P_\sigma)^i_j = \delta_{i\sigma(j)} \qquad \forall i, j \in X.$$

Note that the permutation representation is faithful, such that we have an embedding $S_X \hookrightarrow B(\mathbb{C}^X)$ given by permutation matrices.

Lemma 3.2.3. *Let X and Y be finite sets, $\sigma \in S_X$, $\tau \in S_Y$ and $A \in B(\mathbb{C}^Y, \mathbb{C}^X)$. Then $AP_\tau = P_\sigma A$ if and only if $A^i_{\tau(j)} = A^{\sigma^{-1}(i)}_j$ for all $i \in X$ and $j \in Y$.*

Proof. Using the definition of P_σ and P_τ, we compute

$$(AP_\tau)^i_j = \sum_{k\in Y} A^i_k (P_\tau)^k_j = \sum_{k\in Y} \delta_{k\tau(j)} A^i_k = A^i_{\tau(j)} \qquad \forall i \in X,\, j \in Y,$$

$$(P_\sigma A)^i_j = \sum_{k\in X} (P_\sigma)^i_k A^k_j = \sum_{k\in X} \delta_{i\sigma(k)} A^k_j = A^{\sigma^{-1}(i)}_j \qquad \forall i \in X,\, j \in Y.$$

Thus,

$$AP_\tau = P_\sigma A \iff A^i_{\tau(j)} = A^{\sigma^{-1}(i)}_j \qquad \forall i \in X,\, j \in Y.$$

□

By applying the previous proposition to the incidence matrices of a hypergraph, we obtain precisely the compatibility conditions in Definition 3.2.1.

Proposition 3.2.4. *Let $\Gamma := (V, E)$ be a hypergraph and $(\sigma, \tau) \in S_V \times S_E$. Then*

1. *$\sigma(s(e)) = s(\tau(e))$ for all $e \in E$ if and only if $A_s P_\tau = P_\sigma A_s$,*
2. *$\sigma(r(e)) = r(\tau(e))$ for all $e \in E$ if and only if $A_r P_\tau = P_\sigma A_r$,*

where A_s and A_r are the incidence matrices of Γ.

Proof. We only prove the first statement about the source map s. The second statement about the range map r follows similarly. According to Definition 3.2.3, we have

$$A_s P_\tau = P_\sigma A_s \iff (A_s)^v_{\tau(e)} = (A_s)^{\sigma^{-1}(v)}_e \qquad \forall v \in V,\, e \in E.$$

By the definition of A_s, the right-hand side is equivalent to

$$v \in s(\tau(e)) \iff \sigma^{-1}(v) \in s(e) \qquad \forall v \in V,\, e \in E.$$

Furthermore, $\sigma^{-1}(v) \in s(e)$ can be replaced by $v \in \sigma s(e)$. Thus, we obtain

$$v \in s(\tau(e)) \iff v \in \sigma s(e) \qquad \forall v \in V,\, e \in E,$$

which is equivalent to

$$s(\tau(e)) = \sigma(s(e)) \qquad \forall e \in E.$$

□

Using the characterization of classical hypergraph automorphisms in terms of permutations matrices, we can now define the quantum automorphism group of a hypergraph.

Definition 3.2.5. Let $\Gamma := (V, E)$ be a hypergraph and denote by $\mathcal{A}$ the universal unital C^*-algebra generated by elements u^v_w for all $v, w \in V$ and elements u^e_f for all $e, f \in E$ such that:

1. $u_V := (u^v_w)_{v,w\in V}$ and $u_E := (u^e_f)_{e,f\in E}$ are magic unitaries.
2. $A_s u_E = u_V A_s$ and $A_r u_E = u_V A_r$, where $A_s, A_r \in B(\mathbb{C}^E, \mathbb{C}^V)$ are the incidence matrices of Γ.

Then $\mathrm{Aut}^+(\Gamma) := (\mathcal{A}, u)$ is the *quantum automorphism group* of Γ, where the fundamental representation u defined by

$$u := u_V \oplus u_E = \begin{pmatrix} u_V & 0 \\ 0 & u_E \end{pmatrix} \in \mathcal{A} \otimes B(\mathbb{C}^{V \sqcup E}).$$

Intuitively, we have replaced the permutations $\sigma \in S_V$ and $\tau \in S_E$ in the definition of the classical automorphism group with quantum permutation matrices u_V and u_E. The compatibility conditions between these matrices are now expressed by the intertwining relations from above.

Before we show that the definition of $\mathrm{Aut}^+(\Gamma)$ generalizes the classical automorphism group $\mathrm{Aut}(\Gamma)$, we first comment on the relations in Definition 3.2.5.

Remark 3.2.6. Let $\Gamma := (V, E)$ be a hypergraph and denote by $\mathcal{A}$ the C^*-algebra $C(\mathrm{Aut}^+(\Gamma))$. For $\mathrm{Aut}^+(\Gamma)$ to be a well-defined compact matrix quantum group, the magic unitary relations and intertwiner relations must be compatible with the comultiplication. This means that there exists a unital $*$-homomorphism $\Delta\colon \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$ defined by

$$\Delta(u^{v_1}_{v_2}) = \sum_{w\in V} u^{v_1}_w \otimes u^w_{v_2} \qquad \forall v_1, v_2 \in V,$$

$$\Delta(u^{e_1}_{e_2}) = \sum_{f\in E} u^{e_1}_f \otimes u^f_{e_2} \qquad \forall e_1, e_2 \in E.$$

However, this follows directly from the existence of S_n^+ and the fact that additional intertwiner relations always compatible with the comultiplication, see [89, 96].

Remark 3.2.7. The magic unitary relations for the matrix u_V are given by

$$(u_w^v)^2 = (u_w^v)^* = u_w^v, \quad \sum_{x\in V} u_x^v = \sum_{x\in V} u_v^x = 1 \qquad \forall v, w \in V.$$

Similarly, the magic unitary relations for the matrix u_E are given by

$$(u_f^e)^2 = (u_f^e)^* = u_f^e, \quad \sum_{g\in E} u_g^e = \sum_{g\in e} u_e^g = 1 \qquad \forall e, f \in E,$$

and the intertwiner relations $A_s u_E = u_V A_s$ and $A_r u_E = u_V A_r$ are equivalent to

$$\sum_{\substack{f\in E\\ v\in s(f)}} u_e^f = \sum_{\substack{w\in V\\ w\in s(e)}} u_w^v, \quad \sum_{\substack{f\in E\\ v\in r(f)}} u_e^f = \sum_{\substack{w\in V\\ w\in r(e)}} u_w^v \qquad \forall v \in V,\, e \in E.$$

Since u_V and u_E are unitary representations, Definition 2.5.8 implies that A_s^* and A_r^* are also intertwiners. Thus, we have the additional relations $A_s^* u_V = u_E A_s^*$ and $A_r^* u_V = u_E A_r^*$, which are equivalent to

$$\sum_{\substack{w\in V\\ w\in s(e)}} u_v^w = \sum_{\substack{f\in E\\ v\in s(f)}} u_f^e, \quad \sum_{\substack{w\in V\\ w\in r(e)}} u_v^w = \sum_{\substack{f\in E\\ v\in r(f)}} u_f^e \qquad \forall v \in V,\, e \in E.$$

Note that the intertwiner relations for A_s^* and A_r^* in the previous remark are not only additional relations, but they are equivalent to the original intertwiner relations for A_s and A_r. This fact is a direct consequence of the next proposition and will be used throughout the following sections in the computation of quantum automorphism groups of hypergraphs.

Proposition 3.2.8. *Let V and W be finite dimensional Hilbert spaces and $\mathcal{A}$ be a unital C^*-algebra. Consider two unitaries $u \in \mathcal{A} \otimes B(V)$ and $v \in \mathcal{A} \otimes B(W)$. Then*

$$Tu = vT \iff T^* v = uT^* \qquad \forall T \in B(V, W).$$

Proof. Let $T \in B(V, W)$ with $Tu = vT$. Since T has scalar coefficients with respect to any orthonormal bases of V and W, we have

$$u^* T^* = (Tu)^* = (vT)^* = T^* v^*.$$

Therefore, multiplying on the left by u and on the right by v yields $T^* v = uT^*$. The converse direction follows by replacing T with T^*. ☐

As a first consequence, we obtain that the quantum automorphism group $\mathrm{Aut}^+(\Gamma)$ can alternatively be defined by a single intertwiner relation.

Remark 3.2.9. Consider a quantum automorphism group $\mathrm{Aut}^+(\Gamma)$ with fundamental representation $u := u_V \oplus u_E$, and define the block matrix

$$A := \begin{pmatrix} 0 & A_s \\ A_r^* & 0 \end{pmatrix} \in B(\mathbb{C}^{V \sqcup E}).$$

Then $Au = uA$ is equivalent to $A_s u_E = u_V A_s$ and $A_r^* u_V = u_E A_r^*$, where the second equation is equivalent to $A_r u_E = u_V A_r$ by Definition 3.2.8. Therefore, the relations $A_s u_E = u_V A_s$ and $A_r u_E = u_V A_r$ in Definition 3.2.5 can be expressed as the single intertwiner relation $Au = uA$.

The following proposition finally shows that the quantum automorphism group $\mathrm{Aut}^+(\Gamma)$ generalizes the classical automorphism group $\mathrm{Aut}(\Gamma)$ in the sense of compact matrix quantum groups.

Proposition 3.2.10. *Let Γ be a hypergraph. Then $\operatorname{Spec} C(\mathrm{Aut}^+(\Gamma)) = \mathrm{Aut}(\Gamma)$ as finite groups.*

Proof. Let $\Gamma := (V, E)$ and denote by $\mathcal{A}$ the C^*-algebra $C(\mathrm{Aut}^+(\Gamma))$. Then $\mathrm{Spec}(\mathcal{A})$ is a group with multiplication given by

$$\varphi * \psi := (\varphi \otimes \psi) \circ \Delta \qquad \forall \varphi, \psi \in \operatorname{Spec} \mathcal{A},$$

and it is isomorphic to a subgroup of unitary matrices $G \subseteq B(\mathbb{C}^{V \sqcup E})$ via the correspondence

$$\varphi \in \operatorname{Spec} \mathcal{A} \quad \longleftrightarrow \quad \varphi(u) := \left(\varphi(u^i_j)\right)_{i,j \in V \sqcup E} \in B(\mathbb{C}^{V \sqcup E}),$$

see [83, Proposition 6.1.11]. Furthermore, we have the decomposition $u = u_E \oplus u_V$, such that $\varphi(u) = \varphi(u_E) \oplus \varphi(u_V)$ is given by a pair of matrices $\varphi(u_E)$ and $\varphi(u_V)$. Since u_V and u_E are magic unitaries, $\varphi(u_E)$ and $\varphi(u_V)$ are precisely permutation matrices, which correspond to a pair of permutations $(\sigma, \tau) \in S_V \times S_E$ via the permutation representation in Definition 3.2.2. Definition 3.2.4 then implies that (σ, τ) are precisely automorphisms of Γ. $\square$

3.3. Hypergraphs with maximal quantum symmetry

Before presenting examples of quantum automorphism groups of hypergraphs, we show that these quantum groups are always subgroups of $S_V^+ * S_E^+$.

Proposition 3.3.1. *Let $\Gamma := (V, E)$ be a hypergraph. Then $\mathrm{Aut}^+(\Gamma) \subseteq S_V^+ * S_E^+$.*

Proof. Let u denote the fundamental representation of $\mathrm{Aut}^+(\Gamma)$ and $\widehat{u}$ denote the fundamental representation of $S_V^+ * S_E^+$. Since u_V and u_E are magic unitaries, the universal property of $C(S_V^+ * S_E^+)$ implies the existence of a unital $*$-homomorphism

$$\phi \colon C(S_V^+ * S_E^+) \to C(\mathrm{Aut}^+(\Gamma)),$$

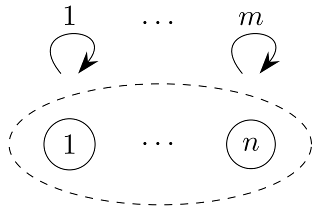


Figure 3.3.: The hypergraph $\Gamma_{n,m}$ from Definition 3.3.2.

$$\widehat{u}^v_w \mapsto u^v_w, \quad \widehat{u}^e_f \mapsto u^e_f \qquad \forall v, w \in V,\, e, f \in E.$$

This $*$-homomorphism is surjective because $C(\mathrm{Aut}^+(\Gamma))$ is generated by the entries of u_V and u_E. Furthermore, it is a morphism of compact quantum groups because

$$\Delta\big(\phi(\widehat{u}^v_w)\big) = \Delta(u^v_w) = \sum_{x\in V} u^v_x \otimes u^x_w = \sum_{x \in V} \phi(\widehat{u}^v_x) \otimes \phi(\widehat{u}^x_w) = (\phi\otimes\phi)\big(\Delta(\widehat{u}^v_w)\big)$$

for all $v, w \in V$. Similarly, we have $\Delta\big(\phi(\widehat{u}^e_f)\big) = \Delta\big(\phi(\widehat{u}^e_f)\big)$ for all $e, f \in E$. Thus, $\mathrm{Aut}^+(\Gamma)$ is a subgroup of $S_V^+ * S_E^+$. □

Next, we construct a concrete family of hypergraphs $\Gamma_{n,m}$ for which the quantum automorphism group is isomorphic to $S_V^+ * S_E^+$. These hypergraphs have maximal quantum symmetries in the sense of Definition 3.2.5 and are illustrated in Figure 3.3.

Definition 3.3.2. Let $n, m \in \mathbb{N}$. Define the hypergraph $\Gamma_{n,m} := (V, E)$ with vertices $V = \{1, \dots, n\}$, edges $E = \{1, \dots, m\}$, and source and range maps defined by

$$s(e) = V, \quad r(e) = V \qquad \forall e \in E.$$

Proposition 3.3.3. *Let $n, m \in \mathbb{N}$ and $\Gamma_{n,m} := (V, E)$ be the hypergraph from Definition 3.3.2. Then $\mathrm{Aut}^+(\Gamma_{n,m}) = S_V^+ * S_E^+$.*

Proof. Let u denote be the fundamental representation of $\mathrm{Aut}^+(\Gamma_{n,m})$ and $\widehat{u}$ denote the fundamental representation of $S_V^+ * S_E^+$. By the proof of Definition 3.3.1, $\mathrm{Aut}^+(\Gamma_{n,m})$ is a subgroup of $S_V^+ * S_E^+$ via the unital $*$-homomorphism

$$\phi\colon C(S_V^+ * S_E^+) \to C(\mathrm{Aut}^+(\Gamma_{n,m})),$$

$$\widehat{u}^v_w \mapsto u^v_w, \quad \widehat{u}^e_f \mapsto u^e_f \qquad \forall v, w \in V,\, e, f \in E.$$

To show the reverse inclusion, we construct the inverse $*$-homomorphism using the universal property of $C(\mathrm{Aut}^+(\Gamma_{n,m}))$. Thus, we must show that $\widehat{u}_V$ and $\widehat{u}_E$ satisfy the relations from Definition 3.2.5. However, $\widehat{u}_V$ and $\widehat{u}_E$ are magic unitaries by definition, and we compute

$$\sum_{\substack{f\in E\\ v \in s(f)}} \widehat{u}^f_e = \sum_{f \in E} \widehat{u}^f_e = 1 = \sum_{w\in V} \widehat{u}^v_w = \sum_{\substack{w \in V\\ w\in s(e)}} \widehat{u}^v_w \qquad \forall v \in V,\, e \in E.$$

This implies $A_s\widehat{u}_E = \widehat{u}_V A_s$ by Definition 3.2.7. Similarly, one shows $A_r\widehat{u}_E = \widehat{u}_V A_r$ by replacing the source map s with the range map r. Thus, the $*$-homomorphism from Definition 3.3.1 is invertible, which shows that $\mathrm{Aut}^+(\Gamma_{n,m})$ and $S_V^+ * S_E^+$ are isomorphic. $\square$

3.4. Dual and opposite hypergraphs

Next, we compute the quantum automorphism groups of the opposite and dual of a hypergraph. Recall from Definition 3.1.14 that the opposite hypergraph Γ^{op} is obtained by interchanging the source and range maps of a hypergraph Γ. In the classical case, both Γ and Γ^{op} have the same automorphism group.

Proposition 3.4.1. *Let $\Gamma := (V, E)$ be a hypergraph. Then* $\mathrm{Aut}(\Gamma) = \mathrm{Aut}(\Gamma^{\mathrm{op}})$.

Proof. The statement follows directly from the fact that

$$\sigma(s(e)) = s(\tau(e)) \iff \sigma(r^{\mathrm{op}}(e)) = r^{\mathrm{op}}(\tau(e)) \qquad \forall e \in E,$$

$$\sigma(r(e)) = r(\tau(e)) \iff \sigma(s^{\mathrm{op}}(e)) = s^{\mathrm{op}}(\tau(e)) \qquad \forall e \in E$$

for all pairs of permutations $(\sigma, \tau) \in S_V \times S_E$. $\square$

The previous proposition generalizes directly to the quantum setting.

Proposition 3.4.2. *Let Γ be a hypergraph. Then* $\mathrm{Aut}^+(\Gamma) = \mathrm{Aut}^+(\Gamma^{\mathrm{op}})$.

Proof. Let $\Gamma := (V, E)$. Denote by u the fundamental representation of $\mathrm{Aut}^+(\Gamma)$ and by $\widehat{u}$ the fundamental representation of $\mathrm{Aut}^+(\Gamma^{\mathrm{op}})$. By definition, we have $A_{s^{\mathrm{op}}} = A_r$ and $A_{r^{\mathrm{op}}} = A_s$, which implies the entries of u and $\widehat{u}$ satisfy the same relations. Therefore, the universal properties of $C(\mathrm{Aut}^+(\Gamma))$ and $C(\mathrm{Aut}^+(\Gamma^{\mathrm{op}}))$ yield a $*$-isomorphism

$$\phi\colon C(\mathrm{Aut}^+(\Gamma)) \to C(\mathrm{Aut}^+(\Gamma^{\mathrm{op}}))$$

$$u_w^v \mapsto \widehat{u}_w^v, \quad u_f^e \mapsto \widehat{u}_f^e \qquad \forall v, w \in V,\ e, f \in E.$$

Furthermore, it is a morphism of compact quantum groups because

$$\Delta\big(\phi(u_w^v)\big) = \sum_{x \in V} \widehat{u}_w^x \otimes \widehat{u}_w^x = (\phi \otimes \phi)\big(\Delta(u_w^v)\big) \qquad \forall v, w \in V,$$

$$\Delta\big(\phi(u_f^e)\big) = \sum_{g \in V} \widehat{u}_g^e \otimes \widehat{u}_f^g = (\phi \otimes \phi)\big(\Delta(u_f^e)\big) \qquad \forall e, f \in E.$$

Thus, $\mathrm{Aut}^+(\Gamma)$ are isomorphic $\mathrm{Aut}^+(\Gamma^{\mathrm{op}})$. $\square$

Next, we consider dual hypergraphs. Recall from Definition 3.1.14 that the dual Γ^* of a hypergraph Γ is obtained by interchanging the vertices and edges. As in the case of opposite hypergraphs, a hypergraph and its dual have isomorphic classical automorphism groups.

Proposition 3.4.3. *Let* Γ *be a hypergraph. Then*

$$\mathrm{Aut}(\Gamma^*) = \{(\tau, \sigma) \mid (\sigma, \tau) \in \mathrm{Aut}(\Gamma)\}.$$

In particular, $\mathrm{Aut}(\Gamma)$ *and* $\mathrm{Aut}(\Gamma^*)$ *are isomorphic.*

Proof. Let $\Gamma := (V, E)$ and $(\sigma, \tau) \in \mathrm{Aut}(\Gamma)$. Then

$$\begin{aligned}\tau(s^*(v)) &= \{\tau(e) \mid e \in E, v \in s(e)\} = \{e \in E \mid v \in s(\tau^{-1}(e))\} \\ &= \{e \in E \mid v \in \sigma^{-1}(s(e))\} = \{e \in E \mid \sigma(v) \in s(e)\} = s^*(\sigma(v))\end{aligned}$$

for all $v \in V$. Similarly, we have $\tau(r^*(v)) = r^*(\sigma(v))$ by replacing s with r. Thus, $(\tau, \sigma) \in \mathrm{Aut}(\Gamma^*)$. Conversely, let $(\tau, \sigma) \in \mathrm{Aut}(\Gamma^*)$. Since $(\Gamma^*)^* = \Gamma$, the previous computation directly implies that $(\sigma, \tau) \in \mathrm{Aut}((\Gamma^*)^*) = \mathrm{Aut}(\Gamma)$. The isomorphism $\mathrm{Aut}(\Gamma)$ between $\mathrm{Aut}(\Gamma^*)$ is given by $(\sigma, \tau) \mapsto (\tau, \sigma)$. □

It is also possible to generalize the previous proposition to the case of quantum groups.

Proposition 3.4.4. *Let* Γ *be a hypergraph. Then* $\mathrm{Aut}^+(\Gamma) = \mathrm{Aut}^+(\Gamma^*)$.

Proof. Let $\Gamma := (V, E)$. Denote by u the fundamental representation of $\mathrm{Aut}^+(\Gamma)$ and by $\widehat{u}$ the fundamental representation of $\mathrm{Aut}^+(\Gamma^*)$. We begin by constructing the $*$-isomorphism

$$\phi\colon C(\mathrm{Aut}^+(\Gamma)) \to C(\mathrm{Aut}^+(\Gamma^*)),$$

$$u^v_w \mapsto \widehat{u}^v_w, \quad u^e_f \mapsto \widehat{u}^e_f \qquad \forall v, w \in V,\, e, f \in E$$

using the universal properties of $C(\mathrm{Aut}^+(\Gamma))$ and $C(\mathrm{Aut}^+(\Gamma^*))$. Hence, we must show that entries of u and $\widehat{u}$ satisfy the same relations. Since u_V, u_E, $\widehat{u}_E$ and $\widehat{u}_V$ are all magic unitaries, it remains to show that

$$A_s u^{(1)} = u^{(2)} A_s \iff A_{s^*} u^{(2)} = u^{(1)} A_{s^*},$$

$$A_r u^{(1)} = u^{(2)} A_r \iff A_{r^*} u^{(2)} = u^{(1)} A_{r^*}$$

for arbitrary magic unitaries $u^{(1)}$ and $u^{(2)}$ indexed by V and E respectively. However, this follows directly from Definition 3.2.8 since

$$(A_{s^*})^e_v = \begin{cases} 1 & \text{if } e \in s^*(v) \\ 0 & \text{otherwise} \end{cases} = \begin{cases} 1 & \text{if } v \in s(e) \\ 0 & \text{otherwise} \end{cases} = (A_s)^v_e = (A_s^T)^e_v \qquad \forall v \in V,\, e \in E,$$

which implies $A_{s^*} = A_s^T = A_s^*$ and similarly $A_{r^*} = A_r^*$. Therefore, the $*$-isomorphism ϕ exists, and it is an isomorphism of compact quantum groups since

$$\Delta\big(\phi(u^v_w)\big) = \sum_{x \in V} \widehat{u}^v_x \otimes \widehat{u}^x_w = \sum_{x \in V} \phi(u^v_x) \otimes \phi(u^x_w) = (\phi \otimes \phi)\big(\Delta(u^v_w)\big)$$

for all $v, w \in V$, and $\Delta\big(\phi(u^e_f)\big) = (\phi \otimes \phi)\big(\Delta(u^e_f)\big)$ for all $e, f \in E$ by a similar computation. □

3.5. Hypergraphs without multiple edges

In contrast to the quantum automorphism groups of graphs by Bichon and Banica in Chapter 4, our quantum automorphism group includes an additional magic unitary u_E for the edges. This magic unitary is necessary to capture quantum symmetries involving multiple edges, see for example the family of hypergraphs in Section 3.3. However, we show that if a hypergraph contains no multiple edges, then the magic unitary u_E becomes redundant, and its entries can be expressed in terms of the entries of u_V.

We begin with the following lemma, which relates the entries of u_E with the entries of u_V.

Lemma 3.5.1. *Let $\Gamma := (V, E)$ be a hypergraph and $X \subseteq V$. Then*

$$\sum_{\substack{f \in E \\ X \subseteq s(f)}} u_e^f = \prod_{v \in X} \sum_{\substack{w \in V \\ w \in s(e)}} u_w^v \qquad \forall e \in E.$$

In particular, the product on the right-hand side commutes. The statement also holds for the range map r.

Proof. Let $e \in E$ and $X = \{v_1, \dots, v_k\}$ with all v_i pairwise distinct. Then

$$\prod_{v \in X} (A_s u_E)_e^v = \prod_{v \in X} \sum_{\substack{f \in E \\ v \in s(f)}} u_e^f = \sum_{\substack{f_1 \in E \\ v_1 \in s(f_1)}} \cdots \sum_{\substack{f_k \in E \\ v_k \in s(f_k)}} u_e^{f_1} \cdots u_e^{f_k}.$$

Since $u_e^{f_1} \cdots u_e^{f_k} = \delta_{f_1 f_2} \cdots \delta_{f_1 f_k} u_e^{f_1}$, it follows that

$$\prod_{v \in X} (A_s u_E)_e^v = \sum_{\substack{f \in E \\ v_1 \in s(f), \dots, v_k \in s(f)}} u_e^f = \sum_{\substack{f \in E \\ X \subseteq s(f)}} u_e^f.$$

On the other hand, applying $A_s u_E = u_V A_s$ to the original expression yields

$$\prod_{v \in X} (A_s u_E)_e^v = \prod_{v \in X} (u_V A_s)_e^v = \prod_{v \in X} \sum_{\substack{w \in V \\ w \in s(e)}} u_w^v.$$

Therefore,

$$\sum_{\substack{f \in E \\ X \subseteq s(f)}} u_e^f = \prod_{v \in X} \sum_{\substack{w \in V \\ w \in s(e)}} u_w^v.$$

Since the left-hand side is independent of the order of the elements in X, the product on the right-hand side commutes. Furthermore, replacing s with r yields the analogous result for the range map r. □

By using an inclusion-exclusion argument, we can strengthen the previous lemma to obtain the equality $X = s(f)$ instead of the inclusion $X \subseteq s(f)$ on the left-hand side.

Lemma 3.5.2. *Let $\Gamma := (V, E)$ be a hypergraph and $X \subseteq V$. Then*

$$\sum_{\substack{f\in E\\ s(f)=X}} u_e^f = \sum_{X\subseteq Y\subseteq V} (-1)^{|Y|-|X|} \prod_{v\in Y} \sum_{\substack{w\in V\\ w\in s(e)}} u_w^v \qquad \forall e \in E.$$

The statement also holds for the range map r.

Proof. Using Definition 3.5.1, we must show that

$$\sum_{\substack{f\in E\\ s(f)=X}} u_e^f = \sum_{X\subseteq Y\subseteq V} (-1)^{|Y|-|X|} \sum_{\substack{f\in E\\ Y\subseteq s(f)}} u_e^f \qquad \forall e \in E.$$

Consider an element u_e^f with $X \subseteq s(f)$ and define $k := |s(f)| - |X|$. Then there are $\binom{k}{\ell}$ subsets Y satisfying $X \subseteq Y \subseteq s(f)$ and $|Y| = |X| + \ell$. Furthermore, each subset contributes a factor of

$$(-1)^{|Y|-|X|} = (-1)^{(|X|+\ell)-|X|} = (-1)^\ell$$

on the right-hand side of the equation. Thus, by the binomial theorem, the total contribution of u_f^e to the right-hand side is

$$\sum_{\ell=0}^{k} \binom{k}{\ell} (-1)^\ell = \big((-1) + 1\big)^k = \begin{cases} 1 & \text{if } k = 0, \\ 0 & \text{if } k > 0. \end{cases}$$

Therefore, the right-hand side contains u_e^f precisely when $k = 0$, which is equivalent to $s(f) = X$. The corresponding result for the range map r follows similarly. □

The previous lemma can now be used to show that the elements of u_E can be expressed in terms of the elements of u_V if the underlying hypergraph contains no multiple edges.

Theorem 3.5.3. *Let $\Gamma := (V, E)$ be a hypergraph without multiple edges. Denote by $C^*(u_V)$ the C^*-algebra generated by u_w^v for all $v, w \in V$. Then $u_f^e \in C^*(u_V)$ for all $e, f \in E$.*

Proof. Let $e, f \in E$. Then

$$\sum_{\substack{g\in E\\ s(g)=s(e)}} u_f^g, \quad \sum_{\substack{g\in E\\ r(g)=r(e)}} u_f^g \in C^*(u_V)$$

by choosing $X = s(e)$ and $X = r(e)$ in Definition 3.5.2. This implies

$$\Bigg(\sum_{\substack{g\in E\\ s(g)=s(e)}} u_f^g \Bigg) \Bigg(\sum_{\substack{g\in E\\ r(g)=r(e)}} u_f^g \Bigg) = \sum_{\substack{g_1,g_2\in E\\ s(g_1)=s(e)\\ r(g_2)=r(e)}} \underbrace{u_e^{g_1} u_e^{g_2}}_{\delta_{g_1 g_2} u_f^{g_1}} = \sum_{\substack{g\in E\\ s(g)=s(e)\\ r(g)=r(e)}} u_f^g \in C^*(u_V).$$

Since Γ contains no multiple edges, we have

$$u_f^e = \sum_{\substack{g \in E \\ s(g)=s(e) \\ r(g)=r(e)}} u_f^g \in C^*(u_V).$$

$\square$

Translating the previous theorem to the setting of quantum groups yields the following two corollaries.

Corollary 3.5.4. *Let* $\Gamma := (V, E)$ *be a hypergraph without multiple edges. Then* $\mathrm{Aut}^+(\Gamma)$ *is a subgroup of* S_V^+*.*

Proof. Denote by u the fundamental representation of $\mathrm{Aut}^+(\Gamma)$ and by $\widehat{u}$ the fundamental representation of S_V^+. Following the proof of Definition 3.3.1, there exists a morphism of compact quantum groups

$$\phi\colon C(S_V^+) \to \mathrm{Aut}^+(\Gamma), \quad \widehat{u}_w^v \mapsto u_w^v \qquad \forall v, w \in V.$$

By Definition 3.5.3, this morphism is surjective, which implies that $\mathrm{Aut}^+(\Gamma)$ is a subgroup of S_V^+. $\square$

Corollary 3.5.5. *Let* $\Gamma := (V, E)$ *be a hypergraph such that* Γ^* *contains no multiple edges. Then* $\mathrm{Aut}^+(\Gamma)$ *is a subgroup of* S_E^+*.*

Proof. By combining Definition 3.4.3 and Definition 3.5.4, we obtain

$$\mathrm{Aut}^+(\Gamma) = \mathrm{Aut}^+(\Gamma^*) \subseteq S_E^+.$$

$\square$

4. Links to quantum symmetries of classical graphs

In the case of classical graphs, there exist two different versions of quantum automorphism groups, which have been introduced by Bichon [13] and Banica [5]. These quantum groups generalize the classical automorphism group of a graph Γ and are constructed by imposing the additional relation $Au = uA$ on a quantum permutation u, where A denotes the adjacency matrix of Γ. Quantum automorphism groups of graphs provide a large class of examples of compact matrix quantum groups and have been further studied for example in [21, 75, 57, 28] and recently been generalized to multigraphs in [45]. Additionally, quantum automorphism groups of graphs have found applications in quantum information theory, where Mančinska and Roberson [64] established a link between quantum symmetries of graphs and non-local graph isomorphism games.

In this chapter, we first introduce the existing quantum automorphism groups for graphs by Bichon and Banica, and a multigraph version by Goswami and Hossain, before comparing these quantum groups to our quantum automorphism group for hypergraphs. Specifically, we show that if a hypergraph arises from a classical directed, simple or multigraph as described in Section 3.1, then our quantum automorphism group for hypergraphs agrees with Bichon's quantum automorphism group or its multigraph version by Goswami and Hossain. Therefore, our quantum automorphism group for hypergraphs can be viewed as a generalization of Bichon's quantum automorphism group for classical graphs.

4.1. Quantum automorphism groups of graphs

In the following, we recall the definition and basic facts about quantum automorphism groups of classical graphs. We begin with the definition of Bichon's quantum automorphism groups before discussing Banica's version and the recent version for multigraphs by Goswami and Hossain. For more information on quantum automorphism groups of graphs, we refer to [76].

Definition 4.1.1. Let $\Gamma := (V, E)$ be a simple or directed graph. Denote by $\mathcal{A}$ the universal unital C^*-algebra generated by elements u_w^v for all $i, j \in V$ satisfying the following relations:

1. $u := (u_j^i)_{i,j \in V}$ is a magic unitary.
2. $Au = uA$, where A is the adjacency matrix of Γ.
3. $u_k^i u_\ell^j = u_\ell^j u_k^i$ for all $i \sim j$ and $k \sim \ell$.

Then $\mathrm{Aut}^+_{\mathrm{Bic}}(\Gamma) := (\mathcal{A}, u)$ is *Bichon's quantum automorphism group* of Γ.

Note that the second relation $Au = uA$ was originally formulated using a different set of relations. However, the following proposition shows that both versions are equivalent.

Proposition 4.1.2. *Let $\Gamma := (V, E)$ be a simple or directed graph, $\mathcal{A}$ be a unital C^*-algebra and $u \in \mathcal{A} \otimes B(\mathbb{C}^V)$ be a magic unitary. Then $Au = uA$ is equivalent to*

$$u^i_k u^j_\ell = u^j_\ell u^i_k = 0 \qquad \forall i \sim j,\ k \nsim \ell,$$

$$u^i_k u^j_\ell = u^j_\ell u^i_k = 0 \qquad \forall i \nsim j,\ k \sim \ell.$$

Proof. See [76, Proposition 2.1.3]. □

Using this reformulation of the intertwiner relations $Au = uA$, we can now prove the following proposition, which will be used later.

Proposition 4.1.3. *Let $\Gamma := (V, E)$ be a simple or directed graph. Denote by u the fundamental representation of $\mathrm{Aut}^+_{\mathrm{Bic}}(\Gamma)$. Then*

$$\sum_{\substack{k,\ell \in V \\ k \sim \ell}} u^i_k u^j_\ell = \sum_{\substack{k,\ell \in V \\ k \sim \ell}} u^k_i u^\ell_j = 1 \qquad \forall i \sim j.$$

Proof. Using Definition 4.1.2 and the fact that u is a magic unitary, we obtain

$$\sum_{\substack{k,\ell \in V \\ k \sim \ell}} u^i_k u^j_\ell = \sum_{k,\ell \in V} u^i_k u^j_\ell = \left(\sum_{k \in V} u^i_k\right)\left(\sum_{\ell \in V} u^j_\ell\right) = 1 \cdot 1 = 1$$

for all $i, j \in V$ with $i \sim j$, see [77]. The second equality follows similarly. □

By dropping Relation 3 in Definition 4.1.1, we obtain of Banica's quantum automorphism group, which is often studied instead of Bichon's version.

Definition 4.1.4. Let $\Gamma := (V, E)$ be a simple or directed graph. Denote by $\mathcal{A}$ the universal unital C^*-algebra generated by elements u^i_j for all $i, j \in V$ satisfying the following relations:

1. $u := (u^i_j)_{i,j \in V}$ is a magic unitary.
2. $Au = uA$, where A is the adjacency matrix of Γ.

Then $\mathrm{Aut}^+_{\mathrm{Ban}}(\Gamma) := (\mathcal{A}, u)$ is *Banica's quantum automorphism group* of Γ.

However, we are primarily interested in Bichon's version and its generalization to multigraphs as recently defined by Goswami and Hossain [45].

Definition 4.1.5. Let $\Gamma := (V, E)$ be a multigraph without isolated vertices. Denote by $\mathcal{A}$ the universal unital C^*-algebra generated by elements u^e_f for all $e, f \in E$ satisfying the following relations:

1. The matrix $u := (u^e_f)_{e,f\in E}$ is a magic unitary.
2. Let $v \in V$ and $e_1, e_2 \in E$. Then

$$\sum_{\substack{f\in E\\ s(f)=v}} u^{e_1}_f = \sum_{\substack{f\in E\\ s(f)=v}} u^{e_2}_f \quad \text{if } s(e_1) = s(e_2),$$

$$\sum_{\substack{f\in E\\ r(f)=v}} u^{e_1}_f = \sum_{\substack{f\in E\\ r(f)=v}} u^{e_2}_f \quad \text{if } r(e_1) = r(e_2).$$

3. Let $e, f \in E$. Then $u^e_f = 0$ if
   - $s(e)$ is neither a source nor a sink and $s(f)$ is a source, or
   - $r(e)$ is neither a source nor a sink and $r(f)$ is a sink.
4. Let $v \in V$ be neither a source nor a sink and $e_1, e_2 \in E$ such that $s(e_1) = r(e_2)$ is neither a source nor a sink. Then

$$\sum_{\substack{f\in E\\ s(f)=v}} u^{e_1}_f = \sum_{\substack{f\in E\\ r(f)=v}} u^{e_2}_f.$$

Then $\mathrm{Aut}^+_{\mathrm{GH,Bic}}(\Gamma) := (\mathcal{A}, u)$ is the *quantum automorphism group by Goswami and Hossain in Bichon's sense* of Γ.

Note that in contrast to the original definition in [45, Definition 4.26], we have added the magic unitary relation $(u^e_f)^2 = u^e_f$ for all $e, f \in E$ and interchanged the conditions in Relation 3.

4.2. Case of directed graphs

We begin by studying the hypergraph quantum symmetries of directed graphs as defined in Definition 3.1.2. Recall from Definition 3.1.11 that we can identify a directed graph $\Gamma := (V, E)$ with a 1-uniform hypergraph without multiple edges by defining the source and range maps as follows:

$$s(v, w) = \{v\}, \quad r(v, w) = \{w\} \qquad \forall (v, w) \in E.$$

In this way, we can apply Definition 3.2.5 to obtain a hypergraph quantum automorphism group $\mathrm{Aut}^+(\Gamma)$. In the following, we show that $\mathrm{Aut}^+(\Gamma)$ coincides with Bichon's quantum automorphism group $\mathrm{Aut}^+_{\mathrm{Bic}}(\Gamma)$.

We begin by reformulating the intertwiner relations $A_s u_E = u_V A_s$ and $A_r u_E = u_V A_r$ in the context of directed graphs.

Lemma 4.2.1. *Let $\Gamma := (V, E)$ be a directed graph, $\mathcal{A}$ be a unital C^*-algebra and*

$$u_V \in \mathcal{A} \otimes B(\mathbb{C}^V), \qquad u_E \in \mathcal{A} \otimes B(\mathbb{C}^E).$$

Then the relations $A_s u_E = u_V A_s$ and $A_r u_E = u_V A_r$ are equivalent to

$$\sum_{\substack{(v_2,w_2)\in E \\ v_0=v_2}} u_{(v_1,w_1)}^{(v_2,w_2)} = u_{v_1}^{v_0}, \qquad \sum_{\substack{(v_2,w_2)\in E \\ v_0=w_2}} u_{(v_1,w_1)}^{(v_2,w_2)} = u_{w_1}^{v_0}$$

for all $v_0 \in V$ and $(v_1, w_1) \in E$.

Proof. Let $v_0 \in V$ and $e := (v_1, w_1) \in E$. Then

$$(A_s u_E)_e^{v_0} = \sum_{\substack{f\in E \\ v_0\in s(f)}} u_e^f = \sum_{\substack{(v_2,w_2)\in E \\ v_0=v_2}} u_{(v_1,w_1)}^{(v_2,w_2)},$$

$$(A_r u_E)_e^{v_0} = \sum_{\substack{f\in E \\ v_0\in r(f)}} u_e^f = \sum_{\substack{(v_2,w_2)\in E \\ v_0=w_2}} u_{(v_1,w_1)}^{(v_2,w_2)}.$$

On the other hand, since the image of s and r contains exactly one element and there are no multiple edges, we have

$$(u_V A_s)_e^{v_0} = \sum_{\substack{w\in V \\ w\in s(e)}} u_w^{v_0} = u_{v_1}^{v_0}, \qquad (u_V A_r)_e^{v_0} = \sum_{\substack{w\in V \\ w\in r(e)}} u_w^{v_0} = u_{w_1}^{v_0}.$$

Therefore, $A_s u_E = u_V A_s$ and $A_r u_E = u_V A_r$ are equivalent to

$$\sum_{\substack{(v_2,w_2)\in E \\ v_0=v_2}} u_{(v_1,w_1)}^{(v_2,w_2)} = u_{v_1}^{v_0}, \qquad \sum_{\substack{(v_2,w_2)\in E \\ v_0=w_2}} u_{(v_1,w_1)}^{(v_2,w_2)} = u_{w_1}^{v_0}$$

for all $v_0 \in V$ and $(v_1, w_1) \in E$. $\square$

Using the previous lemma, the entries of u_E can be expressed in terms of the entries of u_V.

Lemma 4.2.2. *Let $\Gamma := (V, E)$ be a directed graph. Denote by u the fundamental representation of* $\mathrm{Aut}^+(\Gamma)$*. Then*

$$u_{(v_2,w_2)}^{(v_1,w_1)} = u_{v_2}^{v_1} u_{w_2}^{w_1} = u_{w_2}^{w_1} u_{v_2}^{v_1} \qquad \forall (v_1, w_1), (v_2, w_2) \in E.$$

Proof. Let $(v_1, w_1), (v_2, w_2) \in E$. By Definition 4.2.1, we have

$$u_{v_2}^{v_1} = \sum_{\substack{(v_3,w_3)\in E \\ v_1=v_3}} u_{(v_2,w_2)}^{(v_3,w_3)}, \qquad u_{w_2}^{w_1} = \sum_{\substack{(v_4,w_4)\in E \\ w_1=w_4}} u_{(v_2,w_2)}^{(v_4,w_4)},$$

which yields

$$u^{v_1}_{v_2} u^{w_1}_{w_2} = \sum_{\substack{(v_3,w_3)\in E \\ v_1=v_3}} \sum_{\substack{(v_4,w_4)\in E \\ w_1=w_4}} \underbrace{u^{(v_3,w_3)}_{(v_2,w_2)} u^{(v_4,w_4)}_{(v_2,w_2)}}_{\delta_{(v_3,w_3)(v_4,w_4)} u^{(v_3,w_3)}_{(v_2,w_2)}} = \sum_{\substack{(v_3,w_3)\in E \\ v_1=v_3 \\ w_1=w_3}} u^{(v_3,w_3)}_{(v_2,w_2)},$$

$$u^{w_1}_{w_2} u^{v_1}_{v_2} = \sum_{\substack{(v_4,w_4)\in E \\ w_1=w_4}} \sum_{\substack{(v_3,w_3)\in E \\ v_1=v_3}} \underbrace{u^{(v_4,w_4)}_{(v_2,w_2)} u^{(v_3,w_3)}_{(v_2,w_2)}}_{\delta_{(v_4,w_4)(v_3,w_3)} u^{(v_3,w_3)}_{(v_2,w_2)}} = \sum_{\substack{(v_3,w_3)\in E \\ v_1=v_3 \\ w_1=w_3}} u^{(v_3,w_3)}_{(v_2,w_2)}.$$

Since Γ contains no multiple edges, we have

$$u^{(v_1,w_1)}_{(v_2,w_2)} = \sum_{\substack{(v_3,w_3)\in E \\ v_1=v_3 \\ w_1=w_3}} u^{(v_3,w_3)}_{(v_2,w_2)}.$$

Thus,

$$u^{(v_1,w_1)}_{(v_2,w_2)} = u^{v_1}_{v_2} u^{w_1}_{w_2} = u^{w_1}_{w_2} u^{v_1}_{v_2}.$$

$\square$

Now, we can show that our quantum automorphism group $\mathrm{Aut}^+(\Gamma)$ coincides with the quantum automorphism group $\mathrm{Aut}^+_{\mathrm{Bic}}(\Gamma)$ when we identify the magic unitary u_V with the fundamental representation of $\mathrm{Aut}^+_{\mathrm{Bic}}(\Gamma)$.

Theorem 4.2.3. *Let Γ be a directed graph. Then $\mathrm{Aut}^+(\Gamma) = \mathrm{Aut}^+_{\mathrm{Bic}}(\Gamma)$.*

Proof. Let $\Gamma := (V, E)$. Denote by u the fundamental representation of $\mathrm{Aut}^+(\Gamma)$ and by $\widehat{u}$ the fundamental representation of $\mathrm{Aut}^+_{\mathrm{Bic}}(\Gamma)$. Furthermore, define the elements

$$\widehat{u}^{(v_1,w_1)}_{(v_2,w_2)} := \widehat{u}^{v_1}_{v_2} \widehat{u}^{w_1}_{w_2} \qquad \forall (v_1,w_1),(v_2,w_2) \in E,$$

and the matrices $\widehat{u}_V := (\widehat{u}^v_w)_{v,w\in V}$ and $\widehat{u}_E := (\widehat{u}^e_f)_{e,f\in E}$.

Step 1. To prove the statement, we begin by constructing the unital $*$-homomorphism

$$\begin{aligned} \phi\colon C(\mathrm{Aut}^+(\Gamma)) &\to C(\mathrm{Aut}^+_{\mathrm{Bic}}(\Gamma)), \\ u^v_w &\mapsto \widehat{u}^v_w && \forall v,w \in V, \\ u^{(v_1,w_1)}_{(v_2,w_2)} &\mapsto \widehat{u}^{v_1}_{v_2} \widehat{u}^{w_1}_{w_2} && \forall (v_1,w_1),(v_2,w_2) \in E \end{aligned}$$

using the universal property of $C(\mathrm{Aut}^+(\Gamma))$. Therefore, we must show that the matrices $\widehat{u}_V$ and $\widehat{u}_E$ satisfy the relations from Definition 3.2.5. By Definition 4.1.1, $\widehat{u}_V$ is a magic unitary. Moreover, we have

$$\widehat{u}^{v_1}_{v_2} \widehat{u}^{w_1}_{w_2} = \widehat{u}^{w_1}_{w_2} \widehat{u}^{v_1}_{v_2} \qquad \forall (v_1,w_1),(v_2,w_2) \in E,$$

which implies

$$(\widehat{u}^{v_1}_{v_2} \widehat{u}^{w_1}_{w_2})^* = (\widehat{u}^{w_1}_{w_2})^* (\widehat{u}^{v_1}_{v_2})^* = \widehat{u}^{w_1}_{w_2} \widehat{u}^{v_1}_{v_2} = \widehat{u}^{v_1}_{v_2} \widehat{u}^{w_1}_{w_1},$$

$$(\widehat{u}^{v_1}_{v_2}\widehat{u}^{w_1}_{w_2})^2 = \widehat{u}^{v_1}_{v_2}\widehat{u}^{w_1}_{w_2}\widehat{u}^{v_1}_{v_2}\widehat{u}^{w_1}_{w_2} = (\widehat{u}^{v_1}_{v_2})^2(\widehat{u}^{w_1}_{w_2})^2 = \widehat{u}^{v_1}_{v_2}\widehat{u}^{w_1}_{w_2}.$$

Furthermore, the additional relations from Definition 4.1.3 yield

$$\sum_{(v_2,w_2)\in E} \widehat{u}^{v_1}_{v_2}\widehat{u}^{w_1}_{w_2} = \sum_{\substack{v_2,w_2\in V\\(v_2,w_2)\in E}} \widehat{u}^{v_1}_{v_2}\widehat{u}^{w_1}_{w_2} = 1 \qquad \forall (v_1,w_1)\in E,$$

$$\sum_{(v_2,w_2)\in E} \widehat{u}^{v_2}_{v_1}\widehat{u}^{w_2}_{w_1} = \sum_{\substack{v_2,w_2\in V\\(v_2,w_2)\in E}} \widehat{u}^{v_2}_{v_1}\widehat{u}^{w_2}_{w_1} = 1 \qquad \forall (v_1,w_1)\in E.$$

Therefore, $\widehat{u}_E$ is a magic unitary. Next, we verify the intertwiner relation $A_s\widehat{u}_E = \widehat{u}_V A_s$. Let $v_0 \in V$ and $(v_1,w_1)\in E$, and denote by A the adjacency matrix of Γ. Then

$$\sum_{\substack{(v_2,w_2)\in E\\v_0=v_2}} \widehat{u}^{v_2}_{v_1}\widehat{u}^{w_2}_{w_1} = \widehat{u}^{v_0}_{v_1}\sum_{\substack{(v_2,w_2)\in E\\v_0=v_2}} \widehat{u}^{w_2}_{w_1} = \widehat{u}^{v_0}_{v_1}\sum_{w_2\in V} A^{v_0}_{w_2}\widehat{u}^{w_2}_{w_1}.$$

By Definition 4.1.1, we have $A\widehat{u}_V = \widehat{u}_V A$, which implies

$$\widehat{u}^{v_0}_{v_1}\sum_{w_2\in V} A^{v_0}_{w_2}\widehat{u}^{w_2}_{w_1} = \widehat{u}^{v_0}_{v_1}\sum_{w_2\in V} \widehat{u}^{v_0}_{w_2}A^{w_2}_{w_1}$$

Since $\widehat{u}^{v_0}_{v_1}\widehat{u}^{v_0}_{w_2} = \delta_{v_1w_2}\widehat{u}^{v_0}_{v_1}$, it follows that

$$\widehat{u}^{v_0}_{v_1}\sum_{w_2\in V} \widehat{u}^{v_0}_{w_2}A^{w_2}_{w_1} = \sum_{w_2\in V}\delta_{v_1w_2}\widehat{u}^{v_0}_{v_1}A^{w_2}_{w_1} = \widehat{u}^{v_0}_{v_1}A^{v_1}_{w_1} = \widehat{u}^{v_0}_{v_1}.$$

Therefore,

$$\sum_{\substack{(v_2,w_2)\in E\\v_0=v_2}} \widehat{u}^{(v_2,w_2)}_{(v_1,w_1)} = \sum_{\substack{(v_2,w_2)\in E\\v_0=v_2}} \widehat{u}^{v_2}_{v_1}\widehat{u}^{w_2}_{w_1} = \widehat{u}^{v_0}_{v_1},$$

which is equivalent to $A_s\widehat{u}_E = \widehat{u}_V A_s$ by Definition 4.2.1. Similarly, one shows $A_r\widehat{u}_E = \widehat{u}_V A_r$. Thus, the map ϕ exists.

Step 2. Next, we construct the inverse map

$$\psi\colon C(\mathrm{Aut}^+_{\mathrm{Bic}}(\Gamma)) \to C(\mathrm{Aut}^+(\Gamma)), \quad \widehat{u}^v_w \mapsto u^v_w \qquad \forall v,w\in V$$

using the universal property of $C(\mathrm{Aut}^+_{\mathrm{Bic}}(\Gamma))$. Thus, we must show that the matrix u_V satisfies the relations from Definition 4.1.1. First, note that u_V is a magic unitary by Definition 3.2.5. Second, we must show that $Au_V = u_V A$. Observe that $A = A_sA_r^*$ since

$$(A_sA_r^*)^v_w = \sum_{e\in E}(A_s)^v_e(A_r)^w_e = \sum_{e\in E}\delta_{(v,w)e} = \begin{cases}1 & \text{if } (v,w)\in E,\\ 0 & \text{otherwise,}\end{cases} \qquad \forall v,w\in V.$$

Hence,

$$Au_V = A_sA_r^*u_V = A_su_EA_r^* = u_VA_sA_r^* = u_VA,$$

because A_r^* also intertwines u_V and u_E by Definition 3.2.7. Finally, we must verify that

$$u_{v_2}^{v_1} u_{w_2}^{w_1} = u_{w_2}^{w_1} u_{v_2}^{v_1} \qquad \forall (v_1, w_1), (v_2, w_2) \in E.$$

But this follows directly from Definition 4.2.2, since

$$u_{v_2}^{v_1} u_{w_2}^{w_1} = u_{(v_2,w_2)}^{(v_1,w_1)} = u_{w_2}^{w_1} u_{v_2}^{v_1}.$$

Thus, the $*$-homomorphism ψ exists.

Step 3. The maps ϕ and ψ are indeed inverse because

$$\begin{aligned} \widehat{u}_w^v &\longleftrightarrow u_w^v \\ \widehat{u}_{v_2}^{v_1} \widehat{u}_{w_2}^{w_1} &\longleftrightarrow u_{v_2}^{v_1} u_{w_2}^{w_1} = u_{(v_2,w_2)}^{(v_1,w_1)} \end{aligned}$$

by Definition 4.2.2. It remains to show that ψ respects the comultiplication and defines an isomorphism of compact quantum groups. But this follows directly since

$$\Delta\big(\psi(\widehat{u}_w^v)\big) = \sum_{x \in V} u_x^v \otimes u_w^x = \sum_{x \in V} \psi(\widehat{u}_x^v) \otimes \psi(\widehat{u}_w^x) = (\psi \otimes \psi)\big(\Delta(\widehat{u}_w^v)\big)$$

for all $v, w \in V$. □

4.3. Case of simple graphs

Next, we consider simple graphs as defined in Definition 3.1.1. Recall from Definition 3.1.10 that simple graphs can be regarded as 2-uniform undirected hypergraphs without multiple edges by defining the source and range maps as follows:

$$s(\{v, w\}) = \{v, w\}, \qquad r(\{v, w\}) = \{v, w\} \qquad \forall \{v, w\} \in E.$$

In the following, we show that for a simple graph Γ our quantum automorphism group $\mathrm{Aut}^+(\Gamma)$ also coincides with Bichon's quantum automorphism group $\mathrm{Aut}^+_{\mathrm{Bic}}(\Gamma)$.

We begin by reformulating the intertwiner relations $A_s u_E = u_V A_s$ and $A_r u_E = u_V A_r$ in the context of simple graphs.

Lemma 4.3.1. *Let $\Gamma := (V, E)$ be a simple graph, $\mathcal{A}$ be a unital C^*-algebra and*

$$u_V \in \mathcal{A} \otimes B(\mathbb{C}^V), \quad u_E \in \mathcal{A} \otimes B(\mathbb{C}^E)$$

be two unitaries. Then the following are equivalent:

1. $A_s u_E = u_V A_s$ *and* $A_r u_E = u_V A_r$,
2. $\displaystyle\sum_{\substack{v_2 \in V \\ \{v_0, v_2\} \in E}} u_{\{v_1,w_1\}}^{\{v_0,v_2\}} = u_{v_1}^{v_0} + u_{w_1}^{v_0} \qquad \forall v_0 \in V,\ \{v_1, w_1\} \in E,$

3. $\displaystyle\sum_{\substack{v_2\in V\\ \{v_0,v_2\}\in E}} u^{\{v_1,w_1\}}_{\{v_0,v_2\}} = u^{v_1}_{v_0} + u^{w_1}_{v_0} \qquad \forall v_0 \in V,\ \{v_1,w_1\}\in E.$

Proof. Since $A_s = A_r$, we only need to consider $A_s u_E = u_V A_s$. Let $v_0 \in V$ and $e := \{v_1, w_1\} \in E$. Then

$$(A_s u_E)^{v_0}_e = \sum_{f\in E} (A_s)^{v_0}_f u^f_e = \sum_{\substack{f\in E\\ v_0\in f}} u^f_e = \sum_{\substack{v_2\in V\\ \{v_0,v_2\}\in E}} u^{\{v_0,v_2\}}_{\{v_1,w_1\}},$$

$$(u_V A_s)^{v_0}_e = \sum_{w\in V} u^{v_0}_w (A_s)^w_e = \sum_{\substack{w\in V\\ w\in e}} u^{v_0}_w = u^{v_0}_{v_1} + u^{v_0}_{w_1}.$$

Hence, $A_s u_E = u_V A_s$ is equivalent to

$$\sum_{\substack{v_2\in V\\ \{v_0,v_2\}\in E}} u^{\{v_0,v_2\}}_{\{v_1,w_1\}} = u^{v_0}_{v_1} + u^{v_0}_{w_1} \qquad \forall v_0 \in V,\ \{v_1,w_1\}\in E.$$

Similarly, we compute

$$(u_E A_s^*)^e_{v_0} = \sum_{f\in E} u^e_f (A_s^*)^f_{v_0} = \sum_{\substack{f\in E\\ v_0\in f}} u^e_f = \sum_{\substack{v_2\in V\\ \{v_0,v_2\}\in E}} u^{\{v_1,w_1\}}_{\{v_0,v_2\}},$$

$$(A_s^* u_V)^e_{v_0} = \sum_{w\in V} (A_s)^e_w u^w_{v_0} = \sum_{\substack{w\in V\\ w\in e}} u^w_{v_0} = u^{v_1}_{v_0} + u^{w_1}_{v_0}.$$

Thus, $u_E A_s^* = A_s^* u_V$ is equivalent to

$$\sum_{\substack{v_2\in V\\ \{v_0,v_2\}\in E}} u^{\{v_1,w_1\}}_{\{v_0,v_2\}} = u^{v_1}_{v_0} + u^{w_1}_{v_0} \qquad \forall v_0 \in V,\ \{v_1,w_1\}\in E,$$

which is equivalent to $A_s u_E = u_V A_s$ by Definition 3.2.8. □

As in the case of directed graphs, we can use the previous lemma to express the entries of the magic unitary u_E in terms of the entries of u_V.

Lemma 4.3.2. *Let $\Gamma := (V,E)$ be a simple graph. Denote by u the fundamental representation of $\mathrm{Aut}^+(\Gamma)$. Then*

$$u^{\{v_1,w_1\}}_{\{v_2,w_2\}} = u^{v_1}_{v_2} u^{w_1}_{w_2} + u^{v_1}_{w_2} u^{w_1}_{v_2} = u^{v_1}_{v_2} u^{w_1}_{w_2} + u^{w_1}_{v_2} u^{v_1}_{w_2}$$

for all $\{v_1,w_1\},\{v_2,w_2\}\in E$.

Proof. Let $\{v_1, w_1\}, \{v_2, w_2\} \in E$. Since Γ a simple graph, we have $v_1 \neq w_1$, which implies

$$(u^{v_1}_{v_2} + u^{v_1}_{w_2})(u^{w_1}_{v_2} + u^{w_1}_{w_2}) = \underbrace{u^{v_1}_{v_2} u^{w_1}_{v_2}}_{0} + u^{v_1}_{v_2} u^{w_1}_{w_2} + u^{v_1}_{w_2} u^{w_1}_{v_2} + \underbrace{u^{v_1}_{w_2} u^{w_1}_{w_2}}_{0} = u^{v_1}_{v_2} u^{w_1}_{w_2} + u^{v_1}_{w_2} u^{w_1}_{v_2}.$$

On the other hand, Definition 4.3.1 yields

$$\begin{aligned}(u^{v_1}_{v_2} + u^{v_1}_{w_2})(u^{w_1}_{v_2} + u^{w_1}_{w_2}) &= \sum_{\substack{v_3 \in V \\ \{v_1, v_3\} \in E}} u^{\{v_1, v_3\}}_{\{v_2, w_2\}} \sum_{\substack{v_4 \in V \\ \{w_1, v_4\} \in E}} u^{\{w_1, v_4\}}_{\{v_2, w_2\}} \\ &= \sum_{\substack{v_3, v_4 \in V \\ \{v_1, v_3\}, \{w_1, v_4\} \in E}} \underbrace{u^{\{v_1, v_3\}}_{\{v_2, w_2\}} u^{\{w_1, v_4\}}_{\{v_2, w_2\}}}_{\delta_{\{v_1, v_3\}\{w_1, v_4\}} u^{\{v_1, v_3\}}_{\{v_2, w_2\}}} \\ &= u^{\{v_1, w_1\}}_{\{v_2, w_2\}},\end{aligned}$$

where $\{v_1, v_3\} = \{w_1, v_4\}$ and $v_1 \neq w_1$ implies $v_3 = w_1$ in the last step. Thus,

$$u^{\{v_1, w_1\}}_{\{v_2, w_2\}} = (u^{v_1}_{v_2} + u^{v_1}_{w_2})(u^{w_1}_{v_2} + u^{w_1}_{w_2}) = u^{v_1}_{v_2} u^{w_1}_{w_2} + u^{v_1}_{w_2} u^{w_1}_{v_2}.$$

Similarly, we compute

$$(u^{v_1}_{v_2} + u^{w_1}_{v_2})(u^{v_1}_{w_2} + u^{w_1}_{w_2}) = u^{v_1}_{v_2} u^{w_1}_{w_2} + u^{w_1}_{v_2} u^{v_1}_{w_2}$$

and

$$(u^{v_1}_{v_2} + u^{w_1}_{v_2})(u^{v_1}_{w_2} + u^{w_1}_{w_2}) = \sum_{\substack{v_3, v_4 \in V \\ \{v_1, v_3\}, \{v_2, v_4\} \in E}} \underbrace{u^{\{v_1, w_1\}}_{\{v_2, v_3\}} u^{\{v_1, w_1\}}_{\{w_2, v_4\}}}_{\delta_{\{v_2, v_3\}\{w_2, v_4\}} u^{\{v_1, w_1\}}_{\{v_2, v_3\}}} = u^{\{v_1, w_1\}}_{\{v_2, w_2\}}$$

using $v_2 \neq w_2$ and Definition 4.3.1. Therefore,

$$u^{\{v_1, w_1\}}_{\{v_2, w_2\}} = (u^{v_1}_{v_2} + u^{w_1}_{v_2})(u^{v_1}_{w_2} + u^{w_1}_{w_2}) = u^{v_1}_{v_2} u^{w_1}_{w_2} + u^{w_1}_{v_2} u^{v_1}_{w_2}.$$

□

In addition to the previous lemmas, we need the following proposition for general hypergraphs. It states that $u^v_w = 0$ if the vertices v and w are contained in a different number of edges. This proposition will also be used in Section 4.4 when computing the quantum automorphism groups of multigraphs.

Note that this type of relation seems to be useful when computing quantum symmetries of concrete hypergraphs. For example, the similar relation $u^i_j u^k_\ell = 0$ for $d(i, k) \neq d(j, \ell)$ was used in [74, 75] to compute quantum symmetries of classical graphs, where d denotes the distance between vertices.

Proposition 4.3.3. *Let $\Gamma := (V, E)$ be a hypergraph, and denote by u the fundamental representation of* $\mathrm{Aut}^+(\Gamma)$. *Define*

$$N_s(v) := |\{e \in E \mid v \in s(e)\}| \qquad \forall v \in V.$$

Then

$$N_s(v) \cdot u^v_w = u^v_w \cdot N_s(w) \qquad \forall v, w \in V.$$

In particular, if $N_s(v) \neq N_s(w)$*, then* $u^v_w = 0$*. The statement also holds for the range map* r*.*

Proof. Let $v \in V$. By summing both sides of $(A_s u_E)^v_e = (u_V A_s)^v_e$ over all $e \in E$, we obtain

$$\sum_{e \in E} \sum_{f \in E} (A_s)^v_f u^f_e = \sum_{f \in E} (A_s)^v_f \underbrace{\left(\sum_{e \in E} u^f_e \right)}_{1} = \sum_{f \in E} (A_s)^v_f = N_s(v),$$

$$\sum_{e \in E} \sum_{w \in V} u^v_w (A_s)^w_e = \sum_{w \in V} u^v_w \left(\sum_{e \in E} (A_s)^w_e \right) = \sum_{w \in V} u^v_w \cdot N_s(w).$$

Thus,

$$N_s(v) = \sum_{w \in V} u^v_w \cdot N_s(w),$$

which implies

$$N_s(v) \cdot u^v_w = u^v_w \cdot N_s(v) = \sum_{x \in V} \underbrace{u^v_w u^v_x}_{\delta_{wx} u^v_w} \cdot N_s(x) = u^v_w \cdot N_s(w) \qquad \forall v, w \in V.$$

In particular, if $N_s(v) \neq N_s(w)$, then

$$N_s(v) \cdot u^v_w = u^v_w \cdot N_s(w) \iff (N_s(v) - N_s(w)) \cdot u^v_w = 0 \iff u^v_w = 0.$$

The corresponding statement and the proof for the range map r are obtained by replacing s with r. □

We can now show that in the case of simple graphs, our quantum automorphism group coincides with Bichon's quantum automorphism group when we identify the magic unitary u_V with the fundamental representation of $\mathrm{Aut}^+_{\mathrm{Bic}}(\Gamma)$.

Theorem 4.3.4. *Let* Γ *be a simple graph. Then* $\mathrm{Aut}^+(\Gamma) = \mathrm{Aut}^+_{\mathrm{Bic}}(\Gamma)$.

Proof. Denote by u the fundamental representation of $\mathrm{Aut}^+(\Gamma)$ and by $\widehat{u}$ the fundamental representation of $\mathrm{Aut}^+_{\mathrm{Bic}}(\Gamma)$. Furthermore, we define the elements

$$\widehat{u}^{\{v_1,w_1\}}_{\{v_2,w_2\}} := \widehat{u}^{v_1}_{v_2} \widehat{u}^{w_1}_{w_2} + \widehat{u}^{v_1}_{w_2} \widehat{u}^{w_1}_{v_2} \qquad \forall \{v_1, w_1\}, \{v_2, w_2\} \in E,$$

and the corresponding matrices $\widehat{u}_V := (\widehat{u}^v_w)_{v,w \in V}$ and $\widehat{u}_E := (\widehat{u}^e_f)_{e,f \in E}$. Note that the elements $\widehat{u}^{\{v_1,w_1\}}_{\{v_2,w_2\}}$ is well-defined because

$$\widehat{u}^{v_1}_{v_2} \widehat{u}^{w_1}_{w_2} = \widehat{u}^{w_1}_{w_2} \widehat{u}^{v_1}_{v_2} \qquad \forall \{v_1, w_1\}, \{v_2, w_2\} \in E$$

by Definition 4.1.1.

Step 1. First, we construct the unital $*$-homomorphism

$$\begin{aligned} \phi\colon C(\mathrm{Aut}^+(\Gamma)) &\to C(\mathrm{Aut}^+_{\mathrm{Bic}}(\Gamma)), \\ u^v_w &\mapsto \widehat{u}^v_w && \forall v, w \in V, \\ u^e_f &\mapsto \widehat{u}^e_f && \forall e, f \in E \end{aligned}$$

using the universal property of $C(\mathrm{Aut}^+(\Gamma))$. Therefore, we must show that $\widehat{u}_V$ and $\widehat{u}_E$ are magic unitaries that are intertwined by $A_s = A_r$. By Definition 4.1.1, the matrix $\widehat{u}_V$ is a magic unitary, and we compute

$$\begin{aligned} \left(\widehat{u}^{\{v_1,w_1\}}_{\{v_2,w_2\}}\right)^* &= (\widehat{u}^{v_1}_{v_2}\widehat{u}^{w_1}_{w_2})^* + (\widehat{u}^{v_1}_{w_2}\widehat{u}^{w_1}_{v_2})^* \\ &= \widehat{u}^{w_1}_{w_2}\widehat{u}^{v_1}_{v_2} + \widehat{u}^{w_1}_{v_2}\widehat{u}^{v_1}_{w_2} = \widehat{u}^{v_1}_{v_2}\widehat{u}^{w_1}_{w_2} + \widehat{u}^{v_1}_{w_2}\widehat{u}^{w_1}_{v_2} = \widehat{u}^{\{v_1,w_1\}}_{\{v_2,w_2\}} \end{aligned}$$

for all $\{v_1, w_1\}, \{v_2, w_2\} \in E$. Similarly, we compute

$$\begin{aligned} \left(\widehat{u}^{\{v_1,w_1\}}_{\{v_2,w_2\}}\right)^2 &= (\widehat{u}^{v_1}_{v_2}\widehat{u}^{w_1}_{w_2} + \widehat{u}^{v_1}_{w_2}\widehat{u}^{w_1}_{v_2})(\widehat{u}^{v_1}_{v_2}\widehat{u}^{w_1}_{w_2} + \widehat{u}^{v_1}_{w_2}\widehat{u}^{w_1}_{v_2}) \\ &= \widehat{u}^{v_1}_{v_2}\widehat{u}^{w_1}_{w_2}\widehat{u}^{v_1}_{v_2}\widehat{u}^{w_1}_{w_2} + \widehat{u}^{v_1}_{w_2}\widehat{u}^{w_1}_{v_2}\widehat{u}^{v_1}_{w_2}\widehat{u}^{w_1}_{v_2} \\ &= (\widehat{u}^{v_1}_{v_2})^2(\widehat{u}^{w_1}_{w_2})^2 + (\widehat{u}^{v_1}_{w_2})^2(\widehat{u}^{w_1}_{v_2})^2 \\ &= \widehat{u}^{v_1}_{v_2}\widehat{u}^{w_1}_{w_2} + \widehat{u}^{v_1}_{w_2}\widehat{u}^{w_1}_{v_2} \\ &= \widehat{u}^{\{v_1,w_1\}}_{\{v_2,w_2\}} \end{aligned}$$

for all $\{v_1, w_1\}, \{v_2, w_2\} \in E$, where we additionally use the fact that

$$\widehat{u}^{w_1}_{w_2}\widehat{u}^{v_1}_{w_2} = 0, \quad \widehat{u}^{w_1}_{v_2}\widehat{u}^{v_1}_{v_2} = 0,$$

since $v_1 \neq w_1$. Next, show that the rows and columns of $\widehat{u}_E$ sum to 1. Observe that

$$\sum_{\{v_1,w_1\}\in E} \widehat{u}^{v_1}_{v_2}\widehat{u}^{w_1}_{w_2} = \frac{1}{2} \sum_{\substack{v_1,w_1\in V \\ \{v_1,w_1\}\in V}} \widehat{u}^{v_1}_{v_2}\widehat{u}^{w_1}_{w_2} = \frac{1}{2} \qquad \forall \{v_2, w_2\} \in E$$

by Definition 4.1.3. Therefore,

$$\sum_{\{v_1,w_1\}\in E} \widehat{u}^{\{v_1,w_1\}}_{\{v_2,w_2\}} = \sum_{\{v_1,w_1\}\in E} \widehat{u}^{v_1}_{v_2}\widehat{u}^{w_1}_{w_2} + \sum_{\{v_1,w_1\}\in E} \widehat{u}^{v_1}_{w_2}\widehat{u}^{w_1}_{v_2} = \frac{1}{2} + \frac{1}{2} = 1$$

for all $\{v_2, w_2\} \in E$. Similarly,

$$\sum_{\{v_2,w_2\}\in E} \widehat{u}^{\{v_1,w_1\}}_{\{v_2,w_2\}} = \frac{1}{2} + \frac{1}{2} = 1 \qquad \forall \{v_1, w_1\} \in E.$$

Hence, $\widehat{u}_E$ is a magic unitary. It remains to show that $A_s\widehat{u}_E = \widehat{u}_V A_s$. But this follows from Definition 4.3.1 and $A\widehat{u}_V = \widehat{u}_V A$ because

$$\begin{aligned}
\sum_{\substack{v_2\in V\\ \{v_0,v_2\}\in E}} \widehat{u}^{\{v_0,v_2\}}_{\{v_1,w_1\}} &= \sum_{\substack{v_2\in V\\ \{v_0,v_2\}\in E}} \left(\widehat{u}^{v_0}_{v_1}\widehat{u}^{v_2}_{w_1} + \widehat{u}^{v_0}_{w_1}\widehat{u}^{v_2}_{v_1}\right)\\
&= \widehat{u}^{v_0}_{v_1} \sum_{\substack{v_2\in V\\ \{v_0,v_2\}\in E}} \widehat{u}^{v_2}_{w_1} + \widehat{u}^{v_0}_{w_1} \sum_{\substack{v_2\in V\\ \{v_0,v_2\}\in E}} \widehat{u}^{v_2}_{v_1}\\
&= \widehat{u}^{v_0}_{v_1} \sum_{v_2\in V} A^{v_0}_{v_2}\widehat{u}^{v_2}_{w_1} + \widehat{u}^{v_0}_{w_1} \sum_{v_2\in V} A^{v_0}_{v_2}\widehat{u}^{v_2}_{v_1}\\
&= \widehat{u}^{v_0}_{v_1} \sum_{v_2\in V} \widehat{u}^{v_0}_{v_2}A^{v_2}_{w_1} + \widehat{u}^{v_0}_{w_1} \sum_{v_2\in V} \widehat{u}^{v_0}_{v_2}A^{v_2}_{v_1}\\
&= \sum_{v_2\in V} \underbrace{\widehat{u}^{v_0}_{v_1}\widehat{u}^{v_0}_{v_2}}_{\delta_{v_1v_2}\widehat{u}^{v_0}_{v_1}} A_{v_2w_1} + \sum_{v_2\in V} \underbrace{\widehat{u}^{v_0}_{w_1}\widehat{u}^{v_0}_{v_2}}_{\delta_{w_1v_2}\widehat{u}^{v_0}_{w_1}} A_{v_2v_1}\\
&= \widehat{u}^{v_0}_{v_1} + \widehat{u}^{v_0}_{w_1}
\end{aligned}$$

for all $v_0 \in V$ and $\{v_1, w_1\} \in E$. Hence, the map ϕ exists.

Step 2. Next, we construct the inverse map

$$\psi\colon C(\mathrm{Aut}^+_{\mathrm{Bic}}(\Gamma)) \to C(\mathrm{Aut}^+(\Gamma)), \quad \widehat{u}^v_w \mapsto u^v_w \qquad \forall v, w \in V$$

using the universal property of $C(\mathrm{Aut}^+_{\mathrm{Bic}}(\Gamma))$. We must show that u_V satisfies the relations from Definition 4.1.1. By Definition 3.2.5, u_V is a magic unitary. To show that $Au_V = u_V A$, observe that

$$(A_sA_s^*)^v_w = \sum_{e\in E} (A_s)^v_e (A_s)^w_e = |\{e \in E \mid v \in s(e) \wedge w \in s(e)\}| \qquad \forall v, w \in V.$$

Since Γ is a simple graph, each $e \in E$ contains exactly two elements, such that

$$(A_sA_s^*)^v_w = \begin{cases} 0 & \text{if } v \neq w,\ \{v,w\} \notin E,\\ 1 & \text{if } v \neq w,\ \{v,w\} \in E,\\ N_s(v) & \text{if } v = w,\end{cases}$$

where

$$N_s(v) := |\{e \in E \mid v \in s(e)\}|$$

as in Definition 4.3.3. Thus, $A_sA_s^* = A + T$ with $T \in B(\mathbb{C}^V)$ defined by

$$T^v_w = \begin{cases} 0 & \text{if } v \neq w,\\ N_s(v) & \text{if } v = w,\end{cases} \qquad \forall v, w \in V.$$

Definition 4.3.3 shows that $Tu_V = u_V T$, which implies

$$Au_V = A_sA_s^*u_V - Tu_V = A_su_EA_s^* - u_VT = u_VA_sA_s^* - u_VT = u_VA.$$

Finally, we must show that

$$u_{v_2}^{v_1} u_{w_2}^{w_1} = u_{w_2}^{w_1} u_{v_2}^{v_1} \qquad \forall \{v_1, w_1\}, \{v_2, w_2\} \in E.$$

But this follows directly from Definition 4.3.2 since

$$\begin{aligned} u_{v_2}^{v_1} u_{w_2}^{w_1} &= u_{\{v_2,w_2\}}^{\{v_1,w_1\}} - u_{w_2}^{v_1} u_{v_2}^{w_1} \\ &= u_{\{w_2,v_2\}}^{\{w_1,v_1\}} - u_{w_2}^{v_1} u_{v_2}^{w_1} = u_{w_2}^{w_1} u_{v_2}^{v_1} + u_{w_2}^{v_1} u_{v_2}^{w_1} - u_{w_2}^{v_1} u_{v_2}^{w_1} = u_{w_2}^{w_1} u_{v_2}^{v_1}. \end{aligned}$$

Thus, the $*$-homomorphism ψ exists.

Step 3. By definition of ϕ and ψ, we have

$$\begin{aligned} u_w^v &\longleftrightarrow \widehat{u}_w^v \\ u_{\{v_2,w_2\}}^{\{v_1,w_1\}} = u_{v_2}^{v_1} u_{w_2}^{w_1} + u_{w_2}^{v_1} u_{v_2}^{w_1} &\longleftrightarrow \widehat{u}_{v_2}^{v_1} \widehat{u}_{w_2}^{w_1} + \widehat{u}_{w_2}^{v_1} \widehat{u}_{v_2}^{w_1} \end{aligned}$$

for all $v, w \in V$ and $\{v_1, w_1\}, \{v_2, w_2\} \in E$, showing that both maps are indeed inverse. Furthermore, ψ is a morphism compact quantum group since

$$\Delta\big(\psi(\widehat{u}_w^v)\big) = \sum_{x \in V} u_x^v \otimes u_w^x = \sum_{x \in V} \psi(\widehat{u}_x^v) \otimes \psi(\widehat{u}_w^x) = (\psi \otimes \psi)\big(\Delta(\widehat{u}_w^v)\big)$$

for all $v, w \in V$. Hence, ψ is an isomorphism of compact quantum groups, which shows that $\mathrm{Aut}^+(\Gamma)$ and $\mathrm{Aut}^+_{\mathrm{Bic}}(\Gamma)$ are isomorphic. □

4.4. Case of multigraphs

Finally, we consider the case of multigraphs as defined in Definition 3.1.4. Recall from Definition 3.1.12 that we can identify a multigraph $\Gamma := (V, E)$ with source map $s'\colon E \to V$ and range maps $r'\colon E \to V$ with a 1-uniform hypergraphs by defining new source and range maps $s\colon E \to 2^V$ and $r\colon E \to 2^V$ by

$$s(e) = \{s'(e)\}, \quad r(e) = \{r'(e)\} \qquad \forall e \in E.$$

Throughout the rest of this section, we identify the maps s' and r' with the maps s and r. In particular, we write $u_v^{s(e)}$ instead of $u_v^{s'(e)}$ for an edge $e \in E$ and a vertex $v \in V$.

The goal of this section is to show that our quantum automorphism group $\mathrm{Aut}^+(\Gamma)$ coincides with the quantum automorphism group $\mathrm{Aut}^+_{\mathrm{GH,Bic}}(\Gamma)$ by Goswami and Hossain for multigraphs without isolated vertices. We begin by reformulating the intertwiner relations $A_s u_E = u_V A_s$ and $A_r u_E = u_V A_r$ in the context of multigraphs.

Lemma 4.4.1. *Let $\Gamma := (V, E)$ be a multigraph and u denote the fundamental representation of $\mathrm{Aut}^+(\Gamma)$. Then the relations $A_s u_E = u_V A_s$ and $A_r u_E = u_V A_r$ are equivalent to*

$$\sum_{\substack{f \in E \\ s(f) = v}} u_f^e = u_v^{s(e)}, \qquad \sum_{\substack{f \in E \\ r(f) = v}} u_f^e = u_v^{r(e)} \qquad \forall v \in V,\, e \in E.$$

Proof. Consider the equation $A_s u_E = u_V A_s$, which is equivalent to $A_s^* u_V = u_E A_s^*$ by Definition 3.2.8. Let $e \in E$ and $v \in V$. Then a direct computation yields

$$(u_E A_s^*)_v^e = \sum_{\substack{f\in E\\ s(f)=v}} u_f^e, \qquad (A_s^* u_V)_v^e = \sum_{\substack{w\in V\\ s(e)=w}} u_v^w = u_v^{s(e)}$$

Therefore, $A_s^* u_V = u_E A_s^*$ is equivalent to

$$\sum_{\substack{f\in E\\ s(f)=v}} u_f^e = u_v^{s(e)} \qquad \forall v \in V,\, e \in E.$$

The corresponding statement for the range map r follows by replacing s with r in the previous computation. $\square$

To show that our quantum automorphism group coincides with $\mathrm{Aut}^+_{\mathrm{GH,Bic}}(\Gamma)$, we will identify the magic unitary u_E with the fundamental representation of $\mathrm{Aut}^+_{\mathrm{GH,Bic}}(\Gamma)$. However, we need to construct additional elements in $C(\mathrm{Aut}^+_{\mathrm{GH,Bic}}(\Gamma))$ corresponding to the magic unitary u_V.

Definition 4.4.2. Let $\Gamma := (V, E)$ be a multigraph without isolated vertices, and denote by u the fundamental representation of $\mathrm{Aut}^+_{\mathrm{GH,Bic}}(\Gamma)$. Define the elements u_w^v for all $v, w \in V$ as follows:

$$u_w^v = \begin{cases} \displaystyle\sum_{\substack{f\in E\\ s(f)=w}} u_f^e & \text{if there exists } e \in E \text{ with } s(e) = v,\\ \displaystyle\sum_{\substack{f\in E\\ r(f)=w}} u_f^e & \text{if there exists } e \in E \text{ with } r(e) = v. \end{cases}$$

Next, we show that these elements are well-defined and define a quantum permutation of the vertices.

Lemma 4.4.3. *Let $\Gamma := (V, E)$ be a multigraph without isolated vertices. Then the elements u_w^v in Definition 4.4.2 are well-defined. Furthermore, the matrix $u_V := (u_w^v)_{v,w\in V}$ is a magic unitary.*

Proof. First, note that at least one case in Definition 4.4.2 applies to each vertex $v \in V$ since Γ has no isolated vertices. Furthermore, each case is independent of the choice of edge $e \in E$ by Relation 2 in Definition 4.1.5. To show that overlapping cases are well-defined, consider $v, w \in V$ such that v is neither a source nor a sink. If w is neither a source nor a sink, then both cases agree by Relation 4. On the other hand, if w is a source or a sink, then both cases yield $u_w^v = 0$ because the sum is empty in one case, while each $u_f^e = 0$ by Relation 3 in the other case. Thus, the elements u_w^v are well-defined.

Next, we show that the matrix $u_V := (u^v_w)_{v,w\in V}$ is a magic unitary. Let $v, w \in V$, and assume without loss of generality that there exists an edge $e \in E$ with $v = s(e)$. Using Definition 4.4.1 and the fact that $(u^e_f)_{e,f\in E}$ is a magic unitary, we compute

$$(u^v_w)^* = \sum_{\substack{f\in E\\ s(f)=w}} (u^e_f)^* = \sum_{\substack{f\in E\\ s(f)=w}} u^e_f = u^v_w,$$

$$(u^v_w)^2 = \sum_{\substack{f_1\in E\\ s(f_1)=w}} \sum_{\substack{f_2\in E\\ s(f_2)=w}} \underbrace{u^e_{f_1} u^e_{f_2}}_{\delta_{f_1 f_2} u^e_{f_1}} = \sum_{\substack{f\in E\\ s(f)=w}} u^e_f = u^v_w$$

and

$$\sum_{x\in V} u^v_x = \sum_{x\in V} \sum_{\substack{f\in E\\ s(f)=x}} u^e_f = \sum_{f\in E} \underbrace{\sum_{\substack{x\in V\\ s(f)=x}} u^e_f}_{u^e_f} = \sum_{f\in E} u^e_f = 1.$$

To show that the rows of u_V sum to 1, we use an argument from the proof of [89, Theorem 3.1]. By our previous computation, we have

$$(u_V u_V^*)^v_w = \sum_{x\in V} \underbrace{u^v_x (u^w_x)^*}_{\delta_{vw} u^v_x} = \delta_{vw} \sum_{x\in V} u^v_x = \delta_{vw} \qquad \forall v, w \in V.$$

Hence, u_V is right-invertible with $u_V u_V^* = 1$. If we show that u_V is a representation of $\mathrm{Aut}^+_{\mathrm{GH,Bic}}(\Gamma)$, i.e.

$$\Delta(u^v_w) = \sum_{x\in V} u^v_x \otimes u^x_w \qquad \forall v, w \in V,$$

then [95, Proposition 3.2] implies that u_V is also left-invertibe with $u_V^* u_V = 1$. Thus,

$$1 = (u_V^* u_V)^v_v = \sum_{x\in V} (u^x_v)^* u^x_v = \sum_{x\in V} u^x_v \qquad \forall v \in V.$$

Therefore, it remains to show that u_V is a representation of $\mathrm{Aut}^+_{\mathrm{GH,Bic}}(\Gamma)$. Let $v, w \in V$, and assume without loss of generality that there exists an edge $e \in E$ with $v = s(e)$. Then

$$\begin{aligned}\Delta(u^v_w) &= \sum_{\substack{f\in E\\ s(f)=w}} \Delta(u^e_f) = \sum_{\substack{f\in E\\ s(f)=w}} \sum_{g\in E} u^e_g \otimes u^g_f = \sum_{g\in E} u^e_g \otimes \Bigg(\sum_{\substack{f\in E\\ s(f)=w}} u^g_f \Bigg)\\ &= \sum_{g\in E} u^e_g \otimes u^{s(g)}_w,\end{aligned}$$

and on the other hand

$$\sum_{x\in V} u^v_x \otimes u^x_w = \sum_{x\in V} \sum_{\substack{f\in E\\ s(f)=x}} u^e_f \otimes u^x_w = \sum_{f\in E} \sum_{\substack{x\in V\\ s(f)=x}} u^e_f \otimes u^x_w = \sum_{f\in E} u^e_f \otimes u^{s(f)}_w.$$

Thus,

$$\Delta(u^v_w) = \sum_{x \in V} u^v_x \otimes u^x_w \qquad \forall v, w \in V.$$

□

Using the previous lemmas, we can show that our quantum automorphism group agrees coincides with the quantum automorphism group of Goswami and Hossain for multigraphs without isolated vertices.

Theorem 4.4.4. *Let* Γ *be a multigraph without isolated vertices. Then* $\mathrm{Aut}^+(\Gamma) = \mathrm{Aut}^+_{\mathrm{GH,Bic}}(\Gamma)$.

Proof. Let $\Gamma := (V, E)$. Denote by u the fundamental representation of $\mathrm{Aut}^+(\Gamma)$ and by $\widehat{u}$ the fundamental representation of $\mathrm{Aut}^+_{\mathrm{GH,Bic}}(\Gamma)$. Furthermore, define the elements $\widehat{u}^v_w$ for all $v, w \in V$ as in Definition 4.4.2:

$$\widehat{u}^v_w = \begin{cases} \sum\limits_{\substack{f \in E \\ s(f)=w}} \widehat{u}^e_f & \text{if there exists } e \in E \text{ with } s(e) = v, \\ \sum\limits_{\substack{f \in E \\ r(f)=w}} \widehat{u}^e_f & \text{if there exists } e \in E \text{ with } r(e) = v. \end{cases}$$

We begin by constructing the unital $*$-homomorphism

$$\begin{aligned} \phi\colon C(\mathrm{Aut}^+(\Gamma)) &\to C(\mathrm{Aut}^+_{\mathrm{GH,Bic}}(\Gamma)), \\ u^v_w &\mapsto \widehat{u}^v_w && \forall v, w \in V, \\ u^e_f &\mapsto \widehat{u}^e_f && \forall e, f \in E \end{aligned}$$

using the universal property of $C(\mathrm{Aut}^+(\Gamma))$. Therefore, we must show that the matrices

$$\widehat{u}_V := (\widehat{u}^v_w)_{v,w \in V}, \quad \widehat{u}_E := (\widehat{u}^e_f)_{e,f \in E}$$

satisfy the relations of Definition 3.2.5. The matrix $\widehat{u}_E$ is a magic unitary by Relation 1 of Definition 4.1.5, and $\widehat{u}_V$ is a magic unitary by Definition 4.4.3. Next, consider the relation $A_s \widehat{u}_E = \widehat{u}_V A_s$, which is equivalent to $A_s^* \widehat{u}_V = \widehat{u}_E A_s^*$ by Definition 3.2.8. This relation follows directly since

$$(A_s^* \widehat{u}_V)^e_v = \sum_{w \in V} (A_s^*)^e_w \widehat{u}^w_v = \widehat{u}^{s(e)}_v = \sum_{\substack{f \in E \\ s(f)=v}} \widehat{u}^e_f = \sum_{f \in E} \widehat{u}^e_f (A_s^*)^f_v = (\widehat{u}_E A_s^*)^e_v$$

for all $e \in E$ and $v \in V$. Similarly, one shows that $A_r \widehat{u}_E = \widehat{u}_V A_r$. Thus, the $*$-homomorphism ϕ exists. Next, we construct the inverse $*$-homomorphism

$$\psi\colon C(\mathrm{Aut}^+_{\mathrm{GH,Bic}}(\Gamma)) \to C(\mathrm{Aut}^+(\Gamma)), \quad \widehat{u}^e_f \mapsto u^e_f \qquad \forall e, f \in E$$

using the universal property of $C(\mathrm{Aut}^+_{\mathrm{GH,Bic}}(\Gamma))$. Thus, we must show that the entries of u_E satisfy the relations in Relation 4.1.5. By Definition 3.2.5, u_E is a magic unitary. Furthermore, we can use Definition 4.4.1 to compute

$$\sum_{\substack{f\in E\\ s(f)=v}} u_f^{e_1} = u_v^{s(e_1)} = u_v^{s(e_2)} = \sum_{\substack{f\in E\\ s(f)=v}} u_f^{e_2},$$

$$\sum_{\substack{f\in E\\ r(f)=v}} u_f^{e_1} = u_v^{r(e_1)} = u_v^{r(e_2)} = \sum_{\substack{f\in E\\ r(f)=v}} u_f^{e_2}$$

for all $v \in V$ and $e_1, e_2 \in E$ with $s(e_1) = s(e_2)$ or $r(e_1) = r(e_2)$ respectively. Thus, Relation 2 is satisfied. Next, consider Relation 3, and let $e, f \in E$. If $s(e)$ is neither a source nor a sink, and $s(f)$ is a source, then $N_r(s(e)) > 0$ and $N_r(s(f)) = 0$ in the notation of Definition 4.3.3. Therefore, Definition 4.3.3 implies that $u_{s(f)}^{s(e)} = 0$, which yields

$$0 = u_{s(f)}^{s(e)} u_f^e = \sum_{\substack{g\in E\\ s(g)=s(f)}} \underbrace{u_g^e u_f^e}_{\delta_{gf} u_f^e} = u_f^e.$$

Similarly, one shows that $u_{s(f)}^{s(e)} = 0$ if $r(f)$ is a sink. Hence, Relation 3 is satisfied. Finally, Definition 4.4.1 implies

$$\sum_{\substack{f\in E\\ s(f)=v}} u_f^{e_1} = u_v^{s(e_1)} = u_v^{r(e_2)} = \sum_{\substack{f\in E\\ r(f)=v}} u_f^{e_2}$$

for all $v \in V$ and $e_1, e_2 \in E$ with $s(e_1) = r(e_2)$. Thus, Relation 4 holds, and the $*$-homomorphism ϕ exists. The $*$-homomorphisms ϕ and ψ are indeed inverse since

$$\begin{aligned} u_f^e &\longleftrightarrow \widehat{u}_f^e &\qquad \forall e, f \in E,\\ u_w^v &\longleftrightarrow \widehat{u}_w^v &\qquad \forall v, w \in V \end{aligned}$$

by Definition 4.4.1 and Definition 4.4.2. Furthermore, ψ is a morphism of compact quantum groups because

$$\Delta\big(\psi(\widehat{u}_f^e)\big) = \sum_{g\in E} u_g^e \otimes u_f^g = \sum_{g\in E} \psi(\widehat{u}_g^e) \otimes \psi(\widehat{u}_f^g) = (\psi\otimes\psi)\big(\Delta(\widehat{u}_f^e)\big)$$

for all $e, f \in E$. Hence, $\mathrm{Aut}^+(\Gamma) = \mathrm{Aut}^+_{\mathrm{GH,Bic}}(\Gamma)$. □

5. Actions on hypergraph C^*-algebras

Classical graph C^*-algebras are a well-studied class of C^*-algebras that includes many concrete examples such as matrix algebras, the algebra of continuous functions on the circle and the Cuntz algebras. They are defined in terms of an underlying graph Γ and generalize Cuntz-Krieger algebras from [24]. In this context, Schmidt and Weber [77] showed that Banica's quantum automorphism group $\mathrm{Aut}^+_{\mathrm{Ban}}(\Gamma)$ of a finite graph Γ acts maximally on the corresponding graph C^*-algebra $C^*(\Gamma)$.

Recently, Trieb, Weber and Zenner [84] introduced hypergraph C^*-algebras by replacing the graph in the definition of a graph C^*-algebra with a hypergraph. In this chapter, we generalize the result of Schmidt and Weber to this new class of C^*-algebras by showing that our quantum automorphism group $\mathrm{Aut}^+(\Gamma)$ acts naturally on the corresponding hypergraph C^*-algebra $C^*(\Gamma)$. In particular, we recover the action of Schmidt and Weber as special case. Although our action is generally not maximal in the sense of Schmidt and Weber, we are still able to obtain maximality when additionally considering the action of our quantum automorphism group on the opposite and dual hypergraph C^*-algebras $C^*(\Gamma^{\mathrm{op}})$ and $C^*(\Gamma^*)$.

5.1. Graph and hypergraph C^*-algebras

We begin by defining graph C^*-algebras for finite directed graphs. See [70] for further information on graph C^*-algebras and the more general case of infinite graphs.

Definition 5.1.1. Let $\Gamma := (V, E)$ be a directed graph. The *graph C^*-algebra* $C^*(\Gamma)$ is the universal C^*-algebra generated by mutually orthogonal projections p_v for all $v \in V$ and partial isometries s_e with orthogonal ranges for all $e \in E$ such that

1. $s^*_{(v,w)} s_{(v,w)} = p_w$ for all $(v, w) \in E$,
2. $s_{(v,w)} s^*_{(v,w)} \leq p_v$ for all $(v, w) \in E$,
3. $p_v \leq \displaystyle\sum_{(v,w)\in E} s_{(v,w)} s^*_{(v,w)}$ for all $v \in V$ that are not sinks.

By replacing the underlying graph in the previous definition with a hypergraph, Trieb, Weber and Zenner [84] arrived at the following definition of a hypergraph C^*-algebra.

Definition 5.1.2. Let $\Gamma := (V, E)$ be a hypergraph. The *hypergraph C^*-algebra* $C^*(\Gamma)$ is the universal C^*-algebra generated by mutually orthogonal projections p_v for all $v \in V$ and partial isometries s_e for all $e \in E$ such that

1. $s_e^* s_f = \delta_{ef} \sum_{\substack{v \in V \\ v \in r(e)}} p_v$ for all $e, f \in E$,

2. $s_e s_e^* \leq \sum_{\substack{v \in V \\ v \in s(e)}} p_v$ for all $e \in E$,

3. $p_v \leq \sum_{\substack{e \in E \\ v \in s(e)}} s_e s_e^*$ for all $v \in V$ that are not sinks.

If a directed graph is regarded as a hypergraph as in Definition 3.1.11, then the corresponding hypergraph C^*-algebra coincides with the classical graph C^*-algebra. However, hypergraph C^*-algebras also include new examples of non-nuclear C^*-algebras. For further details on the nuclearity of hypergraph C^*-algebras, see the recent work by Schäfer and Weber [78] where the nuclearity of hypergraph C^*-algebras is characterized in terms of minors of the underlying hypergraph.

5.2. Construction of the action

In [77], Schmidt and Weber showed that Banica's quantum automorphism group $\mathrm{Aut}^+_{\mathrm{Ban}}(\Gamma)$ acts maximally on the corresponding graph C^*-algebra $C^*(\Gamma)$ as follows.

Proposition 5.2.1. *Let $\Gamma := (V, E)$ be a directed graph. Then $\mathrm{Aut}^+_{\mathrm{Ban}}(\Gamma)$ acts maximally on $C^*(\Gamma)$ via the action*

$$\alpha\colon C^*(\Gamma) \to C^*(\Gamma) \otimes C(\mathrm{Aut}^+_{\mathrm{Ban}}(\Gamma))$$

defined by

$$\begin{aligned} \alpha(p_v) &= \sum_{w \in V} u^v_w \otimes p_w && \forall v \in V, \\ \alpha(s_{(v_1,w_1)}) &= \sum_{(v_2,w_2) \in E} u^{v_1}_{v_2} u^{w_1}_{w_2} \otimes s_{(v_2,w_2)} && \forall (v_1, w_1) \in E. \end{aligned}$$

Note that we have reversed the order of the tensor legs in the previous proposition to be consistent with the notation in [89] and our definition of an action on a C^*-algebra. See also [52] for further discussions on actions of quantum groups on graph C^*-algebra.

In the following, we generalize this result to hypergraphs by showing that the quantum automorphism group $\mathrm{Aut}^+(\Gamma)$ of a hypergraph Γ acts faithfully on the hypergraph C^*-algebra $C^*(\Gamma)$ from Definition 5.1.2. Here, the action is given by

$$\alpha(p_v) = \sum_{w \in V} p_w \otimes u^w_v \quad \alpha(s_e) = \sum_{f \in E} s_f \otimes u^f_e \qquad \forall v \in V,\ e \in E.$$

We first construct the underlying $*$-homomorphism α, before we show in Definition 5.2.3 that it defines an action of $\mathrm{Aut}^+(\Gamma)$ on $C^*(\Gamma)$.

Lemma 5.2.2. *Let $\Gamma := (V, E)$ be a hypergraph. Denote by u the fundamental representation of $\mathrm{Aut}^+(\Gamma)$. Then there exists a unital $*$-homomorphism*

$$\alpha\colon C^*(\Gamma) \to C^*(\Gamma) \otimes C(\mathrm{Aut}^+(\Gamma))$$

defined by

$$\alpha(p_v) = \sum_{w \in V} p_w \otimes u_v^w \quad \alpha(s_e) = \sum_{f \in E} s_f \otimes u_e^f \qquad \forall v \in V,\, e \in E.$$

Proof. We use the universal property of $C^*(\Gamma)$ to construct the map α. Thus, we must show that $\alpha(p_v)$ are orthogonal projections, $\alpha(s_e)$ are partial isometries and both satisfy the relations from Definition 5.1.2. Recall the magic unitary relations of u_V and u_E, and the intertwiner relations of A_s and A_r from Definition 3.2.7. Then the elements $\alpha(p_v)$ are orthogonal projections because

$$\begin{aligned}\alpha(p_{v_1})\alpha(p_{v_2}) &= \sum_{w_1, w_2 \in V} \underbrace{p_{w_1} p_{w_2}}_{\delta_{w_1 w_2} p_{w_1}} \otimes u_{v_1}^{w_1} u_{v_2}^{w_2} \\ &= \sum_{w \in V} p_w \otimes \underbrace{u_{v_1}^w u_{v_2}^w}_{\delta_{v_1 v_2} u_{v_1}^w} = \delta_{v_1 v_2} \sum_{w \in V} p_w \otimes u_{v_1}^w = \delta_{v_1 v_2}\, \alpha(p_{v_1})\end{aligned}$$

for all $v_1, v_2 \in V$ and

$$\alpha(p_v)^* = \sum_{w \in V} p_w^* \otimes (u_v^w)^* = \sum_{w \in V} p_w \otimes u_v^w = \alpha(p_v)$$

for all $v \in V$. Similarly, we show that $\alpha(s_e)$ are partial isometries, since

$$\begin{aligned}\alpha(s_e)\alpha(s_e)^*\alpha(s_e) &= \sum_{f_1, f_2, f_3 \in E} s_{f_1} s_{f_2} s_{f_3} \otimes \underbrace{u_e^{f_1} (u_e^{f_2})^* u_e^{f_3}}_{\delta_{f_1 f_2} \delta_{f_1 f_3} u_e^{f_1}} \\ &= \sum_{f \in E} \underbrace{s_f s_f^* s_f}_{s_f} \otimes u_e^f = \sum_{f \in E} s_f \otimes u_e^f = \alpha(s_e)\end{aligned}$$

for all $e \in E$. Next, consider Relation 1 from Definition 5.1.2, which states that

$$s_e^* s_f = \delta_{ef} \sum_{\substack{v \in V \\ v \in r(e)}} p_v \qquad \forall e, f \in E.$$

By applying α to the left-hand side, we obtain

$$\begin{aligned}\alpha(s_e)^*\alpha(s_f) &= \sum_{g_1, g_2 \in E} \underbrace{s_{g_1}^* s_{g_2}}_{\text{Rel. 1}} \otimes (u_e^{g_1})^* u_f^{g_2} \\ &= \sum_{g \in E} \sum_{\substack{v \in V \\ v \in r(g)}} p_v \otimes \underbrace{u_e^g u_f^g}_{\delta_{ef} u_e^g} = \delta_{ef} \sum_{v \in V} \sum_{\substack{g \in E \\ v \in r(g)}} p_v \otimes u_e^g.\end{aligned}$$

Using Definition 3.2.7, we have

$$\sum_{\substack{g\in E\\ v\in r(g)}} u_e^g = \sum_{\substack{w\in V\\ w\in r(e)}} u_w^v \qquad \forall v\in V,\, e\in E,$$

which allows us to rewrite $\alpha(s_e)^*\alpha(s_f)$ as

$$\begin{aligned}\alpha(s_e)^*\alpha(s_f) &= \delta_{ef}\sum_{v\in V}\sum_{\substack{w\in V\\ w\in r(e)}} p_v\otimes u_w^v = \delta_{ef}\sum_{\substack{w\in V\\ w\in r(e)}}\sum_{v\in V} p_v\otimes u_w^v\\ &= \delta_{ef}\sum_{\substack{w\in V\\ w\in r(e)}}\alpha(p_w).\end{aligned}$$

Thus, Relation 1 is satisfied. Next, consider Relation 2, which is given by

$$s_e s_e^* \le \sum_{\substack{v\in V\\ v\in s(e)}} p_v \qquad \forall e\in E.$$

By applying α to the left-hand side, we obtain

$$\alpha(s_e)\alpha(s_e)^* = \sum_{f_1,f_2\in E} s_{f_1}s_{f_2}^*\otimes \underbrace{u_e^{f_1}(u_e^{f_2})^*}_{\delta_{f_1f_2}u_e^{f_1}} = \sum_{f\in E} s_f s_f^*\otimes u_e^f.$$

Since each $u_e^f \ge 0$, we can use Definition 2.1.16 with Relation 2 to obtain

$$\alpha(s_e)\alpha(s_e)^* \le \sum_{f\in E}\Bigg(\sum_{\substack{v\in V\\ v\in s(f)}} p_v\Bigg)\otimes u_e^f = \sum_{v\in V}\sum_{\substack{f\in E\\ v\in s(f)}} p_v\otimes u_e^f.$$

By Definition 3.2.7, we have

$$\sum_{\substack{f\in E\\ v\in s(f)}} u_e^f = \sum_{\substack{w\in V\\ w\in s(e)}} u_v^w \qquad \forall v\in V,\, e\in E,$$

such that

$$\alpha(s_e)\alpha(s_e)^* \le \sum_{v\in V}\sum_{\substack{w\in V\\ w\in s(e)}} p_v\otimes u_w^v = \sum_{\substack{w\in V\\ w\in s(e)}}\sum_{v\in V} p_v\otimes u_w^v = \sum_{\substack{w\in V\\ w\in s(e)}}\alpha(p_w).$$

Hence, Relation 2 is satisfied. Finally, consider Relation 3, which states that

$$p_v \le \sum_{\substack{e\in E\\ v\in s(e)}} s_e s_e^*$$

for all $v \in V$ that are not sinks. Let $v \in V$ be not a sink. By Definition 4.3.3, we have $u_w^v = 0$ for all $w \in V$ that are a sink, since $N_s(v) > 0$ if v is not a sink and $N_s(w) = 0$ if w is a sink. Therefore,

$$\alpha(p_v) = \sum_{w \in V} p_w \otimes u_v^w = \sum_{\substack{w \in V \\ w \neq \text{sink}}} p_w \otimes u_v^w.$$

Applying Definition 2.1.16 with Relation 3 yields

$$\alpha(p_v) \leq \sum_{\substack{w \in V \\ w \neq \text{sink}}} \left(\sum_{\substack{e \in E \\ w \in s(e)}} s_e s_e^* \right) \otimes u_v^w = \sum_{e \in E} \sum_{\substack{w \in V \\ w \in s(e)}} s_e s_e^* \otimes u_v^w,$$

where we used the fact that $u_w^v = 0$ if w is a sink. On the other hand, we have

$$\sum_{\substack{f \in E \\ v \in s(f)}} \alpha(s_f)\alpha(s_f)^* = \sum_{e_1, e_2 \in E} \sum_{\substack{f \in E \\ v \in s(f)}} s_{e_1} s_{e_2}^* \otimes \underbrace{u_f^{e_1} (u_f^{e_2})^*}_{\delta_{e_1 e_2} u_f^{e_1}} = \sum_{e \in E} \sum_{\substack{f \in E \\ v \in s(f)}} s_e s_e^* \otimes u_f^e.$$

By Definition 3.2.7, we have

$$\sum_{\substack{w \in V \\ w \in s(e)}} u_v^w = \sum_{\substack{f \in E \\ v \in s(f)}} u_f^e \qquad \forall v \in V,\, e \in E,$$

which implies

$$\alpha(p_v) \leq \sum_{e \in E} \sum_{\substack{w \in V \\ w \in s(e)}} s_e s_e^* \otimes u_v^w = \sum_{e \in E} \sum_{\substack{f \in E \\ v \in s(f)}} s_e s_e^* \otimes u_f^e = \sum_{\substack{f \in E \\ v \in s(f)}} \alpha(s_f)\alpha(s_f)^*.$$

□

Next, we show that the previous $*$-homomorphism α defines a faithful action in the sense of Definition 2.4.1 and Definition 2.4.2.

Theorem 5.2.3. *Let Γ be a hypergraph. Then $\mathrm{Aut}^+(\Gamma)$ acts faithfully on $C^*(\Gamma)$ via the map α from Definition 5.2.2.*

Proof. Let $\Gamma := (V, E)$. Define $\mathcal{B} \subseteq C^*(\Gamma)$ as the $*$-subalgebra generated by p_v for all $v \in V$ and p_e for all $e \in E$. Then $\mathcal{B}$ is dense in $C^*(\Gamma)$, and

$$\alpha(\mathcal{B}) \subseteq \mathcal{B} \otimes \mathcal{O}(\mathrm{Aut}^+(\Gamma))$$

by the definition of α. Next, let $v \in V$. Then

$$(\alpha \otimes \mathrm{id})(\alpha(p_v)) = (\mathrm{id} \otimes \Delta)(\alpha(p_v))$$

since

$$\sum_{w\in V}\alpha(p_w)\otimes u_v^w=\sum_{w_1,w_2\in V}p_{w_2}\otimes u_{w_1}^{w_2}\otimes u_v^{w_1}=\sum_{w_2\in V}p_{w_2}\otimes\Delta(u_v^{w_2}).$$

Furthermore, we compute

$$(\mathrm{id}\otimes\varepsilon)(\alpha(p_v))=(\mathrm{id}\otimes\varepsilon)\Big(\sum_{w\in V}p_w\otimes u_v^w\Big)=\sum_{w\in V}p_w\cdot\delta_{wv}=p_v.$$

The previous computations also show that

$$\begin{aligned}(\alpha\otimes\mathrm{id})(\alpha(s_e))&=(\mathrm{id}\otimes\Delta)(\alpha(s_e)) &&\forall e\in E,\\ (\mathrm{id}\otimes\varepsilon)(\alpha(s_e))&=s_e &&\forall e\in E\end{aligned}$$

by replacing p_v with s_e. Hence,

$$(\mathrm{id}\otimes\alpha)\circ\alpha=(\Delta\otimes\mathrm{id})\circ\alpha,\qquad(\varepsilon\otimes\mathrm{id})\circ\alpha|_{\mathcal{B}}=\mathrm{id}\,.$$

Thus, α defines an action of $\mathrm{Aut}^+(\Gamma)$ on $C^*(\Gamma)$. To show that α is faithful, assume there exists a quotient G of $\mathrm{Aut}^+(\Gamma)$ such that $\alpha|_{C(G)}$ is also an action on $C^*(\Gamma)$. Then

$$\alpha|_{C(G)}(p_v)=\sum_{w\in V}p_w\otimes u_v^w\in C^*(\Gamma)\otimes C(G)\qquad\forall v\in V,$$

$$\alpha|_{C(G)}(s_e)=\sum_{f\in E}s_f\otimes u_e^f\in C^*(\Gamma)\otimes C(G)\qquad\forall e\in E.$$

The representation of $\alpha|_{C(G)}(p_v)$ in terms of p_v is unique since the p_v are linearly independent as orthogonal projections. Thus, $u_v^w\in C(G)$ for all $v,w\in V$. Similarly, $u_f^e\in C(G)$ for all $e,f\in E$ since the s_e are linearly independent as partial isometries with orthogonal ranges. Hence, $C(G)=C(\mathrm{Aut}^+(\Gamma))$, which shows that α is faithful. □

While our action appears to differ from the action of Schmidt and Weber, the following remark shows that it reduces to their action in the special case of classical directed graphs. In particular, this justifies the form of their action retrospectively since the action appears to be non-canonical from the perspective of classical directed graphs.

Remark 5.2.4. Let $\Gamma:=(V,E)$ be a directed graph. By the proof of Definition 4.2.3, we have $\mathrm{Aut}^+(\Gamma)=\mathrm{Aut}^+_{\mathrm{Bic}}(\Gamma)$ via the isomorphism

$$\begin{aligned}u_w^v&\mapsto\widehat{u}_w^v &&\forall v\in V,\\ u_{(w_1,w_2)}^{(v_1,v_2)}&\mapsto\widehat{u}_{w_1}^{v_1}\widehat{u}_{w_2}^{v_2} &&\forall(v_1,v_2),(w_1,w_2)\in E,\end{aligned}$$

where u denotes the fundamental representation of $\mathrm{Aut}^+(\Gamma)$ and $\widehat{u}$ denotes the fundamental representation of $\mathrm{Aut}^+_{\mathrm{Bic}}(\Gamma)$. Under this isomorphism, the action α from Definition 5.2.3

takes the form

$$\alpha(p_v) = \sum_{w \in V} p_w \otimes \widehat{u}_v^w \qquad \forall v \in V,$$
$$\alpha(s_{(v_1,w_1)}) = \sum_{(v_2,w_2)\in E} s_{(v_2,w_2)} \otimes \widehat{u}_{v_1}^{v_2} \widehat{u}_{w_1}^{w_2} \qquad \forall (v_1, w_1) \in E.$$

This coincides with the action in [77]. Therefore, we obtain the action of Schmidt and Weber for $\mathrm{Aut}^+_{\mathrm{Bic}}(\Gamma)$ as a special case. Note that Joardar and Mandal [52] already showed that $\mathrm{Aut}^+_{\mathrm{Bic}}(\Gamma)$ acts on $C^*(\Gamma)$ via this action.

5.3. Maximality of the action

In [77], Schmidt and Weber showed that their action is maximal in the sense that $\mathrm{Aut}^+_{\mathrm{Ban}}(\Gamma)$ is the largest quantum groups acting on the graph C^*-algebra $C^*(\Gamma)$ via the map α described previously. However, if Γ is a classical directed graph, then our quantum automorphism group coincides with $\mathrm{Aut}^+_{\mathrm{Bic}}(\Gamma)$ by Definition 4.2.3, and our action α in Definition 5.2.3 coincides with the action of Schmidt and Weber by Definition 5.2.4. Since $\mathrm{Aut}^+_{\mathrm{Bic}}(\Gamma)$ is a proper subgroup of $\mathrm{Aut}^+_{\mathrm{Ban}}(\Gamma)$ for some directed graphs, our action on hypergraph C^*-algebras is not maximal, even in the special case of directed graphs.

In the following, we show that we obtain maximality of our action by additionally considering a dual action on $C^*(\Gamma')$ defined by

$$\alpha'(p_e) = \sum_{f\in E} p_f \otimes u_e^f, \quad \alpha'(s_v) = \sum_{w\in V} s_w \otimes u_v^w \qquad \forall v \in V,\, e \in E,$$

where Γ' is defined as follows.

Definition 5.3.1. Let $\Gamma := (V, E)$ be a hypergraph. Denote by $\Gamma' := (E, V)$ the hypergraph with source and range maps defined by

$$s'(e) = \{v \in V \mid v \in r(e)\}, \quad r'(e) = \{v \in V \mid v \in s(e)\} \qquad \forall e \in E.$$

Note that $\Gamma' = (\Gamma^{\mathrm{op}})^* = (\Gamma^*)^{\mathrm{op}}$, where Γ^{op} and Γ^* are the opposite and dual hypergraphs defined in Definition 3.1.13 and Definition 3.1.14. From the results in Section 3.4, it follows directly that $\mathrm{Aut}^+(\Gamma)$ acts naturally not only on $C^*(\Gamma)$ but also on $C^*(\Gamma')$.

Proposition 5.3.2. *Let $\Gamma := (V, E)$ be a hypergraph. Then $\mathrm{Aut}^+(\Gamma)$ acts faithfully on $C^*(\Gamma')$ via the action*

$$\alpha\colon C^*(\Gamma') \to C^*(\Gamma') \otimes C(\mathrm{Aut}^+(\Gamma))$$

defined by

$$\alpha(p_e) = \sum_{f\in E} p_f \otimes u_e^f, \quad \alpha(s_v) = \sum_{w\in V} s_w \otimes u_v^w \qquad \forall v \in V,\, e \in E.$$

Proof. Denote by u the fundamental representation of $\mathrm{Aut}^+(\Gamma)$ and by $\widehat{u}$ the fundamental representation $\mathrm{Aut}^+(\Gamma)$. By Definition 3.4.2 and Definition 3.4.3, we have

$$\mathrm{Aut}^+(\Gamma') = \mathrm{Aut}^+\left((\Gamma^*)^{\mathrm{op}}\right) = \mathrm{Aut}^+(\Gamma^*) = \mathrm{Aut}^+(\Gamma)$$

via the correspondence

$$\widehat{u}_V \longleftrightarrow u_E, \quad \widehat{u}_E \longleftrightarrow u_V.$$

The statement follows by applying this isomorphism to the action of $\mathrm{Aut}^+(\Gamma')$ on $C^*(\Gamma')$ described in Definition 5.2.3. □

The main observation for proving maximality is the following lemma, which shows that at least some relations in Definition 3.2.5 can be recovered from the action on $C^*(\Gamma)$.

Lemma 5.3.3. *Let $\Gamma := (V, E)$ be a hypergraph and G be a compact matrix quantum group acting on $C^*(\Gamma)$ via an action*

$$\alpha\colon C^*(\Gamma) \to C^*(\Gamma) \otimes C(G)$$

defined by

$$\alpha(p_v) = \sum_{w\in V} p_w \otimes u_v^w, \quad \alpha(s_e) = \sum_{f\in E} s_f \otimes u_e^f \qquad \forall v \in V,\, e \in E$$

for some elements $u_w^v, u_f^e \in C(G)$. Then $u_V := (u_w^v)_{v,w\in V}$ is a magic unitary. If additionally $u_E := (u_f^e)_{e,f\in E}$ is a magic unitary, then $A_r u_E = u_V A_r$.

Proof. The proof that u_V is a magic unitary is contained in the proof of [89, Theorem 3.1], since the elements p_v are orthogonal projections that sum to 1. For the second part of the statement, assume that u_E is also a magic unitary and consider the relation

$$s_e^* s_e = \sum_{\substack{v\in V\\ v\in r(e)}} p_v \qquad \forall e \in E$$

from Definition 5.1.2. By applying α to the left-hand side and using this relation, we obtain

$$\alpha(s_e^* s_e) = \sum_{f_1, f_2\in E} s_{f_1}^* s_{f_2} \otimes \underbrace{\left(u_e^{f_1}\right)^* u_e^{f_2}}_{\delta_{f_1 f_2} u_e^{f_1}} = \sum_{f\in E} s_f^* s_f \otimes u_e^f = \sum_{f\in E} \sum_{\substack{v\in V\\ v\in r(f)}} p_v \otimes u_e^f.$$

This can be rewritten as

$$\alpha(s_e^* s_e) = \sum_{v\in V} \sum_{f\in E} (A_r)_f^v\, p_v \otimes u_e^f = \sum_{v\in V} p_v \otimes \left(\sum_{f\in E} (A_r)_f^v\, u_e^f \right).$$

Similarly, by applying α to the right-hand side of the original equation, we obtain

$$\begin{aligned}\sum_{\substack{v\in V\\ v\in r(e)}} \alpha(p_v) &= \sum_{\substack{v\in V\\ v\in r(e)}} \sum_{w\in V} p_w \otimes u_v^w = \sum_{v,w\in V} (A_r)_e^v\, p_w \otimes u_v^w \\ &= \sum_{w\in V} p_w \otimes \Bigg(\sum_{v\in V} u_v^w (A_r)_e^v\Bigg).\end{aligned}$$

Therefore,

$$(A_r u_E)_e^v = \sum_{f\in E} (A_r)_f^v\, u_e^f = \sum_{w\in V} u_v^w (A_r)_e^v = (A_r u_V)_e^v \qquad \forall v\in V,\, e\in E$$

by the linear independence of the orthogonal projections p_v, which allows us to compare the terms in the previous sums. This shows $A_r u_E = u_V A_r$. □

By combining the previous lemma with the actions on $C^*(\Gamma)$ and $C^*(\Gamma')$, we can now show that $\mathrm{Aut}^+(\Gamma)$ is the largest quantum group that acts on both $C^*(\Gamma)$ and $C^*(\Gamma')$ in the sense of Definition 5.2.3 and Definition 5.3.2.

Theorem 5.3.4. *Let $\Gamma := (V, E)$ be a hypergraph and G be a compact matrix quantum group acting faithfully on $C^*(\Gamma)$ and $C^*(\Gamma')$ via actions*

$$\alpha_1\colon C^*(\Gamma) \to C^*(\Gamma)\otimes C(G), \quad \alpha_2\colon C^*(\Gamma') \to C^*(\Gamma')\otimes C(G),$$

defined by

$$\begin{aligned}\alpha_1(p_v) &= \sum_{w\in V} p_w \otimes u_v^w, \quad \alpha_1(s_e) = \sum_{f\in E} s_f \otimes u_e^f \qquad \forall v\in V,\, e\in E,\\ \alpha_2(p_e) &= \sum_{f\in E} p_f \otimes u_e^f, \quad \alpha_2(s_v) = \sum_{w\in V} s_w \otimes u_v^w \qquad \forall v\in V,\, e\in E\end{aligned}$$

for some elements $u_w^v, u_f^e \in C(G)$. Then $G \subseteq \mathrm{Aut}^+(\Gamma)$.

Proof. By applying the first part of Definition 5.3.3 to α_1 and α_2, we obtain that the matrices $u_V := (u_w^v)_{v,w\in V}$ and $u_E := (u_f^e)_{e,f\in E}$ are magic unitaries. Therefore, the second part of Definition 5.3.3 yields

$$A_r u_E = u_V A_r, \quad A_{r'} u_V = u_E A_{r'}.$$

Since $A_{r'} = A_s^*$, we have $A_s^* u_V = u_E A_s^*$, which implies $A_s u_E = u_V A_s$ by Definition 3.2.8. This shows that the elements u_w^v and u_f^e satisfy the relations from Definition 3.2.5. Hence, by the universal property of $C(\mathrm{Aut}^+(\Gamma))$, there exists a unital $*$-homomorphism

$$\phi\colon C(\mathrm{Aut}^+(\Gamma)) \to C(G)$$

mapping the generators of $C(\mathrm{Aut}^+(\Gamma))$ to the entries of u_V and u_E. Next, we show that ϕ is a morphism of compact quantum groups. Let $w \in V$. Then

$$\begin{aligned}(\alpha_1 \otimes \mathrm{id})(\alpha_1(w)) &= \sum_{x\in V} \alpha_1(p_x) \otimes u^x_w = \sum_{x\in V}\sum_{v\in V} p_v \otimes u^v_x \otimes u^x_w \\ &= \sum_{v\in V} p_v \otimes \left(\sum_{x\in V} u^v_x \otimes u^x_w\right)\end{aligned}$$

and

$$(\mathrm{id} \otimes \Delta)(\alpha_1(w)) = \sum_{v\in V} p_v \otimes \Delta(u^v_w).$$

Since α_1 is an action, we have $(\alpha_1 \otimes \mathrm{id}) \circ \alpha_1 = (\mathrm{id} \otimes \Delta) \circ \alpha_1$. Furthermore, the elements p_v are linearly independent as orthogonal projections, which implies

$$\Delta(u^v_w) = \sum_{w\in V} u^v_x \otimes u^x_w \qquad \forall v, w \in V.$$

Denote by $\widehat{u}$ the fundamental representation of $\mathrm{Aut}^+(\Gamma)$. Then the previous equation yields

$$\Delta\big(\phi(\widehat{u}^v_w)\big) = \sum_{x\in V} u^v_x \otimes u^x_w = \sum_{x\in V} \phi(\widehat{u}^v_x) \otimes \phi(\widehat{u}^x_w), = (\phi \otimes \phi)\big(\Delta(\widehat{u}^v_w)\big)$$

for all $v, w \in V$. Similarly, we show that

$$\Delta\big(\phi(\widehat{u}^e_f)\big) = (\phi \otimes \phi)\big(\Delta(\widehat{u}^e_f)\big) \qquad \forall e, f \in E$$

using the action α_2. Thus, ϕ is a morphism of compact quantum groups. Furthermore, ϕ is surjective because α_1 and α_2 are faithful. Otherwise, the image of ϕ would define a proper quotient quantum group of G acting on $C^*(\Gamma)$ and $C^*(\Gamma')$ in the same way, which is a contradiction. Therefore, $G \subseteq \mathrm{Aut}^+(\Gamma)$. □

Note that if Γ is an undirected hypergraph, then $\Gamma' = \Gamma^*$. In this case, $\mathrm{Aut}^+(\Gamma)$ is the maximal quantum group that acts faithfully on both $C^*(\Gamma)$ and $C^*(\Gamma^*)$ in a compatible way. Furthermore, because $\mathrm{Aut}^+(\Gamma)$ acts maximally, we can regard $\mathrm{Aut}^+(\Gamma)$ as the quantum symmetry group of $C^*(\Gamma)$ in the sense of Definition 5.3.4.

Part II.

Quantum groups from spatial partitions

6. Combinatorics of spatial partitions

Partitions are combinatorial objects consisting of a row of upper points and a row of lower points that are partitioned into disjoint subsets. They are often visualized as two-dimensional string diagrams and appear in the definition of partition algebras, such as the Temperley-Lieb algebra [82] and the Brauer algebra [7], but are also used in the construction of easy quantum groups by Banica and Speicher [10].

In this chapter, we study the combinatorics of spatial partitions, a generalization of partitions to three dimensions first introduced by Cébron and Weber in [25]. In particular, we generalize their notion of categories of spatial partitions by introducing new base partitions, and we construct various combinatorial functors between categories of spatial partitions that are used in the Chapter 7 and Chapter 8.

Additionally, we present data structures and algorithms for spatial partitions and their basic operations, which were developed in [87] and subsequently implemented in the computer algebra system OSCAR [69] by Volz and the author. In this context, we also prove that natural problems, such as deciding membership in a category of partitions, are algorithmically undecidable in general.

For further information on easy quantum groups and the construction of quantum groups from categories of partitions, see Chapter 7.

6.1. Spatial partitions

We begin by introducing spatial partitions, which are the main combinatorial objects used to define categories of spatial partitions in Section 6.2. In contrast to [25], we focus not only on spatial partitions with white points, but we consider colored partitions as briefly described in [25, Remark 2.7]. In this context, we use the notation from Definition 2.5.9 and denote by $\{\circ, \bullet\}^*$ the set of all finite words in the colors $\circ$ and $\bullet$.

Definition 6.1.1. Let $m \in \mathbb{N}$. A *spatial partition* on m *levels* is a tuple $(x, y, \{B_i\})$, where $x, y \in \{\circ, \bullet\}^*$ and $\{B_i\}$ is a decomposition of the points

$$\{1, \ldots, |x| + |y|\} \times \{1, \ldots, m\}$$

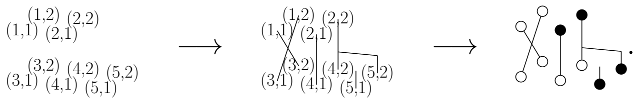
into non-empty disjoint subsets called *blocks*. We call x the *upper colors* and y the *lower colors* of the partition.

Given a spatial partition, we can visualize it as a colored string diagram as follows. First, we draw an upper layer consisting of the points (i, j) with $1 \leq i \leq |x|$ and a lower layer consisting of the points (i, j) with $|x| < i \leq |y|$. In both layers, i increases from the left to the right and j from the front to the back. Then, we connect all points in the same blocks with lines and assign each upper point (i, j) the color x_i and each lower point $(|x| + i, j)$ the color y_i.

Example 6.1.2. Consider a spatial partition on two levels with upper colors $x = \circ\bullet$, lower colors $y = \circ\circ\bullet$ and blocks given by

$$\{(1,1),(3,2)\},\{(1,2),(3,1)\},\{(2,1),(4,1)\},\{(2,2),(4,2),(5,2)\},\{(5,1)\}.$$

Visualizing it as a string diagram yields

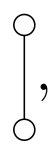
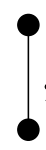
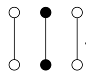


Note that spatial partitions on a single level and with only white points are precisely the usual partitions appearing for example in the definition of diagram algebras, such as the Temperley-Lieb algebra [82] and the Brauer algebra [18], and more generally in the definition of orthogonal easy quantum groups [7].

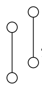
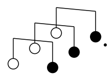
In the following, we denote by $\mathcal{P}^{(m)}$ the set of all spatial partitions on m levels and by $\mathcal{P}^{(m)}(x, y)$ the set of all spatial partitions on m levels with upper colors x and lower colors y. Furthermore, we introduce the following spatial partitions, which will be used throughout the rest of the thesis.

Definition 6.1.3.

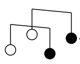
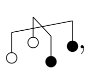
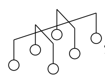
1. Let $x \in \{\circ, \bullet\}^*$. Then we denote by $\mathrm{id}_x \in \mathcal{P}^{(1)}(x, x)$ the *identity partition* on x. It is the spatial partition on a single level with upper and lower colors x, where each upper point is directly connected to the corresponding lower point, e.g.

   $\mathrm{id}_\circ =$ , $\mathrm{id}_\bullet =$ , $\mathrm{id}_{\circ\bullet\circ} =$ .

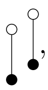
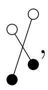
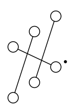
2. Let $p \in \mathcal{P}^{(1)}(x, y)$ be a spatial partition. Then we denote by $p^{(k)} \in \mathcal{P}^{(k)}(x, y)$ the k-fold *amplification* of p. It is obtained by placing k copies of p along the levels, e.g.

   $\mathrm{id}_\circ^{(2)} =$ ,  $^{(3)} =$ .

3. Let $x, y \in \{\circ, \bullet\}$ and $\sigma \in S_m$ be a permutation on $\{1, \ldots, m\}$. Then we denote by $\sigma_{xy} \in \mathcal{P}^{(m)}(1, xy)$ the spatial partition with lower colors xy and blocks given by $\{(1, i), (2, \sigma(i))\}$ for all $1 \leq i \leq m$. For example, we have

   $(1)(2)_{\circ\bullet} =$ , $(12)_{\circ\bullet} =$ , $(132)_{\circ\circ} =$ ,

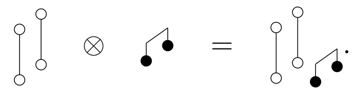
   where the corresponding permutations are written in cycle notation. Similarly, we denote by $\sigma_y^x \in \mathcal{P}^{(m)}(x, y)$ the spatial partition obtained by rotating σ_{xy} such that the upper color is x and the lower color is y, e.g.

   $(1)(2)_\bullet^\circ =$ , $(12)_\bullet^\circ =$ , $(132)_\circ^\circ =$ .

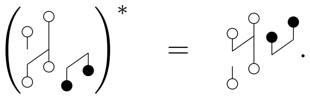
Given spatial partitions on a fixed number of levels m, we can construct new spatial partitions using the following operations.

Definition 6.1.4.

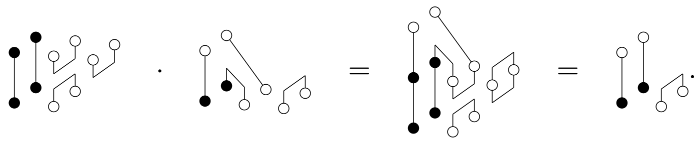
1. Let $p \in \mathcal{P}^{(m)}(x, y)$ and $q \in \mathcal{P}^{(m)}(w, z)$. The *tensor product* $p \otimes q \in \mathcal{P}^{(m)}(xw, yz)$ is obtained by placing p and q side by side, e.g.



2. Let $p \in \mathcal{P}^{(m)}(x, y)$. The *involution* $p^* \in \mathcal{P}^{(m)}(y, x)$ is obtained by swapping the upper and lower points, e.g.



3. Let $p \in \mathcal{P}^{(m)}(y, z)$ and $q \in \mathcal{P}^{(m)}(x, y)$. The *composition* $pq \in \mathcal{P}^{(m)}(x, z)$ is obtained by placing q on top of p and identifying the lower points of q with the upper points of p. Then these points are removed and the resulting blocks are simplified, e.g.

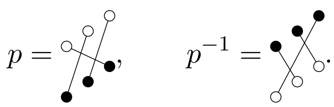


We refer to [73] for additional operations and further information on the combinatorics of partitions in the special case of a single level. Finally, we introduce the following properties of spatial partitions, which will be used later.

Definition 6.1.5. Let $p \in \mathcal{P}^{(m)}(x, y)$ be a spatial partition. Then p is called

1. *invertible* if there exists a spatial partition $p^{-1} \in P^{(m)}(y, x)$ such that $p^{-1}p = \mathrm{id}_x$ and $pp^{-1} = \mathrm{id}_y$.
2. $\boldsymbol{n}$*-graded* for some $\boldsymbol{n} := (n_1, \dots, n_m) \in \mathbb{N}^m$ if $n_k = n_\ell$ for all points (i, k) and (j, ℓ) that belong to the same block in p.
3. *pair partition* if every block of p has size two.

Note that the inverse of a spatial partition $p \in P^{(m)}(x, y)$ is always given by $p^{-1} = p^*$. In this case, p and p^* are both pair partitions whose blocks form a bijection between the corresponding upper and lower points. For example



In particular, if p is invertible, then $|x| = |y|$. In the following, we present a proof of this statement in the case $|x| = |y| = 1$, which will be relevant later.

Lemma 6.1.6. *Let $p \in \mathcal{P}^{(m)}(x, y)$ be invertible with $|x| = |y| = 1$. Then $p = \sigma_y^x$ and $p^{-1} = (\sigma^{-1})_x^y$ for some permutation $\sigma \in S_m$.*

Proof. Following [42], we denote by $t(p)$ the number of through-blocks of p, i.e. the number of blocks of p that contain both an upper and a lower point. Then $t(p) \leq m$ with equality if and only if $p = \sigma_y^x$ for some permutation $\sigma \in S_m$. Furthermore,

$$t(qp) \leq \min(t(q), t(p))$$

for any $q \in \mathcal{P}^{(m)}(y, x)$, see [42, Remark 2.6]. Thus, if p is invertible, then

$$m = t(\mathrm{id}_x^{(m)}) = t(p^{-1} \cdot p) \leq \min(t(p^{-1}), t(p)) \leq t(p) \leq m.$$

This shows $t(p) = m$. Similarly, one obtains $t(p^{-1}) = m$. Therefore, both p and p^{-1} are of the form σ_y^x and $(\sigma^{-1})_x^y$ for some permutation $\sigma \in S_m$. $\square$

6.2. Categories of spatial partitions

Next, we introduce categories of spatial partitions as purely combinatorial objects. These will be used in Section 7.1, where we interpret categories of spatial partitions as representation categories of compact matrix quantum groups.

Definition 6.2.1. A *category of spatial partitions* on m levels is a subset $\mathcal{C} \subseteq \mathcal{P}^{(m)}$ that

1. is closed under composition, tensor product and involution,

2. contains $\mathrm{id}_\circ$ and $\mathrm{id}_\bullet$.

If $\mathcal{C} \subseteq \mathcal{P}^{(m)}$ is a category of spatial partitions, then we denote by $\mathcal{C}(x, y)$ the set of all spatial partitions in $\mathcal{C}$ with upper colors x and lower colors y.

Remark 6.2.2. In contrast to [25], we do not include the base partitions $\sqcap_{\circ\bullet}^{(m)}$ and $\sqcap_{\bullet\circ}^{(m)}$ in the definition. This makes our definition more general and includes all categories of spatial partitions in the sense of Cébron and Weber as special cases. We will come back to the role of these base partitions in Section 6.3.

Remark 6.2.3. Denote by $\mathcal{P} \subseteq \mathcal{P}^{(1)}$ the subset of all spatial partitions on a single level with only white points. A subset $\mathcal{C} \subseteq \mathcal{P}$ is a category of partitions in the sense of [73] if it is closed under composition, involution and tensor product, and contains the uncolored base partitions $\sqcap_{\circ\circ}$ and $|_{\circ}^{\circ}$. Throughout this thesis, we are mostly interested in categories of partitions in the sense of Definition 6.2.1. However, we consider this original notion of categories of partitions in Section 6.5 and Section 6.6. In particular, we will use the notation $\mathcal{C}(k, \ell) := \mathcal{C}(\circ^k, \circ^\ell)$ in this context.

Example 6.2.4. Examples of categories of spatial partitions include:

1. The set $\mathcal{P}^{(m)}$ of all spatial partitions on m levels.

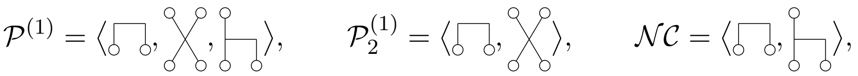
2. The set $\mathcal{P}_2^{(m)}$ of all spatial pair partitions on m levels.
3. The set $\mathcal{NC} \subseteq \mathcal{P}^{(1)}$ of all non-crossing partitions, i.e. partitions that can be drawn without crossing lines in two dimensions.

Additional examples of categories of spatial partitions can be found in [73, 81, 25].

As in the previous examples, categories of spatial partitions are often defined by specifying a property. However, it is also possible to define categories of spatial partition using a generating set.

Definition 6.2.5. Let $\mathcal{C}_0$ be a set of spatial partitions. Then we denote by $\mathcal{C} := \langle \mathcal{C}_0 \rangle$ the category of spatial partitions *generated* by $\mathcal{C}_0$. It is the smallest category of spatial partitions containing $\mathcal{C}_0$ and consists of all finite combinations of elements in $\mathcal{C}_0$ and the base partitions $\mathrm{id}_\circ$ and $\mathrm{id}_\bullet$.

Example 6.2.6. Consider the categories of spatial partitions from Definition 6.2.4. Using generators, we can express these categories as



see [73] for further details. Moreover, generators for the general cases $\mathcal{P}^{(m)}$ and $\mathcal{P}_2^{(m)}$ can be found in [25].

Remark 6.2.7. Interest in easy quantum groups initiated the classification of categories of partitions $\mathcal{C} \subseteq \mathcal{P}$ in [10, 8, 90] and has led to a full classification in [73]. In the case of colored partitions, the classification remains ongoing. See for example [81, 39, 46] and the more recent work by Mang [60, 61, 62, 63]. Similarly, the classification of categories of spatial partitions started in [25, 31] and is still ongoing.

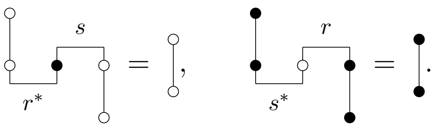
6.3. Duality partitions and rigidity

In the original definition of spatial partition quantum groups in [25], Cébron and Weber included the additional base partitions $\sqcap_{\circ\bullet}^{(m)}$ and $\sqcap_{\bullet\circ}^{(m)}$. These partitions were used in the construction of spatial partitions quantum groups and guaranteed that the representation $\overline{u}$ is unitary. However, as discussed in Section 2.5, it is only required that a conjugate representation $u^\bullet$ exists that is not necessarily given by $\overline{u}$. This is equivalent to the existence of solutions to the conjugate equations in the sense of Definition 2.5.11. Thus, translating the conjugate equations into the setting of spatial partitions yields the following definition.

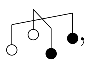
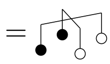
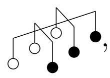
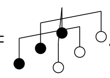
Definition 6.3.1. Let $\mathcal{C} \subseteq \mathcal{P}^{(m)}$ be a category of spatial partitions. Then $\mathcal{C}$ is *rigid* if it contains a pair of duality partitions $r \in \mathcal{C}(1, \circ\bullet)$ and $s \in \mathcal{C}(1, \bullet\circ)$, which satisfy the following conjugate equations:

$$(r^* \otimes \mathrm{id}_\circ) \cdot (\mathrm{id}_\circ \otimes s) = \mathrm{id}_\circ, \qquad (s^* \otimes \mathrm{id}_\bullet) \cdot (\mathrm{id}_\bullet \otimes r) = \mathrm{id}_\bullet\,.$$

In the case of spatial partitions on a single level, the conjugate equations can be visualized diagrammatically as follows:



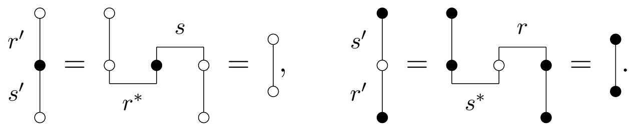
In particular, this shows that $r =$  and $s =$  are the only solutions in the case $m = 1$. Furthermore, one verifies that the duality partitions $r =$ $^{(m)}$ and $s =$ $^{(m)}$ introduced by Cébron and Weber satisfy the conjugate equations. This implies that categories of spatial partitions in the sense of [25] are always rigid. However, for $m \geq 2$, there exist additional solutions given by spatial partitions such as

$r =$ , $s =$  and $r =$ , $s =$ .

The following proposition characterizes all possible solutions of the conjugate equations in the context of spatial partitions.

Proposition 6.3.2. *Duality partitions in the sense of Definition 6.3.1 are precisely of the form $r = \sigma_{\circ\bullet}$ and $s = \sigma^{-1}_{\bullet\circ}$ for some permutation $\sigma \in S_m$.*

Proof. Fix a number of levels m and assume that $r \in \mathcal{P}^{(m)}(1, \circ\bullet)$ and $s \in \mathcal{P}^{(m)}(1, \bullet\circ)$ form a pair of duality partitions. Define new partitions $r' \in \mathcal{P}^{(m)}(\circ, \bullet)$ and $s' \in P^{(m)}(\bullet, \circ)$ by moving the lower points $(1, i)$ for $1 \leq i \leq m$ to the upper layer. Then the conjugate equations (visualized here only for one level) yield



Thus, r' and s' are a pair of inverse partitions, and Definition 6.1.6 implies that they are of the form $r' = \sigma^{\circ}_{\bullet}$ and $s' = (\sigma^{-1})^{\bullet}_{\circ}$ for some permutation $\sigma \in S_m$. Hence, we obtain $r = \sigma_{\circ\bullet}$ and $s = \sigma^{-1}_{\bullet\circ}$ by moving the upper points $(1, i)$ for $1 \leq i \leq m$ back to the lower layer.

Conversely, let $\sigma \in S_m$ and define the partitions $r = \sigma_{\circ\bullet}$ and $s = \sigma^{-1}_{\bullet\circ}$. By tracing the blocks in the first conjugate equation, one verifies that each point $(1, i)$ in r is connected to $(2, \sigma(i))$, which is then connected to $(2, \sigma^{-1}(\sigma(i))) = (2, i)$ in s. Thus, the first conjugate equation

$$(r^* \otimes \mathrm{id}_\circ) \cdot (\mathrm{id}_\circ \otimes s) = \mathrm{id}_\circ$$

is satisfied. Similarly, one verifies that the second conjugate equation is satisfied as well, which shows that r and s form a pair of duality partitions. $\square$

So far, we have only considered categories of spatial partitions $\mathcal{C}$ as purely combinatorial objects. However, we can also regard them as categories in the usual sense by defining the set of objects $\mathrm{Ob}(\mathcal{C}) := \{\circ, \bullet\}^*$ and the set of morphisms

$$\mathrm{Hom}(x, y) := \mathcal{C}(x, y) = \mathcal{C} \cap \mathcal{P}^{(m)}(x, y) \qquad \forall x, y \in \{\circ, \bullet\}^*.$$

Moreover, using the language of categories, the tensor product and involution give categories of spatial partitions the structure of strict monoidal †-categories. See [30] for precise definitions of these terms.

Using this framework, we can now show that the duality of the objects $\circ$ and $\bullet$ in the sense of Definition 6.3.1 implies the duality of x and $\overline{x}$ for all objects $x \in \{\circ, \bullet\}^*$. Thus, our notion of rigidity agrees with the general notion of rigidity for example defined in [30] and [67].

Proposition 6.3.3. *Let $\mathcal{C} \subseteq \mathcal{P}^{(m)}$ be a rigid category of spatial partitions and $x \in \{\circ, \bullet\}^*$. Then $\mathcal{C}$ contains a pair of duality partitions $r \in \mathcal{C}(1, x\overline{x})$ and $s \in \mathcal{C}(1, \overline{x}x)$ satisfying the conjugate equations for x and $\overline{x}$, i.e.*

$$(r^* \otimes \mathrm{id}_x) \cdot (\mathrm{id}_x \otimes s) = \mathrm{id}_x, \qquad (s^* \otimes \mathrm{id}_{\overline{x}}) \cdot (\mathrm{id}_{\overline{x}} \otimes r) = \mathrm{id}_{\overline{x}}.$$

Proof. The statement follows directly by inductively applying [30, Proposition 1.10.7] in the context of spatial partitions. Alternatively, the duality partitions r and s can be explicitly constructed by nesting the duality partitions for $\circ$ and $\bullet$. □

6.4. Spatial partition functors

Next, we introduce functors between categories of spatial partitions, which allow us to transform spatial partitions while preserving their basic operations. Moreover, we construct two concrete examples that will be used in the proof of our main results in Chapter 7 and Chapter 8.

Definition 6.4.1. Let $\mathcal{C} \subseteq \mathcal{P}^{(n)}$ and $\mathcal{D} \subseteq \mathcal{P}^{(m)}$ be categories of spatial partitions. A *spatial partition functor* $F\colon \mathcal{C} \to \mathcal{D}$ consists of a function $F\colon \{\circ, \bullet\}^* \to \{\circ, \bullet\}^*$ and functions $F\colon \mathcal{C}(x, y) \to \mathcal{D}(F(x), F(y))$ for all $x, y \in \{\circ, \bullet\}^*$ satisfying the following properties:

1. $F(1) = 1$ and $F(xy) = F(x)F(y)$ for all $x, y \in \{\circ, \bullet\}^*$.
2. F preserves the category operators, i.e.

$$F(p \cdot q) = F(p) \cdot F(q), \quad F(p^*) = F(p)^*, \quad F(p \otimes q) = F(p) \otimes F(q)$$

   for all (composable) $p, q \in \mathcal{C}$.
3. $F(\mathrm{id}_x) = \mathrm{id}_{F(x)}$ for all $x \in \{\circ, \bullet\}^*$.

Note that a spatial partition functor does not necessarily preserve the number of levels of spatial partitions. Using the language of categories, we additionally introduce the notion of a fully faithful functor.

Definition 6.4.2. Let $F\colon \mathcal{C} \to \mathcal{D}$ be a spatial partition functor. Then F is *fully faithful* if the functions $F\colon \mathcal{C}(x, y) \to \mathcal{D}(F(x), F(y))$ are bijections for all $x, y \in \{\circ, \bullet\}^*$.

A simple example of a spatial partition functor can be obtained by decoloring spatial partitions. Formally, we define $F\colon \mathcal{P}^{(m)} \to \mathcal{P}^{(m)}$ by

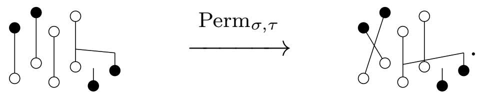
$$F(x) := \circ^{|x|} \qquad\qquad \forall x \in \{\circ, \bullet\}^*,$$
$$F(p) := (\circ^{|x|}, \circ^{|y|}, \{B_i\}) \qquad \forall p := (x, y, \{B_i\}) \in \mathcal{P}^{(m)}.$$

Since F preserves the block structure of spatial partitions, one can verify directly that it satisfies all the properties in Definition 6.4.1 and that F is fully faithful. However, F is not surjective in the sense that its image does not contain $^{(m)}$. In particular, this shows that the image of a spatial partition functor is not necessarily a category of spatial partitions in the sense of Definition 6.2.1.

Next, we present two additional examples of spatial partition functors that will be used in the context of spatial partition quantum groups in Chapter 7 and Chapter 8.

Permuting levels

Let $\sigma, \tau \in S_m$ be permutations. Then the functor $\mathrm{Perm}_{\sigma,\tau}\colon \mathcal{P}^{(m)} \to \mathcal{P}^{(m)}$ is obtained by permuting the levels of white points using σ and the levels of black points using τ. Consider for example $\sigma = (12)$ and $\tau = (1)(2)$. Then



Using the permutation partitions introduced in Definition 6.1.3, we can alternatively define $\mathrm{Perm}_{\sigma,\tau}$ as follows.

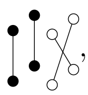
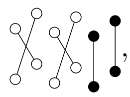
Definition 6.4.3. Let $\sigma, \tau \in S_m$. Define

$$q^{\circ}_{\sigma,\tau} := \sigma^{\circ}_{\circ}, \quad q^{\bullet}_{\sigma,\tau} := \tau^{\bullet}_{\bullet}, \quad q^{xy}_{\sigma,\tau} := q^{x}_{\sigma,\tau} \otimes q^{y}_{\sigma,\tau} \qquad \forall x, y \in \{\circ, \bullet\}^*, \ |xy| > 1.$$

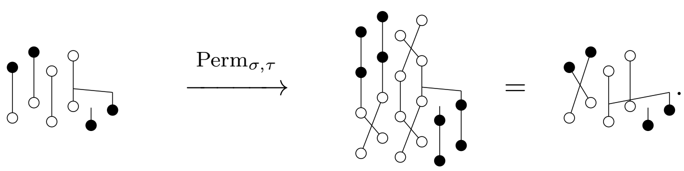
Then the *permutation functor* $\mathrm{Perm}_{\sigma,\tau}\colon \mathcal{P}^{(m)} \to \mathcal{P}^{(m)}$ is defined by

$$\mathrm{Perm}_{\sigma,\tau}(x) = x \qquad\qquad \forall x \in \{\circ, \bullet\}^*,$$
$$\mathrm{Perm}_{\sigma,\tau}(p) = q^{y}_{\sigma,\tau} \cdot p \cdot (q^{x}_{\sigma,\tau})^{-1} \qquad \forall p \in \mathcal{P}^{(m)}(x, y).$$

Example 6.4.4. Consider the spatial partition from the initial example with $\sigma = (12)$ and $\tau = (1)(2)$. Then

$q^{\bullet\circ}_{\sigma,\tau} =$ , $\qquad q^{\circ\circ\bullet}_{\sigma,\tau} =$ ,

which yields



Next, we verify that the permutation functor $\mathrm{Perm}_{\sigma,\tau}$ is indeed a spatial partition functor in the sense of Definition 6.4.1 and that it is additionally fully faithful.

Proposition 6.4.5. $\mathrm{Perm}_{\sigma,\tau}$ *is a fully faithful spatial partition functor.*

Proof. Since $\mathrm{Perm}_{\sigma,\tau}$ preserves colors, it follows immediately that it respects the concatenation of colors. Furthermore, Definition 6.1.6 implies $q_{\sigma,\tau}^* = q_{\sigma,\tau}^{-1}$, which yields

$$\mathrm{Perm}_{\sigma,\tau}(p)^* = ((q_{\sigma,\tau}^x)^{-1})^* \cdot p^* \cdot (q_{\sigma,\tau}^y)^* = \mathrm{Perm}_{\sigma,\tau}(p^*) \qquad \forall p \in \mathcal{P}^{(m)}(x,y).$$

Similarly, one can verify directly that

$$\mathrm{Perm}_{\sigma,\tau}(p_1 \cdot p_2) = \mathrm{Perm}_{\sigma,\tau}(p_1) \cdot \mathrm{Perm}_{\sigma,\tau}(p_2),$$

$$\mathrm{Perm}_{\sigma,\tau}(p_1 \otimes p_2) = \mathrm{Perm}_{\sigma,\tau}(p_1) \otimes \mathrm{Perm}_{\sigma,\tau}(p_2)$$

for all (composable) $p_1, p_2 \in \mathcal{P}^{(m)}$. Hence, $\mathrm{Perm}_{\sigma,\tau}$ respects the category operations. Additionally, we have

$$\mathrm{Perm}_{\sigma,\tau}(\mathrm{id}_x) = q_{\sigma,\tau}^x \cdot \mathrm{id}_x \cdot (q_{\sigma,\tau}^x)^{-1} = \mathrm{id}_x \quad \forall x \in \{\circ, \bullet\}^*,$$

which shows that $\mathrm{Perm}_{\sigma,\tau}$ is spatial partition functor. Finally, one can verify that

$$\mathrm{Perm}_{\sigma,\tau}(\mathrm{Perm}_{\sigma^{-1},\tau^{-1}}(p)) = \mathrm{Perm}_{\sigma^{-1},\tau^{-1}}(\mathrm{Perm}_{\sigma,\tau}(p)) = p \qquad \forall p \in \mathcal{P}^{(m)},$$

which implies that $\mathrm{Perm}_{\sigma,\tau}$ is bijective on the sets $\mathcal{P}^{(m)}(x,y)$. Thus, it is fully faithful. $\square$

Finally, we introduce $\boldsymbol{n}$-graded permutations and show that $\mathrm{Perm}_{\sigma,\tau}$ maps categories of spatial partitions to categories of spatial partitions while preserving rigidity.

Definition 6.4.6. Let $\boldsymbol{n} := (n_1, \dots, n_m) \in \mathbb{N}^m$. A permutation $\sigma \in S_m$ is called ***n*-graded** if $n_i = n_{\sigma(i)}$ for all $1 \le i \le m$.

Proposition 6.4.7. *Let $\sigma, \tau \in S_m$ be $\boldsymbol{n}$-graded permutations and $\mathcal{C} \subseteq \mathcal{P}^{(m)}$ be a $\boldsymbol{n}$-graded rigid category of spatial partitions. Then $\mathrm{Perm}_{\sigma,\tau}(\mathcal{C})$ is a $\boldsymbol{n}$-graded rigid category of spatial partitions.*

Proof. Define $\mathcal{D} := \mathrm{Perm}_{\sigma,\tau}(\mathcal{C})$. Since $\mathrm{Perm}_{\sigma,\tau}$ respects the category operations, it follows directly that $\mathcal{D}$ is closed under composition, tensor product and involution. Furthermore, $\mathrm{Perm}_{\sigma,\tau}(\mathrm{id}_x) = \mathrm{id}_x \in \mathcal{D}$ for all $x \in \{\circ, \bullet\}^*$, which shows that $\mathcal{D}$ is a category of spatial partitions. Since $\mathcal{C}$ is rigid, there exists a pair of duality partitions $r \in \mathcal{C}(1, \circ\bullet)$ and $s \in \mathcal{C}(1, \bullet\circ)$ satisfying the conjugate equations. Then

$$\mathrm{Perm}_{\sigma,\tau}(r) \in \mathcal{D}(1, \circ\bullet), \quad \mathrm{Perm}_{\sigma,\tau}(s) \in \mathcal{D}(1, \bullet\circ)$$

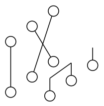
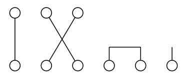
also satisfy the conjugate equations, which implies that $\mathcal{D}$ is rigid. Finally, observe that permuting the levels of a $\boldsymbol{n}$-graded spatial partition by a $\boldsymbol{n}$-graded permutation yields a $\boldsymbol{n}$-graded spatial partition. Thus, $\mathcal{D}$ is also a $\boldsymbol{n}$-graded category of spatial partitions. $\square$

Flattening partitions

Next, we construct the functor $\mathrm{Flat}_{m,z}$ that flattens spatial partitions along the levels and is used in Chapter 8 in the context of projective versions of spatial partition quantum groups. It is motivated by [25, Remark 2.4 & 2.8], where Cébron and Weber consider the bijection

$$\{1,\ldots,k+\ell\}\times\{1,\ldots,m\}\cong\{1,\ldots,m\cdot(k+\ell)\}$$

that identifies a point (i,j) with the point $m\cdot(i-1)+j$. This bijection induces a bijection between the sets of spatial partitions $\mathcal{P}^{(m)}(\circ^k,\circ^\ell)$ and $\mathcal{P}^{(1)}(\circ^{mk},\circ^{m\ell})$ by rearranging the points accordingly, e.g.

 $\in \mathcal{P}^{(3)}(\circ^1,\circ^2) \quad\cong\quad$  $\in \mathcal{P}^{(1)}(\circ^3,\circ^6).$

It respects the composition, tensor product and involution of spatial partitions, but it fails to preserve the base partitions as defined in [25] when applied to categories of spatial partitions.

Next, we generalize the previous bijection to the case of m levels and colored spatial partitions. Note that we reverse the order of black points in the following definition, since in the context of quantum groups, the representation $\overline{u \mathbin{\textcircled{\scriptsize T}} w}$ is equivalent to $\overline{w}\mathbin{\textcircled{\scriptsize T}}\overline{u}$. See Chapter 8 for further information.

Definition 6.4.8. Let $m,d\geq 1$ and $x,y\in\{\circ,\bullet\}^*$ with $k:=|x|$ and $\ell:=|y|$. Define the bijections

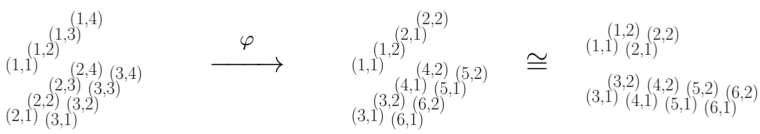
$$\varphi_{m,d}^{x,y}\colon \{1,\ldots,k+\ell\}\times\{1,\ldots,m\cdot d\}\to\{1,\ldots,(k+\ell)\cdot d\}\times\{1,\ldots,m\}$$

$$(i,j+k\cdot m)\mapsto\begin{cases}(i\cdot d-d+k+1,j) & \text{if } (xy)_i=\circ,\\ (i\cdot d-k,j) & \text{if } (xy)_i=\bullet,\end{cases}$$

for all $1\leq i\leq k+\ell$, $1\leq j\leq m$ and $0\leq k<d$.

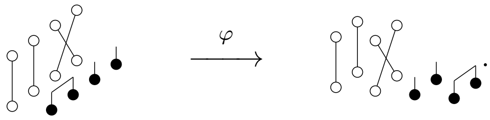
In the special case where $x=\circ^k$, $y=\circ^\ell$ and $m=1$, we obtain exactly the bijection of Cébron and Weber introduced at the beginning of this section. A more general case involving four levels is shown in the following example.

Example 6.4.9. Consider four levels of points with upper colors $x=\circ$ and lower colors $y=\circ\bullet$. Applying $\varphi:=\varphi_{2,2}^{x,y}$ to these points yields



where the relabeled points are rearranged according to our usual convention.

By extending this bijection to the blocks of spatial partitions, we obtain for example



In particular, note that the black points are flattened in reverse order. Using these bijections $\varphi_{m,d}^{x,y}$ from Definition 6.4.8, we can define the functor $\mathrm{Flat}_{m,z}$.

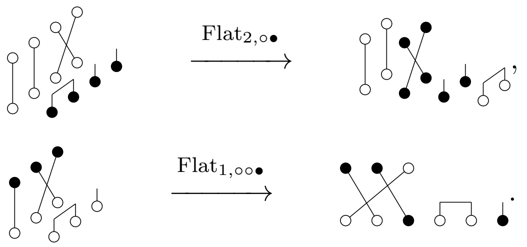
Definition 6.4.10. Let $m \in \mathbb{N}$ and $z \in \{\circ, \bullet\}^*$ with $d := |z| > 1$. The spatial partition functor $\mathrm{Flat}_{m,z} \colon \mathcal{P}^{(m \cdot d)} \to \mathcal{P}^{(m)}$ is defined by

$$\mathrm{Flat}_{m,z}(\circ) = z, \quad \mathrm{Flat}_{m,z}(\bullet) = \overline{z} = \overline{z_d} \cdots \overline{z_1}$$

$$\mathrm{Flat}_{m,z}(p) = \big(\mathrm{Flat}_{m,z}(x), \mathrm{Flat}_{m,z}(y), \{\varphi_{m,d}^{x,y}(B_i)\}\big)$$

for all spatial partitions $p := (x, y, \{B_i\}) \in \mathcal{P}^{(m \cdot d)}$.

If $z = \circ \cdots \circ$, then $\mathrm{Flat}_{m,z}$ preserves colors and only applies the permutations $\varphi_{m,|z|}^{x,y}$ as in the previous example. However, in the general case, we obtain for example



It remains to verify that $\mathrm{Flat}_{m,z}$ indeed satisfies all axioms of a spatial partition functor.

Proposition 6.4.11. *$\mathrm{Flat}_{m,z}$ is a fully faithful spatial partition functor.*

Proof. By definition, $\mathrm{Flat}_{m,z}$ respects the concatenation of colors. Furthermore, $\mathrm{Flat}_{m,z}$ respects compositions and involutions since it permutes the points of a spatial partition only based on the upper colors x and the lower colors y. Moreover, $\mathrm{Flat}_{m,z}$ moves points at back levels in consecutive groups of size m to the front, which implies that it respects tensor products. Finally,

$$\mathrm{Flat}_{m,z}(\mathrm{id}_\circ^{(m \cdot |z|)}) = \mathrm{id}_z^{(m)}, \quad \mathrm{Flat}_{m,z}(\mathrm{id}_\bullet^{(m \cdot |z|)}) = \mathrm{id}_{\overline{z}}^{(m)}$$

shows that $\mathrm{Flat}_{m,z}$ maps identity partitions to identity partitions. Thus, $\mathrm{Flat}_{m,z}$ is a spatial partition functor. Additionally, it is fully faithful since the maps $\varphi_{m,d}^{x,y}$ are bijections. □

The following proposition shows that the preimage under $\mathrm{Flat}_{m,z}$ preserves categories of spatial partitions, including rigidity and gradings. This result will be used in Chapter 8, where it allows us to relate the category of a spatial partition quantum group to the category of its projective versions. Note that the proof relies on our generalization of base partitions and the statement would not hold in the context of spatial partition quantum groups defined by Cébron and Weber.

Proposition 6.4.12. *Let $\boldsymbol{n} := (n_1, \dots, n_m) \in \mathbb{N}^m$ and $z \in \{\circ, \bullet\}^*$ with $d := |z| \geq 1$. If $\mathcal{C} \subseteq \mathcal{P}^{(m)}$ is a $\boldsymbol{n}$-graded rigid category of spatial partitions, then*

$$\mathrm{Flat}_{m,z}^{-1}(\mathcal{C}) := \{p \in \mathcal{P}^{(m\cdot d)} \mid \mathrm{Flat}_{m,z}(p) \in \mathcal{C}\}$$

is a $(\boldsymbol{n} \cdots \boldsymbol{n})$-graded rigid category of spatial partitions, where

$$(\boldsymbol{n} \cdots \boldsymbol{n}) := (n_1, \dots, n_m, \dots, n_1, \dots, n_m) \in \mathbb{N}^{m\cdot d}.$$

Proof. Define $\mathcal{D} := \mathrm{Flat}_{m,z}^{-1}(\mathcal{C})$. Since $\mathrm{Flat}_{m,z}$ respects all spatial partition operations, it follows that $\mathcal{D}$ is closed under composition, involution and tensor product. Similarly, $\mathrm{Flat}_{m,z}$ maps identity partitions to identity partitions, which implies $\mathrm{id}_x \in \mathcal{D}$ for all $x \in \{\circ, \bullet\}^*$. Thus, $\mathcal{D}$ is a category of spatial partitions. Assume $\mathcal{C}$ is rigid. Then Definition 6.3.3 shows that there exist spatial partitions $r \in \mathcal{C}(1, z\overline{z})$ and $s \in \mathcal{C}(1, \overline{z}z)$ satisfying the conjugate equations for z and $\overline{z}$. Since $\mathrm{Flat}_{m,z}$ is fully faithful, there exist $\widetilde{r} \in \mathcal{D}(1, \circ\bullet)$ and $\widetilde{s} \in \mathcal{D}(1, \bullet\circ)$ such that $\mathrm{Flat}_{m,z}(\widetilde{r}) = r$ and $\mathrm{Flat}_{m,z}(\widetilde{s}) = s$. These partitions satisfy the conjugate equations for $\circ$ and $\bullet$, which shows that $\mathcal{D}$ is rigid. Finally, assume $\mathcal{C}$ is $\boldsymbol{n}$-graded. Since $\mathrm{Flat}_{m,z}$ only moves levels in consecutive groups of size m to the front, it follows immediately that $\mathcal{D}$ is $(\boldsymbol{n} \cdots \boldsymbol{n})$-graded. □

6.5. Implementation of partitions

In the following, we present efficient algorithms and data structures for partitions and their basic operators. These have been developed by Volz in [87] under the author's supervision and have been implemented in the computer algebra system OSCAR [69] by Volz and the author. We initially focus on the case of partitions on a single level with only white points, and we discuss how these algorithms and data structures can be generalized to colored partitions and multiple levels at the end of the section.

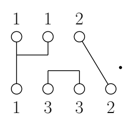
As in Definition 6.2.3, we denote by $\mathcal{P} \subseteq \mathcal{P}^{(1)}$ the set of all partitions on a single level with only white points.

Definition 6.5.1. A partition $p \in \mathcal{P}$ is represented by a tuple (k, ℓ, b), where k is the number of upper points, ℓ is the number of lower points and $b \in \mathbb{N}^{k+\ell}$ encodes the block structure such that two points $1 \leq i, j \leq k + \ell$ are in the same block if and only if $b_i = b_j$.

Example 6.5.2. Consider the partition $p \in \mathcal{P}$ represented by the tuple

$$\big(3,\ 4,\ (1, 1, 2, 1, 3, 3, 2)\big).$$

Then p corresponds to the string diagram



In the following, we identify partitions $p \in \mathcal{P}$ with a corresponding representation (k, ℓ, b). In particular, we write $p := (k, \ell, b) \in \mathcal{P}$ and denote by $|p| := k + \ell$ the *size* of p.

Note that two different representations in the sense of Definition 6.5.1 can correspond to the same partition as string diagram. However, this is precisely the case if both partitions are equivalent in the following sense.

Definition 6.5.3. Let $p := (k, \ell, a), q := (m, n, b) \in \mathcal{P}$ be two partitions. Then p and q are *equivalent* if $k = m$, $\ell = n$ and there exists a bijection

$$\phi\colon \{a_1, \dots, a_{k+\ell}\} \to \{b_1, \dots, b_{k+\ell}\}$$

such that $\phi(a_i) = b_i$ for all $1 \leq i \leq k + \ell$.

Given a partition, it is always possible to normalize its block structure such that the following condition is satisfied.

Definition 6.5.4. Let $p := (k, \ell, b) \in \mathcal{P}$ be a partition. Then p is *normalized* if $b_1 = 1$ and $a_j \leq a_i + 1$ for all $1 \leq i < j \leq k$.

The following algorithm computes the normalization of a given partition using a map data structure. In particular, this allows us to efficiently compare partitions since two partitions are equivalent if and only if their normalized versions agree.

Algorithm 1 Normalize p.

Input: $p = (k, \ell, b)$
Output: an equivalent partition satisfying Definition 6.5.4

$m \leftarrow 1$
$r \leftarrow$ empty map structure
$a \leftarrow (b_1, \dots, b_{k+\ell})$
for $i = 1, \dots, k + \ell$ **do**
    if $r(a_i)$ not defined **then**
        $r(a_i) \leftarrow m$
        $m \leftarrow m + 1$
    $a_i \leftarrow r(a_i)$
return (k, ℓ, a)

Proposition 6.5.5. *Let $p \in \mathcal{P}$ be a partition. Then Algorithm 1 computes an equivalent partition that is normalized in the sense of Definition 6.5.4 using $\mathcal{O}(|p|)$ accesses to a map structure.*

Proof. First, the result is equivalent to p since it has k upper points, ℓ lower points and its block structure is related to p via the bijection r. Moreover, the result is normalized because $r(a_1) = 1$ and $r(a_i) \geq r(a_j) + 1$ for all $1 \leq j < i \leq k + \ell$ by construction. Finally, Algorithm 1 requires $2 \cdot (k+\ell) = \mathcal{O}(|p|)$ accesses to the map structure in the worst case. $\square$

Remark 6.5.6. By implementing the map structure in Algorithm 1 as a balanced search tree, we obtain a worst-case time complexity of $\mathcal{O}(n \log n)$ for $n := |p|$. This can be improved to an average complexity of $\mathcal{O}(n)$ by using a hash map.

Example 6.5.7. Consider the partition

$$p := \big(2,\ 3,\ (1, 3, 1, 5, 5, 3)\big) \in \mathcal{P}.$$

Then p is not normalized since $b_2 = 3 > b_1 + 1$. However, Algorithm 1 computes the map

$$r(1) = 1, \qquad r(3) = 2, \qquad r(5) = 3,$$

which yields the normalized partition

$$\big(2,\ 3,\ (1, 2, 1, 3, 3, 2)\big).$$

In particular, this shows that p is equivalent to the partition from Definition 6.5.2.

Next, we consider the basic operations of partitions. In the case of the tensor product and composition, we will need the following auxiliary procedure to ensure that the blocks of the two input partitions $p := (k, \ell, a)$ and $q := (m, n, b)$ are *disjoint*, i.e.

$$\{a_1, \ldots, a_{k+\ell}\} \cap \{b_1, \ldots, b_{n+m}\} = \emptyset.$$

Algorithm 2 Make the blocks of p disjoint from the blocks of q.

Input: $p = (k, \ell, a)$, $q = (m, n, b)$
Output: a partition equivalent to p with blocks disjoint from q

```
t ← 1 + max{b_i | 1 ≤ i ≤ n + m}
c ← (a_1 + t, ..., a_{k+ℓ} + t)
return (k, ℓ, c)
```

Proposition 6.5.8. *Let $p, q \in \mathcal{P}$ be partitions. Then Algorithm 2 computes a partition equivalent to p with blocks disjoint from q in time $\mathcal{O}(|p| + |q|)$.*

Proof. The result is equivalent to p since it has the same number of upper and lower points, and their blocks are related by the bijection $x \mapsto x + t$. Moreover, the resulting blocks are disjoint from q since $c_i = a_i + t > a_i + b_j \geq b_j$ for all $1 \leq i \leq |p|$ and $1 \leq j \leq |q|$. $\square$

In the following, we assume that without loss of generality the input partitions to our algorithms always have disjoint blocks. This allows us to directly compute the involution and tensor product of partitions by rearranging and merging the corresponding block structures.

Proposition 6.5.9. *Let $p, q \in \mathcal{P}$ be partitions with disjoint blocks. Then*

1. *Algorithm 3 computes the involution p^* in time $\mathcal{O}(|p|)$,*

Algorithm 3 Involution p^*.

Input: $p = (k, \ell, b)$

Output: p^*

1: $a \leftarrow (b_{k+1}, \ldots, b_{k+\ell}, b_1, \ldots, b_k)$

2: **return** (ℓ, k, a)

Algorithm 4 Tensor product $p \otimes q$.

Input: $p = (k, \ell, a)$, $q = (m, n, b)$

Output: $p \otimes q$

1: $c \leftarrow (a_1, \ldots, a_k, b_1, \ldots, b_m, a_{k+1}, \ldots, a_{k+\ell}, b_{m+1}, \ldots, b_{m+n})$

2: **return** $(k + m, \ell + n, c)$

2. *Algorithm 4 computes the tensor product $p \otimes q$ in time $\mathcal{O}(|p| + |q|)$.*

Proof. This follows immediately. $\square$

Next, we consider the composition of partitions. However, we first need to introduce an additional data structure.

Definition 6.5.10. A *union-find structure U* is a data structure representing an equivalence relation on $\mathbb{N}$ that supports the following operations:

1. $union_U(i, j)$: Merge the equivalence classes $[i]$ and $[j]$.
2. $find_U(i)$: Return a canonical representative of the class $[i]$ such that

$$[i] = [j] \iff find_U(i) = find_U(j) \qquad \forall i, j \in \mathbb{N}.$$

Initially, an empty union-find structure represents the equivalence relations given by the singletons $[i] := \{i\}$ for all $i \in \mathbb{N}$.

Remark 6.5.11. A union-find structure can be efficiently implemented using a disjoint-set forest [23], which has a time complexity of $\mathcal{O}(n \log n)$ for n subsequent union and find operations when performing path compression. This can be further improved to $\mathcal{O}(n\alpha(n))$ when additionally performing union by rank, where α denotes the inverse Ackerman function [79].

Using a union-find structure, we can implement the composition of two partitions. As before, we assume that both partitions have disjoint blocks.

Proposition 6.5.12. *Let $p, q \in \mathcal{P}$ be partitions with disjoint blocks that are composable. Then Algorithm 5 computes the composition $p \cdot q$ using $\mathcal{O}(|p| + |q|)$ union and find operations.*

Algorithm 5 Composition $p \cdot q$.

Input: $p = (\ell, m, a)$, $q = (k, \ell, b)$
Output: $p \cdot q$

$U \leftarrow$ empty union-find structure
for $i = 1, \ldots, \ell$ **do**
    $\mathit{union}_U(a_i, b_{k+i})$
$c \leftarrow (b_1, \ldots, b_k, a_{\ell+1}, \ldots, a_{\ell+m})$
for $i = 1, \ldots, k+m$ **do**
    $c_i \leftarrow \mathit{find}_U(c_i)$
return (k, m, c)

Proof. Since the blocks of p and q are disjoint, we can compute the composition by first identify the lower labels of q with the corresponding upper labels of p using *union* operations. We then relabel the upper points of q and lower points of p with new canonical labels using *find*. The runtime of the algorithm is determined by the $\ell = \mathcal{O}(|p|+|q|)$ *union* operations and the $k + m = \mathcal{O}(|p| + |q|)$ *find* operations. $\square$

Remark 6.5.13. When implementing the union-find structure as a disjoint-set forest using path-compression (see Definition 6.5.11), then Algorithm 5 has a worst-case time complexity of $\mathcal{O}(n \log n)$ for $n := |p| + |q|$. Note that it is also possible to achieve a time complexity of $\mathcal{O}(n)$ by first constructing a graph with $\mathcal{O}(n)$ vertices $\{a_1, \ldots, a_{\ell+m}, b_1, \ldots, b_{k+\ell}\}$ and adding the $\mathcal{O}(n)$ edges (a_i, b_{k+i}) for $1 \leq i \leq \ell$ during the *union* operations. Then, we can use a depth-first search to compute all connected components in time $\mathcal{O}(n)$ and select a canonical vertex per component as a final label that is returned during *find*.

Finally, we discuss how the previous algorithms can be extended to colored and spatial partitions. In the case of colored partitions, we can modify Definition 6.5.1 by storing a string of upper and lower colors instead of the corresponding number of points.

Definition 6.5.14. A colored partition $p \in \mathcal{P}^{(1)}$ is represented by a tuple (x, y, b), where $x, y \in \{\circ, \bullet\}^*$ represent the upper and lower colors, and $b \in \mathbb{N}^{|x|+|y|}$ encodes the block structure as in Definition 6.5.1.

Given a colored partition of this form, all previous algorithms can be applied after replacing the number of upper points k and the number of lower points ℓ with $|x|$ and $|y|$ respectively. Moreover, in the precondition of the composition, it no longer suffices to compare only the number of points, but we need to compare their complete color strings.

In the case of spatial partitions, we can use the functor $\mathrm{Flat}_{1,m}$ from Section 6.4 to reduce this case to the case of colored partitions on a single level.

Definition 6.5.15. A spatial partition $p \in \mathcal{P}^{(m)}$ is represented by a pair (m, q), where $m \in \mathbb{N}$ denotes the number of levels and $q \in \mathcal{P}^{(1)}$ represents the colored partition $q := \mathrm{Flat}_{1,m}(p)$.

Since $\mathrm{Flat}_{1,m}$ is fully faithful, we can represent any spatial partition by such a pair. Moreover, $\mathrm{Flat}_{1,m}$ respects all category operations, which allows us to reduce all operations to the case of colored partitions after verifying that the number of levels m coincides.

6.6. Undecidability of partition problems

Consider a category of partitions $\mathcal{C} \subseteq \mathcal{P}$ in the sense of Definition 6.2.3. The following algorithmic problems arise naturally when studying $\mathcal{C}$ or its corresponding quantum group introduced in Chapter 7:

1. Is $p \in \mathcal{C}$ for a partition $p \in \mathcal{P}$?
2. What is the number of partitions in $\mathcal{C}(k, \ell)$?

In this section, we show that there exists a recursively enumerable category of partitions $\mathcal{C}$ for which these problems are undecidable, i.e. there is no Turing machine that solves any of the previous problems for all possible input values.

Here, a category of partitions $\mathcal{C}$ is called *recursively enumerable* if there exists a Turing machine that enumerates all partitions $p \in \mathcal{C}$. Note that if we omit this requirement, the previous statement follows directly from [72], where it is shown that the set of all categories of partitions is uncountable while the set of all Turing machines in countable. However, in this case, it may not even be possible to explicitly describe the category $\mathcal{C}$ or enumerate all its elements.

Our proof of the initial statement relies on a result by Raum and Weber [72] that allows us to perform a reduction from the identity problem for varieties of groups, which was shown to be undecidable by Kleiman in [54]. Thus, we begin by introducing varieties of groups and their corresponding identity problem. See [68] for further information on the general theory of varieties of groups and [53] for algorithmic problems in this context.

In the following, all groups are considered to be multiplicative with 1 as the identity element. Denote by $\mathbb{F}_\infty$ the free group on countably many generators $x_1, x_2, \ldots$. If $w \in \mathbb{F}_\infty$ is a word in the variables $x_1, \ldots, x_n$, then w is an *identity* in a group G if $w(g_1, \ldots, g_n) = 1$ for all $g_1, \ldots, g_n \in G$, where $w(g_1, \ldots, g_n) \in G$ denotes the element obtained by substituting each generator x_i with g_i. Using this notation, we can define varieties of groups as follows.

Definition 6.6.1. Let $W \subseteq \mathbb{F}_\infty$. The *variety of groups* $\mathcal{V}(W)$ defined by W is the class of all groups G such that w is an identity in G for all $w \in W$.

If $\mathcal{V}$ is a variety of groups, then a word $w \in \mathbb{F}_\infty$ is an *identity* in $\mathcal{V}$ if it is an identity in all $G \in \mathcal{V}$. Moreover, we say that a variety of groups $\mathcal{V}(W)$ is *finitely based* if the set W is finite. In this context, Kleiman [54] showed that the identity problem for varieties of groups is undecidable in the following sense.

Proposition 6.6.2. *There exists a finitely based variety of groups $\mathcal{V}(W)$ such that the problem of determining whether a word $w \in \mathbb{F}_\infty$ is an identity in $\mathcal{V}(W)$ is undecidable.*

Proof. See [54] or the proof sketch in [53]. An alternate proof can be found in [1]. □

Next, we recall a general result from the theory of varieties of groups that establishes a bijection between varieties of groups and fully invariant subgroups of $\mathbb{F}_\infty$. Here, a subgroup $H \subseteq G$ is *fully invariant* if $\phi(H) \subseteq H$ for every endomorphism $\phi \in \mathrm{End}(G)$.

Proposition 6.6.3. *There exists a bijection between varieties of groups $\mathcal{V}$ and fully invariant subgroups $H \subseteq \mathbb{F}_\infty$ such that*

$$H = \{w \in \mathbb{F}_\infty \mid w \textit{ is an identity in } \mathcal{V}\}.$$

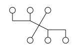
Proof. See [68, Theorem 14.31]. □

Remark 6.6.4. Consider a variety of groups $\mathcal{V}(W)$. By [68, 12.31], the corresponding subgroup $H \subseteq \mathbb{F}_\infty$ in the previous proposition consists of all words that can be obtained by applying a finite number of the following operations to elements of W:

$$v \mapsto v^{-1}, \qquad (v, w) \mapsto vw, \qquad v \mapsto v(u_1, \dots, u_k) \quad \forall u_1, \dots, u_k \in \mathbb{F}_\infty.$$

In particular, this implies that if a variety of groups is finitely based, then the corresponding subgroup H is recursively enumerable.

In [72], Raum and Weber constructed an embedding of varieties of groups into categories of partitions. It is based on a bijection between certain categories of partitions and sS_∞-invariant normal subgroups of $\mathbb{Z}_2^{*\infty}$. Here, $\mathbb{Z}_2^{*\infty}$ denotes the countably infinite free product of $\mathbb{Z}_2$ with canonical generators $a_1, a_2, \dots$ in each factor. Moreover, a subgroup $H \subseteq \mathbb{Z}_2^{*\infty}$ is called *sS_∞-invariant* if it is invariant under all endomorphisms $\phi \in \mathrm{End}(\mathbb{Z}_2^{*\infty})$ of the form $\phi(a_i) = a_{f(i)}$ for some function $f \colon \mathbb{N} \to \mathbb{N}$.

Proposition 6.6.5. *There exists a bijection between sS_∞-invariant normal subgroups $H \subseteq \mathbb{Z}_2^{*\infty}$ and categories of partitions $\mathcal{C} \subseteq \mathcal{P}$ containing*  *such that*

$$\mathcal{C}(0, n) = \{p(i_1, \dots, i_n) \mid i_1, \dots, i_n \in \mathbb{N}^n,\ a_{i_1} \cdots a_{i_n} \in H\} \qquad \forall n \in \mathbb{N}.$$

Here, $p(i_1, \dots, i_n)$ denotes the partition with n lower points, where two points $k, \ell \in \{1, \dots, n\}$ are in the same block if and only if $i_k = i_\ell$.

Proof. See [72, Theorem 4.4 & 4.6]. □

Now, consider the subgroup $E \subseteq \mathbb{Z}_2^{*\infty}$ of all words of even length defined by

$$E := \{a_{i_1} \cdots a_{i_{2k}} \mid k \in \mathbb{N},\ i_1, \dots, i_{2k} \in \mathbb{N}\}.$$

It can be verified that E is isomorphic to $\mathbb{F}_\infty$ via the map

$$\Phi \colon \mathbb{F}_\infty \to E, \quad x_n \mapsto a_1 a_{n+1} \qquad \forall n \in \mathbb{N}.$$

The following proposition shows that fully invariant subgroups of E yield sS_∞-invariant normal subgroups of $\mathbb{Z}_2^{*\infty}$ under this isomorphism. In particular, this proposition corrects the original version in [72], which does not hold in full generality.

Proposition 6.6.6. *The subgroup $E \subseteq \mathbb{Z}_2^{*\infty}$ is sS_∞-invariant and normal. In particular, every fully invariant subgroup $H \subseteq E$ is sS_∞-invariant and normal in $\mathbb{Z}_2^{*\infty}$.*

Proof. Let $a = a_{i_1} \dots a_{i_{2k}} \in E$ and $\phi \in \mathrm{End}(\mathbb{Z}_2^{*\infty})$ of the form $\phi(a_i) = a_{f(i_1)} \dots a_{f(i)}$ for some function $f \colon \mathbb{N} \to \mathbb{N}$. Then $\phi(a) = a_{f(i_1)} \dots a_{f(i_{2k})} \in E$ such that E is sS_∞-invariant. Similarly, let $\phi \in \mathrm{End}(\mathbb{Z}_2^{*\infty})$ be an inner automorphism of the form $\phi(x) = bxb^{-1}$ for some $b \in \mathbb{Z}_2^{*\infty}$. Then, we have $\phi(a) = bab^{-1} \in E$ since both a and bb^{-1} have even length. Since a subgroup is normal if and only if it is invariant under inner automorphism, it follows that E is a normal subgroup.

In particular, this shows that any of the previous endomorphisms restricts to an endomorphism $\phi|_E \in \mathrm{End}(E)$. Thus, any fully invariant subgroup $H \subseteq E$ is also invariant under these endomorphisms and is therefore sS_∞-invariant and normal in $\mathbb{Z}_2^{*\infty}$. $\square$

Therefore, by combining the previous proposition with Definition 6.6.3 and Definition 6.6.5, we obtain an embedding of varieties of groups into the set of categories of partitions.

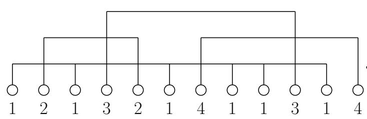
In particular, we can associate to each word $w \in \mathbb{F}_\infty$ a partition $p(w)$ by first applying the isomorphism $\mathbb{F}_\infty \cong E$ to obtain an element $a_{i_1} \dots a_{i_n} \in E$ and then constructing the partition $p(i_1, \dots, i_n)$ as defined in Definition 6.6.5.

Example 6.6.7. Consider the word $w := x_1 x_2 x_1^{-1} x_3^{-1} x_2 x_3 \in \mathbb{F}_\infty$, which corresponds to the element

$$a_1 a_2 a_1 a_3 a_2 a_1 a_4 a_1 a_1 a_3 a_1 a_4 \in \mathbb{Z}_2^{*\infty}$$

under the isomorphism $\mathbb{F}_\infty \cong E$. Then the corresponding partition $p(w)$ is given by

$p(w) =$



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Using the previous embedding of varieties of groups into categories of partitions, we can reduce the identity problem for varieties to the membership problem for categories of partitions.

Lemma 6.6.8. *Let $\mathcal{V}$ be a finitely based variety of groups. Then there exists a recursively enumerable category of partitions $\mathcal{C} \subseteq \mathcal{P}$ such that*

$$w \text{ is an identity in } \mathcal{V} \iff p(w) \in \mathcal{C} \qquad \forall w \in \mathbb{F}_\infty.$$

Proof. By Definition 6.6.3, $\mathcal{V}$ corresponds to a fully invariant subgroup $H \subseteq \mathbb{F}_\infty$ that can be identified with a sS_∞-invariant normal subgroup of $\mathbb{Z}_2^{*\infty}$ using the isomorphism $\mathbb{F}_\infty \cong E$ and Definition 6.6.6. Moreover, this subgroup corresponds to a category of partitions $\mathcal{C}$ by Definition 6.6.5. Following this identification, every identity $w \in H$ yields a partition $p(w) \in \mathcal{C}$. Conversely, if $p(w) \in \mathcal{C}$ for some $w \in \mathbb{F}_\infty$, then $w \in H$ since the mapping in Definition 6.6.5 is bijective.

It remains to show that the category $\mathcal{C}$ is recursively enumerable. Let $w \in H$ and denote by $a(w)$ the corresponding element in E. Then the sets

$$\{(i_1, \dots, i_n) \mid n \in \mathbb{N},\, i_1, \dots, i_n \in \mathbb{N}^n,\, a_{i_1} \cdots a_{i_n} = a(w)\}$$

can be enumerated by generating all words $a_{i_1} \cdots a_{i_n}$ and checking for equality. Since the set H is recursively enumerable by Definition 6.6.4, we can interleave the elements of the previous sets such that their union and the set $\bigcup_{n\in\mathbb{N}} \mathcal{C}(0, n)$ are also recursively enumerable. Because partitions in $\mathcal{C}(k, \ell)$ are precisely the rotated version of partitions in $\mathcal{C}(0, k + \ell)$, it follows that the category $\mathcal{C}$ is recursively enumerable. □

Using the previous lemma, we can finally transfer the undecidability of the identity problem for varieties of groups to the membership problem for categories of partitions.

Theorem 6.6.9. *There exists a recursively enumerable category of partitions $\mathcal{C} \subseteq \mathcal{P}$ such that the problem of determining whether $p \in \mathcal{C}$ for a partition $p \in \mathcal{P}$ is undecidable.*

Proof. Let $\mathcal{V}$ be the variety of groups from Definition 6.6.2 for which the word problem is undecidable, and let $\mathcal{C}$ be the category of partitions obtained from Definition 6.6.8. Then any Turing machine that decides whether $p \in \mathcal{C}$ for arbitrary $p \in \mathcal{P}$ could be used to solve the identity problem for $\mathcal{V}$. Thus, the membership problem for $\mathcal{C}$ is also undecidable. □

Additionally, we obtain as an immediate consequence that the problem of counting partitions of a given size is also undecidable.

Corollary 6.6.10. *There exists a recursively enumerable category of partitions $\mathcal{C} \subseteq \mathcal{P}$ such that there is no Turing machine that computes $|\mathcal{C}(k, \ell)|$ for all $k, \ell \in \mathbb{N}$.*

Proof. Let $\mathcal{C}$ be the category of partitions from Definition 6.6.9. Assume that there exists a Turing machine T that computes $|\mathcal{C}(\cdot, \cdot)|$ and that we are given a partition $p \in \mathcal{P}$. We show that T allows us to determine if $p \in \mathcal{C}$, contradicting Definition 6.6.9. First, we use T to compute the number of partitions $m := |\mathcal{C}(k, \ell)|$, where $k, \ell \in \mathbb{N}$ are the numbers of upper and lower points of p. Next, we start enumerating all possible partitions in $\mathcal{C}$ until we find m distinct partitions $p_1, \dots, p_m \in \mathcal{C}(k, \ell)$ after a finite amount time. Then, we stop the enumeration and decide if $p \in \{p_1, \dots, p_m\} = \mathcal{C}(k, \ell)$. □

Remark 6.6.11. Note that we have only shown that the category $\mathcal{C}$ in the previous theorem is recursively enumerable, while the initial variety of groups is finitely based. It remains open if the category $\mathcal{C}$ is finitely generated and if Definition 6.6.9 holds more generally for finitely generated categories. Using the previous approach, one could try to show that if $H \subseteq \mathbb{Z}_2^{*\infty}$ is a finitely generated a fully invariant subgroup of E, then H is a finitely generated as sS_∞-invariant normal subgroup of $\mathbb{Z}_2^{*\infty}$.

7. Quantum groups based on spatial partitions

Given a category of spatial partitions in the sense of Cébron and Weber [25], we can construct a corresponding compact matrix quantum group by first realizing each spatial partition as a linear operator and then applying Woronowicz's Tannaka-Krein duality to the resulting tensor category. In the case of spatial partitions on one level, this yields precisely unitary easy quantum groups [10, 80], including for example the free orthogonal and free unitary quantum groups [88] and the quantum permutation group [89].

In this chapter, we show that quantum groups can be constructed from categories of spatial partitions containing our new base partitions introduced in Chapter 6. We then describe these quantum groups in terms of universal C^*-algebras and show that the free orthogonal quantum groups $O^+(F_\sigma)$ can be obtained form our new base partitions. Finally, we show that all resulting quantum groups are invariant under permuting levels, which implies that our new base partitions yield the same class of quantum groups as in [25].

Our construction of spatial partition quantum groups follows that in [25]. However, we address additional technicalities that arise from combining colors with our new base partitions.

7.1. Spatial partition quantum groups

Let $n \in \mathbb{N}$. In the following, we use the notation $[n] := \{1, \dots, n\}$ and more generally $[\boldsymbol{n}] := [n_1] \times \cdots \times [n_m]$ for all $\boldsymbol{n} := (n_1, \dots, n_m) \in \mathbb{N}^m$. Additionally, we introduce the Hilbert spaces $\mathbb{C}^{\boldsymbol{n}} := \mathbb{C}^{n_1} \otimes \cdots \otimes \mathbb{C}^{n_m}$ with canonical bases given by

$$e_{\boldsymbol{i}} := e_{i_1} \otimes \cdots \otimes e_{i_m} \qquad \forall \boldsymbol{i} := (i_1, \dots, i_m) \in [\boldsymbol{n}].$$

Moreover, recall from Definition 2.5.9 and Definition 2.5.10 the set $\{\circ, \bullet\}^*$, consisting of all finite words in the colors $\circ$ and $\bullet$, and its use in the context of Hilbert spaces.

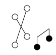
To define spatial partition quantum groups, we need to associate a linear operator $T_p^{(\boldsymbol{n})}$ to each spatial partition $p \in \mathcal{P}^{(m)}$. This requires the following additional notation.

Definition 7.1.1. Let $p \in \mathcal{P}^{(m)}(x, y)$ be a spatial partition and $\boldsymbol{i}_1, \dots, \boldsymbol{i}_{|x|} \in \mathbb{N}^m$, $\boldsymbol{j}_1, \dots, \boldsymbol{j}_{|y|} \in \mathbb{N}^m$ be multi-indices of the form

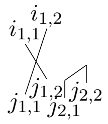
$$\boldsymbol{i}_k = (i_{k,1}, \dots, i_{k,m}), \quad \boldsymbol{j}_k = (j_{k,1}, \dots, j_{k,m})$$

for all $1 \leq k \leq |x|$ and $1 \leq k \leq |y|$ respectively. Label each upper point (k, ℓ) of p with the index $i_{k,\ell}$ and each lower point $(|x| + k, \ell)$ with the index $j_{k,\ell}$. Then define

$$(\delta_p)^{\boldsymbol{i}_1,\ldots,\boldsymbol{i}_{|x|}}_{\boldsymbol{j}_1,\ldots,\boldsymbol{j}_{|y|}} := \begin{cases} 1 & \text{if all points in each block have the same labels,} \\ 0 & \text{otherwise.} \end{cases}$$

Example 7.1.2. Consider the spatial partition $p :=$  $\in \mathcal{P}^{(2)}(\circ, \circ\bullet)$ and the indices $\boldsymbol{i}_1 := (i_{1,1}, i_{1,2}), \boldsymbol{j}_1 := (j_{1,1}, j_{1,2}), \boldsymbol{j}_2 := (j_{2,1}, j_{2,2}) \in \mathbb{N}^2$. Then

$$(\delta_p)^{\boldsymbol{i}_1}_{\boldsymbol{j}_1,\boldsymbol{j}_2} =$$



$$= \delta_{i_{1,1}j_{1,2}}\delta_{i_{1,2}j_{1,1}}\delta_{j_{2,1}j_{2,2}},$$

where $\delta_{k\ell}$ denotes the Kronecker delta.

We can now assign linear operators to spatial partitions as follows.

Definition 7.1.3. Let $p \in \mathcal{P}^{(m)}(x, y)$ be a $\boldsymbol{n}$-graded spatial partition. Define the linear operator

$$T_p^{(\boldsymbol{n})} : (\mathbb{C}^{\boldsymbol{n}})^{\otimes x} \to (\mathbb{C}^{\boldsymbol{n}})^{\otimes y}, \quad \left(T_p^{(\boldsymbol{n})}\right)^{\boldsymbol{i}_1,\ldots,\boldsymbol{i}_\ell}_{\boldsymbol{j}_1,\ldots,\boldsymbol{j}_k} = \left(\delta_p\right)^{\boldsymbol{j}_1,\ldots,\boldsymbol{j}_k}_{\boldsymbol{i}_1,\ldots,\boldsymbol{i}_\ell},$$

where $k := |x|$, $\ell := |y|$ and the coordinates are with respect to the canonical bases

$$\left(e^{x_1}_{\boldsymbol{j}_1} \otimes \cdots \otimes e^{x_k}_{\boldsymbol{j}_k}\right)_{\boldsymbol{j}_1,\ldots,\boldsymbol{j}_k \in [\boldsymbol{n}]}, \quad \left(e^{y_1}_{\boldsymbol{i}_1} \otimes \cdots \otimes e^{y_\ell}_{\boldsymbol{i}_\ell}\right)_{\boldsymbol{i}_1,\ldots,\boldsymbol{i}_\ell \in [\boldsymbol{n}]}$$

of $(\mathbb{C}^{\boldsymbol{n}})^{\otimes x}$ and $(\mathbb{C}^{\boldsymbol{n}})^{\otimes y}$ respectively.

A spatial partition quantum group is a compact matrix quantum group whose intertwiner spaces between tensor powers of its fundamental representation u° and its conjugate representation $u^\bullet$ are spanned by linear operators associated with a category of spatial partitions.

In contrast to [25], the conjugate representation $u^\bullet$ will not necessarily be given by $\overline{u^\circ}$ but depend on the corresponding category of spatial partitions. Furthermore, we incorporate a possible change of basis in the definition such that the notion of spatial partition quantum group is compatible with isomorphisms of compact matrix quantum groups.

Definition 7.1.4. Let G be a compact matrix quantum group with fundamental representation u° on a Hilbert spaces V and a unitary representation $u^\bullet$ on $\overline{V}$. Then G is a *spatial partition quantum group* if there exists a $\boldsymbol{n}$-graded rigid category of spatial partitions $\mathcal{C} \subseteq \mathcal{P}^{(m)}$ and a unitary $Q : \mathbb{C}^{\boldsymbol{n}} \to V$ such that

$$\operatorname{Hom}(u^x, u^y) = \operatorname{span}\left\{Q^{\otimes y} \cdot T_p^{(\boldsymbol{n})} \cdot \left(Q^{-1}\right)^{\otimes x} \mid p \in \mathcal{C}(x, y)\right\} \qquad \forall x, y \in \{\circ, \bullet\}^*.$$

Since the category $\mathcal{C}$ contains a pair of spatial partitions satisfying the conjugate equations, it follows that $u^\bullet$ is conjugate to u°. Before discussing how this conjugate representation $u^\bullet$ depends on $\mathcal{C}$, we first show that for every rigid category of spatial partitions, there exists a corresponding spatial partition quantum group. This requires the following proposition, which states that the mapping $p \mapsto T_p^{(\boldsymbol{n})}$ is almost functorial.

Proposition 7.1.5. *Let* $\boldsymbol{n} := (n_1, \ldots, n_m) \in \mathbb{N}^m$ *and* $p, q \in \mathcal{P}^{(m)}$ *be* $\boldsymbol{n}$*-graded spatial partitions. Then*

1. $T_p^{(\boldsymbol{n})} \otimes T_q^{(\boldsymbol{n})} = T_{p \otimes q}^{(\boldsymbol{n})}$,

2. $\left(T_p^{(\boldsymbol{n})}\right)^* = T_{p^*}^{(\boldsymbol{n})}$,

3. $T_p^{(\boldsymbol{n})} \cdot T_q^{(\boldsymbol{n})} = N^\alpha \cdot T_{pq}^{(\boldsymbol{n})}$, *where* $N := n_1 \cdots n_m$ *and* $\alpha \in \mathbb{N}$ *denotes the number of removed loops when composing p and q (if the composition is defined).*

Proof. See [25, Proposition 3.7]. □

Using the previous proposition, we can show the existence of spatial partition quantum groups using Woronowicz's Tannaka-Krein duality.

Theorem 7.1.6. *Let* $\mathcal{C} \subseteq \mathcal{P}^{(m)}$ *be a* $\boldsymbol{n}$*-graded rigid category of spatial partitions. Then there exists a unique compact matrix quantum group* $G_{\boldsymbol{n}}(\mathcal{C})$ *with fundamental representation* u° *on* $V := \mathbb{C}^{\boldsymbol{n}}$ *and a unitary representation* $u^\bullet$ *on* $\overline{V}$ *such that*

$$\mathrm{Hom}(u^x, u^y) = \mathrm{span}\left\{T_p^{(\boldsymbol{n})} \mid p \in \mathcal{C}(x, y)\right\} \quad \forall x, y \in \{\circ, \bullet\}^*.$$

Proof. Define the linear subspaces

$$\widehat{\mathcal{C}}(x, y) := \mathrm{span}\left\{T_p^{(\boldsymbol{n})} \mid p \in \mathcal{C}(x, y)\right\} \subseteq B(V^{\otimes x}, V^{\otimes y}) \qquad \forall x, y \in \{\circ, \bullet\}^*$$

Then Definition 7.1.5 implies that $\widehat{\mathcal{C}}$ is closed under composition, involution and tensor product as in Definition 2.5.14. Furthermore, one verifies that $T_{\mathrm{id}_x}^{(\boldsymbol{n})} = \mathrm{id}_{V^{\otimes x}} \in \widehat{\mathcal{C}}(x, x)$ for all $x \in \{\circ, \bullet\}^*$. Since $\mathcal{C}$ is rigid, there exists a pair of spatial partitions $r \in \mathcal{C}(1, \circ\bullet)$ and $s \in \mathcal{C}(1, \bullet\circ)$ satisfying the conjugate equations. By Definition 7.1.5, the operators $R := T_r^{(\boldsymbol{n})}$ and $S := T_s^{(\boldsymbol{n})}$ also satisfy the conjugate equations for u° and $u^\bullet$ because no closed loops are removed when composing r and s. Therefore, the subspaces $\widehat{\mathcal{C}}(x, y)$ form a two-colored representation category in the sense of Definition 2.5.14, and the statement follows by applying Definition 2.5.15. □

Remark 7.1.7. As discussed in Section 2.5, the quantum group $G_{\boldsymbol{n}}(\mathcal{C})$ from the previous theorem is uniquely determined by the following universal property. Let H be a compact matrix quantum group with fundamental representation w° on a Hilbert space W and a unitary representation $w^\bullet$ on $\overline{W}$. If there exists a unitary $Q \colon \mathbb{C}^{\boldsymbol{n}} \to W$ such that

$$\mathrm{Hom}(u^x, u^y) \subseteq \left(Q^{-1}\right)^{\otimes y} \cdot \mathrm{Hom}(w^x, w^y) \cdot Q^{\otimes x} \quad \forall x, y \in \{\circ, \bullet\}^*,$$

then H is a subgroup of G via the map $u^\circ \mapsto Q^{-1} w^\circ Q$.

Remark 7.1.8. It follows directly from the definition that a compact matrix quantum group is a spatial partition quantum group if and only if it is equivalent to $G_{\boldsymbol{n}}(\mathcal{C})$ for some $\boldsymbol{n}$-graded rigid category of spatial partitions $\mathcal{C}$.

Example 7.1.9. By applying Definition 7.1.6 to rigid categories of spatial partition $\mathcal{C} \subseteq \mathcal{P}^{(1)}$, we obtain precisely unitary easy quantum groups. These include for example all orthogonal easy quantum groups such as the quantum permutation group S_n^+ and the free orthogonal quantum group O_n^+, but also further examples such as the free unitary quantum group U_n^+. We refer to [10, 73, 80] for further information and additional examples of quantum groups obtained from categories of partitions on one level.

Examples of more general non-easy quantum groups can be found in [25, 31]. These include various product constructions, as well as the projective orthogonal group PO_n corresponding to the category $\mathcal{P}_2^{(2)}$ of all spatial pair partitions on two levels. Moreover, we will construct new examples of spatial partition quantum groups in Section 7.2 and Chapter 8.

Next, we return to the conjugate representations $u^\bullet$ of spatial partition quantum groups and show that $u^\bullet$ can be expressed in terms of u° and the following unitaries $F_\sigma^{(\boldsymbol{n})}$.

Definition 7.1.10. Let $\sigma \in S_m$ be a $\boldsymbol{n}$-graded permutation. Define the linear map

$$F_\sigma^{(\boldsymbol{n})} \colon \mathbb{C}^{\boldsymbol{n}} \to \mathbb{C}^{\boldsymbol{n}},$$

$$F_\sigma^{(\boldsymbol{n})}(e_{i_1} \otimes \cdots \otimes e_{i_m}) = e_{i_{\sigma^{-1}(1)}} \otimes \cdots \otimes e_{i_{\sigma^{-1}(m)}} \quad \forall (i_1, \dots, i_m) \in [\boldsymbol{n}].$$

Note that the mapping $\sigma \mapsto F_\sigma^{(\boldsymbol{n})}$ defines a unitary representation of $\boldsymbol{n}$-graded permutations on $\mathbb{C}^{\boldsymbol{n}}$, i.e. we have

$$F_\sigma^{(\boldsymbol{n})} \cdot F_\tau^{(\boldsymbol{n})} = F_{\sigma\tau}^{(\boldsymbol{n})}, \qquad \big(F_\sigma^{(\boldsymbol{n})}\big)^* = \big(F_\sigma^{(\boldsymbol{n})}\big)^{-1} = F_{\sigma^{-1}}^{(\boldsymbol{n})}$$

for all $\boldsymbol{n}$-graded $\sigma, \tau \in S_m$. Additionally, we denote by $\overline{F}_\sigma^{(\boldsymbol{n})}$ the corresponding conjugate operator.

Using the linear maps $F_\sigma^{(\boldsymbol{n})}$, we can describe the conjugate representation $u^\bullet$ of spatial partition quantum groups of the form $G_{\boldsymbol{n}}(\mathcal{C})$.

Proposition 7.1.11. *Let $\mathcal{C} \subseteq \mathcal{P}^{(m)}$ be a $\boldsymbol{n}$-graded rigid category of spatial partitions containing duality partitions $\sigma_{\circ\bullet}$ and $\sigma_{\bullet\circ}^{-1}$ for some $\sigma \in S_m$. Then the conjugate representation $u^\bullet$ of $G_{\boldsymbol{n}}(\mathcal{C})$ is given by $u^\bullet = \overline{F}_\sigma^{(\boldsymbol{n})}\, \overline{u^\circ}\, \big(\overline{F}_\sigma^{(\boldsymbol{n})}\big)^{-1}$.*

Proof. Let u° denote the fundamental representation of $G_{\boldsymbol{n}}(\mathcal{C})$. Define $r := \sigma_{\circ\bullet}$ and $s := \sigma_{\bullet\circ}^{-1}$. Since r and s form a pair of duality partitions, Definition 7.1.5 implies that $R := T_r^{(\boldsymbol{n})}$ and $S := T_s^{(\boldsymbol{n})}$ satisfy the conjugate equations for u° and $u^\bullet$. Hence, Definition 2.5.13 yields $u^\bullet = F\overline{u^\circ}F^{-1}$, where F is defined by $F_{\boldsymbol{j}}^{\boldsymbol{i}} = R^{\boldsymbol{j},\boldsymbol{i}}$ for all $\boldsymbol{i}, \boldsymbol{j} \in [\boldsymbol{n}]$. We now compute

$$F_{\boldsymbol{j}}^{\boldsymbol{i}} = \big(T_r^{(\boldsymbol{n})}\big)^{\boldsymbol{j},\boldsymbol{i}} = (\delta_r)_{\boldsymbol{j},\boldsymbol{i}} = \delta_{j_1 i_{\sigma(1)}} \cdots \delta_{j_m i_{\sigma(m)}} = \big(F_\sigma^{(\boldsymbol{n})}\big)_{\boldsymbol{j}}^{\boldsymbol{i}}$$

for all $\boldsymbol{i} := (i_1, \dots, i_m), \boldsymbol{j} := (i_1, \dots, j_m) \in [\boldsymbol{n}]$. Therefore, $F = F_\sigma^{(\boldsymbol{n})}$. □

The previous proposition shows that the conjugate representation $u^\bullet$ of $G_{\boldsymbol{n}}(\mathcal{C})$ is uniquely determined by the category $\mathcal{C}$. Since any spatial partition quantum group is equivalent to a quantum group of the form $G_{\boldsymbol{n}}(\mathcal{C})$, it follows that the conjugate representation $u^\bullet$ of a general spatial partition quantum group is uniquely determined by the corresponding category $\mathcal{C}$ and the change of basis Q.

Since the linear maps $\overline{F}_\sigma^{(\boldsymbol{n})}$ are unitary, the previous proposition additionally implies that the representation $\overline{u^\circ} = \big(\overline{F}_\sigma^{(\boldsymbol{n})}\big)^{-1} u^\bullet \overline{F}_\sigma^{(\boldsymbol{n})}$ is also unitary. This shows that $\overline{u^\circ}$ is a conjugate representation of u° and that every spatial partition quantum group is a subgroup of $U_{\boldsymbol{n}}^+$. We will return to this fact in Section 7.3, where we show that the intertwiner spaces of a spatial partition quantum group are still spanned by spatial partitions when replacing $u^\bullet$ by $\overline{u^\circ}$.

7.2. Presentations of spatial partition quantum groups

Next, we describe the C^*-algebras $C(G_{\boldsymbol{n}}(\mathcal{C}))$ from the previous section as universal C^*-algebras. If the category of spatial partitions $\mathcal{C}$ is generated by a finite set of spatial partitions, then we obtain a universal C^*-algebra defined by a finite set of relations. In particular, this allows us to show that the quantum groups $O^+(F_\sigma^{(\boldsymbol{n})})$ are spatial partition quantum groups.

We begin by extending the notation $G_{\boldsymbol{n}}(\mathcal{C})$ to arbitrary sets of spatial partitions that generate a rigid category of spatial partitions.

Definition 7.2.1. Let $\mathcal{C}_0 \subseteq \mathcal{P}^{(m)}$ be a set of $\boldsymbol{n}$-graded spatial partitions with duality partitions $\sigma_{\circ\bullet}, \sigma_{\bullet\circ}^{-1} \in \langle \mathcal{C}_0 \rangle$. Denote by $\mathcal{A}$ the universal unital C^*-algebra generated by the entries of a matrix $u := (u_{\boldsymbol{j}}^{\boldsymbol{i}})_{\boldsymbol{i},\boldsymbol{j} \in [\boldsymbol{n}]}$ satisfying the following relations:

1. $u^\circ := u$ and $u^\bullet := \overline{F}_\sigma^{(\boldsymbol{n})} \overline{u} \big(\overline{F}_\sigma^{(\boldsymbol{n})}\big)^{-1}$ are unitary.
2. $T_p^{(\boldsymbol{n})} u^x = u^y T_p^{(\boldsymbol{n})}$ for all $x, y \in \{\circ, \bullet\}^*$ and $p \in \mathcal{C}_0 \cap \mathcal{P}^{(m)}(x, y)$.

Then $G_{\boldsymbol{n}}(\mathcal{C}_0) := (\mathcal{A}, u)$ is the *spatial partition quantum group defined by* $\mathcal{C}_0$.

First, note that the compact matrix quantum group $G_{\boldsymbol{n}}(\mathcal{C}_0)$ from the previous definition is well-defined. Indeed, $\mathcal{A}$ is generated by the elements $u_{\boldsymbol{j}}^{\boldsymbol{i}}$, and u is unitary and $\overline{u}$ is invertible by the first relations. Furthermore, one verifies that both relations are compatible with the comultiplication, see also Definition 2.2.9 and [96].

Next, we show that the C^*-algebra $C(G_{\boldsymbol{n}}(\mathcal{C}_0))$, and thus the quantum group $G_{\boldsymbol{n}}(\mathcal{C}_0)$, does not depend on the particular choice of duality partitions $\sigma_{\circ\bullet}, \sigma_{\bullet\circ}^{-1} \in \langle \mathcal{C}_0 \rangle$. However, we first need the following lemma.

Lemma 7.2.2. *Let $\mathcal{C}_0 \subseteq \mathcal{P}^{(m)}$ be a set of $\boldsymbol{n}$-graded spatial partitions generating a rigid category of spatial partitions $\mathcal{C} := \langle \mathcal{C}_0 \rangle$. Consider the quantum group $G_{\boldsymbol{n}}(\mathcal{C}_0)$. Then*

$$\operatorname{span}\big\{T_p^{(\boldsymbol{n})} \mid p \in \mathcal{C}(x, y)\big\} \subseteq \operatorname{Hom}(u^x, u^y) \qquad \forall x, y \in \{\circ, \bullet\}^*.$$

Proof. Definition 2.5.8 and Definition 7.1.5 imply that $T^{(\boldsymbol{n})}_{p\otimes q}$, $T^{(\boldsymbol{n})}_{p^*}$ and $T^{(\boldsymbol{n})}_{pq}$ are also intertwiners of $G_{\boldsymbol{n}}(\mathcal{C}_0)$ for all (composable) $p, q \in \mathcal{C}_0$. Therefore, we obtain inductively that $T^{(\boldsymbol{n})}_p$ is an intertwiner for all $p \in \mathcal{C}$. The statement follows by the linearity of the intertwiner spaces, see Definition 2.5.8. $\square$

Consider the quantum group $G_{\boldsymbol{n}}(\mathcal{C}_0)$ and assume that $\mathcal{C} := \langle \mathcal{C}_0 \rangle$ contains a different pair of duality partitions $r := \tau_{\circ\bullet}$ and $s := \tau^{-1}_{\bullet\circ}$ for $\tau \in S_m$. Then the linear maps $R := T^{(\boldsymbol{n})}_r$ and $S := T^{(\boldsymbol{n})}_s$ are intertwiners by the previous proposition, and the proof of Definition 7.1.11 shows that $u^\bullet = \overline{F}^{(\boldsymbol{n})}_\tau \overline{u} \big(\overline{F}^{(\boldsymbol{n})}_\tau\big)^{-1}$. Therefore, $u^\bullet$ and the C^*-algebra $C(G_{\boldsymbol{n}}(\mathcal{C}_0))$ are independent of the particular choice of duality partitions in Definition 7.2.1.

If $\mathcal{C}_0 = \mathcal{C}$ is already a category of spatial partitions, then the notation $G_{\boldsymbol{n}}(\mathcal{C}_0)$ of the previous definition and $G_{\boldsymbol{n}}(\mathcal{C})$ of Definition 7.1.6 overlap. However, the following proposition shows that these quantum group coincide in this case and that more generally $G_{\boldsymbol{n}}(\langle \mathcal{C}_0 \rangle)$ and $G_{\boldsymbol{n}}(\mathcal{C}_0)$ define the same quantum group.

Proposition 7.2.3. *Let $\mathcal{C}_0 \subseteq \mathcal{P}^{(m)}$ be a set of $\boldsymbol{n}$-graded spatial partitions generating a rigid category of spatial partitions $\mathcal{C} := \langle \mathcal{C}_0 \rangle$. Then the compact matrix quantum groups $G_{\boldsymbol{n}}(\mathcal{C})$ and $G_{\boldsymbol{n}}(\mathcal{C}_0)$ from Definition 7.1.6 and Definition 7.2.1 are isomorphic.*

Proof. Denote by u and w the fundamental representations of $G_{\boldsymbol{n}}(\mathcal{C}_0)$ and $G_{\boldsymbol{n}}(\mathcal{C})$ respectively. Then Definition 7.2.2 and the universal property of $G_{\boldsymbol{n}}(\mathcal{C})$ imply the inclusion $G_{\boldsymbol{n}}(\mathcal{C}_0) \subseteq G_{\boldsymbol{n}}(\mathcal{C})$ via the map $w \mapsto u$. Conversely, $\mathcal{C}_0 \subseteq \mathcal{C}$ and Definition 7.1.11 show that w satisfies all the defining relations of u. Thus, the universal property of $C(G_{\boldsymbol{n}}(\mathcal{C}_0))$ yields the inverse inclusion $G_{\boldsymbol{n}}(\mathcal{C}) \subseteq G_{\boldsymbol{n}}(\mathcal{C}_0)$ via the map $u \mapsto w$. Therefore, the two quantum groups are isomorphic. $\square$

Using the description of spatial partition quantum groups as universal C^*-algebras, we can extend Definition 2.2.8 to $\mathbb{C}^{\boldsymbol{n}}$ and construct the free orthogonal quantum groups $O^+(F^{(\boldsymbol{n})}_\sigma)$ from spatial partitions.

Definition 7.2.4. Let $\boldsymbol{n} \in \mathbb{N}^m$ and $F \in B(\mathbb{C}^{\boldsymbol{n}})$ be an invertible operator. Denote by $\iota\colon \overline{\mathbb{C}^{\boldsymbol{n}}} \to \mathbb{C}^{\boldsymbol{n}}$ the linear isomorphism defined by $\iota(\overline{e_{\boldsymbol{i}}}) = e_{\boldsymbol{i}}$ for all $i \in [\boldsymbol{n}]$, and define the universal unital C^*-algebra

$$A_o(F) := C^*(u^{\boldsymbol{i}}_{\boldsymbol{j}} \mid u \text{ is unitary and } u = (F\iota)\, \overline{u}\, (F\iota)^{-1})$$

generated by the entries of a matrix $u := (u^{\boldsymbol{i}}_{\boldsymbol{j}})_{\boldsymbol{i},\boldsymbol{j}\in[n]}$. Then $O^+(F) := (A_o(F), u)$ is the *free orthogonal quantum group* with parameter F.

Proposition 7.2.5. *Let $\sigma \in S_m$ be a $\boldsymbol{n}$-graded permutation and $\mathcal{C} := \big\langle \sigma_{\circ\bullet}, \sigma^{-1}_{\bullet\circ}, \uparrow^{(m)} \big\rangle$. Then $G_{\boldsymbol{n}}(\mathcal{C})$ and $O^+(F^{(\boldsymbol{n})}_\sigma)$ are isomorphic.*

Proof. Denote by $u =: u^\circ$ the fundamental representation of $G_{\boldsymbol{n}}(\mathcal{C})$ and by $w =: w^\circ$ the fundamental representation of $O^+(F^{(\boldsymbol{n})}_\sigma)$. Furthermore, let $\iota\colon \overline{\mathbb{C}^{\boldsymbol{n}}} \to \mathbb{C}^{\boldsymbol{n}}$ be the linear isomorphism defined by $\iota(\overline{e_i}) = e_i$ for all $i \in [\boldsymbol{n}]$. Then u is unitary, and Definition 7.1.11

implies that its conjugate representation $u^\bullet$ is given by $u^\bullet = \overline{F}_\sigma^{(\boldsymbol{n})}\, \overline{u} \left(\overline{F}_\sigma^{(\boldsymbol{n})}\right)^{-1}$. Since $\iota = T^{(\boldsymbol{n})}_{\overset{\bullet}{\underset{\circ}{|}}^{(m)}} \in \mathrm{Hom}(u^\bullet, u^\circ)$, we have $\iota u^\bullet = u^\circ \iota$, which implies

$$u = u^\circ = \iota u^\bullet \iota^{-1} = \left(\iota \overline{F}_\sigma^{(\boldsymbol{n})}\right) \overline{u} \left(\iota \overline{F}_\sigma^{(\boldsymbol{n})}\right)^{-1} = \left(F_\sigma^{(\boldsymbol{n})} \iota\right) \overline{u} \left(F_\sigma^{(\boldsymbol{n})} \iota\right)^{-1}.$$

Thus, the universal property of $C(O^+(F_\sigma^{(\boldsymbol{n})}))$ yields the inclusion $G_{\boldsymbol{n}}(\mathcal{C}) \subseteq O^+(F_\sigma^{(\boldsymbol{n})})$ via the map $w \mapsto u$.

Conversely, define $w^\bullet := \iota^{-1} w \iota$, which is unitary since both w and ι are unitary. Furthermore, $\iota w^\bullet = \iota\iota^{-1} w \iota = w^\circ \iota$, which shows $T^{(\boldsymbol{n})}_{\overset{\bullet}{\underset{\circ}{|}}^{(m)}} = \iota \in \mathrm{Hom}(w^\bullet, w^\circ)$. By the definition of $O^+(F_\sigma^{(\boldsymbol{n})})$ and the previous argument, we also have

$$w^\bullet = \iota^{-1} w \iota = \left(\iota^{-1} F_\sigma^{(\boldsymbol{n})} \iota\right) \overline{w} \left(\iota^{-1} F_\sigma^{(\boldsymbol{n})} \iota\right)^{-1} = \overline{F}_\sigma^{(\boldsymbol{n})}\, \overline{w^\circ} \left(\overline{F}_\sigma^{(\boldsymbol{n})}\right)^{-1}.$$

Hence, Definition 2.5.12 shows that w° and $w^\bullet$ are conjugate via the intertwiners $R \in \mathrm{Hom}(1, w^{\circ\bullet})$ and $S \in \mathrm{Hom}(1, w^{\bullet\circ})$ defined by

$$R^{\boldsymbol{i},\boldsymbol{j}} = (\overline{F}_\sigma^{(\boldsymbol{n})})_{\boldsymbol{i}}^{\boldsymbol{j}}, \quad S^{\boldsymbol{i},\boldsymbol{j}} = (F_{\sigma^{-1}}^{(\boldsymbol{n})})_{\boldsymbol{i}}^{\boldsymbol{j}} \qquad \forall \boldsymbol{i}, \boldsymbol{j} \in [\boldsymbol{n}].$$

As in proof of Definition 7.1.11, this is equivalent to $R = T^{(\boldsymbol{n})}_{\sigma_{\circ\bullet}}$ and $S = T^{(\boldsymbol{n})}_{\sigma^{-1}_{\bullet\circ}}$. Therefore, w satisfies all the defining relations of $C(G_{\boldsymbol{n}}(\mathcal{C}_0))$ with $\mathcal{C}_0 := \{\sigma_{\circ\bullet}, \sigma^{-1}_{\bullet\circ}, \overset{\bullet}{\underset{\circ}{|}}^{(m)}\}$. Since $G_{\boldsymbol{n}}(\mathcal{C}_0) = G_{\boldsymbol{n}}(\mathcal{C})$ by Definition 7.2.3, the universal property of $C(G_{\boldsymbol{n}}(\mathcal{C}_0))$ yields the inverse inclusion $O^+(F_\sigma^{(\boldsymbol{n})}) \subseteq G_{\boldsymbol{n}}(\mathcal{C})$ via the map $u \mapsto w$. $\square$

Note that $O^+(F)$ and $O^+(QFQ^T)$ are isomorphic for any unitary Q, see for example [83, Proposition 6.4.7]. Since $F_\tau^{(\boldsymbol{n})} F_\sigma^{(\boldsymbol{n})} \left(F_\tau^{(\boldsymbol{n})}\right)^T = F_{\tau\sigma\tau^{-1}}^{(\boldsymbol{n})}$, it follows that $O^+(F_\sigma^{(\boldsymbol{n})})$ and $O^+(F_{\tau\sigma\tau^{-1}}^{(\boldsymbol{n})})$ are isomorphic for all $\sigma, \tau \in S_m$. Therefore, $O^+(F_\sigma^{(\boldsymbol{n})})$ depends only on the conjugacy class of σ.

7.3. Permuting levels

In Section 6.4, we introduced the functor $\mathrm{Perm}_{\sigma,\tau}$ that permutes the levels of a spatial partition depending on the color of its points. In the following, we show that $\mathrm{Perm}_{\sigma,\tau}$ leaves the quantum groups $G_{\boldsymbol{n}}(\mathcal{C})$ invariant. This implies that our new duality partitions yield the same class of spatial partition quantum groups as defined by Cébron and Weber in [25].

The main work to prove this result is contained in the following technical lemma about the linear maps $T^{(\boldsymbol{n})}_{\mathrm{Perm}_{\sigma,\tau}(p)}$.

Lemma 7.3.1. *Let $\sigma, \tau \in S_m$ be $\boldsymbol{n}$-graded permutations. Consider a compact matrix quantum group with fundamental representation u° on $\mathbb{C}^{\boldsymbol{n}}$ and unitary representation $u^\bullet$ on $\overline{\mathbb{C}^{\boldsymbol{n}}}$. Define the unitary representations $\hat{u}^\circ := u^\circ$ and $\hat{u}^\bullet := \left(\overline{F}_{\tau\sigma^{-1}}^{(\boldsymbol{n})}\right)^{-1} u^\bullet\, \overline{F}_{\tau\sigma^{-1}}^{(\boldsymbol{n})}$. Then*

$$T^{(\boldsymbol{n})}_{\mathrm{Perm}_{\sigma,\tau}(p)} \in \mathrm{Hom}(u^x, u^y) \iff \left(F_\sigma^{(\boldsymbol{n})}\right)^{\otimes y} T_p^{(\boldsymbol{n})} \left(\left(F_\sigma^{(\boldsymbol{n})}\right)^{-1}\right)^{\otimes x} \in \mathrm{Hom}(\hat{u}^x, \hat{u}^y)$$

for all $p \in \mathcal{P}^{(m)}(x,y)$.

Proof. Let $p \in \mathcal{P}^{(m)}(x,y)$ and recall from Definition 6.4.3 that

$$\mathrm{Perm}_{\tau,\rho}(p) := q^y_{\sigma,\tau} \cdot p \cdot (q^x_{\sigma,\tau})^{-1}.$$

Define the linear operators $Q^z_{\sigma,\tau} := T^{(\boldsymbol{n})}_{q^z_{\sigma,\tau}}$ for all $z \in \{\circ, \bullet\}^*$. Then

$$(Q^\circ_{\sigma,\tau})^{\boldsymbol{i}}_{\boldsymbol{j}} = (\delta_{q^\circ_{\sigma,\tau}})^{\boldsymbol{j}}_{\boldsymbol{i}} = \delta_{j_1 i_{\sigma(1)}} \dots \delta_{j_m i_{\sigma(m)}} = (F^{(\boldsymbol{n})}_\sigma)^{\boldsymbol{i}}_{\boldsymbol{j}}$$

for all $\boldsymbol{i} := (i_1, \dots, i_m), \boldsymbol{j} := (j_1, \dots, j_m) \in [\boldsymbol{n}]$. Hence, $Q^\circ_{\sigma,\tau} = F^{(\boldsymbol{n})}_\sigma$, and similarly one shows that $Q^\bullet_{\sigma,\tau} = \overline{F}^{(\boldsymbol{n})}_\tau$. Since $F^{(\boldsymbol{n})}_\sigma$ and $\overline{F}^{(\boldsymbol{n})}_\tau$ come from a unitary representation, it follows inductively that $Q^z_{\sigma,\tau}$ defines a unitary representation of $\boldsymbol{n}$-graded permutations in $S_m \times S_m$. Thus, we can write

$$T^{(\boldsymbol{n})}_{\mathrm{Perm}_{\tau,\rho}(p)} = Q^y_{\sigma,\tau} \cdot T^{(\boldsymbol{n})}_p \cdot (Q^x_{\sigma,\tau})^{-1} = (Q^y_{\mathrm{id},\tau\sigma^{-1}} \cdot Q^y_{\sigma,\sigma}) \cdot T^{(\boldsymbol{n})}_p \cdot (Q^x_{\mathrm{id},\tau\sigma^{-1}} \cdot Q^x_{\sigma,\sigma})^{-1}.$$

Therefore, $T^{(\boldsymbol{n})}_{\mathrm{Perm}_{\tau,\rho}(p)} \in \mathrm{Hom}(u^x, u^y)$ if and only if

$$\begin{aligned}(Q^y_{\mathrm{id},\tau\sigma^{-1}} \cdot Q^y_{\sigma,\sigma}) \cdot T^{(\boldsymbol{n})}_p \cdot (Q^x_{\mathrm{id},\tau\sigma^{-1}} \cdot Q^x_{\sigma,\sigma})^{-1} \cdot u^x \\ = u^y \cdot (Q^y_{\mathrm{id},\tau\sigma^{-1}} \cdot Q^y_{\sigma,\sigma}) \cdot T^{(\boldsymbol{n})}_p \, (Q^x_{\mathrm{id},\tau\sigma^{-1}} \cdot Q^x_{\sigma,\sigma})^{-1},\end{aligned}$$

which is equivalent to

$$\begin{aligned}Q^y_{\sigma,\sigma} \cdot T^{(\boldsymbol{n})}_p \cdot (Q^x_{\sigma,\sigma})^{-1} \cdot (Q^x_{\mathrm{id},\tau\sigma^{-1}})^{-1} \cdot u^x \cdot Q^x_{\mathrm{id},\tau\sigma^{-1}} \\ = (Q^y_{\mathrm{id},\tau\sigma^{-1}})^{-1} \cdot u^y \cdot Q^y_{\mathrm{id},\tau\sigma^{-1}} \cdot Q^y_{\sigma,\sigma} \cdot T^{(\boldsymbol{n})}_p \cdot (Q^x_{\sigma,\sigma})^{-1}.\end{aligned}$$

Since

$$(F^{(\boldsymbol{n})}_\sigma)^{\otimes z} = Q^z_{\sigma,\sigma}, \quad \hat{u}^z = (Q^z_{\mathrm{id},\tau\sigma^{-1}})^{-1} u^z Q^z_{\mathrm{id},\tau\sigma^{-1}} \qquad \forall z \in \{\circ, \bullet\}^*,$$

we conclude that

$$T^{(\boldsymbol{n})}_{\mathrm{Perm}_{\sigma,\tau}(p)} \in \mathrm{Hom}(u^x, u^y) \iff \left(F^{(\boldsymbol{n})}_\sigma\right)^{\otimes y} T^{(\boldsymbol{n})}_p \left(\left(F^{(\boldsymbol{n})}_\sigma\right)^{-1}\right)^{\otimes x} \in \mathrm{Hom}(\hat{u}^x, \hat{u}^y).$$

□

Using the previous lemma, we can show that the spatial partition functor $\mathrm{Perm}_{\sigma,\tau}$ leaves the quantum groups $G_{\boldsymbol{n}}(\mathcal{C})$ invariant.

Theorem 7.3.2. *Let* $\mathcal{C} \subseteq \mathcal{P}^{(m)}$ *be a* $\boldsymbol{n}$*-graded rigid category of spatial partitions and* $\sigma, \tau \in S_m$ *be* $\boldsymbol{n}$*-graded permutations. Then* $G_{\boldsymbol{n}}(\mathcal{C})$ *and* $G_{\boldsymbol{n}}(\mathrm{Perm}_{\sigma,\tau}(\mathcal{C}))$ *are isomorphic.*

Proof. Let $\mathcal{D} := \mathrm{Perm}_{\sigma,\tau}(\mathcal{C})$. Denote by u° and $u^\bullet$ the fundamental and conjugate representations of $G_{\boldsymbol{n}}(\mathcal{C})$, and by w° and $w^\bullet$ the fundamental and conjugate representations of $G_{\boldsymbol{n}}(\mathcal{D})$. Since the functor $\mathrm{Perm}_{\sigma,\tau}$ is fully faithful and color-preserving, we have

$$\mathrm{Hom}(w^x, w^y) = \mathrm{span}\left\{T_p^{(\boldsymbol{n})} \mid p \in \mathcal{D}(x,y)\right\} = \mathrm{span}\left\{T_{\mathrm{Perm}_{\sigma,\tau}(p)}^{(\boldsymbol{n})} \mid p \in \mathcal{C}(x,y)\right\}$$

for all $x, y \in \{\circ, \bullet\}^*$. Define $\hat{w}^\circ := w^\circ$ and $\hat{w}^\bullet := \left(\overline{F}_{\tau\sigma^{-1}}^{(\boldsymbol{n})}\right)^{-1} w^\bullet\, \overline{F}_{\tau\sigma^{-1}}^{(\boldsymbol{n})}$. Then Definition 7.3.1 yields

$$\mathrm{span}\left\{\left(F_\sigma^{(\boldsymbol{n})}\right)^{\otimes y} T_p^{(\boldsymbol{n})} \left(\left(F_\sigma^{(\boldsymbol{n})}\right)^{-1}\right)^{\otimes x} \mid p \in \mathcal{C}(x,y)\right\} \subseteq \mathrm{Hom}(\hat{w}^x, \hat{w}^y),$$

which is equivalent to

$$\mathrm{Hom}(u^x, u^y) \subseteq \left(\left(F_\sigma^{(\boldsymbol{n})}\right)^{-1}\right)^{\otimes y} \cdot \mathrm{Hom}(\hat{w}^x, \hat{w}^y) \cdot \left(F_\sigma^{(\boldsymbol{n})}\right)^{\otimes x}.$$

Thus, the universal property of $G_{\boldsymbol{n}}(\mathcal{C})$ and $\widehat{w}^\circ = w^\circ$ yield the inclusion

$$G_{\boldsymbol{n}}(\mathcal{D}) \subseteq G_{\boldsymbol{n}}(\mathcal{C}), \quad u^\circ \mapsto \left(F_\sigma^{(\boldsymbol{n})}\right)^{-1} w^\circ F_\sigma^{(\boldsymbol{n})}.$$

Since $\mathcal{C} = \mathrm{Perm}_{\sigma^{-1},\tau^{-1}}(\mathcal{D})$, the same argument yields the inverse inclusion

$$G_{\boldsymbol{n}}(\mathcal{C}) \subseteq G_{\boldsymbol{n}}(\mathcal{D}), \quad w^\circ \mapsto \left(F_{\sigma^{-1}}^{(\boldsymbol{n})}\right)^{-1} u^\circ F_{\sigma^{-1}}^{(\boldsymbol{n})}.$$

Therefore, $G_{\boldsymbol{n}}(\mathcal{C})$ and $G_{\boldsymbol{n}}(\mathcal{D})$ are isomorphic. $\square$

The previous theorem can be useful for determining the quantum groups associated with concrete categories of spatial partitions. However, as an immediate consequence, we obtain that the class of spatial partition quantum groups defined with our new duality partitions coincides with the class of spatial partition quantum groups defined by Cébron and Weber in [25].

Corollary 7.3.3. *Let G be a spatial partition quantum group. Then G is equivalent to $G_{\boldsymbol{n}}(\mathcal{C})$ for some $\boldsymbol{n}$-graded rigid category of spatial partitions $\mathcal{C} \subseteq \mathcal{P}^{(m)}$ containing the duality partitions $\sqcap_{\circ\bullet}^{(m)}$ and $\sqcap_{\bullet\circ}^{(m)}$.*

Proof. As discussed in Definition 7.1.8, any spatial partition quantum group G is equivalent to $G_{\boldsymbol{n}}(\mathcal{D})$ for a $\boldsymbol{n}$-graded rigid category of spatial partitions $\mathcal{D} \subseteq \mathcal{P}^{(m)}$ containing some duality partitions $\sigma_{\circ\bullet}$ and $\sigma_{\bullet\circ}^{-1}$. By the previous theorem, this quantum group is equivalent to $G_{\boldsymbol{n}}(\mathcal{C})$, where $\mathcal{C} := \mathrm{Perm}_{\mathrm{id},\sigma^{-1}}(\mathcal{D})$ contains the duality partitions

$$\mathrm{Perm}_{\mathrm{id},\sigma^{-1}}(\sigma_{\circ\bullet}) = \sqcap_{\circ\bullet}^{(m)}, \qquad \mathrm{Perm}_{\mathrm{id},\sigma^{-1}}(\sigma_{\bullet\circ}^{-1}) = \sqcap_{\bullet\circ}^{(m)}.$$

$\square$

In Section 7.1, we showed that for every spatial partition quantum group, the representation $\overline{u}$ is unitary and conjugate to u. However, it remained an open question whether we could choose $u^{\bullet} = \overline{u}$ and still obtain a spatial partition quantum group. The previous corollary answers this positively. Since $\sqcap_{\circ\bullet}^{(m)} = (\mathrm{id}_{S_m})_{\circ\bullet}$, the previous corollary and Definition 7.1.11 show that any spatial partition quantum group is equivalent to a spatial partition quantum group with

$$u^{\bullet} = \overline{F}^{(\boldsymbol{n})}_{\mathrm{id}_{S_m}} \cdot \overline{u} \cdot \left(\overline{F}^{(\boldsymbol{n})}_{\mathrm{id}_{S_m}}\right)^{-1} = \overline{u}.$$

Therefore, in the context of spatial partition quantum groups, we can always choose $u^{\bullet} = \overline{u}$ without loss of generality.

8. Projective spatial partition quantum groups

Consider a compact matrix quantum group G with fundamental representation u and assume that $\overline{u}$ is unitary. Then $u ⓣ \overline{u}$ is a unitary representation of G that defines a new compact matrix quantum group PG called the projective version of G. If $G \subseteq U_n$ is a classical matrix group, then PG corresponds exactly to the classical projective version

$$PG := G/(G \cap \{\lambda \operatorname{id}_{\mathbb{C}^n} \mid \lambda \in \mathbb{C}\}).$$

See [11, 48, 6] for further information on the projective versions of compact matrix quantum groups.

In this chapter, we show that the class of spatial partition quantum groups is closed under taking projective versions and more generally under taking tensor powers of their fundamental representation u° and their unitary conjugate representation $u^\bullet$. We then use this result to compute the spatial partition quantum groups corresponding to the categories $P_2^{(m)}$ of spatial pair partitions on m levels. Furthermore, it follows that all projective versions of easy quantum groups are spatial partition quantum groups. A result of Gromada [48] then allows us to describe these projective versions explicitly in terms of generators and relations if the underlying quantum group has a degree of reflection two.

8.1. Closure under tensor powers

Before showing that spatial partition quantum groups are closed under projective versions or more generally under tensor powers of their fundamental representation, we first introduce some notation for quantum groups of this form.

Definition 8.1.1. Let $z \in \{\circ, \bullet\}^*$ with $|z| \geq 1$ and G be a compact matrix quantum group with fundamental representation u° and a unitary representation $u^\bullet$. We define the compact matrix quantum group $G^z := (A, w)$, where $A \subseteq C(G)$ is the C^*-algebra generated by the matrix coefficients of $w := u^z$.

Note that if $u^\bullet = \overline{u^\circ}$, then the projective version PG is precisely $G^{\circ\bullet}$. Furthermore, we showed in previous sections that $\overline{u^\circ}$ is unitary for all spatial partition quantum groups and that we can choose $u^\bullet = \overline{u^\circ}$ without loss of generality. Thus, the projective version of every spatial partition quantum group G is well-defined and given by $G^{\circ\bullet}$.

In the following, we show that if G is a spatial partition quantum group with corresponding category $\mathcal{C} \subseteq \mathcal{P}^{(m)}$, then G^z is also a spatial partition quantum group for all $z \in \{\circ, \bullet\}^*$ with $|z| \geq 1$. Moreover, the corresponding category of G^z is given by $\operatorname{Flat}_{m,z}^{-1}(\mathcal{C})$, where

$\mathrm{Flat}_{m,z}$ is the functor from Section 6.4. Here, the fact that $\mathrm{Flat}_{m,z}^{-1}(\mathcal{C})$ is a rigid category of spatial partition relies on our new duality partitions, see Definition 6.4.12.

Since the functor $\mathrm{Flat}_{m,z}^{-1}$ does not preserve colors, our first step is to show that the representations w^x and $u^{\mathrm{Flat}_{m,z}(x)}$ are equivalent for all $x \in \{\circ, \bullet\}^*$, where $w^\circ := u^z$ and $w^\bullet := \overline{u^z}$.

Lemma 8.1.2. *Let $z \in \{\circ, \bullet\}^*$ with $d := |z| \geq 1$. Consider a compact matrix quantum group with fundamental representation u° on a Hilbert space V and a unitary representation $u^\bullet$ on $\overline{V}$. Define $w^\circ := u^z$ and $w^\bullet := \overline{u^z}$, and the unitaries*

$$Q_\circ \colon V^{\otimes z} \to V^{\otimes z}, \quad v_1 \otimes \cdots \otimes v_d \mapsto v_1 \otimes \cdots \otimes v_d \qquad \forall v_1, \ldots, v_d \in V,$$
$$Q_\bullet \colon \overline{V^{\otimes z}} \to V^{\otimes \overline{z}}, \quad \overline{v_1 \otimes \cdots \otimes v_d} \mapsto \overline{v_d} \otimes \cdots \otimes \overline{v_1} \qquad \forall v_1, \ldots, v_d \in V.$$

Then

$$Q_x := \bigotimes_{i=1}^{|x|} Q_{x_i} \in \mathrm{Hom}\left(w^x, u^{\mathrm{Flat}_{m,z}(x)}\right)$$

for all $x \in \{\circ, \bullet\}^*$ and $m \in \mathbb{N}$.

Proof. First, consider the case $x = \circ$. Then $w^\circ = u^z = u^{\mathrm{Flat}_{m,z}(\circ)}$ and $Q_\circ = \mathrm{id}_V^{\otimes z} \in \mathrm{Hom}(u^z, u^z)$. Next, consider the case $x = \bullet$. Choose an orthonormal basis $(v_i)_{i \in I}$ of V, which induces canonical bases on $\overline{V^{\otimes z}}$ and $V^{\otimes \overline{z}}$ indexed by $i_1, \ldots, i_d \in I$. With respect to these bases, the matrix entries of $w^\bullet = \overline{u^z}$ are given by

$$(\overline{u^z})^{i_1,\ldots,i_d}_{j_1,\ldots,j_d} = \left((u^z)^{i_1,\ldots,i_d}_{j_1,\ldots,j_d}\right)^* = \left((u^{z_1})^{i_1}_{j_1} \cdots (u^{z_d})^{i_d}_{j_d}\right)^* = (u^{\overline{z_d}})^{i_d}_{j_d} \cdots (u^{\overline{z_1}})^{i_1}_{j_1}$$

for all $i_1, \ldots, i_d, j_1, \ldots, j_d \in I$. On the other hand, we have $u^{\mathrm{Flat}_{m,z}(\bullet)} = u^{\overline{z}}$ and

$$(u^{\overline{z}})^{i_1,\ldots,i_d}_{j_1,\ldots,j_d} = (u^{\overline{z_d}})^{i_1}_{j_1} \cdots (u^{\overline{z_d}})^{i_d}_{j_d}.$$

Thus, performing a change of basis using $Q_\bullet$ yields $\overline{u^z} = Q_\bullet^{-1} \cdot u^{\overline{z}} \cdot Q_\bullet$, which is equivalent to $Q_\bullet \in \mathrm{Hom}(w^\bullet, u^{\mathrm{Flat}_{m,z}(\bullet)})$.

Finally, consider an arbitrary $x \in \{\circ, \bullet\}^*$. The previous computations show that $Q_{x_i} \in \mathrm{Hom}(w^{x_i}, u^{\mathrm{Flat}_{m,z}(x_i)})$ for all $1 \leq i \leq |x|$. Since intertwiner spaces are closed under tensor products and $\mathrm{Flat}_{m,z}$ is a functor, this implies

$$Q_x := \bigotimes_{i=1}^{|x|} Q_{x_i} \in \mathrm{Hom}\left(\textcircled{\scriptsize T}_{i=1}^{|x|}\, w^{x_i}, \textcircled{\scriptsize T}_{i=1}^{|x|}\, u^{\mathrm{Flat}_{m,z}(x_i)}\right) = \mathrm{Hom}\left(w^x, u^{\mathrm{Flat}_{m,z}(x)}\right).$$

□

Remark 8.1.3. Let $z \in \{\circ, \bullet\}^*$ with $|z| \geq 1$. Consider a compact matrix quantum group G with fundamental representation u° and unitary representation $u^\bullet$. Since both $u^{\overline{z}}$ and $Q_\bullet$ are unitary, the previous lemma implies that

$$w^\bullet := \overline{u^z} = Q_\bullet^{-1} \cdot u^{\overline{z}} \cdot Q_\bullet$$

is unitary. Therefore, by Definition 2.5.12, $w^\bullet = \overline{u^z}$ is conjugate to the fundamental representation $w^\circ = u^z$ of G^z.

Remark 8.1.4. Let u and v be unitary representations of a compact matrix quantum group G. Assume there exists a unitary intertwiner $Q \in \mathrm{Hom}(u, v)$. Then $u = Q^{-1}vQ$, and we have

$$\mathrm{Hom}(u, w) = \mathrm{Hom}(v, w) \cdot Q, \qquad \mathrm{Hom}(w, u) = Q^{-1} \cdot \mathrm{Hom}(w, v)$$

for any unitary representation w of G.

Next, we show that for any spatial partition p the linear maps $T^{(\boldsymbol{n})}_{\mathrm{Flat}_{m,z}(p)}$ and $T^{(\boldsymbol{n}\cdots\boldsymbol{n})}_p$ agree up to a change of basis. Here, $\boldsymbol{n}\cdots\boldsymbol{n}$ denotes the d-fold repetition of $\boldsymbol{n}$ as in Definition 6.4.12.

Lemma 8.1.5. *Let $\boldsymbol{n} \in \mathbb{N}^m$ and $x, y, z \in \{\circ, \bullet\}^*$ with $d := |z| \geq 1$. Then*

$$Q_y^{-1} \cdot T^{(\boldsymbol{n})}_{\mathrm{Flat}_{m,z}(p)} \cdot Q_x = (S^{-1})^{\otimes y} \cdot T^{(\boldsymbol{n}\cdots\boldsymbol{n})}_p \cdot S^{\otimes x}$$

for all $p \in \mathcal{P}^{(m\cdot d)}(x, y)$, where Q_x and Q_y are defined in Definition 8.1.2 using $V = \mathbb{C}^n$ and

$$S \colon (\mathbb{C}^{\boldsymbol{n}})^{\otimes z} \to \mathbb{C}^{\boldsymbol{n}\cdots\boldsymbol{n}},$$
$$e^{z_1}_{\boldsymbol{i}_1} \otimes \cdots \otimes e^{z_d}_{\boldsymbol{j}_d} \mapsto e_{\boldsymbol{i}_1} \otimes \cdots \otimes e_{\boldsymbol{i}_d} \qquad \forall \boldsymbol{i}_1, \ldots, \boldsymbol{i}_d \in [\boldsymbol{n}].$$

Proof. Let $p \in \mathcal{P}^{(m\cdot d)}(x, y)$. Define $k := |x|$, $\ell := |y|$, and let $\boldsymbol{i}_1, \ldots, \boldsymbol{i}_\ell, \boldsymbol{j}_1, \ldots, \boldsymbol{j}_k \in [\boldsymbol{n}\cdots\boldsymbol{n}]$ be of the form

$$\boldsymbol{i}_\alpha := (i_{\alpha,1,1}, \ldots, i_{\alpha,1,m}, \ldots, i_{\alpha,d,1}, \ldots, i_{\alpha,d,m}),$$
$$\boldsymbol{j}_\alpha := (j_{\alpha,1,1}, \ldots, j_{\alpha,1,m}, \ldots, j_{\alpha,d,1}, \ldots, j_{\alpha,d,m}).$$

For $1 \leq \beta \leq d$, define the slices

$$\boldsymbol{i}^{(\beta)}_\alpha := \begin{cases} (i_{\alpha,\beta,1}, \ldots, i_{\alpha,\beta,m}) & \text{if } x_\alpha = \circ, \\ (i_{\alpha,\beta,m}, \ldots, i_{\alpha,\beta,1}) & \text{if } x_\alpha = \bullet, \end{cases} \qquad \boldsymbol{j}^{(\beta)}_\alpha := \begin{cases} (j_{\alpha,\beta,1}, \ldots, j_{\alpha,\beta,m}) & \text{if } y_\alpha = \circ, \\ (j_{\alpha,\beta,m}, \ldots, j_{\alpha,\beta,1}) & \text{if } y_\alpha = \bullet. \end{cases}$$

Then, by the definition of S, we have

$$\left((S^{-1})^{\otimes y} \cdot T^{(\boldsymbol{n}\cdots\boldsymbol{n})}_p \cdot S^{\otimes x}\right)^{\boldsymbol{i}_1,\ldots,\boldsymbol{i}_\ell}_{\boldsymbol{j}_1,\ldots,\boldsymbol{j}_k} = \left(T^{(\boldsymbol{n}\cdots\boldsymbol{n})}_p\right)^{\boldsymbol{i}_1,\ldots,\boldsymbol{i}_\ell}_{\boldsymbol{j}_1,\ldots,\boldsymbol{j}_k} = (\delta_p)^{\boldsymbol{j}_1,\ldots,\boldsymbol{j}_k}_{\boldsymbol{i}_1,\ldots,\boldsymbol{i}_\ell}.$$

On the other hand, the definition of Q_x and Q_y yields

$$\left(Q_y^{-1} \cdot T^{(\boldsymbol{n})}_{\mathrm{Flat}_{m,z}(p)} \cdot Q_x\right)^{\boldsymbol{i}_1,\ldots,\boldsymbol{i}_\ell}_{\boldsymbol{j}_1,\ldots,\boldsymbol{j}_k} = \left(T^{(\boldsymbol{n})}_{\mathrm{Flat}_{m,z}(p)}\right)^{\boldsymbol{i}^{(1)}_1,\ldots,\boldsymbol{i}^{(d)}_1,\ldots,\boldsymbol{i}^{(1)}_\ell,\ldots,\boldsymbol{i}^{(d)}_\ell}_{\boldsymbol{j}^{(1)}_1,\ldots,\boldsymbol{j}^{(d)}_1,\ldots,\boldsymbol{j}^{(1)}_k,\ldots,\boldsymbol{j}^{(d)}_k},$$

where the right-hand side of this equation is given by

$$\left(T^{(\boldsymbol{n})}_{\mathrm{Flat}_{m,z}(p)}\right)^{\boldsymbol{i}^{(1)}_1,\ldots,\boldsymbol{i}^{(d)}_1,\ldots,\boldsymbol{i}^{(1)}_\ell,\ldots,\boldsymbol{i}^{(d)}_\ell}_{\boldsymbol{j}^{(1)}_1,\ldots,\boldsymbol{j}^{(d)}_1,\ldots,\boldsymbol{j}^{(1)}_k,\ldots,\boldsymbol{j}^{(d)}_k} = \left(\delta_{\mathrm{Flat}_{m,z}(p)}\right)^{\boldsymbol{j}^{(1)}_1,\ldots,\boldsymbol{j}^{(d)}_1,\ldots,\boldsymbol{j}^{(1)}_k,\ldots,\boldsymbol{j}^{(d)}_k}_{\boldsymbol{i}^{(1)}_1,\ldots,\boldsymbol{i}^{(d)}_1,\ldots,\boldsymbol{i}^{(1)}_\ell,\ldots,\boldsymbol{i}^{(d)}_\ell}.$$

The statement of the lemma follows, since

$$(\delta_p)^{\boldsymbol{j}_1,\ldots,\boldsymbol{j}_k}_{\boldsymbol{i}_1,\ldots,\boldsymbol{i}_\ell} = \left(\delta_{\mathrm{Flat}_{m,z}(p)}\right)^{\boldsymbol{j}^{(1)}_1,\ldots,\boldsymbol{j}^{(d)}_1,\ldots,\boldsymbol{j}^{(1)}_k,\ldots,\boldsymbol{j}^{(d)}_k}_{\boldsymbol{i}^{(1)}_1,\ldots,\boldsymbol{i}^{(d)}_1,\ldots,\boldsymbol{i}^{(1)}_\ell,\ldots,\boldsymbol{i}^{(d)}_\ell}$$

by the definition of $\mathrm{Flat}_{m,z}$. $\square$

Using the previous two lemmas, we can prove our main theorem and show that quantum groups of the form $G_{\boldsymbol{n}}(\mathcal{C})^z$ are spatial partition quantum groups.

Theorem 8.1.6. *Let $\mathcal{C} \subseteq \mathcal{P}^{(m)}$ be a $\boldsymbol{n}$-graded rigid category of spatial partitions and $z \in \{\circ, \bullet\}^*$ with $|z| \geq 1$. Then $G_{\boldsymbol{n}}(\mathcal{C})^z$ is equivalent to $G_{\boldsymbol{n}\cdots\boldsymbol{n}}(\mathcal{D})$ with $\mathcal{D} := \mathrm{Flat}_{m,z}^{-1}(\mathcal{C})$.*

Proof. Let u° and $u^\bullet$ be the fundamental and conjugate representations of $G_{\boldsymbol{n}}(\mathcal{C})$ and $\hat{w}^\circ$ and $\hat{w}^\bullet$ be the fundamental and conjugate representations of $G_{\boldsymbol{n}\cdots\boldsymbol{n}}(\mathcal{D})$. Furthermore, denote by $w^\circ := (u^\circ)^z$ the fundamental representation of G^z with conjugate representation defined by $w^\bullet := \overline{(u^\circ)^z}$, see Definition 8.1.3. Then Definition 8.1.2, combined with Definition 8.1.4 and the fact that $\mathrm{Flat}_{m,z}$ is fully faithful, implies that

$$\begin{aligned}\mathrm{Hom}(w^x, w^y) &= Q_y^{-1} \cdot \mathrm{Hom}(u^{\mathrm{Flat}_{m,z}(x)}, u^{\mathrm{Flat}_{m,z}(y)}) \cdot Q_x \\ &= \mathrm{span}\left\{Q_y^{-1} \cdot T_p^{(\boldsymbol{n})} \cdot Q_x \mid p \in \mathcal{C}(\mathrm{Flat}_{m,z}(x), \mathrm{Flat}_{m,z}(y))\right\} \\ &= \mathrm{span}\left\{Q_y^{-1} \cdot T_{\mathrm{Flat}_{m,z}(p)}^{(\boldsymbol{n})} \cdot Q_x \mid p \in \mathcal{D}\right\}\end{aligned}$$

for all $x, y \in \{\circ, \bullet\}^*$. By Definition 8.1.5, we have

$$Q_y^{-1} \cdot T_{\mathrm{Flat}_{m,z}(p)}^{(\boldsymbol{n})} \cdot Q_x = (S^{-1})^{\otimes y} \cdot T_p^{(\boldsymbol{n}\cdots\boldsymbol{n})} \cdot S^{\otimes x}$$

for a unitary $S\colon (\mathbb{C}^{\boldsymbol{n}})^{\otimes z} \to \mathbb{C}^{\boldsymbol{n}\ldots\boldsymbol{n}}$, which yields

$$\begin{aligned}\mathrm{Hom}(w^x, w^y) &= \mathrm{span}\left\{(S^{-1})^{\otimes y} \cdot T_p^{(\boldsymbol{n}\cdots\boldsymbol{n})} \cdot S^{\otimes x} \mid p \in \mathcal{D}(x, y)\right\} \\ &= (S^{-1})^{\otimes y} \cdot \mathrm{Hom}(\hat{w}^x, \hat{w}^y) \cdot S^{\otimes x}.\end{aligned}$$

Therefore, $G_{\boldsymbol{n}}(\mathcal{C})^z$ and $G_{\boldsymbol{n}\ldots\boldsymbol{n}}(\mathcal{D})$ are equivalent. □

Since every spatial partition quantum group is equivalent to a quantum group of the form $G_{\boldsymbol{n}}(\mathcal{C})$, we can use the following proposition to extend the previous theorem to all spatial partition quantum groups.

Proposition 8.1.7. *Let G and H be compact matrix quantum groups with fundamental representations u° and $\hat{u}^\circ$, and conjugate representations $u^\bullet$ and $\hat{u}^\bullet$ respectively. If G and H are equivalent, then G^z and H^z are equivalent for all $z \in \{\circ, \bullet\}^*$ with $|z| \geq 1$.*

Proof. Assume that G and H are equivalent. Then there exists a unitary S such that

$$\mathrm{Hom}(u^x, u^y) = S^{\otimes y} \cdot \mathrm{Hom}(\hat{u}^x, \hat{u}^y) \cdot (S^{-1})^{\otimes x} \qquad \forall x, y \in \{\circ, \bullet\}^*.$$

Denote by $w^\circ := (u^\circ)^z$ and $\hat{w}^\circ := (\hat{u}^\circ)^z$ the fundamental representations of G^z and H^z respectively. Then Definition 8.1.3 implies that $w^\bullet := \overline{(u^\circ)^z}$ and $\hat{w}^\bullet := \overline{(\hat{u}^\circ)^z}$ are the corresponding conjugate representations. Let $x, y \in \{\circ, \bullet\}^*$ and define $\widetilde{x} := \mathrm{Flat}_{1,z}(x)$ and $\widetilde{y} := \mathrm{Flat}_{1,z}(y)$. Then, by Definition 8.1.2 and Definition 8.1.4, we have

$$\begin{aligned}\mathrm{Hom}(w^x, w^y) &= Q_y^{-1} \cdot \mathrm{Hom}(u^{\widetilde{x}}, u^{\widetilde{y}}) \cdot Q_x \\ &= Q_y^{-1} \cdot S^{\otimes \widetilde{y}} \cdot \mathrm{Hom}(\hat{u}^{\widetilde{x}}, \hat{u}^{\widetilde{y}}) \cdot (S^{-1})^{\otimes \widetilde{x}} \cdot Q_x \\ &= Q_y^{-1} \cdot S^{\otimes \widetilde{y}} \cdot Q_{\widetilde{y}} \cdot \mathrm{Hom}(\hat{w}^x, \hat{w}^y) \cdot Q_{\widetilde{x}}^{-1} \cdot (S^{-1})^{\otimes \widetilde{x}} \cdot Q_x.\end{aligned}$$

Therefore, G^z and H^z are equivalent. □

Corollary 8.1.8. *Let G be a spatial partition quantum group and $z \in \{\circ, \bullet\}^*$ with $|z| \geq 1$. Then G^z is a spatial partition quantum group.*

Proof. Since G is a spatial partition quantum group, it is equivalent to $G_{\boldsymbol{n}}(\mathcal{C})$ for some $\boldsymbol{n}$-graded rigid category of spatial partitions $\mathcal{C} \subseteq \mathcal{P}^{(m)}$. Then the previous proposition shows that G^z is equivalent to $G_{\boldsymbol{n}}(\mathcal{C})^z$, which is a spatial partition quantum group by Definition 8.1.6. □

As a special case, it follows that the class of spatial partition quantum groups is closed under taking projective versions.

Corollary 8.1.9. *Let G be a spatial partition quantum group. Then PG is a spatial partition quantum group.*

Proof. By Definition 7.3.3 and the discussion at the end of Section 7.3, we can assume that the conjugate representation of G is given by $u^\bullet = \overline{u^\circ}$. In this case, $PG = G^{\circ\bullet}$, which a spatial partition quantum group by Definition 8.1.8. □

8.2. Categories of all spatial pair partitions

As a first application of the previous theorem, we consider the quantum groups associated with the categories $\mathcal{P}_2^{(m)}$ of spatial pair partitions on m levels, i.e. spatial partitions on m levels with every block of size two.

It was shown in [10] that the category $\mathcal{P}_2^{(1)}$ of pair partitions on one level corresponds to the classical orthogonal group O_n. Furthermore, the author showed in [31] that the category $\mathcal{P}_2^{(2)}$ of spatial pair partitions on two levels corresponds to the classical projective orthogonal group PO_n. In the following, we determine quantum groups for the remaining cases $m \geq 3$ and show that $G_{(n,\dots,n)}(P_2^{(m)})$ is equivalent to $O_n^{\circ\cdots\circ}$.

Note that, in contrast to our definition, the categories in [10, 31] are defined in terms of colorless partitions. However, in our setting, $\mathcal{P}_2^{(m)}$ contains the spatial partition $\begin{smallmatrix}\bullet\\ |\\ \circ\end{smallmatrix}^{(m)}$ that corresponds to the C^*-algebraic relations $u^i_j = (u^i_j)^*$, making the generators self-adjoint. Therefore, our quantum groups $G_{(n,\dots,n)}(\mathcal{P}_2^{(m)})$ coincides with the corresponding orthogonal quantum groups in [10, 31]. See also [80] for further details on the relationship between orthogonal and unitary easy quantum groups.

Proposition 8.2.1. *Let $m \geq 1$ and $n \in \mathbb{N}$. Then $G_{(n,\dots,n)}(\mathcal{P}_2^{(m)})$ is equivalent to O_n^z with $z := \circ^m$.*

Proof. Since the functor $\mathrm{Flat}_{1,z}$ does not change the size of blocks, pair partitions are mapped to pair partitions, and we have

$$\mathrm{Flat}_{1,z}^{-1}(\mathcal{P}_2^{(1)}) = \mathcal{P}_2^{(m)}.$$

Since $G_n(\mathcal{P}_2^{(1)}) = O_n$, Definition 8.1.6 implies that $G_{(n,\dots,n)}(\mathcal{P}_2^{(m)})$ is equivalent to O_n^z. □

By relaxing our notion of isomorphism of compact matrix quantum groups, we can provide a more explicit description of the resulting quantum groups $O_n^{\circ\cdots\circ}$.

Proposition 8.2.2. *Let $n \in \mathbb{N}$ and $z := \circ^m$ with $m \geq 1$. Then*

$$O_n^z = \begin{cases} O_n & \text{if } m \text{ is odd,} \\ PO_n & \text{if } m \text{ is even,} \end{cases}$$

as compact quantum groups.

Proof. Denote by u the fundamental representation of O_n. First, assume that m is odd, i.e. $m = 2k+1$ for some $k \in \mathbb{N}$. Consider the inclusion $C(O_n^z) \hookrightarrow C(O_n)$, which is injective and respects the comultiplication. Since u is orthogonal, we compute

$$\begin{aligned} \sum_{j_1,\ldots,j_k \in [n]} u_{j_0}^{i_0} u_{j_1}^1 u_{j_1}^1 \ldots u_{j_k}^1 u_{j_k}^1 &= u_{j_0}^{i_0} \underbrace{\left(\sum_{j_1=1}^n u_{j_1}^1 u_{j_1}^1\right)}_{=1} \cdots \underbrace{\left(\sum_{j_k=1}^n u_{j_k}^1 u_{j_k}^1\right)}_{=1} \\ &= u_{j_0}^{i_0} \in C(O_n^z) \end{aligned}$$

for all $i_0, j_0 \in [n]$. Therefore, the inclusion is surjective and defines an isomorphism of compact quantum groups.

Next, assume that m is even, i.e. $m = 2k$ for some $k \in \mathbb{N}$. Then we have an inclusion $C(O_n^z) \hookrightarrow C(PO_n)$, and we compute

$$\begin{aligned} \sum_{j_2,\ldots,j_k \in [n]} u_{j_0}^{i_0} u_{j_1}^{i_1} u_{j_2}^1 u_{j_2}^1 \ldots u_{j_k}^1 u_{j_k}^1 &= u_{j_0}^{i_0} u_{j_1}^{i_1} \underbrace{\left(\sum_{j_2=1}^n u_{j_2}^1 u_{j_2}^1\right)}_{=1} \cdots \underbrace{\left(\sum_{j_k=1}^n u_{j_k}^1 u_{j_k}^1\right)}_{=1} \\ &= u_{j_0}^{i_0} u_{j_1}^{i_1} \in C(O_n^z) \end{aligned}$$

for all $i_0, j_0, i_1, j_1 \in [n]$. Therefore, the inclusion is surjective and defines an isomorphism of compact quantum groups. □

8.3. Projective easy quantum groups

Consider an easy quantum group G. Then its projective version PG may not be an easy quantum group in general. However, since easy quantum groups form a subclass of spatial partition quantum groups, it follows from Definition 8.1.6 that projective versions of easy quantum groups are spatial partition quantum groups. Therefore, it is always possible to describe projective versions of easy quantum groups in terms of spatial partitions.

If G corresponds to a category of partitions $\mathcal{C}$, then the proof of Definition 8.1.6 shows the category of its projective version PG is given by $\mathrm{Flat}_{1,\circ\bullet}^{-1}(\mathcal{C})$. This is particularly useful if the category $\mathcal{C}$ can be characterized by an abstract property as in the case of spatial pair partitions in the previous section. On the other hand, if the category $\mathcal{C}$ is only given by a set of generators, then obtaining a generating set for the category of its projective version may be difficult in general.

However, in the special case of an orthogonal quantum group with degree of reflection two, Gromada [48] provides a result that allows the conversion of generators of $\mathcal{C}$ to generators of its projective version.

Proposition 8.3.1 ([48])**.** *Let $G \subseteq O^+(F)$ be orthogonal compact matrix quantum group with fundamental representation on $\mathbb{C}^n$ and degree of reflection two. Consider an admissible generating set S of* $\operatorname{Rep}(G)$ *containing the duality morphisms. Then* $\operatorname{Rep}(PG)$ *is generated by S and $\mathrm{id}_{\mathbb{C}^n} \otimes S \otimes \mathrm{id}_{\mathbb{C}^n}$.*

Here, an orthogonal compact matrix quantum group with fundamental representation u has *degree of reflection two* if

$$\operatorname{Hom}(u^{\textcircled{\tiny T} k}, u^{\textcircled{\tiny T} \ell}) = \{0\} \quad \forall k, \ell \in \mathbb{N},\ k + \ell \text{ even}.$$

Furthermore, a linear map $T \in S$ is called *admissible* if T is of the form $T\colon (\mathbb{C}^n)^{\otimes k} \to (\mathbb{C}^n)^{\otimes \ell}$ with k and ℓ even. Note that we can always use the duality morphisms to bring any intertwiner of a degree of reflection two quantum group into admissible form. See [48] or [67] for further details.

Next, we reformulate Definition 8.3.1 in the context of spatial partition quantum groups before applying it to concrete examples of easy quantum groups.

Proposition 8.3.2. *Let $n \in \mathbb{N}$ and $\mathcal{C} \subseteq \mathcal{P}^{(1)}$ be a category of spatial partitions generated by and a set $\mathcal{C}_0 \subseteq \mathcal{P}^{(1)}$ such that*

1. *$\mathcal{C}_0$ contains the partition ,*
2. *every $p \in \mathcal{C}_0$ has only white points,*
3. *if $p \in \mathcal{C}_0$ has upper colors x and lower colors y, then $|x|$ and $|y|$ are even.*

Then $PG_n(\mathcal{C})$ is equivalent to $G_{(n,n)}(\mathcal{D})$, where $\mathcal{D} \subseteq \mathcal{P}^{(2)}$ is generated by $^{(2)}$, $\mathrm{Flat}^{-1}_{1,\circ\bullet}(\mathcal{C}_0)$ *and* $\mathrm{Flat}^{-1}_{1,\circ\bullet}(\mathrm{id}_\circ \otimes \mathcal{C}_0 \otimes \mathrm{id}_\circ)$.

Proof. Since $\mathcal{C}$ contains the partitions and , it follows that $\mathcal{C}$ also contains the partitions and . Thus, Definition 7.2.5 implies that $G_n(\mathcal{C}) \subseteq O_n^+$ is an orthogonal quantum group, and we have $\overline{u} = u$ after identifying $\mathbb{C}^n$ with $\overline{\mathbb{C}^n}$. Moreover, it follows from Definition 7.2.1 that the intertwiners $S := \{T_p^{(n)} \mid p \in \mathcal{C}_0\}$ generate the category $\operatorname{Rep}(G)$ in the sense of [48]. Therefore, we can apply Definition 8.3.1 and obtain that the representation category $\operatorname{Rep}(PG)$ is generated by S and $\mathrm{id}_{\mathbb{C}^n} \otimes S \otimes \mathrm{id}_{\mathbb{C}^n}$ in the sense of [48]. The proof Definition 8.1.6, and in particular Definition 8.1.5, shows that we can use the functor $\mathrm{Flat}_{1,\circ\bullet}$ to translate these new generators back to spatial partitions. Furthermore, we must add the spatial partition $^{(2)}$ to obtain an orthogonal quantum group. Thus, $PG_n(\mathcal{C})$ is equivalent to $G_{(n,n)}(\mathcal{D})$, where $\mathcal{D}$ is generated by $^{(2)}$, $\mathrm{Flat}^{-1}_{1,\circ\bullet}(\mathcal{C}_0)$ and $\mathrm{Flat}^{-1}_{1,\circ\bullet}(\mathrm{id}_\circ \otimes \mathcal{C}_0 \otimes \mathrm{id}_\circ)$. $\square$

Now, we can apply the previous proposition to easy quantum groups with degree of reflection two. Figure 8.1 presents the results for easy quantum groups described in [73] and the appendix of [87]. It contains the generators for the corresponding categories of spatial partitions, as well as their projective versions. Note that the spatial partitions and $^{(2)}$ are included implicitly.

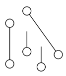
Figure 8.1 does not include the quantum groups S_n, B_n, S_n^+ and B_n^+ because these do not have a degree of reflection two. However, their projective versions depend only on the even part of their respective categories of spatial partitions, which are exactly the categories of S_n', B_n', $S_n'^+$ and $B_n'^+$.

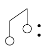
We can now use Figure 8.1 and Definition 7.2.1 to describe the C^*-algebras $\mathcal{A} := C(G_{(n,n)}(\mathcal{D}))$ of the previous projective quantum groups as universal unital C^*-algebras generated by a finite set of relations. Since the fundamental representation of a projective easy quantum group is defined on $\mathbb{C}^n \otimes \overline{\mathbb{C}^n}$, it follows that $\mathcal{A}$ is generated by the entries of a matrix $u := \left(u^{i_1 i_2}_{j_1 j_2}\right)$ with $i_1, i_2, j_1, j_2 \in [n]$. Furthermore, one can verify that the category $\mathcal{D}$ always contains $= (12)_{\circ\circ}$ such that the duality partitions of $\mathcal{D}$ are given by $(12)_{\circ\bullet}$, $(12)_{\bullet\circ}$ Thus, we have $u^\bullet := \overline{F}_{(12)}\overline{u}\overline{F}^{-1}_{(12)}$, which can be written using matrix coefficients as $(u^\bullet)^{i_1 i_2}_{j_1 j_2} = \left(u^{i_2 i_1}_{j_2 j_1}\right)^*$. The C^*-algebra $\mathcal{A}$ then satisfies the relations

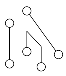
$$uu^* = u^*u = 1, \qquad u^\bullet (u^\bullet)^* = (u^\bullet)^* u^\bullet = 1$$

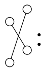
making both u and $u^\bullet$ unitary. Furthermore, the intertwiner $T^{(n,n)}_{(2)}$ yields the relation $T^{(n,n)}_{(2)} u^\bullet = u T^{(n,n)}_{(2)}$ or equivalently $\left(u^{i_2 i_1}_{j_2 j_1}\right)^* = u^{i_1 i_2}_{j_1 j_2}$. Finally, the C^*-algebra $\mathcal{A}$ satisfies the relations $T^{(n,n)}_p u^{\text{\textcircled{$\top$}} k} = u^{\text{\textcircled{$\top$}} \ell} T^{(n,n)}_p$ for all $p \in \mathcal{D}_0$. For the partitions given in Figure 8.1, these relations can be explicitly written as follows, where all indices are quantified over $[n]$.

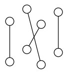
: $1 = \sum_{\ell_1,\ell_2} u^{i_1 i_2}_{\ell_1 \ell_2}$

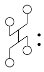
: $u^{i_1 i_4}_{j_1 j_2} = \sum_{\ell_1,\ell_2} u^{i_1 i_2}_{j_1 \ell_1} u^{i_3 i_4}_{\ell_2 j_2}$

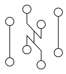
: $\delta_{i_1 i_2} = \sum_{\ell} u^{i_1 i_2}_{\ell \ell}$

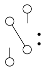
: $\delta_{i_2 i_3} u^{i_1 i_4}_{j_1 j_2} = \sum_{\ell} u^{i_1 i_2}_{j_1 \ell} u^{i_3 i_4}_{\ell j_2}$

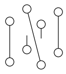
: $u^{i_2 i_1}_{j_1 j_2} = u^{i_1 i_2}_{j_2 j_1}$

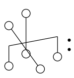
: $u^{i_1 i_3}_{j_1 j_2} u^{i_2 i_4}_{j_3 j_4} = u^{i_3 i_4}_{j_1 j_3} u^{i_1 i_2}_{j_2 j_4}$

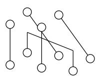
: $\delta_{i_1 i_2} u^{i_1 i_1}_{j_1 j_2} = \delta_{j_1 j_2} u^{i_1 i_2}_{j_1 j_1}$

: $\delta_{i_2 i_3} u^{i_1 i_2}_{j_1 j_2} u^{i_2 i_4}_{j_3 j_4} = \delta_{j_2 j_3} u^{i_1 i_2}_{j_1 j_2} u^{i_3 i_4}_{j_2 j_4}$

: $\sum_k u^{i_2 k}_{j_1 j_2} = \sum_{\ell} u^{i_1 i_2}_{\ell j_1}$

: $\sum_k u^{i_1 i_3}_{j_1 j_2} u^{k i_4}_{j_3 j_4} = \sum_{\ell} u^{i_1 i_2}_{j_1 \ell} u^{i_3 i_4}_{j_2 j_4}$

: $\delta_{i_1 i_4} u^{i_3 i_2}_{j_1 j_2} = \sum_{\ell} u^{i_1 i_2}_{\ell j_2} u^{i_3 i_4}_{j_1 \ell}$

: $\delta_{i_2 i_5} u^{i_1 i_4}_{j_1 j_2} u^{i_3 i_6}_{j_3 j_4} = \sum_{\ell} u^{i_1 i_2}_{j_1 \ell} u^{i_3 i_4}_{j_3 j_2} u^{i_5 i_6}_{\ell j_4}$

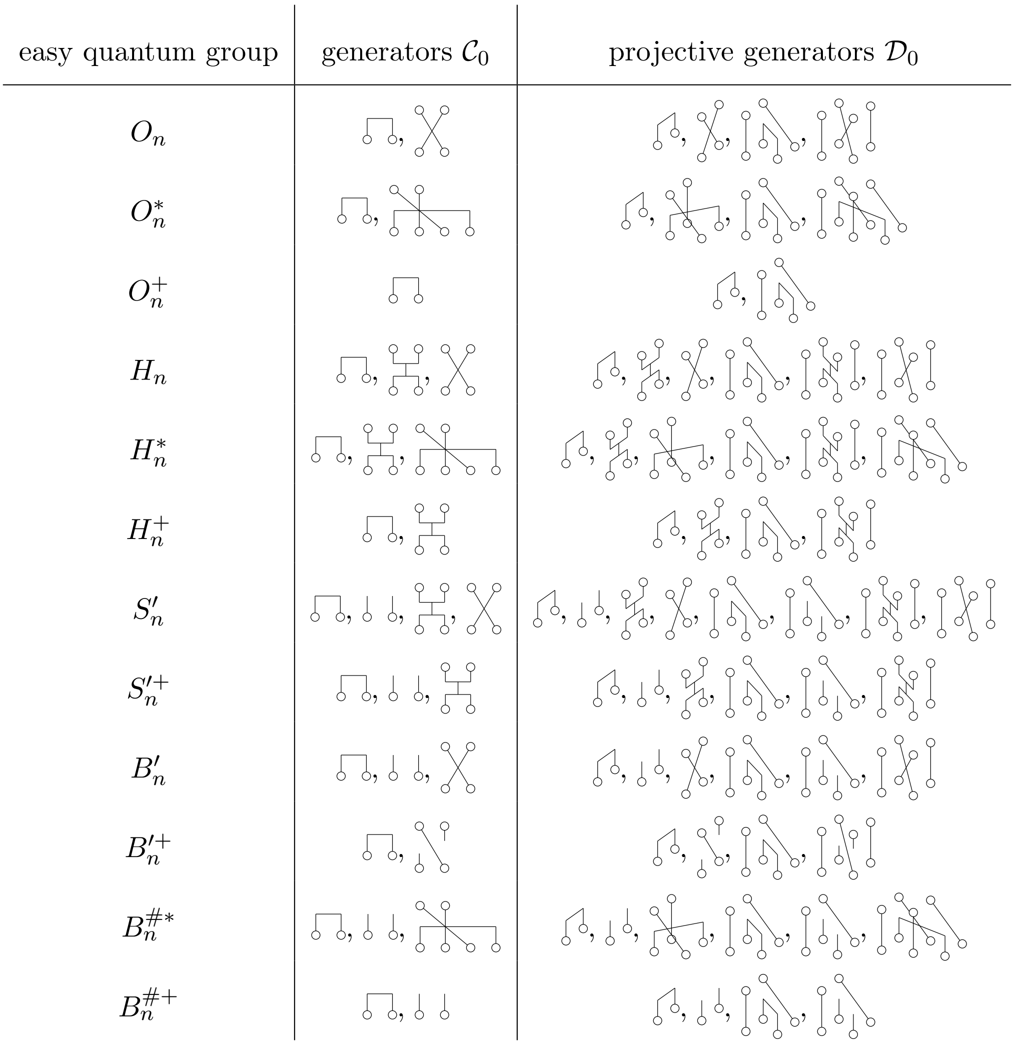


Figure 8.1.: Generators for categories of easy quantum groups and their projective version. The spatial partitions and ${}^{(2)}$ are omitted.

9. Open questions

In this final chapter, we present some remaining open questions related to quantum automorphism groups of hypergraphs and quantum groups based on partitions.

9.1. Quantum symmetries of hypergraphs

In Chapter 3, we constructed a family of hypergraphs with maximal quantum symmetries and determined the quantum automorphism group of opposite and dual hypergraphs. However, it would be interesting to compute the quantum automorphism groups of further examples of hypergraphs. Consider for example a complete hypergraph in the following sense.

Definition 9.1.1. Let V be a finite set. The *complete hypergraph* on V is given by $\Gamma := (V, 2^V \times 2^V)$ with source and range maps defined by

$$s(X,Y) = X, \quad r(X,Y) = Y \qquad \forall X, Y \subseteq V.$$

Since Γ contains no multiple edges, we know that $\mathrm{Aut}^+(\Gamma) \subseteq S_V^+$ by Corollary 3.5.4. However, it remains an open question whether this inclusion is proper or whether $\mathrm{Aut}^+(\Gamma) = S_V^+$ holds.

Question 9.1.2. What is the quantum automorphism group of a complete hypergraph?

Additionally, we showed in Chapter 4 that our quantum automorphism group generalizes Bichon's quantum automorphism group of classical graphs. However, it remains possible that every quantum automorphism group of a hypergraph can be realized as the quantum automorphism group of a possibly larger classical graph.

Question 9.1.3. Let Γ be a hypergraph. Is it possible to construct a classical graph Γ' such that $\mathrm{Aut}^+(\Gamma) = \mathrm{Aut}^+_{\mathrm{Bic}}(\Gamma')$?

It also remains an open question whether there exists a Banica-type version for the quantum automorphism group of hypergraphs in the following sense.

Question 9.1.4. Does there exist an alternative definition of the quantum automorphism group of hypergraphs that reduces to $\mathrm{Aut}^+_{\mathrm{Ban}}(\Gamma)$ in the case of classical directed graphs and that acts naturally on $C^*(\Gamma)$?

Finally, Hahn [51] characterized hypergraphs for which the classical automorphism group of their product is given by the wreath product of their automorphism groups. It might be possible to generalize this result to the setting of quantum automorphism groups.

Question 9.1.5. Can Hahn's result [51] be generalized to the quantum group setting using the free wreath product of Bichon [14]?

9.2. Quantum groups based on partitions

Next, we consider categories of partitions and their corresponding quantum groups. In Section 6.6, we constructed a category of partitions $\mathcal{C} \subseteq \mathcal{P}$ for which the problem of determining if $p \in \mathcal{C}$ for a given partition $p \in \mathcal{P}$ is algorithmically undecidable. While the variety of groups underlying our proof is finitely presented, we were only able to show that the category $\mathcal{C}$ is recursively enumerable. Therefore, it remains an open question whether $\mathcal{C}$ is finite generated as discussed in Definition 6.6.11.

Question 9.2.1. Does there exist a *finitely generated* category of partitions $\mathcal{C} \subseteq \mathcal{P}$ such that the problem of determining whether $p \in \mathcal{C}$ for a given partition $p \in \mathcal{P}$ is algorithmically undecidable?

In addition to the previous question, the membership problem for categories of partitions naturally generalizes to a membership problem for intertwiners of the corresponding quantum groups. In this case, the undecidability still remains an open problem.

Question 9.2.2. Does there exist a category of partitions $\mathcal{C} \subseteq \mathcal{P}$ and a number $n \in \mathbb{N}$ such that determining if

$$T \in \operatorname{span}\left\{T_p^{(n)} \mid p \in \mathcal{C}(k, \ell)\right\}$$

for a given $T \in \mathbb{Z}^{n^\ell \times n^k}$ is algorithmically undecidable?

While the linear maps $T_p^{(n)}$ are linearly independent for non-crossing partitions $p \in \mathcal{NC}$ when n is sufficiently large, this does not hold in general. Therefore, a direct reduction of the previous problem for categories of partitions is not possible, since $p \notin \mathcal{C}$ does not automatically imply that the map $T_p^{(n)}$ is not an intertwiner.

Besides algorithmic problems, the classification of categories of spatial partitions is still ongoing. Since the notion of non-crossing partitions does not directly generalize to spatial partitions, the simplest case appears to be the classification of all categories of spatial pair partition on two levels. In [25], Cébron and Weber started this classification, while the author provided additional examples of such categories in [31]. Moreover, it was shown that the quantum group corresponding to $\mathcal{P}_2^{(2)}$ is given by the projective orthogonal group PO_n. However, it remains an open problem to determine all possible categories and to provide simple descriptions of their corresponding quantum groups.

Question 9.2.3. Determine all categories of spatial partitions $\mathcal{C} \subseteq \mathcal{P}_2^{(2)}$ and describe their corresponding quantum groups $G_{(n,n)}(\mathcal{C})$.

In addition to classifying categories of spatial partitions, it would be interesting to generalize the result of Gromada [48] to the setting of colored partitions. This would allow us to construct spatial partition quantum groups corresponding to projective versions of unitary easy quantum groups using the approach in Section 8.3.

Question 9.2.4. Can the main result of Gromada in [48] be generalized to the context of unitary easy quantum groups with degree of reflection two?

Finally, the author showed in [31] that that category of spatial partitions corresponding to the quantum automorphism groups of the finite quantum space $\mathbb{C}^n \otimes M_N(\mathbb{C})$ can be obtained as the graph of a functor $F\colon \mathcal{NC} \to \mathcal{P}_2^{(2)}$. It would be interesting to study this construction in more detail and use it to obtain additional examples of spatial partition quantum groups.

Question 9.2.5. Given a spatial partition functor $F\colon \mathcal{C} \to \mathcal{D}$. Can the quantum group corresponding to the graph Γ_F in the sense of [31] be described in terms of $G_{\boldsymbol{n}}(\mathcal{C})$ and $G_{\boldsymbol{n}}(\mathcal{D})$? Moreover, what additional examples of spatial partition quantum groups can be constructed in this way?

Bibliography

[1] S. V. Aivazyan. A problem of A. I. Mal'tsev. *Siberian Mathematical Journal*, 29:877–884, 1988. doi:10.1007/BF00972411.

[2] Giorgio Ausiello and Luigi Laura. Directed hypergraphs: Introduction and fundamental algorithms – a survey. *Theoretical Computer Science*, 658:293–306, 2017. doi:10.1016/j.tcs.2016.03.016.

[3] Teodor Banica. Le groupe quantique compact libre U(n). *Communications in Mathematical Physics*, 190:143–172, 1997. doi:10.1007/s002200050237.

[4] Teodor Banica. Symmetries of a generic coaction. *Mathematische Annalen*, 314:763–780, 1999. doi:10.1007/s002080050315.

[5] Teodor Banica. Quantum automorphism groups of homogeneous graphs. *Journal of Functional Analysis*, 224(2):243–280, 2005. doi:10.1016/j.jfa.2004.11.002.

[6] Teodor Banica. *Introduction to Quantum Groups.* Springer, 2023. doi:10.1007/978-3-031-23817-8.

[7] Teodor Banica, Julien Bichon, and Benoît Collins. The hyperoctahedral quantum group. *Journal of the Ramanujan Mathematical Society*, 22(4):345–384, 2007.

[8] Teodor Banica, Stephen Curran, and Roland Speicher. Classification results for easy quantum groups. *Pacific Journal of Mathematics*, 247(1):1–26, 2010. doi:10.2140/pjm.2010.247.1.

[9] Teodor Banica and Debashish Goswami. Quantum isometries and noncommutative spheres. *Communications in Mathematical Physics*, 298:343–356, 2010. doi:10.1007/s00220-010-1060-5.

[10] Teodor Banica and Roland Speicher. Liberation of orthogonal Lie groups. *Advances in Mathematics*, 222(4):1461–1501, 2009. doi:10.1016/j.aim.2009.06.009.

[11] Teodor Banica and Roland Vergnioux. Invariants of the half-liberated orthogonal group. *Annales de l'Institut Fourier*, 60(6):2137–2164, 2010. doi:10.5802/aif.2579.

[12] Claude Berge. *Hypergraphs: Combinatorics of Finite Sets.* North-Holland Mathematical Library. Elsevier Science, 1984.

[13] Julien Bichon. Quantum automorphism groups of finite graphs. *Proceedings of the American Mathematical Society*, 131(3):665–673, 2003. doi:10.1090/S0002-9939-02-06798-9.

[14] Julien Bichon. Free wreath product by the quantum permutation group. *Algebras and Representation Theory*, 7:343–362, 2004. doi:10.1023/B:ALGE.0000042148.97035.ca.

[15] Bruce Blackadar. *Operator Algebras. Theory of C*-Algebras and von Neumann Algebras.* Encyclopaedia of Mathematical Sciences. Springer, 2006. doi:10.1007/3-540-28517-2.

[16] Michael Brannan, Alexandru Chirvasitu, Kari Eifler, Samuel Harris, Vern Paulsen, Xiaoyu Su, and Mateusz Wasilewski. Bigalois extensions and the graph isomorphism game. *Communications in Mathematical Physics*, 375:1777–1809, 2020. doi:10.1007/s00220-019-03563-9.

[17] Michael Brannan, Kari Eifler, Christian Voigt, and Moritz Weber. Quantum Cuntz-Krieger algebras. *Transactions of the American Mathematical Society Series B*, 9:782–826, 2022. doi:10.1090/btran/88.

[18] Richard Brauer. On algebras which are connected with the semisimple continuous groups. *Annals of Mathematics*, 38(4):857–872, 1937. doi:10.2307/1968843.

[19] Nathanial P. Brown and Narutaka Ozawa. *C*-Algebras and Finite-Dimensional Approximations.* Graduate Studies in Mathematics. American Mathematical Society, 2008. doi:10.1090/gsm/088.

[20] Jean-Bernard Bru and Walter Alberto de Siqueira Pedra. *C*-Algebras and Mathematical Foundations of Quantum Statistical Mechanics.* Latin American Mathematics Series. Springer, 2023. doi:10.1007/978-3-031-28949-1.

[21] Arthur Chassaniol. Study of quantum symmetries for vertex-transitive graphs using intertwiner spaces. Preprint at arXiv:1904.00455, 2019. doi:10.48550/arXiv.1904.00455.

[22] Alain Connes. *Noncommutative Geometry.* Academic Press, 1995.

[23] Thomas Cormen, Charles Leiserson, Ronald Rivest, and Clifford Stein. *Introduction to Algorithms, Third Edition.* MIT Press, 2009.

[24] Joachim Cuntz and Wolfgang Krieger. A class of C*-algebras and topological Markov chains. *Inventiones mathematicae*, 56:251–268, 1980. doi:10.1007/BF01390048.

[25] Guillaume Cébron and Moritz Weber. Quantum groups based on spatial partitions. *Annales de la Faculté des sciences de Toulouse*, 32(4):727–768, 2023. doi:10.5802/afst.1750.

[26] Kenneth R. Davidson. *C*-Algebras by Example.* Fields Institute Monographs. American Mathematical Society, 1996. doi:10.1090/fim/006.

[27] Reinhard Diestel. *Graph Theory.* Graduate Texts in Mathematics. Springer, 2025. doi:10.1007/978-3-662-70107-2.

[28] Josse van Dobben de Bruyn, Prem Nigam Kar, David E. Roberson, Simon Schmidt, and Peter Zeman. Quantum automorphism groups of trees. *Journal of Noncommutative Geometry*, 2025. `doi:10.4171/jncg/607`.

[29] Sergio Doplicher and John Roberts. A new duality theory for compact groups. *Inventiones mathematicae*, 98(1):157–218, 1989. `doi:10.1007/BF01388849`.

[30] Pavel Etingof, Shlomo Gelaki, Dmitri Nikshych, and Victor Ostrik. *Tensor Categories*. Mathematical Surveys and Monographs. American Mathematical Society, 2016.

[31] Nicolas Faroß. Spatial pair partitions and applications to finite quantum spaces. Master's thesis, Saarland University, 2022.

[32] Nicolas Faroß. Projective versions of spatial partition quantum groups. Preprint at arXiv:2411.04012, 2024. `doi:10.48550/arXiv.2411.04012`.

[33] Nicolas Faroß. Quantum automorphism groups of hypergraphs. Preprint at arXiv:2405.09894, 2024. `doi:10.48550/arXiv.2405.09894`.

[34] Nicolas Faroß and Simon Schwarz. Gröbner bases for Boolean function minimization (short paper). In *Proceedings of the 8th SC-Square Workshop*, pages 61–68, 2023. URL: `https://ceur-ws.org/Vol-3455/short4.pdf`.

[35] Nicolas Faroß and Simon Schwarz. Gröbner bases for Boolean function minimization. *Mathematics in Computer Science*, page to appear, 2025.

[36] Nicolas Faroß and Thomas Sturm. On the number of real types of univariate polynomials. Preprint at arXiv:2502.04914, 2025. `doi:10.48550/arXiv.2502.04914`.

[37] Nicolas Faroß and Sebastian Volz. Algorithmic problems in categories of partitions. Preprint at arXiv:2502.05373, 2025. `doi:10.48550/arXiv.2502.05373`.

[38] Nicolas Faroß and Moritz Weber. A concrete model for the quantum permutation group on 4 points. *Experimental Mathematics*, pages 1–14, 2024. `doi:10.1080/10586458.2024.2337232`.

[39] Amaury Freslon. On two-coloured noncrossing partition quantum groups. *Transactions of the American Mathematical Society*, 372:4471–4508, 2019. `doi:10.1090/tran/7846`.

[40] Amaury Freslon. *Compact Matrix Quantum Groups and Their Combinatorics*. London Mathematical Society Student Texts. Cambridge University Press, 2023. `doi:10.1017/9781009345705`.

[41] Amaury Freslon, Frank Taipe, and Simeng Wang. Tannaka-Krein reconstruction and ergodic actions of easy quantum groups. *Communications in Mathematical Physics*, 399:105–172, 2023. `doi:10.1007/s00220-022-04555-y`.

[42] Amaury Freslon and Moritz Weber. On the representation theory of partition (easy) quantum groups. *Journal für die reine und angewandte Mathematik*, 720:155–197, 2016. doi:10.1515/crelle-2014-0049.

[43] Giorgio Gallo, Giustino Longo, Stefano Pallottino, and Sang Nguyen. Directed hypergraphs and applications. *Discrete Applied Mathematics*, 42(2–3):177–201, 1993. doi:10.1016/0166-218X(93)90045-P.

[44] Lukas Gerlach, Simon Schwarz, Nicolas Faroß, and Michael Schwarz. Efficient and generic microarchitectural hash-function recovery. In *2024 IEEE Symposium on Security and Privacy*, pages 28–28, 2024. doi:http://doi.org/10.1109/sp54263.2024.00028.

[45] Debashish Goswami and Sk Asfaq Hossain. Quantum symmetry in multigraphs. Preprint at arXiv:2302.08726v3, 2023. doi:10.48550/arXiv.2302.08726.

[46] Daniel Gromada. Classification of globally colorized categories of partitions. *Infinite Dimensional Analysis, Quantum Probability and Related Topics*, 21(4):1850029, 2018. doi:10.1142/S0219025718500297.

[47] Daniel Gromada. *Compact Matrix Quantum Groups and Their Representation Theory*. PhD thesis, Saarland University, 2020. doi:10.22028/D291-32389.

[48] Daniel Gromada. Presentations of projective quantum groups. *Comptes Rendus. Mathématique*, 360:899–907, 2022. doi:10.5802/crmath.353.

[49] Daniel Gromada. Quantum symmetries of Hadamard matrices. *Transactions of the American Mathematical Society*, 377:6341–6377, 2024. doi:10.1090/tran/9153.

[50] Daniel Gromada and Moritz Weber. Intertwiner spaces of quantum group subrepresentations. *Communications in Mathematical Physics*, 376:81–115, 2020. doi:10.1007/s00220-019-03463-y.

[51] Geňa Hahn. The automorphism group of a product of hypergraphs. *Journal of Combinatorial Theory, Series B*, 30(3):276–281, 1981. doi:10.1016/0095-8956(81)90044-7.

[52] Soumalya Joardar and Arnab Mandal. Quantum symmetry of graph C*-algebras associated with connected graphs. *Infinite Dimensional Analysis, Quantum Probability and Related Topics*, 21(3):1850019, 2018. doi:10.1142/S0219025718500194.

[53] Olga Kharlampovich and Mark Sapir. Algorithmic problems in varieties. *International Journal of Algebra and Computation*, 5(4–5):379–602, 1995. doi:10.1142/S0218196795000227.

[54] Yu.G̃. Kleĭman. On identities in groups. *Trudy Moskovskogo Matematicheskogo Obshchestva*, 44:62–108, 1982. URL: https://www.mathnet.ru/eng/mmo417.

[55] Johan Kustermans and Stefaan Vaes. Locally compact quantum groups. *Annales Scientifiques de l'École Normale Supérieure*, 33(6):837–934, 2000. doi:10.1016/S0012-9593(00)01055-7.

[56] Claus Köstler and Roland Speicher. A noncommutative de Finetti theorem: Invariance under quantum permutations is equivalent to freeness with amalgamation. *Communications in Mathematical Physics*, 291:473–490, 2009. doi:10.1007/s00220-009-0802-8.

[57] Viktor Levandovskyy, Christian Eder, Andreas Steenpass, Simon Schmidt, Julien Schanz, and Moritz Weber. Existence of quantum symmetries for graphs on up to seven vertices: A computer based approach. In *Proceedings of the 2022 International Symposium on Symbolic and Algebraic Computation*, pages 311–318, 2022. doi:10.1145/3476446.3535481.

[58] Martino Lupini, Laura Mančinska, and David E. Roberson. Nonlocal games and quantum permutation groups. *Journal of Functional Analysis*, 279(5):108592, 2020. doi:10.1016/j.jfa.2020.108592.

[59] Sara Malacarne. Woronowicz's Tannaka-Krein duality and free orthogonal quantum groups. *Mathematica Scandinavica*, 122(1):151–160, 2018. doi:10.7146/math.scand.a-97320.

[60] Alexander Mang and Moritz Weber. Categories of two-colored pair partitions part I: Categories indexed by cyclic groups. *The Ramanujan Journal*, 53:181–208, 2020. doi:10.1007/s11139-019-00149-w.

[61] Alexander Mang and Moritz Weber. Categories of two-colored pair partitions part II: Categories indexed by semigroups. *Journal of Combinatorial Theory, Series A*, 180:105409, 2021. doi:10.1016/j.jcta.2021.105409.

[62] Alexander Mang and Moritz Weber. Non-hyperoctahedral categories of two-colored partitions part I: New categories. *Journal of Algebraic Combinatorics*, 54:475–513, 2021. doi:10.1007/s10801-020-00998-5.

[63] Alexander Mang and Moritz Weber. Non-hyperoctahedral categories of two-colored partitions part II: All possible parameter values. *Applied Categorical Structures*, 29:951–982, 2021. doi:10.1007/s10485-021-09641-1.

[64] Laura Mančinska and David E. Roberson. Quantum isomorphism is equivalent to equality of homomorphism counts from planar graphs. In *IEEE 61st Annual Symposium on Foundations of Computer Science*, pages 661–672, 2020. doi:10.1109/FOCS46700.2020.00067.

[65] Gerard J. Murphy. *C*-Algebras and Operator Theory*. Elsevier Science, 1990. doi:10.1016/C2009-0-22289-6.

[66] Ion Nechita, Simon Schmidt, and Moritz Weber. Sinkhorn algorithm for quantum permutation groups. *Experimental Mathematics*, 32(1):156–168, 2023. doi:10.1080/10586458.2021.1926005.

[67] Sergey Neshveyev and Lars Tuset. *Compact Quantum Groups and Their Representation Categories.* Cours Spécialisés. Société Mathématique de France, 2013.

[68] Hanna Neumann. *Varieties of Groups.* Ergebnisse der Mathematik und ihrer Grenzgebiete. 2. Folge. Springer, 1967. doi:10.1007/978-3-642-88599-0.

[69] OSCAR – Open Source Computer Algebra Research system, version 1.3.0-dev. https://www.oscar-system.org, 2024.

[70] Iain Raeburn. *Graph Algebras.* CBMS Regional Conference Series in Mathematics. American Mathematical Society, 2005. doi:10.1090/cbms/103.

[71] Sven Raum and Moritz Weber. The combinatorics of an algebraic class of easy quantum groups. *Infinite Dimensional Analysis, Quantum Probability and Related Topics*, 17(3):1450016, 2014. doi:10.1142/S0219025714500167.

[72] Sven Raum and Moritz Weber. Easy quantum groups and quantum subgroups of a semi-direct product quantum group. *Journal of Noncommutative Geometry*, 9(4):1261–1293, 2016. doi:10.4171/jncg/223.

[73] Sven Raum and Moritz Weber. The full classification of orthogonal easy quantum groups. *Communications in Mathematical Physics*, 341:751–779, 2016. doi:10.1007/s00220-015-2537-z.

[74] Simon Schmidt. The Petersen graph has no quantum symmetry. *Bulletin of the London Mathematical Society*, 50(3):395–400, 2018. doi:10.1112/blms.12154.

[75] Simon Schmidt. On the quantum symmetry of distance-transitive graphs. *Advances in Mathematics*, 368:107150, 2020. doi:10.1016/j.aim.2020.107150.

[76] Simon Schmidt. *Quantum Automorphism Groups of Finite Graphs.* PhD thesis, Saarland University, 2020. doi:10.22028/D291-31806.

[77] Simon Schmidt and Moritz Weber. Quantum symmetries of graph C*-algebras. *Canadian Mathematical Bulletin*, 61(4):848–864, 2018. doi:10.4153/CMB-2017-075-4.

[78] Björn Schäfer and Moritz Weber. Nuclearity of hypergraph C*-algebras. Preprint at arXiv:2405.10044, 2024. doi:10.48550/arXiv.2405.10044.

[79] Robert E. Tarjan and Jan van Leeuwen. Worst-case analysis of set union algorithms. *Journal of the ACM*, 31(2):245–281, 19985. doi:10.1145/62.2160.

[80] Pierre Tarrago and Moritz Weber. Unitary easy quantum groups: The free case and the group case. *International Mathematics Research Notices*, 2017(18):5710–5750, 2017. doi:10.1093/imrn/rnw185.

[81] Pierre Tarrago and Moritz Weber. The classification of tensor categories of two-colored noncrossing partitions. *Journal of Combinatorial Theory, Series A*, 154:464–506, 2018. doi:10.1016/j.jcta.2017.09.003.

[82] Neville Temperley and Elliott Lieb. Relations between the 'percolation' and 'colouring' problem and other graph-theoretical problems associated with regular planar lattices: Some exact results for the 'percolation' problem. *Proceedings of the Royal Society A*, 322:251–280, 1971. doi:10.1007/978-3-662-06390-3_31.

[83] Thomas Timmermann. *An Invitation to Quantum Groups and Duality: From Hopf Algebras to Multiplicative Unitaries and Beyond*. EMS textbooks in mathematics. EMS Press, 2008. doi:10.4171/043.

[84] Mirjam Trieb, Moritz Weber, and Dean Zenner. Hypergraph C*-algebras. Preprint at arXiv:2405.09214, 2024. doi:10.48550/arXiv.2405.09214.

[85] Alfons van Daele and Shuzhou Wang. Universal quantum groups. *International Journal of Mathematics*, 7:255–263, 1996. doi:10.1142/S0129167X96000141.

[86] Dan Voiculescu, Alexandru Nica, and Kenneth J. Dykema. *Free Random Variables*. CRM monograph series. American Mathematical Society, 1992.

[87] Sebastian Volz. Design and implementation of efficient algorithms for operations on partitions of sets. Bachelor's thesis, Saarland University, 2023.

[88] Shuzhou Wang. Free products of compact quantum groups. *Communications in Mathematical Physics*, 167:671–692, 1995. doi:10.1007/BF02101540.

[89] Shuzhou Wang. Quantum symmetry groups of finite spaces. *Communications in Mathematical Physics*, 195:195–211, 1998. doi:10.1007/s002200050385.

[90] Moritz Weber. On the classification of easy quantum groups. *Advances in Mathematics*, 245:500–533, 2013. doi:10.1016/j.aim.2013.06.019.

[91] Moritz Weber. Easy quantum groups. In *Free Probability and Operator Algebras*, pages 95–121. EMS Press, 2016. doi:10.4171/165-1/7.

[92] Moritz Weber. Introduction to compact (matrix) quantum groups and Banica-Speicher (easy) quantum groups. *Proceedings – Mathematical Sciences*, 127:881–933, 2017. doi:10.1007/s12044-017-0362-3.

[93] Moritz Weber. Quantum symmetry. *Snapshots of modern mathematics from Oberwolfach*, 5, 2020. doi:10.14760/SNAP-2020-005-EN.

[94] Moritz Weber. Quantum permutation matrices. *Complex Analysis and Operator Theory*, 17(37), 2023. doi:10.1007/s11785-023-01335-x.

[95] Stanisław Woronowicz. Compact matrix pseudogroups. *Communications in Mathematical Physics*, 111:613–665, 1987. doi:10.1007/BF01219077.

[96] Stanisław Woronowicz. Tannaka-Krein duality for compact matrix pseudogroups. twisted SU(N) groups. *Inventiones mathematicae*, 93(1):35–76, 1988. URL: `https://eudml.org/doc/143589`.

[97] Stanisław Woronowicz. A remark on compact matrix quantum groups. *Letters in Mathematical Physics*, 21:35–39, 1991. `doi:10.1007/BF00414633`.