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Multipliers of unitarily invariant spaces and K-contractions

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Sebastian Toth

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Dean of the faculty: Prof. Dr. Roland Speicher

Examination committee: *Chair:*
Prof. Dr. Henryk Zähle

Reviewers:
Prof. Dr. Michael Hartz
Prof. Dr. Moritz Weber
Prof. Dr. Shuaibing Luo

Academic assistant:
Dr. Malte Gerhold

To my family

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Abstract

Let $\mathscr{H}$ be a unitarily invariant regular reproducing kernel Hilbert space consisting of holomorphic functions on the open unit ball in $\mathbb{C}^d$. The aim of the present thesis is to understand certain elements in the multiplier algebra $\mathrm{Mult}(\mathscr{H})$ and to investigate how known results for pure contractions on Hilbert spaces can be transferred to the theory of tuples of commuting operators.

The work essentially consists of three parts:

The first part deals with transfer realizations for K-inner functions, similar to the characteristic functions of pure contractions as introduced by Sz.-Nagy and Foiaș. K-inner functions are a generalization of Bergman-inner functions and of inner functions on the Hardy space. The results shown generalize ideas of Olofsson and Eschmeier. This part is a joint work with Jörg Eschmeier.

The second part contains a uniqueness statement for multiplier functional calculi. It generalizes a uniqueness statement about the $H^\infty(\mathbb{D})$-functional calculus of pure contractions by Miller, Olin, and Thomson for certain tuples of commuting operators. As in the case of pure contractions, we show that the obvious polynomial functional calculus can only be uniquely extended to the corresponding multiplier algebra. This part is a joint work with Michael Hartz.

For the regular unitarily invariant spaces to be studied, the polynomials are contained in the multiplier algebra $\mathrm{Mult}(\mathscr{H})$. The elements in $\mathrm{Mult}(\mathscr{H})$ are in $\mathscr{H}$, bounded and holomorphic, that is in $H^\infty(\mathbb{B}_d) \cap \mathscr{H}$. In the last part, we study elements in the norm-closure of polynomials $A(\mathscr{H}) \subset \mathrm{Mult}(\mathscr{H})$.

Many regular unitarily invariant spaces can be described as radially weighted Besov spaces B^s_ω with an equivalent norm. One advantage is that multiplier functions can be characterized with the help of Carleson measures. A version of this characterization can be found in a paper by Aleman, Hartz, M$^{\mathrm{c}}$Carthy and Richter. Using vanishing Carleson measures, we establish necessary and sufficient conditions for elements to be in the norm-closure of polynomials $A(B^s_\omega) \subset \mathrm{Mult}(B^s_\omega)$. This part is a joint work with Michael Hartz.

Finally, we show that $\mathrm{Mult}(\mathscr{H}) \subsetneq H^\infty(\mathbb{B}_d) \cap \mathscr{H}$ if and only if $A(\mathscr{H}) \subsetneq A(\mathbb{B}_d) \cap \mathrm{Mult}(\mathscr{H})$. Here $A(\mathbb{B}_d)$ is the ball algebra, the set of all holomorphic functions that can be continuously extended to the boundary. The results are motivated by a paper by Fang and Xia on essentially hyponormal multiplication operators on the Drury-Arveson space. The chapter also contains a short proof of the one-function corona theorem for many Banach function spaces.

Zusammenfassung

Sei $\mathscr{H}$ ein unitär invarianter regulärer funktionaler Hilbertraum bestehend aus holomorphen Funktionen auf der offenen Einheitskugel in $\mathbb{C}^d$. Die vorliegende Arbeit beschäftigt sich mit Elementen in der Multiplikatoralgebra $\mathrm{Mult}(\mathscr{H})$ und untersucht, wie man bekannte Resultate für reine Kontraktionen auf Hilberträumen auf die Theorie von Tupeln vertauschender Operatoren übertragen kann.

Die Arbeit besteht im Wesentlichen aus drei Teilen:

Der erste Teil beschäftigt sich mit Transferdarstellungen für K-innere Funktionen. Diese ähnelt der Darstellung für charakteristische Funktionen reiner Kontraktionen, wie sie Sz.-Nagy and Foiaș eingeführt wurde. K- innere Funktionen sind eine Verallgemeinerung von Bergman-Inneren Funktionen und von inneren Funktionen auf dem Hardy Raum. Die gezeigten Resultate verallgemeinern Ideen von Olofsson und Eschmeier. Dieser Teil ist eine gemeinsame Arbeit mit Jörg Eschmeier.

Der zweite Teil enthält eine Eindeutigkeitsaussage für Multiplikator- Funktionalkalküle. Er verallgemeinert eine Eindeutigkeitsaussage über den $H^\infty(\mathbb{D})$-Funktionalkalkül reiner Kontraktionen von Miller, Olin und Thomson für bestimmte Tupel von vertauschenden Operatoren. Wie im Falle reiner Kontraktionen zeigen wir, dass sich der offensichtliche polynomielle Funktionalkalkül nur eindeutig auf die entsprechende Multiplikatoralgebra fortsetzen lässt. Dieser Teil ist eine gemeinsame Arbeit mit Michael Hartz.

Für die untersuchten regulären unitär invarianten Räume sind die Polynome in der Multiplikatoralgebra $\mathrm{Mult}(\mathscr{H})$ enthalten. Die Elemente in $\mathrm{Mult}(\mathscr{H})$ sind in $\mathscr{H}$, beschränkt und holomorph, das heißt in $H^\infty(\mathbb{B}_d) \cap \mathscr{H}$. Im letzten Teil studieren wir Elemente im Normabschluss der Polynome $A(\mathscr{H}) \subset \mathrm{Mult}(\mathscr{H})$.

Viele reguläre unitär invariante Räume lassen sich als radial gewichtete Besov Räume B^s_ω mit äquivalenter Norm darstellen. Ein Vorteil ist, dass sich Multiplikatorfunktionen mit der Hilfe von Carleson-Maßen charakterisieren lassen. Eine Version dieser Charakterisierung findet sich in einer Arbeit von Aleman, Hartz, M$^{\mathrm{c}}$Carthy und Richter. Unter Verwendung verschwindender Carleson-Maße stellen wir notwendige und hinreichende Bedingungen dafür auf, dass Elemente im Normabschluss der Polynome $A(B^s_\omega) \subset \mathrm{Mult}(B^s_\omega)$ liegen. Dieser Teil ist eine gemeinsame Arbeit mit Michael Hartz.

Zum Schluss zeigen wir, dass $\mathrm{Mult}(\mathscr{H}) \subsetneq H^\infty(\mathbb{B}_d) \cap \mathscr{H}$ genau dann, wenn $A(\mathscr{H}) \subsetneq A(\mathbb{B}_d) \cap \mathrm{Mult}(\mathscr{H})$. Dabei bezeichne $A(\mathbb{B}_d)$ die Ball-Algebra, das heißt die Menge aller holomorphen Funktionen, die sich stetig auf den Rand fortsetzen lassen. Die Ergebnisse sind durch ein Paper von Fang und Xia über wesentlich hyponormale Multiplikationsoperatoren auf dem Drury-Arveson Raum motiviert. Das Kapitel enthält auch einen kurzen Beweis für das Einfunktions-Corona Theorem für viele Banachfunktionenräume.

Contents

1. Introduction

Sz.-Nagy's dilation theorem, von Neumann's inequality and Hilbert function spaces

It is a frequent challenge to understand the theory of bounded linear operators $B(H)$ on a Hilbert space H. The theory of contractions known today

$$B_1(H) = \{T \in B(H);\ \|T\| \leq 1\},$$

has been mainly developed by Sz.-Nagy and Foiaș (cf. [SNFBK10]). In the world of contractions, one often considers the defect operator

$$D_T = (\mathrm{id}_H - T^*T)^{1/2}.$$

For the particular case of an isometry $S \in B(H)$, it can be readily seen that $D_S = 0$ and that

$$U = \begin{bmatrix} S & D_{S^*} \\ 0 & S^* \end{bmatrix} : H \oplus H \to H \oplus H \tag{1.1}$$

is a unitary dilation of S. That is,

$$S^n = P_H U^n|_H$$

for all $n \in \mathbb{N}$, where P_H is the orthogonal projection onto H.

On the other hand, every contraction $T \in B_1(H)$ has an isometric dilation

$$S = \begin{bmatrix} T & 0 & 0 & \cdots \\ D_T & 0 & 0 & \cdots \\ 0 & \mathrm{id}_H & 0 & \cdots \\ 0 & 0 & \mathrm{id}_H & \ddots \\ \vdots & \vdots & \ddots & \ddots \end{bmatrix} : \ell^2(H) \to \ell^2(H) \tag{1.2}$$

such that

$$T^n = P_H S^n|_H$$

for all $n \in \mathbb{N}$.

The statements (1.1) and (1.2), yield Sz.-Nagy's dilation theorem from 1953 (see [SNFBK10, Section 5, Chapter I] and [Pau02, Theorem 1.1, Chapter I]):

Theorem (Sz.-Nagy's dilation theorem). *Every contraction $T \in B_1(H)$ has a unitary dilation $U : H' \to H'$ to a Hilbert space H', containing H, such that*

$$T^n = P_H U^n|_H$$

for all $n \in \mathbb{N}$.

Now, let

$$H^2(\mathbb{D}) = \left\{ f \in \mathscr{O}(\mathbb{D});\ \sup_{0<r<1} \int_{\partial\mathbb{D}} |f(rz)|^2 dm(z) < \infty \right\}$$

be the Hardy space on the unit disk

$$\mathbb{D} = \{z \in \mathbb{C};\ |z| < 1\},$$

where m is the normalized Lebesgue-measure. Let $\mathscr{D} = \overline{\operatorname{Im} D_{T^*}}$ be the defect space and let

$$M_z : H^2(\mathbb{D}) \otimes \mathscr{D} \to H^2(\mathbb{D}) \otimes \mathscr{D}$$

be the multiplication operator defined by

$$M_z(f \otimes x) = (zf) \otimes x$$

for $f \otimes x \in H^2(\mathbb{D}) \otimes \mathscr{D}$.

Using the Wold decomposition theorem for isometries (see [SNFBK10, Theorem 1.1, Section 1, Chapter I]) and (1.2), the previous observations yield that every contraction is up to unitarily equivalence, a compression of the direct sum

$$M_z \oplus U$$

to a co-invariant subspace, where $U : \tilde{H} \to \tilde{H}$ is a unitary operator on a Hilbert space $\tilde{H}$.

Besides, it is well-known that the Hardy space is a Hilbert space with an orthonormal basis $(z^n)_{n\in\mathbb{N}}$. The "canonical" isometric isomorphism

$$\ell^2(\mathbb{N}) \overset{\cong}{\to} H^2(\mathbb{D}),\ (a_n)_{n\in\mathbb{N}} \mapsto \sum_{n=0}^{\infty} a_n z^n$$

provides a link between complex analysis and functional analysis. With this identification, the operator

$$M_z : H^2(\mathbb{D}) \to H^2(\mathbb{D}),\ f \mapsto zf$$

is the unilateral shift.

Furthermore, a contraction $T \in B_1(H)$ is called pure, if

$$\mathrm{SOT} - \lim_{n\to\infty} (T^*)^N = 0.$$

It is not difficult to see that M_z is a pure contraction. In fact, every pure contraction $T \in B(H)$ is unitarily equivalent to a compression of the operator

$$M_z : H^2(\mathbb{D}) \otimes \mathscr{D} \to H^2(\mathbb{D}) \otimes \mathscr{D}$$

to a co-invariant subspace.

It can be very useful to consider the Hardy space $H^2(\mathbb{D})$ as a Hilbert function space together with the (reproducing) Szegő kernel

$$K : \mathbb{D} \times \mathbb{D} \to \mathbb{C},\ K(z,w) = \frac{1}{1 - z\overline{w}}.$$

A reproducing kernel Hilbert space or a Hilbert function space is a Hilbert space $\mathscr{H}$ consisting of functions

$$f : X \to \mathbb{C}$$

on a set X such that the point evaluations

$$\delta_x : X \to \mathbb{C},\ f \mapsto f(x) \quad (x \in X)$$

are continuous. Therefore, the theorem of Riesz for Hilbert spaces shows that for every $x \in X$ there is a function

$$k_x : X \to \mathbb{C}$$

in $\mathscr{H}$ such that

$$f(x) = \langle f, k_x \rangle_{\mathscr{H}}$$

for all $f \in \mathscr{H}$. The positive definite mapping

$$K : X \times X \to \mathbb{C},\ K(x,y) = k_y(x),$$

called the reproducing kernel, uniquely determines the space $\mathscr{H}$.

For a reproducing kernel Hilbert space the multiplier algebra

$$\mathrm{Mult}(\mathscr{H}) = \{\varphi : X \to \mathbb{C};\ \varphi \cdot f \in \mathscr{H} \text{ for all } f \in \mathscr{H}\}$$

of $\mathscr{H}$ is of special interest and can be easily seen to be a Banach algebra, when $\mathrm{Mult}(\mathscr{H})$ is equipped with the norm $\|\varphi\|_{\mathrm{Mult}} = \|M_\varphi\|$, where

$$M_\varphi : \mathscr{H} \to \mathscr{H},\ f \to \varphi \cdot f$$

is the bounded multiplication operator associated with the function $\varphi \in \mathrm{Mult}(\mathscr{H})$. The multiplier algebra $\mathrm{Mult}(H^2(\mathbb{D}))$ of $H^2(\mathbb{D})$ coincides with bounded analytic functions $H^\infty(\mathbb{D})$ on the unit disk $\mathbb{D}$.

In several cases in operator theory, one wants to consider the case, when functions can be applied to operators. One can use Sz.-Nagy's dilation theorem to prove von Neumann's inequality (see for example, [Pau02, Corollary 1.1, Chapter I]):

Theorem (von Neumann's inequality)**.** *Let $T \in B(H)$ be a contraction. Then*

$$\|p(T)\| \leq \sup\{|p(z)|;\ z \in \mathbb{D}\} = \|p\|_{\mathrm{Mult}(H^2(\mathbb{D}))}$$

for every polynomial $p \in \mathbb{C}[z]$.

K-contractions

It is an ongoing process in operator theory to evolve Sz.-Nagy dilation theory for commuting tuples of operators

$$T = (T_1, \ldots, T_d) \in B(H)^d.$$

Especially, ideas of Agler (see [Agl82], [Agl85]) from the eighties, firstly developed for a single operator $T \in B(H)$, influenced many of the following works.

Many of the theorems that hold for contractions work for row contractions. A tuple

$$T = (T_1, \ldots, T_d) \in B(H)^d$$

is a row contraction if the operator

$$H^d \to H, (h_l)_{l=1}^d \mapsto \sum_{l=1}^d T_l h_l$$

is a contraction. In the theory of commuting row contractions a helpful multivariable generalization of the Hardy space, is the Drury-Arveson space H_d^2 on the unit ball

$$\mathbb{B}_d = \{(z_1, \ldots, z_d) \in \mathbb{C}^d;\ |z_1|^2 + \ldots + |z_d|^2 < 1\} \subset \mathbb{C}^d$$

with reproducing kernel

$$K : \mathbb{B}_d \times \mathbb{B}_d \to \mathbb{C},\ K(z,w) = \frac{1}{1 - \langle z, w \rangle}.$$

One part of the name “Drury–Arveson space” comes from a 1978 paper by Drury (see [Dru78]), where he proves the analog to von Neumann’s inequality for commuting row contractions:

Theorem (Drury’s inequality)**.** *Let $T = (T_1, \ldots, T_d) \in B(H)^d$ be a commuting row contraction, then*

$$\|p(T_1, \ldots, T_d)\| \leq \|p\|_{\mathrm{Mult}(H_d^2)}$$

for all polynomials $p \in \mathbb{C}[z]$.

A work by Arveson (cf. [Arv98]) brought H_d^2 to prominence.

For the theory of tuples of commuting operators $T = (T_1, \ldots, T_d) \in B(H)^d$, to simplify notation, we use the completely positive map

$$\sigma_T : B(H) \to B(H),\ \sigma_T(X) = \sum_{l=1}^d T_l X T_l^*.$$

A cornerstone for the generalization of Sz.-Nagy dilation theory is the notion of m-hypercontractions. For the definition, the spaces $A_m^2(\mathbb{B}_d)$ $(m \in \mathbb{N}_{>0})$ with reproducing kernel

$$K_m : \mathbb{B}_d \times \mathbb{B}_d \to \mathbb{C},\ K_m(z,w) = \frac{1}{(1 - \langle z, w \rangle)^m}$$

play a central role. A commuting tuple $T = (T_1, \dots, T_d) \in B(H)^d$, fulfilling the positivity conditions

$$\Delta_T^{(n)} = (1 - \sigma_T)^n(\mathrm{id}_H) = \sum_{l=0}^{n} (-1)^l \binom{n}{l} \sigma_T^l(\mathrm{id}_H) \geq 0$$

for $n = 1, \dots, m$ is called an m-hypercontraction. Furthermore, we use the following common notations:

$$C^2 = \frac{1}{K_m}(T) := \frac{1}{K_m}(T, T^*) = \Delta_T^{(m)} \quad (m \in \mathbb{N})$$

and $\mathscr{D} = \overline{\mathrm{Im}\, C}$. Based on the Hardy space case, the tuple $M_z = (M_{z_1}, \dots, M_{z_d})$, consisting of the multiplication operators

$$M_{z_l} : A_m^2(\mathbb{B}_d) \otimes \mathscr{D} \to A_m^2(\mathbb{B}_d) \otimes \mathscr{D},\ f \mapsto z_l f$$

defined by

$$M_{z_l}(f \otimes x) = (z_l f) \otimes x$$

for $f \otimes x \in A_m^2(\mathbb{B}_d) \otimes \mathscr{D}$ and $l = 1, \dots, d$ is sometimes referred to as a weighted shift. A theorem going back to Müller, Vasilescu (cf. [MV93]), extended by Arveson (cf. [Arv98]), proves that m-hypercontractions are, up to unitarily equivalence, compressions to a co-invariant subspace of the operator tuple

$$(M_{z_l} \oplus U_l)_{l=1}^{d}.$$

The tuple $U = (U_1, \dots, U_d) \in B(H)^d$ is a spherical unitary. That is a commuting tuple of bounded normal operators $U = (U_1, \dots, U_d) \in B(H)^d$ such that

$$\sum_{l=1}^{d} U_l U_l^* = \mathrm{id}_H .$$

Analogously to the one-dimensional case, an m-hypercontraction

$$T = (T_1, \dots, T_d) \in B(H)^d$$

is called pure if

$$\mathrm{SOT} - \lim_{n \to \infty} \sigma_T^n(\mathrm{id}_H) = 0.$$

If T is pure, the same model theorem holds, but without the spherical unitary part U.

In continuation of Agler's ideas and the work by Müller and Vasilescu (see [MV93]), Agler and M$^{\text{c}}$Carthy (cf. [AM00b]), Ambrozie, Engliš and Müller (see [AEM02]) and Arazy and Engliš (see [AE03]), develop a general machinery for studying coextensions. Consistent with the previous results, Clouâtre and Hartz (cf. [CH18]) establish operator models for Nevanlinna-Pick spaces. Inspired by these works, Schillo gives a unified approach for operator models (see [Sch18]), which will be used in the present thesis.

Let therefore $\mathscr{H} \subset \mathscr{O}(\mathbb{B}_d)$ be a Hilbert function spaces with reproducing kernel of the form

$$K : \mathbb{B}_d \times \mathbb{B}_d \to \mathbb{C},\ K(z, w) = \sum_{n=0}^{\infty} a_n \langle z, w \rangle^n,$$

where $a_0 = 1$, $a_n > 0$ for $n \geq 1$ and

$$\lim_{n\to\infty} \frac{a_n}{a_{n+1}} = 1.$$

We call such a space unitarily invariant and regular. One can show that

$$\mathbb{C}[z] \subset \mathrm{Mult}(\mathscr{H}) \subset \mathscr{H} \cap H^\infty(\mathbb{B}_d).$$

Suppose in addition that the power series $k(z) = \sum_{n=0}^{\infty} a_n z^n$ has no zeros in $\mathbb{D}$ and

$$\frac{1}{k}(z) = \sum_{n=0} c_n z^n$$

such that the c_n have almost the same sign. A commuting tuple

$$T = (T_1, \ldots, T_d) \in B(H)^d$$

is called K-contraction, if the expression

$$\frac{1}{K}(T) = \frac{1}{K}(T, T^*) = \sum_{n=0} c_n \sigma_T^n(\mathrm{id}_H) \geq 0$$

converges in the strong operator topology and is positive. If $K(z,w) = \frac{1}{1-z\overline{w}}$ $(z, w \in \mathbb{D})$ is the kernel of the Hardy space $H^2(\mathbb{D})$, then $T \in B(H)$ is a K-contraction if and only if

$$\frac{1}{K}(T) = \mathrm{id}_H - TT^* \geq 0,$$

which is equivalent for T to be a contraction.

Similar to the one-dimensional case, we write $C = \frac{1}{K}(T)^{1/2}$ for the generalized defect operator and denote by $\mathscr{D} = \overline{\mathrm{Im}\, C}$ the generalized defect space.

Furthermore, a K-contraction

$$T = (T_1, \ldots, T_d) \in B(H)^d$$

is called pure if T is the compression of the operator tuple $M_z = (M_{z_1}, \ldots, M_{z_d})$ to a co-invariant subspace. For contractions this is equivalent to the fact that

$$(T^*)^N \xrightarrow{\mathrm{SOT}} 0 \text{ for } N \to \infty.$$

It is a frequent challenge to generalize theorems for (pure) contractions to the setting of K-contractions.

K-contractions and K-inner functions

In their studies of contractions Sz.-Nagy and Foiaș, work with characteristic functions (cf. [SNFBK10]). The characteristic function of a pure contraction $T \in B_1(H)$ has the form

$$\theta_T(z) = -T + D_{T^*}(1 - zT^*)^{-1} z D_T \quad (z \in \mathbb{D})$$

and induces an isometric multiplier from to $H^2(\mathbb{D}) \otimes \tilde{\mathscr{D}}$ to $H^2(\mathbb{D}) \otimes \mathscr{D}$, where $\tilde{\mathscr{D}} = \overline{\operatorname{Im} D_T}$ and $\mathscr{D} = \overline{\operatorname{Im} D_{T^*}}$ are the defect spaces. In accordance with Sz.-Nagy's dilation theorem, one can show that a pure contraction T is unitarily equivalent to the compression of the unilateral shift to the co-invariant subspace

$$(H^2(\mathbb{D}) \otimes \mathscr{D}) \ominus (M_{\theta_T}(H^2(\mathbb{D}) \otimes \tilde{\mathscr{D}})).$$

The map θ_T has the properties of an inner function. That is

$$\|\theta_T x\|_{H^2(\mathbb{D}) \otimes \tilde{\mathscr{D}}} = \|x\|_{\tilde{\mathscr{D}}}$$

for all $x \in \tilde{\mathscr{D}}$ and

$$\theta_T(\tilde{\mathscr{D}}) \perp M_z^n(\theta_T(\tilde{\mathscr{D}}))$$

for $n \geq 1$, where we identify $\tilde{\mathscr{D}}$ as a subspace of $H^2(\mathbb{D}) \otimes \tilde{\mathscr{D}}$. By Beurling's theorem, inner functions $\theta : \mathbb{D} \to \mathbb{C}$ characterize the invariant subspaces of the shift operator M_z on the Hardy space $H^2(\mathbb{D})$. Every $\theta \in H^\infty(\mathbb{D})$ with $\|\theta\|_\infty \leq 1$ has a transfer realization of the form

$$\theta(z) = D + C(1 - zA)^{-1} z B \quad (z \in \mathbb{D}),$$

where

$$\left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right) : H \oplus \mathbb{C} \to H \oplus \mathbb{C}$$

is a unitary on a Hilbert space $H \oplus \mathbb{C}$ (see for example, [AM02, Theorem 6.5]).

Motivated by the Hardy space setting and the theory of contractions, Olofsson studies transfer realizations, which are similar to the one for the characteristic function, of Bergman-inner functions (see [Olo06], [Olo07]). The idea of Bergman-inner functions, defined by Olofsson [Olo07], is due to Hedenmalm and his results for wandering subspaces and invariant subspaces of the Bergman shift

$$M_z : L_a^2(\mathbb{D}) \to L_a^2(\mathbb{D});\ f \mapsto zf,$$

where

$$L_a^2(\mathbb{D}) = \left\{ f \in \mathscr{O}(\mathbb{D});\ \int_{\mathbb{D}} |f(z)|^2 dA(z) < \infty \right\}$$

is the Bergman space $A_2^2(\mathbb{D})$, we have already seen above (cf. [Hed91]).

In [Esc18a], Eschmeier generalizes Olofsson's ideas to the multivariable case of the Euclidean unit ball $\mathbb{B}_d$. Chapter 3 is based on a joint work with Eschmeier and appeared in [ET21]. We generalize the previously mentioned paper of Eschmeier for K-contractions.

Let therefore $\mathscr{H}$ be a regular unitarily invariant space with reproducing kernel

$$K : \mathbb{B}_d \times \mathbb{B}_d \to \mathbb{C},\ K(z,w) = k(\langle z,w\rangle).$$

Suppose in addition that the power series $k(z) = \sum_{n=0}^{\infty} a_n z^n$ has no zeros in $\mathbb{D}$ such that

$$\frac{1}{k}(z) = \sum_{n=0} c_n z^n$$

and the c_n have almost the same sign.

Motivated by the notion of Bergman-inner functions a K-inner function is an operator-valued analytic function $W : \mathbb{B}_d \to B(\mathscr{E}_*, \mathscr{E})$ with values in the bounded linear operators $B(\mathscr{E}_*, \mathscr{E})$ between Hilbert spaces $\mathscr{E}_*$ and $\mathscr{E}$ such that

$$\|Wx\|_{\mathscr{H}\otimes\mathscr{E}} = \|x\|_{\mathscr{E}_*}$$

for all $x \in \mathscr{E}_*$ and

$$W(\mathscr{E}_*) \perp M_z^{\alpha}(W(\mathscr{E}_*))$$

for all $\alpha \in \mathbb{N}^d \setminus \{0\}$ (see [BEKS17]).

Now, let $T = (T_1, \dots, T_d) \in B(H)^d$ be a pure K-contraction. One computes that the operator

$$\Delta_T = \mathrm{SOT} - \lim_{N\to\infty} \sum_{n=0}^{N} -c_{n+1}\sigma_T^n(\mathrm{id}_H).$$

is invertible, and that

$$(x,y) = \langle \Delta_T x, y\rangle \quad (x,y \in H)$$

defines a scalar product on H. We write $\tilde{H}$ for H equipped with the norm $\|\cdot\|_T$ and define

$$I_T : H \to \tilde{H},\ x \mapsto x.$$

One checks that $\tilde{T} = (\tilde{T}_1, \dots, \tilde{T}_d) \colon \tilde{H}^d \to H$ is a row contraction. If $C \in B(H, \mathscr{E})$ is any operator with $C^*C = \frac{1}{K}(T)$ and

$$\gamma_\alpha = \frac{(\sum_{l=1}^{d} \alpha_l)!}{\prod_{l=1}^{d} \alpha_l!} \quad \alpha = (\alpha_1, \dots, \alpha_d) \in \mathbb{N}^d$$

then

$$j_C \colon H \to \mathscr{H}\otimes\mathscr{E},\ j_C(x) = \sum_{\alpha\in\mathbb{N}^d} \left(\gamma_\alpha a_{|\alpha|} z^\alpha \otimes (C(T^\alpha)^* x)\right)$$

is a well-defined isometry such that j_C intertwines the tuples $T^* = (T_1^*, \dots, T_d^*) \in B(H)^d$ and $M_z^* = (M_{z_1}^*, \dots, M_{z_d}^*) \in B(\mathscr{H}\otimes\mathscr{E})^d$ componentwise. For our transfer realization we consider bounded linear operators $C \in B(H, \mathscr{E})$, $D \in B(\mathscr{E}_*, \mathscr{E})$ and $B \in B(\mathscr{E}_*, H^d)$ such that

$$\begin{aligned}
&\text{(K1) } C^*C = \frac{1}{K}(T),\\
&\text{(K2) } D^*C + B^*(\oplus\Delta_T)T^* = 0,\\
&\text{(K3) } D^*D + B^*(\oplus\Delta_T)B = \mathrm{id}_{\tilde{\mathscr{D}}},\\
&\text{(K4) } \mathrm{Im}((\oplus j_C)B) \subset M_z^* \mathscr{H}(\mathscr{E}).
\end{aligned}$$

Besides, we use the operator-valued function

$$F_T : \mathbb{B}_d \to B(H),\ F_T(z) = \sum_{n=0}^{\infty} a_{n+1} \left(\sum_{|\alpha|=n} \gamma_\alpha (T^\alpha)^* z^\alpha \right)$$

as well as the row operators

$$Z(w) : H^d \to H, (h_1, \dots h_d) \to \sum_{l=1}^{d} w_l h_l \quad (w \in \mathbb{B}_d).$$

We shall show that (see Theorem 3.2.1):

Theorem. *Let*

$$W : \mathbb{B}_d \to B(\mathscr{E}_*, \mathscr{E})$$

be an operator-valued function between Hilbert spaces $\mathscr{E}_*$ and $\mathscr{E}$ such that

$$W(z) = D + CF_T(z)Z(z)B,$$

where $T \in B(H)^d$ is a pure K-contraction and the matrix operator

$$\left(\begin{array}{c|c} \tilde{T}^* & B \\ \hline C & D \end{array} \right) : H \oplus \mathscr{E}_* \to H^d \oplus \mathscr{E}$$

satisfies the condition (K1)-(K4). Then W is a K-inner function.

If $T \in B_1(H)$ is a contraction and K is the Szegő kernel, then the canonical operator-valued K-inner function belonging to T is the classical characteristic function introduced by Sz.-Nagy and Foiaș. Conversely, we show that (see Theorem 3.2.2):

Theorem. *If $W : \mathbb{B}_d \to B(\mathscr{E}_*, \mathscr{E})$ is a K-inner function, then there exists a pure K-contraction $T \in B(H)^d$ and a matrix operator*

$$\left(\begin{array}{c|c} \tilde{T}^* & B \\ \hline C & D \end{array} \right) \in B(H \oplus \mathscr{E}_*, H^d \oplus \mathscr{E})$$

satisfying the conditions (K1)-(K4) such that

$$W(z) = D + CF_T(z)Z(z)B \quad (z \in \mathbb{B}_d).$$

Uniqueness of multiplier functional calculi

Functional calculi play an important role in the theory of Banach algebras. The analytic or Riesz-Dunford functional calculus of an operator $T \in B(H)$ on a Hilbert space H is an algebra homomorphism

$$\mathscr{O}(U) \to B(H), f \mapsto f(T).$$

It extends the polynomial calculus and is defined for all functions analytic in a neighborhood U of the spectrum $\sigma(T)$. One can ask whether the class of admissible functions can be enlarged by requiring certain properties for the operator T. An example is the continuous functional calculus for normal operators.

Another interesting class of operators or more precisely contractions are completely non-unitary contractions. That is a contraction $T \in B_1(H)$ having no invariant subspaces such that the restriction of the operator to the invariant subspace is unitary. Every contraction $T \in B_1(H)$ can be written as the direct sum $T = U \oplus T_{cnu}$ of a unitary operator U and a completely non-unitary operator T_{cnu}. A special case of such a decomposition is the Wold decomposition for isometries.

Sz.-Nagy and Foiaș show that the analytic functional calculus for completely non-unitary contractions can be extended to $H^\infty(\mathbb{D})$. That is, for every completely non-unitary contraction $T \in B_1(H)$, there exists a bounded and weak-$*$ continuous algebra homomorphism

$$H^\infty(\mathbb{D}) \to B(H),\ \varphi \mapsto \varphi(T).$$

Extending works by Eschmeier (see [Esc97]) and by Clouâtre and Davidson (cf. [CD16]), Bickel, Hartz, and McCarthy establish a multidimensional analog to the classical result of Sz.-Nagy and Foiaș (cf. [BHM18]). They show that:

Theorem (Bickel, Hartz, and McCarthy)**.** *For a completely non-unitary K-contraction*

$$T = (T_1, \dots, T_d) \in B(H)^d,$$

there exists a completely contractive unital algebra homomorphism

$$\pi : \mathrm{Mult}(\mathscr{H}) \to B(H),\ \varphi \mapsto \varphi(T_1, \dots, T_d)$$

with $\pi(z_l) = T_l$*.*

In [MOT86], Miller, Olin, and Thomson study whether any $H^\infty(\mathbb{D})$-calculus for a completely non-unitary contraction T is weak-$*$ continuous and hence unique. In [MOT86, Example 13.4] they give an example of a completely non-unitary contraction $T \in B_1(H)$ for which the polynomial calculus has multiple continuations. For pure contractions a particular class of completely non-unitary contractions, they show the following uniqueness result (see [MOT86, Theorem 13.3]):

Theorem (Miller, Olin, and Thomson). *Let $T \in B_1(H)$ be a pure contraction and let*

$$\pi : H^\infty(\mathbb{D}) \to B(H)$$

be a bounded unital homomorphism with $\pi(z) = T$. Then π is weak-$*$ continuous and therefore agrees with the Sz.-Nagy–Foiaș functional calculus of T.

Chapter 4 is joint work with Hartz. We establish the following analog to Miller, Olin, and Thomson's result for multiplier functional calculi (see Theorem 4.1).

Let therefore $\mathscr{H}$ be a regular unitarily invariant complete Nevanlinna-Pick space with unbounded kernel K. That is, $K : \mathbb{B}_d \times \mathbb{B}_d \to \mathbb{C}$ has the form

$$K(z,w) = \sum_{n=0}^{\infty} a_n \langle z,w\rangle^n = \frac{1}{1 - \sum_{n=1}^{\infty} b_n \langle z,w\rangle^n},$$

where $a_0 = 1$, $a_n > 0$ for $n \geq 1$, $\lim_{n\to\infty} \frac{a_n}{a_{n+1}} = 1$, $\sum_{n=0}^{\infty} a_n = \infty$ and $(b_n)_{n\geq 1}$ is a sequence of non-negative real numbers satisfying $\sum_{n=1}^{\infty} b_n = 1$.

Then, our statement is as follows:

Theorem (Analog to Miller, Olin, and Thomson's theorem). *Let $T = (T_1, \dots, T_d)$ a pure K-contraction and let*

$$\pi : \mathrm{Mult}(\mathscr{H}) \to B(H)$$

be a completely bounded unital algebra homomorphism with $\pi(z_l) = T_l$ for $1 \leq l \leq d$. Then π is weak-$*$ continuous.

Norm-closure of polynomials in the multiplier algebra

Let us take a step back when considering functional calculi for a contraction $T \in B_1(H)$. Due to von Neumann's inequality, the polynomial functional calculus map

$$\mathbb{C}[z] \to B(H),\ p \mapsto p(T)$$

is itself a contraction regarding the supremum-norm. The calculus naturally extends continuously to the norm-closure of the polynomials $A(\mathbb{D})$ in $H^\infty(\mathbb{D})$. The disk algebra $A(\mathbb{D})$ is classically defined as the intersection

$$A(\mathbb{D}) = C(\overline{\mathbb{D}}) \cap \mathscr{O}(\mathbb{D}).$$

Analogously, using Drury's inequality for a commuting row contraction

$$T = (T_1, \dots, T_d) \in B(H)^d,$$

the polynomial functional calculus map in multi-variables

$$\mathbb{C}[z] \to B(H),\ p \mapsto p(T_1, \dots, T_d)$$

is itself a contraction regarding the $\mathrm{Mult}(H^2_d)$-norm. The calculus has a continuous extension to the closure

$$A(H^2_d) = \overline{\mathbb{C}[z]}^{\|\cdot\|_{\mathrm{Mult}}} \subset \mathrm{Mult}(H^2_d).$$

In Chapters 5 and 6, we study this norm-closure of polynomials

$$A(\mathscr{H}) = \overline{\mathbb{C}[z]}^{\|\cdot\|_{\mathrm{Mult}}} \subset \mathrm{Mult}(\mathscr{H})$$

for regular unitarily invariant spaces $\mathscr{H}$.

For a characterization, observe that in the Dirichlet space

$$\mathscr{D} = \left\{ f \in \mathscr{O}(\mathbb{D});\ \int_{\mathbb{D}} |f'(z)|^2 \, dA(z) < \infty \right\},$$

with reproducing kernel

$$K : \mathbb{D} \times \mathbb{D} \to \mathbb{C},\ K(z,w) = \frac{1}{z\overline{w}} \log\left(\frac{1}{1 - z\overline{w}} \right),$$

the multiplier algebra $\mathrm{Mult}(\mathscr{D})$ can be characterized in the following way (see, for example, Theorem 5.1.7 in [EFKMR14]):

Theorem. *A function $\varphi : \mathbb{D} \to \mathbb{C}$ is in $\mathrm{Mult}(\mathscr{D})$ if and only if*

$$\varphi \in H^\infty(\mathbb{D}) \text{ and } \varphi' \in \mathrm{Mult}(\mathscr{D}, L^2_a(\mathbb{D})).$$

Here we use again the notation $L^2_a(\mathbb{D})$ for the Bergman space on the unit disk. In Chapter 5 we obtain the following characterization for the norm-closure of polynomials $A(\mathscr{D})$ (particular case of Theorem 5.1):

Theorem. *A function $\varphi : \mathbb{D} \to \mathbb{C}$ is in $A(\mathscr{D})$ if and only if $\varphi \in A(\mathbb{D})$ and the multiplication operator*

$$M_{\varphi'} : \mathscr{D} \to L^2_a(\mathbb{D}),\ f \to \varphi' \cdot f$$

is compact.

We generalize this idea for radially weighted Besov spaces

$$B^s_\omega = \{ f \in \mathscr{O}(\mathbb{B}_d);\ R^s f \in L^2(\omega dV) \} \quad (s \in \mathbb{R}).$$

In the definition the space B^s_ω the fractional radial derivative $R^s : \mathscr{O}(\mathbb{B}_d) \to \mathscr{O}(\mathbb{B}_d)$,

$$\sum_{\alpha \in \mathbb{N}^d} f_\alpha z^\alpha \mapsto \sum_{n=1}^{\infty} n^s \sum_{|\alpha|=n} f_\alpha z^\alpha$$

is a generalization of the classical radial derivative $R = R^1 : \mathscr{O}(\mathbb{B}_d) \to \mathscr{O}(\mathbb{B}_d)$,

$$f \mapsto \sum_{l=1}^{d} z_l \frac{\partial}{\partial z_l} f$$

and $\omega : \mathbb{B}_d \to \mathbb{R}_{>0}$ in $L^1(dV)$ is a radial weight. For the constant weight $\omega \equiv 1$, these spaces already contain various interesting examples, including the Dirichlet space

$$\mathscr{D} = \{f \in \mathscr{O}(\mathbb{B}_d);\ Rf \in L^2(dA)\},$$

and the Drury-Arveson space

$$H_d^2 = \{f \in \mathscr{O}(\mathbb{B}_d);\ R^{d/2} f \in L^2(dV)\},$$

where equality means equality of spaces with equivalence of norms.

Motivated by the characterization of the multiplier algebra $\mathrm{Mult}(B_\omega^s)$ in [AHMR19], which has its origin in [CF16] and [CFO10], we show that (see Theorem 5.1):

Theorem. *A function $\varphi : \mathbb{B}_d \to \mathbb{C}$ is in $A(B_\omega^s)$ if and only if $\varphi \in A(\mathbb{B}_d)$ and the multiplication operator*

$$M_{R^N\varphi} : B_\omega^s \to B_\omega^{s-N},\ f \to (R^N\varphi)f$$

is compact for $N \geq 1$.

For our further results we use the concept of Carleson measures, which Carleson uses in his solution of the Corona problem. A finite positive Borel measure μ on the unit ball $\mathbb{B}_d$ is called Carleson measure for B_ω^s if and only if $B_\omega^s \subset L^2(\mu)$. In this case, by the closed graph theorem, the linear operator

$$J_\mu : B_\omega^s \to L^2(\mu),\ f \mapsto f$$

is continuous, that is there exists a constant $c(\mu) > 0$ such that

$$\int_{\mathbb{B}_d} |f|^2 d\mu \leq c(\mu) \|f\|_{B_\omega^s}^2 \text{ for all } f \in B_\omega^s.$$

The measure μ is called a vanishing Carleson measure if and only if the linear operator

$$J_\mu : B_\omega^s \to L^2(\mu),\ f \mapsto f$$

is compact. Carleson measures have many applications. Now, suppose that $N \geq s > 0$. Since B_ω^{N-s} can be described as a weighted Bergman space $L_a^2(\omega_{N-s})$, by the previous theorem it is immediate that $\varphi \in A(B_\omega^s)$ if and only if $\varphi \in A(\mathbb{B}_d)$ and

$$\mu_{\varphi,N}(z) = |R^N\varphi(z)|^2 \omega_{N-s}(z) dV(z)$$

is a vanishing Carleson measure for B_ω^s. Besides, to simplify notation, we use the abbreviations:

(a) $B_t^{s,2} = B_t^s$,

(b) $B^{s,p} = B_0^{s,p} = \left\{ f \in \mathscr{O}(\mathbb{B}_d);\ \int_{\mathbb{B}_d} |R^s f(z)|^p \, dV(z) < \infty \right\}$ and

(c) $B^s = B^s_0 = \left\{ f \in \mathscr{O}(\mathbb{B}_d);\ \int_{\mathbb{B}_d} |R^s f(z)|^2 \, dV(z) < \infty \right\}$.

Using the reformulation of the previous theorem in terms of Carleson measures, we obtain the following version of Theorem 5.9 (2) in [BB08] (see Theorem 5.2.17):

Theorem. *Let* $1 < 2s \leq d+1$ *and* $p > \frac{d+1}{s}$. *If*

$$\varphi \in B^{s,p} \cap A(\mathbb{B}_d),$$

then $\varphi \in A(B^s)$.

We will also consider the particular case of the Dirichlet space $\mathscr{D}$. For $f \in \mathscr{D}$ the Sarason function is defined as

$$V_f : \mathbb{D} \to \mathbb{C},\ V_f(z) = 2\langle f, k_z f \rangle_{\mathscr{D}} - \|f\|^2_{\mathscr{D}}.$$

This definition makes sense in several (normalized) complete Nevanlinna-Pick spaces $\mathscr{H}$. In [AHMR18, Theorem 4.5], Aleman, Hartz, M$^{\text{c}}$Carthy and Richter show that if $f \in \mathscr{H}$ and $\operatorname{Re} V_f$ is bounded, then $f \in \operatorname{Mult}(\mathscr{H})$. We show that (see Theorem 5.3.5):

Theorem. *Let* $\varphi \in \mathscr{D}$. *If*

$$\sup_{w \in \mathbb{D}} |\operatorname{Re} V_\varphi(w) - \operatorname{Re} V_{\varphi_r}(rw)| \xrightarrow{r \uparrow 1} 0$$

for $r \uparrow 1$, *then* $\varphi \in A(\mathscr{D})$.

Before, we have already indicated that for the reproducing kernels Hilbert spaces $\mathscr{H}$ with kernel functions $k_w : \mathbb{B}_d \to \mathbb{C}$ ($w \in \mathbb{B}_d$), we want to consider, we always have

$$\operatorname{Mult}(\mathscr{H}) \subset H^\infty \cap \mathscr{H},\ A(\mathscr{H}) \subset \operatorname{Mult}(\mathscr{H}) \cap A(\mathbb{B}_d)$$

and $\|f\|_{\mathscr{H}} + \|f\|_\infty \leq \|f\|_{\text{Mult}}$ for all $f \in \mathscr{H}$. Indeed this basically follows, since $1 \in \mathscr{H}$ and $M^*_\varphi k_w = \overline{\varphi}(w)$ for all $w \in \mathbb{B}_d$ and $\varphi \in \operatorname{Mult}(\mathscr{H})$. In [Arv98] Arveson shows that the supremum-norm $\|\cdot\|_\infty$ does not dominate the operator norm $\|\cdot\|_{\text{Mult}}$ on the Drury-Arveson space H^2_d. Fang and Xia use the multivariable Möbius transformation and this fact in [FX11] to show that

$$A(H^2_d) \subsetneq A(\mathbb{B}_d) \cap \operatorname{Mult}(H^2_d) \subset A(\mathbb{B}_d) \cap H^2_d.$$

Using the one-function Corona theorem for the multiplier algebra of the Drury-Arveson space, the main result in [FX11] shows that there exist functions $\varphi \in \operatorname{Mult}(H^2_d)$ such that the corresponding multiplication operator M_φ is not essentially hyponormal. The one-function Corona theorem applies to a lot of Hilbert or even Banach function spaces $\mathscr{F}$ (see below). It states that, if $\varphi \in \operatorname{Mult}(\mathscr{F})$ and φ is bounded below, then $\frac{1}{\varphi} \in \operatorname{Mult}(\mathscr{F})$.

In [Luo17], Luo obtains similar statements as in Fang and Xia's paper for the Dirichlet space $\mathscr{D}$.

In Chapter 6 we use techniques from [FX11] to establish the following result (see Theorem 6.1):

Theorem. *Let $\mathscr{H}$ be a regular unitarily invariant space. The following are equivalent:*

(i) $\mathrm{Mult}(\mathscr{H}) = H^\infty(\mathbb{B}_d) \cap \mathscr{H}$,

(ii) $A(\mathbb{B}_d) \cap \mathrm{Mult}(\mathscr{H}) = A(\mathbb{B}_d) \cap \mathscr{H}$,

(iii) $A(\mathscr{H}) = A(\mathbb{B}_d) \cap \mathrm{Mult}(\mathscr{H})$,

(iv) $\|M_\varphi\|_e = \|\varphi\|_\infty$ *for all* $\varphi \in \mathrm{Mult}(\mathscr{H}) \cap A(\mathbb{B}_d)$, *where* $\|\cdot\|_e$ *is the essential norm of an operator.*

Similar to Fang and Xia's results, there exists a multiplier function $\varphi \in \mathrm{Mult}(\mathscr{H})$ such that M_φ is not essentially hyponormal, whenever

$$\mathrm{Mult}(\mathscr{H}) \subsetneq \mathscr{H} \cap H^\infty(\mathbb{B}_d)$$

and the one-function Corona theorem holds for $\mathrm{Mult}(\mathscr{H})$.

One-function Corona theorem

Let us now take a closer look at the one-function Corona theorem. In their studies of cyclic vectors in the Drury-Arveson space H^2_d, Richter and Sunkes obtain the one-function Corona theorem for the spaces $A^2_m(\mathbb{B}_d)$ (see [RS16, Theorem 5.4]). In particular, the statement contains the one-function Corona theorem for the Drury-Arveson space H^2_d, appearing in [FX11]. In [CHZ18, Theorem 5 and Corllary 6], Cao, He and Zhu establish the following differentiation formula for the radial derivative:

$$R^N\left(\frac{f}{g}\right) = \frac{(-1)^N}{g^{N+1}} \sum_{l=0}^{N} (-1)^l \binom{N+1}{l} g^l R^N(g^{N-l} f),$$

where $f, g \in \mathscr{O}(\mathbb{B}_d)$, $0 \notin g(\mathbb{B}_d)$ and $N \geq 1$. The formula can be used to prove the one-function Corona theorem for many Banach function spaces (see [LMN20, Lemma 3.1]). In a recent paper, Aleman, Perfekt, Richter, Sundberg and Sunkes obtain a generalized version of the one-function Corona theorem for radially weighted Besov spaces B^N_ω (see Theorem 3.2 in [APR+24] and Theorem 6.2.8):

Theorem. *If* $\varphi, \psi \in \mathrm{Mult}(B^N_\omega)$ *with* $\frac{\varphi}{\psi} \in H^\infty(\mathbb{B}_d)$, *then* $\frac{\varphi^{N+1}}{\psi} \in \mathrm{Mult}(B^N_\omega)$.

We will see that this better version of the one-function Corona theorem also follows from the differentiation formula by Cao, He and Zhu. Similar to the setting in [LMN20, Lemma 3.1], the result is valid for a large variety of Banach function spaces. Examples are radially weighted Besov spaces, Bloch-type spaces, and holomorphic Sobolev-spaces. The differentiation formula by Cao, He, and Zhu can be derived from an application of the binomial theorem (cf. proof of Theorem 6.2.2 and Corollary 6.2.3). The original proof for the formula is more technical.

Let

$$\mathscr{B} = \{f \in \mathscr{O}(\mathbb{B}_d);\ \sup_{z \in \mathbb{B}_d} (1 - |z|^2)|Rf(z)| < \infty\}$$

be the Bloch space. A particular case of Theorem 5.1 in [RS16] shows that $f \in H^2_d \cap \mathscr{B}$ and $\frac{1}{f} \in H^\infty(\mathbb{B}_d)$ imply that $\frac{1}{f} \in H^2_d$. The differentiation formula by Cao, He and Zhu can also be used to establish the following more general version of Theorem 5.1 in [RS16] (see Theorem 6.2.10) for the L^p-versions of standard weighted Besov spaces:

Theorem. *Let* $1 \le p < \infty$, $t > -\frac{1}{p}$ *and* $N \ge 1$.

(a) If $f \in B^{N,p}_t \cap \mathscr{B}$ *and* $\frac{1}{f} \in H^\infty(\mathbb{B}_d)$, *then* $\frac{1}{f} \in B^{N,p}_t$.

(b) If $f, g \in B^{N,p}_t \cap H^\infty(\mathbb{B}_d)$ *and* $\frac{f}{g} \in H^\infty(\mathbb{B}_d)$, *then* $\frac{f^{N+1}}{g} \in B^{N,p}_t \cap H^\infty(\mathbb{B}_d)$.

Concluding remarks

Chapter 2 gives a brief introduction to the basics of the thesis. It contains a sort of crash course on reproducing kernel Hilbert spaces of analytic functions and also provides insight into the theory of radially weighted Besov spaces. The last part of the Chapter is an overview of the theory of K-contractions.

The following sources, which are not listed separately, were also used in preparation of this work [ABR01], [AM00a], [Bar01], [Bar07], [Beu48], [CV78], [CZ92], [Gar07], [Kal09], [Kre14], [Lec95], [Rud91], [Sar89], [Sar90] and [Shi02].

2. Preliminaries

This chapter aims to provide a brief overview of the theoretical framework of unitarily invariant reproducing kernel Hilbert spaces of analytic functions. We also introduce some terminology and notation, which we will use throughout this thesis. In the last part of this chapter, we will take a look at the theory of K-contractions, which we will need for the following results. We start with some notation that we will use throughout the thesis.

2.1. Notation

2.1.1. Multi-indices

Let $d \in \mathbb{N}_{>0}$ be a positive integer.

Notation 2.1.1. Let X be an (abelian) monoid with identity element e. For

$$\alpha = (\alpha_1, \dots, \alpha_d) \in \mathbb{N}^d$$

we use the following notations:

(a) $|\alpha| = \sum_{l=1}^d \alpha_l$,

(b) $\alpha! = \prod_{l=1}^d \alpha_l!$,

(c) $\gamma_\alpha = \frac{|\alpha|!}{\alpha!}$ and

(d) $x^\alpha = \prod_{l=1}^d x_l^{\alpha_l}$ for $x = (x_1, \dots, x_d)$ in X^d , where $x^0 = e$ for all $x \in X$.

Here is a short explanation of how we will use the number γ_α for $\alpha \in \mathbb{N}^d$:

Remark 2.1.2. Let $d \in \mathbb{N}_{>0}$ and let X be a ring. Furthermore, let $x = (x_1, \dots, x_d)$ and $y = (y_1, \dots, y_d)$ be tuples of commuting elements in X. Define the map

$$\sigma_{x,y} : X \to X,\ \sigma(z) = \sum_{l=1}^d x_l z y_l.$$

Then

$$\sigma_{x,y}^n(z) = \sum_{\substack{|\alpha|=n \\ \alpha \in \mathbb{N}^d}} \gamma_\alpha (x^\alpha z y^\alpha) \quad (z \in X)$$

and in particular,

$$\left(\sum_{l=1}^d x_l \right)^n = \sum_{\substack{|\alpha|=n \\ \alpha \in \mathbb{N}^d}} \gamma_\alpha x^\alpha$$

for all $n \in \mathbb{N}_{>0}$.

Sketch of proof. If $l \in \mathbb{N}_{>0}$, then for reasons of readability we use the abbreviation $[l]$ for the set $\{1,\dots,l\}$. Let $n,d \in \mathbb{N}_{>0}$. Consider the letter counting map $\pi : [d]^n \to \{\alpha \in \mathbb{N}^d;\ |\alpha| = n\}$ defined by

$$(\pi((l_1,\dots,l_n)))_i = \operatorname{card}(\{1 \le m \le n;\ l_m = i\}),$$

where $i = 1,\dots,d$. One checks that π is well-defined and surjective. In addition, if $\alpha = \pi((l_1,\dots,l_n))$, then

$$\prod_{m=1}^{n} x_{l_m} = x^\alpha.$$

Now, if $\alpha = (\alpha_1,\dots,\alpha_d) \in \mathbb{N}^d$ with $|\alpha| = n$, a combinatorial argument shows that there exist

$$\gamma_\alpha = \frac{|\alpha|!}{\alpha!} = \binom{n}{\alpha_1} \cdot \binom{n-\alpha_1}{\alpha_2} \cdot \ldots \cdot \binom{n-\sum_{l=1}^{d-1}\alpha_l}{\alpha_d}$$

different tuples $(l_1,\dots,l_n) \in [d]^n$ such that $\pi((l_1,\dots,l_n)) = \alpha$. Thus, we obtain that

$$\begin{aligned}
\sigma_{x,y}^n(z) &= \sum_{l_1,\dots,l_n=1}^{d} \left(\prod_{m=1}^{n} x_{l_m}\right) z \left(\prod_{m=1}^{n} y_{l_m}\right) \\
&= \sum_{(l_1,\dots,l_n)\in[d]^n} \left(\prod_{m=1}^{n} x_{l_m}\right) z \left(\prod_{m=1}^{n} y_{l_m}\right) \\
&= \sum_{(l_1,\dots,l_n)\in[d]^n} x^{\pi((l_1,\dots,l_n))} z y^{\pi((l_1,\dots,l_n))} \\
&= \sum_{\substack{|\alpha|=n \\ \alpha\in\mathbb{N}^d}} \gamma_\alpha (x^\alpha z y^\alpha).
\end{aligned}$$

Using the multiplicative identity of the ring, the additional part is not hard to see. □

2.1.2. Balls and functions

Notation 2.1.3. Let E be a normed space, let $a \in E$ and $r > 0$. We denote by

$$B_E(a,r) = \{x \in E;\ \|x\|_E < r\}$$

the open ball with radius r and center a in E. If $d \in \mathbb{N}_{>0}$ and $E = \mathbb{C}^d$ with the Euclidean norm

$$|z|^2 = |z_1|^2 + \ldots + |z_d|^2 \quad (z = (z_1,\dots,z_d) \in \mathbb{C}^d),$$

we use the notations

(a) $B_d(a,r) = B_{\mathbb{C}^d}(a,r)$, when $a \in \mathbb{C}^d$,

(b) $\mathbb{B}_d(r) = B_d(0,r)$,

(c) $\mathbb{B}_d = B_d(0,1)$,

(d) $D_r(a) = B_1(a,r)$, when $d = 1$ and $a \in \mathbb{C}$,

(e) $\mathbb{D}_r = D_r(0)$,

(f) $\mathbb{D} = D_1(0)$,

(g) and $\mathbb{T} = \partial\mathbb{D} = \{z \in \mathbb{C};\ |z| = 1\}$ for the unit circle in $\mathbb{C}$.

Remark 2.1.4. When not otherwise stated, all functions will be $\mathbb{C}$-valued.

Notation 2.1.5. (a) Let X and Y be sets. We use the notation Y^X for the set of all functions from X to Y.

(b) To shorten notation and when the dimension is clear from the context, we write $\mathbb{C}[z]$ instead of $\mathbb{C}[z_1,\dots,z_d]$ for the polynomials in d complex variables.

(c) Let $\Omega \subset \mathbb{C}^d$ be open and let E be a Banach space. We use the notation $\mathscr{O}(\Omega, E)$ for the set of holomorphic functions defined on Ω with codomain E and abbreviate $\mathscr{O}(\Omega) = \mathscr{O}(\Omega, \mathbb{C})$.

(d) If $\alpha = (\alpha_1,\dots,\alpha_d) \in \mathbb{N}^d$ we use the notation

$$\partial^\alpha = \frac{\partial^{|\alpha|}}{\partial_{z_1}^{\alpha_1}\dots\partial_{z_d}^{\alpha_d}}$$

for the partial derivatives.

The following theorem from the theory of multivariable complex analysis, which can be derived by the one-variable Cauchy integral formula, is well-known. (This is, for example, a particular case of [Esc18b, Satz 2.16].)

Theorem 2.1.6. *Every function $f \in \mathscr{O}(\mathbb{B}_d)$ has a unique homogeneous expansion*

$$f(z) = \sum_{l=0}^{\infty} f_l(z) \quad (z \in \mathbb{B}_d),$$

where each f_l is a homogeneous polynomial, and the series converges normally on $\mathbb{B}_d$. We have

$$f_l(z) = \sum_{|\alpha|=l} \frac{(\partial^\alpha f)(0)}{\alpha!} z^\alpha = \frac{1}{2\pi}\int_0^{2\pi} f(e^{it}z)e^{-ilt}\,dt \quad (l \in \mathbb{N}, z \in \mathbb{B}_d).$$

For $f \in \mathscr{O}(\mathbb{B}_d)$ and $\alpha \in \mathbb{N}^d$ we will often use the abbreviation $f_\alpha = \frac{(\partial^\alpha f)(0)}{\alpha!}$.

Definition 2.1.7. For a complex Hilbert space H and fixed $w = (w_1,\dots,w_d) \in \mathbb{C}^d$ we will use the map

$$Z^{(H)}(w) : H^d \to H,\ (h_1,\dots,h_d) \mapsto \sum_{l=1}^{d} w_l h_l.$$

We abbreviate $Z = Z^{(H)}$, when the Hilbert space is clear from the context.

2.1.3. Estimates and binomial coefficients

Notation 2.1.8. Let $f : X \to \mathbb{R}_{\geq 0}$ and $g : X \to \mathbb{R}_{\geq 0}$ be two positive functions.

(a) We use the notation $f \lesssim g$, if there exists a constant $c > 0$ such that $f \leq cg$.

(b) We use the notation $f \gtrsim g$, if there exists a constant $c > 0$ such that $f \geq cg$.

(c) We use the notation $f \approx g$, if there exist constants $c_1, c_2 > 0$ such that $c_1 g \leq f \leq c_2 g$.

Notation 2.1.9. For $s \in \mathbb{R}$ and $n \in \mathbb{N}$ with $n \geq 1$, we consider the generalized binomial coefficients

$$\binom{s}{n} = \prod_{l=1}^{n} \frac{s-l+1}{l}$$

and set

$$\binom{s}{0} = 1.$$

Remark 2.1.10 (The gamma function). The gamma function $\Gamma : \mathbb{R}_{>0} \to \mathbb{R}$ will be a useful tool for our further studies. Recall, the gamma function can be defined for $s > 0$ by

$$\Gamma(s) = \int_0^\infty t^{s-1} e^{-t} dt,$$

where $\Gamma(n+1) = n!$ holds for all $n \in \mathbb{N}_{>0}$. See for example [AE08, Chapter VI, Section 9] for more details about the gamma function. The Gauss representation formula for $s > 0$ states that

$$\Gamma(s) = \lim_{n\to\infty} \frac{n^s}{n+s} \prod_{l=1}^{n} \frac{l}{l+s-1}$$

(cf. [AE08, Chapter VI, Section 9, Theorem 9.4]). Consequently, it follows for fixed $s > 0$ that there exist constants $c_1(s), c_2(s) > 0$ such that

$$c_1(s)(\Gamma(s)n^{s-1}) \leq (-1)^n \binom{-s}{n} = \binom{n+s-1}{n} \leq c_2(s)(\Gamma(s)n^{s-1})$$

for all $n \in \mathbb{N}$ with $n \geq 1$. Further, we have the following general version of Stirling's asymptotic formula : There exists a function $\theta : \mathbb{R}_{>0} \to (0,1)$ such that

$$\Gamma(s) = \sqrt{\frac{2\pi}{s}} \left(\frac{s}{e}\right)^s e^{\theta(s)/12s},$$

for $s > 0$ (cf. [AE08, Chapter VI, Section 9, Theorem 9.10]). For fixed $s > 0$ we obtain constants $c_3(s), c_4(s) > 0$ such that

$$c_3(s)(\Gamma(s)n^{-s}) \leq \frac{\Gamma(s)\Gamma(n)}{\Gamma(s+n)} \leq c_4(s)(\Gamma(s)n^{-s})$$

for all $n \in \mathbb{N}$ with $n \geq 1$.

2.2. Reproducing kernel Hilbert spaces

Reproducing kernel Hilbert spaces play an important role in complex analysis and functional analysis. We give a brief introduction to the theory of reproducing kernel Hilbert spaces consisting of holomorphic functions on the unit ball $\mathbb{B}_d$ in $\mathbb{C}^d$. Typical examples are the Drury-Arveson space, Bergman spaces or certain classes of Nevanlinna-Pick spaces. We use [Har16], [Har17], [Lan19], [Sch18] and [Wer08] as guidelines. For further reading, we recommend the books [AM02], [PR16].

A Hilbert space $\mathscr{H}$ consisting of functions $f: X \to \mathbb{C}$ on a set X is called reproducing kernel Hilbert space (RKHS) or Hilbert function space, if for each $x \in X$, the point evaluations

$$\delta_x : X \to \mathbb{C},\, x \mapsto f(x)$$

are continuous. Thus, using the Riesz representation theorem, we find for every $x \in X$ a function $k_x \in \mathscr{H}$, such that

$$f(x) = \langle f, k_x \rangle_{\mathscr{H}}$$

for all $f \in \mathscr{H}$.

The reproducing kernel of $\mathscr{H}$ is defined as

$$K : X \times X \to \mathbb{C}, (x,y) \mapsto K(x,y) = k_y(x).$$

Let $\mathscr{E}$ be a Hilbert space. A function $L : X \times X \to \mathbb{B}(\mathscr{E})$ is called positive definite, if for any finite sequence of points $x_1, \dots, x_n$ in X, the $n \times n$ matrix

$$(L(x_l, x_m))_{l,m=1}^n \in B(\mathscr{E}^n)$$

defines a positive operator. Identifying $B(\mathbb{C}) \cong \mathbb{C}$, a reproducing kernel is positive definite. By a theorem of Moore–Aronszajn the converse is also true. See, for example [PR16, Theorem 2.14] for a proof.

Theorem 2.2.1 (Moore–Aronszajn)**.** *Let X be a set and let $K : X \times X \to \mathbb{C}$ be a positive definite function. Then there exists a unique reproducing kernel Hilbert space $\mathscr{H}$ on X, whose reproducing kernel is K.*

We will also need the following well-known characterization of elements in reproducing kernel Hilbert spaces. A proof can be found in [PR16, Theorem 3.11].

Theorem 2.2.2. *Let $\mathscr{H}$ be a reproducing kernel Hilbert space with reproducing kernel $K : X \times X \to \mathbb{C}$. For $f : X \to \mathbb{C}$ the following are equivalent:*

(a) $f \in \mathscr{H}$,

(b) there exists a constant $c \geq 0$ such that

$$K_{f,c} : X \times X \to \mathbb{C},\; (x,y) \mapsto c^2 K(x,y) - f(x)\overline{f(y)}$$

is positive definite.

In this case,

$$\|f\|_{\mathscr{H}} = \inf\{c \geq 0; K_{f,c} \textit{ is positive definite}\}.$$

Convention. *Let $K : X \times X \to \mathbb{C}$ be a positive definite function and $\mathscr{H}$ the corresponding reproducing kernel Hilbert space. Unless otherwise stated, from now on we will always assume that $\mathscr{H}$ has the following property:*

The space $\mathscr{H}$ will be non-degenerate. That is for every $x \in X$ the point evaluation $\delta_x : \mathscr{H} \to \mathbb{C}$ is onto.

Remark 2.2.3. (a) A reproducing kernel Hilbert space $\mathscr{H}$ is non-degenerate if and only if $\mathscr{H}$ has no common zeros. That is

$$\{f(x);\ f \in \mathscr{H}\} \neq \{0\}$$

for all $x \in X$.

(b) Usually, the reproducing kernels $K : X \times X \to \mathbb{C}$ under consideration in this thesis, will be also normalized at a point $x_0 \in X$, that is $K(x, x_0) = 1$ for all $x \in X$. In this case the space $\mathscr{H}$ is clearly non-degenerate.

(c) The space $\mathscr{H}$ is called irreducible if

(i) $K(x,y) \neq 0$ for all $x, y \in X$,

(ii) the kernel functions $k_x : X \to \mathbb{C}$, $k_x(z) = K(z,x)$ and $k_y : X \to \mathbb{C}$, $k_y(z) = K(z,y)$ are linearly independent, if $x \neq y$.

It is clear that irreducibility implies that $\mathscr{H}$ has no common zeros. See also Lemma 1.16 in [Sch18]. For the theory of K-contractions (cf. [Sch18]), which will be used in the following chapters (see also Section 2.5), we will always suppose that $\mathscr{H}$ is irreducible (see [Sch18, 1.38 Lemma]).

2.2.1. Multipliers

The following class of bounded linear operators on reproducing kernel Hilbert spaces is of special interest.

Definition 2.2.4. Let $\mathscr{H}_1$ and $\mathscr{H}_2$ be two reproducing kernel Hilbert spaces consisting of functions $f : X \to \mathbb{C}$ on a set X. The set of multipliers from $\mathscr{H}_1$ to $\mathscr{H}_2$ is defined as

$$\mathrm{Mult}(\mathscr{H}_1, \mathscr{H}_2) = \{\varphi : X \to \mathbb{C};\ \varphi \cdot f \in \mathscr{H}_2 \text{ for all } f \in \mathscr{H}_1\}.$$

By an application of the closed graph theorem, the multiplication operator

$$M_\varphi : \mathscr{H}_1 \to \mathscr{H}_2,\ f \mapsto \varphi \cdot f,$$

is bounded.

One can now consider $\mathrm{Mult}(\mathscr{H}_1,\mathscr{H}_2)$ as a subspace of $B(\mathscr{H}_1,\mathscr{H}_2)$, identifying each multiplier function $\varphi \in \mathrm{Mult}(\mathscr{H}_1,\mathscr{H}_2)$ with the corresponding multiplication operator M_φ.

Using this identification, we equip $\mathrm{Mult}(\mathscr{H}_1,\mathscr{H}_2)$ with the operator topologies

$$\tau_{\|\cdot\|},\ \mathrm{SOT},\ \mathrm{WOT}\ \text{and}\ \tau_w^*$$

on $B(\mathscr{H}_1,\mathscr{H}_2)$.

For the definition of the weak-$*$ topology τ_w^* on $B(\mathscr{H})$, when $\mathscr{H}=\mathscr{H}_1=\mathscr{H}_2$, see Remark 2.3.14 below. For the general case the weak-$*$ topology can be defied in a similar way.

Using point evaluations, it can be readily seen that $\mathrm{Mult}(\mathscr{H}_1,\mathscr{H}_2)$ becomes a complete space with the operator norm

$$\|\varphi\|_{\mathrm{Mult}} := \|\varphi\|_{\mathrm{Mult}(\mathscr{H}_1,\mathscr{H}_2)} := \|M_\varphi\| \quad (\varphi \in \mathrm{Mult}(\mathscr{H}_1,\mathscr{H}_2)).$$

If $\mathscr{H}=\mathscr{H}_1=\mathscr{H}_2$, we use the notation $\mathrm{Mult}(\mathscr{H})=\mathrm{Mult}(\mathscr{H},\mathscr{H})$. With the preceding remarks it is not difficult to see that $(\mathrm{Mult}(\mathscr{H}),\|\cdot\|_{\mathrm{Mult}})$ is a unital commutative Banach algebra, called the multiplier algebra.

Remark 2.2.5. Let $\mathscr{H}_1$ and $\mathscr{H}_2$ be two reproducing kernel Hilbert spaces with reproducing kernels $K_1 : X\times X \to \mathbb{C}$ and $K_2 : X\times X\to\mathbb{C}$ respectively. If $\varphi\in\mathrm{Mult}(\mathscr{H}_1,\mathscr{H}_2)$, one checks that

$$M_\varphi^* K_2(\cdot,x) = \overline{\varphi(x)}K_1(\cdot,x)$$

for all $x\in X$. If $K=K_1=K_2$ and $K(x,x)\neq 0$ for all $x\in X$, it is immediate that

$$\|\varphi\|_\infty = \sup_{x\in X}|\varphi(x)| \le \|M_\varphi\| = \|\varphi\|_{\mathrm{Mult}}.$$

The following characterization of multipliers is well-known (cf. Theorem 5.21 in [PR16]).

Theorem 2.2.6. *Let $\mathscr{H}_1$ and $\mathscr{H}_2$ be two reproducing kernel Hilbert spaces with reproducing kernels $K_1 : X\times X\to\mathbb{C}$ and $K_2 : X\times X\to\mathbb{C}$ respectively. For a map $\varphi : X\to\mathbb{C}$ the following are equivalent:*

(a) $\varphi\in\mathrm{Mult}(\mathscr{H}_1,\mathscr{H}_2)$,

(b) there exists a $c\ge 0$ such that

$$L_{\varphi,c} : X\times X\to\mathbb{C},\ (x,y)\mapsto c^2K_2(x,y)-\varphi(x)\overline{\varphi(y)}K_1(x,y),$$

is positive definite.

In this case, $\|M_\varphi\| = \inf\{c\ge 0;\ L_{\varphi,c}$ is positive definite$\}$.

2.2.2. Pull-backs, subspaces and weak convergence

Sometimes it is very useful to consider the following kind of pullback for reproducing kernel Hilbert spaces. Let $K : X \times X \to \mathbb{C}$ be positive definite and let $\Phi : Y \to X$ be an arbitrary map. Then

$$K_\Phi : Y \times Y \to \mathbb{C},\ (y, y') \mapsto K(\Phi(y), \Phi(y'))$$

is also positive definite and the corresponding reproducing kernel Hilbert space $\mathscr{H}_\Phi$ can be described explicitly as stated in the next theorem (cf. Theorem 5.7 in [PR16]).

Theorem 2.2.7. *Let $\mathscr{H}$ be a reproducing kernel Hilbert space with reproducing kernel $K : X \times X \to \mathbb{C}$ and let $\Phi : Y \to X$ be a function. Then*

$$K_\Phi : Y \times Y \to \mathbb{C},\ (y, y') \mapsto K(\Phi(y), \Phi(y'))$$

is positive definite and the space

$$\mathscr{H}_\Phi = \{f \circ \Phi;\ f \in \mathscr{H}\}$$

with the norm

$$\|h\|_{\mathscr{H}_\Phi} = \inf\{\|f\|_{\mathscr{H}};\ h = f \circ \Phi\}.$$

is the reproducing kernel Hilbert space with kernel K_Φ. Furthermore, the map

$$C(\Phi) : \mathscr{H} \to \mathscr{H}_\Phi,\ f \mapsto f \circ \Phi$$

is a co-isometry with

$$C(\Phi)^* k_y^{(\Phi)} = k_{\Phi(y)}$$

for all $y \in Y$, where

$$k_x : X \to \mathbb{C},\ k_x(z) = K(z, x) \quad (x \in X)$$

and

$$k_y^{(\Phi)} : Y \to \mathbb{C},\ k_y^{(\Phi)}(z) = K_\Phi(z, y) \quad (y \in Y).$$

If $K : X \times X \to \mathbb{C}$ is positive definite and $Y \subset X$, choosing Φ as the inclusion mapping, the following corollary is an immediate consequence of the preceding theorem.

Corollary 2.2.8. *Let $K : X \times X \to \mathbb{C}$ be positive definite with corresponding reproducing kernel Hilbert space $\mathscr{H}$. If $Y \subset X$ is any non-empty subset, let $K|_Y : Y \times Y \to \mathbb{C}$ denote the restriction of K to Y. Then $K|_Y$ is positive definite and the space*

$$\mathscr{H}|_Y = \{f|_Y;\ f \in \mathscr{H}\}$$

with the norm

$$\|h\|_{\mathscr{H}|_Y} = \inf\{\|f\|_{\mathscr{H}};\ h = f|_Y\} \quad (h \in \mathscr{H}|_Y)$$

is the reproducing kernel Hilbert space with kernel $K|_Y$.

We will also need the following well-known theorem about subspaces of reproducing kernel Hilbert spaces. For a proof, see for example [PR16, Theorem 2.5].

Theorem 2.2.9. *Let $\mathscr{H}$ be a reproducing kernel Hilbert space on a set X with reproducing kernel K. Furthermore, let $\mathscr{H}_0 \subset \mathscr{H}$ be a closed subspace and let $P_0 : \mathscr{H} \to \mathscr{H}_0$ be the orthogonal projection onto $\mathscr{H}_0$. Then $\mathscr{H}_0$ is a reproducing kernel Hilbert space on X with reproducing kernel $K_0(x,y) = \langle P_0 k_x, k_y \rangle$.*

We conclude this part with the following statement about pointwise and weak convergence:

Lemma 2.2.10. *Suppose that $\mathscr{H}$, $\mathscr{H}_1$ and $\mathscr{H}_2$ are reproducing kernel Hilbert spaces.*

(a) Let $(f_n)_{n\in\mathbb{N}}$ be a sequence in $\mathscr{H}$. Then the sequence $(f_n)_{n\in\mathbb{N}}$ converges pointwise to a function $f : X \to \mathbb{C}$ and $\sup_{n\in\mathbb{N}} \|f_n\|_{\mathscr{H}} < \infty$ if and only if $f \in \mathscr{H}$ and $f_n \xrightarrow{\tau_w} f$ for $n \to \infty$.

In particular,
$$\|f\|_{\mathscr{H}} \leq \liminf_{n\to\infty} \|f_n\|_{\mathscr{H}}.$$
Additionally, if $\mathscr{H} \subset \mathscr{O}(\mathbb{B}_d)$, the statement is still true, if one replaces pointwise convergence of the sequence $(f_n)_{n\in\mathbb{N}}$ by uniform convergence on compact subsets of $\mathbb{B}_d$.

(b) Suppose that $(\varphi_n)_{n\in\mathbb{N}}$ is a sequence in $\mathrm{Mult}(\mathscr{H}_1, \mathscr{H}_2)$. The following are equivalent:

(i) The sequence $(\varphi_n)_{n\in\mathbb{N}}$ converges pointwise to a function $\varphi : X \to \mathbb{C}$ and $\sup_{n\in\mathbb{N}} \|\varphi_n\|_{\mathrm{Mult}} < \infty$,

(ii) $\varphi \in \mathrm{Mult}(\mathscr{H}_1, \mathscr{H}_2)$ and $M_{\varphi_n} \to M_\varphi$ for $n \to \infty$ in the weak operator topology,

(iii) $\varphi \in \mathrm{Mult}(\mathscr{H}_1, \mathscr{H}_2)$ and $M_{\varphi_n} \to M_\varphi$ for $n \to \infty$ in the weak-$*$ topology.

In particular,
$$\|\varphi\|_{\mathrm{Mult}} \leq \liminf_{n\to\infty} \|\varphi_n\|_{\mathrm{Mult}}.$$
Additionally, if $\mathscr{H}_2 \subset \mathscr{O}(\mathbb{B}_d)$ and $1 \in \mathscr{H}_1$, the statement is still true, if one replaces pointwise convergence of the sequence $(\varphi_n)_{n\in\mathbb{N}}$ by uniform convergence on compact subsets of $\mathbb{B}_d$.

Proof. (a) Suppose that $(f_n)_{n\in\mathbb{N}}$ is a sequence in $\mathscr{H}$ converging pointwise to a function $f : X \to \mathbb{C}$ and that
$$c = \sup_{n\in\mathbb{N}} \|f_n\|_{\mathscr{H}} < \infty.$$
By Theorem 2.2.2, the maps
$$K_{f_n,c} : X \times X \to \mathbb{C},\ (x,y) \mapsto c^2 K(x,y) - f_n(x)\overline{f_n(y)}$$
are positive definite for all $n \in \mathbb{N}$. Let
$$K_{f,c} : X \times X \to \mathbb{C},\ (x,y) \mapsto c^2 K(x,y) - f(x)\overline{f(y)}.$$

Every matrix being the entrywise limit of positive semidefinite matrices is again positive semidefinite and

$$K_{f,c}(x,y) = \lim_{n\to\infty} K_{f_n,c}(x,y)$$

for all $x,y \in X$. So, clearly $K_{f,c}$ is positive definite. Applying the converse implication of Theorem 2.2.2, we deduce that $f \in \mathscr{H}$. As usual let

$$k_x : X \to \mathbb{C},\ k_x(y) = K(y,x)$$

for all $x \in X$. By assumption

$$\lim_{n\to\infty} \langle f_n, k_x \rangle_{\mathscr{H}} = f_n(x) = f(x) = \langle f, k_x \rangle_{\mathscr{H}}$$

for all $x \in X$. Because $\operatorname{span}\{k_x;\ x \in X\} \subset \mathscr{H}$ is dense, we obtain that $f_n \xrightarrow{\tau_w} f$ for $n \to \infty$. Conversely, if $f \in \mathscr{H}$ and $f_n \xrightarrow{\tau_w} f$ for $n \to \infty$, then

$$\lim_{n\to\infty} \langle f_n, k_x \rangle_{\mathscr{H}} = \langle f, k_x \rangle_{\mathscr{H}} = f(x)$$

for all $x \in X$ and

$$\|f\|_{\mathscr{H}} \leq \liminf_{n\to\infty} \|f_n\|_{\mathscr{H}} \leq \sup_{n\in\mathbb{N}} \|f_n\|_{\mathscr{H}} < \infty$$

by the uniform boundedness principle.

For the additional part, let $(f_n)_{n\in\mathbb{N}}$ be a sequence consisting of functions in $\mathscr{O}(\mathbb{B}_d)$ that is bounded on compact subsets of $\mathbb{B}_d$ and that converges pointwise to a holomorphic function $f \in \mathscr{O}(\mathbb{B}_d)$. Since $(f_n)_{n\in\mathbb{N}}$ is bounded on compact subsets, the Cauchy integral formula implies that $(f_n)_{n\in\mathbb{N}}$ is equicontinuous on compact subsets. Pointwise convergence and equicontinuity yield that $(f_n)_{n\in\mathbb{N}}$ converges uniformly on compact subsets to f. So, the only thing remaining to show, is that $(f_n)_{n\in\mathbb{N}}$ is bounded on compact subsets $Q \subset \mathbb{B}_d$, whenever $\sup_{n\in\mathbb{N}} \|f_n\|_{\mathscr{H}} < \infty$. If $z \in \mathbb{B}_d$, then

$$|f_n(z)| = \langle f_n, k_z \rangle_{\mathscr{H}} \leq \left(\sup_{n\in\mathbb{N}} \|f_n\|_{\mathscr{H}} \right) \|k_z\|_{\mathscr{H}}.$$

Let $h \in \mathscr{H}$ and $Q \subset \mathbb{B}_d$ be compact. Since $h \in \mathscr{O}(\mathbb{B}_d)$, it follows that

$$\sup_{z\in Q} |\langle h, k_z \rangle| = \sup_{z\in Q} |h(z)| < \infty$$

and the family $(k_z)_{z\in Q}$ is weakly bounded. By the uniform boundedness principle

$$\sup_{z\in Q} \|k_z\|_{\mathscr{H}} < \infty$$

and the assertion follows.

(b) Part (b) is a straightforward application of part (a) and the uniform boundedness principle. To be more precise, if $f \in \mathscr{H}_1$, use part (a) for the functions $\varphi_n f$ and φf in $\mathscr{H}_2$. Furthermore, observe that the weak operator topology and the weak-$*$ topology coincide on operator-norm bounded sets (cf. A.2.1). □

2.3. Unitarily invariant spaces

In this section, we are interested in some basic results about unitarily invariant reproducing kernel Hilbert spaces on the unit ball $\mathbb{B}_d$ in $\mathbb{C}^d$. We will see that such spaces have some nice properties and provide many interesting examples.

Definition 2.3.1. A reproducing kernel Hilbert space $\mathscr{H}$ with reproducing kernel $K : \mathbb{B}_d \times \mathbb{B}_d \to \mathbb{C}$ is called unitarily invariant, if $K(0,\cdot) \equiv 1$, K is analytic in the first component and

$$K(Uz,Uw) = K(z,w)$$

for all $z,w \in \mathbb{B}_d$ and all unitary maps $U : \mathbb{C}^d \to \mathbb{C}^d$.

A routine verification shows that the reproducing kernel of unitarily invariant spaces can be characterized by power series representations. The following result can be found in [Har17, Lemma 2.2].

Lemma 2.3.2. *Let $\mathscr{H}$ be a reproducing kernel Hilbert space with reproducing kernel $K : \mathbb{B}_d \times \mathbb{B}_d \to \mathbb{C}$. Then $\mathscr{H}$ is unitarily invariant if and only if there exists an analytic function*

$$k\colon \mathbb{D} \to \mathbb{C},\ k(z) = \sum_{n=0}^{\infty} a_n z^n$$

such that $a_0 = 1$, $a_n \geq 0$ for $n \geq 1$ and

$$K(z,w) = k(\langle z,w\rangle)$$

for all $z,w \in \mathbb{B}_d$.

The following proposition about orthonormal basis of unitarily spaces invariant can be found in [GHX04, Proposition 4.1].

Proposition 2.3.3. *Let $\mathscr{H}$ be unitarily invariant space, with reproducing kernel*

$$K : \mathbb{B}_d \times \mathbb{B}_d \to \mathbb{C},\ K(z,w) = \sum_{n=0}^{\infty} a_n \langle z,w\rangle^n,$$

where $a_0 = 1$ and $a_n \geq 0$ for $n \geq 1$. Then $\mathscr{H} \subset \mathscr{O}(\mathbb{B}_d)$ and the family

$$\left(\sqrt{\gamma_\alpha a_{|\alpha|}}\, z^\alpha\right)_{\substack{\alpha\in\mathbb{N}^d \\ a_{|\alpha|}\neq 0}},$$

is an orthonormal basis for $\mathscr{H}$.

Remark 2.3.4. (a) Due to Lemma 2.3.2 and Proposition 2.3.3, the constant functions are elements of $\mathscr{H}$. Furthermore, the set $\mathscr{H} \cap \mathbb{C}[z]$ is densely contained in $\mathscr{H}$.

(b) For $n \in \mathbb{N}$ let

$$\mathbb{H}_n = \left\{ \sum_{|\alpha|=n} p_\alpha z^\alpha;\ p_\alpha \in \mathbb{C} \right\} \subset \mathbb{C}[z]$$

be the space of homogeneous polynomials of degree n. Then $\mathscr{H}$ can be decomposed as

$$\mathscr{H} = \bigoplus_{\{n\in\mathbb{N};\ a_n\neq 0\}} \mathbb{H}_n$$

(c) If $\mathscr{H}$ is a unitarily invariant space, where the reproducing kernel

$$K : \mathbb{B}_d \times \mathbb{B}_d \to \mathbb{C},\ K(z,w) = \sum_{n=0}^{\infty} a_n \langle z, w\rangle^n,$$

is such that $a_0 = 1$ and $a_n > 0$ for $n \geq 1$, then $\mathbb{C}[z] \subset \mathscr{H}$ and thus, $\overline{\mathbb{C}[z]} = \mathscr{H}$ by (a).

Example 2.3.5. (a) For $s > 0$, the spaces $A^2_s(\mathbb{B}_d)$ with reproducing kernel

$$K_s : \mathbb{B}_d \times \mathbb{B}_d \to \mathbb{C},\ K_s(z,w) = \frac{1}{(1-\langle z,w\rangle)^s}$$

are unitarily invariant. Due to Newton's generalized binomial theorem, it is immediate that K_s has the power series representation

$$K_s(z,w) = \sum_{n=0}^{\infty} a_n^{(s)} \langle z,w\rangle^n \quad (z,w \in \mathbb{B}_d),$$

where $a_0 = 1$ and

$$a_n^{(s)} = (-1)^n \binom{-s}{n} = \prod_{l=1}^{n} \frac{s+l-1}{l} > 0$$

for $n \geq 1$. Note, that $A^2_1(\mathbb{B}_d) = H^2_d$ is the Drury-Arveson space, $A^2_d(\mathbb{B}_d) = H^2(\mathbb{B}_d)$ is the Hardy space and $A^2_{d+1}(\mathbb{B}_d) = L^2_a(\mathbb{B}_d)$ is the classical Bergman space.

(b) For $s \in \mathbb{R}$, the spaces $\mathscr{D}_s(\mathbb{B}_d)$ with reproducing kernel

$$K : \mathbb{B}_d \times \mathbb{B}_d \to \mathbb{C},\ K(z,w) = \sum_{n=0}^{\infty} (n+1)^s \langle z,w\rangle^n.$$

are unitarily invariant. In particular, $\mathscr{D}_0(\mathbb{B}_d)$ is the Drury-Arveson space H^2_d and $\mathscr{D}_{-1}(\mathbb{B}_d)$ is the Dirichlet space on the ball. See Examples 2.3.49 and 2.3.64 below.

Remark 2.3.6. If $s > 0$, using the asymptotic formula

$$(-1)^n \binom{-s}{n} \approx (n+1)^{s-1}$$

for $n \in \mathbb{N}$ with $n \geq 1$ (cf. Remark 2.1.10), it follows that $A^2_s(\mathbb{B}_d) = \mathscr{D}_{s-1}(\mathbb{B}_d)$, with equivalence of norms.

2.3.1. Multipliers on unitarily invariant spaces

Given a unitarily invariant space $\mathscr{H}$, we have seen that $\mathbb{C}[z] \cap \mathscr{H}$ is densely contained in $\mathscr{H}$. In the following we want to study the intersection $\mathrm{Mult}(\mathscr{H}) \cap \mathbb{C}[z]$. First, we collect some useful basics, resulting from the unitarily invariance of the kernel function of $\mathscr{H}$.

Notation 2.3.7. (a) If $U : \mathbb{C}^d \to \mathbb{C}^d$ is a unitary map and $f : \mathbb{B}_d \to \mathbb{C}$ is a function on $\mathbb{B}_d$ we set

$$f_U : \mathbb{B}_d \to \mathbb{C}, f_U(z) = f(Uz).$$

(b) For a Hilbert space H let $U(H)$ be the group of all unitary operators in $B(H)$. If $d \in \mathbb{N}_{>0}$ and $H = \mathbb{C}^d$ we simply write $U(d)$ for $U(\mathbb{C}^d)$. Furthermore, let $\mathbb{1}_{\mathbb{C}^d} \in U(d)$ be the unit matrix in $\mathrm{Mat}(d, \mathbb{C})$.

Lemma 2.3.8. *Let $\mathscr{H}$ be a unitarily invariant space and let $U : \mathbb{C}^d \to \mathbb{C}^d$ be a unitary map. Then*

$$\pi_U : \mathscr{H} \to \mathscr{H},\ f \mapsto f_U$$

is a well-defined linear bounded unitary operator.

Proof. Let $K : \mathbb{B}_d \times \mathbb{B}_d \to \mathbb{C}$ be the reproducing kernel of $\mathscr{H}$. Furthermore, let $f \in \mathscr{H}$ and set $c = \|f\|_{\mathscr{H}}$. If $U : \mathbb{C}^d \to \mathbb{C}^d$ is unitary, it follows with Theorem 2.2.2 that $K_{f,U,c}$: $\mathbb{B}_d \times \mathbb{B}_d \to \mathbb{C}$,

$$K_{f,U,c}(z,w) = c^2 K(z,w) - f(Uz)\overline{f(Uw)} = c^2 K(Uz,Uw) - f(Uz)\overline{f(Uw)}$$

is a positive definite function. By the same theorem, we deduce that $f_U \in \mathscr{H}$ with $\|f_U\| \leq \|f\|_{\mathscr{H}}$. Hence π_U is a well-defined linear contraction. Now, using the map π_{U^*}, it can be readily seen that π_U is a unitary with $\pi_U^* = \pi_{U^*}$. □

Theorem 2.3.9. *Let $\mathscr{H}$ be a unitarily invariant space. For a unitary map $U : \mathbb{C}^d \to \mathbb{C}^d$ let $\pi_U : \mathscr{H} \to \mathscr{H}$ be the unitary operator defined as in Lemma 2.3.8. Then the map*

$$\pi : (U(d), \tau_{\|\cdot\|}) \to (U(\mathscr{H}), \mathrm{SOT}),\ U \mapsto \pi_U,$$

is a continuous group homomorphism.

Proof. By the preceding lemma, the operators π_U are unitary and bounded for every unitary matrix $U \in U(d)$. If $U, V \in U(d)$, then clearly

$$\pi(UV) = \pi_{UV} = \pi_U \pi_V = \pi(U)\pi(V)$$

and $\pi(\mathbb{1}_{\mathbb{C}^d}) = \mathrm{id}_{\mathscr{H}}$. Hence, π is a group homomorphism. Next, we show that π is continuous. Multiplication with fixed elements is continuous in $(U(d), \tau_{\|\cdot\|})$ and in $(U(\mathscr{H}), \mathrm{SOT})$ respectively. Since $(U(d), \tau_{\|\cdot\|})$ is metrizable and π is a group homomorphism, it is enough to prove that

$$\lim_{n \to \infty} \|\pi(U_n)f - \pi(\mathbb{1}_{\mathbb{C}^d})f\|_{\mathscr{H}} = 0$$

for all $f \in \mathscr{H}$ and every sequence $(U_n)_{n\in\mathbb{N}}$ in $U(d)$ with $\lim_{n\to\infty} \|U_n - \mathbb{1}_{\mathbb{C}^d}\| = 0$. Let $(U_n)_{n\in\mathbb{N}}$ be such a sequence in $U(d)$ and let $f \in \mathscr{H}$. For $n \in \mathbb{N}$ set $f_n = \pi(U_n)f$. Since

$$\lim_{n\to\infty} \|U_n z - z\|_{\mathbb{C}^d}$$

for all $z \in \mathbb{C}^d$, it follows by continuity of $f \in \mathscr{H}$ that

$$f_n(z) = f(U_n z) \overset{(n\to\infty)}{\longrightarrow} f(z)$$

for all $z \in \mathbb{B}_d$. Using Lemma 2.2.10 and the fact that $\|\pi(U)f\|_{\mathscr{H}} = \|f\|_{\mathscr{H}}$ for all $U \in U(d)$, we deduce that

$$f_n \overset{\tau_w}{\longrightarrow} f$$

as well as $f_n \to f$ in norm for $n \to \infty$ and hence

$$\lim_{n\to\infty} \|\pi(U_n)f - \pi(\mathbb{1}_{\mathbb{C}^d})f\|_{\mathscr{H}} = \lim_{n\to\infty} \|f_n - f\|_{\mathscr{H}} = 0.$$

□

Due to Lemma 2.3.8 and Theorem 2.3.9, we obtain the following theorem.

Theorem 2.3.10. *Let $\mathscr{H}_1$ and $\mathscr{H}_2$ be two unitarily invariant spaces, let $U = (u_{l,m})_{l,m} \in U(d)$ be unitary and let*

$$\pi_l : (U(d), \tau_{\|\cdot\|}) \to (U(\mathscr{H}_l), \mathrm{SOT}),\ U \mapsto \pi_U \quad (l = 1, 2)$$

be the continuous group homomorphisms defined as in Theorem 2.3.9.

(a) The map

$$\Pi_U : B(\mathscr{H}_1, \mathscr{H}_2) \to B(\mathscr{H}_1, \mathscr{H}_2),\ T \mapsto \pi_2(U) \circ T \circ \pi_1(U^*)$$

is an isometric isomorphism with inverse Π_{U^*} such that

$$\Pi_U(K(\mathscr{H}_1, \mathscr{H}_2)) = K(\mathscr{H}_1, \mathscr{H}_2).$$

Additionally, if $\mathscr{H} = \mathscr{H}_1 = \mathscr{H}_2$, the map Π_U is an isometric C^*-algebra isomorphism.

(b) If $\varphi \in \mathrm{Mult}(\mathscr{H}_1, \mathscr{H}_2)$, then $\varphi_U \in \mathrm{Mult}(\mathscr{H}_1, \mathscr{H}_2)$ and $\Pi_U(M_\varphi) = M_{\varphi_U}$. In particular if $\mathscr{H} = \mathscr{H}_1 = \mathscr{H}_2$, it follows for $l = 1, \ldots, d$ that

$$\Pi_U(M_{z_l}) = \sum_{m=1}^{d} u_{l,m} M_{z_m}.$$

(c) For $T \in B(\mathscr{H}_1, \mathscr{H}_2)$ the map

$$\Pi_T : (U(d), \tau_{\|\cdot\|}) \to (B(\mathscr{H}_1, \mathscr{H}_2), \mathrm{SOT}),\ U \mapsto \Pi_U(T)$$

is continuous. Additionally, if T is compact, then $\Pi_T(U)$ is compact for all $U \in U(d)$ and Π_T is continuous with respect to the norm-topology on $B(\mathscr{H}_1, \mathscr{H}_2)$.

To prove the additional statement in part (c), we need the following well-known result (Theorem 2.3.11), which gives us the possibility to construct a norm convergent zero sequence based on a strongly convergent zero sequence.

Theorem 2.3.11. *Let E and F be Banach spaces, let $T \in K(F,E)$ be a compact operator and let $(S_n)_{n\in\mathbb{N}}$ be a sequence of operators in $B(E)$. If*

$$\lim_{n\to\infty} \|S_n x\|_E = 0$$

for all $x \in E$, then

$$\lim_{n\to\infty} \|S_n T\|_{B(F,E)} = 0.$$

Proof. Let $\varepsilon > 0$. Since $\lim_{n\to\infty} \|S_n x\|_E = 0$ for all $x \in E$, it follows by the uniform boundedness principle that

$$c = \sup_{n\in\mathbb{N}} \|S_n\|_{B(E)} < \infty.$$

Let $r = \frac{\varepsilon}{2(c+1)}$. Because

$$\overline{TB_F(0,1)}^E \subset E$$

is compact and

$$\overline{TB_F(0,1)}^E \subset \bigcup_{y\in B_F(0,1)} B_E(Ty,r),$$

there are $y_1,\dots,y_l$ in $B_F(0,1)$ such that

$$\overline{TB_F(0,1)}^E \subset \bigcup_{m=1,\dots,l} B_E(Ty_m,r).$$

Since $\lim_{n\to\infty} \|S_n x\|_E = 0$ for all $x \in E$, there exists a $n_0 \in \mathbb{N}$ such that $\|S_n T y_m\|_E < \frac{\varepsilon}{2}$, for all $n \geq n_0$ and $m = 1,\dots,l$. Let $n \geq n_0$ and let $y \in B_F(0,1)$ be arbitrary. Then there exists a $m_0 \in \{1,\dots,l\}$ such that

$$\|Ty - Ty_{m_0}\|_E < r.$$

We conclude that

$$\|S_n T y\|_E \leq \|S_n T y_{m_0}\|_E + \|S_n\|_{B(E)} \|Ty - Ty_{m_0}\|_E < \varepsilon.$$

See also [Wer00, proof of Theorem II 3.5, page 70]. □

Proof of Theorem 2.3.10. (a) Due to Lemma 2.3.8, the operators

$$\pi_l(U) : \mathscr{H}_l \to \mathscr{H}_l \quad (l = 1,2)$$

are unitary. Thus, part (a) is straightforward.

(b) If $\varphi \in \mathrm{Mult}(\mathscr{H}_1, \mathscr{H}_2)$, then

$$(\Pi_U(M_\varphi)f)(z) = ((\pi_2(U) \circ M_\varphi \circ \pi_1(U^*))f)(z) = \varphi(Uz)f(z)$$

for all $f \in \mathscr{H}_1$ and $z \in \mathbb{B}_d$. Thus, we conclude that $\varphi_U \in \mathrm{Mult}(\mathscr{H}_1, \mathscr{H}_2)$ and $\Pi_U(M_\varphi) = M_{\varphi_U}$. For $l = 1, \dots, d$ consider the coordinate functions

$$z_l : \mathbb{C}^d \to \mathbb{C}, (w_1, \dots, w_d) \mapsto w_l.$$

If $w = (w_d, \dots, w_d)$ is in $\mathbb{C}^d$, then

$$Uw = (\sum_{m=1}^{d} u_{l,m} w_m)_{l=1}^{d}.$$

Hence, it follows for $l = 1, \dots, d$, that

$$z_l(Uw) = \sum_{m=1}^{d} u_{l,m} z_m(w).$$

Thus, one computes for $l = 1, \dots, d$ that

$$\Pi_U(M_{z_l}) = \sum_{m=1}^{d} u_{l,m} M_{z_m}.$$

(c) Since $(U(d), \tau_{\|\cdot\|})$ is metrizable, it suffices to check that Π_T is sequentially continuous. Since multiplication in the strong operator topology is sequentially continuous, taking adjoints in $(U(d), \tau_{\|\cdot\|})$ is continuous and π_1 respectively π_2 are continuous due to Theorem 2.3.9, it follows that Π_T is continuous.

For the additional part of (c) suppose that $T \in K(\mathscr{H}_1, \mathscr{H}_2)$ is compact. Since the compact operators $K(\mathscr{H}_1, \mathscr{H}_2)$ are an ideal of $B(\mathscr{H}_1, \mathscr{H}_2)$, it is immediate that

$$\Pi_T(U) = \pi_2(U) \circ T \circ \pi_1(U^*)$$

is compact for all $U \in U(d)$. If $(U_n)_{n \in \mathbb{N}}$ is a sequence in $U(d)$ and $U \in U(d)$ such that $\lim_{n\to\infty} \|U_n - U\| = 0$, then

$$\pi_2(U_n) \xrightarrow{\text{SOT}} \pi_2(U),$$

as well as,

$$\pi_1(U_n)^* = \pi_1(U_n^*) \xrightarrow{\text{SOT}} \pi_1(U^*) = \pi_1(U)^*$$

for $n \to \infty$. Since

$$\Pi_T(U_n) - \Pi_T(U) = \pi_2(U_n) \circ T \circ (\pi_1(U_n^*) - \pi_1(U^*)) + (\pi_2(U_n) - \pi_2(U^*)) \circ T \circ \pi_1(U^*)$$

and $\|\pi_2(U_n)\| = 1$ for all $n \in \mathbb{N}$, the additional part follows because of Theorem 2.3.11, using the triangle inequality and Schauder's theorem on adjoints of compact operators. □

Notation 2.3.12. For a function $f : \mathbb{B}_d \to \mathbb{C}$ and $\zeta \in \mathbb{T}$ we use the notation

$$f_\zeta : \mathbb{B}_d \to \mathbb{C}, \; f_\zeta(z) = f(\zeta z).$$

Using the fact that the unit circle $\mathbb{T} \subset \mathbb{C}$ is a compact subgroup of $U(d)$ with respect to the representation

$$(\mathbb{T}, \tau_{|\cdot|}) \to (U(d), \tau_{\|\cdot\|}), \ \zeta \mapsto \zeta \mathbb{1}_{\mathbb{C}^d},$$

the following corollary is a consequence of Theorem 2.3.9 and Theorem 2.3.10.

Corollary 2.3.13. *Let $\mathscr{H}, \mathscr{H}_1, \mathscr{H}_2$ be unitarily invariant spaces and let φ be a function in the multiplier space* $\mathrm{Mult}(\mathscr{H}_1, \mathscr{H}_2)$.

(a) The map

$$(\mathbb{T}, \tau_{|\cdot|}) \to (U(\mathscr{H}), \mathrm{SOT}), \ \zeta \mapsto \pi_\zeta,$$

is a continuous group homomorphism, where $\pi_\zeta : \mathscr{H} \to \mathscr{H}$ is the unitary operator with $\pi_\zeta f = f_\zeta$ for all $f \in \mathscr{H}$.

(b) The map

$$\Pi : (\mathbb{T}, \tau_{|\cdot|}) \to (\mathrm{Mult}(\mathscr{H}_1, \mathscr{H}_2), \mathrm{SOT}), \ \zeta \mapsto \varphi_\zeta$$

is continuous and

$$\|\varphi_\zeta\|_{\mathrm{Mult}} = \|\varphi\|_{\mathrm{Mult}}$$

for all $\zeta \in \mathbb{T}$. Additionally, if M_φ is compact, then M_{φ_ζ} is compact for all $\zeta \in \mathbb{T}$ and Π is continuous with respect to the norm-topology on $\mathrm{Mult}(\mathscr{H}_1, \mathscr{H}_2)$.

Using the theorem of Fejér, one can show that the set of trigonometric polynomials is densely contained in the set of continuous functions $C(\mathbb{T})$ on the unit circle $\mathbb{T}$, without using Stone-Weierstraß. For our goals, we need similar results from harmonic analysis to gather some useful facts about unitarily invariant spaces and its multiplier algebras. The results can also be found in [Har12].

In the following let Q be a compact Hausdorff space, let μ be a regular Borel measure on Q and let E be a locally convex Hausdorff space. We denote by E' the space of all continuous linear functionals on E.

Remark 2.3.14. For our studies, we will mainly consider the following locally convex Hausdorff spaces:

(a) If E is a normed space then we consider E with the weak topology τ_w or E' with the weak-$*$ topology τ_{w^*}.

(b) If H is a Hilbert space, then we consider the bounded linear operators $B(H)$ with the weak operator topology WOT and with the strong operator topology SOT.

(c) It is well-known that the dual of $\ell^1(\mathbb{N})$ is $\ell^\infty(\mathbb{N})$. Analogously, if $B(H)$ are the bounded linear operators on a Hilbert space H with orthonormal basis (e_α), then $B(H)$ can be considered as the dual

$$C_1(H)' = B(C_1(H), \mathbb{C})$$

of the Banach space of trace class operators

$$C_1(H) = \{T \in B(H);\ \mathrm{Tr}(|T|) < \infty\}$$

with the norm

$$\|T\|_1 = \mathrm{Tr}(|T|) = \sum_{\alpha} \langle |T| e_\alpha, e_\alpha \rangle_H \quad (T \in C_1(H)),$$

where $|T| = \sqrt{T^*T}$ for $T \in B(H)$. Using this fact, one can define a weak-$*$ topology τ_{w^*} on $B(H)$. For more details see for example [Con00, Chapter 3, 19.2 Theorem and Section 20].

For additional information regarding locally convex Hausdorff spaces, one may refer to the book [SW99]. A brief overview of operator topologies can be found in [Con91, Chapter I, §2].

Definition 2.3.15 (Pettis integral)**.** Suppose that $f : Q \to E$ is a continuous function. If there exists an element $y \in E$ such that

$$x'(y) = \int_Q x'(f) d\mu \qquad \text{for all } x' \in E' \tag{2.1}$$

then y is called the weak or the Pettis integral of f over Q. One usually uses the notation

$$y = \int_Q f d\mu.$$

The Hahn-Banach theorem implies that the integral, if existent, is uniquely determined by (2.1).

Example 2.3.16. Let $E = (B(H), \mathrm{WOT})$ and let $f : Q \to E$ be a continuous function. Define a sesquilinear form

$$H \times H \to \mathbb{C},\ (h_1, h_2) \mapsto \int_Q \langle f(t) h_1, h_2 \rangle d\mu(t)$$

on H. By the uniform boundedness principle, it follows that

$$\sup_{t \in Q} \|f(t)\|_{B(H)} < \infty.$$

Thus, the sesquilinear form is bounded. By the Lax-Milgram theorem (see [Con90, Chapter II, 2.2 Theorem]), there exists an operator $T \in B(H)$ such that

$$\langle T h_1, h_2 \rangle = \int_Q \langle f(t) h_1, h_2 \rangle d\mu(t)$$

for all $h_1, h_2 \in H$. Since the WOT-continuous linear functionals are of the form

$$B(H) \to \mathbb{C};\ T \to \sum_{l=1}^{m} \langle T h_l, \tilde{h}_l \rangle$$

for some vectors $h_1, \ldots, h_m, \tilde{h}_1, \ldots, \tilde{h}_m \in H$ (cf. [Con00, Chapter IX, 5.1 Proposition]), we conclude that

$$T = \int_Q f d\mu.$$

Remark 2.3.17. Suppose that $f : Q \to E$ is continuous and

$$y = \int_Q f d\mu.$$

exists. Suppose that $p : E \to \mathbb{R}_{\geq 0}$ is a continuous seminorm on E. By the Hahn-Banach theorem there must be a continuous functional $x' \in E$ such that $x'(y) = p(y)$ and $|x'(x)| \leq p(x)$ for all $x \in E$. Hence

$$p\left(\int_Q f d\mu\right) = x'\left(\int_Q f d\mu\right) = \int_Q x'(f) d\mu \leq \int_Q p(f) d\mu.$$

The following theorem can be found as a particular case of Theorem 3.27 in [Rud91].

Theorem 2.3.18. *Suppose that for every compact set $M \subset E$ the closure of the convex hull $\overline{\mathrm{co}(M)}$ is again compact. Then the integral*

$$y = \int_Q f d\mu$$

exists for every continuous function $f : Q \to E$ in the sense of Definition 2.3.15.

Example 2.3.19. Let $E = (B(H), \mathrm{WOT})$ or $E = (B(H), \tau_{w*})$. The weak operator topology is coarser than the weak-$*$ topology. By Banach-Alaoglu norm-bounded sets in E are relatively compact. By the uniform boundedness principle WOT-bounded sets are norm-bounded. It follows that every compact set $M \subset E$ is contained in a convex compact set. But then the closure of the convex hull $\overline{\mathrm{co}(M)}$ is again compact.

A locally convex Hausdorff space E is called quasi-complete if every bounded Cauchy net is convergent.

Example 2.3.20. (a) Every Banach space is quasi-complete.

(b) Since a Hilbert space H is complete, it is elementary to check that $E = (B(H), \mathrm{SOT})$ is quasi-complete.

(c) $E = (B(H), \mathrm{WOT})$ is quasi-complete.

(d) $E = (B(H), \tau_{w*})$ is quasi-complete.

The following proposition can be found in [SW99, Chapter II, §4.3].

Proposition 2.3.21. *Let E be a locally convex quasi-complete Hausdorff space, then for every compact set $M \subset E$ the closure of the convex hull $\overline{\mathrm{co}(M)}$ is again compact.*

The following corollary is a consequence.

Theorem 2.3.22. *Suppose that E is quasi-complete. Then the integral*

$$y = \int_Q f d\mu.$$

exists for every continuous function $f : Q \to E$ in the sense of Definition 2.3.15.

In the following, we denote by dm the normalized Lebesgue measure on $\mathbb{T}$.

Definition 2.3.23. A *summability kernel* is a sequence of Lebesgue integrable functions $(K_n)_{n\geq 0}$ on $\mathbb{T}$ with the following properties

(a) $K_n \geq 0$ for all $n \in \mathbb{N}$,

(b) $\int_{\mathbb{T}} K_n(\zeta)dm(\zeta) = 1$,

(c) For all $\delta > 0$, we have $\lim_{n\to\infty} \sup\{|K_n(\zeta)|;\ \zeta \in \mathbb{T}, |1-\zeta| \geq \delta\} = 0$.

Remark 2.3.24. Let $g : \mathbb{T} \to \mathbb{C}$ be a continuous function.

(a) Clearly, $g \in L^2(\mathbb{T})$. So, using the orthonormal basis $(\zeta^n)_{n\in\mathbb{Z}}$ the function g can be represented as $g(\zeta) = \sum_{n\in\mathbb{Z}} \hat{g}(n)\zeta^n$, where $\hat{g}(n)$ $(n \in \mathbb{Z})$ are the Fourier coefficients.

(b) For $n \in \mathbb{N}$ let

$$F_n(\zeta) = \sum_{m=-n}^{n} \frac{n+1-|m|}{n+1}\zeta^n \quad (\zeta \in \mathbb{T})$$

be the Fejér kernels. Then $(F_n)_{n\geq 0}$ is a summability kernel. One computes that

$$\int_{\mathbb{T}} F_n(\zeta)g(\zeta)dm(\zeta) = \frac{1}{n+1}\sum_{l=0}^{n} S_l[g](1),$$

where

$$S_n[g](\zeta) = \sum_{l=-n}^{n} \hat{g}(l)\zeta^l$$

(see [Kat04, Chapter I, Section 2, 2.5]).

(c) For $0 < r < 1$ let

$$P_r(\zeta) = \sum_{n\in\mathbb{Z}} r^{|n|}\zeta^n = \frac{1-r^2}{|1-r\overline{\zeta}|^2} \quad (\zeta \in \mathbb{T})$$

be the the Poisson kernels. Then for every sequence $(r_n)_{n\in\mathbb{N}}$ in $(0,1)$ with

$$\lim_{n\to\infty} r_n = 1$$

the family $(P_{r_n})_{n\geq 0}$ is a summability kernel. One computes that

$$\int_{\mathbb{T}} P_r(\zeta)g(\zeta)dm(z) = \sum_{n\in\mathbb{Z}} \hat{g}(n)r^{|n|};$$

(see [Kat04, Chapter I, Section 2, 2.13]).

In [Kat04, Chapter I, Section 2, 2.2] one can find the following version of Fejér's theorem:

Theorem 2.3.25. *Let $g : \mathbb{T} \to \mathbb{C}$ be continuous and let $(K_n)_{n\geq 0}$ be a summability kernel, then*

$$\lim_{n\to\infty} \int_{\mathbb{T}} K_n(\zeta)g(\zeta)dm(\zeta) = g(1).$$

Remark 2.3.26. Let E be a locally convex quasi-complete Hausdorff space, let $f : \mathbb{T} \to E$ and $g : \mathbb{T} \to \mathbb{C}$ be continuous. A locally convex space is by definition a topological vector space, so scalar multiplication is continuous and thus the function

$$\mathbb{T} \to E,\ \zeta \mapsto g(\zeta)f(\zeta)$$

is continuous. Hence, due to Theorem 2.3.22, the weak integral

$$\int_{\mathbb{T}} g(\zeta)f(\zeta)dm(\zeta)$$

exists in E.

The following corollary is immediate.

Corollary 2.3.27. *Let E be a locally convex quasi-complete Hausdorff space, let $f : \mathbb{T} \to E$ be continuous and let $(K_n)_{n\geq 0}$ be a summability kernel in $C(\mathbb{T})$, then*

$$\lim_{n\to\infty} \int_{\mathbb{T}} K_n(\zeta)f(\zeta)dm(\zeta) = f(1)$$

Proof. By the preceding remark, the integrals exist in the weak sense. Let $p : E \to \mathbb{R}_{\geq 0}$ be a continuous seminorm. The function $g : \mathbb{T} \to \mathbb{C},\ \zeta \mapsto p(f(\zeta) - f(1))$ is continuous with $g(1) = 0$. Because of Theorem 2.3.25 and Remark 2.3.17, we deduce that

$$\begin{aligned} 0 \leq p\left(\int_{\mathbb{T}} K_n(\zeta)f(\zeta)dm(\zeta) - f(1)\right) &= p\left(\int_{\mathbb{T}} K_n(\zeta)(f(\zeta) - f(1))dm(\zeta)\right) \\ \leq \int_{\mathbb{T}} K_n(\zeta)g(\zeta)dm(\zeta) &\longrightarrow g(1) = 0 \end{aligned}$$

for $n \to \infty$. □

In the following, we want to consider locally convex Hausdorff function spaces $\mathscr{F} \subset \mathscr{O}(\mathbb{B}_d)$. In our further studies $\mathscr{F}$ will be a unitarily invariant space $\mathscr{H}$ or a subspace of the multiplier algebra $\operatorname{Mult}(\mathscr{H}_1, \mathscr{H}_2)$, where $\mathscr{H}_1$ and $\mathscr{H}_2$ are unitarily invariant spaces.

Definition 2.3.28. We call a locally convex Hausdorff space $\mathscr{F} \subset \mathscr{O}(\mathbb{B}_d)$ *homogeneous* if

(a) $\mathscr{F}$ is quasi-complete,

(b) the point evaluations $\delta_z : \mathscr{F} \to \mathbb{C},\ f \mapsto f(z)$ are continuous,

(c) for $f \in \mathscr{F}$ and $\zeta \in \mathbb{T}$ the functions $f_\zeta : \mathbb{B}_d \to \mathbb{C}$, $f_\zeta(z) = f(\zeta z)$ belong to $\mathscr{F}$ and the maps
$$\mathbb{T} \to \mathscr{F}, \zeta \mapsto f_\zeta$$
are continuous.

Lemma 2.3.29. *Let $\mathscr{F} \subset \mathscr{O}(\mathbb{B}_d)$ be a Banach space such that the point evaluations $\delta_z : \mathscr{F} \to \mathbb{C}$, $f \mapsto f(z)$ are continuous for all $z \in \mathbb{B}_d$. If*

(a) $\mathbb{C}[z] \subset \mathscr{F}$ is dense and

(b) $f_\zeta \in \mathscr{F}$ with $\|f_\zeta\|_{\mathscr{F}} = \|f\|_{\mathscr{F}}$ for all $f \in \mathscr{F}$ and $\zeta \in \mathbb{T}$,

then $\mathscr{F}$ is a homogeneous space.

Proof. It suffices to show that for all $f \in \mathscr{F}$ the functions
$$\mathbb{T} \to \mathscr{F}, \ \zeta \mapsto f_\zeta$$
are continuous. Since $\|f_\zeta\|_{\mathscr{F}} = \|f\|_{\mathscr{F}}$, it follows that
$$\|f_\zeta - f_\xi\|_{\mathscr{F}} = \|f_{\zeta\overline{\xi}} - f\|_{\mathscr{F}}$$
for all $f \in \mathscr{F}$ and $\zeta, \xi \in \mathbb{T}$. Hence, it is enough to prove that
$$\mathscr{E} = \{f \in \mathscr{F};\ \lim_{\zeta \to 1} \|f_\zeta - f\|_{\mathscr{F}} = 0\}$$
is equal to $\mathscr{F}$. First, we show that $\mathscr{E}$ is closed. Let therefore $f \in \overline{\mathscr{E}}$ and $\varepsilon > 0$. Choose a $g \in \mathscr{E}$ with $\|f - g\|_{\mathscr{F}} < \frac{\varepsilon}{4}$ and a $\delta > 0$ with $\|g_\zeta - g\|_{\mathscr{F}} < \frac{\varepsilon}{4}$ for all $\zeta \in \mathbb{T}$ with $|\zeta - 1| < \delta$. We deduce for all $\zeta \in \mathbb{T}$ with $|\zeta - 1| < \delta$ that
$$\begin{aligned}\|f_\zeta - f\|_{\mathscr{F}} &\le \|f_\zeta - g_\zeta\|_{\mathscr{F}} + \|g_\zeta - g\|_{\mathscr{F}} + \|g - f\|_{\mathscr{F}} \\ &= 2\|f - g\|_{\mathscr{F}} + \|g_\zeta - g\|_{\mathscr{F}} < \varepsilon.\end{aligned}$$
It follows that f is in $\mathscr{E}$ and hence $\mathscr{E}$ is closed. Finally, we show that $\mathscr{E}$ contains all monomials. If z^α $(\alpha \in \mathbb{N}^d)$ is a monomial, then
$$(z^\alpha)_\zeta = (\zeta z)^\alpha.$$
Hence, we obtain
$$\|(z_\alpha)_\zeta - z^\alpha\|_{\mathscr{F}} = |\zeta^\alpha - 1| \|z_\alpha\|_{\mathscr{F}} \longrightarrow 0$$
for $\zeta \to 1$. We conclude that
$$\mathscr{F} = \overline{\mathbb{C}[z]} \subset \mathscr{E} \subset \mathscr{F}.$$
□

Lemma 2.3.30. *Let $\mathscr{H}$ be a unitarily invariant space. The multiplier algebra* Mult$(\mathscr{H})$*, equipped with the strong operator topology, the weak operator topology or the weak-$*$ operator topology, is closed in $B(\mathscr{H})$ and hence quasi-complete.*

Proof. Due to Example 2.3.20, it suffices to show that the multiplier algebra $\mathrm{Mult}(\mathscr{H})$ is closed in $B(\mathscr{H})$ with the strong operator topology, the weak operator topology or the weak-$*$ operator topology. Suppose that $(M_{\varphi_\alpha})_{\alpha\in\mathscr{A}}$ is a net of multiplication operators on $\mathscr{H}$, which converges to an operator $T \in B(\mathscr{H})$ in one of the mentioned operator topologies. It follows, in particular, that $(M_{\varphi_\alpha})_{\alpha\in\mathscr{A}}$ converges to $T \in B(\mathscr{H})$ in the weak operator topology. If

$$K : \mathbb{B}_d \times \mathbb{B}_d \to \mathbb{C},\ K(z,w) = k_w(z)$$

is the reproducing kernel of $\mathscr{H}$, then for all $f \in \mathscr{H}$ and $z \in \mathbb{B}_d$ the identity

$$(Tf)(z) = \langle Tf, k_z\rangle_{\mathscr{H}} = \lim_\alpha \langle M_{\varphi_\alpha} f, k_z\rangle_{\mathscr{H}} = \lim_\alpha \varphi_\alpha(z) f(z)$$

is immediate. Since $1 \in \mathscr{H}$, the net $(\varphi_\alpha)_{\alpha\in\mathscr{A}}$ converges pointwise on $\mathbb{B}_d$ to a function φ. We obtain for all $f \in \mathscr{H}$ and $z \in \mathbb{B}_d$ that

$$(Tf)(z) = \lim_\alpha \varphi_\alpha(z) f(z) = \varphi(z) f(z).$$

Thus, $\varphi \in \mathrm{Mult}(\mathscr{H})$ and $T = M_\varphi$. It follows that

$$\mathrm{Mult}(\mathscr{H}) = \overline{\mathrm{Mult}(\mathscr{H})}^{w^*} = \overline{\mathrm{Mult}(\mathscr{H})}^{\mathrm{SOT}} = \overline{\mathrm{Mult}(\mathscr{H})}^{\mathrm{WOT}}.$$

□

Notation 2.3.31. Let $\mathscr{H}$ be a unitarily invariant space. If $\mathbb{C}[z] \subset \mathrm{Mult}(\mathscr{H})$, we denote by

$$A(\mathscr{H}) = \overline{\mathbb{C}[z]}^{\|\cdot\|_{\mathrm{Mult}}}$$

the norm-closure of the polynomials in $\mathrm{Mult}(\mathscr{H})$.

Lemma 2.3.32. *Let $\mathscr{H}$ be a unitarily invariant space.*

(a) The multiplier algebra $\mathrm{Mult}(\mathscr{H})$, equipped with the strong operator topology, the weak operator topology or the weak-$*$ operator topology, is homogeneous.

(b) If $\mathbb{C}[z] \subset \mathrm{Mult}(\mathscr{H})$, then $A(\mathscr{H})$ equipped with the norm topology is homogeneous.

Proof. (a) Because of Lemma 2.3.30, the multiplier-algebra $\mathrm{Mult}(\mathscr{H})$ is quasi-complete with the corresponding operator topology.

Let $\varphi \in \mathrm{Mult}(\mathscr{H})$. For $\zeta \in \mathbb{T}$ Corollary 2.3.13, part (b) shows that the function

$$\varphi_\zeta : \mathbb{B}_d \to \mathbb{C},\ \varphi_\zeta(z) = \varphi(\zeta z)$$

is in $\mathrm{Mult}(\mathscr{H})$ with $\|\varphi_\zeta\|_{\mathrm{Mult}} = \|\varphi\|_{\mathrm{Mult}}$. Furthermore, the map

$$\mathbb{T} \to \mathrm{Mult}(\mathscr{H}),\ \zeta \to \varphi_\zeta$$

is SOT-continuous. The same map is clearly WOT-continuous. Since every norm-bounded net converges in the weak operator topology if and only if it converges in the weak-$*$ topology (cf. A.2.1), the map is also weak-$*$ continuous.

(b) Let $\zeta \in \mathbb{T}$. If p is a polynomial, then p_ζ is a polynomial. Using the equality

$$\|\varphi_\zeta\|_{\mathrm{Mult}} = \|\varphi\|_{\mathrm{Mult}}$$

for all $\varphi \in \mathrm{Mult}(\mathscr{H})$, it is not difficult to see that $\psi_\zeta \in A(\mathscr{H})$, if $\psi \in A(\mathscr{H})$. Hence, the assertion follows with Lemma 2.3.29. □

Notation 2.3.33. Let $f \in \mathscr{O}(\mathbb{B}_d)$ be a holomorphic function with power series representation $f = \sum_{\alpha\in\mathbb{N}^d} f_\alpha z^\alpha$.

(a) For $l \in \mathbb{N}$, set $f_l = \sum_{|\alpha|=l} f_\alpha z^\alpha$.

(b) For $n \in \mathbb{N}$, we denote by $S_n[f] = \sum_{l=0}^{n} f_l$ the n-th partial sum of $f = \sum_{l=0}^{\infty} f_l$.

(c) For $N \in \mathbb{N}$, we denote by $\sigma_N(f) = \frac{1}{N+1}\sum_{n=0}^{N} S_n[f]$ the Fejér-means of f.

(d) For $0 < r < 1$, we denote by

$$f_r : \mathbb{B}_d \to \mathbb{C},\ z \mapsto f(rz)$$

the radial dilations of f.

Proposition 2.3.34. *Let $\mathscr{F} \subset \mathscr{O}(\mathbb{B}_d)$ be a homogeneous space of analytic functions and let $f = \sum_{\alpha\in\mathbb{N}^d} f_\alpha z^\alpha \in \mathscr{F}$.*

(a) For $n \in \mathbb{N}$ we have

$$f_n = \sum_{|\alpha|=n} f_\alpha z^\alpha = \int_{\mathbb{T}} f_\zeta \zeta^{-n} dm(\zeta) \in \mathscr{F}.$$

For $n < 0$, the integral is zero.

(b) If $(F_N)_{N\geq 0}$ is the Fejér kernel, then

$$\sigma_N(f) = \int_{\mathbb{T}} F_N(\zeta) f_\zeta dm(\zeta) \in \mathscr{F}$$

for all $N \in \mathbb{N}$.

(c) If $0 < r < 1$ and P_r is the Poisson-kernel, then

$$f_r = \int_{\mathbb{T}} P_r(\zeta) f_\zeta dm(\zeta) \in \mathscr{F}.$$

Proof. Due to Remark 2.3.26, the integrals in (a), (b) and (c) exist and are elements of $\mathscr{F}$. Fix $z \in \mathbb{B}_d$. then

$$g_z : \mathbb{T} \to \mathbb{C},\ \zeta \mapsto f(\zeta z)$$

is a continuous function with Fourier coefficients

$$\hat{g}_z(n) = \begin{cases} f_n(z) & \text{if } n \geq 0, \\ 0 & \text{if } n < 0. \end{cases}$$

Part (a): Observe that

$$f_n(z) = \hat{g}_z(n) = \int_{\mathbb{T}} g_z(\zeta)\zeta^{-n} dm(\zeta) = \int_{\mathbb{T}} f(\zeta z)\zeta^{-n} dm(\zeta) = \delta_z \left(\int_{\mathbb{T}} f_\zeta \zeta^{-n} dm(\zeta) \right)$$

for $n \geq 0$, where $\delta_z : \mathscr{F} \to \mathbb{C}$ the point-evaluation in z. Furthermore, the integral

$$\int_{\mathbb{T}} f_\zeta \zeta^{-n} dm(\zeta) = \int_{\mathbb{T}} g_z(\zeta)\zeta^{-n} dm(\zeta) = \hat{g}_z(n)$$

is zero for $n < 0$.

Part (b): Using Remark 2.3.24 part (b) for g_z, we deduce that

$$\begin{aligned} \sigma_N(f)(z) &= \frac{1}{N+1} \sum_{n=0}^{N} S_n[g_z](1) \\ &= \int_{\mathbb{T}} F_N(\zeta) g_z(\zeta) dm(\zeta) = \delta_z \left(\int_{\mathbb{T}} F_N(\zeta) f_\zeta dm(\zeta) \right) \end{aligned}$$

for all $N \in \mathbb{N}$.

Part (c): Using Remark 2.3.24 part (c) for g_z, we obtain

$$f(rz) = \sum_{n=0}^{\infty} f_n(z) r^n = \int_{\mathbb{T}} P_r(\zeta) g_z(\zeta) dm(\zeta) = \delta_z \left(\int_{\mathbb{T}} P_r(\zeta) f_\zeta dm(\zeta) \right)$$

for $0 < r < 1$. □

The following corollary is a consequence of Proposition 2.3.34 and Corollary 2.3.25.

Corollary 2.3.35. *Let $\mathscr{F} \subset \mathscr{O}(\mathbb{B}_d)$ be a homogeneous space.*

(a) If $f \in \mathscr{F}$, then the Fejér-means $(\sigma_n(f))_{n \in \mathbb{N}}$ and the radial dilations $(f_r)_{0<r<1}$ converge in $\mathscr{F}$ to f,

(b) $\mathscr{F}$ contains $\mathscr{F} \cap \mathbb{C}[z]$ as a dense subspace.

Using Lemma 2.3.32 one checks that the next statement is a particular case of Corollary 2.3.35. For ease of reference we state it as a theorem here.

Theorem 2.3.36. *Let $\mathscr{H}$ be a unitarily invariant space.*

(a) *If $f \in \mathrm{Mult}(\mathscr{H})$, then the Fejér-means $(\sigma_n(f))_{n\in\mathbb{N}}$ and the radial dilations $(f_r)_{0<r<1}$ converge in strong operator topology, in the weak operator topology and in the weak-$*$ topology to f.*

In particular, $\mathrm{Mult}(\mathscr{H})$ contains $\mathrm{Mult}(\mathscr{H}) \cap \mathbb{C}[z]$ as a dense subspace with respect to the strong operator topology, the weak operator topology and the weak-$*$ topology.

(b) *Suppose that $\mathbb{C}[z] \subset \mathrm{Mult}(\mathscr{H})$. If $f \in A(\mathscr{H})$, then the Fejér-means $(\sigma_n(f))_{n\in\mathbb{N}}$ and the radial dilations $(f_r)_{0<r<1}$ converge in the multiplier-norm to f.*

2.3.2. Radially weighted Besov spaces

It is a frequent challenge to understand the multiplier algebra of reproducing kernel Hilbert spaces. Often, a function and measure theoretic description of Hilbert function spaces seems to be a useful tool. We consider radially weighted Besov spaces, unitarily invariant spaces, where the Hilbert space norm is equivalent to an L^2-norm of a fractional radial derivative. This description, is roughly speaking, a measure for the smoothness of the Hilbert space functions. Radially weighted Besov spaces contain many interesting examples. Among others the Dirichlet space, Dirichlet-type spaces, Bergman spaces, or the Drury-Arveson space can be described as radially weighted Besov spaces.

The following definitions and results about radially weighted Besov spaces can be found in [Zhu05], [CFO10], [AHMR19] and [RS16], which we also use as guidelines here.

For $s \in \mathbb{R} \setminus \{0\}$ we define $R^s : \mathscr{O}(\mathbb{B}_d) \to \mathscr{O}(\mathbb{B}_d)$,

$$R^s\left(\sum_{n=0}^{\infty} \sum_{|\alpha|=n} f_\alpha z^\alpha\right) = \sum_{n=1}^{\infty} n^s \sum_{|\alpha|=n} f_\alpha z^\alpha$$

The operator R^s is called (fractional) radial derivative and generalizes the radial derivative $R : \mathscr{O}(\mathbb{B}_d) \to \mathscr{O}(\mathbb{B}_d)$,

$$Rf = \sum_{l=1}^{d} z_l \frac{\partial f}{\partial z_l}.$$

Remark 2.3.37. (a) The fractional radial derivative is well-defined. To see this let $s \in \mathbb{R} \setminus \{0\}$ and let $f = \sum_{\alpha\in\mathbb{N}^d} f_\alpha z^\alpha \in \mathscr{O}(\mathbb{B}_d)$. Further, let $0 < r < 1$ and let $z \in \mathbb{B}_d(r)$. By the Cauchy-estimates applied to the holomorphic function $g_z : \mathbb{D}(r^{-1}) \to \mathbb{C}, \zeta \mapsto f(\zeta z)$, we obtain that

$$|f_n(z)| = \left|\sum_{|\alpha|=n} \frac{\partial^\alpha f(0)}{\alpha!} z^\alpha\right| = \frac{1}{2\pi}\left|\int_{\partial\mathbb{D}(r^{-1})} f(\zeta z)\zeta^{-(n+1)} d\zeta\right| \leq \left(\sup_{w\in\mathbb{B}_d(r)} |f(w)|\right) r^n.$$

Using the Weierstrass M-test, we deduce that $R^s f \in \mathscr{O}(\mathbb{B}_d)$ and

$$\sup_{w\in\mathbb{B}_d(r)} |R^s f(w)| \leq \left(\sum_{n=1}^{\infty} n^s r^n\right)\left(\sup_{w\in\mathbb{B}_d(r)} |f(w)|\right).$$

(b) Using the Cauchy product formula, it can be readily seen that there is a similar product rule for the radial derivative as in the classical case. More generally, inductively, one obtains for $f, g \in \mathscr{O}(\mathbb{B}_d)$ and $N \geq 1$ the Leibniz (product) rule

$$R^N(fg) = \sum_{l=0}^{N} \binom{N}{l} R^l f R^{N-l} g.$$

Let dV be the normalized Lebesgue measure on $\mathbb{C}^d$ restricted to $\mathbb{B}_d$ with $dV(\mathbb{B}_d) = 1$. Furthermore, let $d\sigma$ be the surface measure on $\partial\mathbb{B}_d$, normalized so that $d\sigma(\partial\mathbb{B}_d) = 1$.

Remark 2.3.38. (a) For any non-negative measurable function f on $\mathbb{B}_d$, integration in polar coordinates yields that

$$\int_{\mathbb{B}_d} f(z)dV(z) = 2d \int_0^1 \rho^{2d-1} \int_{\partial\mathbb{B}_d} f(\rho\zeta)d\sigma(\zeta)d\rho$$

(see [Rud08, 1.4.3]).

(b) If $\eta \in \mathbb{T}$ and $f \in C(\partial\mathbb{B}_d)$, rotation invariance of $d\sigma$ yields that

$$\int_{\partial\mathbb{B}_d} f(\eta\zeta)d\sigma(\zeta) = \int_{\partial\mathbb{B}_d} f(\zeta)d\sigma(\zeta).$$

A function $\omega \colon \mathbb{B}_d \to \mathbb{R}_{>0}$ is called radial weight if

(a) $\omega \in L^1(dV)$,

(b) for each $0 < r < 1$ the value $\hat{\omega}(r) := \omega(rz)$ is independent of $z \in \partial\mathbb{B}_d$ and

(c) $\int_{|z|>1-\delta} \omega dV > 0$ for all $0 < \delta < 1$.

Condition (c) will assure that radially weighted Besov spaces are Hilbert spaces with bounded point evaluations for all points in $\mathbb{B}_d$. Condition (b) is required to assure unitarily invariance.

With the previous definitions in mind, we define the L^p-version of radially weighted Besov spaces $B_\omega^{s,p}$.

Definition 2.3.39. Let $\omega \colon \mathbb{B}_d \to \mathbb{R}_{>0}$ be a radial weight and let $1 \leq p < \infty$.

(a) Define the radially weighted Bergman-space $L_a^p(\omega)$ by

$$L_a^p(\omega) = \mathscr{O}(\mathbb{B}_d) \cap L^p(\omega dV)$$

with the norm

$$\|f\|_{p,\omega}^p = \frac{1}{\|\omega\|_{L^1(dV)}} \int_{\mathbb{B}_d} |f(z)|^p \omega(z)dV(z).$$

(b) For $s \in \mathbb{R} \setminus \{0\}$, define the space

$$B_\omega^{s,p} = \{f \in \mathscr{O}(\mathbb{B}_d);\ R^s f \in L^p(\omega dV)\}$$

with the norm

$$\|f\|_{p,\omega,s}^p = |f(0)|^p + \|R^s f\|_{p,\omega}^p \quad (f \in B_\omega^{s,p}).$$

We use the notations $B_\omega^s = B_\omega^{s,2}$ and $B_\omega^{0,p} = L_a^p(\omega)$.

For $s \in \mathbb{R}$ the spaces $B_\omega^{s,p}$ are called (holomorphic or analytic) radially weighted Besov spaces.

Lemma 2.3.40. *Let $1 \leq p < \infty$. The norm-topology on the radially weighted Bergman-spaces $L_a^p(\omega)$ is at most finer than the topology of uniform convergence on compact subsets. The spaces $L_a^p(\omega)$ are Banach spaces with continuous point evaluations.*

Proof. Fix $0 < r < 1$, let $f \in L_a^p(\omega)$ and $\varepsilon > 0$. Since $|f|^p$ is uniformly continuous on $\mathbb{B}_d(r)$, there exists a $0 < \delta < 1$ such that

$$\sup_{z \in \mathbb{B}_d(r)} ||f(z)|^p - |f(\rho z)|^p| \leq (1-r)^{-pd} \varepsilon \|f\|_{p,\omega}^p \tag{2.2}$$

for all $1 - \delta < \rho < 1$.

Now, fix a $w \in \mathbb{B}_d(r)$. Using the Cauchy integral formula on the ball [Rud08, 3.2.4] in the first equality and applying Hölder's inequality to the functions f_ρ and 1 in the second inequality, it follows for all $1 - \delta < \rho < 1$ that

$$\begin{aligned} |f(\rho w)|^p &= \left| \int_{\partial \mathbb{B}_d} \frac{f(\rho \zeta)}{(1 - \langle \rho w, \zeta \rangle)^d} d\sigma(\zeta) \right|^p \\ &\leq (1-r)^{-pd} \left(\int_{\partial \mathbb{B}_d} |f(\rho \zeta)| \sigma(\zeta) \right)^p \\ &\leq (1-r)^{-pd} \int_{\partial \mathbb{B}_d} |f(\rho \zeta)|^p \, d\sigma(\zeta). \end{aligned}$$

Hence, inequality (2.2) yields for all $1 - \delta < \rho < 1$ that

$$|f(w)|^p \leq |f(\rho w)|^p + (1-r)^{-pd} \varepsilon \|f\|_{p,\omega}^p \leq (1-r)^{-pd} \int_{\partial \mathbb{B}_d} \left(|f(\rho \zeta)|^p + \varepsilon \|f\|_{p,\omega}^p \right) d\sigma(\zeta).$$

Using polar coordinates, we obtain that

$$\begin{aligned} 0 &\leq \left(\int_{|z| > 1-\delta} \omega(z) dV(z) \right) |f(w)|^p \\ &\leq 2d \int_{1-\delta}^1 \rho^{2d-1} \hat{\omega}(\rho) |f(w)|^p \, d\rho \\ &\leq (1-r)^{-pd} \left(2d \int_{1-\delta}^1 \rho^{2d-1} \hat{\omega}(\rho) \int_{\partial \mathbb{B}_d} \left(|f(\rho \zeta)|^p + \varepsilon \|f\|_{p,\omega}^p \right) d\sigma(\zeta) \, d\rho \right) \\ &\leq (1-r)^{-pd} \left(\int_{\mathbb{B}_d} \left(|f(z)|^p + \varepsilon \|f\|_{p,\omega}^p \right) \omega(z) dV(z) \right) \\ &= (1+\varepsilon) \frac{\|\omega\|_{L^1(dV)}}{(1-r)^{pd}} \|f\|_{p,\omega}^p. \end{aligned}$$

Since $\left(\int_{|z|>1-\delta} \omega(z)dV(z)\right) > 0$, we conclude that

$$L_a^p(\omega) \to H^\infty(\mathbb{B}_d(r)),\ f \mapsto f|_{\mathbb{B}_d(r)}$$

is continuous. It is now immediate that the topology induced by $\|\cdot\|_{p,\omega}$ is at most finer than the topology of uniform convergence on compact subsets. In particular, the point evaluations are continuous for $L_a^p(\omega)$. If $(f_n)_{n\in\mathbb{N}}$ is a sequence in $L_a^p(\omega)$ converging to a function $g \in L^p(\omega dV)$, then $(f_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $L_a^p(\omega)$. We obtain that $(f_n)_{n\in\mathbb{N}}$ converges uniformly on compact subsets to a function f. Then $f \in \mathscr{O}(\mathbb{B}_d)$ and $f = g$ for $\mu = \omega dV$ almost every $z \in \mathbb{B}_d$. Thus, $L_a^p(\omega) \subset L^p(\omega dV)$ is closed and hence a Banach space. □

Proposition 2.3.41. *For $1 \leq p < \infty$ the radially weighted Bergman spaces $L_a^p(\omega)$ are homogeneous Banach spaces in the sense of Definition 2.3.28.*

Proof. Due to Lemma 2.3.40, the spaces $L_a^p(\omega)$ are complete with continuous point evaluations. We use a well-known statement, presumably going back to Lebesgue, similar to the idea of the proof for the classical dominated convergence theorem with Fatou's Lemma. Let $f \in L_a^p(\omega)$ and $\xi \in \mathbb{T}$. Since σ is rotation-invariant, using polar coordinates, it follows that the function $f_\eta : \mathbb{B}_d \to \mathbb{C},\ f_\eta(z) = f(\eta z)$ belongs to $L_a^p(\omega)$ and that

$$\begin{aligned} \|f\|^p_{L_a^p(\omega)} &= \frac{1}{\|\omega\|_{L^1(dV)}} \left(2d \int_0^1 \rho^{2d-1}\hat{\omega}(\rho) \int_{\partial\mathbb{B}_d} |f(\rho\zeta)|^p \, d\sigma(\zeta)\, d\rho\right) \\ &= \frac{1}{\|\omega\|_{L^1(dV)}} \left(2d \int_0^1 \rho^{2d-1}\hat{\omega}(\rho) \int_{\partial\mathbb{B}_d} |f(\eta\rho\zeta)|^p \, d\sigma(\zeta)\, d\rho\right) \quad = \|f_\eta\|^p_{p,\omega}. \end{aligned}$$

For $\eta \in \mathbb{T}$, define the non-negative functions $h_\eta : \mathbb{B}_d \to \mathbb{C}$,

$$h_\eta(z) = 2^p(|f_\eta(z)|^p + |f(z)|^p) - |f_\eta(z) - f(z)|^p.$$

Then

$$\lim_{\eta\to 1} h_\eta(z) = 2^{p+1}|f(z)|^p$$

and

$$\|h_\eta\|_{L_a^1(\omega)} = 2^{p+1}\|f\|^p_{p,\omega} - \|f_\eta - f\|^p_{p,\omega}.$$

Applying Fatou's Lemma to the functions h_η, it follows that

$$0 \leq \limsup_{\eta\to 1} \|f_\eta - f\|^p_{p,\omega} = 0.$$

Hence, we deduce that the map

$$\mathbb{T} \to L_a^p(\omega), \eta \mapsto f_\eta$$

is continuous. □

Lemma 2.3.42. *Let $1 \leq p < \infty$ and let $s \in \mathbb{R}$. The norm-topology on the radially weighted Besov spaces $B^{s,p}_{\omega}$ is at most finer than the topology of uniform convergence on compact subsets. The spaces $B^{s,p}_{\omega}$ are Banach spaces with continuous point evaluations.*

Proof. The case $s = 0$ is Lemma 2.3.40. Let $s \in \mathbb{R} \setminus \{0\}$ and let $Q \subset \mathbb{B}_d$ be compact. Using Remark 2.3.37, we deduce that there exists an $r > 0$ such that

$$|f(z)| = |f_0 + (R^{-s}((R^s f))(z))| \leq |f_0| + \left(\sum_{n=1}^{\infty} n^{-s} r^n \right) \|R^s f\|_Q.$$

for all $z \in Q$. Because of Lemma 2.3.40, the topology induced by $\| \cdot \|_{p,\omega}$ is at most finer than the topology of uniform convergence on compact subsets. Hence, we obtain that the topology induced by $\| \cdot \|_{p,\omega,s}$ is at most finer than the topology of uniform convergence on compact subsets. In particular, the point evaluations are continuous for $B^{s,p}_{\omega}$. Let $(f_n)_{n \in \mathbb{N}}$ be a Cauchy sequence in $B^{s,p}_{\omega}$. By the previous argument, the sequence $(f_n)_{n \in \mathbb{N}}$ converges uniformly on compact subsets to a function f. Then $f \in \mathscr{O}(\mathbb{B}_d)$ and because of Remark 2.3.37, the sequence $(R^s f_n)_{n \in \mathbb{N}}$ converges uniformly on compact subsets of $\mathbb{B}_d$ to $R^s f$. Furthermore, the sequence $(R^s f_n)_{n \in \mathbb{N}}$ is a Cauchy sequence in $L^p_a(\omega)$. Since $L^p_a(\omega)$ is complete, the sequence $(R^s f_n)_{n \in \mathbb{N}}$ converges in $L^p_a(\omega)$ to a function $g \in L^p_a(\omega)$. It is immediate that $R^s f = g$ and $f \in B^{s,p}_{\omega}$. Thus, the spaces $B^{s,p}_{\omega}$ are complete. □

Proposition 2.3.43. *For $1 \leq p < \infty$ and $s \in \mathbb{R}$ the radially weighted Besov spaces $B^{s,p}_{\omega}$ are homogeneous Banach spaces in the sense of Definition 2.3.28.*

Proof. The case $s = 0$ is Proposition 2.3.41. Due to Lemma 2.3.42, the spaces $B^{s,p}_{\omega}$ are complete with continuous point evaluations. If $f \in \mathscr{O}(\mathbb{B}_d)$, $\zeta \in \mathbb{T}$ and $0 < r < 1$, then $(R^s f)_{\zeta} = (R^s f_{\zeta})$ and $(R^s f)_r = (R^s f_r)$. It follows that $\|f_{\zeta}\|_{p,\omega,s} = \|f\|_{p,\omega,s}$ for $f \in B^{s,p}_{\omega}$ and $\zeta \in \mathbb{T}$. Using Proposition 2.3.41 and Corollary 2.3.35, the radial dilations are dense in $L^p_a(\omega)$. Thus, the radial dilations are also dense in $B^{s,p}_{\omega}$. Because of Lemma 2.3.29, the spaces $B^{s,p}_{\omega}$ are homogeneous. □

Using Proposition 2.3.43 one checks that the next statement is a particular case of Corollary 2.3.35. For ease of reference we state it as a theorem here.

Theorem 2.3.44. *For $1 \leq p < \infty$ and $s \in \mathbb{R}$ the radially weighted Besov spaces $B^{s,p}_{\omega}$ are Banach spaces with continuous point evaluations. If $f \in B^{s,p}_{\omega}$, then the Fejér-means $(\sigma_n(f))_{n \in \mathbb{N}}$ and the radial dilations $(f_r)_{0<r<1}$ converge to f. In particular, the polynomials are densely contained in $B^{s,p}_{\omega}$.*

For $p = 2$ we will see that the spaces $B^s_{\omega} = B^{s,p}_{\omega}$ are unitarily invariant reproducing kernel Hilbert spaces.

For $n \in \mathbb{N}$ we define the moments

$$v_n(\omega) = \int_0^1 t^n v(t)\, dt,$$

where $v : [0,1] \to \mathbb{R}_{>0}$,

$$v(t) = \begin{cases} 0, & \text{if } t = 0, \\ \frac{1}{\|\omega\|_{L^1(dV)}} d \cdot t^{d-1} \cdot \hat{\omega}(\sqrt{t}) & \text{if } 0 < t < 1, \\ 1 & \text{if } t = 1. \end{cases}$$

The following lemma can be found in [AHMR19, Lemma 2.1] and is useful for the representations of the reproducing kernel functions of radially weighted Besov spaces.

Lemma 2.3.45. *Let $v, w : [0,1] \to \mathbb{R}_{\geq 0}$ be two non-negative weights in $L^1([0,1])$ such that*

$$\lim_{t \nearrow 1} \frac{v(t)}{w(t)} = 1$$

(with the convention 0/0 = 1). Then

$$\lim_{n \to \infty} \frac{\int_0^1 t^n v(t)\, dt}{\int_0^1 t^n w(t)\, dt} = 1.$$

Remark 2.3.46. For $\alpha, \beta \in \mathbb{N}^d$ Proposition 1.4.8 and Proposition 1.4.9 in [Rud08] show that

$$\int_{\partial \mathbb{B}_d} \zeta^\alpha \overline{\zeta^\beta}\, d\sigma(\zeta) = \left(\frac{\left\| z_1^{|\alpha|} \right\|^2_{H^2(\partial \mathbb{B}_d)}}{\gamma_\alpha} \right) \delta_{\alpha,\beta},$$

where

$$\delta_{\alpha,\beta} = \begin{cases} 1 & \text{if } \alpha = \beta, \\ 0 & \text{else} \end{cases}$$

and

$$\|z_1^n\|^{-2}_{H^2(\partial \mathbb{B}_d)} = \binom{n+d-1}{n} \approx n^{d-1}$$

for $n \in \mathbb{N}$ (see also Remark 2.1.10).

Proposition 2.3.47. *If $f = \sum_{\alpha \in \mathbb{N}^d} f_\alpha z^\alpha \in L^2_a(\omega)$, then*

$$\|f\|^2_{L^2_a(\omega)} = \frac{1}{\|\omega\|_{L^1(dV)}} \int_{\mathbb{B}_d} |f|^2\, \omega dV = \sum_{\alpha \in \mathbb{N}^d} \frac{|f_\alpha|^2}{\gamma_\alpha a_{|\alpha|}(\omega)},$$

where

$$a_n(\omega) = \left(\|z_1^n\|^2_{H^2(\partial \mathbb{B}_d)} v_n(\omega) \right)^{-1} \approx n^{d-1} v_n(\omega)^{-1}$$

for $n \in \mathbb{N}$.

Proof. For $\alpha, \beta \in \mathbb{N}^d$, using polar coordinates, we obtain with the previous Remark 2.3.46, that

$$\begin{aligned}\int_{\mathbb{B}_d} z^\alpha \overline{z^\beta}\, \omega dV &= \left(2d \int_0^1 \rho^{2d+|\alpha|+|\beta|-1} \hat{\omega}(\rho)\, d\rho\right)\left(\int_{\partial\mathbb{B}_d} \zeta^\alpha \overline{\zeta^\beta}\, d\sigma(\zeta)\right) \\ &= \left(d \int_0^1 t^{|\alpha|+d-1} \hat{\omega}(\sqrt{t}) dt\right)\left(\frac{\left\|z_1^{|\alpha|}\right\|^2_{H^2(\partial\mathbb{B}_d)}}{\gamma_\alpha}\right)\delta_{\alpha,\beta} \\ &= \left(\frac{\|\omega\|_{L^1(dV)}}{a_{|\alpha|}(\omega)\gamma_\alpha}\right)\delta_{\alpha,\beta},\end{aligned}$$

where

$$\delta_{\alpha,\beta} = \begin{cases} 1 & \text{if } \alpha = \beta, \\ 0 & \text{else .}\end{cases}$$

If $f = \sum_{\alpha\in\mathbb{N}^d} f_\alpha z^\alpha \in L^2_a(\omega)$ and $0 < r < 1$, this yields

$$\begin{aligned}\int_{\mathbb{B}_d} |f_r|^2\, \omega dV &= \sum_{\alpha\in\mathbb{N}^d}\sum_{\beta\in\mathbb{N}^d} f_\alpha \overline{f_\beta} r^{\alpha+\beta} \int_{\mathbb{B}_d} z^\alpha \overline{z^\beta}\, \omega dV \\ &= \|\omega\|_{L^1(dV)} \left(\sum_{\alpha\in\mathbb{N}^d} \frac{|f_\alpha|^2 r^{2|\alpha|}}{a_{|\alpha|}(\omega)\gamma_\alpha}\right).\end{aligned}$$

Because of Proposition 2.3.41 and Corollary 2.3.35, the radial dilations are densely contained in $L^2_a(\omega)$. Thus, we conclude that

$$\|f\|^2_{L^2_a(\omega)} = \lim_{r\uparrow 1} \|f_r\|^2_{L^2_a(\omega)} = \lim_{r\uparrow 1} \sum_{\alpha\in\mathbb{N}^d} \frac{|f_\alpha|^2 r^{2|\alpha|}}{\gamma_\alpha a_{|\alpha|}(\omega)} = \sum_{\alpha\in\mathbb{N}^d} \frac{|f_\alpha|^2}{\gamma_\alpha a_{|\alpha|}(\omega)}.$$

□

Corollary 2.3.48. *For $s \in \mathbb{R}$ and a radial weight $\omega\colon \mathbb{B}_d \to \mathbb{R}_{>0}$ set $a_0 := a_0(s,\omega) = 1$ and*

$$a_n := a_n(s,\omega) = n^{-2s} a_n(\omega) \approx n^{-2s+d-1} v_n(\omega)^{-1}$$

for $n \geq 1$. Then

$$\lim_{n\to\infty} \frac{a_n}{a_{n+1}} = 1$$

and the spaces B^s_ω are unitarily invariant spaces with reproducing kernel

$$K : \mathbb{B}_d \times \mathbb{B}_d \to \mathbb{C},\ K(z,w) = \sum_{n=0}^{\infty} a_n \langle z,w\rangle^n.$$

Proof. Let $n \in \mathbb{N}$. Lemma 2.3.45 yields that

$$\lim_{n\to\infty} \frac{v_{n+1}(\omega)}{v_n(\omega)} = \lim_{n\to\infty} \frac{\int_0^1 t^{n+1} v(t)\,dt}{\int_0^1 t^n v(t)\,dt} = 1.$$

Secondly, observe that

$$\lim_{n\to\infty} \frac{\|z_1^{n+1}\|^2_{H^2(\partial\mathbb{B}_d)}}{\|z_1^n\|^2_{H^2(\partial\mathbb{B}_d)}} = \lim_{n\to\infty} \left(\frac{n+1}{n}\right)\left(\frac{n+d-1}{n+d}\right) = 1.$$

Hence, one computes that

$$\lim_{n\to\infty} \frac{a_n}{a_{n+1}} = 1.$$

With Proposition 2.3.47 it is not hard to see that $\langle \cdot,\cdot \rangle : B^s_\omega \times B^s_\omega \to \mathbb{C}$,

$$\langle f,g \rangle = \sum_{\alpha\in\mathbb{N}^d} \frac{f_\alpha \overline{g_\alpha}}{a_{|\alpha|}\gamma_\alpha} \qquad \left(f = \sum_{\alpha\in\mathbb{N}^d} f_\alpha z^\alpha, g = \sum_{\alpha\in\mathbb{N}^d} f_\alpha z^\alpha \in B^s_\omega\right)$$

defines a scalar product on B^s_ω such that

$$\|f\|^2_{B^s_\omega} = \langle f,f \rangle.$$

For every $w \in \mathbb{B}_d$ the function $k_w : \mathbb{B}_d \to \mathbb{C}$,

$$k_w(z) = \sum_{n=0}^{\infty} a_n \langle z,w \rangle^n$$

is in B^s_ω such that

$$f(w) = \langle f,k_w \rangle.$$

for all $f \in B^s_\omega$. Using these facts, it follows that B^s_ω is a reproducing kernel Hilbert space with reproducing kernel

$$K : \mathbb{B}_d \times \mathbb{B}_d \to \mathbb{C},\ K(z,w) = \sum_{n=0}^{\infty} a_n \langle z,w \rangle^n.$$

□

Remark 2.3.49. For $t > -1$ the radial weight function

$$\omega^{(t)} : \mathbb{B}_d \to \mathbb{R}_{\geq 0},\ \omega^{(t)}(z) = \left(1-|z|^2\right)^t$$

is called standard weight. For $t > -\frac{1}{2}$, to simplify notation, we write

$$B^s_t = B^s_{\omega^{(2t)}} = \left\{ f \in \mathscr{O}(\mathbb{B}_d);\ \int_{\mathbb{B}_d} |R^s f(z)|^2 (1-|z|^2)^{2t} dV(z) < \infty \right\}$$

for the standard weighted Besov spaces and

$$B^s = B_0^s = \left\{ f \in \mathscr{O}(\mathbb{B}_d);\ \int_{\mathbb{B}_d} |R^s f(z)|^2\, dV(z) < \infty \right\}.$$

for the Besov spaces with constant weight function $\omega^{(0)} \equiv 1$. Let

$$K_{s,t} : \mathbb{B}_d \times \mathbb{B}_d \to \mathbb{C},\ K_{s,t}(z,w) = \sum_{n=0}^{\infty} a_n(s,t)\langle z,w\rangle^n,$$

be the kernel function of the radially weighted Besov space B_t^s, where

$$a_n(s,t) = n^{-2s} a_n\left(\omega^{(2t)}\right) \approx n^{-2s+d-1} v_n\left(\omega^{(2t)}\right)^{-1}.$$

For $n \in \mathbb{N}$ the moments $v_n(\omega^{(2t)})$ of the radial weight $\omega^{(2t)}$ can be computed by the Euler beta integral

$$\begin{aligned} v_n\left(\omega^{(2t)}\right) &= \frac{d}{\|\omega_t\|_{L^1(dV)}} \int_0^1 \rho^{n+d-1}(1-\rho)^{2t}\, d\rho \\ &= \frac{d}{\|\omega^{(2t)}\|_{L^1(dV)}} \frac{\Gamma(n+d)\Gamma(2t+1)}{\Gamma(n+d+2t+1)} \approx (n+1)^{-2t-1} \end{aligned}$$

(see Remark 9.12 (a) in [AE08, Chapter VI, Section 9] and Remark 2.1.10). Hence, we obtain for $n \in \mathbb{N}$ and $s \in \mathbb{R}$ that

$$a_n(s,t) = n^{-2s} a_n\left(\omega^{(2t)}\right) \approx (n+1)^{-2(s-t)+d}.$$

In Example 2.3.5, we considered for $s \in \mathbb{R}$ the spaces $\mathscr{D}_s(\mathbb{B}_d)$ with reproducing kernel

$$K : \mathbb{B}_d \times \mathbb{B}_d \to \mathbb{C},\ K(z,w) = \sum_{n=0}^{\infty} (n+1)^s \langle z,w\rangle^n.$$

It follows that $B_t^s = \mathscr{D}_{-2(s-t)+d}(\mathbb{B}_d)$ with equivalence of norms.

(a) $B^{d/2} = \mathscr{D}_0(\mathbb{B}_d) = H_d^2$ is the Drury-Arveson space.

(b) $B^{(d+1)/2} = \mathscr{D}_{-1}(\mathbb{B}_d)$ is the Dirichlet space on the ball.

(c) If $s \in (\frac{d}{2}, \frac{d+1}{2})$, then $B^s = \mathscr{D}_{-2s+d}(\mathbb{B}_d) = A^2_{d+1-2s}(\mathbb{B}_d)$ are the Dirichlet type spaces.

(d) The space $B^{1/2} = \mathscr{D}_{d-1}(\mathbb{B}_d) = A_d^2(\mathbb{B}_d) = H^2(\partial\mathbb{B}_d)$ is the Hardy space on unit the ball.

(e) For $s < \frac{1}{2}$ we obtain the weighted Bergman spaces

$$B^s = \mathscr{D}_{-2s+d}(\mathbb{B}_d) = L_a^2\left(\omega^{(-2s)}\right)$$

(f) More general,

$$B_t^s = B_{t+r}^{s+r}$$

for all $r > 0$ with equivalence of norms.

In many proofs, it is useful to do an index shift as in (f). The following theorem, which can be found in [AHMR19, Theorem 2.4], generalizes this idea for arbitrary radial weights ω. For $s > 0$ and $z \in \mathbb{B}_d$ define a new radial weight function $\omega_s : \mathbb{B}_d \to \mathbb{R}_{\geq 0}$,

$$\omega_s(z) = \frac{1}{d}|z|^{2-2d} \int_{|w| \geq |z|} \frac{(|w|^2 - |z|^2)^{2s-1}}{\Gamma(2s)} \omega(w) dV(w).$$

Theorem 2.3.50. *Let ω be a radial weight and let $s > 0$. The function ω_s defined in Remark 2.3.49 is again a radial weight and $B_\omega^t = B_{\omega_s}^{t+s}$ with equivalence of norms for all $t \in \mathbb{R}$. In particular, $L_a^2(\omega_s) = B_\omega^{-s}$.*

Remark 2.3.51. Let $s \in \mathbb{R}$ and $N \in \mathbb{N}$ such that $N \geq s > 0$. One advantage of the previous consideration is, that every radially weighted Besov space B_ω^s can be written in the form $B_{\omega_{N-s}}^N$. Since the operator R^N is the higher order of the classical radial derivative $R : \mathscr{O}(\mathbb{B}_d) \to \mathscr{O}(\mathbb{B}_d)$,

$$Rf = \sum_{l=1}^{d} z_l \frac{\partial f}{\partial z_l}.$$

it is sometimes easier to work in this setting (cf. also the product rule in Remark 2.3.37).

Let $s, t \in \mathbb{R}$ with $s < t$ and let $\omega \colon \mathbb{B}_d \to \mathbb{R}_{>0}$ be a radial weight function. Consider the radially weighted Besov spaces B_ω^s and B_ω^t. Due to Corollary 2.3.48, the families

$$\left(\sqrt{\gamma_\alpha a_{|\alpha|}(\omega, s)} z^\alpha\right)_{\alpha \in \mathbb{N}^d} \text{ and } \left(\sqrt{\gamma_\alpha a_{|\alpha|}(\omega, t)} z^\alpha\right)_{\alpha \in \mathbb{N}^d}$$

are orthonormal basis for the spaces B_ω^s and B_ω^t respectively. Since

$$\lim_{n \to \infty} \frac{a_n(\omega, s)}{a_n(\omega, t)} = \lim_{n \to \infty} n^{-2(s-t)} = 0,$$

one obtains the following proposition:

Proposition 2.3.52. *Let $s, t \in \mathbb{R}$ with $t < s$, then $B_\omega^s \subset B_\omega^t$ and the inclusion $i : B_\omega^s \to B_\omega^t$ is compact. In particular, if $f \in B_\omega^s$, then*

$$||R^t f||_{L_a^2(\omega)} \leq ||f||_{B_\omega^t} \lesssim ||f||_{B_\omega^s}.$$

Finally, we are interested in the multiplier algebras of radially weighted Besov spaces.

Remark 2.3.53. (a) Let $\omega : \mathbb{B}_d \to \mathbb{R}_{\geq 0}$ be a radial weight, let $s \in \mathbb{R}$ and let $r > 1$. Clearly, $\mathscr{O}(\mathbb{B}_d(r)) \subset A(\mathbb{B}_d)$, where $A(\mathbb{B}_d)$ is the ball algebra. If $\varphi \in \mathscr{O}(\mathbb{B}_d(r))$, then one can check that $R^t \varphi \in \mathscr{O}(\mathbb{B}_d(r))$ for all $t \in \mathbb{R}$. Hence, it follows with the Leibniz rule (see Remark 2.3.37 and Proposition 2.3.52), that $\varphi \in A(B_\omega^s)$, where $A(B_\omega^s)$ is the norm-closure of polynomials in $\mathrm{Mult}(B_\omega^s)$.

(b) Because of Theorem 2.3.50, we obtain that $\operatorname{Mult}(B^s_\omega) = H^\infty(\mathbb{B}_d)$ for all $s \leq 0$.

The following statement about the containment of multiplier algebras together with Theorem 2.3.50 is an important tool and can be used to prove Theorem 2.3.55. For a proof, see [AHMR19, Corollary 3.8].

Theorem 2.3.54. *Let $\omega : \mathbb{B}_d \to \mathbb{R}_{\geq 0}$ and $\tilde{\omega} : \mathbb{B}_d \to \mathbb{R}_{\geq 0}$ be radial weights and let $s,t,s',t' \in \mathbb{R}$ with $t \leq s$ and $t' - s' \leq t - s$. Then for any pair $\mathscr{E}_1$, $\mathscr{E}_2$ of separable Hilbert spaces,*

$$\operatorname{Mult}(B^s_\omega(\mathscr{E}_1), B^{s'}_{\tilde{\omega}}(\mathscr{E}_2)) \subset \operatorname{Mult}(B^t_\omega(\mathscr{E}_1), B^{t'}_{\tilde{\omega}}(\mathscr{E}_2))$$

and the inclusion is contractive.

For the Dirichlet space $\mathscr{D}$ it is well-known that a function $\varphi : \mathbb{D} \to \mathbb{C}$ is in $\operatorname{Mult}(\mathscr{D})$ if and only if $\varphi \in H^\infty(\mathbb{D})$ and $\varphi' \in \operatorname{Mult}(\mathscr{D}, L^2_a(\mathbb{D}))$. Here $L^2_a(\mathbb{D})$ is the classical Bergman space. See [EFKMR14, Theorem 5.1.7] for details. As explained in [AHMR19], there are similar statements for Besov spaces due to Cascante, Fabrega and Ortega. For the standard weights

$$\omega^{(s)} : \mathbb{B}_d \to \mathbb{R}_{\geq 0},\ \omega^{(s)}(z) = (1 - |z|^2)^s \quad (s > -1),$$

see [OF00], [CFO10]). For general Bekollé-Bonami weights (not necessarily radial) there is a result in [CF16]. The following result can be found in the paper by Aleman, Hartz, M^{c}Carthy and Richter ([AHMR19, Theorem 6.3]).

Theorem 2.3.55. *Let $s \in \mathbb{R}$ and $N \in \mathbb{N}$, then we have*

$$\operatorname{Mult}(B^s_\omega) = \{\varphi \in H^\infty(\mathbb{B}_d);\ R^N\varphi \in \operatorname{Mult}(B^s_\omega, B^{s-N}_\omega)\}$$

and

$$\|\varphi\|_{\operatorname{Mult}(B^s_\omega)} \approx \|R^N\varphi\|_{\operatorname{Mult}(B^s_\omega, B^{s-N}_\omega)} + \|\varphi\|_\infty.$$

In particular

$$R^N : \operatorname{Mult}(B^s_\omega) \to \operatorname{Mult}(B^s_\omega, B^{s-N}_\omega),\ \varphi \to R^N\varphi$$

is a continuous linear operator.

Remark 2.3.56. Consider the one dimensional case $d = 1$, where the radial weight $\omega(z) \equiv 1$ is just one. Then the Dirichlet space $\mathscr{D}$ coincides with B^1_1 and the classical Bergman space $L^2_a(\mathbb{D})$ coincides with B^0_1. Let $\varphi = \sum_{n=0}^\infty \varphi_n z^n \in \mathscr{O}(\mathbb{D})$ be a holomorphic function and let $M_z : L^2_a(\mathbb{D}) \to L^2_a(\mathbb{D})$ the Bergman shift. If

$$\varphi' \in \operatorname{Mult}(\mathscr{D}, L^2_a(\mathbb{D})),$$

using that $R\varphi(z) = z\varphi'(z)$ for all $z \in \mathbb{D}$, we obtain that $R\varphi \in \operatorname{Mult}(\mathscr{D}, L^2_a(\mathbb{D}))$ with multiplication operator $M_{R\varphi} = M_z \circ M_{\varphi'}$. Conversely, let $R\varphi \in \operatorname{Mult}(\mathscr{D}, L^2_a(\mathbb{D}))$. Observe that

$$(M_z^* M_z)\left(\sum_{n=0}^\infty f_n z^n\right) = \sum_{n=0}^\infty \frac{n+1}{n+2} f_n z^n$$

for $f = \sum_{n=0}^{\infty} f_n z^n$. Using the diagonal operator

$$\Delta : L_a^2(\mathbb{D}) \to L_a^2(\mathbb{D}), \sum_{n=0}^{\infty} f_n z^n \mapsto \sum_{n=0}^{\infty} \frac{n+2}{n+1} f_n z^n$$

(cf. also Section 2.4.3), one computes that M_z is left invertible, that is

$$(\Delta \circ M_z^*) \circ M_z = \mathrm{id}_{L_a^2(\mathbb{D})}.$$

Thus, $\varphi' \in \mathrm{Mult}(\mathscr{D}, L_a^2(\mathbb{D}))$ with multiplication operator $M_{\varphi'} = (\Delta \circ M_z^*) \circ M_{R\varphi}$. Consequently, Theorem 2.3.55 is equivalent to previously mentioned result that

$$\mathrm{Mult}(\mathscr{D}) = \{\varphi \in H^\infty(\mathbb{D});\ \varphi' \in \mathrm{Mult}(\mathscr{D}, L_a^2(\mathbb{D}))\},$$

where

$$\|\varphi\|_{\mathrm{Mult}(\mathscr{D})} \approx \|\varphi'\|_{\mathrm{Mult}(\mathscr{D}, L_a^2(\mathbb{D}))} + \|\varphi\|_\infty$$

for all $\varphi \in \mathrm{Mult}(\mathscr{D})$.

2.3.3. The vector-valued case

We will also need the following vector valued version of reproducing kernel Hilbert spaces and multipliers. Let therefore $\mathscr{H}$ be a unitarily invariant space with reproducing kernel

$$K : \mathbb{B}_d \times \mathbb{B}_d \to \mathbb{C},\ K(z,w) = \sum_{n=0}^{\infty} a_n \langle z, w\rangle^n,$$

where $a_0 = 1$ and $a_n > 0$ for $n \geq 1$. Then one may regard $\mathscr{H}(\mathscr{E}) \cong \mathscr{H} \otimes \mathscr{E}$ as a functional Hilbert space of $\mathscr{E}$-valued holomorphic functions on $\mathbb{B}_d$ by identifying an elementary tensor $f \otimes x \in \mathscr{H} \otimes \mathscr{E}$ with the function $z \mapsto f(z)x$. The operator-valued mapping

$$K_{\mathscr{E}} : \mathbb{B}_d \times \mathbb{B}_d \to B(\mathscr{E}),\ K_{\mathscr{E}}(z,w) = K(z,w)\,\mathrm{id}_{\mathscr{E}}$$

is positive definite and defines the reproducing kernel for $\mathscr{H}(\mathscr{E})$.

Similar as in Proposition 2.3.3, one can show that

$$\mathscr{H}(\mathscr{E}) = \left\{ f = \sum_{\alpha \in \mathbb{N}^d} f_\alpha z^\alpha \in \mathscr{O}(\mathbb{B}_d, \mathscr{E});\ \|f\|^2 = \sum_{\alpha \in \mathbb{N}^d} \frac{\|f_\alpha\|_{\mathscr{E}}^2}{a_{|\alpha|}\gamma_\alpha} < \infty \right\}$$

with the scalar product $\langle \cdot, \cdot \rangle_{\mathscr{H}(\mathscr{E})} : \mathscr{H}(\mathscr{E}) \times \mathscr{H}(\mathscr{E}) \to \mathbb{C}$,

$$\left\langle \sum_{\alpha \in \mathbb{N}^d} f_\alpha z^\alpha, \sum_{\alpha \in \mathbb{N}^d} g_\alpha z^\alpha \right\rangle_{\mathscr{H}(\mathscr{E})} = \sum_{\alpha \in \mathbb{N}^d} \frac{\langle f_\alpha, g_\alpha \rangle_{\mathscr{E}}}{a_{|\alpha|}\gamma_\alpha}.$$

Let $\mathscr{E}$, $\mathscr{E}_1$, $\mathscr{E}_2$ be Hilbert spaces and let $\mathscr{H}, \mathscr{H}_1$ and $\mathscr{H}_2$ be unitarily invariant spaces with kernels $K, K_1, K_2 : \mathbb{B}_d \times \mathbb{B}_d \to \mathbb{C}$ respectively. A function

$$\varphi : \mathbb{B}_d \to B(\mathscr{E}_1, \mathscr{E}_2)$$

is called a multiplier between $\mathscr{H}_1(\mathscr{E}_1)$ and $\mathscr{H}_2(\mathscr{E}_2)$, if for every $f \in \mathscr{H}_1(\mathscr{E}_1)$, the function

$$\varphi : \mathbb{B}_d \to \mathscr{E}_2, z \mapsto \varphi(z)f(z)$$

belongs to $\mathscr{H}_2(\mathscr{E}_2)$. The set of all multipliers between $\mathscr{H}_1(\mathscr{E}_1)$ and $\mathscr{H}_2(\mathscr{E}_2)$ is denoted by $\mathrm{Mult}(\mathscr{H}_1(\mathscr{E}_1), \mathscr{H}_2(\mathscr{E}_2))$ and we simply write $\mathrm{Mult}(\mathscr{H}(\mathscr{E})) = \mathrm{Mult}(\mathscr{H}(\mathscr{E}), \mathscr{H}(\mathscr{E}))$. For $\varphi \in \mathrm{Mult}(\mathscr{H}_1(\mathscr{E}_1), \mathscr{H}_2(\mathscr{E}_2))$ an application of the closed graph theorem shows as in the scalar-valued case, that the corresponding multiplication operator,

$$M_\varphi : \mathscr{H}_1(\mathscr{E}_1) \to \mathscr{H}_2(\mathscr{E}_2),\ f \mapsto \varphi f$$

is bounded. Hence, in this vector-valued setting $\mathrm{Mult}(\mathscr{H}_1(\mathscr{E}_1), \mathscr{H}_2(\mathscr{E}_2))$ can also be considered as a subspace of $B(\mathscr{H}_1, \mathscr{H}_2)$, identifying each multiplier function

$$\varphi \in \mathrm{Mult}(\mathscr{H}_1(\mathscr{E}_1), \mathscr{H}_2(\mathscr{E}_2))$$

with the corresponding multiplication operator M_φ. Using this identification, we equip $\mathrm{Mult}(\mathscr{H}_1(\mathscr{E}_1), \mathscr{H}_2(\mathscr{E}_2))$ with the operator topologies $\tau_{\|\cdot\|}$, SOT, WOT and τ_w^*. With the help of point evaluations, it is not difficult to see that $\mathrm{Mult}(\mathscr{H}_1(\mathscr{E}_1), \mathscr{H}_2(\mathscr{E}_2))$ is complete with respect to the norm-topology

$$\|\varphi\|_{\mathrm{Mult}} = \|M_\varphi\| \quad (\varphi \in \mathrm{Mult}(\mathscr{H}_1(\mathscr{E}_1), \mathscr{H}_2(\mathscr{E}_2)))).$$

Fix a map $\varphi \in \mathrm{Mult}(\mathscr{H}_1(\mathscr{E}_1), \mathscr{H}_2(\mathscr{E}_2))$. Similar to the scalar-valued case, it is elementary to check that

$$M_\varphi^* K_2(\cdot, w)x = K_1(\cdot, w)\varphi(w)x$$

for all $w \in \mathbb{B}_d$, $x \in \mathscr{E}_2$ and that the following are equivalent:

(a) $\varphi \in \mathrm{Mult}(\mathscr{H}_1(\mathscr{E}_1), \mathscr{H}_2(\mathscr{E}_2))$,

(b) there exists a $c \geq 0$ such that

$$L_{\varphi,c} : \mathbb{B}_d \times \mathbb{B}_d \to B(\mathscr{E}_2), (z,w) \mapsto c^2 K_2(z,w)\,\mathrm{id}_{\mathscr{E}_2} - \varphi(z)K_1(z,w)\varphi(w)^*$$

is positive definite.

In this case, $\|M_\varphi\| = \inf\{c \geq 0;\ L_{\varphi,c} \text{ is positive definite}\}$.

We need the following particular case of standard characterization of multipliers on reproducing kernel Hilbert spaces (Theorem 2.1 in [Bar11]).

Proposition 2.3.57. *Let $T \in B(\mathscr{H}(\mathscr{E}_1), \mathscr{H}(\mathscr{E}_2))$ and suppose that the operators*

$$M_{z_l} : \mathscr{H} \to \mathscr{H},\ f \mapsto z_l f \quad (l = 1, \ldots, d)$$

are well-defined and bounded. If

$$T(M_{z_l} \otimes \mathrm{id}_{\mathscr{E}_1}) = (M_{z_l} \otimes \mathrm{id}_{\mathscr{E}_2})T$$

for $l = 1, \ldots, d$, then there exists an operator-valued multiplier function

$$\varphi : \mathbb{B}_d \to B(\mathscr{E}_1, \mathscr{E}_2)$$

such that $T = M_\varphi$.

2.3.4. Complete Nevanlinna-Pick spaces

Properties of complete Nevanlinna-Pick spaces are a useful tool to generalize important results of the Hardy space $H^2(\mathbb{D})$, which is a prototype for such spaces. A good introduction is given in the book [AM02]. We give an overview of some ideas given in [Har17]. The following definition can be found in [AM02, Definitions 5.12, 5.13 and Exercise 5.14] and [Sch18, 1.42 Definition].

Definition 2.3.58. Let $\mathscr{H}$ be a reproducing kernel Hilbert space with reproducing kernel $K: X\times X \to \mathbb{C}$. We call $\mathscr{H}$ a complete Nevanlinna-Pick space if, whenever $x_1,\dots,x_n \in X$ and $W_1,\dots,W_n \in B(\ell^2(\mathbb{N}))$ such that

$$((\mathrm{id}_{\ell^2(\mathbb{N})} - W_l W_m^*)K(x_l,x_m))_{l,m=1}^n \in B(\ell^2(\mathbb{N})^n)$$

is positive, then there exists a multiplier φ in the closed unit ball of $\mathrm{Mult}(\mathscr{H}\otimes\ell^2(\mathbb{N}))$ such that

$$\varphi(x_l) = W_l$$

for all $l = 1,\dots,n$.

The next characterization of complete Nevanlinna-Pick spaces due to McCullough-Quiggin and Agler-M$^{\mathrm{c}}$Carthy (cf. Section 7.1 in [AM02] and Theorem 2.1 in [Har17]) is often very useful.

Theorem 2.3.59 (McCullough-Quiggin, Agler-M$^{\mathrm{c}}$Carthy)**.** *Let $\mathscr{H}$ be an irreducible reproducing kernel Hilbert space on a set X with reproducing kernel K which is normalized at a point in X. Then $\mathscr{H}$ is a complete Nevanlinna-Pick space if and only if the Hermitian kernel $F = 1 - 1/K$ is positive definite.*

Let $d \in \mathbb{N}\cup\{\infty\}$ and denote by $\mathbb{B}_\infty$ the unit ball in $\ell^2(\mathbb{N})$. If $d \in \mathbb{N}$, then modulo identification $\mathbb{B}_d \subset \mathbb{B}_\infty$. The positive definite map

$$K^{H^2_d}: \mathbb{B}_d\times\mathbb{B}_d \to \mathbb{C},\ K^{H^2_d}(z,w) = \frac{1}{1-\langle z,w\rangle},$$

is the kernel of the Drury-Arveson space H^2_d, the canonical example for unitarily invariant complete Nevanlinna-Pick spaces. Due to Agler and M$^{\mathrm{c}}$Carthy ([AM02, Theorem 8.2], see also [Har17, Theorem 2.4]) the function space H^2_d is universal.

Theorem 2.3.60 (Agler-M$^{\mathrm{c}}$Carthy)**.** *If $\mathscr{H}$ is an irreducible complete Nevanlinna-Pick space with normalized kernel $K: X\times X\to\mathbb{C}$, then there exists $d\in\mathbb{N}\cup\{\infty\}$ and an embedding $b: X\to\mathbb{B}_d$ such that*

$$K(x,y) = K^{H^2_d}(b(x),b(y)) \quad (x,y\in X).$$

The composition $f\mapsto f\circ b$ defines a unitary operator from $H^2_d|_{b(X)}$ onto $\mathscr{H}$. In this setting, we say that b is an embedding for $\mathscr{H}$.

Suppose that we are in the setting of Theorem 2.3.60. That is, $\mathscr{H}$ is a complete Nevanlinna-Pick space and

$$K : X \times X \to \mathbb{C},\ K(x,y) = \frac{1}{1 - \langle b(x), b(y) \rangle}$$

for some function $b : X \to \mathbb{B}_\infty$. Consider the corresponding row operator

$$b(x) = (b_1(x), b_2(x), \dots) \in B(\ell^2(\mathbb{N}), \mathbb{C})$$

Since

$$K(x,y)(1 - b(x)b(y)^*) = 1 \quad (x, y \in X),$$

we obtain that $b \in \mathrm{Mult}(\mathscr{H} \otimes \ell^2(\mathbb{N}), \mathscr{H})$ with $\|b\|_{\mathrm{Mult}} \leq 1$. Let $y \in X$. Modulo the identification $\mathbb{C} \cong B(\mathbb{C})$, it follows that the function

$$b_{b(y)} : X \to B(\mathbb{C}),\ z \to b(x)b(y)^*,$$

is an element of $\mathrm{Mult}(\mathscr{H})$ with

$$\|b_{b(y)}\|_{\mathrm{Mult}} \leq \|b(y)\|.$$

The following theorem is a consequence.

Theorem 2.3.61. *For $y \in X$ the function*

$$k_y : \mathbb{B}_d \to \mathbb{C},\ k_y(x) = K(x,y)$$

is an element of $\mathrm{Mult}(\mathscr{H})$.

Proof. For all $x, y \in X$ and $b : X \to \mathbb{B}_\infty$ as in the remarks before, we obtain

$$K(x,y) = \sum_{n=0}^{\infty} (b_{b(y)}(x))^n$$

For all $y \in X$ we have $b_{b(y)} \in \mathrm{Mult}(\mathscr{H})$ with $\|b_{b(y)}\|_{\mathrm{Mult}} \leq \|b(y)\|$. Since $\mathrm{Mult}(\mathscr{H})$ is a Banach algebra with pointwise composition as multiplication, we get

$$\|b^n_{b(y)}\|_{\mathrm{Mult}} \leq \|b_{b(y)}\|^n_{\mathrm{Mult}} \leq \|b(y)\|^n.$$

Hence, the sum

$$\sum_{n=0}^{\infty} b^n_{b(y)}$$

converges absolutely in $\mathrm{Mult}(\mathscr{H})$. The map

$$X \times X \to \mathbb{C},\ (x,y) \to K(x,y) - 1 = K(x,y)\left(1 - \frac{1}{K(x,y)}\right)$$

is positive definite by the Schur Product theorem [PR16, Theorem 4.8]. Because $1 \in \mathscr{H}$, the inclusion mapping $\mathrm{Mult}(\mathscr{H}) \hookrightarrow \mathscr{H}$ is well-defined, continuous and linear. Thus, convergence in $\mathrm{Mult}(\mathscr{H})$ yields pointwise convergence. It is immediate that

$$k_y = \sum_{n=0}^{\infty} b_{b(y)}^n$$

is an element of $\mathrm{Mult}(\mathscr{H})$. □

The following lemma (Lemma 2.3 in [Har17]) characterizes unitarily invariant complete Nevanlinna-Pick spaces

Lemma 2.3.62. *Let $d \in \mathbb{N} \cup \{\infty\}$ and let $\mathscr{H}$ be a unitarily invariant space on $\mathbb{B}_d$ with reproducing kernel*

$$K : \mathbb{B}_d \times \mathbb{B}_d \to \mathbb{C},\ K(z,w) = \sum_{n=0}^{\infty} a_n \langle z,w\rangle^n,$$

where $a_0 = 1$ and $a_n \geq 0$ for $n \geq 1$. Suppose that $a_1 > 0$. Then the following are equivalent:

(a) $\mathscr{H}$ is an irreducible complete Nevanlinna-Pick space.

(b) The sequence $(b_n)_{n=1}^{\infty}$ defined by

$$\sum_{n=1}^{\infty} b_n t^n = 1 - \frac{1}{\sum_{n=0}^{\infty} a_n t^n}$$

for t in a neighborhood of 0 is a sequence of non-negative real numbers.

In particular, if (b) holds, then $\mathscr{H}$ is automatically irreducible.

In many cases, one can use the following lemma of Kaluza to show that (unitarily invariant) spaces have the complete Nevanlinna-Pick property (cf. [AM02, Lemma 7.23]).

Lemma 2.3.63 (Kaluza)**.** *Suppose that $a_0 = 1$ and that $a_n > 0$ for $n \geq 1$ such that*

$$\frac{a_n}{a_{n-1}} \leq \frac{a_{n+1}}{a_n}$$

for all $n \geq 1$. Then for all $n \geq 1$ there exist $b_n \geq 0$ such that

$$\sum_{n=1}^{\infty} b_n z^n = 1 - \frac{1}{\sum_{n=0}^{\infty} a_n z^n}.$$

Example 2.3.64. (a) If $s \leq 0$, it follows from Kaluza's Lemma that the Dirichlet type spaces $\mathscr{D}_s(\mathbb{B}_d)$ with reproducing kernel

$$K : \mathbb{B}_d \times \mathbb{B}_d \to \mathbb{C},\ K(z,w) = \sum_{n=0}^{\infty} (n+1)^s \langle z,w\rangle^n$$

are unitarily invariant complete Nevanlinna-Pick spaces.

(b) Let $s \in (0,1]$. If $z \in \mathbb{D}$, by Newton's generalized binomial theorem

$$(1-z)^s = \sum_{n=0}^{\infty} (-1)^n \binom{s}{n} z^n.$$

For $n \geq 1$ one computes that

$$(-1)^n \binom{s}{n} = \prod_{l=1}^{n} \frac{-s+l-1}{l} < 0.$$

Thus, the spaces $A_s^2(\mathbb{B}_d)$ with reproducing kernel

$$K_s : \mathbb{B}_d \times \mathbb{B}_d \to \mathbb{C},\ K(z,w) = \frac{1}{(1-\langle z,w\rangle)^s}$$

are unitarily invariant complete Nevanlinna-Pick spaces. In Remark 2.3.6, we have seen that $A_s(\mathbb{B}_d)$ and $\mathscr{D}_{s-1}(\mathbb{B}_d)$ coincide as vector spaces with equivalence of norms.

(c) Suppose that $t > -1$, $0 \leq r_0 < 1$ and let $\omega : \mathbb{B}_d \to \mathbb{R}_{>0}$ be a radial weight such that $\frac{\omega(z)}{(1-|z|^2)^t}$ is non-decreasing in $|z|$ for all $r_0 < |z| < 1$. If $s \geq \frac{t+d}{2}$, Theorem 1.4 in [AHMR19] shows that the radially weighted Besov spaces B_ω^s, equipped with an equivalent norm, are complete Nevanlinna-Pick spaces.

2.4. Regular unitarily invariant spaces

In this section, we want to assume an additional regularity condition for our space unitarily invariant space $\mathscr{H}$. One big advantage is that for spaces fulfilling this regularity condition, the weighted shift operator tuple is bounded, has closed range and is essentially normal. As before let $\mathscr{H}$ be a unitarily invariant space with reproducing kernel

$$K : \mathbb{B}_d \times \mathbb{B}_d \to \mathbb{C},\ K(z,w) = \sum_{n=0}^{\infty} a_n \langle z,w\rangle^n,$$

where $a_0 = 1$ and $a_n > 0$ for $n \geq 1$.

The unilateral shift operator S on $\ell^2(\mathbb{N})$ is well understood. In many cases, it makes sense to identify S with the operator $M_z : H^2(\mathbb{D}) \to H^2(\mathbb{D})$ that takes a function in $H^2(\mathbb{D})$ and multiplies it by z. Similar to the Hardy space case $H^2(\mathbb{D})$ the multiplication operators

$$M_{z_l} : \mathscr{H} \to \mathscr{H},\ M_{z_l} f = z_l f \quad (l = 1, \dots, d)$$

are of special interest. One can give a boundedness-criterion using the Taylor coefficients $(a_n)_{n\in\mathbb{N}}$. For a proof, see [GHX04, Corollary 4.4].

Lemma 2.4.1. *The operators*

$$M_{z_l} : \mathscr{H} \to \mathscr{H} \quad (l = 1, \dots, d)$$

are bounded if and only if $\sup_{n\in\mathbb{N}} \frac{a_n}{a_{n+1}} < \infty$. *In this case,*

$$\|M_{z_l}\| = \left(\sup_{n\in\mathbb{N}} \frac{a_n}{a_{n+1}}\right)^{1/2}.$$

Recall,

$$\mathscr{H} = \bigoplus_{n=0}^{\infty} \mathbb{H}_n,$$

where

$$\mathbb{H}_n = \left\{ \sum_{|\alpha|=n} p_\alpha z^\alpha;\ p_\alpha \in \mathbb{C} \right\} \subset \mathbb{C}[z]$$

are the spaces consisting of all homogeneous polynomials of degree n. Denote by $P_n : \mathscr{H} \to \mathscr{H}$ the orthogonal projections onto $\mathbb{H}_n$.

For the next lemma set $a_n = 0$ for all negative integers n and $\gamma_\alpha = 0$ for all $\alpha \in \mathbb{Z}^d$ with $\alpha_l < 0$ for some $l \in \{1,\ldots,d\}$. Using elementary calculations (see Proposition 4.3 in [GHX04] and 1.29 Lemma in [Sch18]) one can show that

Lemma 2.4.2. *(a)* $\left(M_z^\beta\right)^* z^\alpha = \left(\frac{\gamma_{\alpha-\beta}}{\gamma_\alpha} \frac{a_{|\alpha-\beta|}}{a_{|\alpha|}}\right) z^{\alpha-\beta}$ *for* $\alpha,\beta \in \mathbb{N}^d$,

(b) $\left(M_{z_l} M_{z_l}^*\right) z^\alpha = \left(\frac{\alpha_l}{|\alpha|} \frac{a_{|\alpha|-1}}{a_{|\alpha|}}\right) z^\alpha$ *for* $\alpha \in \mathbb{N}^d$ *and* $l = 1,\ldots,d$,

(c) $\left(M_{z_l}^* M_{z_l}\right) z^\alpha = \left(\frac{\alpha_l+1}{|\alpha|+1} \frac{a_{|\alpha|}}{a_{|\alpha|+1}}\right) z^\alpha$ *for* $\alpha \in \mathbb{N}^d$ *and* $l = 1,\ldots,d$,

(d) $\sum_{l=1}^d M_{z_l} M_{z_l}^* = \mathrm{SOT} - \sum_{n=1}^\infty \left(\frac{a_{n-1}}{a_n}\right) P_n$,

(e) $\sum_{l=1}^d M_{z_l}^* M_{z_l} = \mathrm{SOT} - \sum_{n=1}^\infty \left(\frac{n+d}{n+1} \frac{a_n}{a_{n+1}}\right) P_n$.

Together with Theorem A.1.4 (cf. also Theorem 2.5 in [Wer08]) one obtains the following lemma:

Lemma 2.4.3. *Let* $M_z : \mathscr{H}^d \to \mathscr{H}$ *be bounded. Then the following are equivalent:*

(a) $M_z : \mathscr{H}^d \to \mathscr{H}$ *has closed range,*

(b) $\operatorname{Im} M_z = \{f \in \mathscr{H};\ f(0) = 0\}$,

(c) $\inf_{n\in\mathbb{N}} \frac{a_n}{a_{n+1}} > 0$.

Definition 2.4.4. We call a unitarily invariant space $\mathscr{H}$ regular if

$$\lim_{n\to\infty} \frac{a_n}{a_{n+1}} = 1.$$

Remark 2.4.5. If $\mathscr{H}$ is a unitarily invariant regular space and hence

$$\lim_{n\to\infty} \frac{a_n}{a_{n+1}} = 1,$$

then clearly

$$\sup_{n\in\mathbb{N}} \frac{a_n}{a_{n+1}} < \infty \text{ and } \inf_{n\in\mathbb{N}} \frac{a_n}{a_{n+1}} > 0.$$

Thus, the row operator $M_z : \mathscr{H}^d \to \mathscr{H}$ is bounded and has closed range.

Example 2.4.6. (a) In Example 2.3.5 we considered for $s > 0$, the unitarily invariant spaces $A^2_s(\mathbb{B}_d)$ with reproducing kernel $K_s : \mathbb{B}_d \times \mathbb{B}_d \to \mathbb{C}$,

$$K_s(z,w) = \frac{1}{(1-\langle z,w\rangle)^s} = \sum_{n=0}^{\infty} (-1)^n \binom{-s}{n} \langle z,w\rangle^n,$$

where $a_0 = 1$ and

$$a_n^{(s)} = (-1)^n \binom{-s}{n} = \prod_{l=1}^{n} \frac{s+l-1}{l} > 0$$

for $n \in \mathbb{N}_{>0}$. With the Gauss representation formula

$$\Gamma(s) = \lim_{n\to\infty} \frac{n^{s-1}}{(-1)^n\binom{-s}{n}} \qquad (s > 0)$$

(see Remark 2.1.10), it follows that

$$\lim_{n\to\infty} \frac{(-1)^n\binom{-s}{n}}{(-1)^{n+1}\binom{-s}{n+1}} = 1.$$

Thus, the spaces $A^2_s(\mathbb{B}_d)$ are regular. Additionally, the function

$$k : \mathbb{D} \to \mathbb{C},\ k(z) = \frac{1}{(1-z)^s}$$

has no zeros in $\mathbb{D}$.

(b) For $s \in \mathbb{R}$, let $\mathscr{D}_s(\mathbb{B}_d)$ be the unitarily invariant spaces with reproducing kernel

$$K : \mathbb{B}_d \times \mathbb{B}_d \to \mathbb{C},\ K(z,w) = \sum_{n=0}^{\infty} (n+1)^s \langle z,w\rangle^n.$$

It is clear that

$$\lim_{n\to\infty} \frac{(n+1)^s}{(n+2)^s} = 1.$$

Thus, the spaces $\mathscr{D}_s(\mathbb{B}_d)$ are regular.

(c) Let $s \in \mathbb{R}$ and let $\omega : \mathbb{B}_d \to \mathbb{R}_{>0}$ be a radial weight. In Corollary 2.3.48 we have seen that the radially weighted Besov spaces B^s_ω are unitarily invariant and regular.

2.4.1. Multivariable spectrum

The multivariable spectrum of a tuple of commuting bounded linear operators on a regular unitarily invariant space extends the spectrum of a single bounded linear operator. It can be a useful tool, since there is a generalization of the Riesz-Dunford or analytic functional calculus for tuples of commuting operators. We only give a brief introduction. For further background we recommend [EP96], [Mue07], [Wer08] which we also use as guidelines here.

We start with the Taylor spectrum. Let $\Lambda[x]$ be the free complex algebra generated by d indeterminates $x_1,\dots,x_d$, where the multiplication relation $\wedge$ satisfies the anti-commutative relations

$$x_l \wedge x_m = -x_m \wedge x_l \quad (l,m = 1,\dots,d).$$

Then $x_l \wedge x_l = 0$ and the set

$$\begin{aligned}\Lambda[s] &= \left\{ \sum_{1\le l_1<\ldots<l_p\le d} c_{l_1,\dots,l_p}(x_{l_1}\wedge\ldots\wedge x_{l_p});\ p=0,\dots,d, c_{l_1,\dots,l_p}\in\mathbb{C} \right\}\\ &= \left\{ \sum_{M\subset\{1,\dots,d\}} c_M x_M;\ c_M\in\mathbb{C}\right\},\end{aligned}$$

where

$$x_M = x_{l_1}\wedge\ldots\wedge x_{l_p} \quad \text{for} \quad M=\{l_1,\dots,l_p\},$$

can be considered as a Hilbert space with orthonormal basis

$$(x_{l_1}\wedge\ldots\wedge x_{l_p};\ ,p=0,\dots,d, 1\le l_1<\ldots<l_p\le d).$$

For $l=1,\dots,d$ define the linear operators $S_l:\Lambda[x]\to\Lambda[x]$ by

$$S_l\left(\sum_{M\subset\{1,\dots,d\}} c_M x_M\right) = \sum_{M\subset\{1,\dots,d\}} c_M(x_l\wedge x_M)$$

Then $S_lS_m = -S_mS_l$ and $S_l^2 = 0$ for $l,m = 1,\dots d$.

For a Hilbert space H let $\Lambda[x,H] = \Lambda[x]\otimes H$. Then $\Lambda[x,H]$ decomposes for $p=0,\dots d$ into the spaces

$$\Lambda^p[x,H] = \left\{ \sum_{1\le l_1<\ldots<l_p\le d} h_{l_1,\dots,l_p}(x_{l_1}\wedge\ldots\wedge x_{l_p});\ h_{l_1,\dots,l_p}\in H\right\}\subset\Lambda[s]$$

of degree p, that is $\Lambda[x,H] = \bigoplus_{p=0}^d \Lambda^p[x,H]$.

For a tuple of commuting operators $T = (T_1, \dots T_d) \in B(H)^d$, denote by $\delta_T : \Lambda[x,H] \to \Lambda[x,H]$ the operator defined by

$$\delta_T = \sum_{l=1}^{d} S_l \otimes T_l.$$

Observe that $\delta_T(\Lambda^p[x,H]) \subset \Lambda^{p+1}[x,H]$. We define for $p = 0, \dots, d-1$ the operators $\delta_T^p : \Lambda^p[x,H] \to \Lambda^{p+1}[x,H]$ as the restrictions $\delta_T^p = \delta_T|_{\Lambda^p[x,H]}$. Since the operators T_l $(l = 1, \dots, d)$ commute we have $\delta_T^{p+1}\delta_T^p = 0$. So, the operators δ_T^p form a complex

$$K^\bullet(T,H) : 0 \longrightarrow \Lambda^0[x,H] \xrightarrow{\delta_T^0} \Lambda^1[x,H] \xrightarrow{\delta_T^1} \cdots \xrightarrow{\delta_T^{d-1}} \Lambda^d[x,H] \longrightarrow 0.$$

The complex $K^\bullet(T,H)$ is called the Koszul complex of T.

With these notions, we define the Taylor spectrum.

Definition 2.4.7. Let $T = (T_1, \dots, T_d) \in B(H)^d$ be a tuple of commuting operators, then

$$\sigma(T) = \{z \in \mathbb{C}^d;\ K^\bullet(z - T, H) \text{ is not exact}\}$$

is called the Taylor spectrum of T.

As in the one-dimensional case, there is a spectral mapping theorem for a tuple of commuting operators. This is for example, a particular case of Theorem 2.5.10 in [EP96].

Theorem 2.4.8. *Let $T = (T_1, \dots, T_d) \in B(H)^d$ be a tuple of commuting operators and $p : \mathbb{C}^d \to \mathbb{C}$ a polynomial then*

$$\sigma(p(T)) = p(\sigma(T)).$$

Next, let us consider the case of commutative unital Banach algebras.

Definition 2.4.9. Let $\mathscr{A}$ be a commutative unital Banach algebra. A linear functional $\chi : \mathscr{A} \to \mathbb{C}$ is called a character or multiplicative if $\chi(1_{\mathscr{A}}) = 1$ and $\chi(xy) = \chi(x)\chi(y)$ for all $x, y \in \mathscr{A}$. We denote by

$$\mathscr{M}(\mathscr{A}) = \{\chi : \mathscr{A} \to \mathbb{C} \text{ is a character}\}$$

the maximal ideal space of $\mathscr{A}$.

The following definition can be found in [Mue07, Chapter I, Section 2, Definition 14].

Definition 2.4.10. Let $\mathscr{A}$ be a commutative unital Banach algebra and let $x = (x_1, \dots, x_d)$ be a tuple with elements in $\mathscr{A}$. The joint spectrum is the set

$$\sigma_{\text{joint}}^{\mathscr{A}}(x) = \{(\chi(x_1), \dots, \chi(x_d));\ \chi \in \mathscr{M}(\mathscr{A})\} \subset \mathbb{C}^d.$$

Notation 2.4.11. Let $\mathscr{A}$ be a unital Banach algebra and let $x = (x_1, \dots, x_d)$ be a tuple of commuting elements in $\mathscr{A}$.

(a) We denote by $\langle x \rangle$ the unital norm closed (commutative) algebra generated by the elements $x_1, \ldots, x_d$.

(b) For reasons of readability, we use the notation

$$\sigma_{joint}(x) = \sigma_{joint}^{\langle x \rangle}(x).$$

Remark 2.4.12. If $T \in B(H)$, it might happen that

$$\sigma(T) \subsetneq \sigma_{joint}^{\langle T \rangle}(T) = \sigma_{joint}(T).$$

In fact, we will see that $\sigma_{joint}(T)$ is the polynomially convex hull of $\sigma(T)$.

Let $\mathscr{A}$ and $\mathscr{B}$ be a commutative unital Banach algebras. If $\pi : \mathscr{B} \to \mathscr{A}$ is a unital algebra homomorphism, it is not difficult to see that

$$\{\chi \circ \pi;\ \chi \in \mathscr{M}(\mathscr{A})\} \subset \mathscr{M}(\mathscr{B}).$$

Hence, one obtains the following proposition.

Proposition 2.4.13. *Let $\mathscr{A}$ and $\mathscr{B}$ be a commutative unital Banach algebras and let $x = (x_1, \ldots, x_n)$ be a tuple with elements in $\mathscr{B}$. If $\pi : \mathscr{B} \to \mathscr{A}$ is a unital algebra homomorphism, then*

$$\sigma_{joint}^{\mathscr{A}}(\pi(x)) \subset \sigma_{joint}^{\mathscr{B}}(x).$$

In particular, if $\pi : \mathscr{B} \to \mathscr{A}$ is invertible, then

$$\sigma_{joint}^{\mathscr{A}}(\pi(x)) = \sigma_{joint}^{\mathscr{B}}(x).$$

Remark 2.4.14. (a) Let $\mathscr{A}$ be a commutative unital Banach algebra, let $\mathscr{B}$ be a unital subalgebra and let $x = (x_1, \ldots, x_d)$ be a tuple with elements in $\mathscr{B}$. Since the inclusion mapping

$$\pi_{\text{inclusion}} : \mathscr{B} \hookrightarrow \mathscr{A},\ x \to x$$

is obviously a unital algebra homomorphism, it follows that

$$\sigma_{\text{joint}}^{\mathscr{A}}(x) = \sigma_{\text{joint}}^{\mathscr{A}}(\pi_{\text{inclusion}}(x)) \subset \sigma_{\text{joint}}^{\mathscr{B}}(x).$$

(b) Let $\mathscr{A}$ be a unital Banach algebra and let $x = (x_1, \ldots, x_d)$ be a tuple with commuting elements in $\mathscr{A}$, then

$$\sigma_{joint}(x) \subset \sigma(x_1) \times \ldots \times \sigma(x_d).$$

In addition, if $\mathscr{A}$ is commutative, it follows from part (a) that

$$\sigma_{joint}^{\mathscr{A}}(x) \subset \sigma_{joint}(x).$$

(c) If $\mathscr{H}$ is a reproducing kernel Hilbert space consisting of $\mathbb{C}$-valued functions on a set X and $\varphi_1, \ldots, \varphi_d$ are elements in the multiplier algebra $\text{Mult}(\mathscr{H})$ with corresponding multiplication operators $M_{\varphi_1}, \ldots, M_{\varphi_d}$ in $B(\mathscr{H})$, one obtains

$$\sigma_{joint}^{\text{Mult}(\mathscr{H})}((\varphi_1, \ldots, \varphi_d)) \subset \sigma_{joint}((M_{\varphi_1}, \ldots, M_{\varphi_d})).$$

Proposition 2.4.15. *Let $x = (x_1, \dots, x_d)$ be a tuple with commuting elements in a Banach algebra $\mathscr{A}$. Then the mapping*

$$\pi : \mathscr{M}(\langle x\rangle) \to \sigma_{joint}(x), \chi \mapsto (\chi(x_1), \dots \chi(x_d))$$

is a homeomorphism.

Proof. The maximal ideal space $\mathscr{M}(\langle x\rangle)$ is compact and π is continuous and onto. It suffices to prove that π is one to one. This is immediate, since $\pi(\chi_1) = \pi(\chi_2)$ implies that $\chi_1(x_l) = \chi_2(x_l)$ for $\chi_1, \chi_2 \in M(\langle x\rangle)$ and $l = 1, \dots, d$. See also [Mue07, Theorem 16, Chapter I]. □

As in the one-dimensional case, the following spectral mapping theorem is immediate.

Theorem 2.4.16. *Let $\mathscr{A}$ be a commutative unital Banach algebra, let $x = (x_1, \dots, x_d)$ be a tuple with elements in $\mathscr{A}$ and let $p = (p_1, \dots, p_m)$ be a tuple of polynomials in $\mathbb{C}[z_1, \dots, z_d]$, then*

$$\sigma^{\mathscr{A}}_{joint}(p(x)) = p(\sigma^{\mathscr{A}}_{joint}(x)).$$

Notation 2.4.17. Let $Q \subset \mathbb{C}^d$ be compact, we denote by

$$\hat{Q} = \left\{ z \in \mathbb{C}^d;\ |p(z)| \le \sup_{\zeta \in Q} |p(\zeta)| \text{ for all } p \in \mathbb{C}[z] \right\}$$

the polynomially convex hull of Q. The set Q is called polynomially convex if and only if $Q = \hat{Q}$.

Remark 2.4.18. (a) If $Q_1, Q_2 \subset \mathbb{C}^d$ are compact with $Q_1 \subset Q_2$, then $\hat{Q}_1 \subset \hat{Q}_2$.

(b) Compact convex sets in $\mathbb{C}^d$ are polynomially convex. In particular, if $a \in \mathbb{C}^d$ and $r > 0$, the closed Euclidean ball $\overline{B}_d(a, r)$ is polynomially convex.

Proposition 2.4.19. *Let $x = (x_1, \dots, x_d)$ be a tuple with commuting elements in a Banach algebra $\mathscr{A}$. Then $\sigma_{joint}(x)$ is polynomially convex.*

Proof. Fix a point w in the polynomially convex hull $\hat{\sigma}_{joint}(x)$. Then

$$\begin{aligned} |p(w)| &\le \sup\{|p(z)|;\ z \in \sigma_{joint}(x)\} \\ &= \sup\{|z|;\ z \in \sigma_{joint}(p(x))\} \le \|p(x)\| \end{aligned}$$

for each polynomial $p \in \mathbb{C}[z]$. It follows that

$$\delta_w|_{\mathbb{C}[x]} : \{p(x);\ p \in \mathbb{C}[z]\} \to \mathbb{C}, p(x) \mapsto p(w)$$

is well-defined and continuous. Since

$$\{p(x);\ p \in \mathbb{C}[z]\} \subset \langle x\rangle$$

is dense, the point evaluation

$$\delta_w : \langle x\rangle \to \mathbb{C},$$

is an element of $\mathscr{M}(\langle x\rangle)$. Hence,

$$w = \delta_w(x) \in \sigma_{joint}(x)$$

and $\sigma_{joint}(x)$ is polynomially convex (cf. [Mue07, Theorem 18, Chapter I]). □

Notation 2.4.20. For a bounded linear operator $T \in B(H)$ denote by $\rho(T)$ the spectral radius of T.

Remark 2.4.21. Let H and $\tilde{H}$ be Hilbert spaces and let $T = (T_1, \dots, T_d) \in B(H)^d$ and $S = (S_1, \dots, S_d) \in B(\tilde{H})^d$ be tuples of commuting operators.

(a) By the preceding spectral mapping theorems, we obtain that

$$\hat{\sigma}(T) = \left\{ z \in \mathbb{C}^d;\ |p(z)| \leq \rho(p(T)) \text{ for all } p \in \mathbb{C}[z] \right\} = \hat{\sigma}_{joint}(T) = \sigma_{joint}(T),$$

In particular, if $\sigma(T)$ is (polynomially) convex, it follows that

$$\sigma(T) = \sigma_{joint}(T).$$

(b) Suppose that

$$\|p(T)\|_H \leq \|p(S)\|_{\tilde{H}}$$

for all $p \in \mathbb{C}[z]$. By the spectral radius formula

$$\rho(p(T)) \leq \rho(p(S))$$

for all $p \in \mathbb{C}[z]$. Thus, we conclude that

$$\hat{\sigma}(T) \subset \hat{\sigma}(S).$$

Due to Remark 2.4.21, part (b), we obtain the following lemma:

Lemma 2.4.22. *Let H and $\tilde{H}$ be Hilbert spaces and let $T = (T_1, \dots, T_d) \in B(H)^d$ and $S = (S_1, \dots, S_d) \in B(\tilde{H})^d$ be tuples of commuting operators. Suppose that there exists an isometry $V : H \to \tilde{H}$ such that $T_l^n = V^* S_l^n V$ for all $n \in \mathbb{N}$ and all $l \in \{1, \dots, d\}$. Then*

$$\hat{\sigma}(T) \subset \hat{\sigma}(S).$$

Proof. Since $V : H \to \tilde{H}$ is linear, using the intertwining relation $T_l^n = V^* S_l^n V$ for all $n \in \mathbb{N}$ and all $l \in \{1, \dots, d\}$, it follows that

$$p(T) = V^* p(S) V$$

for all polynomials $p \in \mathbb{C}[z]$. Since V is an isometry, we obtain

$$\|p(T)\|_H = \|V^* p(S) V\|_H \leq \|p(S)\|_{\tilde{H}}$$

for all $p \in \mathbb{C}[z]$. Hence, the assertion follows with Remark 2.4.21, part (b). □

Remark 2.4.23. Let $T = (T_1, \dots, T_d) \in B(H)^d$ be a tuple of commuting operators such that $\hat{\sigma}(T) \subset \overline{\mathbb{B}_d}$. By the spectral mapping theorem for the multivariable spectrum and the properties of the Riesz-Dunford functional calculus, we obtain the following statements for $S = (T_1, \dots, T_d)$ or $S = (T_1^*, \dots, T_d^*)$:

(a) Let $f \in \mathscr{O}(\mathbb{D})$ with power series representation $f = \sum_{n=0}^{\infty} f_n z^n$. Then for all $w \in \mathbb{B}_d$ the power series

$$\sum_{\alpha \in \mathbb{N}^d} f_{|\alpha|} \gamma_\alpha w^\alpha S^{*\alpha} \in B(H)$$

converges in norm and the function

$$f_S : \mathbb{B}_d \to B(H),\ w \mapsto \sum_{\alpha \in \mathbb{N}^d} f_{|\alpha|} \gamma_\alpha S^{*\alpha} w^\alpha$$

is holomorphic. In particular, since taking adjoints is continuous with respect to the norm-topology in $B(H)$, it follows that $f_S(w)^* = f_{S^*}(\overline{w})$.

(b) If $f, g \in \mathscr{O}(\mathbb{D})$, then $(fg)_S = f_S g_S$.

2.4.2. Multivariable spectrum of the weighted shift

In the last part of this section we consider the multivariable spectrum of the weighted shift operator tuple $M_z = (M_{z_1}, \dots, M_{z_d}) \in B(\mathscr{H})^d$ on a unitarily invariant regular space $\mathscr{H}$. As a particular case of Theorem 4.6 in [GHX04], one obtains the following theorem:

Theorem 2.4.24. *The Taylor spectrum of $M_z \in B(\mathscr{H})^d$ is given by*

$$\sigma(M_z) = \overline{\mathbb{B}_d}.$$

For the case $d = 1$ see also [Shi74] , in particular Proposition 15 and [Con00, 27.7 Proposition]. For convenience we give a slightly more elementary proof as in [GHX04], to show that

$$\sigma(M_z) = \sigma_{joint}(M_z) = \overline{\mathbb{B}_d}.$$

If one feels more comfortable with it, one can only use the joint spectrum for the results in the following chapters.

Lemma 2.4.25. $\sigma(M_{z_l}) = \overline{\mathbb{D}}$ *for* $l = 1, \dots, d$.

Proof. First, let us prove that $\sigma(M_{z_1}) = \overline{\mathbb{D}}$. We show that 1 is an upper bound for the spectral radius $\rho(M_{z_1})$ of the multiplication operator $M_{z_1} : \mathscr{H} \to \mathscr{H}$. By the spectral radius formula

$$\rho(M_{z_1}) = \lim_{N \to \infty} \|M_{z_1}^N\|^{1/N}.$$

For $N \in \mathbb{N}$ and $\beta \in \mathbb{N}^d$ we have

$$M_{z_1}^N z^\beta = z_1^N z^\beta.$$

Hence, if $\alpha = (N, 0, \dots, 0) \in \mathbb{N}^d$ and $\beta \in \mathbb{N}^d$ with $|\beta| = n$, it follows that

$$\|M_{z_1}^N z^\beta\|^2 = \frac{\gamma_\beta}{\gamma_{\alpha+\beta}} \left(\frac{a_n}{a_{n+N}} \right) \|z^\beta\|^2 \leq \sup_{n \in \mathbb{N}} \left(\frac{a_n}{a_{n+N}} \right) \|z^\beta\|^2.$$

By orthogonality
$$\|M_{z_1}^N\| \le \sup_{n\in\mathbb{N}} \left(\frac{a_n}{a_{n+N}}\right)^{1/2}.$$
Now, let $b_n = \frac{a_n}{a_{n+1}}$ for $n \in \mathbb{N}$. Then we have that
$$\|M_{z_1}^N\| \le \sup_{n\in\mathbb{N}} \left(\frac{a_n}{a_{n+N}}\right)^{1/2} = \sup_{n\in\mathbb{N}} \left(\prod_{l=0}^{N} b_{n+l}\right)^{1/2}$$
for $N \ge 1$ and
$$\lim_{l\to\infty}\left(\sup_{n\in\mathbb{N}} b_{n+l}\right) = \limsup_{l\to\infty} b_l = \lim_{l\to\infty}\frac{a_l}{a_{l+1}} = 1.$$
Thus, it follows that
$$\lim_{N\to\infty}\frac{1}{N}\sum_{l=1}^{N}\left(\sup_{n\in\mathbb{N}} b_{n+l}\right) = 1.$$
By the arithmetic-mean-geometric-mean inequality, we conclude that
$$\left(\prod_{l=1}^{N} b_{n+l}\right)^{\frac{1}{N}} \le \frac{1}{N}\sum_{l=1}^{N} b_{n+l}$$
for all $N, n \in \mathbb{N}$ with $N \ge 1$, so
$$\|M_{z_1}^N\|^{1/N} = \sup_{n\in\mathbb{N}}\left(\prod_{l=1}^{N} b_{n+l}\right)^{\frac{1}{2N}} \le \left(\frac{1}{N}\sum_{l=1}^{N}\left(\sup_{n\in\mathbb{N}} b_{n+l}\right)\right)^{1/2}$$
for all $N \ge 1$. We obtain
$$\rho(M_{z_1}) = \lim_{N\to\infty}\|M_{z_1}^N\|^{1/N} \le 1.$$
For every $w \in \mathbb{D}$ the point evaluation
$$\delta_{(w,0,\ldots,0)} : \mathrm{Mult}(\mathscr{H}) \to \mathbb{C}$$
is a well-defined character. But then
$$w = \delta_{(w,0,\ldots,0)}(M_{z_1}) \in \sigma(M_{z_1}).$$
and thus, $\sigma(M_{z_1}) = \overline{\mathbb{D}}$. Analogously, one proves that $\sigma(M_{z_l}) = \overline{\mathbb{D}}$ for $l = 2, \ldots, d$. □

Proof of Theorem 2.4.24. Let $U = (u_{l,m})_{l,m=1,\ldots,d} \in U(d)$ be a unitary operator on $\mathbb{C}^d$. The norm-closure of polynomials $A(\mathscr{H})$ in $\mathrm{Mult}(\mathscr{H})$ is the unital norm-closed algebra $\langle M_z\rangle$ generated by $M_{z_1}, \ldots, M_{z_d}$. Let $\Pi_U : B(\mathscr{H}) \to B(\mathscr{H})$ be the unital invertible algebra homomorphism defined in Theorem 2.3.10. By the previous computation we deduce that $\Pi_U(\langle M_z\rangle) = \langle M_z\rangle$. It follows by Theorem 2.3.10 that
$$\Pi_U(A(\mathscr{H})) = A(\mathscr{H}).$$

As a particular case of Theorem 2.4.16, we obtain that

$$\sigma_{joint}(M_z) = \sigma(\Pi_U(M_z)) = U(\sigma_{joint}(M_z)).$$

Now if $w \in \sigma_{joint}(M_z) \subset \mathbb{C}^d$ and $\|w\|_2^2 = \sum_{l=1}^d |w_l|^2$, choose a unitary operator $U : \mathbb{C}^d \to \mathbb{C}^d$ such that

$$U^*\left(\frac{w}{\|w\|_2}\right) = e_1,$$

where $e_1 = (1,0,\dots,0) \in \mathbb{C}^d$ is the first basis vector. Since

$$\|w\|_2 e_1 = Uw \in \sigma_{joint}(M_z)$$

and

$$\sigma_{joint}(M_z) \subset \sigma(M_{z_1}) \times \dots \times \sigma(M_{z_d}),$$

it is immediate that

$$\|w\|_2 e_1 \in \sigma(M_{z_1}) \times \{0\} \times \dots \times \{0\}.$$

Because of Lemma 2.4.25, we have that $\sigma(M_{z_1}) = \overline{\mathbb{D}}$ and thus, $\|w\|_2 \leq 1$. We deduce, that

$$\sigma_{joint}(M_z) \subset \overline{\mathbb{B}_d}.$$

For $w = (w_1, \dots, w_d) \in \mathbb{B}_d$ the operator

$$(M_{z_1} - w_1 \operatorname{id}_{\mathscr{H}}, \dots, M_{z_d} - w_d \operatorname{id}_{\mathscr{H}}) : \mathscr{H}^d \to \mathscr{H}$$

is clearly not surjective. This means that the Koszul complex in the last position is not exact. Hence, we conclude that

$$\mathbb{B}_d \subset \sigma(M_z) \subset \sigma_{joint}(M_z) \subset \overline{\mathbb{B}_d}$$

Since $\sigma(M_z)$ is closed, it is immediate that

$$\overline{\mathbb{B}_d} = \sigma(M_z) = \sigma_{joint}(M_z).$$

□

Let H be a Hilbert space with orthonormal basis $(e_n)_{n\in\mathbb{N}}$. Let $S : H \to H$ be the unilateral shift operator with $Se_n = e_{n+1}$. From the theory of C^*-Algebras it is well-known that the Toeplitz sequence

$$0 \longrightarrow K(H) \longrightarrow C^*(S) \longrightarrow C(\partial\mathbb{D}) \longrightarrow 0$$

is exact (see for example [Dou98, 7.23 Theorem]). We consider a more general result for the weighted shift operator tuple M_z on regular unitarily invariant spaces $\mathscr{H}$.

Proposition 2.4.26. *Let $\mathscr{H}$ be a unitarily invariant space, not necessarily regular, such that the multiplication operators of the coordinate functions*

$$M_{z_l} : \mathscr{H} \to \mathscr{H},\ f \mapsto z_l f \quad (l = 1, \ldots, d)$$

are well-defined and bounded. Then the C^*- algebra $C^*(M_z)$, generated by the operators $\{\mathrm{id}_{\mathscr{H}}, M_{z_1}, \ldots, M_{z_d}\}$, is irreducible. That is, if $P : \mathscr{H} \to \mathscr{H}$ is a non-zero orthogonal projection such that

$$PT = TP \quad \textit{for all } T \in C^*(M_z),$$

then $P = \mathrm{id}_{\mathscr{H}}$. Consequently, if

$$\emptyset \neq K(\mathscr{H}) \cap C^*(M_z),$$

then $K(\mathscr{H}) \subset C^*(M_z)$.

Proof. Let $P : \mathscr{H} \to \mathscr{H}$ be a non-zero orthogonal projection such that

$$PT = TP \quad \text{for all } T \in C^*(M_z)$$

Because of Proposition 2.3.57, there exist functions $\varphi, \psi \in \mathrm{Mult}(\mathscr{H})$ such that $P = M_\varphi$ and $P^* = M_\psi$. But, then

$$M_\varphi k_w = M_\psi^* k_w = \overline{\psi(w)} k_w$$

for all $w \in \mathbb{B}_d$. Therefore, φ and ψ are constant and P must be the identity. It is well-known from the theory of C^*-algebras (see [Con00, Chapter 3, 16.8 Corollary]), that if $\mathscr{B} \subset B(\mathscr{H})$ is an irreducible C^*-algebra and

$$\emptyset \neq K(\mathscr{H}) \cap \mathscr{B},$$

then $K(\mathscr{H}) \subset \mathscr{B}$. □

Remark 2.4.27. The C^*- algebra $C^*(M_z)$ is sometimes also called Toeplitz algebra.

Remark 2.4.28. Let $\mathscr{H}$ be regular.

(a) Due to Lemma 2.4.2, the d-tuple $M_{z_1}, \ldots, M_{z_d}$ acting on $\mathscr{H}$ is essentially normal and the operator

$$\mathrm{id}_{\mathscr{H}} - \sum_{l=1}^{d} M_{z_l} M_{z_l}^*$$

is compact.

(b) Due to Proposition 2.4.26, it follows that $K(\mathscr{H}) \subset C^*(M_z)$.

(c) The C^*-subalgebra $C^*(M_z)/K(\mathscr{H})$ of the Calkin-algebra $C(\mathscr{H}) = B(H)/K(H)$ is commutative by the Fuglede–Putnam theorem [Con90, Theorem 6.7, Chapter IX].

The following statement can be found as a particular case of Theorem 4.6 in [GHX04].

Theorem 2.4.29 (Guo, Hu, Xu)**.** *If $\mathscr{H}$ is regular, ι is the inclusion mapping and π is a unital $*$-homomorphism uniquely determined by $\pi(M_{z_l}) = z_l|_{\partial\mathbb{B}_d}$ for $l = 1,\dots,d$, the sequence of C^*-algebras*

$$0 \longrightarrow K(\mathscr{H}) \xrightarrow{\ \iota\ } C^*(M_z) \xrightarrow{\ \pi\ } C(\partial\mathbb{B}_d) \longrightarrow 0$$

is exact. Since modulo identification

$$A(\mathscr{H}) = \overline{\mathbb{C}[z]}^{\|\cdot\|_{\mathrm{Mult}}} \subset C^*(M_z),$$

it follows for all $\varphi \in A(\mathscr{H})$, that

$$\|M_\varphi\|_e = \inf_{S\in K(\mathscr{H})} \|M_\varphi + S\|_{B(\mathscr{H})} = \|\varphi\|_\infty.$$

Proof. Let $\mathscr{A} = C^*(M_z)/K(\mathscr{H})$ be the commutative C^*-subalgebra of the Calkin-algebra $C(\mathscr{H}) = B(H)/K(H)$. By Gelfand representation theory

$$\mathscr{A} \cong C(\mathscr{M}(A)),$$

where $\mathscr{M}(A)$ is the maximal ideal space. Let $\chi : \mathscr{A} \to \mathbb{C}$ be a character. Since the operator

$$\mathrm{id}_{\mathscr{H}} - \sum_{l=1}^{d} M_{z_l} M_{z_l}^* \in K(\mathscr{H})$$

is compact, it is immediate that

$$|\chi([M_{z_1}])|^2 + \ldots + |\chi([M_{z_d}])|^2 - 1 = \chi\left(\left[\mathrm{id}_{\mathscr{H}} - \sum_{l=1}^{d} M_{z_l} M_{z_l}^*\right]\right) = 0.$$

It suffices to prove that the mapping

$$\psi : \mathscr{M}(\mathscr{A}) \to \partial\mathbb{B}_d, \chi \mapsto (\chi([M_{z_1}]),\dots \chi([M_{z_d}]))$$

is a homeomorphism. Since $\mathscr{M}(\mathscr{A})$ is compact, it is enough to show that ψ is bijective and continuous. It is not difficult to see that ψ is injective and continuous, so it remains to prove that ψ is onto. Due to Theorem 2.3.10, a unitary matrix $U = (u_{l,m})_{l,m} \in U(d)$ on $\mathbb{C}^d$ induces a C^*-algebra isomorphism

$$\Pi_U : \mathscr{A} \to \mathscr{A}$$

with

$$\Pi_U([M_{z_l}]) = \sum_{m=1}^{d} u_{l,m}[M_{z_m}].$$

Hence, it follows for every $\chi \in \mathscr{M}(\mathscr{A})$ that $\chi \circ \Pi_U \in M(\mathscr{A})$, with

$$(\chi \circ \Pi_U)([M_{z_l}]) = \sum_{m=1}^{d} u_{l,m}\chi([M_{z_m}]).$$

Thus, since $\mathscr{M}(\mathscr{A})$ is not empty, we deduce that ψ is onto. □

2.4.3. Diagonal operators and the Cauchy dual

In this section, we introduce two diagonal operators, which will be used in Chapter 3. Furthermore, we will see how they are related to the so-called Cauchy dual of the weighted shift operator tuple on unitarily invariant spaces on the ball. We also give a short explanation of how the Cauchy dual of a single weighted shift operator is related to the Cauchy dual of the corresponding unitarily invariant space on the disk. Now, let $\mathscr{H}$ be a unitarily invariant space with reproducing kernel

$$K\colon \mathbb{B}_d \times \mathbb{B}_d \to \mathbb{C},\ K(z,w) = \sum_{n=0}^{\infty} a_n \langle z, w\rangle^n$$

such that $a_0 = 1$, $a_n > 0$ for $n \geq 1$ and

$$\sup_{n\in\mathbb{N}} \frac{a_n}{a_{n+1}} < \infty,\ \inf_{n\in\mathbb{N}} \frac{a_n}{a_{n+1}} > 0.$$

Suppose that the analytic function $k(z) = \sum_{n=0}^{\infty} a_n z^n$ has no zeros in $\mathbb{D}$ and denote by c_n $(n \in \mathbb{N})$ the coefficients of $\frac{1}{k}$. Additionally, suppose that almost all the coefficients c_n have the same sign.

Example 2.4.30. (a) For $s > 0$ the spaces $A^2_s(\mathbb{B}_d)$ provide examples for such spaces.

(b) The unitarily invariant complete Nevanlinna-Pick spaces as they occur, for example in 2.3.5 are also spaces of this type. See also Example 2.4.6.

In the following we consider the operators

$$\delta\colon \mathscr{H} \to \mathscr{H},\ \delta\left(\sum_{n=0}^{\infty} \sum_{|\alpha|=n} f_\alpha z^\alpha\right) = f_0 + \sum_{n=1}^{\infty} \frac{a_n}{a_{n-1}} \sum_{|\alpha|=n} f_\alpha z^\alpha$$

and

$$\Delta\colon \mathscr{H} \to \mathscr{H},\ \Delta\left(\sum_{n=0}^{\infty} \sum_{|\alpha|=n} f_\alpha z^\alpha\right) = \sum_{n=0}^{\infty} \frac{a_{n+1}}{a_n} \sum_{|\alpha|=n} f_\alpha z^\alpha.$$

By definition, Δ and δ are diagonal operators with respect to the orthogonal decomposition $\mathscr{H} = \oplus_{n=0}^{\infty} \mathbb{H}_n$ of $\mathscr{H}$ into the spaces $\mathbb{H}_n$ of all homogeneous polynomials of degree n. Our hypotheses on the sequence $\left(\frac{a_n}{a_{n+1}}\right)$ imply that δ and Δ are invertible positive operators on $\mathscr{H}$. An elementary calculation shows that

$$\delta M_{z_l} = M_{z_l} \Delta$$

for $l = 1, \ldots, d$.

Consider the completely-positive map

$$\sigma_{M_z}\colon B(\mathscr{H}) \to B(\mathscr{H}),\ X \mapsto \sum_{l=0}^{d} M_{z_l} X M_{z_l}^*.$$

In [Lan19, Theorem 2.2.2], Langendörfer proves a result, that is similar to a result due to Chen (cf. [Che12, Proposition 2.1 and Lemma 2.2]): If almost all the coefficients c_n have the same sign, then the limit

$$\mathrm{SOT}-\lim_{N\to\infty}\sum_{n=0}^{N}-c_{n+1}\sigma_{M_z}^n(\mathrm{id}_{\mathscr{H}})$$

exists. In this case, one can show (cf. Theorem 2.2.6 in [Lan19]) the following identity

$$\Delta=\mathrm{SOT}-\lim_{N\to\infty}\sum_{n=0}^{N}(-c_{n+1})\sigma_{M_z}^n(\mathrm{id}_{\mathscr{H}}).$$

We want to use the matrix operator defined by

$$M_z^*M_z=(M_{z_l}^*M_{z_m})_{1\le l,m\le d}\in B\left(\mathscr{H}^d\right).$$

Since the row operator $M_z\colon \mathscr{H}^d\to\mathscr{H}$ has closed range, the operator

$$M_z^*M_z\colon\ \mathrm{Im}\,M_z^*\to\mathrm{Im}\,M_z^*$$

is invertible (see A.1.4). We denote its inverse by $(M_z^*M_z)^{-1}$.

Remark 2.4.31 (Cauchy dual). Suppose for a moment that we are in the one dimensional case $d=1$ and that $\mathscr{H}$ is regular. Then $\mathrm{Im}\,M_z^*=\mathscr{H}$ and $M_z^*:\mathscr{H}\to\mathscr{H}$ is right invertible with inverse $M_z\left(M_z^*M_z\right)^{-1}$.

Since $\sum_{n=0}^{\infty}\frac{|w|^{2n}}{a_n}<\infty$ for all $w\in\mathbb{D}$, it follows that $\mathscr{H}$ contains the Cauchy or Szegő kernel functions

$$s_w:\mathbb{D}\to\mathbb{D},\ s_w(z)=\frac{1}{1-z\overline{w}}.$$

Let

$$\mathscr{H}'=\left\{\sum_{n=0}^{\infty}f_nz^n\in\mathscr{O}(\mathbb{D});\ \sum_{n=0}^{\infty}a_n|f_n|^2<\infty\right\}$$

be the corresponding reproducing kernel Hilbert space with reproducing kernel $K':\mathbb{D}\times\mathbb{D}\to\mathbb{C}$,

$$K'(z,w)=\sum_{n=0}^{\infty}\frac{(z\overline{w})^n}{a_n}.$$

The operator $U:\mathscr{H}\to\mathscr{H}'$, which is uniquely determined by

$$z^n\mapsto\frac{z^n}{a_n}\qquad(n\in\mathbb{N})$$

is unitary. For $w\in\mathbb{D}$ and $f=\sum_{n=0}^{\infty}f_nz^n$ in $\mathscr{H}$ one computes that

$$(Uf)(w)=\left(\sum_{n=0}^{\infty}U(f_nz^n)\right)(w)=\sum_{n=0}^{\infty}\frac{f_nw^n}{a_n}=\left\langle\sum_{n=0}^{\infty}f_nz^n,\sum_{n=0}^{\infty}\overline{w}^nz^n\right\rangle_{\mathscr{H}}=\langle f,s_w\rangle_{\mathscr{H}}.$$

With respect to the pairing

$$\langle f,g\rangle_{\mathscr{H}\times\mathscr{H}'} = \langle f,U^*g\rangle_{\mathscr{H}} \quad \left(f\in\mathscr{H}, g\in\mathscr{H}'\right).$$

the space $\mathscr{H}'$ is called the Cauchy dual of $\mathscr{H}$. For all $w\in\mathbb{D}$ we have

$$U^*k'_w = s_w,$$

where

$$k'_w:\mathbb{B}_d\to\mathbb{C},\ k'_w(z)=K'(z,w) \quad (w\in\mathbb{D}).$$

Hence, we obtain

$$\langle f,k'_w\rangle_{\mathscr{H}\times\mathscr{H}'} = \langle f,s_w\rangle_{\mathscr{H}}.$$

In addition, if $p=\sum_{n=0}^N p_n z^n$ and $q=\sum_{n=0}^N q_n z^n$ are polynomials, then the pairing between H and H' is given by

$$\begin{aligned}\langle p,q\rangle_{\mathscr{H}\times\mathscr{H}'} = \langle p,U^*q\rangle_{\mathscr{H}} &= \left\langle \sum_{n=0}^N p_n z^n, \sum_{n=0}^N a_n q_n z^n\right\rangle_{\mathscr{H}} = \sum_{n=0}^n p_n\overline{q_n}\\ &= \langle p,q\rangle_{H^2(\mathbb{D})}.\end{aligned}$$

Hence, $\mathscr{H}'$ is a dual space via "$H^2(\mathbb{D})$-duality", which is thus often called Cauchy dual of $\mathscr{H}$.

The Bergman space $L^2_a(\mathbb{D})$ with reproducing kernel $K^{L^2_a(\mathbb{D})}:\mathbb{D}\times\mathbb{D}\to\mathbb{C}$,

$$K^{L^2_a(\mathbb{D})}(z,w) = \frac{1}{(1-z\overline{w})^2} = \sum_{n=0}^\infty (n+1)(z\overline{w})^n$$

can be considered as the Cauchy dual of the Dirichlet space $\mathscr{D}$ with reproducing kernel $K^{\mathscr{D}}:\mathbb{D}\times\mathbb{D}\to\mathbb{C}$,

$$K^{\mathscr{D}}(z,w) = \frac{1}{z\overline{w}}\log\left(\frac{1}{1-z\overline{w}}\right) = \sum_{n=0}^\infty \frac{1}{n+1}(z\overline{w})^n$$

and vice versa. For a motivation and a definition on Banach function spaces see [ARR98, Section 5].

Now consider the operators

$$T:\mathscr{H}\to\mathscr{H},\ f\mapsto M_z f, \quad T':\mathscr{H}\to\mathscr{H},\ f\mapsto M_z\left(M_z^*M_z\right)^{-1}f$$

and

$$S:\mathscr{H}'\to\mathscr{H}',\ f\mapsto M_z f.$$

Since

$$(UT')z^n = U\left(\frac{a_{n+1}}{a_n}z^{n+1}\right) = \frac{z^{n+1}}{a_n} = (SU)z^n$$

for all $n\in\mathbb{N}$, it follows that

$$UT' = SU.$$

Hence, the following diagram commutes

$$\begin{array}{ccc} \mathscr{H} & \xrightarrow{T'} & \mathscr{H} \\ \downarrow U & & \downarrow U \\ \mathscr{H}' & \xrightarrow{S} & \mathscr{H}' \end{array}$$

and

$$\langle T'f,g\rangle_{\mathscr{H}\times\mathscr{H}'} = \langle T'f,U^*g\rangle_{\mathscr{H}} = \langle f,U^*S^*g\rangle_{\mathscr{H}} = \langle f,S^*g\rangle_{\mathscr{H}\times\mathscr{H}'}$$

for all $f \in \mathscr{H}$ and $g \in \mathscr{H}'$. That is why in the one dimensional setting the operator $M_z' = M_z\left(M_z^*M_z\right)^{-1}$ is sometimes called the Cauchy dual of M_z. The name Cauchy dual for operators of this type presumably goes back to Shimorin (see [Shi01]). On the other hand, operators of this type have been considered much earlier by other authors. If $L : \mathscr{H} \to \mathscr{H}$ with $Lf(z) = \frac{f(z)-f(0)}{z}$ is the backward shift then

$$M_z'f = M_z(M_z^*M_z)^{-1}f = L^*f.$$

for all $f \in \mathscr{H}$. Using this fact, the previous explanation can already be found for a larger class of spaces in [ARR98, Proposition 5.2].

The following lemma gives a possibility to construct the Cauchy dual of the operator $M_z : \mathscr{H}^d \to \mathscr{H}$ in the multivariable setting using the diagonal operator δ.

Lemma 2.4.32. *For $f \in \mathscr{H}$, we have*

$$(M_z^*M_z)^{-1}(M_z^*f) = M_z^*\delta f = (\oplus\Delta)M_z^*f.$$

In particular, the row operator

$$\delta M_z \colon \mathscr{H}^d \to \mathscr{H}$$

extends

$$M_z\left(M_z^*M_z\right)^{-1} : \operatorname{Im} M_z^* \to \mathscr{H}$$

by 0 *on* $(\operatorname{Im}(M_z^*))^{\perp}$.

Proof. The column operator M_z^* annihilates the constant functions. Thus, we may suppose that $f(0) = 0$. Due to Lemma 2.4.2 the operator $M_zM_z^*$ acts as

$$M_zM_z^*\left(\sum_{n=0}^{\infty} f_n\right) = \sum_{n=1}^{\infty}\left(\frac{a_{n-1}}{a_n}\right)f_n,$$

with respect to the orthogonal decomposition $\mathscr{H} = \bigoplus_{n=0}^{\infty}\mathbb{H}_n$. Hence, $M_zM_z^*\delta f = f$ and

$$\left(M_z^*M_z\right)^{-1}M_z^*f = \left(M_z^*M_z\right)^{-1}\left(M_z^*M_z\right)M_z^*\delta f = M_z^*\delta f.$$

Using that

$$\delta M_{z_l} = M_{z_l}\Delta,$$

we get that $M_z^*\delta = (\oplus\Delta)M_z^*$. Since any two diagonal operators and in particular $M_zM_z^*$ and δ commute, it follows that $M_z\left(M_z^*M_z\right)^{-1}M_z^* = (M_zM_z^*)\delta = (\delta M_z)M_z^*$. Hence, the second assertion follows. □

As in the one dimensional case, we call the operator defined by

$$M_z' = \delta M_z \in B\left(\mathscr{H}^d, \mathscr{H}\right)$$

the Cauchy dual of M_z.

Remark 2.4.33. The preceding proof shows in particular that the orthogonal projection of $\mathscr{H}$ onto $\operatorname{Im} M_z$ acts as

$$P_{\operatorname{Im} M_z} = M_z(M_z^* M_z)^{-1} M_z^* = \delta(M_z M_z^*) = P_{\mathbb{C}^\perp},$$

where $\mathbb{C} \subset \mathscr{H}$ is regarded as the closed subspace consisting of all constant functions.

Remark 2.4.34. Suppose that $\mathscr{E}$ is an arbitrary Hilbert space. It is clear that the previous results for the operators M_z, δ and Δ on $\mathscr{H}$ also apply to the operators $M_z = M_z^{\mathscr{E}}$, $\delta = \delta \otimes \operatorname{id}_{\mathscr{E}}$ and $\Delta = \Delta \otimes \operatorname{id}_{\mathscr{E}}$ on $H(\mathscr{E})$ modulo the identification $\mathscr{H}(\mathscr{E}) \cong \mathscr{H} \otimes \mathscr{E}$.

2.5. K-contractions

The purpose of this section is to gather definitions and results from the theory of K-contractions. A K-contraction is a generalization of a contraction $T \in B(H)$ on a Hilbert space H. The idea presumably goes back to Agler and has been developed over the years. We mainly follow results from Schillo's PhD thesis (cf. [Sch18]) here.

We use the relationship between the Hardy space $H^2(\mathbb{D})$ and contractions as a motivation. Let

$$K : \mathbb{D} \times \mathbb{D} \to \mathbb{C},\ K(z,w) = \frac{1}{1 - z\overline{w}},$$

be the reproducing kernel of the Hardy space $H^2(\mathbb{D})$. An operator $T \in B(H)$ is a contraction if and only if

$$D_{T^*}^2 = \frac{1}{K}(T) = \operatorname{id}_H - TT^* \geq 0.$$

The closure

$$\mathscr{D} = \overline{\operatorname{Im}(D_{T^*})}$$

is called the defect space of a contraction.

Sz.-Nagy dilation theory, a contraction $T \in B(H)$ is unitarily equivalent to a compression of the direct sum

$$M_z \oplus U,$$

where

$$M_z : H^2(\mathscr{D}) \to H^2(\mathscr{D}),\ f \mapsto zf.$$

We now want to analyze which operators can be modeled analogously for more general reproducing kernels.

First ideas of this kind can be found in a work by Agler (see [Agl82]):

Suppose that $\mathscr{H} \subset \mathscr{O}(\mathbb{D})$ is a Hilbert function space with reproducing kernel $K : \mathbb{D} \times \mathbb{D} \to \mathbb{C}$ such that

$$\hat{K} : \mathbb{D} \times \mathbb{D} \to \mathbb{C}, (z,w) \mapsto K(z,\overline{w})$$

is analytic, has no zeros and

$$M_z : \mathscr{H} \to \mathscr{H},\ f \mapsto zf$$

is bounded. Let $T \in B(H)$ be an operator on a separable Hilbert space H with spectrum

$$\sigma(T) \subset \mathbb{D}.$$

Since $\sigma(T)$ is compact, there exists an $0 < r < 1$ such that

$$\sigma(T) \subset \mathbb{D}(r).$$

For $r < s < 1$ define

$$\frac{1}{K}(T) = \frac{1}{\hat{K}}(T,T^*) = \int_{|w|=s} \int_{|z|=s} \frac{1}{\hat{K}}(z,w)(w-T^*)^{-1}(z-T)^{-1} dz dw.$$

Under suitable assumptions on the space $\mathscr{H}$, Agler proves that T co-extends to a direct sum of copies of M_z if and only if $\frac{1}{K}(T) \geq 0$ (see [Agl82, 2.3 Theorem]).

In a higher-dimensional setting, Agler M^{c}Carthy (cf. [AM00b]), Ambrozie, Engliš and Müller (cf. [AEM02]) and Arazy and Engliš (cf. [AE03]) extend these ideas. For Nevanlinna-Pick spaces, there are characterizations by Clouâtre and Hartz (see [CH18]). One of the difficulties is making sense of the expression $(1/K)(T)$. There is a unified approach for unitarily invariant spaces by Schillo (see [Sch18]), which we use as a guideline here.

For the definition, let $\mathscr{H}$ be a unitarily invariant space with reproducing kernel

$$K \colon \mathbb{B}_d \times \mathbb{B}_d \to \mathbb{C},\ K(z,w) = \sum_{n=0}^{\infty} a_n \langle z,w \rangle^n$$

such that $a_0 = 1$, $a_n > 0$ for $n \geq 1$ and

$$\sup_{n \in \mathbb{N}} \frac{a_n}{a_{n+1}} < \infty.$$

Suppose that the analytic function

$$k : \mathbb{D} \to \mathbb{C},\ k(z) = \sum_{n=0}^{\infty} a_n z^n$$

has no zeros in $\mathbb{D}$. Denote by c_n $(n \in \mathbb{N})$ the coefficients of the holomorphic function

$$\frac{1}{k} : \mathbb{D} \to \mathbb{C}$$

Additionally, suppose that almost all the coefficients c_n have the same sign.

Example 2.5.1. We have already seen the following examples in 2.4.30:

(a) For $s > 0$ the spaces $A_s^2(\mathbb{B}_d)$,

(b) the complete Nevanlinna-Pick spaces as they occur in Example 2.3.5.

Let $T = (T_1, \dots T_d) \in B(H)^d$ be a commuting operator tuple on a Hilbert space H. For $N \in \mathbb{N}$ let

$$\left(\frac{1}{K}\right)_N (z,w) = \sum_{n=0}^{N} c_n \langle z,w\rangle^n = \sum_{|\alpha|\leq N} c_{|\alpha|}\gamma_\alpha z^\alpha \overline{w}^\alpha \quad (z,w \in \mathbb{B}_d)$$

be the N-th partial sum of $1/K$. We define

$$\left(\frac{1}{K}\right)_N (T) = \sum_{n=0}^{N} c_n \sigma_T^n(\mathrm{id}_H) = \sum_{|\alpha|\leq N} c_{|\alpha|}\gamma_\alpha T^\alpha (T^\alpha)^*$$

for all $N \in \mathbb{N}$, where

$$\sigma_T : B(H) \to B(H),\ X \mapsto \sum_{l=0}^{d} T_l X T_l^*.$$

Definition 2.5.2. The commuting tuple $T \in B(H)^d$ is called K-contraction if

$$\frac{1}{K}(T) = \mathrm{SOT} - \lim_{N\to\infty} \left(\frac{1}{K}\right)_N (T)$$

exists and defines a positive operator.

Notation 2.5.3. Motivated by the definition of the defect operator and the defect space of a contraction, we call

$$C = \left(\frac{1}{K}(T)\right)^{\frac{1}{2}}$$

the defect operator and

$$\mathscr{D} = \mathscr{D}_{T^*} = \overline{\mathrm{Im}\, C}$$

the defect space of a K-contraction.

Remark 2.5.4. Due to [MV93, Lemma 2] a commuting tuple $T = (T_1, \dots, T_d) \in B(H)^d$ is an m-hypercontraction if and only if T is a $K^{(1)}$ and a $K^{(m)}$-contraction, where

$$K^{(l)} : \mathbb{B}_d \times \mathbb{B}_d \to \mathbb{C},\ K^{(l)}(z,w) = \frac{1}{(1-\langle z,w\rangle)^l} \quad (l > 0).$$

There are sufficient conditions for the weighted shift operator tuple

$$M_z = (M_{z_1}, \dots, M_{z_d}) \in B(\mathscr{H})^d$$

to be a K-contraction. The following proposition, which originates from [AEM02, Proposition 13] and can also be found in [Sch18, Proposition 2.9], reduces the problem to the bare existence of $\frac{1}{K}(M_z)$.

Proposition 2.5.5. *Suppose that*

$$\frac{1}{K}(M_z) = \text{SOT} - \lim_{N\to\infty} \left(\frac{1}{K}\right)_N (M_z)$$

exists. Then $M_z \in B(\mathscr{H})^d$ is a K-contraction and

$$\frac{1}{K}(M_z) = P_{\mathbb{C}}$$

coincides with projection

$$P_{\mathbb{C}} : \mathscr{H} \to \mathscr{H},\ f \mapsto f(0)$$

onto the constant functions.

Using the previous proposition, one obtains the following result due to Chen:

Theorem 2.5.6 (Chen)**.** *Suppose that there exists a natural number $p \in \mathbb{N}$ such that $c_n \geq 0$ for all $n \geq p$ or $c_n \leq 0$ for all $n \geq p$ holds. Then M_z is a K-contraction with*

$$\frac{1}{K}(M_z) = P_{\mathbb{C}}$$

and $\sum_{n=0}^{\infty} c_n$ converges absolutely.

See [Che12, Proposition 2.1 and Lemma 2.2] and [Sch18, Proposition 2.10].

Remark 2.5.7. We required at the beginning of the chapter that the c_n have almost the same sign. According to the previous theorem, in our context M_z is always a K-contraction with

$$\frac{1}{K}(M_z) = P_{\mathbb{C}}$$

and $\sum_{n=0}^{\infty} c_n$ converges absolutely.

The following theorem about the Taylor spectrum of K-contractions can be found in [CH18, Lemma 5.3].

Theorem 2.5.8. *Let $K : \mathbb{B}_d \times \mathbb{B}_d \to \mathbb{C}$ be a unitarily invariant complete Nevanlinna-Pick kernel with radius of convergence 1. If $T = (T_1, \ldots, T_d)$ is a K-contraction, then*

$$\sigma(T) \subset \sigma_{joint}(T) \subset \overline{\mathbb{B}_d},$$

where $\sigma(T) \subset \mathbb{C}^d$ is the Taylor spectrum.

A contraction $T \in B(H)$ is called a pure or of class $C_{\cdot 0}$ if and only if $(T^*)^N \to 0$ for $N \to \infty$ in the strong operator topology. If

$$K : \mathbb{D} \times \mathbb{D} \to \mathbb{C},\ K(z,w) = \frac{1}{1 - z\overline{w}}$$

is the Szegő kernel, the pureness condition $\text{SOT} - \lim_{N\to\infty}(T^*)^N \to 0$ is equivalent to

$$\text{SOT} - \lim_{N\to\infty} \left(\text{id}_H - \sum_{n=0}^{N} a_n \sigma_T^n \left(\frac{1}{K}(T)\right)\right) = \text{SOT} - \lim_{N\to\infty} T^N (T^*)^N = 0.$$

Definition 2.5.9. We call a K-contraction $T \in B(H)^d$ pure if

$$\text{SOT} - \lim_{N\to\infty} \left(\mathrm{id}_H - \sum_{n=0}^{N} a_n \sigma_T^n \left(\frac{1}{K}(T) \right) \right) = 0.$$

Remark 2.5.10. Suppose that the operator tuple $T \in B(H)^d$ is an m-hypercontraction. Using the Taylor functional calculus, one can show that the previous definition of pureness for $T \in B(H)^d$ as a $K^{(m)}$-contraction, coincides with the classical definition of pureness

$$\text{SOT} - \lim_{N\to\infty} \sigma_T^N(\mathrm{id}_H) = 0$$

for row contractions (see [Sch18, Theorem 3.51]).

Example 2.5.11. Suppose that $M_z = (M_{z_1}, \dots, M_{z_d}) \in B(\mathscr{H})^d$ is a K-contraction. Similar to Proposition 2.12 in [Sch18] one checks, that M_z is pure.

Proof. Since M_z is a K-contraction, it follows with Proposition 2.5.5 that

$$\frac{1}{K}(M_z) = P_{\mathbb{C}}.$$

Let $\mathscr{H} = \bigoplus_{n\in\mathbb{N}} \mathbb{H}_n$ the orthogonal decomposition of $\mathscr{H}$ into the space of homogeneous polynomials of degree n. Since $P_{\mathbb{C}} = P_{\mathbb{H}_0}$ is the orthogonal projection onto the constant functions, a straightforward computation shows that

$$\left(\mathrm{id}_{\mathscr{H}} - \sum_{n=0}^{N} a_n \sigma_{M_z}^n (P_{\mathbb{C}}) \right) k_w = \left(\mathrm{id}_{\mathscr{H}} - \sum_{n=0}^{N} P_{\mathbb{H}_n} \right) k_w$$

for all $N \in \mathbb{N}$ and $w \in \mathbb{B}_d$. Since linear combinations of the kernel functions

$$k_w : \mathbb{B}_d \to \mathbb{C},\ k_w(z) = K(z,w) \quad (w \in \mathbb{B}_d)$$

are dense in $\mathscr{H}$, one obtains that M_z is pure. □

Lemma 2.5.12. *Let $T \in B(H)^d$ be a commuting tuple of operators on a Hilbert space H and suppose that there exists a Hilbert space $\mathscr{E}$ an isometry $V : H \to \mathscr{H}(\mathscr{E})$ such that*

$$VT_l^* = M_{z_l}^* V$$

for all $l = 1, \dots, d$. If $P_{\mathscr{E}} : \mathscr{H}(\mathscr{E}) \to \mathscr{H}(\mathscr{E})$ is the orthogonal projection onto the constant functions, we use the notation

$$C_V = P_{\mathscr{E}} V.$$

Then, we have

(a) $Vh = \sum_{\alpha\in\mathbb{N}^d} a_{|\alpha|} \gamma_\alpha C_V (T^\alpha)^* h z^\alpha$ *for all* $h \in H$,

(b) $V^*(f) = \sum_{\alpha\in\mathbb{N}^d} T^\alpha C_V^* f_\alpha$ *for all* $f = \sum_{\alpha\in\mathbb{N}^d} f_\alpha z^\alpha \in \mathscr{H}(\mathscr{E})$,

(c) *T is a pure K-contraction with $C_V^* C_V = \frac{1}{K}(T) \geq 0$.*

Proof. Let $\alpha \in \mathbb{N}^d$ and $x \in \mathscr{E}$. Using the intertwining property

$$VT_l^* = M_{z_l}^* V,$$

we deduce that

$$V^*(x \otimes z^\alpha) = V^* M_z^\alpha P_{\mathscr{E}} x = T^\alpha C_V^* x.$$

Since the adjoint $V^* \colon \mathscr{H}(\mathscr{E}) \to H$ is continuous, it follows that

$$V^*(f) = \sum_{\alpha \in \mathbb{N}^d} T^\alpha C_V^* f_\alpha.$$

for all $f = \sum_{\alpha \in \mathbb{N}^d} f_\alpha z^\alpha \in \mathscr{H}(\mathscr{E})$. If $h \in H$, then the previous computation shows that

$$\begin{aligned} \langle Vh, f_\alpha z^\alpha \rangle_{\mathscr{H}(\mathscr{E})} &= \langle h, T^\alpha C_V^* f_\alpha \rangle_H \\ &= \langle C_V (T^\alpha)^* h, f_\alpha \rangle_{\mathscr{E}} \\ &= \left\langle a_{|\alpha|} \gamma_\alpha C_V (T^\alpha)^* z^\alpha h, f_\alpha z^\alpha \right\rangle_{\mathscr{H}(\mathscr{E})}. \end{aligned}$$

Thus, we obtain

$$Vh = \sum_{\alpha \in \mathbb{N}^d} a_{|\alpha|} \gamma_\alpha C_V (T^\alpha)^* h z^\alpha.$$

Since $\frac{1}{K}(M_z) = P_{\mathscr{E}}$ (see Proposition 2.5.5), we conclude that

$$C_V^* C_V = V^* P_{\mathscr{E}} V = V^* \frac{1}{K}(M_z) V = \frac{1}{K}(T) \geq 0$$

and T is a K-contraction. Furthermore, pureness of M_z yields that

$$\begin{aligned} &\mathrm{SOT} - \lim_{N \to \infty} \left(\mathrm{id}_H - \sum_{n=0}^{N} a_n \sigma_T^n \left(\frac{1}{K}(T) \right) \right) \\ &= \mathrm{SOT} - \lim_{N \to \infty} V^* \left(\mathrm{id}_{\mathscr{H}(\mathscr{E})} - \sum_{n=0}^{N} a_n \sigma_{M_z}^n \left(\frac{1}{K}(M_z) \right) \right) V = 0 \end{aligned}$$

and thus, T is pure. □

Lemma 2.5.13. *Let $T \in B(H)^d$ and $S \in B(\tilde{H})^d$ be commuting tuples on Hilbert spaces H and $\tilde{H}$, respectively, and suppose that there exists an isometry $V : H \to \tilde{H}$ such that*

$$VT_l^* = S_l^* V$$

for all $l = 1, \ldots, d$. If S is a (pure) K-contraction, then T is a (pure) K-contraction.

Proof. The assertion follows similarly to the proof of the previous theorem using the relations

$$V^*V = \mathrm{id}_H \text{ and } VT_l^* = S_l^* V$$

for all $l = 1, \ldots, d$. (For more details, see also Lemma 2.13 in [Sch18].) □

It is possible to define a minimal dilation map j for pure K-contractions, which intertwines the tuples M_z^* and T^* componentwise. The following construction already appears in [AE03, Theorem 1.3] and has its roots in [AEM02].

Theorem 2.5.14. *Let $T \in B(H)^d$ be a pure K-contraction. Then*

$$j\colon H \to \mathscr{H}(\mathscr{D}),\ j(h) = \sum_{\alpha \in \mathbb{N}^d} a_{|\alpha|} \gamma_\alpha C (T^\alpha)^* z^\alpha$$

is a well-defined isometry such that

$$jT_l^* = M_{z_l}^* j \quad (l = 1, \ldots, d).$$

In fact, this follows because

$$j^* j = \mathrm{SOT} - \lim_{N \to \infty} \left(\sum_{n=0}^{N} a_n \sigma_T^n \left(\frac{1}{K}(T) \right) \right) = \mathrm{id}_H .$$

For a detailed proof, see [Sch18, Proposition 2.6].

Remark 2.5.15. Suppose that

$$K : \mathbb{B}_d \times \mathbb{B}_d \to \mathbb{C},\ K(z,w) = \sum_{n=0}^{\infty} a_n \langle z, w \rangle^n$$

is a complete unitarily invariant Nevanlinna-Pick kernel such that M_z is bounded and let $T \in B(H)^d$ be a K-contraction. In this case, one computes that

$$0 \leq \sum_{n=0}^{N} a_n \sigma_T^n \left(\frac{1}{K}(T) \right) \leq \mathrm{id}_H$$

for all $N \in \mathbb{N}$. Hence,

$$j\colon H \to \mathscr{H}(\mathscr{D}),\ j(h) = \sum_{\alpha \in \mathbb{N}^d} a_{|\alpha|} \gamma_\alpha C (T^\alpha)^* z^\alpha$$

is a well-defined contraction such that

$$jT_l^* = M_{z_l}^* j \quad (l = 1, \ldots, d).$$

(see [Sch18, Corollary 2.4 and Proposition 2.6] and [BJ23a, Lemma 4.1 and Corollary 4.2].)

Lemma 2.5.12 and Theorem 2.5.14 yield the following characterization, which can also be found in [Sch18, Theorem 2.15]:

Theorem 2.5.16. *Let $T \in B(H)^d$ be a commuting tuple. The following statements are equivalent:*

(a) T is a pure K-contraction,

(b) T is a K-contraction and

$$j\colon H \to \mathscr{H}(\mathscr{D}),\ j(h) = \sum_{\alpha\in\mathbb{N}^d} a_{|\alpha|}\gamma_\alpha C(T^\alpha)^* z^\alpha$$

is a well-defined isometry such that

$$jT_l^* = M_{z_l}^* j \quad (l = 1,\dots,d),$$

(c) there exists a Hilbert space $\mathscr{E}$ and an isometry $V : H \to \mathscr{H}(\mathscr{E})$ such that

$$VT_l^* = M_{z_l}^* V$$

for all $l = 1,\dots,d$.

Using Theorem 2.4.24 and Lemma 2.4.22, we obtain the following lemma for the multivariable spectrum of pure K-contractions:

Lemma 2.5.17. *Let $\mathscr{H}$ be regular and let*

$$T = (T_1,\dots,T_d) \in B(H)^d$$

be a pure K-contraction. Then

$$\sigma(T) \subset \sigma_{joint}(T) \subset \overline{\mathbb{B}_d}.$$

Remark 2.5.18. Let $T = (T_1,\dots,T_d) \in B(H)^d$ be a K-contraction. Suppose that we are in one of the following cases:

(a) $\mathscr{H}$ is a complete unitarily invariant Nevanlinna-Pick space, where M_z is bounded, and the kernel function has radius of convergence 1.

(b) $\mathscr{H}$ is regular and T is pure.

Due to Lemma 2.5.17, Theorem 2.5.8 and Remark 2.4.23, we obtain that the function

$$k_T : \mathbb{B}_d \to B(H),\ z \mapsto \sum_{\alpha\in\mathbb{N}^d} \gamma_\alpha a_{|\alpha|}(T^\alpha)^* z^\alpha$$

is well defined and analytic and the intertwining contraction

$$j\colon H \to \mathscr{H}(\mathscr{D}),\ j(h) = \sum_{\alpha\in\mathbb{N}^d} a_{|\alpha|}\gamma_\alpha C(T^\alpha)^* z^\alpha$$

defined in Theorem 2.5.14 can be written as

$$j(h)(z) = Ck_T(z)h \quad (z \in \mathbb{B}_d, h \in H)$$

(cf. transfer realization of the characteristic function for pure contractions in the introductory part of Chapter 3).

3. K-inner functions and Wandering subspaces

The contents of this chapter are a joint work with Jörg Eschmeier and appear in [ET21].

We establish a realization formula for K- inner functions. The transfer realization is similar to the one of characteristic functions introduced by Sz.-Nagy and Foiaș.

To be more specific, let us recall some facts about the characteristic function of a pure contraction $T \in B(H)$ with defect operators $D_T = (1 - T^*T)^{1/2}$ and $D_{T^*} = (1 - TT^*)^{1/2}$ and defect spaces $\tilde{\mathscr{D}} = \overline{\operatorname{Im} D_T}$ and $\mathscr{D} = \overline{\operatorname{Im} D_{T^*}}$. The characteristic function $\theta_T : \mathbb{D} \to B(\tilde{\mathscr{D}}, \mathscr{D})$ is defined by

$$\theta_T(z) = -T + D_{T^*}(1 - zT^*)^{-1} z D_T \quad (z \in \mathbb{D}).$$

On the other hand if

$$M_z : H^2(\mathbb{D}) \to H^2(\mathbb{D})$$

is the unilateral shift, the map $j \colon H \to \mathscr{H}(\mathscr{D})$ defined by

$$j(h)(z) = D_{T^*}(1 - zT^*)^{-1} h$$

for $h \in H$ and $z \in \mathbb{D}$ is a well-defined isometry with intertwining property

$$jT^* = M_z^* j$$

(see Theorem 2.5.14 and Remark 2.5.18).

Therefore one computes that $\theta_T : \mathbb{D} \to B(\tilde{\mathscr{D}}, \mathscr{D})$ is a contractive multiplier from $H^2(\tilde{\mathscr{D}})$ to $H^2(\mathscr{D})$ with

$$M_{\theta_T} M_{\theta_T}^* + jj^* = \mathrm{id}_{H^2(\mathscr{D})}$$

and

$$\theta_T x = -Tx + M_z j(D_T x)$$

for all $x \in \mathscr{D}$. Since j intertwines M_z^* and T^*, it also follows that

$$M = M_{\theta_T}\left(\operatorname{Ker}(M_{\theta_T})^{\perp}\right) = (\operatorname{Im} j)^{\perp}.$$

is invariant for M_z and that T is unitarily equivalent to the compression of M_z to the space $H^2(\mathscr{D}) \ominus M$. Besides, we have that

$$\|\theta x\|^2_{H^2(\mathscr{D})} = \|x\|^2_{\mathscr{D}}$$

for all $x \in \mathscr{D}$ and
$$W(M) = M \ominus M_z M = \theta_T(\tilde{\mathscr{D}} \cap \mathrm{Ker}(\theta_T)^{\perp}).$$
The closed subspace $W(M)$ satisfies

$$W(M) \perp M_z^n W(M)$$

for all $n \geq 1$.

Let us take a closer look at spaces with this property. If $S \in B(H)$ is a bounded linear operator, we call closed subspaces $\mathscr{W} \subset H$ with the property

$$\mathscr{W} \perp S^n \mathscr{W} \quad (n \geq 1)$$

wandering subspaces for S. If S is an isometry, this definition coincides with the more common definition of wandering subspaces

$$S^m \mathscr{W} \perp S^n \mathscr{W} \quad (n, m \in \mathbb{N} \text{ with } m \neq n),$$

as it occurs in the Wold decomposition theorem (see [SNFBK10, Theorem 1.1, Section 1, Chapter I]). Wandering subspaces usually arise in the following way: Given any S-invariant subspace M, the space

$$W_S(M) = M \ominus SM$$

is a wandering subspace for S. Now the idea is if

$$M = \bigvee_{n \geq 0} (S^n h;\, h \in W_S(M))$$

all the information about the invariant subspace M is contained in the wandering subspace $W_S(M)$. Conversely, observe that, if $\mathscr{W}$ is a wandering subspace for S and

$$M = \bigvee_{n \geq 0} (S^n h;\, h \in \mathscr{W}),$$

then obviously
$$\mathscr{W} = W_S(M).$$
Recall, in the particular case, when S is an isometry with S-invariant subspace M, it follows by the Wold decomposition theorem that

$$M = \bigvee_{n \geq 0} (S^n h;\, h \in W_S(M)) \oplus \bigcap_{n=0}^{\infty} S^n M$$

and if $S = M_z : H^2(\mathbb{D}) \to H^2(\mathbb{D})$ is the unilateral shift, then

$$M = \bigvee_{n \geq 0} (S^n h;\, h \in W_S(M)).$$

By a well-known theorem of Beurling every non-zero M_z-invariant subspace M has the form $M_\theta H^2(\mathbb{D})$, for function a $\theta \in H^2(\mathbb{D})$ with unit norm in the wandering subspace $W_S(M)$. This is equivalent to the fact that θ is inner. We have seen that the characteristic function $\theta_T : \mathbb{D} \to B(\mathscr{D}_T, \mathscr{D})$ of a pure contraction $T \in B(H)$ has the properties of an inner function.

Furthermore, to study invariant subspaces M for the Bergman shift

$$M_z : L^2_a(\mathbb{D}) \to L^2_a(\mathbb{D}),\ f \mapsto zf$$

on the Bergman space $L^2_a(\mathbb{D})$, Hedenmalm uses elements in the wandering subspaces $W(M) = M \ominus M_z M$ with norm $\|\theta\|_{L^2_a(\mathbb{D})} = 1$. If A is a Bergman space zero set with corresponding invariant subspace

$$M_A = \{f \in L^2_a(\mathbb{D});\ f|_A \equiv 0\},$$

the Bergman-inner functions G_A in the wandering subspaces $W(M_A) = M_A \ominus M_z M_A$ are canonical zero-divisors in Bergman factorization theory. Due to a theorem of Aleman, Richter, and Sundberg (cf. [ARS96]) every M_z-invariant space M in the Bergman space with corresponding wandering subspace $W(M) = M \ominus M_z M$ is characterized by

$$M = \bigvee_{n \geq 0} \left(M_z^n h;\ h \in W(M)\right).$$

See also the introduction of [Shi01] for a motivation for wandering subspaces.

Let us come back to the characteristic function θ_T. Using that $TD_T = D_{T^*}T$ one computes that the corresponding transfer matrix

$$\left(\begin{array}{c|c} T^* & D_T \\ \hline D_{T^*} & -T \end{array}\right)$$

of

$$\theta_T(z) = -T + D_{T^*}(1 - zT^*)^{-1} z D_T \quad (z \in \mathbb{D}).$$

is unitary. One can show that each function θ in the unit ball of $H^\infty(\mathbb{D})$ and hence any inner function has a transfer realization

$$\theta(z) = D + C(1 - zA)^{-1} zB \quad (z \in \mathbb{D})$$

similar to the characteristic function, where

$$\left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right) : H \oplus \mathbb{C} \to H \oplus \mathbb{C}$$

a unitary on a Hilbert space $H \oplus \mathbb{C}$.

More general adapted transfer realizations for functions in the unit ball of the multiplier algebra of complete Nevanlinna-Pick spaces play an important role. They can be used to describe the Pick interpolation problem in terms of a positive semidefinite matrix (see [AM02, Theorem 8.33]). They are also useful for the proof of Leech's Theorem (see [AM02, Theorem 8.57]) for Pick spaces, which we will use in Chapter 4. Transfer functions first appeared in system or control theory and are widely used in electronic engineering.

For this chapter we always suppose that we are in the following setting, which covers also the Hardy space $H^2(\mathbb{D})$ and the Bergman space $L_a^2(\mathbb{D})$ on the disk:

The function

$$k\colon \mathbb{D} \to \mathbb{C},\ k(z) = \sum_{n=0}^{\infty} a_n z^n$$

is analytic without zeros, such that $a_0 = 1$, $a_n > 0$ for all $n \in \mathbb{N}_{>0}$ and

$$\sup_{n\in\mathbb{N}} \frac{a_n}{a_{n+1}} < \infty,\ \inf_{n\in\mathbb{N}} \frac{a_n}{a_{n+1}} > 0.$$

We denote by c_n $(n \in \mathbb{N})$ the coefficients of $\frac{1}{k}$. Additionally, suppose that almost all the coefficients c_n have the same sign. The reproducing kernel

$$K\colon \mathbb{B}_d \times \mathbb{B}_d \to \mathbb{C},\ K(z,w) = k(\langle z,w\rangle)$$

defines an unitarily invariant reproducing kernel Hilbert space $\mathscr{H}$ such that the row operator $M_z\colon \mathscr{H}^d \to \mathscr{H}$ is bounded and has closed range (see Chapter 2).

Motivated by the observations of Hedenmalm for wandering and invariant subspaces of the Bergman shift and by a survey paper of Ball and Cohen (cf. [BC91]), Olofsson introduces the concept of Bergman-inner functions for the spaces $A_m^2(\mathbb{D})$ with reproducing kernels

$$K_m : \mathbb{D} \times \mathbb{D} \to \mathbb{C},\ K_m(z,w) = \frac{1}{(1-\langle z,w\rangle)^m} \quad (m \in \mathbb{N}, m > 0)$$

(see [Olo06],[Olo07]). (Recall, K_2 is the kernel of the Bergman space $L_a^2(\mathbb{D})$). He studies corresponding transfer realizations similar to the one for characteristic functions. We generalize a former paper of Eschmeier (cf. [Esc18a]), which is the multivariable generalization of the papers [Olo06] and [Olo07] by Olofsson for the spaces $A_m^2(\mathbb{B}_d)$ with reproducing kernels of the form

$$K_m : \mathbb{B}_d \times \mathbb{B}_d \to \mathbb{C},\ K_m(z,w) = \frac{1}{(1-\langle z,w\rangle)^m} \quad (m \in \mathbb{N}, m > 0).$$

Observe therefore that the idea of Bergman-inner functions can be reformulated for all unitarily invariant kernels K as above by the notion of so called K-inner functions (cf. [BEKS17]). A K-inner function is an operator-valued analytic function
$W\colon \mathbb{B}_d \to B(\tilde{\mathscr{D}}, \mathscr{D})$ with $Wx \in \mathscr{H}(\mathscr{D})$, $\|Wx\|_{\mathscr{H}(\tilde{\mathscr{D}})} = \|x\|_{\mathscr{D}}$ for all $x \in \tilde{\mathscr{D}}$ and

$$W(\tilde{\mathscr{D}}) \perp M_z^{\alpha}\left(W(\tilde{\mathscr{D}})\right) \text{ for all } \alpha \in \mathbb{N}^d \setminus \{0\}.$$

In the case that $M_z \in B(\mathscr{H})^d$ is a row contraction, one can show that each K-inner function $W : \mathbb{B}_d \to B(\tilde{\mathscr{D}}, \mathscr{D})$ defines a contractive multiplier

$$M_W : H^2_d(\tilde{\mathscr{D}}) \to \mathscr{H}(\mathscr{D}),\ f \to Wf$$

(see [BEKS17, Theorem 6.2]). We also want to highlight that in recent papers (cf. [BJ23a], [BJ23b]), Bhattacharyy and Jindal work on characteristic functions for K-contractions, where K has a complete Nevanlinna-Pick factor. Restrictions of such characteristic functions are K-inner.

For our transfer realization, we us the concept of K-contractions that have already seen in Section 2.5. For convenience, we recall the important things that will be used in this chapter. If $T = (T_1, \dots, T_d) \in B(H)^d$ is a tuple of commuting operators let

$$\sigma_T : B(H) \to B(H),\ X \mapsto \sum_{l=0}^{d} T_l X T_l^*.$$

As in Definition 2.5.2 we call T a K-contraction if

$$\left(\frac{1}{K}\right)(T) = \sum_{n=0}^{\infty} c_n \sigma_T^n(\mathrm{id}_H)$$

converges in the strong operator topology and defines a positive operator. We use the notations

$$C = \left(\frac{1}{K}(T)\right)^{\frac{1}{2}}$$

and

$$\mathscr{D} = \mathscr{D}_{T^*} = \overline{\operatorname{Im} C}.$$

Further, we use the following characterization of pure K-contractions (see also Theorem 2.5.16):

Theorem. *Let $T \in B(H)^d$ be a commuting tuple. The following statements are equivalent:*

(a) T is a pure K-contraction,

(b) there exists a Hilbert space $\mathscr{E}$ and an isometry $V : H \to \mathscr{H}(\mathscr{E})$ such that

$$VT_l^* = M_{z_l}^* V$$

for all $l = 1, \dots, d$.

(c) T is a K-contraction and

$$j\colon H \to \mathscr{H}(\mathscr{D}),\ j(h) = \sum_{\alpha \in \mathbb{N}^d} a_{|\alpha|} \gamma_\alpha C (T^\alpha)^* z^\alpha$$

is a well-defined isometry such that

$$jT_l^* = M_{z_l}^* j \quad (l = 1, \dots, d).$$

If T is a pure K-contraction the adjoint $j^*\colon \mathscr{H}(\mathscr{D}) \to H$ has the representation

$$j^*\left(\sum_{\alpha\in\mathbb{N}^d} f_\alpha z^\alpha\right) = \sum_{\alpha\in\mathbb{N}^d} T^\alpha C f_\alpha.$$

(see Lemma 2.5.12).

First, we will characterize wandering subspaces associated with the restriction of M_z to the invariant subspace $M = (\operatorname{Im} j)^\perp$ by K-inner functions. For this purpose we will use the diagonal operators

$$\delta\colon \mathscr{H}(\mathscr{D}) \to \mathscr{H}(\mathscr{D}),\ \delta\left(\sum_{n=0}^{\infty}\sum_{|\alpha|=n} f_\alpha z^\alpha\right) = f_0 + \sum_{n=1}^{\infty}\frac{a_n}{a_{n-1}}\sum_{|\alpha|=n} f_\alpha z^\alpha$$

and

$$\Delta\colon \mathscr{H}(\mathscr{D}) \to \mathscr{H}(\mathscr{D}),\ \Delta\left(\sum_{n=0}^{\infty}\sum_{|\alpha|=n} f_\alpha z^\alpha\right) = \sum_{n=0}^{\infty}\frac{a_{n+1}}{a_n}\sum_{|\alpha|=n} f_\alpha z^\alpha$$

that we have introduced in Subsection 2.4.3. An elementary calculation shows that

$$\delta M_{z_l} = M_{z_l}\Delta$$

for $l = 1,\ldots,d$. Our hypotheses on the sequence $\left(\frac{a_n}{a_{n+1}}\right)$ imply that δ and Δ are invertible positive operators on $\mathscr{H}(\mathscr{D})$.

Besides, in [Lan19, Theorem 2.2.2], it is shown that the limit

$$\mathrm{SOT} - \lim_{N\to\infty}\sum_{n=0}^{N} -c_{n+1}\sigma_{M_z}^n(\mathrm{id}_{\mathscr{H}(\mathscr{D})})$$

exists, if almost all of the coefficients c_n have the same sign. Thus, by [Lan19, Theorem 2.2.6], we also have

$$\Delta = \mathrm{SOT} - \lim_{N\to\infty}\sum_{n=0}^{N}(-c_{n+1})\sigma_{M_z}^n(\mathrm{id}_{\mathscr{H}(\mathscr{D})}).$$

We define the operator Δ_T by

$$\Delta_T = j^*\Delta j.$$

Because Δ is invertible, we will see later in this chapter that the operator

$$\Delta_T = \mathrm{SOT} - \lim_{N\to\infty}\sum_{n=0}^{N} -c_{n+1}\sigma_T^n(\mathrm{id}_H).$$

is also invertible and that

$$(x,y) = \langle \Delta_T x, y\rangle \quad (x,y\in H)$$

defines a scalar product on H. We write $\tilde{H}$ for H equipped with the norm $\|\cdot\|_T$ and define

$$I_T : H \to \tilde{H},\ x \mapsto x.$$

One checks that $\tilde{T} = (\tilde{T}_1, \ldots, \tilde{T}_d) : \tilde{H}^d \to H$ is a row contraction. If $C \in B(H, \mathscr{E})$ is any operator with $C^*C = \frac{1}{K}(T)$, then

$$j_C : H \to \mathscr{H}(\mathscr{E}),\ j_C(x) = \sum_{\alpha \in \mathbb{N}^d} a_{|\alpha|} \gamma_\alpha (C(T^\alpha)^* x) z^\alpha$$

is a well-defined isometry such that j_C intertwines the tuples $T^* = (T_1^*, \ldots, T_d^*) \in B(H)^d$ and $M_z^* = (M_{z_1}^*, \ldots, M_{z_d}^*) \in B(\mathscr{H}(\mathscr{E}))^d$ componentwise. Suppose now that $\mathscr{H}$ is regular, that is $\lim_{n\to\infty} \frac{a_n}{a_{n+1}} = 1$. We shall consider bounded linear operators $C \in B(H, \mathscr{E})$, $D \in B(\mathscr{E}_*, \mathscr{E})$ and $B \in B(\mathscr{E}_*, H^d)$ such that

$$\begin{aligned}
&\text{(K1) } C^*C = \frac{1}{K}(T),\\
&\text{(K2) } D^*C + B^*(\oplus \Delta_T) T^* = 0,\\
&\text{(K3) } D^*D + B^*(\oplus \Delta_T) B = \mathrm{id}_{\tilde{\mathscr{D}}},\\
&\text{(K4) } \mathrm{Im}((\oplus j_C) B) \subset M_z^* \mathscr{H}(\mathscr{E}).
\end{aligned}$$

For our transfer realization we use the operator-valued function

$$F_T : \mathbb{B}_d \to B(H),\ F_T(z) = \sum_{n=0}^{\infty} a_{n+1} \left(\sum_{|\alpha|=n} \gamma_\alpha (T^\alpha)^* z^\alpha \right)$$

as well as the row operators

$$Z(w) : H^d \to H, (h_1, \ldots h_d) \to \sum_{l=1}^{d} w_l h_l \quad (w \in \mathbb{B}_d).$$

Finally, we show that each K-inner function $W : \mathbb{B}_d \to B(\mathscr{E}_*, \mathscr{E})$ admits a realization as a transfer function of the form

$$W(z) = D + CF_T(z) Z(z) B.$$

Conversely, each function that admits such a realization defines a K-inner function.

3.1. Wandering subspaces

Let $T = (T_1, \ldots, T_d) \in B(H)^d$ be a pure K-contraction. Since the isometry

$$j : H \to \mathscr{H}(\mathscr{D}),\ j(h) = \sum_{\alpha \in \mathbb{N}^d} a_{|\alpha|} \gamma_\alpha C (T^\alpha)^* z^\alpha$$

intertwines $T^* = (T_1^*, \dots, T_d^*)$ and $M_z^* = (M_{z_1}^*, \dots, M_{z_d}^*)$ componentwise, the space

$$M = \mathscr{H}(\mathscr{D}) \ominus \operatorname{Im} j = (\operatorname{Im} j)^\perp = \operatorname{Ker} j^*$$

is invariant for $M_z = (M_{z_1}, \dots, M_{z_d}) \in B(\mathscr{H}(\mathscr{D}))^d$. In the following, we show that the wandering subspace of M_z restricted to M can be described in terms of a suitable K-inner function.

We call a closed subspace $\mathscr{W} \subset H$ a wandering subspace for a commuting tuple $S = (S_1, \dots, S_d) \in B(H)^d$, if

$$\mathscr{W} \perp S^\alpha \mathscr{W} \quad (\alpha \in \mathbb{N}^d \setminus \{0\}).$$

The space $\mathscr{W}$ is called a generating wandering subspace for S, if in addition

$$H = \bigvee \left(S^\alpha \mathscr{W};\, \alpha \in \mathbb{N}^d \right).$$

For each closed S-invariant subspace $L \subset H$, the space

$$W_S(L) = L \ominus \left(\sum_{l=1}^d S_l L \right) = \bigcap_{l=1}^d (L \ominus S_l L)$$

is a wandering subspace for S. The space $W_S(L)$ is usually called the wandering subspace associated with S on L. If $\mathscr{W}$ is a generating wandering subspace for S, an elementary argument shows that necessarily $\mathscr{W} = W_S(H)$.

In the following, we write

$$W(M) = M \ominus \left(\sum_{l=1}^d M_{z_l} M \right)$$

for the wandering subspace associated with the restriction of M_z to the invariant subspace $M = (\operatorname{Im} j)^\perp$.

The Cauchy dual

$$M_z' = \delta M_z \in B(\mathscr{H}(\mathscr{D})^d, \mathscr{H}(\mathscr{D}))$$

extends the operator

$$M_z \left(M_z^* M_z \right)^{-1} :\, \operatorname{Im} M_z^* \to \mathscr{H}(\mathscr{D})$$

(see Lemma 2.4.32).

The proof of Lemma 2.4.32 shows that the orthogonal projection

$$P_{\operatorname{Im} M_z} : \mathscr{H}(\mathscr{D}) \to \mathscr{H}(\mathscr{D})$$

onto $\operatorname{Im} M_z$ acts as

$$P_{\operatorname{Im} M_z} = M_z (M_z^* M_z)^{-1} M_z^* = \delta(M_z M_z^*) = P_{\mathscr{D}^\perp}, \tag{3.1}$$

where $\mathscr{D} \subset \mathscr{H}(\mathscr{D})$ is regarded as the closed subspace consisting of all constant functions.

As in the case of m-hypercontractions (cf. [Esc18a]), we give a characterization of the wandering subspace $W(M)$ of $M = (\operatorname{Im} j)^{\perp}$. Analogously to [Esc18a], one can show the following properties of the wandering subspace:

Theorem 3.1.1. *A function $f = \sum_{\alpha \in \mathbb{N}^d} f_\alpha z^\alpha \in \mathscr{H}(\mathscr{D})$ is an element of the wandering subspace $W(M)$ if and only if*

$$f = f_0 + M_z'(jx_l)_{l=1}^d$$

for vectors $f_0 \in \mathscr{D}$, $x_1, \ldots, x_d \in H$ with $(jx_l)_{l=1}^d \in \operatorname{Im} M_z^*$ and

$$Cf_0 + T(\Delta_T x_l)_{l=1}^d = 0.$$

In this case, $(jx_l)_{l=1}^d = M_z^* f$.

Proof. We follow the proof for m- hypercontractions, given in [Esc18a]. Since j is an isometry, the operator

$$\mathrm{id}_{\mathscr{H}(\mathscr{D})} - jj^* : \mathscr{H}(\mathscr{D}) \to \mathscr{H}(\mathscr{D})$$

is the orthogonal projection onto

$$M = (\operatorname{Im} j)^{\perp} = \operatorname{Ker} j^*$$

A function $f = \sum_{\alpha \in \mathbb{N}^d} f_\alpha z^\alpha \in \mathscr{H}(\mathscr{D})$ belongs to the wandering subspace

$$W(M) = \bigcap_{l=1}^{d} (M \ominus M_{z_l} M)$$

if and only if $j^* f = 0$ and

$$(\mathrm{id}_{\mathscr{H}(\mathscr{D})} - jj^*) M_{z_l}^* f = 0$$

for $l = 1, \ldots, d$. Using Equation (3.1), we obtain for $(x_l)_{l=1}^d \in H^d$ and $f = \sum_{\alpha \in \mathbb{N}^d} f_\alpha z^\alpha \in \mathscr{H}(\mathscr{D})$ with $(jx_l)_{l=1}^d = M_z^* f$, that

$$\begin{aligned} j^* f &= j^*(f_0 + \delta M_z M_z^* f) \\ &= Cf_0 + j^* M_z (\Delta j x_l)_{l=1}^d \\ &= Cf_0 + T(j^* \Delta j x_l)_{l=1}^d \\ &= Cf_0 + T(\Delta_T x_l)_{l=1}^d. \end{aligned}$$

Thus, if $f \in W(M)$, then $(x_l)_{l=1}^d = (j^* M_{z_l}^* f)_{l=1}^d$ defines a tuple in H^d with $(jx_l)_{l=1}^d = M_z^* f$ such that

$$Cf_0 + T(\Delta_T x_l)_{l=1}^d = j^* f = 0$$

and

$$f = f_0 + (f - f_0) = f_0 + M_z (M_z^* M_z)^{-1} M_z^* f = f_0 + M_z'(jx_l)_{l=1}^d.$$

Conversely, suppose that

$$f = f_0 + M_z'(jx_l)_{l=1}^d$$

with $f_0 \in \mathscr{D}$, $x_1, \ldots, x_d$ as in the assumption. Using Lemma 2.4.32, we find that

$$M_z^* f = M_z^* M_z (M_z^* M_z)^{-1} (jx_l)_{l=1}^d = (jx_l)_{l=1}^d.$$

Since j is an isometry, it follows that

$$jj^* M_{z_l}^* f = jx_l = M_{z_l}^* f$$

for $l = 1, \ldots, d$. Because

$$j^* f = Cf_0 + T(\Delta_T x_l)_{l=1}^d = 0,$$

we conclude that $f \in W(M)$. □

Lemma 3.1.2. *Let $T \in B(H)^d$ be a pure K-contraction and let*

$$f = f_0 + M_z'(jx_l)_{l=1}^d$$

be a representation of a function $f = \sum_{\alpha \in \mathbb{N}^d} f_\alpha z^\alpha \in W(M)$ as in Theorem 3.1.1. Then we have

$$\|f\|^2 = \|f_0\|^2 + \sum_{l=1}^d \langle \Delta_T x_l, x_l \rangle.$$

Proof. Because $\operatorname{Im} M_z$ is closed, it follows that

$$\operatorname{Im} M_z = \operatorname{Im}(M_z M_z^*)$$

(see Theorem A.1.4). Using Equation (3.1), we obtain

$$\operatorname{Im} M_z' = \operatorname{Im}(\delta M_z) = \operatorname{Im}\left(\delta M_z M_z^*\right) = \operatorname{Im} M_z = \mathscr{D}^\perp.$$

Since $(jx_l)_{l=1}^d = M_z^* f$ and $M_z(M_z^* M_z)^{-1} = M_z'|_{\operatorname{Im} M_z^*} = \delta M_z|_{\operatorname{Im} M_z^*}$, we conclude that

$$\begin{aligned}
\|f\|^2 - \|f_0\|^2 &= \|M_z'(jx_l)_{l=1}^d\|^2 \\
&= \|M_z(M_z^* M_z)^{-1}(jx_l)_{l=1}^d\|^2 \\
&= \langle (M_z^* M_z)^{-1} M_z^* f, (jx_l)_{l=1}^d \rangle \\
&= \langle (\oplus j^*) M_z^* \delta f, (x_l)_{l=1}^d \rangle \\
&= \langle (j^* \Delta j x_l)_{l=1}^d, (x_l)_{l=1}^d \rangle.
\end{aligned}$$

Because $\Delta_T = j^* \Delta j$, the assertion follows. □

As in [Esc18a], we modify the Hilbert space H using the operator Δ_T. Let $T \in B(H)^d$ be a pure K-contraction. Then $\Delta_T = j^* \Delta j$ is a positive operator. We obtain

$$\langle \Delta_T x, x \rangle = \|\Delta^{\frac{1}{2}} jx\|^2 \geq \|\Delta^{-\frac{1}{2}}\|^{-2} \|jx\|^2 = \|\Delta^{-1}\|^{-1} \|x\|^2$$

for all $x \in H$. For the last equation, we have used the C^*-identity and the fact that j is an isometry. Hence, $\Delta_T \in B(H)$ is invertible and

$$(x,y) = \langle \Delta_T x, y \rangle$$

defines a scalar product on H. Since j is an isometry, one computes that the induced norm $\|\cdot\|_T$ is equivalent to the original norm with

$$\|\Delta^{\frac{1}{2}}\|\|x\| \geq \|x\|_T \geq \|\Delta^{-\frac{1}{2}}\|^{-1}\|x\| \quad (x \in H).$$

We write $\tilde{H}$ for H equipped with the norm $\|\cdot\|_T$. Then

$$I_T : H \to \tilde{H},\ x \mapsto x,$$

is an invertible bounded operator such that

$$\langle I_T^* I_T x, y \rangle = \langle \Delta_T x, y \rangle \quad (x, y \in H).$$

Hence, $I_T^* I_T x = \Delta_T x$ for $x \in H$. Let $\tilde{T} = (\tilde{T}_1, \ldots, \tilde{T}_d) \colon \tilde{H}^d \to H$ be the row operator with components $\tilde{T}_l = T_l \circ I_T^* \in B(\tilde{H}, H)$. Then

$$\begin{aligned}\tilde{T}\tilde{T}^* &= \sum_{l=1}^{d} T_l (I_T^* I_T) T_l^* = \sigma_T(\Delta_T) = \sigma_T(j^* \Delta j) = j^* M_z (\oplus \Delta) M_z^* j \\ &= j^*(\delta M_z M_z^*) j = j^* P_{\mathscr{D}^\perp} j.\end{aligned}$$

Thus, $\tilde{T}$ is a contraction. As in [Olo06], we use the defect operators

$$\begin{aligned}D_{\tilde{T}} &= (\mathrm{id}_{\tilde{H}^d} - \tilde{T}^* \tilde{T})^{1/2} \in B(\tilde{H}^d), \\ D_{\tilde{T}^*} &= (\mathrm{id}_H - \tilde{T}\tilde{T}^*)^{1/2} = (j^* P_{\mathscr{D}} j)^{1/2} = C \in B(H).\end{aligned}$$

The identity $(j^* P_{\mathscr{D}} j)^{1/2} = C$ follows from the definition of j in Theorem 2.5.14 and the representation of j^* explained in Lemma 2.5.12. We write $\mathscr{D}_{\tilde{T}} = \overline{D_{\tilde{T}} \tilde{H}^d} \subset H^d$ and $\mathscr{D}_{\tilde{T}^*} = \overline{D_{\tilde{T}^*} H} = \mathscr{D}$ for the defect spaces of $\tilde{T}$. As in the one-dimensional case (cf. [SNFBK10, Chapter I, 3.1]) it is elementary to check that $\tilde{T} D_{\tilde{T}} = D_{\tilde{T}^*} \tilde{T}$ and that

$$U = \left(\begin{array}{c|c} \tilde{T} & D_{\tilde{T}^*} \\ \hline D_{\tilde{T}} & -\tilde{T}^* \end{array} \right) : \tilde{H}^d \oplus \mathscr{D}_{\tilde{T}^*} \to H \oplus \mathscr{D}_{\tilde{T}}$$

is a well-defined unitary operator.

We will now define a holomorphically parametrized family

$$W_T(z) \in B(\tilde{\mathscr{D}}, \mathscr{D}) \quad (z \in \mathbb{B}_d)$$

of operators on the subspace

$$\tilde{\mathscr{D}} = \{ y \in \mathscr{D}_{\tilde{T}};\ (\oplus j I_T^{-1}) D_{\tilde{T}} y \in \operatorname{Im} M_z^* \} \subset \mathscr{D}_{\tilde{T}}$$

such that

$$W(M) = \{W_T x;\ x \in \tilde{\mathscr{D}}\},$$

where $W_T x\colon \mathbb{B}_d \to \mathscr{D}$ $(x \in \tilde{\mathscr{D}})$ acts as $(W_T x)(z) = W_T(z)x$. We equip $\tilde{\mathscr{D}}$ with the norm $\|y\| = \|y\|_{\tilde{H}^d}$, that it inherits as a subspace $\tilde{\mathscr{D}} \subset \tilde{H}^d$. In Lemma 3.1.4 we will show that $\tilde{\mathscr{D}} \subset \tilde{H}^d$ is closed.

In the following, suppose that $\mathscr{H}$ is regular. That is

$$\lim_{n\to\infty} \frac{a_n}{a_{n+1}} = 1$$

(see Section 2.4; in particular, Definition 2.4.4 and Example 2.4.6). If $T = (T_1, \dots, T_d) \in B(H)^d$ is a pure K-contraction, then we have seen in Lemma 2.5.17 that

$$\sigma(T) \subset \sigma_{joint}(T) \subset \overline{\mathbb{B}_d}.$$

For our transfer realization, we use the maps

$$F_T(z) = \sum_{n=0}^{\infty} a_{n+1} \left(\sum_{|\alpha|=n} \gamma_\alpha (T^\alpha)^* z^\alpha \right) \quad (z \in \mathbb{B}_d)$$

as well as the row operators

$$Z(w) : H^d \to H, (h_1, \dots h_d) \to \sum_{l=1}^{d} w_l h_l \quad (w \in \mathbb{B}_d).$$

As in Remark 2.4.23, it follows that $F_T : \mathbb{B}_d \to B(H)$ is a well-defined operator-valued function.

Lemma 3.1.3. *For $(x_l)_{l=1}^d \in H^d$, the identity*

$$CF_T(z)Z(z)(x_l)_{l=1}^d = (\delta M_z (jx_l)_{l=1}^d)(z)$$

holds for all $z \in \mathbb{B}_d$.

Proof. For $(x_l)_{l=1}^d \in H^d$,

$$\begin{aligned}
\delta M_z (jx_l)_{l=1}^d &= \sum_{l=1}^{d} \delta M_{z_l} \sum_{n=0}^{\infty} a_n \left(\sum_{|\alpha|=n} \gamma_\alpha C(T^\alpha)^* x_l z^\alpha \right) \\
&= \sum_{l=1}^{d} \sum_{n=0}^{\infty} a_n \delta \left(\sum_{|\alpha|=n} \gamma_\alpha C(T^\alpha)^* x_l z^{\alpha+e_l} \right) \\
&= \sum_{l=1}^{d} \sum_{n=0}^{\infty} a_{n+1} \sum_{|\alpha|=n} \gamma_\alpha C(T^\alpha)^* x_l z^{\alpha+e_l},
\end{aligned}$$

where the series converge in $\mathscr{H}(\mathscr{D})$. Since the point evaluations are continuous on $\mathscr{H}(\mathscr{D})$, we obtain

$$\begin{aligned}\left(\delta M_z (jx_l)_{l=1}^d\right)(z) &= \sum_{n=0}^{\infty} a_{n+1} \sum_{|\alpha|=n} \gamma_\alpha C(T^\alpha)^* \left(\sum_{l=1}^d z_l x_l\right) z^\alpha \\ &= CF_T(z)Z(z)(x_l)_{l=1}^d\end{aligned}$$

for all $z \in \mathbb{B}_d$. □

To improve our criterion for a function $f \in \mathscr{H}(\mathscr{D})$ belonging to the wandering subspace $W(M)$ of $M = (\operatorname{Im} j)^\perp$ we give the following lemma, which is similar to the case of an m-hypercontraction with $C_{\cdot 0}$ property (see [Esc18a, Theorem 7])

Lemma 3.1.4. *Let $T \in B(H)^d$ be a pure K-contraction. Then a function*

$$f = \sum_{\alpha \in \mathbb{N}^d} f_\alpha z^\alpha \in \mathscr{H}(\mathscr{D})$$

belongs to the wandering subspace $W(M)$ if and only if there is a vector $y \in \tilde{\mathscr{D}}$ with

$$f = -\tilde{T}y + M_z'(\oplus j I_T^{-1}) D_{\tilde{T}} y.$$

In this case, $\|f\|^2 = \|y\|^2_{\tilde{H}^d}$ and in particular, $\tilde{\mathscr{D}} \subset \tilde{H}$ is closed.

Proof. Due to Theorem 3.1.1, a function $f = \sum_{\alpha \in \mathbb{N}^d} f_\alpha z^\alpha \in \mathscr{H}(\mathscr{D})$ belongs to $W(M)$ if and only if it is of the form

$$f = f_0 + M_z'(jx_l)_{l=1}^d$$

with $f_0 \in \mathscr{D}$ and $x_1, \dots, x_d \in H$ such that $(jx_l)_{l=1}^d \in \operatorname{Im} M_z^*$ and

$$\tilde{T}(I_T x_l)_{l=1}^d + D_{\tilde{T}^*} f_0 = 0.$$

Since $f_0 \in \mathscr{D} = \mathscr{D}_{\tilde{T}^*} = \overline{D_{\tilde{T}^*} H}$ and $D_{\tilde{T}} \tilde{T}^* = \tilde{T}^* D_{\tilde{T}^*}$, it follows that $\tilde{T}^* f_0 \in \mathscr{D}_{\tilde{T}}$. If

$$y = D_{\tilde{T}}(I_T x_l)_{l=1}^d - \tilde{T}^* f_0 \in \mathscr{D}_{\tilde{T}},$$

then

$$U \begin{pmatrix} (I_T x_l)_{l=1}^d \\ f_0 \end{pmatrix} = \begin{pmatrix} \tilde{T} & D_{\tilde{T}^*} \\ D_{\tilde{T}} & -\tilde{T}^* \end{pmatrix} \begin{pmatrix} (I_T x_l)_{l=1}^d \\ f_0 \end{pmatrix} = \begin{pmatrix} 0 \\ y \end{pmatrix}.$$

We obtain

$$\begin{pmatrix} (I_T x_l)_{l=1}^d \\ f_0 \end{pmatrix} = U^* \begin{pmatrix} 0 \\ y \end{pmatrix} = \begin{pmatrix} \tilde{T}^* & D_{\tilde{T}} \\ D_{\tilde{T}^*} & -\tilde{T} \end{pmatrix} \begin{pmatrix} 0 \\ y \end{pmatrix} = \begin{pmatrix} D_{\tilde{T}} y \\ -\tilde{T} y \end{pmatrix}.$$

But then

$$\oplus j I_T^{-1} D_{\tilde{T}} y = (jx_l)_{l=1}^d \in \operatorname{Im} M_z^*.$$

It follows that $y \in \tilde{\mathscr{D}}$ and

$$f = f_0 + M_z'(jx_l)_{l=1}^d = -\tilde{T}y + M_z'(\oplus jI_T^{-1})D_{\tilde{T}}y.$$

Conversely, let

$$f = -\tilde{T}y + M_z'(\oplus jI_T^{-1})D_{\tilde{T}}y$$

with

$$y \in \tilde{\mathscr{D}} = \{z \in \mathscr{D}_{\tilde{T}};\ (\oplus jI_T^{-1})D_{\tilde{T}}z \in \operatorname{Im} M_z^*\}.$$

Then

$$f_0 = -\tilde{T}y \in \mathscr{D} \text{ and } (x_l)_{l=1}^d = (\oplus I_T^{-1})D_{\tilde{T}}y \in H^d$$

yield a representation

$$f = f_0 + M_z'(jx_l)_{l=1}^d$$

as in Theorem 3.1.1, since then

$$Cf_0 + T(\Delta_T x_l)_{l=1}^d = -D_{\tilde{T}^*}\tilde{T}y + \tilde{T}D_{\tilde{T}^*}y = 0.$$

Using Lemma 3.1.2, we find that

$$\begin{aligned}\|f\|^2 &= \|f_0\|^2 + \sum_{l=1}^d \langle \Delta_T x_l, x_l\rangle = \|\tilde{T}y\|^2 + \sum_{l=1}^d \|I_T x_l\|_{\tilde{H}}^2 \\ &= \|\tilde{T}y\|^2 + \|D_{\tilde{T}}y\|_{\tilde{H}^d}^2 = \|y\|_{\tilde{H}^d}^2.\end{aligned}$$

□

In the following $\tilde{\mathscr{D}}$ will always be equipped with the norm $\|y\| = \|y\|_{\tilde{H}^d}$ that it inherits as a closed subspace $\tilde{\mathscr{D}} \subset \tilde{H}^d$. Due to Lemma 3.1.3, the map $W_T : \mathbb{B}_d \to B(\tilde{\mathscr{D}}, \mathscr{D})$,

$$\begin{aligned}W_T(z)(x) &= -T(\oplus \Delta_T I_T^{-1})x + CF_T(z)Z(\oplus I_T^{-1})D_{\tilde{T}}x \\ &= -\tilde{T}x + CF_T(z)Z(\oplus I_T^{-1})D_{\tilde{T}}x\end{aligned}$$

defines an analytic operator-valued function.

Theorem 3.1.5. *Let $T \in B(H)^d$ be a pure K-contraction. Then*

$$W(M) = \{W_T x;\ x \in \tilde{\mathscr{D}}\}$$

and $\|W_T x\| = \|x\|$ for $x \in \tilde{\mathscr{D}}$.

Proof. For $x \in \tilde{\mathscr{D}}$, Lemma 3.1.3 implies that

$$W_T x = -\tilde{T}x + \delta M_z(\oplus jI_T^{-1})D_{\tilde{T}}x = -\tilde{T}x + M_z'(\oplus jI_T^{-1})D_{\tilde{T}}x.$$

Thus, the assertion follows using Lemma 3.1.4. □

We have seen that $W_T : \mathbb{B}_d \to B(\tilde{\mathscr{D}}, \mathscr{D})$ is an operator-valued analytic function with

$$W_T x \in \mathscr{H}(\mathscr{D}) \text{ and } \|W_T x\| = \|x\|$$

for all $x \in \tilde{\mathscr{D}}$ as well as

$$W_T(\tilde{\mathscr{D}}) \perp M_z^\alpha\left(W_T(\tilde{\mathscr{D}})\right) \text{ for all } \alpha \in \mathbb{N}^d \setminus \{0\}.$$

Thus $W_T : \mathbb{B}_d \to B(\tilde{\mathscr{D}}, \mathscr{D})$ is a K-inner functions with $W_T(\tilde{\mathscr{D}}) = W(M)$.

3.2. K-inner functions

In the previous section we saw that the K-inner function $W_T\colon \mathbb{B}_d \to B(\tilde{\mathscr{D}},\mathscr{D})$ associated with a pure K-contraction $T \in B(H)^d$ has the form

$$W_T(z) = D + CF_T(z)Z(z)B,$$

where $C = \left(\frac{1}{K}(T)\right)^{\frac{1}{2}} \in B(H,\mathscr{D})$, $D = -\tilde{T} \in B(\tilde{\mathscr{D}},\mathscr{D})$ and $B = (\oplus I_T^{-1})D_{\tilde{T}} \in B(\tilde{\mathscr{D}},H^d)$. An elementary calculation using the definitions and the intertwining relation $\tilde{T}D_{\tilde{T}} = D_{\tilde{T}^*}\tilde{T}$ shows that the operators T, B, C, D satisfy the conditions

$$\begin{aligned}
&(\text{K1})\ C^*C = \frac{1}{K}(T),\\
&(\text{K2})\ D^*C + B^*(\oplus\Delta_T)T^* = 0,\\
&(\text{K3})\ D^*D + B^*(\oplus\Delta_T)B = \mathrm{id}_{\tilde{\mathscr{D}}},\\
&(\text{K4})\ \mathrm{Im}((\oplus j)B) \subset M_z^*\mathscr{H}(\mathscr{D}).
\end{aligned}$$

Let $\mathscr{E}$ be a Hilbert space and $C \in B(H,\mathscr{E})$ any operator with $C^*C = \frac{1}{K}(T)$. Then it follows exactly as in the proof of Theorem 2.5.14 that the map

$$j_C\colon H \to \mathscr{H}(\mathscr{E}),\ j_C(x) = \sum_{\alpha\in\mathbb{N}^d} a_{|\alpha|}\gamma_\alpha(C(T^\alpha)^*x)z^\alpha$$

is a well-defined isometry, such that j_C intertwines the tuples $T^* = (T_1^*,\dots,T_d^*) \in B(H)^d$ and $M_z^* = (M_{z_1}^*,\dots,M_{z_d}) \in B(\mathscr{H}(\mathscr{E}))^d$ componentwise.

Our next aim is to show that any matrix operator

$$\left(\begin{array}{c|c} \tilde{T}^* & B \\ \hline C & D \end{array}\right)\colon H\oplus\mathscr{E}_* \to H^d\oplus\mathscr{E},$$

where T is a pure K-contraction and T, B, C, D satisfy the conditions (K1)-(K4) with $(\tilde{\mathscr{D}},\mathscr{D})$ replaced by $(\mathscr{E}_*,\mathscr{E})$ and (K4) replaced by

$$\mathrm{Im}((\oplus j_C)B) \subset M_z^*\mathscr{H}(\mathscr{E})$$

gives rise to a K-inner function $W\colon \mathbb{B}_d \to B(\mathscr{E}_*,\mathscr{E})$ with transfer realization

$$W(z) = D + CF_T(z)Z(z)B.$$

Conversely, we prove that under a natural condition under the kernel K, each K-inner function is of this form.

In the one dimensional case $d=1$, condition $(K4)$ can be omitted, since then M_z^* is surjective. Using the equation $\frac{1-z^n}{1-z} = \sum_{l=0}^{n-1} z^l$, one computes that

$$\frac{1}{z}\left(\frac{1}{(1-z)^n} - 1\right) = \frac{1-(1-z)^n}{z(1-z)^n} = \sum_{l=1}^{n}(1-z)^{-l}.$$

Using the representation

$$\text{SOT} - \lim_{N\to\infty} \sum_{n=0}^{N} -c_{n+1}\sigma_{M_z}^{n}(\text{id}_{\mathscr{H}(\mathscr{D})})$$

of the diagonal operator Δ and Lemma 4 in [Esc18a], one can check that the next theorem is a generalization of Theorem 2.1 in [Olo07] and Theorem 11 in [Esc18a].

Theorem 3.2.1. *Let*

$$W : \mathbb{B}_d \to B(\mathscr{E}_*, \mathscr{E})$$

be an operator-valued function between Hilbert spaces $\mathscr{E}_*$ and $\mathscr{E}$ such that

$$W(z) = D + CF_T(z)Z(z)B,$$

where $T \in B(H)^d$ is a pure K-contraction and the matrix operator

$$\left(\begin{array}{c|c} \tilde{T}^* & B \\ \hline C & D \end{array}\right) : H \oplus \mathscr{E}_* \to H^d \oplus \mathscr{E}$$

satisfies the condition (K1)-(K4). Then W is a K-inner function.

Proof. Because the isometry j_C intertwines the tuples T^* and M_z^* componentwise, the space

$$M = \mathscr{H}(\mathscr{E}) \ominus \text{Im}\, j_C \subset \mathscr{H}(\mathscr{E})$$

is a closed M_z-invariant subspace. Let $x \in \mathscr{E}_*$ be a fixed vector. By condition (K4) there is a function $f \in \mathscr{H}(\mathscr{E})$ with

$$(\oplus j_C)Bx = M_z^* f.$$

It follows exactly as in the proof of Lemma 3.1.3, that

$$CF_T(z)Z(z)Bx = (\delta M_z(\oplus j_C)Bx)(z) = (\delta M_z M_z^* f)(z)$$

for all $z \in \mathbb{B}_d$. Using that $\delta M_z M_z^* = P_{\text{Im}\, M_z}$, $\delta M_z = M_z(\oplus\Delta)$, $(\oplus j_C)Bx = M_z^* f$ and (K3), we find that

$$\begin{aligned}
\|Wx\|^2_{\mathscr{H}(\mathscr{E})} - \|Dx\|^2 &= \|P_{\text{Im}\, M_z} Wx\|^2 \\
&= \langle \delta M_z M_z^* f, f\rangle_{\mathscr{H}(\mathscr{E})} \\
&= \langle (\oplus\Delta) M_z^* f, M_z^* f\rangle_{\mathscr{H}(\mathscr{E})} \\
&= \langle \oplus(j_C^* \Delta j_C)Bx, Bx\rangle_{H^d} \\
&= \langle (\oplus\Delta_T)Bx, Bx\rangle_{H^d} \\
&= \langle (\text{id}_{\mathscr{E}_*} - D^*D)x, x\rangle \\
&= \|x\|^2 - \|Dx\|^2.
\end{aligned}$$

Hence, the map

$$\mathscr{E}_* \to \mathscr{H}(\mathscr{E}),\ x \mapsto Wx,$$

is a well-defined isometry. By the second part of Lemma 2.4.32, we obtain

$$M_z^*(Wx) = M_z^* \delta M_z M_z^* f = M_z^* f = (\oplus j_C)Bx.$$

Hence, we derive that

$$(\mathrm{id}_{\mathscr{H}(\mathscr{E})} - j_C j_C^*) M_{z_l}^*(Wx) = 0$$

for $l = 1, \cdots, d$. For x and f condition (K2) implies that

$$\begin{aligned} j_C^*(Wx) &= C^*Dx + j_C^*(\delta M_z M_z^* f) \\ &= C^*Dx + j_C^*(M_z(\oplus\Delta)M_z^* f) \\ &= C^*Dx + T(\oplus j_C^*\Delta j_C)Bx \\ &= C^*Dx + T(\oplus\Delta_T)Bx \\ &= 0. \end{aligned}$$

Since

$$\mathrm{id}_{\mathscr{H}(\mathscr{E})} - jj^* : \mathscr{H}(\mathscr{E}) \to \mathscr{H}(\mathscr{E})$$

is the orthogonal projection onto $M = \mathrm{Ker}\, j_C^*$, it follows that

$$W(\mathscr{E}_*) \subset W(M) = \bigcap_{l=1}^{d} (M \ominus M_{z_l} M).$$

This yields that

$$W(\mathscr{E}_*) \perp M_z^\alpha(W(\mathscr{E}_*))$$

for all $\alpha \in \mathbb{N}^d \setminus \{0\}$. □

The next theorem is a generalization of Theorem 4.2 in [Olo07] and Theorem 12 in [Esc18a]. We show that each K-inner function $W : \mathbb{B}_d \to B(\mathscr{E}_*, \mathscr{E})$ has the form described in Theorem 3.2.1. In the proof we shall use a uniqueness result for minimal K-dilations whose proof we postpone to Section 3.3.

Theorem 3.2.2. *If $W : \mathbb{B}_d \to B(\mathscr{E}_*, \mathscr{E})$ is a K-inner function, then there exists a pure K-contraction $T \in B(H)^d$ and a matrix operator*

$$\left(\begin{array}{c|c} \tilde{T}^* & B \\ \hline C & D \end{array}\right) \in B(H \oplus \mathscr{E}_*, H^n \oplus \mathscr{E})$$

satisfying the conditions (K1)-(K4) such that

$$W(z) = D + CF_T(z)Z(z)B \quad (z \in \mathbb{B}_d).$$

Proof. Since W is K-inner, the space

$$\mathscr{W} = W(\mathscr{E}_*) \subset \mathscr{H}(\mathscr{E})$$

is a wandering subspace for $M_z = (M_{z_1}, \dots, M_{z_d}) \in B(\mathscr{H}(\mathscr{E}))^d$. We denote by

$$\mathscr{S} = \bigvee_{\alpha \in \mathbb{N}^d} M_z^\alpha \mathscr{W} \subset \mathscr{H}(\mathscr{E})$$

the smallest closed M_z-invariant subspace of $\mathscr{H}(\mathscr{E})$ containing $\mathscr{W}$. The compression $T = P_H M_z|_H$ to the M_z^*-invariant subspace $H = \mathscr{H}(\mathscr{E}) \ominus \mathscr{S}$ is easily seen to be a pure K-contraction (cf. Definition 2.5.9 and Example 2.5.11). Let $\mathscr{R} \subset \mathscr{H}(\mathscr{E})$ be the smallest reducing subspace for $M_z \in B(\mathscr{H}(\mathscr{E}))^d$ that contains H. Because of Lemma 3.3.7, it follows that

$$\mathscr{R} = \bigvee_{\alpha \in \mathbb{N}^d} z^\alpha (\mathscr{R} \cap \mathscr{E}) = \mathscr{H}(\mathscr{R} \cap \mathscr{E}).$$

Thus, the inclusion map $i\colon H \to \mathscr{H}(\mathscr{R} \cap \mathscr{E})$ is a minimal K-dilation for T. Let $j\colon H \to \mathscr{H}(\mathscr{D})$ be the K-dilation of the pure K-contraction $T \in B(H)^d$, defined in Theorem 2.5.14. Since also j is a minimal K-dilation for T (Corollary 3.3.8), it follows from Corollary 3.3.6 that there is a unitary operator $U\colon \mathscr{D} \to \mathscr{R} \cap \mathscr{E}$ such that

$$i = (\mathrm{id}_{\mathscr{H}} \otimes U) j.$$

Define $\hat{\mathscr{E}} = \mathscr{E} \ominus (\mathscr{R} \cap \mathscr{E})$. By construction

$$\mathscr{H}(\hat{\mathscr{E}}) = \mathscr{H}(\mathscr{E}) \ominus \mathscr{H}(\mathscr{R} \cap \mathscr{E}) = \mathscr{H}(\mathscr{E}) \ominus \mathscr{R} \subset \mathscr{S}$$

is the largest reducing subspace for $M_z \in B(\mathscr{H}(\mathscr{E}))^d$ contained in $\mathscr{S}$. In particular, the space $\mathscr{S}$ admits the orthogonal decomposition

$$\mathscr{S} = \mathscr{H}(\hat{\mathscr{E}}) \oplus (\mathscr{S} \cap \mathscr{H}(\hat{\mathscr{E}})^\perp) = \mathscr{H}(\hat{\mathscr{E}}) \oplus (\mathscr{H}(\mathscr{R} \cap \mathscr{E}) \ominus \mathscr{S}^\perp).$$

We complete the proof by comparing the given K-inner function $W\colon \mathbb{B}_d \to B(\mathscr{E}_*, \mathscr{E})$ with the K-inner function $W_T\colon \mathbb{B}_d \to B(\tilde{\mathscr{D}}, \mathscr{D})$ associated with the pure K-contraction $T \in B(H)^d$. For this purpose, let us define the M_z-invariant subspace

$$M = \mathscr{H}(\mathscr{D}) \ominus \operatorname{Im} j$$

and its wandering subspace

$$W(M) = M \ominus \left(\sum_{l=1}^{d} z_l M \right)$$

as in Section 3.1. Using the identity $i = (\mathrm{id}_{\mathscr{H}} \otimes U) j$, one obtains that

$$\mathrm{id}_{\mathscr{H}} \otimes U\colon M \to \mathscr{H}(\mathscr{R} \cap \mathscr{E}) \ominus \mathscr{S}^\perp = \mathscr{H}(\mathscr{R} \cap \mathscr{E}) \cap \mathscr{S}$$

defines a unitary operator that intertwines the restrictions of M_z to both sides componentwise. Consequently, we obtain the orthogonal decomposition

$$\begin{aligned} \mathscr{W} &= W_{M_z}(\mathscr{S}) = W_{M_z}(\mathscr{H}(\hat{\mathscr{E}})) \oplus W_{M_z}(\mathscr{H}(\mathscr{R} \cap \mathscr{E}) \cap \mathscr{S}) \\ &= \hat{\mathscr{E}} \oplus (\mathrm{id}_{\mathscr{H}} \otimes U) W(M). \end{aligned}$$

Let $W_T : \mathbb{B}_d \to B(\tilde{\mathscr{D}}, \mathscr{D})$ be the K-inner function, associated with the pure K-contraction $T \in B(H)^d$. Then W_T admits a transfer realization of the form

$$W_T(z) = D + CF_T(z)Z(z)B \quad (z \in \mathbb{B}_d)$$

with matrix representation

$$\left(\begin{array}{c|c} \tilde{T}^* & B \\ \hline C & D \end{array}\right) \in B(H \oplus \tilde{\mathscr{D}}, H^d \oplus \mathscr{D})$$

such that the operators

$$C = \left(\frac{1}{K}(T)\right)^{\frac{1}{2}} \in B(H, \mathscr{D}), D = -\tilde{T} \in B(\tilde{\mathscr{D}}, \mathscr{D}) \text{ and } B = (\oplus I_T^{-1})D_{\tilde{T}} \in B(\tilde{\mathscr{D}}, H^d)$$

satisfy the conditions (K1)-(K4). Furthermore, the wandering subspace associated with M_z on M is characterized by

$$W(M) = \{W_T x;\, x \in \tilde{\mathscr{D}}\}$$

(see the beginning of Section 3.2 and Theorem 3.1.5). Let us denote by

$$P_1 : \mathscr{W} \to \hat{\mathscr{E}} \text{ and } P_2 : \mathscr{W} \to (\mathrm{id}_{\mathscr{H}} \otimes U)W(M)$$

the orthogonal projections onto $\hat{\mathscr{E}}$ and onto $\mathrm{id}_{\mathscr{H}} \otimes U)W(M)$ respectively. The K-inner functions $W : \mathbb{B}_d \to B(\mathscr{E}_*, \mathscr{E})$ and $W_T : \mathbb{B}_d \to B(\tilde{\mathscr{D}}, \mathscr{D})$ induce unitary operators

$$\mathscr{E}_* \to \mathscr{W},\ x \mapsto Wx$$

and

$$\tilde{\mathscr{D}} \to W(M)\ x \mapsto W_T x.$$

We define surjective bounded linear operators by

$$U_1 : \mathscr{E}_* \to \hat{\mathscr{E}},\ U_1 x = P_1 W x$$

and

$$U_2 : \mathscr{E}_* \to \tilde{\mathscr{D}},\ U_2 x = \tilde{x} \text{ if } (\mathrm{id}_{\mathscr{H}} \otimes U)W_T \tilde{x} = P_2 W x.$$

By construction the column operator

$$(U_1, U_2) : \mathscr{E}_* \to \hat{\mathscr{E}} \oplus \tilde{\mathscr{D}}$$

defines an isometry such that

$$W(z)x = U_1 x + UW_T(z)U_2 x = (U_1 + UDU_2)x + (UC)F_T(z)Z(BU_2)x$$

for $z \in \mathbb{B}_d$ and $x \in \mathscr{E}_*$. To complete the proof we show that the operators

$$T \in B(H^d, H),\ \tilde{B} = BU_2 \in B(\mathscr{E}_*, H^d),\ \tilde{C} = UC \in B(H, \mathscr{E})$$
$$\text{and } \tilde{D} = (U_1 + UDU_2) \in B(\mathscr{E}_*, \mathscr{E})$$

satisfy the conditions (K1)-(K4). Using that $C^*C = \frac{1}{K}(T)$ and $D^*C + B^*(\oplus\Delta_T)T^* = 0$, it is readily seen that

$$\tilde{C}^*\tilde{C} = C^*U^*UC = C^*C = \frac{1}{K}(T)$$

and

$$\begin{aligned}\tilde{D}^*\tilde{C} &= U_2^*D^*U^*UC = U_2^*D^*C \\ &= -U_2^*B^*(\oplus\Delta_T)T^* = -\tilde{B}^*(\oplus\Delta_T)T^*.\end{aligned}$$

To verify condition (K3) note that, using the identification $\mathscr{E} = \hat{\mathscr{E}} \oplus (\mathscr{R} \cap \mathscr{E})$, $\tilde{D}$ acts as the column operator

$$\tilde{D} = (U_1, UDU_2) \colon \mathscr{E}_* \to \hat{\mathscr{E}} \oplus (\mathscr{R} \cap \mathscr{E}).$$

With $D^*D + B^*(\oplus\Delta_T)B = \mathrm{id}_{\tilde{\mathscr{D}}}$ we obtain that

$$\begin{aligned}\tilde{D}^*\tilde{D} &= U_1^*U_1 + U_2^*D^*U^*UDU_2 \\ &= U_1^*U_1 + U_2^*U_2 - U_2^*B^*(\oplus\Delta_T)BU_2 \\ &= \mathrm{id}_{\mathscr{E}_*} - \tilde{B}^*(\oplus\Delta_T)\tilde{B}.\end{aligned}$$

Since $j_{\tilde{C}} = i = (\mathrm{id}_{\mathscr{H}} \otimes U)j$ and $\mathrm{Im}(U_2) = \tilde{D}$, it follows from

$$\mathrm{Im}((\oplus j)B) \subset M_z^*\mathscr{H}(\mathscr{D})$$

that

$$\left(\oplus j_{\tilde{C}}\right)\tilde{B}x = (\oplus\,\mathrm{id}_{\mathscr{H}} \otimes U)(\oplus j)B(U_2x) \in M_z^*\mathscr{H}(\mathscr{E})$$

for all $x \in \mathscr{E}_*$. Thus, the K-inner function $W \colon \mathbb{B}_d \to B(\mathscr{E}_*, \mathscr{E})$ admits a matrix representation of the claimed form. □

Finally, we want to indicate how the theory of characteristic functions is related to the theory of K-inner functions.

Remark 3.2.3. Let $\mathscr{H}$ be a unitarily invariant space with kernel K, such that

$$M_z = (M_{z_1}, \ldots, M_{z_d}) \in B(\mathscr{H})^d$$

is bounded and a pure K-contraction. Let $\tilde{\mathscr{H}}$ be a unitarily invariant Nevanlinna-Pick space with kernel $\tilde{K}$ such that

$$M_z = (M_{z_1}, \ldots, M_{z_d}) \in B(\tilde{\mathscr{H}})^d$$

is bounded and let $\mathscr{E}$ be an arbitrary Hilbert space. Furthermore, let $T = (T_1, \ldots, T_d) \in B(H)^d$ be a pure K-contraction with corresponding intertwining isometry $j : H \to \mathscr{H}(\mathscr{D})$ and let $\theta_T : \mathbb{B}_d \to B(\mathscr{E}, \mathscr{D})$ be a function in $\mathrm{Mult}(\tilde{\mathscr{H}}(\mathscr{E}), \mathscr{H}(\mathscr{D}))$ with multiplication operator

$$M_{\theta_T} : \tilde{\mathscr{H}}(\mathscr{E}) \to \mathscr{H}(\mathscr{D}),$$

such that

$$M_{\theta_T}M_{\theta_T}^* + jj^* = \mathrm{id}_{\mathscr{H}(\mathscr{D})}.$$

It is immediate that

$$M_{\theta_T} : \tilde{\mathscr{H}}(\mathscr{E}) \to \mathscr{H}(\mathscr{D}),$$

is partially isometric multiplier and

$$M_{\theta_T}(\mathrm{Ker}(\theta_T)^{\perp}) = M = (\mathrm{Im}\, j)^{\perp} \in \mathrm{Lat}(M_z, \mathscr{H}(\mathscr{D})).$$

Let

$$\mathscr{F} = \mathscr{E} \cap (\mathrm{Ker}\, \theta_T)^{\perp} \subset \mathscr{E}.$$

Then, clearly

$$\|\theta_T x\|_{\mathscr{H}(\mathscr{D})} = \|M_{\theta_T} x\|_{\mathscr{H}(\mathscr{D})} = \|x\|_{\tilde{\mathscr{H}}(\mathscr{E})} = \|x\|_{\mathscr{E}} = \|x\|_{\mathscr{F}}$$

for all $x \in \mathscr{F}$. Furthermore, we obtain that

$$\langle M_{\theta_T} f, M_{z_l} M_{\theta_T} g\rangle_{\mathscr{H}(\mathscr{D})} = \langle M_{\theta_T} f, M_{\theta_T} M_{z_l} g\rangle_{\mathscr{H}(\mathscr{D})} = \langle f, M_{z_l} g\rangle_{\tilde{\mathscr{H}}(\mathscr{E})}$$

for all $l = 1, \dots, d$, $f \in \mathrm{Ker}(\theta_T)^{\perp}$ and $g \in \tilde{\mathscr{H}}(\mathscr{E})$. Using that $\mathscr{E} = \tilde{\mathscr{H}}(\mathscr{E}) \ominus \mathrm{Im}\, M_z$, it follows that

$$W(M) = M \ominus \left(\sum_{l=1}^{d} M_{z_l} M \right) = \theta_T(\mathscr{E} \cap (\mathrm{Ker}\, \theta_T)^{\perp}) = \theta_T(\mathscr{F}).$$

Hence,

$$\mathbb{B}_d \to B(\mathscr{F}, \mathscr{D}),\ z \mapsto \theta_T(z)|_{\mathscr{F}}$$

is a K-inner function (see Theorem 5.2 in [BEKS17]). Let $W_T : \mathbb{B}_d \to B(\tilde{\mathscr{D}}, \mathscr{D})$,

$$W_T(z)(x) = -\tilde{T}x + CF_T(z)Z(\oplus I_T^{-1})D_{\tilde{T}}x$$

be the K-inner functions as before such that $W_T(\tilde{\mathscr{D}}) = W(M)$.

For each vector $x \in \tilde{\mathscr{D}}$, there is a unique element $y_x \in \mathscr{F}$ with $W_T x = \theta_T y_x$. The induced map

$$\tilde{\mathscr{D}} \to \mathscr{F},\ x \to y_x,$$

is unitary and modulo the isometry

$$i : \tilde{\mathscr{D}} \to \mathscr{E}, x \to y_x$$

the function θ_T extends the K-inner function W_T. The characteristic functions in [Esc18a], [BJ23a] and [BJ23b] are functions of this type.

3.3. Minimal K-dilations

In this section, we generalize a result from [BEKS17] about uniqueness of minimal dilations for pure row contractions, which we have used for Theorem 3.2.2. For the proof we use the uniqueness of minimal Stinespring dilations. Similar approaches can also be found in [Kla16] and [Sch18]. We also give an alternative proof for the uniqueness of minimal K-dilations using a lurking isometry argument due to Abadias, Bello and Yakubovich (see [ABY21, Theorem 1.12 and Section 3]).

Let $\Pi : \mathscr{B} \to B(H)$ be a completely positive map from a unital C^*-Algebra $\mathscr{B}$ to the bounded linear operators $B(H)$ on a Hilbert space H. By the Stinespring dilation theorem, $\Pi : \mathscr{B} \to B(H)$ has a (minimal) Stinespring representation (V, π), where π is a representation of $\mathscr{B}$ on a Hilbert space H_π and $V \in B(H, H_\pi)$ (see [Pau02, Theorem 4.1]). The pair (V, π), is called Stinespring representation for Π if

$$\Pi(x) = V^* \pi(x) V$$

for every $x \in \mathscr{B}$. If in addition

$$H_\pi = \bigvee \{\pi(x)Vh : x \in \mathscr{B}, h \in H\}$$

the pair (V, π) is called minimal.

Let $\mathscr{A}$ be a unital subalgebra of a unital C^*-algebra $\mathscr{B}$ and $\Pi \colon \mathscr{B} \to B(H)$ a completely positive unital map. The map Π is called an $\mathscr{A}$-morphism if $\Pi(1_{\mathscr{B}}) = \mathrm{id}_H$ and $\Pi(ax) = \Pi(a)\Pi(x)$ for $a \in \mathscr{A}$ and $x \in \mathscr{B}$. Suppose in addition that

$$\mathscr{B} = \overline{\mathrm{span}}^{\|\cdot\|}\{\mathscr{A}\mathscr{A}^*\}.$$

Arveson shows that every unitary operator, intertwining two $\mathscr{A}$-morphisms

$$\Pi_l \colon \mathscr{B} \to B(H_l) \quad (l = 1, 2)$$

pointwise on $\mathscr{A}$, extends to a unitary operator, intertwining the minimal Stinespring representations of Π_1 and Π_2 (cf. [Arv98, Lemma 8.6]).

Let $\mathscr{B}$ be a von Neumann algebra and $\mathscr{A}$ a unital C^*-subalgebra such that

$$\mathscr{B} = \overline{\mathrm{span}}^{w^*}\{\mathscr{A}\mathscr{A}^*\}.$$

Suppose that the $\mathscr{A}$-morphisms

$$\Pi_l \colon \mathscr{B} \to B(H_l) \quad (l = 1, 2)$$

are weak-$*$ continuous. Straightforward modifications of the arguments given in [Arv98], show that Arveson's result remains true in this modified setting.

Theorem 3.3.1. *Let $\mathscr{B}$ be a von Neumann algebra and let $\mathscr{A} \subset \mathscr{B}$ be a unital subalgebra such that*

$$\mathscr{B} = \overline{\text{span}}^{w^*}\{\mathscr{A}\mathscr{A}^*\}.$$

For $l = 1,2$, let $\Pi_l : \mathscr{B} \to B(H_l)$ be a weak-$*$ continuous $\mathscr{A}$-morphism and let (π_l, V_l, H_{π_l}) be the minimal Stinespring representations for Π_l. For every unitary operator $U : H_1 \to H_2$ with

$$U\Pi_1(a) = \Pi_2(a)U \quad (a \in \mathscr{A}),$$

there is a unique unitary operator $W : H_{\pi_1} \to H_{\pi_2}$ with $WV_1 = V_2U$ and $W\pi_1(x) = \pi_2(x)W$ for all $x \in \mathscr{B}$.

Since this version of Arveson's result follows in the same way as the original one (cf. [Arv98, Lemma 8.6]), we omit the details.

Definition 3.3.2. Let $T \in B(H)^d$ be a pure K-contraction. A K-dilation of T is an isometry $V : H \to \mathscr{H}(\mathscr{E})$ together with the operator tuple M_z such that

$$VT_l^* = M_{z_l}^* V$$

for $l = 1,\dots,d$. We call a K-dilation of T minimal if the only reducing subspace for M_z that contains $\operatorname{Im} V$ is $\mathscr{H}(\mathscr{E})$.

We will see that an application of Theorem 3.3.1 yields that minimal K-dilations are uniquely determined. We will use the following theorem:

Theorem 3.3.3. *Suppose that $\frac{1}{K}(M_z)$ exists. Then $B(\mathscr{H})$ coincides with the von Neumann algebra $W^*(M_z)$ generated by $M_{z_1},\dots,M_{z_d}$.*

Proof. Since the von Neumann algebra generated by the compact operators $K(\mathscr{H})$ is all of $B(\mathscr{H})$, it suffices to prove that $K(\mathscr{H}) \subset W^*(M_z)$. Because of Proposition 2.4.26, the C^*-algebra $C^*(M_z)$ is irreducible. Thus, clearly $W^*(M_z)$ is irreducible. So, by a well-known result on C^*- algebras it is enough to show that

$$W^*(M_z) \cap K(\mathscr{H}) \neq \emptyset.$$

Using Proposition 2.5.5, it is immediate that

$$P_{\mathbb{C}} = \frac{1}{K}(M_z) = \text{SOT} - \sum_{n=0}^{\infty} c_n \sigma_{M_z}^n(\text{id}_{\mathscr{H}}) \in W^*(M_z) \cap K(\mathscr{H})$$

and the assertion follows. ☐

Suppose now that $\frac{1}{K}(M_z)$ exists and set $\mathscr{A} = \{M_p : \mathscr{H} \to \mathscr{H};\ p \in \mathbb{C}[z]\}$. Because of Theorem 3.3.3, it follows that

$$\overline{\text{span}}^{w^*}\{\mathscr{A}\mathscr{A}^*\} = W^*(M_z) = B(\mathscr{H}).$$

Remark 3.3.4. Let $T \in B(H)^d$ be a pure K-contraction and let $V: H \to \mathscr{H}(\mathscr{E})$ be a K-dilation of T. The unital C^*-homomorphism

$$\pi: B(\mathscr{H}) \to B(\mathscr{H}(\mathscr{E})),\, X \mapsto X \otimes \mathrm{id}_{\mathscr{E}}$$

together with the isometry $V: H \to \mathscr{H}(\mathscr{E})$ is a Stinespring representation for the completely positive map

$$\Pi: B(\mathscr{H}) \to B(H),\, \Pi(X) = V^*(X \otimes \mathrm{id}_{\mathscr{E}})V.$$

By definition the K-dilation $V: H \to \mathscr{H}(\mathscr{E})$ is minimal if and only if

$$\bigvee_{X \in W^*(M_z)} \pi(X)(VH) = \bigvee_{X \in B(\mathscr{H})} \pi(X)(VH) = \mathscr{H}(\mathscr{E}).$$

Hence, the K-dilation $V: H \to \mathscr{H}(\mathscr{E})$ is minimal if and only if (π, V) as a Stinespring representation of Π is minimal.

Lemma 3.3.5. *The completely positive map*

$$\Pi: B(\mathscr{H}) \to B(H),\, \Pi(X) = V^*(X \otimes \mathrm{id}_{\mathscr{E}})V$$

is weak-$*$ continuous and an $\mathscr{A}$-morphism, that is,

$$\Pi(M_p X) = \Pi(M_p)\Pi(X)$$

for all $X \in B(\mathscr{H})$ and $p \in \mathbb{C}[z]$.

Proof. The intertwining property of V yields that

$$p(T)V^* = V^*(M_p \otimes \mathrm{id}_{\mathscr{E}})$$

and

$$p(T) = \Pi(M_p)$$

for all $p \in \mathbb{C}[z]$. Thus, it follows that

$$\Pi(M_p X) = V^*(M_p \otimes \mathrm{id}_{\mathscr{E}})(X \otimes \mathrm{id}_{\mathscr{E}})V = p(T)\Pi(X) = \Pi(M_p)\Pi(X)$$

for all $p \in \mathbb{C}[z]$ and $X \in B(\mathscr{H})$. Hence, Π is an $\mathscr{A}$-morphism. Standard duality for Banach space operators shows that Π is weak-$*$ continuous. Indeed, as an application of Krein-Smulian's theorem (Lemma A.2.2) one only has to check that $w^* - \lim_\alpha V^*(X_\alpha \otimes \mathrm{id}_{\mathscr{E}})V = V^*(X \otimes \mathrm{id}_{\mathscr{E}})V$ for each norm-bounded net (X_α) in $B(\mathscr{H})$ with $w^* - \lim_\alpha X_\alpha = X$. To complete the argument, it suffices to recall that on norm-bounded sets the weak-$*$ topology and the weak operator topology coincide. Thus, we have shown that Π is a weak-$*$ continuous $\mathscr{A}$-morphism. □

Corollary 3.3.6. *If $V_l: H \to \mathscr{H}(\mathscr{E}_l)$ $(l = 1, 2)$ are two minimal K-dilations of a pure K-contraction $T \in B(H)^d$, then there is a unitary operator $U \in B(\mathscr{E}_1, \mathscr{E}_2)$ such that $V_2 = (\mathrm{id}_{\mathscr{H}} \otimes U)V_1$*

Proof. Remark 3.3.4 and Lemma 3.3.5, preceding the corollary, show that the maps

$$\Pi_l\colon B(\mathscr{H}) \to B(H),\ \Pi_l(X) = V_l^*(X \otimes \mathrm{id}_{\mathscr{E}_l})V_l \quad (l = 1,2)$$

are weak-$*$ continuous $\mathscr{A}$-morphisms with minimal Stinespring representations

$$\pi_l\colon B(\mathscr{H}) \to B(\mathscr{H}(\mathscr{E}_l)),\ \pi_l(X) = X \otimes \mathrm{id}_{\mathscr{E}_l} \quad (l = 1,2).$$

Because of Theorem 3.3.1, there is a unitary operator $W\colon \mathscr{H}(\mathscr{E}_1) \to \mathscr{H}(\mathscr{E}_2)$ with $WV_1 = V_2$ and $W(X \otimes \mathrm{id}_{\mathscr{E}_1}) = (X \otimes \mathrm{id}_{\mathscr{E}_2})W$ for all $X \in B(\mathscr{H})$. In particular, the unitary operator W satisfies the intertwining relations

$$W(M_{z_l} \otimes \mathrm{id}_{\mathscr{E}_1}) = (M_{z_l} \otimes \mathrm{id}_{\mathscr{E}_2})W \quad (l = 1,\dots,d).$$

Due to Proposition 2.3.57, there exist operator-valued functions

$$A\colon \mathbb{B}_d \to B(\mathscr{E}_1,\mathscr{E}_2) \text{ and } B\colon \mathbb{B}_d \to B(\mathscr{E}_2,\mathscr{E}_1)$$

such that $Wf = Af$ and $W^*g = Bg$ for $f \in \mathscr{H}(\mathscr{E}_1)$ and $g \in \mathscr{H}(\mathscr{E}_2)$ (see also [Sch18, Proposition 4.5]). It follows that $A(z)B(z) = \mathrm{id}_{\mathscr{E}_2}$ and $B(z)A(z) = \mathrm{id}_{\mathscr{E}_1}$ for $z \in \mathbb{B}_d$. Since

$$K(z,w)x = (WW^*K(\cdot,w)x)(z) = A(z)A(w)^*K(z,w)x$$

for $z,w \in \mathbb{B}_d$ and $x \in \mathscr{E}_2$, we find that $A(z)A(w)^* = \mathrm{id}_{\mathscr{E}_2}$ for $z,w \in \mathbb{B}_d$. But then the constant value $A(z) \equiv U \in B(\mathscr{E}_1,\mathscr{E}_2)$ is a unitary operator with $W = \mathrm{id}_{\mathscr{H}} \otimes U$. □

We proceed by characterizing minimal K-dilations of pure K-contractions. To prepare this result we first identify the M_z-reducing subspaces of $\mathscr{H}(\mathscr{E})$.

Lemma 3.3.7. *Let $M \subset \mathscr{H}(\mathscr{E})$ be a closed linear subspace. If M is reducing for $M_z \in B(\mathscr{H}(\mathscr{E}))^d$, then $P_{\mathscr{E}}M \subset M$ and*

$$M = \bigvee_{\alpha \in \mathbb{N}^d} z^\alpha (M \cap \mathscr{E}) = \mathscr{H}(M \cap \mathscr{E}).$$

Proof. The hypothesis implies that M is reducing for the von Neumann algebra $W^*(M_z)$ generated by $M_{z_1},\dots M_{z_d}$. Due to Theorem 3.3.3, it follows that $B(\mathscr{H}) = W^*(M_z)$ and thus, $B(\mathscr{H})M \subset M$. In particular, it follows that

$$(P_{\mathscr{E}}M_z^{*\beta})M \subset P_{\mathscr{E}}M = \mathscr{E} \cap M$$

for all $\beta \in \mathbb{N}^d$. Let $f = \sum_{\alpha \in \mathbb{N}^d} f_\alpha z^\alpha \in M$ be arbitrary. For every $\beta \in \mathbb{N}^d$ it follows from Lemma 2.4.2 that

$$f_\beta = (\gamma_\beta a_{|\beta|})(P_{\mathscr{E}}M_z^{*\beta})f \in \mathscr{E} \cap M.$$

The observation that

$$f = \sum_{\alpha \in \mathbb{N}^d} f_\alpha z^\alpha \in \bigvee_{\alpha \in \mathbb{N}^d} z^\alpha (M \cap \mathscr{E}) = \mathscr{H}(M \cap \mathscr{E})$$

completes the proof. □

Corollary 3.3.8. *Let $T \in B(H)^d$ be a pure K-contraction and let*

$$V : H \to \mathscr{H}(\mathscr{E})$$

be a K-dilation. Consider the operator

$$C_V : H \to \mathscr{E},\ h \mapsto P_{\mathscr{E}} V h.$$

and denote by $\mathscr{D} = \overline{\operatorname{Im} C_V} \subset \mathscr{E}$ the closure of its range. Then $\operatorname{Im} V \subset \mathscr{H}(\mathscr{D})$ and V is minimal if and only if $\mathscr{D} = \mathscr{E}$. In particular, the operator

$$V_{min} : H \to \mathscr{H}(\mathscr{D}),\ h \mapsto Vh$$

is a minimal K-dilation for T.

Proof. Clearly, $\mathscr{H}(\mathscr{D})$ is reducing for M_z. For $h \in H$ Lemma 2.5.12 yields that

$$Vh = \sum_{\alpha \in \mathbb{N}^d} \gamma_\alpha a_{|\alpha|} C_V (T^\alpha)^* h z^\alpha \in \mathscr{H}(\mathscr{D}).$$

Hence, $\operatorname{Im} V \subset \mathscr{H}(\mathscr{D})$. If V is minimal we obtain that

$$\mathscr{H}(\mathscr{D}) = \mathscr{H}(\mathscr{E})$$

and thus,

$$\mathscr{E} = P_{\mathscr{E}} \mathscr{H}(\mathscr{E}) = P_{\mathscr{E}} \mathscr{H}(\mathscr{D}) = \mathscr{D} = \overline{\operatorname{Im} C_V}.$$

For the converse direction, suppose that $\mathscr{D} = \overline{\operatorname{Im} C_V} = \mathscr{E}$ and that $\operatorname{Im} V \subset M$ is a reducing subspace for $M_z \in B(\mathscr{H}(\mathscr{E}))^d$. We know from Lemma 3.3.7 that

$$M = \mathscr{H}(M \cap \mathscr{E}).$$

and that

$$\operatorname{Im} C_V = \operatorname{Im}(P_{\mathscr{E}} V) \subset P_{\mathscr{E}}(M) \subset M \cap \mathscr{E}.$$

Thus, if $\mathscr{D} = \mathscr{E}$, it follows that

$$\mathscr{H}(\mathscr{E}) = \mathscr{H}(\mathscr{D}) \subset \mathscr{H}(M \cap \mathscr{E}) = M.$$

The additional part is immediate. □

We conclude this section with a more straightforward proof for Corollary 3.3.6, using a lurking isometry argument due to Abadias, Bello and Yakubovich (see [ABY21]).

Proof. Let $V_l : H \to \mathscr{H}(\mathscr{E}_l)$ $(l = 1, 2)$ be two minimal K-dilations of a pure K-contraction $T \in B(H)^d$. Set $C_l = P_{\mathscr{E}_l} V_l$. Due to Lemma 2.5.12, we obtain for $h \in H$ that

$$V_l h = \sum_{\alpha \in \mathbb{N}^d} a_{|\alpha|} \gamma_\alpha C_l (T^\alpha)^* h z^\alpha$$

and

$$C_l^* C_l = \frac{1}{K}(T).$$

Hence, $\|C_1 h\| = \|C_2 h\|$ for all $h \in H$. Since the K-dilations $V_l \colon H \to \mathscr{H}(\mathscr{E}_l)$ are minimal, it follows from Corollary 3.3.8 that $\overline{\operatorname{Im} C_l} = \mathscr{E}_l$ for $l = 1,2$. Thus, the isometry $W : \operatorname{Im} C_1 \to \mathscr{E}_2$, $C_1 h \mapsto C_2 h$ is well-defined and extends to a unitary operator $U \colon \mathscr{E}_1 \to \mathscr{E}_2$. Furthermore, we deduce that

$$\begin{aligned}(\mathrm{id}_{\mathscr{H}} \otimes U) V_1 h &= \sum_{\alpha \in \mathbb{N}^d} a_{|\alpha|} \gamma_\alpha U C_1 (T^\alpha)^* h z^\alpha \\ &= \sum_{\alpha \in \mathbb{N}^d} a_{|\alpha|} \gamma_\alpha C_2 (T^\alpha)^* h z^\alpha \\ &= V_2 h\end{aligned}$$

for all $h \in H$. □

4. Uniqueness of multiplier calculus for pure K-contractions

The contents of this chapter are joint work with Michael Hartz.

Let $T \in B(H)$ be a completely non-unitary contraction, that is an operator of norm at most 1 without unitary direct summand. A fundamental result of Sz.-Nagy and Foiaș shows that the obvious polynomial functional calculus

$$\Pi : \mathbb{C}[z] \to B(H), \quad p \mapsto p(T),$$

extends to a weak-$*$ continuous, (completely) contractive algebra homomorphism on $H^\infty(\mathbb{D})$.

Furthermore, it is a well-known fact that the polynomials form a weak-$*$ dense subalgebra of $H^\infty(\mathbb{D})$ (see for example Theorem 2.3.36). Thus, if the extension of the polynomial functional calculus Π to $H^\infty(\mathbb{D})$ is weak-$*$ continuous, it is clearly unique. Miller, Olin, and Thomson give an example (cf. [MOT86, Example 13.4]) of a completely non-unitary contraction T for which the polynomial functional calculus Π admits multiple extensions to a norm continuous algebra homomorphism on $H^\infty(\mathbb{D})$.

On the other hand, assume that $T \in B(H)$ is not only completely non-unitary but also pure, that is $(T^*)^N \to 0$ for $N \to \infty$ in the strong operator topology. In [MOT86, Theorem 13.3], Miller, Olin, and Thomson establish the following uniqueness result:

Theorem (Miller, Olin, and Thomson)**.** *Let $T \in B(H)$ be a pure contraction and let $\pi : H^\infty(\mathbb{D}) \to B(H)$ be a bounded unital homomorphism with $\pi(z) = T$. Then π is weak-$*$ continuous and hence agrees with the Sz.-Nagy–Foiaș functional calculus of T.*

The goal of this chapter is to establish an analog of Miller, Olin and Thomson's theorem for multiplier functional calculi of pure K-contractions.

For the rest of this chapter let therefore $\mathscr{H}$ be a regular unitarily invariant complete Nevanlinna-Pick space with kernel $K : \mathbb{B}_d \times \mathbb{B}_d \to \mathbb{C}$,

$$K(z,w) = \sum_{n=0}^{\infty} a_n \langle z, w\rangle^n = \frac{1}{1 - \sum_{n=1}^{\infty} b_n \langle z, w\rangle^n},$$

where $a_0 = 1$, $a_n > 0$ for all $n \in \mathbb{N}$, $\lim_{n\to\infty} \frac{a_n}{a_{n+1}} = 1$ and $(b_n)_{n\geq 1}$ is a sequence of non-negative real numbers satisfying $\sum_{n=1}^{\infty} b_n \leq 1$.

Examples are the spaces $A_s^2(\mathbb{B}_d)$, where $s \in (0,1)$, with reproducing kernel

$$K_s : \mathbb{B}_d \times \mathbb{B}_d \to \mathbb{C},\ K_s(z,w) = \frac{1}{(1-\langle z,w\rangle)^s},$$

and the spaces $\mathscr{D}_s(\mathbb{B}_d)$, where $s \leq 0$, with reproducing kernel

$$K : \mathbb{B}_d \times \mathbb{B}_d \to \mathbb{C},\ K(z,w) = \sum_{n=0}^{\infty} (n+1)^s \langle z,w\rangle^n.$$

If $s \in (0,1)$, then $A_s^2(\mathbb{B}_d)$ and $\mathscr{D}_{s-1}(\mathbb{B}_d)$ coincide as vector spaces with equivalence of norms. See Examples 2.3.64 and 2.4.6.

For a tuple of commuting operators $T = (T_1,\ldots,T_d) \in B(H)^d$ let

$$\sigma_T : B(H) \to B(H),\ X \mapsto \sum_{l=0}^{d} T_l X T_l^*.$$

We have seen in Definition 2.5.2 that a tuple of commuting operators T is called K-contraction if

$$\left(\frac{1}{K}\right)(T) = \mathrm{id}_H - \sum_{n=1}^{\infty} b_n \sigma_T^n(\mathrm{id}_H)$$

converges in the strong operator topology and defines a positive operator. Motivated by the definition of the defect operator and the defect space of a contraction, we called

$$C = \left(\frac{1}{K}(T)\right)^{\frac{1}{2}}$$

the defect operator and

$$\mathscr{D} = \mathscr{D}_{T^*} = \overline{\operatorname{Im} C}$$

the defect space of a K-contraction. Further, we considered in Theorem 2.5.16 the following characterization of pure K-contractions.

Theorem. *Let $T \in B(H)^d$ be a commuting tuple. The following statements are equivalent:*

(a) T is a pure K-contraction,

(b) there exists a Hilbert space $\mathscr{E}$ and an isometry $V : H \to \mathscr{H}(\mathscr{E})$ such that

$$VT_l^* = M_{z_l}^* V$$

for all $l = 1,\ldots,d$.

(c) T is a K-contraction and

$$j\colon H \to \mathscr{H}(\mathscr{D}),\ j(h) = \sum_{\alpha\in\mathbb{N}^d} a_{|\alpha|}\gamma_\alpha C(T^\alpha)^* z^\alpha$$

is a well-defined isometry such that

$$jT_l^* = M_{z_l}^* j \quad (l = 1,\ldots,d).$$

In [Esc97], Eschmeier extends the Sz.-Nagy-Foiaș $H^\infty(\mathbb{D})$-functional calculus for completely non-unitary contractions to the multivariable setting. He establishes $H^\infty(\mathbb{B}_d)$-functional calculi for completely non-unitary tuples of commuting operators satisfying von Neumann's inequality over the unit ball $\mathbb{B}_d$. Clouâtre and Davidson show that completely non-unitary commuting row contractions admit functional calculi for the multiplier algebra of the Drury-Arveson space H^2_d (cf. [CD16]). Bickel, M^c^Carthy and Hartz generalize these ideas for a larger class of commuting operator tuples and multiplier algebras of reproducing kernel Hilbert spaces on unit ball $\mathbb{B}_d$ (see [BHM18]).

The next theorem is a particular case of Proposition 3.5 in [BHM18].

Theorem (Bickel, M^c^Carthy and Hartz)**.** *If T is a pure K-contraction, then there exists a completely contractive algebra homomorphism*

$$\pi\colon \operatorname{Mult}(\mathscr{H}) \to B(H)$$

with $\pi(1) = \mathrm{id}_H$ and $\pi(z_l) = T_l$ for $l = 1, \ldots, d$ such that π is weak-$*$ continuous. In particular, if

$$\tilde{\pi}\colon \operatorname{Mult}(\mathscr{H}) \to B(H)$$

is any weak-$*$ continuous map extending the polynomial calculus

$$\mathbb{C}[z] \to B(H),\ p \mapsto p(T)$$

for T, then $\tilde{\pi} = \pi$.

Proof. Let T be a pure K-contraction and let $j : H \to \mathscr{H}(\mathscr{D})$ be the intertwining isometry defined in Theorem 2.5.14. Using Lemma 3.3.5, it follows that

$$\pi : B(\mathscr{H}) \to B(H),\ X \mapsto j^*(X \otimes \mathrm{id}_{\mathscr{D}})j$$

is weak-$*$ continuous, completely positive, $\Pi(\mathrm{id}_{\mathscr{H}}) = \mathrm{id}_H$, $\Pi(M_{z_l}) = T_l$ for $l = 1, \ldots, d$ and

$$\Pi(M_p X) = \Pi(M_p)\Pi(X)$$

for all $p \in \mathbb{C}[z]$ and $X \in B(\mathscr{H})$. Since $\mathbb{C}[z]$ is weak-$*$ dense in $\operatorname{Mult}(\mathscr{H})$ (see Theorem 2.3.36) and multiplication on $B(\mathscr{H})$ and $B(H)$ is separately weak-$*$ continuous, we deduce that

$$\Pi(M_\varphi X) = \Pi(M_\varphi)\Pi(X)$$

for all $\varphi \in \operatorname{Mult}(\mathscr{H})$ and $X \in B(\mathscr{H})$. Since $\|j\| \leq 1$, using the definition of π, it can be readily seen that π is completely contractive. The additional part is immediate, using that $\operatorname{Mult}(\mathscr{H}) = \overline{\mathbb{C}[z]}^{w^*}$. □

The previous theorem about pure K-contractions generalizes the $H^\infty(\mathbb{D})$-calculus for pure contractions. For generalizations of the completely non-unitary case we recommend the mentioned paper [BHM18] by Bickel, M^c^Carthy and Hartz.

The main result in this chapter will be as follows:

Theorem 4.1 (Analog to Miller, Olin, and Thomson's theorem)**.** *Let $\mathscr{H}$ be a regular unitarily invariant complete Nevanlinna-Pick space on $\mathbb{B}_d$ with unbounded kernel K, let $T = (T_1,\dots,T_d)$ be a pure K-contraction and let $\pi : \mathrm{Mult}(\mathscr{H}) \to B(H)$ be a completely bounded unital homomorphism with $\pi(z_l) = T_l$ for $1 \le l \le d$. Then π is weak-$*$ continuous.*

As in the $H^\infty(\mathbb{D})$-case, we have

$$\overline{\mathbb{C}[z]}^{w^*} = \mathrm{Mult}(\mathscr{H})$$

(see Theorem 2.3.36). Hence, similarly to Miller, Olin, and Thomson's Theorem, the result (Theorem 4.1) can be understood as a uniqueness statement for the multiplier functional calculus of pure K-contractions without an a priori weak-$*$ continuity assumption.

Our goal in the following two sections is to characterize pure K-contractions for complete Nevanlinna–Pick kernels K with the help of a "Schur-type" product for infinite tuples. This characterization is similar to the original definition $\mathrm{SOT}-\lim_{N\to\infty} T^{*N} \to 0$ of pure contractions $T \in B(H)$.

4.1. A product for infinite tuples

In this section, we will introduce a "Schur-type" product for row operators. The most basic case occurs when

$$T = \begin{bmatrix} T_1 & T_2 & \cdots & T_n \end{bmatrix} \in B(H \otimes \mathbb{C}^n, H)$$

and

$$S = \begin{bmatrix} S_1 & S_2 & \cdots & S_m \end{bmatrix} \in B(H \otimes \mathbb{C}^m, H).$$

In this case, we define

$$T \odot S = \begin{bmatrix} T_1 S_1 & T_1 S_2 & \cdots & T_1 S_m & T_2 S_1 & \cdots & T_n S_m \end{bmatrix},$$

which can be regarded as an operator from $H \otimes \mathbb{C}^{nm}$ into H.

We also have to deal with infinite rows and iterated $\odot$-products, which makes it more convenient to define $\odot$ in an basis independent fashion.

Definition 4.1.1. Let $H, \mathscr{E}_1, \mathscr{E}_2$ be Hilbert spaces and let $T \in B(H \otimes \mathscr{E}_1, H)$ and $S \in B(H \otimes \mathscr{E}_2, H)$. We define

$$T \odot S = T(S \otimes \mathrm{id}_{\mathscr{E}_1}) \in B(H \otimes \mathscr{E}_1 \otimes \mathscr{E}_2, H).$$

We set $T^{\odot n} = T \odot \cdots \odot T$ (n-times), which is an operator in $B(H \otimes \mathscr{E}^{\otimes n}, H)$.

If H and $\mathscr{E}$ are Hilbert spaces and $T \in B(H \otimes \mathscr{E}, H)$, it is sometimes useful to consider the map

$$\sigma_T : B(H) \to B(H),\ X \mapsto T(X \otimes \mathrm{id}_{\mathscr{E}})T^*.$$

The following properties of the $\odot$-product are immediate from the definition and will be used without further reference. In particular, associativity shows that the power $T^{\odot n}$ above is well-defined.

Lemma 4.1.2. *Let $H, \mathscr{E}_1, \mathscr{E}_2, \mathscr{E}_3$ be Hilbert spaces. Furthermore, let*

$$T, T_1, T_2 \in B(H \otimes \mathscr{E}_1, H),\ S, S_1, S_2 \in B(H \otimes \mathscr{E}_2, H) \text{ and } R \in B(H \otimes \mathscr{E}_3, H).$$

Then, we have

(a) (Associativity) $T \odot (S \odot R) = (T \odot S) \odot R$.

(b) (Distributivity / Bilinearity) $(T_1 + T_2) \odot S = T_1 \odot S + T_2 \odot S$ and $T \odot (S_1 + S_2) = T \odot S_2 + T \odot S_2$. Moreover, $(\lambda T) \odot S = \lambda(T \odot S) = T \odot (\lambda S)$ for $\lambda \in \mathbb{C}$.

(c) (Continuity) $\|T \odot S\| \leq \|T\|\|S\|$.

(d) $(T_1 \odot S_1)(T_2 \odot S_2)^* = T_1(S_1 S_2^* \otimes \mathrm{id}_{\mathscr{E}_1})T_2^*$. In particular, $T^{\odot n}(T^{\odot n})^* = \sigma_T^n(\mathrm{id}_H)$.

(e) If $z, w \in B(\mathscr{E}, \mathbb{C})$, then $z^{\odot n}(w^{\odot n})^* = (zw^*)^n$ for $n \geq 1$.

(f) (Compatibility with multiplication operators) Let $\mathscr{H}$ be a reproducing kernel Hilbert space and let $\varphi \in \mathrm{Mult}(\mathscr{H} \otimes \mathscr{E}_1, \mathscr{H})$ and $\psi \in \mathrm{Mult}(\mathscr{H} \otimes \mathscr{E}_2, \mathscr{H})$ be row multipliers. Define $(\varphi \odot \psi)(z) = \varphi(z) \odot \psi(z)$. Then $\varphi \odot \psi \in \mathrm{Mult}(\mathscr{H} \otimes \mathscr{E}_2 \otimes \mathscr{E}_1)$ and $M_{\varphi \odot \psi} = M_\varphi \odot M_\psi$.

Proof. (a) Using the canonical identification of $(\mathscr{E}_1 \otimes \mathscr{E}_2) \otimes \mathscr{E}_3$ with $\mathscr{E}_1 \otimes (\mathscr{E}_2 \otimes \mathscr{E}_3)$, we find that

$$\begin{aligned} T \odot (S \odot R) = T((S(R \otimes \mathrm{id}_{\mathscr{E}_2})) \otimes \mathrm{id}_{\mathscr{E}_1}) &= T(S \otimes \mathrm{id}_{\mathscr{E}_1})(R \otimes \mathrm{id}_{\mathscr{E}_2} \otimes \mathrm{id}_{\mathscr{E}_1}) \\ &= (T \odot S) \odot R. \end{aligned}$$

(b) and (c) are obvious.

(d) follows from a straightforward computation.

(e) The proof is by induction on n, with a trivial base step $n = 1$. If $n \geq 2$ and the assertion has been shown for $n - 1$, then using associativity in the first step and the inductive hypothesis in the second, we obtain

$$z^{\odot n}(w^{\odot n})^* = z(z^{\odot n-1} \otimes \mathrm{id}_{\mathscr{E}})(w^{\odot n-1} \otimes \mathrm{id}_{\mathscr{E}})^* w^* = z((zw^*)^{n-1} \otimes \mathrm{id}_{\mathscr{E}})w^* = (zw^*)^n,$$

as $zw^* \in \mathbb{C}$, establishing the claim for n.

(f) By (c), $M_\varphi \odot M_\psi \in B(\mathscr{H} \otimes \mathscr{E}_2 \otimes \mathscr{E}_1, \mathscr{H})$. If $f \in \mathscr{H}, x \in \mathscr{E}_1, y \in \mathscr{E}_2$, then

$$\begin{aligned} (M_\varphi \odot M_\psi)(f \otimes y \otimes x)(z) = M_\varphi(M_\psi(f \otimes y) \otimes x)(z) &= \varphi(z)(\psi(z)(f(z)y)x) \\ &= (\varphi(z) \odot \psi(z))(f(z)y \otimes x), \end{aligned}$$

so $M_\varphi \odot M_\psi$ is given by multiplication with $\varphi \odot \psi$. □

Since the $\odot$-product reduces to the ordinary product of operators when the coefficient Hilbert spaces $\mathscr{E}_1 = \mathscr{E}_2 = \mathbb{C}$, it is in general not commutative.

Lemma 4.1.3. *Let H_1, H_2 and $\mathscr{E}$ be Hilbert spaces.*

(a) Let $S \in B(H_1 \otimes \mathscr{E}, H_1)$, $T \in B(H_2 \otimes \mathscr{E}, H_2)$ and $R \in B(H_1, H_2)$. If

$$RS = T(R \otimes \mathrm{id}_{\mathscr{E}}),$$

then

$$RS^{\odot n} = T^{\odot n}(R \otimes \mathrm{id}_{\mathscr{E}^{\otimes n}}).$$

(b) Let $d \in \mathbb{N} \cup \{\infty\}$. Using the notation $\mathbb{C}^\infty = \ell^2(\mathbb{N})$, let $T = [T_1, \ldots, T_d] : H \otimes \mathbb{C}^d \to H$ be a bounded row operator, with

$$T\left(h \otimes (z_l)_{l=1}^d\right) = \sum_{l=1}^d T_l(z_l h)$$

for $h \in H$ and $(z_l)_{l=1}^d \in \mathbb{C}^d$. If $h \in H$ and $(z^{(m)})_{m=1}^n$ is a family in $\mathbb{C}^d$ with $z^{(m)} = \left(z_l^{(m)}\right)_{l=1}^d$ for $m = 1, \ldots, n$, then

$$T^{\odot n}\left(h \otimes \left(\otimes_{m=1}^n z^{(m)}\right)\right) = \sum_{l_n, \ldots, l_1 = 1}^d (T_{l_n} \cdot \ldots \cdot T_{l_1})\left(\left(z_{l_n}^{(n)} \cdot \ldots \cdot z_{l_1}^{(1)}\right) h\right)$$

In particular

$$\mathrm{Im}(T^{\odot n}) \subset \bigvee \{T_{l_1} \cdot \ldots \cdot T_{l_n} h;\ h \in H, (l_1, \ldots, l_n) \in [d]^n\},$$

where

$$[d] = \begin{cases} \{1, \ldots, d\} \text{ for } d \geq 1 \text{ and} \\ \mathbb{N}_{>0} \text{ for } d = \infty. \end{cases}$$

Proof. (a) Proof by induction on n, with trivial base step $n = 1$. Suppose that $n \geq 2$ and the assertion has been shown for $n-1$. Then using associativity in the first and fourth step and the inductive hypothesis in the third, we obtain

$$\begin{aligned} RS^{\odot n} &= RS(S^{\odot n-1} \otimes \mathrm{id}_{\mathscr{E}}) = T((RS^{\odot n-1}) \otimes \mathrm{id}_{\mathscr{E}}) = T((T^{\odot n-1}(R \otimes \mathrm{id}_{\mathscr{E}^{\otimes n-1}})) \otimes \mathrm{id}_{\mathscr{E}})) \\ &= T^{\odot n}(R \otimes \mathrm{id}_{\mathscr{E}^{\otimes n}}). \end{aligned}$$

(b) We proceed by induction on n. The base step $n = 1$ is trivial. If $n \geq 2$ and the assertion has been shown for $n-1$, let $h \in H$ and $z^{(m)} = \left(z_l^{(m)}\right)_{l=1}^d \in \mathbb{C}^d$ for $m = 1, \ldots, n$.

Then using associativity in the first and fourth step and the inductive hypothesis in the third, we deduce that

$$
\begin{aligned}
T^{\odot n}\left(h\otimes\left(\otimes_{m=1}^{n} z^{(m)}\right)\right) &= T\left(T^{\odot n-1}\left(h\otimes\left(\otimes_{m=1}^{n-1} z^{(m)}\right)\right)\otimes z^{n}\right)\\
&= \sum_{l_n=1}^{d} T_{l_n}(T^{\odot n-1}\left(h\otimes\left(\otimes_{m=1}^{n-1} z^{(m)}\right)\right) z_{l_n}^{(n)}\\
&= \sum_{l_n=1}^{d} T_{l_n}\left(\sum_{l_{n-1},\dots,l_1=1}^{d}(T_{l_{n-1}}\cdot\dots\cdot T_{l_1})\left(\left(z_{l_{n-1}}^{(n-1)}\cdot\dots\cdot z_{l_1}^{(1)}\right)h\right)\right) z_{l_n}^{(n)}\\
&= \sum_{l_n,\dots,l_1=1}^{d}(T_{l_n}\cdot\dots\cdot T_{l_1})\left(\left(z_{l_n}^{(n)}\cdot\dots\cdot z_{l_1}^{(1)}\right)h\right).
\end{aligned}
$$

The additional part follows, since

$$
\sum_{l_n,\dots,l_1=1}^{d}(T_{l_n}\cdot\dots\cdot T_{l_1})\left(\left(z_{l_n}^{(n)}\cdot\dots\cdot z_{l_1}^{(1)}\right)h\right)\in\bigvee\{T_{l_1}\cdot\dots\cdot T_{l_n}h;\ h\in H,(l_1,\dots,l_n)\in[d]^n\},
$$

using that $T^{\odot n}$ is linear and bounded and the fact that linear combinations of elementary tensors are dense in $H\otimes(\mathbb{C}^d)^{\otimes n}$. □

4.1.1. Pureness of K-contractions with Nevanlinna-Pick kernel

We now want to characterize pure K-contractions for complete Nevanlinna–Pick kernels K with the help of the product for infinite tuples from the previous section.

Definition 4.1.4. A tuple $U=(U_1,\dots,U_d)\in B(\tilde{H})^d$ is called a spherical unitary if each U_l is normal and $\sum_{l=1}^{d}U_lU_l^*=\mathrm{id}_{\tilde{H}}$ or equivalently $\sum_{l=1}^{d}U_lU_l^*=\sum_{l=1}^{d}U_l^*U_l=\mathrm{id}_{\tilde{H}}$.

We need the following theorem by Clouâtre and Hartz (see Theorem 5.6 in [CH18]):

Theorem 4.1.5 (Clouâtre, Hartz)**.** *Let $\mathscr{H}$ be a regular unitarily invariant complete Nevanlinna-Pick space. Let $T=(T_1,\dots,T_d)\in B(H)^d$ be a commuting tuple of operators on a Hilbert space. Denote by $M_z=(M_{z_1},\dots,M_{z_d})\in B(\mathscr{H})^d$ the tuple of operators of multiplication by the coordinate functions. Then the following assertions are equivalent:*

(a) T is a K-contraction

(b) There exist Hilbert spaces $\mathscr{E}$, $\tilde{H}$, a spherical unitary $U\in B(\tilde{H})^d$, and an isometry $V^{(U)}:H\to\mathscr{H}(\mathscr{E})\oplus\tilde{H}$ such that

$$V^{(U)}T_l^*=(M_{z_l}\oplus U_l)^*V^{(U)}$$

for all $l=1,\dots,d$

For a proof see also Remark 2.5.15 and [Sch18, Theorem 2.25].

Notation 4.1.6. For a Hilbert space $\mathscr{E}$ denote by

$$F(\mathscr{E}) = \bigoplus_{n=0}^{\infty} \mathscr{E}^{\otimes n}$$

the corresponding (full) Fock-space.

Remark 4.1.7. (a) For $n \geq 1$, let S_n be the symmetric group on n letters. For a permutation $\sigma \in S_n$, let $U_\sigma : \mathscr{E}^{\otimes n} \to \mathscr{E}^{\otimes n}$ be the unique unitary operator satisfying

$$U_\sigma(x_1 \otimes \ldots \otimes x_n) = x_{\sigma^{-1}(1)} \otimes \ldots \otimes x_{\sigma^{-1}(n)} \quad (x_1 \ldots, x_n \in \mathscr{E}).$$

The symmetric n-fold tensor power of $\mathscr{E}$ is defined by

$$\mathscr{E}^{\otimes^{\text{sym}}(n)} = \{x \in \mathscr{E}^{\otimes n};\ U_\sigma x = x\}.$$

The Hilbert space

$$F_{\text{sym}}(\mathscr{E}) = \bigoplus_{n=0}^{\infty} \mathscr{E}^{\otimes^{\text{sym}}(n)} \subset F(\mathscr{E})$$

is called the symmetric Fock-space over $\mathscr{E}$. Denote by P_{sym} the orthogonal projection from $F(\mathscr{E})$ onto $F_{\text{sym}}(\mathscr{E})$.

(b) Let $e_1, \ldots, e_d$ be the canonical orthogonal basis for $\mathbb{C}^d$. In [Arv98, Proposition 2.13], Arveson shows that there exists a unique unitary operator $W : H_d^2 \to F_{\text{sym}}(\mathbb{C}^d)$ such that $W(1) = 1$ and

$$W(z_1, \ldots, z_{i_n}) = \otimes^{\text{sym}}_{l=1,\ldots,n} e_{i_l}, \quad (n \geq 1).$$

Furthermore, if $S_1, \ldots, S_d$ on $F_{\text{sym}}(\mathbb{C}^d)$ are defined by

$$S_l x = P_{\text{sym}}(e_l \otimes x), \quad x \in F_{\text{sym}}(\mathbb{C}^d),$$

then

$$W M_{z_l} = M_{z_l} W \quad (l = 1, \ldots, d).$$

(c) Let $T = (T_1, \ldots, T_d) \in B(H)^d$ be a commuting row contraction, let

$$C = \left(1 - \sum_{l=1}^{d} T_l T_l^*\right)^{1/2}$$

be the corresponding defect operator and denote by $\mathscr{D} = \overline{\operatorname{Im} C}$ the defect space. Furthermore, let $j : H \to H^2(\mathscr{D})$,

$$j(h) = \sum_{\alpha \in \mathbb{N}^d} \gamma_\alpha C (T^\alpha)^* h z^\alpha$$

be the contraction with

$$j T_l^* = M_{z_l}^* j \quad (l = 1, \ldots, d),$$

defined in Theorem 2.5.14 and Remark 2.5.15. Fix $w \in \mathbb{B}_d$. With the notation introduced in Definition 2.1.7, one computes that

$$j(h)(w) = \sum_{n=0}^{\infty} CZ(w)^{\odot n} \left(T^{\odot n}\right)^* h.$$

Since the operators T_l commute, it follows modulo identification that

$$\left(T^{\odot n}\right)^* h \in \left(\mathbb{C}^d\right)^{\otimes^{\mathrm{sym}}(n)} \otimes H.$$

Hence,

$$(W \otimes \mathrm{id}_{\mathscr{D}})\, j(h) = \oplus_{n=0}^{\infty} \left(\mathrm{id}_{(\mathbb{C}^d)^{\otimes \mathrm{sym}(n)}} \otimes C\right) \left(T^{\odot n}\right)^* h$$

and one can check that j is an isometry if and only if

$$\mathrm{SOT} - \lim_{n\to\infty} (T^{\odot n})^* = 0.$$

Remark 4.1.8. (a) Let $z, w \in \mathbb{B}_d$ and let Z be as in Definition 2.1.7. Let $N \geq 1$ and

$$b^{(N)}(z) = \left(\sqrt{b_1} Z(z), \sqrt{b_2} Z(z)^{\odot 2}, \ldots, \sqrt{b_N} Z(z)^{\odot N}, 0, \ldots\right).$$

Using Lemma 4.1.2 and

$$Z(z)Z(w)^* = \langle z, w\rangle\, \mathrm{id}_{\mathbb{C}},$$

one computes that

$$b^{(N)}(z) b^{(N)}(w)^* = \sum_{n=1}^{N} b_n \langle z, w\rangle^N.$$

By a well-known argument form the theory of infinite operator matrices (Corollary A.3.3), it follows that $b : \mathbb{B}_d \to B\left(F(\mathbb{C}^d), \mathbb{C}\right)$,

$$b(z) = \left(\sqrt{b_1} Z(z), \sqrt{b_2} Z(z)^{\odot 2}, \ldots\right)$$

is well-defined and that

$$K(z,w) = \frac{1}{1 - b(z)b(w)^*}$$

for all $z, w \in \mathbb{B}_d$. In particular, since $L_b : \mathbb{B}_d \times \mathbb{B}_d \to \mathbb{C}$,

$$L_b(z,w) = K(z,w)(1 - b(z)b(w)^*) = 1$$

is positive definite, we conclude that $b \in \mathrm{Mult}(\mathscr{H} \otimes F(\mathbb{C}^d), \mathscr{H})$ is contractive.

(b) Let $T \in B(H)^d$ be a K-contraction. Let $N \geq 1$ and

$$b^{(N)}(T) = \left(\sqrt{b_1} T, \sqrt{b_2} T^{\odot 2}, \ldots, \sqrt{b_N} T^{\odot N}, 0, \ldots\right).$$

Using Lemma 4.1.2, one computes that

$$b^{(N)}(T)b^{(N)}(T)^* = \sum_{n=0}^{N} b_n \sigma_T^n(\mathrm{id}_H) = 1 - \left(\frac{1}{K}\right)_N (T).$$

A well-known argument form the theory of infinite operator matrices (cf. Corollary A.3.3) yields that

$$b(T) = \left(\sqrt{b_1}T, \sqrt{b_2}T^{\odot 2}, \sqrt{b_3}T^{\odot 3}, \dots\right)$$

is an operator in $B(H \otimes F\left(\mathbb{C}^d\right), H)$ with $\|b(T)\| \leq 1$. Conversely, let

$$T = (T_1, \dots, T_d) \in B(H)^d$$

be a tuple of commuting operators such that

$$b(T) = \left(\sqrt{b_1}T, \sqrt{b_2}T^{\odot 2}, \sqrt{b_3}T^{\odot 3}, \dots\right)$$

is a well-defined operator in $B(H \otimes F\left(\mathbb{C}^d\right), H)$ with $\|b(T)\| \leq 1$. One checks that

$$\left(\frac{1}{K}\right)(T) = 1 - b(T)b(T)^* \geq 0.$$

Hence, $T \in B(H)^d$ is a K-contraction.

Remark 4.1.9. (a) Since the point evaluations of $\mathscr{H}$ are continuous, it is not difficult to check that

$$M_{b^{(N)}} = b^{(N)}(M_z) \quad (N \in \mathbb{N}_{>0}) \text{ and } M_b = b(M_z).$$

(b) Suppose that $U \in B(\tilde{H})^d$ is a spherical unitary. Since $(b_n)_{n\in\mathbb{N}}$ is a sequence of non-negative real numbers satisfying $\sum_{n=1}^{\infty} b_n \leq 1$, it is also readily seen that

$$b(U) = \left(\sqrt{b_1}U, \sqrt{b_2}U^{\odot 2}, \sqrt{b_3}U^{\odot 3}, \dots\right)$$

is a well-defined operator in $B(H \otimes F\left(\mathbb{C}^d\right), H)$ with $\|b(U)\| \leq 1$.

(c) Let T be a K-contraction. Due to Theorem 4.1.5, there exist Hilbert spaces $\mathscr{E}$, $\tilde{H}$, a spherical unitary $U \in B(\tilde{H})^d$ and an isometry $V^{(U)} : H \to \mathscr{H}(\mathscr{E}) \oplus \tilde{H}$ such that

$$V^{(U)} T_l^* = (M_{z_l} \oplus U_l)^* V^{(U)}$$

for all $l = 1, \dots, d$. If $M_z = (M_{z_l})_{l=1}^d$, then part (a) of Lemma 4.1.3 yields that

$$\left(V^{(U)} \otimes \mathrm{id}_{(\mathbb{C}^d)^{\otimes n}}\right)(T^{\odot n})^* = ((M_z \oplus U)^{\odot n})^* V^{(U)}.$$

This implies for $N \in \mathbb{N}$ that

$$\left(V^{(U)} \otimes \mathrm{id}_{F(\mathbb{C}^d)}\right) b^{(N)}(T)^* = (b^{(N)}(M_z) \oplus b_N(U))^* V^{(U)}.$$

Since $b^{(N)}(T) \xrightarrow{SOT} b(T)$, $b^{(N)}(M_z) \xrightarrow{SOT} b(M_z)$ and $b^{(N)}(U) \xrightarrow{SOT} b(U)$ for $N \to \infty$ (cf. Corollary A.3.3), it follows that

$$\left(V^{(U)} \otimes \mathrm{id}_{F(\mathbb{C}^d)}\right) b(T)^* = (b(M_z) \oplus b(U))^* V^{(U)}.$$

Part (a) of Lemma 4.1.3 implies that

$$\left(V^{(U)} \otimes \mathrm{id}_{F(\mathbb{C}^d)^{\otimes n}}\right) (b(T)^{\odot n})^* = (b(M_z))^{\odot n} \oplus b(U))^{\odot n})^* V^{(U)}$$

for $n \geq 1$. In particular, if T is a pure K-contraction, that is if $\tilde{H} = 0$, then

$$\left(V^{(U)} \otimes \mathrm{id}_{F(\mathbb{C}^d)^{\otimes n}}\right) (b(T)^{\odot n})^* = (b(M_z)^{\odot n})^* V^{(U)}.$$

for $n \geq 1$.

Lemma 4.1.10. *Let $M_z \in B(\mathscr{H})^d$ be the weighted shift, then*

$$\mathrm{SOT} - \lim_{n\to\infty} (b(M_z)^{\odot n})^* = 0.$$

Proof. Let $n \in \mathbb{N}$, then

$$(b(M_z)^{\odot n})^* = (M_b^{\odot n})^* = (M_{b^{\odot n}})^*$$

and

$$\|(b(M_z)^{\odot n})^*\| \leq \|M_b\|^n \leq 1.$$

Fix $z \in \mathbb{B}_d$. It follows that

$$(b(M_z)^{\odot n})^* k_z = (M_{b^{\odot n}})^* k_z = b(z)^{\odot n} k_z$$

where $b(z) : \mathscr{H} \otimes F\left(\mathbb{C}^d\right) \to \mathscr{H}$,

$$b(z) = \left(\sqrt{b_1} Z(z), \sqrt{b_2} Z(z)^{\odot 2}, \ldots\right)$$

with $\|b(z)\|_{\mathscr{H} \otimes F\left(\mathbb{C}^d\right) \to \mathscr{H}} < 1$. Since

$$0 \leq \lim_{n\to\infty} \|b(z)^{\odot n}\| \leq \lim_{n\to\infty} \|b(z)\|^n = 0,$$

we deduce that

$$\lim_{n\to\infty} (b(M_z)^{\odot n})^* h = 0$$

for all

$$h \in \mathrm{span}\{k_z;\, z \in \mathbb{B}_d\}.$$

Using that $\|(b(M_z)^{\odot n})^*\| \leq 1$ for all $n \geq 1$ and the fact that $\mathrm{span}\{k_z;\, z \in \mathbb{B}_d\} \subset \mathscr{H}$ is dense, we conclude that

$$\lim_{n\to\infty} \|(b(M_z)^{\odot n})^* f\| = 0$$

for all $f \in \mathscr{H}$. □

Lemma 4.1.11. *Let $T \in B(H)^d$ be a pure K-contraction. Then*

$$\mathrm{SOT} - \lim_{n\to\infty} (b(T)^{\odot n})^* = 0.$$

Proof. Let T be a pure K-contraction. Let $j : H \to \mathscr{H}(\mathscr{D})$ be the intertwining isometry, defined in Theorem 2.5.14 with $jT_l^* = M_{z_l}^* j$ for $l = 1, \dots, d$. With Remark 4.1.9, it follows that

$$(b(T)^{\odot n})^* = \left(j^* \otimes \mathrm{id}_{F(\mathbb{C}^d)^{\otimes n}} \right) (b(M_z)^{\odot n})^* j$$

for all $n \in \mathbb{N}$ with $n \geq 1$. Since

$$\mathrm{SOT} - \lim_{n\to\infty} (b(M_z)^{\odot n})^* = 0,$$

the assertion follows from Lemma 4.1.10. □

Next we want to show that the converse of Lemma 4.1.11 is also true. Namely, if

$$\mathrm{SOT} - \lim_{n\to\infty} (b(T)^{\odot n})^* = 0,$$

for a K-contraction T, then T is pure.

Lemma 4.1.12. *Let $\mathscr{H}$ be a regular unitarily invariant complete Nevanlinna-Pick space with reproducing kernel $K : \mathbb{B}_d \times \mathbb{B}_d \to \mathbb{C}$,*

$$K(z,w) = \sum_{n=0}^{\infty} a_n \langle z, w\rangle^n = \frac{1}{1 - \sum_{n=1}^{\infty} b_n \langle z, w\rangle^n},$$

where $\sum_{n=0}^{\infty} a_n = \infty$. If $T \in B(H)^d$ is a K-contraction with

$$\mathrm{SOT} - \lim_{n\to\infty} (b(T)^{\odot n})^* = 0,$$

then T is pure.

Proof. Suppose that T is a K-contraction with

$$\mathrm{SOT} - \lim_{n\to\infty} (b(T)^{\odot n})^* = 0.$$

Due to Theorem 4.1.5, there exist Hilbert spaces $\mathscr{E}$, $\tilde{H}$, a spherical unitary $U \in B(\tilde{H})^d$ and an isometry $V^{(U)} : H \to \mathscr{H}(\mathscr{E}) \oplus \tilde{H}$ such that

$$V^{(U)} T_l^* = (M_{z_l} \oplus U_l)^* V^{(U)}$$

for all $l = 1, \dots, d$. If $\sum_{n=0}^{\infty} a_n = \infty$, using that $a_n > 0$ and $b_n > 0$ for all $n \in \mathbb{N}$, the theorem on convergence of a monotone series yields that

$$\sum_{n=0}^{\infty} b_n = 1 - \frac{1}{\sum_{n=0}^{\infty} a_n} = 1.$$

Hence, if $U \in B(\tilde{H})^d$ is a spherical unitary, then

$$\begin{aligned} b(U)b(U)^* &= \sum_{n=1}^{\infty} b_n \sum_{|\alpha|=n} \frac{n!}{\alpha!} U^\alpha U^{\alpha *} \\ &= \sum_{n=1}^{\infty} b_n \left(\sum_{l=1}^{d} U_l U_l^* \right)^n \\ &= \sum_{n=1}^{\infty} b_n \operatorname{id}_{\tilde{H}} \\ &= \operatorname{id}_{\tilde{H}} . \end{aligned}$$

It follows inductively that

$$\begin{aligned} (b(U)^{\odot n})(b(U)^{\odot n})^* &= b(U)((b(U)^{\odot n-1}) \left(b(U)^{\odot n-1})^* \otimes \operatorname{id}_{F(\mathbb{C}^d)} \right) b(U)^* \\ &= b(U) \left(\operatorname{id}_{\tilde{H}} \otimes \operatorname{id}_{F(\mathbb{C}^d)} \right) b(U)^* \\ &= b(U)b(U)^* \\ &= \operatorname{id}_{\tilde{H}} . \end{aligned}$$

Because of Lemma 4.1.10,

$$\mathrm{SOT} - \lim_{n\to\infty} (b(M_z)^{\odot n})^* = 0.$$

Hence, by assumption

$$\begin{aligned} \left\| P_{\tilde{H}} V^{(U)} h \right\| &= \lim_{n\to\infty} \left\| ((b(M_z)^{\odot n})^* \oplus (b(U)^{\odot n})^*) V^{(U)} h \right\| \\ &= \lim_{n\to\infty} \left\| \left(V^{(U)} \oplus \operatorname{id}_{F(\mathbb{C}^d)^{\otimes n}} \right) (b(T)^{\odot n})^* h \right\| \\ &= \lim_{n\to\infty} \|(b(T)^{\odot n})^* h\| \\ &= 0 \end{aligned}$$

for all $h \in H$. Thus, $\operatorname{Im}(V^{(U)}) \perp \tilde{H}$ and the pureness condition follows from Theorem 2.5.16 with the isometry $V = V^{(U)}$. □

Remark 4.1.13. In the case $\sum_{n=0}^{\infty} a_n < \infty$ every spherical unitary U is a pure K-contraction. Since M_z is also pure K-contraction, every K-contraction is pure (cf. Proposition 2.19, Lemma 2.20, Lemma 2.21, Theorem 2.25 and Theorem 3.22 in [Sch18]).

By the preceding lemmas (Lemma 4.1.11 and Lemma 4.1.12) and Remark 4.1.9 we obtain the following theorem, which characterizes pure K-contractions.

Theorem 4.1.14. *Let $\mathscr{H}$ be a regular unitarily invariant complete Pick space with reproducing kernel $K\colon \mathbb{B}_d \times \mathbb{B}_d \to \mathbb{C}$,*

$$K(z,w) = \sum_{n=0}^{\infty} a_n \langle z,w \rangle^n = \frac{1}{1 - \sum_{n=1}^{\infty} b_n \langle z,w \rangle^n}.$$

Then $T \in B(H)^d$ is a pure K-contraction if and only if

$$b(T) = \left(\sqrt{b_1}T, \sqrt{b_2}T^{\odot 2}, \sqrt{b_3}T^{\odot 3}, \dots\right)$$

is a well-defined operator in $B(H \otimes F\left(\mathbb{C}^d\right), H)$ with $\|b(T)\| \leq 1$ and

$$\mathrm{SOT} - \lim_{n\to\infty} (b(T)^{\odot n})^* = 0.$$

In particular, it follows immediately that $T \in B(H)^d$ is a pure K-contraction if

$$\frac{1}{K}(T) > 0.$$

4.2. Row multipliers with vanishing tails

If

$$K : \mathbb{B}_d \times \mathbb{B}_d \to \mathbb{C},\ K(z,w) = \frac{1}{1 - b(z)b(w)^*}$$

is a unitarily invariant complete Pick kernel, then the row multiplier b has in general infinitely many non-zero entries. This complicates the proof of the main result. We overcome this obstacle by showing that under the assumptions of Theorem 4.1, the tails of the multiplier b tend to zero in multiplier norm, so that b can be approximated by finite row multipliers.

It turns out to be convenient to mostly work in a coordinate-free fashion, which leads to the following definition: If $\mathscr{E}$ is a Hilbert space, let

$$B_0(H \otimes \mathscr{E}, H) = \bigvee \{T \otimes u : T \in B(H), u \in B(\mathscr{E}, \mathbb{C})\} \subset B(H \otimes \mathscr{E}, H),$$

where $\bigvee$ denotes the norm closed linear span. We think of operators of the form $T \otimes u$ as analogues of rank one operators, and $B_0(H \otimes \mathscr{E}, H)$ as analogues of compact operators.

The following lemma explains the notion that operators in $B_0\left(H \otimes F(\mathbb{C}^d), H\right)$ have vanishing tails.

Lemma 4.2.1. *Let*

$$T = \begin{bmatrix} T_1 & T_2 & \cdots \end{bmatrix} \in B\left(H \otimes F(\mathbb{C}^d), H\right),$$

with

$$T_l \in B\left(H \otimes (\mathbb{C}^d)^{\otimes l}, H\right) \quad (l \in \mathbb{N}).$$

Then

$$T \in B_0\left(H \otimes F(\mathbb{C}^d), H\right),$$

if and only if

$$\lim_{N\to\infty} \| \begin{bmatrix} 0 & \cdots & 0 & T_{N+1} & T_{N+2} & \cdots \end{bmatrix} \| = 0.$$

Proof. Let $l \in \mathbb{N}_{>0}$. Since $(\mathbb{C}^d)^{\otimes l}$ is finite dimensional, the operator

$$T_l \in B\left(H \otimes (\mathbb{C}^d)^{\otimes l}, H\right)$$

can be written in the form

$$T_l = \sum_{m=1}^{d \cdot l} T_l^{(m)} \otimes u_m$$

where $T_l^{(m)} \in B(H)$ and $u_m \in B\left((\mathbb{C}^d)^{\otimes l}, \mathbb{C}\right)$ for $m = 1, \ldots, l \cdot d$. Thus, it is immediate that

$$\begin{bmatrix} T_1 & \cdots & T_N & 0 & \cdots \end{bmatrix} \in B_0\left(H \otimes F(\mathbb{C}^d), H\right).$$

If the tails of T converge to 0, then T is the norm limit of the operators above, and hence belongs to $B_0\left(H \otimes F(\mathbb{C}^d), H\right)$.

Conversely, suppose that $T \in B_0\left(H \otimes F(\mathbb{C}^d), H\right)$ and let $\varepsilon > 0$. Then there exists $S = \sum_{l=1}^n S_l \otimes u_l$ with $\|T - S\| \leq \varepsilon$, where $S_l \in B(H), \|S_l\| \leq 1$ and $u_l \in B\left(F(\mathbb{C}^d), \mathbb{C}\right)$. By definition of $F(\mathbb{C}^d) \cong B\left(F(\mathbb{C}^d), \mathbb{C}\right)$, there is an $N_0 \in \mathbb{N}$ and

$$v_l = \left(v_l^{(m)}\right)_{m=1}^{N_0} \in B\left(\bigoplus_{m=1}^{N_0} (\mathbb{C}^d)^{\otimes m}, \mathbb{C}\right)$$

so that $\|u_l - v_l\| \leq \varepsilon / n$ for $l = 1, \ldots, n$. For $m = 1, \ldots, N_0$ let

$$R_m = \sum_{l=1}^n S_l \otimes v_l^{(m)} \in B\left(H \otimes (\mathbb{C}^d)^{\otimes m}, H\right)$$

and define

$$R = \begin{bmatrix} R_1 & R_2 & \cdots & R_{N_0} & 0 & \cdots \end{bmatrix}.$$

Then $\|R - T\| \leq 2\varepsilon$ and $R_N = 0$ for $N > N_0$. Hence, we obtain

$$\begin{aligned} \limsup_{N \to \infty} \| \begin{bmatrix} 0 & \cdots & 0 & T_{N+1} & T_{N+2} \end{bmatrix} \| &\leq 2\varepsilon + \limsup_{N \to \infty} \| \begin{bmatrix} 0 & \cdots & 0 & R_{N+1} & R_{N+2} \end{bmatrix} \| \\ &= 2\varepsilon. \end{aligned}$$

Since $\varepsilon > 0$ was arbitrary, it follows that the tails of T tend to zero. □

We also require the following general operator theoretic result, which is undoubtedly known. If $T \in B(H)$ is an bounded linear operator on a Hilbert space H, we denote by $\|T\|_e$ the essential norm of T.

Lemma 4.2.2. *Let (T_n) be an increasing sequence of self-adjoint operators on a Hilbert space that converges to T in the strong operator topology. Then* $\lim_{n \to \infty} \|T - T_n\| = 0$ *if and only if* $\lim_{n \to \infty} \|T - T_n\|_e = 0$.

Proof. To establish the non-trivial direction, it suffices to prove the following variant: If (T_n) is a decreasing sequence of positive contractions tending to 0 in SOT and satisfying $\lim_{n\to\infty} \|T_n\|_e = 0$, then $\lim_{n\to\infty} \|T_n\| = 0$. Indeed, the lemma follows by applying this variant to a suitably rescaled version of the sequence $(T - T_n)$.

To prove the variant, let $\varepsilon > 0$ and let $N \in \mathbb{N}$ so that $\|T_N\|_e < \varepsilon$. Then there exists a compact operator K so that $\|T_N - K\| < \varepsilon$. Since K is compact, we may choose a finite rank projection P so that $\|K(1-P)\| < \varepsilon$; whence

$$\|T_N(1-P)\| \leq \|(T_N - K)(1-P)\| + \|K(1-P)\| < 2\varepsilon.$$

Since (T_n) is a decreasing sequence of positive contractions, $T_n^2 \leq T_n \leq T_N$ for $n \geq N$. Along with the C^*-identity, this applies for $n \geq N$ the estimate

$$\|T_n(1-P)\|^2 = \|(1-P)T_n^2(1-P)\| \leq \|(1-P)T_N(1-P)\| < 2\varepsilon. \tag{4.1}$$

Finally, since (T_n) converges to 0 in SOT and since P has finite rank, one easily checks that $\lim_{n\to\infty} \|T_n P\| = 0$. In combination with (4.1), this yields

$$\limsup_{n\to\infty} \|T_n\| \leq \limsup_{n\to\infty} \|T_n(1-P)\| + \limsup_{n\to\infty} \|T_n P\| \leq \sqrt{2\varepsilon}.$$

Thus, (T_n) converges to 0 in operator norm. □

Simple examples of sequences of finite rank projections show that in the preceding lemma, both monotonicity of the sequence (T_n) and the assumption that the SOT-limit agrees with the limit in the Calkin algebra cannot be dropped.

We are now in the position to prove the result alluded to at the beginning of this section.

Proposition 4.2.3. *Let $\mathscr{H}$ be a regular unitarily invariant complete Pick space on with kernel*

$$K : \mathbb{B}_d \times \mathbb{B}_d \to \mathbb{C},\ K(z,w) = \frac{1}{1 - b(z)b(w)^*}$$

as above. Then $M_b \in B_0(\mathscr{H} \otimes F(\mathbb{C}^d), \mathscr{H})$ if and only if K is unbounded.

Proof. Let

$$b_N(z) = (b - b^{(N)})(z) = (0, \ldots, 0\sqrt{b_N} Z(z)^{\odot N}, \sqrt{b_{N+1}} Z(z)^{\odot N+1}, \ldots)$$

Due to Lemma 4.2.1, it suffices to show that

$$\lim_{N\to\infty} \|b_N\|_{\mathrm{Mult}} = \|b_N(M_z)\| = 0$$

if and only if K is unbounded.

The C^*-identity shows that

$$\|b_N(M_z)\| = \|M_{b_N} M_{b_N}^*\| = \left\| \sum_{n=N}^{\infty} b_n M_z^{\odot n} (M_z^{\odot n})^* \right\|.$$

If $P_{\mathbb{C}}$ denotes the orthogonal projection onto the constant functions, then it is well-known and easy to see that

$$1 - P_{\mathbb{C}} = M_b M_b^* = \sum_{n=1}^{\infty} b_n M_z^{\odot n} (M_z^{\odot n})^*.$$

Here, the preceding two sums converge in the strong operator topology (see Theorem 2.5.6). It follows that

$$\lim_{N \to \infty} \|b_N(M_z)\| = 0,$$

if and only if

$$\lim_{N \to \infty} \left\| 1 - P_{\mathbb{C}} - \sum_{n=1}^{N-1} b_n M_z^{\odot n} (M_z^{\odot n})^* \right\| = 0.$$

In turn, Lemma 4.2.2 shows that this happens if and only if the essential norms satisfy

$$\lim_{N \to \infty} \left\| 1 - \sum_{n=1}^{N-1} b_n M_z^{\odot n} (M_z^{\odot n})^* \right\|_e = 0. \tag{4.2}$$

Since $\mathscr{H}$ is regular, the quotient of $C^*(A(\mathscr{H}))/K(\mathscr{H})$ is $*$-isomorphic to $C(\partial \mathbb{B}_d)$, via the map sending M_{z_i} to z_i (see Theorem 2.4.29). Thus,

$$M_z^{\odot n} (M_z^{\odot n})^* = \sum_{|\alpha|=n} \gamma_\alpha M_z^\alpha (M_z^\alpha)^* \equiv \sum_{|\alpha|=n} \gamma_\alpha z^\alpha \overline{z}^\alpha = \|z\|^{2n} = 1 \mod K(\mathscr{H})$$

so that the left-hand side of equation (4.2) is equals $1 - \sum_{n=1}^{N-1} b_n$. It follows that

$$\lim_{N \to \infty} \|b_N(M_z)\| = 0,$$

if and only if $\sum_{n=1}^{\infty} b_n = 1$, which in turn is equivalent to unboundedness of K. □

4.3. Factoring out zeros

The proof of the theorem of Miller, Olin, and Thomson crucially uses the fact that in $H^\infty(\mathbb{D})$, one can divide out zeros. More precisely, if $\varphi \in H^\infty(\mathbb{D})$ has a zero of order N at the origin, then $\varphi = z^N \psi$ for some function $\psi \in H^\infty(\mathbb{D})$ with $\|\psi\|_\infty \le \|\varphi\|_\infty$. In the Drury-Arveson space, a version of Leech's theorem yields the following replacement, which can be regarded as a solution of Gleason's problem in $\operatorname{Mult}(H_d^2)$: If $\varphi \in \operatorname{Mult}(H_d^2)$ with $\varphi(0) = 0$, then

$$\varphi = \begin{bmatrix} z_1 & \cdots & z_d \end{bmatrix} \begin{bmatrix} \varphi_1 \\ \vdots \\ \varphi_d \end{bmatrix},$$

where each $\varphi_l \in \operatorname{Mult}(H_d^2)$ and the column has multiplier norm at most $\|\varphi\|_{\operatorname{Mult}(H_d^2)}$ (see [GRS05, Corollary 4.2]) and its proof. This procedure can be iterated to factor out zeros of higher order.

For more general spaces, the characterization of pure K-contractions in Theorem 4.1.14 suggests that one should aim to factor out the row b, rather than the row of coordinate functions. This is accomplished in the following result:

Proposition 4.3.1. *Let $\mathscr{H}$ be a regular unitarily invariant complete Nevanlinna-Pick space, whose kernel*

$$K : \mathbb{B}_d \times \mathbb{B}_d \to \mathbb{C},\ K(z,w) = \frac{1}{1 - b(z)b(w)^*}$$

is unbounded. Let $N \in \mathbb{N}$ and $\varepsilon > 0$. Then there exists $M \geq N$ such that whenever $\varphi \in \mathrm{Mult}(\mathscr{H})$ has a zero of order M at 0, there exists $\psi \in \mathrm{Mult}\left(\mathscr{H}, \mathscr{H} \otimes F\left(\mathbb{C}^d\right)^{\otimes N}\right)$ with

$$\varphi = b^{\odot N}\psi$$

and

$$\|\psi\|_{\mathrm{Mult}} \leq (1+\varepsilon)\|\varphi\|_{\mathrm{Mult}}.$$

Before we start with some lemmas that are helpful for the proof of Proposition 4.3.1, we first want to state a corollary, which is an immediate consequence of Proposition 4.3.1.

For $f = \sum_{\alpha \in \mathbb{N}^d} f_\alpha z^\alpha \in \mathscr{O}(\mathbb{B}_d)$ and $N \in \mathbb{N}$ let

$$S_N[f] = \sum_{|\alpha| \leq N} f_\alpha z^\alpha$$

be the N-th partial sum of f.

Corollary 4.3.2. *Let $\mathscr{H}$ be a regular unitarily invariant complete Nevanlinna-Pick space, whose kernel*

$$K : \mathbb{B}_d \times \mathbb{B}_d \to \mathbb{C},\ K(z,w) = \frac{1}{1 - b(z)b(w)^*}$$

is unbounded. Let $N \in \mathbb{N}$ and $\varepsilon > 0$. Then there exists $M \geq N$ such that whenever $\varphi \in \mathrm{Mult}(\mathscr{H})$, there exists $\psi \in \mathrm{Mult}\left(\mathscr{H}, \mathscr{H} \otimes F\left(\mathbb{C}^d\right)^{\otimes N}\right)$ with

$$\varphi = S_{M-1}[\varphi] + b^{\odot N}\psi$$

and

$$\|\psi\|_{\mathrm{Mult}} \leq (1+\varepsilon)\|\varphi - S_{M-1}[\varphi]\|_{\mathrm{Mult}}.$$

The proof of Proposition 4.3.1 will occupy the remainder of this section. Thus, throughout this section, we assume the setting of Proposition 4.3.1. We will require two truncations of the kernel K.

First, for $N \in \mathbb{N}$ with $N \geq 1$, let

$$K_N : \mathbb{B}_d \times \mathbb{B}_d \to \mathbb{C},\ K_N(z,w) = \sum_{n=0}^{N-1} (b(z)b(w)^*)^n.$$

Let $\mathscr{H}_N$ be the reproducing kernel Hilbert space with kernel K_N.

Second, we set

$$K_N^{\perp} : \mathbb{B}_d \times \mathbb{B}_d \to \mathbb{C},\ K_N^{\perp}(z,w) = \sum_{n=N}^{\infty} (b(z)b(w)^*)^n.$$

This truncation will be relevant for factoring out powers of the row b. This can be seen from the next lemma, which is a consequence of Leech's theorem for complete Nevanlinna-Pick spaces.

Lemma 4.3.3. *Let $N \in \mathbb{N}$ and $c > 0$. A function $\varphi \in \operatorname{Mult}(\mathscr{H})$ admits a factorization of the form*

$$\varphi = b^{\odot N}\psi$$

for some $\psi \in \operatorname{Mult}\left(\mathscr{H}, F\mathscr{H} \otimes F\left(\mathbb{C}^d\right)^{\otimes N}\right)$ with $\|\psi\|_{\mathrm{Mult}} \leq c$ if and only if

$$L_{\varphi,c} : \mathbb{B}_d \times \mathbb{B}_d \to \mathbb{C},\ (z,w) \mapsto c^2 K_N^{\perp}(z,w) - K(z,w)\varphi(z)\overline{\varphi(w)}$$

is positive definite .

Proof. Due to part (e) of Lemma 4.1.2 and by the definition of the kernel functions $K_N^{\perp}$, we deduce that

$$K_N^{\perp}(z,w) = K(z,w)(b(z)b(w)^*)^N = K(z,w)b(z)^{\odot N}(b(w)^{\odot N})^*, \tag{4.3}$$

for all $z,w \in \mathbb{B}_d$. Thus, we obtain that

$$L_{\varphi,c}(z,w) = K(z,w)\left(c^2 b(z)^{\odot N}(b(w)^{\odot N})^* - \varphi(z)\overline{\varphi(w)}\right).$$

for all $z,w \in \mathbb{B}_d$. By Leech's theorem for complete Nevanlinna-Pick spaces (cf. [AM02, Theorem 8.57]), the function $L_{\varphi,c}$ is positive definite if and only if there exists $\psi \in \operatorname{Mult}\left(\mathscr{H}, \mathscr{H} \otimes F\left(\mathbb{C}^d\right)^{\otimes N}\right)$ with

$$\|\psi\|_{\mathrm{Mult}} \leq c \text{ and } \varphi = b^{\odot N}\psi.$$

□

For $N \in \mathbb{N}$ with $N \geq 1$ define the closed subspaces

$$\widetilde{\mathscr{H}_N} = \left\{ \sum_{\alpha \in \mathbb{N}^d} f_\alpha z^\alpha \in \mathscr{H};\ f_\alpha = 0 \text{ for } 0 \leq |\alpha| \leq N-1 \right\} \subset \mathscr{H}$$

Denote by $\widetilde{P}_N : \mathscr{H} \to \mathscr{H}$ the orthogonal projection onto $\widetilde{\mathscr{H}_N}$ and by $\widetilde{P}_N^{\perp} : \mathscr{H} \to \mathscr{H}$ the orthogonal projection onto $\widetilde{\mathscr{H}_N}^{\perp}$. If $f = \sum_{\alpha \in \mathbb{N}^d} f_\alpha z^\alpha$ is an element in $\mathscr{H}$, the orthogonal projections $\widetilde{P}_N$ and $\widetilde{P}_N^{\perp}$ act as

$$\widetilde{P}_N(f) = \sum_{\substack{|\alpha| \geq N \\ \alpha \in \mathbb{N}^d}} f_\alpha z^\alpha$$

and

$$\widetilde{P}_N^{\perp}(f) = \sum_{\substack{|\alpha| \le N-1 \\ \alpha \in \mathbb{N}^d}} f_\alpha z^\alpha$$

Due to Theorem 2.2.9, the function $\widetilde{K}_N : \mathbb{B}_d \times \mathbb{B}_d \to \mathbb{C}$,

$$\widetilde{K}_N(z,w) = \langle \widetilde{P}_N(k_z), k_w \rangle = \sum_{n=N}^{\infty} a_n \langle z, w \rangle^n$$

is the reproducing kernel of $\widetilde{\mathscr{H}_N}$ and $\widetilde{K}_N^{\perp} : \mathbb{B}_d \times \mathbb{B}_d \to \mathbb{C}$,

$$\widetilde{K}_N^{\perp}(z,w) = \langle \widetilde{P}_N^{\perp}(k_z), k_w \rangle = \sum_{n=0}^{N-1} a_n \langle z, w \rangle^n$$

is the reproducing kernel of $\widetilde{\mathscr{H}_N^{\perp}}$. If

$$\varphi \in \widetilde{\mathscr{H}_N} \cap \mathrm{Mult}(\mathscr{H}) \text{ and } f \in \mathscr{H},$$

then obviously $\varphi \cdot f \in \widetilde{\mathscr{H}_N}$ and

$$\|\varphi f\|_{\widetilde{\mathscr{H}_N}} = \|\varphi f\|_{\mathscr{H}} \le \|\varphi\|_{\mathrm{Mult}(\mathscr{H})} \|f\|_{\mathscr{H}}.$$

Thus, $\varphi \in \mathrm{Mult}(\mathscr{H}, \widetilde{\mathscr{H}_N})$ with

$$\|\varphi\|_{\mathrm{Mult}(\mathscr{H}, \widetilde{\mathscr{H}_N})} \le \|\varphi\|_{\mathrm{Mult}(\mathscr{H})}.$$

The key to prove Proposition 4.3.1, is to compare the kernels $K_N^{\perp}$ and $\widetilde{K}_N$. We need the following lemma:

Lemma 4.3.4. *Let $S, T \in B_0(H \otimes \mathscr{E}, H)$ and let $R \in K(H)$. Then $T(R \otimes \mathrm{id}_{\mathscr{E}})S^* \in K(H)$.*

Proof. Suppose first that $T = A \otimes u$ and $S = B \otimes v$ with $A, B \in B(H)$ and $u, v \in B(\mathscr{E}, \mathbb{C})$. Then

$$T(R \otimes \mathrm{id}_{\mathscr{E}})S^* = (A \otimes u)(R \otimes \mathrm{id}_{\mathscr{E}})(B^* \otimes v^*) = (uv^*)(ARB^*),$$

which is compact since R is compact. (Note that $uv^* \in \mathbb{C}$.) Since $B_0(H \otimes \mathscr{E}, H)$ is the closed linear span of operators of the above form, the claim follows. □

In the case of the Drury-Arveson space, $\mathscr{H}_N = \widetilde{\mathscr{H}_N^{\perp}}$ is finite dimensional. In general, $\mathscr{H}_N$ need not be finite dimensional. However, we still have the following lemma:

Lemma 4.3.5. *The inclusion $i : \mathscr{H}_N \hookrightarrow \mathscr{H}$ is contractive and compact.*

Proof. Since $K_N^{\perp} : \mathbb{B}_d \times \mathbb{B}_d \to \mathbb{C}$,

$$K_N^{\perp}(z,w) = \sum_{n=N}^{\infty} \langle b(z), b(w) \rangle^n$$

is a positive definite function and $K_N^{\perp} = K - K_N$, it follows that the inclusion i is contractive.

To see compactness of i, it suffices to establish compactness of ii^* by basic functional analysis. To this end, we will show that $ii^* = (I - M_{b^{\odot N}} M_{b^{\odot N}}^*)$. Because $i^* K(\cdot, w) = K_N(\cdot, w)$, it follows that

$$\langle ii^* K(\cdot, w), K(\cdot, z)\rangle_{\mathscr{H}} = \langle K_N(\cdot, w), K_N(\cdot, z)\rangle_{\mathscr{H}_N} = K_N(z, w)$$

On the other hand,

$$\begin{aligned}\langle (I - M_{b^{\odot N}} M_{b^{\odot N}}^*) K(\cdot, w), K(\cdot, z)\rangle_{\mathscr{H}} &= K(z,w) - \langle (b^{\odot N}(w))^* K(\cdot, w), (b^{\odot N}(z))^* K(\cdot, z)\rangle \\ &= K(z,w) - K(z,w) b^{\odot N}(z) (b^{\odot N}(w))^* \\ &= K(z,w)\left(1 - (b(z)b(w)^*)^N\right) \\ &= K_N(z,w).\end{aligned}$$

Combining the preceding two equations, we see that

$$ii^* = (I - M_{b^{\odot N}} M_{b^{\odot N}}^*), \tag{4.4}$$

as claimed.

Next, we will show by induction on N that the operator ii^* is compact. If $N = 1$, then $K_1 = 1$, so $\mathscr{H}_1$ is the space of all constant functions. In particular, ii^* has finite rank and is hence compact. (Indeed, ii^* is the orthogonal projection onto the constant functions.) If the assertion has been shown for $N-1$, then from (4.4) and part (d) of Lemma 4.1.2, we infer that

$$ii^* = \left(I - M_{b^{\odot N}} M_{b^{\odot N}}^*\right) = I - M_b M_b^* + M_b \left(\left(I - M_{b^{\odot N-1}} M_{b^{\odot N-1}}^*\right) \otimes \mathrm{id}_{F(\mathbb{C}^d)}\right) M_b^*.$$

The operator $I - M_b M_b^*$ is compact by the base case $N = 1$. To see compactness of the last summand above, note that $I - M_{b^{\odot N-1}} M_{b^{\odot N-1}}^*$ is compact by the inductive hypothesis and $M_b \in B_0\left(\mathscr{H} \otimes F(\mathbb{C}^d), \mathscr{H}\right)$ by Lemma 4.2.1. So, the last summand is compact by Lemma 4.3.4. □

Lemma 4.3.6. *Let $N \in \mathbb{N}$ with $N \geq 2$ and $\varepsilon > 0$. Then there exists $M \geq N$ such that*

$$((1+\varepsilon) K_N^{\perp} - \widetilde{K}_M) : \mathbb{B}_d \times \mathbb{B}_d \to \mathbb{C}$$

is positive definite.

Proof. Let $N \in \mathbb{N}$ with $N \geq 2$ and $\varepsilon > 0$. The kernel $K_N : \mathbb{B}_d \times \mathbb{B}_d \to \mathbb{C}$ of the space $\mathscr{H}_N$ is unitarily invariant. Thus, by Lemma 2.3.2 it has a power series representation

$$K_N(z, w) = \sum_{n=0}^{\infty} d_n \langle z, w\rangle^n \quad (z, w \in \mathbb{B}_d),$$

with $d_n \geq 0$ for $n \in \mathbb{N}$. Due to Proposition 2.3.3, the family

$$\left(\sqrt{d_n}\binom{n}{\alpha}^{1/2} z^\alpha\right)_{\alpha\in\mathbb{N}^d}$$

is a orthonormal basis of $\mathscr{H}_N$ and

$$\left(\sqrt{a_n}\binom{n}{\alpha}^{1/2} z^\alpha\right)_{\alpha\in\mathbb{N}^d}$$

is a the orthonormal basis of $\mathscr{H}$. The inclusion $i : \mathscr{H}_N \hookrightarrow \mathscr{H}$ is a diagonal operator with

$$i\left(\sqrt{d_n}\binom{n}{\alpha}^{1/2} z^\alpha\right) = \left(\sqrt{\frac{d_n}{a_n}}\right)\left(\sqrt{a_n}\binom{n}{\alpha}^{1/2} z^\alpha\right)$$

By Lemma 4.3.5, the map $i : \mathscr{H}_N \hookrightarrow \mathscr{H}$ is contractive and compact. A well-known argument on diagonal operators yields that

$$a_n - d_n \geq 0$$

for all $n \in \mathbb{N}$ and

$$\lim_{n\to\infty} \frac{d_n}{a_n} = 0.$$

In particular, there exists an $M \geq N$ such that

$$0 \leq \frac{d_n}{a_n} < \frac{\varepsilon}{\varepsilon+1}$$

for all $n \geq M$. Hence, we deduce that

$$\varepsilon a_n - (\varepsilon+1)d_n \geq 0$$

for all $n \geq M$. Using that $K_N^\perp = K - K_N$, one computes that

$$((1+\varepsilon)K_N^\perp - \widetilde{K}_M)(z,w) = (\varepsilon+1)\sum_{n=0}^{M-1}(a_n - d_n)\langle z,w\rangle^n + \sum_{n=M}^{\infty}(\varepsilon a_n - (\varepsilon+1)d_n)\langle z,w\rangle^n$$

for all $z, w \in \mathbb{B}_d$. Thus, it follows by Lemma 2.3.2 that

$$(1+\varepsilon)K_N^\perp - \widetilde{K}_M$$

is positive definite. □

Proof of Proposition 4.3.1. Let $N \in \mathbb{N}$ and $\varepsilon > 0$. By Lemma 4.3.6 there exists $M \geq N$ such that

$$((1+\varepsilon)^2 K_N^\perp - \widetilde{K}_M) : \mathbb{B}_d \times \mathbb{B}_d \to \mathbb{C},$$

is positive definite. Suppose that $\varphi \in \mathrm{Mult}(\mathscr{H})$ has a zero of order M at 0. Without loss of generality let us assume that $\|\varphi\|_{\mathrm{Mult}} = 1$ otherwise replace φ by $\frac{\varphi}{\|\varphi\|_{\mathrm{Mult}}}$ if $\varphi \neq 0$. Then $\varphi \in \mathrm{Mult}(\mathscr{H}, \widetilde{\mathscr{H}}_M)$ with multiplier norm less then 1. Due to Theorem2.2.6, the map

$$\widetilde{L}_{\varphi,1} : \mathbb{B}_d \times \mathbb{B}_d \to \mathbb{C},\ (z,w) \mapsto \widetilde{K}_M(z,w) - K(z,w)\varphi(z)\overline{\varphi(w)}$$

is positive definite. It follows that

$$L_{\varphi,1+\varepsilon} : \mathbb{B}_d \times \mathbb{B}_d \to \mathbb{C},\ L_{\varphi,1+\varepsilon}(z,w) = (1+\varepsilon)^2 K_N^{\perp}(z,w) - K(z,w)\varphi(z)\overline{\varphi(w)}$$

is positive definite. By Lemma 4.3.3 the multiplier $\varphi \in \mathrm{Mult}(\mathscr{H})$ admits a factorization of the form

$$\varphi = b^{\odot N}\psi$$

for some $\psi \in \mathrm{Mult}\left(\mathscr{H}, \mathscr{H} \otimes F\left(\mathbb{C}^d\right)^{\otimes N}\right)$ with

$$\|\psi\|_{\mathrm{Mult}} \leq (1+\varepsilon)\|\varphi\|_{\mathrm{Mult}},$$

which completes the proof. □

4.4. Analog to Miller, Olin, and Thomson's theorem

In this section we want to prove the main result. We use a well-known consequence of the Krein-Smulian theorem.

Lemma 4.4.1. *Let H_1, H_2 be Hilbert spaces. Furthermore, let H_1 be separable, $\mathscr{A} \subset B(H_1)$ weak-$*$ closed and $\pi\colon \mathscr{A} \to B(H_2)$ linear and bounded. The map π is weak-$*$ continuous if and only if $\mathrm{WOT} - \lim_{n\to\infty} \pi(T_n) = 0$ for every sequence $(T_n)_{n\in\mathbb{N}}$ in the unit ball $B_1(\mathscr{A})$ of $\mathscr{A}$ with $\mathrm{WOT} - \lim_{n\to\infty} T_n = 0$.*

Proof. The map π is linear. Hence, π is weak-$*$ continuous if and only if π is weak-$*$ continuous at zero. By an application of the Krein-Smulian-Theorem π is weak-$*$ continuous if and only if

$$\pi\colon B_1(\mathscr{A}) \to B(H_2)$$

is weak-$*$ continuous [confer Lemma A.2.3]. Let $C_1(H_1)$ be the trace class operators on H_1 and let ${}^{\perp}\mathscr{A}$ the annihilator of $\mathscr{A}$. Because $\mathscr{A} \subset B(H_1)$ is weak-$*$ closed, it follows that

$$\mathscr{A} \cong \left({}^{C_1(H_1)}/{}_{{}^{\perp}\mathscr{A}}\right)'.$$

Since H_1 is separable $C_1(H_1)$ is separable. By [Con90, §5 Theorem 5.1] the topological space $(B_1(\mathscr{A}), \tau_w^*)$ is metrizable. Hence, $\pi\colon B_1(\mathscr{A}) \to B(H_2)$ is weak-$*$ continuous if and only if $w^* - \lim_{n\to\infty} \pi(T_n) = 0$ for every sequence $(T_n)_{n\in\mathbb{N}}$ in $B_1(\mathscr{A})$ with $w^* - \lim_{n\to\infty} T_n = 0$. Since every bounded net converges to zero in the weak operator topology if and only if it converges to zero in the weak-$*$ topology the assertion follows. □

We use the following two lemmas, that will simplify the proof of our main result:

Lemma 4.4.2. *Let $\mathscr{H}$ be a regular unitarily invariant space such that*

$$\mathbb{C}[z] \subset \mathrm{Mult}(\mathscr{H}).$$

Let $(\varphi_n)_{n\in\mathbb{N}}$ be a sequence in $\mathrm{Mult}(\mathscr{H})$ with

$$\mathrm{WOT} - \lim_{n\to\infty} \varphi_n = 0$$

and fix $N \in \mathbb{N}$. Then $(S_N[\varphi_n])_{n\in\mathbb{N}}$ is a sequence in $\mathrm{Mult}(\mathscr{H})$ with

$$\lim_{n\to\infty} \|S_N[\varphi_n]\|_{\mathrm{Mult}} = 0.$$

Proof. For $n \in \mathbb{N}$ let $\varphi_n = \sum_{\alpha\in\mathbb{N}^d} \varphi_\alpha^{(n)} z^\alpha$ be in $\mathrm{Mult}(\mathscr{H})$ such that

$$\mathrm{WOT} - \lim_{n\to\infty} \varphi_n = 0.$$

Fix $N \in \mathbb{N}$. Since $\mathbb{C}[z] \subset \mathrm{Mult}(\mathscr{H})$, we have

$$S_N[\varphi_n] = \sum_{0\leq|\alpha|\leq N} \varphi_\alpha^{(n)} z^\alpha \in \mathrm{Mult}(\mathscr{H})$$

for all $n \in \mathbb{N}$. Because

$$\mathrm{WOT} - \lim_{n\to\infty} \varphi_n = 0,$$

it follows that $\lim_{n\to\infty}\langle \varphi_n, z^\alpha\rangle_{\mathscr{H}} = 0$ and thus

$$\lim_{n\to\infty} \varphi_\alpha^{(n)} = 0$$

for every $\alpha \in \mathbb{N}^d$. We conclude that

$$\lim_{n\to\infty} \left(\sum_{0\leq|\alpha|\leq N} \left|\varphi_\alpha^{(n)}\right| \right) = 0.$$

Hence, if $c = \sup_{0\leq|\alpha|\leq N} \|M_z^\alpha\|$, then

$$0 \leq \|S_N[\varphi_n]\|_{\mathrm{Mult}} \leq c \left(\sum_{0\leq|\alpha|\leq N} \left|\varphi_\alpha^{(n)}\right| \right) \to 0$$

for $n \to \infty$. □

The pureness condition of a K-contraction plays a central role in our proof of the main result. In contrast to the classical result, to deal with problems of convergence, we will use the assumption that the algebra homomorphism is completely contractive.

Lemma 4.4.3. *Let $\mathscr{H}$ be a regular unitarily invariant complete Nevanlinna-Pick space, whose kernel*

$$K : \mathbb{B}_d \times \mathbb{B}_d \to \mathbb{C},\; K(z,w) = \frac{1}{1 - b(z)b(w)^*}$$

is unbounded. Let $T \in B(H)^d$ be a K-contraction and let

$$\pi\colon \operatorname{Mult}(\mathscr{H}) \to B(H)$$

be a completely bounded algebra homomorphism with $\pi(z_l) = T_l$ for $l = 1, \ldots, d$. Furthermore, let $N \in \mathbb{N}$ and

$$\psi = \begin{bmatrix} \psi_1 \\ \vdots \\ \psi_n \\ \vdots \end{bmatrix},$$

in $\operatorname{Mult}\left(\mathscr{H}, \mathscr{H} \otimes F\left(\mathbb{C}^d\right)^{\otimes N}\right)$. If we denote by

$$\pi(\psi) = \begin{bmatrix} \pi(\psi_1) \\ \vdots \\ \pi(\psi_n) \\ \vdots \end{bmatrix},$$

then

$$\pi(b^{\odot N}\psi) = b(T)^{\odot N}\pi(\psi)$$

and in particular if $h_1, h_2 \in H$ with $\|h_1\| = 1$, then

$$\left|\langle \pi(b^{\odot N}\psi)h_1, h_2\rangle\right| \le \|\pi\|_{cb}\|\psi\|_{\text{Mult}}\|(b(T)^{\odot N})^* h_2\|.$$

Proof. For $N \in \mathbb{N}$ embed $b^{\odot N}$ in an infinite matrix where the first row coincides with $b^{\odot N}$ and the other entries are zero. Then embed ψ in an infinite matrix where the first column coincides with ψ and the rest is zero. Since M_b has a vanishing tail due to Proposition 4.2.3, using the properties of $\odot$ and Lemma 4.2.1, the infinite matrices induced by $b^{\odot N}$ can be approximated in norm the topology by finite matrices. Because π is completely bounded and $\pi(z_l) = T_l$ for $l = 1, \ldots, d$, we conclude modulo identification that

$$\pi(b^{\odot N}\psi) = b(T)^{\odot N}\pi(\psi).$$

We obtain

$$\|\pi(\psi)h_1\|_{H \otimes F\left(\mathbb{C}^d\right)^{\otimes N}} \le \|\pi(\psi)\| \le \|\pi\|_{cb}\|\psi\|_{\text{Mult}}.$$

Using the Cauchy Schwarz inequality, it is immediate that

$$\begin{aligned}\left|\langle \pi(b^{\odot N}\psi)h_1, h_2\rangle\right| &= \left|\langle \pi(\psi)x, (b(T)^{\odot N})^* h_2\rangle\right| \\ &\le \|\pi\|_{cb}\|\psi\|_{\text{Mult}}\|(b(T)^{\odot N})^* h_2\|.\end{aligned}$$

□

We can now prove Theorem 4.1. For convenience we restate it here.

Theorem 4.4.4. *Let $\mathscr{H}$ be a regular unitarily invariant complete Nevanlinna-Pick space, whose kernel*

$$K : \mathbb{B}_d \times \mathbb{B}_d \to \mathbb{C},\ K(z,w) = \frac{1}{1 - b(z)b(w)^*}$$

is unbounded. If $T \in B(H)^d$ is a pure K-contraction and

$$\pi\colon \operatorname{Mult}(\mathscr{H}) \to B(H)$$

is a completely contractive algebra homomorphism with $\pi(1) = \mathrm{id}_H$ and $\pi(z_l) = T_l$ for $l = 1, \dots, d$, then π is weak-$*$ continuous.

Before we start with the proof, we want to consider the case, when $H = \mathbb{C}$.

Remark 4.4.5. Let $\mathscr{H}$ be a regular unitarily invariant complete Nevanlinna-Pick space with reproducing kernel $K : \mathbb{B}_d \times \mathbb{B}_d \to \mathbb{C}$,

$$K(z,w) = \sum_{n=0}^{\infty} a_n \langle z, w\rangle^n = \frac{1}{1 - \sum_{n=1}^{\infty} b_n \langle z, w\rangle^n},$$

where $a_0 = 1$, $a_n > 0$ for $n \geq 1$, $\lim_{n\to\infty} \frac{a_n}{a_{n+1}} = 1$ and $(b_n)_{n\geq 1}$ is a sequence of non-negative real numbers satisfying $\sum_{n=1}^{\infty} b_n \leq 1$.

(a) In [Har17, Proposition 8.5], Hartz shows that the regularity condition

$$\lim_{n\to\infty} \frac{a_n}{a_{n+1}} = 1,$$

implies that Gleason's problem can be solved in $\operatorname{Mult}(\mathscr{H})$. That is, given $w \in \mathbb{B}_d$ and $\varphi \in \operatorname{Mult}(\mathscr{H})$, there are $\varphi_1, \dots, \varphi_d \in \operatorname{Mult}(\mathscr{H})$ such that

$$\varphi - \varphi(w) = \sum_{l=1}^{d} (z_l - w_l)\varphi_l. \tag{4.5}$$

(b) Let $\mathscr{M}(\operatorname{Mult}(\mathscr{H}))$ be the maximal ideal space of the unital commutative Banach algebra $\operatorname{Mult}(\mathscr{H})$. Because $\lim_{n\to\infty} a_n/a_{n+1} = 1$, one computes that

$$\sigma_{joint}(M_z) = \{(\chi(z_1), \dots, \chi(z_n));\ \chi \in \operatorname{Mult}(\mathscr{H}))\} = \overline{\mathbb{B}_d}$$

(confer Theorem 2.4.24).

(c) Let $\chi \in \mathscr{M}(\operatorname{Mult}(\mathscr{H}))$. If

$$w = (\chi(z_1), \dots, \chi(z_d))) \in \mathbb{B}_d,$$

Equation (4.5) yields that

$$\chi(\varphi - \varphi(w)) = \sum_{l=1}^{d} (\chi(z_l) - w_l)\chi(\varphi_l) = 0$$

for all $\varphi \in \mathrm{Mult}(\mathscr{H})$. Thus, $\chi : \mathrm{Mult}(\mathscr{H}) \to \mathbb{C}$ coincides with the point evaluation $\delta_w : \mathrm{Mult}(\mathscr{H}) \to \mathbb{C}$,

$$\delta_w(\varphi) = \varphi(w).$$

It is not difficult to see that point evaluations are weak-$*$ -continuous.

(d) It is well-known (see [Dou98, 2.22 Proposition] and [Pau02, Proposition 3.8]) that the completely contractive unital algebra homomorphisms

$$\pi\colon\ \mathrm{Mult}(\mathscr{H}) \to B(\mathbb{C}) \cong \mathbb{C}$$

are precisely the elements of $\mathscr{M}(\mathrm{Mult}(\mathscr{H}))$.

(e) Let $\chi \in \mathscr{M}(\mathrm{Mult}(\mathscr{H}))$. Since $b_n \geq 0$ for all $n \in \mathbb{N}$ and $\sum_{n=1}^{\infty} b_n \leq 1$, it follows that

$$\chi(z) = (\chi(z_1), \dots, \chi(z_d)) \in \mathbb{C}^d \cong B(\mathbb{C}^d)$$

is a K-contraction. If

$$w = (\chi(z_1), \dots, \chi(z_d))) \in \mathbb{B}_d,$$

it follows from Theorem 4.1.14 that

$$\chi(z) = (\chi(z_1), \dots, \chi(z_d)) \in \mathbb{C}^d \cong B(\mathbb{C}^d)$$

is a pure K-contraction. If $\sum_{n=0}^{\infty} a_n = \infty$ and $\chi(z)$ is a pure, then $\chi(z) \in \mathbb{B}_d$.

Due to the previous remarks, the proof of Theorem 4.1 is immediate in case, when $H = \mathbb{C}$.

Proof of Theorem 4.1. Let $(\varphi_n)_{n\in\mathbb{N}}$ be a sequence in the unit ball

$$B_1(\mathrm{Mult}(\mathscr{H})) = \{\varphi \in \mathrm{Mult}(\mathscr{H});\ \|\varphi\|_{\mathrm{Mult}} \leq 1\}$$

such that

$$\mathrm{WOT} - \lim_{n\to\infty} \varphi_n = 0.$$

Since $\mathscr{H}$ is separable and $\mathrm{Mult}(\mathscr{H}) \subset B(\mathscr{H})$ is weak-$*$ closed, Lemma 4.4.1 shows that it is sufficient to prove that $\mathrm{WOT} - \lim_{n\to\infty} \pi(\varphi_n) = 0$.

Let $h_1, h_2 \in H$. We want to show that $\lim_{n\to\infty}\langle \pi(\varphi_n)h_1, h_2\rangle = 0$. With out loss of generality we may assume that $\|h_1\| = 1$.

Fix $N \geq 2$.

Using Corollary 4.3.2 we find a $M \geq N$ such that for every $n \in \mathbb{N}$ there exists $\psi_n \in \mathrm{Mult}\left(\mathscr{H}, \mathscr{H} \otimes F\left(\mathbb{C}^d\right)^{\otimes N}\right)$ with

$$\varphi_n = S_{M-1}[\varphi_n] + b^{\odot N}\psi_n$$

and

$$\|\psi_n\|_{\mathrm{Mult}} \leq 2\|\varphi_n - S_{M-1}[\varphi_n]\|_{\mathrm{Mult}}.$$

Because of Lemma 4.4.3, it follows that

$$\lim_{n\to\infty} \|S_{M-1}[\varphi_n]\|_{\mathrm{Mult}} = 0.$$

Because $\|\varphi_n\|_{\mathrm{Mult}} \leq 1$ for $n \in \mathbb{N}$, we have that

$$\limsup_{n\to\infty} \|\psi_n\|_{\mathrm{Mult}} \leq 2 \limsup_{n\to\infty} \|\varphi_n\|_{\mathrm{Mult}} \leq 2.$$

Lemma 4.4.2 yields that

$$\limsup_{n\to\infty} |\langle \pi(\varphi_n)h_1, h_2\rangle| = \limsup_{n\to\infty} \left|\langle \pi(b^{\odot N}\psi_n)h_1, h_2\rangle\right| \leq 2\|(b(T)^{\odot N})^* h_2\|.$$

Since T is a pure K-contraction, it follows with Theorem 4.1.14 that

$$\mathrm{SOT} - \lim_{N\to\infty} (b(T)^{\odot N})^* = 0.$$

Hence, we obtain

$$0 \leq \limsup_{n\to\infty} |\langle \pi(\varphi_n)h_1, h_2\rangle| = 0.$$

□

4.5. The bounded case

Suppose that $\mathscr{H}$ is a complete Nevanlinna-Pick space with reproducing kernel of the form

$$K : \mathbb{B}_d \times \mathbb{B}_d \to \mathbb{C}, K(z,w) = \sum_{n=0}^{\infty} a_n \langle z, w\rangle^n,$$

where $a_0 = 1$, $a_n > 0$ for $n \geq 1$, $\lim_{n\to\infty} \frac{a_n}{a_{n+1}} = 1$ and $\sum_{n=0}^{\infty} a_n < \infty$.

Remark 4.5.1. (a) Because $\sum_{n=0}^{\infty} a_n < \infty$, the kernel K extends to the positive definite function

$$\tilde{K} : \overline{\mathbb{B}_d} \times \overline{\mathbb{B}_d} \to \mathbb{C},\ \tilde{K}(z,w) = \sum_{n=0}^{\infty} a_n \langle z, w\rangle^n,$$

where the corresponding reproducing kernel Hilbert space $\tilde{\mathscr{H}}$ is a subset of $A(\mathbb{B}_d)$. Using Corollary 2.2.8, it can be readily seen that every function $f \in \mathscr{H}$ has a unique extension in $\tilde{\mathscr{H}}$.

(b) It follows that for every $w \in \overline{\mathbb{B}_d}$ the point evaluation $\delta_w : \mathrm{Mult}(\mathscr{H}) \to \mathbb{C}$ is a well-defined character.

(c) Since $\sum_{n=0}^{\infty} a_n < \infty$, by Remark 4.1.13, every K-contraction is pure.

We want to distinguish between two cases:

First case: Suppose that $\mathrm{Mult}(\mathscr{H}) = \mathscr{H}$.

Let $T \in B(H)^d$ be a pure K-contraction and let

$$\pi\colon \mathrm{Mult}(\mathscr{H}) \to B(H)$$

be a continuous algebra homomorphism with $\pi(1) = \mathrm{id}_H$ and $\pi(z_l) = T_i$ for $l = 1, \ldots, d$. By the open mapping theorem the norms of $\mathscr{H}$ and $\mathrm{Mult}(\mathscr{H})$ are equivalent. Thus, since the polynomials $\mathbb{C}[z]$ are dense in $\mathscr{H}$, they are dense in $\mathrm{Mult}(\mathscr{H})$. Hence, the map π is uniquely determined by $\pi(1) = \mathrm{id}_H$ and $\pi(z_l) = T_l$ for $l = 1, \ldots, d$. In particular, π is weak-$*$ continuous.

Example 4.5.2. If $d = 1$, Shields establishes in [Shi74, Chapter 9 on page 92] sufficient and necessary conditions on the coefficients a_n such that $M_z \in B(\mathscr{H})$ is strictly cyclic. This is equivalent to the fact that $\mathrm{Mult}(\mathscr{H}) = \mathscr{H}$. For example:

If $s > 1$ and $\mathscr{D}_{-s}$ is the unitarily invariant space with reproducing kernel

$$K : \mathbb{D} \times \mathbb{D} \to \mathbb{C},\ K(z, w) = \sum_{n=0}^{\infty} \frac{z\overline{w}}{(n+1)^s},$$

Shields proves that $M_z \in B(\mathscr{H})$ is strictly cyclic and thus $\mathrm{Mult}(\mathscr{D}_{-s}) = \mathscr{D}_{-s}$.

Second case: Suppose that $\mathrm{Mult}(\mathscr{H}) \subsetneq \mathscr{H}$.

We have already seen in Remark 4.4.5 that:

(a) The completely contractive unital algebra homomorphisms

$$\pi\colon \mathrm{Mult}(\mathscr{H}) \to B(\mathbb{C}) \cong \mathbb{C}$$

are precisely the elements of the maximal ideal space $\mathscr{M}(\mathrm{Mult}(\mathscr{H}))$.

(b) If $\chi : \mathrm{Mult}(\mathscr{H}) \to \mathbb{C}$ is an element of the maximal ideal space $\mathscr{M}(\mathrm{Mult}(\mathscr{H}))$, then

$$\chi(z) = (\chi(z_1), \ldots, \chi(z_d)) \in \mathbb{C}^d \cong B(\mathbb{C}^d)$$

is a pure K-contraction.

(c) If $\chi \in \mathscr{M}(\mathrm{Mult}(\mathscr{H}))$ with

$$w = (\chi(z_1), \dots, \chi(z_d)) \in \mathbb{B}_d,$$

the Gleason problem implies that χ coincides with the point evaluation $\delta_w : \mathscr{H} \to \mathbb{C}$.

Let $f \in \mathscr{H} \setminus \mathrm{Mult}(\mathscr{H})$.

Due to Theorem 3.1 in [AHMR17], there are $\varphi, \psi \in \mathrm{Mult}(\mathscr{H})$ with $\psi(\overline{\mathbb{B}_d}) \subset \mathbb{C} \setminus \{0\}$ such that $f = \frac{\varphi}{\psi}$.

The assumption $f \in \mathscr{H} \setminus \mathrm{Mult}(\mathscr{H})$ implies that $\frac{1}{\psi} \notin \mathrm{Mult}(\mathscr{H})$. Hence, there exists a character $\chi : \mathrm{Mult}(\mathscr{H}) \to \mathbb{C}$ with $\chi(\psi) = 0$.

Let

$$w = (\chi(z_1), \dots, \chi(z_d)) \in \overline{\mathbb{B}_d},$$

Because $\psi(\overline{\mathbb{B}_d}) \subset \mathbb{C} \setminus \{0\}$, we conclude that $\delta_w : \mathrm{Mult}(\mathscr{H}) \to \mathbb{C}$ and $\chi : \mathrm{Mult}(\mathscr{H}) \to \mathbb{C}$ are two different completely contractive algebra homomorphisms with $\pi(1) = \mathrm{id}_H$ and $\pi(z_l) = w_l$ for $l = 1, \dots, d$. In particular, this yields that $w \in \partial\mathbb{B}_d$.

Since the polynomials form a weak-$*$ dense subalgebra of $\mathrm{Mult}(\mathscr{H})$ (confer Theorem 2.3.36) and $\delta_w : \mathrm{Mult}(\mathscr{H}) \to \mathbb{C}$ is weak-$*$ continuous, it follows that χ cannot be weak-$*$ continuous.

Example 4.5.3. Proposition 5.4 and Theorem 5.5 in [AHMR17], show that there exist regular unitarily invariant complete Nevanlinna-Pick spaces $\mathscr{H}$ with reproducing kernel of the form

$$K : \mathbb{B}_d \times \mathbb{B}_d \to \mathbb{C},\ K(z,w) = \sum_{n=0}^{\infty} a_n \langle z, w \rangle^n,$$

where $a_0 = 1$, $a_n > 0$ for $n \geq 1$, $\lim_{n\to\infty} a_n/a_{n+1} = 1$ and $\sum_{n=0}^{\infty} a_n < \infty$ such that

$$\mathrm{Mult}(\mathscr{H}) \subsetneq \mathscr{H}.$$

Spaces of this type are called Salas spaces. According to the previous explanations, the statement in Theorem 4.1 is false for the Salas spaces.

5. Norm-closure of polynomials

The contents of this chapter are a joint work with Michael Hartz.

For $s \in \mathbb{R}$ and a radial weight function $\omega : \mathbb{B}_d \to \mathbb{R}_{\geq 0}$ let

$$B^s_\omega = \{f \in \mathscr{O}(\mathbb{B}_d);\ R^s f \in L^2(\omega dV)\}$$

be a radially weighted Besov space, as in Subsection 2.3.2.

The algebra $A(\mathbb{D})$ of continuous functions on the closed unit disk that are analytic on the interior part, the so-called disk algebra, is often a useful tool in function theory and functional analysis. The algebra can be considered as the norm-closure of polynomials in $H^\infty(\mathbb{D})$, which is the multiplier algebra of the Hardy space $H^2(\mathbb{D})$. It can be checked that, for example, infinite Blaschke products are contained in $H^\infty(\mathbb{D})$ but not in $A(\mathbb{D})$ and hence $A(\mathbb{D}) \subsetneq H^\infty(\mathbb{D})$. Given any radially weighted Besov space B^s_ω, one can consider a generalized concept, namely the norm-closure of polynomials $A(B^s_\omega)$ in the multiplier algebra $\mathrm{Mult}(B^s_\omega)$.

In Subsection 2.3.2, we consider the following characterization for the multiplier algebra $\mathrm{Mult}(B^s_\omega)$ (see Theorem 2.3.55):

Theorem. *A function $\varphi \in \mathscr{O}(\mathbb{B}_d)$ is an element of the multiplier algebra $\mathrm{Mult}(B^s_\omega)$ if and only if $\varphi \in H^\infty(\mathbb{B}_d)$ and $R^N\varphi \in \mathrm{Mult}(B^s_\omega, B^{s-N}_\omega)$ for $N \geq 1$. In this case,*

$$\|\varphi\|_{\mathrm{Mult}(B^s_\omega)} \approx \|R^N\varphi\|_{\mathrm{Mult}(B^s_\omega, B^{s-N}_\omega)} + \|\varphi\|_\infty.$$

The idea is due to Cascante, Fabrega and Ortega, which has been adapted by Aleman, Hartz, M[c]Carthy and Richter (see [AHMR19, Theorem 6.3]) to the case of radially weighted Besov spaces B^s_ω.

Motivated by the characterization of the multiplier algebra, we establish a similar characterization for the norm-closure of polynomials $A(B^s_\omega)$.

Theorem 5.1. *Let $N \geq 1$. A function $\varphi : \mathbb{B}_d \to \mathbb{C}$ is an element of the norm-closure*

$$A(B^s_\omega) = \overline{\mathbb{C}[z]}^{\|\cdot\|_{\mathrm{Mult}}} \subset \mathrm{Mult}(B^s_\omega),$$

if and only if φ is an element of the ball algebra $A(\mathbb{B}_d)$ and the operator

$$B^s_\omega \to B^{s-N}_\omega, f \mapsto (R^N\varphi) f$$

is compact.

Besides, one can also reformulate the previous result in terms of (vanishing) Carleson measures (for a definition see Section 5.2). Now, to simplify notation, we abbreviate

$$B^{s,p} = \left\{ f \in \mathscr{O}(\mathbb{B}_d);\ \int_{\mathbb{B}_d} |R^s f(z)|^p \, dV(z) < \infty \right\}$$

for the radially Besov spaces with the constant weight $\omega \equiv 1$ and $B^s = B^{s,2}$. We use the reformulation of Theorem 5.1 in terms of vanishing Carleson measures and obtain the following version of Theorem 5.9 (2) in [BB08] (see Theorem 5.2.17):

Theorem. *Let* $1 < 2s \leq d+1$ *and* $p > \frac{d+1}{s}$. *If*

$$\varphi \in B^{s,p} \cap A(\mathbb{B}_d),$$

then $\varphi \in A(B^s)$.

We conclude the chapter by considering the case of the Dirichlet space $\mathscr{D}$. For $f \in \mathscr{D}$ the Sarason function is defined as

$$V_f : \mathbb{D} \to \mathbb{C},\ V_f(z) = 2\langle f, k_z f \rangle_{\mathscr{D}} - \|f\|^2_{\mathscr{D}}.$$

As a particular case of [AHMR18, Theorem 4.5], it follows that if $\operatorname{Re} V_f$ is bounded, then $f \in \operatorname{Mult}(\mathscr{H})$. Using vanishing Carleson measures and Theorem 5.1, we show that (see Theorem 5.3.5):

Theorem. *Let* $\varphi \in \mathscr{D}$. *If*

$$\sup_{w \in \mathbb{D}} |\operatorname{Re} V_\varphi(w) - \operatorname{Re} V_{\varphi_r}(rw)| \xrightarrow{r \uparrow 1} 0$$

for $r \uparrow 1$, *then* $\varphi \in A(\mathscr{D})$.

5.1. A characterization for radially weighted Besov spaces

The goal of this section is to prove Theorem 5.1. We use the characterization of the multiplier algebra $\operatorname{Mult}(B^s_\omega)$ in Theorem 2.3.55 and the theory of homogeneous spaces, that we introduced in Section 2.3.

For a function $f : \mathbb{B}_d \to \mathbb{C}$, $\zeta \in \mathbb{T}$ and $0 < r < 1$ we use the notations

$$f_\zeta : \mathbb{B}_d \to \mathbb{C},\ f_\zeta(z) = f(\zeta z)$$

and

$$f_r : \mathbb{B}_d\left(\frac{1}{r}\right) \to \mathbb{C},\ z \to f(rz).$$

(see Notations 2.3.12 and 2.3.33).

In Definition 2.3.28, we call a locally convex Hausdorff space $\mathscr{F} \subset \mathscr{O}(\mathbb{B}_d)$ homogeneous if

(a) $\mathscr{F}$ is quasi-complete,

(b) the point evaluations $\delta_z : \mathscr{F} \to \mathbb{C},\ f \mapsto f(z)$ are continuous,

(c) for $f \in \mathscr{F}$ and $\zeta \in \mathbb{T}$ the functions $f_\zeta : \mathbb{B}_d \to \mathbb{C},\ f_\zeta(z) = f(\zeta z)$ belong to $\mathscr{F}$ and the maps

$$\mathbb{T} \to \mathscr{F}, \zeta \mapsto f_\zeta$$

are continuous.

Remark 5.1.1. Let $f \in A(\mathbb{B}_d)$. Since

$$A(\mathbb{B}_d) = C(\overline{\mathbb{B}_d}) \cap \mathscr{O}(\mathbb{B}_d),$$

it follows that

$$\lim_{r\to 1} \|f_r - f\|_\infty = 0$$

and hence by Taylor expansion

$$A(\mathbb{B}_d) = \overline{\mathbb{C}[z]}^{\|\cdot\|_\infty} \subset H^\infty(\mathbb{B}_d).$$

Because

$$\mathrm{Mult}(H^2(\partial\mathbb{B}_d)) = H^\infty(\mathbb{B}_d),$$

we deduce that $A(\mathbb{B}_d) = A(H^2(\partial\mathbb{B}_d))$. With Lemma 2.3.32, part (c), it is immediate that the ball algebra $A(\mathbb{B}_d)$ is homogeneous in the sense of Definition 2.3.28. So if you want to be more precise with polynomial approximation, you can also approximate f by the Fejér-means $\sigma_n(f)$ in the supremum norm (see Theorem 2.3.36 and Notation 2.3.33).

For an arbitrary $N \in \mathbb{N}$ with $N \geq 1$, consider the set

$$\mathscr{F} = \{\varphi \in A(\mathbb{B}_d);\ B^s_\omega \to B^{s-N}_\omega, f \mapsto (R^N\varphi)f \text{ is compact}\} \subset \mathrm{Mult}(B^s_\omega)$$

(see Theorem 2.3.55). We will see that

$$A(B^s_\omega) = \mathscr{F}.$$

It is enough to show that $A(B^s_\omega) \subset \mathscr{F}$ and that $\mathscr{F}$ equipped with the norm-topology is a homogeneous space in the sense of Definition 2.3.28. This yields that the polynomials are dense in $\mathscr{F}$.

Proposition 5.1.2. *The space $\mathscr{F}$ is norm-closed in* $\mathrm{Mult}(B^s_\omega)$ *and* $A(B^s_\omega) \subset \mathscr{F}$.

Proof. We show that $\mathscr{F}$ is closed. Because of Theorem 2.3.55, the maps

$$\hat{R}^N : \mathrm{Mult}(B^s_\omega) \to B(B^s_\omega, B^{s-N}_\omega),\ \varphi \mapsto M_{R^N\varphi}$$

and

$$i_\infty : \mathrm{Mult}(B^s_\omega) \to H^\infty(\mathbb{B}_d),\ \varphi \mapsto \varphi$$

are bounded linear operators. Let $K(B^s_\omega, B^{s-N}_\omega)$ be the Banach space of the compact operators from B^s_ω to B^{s-N}_ω. Clearly,

$$\mathscr{F} = (\hat{R}^N)^{-1}\left(K(B^s_\omega, B^{s-N}_\omega)\right) \cap (i_\infty)^{-1}\left(A(\mathbb{B}_d)\right).$$

Since $K(B^s_\omega, B^{s-N}_\omega) \subset B(B^s_\omega, B^{s-N}_\omega)$ and $A(\mathbb{B}_d) \subset H^\infty(\mathbb{B}_d)$ are closed, it follows that $\mathscr{F} \subset \mathrm{Mult}(B^s_\omega)$ is closed.

It remains to show that $A(B^s_\omega) \subset \mathscr{F}$. For all $p \in \mathbb{C}[z]$, we have that $R^N p \in \mathbb{C}[z]$. By Corollary 2.3.52 the inclusion $i : B^s_\omega \to B^{s-N}_\omega$ is compact. Hence, it follows that $B^s_\omega \to B^{s-N}_\omega, f \mapsto (R^N p) f$ is compact as the composition of a bounded and a compact operator. This finishes the proof, since then

$$A(B^s_\omega) = \overline{\mathbb{C}[z]}^{\|\cdot\|_{\mathrm{Mult}}} \subset \mathscr{F}.$$

□

Proposition 5.1.3. *The norm-closed space $\mathscr{F} \subset \mathrm{Mult}(B^s_\omega)$ is a homogeneous space in the sense of Definition 2.3.28.*

Proof. (a) For all $z \in \mathbb{B}_d$ the point evaluations $\delta_z : \mathrm{Mult}(B^s_\omega) \to \mathbb{C}$, $\varphi \to \varphi(z)$ are characters. In particular, the restrictions $\delta_z|_{\mathscr{F}}$ are continuous.

(b) Let $\varphi \in \mathscr{F}$. Then $M_{R^N\varphi} : B^s_\omega \to B^{s-N}_\omega$ is compact. By Corollary 2.3.13, the multiplication operator $M_{(R^N\varphi)_\zeta}$ $(\zeta \in \mathbb{T})$ is compact and

$$\mathbb{T} \to \mathrm{Mult}(B^s_\omega, B^{s-N}_\omega),\ \zeta \mapsto (R^N\varphi)_\zeta$$

is continuous with respect to the norm-topology. Since $A(\mathbb{B}_d) \subset C(\overline{\mathbb{B}_d})$, the map

$$\mathbb{T} \to A(\mathbb{B}_d),\ \zeta \mapsto \varphi_\zeta$$

is continuous. Observe that $(R^N\varphi)_\zeta = R^N(\varphi_\zeta)$ for all $\zeta \in \mathbb{T}$. Due to Theorem 2.3.55, there exists a $c > 0$ such that

$$\|\varphi_\zeta - \varphi_\eta\|_{\mathrm{Mult}} \leq c(\|(R^N\varphi)_\zeta - (R^N\varphi)_\eta\|_{\mathrm{Mult}(B^s_\omega, B^{s-N}_\omega)} + \|\varphi_\zeta - \varphi_\eta\|_\infty)$$

for all $\eta, \zeta \in \mathbb{T}$. Hence, the map

$$\mathbb{T} \to \mathscr{F},\ \zeta \mapsto \varphi_\zeta$$

is continuous with respect to the norm-topology.

□

Using the previous Propositions 5.1.2 and 5.1.3, Theorem 5.1 is immediate. For convenience we restate it here.

Theorem 5.1.4. *Let $N \geq 1$. A function $\varphi : \mathbb{B}_d \to \mathbb{C}$ is an element of the norm-closure*

$$A(B^s_\omega) = \overline{\mathbb{C}[z]}^{\|\cdot\|_{\mathrm{Mult}}} \subset \mathrm{Mult}(B^s_\omega),$$

if and only if φ is an element of the ball algebra $A(\mathbb{B}_d)$ and the operator

$$B^s_\omega \to B^{s-N}_\omega, f \mapsto (R^N\varphi)f$$

is compact.

In special cases there are sometimes criteria, which can be easily checked, proving that a function is in the polynomial norm-closure $A(B^s_\omega)$. In Section 2.3, Example 2.3.5 we introduced for $s \in \mathbb{R}$ the unitarily invariant spaces $\mathscr{D}_s(\mathbb{B}_d)$ with reproducing kernel

$$K : \mathbb{B}_d \times \mathbb{B}_d \to \mathbb{C},\ K(z,w) = \sum_{n=0}^{\infty} (n+1)^s \langle z,w\rangle^n.$$

The space $\mathscr{D}_0(\mathbb{B}_d)$ is the Drury-Arveson space H^2_d and $\mathscr{D}_{-1}(\mathbb{D})$ is the Dirichlet space $\mathscr{D}$. The spaces $\mathscr{D}_s(\mathbb{B}_d)$ can be described as radially weighted Besov spaces B^s_ω (see Example 2.3.49).

Using the unitarily invariant space $\mathscr{D}^0_{-1}(\mathbb{B}_d)$ with reproducing kernel

$$K : \mathbb{B}_d \times \mathbb{B}_d \to \mathbb{C},\ K(z,w) = \sum_{n=0}^{\infty} \frac{1}{(n+1)\log(n+2)} \langle z,w\rangle^n,$$

Aleman, Hartz, M$^{\mathrm{c}}$Carthy and Richter established in [AHMR23, Lemma 14.7] sufficient conditions for being an element of $A(\mathscr{D}_s(\mathbb{B}_d))$.

Lemma 5.1.5. *(a) There is a constant $c > 0$ such that*

$$\|fR\varphi\|_{\mathscr{D}_1(\mathbb{B}_d)} \leq c\|\varphi\|_{\mathscr{D}^0_{-1}(\mathbb{B}_d)}\|f\|_{\mathscr{D}_{-1}(\mathbb{B}_d)}$$

for all $f \in \mathscr{D}_{-1}(\mathbb{B}_d)$, $\varphi \in \mathscr{D}^0_{-1}(\mathbb{B}_d)$.

(b) If $s < 1$, then there is a $c > 0$ such that

$$\|fR\varphi\|_{\mathscr{D}_{-s+2}(\mathbb{B}_d)} \leq c\|\varphi\|_{\mathscr{D}_{-1}(\mathbb{B}_d)}\|f\|_{\mathscr{D}_s}$$

for all $f \in \mathscr{D}_{-s}(\mathbb{B}_d)$, $\varphi \in \mathscr{D}_{-1}(\mathbb{B}_d)$.

The proof of the Lemma is an clever application of the characterization elements in reproducing kernel Hilbert spaces (see Theorem 2.2.2) and the characterization of multipliers (see Theorem 2.2.6) plus the Schur product theorem (see [PR16, Theorem 4.8]). For a detailed proof see [AHMR23, Lemma 14.7].

We obtain the $A(\mathscr{H})$-analogue of Theorem 14.8 in [AHMR23]. For the Dirichlet space, the result has its origin in a work by Brown and Shields (cf. [BS84, Proposition 18]). For the other cases, a statement of this type already appeared in [BB08, Corollary 5.11].

Theorem 5.1.6 (Brown-Shields, Beatrous-Burbea)**.** *(a) If $\varphi \in \mathscr{D}^0_{-1}(\mathbb{B}_d)$, then*

(i) $\varphi \in \mathrm{Mult}(\mathscr{D}_{-1}(\mathbb{B}_d))$ if and only if $\varphi \in H^\infty(\mathbb{B}_d)$,

(ii) $\varphi \in A(\mathscr{D}_{-1}(\mathbb{B}_d)))$ if and only if $\varphi \in A(\mathbb{B}_d)$.

(b) If $s < 1$ and $\varphi \in \mathscr{D}_{-1}(\mathbb{B}_d)$, then

(i) $\varphi \in \mathrm{Mult}(\mathscr{D}_{-s}(\mathbb{B}_d))$ if and only if $\varphi \in H^\infty(\mathbb{B}_d)$,

(ii) $\varphi \in A(\mathscr{D}_{-s}(\mathbb{B}_d))$ if and only if $\varphi \in A(\mathbb{B}_d)$.

Proof. The proof is a consequence of Lemma 5.1.5 and is similar to the arguments in Theorem 14.8 in [AHMR23]. Hence, we omit the details here. □

5.2. Vanishing Carleson measures

In this section, we consider Carleson measure conditions. First, we start with a motivation:

Remark 5.2.1. Let $N \geq s > 0$ and let $\omega : \mathbb{B}_d \to \mathbb{R}_{\geq 0}$ be a radial weight function. Due to Theorem 2.3.50, the space B^s_ω coincides $B^N_{\omega_{N-s}}$ with equivalence of norms. Hence, a function $f \in B^s_\omega$ induces a finite measure

$$\mu_{f,N}(z) = |R^N f(z)|^2 \omega_{N-s}(z) dV(z)$$

on $\mathbb{B}_d$. Let $\varphi \in B^s_\omega$. One checks that

$$R^N \varphi \in \mathrm{Mult}(B^s_\omega, B^{s-N}_\omega),$$

if and only if the linear operator

$$J_{\mu_{\varphi,N}} : B^s_\omega \to L^2(\mu),\ f \to f$$

is bounded, that is there exits a constant $c(\mu) > 0$ such that

$$\int_{\mathbb{B}_d} |f|^2 d\mu \leq c(\mu) \|f\|^2_{B^s_\omega} \text{ for all } f \in B^s_\omega.$$

In particular

$$\|R^N \varphi\|_{\mathrm{Mult}(B^s_\omega, B^{s-N}_\omega)} \approx \|J_{\mu_{\varphi,N}}\|.$$

This is to say $\mu_{\varphi,N}$ is a Carleson measure for B^s_ω. The induced multiplication operator

$$M_{R^N\varphi} : B^s_\omega \to B^{s-N}_\omega$$

is compact, if and only if $J_{\mu_{\varphi,N}}$ is compact. The measure $\mu_{\varphi,N}$ is then called vanishing Carleson measure for B^s_ω. Because of Theorem 2.3.55, it is immediate that $\varphi \in \mathrm{Mult}(B^s_\omega)$ if and only if $\varphi \in H^\infty(\mathbb{B}_d)$ and $\mu_{\varphi,N}$ is a Carleson measure for B^s_ω. Using Theorem 5.1, we conclude that $\varphi \in A(B^s_\omega)$, if and only if $\varphi \in A(\mathbb{B}_d)$ and $\mu_{\varphi,N}$ is a vanishing Carleson measure for B^s_ω.

Carleson measures characterize all interpolating sequences for $H^\infty(\mathbb{D})$ (see for example [AM02, Chapter 9]). Carleson uses Carleson measures in his solution of the Corona problem. The measures appear in several contexts in harmonic analysis and for different function spaces. For convenience, we restate the definition of a Carleson measure here again:

Definition 5.2.2. Let $\mathscr{H} \subset \mathscr{O}(\mathbb{B}_d)$ be a reproducing kernel Hilbert space. A finite positive Borel measure μ on the unit ball $\mathbb{B}_d$ is called Carleson measure for $\mathscr{H}$ if and only if $\mathscr{H} \subset L^2(\mu)$. In this case, by the closed graph theorem, the linear operator

$$J_\mu : \mathscr{H} \to L^2(\mu),\ f \mapsto f$$

is continuous. The norm $\|J_\mu\|$ is called Carleson constant. The measure μ is called compact or vanishing Carleson measure if and only if the linear operator

$$J_\mu : \mathscr{H} \to L^2(\mu),\ f \mapsto f$$

is compact.

In the following, let $\mathscr{H} \subset \mathscr{O}(\mathbb{B}_d)$ be a reproducing kernel Hilbert space of holomorphic functions with reproducing kernel $K : \mathbb{B}_d \times \mathbb{B}_d \to \mathbb{C}$. Furthermore, let μ be a finite positive Borel measure on $\mathbb{B}_d$.

There is a a well-known characterization for Carleson measures μ on reproducing kernel Hilbert spaces $\mathscr{H}$ by positive bounded integral operators associated to the real part of the kernel function K on $L^2(\mu)$ due to Arcozzi, Rochberg and Sawyer (cf. [ARS08, Lemma 24]). We adapt the argument and obtain a sufficient condition, which assures that μ is a a vanishing Carleson measure for $\mathscr{H}$.

Remark 5.2.3. For a Borel measurable set $M \subset \mathbb{B}_d$ denote by $\mathbb{1}_M : \mathbb{B}_d \to \mathbb{R}_{\geq 0}$,

$$\mathbb{1}_M(z) = \begin{cases} 1, & \text{if } z \in M, \\ 0 & \text{else} \end{cases}$$

the corresponding characteristic function. One checks that, the induced linear operator

$$P_M : L^2(\mu) \to L^2(\mu),\ h \to \mathbb{1}_M h$$

is an orthogonal projection.

The following definition characterizes compact operators between Hilbert spaces, as will be seen in Proposition 5.2.5. See for example [Con90, 3.2 Definition, 3.3 Proposition, Chapter VI, § 3 Compact Operators, page 173] for a proof of Proposition 5.2.5.

Definition 5.2.4. An operator $T : H_1 \to H_2$ between two Hilbert spaces H_1 and H_2 is completely continuous, if it follows for every sequence $(x_n)_{n\in\mathbb{N}}$ in H_1 with $x_n \xrightarrow{\tau_w} 0$, that $\|Tx_n\|_{H_2} \xrightarrow{n} 0$.

Proposition 5.2.5. *An operator $T : H_1 \to H_2$ between two Hilbert spaces H_1 and H_2 is compact, if and only if it is completely continuous.*

Lemma 5.2.6. *Let $Q \subset \mathbb{B}_d$ be compact, then the operator*

$$J_\mu(Q) : \mathscr{H} \to L^2(\mu),\ f \to \mathbb{1}_Q f$$

is well-defined and compact.

Proof. If $f \in \mathscr{H}$, then f is holomorphic and $f\mathbb{1}_Q$ is bounded. Thus, it follows in particular that $f\mathbb{1}_Q \in L^2(\mu)$ and

$$J_\mu(Q) : \mathscr{H} \to L^2(\mu),\ f \to \mathbb{1}_Q f$$

is well-defined. If $f_n \xrightarrow{\tau_w} f$ for $n \to \infty$ in $\mathscr{H}$, then we obtain by Lemma 2.2.10 that

$$0 \leq \limsup_{n\to\infty} \|\mathbb{1}_Q(f_n - f)\|_{L^2(\mu)} \leq \mu(Q)^{1/2} \lim_{n\to\infty} \|f_n - f\|_\infty = 0.$$

Due to Proposition 5.2.5, the operator T_Q is compact. □

Remark 5.2.7. (a) Because of Lemma 5.2.6, the operators

$$J_\mu(r) : \mathscr{H} \to L^2(\mu),\ f \to \mathbb{1}_{\overline{\mathbb{B}_d(r)}} f \quad (0 < r < 1)$$

are well-defined and compact.

(b) For $0 < r < 1$ consider the orthogonal projections

$$P_r : L^2(\mu) \to L^2(\mu),\ h \mapsto \mathbb{1}_{\overline{\mathbb{B}_d(r)}} h.$$

Fix $w \in \mathbb{B}_d$ and let $h \in L^2(\mu)$. Then there exists an $0 < r_0 < 1$ such that $w \in \mathbb{B}_d(r_0)$. Hence, we have

$$\left| h(w) - \mathbb{1}_{\overline{\mathbb{B}_d(r)}}(w) h(w) \right| = 0$$

for all $0 < r < r_0$. Since

$$\left| h(z) - \mathbb{1}_{\overline{\mathbb{B}_d(r)}}(z) h(z) \right| \leq |h(z)|$$

for all $z \in \mathbb{B}_d$ and $0 < r < 1$, dominated convergence yields that

$$P_r \xrightarrow{SOT} \mathrm{id}_{L^2(\mu)}$$

for $r \uparrow 1$.

Proposition 5.2.8. *(a) The measure μ is a Carleson measure for $\mathscr{H}$ if and only if*

$$c(\mu) = \sup_{0<r<1} \|J_\mu(r)\| < \infty.$$

In particular, $J_\mu(r) \xrightarrow{\mathrm{SOT}} J_\mu$ for $r \uparrow 1$ and $\|J_\mu\| = c(\mu)$.

(b) *The measure μ is a vanishing Carleson measure for $\mathscr{H}$ if and only if* $c(\mu) = \sup_{0<r<1} \|J_\mu(r)\| < \infty$ *and* $\lim_{r\uparrow 1} \|J_\mu(r) - J_\mu\| = 0$.

Proof. Suppose that μ is a Carleson measure. By the previous Remark 5.2.7, the orthogonal projections

$$P_r : L^2(\mu) \to L^2(\mu_r),\ f \mapsto \mathbb{1}_{\overline{\mathbb{B}_d(r)}} f$$

converge in the strong operator topology for $r \uparrow 1$ to the identity operator on $L^2(\mu)$. Thus,

$$J_\mu(r) = P_r J_\mu \xrightarrow{\text{SOT}} J_\mu$$

for $r \uparrow 1$. Since $\|J_\mu(r)\| \leq \|J_\mu\|$, we deduce that

$$c(\mu) = \sup_{0<r<1} \|J_\mu(r)\| < \infty$$

and we have $\|J_\mu\| = c(\mu)$. If μ is a vanishing Carleson measure and J_μ is compact, it follows from Theorem 2.3.11 that

$$\|J_\mu(r) - J_\mu\| = \|(\mathrm{id}_{L^2(\mu)} - P_r) J_\mu\| \to 0$$

for $r \uparrow 1$. Conversely, suppose that $c(\mu) = \sup_{0<r<1} \|J_\mu(r)\| < \infty$. By monotone convergence

$$\|h\|_{L^2(\mu)} = \sup_{0<r<1} \|J_\mu(r)h\|_{L^2(\mu)} < \infty$$

for all $h \in \mathscr{H}$. Hence, J_μ is well-defined. Using Lemma 2.2.10, the closed graph theorem implies that J_μ is bounded. Due to Remark 5.2.7, the linear operators

$$J_\mu(r) : \mathscr{H} \to L^2(\mu),\ f \to \mathbb{1}_{\overline{\mathbb{B}_d(r)}} f$$

are compact. So if $\lim_{r\uparrow 1} \|J_\mu(r) - J_\mu\| = 0$, we deduce that J_μ is compact. □

Lemma 5.2.9. *Let $T : L^2(\mu) \to L^2(\mu)$ be a positive bounded linear operator. If*

$$\|T\|_{Re} := \sup\{\langle Th, h\rangle_{L^2(\mu)};\ h \in L^2(\mu) \text{ with } h(\mathbb{B}_d) \subset \mathbb{R} \text{ and } \|h\|_{L^2(\mu)} \leq 1\},$$

then

$$\|T\| \leq 4\|T\|_{\mathrm{Re}}.$$

Proof. Let $T : L^2(\mu) \to L^2(\mu)$ be a positive linear operator such that

$$\|T\|_{\mathrm{Re}} := \sup\{\langle Th, h\rangle_{L^2(\mu)};\ h \in L^2(\mu) \text{ with } h(\mathbb{B}_d) \subset \mathbb{R} \text{ and } \|h\|_{L^2(\mu)} \leq 1\} < \infty.$$

and let $h \in L^2(\mu)$ with $\|h\|_{L^2(\mu)} \leq 1$. Set $h_1 = \mathrm{Re}(h)$ and $h_2 = \mathrm{Im}(h)$. Then $h_l \in L^2(\mu)$ with $h_l(\mathbb{B}_d) \subset \mathbb{R}$ $(l = 1, 2)$ and $\max_{l=1,2} \|h_l\|_{L^2(\mu)} \leq 1$. Using the Cauchy-Schwarz inequality, we obtain

$$\left|\langle Th_2, h_1\rangle_{L^2(\mu)} - \langle Th_1, h_2\rangle_{L^2(\mu)}\right| \leq 2\left(\max_{l=1,2} \langle Th_l, h_l\rangle_{L^2(\mu)}\right).$$

Thus, since T is positive, we conclude that

$$\begin{aligned}0 \leq \langle Th,h\rangle_{L^2(\mu)} &= \langle T(h_1+ih_2),h_1+ih_2\rangle_{L^2(\mu)} \\ &= \langle Th_1,h_1\rangle_{L^2(\mu)} + \langle Th_2,h_2\rangle_{L^2(\mu)} + i(\langle Th_2,h_1\rangle_{L^2(\mu)} - \langle Th_1,h_2\rangle_{L^2(\mu)}) \\ &\leq 4\left(\max_{l=1,2}\langle Th_l,h_l\rangle_{L^2(\mu)}\right) \\ &\leq 4\|T\|_{\mathrm{Re}}.\end{aligned}$$

Using again that T is positive, this yields

$$\|T\| = \sup_{\|h\|_{L^2(\mu)}\leq 1}\left(\langle Th,h\rangle_{L^2(\mu)}\right) \leq 4\|T\|_{\mathrm{Re}}.$$

□

Lemma 5.2.10. *Let $\mathscr{H}$ be a Hilbert function space with reproducing kernel $K : \mathbb{B}_d \times \mathbb{B}_d \to \mathbb{C}$ and let μ be a finite measure. Let $M \subset \mathbb{B}_d$ be Borel measurable, and suppose that the induced operator*

$$J_\mu(M) : \mathscr{H} \to L^2(\mu),\ f \mapsto \mathbb{1}_M f$$

is well-defined and bounded, then

$$\|J_\mu(M)\|^2 \leq 16 \sup_{w\in M}\int_M |\mathrm{Re}(K(z,w))|d\mu(z) \in [0,\infty].$$

Proof. Without loss of generality, we may assume that

$$\sup_{w\in M}\int_M |\mathrm{Re}(K(z,w))|d\mu(z) < \infty$$

is finite. For $h \in L^2(\mu)$ we have

$$\begin{aligned}(J_\mu(M)^*h)(z) &= \langle J_\mu(M)^*h,k_z\rangle_{\mathscr{H}} \\ &= \langle h,J_\mu(M)k_z\rangle_{L^2(\mu)} \\ &= \int_M K(z,w)h(w)d\mu(w).\end{aligned}$$

Let $h \in L^2(\mu)$ with $h(\mathbb{B}_d) \subset \mathbb{R}$ and $\|h\|_{L^2(\mu)} \leq 1$ and define

$$f_h : \mathbb{B}_d \times \mathbb{B}_d \to \mathbb{C},\ f_h(z,w) = \mathbb{1}_{M\times M}(z,w)|\mathrm{Re}\,K(z,w)|^{1/2}|h(w)|.$$

Then $f_h \in L^2(\mu^{\otimes 2})$. Using that $\mathrm{Re}\,K(z,w) = \mathrm{Re}\,\overline{K(z,w)} = \mathrm{Re}\,K(w,z)$ and hence

$$f_h(w,z) = \mathbb{1}_{M\times M}(z,w)|\mathrm{Re}\,K(z,w)|^{1/2}|h(z)|$$

for all $z, w \in \mathbb{B}_d$, we obtain by Cauchy-Schwarz that

$$\begin{aligned}\langle J_\mu(M)J_\mu(M)^* h, h\rangle_{L^2(\mu)} &= \operatorname{Re}\left(\int_{\mathbb{B}_d}\int_{\mathbb{B}_d} \mathbb{1}_{M\times M}(z,w)K(z,w)h(w)\overline{h(z)}d\mu(w)d\mu(z)\right)\\ &= \int_{\mathbb{B}_d}\int_{\mathbb{B}_d} \mathbb{1}_{M\times M}(z,w)\operatorname{Re}(K(z,w))h(w)h(z)d\mu(w)d\mu(z)\\ &\leq \int_{\mathbb{B}_d\times\mathbb{B}_d} f_h(z,w)f_h(w,z)d\mu^{\otimes 2}(w,z)\\ &\leq \|f_h\|^2_{L^2(\mu^{\otimes 2})}.\end{aligned}$$

Since

$$\|f_h\|^2_{L^2(\mu^{\otimes 2})} = \int_{\mathbb{B}_d} \mathbb{1}_M(w)|h(w)|^2\left(\int_M |\operatorname{Re}(K(z,w)|d\mu(z)\right)\mu(w),$$

using Lemma 5.2.9, it is immediate that

$$\|J_\mu(M)\|^2 \leq 16\sup_{w\in M}\int_M |\operatorname{Re}(K(z,w))|d\mu(z).$$

□

We obtain the following theorem, where part (a) is a Corollary of the earlier mentioned characterization of Carleson measures (see [ARS08, Lemma 24]) by Arcozzi, Rochberg and Sawyer.

Theorem 5.2.11. *Let $\mathscr{H}$ be a Hilbert function space with reproducing kernel $K : \mathbb{B}_d \times \mathbb{B}_d \to \mathbb{C}$ and let μ be a finite positive Borel measure.*

(a) If

$$\|\mu\|_K^2 := \sup_{w\in\mathbb{B}_d}\int_{\mathbb{B}_d} |\operatorname{Re}(K(z,w))|d\mu(z) < \infty,$$

then μ is a Carleson measure for $\mathscr{H}$ and

$$\|J_\mu\| \leq 4\|\mu\|_K.$$

(b) If

$$\sup_{w\in\mathbb{B}_d}\int_{\mathbb{B}_d\setminus\overline{\mathbb{B}_d(r)}} |\operatorname{Re}(K(z,w))|d\mu(z) \to 0$$

for $r \uparrow 1$, then μ is a vanishing Carleson measure for $\mathscr{H}$.

Proof. (a) Because of Lemma 5.2.6, the linear operators

$$J_\mu(r) : \mathscr{H} \to L^2(\mu),\ f \mapsto f\mathbb{1}_{\overline{\mathbb{B}_d(r)}} \quad (0 < r < 1)$$

are bounded. Using Lemma 5.2.10, it follows for $0 < r < 1$ that

$$\|J_\mu(r)\|^2 \leq 16\|\mu\|_K^2 = 16\sup_{w\in\mathbb{B}_d}\int_{\mathbb{B}_d} |\operatorname{Re}(K(z,w))|d\mu(z).$$

Thus, Proposition 5.2.8 yields that μ is a Carleson measure, where

$$\|J_\mu\|^2 \leq 16\|\mu\|_K^2.$$

(b) If

$$\sup_{w\in\mathbb{B}_d} \int_{\mathbb{B}_d\setminus\overline{\mathbb{B}_d(r)}} |\mathrm{Re}(K(z,w))| d\mu(z) \to 0$$

for $r \uparrow 1$, it is straightforward to check that

$$\|\mu\|_K^2 := \sup_{w\in\mathbb{B}_d} \int_{\mathbb{B}_d} |\mathrm{Re}(K(z,w))| d\mu(z) < \infty.$$

By part (a) J_μ is a Carleson measure. Due to Proposition 5.2.8, it remains to show that

$$\lim_{r\uparrow 1} \|J_\mu(r) - J_\mu\| = 0.$$

For $0 < r < 1$ and $f \in \mathscr{H}$ it is immediate that

$$(J_\mu - J_\mu(r))f = \mathbb{1}_{\mathbb{B}_d\setminus\overline{\mathbb{B}_d(r)}} f$$

The operator

$$J_\mu^{(1-r)} : \mathscr{H} \to L^2(\mu),\ f \mapsto (J_\mu - J_\mu(r))f$$

is well-defined and bounded. Applying Lemma 5.2.10 to the operators $J_\mu^{(1-r)}$, we obtain that

$$\|J_\mu - J_\mu(r)\|^2 \leq 16 \sup_{w\in\mathbb{B}_d} \int_{\mathbb{B}_d\setminus\overline{\mathbb{B}_d(r)}} |\mathrm{Re}(K(z,w))| d\mu(z) \to 0$$

for $r \uparrow 1$. □

Remark 5.2.12. Let H be a separable Hilbert space with orthonormal basis $(e_n)_{n\in\mathbb{N}}$ and let $(d_n)_{n\in\mathbb{N}}$ be a sequence in $\mathbb{C}$. The previous theorem is reminiscent of the condition that a diagonal operator $D : H \to H$ defined by

$$D(e_n) = d_n e_n \quad (n \in \mathbb{N})$$

is bounded if and only if $\sup_{n\in\mathbb{N}} |d_n| < \infty$ and compact if and only if $\lim_{n\to\infty} d_n = 0$.

Using Theorems 5.2.11, 2.3.55, 5.1 and 2.3.50, the following theorem is immediate.

Theorem 5.2.13. *Let $\omega : \mathbb{B}_d \to \mathbb{R}_{\geq 0}$ be a radial weight, let $N \in \mathbb{N}_{>0}$ and let $\mathscr{H}$ a be Hilbert function space with reproducing kernel*

$$K : \mathbb{B}_d \times \mathbb{B}_d \to \mathbb{C}$$

such that $\mathscr{H} = B_\omega^N$ with equivalence of norms.

(a) *If* $\varphi \in \mathscr{H} \cap H^\infty(\mathbb{B}_d)$ *such that*

$$\|\mu_{\varphi,N}\|_K = \sup_{w \in \mathbb{B}_d} \int_{\mathbb{B}_d} |\operatorname{Re}(K(z,w))||R^N \varphi(z)|^2 \omega(z) dV(z) < \infty,$$

then $\varphi \in \operatorname{Mult}(\mathscr{H})$ *with*

$$\|\varphi\|_{\mathrm{Mult}} \lesssim (\|\mu_{\varphi,N}\|_K^2 + \|\varphi\|_\infty^2)^{1/2}.$$

(b) *If* $\varphi \in \mathscr{H} \cap A(\mathbb{B}_d)$ *such that*

$$\sup_{w \in \mathbb{B}_d} \int_{\mathbb{B}_d \setminus \overline{\mathbb{B}_d(r)}} |\operatorname{Re}(K(z,w))||R^N \varphi(z)|^2 \omega(z) dV(z) \to 0$$

for $r \uparrow 1$, *then* $\varphi \in A(\mathscr{H})$.

Remark 5.2.14. For $z \in \mathbb{D}$ we have

$$\operatorname{Re}\left(\frac{1}{1-z}\right) = \frac{1-\operatorname{Re}(z)}{|1-z|^2} \geq 0.$$

Suppose that $K : \mathbb{B}_d \times \mathbb{B}_d \to \mathbb{C}$ is a (unitarily invariant) complete Nevanlinna-Pick kernel. Due to Theorem 2.3.60, there exists an $n \in \mathbb{N} \cup \{\infty\}$ and an embedding $b : \mathbb{B}_d \to \mathbb{B}_n$ such that

$$|\operatorname{Re} K(z,w)| = \operatorname{Re} K(z,w) = \frac{1-\operatorname{Re}(\langle b(z), b(w)\rangle)}{|1-\langle b(z), b(w)\rangle|^2} \geq 0 \quad (z,w \in \mathbb{B}_d).$$

Thus, one may replace the absolute value of the real part in the previous theorem by the real part itself.

Let

$$K_0 : \mathbb{B}_d \times \mathbb{B}_d \to \mathbb{C},\ K_0(z,w) = \frac{1}{\langle z,w\rangle} \log\left(\frac{1}{(1-\langle z,w\rangle)}\right)$$

be the kernel of the Dirichlet space on the ball and for $0 < s < d$ set

$$K_s : \mathbb{B}_d \times \mathbb{B}_d \to \mathbb{C},\ K_s(z,w) = \frac{1}{(1-\langle z,w\rangle)^s}.$$

Lemma 5.2.15. *Let* $1 < 2s \leq d+1$. *If* $p > \frac{d+1}{s}$, *then*

$$\sup_{w \in \mathbb{B}_d} \int_{\mathbb{B}_d} |K_{-2s+d+1}(z,w)|^{\frac{p}{p-2}} dV(z) < \infty.$$

Proof. We first consider the case $2s = d+1$. Since

$$\left|\frac{1}{z}\log\left(\frac{1}{1-z}\right)\right| \leq 2\log\left(\frac{2}{|1-z|}\right) + 4\pi$$

for all $z \in \mathbb{D}$ (see Lemma A.4.2), one computes for all $n \in \mathbb{N}$ and $z, w \in \mathbb{B}_d$ that

$$0 \leq |K_0(z,w)|^n \lesssim \log\left(\frac{2}{|1-\langle z,w\rangle|}\right)^n.$$

Using that

$$\log(x)^n \leq n!x$$

for all $x \geq 1$, one checks for all $n \in \mathbb{N}_{>0}$ and $z, w \in \mathbb{B}_d$ that

$$0 \leq |K_0(z,w)|^n \lesssim \frac{1}{|1-\langle z,w\rangle|}.$$

By [Rud08, 1.4.10. Proposition, 1 Preliminaries] the function $J_1 : \mathbb{B}_d \to \mathbb{C}$,

$$w \mapsto \int_{\mathbb{B}_d} \frac{1}{|1-\langle z,w\rangle|} dV(z)$$

is bounded. We deduce that

$$\sup_{w\in\mathbb{B}_d} \int_{\mathbb{B}_d} |K_0(z,w)|^n dV(z) < \infty$$

for all $n \in \mathbb{N}$. It follows for all $p > 2$ that

$$\sup_{w\in\mathbb{B}_d} \int_{\mathbb{B}_d} |K_0(z,w)|^{\frac{p}{p-2}} dV(z) < \infty.$$

Now, let $t = -2s + d + 1$ and consider

$$K_t : \mathbb{B}_d \times \mathbb{B}_d \to \mathbb{C},\ K_t(z,w) = \frac{1}{(1-\langle z,w\rangle)^t}.$$

By [Rud08, 1.4.10. Proposition, 1 Preliminaries], the function $J_{t\frac{p}{p-2}} : \mathbb{B}_d \to \mathbb{C}$,

$$w \mapsto \int_{\mathbb{B}_d} |K_t(z,w)|^{\frac{p}{p-2}} dV(z)$$

is bounded if

$$t\frac{p}{p-2} = \frac{p}{p-2}(-2s+d+1) < d+1.$$

This is the case if and only if $p > \frac{d+1}{s}$. □

For $1 \leq p < \infty$, $t > -\frac{1}{p}$ and $s \in \mathbb{R}$ let

$$B_t^{s,p} = \left\{ f \in \mathscr{O}(\mathbb{B}_d);\ \int_{\mathbb{B}_d} |R^s f(z)|^p \left(1-|z|^2\right)^{pt} dV(z) < \infty \right\}$$

be the L^p-version of the standard weighted Besov spaces with the norm

$$\|f\|^p_{B^{s,p}_t} = |f(0)|^p + \|R^s f\|^p_{L^p_a(\omega^{(pt)})} \quad (f \in B^{s,p}_t),$$

where $\omega^{(pt)} : \mathbb{B}_d \to \mathbb{R}_{\geq 0}$,

$$\omega^{(pt)}(z) = (1-|z|^2)^{pt}.$$

(see Definition 2.3.39). To simplify notation, we use the abbreviations:

(a) $B^{s,2}_t = B^s_t$,

(b) $B^{s,p} = B^{s,p}_0 = \left\{ f \in \mathscr{O}(\mathbb{B}_d);\ \int_{\mathbb{B}_d} |R^s f(z)|^p \, dV(z) < \infty \right\}$ and

(c) $B^s = B^s_0 = \left\{ f \in \mathscr{O}(\mathbb{B}_d);\ \int_{\mathbb{B}_d} |R^s f(z)|^2 \, dV(z) < \infty \right\}$.

In the Hilbert space case, when $p = 2$, we saw in Remark 2.3.49 (confer also Theorem 2.3.50) that one can do the following index shift

$$B^s_t = B^{s+r}_{t+r} \quad (r > 0).$$

For this standard weighted Besov spaces, there is also a L^p-version of this result. It is clear that it suffices to prove the statement for standard weighted Bergman spaces. For a proof, see [BB89, Theorem 5.12] and [Zhu05, Exercise 2.6].

Theorem 5.2.16 (Beatrous and Burbea)**.** *For $s \in \mathbb{R}$ and $r > 0$ we have*

$$B^{s,p}_t = B^{s+r,p}_{t+r}$$

with equivalence of norms.

We obtain the following version of Theorem 5.9 (2) in [BB08].

Theorem 5.2.17. *Let $1 < 2s \leq d+1$ and $p > \frac{d+1}{s}$. If*

$$\varphi \in B^{s,p} \cap H^\infty(\mathbb{B}_d),$$

then

(a) $\varphi \in \mathrm{Mult}(B^s)$ *with* $\|\varphi\|^2_{\mathrm{Mult}} \lesssim \|\varphi\|^2_{B^{s,p}} + \|\varphi\|^2_\infty$ *and*

(b) $\varphi \in A(B^s)$ *if and only if* $\varphi \in A(\mathbb{B}_d)$.

Proof. Set $A^2_0(\mathbb{B}_d) = \mathscr{D}_{-1}(\mathbb{B}_d)$ and let $N \geq s$. Due to Remark 2.3.49, we obtain that

$$B^s = B^N_{N-s} = \mathscr{D}_{-2s+d}(\mathbb{B}_d) = A^2_{-2s+d+1}(\mathbb{B}_d) \tag{5.1}$$

with equivalence of norms. Because of Theorem 5.2.13, it suffices to prove that

$$\sup_{w \in \mathbb{B}_d} \int_{\overline{\mathbb{B}_d \setminus \mathbb{B}_d(r)}} |K_{-2s+d+1}(z,w)||R^N \varphi(z)|^2 (1-|z|^2)^{2(N-s)} dV(z) \to 0$$

for $r \uparrow 1$. Since $p > \frac{d+1}{s}$, it follows from Lemma 5.2.15 that

$$c = \sup_{w \in \mathbb{B}_d} \int_{\mathbb{B}_d} |K_{-2s+d+1}(z,w)|^{\frac{p}{p-2}} dV(z) < \infty.$$

Fix $w \in \mathbb{B}_d$ and $0 \leq r < 1$. We use the notation $\mathbb{B}_d(0) = \emptyset$ and apply Hölder's inequality with Hölder conjugates $p' = \frac{p}{p-2}$ and $q' = \frac{p}{2}$ to the functions $f_{p'} : \mathbb{B}_d, \to \mathbb{R}_{\geq 0}$,

$$f_{p'}(z) = |K_{-2s+d+1}(z,w)|$$

and $g_{q'} : \mathbb{B}_d, \to \mathbb{R}_{\geq 0}$,

$$g_{q'}(z) = \mathbb{1}_{\mathbb{B}_d \setminus \overline{\mathbb{B}_d(r)}}(z) |R^N \varphi(z)|^2 (1-|z|^2)^{2(N-s)}.$$

It follows that

$$\begin{aligned} &\int_{\mathbb{B}_d \setminus \overline{\mathbb{B}_d(r)}} |K_{-2s+d+1}(z,w)| |R^N \varphi(z)|^2 (1-|z|^2)^{2(N-s)} dV(z) \\ &\leq c^{\frac{p-2}{p}} \left(\int_{\mathbb{B}_d \setminus \overline{\mathbb{B}_d(r)}} |R^N \varphi(z)|^p (1-|z|^2)^{p(N-s)} dV(z) \right)^{1/p}. \end{aligned}$$

Due to Theorem 5.2.16, we have that

$$B^{s,p} = B^{N,p}_{N-s}$$

with equivalence of norms. Since $\varphi \in B^{s,p}$, dominated convergence yields

$$\int_{\mathbb{B}_d \setminus \overline{\mathbb{B}_d(r)}} |R^N \varphi(z)|^p (1-|z|^2)^{p(N-s)}(z) dV(z) \to 0.$$

for $r \uparrow 1$. Hence,

$$\sup_{w \in \mathbb{B}_d} \int_{\mathbb{B}_d \setminus \overline{\mathbb{B}_d(r)}} |\operatorname{Re}(K_{-2s+d+1}(z,w))| |R^N \varphi(z)|^2 (1-|z|^2)^{2(N-s)} dV(z) \to 0$$

for $r \uparrow 1$. Similarly, one obtains that

$$\|\varphi\|^2_{\mathrm{Mult}} \lesssim \|\varphi\|^2_{B^{s,p}} + \|\varphi\|^2_\infty.$$

□

Remark 5.2.18. Let $1 < 2s \leq d+1$, let $p > \frac{d+1}{s}$ and let

$$\varphi \in B^{s,p} \cap H^\infty(\mathbb{B}_d).$$

In the proof of Theorem 5.2.17, we saw that

$$\sup_{w \in \mathbb{B}_d} \int_{\mathbb{B}_d \setminus \overline{\mathbb{B}_d(r)}} |K_{-2s+d+1}(z,w)| |R^N \varphi(z)|^2 (1-|z|^2)^{2(N-s)} dV(z) \to 0$$

for $r \uparrow 1$. In particular, there are non-trivial examples for condition (b) in Theorem 5.2.11.

Let $-d < s' \leq 1$ and $s = \frac{s'+d}{2}$. Because of Remark 2.3.49, we have that $B^s = \mathscr{D}_{-s'}(\mathbb{B}_d)$. Thus, we obtain the following reformulation of Theorem 5.2.17:

Corollary 5.2.19. *Let $-d < s' \leq 1$, $s = \frac{s'+d}{2}$ and $p > 2\left(\frac{d+1}{d+s'}\right)$. If $\varphi \in B^{s,p}$, then*

(i) $\varphi \in \mathrm{Mult}(\mathscr{D}_{-s'}(\mathbb{B}_d))$ with $\|\varphi\|^2_{\mathrm{Mult}} \lesssim \|\varphi\|^2_{B^{s,p}} + \|\varphi\|^2_\infty$ and

(ii) $\varphi \in A(\mathscr{D}_{-s'}(\mathbb{B}_d))$ if and only if $\varphi \in A(\mathbb{B}_d)$.

The following proposition can be used to compare the sufficient condition in Corollary 5.2.19 with the sufficient condition in Theorem 5.1.6.

Proposition 5.2.20. *Let $-d < s' \leq 1$, $s = \frac{s'+d}{2}$, $p > 2\left(\frac{d+1}{d+s'}\right)$ and $p' = \frac{p-2}{p}$. Then*

$$\frac{1-s'}{d+1} < p' < 1$$

and we have that

$$\mathscr{D}_{-s'-(d+1)p'}(\mathbb{B}_d) \subset B^{s,p} \subsetneq \mathscr{D}_{-s'-p'+\varepsilon}(\mathbb{B}_d)$$

for all $\varepsilon > 0$, where the inclusions are continuous.

Proof. Let

$$t = \frac{(d+1)p'-1}{2} = d\left(\frac{1}{2} - \frac{1}{p}\right) - \frac{1}{p},$$

then $s' + 2t + 1 = s' + (d+1)p'$. Using Remark 2.3.49, we deduce that

$$\mathscr{D}_{-s'-(d+1)p'}(\mathbb{B}_d) = B^{\frac{s'+d+2t+1}{2}} = R^{\frac{s'+d}{2}+t} H^2(\partial\mathbb{B}_d)$$

with equivalence of norms. Due to Theorem 4.48 in [Zhu05], it follows that

$$\mathscr{D}_{-s'-(d+1)p'}(\mathbb{B}_d) \subset B^{s+t,p}_t,$$

where the inclusion is continuous. Because of Theorem 5.2.16, it is immediate that

$$\mathscr{D}_{-s'-(d+1)p'}(\mathbb{B}_d) \subset B^{s,p}.$$

Let $\varepsilon > 0$. Due to Theorem 70 in [ZZ08] with $\alpha = 0$ and $q = 2$, we obtain for $t' = \frac{-p'+\varepsilon}{2}$ that

$$B^{s,p} \subsetneq B^s_{t'} = \mathscr{D}_{-s'+2t'}(\mathbb{B}_d) = \mathscr{D}_{-s'-p'+\varepsilon}(\mathbb{B}_d),$$

where the inclusion is continuous. □

Remark 5.2.21. (a) Let $-d < s' \leq 1$. If $t > 1$ and

$$\varphi \in \mathscr{D}_{-t}(\mathbb{B}_d) \subset A(\mathbb{B}_d),$$

Corollary 5.2.19 and Proposition 5.2.20 yield that $\varphi \in A(\mathscr{D}_{-s'}(\mathbb{B}_d))$.

(b) For the Dirichlet space $\mathscr{D}_{-1}(\mathbb{B}_d) = B^{\frac{d+1}{2}}$ on the ball let $p > 2$, $p' = \frac{p-2}{p}$ and $\varepsilon = \frac{p'}{2}$. Using Proposition 5.2.20, we conclude that

$$B^{\frac{d+1}{2},p} \subset \mathscr{D}_{-1-p'/2}(\mathbb{B}_d) \subset \mathscr{D}^0_{-1}(\mathbb{B}_d) \cap A(\mathbb{B}_d).$$

Thus, in this case, Theorem 5.2.17 follows from Theorem 5.1.6, part (a).

(c) Theorem 5.2.17 and Theorem 5.1.6 seem to give conditions that cannot be derived from another.

5.3. A special case: The Dirichlet space on the unit disk

Let dA be the normalized area measure on the unit disk $\mathbb{D}$, let

$$\mathscr{D} = \left\{ f \in \mathscr{O}(\mathbb{D});\ \int_{\mathbb{D}} |f'(z)|^2 dA(z) < \infty \right\}$$

be the Dirichlet space with reproducing kernel

$$K : \mathbb{D} \times \mathbb{D} \to \mathbb{C},\ K(z,w) = \frac{1}{z\overline{w}} \log\left(\frac{1}{1 - z\overline{w}}\right)$$

and for $1 \leq p < \infty$ let

$$L_a^p(\mathbb{D}) = \left\{ f \in \mathscr{O}(\mathbb{D});\ \int_{\mathbb{D}} |f(z)|^p dA(z) < \infty \right\}$$

be the Bergman spaces on the unit disk $\mathbb{D}$.

The book [EFKMR14] by El-Fallah, Kellay, Mashregi and Ransford gives a good introduction to the theory of the Dirichlet space. We use in particular Chapter 5 as guidelines here.

The multiplier algebra $\mathrm{Mult}(\mathscr{D})$ of the Dirichlet space can be characterized in the following way (see, for example Theorem 5.1.7 in [EFKMR14]):

Theorem 5.3.1. *A function $\varphi : \mathbb{D} \to \mathbb{C}$ is in $\mathrm{Mult}(\mathscr{D})$ if and only if*

$$\varphi \in H^\infty(\mathbb{D}) \text{ and } \varphi' \in \mathrm{Mult}(\mathscr{D}, L_a^2(\mathbb{D})).$$

This is also a special case of Theorem 2.3.55. Now, let $\varphi \in \mathscr{D}$. One checks that

$$\varphi' \in \mathrm{Mult}(\mathscr{D}, L_a^2(\mathbb{D})),$$

if and only if the finite measure

$$\mu_{\varphi'}(z) = |\varphi'(z)|^2 dA(z)$$

is a Carleson measure for $\mathscr{D}$. The multiplication operator

$$M_{\varphi'} : \mathscr{D} \to L_a^2(\mathbb{D})$$

is compact if and only if $\mu_{\varphi'}$ is vanishing (see Remark 5.2.1).

Using Remark 2.3.56 one obtains the following reformulation of Theorem 5.2.13 in the Dirichlet space case.

Theorem 5.3.2. *(a) If $\varphi \in \mathscr{D} \cap H^\infty(\mathbb{D})$ such that*

$$\|\mu_{\varphi'}\|_K = \sup_{w\in\mathbb{D}} \int_{\mathbb{D}} |\mathrm{Re}(K(z,w))| \varphi'(z)|^2 dA(z) < \infty,$$

then $\varphi \in \mathrm{Mult}(\mathscr{D})$ with

$$\|\varphi\|_{\mathrm{Mult}} \lesssim (\|\mu_{\varphi'}\|_K^2 + \|\varphi\|_\infty^2)^{1/2}.$$

(b) If $\varphi \in \mathscr{D} \cap A(\mathbb{D})$ such that

$$\sup_{w\in\mathbb{D}} \int_{\mathbb{D}\setminus\overline{\mathbb{D}(r)}} |\mathrm{Re}(K(z,w))||\varphi'(z)|^2 dA(z) \to 0$$

for $r \uparrow 1$, then $\varphi \in A(\mathscr{D})$.

By Remark 5.2.21 we can reformulate Corollary 5.2.19 as:

Theorem 5.3.3 (Brown and Shields)**.** *Let $p > 2$. If $\varphi' \in L_a^p(\mathbb{D})$, then $\varphi \in A(\mathscr{D})$.*

The second statement presumably appeared the first time in Proposition 19 in [BS84].

Recall, the Dirichlet space $\mathscr{D}$ is a regular unitarily invariant complete Nevanlinna-Pick space (see Lemma 2.3.62 and Kaluza's Lemma 2.3.63).

For unitarily invariant complete Nevanlinna-Pick spaces $\mathscr{H}$ with reproducing kernel $K : \mathbb{B}_d \times \mathbb{B}_d \to \mathbb{C}$ the functions

$$k_w : \mathbb{B}_d \to \mathbb{C},\ k_w(z) = K(z,w) \quad (z \in \mathbb{B}_d)$$

are elements of $\mathrm{Mult}(\mathscr{H})$ (see Theorem 2.3.61). For $f \in \mathscr{H}$ the Sarason function $V_f : X \to \mathbb{C}$ is defined as

$$V_f(z) = 2\langle f, k_z f\rangle_{\mathscr{H}} - \|f\|_{\mathscr{H}}^2.$$

A straightforward computation shows that if $f : \mathbb{D} \to \mathbb{C}$ is in the Hardy space $H^2(\mathbb{D})$ and

$$K : \mathbb{D} \times \mathbb{D} \to \mathbb{C},\ K(z,w) = \frac{1}{1 - z\overline{w}}$$

is the Szegő kernel, then

$$\mathrm{Re}\, V_f(z) = \int_{\mathbb{T}} \frac{1 - |z|^2}{\left|1 - z\overline{\xi}\right|^2} |f^*(\xi)|^2 dm(\xi) = P[|f^*|^2](z) \quad (z \in \mathbb{D}),$$

is just the Poisson integral of $|f^*|^2$, where

$$f^* \in H^2(\mathbb{T}) = \left\{h \in L^2(\mathbb{T});\ \hat{h}(n) = 0 \text{ for all } n \leq 0\right\}$$

is the radial limit of f, that exists for almost every $z \in \mathbb{T}$. Thus, it follows that

$$0 \leq |f(z)|^2 \leq P[|f^*|^2](z) = \mathrm{Re}\, V_f(z) \quad (z \in \mathbb{D}).$$

This inequality continues to hold for arbitrary (unitarily invariant) complete Nevanlinna-Pick spaces $\mathscr{H}$, that is

$$0 \leq |f(z)|^2 \leq \frac{\|k_z f\|_{\mathscr{H}}}{\|k_z\|_{\mathscr{H}}} \leq \operatorname{Re} V_f(z) \quad (f \in \mathscr{H}, z \in \mathbb{B}_d)$$

(see [GRS02, Section 2] and [AHMR18, Section 3.2]). Suppose now that the unitarily invariant complete Nevanlinna-Pick space $\mathscr{H}$ is a standard weighted Besov space B_t^N $t > -\frac{1}{2}$ $(N \in \mathbb{N})$. Aleman, Hartz, M^{c}Carthy and Richter give the following sufficient condition for a function $f \in \mathscr{H}$ to be in the multiplier algebra $\operatorname{Mult}(\mathscr{H})$ (particular case of [AHMR18, Theorem 4.5]):

Theorem 5.3.4. *Let* $\varphi \in \mathscr{H}$ *such that* $\operatorname{Re} V_\varphi : \mathbb{B}_d \to \mathbb{R}_{\geq 0}$ *is bounded. Then* $\varphi \in \operatorname{Mult}(\mathscr{H})$ *and*

$$\|\varphi\|_{\mathrm{Mult}} \lesssim \|\operatorname{Re} V_\varphi\|_\infty^N.$$

Let $f \in \mathscr{D}$. As in [Shi98, Proposition 3 and Corollary 4]), one can show that

$$\operatorname{Re} V_f(w) = \|f\|_{H^2(\mathbb{D})}^2 + P[|f^*|^2](z) + 2\int_{\mathbb{D}} \operatorname{Re}(K(z,w))|f'(z)|^2 dA(z) - \|f\|_{\mathscr{D}}^2 \quad (w \in \mathbb{D}).$$

Hence, it follows that $\operatorname{Re} V_f$ is bounded if and only if $f \in H^\infty(\mathbb{D})$ and

$$\sup_{w\in\mathbb{D}} \int_{\mathbb{D}} \operatorname{Re}(K(z,w))|f'(z)|^2 dA(z) < \infty.$$

A computation, using the transformation formula, yields that

$$\begin{aligned}
&\operatorname{Re} V_f(z) - \operatorname{Re} V_{f_r}(rz) \\
&= (\|f\|_{H^2(\mathbb{D})}^2 - \|f_r\|_{H^2(\mathbb{D})}^2) + (\|f_r\|_{\mathscr{D}}^2 - \|f\|_{\mathscr{D}}^2) + (P[|f^*|^2](z) - P[|f_r^*|^2](rz)) \\
&\quad + 2\left(\int_{\mathbb{D}\setminus\mathbb{D}_r} \operatorname{Re}(K(w,z))|f'(w)|^2 dA(w)\right).
\end{aligned}$$

If $f \in A(\mathbb{D})$, we conclude that

$$\begin{aligned}
&\sup_{z\in\mathbb{D}} |P[|f|^2](z) - P[|f_r|^2](rz)| \\
&\leq \sup_{z\in\mathbb{D}} |(P[|f|^2] - P[|f|^2]_r)(z)| + \sup_{z\in\mathbb{D}} |(P[|f|^2] - P[|f_r|^2])(rz)| \\
&\leq \sup_{z\in\mathbb{D}} |(P[|f|^2] - P[|f|^2]_r)(z)| + \sup_{z\in\mathbb{T}} ||f|^2 - |f_r|^2| \\
&\xrightarrow{r\uparrow 1} 0.
\end{aligned}$$

Thus, it follows that

$$\sup_{z\in\mathbb{D}} |\operatorname{Re} V_f(z) - \operatorname{Re} V_{f_r}(rz)| \xrightarrow{r\uparrow 1} 0,$$

if and only if $f \in A(\mathbb{D})$ and

$$\sup_{z\in\mathbb{D}} \left(\int_{\mathbb{D}\setminus\mathbb{D}_r} \operatorname{Re}(K(w,z))|f'(w)|^2 dA(w) \right) \xrightarrow{r\uparrow 1} 0.$$

We obtain the following equivalent formulation of Theorem 5.3.2:

Theorem 5.3.5. *Let $\varphi \in \mathscr{D}$.*

(a) If

$$\sup_{w\in\mathbb{D}} \operatorname{Re} V_\varphi(w) < \infty,$$

then $\varphi \in \operatorname{Mult}(\mathscr{D})$ with

$$\|\varphi\|_{\mathrm{Mult}} \lesssim \|\operatorname{Re} V_\varphi\|_\infty.$$

(b) If

$$\sup_{w\in\mathbb{D}} |\operatorname{Re} V_\varphi(w) - \operatorname{Re} V_{\varphi_r}(rw)| \xrightarrow{r\uparrow 1} 0$$

for $r \uparrow 1$, then $\varphi \in A(\mathscr{D})$.

5.3.1. Vanishing Carleson measures and Carleson-boxes

For Hardy spaces $H^p(\mathbb{D})$ and the Dirichlet space, it is possible to give a geometric characterization for Carleson measures by so-called Carleson-boxes. For an given arc $I \subset \mathbb{T}$, the corresponding Carleson-box is defined by

$$S(I) = \{re^{i\theta};\ e^{i\theta} \in I, 1 - |I| < r < 1\},$$

where $|I|$ denotes the arclength of I. Carleson showed that

$$\int_{\mathbb{D}} |f|^p d\mu \le C\|f\|^p_{H^p(\mathbb{D})} \quad (f \in H^p(\mathbb{D})),$$

for $p \ge 1$ if and only if

$$\mu(S(I)) = O(|I|).$$

There is a similar but more complicated characterization of Carleson measures for the Dirichlet space using the logarithmic capacity.

The logarithmic capacity $c : \mathscr{B}(\mathbb{T}) \to \mathbb{R}_{\ge 0}$ is a non-negative monotonic function, defined on Borel-subsets of the unit circle (see [EFKMR14, Chapter 2]). One can use the term as a black box here, if one does not want to go into details.

For a finite positive Borel measure μ on the open unit disk, define

$$|||\mu||| := \sup \left(\frac{\mu(\cup_{n=1}^l S(I_n))}{c(\cup_{n=1}^l I_n)};\ I_n \in \mathscr{I} \text{ and } l \in \mathbb{N}_{>0} \right) \in [0,\infty].$$

Let $\mathscr{I}$ be the set of all arcs in the unit circle $\mathbb{T} \subset \mathbb{C}$.

In [Ste80, Theorem 2.3] (see also [EFKMR14, Theorem 5.2.6]), Stegenga shows that:

Theorem 5.3.6 (Stegenga)**.** *A finite positive Borel measure μ on the open unit disk is a Carleson measure if and only if $|||\mu||| < \infty$. In this case, the Carleson operator*

$$J_\mu : \mathscr{H} \to L^2(\mu),\ f \mapsto f$$

is bounded with

$$\|J_\mu\| \approx |||\mu|||.$$

The characterization of vanishing Carleson measures in Proposition 5.2.8 yields the following version of Theorem 5.3.6 for vanishing Carleson measures:

Theorem 5.3.7. *A measure μ on the open unit disk is a compact Carleson measure for the Dirichlet space if and only if*

$$\sup\left(\frac{\mu(\cup_{n=1}^{l} S(I_n))}{c(\cup_{n=1}^{l} I_n)};\ I_n \in \mathscr{I} \text{ with } |I_n| \leq t \text{ and } l \in \mathbb{N}_{>0}\right) \to 0 \text{ for } t \to 0.$$

Proof. If

$$\sup\left(\frac{\mu(\cup_{n=1}^{l} S(I_n))}{c(\cup_{n=1}^{l} I_n)};\ I_n \in \mathscr{I} \text{ with } |I_n| \leq t \text{ and } l \in \mathbb{N}_{>0}\right) \to 0 \text{ for } t \to 0,$$

then μ is a Carleson measure for $\mathscr{D}$ by Theorem 5.3.6. For $0 < r < 1$ let $J_\mu(r) : \mathscr{D} \to L^2(\mu)$,

$$f \mapsto \mathbb{1}_{\overline{\mathbb{D}(r)}} f$$

be the compact operators defined as in Remark 5.2.7. It is immediate that

$$\nu_r = \mathbb{1}_{\mathbb{D}\setminus\overline{\mathbb{D}(r)}}\mu$$

is a Carleson measure for $\mathscr{D}$ with Carleson operator

$$J_{\nu_r} = J_\mu - J_\mu(r).$$

For an interval $I \in \mathscr{I}$, it follows from the definition of a Carleson-box $S(I)$, that there is a finite set of intervals $(I_m)_{m=1}^{n}$ in $\mathscr{I}$ with $|I_m| \leq 1 - r$ such that

$$S(I) \cap (\mathbb{D} \setminus \mathbb{D}_r) \subset \bigcup_{m=1}^{n} S(I_m).$$

Applying Theorem 5.3.6 to the measure ν_r, we obtain that

$$\begin{aligned}\|J_\mu - J_\mu(r)\| &\approx \sup\left(\frac{\mu((\cup_{n=1}^{l} S(I_n)) \cap \mathbb{D} \setminus \mathbb{D}_r)}{c(\cup_{n=1}^{l} I_n)};\ I_n \in \mathscr{I} \text{ and } l \in \mathbb{N}_{>0}\right) \\ &= \sup\left(\frac{\mu(\cup_{n=1}^{l} S(I_n))}{c(\cup_{n=1}^{l} I_n)};\ I_n \in \mathscr{I} \text{ with } |I| \leq 1 - r \text{ and } l \in \mathbb{N}_{>0}\right).\end{aligned}$$

Thus, Proposition 5.2.8 yields that the measure μ is a compact Carleson measure if and only if

$$\sup\left(\frac{\mu(\cup_{n=1}^{l} S(I_n))}{c(\cup_{n=1}^{l} I_n)}; I_n \in \mathscr{I} \text{ with } |I| \leq t \text{ and } l \in \mathbb{N}_{>0}\right) \to 0 \text{ for } t \to 0.$$

□

The previous statement seems to be a little complicated to check. We want to consider a sufficient geometric condition for vanishing Carleson measures in terms of a one-box condition.

If μ is a positive Borel measure on the open unit disk and $\phi : (0,1) \to \mathbb{R}_{>0}$ is increasing with $\int_0^1 \frac{\phi(t)}{t} dt < \infty$ define

$$|||\mu|||_\phi = \sup\left(\frac{\mu(S(I))}{\phi(|I|)}; I \in \mathscr{I}\right).$$

A theorem by Wynn (see Theorem 5.2.5 (ii) in [EFKMR14]) shows that:

Theorem 5.3.8 (Wynn)**.** *A finite positive Borel measure μ on the open unit disk is a Carleson measure, if $|||\mu|||_\phi < \infty$. In this case, the Carleson operator*

$$J_\mu : \mathscr{H} \to L^2(\mu),\ f \mapsto f$$

is bounded by

$$\|J_\mu\| \lesssim |||\mu|||_\phi.$$

We obtain the modified version of Wynn's result for vanishing Carleson measures.

Theorem 5.3.9. *A sufficient condition for a measure μ on the open unit disk to be a compact Carleson measure for the Dirichlet space is that*

$$\sup\left(\frac{\mu(S(I))}{\phi(|I|)}; I \in \mathscr{I} \text{ with } |I| \leq t\right) \to 0 \text{ for } t \to 0,$$

where $\phi : (0,1) \to \mathbb{R}_{>0}$ is increasing with $\int_0^1 \frac{\phi(t)}{t} dt < \infty$.

Proof. Suppose that

$$\sup\left(\frac{\mu(S(I))}{\phi(|I|)}; I \in \mathscr{I} \text{ with } |I| \leq t\right) \to 0 \text{ for } t \to 0.$$

Because of Theorem 5.3.8, it follows that μ is a Carleson measure for $\mathscr{D}$. For $0 < r < 1$ let $J_\mu(r) : \mathscr{D} \to L^2(\mu)$,

$$f \mapsto \mathbb{1}_{\overline{\mathbb{D}(r)}} f$$

be the compact operators defined as in Remark 5.2.7. It is immediate that

$$\nu_r = \mathbb{1}_{\mathbb{D}\setminus\overline{\mathbb{D}(r)}}\mu$$

is a Carleson measure for $\mathscr{D}$ with Carleson operator

$$J_{\nu_r} = J_\mu - J_\mu(r).$$

Let $\varepsilon > 0$. Then there exists a $t > 0$ such that

$$\frac{\nu_r(S(I))}{\phi(|I|)} \leq \frac{\mu(S(I))}{\phi(|I|)} < \varepsilon$$

for all $I \in \mathscr{I}$ with $|I| < t$. Since ϕ is increasing, we deduce that $\phi(t) < \phi(|I|)$ for all $I \in \mathscr{I}$ with $|I| > t$. Hence,

$$\frac{\nu_r(S(I))}{\phi(|I|)} \leq \frac{\mu(\mathbb{D}\setminus\mathbb{D}_r)}{\phi(t)}$$

for all $I \in \mathscr{I}$ with $|I| > t$. This yields

$$\limsup_{r\uparrow 1} |||\nu_r|||_\phi \leq \limsup_{r\uparrow 1}\left(\max\left(\varepsilon, \frac{\mu(\mathbb{D}\setminus\mathbb{D}_r)}{\phi(t)}\right)\right) = \varepsilon$$

and thus,

$$0 \leq \limsup_{r\uparrow 1} \|J_\mu - J_\mu(r)\| \lesssim \limsup_{r\uparrow 1} |||\nu_r||| = 0.$$

Due to Proposition 5.2.8, the measure μ is a compact Carleson measure. $\square$

6. Norm-closure of polynomials and one-function Corona theorem

In the following, let $\mathscr{H}$ be a regular unitarily invariant space with reproducing kernel

$$K : \mathbb{B}_d \times \mathbb{B}_d \to \mathbb{C},\ K(z,w) = \sum_{n=0}^{\infty} a_n \langle z,w\rangle^n$$

such that $a_0 = 1$, $a_n > 0$ for $n \geq 1$ and $\lim_{n\to\infty} \frac{a_n}{a_{n+1}} = 1$. In this case,

$$\mathbb{C}[z] \subset \mathrm{Mult}(\mathscr{H}) \subset H^\infty(\mathbb{B}_d) \cap \mathscr{H},$$

where the second inclusion is continuous. It follows that

$$A(\mathscr{H}) = \overline{\mathbb{C}[z]}^{\|\cdot\|_{\mathrm{Mult}}} \subset A(\mathbb{B}_d) \cap \mathrm{Mult}(\mathscr{H}).$$

For the Hardy space $H^2(\partial\mathbb{B}_d)$ and the Bergman space $L^2_a(\mathbb{B}_d)$ we have that

$$\mathrm{Mult}(\mathscr{H}) = H^\infty(\mathbb{B}_d) \subset \mathscr{H},$$

and thus,

$$A(\mathscr{H}) = A(\mathbb{B}_d),$$

but this is not true in general. It is well-known that, if $\mathscr{H}$ is the Drury-Arveson space H^2_d or the Dirichlet space $\mathscr{D}$, then

$$\mathrm{Mult}(\mathscr{H}) \subsetneq H^\infty(\mathbb{B}_d) \cap \mathscr{H}.$$

In Chapter 5 we have seen sufficient and necessary conditions for a function to be in the norm-closure $A(\mathscr{H})$. We now want to analyze in which cases

$$A(\mathscr{H}) \subsetneq A(\mathbb{B}_d) \cap \mathrm{Mult}(\mathscr{H}).$$

In [FX11], Fang and Xia show that there exist multiplication operators in the Drury-Arveson space H^2_d that are not essentially hyponormal. For their result, they prove that

$$A(H^2_d) \subsetneq A(\mathbb{B}_d) \cap \mathrm{Mult}(H^2_d).$$

In [Luo17], Lou establishes the same result for the Dirichlet space $\mathscr{D}$. Similarly, he uses the fact that

$$A(\mathscr{D}) \subsetneq A(\mathbb{B}_d) \cap \mathrm{Mult}(\mathscr{D}).$$

Extracting arguments from Fang and Xia's paper, we obtain the following theorem:

Theorem 6.1. *Let $\mathscr{H}$ be a regular unitarily invariant space, then the following are equivalent:*

(i) $\mathrm{Mult}(\mathscr{H}) = H^\infty(\mathbb{B}_d) \cap \mathscr{H}$,

(ii) $A(\mathbb{B}_d) \cap \mathrm{Mult}(\mathscr{H}) = A(\mathbb{B}_d) \cap \mathscr{H}$,

(iii) $A(\mathscr{H}) = A(\mathbb{B}_d) \cap \mathrm{Mult}(\mathscr{H})$,

(iv) $\|M_\varphi\|_e = \|\varphi\|_\infty$ *for all* $f \in \mathrm{Mult}(\mathscr{H}) \cap A(\mathbb{B}_d)$, *where* $\|\cdot\|_e$ *is the essential norm of an operator.*

Now, suppose that $\mathrm{Mult}(\mathscr{H}) \subsetneq H^\infty(\mathbb{B}_d) \cap \mathscr{H}$ and suppose that the one-function Corona theorem can be applied in $\mathscr{H}$. That is:

Theorem (One-function Corona theorem)**.** *If* $\varphi \in \mathrm{Mult}(\mathscr{H})$ *and* $1/\varphi \in H^\infty(\mathbb{B}_d)$, *then* $1/\varphi \in \mathrm{Mult}(\mathscr{H})$.

Using Theorem 6.1, one can show in the same way as in [FX11, Proposition 3.3] that there exists a $\varphi \in \mathrm{Mult}(\mathscr{H})$ such that M_φ is not essentially hyponormal.

In Theorem 6.2.8, we will see that the one-function Corona theorem can be applied in many function spaces (not necessarily unitarily invariant Hilbert spaces).

In a recent paper [APR+24] about cylicity in weighted Besov spaces, Aleman, Perfekt, Richter, Sundberg, and Sunkes obtain a generalized version of the one-function Corona theorem for radially weighted Besov (Hilbert) spaces:

Theorem (Aleman, Perfekt, Richter, Sundberg, and Sunkes)**.** *If* $\varphi, \psi \in \mathrm{Mult}(B^N_\omega)$ *with* $\frac{\varphi}{\psi} \in H^\infty(\mathbb{B}_d)$, *then* $\frac{\varphi^{N+1}}{\psi} \in \mathrm{Mult}(B^N_\omega)$

Similarly to a paper by Lindström, Miihkinen, and Norrbo (cf. [LMN20, Lemma 3.1]), it is not difficult to check that this general version is also valid for many other Banach function spaces of this type (see Theorem 6.2.8). Examples are L^p-versions of radially weighted Besov spaces, Hardy Sobolev spaces, and Bloch-type spaces. The proof of the generalized one-function Corona Theorem is straightforward, using the differentiation formula

$$R^N\left(\frac{f}{g}\right) = \frac{(-1)^N}{g^{N+1}} \sum_{l=0}^{N} (-1)^l \binom{N+1}{l} g^l R^N(g^{N-l} f),$$

where $f, g \in \mathscr{O}(\mathbb{B}_d)$, $0 \notin g(\mathbb{B}_d)$ and $N \geq 1$. The formula is due to Cao, He, and Zhu (see [CHZ18, Theorem 5, Corollary 6 and Proposition 7]). The original proof of the formula is technical, but it turns out that it also follows from an application of the binomial theorem.

Let

$$\mathscr{B} = \left\{ f \in \mathscr{O}(\mathbb{B}_d);\ \sup_{z \in \mathbb{B}_d} (1 - |z|^2)|Rf(z)| < \infty \right\}$$

be the Bloch space. A particular case of Theorem 5.1 in [RS16] shows that $f \in H^2_d \cap \mathscr{B}$ and $\frac{1}{f} \in H^\infty(\mathbb{B}_d)$ imply that $\frac{1}{f} \in H^2_d$. We conclude the chapter with another application of the differentiation formula by Cao, He, and Zhu. We establish a generalized version of Theorem 5.1 in [RS16] (see Theorem 6.2.10) for the L^p-versions of standard weighted Besov spaces.

6.1. Norm-closure of polynomials, ball algebra and essential hyponormality

The goal of this section is to prove Theorem 6.1.

Throughout the section, we use the notation

$$\|T\|_e = \inf\{\|T+X\|;\ X \in K(H)\}$$

for the essential norm of a bounded operator $T \in B(H)$ on a Hilbert space H. We start with the following corollary, which is a consequence of Theorem 2.3.11.

Corollary 6.1.1. *Let $T \in B(H)$ and let $(S_n)_{n\in\mathbb{N}}$ be a sequence of self-adjoint operators in $B(H)$ such that $S_n \xrightarrow{\text{SOT}} 0$ for $n \to \infty$. Then*

$$\limsup_{n\to\infty} \|S_n T\| \leq \left(\limsup_{n\to\infty} \|S_n\|\right) \|T\|_e$$

and

$$\limsup_{n\to\infty} \|T S_n\| \leq \left(\limsup_{n\to\infty} \|S_n\|\right) \|T\|_e.$$

Proof. Using Theorem 2.3.11 and the fact that $S_n \overset{\text{SOT}}{\to} 0$ for $n \to \infty$, we deduce that

$$\limsup_{n\to\infty} \|S_n T\| = \limsup_{n\to\infty} \|S_n(T+X)\| \leq \left(\limsup_{n\to\infty} \|S_n\|\right) \|T+X\|$$

for all compact operators $X \in K(H)$. This yields

$$\limsup_{n\to\infty} \|S_n T\| \leq \left(\limsup_{n\to\infty} \|S_n\|\right) \|T\|_e.$$

The second inequality follows by applying the first one to T^* and by using that

$$\|T S_n\| = \|S_n T^*\| \text{ and } \|T\|_e = \|T^*\|_e.$$

□

Since $\mathscr{H}$ is regular, due to Theorem 2.4.29, it follows that,

$$\|M_\varphi\|_e = \|\varphi\|_\infty$$

for all $\varphi \in A(\mathscr{H})$. Together with the next result, this is a crucial step in the proof of Theorem 6.1. The following statement is a gliding hump argument that gives a possibility to construct a SOT-convergent series by thinning out a suitable zero sequence.

Theorem 6.1.2. *Let H be a Hilbert space and let $(Q_l)_{l\in\mathbb{N}}$ be a sequence of pairwise orthogonal projections with finite range such that*

$$P_m = \sum_{l=0}^{m} Q_l \xrightarrow{\text{SOT}} \mathrm{id}_H$$

for $m \to \infty$. Suppose that $(T_n)_{n\in\mathbb{N}}$ is a sequence in $B(H)$ such that

(1) $T_n \xrightarrow{\text{SOT}} 0$ *for* $n \to \infty$,

(2) $\|T_n\|_e \longrightarrow 0$ *for* $n \to \infty$ *and*

(3) $T_n(\mathrm{id}_H - P_m) = (\mathrm{id}_H - P_m)T_n(\mathrm{id}_H - P_m)$ *for all* $m,n \in \mathbb{N}$.

Then, for every $\varepsilon > 0$ the sequence $(T_n)_{n\in\mathbb{N}}$ has a subsequence $(S_n)_{n\in\mathbb{N}}$, such that

$$S = \sum_{n=0}^{\infty} S_n$$

is a SOT*-convergent series with*

$$\|S\| \leq \sup_{n\in\mathbb{N}} \|T_n\| + \varepsilon.$$

Additionally, if $\inf_{n\in\mathbb{N}} \|T_n\| > 0$, *then one can choose $(S_n)_{n\in\mathbb{N}}$, such that*

$$\|S\|_e > \sum_{n=0}^{\infty} \|S_n\|_e.$$

Proof. Let $\varepsilon > 0$ and let $(T_n)_{n\in\mathbb{N}}$ be a sequence in $B(H)$ such that the conditions $(1),(2)$ and (3) hold.

By (1) and the fact that the projections P_m have finite range, it follows that

$$\lim_{n\to\infty} \|T_n P_m\| = 0$$

for all $m \in \mathbb{N}$. Hence, by passing to a subsequence of $(T_n)_{n\in\mathbb{N}}$, we may assume without loss of generality that

$$\lim_{n\to\infty} \|T_n P_n\| = 0.$$

Using Corollary 6.1.1 for T_n and the sequence $(\mathrm{id}_H - P_m)_{m\in\mathbb{N}}$, we can find a map $r : \mathbb{N} \to \mathbb{N}$, such that

(i) $\|T_n(\mathrm{id}_H - P_{r(n)})\| + \|(\mathrm{id}_H - P_{r(n)})T_n\| \leq 3\|T_n\|_e$ and

(ii) $r(n+1) > r(n)$

for all $n \in \mathbb{N}$. Now define for each $n \in \mathbb{N}$ the operators

$$\tilde{A}_n = T_nP_n + (\mathrm{id}_H - P_{r(n)})T_n(\mathrm{id}_H - P_n) + (P_{r(n)} - P_n)T_n(\mathrm{id}_H - P_{r(n)})$$

and

$$\tilde{B}_n = (P_{r(n)} - P_n)T_n(P_{r(n)} - P_n).$$

Using (3), one computes that

$$T_n = \tilde{A}_n + \tilde{B}_n$$

for all $n \in \mathbb{N}$. Hence, it is enough to show that $(\tilde{A}_n)_{n\in\mathbb{N}}$ and $(\tilde{B}_n)_{n\in\mathbb{N}}$ admit subsequences $(A_n)_{n\in\mathbb{N}}$ and $(B_n)_{n\in\mathbb{N}}$, such that $\sum_{n=0}^{\infty} A_n$ converges in norm, $\sum_{n=0}^{\infty} B_n$ converges in the strong operator topology and

$$\left\| \mathrm{SOT} - \sum_{n=0}^{\infty} (A_n + B_n) \right\| \leq \sup_{n\in\mathbb{N}} \|T_n\| + \varepsilon.$$

Obviously, by (2), using the definition of the operators $\tilde{A}_n$, it follows that

$$\begin{aligned} 0 \leq \|\tilde{A}_n\| &\leq \|T_nP_n\| + \|(\mathrm{id}_H - P_{r(n)})T_n\| + \|T_n(\mathrm{id}_H - P_{r(n)})\| \\ &\leq \|T_nP_n\| + 3\|T_n\|_e \to 0 \end{aligned}$$

for $n \to \infty$. So, it is not difficult to see that $(\tilde{A}_n)_{n\in\mathbb{N}}$ has a subsequence $(A_n)_{n\in\mathbb{N}}$, such that

$$\sum_{n=0}^{\infty} \|A_n\| < \varepsilon.$$

Thus, it is enough to find a subsequence $(B_n)_{n\in\mathbb{N}}$ of $(\tilde{B}_n)_{n\in\mathbb{N}}$ such that $\mathrm{SOT} - \sum_{n=0}^{\infty} B_n$ exists and

$$\left\| \sum_{n=0}^{\infty} B_n \right\| \leq \sup_{n\in\mathbb{N}} \|T_n\|.$$

To find a desired subsequence, define a map $s : \mathbb{N} \to \mathbb{N}$ such that $s(n+1) > r(s(n))$ for all $n \in \mathbb{N}$. Since

$$P_{r(s(n))} - P_{s(n)} = \sum_{l=s(n)+1}^{r(s(n))} Q_l$$

and because the projections Q_l are pairwise orthogonal, observe that

$$\left(P_{r(s(n))} - P_{s(n)}\right) \perp \left(P_{r(s(m))} - P_{s(m)}\right)$$

for all $n, m \in \mathbb{N}$ with $n \neq m$. Fix $x \in H$. We deduce that

$$\begin{aligned} \left\| \sum_{n=M}^{N} \tilde{B}_{s(n)}x \right\|^2 &= \left\| \sum_{n=M}^{N} (P_{r(s(n))} - P_{s(n)})T_{s(n)}(P_{r(s(n))} - P_{s(n)})x \right\|^2 \\ &\leq \left(\sup_{n\in\mathbb{N}} \|T_n\| \right)^2 \sum_{n=M}^{N} \left\| (P_{r(s(n))} - P_{s(n)})x \right\|^2 \end{aligned}$$

for $N, M \in \mathbb{N}$. Since $\sum_{n=0}^{N} \left\|(P_{r(s(n))} - P_{s(n)})x\right\|^2 \leq \|x\|^2$ for all $N \in \mathbb{N}$, it follows that $\sum_{n=0}^{\infty} \tilde{B}_{s(n)}$ is a SOT-convergent series with

$$\left\| \sum_{n=0}^{\infty} \tilde{B}_{s(n)} \right\| \leq \sup_{n\in\mathbb{N}} \|T_n\|.$$

Thus, the sequence $(B_n)_{n\in\mathbb{N}}$ with $B_n = \tilde{B}_{s(n)}$ has the desired properties.

Suppose in addition that $\inf_{n\in\mathbb{N}} \|T_n\| > c > 0$. Then using the previous arguments one can find a subsequence $(S_n = A_n + B_n)_{n\in\mathbb{N}}$ of $(T_n = \tilde{A}_n + \tilde{B}_n)_{n\in\mathbb{N}}$, such that $\sum_{n=0}^{\infty} A_n$ converges in norm with $\sum_{n=0}^{\infty} \|A_n\| < c/4$, $\sum_{n=0}^{\infty} B_n$ converges in the strong operator topology and such that

$$\sum_{n=0}^{\infty} \|S_n\|_e < c/4.$$

Let $M_n = \overline{\operatorname{Im}(B_n)}$. By a similar definition of the operators B_n as before, one can achieve that:

(i) $M_n \perp M_m$ for all $n, m \in \mathbb{N}$ with $n \neq m$.

(ii) If P_{M_n} is the orthogonal projection onto M_n, then $\mathrm{SOT} - \lim_{n\to\infty} P_{M_n} = 0$.

Since $\lim_{n\to\infty} \|A_n\| = 0$, it follows that

$$\limsup_{n\to\infty} \|B_n\| = \limsup_{n\to\infty} (\|A_n\| + \|B_n\|) \geq \limsup_{n\to\infty} \|S_n\| > c.$$

But then, Corollary 6.1.1 yields that

$$\left\| \sum_{l=0}^{\infty} B_l \right\|_e \geq \limsup_{n\to\infty} \left\| P_{M_n} \left(\sum_{l=0}^{\infty} B_l \right) \right\| = \limsup_{n\to\infty} \|B_n\| > c.$$

Thus, if $S = \mathrm{SOT} - \sum_{n=0}^{\infty} S_n$, then

$$\|S\|_e \geq \left| \left\| \sum_{n=0}^{\infty} B_n \right\|_e - \left\| \sum_{n=0}^{\infty} A_n \right\|_e \right| \geq c/2 > \sum_{n=0}^{\infty} \|S_n\|_e.$$

□

Proposition 6.1.3. *Let $(\varphi_n)_{n\in\mathbb{N}}$ be a sequence in* $\operatorname{Mult}(\mathscr{H})$ *with $\sup_{n\in\mathbb{N}} \|\varphi_n\|_{\mathrm{Mult}} < \infty$ and $\lim_{n\to\infty} \|\varphi_n\|_{\mathscr{H}} = 0$, then*

$$M_{\varphi_n} \xrightarrow{\mathrm{SOT}} 0 \text{ for } n \to \infty.$$

Proof. Set $c = \sup_{n\in\mathbb{N}} \|\varphi_n\|_{\mathrm{Mult}} < \infty$. Let $\varepsilon > 0$ and $f \in \mathscr{H}$ be arbitrary. Choose a $p \in \mathbb{C}[z]$ with $\|f - p\|_{\mathscr{H}} < \frac{\varepsilon}{2c}$. Since

$$\|\varphi_n\|_{\mathscr{H}} \to 0 \text{ for } n \to \infty,$$

there exists an $n_0 \in \mathbb{N}$, such that

$$\|\varphi_n\|_{\mathscr{H}} < \frac{\varepsilon}{2(\|M_p\|+1)}$$

for all $n \geq n_0$. It follows that

$$\begin{aligned} &\|M_{\varphi_n} f\|_{\mathscr{H}} \\ &\leq \|\varphi_n(f-p)\|_{\mathscr{H}} + \|M_p \varphi_n\|_{\mathscr{H}} \\ &\leq c\|f-p\|_{\mathscr{H}} + \|M_p\|\|\varphi_n\|_{\mathscr{H}} \\ &< \varepsilon \end{aligned}$$

for all $n \geq n_0$. □

The following theorem is a first part of Theorem 6.1.

Theorem 6.1.4. *If*

$$\mathrm{Mult}(\mathscr{H}) \cap A(\mathbb{B}_d) \subsetneq \mathscr{H} \cap A(\mathbb{B}_d),$$

then there exists a $\varphi \in \mathrm{Mult}(\mathscr{H}) \cap A(\mathbb{B}_d)$ *with*

$$\|M_\varphi\|_e > \|\varphi\|_\infty.$$

In particular, $\varphi \notin A(\mathscr{H})$.

Proof. Recall that

$$\|M_\psi\|_e = \|\psi\|_\infty$$

for all $\psi \in A(\mathscr{H}) = \overline{\mathbb{C}[z]}^{\|\cdot\|_{\mathrm{Mult}}}$. Thus, for the construction of the function φ, it is enough to find a sequence of polynomials $(q_n)_{n\in\mathbb{N}}$ such that the limit

$$\varphi = \mathrm{SOT} - \sum_{n=0}^{\infty} q_n$$

exists and such that

$$\|\varphi\|_\infty \leq \sum_{n=0}^{\infty} \|M_{q_n}\|_\infty = \sum_{n=0}^{\infty} \|M_{q_n}\|_e < \|M_\varphi\|_e.$$

To do so, we want to use Theorem 6.1.2. By assumption there exists an

$$f \in (\mathscr{H} \cap A(\mathbb{B}_d)) \setminus (\mathrm{Mult}(\mathscr{H}) \cap A(\mathbb{B}_d)).$$

For $N \in \mathbb{N}_{>0}$ let $\sigma_N(f) = \frac{1}{N+1}\sum_{m=0}^{N}\sum_{l=0}^{m} f_l$ be the Fejér-means of f. Then

$$\|\sigma_N(f) - f\|_{\mathscr{H}} + \|\sigma_N(f) - f\|_\infty \to 0 \text{ for } N \to \infty.$$

Using Lemma 2.2.10, we conclude that

$$\sup_{N\in\mathbb{N}} \|M_{\sigma_N(f)}\| = \infty$$

For $n \in \mathbb{N}$ set

$$p_n(z) = \frac{\sigma_n(f)(z)}{\|M_{\sigma_N(f)}\|} \in \mathbb{C}[z].$$

Since

$$\sup_{n\in\mathbb{N}} \|\sigma_n(f)\|_{\mathscr{H}} < \infty, \ \sup_{n\in\mathbb{N}} \|\sigma_n(f)\|_\infty < \infty \text{ and } \sup_{n\in\mathbb{N}} \|M_{\sigma_N(f)}\| = \infty,$$

it follows that

$$\|p_n\|_{\mathscr{H}} + \|p_n\|_\infty \to 0 \text{ for } n \to \infty.$$

By construction, $\|p_n\|_{\mathrm{Mult}(\mathscr{H})} = 1$ for all $n \in \mathbb{N}$ and

$$\|M_{p_n}\|_e = \|p_n\|_\infty \to 0 \text{ for } n \to \infty.$$

Since $\|p_n\|_{\mathscr{H}} \to 0$ for $n \to \infty$, it follows by Proposition 6.1.3 that

$$M_{p_n} \xrightarrow{\mathrm{SOT}} 0 \text{ for } n \to \infty.$$

For $n \in \mathbb{N}$ let $P_n : \mathscr{H} \to \mathscr{H}$ be the orthogonal projection onto the finite-dimensional spaces

$$E_n = \operatorname{span}\{z^\alpha;\ \alpha \in \mathbb{N}^d \text{ with } 0 \le |\alpha| \le n\} \subset \mathscr{H}$$

Since

$$M_\psi(\mathrm{id}_{\mathscr{H}} - P_n) = (\mathrm{id}_{\mathscr{H}} - P_n)M_\psi$$

for all $\psi \in \mathrm{Mult}(\mathscr{H})$, we can apply Theorem 6.1.2 to the sequence of operators $T_n = M_{p_n}$ $(n \in \mathbb{N})$. Hence, we get a subsequence $(q_n)_{n\in\mathbb{N}}$ of $(p_n)_{n\in\mathbb{N}}$ such that $S = \sum_{n=0}^\infty M_{q_n}$ is a SOT-convergent series and

$$\|S\|_e > \sum_{n=0}^\infty \|M_{q_n}\|_e = \sum_{n=0}^\infty \|q_n\|_\infty.$$

Since $\mathrm{Mult}(\mathscr{H})$ is SOT-closed and

$$\sum_{n=0}^\infty \|q_n\|_\infty < \|S\|_e,$$

we conclude that $\varphi = \sum_{n=0}^\infty q_n$ is an element of $\mathrm{Mult}(\mathscr{H}) \cap A(\mathbb{B}_d)$ with $S = M_\varphi$. Then

$$\|M_\varphi\|_e = \|S\|_e > \sum_{n=0}^\infty \|q_n\|_\infty \ge \|\varphi\|_\infty.$$

□

We can now prove Theorem 6.1. For convenience, we restate it here.

Theorem 6.1.5. *The following are equivalent:*

(i) $\mathrm{Mult}(\mathscr{H}) = H^\infty(\mathbb{B}_d) \cap \mathscr{H}$,

(ii) $A(\mathbb{B}_d)\cap \mathrm{Mult}(\mathscr{H}) = A(\mathbb{B}_d)\cap \mathscr{H}$,

(iii) $A(\mathscr{H}) = A(\mathbb{B}_d)\cap \mathrm{Mult}(\mathscr{H})$,

(iv) $\|M_\varphi\|_e = \|\varphi\|_\infty$ *for all* $\varphi \in \mathrm{Mult}(\mathscr{H})\cap A(\mathbb{B}_d)$.

Proof. (i)⇒(ii): Let $\mathrm{Mult}(\mathscr{H}) = H^\infty(\mathbb{B}_d)\cap \mathscr{H}$. It follows that

$$\mathrm{Mult}(\mathscr{H})\cap A(\mathbb{B}_d) = \mathscr{H}\cap A(\mathbb{B}_d).$$

(ii)⇒(iii): Let $\mathrm{Mult}(\mathscr{H})\cap A(\mathbb{B}_d) = \mathscr{H}\cap A(\mathbb{B}_d)$. By the open mapping theorem

$$\|\cdot\|_{\mathscr{H}} + \|\cdot\|_\infty \approx \|\cdot\|_{\mathrm{Mult}(\mathscr{H})}.$$

For $n\in\mathbb{N}$ let $\sigma_n(\varphi)$ be the Fejér-means of $\varphi \in \mathrm{Mult}(\mathscr{H})\cap A(\mathbb{B}_d)$. We obtain that

$$\|\varphi-\sigma_n(\varphi)\|_{\mathrm{Mult}(\mathscr{H})} \approx \|\varphi-\sigma_n(\varphi)\|_{\mathscr{H}} + \|\varphi-\sigma_n(\varphi)\|_\infty \longrightarrow 0 \text{ for } n\to\infty$$

and thus, $\varphi\in A(\mathscr{H})$. On the other hand, since

$$\|\cdot\|_{\mathscr{H}} + \|\cdot\|_\infty \lesssim \|\cdot\|_{\mathrm{Mult}},$$

we have $A(\mathscr{H})\subset A(\mathbb{B}_d)\cap \mathrm{Mult}(\mathscr{H})$

(iii)⇒(iv): Let $\varphi \in \mathrm{Mult}(\mathscr{H})\cap A(\mathbb{B}_d) = A(\mathscr{H})$. Since $\mathscr{H}$ is regular, it follows that $\|M_\varphi\|_e = \|\varphi\|_\infty$ for all $\varphi\in \mathrm{Mult}(\mathscr{H})\cap A(\mathbb{B}_d)$.

(iv)⇒(i): Suppose that (iv) holds. Using Theorem 6.1.4, we deduce that $\mathrm{Mult}(\mathscr{H})\cap A(\mathbb{B}_d) = \mathscr{H}\cap A(\mathbb{B}_d)$. Using the open mapping theorem, we obtain that

$$\|\varphi\|_{\mathscr{H}} + \|\varphi\|_\infty \approx \|\varphi\|_{\mathrm{Mult}}$$

for all $\varphi\in\mathscr{H}\cap A(\mathbb{B}_d)$. Due to Lemma 2.2.10, it follows that

$$\sup_{n\in\mathbb{N}} \|\sigma_n(\varphi)\|_{\mathrm{Mult}} \approx \sup_{n\in\mathbb{N}} \left(\|\sigma_n(\varphi)\|_{\mathscr{H}} + \|\sigma_n(\varphi)\|_\infty\right) < \infty.$$

Using Lemma 2.2.10 again, we conclude that $H^\infty(\mathbb{B}_d)\cap\mathscr{H} = \mathrm{Mult}(\mathscr{H})$. □

An operator $T\in B(H)$ is called essentially hyponormal, if there is a compact self-adjoint operator X such that

$$T^*T - TT^* + X \geq 0.$$

Remark 6.1.6. If $T\in B(H)$ is essentially hyponormal, one can check that the spectral radius $\rho(T)$ is an upper bound for the essential norm of T, that is

$$\|T\|_e \leq \rho(T),$$

(see proof of Proposition 3.3 in [FX11]). Suppose now that $H=\mathscr{H}$ is a (regular) unitarily invariant space such that $\frac{1}{\varphi}\in\mathrm{Mult}(\mathscr{H})$, whenever $\varphi\in\mathrm{Mult}(\mathscr{H})$ and $\frac{1}{\varphi}\in H^\infty(\mathbb{B}_d)$.

This is often called one-function Corona theorem. In this case, it follows for every $\varphi \in \mathrm{Mult}(\mathscr{H})$ that

$$\rho(M_\varphi) \leq \|\varphi\|_\infty.$$

Hence, if $\varphi \in \mathrm{Mult}(\mathscr{H})$ such that $M_\varphi : \mathscr{H} \to \mathscr{H}$ is essentially hyponormal, then

$$\|M_\varphi\|_e \leq \|\varphi\|_\infty.$$

We obtain a generalization of the result of Fang and Xia, establishing the existence of multiplication operators on the Drury-Arveson space $\mathscr{H} = H^2_d$ that are not essentially hyponormal (cf. [FX11]).

Theorem 6.1.7. *Suppose that $\mathscr{H}$ is a unitarily invariant space such that the one-function Corona theorem 6.2.8 holds true and such that $\mathrm{Mult}(\mathscr{H}) \subsetneq \mathscr{H} \cap H^\infty(\mathbb{B}_d)$. Then there exists a $\varphi \in \mathrm{Mult}(\mathscr{H})$ such that M_φ is not essentially hyponormal.*

Proof. This is a straightforward application of Remark 6.1.6, Theorem 6.1.4 and Theorem 6.1. □

6.1.1. Examples

In this section, we want to consider regular unitarily invariant spaces $\mathscr{H}$ with reproducing kernel

$$K : \mathbb{B}_d \times \mathbb{B}_d \to \mathbb{C},\ K(z,w) = k_w(z),$$

where

$$\mathrm{Mult}(\mathscr{H}) \subsetneq H^\infty(\mathbb{B}_d) \cap \mathscr{H}.$$

For all the given examples, one can show in addition that the one-function Corona theorem 6.2.8 can be applied. Hence, due to Theorem 6.1.7, there exist multiplication operators $M_\varphi : \mathscr{H} \to \mathscr{H}$, that are not essentially hyponormal.

In the following, we denote by

$$\hat{k}_z = \frac{k_z}{\|k_z\|_{\mathscr{H}}} \qquad (z \in \mathbb{B}_d)$$

the normalized kernel functions and assume that $k_z \in \mathrm{Mult}(\mathscr{H})$ for all $z \in \mathbb{B}_d$.

Remark 6.1.8. (i) If $k_z \in \mathrm{Mult}(\mathscr{H})$ for all $z \in \mathbb{B}_d$, we have

$$\mathrm{Mult}(\mathscr{H}) \subset \left\{ f \in \mathscr{H};\ \sup_{z \in \mathbb{B}_d} \|f\hat{k}_z\|_{\mathscr{H}} < \infty \right\} \subset \mathscr{H} \cap H^\infty(\mathbb{B}_d).$$

For the second inclusion, observe that

$$f(z) = \langle f\hat{k}_z, \hat{k}_z \rangle_{\mathscr{H}} \qquad (z \in \mathbb{B}_d)$$

for all $f \in \mathscr{H}$.

(ii) Let $d \in \mathbb{N}$ with $d \geq 2$. In [FX11], Fang and Xia show that

$$\mathrm{Mult}(H_d^2) \subsetneqq \left\{ f \in H_d^2;\ \sup_{z \in \mathbb{B}_d} \|f\hat{k}_z\|_{H_d^2} < \infty \right\}.$$

(iii) In the case of the Dirichlet space $\mathscr{D}$, Stegenga proves in [Ste80, Section 4] that there exists a function $f \in H^\infty(\mathbb{D}) \cap \mathscr{D}$ such that

$$\sup \left(\log \frac{1}{|I|} \mu(S(I));\ I \subset \mathbb{T} \text{ interval} \right) < \infty, \tag{6.1}$$

for $d\mu = |f'|^2 dA$, but $d\mu$ is not a Carleson measure. In particular, f is not in the multiplier algebra of the Dirichlet space. Since testing on kernel functions is equivalent to the geometric statement in terms of the one-box condition 6.1 (see Theorem A.4.1), it follows that

$$\mathrm{Mult}(\mathscr{D}) \subsetneqq \left\{ f \in \mathscr{D};\ \sup_{z \in \mathbb{D}} \|f\hat{k}_z\|_{\mathscr{D}} < \infty \right\}.$$

In Theorem 5.1.6 in [EFKMR14], there is an explicit example, proving that

$$\mathrm{Mult}(\mathscr{D}) \subsetneqq \mathscr{D} \cap H^\infty(\mathbb{D}).$$

Notation 6.1.9. For $s > 0$ and $w \in \mathbb{B}_d$ let $\hat{k}_w^s : \mathbb{B}_d \to \mathbb{C}$,

$$\hat{k}_w^s(z) = \frac{(1 - \|w\|^2)^{s/2}}{(1 - \langle z, w \rangle)^s}.$$

be the normalized kernel functions of the spaces $A_s^2(\mathbb{B}_d)$. We denote by $\|\cdot\|_s$ the norm of $A_s^2(\mathbb{B}_d)$.

In the following, we will prove for $0 < s < d$ that

$$\left\{ f \in A_s^2(\mathbb{B}_d);\ \sup_{z \in \mathbb{B}_d} \|f\hat{k}_z^s\|_s < \infty \right\} \subsetneqq A_s^2(\mathbb{B}_d) \cap H^\infty(\mathbb{B}_d).$$

Remark 6.1.10. Let $\log : \mathbb{C} \setminus (-\infty, 0] \to \mathbb{C}$ be the principle branch of the logarithm. For $z, w \in \mathbb{D}$, using polar coordinates, one checks that $(1-z)(1-w) \in \mathbb{C} \setminus (-\infty, 0]$. Thus, $\log(1-z) + \log(1-w) = \log((1-z)(1-w))$. Since

$$(1-z)^s = \exp(s \log(1-z)) \text{ for all } z \in \mathbb{D} \text{ and } s \in \mathbb{R}.$$

we have

$$(1-z)^s (1-z)^t = (1-z)^{s+t}$$

and

$$(1-z)^s (1-w)^s = ((1-z)(1-w))^s$$

for all $z, w \in \mathbb{D}$ and $s, t \in \mathbb{R}$.

Notation 6.1.11. For $a \in \mathbb{B}_d \setminus \{0\}$ let $\varphi_a : \overline{\mathbb{B}_d} \to \overline{\mathbb{B}_d}$ be the Möbius transform in the unit ball (for a definition see [Rud08, Section 2.2.1]). The function $\varphi_a : \overline{\mathbb{B}_d} \to \overline{\mathbb{B}_d}$ has similar properties as in the one dimensional case. One computes that, $(\varphi_a \circ \varphi_a)(z) = z$, $\varphi_a(0) = a$, $\varphi_a(a) = 0$ and

$$1 - \langle \varphi_a(z), \varphi_a(w) \rangle = \frac{(1 - \langle a, a \rangle)(1 - \langle z, w \rangle)}{(1 - \langle z, a \rangle)(1 - \langle a, w \rangle)}$$

for all $z, w \in \overline{\mathbb{B}_d}$ (see Theorem 2.2.2 in [Rud08]).

Lemma 6.1.12. *For $a \in \mathbb{B}_d \setminus \{0\}$ and $s > 0$ the weighted composition operator*

$$U_a : A^2_s(\mathbb{B}_d) \to A^2_s(\mathbb{B}_d),\ f \mapsto (f \circ \varphi_a)\hat{k}^s_a$$

is a well-defined unitary. Moreover, one computes for all $\psi \in \mathrm{Mult}(A^2_s(\mathbb{B}_d))$ that

$$U_a M_\psi U_a^* = M_{\psi \circ \varphi_a} \tag{6.2}$$

and hence,

$$\|\psi \circ \varphi_a\|_{\mathrm{Mult}} = \|\psi\|_{\mathrm{Mult}}, \tag{6.3}$$

as well as

$$\|\psi \circ \varphi_a\|_s = \|U_a(\psi \circ \varphi_a)\|_s = \|\psi \hat{k}^s_a\|_s. \tag{6.4}$$

Proof. Let $a \in \mathbb{B}_d \setminus \{0\}$ and $s > 0$. Then

$$\hat{k}^s_{\varphi_a(w)}(\varphi_a(z))\hat{k}^s_a(z) = \frac{(1 - \|\varphi_a(w)\|^2)^{s/2}(1 - \langle a, w \rangle)^s}{(1 - \|a\|^2)^{s/2}(1 - \|w\|^2)^{s/2}} \hat{k}^s_w(z)$$

for all $z, w \in \mathbb{B}_d$ and hence,

$$\langle (\hat{k}^s_{\varphi_a(x)} \circ \varphi_a)\hat{k}^s_a, (\hat{k}^s_{\varphi_a(y)} \circ \varphi_a)\hat{k}^s_a \rangle = \langle \hat{k}^s_{\varphi_a(x)}, \hat{k}^s_{\varphi_a(y)} \rangle$$

for all $x, y \in \mathbb{B}_d$. Thus, $\varphi_a \circ \varphi_a = \mathrm{id}_{\mathbb{B}_d}$ yields that

$$\langle (\hat{k}^s_x \circ \varphi_a)\hat{k}^s_a, (\hat{k}^s_y \circ \varphi_a)\hat{k}^s_a \rangle = \langle \hat{k}^s_x, \hat{k}^s_y \rangle$$

for all $x, y \in \mathbb{B}_d$. Since the set of kernel functions is total in $\mathcal{H}$, the assertion follows. □

Following the ideas of [FX11] and using the proof of Proposition 9.7 in [AHMR22] we obtain:

Proposition 6.1.13. *For $0 < s < d$ there exist sequences $(w_n)_{n \in \mathbb{N}}$ in $\mathbb{B}_d$ and $(\psi_n)_{n \in \mathbb{N}}$ in $\mathrm{Mult}(A^2_s(\mathbb{B}_d)) \cap A(\mathbb{B}_d)$ such that $\|\psi_n \hat{k}^s_{w_n}\|_s \to \infty$ for $n \to \infty$ and*

$$\sup_{n \in \mathbb{N}} (\|\psi_n\|_s + \|\psi_n\|_\infty) \leq 1.$$

Proof. As in proof of Proposition 9.7 in [AHMR22] one can find sequence $(p_n)_{n\in\mathbb{N}}$ in $\mathbb{C}[z]$ such that $\|p_n\|_s = 1$ for all $n \in \mathbb{N}$ and additionally $\|p_n\|_\infty \to 0$ for $n \to \infty$. Equation (6.4) shows that

$$\|p_n \circ \varphi_z\|_s = \|p_n \hat{k}_z^s\|_s$$

for all $z \in \mathbb{B}_d \setminus \{0\}$ and $n \in \mathbb{N}$. Now choose a sequence $(z_l)_{l\in\mathbb{N}}$ in $\mathbb{B}_d \setminus \{0\}$ with $\|z_l\| \uparrow 1$. Then $\hat{k}_{z_l}^s \xrightarrow{\tau_w} 0$ for $l \to \infty$ and hence

$$\limsup_{l\to\infty} \|p_n \circ \varphi_{z_l}\|_s = \limsup_{l\to\infty} \|p_n \hat{k}_{z_l}^s\|_s = \limsup_{l\to\infty} \|(M_{p_n} + S)\hat{k}_{z_l}^s\|_s$$

for every compact operator $S \in K(A_s^2(\mathbb{B}_d))$. Since $A_s^2(\mathbb{B}_d)$ is regular, this implies that

$$\limsup_{l\to\infty} \|p_n \circ \varphi_{z_l}\|_s \le \|M_{p_n}\|_e = \|p_n\|_\infty.$$

Thus, we can choose a sequence $(w_n)_{n\in\mathbb{N}}$ in $\mathbb{B}_d$ such that

$$\|p_n \circ \varphi_{w_n}\|_s \le 2\|p_n\|_\infty.$$

Consider the functions

$$\psi_n = \frac{p_n \circ \varphi_{w_n}}{3\|p_n\|_\infty} \in \mathrm{Mult}(A_s^2(\mathbb{B}_d)) \cap A(\mathbb{B}_d).$$

Because $\|p_n \circ \varphi_{w_n}\|_\infty = \|p_n\|_\infty$ for all $n \in \mathbb{N}$, we compute that

$$\sup_{n\in\mathbb{N}} (\|\psi_n\|_s + \|\psi_n\|_\infty) \le 1$$

and that

$$\|\psi_n \hat{k}_{w_n}\|_s = \frac{\|U_{w_n} p_n\|_s}{3\|p_n\|_\infty} = \frac{\|p_n\|_s}{3\|p_n\|_\infty} = \frac{1}{3\|p_n\|_\infty} \to \infty$$

for $n \to \infty$. □

Theorem 6.1.14. *Let* $0 < s < d$, *then*

$$\left\{ f \in A_s^2(\mathbb{B}_d);\ \sup_{z\in\mathbb{B}_d} \|f\hat{k}_z^s\|_s < \infty \right\} \subsetneq A_s^2(\mathbb{B}_d) \cap H^\infty(\mathbb{B}_d).$$

Proof. The operator

$$M_{\hat{k}_z^s} : (A_s^2(\mathbb{B}_d) \cap A(\mathbb{B}_d), \|\cdot\|_s + \|\cdot\|_\infty) \to (A_s^2(\mathbb{B}_d), \|\cdot\|_s),\ f \mapsto f\hat{k}_z^s$$

is well-defined and bounded. Suppose that

$$\left\{ f \in A_s^2(\mathbb{B}_d);\ \sup_{z\in\mathbb{B}_d} \|f\hat{k}_z^s\|_s < \infty \right\} = A_s^2(\mathbb{B}_d) \cap H^\infty(\mathbb{B}_d).$$

By the uniform boundedness principle,

$$\sup_{z\in\mathbb{B}_d} \left\| M_{\hat{k}_z^s} \right\| < \infty.$$

Due to the previous Proposition 6.1.13, there exist sequences $(w_n)_{n\in\mathbb{N}}$ in $\mathbb{B}_d$ and $(\psi_n)_{n\in\mathbb{N}}$ in $\operatorname{Mult}(A_s^2(\mathbb{B}_d)) \cap A(\mathbb{B}_d)$ such that $\sup_{n\in\mathbb{N}}(\|\psi_n\|_s + \|\psi_n\|_\infty) \leq 1$ and

$$\sup_{n\in\mathbb{N}} \left\| M_{\hat{k}_{w_n}^s} \right\| \geq \|\psi_n \hat{k}_{w_n}^s\|_s \to \infty \text{ for } n \to \infty,$$

a contradiction. □

Corollary 6.1.15. *Let* $0 < s < d$ *and* $\mathscr{H} = A_s^2(\mathbb{B}_d)$*, then*

$$\operatorname{Mult}(\mathscr{H}) \subsetneqq \mathscr{H} \cap H^\infty(\mathbb{B}_d)$$

and

$$A(\mathscr{H}) \subsetneq \operatorname{Mult}(\mathscr{H}) \cap A(\mathbb{B}_d).$$

6.2. One-function Corona theorem

Let $\mathscr{M}(H^\infty(\mathbb{D}))$ be the maximal ideal space of the unital commutative Banach algebra $H^\infty(\mathbb{D})$ (cf. Definition 2.4.9). It is well-known that the point evaluations $\delta_z : H^\infty(\mathbb{D}) \to \mathbb{C}$ are elements of $\mathscr{M}(H^\infty(\mathbb{D}))$ and that

$$\{\chi(z);\ \chi \in \mathscr{M}(H^\infty(\mathbb{D}))\} = \overline{\mathbb{D}}$$

(see also Theorem 2.4.24). Basic function theory shows that for every $\varphi \in H^\infty(\mathbb{D})$ and every $a \in \mathbb{D}$, there exists a function $\psi_a \in H^\infty(\mathbb{D})$ such that

$$\varphi - \varphi(a) = (z-a)\psi_a.$$

Using these two facts, one computes for every open disk $D_r(a) \subset \mathbb{D}$, that the corresponding set

$$\{\chi \in \mathscr{M}(H^\infty(\mathbb{D}));\ \chi(z) \in D_r(a)\} = \{\delta_z;\ z \in D_r(a)\} \subset \mathscr{M}(H^\infty(\mathbb{D}))$$

is an open neighborhood of the point evaluation $\delta_a : H^\infty(\mathbb{D}) \to \mathbb{C}$ in the weak-$*$ topology. Whence the unit disk $\mathbb{D}$ can be considered as an open subset of $\mathscr{M}(H^\infty(\mathbb{D}))$. Because $\mathscr{M}(H^\infty(\mathbb{D}))$ is compact by Banach-Alaoglu, the open subset $\mathbb{D}$ is properly contained in $\mathscr{M}(H^\infty(\mathbb{D}))$. The complement $\mathscr{M}(H^\infty(\mathbb{D})) \setminus \overline{\mathbb{D}}$ is called Corona in [New59]. The Corona theorem, conjectured by Kakutani in 1941 (see [Kak41]), states that the open unit disc $\mathbb{D}$ is dense in $\mathscr{M}(H^\infty(\mathbb{D}))$, or equivalently that the Corona is empty. A first proof is given by Carleson in 1962 (cf. [Car62]). For his proof, Carleson uses the following function theoretic reformulation of the Corona theorem:

For all $n \in \mathbb{N}$ with $n \geq 1$ and $\varphi_1, \ldots, \varphi_n \in H^\infty(\mathbb{D})$ the following are equivalent:

(a) There exists a $\delta > 0$ such that $|\varphi_1(z)| + \cdots + |\varphi_n(z)| \geq \delta$ for all $z \in \mathbb{D}$,

(b) there exist $\psi_1, \ldots, \psi_n \in H^\infty(\mathbb{D})$ such that $\psi_1(z)\varphi_1(z) + \cdots + \psi_n(z)\varphi_n(z) = 1$ for all $z \in \mathbb{D}$.

See [CQ15] for additional information and a proof of the Corona theorem, using the previous statement and ideas of Hörmander and Wolff.

It is a frequent challenge to prove the Corona theorem for the multiplier algebra of reproducing kernel Hilbert spaces $\mathscr{H}$ with unitarily invariant kernel functions on the unit ball in $\mathbb{C}^d$. That is, to prove that the unit ball $\mathbb{B}_d$ is dense in the spectrum $\mathscr{M}(\mathrm{Mult}(\mathscr{H}))$ of the commutative unital Banach algebra $\mathrm{Mult}(\mathscr{H})$. This seems to be a deep question. In [CSW11], Costea, Sawyer, and Wick establish the Corona theorem for the Drury-Arveson space and other holomorphic Besov-Sobolev spaces on the unit ball in $\mathbb{C}^d$.

Let $\mathscr{F} \subset \mathscr{O}(\mathbb{B}_d)$ be a Banach function space such that

(a) the point evaluations are continuous,

(b) the constant functions are contained in $\mathscr{F}$.

Let $\varphi : \mathbb{B}_d \to \mathbb{C}$ be a function in

$$\mathrm{Mult}(\mathscr{F}) = \{\, \psi : \mathbb{B}_d \to \mathbb{C};\ \psi \cdot f \in \mathscr{F}\}.$$

We now want to analyze whether $\frac{1}{\varphi} \in H^\infty(\mathbb{B}_d)$ already implies that

$$\frac{1}{\varphi} \in \mathrm{Mult}(\mathscr{F}).$$

This can be considered the function theoretic reformulation of the Corona theorem for one function.

Let $\Omega \subset \mathbb{C}^d$ be open and let $R : \mathscr{O}(\Omega) \to \mathscr{O}(\Omega)$,

$$Rf = \sum_{l=1}^{d} z_l \frac{\partial f}{\partial z_l}$$

be the radial derivative defined as in Section 2.3.2. Since the partial derivatives $\frac{\partial}{\partial z_l}$ are linear, the operator $R : \mathscr{O}(\Omega) \to \mathscr{O}(\Omega)$ is linear and the product rule for $\frac{\partial}{\partial z_l}$ yields that

$$R(fg) = fR(g) + R(f)g \text{ for all } f, g \in \mathscr{O}(\Omega).$$

Using the binomial theorem, we give a short proof for a differentiation formula for the radial derivative, which has been established in [CHZ18, Theorem 5 and Corllary 6]. The formula is a useful tool to prove a generalized version of the Corona theorem for one function, as it appears in [APR+24, Theorem 3.2] (cf. [CHZ18, Proposition 7] and [LMN20, Lemma 3.1]).

Lemma 6.2.1. *Let $\Omega \subset \mathbb{C}^d$ be open, $g \in \mathscr{O}(\Omega)$ and $N \in \mathbb{N}$. For every function $f \in \mathscr{O}(\Omega)$ there exists a function $h \in \mathscr{O}(\Omega)$ such that*

$$R^N\left(g^{N+1}f\right) = gh.$$

Proof. Proof by induction on N for all $g \in \mathscr{O}(\Omega)$. The case $N = 0$ is trivial with $h = f$. Suppose that the assertion holds for $N \in \mathbb{N}$ and define

$$\tilde{f} = (N+2)R(g)f + gR(f).$$

By assumption there exists a function $\tilde{h} = h(\tilde{f}, N) \in \mathscr{O}(\Omega)$ such that

$$R^{N+1}\left(g^{N+2}f\right) = R^N\left(R\left(g^{N+2}\right)f + g^{N+2}R(f)\right) = R^N\left(g^{N+1}\tilde{f}\right) = g\tilde{h}.$$

Thus, the assertion follows. □

The next statement is the differentiation formula due to Cao, He, and Zhu (see [CHZ18, Theorem 5]). We give a new proof here.

Theorem 6.2.2 (Cao, He, and Zhu)**.** *Let $\Omega \subset \mathbb{C}^d$ be open, $f, g \in \mathscr{O}(\Omega)$ and $N \in \mathbb{N}$, then*

$$\sum_{l=0}^{N+1} (-1)^l \binom{N+1}{l} g^l R^N(g^{N+1-l}f) = 0.$$

Proof. Fix $w \in \Omega$. Using the linearity of R^N, the binomial theorem yields for all $z \in \Omega$ that

$$\begin{aligned} R^N\left((g - g(w))^{N+1}f\right)(z) &= R^N\left(\sum_{l=0}^{N+1}(-1)^l\binom{N+1}{l}g(w)^l g^{N+1-l}f\right)(z) \\ &= \sum_{l=0}^{N+1}(-1)^l\binom{N+1}{l}g(w)^l R^N(g^{N+1-l}f)(z). \end{aligned}$$

On the other hand, due to Lemma 6.2.1, there exists a function $h \in \mathscr{O}(\Omega)$ such that

$$R^N\left((g - g(w))^{N+1}f\right) = (g - g(w))h.$$

In particular,

$$R^N\left((g - g(w))^{N+1}f\right)(w) = 0$$

and hence

$$\sum_{l=0}^{N+1}(-1)^l\binom{N+1}{l}g(w)^l R^N(g^{N+1-l}f)(w) = 0.$$

□

The following quotient rule is a useful reformulation of the previous Theorem 6.2.2 (see [CHZ18, Corllary 6]).

Corollary 6.2.3 (Cao, He, and Zhu)**.** *Let $\Omega \subset \mathbb{C}^d$ be open. Let $f, g \in \mathscr{O}(\Omega)$ and suppose that g is non- vanishing on Ω, then*

$$R^N\left(\frac{f}{g}\right) = \frac{(-1)^N}{g^{N+1}} \sum_{l=0}^{N} (-1)^l \binom{N+1}{l} g^l R^N(g^{N-l} f).$$

Proof. Because of Theorem 6.2.2, it is immediate that

$$R^N(f) = -\frac{(-1)^{N+1}}{g^{N+1}} \sum_{l=0}^{N} (-1)^l \binom{N+1}{l} g^l R^N(g^{N+1-l} f).$$

Replacing f by f/g one obtains the desired result. □

Using the identity theorem we also obtain the following reformulation, where the previous Corollary is the case, when h is the constant function 1:

Corollary 6.2.4. *Let $\Omega \subset \mathbb{C}^d$ be open and connected. Let $f, g, h \in \mathscr{O}(\Omega)$ and suppose that g is non- vanishing on Ω, then*

$$R^N\left(\frac{h^{N+1} f}{g}\right) = (-1)^N \sum_{l=0}^{N} (-1)^l \binom{N+1}{l} \left(\frac{h}{g}\right)^{N+1-l} R^N(g^{N+1-l} h^l f).$$

Proof. Define

$$\Omega_h = \{z \in \Omega;\ h(z) \neq 0\}.$$

Using Corollary 6.2.3 for the functions $\tilde{f} : \Omega \to \mathbb{C}$, $\tilde{f}(z) = (h^N f)(z)$ and $\tilde{g} : \Omega \to \mathbb{C}$, $\tilde{g}(z) = \frac{g}{h}(z)$, we conclude that

$$R^N\left(\frac{h^{N+1} f}{g}\right)(z) = (-1)^N \sum_{l=0}^{N} (-1)^l \binom{N+1}{l} \left(\frac{h}{g}\right)^{N+1-l} R^N(g^{N+1-l} h^l f)(z)$$

for all $z \in \Omega_h$. Since $\Omega_h \subset \Omega$ is open, the identity theorem for holomorphic functions (see for example [Esc18b, Satz 2.3]) yields equality for all $z \in \Omega$. □

Now, denote by $H^\infty(\mathbb{B}_d)$ the space of the bounded holomorphic functions on $\mathbb{B}_d$.

Remark 6.2.5. Let $\mathscr{F} \subset \mathscr{O}(\mathbb{B}_d)$ be a Banach space such that the constant functions are contained in $\mathscr{F}$. Suppose that the topology induced by $\|\cdot\|_{\mathscr{F}}$ is at most finer than the topology of uniform convergence on compact subsets.

(a) For all $z \in \mathbb{B}_d$ the point evaluations $\delta_z : \mathscr{F} \to \mathbb{C}$, $f \mapsto f(z)$ are well-defined and bounded.

(b) The multiplier algebra is defined as

$$\mathrm{Mult}(\mathscr{F}) = \{\varphi : \mathbb{B}_d \to \mathbb{C};\ \varphi \cdot f \in \mathscr{F}\}.$$

By an application of the closed graph theorem, the multiplication operator

$$M_\varphi : \mathscr{F} \to \mathscr{F},\ f \mapsto \varphi \cdot f,$$

is bounded and $\mathrm{Mult}(\mathscr{F})$ is a unital commutative Banach algebra with the induced norm

$$\|\varphi\|_{\mathrm{Mult}} = \|M_\varphi\| \quad (\varphi \in \mathrm{Mult}(\mathscr{F})).$$

(c) The point evaluations $\delta_z : \mathrm{Mult}(\mathscr{F}) \to \mathbb{C}$, $\varphi \mapsto \varphi(z)$ are characters on the unital commutative Banach algebra $\mathrm{Mult}(\mathscr{F})$. Thus,

$$|\varphi(z)| = |\delta_z(\varphi)| \leq \|\varphi\|_{\mathrm{Mult}}$$

for all $z \in \mathbb{B}_d$ and $\varphi \in \mathrm{Mult}(\mathscr{F})$. It follows that $\mathrm{Mult}(\mathscr{F}) \subset H^\infty(\mathbb{B}_d)$ and the inclusion is continuous.

Suppose that we are in the following setting (similar to the one in [LMN20]):

(a) Let $\mathscr{E} \subset \mathscr{O}(\mathbb{B}_d)$ be a Banach space with the following properties:

 (i) The constant functions are contained in $\mathscr{E}$.

 (ii) $H^\infty(\mathbb{B}_d) \subset \mathrm{Mult}(\mathscr{E})$.

 (iii) The topology induced by $\|\cdot\|_{\mathscr{E}}$ is at most finer than the topology of uniform convergence on compact subsets.

(b) Let $N \in \mathbb{N}$. Suppose that

$$\mathscr{F} = \left\{ f \in \mathscr{O}(\mathbb{B}_d);\, R^N(f) \in \mathscr{E} \right\}$$

is a Banach space with the norm

$$\|f\|_{\mathscr{F}} = |f(0)| + \|R^N f\|_{\mathscr{E}} \quad (f \in \mathscr{F}).$$

Remark 6.2.6. (a) Due to the previous definition, the constant functions are contained in $\mathscr{F}$.

(b) Using the definitions of $\mathscr{F}$ and $\mathscr{E}$, it follows similarly to the proof of Lemma 2.3.42 that the topology induced by $\|\cdot\|_{\mathscr{F}}$ is at most finer than the topology of uniform convergence on compact subsets.

For $1 \leq p < \infty$, $t > -\frac{1}{p}$ and $s \in \mathbb{R}$, we use here again the notation

$$B_t^{s,p} = \left\{ f \in \mathscr{O}(\mathbb{B}_d);\, \int_{\mathbb{B}_d} |R^s f(z)|^p (1-|z|^2)^{pt} dV(z) < \infty \right\}$$

for the L^p-versions of the standard weighted Besov spaces with the norm

$$\|f\|_{B_t^{s,p}}^p = |f(0)|^p + \|R^s f\|_{L_a^p(\omega^{(pt)})}^p \quad (f \in B_t^{s,p}),$$

where $\omega^{(pt)} : \mathbb{B}_d \to \mathbb{R}_{\geq 0}$,

$$\omega^{(pt)}(z) = (1-|z|^2)^{pt}$$

(see Definition 2.3.39).

Example 6.2.7. (a) For $1 \le p < \infty$ and $N \in \mathbb{N}$, the radially weighted Besov spaces

$$B_\omega^{N,p} = \{f \in \mathscr{O}(\mathbb{B}_d);\ R^N f \in L_a^p(\omega)\}$$

we introduced in Section 2.3.2 provide an example for the space $\mathscr{F}$, where $\mathscr{E} = L_a^p(\omega)$ is the radially weighted Bergman space.

(i) Using Theorem 2.3.50, any radially weighted Besov space B_ω^s can be described as the space $B_{\omega_{N-s}}^N$, where $N \ge s$ with equivalence of norms.

(ii) Let $t > -\frac{1}{p}$. Due to Theorem 5.2.16, the spaces $B_t^{s,p}$ coincide with the spaces $B_{t+N-s}^{N,p}$, where $N \ge s$ with equivalence of norms.

(b) All other examples in [LMN20], like the Bloch-type spaces and the Hardy Sobolev spaces, are covered by the previous setting.

We obtain the following version of Theorem 3.2 in [APR+24] by Aleman, Perfekt, Richter, Sundberg and Sunkes:

Theorem 6.2.8. *If $\varphi, \psi \in \mathrm{Mult}(\mathscr{F})$ and $\frac{\varphi}{\psi} \in H^\infty(\mathbb{B}_d)$ with $c = \left\| \frac{\varphi}{\psi} \right\|_\infty$, then $\frac{\varphi^{N+1}}{\psi} \in \mathrm{Mult}(\mathscr{F})$ with*

$$\left\| \frac{\varphi^{N+1}}{\psi} \right\|_{\mathrm{Mult}} \le c\|\varphi\|_\infty^N + (c\|\psi\|_{\mathrm{Mult}} + \|\varphi\|_{\mathrm{Mult}})^{N+1}.$$

Proof. Let $f \in \mathscr{F}$. Using Corollary 6.2.4, we conclude that

$$R^N\left(\frac{\varphi^{N+1} f}{\psi}\right) = (-1)^N \sum_{l=0}^{N} (-1)^l \binom{N+1}{l} \left(\frac{\varphi}{\psi}\right)^{N+1-l} R^N(\psi^{N+1-l}\varphi^l f). \tag{6.5}$$

Since $\varphi, \psi \in \mathrm{Mult}(\mathscr{F})$, it follows that the functions $\psi^{N+1-l}\varphi^l f$ belong to $\mathscr{F}$, which means that $R^N(\psi^{N+1-l}\varphi^l f) \in \mathscr{E}$. By assumption $\frac{\varphi}{\psi} \in H^\infty(\mathbb{B}_d)$. Because $\mathrm{Mult}(\mathscr{E}) = H^\infty(\mathbb{B}_d)$, we obtain $R^N\left(\frac{\varphi^{N+1} f}{\psi}\right) \in \mathscr{E}$ and thus, $\frac{\varphi^{N+1} f}{\psi} \in \mathscr{F}$. Hence, it is clear that $\frac{\varphi^{N+1}}{\psi} \in \mathrm{Mult}(\mathscr{F})$. Using the binomial theorem, Equation 6.5 yields that

$$\begin{aligned}\left\| \frac{\varphi^{N+1} f}{\psi} \right\|_{\mathscr{F}} &\le \left(c\|\varphi\|_\infty^N + \sum_{l=0}^{N} \binom{N+1}{l} (c\|\psi\|_{\mathrm{Mult}})^{N+1-l} \|\varphi\|_{\mathrm{Mult}}^l \right) \|f\|_{\mathscr{F}} \\ &\le \left(c\|\varphi\|_\infty^N + (c\|\psi\|_{\mathrm{Mult}} + \|\varphi\|_{\mathrm{Mult}})^{N+1} \right) \|f\|_{\mathscr{F}}\end{aligned}$$

for all $f \in \mathscr{F}$ and we obtain the desired estimation for the multiplier norm. □

Define a norm on the space $\mathscr{A} = B_t^{N,p} \cap H^\infty(\mathbb{B}_d)$ by

$$\|\cdot\|_{\mathscr{A}} : \mathscr{A} \to \mathbb{R}_{\ge 0},\ \|f\|_{\mathscr{A}} = \|f\|_{B_t^{N,p}} + \|f\|_\infty.$$

Furthermore, let

$$\mathscr{B} = \left\{ f \in \mathscr{O}(\mathbb{B}_d);\ \sup_{z\in\mathbb{B}_d} (1-|z|^2)|Rf(z)| < \infty \right\}$$

be the Bloch space (see [Zhu05, Chapter 3]). We have the following lemma:

Lemma 6.2.9. *Let* $1 \le p < \infty$, $t > -\frac{1}{p}$ *and* $N \ge 1$.

(a) If $f,g \in B_t^{N,p} \cap \mathscr{B}$ *and* $\frac{1}{f}, \frac{1}{g} \in H^\infty(\mathbb{B}_d)$, *then*

$$\int_{\mathbb{B}_d} \left| \frac{R^N(fg)}{fg}(z) \right|^p (1-|z|^2)^{pt} dV(z) < \infty.$$

(b) The space $\mathscr{A} = B_t^{N,p} \cap H^\infty(\mathbb{B}_d)$ *is an algebra and there exists a* $c > 0$ *such that* $\mathscr{A}$ *becomes a Banach algebra with the norm*

$$\|\cdot\|_{\mathscr{A},c} : \mathscr{A} \to \mathbb{R}_{\ge 0},\ \|f\|_{\mathscr{A},c} = c\|f\|_{\mathscr{A}}.$$

We follow the ideas for the proof of Proposition 4 in [CHZ18].

Proof. For part (a) suppose that $f,g \in B_t^{N,p} \cap \mathscr{B}$ and $\frac{1}{f}, \frac{1}{g} \in H^\infty(\mathbb{B}_d)$. By the Leibniz product rule, it follows that

$$R^N(fg) = \sum_{l=0}^{N} \binom{N}{l} R^l(f) R^{N-l}(g).$$

Hence, it suffices to prove that

$$\int_{\mathbb{B}_d} \left| \frac{(R^l f R^{N-l} g)}{fg}(z) \right|^p (1-|z|^2)^{pt} dV(z) < \infty$$

for $0 \le l \le N$. The two terms corresponding to $l = 0$ and $l = N$ are disposed of easily, as both f and g are bounded from below. For $0 < m < N$ define $p'(m) = N/m$. For $h \in B_t^{N,p} \cap \mathscr{B}$ consider the integral

$$I_m(h) = \int_{\mathbb{B}_d} |R^m h(z)|^{pp'(m)} (1-|z|^2)^{pt} dV(z).$$

Observe that $p'(m)(N-m) = N(p'(m)-1)$. Due to Theorem 5.2.16, it follows that

$$I_m(h) \approx \int_{\mathbb{B}_d} |R^N h(z)|^{pp'(m)} (1-|z|^2)^{pt+pN(p'(m)-1)} dV(z).$$

Since $h \in B_t^{N,p} \cap \mathscr{B}$, we deduce that

$$\sup_{z\in\mathbb{B}_d} \left((1-|z|^2)^{pN(p'(m)-1)} |R^N h(z)|^{p(p'(m)-1)} \right) \lesssim \|h\|_{\mathscr{B}}^{p(p'(m)-1)}$$

is bounded.

(See proofs of Theorems 3.4 and 3.5 in [Zhu05] and note that there are coefficients c_α $0 \leq |\alpha| \leq N$ such that

$$R^N f(z) = \sum_{0 \leq \alpha \leq N} c_\alpha z^\alpha (\partial^\alpha f)(z) \quad (f \in \mathscr{O}(\mathbb{B}_d), z \in \mathbb{B}_d).)$$

Thus, we obtain that

$$I_m(h) \lesssim \|h\|_{\mathscr{B}}^{p(p'(m)-1)} \int_{\mathbb{B}_d} |R^N h(z)|^p (1-|z|^2)^{pt} dV(z) = \|h\|_{\mathscr{B}}^{p(p'(m)-1)} \|h\|_{B_t^{N,p}}^p.$$

Fix $0 < l < N$. For $\tilde{p} = \frac{N}{l} = p'(l)$ and $\tilde{q} = \frac{N}{N-l} = p'(N-l)$, we use Hölder's inequality and obtain

$$\int_{\mathbb{B}_d} \left|(R^l f R^{N-l} g)(z)\right|^p (1-|z|^2)^{pt} dV(z) \leq I_l(f)^{\frac{l}{N}} I_{N-l}(g)^{\frac{N-l}{N}}.$$

Using that f and g are bounded from below, we deduce that

$$\begin{aligned}\int_{\mathbb{B}_d} &\left|\frac{(R^l f R^{N-l} g)}{fg}(z)\right|^p (1-|z|^2)^{pt} dV(z) \\ &\lesssim \left(\|g\|_{\mathscr{B}} \|f\|_{B_t^{N,p}}\right)^{l\frac{p}{N}} \left(\|f\|_{\mathscr{B}} \|g\|_{B_t^{N,p}}\right)^{(N-l)\frac{p}{N}} < \infty.\end{aligned}$$

For part (b) let $f, g \in \mathscr{A} = B_t^{N,p} \cap H^\infty(\mathbb{B}_d)$. It is well-known that $H^\infty(\mathbb{B}_d) \subset \mathscr{B}$ and that $\|h\|_{\mathscr{B}} \leq \|h\|_\infty$ for all $h \in H^\infty(\mathbb{B}_d)$. It follows for all $0 \leq l \leq N$ precisely in the same way as in the proof of part (a) that

$$\begin{aligned}\int_{\mathbb{B}_d} &\left|(R^l f R^{N-l} g)(z)\right|^p (1-|z|^2)^{pt} dV(z) \\ &\lesssim \left(\|g\|_\infty \|f\|_{B_t^{N,p}}\right)^{l\frac{p}{N}} \left(\|f\|_\infty \|g\|_{B_t^{N,p}}\right)^{(N-l)\frac{p}{N}} < \infty.\end{aligned}$$

Thus, by the binomial theorem

$$\begin{aligned}\int_{\mathbb{B}_d} &\left|R^N(fg)(z)\right|^p (1-|z|^2)^{pt} dV(z) \\ &\lesssim \sum_{l=0}^N \binom{N}{l} \left(\|g\|_\infty \|f\|_{B_t^{N,p}}\right)^{l\frac{p}{N}} \left(\|f\|_\infty \|g\|_{B_t^{N,p}}\right)^{(N-l)\frac{p}{N}} \\ &= (\|g\|_\infty \|f\|_{B_t^{N,p}} + \|f\|_\infty \|g\|_{B_t^{N,p}})^p \\ &\leq (\|f\|_{\mathscr{A}})^p (\|g\|_{\mathscr{A}})^p.\end{aligned}$$

Hence, there exists a $c > 0$ such that

$$\|fg\|_{\mathscr{A}} \leq c \|f\|_{\mathscr{A}} \|g\|_{\mathscr{A}},$$

which is equivalent to the fact that

$$\|fg\|_{\mathscr{A},c} \leq \|f\|_{\mathscr{A},c} \|g\|_{\mathscr{A},c}.$$

□

In [RS16], Richter and Sunkes show that every function $f \in H^2_d \cap \mathscr{B}$ in the Drury Arveson space with $\frac{1}{f} \in H^\infty(\mathbb{B}_d)$ is cyclic. Indeed, they use the fact that, in this case, $\frac{1}{f} \in H^2_d$. In [APR+24], this result has been improved, showing that every function $f \in H^2_d$, being bounded below, is cyclic.

Using a similar argument as in the proof of Theorem 6.2.8, one obtains the following version of Theorem 5.1 in [RS16]:

Theorem 6.2.10. *Let $1 \leq p < \infty$, $t > -\frac{1}{p}$ and $N \geq 1$.*

(a) If $f \in B^{N,p}_t \cap \mathscr{B}$ and $\frac{1}{f} \in H^\infty(\mathbb{B}_d)$, then $\frac{1}{f} \in B^{N,p}_t$.

(b) If $g, h \in B^{N,p}_t \cap H^\infty(\mathbb{B}_d)$ and $\frac{h}{g} \in H^\infty(\mathbb{B}_d)$ with $c = \left\|\frac{h}{g}\right\|_\infty$, then $\frac{h^{N+1}}{g} \in B^{N,p}_t \cap H^\infty(\mathbb{B}_d)$ and

$$\left\|\frac{h^{N+1}}{g}\right\|_{\mathscr{A}} \lesssim c\|h\|^N_\infty + (c\|h\|_{\mathscr{A}} + \|g\|_{\mathscr{A}})^{N+1},$$

where

$$\|f\|_{\mathscr{A}} = \|f\|_{B^{N,p}_t} + \|f\|_\infty \quad (f \in B^{s,p}_t \cap H^\infty(\mathbb{B}_d)).$$

Proof. (a) Let $f \in B^{N,p}_t \cap \mathscr{B}$ and $\frac{1}{f} \in H^\infty(\mathbb{B}_d)$. Using Corollary 6.2.3, it follows that

$$R^N\left(\frac{1}{f}\right) = -\sum_{l=0}^{N} \binom{N+1}{l} (-f)^{-(N+1-l)} R^N(f^{N+1-l}).$$

By applying Lemma 6.2.9 inductively, we conclude that $\frac{1}{f} \in B^{N,p}_t$.

(b) Let $g, h \in B^{N,p}_t \cap H^\infty(\mathbb{B}_d)$ and $\frac{h}{g} \in H^\infty(\mathbb{B}_d)$. Using Lemma 6.2.9 part (b) and the fact that

$$R^N\left(\frac{h^{N+1}}{g}\right) = (-1)^N \sum_{l=0}^{N} (-1)^l \binom{N+1}{l} \left(\frac{h}{g}\right)^{N+1-l} R^N(g^{N+1-l}h^l)$$

by Corollary 6.2.4, the result follows exactly in the same way as in the proof of Theorem 6.2.8. □

A. Appendix

A.1. Operators with closed range and Moore-Penrose pseudoinverse

For convenience, we want to recall a well-known statement about bounded linear operators on Hilbert spaces with closed range and generalized inverses (Moore-Penrose), which fits nicely within the theory used in Chapter 3. There is a whole theory on Moore-Penrose pseudoinverse (also unbounded) operators that has its origin in $n \times m$-matrices, but we will only need the following special case:

Definition A.1.1. Let $H, \tilde{H}$ be Hilbert spaces and let $T : H \to \tilde{H}$ be a bounded linear operator. We call a bounded linear operator $T^+ : \tilde{H} \to H$ Moore-Penrose pseudoinverse of T if and only if

(i) $TT^+T = T$

(ii) $T^+TT^+ = T^+$

(iii) $(TT^+)^* = TT^+$

(iv) $(T^+T)^* = T^+T$

Example A.1.2. For a partial isometry $V : H \to \tilde{H}$, the adjoint $V^* : \tilde{H} \to H$ is a Moore-Penrose pseudoinverse of V.

The following result is easily verified:

Theorem A.1.3. *Let $H, \tilde{H}$ be Hilbert spaces and let $T : H \to \tilde{H}$ be a bounded linear operator. Then*

$$\mathrm{Im}(T^*)^\perp = \mathrm{Ker}(T) = \mathrm{Ker}(T^*T) = \mathrm{Im}(T^*T)^\perp$$

and

$$\overline{\mathrm{Im}(T)} = \mathrm{Ker}(T^*)^\perp = \mathrm{Ker}(TT^*)^\perp = \overline{\mathrm{Im}(TT^*)}.$$

For a proof see, for example [Con90, Chapter II, 2.19 Theorem].

Theorem A.1.4. *Let $H, \tilde{H}$ be Hilbert spaces and let $T : H \to \tilde{H}$ be a bounded linear operator. Then the following are equivalent:*

(a) $\mathrm{Im}(T)$ *is closed,*

(b) $\mathrm{Im}(T^*)$ *is closed,*

(c) $\mathrm{Im}(T^*T)$ *is closed,*

(d) $T^*T : \mathrm{Im}(T^*) \to \mathrm{Im}(T^*)$ *is invertible,*

(e) T *has a Moore-Penrose pseudoinverse* T^+*.*

Furthermore,

1. *the Moore-Penrose pseudoinverse is uniquely determined by* $T^+ = (T^*T)^{-1}T^*$*,*
2. *the operator* $TT^+ : \tilde{H} \to \tilde{H}$ *is the orthogonal projection onto* $\mathrm{Im}(T)$*,*
3. *the operator* $T^+T : H \to H$ *is the orthogonal projection onto* $\mathrm{Im}(T^*)$*.*

Proof. (a) implies (b): By the open mapping theorem, the bounded linear operator

$$S_T : \mathrm{Ker}(T)^{\perp} \to \mathrm{Im}(T),\ h \to Th$$

has a bounded inverse S_T^{-1}. Let

$$h_0 \in \overline{\mathrm{Im}(T^*)} = \mathrm{Ker}(T)^{\perp} \subset H$$

(see Theorem A.1.3). The induced mapping

$$\varphi_{h_0} : \mathrm{Im}(T) \to \mathbb{C},\ Th \mapsto \langle S_T^{-1}Th, h_0\rangle_H$$

is well-defined, linear, and bounded. Hence, there exists an $\tilde{h}_0 \in \mathrm{Im}(T) \subset \tilde{H}$ such that

$$\varphi_{h_0}(Th) = \langle Th, \tilde{h}_0\rangle_{\tilde{H}}.$$

for all $h \in H$. We obtain that

$$\langle h, T^*\tilde{h}_0\rangle_{\mathrm{Ker}(T)^{\perp}} = \varphi_{h_0}(Th) = \langle S_T^{-1}Th, h_0\rangle_{\mathrm{Ker}(T)^{\perp}} = \langle h, h_0\rangle_{\mathrm{Ker}(T)^{\perp}}.$$

for all $h \in \mathrm{Ker}(T)^{\perp}$. Thus,

$$h_0 = T^*\tilde{h}_0 \in \mathrm{Im}(T^*).$$

(a) and (b) imply (c): Since $\mathrm{Im}(T)$ is closed, it follows from Theorem A.1.3 that $\mathrm{Im}(T) = \mathrm{Ker}(T^*)^{\perp}$. Hence, if $h_0 \in \mathrm{Im}(T^*)$, there exists an $h \in H$ such that $h_0 = T^*Th$ and thus,

$$\mathrm{Im}(T^*T) = \mathrm{Im}(T^*)$$

is closed.

(c) implies (d): Due to Theorem A.1.3,

$$\overline{\mathrm{Im}(T^*)} = \overline{\mathrm{Im}(T^*T)}.$$

Since $\mathrm{Im}(T^*T)$ is closed, we conclude that

$$\mathrm{Ker}(T^*T)^{\perp} = \mathrm{Im}(T^*T) = \mathrm{Im}(T^*) = \mathrm{Ker}(T)^{\perp}.$$

Then $T^*T : \mathrm{Im}(T^*) \to \mathrm{Im}(T^*)$ is injective, has a closed range, and is therefore invertible.

(d) implies (e): Set $T^+ : \tilde{H} \to H$, $\tilde{h} \mapsto (T^*T)^{-1}T^*\tilde{h}$. Then it is straightforward to check that the properties of a Moore-Penrose pseudoinverse hold.

(e) implies (a): Suppose that T has a Moore-Penrose pseudoinverse T^+. Since

$$TT^+T = T,$$

we have that $\mathrm{Im}(TT^+) = \mathrm{Im}(T)$. Using the properties (i) and (iii) of a Moore-Penrose pseudoinverse, it follows immediately that $\mathrm{Im}(T)$ is closed and that $TT^+ : \tilde{H} \to \tilde{H}$ is the orthogonal projection onto $\mathrm{Im}(T)$.

Furthermore, using the properties (i),(ii),(iv) of a Moore-Penrose pseudoinverse it is not difficult to see that $T^+T : H \to H$ is the orthogonal projection onto $\mathrm{Im}(T^*)$.

For the uniqueness statement, let $T_1^+, T_2^+ : \tilde{H} \to H$ be two Moore-Penrose pseudoinverses of T. Then by the previous discussion $TT_1^+ = TT_2^+$ and $T_1^+T = T_2^+T$. Thus, we obtain that

$$T_1^+ = T_1^+TT_1^+ = T_1^+TT_2^+TT_1^+ = T_2^+TT_2^+TT_2^+ = T_2^+TT_2^+ = T_2^+.$$

□

A.2. Weak-star continuity and Krein-Smulian

To study the weak-$*$ continuity of homomorphisms between operator spaces, it is sometimes helpful to use a theorem due to Krein-Smulian.

Let E, F be Banach spaces. We denote by $E' = B(E, \mathbb{C})$ the dual space of E and by $F' = B(F, \mathbb{C})$ the dual space of F.

It is well-known that the space of all bounded linear operators $B(H)$ is the dual of the trace class operators $C_1(H)'$ (see Remark 2.3.14, part (c)).

The following ideas are well-known and can be found, for example, in [Eve08]. For convenience, we recall the most important results here again.

Proposition A.2.1. *Let H be a Hilbert space and let $(T_\alpha)_{\alpha\in A}$ be a* WOT*-bounded net in $B(H)$. For $T \in B(H)$ the following are equivalent*

(a) $(T_\alpha)_{\alpha\in A}$ converges in the weak operator topology to T.

(b) $(T_\alpha)_{\alpha\in A}$ converges in the weak-$*$-topology to T.

Proof. WOT-bounded sets in $B(H)$ are norm-bounded by the uniform boundedness principle. Consequently, we obtain the desired result, using the well-known fact that the weak operator topology and the weak-$*$-topology coincide on norm-bounded sets (c.f. [Con00, Chapter 3, Proposition 20.1]). □

Denote by

$$\mathrm{ball}(F') = \{y' \in F';\ \|y'\|_{F'} \leq 1\}$$

the closed unit ball. We have the following lemma:

Lemma A.2.2. *Let E, F be Banach spaces and let $T : (F', \|\cdot\|_{F'}) \to (E', \|\cdot\|_{E'})$ be bounded and linear. Then the following are equivalent:*

(a) T is weak-$*$ continuous,

(b) $T : (\mathrm{ball}(F'), \tau_{w^*}) \to (E', \tau_{w^*})$ is continuous.

If H and $\tilde{H}$ are Hilbert spaces and $B(H) = C_1(H)', B(\tilde{H}) = C_1(\tilde{H})'$ are the dual spaces of the trace class operators, then (a) and (b) are equivalent to

(c) $T : (\mathrm{ball}(B(\tilde{H})), \mathrm{WOT}) \to (B(H), \mathrm{WOT})$ is continuous.

Proof. (a) implies (b) is trivial, since the continuity of $T : (\mathrm{ball}(F'), \tau_w^*) \to (E', \tau_{w^*})$ follows immediately from the weak-$*$-continuity of T.

For (b) implies (a) suppose that $T : (\mathrm{ball}(F'), \tau_{w^*}) \to (E', \tau_w^*)$ is continuous. Let $(y'_\alpha)_{\alpha \in A}$ be converging in the weak-$*$ topology of F' to $y' \in F'$, that is

$$y'_\alpha(y) \overset{\alpha}{=\!\!\longrightarrow} y'(y)$$

for all $y \in F$. For the continuity of $T : (\mathrm{ball}(F'), \tau_{w^*}) \to (E', \tau_{w^*})$ we have to show that $(Ty'_\alpha)_{\alpha \in A}$ converges in the weak-$*$ topology of E' to $Ty' \in E'$, that is

$$(Ty'_\alpha)(x) \overset{\alpha}{\longrightarrow\!\!=} (Ty')(x)$$

for all $x \in E$. So, continuity of $T : (\mathrm{ball}(F'), \tau_{w^*}) \to (E', \tau_{w^*})$ follows if and only if the linear functionals

$$\langle x, T(\cdot)\rangle : (F', \tau_{w^*}) \to \mathbb{C},\ y' \mapsto (Ty')(x)$$

induced by the points $x \in E$ are continuous. A well-known statement [Con90, Chapter IV; 3.1 Theorem], yields that the linear form

$$\langle x, T(\cdot)\rangle : (F', \tau_{w^*}) \to \mathbb{C}$$

is continuous if and only if $\mathrm{Ker}(\langle x, T(\cdot)\rangle)$ is weak-$*$ closed. By assumption, the map

$$T : (\mathrm{ball}(F'), \tau_{w^*}) \to \to (E', \tau_w^*)$$

is continuous and hence

$$\mathrm{Ker}(\langle x, T(\cdot)\rangle) \cap \mathrm{ball}(F')$$

is weak-$*$ closed. By the Banach-Dieudonné theorem (cf. [Con90, Chapter V; 12.6 Corollary]), which follows as a corollary of the Krein-Smulian theorem, we obtain that $\mathrm{Ker}(\langle x, T(\cdot)\rangle)$ is weak-$*$ closed. Thus, $T : (F', \tau_{w^*}) \to (E', \tau_w^*)$ is continuous.

If H and $\tilde{H}$ are two separable Hilbert spaces and $E = C_1(H)$ and $F = C_1(\tilde{H})$, the equivalence follows by Proposition A.2.1 together with the fact that the weak-$*$ topology is finer than the weak operator topology and the fact that $T : (F', \|\cdot\|_{F'}) \to (E', \|\cdot\|_{E'})$ is bounded. □

Using Lemma A.2.2, the following corollary is immediate:

Corollary A.2.3. *Let H and $\tilde{H}$ be two Hilbert spaces, and let $T : B(\tilde{H}) \to B(H)$ be linear and bounded. If $T : (B(\tilde{H}), \mathrm{WOT}) \to (B(H), \mathrm{WOT})$ is continuous, then $T : (B(\tilde{H}), \tau_{w^*}) \to (B(H), \tau_{w^*})$ is continuous.*

A.3. Infinite operator matrices

This section contains some basic facts about infinite operator matrices, as they appear in Chapter 4.

Let $(H_l)_{l\in\mathbb{N}}$ and $(\tilde{H}_m)_{m\in\mathbb{N}}$ be families of Hilbert spaces and let

$$H = \bigoplus_{l\in\mathbb{N}} H_l \quad \text{and} \quad \tilde{H} = \bigoplus_{m\in\mathbb{N}} \tilde{H}_m$$

be the corresponding direct sums. Consider for a family of operators

$$T_{l,m} \in B(H_l, \tilde{H}_m) \quad ((l,m) \in \mathbb{N}\times\mathbb{N})$$

the infinite matrix

$$T = (T_{l,m})_{(l,m)}$$

and for $N \in \mathbb{N}$ the finite matrices

$$T^{(N)} = (T_{l,m})_{0\le l\le N, 0\le m\le N}.$$

The finite matrices $T^{(N)}$ induce bounded linear operators $T^{(N)} : H \to \tilde{H}$ with

$$(T^{(N)}h)_l = \begin{cases} \sum_{m=0}^{N} T_{m,l} h_m & \text{if } 0 \le l \le N, \\ 0, & \text{else} \end{cases}$$

for

$$h = (h_m)_{m\in\mathbb{N}} \in \bigoplus_{l\in\mathbb{N}} H_l.$$

Theorem A.3.1. *Suppose that*

$$c = \sup_{N\in\mathbb{N}} \left\| T^{(N)} \right\| < \infty.$$

Then the linear operator $T : H \to \tilde{H}$, defined by

$$Th = \lim_{N\to\infty} T^{(N)} h$$

is well-defined and bounded with $\|T\| = c$ and

$$T^* \tilde{h} = \lim_{N\to\infty} \left(T^{(N)}\right)^* \tilde{h}$$

for all $\tilde{h} \in \tilde{H}$.

Proof. For $N \in \mathbb{N}$ let $P_N : H \to H$ be the orthogonal projection onto the closed subspace $\bigoplus_{l=0}^{N} H_l$ and let $\tilde{P}_N : \tilde{H} \to \tilde{H}$ be the orthogonal projection onto the closed subspace $\bigoplus_{l=0}^{N} \tilde{H}_l$. Then

$$P_N \xrightarrow{\text{SOT}} \mathrm{id}_H, \quad \tilde{P}_N \xrightarrow{\text{SOT}} \mathrm{id}_{\tilde{H}}$$

for $N \to \infty$ and $T^{(N)} = \tilde{P}_N T^{(N)} P_N$ for all $N \in \mathbb{N}$. Let $N \geq M \geq 0$, then

$$\tilde{P}_M T^{(N)} P_M = T^{(M)} \tag{A.1}$$

and thus

$$T^{(N)} - T^{(M)} = \tilde{P}_N T^{(N)} P_N - \tilde{P}_M T^{(N)} P_M = (\tilde{P}_N - \tilde{P}_M) T^{(N)} P_N + \tilde{P}_M T^{(N)} (P_N - P_M)$$

Using Cauchy Schwarz, we obtain that

$$|\langle (T^{(M)} - T^{(N)}) h, \tilde{h} \rangle_{\tilde{H}}| = c \|h\|_H \|(\tilde{P}_N - \tilde{P}_M)\tilde{h}\|_{\tilde{H}} + c \|\tilde{h}\|_{\tilde{H}} \|(P_N - P_M)h\|_H$$

for all $h \in H$ and $\tilde{h} \in \tilde{H}$. Thus, it is immediate that $(T^{(N)})_{N \in \mathbb{N}}$ is a bounded WOT-Cauchy sequence. Since $(B(H), \mathrm{WOT})$ is quasi-complete (see Example 2.3.20), it follows that $(T^{(N)})_{N \in \mathbb{N}}$ converges in the weak operator topology to a bounded linear operator $T : H \to \tilde{H}$. Using (A.1), it can be easily verified that

$$T^{(N)} = \tilde{P}_N T P_N$$

for all $N \in \mathbb{N}$. Then

$$T - T^{(N)} = (\mathrm{id}_{\tilde{H}} - \tilde{P}_N) T P_N + \tilde{P}_N T (\mathrm{id}_H - P_N)$$

and hence

$$Th = \lim_{N \to \infty} T^{(N)} h$$

for all $h \in H$. Since $(T^*)^{(N)} = P_N T^* \tilde{P}_N$ for all $N \in \mathbb{N}$ we obtain in the same way that

$$T^* \tilde{h} = \lim_{N \to \infty} \left(T^{(N)} \right)^* \tilde{h}$$

for all $\tilde{h} \in \tilde{H}$. Furthermore, we conclude that

$$c = \sup_{N \in \mathbb{N}} \left\| T^{(N)} \right\| = \sup_{N \in \mathbb{N}} \left\| \tilde{P}_N T P_N \right\| \leq \|T\| \leq \sup_{N \in \mathbb{N}} \left\| T^{(N)} \right\| = c.$$

□

Remark A.3.2. In general, it is not possible to approximate T by $T^{(N)}$ in norm. Consider the Hilbert space of the square summable sequences $\ell^2(\mathbb{N})$ and the infinite matrix $T = (t_{l,m})_{(l,m)}$ with $t_{l,l} = 1$ and $t_{l,m} = 0$, when $l \neq m$. Then one can check that $T = \mathrm{id}_{\ell^2(\mathbb{N})}$ and that

$$T^{(N)} = P_N : \ell^2(\mathbb{N}) \to \ell^2(\mathbb{N})$$

are the orthogonal projections on the finite dimensional subspaces $E_N = \{e_n;\ 0 \le n \le N\}$, where $(e_n)_{n\in\mathbb{N}}$ is the canonical orthonormal basis of $\ell^2(\mathbb{N})$. In this case,

$$T^{(N)} = P_N \overset{\text{SOT}}{\longrightarrow} \mathrm{id}_{\ell^2(\mathbb{N})} = T,$$

but

$$\left\| T - T^{(N)} \right\| = \| \mathrm{id}_{\ell^2(\mathbb{N})} - P_N \| = 1.$$

For a sequence of operators $S_l : H_l \to \tilde{H}_0$ consider the infinite row

$$S = (S_1, \dots, S_N, S_{N+1} \dots)$$

Let

$$T_S = (T_{l,m})_{(l,m)\in\mathbb{N}\times\mathbb{N}}$$

be the infinite matrix where $(T_S)_{0,m} = S_m$ for $m \in \mathbb{N}$ and $(T_S)_{l,m} = 0$ else. For $N \in \mathbb{N}$ define

$$S^{(N)} = (S_1, \dots, S_N, 0, 0, \dots),$$

then $T_S^{(N)} = T_{S^{(N)}}$. Using the orthogonal projection $\tilde{P}_0 : \bigoplus_{l\in\mathbb{N}} \tilde{H}_l \to \tilde{H}_0$ onto the space $\tilde{H}_0$ and the inclusion map $\tilde{i} : \tilde{H}_0 \to \bigoplus_{l\in\mathbb{N}} \tilde{H}_l$ the following corollary is a direct consequence of Theorem A.3.1.

Corollary A.3.3. *Suppose that* $c = \sup_{N\in\mathbb{N}} \left\| S^{(N)} \right\| < \infty$. *Then the row operator*

$$S : \bigoplus_{l\in\mathbb{N}} H_l \to \tilde{H}_0,\ (Sh_l)_{l\in\mathbb{N}} = \sum_{l=0}^{\infty} S_l h_l$$

is well-defined and bounded with $\|S\| = c$, $S^{(N)} \overset{\text{SOT}}{\longrightarrow} S$ *and* $\left(S^{(N)}\right)^* \overset{\text{SOT}}{\longrightarrow} S^*$ *for* $N \to \infty$.

A.4. One-box conditions for the Dirichlet space

Let

$$K : \mathbb{D}\times\mathbb{D} \to \mathbb{C},\ K(z,w) = \frac{1}{z\overline{w}} \log\left(\frac{1}{1 - z\overline{w}}\right)$$

be the reproducing kernel of the Dirichlet space $\mathscr{D}$. Let $w \in \mathbb{D}$. As always, we use the notations

$$k_w : \mathbb{D} \to \mathbb{C},\ k_w(z) = K(z,w)$$

and

$$\hat{k}_w = \frac{k_w}{\|k_w\|_{\mathscr{D}}}.$$

Given an arc $I \subset \mathbb{T}$, let

$$S(I) = \{re^{i\theta};\ e^{i\theta} \in I, 1 - |I| < r < 1\},$$

be the corresponding Carleson-box, where $|I|$ denotes the arclength of I.

In Remark 6.1.8 we consider a statement due to Stegenga, which is equivalent to the fact that:

$$\mathrm{Mult}(\mathscr{D}) \subsetneq \left\{ f \in \mathscr{D};\ \sup_{z\in\mathbb{D}} \|f\hat{k}_z\|_{\mathscr{D}} < \infty \right\}.$$

To see the equivalence, we use the next theorem. I would like to thank Dr. Nikolaos Chalmoukis, who communicated the ideas for the proof.

Theorem A.4.1. *Let μ be a finite Borel measure on $\mathbb{D}$ then*

$$\sup\left(\log\frac{1}{|I|}\mu(S(I));\ I\subset\mathbb{T}\ \textit{interval}\right) < \infty$$

if and only if there exists a $C>0$ such that

$$\int_{\mathbb{D}} |k_w|^2 d\mu(z) \le C\|k_w\|_{\mathscr{D}}^2.$$

This result is undoubtedly well-known to experts, but for convenience, we sketch important ideas for the proof here. We start with some useful lemmas first.

Lemma A.4.2. *If $z\in\mathbb{D}$, then*

$$\left|\frac{1}{z}\log\left(\frac{1}{1-z}\right)\right| \le 2\log\left(\frac{2}{|1-z|}\right)+4\pi \le 2\log\left(\frac{1}{|1-z|}\right)+18.$$

Proof. Let $\log : \mathbb{C}\setminus(-\infty,0] \to \{z\in\mathbb{C};\ \mathrm{Im}(z)\in(-\pi,\pi)\}$,

$$z\mapsto \log|z| + i\arg_{-\pi}(z).$$

be the principle branch of the complex logarithm. If $z\in\mathbb{D}$, then $\frac{1}{1-z}\in\mathbb{C}\setminus(-\infty,0]$. Thus,

$$\left|\log\left(\frac{1}{1-z}\right)\right| \le \log\left(\frac{2}{|1-z|}\right)+2\pi.$$

Using the inequality $\frac{1}{2}|z| \le |\log\left(\frac{1}{1-z}\right)| \le 2|z|$ for $|z|<\frac{1}{2}$, we obtain that

$$\left|\frac{1}{z}\log\left(\frac{1}{1-z}\right)\right| \le 2\log\left(\frac{2}{|1-z|}\right)+4\pi \le 2\log\left(\frac{1}{|1-z|}\right)+18.$$

□

Lemma A.4.3. *Let $\theta\in(-\pi,\pi]$, then $|\theta| < \frac{\pi}{2}|1-e^{i\theta}|$.*

Proof. The length of the semicircle with diameter $|1-e^{i\theta}|$ is larger then the length of the arc of the segment with angle θ. □

Lemma A.4.4. *Let $w \in \mathbb{D}$ with $\frac{1}{2} < |w| < 1$ and $0 < t \leq \frac{1}{2}$. Then the set*

$$A = \{z \in \mathbb{D};\ |1 - z\overline{w}| \leq t\}$$

is contained in a Carleson-box $S(I)$, where $I = I(\frac{w}{|w|}, 2t) \subset \mathbb{T}$ is the arc with midpoint $\frac{w}{|w|}$ and length $|I| = 2t$.

Proof. Fix $w \in \mathbb{D}$ with $\frac{1}{2} < |w| < 1$ and observe that

$$1 - |z| \leq \frac{1}{|w|} - |z| \leq \left| z - \frac{1}{\overline{w}} \right| = \frac{|1 - z\overline{w}|}{|w|} < 2t$$

for all $z \in A$. Thus, $1 - 2t < |z|$ for all $z \in A$. Now fix $z \in A$. A computation shows that

$$\begin{aligned} \left| \frac{z}{|z|} - \frac{w}{|w|} \right| = \left| 1 - \frac{z\overline{w}}{|zw|} \right| &\leq |1 - z\overline{w}| + \left| z\overline{w} - \frac{z\overline{w}}{|zw|} \right| \\ &= |1 - z\overline{w}| + |1 - |z\overline{w}|| \leq 2|1 - z\overline{w}| \leq 2t. \end{aligned}$$

Due to the previous lemma, we obtain for $\theta = \arg(z)$ and $\theta_0 = \arg(w)$ in $(-\pi, \pi]$ that

$$|\theta - \theta_0| \leq \frac{\pi}{2} |e^{i(\theta - \theta_0)} - 1| \leq \frac{\pi}{2} |e^{i\theta} - e^{i\theta_0}| \leq \pi t.$$

It follows that $\theta \in I = (\theta_0 - 2\pi t, \theta_0 + 2\pi t)$, where $|I| = 2t$. □

We use the Kolmogorov equality to show that the one-box condition implies the Carleson measure condition for the kernel functions. The converse implication can be found in [EFKMR14, Theorem 5.2.5 (i)].

Proof of Theorem A.4.1. Let μ be a finite Borel measure on $\mathbb{D}$. First, suppose that the one-box condition

$$\sup \left(\log \frac{1}{|I|} \mu(S(I));\ I \subset \mathbb{T} \text{ interval} \right) < \infty$$

holds true. Since

$$\sup \left(\frac{|k_w(z)|^2}{\|k_w\|_{\mathbb{D}}^2};\ |z| < 1, |w| \leq \frac{19}{20} \right) < \infty,$$

we can assume without loss of generality that $\frac{79}{80} < |w| < 1$. If $f \in L^p(\mu)$, then using Tonelli, we obtain

$$\begin{aligned} &\int_{\mathbb{D}} |f(z)|^p d\mu(z) \\ &= \int_{\mathbb{D}} \int_{[0,\infty]} \chi_{\{t \leq |f(z)|\}}(t) p t^{p-1} d\lambda(t) d\mu(z) \\ &= p \int_{[0,\infty]} \int_{\mathbb{D}} \chi_{\{t \leq |f(z)|\}}(t) d\mu(z) t^{p-1} d\lambda(t) \\ &= p \int_{[0,\infty]} \mu(\{t \leq |f(z)|\}) t^{p-1} d\lambda(t). \end{aligned}$$

Thus,

$$\int_{\mathbb{D}} |k_w(z)|^2 d\mu(z) = 2\int_{[0,\|k_w\|_\infty]} \mu(\{t \le |k_w(z)|\}) t d\lambda(t)$$
$$= 2\|k_w\|_\infty^2 \int_{[0,1]} \mu(\{\|k_w\|_\infty t \le |k_w(z)|\}) t d\lambda(t).$$

Observe that

$$\|k_w\|_{\mathscr{D}}^2 \le \|k_w\|_\infty \le 5\|k_w\|_{\mathscr{D}}^2.$$

As mentioned above, one can compute that

$$18 + 2\log\frac{1}{|1-z\overline{w}|} \ge |k_w(z)|.$$

Since $18 + 2\log\frac{1}{|1-z\overline{w}|} \ge t$ if and only if $|1-z\overline{w}| \le e^{-\frac{t}{2}+9}$, we obtain

$$\{z \in \mathbb{D};\ |k_w(z)| > t\|k_w\|_\infty\} \subset \{z \in \mathbb{D};\ |1-z\overline{w}| \le e^{-\frac{\|k_w\|_\infty}{2}t+9}\}.$$

Because $\frac{79}{80} < |w| < 1$, one computes that $\|k_w\|_\infty \ge 80$ and if $\frac{1}{2} < t < 1$, then $-\frac{\|k_w\|_\infty}{2}t + 9 \le -11$. Hence,

$$\{z \in \mathbb{D};\ |1-z\overline{w}| \le e^{-\frac{\|k_w\|_\infty}{2}t+9}\} \subset S(I),$$

where $S(I)$ is a Carleson-box of an interval I of length $|I| \le 2e^{-\frac{\|k_w\|_\infty}{2}t+9}$ and center $\frac{w}{|w|} \in \mathbb{T}$. Since

$$\mu(S(I)) \lesssim \left(\log\left(\frac{1}{|I|}\right)\right)^{-1},$$

it is immediate that there exists a $c > 0$ such that

$$\mu(\{\|k_w\|_\infty t \le |k_w(z)|\}) \le \mu(\{z \in \mathbb{D};\ |1-z\overline{w}| \le e^{-\frac{\|k_w\|_\infty}{2}t+9}\}) \le \frac{c}{\|k_w\|_\infty t}.$$

Hence,

$$\int_{\mathbb{D}} |k_w(z)|^2 d\mu(z)$$
$$= 2\|k_w\|_\infty^2 \int_{[0,1]} \mu(\{\|k_w\|_\infty t \le |k_w(z)|\}) t d\lambda(t)$$
$$\lesssim \|k_w\|_\infty \le 5\|k_w\|_{\mathscr{D}}^2.$$

For the proof of the converse implication, see [EFKMR14, Theorem 5.2.5 (i)]. □

Bibliography

[ABR01] Sheldon Axler, Paul Bourdon, and Wade Ramey. *Harmonic function theory*, volume 137 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, second edition, 2001.

[ABY21] Luciano Abadias, Glenier Bello, and Dmitry Yakubovich. Operator inequalities, functional models and ergodicity. *J. Math. Anal. Appl.*, 498(2):Paper No. 124984, 39, 2021.

[AE03] Jonathan Arazy and Miroslav Engliš. Analytic models for commuting operator tuples on bounded symmetric domains. *Trans. Amer. Math. Soc.*, 355(2):837–864, 2003.

[AE08] Herbert Amann and Joachim Escher. *Analysis. II*. Birkhäuser Verlag, Basel, 2008. Translated from the 1999 German original by Silvio Levy and Matthew Cargo.

[AEM02] Calin-Grigore Ambrozie, Miroslav Engliš, and Vladimir Müller. Operator tuples and analytic models over general domains in $\mathbb{C}^n$. *J. Operator Theory*, 47(2):287–302, 2002.

[Agl82] Jim Agler. The Arveson extension theorem and coanalytic models. *Integral Equations Operator Theory*, 5(5):608–631, 1982.

[Agl85] Jim Agler. Hypercontractions and subnormality. *J. Operator Theory*, 13(2):203–217, 1985.

[AHMR17] Alexandru Aleman, Michael Hartz, John E. McCarthy, and Stefan Richter. The Smirnov class for spaces with the complete Pick property. *J. Lond. Math. Soc. (2)*, 96(1):228–242, 2017.

[AHMR18] Alexandru Aleman, Michael Hartz, John E. McCarthy, and Stefan Richter. Factorizations induced by complete Nevanlinna-Pick factors. *Adv. Math.*, 335:372–404, 2018.

[AHMR19] Alexandru Aleman, Michael Hartz, John E. McCarthy, and Stefan Richter. Radially weighted Besov spaces and the Pick property. In *Analysis of operators on function spaces*, Trends Math., pages 29–61. Birkhäuser, Cham, 2019.

[AHMR22] Alexandru Aleman, Michael Hartz, John E. McCarthy, and Stefan Richter. Multiplier tests and subhomogeneity of multiplier algebras. *Doc. Math.*, 27:719–764, 2022.

[AHMR23] Alexandru Aleman, Michael Hartz, John E. McCarthy, and Stefan Richter. Free outer functions in complete Pick spaces. *Trans. Amer. Math. Soc.*, 376(3):1929–1978, 2023.

[AM00a] Jim Agler and John E. McCarthy. Complete Nevanlinna-Pick kernels. *J. Funct. Anal.*, 175:111–124, 2000.

[AM00b] Jim Agler and John E. McCarthy. Nevanlinna-Pick kernels and localization. In *Operator theoretical methods (Timişoara, 1998)*, pages 1–20. Theta Found., Bucharest, 2000.

[AM02] Jim Agler and John E. McCarthy. *Pick interpolation and Hilbert function spaces*, volume 44 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2002.

[APR$^+$24] Alexandru Aleman, Karl-Mikael Perfekt, Stefan Richter, Carl Sundberg, and James Sunkes. Cyclicity in the Drury-Arveson space and other weighted Besov spaces. *Trans. Amer. Math. Soc.*, 377(2):1273–1298, 2024.

[ARR98] Alexandru Aleman, Stefan Richter, and William T. Ross. Pseudocontinuations and the backward shift. *Indiana Univ. Math. J.*, 47(1):223–276, 1998.

[ARS96] Alexandru Aleman, Stefan Richter, and Carl Sundberg. Beurling's theorem for the Bergman space. *Acta Math.*, 177(2):275–310, 1996.

[ARS08] N. Arcozzi, R. Rochberg, and E. Sawyer. Carleson measures for the Drury-Arveson Hardy space and other Besov-Sobolev spaces on complex balls. *Adv. Math.*, 218(4):1107–1180, 2008.

[Arv98] William Arveson. Subalgebras of C^*-algebras. III. Multivariable operator theory. *Acta Math.*, 181(2):159–228, 1998.

[Bar01] Christoph Barbian. Positivitätsbedingungen funktionaler Hilberträume und Anwendungen in der mehrdimensionalen Operatorentheorie. Diplomarbeit, Universität des Saarlandes, FR 6.1-Mathematik, AG Eschmeier, Universität des Saarlandes, Saarbrücken, 2001.

[Bar07] Christoph Barbian. *Beurling-type representation of invariant subspaces in reproducing kernel Hilbert spaces.* Dissertation, Universität des Saarlandes, FR 6.1-Mathematik, AG Eschmeier, Universität des Saarlandes, Saarbrücken, 2007.

[Bar11] Christoph Barbian. A characterization of multiplication operators on reproducing kernel Hilbert spaces. *J. Operator Theory*, 65(2):235–240, 2011.

[BB89] Frank Beatrous and Jacob Burbea. Holomorphic Sobolev spaces on the ball. *Dissertationes Math. (Rozprawy Mat.)*, 276:60, 1989.

[BB08] Frank Beatrous and Jacob Burbea. On multipliers for Hardy-Sobolev spaces. *Proc. Amer. Math. Soc.*, 136(6):2125–2133, 2008.

[BC91] Joseph A. Ball and Nir Cohen. de Branges-Rovnyak operator models and systems theory: a survey. In *Topics in matrix and operator theory (Rotterdam, 1989)*, volume 50 of *Oper. Theory Adv. Appl.*, pages 93–136. Birkhäuser, Basel, 1991.

[BEKS17] Madhuchhanda Bhattacharjee, Jörg Eschmeier, Dinesh K. Keshari, and Jaydeb Sarkar. Dilations, wandering subspaces, and inner functions. *Linear Algebra Appl.*, 523:263–280, 2017.

[Beu48] Arne Beurling. On two problems concerning linear transformations in Hilbert space. *Acta Math.*, 81:239–255, 1948.

[BHM18] Kelly Bickel, Michael Hartz, and John E. McCarthy. A multiplier algebra functional calculus. *Trans. Amer. Math. Soc.*, 370(12):8467–8482, 2018.

[BJ23a] Tirthankar Bhattacharyya and Abhay Jindal. Complete Nevanlinna-Pick kernels and the characteristic function. *Adv. Math.*, 426:Paper No. 109089, 25, 2023.

[BJ23b] Tirthankar Bhattacharyya and Abhay Jindal. Kernels with complete Nevanlinna-Pick factors and the characteristic function, 2023.

[BS84] Leon Brown and Allen L. Shields. Cyclic vectors in the Dirichlet space. *Trans. Amer. Math. Soc.*, 285(1):269–303, 1984.

[Car62] Lennart Carleson. Interpolations by bounded analytic functions and the corona problem. *Ann. of Math. (2)*, 76:547–559, 1962.

[CD16] Raphaël Clouâtre and Kenneth R. Davidson. Absolute continuity for commuting row contractions. *J. Funct. Anal.*, 271(3):620–641, 2016.

[CF16] Carme Cascante and Joan Fàbrega. Bilinear forms on weighted Besov spaces. *J. Math. Soc. Japan*, 68(1):383–403, 2016.

[CFO10] Carme Cascante, Joan Fàbrega, and Joaquín M. Ortega. On weighted Toeplitz, big Hankel operators and Carleson measures. *Integral Equations Operator Theory*, 66(4):495–528, 2010.

[CH18] Raphaël Clouâtre and Michael Hartz. Multiplier algebras of complete Nevanlinna-Pick spaces: dilations, boundary representations and hyperrigidity. *J. Funct. Anal.*, 274(6):1690–1738, 2018.

[Che12] Yong Chen. Quasi-wandering subspaces in a class of reproducing analytic Hilbert spaces. *Proc. Amer. Math. Soc.*, 140(12):4235–4242, 2012.

[CHZ18] Guangfu Cao, Li He, and Kehe Zhu. Spectral theory of multiplication operators on Hardy-Sobolev spaces. *J. Funct. Anal.*, 275(5):1259–1279, 2018.

[Con90] John B. Conway. *A course in functional analysis*, volume 96 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, second edition, 1990.

[Con91] John B. Conway. *The theory of subnormal operators*, volume 36 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 1991.

[Con00] John B. Conway. *A course in operator theory*, volume 21 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2000.

[CQ15] Denis Choimet and Hervé Queffélec. *Twelve landmarks of twentieth-century analysis*. Cambridge University Press, New York, 2015. Illustrated by Michaël Monerau, Translated from the 2009 French original by Danièle Gibbons and Greg Gibbons, With a foreword by Gilles Godefroy.

[CSW11] Şerban Costea, Eric T. Sawyer, and Brett D. Wick. The corona theorem for the Drury-Arveson Hardy space and other holomorphic Besov-Sobolev spaces on the unit ball in $\mathbb{C}^n$. *Anal. PDE*, 4(4):499–550, 2011.

[CV78] Zoia Ceauşescu and Florian-Horia Vasilescu. Tensor products and Taylor's joint spectrum. *Studia Math.*, 62(3):305–311, 1978.

[CZ92] Muneo Cho and Wieslaw Zelazko. On geometric spectral radius of commuting n-tuples of operators. *Hokkaido Math. J.*, 21(2):251–258, 1992.

[Dou98] Ronald G. Douglas. *Banach algebra techniques in operator theory*, volume 179 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, second edition, 1998.

[Dru78] Stephen W. Drury. A generalization of von Neumann's inequality to the complex ball. *Proc. Amer. Math. Soc.*, 68(3):300–304, 1978.

[EFKMR14] Omar El-Fallah, Karim Kellay, Javad Mashreghi, and Thomas Ransford. *A primer on the Dirichlet space*, volume 203 of *Cambridge Tracts in Mathematics*. Cambridge University Press, Cambridge, 2014.

[EP96] Jörg Eschmeier and Mihai Putinar. *Spectral decompositions and analytic sheaves*, volume 10 of *London Mathematical Society Monographs. New Series*. The Clarendon Press, Oxford University Press, New York, 1996. Oxford Science Publications.

[Esc97] Jörg Eschmeier. Invariant subspaces for spherical contractions. *Proc. London Math. Soc. (3)*, 75(1):157–176, 1997.

[Esc18a] Jörg Eschmeier. Bergman inner functions and m-hypercontractions. *J. Funct. Anal.*, 275(1):73–102, 2018.

[Esc18b] Jörg Eschmeier. *Funktionentheorie mehrerer Veränderlicher*. Springer-Verlag, 2018.

[ET21] Jörg Eschmeier and Sebastian Toth. K-inner functions and K-contractions. In *Operator theory, functional analysis and applications*, volume 282 of *Oper. Theory Adv. Appl.*, pages 157–181. Birkhäuser/Springer, Cham, 2021.

[Eve08] Kevin Everad. Über sphärische Isometrien und exakte Sequenzen von Toeplitzalgebren. Diplomarbeit, Saarland University, 2008.

[FX11] Quanlei Fang and Jingbo Xia. Multipliers and essential norm on the Drury-Arveson space. *Proc. Amer. Math. Soc.*, 139(7):2497–2504, 2011.

[Gar07] John B. Garnett. *Bounded analytic functions*, volume 236 of *Graduate Texts in Mathematics*. Springer, New York, first edition, 2007.

[GHX04] Kunyu Guo, Junyun Hu, and Xianmin Xu. Toeplitz algebras, subnormal tuples and rigidity on reproducing $\mathbf{C}[z_1,\dots,z_d]$-modules. *J. Funct. Anal.*, 210(1):214–247, 2004.

[GRS02] Devin C. V. Greene, Stefan Richter, and Carl Sundberg. The structure of inner multipliers on spaces with complete Nevanlinna-Pick kernels. *J. Funct. Anal.*, 194(2):311–331, 2002.

[GRS05] Jim Gleason, Stefan Richter, and Carl Sundberg. On the index of invariant subspaces in spaces of analytic functions of several complex variables. *J. Reine Angew. Math.*, 587:49–76, 2005.

[Har12] Michael Hartz. Universal operator algebras for commuting row contractions. Master thesis, Saarland University, 2012.

[Har16] Michael Hartz. *Nevanlinna-Pick Spaces and Dilations*. Phd thesis, University of Waterloo, 2016.

[Har17] Michael Hartz. On the isomorphism problem for multiplier algebras of Nevanlinna-Pick spaces. *Canad. J. Math.*, 69(1):54–106, 2017.

[Hed91] Haakan Hedenmalm. A factorization theorem for square area-integrable analytic functions. *J. Reine Angew. Math.*, 422:45–68, 1991.

[Kak41] Shizuo Kakutani. Concrete representation of abstract (M)-spaces. (A characterization of the space of continuous functions.). *Ann. of Math. (2)*, 42:994–1024, 1941.

[Kal09] Manuel Kaluza. Bergmanräume über der Einheitskugel in $\mathbb{C}^n$. Diplomarbeit, Universität des Saarlandes, 2009.

[Kat04] Yitzhak Katznelson. *An introduction to harmonic analysis*. Cambridge Mathematical Library. Cambridge University Press, Camb., third edition, 2004.

[Kla16] Valerie Klauk. Dilatationssätze für m-Hyperkontraktionen. Masterarbeit, Universität des Saarlandes, 2016.

[Kre14] Elena Kreuzer. Toeplitz Extensions and Berezin transforms. Master thesis, Universität des Saarlandes, 2014.

[Lan19] Sebastian Langendörfer. On unitarily invariant spaces and Cowen-douglas theory. Phd thesis, Saarland University, 2019.

[Lec95] Jarosł aw Lech. Essentially normal multiplication operators on the Dirichlet space. *Michigan Math. J.*, 42(1):127–140, 1995.

[LMN20] Mikael Lindström, Santeri Miihkinen, and David Norrbo. Unified approach to spectral properties of multipliers. *Taiwanese J. Math.*, 24(6):1471–1495, 2020.

[Luo17] Shuaibing Luo. Essentially hyponormal multiplication operators on the Dirichlet space. *Arch. Math. (Basel)*, 108(3):283–291, 2017.

[MOT86] Thomas L. Miller, Robert F. Olin, and James E. Thomson. Subnormal operators and representations of algebras of bounded analytic functions and other uniform algebras. *Mem. Amer. Math. Soc.*, 63(354):vi+125, 1986.

[Mue07] Vladimir Mueller. *Spectral theory of linear operators and spectral systems in Banach algebras*, volume 139 of *Operator Theory: Advances and Applications*. Birkhäuser Verlag, Basel, second edition, 2007.

[MV93] Vladimir Mueller and Florian-Horia Vasilescu. Standard models for some commuting multioperators. *Proc. Amer. Math. Soc.*, 117(4):979–989, 1993.

[New59] Donald J. Newman. Some remarks on the maximal ideal structure of H^∞. *Ann. of Math. (2)*, 70:438–445, 1959.

[OF00] Joaquín M. Ortega and Joan Fàbrega. Pointwise multipliers and decomposition theorems in analytic Besov spaces. *Math. Z.*, 235(1):53–81, 2000.

[Olo06] Anders Olofsson. A characteristic operator function for the class of n-hypercontractions. *J. Funct. Anal.*, 236(2):517–545, 2006.

[Olo07] Anders Olofsson. Operator-valued Bergman inner functions as transfer functions. *Algebra i Analiz*, 19(4):146–173, 2007.

[Pau02] Vern Paulsen. *Completely bounded maps and operator algebras*, volume 78 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 2002.

[PR16] Vern I. Paulsen and Mrinal Raghupathi. *An introduction to the theory of reproducing kernel Hilbert spaces*, volume 152 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 2016.

[RS16] Stefan Richter and James Sunkes. Hankel operators, invariant subspaces, and cyclic vectors in the Drury-Arveson space. *Proc. Amer. Math. Soc.*, 144(6):2575–2586, 2016.

[Rud91] Walter Rudin. *Functional analysis*. International Series in Pure and Applied Mathematics. McGraw-Hill, Inc., New York, second edition, 1991.

[Rud08] Walter Rudin. *Function theory in the unit ball of* $\mathbb{C}^n$. Classics in Mathematics. Springer-Verlag, Berlin, 2008. Reprint of the 1980 edition.

[Sar89] Donald Sarason. Exposed points in H^1. I. In *The Gohberg anniversary collection, Vol. II (Calgary, AB, 1988)*, volume 41 of *Oper. Theory Adv. Appl.*, pages 485–496. Birkhäuser, Basel, 1989.

[Sar90] Donald Sarason. Exposed points in H^1. II. In *Topics in operator theory: Ernst D. Hellinger memorial volume*, volume 48 of *Oper. Theory Adv. Appl.*, pages 333–347. Birkhäuser, Basel, 1990.

[Sch18] Dominik Schillo. *K-contractions and perturbations of Toeplitz operators*. Phd thesis, Saarland University, 2018.

[Shi74] Allen Shields. Weighted shift operators and analytic function theory. *Topics in operator theory*, pages 49–128b, 1974.

[Shi98] Serguei Shimorin. Reproducing kernels and extremal functions in Dirichlet-type spaces. *Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI)*, 255(Issled. po Lineĭn. Oper. i Teor. Funkts. 26):198–220, 254, 1998.

[Shi01] Serguei Shimorin. Wold-type decompositions and wandering subspaces for operators close to isometries. *J. Reine Angew. Math.*, 531:147–189, 2001.

[Shi02] Serguei Shimorin. Complete Nevanlinna-Pick property of Dirichlet-type spaces. *J. Funct. Anal.*, 191:276–296, 2002.

[SNFBK10] Béla Sz.-Nagy, Ciprian Foias, Hari Bercovici, and László Kérchy. *Harmonic analysis of operators on Hilbert space*. Universitext. Springer, New York, second edition, 2010.

[Ste80] David A. Stegenga. Multipliers of the Dirichlet space. *Illinois J. Math.*, 24(1):113–139, 1980.

[SW99] Helmut H. Schaefer and M. P. Wolff. *Topological vector spaces*, volume 3 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, second edition, 1999.

[Wer00] Dirk Werner. *Funktionalanalysis*. Springer-Verlag, Berlin, extended edition, 2000.

[Wer08] Michael Wernet. Wesentlich normale Hilbertmoduln über Gebieten in $\mathbb{C}^d$. Diplomarbeit, Saarland University, 2008.

[Zhu05] Kehe Zhu. *Spaces of holomorphic functions in the unit ball*, volume 226 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 2005.

[ZZ08] Ruhan Zhao and Kehe Zhu. Theory of Bergman spaces in the unit ball of $\mathbb{C}^n$. *Mém. Soc. Math. Fr. (N.S.)*, (115):vi+103, 2008.