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Splitting-type variational problems with asymmetrical growth conditions

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Abstract
Splitting-type variational problems

$$\int_\Omega \sum_{i=1}^n f_i(\partial_i w)\,\mathrm{d}x \to \min$$

with superlinear growth conditions are studied by assuming

$$h_i(t) \le f_i''(t) \le H_i(t) \qquad (*)$$

with suitable functions $h_i, H_i\colon \mathbb{R} \to \mathbb{R}^+$, $i = 1, \ldots, n$, measuring the growth and ellipticity of the energy density. Here, as the main feature, we do not impose a symmetric behaviour like $h_i(t) \approx h_i(-t)$ and $H_i(t) \approx H_i(-t)$ for large $|t|$. Assuming quite weak hypotheses on the functions appearing in $(*)$, we establish higher integrability of $|\nabla u|$ for local minimizers $u \in L^\infty(\Omega)$ by using a Caccioppoli-type inequality with some power weights of negative exponent.

Keywords Splitting-type variational problems · Asymmetrical growth conditions · Non-uniform ellipticity

Mathematics Subject Classification 49N60 · 49N99 · 35J45

1 Introduction

Suppose that $\Omega \subset \mathbb{R}^n$ is a bounded Lipschitz domain and consider the variational integral

$$J[w] := \int_\Omega f(\nabla w)\,\mathrm{d}x \tag{1.1}$$

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of splitting-type, i.e.

$$f : \mathbb{R}^n \to \mathbb{R}\,, \quad f(Z) = \sum_{i=1}^{n} f_i(Z_i) \tag{1.2}$$

with strictly convex functions $f_i : \mathbb{R} \to \mathbb{R}$ of class $C^2(\mathbb{R})$, $i = 1, \ldots, n$, satisfying in addition some suitable superlinear growth and ellipticity conditions.

Problem (1.1), (1.2) serves as a prototype for non-uniformly elliptic variational problems. After Giaquinta's counterexample [1] and the pioneering work of, e.g., Marcellini [2, 3], Acerbi and Fusco [4], Fusco and Sbordone [5] and many others it is well understood that the ratio of the highest and the lowest eigenvalue of $D^2 f$ is the crucial quantity for proving the regularity of solutions. The reader will find an extensive overview including different settings of non-uniformly elliptic variational problems in the recent paper [6]. Without going into further details we refer to the series of references given in this paper. We just like to finish this short considerations with the remark, that the unbounded counterexamples constructed by Giaquinta et al. are complemented, e.g., by the work of Fusco and Sbordone [7].

In Section 1.3 of [6], the authors consider general growth conditions which, roughly speaking, means that the energy density f is controlled in the sense of

$$g(|Z|)|\xi|^2 \le D^2 f(Z)(\xi, \xi)\,, \quad \left|D^2 f(Z)\right| \le G(|Z|)\,, \tag{1.3}$$

with suitable functions g, G: $\mathbb{R}_0^+ \to \mathbb{R}^+$. Then, under appropriate assumptions on g, G, a general approach to regularity theory is given in [6].

Our note is motivated by the observation, that in (1.2) there is no obvious reason to assume some kind of symmetry for the functions f_i, i.e. in general we have $f_i(t) \neq f_i(-t)$ and, as one model case, we just consider ($q_i^{\pm} > 1$, $i = 1, \ldots, n$)

$$f_i(t) \approx |t|^{q_i^-} \quad \text{if} \quad t \ll -1\,, \qquad f_i(t) \approx |t|^{q_i^+} \quad \text{if} \quad t \gg 1\,. \tag{1.4}$$

Then, both for $t \ll -1$ and for $t \gg 1$, the functions f_i just behave like a uniform power of $|t|$. Nevertheless, the power q_i^- enters the left-hand side of (1.3) and q_i^+ is needed on the right-hand side of (1.3).

This motivates to study the model case (1.2) and to establish regularity results for solutions under the weaker assumption

$$h_i(t) \le f_i''(t) \le H_i(t) \quad t \in \mathbb{R}\,, \tag{1.5}$$

with suitable functions h_i, $H_i : \mathbb{R} \to \mathbb{R}^+$, $i = 1, \ldots n$.

There is another quite subtle difficulty in studying regularity of solutions to splitting-type variational problems: in [8] the authors consider variational integrals of the form ($1 \le k < n$)

$$I[w, \Omega] = \int_\Omega \left[f\,(\partial_1 w, \ldots, \partial_k w) + g\,(\partial_{k+1} w, \ldots, \partial_n w) \right] \mathrm{d}x\,, \tag{1.6}$$

where f and g are of p and q-growth, respectively ($p, q > 1$). Then the regularity of bounded solutions follows in the sense of [8], Theorem 1.1, without any further condition relating p and q. The proof argues step by step and works since the energy density splits into two parts. If, as supposed in (1.2), the energy density splits in more than two components, then one has to be more careful dealing with the exponents and some more restrictive (but still quite weak) assumptions have to be made. In this sense Remark 1.3 of [8] might be a little bit misleading. We note that a splitting structure into two components as supposed in (1.6) is also assumed, e.g., in [9] and related papers.

In the following we consider the variational integral (1.1), (1.2) defined on the energy class

$$E_f(\Omega) := \left\{ w \in W^{1,1}(\Omega) : \int_\Omega f(\nabla w)\,\mathrm{d}x < \infty \right\}.$$

We are interested in local minimizers $u\colon \Omega \to \mathbb{R}$ of class $E_f(\Omega)$, i.e. it holds that

$$\int_\Omega f(\nabla u)\,\mathrm{d}x \le \int_\Omega f(\nabla w)\,\mathrm{d}x \tag{1.7}$$

for all $w \in E_f(\Omega)$ such that $\mathrm{spt}(u - w) \Subset \Omega$.

Notation. We will always denote by $q_i^+ > 1$, $q_i^- > 1$, $1 \le i \le n$, real exponents and we let for fixed $1 \le i \le n$

$$\underline{q}_i := \min\left\{q_i^\pm\right\}, \quad \overline{q}_i := \max\left\{q_i^\pm\right\}. \tag{1.8}$$

Moreover, we let

$$\Gamma :\ [0, \infty) \to \mathbb{R}, \quad \Gamma(t) = 1 + t^2.$$

Recalling the idea sketched in (1.4), (1.5) we denote by h_i and H_i, $i = 1, \dots, n$, functions $\mathbb{R} \to \mathbb{R}^+$ such that with positive constants $\underline{a}_i$, $\overline{a}_i$

$$\left.\begin{array}{ll} \underline{a}_i \Gamma^{\frac{q_i^- - 2}{2}}(|t|) & \text{if } t < -1 \\ \underline{a}_i \Gamma^{\frac{q_i^+ - 2}{2}}(|t|) & \text{if } t > 1 \end{array}\right\} \le h_i(t) \tag{1.9}$$

and

$$H_i(t) \le \left\{\begin{array}{ll} \overline{a}_i \Gamma^{\frac{q_i^- - 2}{2}}(|t|) & \text{if } t < -1 \\ \overline{a}_i \Gamma^{\frac{q_i^+ - 2}{2}}(|t|) & \text{if } t > 1 \end{array}\right. . \tag{1.10}$$

We consider functions $f_i\colon \mathbb{R} \to [0, \infty)$ of class $C^2(\mathbb{R})$, $i = 1, \dots, n$, such that for all $t \in \mathbb{R}$

$$h_i(t) \le f_i''(t) \le H_i(t) \tag{1.11}$$

and note that (1.11) immediately implies for all $i \in \{1, \dots, n\}$ with constants $b_i > 0$

$$|f_i'(t)| \le b_i \left\{\begin{array}{l} \Gamma^{\frac{q_i^- - 1}{2}}(|t|) \text{ if } t < -1 \\ \Gamma^{\frac{q_i^+ - 1}{2}}(|t|) \text{ if } t > 1 \end{array}\right\}. \tag{1.12}$$

Moreover we obtain (maybe up to additive constants) for all $i = 1, \dots, n$ with constants $\underline{c}_i$, $\overline{c}_i > 0$

$$\underline{c}_i \left\{\begin{array}{l} \Gamma^{\frac{q_i^-}{2}}(|t|) \text{ if } t < -1 \\ \Gamma^{\frac{q_i^+}{2}}(|t|) \text{ if } t > 1 \end{array}\right\} \le f_i(t) \le \overline{c}_i \left\{\begin{array}{r} \Gamma^{\frac{q_i^-}{2}}(|t|);\text{ if } t < -1 \\ \Gamma^{\frac{q_i^+}{2}}(|t|) \text{ if } t > 1 \end{array}\right\}. \tag{1.13}$$

With this notation our main result reads as follows.

Theorem 1.1 *Suppose that for* $i = 1, \ldots, n$ *the functions* $f_i : \mathbb{R} \to [0, \infty)$ *are of class* $C^2(\mathbb{R})$ *and satisfy* (1.11) *with* h_i, H_i *given in* (1.9), (1.10) *for* $q_i^{\pm} > 1$.

With the notation (1.8) *we assume in addition:*

(i) in the case $n = 2$ *and* $\overline{q}_2 > 2$ *we suppose that*

$$\overline{q}_2 < 2\underline{q_1} + 2. \tag{1.14}$$

By (1.14) *we may choose* $\rho_1 > 1$ *such that*

$$\rho_1 < 2\frac{\underline{q}_1}{\overline{q}_2 - 2}$$

and we further suppose that

$$\overline{q}_1 < \underline{q}_2[2 + \rho_1] + 2. \tag{1.15}$$

In the case $n = 2$ *and* $1 \le \overline{q}_2 \le 2$ *we may take any* $\rho_1 < \infty$ *which follows from* (5.1) *with* $\theta_j = 0$.

(ii) in the case $n \ge 3$ *suppose that we have for every fixed* $1 \le i \le n$

$$\overline{q}_j < 2\underline{q}_i + 2 \quad \text{for all} \quad i < j \le n, \tag{1.16}$$

$$\overline{q}_j < 3\underline{q}_i + 2 \quad \text{for all} \quad 1 \le j < i. \tag{1.17}$$

If $u \in L^\infty(\Omega) \cap E_f(\Omega)$ *denotes a local minimizer of* (1.1), (1.2), *i.e. of*

$$J[w] = \int_\Omega \left[\sum_{i=1}^n f_i\,(\partial_i w) \right] \mathrm{d}x,$$

then for every $1 \le i \le n$, *for some* $\delta > 1/2$ *and for any ball* $B_{2r}(x_0) \Subset \Omega$

$$\int_{B_r(x_0)} f_i(\partial_i u)\Gamma^\delta(|\partial_i u|)\,\mathrm{d}x \le c \tag{1.18}$$

with a finite local constant c.

Remark 1.1 i) In the two dimensional case as discussed in [8] we have $p = q_2 \le q_1 = q$ and $\overline{q}_j = \underline{q}_j = q_j$, $j = 1, 2$.

In this case (1.14) gives no restriction on the exponents since we have chosen w.l.o.g. $p \le q$.

In an analogous way it is possible to renumber the exponents in the general form (1.16), (1.17) (or in the same spirit (2.5), (2.6) of Theorem 2.1 below) in order to have these conditions after reordering.

For (1.15) we observe

$$p\left[2 + \frac{2q}{p-2}\right] + 2 = \frac{2pq}{p-2} + 2(p+1) > 2q,$$

hence (1.15) as well does not restrict the class of admissible exponents p and q.

Here and in the following we always suppose w.l.o.g. that we have (5.1) (see Section 2 for the precise notation) since otherwise no further hypotheses on the exponents have to be made.

ii) Of course with an explicit choice of ρ_1 and the parameter ρ_2 (see (5.11)) the choice of δ in (1.18) can be made precise. Moreover, given (1.18) one may iterate the arguments in order to improve the integrability results. We leave the details to the reader.

iii) In the case $n \geq 3$ we could choose parameters $\rho_{i,j\neq i}$ as outlined by the choice of ρ_1 and ρ_2 in the case $n = 2$. We prefer the much simpler formulation of (1.16) and (1.17).

Theorem 1.1 describes the typical situation we have in mind. The proof however is not limited to this particular case which leads to the generalized version stated in Theorem 2.1 below.

Although the algebraic choice of parameters in the general form appears somehow involved, we prefer the general formulation since it clearly indicates the idea of the proof. We start with an Ansatz involving both $f_i(\partial_i u)$ and $\Gamma(|\partial_i u|)$ where the asymmetric structure enters by exploiting the structure of f_i combined with the relation to its derivatives. This Ansatz leads to the first general inequalities and we end up with some mixed terms which have to be discussed in the last section. There we combine a careful pointwise analysis with an iteration procedure which generalizes the arguments given in [8]. We note that even in the symmetric case splitting into more than 2 groups the known results are generalized by our Theorems.

In Sect. 3 we shortly sketch a regularization procedure via Hilbert-Haar solutions while Sect. 4 presents the main inequalities for the iteration procedure of Sect. 5. This completes the proof of Theorem 2.1 and hence Theorem 1.1.

2 Precise assumptions on *f*

The suitable larger class of admissible energy densities is given by the following assumption.

Assumption 2.1 The energy density f,

$$f : \mathbb{R}^n \to \mathbb{R}, \quad f(Z) = \sum_{i=1}^{n} f_i(Z_i),$$

introduced in (1.2) is supposed to satisfy the following hypotheses.

(i) The function $f_i : \mathbb{R} \to [0, \infty)$, $i = 1, \ldots, n$, is of class $C^2(\mathbb{R})$ and for all $t \in \mathbb{R}$ we have $f_i''(t) > 0$.
For $1 \leq i \leq n$ we suppose superlinear growth in the sense of

$$\lim_{t\to\pm\infty} \left|f_i'(t)\right| = \infty$$

and at most of polynomial growth in the sense that for some $s > 0$ we have for $|t|$ sufficiently large

$$f_i(t) \leq c|t|^s \quad \text{with a finite constant } c\,.$$

(ii) For $i \in \{1, \ldots, n\}$ and for

$$0 \leq \theta_i < \frac{1}{2} \tag{2.1}$$

we suppose that for all $|t|$ sufficiently large

$$c_1\Gamma^{1-\theta_i}(|t|) f_i''(t) \leq f_i(t) \leq c_2 f_i''(t)\Gamma^{1+\theta_i}(|t|)\,, \tag{2.2}$$

$$\left|f_i'(t)\right|^2 \leq c_3 f_i''(t) f_i(t) \Gamma^{\theta_i}(|t|), \tag{2.3}$$

where c_1, c_2 and c_3 denote positive constants.

(iii) We let

$$\Gamma^{\frac{q_i^\pm}{2}}(t) = \left\{ \begin{array}{l} \Gamma^{\frac{q_i^-}{2}}(|t|) \text{ if } t < 0 \\ \Gamma^{\frac{q_i^+}{2}}(|t|) \text{ if } t \geq 0 \end{array} \right\}.$$

and suppose that f_i, $i = 1, \ldots, n$, satisfies with $q_i^\pm > 1$, with positive constants c_4, c_5 and for $|t|$ sufficiently large

$$c_4 \Gamma^{\frac{q_i^\pm}{2}}(|t|) \leq f_i(t) \leq c_5 \Gamma^{\frac{q_i^\pm}{2}}(|t|). \tag{2.4}$$

Remark 2.1 1. If f_i is a power growth function like, e.g., $f_i(t) = (1+t^2)^{p_i/2}$, $p_i > 1$ fixed, then we have

$$c\Gamma(|t|) f_i''(t) \leq f_i(t) \leq c\Gamma(|t|) f_i''(t),$$

i.e. (2.2) with $\theta_i = 0$. Our asymmetric model case given by (1.9)–(1.13) as well is an admissible choice satisfying (2.2).

2. By convexity it is well known (see, e.g., [10], exercise 1.5.9, p. 53) that if a convex function is at most of growth rate s, then we have at most the growth rate $s - 1$ for its derivative. Hence, (2.4) together with the right-hand side of (2.2) imply (2.3).
3. The condition (2.2) with $\theta_i < 1/2$ formally corresponds with the condition $q < p + 2$ in the standard (p, q)-case (see, e.g., [11], Chapter 5, and the references quoted therein).

Theorem 2.1 *Suppose that we have Assumption 2.1 with $q_i^\pm > 1$, $i = 1, \ldots, n$. With the above notation we assume in addition that we have for every fixed $1 \leq i \leq n$*

$$\overline{q}_j < (1 - \theta_j)\left[2\left(\underline{q}_i - 2\theta_i\right) + 2\right] \quad \textit{for all} \quad i < j \leq n, \tag{2.5}$$

$$\overline{q}_j < (1 - \theta_j)\left[(2 + \tau)\left(\underline{q}_i - 2\theta_i\right) + 2\right], \quad \tau := \frac{1 - 2\theta_j}{1 - \theta_j}, \quad \textit{for all} \quad 1 \leq j < i. \tag{2.6}$$

If $u \in L^\infty(\Omega) \cap E_f(\Omega)$ denotes a local minimizer of (1.1), (1.2), i.e. of

$$J[w] = \int_\Omega \left[\sum_{i=1}^n f_i(\partial_i w)\right] \mathrm{d}x,$$

then for every $1 \leq i \leq n$, for some $\delta > 1/2$ and for any ball $B_{2r}(x_0) \Subset \Omega$ we have

$$\int_{B_r(x_0)} f_i(\partial_i u) \Gamma^{\delta - \theta_i}(|\partial_i u|) \, \mathrm{d}x \leq c \tag{2.7}$$

with a finite local constant c.

Remark 2.2 In particular we note that (2.5), (2.6) reduce to (1.16), (1.17) if θ_i is equal to zero.

3 Some remarks on regularization

We have to start with a regularization procedure such that the expressions given below are well defined. We follow Section 2 of [8] and fix a ball $D \Subset \Omega$. If u denotes the local minimizer in the sense of (1.7) and if $\varepsilon > 0$ is sufficiently small, we consider the mollification $(u)_\varepsilon$ of u w.r.t. the radius ε. We consider the Dirichlet-problem

$$\int_D \sum_{i=1}^n f_i(\partial_i w)\, \mathrm{d}x \to \min$$

among all Lipschitz mappings $\overline{D} \to \mathbb{R}$ with boundary data $(u)_\varepsilon$. According to, e.g., [12], there exists a unique (Hilbert-Haar) solution u_ε to this problem.

Exactly as outlined in [8], Lemma 2.1 and Lemma 2.2, we obtain:

Lemma 3.1 *Let* $\underline{q} := \min_{1\le i\le n} \underline{q}_i$

i) We have as $\varepsilon \to 0$

$$u_\varepsilon \rightharpoondown u \quad \text{in } W^{1,\underline{q}}(D)\,, \qquad \int_D \sum_{i=1}^n f_i(\partial_i u_\varepsilon)\, \mathrm{d}x \to \int_D \sum_{i=1}^n f_i(\partial_i u)\, \mathrm{d}x\,.$$

ii) There is a finite constant $c > 0$ *not depending on* ε *such that*

$$\|u_\varepsilon\|_{L^\infty(D)} \le c\,.$$

iii) For any $\alpha < 1$ *we have* $u_\varepsilon \in C^{1,\alpha}(D) \cap W^{2,2}_{\mathrm{loc}}(D)$.

We then argue as follows: consider a local minimizer u of (1.1), (1.2) and the approximating sequence $\{u_\varepsilon\}$ minimizing the functional

$$J[w, D] := \int_D \sum_{i=1}^n f_i\,(\partial_i w_i)\; \mathrm{d}x \tag{3.1}$$

w.r.t. the data $(u)_\varepsilon$. In particular we have a sequence of local $J[w, D]$-minimizers. We apply the a priori results of the next section to u_ε and Theorem 1.1 follows from Lemma 3.1 passing to the limit $\varepsilon \to 0$.

4 General inequalities

The main result of this section is Proposition 4.2 which is not depending on the hypotheses made in Assumption 2.1, iii).

We will rely on the following variant of Caccioppoli's inequality which was first introduced in [13]. We also refer to Section 6 of [14] on Caccioppoli-type inequalities involving powers with negative exponents, in particular we refer to Proposition 6.1.

Lemma 4.1 *Fix* $l \in \mathbb{N}$ *and suppose that* $\eta \in C_0^\infty(D)$, $0 \le \eta \le 1$. *If we consider a local minimizer* $u \in W^{1,\infty}_{\mathrm{loc}}(D) \cap W^{2,2}_{\mathrm{loc}}(D)$ *of the variational functional*

$$I[w] = \int_D g(\nabla w)\, \mathrm{d}x$$

with energy density $g\colon \mathbb{R}^n \to \mathbb{R}$ *of class* C^2 *satisfying* $D^2g(Z)(Y,Y) > 0$ *for all* $Y, Z \in \mathbb{R}^n$, *then for any fixed* $i \in \{1,\dots,n\}$ *we have*

$$\int_D D^2 g(\nabla u)\,(\nabla\partial_i u, \nabla\partial_i u)\,\eta^{2l}\Gamma^{\beta}(|\partial_i u|)\,\mathrm{d}x \\ \leq c\int_D D^2 g(\nabla u)(\nabla\eta,\nabla\eta)\eta^{2l-2}\Gamma^{1+\beta}(|\partial_i u|)\,\mathrm{d}x$$

for any $\beta > -1/2$.

To the end of our note we always consider a fixed ball

$$B = B_{2r}(x_0) \Subset D\,.$$

With this notation we have the following auxiliary proposition.

Proposition 4.1 *Suppose that we have i) of Assumption 2.1 and let* $\eta \in C_0^\infty(B)$, $0 \leq \eta \leq 1$, $\eta \equiv 1$ *on* $B_r(x_0)$, $|\nabla\eta| \leq c/r$. *Moreover, we assume that* $u \in L^\infty(D)\cap W^{1,\infty}_{\mathrm{loc}}(D)\cap W^{2,2}_{\mathrm{loc}}(D)$.

Then we have for fixed $\gamma \in \mathbb{R}$, *for all* $k > 0$ *sufficiently large and for* $i = 1, \dots, n$ *the starting inequalities (no summation w.r.t.* i*)*

$$\int_B f_i(\partial_i u)\Gamma^{1+\gamma}(|\partial_i u|)\eta^{2k}\,\mathrm{d}x \leq c\Bigg[1 + \int_B |\partial_i\partial_i u|\Gamma^{\gamma}(|\partial_i u|)\,f_i(\partial_i u)\eta^{2k}\,\mathrm{d}x \\ + \int_B |\partial_i\partial_i u|\,|f_i'|(\partial_i u)\,\Gamma^{\frac{1}{2}+\gamma}(|\partial_i u|)\eta^{2k}\,\mathrm{d}x\Bigg]\,, \tag{4.1}$$

where the constant may depend on $\|u\|_{L^\infty}$ *and on* $r > 0$.

Remark 4.1
- (i) The idea of the proof of Proposition 4.1 is based on an integration by parts using the boundedness of u. An Ansatz of this kind was already made by Choe [15], where all relevant quantities are depending on $|\nabla u|$. Here the main new feature is to work with the energy density f which is not depending on the modulus of ∇u.
- (ii) We note that for the proof of Proposition 4.1 no minimizing property of u is needed.

Proof of Proposition 4.1 With $i \in \{1,\dots,n\}$ fixed we obtain using an integration by parts

$$\begin{aligned}\int_B f_i(\partial_i u)\ &\Gamma^{1+\gamma}(|\partial_i u|)\eta^{2k}\,\mathrm{d}x \\ &= \int_B |\partial_i u|^2 f_i(\partial_i u)\Gamma^{\gamma}(|\partial_i u|)\eta^{2k}\,\mathrm{d}x + \int_B f_i(\partial_i u)\Gamma^{\gamma}(|\partial_i u|)\eta^{2k}\,\mathrm{d}x \\ &= -\int_B u\partial_i\Big[\partial_i u f_i(\partial_i u)\Gamma^{\gamma}(|\partial_i u|)\eta^{2k}\Big]\,\mathrm{d}x + \int_B f_i(\partial_i u)\Gamma^{\gamma}(|\partial_i u|)\eta^{2k}\,\mathrm{d}x \\ &\leq c\int_B |\partial_i\partial_i u|\Gamma^{\gamma}(|\partial_i u|)\,f_i(\partial_i u)\eta^{2k}\,\mathrm{d}x \\ &\quad + c\int_B |\partial_i\partial_i u|\,|\partial_i u|\,|f_i'|(\partial_i u)\,\Gamma^{\gamma}(|\partial_i u|)\eta^{2k}\,\mathrm{d}x \\ &\quad + c\int_B |\partial_i u|\,f_i(\partial_i u)\Gamma^{\gamma}(|\partial_i u|)\eta^{2k-1}|\partial_i\eta|\,\mathrm{d}x + \int_B f_i(\partial_i u)\Gamma^{\gamma}(|\partial_i u|)\eta^{2k}\,\mathrm{d}x \\ &= I_{1,i} + I_{2,i} + I_{3,i} + I_{4,i}\,.\end{aligned} \tag{4.2}$$

In (4.2) we discuss $I_{3,i}$: for $\varepsilon > 0$ sufficiently small we estimate

$$\begin{aligned} I_{3,i} &\le \int_B |\partial_i u| f_i^{\frac{1}{2}}(\partial_i u)\Gamma^{\frac{\gamma}{2}}(|\partial_i u|)\eta^k f_i^{\frac{1}{2}}(\partial_i u)\Gamma^{\frac{\gamma}{2}}(|\partial_i u|)\eta^{k-1}|\nabla\eta|\,\mathrm{d}x \\ &\le \varepsilon \int_B |\partial_i u|^2 f_i(\partial_i u)\Gamma^{\gamma}(|\partial_i u|)\eta^{2k} dx \\ &\quad + c(\varepsilon, r)\int_B f_i(\partial_i u)\Gamma^{\gamma}(|\partial_i u|)\eta^{2k-2}\,\mathrm{d}x\,. \end{aligned} \tag{4.3}$$

The first integral on the right-hand side of (4.3) is absorbed in the left-hand side of (4.2), i.e.

$$\begin{aligned} &\int_B f_i(\partial_i u)\Gamma^{1+\gamma}(|\partial_i u|)\eta^{2k}\,\mathrm{d}x \\ &\qquad \le I_{1,i} + I_{2,i} + c(\varepsilon, r)\int_B f_i(\partial_i u)\Gamma^{\gamma}(|\partial_i u|)\eta^{2k-2}\,\mathrm{d}x \\ &\qquad\quad + \int_B f_i(\partial_i u)\Gamma^{\gamma}(|\partial_i u|)\eta^{2k}\,\mathrm{d}x \\ &\qquad \le I_{1,i} + I_{2,i} + c(\varepsilon, r)\int_B f_i(\partial_i u)\Gamma^{\gamma}(|\partial_i u|)\eta^{2k-2}\,\mathrm{d}x\,. \end{aligned} \tag{4.4}$$

Discussing the remaining integral we recall that the function $f_i(t)\Gamma^{1+\gamma}(|t|)$ is at most of polynomial growth, hence we may apply the auxiliary Lemma 4.2 below to the functions $\varphi(t) = f_i(t)\Gamma^{\gamma}(|t|)$ and $\psi(t) := f_i(t)\Gamma^{1+\gamma}(|t|)$ with the result that for some $\rho > 1$ and for all $t \in \mathbb{R}$

$$f_i(t)\Gamma^{\gamma}(|t|) \le c\big[f_i(t)\Gamma^{1+\gamma}(|t|)\big]^{\frac{1}{\rho}} + c \tag{4.5}$$

with a suitable finite constant c.

With (4.5) we estimate for $\tilde{\varepsilon} > 0$ sufficiently small and for $k > \rho^* = \rho/(\rho - 1)$

$$\begin{aligned} &c(\varepsilon, r)\int_B f_i(\partial_i u)\;\;\Gamma^{\gamma}(|\partial_i u|)\eta^{2k-2}\,\mathrm{d}x \\ &\qquad \le c(\varepsilon, r)\int_B \big[f_i(\partial_i u)\Gamma^{1+\gamma}(|\partial_i u|)\big]^{\frac{1}{\rho}}\eta^{\frac{2k}{\rho}}\eta^{\frac{2k}{\rho^*}-2}\,\mathrm{d}x + c \\ &\qquad \le \tilde{\varepsilon}\int_B f_i(\partial_i u)\Gamma^{1+\gamma}(|\partial_i u|)\eta^{2k}\,\mathrm{d}x + c(\tilde{\varepsilon}, \varepsilon, r)\int_B \eta^{2(k-\rho^*)}\,\mathrm{d}x + c\,. \end{aligned} \tag{4.6}$$

The inequalities (4.4) and (4.6) complete the proof of the proposition by absorbing the first integral on the right-hand side of (4.6) in the left-hand side of (4.4). □

It remains to give an elementary proof of the following auxiliary Lemma.

Lemma 4.2 *For $m \in \mathbb{N}$ we consider functions φ, ψ: $\mathbb{R}^m \to [0, \infty)$ such that $\psi(X) \le c\Gamma^{\tau}(|X|)$ for some $\tau > 0$ and for all $X \in \mathbb{R}^m$. Suppose that we have for some $\varepsilon > 0$ and for all $X \in \mathbb{R}^n$*

$$\varphi(X) \le c\Gamma^{-\varepsilon}(|X|)\psi(X)\,.$$

Then there exists a real number $\rho > 1$ and a constant $C > 0$ such that

$$\varphi(X) \le \big[\psi(X)\big]^{\frac{1}{\rho}} + C\,.$$

Proof Let $\delta := \varepsilon/\tau$, w.l.o.g. $\delta < 1$, i.e. for all $X \in \mathbb{R}^m$

$$1 + \psi^\delta(X) \le 1 + c\Gamma^\varepsilon(|X|) \le (1+c)\Gamma^\varepsilon(|X|)\,,$$

hence we have by assumption

$$\begin{aligned}\varphi(X) &\le c\big[1+\psi^\delta(X)\big]^{-1}\psi(X)\\ &\le \left\{\begin{array}{cc} c & \text{if } \psi^\delta(X)\le 1\\ c\psi^{1-\delta}(X) & \text{if } \psi^\delta(X) > 1\end{array}\right\}.\end{aligned}$$

The lemma follows with the choice $\rho = 1/(1-\delta)$. □

With the help of Proposition 4.1 we now establish the main inequality of this section.

Proposition 4.2 *Suppose that we have Assumption 2.1 and let $\eta \in C_0^\infty(B)$, $0 \le \eta \le 1$, $\eta \equiv 1$ on $B_r(x_0)$, $|\nabla\eta| \le c/r$. Moreover, we assume that $u \in L^\infty(D) \cap W^{1,\infty}_{\mathrm{loc}}(D) \cap W^{2,2}_{\mathrm{loc}}(D)$ is a local minimizer of (3.1).*

For $i \in \{1,\dots,n\}$ we choose σ_i satisfying (recall $0 < \theta_i < 1/2$)

$$\theta_i < \sigma_i < 1/2\,.$$

Moreover, again for $i = 1, \dots, n$ we choose arbitrary real numbers $\gamma_i > -1$, $\beta_i > -1/2$ subject to the condition

$$\gamma_i + \sigma_i =: \beta_i > -\frac{1}{2}\,. \tag{4.7}$$

Then we have for any sufficiently large real number $k > 0$

$$\begin{aligned}&\int_B f_i(\partial_i u)\quad \Gamma^{1+\gamma_i}(|\partial_i u|)\eta^{2k}\,\mathrm{d}x\\ &\qquad\le c\Bigg[1+\sum_{j\ne i}\int_B f_j''(\partial_j u)\Gamma^{1+\beta_i}(|\partial_i u|)\eta^{2k-2}\,\mathrm{d}x\Bigg].\end{aligned} \tag{4.8}$$

Proof We recall the starting inequality (4.1),

$$\int_B f_i(\partial_i u)\Gamma^{1+\gamma_i}(|\partial_i u|)\eta^{2k}\,\mathrm{d}x \le c\Big[1 + I_{1,i} + I_{2,i}\Big]\,, \tag{4.9}$$

where we fix $i \in \{1,\dots,n\}$. We estimate for fixed β_i as above

$$\begin{aligned}I_{1,i} &= \int_B |\partial_i\partial_i u|\big[f_i''(\partial_i u)\big]^{\frac12}\Gamma^{\frac{\beta_i}{2}}(|\partial_i u|)\big[f_i''(\partial_i u)\big]^{-\frac12}\Gamma^{-\frac{\beta_i}{2}}(|\partial_i u|)\\ &\qquad\cdot\Gamma^{\gamma_i}(|\partial_i u|)\,f_i(\partial_i u)\eta^{2k}\,\mathrm{d}x\\ &\le c\int_B f_i''(\partial_i u)|\partial_i\partial_i u|^2\Gamma^{\beta_i}(|\partial_i u|)\eta^{2k}\,\mathrm{d}x\\ &\quad + c\int_B \big[f_i''(\partial_i u)\big]^{-1}\Gamma^{\gamma_i-\sigma_i}(|\partial_i u|)\,f_i^2(\partial_i u)\eta^{2k}\,\mathrm{d}x\,.\end{aligned} \tag{4.10}$$

The second integral on the right-hand side of (4.10) is handled with the help of the right-hand side of (2.2) using in addition Lemma 4.2 (recalling $\sigma_i > \theta_i$)

$$
\begin{aligned}
\int_B \big[f_i''(\partial_i u)\big]^{-1} \; & \Gamma^{\gamma_i-\sigma_i}(|\partial_i u|) f_i^2(\partial_i u)\eta^{2k}\,\mathrm{d}x \\
& \le \int_B \Big[f_i(\partial_i u)\Gamma^{1+\gamma_i-(\sigma_i-\theta_i)}(|\partial_i u|)\Big]\eta^{2k}\,\mathrm{d}x \\
& \le \int_B \Big[f_i(\partial_i u)\Gamma^{1+\gamma_i}(|\partial_i u|)\Big]^{\frac{1}{\rho}}\eta^{\frac{2k}{\rho}}\eta^{\frac{2k}{\rho*}}\,\mathrm{d}x + c \\
& \le \varepsilon \int_B f_i(\partial_i u)\Gamma^{1+\gamma_i}(|\partial_i u|)\eta^{2k}\,\mathrm{d}x + c(\varepsilon, r)\,. \qquad (4.11)
\end{aligned}
$$

Absorbing terms it is shown up to now (using (4.9)– (4.11))

$$
\begin{aligned}
\int_B f_i(\partial_i u) \; & \Gamma^{1+\gamma_i}(|\partial_i u|)\eta^{2k}\,\mathrm{d}x \\
& \le c\Big[1 + \int_B f_i''(\partial_i u)|\partial_i\partial_i u|^2\Gamma^{\beta_i}(|\partial_i u|)\eta^{2k}\,\mathrm{d}x + I_{2,i}\Big]. \qquad (4.12)
\end{aligned}
$$

Let us consider $I_{2,i}$, $i \in \{1, \dots, n\}$. With $\beta_i > -1/2$ as above we have

$$
\begin{aligned}
I_{2,i} &= \int_B |\partial_i\partial_i u|\big[f_i''(\partial_i u)\big]^{\frac{1}{2}}\Gamma^{\frac{\beta_i}{2}}(|\partial_i u|)\big[f_i''(\partial_i u)\big]^{-\frac{1}{2}}\Gamma^{-\frac{\beta_i}{2}}(|\partial_i u|) \\
&\quad \cdot\Gamma^{\frac{1}{2}+\gamma_i}(|\partial_i u|)|f_i'|(\partial_i u)\eta^{2k}\,\mathrm{d}x \\
&\le c\int_B f_i''(\partial_i u)|\partial_i\partial_i u|^2\Gamma^{\beta_i}(|\partial_i u|)\eta^{2k}\,\mathrm{d}x \\
&\quad + c\int_B \big[f_i''(\partial_i u)\big]^{-1}\Gamma^{1+\gamma_i-\sigma_i}(|\partial_i u|)|f_i'|^2(\partial_i u)\eta^{2k}\,\mathrm{d}x\,. \qquad (4.13)
\end{aligned}
$$

The first integral on the right-hand side of (4.13) already occurs in (4.12) and the second one is handled with (2.3) and Lemma 4.2 (recalling $\sigma_i > \theta_i$)

$$
\begin{aligned}
\int_B \big[f_i''(\partial_i u)\big]^{-1} & \Gamma^{1+\gamma_i-\sigma_i}(|\partial_i u|)|f_i'|^2(\partial_i u)\eta^{2k}\,\mathrm{d}x \\
& \le \int_B f_i(\partial_i u)\Gamma^{1+\gamma_i-(\sigma_i-\theta_i)}(|\partial_i u|)\eta^{2k}\,\mathrm{d}x \\
& \le \int_B \Big[f_i(\partial_i u)\Gamma^{1+\gamma_i}(|\partial_i u|)\Big]^{\frac{1}{\rho}}\eta^{\frac{2k}{\rho}}\eta^{\frac{2k}{\rho*}}\,\mathrm{d}x + c \\
& \le \varepsilon\int_B f_i(\partial_i u)\Gamma^{1+\gamma_i}(|\partial_i u|)\eta^{2k}\,\mathrm{d}x + c(\varepsilon, r) \qquad (4.14)
\end{aligned}
$$

and once more the integral on the right-hand side is absorbed.

To sum up, (4.12) implies with the help of (4.13) and (4.14) for $i = 1, \dots, n$

$$
\begin{aligned}
\int_B & f_i(\partial_i u)\Gamma^{1+\gamma_i}(|\partial_i u|)\eta^{2k}\,\mathrm{d}x \\
& \le c\Big[1 + \int_B f_i''(\partial_i u)|\partial_i\partial_i u|^2\Gamma^{\beta_i}(|\partial_i u|)\eta^{2k}\,\mathrm{d}x\Big]. \qquad (4.15)
\end{aligned}
$$

Discussing the right-hand side of (4.15) we apply Lemma 4.1, where we let $f(Z) = \sum_{j=1}^n f_j(Z_j)$ and fix $i \in \{1, \dots, n\}$:

$$\begin{aligned}\int_B f_i''(\partial_i u)|\partial_i\partial_i u|^2 \Gamma^{\beta_i}(|\partial_i u|)\eta^{2k}\,\mathrm{d}x & \\ &\leq c\int_B D^2 f(\nabla u)\big(\partial_i \nabla u, \partial_i \nabla u\big)\Gamma^{\beta_i}(|\partial_i u|)\eta^{2k}\,\mathrm{d}x\\ &\leq c\int_B D^2 f(\nabla u)\big(\nabla\eta, \nabla\eta\big)\Gamma^{1+\beta_i}(|\partial_i u|)\eta^{2k-2}\,\mathrm{d}x\\ &\leq c(r)\sum_{j=1}^n \int_B f_j''(\partial_j u)\Gamma^{1+\beta_i}(|\partial_i u|)\eta^{2k-2}\,\mathrm{d}x\,. \end{aligned}\tag{4.16}$$

For $j = i$ on the right-hand side of (4.16) we now apply the left-hand side of (2.2) and again Lemma 4.2 with the result (recall $\theta_i, \sigma_i < 1/2$)

$$\begin{aligned}\int_B f_i''(\partial_i u)\Gamma^{1+\beta_i}(|\partial_i u|)\eta^{2k-2}\,\mathrm{d}x & \\ &\leq \int_B f_i(\partial_i u)\Gamma^{\gamma_i+\theta_i+\sigma_i}(|\partial_i u|)\eta^{2k-2}\,\mathrm{d}x\\ &\leq \int_B \Big[f_i(\partial_i u)\Gamma^{1+\gamma_i}(|\partial_i u|)\Big]^{\frac{1}{\rho}}\eta^{\frac{2k}{\rho}}\eta^{\frac{2k}{\rho^*}-2}\,\mathrm{d}x + c\\ &\leq \varepsilon\int_B f_i(\partial_i u)\Gamma^{1+\gamma_i}(|\partial_i u|)\eta^{2k}\,\mathrm{d}x + c(\varepsilon, r)\,. \end{aligned}\tag{4.17}$$

Note that the integral on the right-hand side of (4.17) can be absorbed in the left-hand side of (4.15). This proves Proposition 4.2. □

5 Iteration

Step 1 - preliminaries. We start with an elementary proposition recalling and relating the relevant parameters of the problem.

Proposition 5.1 *With $q_i^{\pm}$, $\underline{q}_i$, $\overline{q}_i$, θ_i, $-1/2 < \beta_i = \gamma_i + \sigma_i$, $i = 1, \dots, n$, as above we further let*

$$\omega_i^{\pm} := \frac{q_i^{\pm}}{2} + \gamma_i\,, \quad i \in \{1, \dots, n\}\,.$$

W.l.o.g. (since otherwise the claim (5.3) trivially holds on account of $\sigma_j < 1/2$) we assume that we have for $j \in \{1, \dots, n\}$

$$\overline{q}_j > 2(1-\theta_j)\,.\tag{5.1}$$

We fix $\tau \geq 0$, $i, j \in \{1, \dots, n\}$ and suppose in addition to (4.7) that γ_i (and β_i) are given such that

$$1 + \gamma_i < \frac{\underline{q}_i\big(1-\theta_j\big)}{2}\frac{2+\tau}{\overline{q}_j - 2\big(1-\theta_j\big)} - \sigma_i\big(1-\theta_j\big)\frac{\tau + \frac{\overline{q}_j}{(1-\theta_j)}}{\overline{q}_j - 2\big(1-\theta_j\big)}\tag{5.2}$$

This yields (for any combination of $q_j^\pm$ and $q_i^\pm$, recall $\sigma_i < 1/2$, hence $\omega_i^\pm - \beta_i > 0$)

$$q_j^\pm \frac{1+\beta_i}{\omega_i^\pm - \beta_i} < 2\left(1-\theta_j\right) \frac{1 + \frac{q_i^\pm}{2} + \gamma_i}{\omega_i^\pm - \beta_i} + \tau\left(1-\theta_j\right) . \tag{5.3}$$

Proof We note that

$$1+\gamma_i < \frac{\underline{q}_i\left(1-\theta_j\right)}{2} \frac{2+\tau}{\overline{q}_j - 2\left(1-\theta_j\right)} - \sigma_i\left(1-\theta_j\right) \frac{\tau + \overline{q}_j/\left(1-\theta_j\right)}{\overline{q}_j - 2\left(1-\theta_j\right)} ,$$

is equivalent to

$$(1+\gamma_i)\left[\overline{q}_j - 2\left(1-\theta_j\right)\right] < \underline{q}_i\left(1-\theta_j\right) + \tau\left[\frac{\underline{q}_i\left(1-\theta_j\right)}{2} - \sigma_i\left(1-\theta_j\right)\right] - \sigma_i \overline{q}_j .$$

Writing this in the form

$$\overline{q}_j(1+\beta_i) < 2\left(1-\theta_j\right)\left[1+\gamma_i+\frac{\underline{q}_i}{2}\right] + \tau\left(1-\theta_j\right)\left[\frac{\underline{q}_i}{2} - \sigma_i\right]$$

and recalling that we have by definition $\omega_i^\pm - \beta_i = (q_i^\pm/2) - \sigma_i$ we obtain as an equivalent inequality

$$\overline{q}_j \frac{1+\beta_i}{\omega_i^\pm - \beta_i} < 2\left(1-\theta_j\right) \frac{1+\frac{\underline{q}_i}{2}+\gamma_i}{\omega_i^\pm - \beta_i} + \tau\left(1-\theta_j\right) \frac{\underline{q}_i - 2\sigma_i}{q_i^\pm - 2\sigma_i} . \square$$

Up to now no relation between q_i^+ and q_i^- was needed due to our particular Ansatz depending on t instead of $|t|$.

Step 2 - main inequality. To complete the proofs of Theorem 1.1 and Theorem 2.1 it remains to handle the mixed terms on the right-hand side of (4.8). Here, of course, it is no longer possible to argue with the structure conditions for fixed i, i.e. to argue with $q_i^\pm$ separated from each other in disjoint regions.

Throughout the rest of this section we suppose that the assumptions of Theorem 2.1 are satisfied.

Consider a set $U \subset \Omega$ and a C^1-function $v\colon \Omega \to \mathbb{R}$. We let for any $i \in \{1, \ldots, n\}$

$$U \cap [\partial_i v \geq 0] =: U_i^+[v] =: U_i^+ , \qquad U \cap [\partial_i v < 0] =: U_i^-[v] =: U_i^- ,$$

in particular U can be written as the disjoint union

$$U = U_i^+ \cup U_i^-$$

for every $1 \leq i \leq n$.

Using this notation, recalling Proposition 4.2 and the left-hand side of (2.2) we have for every $1 \leq i \leq n$

$$\begin{aligned} &\int_B f_i(\partial_i u)\Gamma^{1+\gamma_i}(|\partial_i u|)\eta^{2k}\,\mathrm{d}x \\ &\quad \leq c\left[1 + \sum_{j\neq i}\int_B f_j''(\partial_j u)\Gamma^{1+\beta_i}(|\partial_i u|)\eta^{2k-2}\,\mathrm{d}x\right] \\ &\quad \leq c\left[1 + \sum_{j\neq i}\int_B f_j(\partial_j u)\Gamma^{\theta_j-1}(|\partial_j u|)\Gamma^{1+\beta_i}(|\partial_i u|)\eta^{2k-2}\,\mathrm{d}x\right]. \end{aligned} \tag{5.4}$$

Fix $i \in \{1, \dots, n\}$ and let

$$\kappa_i^\pm = \frac{1+\omega_i^\pm}{1+\beta_i}\,, \qquad \hat{\kappa}_i^\pm = \frac{1+\omega_i^\pm}{\omega_i^\pm - \beta_i}\,.$$

For the choice of exponents we observe

$$1+\omega_i^\pm > 1+\beta_i \quad \Leftrightarrow \quad \frac{q_i^\pm}{2} > \sigma_i$$

which follows from $\sigma_i < 1/2$.

We obtain for fixed $1 \le i \le n$ and for $\varepsilon > 0$ sufficiently small (note that the ball B is divided into two parts w.r.t. the function $\partial_i u$)

$$\begin{aligned}
&\sum_{j\neq i} \int_B f_j(\partial_j u)\Gamma^{\theta_j - 1}(|\partial_j u|)\Gamma^{1+\beta_i}(|\partial_i u|)\eta^{2k-2}\,\mathrm{d}x \\
&\quad \le c \sum_{j\neq i}\sum_{\pm} \int_{B_i^\pm} \big(1+f_j(\partial_j u)\big)\Gamma^{\theta_j-1}(|\partial_j u|)\Gamma^{1+\beta_i}(|\partial_i u|)\eta^{2k-2}\,\mathrm{d}x \\
&\quad \le \sum_{j\neq i}\sum_{\pm} \Bigg[\varepsilon \int_{B_i^\pm} \Gamma(|\partial_i u|)^{1+\omega_i^\pm}\eta^{2k}\,\mathrm{d}x \\
&\quad + c(\varepsilon)\int_{B_i^\pm} \big(1+f_j(\partial_j u)\big)^{\frac{1+\omega_i^\pm}{\omega_i^\pm-\beta_i}} \Gamma^{(\theta_j-1)\frac{1+\omega_i^\pm}{\omega_i^\pm-\beta_i}}(|\partial_j u|)\,\mathrm{d}x\Bigg].
\end{aligned} \tag{5.5}$$

By (2.4) we have on $B_i^\pm$ for $|\partial_i u|$ sufficiently large $\Gamma(|\partial_i u|)^{q_i^\pm/2} \le c f_i(\partial_i u)$, hence by the definition of $\omega_i^\pm$

$$\varepsilon \sum_{j\neq i}\sum_{\pm}\int_{B_i^\pm} \Gamma(|\partial_i u|)^{1+\omega_i^\pm}\eta^{2k}\,\mathrm{d}x \le c(n-1)\varepsilon \int_B f_i(\partial_i u)\Gamma^{1+\gamma_i}(|\partial_i u|)\eta^{2k}\,\mathrm{d}x + c \tag{5.6}$$

and, as usual, the integral on the right-hand side can be absorbed in (5.4).

We will finally show with the help of an iteration procedure that for every $1 \le i \le n$

$$\sum_{j\neq i}\sum_{\pm}\int_{B_i^\pm} \big(1+f_j(\partial_j u)\big)^{\frac{1+\omega_i^\pm}{\omega_i^\pm-\beta_i}} \Gamma^{(\theta_j-1)\frac{1+\omega_i^\pm}{\omega_i^\pm-\beta_i}}(|\partial_j u|)\,\mathrm{d}x \le c\,, \tag{5.7}$$

which completes the proof of Theorem 2.1 since we have by (5.4), (5.5) and (5.6) for every $1 \le i \le n$

$$\begin{aligned}
&\int_B f_i(\partial_i u)\Gamma^{1+\gamma_i}(|\partial_i u|)\eta^{2k}\,\mathrm{d}x \\
&\qquad \le c\Bigg[1 + \sum_{j\neq i}\sum_{\pm}\int_{B_i^\pm} \big(1+f_j(\partial_j u)\big)^{\frac{1+\omega_i^\pm}{\omega_i^\pm-\beta_i}} \Gamma^{(\theta_j-1)\frac{1+\omega_i^\pm}{\omega_i^\pm-\beta_i}}(|\partial_j u|)\,\mathrm{d}x\Bigg].
\end{aligned} \tag{5.8}$$

In order to establish (5.7) let us suppose that (5.2) is true with a real number $\tau \ge 0$. Then we may apply Proposition 5.1 and (5.3) implies in the case $\overline{q}_j > 2\big(1-\theta_j\big)$ (recall

$f_j \approx c\Gamma^{q_j^\pm/2}$)

$$\Gamma^{-(1-\theta_j)\frac{1+\omega_i^\pm}{\omega_i^\pm-\beta_i}-(1-\theta_j)\frac{\tau}{2}}(|\partial_j u|) \le c\big(1+f_j(\partial_j u)\big)^{-\frac{1+\beta_i}{\omega_i^\pm-\beta_i}}$$
$$= c\big(1+f_j(\partial_j u)\big)^{1-\frac{1+\omega_i^\pm}{\omega_i^\pm-\beta_i}}.$$

Thus we obtain

$$\big(1+f_j(\partial_j u)\big)^{\frac{1+\omega_i^\pm}{\omega_i^\pm-\beta_i}}\Gamma^{(\theta_j-1)\frac{1+\omega_i^\pm}{\omega_i^\pm-\beta_i}}(|\partial_j u|) \le c\big(1+f_j(\partial_j u)\big)\Gamma^{(1-\theta_j)\frac{\tau}{2}}(|\partial_j u|). \tag{5.9}$$

We note that (5.9) is formulated uniformly w.r.t. the index j and the symbol $\pm$ is just related to $\partial_i u$.

In the case $\overline{q}_j \le 2\left(1-\theta_j\right)$ we have

$$(1+f_j(\partial_j u)) \le c\Gamma^{1-\theta_j}(|\partial_j u|),$$

hence

$$\big(1+f_j(\partial_j u)\big)^{\frac{1+\omega_i^\pm}{\omega_i^\pm-\beta_i}}\Gamma^{(\theta_j-1)\frac{1+\omega_i^\pm}{\omega_i^\pm-\beta_i}}(|\partial_j u|) \le c$$

and (5.9) holds for any $\tau \ge 0$.

Inequality (5.9) is the main tool for the iteration procedure leading to the claim (5.7).

The strategy is the following: we start with $i = 1$ and use (5.2) with an appropriate real number γ_1 which gives (5.9) for $i = 1$ and $2 \le j \le n$. In this first step $\tau = 0$ is chosen such that we have a priori integrability on the right-hand side.

For $i = 2$ and $3 \le j \le n$ the same is done in the next step. We note that for $j = 1$ we may benefit from the integrability obtained before with $i = 1$, i.e. we may choose an appropriate $\tau > 0$ in (5.2).

The iteration is done with the case $i = n$.

Step 3 - proof of Theorem 1.1, (i). Let us start with the easiest case i) of Theorem 1.1, i.e. $n = 2$, $\theta_i = 0$, $i = 1, 2$.

On account of $\theta_1 = 0$ we may choose σ_1 arbitrarily small and (5.2) for $\overline{q}_2 > 2$ and $\tau = 0$ becomes with $\rho_1 > 1$

$$1+\gamma_1 =: \frac{1}{2}\rho_1 < \frac{\underline{q}_1}{\overline{q}_2-2}. \tag{5.10}$$

We note that (5.10) is satisfied with some $\rho_1 > 1$ if we have

$$\overline{q}_2 < 2\underline{q}_1 + 2,$$

which corresponds to our assumption (1.14).

This implies by (5.8) and (5.9)

$$\int_B f_1(\partial_1 u)\Gamma^{\frac{\rho_1}{2}}(|\partial_1 u|)\eta^{2k}\,\mathrm{d}x \le c.$$

As a consequence, in the second step we consider (5.2) with

$$\tau < \rho_1 < 2\frac{\underline{q}_1}{\overline{q}_2-2}.$$

This leads to the condition ($\rho_2 > 1$)

$$\frac{1}{2} < 1 + \gamma_2 =: \frac{1}{2}\rho_2 < \frac{\underline{q}_2}{2}\frac{2+\rho_1}{\overline{q}_1 - 2}\,. \tag{5.11}$$

This condition is satisfied for an appropriate ρ_2 if we suppose that

$$\overline{q}_1 < \underline{q}_2[2+\rho_1] + 2$$

with ρ_1 satisfying (5.10). This proves Theorem 1.1 *i*).

Step 4 - iteration and proofs of the theorems.

We first note that we have (5.2) with

$$\gamma_i + \sigma_i = \beta_i$$

for β_i sufficiently close to $-1/2$ if

$$\overline{q}_j < (2+\tau)\underline{q}_i(1-\theta_j) + 2(1-\theta_j)(1-(2+\tau)\sigma_i)\,.$$

and for $\sigma_i > \theta_i$ sufficiently close to θ_i we are led to

$$\overline{q}_j < (2+\tau)\underline{q}_i(1-\theta_j) + 2(1-\theta_j)(1-(2+\tau)\theta_i)\,. \tag{5.12}$$

<u>$i = 1$.</u>

Again with β_i sufficiently close to $-1/2$ (here for $i = 1$) and choosing $\sigma_i > \theta_i$ sufficiently close to θ_i ($i = 1$), (5.2) is valid with the choice $\tau = 0$ if we have (recall (5.12))

$$\overline{q}_j < (1-\theta_j)\left[2\left(\underline{q}_i - 2\theta_i\right) + 2\right] \qquad \text{for all} \qquad 2 \le j \le n\,, \tag{5.13}$$

i.e. in the particular case $i = 1$ under consideration

$$\overline{q}_j < \left(1-\theta_j\right)\left[2\left(\underline{q}_1 - 2\theta_1\right) + 2\right] \qquad \text{for all} \qquad 2 \le j \le n\,,$$

and this is just assumption (2.5) for $i = 1$.

From (5.2) we deduce (5.9) for $i = 1$, $\tau = 0$, and (5.7) follows from (5.9) for $i = 1$ and for all $2 \le j \le n$ with the choice $\tau = 0$

$$\begin{aligned}
&\sum_{j\neq 1}\sum_{\pm}\int_{B_i^{\pm}} \left(1 + f_j(\partial_j u)\right)^{\frac{1+\omega_1^{\pm}}{\omega_1^{\pm}-\beta_1}} \Gamma^{(\theta_j - 1)\frac{1+\omega_1^{\pm}}{\omega_1^{\pm}-\beta_1}}(|\partial_j u|)\,\mathrm{d}x \\
&\le c\int_B \left(1 + f_j(\partial_j u)\right)\mathrm{d}x \le c\,,
\end{aligned} \tag{5.14}$$

Returning to (5.8) we insert (5.14) which yields (w.l.o.g. $1 + \gamma_1 = 1 + \beta_1 - \sigma_1 > \delta - \theta_1$ for some $\delta > 1/2$)

$$\int_B f_1(\partial_1 u)\Gamma^{\delta-\theta_1}(|\partial_1 u|)\eta^{2k}\,\mathrm{d}x \le c \tag{5.15}$$

for some $\delta > 1/2$

<u>$1 < i \le n$.</u>

Suppose that we have (5.13) (again compare (2.5)) in the sense

$$\overline{q}_j < (1-\theta_j)\left[2\left(\underline{q}_i - 2\theta_i\right) + 2\right] \qquad \text{for} \qquad i+1 \le j \le n\,. \tag{5.16}$$

With the same argument leading to (5.14) we have for all $i+1 \le j \le n$

$$\sum_{j>i}\sum_{\pm}\int_{B_i^\pm}\big(1+f_j(\partial_j u)\big)^{\frac{1+\omega_i^\pm}{\omega_i^\pm-\beta_i}}\Gamma^{(\theta_j-1)\frac{1+\omega_i^\pm}{\omega_i^\pm-\beta_i}}(|\partial_j u|)\,\mathrm{d}x \le c. \tag{5.17}$$

Moreover, we suppose that by iteration we have (5.15) for $1 \le j < i$, i.e. by decreasing radii in this finite iteration, if necessary, we have w.l.o.g.

$$\int_B f_j(\partial_j u)\Gamma^{\delta-\theta_j}(|\partial_j u|)\,\mathrm{d}x \le c, \quad 1 \le j < i, \tag{5.18}$$

for some $\delta > 1/2$.

Then we return to (5.2) with the choice $\tau = (1-2\theta_j)/(1-\theta_j)$. For β_i sufficiently close to $-1/2$ and σ_i sufficiently close to θ_i we are lead to the condition (recall (5.12))

$$\overline{q}_j < \big(1-\theta_j\big)\Big[(2+\tau)\Big(\underline{q}_i - 2\theta_i\Big)+2\Big], \quad \tau = \frac{1-2\theta_j}{1-\theta_j}, \tag{5.19}$$

$1 \le j < i$, and (5.19) is just the assumption (2.6).

With (5.2) we again have (5.9), now with $\tau = (1-2\theta_j)/(1-\theta_j)$, hence

$$\begin{aligned}\sum_{j<i}\sum_{\pm}\int_{B_i^\pm}\big(1+f_j(\partial_j u)\big)^{\frac{1+\omega_i^\pm}{\omega_i^\pm-\beta_i}}\Gamma^{(\theta_j-1)\frac{1+\omega_i^\pm}{\omega_i^\pm-\beta_i}}(|\partial_j u|)\,\mathrm{d}x\\ \le c\sum_{j<i}\int_B\big(1+f_j(\partial_j u)\big)\Gamma^{\delta-\theta_j}(|\partial_j u|)\,\mathrm{d}x \le c\,,\end{aligned} \tag{5.20}$$

where the last estimate follows from (5.18) for some $\delta > 1/2$.

With (5.17) and (5.20) one has

$$\sum_{j\neq i}\sum_{\pm}\int_{B_i^\pm}\big(1+f_j(\partial_j u)\big)^{\frac{1+\omega_i^\pm}{\omega_i^\pm-\beta_i}}\Gamma^{(\theta_j-1)\frac{1+\omega_i^\pm}{\omega_i^\pm-\beta_i}}(|\partial_j u|)\,\mathrm{d}x \le c\,, \tag{5.21}$$

which exactly as in the case $i=1$ shows for some sufficiently small $\delta > 1/2$

$$\int_B f_i(\partial_i u)\Gamma^{\delta-\theta_i}(|\partial_i u|)\eta^{2k}\,\mathrm{d}x \le c\,, \tag{5.22}$$

hence with (5.22) we proceed one step in the iteration of (5.19). This completes the proof of Theorem 2.1. □

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Conflict of interest On behalf of all authors, the corresponding author states that there is no conflict of interest.

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