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Analytical study of a generalised Dirichlet–Neumann operator and application to three-dimensional water waves on Beltrami flows

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Abstract

We consider three-dimensional doubly periodic steady water waves with vorticity, under the action of gravity and surface tension; in particular we consider so-called Beltrami flows, for which the velocity field and the vorticity are collinear. We adapt a recent formulation of the corresponding problem for localised waves which involves a generalisation of the classical Dirichlet–Neumann operator. We study this operator in detail, extending some well-known results for the classical Dirichlet–Neumann operator, such as the Taylor expansion in homogeneous powers of the wave profile, the computation of its differential and the asymptotic expansion of its associated symbol. A new formulation of the problem as a single equation for the wave profile is also presented and discussed in a similar vein. As an application of these results we prove existence of doubly periodic gravity-capillary steady waves and construct approximate doubly periodic gravity steady waves.

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1. Introduction

This paper is concerned with three-dimensional doubly periodic steady water waves with vorticity, under the action of gravity and surface tension. Irrotational water waves have been studied extensively, both in two and three dimensions (see the survey paper by Haziot et al. [14] and references therein); fewer results are available for non-zero vorticity, although it may be significant for modelling the interaction of three-dimensional waves with non-uniform currents. We restrict ourselves to Beltrami fields, in which the velocity field $\boldsymbol{u}$ and the vorticity $\operatorname{curl} \boldsymbol{u}$ are collinear, so that $\operatorname{curl} \boldsymbol{u} = \alpha \boldsymbol{u}$; more precisely, we consider the so-called strong Beltrami fields, for which the proportionality factor α is a constant (this case appears to be the most relevant, since Enciso and Peralta-Salas [10] proved that Beltrami fields with non-constant proportionality factors are 'rare' in a topological sense).

The importance of Beltrami fields in the context of ideal fluids, and more precisely in the context of stationary Euler flows, was highlighted by Arnold [3] and Arnold and Khesin [4]: indeed, Arnold's structure theorem ensures that, under suitable technical assumptions, a smooth stationary solution to the three-dimensional Euler equation is either integrable or a Beltrami field. It is thus natural to expect that more complex dynamics (usually associated to turbulent flows in physical literature) in stationary fluids are related to Beltrami fields (see Monchaux et al. [22]). The dynamics of Beltrami fields, and in particular the dynamics of the so-called ABC flows, have been numerically studied by Hénon [15] and Dombre et al. [9]. Such studies lead to the conjecture that Beltrami fields should exhibit chaotic dynamics together with a positive measure set of invariant tori, much like the restriction to an energy level of a typical mechanical system with two degrees of freedom; recently Enciso, Peralta-Salas and Romaniega [11] proved that with probability one a random Beltrami field in $\mathbb{R}^3$ exhibits chaotic regions that coexist with invariant tori of complicated topology.

There has recently been some interest in variational formulations of the three-dimensional steady water-wave problem with relative velocities given by Beltrami fields. We mention a recent variational formulation by Lokharu and Wahlén [20] for doubly periodic waves which is valid under general assumptions on the wave profile (including for example the case of overhanging wave profiles). More recently, Groves and Horn [12] gave another variational formulation for localised waves (solitary waves) under the more classical assumption that the free surface is given by the graph of an unknown function η depending only on the horizontal directions. Their formulation, which can be considered as a generalisation of an alternative variational framework for three-dimensional irrotational water waves by Benjamin [5, §6.6], is not only more explicit, but it allows one to recover the classical Zakharov–Craig–Sulem formulation of steady water waves in the irrotational case $\alpha = 0$. Moreover, this formulation leads naturally to the definition of a generalised Dirichlet–Neumann operator $H(\eta)$ which reduces to the classical Dirichlet–Neumann operator in the irrotational case.

In this paper we perform an analytical study of the generalised Dirichlet–Neumann operator (whose definition is subtly different in the present context of doubly periodic waves) and of a related operator appearing in a new single equation formulation of the problem, extending some well-known results for the classical Dirichlet–Neumann operator, such as the Taylor expansion in homogeneous powers of the profile η by Craig and Sulem [8], the computation of its differential by Lannes [18, §3.3], and the asymptotic expansion of its associated symbol (see Alazard and Métivier [2, §2.4]).

As an application of the above results, we prove the existence of doubly periodic gravity-capillary waves by Lyapunov–Schmidt reduction, recovering a result recently given by Lokharu, Seth and Wahlén [19]. We also show how the reduction can be formally carried out in the absence of surface tension and thus compute approximate doubly periodic gravity waves in the form of formal power series. The failure of the Lyapunov–Schmidt reduction for gravity waves is due to the presence of small divisors when attempting to invert the relevant linear operator. This problem has been overcome for irrotational waves by Iooss and Plotnikov [16,17] using Nash-Moser theory; its treatment for Beltrami flows is deferred to a future article.

1.1. The hydrodynamic problem

We consider an incompressible inviscid fluid occupying a three-dimensional domain with flat bottom, under the action of gravity and surface tension. We study steady water waves, namely a fluid flow in which the velocity field and the free-surface profile are stationary with respect to a uniformly translating frame. In this moving frame, the fluid domain can be parametrized by

$$D_\eta := \{(\boldsymbol{x}', z) \in \mathbb{R}^2 \times \mathbb{R} \colon -h < z < \eta(\boldsymbol{x}')\},$$

so that the free surface is given by the graph of an unknown function $\eta \colon \mathbb{R}^2 \to (-h, \infty)$, and $h > 0$ is the depth of the fluid. We consider a so-called strong Beltrami flow, in which the velocity field $\boldsymbol{u} \colon \overline{D_\eta} \to \mathbb{R}^3$ and the vorticity $\operatorname{curl}\boldsymbol{u}$ are collinear, that is $\operatorname{curl}\boldsymbol{u} = \alpha\boldsymbol{u}$ for some constant α. The equations describing the flow are given by

$$\operatorname{div}\boldsymbol{u} = 0 \qquad \text{in } D_\eta, \tag{1.1}$$

$$\operatorname{curl}\boldsymbol{u} = \alpha\boldsymbol{u} \qquad \text{in } D_\eta, \tag{1.2}$$

$$\boldsymbol{u}\cdot\boldsymbol{e}_3 = 0 \qquad \text{at } z = -h, \tag{1.3}$$

$$\boldsymbol{u}\cdot\boldsymbol{n} = 0 \qquad \text{at } z = \eta, \tag{1.4}$$

$$\frac{1}{2}|\boldsymbol{u}|^2 + g\eta - \beta\left(\frac{\eta_x}{(1+|\nabla\eta|^2)^{1/2}}\right)_x - \beta\left(\frac{\eta_y}{(1+|\nabla\eta|^2)^{1/2}}\right)_y = \frac{1}{2}|\boldsymbol{c}|^2 \qquad \text{at } z = \eta, \tag{1.5}$$

where $\nabla\eta := (\eta_x, \eta_y)^T$, g is the acceleration due to gravity, β is the coefficient of surface tension, $\boldsymbol{c} := (c_1, c_2)^T$ is the wave velocity, $\boldsymbol{e}_3 := (0, 0, 1)^T$ and

$$\boldsymbol{n} := \frac{1}{1+|\nabla\eta|^2}\boldsymbol{N}, \qquad \boldsymbol{N} := (-\eta_x, -\eta_y, 1)^T$$

denotes the outward unit normal vector. We discuss doubly periodic solutions to (1.1)–(1.5), that is solutions which satisfy

$$\eta(\boldsymbol{x}' + \boldsymbol{\lambda}) = \eta(\boldsymbol{x}'), \qquad \boldsymbol{u}(\boldsymbol{x}' + \boldsymbol{\lambda}, z) = \boldsymbol{u}(\boldsymbol{x}', z)$$

for every $\boldsymbol{\lambda} \in \Lambda$, where Λ is the lattice given by

$$\Lambda := \{\boldsymbol{\lambda} = m_1\boldsymbol{\lambda}_1 + m_2\boldsymbol{\lambda}_2 \colon m_1, m_2 \in \mathbb{Z}\}$$

for two linearly independent vectors $\boldsymbol{\lambda}_1$, $\boldsymbol{\lambda}_2$. The functions η and $\boldsymbol{u}$ are therefore defined on the periodic domains $\mathbb{R}^2/\Lambda$ and (with a slight abuse of notation) D_η/Λ.

A 'trivial solution' of (1.1)–(1.5) is given by $(0, \boldsymbol{u}^\star)$, where $\boldsymbol{u}^\star$ is the two-parameter family of laminar flows

$$\begin{aligned}\boldsymbol{u}^\star &:= c_1\boldsymbol{u}^{(1)} + c_2\boldsymbol{u}^{(2)}, \quad c_1, c_2 \in \mathbb{R},\\ \boldsymbol{u}^{(1)} &:= (\cos(\alpha z), -\sin(\alpha z), 0)^T,\\ \boldsymbol{u}^{(2)} &:= (\sin(\alpha z), \cos(\alpha z), 0)^T.\end{aligned}$$

We consider solutions $(\eta, \boldsymbol{u})$ of (1.1)–(1.5) which are small perturbations of $(0, \boldsymbol{u}^\star)$; setting $\boldsymbol{v} = \boldsymbol{u} - \boldsymbol{u}^\star$ and representing the velocity field $\boldsymbol{v}$ by a solenoidal vector potential $\boldsymbol{A}$, we seek solutions $(\eta, \boldsymbol{A})$ of the equations

$$\operatorname{div} \boldsymbol{A} = 0 \qquad \text{in } D_\eta, \tag{1.6}$$

$$\operatorname{curl}\operatorname{curl} \boldsymbol{A} = \alpha \operatorname{curl} \boldsymbol{A} \qquad \text{in } D_\eta, \tag{1.7}$$

$$\boldsymbol{A} \times \boldsymbol{e}_3 = \boldsymbol{0} \qquad \text{at } z = -h, \tag{1.8}$$

$$\boldsymbol{A} \cdot \boldsymbol{n} = 0 \qquad \text{at } z = \eta, \tag{1.9}$$

$$\operatorname{curl} \boldsymbol{A} \cdot \boldsymbol{n} + \boldsymbol{u}^\star \cdot \boldsymbol{n} = 0 \qquad \text{at } z = \eta, \tag{1.10}$$

$$\frac{1}{2}|\operatorname{curl} \boldsymbol{A}|^2 + \operatorname{curl} \boldsymbol{A} \cdot \boldsymbol{u}^\star + g\eta - \beta\left(\frac{\eta_x}{(1+|\nabla\eta|^2)^{1/2}}\right)_x - \beta\left(\frac{\eta_y}{(1+|\nabla\eta|^2)^{1/2}}\right)_y = 0 \text{ at } z = \eta. \tag{1.11}$$

Note that (1.1)–(1.3) are implied by (1.6)–(1.8), while (1.4), (1.5) are equivalent to (1.10), (1.11); furthermore $\boldsymbol{u}^\star = \operatorname{curl} \boldsymbol{A}^\star$, where

$$\begin{aligned}\boldsymbol{A}^\star &:= \frac{c_1}{\alpha}\boldsymbol{A}^{(1)} + \frac{c_2}{\alpha}\boldsymbol{A}^{(2)},\\ \boldsymbol{A}^{(1)} &:= (\cos(\alpha z) - 1, -\sin(\alpha z), 0)^T,\\ \boldsymbol{A}^{(2)} &:= (\sin(\alpha z), \cos(\alpha z) - 1, 0)^T.\end{aligned}$$

Remark 1.1. In the irrotational case $\alpha = 0$ we can write $\operatorname{curl} \boldsymbol{A} = \operatorname{grad} \varphi$ for a scalar potential φ, so that (1.6)–(1.11) becomes the classical steady water-wave problem

$$\Delta\varphi = 0 \qquad \text{in } D_\eta,$$

$$\partial_n\varphi = 0 \qquad \text{at } z = -h,$$

$$(1+|\nabla\eta|^2)^{\frac{1}{2}}\partial_n\varphi = \boldsymbol{c} \cdot \nabla\eta \qquad \text{at } z = \eta,$$

$$\frac{1}{2}|\operatorname{grad}\varphi|^2 + \boldsymbol{c} \cdot (\varphi_x, \varphi_y)^T + g\eta - \beta\left(\frac{\eta_x}{(1+|\nabla\eta|^2)^{1/2}}\right)_x - \beta\left(\frac{\eta_y}{(1+|\nabla\eta|^2)^{1/2}}\right)_y = 0 \text{ at } z = \eta.$$

1.2. The formulation

Let $\boldsymbol{F} = (F_1, F_2, F_3)^T$ be a three-dimensional vector field, and denote by $\boldsymbol{F}_\mathrm{h} = (F_1, F_2)^T$ its horizontal component and by $\boldsymbol{F}_\parallel = \boldsymbol{F}_\mathrm{h} + F_3 \nabla\eta|_{z=\eta}$ the horizontal component of its tangential part at $z = \eta$. Let $\boldsymbol{f} = (f_1, f_2)^T$ be a two-dimensional vector field and write $\boldsymbol{f}^\perp = (f_2, -f_1)^T$. According to the Hodge–Weyl decomposition for doubly periodic vector fields on $\mathbb{R}^2$ (see Majda and Bertozzi [21, Proposition 1.18]) we have

$$\boldsymbol{f} = \boldsymbol{\gamma} + \nabla\Phi + \nabla^\perp\Psi, \tag{1.12}$$
$$\boldsymbol{\gamma} := \langle \boldsymbol{f} \rangle, \quad \Phi := \Delta^{-1}(\nabla \cdot \boldsymbol{f}), \quad \Psi := \Delta^{-1}(\nabla^\perp \cdot \boldsymbol{f}),$$

where $\langle \boldsymbol{f} \rangle$ denotes the mean value of $\boldsymbol{f}$ over one periodic cell, $\nabla := (\partial_x, \partial_y)^T$, $\nabla^\perp := (\partial_y, -\partial_x)^T$ and Δ^{-1} is the two-dimensional periodic Newtonian potential.

Equations (1.6)–(1.11) can be reformulated in terms of η and the mean-value and gradient-potential parts of $(\operatorname{curl}\boldsymbol{A})_\parallel$ using the following procedure. Fix $\boldsymbol{\gamma}$ and Φ, let $\boldsymbol{A}$ be the unique solution of the boundary-value problem

$$\operatorname{div}\boldsymbol{A} = 0 \quad \text{in } D_\eta, \tag{1.13}$$
$$\operatorname{curl}\operatorname{curl}\boldsymbol{A} = \alpha \operatorname{curl}\boldsymbol{A} \quad \text{in } D_\eta, \tag{1.14}$$
$$\boldsymbol{A} \times \boldsymbol{e}_3 = \boldsymbol{0} \quad \text{at } z = -h, \tag{1.15}$$
$$\boldsymbol{A} \cdot \boldsymbol{n} = 0 \quad \text{at } z = \eta, \tag{1.16}$$
$$(\operatorname{curl}\boldsymbol{A})_\parallel = \boldsymbol{\gamma} + \nabla\Phi - \alpha\nabla^\perp\Delta^{-1}(\nabla \cdot \boldsymbol{A}_\parallel^\perp) \quad \text{at } z = \eta, \tag{1.17}$$

and define the *generalised Dirichlet–Neumann operator* by the formula

$$H(\eta)(\boldsymbol{\gamma}, \Phi) := \operatorname{curl}\boldsymbol{A} \cdot \boldsymbol{N}|_{z=\eta} = \nabla \cdot \boldsymbol{A}_\parallel^\perp. \tag{1.18}$$

(Note that $\Psi = \Delta^{-1}(\nabla^\perp \cdot (\operatorname{curl}\boldsymbol{A})_\parallel)$ is necessarily given by $\Psi = -\alpha\,\Delta^{-1}(\nabla \cdot \boldsymbol{A}_\parallel^\perp)$ because

$$\begin{aligned}\Psi &= -\Delta^{-1}(\nabla \cdot \operatorname{curl}\boldsymbol{A}_\parallel^\perp) = -\Delta^{-1}(\operatorname{curl}\operatorname{curl}\boldsymbol{A} \cdot \boldsymbol{N}\big|_{z=\eta}) = -\alpha\,\Delta^{-1}(\operatorname{curl}\boldsymbol{A} \cdot \boldsymbol{N}\big|_{z=\eta}) \\ &= -\alpha\,\Delta^{-1}(\nabla \cdot \boldsymbol{A}_\parallel^\perp),\end{aligned} \tag{1.19}$$

in which the vector identity $\operatorname{curl}\boldsymbol{F} \cdot \boldsymbol{N}\big|_{z=\eta} = \nabla \cdot \boldsymbol{F}_\parallel^\perp$ has been used.)

Proposition 1.2. *Equations* (1.10) *and* (1.11) *are equivalent to*

$$H(\eta)(\boldsymbol{\gamma}, \Phi) + \boldsymbol{u}^\star \cdot \boldsymbol{N}|_{z=\eta} = 0, \tag{1.20}$$

$$\begin{aligned}&\frac{1}{2}|\boldsymbol{K}(\eta)(\boldsymbol{\gamma}, \Phi)|^2 - \frac{(H(\eta)(\boldsymbol{\gamma}, \Phi) + \boldsymbol{K}(\eta)(\boldsymbol{\gamma}, \Phi)\cdot\nabla\eta)^2}{2(1 + |\nabla\eta|^2)} \\ &\qquad + \boldsymbol{K}(\eta)(\boldsymbol{\gamma}, \Phi) \cdot \boldsymbol{u}_\mathrm{h}^\star|_{z=\eta} + g\eta - \beta\left(\frac{\eta_x}{(1 + |\nabla\eta|^2)^{1/2}}\right)_x - \beta\left(\frac{\eta_y}{(1 + |\nabla\eta|^2)^{1/2}}\right)_y = 0,\end{aligned} \tag{1.21}$$

where

$$\boldsymbol{K}(\eta)(\boldsymbol{\gamma}, \Phi) := \boldsymbol{\gamma} + \nabla\Phi - \alpha\nabla^{\perp}\Delta^{-1}(H(\eta)(\boldsymbol{\gamma}, \Phi)).$$

This proposition, which is established by an elementary calculation, shows that the mathematical problem reduces to solving (1.20) and (1.21) for η and Φ (with an arbitrary choice of $\boldsymbol{\gamma}$); the velocity field $\boldsymbol{v} = \operatorname{curl} \boldsymbol{A}$ is recovered by solving (1.13)–(1.17). The method was first given in the context of solitary waves (with a slightly different Hodge–Weyl decomposition for spatially extended functions) by Groves and Horn [12]; note however the spurious extra term in the statement of the equations in that reference.

Remark 1.3. In the irrotational case $\alpha = 0$ one finds that $\operatorname{curl} \boldsymbol{A} = \operatorname{grad} \varphi$, where φ is the unique harmonic function such that $\varphi_n|_{z=-h} = 0$ and $\varphi|_{z=\eta} = \Phi$, so that $\boldsymbol{\gamma} = \boldsymbol{0}$ (because $(\operatorname{grad} \phi)_\parallel = \nabla(\phi|_{z=\eta})$) and

$$H(\eta)(\boldsymbol{0}, \Phi) = \nabla\varphi \cdot \boldsymbol{N}|_{z=\eta} = G(\eta)\Phi,$$

where $G(\eta)$ is the classical Dirichlet–Neumann operator. Furthermore, equations (1.20), (1.21) reduce to

$$\begin{aligned} &G(\eta)\Phi + \boldsymbol{c} \cdot \nabla\eta = 0, \\ &\tfrac{1}{2}|\nabla\Phi|^2 - \frac{(G(\eta)\Phi + \nabla\eta \cdot \nabla\Phi)^2}{2(1+|\nabla\eta|^2)} - \boldsymbol{c} \cdot \nabla\Phi + g\eta \\ &\qquad - \beta\left(\frac{\eta_x}{(1+|\nabla\eta|^2)^{\frac{1}{2}}}\right)_x - \beta\left(\frac{\eta_y}{(1+|\nabla\eta|^2)^{1/2\frac{1}{2}}}\right)_y = 0, \end{aligned}$$

so that we recover the Zakharov–Craig–Sulem formulation of the steady water-wave problem (see Zakharov [27] and Craig and Sulem [8]).

We proceed by specialising to $\boldsymbol{\gamma} = \boldsymbol{0}$, writing $\boldsymbol{c} = \boldsymbol{c}_0 + \boldsymbol{\mu}$, where $\boldsymbol{c}_0 = (c_{10}, c_{20})^T$ is a reference wave velocity to be chosen later, so that

$$\boldsymbol{u}^\star = (c_{10} + \mu_1)\boldsymbol{u}^{(1)} + (c_{20} + \mu_2)\boldsymbol{u}^{(2)},$$

and reducing equations (1.20), (1.21) to a single equation for η (see Oliveras and Vasan [24] for a derivation of the corresponding single-equation formulation for irrotational water waves). Eliminating Φ from (1.21) using (1.20), we find that

$$\begin{aligned} J(\eta, \boldsymbol{\mu}) := \frac{1}{2}|\boldsymbol{T}(\eta)|^2 - \frac{(-\underline{\boldsymbol{u}}^\star \cdot \boldsymbol{N} + \boldsymbol{T}(\eta) \cdot \nabla\eta)^2}{2(1+|\nabla\eta|^2)} + \boldsymbol{T}(\eta) \cdot \underline{\boldsymbol{u}}_{\mathrm{h}}^\star + g\eta \\ - \beta\left(\frac{\eta_x}{(1+|\nabla\eta|^2)^{1/2}}\right)_x - \beta\left(\frac{\eta_y}{(1+|\nabla\eta|^2)^{1/2}}\right)_y = 0, \end{aligned}$$

where

$$\boldsymbol{T}(\eta) := -\nabla\left(H(\eta)(\mathbf{0},\cdot)^{-1}(\underline{\boldsymbol{u}}^\star \cdot \boldsymbol{N})\right) + \alpha\,\nabla^\perp \Delta^{-1}(\underline{\boldsymbol{u}}^\star \cdot \boldsymbol{N})$$

and the underscore denotes evaluation at $z = \eta$.

Remark 1.4. Let S_0 be the reflection

$$S_0\eta(\boldsymbol{x}') := \eta(-\boldsymbol{x}'),$$

and $T_{\boldsymbol{v}'}$ be the translation

$$T_{\boldsymbol{v}'}\eta(\boldsymbol{x}') := \eta(\boldsymbol{x}' + \boldsymbol{v}').$$

The mapping J is equivariant with respect to both S_0 and $T_{\boldsymbol{v}'}$, that is

$$J(T_{\boldsymbol{v}'}\eta, \boldsymbol{\mu}) = T_{\boldsymbol{v}'}J(\eta, \boldsymbol{\mu}), \qquad J(S_0\eta, \boldsymbol{\mu}) = S_0 J(\eta, \boldsymbol{\mu}).$$

The operator $\boldsymbol{T}(\eta)$ can be defined more rigorously in terms of a boundary-value problem. Noting that $\underline{\boldsymbol{u}}^\star \cdot \boldsymbol{N} = \nabla \cdot \boldsymbol{S}(\eta)^\perp$, where

$$\boldsymbol{S}(\eta) := \frac{c_1}{\alpha}\begin{pmatrix}\cos(\alpha\,\eta) - 1\\ -\sin(\alpha\,\eta)\end{pmatrix} + \frac{c_2}{\alpha}\begin{pmatrix}\sin(\alpha\,\eta)\\ \cos(\alpha\,\eta) - 1\end{pmatrix}, \tag{1.22}$$

we can define

$$\boldsymbol{T}(\eta) := \boldsymbol{M}(\eta)(\mathbf{0}, \boldsymbol{S}(\eta)),$$

where

$$\boldsymbol{M}(\eta)(\boldsymbol{\gamma}, \boldsymbol{g}) := -(\operatorname{curl}\boldsymbol{B})_\parallel,$$

and $\boldsymbol{B}$ solves the boundary-value problem

$$\operatorname{curl}\operatorname{curl}\boldsymbol{B} = \alpha\operatorname{curl}\boldsymbol{B} \qquad \text{in } D_\eta, \tag{1.23}$$

$$\operatorname{div}\boldsymbol{B} = 0 \qquad \text{in } D_\eta, \tag{1.24}$$

$$\boldsymbol{B} \times \boldsymbol{e}_3 = \mathbf{0} \qquad \text{at } z = -h, \tag{1.25}$$

$$\boldsymbol{B} \cdot \boldsymbol{n} = 0 \qquad \text{at } z = \eta, \tag{1.26}$$

$$\nabla \cdot \boldsymbol{B}_\parallel^\perp = \nabla \cdot \boldsymbol{g}^\perp \qquad \text{at } z = \eta, \tag{1.27}$$

$$\langle (\operatorname{curl}\boldsymbol{B})_\parallel \rangle = \boldsymbol{\gamma}. \tag{1.28}$$

Any solution to this boundary-value problem satisfies

$$(\operatorname{curl}\boldsymbol{B})_\parallel = \boldsymbol{\gamma} + \nabla\Phi - \alpha\nabla^\perp\Delta^{-1}(\nabla \cdot \boldsymbol{B}_\parallel^\perp)$$

for some Φ (see equation (1.19)), so that $\Phi = H(\eta)(\boldsymbol{\gamma}, \cdot)^{-1}\nabla \cdot \boldsymbol{g}^\perp$ and

$$-(\operatorname{curl} \boldsymbol{B})_\| = -\boldsymbol{\gamma} - \nabla(H(\eta)(\boldsymbol{\gamma}, \cdot)^{-1}\nabla \cdot \boldsymbol{g}^\perp) + \alpha\nabla^\perp\Delta^{-1}(\nabla \cdot \boldsymbol{g}^\perp).$$

A rigorous treatment of the boundary-value problems (1.13)–(1.17) and (1.23)–(1.28) is given in Section 2.1 using a traditional weak/strong-solution approach.

1.3. Analytical results for the operators H and M

We write functions $f: \mathbb{R}^2/\Lambda \to \mathbb{R}$ as Fourier series

$$f(\boldsymbol{x}') = \sum_{\boldsymbol{k}\in\Lambda'} \hat{f}_{\boldsymbol{k}} \mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}'},$$

where Λ' is the dual lattice to Λ; the Fourier coefficients $\hat{f}_{\boldsymbol{k}}$ are given by

$$\hat{f}_{\boldsymbol{k}} = \frac{1}{|\Omega|}\int_\Omega f(\boldsymbol{x}')\mathrm{e}^{-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}'}\,\mathrm{d}\boldsymbol{x}',$$

where Ω is the parallelogram built with $\boldsymbol{\lambda}_1, \boldsymbol{\lambda}_2$. We write $\boldsymbol{k} = (k_1, k_2)^T$ and work in the Sobolev spaces

$$H^s(\mathbb{R}^2/\Lambda) := \left\{ f \in L^2(\mathbb{R}^2/\Lambda) : \|f\|_s^2 := \sum_{\boldsymbol{k}\in\Lambda'} \left(1+|\boldsymbol{k}|^2\right)^s |\hat{f}_{\boldsymbol{k}}|^2 < \infty \right\}, \qquad s \geq 0,$$

and their subspaces

$$\mathring{H}^s(\mathbb{R}^2/\Lambda) := \{ f \in H^s(\mathbb{R}^2) : \hat{f}_{\boldsymbol{0}} = 0 \}$$

of functions with zero mean, noting that the Hodge–Weyl decomposition (1.12) of a function $\boldsymbol{f} \in H^s(\mathbb{R}^2/\Lambda)^2$ is given by

$$\mathbb{R}^2 \ni \boldsymbol{\gamma} = (\hat{f}_{1\boldsymbol{0}}, \hat{f}_{2\boldsymbol{0}})^T,$$

$$\mathring{H}^{s+1}(\mathbb{R}^2/\Lambda)^2 \ni \Phi = -\sum_{\substack{\boldsymbol{k}\in\Lambda' \\ \boldsymbol{k}\neq\boldsymbol{0}}} \left(\frac{\mathrm{i}k_1\hat{f}_{1\boldsymbol{k}} + \mathrm{i}k_2\hat{f}_{2\boldsymbol{k}}}{|\boldsymbol{k}|^2} \right) \mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}'},$$

$$\mathring{H}^{s+1}(\mathbb{R}^2/\Lambda)^2 \ni \Psi = -\sum_{\substack{\boldsymbol{k}\in\Lambda' \\ \boldsymbol{k}\neq\boldsymbol{0}}} \left(\frac{\mathrm{i}k_2\hat{f}_{1\boldsymbol{k}} - \mathrm{i}k_1\hat{f}_{2\boldsymbol{k}}}{|\boldsymbol{k}|^2} \right) \mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}'}.$$

In Section 2.2 we show that the solutions to the boundary-value problems (1.13)–(1.17) and (1.23)–(1.28) depend analytically upon η and use this result to deduce that the same is true of $H(\eta)$ and $\boldsymbol{M}(\eta)$. We proceed by 'flattening' the fluid domain by means of the transformation $\Sigma: D_0 \to D_\eta$ given by

$$\Sigma\colon (\boldsymbol{x}', v) \mapsto (\boldsymbol{x}', v + \sigma(\boldsymbol{x}', v)), \qquad \sigma(\boldsymbol{x}', v) := \eta(\boldsymbol{x}')(1 + v/h)$$

which transforms the boundary-value problems for $\boldsymbol{A}$ and $\boldsymbol{B}$ into equivalent problems for $\tilde{\boldsymbol{A}} := \boldsymbol{A} \circ \Sigma$ and $\tilde{\boldsymbol{B}} := \boldsymbol{B} \circ \Sigma$ in the fixed domain D_0 (equations (2.16)–(2.20) and (2.21)–(2.26) respectively). The spatially extended version of the boundary-value problem for $\tilde{\boldsymbol{A}}$ was studied by Groves and Horn [12, §4] under the following non-resonance condition.

(NR) The restrictions

$$\begin{cases} |\boldsymbol{k}| \neq |\alpha|, \\ h\sqrt{\alpha^2 - |\boldsymbol{k}|^2} \notin \frac{\pi}{2}\mathbb{N}, \quad \text{if } |\boldsymbol{k}| < |\alpha|, \end{cases}$$

hold for each $\boldsymbol{k} \in \Lambda'$.

Their analysis in the present context leads to the first statement in the following theorem; the second is deduced from it. Condition (NR) is a blanket hypothesis in Sections 2.3, 2.4, 3 and 4, which rely upon these theorems.

Theorem 1.5. *Suppose that $s \geq 2$, and assume that the non-resonance condition (NR) holds. There exists an open neighbourhood U of the origin in $H^{s+\frac{1}{2}}(\mathbb{R}^2/\Lambda)$ such that*

(i) *the boundary-value problem (2.16)–(2.20) has a unique solution $\tilde{\boldsymbol{A}} = \tilde{\boldsymbol{A}}(\eta, \boldsymbol{\gamma}, \Phi)$ in $H^s(D_0/\Lambda)^3$ which depends analytically upon $\eta \in U$, $\boldsymbol{\gamma} \in \mathbb{R}^2$ and $\Phi \in \mathring{H}^{s-\frac{1}{2}}(\mathbb{R}^2/\Lambda)$ (and linearly upon $(\boldsymbol{\gamma}, \Phi)$);*
(ii) *the boundary-value problem (2.21)–(2.26) has a unique solution $\tilde{\boldsymbol{B}} = \tilde{\boldsymbol{B}}(\eta, \boldsymbol{\gamma}, \boldsymbol{g})$ in $H^s(D_0/\Lambda)^3$ which depends analytically upon $\eta \in U$ and $\boldsymbol{g} \in H^{s-\frac{3}{2}}(\mathbb{R}^2)^2$ (and linearly upon $(\boldsymbol{\gamma}, \boldsymbol{g})$).*

The analyticity of H, $\boldsymbol{M}$ and $\boldsymbol{T}$ follows from Theorem 1.5 and the facts that

$$H(\eta)(\boldsymbol{\gamma}, \Phi) = \nabla \cdot \tilde{\boldsymbol{A}}_{\|}^{\perp}, \qquad \boldsymbol{M}(\eta)(\boldsymbol{\gamma}, \boldsymbol{g}) = -(\operatorname{curl}^\sigma \tilde{\boldsymbol{B}})_{\|}, \tag{1.29}$$

and $\boldsymbol{T}(\eta) = \boldsymbol{M}(\eta)(\boldsymbol{0}, \boldsymbol{S}(\eta))$, where

$$\operatorname{curl}^\sigma \tilde{\boldsymbol{B}}(\boldsymbol{x}', v) := (\operatorname{curl} \boldsymbol{B}) \circ \Sigma(\boldsymbol{x}', v).$$

Theorem 1.6. *Suppose that $s \geq 2$, and assume that the non-resonance condition (NR) holds. There exists an open neighbourhood U of the origin in $H^{s+\frac{1}{2}}(\mathbb{R}^2/\Lambda)$ such that $\eta \mapsto H(\eta)$, $\eta \mapsto \boldsymbol{M}(\eta)$ and $\eta \mapsto \boldsymbol{T}(\eta)$ are analytic mappings $U \to L(\mathbb{R}^2 \times \mathring{H}^{s-\frac{1}{2}}(\mathbb{R}^2/\Lambda), \mathring{H}^{s-\frac{3}{2}}(\mathbb{R}^2/\Lambda))$, $U \to L(\mathbb{R}^2 \times H^{s-\frac{1}{2}}(\mathbb{R}^2/\Lambda)^2, H^{s-\frac{3}{2}}(\mathbb{R}^2/\Lambda)^2)$ and $U \to H^{s-\frac{3}{2}}(\mathbb{R}^2/\Lambda)^2$ respectively.*

In Section 2.3 we turn to the differentials of $H(\eta)$ and $\boldsymbol{M}(\eta)$. Applying the operator $\mathrm{d} - \mathrm{d}\sigma\, \partial_v^\sigma$, where $\partial_v^\sigma = (1 + \partial_v \sigma)^{-1}\partial_v$, to equations (1.29) shows that

$$\mathrm{d}H[\eta](\delta\eta)(\boldsymbol{\gamma},\Phi)=\nabla\cdot\tilde{\boldsymbol{C}}_{\|}^{\perp}+\partial_v^{\sigma}\,\mathrm{curl}^{\sigma}\,\tilde{\boldsymbol{A}}\cdot\boldsymbol{N}|_{v=0}\delta\eta-(\mathrm{curl}^{\sigma}\,\tilde{\boldsymbol{A}})_{\mathrm{h}}\cdot\nabla\delta\eta,$$
$$\mathrm{d}\boldsymbol{M}[\eta](\delta\eta)(\boldsymbol{\gamma},\boldsymbol{g})=-(\mathrm{curl}^{\sigma}\,\tilde{\boldsymbol{D}})_{\|}-\delta\eta(\partial_v^{\sigma}\,\mathrm{curl}^{\sigma}\,\tilde{\boldsymbol{B}})_{\|}-(\mathrm{curl}^{\sigma}\,\tilde{\boldsymbol{B}})_3|_{v=0}\nabla\delta\eta,$$

where $\tilde{\boldsymbol{C}}=(\mathrm{d}\tilde{\boldsymbol{A}}-\mathrm{d}\sigma\,\partial_v^{\sigma}\tilde{\boldsymbol{A}})$ and $\tilde{\boldsymbol{D}}=(\mathrm{d}\tilde{\boldsymbol{B}}-\mathrm{d}\sigma\,\partial_v^{\sigma}\tilde{\boldsymbol{B}})$. Careful inspection of the boundary-value problems for $\tilde{\boldsymbol{C}}$ and $\tilde{\boldsymbol{D}}$ (which are obtained by applying $\mathrm{d}-\mathrm{d}\sigma\,\partial_v^{\sigma}$ to the boundary-value problems for $\tilde{\boldsymbol{A}}$ and $\tilde{\boldsymbol{B}}$) yields the following result. Note the increased regularity requirement due to the double application of $H(\eta)$ and $\boldsymbol{M}(\eta)$ in the formulae.

Theorem 1.7. *Suppose that $s\geq 3$.*

(i) *The differential of the operator $H(\cdot)\colon U\to L(\mathbb{R}^2\times\mathring{H}^{s-\frac{1}{2}}(\mathbb{R}^2/\Lambda),\mathring{H}^{s-\frac{3}{2}}(\mathbb{R}^2/\Lambda))$ is given by*

$$\begin{aligned}&\mathrm{d}H[\eta](\delta\eta)(\boldsymbol{\gamma},\Phi)\\&\quad=H(\eta)\Big(-\alpha\langle(\boldsymbol{K}(\eta)(\boldsymbol{\gamma},\Phi)-u\nabla\eta)^{\perp}\delta\eta\rangle,-\alpha\Delta^{-1}\nabla\cdot((\boldsymbol{K}(\eta)(\boldsymbol{\gamma},\Phi)-u\nabla\eta)^{\perp}\delta\eta)\\&\qquad\qquad-u\delta\eta+\langle u\delta\eta\rangle\Big)\\&\qquad\qquad-\nabla\cdot((\boldsymbol{K}(\eta)(\boldsymbol{\gamma},\Phi)-u\nabla\eta)\delta\eta),\end{aligned}$$

where

$$u=\frac{\boldsymbol{K}(\eta)(\boldsymbol{\gamma},\Phi)\cdot\nabla\eta+H(\eta)(\boldsymbol{\gamma},\Phi)}{1+|\nabla\eta|^2}.$$

(ii) *The differential of the operator $\boldsymbol{M}(\cdot)\colon U\to L(\mathbb{R}^2\times H^{s-\frac{1}{2}}(\mathbb{R}^2/\Lambda)^2,H^{s-\frac{3}{2}}(\mathbb{R}^2/\Lambda)^2)$ is given by*

$$\begin{aligned}&\mathrm{d}\boldsymbol{M}[\eta](\delta\eta)(\boldsymbol{\gamma},\boldsymbol{g})\\&\quad=\boldsymbol{M}(\eta)\Big(\alpha\langle(\boldsymbol{M}(\eta)(\boldsymbol{\gamma},\boldsymbol{g})+u\nabla\eta)^{\perp}\delta\eta\rangle,(\boldsymbol{M}(\eta)(\boldsymbol{\gamma},\boldsymbol{g})+u\nabla\eta)^{\perp}\delta\eta\Big)\\&\qquad-\nabla(u\delta\eta)+\alpha(\boldsymbol{M}(\eta)(\boldsymbol{\gamma},\boldsymbol{g})+u\delta\eta)^{\perp}\delta\eta,\end{aligned}$$

where

$$u=\frac{\nabla\cdot\boldsymbol{g}^{\perp}-\boldsymbol{M}(\eta)(\boldsymbol{\gamma},\boldsymbol{g})\cdot\nabla\eta}{1+|\nabla\eta|^2}.$$

In Section 2.4 we show how to use recursion formulae to compute the terms in the Taylor expansions

$$H(\eta)=\sum_{j=0}^{\infty}H_j(\eta),\qquad\boldsymbol{M}(\eta)=\sum_{j=0}^{\infty}\boldsymbol{M}_j(\eta)\tag{1.30}$$

of $H(\eta)$ and $\boldsymbol{M}(\eta)$ at $\eta = 0$ systematically, where $H_j(\eta)$ and $\boldsymbol{M}_j(\eta)$ are homogeneous of degree j in η (compare with the recursion formulae for the Taylor expansion of the Dirichlet–Neumann operator appearing in the irrotational case given by Craig and Sulem [8]). The recursion formulae are derived by substituting the expansions (1.30) into the expressions for $\mathrm{d}H[\eta](\eta)(\boldsymbol{\gamma}, \Phi)$ and $\mathrm{d}\boldsymbol{M}[\eta](\eta)(\boldsymbol{\gamma}, \boldsymbol{g})$ given by Theorem 1.7, and equating terms of equal homogeneity in η. The individual terms in the series are computed as functions of H_0 and $\boldsymbol{M}_0$ using the recursion formulae, and straightforward calculations using Fourier series show that

$$H_0(\boldsymbol{\gamma}, \Phi) = D^2\, \mathtt{t}(D)\, \Phi, \qquad \boldsymbol{M}_0(\boldsymbol{\gamma}, \boldsymbol{g}) = -\boldsymbol{\gamma} + \frac{1}{D^2}\left(\alpha\, \boldsymbol{D}^{\perp} + \boldsymbol{D}\, \mathtt{c}(D)\right) \boldsymbol{D} \cdot \boldsymbol{g}^{\perp},$$

where

$$\mathtt{c}(|\boldsymbol{k}|) := \begin{cases} \sqrt{\alpha^2 - |\boldsymbol{k}|^2}\, \cot(h\sqrt{\alpha^2 - |\boldsymbol{k}|^2}), & \text{if } |\boldsymbol{k}| < |\alpha|, \\ \sqrt{|\boldsymbol{k}|^2 - \alpha^2}\, \coth(h\sqrt{|\boldsymbol{k}|^2 - \alpha^2}), & \text{if } |\boldsymbol{k}| > |\alpha|, \end{cases}$$

$$\mathtt{t}(|\boldsymbol{k}|) := \begin{cases} \dfrac{\tan(h\sqrt{\alpha^2 - |\boldsymbol{k}|^2})}{\sqrt{\alpha^2 - |\boldsymbol{k}|^2}}, & \text{if } |\boldsymbol{k}| < |\alpha|, \\ \dfrac{\tanh(h\sqrt{|\boldsymbol{k}|^2 - \alpha^2})}{\sqrt{|\boldsymbol{k}|^2 - \alpha^2}}, & \text{if } |\boldsymbol{k}| > |\alpha|, \end{cases}$$

and

$$\boldsymbol{D} = (D_1, D_2)^T = -\mathrm{i}\nabla, \qquad D = |\boldsymbol{D}|.$$

Explicit formulae for H_0, H_1, H_2 and $\boldsymbol{M}_0$, $\boldsymbol{M}_1$, $\boldsymbol{M}_2$ are computed in Section 2.4.

Remark 1.8. This method leads to formulae involving ever more derivatives of η in the individual terms in the formulae for $H_j(\eta)$ and $\boldsymbol{M}_j(\eta)$; the overall validity of the formulae arises from subtle cancellations between the terms (see Nicholls and Reitich [23, §2.2] for a discussion of this phenomenon in the context of the classical Dirichlet–Neumann operator).

1.4. Pseudodifferential calculus for the operators H and M

In Section 3 we fix $\eta \in C^\infty(\mathbb{R}^2/\Lambda)$, prove that $H(\eta)(\boldsymbol{0}, \cdot)$ and $\boldsymbol{M}(\eta)(\boldsymbol{0}, \cdot)$ are smooth perturbations of properly supported pseudodifferential operators, and compute their asymptotic expansions.

Following Alazard, Burq and Zuily [1], we begin by introducing a localising transform (which differs from the flattening transform used in Section 2). Choose $\delta > 0$ so that the fluid domain D_η contains the strip

$$\Omega_\delta := \{(\boldsymbol{x}', z) \in \mathbb{R}^2 \times \mathbb{R} : \eta(\boldsymbol{x}') - \delta h \le z < \eta(\boldsymbol{x}')\}$$

for $\eta \in U$ and define $\hat{\Sigma}\colon D_0 \to \Omega_\delta$ by

$$\hat{\Sigma}\colon (\boldsymbol{x}', w) \mapsto (\boldsymbol{x}', \varrho(\boldsymbol{x}', w)), \qquad \varrho(\boldsymbol{x}', w) := \delta w + \eta(\boldsymbol{x}').$$

This transform converts the equation

$$-\Delta \boldsymbol{U} = \alpha \operatorname{curl} \boldsymbol{U} \qquad \text{in } \Omega_\delta$$

into

$$-\Delta^\varrho \hat{\boldsymbol{U}} - \alpha \operatorname{curl}^\varrho \hat{\boldsymbol{U}} = \boldsymbol{0} \qquad \text{in } D_0,$$

where

$$\operatorname{curl}^\varrho \hat{\boldsymbol{U}}(\boldsymbol{x}', w) = (\operatorname{curl} \boldsymbol{U}) \circ \hat{\Sigma}(\boldsymbol{x}', w), \qquad \Delta^\varrho \hat{\boldsymbol{U}} = (\Delta \boldsymbol{U}) \circ \hat{\Sigma}(\boldsymbol{x}', w),$$

which we write as

$$L\hat{\boldsymbol{U}} = \boldsymbol{0} \tag{1.31}$$

(the explicit formula for L is given in Section 3.1). We proceed by implementing Treves's factorisation method (Treves [26, Ch. III, §3]) and examining its consequences for solutions of equation (1.31).

Lemma 1.9. *There are properly supported operators M, $N \in \Psi^1(\mathbb{R}^2/\Lambda)$ such that*

(i) $L - a(\partial_w I - N)(\partial_w I - M) \in \Psi^{-\infty}(\mathbb{R}^2/\Lambda)$, *where* $a = (1 + |\nabla\eta|^2)/\delta^2$,
(ii) *the principal symbols* $\mathtt{M}^{(1)}$, $\mathtt{N}^{(1)}$ *of* M, N *take the form* $\mathtt{M}^{(1)} = \mathtt{m}^{(1)}\mathbb{I}_3$, $\mathtt{N}^{(1)} = \mathtt{n}^{(1)}\mathbb{I}_3$, *where the scalar-valued symbols* $\mathtt{m}^{(1)}$, $-\mathtt{n}^{(1)} \in S^1(\mathbb{R}^2/\Lambda)$ *are strongly elliptic.*

Lemma 1.10. *Any function $\hat{\boldsymbol{U}} \in H^2(D_0/\Lambda)^3$ with $L\hat{\boldsymbol{U}} = 0$ in D_0 satisfies*

$$\partial_w \hat{\boldsymbol{U}} = M\hat{\boldsymbol{U}} + R_\infty \hat{\boldsymbol{U}} \qquad \text{at } w = 0,$$

where the symbol R_∞ denotes a linear function of its argument whose range lies in $C^\infty(\mathbb{R}^2/\Lambda)^3$.

Let $s \geq 2$, $\Phi \in \mathring{H}^{s-\frac{1}{2}}(\mathbb{R}^2/\Lambda)$ and $\tilde{\boldsymbol{A}} \in H^s(D_0/\Lambda)^3$ be the function defining $H(\eta)(\boldsymbol{0}, \Phi)$ (see equation (1.29)). The variable

$$\hat{\boldsymbol{A}}(\boldsymbol{x}', w) := \tilde{\boldsymbol{A}}(\boldsymbol{x}', v), \qquad w := \frac{1}{\delta h}(h + \eta)v$$

satisfies (1.31) and hence

$$\partial_w \hat{\boldsymbol{A}}|_{w=0} = M\hat{\boldsymbol{A}}|_{w=0} + R_\infty \Phi \tag{1.32}$$

(see Lemma 1.10, noting that $\hat{\boldsymbol{A}}$ is a linear function of Φ), together with

$$\hat{A}_3 = \eta_x \hat{A}_1 + \eta_y \hat{A}_2 \qquad \text{at } w = 0, \tag{1.33}$$

$$(\operatorname{curl}^\varrho \hat{\boldsymbol{A}})_\parallel = \nabla\Phi - \alpha \nabla^\perp \Delta^{-1}(\nabla \cdot \hat{\boldsymbol{A}}_\parallel^\perp) \qquad \text{at } w = 0. \tag{1.34}$$

Eliminating $\partial_w \hat{\boldsymbol{A}}$ using (1.32), we find from (1.33), (1.34) that

$$\hat{\boldsymbol{A}}|_{w=0} = Z\Phi + R_\infty \Phi, \tag{1.35}$$

where $\mathtt{Z} \in S^0(\mathbb{R}^2/\Lambda)$ and $Z = \operatorname{Op} \mathtt{Z}$. Finally, inserting $\hat{\boldsymbol{A}}|_{w=0}$ and $\partial_w \hat{\boldsymbol{A}}|_{w=0}$ from (1.32), (1.34) into

$$H(\eta)(\boldsymbol{0}, \Phi) = \hat{A}_{2x} + \eta_y \hat{A}_{3x} - \hat{A}_{1y} - \eta_x \hat{A}_{3y}\big|_{w=0}$$

shows that

$$H(\eta)(\boldsymbol{0}, \Phi) = \operatorname{Op} \lambda_\alpha \Phi + R_\infty \Phi,$$

where $\lambda_\alpha \in S^1(\mathbb{R}^2/\Lambda)$. The asymptotic expansions

$$\mathtt{Z} \sim \sum_{j \le 0} \mathtt{Z}^{(j)}, \qquad \lambda_\alpha \sim \sum_{j \le 1} \lambda_\alpha^{(j)}$$

can be determined recursively by substituting

$$\hat{\boldsymbol{A}}|_{w=0} = Z\Phi + R_\infty \Phi, \qquad \partial_w \hat{\boldsymbol{A}}|_{w=0} = MZ\Phi + R_\infty \Phi$$

into (1.33), (1.34). These calculations are performed in Section 3.2 and summarised in the following theorem.

Theorem 1.11. *Suppose $s \ge 2$, $\eta \in C^\infty(\mathbb{R}^2/\Lambda)$ and $\Phi \in \mathring{H}^{s-1/2}(\mathbb{R}^2/\Lambda)$. We have that*

$$H(\eta)(\boldsymbol{\gamma}, \Phi) = H(\eta)(\boldsymbol{\gamma}, 0) + H(\eta)(\boldsymbol{0}, \Phi),$$

the first term of which belongs to $C^\infty(\mathbb{R}^2/\Lambda)$, and

$$H(\eta)(\boldsymbol{0}, \Phi) = \operatorname{Op} \lambda_\alpha \Phi + R_\infty \Phi,$$

where $\lambda_\alpha \in S^1(\mathbb{R}^2/\Lambda)$ and $R_\infty \Phi \in C^\infty(\mathbb{R}^2/\Lambda)$. The symbol λ_α admits the asymptotic expansion

$$\lambda_\alpha \sim \lambda_\alpha^{(1)} + \lambda_\alpha^{(0)} + \cdots,$$

in which $\lambda_\alpha^{(j)}(\boldsymbol{x}', \boldsymbol{k})$ is homogeneous of degree j in $\boldsymbol{k}$. Moreover

$$\lambda_\alpha^{(1)}(\boldsymbol{x}', \boldsymbol{k}) = \lambda^{(1)}(\boldsymbol{x}', \boldsymbol{k}),$$

$$\lambda_\alpha^{(0)}(\boldsymbol{x}', \boldsymbol{k}) := \lambda^{(0)}(\boldsymbol{x}', \boldsymbol{k}) + \alpha \frac{(\boldsymbol{k} \cdot \nabla\eta)(\boldsymbol{k} \cdot \nabla^\perp \eta)}{|\boldsymbol{k}|^2},$$

where

$$\lambda^{(1)}(\boldsymbol{x}', \boldsymbol{k}) := \sqrt{(1 + |\nabla\eta|^2)|\boldsymbol{k}|^2 - (\boldsymbol{k} \cdot \nabla\eta)^2},$$

$$\lambda^{(0)}(\boldsymbol{x}', \boldsymbol{k}) := \frac{1 + |\nabla\eta|^2}{2\lambda^{(1)}} \left(\nabla \cdot (\mathtt{m}^{(1)} \nabla\eta) + \mathrm{i}\nabla_{\boldsymbol{k}} \lambda^{(1)} \cdot \nabla \mathtt{m}^{(1)}\right), \qquad \mathtt{m}^{(1)}(\boldsymbol{x}', \boldsymbol{k}) := \frac{\mathrm{i}\boldsymbol{k} \cdot \nabla\eta + \lambda^{(1)}}{1 + |\nabla\eta|^2},$$

are the principal and sub-principal symbols of the classical Dirichlet–Neumann operator.

The corresponding result for $M(\eta)$ is obtained in a similar fashion in Section 3.3.

Theorem 1.12. *Suppose $s \geq 2$, $\eta \in C^\infty(\mathbb{R}^2/\Lambda)$ and $\boldsymbol{g} \in H^{s-1/2}(\mathbb{R}^2/\Lambda)^2$. We have that*

$$\boldsymbol{M}(\eta)(\boldsymbol{\gamma}, \boldsymbol{g}) = \boldsymbol{M}(\eta)(\boldsymbol{\gamma}, \boldsymbol{0}) + \boldsymbol{M}(\eta)(\boldsymbol{0}, \boldsymbol{g}),$$

the first term of which belongs to $C^\infty(\mathbb{R}^2/\Lambda)^2$, and

$$\boldsymbol{M}(\eta)(\boldsymbol{0}, \boldsymbol{g}) = \operatorname{Op} \nu_\alpha \boldsymbol{g} + R_\infty \boldsymbol{g},$$

where $\nu_\alpha \in S^1(\mathbb{R}^2/\Lambda)$ and $R_\infty \boldsymbol{g} \in C^\infty(\mathbb{R}^2/\Lambda)^2$. The symbol ν_α admits the asymptotic expansion

$$\nu_\alpha \sim \nu_\alpha^{(1)} + \nu_\alpha^{(0)} + \cdots,$$

in which $\nu_\alpha^{(j)}(\boldsymbol{x}', \boldsymbol{k})$ is homogeneous of degree j in $\boldsymbol{k}$. Moreover

$$\nu_\alpha^{(1)}(\boldsymbol{x}', \boldsymbol{k})\boldsymbol{g} = \boldsymbol{k}\frac{(\boldsymbol{k} \cdot \boldsymbol{g}^\perp)}{\lambda^{(1)}}, \qquad \nu_\alpha^{(0)}(\boldsymbol{x}', \boldsymbol{k})\boldsymbol{g} = \begin{pmatrix} \zeta_1(\boldsymbol{x}', \boldsymbol{k}) \\ \zeta_2(\boldsymbol{x}', \boldsymbol{k}) \end{pmatrix}(\boldsymbol{k} \cdot \boldsymbol{g}^\perp),$$

where

$$\begin{aligned}
\zeta_1(\boldsymbol{x}', \boldsymbol{k}) = {}& \frac{\mathrm{i}}{2(\lambda^{(1)})^5}\Big(k_1^2(-1 + 2\eta_y^2)\eta_x - k_1k_2\eta_y(3 + 4\eta_x^2) + 2k_2^2\eta_x(1 + \eta_x^2) + \mathrm{i}k_1\lambda^{(1)}\Big) \\
& \times \Big(k_1^2\eta_{yy} - 2k_1k_2\eta_{xy} + k_2^2\eta_{xx}\Big) + \frac{\alpha}{(\lambda^{(1)})^2}\Big(k_2(1 + \eta_x^2) - k_1\eta_x\eta_y\Big), \\
\zeta_2(\boldsymbol{x}', \boldsymbol{k}) = {}& \frac{\mathrm{i}}{2(\lambda^{(1)})^5}\Big(2k_1^2\eta_y(1 + \eta_y^2) - k_1k_2\eta_x(3 + 4\eta_y^2) + k_2^2\eta_y(-1 + 2\eta_x^2) + \mathrm{i}k_2\lambda^{(1)}\Big) \\
& \times \Big(k_1^2\eta_{yy} - 2k_1k_2\eta_{xy} + k_2^2\eta_{xx}\Big) + \frac{\alpha}{(\lambda^{(1)})^2}\Big(-k_1(1 + \eta_y^2) + k_2\eta_x\eta_y\Big).
\end{aligned}$$

1.5. Construction of approximate solutions

In Section 4 we construct approximate solutions of

$$J(\eta, \boldsymbol{\mu}) = 0 \tag{1.36}$$

for $\beta \geq 0$ in the form of power series and moreover prove their convergence for $\beta > 0$. The solutions have wave velocity $\boldsymbol{c}$ close to a reference value $\boldsymbol{c}_0$ chosen such that the following transversality condition holds; we refer to Lokharu, Seth and Wahlén [19] for a detailed geometrical investigation of this condition (see in particular condition (3.7) and Proposition 3.3 in that reference).

(T) The only solutions $\boldsymbol{k} \in \Lambda'$ of the *dispersion relation*

$$\rho(\boldsymbol{k}, \boldsymbol{c}_0, \beta) := \left[g + \beta\, |\boldsymbol{k}|^2 - \frac{\alpha}{|\boldsymbol{k}|^2}\, (\boldsymbol{c}_0 \cdot \boldsymbol{k})(\boldsymbol{k}^\perp \cdot \boldsymbol{c}_0) \right] |\boldsymbol{k}|^2 \mathtt{t}(|\boldsymbol{k}|) - (\boldsymbol{c}_0 \cdot \boldsymbol{k})^2 = 0$$

are $\boldsymbol{k} = \boldsymbol{0}$, $\pm\boldsymbol{k}_1$, $\pm\boldsymbol{k}_2$, where $\boldsymbol{k}_1$ and $\boldsymbol{k}_2$ are the generators of the lattice Λ'. Furthermore, the vectors $\nabla_{\boldsymbol{c}}\, \rho(\boldsymbol{k}_1, \boldsymbol{c}_0, \beta)$ and $\nabla_{\boldsymbol{c}}\, \rho(\boldsymbol{k}_2, \boldsymbol{c}_0, \beta)$ are linearly independent.

We consider J as a locally analytic mapping $X_s^\beta \times \mathbb{R}^2 \to H^s(\mathbb{R}^2/\Lambda)$ for a sufficiently large value of s, where

$$X_s^\beta := \begin{cases} H^{s+2}(\mathbb{R}^2/\Lambda), & \text{if } \beta > 0, \\ H^{s+1}(\mathbb{R}^2/\Lambda), & \text{if } \beta = 0, \end{cases}$$

and proceed to investigate the kernel and range of

$$J_{10}(\eta) := \mathrm{d}_1 J[0, \boldsymbol{0}](\eta) = \boldsymbol{T}_1(\eta) \cdot \boldsymbol{c}_0 + g\eta - \beta\Delta\eta.$$

Writing

$$\eta(\boldsymbol{x}') = \sum_{\boldsymbol{k} \in \Lambda'} \hat{\eta}_{\boldsymbol{k}} \mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}'},$$

so that

$$(J_{10}\eta)(\boldsymbol{x}') = g\hat{\eta}_{\boldsymbol{0}} + \sum_{\boldsymbol{k} \in \Lambda' \setminus \{\boldsymbol{0}\}} \frac{\mathtt{c}(|\boldsymbol{k}|)}{|\boldsymbol{k}|^2} \rho(\boldsymbol{k}, \boldsymbol{c}_0, \beta) \hat{\eta}_{\boldsymbol{k}} \mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}'},$$

we find that

$$\ker(J_{10}) = \{A\mathrm{e}^{\mathrm{i}\boldsymbol{k}_1\cdot\boldsymbol{x}'} + B\mathrm{e}^{\mathrm{i}\boldsymbol{k}_2\cdot\boldsymbol{x}'} + \bar{A}\mathrm{e}^{-\mathrm{i}\boldsymbol{k}_1\cdot\boldsymbol{x}'} + \bar{B}\mathrm{e}^{-\mathrm{i}\boldsymbol{k}_2\cdot\boldsymbol{x}'} : A, B \in \mathbb{C}\},$$

since $\rho(\boldsymbol{k}, \boldsymbol{c}_0, \beta) = 0$ if and only if $\boldsymbol{k} = \boldsymbol{0}, \pm\boldsymbol{k}_1, \pm\boldsymbol{k}_2$. The operator J_{10} is formally invertible if $\hat{f}_{\pm\boldsymbol{k}_1} = \hat{f}_{\pm\boldsymbol{k}_2} = 0$ with formal inverse given by

$$(J_{10}^{-1} f)(\boldsymbol{x}') = \frac{1}{g}\hat{f}_{\boldsymbol{0}} + \sum_{\substack{\boldsymbol{k} \in \Lambda' \\ \boldsymbol{k} \neq \boldsymbol{0}, \pm\boldsymbol{k}_1, \pm\boldsymbol{k}_2}} \frac{|\boldsymbol{k}|^2}{\mathtt{c}(|\boldsymbol{k}|)\rho(\boldsymbol{k}, \boldsymbol{c}_0, \beta)} \hat{f}_{\boldsymbol{k}} \mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}'}.$$

For $\beta > 0$ we find that $\rho(\boldsymbol{k}, \boldsymbol{c}_0, \beta) \gtrsim |\boldsymbol{k}|^3$ for sufficiently large $|\boldsymbol{k}|$, so that the above series converges in $H^{s+2}(\mathbb{R}^2/\Lambda)$ for $f \in H^s(\mathbb{R}^2/\Lambda)$; it follows that $J_{10}\colon H^{s+2}(\mathbb{R}^2/\Lambda) \to H^s(\mathbb{R}^2/\Lambda)$ is Fredholm with index 0. In contrast $\rho(\boldsymbol{k}, \boldsymbol{c}_0, 0)$ is not bounded from below as $|\boldsymbol{k}| \to \infty$, so that the above formula does not define a bounded operator from $H^s(\mathbb{R}^2/\Lambda)$ to $H^{s+1}(\mathbb{R}^2/\Lambda)$ for any s. We therefore proceed formally, noting that the procedure is rigorously valid for $\beta > 0$.

To apply the Lyapunov–Schmidt reduction to equation (1.36) we write $\eta = \eta_1 + \eta_2$, where

$$\eta_1 = A\mathrm{e}^{\mathrm{i}\boldsymbol{k}_1\cdot\boldsymbol{x}'} + B\mathrm{e}^{\mathrm{i}\boldsymbol{k}_2\cdot\boldsymbol{x}'} + \bar{A}\mathrm{e}^{-\mathrm{i}\boldsymbol{k}_1\cdot\boldsymbol{x}'} + \bar{B}\mathrm{e}^{-\mathrm{i}\boldsymbol{k}_2\cdot\boldsymbol{x}'}$$

and $\eta_2 \in \ker(J_{10})^\perp$. Noting that S_0 and $T_{\boldsymbol{v}'}$ act on the coordinates $(A, B, \bar{A}, \bar{B})$ as

$$S_0(A, B, \bar{A}, \bar{B}) = (\bar{A}, \bar{B}, A, B), \qquad T_{\boldsymbol{v}'}(A, B, \bar{A}, \bar{B}) = (A\mathrm{e}^{\mathrm{i}\boldsymbol{k}_1\cdot\boldsymbol{v}'}, B\mathrm{e}^{\mathrm{i}\boldsymbol{k}_2\cdot\boldsymbol{v}'}, \bar{A}\mathrm{e}^{-\mathrm{i}\boldsymbol{k}_1\cdot\boldsymbol{v}'}, \bar{B}\mathrm{e}^{-\mathrm{i}\boldsymbol{k}_2\cdot\boldsymbol{v}'}),$$

and that the reduced equation remains equivariant under these symmetries, we show that (1.36) is locally equivalent to

$$A g_1(|A|^2, |B|^2, \boldsymbol{\mu}) = 0,$$
$$B g_2(|A|^2, |B|^2, \boldsymbol{\mu}) = 0,$$

where g_1, g_2 are real-valued locally analytic functions which vanish at the origin. The following result is obtained from the analytic implicit-function theorem and the transversality condition (T).

Proposition 1.13. *There exist $\varepsilon > 0$ and analytic functions $\mu_i\colon B_\varepsilon(\mathbf{0}, \mathbb{R}^2) \to \mathbb{R}$, $i = 1, 2$ such that $\mu_i(0, 0) = 0$ and $(|A|^2, |B|^2, \mu_1(|A|^2, |B|^2), \mu_2(|A|^2, |B|^2))$ is the unique local solution of $g_i(|A|^2, |B|^2, \boldsymbol{\mu}) = 0$, $i = 1, 2$.*

Our main result now follows by substituting $\boldsymbol{\mu} = \boldsymbol{\mu}(|A|^2, |B|^2)$ into $\eta = \eta_1 + \eta_2(\eta_1, \boldsymbol{\mu})$.

Theorem 1.14. *Suppose that $\beta > 0$. There exist $\varepsilon > 0$, a neighbourhood V of the origin in $X_s^\beta \times \mathbb{R}^2$ and analytic functions $\mu_1, \mu_2\colon B_\varepsilon(\mathbf{0}, \mathbb{R}^2) \to \mathbb{R}$ and $\eta\colon B_\varepsilon(\mathbf{0}, \mathbb{C}^4) \to X_s^\beta$ such that*

$$\begin{aligned}\{(\eta, \boldsymbol{\mu}) \in X_s^\beta \times \mathbb{R}^2 \colon J(\eta, \boldsymbol{\mu}) = 0,\ \eta \neq 0\} \cap V \\ = \{(\eta(A, B, \bar{A}, \bar{B}), \boldsymbol{\mu}(|A|^2, |B|^2)) \colon (A, B, \bar{A}, \bar{B}) \in B_\varepsilon'(\mathbf{0}, \mathbb{C}^4)\};\end{aligned}$$

furthermore $\boldsymbol{\mu}(0, 0) = \mathbf{0}$ and

$$\eta(\boldsymbol{x}') = A\mathrm{e}^{\mathrm{i}\boldsymbol{k}_1\cdot\boldsymbol{x}'} + B\mathrm{e}^{\mathrm{i}\boldsymbol{k}_2\cdot\boldsymbol{x}'} + \bar{A}\mathrm{e}^{-\mathrm{i}\boldsymbol{k}_1\cdot\boldsymbol{x}'} + \bar{B}\mathrm{e}^{-\mathrm{i}\boldsymbol{k}_2\cdot\boldsymbol{x}'} + O(|(A, B, \bar{A}, \bar{B})|^2).$$

The terms in the expansions

$$\eta = A\mathrm{e}^{\mathrm{i}\boldsymbol{k}_1\cdot\boldsymbol{x}'} + B\mathrm{e}^{\mathrm{i}\boldsymbol{k}_2\cdot\boldsymbol{x}'} + \bar{A}\mathrm{e}^{-\mathrm{i}\boldsymbol{k}_1\cdot\boldsymbol{x}'} + \bar{B}\mathrm{e}^{-\mathrm{i}\boldsymbol{k}_2\cdot\boldsymbol{x}'} + \sum_{i+j+k+l \geq 2} \eta_{ijkl} A^i B^j \bar{A}^k \bar{B}^l$$

and

$$\mu_i = \sum_{j+k \geq 1} \mu_{i,jk} |A|^{2j} |B|^{2k}, \qquad i = 1, 2,$$

can be determined recursively by substituting these expressions into (1.36) and equating monomials in $(A, B, \bar{A}, \bar{B})$. Note that the series can be computed to any order for $\beta \geq 0$ but their convergence has been established only for $\beta > 0$. The coefficients η_{ijkl} for $i + j + k + \ell = 2$ and $\mu_{1,jk}, \mu_{2,jk}$ for $j + k = 1$ are computed in Section 4.

2. The operators $H(\eta)$ and $M(\eta)$

In this section we study the operators $H(\eta)$ and $\boldsymbol{M}(\eta)$ defined by

$$H(\eta)(\boldsymbol{\gamma}, \Phi) := \nabla \cdot \boldsymbol{A}_{\parallel}^{\perp}, \qquad \boldsymbol{M}(\eta)(\boldsymbol{\gamma}, \boldsymbol{g}) := -(\operatorname{curl} \boldsymbol{B})_{\parallel}, \tag{2.1}$$

where $\boldsymbol{A}$ and $\boldsymbol{B}$ are the solutions to the boundary-value problems

$$\operatorname{curl}\operatorname{curl} \boldsymbol{A} = \alpha \operatorname{curl} \boldsymbol{A} \qquad \text{in } D_\eta, \tag{2.2}$$

$$\operatorname{div} \boldsymbol{A} = 0 \qquad \text{in } D_\eta, \tag{2.3}$$

$$\boldsymbol{A} \times \boldsymbol{e}_3 = \boldsymbol{0} \qquad \text{at } z = -h, \tag{2.4}$$

$$\boldsymbol{A} \cdot \boldsymbol{n} = 0 \qquad \text{at } z = \eta, \tag{2.5}$$

$$(\operatorname{curl} \boldsymbol{A})_{\parallel} = \boldsymbol{\gamma} + \nabla\Phi - \alpha \nabla^{\perp} \Delta^{-1} (\nabla \cdot \boldsymbol{A}_{\parallel}^{\perp}) \qquad \text{at } z = \eta \tag{2.6}$$

and

$$\operatorname{curl}\operatorname{curl} \boldsymbol{B} = \alpha \operatorname{curl} \boldsymbol{B} \qquad \text{in } D_\eta, \tag{2.7}$$

$$\operatorname{div} \boldsymbol{B} = 0 \qquad \text{in } D_\eta, \tag{2.8}$$

$$\boldsymbol{B} \times \boldsymbol{e}_3 = \boldsymbol{0} \qquad \text{at } z = -h, \tag{2.9}$$

$$\boldsymbol{B} \cdot \boldsymbol{n} = 0 \qquad \text{at } z = \eta, \tag{2.10}$$

$$\nabla \cdot \boldsymbol{B}_{\parallel}^{\perp} = \nabla \cdot \boldsymbol{g}^{\perp}, \tag{2.11}$$

$$\langle (\operatorname{curl} \boldsymbol{B})_{\parallel} \rangle = \boldsymbol{\gamma}. \tag{2.12}$$

2.1. Weak and strong solutions

We first suppose that η is a fixed function in $W^{2,\infty}(\mathbb{R}^2/\Lambda)$ with $\inf \eta > -h$ and present a traditional weak/strong-solution approach to the boundary-value problems (2.2)–(2.6) and (2.7)–(2.12), working with the standard spaces $\mathscr{D}(D_\eta/\Lambda)^3$ and $\mathscr{D}(\overline{D_\eta/\Lambda})^3$ of periodic test functions, the Sobolev spaces $L^2(D_\eta/\Lambda)^3$ and $H^1(D_\eta/\Lambda)^3$, and the closed subspace

$$\mathcal{X}_\eta = \{\boldsymbol{F} \in H^1(D_\eta/\Lambda)^3 \colon \boldsymbol{F} \times \boldsymbol{e}_3|_{z=-h} = \boldsymbol{0},\ \boldsymbol{F} \cdot \boldsymbol{n}|_{z=\eta} = 0\}$$

of $H^1(D_\eta/\Lambda)^3$.

Definition 2.1.

(i) A weak solution of (2.2)–(2.6) is a function $\boldsymbol{A} \in \mathcal{X}_\eta$ such that

$$\int_\Omega \int_{-h}^{\eta} (\operatorname{curl} \boldsymbol{A} \cdot \operatorname{curl} \boldsymbol{C} - \alpha \operatorname{curl} \boldsymbol{A} \cdot \boldsymbol{C} + \operatorname{div} \boldsymbol{A}\, \operatorname{div} \boldsymbol{C}) - \alpha \int_\Omega \nabla \Delta^{-1} (\nabla \cdot \boldsymbol{A}_{\parallel}^{\perp}) \cdot \boldsymbol{C}_{\parallel}$$

$$= \int_\Omega (\boldsymbol{\gamma}^{\perp} + \nabla^{\perp}\Phi) \cdot \boldsymbol{C}_{\parallel} \tag{2.13}$$

for all $\boldsymbol{C} \in \mathcal{X}_\eta$, while a strong solution has the additional regularity requirement that $\boldsymbol{A} \in H^2(D_\eta/\Lambda)^3$, is solenoidal and satisfies (2.2) in $L^2(D_\eta/\Lambda)^3$ and (2.6) in $H^{\frac{1}{2}}(\mathbb{R}^2/\Lambda)^2$.

(ii) A weak solution of (2.7)–(2.12) is a function $\boldsymbol{B} \in \mathcal{X}_\eta$ which satisfies (2.11) and

$$\int_\Omega \int_{-h}^{\eta} (\operatorname{curl} \boldsymbol{B} \cdot \operatorname{curl} \boldsymbol{D} - \alpha \operatorname{curl} \boldsymbol{B} \cdot \boldsymbol{D} + \operatorname{div} \boldsymbol{B}\ \operatorname{div} \boldsymbol{D}) = \int_\Omega (\boldsymbol{\gamma}^\perp + \alpha \nabla \Delta^{-1}(\nabla \cdot \boldsymbol{g}^\perp)) \cdot \boldsymbol{D}_\parallel \quad (2.14)$$

for all $\boldsymbol{D} \in \mathcal{X}_\eta^0$, where

$$\mathcal{X}_\eta^0 := \{\boldsymbol{F} \in \mathcal{X}_\eta :\ \nabla \cdot \boldsymbol{F}_\parallel^\perp = 0\},$$

while a strong solution has the additional regularity requirement that $\boldsymbol{B}$ lies in $H^2(D_\eta/\Lambda)^3$, is solenoidal, satisfies (2.12) and satisfies (2.7) in $L^2(D_\eta/\Lambda)^3$.

The existence of weak and strong solutions is established in Lemmata 2.5 and 2.6 below, whose proofs rely upon the following technical results (see Groves and Horn [12, §4(b)]).

Proposition 2.2.

(i) *The space $\mathcal{X}_\eta$ coincides with*

$$\begin{aligned}\{\boldsymbol{F} \in L^2(D_\eta/\Lambda)^3 :\ & \operatorname{curl} \boldsymbol{F} \in L^2(D_\eta/\Lambda)^3,\ \operatorname{div} \boldsymbol{F} \in L^2(D_\eta/\Lambda),\\ & \boldsymbol{F} \times \boldsymbol{e}_3|_{z=-h} = \boldsymbol{0},\ \boldsymbol{F} \cdot \boldsymbol{n}|_{z=\eta} = 0\}\end{aligned}$$

and the function $\boldsymbol{F} \mapsto (\|\operatorname{curl} \boldsymbol{F}\|_{L^2(D_\eta/\Lambda)^3}^2 + \|\operatorname{div} \boldsymbol{F}\|_{L^2(D_\eta/\Lambda)}^2)^{\frac{1}{2}}$ is equivalent to its usual norm.

(ii) *The spaces*

$$\begin{aligned}\{\boldsymbol{F} \in L^2(D_\eta/\Lambda)^3 :\ & \operatorname{curl} \boldsymbol{F} \in L^2(D_\eta/\Lambda)^3,\ \operatorname{div} \boldsymbol{F} \in L^2(D_\eta/\Lambda),\\ & \boldsymbol{F} \times \boldsymbol{e}_3|_{z=-h} \in H^{\frac{1}{2}}(\mathbb{R}^2/\Lambda)^3,\ \boldsymbol{F} \cdot \boldsymbol{N}|_{z=\eta} \in H^{\frac{1}{2}}(\mathbb{R}^2/\Lambda)\},\\ \{\boldsymbol{F} \in L^2(D_\eta/\Lambda)^3 :\ & \operatorname{curl} \boldsymbol{F} \in L^2(D_\eta/\Lambda)^3,\ \operatorname{div} \boldsymbol{F} \in L^2(D_\eta/\Lambda),\\ & \boldsymbol{F} \times \boldsymbol{e}_3|_{z=-h} \in H^{\frac{1}{2}}(\mathbb{R}^2/\Lambda)^3,\ \boldsymbol{F}_\parallel^\perp \in H^{\frac{1}{2}}(\mathbb{R}^2/\Lambda)^2\},\\ \{\boldsymbol{F} \in L^2(D_\eta/\Lambda)^3 :\ & \operatorname{curl} \boldsymbol{F} \in L^2(D_\eta/\Lambda)^3,\ \operatorname{div} \boldsymbol{F} \in L^2(D_\eta/\Lambda),\\ & \boldsymbol{F} \cdot \boldsymbol{e}_3|_{z=-h} \in H^{\frac{1}{2}}(\mathbb{R}^2/\Lambda),\ \boldsymbol{F} \cdot \boldsymbol{N}|_{z=\eta} \in H^{\frac{1}{2}}(\mathbb{R}^2/\Lambda)\},\\ \{\boldsymbol{F} \in L^2(D_\eta/\Lambda)^3 :\ & \operatorname{curl} \boldsymbol{F} \in L^2(D_\eta/\Lambda)^3,\ \operatorname{div} \boldsymbol{F} \in L^2(D_\eta/\Lambda),\\ & \boldsymbol{F} \cdot \boldsymbol{e}_3|_{z=-h} \in H^{\frac{1}{2}}(\mathbb{R}^2/\Lambda),\ \boldsymbol{F}_\parallel^\perp \in H^{\frac{1}{2}}(\mathbb{R}^2/\Lambda)^2\}\end{aligned}$$

coincide with $H^1(D_\eta/\Lambda)^3$.

(iii) *The space*

$$\{\boldsymbol{F} \in L^2(D_\eta/\Lambda)^3 \colon \operatorname{curl} \boldsymbol{F} \in H^1(D_\eta/\Lambda)^3,\ \operatorname{div} \boldsymbol{F} \in H^1(D_\eta/\Lambda),\\ \boldsymbol{F} \times \boldsymbol{e}_3|_{z=-h} = \boldsymbol{0},\ \boldsymbol{F} \cdot \boldsymbol{n}|_{z=\eta} = 0\}$$

coincides with $\{\boldsymbol{F} \in H^2(D_\eta/\Lambda)^3 \colon \boldsymbol{F} \times \boldsymbol{e}_3|_{z=-h} = \boldsymbol{0},\ \boldsymbol{F} \cdot \boldsymbol{n}|_{z=\eta} = 0\}$.

Proposition 2.3.

(i) *It follows from the formula*

$$\int_\Omega \int_{-h}^{\eta} (\boldsymbol{F} \cdot \operatorname{curl} \boldsymbol{G} - \operatorname{curl} \boldsymbol{F} \cdot \boldsymbol{G}) = \int_\Omega \boldsymbol{F}_\parallel^\perp \cdot \boldsymbol{G}_\parallel,$$

where

$$\boldsymbol{F} \in \mathscr{D}(\overline{D_\eta/\Lambda})^3,\ \boldsymbol{G} \in H^1(D_\eta/\Lambda)^3, \boldsymbol{G}|_{z=-h} = \boldsymbol{0},$$

that the mapping $\boldsymbol{F} \mapsto \boldsymbol{F}_\parallel^\perp$ *defined on* $\mathscr{D}(\overline{D_\eta/\Lambda})^3$ *extends to a continuous linear mapping* $\{\boldsymbol{F} \in L^2(D_\eta/\Lambda)^3 \colon \operatorname{curl} \boldsymbol{F} \in L^2(D_\eta/\Lambda)^3\} \to H^{-\frac{1}{2}}(\mathbb{R}^2/\Lambda)^2$, *where the former space is equipped with the norm* $\boldsymbol{F} \mapsto (\|\boldsymbol{F}\|^2_{L^2(D_\eta/\Lambda)^3} + \|\operatorname{curl} \boldsymbol{F}\|^2_{L^2(D_\eta/\Lambda)^3})^{\frac{1}{2}}$.

(ii) *It follows from the formula*

$$\int_\Omega \int_{-h}^{\eta} \operatorname{curl} \boldsymbol{F} \cdot \operatorname{grad} \phi = \int_\Omega \operatorname{curl} \boldsymbol{F} \cdot \boldsymbol{N} \phi|_{z=\eta},$$

where

$$\boldsymbol{F} \in \mathscr{D}(\overline{D_\eta/\Lambda})^3,\ \phi \in H^1(D_\eta/\Lambda), \phi|_{z=-h} = 0,$$

that the mapping $\boldsymbol{F} \mapsto \operatorname{curl} \boldsymbol{F} \cdot \boldsymbol{N}|_{z=\eta}$ *defined on* $\mathscr{D}(\overline{D_\eta/\Lambda})^3$ *extends to a continuous linear mapping* $\{\boldsymbol{F} \in L^2(D_\eta)^3 \colon \operatorname{div} \boldsymbol{F} \in L^2(D_\eta)\} \to H^{-\frac{1}{2}}(\mathbb{R}^2)^2$, *where the former space is equipped with the norm* $\boldsymbol{F} \mapsto (\|\boldsymbol{F}\|^2_{L^2(D_\eta)^3} + \|\operatorname{div} \boldsymbol{F}\|^2_{L^2(D_\eta)})^{\frac{1}{2}}$.

Proposition 2.4. *The boundary-value problem*

$$\begin{aligned} \Delta \phi &= F && \text{in } D_\eta, \\ \partial_n \phi &= f && \text{at } z = \eta, \\ \phi &= 0 && \text{at } z = -h \end{aligned}$$

has a unique solution $\phi \in H^2(D_\eta/\Lambda)$ *for each* $F \in L^2(D_\eta/\Lambda)$ *and* $f \in H^{\frac{1}{2}}(S_\eta/\Lambda)$.

Lemma 2.5.

(i) *For all sufficiently small values of* $|\alpha|$ *the boundary-value problem* (2.2)–(2.6) *admits a unique weak solution for each* $\boldsymbol{\gamma} \in \mathbb{R}^2$ *and* $\Phi \in \mathring{H}^{\frac{1}{2}}(\mathbb{R}^2/\Lambda)$. *The weak solution is solenoidal and satisfies* (2.2) *in the sense of distributions and* (2.6) *in* $H^{-\frac{1}{2}}(\mathbb{R}^2/\Lambda)^2$.

(ii) *For all sufficiently small values of* $|\alpha|$ *the boundary-value problem* (2.7)–(2.12) *admits a unique weak solution for each* $\boldsymbol{\gamma} \in \mathbb{R}^2$ *and* $\boldsymbol{g} \in H^{\frac{1}{2}}(\mathbb{R}^2/\Lambda)^2$. *The weak solution is solenoidal and satisfies* (2.7) *in the sense of distributions.*

Proof.

(i) The estimates

$$\left|\int_\Omega \int_{-h}^{\eta} \operatorname{curl} \boldsymbol{A} \cdot \boldsymbol{C}\right| \lesssim \|\boldsymbol{A}\|_{H^1(D_\eta/\Lambda)^3} \|\boldsymbol{C}\|_{H^1(D_\eta/\Lambda)^3},$$

$$\left|\int_\Omega \nabla\Delta^{-1}(\nabla \cdot \boldsymbol{A}_\parallel^\perp) \cdot \boldsymbol{C}_\parallel\right| \lesssim \|\boldsymbol{A}_\parallel\|_0 \|\boldsymbol{C}_\parallel\|_0 \lesssim \|\boldsymbol{A}|_{z=\eta}\|_{\frac{1}{2}} \|\boldsymbol{C}|_{z=\eta}\|_{\frac{1}{2}}$$

$$\lesssim \|\boldsymbol{A}\|_{H^1(D_\eta/\Lambda)^3} \|\boldsymbol{C}\|_{H^1(D_\eta/\Lambda)^3}$$

and Proposition 2.2(i) imply that for sufficiently small values of $|\alpha|$ the left-hand side of (2.13) is a continuous, coercive, bilinear form $\mathcal{X}_\eta \times \mathcal{X}_\eta \to \mathbb{R}$, while the estimate

$$\left|\int_\Omega (\boldsymbol{\gamma}^\perp + \nabla^\perp \Phi) \cdot \boldsymbol{C}_\parallel\right| \lesssim (|\boldsymbol{\gamma}^\perp| + \|\nabla^\perp \Phi\|_{-\frac{1}{2}}) \|\boldsymbol{C}|_{z=\eta}\|_{\frac{1}{2}} \lesssim (|\boldsymbol{\gamma}| + \|\Phi\|_{\frac{1}{2}}) \|\boldsymbol{C}\|_{H^1(D_\eta/\Lambda)^3}$$

shows that its right-hand side is a continuous, bilinear form $(\mathbb{R}^2 \times \mathring{H}^{\frac{1}{2}}(\mathbb{R}^2/\Lambda)) \times \mathcal{X}_\eta \to \mathbb{R}$. The existence of a unique solution $\boldsymbol{A} \in \mathcal{X}_\eta$ now follows from the Lax-Milgram lemma.

Let $\phi_{\boldsymbol{A}} \in H^2(D_\eta/\Lambda)$ be the unique function satisfying $\Delta \phi_{\boldsymbol{A}} = \operatorname{div} \boldsymbol{A}$ in D_η with boundary conditions $\partial_n \phi_{\boldsymbol{A}}|_{y=\eta} = 0$, $\phi_{\boldsymbol{A}}|_{z=-h} = 0$ (see Proposition 2.4). It follows that $\boldsymbol{C} = \operatorname{grad} \phi_{\boldsymbol{A}} \in \mathcal{X}_\eta$ and hence

$$\int_\Omega \int_{-h}^{\eta} \big(-\alpha \operatorname{curl} \boldsymbol{A} \cdot \operatorname{grad} \phi_{\boldsymbol{A}} + (\operatorname{div} \boldsymbol{A})^2\big) - \alpha \int_\Omega \nabla\Delta^{-1}(\nabla \cdot \boldsymbol{A}_\parallel^\perp) \cdot \nabla(\phi_{\boldsymbol{A}}|_{z=\eta}) = 0$$

(because $\boldsymbol{C}_\parallel = \nabla(\phi_{\boldsymbol{A}}|_{z=\eta})$, which is orthogonal to $\boldsymbol{\gamma}^\perp$ and $\nabla^\perp \Phi$). Since

$$\int_\Omega \int_{-h}^{\eta} \operatorname{curl} \boldsymbol{A} \cdot \operatorname{grad} \phi_{\boldsymbol{A}} = \int_\Omega \operatorname{curl} \boldsymbol{A} \cdot \boldsymbol{N}\, \phi_{\boldsymbol{A}}|_{z=\eta} = \int_\Omega \nabla \cdot \boldsymbol{A}_\parallel^\perp\, \phi_{\boldsymbol{A}}|_{z=\eta}$$
$$= -\int_\Omega \nabla \Delta^{-1}(\nabla \cdot \boldsymbol{A}_\parallel^\perp) \cdot \nabla(\phi_{\boldsymbol{A}}|_{z=\eta})$$

(see Proposition 2.3(ii)), one concludes that $\operatorname{div} \boldsymbol{A} = 0$.

Choosing $\boldsymbol{C} \in \mathscr{D}(D_\eta/\Lambda)^3$, one finds that $\boldsymbol{A}$ solves (2.2) in the sense of distributions and hence that $\operatorname{curl}\operatorname{curl} \boldsymbol{A} \in L^2(D_\eta)^3$. It follows that $(\operatorname{curl} \boldsymbol{A})_\parallel^\perp \in H^{-\frac{1}{2}}(\mathbb{R}^2)^2$ (Proposition 2.3(i)) and

$$\int_\Omega \int_{-h}^{\eta} (\operatorname{curl}\operatorname{curl} \boldsymbol{A} - \alpha \operatorname{curl} \boldsymbol{A}) \cdot \boldsymbol{C} + \int_\Omega \Big((\operatorname{curl} \boldsymbol{A})_\parallel^\perp - \boldsymbol{\gamma}^\perp - \nabla^\perp \Phi - \alpha \nabla \Delta^{-1}(\nabla \cdot \boldsymbol{A}_\parallel^\perp) \Big) \cdot \boldsymbol{C}_\parallel = 0.$$

One concludes that (2.6) holds in $H^{-\frac{1}{2}}(\mathbb{R}^2/\Lambda)^2$.

(ii) Let $\boldsymbol{F} \in H^1(D_\eta/\Lambda)^3$ be a function such that $\boldsymbol{F}_\parallel = \boldsymbol{g}$ and $\boldsymbol{F} \times \boldsymbol{e}_3|_{z=-h} = \boldsymbol{0}$, and let $\phi_{\boldsymbol{F}} \in H^2(D_\eta/\Lambda)$ be the unique function satisfying $\Delta \phi_{\boldsymbol{F}} = \operatorname{div} \boldsymbol{F}$ in D_η with boundary conditions $\partial_n \phi_{\boldsymbol{F}}|_{z=\eta} = \boldsymbol{F} \cdot \boldsymbol{n}$, $\phi_{\boldsymbol{F}}|_{z=-h} = 0$ (see Proposition 2.4). It follows that $\boldsymbol{G} := \boldsymbol{F} - \operatorname{grad} \phi_{\boldsymbol{F}}$ satisfies $\operatorname{div} \boldsymbol{G} = 0$, $\boldsymbol{G} \cdot \boldsymbol{n}|_{z=\eta} = 0$ and $\nabla \cdot \boldsymbol{G}_\parallel^\perp = \nabla \cdot \boldsymbol{g}^\perp$ because $\nabla \cdot (\operatorname{grad} \phi_{\boldsymbol{F}})_\parallel^\perp = \nabla \cdot \nabla(\phi_{\boldsymbol{F}}|_{z=\eta})^\perp = 0$. We accordingly seek $\boldsymbol{C} \in \mathcal{X}_\eta^0$ such that

$$\int_\Omega \int_{-h}^{\eta} (\operatorname{curl} \boldsymbol{C} \cdot \operatorname{curl} \boldsymbol{D} - \alpha \operatorname{curl} \boldsymbol{C} \cdot \boldsymbol{D} + \operatorname{div} \boldsymbol{C} \operatorname{div} \boldsymbol{D})$$
$$= -\int_\Omega \int_{-h}^{\eta} (\operatorname{curl} \boldsymbol{G} \cdot \operatorname{curl} \boldsymbol{D} - \alpha \operatorname{curl} \boldsymbol{G} \cdot \boldsymbol{D}) + \int_\Omega (\boldsymbol{\gamma}^\perp + \alpha \nabla \Delta^{-1}(\nabla \cdot \boldsymbol{G}^\perp)) \cdot \boldsymbol{D}_\parallel \tag{2.15}$$

for all $\boldsymbol{D} \in \mathcal{X}_\eta^0$, so that $\boldsymbol{B} = \mathbf{C} + \boldsymbol{G}$ is a weak solution of (2.7)–(2.12).

For sufficiently small values of $|\alpha|$ the left-hand side of (2.15) is a continuous, coercive, bilinear form $\mathcal{X}_\eta^0 \times \mathcal{X}_\eta^0 \to \mathbb{R}$, while the right-hand side is a continuous, bilinear form $(\mathbb{R}^2 \times \mathcal{X}_\eta) \times \mathcal{X}_\eta^0 \to \mathbb{R}$. The existence of a unique function $\boldsymbol{C} \in \mathcal{X}_\eta^0$ satisfying (2.15) for all $\boldsymbol{D} \in \mathcal{X}_\eta^0$ now follows from the Lax-Milgram lemma, and the corresponding weak solution $\boldsymbol{B}$ to (2.7)–(2.12) is unique since the difference between two weak solutions satisfies (2.15) with $\boldsymbol{\gamma} = \boldsymbol{0}$ and $\boldsymbol{g} = \boldsymbol{0}$ for all $\boldsymbol{D} \in \mathcal{X}_\eta^0$ and is therefore zero.

Let $\phi_{\boldsymbol{B}} \in H^2(D_\eta/\Lambda)$ be the unique function satisfying $\Delta \phi_{\boldsymbol{B}} = \operatorname{div} \boldsymbol{B}$ in D_η with boundary conditions $\partial_n \phi_{\boldsymbol{B}}|_{z=\eta} = 0$, $\phi_{\boldsymbol{B}}|_{z=-h} = 0$ (see Proposition 2.4). Substituting $\boldsymbol{D} = \operatorname{grad} \phi_{\boldsymbol{B}} \in \mathcal{X}_\eta^0$ into (2.14), we find that

$$\int_\Omega \int_{-h}^{\eta} \big(-\alpha \operatorname{curl} \boldsymbol{B} \cdot \operatorname{grad} \phi_{\boldsymbol{B}} + (\operatorname{div} \boldsymbol{B})^2 \big) = \alpha \int_\Omega \nabla \Delta^{-1}(\nabla \cdot \boldsymbol{B}_\parallel^\perp) \cdot \nabla(\phi_{\boldsymbol{B}}|_{z=\eta}),$$

and since

$$-\int_\Omega \int_{-h}^{\eta} \operatorname{curl} \boldsymbol{B} \cdot \operatorname{grad} \phi_{\boldsymbol{B}} = -\int_\Omega \nabla \cdot \boldsymbol{B}_\parallel^\perp \, \phi_{\boldsymbol{B}}|_{z=\eta} = \int_\Omega \nabla \Delta^{-1}(\nabla \cdot \boldsymbol{B}_\parallel^\perp) \cdot \nabla(\phi_{\boldsymbol{B}}|_{z=\eta}),$$

one concludes that $\operatorname{div} \boldsymbol{B} = 0$.

Finally, taking $\boldsymbol{D} \in \mathscr{D}(D_\eta/\Lambda)^3$ in (2.14), we find that $\boldsymbol{B}$ satisfies (2.7) in the sense of distributions. □

Lemma 2.6.

(i) *Suppose that* $\boldsymbol{\gamma} \in \mathbb{R}^2$ *and* $\Phi \in \mathring{H}^{\frac{3}{2}}(\mathbb{R}^2/\Lambda)$. *Any weak solution* $\boldsymbol{A}$ *of* (2.2)–(2.6) *is in fact a strong solution.*

(ii) *Suppose that* $\boldsymbol{\gamma} \in \mathbb{R}^2$ *and* $\boldsymbol{g} \in H^{\frac{3}{2}}(\mathbb{R}^2/\Lambda)^2$. *Any weak solution* $\boldsymbol{B}$ *of* (2.7)–(2.12) *is in fact a strong solution.*

Proof.

(i) Recall that $\operatorname{curl}\operatorname{curl} \boldsymbol{A} \in L^2(D_\eta/\Lambda)^3$ and

$$(\operatorname{curl} \boldsymbol{A})_\parallel^\perp = \boldsymbol{\gamma}^\perp + \nabla^\perp \Phi + \alpha \nabla \Delta^{-1}(\nabla \cdot \boldsymbol{A}_\parallel^\perp)$$

holds in $H^{-\frac{1}{2}}(\mathbb{R}^2/\Lambda)^2$; hence $(\operatorname{curl} \boldsymbol{A})_\parallel^\perp \in H^{\frac{1}{2}}(\mathbb{R}^2/\Lambda)^2$ (because the right-hand side of this equation belongs to $H^{\frac{1}{2}}(\mathbb{R}^2/\Lambda)^2$). Since $0 = \operatorname{div}\operatorname{curl} \boldsymbol{A} \in L^2(D_\eta/\Lambda)$ and $\operatorname{curl} \boldsymbol{A} \cdot \boldsymbol{e}_3|_{z=-h} = 0$ it follows that $\operatorname{curl} \boldsymbol{A} \in H^1(D_\eta/\Lambda)^3$ (Proposition 2.2(ii)), and furthermore $\operatorname{curl} \boldsymbol{A} \in H^1(D_\eta/\Lambda)^3$, $0 = \operatorname{div} \boldsymbol{A} \in H^1(D_\eta/\Lambda)$ with $\boldsymbol{A} \times \boldsymbol{e}_3|_{z=-h} = \boldsymbol{0}$, $\boldsymbol{A} \cdot \boldsymbol{n}|_{z=\eta} = 0$ imply that $\boldsymbol{A} \in H^2(D_\eta/\Lambda)^3$ (Proposition 2.2(iii)). Finally note that (2.2) holds in $L^2(D_\eta/\Lambda)^3$ because it holds in the sense of distributions and $\boldsymbol{A} \in H^2(D_\eta/\Lambda)^3$.

(ii) Clearly $0 = \operatorname{div}\operatorname{curl} \boldsymbol{B} \in L^2(D_\eta/\Lambda)$ and $\operatorname{curl}\operatorname{curl} \boldsymbol{B} \in L^2(D_\eta/\Lambda)^3$ because (2.7) is satisfied in the sense of distributions and $\operatorname{curl} \boldsymbol{B} \in L^2(D_\eta/\Lambda)^3$; furthermore

$$\operatorname{curl} \boldsymbol{B} \cdot \boldsymbol{N}|_{z=\eta} = \nabla \cdot \boldsymbol{B}_\parallel^\perp = \nabla \cdot \boldsymbol{g}^\perp \in \mathring{H}^{\frac{1}{2}}(\mathbb{R}^2/\Lambda), \qquad \operatorname{curl} \boldsymbol{B} \cdot \boldsymbol{e}_3|_{z=-h} = 0,$$

so that $\operatorname{curl} \boldsymbol{B} \in H^1(D_\eta/\Lambda)^3$ by Proposition 2.2(ii). Next note that $\operatorname{curl} \boldsymbol{B} \in H^1(D_\eta/\Lambda)^3$, $0 = \operatorname{div} \boldsymbol{B} \in H^1(D_\eta/\Lambda)$ with $\boldsymbol{B} \times \boldsymbol{e}_3|_{z=-h} = \boldsymbol{0}$, $\boldsymbol{B} \cdot \boldsymbol{n}|_{z=\eta} = 0$ implies that $\boldsymbol{B} \in H^2(D_\eta/\Lambda)^3$ by Proposition 2.2(iii), and (2.7) holds in $L^2(D_\eta/\Lambda)^3$ because it holds in the sense of distributions and $\boldsymbol{B} \in H^2(D_\eta/\Lambda)^3$. Finally

$$\underbrace{\int_\Omega \int_{-h}^{\eta} (\operatorname{curl}\operatorname{curl} \boldsymbol{B} - \alpha \operatorname{curl} \boldsymbol{B}) \cdot \boldsymbol{D}}_{=0} + \int_\Omega \Big((\operatorname{curl} \boldsymbol{B})_\parallel^\perp - \boldsymbol{\gamma}^\perp - \alpha \nabla \Delta^{-1}(\nabla \cdot \boldsymbol{B}_\parallel^\perp)\Big) \cdot \boldsymbol{D}_\parallel = 0$$

for all $\boldsymbol{D} \in \mathcal{X}_\eta^0$, which implies that $(\operatorname{curl} \boldsymbol{B})_\parallel^\perp = \boldsymbol{\gamma}^\perp + \nabla^\perp \Phi + \alpha \nabla \Delta^{-1}(\nabla \cdot \boldsymbol{B}_\parallel^\perp)$ for some $\Phi \in \mathring{H}^{\frac{3}{2}}(\mathbb{R}^2/\Lambda)$ and in particular that (2.12) holds. □

Remark 2.7. Suppose that $\boldsymbol{B} \in H^2(D_\eta/\Lambda)^3$ satisfies (2.7)–(2.12). The orthogonal gradient part of $(\operatorname{curl}\mathbf{B})_\parallel$ is equal to $-\alpha\,\nabla^\perp\Delta^{-1}(\nabla\cdot\boldsymbol{B}_\parallel^\perp)$.

Corollary 2.8. *The formulae* (2.1) *define linear operators* $H(\eta)\colon \mathbb{R}^2\times\mathring{H}^{\frac{3}{2}}(\mathbb{R}^2/\Lambda)\to\mathring{H}^{\frac{1}{2}}(\mathbb{R}^2/\Lambda)$ *and* $\boldsymbol{M}(\eta)\colon \mathbb{R}^2\times H^{\frac{3}{2}}(\mathbb{R}^2/\Lambda)^2\to H^{\frac{1}{2}}(\mathbb{R}^2/\Lambda)^2$.

2.2. Analyticity

In this section we show that improved regularity of η, Φ and $\boldsymbol{g}$ leads to improved regularity of the solution to the boundary-value problems (2.2)–(2.6) and (2.7)–(2.12) and use this result to deduce that $H(\eta)$ and $\boldsymbol{M}(\eta)$ depend analytically upon η (see Theorem 2.11(i) below for a precise statement). We proceed by transforming (2.2)–(2.6) and (2.7)–(2.12) into equivalent boundary-value problems in the fixed domain D_0 by means of the following flattening transformation. Define $\Sigma\colon D_0\to D_\eta$ by

$$\Sigma\colon (\boldsymbol{x}',v)\mapsto(\boldsymbol{x}',v+\sigma(\boldsymbol{x}',v)),\qquad \sigma(\boldsymbol{x}',v):=\eta(\boldsymbol{x}')(1+v/h),$$

and for $f\colon D_\eta\to\mathbb{R}$ and $\boldsymbol{F}\colon D_\eta\to\mathbb{R}^3$ write $\tilde{f}=f\circ\Sigma$, $\tilde{\boldsymbol{F}}=\boldsymbol{F}\circ\Sigma$ and use the notation

$$\begin{aligned}
\operatorname{grad}^\sigma\tilde{f}(\boldsymbol{x}',v)&:=(\operatorname{grad}f)\circ\Sigma(\boldsymbol{x}',v),\\
\operatorname{div}^\sigma\tilde{f}(\boldsymbol{x}',v)&:=(\operatorname{div}f)\circ\Sigma(\boldsymbol{x}',v),\\
\operatorname{curl}^\sigma\tilde{\boldsymbol{F}}(\boldsymbol{x}',v)&:=(\operatorname{curl}\boldsymbol{F})\circ\Sigma(\boldsymbol{x}',v),\\
\Delta^\sigma\tilde{f}(\boldsymbol{x}',v)&:=(\Delta f)\circ\Sigma(\boldsymbol{x}',v)
\end{aligned}$$

and more generally

$$\partial_x^\sigma:=\partial_x-\frac{\partial_x\sigma}{1+\partial_v\sigma}\partial_v,\qquad \partial_y^\sigma:=\partial_y-\frac{\partial_y\sigma}{1+\partial_v\sigma}\partial_v,\qquad \partial_v^\sigma:=\frac{\partial_v}{1+\partial_v\sigma}.$$

Remark 2.9. The flattened versions of the operators curl, div, grad and Δ applied to $\tilde{\boldsymbol{F}}(x,y,v)=\boldsymbol{F}(x,y,z)$ and to $\tilde{f}(x,y,v)=f(x,y,z)$ are given explicitly by

$$\begin{aligned}
\operatorname{curl}^\sigma\tilde{\boldsymbol{F}}&=\operatorname{curl}\tilde{\boldsymbol{F}}-\frac{\eta}{\eta+h}(-\partial_v\tilde{F}_2,\partial_v\tilde{F}_1,0)^T\\
&\qquad-\frac{h+v}{\eta+h}(\eta_y\,\partial_v\tilde{F}_3,-\eta_x\,\partial_v\tilde{F}_3,\eta_x\,\partial_v\tilde{F}_2-\eta_y\,\partial_v\tilde{F}_1)^T,\\
\operatorname{div}^\sigma\tilde{\boldsymbol{F}}&=\operatorname{div}\tilde{\boldsymbol{F}}-\frac{h+v}{\eta+h}(\eta_x\,\partial_v\tilde{F}_1+\eta_y\,\partial_v\tilde{F}_2)-\frac{\eta}{\eta+h}\partial_v\tilde{F}_3,\\
\operatorname{grad}^\sigma\tilde{f}&=\operatorname{grad}\tilde{f}-\frac{h+v}{\eta+h}(\eta_x\,\partial_v\tilde{f},\eta_y\,\partial_v\tilde{f},0)^T-\frac{\eta}{\eta+h}(0,0,\partial_v\tilde{f})^T,\\
\Delta^\sigma\tilde{f}&=\Delta\tilde{f}-2\frac{h+v}{\eta+h}(\eta_x\,\partial_{vx}^2\tilde{f}+\eta_y\,\partial_{vy}^2\tilde{f})-\frac{h+v}{\eta+h}(\eta_{xx}+\eta_{yy})\partial_v\tilde{f}
\end{aligned}$$

$$+ 2\frac{h+v}{(\eta+h)^2}(\,(\eta_x)^2+(\eta_y)^2\,)\partial_v\tilde{f}+\left(\frac{h+v}{\eta+h}\right)^2(\,(\eta_x)^2+(\eta_y)^2\,)\partial_v^2\tilde{f}$$
$$-\frac{\eta^2+2h\eta}{(\eta+h)^2}\partial_v^2\tilde{f}.$$

Equations (2.2)–(2.6) are equivalent to the flattened boundary-value problem

$$\operatorname{curl}^\sigma \operatorname{curl}^\sigma \tilde{\boldsymbol{A}} - \alpha \operatorname{curl}^\sigma \tilde{\boldsymbol{A}} = \boldsymbol{0} \qquad \text{in } D_0, \tag{2.16}$$
$$\operatorname{div}^\sigma \boldsymbol{A} = 0 \qquad \text{in } D_0, \tag{2.17}$$
$$\tilde{\boldsymbol{A}} \times \boldsymbol{e}_3 = \boldsymbol{0} \qquad \text{at } v=-h, \tag{2.18}$$
$$\tilde{\boldsymbol{A}} \cdot \boldsymbol{N} = 0 \qquad \text{at } v=0, \tag{2.19}$$
$$(\operatorname{curl}^\sigma \tilde{\boldsymbol{A}})_\parallel = \boldsymbol{\gamma} + \nabla\Phi - \alpha\nabla^\perp\Delta^{-1}(\nabla\cdot\tilde{\boldsymbol{A}}_\parallel^\perp) \qquad \text{at } v=0, \tag{2.20}$$

in terms of which

$$H(\eta)(\boldsymbol{\gamma},\Phi) = \nabla\cdot\tilde{\boldsymbol{A}}_\parallel^\perp,$$

while equations (2.7)–(2.12) are equivalent to the flattened boundary-value problem

$$\operatorname{curl}^\sigma \operatorname{curl}^\sigma \tilde{\boldsymbol{B}} - \alpha \operatorname{curl}^\sigma \tilde{\boldsymbol{B}} = \boldsymbol{0} \qquad \text{in } D_0, \tag{2.21}$$
$$\operatorname{div}^\sigma \boldsymbol{B} = 0 \qquad \text{in } D_0, \tag{2.22}$$
$$\tilde{\boldsymbol{B}} \times \boldsymbol{e}_3 = \boldsymbol{0} \qquad \text{at } v=-h, \tag{2.23}$$
$$\tilde{\boldsymbol{B}} \cdot \boldsymbol{N} = 0 \qquad \text{at } v=0, \tag{2.24}$$
$$\nabla\cdot\tilde{\boldsymbol{B}}_\parallel^\perp = \nabla\cdot\boldsymbol{g}^\perp, \tag{2.25}$$
$$\langle(\operatorname{curl}^\sigma \tilde{\boldsymbol{B}})_\parallel\rangle = \boldsymbol{\gamma}, \tag{2.26}$$

in terms of which

$$\boldsymbol{M}(\eta)(\boldsymbol{\gamma},\boldsymbol{g}) = -(\operatorname{curl}^\sigma \tilde{\boldsymbol{B}})_\parallel;$$

note that the orthogonal gradient part of $(\operatorname{curl}^\sigma \tilde{\boldsymbol{B}})_\parallel$ is equal to $-\alpha\,\nabla^\perp\Delta^{-1}(\nabla\cdot\tilde{\boldsymbol{B}}_\parallel^\perp)$ for any solution $\tilde{\boldsymbol{B}} \in H^2(D_0/\Lambda)^3$ of (2.21)–(2.25). The spatially extended version of the first of the above boundary-value problems was studied by Groves and Horn [12, §4], whose analysis in particular leads to the following result in the present context.

Theorem 2.10. *Suppose that $s \geq 2$, and assume that the non-resonance condition (NR) holds. There exists an open neighbourhood U of the origin in $H^{s+\frac{1}{2}}(\mathbb{R}^2/\Lambda)$ such that the boundary-value problem (2.16)–(2.20) has a unique solution $\tilde{\boldsymbol{A}} = \tilde{\boldsymbol{A}}(\eta,\boldsymbol{\gamma},\Phi)$ in $H^s(D_0/\Lambda)^3$ which depends analytically upon $\eta \in U$, $\boldsymbol{\gamma} \in \mathbb{R}^2$ and $\Phi \in \mathring{H}^{s-\frac{1}{2}}(\mathbb{R}^2/\Lambda)$ (and linearly upon $(\boldsymbol{\gamma},\Phi)$).*

The corresponding result for the boundary-value problem (2.21)–(2.26), together with the analyticity of the operators H and M, is now readily deduced.

Theorem 2.11. *Suppose that $s \geq 2$, and assume that the non-resonance condition (NR) holds. There exists an open neighbourhood U of the origin in $H^{s+\frac{1}{2}}(\mathbb{R}^2/\Lambda)$ such that*

(i) *$\eta \mapsto H(\eta)$ and $\eta \mapsto \boldsymbol{M}(\eta)$ are analytic mappings $U \to L(\mathbb{R}^2 \times \mathring{H}^{s-\frac{1}{2}}(\mathbb{R}^2/\Lambda), \mathring{H}^{s-\frac{3}{2}}(\mathbb{R}^2/\Lambda))$ and $U \to L(\mathbb{R}^2 \times H^{s-\frac{1}{2}}(\mathbb{R}^2/\Lambda)^2, H^{s-\frac{3}{2}}(\mathbb{R}^2/\Lambda)^2)$ respectively;*
(ii) *the boundary-value problem (2.21)–(2.26) has a unique solution $\tilde{\boldsymbol{B}} = \tilde{\boldsymbol{B}}(\eta, \boldsymbol{\gamma}, \boldsymbol{g})$ in $H^s(D_0/\Lambda)^3$ which depends analytically upon $\eta \in U$ and $\boldsymbol{g} \in H^{s-\frac{3}{2}}(\mathbb{R}^2)^2$ (and linearly upon $(\boldsymbol{\gamma}, \boldsymbol{g})$).*

Proof. The analyticity of $H(\cdot)\colon U \to L(\mathbb{R}^2 \times \mathring{H}^{s-\frac{1}{2}}(\mathbb{R}^2/\Lambda), \mathring{H}^{s-\frac{3}{2}}(\mathbb{R}^2/\Lambda))$ follows from Theorem 2.10 and equation (2.1), and it follows that the formula

$$V(\eta)\begin{pmatrix}\boldsymbol{\gamma}\\ \Phi\end{pmatrix} = \begin{pmatrix}\boldsymbol{\gamma}\\ H(\eta)(\boldsymbol{\gamma}, \Phi)\end{pmatrix}$$

defines an analytic function $V\colon U \to L(\mathbb{R}^2 \times \mathring{H}^{s-\frac{1}{2}}(\mathbb{R}^2/\Lambda), \mathbb{R}^2 \times \mathring{H}^{s-\frac{3}{2}}(\mathbb{R}^2/\Lambda))$. A straightforward calculation shows that

$$V(0)\begin{pmatrix}\boldsymbol{\gamma}\\ \Phi\end{pmatrix} = \begin{pmatrix}\boldsymbol{\gamma}\\ D^2\mathtt{t}(D)\end{pmatrix},$$

and $V(0) \in L(\mathbb{R}^2 \times \mathring{H}^{s-\frac{1}{2}}(\mathbb{R}^2/\Lambda), \mathbb{R}^2 \times \mathring{H}^{s-\frac{3}{2}}(\mathbb{R}^2/\Lambda))$ is an isomorphism because

$$\lim_{|\boldsymbol{k}|\to\infty} \frac{|\boldsymbol{k}|}{|\boldsymbol{k}|^2\mathtt{t}(|\boldsymbol{k}|)} = 1.$$

One concludes that $V(\eta) \in L(\mathbb{R}^2 \times \mathring{H}^{s-\frac{1}{2}}(\mathbb{R}^2/\Lambda), \mathbb{R}^2 \times \mathring{H}^{s-\frac{3}{2}}(\mathbb{R}^2/\Lambda))$ is an isomorphism for each $\eta \in U$ and that $V(\eta)^{-1} \in L(\mathbb{R}^2 \times \mathring{H}^{s-\frac{3}{2}}(\mathbb{R}^2/\Lambda), \mathbb{R}^2 \times \mathring{H}^{s-\frac{1}{2}}(\mathbb{R}^2/\Lambda))$ also depends analytically upon $\eta \in U$. Clearly

$$V(\eta)^{-1} = \begin{pmatrix}\mathbb{I}_2\\ W_2(\eta)\end{pmatrix}$$

for some analytic function $W_2\colon U \to L(\mathbb{R}^2 \times \mathring{H}^{s-\frac{3}{2}}(\mathbb{R}^2/\Lambda), \mathring{H}^{s-\frac{1}{2}}(\mathbb{R}^2/\Lambda))$.

Observe that $\tilde{\boldsymbol{B}}(\eta, \boldsymbol{\gamma}, \boldsymbol{g}) := \tilde{\boldsymbol{A}}(\eta, \boldsymbol{\gamma}, \Phi)$ with $\Phi = W_2(\eta)(\boldsymbol{\gamma}, \nabla\cdot\boldsymbol{g}^\perp)$ depends analytically upon η, $\boldsymbol{\gamma}$ and $\boldsymbol{g}$, and solves (2.21)–(2.26) since by construction

$$\nabla\cdot\boldsymbol{g}^\perp = H(\eta)(\boldsymbol{\gamma}, \Phi) = \nabla\cdot\tilde{\boldsymbol{A}}(\eta, \boldsymbol{\gamma}, \Phi)_\parallel^\perp = \nabla\cdot\tilde{\boldsymbol{B}}(\eta, \boldsymbol{\gamma}, \Phi)_\parallel^\perp.$$

The uniqueness of this solution follows by noting that any other solution $\tilde{\boldsymbol{B}}(\eta, \boldsymbol{\gamma}, \boldsymbol{g})$ is equal to $\tilde{\boldsymbol{A}}(\eta, \boldsymbol{\gamma}, \Phi)$ with $\Phi = \Delta^{-1}\nabla\cdot(\mathrm{curl}^\sigma\,\boldsymbol{B})_\parallel$, so that

$$H(\eta)(\boldsymbol{\gamma}, \Phi) = \nabla\cdot\tilde{\boldsymbol{A}}(\eta, \boldsymbol{\gamma}, \Phi)_\parallel^\perp = \nabla\cdot\tilde{\boldsymbol{B}}(\eta, \boldsymbol{\gamma}, \boldsymbol{g})_\parallel^\perp = \nabla\cdot\boldsymbol{g}^\perp,$$

that is $\Phi = W_2(\eta)(\boldsymbol{\gamma}, \nabla\cdot\boldsymbol{g}^\perp)$. Finally, the analyticity of $\boldsymbol{M}$ follows from the calculation

$$\begin{aligned}\boldsymbol{M}(\eta)(\boldsymbol{\gamma},\boldsymbol{g}) &= -(\operatorname{curl}^\sigma \tilde{\boldsymbol{B}}(\eta,\boldsymbol{\gamma},\boldsymbol{g}))_\parallel \\ &= -(\operatorname{curl}^\sigma \tilde{\boldsymbol{A}}(\eta,\boldsymbol{\gamma},\Phi))_\parallel \\ &= -\boldsymbol{\gamma} - \nabla\Phi + \alpha\,\nabla^\perp\Delta^{-1}(\nabla\cdot\boldsymbol{g}^\perp)\end{aligned}$$

with $\Phi = W_2(\eta)(\boldsymbol{\gamma}, \nabla\cdot\boldsymbol{g}^\perp)$. □

Remark 2.12. It follows from the proof of Theorem 2.11 that

$$\begin{aligned}H(0)(\boldsymbol{\gamma},\Phi) &= D^2\,\mathtt{t}(D)\,\Phi,\\ \boldsymbol{M}(0)(\boldsymbol{\gamma},\boldsymbol{g}) &= -\boldsymbol{\gamma} - \nabla\left(\frac{1}{D^2\mathtt{t}(D)}\nabla\cdot\boldsymbol{g}^\perp\right) + \alpha\,\nabla^\perp\Delta^{-1}\left(\nabla\cdot\boldsymbol{g}^\perp\right)\\ &= -\boldsymbol{\gamma} + \frac{1}{D^2}\Big(\alpha\,\mathbf{D}^\perp + \mathbf{D}\,\mathtt{c}(D)\Big)\,\boldsymbol{D}\cdot\boldsymbol{g}^\perp.\end{aligned}$$

We conclude this section by recording the following flattened version of Proposition 2.4, which is established by the methods used by Groves and Horn [12, §4(c)].

Proposition 2.13. *Suppose that $s \geq 2$. There exists an open neighbourhood U of the origin in $H^{s+1/2}(\mathbb{R}^2/\Lambda)$ such that the boundary-value problem*

$$\begin{aligned}\Delta^\sigma\phi &= F && \text{in } D_0,\\ \operatorname{grad}^\sigma u\cdot\boldsymbol{N} &= f && \text{at } v=0,\\ \phi &= 0 && \text{at } v=-h\end{aligned}$$

has a unique solution $\phi \in H^s(D_0/\Lambda)$ which depends analytically upon $\eta \in U$, $F \in H^{s-2}(D_0/\Lambda)$ and $f \in H^{s-\frac{3}{2}}(\mathbb{R}^2/\Lambda)$ (and linearly upon F and f).

2.3. Differentials

In this section we derive useful formulae for the differentials $\mathrm{d}H[\eta](\delta\eta)(\boldsymbol{\gamma},\Phi)$ and $\mathrm{d}\boldsymbol{M}[\eta](\delta\eta)(\boldsymbol{\gamma},\boldsymbol{g})$, where $\eta \in U$, $\boldsymbol{\gamma} \in \mathbb{R}^2$, $\Phi \in H^{s-3/2}(\mathbb{R}^2/\Lambda)$ and $\boldsymbol{g} \in H^{s-1/2}(\mathbb{R}^2/\Lambda)^2$, so that $\tilde{\boldsymbol{A}}, \tilde{\boldsymbol{B}} \in H^s(D_0/\Lambda)^3$ (in the notation of Section 2.2), working under the stronger condition $s \geq 3$ and again assuming the non-resonance condition (NR). Recall the identity

$$\mathrm{d}(\partial_x^\sigma f) = \partial_x^\sigma(\mathrm{d}f - \mathrm{d}\sigma\,\partial_v^\sigma f) + \mathrm{d}\sigma\,\partial_v^\sigma\partial_x^\sigma f, \tag{2.27}$$

where ∂_x can be replaced by ∂_y or ∂_v and d can be any linearisation operator (see Castro and Lannes [6, Eq. (3.41)]); the quantity $\mathrm{d}f - \mathrm{d}\sigma\,\partial_v^\sigma f$ is called *Alinhac's good unknown*.

We proceed by finding a boundary-value problem for $\tilde{\boldsymbol{C}} := (\mathrm{d}\tilde{\boldsymbol{A}} - \mathrm{d}\sigma\,\partial_v^\sigma\tilde{\boldsymbol{A}}) \in H^{s-1}(D_0/\Lambda)^3$, where $\mathrm{d} = d_\eta$, observing that $H(\cdot)\colon U \to L(\mathbb{R}^2 \times \mathring{H}^{s-\frac{1}{2}}(\mathbb{R}^2/\Lambda), \mathring{H}^{s-\frac{3}{2}}(\mathbb{R}^2/\Lambda))$ and $\tilde{\boldsymbol{A}}\colon U \to L(\mathbb{R}^2 \times \mathring{H}^{s-\frac{1}{2}}(\mathbb{R}^2/\Lambda, H^s(D_0/\Lambda)^3)$ are analytic. Applying (2.27) with $\mathrm{d} = d_\eta$ to

$$H(\eta)(\boldsymbol{\gamma},\Phi) = \nabla\cdot\tilde{\boldsymbol{A}}_\parallel^\perp,$$

and to equations (2.16)–(2.20), we find that

$$\mathrm{d}H[\eta](\delta\eta)(\boldsymbol{\gamma},\Phi) = \nabla\cdot\tilde{\boldsymbol{C}}_{\parallel}^{\perp} + \partial_v^{\sigma}\,\mathrm{curl}^{\sigma}\,\tilde{\boldsymbol{A}}\cdot\boldsymbol{N}|_{v=0}\delta\eta - (\mathrm{curl}^{\sigma}\,\tilde{\boldsymbol{A}})_{\mathrm{h}}\cdot\nabla\delta\eta$$

where

$$\mathrm{curl}^{\sigma}\,\mathrm{curl}^{\sigma}\,\tilde{\boldsymbol{C}} - \alpha\,\mathrm{curl}^{\sigma}\,\tilde{\boldsymbol{C}} = \boldsymbol{0} \qquad \text{in } D_0, \tag{2.28}$$

$$\mathrm{div}^{\sigma}\,\tilde{\boldsymbol{C}} = 0 \qquad \text{in } D_0, \tag{2.29}$$

$$\tilde{\boldsymbol{C}}\times\boldsymbol{e}_3 = \boldsymbol{0} \qquad \text{at } v=-h, \tag{2.30}$$

$$\tilde{\boldsymbol{C}}\cdot\boldsymbol{N} = \nabla\delta\eta\cdot\tilde{\boldsymbol{A}}_{\mathrm{h}} - \delta\eta\,\partial_v^{\sigma}\tilde{\boldsymbol{A}}\cdot\boldsymbol{N} \qquad \text{at } v=0, \tag{2.31}$$

and

$$\begin{aligned}(\mathrm{curl}^{\sigma}\,\tilde{\boldsymbol{C}})_{\parallel}^{\perp} &= -\delta\eta\partial_v^{\sigma}(\mathrm{curl}^{\sigma}\,\tilde{\boldsymbol{A}})_{\mathrm{h}} - \delta\eta\partial_v^{\sigma}(\mathrm{curl}^{\sigma}\,\tilde{\boldsymbol{A}})_3|_{v=0}\nabla\eta \\ &\quad - (\mathrm{curl}^{\sigma}\,\tilde{\boldsymbol{A}})_3|_{v=0}\nabla\delta\eta - \alpha\nabla^{\perp}\Delta^{-1}(\nabla\cdot\tilde{\boldsymbol{C}}_{\parallel}^{\perp} - \nabla\cdot((\mathrm{curl}^{\sigma}\,\tilde{\boldsymbol{A}})_{\mathrm{h}})\delta\eta).\end{aligned} \tag{2.32}$$

(Equation (2.31) can be rewritten as

$$\tilde{\boldsymbol{C}}\cdot\boldsymbol{N} = \nabla\cdot(\tilde{\boldsymbol{A}}_{\mathrm{h}}\delta\eta)$$

because $\mathrm{div}^{\sigma}\,\tilde{\boldsymbol{A}}|_{v=0} = 0$ implies that

$$\partial_v^{\sigma}\tilde{\boldsymbol{A}}\cdot\boldsymbol{N}|_{v=0} = -\nabla\cdot\tilde{\boldsymbol{A}}_{\mathrm{h}}.\ )$$

Using the relation

$$-(\partial_v^{\sigma}\,\mathrm{curl}\,\tilde{\boldsymbol{A}})_{\parallel} = -\nabla(\mathrm{curl}\,\tilde{\boldsymbol{A}})_3|_{v=0} - \alpha(\mathrm{curl}^{\sigma}\,\tilde{\boldsymbol{A}})_{\mathrm{h}}^{\perp}$$

we can rewrite equation (2.32) as

$$\begin{aligned}(\mathrm{curl}^{\sigma}\,\tilde{\boldsymbol{C}})_{\parallel}^{\perp} &= -\nabla((\mathrm{curl}^{\sigma}\,\tilde{\boldsymbol{A}})_3|_{v=0}\delta\eta) - \alpha(\mathrm{curl}^{\sigma}\,\tilde{\boldsymbol{A}})_{\mathrm{h}}^{\perp}\delta\eta \\ &\qquad - \alpha\nabla^{\perp}\Delta^{-1}\nabla\cdot\tilde{\boldsymbol{C}}_{\parallel}^{\perp} + \alpha\nabla^{\perp}\Delta^{-1}\nabla^{\perp}\cdot((\mathrm{curl}^{\sigma}\,\tilde{\boldsymbol{A}})_{\mathrm{h}}^{\perp}\delta\eta) \\ &= -\alpha\langle(\mathrm{curl}^{\sigma}\,\tilde{\boldsymbol{A}})_{\mathrm{h}}^{\perp}\delta\eta\rangle - \alpha\nabla\Delta^{-1}\nabla\cdot((\mathrm{curl}^{\sigma}\,\tilde{\boldsymbol{A}})_{\mathrm{h}}^{\perp}\delta\eta) \\ &\qquad - \nabla((\mathrm{curl}^{\sigma}\,\tilde{\boldsymbol{A}})_3|_{v=0}\delta\eta) - \alpha\nabla^{\perp}\Delta^{-1}\nabla\cdot\tilde{\boldsymbol{C}}_{\parallel}^{\perp},\end{aligned}$$

and writing $\tilde{\boldsymbol{C}} = \tilde{\boldsymbol{C}}' + \mathrm{grad}^{\sigma}\,\varphi$, where $\varphi\in H^s(D_0)$ is the unique solution of the boundary-value problem

$$\begin{aligned}\Delta^{\sigma}\varphi &= 0 && \text{in } D_0, \\ \mathrm{grad}^{\sigma}\,\varphi\cdot\boldsymbol{N} &= \nabla\cdot(\tilde{\boldsymbol{A}}_{\mathrm{h}}\delta\eta) && \text{at } v=0, \\ \varphi &= 0 && \text{at } v=-h\end{aligned}$$

(see Proposition 2.13), one finds that

$$\mathrm{d}H[\eta](\delta\eta)(\boldsymbol{\gamma},\Phi)=\nabla\cdot\tilde{\boldsymbol{C}}_{\parallel}^{\prime\perp}+\partial_v^\sigma\operatorname{curl}^\sigma\tilde{\boldsymbol{A}}\cdot\boldsymbol{N}|_{v=0}\delta\eta-(\operatorname{curl}^\sigma\tilde{\boldsymbol{A}})_{\mathrm{h}}\cdot\nabla\delta\eta,$$

where

$$\begin{aligned}\operatorname{curl}^\sigma\operatorname{curl}^\sigma\tilde{\boldsymbol{C}}'-\alpha\operatorname{curl}^\sigma\tilde{\boldsymbol{C}}'&=\boldsymbol{0} &&\text{in } D_0,\\ \operatorname{div}^\sigma\tilde{\boldsymbol{C}}'&=0 &&\text{in } D_0,\\ \tilde{\boldsymbol{C}}'\times\boldsymbol{e}_3&=\boldsymbol{0} &&\text{at } v=-h,\\ \tilde{\boldsymbol{C}}'\cdot\boldsymbol{N}&=0 &&\text{at } v=0,\end{aligned}$$

and

$$\begin{aligned}(\operatorname{curl}^\sigma\tilde{\boldsymbol{C}}')_{\parallel}^{\perp}=&-\alpha\langle(\operatorname{curl}^\sigma\tilde{\boldsymbol{A}})_{\mathrm{h}}^{\perp}\delta\eta\rangle-\alpha\nabla\Delta^{-1}\nabla\cdot((\operatorname{curl}^\sigma\tilde{\boldsymbol{A}})_{\mathrm{h}}^{\perp}\delta\eta)\\ &-\nabla((\operatorname{curl}^\sigma\tilde{\boldsymbol{A}})_3|_{v=0}\delta\eta)-\alpha\nabla^{\perp}\Delta^{-1}\nabla\cdot\tilde{\boldsymbol{C}}_{\parallel}^{\prime\perp}.\end{aligned}$$

It follows that

$$\begin{aligned}&\mathrm{d}H[\eta](\delta\eta)(\boldsymbol{\gamma},\Phi)\\ &=H(\eta)\Big(-\alpha\langle(\operatorname{curl}^\sigma\tilde{\boldsymbol{A}})_{\mathrm{h}}^{\perp}\delta\eta\rangle,-\alpha\Delta^{-1}\nabla\cdot((\operatorname{curl}^\sigma\tilde{\boldsymbol{A}})_{\mathrm{h}}^{\perp}\delta\eta)\\ &\qquad-(\operatorname{curl}^\sigma\tilde{\boldsymbol{A}})_3|_{v=0}\delta\eta+\langle(\operatorname{curl}^\sigma\tilde{\boldsymbol{A}})_3|_{v=0}\delta\eta\rangle\Big)\\ &\quad-\nabla\cdot((\operatorname{curl}^\sigma\tilde{\boldsymbol{A}})_{\mathrm{h}}\delta\eta),\end{aligned}$$

and we obtain our final theorem by setting $u:=(\operatorname{curl}^\sigma\tilde{\boldsymbol{A}})_3|_{v=0}$.

Theorem 2.14. *Suppose that $s\geq 3$ and that the non-resonance condition (NR) holds. The differential of the operator $H(\cdot)\colon U\to L(\mathbb{R}^2\times\mathring{H}^{s-\frac{1}{2}}(\mathbb{R}^2/\Lambda),\mathring{H}^{s-\frac{3}{2}}(\mathbb{R}^2/\Lambda))$ is given by*

$$\begin{aligned}&\mathrm{d}H[\eta](\delta\eta)(\boldsymbol{\gamma},\Phi)\\ &=H(\eta)\Big(-\alpha\langle(\boldsymbol{K}(\eta)(\boldsymbol{\gamma},\Phi)-u\nabla\eta)^{\perp}\delta\eta\rangle,\\ &\qquad-\alpha\Delta^{-1}\nabla\cdot((\boldsymbol{K}(\eta)(\boldsymbol{\gamma},\Phi)-u\nabla\eta)^{\perp}\delta\eta)-u\delta\eta+\langle u\delta\eta\rangle\Big)\\ &\quad-\nabla\cdot((\boldsymbol{K}(\eta)(\boldsymbol{\gamma},\Phi)-u\nabla\eta)\delta\eta),\end{aligned}$$

where

$$\boldsymbol{K}(\eta)(\boldsymbol{\gamma},\Phi)=\boldsymbol{\gamma}+\nabla\Phi-\alpha\nabla^{\perp}\Delta^{-1}H(\eta)(\boldsymbol{\gamma},\Phi),\qquad u=\frac{\boldsymbol{K}(\eta)(\boldsymbol{\gamma},\Phi)\cdot\nabla\eta+H(\eta)(\gamma,\Phi)}{1+|\nabla\eta|^2}.$$

The corresponding result for $\boldsymbol{M}(\eta)$ is obtained in a similar fashion.

Theorem 2.15. *Suppose that $s \geq 3$ and that the non-resonance condition (NR) holds. The differential of the operator $\boldsymbol{M}(\cdot)\colon U \to L(\mathbb{R}^2 \times H^{s-\frac{1}{2}}(\mathbb{R}^2/\Lambda)^2, H^{s-\frac{3}{2}}(\mathbb{R}^2/\Lambda)^2)$ is given by*

$$\begin{aligned}
&\mathrm{d}\boldsymbol{M}[\eta](\delta\eta)(\boldsymbol{\gamma}, \boldsymbol{g}) \\
&\quad = \boldsymbol{M}(\eta)\left(\alpha\langle(\boldsymbol{M}(\eta)(\boldsymbol{\gamma}, \boldsymbol{g}) + u\nabla\eta)^{\perp}\delta\eta\rangle, (\boldsymbol{M}(\eta)(\boldsymbol{\gamma}, \boldsymbol{g}) + u\nabla\eta)^{\perp}\delta\eta\right) \\
&\qquad - \nabla(u\delta\eta) + \alpha(\boldsymbol{M}(\eta)(\boldsymbol{\gamma}, \boldsymbol{g}) + u\delta\eta)^{\perp}\delta\eta,
\end{aligned}$$

where

$$u = \frac{\nabla\cdot\boldsymbol{g}^{\perp} - \boldsymbol{M}(\eta)(\boldsymbol{\gamma}, \boldsymbol{g})\cdot\nabla\eta}{1+|\nabla\eta|^2}.$$

2.4. Taylor expansions

The terms in the expansion

$$H(\eta) = \sum_{k=0}^{\infty} H_k(\eta), \tag{2.33}$$

where $H_k(\eta)$ is homogeneous of degree k in η, can be calculated recursively from the equation

$$\begin{aligned}
\mathrm{d}H[\eta](\eta)(\boldsymbol{\gamma}, \Phi) = H(\eta)\Big(&-\alpha\,\langle(\boldsymbol{K}(\eta)(\boldsymbol{\gamma}, \Phi) - u\,\nabla\eta)^{\perp}\,\eta\rangle, \\
&-\alpha\,\Delta^{-1}\nabla\cdot(\boldsymbol{K}(\eta)(\boldsymbol{\gamma}, \Phi) - u\,\nabla\eta)^{\perp}\,\eta - u\,\eta + \langle u\,\eta\rangle\Big) \\
&- \nabla\cdot((\boldsymbol{K}(\eta)(\boldsymbol{\gamma}, \Phi) - u\,\nabla\eta)\,\eta)\,,
\end{aligned}$$

(see Theorem 2.14), and the explicit formula

$$H_0(\boldsymbol{\gamma}, \Phi) = D^2\,\mathtt{t}(D)\,\Phi$$

(see Remark 2.12); these results hold under the non-resonance condition (NR). (Note that we suppress the argument in the η-independent terms in Taylor series of this kind).

Expanding

$$\boldsymbol{K}(\eta)(\boldsymbol{\gamma}, \Phi) = \sum_{k=0}^{\infty} \boldsymbol{K}_k(\eta)(\boldsymbol{\gamma}, \Phi), \qquad u(\eta)(\boldsymbol{\gamma}, \Phi) = \sum_{k=0}^{\infty} u_k(\eta)(\boldsymbol{\gamma}, \Phi), \tag{2.34}$$

we find that

$$\begin{aligned}
&\boldsymbol{K}_0(\boldsymbol{\gamma}, \Phi) = \boldsymbol{\gamma} + \nabla\Phi - \alpha\,\nabla^{\perp}\,\Delta^{-1}H_0(\boldsymbol{\gamma}, \Phi), \\
&u_0(\boldsymbol{\gamma}, \Phi) = H_0(\boldsymbol{\gamma}, \Phi),
\end{aligned}$$

and

$$\boldsymbol{K}_k(\eta)(\boldsymbol{\gamma},\Phi) = -\alpha\,\nabla^\perp\,\Delta^{-1} H_k(\eta)(\boldsymbol{\gamma},\Phi),$$

$$u_k(\eta)(\boldsymbol{\gamma},\Phi) = \begin{cases} (-1)^{k/2}|\nabla\eta|^k\, H_0(\boldsymbol{\gamma},\Phi) \\ \quad + \displaystyle\sum_{i+2j=k} (\boldsymbol{K}_{i-1}(\eta)(\boldsymbol{\gamma},\Phi)\cdot\nabla\eta + H_i(\eta)(\boldsymbol{\gamma},\Phi))\,(-1)^j|\nabla\eta|^{2j}, & \text{if } k\in 2\mathbb{N}, \\ \displaystyle\sum_{i+2j=k} (\boldsymbol{K}_{i-1}(\eta)(\boldsymbol{\gamma},\Phi)\cdot\nabla\eta + H_i(\eta)(\boldsymbol{\gamma},\Phi))\,(-1)^j|\nabla\eta|^{2j}, & \text{if } k\notin 2\mathbb{N}, \end{cases}$$

for $k \geq 1$, and inserting the expansions (2.33) and (2.34) into the formula for $\mathrm{d}H[\eta](\eta)$ yields

$$\begin{aligned} &\sum_{k\geq 0} k H_k(\eta)(\boldsymbol{\gamma},\Phi) \\ &\quad = \sum_{k\geq 0} H_k(\eta)\Big(-\alpha\,\langle \boldsymbol{K}_0(\boldsymbol{\gamma},\Phi)^\perp\eta\rangle, -\alpha\,\Delta^{-1}\nabla\cdot\big(\boldsymbol{K}_0(\boldsymbol{\gamma},\Phi)^\perp\eta\big) - H_0(\boldsymbol{\gamma},\Phi)\eta + \langle H_0(\boldsymbol{\gamma},\Phi)\eta\rangle\Big) \\ &\qquad + \sum_{k\geq 1}\sum_{j=1}^{k} H_{k-j}(\eta)\Big(-\alpha\,\langle\big(\boldsymbol{K}_j(\eta)(\boldsymbol{\gamma},\Phi) - u_{j-1}(\eta)(\boldsymbol{\gamma},\Phi)\,\nabla\eta\big)^\perp\,\eta\rangle, \\ &\qquad\qquad -\alpha\,\Delta^{-1}\nabla\cdot\Big(\big(\boldsymbol{K}_j(\eta)(\boldsymbol{\gamma},\Phi) - u_{j-1}(\eta)(\boldsymbol{\gamma},\Phi)\nabla\eta\big)^\perp\,\eta\Big) \\ &\qquad\qquad - u_j(\eta)(\boldsymbol{\gamma},\Phi)\eta + \langle u_j(\eta)(\boldsymbol{\gamma},\Phi)\eta\rangle\Big) \\ &\qquad - \nabla\cdot\big(\boldsymbol{K}_0(\boldsymbol{\gamma},\Phi)\eta\big) - \sum_{k\geq 1}\nabla\cdot\big((\boldsymbol{K}_k(\eta)(\boldsymbol{\gamma},\Phi) - u_{k-1}(\eta)(\boldsymbol{\gamma},\Phi)\,\nabla\eta)\,\eta\big), \end{aligned}$$

so that

$$\begin{aligned} &H_1(\eta)(\boldsymbol{\gamma},\Phi) \\ &\quad = H_0\Big(-\alpha\,\langle\boldsymbol{K}_0(\boldsymbol{\gamma},\Phi)^\perp\eta\rangle, -\alpha\,\Delta^{-1}\nabla\cdot\boldsymbol{K}_0(\boldsymbol{\gamma},\Phi)^\perp\eta - H_0(\boldsymbol{\gamma},\Phi)\,\eta + \langle H_0(\boldsymbol{\gamma},\Phi)\,\eta\rangle\Big) \\ &\qquad - \nabla\cdot(\boldsymbol{K}_0(\boldsymbol{\gamma},\Phi)\eta), \end{aligned}$$

and

$$\begin{aligned} &H_k(\eta)(\boldsymbol{\gamma},\Phi) \\ &\quad = \frac{1}{k}\Big\{H_{k-1}(\eta)\Big(-\alpha\,\langle\boldsymbol{K}_0(\boldsymbol{\gamma},\Phi)^\perp\eta\rangle, -\alpha\,\Delta^{-1}\nabla\cdot\big(\boldsymbol{K}_0(\boldsymbol{\gamma},\Phi)^\perp\eta\big) - H_0(\boldsymbol{\gamma},\Phi)\eta + \langle H_0(\boldsymbol{\gamma},\Phi)\eta\rangle\Big) \\ &\qquad + \sum_{j=0}^{k-2} H_j(\eta)\Big(-\alpha\,\langle\big(\boldsymbol{K}_{k-1-j}(\eta)(\boldsymbol{\gamma},\Phi) - u_{k-2-j}(\eta)(\boldsymbol{\gamma},\Phi)\,\nabla\eta\big)^\perp\,\eta\rangle, \\ &\qquad\qquad -\alpha\,\Delta^{-1}\nabla\cdot\Big(\big(\boldsymbol{K}_{k-1-j}(\eta)(\boldsymbol{\gamma},\Phi) - u_{k-2-j}(\eta)(\boldsymbol{\gamma},\Phi)\nabla\eta\big)^\perp\,\eta\Big) \\ &\qquad\qquad - u_{k-1-j}(\eta)(\boldsymbol{\gamma},\Phi)\eta + \langle u_{k-1-j}(\eta)(\boldsymbol{\gamma},\Phi)\eta\rangle\Big) \\ &\qquad - \nabla\cdot\big((\boldsymbol{K}_{k-1}(\eta)(\boldsymbol{\gamma},\Phi) - u_{k-2}(\eta)(\boldsymbol{\gamma},\Phi)\,\nabla\eta)\,\eta\big)\Big\}, \end{aligned}$$

for $k \geq 2$.

In particular, we find that

$$H_0(\boldsymbol{\gamma},\Phi)=H_0\Phi,$$
$$H_1(\eta)(\boldsymbol{\gamma},\Phi)=-\alpha\,H_0\Delta^{-1}\nabla\cdot\boldsymbol{K}_0(\boldsymbol{\gamma},\Phi)^{\perp}\eta-H_0(\eta\,H_0\Phi)-\nabla\cdot(\boldsymbol{K}_0(\boldsymbol{\gamma},\Phi)\eta)\,,$$

where $H_0=D^2\mathtt{t}(D)$ and $\boldsymbol{K}_0(\boldsymbol{\gamma},\Phi)=\boldsymbol{\gamma}+\nabla\Phi-\alpha\,\nabla^{\perp}\Delta^{-1}\,H_0\Phi$, and that

$$\begin{aligned}
&H_2(\eta)(\boldsymbol{\gamma},\Phi)\\
&=\frac{1}{2}\Big\{H_1(\eta)\Big(-\alpha\,\langle\boldsymbol{K}_0(\boldsymbol{\gamma},\Phi)^{\perp}\eta\rangle,-\alpha\,\Delta^{-1}\nabla\cdot\Big[\boldsymbol{K}_0(\boldsymbol{\gamma},\Phi)^{\perp}\eta\Big]-\eta H_0\Phi+\langle\eta H_0\Phi\rangle\Big)\\
&\qquad+H_0\Big(-\alpha\,\Delta^{-1}\nabla\cdot\big((\boldsymbol{K}_1(\eta)(\boldsymbol{\gamma},\Phi)-H_0\Phi\nabla\eta)^{\perp}\,\eta\big)-u_1(\eta)(\boldsymbol{\gamma},\Phi)\eta\Big)\\
&\qquad-\nabla\cdot\big((\boldsymbol{K}_1(\eta)(\boldsymbol{\gamma},\Phi)-H_0\Phi\,\nabla\eta)\,\eta\big)\Big\}\\
&=\frac{1}{2}\Big\{-\alpha\,H_0\Delta^{-1}\nabla\cdot\Big(\boldsymbol{K}_0\Big(-\alpha\,\langle\boldsymbol{K}_0(\boldsymbol{\gamma},\Phi)^{\perp}\eta\rangle,-\alpha\,\Delta^{-1}\nabla\cdot\Big(\boldsymbol{K}_0(\boldsymbol{\gamma},\Phi)^{\perp}\eta\Big)\\
&\qquad\qquad-\eta\,H_0\Phi+\langle\eta\,H_0\Phi\rangle\Big)^{\perp}\eta\Big)\\
&\qquad+\alpha\,H_0\Big(\eta\,H_0\Delta^{-1}\nabla\cdot(\boldsymbol{K}_0(\boldsymbol{\gamma},\Phi)^{\perp}\eta)\Big)+H_0(\eta\,H_0(\eta\,H_0\Phi))\\
&\qquad-\nabla\cdot\Big(\boldsymbol{K}_0\Big(-\alpha\,\langle\boldsymbol{K}_0(\boldsymbol{\gamma},\Phi)^{\perp}\eta\rangle,-\alpha\,\Delta^{-1}\nabla\cdot\Big(\boldsymbol{K}_0(\boldsymbol{\gamma},\Phi)^{\perp}\eta\Big)-H_0(\boldsymbol{\gamma},\Phi)\eta\Big)\,\eta\Big)\\
&\qquad-\alpha H_0\Delta^{-1}\nabla\cdot\Big(\eta\Big(-\alpha\nabla^{\perp}\Delta^{-1}\Big(-\alpha H_0\Delta^{-1}\nabla\cdot\boldsymbol{K}_0(\boldsymbol{\gamma},\Phi)^{\perp}\eta\\
&\qquad\qquad-H_0(\eta H_0\Phi)-\nabla\cdot(\boldsymbol{K}_0(\boldsymbol{\gamma},\Phi)\eta)\Big)-H_0\Phi\nabla\eta\Big)^{\perp}\Big)\\
&\qquad-H_0\Big(\eta\,\boldsymbol{K}_0(\boldsymbol{\gamma},\Phi)\cdot\nabla\eta-\alpha\eta\,H_0\Delta^{-1}\nabla\cdot\Big(\boldsymbol{K}_0(\boldsymbol{\gamma},\Phi)^{\perp}\eta\Big)\\
&\qquad\qquad-\eta\,H_0(\eta\,H_0\Phi)-\eta\,\nabla\cdot(\boldsymbol{K}_0(\boldsymbol{\gamma},\Phi)\eta)\Big)\\
&\qquad+\nabla\cdot\Big(\Big(\alpha\nabla^{\perp}\Delta^{-1}\Big(-\alpha\,H_0\Delta^{-1}\nabla\cdot\boldsymbol{K}_0(\boldsymbol{\gamma},\Phi)^{\perp}\eta-H_0(\eta\,H_0\Phi)-\nabla\cdot(\boldsymbol{K}_0(\boldsymbol{\gamma},\Phi)\eta)\Big)\\
&\qquad\qquad+H_0\Phi\,\nabla\eta\Big)\,\eta\Big)\Big\}.
\end{aligned}$$

Remark 2.16. For $\alpha=0$ we recover the formulae for the classical Dirichlet–Neumann operator, in particular

$$H_0(\boldsymbol{\gamma}, \Phi) = H_0\Phi,$$

$$H_1(\eta)(\boldsymbol{\gamma}, \Phi) = -H_0(\eta\, H_0\Phi) - \nabla\cdot(\eta\,\nabla\Phi),$$

$$H_2(\eta)(\boldsymbol{\gamma}, \Phi) = H_0(\eta\, H_0(\eta\, H_0\Phi)) + \frac{1}{2}\,H_0(\eta^2\,\Delta\Phi) + \frac{1}{2}\,\Delta(\eta^2\,H_0\Phi),$$

where $H_0 = D\tanh(hD)$.

Similarly, the terms in the expansion

$$\boldsymbol{M}(\eta) = \sum_{k=0}^{\infty} \boldsymbol{M}_k(\eta), \tag{2.35}$$

where $\boldsymbol{M}_k(\eta)$ is homogeneous of degree k in η, can be calculated recursively from the equation

$$\begin{aligned}\mathrm{d}\boldsymbol{M}[\eta](\eta)(\boldsymbol{\gamma}, \boldsymbol{g}) = \boldsymbol{M}(\eta)\Big(\alpha\,\langle(\boldsymbol{M}(\eta)(\boldsymbol{\gamma}, \boldsymbol{g}) + u\,\nabla\eta)^{\perp}\,\eta\rangle, (\boldsymbol{M}(\eta)(\boldsymbol{\gamma}, \boldsymbol{g}) + u\,\nabla\eta)^{\perp}\,\eta\Big)\\ - \nabla(u\,\eta) + \alpha\,(\boldsymbol{M}(\eta)(\boldsymbol{\gamma}, \boldsymbol{g}) + u\,\nabla\eta)^{\perp}\,\eta\end{aligned}$$

(see Theorem 2.15) and the explicit formula

$$\boldsymbol{M}_0(\boldsymbol{\gamma}, \boldsymbol{g}) = -\boldsymbol{\gamma} + \frac{1}{D^2}\Big(\alpha\,\boldsymbol{D}^{\perp} + \boldsymbol{D}\,\mathrm{c}(D)\Big)\,\boldsymbol{D}\cdot\boldsymbol{g}^{\perp}$$

(see Remark 2.12).

Expanding

$$u(\boldsymbol{\gamma}, \boldsymbol{g}) = \sum_{k=0}^{\infty} u_k(\eta)(\boldsymbol{\gamma}, \boldsymbol{g}), \tag{2.36}$$

we find that

$$u_0(\boldsymbol{\gamma}, \boldsymbol{g}) = \nabla\cdot\boldsymbol{g}^{\perp}$$

and

$$u_k(\eta)(\boldsymbol{\gamma}, \boldsymbol{g}) = \begin{cases} (-1)^{k/2}|\nabla\eta|^k\,(\nabla\cdot\boldsymbol{g}^{\perp}) - \displaystyle\sum_{i+2j=k-1}(M_i(\eta)(\boldsymbol{\gamma}, \boldsymbol{g})\cdot\nabla\eta)\,(-1)^j|\nabla\eta|^{2j}, & \text{if } k\in 2\mathbb{N},\\ -\displaystyle\sum_{i+2j=k-1}(M_i(\eta)(\boldsymbol{\gamma}, \boldsymbol{g})\cdot\nabla\eta)\,(-1)^j|\nabla\eta|^{2j}, & \text{if } k\notin 2\mathbb{N},\end{cases}$$

for $k\geq 1$, and inserting the expansions (2.35) and (2.36) into the formula for $\mathrm{d}\boldsymbol{M}[\eta](\eta)$ yields

$$\begin{aligned}
&\sum_{k\geq 0} k\,\boldsymbol{M}_k(\eta)(\boldsymbol{\gamma},\boldsymbol{g})\\
&\quad=\sum_{k\geq 0}\boldsymbol{M}_k(\eta)\Big(\alpha\,\langle \boldsymbol{M}_0(\boldsymbol{\gamma},\boldsymbol{g})^{\perp}\eta\rangle,\boldsymbol{M}_0(\boldsymbol{\gamma},\boldsymbol{g})^{\perp}\eta\Big)\\
&\qquad+\sum_{k\geq 1}\sum_{j=1}^{k}\boldsymbol{M}_{k-j}(\eta)\Bigg(\alpha\,\langle\big(\boldsymbol{M}_j(\eta)(\boldsymbol{\gamma},\boldsymbol{g})+u_{j-1}(\eta)(\boldsymbol{\gamma},\boldsymbol{g})\,\nabla\eta\big)^{\perp}\,\eta\rangle,\\
&\qquad\qquad\big(\boldsymbol{M}_j(\eta)(\boldsymbol{\gamma},\boldsymbol{g})+u_{j-1}(\eta)(\boldsymbol{\gamma},\boldsymbol{g})\,\nabla\eta\big)^{\perp}\,\eta\Bigg)\\
&\qquad-\sum_{k\geq 0}\nabla\,(u_k(\eta)(\boldsymbol{\gamma},\boldsymbol{g})\eta)+\alpha\,\boldsymbol{M}_0(\boldsymbol{\gamma},\boldsymbol{g})^{\perp}\eta+\alpha\sum_{k\geq 1}(\boldsymbol{M}_k(\eta)(\boldsymbol{\gamma},\boldsymbol{g})+u_{k-1}(\eta)(\boldsymbol{\gamma},\boldsymbol{g})\,\nabla\eta)^{\perp}\,\eta,
\end{aligned}$$

so that

$$\boldsymbol{M}_1(\eta)(\boldsymbol{\gamma},\boldsymbol{g})=\boldsymbol{M}_0\Bigg(\alpha\,\langle \boldsymbol{M}_0(\boldsymbol{\gamma},\boldsymbol{g})^{\perp}\,\eta\rangle,\boldsymbol{M}_0(\boldsymbol{\gamma},\boldsymbol{g})^{\perp}\,\eta\Bigg)-\nabla((\nabla\cdot\boldsymbol{g}^{\perp})\,\eta)+\alpha\,\boldsymbol{M}_0(\boldsymbol{\gamma},\boldsymbol{g})^{\perp}\,\eta,$$

and

$$\begin{aligned}
&\boldsymbol{M}_k(\eta)(\boldsymbol{\gamma},\boldsymbol{g})\\
&\quad=\frac{1}{k}\Bigg\{\boldsymbol{M}_{k-1}(\eta)\Big(\alpha\,\langle \boldsymbol{M}_0(\boldsymbol{\gamma},\boldsymbol{g})^{\perp}\eta\rangle,\boldsymbol{M}_0(\boldsymbol{\gamma},\boldsymbol{g})^{\perp}\eta\Big)\\
&\qquad+\sum_{j=0}^{k-2}\boldsymbol{M}_j(\eta)\Bigg(\alpha\,\langle\big(\boldsymbol{M}_{k-1-j}(\eta)(\boldsymbol{\gamma},\boldsymbol{g})+u_{k-2-j}(\eta)(\boldsymbol{\gamma},\boldsymbol{g})\,\nabla\eta\big)^{\perp}\,\eta\rangle,\\
&\qquad\qquad\big(\boldsymbol{M}_{k-1-j}(\eta)(\boldsymbol{\gamma},\boldsymbol{g})+u_{k-2-j}(\eta)(\boldsymbol{\gamma},\boldsymbol{g})\,\nabla\eta\big)^{\perp}\,\eta\Bigg)\\
&\qquad-\nabla\,(u_{k-1}(\eta)(\boldsymbol{\gamma},\boldsymbol{g})\eta)+\alpha\,\,(\boldsymbol{M}_{k-1}(\eta)(\boldsymbol{\gamma},\boldsymbol{g})+u_{k-2}(\eta)(\boldsymbol{\gamma},\boldsymbol{g})\,\nabla\eta)^{\perp}\,\eta\Bigg\}.
\end{aligned}$$

In particular, we find that

$$\begin{aligned}
&\boldsymbol{M}_2(\eta)(\boldsymbol{\gamma},\boldsymbol{g})\\
&=\frac{1}{2}\Bigg\{\boldsymbol{M}_0\Big(\alpha\langle \boldsymbol{M}_0(\alpha\langle \boldsymbol{M}_0(\boldsymbol{\gamma},\boldsymbol{g})^{\perp}\eta\rangle,\boldsymbol{M}_0(\boldsymbol{\gamma},\boldsymbol{g})^{\perp}\eta)^{\perp}\eta\rangle,\boldsymbol{M}_0(\alpha\langle \boldsymbol{M}_0(\boldsymbol{\gamma},\boldsymbol{g})^{\perp}\eta\rangle,\boldsymbol{M}_0(\boldsymbol{\gamma},\boldsymbol{g})^{\perp}\eta)^{\perp}\eta\Big)\\
&\qquad+\nabla(\eta\nabla\cdot(\boldsymbol{M}_0(\boldsymbol{\gamma},\boldsymbol{g})\eta))+\alpha\boldsymbol{M}_0\Big(\alpha\langle \boldsymbol{M}_0(\boldsymbol{\gamma},\boldsymbol{g})^{\perp}\eta\rangle,\boldsymbol{M}_0(\boldsymbol{\gamma},\boldsymbol{g})^{\perp}\eta\Big)^{\perp}\eta\\
&\qquad+\boldsymbol{M}_0\Bigg(\alpha\Bigg\langle\eta\boldsymbol{M}_0\Big(\alpha\langle \boldsymbol{M}_0(\boldsymbol{\gamma},\boldsymbol{g})^{\perp}\eta\rangle,\boldsymbol{M}_0(\boldsymbol{\gamma},\boldsymbol{g})^{\perp}\eta\Big)^{\perp}-\eta\nabla^{\perp}((\nabla\cdot\boldsymbol{g}^{\perp})\eta)\\
&\qquad\qquad-\alpha\boldsymbol{M}_0(\boldsymbol{\gamma},\boldsymbol{g})\eta^2+\eta(\nabla\cdot\boldsymbol{g}^{\perp})\nabla^{\perp}\eta\Bigg\rangle,\\
&\qquad\quad\eta\boldsymbol{M}_0\Big(\alpha\langle \boldsymbol{M}_0(\boldsymbol{\gamma},\boldsymbol{g})^{\perp}\eta\rangle,\boldsymbol{M}_0(\boldsymbol{\gamma},\boldsymbol{g})^{\perp}\eta\Big)^{\perp}-\eta\nabla^{\perp}((\nabla\cdot\boldsymbol{g}^{\perp})\eta)\\
&\qquad\qquad-\alpha\boldsymbol{M}_0(\boldsymbol{\gamma},\boldsymbol{g})\eta^2+\eta(\nabla\cdot\boldsymbol{g}^{\perp})\nabla^{\perp}\eta\Bigg)
\end{aligned}$$

$$+\nabla(\eta \boldsymbol{M}_0(\boldsymbol{\gamma},\boldsymbol{g})\cdot\nabla\eta)$$
$$+\alpha\eta \boldsymbol{M}_0\left(\alpha\langle \boldsymbol{M}_0(\boldsymbol{\gamma},\boldsymbol{g})^\perp\eta\rangle, \boldsymbol{M}_0(\boldsymbol{\gamma},\boldsymbol{g})^\perp\eta\right)^\perp$$
$$-\alpha\eta\nabla^\perp((\nabla\cdot\boldsymbol{g}^\perp)\eta)-\alpha^2\boldsymbol{M}_0(\boldsymbol{\gamma},\boldsymbol{g})\eta^2+\alpha\eta(\nabla\cdot\boldsymbol{g}^\perp)\nabla^\perp\eta\Big\}.$$

Finally, the terms in the Taylor expansion

$$\boldsymbol{T}(\eta)=\sum_{k=0}^{\infty}\boldsymbol{T}_k(\eta),$$

where $\boldsymbol{T}_k(\eta)$ is homogeneous of degree k in η, can be computed from the formula $\boldsymbol{T}(\eta)=\boldsymbol{M}(\eta)(\boldsymbol{0},\boldsymbol{S}(\eta))$ using the expansion of $\boldsymbol{M}(\eta)$ derived above and the corresponding expansion

$$\boldsymbol{S}(\eta)=\sum_{k=1}^{\infty}\boldsymbol{S}_k(\eta)$$

of $\boldsymbol{S}(\eta)$, where

$$\boldsymbol{S}_k(\eta):=\begin{cases}(-1)^{\frac{k}{2}}\,\dfrac{\alpha^{k-1}\eta^k}{k!}\boldsymbol{c}, & \text{if } k\in 2\mathbb{N},\\[2ex] (-1)^{\frac{k-1}{2}}\,\dfrac{\alpha^{k-1}\eta^k}{k!}\boldsymbol{c}^\perp, & \text{if } k\notin 2\mathbb{N}.\end{cases}$$

Clearly $\boldsymbol{T}_0=\boldsymbol{0}$, $\boldsymbol{T}_k(\eta)=\sum\limits_{j=1}^{k}\boldsymbol{M}_{k-j}(\eta)(\boldsymbol{0},\boldsymbol{S}_j(\eta))$ for $k\geq 1$, and because

$$\boldsymbol{S}_1(\eta)=\eta\boldsymbol{c}^\perp,\quad \boldsymbol{S}_2(\eta)=-\frac{\alpha}{2}\eta^2\boldsymbol{c},\quad \boldsymbol{S}_3(\eta)=-\frac{\alpha^2}{6}\eta^3\boldsymbol{c}^\perp,$$

and

$$\boldsymbol{M}_0(\boldsymbol{0},\boldsymbol{g})=\boldsymbol{L}_1\boldsymbol{g},$$
$$\boldsymbol{M}_1(\eta)(\boldsymbol{0},\boldsymbol{g})=-\alpha\,\langle\eta\,\boldsymbol{L}_2\rangle+\boldsymbol{L}_1\big(\eta\,(\boldsymbol{L}_2\boldsymbol{g})\big)-\eta\,\boldsymbol{L}\boldsymbol{g}-\nabla\eta\,\nabla\cdot\boldsymbol{g}^\perp,$$
$$\boldsymbol{M}_2(\eta)(\boldsymbol{0},\boldsymbol{g})=\frac{1}{2}\Big\{2\alpha\boldsymbol{L}_1\big(\langle\eta\boldsymbol{L}_1\boldsymbol{g}\rangle\eta\big)+2\alpha^2\langle\eta\boldsymbol{L}_1\boldsymbol{g}\rangle\eta+2\boldsymbol{L}_1\big(\eta\boldsymbol{L}_2(\eta\boldsymbol{L}_2\boldsymbol{g})\big)$$
$$-\eta\boldsymbol{L}(\eta\boldsymbol{L}_2\boldsymbol{g})+\nabla\eta\nabla\cdot(\eta\boldsymbol{L}_1\boldsymbol{g})-\boldsymbol{L}_1\big(\eta^2(\boldsymbol{L}\boldsymbol{g})^\perp\big)$$
$$+\nabla\big(\eta\boldsymbol{L}_1\boldsymbol{g}\cdot\nabla\eta\big)+\alpha\eta\Big(\boldsymbol{L}_2(\eta\boldsymbol{L}_2\boldsymbol{g})-\eta(\boldsymbol{L}\boldsymbol{g})^\perp\Big)$$
$$-\alpha\Big\langle\Big(2\alpha\langle\eta\boldsymbol{L}_1\,\boldsymbol{g}\rangle+2\boldsymbol{L}_2(\eta\boldsymbol{L}_2\boldsymbol{g})-\eta(\boldsymbol{L}\boldsymbol{g})^\perp\Big)\eta\Big\rangle\Big\},$$

where

$$\boldsymbol{L}\boldsymbol{g} := \frac{1}{D^2}\left((\alpha^2 - D^2)\boldsymbol{D} - \alpha\,\mathrm{c}(D)\,\boldsymbol{D}^\perp\right)\boldsymbol{D}\cdot\boldsymbol{g}^\perp,$$

$$\boldsymbol{L}_1\boldsymbol{g} := \frac{1}{D^2}\left(\alpha\,\boldsymbol{D}^\perp + \boldsymbol{D}\,\mathrm{c}(D)\right)\boldsymbol{D}\cdot\boldsymbol{g}^\perp,$$

$$\boldsymbol{L}_2\boldsymbol{g} := \frac{1}{D^2}\left(-\alpha\,\boldsymbol{D} + \boldsymbol{D}^\perp\mathrm{c}(D)\right)\boldsymbol{D}\cdot\boldsymbol{g}^\perp,$$

we find in particular that

$$\begin{aligned}
\boldsymbol{T}_1(\eta) &= \boldsymbol{M}_0(\boldsymbol{0}, \boldsymbol{S}_1(\eta)) \\
&= \boldsymbol{L}_1(\eta\,\boldsymbol{c}^\perp), \\
\boldsymbol{T}_2(\eta) &= \boldsymbol{M}_1(\eta)(\boldsymbol{0}, \boldsymbol{S}_0(\eta)) + \boldsymbol{M}_0(\eta)(\boldsymbol{0}, \boldsymbol{S}_1(\eta)) \\
&= -\tfrac{1}{2}\alpha\boldsymbol{L}_1(\eta^2\,\boldsymbol{c}) + \boldsymbol{L}_1\big(\eta\,\boldsymbol{L}_2(\eta\boldsymbol{c}^\perp)\big) - \eta\,\boldsymbol{L}(\eta\boldsymbol{c}^\perp) + \nabla\eta\,\nabla\cdot(\eta\boldsymbol{c}) - \alpha\big\langle\eta\,\boldsymbol{L}_2(\eta\boldsymbol{c}^\perp)\big\rangle, \\
\boldsymbol{T}_3(\eta) &= \boldsymbol{M}_2(\eta)(\boldsymbol{0}, \boldsymbol{S}_1(\eta)) + \boldsymbol{M}_1(\eta)(\boldsymbol{0}, \boldsymbol{S}_2(\eta)) + \boldsymbol{M}_0(\boldsymbol{0}, \boldsymbol{S}_3(\eta)) \\
&= \frac{1}{2}\Big\{2\alpha\boldsymbol{L}_1\big(\langle\eta\boldsymbol{L}_1(\eta\,\boldsymbol{c}^\perp)\rangle\eta\big) + 2\alpha^2\langle\eta\boldsymbol{L}_1(\eta\,\boldsymbol{c}^\perp)\rangle\eta + 2\boldsymbol{L}_1\big(\eta\boldsymbol{L}_2(\eta\boldsymbol{L}_2(\eta\,\boldsymbol{c}^\perp))\big) \\
&\qquad - \eta\boldsymbol{L}(\eta\boldsymbol{L}_2(\eta\,\boldsymbol{c}^\perp)) + \nabla\eta\nabla\cdot(\eta\boldsymbol{L}_1(\eta\,\boldsymbol{c}^\perp)) - \boldsymbol{L}_1\big(\eta^2(\boldsymbol{L}(\eta\,\boldsymbol{c}^\perp))^\perp\big) \\
&\qquad + \nabla\big(\eta\boldsymbol{L}_1(\eta\,\boldsymbol{c}^\perp)\cdot\nabla\eta\big) + \alpha\eta\Big(\boldsymbol{L}_2(\eta\boldsymbol{L}_2(\eta\,\boldsymbol{c}^\perp)) - \eta(\boldsymbol{L}(\eta\,\boldsymbol{c}^\perp))^\perp\Big) \\
&\qquad - \alpha\Big\langle\Big(2\alpha\langle\eta\boldsymbol{L}_1\,(\eta\,\boldsymbol{c}^\perp)\rangle + 2\boldsymbol{L}_2(\eta\boldsymbol{L}_2(\eta\,\boldsymbol{c}^\perp)) - \eta(\boldsymbol{L}(\eta\,\boldsymbol{c}^\perp))^\perp\Big)\eta\Big\rangle\Big\} \\
&\quad + \frac{\alpha^2}{2}\langle\eta\boldsymbol{L}_2(\eta^2\,\boldsymbol{c})\rangle - \frac{\alpha}{2}\boldsymbol{L}_1(\eta\boldsymbol{L}_2(\eta^2\,\boldsymbol{c})) + \frac{\alpha}{2}\eta\boldsymbol{L}(\eta^2\boldsymbol{c}) + \frac{\alpha}{2}\nabla\eta\nabla\cdot(\eta^2\boldsymbol{c})^\perp - \frac{\alpha^2}{6}\boldsymbol{L}_1(\eta^3\boldsymbol{c}^\perp).
\end{aligned}$$

3. Description of $H(\eta)$ and $\boldsymbol{M}(\eta)$ as pseudodifferential operators

3.1. Flattening and factorisation

Choose $\eta \in C^\infty(\mathbb{R}^2/\Lambda)$. In this section we prove that $H(\eta)$ and $\boldsymbol{M}(\eta)$ are smooth perturbations of properly supported pseudodifferential operators and compute their asymptotic expansions, working under the non-resonance condition (NR). We begin by introducing a flattening transform (which differs from that used in Section 2). Choose $\delta > 0$ so that the fluid domain D_η contains the strip

$$\Omega_\delta := \{(\boldsymbol{x}', z) \in \mathbb{R}^2 \times \mathbb{R} \colon \eta(\boldsymbol{x}') - \delta h \le z < \eta(\boldsymbol{x}')\}$$

for $\eta \in U$ and define $\hat{\Sigma} \colon D_0 \to \Omega_\delta$ by

$$\hat{\Sigma} \colon (\boldsymbol{x}', w) \mapsto (\boldsymbol{x}', \varrho(\boldsymbol{x}', w)), \;\; \varrho(\boldsymbol{x}', w) := \delta w + \eta(\boldsymbol{x}').$$

For $f\colon D_\eta \to \mathbb{R}$ and $\boldsymbol{F}\colon D_\eta \to \mathbb{R}^3$ we write $\hat{f} = f \circ \hat{\Sigma}$, $\hat{\boldsymbol{F}} = \boldsymbol{F} \circ \hat{\Sigma}$ and use the notation

$$\begin{aligned}
\operatorname{grad}^\varrho \hat{f}(\boldsymbol{x}', w) &:= (\operatorname{grad} f) \circ \hat{\Sigma}(\boldsymbol{x}', w),\\
\operatorname{div}^\varrho \hat{f}(\boldsymbol{x}', w) &:= (\operatorname{div} f) \circ \hat{\Sigma}(\boldsymbol{x}', w),\\
\operatorname{curl}^\varrho \hat{\boldsymbol{F}}(\boldsymbol{x}', w) &:= (\operatorname{curl} \boldsymbol{F}) \circ \hat{\Sigma}(\boldsymbol{x}', w),\\
\Delta^\varrho \hat{f} &:= (\Delta f) \circ \hat{\Sigma}(\boldsymbol{x}', w)
\end{aligned}$$

and more generally

$$\partial_x^\varrho := \partial_x - \frac{\partial_x \eta}{\delta}\partial_w, \quad \partial_y^\varrho := \partial_y - \frac{\partial_y \eta}{\delta}\partial_w, \quad \partial_w^\varrho := \frac{\partial_w}{\delta}.$$

Remark 3.1. The flattened versions of the operators curl, div and Δ applied to $\hat{\boldsymbol{F}}(x, y, w) = \boldsymbol{F}(x, y, z)$ and to $\hat{f}(x, y, w) = f(x, y, z)$ are given explicitly by

$$\begin{aligned}
\operatorname{curl}^\varrho \hat{\boldsymbol{F}} &= (\partial_y^\varrho \hat{F}_3 - \partial_w^\varrho \hat{F}_2, -\partial_x^\varrho \hat{F}_3 + \partial_w^\varrho \hat{F}_1, \partial_x^\varrho \hat{F}_2 - \partial_y^\varrho \hat{F}_1)^T\\
&= \operatorname{curl} \hat{\boldsymbol{F}} - \frac{1}{\delta}(\eta_y\, \partial_w \hat{F}_3, -\eta_x\, \partial_w \hat{F}_3, \eta_x\, \partial_w \hat{F}_2 - \eta_y\, \partial_w \hat{F}_1)^T\\
&\quad - \left(\frac{1}{\delta} - 1\right)(\partial_w \hat{F}_2, -\partial_w \hat{F}_1, 0)^T,\\
\operatorname{div}^\varrho \hat{\boldsymbol{F}} &= \partial_x^\varrho \hat{F}_1 + \partial_y^\varrho \hat{F}_2 + \partial_w^\varrho \hat{F}_3\\
&= \partial_x \hat{F}_1 + \partial_y \hat{F}_2 + \frac{1}{\delta}\left(-\eta_x\, \partial_w \hat{F}_1 - \eta_y\, \partial_w \hat{F}_2 + \partial_w \hat{F}_3\right),\\
-\Delta^\varrho \hat{f} &= -(\partial_x^\varrho)^2 \hat{f} - (\partial_y^\varrho)^2 \hat{f} - (\partial_w^\varrho)^2 \hat{f}\\
&= -\Delta \hat{f} + \frac{2}{\delta}\left(\eta_x\, \partial_{xw}^2 \hat{f} + \eta_y\, \partial_{yw}^2 \hat{f}\right) - \left(\frac{1}{\delta^2} - 1\right)\partial_w^2 \hat{f}\\
&\quad + \frac{1}{\delta}(\eta_{xx} + \eta_{yy})\,\partial_w \hat{f} - \frac{1}{\delta^2}\,(\eta_x^2 + \eta_y^2)\,\partial_w^2 \hat{f}.
\end{aligned}$$

The flattening transform converts the equation

$$-\Delta \boldsymbol{F} = \alpha \operatorname{curl} \boldsymbol{F} \qquad \text{in } \Omega_\delta$$

into

$$-\Delta^\varrho \hat{\boldsymbol{F}} - \alpha \operatorname{curl}^\varrho \hat{\boldsymbol{F}} = \boldsymbol{0} \qquad \text{in } D_0,$$

which is equivalent to the system

$$L\hat{\boldsymbol{F}} = \boldsymbol{0}, \tag{3.1}$$

where $L := aI\partial_w^2 + L_1\partial_w + L_0$ with

$$L_1 := \begin{pmatrix} \boldsymbol{b}\cdot\nabla - c & -\frac{\alpha}{\delta} & -\frac{\alpha}{\delta}\eta_y \\ \frac{\alpha}{\delta} & \boldsymbol{b}\cdot\nabla - c & \frac{\alpha}{\delta}\eta_x \\ \frac{\alpha}{\delta}\eta_y & -\frac{\alpha}{\delta}\eta_x & \boldsymbol{b}\cdot\nabla - c \end{pmatrix}, \qquad L_0 := \begin{pmatrix} \Delta & 0 & \alpha\partial_y \\ 0 & \Delta & -\alpha\partial_x \\ -\alpha\partial_y & \alpha\partial_x & \Delta \end{pmatrix}$$

and

$$a := \frac{1+|\nabla\eta|^2}{\delta^2}, \quad \boldsymbol{b} := -\frac{2\nabla\eta}{\delta}, \quad c := \frac{\Delta\eta}{\delta}.$$

Lemma 3.2. *There are properly supported operators M, $N \in \Psi^1(\mathbb{R}^2/\Lambda)$ such that*

(i) $L - a(\partial_w I - N)(\partial_w I - M) \in \Psi^{-\infty}(\mathbb{R}^2/\Lambda)$,
(ii) *the principal symbols $\mathtt{M}^{(1)}$, $\mathtt{N}^{(1)}$ of M, N take the form $\mathtt{M}^{(1)} = \mathtt{m}^{(1)}\mathbb{I}_3$, $\mathtt{N}^{(1)} = \mathtt{n}^{(1)}\mathbb{I}_3$, where the scalar-valued symbols $\mathtt{m}^{(1)}$, $-\mathtt{n}^{(1)} \in S^1(\mathbb{R}^2/\Lambda)$ are strongly elliptic.*

Proof. Because

$$L - a(\partial_w I - N)(\partial_w I - M) = (L_1 + a(M+N))\partial_w + (L_0 - aNM)$$

we set

$$N = -a^{-1}L_1 - M \tag{3.2}$$

and seek M with

$$L_0 + L_1 M + aM^2 = 0$$

by constructing a symbol $\mathtt{M} \in S^1(\mathbb{R}^2/\Lambda)$ such that

$$\begin{pmatrix} -|\boldsymbol{k}|^2 & 0 & \alpha\mathrm{i}k_2 \\ 0 & -|\boldsymbol{k}|^2 & -\alpha\mathrm{i}k_1 \\ -\alpha\mathrm{i}k_2 & \alpha\mathrm{i}k_1 & -|\boldsymbol{k}|^2 \end{pmatrix} + \begin{pmatrix} \mathrm{i}\boldsymbol{b}\cdot\boldsymbol{k} - c & -\frac{\alpha}{\delta} & -\frac{\alpha}{\delta}\eta_y \\ \frac{\alpha}{\delta} & \mathrm{i}\boldsymbol{b}\cdot\boldsymbol{k} - c & \frac{\alpha}{\delta}\eta_x \\ \frac{\alpha}{\delta}\eta_y & -\frac{\alpha}{\delta}\eta_x & \mathrm{i}\boldsymbol{b}\cdot\boldsymbol{k} - c \end{pmatrix}\mathtt{M} + (\boldsymbol{b}\cdot\nabla)\mathtt{M}$$
$$+ a\sum_{\boldsymbol{\alpha}\in\mathbb{N}_0^2} \partial_{k_1}^{\alpha_1}\partial_{k_2}^{\alpha_2}\mathtt{M}\, D_1^{\alpha_1}D_2^{\alpha_2}\mathtt{M} \sim 0$$

and

$$\mathtt{M} \sim \sum_{j\le 1}\mathtt{M}^{(j)},$$

where $\mathtt{M}^{(j)} \in S^j(\mathbb{R}^2/\Lambda)$.

We proceed by computing the terms in the asymptotic expansion of $\mathtt{M}$ inductively.

- Obviously

$$-|\boldsymbol{k}|^2\mathbb{I}_3+\mathrm{i}\boldsymbol{b}\cdot\boldsymbol{k}\mathtt{M}^{(1)}+a(\mathtt{M}^{(1)})^2=0,$$

so that

$$\mathtt{M}^{(1)}=\delta\mathtt{m}^{(1)}\mathbb{I}_3,$$

where

$$\mathtt{m}^{(1)}(\boldsymbol{x}',\boldsymbol{k}):=\frac{\mathrm{i}\boldsymbol{k}\cdot\nabla\eta+\lambda^{(1)}}{1+|\nabla\eta|^2},\qquad \lambda^{(1)}(\boldsymbol{x}',\boldsymbol{k}):=\sqrt{(1+|\nabla\eta|^2)|\boldsymbol{k}|^2-(\boldsymbol{k}\cdot\nabla\eta)^2}.$$

Note that $\lambda^{(1)}$ is the leading order symbol of the classical Dirichlet–Neumann operator.

- The subprincipal symbol of $\mathtt{M}$ is found from the equation

$$\begin{aligned}\alpha\begin{pmatrix}0&0&\mathrm{i}k_2\\0&0&-\mathrm{i}k_1\\-\mathrm{i}k_2&\mathrm{i}k_1&0\end{pmatrix}-\left(c\mathbb{I}_3+\frac{\alpha}{\delta}\begin{pmatrix}0&1&\eta_y\\-1&0&-\eta_x\\-\eta_y&\eta_x&0\end{pmatrix}\right)\mathtt{M}^{(1)}\\+\mathrm{i}\boldsymbol{b}\cdot\boldsymbol{k}\mathtt{M}^{(0)}+a\mathtt{M}^{(0)}\mathtt{M}^{(1)}+a\mathtt{M}^{(1)}\mathtt{M}^{(0)}\\+(\boldsymbol{b}\cdot\nabla)\mathtt{M}^{(1)}-\mathrm{i}a\partial_{k_1}\mathtt{M}^{(1)}\partial_x\mathtt{M}^{(1)}-\mathrm{i}a\partial_{k_2}\mathtt{M}^{(1)}\partial_y\mathtt{M}^{(1)}=0,\end{aligned}$$

which yields

$$\mathtt{M}^{(0)}=\delta\mathtt{m}^{(0)}\mathbb{I}_3+\delta\mathtt{M}_1^{(0)},$$

where

$$\begin{aligned}\mathtt{m}^{(0)}(\boldsymbol{x}',\boldsymbol{k})&:=\frac{1}{2\lambda^{(1)}}\left(\nabla\cdot(\mathtt{m}^{(1)}\nabla\eta)+\mathrm{i}\nabla_{\boldsymbol{k}}\lambda^{(1)}\cdot\nabla\mathtt{m}^{(1)}\right),\\ \mathtt{M}_1^{(0)}(\boldsymbol{x}',\boldsymbol{k})&:=\frac{\alpha}{2\lambda^{(1)}}\begin{pmatrix}0&\mathtt{m}^{(1)}&-\mathrm{i}k_2+\mathtt{m}^{(1)}\eta_y\\-\mathtt{m}^{(1)}&0&\mathrm{i}k_1-\mathtt{m}^{(1)}\eta_x\\\mathrm{i}k_2-\mathtt{m}^{(1)}\eta_y&-\mathrm{i}k_1+\mathtt{m}^{(1)}\eta_x&0\end{pmatrix}.\end{aligned}$$

- Suppose that $\mathtt{M}^{(j)}$ has been calculated for $j=1,0,\dots-j_0$ for some $j_0\geq 0$. The term $\mathtt{M}^{(-j_0-1)}$ can be found from the equation

$$(2a\delta\mathtt{m}^{(1)}+\mathrm{i}\boldsymbol{b}\cdot\boldsymbol{k})\mathtt{M}^{(-j_0-1)}=\tilde{\mathtt{M}}^{(-j_0)},$$

where $\tilde{\mathtt{M}}^{(-j_0)}\in S^{(-j_0)}(\mathbb{R}^2/\Lambda)$ is given by

$$
\begin{aligned}
\tilde{\mathtt{M}}^{(-j_0)}(\boldsymbol{x}', \boldsymbol{k}) = &\left(c\mathbb{I}_3 + \frac{\alpha}{\delta}\begin{pmatrix} 0 & 1 & \eta_y \\ -1 & 0 & -\eta_x \\ -\eta_y & \eta_x & 0 \end{pmatrix}\right)\mathtt{M}^{(-j_0)} - \boldsymbol{b}\cdot\nabla\mathtt{M}^{(-j_0)} \\
&- a\sum_{\substack{j_1, j_2 \le 0 \\ |\boldsymbol{\alpha}|+j_1+j_2=j_0}} \partial_{k_1}^{\alpha_1}\partial_{k_2}^{\alpha_2}\mathtt{M}^{(-j_1)} D_1^{\alpha_1} D_2^{\alpha_2}\mathtt{M}^{(-j_2)} - a\sum_{|\boldsymbol{\alpha}|=j_0+2} \partial_{k_1}^{\alpha_1}\partial_{k_2}^{\alpha_2}\mathtt{M}^{(1)} D_1^{\alpha_1} D_2^{\alpha_2}\mathtt{M}^{(1)} \\
&- a\sum_{|\boldsymbol{\alpha}|=j_0+1} \partial_{k_1}^{\alpha_1}\partial_{k_2}^{\alpha_2}\mathtt{M}^{(1)} D_1^{\alpha_1} D_2^{\alpha_2}\mathtt{M}^{(0)} - a\sum_{|\boldsymbol{\alpha}|=j_0+1} \partial_{k_1}^{\alpha_1}\partial_{k_2}^{\alpha_2}\mathtt{M}^{(0)} D_1^{\alpha_1} D_2^{\alpha_2}\mathtt{M}^{(1)},
\end{aligned}
$$

so that

$$\mathtt{M}^{(-j_0-1)} = \frac{\delta}{2\lambda^{(1)}}\tilde{\mathtt{M}}^{(-j_0)}.$$

The construction is completed by noting that there exists a symbol $\mathtt{M} \in S^1(\mathbb{R}^2/\Lambda)$ such that

$$\mathtt{M} \sim \sum_{j \le 1} \mathtt{M}^{(j)}$$

(see Shubin [25, §3.3]).

Defining $M = \operatorname{Op}\mathtt{M}$ and N by equation (3.2), we find that $M, N \in \Psi^1(\mathbb{R}^2/\Lambda)$. The terms in the asymptotic expansion

$$\mathtt{N} \sim \sum_{j \le 1} \mathtt{N}^{(j)}$$

of N are readily computed using (3.2); in particular we find that

$$\mathtt{N}^{(1)} = \delta\mathtt{n}^{(1)}\mathbb{I}_3, \qquad \mathtt{n}^{(1)}(\boldsymbol{x}', \boldsymbol{k}) := \frac{\mathrm{i}\boldsymbol{k}\cdot\nabla\eta - \lambda^{(1)}}{1+|\nabla\eta|^2}.$$

Finally, note that

$$\operatorname{Re}\mathtt{m}^{(1)}(\boldsymbol{x}', \boldsymbol{k}) = -\operatorname{Re}\mathtt{n}^{(1)}(\boldsymbol{x}', \boldsymbol{k}) = \frac{\delta\lambda^{(1)}}{1+|\nabla\eta|^2} \ge \frac{\delta|\boldsymbol{k}|}{1+\max|\nabla\eta|^2} \gtrsim \langle\boldsymbol{k}\rangle$$

for sufficiently large $|\boldsymbol{k}|$, so that $\mathtt{m}^{(1)}$, $-\mathtt{n}^{(1)}$ are strongly elliptic. □

Theorem 3.5 below gives information on the Neumann boundary data of a solution to (3.1). It is proved using Lemmata 3.3 and 3.4 below, the former of which is an existence result for an abstract heat equation (see Treves [26, Ch. III §1] for a more general theory).

Lemma 3.3. *Suppose that $T > 0$, Γ is a full rank lattice in $\mathbb{R}^{n-1}$ and $A \in \Psi^m(\mathbb{R}^{n-1}/\Gamma)$ for some $m \in \mathbb{N}$ is a properly supported pseudodifferential operator whose principal symbol $\mathtt{A}^{(m)}$ takes the form $\mathtt{A}^{(m)} = \mathtt{a}^{(m)}\mathbb{I}_n$, where the scalar-valued symbol $\mathtt{a}^{(m)} \in S^m(\mathbb{R}^{n-1}/\Gamma)$ is strongly elliptic.*

There is a properly supported pseudodifferential operator $P \in \Psi^{0,m}([T_0, T_0+T]; \mathbb{R}^{n-1}/\Gamma)$ which satisfies

$$
\begin{aligned}
&\partial_t P + AP \in \Psi^{-\infty}([T_0, T_0+T]; \mathbb{R}^{n-1}/\Gamma), \\
&P|_{t=T_0} = I.
\end{aligned}
$$

In particular, any solution of the initial-value problem

$$\partial_t \hat{\boldsymbol{U}} + A\hat{\boldsymbol{U}} = \hat{\boldsymbol{F}}, \qquad t \in [T_0, T_0 + T],$$
$$\hat{\boldsymbol{U}}|_{t=T_0} = \hat{\boldsymbol{U}}_0,$$

where $\hat{\boldsymbol{F}} \in C^\infty([T_0, T_0 + T]; C^\infty(\mathbb{R}^{n-1}/\Gamma)^n)$ *and* $\hat{\boldsymbol{U}}_0 \in C^\infty(\mathbb{R}^{n-1}/\Gamma)^n$*, belongs to* $C^\infty([T_0, T_0 + T]; C^\infty(\mathbb{R}^{n-1}/\Gamma)^n)$.

Lemma 3.4. *Suppose that* $T > 0$*,* Γ *is a full rank lattice in* $\mathbb{R}^{n-1}$ *and* $\mathcal{P}$ *is a linear differential operator of order* m *in the variables* $(\boldsymbol{z}, t) \in \mathbb{R}^n$ *of the form*

$$\mathcal{P} = \mathbb{I}_n \partial_t^m + \sum_{\substack{|\boldsymbol{\alpha}| \le m \\ \alpha_n \le m-1}} A_{\boldsymbol{\alpha}}(\boldsymbol{z}) \partial^{\boldsymbol{\alpha}},$$

where $\boldsymbol{\alpha} \in \mathbb{N}_0^n$*,* $\partial^{\boldsymbol{\alpha}} = \partial_{z_1}^{\alpha_1}, \ldots, \partial_{z_{n-1}}^{\alpha_{n-1}} \partial_t^{\alpha_n}$ *and the coefficients of the matrix* $A_{\boldsymbol{\alpha}}(\boldsymbol{z})$ *are functions of* $\boldsymbol{z}$ *of class* $C^\infty(\mathbb{R}^{n-1}/\Gamma)$*. Any solution* $\hat{\boldsymbol{U}} \in H^{m-1}(\mathbb{R}^{n-1}/\Gamma \times (T_0, T_0 + T))^n$ *of* $\mathcal{P}\hat{\boldsymbol{U}} = \boldsymbol{0}$ *lies in* $C^\infty([T_0, T_0 + T]; \mathscr{D}'(\mathbb{R}^{n-1}/\Gamma)^n)$.

Proof. Suppose that $\mathcal{P}\hat{\boldsymbol{U}} = \boldsymbol{0}$. A straightforward argument using Fubini's theorem shows that $\hat{\boldsymbol{U}}$ lies in $H^{m-1}((T_0, T_0 + T); L^2(\mathbb{R}^{n-1}/\Gamma)^n)$, and the next step is to show inductively that $\hat{\boldsymbol{U}}$ in fact lies in $H^{m-1+k}((T_0, T_0 + T); H^{-km}(\mathbb{R}^{n-1}/\Gamma)^n)$ for every $k \in \mathbb{N}_0$.

To this end let $\ell_1 \in \mathbb{N}_0$, $\ell_2 \in \mathbb{Z}$ and $a \in C^\infty(\mathbb{R}^{n-1}/\Gamma)$, and observe that the mappings $w \mapsto \partial_t w$ and $w \mapsto \partial_{z_j} w$ induce continuous linear operators $H^{\ell_1}((T_0, T_0+T); H^{\ell_2}(\mathbb{R}^{n-1}/\Gamma)) \to H^{\ell_1-1}((T_0, T_0 + T); H^{\ell_2}(\mathbb{R}^{n-1}/\Gamma))$ and $H^{\ell_1}((T_0, T_0 + T); H^{\ell_2}(\mathbb{R}^{n-1}/\Gamma)) \to H^{\ell_1}((T_0, T_0 + T); H^{\ell_2-1}(\mathbb{R}^{n-1}/\Gamma))$ respectively, while the mapping $w \mapsto aw$ induces a continuous linear operator $H^{\ell_1}((T_0, T_0 + T); H^{\ell_2}(\mathbb{R}^{n-1}/\Gamma)) \to H^{\ell_1}((T_0, T_0 + T); H^{\ell_2}(\mathbb{R}^{n-1}/\Gamma))$. It follows that the formula

$$\mathcal{Q}\hat{\boldsymbol{U}} = -\sum_{\substack{|\boldsymbol{\alpha}| \le m \\ \alpha_n \le m-1}} A_{\boldsymbol{\alpha}}(\boldsymbol{z}) \partial^{\boldsymbol{\alpha}} \hat{\boldsymbol{U}}$$

defines a continuous linear operator $H^{m-1+k}((T_0, T_0 + T); H^{-km}(\mathbb{R}^{n-1}/\Gamma)^n) \to H^k((T_0, T_0 + T); H^{-(k+1)m}(\mathbb{R}^{n-1}/\Gamma)^n)$ for each $k \in \mathbb{N}_0$.

Returning to the induction, let $k \in \mathbb{N}_0$ and suppose that $\hat{\boldsymbol{U}}$ belongs to $H^{m-1+k}((T_0, T_0 + T); H^{-km}(\mathbb{R}^{n-1}/\Gamma)^n)$ satisfies $P\hat{\boldsymbol{U}} = 0$, so that $\partial_t^m \hat{\boldsymbol{U}} = \mathcal{Q}\hat{\boldsymbol{U}}$ in $\mathscr{D}'((T_0, T_0 + T); H^{-(k+1)m}(\mathbb{R}^{n-1}/\Gamma)^n)$. The above argument implies that $\partial_t^m \hat{\boldsymbol{U}} \in H^k((T_0, T_0 + T); H^{-(k-1)m}(\mathbb{R}^{n-1}/\Gamma)^n)$, and since

$$\hat{\boldsymbol{U}} \in H^{m-1+k}((T_0, T_0 + T); H^{-km}(\mathbb{R}^{n-1}/\Gamma)^n) \subseteq H^{m-1+k}((T_0, T_0 + T); H^{-(k+1)m}(\mathbb{R}^{n-1}/\Gamma)^n),$$

we conclude that $\hat{\boldsymbol{U}} \in H^{m+k}((T_0, T_0 + T); H^{-(k+1)m}(\mathbb{R}^{n-1}/\Gamma)^n)$.

Finally, choose $\ell \in \mathbb{N}$ with $\ell \geq m-2$ and set $k = \ell - m + 2$, so that

$$\begin{aligned}\hat{\boldsymbol{U}} &\in H^{\ell+1}((T_0, T_0+T); H^{-(\ell-m+2)m}(\mathbb{R}^{n-1}/\Gamma)^n)\\ &\subseteq C^{\ell}([T_0, T_0+T]; H^{-(\ell-m+2)m}(\mathbb{R}^{n-1}/\Gamma)^n)\\ &\subseteq C^{\ell}([T_0, T_0+T]; \mathscr{D}'(\mathbb{R}^{n-1}/\Gamma)^n).\end{aligned}$$

However this result holds for arbitrarily large $\ell \in \mathbb{N}$, so that $\hat{\boldsymbol{U}}$ belongs to $C^{\infty}([T_0, T_0+T]; \mathscr{D}'(\mathbb{R}^{n-1}/\Gamma)^n)$. □

Theorem 3.5. *Any function $\hat{\boldsymbol{U}} \in H^2(D_0/\Lambda)^3$ with $L\hat{\boldsymbol{U}} = 0$ in D_0 satisfies*

$$\partial_w \hat{\boldsymbol{U}} = M\hat{\boldsymbol{U}} + R_\infty \hat{\boldsymbol{U}} \qquad \text{at } w = 0,$$

where the symbol R_∞ denotes a linear function of its argument whose range lies in $C^{\infty}(\mathbb{R}^2/\Lambda)^3$.

Proof. The equation

$$L\hat{\boldsymbol{U}} = \boldsymbol{0} \tag{3.3}$$

is equivalent to the coupled equations

$$(\partial_w I - M)\hat{\boldsymbol{U}} = \hat{\boldsymbol{U}}_1, \tag{3.4}$$

$$(\partial_w I - N)\hat{\boldsymbol{U}}_1 = -R_\infty \hat{\mathbf{U}} \tag{3.5}$$

(the smoothing operator in equation (3.5) in fact lies in $\Psi^{-\infty}(\mathbb{R}^2/\Lambda)$).

By elliptic regularity theory $\hat{\boldsymbol{U}} \in C^{\infty}(D_0/\Lambda)^3 \cong C^{\infty}((-h, 0); C^{\infty}(\mathbb{R}^2/\Lambda)^3)$, and it follows from equation (3.4) that $\hat{\boldsymbol{U}}_1 \in C^{\infty}((-h, 0); C^{\infty}(\mathbb{R}^2/\Lambda)^3)$; in particular $\hat{\boldsymbol{U}}_1|_{w=-\frac{1}{2}h} \in C^{\infty}(\mathbb{R}^2/\Lambda)^3$. Furthermore, applying Lemma 3.4 to (3.3) shows that $\hat{\boldsymbol{U}} \in C^{\infty}([-h, 0]; \mathscr{D}'(\mathbb{R}^2/\Lambda)^3)$, so that $R_\infty \hat{\boldsymbol{U}} \in C^{\infty}([-h, 0]; C^{\infty}(\mathbb{R}^2/\Lambda)^3)$. Applying Lemma 3.3 to equation (3.5) for $w \in [-\frac{1}{2}h, 0]$, we thus find that $\hat{\boldsymbol{U}}_1 \in C^{\infty}([-\frac{1}{2}h, 0]; C^{\infty}(\mathbb{R}^2/\Lambda)^3)$. Finally, equation (3.4) shows that

$$\partial_w \hat{\boldsymbol{U}} = M\hat{\boldsymbol{U}} + \hat{\boldsymbol{U}}_1 \qquad \text{at } w = 0$$

because $\hat{\boldsymbol{U}}_1$ is a linear function of $\hat{\boldsymbol{U}}$. □

3.2. The operator $H(\eta)$

Let $s \geq 2$, $\Phi \in \mathring{H}^{s-\frac{1}{2}}(\mathbb{R}^2/\Lambda)$ and $\tilde{\boldsymbol{A}} \in H^s(D_0/\Lambda)^3$ be the unique solution of the boundary-value problem (2.16)–(2.20) with $\boldsymbol{\gamma} = \boldsymbol{0}$. The variable

$$\hat{\boldsymbol{A}}(\boldsymbol{x}', w) := \tilde{\boldsymbol{A}}(\boldsymbol{x}', v), \qquad w := \frac{1}{\delta h}(h + \eta)v$$

satisfies

$$L\hat{\boldsymbol{A}} = \boldsymbol{0} \qquad \text{in } D_0,$$

and

$$\begin{aligned} \hat{\boldsymbol{A}} \cdot \boldsymbol{N} &= 0 && \text{at } w = 0, \\ (\text{curl}^{\varrho} \hat{\boldsymbol{A}})_{\parallel} &= \nabla\Phi - \alpha \nabla^{\perp} \Delta^{-1} (\nabla \cdot \hat{\boldsymbol{A}}_{\parallel}^{\perp}) && \text{at } w = 0, \end{aligned}$$

which can be written explicitly as

$$\eta_x \hat{A}_1 + \eta_y \hat{A}_2 - \hat{A}_3 = 0 \qquad \text{at } w = 0, \tag{3.6}$$

$$\begin{aligned} &\hat{A}_{3y} - \frac{\eta_y}{\delta}\partial_w \hat{A}_3 - \frac{1}{\delta}\partial_w \hat{A}_2 + \eta_x \left(\hat{A}_{2x} - \hat{A}_{1y} - \frac{1}{\delta}\eta_x \partial_w \hat{A}_2 + \frac{1}{\delta}\eta_y \partial_w \hat{A}_1 \right) \\ &\quad + \alpha \Delta^{-1} (\hat{A}_{2xy} + \eta_{yy} \hat{A}_{3x} + \eta_y \hat{A}_{3xy} - \hat{A}_{1yy} - \eta_{xy} \hat{A}_{3y} - \eta_x \hat{A}_{3yy}) = \Phi_x \qquad \text{at } w = 0, \end{aligned} \tag{3.7}$$

$$\begin{aligned} &-\hat{A}_{3x} + \frac{\eta_x}{\delta}\partial_w \hat{A}_3 + \frac{1}{\delta}\partial_w \hat{A}_1 + \eta_y \left(\hat{A}_{2x} - \hat{A}_{1y} - \frac{1}{\delta}\eta_x \partial_w \hat{A}_2 + \frac{1}{\delta}\eta_y \partial_w \hat{A}_1 \right) \\ &\quad + \alpha \Delta^{-1} (-\hat{A}_{2xx} - \eta_{xy} \hat{A}_{3x} - \eta_y \hat{A}_{3xx} + \hat{A}_{1xy} + \eta_{xx} \hat{A}_{3y} + \eta_x \hat{A}_{3xy}) = \Phi_y \qquad \text{at } w = 0. \end{aligned} \tag{3.8}$$

Substituting

$$\hat{A}_3 = \eta_x \hat{A}_1 + \eta_y \hat{A}_2 \tag{3.9}$$

(see equation (3.6)) and

$$\partial_w \hat{\boldsymbol{A}}|_{w=0} = M \hat{\boldsymbol{A}}|_{w=0} + R_\infty \Phi \tag{3.10}$$

(see Lemma 3.5, noting that $\hat{\boldsymbol{A}}$ is a linear function of Φ) into equations (3.7), (3.8), we find that

$$P \begin{pmatrix} \hat{A}_1|_{w=0} \\ \hat{A}_2|_{w=0} \end{pmatrix} = \begin{pmatrix} \Phi_x \\ \Phi_y \end{pmatrix} + R_\infty \Phi, \tag{3.11}$$

where $P \in \Psi^1(\mathbb{R}^2/\Lambda)$ is a properly supported pseudodifferential operator with principal symbol

$$\mathtt{P}^{(1)}(\boldsymbol{x}', \boldsymbol{k}) = \begin{pmatrix} 0 & -(1 + |\nabla\eta|^2)\mathtt{m}^{(1)} + \mathrm{i}\boldsymbol{k} \cdot \nabla\eta \\ (1 + |\nabla\eta|^2)\mathtt{m}^{(1)} - \mathrm{i}\boldsymbol{k} \cdot \nabla\eta & 0 \end{pmatrix}.$$

Observe that $\mathtt{P}^{(1)}$ is invertible for $|\boldsymbol{k}| \neq 0$, so that P is elliptic and hence admits a parametrix $Q \in \Psi^{-1}(\mathbb{R}^2/\Lambda)$ such that $PQ - I \in \Psi^{-\infty}(\mathbb{R}^2/\Lambda)$ (see Grubb [13, Theorem 7.18]). We thus find from equation (3.11) that

$$\begin{pmatrix} \hat{A}_1|_{w=0} \\ \hat{A}_2|_{w=0} \end{pmatrix} = Q \begin{pmatrix} \Phi_x \\ \Phi_y \end{pmatrix} + R_\infty \Phi,$$

and appending (3.9) to this equation yields

$$\hat{\boldsymbol{A}}|_{w=0} = Z\Phi + R_\infty \Phi, \tag{3.12}$$

where $\mathtt{Z} \in S^0(\mathbb{R}^2/\Lambda)$ and $Z = \operatorname{Op} \mathtt{Z}$.

We have that

$$H(\eta)(\boldsymbol{\gamma}, \Phi) = \underbrace{H(\eta)(\boldsymbol{\gamma}, 0)}_{\in C^\infty(\mathbb{R}^2/\Lambda)} + H(\eta)(\mathbf{0}, \Phi),$$

and in the new coordinates

$$H(\eta)(\mathbf{0}, \Phi) = \hat{A}_{2x} + \eta_y \hat{A}_{3x} - \hat{A}_{1y} - \eta_x \hat{A}_{3y}\big|_{w=0}. \tag{3.13}$$

Inserting $\hat{\boldsymbol{A}}|_{w=0}$ and $\partial_w \hat{\boldsymbol{A}}|_{w=0}$ from (3.10), (3.12) into this formula shows that

$$H(\eta)(\mathbf{0}, \Phi) = \operatorname{Op} \lambda_\alpha \Phi + R_\infty \Phi,$$

where $\lambda_\alpha \in S^1(\mathbb{R}^2/\Lambda)$. The asymptotic expansions

$$\mathtt{Z} \sim \sum_{j \le 0} \mathtt{Z}^{(j)}, \qquad \lambda_\alpha \sim \sum_{j \le 1} \lambda_\alpha^{(j)}$$

can be determined recursively by substituting

$$\hat{\boldsymbol{A}}|_{w=0} = Z\Phi + R_\infty \Phi, \qquad \partial_w \hat{\boldsymbol{A}}|_{w=0} = MZ\Phi + R_\infty \Phi$$

into (3.6)–(3.8).

3.2.1. Principal symbol

Equating the order 0 terms in (3.6) and order 1 terms in (3.7), (3.8) yields the equations

$$\eta_x \mathtt{Z}_1^{(0)} + \eta_y \mathtt{Z}_2^{(0)} - \mathtt{Z}_3^{(0)} = 0, \tag{3.14}$$

$$(\mathrm{i}k_2 - \eta_y \mathtt{m}^{(1)})\mathtt{Z}_3^{(0)} + (-\mathtt{m}^{(1)} + \mathrm{i}k_1\eta_x - \eta_x^2 \mathtt{m}^{(1)})\mathtt{Z}_2^{(0)} + (-\mathrm{i}k_2\eta_x + \eta_x\eta_y \mathtt{m}^{(1)})\mathtt{Z}_1^{(0)} = \mathrm{i}k_1, \tag{3.15}$$

$$(-\mathrm{i}k_1 + \eta_x \mathtt{m}^{(1)})\mathtt{Z}_3^{(0)} + (\mathtt{m}^{(1)} - \mathrm{i}k_2\eta_y + \eta_y^2 \mathtt{m}^{(1)})\mathtt{Z}_1^{(0)} + (\mathrm{i}k_1\eta_y - \eta_x\eta_y \mathtt{m}^{(1)})\mathtt{Z}_2^{(0)} = \mathrm{i}k_2. \tag{3.16}$$

Substituting for $\mathtt{Z}_3^{(0)}$ from (3.14) into (3.15), one finds that

$$(\mathrm{i}k_2 - \eta_y \mathtt{m}^{(1)})(\eta_x \mathtt{Z}_1^{(0)} + \eta_y \mathtt{Z}_2^{(0)}) + (-\mathtt{m}^{(1)} + \mathrm{i}k_1\eta_x - \eta_x^2 \mathtt{m}^{(1)})\mathtt{Z}_2^{(0)} + (-\mathrm{i}k_2\eta_x + \eta_x\eta_y \mathtt{m}^{(1)})\mathtt{Z}_1^{(0)} = \mathrm{i}k_1,$$

so that

$$\underbrace{(-\mathtt{m}^{(1)}(1+|\nabla\eta|^2)+\mathrm{i}\boldsymbol{k}\cdot\nabla\eta)}_{=-\lambda^{(1)}}\mathtt{Z}_2^{(0)}=\mathrm{i}k_1$$

and hence

$$\mathtt{Z}_2^{(0)}(\boldsymbol{x}',\boldsymbol{k})=-\frac{\mathrm{i}k_1}{\lambda^{(1)}}.$$

Similarly, substituting for $\mathtt{Z}_3^{(0)}$ from (3.14) into (3.16) yields

$$\mathtt{Z}_1^{(0)}(\boldsymbol{x}',\boldsymbol{k})=\frac{\mathrm{i}k_2}{\lambda^{(1)}},$$

and it follows from (3.14) that

$$\mathtt{Z}_3^{(0)}(\boldsymbol{x}',\boldsymbol{k})=-\frac{\mathrm{i}(\boldsymbol{k}\cdot\nabla^{\perp}\eta)}{\lambda^{(1)}}.$$

Equating terms of order 1 in equation (3.13), we find that

$$\begin{aligned}\lambda_\alpha^{(1)}(\boldsymbol{x}',\boldsymbol{k})&=\mathrm{i}k_1\mathtt{Z}_2^{(0)}+\eta_y\mathrm{i}k_1\mathtt{Z}_3^{(0)}-\mathrm{i}k_2\mathtt{Z}_1^{(0)}-\eta_x\mathrm{i}k_2\mathtt{Z}_3^{(0)}\\&=\mathrm{i}(\boldsymbol{k}\cdot\nabla^{\perp}\eta)\mathtt{Z}_3^{(0)}+\mathrm{i}k_1\mathtt{Z}_2^{(0)}-\mathrm{i}k_2\mathtt{Z}_1^{(0)}\\&=\frac{1}{\lambda^{(1)}}\underbrace{((\boldsymbol{k}\cdot\nabla^{\perp}\eta)^2+|\boldsymbol{k}|^2)}_{=(\lambda^{(1)})^2}\\&=\lambda^{(1)};\end{aligned}$$

the principal symbol of the generalised Dirichlet–Neumann operator is thus the same as the principal symbol of the classical Dirichlet–Neumann operator.

3.2.2. Sub-principal symbol

Equating the order -1 terms in (3.6) and the order 0 terms in (3.7), (3.8) yields the equations

$$\eta_x\mathtt{Z}_1^{(-1)}+\eta_y\mathtt{Z}_2^{(-1)}-\mathtt{Z}_3^{(-1)}=0, \tag{3.17}$$

$$(\mathrm{i}k_2-\eta_y\mathtt{m}^{(1)})\mathtt{Z}_3^{(-1)}+(-\mathtt{m}^{(1)}+\mathrm{i}k_1\eta_x-\eta_x^2\mathtt{m}^{(1)})\mathtt{Z}_2^{(-1)}+(-\mathrm{i}k_2\eta_x+\eta_x\eta_y\mathtt{m}^{(1)})\mathtt{Z}_1^{(-1)}=-\mathtt{F}_1-\alpha\mathtt{F}_3, \tag{3.18}$$

$$(-\mathrm{i}k_1+\eta_x\mathtt{m}^{(1)})\mathtt{Z}_3^{(-1)}+(\mathtt{m}^{(1)}-\mathrm{i}k_2\eta_y+\eta_y^2\mathtt{m}^{(1)})\mathtt{Z}_1^{(-1)}+(\mathrm{i}k_1\eta_y-\eta_x\eta_y\mathtt{m}^{(1)})\mathtt{Z}_2^{(-1)}=-\mathtt{F}_2-\alpha\mathtt{F}_4, \tag{3.19}$$

where

$$
\begin{aligned}
\mathtt{F}_1(\boldsymbol{x}', \boldsymbol{k}) &= \eta_{xy}\mathtt{Z}_1^{(0)} + \eta_{yy}\mathtt{Z}_2^{(0)} + \nabla\mathtt{Z}_2^{(0)} \cdot \nabla\eta \\
&\quad + [\eta_y(\mathtt{Z}_1^{(0)}\nabla\eta_x + \mathtt{Z}_2^{(0)}\nabla\eta_y) + (1+|\nabla\eta|^2)\nabla\mathtt{Z}_2^{(0)}] \cdot \mathrm{i}\nabla_{\boldsymbol{k}}\mathtt{m}^{(1)} \\
&\quad - (1+|\nabla\eta|^2)\mathtt{Z}_2^{(0)} m^{(0)}, \\
\mathtt{F}_2(\boldsymbol{x}', \boldsymbol{k}) &= -\eta_{xx}\mathtt{Z}_1^{(0)} - \eta_{xy}\mathtt{Z}_2^{(0)} - \nabla\eta \cdot \nabla\mathtt{Z}_1^{(0)} \\
&\quad - [\eta_x(\mathtt{Z}_1^{(0)}\nabla\eta_x + \mathtt{Z}_2^{(0)}\nabla\eta_y) + (1+|\nabla\eta|^2)\nabla\mathtt{Z}_1^{(0)}] \cdot \mathrm{i}\nabla_{\boldsymbol{k}}\mathtt{m}^{(1)} \\
&\quad + (1+|\nabla\eta|^2)\mathtt{Z}_1^{(0)} m^{(0)}, \\
\mathtt{F}_3(\boldsymbol{x}', \boldsymbol{k}) &= \frac{1}{2}\Big((1+\eta_x^2)\mathtt{Z}_1^{(0)} + \eta_x\eta_y\mathtt{Z}_2^{(0)}\Big) \\
&\quad - \frac{k_2}{|\boldsymbol{k}|^2}\Big(\Big(k_2 - \eta_x(\boldsymbol{k}\cdot\nabla\eta^\perp)\Big)\mathtt{Z}_1^{(0)} - \Big(k_1 + \eta_y(\boldsymbol{k}\cdot\nabla\eta^\perp)\Big)\mathtt{Z}_2^{(0)}\Big), \\
\mathtt{F}_4(\boldsymbol{x}', \boldsymbol{k}) &= \frac{1}{2}\Big(\eta_x\eta_y\mathtt{Z}_1^{(0)} + (1+\eta_y^2)\mathtt{Z}_2^{(0)}\Big) \\
&\quad - \frac{k_1}{|\boldsymbol{k}|^2}\Big(\Big(-k_2 + \eta_x(\boldsymbol{k}\cdot\nabla\eta^\perp)\Big)\mathtt{Z}_1^{(0)} + \Big(k_1 + \eta_y(\boldsymbol{k}\cdot\nabla\eta^\perp)\Big)\mathtt{Z}_2^{(0)}\Big).
\end{aligned}
$$

Substituting for $\mathtt{Z}_3^{(-1)}$ from (3.17) into (3.18)–(3.19), we obtain

$$
\mathtt{Z}_1^{(-1)}(\boldsymbol{x}', \boldsymbol{k}) = -\frac{\mathtt{F}_2 + \alpha\mathtt{F}_4}{\lambda^{(1)}}, \qquad \mathtt{Z}_2^{(-1)}(\boldsymbol{x}', \boldsymbol{k}) = \frac{\mathtt{F}_1 + \alpha\mathtt{F}_3}{\lambda^{(1)}}
$$

and hence

$$
\mathtt{Z}_3^{(-1)}(\boldsymbol{x}', \boldsymbol{k}) = \frac{1}{\lambda^{(1)}}\nabla\eta^\perp \cdot (\mathtt{F}_1 + \alpha\mathtt{F}_3, \mathtt{F}_2 + \alpha\mathtt{F}_4)^T.
$$

Equating terms of order 1 in equation (3.13), we find that

$$
\begin{aligned}
\lambda_\alpha^{(0)}(\boldsymbol{x}', \boldsymbol{k}) &= \partial_x\mathtt{Z}_2^{(0)} + \mathrm{i}k_1\mathtt{Z}_2^{(-1)} + \eta_y\partial_x\mathtt{Z}_3^{(0)} + \mathrm{i}k_1\eta_y\mathtt{Z}_3^{(-1)} \\
&\qquad - \partial_y\mathtt{Z}_1^{(0)} - \mathrm{i}k_2\mathtt{Z}_1^{(-1)} - \eta_x\partial_y\mathtt{Z}_3^{(0)} - \mathrm{i}k_2\eta_x\mathtt{Z}_3^{(-1)} \\
&= \partial_x\mathtt{Z}_2^{(0)} - \partial_y\mathtt{Z}_1^{(0)} + \eta_y\partial_x\mathtt{Z}_3^{(0)} - \eta_x\partial_y\mathtt{Z}_3^{(0)} \\
&\qquad + \frac{\mathrm{i}k_1 + \mathrm{i}\eta_y(k\cdot\nabla^\perp\eta)}{\lambda^{(1)}}\mathtt{F}_1 - \frac{-\mathrm{i}k_2 + \mathrm{i}\eta_x(k\cdot\nabla^\perp\eta)}{\lambda^{(1)}}\mathtt{F}_2 \\
&\qquad + \alpha\left[\frac{\mathrm{i}k_1 + \mathrm{i}\eta_y(\boldsymbol{k}\cdot\nabla^\perp\eta)}{\lambda^{(1)}}\mathtt{F}_3 - \frac{-\mathrm{i}k_2 + \mathrm{i}\eta_x(\boldsymbol{k}\cdot\nabla^\perp\eta)}{\lambda^{(1)}}\mathtt{F}_4\right] \\
&= \lambda^{(0)} + \alpha\frac{(\boldsymbol{k}\cdot\nabla\eta)(\boldsymbol{k}\cdot\nabla^\perp\eta)}{|\boldsymbol{k}|^2},
\end{aligned}
$$

where

$$
\lambda^{(0)}(\boldsymbol{x}', \boldsymbol{k}) := \frac{1+|\nabla\eta|^2}{2\lambda^{(1)}}\Big(\nabla\cdot(\mathtt{m}^{(1)}\nabla\eta) + \mathrm{i}\nabla_{\boldsymbol{k}}\lambda^{(1)}\cdot\nabla\mathtt{m}^{(1)}\Big)
$$

is the sub-principal symbol of the classical Dirichlet–Neumann operator (see Alazard, Burq and Zuily [1, Eq. (3.11)]).

3.3. The operator $M(\eta)$

Let $s \geq 2$, $\boldsymbol{g} \in H^{s-\frac{1}{2}}(\mathbb{R}^2/\Lambda)^2$ and $\tilde{\boldsymbol{B}} \in H^s(D_0/\Lambda)^3$ be the unique solution of the boundary-value problem (2.21)–(2.26) with $\boldsymbol{\gamma} = \boldsymbol{0}$. The variable

$$\hat{\boldsymbol{B}}(\boldsymbol{x}', w) := \tilde{\boldsymbol{B}}(\boldsymbol{x}', v), \qquad w := \frac{1}{\delta h}(h+\eta)v$$

satisfies

$$L\hat{\boldsymbol{B}} = \boldsymbol{0} \qquad \text{in } D_0,$$

and

$$\operatorname{div}^{\varrho} \hat{\boldsymbol{B}} = 0 \qquad \text{at } w = 0, \tag{3.20}$$

$$\hat{\boldsymbol{B}} \cdot \boldsymbol{N} = 0 \qquad \text{at } w = 0, \tag{3.21}$$

$$\nabla \cdot \hat{\boldsymbol{B}}_{||}^{\perp} = \nabla \cdot \boldsymbol{g}^{\perp} \qquad \text{at } w = 0, \tag{3.22}$$

$$\langle (\operatorname{curl}^{\varrho} \hat{\boldsymbol{B}})_{\|} \rangle = \boldsymbol{0}. \tag{3.23}$$

(equation (3.20) actually holds in $\bar{D}_0$). The boundary conditions (3.20)–(3.22) can be written more explicitly as

$$\hat{B}_{1x} + \hat{B}_{2y} + \frac{1}{\delta}(-\eta_x \hat{B}_{1w} - \eta_y \hat{B}_{2w} + \hat{B}_{3w}) = 0 \qquad \text{at } w = 0, \tag{3.24}$$

$$\eta_x \hat{B}_1 + \eta_y \hat{B}_2 - \hat{B}_3 = 0 \qquad \text{at } w = 0, \tag{3.25}$$

$$(\hat{B}_2 + \hat{B}_3\eta_y)_x - (\hat{B}_1 + \hat{B}_3\eta_x)_y = g_{2x} - g_{1y} \qquad \text{at } w = 0. \tag{3.26}$$

Substituting

$$\hat{B}_3 = \eta_x \hat{B}_1 + \eta_y \hat{B}_2 \tag{3.27}$$

(see equation (3.25)) and

$$\partial_w \hat{\boldsymbol{B}}|_{w=0} = M\hat{\boldsymbol{B}}|_{w=0} + R_\infty \boldsymbol{g} \tag{3.28}$$

(see Lemma 3.5, noting that $\boldsymbol{B}$ is a linear function of $\boldsymbol{g}$) into equations (3.24), (3.26), we find that

$$P\begin{pmatrix} \hat{B}_1|_{w=0} \\ \hat{B}_2|_{w=0} \end{pmatrix} = \begin{pmatrix} 0 \\ g_{2x} - g_{1y} \end{pmatrix} + R_\infty \boldsymbol{g}, \tag{3.29}$$

where $P \in \Psi^1(\mathbb{R}^2/\Lambda)$ is a properly supported pseudodifferential operator with principal symbol

$$\mathtt{P}^{(1)}(\boldsymbol{x}',\boldsymbol{k})=\begin{pmatrix}\mathrm{i}k_1 & \mathrm{i}k_2\\ \mathrm{i}\eta_x\eta_y k_1-\mathrm{i}(1+\eta_x^2)k_2 & \mathrm{i}(1+\eta_y^2)k_1-\mathrm{i}\eta_x\eta_y k_2\end{pmatrix}.$$

Observe that $\mathtt{P}^{(1)}(\boldsymbol{x}',\boldsymbol{k})$ is invertible for $|\boldsymbol{k}|\neq 0$, so that P is elliptic and hence admits a parametrix $Q\in\Psi^{-1}(\mathbb{R}^2/\Lambda)$ such that $PQ-I\in\Psi^{-\infty}(\mathbb{R}^2/\Lambda)$. We thus find from equation (3.29) that

$$\begin{pmatrix}\hat{B}_1|_{w=0}\\ \hat{B}_2|_{w=0}\end{pmatrix}=Q\begin{pmatrix}0\\ g_{2x}-g_{1y}\end{pmatrix}+R_\infty\boldsymbol{g},$$

and appending (3.27) to this equation yields

$$\hat{\boldsymbol{B}}|_{w=0}=Z\boldsymbol{g}+R_\infty\boldsymbol{g}, \tag{3.30}$$

where $\mathtt{Z}\in S^0(\mathbb{R}^2/\Lambda)$ and $Z=\operatorname{Op}\mathtt{Z}$.

We have that

$$\boldsymbol{M}(\eta)(\boldsymbol{\gamma},\boldsymbol{g})=\underbrace{\boldsymbol{M}(\eta)(\boldsymbol{\gamma},\boldsymbol{0})}_{\in C^\infty(\mathbb{R}^2/\Lambda)^2}+\boldsymbol{M}(\eta)(\boldsymbol{0},\boldsymbol{g}),$$

and in the new coordinates

$$\begin{aligned}\boldsymbol{M}(\eta)(\boldsymbol{0},\boldsymbol{g})=&-\begin{pmatrix}\tilde{B}_{3y}\\ -\tilde{B}_{3x}\end{pmatrix}+\frac{1}{\delta}\tilde{B}_{3w}\nabla^\perp\eta+\frac{1}{\delta}\begin{pmatrix}\tilde{B}_{2w}\\ -\tilde{B}_{1w}\end{pmatrix}-(\tilde{B}_{2x}-\tilde{B}_{1y})\nabla\eta\\ &+\frac{1}{\delta}(\eta_x\tilde{B}_{2w}-\eta_y\tilde{B}_{1w})\nabla\eta\bigg|_{w=0}.\end{aligned} \tag{3.31}$$

Inserting $\hat{\boldsymbol{B}}|_{w=0}$ and $\partial_w\hat{\boldsymbol{B}}|_{w=0}$ from (3.28), (3.30) into this formula shows that

$$\boldsymbol{M}(\eta)(\boldsymbol{0},\boldsymbol{g})=\operatorname{Op}\nu_\alpha\boldsymbol{g}+R_\infty\boldsymbol{g},$$

where $\nu_\alpha\in S^1(\mathbb{R}^2/\Lambda)$. The asymptotic expansions

$$\mathtt{Z}\sim\sum_{j\le 0}\mathtt{Z}^{(j)},\qquad \nu_\alpha\sim\sum_{j\le 1}\nu_\alpha^{(j)}$$

can be determined recursively by substituting

$$\hat{\boldsymbol{B}}|_{w=0}=Z\boldsymbol{g}+R_\infty\boldsymbol{g},\qquad \partial_w\hat{\boldsymbol{B}}|_{w=0}=MZ\boldsymbol{g}+R_\infty\boldsymbol{g}$$

into (3.24)–(3.26).

Remark 3.6. The asymptotic expansion of ν_α can also be determined from the formula

$$\boldsymbol{M}(\eta)(\boldsymbol{0},\boldsymbol{g}):=-\nabla\big(H(\eta)(\boldsymbol{0},\cdot)^{-1}(\nabla\cdot\boldsymbol{g}^\perp)\big)+\alpha\nabla^\perp\Delta^{-1}(\nabla\cdot\boldsymbol{g}^\perp)$$

and the asymptotic expansion of the symbol λ_α of $H(\eta)(\boldsymbol{0},\cdot)$.

3.3.1. Principal symbol

Equating terms of order 1 in equations (3.24), (3.26) and order 0 in equation (3.25) yields

$$\begin{aligned}
\mathrm{i}k_1\, \mathtt{z}_{11}^{(0)} + \mathrm{i}k_2 \mathtt{z}_{21}^{(0)} - \eta_x\, \mathtt{m}^{(1)} \mathtt{z}_{11}^{(0)} - \eta_y\, \mathtt{m}^{(1)} \mathtt{z}_{21}^{(0)} + \mathtt{m}^{(1)} \mathtt{z}_{31}^{(0)} &= 0,\\
\mathrm{i}k_1\, \mathtt{z}_{12}^{(0)} + \mathrm{i}k_2 \mathtt{z}_{22}^{(0)} - \eta_x\, \mathtt{m}^{(1)} \mathtt{z}_{12}^{(0)} - \eta_y\, \mathtt{m}^{(1)} \mathtt{z}_{22}^{(0)} + \mathtt{m}^{(1)} \mathtt{z}_{32}^{(0)} &= 0,\\
\eta_x \mathtt{z}_{11}^{(0)} + \eta_y \mathtt{z}_{21}^{(0)} - \mathtt{z}_{31}^{(0)} &= 0,\\
\eta_x \mathtt{z}_{12}^{(0)} + \eta_y \mathtt{z}_{22}^{(0)} - \mathtt{z}_{32}^{(0)} &= 0,\\
\mathrm{i}k_1 \mathtt{z}_{21}^{(0)} - \mathrm{i}k_2 \mathtt{z}_{11}^{(0)} + \mathrm{i}(\boldsymbol{k}\cdot\nabla^{\perp}\eta)\mathtt{z}_{31}^{(0)} &= -\mathrm{i}k_2,\\
\mathrm{i}k_1 \mathtt{z}_{22}^{(0)} - \mathrm{i}k_2 \mathtt{z}_{12}^{(0)} + \mathrm{i}(\boldsymbol{k}\cdot\nabla^{\perp}\eta)\mathtt{z}_{32}^{(0)} &= \mathrm{i}k_1,
\end{aligned}$$

whose unique solution is

$$\begin{aligned}
\mathtt{z}_{11}^{(0)}(\boldsymbol{x}',\boldsymbol{k}) &= \frac{k_2^2}{(\lambda^{(1)})^2}, & \mathtt{z}_{12}^{(0)}(\boldsymbol{x}',\boldsymbol{k}) &= -\frac{k_1k_2}{(\lambda^{(1)})^2},\\
\mathtt{z}_{21}^{(0)}(\boldsymbol{x}',\boldsymbol{k}) &= -\frac{k_1k_2}{(\lambda^{(1)})^2}, & \mathtt{z}_{22}^{(0)}(\boldsymbol{x}',\boldsymbol{k}) &= \frac{k_1^2}{(\lambda^{(1)})^2},\\
\mathtt{z}_{31}^{(0)}(\boldsymbol{x}',\boldsymbol{k}) &= -\frac{k_2(\boldsymbol{k}\cdot\nabla^{\perp}\eta)}{(\lambda^{(1)})^2}, & \mathtt{z}_{32}^{(0)}(\boldsymbol{x}',\boldsymbol{k}) &= \frac{k_1(\boldsymbol{k}\cdot\nabla^{\perp}\eta)}{(\lambda^{(1)})^2}.
\end{aligned}$$

Equating terms of order 1 in equation (3.31), we find that

$$\begin{aligned}
\nu_\alpha^{(1)}(\boldsymbol{x}',\boldsymbol{k})\boldsymbol{g} = {}& \begin{pmatrix} -\mathrm{i}k_2(\mathtt{z}_{31}^{(0)} g_1 + \mathtt{z}_{32}^{(0)} g_2) \\ \mathrm{i}k_1(\mathtt{z}_{31}^{(0)} g_1 + \mathtt{z}_{32}^{(0)} g_2) \end{pmatrix} + \mathtt{m}^{(1)}\,(\mathtt{z}_{31}^{(0)} g_1 + \mathtt{z}_{32}^{(0)} g_2)\nabla^{\perp}\eta\\
&+ \begin{pmatrix} \mathtt{m}^{(1)}\,(\mathtt{z}_{21}^{(0)} g_1 + \mathtt{z}_{22}^{(0)} g_2) \\ -\mathtt{m}^{(1)}\,(\mathtt{z}_{11}^{(0)} g_1 + \mathtt{z}_{12}^{(0)} g_2) \end{pmatrix}\\
&- \Big[\mathrm{i}k_1\,(\mathtt{z}_{21}^{(0)} g_1 + \mathtt{z}_{22}^{(0)} g_2) - \mathrm{i}k_2\,(\mathtt{z}_{11}^{(0)} g_1 + \mathtt{z}_{12}^{(0)} g_2)\Big]\nabla\eta\\
&+ \Big[\eta_x \mathtt{m}^{(1)}\,(\mathtt{z}_{21}^{(0)} g_1 + \mathtt{z}_{22}^{(0)} g_2) - \eta_y\,\mathtt{m}^{(1)}\,(\mathtt{z}_{11}^{(0)} g_1 + \mathtt{z}_{12}^{(0)} g_2)\Big]\nabla\eta.
\end{aligned}$$

The first component of $\nu_\alpha^{(1)}\boldsymbol{g}$ can be rewritten as

$$\begin{aligned}
&\eta_x(\mathrm{i}k_2 - \eta_y\,\mathtt{m}^{(1)})\,(\mathtt{z}_{11}^{(0)} g_1 + \mathtt{z}_{12}^{(0)} g_2) + \Big(-\mathrm{i}k_1\,\eta_x + (1+\eta_x^2)\mathtt{m}^{(1)}\Big)\,(\mathtt{z}_{21}^{(0)} g_1 + \mathtt{z}_{22}^{(0)} g_2)\\
&\qquad + (-\mathrm{i}k_2 + \eta_y \mathtt{m}^{(1)})\,(\mathtt{z}_{31}^{(0)} g_1 + \mathtt{z}_{32}^{(0)} g_2)\\
&= \Big[\eta_x(\mathrm{i}k_2 - \eta_y \mathtt{m}^{(1)})\mathtt{z}_{11}^{(0)} + (-\mathrm{i}k_1\eta_x + (1+\eta_x^2)\mathtt{m}^{(1)})\mathtt{z}_{21}^{(0)} + (-\mathrm{i}k_2 + \eta_y \mathtt{m}^{(1)})\mathtt{z}_{31}^{(0)}\Big] g_1\\
&\qquad + \Big[\eta_x(\mathrm{i}k_2 - \eta_y \mathtt{m}^{(1)})\mathtt{z}_{12}^{(0)} + (-\mathrm{i}k_1\eta_x + (1+\eta_x^2)\mathtt{m}^{(1)})\mathtt{z}_{22}^{(0)} + (-\mathrm{i}k_2 + \eta_y \mathtt{m}^{(1)})\mathtt{z}_{32}^{(0)}\Big] g_2\\
&= -\frac{k_1k_2}{\lambda^{(1)}} g_1 + \frac{k_1^2}{\lambda^{(1)}} g_2\\
&= \frac{k_1}{\lambda^{(1)}}(\boldsymbol{k}\cdot\boldsymbol{g}^{\perp}),
\end{aligned}$$

and in the same way we find that the second component of $\nu_\alpha^{(1)}(\boldsymbol{x}',\boldsymbol{k})\boldsymbol{g}$ is $\dfrac{k_2}{\lambda^{(1)}}(\boldsymbol{k}\cdot\boldsymbol{g}^\perp)$; altogether we obtain

$$\nu_\alpha^{(1)}(\boldsymbol{x}',\boldsymbol{k})\boldsymbol{g}=\boldsymbol{k}\frac{(\boldsymbol{k}\cdot\boldsymbol{g}^\perp)}{\lambda^{(1)}}.$$

3.3.2. Sub-principal symbol

Equating terms of order 0 in equations (3.24), (3.26) and order −1 in equation (3.25) yields

$$\begin{aligned}
\mathrm{i}k_1\mathtt{Z}_{11}^{(-1)}+\mathrm{i}k_2\mathtt{Z}_{21}^{(-1)}+k_2\mathtt{G}_1&=0,\\
\mathrm{i}k_1\mathtt{Z}_{12}^{(-1)}+\mathrm{i}k_2\mathtt{Z}_{22}^{(-1)}+k_1\mathtt{G}_2&=0,\\
\mathtt{Z}_{31}^{(-1)}&=\eta_x\mathtt{Z}_{11}^{(-1)}+\eta_y\mathtt{Z}_{21}^{(-1)},\\
\mathtt{Z}_{32}^{(-1)}&=\eta_x\mathtt{Z}_{12}^{(-1)}+\eta_y\mathtt{Z}_{22}^{(-1)},\\
(-\mathrm{i}k_2+\mathrm{i}(\boldsymbol{k}\cdot\nabla^\perp\eta)\eta_x)\mathtt{Z}_{11}^{(-1)}+(\mathrm{i}k_1+\mathrm{i}(\boldsymbol{k}\cdot\nabla^\perp\eta)\eta_y)\mathtt{Z}_{21}^{(-1)}+k_2\mathtt{G}_3&=0,\\
(-\mathrm{i}k_2+\mathrm{i}(\boldsymbol{k}\cdot\nabla^\perp\eta)\eta_x)\mathtt{Z}_{12}^{(-1)}+(\mathrm{i}k_1+\mathrm{i}(\boldsymbol{k}\cdot\nabla^\perp\eta)\eta_y)\mathtt{Z}_{22}^{(-1)}+k_1\mathtt{G}_4&=0,
\end{aligned}$$

where

$$\begin{aligned}
\mathtt{G}_1(\boldsymbol{x}',\boldsymbol{k})&=\frac{2(\boldsymbol{k}\cdot\nabla^\perp\lambda^{(1)})}{(\lambda^{(1)})^3}-\frac{\mathrm{i}}{(\lambda^{(1)})^2}\Big[k_2(\nabla_{\boldsymbol{k}}\cdot\nabla\eta_x)-k_1(\nabla_{\boldsymbol{k}}\mathtt{m}^{(1)}\cdot\nabla\eta_y)\Big]+\frac{\mathrm{i}\alpha}{2\lambda^{(1)}},\\
\mathtt{G}_2(\boldsymbol{x}',\boldsymbol{k})&=-\frac{2(\boldsymbol{k}\cdot\nabla^\perp\lambda^{(1)})}{(\lambda^{(1)})^3}+\frac{\mathrm{i}}{(\lambda^{(1)})^2}\Big[k_2(\nabla_{\boldsymbol{k}}\mathtt{m}^{(1)}\cdot\nabla\eta_x)-k_1(\nabla_{\boldsymbol{k}}\mathtt{m}^{(1)}\cdot\nabla\eta_y)\Big]-\frac{\mathrm{i}\alpha}{2\lambda^{(1)}},\\
\mathtt{G}_3(\boldsymbol{x}',\boldsymbol{k})&=\frac{1}{(\lambda^{(1)})^3}\Big[2(1+\eta_y^2)\partial_x\lambda^{(1)}k_1+2(1+\eta_x^2)\partial_y\lambda^{(1)}k_2-2\eta_x\eta_y(\partial_x\lambda^{(1)}k_2+\partial_y\lambda^{(1)}k_1)\\
&\qquad+(\eta_y\eta_{xx}-\eta_x\eta_{xy})\lambda^{(1)}k_2-(\eta_y\eta_{xy}-\eta_x\eta_{yy})\lambda^{(1)}k_1\Big],\\
\mathtt{G}_4(\boldsymbol{x}',\boldsymbol{k})&=\frac{1}{(\lambda^{(1)})^3}\Big[-2(1+\eta-y^2)\partial_x\lambda^{(1)}k_1-2(1+\eta_x^2)\partial_y\lambda^{(1)}k_2\\
&\qquad+2\eta_x\eta_y(\partial_x\lambda^{(1)}k_2+\partial_y\lambda^{(1)}k_1)\\
&\qquad-(\eta_y\eta_{xx}-\eta_x\eta_{xy})\lambda^{(1)}k_2+(\eta_y\eta_{xy}-\eta_x\eta_{yy})\lambda^{(1)}k_1\Big],
\end{aligned}$$

whose unique solution is

$$\begin{aligned}
\mathtt{Z}_{11}^{(-1)}(\boldsymbol{x}',\boldsymbol{k})&=\frac{\mathrm{i}k_2}{(\lambda^{(1)})^2}\Big((k_1+(\boldsymbol{k}\cdot\nabla^\perp\eta)\eta_y)-k_2\mathtt{G}_3\Big),\\
\mathtt{Z}_{12}^{(-1)}(\boldsymbol{x}',\boldsymbol{k})&=\frac{\mathrm{i}\mathtt{G}_2}{(\lambda^{(1)})^2}\Big((\lambda^{(1)})^2-k_2(k_2-(\boldsymbol{k}\cdot\nabla^\perp\eta)\eta_x)\Big)-\frac{\mathrm{i}k_1k_2\mathtt{G}_4}{(\lambda^{(1)})^2},\\
\mathtt{Z}_{21}^{(-1)}(\boldsymbol{x}',\boldsymbol{k})&=\frac{\mathrm{i}\mathtt{G}_1}{(\lambda^{(1)})^2}\Big((\lambda^{(1)})^2-k_1(k_1+(\boldsymbol{k}\cdot\nabla^\perp\eta)\eta_y)\Big)+\frac{\mathrm{i}k_1k_2\mathtt{G}_3}{(\lambda^{(1)})^2},
\end{aligned}$$

$$\mathrm{Z}_{22}^{(-1)}(\boldsymbol{x}',\boldsymbol{k}) = \frac{\mathrm{i}k_1}{(\lambda^{(1)})^2}\Big((k_2-(\boldsymbol{k}\cdot\nabla^\perp\eta)\eta_x)\mathrm{G}_2 + k_1\mathrm{G}_4\Big),$$
$$\mathrm{Z}_{31}^{(-1)}(\boldsymbol{x}',\boldsymbol{k}) = \eta_x\mathrm{Z}_{11}^{(-1)} + \eta_y\mathrm{Z}_{21}^{(-1)},$$
$$\mathrm{Z}_{32}^{(-1)}(\boldsymbol{x}',\boldsymbol{k}) = \eta_x\mathrm{Z}_{12}^{(-1)} + \eta_y\mathrm{Z}_{22}^{(-1)}.$$

Inserting these formulae into the zeroth order part of (3.31), we find after a lengthy but straightforward computation that

$$\nu_\alpha^{(0)}(\boldsymbol{x}',\boldsymbol{k})\boldsymbol{g} = \begin{pmatrix}\zeta_1(\boldsymbol{x}',\boldsymbol{k})\\ \zeta_2(\boldsymbol{x}',\boldsymbol{k})\end{pmatrix}(\boldsymbol{k}\cdot\boldsymbol{g}^\perp),$$

where

$$\begin{aligned}\zeta_1(\boldsymbol{x}',\boldsymbol{k}) = {}& \frac{\mathrm{i}}{2(\lambda^{(1)})^5}\Big(k_1^2(-1+2\eta_y^2)\eta_x - k_1k_2\eta_y(3+4\eta_x^2) + 2k_2^2\eta_x(1+\eta_x^2) + \mathrm{i}k_1\lambda^{(1)}\Big)\\ &\times\Big(k_1^2\eta_{yy} - 2k_1k_2\eta_{xy} + k_2^2\eta_{xx}\Big) + \frac{\alpha}{(\lambda^{(1)})^2}\Big(k_2(1+\eta_x^2) - k_1\eta_x\eta_y\Big),\\ \zeta_2(\boldsymbol{x}',\boldsymbol{k}) = {}& \frac{\mathrm{i}}{2(\lambda^{(1)})^5}\Big(2k_1^2\eta_y(1+\eta_y^2) - k_1k_2\eta_x(3+4\eta_y^2) + k_2^2\eta_y(-1+2\eta_x^2) + \mathrm{i}k_2\lambda^{(1)}\Big)\\ &\times\Big(k_1^2\eta_{yy} - 2k_1k_2\eta_{xy} + k_2^2\eta_{xx}\Big) + \frac{\alpha}{(\lambda^{(1)})^2}\Big(-k_1(1+\eta_y^2) + k_2\eta_x\eta_y\Big).\end{aligned}$$

4. Approximate solutions

In this section we construct approximate solutions of

$$J(\eta,\boldsymbol{\mu}) = 0 \tag{4.1}$$

for $\beta \geq 0$ in the form of power series and moreover prove their convergence for $\beta > 0$; the solutions have wave velocity $\boldsymbol{c}$ close to a reference value $\boldsymbol{c}_0$ chosen such that the transversality condition (T) holds. Assuming that the non-resonance condition (NR) also holds, we consider J as a locally analytic mapping $X_s^\beta \times \mathbb{R}^2 \to H^s(\mathbb{R}^2/\Lambda)$ for a sufficiently large value of s, where

$$X_s^\beta := \begin{cases} H^{s+2}(\mathbb{R}^2/\Lambda), & \text{if } \beta > 0,\\ H^{s+1}(\mathbb{R}^2/\Lambda), & \text{if } \beta = 0.\end{cases}$$

Our strategy is to perform a Lyapunov–Schmidt reduction, and we therefore proceed to investigate the kernel and range of

$$J_{10}(\eta) := \mathrm{d}_1 J[0,\boldsymbol{0}](\eta) = \boldsymbol{T}_1(\eta)\cdot\boldsymbol{c}_0 + g\eta - \beta\Delta\eta.$$

Write

$$\eta(\boldsymbol{x}') = \sum_{\boldsymbol{k}\in\Lambda'}\hat{\eta}_{\boldsymbol{k}}\mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}'},$$

so that

$$(J_{10}\eta)(\boldsymbol{x}') = g\hat{\eta}_{\boldsymbol{0}} + \sum_{\boldsymbol{k}\in\Lambda'\setminus\{\boldsymbol{0}\}} \frac{\mathrm{c}(|\boldsymbol{k}|)}{|\boldsymbol{k}|^2}\rho(\boldsymbol{k},\boldsymbol{c}_0,\beta)\hat{\eta}_{\boldsymbol{k}}\mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}'}.$$

The equation $J_{10}\eta = 0$ is equivalent to

$$\rho(\boldsymbol{k},\boldsymbol{c}_0,\beta)\hat{\eta}_{\boldsymbol{k}} = 0$$

for $\boldsymbol{k} \in \Lambda' \setminus \{\boldsymbol{0}\}$, which by assumption has non-trivial solutions if and only if $\boldsymbol{k} = \pm\boldsymbol{k}_1, \pm\boldsymbol{k}_2$; it follows that

$$\ker(J_{10}) = \{A\mathrm{e}^{\mathrm{i}\boldsymbol{k}_1\cdot\boldsymbol{x}'} + B\mathrm{e}^{\mathrm{i}\boldsymbol{k}_2\cdot\boldsymbol{x}'} + \bar{A}\mathrm{e}^{-\mathrm{i}\boldsymbol{k}_1\cdot\boldsymbol{x}'} + \bar{B}\mathrm{e}^{-\mathrm{i}\boldsymbol{k}_2\cdot\boldsymbol{x}'} :\, A, B \in \mathbb{C}\}.$$

We next consider the range of J_{10}. Let

$$f(\boldsymbol{x}') = \sum_{\boldsymbol{k}\in\Lambda'} \hat{f}_{\boldsymbol{k}}\mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}'} \in H^s(\mathbb{R}^2/\Lambda).$$

The equation $J_{10}\eta = f$ is equivalent to

$$g\hat{\eta}_{\boldsymbol{0}} = \hat{f}_{\boldsymbol{0}}$$

and

$$\frac{\mathrm{c}(|\boldsymbol{k}|)}{|\boldsymbol{k}|^2}\rho(\boldsymbol{k},\boldsymbol{c}_0,\beta)\hat{\eta}_{\boldsymbol{k}} = \hat{f}_{\boldsymbol{k}} \tag{4.2}$$

for $\boldsymbol{k} \in \Lambda' \setminus \{\boldsymbol{0}\}$. Obviously

$$\hat{\eta}_{\boldsymbol{0}} = \frac{1}{g}\hat{f}_{\boldsymbol{0}}, \tag{4.3}$$

while for $\boldsymbol{k} \neq \pm\boldsymbol{k}_1, \pm\boldsymbol{k}_2$ equation (4.2) has the unique solution

$$\hat{\eta}_{\boldsymbol{k}} = \frac{|\boldsymbol{k}|^2}{\mathrm{c}(|\boldsymbol{k}|)\rho(\boldsymbol{k},\boldsymbol{c}_0,\beta)}\hat{f}_{\boldsymbol{k}}, \tag{4.4}$$

and for $\boldsymbol{k} = \pm\boldsymbol{k}_1, \pm\boldsymbol{k}_2$ it is solvable if and only if $\hat{f}_{\pm\boldsymbol{k}_1} = \hat{f}_{\pm\boldsymbol{k}_2} = 0$. For $\beta > 0$ we find that $\rho(\boldsymbol{k},\boldsymbol{c}_0,\beta) \gtrsim |\boldsymbol{k}|^3$ for sufficiently large $|\boldsymbol{k}|$, so that the series

$$\sum_{\substack{\boldsymbol{k}\in\Lambda' \\ \boldsymbol{k}\neq\pm\boldsymbol{k}_1,\pm\boldsymbol{k}_2}} \hat{\eta}_{\boldsymbol{k}}\mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}'},$$

where $\hat{\eta}_{\boldsymbol{k}}$ is given by (4.3), (4.4), converges in $H^{s+2}(\mathbb{R}^2/\Lambda)$. It follows that $J_{10}\colon H^{s+2}(\mathbb{R}^2/\Lambda) \to H^s(\mathbb{R}^2/\Lambda)$ is Fredholm with index 0, where

$$\mathrm{ran}(J_{10}) = \{f \in H^s(\mathbb{R}^2/\Lambda)\colon \hat{f}_{\pm\boldsymbol{k}_1} = \hat{f}_{\pm\boldsymbol{k}_2} = 0\}$$

and $J_{10}^{-1}\colon \mathrm{ran}(J_{10}) \to H^{s+2}(\mathbb{R}^2/\Lambda)$ is given by (4.3), (4.4). In contrast $\rho(\boldsymbol{k}, \boldsymbol{c}_0, 0)$ is not bounded from below as $|\boldsymbol{k}| \to \infty$, so that (4.3), (4.4) does not define a bounded operator from $H^s(\mathbb{R}^2/\Lambda)$ to $H^{s+1}(\mathbb{R}^2/\Lambda)$ for any s. We therefore proceed formally, noting that the procedure is rigorously valid for $\beta > 0$.

To apply the Lyapunov–Schmidt reduction let Π be the orthogonal projection of $H^s(\mathbb{R}^2/\Lambda)$ onto $\ker(J_{10})$ with respect to the $L^2(\mathbb{R}^2/\Lambda)$ inner product $\langle \cdot\,, \cdot \rangle$. Write $\eta = \eta_1 + \eta_2$, where

$$\eta_1 = A\mathrm{e}^{\mathrm{i}\boldsymbol{k}_1\cdot\boldsymbol{x}'} + B\mathrm{e}^{\mathrm{i}\boldsymbol{k}_2\cdot\boldsymbol{x}'} + \bar{A}\mathrm{e}^{-\mathrm{i}\boldsymbol{k}_1\cdot\boldsymbol{x}'} + \bar{B}\mathrm{e}^{-\mathrm{i}\boldsymbol{k}_2\cdot\boldsymbol{x}'},$$

and $\eta_2 \in \ker(J_{10})^\perp = (I - \Pi)X_s^\beta$, and decompose (4.1) as

$$\Pi J(\eta_1 + \eta_2, \boldsymbol{\mu}) = 0, \tag{4.5}$$

$$(I - \Pi) J(\eta_1 + \eta_2, \boldsymbol{\mu}) = 0. \tag{4.6}$$

The linearisation of $(I - \Pi)J$ at 0 is

$$(I - \Pi)J_{10}\colon (I - \Pi)X_s^\beta \to (I - \Pi)H^s(\mathbb{R}^2/\Lambda).$$

For $\beta > 0$ this operator is an isomorphism (see above) and we can solve (4.6) to determine η_2 as a locally analytic function of η_1 and $\boldsymbol{\mu}$; substituting $\eta_2 = \eta_2(\eta_1, \mu)$ into (4.5) yields the reduced equation

$$\Pi J(\eta_1 + \eta_2(\eta_1, \boldsymbol{\mu}), \boldsymbol{\mu}) = 0. \tag{4.7}$$

Note that $\eta_2 = \mathcal{O}(|(\eta_1, \boldsymbol{\mu})||\eta_1|)$ and the left-hand side of equation (4.7) is also $\mathcal{O}(|(\eta_1, \boldsymbol{\mu})||\eta_1|)$ because $\Pi J_{10}(\eta_1 + \eta_2(\eta_1, \boldsymbol{\mu})) = 0$. For $\beta = 0$ we can only formally solve (4.6) for η_2 as a function of η_1 and $\boldsymbol{\mu}$.

We proceed to solve equation (4.7), which can be written as

$$\langle J(\eta_1 + \eta_2(\eta_1, \boldsymbol{\mu}), \boldsymbol{\mu}), \mathrm{e}^{\mathrm{i}\boldsymbol{k}_i\cdot\boldsymbol{x}'} \rangle = 0, \qquad i = 1, 2,$$

because J is real-valued. We write these equations as

$$f_1(A, B, \bar{A}, \bar{B}, \boldsymbol{\mu}) = 0,$$

$$f_2(A, B, \bar{A}, \bar{B}, \boldsymbol{\mu}) = 0,$$

and note that

$$f_j(A, B, \bar{A}, \bar{B}, \boldsymbol{\mu}) = O(|(A, B, \boldsymbol{\mu})||(A, B)|).$$

Recall that J is equivariant with respect to the symmetries S_0 and $T_{\boldsymbol{v}'}$ (see Remark 1.4), which act on the coordinates $(A, B, \bar{A}, \bar{B})$ as

$$S_0(A, B, \bar{A}, \bar{B}) = (\bar{A}, \bar{B}, A, B), \qquad T_{\boldsymbol{v}'}(A, B, \bar{A}, \bar{B}) = (A\mathrm{e}^{\mathrm{i}\boldsymbol{k}_1\cdot\boldsymbol{v}'}, B\mathrm{e}^{\mathrm{i}\boldsymbol{k}_2\cdot\boldsymbol{v}'}, \bar{A}\mathrm{e}^{-\mathrm{i}\boldsymbol{k}_1\cdot\boldsymbol{v}'}, \bar{B}\mathrm{e}^{-\mathrm{i}\boldsymbol{k}_2\cdot\boldsymbol{v}'}),$$

so that the reduced equation remains equivariant under these symmetries, that is

$$\begin{aligned} f_1(A\mathrm{e}^{\mathrm{i}\boldsymbol{k}_1\cdot\boldsymbol{v}'}, B\mathrm{e}^{\mathrm{i}\boldsymbol{k}_2\cdot\boldsymbol{v}'}, \bar{A}\mathrm{e}^{-\mathrm{i}\boldsymbol{k}_1\cdot\boldsymbol{v}'}, \bar{B}\mathrm{e}^{-\mathrm{i}\boldsymbol{k}_2\cdot\boldsymbol{v}'}, \boldsymbol{\mu}) &= \mathrm{e}^{\mathrm{i}\boldsymbol{k}_1\cdot\boldsymbol{v}'} f_1(A, B, \bar{A}, \bar{B}, \boldsymbol{\mu}),\\ f_2(A\mathrm{e}^{\mathrm{i}\boldsymbol{k}_1\cdot\boldsymbol{v}'}, B\mathrm{e}^{\mathrm{i}\boldsymbol{k}_2\cdot\boldsymbol{v}'}, \bar{A}\mathrm{e}^{-\mathrm{i}\boldsymbol{k}_1\cdot\boldsymbol{v}'}, \bar{B}\mathrm{e}^{-\mathrm{i}\boldsymbol{k}_2\cdot\boldsymbol{v}'}, \boldsymbol{\mu}) &= \mathrm{e}^{\mathrm{i}\boldsymbol{k}_2\cdot\boldsymbol{v}'} f_2(A, B, \bar{A}, \bar{B}, \boldsymbol{\mu}),\\ f_1(\bar{A}, \bar{B}, A, B, \boldsymbol{\mu}) &= \bar{f}_1(A, B, \bar{A}, \bar{B}, \boldsymbol{\mu}),\\ f_2(\bar{A}, \bar{B}, A, B, \boldsymbol{\mu}) &= \bar{f}_2(A, B, \bar{A}, \bar{B}, \boldsymbol{\mu}). \end{aligned}$$

It follows that

$$f_1(A, B, \bar{A}, \bar{B}, \boldsymbol{\mu}) = Ag_1(|A|^2, |B|^2, \boldsymbol{\mu}), \tag{4.8}$$

$$f_2(A, B, \bar{A}, \bar{B}, \boldsymbol{\mu}) = Bg_2(|A|^2, |B|^2, \boldsymbol{\mu}), \tag{4.9}$$

where g_1, g_2 are real-valued locally analytic functions which vanish at the origin.

Solutions to equations (4.8), (4.9) with $A \neq 0$, $B = 0$, such that

$$g_1(|A|^2, 0, \boldsymbol{\mu}) = 0,$$

lead to solutions of (4.1) of the form $\eta = \eta_1 + \eta_2(\eta_1, \boldsymbol{\mu})$ with $\eta_1 = A\mathrm{e}^{\mathrm{i}\boldsymbol{k}_1\cdot\boldsymbol{x}'} + \overline{A}\mathrm{e}^{-\mathrm{i}\boldsymbol{k}_1\cdot\boldsymbol{x}'}$, so that η depends on the single variable $\tilde{x} := \boldsymbol{k}_1 \cdot \boldsymbol{x}'$. Such waves are often called $2\frac{1}{2}$-*dimensional waves* since they only depend upon one horizontal variable $\tilde{x}$. Similarly, solutions to (4.8), (4.9) with $A = 0$, $B \neq 0$ give rise to $2\frac{1}{2}$-dimensional waves depending on the single horizontal variable $\boldsymbol{k}_2 \cdot \boldsymbol{x}'$. We refer to Lokharu, Seth and Wahlén [20, Section 1.2.2] for a more detailed discussion on $2\frac{1}{2}$-dimensional waves. Fully three-dimensional waves are found by assuming that $A \neq 0$ and $B \neq 0$, in which case (4.8), (4.9) are equivalent to

$$g_1(|A|^2, |B|^2, \boldsymbol{\mu}) = 0, \tag{4.10}$$

$$g_2(|A|^2, |B|^2, \boldsymbol{\mu}) = 0. \tag{4.11}$$

Proposition 4.1. *There exist $\varepsilon > 0$ and analytic functions $\mu_i\colon B_\varepsilon(\mathbf{0}, \mathbb{R}^2) \to \mathbb{R}$, $i = 1, 2$ such that $\mu_i(0, 0) = 0$ and $(|A|^2, |B|^2, \mu_1(|A|^2, |B|^2), \mu_2(|A|^2, |B|^2))$ is the unique local solution of (4.10), (4.11).*

Proof. Write equations (4.10), (4.11) as

$$a_1\mu_1 + a_2\mu_2 + \mathcal{O}(|(|A|^2, |B|^2)| + |(|A|^2, |B|^2, \boldsymbol{\mu})|^2) = 0, \tag{4.12}$$

$$b_1\mu_1 + b_2\mu_2 + \mathcal{O}(|(|A|^2, |B|^2)| + |(|A|^2, |B|^2, \boldsymbol{\mu})|^2) = 0, \tag{4.13}$$

where

$$a_1 = \langle J_{11}e^{i\boldsymbol{k}_1\cdot\boldsymbol{x}'}, e^{i\boldsymbol{k}_1\cdot\boldsymbol{x}'}\rangle, \qquad b_1 = \langle J_{11}e^{i\boldsymbol{k}_2\cdot\boldsymbol{x}'}, e^{i\boldsymbol{k}_2\cdot\boldsymbol{x}'}\rangle,$$
$$a_2 = \langle J_{12}e^{i\boldsymbol{k}_1\cdot\boldsymbol{x}'}, e^{i\boldsymbol{k}_1\cdot\boldsymbol{x}'}\rangle, \qquad b_2 = \langle J_{12}e^{i\boldsymbol{k}_2\cdot\boldsymbol{x}'}, e^{i\boldsymbol{k}_2\cdot\boldsymbol{x}'}\rangle,$$

and $J_{11} = \partial_{\mu_1} \mathrm{d}_1 J[0, \boldsymbol{\mu}]|_{\boldsymbol{\mu}=0}$, $J_{12} = \partial_{\mu_2} \mathrm{d}_1 J[0, \boldsymbol{\mu}]|_{\boldsymbol{\mu}=0}$. A short calculation shows that

$$\frac{\partial}{\partial c_1}\rho(\boldsymbol{k}, \boldsymbol{c}_0, \beta) = \frac{|\boldsymbol{k}|^2}{\mathsf{c}(|\boldsymbol{k}|)}\langle J_{11}e^{i\boldsymbol{k}\cdot\boldsymbol{x}'}, e^{i\boldsymbol{k}\cdot\boldsymbol{x}'}\rangle, \qquad \frac{\partial}{\partial c_2}\rho(\boldsymbol{k}, \boldsymbol{c}_0, \beta) = \frac{|\boldsymbol{k}|^2}{\mathsf{c}(|\boldsymbol{k}|)}\langle J_{12}e^{i\boldsymbol{k}\cdot\boldsymbol{x}'}, e^{i\boldsymbol{k}\cdot\boldsymbol{x}'}\rangle,$$

and hence

$$a_1 = \frac{\mathsf{c}(|\boldsymbol{k}_1|)}{|\boldsymbol{k}_1|^2}\frac{\partial}{\partial c_1}\rho(\boldsymbol{k}_1, \boldsymbol{c}_0, \beta), \qquad b_1 = \frac{\mathsf{c}(\boldsymbol{k}_2)}{|\boldsymbol{k}_2|^2}\frac{\partial}{\partial c_1}\rho(\boldsymbol{k}_2, \boldsymbol{c}_0, \beta),$$
$$a_2 = \frac{\mathsf{c}(\boldsymbol{k}_1)}{|\boldsymbol{k}_1|^2}\frac{\partial}{\partial c_2}\rho(\boldsymbol{k}_1, \boldsymbol{c}_0, \beta), \qquad b_2 = \frac{\mathsf{c}(\boldsymbol{k}_2)}{|\boldsymbol{k}_2|^2}\frac{\partial}{\partial c_2}\rho(\boldsymbol{k}_2, \boldsymbol{c}_0, \beta).$$

Equations (4.12), (4.13) can be locally solved for μ_1, μ_2 as functions of $|A|^2, |B|^2$ by the implicit function theorem provided that

$$\det\begin{pmatrix} a_1 & a_2 \\ b_1 & b_2 \end{pmatrix} \neq 0.$$

The above formulae show that this condition holds if and only if $\nabla_{\boldsymbol{c}}\, \rho(\boldsymbol{k}_1, \boldsymbol{c}_0, \beta)$ and $\nabla_{\boldsymbol{c}}\, \rho(\boldsymbol{k}_2, \boldsymbol{c}_0, \beta)$ are linearly independent. □

Our main result now follows by substituting $\boldsymbol{\mu} = \boldsymbol{\mu}(|A|^2, |B|^2)$ into $\eta = \eta_1 + \eta_2(\eta_1, \boldsymbol{\mu})$.

Theorem 4.2. *Suppose that $\beta > 0$. There exist $\varepsilon > 0$, a neighbourhood V of the origin in $X_s^\beta \times \mathbb{R}^2$ and analytic functions $\mu_1, \mu_2\colon B_\varepsilon(\mathbf{0}, \mathbb{R}^2) \to \mathbb{R}$ and $\eta\colon B_\varepsilon(\mathbf{0}, \mathbb{C}^4) \to X_s^\beta$ such that*

$$\{(\eta, \boldsymbol{\mu}) \in X_s^\beta \times \mathbb{R}^2\colon J(\eta, \boldsymbol{\mu}) = 0,\ \eta \neq 0\} \cap V$$
$$= \{(\eta(A, B, \bar{A}, \bar{B}), \boldsymbol{\mu}(|A|^2, |B|^2))\colon (A, B, \bar{A}, \bar{B}) \in B'_\varepsilon(\mathbf{0}, \mathbb{C}^4)\};$$

furthermore $\boldsymbol{\mu}(0, 0) = \mathbf{0}$ and

$$\eta(\boldsymbol{x}') = Ae^{i\boldsymbol{k}_1\cdot\boldsymbol{x}'} + Be^{i\boldsymbol{k}_2\cdot\boldsymbol{x}'} + \bar{A}e^{-i\boldsymbol{k}_1\cdot\boldsymbol{x}'} + \bar{B}e^{-i\boldsymbol{k}_2\cdot\boldsymbol{x}'} + O(|(A, B, \bar{A}, \bar{B})|^2).$$

Remarks 4.3.

(i) Elements of the solution set $\{(\eta(A, B, \bar{A}, \bar{B}), \boldsymbol{\mu}(|A|^2, |B|^2))\colon (A, B, \bar{A}, \bar{B}) \in B'_\varepsilon(\mathbf{0}, \mathbb{C}^4)\}$ with $A = 0$ or $B = 0$ are $2\frac{1}{2}$-dimensional waves (see above).

(ii) Elements of the solution set $\{(\eta(A,B,\bar{A},\bar{B}), \boldsymbol{\mu}(|A|^2,|B|^2))\colon (A,B,\bar{A},\bar{B}) \in B'_\varepsilon(\mathbf{0},\mathbb{C}^4)\}$ with A, $B \in \mathbb{R}$ are waves which are invariant under the reflection S_0. Note that it is possible to restrict to such solutions before performing the Lyapunov–Schmidt reduction; this approach was taken by Craig and Nicholls [7] in a similar study of irrotational travelling waves.

The terms in the series

$$\eta = A\mathrm{e}^{\mathrm{i}\boldsymbol{k}_1\cdot\boldsymbol{x}'} + B\mathrm{e}^{\mathrm{i}\boldsymbol{k}_2\cdot\boldsymbol{x}'} + \bar{A}\mathrm{e}^{-\mathrm{i}\boldsymbol{k}_1\cdot\boldsymbol{x}'} + \bar{B}\mathrm{e}^{-\mathrm{i}\boldsymbol{k}_2\cdot\boldsymbol{x}'} + \sum_{i+j+k+l\geq 2} \eta_{ijkl} A^i B^j \bar{A}^k \bar{B}^l$$

and

$$\mu_i = \sum_{j+k\geq 1} \mu_{i,jk} |A|^{2j}|B|^{2k}, \qquad i=1,2,$$

can be determined recursively by substituting these expressions into (4.1) and equating monomials in $(A,B,\bar{A},\bar{B})$. Note that the series can be computed to any order for $\beta \geq 0$ but their convergence has been established only for $\beta > 0$.

- We find that

$$\eta_{2,2}(\eta_1) = \sum_{i+j+k+l=2} \eta_{2,ijkl} A^i B^j \bar{A}^k \bar{B}^l$$

satisfies the equation

$$J_{10}\eta_{2,2} = -J_{20}(\eta_1,\eta_1),$$

where $J_{20} := \frac{1}{2}\mathrm{d}_1^2 J[0,\mathbf{0}]$, so that

$$\begin{aligned} J_{20}(\eta_1,\eta_1) = {} & A^2 J_{20}(\mathrm{e}^{\mathrm{i}\boldsymbol{k}_1\cdot\boldsymbol{x}'},\mathrm{e}^{\mathrm{i}\boldsymbol{k}_1\cdot\boldsymbol{x}'}) + 2AB\,J_{20}(\mathrm{e}^{\mathrm{i}\boldsymbol{k}_1\cdot\boldsymbol{x}'},\mathrm{e}^{\mathrm{i}\boldsymbol{k}_2\cdot\boldsymbol{x}'}) + 2|A|^2 J_{20}(\mathrm{e}^{\mathrm{i}\boldsymbol{k}_1\cdot\boldsymbol{x}'},\mathrm{e}^{-\mathrm{i}\boldsymbol{k}_1\cdot\boldsymbol{x}'}) \\ & + 2A\bar{B}\,J_{20}(\mathrm{e}^{\mathrm{i}\boldsymbol{k}_1\cdot\boldsymbol{x}'},\mathrm{e}^{-\mathrm{i}\boldsymbol{k}_2\cdot\boldsymbol{x}'}) + B^2 J_{20}(\mathrm{e}^{\mathrm{i}\boldsymbol{k}_2\cdot\boldsymbol{x}'},\mathrm{e}^{\mathrm{i}\boldsymbol{k}_2\cdot\boldsymbol{x}'}) \\ & + 2\bar{A}B\,J_{20}(\mathrm{e}^{-\mathrm{i}\boldsymbol{k}_1\cdot\boldsymbol{x}'},\mathrm{e}^{\mathrm{i}\boldsymbol{k}_2\cdot\boldsymbol{x}'}) \\ & + 2|B|^2 J_{20}(\mathrm{e}^{\mathrm{i}\boldsymbol{k}_2\cdot\boldsymbol{x}'},\mathrm{e}^{-\mathrm{i}\boldsymbol{k}_2\cdot\boldsymbol{x}'}) + \bar{A}^2 J_{20}(\mathrm{e}^{-\mathrm{i}\boldsymbol{k}_1\cdot\boldsymbol{x}'},\mathrm{e}^{-\mathrm{i}\boldsymbol{k}_1\cdot\boldsymbol{x}'}) \\ & + 2\bar{A}\bar{B}\,J_{20}(\mathrm{e}^{-\mathrm{i}\boldsymbol{k}_1\cdot\boldsymbol{x}'},\mathrm{e}^{-\mathrm{i}\boldsymbol{k}_2\cdot\boldsymbol{x}'}) \\ & + \bar{B}^2 J_{20}(\mathrm{e}^{-\mathrm{i}\boldsymbol{k}_2\cdot\boldsymbol{x}'},\mathrm{e}^{-\mathrm{i}\boldsymbol{k}_2\cdot\boldsymbol{x}'}). \end{aligned}$$

For $\boldsymbol{\ell}, \boldsymbol{k}$ with $\boldsymbol{k} \neq -\boldsymbol{\ell}$ we find that

$$\begin{aligned} & J_{20}(\mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}'},\mathrm{e}^{\mathrm{i}\boldsymbol{\ell}\cdot\boldsymbol{x}'}) \\ & = \underbrace{\left[\frac{1}{2}\mathrm{T}_{10}(\boldsymbol{k})\cdot\mathrm{T}_{10}(\boldsymbol{\ell}) + \frac{1}{2}(\boldsymbol{k}\cdot\boldsymbol{c}_0)(\boldsymbol{\ell}\cdot\boldsymbol{c}_0) + \mathrm{T}_{20,2}(\boldsymbol{k},\boldsymbol{\ell})\cdot\boldsymbol{c}_0 + \frac{\alpha}{2}(\mathrm{T}_{10}(\boldsymbol{k}) + \mathrm{T}_{10}(\boldsymbol{\ell}))\cdot\boldsymbol{c}_0^\perp\right]}_{=:\,\mathrm{p}_{20,2}(\boldsymbol{k},\boldsymbol{\ell})} \\ & \quad \times \mathrm{e}^{\mathrm{i}(\boldsymbol{k}+\boldsymbol{\ell})\cdot\boldsymbol{x}'}, \end{aligned}$$

while

$$J_{20}(\mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}'},\mathrm{e}^{-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}'})=\underbrace{\frac{1}{2}|\mathtt{T}_{10}(\boldsymbol{k})|^2-\frac{1}{2}(\boldsymbol{k}\cdot\boldsymbol{c}_0)^2+\mathtt{T}_{20,1}(\boldsymbol{k})\cdot\boldsymbol{c}_0+\alpha\mathtt{T}_{10}(\boldsymbol{k})\cdot\boldsymbol{c}_0^\perp}_{=:\,\mathtt{p}_{20,1}(\boldsymbol{k})},$$

where

$$\begin{aligned}
\mathtt{T}_{10}(\boldsymbol{k})&=-\Big(\alpha\,\boldsymbol{k}^\perp+\boldsymbol{k}\,\mathtt{c}(|\boldsymbol{k}|)\Big)\frac{\boldsymbol{c}_0\cdot\boldsymbol{k}}{|\boldsymbol{k}|^2},\\
\mathtt{T}_{20,2}(\boldsymbol{k},\boldsymbol{\ell})&=\frac{1}{2|\boldsymbol{k}+\boldsymbol{\ell}|^2}\Big(\alpha(\boldsymbol{k}+\boldsymbol{\ell})^\perp+(\boldsymbol{k}+\boldsymbol{\ell})\mathtt{c}(|\boldsymbol{k}+\boldsymbol{\ell}|)\Big)\\
&\qquad\times\Bigg[\alpha(\boldsymbol{k}+\boldsymbol{\ell})\cdot\boldsymbol{c}_0+\alpha(\boldsymbol{k}\cdot\boldsymbol{\ell}^\perp)\left(\frac{\boldsymbol{c}_0\cdot\boldsymbol{\ell}}{|\boldsymbol{\ell}|^2}-\frac{\boldsymbol{c}_0\boldsymbol{k}}{|\boldsymbol{k}|^2}\right)\\
&\qquad\quad+(\boldsymbol{k}+\boldsymbol{\ell})\cdot\left(\frac{\boldsymbol{c}_0\cdot\boldsymbol{\ell}}{|\boldsymbol{\ell}|^2}\mathtt{c}(|\boldsymbol{\ell}|)\boldsymbol{\ell}+\frac{\boldsymbol{c}_0\cdot\boldsymbol{k}}{|\boldsymbol{k}|^2}\mathtt{c}(|\boldsymbol{k}|)\boldsymbol{k}\right)\Bigg]\\
&\qquad+\Big((\alpha^2-|\boldsymbol{\ell}|^2)\boldsymbol{\ell}-\alpha\mathtt{c}(|\boldsymbol{\ell}|)\boldsymbol{\ell}^\perp\Big)\frac{\boldsymbol{c}_0\cdot\boldsymbol{\ell}}{2|\boldsymbol{\ell}|^2}+\Big((\alpha^2-|\boldsymbol{k}|^2)\boldsymbol{k}-\alpha\mathtt{c}(|\boldsymbol{k}|)\boldsymbol{k}^\perp\Big)\frac{\boldsymbol{c}_0\cdot\boldsymbol{k}}{2|\boldsymbol{k}|^2}\\
&\qquad-\frac{1}{2}\boldsymbol{k}(\boldsymbol{c}_0\cdot\boldsymbol{\ell})-\frac{1}{2}\boldsymbol{\ell}(\boldsymbol{c}_0\cdot\boldsymbol{k}),\\
\mathtt{T}_{20,1}(\boldsymbol{k})&=\alpha(\alpha\boldsymbol{k}-\mathtt{c}(|\boldsymbol{k}|)\boldsymbol{k}^\perp).
\end{aligned}$$

The solution of the equation

$$J_{10}f\mathrm{e}^{\mathrm{i}(\boldsymbol{\ell}+\boldsymbol{k})\cdot\boldsymbol{x}'}=\mathtt{p}_{20,2}(\boldsymbol{k},\boldsymbol{\ell})\mathrm{e}^{\mathrm{i}(\boldsymbol{\ell}+\boldsymbol{k})\cdot\boldsymbol{x}'},\qquad \boldsymbol{k}\neq-\boldsymbol{\ell},$$

is

$$f=\underbrace{\frac{|\boldsymbol{k}+\boldsymbol{\ell}|^2}{\mathtt{c}(|\boldsymbol{k}+\boldsymbol{\ell}|)\rho(\boldsymbol{k}+\boldsymbol{\ell},\boldsymbol{c}_0,\beta)}\mathtt{p}_{20,2}(\boldsymbol{k},\boldsymbol{\ell})}_{=:\,\mathtt{q}_{20,2}(\boldsymbol{\ell},\boldsymbol{k})},$$

while the solution of

$$J_{10}f=\mathtt{p}_{20,1}(\boldsymbol{k})$$

is simply

$$f=\frac{1}{g}\mathtt{p}_{20,1}(\boldsymbol{k}).$$

Altogether we find that

$$\begin{aligned}
&\eta_{2,2000} = -\mathrm{q}_{20,2}(\boldsymbol{k}_1, \boldsymbol{k}_1)\mathrm{e}^{2\mathrm{i}\boldsymbol{k}_1\cdot\boldsymbol{x}'}, && \eta_{2,0020} = \overline{\eta_{2,2000}},\\
&\eta_{2,1100} = -2\mathrm{q}_{20,2}(\boldsymbol{k}_1, \boldsymbol{k}_2)\mathrm{e}^{\mathrm{i}(\boldsymbol{k}_1+\boldsymbol{k}_2)\cdot\boldsymbol{x}'}, && \eta_{2,0011} = \overline{\eta_{2,1100}},\\
&\eta_{2,1010} = -\frac{2}{g}\mathrm{p}_{20,1}(\boldsymbol{k}_1), &&\\
&\eta_{2,1001} = -2\mathrm{q}_{20,2}(\boldsymbol{k}_1, -\boldsymbol{k}_2)\mathrm{e}^{\mathrm{i}(\boldsymbol{k}_1-\boldsymbol{k}_2)\cdot\boldsymbol{x}'}, && \eta_{2,0110} = \overline{\eta_{2,1001}},\\
&\eta_{2,0200} = -\mathrm{q}_{20,2}(\boldsymbol{k}_2, \boldsymbol{k}_2)\mathrm{e}^{2\mathrm{i}\boldsymbol{k}_2\cdot\boldsymbol{x}'}, && \eta_{2,0002} = \overline{\eta_{2,0200}},\\
&\eta_{2,0101} = -\frac{2}{g}\mathrm{p}_{20,1}(\boldsymbol{k}_2). &&
\end{aligned}$$

- Expanding (4.10), (4.11) further as

$$\begin{aligned}
a_1\mu_1 + a_2\mu_2 + a_3|A|^2 + a_4|B|^2 + \mathcal{O}(|(|A|^2, |B|^2, \boldsymbol{\mu})|^2) = 0,\\
b_1\mu_1 + b_2\mu_2 + b_3|A|^2 + b_4|B|^2 + \mathcal{O}(|(|A|^2, |B|^2, \boldsymbol{\mu})|^2) = 0,
\end{aligned}$$

we find that

$$\begin{aligned}
\mu_1(|A|^2, |B|^2) &= -\frac{a_3b_2 - a_2b_3}{a_1b_2 - b_1a_2}|A|^2 - \frac{a_4b_2 - a_2b_4}{a_1b_2 - b_1a_2}|B|^2 + \mathcal{O}(|(|A|^2, |B|^2)|^2),\\
\mu_2(|A|^2, |B|^2) &= -\frac{a_1b_3 - a_3b_1}{a_1b_2 - b_1a_2}|A|^2 - \frac{a_1b_4 - a_4b_1}{a_1b_2 - b_1a_2}|B|^2 + \mathcal{O}(|(|A|^2, |B|^2)|^2).
\end{aligned}$$

The coefficients a_3, a_4, b_3, b_4 are given by

$$\begin{aligned}
a_3 &= \Big\langle 2J_{20}(\mathrm{e}^{\mathrm{i}\boldsymbol{k}_1\cdot\boldsymbol{x}'}, \eta_{2,1010}) + 2J_{20}(\mathrm{e}^{-\mathrm{i}\boldsymbol{k}_1\boldsymbol{x}'}, \eta_{2,2000}) + 3J_{30}(\mathrm{e}^{\mathrm{i}\boldsymbol{k}_1\cdot\boldsymbol{x}'}, \mathrm{e}^{\mathrm{i}\boldsymbol{k}_1\cdot\boldsymbol{x}'}, \mathrm{e}^{-\mathrm{i}\boldsymbol{k}_1\cdot\boldsymbol{x}'}), \mathrm{e}^{\mathrm{i}\boldsymbol{k}_1\cdot\boldsymbol{x}'}\Big\rangle\\
&= -\frac{4}{g}\mathrm{p}_{20,1}(\boldsymbol{k}_1)\mathrm{p}_{20,2}(\boldsymbol{k}_1, \boldsymbol{0}) - 2\mathrm{q}_{20,2}(\boldsymbol{k}_1, \boldsymbol{k}_1)\mathrm{p}_{20,2}(-\boldsymbol{k}_1, 2\boldsymbol{k}_1) + 3\mathrm{p}_{30,1}(\boldsymbol{k}_1),\\
a_4 &= \Big\langle 2J_{20}(\mathrm{e}^{\mathrm{i}\boldsymbol{k}_1\cdot\boldsymbol{x}'}, \eta_{2,0101}) + 2J_{20}(\mathrm{e}^{\mathrm{i}\boldsymbol{k}_2\cdot\boldsymbol{x}'}, \eta_{2,1001}) + 2J_{20}(\mathrm{e}^{-\mathrm{i}\boldsymbol{k}_2\cdot\boldsymbol{x}'}, \eta_{2,1100})\\
&\qquad + 6J_{30}(\mathrm{e}^{\mathrm{i}\boldsymbol{k}_1\cdot\boldsymbol{x}'}, \mathrm{e}^{\mathrm{i}\boldsymbol{k}_2\cdot\boldsymbol{x}'}, \mathrm{e}^{-\mathrm{i}\boldsymbol{k}_2\cdot\boldsymbol{x}'}), \mathrm{e}^{\mathrm{i}\boldsymbol{k}_1\cdot\boldsymbol{x}'}\Big\rangle\\
&= -\frac{4}{g}\mathrm{p}_{20,1}(\boldsymbol{k}_2)\mathrm{p}_{20,2}(\boldsymbol{k}_1, \boldsymbol{0}) - 4\mathrm{q}_{20,2}(\boldsymbol{k}_1, -\boldsymbol{k}_2)\mathrm{p}_{20,2}(\boldsymbol{k}_2, \boldsymbol{k}_1 - \boldsymbol{k}_2)\\
&\qquad - 4\mathrm{q}_{20,2}(\boldsymbol{k}_1, \boldsymbol{k}_2)\mathrm{p}_{20,2}(-\boldsymbol{k}_2, \boldsymbol{k}_1 + \boldsymbol{k}_2) + 6\mathrm{p}_{30,2}(\boldsymbol{k}_1, \boldsymbol{k}_2),\\
b_3 &= \Big\langle 2J_{20}(\mathrm{e}^{\mathrm{i}\boldsymbol{k}_2\cdot\boldsymbol{x}'}, \eta_{2,1010}) + 2J_{20}(\mathrm{e}^{\mathrm{i}\boldsymbol{k}_1\cdot\boldsymbol{x}'}, \eta_{2,0110}) + 2J_{20}(\mathrm{e}^{-\mathrm{i}\boldsymbol{k}_1\cdot\boldsymbol{x}'}, \eta_{2,1100})\\
&\qquad + 6J_{30}(\mathrm{e}^{\mathrm{i}\boldsymbol{k}_2\cdot\boldsymbol{x}'}, \mathrm{e}^{\mathrm{i}\boldsymbol{k}_1\cdot\boldsymbol{x}'}, \mathrm{e}^{-\mathrm{i}\boldsymbol{k}_1\cdot\boldsymbol{x}'}), \mathrm{e}^{\mathrm{i}\boldsymbol{k}_2\cdot\boldsymbol{x}'}\Big\rangle\\
&= -\frac{4}{g}\mathrm{p}_{20,1}(\boldsymbol{k}_1)\mathrm{p}_{20,2}(\boldsymbol{k}_2, \boldsymbol{0}) - 4\mathrm{q}_{20,2}(-\boldsymbol{k}_1, \boldsymbol{k}_2)\mathrm{p}_{20,2}(\boldsymbol{k}_1, \boldsymbol{k}_2 - \boldsymbol{k}_1)\\
&\qquad - 4\mathrm{q}_{20,2}(\boldsymbol{k}_1, \boldsymbol{k}_2)\mathrm{p}_{20,2}(-\boldsymbol{k}_1, \boldsymbol{k}_1 + \boldsymbol{k}_2) + 6\mathrm{p}_{30,2}(\boldsymbol{k}_2, \boldsymbol{k}_1),
\end{aligned}$$

$$b_4 = \Big\langle 2J_{20}(\mathrm{e}^{\mathrm{i}\boldsymbol{k}_2\cdot\boldsymbol{x}'}, \eta_{2,0101}) + 2J_{20}(\mathrm{e}^{-\mathrm{i}\boldsymbol{k}_2\cdot\boldsymbol{x}'}, \eta_{2,0200}) + 3J_{30}(\mathrm{e}^{\mathrm{i}\boldsymbol{k}_2\cdot\boldsymbol{x}'}, \mathrm{e}^{\mathrm{i}\boldsymbol{k}_2\cdot\boldsymbol{x}'}, \mathrm{e}^{-\mathrm{i}\boldsymbol{k}_2\cdot\boldsymbol{x}'}), \mathrm{e}^{\mathrm{i}\boldsymbol{k}_2\cdot\boldsymbol{x}'}\Big\rangle$$
$$= -\frac{4}{g}\mathtt{p}_{20,1}(\boldsymbol{k}_2)\mathtt{p}_{20,2}(\boldsymbol{k}_2,\boldsymbol{0}) - 2\mathtt{q}_{20,2}(\boldsymbol{k}_2,\boldsymbol{k}_2)\mathtt{p}_{20,2}(-\boldsymbol{k}_2, 2\boldsymbol{k}_2) + 3\mathtt{p}_{30,1}(\boldsymbol{k}_2),$$

where $J_{30} = \frac{1}{3!}\mathrm{d}_1^3 J[0,\boldsymbol{0}]$.

One finds that

$$J_{30}(\mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}'}, \mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}'}, \mathrm{e}^{-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}'}) = \mathtt{p}_{30,1}(\boldsymbol{k})\mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}'}, \qquad \boldsymbol{k}\neq\boldsymbol{0},$$
$$J_{30}(\mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}'}, \mathrm{e}^{\mathrm{i}\boldsymbol{\ell}\cdot\boldsymbol{x}'}, \mathrm{e}^{-\mathrm{i}\boldsymbol{\ell}\cdot\boldsymbol{x}'}) = \mathtt{p}_{30,2}(\boldsymbol{k},\boldsymbol{\ell})\mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}'}, \qquad \boldsymbol{k}\neq-\boldsymbol{\ell},$$

where

$$\mathtt{p}_{30,1}(\boldsymbol{k}) = \frac{2}{3}\mathtt{T}_{10}(\boldsymbol{k})\cdot\mathtt{T}_{20,1}(\boldsymbol{k}) + \frac{1}{3}\mathtt{T}_{10}(\boldsymbol{k})\cdot\mathtt{T}_{20,2}(\boldsymbol{k},\boldsymbol{k})$$
$$-\frac{1}{3}\alpha(\boldsymbol{c}_0\cdot\boldsymbol{k})(\boldsymbol{c}_0^\perp\cdot\boldsymbol{k}) - \frac{1}{3}(\boldsymbol{c}_0\cdot\boldsymbol{k})(\mathtt{T}_{10}(\boldsymbol{k})\cdot\boldsymbol{k})$$
$$-\frac{\alpha^2}{2}\mathtt{T}_{10}(\boldsymbol{k})\cdot\boldsymbol{c}_0 + \frac{\alpha}{3}\left(2\boldsymbol{c}_0^\perp\cdot\mathtt{T}_{20,1}(\boldsymbol{k}) + \boldsymbol{c}_0^\perp\cdot\mathtt{T}_{20,2}(\boldsymbol{k},\boldsymbol{k})\right) + \mathtt{T}_{30,1}(\boldsymbol{k})\cdot\boldsymbol{c}_0 - \frac{\beta}{2}|\boldsymbol{k}|^4,$$

$$\mathtt{T}_{30,1}(\boldsymbol{k}) = -\frac{1}{3}\boldsymbol{r}_1(\boldsymbol{k})\frac{\boldsymbol{c}_0\cdot\boldsymbol{k}}{|\boldsymbol{k}|^2}\mathtt{c}(|\boldsymbol{k}|)\mathtt{c}(2|\boldsymbol{k}|) - \frac{1}{12}\boldsymbol{r}_2(2\boldsymbol{k})\frac{\boldsymbol{c}\cdot\boldsymbol{k}}{|\boldsymbol{k}|^2}\mathtt{c}(|\boldsymbol{k}|) - \frac{1}{6}\boldsymbol{k}(\boldsymbol{c}_0\cdot\boldsymbol{k})\mathtt{c}(|\boldsymbol{k}|)$$
$$-\frac{1}{2}\boldsymbol{r}_1(\boldsymbol{k})(\alpha^2-|\boldsymbol{k}|^2)\frac{\boldsymbol{c}_0\cdot\boldsymbol{k}}{|\boldsymbol{k}|^2} + \frac{\alpha}{3}\boldsymbol{r}_2(\boldsymbol{k})^\perp\frac{\boldsymbol{c}_0\cdot\boldsymbol{k}}{|\boldsymbol{k}|^2}$$
$$+\frac{\alpha}{12}\boldsymbol{r}_1(2\boldsymbol{k})^\perp\frac{\boldsymbol{c}_0\cdot\boldsymbol{k}}{|\boldsymbol{k}|^2}\mathtt{c}(|\boldsymbol{k}|) + \frac{\alpha}{6}\boldsymbol{r}_2(\boldsymbol{k})^\perp\frac{\boldsymbol{c}_0\cdot\boldsymbol{k}}{|\boldsymbol{k}|^2} + \frac{\alpha}{6}\boldsymbol{r}_1(\boldsymbol{k})\frac{\boldsymbol{c}_0^\perp\cdot\boldsymbol{k}}{|\boldsymbol{k}|^2}\mathtt{c}(|\boldsymbol{k}|)$$
$$+\frac{\alpha}{12}\boldsymbol{r}_2(2\boldsymbol{k})\frac{\boldsymbol{c}_0^\perp\cdot\boldsymbol{k}}{|\boldsymbol{k}|^2} + \frac{\alpha}{6}\boldsymbol{k}_1(\boldsymbol{c}_0^\perp\cdot\boldsymbol{k}) + \frac{\alpha^2}{6}\boldsymbol{r}_1(\boldsymbol{k})\frac{\boldsymbol{c}_0\cdot\boldsymbol{k}}{|\boldsymbol{k}|^2}$$

and

$$\mathtt{p}_{30,2}(\boldsymbol{k},\boldsymbol{\ell}) = \frac{1}{3}\left(\mathtt{T}_{10}(\boldsymbol{k})\cdot\mathtt{T}_{20,1}(\boldsymbol{k}) + \mathtt{T}_{10}(\boldsymbol{\ell})\cdot\mathtt{T}_{20,2}(\boldsymbol{k},-\boldsymbol{\ell}) + \mathtt{T}_{10}(\boldsymbol{\ell})\cdot\mathtt{T}_{20,2}(\boldsymbol{k},\boldsymbol{\ell})\right)$$
$$-\frac{\alpha}{3}(\boldsymbol{c}_0\cdot\boldsymbol{\ell})(\boldsymbol{c}_0^\perp\cdot\boldsymbol{\ell})$$
$$-\frac{1}{3}(\boldsymbol{c}_0\cdot\boldsymbol{\ell})(\mathtt{T}_{10}(\boldsymbol{k})\cdot\boldsymbol{\ell}) - \frac{\alpha^2}{6}\left(\mathtt{T}_{10}(\boldsymbol{k})\cdot\boldsymbol{c}_0 + 2\mathtt{T}_{10}(\boldsymbol{\ell})\cdot\boldsymbol{c}_0\right) + \frac{\alpha}{3}\big(\boldsymbol{c}_0^\perp\cdot\mathtt{T}_{20,1}(\boldsymbol{\ell})$$
$$+\boldsymbol{c}_0^\perp\cdot\mathtt{T}_{20,2}(\boldsymbol{k},-\boldsymbol{\ell}) + \boldsymbol{c}_0^\perp\cdot\mathtt{T}_{20,2}(\boldsymbol{k},\boldsymbol{\ell})\big) + \mathtt{T}_{30,2}(\boldsymbol{k},\boldsymbol{\ell})\cdot\boldsymbol{c}_0$$
$$-\frac{\beta}{6}\left(|\boldsymbol{k}|^2|\boldsymbol{\ell}|^2 + 2(\boldsymbol{k}\cdot\boldsymbol{\ell})^2\right),$$

$$
\begin{aligned}
\mathtt{T}_{30,2}(\boldsymbol{k},\boldsymbol{\ell}) = &-\frac{1}{6}\boldsymbol{r}_1(\boldsymbol{k})\frac{\boldsymbol{k}}{|\boldsymbol{k}|^2}\cdot\Bigg(\boldsymbol{r}_1(\boldsymbol{k}-\boldsymbol{\ell})\bigg[\frac{\boldsymbol{k}-\boldsymbol{\ell}}{|\boldsymbol{k}-\boldsymbol{\ell}|^2}\cdot\boldsymbol{r}_3(\boldsymbol{k},\boldsymbol{\ell})\bigg] \\
&+\boldsymbol{r}_1(\boldsymbol{k}+\boldsymbol{\ell})\bigg[\frac{\boldsymbol{k}+\boldsymbol{\ell}}{|\boldsymbol{k}+\boldsymbol{\ell}|^2}\cdot\boldsymbol{r}_3(\boldsymbol{k},\boldsymbol{\ell})\bigg]\Bigg) \\
&-\frac{1}{12}\boldsymbol{r}_2(\boldsymbol{k}-\boldsymbol{\ell})\bigg[\frac{\boldsymbol{k}-\boldsymbol{\ell}}{|\boldsymbol{k}-\boldsymbol{\ell}|^2}\cdot\boldsymbol{r}_3(\boldsymbol{k},\boldsymbol{\ell})\bigg]-\frac{1}{12}\boldsymbol{r}_2(\boldsymbol{k}+\boldsymbol{\ell})\bigg[\frac{\boldsymbol{k}+\boldsymbol{\ell}}{|\boldsymbol{k}+\boldsymbol{\ell}|^2}\cdot\boldsymbol{r}_3(\boldsymbol{k},\boldsymbol{\ell})\bigg] \\
&-\frac{1}{6}\boldsymbol{\ell}\bigg[\boldsymbol{\ell}\cdot\boldsymbol{r}_3(\boldsymbol{k},\boldsymbol{\ell})\bigg]-\frac{1}{6}\boldsymbol{r}_1(\boldsymbol{k})\frac{\boldsymbol{k}}{|\boldsymbol{k}|^2}\cdot\bigg[2\boldsymbol{r}_2(\boldsymbol{\ell})\frac{\boldsymbol{c}_0\cdot\boldsymbol{\ell}}{|\boldsymbol{\ell}|^2}+\boldsymbol{r}_2(\boldsymbol{k})\frac{\boldsymbol{c}_0\cdot\boldsymbol{k}}{|\boldsymbol{k}|^2}\bigg] \\
&+\frac{1}{6}\boldsymbol{k}\bigg[\boldsymbol{k}\cdot\boldsymbol{r}_1(\boldsymbol{\ell})\frac{\boldsymbol{c}_0\cdot\boldsymbol{\ell}}{|\boldsymbol{\ell}|^2}\bigg]+\frac{\alpha}{6}\boldsymbol{r}_2(\boldsymbol{\ell})^\perp\frac{\boldsymbol{c}_0\cdot\boldsymbol{\ell}}{|\boldsymbol{\ell}|^2} \\
&+\frac{\alpha}{12}\Bigg(\boldsymbol{r}_1(\boldsymbol{k}-\boldsymbol{\ell})^\perp\bigg[\frac{\boldsymbol{k}-\boldsymbol{\ell}}{|\boldsymbol{k}-\boldsymbol{\ell}|^2}\cdot\boldsymbol{r}_3(\boldsymbol{k},\boldsymbol{\ell})\bigg]+\boldsymbol{r}_2(\boldsymbol{\ell})^\perp\frac{\boldsymbol{c}_0\cdot\boldsymbol{\ell}}{|\boldsymbol{\ell}|^2}+\boldsymbol{r}_2(\boldsymbol{k})^\perp\frac{\boldsymbol{c}_0\cdot\boldsymbol{k}}{|\boldsymbol{k}|^2}\Bigg) \\
&+\frac{\alpha}{12}\Bigg(\boldsymbol{r}_1(\boldsymbol{k}+\boldsymbol{\ell})^\perp\bigg[\frac{\boldsymbol{k}+\boldsymbol{\ell}}{|\boldsymbol{k}+\boldsymbol{\ell}|^2}\cdot\boldsymbol{r}_3(\boldsymbol{k},\boldsymbol{\ell})\bigg]+\frac{1}{|\boldsymbol{k}|^2}\boldsymbol{r}_2(\boldsymbol{k})^\perp\frac{\boldsymbol{c}_0\cdot\boldsymbol{k}}{|\boldsymbol{k}|^2}+\boldsymbol{r}_2(\boldsymbol{\ell})^\perp\frac{\boldsymbol{c}_0\cdot\boldsymbol{\ell}}{|\boldsymbol{\ell}|^2}\Bigg) \\
&+\frac{\alpha}{6}\boldsymbol{r}_1(\boldsymbol{k})\frac{\boldsymbol{k}}{|\boldsymbol{k}|^2}\cdot\Bigg(\boldsymbol{r}_1(\boldsymbol{k}-\boldsymbol{\ell})\frac{\boldsymbol{c}_0^\perp\cdot(\boldsymbol{k}-\boldsymbol{\ell})}{|\boldsymbol{k}-\boldsymbol{\ell}|^2}+\boldsymbol{r}_1(\boldsymbol{k}+\boldsymbol{\ell})\frac{\boldsymbol{c}_0^\perp\cdot(\boldsymbol{k}+\boldsymbol{\ell})}{|\boldsymbol{k}+\boldsymbol{\ell}|^2}\Bigg) \\
&+\frac{\alpha}{6}\boldsymbol{r}_2(\boldsymbol{k}-\boldsymbol{\ell})\frac{\boldsymbol{c}_0\cdot(\boldsymbol{k}-\boldsymbol{\ell})}{|\boldsymbol{k}-\boldsymbol{\ell}|^2}+\frac{\alpha}{6}\boldsymbol{r}_2(\boldsymbol{k}+\boldsymbol{\ell})\frac{\boldsymbol{c}_0\cdot(\boldsymbol{k}+\boldsymbol{\ell})}{|\boldsymbol{k}+\boldsymbol{\ell}|^2} \\
&+\frac{\alpha}{3}\boldsymbol{\ell}(\boldsymbol{c}_0^\perp\cdot\boldsymbol{\ell})+\frac{\alpha^2}{6}\boldsymbol{r}_1(\boldsymbol{k})\frac{\boldsymbol{c}_0\cdot\boldsymbol{k}}{|\boldsymbol{k}|^2}-\frac{\alpha}{6}\boldsymbol{r}_1(\boldsymbol{k})\bigg[\frac{\boldsymbol{k}}{|\boldsymbol{k}|^2}\cdot\boldsymbol{r}_1(\boldsymbol{\ell})^\perp\bigg]\frac{\boldsymbol{c}_0\cdot\boldsymbol{\ell}}{|\boldsymbol{\ell}|^2} \\
&-\frac{\alpha^2}{6}\boldsymbol{r}_1(\boldsymbol{\ell})\frac{\boldsymbol{c}_0\cdot\boldsymbol{\ell}}{|\boldsymbol{\ell}|^2},
\end{aligned}
$$

with

$$
\begin{aligned}
\boldsymbol{r}_1(\boldsymbol{k}) &= \alpha\boldsymbol{k}^\perp+\boldsymbol{k}\mathtt{c}(|\boldsymbol{k}|), \\
\boldsymbol{r}_2(\boldsymbol{k}) &= \boldsymbol{k}(\alpha^2-|\boldsymbol{k}|^2)-\alpha\boldsymbol{k}^\perp\mathtt{c}(|\boldsymbol{k}|), \\
\boldsymbol{r}_3(\boldsymbol{k},\boldsymbol{\ell}) &= \boldsymbol{r}_1(\boldsymbol{k})\frac{\boldsymbol{c}_0\cdot\boldsymbol{k}}{|\boldsymbol{k}|^2}+\boldsymbol{r}_1(\boldsymbol{\ell})\frac{\boldsymbol{c}_0\cdot\boldsymbol{\ell}}{|\boldsymbol{\ell}|^2}.
\end{aligned}
$$

Data availability

No data was used for the research described in the article.

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