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## Regular Article

# Analytic capacities in Besov spaces <sup>☆</sup>



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## ABSTRACT

We derive new estimates on analytic capacities of finite sequences in the unit disc in Besov spaces with zero smoothness, which sharpen the estimates obtained by N.K. Nikolski in 2005 and, for a range of parameters, are optimal. The work is motivated both from the perspective of complex analysis by the description of sets of zeros/uniqueness, and from the one of matrix analysis/operator theory by estimates on norms of inverses.

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### 1. Introduction

Let  $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$  be the open unit disk, let  $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$  be its boundary and  $\mathbb{D}_* = \mathbb{D} \setminus \{0\}$ . We denote by  $\mathcal{Hol}(\mathbb{D})$  the space of analytic functions on  $\mathbb{D}$ , equipped with the topology of local uniform convergence. Let  $X$  be a Banach space that is continuously contained in  $\mathcal{Hol}(\mathbb{D})$  and that contains the polynomials. Given a finite sequence  $\sigma = (\lambda_1, \dots, \lambda_N) \in \mathbb{D}_*^N$ , Nikolski [10] defined the  $X$ -zero capacity of  $\sigma$  as

$$\text{cap}_X(\sigma) = \inf\{\|f\|_X : f(0) = 1, f|_\sigma = 0\},$$

where  $f|_\sigma = 0$  means that  $f(\lambda_i) = 0$  for all  $i = 1, \dots, N$  taking into account possible multiplicities. Namely, if  $\sigma = (\lambda_1, \dots, \lambda_1, \lambda_2, \dots, \lambda_2, \dots, \lambda_s, \dots, \lambda_s) \in \mathbb{D}_*^N$ , where each  $\lambda_i$  is repeated according to its multiplicity  $m_i \geq 1$ , then  $f|_\sigma = 0$  means that

$$f(\lambda_i) = f'(\lambda_i) = f''(\lambda_i) = \dots = f^{(m_i-1)}(\lambda_i) = 0, \quad i = 1, \dots, s.$$

The latter quantity is closely related on one hand to the problem of uniqueness sets for the function space  $X$  and on the other hand to condition numbers of large matrices and inverses, as observed by Nikolski [10, Section 1]. We briefly review these connections here.

#### 1.1. Motivation from complex analysis: sets of zeros/uniqueness

From the point of view of complex analysis, the  $X$ -zero capacities are closely related to the problem of characterizing uniqueness sets for the function space  $X$ ; here  $\sigma$  is said to be a uniqueness set for  $X$  if  $f \in X, f|_\sigma = 0 \implies f = 0$ . Following [10], assume that the function space  $X$  satisfies the following Fatou property: if  $f_n \in X, \sup_n \|f_n\|_X < \infty$  and  $\lim_{n \rightarrow \infty} f_n(z) = f(z)$  for  $z \in \mathbb{D}$ , then  $f \in X$ . Then it is not hard to see that an infinite sequence  $\sigma = (\lambda_i)_{i \geq 1} \in \mathbb{D}_*^\infty$  is a uniqueness sequence for  $X$  if and only if

$$\sup_N \{\text{cap}_X(\sigma_N)\} = \infty, \tag{1.1}$$

where  $\sigma_N = (\lambda_i)_{i=1}^N$  is the truncation of  $\sigma$  of order  $N$ . For example, let  $X$  be the algebra  $H^\infty$  of bounded holomorphic functions in  $\mathbb{D}$  endowed with the norm  $\|f\|_{H^\infty} = \sup_{\zeta \in \mathbb{D}} |f(\zeta)|$ . It is known [10, Theorem 3.12] that given  $\sigma_N = (\lambda_i)_{i=1}^N \in \mathbb{D}_*^N$ ,

$$\text{cap}_{H^\infty}(\sigma_N) = \frac{1}{\prod_{i=1}^N |\lambda_i|}. \tag{1.2}$$

Denoting by

$$B = B_{\sigma_N} = \prod_{i=1}^N \frac{z - \lambda_i}{1 - \overline{\lambda_i}z}$$

the finite Blaschke product associated with  $\sigma_N$ , observe that the right-hand side in (1.2) is achieved by the test function  $f = B/B(0)$ , which is admissible for the conditions in the infimum defining the capacity of  $\sigma_N$ . Thus, an application of the above criterion (1.1) leads to the well-known Blaschke condition: an infinite sequence  $\sigma = (\lambda_i)_{i \geq 1} \in \mathbb{D}_*^\infty$  is a uniqueness sequence for  $H^\infty$  if and only if

$$\sum_{i \geq 1} (1 - |\lambda_i|) = \infty.$$

1.2. Motivation in operator theory/matrix analysis

Let  $T$  be an invertible operator acting on a Banach space or an  $N \times N$  invertible matrix with complex entries acting on  $\mathbb{C}^N$  equipped with some norm. We seek upper bounds on the norm of the inverse  $T^{-1}$ . Assume that the minimal polynomial of  $T$  is given by

$$m(z) = m_T(z) = \prod_{i=1}^N (z - \lambda_i),$$

where  $\sigma = (\lambda_i)_{i=1}^N \in \mathbb{D}_*^N$  and we assumed for simplicity that  $\deg m_T = N$ . Following [10], assume that our Banach space  $X \subset \mathcal{H}ol(\mathbb{D})$  is in fact an algebra, and write  $A = X$ . Assume further that

- (1)  $T$  admits a  $C$ -functional calculus on  $A$ , i.e. there exists a bounded homomorphism  $f \mapsto f(T)$  extending the polynomial functional calculus and a constant  $C > 0$  such that

$$\|f(T)\| \leq C \|f\|_A, \quad f \in A;$$

- (2) the shift operator  $S : f \mapsto zf$ , the backward shift operator  $S^* : f \mapsto \frac{f-f(0)}{z}$  and the generalized backward shift operators  $f \mapsto \frac{f-f(\lambda)}{z-\lambda}$  are bounded on  $A$  for all  $\lambda \in \mathbb{D}$ .

These assumptions are mild and satisfied by all the algebras  $A$  considered below. Noticing that the analytic polynomial  $P = \frac{m(0)-m}{zm(0)}$  interpolates the function  $\frac{1}{z}$  on  $\sigma$  we observe that

$$T^{-1} = P(T) = (P + mh)(T)$$

for any  $h \in A$ . Applying assumption (1) to the above operator we obtain

$$\|T^{-1}\| \leq C \|P + mh\|_A$$

and taking the infimum over all  $h \in A$  and using our assumptions on  $A$ , we get

$$\|T^{-1}\| \leq C \inf \left\{ \|g\|_A : g|_\sigma = P|_\sigma = \frac{1}{z} \Big|_\sigma \right\}. \tag{1.3}$$

Now, if  $f \in A$  satisfies  $f(0) = 1$  and  $f|_\sigma = 0$ , then  $g := S^*(1 - f) = -S^*(f)$  is admissible for the last infimum, and so

$$\|T^{-1}\| \leq C \|S^*\|_{A \rightarrow A} \text{cap}_A(\sigma). \tag{1.4}$$

In particular (1.3) and (1.4) are applied (among other situations) in [10] to the cases of:

- Hilbert space contractions,  $A$  the disc algebra and  $C = 1$ ;
- Banach space contractions,  $A$  the Wiener algebra of absolutely convergent Taylor/Fourier series,

$$A = W = \left\{ f = \sum_{k \geq 0} \hat{f}(k) z^k \in \mathcal{H}ol(\mathbb{D}) : \|f\|_W = \sum_{k \geq 0} |\hat{f}(k)| < \infty \right\},$$

and once again  $C = 1$ ;

- Tadmor–Ritt type matrices or power-bounded matrices on Hilbert spaces and  $A$  the Besov algebra

$$A = \mathcal{B}_{\infty,1}^0 = \left\{ f \in \mathcal{H}ol(\mathbb{D}) : \|f\|_{\mathcal{B}_{\infty,1}^0} = |f(0)| + \int_0^1 \|f'_\rho\|_{L^\infty(\mathbb{T})} d\rho < \infty \right\},$$

where  $f_\rho(\zeta) = f(\rho\zeta)$ ,  $\zeta \in \mathbb{T}$ .

*Outline of the paper* In Section 2 below we first review Nikolski’s upper estimates on  $\text{cap}_X(\sigma)$  where  $X$  is a general Besov space  $\mathcal{B}_{p,q}^s$ ,  $s \geq 0$ ,  $(p, q) \in [1, \infty]^2$ , see below for their definition. We also relate the special case  $(p, q) = (\infty, 1)$  to applications in operator theory/matrix analysis and especially to Schäffer’s question on norms of inverses.

In Section 3 we formulate the main results of the paper. Theorem 2, which corresponds to the special case  $(p, q) = (\infty, 1)$ , exhibits an explicit sequence  $\sigma^*$  for which we derive a quantitative lower bound on  $\text{cap}_{\mathcal{B}_{\infty,1}^0}(\sigma^*)$  and thereby *almost* prove the sharpness of Nikolski’s upper bound in this case. Theorem 3 improves Nikolski’s upper bounds on  $\text{cap}_{\mathcal{B}_{p,q}^0}(\sigma)$  for a range of parameters, while in Theorem 4 the sharpness of these new bounds is discussed.

In Section 4 we prove Lemma 6, which is our main tool for bounding the capacities from below. In Section 5 we prove Theorem 2. The proofs of the lower bounds in Theorem 4 are provided in Section 6. Finally, in Section 7 we prove the upper bounds stated in Theorem 3. The proof is based on estimates of Besov norms of finite Blaschke products (Proposition 10) which may be of independent interest.

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## 2. Known results and open questions

### 2.1. Capacities in Besov spaces

The case where  $X$  is an analytic Besov space  $X = \mathcal{B}_{p,q}^s$  is considered in [10]. Let  $s \geq 0$ ,  $1 \leq p, q \leq \infty$  and let

$$\mathcal{B}_{p,q}^s = \left\{ f \in \text{Hol}(\mathbb{D}) : \|f\|_{\mathcal{B}_{p,q}^s}^* = \left( \int_0^1 ((1-\rho)^{m-s-1/q} \|f_\rho^{(m)}\|_{L^p(\mathbb{T})})^q d\rho \right)^{1/q} < \infty \right\},$$

where  $f_\rho^{(m)}(\zeta) = f^{(m)}(\rho\zeta)$ ,  $m$  being a nonnegative integer such that  $m > s$  (the choice of  $m$  is not essential and the norms for different  $m$ -s are equivalent). We need to make the obvious modification for  $q = \infty$ . The space  $\mathcal{B}_{p,q}^s$  equipped with the norm

$$\|f\|_{\mathcal{B}_{p,q}^s} = \sum_{k=0}^{m-1} |f^{(k)}(0)| + \|f\|_{\mathcal{B}_{p,q}^s}^*$$

is a Banach space. We refer to [5,13,20] for general properties of Besov spaces. Note that for  $1 \leq q < \infty$  we have  $f_\rho \rightarrow f$  in the norm of  $\mathcal{B}_{p,q}^s$  as  $\rho \rightarrow 1-$ .

In the present paper we deal with Besov spaces with zero smoothness  $s = 0$ . In this case we take  $m = 1$  and

$$\|f\|_{\mathcal{B}_{p,q}^0}^* = \left( \int_0^1 (1-\rho)^{q-1} \|f'_\rho\|_{L^p(\mathbb{T})}^q d\rho \right)^{1/q}, \quad 1 \leq q < \infty,$$

$$\|f\|_{\mathcal{B}_{p,\infty}^0}^* = \sup_{0 < \rho < 1} (1-\rho) \|f'_\rho\|_{L^p(\mathbb{T})}.$$

Note that  $\mathcal{B}_{\infty,\infty}^0$  coincides with the classical Bloch space.

It is shown [10, Theorem 3.26] that given  $1 \leq p, q \leq \infty$ ,  $s > 0$  and  $\sigma \in \mathbb{D}_*^N$  the following upper estimate holds

$$\text{cap}_{\mathcal{B}_{p,q}^s}(\sigma) \leq c \frac{N^s}{\prod_{i=1}^N |\lambda_i|},$$

where  $c = c(s, q)$ , and that if  $s = 0$  then

$$\text{cap}_{\mathcal{B}_{p,q}^0}(\sigma) \leq c \frac{(\log N)^{1/q}}{\prod_{i=1}^N |\lambda_i|}, \quad (2.1)$$

where  $c > 0$  is a numerical constant. It is also shown that for  $s > 0$  these estimates are asymptotically sharp in the following sense [10, Theorem 3.31]: there exist constants  $c = c(s, p, q) > 0$  and  $K = K(s, p, q) > 0$  such that for any  $\sigma = (\lambda_1, \dots, \lambda_N) \in \mathbb{D}_*^N$ ,  $s > 0$ ,  $1 \leq p, q \leq \infty$ ,

$$\text{cap}_{\mathcal{B}_{p,q}^s}(\sigma) \geq c \frac{N^s}{\prod_{i=1}^N |\lambda_i|} \left( 1 + K - \prod_{i=1}^N (1 + |\lambda_i|) \right).$$

The sharpness of the upper bound in (2.1) is left as an open question in [10].

## 2.2. Norms of inverses and Schäffer's question

Let  $\|\cdot\|$  denote the operator norm induced on  $\mathcal{M}_N$ , the space of complex  $N \times N$  matrices, by a Banach space norm on  $\mathbb{C}^N$ . What is the smallest constant  $\mathcal{S}_N$  so that

$$|\det T| \cdot \|T^{-1}\| \leq \mathcal{S}_N \|T\|^{N-1}$$

holds for any invertible matrix  $T \in \mathcal{M}_N$  and any operator norm  $\|\cdot\|$ ? Schäffer [17, Theorem 3.8] proved that

$$\mathcal{S}_N \leq \sqrt{eN},$$

but he conjectured that  $\mathcal{S}_N$  should in fact be bounded, as it is the case for Hilbert space. This conjecture was disproved in the early 90's by E. Gluskin, M. Meyer, and A. Pajor [7]. Later, Queffélec [15] showed that the  $\sqrt{N}$  bound is essentially optimal for arbitrary Banach spaces, but both arguments are non-constructive. An explicit construction giving a  $\sqrt{N}$  lower bound was recently given in [19]. For a detailed account on the history of Schäffer's question, the reader is referred to [19]. A key tool in the works cited above is the equality

$$\mathcal{S}_N = \sup_{(\lambda_1, \dots, \lambda_N) \in \mathbb{D}^N} \prod_{i=1}^N |\lambda_i| (\text{cap}_W(\lambda_1, \dots, \lambda_N) - 1), \quad (2.2)$$

due to Gluskin, Meyer and Pajor. It connects Schäffer's question to capacity in the Wiener algebra and shows that (1.4) is essentially sharp in this case.

It is natural to consider Schäffer's question for operator classes different from Hilbert or Banach space contractions. In particular, following [10], we may consider the following classes, which admit a Besov  $\mathcal{B}_{\infty,1}^0$ -functional calculus.

(1) Power bounded operators on Hilbert space, i.e. operators  $T$  on Hilbert space satisfying

$$\sup_{k \geq 0} \|T^k\| = C_{pb} < \infty.$$

Peller [14] proved that  $\|f(T)\| \leq k_G C_{pb}^2 \|f\|_{\mathcal{B}_{\infty,1}^0}$  for every analytic polynomial  $f$ , where  $k_G$  is the Grothendieck constant. Combining (1.4) with Nikolski’s upper estimate (2.1) for  $q = 1$ , we obtain the upper bounds

$$\|T^{-1}\| \leq c_1 \cdot \text{cap}_{\mathcal{B}_{\infty,1}^0}(\lambda_1, \dots, \lambda_N) \leq c_3 \frac{k_G C_{pb}^2 \log N}{\prod_{i=1}^N |\lambda_i|}, \tag{2.3}$$

where  $c_1 > 0$  is an absolute constant and  $(\lambda_i)_{i=1}^N$  is the sequence of eigenvalues of  $T$ .

(2) Tadmor–Ritt operators on Banach space, i.e. operators  $T$  acting on a Banach space and satisfying the resolvent estimate

$$\sup_{|\zeta| > 1} |\zeta - 1| \|(\zeta - T)^{-1}\| = C_{TR} < \infty.$$

According to P. Vitse’s functional calculus [22, Theorem 2.5] we have  $\|f(T)\| \leq 300 C_{TR}^5 \|f\|_{\mathcal{B}_{\infty,1}^0}$  for every analytic polynomial  $f$ , and following the same reasoning as above this yields

$$\|T^{-1}\| \leq c_2 \cdot \text{cap}_{\mathcal{B}_{\infty,1}^0}(\lambda_1, \dots, \lambda_N) \leq c_2 \frac{300 C_{TR}^5 \log N}{\prod_{i=1}^N |\lambda_i|}, \tag{2.4}$$

where  $c_2 > 0$  is an absolute constant. In fact, thanks to work of Schwenninger [18], the dependence on  $C_{TR}$  can be improved from  $C_{TR}^5$  to  $C_{TR}(\log C_{TR} + 1)$ .

The sharpness of the right-hand side in (2.3) and (2.4) is an open question both from the point of view of operators/matrices and from the one of capacities. Note that we have the following (strict) inclusions:

$$W \subset \mathcal{B}_{\infty,1}^0 \subset H^\infty \tag{2.5}$$

(see [5,13] or [11, Section B.8.7]). Observe that  $\mathcal{B}_{\infty,1}^0$  is actually contained in the disc algebra. From the perspective of capacities (2.5) implies that for any sequence  $\sigma = (\lambda_1, \dots, \lambda_N) \in \mathbb{D}_*^N$  we have

$$\text{cap}_{H^\infty}(\sigma) \leq c_3 \text{cap}_{\mathcal{B}_{\infty,1}^0}(\sigma) \leq c_4 \text{cap}_W(\sigma) \tag{2.6}$$

where  $c_3, c_4 > 0$  are absolute constants. Observe that in view of (2.6) and (2.2) any sequence  $\sigma \in \mathbb{D}_*^N$  such that  $\prod_{i=1}^N |\lambda_i| \cdot \text{cap}_{\mathcal{B}_{\infty,1}^0}(\sigma)$  grows unboundedly in  $N$  will automatically give a counterexample to Schäffer’s original question.

### 3. Main results

Throughout this paper, we will use the following standard notation. For two positive functions  $f, g$  we say that  $f$  is dominated by  $g$ , denoted by  $f \lesssim g$ , if there is a constant  $c > 0$  such that  $f \leq cg$  for all admissible variables. We say that  $f$  and  $g$  are comparable, denoted by  $f \asymp g$ , if both  $f \lesssim g$  and  $g \lesssim f$ .

The main goals of this paper are to

- (1) Provide an example of a sequence  $\sigma^* = (\lambda_1, \dots, \lambda_N) \in \mathbb{D}_*^N$  such that  $\prod_{i=1}^N |\lambda_i| \cdot \text{cap}_{\mathcal{B}_{\infty,1}^0}(\sigma^*)$  almost (up to a double logarithmic factor) approaches Nikolski's upper bound  $\log N$ .
- (2) Improve Nikolski's upper bound (2.1) on  $\prod_{i=1}^N |\lambda_i| \cdot \text{cap}_{\mathcal{B}_{p,q}^0}(\sigma)$  identifying three regions of  $(p, q) \in [1, \infty]^2$  with a different behavior of this quantity (see Theorem 3 below). For all  $(p, q)$  with  $p \neq \infty$  our estimates give a smaller growth than the estimates in [10], and for a range of parameters, namely for  $1 \leq q \leq p < \infty$  and  $p \geq 2$ , they are best possible.

#### 3.1. A lower estimate on $\text{cap}_{\mathcal{B}_{\infty,1}^0}(\sigma)$

Our approach to bounding  $\text{cap}_{\mathcal{B}_{\infty,1}^0}(\sigma)$  from below uses duality. To estimate  $\text{cap}_{\mathcal{B}_{\infty,1}^0}(\sigma)$  from below, we estimate the Besov seminorm in  $\mathcal{B}_{1,\infty}^0$  of finite Blaschke products from above. The key inequality, which will be proved in Lemma 6, is

$$\text{cap}_{\mathcal{B}_{\infty,1}^0}(\sigma) \gtrsim \frac{1}{\prod_{i=1}^N |\lambda_i|} \frac{1 - \prod_{i=1}^N |\lambda_i|^2}{\|B\|_{\mathcal{B}_{1,\infty}^0}^*}, \tag{3.1}$$

where  $\sigma = (\lambda_1, \dots, \lambda_N)$  is an arbitrary sequence in  $\mathbb{D}_*^N$ , and  $B = B_\sigma$  is the finite Blaschke product associated to  $\sigma$ . To conclude we consider  $n \geq 2$  and for  $k = 1, \dots, n$  we put

$$\sigma_k = (r_k^{(n)} e^{2i\pi j/2^k})_{j=1}^{2^k} \in \mathbb{D}_*^{2^k}, \quad r_k^{(n)} = (1 - 1/n)^{2^{-k}}.$$

We put  $N = \sum_{k=1}^n 2^k \asymp 2^n$  and define the sequence  $\sigma^* = (\lambda_1, \dots, \lambda_N) \in \mathbb{D}_*^N$  by

$$\sigma^* = (\sigma_1, \sigma_2, \dots, \sigma_n). \tag{3.2}$$

Denoting by  $B^*$  the Blaschke product associated with  $\sigma^*$  we have

$$B^*(z) = \prod_{k=1}^n \frac{z^{2^k} - a}{1 - az^{2^k}}, \tag{3.3}$$

where  $a = 1 - \frac{1}{n}$ . We will prove the following result.



**Proposition 1.** *The Blaschke product  $B^*$  satisfies*

$$\|B^*\|_{\mathcal{B}_{1,\infty}^*}^* \lesssim \frac{\log \log N}{\log N}. \tag{3.4}$$

Taking into account that  $\prod_{j=1}^N |\lambda_j| \leq e^{-1}$  and combining (3.1) with (3.4) we obtain the following theorem.

**Theorem 2.** *Let  $\sigma^* \in \mathbb{D}_*^N$  and  $B^*$  be defined by (3.2) and (3.3). Then*

$$\prod_{i=1}^N |\lambda_i| \cdot \text{cap}_{\mathcal{B}_{\infty,1}^0}(\sigma^*) \gtrsim \frac{\log N}{\log \log N}.$$

As a consequence regarding Schäffer’s question, Theorem 2 implies (taking into account (2.6)) that

$$\prod_{i=1}^N |\lambda_i| \cdot \text{cap}_W(\sigma^*) \gtrsim \frac{\log N}{\log \log N}.$$

From this, following arguments in [19], one obtains another explicit counterexample to Schäffer’s question, acting as multiplication by  $z$  on the quotient  $W/B^*W$  of the Wiener algebra. One can identify the dual space of  $W/B^*W$  with the space of rational functions of degree at most  $N$  with poles at  $1/\bar{\lambda}_j$  for  $j = 1, \dots, N$ , equipped with the supremum norm of the Taylor coefficients. Then, as in [19, Theorem 8], one obtains another explicit matrix that serves as a counterexample to Schäffer’s question.

3.2. *Upper bounds on  $\text{cap}_{\mathcal{B}_{p,q}^0}(\sigma)$  for general values of  $(p, q) \in [1, \infty]^2$*

In the following statements the constants in  $\lesssim$  relations may depend on  $p, q$ , but not on  $N$ .

**Theorem 3.** *Given  $(p, q) \in [1, \infty]^2$  and  $\sigma = (\lambda_1, \dots, \lambda_N) \in \mathbb{D}_*^N$ , the following upper estimates on  $\text{cap}_{\mathcal{B}_{p,q}^0}(\sigma)$  hold depending on the region to which  $(p, q)$  belongs.*

1) *If  $(p, q) \in [1, 2]^2$  (Region I), then*

$$\text{cap}_{\mathcal{B}_{p,q}^0}(\sigma) \lesssim \frac{(\log N)^{1/q-1/2}}{\prod_{i=1}^N |\lambda_i|}.$$

2) *If  $1 \leq p \leq q \leq \infty$  and  $q \geq 2$  (Region II), then*

$$\text{cap}_{\mathcal{B}_{p,q}^0}(\sigma) \lesssim \frac{1}{\prod_{i=1}^N |\lambda_i|}.$$

3) If  $1 \leq q \leq p \leq \infty$  and  $p \geq 2$  (Region III), then

$$\text{cap}_{\mathcal{B}_{p,q}^0}(\sigma) \lesssim \frac{(\log N)^{1/q-1/p}}{\prod_{i=1}^N |\lambda_i|}.$$

**Remark.** The upper bound in part 2 of Theorem 3 is attained by any sequence  $\sigma = (\lambda_1, \dots, \lambda_N) \in \mathbb{D}_*^N$  such that  $|\lambda_i| \geq 1 - 1/N$  for all  $i = 1, \dots, N$ .

3.3. Lower estimates on  $\text{cap}_{\mathcal{B}_{p,q}^0}(\sigma^*)$

In the following theorem we derive quantitative lower estimates on  $\text{cap}_{\mathcal{B}_{p,q}^0}(\sigma^*)$  for  $1 \leq q \leq p \leq \infty$ . This proves, in particular, the sharpness of Theorem 3 for  $(p, q)$  in Region III if  $p < \infty$ .

**Theorem 4.** Let  $\sigma^* \in \mathbb{D}_*^N$  and  $B^*$  be defined by (3.2) and (3.3), and let  $(p, q) \in [1, \infty]^2$  be such that  $1 \leq q \leq p \leq \infty$ . Then

$$\prod_{i=1}^N |\lambda_i| \cdot \text{cap}_{\mathcal{B}_{p,q}^0}(\sigma^*) \gtrsim (\log N)^{1/q-1/p}, \quad p < \infty, \tag{3.5}$$

$$\prod_{i=1}^N |\lambda_i| \cdot \text{cap}_{\mathcal{B}_{\infty,q}^0}(\sigma^*) \gtrsim \frac{(\log N)^{1/q}}{\log \log N}. \tag{3.6}$$

In particular, for  $(p, q)$  in Region III and  $p < \infty$ ,

$$\prod_{i=1}^N |\lambda_i| \cdot \text{cap}_{\mathcal{B}_{p,q}^0}(\sigma^*) \asymp (\log N)^{1/q-1/p}. \tag{3.7}$$

However, for  $1 \leq q \leq p < 2$  there is still a certain gap between the upper and lower estimates for the capacities:

$$(\log N)^{1/q-1/p} \lesssim \prod_{i=1}^N |\lambda_i| \cdot \text{cap}_{\mathcal{B}_{p,q}^0}(\sigma^*) \lesssim (\log N)^{1/q-1/2}.$$

Let us consider the diagonal case  $1 \leq q = p < 2$ . Rudin [16] showed that there exists a Blaschke product that is not contained in  $\mathcal{B}_{1,1}^0$ , see also [12]. Vinogradov [21, Theorem 3.11] extended Rudin’s result to  $\mathcal{B}_{p,p}^0$  for  $p \in (0, 2)$ . These results perhaps suggest that the expression in the middle might be unbounded for  $1 \leq q = p < 2$ . Indeed, unboundedness would follow if we knew that there are Blaschke sequences that are not zero sets for  $\mathcal{B}_{p,p}^0$ . However, the existence of such Blaschke sequences appears to be an open question. Results about zero sets for  $\mathcal{B}_{p,p}^0$ , also for  $p > 2$ , can be found in [6].

Instead, we will give a different, qualitative argument showing that, in case  $1 \leq q = p < 2$ , the expression in the middle may be unbounded.

**Theorem 5.** For each  $N \in \mathbb{N}$  there exists a finite sequence  $\sigma_N \in \mathbb{D}_*^N$  such that for all  $1 \leq p < 2$ , we have

$$\lim_{N \rightarrow \infty} \prod_{\lambda \in \sigma_N} |\lambda| \cdot \text{cap}_{\mathcal{B}_{p,p}^0}(\sigma_N) = \infty.$$

It will be convenient to extend the definition of  $\text{cap}_{\mathcal{B}_{p,q}^s}(\sigma)$  to possibly infinite sequences  $\sigma$  in the obvious way. The infimum over the empty set is understood to be  $+\infty$ , so that  $\text{cap}_{\mathcal{B}_{p,q}^s}(\sigma) = +\infty$  in case  $\sigma$  is a uniqueness set for  $\mathcal{B}_{p,q}^s$ . Our approach to bound  $\text{cap}_{\mathcal{B}_{p,q}^0}(\sigma)$  from below is based on a duality method. Namely, the key step of the proof is the following lemma:

**Lemma 6.** Given  $1 \leq p, q \leq \infty$  and a finite sequence  $\sigma$  in  $\mathbb{D}_*$ , we have

$$\prod_{\lambda \in \sigma} |\lambda| \cdot \text{cap}_{\mathcal{B}_{p,q}^0}(\sigma) \gtrsim \frac{1 - \prod_{\lambda \in \sigma} |\lambda|^2}{\|B_\sigma\|_{\mathcal{B}_{p',q'}^0}^*},$$

where  $B_\sigma$  is the Blaschke product with the zero set  $\sigma$  and  $p', q'$  are the exponents conjugate to  $p, q$ . The same estimate is true for arbitrary Blaschke sequences  $\sigma$  in  $\mathbb{D}_*$  in case  $1 \leq p = q \leq 2$ .

To prove the lower estimate (3.5) it remains to apply Lemma 6 to  $\sigma = \sigma^*$  and estimate from above the Besov seminorm of  $B^*$ . Namely we prove the following.

**Proposition 7.** If  $1 \leq p \leq q \leq \infty$ , then

$$\begin{aligned} \|B^*\|_{\mathcal{B}_{p,q}^0}^* &\lesssim \frac{1}{(\log N)^{1/p-1/q}}, \quad p > 1, \\ \|B^*\|_{\mathcal{B}_{1,q}^0}^* &\lesssim \frac{\log \log N}{(\log N)^{1-1/q}}. \end{aligned}$$

The idea of the proof of Theorem 5 is also to use duality. In case  $p = 1$ , the dual norm turns out to be the Bloch semi-norm. An obstacle to this strategy is a result of Baranov, Kayumov, and Nasyrov [4], according to which the Bloch semi-norm of finite Blaschke products is bounded below by a universal constant. Instead, we will work with infinite Blaschke products, and carry out an approximation argument.

#### 4. Proof of Lemma 6

We first prove Lemma 6. Let  $\langle \cdot, \cdot \rangle$  denote the Cauchy sesquilinear form: given two functions  $g \in H^p$  and  $h \in H^{p'}$ , let

$$\langle h, g \rangle = \int_{\mathbb{T}} h(z) \overline{g(z)} \, dm(z),$$

where  $m$  denotes the normalized Lebesgue measure on  $\mathbb{T}$ . We require the following basic duality result for Besov spaces.

**Lemma 8.** *Let  $1 \leq p, q \leq \infty$ . There exists a constant  $C \geq 0$  such that for all functions  $f$  and  $g$  that are analytic in a neighborhood of  $\overline{\mathbb{D}}$ , we have*

$$|\langle f, g \rangle| \leq |f(0)||g(0)| + C\|f\|_{\mathcal{B}_{p,q}^0}^* \|g\|_{\mathcal{B}_{p',q'}^0}^*,$$

where  $p', q'$  are the exponents conjugate to  $p, q$ .

**Proof.** Denote by  $(h, g)$  the scalar product on the Bergman space  $A^2$  defined by

$$(h, g) = \int_{\mathbb{D}} h(u)\overline{g(u)}d\mathcal{A}(u), \quad h, g \in A^2,$$

where  $d\mathcal{A}(u) = \frac{dx dy}{\pi}$  is the normalized planar Lebesgue measure on  $\mathbb{D}$ . We recall the simplest form of Green’s formula,

$$\langle \phi, \psi \rangle = (\phi', S^*\psi) + \phi(0)\overline{\psi(0)}, \tag{4.1}$$

where  $S^*$  is the backward shift operator  $S^*f = (f - f(0))/z$  and  $\varphi, \psi$  are functions that are analytic in a neighborhood of  $\overline{\mathbb{D}}$ . We will also need to use the following integral formula. Recall that the fractional differentiation operator  $D_\alpha$ ,  $-1 < \alpha < \infty$ , is defined by  $D_\alpha(z^j) = \frac{\Gamma(j+2+\alpha)}{(j+1)\Gamma(2+\alpha)}z^j$ ,  $j = 0, 1, 2, \dots$ , and extends linearly and continuously to the whole space  $\mathcal{H}ol(\mathbb{D})$ . Then, for functions  $f, g$  analytic in a neighborhood of  $\overline{\mathbb{D}}$  and  $-1 < \alpha < \infty$ , we have

$$\int_{\mathbb{D}} f(u)\overline{g(u)}d\mathcal{A}(u) = (\alpha + 1) \int_{\mathbb{D}} D_\alpha f(u)\overline{g(u)}(1 - |u|^2)^\alpha d\mathcal{A}(u), \tag{4.2}$$

see [8, Lemma 1.20].

Let  $f, g$  be analytic in a neighborhood of  $\overline{\mathbb{D}}$ . Applying (4.1) we get

$$\langle f, g \rangle = (f', S^*g) + f(0)\overline{g(0)}.$$

Then we apply (4.2) to  $\overline{(f', S^*g)} = (S^*g, f')$  with  $\alpha = 1$ :

$$\begin{aligned} (S^*g, f') &= 2 \int_{\mathbb{D}} D_1(S^*g)(u)\overline{f'(u)}(1 - |u|^2) d\mathcal{A}(u) \\ &= 2 \int_0^1 \rho(1 - \rho^2) \left( \int_{\mathbb{T}} D_1(S^*g)(\rho z)\overline{f'(\rho z)}dm(z) \right) d\rho. \end{aligned}$$

By Hölder’s inequality

$$\left| \int_{\mathbb{T}} D_1(S^*g)(\rho z) \overline{f'(\rho z)} dm(z) \right| \leq \|f'_\rho\|_{L^p} \|(D_1(S^*g))_\rho\|_{L^{p'}}.$$

Since  $D_1(S^*g) = \frac{1}{2}(S^*g + g')$  and  $(S^*g)(z) = \frac{1}{z} \int_0^1 tg'(tz) dt$ , it follows that  $\|D_1(S^*g)\|_{L^{p'}} \lesssim \|g'\|_{L^{p'}}$ . The preceding estimates therefore give

$$|(S^*g, f')| \lesssim \int_0^1 (1 - \rho) \|f'_\rho\|_{L^p} \|g'_\rho\|_{L^{p'}} d\rho.$$

Then (again by Hölder’s inequality) we get

$$|(S^*g, f')| \lesssim \|f\|_{\mathcal{B}_{p,q}^0}^* \|g\|_{\mathcal{B}_{p',q'}^0}^*,$$

as desired.  $\square$

**Proof of Lemma 6.** Suppose first that  $\sigma$  is a finite sequence in  $\mathbb{D}_*$ , say  $|\sigma| = N$ . Let  $f$  be a function that is analytic in a neighborhood of  $\overline{\mathbb{D}}$  such that  $f(0) = 1$  and  $f|_\sigma = 0$ . Then we have (writing  $B = B_\sigma$ )

$$\langle f, B \rangle = \frac{f(0)}{B(0)} = \frac{1}{\prod_{\lambda \in \sigma} \lambda}.$$

On the other hand, Lemma 8 shows that

$$|\langle f, B \rangle| \leq |f(0)| |B(0)| + C \|f\|_{\mathcal{B}_{p,q}^0}^* \|B\|_{\mathcal{B}_{p',q'}^0}^* = \prod_{\lambda \in \sigma} |\lambda| + C \|f\|_{\mathcal{B}_{p,q}^0}^* \|B\|_{\mathcal{B}_{p',q'}^0}^*.$$

Thus,

$$\prod_{\lambda \in \sigma} |\lambda| \cdot \|f\|_{\mathcal{B}_{p,q}^0}^* \geq \frac{1 - \prod_{\lambda \in \sigma} |\lambda|^2}{C \|B\|_{\mathcal{B}_{p',q'}^0}^*}. \tag{4.3}$$

Now, let  $f \in \mathcal{B}_{p,q}^0$  be an arbitrary function such that  $f(0) = 1$  and  $f|_\sigma = 0$ . Let  $0 < r < 1$  be such that  $\frac{1}{r}\sigma \subset \mathbb{D}$ . Then  $f_r$  vanishes on  $\frac{1}{r}\sigma$ , hence by what has already been proved,

$$r^N \prod_{\lambda \in \sigma} |\lambda| \cdot \|f_r\|_{\mathcal{B}_{p,q}^0}^* \geq \frac{1 - r^{2N} \prod_{\lambda \in \sigma} |\lambda|^2}{C \|B_{\frac{1}{r}\sigma}\|_{\mathcal{B}_{p',q'}^0}^*}.$$

Recall that  $\|f_r\|_{\mathcal{B}_{p,q}^0}^* \leq \|f\|_{\mathcal{B}_{p,q}^0}^*$ . Moreover,  $B_{\frac{1}{r}\sigma}$  converges to  $B_\sigma$  uniformly in a neighborhood of  $\overline{\mathbb{D}}$  as  $r \rightarrow 1$ . So taking the limit  $r \rightarrow 1$ , we conclude that (4.3) holds for arbitrary

$f \in \mathcal{B}_{p,q}^0$  satisfying  $f(0) = 1$  and  $f|_\sigma = 0$ . Taking the infimum over all admissible functions  $f$ , we obtain the lemma for finite sequences.

Let now  $1 \leq p = q \leq 2$  and let  $\sigma$  be a possibly infinite Blaschke sequence. Let  $B = B_\sigma$  and let  $f \in \mathcal{B}_{p,p}^0$  be a function vanishing on  $\sigma$  with  $f(0) = 1$ . We apply Lemma 8 to the functions  $f_r$  and  $B_r$  to obtain the bound

$$|\langle f_r, B_r \rangle| \leq |B(0)| + C \|f\|_{\mathcal{B}_{p,p}^0}^* \|B\|_{\mathcal{B}_{q,q}^0}^*$$

for all  $r < 1$ .

The classical Littlewood–Paley inequality shows that  $\mathcal{B}_{p,p}^0 \subset H^p \subset H^1$  (see [9, Theorem 6] and also [21, Lemma 1.4]), so  $f_r \rightarrow f$  in the norm of  $H^1$ . Moreover,  $B \in H^\infty$  and  $B_r \rightarrow B$  weak-\* in  $H^\infty$ . From this, it follows that

$$\langle f, B \rangle - \langle f_r, B_r \rangle = \langle f, B - B_r \rangle + \langle f - f_r, B_r \rangle \rightarrow 0$$

as  $r \rightarrow 1$ . Thus,

$$|\langle f, B \rangle| \leq |B(0)| + C \|f\|_{\mathcal{B}_{p,p}^0}^* \|B\|_{\mathcal{B}_{p',p'}^0}^*.$$

Using that  $f \in H^1$  vanishes on  $\sigma$ , we may factor  $f = Bg$  for some  $g \in H^1$ . Then

$$\langle f, B \rangle = \langle g, 1 \rangle = g(0) = \frac{1}{B(0)}.$$

Combining the last two formulas and taking the infimum over all admissible  $f \in \mathcal{B}_{p,p}^0$  again yields the desired inequality.  $\square$

### 5. Proof of Theorem 2

#### 5.1. Proof of Proposition 1

For simplicity we write  $B$  instead of  $B^*$  throughout the proof. Then  $N = \deg B \asymp 2^n$ . For the zeros  $z_1, \dots, z_N$  of  $B$  we have

$$\prod_{j=1}^N |z_j| = a^n < e^{-1}.$$

For  $z \in \mathbb{D}$ ,  $|z| = r$ , we have

$$|B'(z)| \leq \sum_{k=1}^n 2^k r^{2^k-1} \frac{1-a^2}{|1-az^{2^k}|^2}. \tag{5.1}$$

Using that  $\|(1-bz^N)^{-1}\|_{H^2}^2 = (1-b^2)^{-1}$  for  $b \in [0, 1)$ , we find that

$$(1 - r) \int_0^{2\pi} |B'(re^{it})| dt \leq 2\pi \sum_{k=1}^n 2^k r^{2^k-1} \frac{(1-r)(1-a^2)}{1-a^2 r^{2^{k+1}}} \lesssim \frac{1}{n} \sum_{k=1}^n 2^k r^{2^k-1} \frac{1-r}{1-ar^{2^k}}.$$

Let us first estimate this quantity for  $0 \leq r \leq \frac{1}{2}$ . In this case

$$\frac{1}{n} \sum_{k=1}^n 2^k r^{2^k-1} \frac{1-r}{1-ar^{2^k}} \lesssim \frac{1}{n} \sum_{k=1}^n 2^{k-2^k} \lesssim \frac{1}{n}.$$

From now on we assume that  $r = 1 - \frac{1}{2^s}$ , where  $s \geq 1$ , and we write

$$\frac{1}{n} \sum_{k=1}^n 2^k r^{2^k-1} \frac{1-r}{1-ar^{2^k}} = \frac{1}{n} \sum_{k=1}^{[s]} 2^k r^{2^k-1} \frac{1-r}{1-ar^{2^k}} + \frac{1}{n} \sum_{k=[s]+1}^n 2^k r^{2^k-1} \frac{1-r}{1-ar^{2^k}} = S_1 + S_2.$$

Since  $(1-x)^t < e^{-tx}$ ,  $0 < x < 1$ ,  $t > 0$ , we have

$$r^{2^k} = \left(1 - \frac{1}{2^s}\right)^{2^k} < e^{-2^{k-s}}.$$

Therefore, for  $k \geq [s] + 1$  we have  $r^{2^k} < e^{-1}$  and so

$$S_2 \lesssim \frac{1}{n} \sum_{k=[s]+1}^n 2^{k-s} e^{-2^{k-s}} \lesssim \frac{1}{n}.$$

For  $k \leq [s]$  we use the inequality

$$r^{-2^k} - a > e^{2^{k-s}} - 1 + \frac{1}{n} > 2^{k-s} + \frac{1}{n}.$$

Thus,

$$S_1 \lesssim \frac{1}{n} \sum_{k=1}^{[s]} 2^k \frac{1-r}{r^{-2^k} - a} < \frac{1}{n} \sum_{k=1}^{[s]} \frac{2^{k-s}}{2^{k-s} + \frac{1}{n}}.$$

We split this sum into two more sums, over  $k$  such that  $2^{k-s} < \frac{1}{n}$  and  $2^{k-s} \geq \frac{1}{n}$ . Then we have

$$S_1 \lesssim \frac{1}{n} \sum_{1 \leq k < s - \frac{\log n}{\log 2}} 2^{k-s} \cdot n + \frac{1}{n} \sum_{s - \frac{\log n}{\log 2} \leq k \leq [s]} 2^{k-s} \cdot 2^{s-k} \lesssim 2^{-\frac{\log n}{\log 2}} + \frac{\log n}{n \log 2} \lesssim \frac{1}{n} + \frac{\log n}{n \log 2}.$$

Thus, we have shown that

$$\|B\|_{B_{1,\infty}^0}^* \lesssim \frac{1}{n} + \frac{\log n}{n \log 2} \lesssim \frac{\log \log N}{\log N}. \quad \square$$

5.2. Proof of Theorem 2

Applying Lemma 6 to  $\sigma = \sigma^*$  with  $(p, q) = (\infty, 1)$  we obtain

$$\prod_{i=1}^N |\lambda_i| \cdot \text{cap}_{\mathcal{B}_{\infty,1}^0}(\sigma^*) \gtrsim \frac{1}{\|B\|_{\mathcal{B}_{1,\infty}^0}^*}$$

because  $\prod_{i=1}^N |\lambda_i| = a^n < e^{-1}$ . It remains to apply Proposition 1.  $\square$

6. Proofs of Theorem 4 and Theorem 5

6.1. Proof of Proposition 7

As in the proof of Proposition 1, for simplicity we write  $B$  instead of  $B^*$  throughout the proof.

Step 1: the case  $q = \infty$ . Note that the case  $p = 1$  is already covered by Proposition 1. We start with the case  $1 < p \leq 2$ . We have to prove that

$$\sup_{0 \leq r < 1} (1-r) \left( \int_0^{2\pi} |B'(re^{it})|^p dt \right)^{1/p} \lesssim \frac{1}{(\log N)^{1/p}}. \tag{6.1}$$

It follows from (5.1) that

$$I := (1-r)^p \int_0^{2\pi} |B'(re^{it})|^p dt \lesssim \int_0^{2\pi} \left( \frac{1}{n} \sum_{k=1}^n 2^k r^{2^k-1} \frac{(1-r)}{|1 - ar^{2^k} e^{i2^k t}|^2} \right)^p dt.$$

Since for  $0 < p/2 \leq 1$  and any  $a_k \geq 0$  one has

$$\left( \sum_k a_k \right)^p \leq \left( \sum_k a_k^{p/2} \right)^2,$$

we conclude that

$$\begin{aligned} I &\leq \frac{1}{n^p} \int_0^{2\pi} \left( \sum_{k=1}^n \frac{(2^k r^{2^k-1} (1-r))^{p/2}}{|1 - ar^{2^k} e^{i2^k t}|^p} \right)^2 dt \\ &\lesssim \frac{1}{n^p} \int_0^{2\pi} \sum_{k \leq j \leq n} \frac{(2^k r^{2^k-1} (1-r))^{p/2} (2^j r^{2^j-1} (1-r))^{p/2}}{|1 - ar^{2^k} e^{i2^k t}|^p |1 - ar^{2^j} e^{i2^j t}|^p} dt \\ &\lesssim \frac{1}{n^p} \int_0^{2\pi} \sum_{k \leq j \leq n} \frac{(2^k r^{2^k-1} (1-r))^{p/2} (2^j r^{2^j-1} (1-r))^{p/2}}{|1 - ar^{2^k} e^{i2^k t}|^p (1 - ar^{2^j})^p} dt. \end{aligned}$$



After integration with respect to  $t$  and using a well-known estimate of Forelli and Rudin (see [8, Theorem 1.7]) we get

$$\begin{aligned}
 I &\lesssim \frac{1}{n^p} \sum_{k \leq j \leq n} \frac{(2^k r^{2^k-1}(1-r))^{p/2} (2^j r^{2^j-1}(1-r))^{p/2}}{(1-ar^{2^k})^{p-1} (1-ar^{2^j})^p} \\
 &\lesssim \frac{1}{n^p} \sum_{k \leq j \leq n} \frac{(2^k r^{2^k-1}(1-r))^{p/2} (2^j r^{2^j-1}(1-r))^{p/2}}{(1-ar^{2^k})^{p-1/2} (1-ar^{2^j})^{p-1/2}} \\
 &\lesssim \frac{1}{n^p} \left( \sum_{k=1}^n \frac{(2^k r^{2^k-1}(1-r))^{p/2}}{(1-ar^{2^k})^{p-1/2}} \right)^2.
 \end{aligned}$$

Thus, we need to show that

$$S = \frac{1}{n^{p/2}} \sum_{k=1}^n \frac{(2^k r^{2^k-1}(1-r))^{p/2}}{(1-ar^{2^k})^{p-1/2}} \leq \frac{1}{\sqrt{n}}.$$

If  $r \leq 1/2$ , then, clearly,  $S \lesssim n^{-p/2} \leq n^{-1/2}$ . Now, let  $r = 1 - 1/2^s$  where  $s \geq 1$ . If  $k \geq [s] + 1$ , then  $r^{2^k} < e^{-2^{k-s}} \leq 1/e$  and

$$\frac{1}{n^{p/2}} \sum_{k=[s]+1}^n \frac{(2^k r^{2^k-1}(1-r))^{p/2}}{(1-ar^{2^k})^{p-1/2}} \lesssim \frac{1}{n^{p/2}} \sum_{k=[s]+1}^n (2^{k-s} e^{-2^{k-s}})^{p/2} \lesssim \frac{1}{n^{p/2}}.$$

Note that  $r^{2^k} = (1 - 1/2^s)^{2^k} \geq (1 - 1/2)^2$  for  $k \leq [s]$  and, therefore, as in the proof of Proposition 1,

$$|1 - ar^{2^k}| \gtrsim r^{-2^k} - a = r^{-2^k} - 1 + 1/n \geq 2^{k-s} + 1/n.$$

As in the proof of Proposition 1 we split the sum into two parts. For  $1 \leq k < s - \frac{\log n}{\log 2}$  we have  $2^{k-s} < 1/n$  and, therefore,

$$\begin{aligned}
 \frac{1}{n^{p/2}} \sum_{k < s - \log n / \log 2} \frac{(2^k r^{2^k-1}(1-r))^{p/2}}{(1-ar^{2^k})^{p-1/2}} &\lesssim \frac{1}{n^{p/2}} n^{p-1/2} \sum_{k < s - \log n / \log 2} 2^{(k-s)p/2} \\
 &\lesssim n^{p/2-1/2} 2^{(-\log n / \log 2)p/2} = n^{-1/2}.
 \end{aligned}$$

Finally, for  $s - \frac{\log n}{\log 2} \leq k \leq [s]$  we have  $2^{k-s} \geq 1/n$  and so

$$\begin{aligned}
 \frac{1}{n^{p/2}} \sum_{s - \log n / \log 2 \leq k \leq [s]} \frac{(2^k r^{2^k-1}(1-r))^{p/2}}{(1-ar^{2^k})^{p-1/2}} &\lesssim \frac{1}{n^{p/2}} \sum_{s - \log n / \log 2 \leq k \leq [s]} 2^{(k-s)p/2} 2^{(s-k)(p-1/2)} \\
 &\lesssim n^{-p/2} 2^{(\log n / \log 2)(p-1)/2} = n^{-1/2}.
 \end{aligned}$$

Thus, we have shown that  $S \leq n^{-1/2}$  for  $1 < p \leq 2$ , and so

$$I = (1 - r)^p \int_0^{2\pi} |B'(re^{it})|^p dt \lesssim S^2 \lesssim \frac{1}{\log N}.$$

The estimate remains true for  $p > 2$  since by the Schwarz–Pick inequality, we have  $|B'(re^{it})|^p \leq (1 - r^2)^{2-p} |B'(re^{it})|^2$ .

Step 2: the case  $1 \leq p \leq q < \infty$ . We have to show that

$$\int_0^1 (1 - r)^{q-1} \left( \int_0^{2\pi} |B'(re^{it})|^p dt \right)^{q/p} dr \lesssim \frac{1}{(\log N)^{q/p-1}}$$

(respectively  $\lesssim \frac{(\log \log N)^q}{(\log N)^{q-1}}$  in case  $p = 1$ ). It follows from (6.1) that for  $1 < p < \infty$

$$\int_0^{1-1/N^2} (1 - r)^{q-1} \left( \int_0^{2\pi} |B'(re^{it})|^p dt \right)^{q/p} dr \lesssim \frac{1}{(\log N)^{q/p}} \int_0^{1-1/N^2} \frac{dr}{1 - r} \lesssim \frac{1}{(\log N)^{q/p-1}},$$

while for  $p = 1$  we have by Proposition 1 that

$$\int_0^{1-1/N^2} (1 - r)^{q-1} \left( \int_0^{2\pi} |B'(re^{it})| dt \right)^q dr \lesssim \frac{(\log \log N)^q}{(\log N)^{q-1}}.$$

On the other hand, note that, since  $\int_0^{2\pi} |B'(re^{it})| dt \leq \int_0^{2\pi} |B'(e^{it})| dt = 2\pi N$ , we have by the Schwarz–Pick inequality

$$\int_0^{2\pi} |B'(re^{it})|^p dt \leq \frac{1}{(1 - r^2)^{p-1}} \int_0^{2\pi} |B'(re^{it})| dt \leq \frac{2\pi N}{(1 - r^2)^{p-1}}.$$

Therefore,

$$\begin{aligned} \int_{1-1/N^2}^1 (1 - r)^{q-1} \left( \int_0^{2\pi} |B'(re^{it})|^p dt \right)^{q/p} dr &\lesssim N^{q/p} \int_{1-1/N^2}^1 (1 - r)^{q-1-q(p-1)/p} dr \\ &= N^{q/p} \int_{1-1/N^2}^1 (1 - r)^{q/p-1} \lesssim N^{-q/p} = o\left(\frac{1}{(\log N)^{q/p-1}}\right). \end{aligned}$$

Combining the above estimates we come to the conclusion of the proposition.  $\square$

6.2. Proof of Theorem 4

As in the proof of Theorem 2, we apply Lemma 6 to  $\sigma = \sigma^*$ . This gives

$$\prod_{i=1}^N |\lambda_i| \cdot \text{cap}_{\mathcal{B}_{p,q}^0}(\sigma^*) \gtrsim \frac{1}{\|B\|_{\mathcal{B}_{p',q'}^0}^*}$$

again because  $\prod_{i=1}^N |\lambda_i| = a^n < e^{-1}$ . It remains to apply Proposition 7 (with  $p', q'$  in place of  $p, q$ ) to prove the lower bounds (3.5) and (3.6). The upper estimate in (3.7) follows from Theorem 3.  $\square$

6.3. Proof of Theorem 5

To pass from infinite to finite Blaschke sequences, we require the following continuity property of capacity.

**Lemma 9.** *Let  $1 \leq p, q \leq \infty$ . Let  $\sigma \subset \mathbb{D} \setminus \{0\}$  be an infinite sequence. For  $N \in \mathbb{N}$ , let  $\sigma_N$  consist of the first  $N$  points of  $\sigma$ . Then*

$$\text{cap}_{\mathcal{B}_{p,q}^0}(\sigma) = \lim_{N \rightarrow \infty} \text{cap}_{\mathcal{B}_{p,q}^0}(\sigma_N).$$

**Proof.** For simplicity, we abbreviate  $\text{cap} = \text{cap}_{\mathcal{B}_{p,q}^0}$ . The inequality  $\text{cap}(\sigma_N) \leq \text{cap}(\sigma)$  is trivial, so it suffices to show that  $\text{cap}(\sigma) \leq \liminf_{N \rightarrow \infty} \text{cap}(\sigma_N)$ . Clearly, we may assume that the limit inferior is finite.

If  $c > \liminf_{N \rightarrow \infty} \text{cap}(\sigma_N)$ , then by definition of capacity, there exist a sequence  $(N_k)$  tending to infinity and functions  $f_{N_k} \in \mathcal{B}_{p,q}^0$  such that  $f_{N_k}$  vanishes on  $\sigma_{N_k}$ ,  $f_{N_k}(0) = 1$  and  $\|f\|_{\mathcal{B}_{p,q}^0} \leq c$  for all  $k$ . Then  $(f_{N_k})_k$  is a normal family, so a subsequence converges locally uniformly on  $\mathbb{D}$  to a holomorphic function  $f$ . By Fatou's lemma,  $f \in \mathcal{B}_{p,q}^0$  with  $\|f\|_{\mathcal{B}_{p,q}^0} \leq c$ , and  $f$  vanishes on  $\sigma$  and  $f(0) = 1$ . Thus,  $\text{cap}(\sigma) \leq c$ .  $\square$

**Proof of Theorem 5.** Let  $2 < q < \infty$ . Then, for  $f \in \mathcal{B}_{\infty,\infty}^0 \cap \mathcal{B}_{2,2}^0$ , we have

$$\int_0^1 (1 - \rho)^{q-1} \|f'_\rho\|_{L^q}^q d\rho \leq \int_0^1 (1 - \rho)^{q-2} \|f'_\rho\|_{L^\infty}^{q-2} (1 - \rho) \|f'_\rho\|_{L^2}^2 d\rho \leq \|f\|_{\mathcal{B}_{\infty,\infty}^0}^{*q-2} \|f\|_{\mathcal{B}_{2,2}^0}^{*2}.$$

Recalling that  $\mathcal{B}_{2,2}^0 = H^2$  with equivalence of norms, we conclude that there exists a constant  $C > 0$  so that for any Blaschke product  $B$ , we have

$$\|B\|_{\mathcal{B}_{q,q}^0}^* \leq C^{1/q} \|B\|_{\mathcal{B}_{\infty,\infty}^0}^{*1-\frac{2}{q}}.$$

(This also follows from the Littlewood–Paley inequality.) Note that the inequality trivially holds for  $q = \infty$  as well.

Let  $\varepsilon > 0$ . It follows from a theorem of Aleksandrov, Anderson, and Nicolau [1, Theorem 2] that there exists an infinite Blaschke product  $B$  with  $\|B\|_{\mathcal{B}_{\infty,\infty}^0}^* \leq \varepsilon$ . By precomposing  $B$  with a conformal automorphism, we may assume that  $|B(0)| = \frac{1}{2}$ . By the preceding estimate with  $q = p'$ , we have  $\|B\|_{\mathcal{B}_{p',p'}^0}^* \leq C^{1/p'} \varepsilon^{1-\frac{2}{p'}}$ , so Lemma 6 implies that there exists a constant  $c > 0$  such that

$$\frac{1}{2} \text{cap}_{\mathcal{B}_{p,p}^0}(\sigma) \geq c\varepsilon^{\frac{2}{p'}-1}.$$

We distinguish two cases. If  $\text{cap}_{\mathcal{B}_{p,p}^0}(\sigma) < \infty$ , then Lemma 9 yields a finite subsequence  $\sigma' \subset \sigma$  such that  $\text{cap}_{\mathcal{B}_{p,p}^0}(\sigma') \geq \frac{1}{2} \text{cap}_{\mathcal{B}_{p,p}^0}(\sigma)$ . Note that  $\prod_{\lambda \in \sigma'} |\lambda| \geq \prod_{\lambda \in \sigma} |\lambda| = 1/2$ , so

$$\prod_{\lambda \in \sigma'} |\lambda| \cdot \text{cap}_{\mathcal{B}_{p,p}^0}(\sigma') \geq c\varepsilon^{\frac{2}{p'}-1}.$$

If  $\text{cap}_{\mathcal{B}_{p,p}^0}(\sigma) = \infty$ , then Lemma 9 directly yields a finite sequence  $\sigma' \subset \sigma$  satisfying the last inequality. Since  $\varepsilon > 0$  was arbitrary, the result follows.  $\square$

### 7. Proofs of the upper bounds

An obvious choice of a test function for an estimate of the capacity from above is  $f = B_\sigma/B_\sigma(0) = (-1)^N (\prod_{i=1}^N \lambda_i)^{-1} B_\sigma$ ,  $\sigma = (\lambda_1, \dots, \lambda_N)$ . Surprisingly, this function gives sharp estimates in many situations. To use  $f$  as a test function in the proof of Theorem 3 we need first to obtain some estimates for the norms of finite Blaschke products in Besov spaces which may be of independent interest.

#### 7.1. Estimates for the $\mathcal{B}_{p,q}^0$ norms of finite Blaschke products

**Proposition 10.** *Let  $(p, q) \in [1, \infty]^2$ ,  $\sigma = (\lambda_1, \dots, \lambda_N) \in \mathbb{D}^N$  and  $B = B_\sigma$ .*

1) *If  $(p, q) \in [1, 2]^2$ , then*

$$\|B\|_{\mathcal{B}_{p,q}^0} \lesssim (\log N)^{1/q-1/2}.$$

2) *If  $1 \leq p \leq q \leq \infty$  and  $q \geq 2$ , then*

$$\|B\|_{\mathcal{B}_{p,q}^0} \lesssim 1.$$

3) *If  $1 \leq q \leq p < \infty$  and  $p \geq 2$ , then*

$$\|B\|_{\mathcal{B}_{p,q}^0} \lesssim (\log N)^{1/q-1/p}.$$

4) *If  $1 \leq q < \infty$  and  $p = \infty$ , then*

$$\|B\|_{\mathcal{B}_{p,q}^0} \lesssim N^{1/q}.$$

The constants in the relations  $\lesssim$  depend only on  $p, q$ , but not on  $N$  and  $\sigma$ . Moreover, in all inequalities the dependence on the growth on  $N$  is best possible.

Note that there is an essential difference between the case  $p < \infty$  and  $p = \infty$ , where the growth is much faster.

In the proof of Proposition 10 we will need several simple estimates, the first of which can be found in [3]. We will give their proofs for the sake of completeness.

**Lemma 11.** *Let  $B$  be a Blaschke product of degree  $N$ . Then*

$$\int_0^1 (1 - \rho)^{p-1} \int_0^{2\pi} |B'(\rho e^{it})|^p dt d\rho \lesssim \begin{cases} (\log N)^{1-p/2}, & 1 \leq p \leq 2, \\ 1, & 2 \leq p < \infty, \end{cases} \tag{7.1}$$

and, for  $\rho \in [0, 1)$  and  $1 \leq p < \infty$ ,

$$\int_0^{2\pi} |B'(\rho e^{it})|^p dt \lesssim \frac{N}{(1 - \rho)^{p-1}}. \tag{7.2}$$

**Proof.** Since  $\int_0^{2\pi} |B'(\rho e^{it})| dt \leq 2\pi N$ ,  $\rho \in [0, 1]$ , we conclude that

$$\int_{1-1/N}^1 \int_0^{2\pi} |B'(\rho e^{it})| dt d\rho \lesssim 1.$$

Note that

$$\int_0^1 \rho(1 - \rho^2) \int_0^{2\pi} |f'(\rho e^{it})|^2 dt d\rho = \pi \sum_{n=1}^{\infty} \frac{n}{n+1} |a_n|^2 \leq \pi \|f\|_{H^2}^2 \leq \pi \|f\|_{H^\infty}^2 \tag{7.3}$$

for any function  $f(z) = \sum_{n \geq 0} a_n z^n$  in the Hardy space  $H^2$ . Therefore,

$$\begin{aligned} & \int_0^{1-1/N} \int_0^{2\pi} |B'(\rho e^{it})| dt d\rho \\ & \lesssim \left( \int_0^{1-1/N} \int_0^{2\pi} (1 - \rho) |B'(\rho e^{it})|^2 dt d\rho \right)^{1/2} \left( \int_0^{1-1/N} \int_0^{2\pi} \frac{dt d\rho}{1 - \rho} \right)^{1/2} \\ & \lesssim (\log N)^{1/2}. \end{aligned}$$

Thus, the inequality is already proved for  $p = 2$  (simply apply (7.3) to  $B$ ) and  $p = 1$ . For  $1 < p < 2$  inequality (7.1) follows from the Hölder inequality with exponents  $(p - 1)^{-1}$

and  $(2 - p)^{-1}$  (note that  $p = 2(p - 1) + 2 - p$ ) and the estimates for exponents 1 and 2. Finally, for  $p > 2$  it follows from the estimate  $(1 - |z|^2)|B'(z)| \leq 1$ ,  $z \in \mathbb{D}$ , that  $(1 - \rho)^{p-1} \|B'_\rho\|_{L^p(\mathbb{T})}^p \leq (1 - \rho) \|B'_\rho\|_{L^2(\mathbb{T})}^2$  and we can again apply (7.3).

The estimate (7.2) is obvious:

$$\int_0^{2\pi} |B'(\rho e^{it})|^p dt \lesssim \frac{1}{(1 - \rho)^{p-1}} \int_0^{2\pi} |B'(\rho e^{it})| dt \lesssim \frac{N}{(1 - \rho)^{p-1}}. \quad \square$$

**Proof of Proposition 10.** Let  $1 \leq p < \infty$ . Then it follows from (7.1) that

$$(\|B\|_{\mathcal{B}_{p,p}^0}^*)^p = 2\pi \int_0^1 (1 - \rho)^{p-1} \int_0^{2\pi} |B'(\rho e^{it})|^p dt d\rho \lesssim \begin{cases} (\log N)^{1-p/2}, & 1 \leq p \leq 2 \\ 1, & p \geq 2. \end{cases}$$

Also trivially  $\|B\|_{\mathcal{B}_{\infty,\infty}^0} \lesssim 1$ .

Now let  $1 \leq q \leq p < \infty$ . We write

$$\begin{aligned} \frac{1}{2\pi} (\|B\|_{\mathcal{B}_{p,q}^0}^*)^q &= \int_0^{1-1/N} (1 - \rho)^{q-1} \left( \int_0^{2\pi} |B'(\rho e^{it})|^p dt \right)^{q/p} d\rho \\ &+ \int_{1-1/N}^1 (1 - \rho)^{q-1} \left( \int_0^{2\pi} |B'(\rho e^{it})|^p dt \right)^{q/p} d\rho = I_1 + I_2. \end{aligned}$$

We show that  $I_2 \lesssim 1$ . Indeed, applying (7.2) we get

$$I_2 \lesssim N^{q/p} \int_{1-1/N}^1 (1 - \rho)^{q-1} (1 - \rho)^{-q(1-1/p)} d\rho = N^{q/p} \int_{1-1/N}^1 (1 - \rho)^{-1+q/p} d\rho \lesssim 1.$$

To estimate  $I_1$ , we apply the Hölder inequality with exponents  $p/q$  and  $p/(p - q)$  to get (with an obvious modification for  $p = q$ )

$$I_1 \leq \left( \int_0^{1-1/N} (1 - \rho)^{p-1} \int_0^{2\pi} |B'(\rho e^{it})|^p dt d\rho \right)^{\frac{q}{p}} \left( \int_0^{1-1/N} \frac{d\rho}{1 - \rho} \right)^{\frac{p-q}{p}}.$$

Hence, for  $1 < p \leq 2$ ,

$$I_1 \lesssim (\log N)^{\frac{q}{p}(1-\frac{p}{2})+\frac{p-q}{p}} = (\log N)^{1-\frac{q}{2}},$$

while for  $p > 2$

$$I_1 \lesssim (\log N)^{\frac{p-q}{p}}.$$

Thus, we have proved 3) and 1) for the case  $p \geq q$ .

If  $1 \leq p < q < \infty$  we simply have

$$\begin{aligned} \|B\|_{\mathcal{B}_{p,q}^0}^* &= \left( \int_0^1 (1-\rho)^{q-1} \|B'_\rho\|_{L^p(\mathbb{T})}^q d\rho \right)^{1/q} \\ &\lesssim \left( \int_0^1 (1-\rho)^{q-1} \|B'_\rho\|_{L^q(\mathbb{T})}^q d\rho \right)^{1/q} \lesssim \begin{cases} (\log N)^{1/q-1/2}, & 1 \leq q \leq 2, \\ 1, & q \geq 2. \end{cases} \end{aligned}$$

The case  $q = \infty$  is trivial by the Schwarz–Pick lemma. The proof of the statements 1)–3) is completed.

4) Consider the case  $p = \infty$ . If  $B(z) = \prod_{j=1}^N \frac{z-\lambda_j}{1-\bar{\lambda}_j z}$ , then

$$B'(z) = \sum_{j=1}^N \hat{B}_j(z) \frac{1-|\lambda_j|^2}{(1-\bar{\lambda}_j z)^2},$$

where  $\hat{B}_j(z) = \prod_{k \neq j} \frac{z-\lambda_k}{1-\bar{\lambda}_k z}$ . Hence,

$$\|B'_\rho\|_\infty \lesssim \sum_{j=1}^N \frac{1-|\lambda_j|^2}{(1-|\lambda_j|\rho)^2}$$

and, using again the fact that  $\|B'_\rho\|_\infty \leq (1-\rho)^{-1}$ , we get

$$(\|B\|_{\mathcal{B}_{\infty,q}^0}^*)^q = \int_0^1 (1-\rho)^{q-1} \|B'_\rho\|_\infty^q d\rho \leq \int_0^1 \|B'_\rho\|_\infty d\rho \lesssim N.$$

Let us show that all estimates are sharp. The growth  $(\log N)^{1/q-1/p}$  in the case 3) is achieved by the product  $B^*$  defined by (3.3). Indeed, for  $1 \leq q \leq p < \infty$ ,

$$\|B^*\|_{\mathcal{B}_{p,q}^0}^* \geq \prod_{i=1}^N |\lambda_i| \cdot \text{cap}_{\mathcal{B}_{p,q}^0}(\sigma^*) \gtrsim (\log N)^{1/q-1/p}$$

by Theorem 4. In the case 2) the optimality of the estimate can be already seen on  $B(z) = z^N$ .

For the case  $1 \leq p, q \leq 2$  one can use an example of a Blaschke product constructed in [3]: there exists a Blaschke product of order  $N$  such that

$$\int_0^{1-1/N} \int_0^{2\pi} |B'(\rho e^{it})| dt d\rho \geq c\sqrt{\log N},$$

where  $c > 0$  is an absolute constant; see the end of Section 2 in [3]. This construction is based on deep results of R. Bañuelos and C.N. Moore [2] related to Makarov’s law of the iterated logarithm. An easy application of the Hölder inequality shows that for  $1 \leq p, q \leq 2$

$$\int_0^{1-1/N} (1 - \rho)^{q-1} \|B'_\rho\|_{L^p(\mathbb{T})}^q d\rho \gtrsim (\log N)^{1-q/2}.$$

Finally, let us show that the estimate in the case 4) also is best possible. Take  $\lambda_j = 1 - 2^{-j}$ . Since the sequence  $(\lambda_j)$  is an interpolating sequence for  $H^\infty$ , there exists  $\delta > 0$  such that

$$\prod_{k \neq j} \left| \frac{\lambda_k - \lambda_j}{1 - \overline{\lambda_k} \lambda_j} \right| \geq \delta$$

for all  $j$ . Thus, if  $B$  denotes the Blaschke product with zeros  $\lambda_1, \dots, \lambda_N$ , then

$$|B'(\lambda_j)| \geq \frac{\delta}{1 - |\lambda_j|^2} \gtrsim 2^j$$

for  $j = 1, \dots, N$ . It follows that  $\|B'_\rho\|_\infty \geq 2^j$  for  $\rho \geq \lambda_j$  and thus

$$\|B\|_{\mathcal{B}_{\infty,q}^0}^q \gtrsim \sum_{j=1}^{N-1} \int_{\lambda_j}^{\lambda_{j+1}} (1 - \rho)^{q-1} \|B'_\rho\|_\infty^q d\rho \gtrsim N. \quad \square$$

### 7.2. Proof of Theorem 3

Consider the simplest test function

$$f = \frac{B_\sigma}{B_\sigma(0)} = (-1)^N \frac{B_\sigma}{\prod_{i=1}^N \lambda_i}.$$

Then

$$\text{cap}_{\mathcal{B}_{p,q}^0}(\sigma) \leq \frac{\|B_\sigma\|_{\mathcal{B}_{p,q}^0}}{\prod_{i=1}^N |\lambda_i|},$$

and the statement follows from Proposition 10 in all cases except  $p = \infty$  (in Region III), where an application of Proposition 10 will lead to a substantially worse growth order.



To treat the case  $p = \infty$  we use the test function from [10]. Put  $r = 1 - 1/N$  and consider the finite Blaschke product  $\tilde{B}$  with zeros  $r\lambda_1, \dots, r\lambda_N$ ,

$$\tilde{B}(z) = \prod_{i=1}^N \frac{z - r\lambda_i}{1 - r\overline{\lambda_i}z}.$$

Let

$$f(z) = (-1)^N \frac{\tilde{B}(rz)}{r^N \prod_{i=1}^N \lambda_i}.$$

Clearly  $f$  satisfies  $f(0) = 1$  and  $f(\lambda_i) = 0$  for  $i = 1, \dots, n$ . Since  $r^N \asymp 1$ , we have

$$\prod_{i=1}^N |\lambda_i| \cdot |f'(\rho e^{it})| \lesssim |\tilde{B}'(r\rho e^{it})| \lesssim \frac{1}{1 - r\rho}.$$

Then, for  $1 \leq q < \infty$ ,

$$\left( \prod_{i=1}^N |\lambda_i| \right)^q \cdot \|f\|_{\mathcal{B}_{\infty, q}^0}^q \lesssim \int_0^{1-1/N} \frac{d\rho}{1 - \rho} + \int_{1-1/N}^1 \frac{d\rho}{1 - r} \lesssim \log N.$$

Theorem 3 is proved.  $\square$

### Data availability

No data was used for the research described in the article.

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