Free Probability Meets Topological Recursion

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Abstract

The subject of this thesis lies at the intersection of free probability, Hurwitz theory and topological recursion. It is based on the following collaborations:

[BCGF$^+$23] Higher order free probability vs. topological recursion: We derive functional relations for moment and cumulant generating series in higher order free probability. This solves an open problem posed about 15 years ago. We extend the combinatorics of higher order freeness and propose a new extension of free probability theory. Our results led also to new insights in topological recursion.

[HvIL22] Quantum curves and monotone Hurwitz numbers: We introduce monotone and strictly monotone Hurwitz numbers over an arbitrary base curve and derive a differential equation for their partition function, and we study the semiclassical limit of the quantum curve. Surprisingly, we recover the spectral curve for the Möbius function of higher order free probability. The latter emphasizes the relation between the two subjects and led to the breakthrough in [BCGF$^+$23].

Abstrakt

Das Thema dieser Arbeit ist in den Gebieten der freien Wahrscheinlichkeitstheorie, Hurwitz Theorie und topologischer Rekursion angesiedelt. Sie basiert auf den folgenden Kollaborationen:

[BCGF⁺23] Freie Wahrscheinlichkeit in höherer Ordnung und topologische Rekursion: Wir leiten funktionale Relationen für Momenten und Kumulanten erzeugende Funktionen in der freien Wahrscheinlichkeitstheorie in höheren Ordnung her. Dies löst ein seit ca. 15 Jahren offenes Problem. Wir formulieren eine Erweiterung der freien Unabhängigkeit und ihrer Kombinatorik. Unsere Resultate führen zu neuen Erkenntnissen in der topologischen Rekursion.

[HvIL22] Quantenkurven und monotone Hurwitz-Zahlen: Wir führen Hurwitz Zahlen über Kurven von höherem Geschlecht ein und leiten eine Differentialgleichung für ihre Partitionsfunktionen her und bestimmen den semiclassical limit der Quantenkurve. Überraschenderweise finden wir eine spektrale Kurve für die Möbius Funktion der freien Wahrscheinlichkeit in höherer Ordnung. Dieses Resultat bekräftigt den Zusammenhang der Gebiete und führte zu den Untersuchungen in [BCGF⁺23].

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Contents

Introduction

This thesis summarizes results obtained by the author within the scope of the two projects

i) Random matrices and Hurwitz numbers,

ii) Topological recursion and free probability.

Those projects constitute the projects A14 of the SFB-TRR 195 "Symbolic tools in mathematics and their applications", project i) during the first and ii) during the second funding period. The subject of this thesis sits at the intersection of the areas free probability, Hurwitz theory and topological recursion. We start by providing background on each topic and afterward explain the results of the author.

Free probability

Free probability was introduced by D. V. Voiculescu in the '80s with the intention to tackle the famous isomorphism problem of the free group factors [Voi85]. It constitutes a noncommutative analogue of classical probability theory and thus allows a probabilistic study of noncommuting random variables. A main feature in classical probability is the notion of independence, and free probability admits an analogue called *free independence*. Despite that free independence is quite different from the classical notion, the theories show a lot of similarities.

Since then, free probability gained traction from various other fields such as mathematical physics and combinatorics. It even found interest in applied sciences such as the study of wireless networks. Most notable is the relation to random matrices discovered by Voiculescu in [Voi91]. He showed that if $X_N = (X_N^{(1)}, \dots, X_N^{(d))})$ are unitarily invariant random matrices in a general position, then the limit of the joint moments

$$\varphi\bigg(\Big(X_N^{(i_1)}\Big)^{k_1} \dots \Big(X_N^{(i_r)}\Big)^{k_r}\bigg) = \lim_{N\to\infty} \mathbb{E}\bigg[\operatorname{tr}\bigg(\Big(X_N^{(i_1)}\Big)^{k_1} \dots \Big(X_N^{(i_r)}\Big)^{k_r}\bigg)\bigg]$$

can be computed from the limits of the moments of individual matrices $\mathbb{E}[\operatorname{tr}(X_N^{(i)})]$. This computation rule is precisely the notion of free independence in free probability theory.

A *noncommutative probability space* $(\mathcal{A}, \varphi)$ consists of a unital algebra $\mathcal{A}$ over $\mathbb{C}$ and a unital linear functional $\varphi\colon \mathcal{A} \to \mathbb{C}$. By Voiculescu's discoveries, we may think of the algebra as the limit of the random matrix ensemble and the functional to be the limit of $\mathbb{E} \circ \operatorname{tr}_N$. The notion of freeness is defined as follows. Let $(\mathcal{A}, \varphi)$ be a noncommutative probability space and $\mathcal{A}_1, \mathcal{A}_2 \subseteq \mathcal{A}$ unital subalgebras. We say $\mathcal{A}_1$ and $\mathcal{A}_2$ are *freely* *independent* or simply *free*, if whenever we take $n \in \mathbb{N}$ and elements $a_1, \ldots, a_n \in \mathcal{A}_1$ and $a'_1, \ldots, a'_n \in \mathcal{A}_2$ with $\varphi(a_i) = 0 = \varphi(a'_i)$ for $i = 1, \ldots, n$, then we have

$$\varphi(a_1 a'_1 \ldots a_n a'_n) = 0.$$

Loosely speaking, the expectation of an alternating product of centered elements vanishes. The notion for the case of more than two algebras is defined similarly, i.e. the expectation of a product of centered elements vanishes if elements have only neighbours from different algebras. Thus, freeness allows for an effective computation of the distribution of, for example, the sum or the product of two random variables which are free from each other. In the case where the random variables admit a distribution given by probability measures μ, ν, this corresponds to the *free additive convolution* $\mu \boxplus \nu$ ([Voi86]) and *free multiplicative convolution* $\mu \boxtimes \nu$ ([Voi87]). These are analogues of the convolution $\mu * \nu$ in classical probability theory.

The main results of this thesis are formulas that generalize the relation between *Voiculescu's $\mathcal{R}$-transform* and the *Cauchy transform* [Voi86]. Consider a probability distribution μ of a random variable, then its Cauchy transform is the analytic function

$$G_\mu \colon \mathbb{C}^+ \to \mathbb{C}^-, \quad G_\mu(z) = \int_{\mathbb{R}} \frac{\mathrm{d}\mu(t)}{z - t}.$$

If μ is compactly supported, it admits an expansion at ∞:

$$G_\mu(z) = \sum_{n=0}^{\infty} \frac{m_n}{z^{n+1}}, \quad m_n = \int_{\mathbb{R}} t^n \mathrm{d}\mu(t). \tag{0.0.1}$$

Note that the coefficients are precisely the moments of the probability measure μ, i.e. G_μ is a generating function of the moments of μ. Then Voiculescu introduced the $\mathcal{R}$-transform, it is determined by the *$\mathcal{R}$-transform formula*

$$\mathcal{R}_\mu(G_\mu(z)) + \frac{1}{G_\mu(z)} = z. \tag{0.0.2}$$

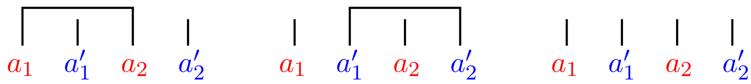
He proved the important fact that the $\mathcal{R}$-transform linearizes the free additive convolution

$$\mathcal{R}_{\mu \boxplus \nu}(z) = \mathcal{R}_\mu(z) + \mathcal{R}_\nu(z).$$

Thus it does not only provide an effective tool to compute the distribution of the sum of two free random variables, it is also the noncommutative analogue of the logarithm of the Fourier-transform which linearizes $\mu * \nu$.

In 1994 R. Speicher discovered that many of Voiculescu's results can be recovered from a purely combinatorial standpoint [Spe94]. Speicher introduced the moment-cumulant formalism for free probability via multiplicative functions on the lattice of noncrossing partitions. First, note the combinatorial flavour of freeness by considering the following example. Let $\{a_1, a_2\}$ be free from $\{a'_1, a'_2\}$. Then they satisfy the following relation:

$$\varphi(a_1 a'_1 a_2 a'_2) = \varphi(a_1 a_2)\varphi(a'_1)\varphi(a'_2) + \varphi(a_1)\varphi(a_2)\varphi(a'_1 a'_2) - \varphi(a_1)\varphi(a_2)\varphi(a'_1)\varphi(a'_2).$$

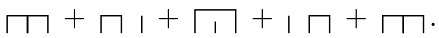


Example of the free computation rule.

Note that on the right-hand side there is no term of the form $\varphi(a_1 a_2)\varphi(a'_1 a'_2)$, since the corresponding partition has a crossing. Speicher put free probability theory in the framework of the incidence algebra formalism of Rota and developed a theory of cumulants in analogy to the classical case. One can interpret the functional φ as a multiplicative function on the lattice of noncrossing partitions and define the *free cumulants* κ_n via convolution with the Möbius function of the lattice. The relations of this convolution can be written down explicitly:

$$\begin{aligned} \varphi(a_1) &= \kappa_1(a_1) \\ \varphi(a_1 a_2) &= \kappa_2(a_1, a_2) + \kappa_1(a_1)\kappa_1(a_2) \\ \varphi(a_1 a_2 a_3) &= \kappa_3(a_1, a_2, a_3) + \kappa_2(a_1, a_2)\kappa_1(a_3) + \kappa_2(a_1, a_3)\kappa_1(a_2) \\ &\quad + \kappa_2(a_2, a_3)\kappa_1(a_1) + \kappa_1(a_1)\kappa_1(a_2)\kappa_1(a_3) \\ &\vdots \end{aligned} \tag{0.0.3}$$

The formulas in (0.0.3) actually express $\varphi(a_1 \dots a_n)$ as sums over noncrossing partitions on n points. E.g., for $n = 3$, the terms on the right-hand side of (0.0.3) correspond to



One of the main features of $\kappa_n \colon \mathcal{A}^n \to \mathbb{C}$ is the fact that they capture the notion of freeness particularly well. If two subalgebras $\mathcal{A}_1, \mathcal{A}_2$ are free, then $\kappa_n(a_1, \dots, a_n)$ vanishes if not all a_i come from the same algebra. This fact remains true for more than two algebras free from each other. The latter is called the *vanishing of mixed cumulants*. Another important discovery is that the combinatorial moment-cumulant relations are equivalent to Voiculescu's $\mathcal{R}$-transform formulas. Let $a \in \mathcal{A}$ be a noncommutative random variable. If we define the formal power series

$$G(x) = \frac{1}{x} + \sum_{n=1}^{\infty} \frac{\varphi(a^n)}{x^{n+1}}, \quad \mathcal{R}(x) = 1 + \sum_{n=1}^{\infty} \kappa_n(a, \dots, a) x^{n-1},$$

then the relations (0.0.3) are equivalent to

$$\mathcal{R}(G(x)) + \frac{1}{G(x)} = x.$$

Note, that by this observation, Speicher gave Voiculescu's $\mathcal{R}$-transform the meaning of a generating function similar to (0.0.1). Further, note that the vanishing of mixed cumulants correspond to the fact that the $\mathcal{R}$-transform linearizes the free convolution.

Higher order free probability

In a series of papers [MS06, MŚS07, CMSS07], B. Collins, J. Mingo, P. Śniady and R. Speicher extended free probability to second and higher order free probability theory. Their goal was to study more refined questions in random matrix theory. For example, second order free probability describes the fluctuations of the eigenvalues around their limit. The information regarding the fluctuations is described by the asymptotic behaviour of the covariances

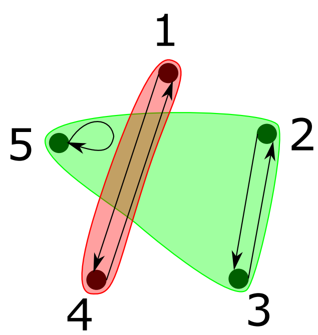
$$\operatorname{cov}(\operatorname{Tr}(X_N^{r_1}), \operatorname{Tr}(X_N^{r_2})), \tag{0.0.4}$$

where Tr is the unnormalized trace. They add an abstract framework to the random matrix perspective, extending the theory of Voiculescu. For example, they introduce a free moment-cumulant formalism and a notion of freeness, which captures the limiting behaviour of the quantities (0.0.4). This extended theory is called *second order free probability*. More generally, they introduce n-th order contributions by the asymptotic behaviour of

$$\mathrm{k}_n(\operatorname{Tr}(X_N^{r_1}), \ldots, \operatorname{Tr}(X_N^{r_n})),$$

where k_n are classical cumulants. Thus, the notion of a *higher order noncommutative probability* space $(\mathcal{A}, \boldsymbol{\varphi})$ consists of a unital algebra $\mathcal{A}$ and a family $\boldsymbol{\varphi} = (\varphi_n)_{n=1}^{\infty}$ of functions $\varphi_n \colon \mathcal{A}^n \to \mathbb{C}$, these correspond to the limits of the higher order classical cumulants of traces.

Similarly to the work [Spe94], Collins, Mingo, Śniady and Speicher develop a combinatorial formalism by multiplicative functions. The functions are not just defined on the set of noncrossing partitions, but on more complicated objects called partitioned permutations. As the name indicates, the latter are tuples $(\mathcal{V}, \pi)$, consisting of a partition $\mathcal{V}$ and a permutation $\pi \in S(d)$ such that the cycles of π are partitioned by $\mathcal{V}$.



Visualization of the partitioned permutation $(\{\{1,4\},\{2,3,5\}\},(1,4)(2,3)(5))$.

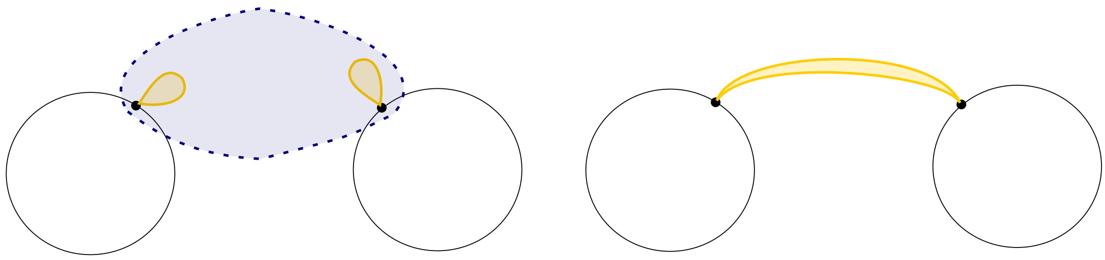
Despite not being a lattice, the set of partitioned permutations admits a theory of multiplicative functions similar to the noncrossing partitions. This setting is indeed a generalization of [Spe94] and recovers the first order theory. The *higher order free* *cumulants* are functions

$$\kappa_{r_1,\dots,r_n} \colon \mathcal{A}^{r_1} \times \mathcal{A}^{r_2} \times \cdots \times \mathcal{A}^{r_n} \to \mathbb{C}.$$

We can state the moment-cumulant relations explicitly:

$$\begin{aligned}
\varphi_{1,1}[a_1; a_3] &\coloneqq \varphi_2(a_1, a_2) = \kappa_{1,1}(a_1; a_2) + \kappa_2(a_1, a_2) \\
\varphi_{2,1}[a_1, a_2; a_3] &\coloneqq \varphi_2(a_1 a_2, a_3) = \kappa_{2,1}(a_1, a_2; a_3) + \kappa_1(a_1)\kappa_{1,1}(a_2, a_3) + \kappa_1(a_2)\kappa_{1,1}(a_1, a_3) \\
&\qquad + \kappa_1(a_1)\kappa_2(a_2, a_3) + \kappa_1(a_2)\kappa_2(a_1, a_3) + 2\kappa_3(a_1, a_2, a_3) \\
&\vdots
\end{aligned}$$

Similar to first order, the right-hand side can be seen to be a sum over certain planar diagrams. In order n, they are given by connecting n circles with each other. Unlike in first order, the diagrams have two different kinds of possible connections: One corresponding to the partition part and one corresponding to the permutation part of a partitioned permutation.



Graphical representation of $\kappa_{1,1}$ (left) and κ_2 (right).

One of the main results of [CMSS07] is an analogue of the functional relation for the generating functions between second order moments and cumulants. For $a \in \mathcal{A}$, we define

$$G(x_1, x_2) = \sum_{r_1, r_2 = 1}^{\infty} m_{k_1,k_2} \frac{1}{x_1^{r_1}} \frac{1}{x_2^{r_2}}, \qquad R(x_1, x_2) = \sum_{r_1, r_2 =}^{\infty} \kappa_{r_1,r_2} x_1^{r_1 - 1} x_2^{r_2 - 1}$$

where

$$m_{r_1,r_2} = \varphi_2(a^{r_1}, a^{r_2}), \text{ and } \kappa_{r_1,r_2} = \kappa_{r_1,r_2}(\underbrace{a, \dots, a}_{r_1}; \underbrace{a, \dots, a}_{r_2}),$$

then it holds

$$G(x_1, x_2) = G'(x_1)G'(x_2)\left(\mathcal{R}(G(x_1), G(x_2)) + \frac{1}{(G(x_1) - G(x_2))^2}\right) - \frac{1}{(x_1 - x_2)^2}.$$

Although the combinatorial description for higher order free cumulants is well-developed, the authors of [CMSS07] have not been able to find analogous functional relations beyond

$n = 2$. The reason is that the proof of such relations relies on a tedious case by case analysis of combinatorial diagrams, and their complexity increases with the number of circles. Still, a theory of higher order freeness is developed in [CMSS07], it is defined by the vanishing of mixed higher cumulants. Further, it is proven that this new notion of free independence is sensible: Elements in $\mathbb{C} \cong 1_{\mathcal{A}}\mathbb{C}$ are free from everything and freeness does not depend on the choice of generators.

The main results of [BCGF$^+$23] presented in this thesis are the derivation of the missing functional relations for $n \geq 3$ and a non-planar extension of higher order free probability theory of [CMSS07].

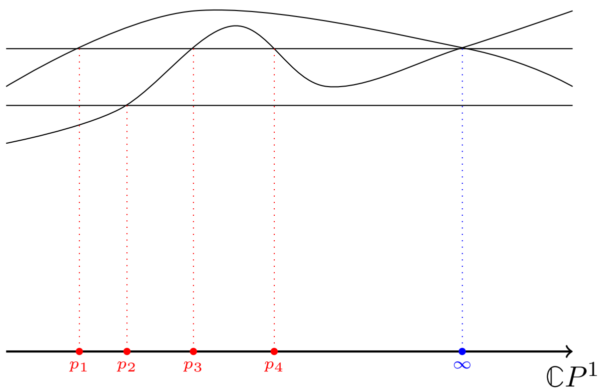
Relating the combinatorics of higher order free probability with monotone Hurwitz numbers was crucial for the success of [BCGF$^+$23].

Monotone Hurwitz Numbers

Hurwitz numbers have been studied first by A. Hurwitz [Hur01]. They count the number of ramified coverings of the Riemann sphere $\mathbb{C}P^1$. Hurwitz already discovered that the covers can be described by factorizations in the symmetric group. A ramified genus g cover of degree d with ramification point over ∞ and only simple ramification points elsewhere is given by a factorization

$$\tau_1 \dots \tau_r \sigma = \mathrm{id}$$

in the symmetric group $S(d)$. The τ_i are transpositions for $i = 1, \dots, r$ and describe the simple ramification points where two sheets meet. The number $r = 2g-2+n+d$ is given by the Riemann-Hurwitz formula. On the other hand, the cycle type of σ determines the ramification at infinity, i.e. how the d sheets meet above ∞. The past two decades have seen numerous discoveries linking Hurwitz numbers to diverse disciplines in mathematics, including operator theory, integrable systems, random matrix models, tropical geometry, and many more.



Schematic illustration of a ramified cover of degree 4:
Over ∞ three sheets meet, which is not a simple ramification;
Over p_1, p_2, p_3, p_4 two sheets meet, these are simple ramification points.

For this thesis the most important type of Hurwitz numbers are the monotone Hurwitz numbers, which count factorizations such that the τ_i satisfy an additional monotonicity condition. These types of Hurwitz numbers have been discovered as coefficients in the expansion of the Harish-Chandra-Itzykson-Zuber-Integral integral by I. P. Goulden, M. Guay-Paquet and J. Novak [GGPN13, GGPN14]. They are also related to the Weingarten function of [Col03], which was used to motivate the introduction of higher order free cumulants in [CMSS07, Theorem 4.4]. This relation has been rediscovered by G. Borot and E. Garcia-Failde in the study of matrix models for ordinary and fully simple maps [BGF20] in the context of topological recursion.

In general, Hurwitz numbers are an active research topic in topological recursion. Many kinds of Hurwitz numbers satisfy the topological recursion [BEMS11, DLN12, DDM17, BKW23]. Within the scope of the first project "Random matrices and Hurwitz numbers" of the SFB TRR-195 the author studied generalizations of monotone Hurwitz numbers and their relation to topological recursion and free probability leading to the publication [HvIL22]. There the authors noticed that the Möbius function of higher order free probability is given by alternating monotone Hurwitz numbers and satisfy topological recursion. During this time G. Borot, S. Charbonnier, N. Do and E. Garcia-Failde [BCDGF19] proved that the generating functions of ordinary and fully simple maps are related via monotone Hurwitz numbers. These new insights drew the attention of the author of this thesis more towards the study of the connection between the moment-cumulant formalism in free probability and the $x - y$ duality in topological recursion.

Topological recursion

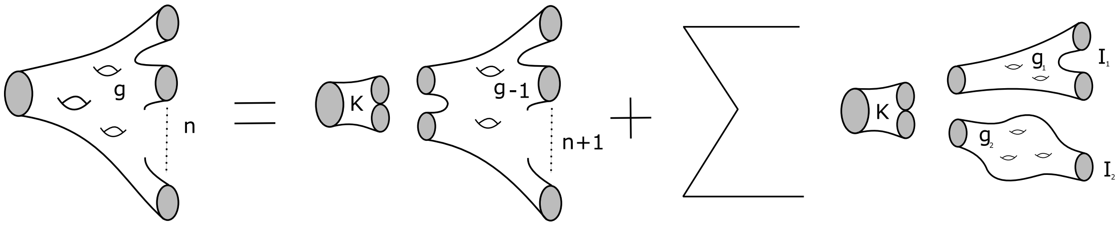
The Chekhov-Eynard-Oratin (CEO) topological recursion is a recursive procedure that produces meromorphic n-forms $\omega_{g,n}$ on a Riemann surface. The input data (Σ, x, y, B) consists of

i) a Riemann surface Σ,

ii) meromorphic functions $x, y \colon \Sigma \to \mathbb{C}$ with prescribed pole behaviour.

iii) a bi-differential B on $\Sigma \times \Sigma$ with prescribed pole behaviour.

From this input data, one defines the initial values $\omega_{0,1} = y\mathrm{d}x$ and $\omega_{0,2} = B$, then the topological recursion computes $\omega_{g,n}$ by recursion on $-\chi = 2g - 2 + n$. We do not state the explicit formula here, but let us note that it can be schematically represented by topological surfaces. The differential form $\omega_{g,n}$ is represented by a surface of genus g and n boundaries. It is computed by terms represented by smaller negative Euler characteristic $-\chi$ and by gluing pairs of pants to the latter.



Schematic representation of topological recursion.

This formula was first discovered in random matrix theory [CEO06] and later formulated as an independent theory [EO07]. Surprisingly, the same recursive formula has been discovered to compute different invariants from various areas in mathematics and mathematical physics. Some examples are Hurwitz numbers [BM07, BEMS11], Gromov-Witten invariants [EO15] and knot invariants [BE15, DBPSS17]. Furthermore, the invariants $\omega_{g,n}$ satisfy a lot of nice properties [Eyn16]. The most interesting one for us is called symplectic invariance. Two spectral curves (Σ, x, y, B), $(\widetilde{\Sigma}, \widetilde{x}, \widetilde{y}, \widetilde{B})$ are called symplectically equivalent if there is a map $(x, y) \mapsto (\widetilde{x}, \widetilde{y})$ compatible with the initial data, such that $\mathrm{d}x \wedge \mathrm{d}y = \mathrm{d}\widetilde{x} \wedge \mathrm{d}\widetilde{y}$. In this case, it is expected that the invariants of both curves are related to each other. The most interesting transformation has been the so-called $x-y$ swap that interchanges the roles of x and y, that is $\widetilde{x} = y$ and $\widetilde{y} = x$ (see [EO13]). This transformation has been intensely studied by G. Borot and E. Garcia-Failde [BGF20]. They discovered a combinatorial example of the $x-y$ duality that allowed for deeper insights into this mysterious transformation. More precisely, they discovered that the $x-y$ transformation of the spectral curve of ordinary maps computes another type of maps, called fully simple maps. In addition, they found a remarkable relation between the generating functions of ordinary and fully simple maps. Let us denote by $W_1(x)$ and $X_1(w)$ the generating series of ordinary and fully simple discs and by $W_2(x_1, x_2)$ and $X_2(w_1, w_2)$ the generating series of ordinary and fully simple cylinders. Then

$$W_1(X_1(x)) = x$$

and

$$W_2(x_1, x_2) = W'(x_1)W'(x_2)\Bigg(X_2(W(x_1), W(x_2)) + \frac{1}{(W(x_1) - W(x_2))^2}\Bigg) - \frac{1}{(x_1 - x_2)^2},$$

These are exactly the relations of Voiculescu and Speicher and of Collins, Mingo, Śniady, Speicher in free probability. These developments motivated the author to shift his focus from "Random matrices and Hurwitz numbers", towards the connection of free probability to topological recursion. The goal of the project "Topological recursion and free probability" within the SFB TRR-195 was to establish a concrete connection of the moment-cumulant formalism of free probability and the $x-y$ duality of topological recursion. This was achieved in collaboration with researchers from topological recursion in the main publication [BCGF$^+$23].

Outline of the main results

Free probability and topological recursion

Chapter 2 is the main part of this thesis, and explains the derivation of the higher order functional relations between the moment and cumulant generating functions. This solves an open problem posed in [CMSS07]. The chapter is based on the results of the author of this thesis and his coauthors in [BCGF$^+$23].

We start the chapter by explaining higher order freeness and its origin in [CMSS07]. In particular, we explain the combinatorics of partitioned permutations and the theory of multiplicative functions on the set of partitioned permutations. We conclude Section 2.1 with a discussion of the proof of [CMSS07] for the second order functional relations. Afterward, in Section 2.4, we explain the obstacles of deriving the functional relations beyond second order. In Section 2.2, we explain the extension of the original theory to a higher genus. This idea for this first step is based on discussions between J. Mingo and the author of this thesis, during a stay in Montreal within the scope of the program "New Developments in Free Probability and Applications" at CRM. The idea was to add higher genus contributions to the framework of [CMSS07], by allowing non-planar contributions into the combinatorics of partitioned permutations. The key point here is to remove the planarity condition in the product of partitioned permutations. However, in this setting, it is important to keep track of the non-planar contributions when dealing with multiplicative functions. This is done by extending the range of the functions from $\mathbb{C}$ to $\mathbb{C}[\![\hbar]\!]$. The formal parameter $\hbar$ controls the higher genus contributions. This extended theory still evolves alongside the first order theory of [Spe94]. In particular, we can define extended cumulant functions via convolution, these also take values in $\mathbb{C}[\![\hbar]\!]$ now. At the same time, the author studied certain Hurwitz numbers within the scope of the publication [HvIL22]. During this project, he realized that the values of the extended Möbius function agree with signed monotone Hurwitz numbers, this fact is discussed in Section 2.2.1. This link put even more emphasis on the connection of free probability and topological recursion.

Meanwhile, Borot and Garcia-Failde studied ordinary and fully simple maps to explore the $x - y$ duality in topological recursion. They discovered that their generating series satisfy the same functional relations as the moments and cumulants in second order free probability [BGF20]. Then in [BCDGF19] it has been discovered that the generating series of ordinary and fully simple maps with prescribed boundary conditions are related via monotone Hurwitz numbers,

$$\mathrm{Map}(\lambda) = \sum_{\mu \vdash |\lambda|} H^{<}(\lambda, \mu)\mathrm{FSMap}(\mu). \tag{0.0.5}$$

These observations together with the results of [HvIL22] lead the author of this thesis to prove that an analogue of (0.0.5) in the extended higher order free probability setting is equivalent to the higher genus moment-cumulant relations. A discussion during the workshop "Noncommutative geometry meets topological recursion" in Münster regarding this result was the starting point of the collaboration [BCGF$^+$23]. We call (0.0.5) the *master relation* and show that it can be reformulated in multiple ways. We discuss its avatars in Section 2.5.1. In particular, it has a nice description in the Fock space formalism of Bychkov, Dunin-Barkowski, Kazarian and Shadrin in the earlier papers [BDBKS22, BDBKS23]. S. Shadrin is part of the collaboration [BCGF$^+$23] bringing this expertise to the project. Concretely, the master relation can be formulated as an operator equation for partition functions in the bosonic Fock space, involving an operator D which is related to Hurwitz numbers. We discuss the relation between multiplicative functions on the set of partitioned permutations and partition functions in the bosonic Fock space in Section 2.3. The following theorem is a refined and extended version of the original results of the author.

Theorem (Theorem 2.5.2, [BCGF$^+$23])**.**
Consider two topological partition functions Z^φ, Z^κ (or equivalently multiplicative functions φ, κ) and $d \in \mathbb{N}$. The following four properties are equivalent:

i) $Z^\varphi(\lambda) = z_\lambda \sum_{\nu \vdash d} H^{<}(\lambda, \nu) Z^\kappa(\nu)$ holds for any $\lambda \vdash d$;

ii) $\varphi = \zeta_\hbar \circledast \kappa$ holds between functions on $\mathcal{PS}(d)$;

iii) $Z^\kappa(\nu) = z_\nu \sum_{\lambda \vdash d} H^{\leq}(\nu, \lambda) Z^\varphi(\lambda)$ holds for any $\nu \vdash d$;

iv) $\kappa = \mu_\hbar \circledast \varphi$ holds between functions on $\mathcal{PS}(d)$.

Besides, the property $Z^\varphi = \mathsf{D} Z^\kappa$ is equivalent to any of these conditions simultaneously for all $d > 0$.

This is one key element in the derivation of the functional relations of higher order free probability and in understanding the relation between topological recursion and free probability.

Let us now state the functional relations that answer the problem posed in [CMSS07] about 15 years ago.

Theorem (Theorem 2.4.1)**.**
Let $\varphi, \kappa \colon \mathcal{PS} \to \mathbb{C}$ be multiplicative functions on $\mathcal{PS}$ with values in $\mathbb{C}$, satisfying the moment-cumulant relations $\varphi = \zeta * \kappa$. Then under the change of variables $x_i = w_i / C(w_i)$ and for $n \geq 3$, we have:

$$M_n(x_1, \ldots, x_n) = \sum_{s_1, \ldots, s_n \geq 0} \sum_{T \in \mathcal{G}_{0,n}(\mathbf{s}+1)} \prod_{i=1}^{n} \vec{\mathsf{O}}^\vee_{r_i}(w_i) \prod_{I \in \mathcal{I}(T)}' C_{\# I}(w_I).$$

The formula expresses the moment generating function $M(x_1, \ldots, x_n)$ as a sum over bicoloured labelled trees $\mathcal{G}_{0,n}$ with prescribed valencies. The coloured vertices correspond to operator weights that are applied to the cumulant generating functions $C(x_1, \ldots, x_k)$ with $k \leq n$. We explain the formula in detail and provide examples in Section 2.4. It is also important to note that this formula only includes genus zero, and therefore solves the problem posed in [CMSS07]. Actually, it is a special case of a more general formula including higher genus contributions. Thus, we do not only solve the problem posed by Collins, Mingo, Śniady and Speicher, but also the extension of the problem in higher genus. The latter is discussed in Section 2.4 too. The relations are derived via techniques developed in [BDBKS22] and the calculations we present in Section 2.5 are merely rewriting the proof of [BDBKS23]. The derivation of the formula in Theorem 2.4.1 is split into three parts. First, we prove Theorem 2.5.2 in Section 2.5.1. From there we proceed to present the techniques of [BDBKS22, BDBKS23] to obtain the general formula including higher genus in Section 2.5.2. We conclude by extracting the genus zero part of the general formula to obtain Equation (2.7.6) in Section 2.5.3.

In analogy to [CMSS07], the new extended functional relation and the extended multiplication for partitioned permutations can be used to introduce a more general version of free probability theory. We call it *surfaced free probability*; it is inspired by [CMSS07, Appendix 9]. The combinatorial framework is given by *surfaced permutations*, these are partitioned permutations endowed with a genus. All the notions of [CMSS07] can be extended. Naturally, the introduction of new cumulants comes with a notion of freeness via vanishing of mixed free cumulants. We call the latter (g, n)-freeness and show that this definition is reasonable:

- In Section 2.6.4 we show that constants are free from everything and that the notion does not depend on the choice of generators, i.e. freeness of sets carries over to the algebras generated by the sets.
- In Section 2.6.5 we prove (∞, ∞)-asymptotic freeness of two independent ensembles of random matrices, one of which is unitarily invariant.
- Our $(0, 1)$-freeness recovers Voiculescu's free independence and $(0, \infty)$-freeness recovers freeness of all orders; furthermore, if we allow half integer genus, surprisingly $(\frac{1}{2}, 1)$-freeness retrieves the notion of infinitesimal freeness of [FN10] and [BS12]. The latter is discussed in Section 2.6.3.

Finally, we reformulate our main formula in the language of differentials and propose that these relations explain the conjectured relation between the invariants of two $x-y$ symplectically equivalent spectral curves. This is formulated in Conjecture 2.7.2 in Section 2.7.

Conjecture (Conjecture 2.7.2)**.**
Let $\mathcal{C}$ be a compact Riemann surface, x, w are meromorphic functions on $\mathcal{C}$ such that $\mathrm{d}x$ and $\mathrm{d}w$ do not have common zeroes, and $\mathcal{B}$ is a fundamental bidifferential of the second kind. We call $\omega_{g,n}$ the differentials obtained from the topological recursion with the spectral curve $(\mathcal{C}, x, w, \mathcal{B})$, and $\omega^\vee_{g,n}$ the ones associated to the spectral curve $(\mathcal{C}, w, x, \mathcal{B})$, and define $\widetilde{\omega}_{0,2} = \widetilde{\omega}^\vee_{0,2} = \mathcal{B}$. Then, these differentials will satisfy for all $2g - 2 + n \geq 0$ the functional relations of Theorem 2.4.8 (after they are converted to relations between meromorphic differentials on $\mathcal{C}$).

The latter sheds light on the previously not fully understood $x-y$ duality in topological recursion. In the meantime, the conjecture has been proven in [ABDB$^+$22] by S. Shadrin and coauthors.

Monotone Hurwitz numbers and topological recursion

The second part of the thesis, Chapter 3, summarizes the results of the author while working on the project "Random matrices and Hurwitz numbers" of the SFB TRR-195. Although presented as the last chapter of this thesis, it is chronologically prior to the results of Chapter 2. In this first phase, the author studied the relation between Hurwitz numbers and topological recursion. The monotone Hurwitz numbers are deeply related to random matrix theory. In particular, they have been introduced in the study of the HCIZ-integral [GGPN13, GGPN14]. See also [Nov20, Nov22]. On the other hand, many types of Hurwitz numbers satisfy topological recursion. Thus, they are particularly interesting when studying the connection of free probability and topological recursion.

The results of this chapter are based on the publication [HvIL22]. We derive a quantum curve for a newly introduced type of Hurwitz numbers: A *partition function* is a formal power series in infinitely many variables $p_1, p_2, p_3, \dots$ given by

$$Z = \exp\left(\sum_{\substack{g \geq 0 \\ n \geq 1}} \frac{\hbar^{2g-2+n}}{n!} \sum_{\mu_1,\dots,\mu_n=1}^{\infty} F_{g,\mu_1,\dots,\mu_n} p_{\mu_1} \cdots p_{\mu_n} \right).$$

Typically the $F_{g,\mu_1,\dots,\mu_n}$ are chosen to be invariants from combinatorics or enumerative geometry, for instance Hurwitz numbers. Then a *quantum curve* is a differential equation for the *principle specialization* $\Psi(x,h)$ of a partition function. The latter is defined by replacing the variables p_i by x^i. Quantum curves are closely related to the topological recursion. For example, the simple Hurwitz numbers satisfy topological recursion with the spectral curve (see [EMS11])

$$y - xe^y = 0$$

and a quantum curve of the Hurwitz numbers is given by

$$(\hat{\mathsf{y}} - \hat{\mathsf{x}} e^{\hat{\mathsf{y}}})\Psi(x,\hbar) = 0$$

where $\hat{\mathsf{x}} = x\cdot$ is a multiplication operator and $\hat{\mathsf{y}} = -\hbar \frac{\mathrm{d}}{\mathrm{d}x}$ a differential operator. Thus, formally replacing the noncommuting operators by commuting variables x, y, one recovers the initial data of topological recursion. This procedure is sometimes called dequantization. Vice versa, replacing x, y by $\hat{\mathsf{x}}, \hat{\mathsf{y}}$ is sometimes called a quantization. Of course, by the noncommutativity of the operators, there is no unique way to quantize a spectral curve.

In the same spirit, a quantum curve for Hurwitz numbers over a higher base curve was computed in [LMS13]. These numbers count ramified covers of a Riemann surface with genus $h \geq 1$ instead of $\mathbb{C}P^1$. Inspired by their results, we introduced and studied monotone versions of the numbers of [LMS13] in [HvIL22]. The main results are the computation of the following two families of quantum curves.

Theorem (Theorem 3.3.5).
The partition function $Z_h^{\leq}$ of the monotone base h Hurwitz numbers, resp. its principle specialization $\Psi_h^{\leq}(x,\hbar)$, satisfies the differential equation

$$[\hat{\mathsf{x}}\hat{\mathsf{y}}^2 + \hat{\mathsf{y}} + (\hat{\mathsf{y}}\hat{\mathsf{x}})^{2h}]\Psi_h^{\leq}(x,\hbar) = 0,$$

where $\hat{\mathsf{x}} = x\cdot$ and $\hat{\mathsf{y}} = -\hbar\frac{\partial}{\partial x}$.

Theorem (Theorem 3.3.6).
The partition function $Z_h^{<}$ of the strictly monotone base h Hurwitz numbers, resp. its principle specialization $\Psi_h^{<}(x,\hbar)$, satisfies the differential equation

$$[\hat{\mathsf{y}} + (1 - \hat{\mathsf{x}}\hat{\mathsf{y}})(\hat{\mathsf{y}}\hat{\mathsf{x}})^{2h}]\Psi_h^{\leq}(x,\hbar) = 0,$$

where $\hat{\mathsf{x}} = x\cdot$ and $\hat{\mathsf{y}} = -\hbar\frac{\partial}{\partial x}$.

In particular, for $h = 0$, we recover the quantum curves of [DDM17] and [DM14]; this is discussed in Section 3.3. Motivated by the *quantum curve – spectral curve correspondence*, we tried to run topological recursion on the curves obtained by replacing the operators in Theorem 3.3.5 and Theorem 3.3.6 by commuting variables. Surprisingly, it turned out that topological recursion on the spectral curve obtained for $h = 1$ in Theorem 3.3.6

$$y(1 + yx^2 - y^2x^3) = 0$$

computes signed simple monotone Hurwitz numbers instead of elliptic ($h = 1$) strictly monotone Hurwitz numbers. This observation indicates a combinatorial relation between those two quantities. More importantly, the signed simple Hurwitz numbers satisfy topological recursion and agree with the values of the Möbius function for partitioned permutations of higher genus. This new relation of free probability to the theory of topological recursion and partition functions motivated the author of this thesis to further explorations, ultimately leading to the breakthrough in [BCGF+23].

1 Preliminaries

1.1 Free probability and random matrices

In this chapter, we give an introduction to free probability. We will loosely follow the literature of [NS06], [MS17] and the lecture notes [Spe19].

Free probability is the study of noncommutative random variables and their distribution. These notions are noncommutative analogues of the objects in classical probability theory, that is, classical random variables in a classical probability space and their distribution. As the name indicates, the noncommutative random variables in free probability do not need to commute with each other, thus their probabilistic behaviour is not captured by the classical theory. The central concept of independence in the classical setting has a noncommutative analogue called free independence in free probability. We start this section by introducing these main features of free probability and discuss some examples in Section 1.1.1.

A particularly important quantity for this thesis are the free cumulants. These pose noncommutative versions of the classical cumulants in classical probability theory and add a combinatorial flavour to the probabilistic theory. Whereas the classical cumulants are defined via partitions, in free probability the free cumulants relate to moments via noncrossing partitions. We introduce the moment-cumulant formalism of free probability in Section 1.1.2.

Since free probability was originally introduced to study the isomorphism problem of the free group factors in operator theory [Voi85], operator algebras constitute a natural example for noncommutative probability spaces. More pertinent for this thesis is free probability in the context of random matrices. It was discovered by Voiculescu ([Voi91]) that the asymptotic behaviour of random matrices is also described by the freeness rule coming from the free group factor problem. We will discuss this matter in Section 1.1.3.

Finally, we introduce an extension of Voiculescu's free probability, called infinitesimal freeness. It was introduced by Belinschi and Shlyakhtenko in an analytic framework with the intention to complement the theory of Type B freeness of [BGN03]. In 2010 Février and Nica gave a combinatorial description of infinitesimal freeness. We introduce this combinatorial approach in Section 1.1.4.

1.1.1 Free Probability Theory

In this section, we introduce the foundations of free probability. We start by defining noncommutative probability spaces.

Definition 1.1.1.

i) A *noncommutative probability space* $(\mathcal{A}, \varphi)$ consists of a unital complex algebra $\mathcal{A}$ and a unital linear functional $\varphi\colon \mathcal{A} \to \mathbb{C}$. We call the elements $a \in \mathcal{A}$ *noncommutative random variables*.

ii) A noncommutative probability space $(\mathcal{A}, \varphi)$ is called a $*$-*probability space* if $\mathcal{A}$ is a $*$-algebra and φ is *positive*, i.e.

$$\varphi(aa^*) \geq 0, \qquad \text{for all } a \in \mathcal{A}.$$

Further, a positive linear functional φ is called a *state*.

iii) A noncommutative probability space $(\mathcal{A}, \varphi)$ is called a C^*-*probability space* if $\mathcal{A}$ is a C^*-algebra and φ a state.

iv) Let $B(H)$ be the bounded linear operators on a Hilbert space H. A noncommutative probability space $(\mathcal{A}, \varphi)$ is called a W^*-*probability space* if $\mathcal{A} \subset B(H)$ is a von Neumann algebra and φ is a state with the properties of being

- *normal:* φ is continuous with respect to the weak operator topology on $B(H)$,
- *faithful:* $\varphi(aa^*) = 0$ implies $a = 0$ for all $x \in \mathcal{A}$.

We call φ *normal faithful state*.

v) A W^*-probability space $(\mathcal{A}, \varphi)$ is called a *tracial* W^*-*probability space*, if φ is a *trace*, i.e.

$$\varphi(ab) = \varphi(ba), \qquad \text{for all } a, b \in \mathcal{A}.$$

Let us consider the following examples of noncommutative probability spaces.

Example.

i) Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a classical probability space. We set $\mathcal{A} = L^\infty(\Omega, \mathbb{P})$ to be the bounded measurable functions $f\colon \Omega \to \mathbb{C}$ and

$$\varphi(f) = \mathbb{E}[f] = \int_\Omega f(x) \mathrm{d}\mathbb{P}(x),$$

then $(\mathcal{A}, \varphi)$ is a W^*-probability space.

ii) Let $N \in \mathbb{N}$ be a natural number and denote by $M_N(\mathbb{C})$ the complex $N \times N$ matrices. We define the *normalized trace* by

$$\mathrm{tr}(A) = \frac{1}{N} \sum_{i=1}^{N} A_{ii}, \qquad \text{for all } A \in M_N(\mathbb{C}),$$

then $(M_N(\mathbb{C}), \mathrm{tr})$ is a C^*-probability space.

iii) We can combine the two examples above and define a C^*-probability space

$$(\mathcal{A}, \varphi) = (M_N(\mathbb{C}) \otimes L^\infty(\Omega, \mathbb{P}), \operatorname{tr} \otimes \mathbb{E}).$$

Definition 1.1.2.
Let $(\mathcal{A}, \varphi)$ be a noncommutative probability space.

i) Let $a_1, \ldots, a_n \in \mathcal{A}$ be noncommutative random variables, we define their *joint distribution* to be the set

$$\mu_{a_1,\ldots,a_n} = \Big\{ \varphi(p(a_1, \ldots, a_n)) \colon p \in \mathbb{C}\langle x_1, x_2, \ldots, x_n \rangle \Big\},$$

where $\mathbb{C}\langle x_1, x_2, \ldots, x_n\rangle$ is the unital free algebra over $\mathbb{C}$ generated by variables $x_1, \ldots, x_n$, it is called the *ring of noncommutative polynomials.*

ii) An element $a \in A$ is called *centered* if

$$\varphi(a) = 0.$$

Remark 1.1.3.
We can interpret the joint distribution $\mu_{a_1,\ldots,a_n}$ of $a_1, \ldots, a_n$ as a function $\bar{\mu}_{a_1,\ldots,a_n}$ via

$$\overline{\mu}_{a_1,\ldots,a_n} \colon \mathbb{C}\langle x_1, \ldots, x_n \rangle \to \mathbb{C}, \quad p(x_1, \ldots, x_n) \mapsto \varphi(p(a_1, \ldots, a_n)).$$

Due to linearity, the functional is determined by the values of $\overline{\mu}_{a_1,\ldots,a_n}$ on monomials

$$\varphi(a_{i_1} \ldots a_{i_k}),$$

where $k \in \mathbb{N}, i_1, \ldots i_k \in \{1, \ldots, n\}$.

Note that the latter is a purely combinatorial description of the joint distribution of several noncommutative random variables. In some cases, there is more (analytic) structure, consider the following definition.

Definition 1.1.4.
Let $(\mathcal{A}, \varphi)$ be a $*$-probability space and $a^* = a \in \mathcal{A}$ be a self-adjoint element. If there exists a probability measure μ on $\mathbb{R}$ such that

$$\varphi(a^k) = \int_{\mathbb{R}} t^k \mathrm{d}\mu(t),$$

we say a admits an *analytic distribution* μ.

Example.

- A self-adjoint element $S \in \mathcal{A}$ in a $*$-probability space $(\mathcal{A}, \varphi)$ is called a *semicircular random variable of variance* $\sigma > 0$ if

$$\varphi(S^k) = \begin{cases} 0 & \text{if } k \text{ is odd,} \\ \sigma^{2n} C_n & \text{if } k = 2n, \end{cases}$$

where

$$C_n = \frac{1}{n+1}\binom{2n}{n}$$

are the *Catalan numbers*. Note that a semicircular random variable admits an analytic distribution since

$$\varphi(S^k) = \frac{2}{\pi a^2}\int_{-a}^{a} x^k \sqrt{a^2 - x^2}\mathrm{d}x,$$

where $a = 2\sqrt{\sigma}$.

- If $A_N \in M_N(\mathbb{C})$ is a hermitian matrix in the noncommutative probability space $(M_N(\mathbb{C}), \mathrm{tr})$, then

  $$\mathrm{tr}(A_N^k) = \frac{1}{N}\sum_{i=1}^{N} \lambda_i^k = \int_{\mathbb{R}} t^k \mathrm{d}\mu(t),$$

  where

  $$\mu = \frac{1}{N}\sum_{i=1}^{N} \delta_{\lambda_i}$$

  is the sum of point measures at the N, not necessarily distinct, eigenvalues λ_i of A_N with mass $\frac{1}{N}$.

The noncommutative analogue of independence of random variables in classical probability theory is the notion of freeness, given in the following definition.

Definition 1.1.5.
Let $(\mathcal{A}, \varphi)$ be a noncommutative probability space.

i) Let $(\mathcal{A}_i)_{i\in I}$ be a family of unital subalgebras of $\mathcal{A}$. Then $(\mathcal{A}_i)_{i\in I}$ are called *freely independent* or just *free*, if for any $k \in \mathbb{N}$

$$\varphi(a_1 \dots a_k) = 0,$$

whenever $a_i \in \mathcal{A}_{j_i}$ are centered random variables, and the indices $j_i \in I$ satisfy $j_1 \neq j_2 \neq \dots \neq j_{k-1} \neq j_k$.

ii) Let $(\mathcal{S}_i)_{i\in I}$ be a family of subsets of $\mathcal{A}$. Then $(S_i)_{i\in I}$ are called *freely independent* or just *free* if the unital algebras $(\mathrm{Alg}_{\mathbb{C}}(1_{\mathcal{A}}, S_i))_{i\in I}$ generated by the S_i are freely independent.

iii) Let $(a_i)_{i\in I}$ be a family of noncommutative random variables in $\mathcal{A}$. Then $(a_i)_{i\in I}$ are *freely independent* or just *free* if $(\{a_i\})_{i\in I}$ are freely independent.

Remark 1.1.6.
It is important to note that freeness provides a rule to compute mixed moments of free random variables, let us consider the following examples.

Example.
Let $a, b \in \mathcal{A}$ be free from each other in a noncommutative probability space $(\mathcal{A}, \varphi)$, then $\tilde{a} = a - \varphi(a)1_{\mathcal{A}}$ and $\tilde{b} = b - \varphi(b)1_{\mathcal{A}}$ are centered and free from each other and we have

$$0 = \varphi(\tilde{a}\tilde{b}) = \varphi(ab) - \varphi(a)\varphi(b)$$

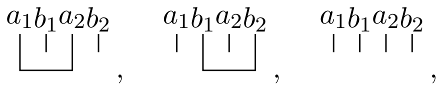
thus $\varphi(ab) = \varphi(a)\varphi(b)$. Another example is given by

$$0 = \varphi(\tilde{a}_1\tilde{b}\tilde{a}_2) \implies \varphi(a_1ba_2) = \varphi(a_1a_2)\varphi(b),$$

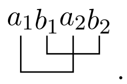
if $\{a_1, a_2\}$ is free from $\{b\}$. Let us consider a last example, we have

$$\varphi(a_1b_1a_2b_2) = \varphi(a_1a_2)\varphi(b_1)\varphi(b_2) + \varphi(a_1)\varphi(a_2)\varphi(b_1b_2) - \varphi(a_1)\varphi(a_2)\varphi(b_1)\varphi(b_2), \tag{1.1.1}$$

where we assumed $\{a_1, a_2\}$ to be free from $\{b_1, b_2\}$. Note the combinatorial flavour of the terms on the right-hand side of (1.1.1),



where the blocks indicate how the terms are collected in the expectations. Note that the term $\varphi(a_1a_2)\varphi(b_1b_2)$ does not appear since it would correspond to the crossing partition



We will discuss the combinatorics of free probability theory in the next subsection.

Remark 1.1.7.
The notion of freeness is not really compatible with commutativity, as it can be shown that if commuting variables are free from each other one of them must be a constant.

1.1.2 Combinatorics of free probability and free cumulants

In the previous section, we have seen that the rule for computing mixed moments of free random variables has a combinatorial flavour. We want to give an introduction to the combinatorics of (first order) free probability of [Spe94]. We start by recalling some notation.

Notation 1.1.8.
Let $d \in \mathbb{N}$ be an integer.

i) We denote the set $\{1, \ldots, d\}$ of the first d natural numbers by $[d]$.

ii) Let $d \in \mathbb{N}$, then $\mathcal{V} = \{B_1, \ldots, B_r\}$ is called a *partition of* $[d]$ if
   - $1 \leq r \leq d$,
   - $\emptyset \neq B_i \subset [d]$ for $i = 1, \ldots, r$,
   - $B_i \cap B_j = \emptyset$ for $i, j = 1, \ldots, r$ with $i \neq j$, and
   - $\bigcup_{i=1}^{r} B_i = [d]$.

   We call B_i, $i = 1, \ldots, r$, the *blocks* of $\mathcal{V}$. We denote the set of all partitions of $[d]$ by $\mathcal{P}(d)$.

iii) Let $\mathcal{V} = \{B_1, \ldots, B_r\} \in \mathcal{P}(d)$ be a partition of $[d]$, then we denote by $\#\mathcal{V} = r$ its number of blocks. Further, we define its *colength* by $|\mathcal{V}| = d - \#\mathcal{V}$.

iv) We say a partition $\mathcal{V} \in \mathcal{P}(d)$ has a *crossing*, if there are distinct blocks B_1, B_2 of $\mathcal{V}$ and elements $i_1, i_2 \in B_1$ and $j_1, j_2 \in B_2$ such that $1 \leq i_1 < j_1 < i_2 < j_2 \leq d$. We denote the set of partitions without crossings by $\mathcal{NC}(d) \subseteq \mathcal{P}(d)$ and call it the set of *noncrossing partitions on* $[d]$.

v) Given two set partitions $\mathcal{V}, \mathcal{W} \in \mathcal{P}(d)$, we write $\mathcal{V} \leq \mathcal{W}$ if for every block, $B \in \mathcal{V}$, there is a block $B' \in \mathcal{W}$ such that $B \subseteq B'$.

Remark 1.1.9.
The notion of a partition of a set in Notation 1.1.8 should not be confused with the partition of an integer in Notation 1.2.1. Usually it is clear from the context which partition we mean. Additionally, we use very distinctive notation throughout this thesis:

i) Set partitions are usually denoted by calligraphic letters $\mathcal{V} \in \mathcal{P}(d)$.

ii) Integer partitions are usually denoted by lower case Greek letters $\lambda \vdash d$ (see Notation 1.2.1).

The introduction of $\leq$ on $\mathcal{NC}(d)$ makes it into a lattice:

Proposition 1.1.10.
Let $d \in \mathbb{N}$.

i) For any $\mathcal{V}, \mathcal{W} \in \mathcal{NC}(d)$ there is a unique smallest element w.r.t. $\leq$, denoted by $\mathcal{V} \vee \mathcal{W}$, with the defining property

$$\mathcal{V}, \mathcal{W} \leq \mathcal{V} \vee \mathcal{W}.$$

   We call $\mathcal{V} \vee \mathcal{W}$ the *join* of $\mathcal{V}$ and $\mathcal{W}$.

ii) For any $\mathcal{V}, \mathcal{W} \in \mathcal{NC}(d)$ there is a unique largest element w.r.t. $\leq$, denoted by $\mathcal{V} \wedge \mathcal{W}$, with the defining property

$$\mathcal{V} \wedge \mathcal{W} \leq \mathcal{V}, \mathcal{W}.$$

We call $\mathcal{V} \wedge \mathcal{W}$ the *meet* of $\mathcal{V}$ and $\mathcal{W}$.

Consequently, $\mathcal{NC}(d)$ is a lattice.

In the reminder of the section we present the key points of the incidence algebra formalism of Rota et al. tailored to the poset $\mathcal{NC}(d)$; see [DRS72].

Proposition 1.1.11.
Consider the partially ordered set $\mathcal{NC}(d)$.

i) We define

$$\mathcal{NC}(d)^{(2)} = \{(\mathcal{V}, \mathcal{W}) \in \mathcal{NC}(d) \times \mathcal{NC}(d) \colon \mathcal{V} \leq \mathcal{W}\}.$$

Moreover for functions $f, g \colon \mathcal{NC}(d)^{(2)} \to \mathbb{C}$, we define their convolution by

$$f * g \colon \mathcal{NC}(d)^{(2)} \to \mathbb{C}, \quad f * g(\mathcal{V}, \mathcal{W}) = \sum_{\mathcal{V} \leq \mathcal{U} \leq \mathcal{W}} f(\mathcal{V}, \mathcal{U}) g(\mathcal{U}, \mathcal{W}).$$

ii) Let $f \colon \mathcal{N}(d) \to \mathbb{C}, g \colon \mathcal{NC}(d)^{(2)} \to \mathbb{C}$ be functions, then we define their convolution by

$$f * g \colon \mathcal{NC}(d)^{(2)} \to \mathbb{C}, \quad f * g(\mathcal{V}, \mathcal{W}) = \sum_{\mathcal{V} \leq \mathcal{W}} f(\mathcal{V}) g(\mathcal{V}, \mathcal{W}).$$

Proposition 1.1.12.
i) We define the *delta* and *zeta function*, $\delta, \zeta \colon \mathcal{NC}(d)^{(2)} \to \mathbb{C}$, by $\zeta \equiv 1$ and

$$\delta(\mathcal{V}, \mathcal{W}) = \begin{cases} 1 & \text{if } \mathcal{V} = \mathcal{W}, \\ 0 & \text{otherwise.} \end{cases}$$

ii) There is a function $\mu \colon \mathcal{NC}(d)^{(2)} \to \mathbb{C}$ such that

$$\mu * \zeta = \zeta * \mu = \delta,$$

it is called the *Möbius function* of the lattice $\mathcal{NC}(d)$.

Proposition 1.1.13 (Möbius inversion)**.**
Let $f, g \colon \mathcal{NC}(d) \to \mathbb{C}$ be functions, then we have the equivalence

$$f = g * \zeta \iff g = f * \mu,$$

it is called the *Möbius inversion.*

This language was used by Speicher to define the free cumulants in [Spe94].

Proposition 1.1.14 ([Spe94])**.**
Let $(\mathcal{A}, \varphi)$ be a noncommutative probability space. We define for any $n \in \mathbb{N}$ a multilinear functional

$$\varphi_n \colon \mathcal{A}^n \to \mathbb{C}, \quad (a_1, \ldots, a_n) \mapsto \varphi(a_1 \cdot a_2 \cdots a_n)$$

and extend these to a family of multilinear functionals $(\varphi_{\mathcal{V}})_{\mathcal{V} \in \mathcal{NC}(d)}$ by

$$(a_1, \ldots, a_d) \mapsto \varphi_{\mathcal{V}}(a_1, \ldots, a_d) \coloneqq \prod_{B \in \mathcal{V}} \varphi_{\#B}\Big(a_i \colon i \in B\Big),$$

where $(a_i : i \in B)$ denotes the product of the a_i with $i \in B$ in increasing order. Then we can define a family of functionals $(\kappa_{\mathcal{V}})_{\mathcal{V} \in \mathcal{NC}(d)}$ on $\mathcal{A}^d$ by

$$(a_1, \ldots, a_d) \mapsto \kappa_{\mathcal{V}}(a_1, \ldots, a_d) \coloneqq \sum_{\mathcal{U} \leq \mathcal{V}} \varphi_{\mathcal{U}}(a_1, \ldots, a_d) \mu(\mathcal{U}, \mathcal{V}).$$

We call the collection of $\kappa_{\mathcal{V}}$ for all $d \in \mathcal{N}$ the *free cumulants*. The free cumulants are linear in its entries and determine the moments via Möbius inversion, i.e.

$$\varphi = \kappa * \zeta.$$

More precisely, on each level $d \in \mathbb{N}$ we have

$$\begin{aligned} \varphi_{\mathcal{V}}(a_1, \ldots, a_d) &= \sum_{\mathcal{U} \leq \mathcal{V}} \kappa_{\mathcal{U}}(a_1, \ldots, a_d) \zeta(\mathcal{U}, \mathcal{V}) \\ &= \sum_{\mathcal{U} \leq \mathcal{V}} \kappa_{\mathcal{U}}(a_1, \ldots, a_d). \end{aligned} \tag{1.1.2}$$

Example.
Consider a semicircular random variable of variance 1. Recall that its moments are given by the Catalan numbers. These count the number of noncrossing pairings on $\mathcal{NC}_2(2k)$, we have

$$\varphi(S^{2k}) = C_k = \sum_{\mathcal{V} \in \mathcal{NC}_2(2k)} \prod_{B \in \mathcal{V}} 1.$$

Comparing with (1.1.2) for $\mathcal{V} = 1_{2k}$ we obtain

$$\kappa_n(S, \ldots, S) = \delta_{n,2}.$$

Now we are ready to state freeness in terms of the cumulants.

Theorem 1.1.15 ([Spe94]).
Let $(\mathcal{A}, \varphi)$ be a noncommutative probability space and $(\mathcal{A}_i)_{i\in I}$ be a family of unital subalgebras of $\mathcal{A}$. Then the following two statements are equivalent.

i) The family $(\mathcal{A}_i)_{i\in I}$ is freely independent in $(\mathcal{A}, \varphi)$.

ii) All mixed free cumulants vanish, that is for every $d \geq 2$, $i\colon [d] \to I$ and $a_j \in \mathcal{A}_{i(j)}$ for $j = 1, \dots, d$ we have $\kappa(a_1, \dots, a_d) = 0$ if i is not constant, i.e. there are at least two different $j_1, j_2 \in [d]$ with $i(j_1) \neq i(j_2)$.

An efficient way to deal with the distribution of a single element is to consider its generating series of moments or resp. cumulants. Thus, let us introduce the following objects.

Notation 1.1.16.
We denote by $\mathbb{C}^{\mathbb{Z}_{\geq 0}}$ the set of functions $\mathbb{Z}_{\geq 0} \to \mathbb{C}$. For $f, g \in \mathbb{C}^{\mathbb{Z}_{\geq 0}}$ we define the operations

$$f + g\colon \mathbb{Z}_{\geq 0} \to \mathbb{C}, \quad n \mapsto f(n) + g(n),$$
$$f \cdot g\colon \mathbb{Z}_{\geq 0} \to \mathbb{C}, \quad n \mapsto \sum_{k=0}^{n} f(k)g(n-k).$$

Then $(\mathbb{C}^{\mathbb{Z}_{\geq 0}}, +, \cdot)$ is a commutative ring with zero element $f \equiv 0$ and multiplicative identity

$$f\colon \mathbb{Z}_{\geq 0} \to \mathbb{C}, \quad n \mapsto \begin{cases} 1 & \text{if } n = 0, \\ 0 & \text{otherwise.} \end{cases}$$

We denote this ring by $\mathbb{C}[\![x]\!]$ and call it the *ring of formal power series in the indeterminate x*. An element $f \in \mathbb{C}^{\mathbb{Z}_{\geq 0}}$ will be denoted by

$$\sum_{n=0}^{\infty} f(n)x^n.$$

A family $(a_n)_{n\in\mathbb{N}}$ of complex number uniquely determines a function $a \in \mathbb{C}^{\mathbb{Z}_{\geq 0}}$ via $a(n) = a_n$. We call the corresponding element

$$a = \sum_{n=0}^{\infty} a_n x^n \in \mathbb{C}[\![x]\!]$$

a *generating series for the family* $(a_n)_{n\in\mathbb{N}}$ and use the notation $[x^n]a = a(n) = a_n$. For details, see [Art10].

Remark 1.1.17.
Later in this thesis, we will encounter power series in more than one variable. These can be constructed inductively by the procedure in Notation 1.1.16 and we will adopt the notations from the case of a single variable. We refer to [Sam23] for more details.

Definition 1.1.18.
Let $(\mathcal{A}, \varphi)$ be a noncommutative probability space and $a \in \mathcal{A}$. Then we define the *moment generating series*

$$M_a(x) = 1 + \sum_{r=1}^{\infty} \varphi(a^r)x^r,$$

and the *cumulant generating series*

$$C_a(x) = 1 + \sum_{r=1}^{\infty} \kappa_r(a, \dots, a)x^r.$$

Remark 1.1.19.

i) Sometimes we omit the subscript in $M_a(x)$ resp. $C_a(x)$ if it is clear what element we are talking about.

ii) It is also conventional to abbreviate

$$\kappa_r(a, \dots, a) = \kappa_r^a, \quad \varphi(a^r) = m_r^a.$$

Finally, let us state the manifestation of the moment-cumulant relations in terms of the generating functions $M(x)$ and $C(x)$, called the (moment-cumulant) functional relations. These first order relations go back to [Spe94] and recover the relations of [Voi86] in a purely combinatorial language.

Theorem 1.1.20 (First order functional relations)**.**
Let $a \in \mathcal{A}$ be a noncommutative random variable in a noncommutative probability space $(\mathcal{A}, \varphi)$, then the following statements are equivalent:

i) We have $m = \zeta * \kappa$ as functions on $\mathcal{NC}(d)$ on every level of d, that is

$$m_d^a = \sum_{\mathcal{V} \in \mathcal{NC}(d)} \kappa_{\mathcal{V}}^a$$

for every $d \in \mathbb{N}$.

ii) We have

$$m_d^a = \sum_{s=1}^{d} \sum_{\substack{r_1, \dots, r_n \geq 0 \\ r_1 + \dots + r_s = d-s}} \kappa_s^a m_{r_1}^a \dots m_{r_s}^a.$$

iii) The generating functions of m_r^a and κ_r^a satisfy the functional relation

$$C_a(xM_a(x)) = M_a(x).$$

iv) The generating functions of m_r^a and κ_r^a satisfy the functional relation

$$M_a\Big(\frac{x}{C_a(x)}\Big) = C_a(x).$$

Example.
For a semicircular variable S of variance 1 we already know that $\kappa_n^S = \delta_{n,2}$ and thus

$$C_S(x) = 1 + x^2.$$

By the functional relations we obtain

$$1 + x^2 M_S(x)^2 = M_S(x).$$

Solving for $M_S(x)$ yields

$$M_S(x) = \frac{1 \pm \sqrt{1-4x^2}}{2x^2}.$$

The asymptotic behaviour as $x \to 0$ shows that we must take the minus sign.

Remark 1.1.21.
Let $a \in \mathcal{A}$ be a noncommutative random variable in a noncommutative probability space $(\mathcal{A}, \varphi)$. We define the *Cauchy-* and *$\mathcal{R}$-transform* by

$$G_a(x) = \frac{1}{x} M_a\Big(\frac{1}{x}\Big) \quad \text{and} \quad \mathcal{R}_a(x) = \frac{C_a(x) - 1}{x},$$

respectively. Then the relations of Theorem 1.1.20 are equivalent to

$$G_a\Big(\mathcal{R}_a(x) + \frac{1}{x}\Big) = x \quad \text{and} \quad \mathcal{R}_a(G_a(x)) + \frac{1}{G_a(x)} = x.$$

Further let $b \in \mathcal{A}$ be a noncommutative random variable free from a. Then the vanishing of mixed cumulants implies

$$\mathcal{R}_{a+b}(x) = \mathcal{R}_a(x) + \mathcal{R}_b(x).$$

Thus, Speicher recovered the analytic results of Voiculescu in [Voi86] in purely combinatorial terms.

The functional relations between the generating series of moments and cumulants in higher order constitute an integral part of this thesis, see Chapter 2. Prior to the collaboration of the author with G. Borot, E. Garcia-Failde, S. Charbonnier and S. Shadrin, these functional relations were unknown in order higher than 2.

1.1.3 Limiting eigenvalue distributions

Voiculescu discovered that the limiting behaviour of the eigenvalue distribution of random matrices [Voi91] is captured by the notion of free independence in free probability. We give a short overview on the random matrix perspective of free probability. This section serves as a motivation for later concepts in Chapter 2. For a detailed introduction to random matrix theory see [Spe20].

Notation 1.1.22.

i) We denote by $M_N(\mathbb{C})$ the unital algebra of $N \times N$ matrices with coefficients in $\mathbb{C}$ and by I_N the identity matrix in $M_N(\mathbb{C})$.

ii) We denote by Tr_N the trace

$$\mathrm{Tr}_N \colon M_N(\mathbb{C}) \to \mathbb{C}, \quad A = (A_{ij})_{i,j=1}^N \mapsto \sum_{i=1}^{N} A_{ii}$$

and by tr_N the normalized trace $\mathrm{tr}_N \equiv \frac{1}{N} \mathrm{Tr}_N$. Sometimes we omit the subscript N.

iii) Let $A = (A_N)_{N\in\mathbb{N}}$ and $B = (B_N)_{N\in\mathbb{N}}$ be $N \times N$ random matrices.

- A and B are called *independent* if for each $N \in \mathbb{N}$ all entries of A_N are independent from all entries of B_N.
- A is called *unitarily invariant*, if for each $N \in \mathbb{N}$ the joint distribution of the entries does not change if we conjugate the random matrix A with an arbitrary unitary $N \times N$ matrix.

iv) Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a classical probability space and X a random variable. Then we define the *expectation* $\mathbb{E}[\mathrm{X}]$ of X by

$$\mathbb{E}[X] = \int_{\Omega} X \mathrm{d}\mathbb{P}.$$

Furthermore, if X_1, X_2 are two random variables such that $\mathbb{E}[|X_1|], \mathbb{E}[|X_2|] < \infty$, we define their covariance by

$$\mathrm{cov}(X, Y) = \mathbb{E}[(X_1 - \mathbb{E}[X_1])(X_2 - \mathbb{E}[X_2])].$$

v) In classical probability, there is a notion of classical cumulants. These are usually defined via the classical moment generating function, but they can also be defined combinatorially: let $(\Omega, \mathcal{F}, \mathbb{P})$ be a classical probability space and $X_1, \ldots, X_n$ be random variables with finite moments. Then we define the *joint classical n-cumulant* k_n *of* $X_1, \ldots, X_n$ by

$$\mathrm{k}_n(X_1, \ldots, X_n) = \sum_{\mathcal{V}\in\mathcal{P}(n)} (\#\mathcal{V} - 1)!(-1)^{\#\mathcal{V}-1} \prod_{B\in\mathcal{V}} \mathbb{E}\Bigg[\prod_{i\in B} X_i\Bigg].$$

Equivalently, we have

$$\mathbb{E}[X_1 \dots X_n] = \sum_{\mathcal{V} \in \mathcal{P}(n)} \prod_{B \in \mathcal{V}} \mathrm{k}_{\#B}(X_i \colon i \in B).$$

See [Shi16] for more details on cumulants in classical probability.

Example 1.1.23.
We have $\mathrm{k}_1(X) = \mathbb{E}[X]$ and

$$\mathrm{k}_2(X_1, X_2) = \mathbb{E}[X_1 X_2] - \mathbb{E}[X_1]\mathbb{E}[X_2] = \mathrm{cov}(X_1, X_2)$$

Furthermore, if we consider the noncommutative probability space $(L^\infty, \mathbb{E})$, then we have

$$\mathrm{k}_1(X) = \kappa_1(X), \quad \mathrm{k}_2(X_1, X_2) = \kappa_2(X_1, X_2) \quad \text{and} \quad \mathrm{k}_3(X_1, X_2, X_3) = \kappa_3(X_1, X_2, X_3),$$

since $\mathcal{P}(1) = \mathcal{NC}(1)$, $\mathcal{P}(2) = \mathcal{NC}(2)$ and $\mathcal{P}(3) = \mathcal{NC}(3)$. Moreover, for $n \geq 4$ we have a proper inclusion $\mathcal{NC}(n) \subsetneqq \mathcal{P}(n)$ and hence the corresponding classical and free cumulants are distinct.

Definition 1.1.24.
Let $A = (A_N)_{N \in \mathbb{N}}$ be a sequence of $N \times N$ random matrices. The sequence $(A_N)_{N \in \mathbb{N}}$ has a *(first order) limiting eigenvalue distribution if*

$$\lim_{N \to \infty} \mathbb{E}[\mathrm{tr}(A_N^r)] =: \varphi_r^A$$

exists for all $r \in \mathbb{N}$ and

$$\lim_{N \to \infty} \mathrm{k}_n[\mathrm{tr}(A_N^{r_1}), \dots, \mathrm{tr}(A_N^{r_n})] = 0$$

for any $n \geq 2$ and $r_1, \dots, r_n \in \mathbb{N}$, where k_n are classical cumulants.

As an example, let us state Wigner's famous semicircular law for the GUE.

Definition 1.1.25.
The *Gaussian unitary ensemble (GUE)* $(A_N)_{N \in \mathbb{N}}$ is the collection of $N \times N$ random matrices $A_N = \frac{1}{\sqrt{N}}(A_{ij}^N)_{i,j=1}^N$ where

i) A_N is self-adjoint, i.e. $A_{ij}^N = \overline{A_{ji}^N}$ for all $i, j = 1, \dots N$,

ii) $\{A_{ij} \colon i \geq j\}$ are independent, and

iii) A_{ij} is a standard Gaussian random variable, which is complex for $i \neq j$ and real for $i = j$.

Theorem 1.1.26 (Wigner's semicircle law for GUE)**.**
The GUE has a limiting eigenvalue distribution given by the semicircular law. We have

$$\lim_{N \to \infty} \mathbb{E}[\mathrm{tr}(A_N^r)] = \frac{1}{2\pi} \int_{-2}^{2} x^r \sqrt{1 - x^2} \mathrm{d}x = \begin{cases} C_n & \text{if } r = 2n \text{ for some } n \in \mathbb{N}, \\ 0 & \text{otherwise.} \end{cases}$$

Finally, we explain the appearance of freeness in random matrix theory.

Definition 1.1.27.
Let $A = (A_N)_{N\in\mathbb{N}}$, $B = (B_N)_{N\in\mathbb{N}}$ be sequences of random matrices having (first order) limiting distributions. A and B are called *asymptotically free* if

$$\lim_{N\to\infty} \mathbb{E}\Big[\operatorname{tr}\Big(\big(A_N^{r_1} - I_N\varphi^A_{r_1}\big)\big(B_N^{s_1} - I_N\varphi^B_{s_1}\big)\dots\big(A_N^{r_n} - I_N\varphi^A_{r_n}\big)\big(B_N^{s_n} - I_N\varphi^B_{s_n}\big)\Big)\Big] = 0,$$

for all $n \in \mathbb{N}$ and all $r_1, s_1 \dots r_n, s_n \in \mathbb{N}$.

Theorem 1.1.28 ([Voi91])**.**
Let $A = (A_N)_{N\in\mathbb{N}}, B = (B_N)_{N\in\mathbb{N}}$ be independent sequences of random matrices having (first order) limiting distributions, one of them being unitarily invariant. Then, A and B are asymptotically free.

1.1.4 Infinitesimal freeness

In Section 1.1.2, we have discussed the combinatorial description of free probability in terms of noncrossing partitions. The noncrossing partition have an analogue called type B noncrossing partitions [Rei97]. In [BGN03], Biane et al. used these partitions to define free independence of type B. Broadly speaking, they replaced the noncrossing partitions by their type B analogue and developed a corresponding cumulant formalism. Later in [BS12], Belinschi and Shlyakhtenko studied the analytic side of type B free probability. In this context, they introduced the notion of infinitesimal laws, which are loosely speaking the derivative of the distribution functional (law) $\overline{\mu}$ defined in Remark 1.1.3. Thus, in infinitesimal freeness, the random variables are described by their distribution $\overline{\mu}$ and additionally by its derivative $\overline{\mu}'$. In [FN10], Février and Nica developed a combinatorial theory tailored to the analytic framework of [BS12], it is called infinitesimal free probability. We discuss the key points of [FN10] in this section.

Definition 1.1.29.

i) An *infinitesimal noncommutative probability space* (INCPS) $(\mathcal{A}, \varphi, \varphi')$ consists of a noncommutative probability space $(\mathcal{A}, \varphi)$ and a linear functional $\varphi' : \mathcal{A} \to \mathcal{C}$ with $\varphi'(1_\mathcal{A})$=0.

ii) Let $(\mathcal{A}, \varphi, \varphi')$ be an INCPS and $(\mathcal{A}_i)_{i\in I}$ be a family of unital subalgebras. Then $(\mathcal{A}_i)_{i\in I}$ are called *infinitesimally free*, if for any $k \in \mathbb{N}$

$$\varphi(a_1 \dots a_k) = 0,$$
$$\varphi'(a_1 \dots a_k) = \sum_{i=1}^{k} \varphi(a_1 \dots a_{i-1}\varphi'(a_i)a_{i+1} \dots a_k),$$

whenever $a_i \in \mathcal{A}_{j_i}$ are centered random variables, $j_i \in I$ and the indices satisfy $j_1 \neq j_2 \neq \dots \neq j_{k-1} \neq j_k$.

Similarly to the theory of Speicher [Spe94] we want to consider φ and φ' as functions on $\mathcal{NC}(d)$. But recall that φ' should be thought of as the derivative of φ, hence we need to introduce a Leibniz rule when dealing with multiplicative functions.

Definition 1.1.30.
Let $(\mathcal{A}, \varphi, \varphi')$ be an INCPS. We define for any $d \in \mathbb{N}$ multilinear functionals

$$\varphi_d \colon \mathcal{A}^d \to \mathbb{C}, \quad (a_1, \dots, a_d) \mapsto \varphi(a_1 \cdot a_2 \dots a_d),$$
$$\varphi'_d \colon \mathcal{A}^d \to \mathbb{C}, \quad (a_1, \dots, a_d) \mapsto \varphi'(a_1 \cdot a_2 \dots a_d)$$

and extend them to a family of d-linear functionals $(\varphi_{\mathcal{V}})_{\mathcal{V} \in \mathcal{NC}(d)}$ and $(\varphi'_{\mathcal{V}})_{\mathcal{V} \in \mathcal{NC}(d)}$ by

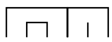
$$(a_1, \dots, a_d) \mapsto \varphi_{\mathcal{V}}(a_1, \dots, a_d) \coloneqq \prod_{B \in \mathcal{V}} \varphi_{\# B}(a_i \colon i \in B),$$
$$(a_1, \dots, a_d) \mapsto \varphi'_{\mathcal{V}}(a_1, \dots, a_d) \coloneqq \prod_{B \in \mathcal{V}} \varphi'_{\# B}(a_i \colon i \in B).$$

Furthermore, for any $d \in \mathbb{N}$ and $\mathcal{V} \in \mathcal{NC}(d)$ we define

$$\partial \varphi_{\mathcal{V}} \colon \mathcal{A}^d \to \mathbb{C}, \quad (a_1, \dots, a_d) \mapsto \partial \varphi(a_1, \dots, a_d) = \sum_{B \in \mathcal{V}} \partial \varphi_B(a_1, \dots, a_d),$$

where

$$\partial \varphi_B = \varphi'_{\# B}(a_i \colon i \in B) \prod_{\substack{V \in \mathcal{V} \\ V \neq B}} \varphi_{\# V}(a_i \colon i \in V).$$

Example 1.1.31.
Consider $\mathcal{V} = \{\{1,4,6\}, \{2,3\}, \{5\}\} =$ , then

$$\begin{aligned}\partial \varphi_{\mathcal{V}}(a_1 \dots a_6) &= \partial \varphi_{\{1,4,6\}}(a_1 \dots a_6) + \partial \varphi_{\{2,3\}}(a_1 \dots a_6) + \partial \varphi_{\{5\}}(a_1 \dots a_6) \\ &= \varphi'(a_1 a_4 a_6) \varphi(a_2 a_3) \varphi(a_5) + \varphi(a_1 a_4 a_6) \varphi'(a_2 a_3) \varphi(a_5) \\ &\quad + \varphi(a_1 a_4 a_6) \varphi(a_2 a_3) \varphi'(a_5).\end{aligned}$$

Now, we are ready to define the infinitesimal free cumulants.

Definition 1.1.32.
Let $(\mathcal{A}, \varphi, \varphi')$ be an INCPS. We define the *infinitesimal free cumulants* by the equations

$$\varphi'(a_1 \dots a_d) = \sum_{\mathcal{V} \in \mathcal{NC}(d)} \partial \kappa_{\mathcal{V}}(a_1, \dots, a_d), \tag{1.1.3}$$

equivalently

$$\kappa'(a_1, \dots, a_d) = \sum_{\mathcal{V} \in \mathcal{NC}(d)} \mu(\mathcal{V}, \mathbf{1}_d) \partial \varphi_{\mathcal{V}}(a_1, \dots, a_d).$$

Example 1.1.33.
We have

$$\varphi_1'(a_1) = \kappa_1'(a_1)$$

and for $d = 2$

$$\varphi_2'(a_1 a_2) = \kappa_2'(a_1, a_2) + \kappa_1'(a_1)\kappa_1(a_2) + \kappa_1(a_1)\kappa_1'(a_2).$$

Remark 1.1.34.
Another way of viewing the moment-cumulant relation for infinitesimal freeness is to think of the sum

$$\varphi'(a_1 \dots a_d) = \sum_{\mathcal{V} \in \mathcal{NC}(d)} \partial\kappa_{\mathcal{V}}(a_1, \dots, a_d)$$

as the sum over all noncommutative partitions with one special block, which corresponds to κ'. This idea is similar to our higher genus development in Section 2.2.

We also have the following analogue of the vanishing of mixed cumulants.

Theorem 1.1.35.
Let $(\mathcal{A}, \varphi, \varphi')$ be an INCPS and $(\mathcal{A}_i)_{i \in I}$ be a family of unital subalgebras of $\mathcal{A}$. Then the following statements are equivalent:

i) The family $(\mathcal{A}_i)_{i \in I}$ is infinitesimally free in $(\mathcal{A}, \varphi, \varphi')$.

ii) All mixed free and infinitesimal free cumulants vanish, that is for every $d \geq 2$, $i\colon [d] \to I$ and $a_j \in \mathcal{A}_{i(j)}$ for $i = 1, \dots, d$ we have

$$\kappa'(a_1, \dots, a_d) = \kappa(a_1, \dots, a_d) = 0$$

if i is not constant, i.e. there are at least two different $j_1, j_2 \in [d]$ with $i(j_1) \neq i(j_2)$.

Moreover, we can express the moment-cumulant formula in terms of generating series.

Proposition 1.1.36.
Let $(\mathcal{A}, \varphi)$ be a noncommutative probability space and $a \in \mathcal{A}$. Recall the generating series

$$M_a(x) = 1 + \sum_{r=1}^{\infty} \varphi(a^r)x^r \quad \text{and} \quad C_a(x) = 1 + \sum_{r=1}^{\infty} \kappa_r(a, \dots, a)x^r.$$

Further we define

$$M_a'(x) = 1 + \sum_{r=1}^{\infty} \varphi'(a^r)x^r \quad \text{and} \quad C_a'(x) = 1 + \sum_{r=1}^{\infty} \kappa_r'(a, \dots, a)x^r.$$

Then we have the functional relation

$$M_a'\left(\frac{x}{C_a(x)}\right) = \frac{C_a'(x)}{C_a(x)\frac{\mathrm{d}}{\mathrm{d}x}\frac{x}{C_(x)}}.$$

Remark 1.1.37.
A concise way to write this equation is in terms of differentials: if we put $w = \frac{x}{C_a(x)}$, we have

$$M_a'(w)\frac{\mathrm{d}w}{w} = C_a'(x)\frac{\mathrm{d}x}{x}.$$

1.2 Fock spaces and the boson-fermion correspondence

In this subsection, we will give a brief introduction to the boson-fermion correspondence and recall some results of the theory of symmetric functions. The first part is a short exposition of basic results covered in [Mac15], see also [Sav22] for an introduction. Furthermore, see [Kac90] and [KRR13] for a classical Lie algebraic perspective and [MJD00] for an introduction from the viewpoint of integrable hierarchies.

The boson-fermion correspondence expresses the representation of the Heisenberg algebra on the bosonic Fock space in terms of the fermionic Fock space and vice versa the representation of the Clifford algebra on the fermionic Fock space in terms of the bosonic picture. These actions can also be expressed in terms of the projective representation of

$$\widehat{\mathfrak{gl}}_\infty := \left\{(A_{i,j})_{i,j\in\mathbb{Z}+\frac{1}{2}} \colon A_{i,j} \in \mathbb{C},\, \exists K \geq 0 \colon A_{i,j} = 0 \text{ for all } |i-j| > K\right\},$$

i.e. the Lie algebra of infinite matrices having a finite number of nonzero diagonals. This correspondence allows us to deal with (difficult) equations involving differential operators in terms of matrices. It is the main tool in Section 2.5 to obtain the higher order relations for free probability.

1.2.1 Symmetric functions

Essentially, the bosonic Fock space is the ring of symmetric functions. Thus, before we discuss the boson-fermion correspondence, let us recall some theory of symmetric functions. For a complete introduction, see [Mac98] and [Sav22].

Notation 1.2.1.

i) We denote the *symmetric group* acting on d elements by $S(d)$. By convention, we define $S(0) = \{\emptyset\}$.

ii) Let $\pi \in S(d)$ be a permutation. We denote by $\#\pi$ the number of its cycles and define the *colength* of $\pi \in S(d)$ by $|\pi| = d - \#\pi$.

iii) Let $d \in \mathbb{N}$. We call $\lambda = (\lambda_1, \dots, \lambda_r)$ *partition of d* if
   - $1 \leq r \leq d$,
   - $\lambda_i \geq \lambda_{i+1}$ for $i = 1, \dots, r-1$, and
   - $\sum_{i=1}^r \lambda_i = d$.

In this case, we write $\lambda \vdash d$ and call $\ell(\lambda) = r$ the *length of* λ. By convention, we define $\emptyset$ to be the only partition of 0.

iv) If $\mathbf{r} = (r_1, \dots, r_\ell)$ is a sequence of positive integers, we denote by $\lambda(\mathbf{r})$ this sequence written in decreasing order. Furthermore, we define a permutation by

$$\gamma_{r_1,\dots,r_\ell} := (1, \dots, r_1)(r_1 + 1, \dots, r_1 + r_2) \cdots (r_1 + \cdots + r_{\ell-1} + 1, \dots, r_1 + \cdots + r_\ell).$$

In particular we may use the notation γ_λ when λ is a partition of a nonnegative integer.

v) Conversely, if σ is a permutation, we denote by $\lambda(\sigma)$ the sequence of lengths of the cycles of σ, in weakly decreasing order.

vi) Given $\lambda \vdash d$, let $C_\lambda \subseteq S(d)$ be the conjugacy class of γ_λ, that is, the set of permutations $\sigma \in S[d]$ such that $\lambda(\sigma) = \lambda$, and

$$z_\lambda = \frac{d!}{\# C_\lambda} = \prod_{i=1}^{\ell(\lambda)} \lambda_i \prod_{j \geq 1} m_j(\lambda)!, \tag{1.2.1}$$

where $m_j(\lambda)$ is the number of occurrences of j in the sequence λ.

vii) The irreducible representations of the symmetric group $S(d)$ can be parameterized by partitions $\lambda \vdash d$. If $\lambda \vdash d$ is a partition, we denote the corresponding character by χ_λ and its value on $\pi \in C_\mu$ by $\chi_\lambda(\mu)$. Note that χ_λ is constant on the conjugacy classes of $S(d)$. See [Sav22, Chapter 6] for details on representation theory and characters of finite groups.

Definition 1.2.2.
Let $n \in \mathbb{N}$, we denote by Λ_n the ring of polynomials in n indeterminates $x_1, \dots, x_n$, which are invariant under the action of the symmetric group $S(n)$ on the indices of the x_i, $i = 1, \dots, n$. More precisely,

$$\Lambda_n := \mathbb{C}[x_1, \dots, x_n]^{S(n)}.$$

The ring of symmetric functions is the inverse limit

$$\Lambda := \varprojlim_{n \in \mathbb{N}} \Lambda_n$$

in the category of graded $\mathbb{C}$ algebras, where projections of the limit are given by

$$f_{m,n} \colon \Lambda_m \to \Lambda_n, \quad p(x_1, \dots, x_m) \mapsto p(x_1, \dots, x_n, 0, \dots, 0)$$

for $m \geq n$.

Remark 1.2.3.
The algebra Λ can also be obtained by the following construction. Denote by Λ_n^k the symmetric polynomials of homogeneous degree k in n variables, that is

$$\Lambda_n^k = (\mathbb{C}[x_1, \dots, x_n]^k)^{S(n)}.$$

We can take the inverse limits w.r.t. the projections from the definition before and write

$$\Lambda^k = \varprojlim_{n \in \mathbb{N}} \Lambda_n^k.$$

One can show that

$$\Lambda = \bigoplus_{k \in \mathbb{N}} \Lambda^k.$$

From this construction it is clear that Λ consists of infinite sums of monomials with bounded degree.

Example.
For $n, k \in \mathbb{N}$ we have

$$p_k(x_1, \dots, x_n) = \sum_{i=1}^{n} x_i^k \in \Lambda_n^k,$$

which means in the limit Λ we have an element that restricts to these elements. It is given by

$$p_k = \sum_{i=1}^{\infty} x_i^k,$$

which clearly is a sum of monomials of degree k. We will encounter more examples in the following.

Let us list the following important symmetric polynomials and functions.

Definition 1.2.4.
Let $k \in \mathbb{N}$, then we define the following symmetric polynomials.

i) We define the *elementary symmetric polynomials* $e_k(x_1, \dots, x_n) \in \Lambda_n^k$ by

$$e_k(x_1, \dots, x_n) = \begin{cases} \sum\limits_{1 \le i_1 < \dots < i_k \le n} x_{i_1} \dots x_{i_k} & \text{if } k < n, \\ 0 & \text{otherwise.} \end{cases}$$

The *elementary symmetric functions* $e_k \in \Lambda$ are given by

$$e_k = \sum_{1 \le i_1 < i_2 \dots < i_k < \infty} x_{i_1} \dots x_{i_k}.$$

ii) We define the *complete homogeneous symmetric polynomials* $h_k(x_1, \dots, x_n) \in \Lambda_n^k$ by

$$h_k(x_1, \dots, x_n) = \sum_{1 \le i_1 \le \dots \le i_k \le n} x_{i_1} \dots x_{i_k}.$$

The *complete homogeneous symmetric functions* $h_k \in \Lambda$ are given by

$$h_k = \sum_{1 \le i_1 \le i_2 \dots < i_k < \infty} x_{i_1} \dots x_{i_k}.$$

iii) We define the *power sum symmetric polynomials* $p_k(x_1, \dots, x_n) \in \Lambda_n^k$ by

$$p_k(x_1, \dots, x_n) = \sum_{i=1}^{n} x_i^k.$$

The *power sum symmetric functions* $p_k \in \Lambda$ are given by

$$p_k = \sum_{i=1}^{\infty} x_i^k.$$

By convention, we set $h_0 = e_0 = 1$.

Let us recall the following lemma, describing the generating series of the elementary and complete homogeneous polynomials.

Lemma 1.2.5.
We have

$$\sum_{k=0}^{\infty} e_k(x_1, \dots, x_n) t^k = \prod_{i=1}^{n} (1 + t x_n),$$

and

$$\sum_{k=0}^{\infty} h_k(x_1, \dots, x_n) t^k = \prod_{i=1}^{n} \frac{1}{(1 + t x_n)}.$$

All of these functions form a basis of Λ in the following sense.

Theorem 1.2.6.
Let λ be a partition, and g_k any of the families of functions in Definition 1.2.4. We define

$$g_\lambda = g_{\lambda_1} \cdots g_{\lambda_{\ell(\lambda)}},$$

then $\{g_\lambda \colon \lambda \vdash d,\ d \ge 0\}$ is a linear basis of Λ. In particular

$$\Lambda = \mathbb{C}[g_1, g_2, g_3 \dots].$$

Remark 1.2.7.
Theorem 1.2.6 for $g_k = e_k$ is sometimes called the *Fundamental theorem of symmetric functions.*

There is another important basis, which is given by the Schur functions.

Definition 1.2.8.
We define the *elementary Schur functions* by the generating function

$$\sum_{k=0}^{\infty} s_k z^k = \exp\left(\sum_{i=1}^{\infty} \frac{p_i}{i} z^i\right).$$

For any partition λ we define the *Schur function* associated to λ by

$$\begin{aligned} s_\lambda &= \det\left((s_{\lambda_i+j-i})_{1\le i,j\le \ell(\lambda)}\right) \\ &= \det\begin{pmatrix} s_{\lambda_1} & s_{\lambda_1-1} & \cdots & s_{\lambda_1+\ell(\lambda)-1} \\ s_{\lambda_2-1} & s_{\lambda_2} & \cdots & s_{\lambda_1+\ell(\lambda)-2} \\ \vdots & & \ddots & \vdots \\ s_{\lambda_{\ell(\lambda)}-\ell(\lambda)} & s_{\lambda_{\ell(\lambda)}-\ell(\lambda)+1} & \cdots & s_{\lambda_{\ell(\lambda)}} \end{pmatrix}. \end{aligned}$$

Since the Schur functions are elements of Λ, they can be projected onto Λ_n. The image in Λ_n is called *Schur polynomials.*

Theorem 1.2.9.
The Schur functions $\{s_\lambda\colon \lambda \vdash d,\, d \ge 0\}$ form a basis of Λ.

Proposition 1.2.10.
Let $d \in \mathbb{N}$, $\lambda, \mu \vdash d$ partitions, and we denote

$$\delta_{\lambda,\mu} = \begin{cases} 1 & \text{if } \lambda = \mu, \\ 0 & \text{otherwise.} \end{cases}$$

Let

$$\langle p_\mu, p_\lambda\rangle \coloneqq z_\lambda \delta_{\mu,\lambda},$$

where z_λ is defined in (1.2.1). This induces an inner product on Λ, called the *Hall inner product.* Moreover, the Schur functions are orthonormal w.r.t. the Hall inner product, that is

$$\langle s_\mu, s_\lambda\rangle = \delta_{\lambda,\mu}.$$

Lemma 1.2.11.
We have the following formulas for the change of basis between the power sum basis and the Schur basis of Λ

$$p_\mu = \sum_{\lambda\vdash|\mu|} \chi_\lambda(\mu) s_\lambda \quad \text{and} \quad s_\lambda = \sum_{\mu\vdash|\lambda|} \frac{\chi_\lambda(\mu)}{z_\mu},$$

where $\chi_\lambda(\mu)$ are the irreducible characters of the symmetric group.

1.2.2 Bosonic Fock space

Actually, we have already defined the bosonic Fock space, namely by the ring of symmetric functions Λ. We now want to put more emphasis on the operators $\mathrm{End}(\Lambda)$. In particular, we want to study certain Lie algebra representations on the bosonic and fermionic Fock space. These play a key role in the boson-fermion correspondence.

Definition 1.2.12.
We denote the *bosonic Fock space* by

$$\mathscr{B} = \varprojlim_{k\in\mathbb{N}} \mathbb{C}[p_1,\dots,p_k] = \mathbb{C}[p_1,p_2,p_3,\dots],$$

it contains the special element $|\rangle \coloneqq 1 \in \mathbb{C}$ called the *vacuum* and admits a linear form $\langle| \colon \mathscr{B} \to \mathbb{C}$ that extracts the constant term of an element by evaluating at $p_i = 0$ for all $i \in \mathbb{N}$. It is called the *covacuum*.

Definition 1.2.13.
A complex *Lie algebra* $(\mathfrak{g}, [\cdot,\cdot])$ consists of a complex vector space $\mathfrak{g}$ together with a *Lie bracket* $[\cdot,\cdot] \colon \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$, where

i) $[\cdot,\cdot]$ is bilinear,

ii) $[x,x] = 0$ for all $x \in \mathfrak{g}$, and

iii) $[\cdot,\cdot]$ satisfies the *Jacobi identity*

$$[x,[y,z]] + [z,[x,y]] + [y,[z,x]] = 0.$$

Remark 1.2.14.
For Lie algebras over $\mathbb{C}$, property ii) is equivalent to $[x,y] = -[y,x]$ for all $x, y \in \mathfrak{g}$, hence it is sometimes called *skew symmetry*.

Example.
Let V be a complex vector space and denote by $\mathrm{End}(V)$ the complex vector space of linear maps $V \to V$. Then

$$[f,g] \colon \mathrm{End}(V) \times \mathrm{End}(V) \to \mathrm{End}(V), \quad (f,g) \mapsto [f,g] \coloneqq f \circ g - g \circ f$$

defines a Lie bracket on $\mathrm{End}(V)$ and thus makes it into a Lie algebra. In fact, every associative complex algebra $\mathcal{A}$ can be made into a Lie algebra by defining the Lie bracket to be the *commutator*

$$[a,b] \coloneqq ab - ba,$$

for all $a, b \in \mathcal{A}$.

Definition 1.2.15.
Let $(\mathfrak{g}, [\cdot,\cdot]_{\mathfrak{g}})$, $(\mathfrak{h}, [\cdot,\cdot]_{\mathfrak{h}})$ be complex Lie algebras.

i) A *Lie algebra morphism* $\Phi\colon \mathfrak{g} \to \mathfrak{h}$ is $\mathbb{C}$-linear map such that

$$\Phi([x,y]_{\mathfrak{g}}) = [\Phi(x), \Phi(y)]_{\mathfrak{h}}$$

for all $x, y \in \mathfrak{g}$.

ii) Let V be a complex vector space. A *representation of* $\mathfrak{g}$ *on* V is a Lie algebra morphism

$$\Phi\colon \mathfrak{g} \to \mathrm{End}(V).$$

A particularly important set of operators is described by the action of Heisenberg algebra on $\mathscr{B}$.

Definition 1.2.16.
The *(oscillator) Heisenberg algebra* $\mathcal{H}$ is the complex unital Lie algebra generated by $\{a_n\colon n \in \mathbb{Z}\}$ with relations

$$\begin{aligned}
&[1_{\mathcal{H}}, a_n] = 0, \quad \forall n \in \mathbb{Z}, \\
&[a_n, a_m] = \delta_{m,-n}, \quad \forall m, n \in \mathbb{Z}.
\end{aligned}$$

Lemma 1.2.17.
The Heisenberg algebra $\mathcal{H}$ has a representation on the bosonic Fock space $\mathscr{B}$ via

$$1_{\mathcal{H}} \mapsto \mathrm{id}_B, \quad a_n \mapsto \mathsf{J}_n = \begin{cases} n\partial_{p_n} & \text{if } n > 0, \\ p_{-n} & \text{if } n < 0, \\ 0 & \text{if } n = 0. \end{cases}$$

This settles the most important notions on the bosonic side of the boson-fermion correspondence, we proceed by explaining the fermionic side.

1.2.3 Fermionic Fock space

In this section, we introduce the fermionic Fock space via the so-called semi-infinite wedge formalism. Let us denote by $\mathbb{Z} + \frac{1}{2} = \{z + \frac{1}{2}\colon z \in \mathbb{Z}\}$ the set of half integers.

Definition 1.2.18.
Let V be an infinite dimensional complex vector space with basis $\{v_n\colon n \in \mathbb{Z} + \frac{1}{2}\}$. In other words,

$$V = \bigoplus_{i \in \mathbb{Z} + \frac{1}{2}} \mathbb{C} v_i.$$

The *fermionic Fock space* or *semi-infinite wedge space*

$$\mathscr{F} = \bigwedge^{\frac{\infty}{2}} V$$

is the complex vector space spanned by

$$v = v_{i_1} \wedge v_{i_2} \wedge v_{i_3} \dots \tag{1.2.2}$$

such that there exists $k \in \mathbb{N}$ and $K \geq 1$ such that $i_k = c - k + \frac{1}{2}$ for all $k \geq K$. The integer c is called the *charge of* v.

Remark 1.2.19.

i) The fermionic Fock space originates in physics and is motivated by *Dirac's electron sea*, see [KRR13].

ii) By the properties of the exterior product, we will always assume that the i_j are strictly descending, i.e. $i_1 > i_2 > i_3 \dots$ at cost of changing the sign. Moreover, it is clear that any v in (1.2.2) such that two indices agree, $i_{j_1} = i_{j_2}$, will vanish.

iii) The charge yields a decomposition of the space

$$\mathscr{F} = \bigoplus_{c \in \mathbb{Z}} \mathscr{F}^{(c)},$$

where $\mathscr{F}^{(c)}$ is the space spanned by elements of charge c.

iv) We can define an inner product on $\mathscr{F}$ by declaring the elements ψ in (1.2.2) and under the convention ii) to be orthogonal.

Definition 1.2.20.
Let $\lambda \vdash d \in \mathbb{N}$ be a partition, then we define

$$v_\lambda = v_{\lambda_1 - \frac{1}{2}} \wedge v_{\lambda_2 - \frac{3}{2}} \wedge \dots \wedge v_{\lambda_{\ell(\lambda)} - \frac{2\ell(\lambda)+1}{2}} \wedge v_{-\frac{2\ell(\lambda)+3}{2}} \wedge v_{-\frac{2\ell(\lambda)+5}{2}} \dots,$$

furthermore we call the element

$$|\rangle \coloneqq v_\emptyset = \psi_0 = v_{-\frac{1}{2}} \wedge v_{-\frac{3}{2}} \wedge v_{-\frac{5}{2}} \wedge \dots$$

the *vacuum*.

The v_λ have charge 0 and span the charge zero sector $\mathscr{F}^{(0)}$ of $\mathscr{F}$.

Proposition 1.2.21.
The set $\{v_\lambda \colon \lambda \vdash d, \forall d \in \mathbb{N}\}$ is a linear basis of $\mathscr{F}^{(0)}$, i.e.

$$\mathscr{F}^{(0)} = \bigoplus_{k \in \mathbb{N}} \bigoplus_{\lambda \vdash k} \mathbb{C} v_\lambda.$$

Remark 1.2.22.
More generally, for $c \in \mathbb{Z} \setminus \{0\}$ and $\lambda \vdash d$ one can define

$$v_\lambda^{(c)} = v_{\lambda_1 - \frac{1}{2} + c} \wedge v_{\lambda_2 - \frac{3}{2} + c} \wedge \cdots \wedge v_{\lambda_{\ell(\lambda)} - \frac{2\ell(\lambda)+1}{2} + c} \wedge v_{-\frac{2\ell(\lambda)+3}{2} + c} \wedge v_{-\frac{2\ell(\lambda)+5}{2} + c} \cdots$$

which then form a basis of $\mathscr{F}^{(c)}$.

Definition 1.2.23.
For any $i \in \mathbb{Z} + \frac{1}{2}$ we define the *creation operators*

$$\psi_i \colon \mathscr{F} \to \mathscr{F}, \quad v_{i_1} \wedge v_{i_2} \wedge v_{i_3} \wedge \cdots \mapsto v_i \wedge v_{i_1} \wedge v_{i_2} \wedge v_{i_3} \wedge \ldots$$

and *annihilation operators*

$$\psi_i^* \colon \mathscr{F} \to \mathscr{F},$$
$$v_{i_1} \wedge v_{i_2} \wedge v_{i_3} \wedge \cdots \mapsto \begin{cases} (-1)^{k-1} v_{i_1} \wedge v_{i_2} \wedge \ldots v_{i_{k-1}} \wedge v_{i_{k+1}} \wedge \ldots & \text{if } \exists k \in \mathbb{N} \colon i_k = i, \\ 0 & \text{otherwise.} \end{cases}$$

Both are elements of $\mathrm{End}(\mathcal{F})$.

Lemma 1.2.24.

i) The operators ψ_i, ψ_i^* satisfy

$$\psi_i \psi_j + \psi_j \psi_i = 0, \quad \psi_i^* \psi_j^* + \psi_j^* \psi_i^* = 0, \quad \psi_i \psi_j^* + \psi_j \psi_i^* = \delta_{i,j}.$$

The unital algebra generated by these operators is called the *infinite rank Clifford algebra*, we denote it by $\mathcal{C}$.

ii) The operators ψ_i increase the charge while ψ_i^* decrease it, that is

$$\psi_i[\mathscr{F}^{(c)}] = \mathscr{F}^{(c+1)}, \quad \psi_i^*[\mathscr{F}^{(c)}] = \mathscr{F}^{(c-1)}.$$

iii) The operators ψ_i and ψ_i^* are adjoint w.r.t. the natural inner product on $\mathscr{F}$.

The following operators are a central object in the boson-fermion correspondence.

Definition 1.2.25.
We define the operators

$$\Lambda_r = \sum_{k \in \mathbb{Z} + \frac{1}{2}} : \psi_{k-r} \psi_k^* :$$

where $: \ :$ denotes the *ordered product* defined by

$$: \psi_i \psi_j^* : \ = \begin{cases} \psi_i \psi_j^* & \text{if } j > 0, \\ \psi_j^* \psi_i & \text{if } j < 0. \end{cases}$$

Remark 1.2.26.

i) By elementary calculations, one can see that the Λ_r are well-defined and contained in $\mathrm{End}(\mathscr{F})$. In particular, they preserve the charge, i.e.

$$\Lambda_r|_{\mathscr{F}^c} \in \mathrm{End}(\mathscr{F}^{(c)})$$

for any $c \in \mathbb{Z}$. But also note that Λ_r are not elements in the Clifford algebra representation.

ii) The operators Λ_r have a nice combinatorial description in terms of Maya diagrams and Young diagrams, see for example [RZ16].

As mentioned, the operators Λ_r are not contained in the Clifford algebra, but we introduce the following algebra to overcome this problem.

Definition 1.2.27.
We define the following Lie algebra of infinite matrices,

$$\widehat{\mathfrak{gl}}_\infty := \left\{ (A_{i,j})_{i,j\in\mathbb{Z}+\frac{1}{2}} \colon A_{i,j} \in \mathbb{C},\, \exists K \geq 0 \colon A_{i,j} = 0 \,\text{for all}\, |i-j| > K \right\},$$

where the Lie bracket is given by the matrix commutator. Note that matrix units

$$E_{ij} = (\delta_{k,i}\delta_{l,j})_{k,l\in\mathbb{Z}+\frac{1}{2}}, \quad i,j \in \mathbb{Z} + \frac{1}{2}.$$

are contained in $\widehat{\mathfrak{gl}}_\infty$ and that the elements

$$L_r = \sum_{k\in\mathbb{Z}+\frac{1}{2}} E_{k-r,k}$$

span a commutative subalgebra algebra of $\widehat{\mathfrak{gl}}_\infty$.

Remark 1.2.28.
There is a natural action of matrix algebras on the wedge space via

$$A(v_{i_1} \wedge v_{i_2} \wedge v_{i_3} \cdots = (Av_{i_1}) \wedge v_{i_2} \wedge v_{i_3} \cdots + v_{i_1} \wedge (Av_{i_2}) \wedge v_{i_3} \cdots + \dots \tag{1.2.3}$$

but by the possible infinite diagonals of elements in $\widehat{\mathfrak{gl}}_\infty$ we may run into problems. Consider

$$T = \sum_{k\in\mathbb{Z}+\frac{1}{2}} \lambda_k E_{k,k} \in \widehat{\mathfrak{gl}}_\infty$$

for $\lambda_k \in \mathbb{C}$, then

$$T(v_\emptyset) = \Bigg(\sum_{k\in\mathbb{Z}+\frac{1}{2}} \lambda_k \Bigg) v_\emptyset.$$

Depending on the choice of λ_k the sum may not converge. This issue can be fixed by declaring

$$\hat{E}_{i,j}v = \begin{cases} \big(E_{i,i} - \mathrm{Id}_{\mathcal{F}^{(0)}}\big)(v) & \text{if } i = j, \\ E_{i,j}(v) & \text{if } i \neq j, \end{cases}$$

where the right-hand side is given by the action defined in (1.2.3). Unfortunately, this assignment does not define a Lie algebra representation, since the Lie bracket is not compatible in the sense of Definition 1.2.15. Still, we can make sense of this assignment in terms of a projective representation [KRR13]. Equivalently, the latter can be described by a representation of a central extension of $\widehat{\mathfrak{gl}}_\infty$, that is a representation of the Lie algebra

$$\widetilde{\mathfrak{gl}}_\infty = \widehat{\mathfrak{gl}}_\infty \oplus \mathbb{C} \cdot \mathfrak{c}$$

with bracket

$$[A + z\mathfrak{c}, B + w\mathfrak{c}]_{\widetilde{\mathfrak{gl}}_\infty} = AB - BA + \alpha(A, B) \cdot \mathfrak{c}$$

for all $A + z\mathfrak{c}, B + w\mathfrak{c} \in \widetilde{\mathfrak{gl}}_\infty$. Here, α is given by extending

$$\alpha(E_{ij}, E_{kl}) = \begin{cases} 1 & \text{if } k = j,\, l = i \text{ and } i \leq 0,\, j \geq 1, \\ -1 & \text{if } k = j,\, l = i \text{ and } j \leq 0,\, i \geq 1, \\ 0 & \text{otherwise.} \end{cases}$$

For more details see [KRR13].

Proposition 1.2.29.
As elements in $\mathrm{End}(\mathscr{F}^{(0)})$, we have

$$\hat{E}_{i,j} =: \psi_i \psi_j^* : \quad \text{and} \quad \hat{L}_r = \Lambda_r.$$

Remark 1.2.30.
Similarly, the algebra $\widetilde{\mathfrak{gl}}_\infty$ can be represented on $\mathscr{F}^{(c)}$; we are mainly interested in $\mathscr{F}^{(0)}$.

1.2.4 Boson-fermion correspondence

We are now ready to state boson-fermion correspondence. The latter consists of two parts, sometimes called the bosonization and fermionization. The first part carries the Heisenberg structure to the bosonic Fock space $\mathscr{B}$. Vice versa, the second part deploys the Clifford algebra on $\mathscr{B}$. For a detailed exposition tailored to the theory of integrable systems see [MJD00] and for a discussion from the point of view of symmetric functions see [Sav22, Chapter 5].

Theorem 1.2.31 (Bosonization)**.**
The map

$$\phi^{(c)} \colon \mathscr{F}^{(c)} \to \mathscr{B}, \quad v_\lambda^{(c)} \mapsto s_\lambda$$

is an isometric isomorphism and for any $r \neq 0$ the diagrams

$$\begin{array}{ccc} \mathscr{F}^{(c)} & \xrightarrow{\hat{L}_r} & \mathscr{F}^{(c)} \\ {\scriptstyle \phi^{(c)}}\downarrow & & \downarrow{\scriptstyle \phi^{(c)}} \\ \mathscr{B} & \xrightarrow{\mathsf{J}_r} & \mathscr{B} \end{array}$$

commute. In particular, the map

$$\mathcal{H} \to \mathrm{End}(\mathscr{F}^{(c)}), \quad \mathsf{J}_r \mapsto \Lambda_r$$

induces a representation of the Heisenberg algebra $\mathcal{H}$ on $\mathscr{F}^{(c)}$.

Note that every subspace of charge c is isomorphic to a copy of the bosonic Fock space. These isomorphisms can be collected by introducing a dummy, which captures the charge.

Corollary 1.2.32.
We have an isomorphism

$$\Phi \colon \mathscr{F} = \bigoplus_{c \in \mathbb{Z}} \mathscr{F}^{(c)} \to \mathscr{B}\,[q^{-1}, q] =: \mathscr{B}_q, \quad v_\lambda^{(c)} \mapsto q^c s_\lambda.$$

For the second part of the boson-fermion correspondence, we need to introduce the generating functions of the annihilation and creation operators.

Definition 1.2.33.
We define the generating series

$$\Psi(x) = \sum_{k \in \mathbb{Z} + \frac{1}{2}} x^{-k-\frac{1}{2}} \psi_k \quad \text{and} \quad \Psi^*(x) = \sum_{k \in \mathbb{Z} + \frac{1}{2}} x^{-k-\frac{1}{2}} \psi_k^*.$$

Then we have the following second part of the boson-fermion correspondence, sometimes called fermionization.

Theorem 1.2.34 (Fermionization)**.**
Under the map $\Theta \colon \mathrm{End}(\mathscr{F}) \to \mathrm{End}(\mathscr{B}_q)$ induced by Φ, the generating functions Ψ, Ψ^* have the following form:

$$\begin{aligned} A(x) \coloneqq \Theta(\Psi(x)) &= Tq \exp\Big(-\sum_{m \geq 0} \frac{x^{-m}}{m} \mathsf{J}_{-m} \Big) \exp\Big(\sum_{m \geq 0} \frac{x^m}{m} \mathsf{J}_m \Big) \\ &= Tq \exp\Big(-\sum_{m \geq 0} x^{-m} \frac{p_m}{m} \Big) \exp\Big(\sum_{m \geq 0} x^m \partial_m \Big), \end{aligned}$$

and

$$A^*(x) \coloneqq \Theta(\Psi(x)) = Tq^{-1} \exp\left(\sum_{m\geq 0} \frac{x^m}{m} \mathsf{J}_{-m}\right) \exp\left(-\sum_{m\geq 0} \frac{x^{-m}}{m} \mathsf{J}_m\right)$$
$$= Tq^{-1} \exp\left(\sum_{m\geq 0} x^m \frac{p_m}{m}\right) \exp\left(-\sum_{m\geq 0} x^{-m} \partial_m\right),$$

where T is the map that replaces $q \mapsto x^{-1}q$. More precisely, if we denote

$$a_k = [x^{-k-\frac{1}{2}}]A(x), \quad a_k^* = [x^{-k-\frac{1}{2}}]A^*(x),$$

then the map

$$\mathcal{C} \to \mathrm{End}(\mathscr{B}_q), \quad \psi_i \mapsto a_i, \quad \psi_i^* \mapsto a_i^* \quad \forall i \in \mathbb{Z}$$

is a representation of $\mathcal{C}$ on $\mathscr{B}$ and for every $r \in \mathbb{Z}$ the diagram

$$\begin{array}{ccc} \mathscr{F} & \xrightarrow{\psi_r \,/\, \psi_r^*} & \mathscr{F} \\ {\scriptstyle\Phi}\downarrow & & \downarrow{\scriptstyle\Phi} \\ \mathscr{B}_q & \xrightarrow{a_r \,/\, a_r^*} & \mathscr{B}_q \end{array}$$

commutes.

The most important consequence for us is the representation of the following generating series of operators.

Proposition 1.2.35.
Under the boson-fermion correspondence we have

$$\sum_{k,l\in\mathbb{Z}+\frac{1}{2}} x^l y^{-k} \hat{E}_{k,l} = \frac{y^{\frac{1}{2}}}{x^{\frac{1}{2}}} : \Psi(x)\Psi^*(y^{-1}):$$
$$= x^{\frac{1}{2}} y^{\frac{1}{2}} \frac{\exp\left(\sum_{i>0}(y^{-i} - x^{-i})\frac{p_i}{i}\right) \exp\left(\sum_{i>0}(x^i - y^i)\partial_m\right) - 1}{x - y}.$$

Furthermore, under the transformation $x = ze^{\frac{u}{2}}, y = ze^{-\frac{u}{2}}$ and with the notation

$$\varsigma(z) = \frac{e^{\frac{z}{2}} - e^{-\frac{z}{2}}}{z},$$

we have

$$\sum_{m\in\mathbb{Z}} z^m \sum_{k\in\mathbb{Z}+\frac{1}{2}} e^{u\left(k-\frac{1}{2}\right)} \hat{E}_{k-i,k} = \frac{\exp\left(\sum_{m>0} u\varsigma(iu) z^{-i} \mathsf{J}_{-i}\right) \exp\left(\sum_{i>0} u\varsigma(iu) z^i \mathsf{J}_i\right)}{u\varsigma(u)}. \tag{1.2.4}$$

Remark 1.2.36.
Typically, we will deal with partition functions throughout Chapter 2 (see Section 2.3). These are elements in $\mathbb{C}[\![p_1, p_2, p_3, \dots]\!]$, that is, power series in infinitely many variables. We will abuse notation and write $\mathscr{B} = \mathbb{C}[\![p_1, p_2, p_3, \dots]\!]$. We further extend the bosonic Fock space by the formal parameter $\hbar$,

$$\mathscr{B}_\hbar = \mathscr{B} \otimes \mathbb{Q}((\hbar)).$$

Note as a $\mathbb{C}(\!(\hbar)\!)$-module, $\mathscr{B}_\hbar$ admits a Schauder basis given by the Schur polynomials s_λ. The operators defined in this section still apply entrywise, and we will use the result of the boson-fermion correspondence for the charge 0 sector.

1.3 Monotone Hurwitz numbers

Hurwitz numbers originates in enumerative geometry, counting ramified morphisms between Riemann surfaces. For a fixed compact Riemann surface S of genus h, Hurwitz numbers count holomorphic maps $\pi : S' \to S$ (up to isomorphism), where S' is a compact Riemann surface of genus g, such that

- π has ramification profile $\mu^1, \dots, \mu^n$ over n arbitrary, but fixed points on S, and
- each map is weighted by $\frac{1}{|\mathrm{Aut}(\pi)|}$.

Hurwitz used the monodromy representations for the holomorphic maps (see [Hur91, Hur01]) to count these morphisms via factorizations in the symmetric group. We want to summarize the most important facts needed for this thesis.

Definition 1.3.1.
Let $g, h \geq 0$ be non-negative integers, d a positive integer and $\mu = (\mu^1, \dots, \mu^n)$ a tuple of partitions of d. Let

$$2g - 2 = d \cdot (2h - 2) + \sum_{j=1}^{n} \left(|\mu^j| - \ell(\mu^j)\right),$$

we call a collection $(\sigma_1, \dots, \sigma_n, \alpha_1, \beta_1, \dots, \alpha_h, \beta_h)$ of permutations in $S(d)$ a *factorization of type* (h, g, d, μ) if the following conditions are satisfied:

(H1) $\lambda(\sigma_i) = \mu^i$, i.e. $\sigma \in C_{\mu^i}$;

(H2) $\sigma_1 \cdots \sigma_n = [\alpha_1, \beta_1] \cdots [\alpha_h, \beta_h]$;

If additionally we have

(H3) $\langle \sigma_1, \dots, \sigma_n, \alpha_1, \beta_1, \dots, \alpha_h, \beta_h \rangle$ acts transitively on the set $\{1, 2, \dots, d\}$, we call the factorization *connected*. Denote by $\mathcal{F}(h,g,d,\mu)$ and $\mathcal{F}^\circ(h,g,d,\mu)$ the set of factorizations of type (h,g,d,μ) and the set of connected factorizations of type (h,g,d,μ), respectively. The *Hurwitz numbers* and *connected Hurwitz numbers* are defined by

$$H_{h,g}(\mu^1,\dots,\mu^n) = \frac{1}{d!}|\mathcal{F}(h,g,d,\mu)| \quad \text{respectively} \quad H^\circ_{h,g}(\mu^1,\dots,\mu^n) = \frac{1}{d!}|\mathcal{F}^\circ(h,g,d,\mu)|.$$

Let us state some special cases of these numbers.

Definition 1.3.2.

i) Consider $\mu = (\mu^1, T, \dots, T)$, where $T = (2,1,\dots,1)$ is the partition corresponding to transpositions. Then we call

$$H_{h,g}(\mu^1) = H_{h,g}(\mu^1, T, \dots, T)$$

simple base h Hurwitz numbers. Note, the number $r = n-1$ of transpositions is given by

$$r = 2g - 2 + \ell(\mu^1) - d(2h-1).$$

For $h = 0$, that is a base curve of genus 0, we obtain the *simple Hurwitz numbers.*

ii) If we take $\mu = (\mu^1, \mu^2, T\dots, T)$ and $h = 0$, we obtain the *double Hurwitz numbers.*

In this thesis, we are particularly interested in *(strictly) monotone* versions of Hurwitz numbers. The monotone version of simple Hurwitz numbers has been introduced by Goulden, Guay-Paquet and Novak, see [GGPN13] and[GGPN14]. They appear naturally in the study of the genus expansion of the *Harish-Chandra-Itzykson-Zuber-integral* and are given by imposing a monotonicity condition on factorizations that involve transpositions. More precisely, we say a collection of transpositions $\tau_1 = (s_1, t_1), \dots, \tau_n = (s_n, t_n)$ with $s_i < t_i$ for all $i = 1, \dots, n$, satisfies the *monotonicity condition* or call them *monotone* if

(M1) $t_i \leq t_{i+1}$ for $i = 1, \dots, r-1$,

we call them *strictly monotone* if

(M2) $t_i < t_{i+1}$ for $i = 1, \dots, r-1$.

If we have a factorization involving a (strictly) monotone product of transpositions, we say it is a (strictly) monotone factorization. Now we are ready to introduce the (strictly) monotone versions of Hurwitz numbers.

Definition 1.3.3.
Let $\lambda, \nu \vdash d$ be integer partitions and $r \geq 0$.

i) The *monotone double Hurwitz number* $H^{\leq}_r(\lambda,\nu)$ is $\frac{1}{d!}$ times the number of tuples $(\alpha, \tau_1, \dots, \tau_r, \beta)$ of permutations in $S(d)$ such that:

- $\alpha \in C_\lambda$ and $\beta \in C_\nu$;
- $\tau_1, \dots, \tau_r$ satisfies (M1), that is $\tau_1, \dots, \tau_r$ is a monotone sequence of transpositions; and
- $\alpha \circ \tau_1 \circ \cdots \circ \tau_r \circ \beta = \mathrm{id}$.

If the factorization additionally satisfies the connectedness condition (H3), we call the number *connected monotone double Hurwitz number* and denote it by $H_r^{\circ,\leq}(\lambda, \nu)$.

ii) The *(connected) strictly monotone Hurwitz number* ($H_r^{\circ,<}(\lambda, \nu)$) $H_r^{<}(\lambda, \nu)$ is defined analogously by replacing condition (M1) with (M2).

We also introduce the generating series

$$
\begin{aligned}
H^{<}(\lambda, \nu) &= \sum_{r=0}^{d-1} \hbar^r \, H_r^{<}(\lambda, \nu) \in \mathbb{Q}[\hbar]\,, \\
H^{\leq}(\lambda, \nu) &= \sum_{r \geq 0} (-\hbar)^r \, H_r^{\leq}(\lambda, \nu) \in \mathbb{Q}[\![\hbar]\!]\,.
\end{aligned}
\tag{1.3.1}
$$

Remark 1.3.4.

i) Similarly as in the nonmonotonic case, we can define a genus $g \geq 0$ via the Riemann-Hurwitz formula

$$r = 2g - 2 + \ell(\lambda) + \ell(\nu).$$

This justifies the notation $H^g(\lambda, \nu)$, which is sometimes encountered in the literature.

ii) In the literature, sometimes the monotone Hurwitz numbers are referred to as *weakly* monotone Hurwitz numbers to emphasize the distinction from the strictly monotone Hurwitz numbers.

iii) The strictly monotone Hurwitz numbers are also called *Grothendieck dessins d'enfant Hurwitz numbers* according to their relation to dessins d'enfants, see [ALS16] and Proposition 1.3.11.

iv) The special case $\lambda = (1^d)$, or equivalently $\alpha = e \in S(d)$, in Definition 1.3.3 is called *simple* (strictly) monotone Hurwitz number and denoted by $H_r^{\leq}(\nu)$ ($H_r^{<}(\nu)$).

Definition 1.3.5 ([HvIL22])**.**
Let $\lambda \vdash d$ be partitions and $r, h, g \geq 0$ such that

$$r = 2g - 2 - d(2h - 1) + \ell(\lambda).$$

i) The *monotone simple base h Hurwitz number* $H_{g,h}^{\leq}(\lambda)$ is $\frac{1}{d!}$ times the number of tuples $(\tau_1, \dots, \tau_r, , \alpha_1, \beta_1, \dots, \alpha_h, \beta_h)$ of permutations in $S(d)$ such that:

- $\sigma \in C_\lambda$;
- $\tau_1, \dots, \tau_r$ satisfies M1, that is $\tau_1, \dots, \tau_r$ is a monotone sequence of transpositions; and
- $\circ\tau_1 \circ \cdots \circ \tau_r \circ \beta = [\alpha_1, \beta_1] \dots [\alpha_h, \beta_h]$.

If the factorization additionally satisfies the connectedness condition (H3), we call the number *connected monotone simple base h Hurwitz number* and denote it by $H^{\circ,\leq}_{g,h}(\lambda)$.

ii) The *(connected) strictly monotone Hurwitz number* ($H^{\circ,<}_{g,h}(\lambda)$) $H^{<}_{g,h}(\lambda, \nu)$ is defined analogously by replacing condition (M1) with (M2).

The Hurwitz numbers are in a close relationship to the symmetric functions of Section 1.2.1 via the group algebra $\mathbb{C}[S(d)]$.

Definition 1.3.6.
Let G be a finite group. Then we define the *group algebra of G over* $\mathbb{C}$ to be the free vector space generated by basis vectors $(v_g)_{g\in G}$:

$$\mathbb{C}[G] := \operatorname{span}_{\mathbb{C}}(\{v_g : g \in G\}) = \bigoplus_{g\in G} \mathbb{C}v_g.$$

For convenience, we identify $v_g = g$ and denote elements of $\mathbb{C}[G]$ by

$$\sum_{g\in G} \alpha_g \cdot g,$$

where $\alpha_g \in \mathbb{C}$ for every $g \in G$. The algebra structure is induced by the group operation

$$v_g \cdot v_h = v_{gh}, \quad \forall g, h \in G$$

and extended bilinearly to the vector space $\mathbb{C}[G]$.

We are particularly interested in the group algebra $\mathbb{C}[S(d)]$ for $d \in \mathbb{N}$. An important class of elements in the group algebra of the symmetric group are the so called *Jucys–Murphy* elements.

Definition 1.3.7.
The *Jucys–Murphy* elements J_k, are the elements in $\mathbb{C}[S(d)]$ defined by

$$J_k = \sum_{i=1}^{k-1} (i, k), \quad \text{for } 2 \leq k \leq d.$$

It is a classical result [Juc74, Mur81] that the evaluation of symmetric polynomials in the Jucys–Murphy elements are contained in the center of the group algebra $\mathcal{Z}(\mathbb{C}[S(d)])$.

In particular, if we recall the elementary symmetric and complete symmetric polynomials from Definition 1.2.4, we have

$$e_k(J_2, \dots, J_d) = \sum_{2 \le i_1 < \dots < i_k \le d} J_{i_1} \dots J_{i_k} \in \mathcal{Z}(\mathbb{C}[S(d)])$$

and

$$h_k(J_2, \dots, J_d) = \sum_{2 \le i_1 \le \dots \le i_k \le d} J_{i_1} \dots J_{i_k} \in \mathcal{Z}(\mathbb{C}[S(d)]).$$

By definition we have the following relation to the (strictly) monotone Hurwitz numbers.

Proposition 1.3.8.
We have

$$H_r^{<}(\lambda, \nu) = \frac{1}{d!} \cdot [\mathrm{id}]\ C_\lambda C_\nu\, \mathsf{e}_r(J_2, \dots, J_d)$$

and

$$H_r^{\le}(\lambda, \nu) = \frac{1}{d!} \cdot [\mathrm{id}]\ C_\lambda C_\nu\, \mathsf{h}_r(J_2, \dots, J_d)\,.$$

where the operation [id] stands for the extraction of the coefficient of the identical permutation e in the group algebra, and C_λ is the conjugacy class of permutations of cycle type λ seen as the element

$$\sum_{\sigma \in C_\lambda} \sigma \in \mathbb{C}[S(d)].$$

Lemma 1.3.9.
For any $d \in \mathbb{N}$ and $\lambda, \nu \vdash d$, we have:

$$\sum_{\rho \vdash d} z_\lambda H^{<}(\lambda, \rho) \cdot z_\rho H^{\le}(\rho, \nu) = \delta_{\lambda,\nu}$$

and

$$\sum_{\rho \vdash d} z_\lambda H^{\le}(\lambda, \rho) \cdot z_\rho H^{<}(\rho, \nu) = \delta_{\lambda,\nu}.$$

We give the proof of this classical result, since it is used in one of the main steps of proving the higher order moment-cumulant relations in Theorem 2.5.2.

Proof. Using Proposition 1.3.8, we rewrite the generating series of (strictly) monotone Hurwitz numbers from Definition 1.3.3 and use Lemma 1.2.5 for the generating functions of the elementary and homogeneous symmetric polynomials, we find

$$\begin{aligned} H^{<}(\lambda, \nu) &= \frac{1}{d!} \cdot [\mathrm{id}]\ C_\lambda C_\nu \prod_{k=2}^{d} (1 + \hbar J_k)\,, \\ H^{\le}(\lambda, \nu) &= \frac{1}{d!} \cdot [\mathrm{id}]\ C_\lambda C_\nu \frac{1}{\prod_{k=2}^{d} (1 + \hbar J_k)}\,. \end{aligned} \tag{1.3.2}$$

In general, if B is in the center of $\mathbb{C}[S(d)]$, we have

$$\frac{1}{d!} \cdot [\mathrm{id}]\ C_\lambda C_\nu B = [C_\lambda]\ \frac{C_\nu B}{z_\lambda} = [C_\nu]\ \frac{C_\lambda B}{z_\nu}\,,$$

where we recall $z_\lambda = \frac{d!}{\# C_\lambda}$. We use this relation to compute for any $\lambda, \nu \vdash d$:

$$\begin{aligned}
\delta_{\lambda,\nu} &= \frac{z_\lambda}{d!} \cdot [\mathrm{id}]\ C_\lambda C_\nu = \frac{z_\lambda}{d!} \cdot [\mathrm{id}]\ C_\lambda \prod_{k=2}^{d} (1 + \hbar J_k) \cdot C_\nu \frac{1}{\prod_{k=2}^{d}(1+\hbar J_k)} \\
&= \frac{z_\lambda}{d!} \cdot [\mathrm{id}] \Bigg(\sum_{\rho,\rho' \vdash d} H^{<}(\lambda,\rho)\, z_\rho C_\rho \cdot H^{\leq}(\rho',\nu) z_{\rho'} C_{\rho'} \Bigg) \\
&= \sum_{\rho \vdash d} z_\lambda H^{<}(\lambda,\rho) \cdot z_\rho H^{\leq}(\rho,\nu)\,.
\end{aligned}$$

The second relation is proved similarly. □

Definition 1.3.10 ([ALS16]).
Let $d \geq 0$ and $\lambda, \nu \vdash d$.

i) The *disconnected free single Hurwitz number* $H_r^{|}(\lambda,\nu)$ is $\frac{1}{d!}$ times the number of triples (α, σ, β) of permutations of $[d]$ such that

- $\alpha \in C_\lambda$ and $\beta \in C_\nu$;
- $\sigma \in S(d)$ has colength r, equivalently $\#\sigma = d - r$; and
- $\alpha \circ \sigma \circ \beta = \mathrm{id}$.

In other words, in the group algebra we have

$$H_r^{|}(\lambda,\nu) = \frac{1}{d!} \cdot [\mathrm{id}]\ C_\lambda C_\nu \sum_{\substack{\rho \vdash d \\ \ell(\rho) = d - r}} C_\rho\,. \tag{1.3.3}$$

If we require the additional condition

- $\{\alpha, \beta, \sigma\}$ generates a transitive subgroup of $S(d)$,

we call the resulting number the *connected free single Hurwitz number* and denote it by $H_r^{|,\circ}(\lambda,\nu)$.

ii) The *free group Hurwitz number* $H_r^{||}(\lambda,\nu)$ is $\frac{1}{d!}$ times the weighted count of tuples $(\alpha, \sigma_1, \ldots, \sigma_k, \beta)$ from $k = 1, \ldots, r$ of permutations of $[d]$ such that

- $\alpha \in C_\lambda$ and $\beta \in C_\nu$;
- $\sigma_i \in S(d)$, with $\sum_i |\sigma_i| = r$ equivalently $\sum \#\sigma = dk - r$; and
- $\alpha \circ \sigma_1 \circ \cdots \circ \sigma_k \circ \beta = \mathrm{id}$,

where the weight is $(-1)^{r+k}$. In other words, in the group algebra we have

$$H_r^{||}(\lambda,\nu) = \frac{1}{d!} \cdot [\mathrm{id}]\ C_\lambda C_\nu \sum_{k=1}^{r} (-1)^{k+r} \sum_{\substack{\rho_1,\dots,\rho_k \vdash d \\ \sum \ell(\rho_i) = dk-r}} \prod_{i=1}^{k} C_{\rho_i}\,.$$

If we require the additional condition

- $\{\alpha,\beta,\sigma\}$ generates a transitive subgroup of $S(d)$),

we call the resulting number the *connected free group Hurwitz number* and denote it by $H_r^{||,\circ}(\lambda,\nu)$.

Proposition 1.3.11 ([ALS16]**).**
We have

$$H_r^{|}(\lambda,\nu) = H_r^{<}(\lambda,\nu) \quad \text{and} \quad H_r^{||}(\lambda,\nu) = H_r^{\leq}(\lambda,\nu).$$

Remark 1.3.12.
The latter result can be proved using either techniques from symmetric functions and its avatar $\mathbb{C}[S(d)]$ ([ALS16]), or by the elementary observation that every permutation admits a unique factorization into strictly monotone transpositions [BCDGF19]. We give a proof of the monotone case via the Möbious function and its recursion in Section 2.2.1.

1.4 Topological recursion

Topological recursion is a recursive procedure that computes a family of symmetric multidifferentials $(\omega_{g,n})_{g\geq 0, n\geq 1}$. Given the information of a spectral curve encoding the information of the initial values $\omega_{0,1}$ and $\omega_{0,2}$, it computes $\omega_{g,n}$ from $\omega_{g',n'}$ recursively on the Euler characteristic $-\chi(g,n) = 2g-2+n$. It originates in random matrix theory, where it computes the topological expansion of correlators of matrix integrals, see e.g. [Eyn05, CEO06, CE06]. Later, it has been detached from random matrix theory and formalized in [EO07]. Surprisingly, the same recursion formula for different initial data has been found to produce many other quantities from different subjects such as enumerative geometry, Gromov-Witten theory, integrable systems, and knot theory. In recent developments the original formulation of topological recursion has been generalized to find even more applications, e.g. via *blobbed topological recursion* [Bor15, BS17] by G. Borot and S. Shadrin or via *Quantum Airy structures* by M. Kontsevich and Y. Soibelman [KS17, ABCO17], see also [Bor17] for an introductory course. We however will only briefly discuss the original Checkov-Eynard-Oratin formulation, for an overview we refer to [Eyn14], for a lecture series to [Ora17] and for a full exposition to the textbook [Eyn16]. We start by defining the initial data, the spectral curve.

Definition 1.4.1.
A *spectral curve* (Σ, x, y, B) consists of

i) a Riemann surface Σ, not necessarily connected or compact;

ii) two function $x, y\colon \sigma \to \mathbb{C}$ such that $\mathrm{d}x$ is a meromorphic 1-form having finitely many zeros that are simple. We call these points *ramification points of* x. Further, we require $\mathrm{d}y$ to not vanish at the ramification points of x; and

iii) a symmetric meromorphic 2-form $B\colon \Sigma\times\Sigma \to \mathbb{C}$ with a double pole on the diagonal and no further singularities. That is

$$B(z_1, z_2) \sim \frac{\mathrm{d}z_1\mathrm{d}z_2}{(z_1 - z_2)^2} + f(z_1, z_2)$$

where f is holomorphic.

Remark 1.4.2.

i) The functions x, y in the definition of a spectral curve encode the 1-form $\omega_{0,1} = y\mathrm{d}x$ and B encodes the initial 2-form $\omega_{0,2} = B$. Thus, sometimes the spectral curve is equivalently defined by the data $(\Sigma, x, \omega_{0,1}, \omega_{0,2})$.

ii) In this thesis we are mostly interested in the case when Σ is *rational*, that is Σ is isomorphic to the Riemann sphere, $\Sigma \cong \mathbb{C}P^1$. Then it is known that (under some normalization assumption) the properties of B in Definition 1.4.1 determine B uniquely, and

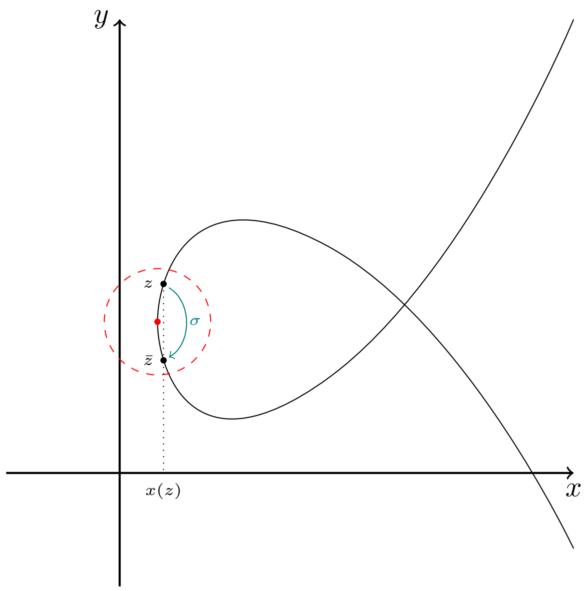
$$B(z_1, z_2) = \frac{\mathrm{d}z_1\mathrm{d}z_2}{(z_1 - z_2)^2}.$$

This 2-form is often called the *Bergman kernel*. Moreover, $x(z), y(z)$ are rational functions and hence they satisfy an algebraic equation

$$E(x, y) = 0,$$

which motivates the name spectral *curve*.

iii) Since $\mathrm{d}x$ has only simple zeros p, x is two-to-one in a neighbourhood of p and there is an involution σ_p interchanging the sheets locally above p. We illustrate the situation in the following figure.



Local involution (teal) in a neighbourhood of branch point p (red) mapping $z \mapsto \bar{z} := \sigma_p(z)$.

Definition 1.4.3 (Topological recursion)**.**
Given a spectral curve (Σ, x, y, B) we denote $\omega_{0,1} = y\mathrm{d}x$, $\omega_{0,2} = B$. For $g \geq 0, n \geq 1$ with $-\chi = 2g-2+n > 0$, we define the symmetric differential forms $\omega_{g,n}$ on Σ^n by the recursion

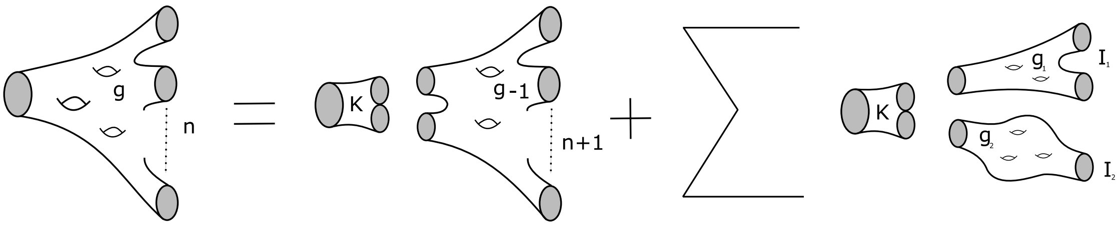
$$\begin{aligned}\omega_{g,n}(z_1,\ldots,z_n) = \sum_{\substack{p\in\Sigma \\ \mathrm{d}x(p)=0}} \operatorname*{Res}_{z\to p} K_p(z,z_1)\Big(&\omega_{g-1,n+1}(z,\sigma_p(z),z_2,\ldots,z_n) \\ &+ \sum_{\substack{g_1+g_2=g \\ I_1\sqcup I_2=[z_n]}}^{\text{stable}} \omega_{g_1,\#I_1}(z,z_{I_1})\omega_{g_2,\#I_2}(\sigma_p(z),z_{I_2})\Big)\end{aligned} \tag{1.4.1}$$

where we denote $[z_n] = \{z_1,\ldots,z_n\}$ and "stable" means we exclude the cases $(g_j, I_j) = (0,\emptyset)$, that is the cases where either of the factors in the sum is equal to $\omega_{0,1}$. Furthermore

$$K_p(z_1,z) = \frac{\int_{\sigma_p(z)}^{z}\omega_{0,2}(z_1,\cdot)}{2(\omega_{0,1}(z)-\omega_{0,1}(\sigma_p(z)))}$$

is called the *recursion kernel.*

The following picture demonstrates the structure of the recursion formula. The idea is that $\omega_{g,n}$ is understood as a genus g surface with n boundaries and can be obtained by gluing a pair of pants (the recursion kernel K) to a (possibly disconnected) surface with $-\chi(g',n') < -\chi(g,n)$ in (1.4.1). Hence, the name *topological recursion.*



Schematic representation of topological recursion.

Let us list some properties of topological recursion.

Theorem 1.4.4 ([EO07]).
The correlators $\omega_{g,n}$ of the topological recursion have the following properties.

i) *Symmetry:* The $\omega_{g,n}(z_1, \dots, z_n)$ are symmetric in its variables z_i.[1]

ii) *Pole behaviour:* If $2g-2+n>0$, then $\omega_{g,n}$ is meromorphic in each variable, with poles only at the ramification points of degree at most $6g-6+2n+2$ and with vanishing residue.

iii) *Homogenity:* For $\lambda \neq 0$ and $2g-2+n>0$, under interchanging y with λy in the spectral curve, the $\omega_{g,n}$ turn into $\lambda^{2-2g+n}\omega_{g,n}$.

iv) *Dilaton equation:* For $2g-2+n>0$, we have

$$\sum_{\substack{p\in\Sigma \\ \mathrm{d}x(p)=0}} \operatorname*{Res}_{z\to p} \omega_{g,n+1}(z_1,\dots,z_n,z)\Phi(z) = (2g-2+n)\omega_{g,n}(z_1,\dots,z_n),$$

where Φ is such that $\mathrm{d}\Phi = \omega_{0,1} = y\mathrm{d}x$.

Finally, let us list and discuss some examples.

Example 1.4.5.

- *TR for ordinary maps/Formal 1-matrix model:* Consider the formal 1-matrix integral [Eyn16]

$$\begin{aligned} Z &= \int_{\text{formal}} \exp\Big(-\frac{N}{t}\Big(\frac{\mathrm{Tr}(M^2)}{2} - \sum_{j=3}^{d} \frac{t_j}{j}\,\mathrm{Tr}(M^j)\Big)\Big)\mathrm{d}M \\ &= \int_{\text{formal}} \exp\Big(\frac{-N}{t}\,\mathrm{Tr}\,V(M)\Big)\mathrm{d}M \end{aligned}$$

for formal variables $t, t_1, t_3, t_4 \dots t_d, N$. It is formally defined via interchanging the expansion of the exponential and the integration. It is considered as a formal series. Note that in general such an expression does not converge, again see [Eyn16] for a

[1]Note that in (1.4.1) z_1 seems to play a special role.

rigorous discussion. But it turns out that Z is a generating series of maps without boundaries and bounded degree of internal faces,

$$Z = \sum_{\Xi \text{ closed maps}} \left(\frac{N}{t}\right)^{\chi(\Xi)} t_3^{n_3} \dots t_d^{n_d} \frac{t^{v(\Xi)}}{\#\mathrm{Aut}(\Xi)},$$

where n_i is the number of i-gons of Ξ and $v(\Xi)$ the number of vertices. Then the topological recursion (1.4.1), with initial data

$$\Sigma = \mathbb{C}P^1, \quad x(z) = \alpha + \gamma\Big(\frac{1}{z} + z\Big), \quad y(z)\mathrm{d}x(z) = \omega_{0,1}(z) = W_{0,1}(x(z))\mathrm{d}x(z),$$

where α, γ are some formal parameters depending on $t_1, \dots, t_n$ and

$$B(z_1, z_2) = \frac{\mathrm{d}z_1 \mathrm{d}z_2}{(z_1 - z_2)^2},$$

computes the generating series of maps of genus g with n boundaries. More precisely

$$\begin{aligned}\omega_{g,n}(z_1, \dots, z_n) = W_{g,n}(x(z_1), \dots, x(z_n))\mathrm{d}x(z_1) \dots \mathrm{d}x(z_n) \\ + \delta_{g,0}\delta_{n,2} \frac{\mathrm{d}x(z_1)\mathrm{d}x(z_2)}{(x(z_1) - x(z_2))^2} - \delta_{g,0}\delta_{n,1} \frac{V'(x(z_1))\mathrm{d}x(z_1)}{2},\end{aligned}$$

where

$$W_{g,n}(x_1, \dots, x_n) = \sum_{v=1}^{\infty} t^v \sum_{\Xi \in \mathbb{M}_{g,n}(v)} \frac{t_1^{n_1(\Xi)} \dots t_d^{n_d(\Xi)}}{x_1^{l_1(\Xi)} \dots x_n^{l_n(\Xi)}}$$

and where

i) $\mathbb{M}_{g,n}(v)$ is the set of maps of genus g having n boundaries and v vertices and internal faces of degree less than d,

ii) $l_i(\Xi)$ is the length of the i-th boundary for $i = 1, \dots, n$, and

iii) $n_i(\Xi)$ is the number of internal (unmarked) faces of degree i for $i = 1, \dots, d$.

- *Gaussian matrix model:* We present some calculations in a case where the spectral curve is very simple. Consider the functions

$$\begin{aligned} x(z) &= z + \frac{1}{z}, \\ y(z) &= \frac{1}{2}\Big(\frac{1}{z} - z\Big) = -\frac{1}{2}\sqrt{x^2 - 4}, \end{aligned} \tag{1.4.2}$$

and

$$B(z_1, z_2) = \frac{\mathrm{d}z_1 \mathrm{d}z_2}{(z_1 - z_2)^2},$$

then this initial data corresponds to the case $V(x) = \frac{t_2}{2}x^2$ above, but we omit the formal variables t, t_2. We compute some genus zero invariants of the Gaussian random matrix model. These are known to count planar pairings. Recall that ramification points are given by the zeroes of

$$\mathrm{d}x(z) = x'(z)\mathrm{d}z = \Big(1 - \frac{1}{z^2}\Big)\mathrm{d}z,$$

i.e., by $p = \pm 1$. Moreover, the (global) involution is given by $z \mapsto \frac{1}{z}$ for both ramification points. We compute

$$\begin{aligned}\omega_{0,1}(z) &= y(z)x'(z)\mathrm{d}z\\ \omega_{0,1}\Big(\frac{1}{z}\Big) &= y\Big(\frac{1}{z}\Big)x'\Big(\frac{1}{z}\Big)\mathrm{d}\frac{1}{z}\\ &= -y(z)x'(z)\mathrm{d}z\end{aligned}$$

and

$$w_{0,2}(z_1, z_2) = \frac{\mathrm{d}z_1\mathrm{d}z_2}{(z_1 - z_2)^2}.$$

Thus the recursion kernel is given by

$$\begin{aligned}K_{\pm 1}(z_1, z) &= \frac{\int_{\frac{1}{z}}^{z} \omega_{0,2}(z_1, \cdot)}{2(w_{0,1}(z) - w_{0,1}(\frac{1}{z}))}\\ &= \int_{\frac{1}{z}}^{z} \frac{\mathrm{d}z_1}{(z_1 - z)^2}\mathrm{d}z \frac{1}{4y(z)x'(\frac{1}{z})\mathrm{d}\frac{1}{z}}\\ &= \left(\frac{1}{z_1 - z} - \frac{1}{z_1 - \frac{1}{z}}\right)\frac{\mathrm{d}z_0}{4y(z)(1 - z^2)\mathrm{d}\frac{1}{z}}.\end{aligned}$$

Then by the recursion formula (1.4.1) we have

$$\omega_{0,3}(z_1, z_2, z_3) = \sum_{p=\pm 1} \operatorname*{Res}_{z\to p} K_p(z_1, z)\Bigg(\sum_{\substack{g_1+g_2=g\\ I_1\sqcup I_2=[z_n]}}^{\text{stable}} \omega_{g_1,\#I_1}(z, z_{I_1})\omega_{g_2,\#I_2}(\sigma_p(z), z_{I_2})\Bigg).$$

Recall, that the stable sum forbids $\omega_{0,1}$ as a factor in the sum. Thus, the only terms in the sum are given by $I_1 = \{z_i\}, I_2 = [z_2] \setminus I_2$ for $i = 2, 3$ and we obtain

$$\begin{aligned}\omega_{0,3}(z_1, z_2, z_3) = \sum_{p=\pm 1} \operatorname*{Res}_{z\to p} &\left(\frac{1}{z_1 - z} - \frac{1}{z_1 - \frac{1}{z}}\right)\frac{\mathrm{d}z_1}{4y(z)(1 - z^2)\mathrm{d}\frac{1}{z}}\\ &\times \left[\omega_{0,2}(z, z_2)w_{0,2}\Big(\frac{1}{z}, z_3\Big) + \omega_{0,2}(z, z_3)\omega_{0,2}\Big(\frac{1}{z}, z_2\Big)\right].\end{aligned}$$

We compute the residues

$$\operatorname*{Res}_{z\to\pm1}\left(\frac{1}{z_1-z}-\frac{1}{z_1-\frac{1}{z}}\right)\frac{\mathrm{d}z_1}{4y(z)(1-z^2)\mathrm{d}\frac{1}{z}}\left[\frac{\mathrm{d}z\mathrm{d}z_2\mathrm{d}\frac{1}{z}\mathrm{d}z_3}{(z-z_2)^2(\frac{1}{z}-z_3)^2}+\frac{\mathrm{d}z\mathrm{d}z_2\mathrm{d}\frac{1}{z}\mathrm{d}z_3}{(z-z_2)^2(\frac{1}{z}-z_2)^2}\right]$$

$$=\operatorname*{Res}_{z\to\pm1}\Bigg(\underbrace{\frac{-1}{z(z_1-z)(z_1-\frac{1}{z})}}_{\text{no pole at }\pm1}\Bigg)\frac{\mathrm{d}z_1}{4\underbrace{y(z)}_{\substack{\text{simple zeroes}\\ \text{at }\pm1}}}\Bigg[\underbrace{\frac{\mathrm{d}z\mathrm{d}z_2\mathrm{d}z_3}{(z-z_2)^2(\frac{1}{z}-z_3)^2}+\frac{\mathrm{d}z\mathrm{d}z_2\mathrm{d}z_3}{(z-z_3)^2(\frac{1}{z}-z_1)^2}}_{\text{no poles at }\pm1}\Bigg]$$

$$=\left(\frac{-\mathrm{d}z_1\mathrm{d}z_2\mathrm{d}z_3}{4z(z_1-z)(z_1-\frac{1}{z})}\right)\left[\frac{1}{(z-z_2)^2(\frac{1}{z}-z_3)^2}+\frac{1}{(z-z_3)^2(\frac{1}{z}-z_2)^2}\right]\Bigg|_{z=\pm1}\operatorname*{Res}_{z\to\pm1}\frac{\mathrm{d}z}{y(z)}$$

and

$$\operatorname*{Res}_{z\to\pm1}\frac{\mathrm{d}z}{y(z)}=\frac{1}{y'(\pm1)}.$$

Moreover

$$\begin{aligned}&\left(\frac{-\mathrm{d}z_1\mathrm{d}z_2\mathrm{d}z_3}{4z(z_1-z)(z_1-\frac{1}{z})}\right)\left[\frac{1}{(z-z_2)^2(\frac{1}{z}-z_3)^2}+\frac{1}{(z-z_3)^2(\frac{1}{z}-z_2)^2}\right]\Bigg|_{z=\pm1}\\ &=\frac{\mathrm{d}z_1\mathrm{d}z_2\mathrm{d}z_3}{4(z_1\mp1)(z_1\mp1)}\left[\frac{1}{(\pm1-z_2)^2(\pm1-z_3)^2}+\frac{1}{(\pm1-z_3)^2(\pm1-z_2)^2}\right]\\ &=\mp\frac{1}{2}\prod_{i=1}^{3}\left(\frac{1}{z_i\mp1}\right)\end{aligned}$$

and finally

$$\omega_3^{(0)}(z_1,z_2,z_3)=\frac{1}{2y'(-1)}\prod_{i=1}^{3}\frac{1}{(z_i+1)^2}-\frac{1}{2y'(1)}\prod_{i=1}^{3}\frac{1}{(z_i-1)^2}.$$

Let us verify that these invariants are generating series for Catalan numbers. We extract the coefficients by taking residues. In first order we have a shift by $\frac{V'(x(z))}{2}=\frac{x(z)}{2}$ and thus

$$\begin{aligned}m_{2l}^{(0)}&:=-\operatorname*{Res}_{z\to\infty}\Big((x(z))^{2l}\omega_1^{(0)}(z)-\frac{1}{2}V'(x(z))\mathrm{d}x(z)\Big)\\ &=-\operatorname*{Res}_{z\to\infty}\Big(z+\frac{1}{z}\Big)^{2l}\Big(y(z)+\frac{x(z)}{2}\Big)x'(z)\mathrm{d}z\\ &=-\operatorname*{Res}_{z\to\infty}\sum_{k=0}^{2l}\binom{2l}{k}z^{2(l-k)}\frac{1}{z}(1-\frac{1}{z^2})\mathrm{d}z\\ &=\operatorname*{Res}_{z\to0}\frac{1}{z^2}\sum_{k=0}^{2l}\binom{2l}{k}z^{2(k-l)}z(1-z^2)\end{aligned}$$

$$
\begin{aligned}
&= \operatorname*{Res}_{z\to 0} \sum_{k=0}^{2l} \binom{2l}{k} z^{2(k-l)-1} - z^{2(k-l)+1} \\
&= \left[\binom{2l}{l} - \binom{2l}{l-1} \right] \\
&= C_l
\end{aligned}
$$

where $C_l = \frac{1}{l+1}\binom{2l}{l}$ is the l-Catalan number. A similar computation shows that m_{2l+1} vanishes. Let us compute the coefficients of the second order,

$$
\begin{aligned}
m^{(0)}_{2l_1,2l_2} &= \operatorname*{Res}_{z_1,z_2\to\infty} x(z_1)^{2l_1} x(z_2)^{2l_2} \omega_{0,2}(z_1,z_2) \\
&= \operatorname*{Res}_{z_1,z_2\to\infty} \left(z_2 + \frac{1}{z_2}\right)^{2l_2} \left(z_1 + \frac{1}{z_1}\right)^{2l_1} \frac{\mathrm{d}z_1 \mathrm{d}z_2}{(z_1-z_2)^2} \\
&= \operatorname*{Res}_{z_1,z_2\to\infty} \left(z_2 + \frac{1}{z_2}\right)^{2l_2} \left(\sum_{j=0}^{2l_1} \binom{2l_1}{j} z_1^{2(j-l_1)} \right) \frac{1}{z_1^2} \frac{1}{(1-\frac{z_2}{z_1})^2} \\
&= \operatorname*{Res}_{z_1,z_2\to\infty} \left(z_2 + \frac{1}{z_2}\right)^{2l_2} \frac{1}{z_1^2} \sum_{j=0}^{2l_1} \binom{2l_1}{j} z_1^{2(j-l_1)} \left(\sum_{n=0}^{\infty} \left(\frac{z_2}{z_1}\right)^n \right)^2 \\
&= \operatorname*{Res}_{z_1,z_2\to\infty} \left(z_2 + \frac{1}{z_2}\right)^{2l_2} \frac{1}{z_1^2} \sum_{j=0}^{2l_1} \binom{2l_1}{j} z_1^{2(j-l_1)} \sum_{n=0}^{\infty} \sum_{k=0}^{n} \left(\frac{z_2}{z_1}\right)^k \left(\frac{z_2}{z_1}\right)^{n-k} \\
&= \operatorname*{Res}_{z_1,z_2\to\infty} (z_2 + \frac{1}{z_2})^{2l_2} \frac{1}{z_1^2} \sum_{j=0}^{2l_1} \binom{2l_1}{j} z_1^{2(j-l_1)} \sum_{n=0}^{\infty} (n+1) \left(\frac{z_2}{z_1}\right)^n \\
&= -\operatorname*{Res}_{z_2\to\infty} \left(z_2 + \frac{1}{z_2}\right)^{2l_2} \operatorname*{Res}_{z_1\to 0} \frac{1}{z_1^2} z_1^2 \sum_{j=0}^{2l_1} \binom{2l_1}{j} z_1^{l_1-j} \sum_{n=0}^{\infty} (n+1)(z_2 z_1)^n \\
&= -\operatorname*{Res}_{z_2\to\infty} \left(z_2 + \frac{1}{z_2}\right)^{2l_2} \operatorname*{Res}_{z_1\to 0} \sum_{j=0}^{2l_1} \sum_{n=0}^{\infty} \binom{2l_1}{j} (n+1) z_2^n z_1^{l_1-j+n}.
\end{aligned}
$$

Now as we take the residue $z_1 \to 0$, we take only the coefficient of z_1^{-1}, i.e. for every $0 \le j \le 2l_1$ we have

$$
2(l_1 - j) + n = -1 \iff n = 2(j - l_1) - 1.
$$

Since $n \ge 0$ we find that $j > l_1$. We continue the calculation

$$
\begin{aligned}
&-\operatorname*{Res}_{z_2\to\infty} \left(z_2 + \frac{1}{z_2}\right)^{2l_2} \operatorname*{Res}_{z_1\to 0} \sum_{j=0}^{2l_1} \sum_{n=0}^{\infty} \binom{2l_1}{j} (n+1) z_2^n z_1^{l_1-j+n} \\
&\quad = -\operatorname*{Res}_{z_2\to\infty} (z_2 + \frac{1}{z_2})^{2l_2} \sum_{j=l+1}^{2l_1} \binom{2l_1}{j} (2(j-l_1)) z_2^{2(j-l_1)-1}
\end{aligned}
$$

$$
\begin{aligned}
&= -\operatorname*{Res}_{z_2\to\infty} \sum_{k=0}^{2l_2} \binom{2l_2}{k} z_2^{2(k-l_2)} \sum_{j=l+1}^{2l_1} \binom{2l_1}{j} (2(j-l_1)) z_2^{2(j-l_1)-1} \\
&= \operatorname*{Res}_{z_2\to\infty} \frac{1}{z_2^2} \sum_{k=0}^{2l_2} \binom{2l_2}{k} z_2^{2(l_2-k)} \sum_{j=l+1}^{2l_1} \binom{2l_1}{j} (2(j-l_1)) z_2^{2(l_1-j)+1} \\
&= \operatorname*{Res}_{z_2\to\infty} \sum_{k=0}^{2l_2} \sum_{j=l+1}^{2l_1} \binom{2l_2}{k} \binom{2l_1}{j} (2(j-l_1)) z_2^{2(l_1+l_2-j-k)-1}.
\end{aligned}
$$

Again we compute the residue,

$$2(l_1+l_2-k-j)-1=-1 \iff l_1+l_2-k=j.$$

Without loss of generality, we put $l_1 \geq l_2$ and find that

$$l_1+1 \leq j = l_1+l_2-k \leq 2l_1 \iff l_2-l_1 \leq k \leq l_2-1,$$

where $l_2 - l_1 \leq k$ is redundant since we have $k \geq 0$ anyway. Finally, we have

$$
\begin{aligned}
m^{(0)}_{2l_1,2l_2} &= \operatorname*{Res}_{z_2\to\infty} \sum_{k=0}^{2l_2} \sum_{j=l+1}^{2l_1} \binom{2l_2}{k} \binom{2l_1}{j} (2(j-l_1)) z_2^{2(l_1+l_2-j-k)-1} \\
&= \sum_{k=0}^{l_2-1} \binom{2l_2}{k} \binom{2l_1}{l_1+l_2-k} (2(l_2-k)),
\end{aligned}
$$

which can be seen to be equal to two times

$$C_{l_1,l_2} = \frac{2l_1l_2}{l_1+l_2} \binom{2l_1-1}{l_1} \binom{2l_2-1}{l_2},$$

the number of annular noncrossing partitions. They agree with the number of annular pairings on $2l_1, 2l_2$ points, see [MN04, Corollary 6.8, Remark 6.9]. Finally, let us compute the coefficients of $\omega_3^{(0)}(z_0, z_1, z_2)$, the first quantity computed by the TR. Recall that

$$y'(z)\Big|_{z=\pm 1} = \left(\frac{\mathrm{d}}{\mathrm{d}z} \frac{1}{2}\left(\frac{1}{z} - z\right)\right)\Big|_{z=\pm 1} = -1$$

and

$$\omega_{0,3}(z_1,z_2,z_3) = \frac{1}{2y'(-1)} \prod_{i=1}^{3} \frac{1}{(z_i+1)^2} - \frac{1}{2y'(1)} \prod_{i=1}^{3} \frac{1}{(z_i-1)^2}$$

Thus we compute

$$\begin{aligned} m^{(0)}_{l_1,l_2,l_3} &= \operatorname*{Res}_{z_1,z_2,z_3\to\infty} x(z_1)^{l_1}x(z_2)^{l_2}x(z_3)^{l_3}\omega^{(0)}_3(z_1,z_2,z_3) \\ &= \frac{1}{2y'(1)}\left[\prod_{i=1}^{3}\operatorname*{Res}_{z_i\to\infty}\frac{x(z_i)^{l_i}}{(z_i-1)^2} - \prod_{i=1}^{3}\operatorname*{Res}_{z_i\to\infty}\frac{x(z_i)^{l_i}}{(z_i+1)^2}\right], \end{aligned}$$

hence for $j=\pm 1$ and a circle Γ with radius $r>1$ we compute by partial integration

$$\begin{aligned} \operatorname*{Res}_{z\to\infty}\frac{x(z)^l}{(z+j)^2} &= \frac{-1}{2\pi i}\int_\Gamma \frac{\left(\frac{1}{z}+z\right)^l}{(z+j)^2}\mathrm{d}z \\ &= \frac{-1}{2\pi i}\Bigg(-\underbrace{\left[\frac{\left(\frac{1}{z}+z\right)^l}{(z+j)}\right]^{\Gamma(1)}_{\Gamma(0)}}_{=0\,,\Gamma\text{ is closed}} + l\int_\Gamma \frac{\left(\frac{1}{z}+z\right)^{l-1}\left(1-\frac{1}{z^2}\right)}{(z+j)}\mathrm{d}z\Bigg) \\ &= -l\operatorname*{Res}_{z\to\infty}\left(\frac{1}{z}+z\right)^{l-1}\frac{(z+1)(z-1)}{z^2(z+j)} \\ &= -l\operatorname*{Res}_{z\to\infty}(\frac{1}{z}+z)^{l-1}\frac{z-j}{z^2} \\ &= -l\operatorname*{Res}_{z\to\infty}\sum_{m=0}^{l-1}\binom{l-1}{m}z^{l-1-2m}\left(\frac{z-j}{z^2}\right) \\ &= -l\left[\operatorname*{Res}_{z\to\infty}\sum_{m=0}^{l-1}\binom{l-1}{m}z^{l-2-2m} - j\sum_{m=0}^{l-1}\binom{l-1}{m}z^{l-3-2m}\right] \\ &= -l\left[\sum_{m=0}^{l-1}\binom{l-1}{m}(-\delta_{l-2-2m,-1}) - j\sum_{m=0}^{l-1}\binom{l-1}{m}(-\delta_{l-3-2m,-1})\right]. \end{aligned}$$

The nonzero contribution of the Kronecker delta are

$$l-2-2m=-1 \iff m=\frac{l-1}{2} \qquad \text{and} \qquad l-3-2m=-1 \iff m=\frac{l-2}{2},$$

so the first term only contributes if l is odd and the second only if l is even. Hence, we find

$$\begin{aligned} \operatorname*{Res}_{z\to\infty}\frac{x(z)^l}{(z+j)^2} &= \begin{cases} -l\binom{2q}{q} & \text{if } l=2q+1 \\ j\cdot l\binom{2q-1}{q-1} & \text{if } l=2q \end{cases} \\ &= -j^{l-1}\begin{cases} l\binom{2q}{q} & \text{if } l=2q+1 \\ l\binom{2q-1}{q-1} & \text{if } l=2q \end{cases} \\ &=: -j^{l-1}c_l. \end{aligned}$$

Thus we find

$$
\begin{aligned}
m_{l_1,l_2,l_3} &= \left[(-1)^{l_1+l_2+l_3-3} - 1\right] \\
&= -\frac{c_{l_1}c_{l_2}c_{l_3}}{2}\left[(-1)^{l_1+l_2+l_3-2} + 1\right] \\
&= -\frac{c_{l_1}c_{l_2}c_{l_3}}{2}\left[(-1)^{l_1+l_2+l_3} + 1\right]
\end{aligned}
$$

and in particular

$$
\begin{aligned}
m_{2l_1,2l_2,2l_3} &= 4\left(l_1\binom{2l_1-1}{l_1-1}l_2\binom{2l_2-1}{l_2-1}l_3\binom{2l_3-1}{l_3-1}\right) \\
&=: 4C_{l_1,l_2,l_3}.
\end{aligned}
$$

This is the number of noncrossing pairings on three circles with $2l_1, 2l_2, 2l_3$ points.

- *Simple Hurwitz numbers:* $\Sigma = \mathbb{C}P^1$, $x(z) = \ln(z) - z$, $y(z) = z$, $B(z_1, z_2) = \frac{\mathrm{d}z_1\mathrm{d}z_2}{(z_1-z_2)^2}$, then

$$
\omega_{g,n}(z_1, \dots, z_n) = \sum_{\ell(\mu)=n} H_g(\mu_1, \dots, \mu_n) \prod_{i=1}^n \mu_i e^{\mu_i x(z_i)} \mathrm{d}x(z_i).
$$

  See [EMS11, BEMS11].

- *Monotone Hurwitz numbers:* $\Sigma = \mathbb{C}P^1$, $x(z) = \frac{z-1}{z^2}$, $y(z) = -z$ and $B(z_1, z_2) = \frac{\mathrm{d}z_1\mathrm{d}z_2}{(z_1-z_2)^2}$, then

$$
\omega_{g,n}(z_1, \dots, z_n) = \sum_{\mu_1,\dots,m_n=1}^{\infty} \prod_{i=1}^n \mu_i x_i^{\mu_i-1} \mathrm{d}x_i.
$$

  See [DN18].

- *Bousquet-Mélou–Schaeffer numbers:*[2] $\Sigma = \mathbb{C}P^1$, $x(z) = \frac{(1+z)^m}{z}$, $y(z) = \frac{-z}{(1+z)^m}$, $B(z_1, z_2) = \frac{\mathrm{d}z_1\mathrm{d}z_2}{(z_1-z_2)^2}$, then

$$
\omega_{g,n}(z_1, \dots, z_n) = \sum_{\mu_1,\dots,\mu_n=1}^{\infty} b_g(\mu_1, \dots, \mu_n) \prod_{i=1}^n \mathrm{d}X_i^{\mu_i}
$$

  where $X = \frac{1}{x}$, see [BDBS20]. Note also that the Bousquet-Mélou–Schaeffer numbers are related to higher order free probability [CMSS07, Notation 5.17].

Let us finish with a note on *symplectic invariance.*

[2]Technically, these numbers do not satisfy the CEO topological recursion defined in Definition 1.4.3 but the Bouchard-Eynard formulation of [BE13].

Definition 1.4.6.

i) Let (Σ, x, y, B) be a spectral curve, $g \geq 2$ and Φ be a primitive of $\omega_{0,1}$, that is $\mathrm{d}\Phi = \omega_{0,1}$, then the *symplectic invariants*[3] $\mathcal{F}_g$ are defined by

$$\mathcal{F}_g = \omega_{g,0} = \frac{1}{2g-2} \sum_{\substack{p \in \Sigma \\ \mathrm{d}x(p)=0}} \operatorname*{Res}_{z \to p} (\omega_{g,1}(z)\Phi(z)).$$

ii) Two spectral curves $(\Sigma, x, y, B), (\Sigma, \widetilde{x}, \widetilde{y}, B)$ are said to be symplectically invariant if the symplectic form is preserved, that is

$$|\mathrm{d}x \wedge \mathrm{d}y| = |\mathrm{d}\widetilde{x} \wedge \mathrm{d}\widetilde{y}|.$$

It was known for a while [EO07] that the symplectic transforms

i) $(x, y) \mapsto (x, y + R(x))$, where R is a rational function,

ii) $(x, y) \mapsto (\frac{ax+c}{cx+d}, (cx+d)^2 y)$, where $ad - bc = 1$,

preserve the $\mathcal{F}_g$. The $x - y$ *swap*,

$$\widetilde{x} = y, \quad \widetilde{y} = x,$$

has turned out to be the most interesting case, see [EO13]. The first combinatorial accessible example of this $x-y$ swap has been introduced by G. Borot and E. Garcia-Failde [BGF20] and since then has been heavily studied [BCDGF19, BCGF21, BDBKS23]. In particular, we conjectured in [BCGF+23] that the higher order higher genus functional relations for moments and cumulants describe the relations between $\omega_{g,n}$ and $\widetilde{\omega}_{g,n}$ where $\widetilde{\omega}_{g,n}$ are the invariants computed from the spectral curve after the $x - y$ swap; see Conjecture 2.7.2. Indeed, the conjecture has been answered positively in [ABDB+22].

[3] There are also invariants $\mathcal{F}_0$ and $\mathcal{F}_1$, however their definitions are more involved see [EO07, Eyn16].

2 Higher order and surfaced free probability

This chapter is an exposition of recent developments in higher order free and surfaced free probability. We explain our results of the paper [BCGF+23], which solve the problem of finding the functional relations between higher order Cauchy and $\mathcal{R}$-transform.

Higher order free probability was introduced in the series of papers [MN04], [MS06] and [MŚS07]. The second order theory is motivated by the study of fluctuations in random matrices. These quantities can be described by covariances of traces of powers of the random matrix. Furthermore, in examples, it has been discovered that their limits show a similar behaviour as the limiting eigenvalue distribution in Voiculescu's first order free probability theory; [MN04, MS06]. In particular, it turns out that joint covariances of traces of several matrices in a general position can be computed by the knowledge of the covariances of traces of the individual ones. This fact justifies the name second order freeness. Thus, we start this chapter by explaining second order free probability, and afterward continue explaining the motivation of higher order free probability from random matrix theory.

In first order free probability, an efficient tool to deal with joint distributions of free random variables is the $\mathcal{R}$ transform; the generating series of cumulants. For instance, to compute the distribution of the sum $a + b$ of two free random variables, it is particularly well-suited. Thus, the authors of [CMSS07] introduce a combinatorial framework of higher order free cumulants via so-called partitioned permutations. Despite the fact that partitioned permutations do not form a lattice, the theory of multiplicative functions evolves in parallel to the incidence algebra formalism of the noncrossing partitions. We explain this combinatorial setting in Section 2.1.2. In particular, we state the second order functional relations discovered in [CMSS07]. In fact, the second order theory with most of its features is easily expanded to a more general n-th order theory. However, the functional relations for the moment-cumulant formalism beyond second order could not be derived by Collins, Mingo, Śniady and Speicher. We explain the obstacles in their approach in Section 2.1.4.

The main result of this thesis is the derivation of the missing functional relations in higher order free probability. Thus, the main task in this chapter is to explain our paper [BCGF+23]. We start by introducing an extension of the setting of [CMSS07] to a higher genus in Section 2.2. This is necessary to relate multiplicative functions to so-called partition functions in the Fock space in Section 2.3. Furthermore, the extension allows us to relate the Möbius function to monotone Hurwitz numbers in Section 2.2.1. In fact, the latter observation was discovered while working on the paper [HvIL22], which will be discussed in Chapter 3. At a first glance, the new functional relations are very complicated formulas. But they are actually combinatorial equations involving a sum over certain bicoloured trees, and thus become more accessible when explained graphically.

We present and explain some examples of the formula in Section 2.4. Afterward, we want to address the proof of our main results, which splits into three major steps. These three steps are explained in the three subsections of Section 2.5. At this point, the relation of multiplicative functions and partition functions in the Fock space is explained, but we are lacking a reformulation of the moment-cumulant convolution formulas in terms of the Fock space language. Hence, the first step is to reformulate the relations in terms of an operator equation. Then, the second step is to manipulate the operators in the Fock space, using the techniques of [BDBKS22, BDBKS23]. This establishes extended functional relations in all genera, and we conclude by extracting the genus zero sector to derive the functional relations in higher order free probability and thus answer the question of [CMSS07].

Naturally, our extension of the moment-cumulant formalism to higher genus comes with a notion of freeness by the vanishing of mixed cumulants. We explain this new extension of free probability in Section 2.6, we call it surfaced free probability; following [CMSS07, Appendix 9]. In order to show that this definition is sensible, we prove important properties of freeness, for instance that freeness is independent of the choice of generators of an algebra. Furthermore, we recover many known instances of freeness, such as Voiculescu's first order free probability, freeness of all order of Collins, Mingo, Śniady and Speicher and surprisingly also the notion of infinitesimal freeness of Février and Nica.

Finally, in Section 2.7, we explain how our results relate to the theory of topological recursion. In particular, we propose that our newly discovered relations in higher order and higher genus can be reformulated in the language of differential forms and then describe the $x - y$ duality in topological recursion. We conclude this chapter with further questions regarding the connection of free probability with topological recursion.

We want to emphasize that we changed the notation from [BCGF$^+$23] to present the results from a perspective catered towards free probability and to align it with the rest of the notation.

2.1 Introduction and prior work

In this section, we recall some key results of higher order free probability. Mostly, we will omit the proofs, they can be found in [MS06], [MŚS07] and [CMSS07].

The motivation for higher order free probability was to capture a more refined behaviour of the eigenvalue distribution, namely the fluctuation around the limiting distribution. Our starting point is the observation that the fluctuation of the moments

$$\mathrm{tr}(A_N^r) - \lim_{N\to\infty} \mathbb{E}[\mathrm{tr}\, A_N^r]$$

of a random matrix $A = (A_N)_{N\in\mathbb{N}}$ are often asymptotically Gaussian of order $\frac{1}{N}$, and the information is captured in the quantity

$$\mathrm{k}_2(\mathrm{Tr}(A_N^r), \mathrm{Tr}(A_N^s)),$$

where k_2 is a classical cumulant; see Notation 1.1.22. Mingo and Speicher studied the limiting behaviour of these classical cumulants of traces in the case of Whishard matrices. Their study of these examples led them to the following definition of a second order limiting distribution; cf. Definition 1.1.24.

Definition 2.1.1.
Let $A = (A_N)_{N\in\mathbb{N}}$ be a sequence of $N \times N$ random matrices. The sequence $(A_N)_{N\in\mathbb{N}}$ has a *second order limiting distribution* if

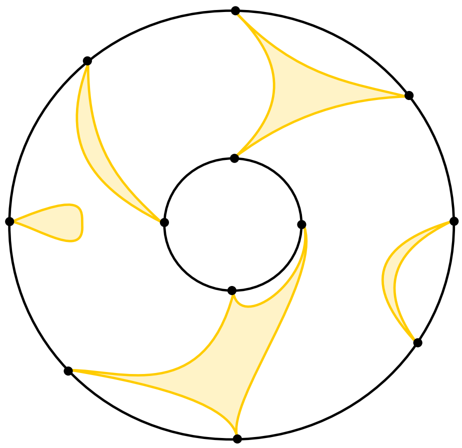
$$\varphi_r^A := \lim_{N\to\infty} \mathrm{k}_1[\mathrm{tr}(A_N^r)] \quad \text{and} \quad \varphi_{r_1,r_2}^A := \lim_{N\to\infty} \mathrm{k}_2(\mathrm{Tr}(A_N^{r_1}), \mathrm{Tr}(A_N^{r_2}))$$

exists for all $r, r_1, r_2 \in \mathbb{N}$ and

$$\lim_{N\to\infty} \mathrm{k}_n[\mathrm{Tr}(A_N^{r_1}), \ldots, \mathrm{Tr}(A_N^{r_n})] = 0$$

for any $n \geq 3$ and $r_1, \ldots, r_n \in \mathbb{N}$.

Furthermore, they discovered that the limiting second order distribution of Whishart matrices can be described by annular diagrams.



Example of an annular diagram.

Inspired by first order freeness, the question arose whether there is a notion of second order freeness, i.e. a way of computing the limiting mixed second order moments of matrices A_N, B_N provided we know the second order limiting distribution of the individual matrices A_N and B_N. Using the combinatorial description in terms of annular diagrams, they proposed the following notion of asymptotic second order freeness.

Definition 2.1.2 ([MN04], [CMSS07])**.**
Let $A = (A_N)_{N\in\mathbb{N}}, B = (B_N)_{N\in\mathbb{N}}$ be random matrices having a second order limiting distribution. We say A_N and B_N are *asymptotically free* if

$$\lim_{N\to\infty} \mathrm{k}_2(\mathrm{Tr}(A_N^{r_1}) - \varphi_{r_1}^A, \mathrm{Tr}(B_N^{r_2}) - \varphi_{r_2}^B) = 0 \quad \text{for all } r_1, r_2 \in \mathbb{N}$$

and

$$\lim_{N\to\infty} \kappa_2(\mathrm{Tr}(\Omega_{s_1}(A_N)\Omega_{r_1}(B_N)\dots\Omega_{s_n}(B_N)), \mathrm{Tr}(\Omega_{\tilde{s}_1}(A_N)\Omega_{\tilde{s}_1}(B_N)\dots\Omega_{\tilde{s}_m}(B_N)))$$
$$= \delta_{m,n}\sum_{i=1}^{n}\prod_{j=1}^{n}(\varphi^A_{r_{j+i}+\tilde{r}_j} - \varphi^A_{r_{j+i}}\varphi^A_{\tilde{r}_j})(\varphi^B_{s_{j+i}+\tilde{s}_j} - \varphi^B_{s_{j+i}}\varphi^B_{\tilde{s}_j}),$$

where we denote $\Omega_r(A_N) = \mathrm{Tr}(A_N^{r_1}) - \varphi^A_{r_1}$ ($\Omega_r(B_N)$ analogously) for all $m, n \in \mathbb{N}$ and $r_1, s_1, \dots, r_n, s_n \in \mathbb{N}$, $\tilde{r}_1, \tilde{s}_1, \dots, \tilde{r}_m, \tilde{s}_m \in \mathbb{N}$.

This motivates the following abstract definition of a second order noncommutative probability space and second order freeness. In the following definition, φ_2 plays the role of the limit of $\kappa_2(\cdot,\cdot)$ in Definition 2.1.2.

Definition 2.1.3 ([MŚS07], [CMSS07])**.**

i) A *second order noncommutative probability space* $(\mathcal{A}, \varphi, \varphi_2)$ consists of the data of a tracial noncommutative probability space $(\mathcal{A}, \varphi)$, endowed with a symmetric bilinear function

$$\varphi_2 \colon \mathcal{A} \times \mathcal{A} \to \mathbb{C}$$

that is tracial in both arguments, and it holds that

$$\varphi_2(1, a) = \varphi_2(a, 1) = 0,$$

for all $a \in \mathcal{A}$.

ii) Let $(\mathcal{A}, \varphi, \varphi_2)$ be a second order noncommutative probability space. We say that the subalgebras $\mathcal{A}_1, \dots, \mathcal{A}_s \subset \mathcal{A}$ are *free of second order* if

$$\varphi_2(a, b) = 0,$$

where a and b are centered and belong to different algebras $\mathcal{A}_{j_1}, \mathcal{A}_{j_2}$. Furthermore, any centered cyclically alternating tuples $(a_1, \dots, a_n)$, $(b_1, \dots, b_m)$ satisfy

$$\varphi_2(a_1, \dots, a_n; b_1, \dots, b_m) = \delta_{m,n}\sum_{i=1}^{n}\varphi(a_1 b_{1+i})\dots\varphi(a_n b_{n+i}),$$

where the indices are understood modulo n.

Similar to the first order case, the freeness rule yields a formula for computing mixed second order moments from individual ones, but in an inefficient way. Recall that in first order, Speicher developed the free moment-cumulant formalism to overcome this problem. In [CMSS07] the authors extend the first order moment-cumulant formalism to second and higher orders. Using the second order machinery, they derive the following second order analogue of Voiculescu's asymptotic freeness in first order; see Theorem 1.1.28.

Theorem 2.1.4 ([MŚS07]).
Let $A = (A_N)_{N\in\mathbb{N}}, B = (B_N)_{N\in\mathbb{N}}$ be independent random matrices having a second order limiting distribution, with at least one of them being unitarily invariant. Then A and B are asymptotically free of second order in the sense of Definition 2.1.2.

Furthermore, they derive analogues of the $\mathcal{R}$-transform formula in second order. We state their main results here in advance, without giving the precise definition of the second order free cumulants.

Theorem 2.1.5 ([CMSS07]).
Let $(\mathcal{A}, \varphi, \varphi_2)$ be a second order noncommutative probability space and $a \in \mathcal{A}$. We denote $\varphi_r^a = \varphi(a^r)$, $\varphi_{r,s}^a = \varphi_2(a^r, a^s)$ and let $k_r^a, \kappa_{r,s}^a$ be the first and second order cumulants, then we denote

$$C(x) := 1 + \sum_{i=1}^{\infty} \kappa_i^a x^i, \qquad M(x) := 1 + \sum_{i=1}^{\infty} \varphi_i^a x^i$$

and

$$C(x_1, x_2) := \sum_{i_1,i_2=1}^{\infty} \kappa_{i_1,i_2}^a x_1^{i_1} x_2^{i_2}, \qquad M(x_1, x_2) := \sum_{i_1,i_2=1}^{\infty} \varphi_{i_1,i_2}^a x_1^{i_1} x_2^{i_2}.$$

Then the generating series satisfy the functional relation in second order

$$\begin{aligned} M(x_1, x_2) &= C(x_1 M(x_1), x_2 M(x_2)) \frac{\partial_{x_1}(x_1 M(x_1))}{M(x_1)} \frac{\partial_{x_2}(x_2 M(x_2))}{M(x_2)} \\ &\quad + x_1 x_2 \left(\frac{\partial_{x_1}(x_1 M(x_1)) \partial_{x_2}(x_2 M(x_2))}{(x_1 M(x_1) - x_2 M(x_2))^2} - \frac{1}{(x_1 - x_2)^2} \right). \end{aligned}$$

It also turns out that the second order cumulants capture freeness in an easier way than the moments from Definition 2.1.3, by *vanishing of mixed second order cumulants.* This implies the additivity of the free cumulants

Theorem 2.1.6 ([CMSS07]).
Let $(\mathcal{A}, \varphi, \varphi_2)$ be a second order noncommutative probability space and $a, b \in \mathcal{A}$ be noncommutative random variables, which are free of second order. Then

$$\kappa_r^{a+b} = \kappa_r^a + \kappa_r^b \quad \text{and} \quad \kappa_{r,s}^{a+b} = \kappa_{r,s}^a + \kappa_{r,s}^b$$

where the κ are the limiting free cumulants.

Remark 2.1.7.
Theorem 2.1.5 can be formulated in terms of the Cauchy- and R-transforms. With Remark 1.1.21 and

$$G(x_1, x_2) = \frac{M\left(\frac{1}{x_1}, \frac{1}{x_1}\right)}{x_1 x_2} \quad \text{and} \quad \mathcal{R}(x_1, x_2) = \frac{C(x_1, x_2)}{x_1 x_2},$$

we have ([CMSS07, Corollary 6.4]):

$$G(x_1, x_2) = G'(x_1)G'(x_2)\left(\mathcal{R}(G(x_1), G(x_2)) + \frac{1}{(G(x_1) - G(x_2))^2}\right) - \frac{1}{(x_1 - x_2)^2}.$$

More generally, they introduce higher order noncommutative probability spaces. These are spaces equipped with even more functionals φ_n, $n \in \mathbb{N}$, and are motivated by limits of higher order classical cumulants of traces:

$$\mathrm{k}_n(\mathrm{Tr}(A^{r_1}), \dots, \mathrm{Tr}(A^{r_n})).$$

The higher order free moment-cumulant formalism is then described by combinatorial objects called *partitioned permutations*. We explain the theory of multiplicative functions on the set of partitioned permutations in Section 2.1.2 and the motivation from random matrices in the following section.

2.1.1 Motivation from random matrices

Although the main result of [CMSS07] only proves functional relations for the moment and cumulant generating series for second order (and reproves the R-transform formula of Voiculescu for first order (Theorem 1.1.20), they introduce the moment-cumulant formalism for all orders. As in the previous section, we have seen that the second order moments are motivated by the limits of second classical cumulant of traces of a random matrix $A = (A_N)_{N\in\mathbb{N}}$. The natural extension is to take limits of the n-th classical cumulants of traces

$$\lim_{N\to\infty} N^{n-2}\mathrm{k}_n(\mathrm{Tr}(A_N^{r_1}), \dots, \mathrm{Tr}(A_N^{r_n})), \quad n, r_1, \dots, r_n \in \mathbb{N}, \tag{2.1.1}$$

where the convergence order is motivated by computations for Gaussian and Wishart ensembles. Given random matrices $A_1, \dots, A_d$, $d \in \mathbb{N}$, the joint higher order moments can be encoded using permutations in the symmetric group. Consider a cycle decomposition of a permutation $\pi = c_1 \dots c_n \in S(d)$ and set

$$A|_c = A_{i_1}A_{i_2}\dots A_{i_r}$$

for a cycle of length $r \in \mathbb{N}$ and $c = (i_1, \dots, i_r)$, where $i_1, \dots i_r \in [d]$ are pairwise distinct. We define

$$\varphi(\pi)[A_1, \dots, A_d] := \varphi_n(A|_{c_1}, \dots, A|_{c_n}) := k_n(\mathrm{Tr}(A|_{c_1}), \dots, \mathrm{Tr}(A|_{c_n})).$$

A posteriori we can see from Theorem 2.1.6 that we need to deal with products of different orders, thus it necessitates to develop a theory that can capture all levels $k \leq n$ if we want to speak of freeness of order n. We will deal with these products by the notion of *partitioned permutations*. Before we continue our motivation, we introduce some notation.

Notation 2.1.8.

i) Let $d \in \mathbb{N}$, then we denote $\mathbf{1}_d = \{\{1, \dots, d\}\} \in \mathcal{P}(d)$.

ii) Given a permutation $\sigma \in S(d)$, we denote by $\mathbf{0}_\sigma \in \mathcal{PS}(d)$ the partition given by the orbits of σ. Sometimes we omit the index σ for convenience.

Given a permutation $\pi \in S(d)$, we introduce a partition $\mathcal{V} \in \mathcal{P}(d)$ that partitions the cycles of π. More concretely, we require $\pi(B) \subseteq B$ for all $B \in \mathcal{V}$. In other words, the moved points of each cycle of π are contained in exactly one block of $\mathcal{V}$. In this case, we have $\mathbf{0}_\pi \leq \mathcal{V}$ and often write $\pi \leq \mathcal{V}$. Let us illustrate this idea by the following example.

Example 2.1.9.
Consider $\pi = (123)(4)(56)(789)$, then

$$\pi \leq \{\{1,2,3,4\},\{5,6\},\{7,8,9\}\} \text{ but } \pi \not\leq \{\{1,2\},\{3,4,5,6,7,8,9\}\},$$

since the cycle (123) does not leave the blocks invariant in the last example.

Then we can patch higher order moments together via

$$\varphi(\mathcal{V},\pi)[A_1,\dots,A_d] = \prod_{B\in\mathcal{V}} \varphi(\pi)|_B[A_1,\dots,A_d|_B] \tag{2.1.2}$$

for any $\pi \leq \mathcal{V}$, where $A_1,\dots,A_d|_B$ means we take only indices that are contained in the block B. Let us illustrate this procedure again by an example.

Example 2.1.10.
Take $\pi \leq \mathcal{V}$ from the last example, i.e.

$$(123)(4)(56)(789) \leq \{\{1,2,3,4\},\{5,6\},\{7,8,9\}\},$$

then

$$\begin{aligned}\varphi(\mathcal{V},\pi)[A_1,\dots,A_9] &= \varphi_2(A_1A_2A_3,A_4)\varphi(A_5A_6)\varphi(A_7A_8A_9)\\ &= k_2(\mathrm{Tr}(A_1A_2A_3),\mathrm{Tr}(A_4))k_1(\mathrm{Tr}(A_5A_6))k_1(\mathrm{Tr}(A_7A_8A_9)).\end{aligned}$$

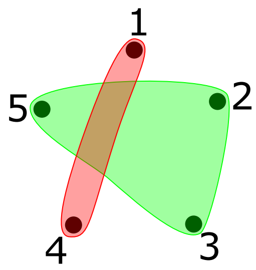
Remark 2.1.11.
Let $\pi \leq \mathcal{V}$ with $\pi \in S(d)$ and $\mathcal{V} \neq \mathbf{1}_d$, then one may think of $\varphi(\mathcal{V},\pi)[A_1,\dots,A_d]$ as a *disconnected* higher order moment. On the other hand, if $\mathcal{V} = \mathbf{1}_d$, we think of $\varphi(\mathcal{V},\pi)[A_1,\dots,A_d]$ as *connected* moments. This interpretation will become more evident in Section 2.5.

The latter discussion motivates the definition of the main combinatorial tool in higher order free probability, the notion of *partitioned permutations*.

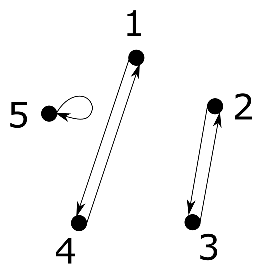
Definition 2.1.12.
A *partitioned permutation* of d elements is a pair $(\mathcal{V}, \pi)$, where $\mathcal{V} \in \mathcal{P}(d)$ and $\pi \in S(d)$, such that $\mathbf{0}_\pi \leq \mathcal{V}$. We denote by $\mathcal{PS}(d)$ the set of partitioned permutations of d elements and write $\mathcal{PS} = \bigcup_{d\geq 1} \mathcal{PS}(d)$.

Before we continue, let us give some visualisation of this definition in the following example.

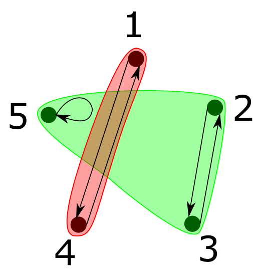
Example 2.1.13.
One may visualize the partitioned permutation $(\mathcal{V}, \pi)$, with $\mathcal{V} = \{\{1,4\},\{2,3,5\}\}$ and $\pi = (14)(23)(5)$, by the following diagrams. The cycles of π are contained in the blocks of $\mathcal{V}$, we are indeed partitioning the cycles of the permutation.



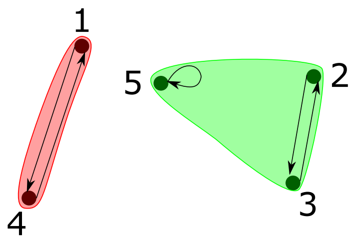
$\mathcal{V} = \{\{1,4\},\{2,3,5\}\}$



$\pi = (14)(23)(5)$



$(\mathcal{V}, \pi)$



$(\mathcal{V}, \pi)$

In the last picture we want to emphasize the disconnectedness of the quantity $\varphi(\mathcal{V}, \pi)$. Also, cf. Example 2.1.22.

By calculations in the unitary group, more precisely by the so-called *Weingarten calculus* (see [Col03]), the authors of [CMSS07] obtain the following theorem, which defines the asymptotic free cumulants of higher order.

Theorem 2.1.14 ([CMSS07])**.**

i) Let $n \in \mathbb{N}$ and $A_1 = (A_1^N)_{N\in\mathbb{N}}, \ldots, A_n = (A_n^N)_{N\in\mathbb{N}}$ be random matrices, then we define *correlation moments*[1] by

$$\varphi_n^N(A_1, \ldots, A_n) \coloneqq \mathrm{k}_n(\mathrm{Tr}(A_1^N), \ldots, \mathrm{Tr}(A_n^N))$$

and their extension to partitioned permutations $(\mathcal{V}, \pi) \in \mathcal{PS}$ as in (2.1.2). The corresponding cumulant functions are defined by

$$\kappa^N(\mathcal{V}, \pi)[A_1, \ldots, A_n] = \sum_{(\mathcal{W},\sigma)\in\mathcal{PS}(n)} \varphi^N(\mathcal{W}, \sigma)[A_1, \ldots, A_n]\mathrm{C}^N_{\mathbf{0}_\pi\vee\mathcal{W},\mathcal{V}}(\sigma\pi^{-1}), \tag{2.1.3}$$

where

$$\mathrm{C}^N_{\mathbf{0}_\pi\vee\mathcal{W},\mathcal{V}}(\sigma\pi^{-1}) = \sum_{\substack{U\in\mathcal{P}(n)\\ \mathcal{V}\geq\mathcal{U}\geq\mathbf{0}_\pi\vee\mathcal{W}}} \mathrm{M\ddot{o}b}(\mathcal{U}, \mathcal{V})\mathrm{Wg}(\mathcal{U}, \sigma\pi^{-1})$$

is the relative cumulant of the Weingarten function. If the $A_k^N = (A_{i,j}^{(k)})_{i,j=1}^N$ for $k = 1, \ldots, n$ are unitarily invariant and $n \leq N$, then

$$\kappa(\mathbf{1}_n, \pi)[A_1, \ldots, A_n] = \mathrm{k}_n(A^{(1)}_{i_1,i_{\pi(1)}}, \ldots, A^{(n)}_{i_n,i_{\pi(n)}}),$$

for distinct $i_1, \ldots, i_n$.

ii) Let $A_1 = (A_1^N)_{N\in\mathbb{N}}, \ldots, A_n = (A_n^N)_{N\in\mathbb{N}}$ be unitarily invariant random matrices and $B_1 = (B_1^N)_{N\in\mathbb{N}}, \ldots, B_n = (B_n^N)_{N\in\mathbb{N}}$ independent from $A_1, \ldots, A_d$. Then

$$\varphi^N(\mathcal{U}, \gamma)[A_1B_1, \ldots, A_nB_n] = \sum_{\substack{(\mathcal{V},\pi),(\mathcal{W},\sigma)\in\mathcal{PS}(n)\\ \mathcal{V}\vee\mathcal{W}=\mathcal{U},\gamma=\pi\sigma}} \kappa^N(\mathcal{V}, \pi)[A]\varphi^N(\mathcal{W}, \sigma)[B], \tag{2.1.4}$$

where we abbreviate $A = A_1, \ldots, A_n$ and $B = B_1, \ldots, B_n$.

Remark 2.1.15.
Instead of (2.1.3) one can write the defining equations implicitly as

$$\varphi^N(\mathcal{U}, \gamma)[A_1, \ldots, A_n] = \sum_{\substack{(\mathcal{V},\pi)\in\mathcal{PS}(n)\\ \mathcal{V}\vee\gamma\pi^{-1}=\mathcal{U}}} \kappa^{(N)}[A_1, \ldots, A_n](\mathcal{V}, \pi)N^{\#\gamma\pi^{-1}}. \tag{2.1.5}$$

Recall from Equation (2.1.1) that we assume the leading order of the correlation moments to be N^{2-n}. Then we can deduce the convergence order for the cumulant functions.

[1]Note despite being defined via classical cumulants, these quantities play the role of moments in this theory.

Theorem 2.1.16.
Let $A_1 = (A_1^N)_{N\in\mathbb{N}}, \ldots, A_d = (A_d^N)_{N\in\mathbb{N}}$ be random matrices such that the limits

$$\lim_{N\to\infty} N^{n-2} k_n(p_1(A_1^N, \ldots, A_d^N), \ldots, p_n(A_1^N, \ldots, A_d^N))$$

exist for all $n \in \mathbb{N}$ and all choices of polynomials $p_1, \ldots, p_n \in \mathbb{C}\langle z_1, \ldots, z_d\rangle$ in d non-commuting variables. Then we say $A_1, \ldots, A_d$ *have a (joint) limiting distribution* and we have that for the cumulants of (2.1.3) the limits

$$\lim_{N\to\infty} N^{n-2\#\mathcal{V}+\#\pi} \kappa_d^N(\mathcal{V}, \pi)[A_1, \ldots, A_d]$$

exist for any $(\mathcal{V}, \pi) \in \mathcal{PS}(d)$.

Remark 2.1.17.
If $A_1 = (A_1^N)_{N\in\mathbb{N}}, \ldots, A_d = (A_d^N)_{N\in\mathbb{N}}$ have a joint limiting distribution, then $\varphi^N(\mathcal{V}, \pi)$ is of order $N^{2\#\mathcal{V}-\#\pi}$.

Therefore, we find for the leading order the equation in Equation (2.1.4)

$$|\gamma| - 2|\mathcal{U}| = (|\pi| - 2|\mathcal{V}|) + (|\sigma| - 2|\mathcal{W}|),$$

where $\mathcal{U} = \mathcal{V} \vee \mathcal{V}$ and $\gamma = \pi\sigma$.

Remark 2.1.18.
Let us also mention here that the results of [CMSS07] were rediscovered and connected to recent developments. Theorem 2.1.14 was rediscovered in [BGF20] and formulated in terms of (strictly) monotone Hurwitz numbers. Moreover, they discovered that combinatorial maps and a special variant, fully simple maps, satisfy the relations of Theorem 2.1.5. These developments started the interest in connecting free probability theory to the theory of topological recursion and started the collaboration [BCGF$^+$23].

2.1.2 Multiplicative functions on $\mathcal{PS}$

In the last section, we have seen that the higher order limiting distributions admit a description in terms of so-called partitions permutations. Furthermore, the leading order can be expressed in terms of the involved partitioned permutations. These observations motivate the following combinatorial framework for higher order free probability.

Definition 2.1.19.
Let $d \geq 1$ be an natural number.

i) Let $(\mathcal{V}, \pi) \in \mathcal{PS}(d)$ be a partitioned permutation, then we define its *colength* by

$$|(\mathcal{V}, \pi)| := 2|\mathcal{V}| - |\pi|.$$

ii) Let $(\mathcal{V},\pi),(\mathcal{W},\sigma)\in\mathcal{PS}(d)$ be partitioned permutations. We define their product by

$$(\mathcal{V},\pi)\cdot(\mathcal{W},\sigma)=\begin{cases}(\mathcal{V}\vee\mathcal{W},\pi\sigma) & \text{if } |(\mathcal{V},\pi)|+|(\mathcal{W},\sigma)|=|(\mathcal{V}\vee\mathcal{W},\pi\sigma)|,\\ 0 & \text{otherwise.}\end{cases}$$

We will usually omit $\cdot$ and just write $(\mathcal{V},\pi)(\mathcal{W},\sigma)$ for the product of two partitioned permutations.

Remark 2.1.20.
The product of partitioned permutations is associative and for $d>1$ not commutative.

Lemma 2.1.21.
i) The multiplication of partitioned permutation is associative and $(\mathbf{0}_e,e)$ is the neutral element, where $e\in S(n)$ is the identical permutation.

ii) The colength of a partitioned permutation $(\mathcal{V},\pi)$ is a nonnegative integer, and it holds

$$\begin{aligned}|(\mathbf{0}_\pi,\pi)|&=|\pi|,\\ |(\mathcal{V},\pi)|&=\sum_{B\in\mathcal{V}}|(\mathbf{1}_{\#B},\pi|_B)|,\end{aligned}\tag{2.1.6}$$

where we have made a choice of bijections $[\#B]\to B$ to consider $(\mathbf{1}_{\#B},\alpha_{|_B})\in\mathcal{PS}(\#B)$, but its colength appearing on the right-hand side is independent of these choices.

iii) We have a triangle inequality

$$|(\mathcal{V}\vee\mathcal{W},\pi\sigma)|\le|(\mathcal{V},\pi)|+|(\mathcal{W},\sigma)|,$$

more precisely there is an integer $g\ge 0$ such that

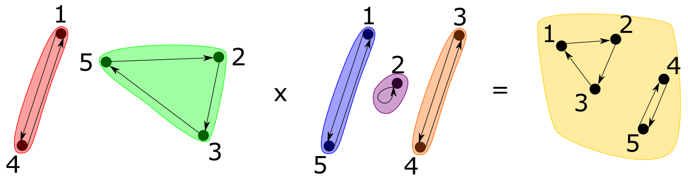
$$|(\mathcal{V}\vee\mathcal{W},\pi\sigma)|=|(\mathcal{V},\pi)|+|(\mathcal{W},\sigma)|-2g.\tag{2.1.7}$$

Let us discuss some examples to clarify the definition of partitioned permutations and provide some graphical interpretation.

Example 2.1.22.
The condition in Definition 2.1.19 should be interpreted as a planarity condition, consider $\pi_1=(14)(235)$, $\pi_2=(15)(2)(34)$, $(\mathbf{0}_{\pi_1}\vee\mathbf{0}_{\pi_2},\pi_1\pi_2)=(\mathbf{1}_5,(123)(45))$. Then we have

$$\begin{aligned}|(\mathbf{0}_{\pi_1},\pi_1)|&=|\pi_1|=5-2=3,\\ |(\mathbf{0}_{\pi_2},\pi_2)|&=|\pi_2|=5-3=2,\\ |(\mathbf{1}_5,(123)(45))|&=2(5-1)-(5-2)=5.\end{aligned}$$

Thus the condition in Definition 2.1.19 is satisfied, and the product is indeed defined to be nonzero. Let us visualize this situation with the following diagram.



Planar product of partitioned permutations.

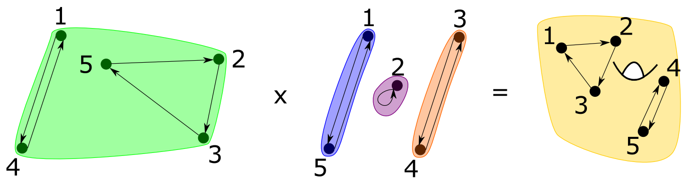
On the other hand, we may consider the case

$$(\mathbf{1}_5, \pi_1)(\mathbf{0}_{\pi_2}, \pi_2) = 0$$

since

$$|(\mathbf{1}_5, \pi_1)| + |(\mathbf{0}_{\pi_2}, \pi_2)| = 5 + 2 = 7 > 5 = |(\mathbf{1}_5, (123)(45))|.$$

This case should be considered as non-planar and visualized as follows.



Non-planar product of partitioned permutations.

Since in both examples the partition of the product is $\mathbf{1}_d$, one should interpret them as *connected* objects and the value $\varphi(\mathbf{1}_5, (123)(45))$ as a *connected* moment.

Remark 2.1.23.
The authors of [CMSS07] did not consider non-planar products, since they vanish in the limit in the random matrix case. Therefore, they are not part of classical (higher order) free probability theory. Allowing such products was the key for realizing the connection between the Fock space language of [BDBKS22, BDBKS23] and free probability. Their machinery provided a solution for the problem of finding higher order functional relations. We will discuss the more general framework in the Section 2.2.

With this framework of partitioned permutations, we can rephrase the leading order part of Equation (2.1.4):

$$\varphi(\mathcal{U}, \gamma)[A_1B_1, \ldots, A_nB_n] = \sum_{\substack{(\mathcal{V},\pi)(\mathcal{W},\sigma)=(\mathcal{U},\gamma) \\ (\mathcal{V},\pi),(\mathcal{W},\sigma)\in\mathcal{PS}(n)}} \kappa(\mathcal{V}, \pi)[A]\varphi(\mathcal{W}, \sigma)[B],$$

where we left out the superscript to indicate that we deal with the leading order terms. Despite the fact that the partitioned permutations do not form a lattice, the theory of *multiplicative functions* can be developed surprisingly in parallel to the first order. We continue by explaining the theory of [CMSS07].

Definition 2.1.24.
Let $d \in \mathbb{N}$, $f, g \colon \mathcal{PS}(d) \to \mathbb{C}$ be functions and $(\mathcal{V}, \pi), (\mathcal{W}, \sigma), (\mathcal{U}, \gamma) \in \mathcal{PS}(d)$.

i) We say the function f is *multiplicative* if

$$f(\mathcal{V}, \pi) = \prod_{B \in \mathcal{V}} f(\mathbf{1}_{\#B}, \pi|_B)$$

and values $f(1_d, \pi)$ only depend on the conjugacy class of $\pi \in S(d)$.

ii) We define the *convolution* of f, g to be the function defined by

$$f * g(\mathcal{U}, \gamma) = \sum_{(\mathcal{V},\pi)(\mathcal{W},\sigma)=(\mathcal{U},\gamma)} f(\mathcal{V}, \pi) g(\mathcal{W}, \sigma).$$

Comparing the leading order in Equation (2.1.5) we find that

$$\#\gamma - 2\#\mathcal{U} = \#\pi - 2\#\mathcal{V} - n + \#\gamma\pi^{-1}.$$

Writing $\sigma = \gamma\pi^{-1}$ and subtracting n we obtain

$$|(\mathcal{U}, \gamma)| = |(\mathcal{V}, \pi)| + |(\mathbf{0}_\sigma, \sigma)|.$$

Thus in the limit (2.1.5) reads

$$\varphi(\mathcal{U}, \gamma)[A_1, \ldots, A_n] = \sum_{(\mathcal{V},\pi)(\mathbf{0}_\sigma,\sigma)=(\mathcal{U},\gamma)} \kappa(\mathcal{V}, \pi)[A_1, \ldots, A_n] \tag{2.1.8}$$

and motivates the following definitions.

Definition 2.1.25.
We define the following functions.

i) We define the *delta-function* $\delta \colon \mathcal{PS} \to \mathbb{C}$ by

$$\delta(\mathcal{U}, \pi) = \begin{cases} 1 & \text{if } (\mathcal{U}, \pi) = (\mathbf{0}_e, e), \\ 0 & \text{otherwise.} \end{cases}$$

ii) We define the *zeta-function* $\zeta \colon \mathcal{PS} \to \mathbb{C}$ by

$$\zeta(\mathcal{U}, \pi) = \begin{cases} 1 & \text{if } \mathcal{U} = \mathbf{0}_\pi, \\ 0 & \text{otherwise.} \end{cases}$$

Lemma 2.1.26.

i) The delta function is the neutral element w.r.t. the convolution of Definition 2.1.24,

$$\delta * f = f * \delta = f,$$

for any function $f\colon \mathcal{PS} \to \mathbb{C}$.

ii) The zeta-function is invertible w.r.t. to the convolution of Definition 2.1.24. We denote its inverse by $\mu\colon \mathcal{PS} \to \mathbb{C}$ and call it the *Möbius-function on $\mathcal{PS}$*. Formally,

$$\mu * \zeta = \zeta * \mu = \delta.$$

Definition 2.1.27.
Let $\varphi, \kappa\colon \mathcal{PS} \to \mathbb{C}$ be functions. We say φ, κ *satisfy the (higher order) moment-cumulant relations* if

$$\varphi = \kappa * \zeta,$$

or equivalently

$$\kappa = \varphi * \mu.$$

2.1.3 Proof of Theorem 2.1.5

The derivation of the second order functional relations in [CMSS07] relies on investigating the combinatorics of partitioned permutations. By Definition 2.1.27 the second order moment-cumulant relations are given by

$$\begin{aligned}\varphi(\mathbf{1}_d, \gamma_{r_1,r_2}) &= \kappa * \zeta(\mathbf{1}_d, \gamma_{r_1,r_2}) \\ &= \sum_{(\mathcal{V},\pi)(\mathcal{W},\sigma)=(\mathbf{1}_d,\gamma_{r_1,r_2})} \kappa(\mathcal{V},\pi)\zeta(\mathcal{W},\sigma) \\ &= \sum_{(\mathcal{V},\pi)(\mathbf{0}_\sigma,\sigma)=(\mathbf{1}_d,\gamma_{r_1,r_2})} \kappa(\mathcal{V},\pi),\end{aligned}$$

where $n = r_1 + r_2$. Thus, in order to understand the right-hand side, we must investigate the possible factorizations

$$(\mathcal{V},\pi)(\mathbf{0}_\sigma,\sigma) = (\mathbf{1}_d, \gamma_{r_1,r_2}). \tag{2.1.9}$$

The following classes of partitioned permutations are particularly important when characterizing the solutions of (2.1.9).

Definition 2.1.28.
Let $d, r_1, r_2 \in \mathbb{N}$ such that $d = r_1 + r_2$ and $\gamma \coloneqq \gamma_{r_1,r_2}$. We define the following sets of (partitioned) permutations.

i) The set of $(\mathbf{1}_d, \gamma_{r_1,r_2})-$noncrossing partitioned permutations

$$\mathcal{PS}_{NC}(r_1, r_2) = \{(\mathcal{V}, \pi) \in \mathcal{PS}(d) \colon |(\mathcal{V}, \pi)| + |(\mathbf{0}_{\pi^{-1}\gamma}, \pi^{-1}\gamma)| = d\}.$$

ii) The set

$$\mathcal{S}_{NC}(r_1, r_2) = \{\pi \in S_d \colon |\pi| + |\gamma_{r_1,r_2}\pi^{-1}| = d \text{ and } \langle \pi, \gamma_{r_1,r_2}\rangle \text{ is transitive}\}.$$

Recalling that the product of partitioned permutations must satisfy

$$|(\mathcal{V}, \pi)| + |(\mathcal{W}, \sigma)| = |(\mathcal{U}, \gamma)|,$$

a careful case-by-case analysis of $(\mathcal{U}, \gamma) = (\mathbf{1}_d, \gamma_{r_1,r_2})$ in [CMSS07] yields the following result.

Proposition 2.1.29.
For $r_1, r_2 \in \mathbb{N}$, the solutions of the equation

$$(\mathbf{1}_{r_1+r_2}, \gamma_{r_1,r_2}) = (\mathcal{V}, \pi) \cdot (\mathcal{W}, \sigma)$$

are exactly of the following forms:

i) we have

$$(\mathbf{1}_{r_1+r_2}, \gamma_{r_1,r_2}) = (\mathbf{0}_\pi, \pi) \cdot (\mathbf{0}_{\pi^{-1}\gamma_{r_1,r_2}}, \pi^{-1}\gamma_{r_1,r_2}),$$

where $\pi \in \mathcal{S}_{NC}(r_1, r_2)$;

ii) we have

$$(\mathbf{1}_{r_1+r_2}, \gamma_{r_1,r_2}) = (\mathcal{V}, \pi) \cdot (\mathbf{0}_{\pi^{-1}\gamma_{r_1,r_2}}, \pi^{-1}\gamma_{r_1,r_2}),$$

where $\pi = \pi_1 \times \pi_2 \in \mathcal{NC}(r_1) \times \mathcal{NC}(r_2)$ and $\#\mathcal{V} + 1 = \#\pi$, where $\mathcal{V}$ contains a block that consists of the moved points of two cycles, one of π_1 and one of π_2;

iii) we have

$$(\mathbf{1}_{r_1+r_2}, \gamma_{r_1,r_2}) = (\mathbf{0}_\pi, \pi) \cdot (\mathcal{W}, \pi^{-1}\gamma_{r_1,r_2}),$$

where $\pi^{-1}\gamma_{r_1,r_2} = \sigma_1 \times \sigma_2 \in \mathcal{NC}(r_1) \times \mathcal{NC}(r_2)$ and $\#\mathcal{W} + 1 = \#\sigma$, where $\mathcal{V}$ contains a block that consists of the moved points of two cycles, one of σ_1 and one of σ_2.

Later, we want to discuss the difficulties of [CMSS07] of deriving functional relations for the higher order moment-cumulant formalism in $n \geq 3$. Thus, we sketch the proof of Proposition 2.1.29.

Sketch of the proof. First note that

$$|(\mathcal{V},\pi)| + |(\mathcal{W},\sigma)| = |(\mathcal{V} \vee \mathcal{W}, \pi\sigma)|$$

can be rewritten as

$$(|\mathcal{V}| - |\pi|) + (|\mathcal{W}| - |\sigma|) + (|\mathcal{V}| + |\mathcal{W}| - |\mathcal{V} \vee \mathcal{W}|) = |\mathbf{1}_{r_1+r_2}| - |\gamma_{r_1,r_2}| = 1. \quad (2.1.10)$$

Since every term in brackets on the left-hand side must be positive, we have the following possibilities

i) $|\mathcal{V}| = |\pi|, \quad |\mathcal{W}| = |\sigma|, \quad |\mathcal{V}| + |\mathcal{W}| = |\mathcal{V} \vee \mathcal{W}| + 1;$

ii) $|\mathcal{V}| = |\pi| + 1, \quad |\mathcal{W}| = |\sigma|, \quad |\mathcal{V}| + |\mathcal{W}| = |\mathcal{V} \vee \mathcal{W}|;$

iii) $|\mathcal{V}| = |\pi|, \quad |\mathcal{W}| = |\sigma| + 1, \quad |\mathcal{V}| + |\mathcal{W}| = |\mathcal{V} \vee \mathcal{W}|.$

Each of these cases corresponds to one of the claimed factorizations. □

Remark 2.1.30.

i) The factorizations of the cycle $(1,\gamma_r) = (\mathcal{V},\pi)(\mathcal{W},\sigma)$ can be identified with the noncrossing partitions and hence recover the theory of Speicher; [Spe94].

ii) The assertion of Proposition 2.1.29 can be formulated as

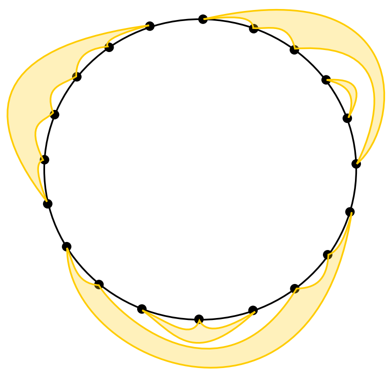
$$\mathcal{PS}_{NC}(r_1,r_2) = \mathcal{S}_{NC}(r_1,r_2) \sqcup \mathcal{T}_{NC}(r_1,r_2),$$

where $\mathcal{T}_{NC}(r_1,r_2)$ consists of the partitioned permutations $(\mathcal{V},\pi)$ described in Proposition 2.1.29 part ii).

iii) One can easily adapt the definitions and the proof of Proposition 2.1.29 to obtain a result for arbitrary $\gamma_{r_1,\dots,r_n}$, where $n, r_1, \dots, r_n \in \mathbb{N}$. We will discuss this matter in Section 2.1.4.

Example 2.1.31.

Let us also give a visual interpretation of the result above. As mentioned above, for one cycle we obtain the combinatorics of the classical results of [Spe94].

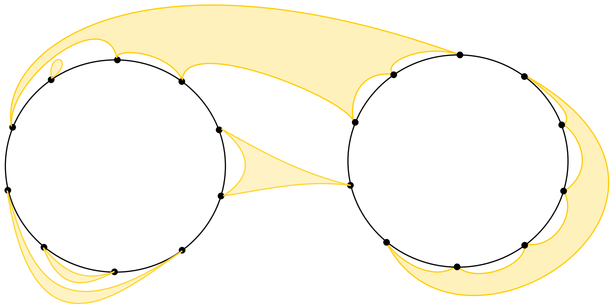


Elements of $\mathcal{NC}(r)$.

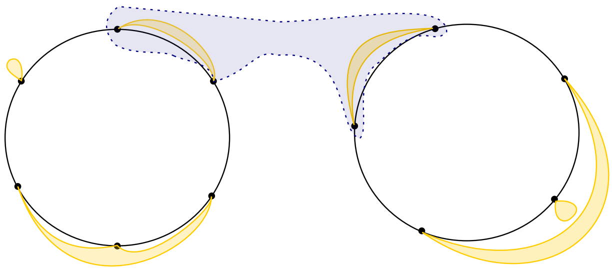
The diagram corresponds to the formula

$$\varphi_r = \varphi(\mathbf{1}_r, \gamma_r) = \sum_{\pi \in \mathcal{NC}(r)} \zeta(\mathbf{0}_\pi, \pi)\kappa(\mathbf{0}_{\pi^{-1}\gamma}) = \sum_{\pi \in \mathcal{NC}(r)} \kappa_\pi.$$

In second order case, we need to study the following diagrams.



Elements of $\mathcal{S}_{NC}(r_1, r_2)$.



Elements of $\mathcal{T}_{NC}(r_1, r_2)$: $\pi_1 \times \pi_2$ connected by a partition $\mathcal{V}$.

These diagrams correspond to the moment-cumulant formula

$$\varphi_{r_1,r_2} = \varphi(\mathbf{1}_{r_1+r_2}, \gamma_{r_1,r_2}) = \sum_{\pi \in \mathcal{S}_{NC}(r_1,r_2)} \kappa(\mathbf{0}_\pi, \pi) + \sum_{\substack{\pi_1 \times \pi_2 \in \mathcal{NC}(r_1) \times \mathcal{NC}(r_2) \\ \mathcal{V} \text{ connects two cycles}}} \kappa(\mathcal{V}, \pi_1 \times \pi_2).$$

We conclude this section by sketching the proof of the second order functional relations of [CMSS07].

Sketch of the proof of Theorem 2.1.5. Recall that we are interested in functional relations between the generating functions

$$C(x) := 1 + \sum_{i=1}^{\infty} \kappa_i^a x^i, \qquad M(x) := 1 + \sum_{i=1}^{\infty} \varphi_i^a x^i$$

and

$$C(x_1,x_2) \coloneqq \sum_{i_1,i_2=1}^{\infty} \kappa^a_{i_1,i_2} x_1^{i_1} x_2^{i_2}, \qquad M(x_1,x_2) \coloneqq \sum_{i_1,i_2=1}^{\infty} \varphi^a_{i_1,i_2} x_1^{i_1} x_2^{i_2}.$$

Thus the strategy is to multiply the equation

$$\varphi_{r_1,r_2} = \varphi(\mathbf{1}_{r_1+r_2}, \gamma_{r_1,r_2}) = \sum_{\pi \in \mathcal{S}_{NC}(r_1,r_2)} \kappa(\mathbf{0}_\pi, \pi) + \sum_{\substack{\pi_1\times\pi_2 \in \mathcal{NC}(r_1)\times\mathcal{NC}(r_2) \\ \mathcal{V} \text{ connects two cycles}}} \kappa(\mathcal{V}, \pi_1 \times \pi_2) \tag{2.1.11}$$

by powers of formal variables $x_1^{r_1}, x_2^{r_2}$ and summing over $r_1, r_2 \geq 1$. But to recover the base levels κ_{r_1,r_2} and κ_r from the summands in each of the sums of (2.1.11), we must unwind the multiplicative structure of $\kappa(\mathbf{0}_\pi, \pi)$ and $\kappa(\mathcal{V}, \pi_1 \times \pi_2)$. A term in the second sum must be of the form

$$\kappa_{s_1,s_2}\kappa_{a_1}\dots\kappa_{a_{s_1}}\kappa_{b_1}\dots\kappa_{b_{s_2}},$$

where $a_1 + \cdots + a_{s_1} + s_1 = r_1$ and $b_1 + \cdots + b_{s_2} + s_2 = r_2$. It is easy to see that this kind of terms will yield the expression

$$C(x_1M(x_1), x_2M(x_2)) \frac{\partial_{x_1}(x_1M(x_1))}{M(x_1)} \frac{\partial_{x_2}(x_2M(x_2))}{M(x_2)}$$

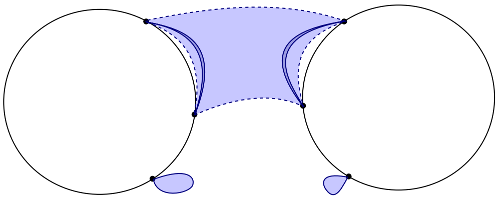
when multiplying $x_1^{r_1}, x_2^{r_2}$ and summing over $r_1, r_2 \geq 1$. The leftover term is the sum over $\pi \in \mathcal{S}_{NC}(r_1, r_2)$, which can be reduced in two steps. First we can rearrange the terms in a way that we only sum over π that have only *through-cycles*, that is, every cycle of π contains elements of $\{1,\dots,r_1\}$ as well as elements from $\{r_1+1,\dots,r_1+r_2\}$. The set of all π that only have through-cycles is denoted by $\mathcal{S}^{\text{all}}_{NC}(r_1,r_2)$. This idea can be formulated by introducing a new multiplicative function via

$$\tilde{\kappa}_r = \kappa_r,$$
$$\tilde{\kappa}_{r_1,r_2} = \tilde{\kappa}(\mathbf{1}_{r_1+r_2}, \gamma_{r_1,r_2}) = \sum_{\pi \in \mathcal{S}^{\text{all}}_{NC}(r_1,r_2)} \kappa(\mathbf{0}_\pi, \pi),$$

which leads to

$$\sum_{\pi \in \mathcal{S}_{NC}(r_1,r_2)} \kappa(\mathbf{0}_\pi, \pi) = \sum_{\substack{\pi_1\times\pi_2 \in \mathcal{NC}(r_1)\times\mathcal{NC}(r_2) \\ \mathcal{V} \text{ connects two cycles}}} \tilde{\kappa}(\mathcal{V}, \pi). \tag{2.1.12}$$

Let us give a small example to clarify the idea of (2.1.12). Consider the partitioned permutation $(\mathcal{V}, \pi) = (\{\{1,2,4,5\}\{3\}\{6\}\}, (12)(45)(3)(6))$ in $\mathcal{NC}(3) \times \mathcal{NC}(3)$, understood as permutations of the sets $\{1,2,3\}$ and $\{4,5,6\}$.

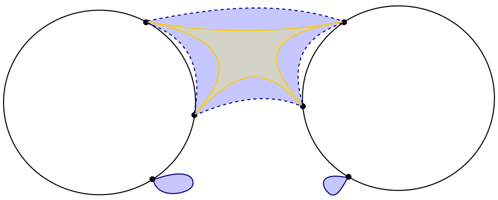


$(\mathcal{V}, \pi)$, dashed lines represent the block connection of $\mathcal{V}$.

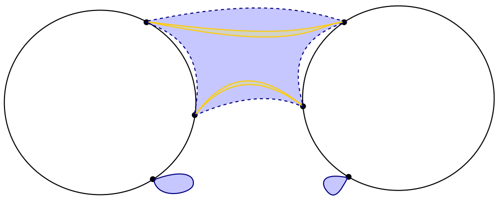
Then on the right-hand side of (2.1.12) we have the terms

$$
\begin{aligned}
\tilde{\kappa}(\mathcal{V}, \pi) &= \tilde{\kappa}(\mathbf{1}_4, (12)(45))\kappa(\mathbf{1}_1, (3))\kappa(\mathbf{1}_1, (6)) \\
&= \big(\kappa(\mathbf{0}, (1245)) + \kappa(\mathbf{0}, (1254)) + \kappa(\mathbf{0}, (14)(25)) \\
&\quad + \kappa(\mathbf{0}, (15)(24))\big)\kappa(\mathbf{1}, (3))\big)\kappa(\mathbf{1}, (6)) \\
&= \kappa(\mathbf{0}, (1245)(3)(6)) + \kappa(\mathbf{0}, (1254)(3)(6)) + \kappa(\mathbf{0}, (15)(24)(3)(6)) \\
&\quad + \kappa(\mathbf{0}, (14)(25)(3)(6)).
\end{aligned}
$$

Roughly speaking we replace the one block connection of an element in $\mathcal{T}_{NC}(r_1, r_2)$ by all possible through-cycles.

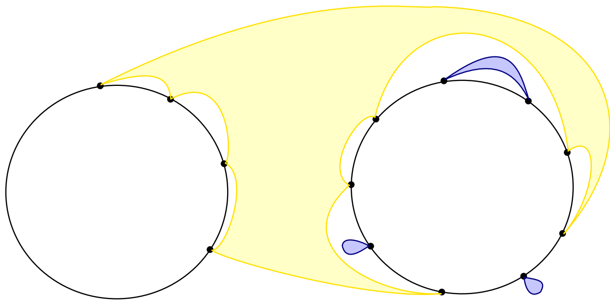


Putting a 4-cycle in the block.



Putting two pairings in the block.

We continue with the proof. Assume we have $\pi \in \mathcal{S}_{NC}(r_1, r_2)$ with a through-cycle c of length $s = a + b$, where a is the number of points of c in $\{1, \ldots, r_1\}$ and b the number of points of c in $\{r_1+1, \ldots, r_1+r_2\}$. Then, for the non-through cycles, there are only finitely many possibilities left. These are characterized by a choice of noncrossing permutations $\pi_1, \ldots, \pi_{\tilde{a}} \in \mathcal{NC}(r_1 - a)$ and $\sigma_1 \ldots \sigma_{\tilde{b}} \in \mathcal{NC}(r_2 - b)$, with $\tilde{a} \leq a$, $\tilde{b} \leq b$, on the remaining points with the restriction that they cannot cross the through cycles. Let us illustrate that by the following diagram.



Blue: possible non crossing configuration of cycles on just one of the circles.
Yellow: through cycle connection.

Thus, one gets (similar to the first case) a contribution

$$\tilde{\kappa}_{s_1,s_2}\kappa_{a_1}\dots\kappa_{a_{s_1}}\kappa_{b_1}\dots\kappa_{b_{s_2}},$$

which will correspond to a term

$$\tilde{C}(x_1 M(x_1), x_2 M(x_2))$$

and we are left with analyzing the function

$$\tilde{C}(x_1, x_2) = \sum_{r_1,r_2} \tilde{\kappa}_{r_1,r_2} x_1^{r_1} x_2^{r_2}.$$

By a counting argument for the elements in $\mathcal{S}_{NC}^{\text{all}}(r_1, r_2)$, we have

$$\tilde{\kappa}_{r_1,r_2} = r_2 \sum_{r\geq 1} \sum_{\substack{a_1,\dots,a_r\geq 1 \\ a_1+\dots+a_r=r_1}} \sum_{\substack{b_1,\dots,b_r\geq 1 \\ b_1+\dots+b_r=r_2}} r_1 \kappa_{a_1+b_1}\dots\kappa_{a_r+b_r}$$

and we obtain the formula

$$\tilde{C}(x_1, x_2) = x_1 x_2 \partial_{x_1}\partial_{x_1} \log\left(\frac{x_1 C(x_2) - x_2 C(x_1)}{x_1 - x_2}\right).$$

Putting everything together yields the theorem. □

2.1.4 Functional relations beyond $n = 2$

In this section, we want to briefly review the problems in order bigger than two. Recall that the first step towards deriving the functional relation in [CMSS07] for order two was investigating the possible factorizations (2.1.9). For the case $n = 3$, the equation is given by

$$(\mathcal{V}, \pi)(\mathbf{0}_\pi, \pi) = (\mathbf{1}_{r_1+r_2+r_3}, \gamma_{r_1,r_2,r_3}).$$

We denote the obvious generalizations of the sets $\mathcal{PS}_{NC}(r_1, r_2)$ and $\mathcal{S}_{NC}(r_1, r_2)$ by $\mathcal{PS}_{NC}(r_1, \dots, r_n)$ and $\mathcal{S}_{NC}(r_1, \dots, r_n)$ which are defined by replacing γ_{r_1,r_2} by $\gamma_{r_1,\dots,r_n}$ in Definition 2.1.28. As already noted in [CMSS07], Proposition 2.1.29 can be generalized to arbitrary n. Let us discuss $n = 3$.

Proposition 2.1.32.
Let $r_1, r_2, r_3 \in \mathbb{N}$ and denote $d = r_1 + r_2 + r_3$, $\gamma = \gamma_{r_1,r_2,r_3}$. The solutions of the equation

$$(\mathbf{1}_d, \gamma) = (\mathcal{V}, \pi) \cdot (\mathcal{W}, \sigma)$$

are exactly of the following forms.

i) We have

$$(\mathbf{1}_d, \gamma) = (0, \pi) \cdot (0, \pi^{-1}\gamma),$$

where $\pi \in \mathcal{S}_{NC}(r_1, r_2, r_3)$.

ii) We have

$$(\mathbf{1}_d, \gamma) = (\mathcal{V}, \pi) \cdot (0, \pi^{-1}\gamma),$$

where either

a) $\pi \in \mathcal{NC}(r_1) \times \mathcal{NC}(r_2) \times \mathcal{NC}(r_3)$, $|\mathcal{V}| = |\pi| + 2$ and $\mathcal{V}$ connects three cycles $c_i \in \mathcal{NC}(r_i)$, $i = 1, 2, 3$ of π

or

b) $\pi \in \mathcal{S}_{NC}(r_i, r_j) \times \mathcal{NC}(r_k)$, $|\mathcal{V}| = |\pi| + 1$ and $\mathcal{V}$ connects two cycles $c_1 \in \mathcal{S}_{NC}(r_i, r_j)$, $c_2 \in \mathcal{NC}(r_k)$ of π.

iii) We have

$$(\mathbf{1}_d, \gamma) = (0, \pi) \cdot (\mathcal{W}, \pi^{-1}\gamma),$$

a) $\pi \in \mathcal{NC}(r_1) \times \mathcal{NC}(r_2) \times \mathcal{NC}(r_3)$, $|\mathcal{W}| = |\pi^{-1}\gamma| + 2$ and $\mathcal{W}$ connects three cycles $c_i \in \mathcal{NC}(r_i)$, $i = 1, 2, 3$ of $\pi^{-1}\gamma$

or

a) $\pi \in \mathcal{S}_{NC}(r_i, r_j) \times \mathcal{NC}r_k)$, $|\mathcal{W}| = |\pi^{-1}\gamma| + 1$ and $\mathcal{W}$ connects two cycles $c_1 \in \mathcal{S}_{NC}(r_i, r_j)$, $c_2 \in \mathcal{NC}(r_k)$ of $\pi^{-1}\gamma$.

iv) We have

$$(\mathbf{1}_d, \gamma) = (\mathcal{V}, \pi) \cdot (\mathcal{W}, \pi^{-1}\gamma),$$

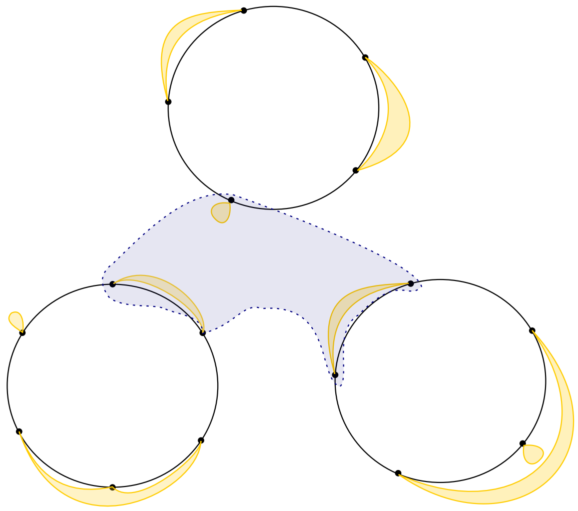
where $\pi \in \mathcal{NC}(r_1) \times \mathcal{NC}(r_2) \times \mathcal{NC}(r_3)$, $|\mathcal{V}| = |\pi| + 1$ and $\mathcal{V}$ connects two cycles $c_i \in \mathcal{NC}(r_i), c_j \in \mathcal{NC}(r_j)$ of π and $|\mathcal{W}| = |\pi^{-1}\gamma| + 1$ and $\mathcal{W}$ connects two cycles $c_i \in \mathcal{NC}(r_i), c_j \in \mathcal{NC}(r_j)$ of $\pi^{-1}\gamma$.

Remark 2.1.33.
Note that the case iv) in Proposition 2.1.32 does not contribute anything to the moment-cumulant equation, since the zeta function yields zero, $\zeta(\mathcal{V}, \pi) = 0$, if $\mathcal{V} \neq \mathbf{0}_\pi$.

We discuss the diagrammatic interpretation of Proposition 2.1.32 with the following examples.

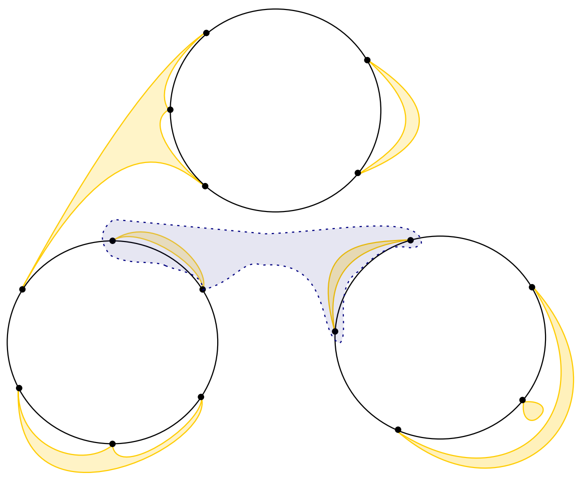
Example 2.1.34.

- Case ii) a): the following kind of diagrams give a *true* third order (cumulant) contribution to the moment-cumulant formula. They have a *partition connection* of order three.



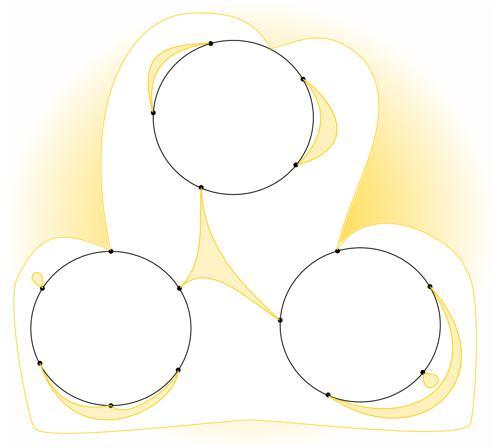
Contribution: $\kappa_{2,2,1}\kappa_3\kappa_2^3\kappa_1^2$.

- Case ii) b) involves the following kind of diagrams, giving second and first order (cumulant) contributions to the formula. They only have a *partition connection* of order two.

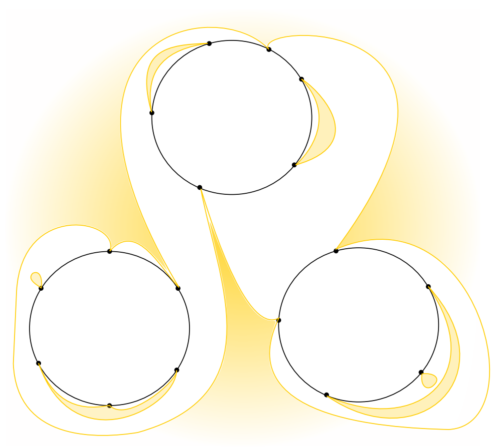


Contribution: $\kappa_{2,2}\kappa_4\kappa_3\kappa_2^2\kappa_1$.

- The case i) is the most complicated, since the *first order connections* are coming from the permutation. The following two examples have cycle connections to all three orbits of γ_{r_1,r_2,r_3}.

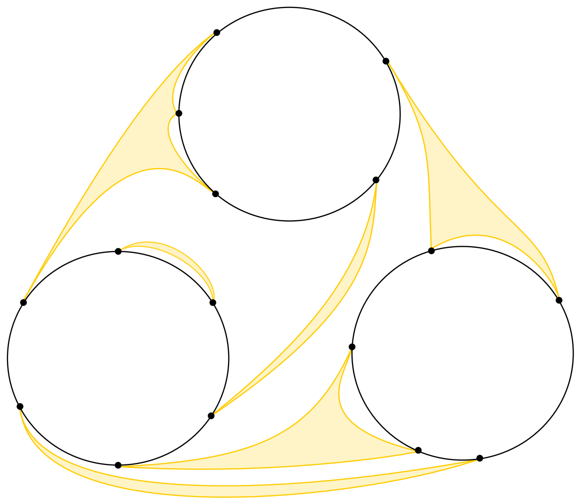


Two cycles connecting three orbits (circles).



One cycle connecting three orbits, but touching the upper one twice.

Note that in the right picture the outermost cycle connecting all three circles forbids connections between the bottom two circles (such would disturb the planarity), where on the configuration on the left there is no such restriction. But there might also be permutations that do not involve cycles permuting points of all three orbits of γ at all.



Example with no cycles that connect all three circles.

Now the cases ii) (or iii)) a) and b) in Proposition 2.1.32, can be dealt with similarly as in the case $n = 2$. If they give a *true* third order contributions, then we have summands of type

$$\kappa_{s_1,s_2,s_3}\kappa_{k_1}\dots\kappa_{k_m}, \quad m, s_1, s_2, s_3, k_1, \dots, k_m \in \mathbb{N}$$

which sums to

$$C_3(x_1M(x_1), x_2M(x_2), x_3M(x_3))$$

in the functional relation (see. Example 2.4.3), where C_3 is the generating series of third order cumulants. On the other hand in case ii) a) and b) there are factorizations that combine a second order with first order terms, which will contribute as a summand

$$C_2(x_iM(x_i), x_jM(x_j)) + \frac{x_iM(x_i)x_kM(x_k)}{(x_iM(x_i) - x_kM(x_k))^2}$$

and

$$C_2(x_iM(x_i), x_jM(x_j))C_2(x_iM(x_i), x_kM(x_k)).$$

As already stated in [CMSS07], it turns out that the hardest part is to mimic the proof for the contributions of elements in $\mathcal{S}_{NC}(r_1, r_2, r_2)$. Still, one can reduce the discussion to $\mathcal{S}_{NC}^{\text{all}}(r_1, r_2, r_3)$, elements consisting of only through cycles. But the analysis of the latter becomes a tedious case by case analysis, which can be seen from part iii) of the last example.

Let us finish by a final remark on the situation beyond $n = 3$ and combinatorial approaches towards reproving our formula in [BCGF$^+$23].

Remark 2.1.35.

i) For $n = 3$ it is still doable to study the elements in $\mathcal{S}_{NC}^{\text{all}}(r_1, r_2, r_3)$ and the author is confident that his computations recover the formula for $n = 3$, but this will not be not part of the thesis and is still a work in progress. For $n > 3$ it is still necessary to unwrap the conceptional simplifications of the term corresponding to $\mathcal{S}_{NC}^{\text{all}}(r_1, \dots, r_n)$ and it seems out of reach to do the case by case analysis without the latter.

ii) Let us also emphasize that recently there has been a fruitful combinatorial approach by L. Lionni [Lio22] using *hypermaps* to study the moment-cumulant relations. In particular, there is also a proof for $n = 3$ and a reformulation for general n that reduces the proof to understanding terms from so called *non-separable hypermaps.* The latter, from the point of view of partitioned permutations, correspond to understanding $\mathcal{S}_{NC}^{\text{all}}(r_1, \dots, r_n)$.

2.2 Extension to higher genus

In this section, we will present a generalization of the theory of [CMSS07]. The idea is to include more information of the distribution of noncommutative random variables, inspired from sub-leading orders in random matrix theory. In this way, the connection to the notion of topological partition functions in the bosonic Fock space is more apparent. This connection will then be discussed in the next section, Section 2.3. Recall from the random matrix case (2.1.3) or (2.1.5), that the authors of [CMSS07] were only interested in the leading order terms, i.e. the terms

$$\lim_{N\to\infty} N^{-|(\mathcal{U},\gamma)|}\varphi^{(N)}(\mathcal{U},\gamma)[A_1,\dots,A_n].$$

But $\varphi^{(N)}(\mathcal{U},\gamma)[A_1,\dots,A_n]$ typically admits an expansion in N, *called genus expansion.* Thus, to capture the subleading orders of N, we will extend the notion of multiplicative functions of $f\colon \mathcal{PS}\to\mathbb{C}$ by allowing

$$f\colon \mathcal{PS}\to\mathbb{C}[\![\hbar]\!]$$

for some formal parameter $\hbar$. The idea behind this is to model the random matrix behaviour in subleading orders, i.e. to enforce a genus expansion on the abstract level

$$\varphi_{r_1,\dots,r_n} = \varphi(\mathbf{1}_d,\gamma_{r_1,\dots,r_n}) = \sum_{g\geq 0}\varphi^{[g]}(\mathbf{1}_d,\gamma_{r_1,\dots,r_n})\hbar^{|(\mathbf{1}_d,\gamma_{r_1,\dots,r_n})|+2g}, \tag{2.2.1}$$

$$\kappa_{r_1,\dots,r_n} = \kappa(\mathbf{1}_d,\gamma_{r_1,\dots,r_n}) = \sum_{g\geq 0}\kappa^{[g]}(\mathbf{1}_d,\gamma_{r_1,\dots,r_n})\hbar^{|(\mathbf{1}_d,\gamma_{r_1,\dots,r_n})|+2g}, \tag{2.2.2}$$

where $d\in\mathbb{N}$, $\gamma\in S(d)$. Additionally, we must allow the product of two partitioned permutations to generate a higher genus (cf. Theorem 2.1.14 ii) and Lemma 2.1.21 iii)). Thus, the following definitions extend Definition 2.1.24 and Definition 2.1.25.

Definition 2.2.1.
Let $(\mathcal{U},\gamma),(\mathcal{V},\pi),(\mathcal{W},\sigma)\in\mathcal{PS}(d)$ be partitioned permutations and $f_1,f_2\colon\mathcal{PS}(d)\to\mathbb{C}[\![\hbar]\!]$ two functions.

i) We define the *extended multiplication* $\odot$ of partitioned permutations to be

$$(\mathcal{V},\pi)\odot(\mathcal{W},\sigma) := (\mathcal{V}\vee\mathcal{W},\pi\sigma).$$

ii) The *extended convolution* of two functions is

$$(f_1\circledast f_2)(\mathcal{U},\gamma) := \sum_{(\mathcal{V},\pi)\odot(\mathcal{W},\sigma)=(\mathcal{U},\gamma)} f_1(\mathcal{V},\pi)\,f_2(\mathcal{W},\sigma). \tag{2.2.3}$$

Definition 2.2.2.
We define the following extensions.

i) The *extended zeta function* $\zeta_\hbar\colon\mathcal{PS}\to\mathbb{C}[\![\hbar]\!]$ is

$$\zeta_\hbar(\mathcal{V},\pi) := \begin{cases}\hbar^{|\pi|} & \text{if } \mathcal{V}=\mathbf{0}_\pi,\\ 0 & \text{otherwise.}\end{cases}$$

ii) The *extended Möbius function* $\mu_\hbar\colon \mathcal{PS} \to \mathbb{C}[\![\hbar]\!]$ is uniquely determined by

$$\mu_\hbar \circledast \zeta_\hbar = \zeta_\hbar \circledast \mu_\hbar = \delta,$$

where δ is the delta function of Definition 2.1.25 but understood as a function $\mathcal{PS} \to \mathbb{C}[\![\hbar]\!]$.

Lemma 2.2.3.
The Möbius function $\mu_\hbar$ exists.

Proof. For the existence of $\mu_\hbar$, we view functions $\mathcal{PS} \to \mathbb{C}[\![\hbar]\!]$ (and their $\circledast$ convolution) as elements of power series over the group ring $\mathbb{C}[\mathcal{PS}(d)][\![\hbar]\!]$ (with product induced by $\odot$). In particular, we have

$$\zeta_\hbar = \sum_{\pi \in S(d)} \hbar^{|\pi|} \, (\mathbf{0}_\pi, \pi).$$

Its *constant term*, i.e. the coefficient of $\hbar^0$ is exactly $(\mathbf{0}_e, e)$, since the only permutation with zero colength is the identity. Since $(\mathbf{0}_e, e)$ is the unit for $\odot$, $\zeta_\hbar$ is invertible as a power series in $\mathbb{C}[\mathcal{PS}(d)][\![\hbar]\!]$. We denote its inverse by $\mu_\hbar$. □

Definition 2.2.4.
A function $f\colon \mathcal{PS} \to \mathbb{C}[\hbar]$ is called *multiplicative* if for any $d \in \mathbb{N}$ and $\pi \in S(d)$, the value $f(\mathbf{1}_d, \pi)$ depends only on the conjugacy class of π, and for any $(\mathcal{V}, \pi) \in \mathcal{PS}$ we have

$$f(\mathcal{V}, \pi) = \prod_{B \in \mathcal{B}} f(\mathbf{1}_{\#B}, \pi_{|B}).$$

Lemma 2.2.5.

i) The convolution of two multiplicative functions is again multiplicative.

ii) The zeta function and the Möbius function are multiplicative.

Proof.

i) Let $(\mathcal{V}, \pi)(\mathcal{W}, \sigma) = (\mathcal{U}, \gamma)$, i.e. $\mathcal{V} \vee \mathcal{W} = \mathcal{U}$ and $\gamma = \pi\sigma$, then we have $\mathcal{V}, \mathcal{W} \leq \mathcal{U}$. Thus, the blocks of $\mathcal{V}, \mathcal{W}$ must be contained in the blocks of $\mathcal{U}$. By the multiplicativity of f, g we get

$$\begin{aligned} f(\mathcal{V}, \pi) g(\mathcal{W}, \sigma) &= \prod_{V \in \mathcal{V}} f(V, \pi|_V) \prod_{W \in \mathcal{W}} g(W, \sigma|_W) \\ &= \prod_{U \in \mathcal{U}} \prod_{V \subset U} f(V, \pi|_V) \prod_{W \subset U} g(W, \sigma|_W). \end{aligned}$$

The blocks $U \in \mathcal{U}$ must satisfy

$$U = \Bigg(\bigcup_{\substack{V \in \mathcal{V} \\ V \subset U}} V \Bigg) \vee \Bigg(\bigcup_{\substack{W \in \mathcal{W} \\ W \subset U}} W \Bigg) =: S_{\mathcal{V}}(U) \vee T_{\mathcal{W}}(U),$$

If we would have $S_{\mathcal{V}}(U) \vee T_{\mathcal{W}}(U) < \mathbf{1}_U$ then either there is at least one point $i \in U$ which is not contained in any of the V's and W's in the union. But there must be blocks V', W' containing i and $V', W' \not\subset U$ and by definition there must be a block U' of $\mathcal{U}$ containing both. Then U' must contain i and as the blocks of $\mathcal{U}$ must be disjoint it agrees with U, which is a contradiction. Moreover, we know that π and σ decompose respecting the blocks of $\mathcal{V}$ resp. $\mathcal{W}$, such that $\pi = \prod_{V \in \mathcal{V}} \pi|_V$ with $\pi|_V$ pairwise disjoint, similarly for σ. Thus, we have $\pi|_U = \prod_{V \in \mathcal{V}, V \subset U} \pi|_V = \pi_{S_{\mathcal{V}}}$, $\sigma|_U = \sigma|_{T_{\mathcal{W}}(U)}$ and since $\sigma(U) = U$ we must have $\gamma|_U = \pi\sigma|_U = \pi|_U \sigma|_U = \pi|_{S_{\mathcal{V}}} \sigma|_{T_{\mathcal{W}}(U)}$. Thus, we have shown that

$$(S_{\mathcal{V}}(U), \pi|_{S_{\mathcal{V}}(U)})(T_{\mathcal{W}}(U), \sigma|_{T_{\mathcal{W}}(U)}) = (U, \gamma|_U)$$

and vice versa every factorization for a restriction to a block,

$$(U, \gamma|_U) = (V_0(U), \pi_0)(W_0(U), \sigma_0),$$

gives rise to a factorization $(\mathcal{U}, \gamma) = (\mathcal{V}, \pi)(\mathcal{W}, \sigma)$ via $\mathcal{V} = \{V_0(U) \colon U \in \mathcal{U}\}$ and $\pi = \prod_{U \in \mathcal{U}} \pi|_U$ and similarly for $(\mathcal{W}, \sigma)$. Thus,

$$\begin{aligned}
f \circledast g\,(\mathcal{U}, \gamma) &= \sum_{(\mathcal{U},\gamma)=(\mathcal{V},\pi)(\mathcal{W},\sigma)} f(\mathcal{V}, \pi) g(\mathcal{W}, \sigma) \\
&= \sum_{(\mathcal{U},\gamma)=(\mathcal{V},\pi)(\mathcal{W},\sigma)} \prod_{U \in \mathcal{U}} \prod_{V \subset U} f(V, \pi|_V) \prod_{W \subset U} g(W, \sigma|_W) \\
&= \sum_{(\mathcal{U},\gamma)=(\mathcal{V},\pi)(\mathcal{W},\sigma)} \prod_{U \in \mathcal{U}} f(S_{\mathcal{V}}(U), \pi|_{S_{\mathcal{V}}}(U)) g(T_{\mathcal{W}}(U)), \sigma|_{T_{\mathcal{W}}(U)}) \\
&= \prod_{U \in \mathcal{U}} \sum_{(V_0,\pi_0)(W_0,\sigma_0)=(U,\gamma|_U)} f(V_0, \pi_0) g(W_0, \sigma_0) \\
&= \prod_{U \in \mathcal{U}} f \circledast g\,(U, \gamma|_U).
\end{aligned}$$

The fact that the value only depends on the conjugacy class is clear by

$$\gamma = \pi\sigma \iff \tau^{-1}\gamma\tau = \tau^{-1}\pi\tau\tau^{-1}\sigma\tau$$

and the fact that f, g are multiplicative, i.e. their value only depends on the conjugacy class of the argument.

ii) It is easy to see that the zeta function is multiplicative: let $(\mathcal{V}, \pi)$ be a partitioned permutation with $\pi = c_1 \dots c_r \in S(d)$, where $c_1 \dots c_r$ is the cycle decomposition of π into cycles of length $s_1, \dots, s_r \in \mathbb{N}$. If $\mathcal{V} \neq \mathbf{0}_\pi$, then, $\zeta(\mathcal{V}, \pi) = 0$ and since there is a block B containing at least two cycles of π, $\zeta(B, \pi|_B) = 0$. On the other hand if $\mathcal{V} = \mathbf{0}_\pi$, then

$$\zeta(\mathbf{0}_\pi, \pi) = \hbar^{|\pi|} = \hbar^{d - \#\pi} = \hbar^{s_1 + \dots + s_r - r} = \hbar^{|c_1| + \dots + |c_r|} = \prod_{i=1}^{r} \hbar^{|c_i|} = \prod_{B \in \mathbf{0}_\pi} \zeta(\mathbf{1}_B, \pi|_B).$$

Now recall hat $\mu_\hbar$ is the inverse of $\zeta_\hbar$ in $\mathbb{C}[\mathcal{PS}(d)][\![\hbar]\!]$ w.r.t. $\circledast$. These relations can be written down explicitly: we write $\mu_{r_1,\dots,r_n} = \mu(\mathbf{1}_d, r_1, \dots, r_n)$ if $r_1+\cdots+r_n = d$, then we have

$$\begin{aligned}
1 &= \mu_1\\
0 &= \hbar\mu_2 + \mu_{1,1}\\
0 &= \mu_2 + \mu(\mathbf{0}_2, (1)(2)) + \hbar\mu_{1,1}\\
0 &= \mu_{1,1,1} + 3\hbar\mu_{2,1} + 2\hbar^2\mu_3\\
&\ \ \vdots
\end{aligned}$$

In general we have

$$\begin{aligned}
0 &= \sum_{(\mathbf{0}_\pi,\pi)(\mathcal{V},\sigma)=(1_d,\gamma_{r_1,\dots,r_n})} \hbar^{|\pi|}\mu(\mathcal{V},\sigma)\\
&= \mu_{r_1,\dots,r_n} + \sum_{\substack{(\mathbf{0}_\pi,\pi)(\mathcal{V},\sigma)=(1_d,\gamma_{r_1,\dots,r_n})\\ (\mathbf{0}_\pi,\pi)\neq(\mathbf{0}_e,e)}} \hbar^{|\pi|}\mu(\mathcal{V},\sigma).
\end{aligned}$$

If we expand these equations multiplicatively, they determine the values $\mu_{r_1,\dots,r_n}$ uniquely. Thus we may define a function $M\colon \mathcal{PS} \to \mathbb{C}[\![\hbar]\!]$ by these relations on $(1_d, \gamma_{r_1,\dots,r_n})$ and on $(\mathcal{V},\pi)$ by multiplicativity:

$$M(\mathcal{V},\pi) := \prod_{B\in\mathcal{V}} M(\mathbf{1}_B, \pi|_B).$$

Then we have

$$\begin{aligned}
\zeta \circledast M(\mathcal{V},\pi) &= \prod_{B\in\mathcal{V}} \zeta \circledast M(\mathbf{1}_B, \pi|_B)\\
&= \prod_{B\in\mathcal{V}} \delta(\mathbf{1}_B, \pi|_B)\\
&= \delta(\mathcal{V},\pi),
\end{aligned}$$

where we used ii) in the first equality and the defining relations of M in the second equality. Thus M is the convolution inverse and must agree with μ. In particular μ is multiplicative.

□

Moreover, as in the genus 0 case, the convolution of two multiplicative functions is commutative:

Lemma 2.2.6.
Let $f, g\colon PS \to \mathbb{C}[\![\hbar]\!]$ be two multiplicative functions. Then

$$f_1 \circledast f_2 = f_2 \circledast f_1.$$

Proof. Let $(\mathcal{V},\pi),(\mathcal{W},\sigma),(\mathcal{U},\gamma)\in\mathcal{PS}$ be partitioned permutations, such that $(\mathcal{V},\pi)\odot(\mathcal{W},\sigma)=(\mathcal{U},\gamma)$. Then $\mathcal{U}=\mathcal{V}\vee\mathcal{W}$ and $\pi\circ\sigma=\gamma$. This can also be written $\mathcal{U}=\mathcal{W}\vee\mathcal{V}$ and $\sigma^{-1}\circ\pi^{-1}=\gamma^{-1}$. The support of cycles of a permutation and its inverse are the same: $\mathbf{0}_\pi=\mathbf{0}_{\pi^{-1}}$, etc. Therefore, $(\mathcal{V},\pi^{-1}),(\mathcal{W},\sigma^{-1}),(\mathcal{U},\gamma^{-1})$ are still partitioned permutations and $(\mathcal{W},\sigma^{-1})\odot(\mathcal{V},\pi^{-1})=(\mathcal{U},\gamma^{-1})$. Since π and π^{-1} are conjugated, a multiplicative function takes the same values on $(\mathcal{V},\pi)$ and on $(\mathcal{V},\pi^{-1})$. Then, relabelling $(\mathcal{V},\pi)$ into $(\mathcal{W},\sigma^{-1})$ and $(\mathcal{W},\sigma)$ into $(\mathcal{V},\pi^{-1})$ in equation (2.2.3) shows that $f\circledast g=g\circledast f$. □

Definition 2.2.7.
Let $\varphi,\kappa\colon\mathcal{PS}\to\mathbb{C}[\![\hbar]\!]$ be functions. We say φ and κ satisfy the *extended or all genera cumulant relations* if

$$\varphi=\zeta_\hbar\circledast\kappa$$

or equivalently

$$\kappa=\mu_\hbar\circledast\varphi.$$

Lemma 2.2.8.
Let $\varphi,\kappa\colon\mathcal{PS}\to\mathbb{C}[\![\hbar]\!]$ be two multiplicative functions with

$$\varphi(\mathcal{V},\pi)=\hbar^{|(\mathcal{V},\pi)|}\varphi_0(\mathcal{V},\pi)+o(\hbar^{|(\mathcal{V},\pi)|})\text{ and }\kappa(\mathcal{V},\pi)=\hbar^{|(\mathcal{V},\pi)|}\kappa_0(\mathcal{V},\pi)+o(\hbar^{|(\mathcal{V},\pi)|})$$

for any $(\mathcal{V},\pi)\in\mathcal{PS}$. Then φ_0,κ_0 define multiplicative functions $\varphi_0,\kappa_0\colon\mathcal{PS}\to\mathbb{C}$ and the relation $\varphi=\zeta_\hbar\circledast\kappa$ implies $\varphi_0=\zeta*\kappa_0$ in the sense of [CMSS07].

Proof. Let $d\geq 0$ and $(\mathcal{U},\gamma)\in\mathcal{PS}(d)$ then by definition we have

$$\begin{aligned}\varphi(\mathcal{U},\gamma)&=\zeta_\hbar\circledast\kappa(\mathcal{U},\gamma)\\&=\sum_{(\mathcal{V},\pi)(\mathcal{W},\sigma)=(\mathcal{U},\gamma)}\zeta_\hbar(\mathcal{V},\pi)\kappa(\mathcal{W},\sigma)\\&=\sum_{(\mathbf{0}_\pi,\pi)(\mathcal{W},\sigma)=(\mathcal{U},\gamma)}\hbar^{|\pi|}\kappa(\mathcal{W},\sigma)\end{aligned}$$

Hence, if we take the $\hbar^{|(\mathcal{U},\gamma)|}$ coefficient, we obtain

$$\begin{aligned}\varphi_0(\mathcal{U},\gamma)&=[\hbar^{|(\mathcal{U},\gamma)|}]\sum_{(\mathbf{0}_\pi,\pi)(\mathcal{W},\sigma)=(\mathcal{U},\gamma)}\hbar^{|\pi|}\kappa(\mathcal{W},\sigma)\\&=[\hbar^{|(\mathcal{U},\gamma)|}]\Bigg(\sum_{(\mathbf{0}_\pi,\pi)(\mathcal{W},\sigma)=(\mathcal{U},\gamma)}\kappa_0(\mathcal{W},\sigma)\hbar^{|\pi|+|(\mathcal{W},\sigma)|}+o(|\pi|+|(\mathcal{W},\sigma)|)\Bigg)\\&=\sum_{\substack{(\mathbf{0}_\pi,\pi)(\mathcal{W},\sigma)=(\mathcal{U},\gamma)\\|\pi|+|(\mathcal{W},\sigma)|=|(\mathcal{U},\gamma)|}}\kappa_0(\mathcal{W},\sigma)\\&=\zeta*\kappa(\mathcal{U},\gamma),\end{aligned}$$

since $|(\mathcal{U},\gamma)|=|(\mathbf{0}_\pi,\pi)(\mathcal{W},\sigma)|\leq|\pi|+|(\mathcal{W},\sigma)|$. □

2.2.1 Hurwitz numbers and the Möbius function

There is a close relationship between the Möbius function and (monotone) Hurwitz numbers. This is not surprising, since the Möbius function originates in the calculations for unitary random matrices (c.f. Theorem 2.1.14) using Weingarten calculus. More precisely, the Möbius function is given by the cumulants of the Weingarten function, see [GGPN14]. Also, there is a nice exposition of a more general case for random tensors in [CGL23]. In this section, we want to study the relationship of free probability to Hurwitz numbers from the viewpoint of partitioned permutations. This will be important in Chapter 3, which is based on the collaboration of the author and his coauthors in [HvIL22]. Furthermore, the connection of the zeta and Möbius function to (strictly monotone) Hurwitz numbers and their generating function is one key ingredient for the proof of our main result in [BCGF$^+$23].

Lemma 2.2.9.
It holds that

$$\mu(\mathcal{V},\pi) = \delta(\mathcal{V},\pi) + \sum_{k=1}^{\infty}(-1)^k \sum_{\substack{(\mathbf{0}_{\sigma_1},\sigma_1)\cdots(\mathbf{0}_{\sigma_k},\sigma_k)=(\mathcal{V},\pi)\\ \sigma_i\neq e}} \hbar^{|\sigma_1|+\cdots+|\sigma_k|}$$

for any $(\mathcal{V},\pi)\in\mathcal{PS}$.

Proof. Let $d\geq 1$ and recall the geometric expansion in $\mathbb{C}[\mathcal{PS}(d)][\![\hbar]\!]$, then we have

$$\begin{aligned}
\mu = \zeta^{-1} &= (\delta + (\zeta-\delta))^{-1}\\
&= \delta + \sum_{k=1}^{\infty}(-1)^k(\zeta-\delta)^k\\
&= \delta + \sum_{k=1}^{\infty}(-1)^k\Bigg(\sum_{(\mathcal{W},\sigma)\in\mathcal{PS}(d),\pi\neq e}\zeta(\mathcal{W},\sigma)(\mathcal{W},\sigma)\Bigg)^k\\
&= \delta + \sum_{k=1}^{\infty}(-1)^k\Bigg(\sum_{(\mathbf{0}_\pi,\pi)\in\mathcal{PS}(d),\pi\neq e}\hbar^{|\pi|}(\mathbf{0}_\pi,\pi)\Bigg)^k\\
&= \delta + \sum_{k=1}^{\infty}(-1)^k\sum_{\substack{(\mathbf{0}_{\sigma_1},\sigma_1)\cdots(\mathbf{0}_{\sigma_k},\sigma_k)\in\mathcal{PS}(d)\\ \sigma_i\neq e}}\hbar^{|\sigma_1|+\cdots+|\sigma_k|}(\mathbf{0}_{\sigma_1},\sigma_1)\cdots(\mathbf{0}_{\sigma_k},\sigma_k)\\
&= \delta + \sum_{(\mathcal{V},\pi)\in\mathcal{PS}(d)}(\mathcal{V},\pi)\sum_{k=1}^{\infty}(-1)^k\sum_{\substack{(\mathbf{0}_{\sigma_1},\sigma_1)\cdots(\mathbf{0}_{\sigma_k},\sigma_k)=(\mathcal{V},\pi)\\ \sigma_i\neq e}}\hbar^{|\sigma_1|+\cdots+|\sigma_k|}.
\end{aligned}$$

Thus, taking the coefficient for fixed $(\mathcal{V},\pi)$, we obtain

$$\mu(\mathcal{V},\pi) = [(\mathcal{V},\pi)]\mu = \delta(\mathcal{V},\pi) + \sum_{k=1}^{\infty}(-1)^k\sum_{\substack{(\mathbf{0}_{\sigma_1},\sigma_1)\cdots(\mathbf{0}_{\sigma_k},\sigma_k)=(\mathcal{V},\pi)\\ \sigma_i\neq e}}\hbar^{|\sigma_1|+\cdots+|\sigma_k|}.$$

□

This formula enables us to identify μ as the generating function of a weighted version of the free group Hurwitz numbers of Definition 1.3.10.

Corollary 2.2.10.
The function μ is the generating function of alternating simple monotone free group Hurwitz numbers, i.e. let $\mu = (n_1, \dots, n_l) \vdash d$ be a partition, then

$$\mu(\mathbf{1}_d, \gamma_{n_1,\dots,n_l}) = \sum_{r=0}^{\infty} \hbar^k C_k^{||}(n_1, \dots, n_l),$$

where

$$C_k^{||}(n_1, \dots, n_l) = (-1)^k z_\mu H_k^{||}((1, \dots 1), (n_1, \dots, n_l)).$$

Proof. If we take $d > 1$ and $(\mathcal{V}, \pi) = (\mathbf{1}_d, \gamma_{n_1,\dots,n_l}) \in \mathcal{PS}(d)$ in Lemma 2.2.9 we find

$$\mu(\mathbf{1}_d, \gamma_{n_1,\dots,n_l}) = \sum_{k=1}^{\infty} (-1)^k \sum_{\substack{(\mathbf{0}_{\sigma_1}, \sigma_1)\cdots(\mathbf{0}_{\sigma_k}, \sigma_k) = (\mathbf{1}_d, \gamma_{n_1,\dots,n_l}) \\ \sigma_i \neq e}} \hbar^{|\sigma_1| + \cdots + |\sigma_k|}.$$

For any fixed value $r = |\sigma_1| + \cdots + |\sigma_k|$ we have

$$k \leq |\sigma_1| + \cdots + |\sigma_k| = r,$$

so only finitely many values of k, namely $k = 1, \dots, r$, contribute to the coefficient of $\hbar^r$. Thus, we may sum over r instead. We have

$$\mu(\mathbf{1}_d, \gamma_{n_1,\dots,n_l}) = \sum_{r=1}^{\infty} \hbar^r \sum_{k=1}^{r} (-1)^k \sum_{\substack{(\mathbf{0}_{\sigma_1}, \sigma_1)\cdots(\mathbf{0}_{\sigma_k}, \sigma_k) = (\mathbf{1}_d, \gamma_{n_1,\dots,n_l}) \\ \sigma_i \neq e, |\sigma_1| + \cdots + |\sigma_k| = r}} 1,$$

where the last sum counts factorizations of $\gamma_{n_1,\dots,n_l}$, i.e. of a fixed permutation of given cycle type $(n_1, \dots, n_l)$ (w.l.o.g. $n_{i+1} \geq n_i$), such that

- $|\sigma_1| + \cdots + |\sigma_k| = r$ or equivalently $\#\sigma_1 + \cdots + \#\sigma_k = dk - r$, and
- $\mathbf{0}_{\sigma_1} \vee \cdots \vee \mathbf{0}_{\sigma_k} = \mathbf{1}_d$ or in other words, $\{\sigma_1, \dots, \sigma_k, \gamma_{n_1,\dots,n_k}\}$ is a transitive subgroup of $S(d)$.

Hence, we continue

$$\begin{aligned}
\mu(\mathbf{1}_d, \gamma_{n_1,\dots,n_l}) &= \sum_{r=1}^{\infty} \hbar^r \sum_{k=1}^{r} (-1)^k \sum_{\substack{(\mathbf{0}_{\sigma_1},\sigma_1)\cdots(\mathbf{0}_{\sigma_k},\sigma_k)=(\mathbf{1}_d,\gamma_{n_1,\dots,n_l}) \\ \sigma_i \neq e, |\sigma_1|+\cdots+|\sigma_k|=r}} 1 \\
&= \sum_{r=1}^{\infty} \hbar^r (-1)^r \sum_{k=1}^{r} (-1)^{k+r} \sum_{\substack{(\mathbf{0}_{\sigma_1},\sigma_1)\cdots(\mathbf{0}_{\sigma_k},\sigma_k)=(\mathbf{1}_d,\gamma_{n_1,\dots,n_l}) \\ \sigma_i \neq e, |\sigma_1|+\cdots+|\sigma_k|=r}} 1 \\
&= \sum_{r=1}^{\infty} \hbar^r (-1)^r \sum_{k=1}^{r} (-1)^{k+r} \sum_{\substack{(\mathbf{0}_{\sigma_1},\sigma_1)\cdots(\mathbf{0}_{\sigma_k},\sigma_k)=(\mathbf{1}_d,\gamma_{n_1,\dots,n_l}) \\ \sigma_i \neq e, |\sigma_1|+\cdots+|\sigma_k|=r}} \\
&= \sum_{r=1}^{\infty} \hbar^r (-1)^r z_\mu H^{||,\circ}((1,\dots,1),(n_1,\dots,n_l)).
\end{aligned}$$

The extra factor z_μ takes into account that in the definition of $H^{||,\circ}(\lambda,\mu)$ we count factorizations of an arbitrary element in the conjugacy class C_μ; here we only consider one fixed element, namely $\gamma_{r_1,\dots,r_n}$. Furthermore, we need to compensate the factor $\frac{1}{d!}$ in the definition of the Hurwitz numbers. □

Of course, from the last lemma together with Proposition 1.3.11, we could conclude that μ enumerates monotone Hurwitz numbers. We prefer to prove this fact by a cut-and-join equation, obtained by ideas from [CMSS07]. More precisely, we will extend their Theorem 5.22.

Lemma 2.2.11.
Let $\gamma \in S(d)$ be such that it does not fix 1. Then we have

$$-\mu(\mathcal{U},\gamma) = \sum_{\substack{(\mathbf{0}_\tau,\tau)(\mathcal{V},\pi)=(\mathcal{U},\gamma) \\ \tau=(1,p)\in S(d))}} \hbar\mu(\mathcal{V},\pi) = \sum_{p=1}^{d} \sum_{(\mathbf{0}_{(1,p)},(1,p))(\mathcal{V},\pi)=(\mathcal{U},\gamma)} \hbar\mu(\mathcal{V},\pi),$$

i.e. we can express μ recursively by factoring a transposition that maps 1 to some $2 \le p \le d$.

Proof. The proof uses the same technique as in [CMSS07], we reproduce the main steps here. Recall that by Lemma 2.2.9

$$\mu(\mathcal{V},\pi) = \delta(\mathcal{V},\pi) + \sum_{k=1}^{\infty} (-1)^k \sum_{\substack{(\mathbf{0}_{\sigma_1},\sigma_1)\cdots(\mathbf{0}_{\sigma_k},\sigma_k)=(\mathcal{V},\pi) \\ \sigma_i \neq e}} \hbar^{|\sigma_1|+\cdots+|\sigma_k|},$$

i.e. the value of μ depends on factorizations of length k with a sign. The formula we want to prove is merely the realization of some cancellations within this sum. Therefore, we denote by S_k the set of factorizations of $(\mathcal{U},\gamma)$ of length k and we define the following subsets of S_k

$$S_k^q = \left\{ (\mathbf{0}_{\pi_1},\pi_1)\dots(\mathbf{0}_{\pi_k},\pi_k) = (\mathcal{U},\gamma) \;\middle|\; \begin{array}{l} q \text{ is the minimal } p \text{ such that} \\ \pi_p(1)\neq 1 \text{ and } \pi_p \text{ is a transposition} \end{array} \right\}.$$

In [CMSS07] it was shown that

$$\bigcup_{q\geq 2} S_k^q \cong S_{k-1} \setminus \bigcup_{q\geq 1} S_{k-1}^q := \hat{S}_k, \tag{2.2.4}$$

i.e. there is a bijection between the factorizations where there is minimal $q \geq 1$ such that $\pi_q = (1,p)$ is a transposition and the set of length $k-1$ factorization which contain no π_i of the latter form, i.e. there is no minimal q such that π_i interchanges 1 with some $2 \leq p \leq d$. Since the sign in (2.2.4) is determined by the length of the factorization, we only need to see that the $\hbar$ contribution cancels. But this is also clear from the explicit bijection in [CMSS07], we have

$$(\mathbf{0}_{\pi_1},\pi_1),\dots(\mathbf{0}_{\pi_k},\pi_k) \in S_k^q \mapsto (\mathbf{0}_{\pi_1},\pi_1),\dots,(\mathbf{0}_{\pi_{q-1}\pi_q},\pi_{q-1}\pi_q),\dots,(\mathbf{0}_{\pi_k},\pi_k) \in \hat{S}_k.$$

Since by definition of the set S_k^q, π_{q-1} fixes 1, and π_q is a transposition not fixing 1, π_q joins two cycles of π_{q-1}, i.e. we have

$$|\pi_{q-1}\pi_q| = |\pi_{q-1}| + 1 = |\pi_{q-1}| + |\pi_q|,$$

Finally, this gives the following equation

$$\begin{aligned}
\mu(\mathcal{U},\gamma) &= \sum_{k=1}^{\infty}(-1)^k \sum_{\substack{(\mathbf{0}_{\pi_1},\pi_1)\cdots(\mathbf{0}_{\pi_k},\pi_k)=(\mathcal{U},\gamma)\\ \pi_i\neq e}} \hbar^{|\pi_1|+\dots+|\pi_k|} \\
&= \sum_{k=2}^{\infty}(-1)^k \sum_{p\geq 2} \sum_{\substack{(\mathbf{0},(1,p))(\mathbf{0}_{\pi_2},\pi_2)\cdots(\mathbf{0}_{\pi_k},\pi_k)=(\mathcal{U},\gamma)\\ \pi_i\neq e}} \hbar^{1+|\pi_2|+\dots+|\pi_k|} \\
&= -\sum_{p\geq 2} \sum_{(\mathbf{0},(1,p))(\mathcal{V},\pi)=(\mathcal{U},\gamma)} \hbar \sum_{k=2}^{\infty}(-1)^{k-1} \sum_{\substack{(\mathbf{0}_{\pi_2},\pi_2)\cdots(\mathbf{0}_{\pi_k},\pi_k)=(\mathcal{V},\pi)\\ \pi_i\neq e}} \hbar^{|\pi_2|+\dots+|\pi_k|} \\
&= -\sum_{p\geq 2} \sum_{(\mathbf{0},(1,p))(\mathcal{V},\pi)=(\mathcal{U},\gamma)} \hbar\mu(\mathcal{V},\pi).
\end{aligned}$$

□

Remark 2.2.12.
Note that in the proof, it was crucial to assume that γ does not fix 1. Otherwise, the sets S_k^q could have been empty. But we can fix that problem and extend it to cases where γ

fixes 1. Assume we have a $\gamma \in S(d)$ that fixes 1 and is not the identity. Assume it does not fix $j \in [d]$ and denote $\tau = (1, j)$. If we denote $\mathcal{U}_\tau$ the partition where we apply τ to blocks of $\mathcal{U}$. Then by the multiplicativity

$$\mu(\mathcal{U}_\tau, \tau\gamma\tau) = \mu(\mathcal{U}, \gamma).$$

and since $\tau^{-1}\gamma\tau$ does not fix 1 but j we can apply the lemma to it instead.

The only case left yet, is the case of $\gamma = e \in S(d)$. But still, we can prove the following.

Lemma 2.2.13.
It holds

$$-\mu(\mathbf{1}_d, e) = \sum_{k=2}^{d} \sum_{\substack{(\mathbf{0}_\tau,\tau)(\mathcal{V},\pi)=(\mathbf{1}_n,e)\\ \tau=(1,p)\in S(d))}} \hbar\mu(\mathcal{V}, \pi)$$

Proof. Given a factorization $(\mathbf{0}_{\pi_1}, \pi_1)\cdots(\mathbf{0}_{\pi_k}, \pi_k) = (\mathbf{1}_d, e)$ we must have

$$\mathbf{0}_{\pi_1} \vee \cdots \vee \mathbf{0}_{\pi_k} = \mathbf{1}_d,$$

thus there must be $i = 1, \ldots, k$ such that π_i moves 1. Hence, we can follow the proof of Lemma 2.2.11, i.e. use the bijection (2.2.4). □

With these results at hand, we have the following recursion.

Theorem 2.2.14.
Let $n_1, \ldots, n_k \in \mathbb{N}$, $d = n_1 + \cdots + n_k$, $N = \{2, \ldots, k\}$ and

$$\gamma_{n_1,\ldots,n_k} = (1, \ldots, n_1)(n_1 + 1, \ldots, n_1 + n_2)\ldots(d - n_k + 1, \ldots, d).$$

Furthermore, for $I \subset N$, we define $M(I) = \sum_{i\in I} n_i$. Then it holds that

$$\begin{aligned} -\mu(\mathbf{1}_d,\gamma_{n_1,\ldots,n_k}) &= \hbar\sum_{j=2}^{k} n_j\mu(\mathbf{1}_d, \gamma_{n_1+n_j,n_2,\ldots,\check{n}_j,\ldots,n_k}) \\ &+ \sum_{\alpha+\beta=n_1} \hbar\mu(\mathbf{1}_d, \gamma_{\alpha,\beta,n_2,\ldots,n_k}) + \hbar\sum_{I\sqcup J=N} \mu(\mathbf{1}_{M(I)+\alpha}, \gamma_{\alpha,I})\mu(\mathbf{1}_{M(J)+\beta}, \gamma_{\beta,J}), \end{aligned}$$

where for $I \sqcup J = N$ and $\alpha + \beta = n_1$, and for $\alpha \geq \beta$ (for $\beta \geq \alpha$ analogously)

$$\gamma_{\alpha,I} = (1, \ldots, \alpha)\prod_{l\in I}(n_1 + \cdots + n_{l-1} + 1, \ldots, n_1 + \cdots + n_l)$$

and

$$\gamma_{\beta,J} = (\alpha + 1, \ldots, n_1)\prod_{l\in J}(n_1 + \cdots + n_{l-1} + 1, \ldots, n_1 + \cdots + n_l).$$

and $\check{j}$ denotes the fact that we omit the index j.

Proof. Before we start, let us introduce the notation

$$\begin{aligned}\gamma_{n_1,\dots,n_k} &= (1,\dots,n_1)(n_1+1,\dots,n_1+n_2)\dots(d-n_k+1,\dots,d)\\ &=: c_1\dots c_k.\end{aligned}$$

The proof is similar to a standard cut-and-join analysis. By Lemma 2.2.11, we have

$$-\mu(\mathbf{1}_d,\gamma_{n_1,\dots,n_k}) = \sum_{p=2}^{n} \sum_{\substack{(\mathbf{0}_\tau,\tau)(\mathcal{V},\pi)=(\mathbf{1}_d,\gamma_{n_1,\dots,n_k})\\ \tau=(1,p)\in S_n}} \hbar\mu(\mathcal{V},\pi).$$

This means we have to investigate the factorizations $(\mathbf{0}_\tau,\tau)(\mathcal{V},\pi)=(\mathbf{1}_d,\gamma_{n_1,\dots,n_k})$ with $\tau=(1,p)$. We have the following cases.

- First assume that $\mathcal{V}=\mathbf{1}_d$ and we have $\tau\pi=\gamma_{n_1,\dots,n_k}=:\gamma$, i.e. $\pi=\tau\gamma_{n_1,\dots,n_k}$. Then there are two possibilities, either $p>n_1$ in which case τ joins two cycles of γ or $2\le p\le n_1$, i.e. τ cuts the first cycle into two cycles of π. In the first case there must be $j=2,\dots,k$ such that

  $$n_1+\dots+n_{j-1}+1\le p\le n_1+\dots+n_j=:n^{(j)}.$$

  Then

  $$\tau\gamma=(1,\dots,n_1,p,p+1,\dots,n^{(j)},\dots,p-1)\prod_{\substack{l=2\\ l\neq j}}^{k} c_l,$$

  i.e. by conjugation invariance we get a contribution

  $$\hbar\mu(\mathbf{1}_d,\tau\gamma)=\hbar\mu(\mathbf{1}_d,\gamma_{n_1+n_j,n_2,\dots,\check{n}_j,\dots,n_k}).$$

  Moreover we get n_j such contributions since there are n_j possibilities for

  $$n^{(j-1)}+1\le p\le n^{(j)}.$$

  This gives the first summand

  $$\hbar\sum_{j=2}^{k} n_j\mu(\mathbf{1}_d,\gamma_{n_1+n_j,n_2,\dots,\check{n}_j,\dots,n_k}).$$

  If $1\le p\le n_1$ then we have

  $$\tau\gamma=(1,\dots,p-1)(p,\dots,n_1)\prod_{l=2}^{m} c_l$$

  and if we put $\alpha=p-1,\beta=n_1-p-1$ we get a contribution

  $$\mu(\mathbf{1}_d,\tau\gamma)=\mu(\mathbf{1}_d,\gamma_{\alpha,\beta,n_2,\dots,n_k}).$$

  Since all $1\le p\le n_1$ give such contribution we get all possibilities for $\alpha+\beta=n_1$, i.e. the sum

  $$\sum_{\alpha+\beta=k_1}\hbar\mu(\mathbf{1}_d,\gamma_{\alpha,\beta,n_2,\dots,n_k}).$$

- If $\mathcal{V} \neq \mathbf{1}_d$ then $\mathcal{V}$ can at most have 2 blocks, since τ can only connect 2 blocks in $\mathbf{0}_\tau \vee \mathcal{V} = \mathbf{1}_d$. Thus, we may write $\mathcal{V} = \{V_1, V_2\}$. Moreover, in this case τ must be a cut for the cycles of γ in $\pi = \tau\gamma$: if not, then π must be of the form

$$\tau\gamma = (1, \ldots, n_1, p, p+1, \ldots, n^{(j)}, \ldots, p-1) \prod_{\substack{l=2 \\ l \neq j}}^{k} c_l$$

hence the first cycle of $\tau\gamma$ either respects V_1 or V_2. Then τ must respect the same block. This implies $\mathbf{0}_\tau \vee \mathcal{V} = \mathcal{V} \neq \mathbf{1}_d$. Thus, τ cuts the first cycle of γ:

$$\pi = \underbrace{(1, \ldots, \alpha)}_{a} \underbrace{(\alpha+1, \ldots, \alpha+\beta)}_{b} \prod_{l=2}^{k} c_l,$$

and the cycles a and b must respect different blocks (otherwise we would again end up with $\mathbf{0}_\tau \vee \mathcal{V} \neq \mathbf{1}_d$), let us assume a respects V_1. Let I be the set of indices $j = 2, \ldots, k$ such that the j-th cycles c_j of γ respects V_1 and let J the set of indices that correspond to cycles that respect V_2. Then μ factorizes as

$$\mu(\mathcal{V}, \pi) = \mu\Big(V_1, (1, \ldots, \alpha) \prod_{j \in I} c_j\Big) \mu\Big(V_2, (\alpha+1, \ldots, \alpha+\beta) \prod_{j \in J} c_j\Big)$$

and by conjugacy invariance we have

$$\hbar \sum_{I \sqcup J = N} \mu(\mathbf{1}_{M(I)+\alpha}, \gamma_{\alpha,I}) \mu(\mathbf{1}_{M(J)+\beta}, \gamma_{\beta,J}).$$

Taking all possible cuts for c_1 into consideration, we get the desired sum

$$\sum_{\alpha+\beta=n_1} \hbar \sum_{I \sqcup J = N} \mu(\mathbf{1}_{M(I)+\alpha}, \gamma_{\alpha,I}) \mu(\mathbf{1}_{M(J)+\beta}, \gamma_{\beta,J}).$$

□

As already mentioned, we discuss topological recursion for these numbers in Chapter 3, thus it is natural to write the generating functions in terms of a tuple (g, n) of positive integers. First, we have the following observation.

Lemma 2.2.15.
We have

$$\mu(\mathcal{U}, \gamma) = \hbar^{|(\mathcal{U},\gamma)|} \sum_{g \geq 0} \mu^{[g]}(\mathcal{U}, \gamma) \hbar^{2g}.$$

Proof. Recall the geometric series expansion of μ:

$$\mu(\mathcal{U},\gamma)=\delta(\mathcal{U},\gamma)+\sum_{k=1}^{\infty}(-1)^k\sum_{\substack{(\mathbf{0}_{\pi_1},\pi_1)\cdots(\mathbf{0}_{\pi_k},\pi_k)=(\mathcal{U},\gamma)\\ \pi_i\neq e}}\hbar^{|\pi_1|+\cdots+|\pi_k|}.$$

Then for any factorization $(\mathbf{0}_{\pi_1},\pi_1)\cdots(\mathbf{0}_{\pi_k},\pi_k)=(\mathcal{U},\gamma)$ we have by the triangle inequality

$$|\pi_1|+\cdots+|\pi_k|=|(\mathbf{0}_{\pi_1},\pi_1)|+\cdots+|(\mathbf{0}_{\pi_k},\pi_k)|\geq|(\mathcal{U},\gamma)|,$$

i.e. the lowest order of $\hbar$ we can get is indeed $\hbar^{|(\mathcal{U},\gamma)|}$. The part about the exponent being even up to $\hbar^{|(\mathcal{U},\gamma)|}$ follows from the fact that for a factorization

$$|(\mathcal{V},\pi)|+|(\mathcal{W},\sigma)|-|(\mathcal{V}\vee\mathcal{W},\pi\sigma)|$$

is even. We prove this by induction on the number k of factors of π. We already know the case $k=2$. So given the factorization above into $\pi_1,\dots,\pi_k$, then

$$|\pi_2|+\cdots+|\pi_k|-|(\mathbf{0}_{\pi_2}\vee\cdots\vee\mathbf{0}_{\pi_k},\pi_2\cdots\pi_k)|=2g_1$$

by induction hypothesis and by the case $k=2$ we have

$$|\pi_1|+|(\mathbf{0}_{\pi_2}\vee\cdots\vee\mathbf{0}_{\pi_k},\pi_2\cdots\pi_k)|-|(\mathcal{V},\gamma)|=2g_2.$$

Putting things together, we find

$$|\pi_1|+|\pi_2|+\cdots+|\pi_k|-|(\mathcal{V},\gamma)|=2(g_1+g_2).$$

This finally proves that we can write μ as claimed. □

Theorem 2.2.16.
Let $d\in\mathbb{N}$ and $\lambda=(n_1,\dots,n_k)\vdash d$ a partition. Then the values $\mu_{g,k}:=\mu^g(\mathbf{1}_d,\gamma_{n_1,\dots,n_k})$ are given by the alternating simple monotone Hurwitz numbers via

$$\mu_{g,k}(\lambda)=(-1)^{d+k}z_\mu H^{\leq}_{g,k}(\mu).$$

Proof. We insert the expansion of Lemma 2.2.15 in the recursion of Theorem 2.2.14 and get

$$\begin{aligned}-\hbar^{|(\mathbf{1}_d,\gamma_{n_1,\dots,n_k})|}&\sum_{g\geq0}\mu^{[g]}(\mathbf{1}_d,\gamma_{n_1,\dots,n_k})\hbar^{2g}=\\ &\hbar^{|(\mathbf{1}_d,\gamma_{n_1+n_j,n_2,\dots,\check{n}_j,\dots,n_k})|+1}\sum_{j=2}^{k}\sum_{g\geq0}n_j\mu^{[g]}(\mathbf{1}_d,\gamma_{n_1+n_j,n_2,\dots,\check{n}_j,\dots,n_k})\hbar^{2g}\\ &+\hbar\sum_{\alpha+\beta=n_1}\hbar^{|(\mathbf{1}_d,\gamma_{\alpha,\beta,n_2,\dots,n_k})|}\sum_{g\geq0}\mu^{[g]}(\mathbf{1}_d,\gamma_{\alpha,\beta,n_2,\dots,n_k})\hbar^{2g}\end{aligned}$$

$$+\hbar \sum_{\alpha+\beta=n_1} \sum_{I\sqcup J=N} \hbar^{|(\mathbf{1}_{M(I)+\alpha},\gamma_{\alpha,I})|} \sum_{g\geq 0} \mu^{[g]}(\mathbf{1}_{M(I)+\alpha},\gamma_{\alpha,I})\hbar^{2g}$$
$$\times \hbar^{|(\mathbf{1}_{M(J)+\beta},\gamma_{\beta,J})|} \sum_{g\geq 0} \mu^{[g]}(\mathbf{1}_{M(J)+\beta},\gamma_{\beta,J})\hbar^{2g}.$$

Now note that

$$\begin{aligned}
|(\mathbf{1}_d,\gamma_{n_1,\dots,n_k})| &= 2(d-1)-(d-k) = d-2+k \\
|(\mathbf{1}_d,\gamma_{n_1+n_j,\dots,n_k})| &= 2(d-1)-(d-(k-1)) = |(\mathbf{1}_d,\gamma_{n_1,\dots,n_k})|-1 \\
|(\mathbf{1}_d,\gamma_{\alpha,\beta,\dots,n_k}) &= 2(d-1)-(d-(k+1)) = |(\mathbf{1}_d,\gamma_{n_1,\dots,n_k})|+1
\end{aligned}$$

and

$$\begin{aligned}
|(\mathbf{1}_{|I|+1},\gamma_{\alpha,I})|+|(\mathbf{1}_{|J|+1},\gamma_{\beta,I})| &= (2(M(I)+\alpha-1))-(M(I)+\alpha-(|I|+1)) \\
&\quad + (2(M(J)+\beta-1))-(M(J)+\beta-(|J|+1)) \\
&= \underbrace{M(I)+M(J)+\alpha+\beta}_{d}+\underbrace{|I|+|J|}_{k-1}-2 \\
&= |(\mathbf{1}_d,\gamma_{n_1,\dots,n_k})|-1.
\end{aligned}$$

So dividing by $\hbar^{|(\mathbf{1}_d,\gamma_{n_1,\dots,n_k})|}$ we get

$$\begin{aligned}
\sum_{g\geq 0} \mu^{[g]}(\mathbf{1}_d,\gamma_{n_1,\dots,n_k})\hbar^{2g} &= \sum_{g\geq 0}\sum_{j=2}^{k} n_j \mu^{[g]}(\mathbf{1}_d,\gamma_{n_1+n_j,n_2,\dots,\check{n}_j,\dots,n_k})\hbar^{2g} \\
&+ \sum_{g\geq 0}\sum_{\alpha+\beta=n_1} \mu^{[g]}(\mathbf{1}_d,\gamma_{\alpha,\beta,n_2,\dots,n_k})d^{2(g-1)} \\
&+ \sum_{g\geq 0}\sum_{g_1+g_2=g}\sum_{\alpha+\beta=n_1}\sum_{I\sqcup J=N} \mu^{[g_1]}(\mathbf{1}_{M(I)+\alpha},\gamma_{\alpha,I})\mu^{[g_2]}(\mathbf{1}_{M(J)+\beta}.\gamma_{\beta,J})\hbar^{2(g_1+g_2)}
\end{aligned}$$

By taking the coefficient of $\hbar^{2g}$ we find

$$\begin{aligned}
\mu^{[g]}(\mathbf{1}_d,\gamma_{n_1,\dots,n_k}) &= \sum_{j=2}^{k} n_j \mu^{[g]}(\mathbf{1}_d,\gamma_{n_1+n_j,n_2,\dots,\check{n}_j,\dots,n_k}) \\
&+ \sum_{\alpha+\beta=n_1} \mu^{[g-1]}(\mathbf{1}_d,\gamma_{\alpha,\beta,n_2,\dots,n_k}) \\
&+ \sum_{\alpha+\beta=n_1}\sum_{g_1+g_2=g}\sum_{I\sqcup J=N} \mu^{[g_1]}(\mathbf{1}_{M(J)+\alpha},\gamma_{\alpha,I})\mu^{[g_2]}(\mathbf{1}_{M(J)+\beta},\gamma_{\beta,J}).
\end{aligned}$$

which is exactly the recursion formula for alternating monotone Hurwitz numbers (see [HvIL22] or Chapter 3). Thus, since the recursion and the initial value determine the numbers uniquely, both Hurwitz numbers and values of the Möbius function must agree. □

Remark 2.2.17.

i) Technically, we reproved Proposition 1.3.11. There are several possibilities to prove this fact, in particular using elementary combinatorics in $S(d)$, which is of course the main ingredient here as well.

ii) Furthermore, we see that the Möbius inversion on partitioned permutations is merely an avatar of Lemma 1.3.9.

2.3 Topological partition functions vs multiplicative functions

The combinatorial proof of the functional relations could not be extended to $n \geq 3$ yet. The derivation for these higher order functional relations in [BCGF$^+$23] uses a reformulation of the moment-cumulant formalism into the language of operators on the (bosonic) Fock space. In this framework, we can use the toolbox [BDBKS22] to answer this problem which was posed in [CMSS07]. More precisely, the information of an extended multiplicative function can be encoded in a so-called partition function. These partition functions are generating series in infinitely many variables and can be manipulated by operators on the Fock space.

Recall the definition of the bosonic Fock space

$$\mathscr{B} = \mathbb{C}[\![p_1, p_2 \dots]\!], \quad \mathscr{B}_\hbar = \mathbb{C}[\![p_1, p_2 \dots]\!](\!(\hbar)\!).$$

Let us note, that for an element $F \in \mathscr{B}$ (resp. $\mathscr{B}_\hbar$) with no constant term, $e^F \in B_\hbar$ is well-defined. Moreover, if $\lambda = (\lambda_1, \dots, \lambda_{\ell(\lambda)}) \vdash d$ is a partition, we define

$$p_\lambda = p_{\lambda_1} \cdots p_{\lambda_{\ell(\lambda)}}$$

If $\sigma \in S(d)$ is a permutation, we write $[p_\sigma]\, F$ for the coefficient of $p_{\lambda(0_\sigma)}$ in F, with the convention that $p_\emptyset = 1$. Further, we denote $[p_\emptyset]\, F = \langle | F$. Recall the definition

$$z_\lambda := \prod_{i=1}^{\ell(\lambda)} \lambda_i \prod_{j \geq 1} m_j(\lambda)!,$$

where $m_j(\lambda)$ is the number of occurrences of j in λ.

Definition 2.3.1.
A *topological partition function* is an element of $\mathcal{B}_\hbar$ of the form $Z = e^F$, where

$$F = \left(\sum_{\substack{g \in \mathbb{Z}_{\geq 0} \\ n \in \mathbb{N}}} \hbar^{2g-2+n} F_{g,n} \right) \in \mathcal{B}_\hbar,$$

and $F_{g,n} \in \mathcal{B}$ has no constant term, i.e. $\langle | F = 0$.

Recall that our motivation was to collect higher genus terms for matrix models (2.2.1) and thus a multiplicative function $\varphi \in \mathcal{PS} \to \mathbb{C}[\![\hbar]\!]$ is determined by a collection of numbers $\varphi^{[g]}_{r_1,\dots,r_n}$ arranged in a generating function $\varphi(\mathbf{1}_{r_1+\dots+r_n}, \gamma_{r_1,\dots,r_n})$ in $\hbar$. Having again (2.2.1) and Lemma 2.2.8 in mind, we assume that $\varphi(\mathcal{U},\gamma) = o(\hbar^{|(\mathcal{U},\gamma)|})$. We have

$$\varphi(\mathbf{1}_d, \gamma_{r_1,\dots,r_n}) = \hbar^{d+n-2} \sum_{g\geq 0} \varphi^{[g]}(\mathbf{1}_d, \gamma_{r_1,\dots,r_n})\hbar^{2g} = \hbar^d \sum_{g\geq 0} \varphi^{[g]}(\mathbf{1}_d, \gamma_{r_1,\dots,r_n})\hbar^{n+2g-2},$$

where $d = r_1 + \cdots + r_n$. Naturally we would like to define

$$F^{\varphi} = \sum_{r_1,\dots,r_n\geq 1} \hbar^{-d} \varphi(\mathbf{1}_{r_1+\dots+r_n}, \gamma_{r_1,\dots,r_n}) p_{r_1} \cdots p_{r_n},$$

but since multiplicative functions are invariant w.r.t. conjugation we need to introduce a symmetry factor so that we do not count the same information multiple times. Given lengths $r_1, \dots, r_n$, we have $n!$ ways to label the cycles of a permutation and $r_1 \cdots r_n$ ways for the elements respecting the cyclic order. Then

$$F^{\varphi} = \sum_{r_1,\dots,r_n\geq 1} \hbar^{-d} \frac{\varphi(\mathbf{1}_{r_1+\dots+r_n}, \gamma_{r_1,\dots,r_n})}{n! r_1 \cdots r_n} p_{r_1} \cdots p_{r_n} = \sum_{\substack{d\geq 1\\ \lambda\vdash d}} \hbar^{-d} \varphi(\mathbf{1}_d, \gamma_\lambda) \frac{p_\lambda}{z_\lambda}. \tag{2.3.1}$$

This defines a topological partition function $Z^{\varphi} = e^{F^{\varphi}}$ and we can write

$$Z = 1 + \sum_{\substack{d\geq 1\\ \lambda\vdash d}} \hbar^{-d} Z(\lambda) \frac{p_\lambda}{z_\lambda}. \tag{2.3.2}$$

The relation between the coefficients $Z(\lambda)$ and the multiplicative function is

$$Z(\lambda) = \sum_{\substack{\mathcal{V}\in\mathcal{P}(d)\\ \mathbf{0}_\lambda\leq\mathcal{V}}} \varphi(\mathcal{V}, \gamma_\lambda). \tag{2.3.3}$$

Note, the 1 in (2.3.2) may be absorbed by allowing the empty partition in the sum while setting $Z(\emptyset) = 1$.

Vice versa, consider a topological partition function $Z = e^F$, we can associate the unique multiplicative function $f_{Z,\hbar} \colon \mathcal{PS} \to \mathbb{C}[\![\hbar]\!]$ such that

$$F = \sum_{\substack{d\geq 1\\ \lambda\vdash d}} \hbar^{-d} f_{Z,\hbar}(\mathbf{1}_d, \gamma_\lambda) \frac{p_\lambda}{z_\lambda}. \tag{2.3.4}$$

In other words, for $(\mathcal{V},\gamma) \in \mathcal{PS}(d)$ we have

$$\begin{aligned} f_{Z,\hbar}(\mathcal{V},\gamma) &\coloneqq \hbar^d \prod_{B\in\mathcal{V}} [p_{\gamma_{|B}}] z_{\gamma_{|B}} F \\ &= \hbar^{|(\mathcal{V},\gamma)|} \sum_{g\colon \mathcal{V}\to\mathbb{N}} \hbar^{2\sum_{B\in\mathcal{V}} g(B)} \prod_{B\in\mathcal{V}} [p_{\gamma_{|B}}] z_{\gamma_{|B}} F_{g(V),\ell(\gamma_{|B})} \\ &\coloneqq \hbar^{|(\mathcal{V},\gamma)|} \sum_{g\colon \mathcal{V}\to\mathbb{N}} \hbar^{2\sum_{B\in\mathcal{V}} g(B)} f^{[g]}_Z(\mathcal{V},\gamma). \end{aligned} \tag{2.3.5}$$

Here in the second and the third formula, the sum runs over all possible assignments of non-negative integers $g(B)$ to $B \in \mathcal{V}$. In the way described above, the multiplicative function completely determines the partition function and vice versa.

Remark 2.3.2.
These definitions involve conventions regarding powers of $\hbar$ and prefactors z_λ as explained above. These are motivated by the following guiding principles from analytic combinatorics. Partition functions are commonly used to encode and manipulate generating function of quantities counting algebraic and topological objects and structures, e.g. Hurwitz numbers or maps, see e.g. [KL15], [BDBKS23]. Particularly, the power of $\hbar$ in topological partition functions behaves like the Euler characteristic, the latter being additive under disjoint union. This is compatible with an interpretation of F as a generating series of connected objects, while $Z = e^F$ generates disconnected objects. Thus, we want the contribution of type (g, n), of genus g with n cycles, to appear in F with $\hbar^{2g-2+n}$. Again, for the multiplicative function the power of $\hbar$ is dictated by (2.2.1) and Lemma 2.2.8, we want $\varphi(\mathcal{V}, \gamma)$ to have leading order $\hbar^{|(\mathcal{V},\gamma)|}$. In particular, due to $|(\mathbf{1}_d, \gamma_\lambda)| = d - 2 + \ell(\lambda)$ and (2.3.5), the (g, n)-part contributes to $\varphi(\mathbf{1}_d, \gamma_\lambda)$ with $\hbar^{2g-2+\ell(\lambda)+d}$. To respect the principle, we canceled the $\hbar^d$ in (2.3.5) to define F in (2.3.4).

Now recall from Section 1.2 that the Heisenberg algebra acts on the Fock space via the operators

$$1_{\mathcal{H}} \mapsto \mathrm{id}_B, \quad a_n \mapsto \mathsf{J}_n = \begin{cases} n\partial_{p_n} & \text{if } n > 0, \\ p_{-n} & \text{if } n < 0, \\ 0 & \text{if } n = 0. \end{cases}$$

We collect them in two generating series

$$\mathsf{J}(x) = \sum_{k>0} x^k \mathsf{J}_k \quad \text{and} \quad \widetilde{\mathsf{J}}(x) = \sum_{k\in\mathbb{Z}} x^k \mathsf{J}_k.$$

Thus, given a topological partition function (or equivalently multiplicative function) $Z = e^F$, we may describe it as an operator on the Fock space via

$$\mathsf{F} = \sum_{\substack{n\geq 1 \\ g\geq 0}} \frac{\hbar^{2g-2+n}}{n!} \sum_{r_1,\dots,r_n>0} F_{g;r_1,\dots,r_n} \prod_{i=1}^{n} \frac{\mathsf{J}_{-r_i}}{r_i} \tag{2.3.6}$$

and hence

$$Z = \exp \mathsf{F}|\rangle.$$

The main result relevant for higher order free probability is a functional relation between the generating series of higher order moments and cumulants of fixed order, rather than organized in a topological partition function. Thus, we have the following definition.

Definition 2.3.3.
Let $Z = e^F$ be a partition function. We define the *n-point functions* by

$$G_n(x_1, \dots, x_n) = \hbar^{-1}\delta_{n,1} + \sum_{g\geq 0} \sum_{r_1,\dots,r_n>0} \hbar^{2g-2+n} F_{g;r_1,\dots,r_n} x_1^{r_1} \cdots x_n^{r_n} \tag{2.3.7}$$

and the *shifted n-point functions* by

$$\widetilde{G}_n(x_1, \dots, x_n) = G_n(x_1, \dots, x_n) + \delta_{n,2} \frac{x_1 x_2}{(x_1 - x_2)^2}.$$

If we want to emphasize that the n-point function depends on a multiplicative function $\varphi\colon \mathcal{PS} \to \mathbb{C}[\![\hbar]\!]$, i.e. $F_{g,n}$ is given by (2.3.1), we denote it by G^{φ}.

Remark 2.3.4.
By definition, the n-point functions admit a genus expansion

$$G_n = \sum_{g\geq 0} \hbar^{2g-2+n} G_{g,n}(x_1, \dots, x_n),$$
$$\widetilde{G}_n = \sum_{g\geq 0} \hbar^{2g-2+n} \widetilde{G}_{g,n}(x_1, \dots, x_n).$$

We can also express these quantities in terms of Fock space operators, more precisely in terms of vacuum expectation values. Therefore, let us introduce the following inclusion-exclusion formulas.

Lemma 2.3.5.
There is a linear map $\langle| \cdot |\rangle \colon \mathrm{GL}(\mathscr{B}_\hbar) \to \mathbb{C}[\![\hbar]\!]$ given by

$$A \mapsto \langle| A |\rangle = [1]A(1).$$

We call it *vacuum expectation value*. Moreover, we can define the *connected vacuum expectation value* $\langle| \cdot |\rangle^\circ \colon \mathrm{GL}(\mathscr{B}_\hbar) \to \mathbb{C}[\![\hbar]\!]$ by the relations

$$\langle| \mathsf{A}_1 \cdots \mathsf{A}_n e^{\mathsf{F}} |\rangle = \sum_{\mathcal{I}\in\mathcal{P}(n)} \prod_{I\in\mathcal{I}} \Big\langle\Big| \prod_{i\in I} \mathsf{A}_i \cdot e^{\mathsf{F}} \Big|\Big\rangle^\circ, \tag{2.3.8}$$

$$\langle| \mathsf{A}_1 \cdots \mathsf{A}_n \cdot e^{\mathsf{F}} |\rangle^\circ = \sum_{\mathcal{I}\in\mathcal{P}(n)} (-1)^{\#\mathcal{I}-1} (\#\mathcal{I} - 1)! \prod_{I\in\mathcal{I}} \Big\langle\Big| \prod_{i\in I} \mathsf{A}_i e^{\mathsf{F}} \Big|\Big\rangle \tag{2.3.9}$$

or equivalently

$$\langle| \mathsf{A}_1 \cdots \mathsf{A}_n \cdot e^{\mathsf{F}} |\rangle^\circ := \partial_{t_1} \cdots \partial_{t_n} \ln \Big(\langle| e^{t_1 \mathsf{A}_1} \cdots e^{t_n \mathsf{A}_n} e^{\mathsf{F}} |\rangle \Big) \Big|_{t_1=0,\dots,t_n=0}. \tag{2.3.10}$$

Proof. Note that the first two statements in Lemma 2.3.5 are equivalent by Möbius inversion on the lattice $\mathcal{P}(n)$. Thus, we prove their equivalence to (2.3.10). We start by proving that

$$\partial_{t_1}\dots\partial_{t_n}\ln\langle|e^{t_1\mathsf{A}_1}\dots e^{t_n\mathsf{A}_n}\mathsf{B}|\rangle = \sum_{l=1}^{n}\sum_{k=l}^{\infty}(-1)^{k-1}\frac{(k-1)!}{(k-l)!}\Big[\langle|e^{t_1\mathsf{A}_1}\cdots e^{t_n\mathsf{A}_n}\mathsf{B}|\rangle-1\Big]^{k-l} \sum_{\substack{\mathcal{I}\in\mathcal{P}(n)\\ \#\mathcal{I}=l}}\prod_{J\in\mathcal{I}}\Big[\Big(\prod_{i\in J}\partial_{t_i}\Big)\langle|e^{t_1\mathsf{A}_1}\cdots e^{t_n\mathsf{A}_n}\mathsf{B}|\rangle\Big] \tag{2.3.11}$$

for any B independent of $t_1,\dots,t_n$. For $n=1$ we have by expanding the logarithm

$$\begin{aligned}\partial_{t_1}\ln\langle|e^{A_1t_1}\mathsf{B}|\rangle &= \partial_{t_1}\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k}\Big[\langle|e^{A_1t_1}\mathsf{B}|\rangle-1\Big]^{k}\\ &=\sum_{k=1}^{\infty}(-1)^{k-1}\Big[\langle|e^{A_1t_1}\mathsf{B}|\rangle-1\Big]^{k-1}\partial_{t_1}\langle|e^{A_1t_1}\mathsf{B}|\rangle.\end{aligned}$$

Now for any $n>1$ we have by induction hypothesis for $\mathsf{B}'=e^{t_nA_n}\mathsf{B}$,

$$\begin{aligned}&\partial_{t_1}\dots\partial_{t_n}\ln\langle|e^{t_1\mathsf{A}_1}\dots e^{t_n\mathsf{A}_n}\mathsf{B}|\rangle\\
&=\partial_{t_n}\sum_{l=1}^{n-1}\sum_{k=l}^{\infty}(-1)^{k-1}\frac{(k-1)!}{(k-l)!}\Big[\langle|e^{t_1\mathsf{A}_1}\cdots e^{t_n\mathsf{A}_n}\mathsf{B}|\rangle-1\Big]^{k-l}\\
&\quad\sum_{\substack{\mathcal{I}\in\mathcal{P}(n-1)\\ \#\mathcal{I}=l}}\prod_{J\in\mathcal{I}}\Big[\Big(\prod_{i\in J}\partial_{t_i}\Big)\langle|e^{t_1\mathsf{A}_1}\cdots e^{t_n\mathsf{A}_n}\mathsf{B}|\rangle\Big]\\
&=\sum_{l=1}^{n-1}\sum_{k=l+1}^{\infty}(-1)^{k-1}\frac{(k-1)!}{(k-(l+1))!}\Big[\langle|e^{t_1\mathsf{A}_1}\cdots e^{t_n\mathsf{A}_n}\mathsf{B}|\rangle-1\Big]^{k-(l+1)}\\
&\quad\partial_{t_n}\langle|e^{t_1\mathsf{A}_1}\cdots e^{t_n\mathsf{A}_n}\mathsf{B}|\rangle\sum_{\substack{\mathcal{I}\in\mathcal{P}(n-1)\\ \#\mathcal{I}=l}}\Big[\prod_{J\in\mathcal{I}}\Big(\prod_{i\in J}\partial_{t_i}\Big)\langle|e^{t_1\mathsf{A}_1}\cdots e^{t_n\mathsf{A}_n}\mathsf{B}|\rangle\Big]\\
&\quad+\sum_{l=1}^{n-1}\sum_{k=l}^{\infty}(-1)^{k-1}\frac{(k-1)!}{(k-l)!}\Big[\langle|e^{t_1\mathsf{A}_1}\cdots e^{t_n\mathsf{A}_n}\mathsf{B}|\rangle-1\Big]^{k-l}\\
&\quad\sum_{\substack{\mathcal{I}\in\mathcal{P}(n-1)\\ \#\mathcal{I}=l}}\partial_{t_n}\Big[\prod_{J\in\mathcal{I}}\Big(\prod_{i\in J}\partial_{t_i}\Big)\langle|e^{t_1\mathsf{A}_1}\cdots e^{t_n\mathsf{A}_n}\mathsf{B}|\rangle\Big].\end{aligned}$$

Now in the first sum $\mathcal{I}_1=\mathcal{I}\sqcup\{\{n\}\}\in\mathcal{P}(n)$ with $\#\mathcal{I}_1=l+1$ and in the second we apply the product rule and extend one of the blocks of $\mathcal{I}$ by the element n and hence also get a sum over $\mathcal{I}_2\in\mathcal{P}(n)$ having n not as a singleton and $\#\mathcal{I}_2=l$. Thus, for the first term with shifting the index

$$\sum_{l=1}^{n-1}\sum_{k=l+1}^{\infty}(-1)^{k-1}\frac{(k-1)!}{(k-(l+1))!}\Big[\langle|e^{t_1\mathsf{A}_1}\cdots e^{t_n\mathsf{A}_n}\mathsf{B}|\rangle-1\Big]^{k-(l+1)}$$
$$\partial_{t_n}\langle|e^{t_1\mathsf{A}_1}\cdots e^{t_n\mathsf{A}_n}\mathsf{B}|\rangle\sum_{\substack{\mathcal{I}\in\mathcal{P}(n-1)\\ \#\mathcal{I}=l}}\Big[\prod_{i\in\mathcal{I}}\partial_{t_i}\Big]\langle|e^{t_1\mathsf{A}_1}\cdots e^{t_n\mathsf{A}_n}\mathsf{B}|\rangle$$
$$=\sum_{l=2}^{n}\sum_{k=l}^{\infty}(-1)^{k-1}\frac{(k-1)!}{(k-l)!}\Big[\langle|e^{t_1\mathsf{A}_1}\cdots e^{t_n\mathsf{A}_n}\mathsf{B}|\rangle-1\Big]^{k-l}$$
$$\sum_{\substack{\mathcal{I}\sqcup\{\{n\}\}=\mathcal{I}_1\in\mathcal{P}(n)\\ \#\mathcal{I}=l-1}}\Big[\prod_{i\in\mathcal{I}_1}\partial_{t_i}\Big]\langle|e^{t_1\mathsf{A}_1}\cdots e^{t_n\mathsf{A}_n}\mathsf{B}|\rangle$$

and for the second one

$$\sum_{l=1}^{n-1}\sum_{k=l}^{\infty}(-1)^{k-1}\frac{(k-1)!}{(k-l)!}\Big[\langle|e^{t_1\mathsf{A}_1}\cdots e^{t_n\mathsf{A}_n}\mathsf{B}|\rangle-1\Big]^{k-l}$$
$$\sum_{\substack{\mathcal{I}\in\mathcal{P}(n-1)\\ \#\mathcal{I}=l}}\partial_{t_n}\Big[\prod_{J\in\mathcal{I}}\Big(\prod_{i\in J}\partial_{t_i}\Big)\langle|e^{t_1\mathsf{A}_1}\cdots e^{t_n\mathsf{A}_n}\mathsf{B}|\rangle\Big]$$
$$=\sum_{l=1}^{n-1}\sum_{k=l}^{\infty}(-1)^{k-1}\frac{(k-1)!}{(k-l)!}\Big[\langle|e^{t_1\mathsf{A}_1}\cdots e^{t_n\mathsf{A}_n}\mathsf{B}|\rangle-1\Big]^{k-l}$$
$$\sum_{\substack{\mathcal{I}\in\mathcal{P}(n),\,\#\mathcal{I}=l\\ n\text{ is not a singleton}}}\Big[\prod_{J\in\mathcal{I}}\Big(\prod_{i\in J}\partial_{t_i}\Big)\langle|e^{t_1\mathsf{A}_1}\cdots e^{t_n\mathsf{A}_n}\mathsf{B}|\rangle\Big].$$

Besides the terms $l=n$ in the first equation and the $l=1$ term in the second we can combine both sums since they complement each other and find precisely the desired formula. Further, we have for $\mathsf{B}=e^{\mathsf{F}}$

$$\langle|e^{t_1\mathsf{A}_1}\cdots e^{t_n\mathsf{A}_n}e^{\mathsf{F}}|\rangle|_{t_1=0,\dots,t_n=0}=1$$

and

$$\partial_{t_{i_1}}\dots\partial_{t_{i_k}}\langle|e^{t_1\mathsf{A}_1}\cdots e^{t_n\mathsf{A}_n}e^{\mathsf{F}}|\rangle=\partial_{t_{i_1}}\dots\partial_{t_{i_k}}\sum_{r_1,\dots,r_n=0}^{\infty}\frac{t_1^{r_1}\dots t_n^{r_n}}{r_1!\dots r_n!}\langle|\mathsf{A}_1^{r_1}\dots\mathsf{A}_n^{r_n}e^{\mathsf{F}}|\rangle$$
$$=\sum_{\substack{r_j=0,\,j\neq i_s\\ j=1,\dots,n\\ s=1,\dots,k}}^{\infty}\sum_{\substack{r_j=1,\,j=i_s\\ j=1,\dots,n\\ s=1,\dots,k}}^{\infty}\prod_{j\neq i_s}\frac{t_1^{r_j}}{r_j!}\prod_{j=i_s}\frac{t_1^{r_j-1}}{(r_j-1)!}\langle|\mathsf{A}_1^{r_1}\dots\mathsf{A}_n^{r_n}e^{\mathsf{F}}|\rangle.$$

Thus

$$\Big[\langle|e^{t_1\mathsf{A}_1}\cdots e^{t_n\mathsf{A}_n}e^{\mathsf{F}}|\rangle-1\Big]^{k-l}\Big|_{t_1=0,\dots,t_n=0}=\delta_{k-l,0},$$

and

$$\partial_{t_{i_1}} \dots \partial_{t_{i_k}} \langle | e^{t_1 \mathsf{A}_1} \cdots e^{t_n \mathsf{A}_n} e^{\mathsf{F}} | \rangle \Big|_{t_1=0,\dots,t_n=0} = \langle | \mathsf{A}_{i_1}^{r_{i_1}} \dots \mathsf{A}_{i_k}^{r_{i_k}} e^{\mathsf{F}} | \rangle.$$

The latter together with (2.3.11) yields the equivalence of (2.3.8) and (2.3.10). □

Then we have the following description of the n-point function in terms of the Fock space operators.

Lemma 2.3.6.
It holds

$$G_n(x_1,\dots,x_n) = \hbar^{-1}\delta_{n,1} + \Big\langle \Big| \prod_{i=1}^{n} \mathsf{J}(x_i) \cdot e^{\mathsf{F}} \Big| \Big\rangle^{\circ},$$
$$\widetilde{G}_n(x_1,\dots,x_n) = \hbar^{-1}\delta_{n,1} + \Big\langle \Big| \prod_{i=1}^{n} \widetilde{\mathsf{J}}(x_i) \cdot e^{\mathsf{F}} \Big| \Big\rangle^{\circ}.$$

Proof. By Leibniz rule

$$\partial_{p_{r_1}} \dots \partial_{p_{r_n}} e^F = \sum_{\mathcal{I} \in \mathcal{P}(n)} \prod_{J \in \mathcal{I}} \Big(\prod_{i \in J} \partial_{p_{r_i}} F \Big) e^F$$

and we write $F_{g;r_{j_1},\dots,r_{j_l}} = F_{g;J}$ if $J = \{j_1,\dots,j_l\}$, then by (2.3.6)

$$\langle | \mathsf{J}_{r_1} \dots \mathsf{J}_{r_n} e^F | \rangle = \prod_{\mathcal{I} \in \mathcal{P}(n)} \prod_{J \in \mathcal{I}} \Big(\sum_{g \geq 0} \hbar^{2g-2+\#J} F_{g;J} \Big).$$

By inclusion-exclusion we have

$$\langle | \mathsf{J}_{r_1} \dots \mathsf{J}_{r_n} e^F | \rangle^{\circ} = \sum_{g \geq 0} \hbar^{2g-2+n} F_{g;r_1,\dots,r_n},$$

which proves the first equation. For the second equation, we use the ideas of the proof of [BDBKS22] and give a more detailed exposition. We denote

$$\tilde{\mathsf{J}}(x) = \mathsf{J}(x) + \sum_{k>0} x^{-k} \mathsf{J}_{-k} =: \mathsf{J}_+(x) + \mathsf{J}_-(x)$$

and note that by the Leibniz rule we have

$$\begin{aligned} \mathsf{J}_+(x_i)\mathsf{J}_-(x_j)P &= \mathsf{J}_+(x_i) \sum_{k>0} x_j^{-k} p_k P \\ &= \Big(\sum_{k>0} x_j^{-k} \mathsf{J}_+(x_i) p_k \Big) P + \Big(\sum_{k>0} x_j^{-k} p_k \Big) \mathsf{J}_+(x_i) P \end{aligned}$$

$$
\begin{aligned}
&= \sum_{k>0} k\left(\frac{x_i}{x_j}\right)^k P + \mathsf{J}_-(x_i)\mathsf{J}_+(x_i)P \\
&= \frac{x_i x_j}{(x_i - x_j)^2} P + \mathsf{J}_-(x_i)\mathsf{J}_+(x_i)P,
\end{aligned}
$$

for any $P \in \mathscr{B}_\hbar$. In other words

$$[\mathsf{J}_+(x_i), \mathsf{J}_-(x_j)] = \sum_{k>0} k\left(\frac{x_i}{x_j}\right)^k = \frac{x_i x_j}{(x_i - x_j)^2}.$$

Note that $\langle|\mathsf{J}_-(x_j)P = 0$ for any P. Thus, if we write

$$\langle|\tilde{\mathsf{J}}(x_1)\dots\tilde{\mathsf{J}}(x_n)e^{\mathsf{F}}|\rangle = \langle|(\mathsf{J}_+(x_1) + \mathsf{J}_-(x_1))\dots(\mathsf{J}_+(x_n) + \mathsf{J}_-(x_n))e^{\mathsf{F}}|\rangle \tag{2.3.12}$$

the strategy for the proof is to use the commutation relation to bring all the J_- to the left, then these will vanish but may create a term $\frac{x_i x_j}{(x_i - x_j)^2}$. Let us illustrate this step in a small example, we compute

$$(\mathsf{J}_+(x_1) + \mathsf{J}_-(x_1))(\mathsf{J}_+(x_2) + \mathsf{J}_-(x_2))(\mathsf{J}_+(x_3) + \mathsf{J}_-(x_3)),$$

we find

$$
\begin{aligned}
\mathsf{J}_+(x_1)\mathsf{J}_+(x_2)\mathsf{J}_+(x_3) + \mathsf{J}_+(x_1)\mathsf{J}_+(x_2)\mathsf{J}_-(x_3) + \mathsf{J}_+(x_1)\mathsf{J}_-(x_2)\mathsf{J}_+(x_3)& \\
+\mathsf{J}_+(x_1)\mathsf{J}_-(x_2)\mathsf{J}_-(x_3) + \text{ terms starting with } \mathsf{J}_-(x_1)&.
\end{aligned}
\tag{2.3.13}
$$

The terms starting with J_- will vanish in (2.3.12). For the others we expand

$$
\begin{aligned}
\mathsf{J}_+(x_1)\mathsf{J}_+(x_2)\mathsf{J}_-(x_3) &= \mathsf{J}_+(x_1)\mathsf{J}_-(x_3)\mathsf{J}_+(x_2) + \frac{x_2 x_3}{(x_2 - x_3)^2}\mathsf{J}_+(x_1) \\
&= \mathsf{J}_-(x_3)\mathsf{J}_+(x_1)\mathsf{J}_+(x_2) + \frac{x_1 x_3}{(x_1 - x_3)^2}\mathsf{J}_+(x_2) + \frac{x_2 x_3}{(x_2 - x_3)^2}\mathsf{J}_+(x_1) \\
\mathsf{J}_+(x_1)\mathsf{J}_-(x_2)\mathsf{J}_+(x_3) &= \mathsf{J}_-(x_2)\mathsf{J}_+(x_1)\mathsf{J}_+(x_3) + \frac{x_1 x_2}{(x_1 - x_2)^2}\mathsf{J}_+(x_3) \\
\mathsf{J}_+(x_1)\mathsf{J}_-(x_2)\mathsf{J}_-(x_3) &= \mathsf{J}_-(x_2)\mathsf{J}_+(x_1)\mathsf{J}_-(x_3) + \frac{x_1 x_2}{(x_1 - x_2)^2}\mathsf{J}_-(x_3),
\end{aligned}
$$

thus only the first three terms in (2.3.13) contribute when we apply $\langle|\cdots e^{\mathsf{F}}|\rangle$. We get

$$
\begin{aligned}
\left\langle\middle|\prod_{i=1}^{3}(\mathsf{J}_+(x_i) + \mathsf{J}_-(x_i))e^{\mathsf{F}}\middle|\right\rangle &= \langle|\mathsf{J}_+(x_1)\mathsf{J}_+(x_2)\mathsf{J}_+(x_3)e^{\mathsf{F}}|\rangle + \frac{x_1 x_3}{(x_1 - x_3)^2}\langle|\mathsf{J}_+(x_2)e^{\mathsf{F}}|\rangle \\
&+ \frac{x_2 x_3}{(x_2 - x_3)^2}\langle|\mathsf{J}_+(x_1)e^{\mathsf{F}}|\rangle + \frac{x_1 x_2}{(x_1 - x_2)^2}\langle|\mathsf{J}_+(x_3)e^{\mathsf{F}}|\rangle.
\end{aligned}
$$

In the next step we can rewrite the formula with the inclusion-exclusion formula and collect terms, we continue

$$\begin{aligned}
\Big\langle\Big|\prod_{i=1}^{3}(\mathsf{J}_+(x_i)+\mathsf{J}_-(x_i))e^{\mathsf{F}}\Big|\Big\rangle &= \sum_{\mathcal{I}\in\mathcal{P}(3)}\prod_{J\in\mathcal{I}}\langle|\prod_{i\in J}\mathsf{J}_+(x_i)e^{\mathsf{F}}|\rangle^\circ + \frac{x_1x_3}{(x_1-x_3)^2}\langle|\mathsf{J}_+(x_2)e^{\mathsf{F}}|\rangle \\
&+ \frac{x_2x_3}{(x_2-x_3)^2}\langle|\mathsf{J}_+(x_1)e^{\mathsf{F}}|\rangle + \frac{x_1x_2}{(x_1-x_2)^2}\langle|\mathsf{J}_+(x_3)e^{\mathsf{F}}|\rangle \\
&= \langle|\mathsf{J}_+(x_1)\mathsf{J}_+(x_2)\mathsf{J}_+(x_3)e^{\mathsf{F}}|\rangle^\circ + \prod_{i=1}^{3}\langle|\mathsf{J}_+(x_i)e^{\mathsf{F}}|\rangle^\circ \\
&+ \langle|\mathsf{J}_+(x_1)e^{\mathsf{F}}|\rangle^\circ\langle|\mathsf{J}_+(x_2)\mathsf{J}_+(x_3)e^{\mathsf{F}}|\rangle^\circ + \langle|\mathsf{J}_+(x_2)e^{\mathsf{F}}|\rangle^\circ\langle|\mathsf{J}_+(x_1)\mathsf{J}_+(x_3)e^{\mathsf{F}}|\rangle^\circ \\
&+ \langle|\mathsf{J}_+(x_3)e^{\mathsf{F}}|\rangle^\circ\langle|\mathsf{J}_+(x_1)\mathsf{J}_+(x_2)e^{\mathsf{F}}|\rangle^\circ + \frac{x_1x_3}{(x_1-x_3)^2}\langle|\mathsf{J}_+(x_2)e^{\mathsf{F}}|\rangle^\circ \\
&+ \frac{x_2x_3}{(x_2-x_3)^2}\langle|\mathsf{J}_+(x_1)e^{\mathsf{F}}|\rangle^\circ + \frac{x_1x_2}{(x_1-x_2)^2}\langle|\mathsf{J}_+(x_3)e^{\mathsf{F}}|\rangle^\circ \\
&= \langle|\mathsf{J}_+(x_1)\mathsf{J}_+(x_2)\mathsf{J}_+(x_3)e^{\mathsf{F}}|\rangle^\circ + \prod_{i=1}^{3}\langle|\mathsf{J}_+(x_i)e^{\mathsf{F}}|\rangle \\
&+ \langle|\mathsf{J}_+(x_1)e^{\mathsf{F}}|\rangle^\circ\left(\langle|\mathsf{J}_+(x_2)\mathsf{J}_+(x_3)e^{\mathsf{F}}|\rangle^\circ + \frac{x_2x_3}{(x_2-x_3)^2}\right) \\
&+ \langle|\mathsf{J}_+(x_2)e^{\mathsf{F}}|\rangle^\circ\left(\langle|\mathsf{J}_+(x_1)\mathsf{J}_+(x_3)e^{\mathsf{F}}|\rangle^\circ + \frac{x_1x_3}{(x_1-x_3)^2}\right) \\
&+ \langle|\mathsf{J}_+(x_3)e^{\mathsf{F}}|\rangle^\circ\left(\langle|\mathsf{J}_+(x_1)\mathsf{J}_+(x_2)e^{\mathsf{F}}|\rangle^\circ + \frac{x_1x_2}{(x_1-x_2)^2}\right) \\
&= \sum_{\mathcal{I}\in\mathcal{P}(3)}\prod_{J\in\mathcal{I}}\left(\langle|\prod_{i\in J}\mathsf{J}_+(x_i)e^{\mathsf{F}}|\rangle^\circ + B(J)\right)
\end{aligned}$$

where we define

$$B(J) = \begin{cases} \frac{x_ix_j}{(x_i-x_j)^2} & \text{if } \#J = 2 \text{ and } J=\{i,j\}, \\ 0 & \text{if } \#J \neq 2. \end{cases}$$

Moreover, we used

$$\langle|J_+(x_i)e^{\mathsf{F}}|\rangle = \langle|J_+(x_i)e^{\mathsf{F}}|\rangle^\circ.$$

By the inclusion-exclusion principle, we must have

$$\begin{aligned}
\Big\langle\Big|\prod_{i=1}^{j}(\mathsf{J}_+(x_i)+\mathsf{J}_-(x_i))e^{\mathsf{F}}\Big|\Big\rangle^\circ &= \Big\langle\Big|\prod_{i=1}^{j}\mathsf{J}_+(x_i)e^{\mathsf{F}}\Big|\Big\rangle^\circ + \delta_{j,2}\frac{x_1x_2}{(x_1-x_2)^2} \\
&= G_j(x_1,\dots,x_j) + \delta_{j,2}\frac{x_1x_2}{(x_1-x_2)^2}
\end{aligned}$$

for $j = 1, 2, 3$. Now we do the computation for all n, the same argument yields the assertion. Since $\langle|$ kills every $\mathsf{J}_-(x_i)$, every contribution is characterized by k, the number of $\mathsf{J}_-(x_i)$ in the product. These will get permuted past every $\mathsf{J}_+(x_j)$, $j < i$, left in the product. After finitely many steps we arrive at a product that has terms $\frac{x_{j_s}x_{i_s}}{(x_{j_s}-x_{i_s})^2}$, $s = 1, \ldots, k$ and a product of $n-2k$ terms $\mathsf{J}_+(x_{l_r})$, $r = 1, \ldots, n-2k$. If $k > \lfloor \frac{n}{2} \rfloor$ we have more J_- in a product than J_+ and hence we will be left with a product having $\mathsf{J}_-(x_{l_r})$ and no $\mathsf{J}_+(x_{l_r})$. These will vanish when taking the vacuum expectation. Thus at the bottom line, every contribution is characterized by a set of indices $\mathcal{K}$ with cardinality $n-2k$ and sets $\{i_s, j_s\}$, $s = 1, \ldots, k$. Thus, we get

$$\Big\langle \Big| \prod_{i=1}^{n} (\mathsf{J}_+(x_i) + \mathsf{J}_-(x_i)) e^{\mathsf{F}} \Big| \Big\rangle = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \sum_{\mathcal{K} \sqcup \bigsqcup_{s=1}^{k} \{i_s, j_s\} = [n]} \prod_{s=1}^{k} \frac{x_{i_s} x_{j_s}}{(x_{i_s} - x_{j_s})^2} \Big\langle \Big| \prod_{j \in \mathcal{K}} \mathsf{J}_+(x_j) e^{F} \Big| \Big\rangle. \tag{2.3.14}$$

By inclusion-exclusion we can expand the $k = 0$ term

$$\begin{aligned} \langle | \mathsf{J}_+(x_1) \ldots \mathsf{J}_+(x_n) | \rangle = & \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \sum_{[n] = \mathcal{K} \sqcup \{i_s, j_s \colon s = 1, \ldots, k\}} \sum_{J \in \mathcal{P}(\mathcal{K})} \Big\langle \Big| \prod_{j \in J} \mathsf{J}_+(x_j) e^{\mathsf{F}} \Big| \Big\rangle^{\circ} \\ & \sum_{I \in \mathcal{P}(\{i_1, j_1, \ldots, i_s, j_s\})} \Big\langle \Big| \prod_{i \in I} \mathsf{J}_+(x_i) e^{\mathsf{F}} \Big| \Big\rangle^{\circ} \\ = & \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \sum_{[n] = \mathcal{K} \sqcup \{i_s, j_s \colon s = 1, \ldots, k\}} \Big\langle \Big| \prod_{j \in \mathcal{K}} \mathsf{J}_+(x_j) e^{\mathsf{F}} \Big| \Big\rangle \\ & \sum_{I \in \mathcal{P}(\{i_1, j_1, \ldots, i_s, j_s\})} \Big\langle \Big| \prod_{i \in I} \mathsf{J}_+(x_i) e^{\mathsf{F}} \Big| \Big\rangle^{\circ}, \end{aligned}$$

where $\mathcal{P}(\{i_1, j_1, \ldots, i_s, j_s\})$ denotes the set of partitions of the set $\{i_1, j_1, \ldots, i_s, j_s\}$. Now in the second factor there must be I that consists only of pairs $\bigsqcup_{s=1}^{k} \{i_s, j_s\}$, which we can combine with the term in (2.3.14) and get the desired

$$\Big\langle \Big| \prod_{i=1}^{n} (\mathsf{J}_+(x_i) + \mathsf{J}_-(x_i)) e^{\mathsf{F}} \Big| \Big\rangle = \sum_{\mathcal{J} \in \mathcal{P}(n)} \prod_{I \in \mathcal{J}} \bigg(\Big\langle \Big| \prod_{j \in I} \mathsf{J}_+(x_j) e^{\mathsf{F}} \Big| \Big\rangle^{\circ} + B(J) \bigg).$$

Finally by inclusion-exclusion the connected objects must be given by

$$\begin{aligned} \Big\langle \Big| \prod_{i=1}^{n} (\mathsf{J}_+(x_i) + \mathsf{J}_-(x_i)) e^{\mathsf{F}} \Big| \Big\rangle^{\circ} &= \Big\langle \Big| \prod_{i=1}^{n} \mathsf{J}_+(x_i) e^{\mathsf{F}} \Big| \Big\rangle^{\circ} + \delta_{2,n} B(x_1, x_2) \\ &= G_n(x_1, \ldots, x_n) + \delta_{2,n} \frac{x_1 x_2}{(x_1 - x_2)^2} \\ &= \widetilde{G}_n(x_1, \ldots, x_n). \end{aligned}$$

□

2.4 Functional relations for $n > 2$, statement and examples

In this section we want to discuss the main result, the functional relations between higher order moment and cumulant generating series, and delay its proof to the next section. We mainly focus on the structure of the formula and give examples. Consider two multiplicative functions $\varphi, \kappa \colon \mathcal{PS} \to \mathbb{C}$, then we define

$$\begin{aligned} M_n(x_1, \dots, x_n) &= \sum_{s_1,\dots,s_n \geq 1} \varphi_{s_1,\dots,s_n} x_1^{s_1} \dots x_n^{s_n}, \\ C_n(x_1, \dots, x_n) &= \sum_{s_1,\dots,s_n \geq 1} \kappa_{s_1,\dots,s_n} x_1^{s_1} \dots x_n^{s_n}, \end{aligned}$$

where we usually just write $M(x_1)$ and $C(x_1)$ for the $n = 1$ case. Then the following theorem gives the functional relations in higher order free probability that solve the problem posed in [CMSS07].

Theorem 2.4.1.
Let $\varphi, \kappa \colon \mathcal{PS} \to \mathbb{C}$ be multiplicative functions on $\mathcal{PS}$ with values in $\mathbb{C}$ satisfying the moment-cumulant relations $\varphi = \zeta * \kappa$. Then under the change of variables $x_i = w_i / C(w_i)$ and for $n \geq 3$, we have:

$$M_n(x_1, \dots, x_n) = \sum_{r_1,\dots,r_n \geq 0} \sum_{T \in \mathcal{G}_{0,n}(\mathbf{r}+1)} \prod_{i=1}^{n} \vec{\mathsf{O}}^{\kappa}_{r_i}(w_i) \prod_{I \in \mathcal{I}(T)}{}' C_{\#I}(w_I)\,, \tag{2.4.1}$$

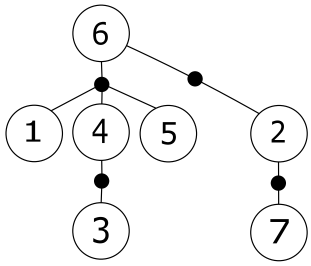
where:

- $\mathcal{G}_{0,n}(\mathbf{r}+1)$ is the set of bicoloured trees with white vertices labeled from 1 to n having valency $r_1 + 1, \dots, r_n + 1$, and without univalent black vertices.
- The weight $\vec{\mathsf{O}}^{\kappa}_{s_i}(w_i)$ of the i-th white vertex is a differential operator acting on the variable w_i,

  $$\vec{\mathsf{O}}^{\kappa}_{r}(w) = \sum_{m \geq 0} (P^{\kappa}(w) w \partial_w)^m P^{\kappa}(w) \cdot [v^m] \left(\partial_y + \frac{v}{y}\right)^r \cdot 1 \Big|_{y = G^{\kappa}_{0,1}(w)}\,, \tag{2.4.2}$$

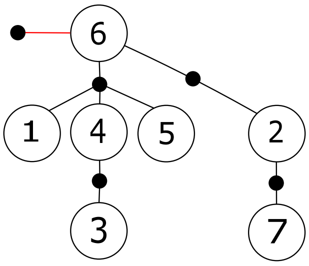
  where $P^{\kappa}(w_i) = \frac{\mathrm{d} \ln w_i}{\mathrm{d} \ln x_i}$. Note, the expression in (2.4.2) only involves $C(w_i)$.
- $\mathcal{I}(T)$ is the set of black vertices, identified with the subset of white vertices they connect to.
- $\prod'$ means that every occurrence of $C_2(w_i, w_j)$ should be replaced with $C_2(w_i, w_j) + \frac{w_i w_j}{(w_i - w_j)^2}$.
- For a given monomial in the x_i, $i = 1, \dots, n$, only finitely many terms of the right-hand side contribute.

Example 2.4.2.
i) First we want to discuss the set $\mathcal{G}_{0,n}(\mathbf{r}+1)$ and its elements. The following is an element of $\mathcal{G}_{0,7}(\mathbf{r}+1)$, with $\mathbf{r} = (0, 1, 0, 1, 0, 1, 0)$.

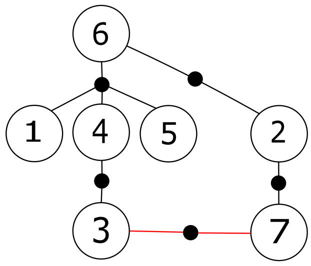


An element of $\mathcal{G}_{0,7}(\mathrm{r}+1)$ with $\mathrm{r}=(0,1,0,1,0,1,0)$.

On the other hand, the following graphs are not elements of $\mathcal{G}_{0,n}(\mathbf{s}+1)$ for any n and r.



No black vertices of valency < 2 allowed.

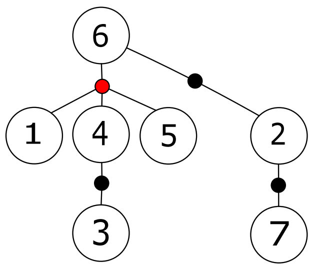


Trees do not admit cycles.

ii) Let us illustrate how the black vertex weight is applied for the admissible tree in i). Every factor in the expression

$$\prod_{I\in\mathcal{I}(T)}' C_{\#I}(w_I)$$

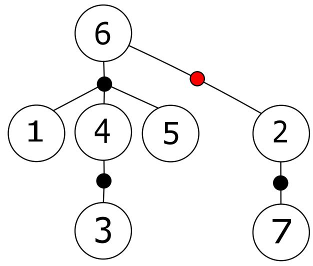
corresponds to a black vertex of the graph:



The highlighted vertex above corresponds to a factor

$$C_4(w_1,w_4,w_5,w_6).$$

When we have a black vertex with valency two



we need an additional correction term

$$\tilde{C}_2(w_2, w_6) = \left(C_2(w_2, w_6) + \frac{w_2 w_6}{(w_2 - w_6)^2} \right).$$

If we proceed with every black vertex we find the weight

$$\prod_{I \in \mathcal{I}(T)}' C_{\#I}(w_I) = C_4(w_1, w_4, w_5, w_6)\tilde{C}_2(w_2, w_6)\tilde{C}_2(w_2, w_7)\tilde{C}_2(w_3, w_4).$$

iii) For the last part of the formula, let us have a closer look at the operators $\vec{\mathsf{O}}^\kappa_r$ associated to the white vertices. We present the formulas for $r = 0, 1$. We have

$$\begin{aligned}
\vec{\mathsf{O}}^\kappa_0(w) &= \sum_{m \geq 0} \left(\frac{\mathrm{d}\ln w}{\mathrm{d}\ln x} w\partial_w \right)^m \frac{\mathrm{d}\ln y}{\mathrm{d}\ln x} [v^m] \left(\partial_y + \frac{v}{y} \right)^0 \cdot 1 \Big|_{y = C_1(w)} \\
&= \sum_{m \geq 0} \left(\frac{\mathrm{d}\ln w}{\mathrm{d}\ln x} w\partial_w \right)^m \frac{\mathrm{d}\ln w}{\mathrm{d}\ln x} [v^m] \cdot 1 \Big|_{y = C_1(w)} \\
&= \frac{\mathrm{d}\ln w}{\mathrm{d}\ln x} = \frac{x}{w}\frac{\mathrm{d}w}{\mathrm{d}x} = \frac{1}{C_1(w)\frac{\mathrm{d}x}{\mathrm{d}w}} = \frac{1}{C_1(w)x'(w)}.
\end{aligned}$$

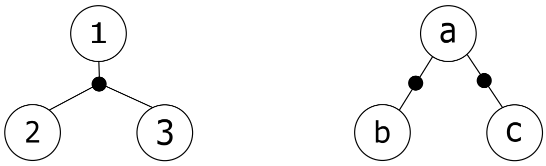
and

$$\begin{aligned}
\vec{\mathsf{O}}^\kappa_1(w) &= \sum_{m \geq 0} \left(\frac{\mathrm{d}\ln w}{\mathrm{d}\ln x} w\partial_w \right)^m \frac{\mathrm{d}\ln w}{\mathrm{d}\ln x} [v^m] \left(\partial_y + \frac{v}{y} \right)^1 \cdot 1 \Big|_{y = C_1(w)} \\
&= \sum_{m \geq 0} \left(\frac{\mathrm{d}\ln w}{\mathrm{d}\ln x} w\partial_w \right)^m \frac{\mathrm{d}\ln w}{\mathrm{d}\ln x} [v^m] \frac{v}{y} \cdot 1 \Big|_{y = C_1(w)} \\
&= \left(\frac{\mathrm{d}\ln w}{\mathrm{d}\ln x} w\partial_w \right) \frac{\mathrm{d}\ln w}{\mathrm{d}\ln x} \frac{1}{C_1(w)} = \frac{w}{C_1(w)x'(w)} \frac{\partial}{\partial w} \frac{1}{C_1(w)^2 x'(w)}.
\end{aligned}$$

Example 2.4.3.
Let us discuss the case $n = 3$ in Theorem 2.4.1. The formula is the following:

$$M_3(x_1, x_2, x_3) = \sum_{r_1, r_2, r_3 \in \mathbb{N}} \sum_{T \in \mathcal{G}_{0,3}(\mathbf{r}+\mathbf{1})} O^\kappa_{r_i}(w_i) \prod_{I \in \mathcal{I}(T)}' C_{\#I}(w_I).$$

The only trees that contribute are the following two.



r = (0, 0, 0)   r = (1, 0, 0), (0, 1, 0), (0, 0, 1)

- The first one has only one black vertex of valency three, which corresponds to $C_3(w_1, w_2, w_3)$. All white vertices have valency 1, which means only $\vec{\mathsf{O}}_0(w_i)$, for $i = 1, 2, 3$, will be applied to $C_3(w_1, w_2, w_3)$.
- The diagram on the right reflects three contributions, that depend on the choice of the label for the vertex a. In any case it consists of two black vertices, they give a contribution $\tilde{C}(w_a, w_b)$ and $\tilde{C}(w_a, w_c)$. The white vertices b, c have valency one and thus will give the operators $\vec{\mathsf{O}}_0(w_b), \vec{\mathsf{O}}_0(w_c)$. The vertex a has valency two and thus will correspond to an operator $\vec{\mathsf{O}}_1(w_a)$.

We take the formulas of Example 2.4.2 iii), then all together this yields

$$\begin{aligned} M_3(x_1, x_2, x_3) = &\frac{C_3(w_1, w_2, w_3)}{\prod_{i=1}^3 C_1(w_i)x'(w_i)} \\ &+ \frac{w_1}{\prod_{i=1}^3 C_1(w_i)x'(w_i)} \frac{\partial}{\partial w_1} \frac{\tilde{C}_2(w_1, w_2)\tilde{C}_2(w_1, w_3)}{C_1^2(w_1)x'(w_1)} \\ &+ \frac{w_2}{\prod_{i=1}^3 C_1(w_i)x'(w_i)} \frac{\partial}{\partial w_2} \frac{\tilde{C}_2(w_1, w_2)\tilde{C}_2(w_2, w_3)}{C_1^2(w_2)x'(w_2)} \\ &+ \frac{w_3}{\prod_{i=1}^3 C_1(w_i)x'(w_i)} \frac{\partial}{\partial w_3} \frac{\tilde{C}_2(w_1, w_3)\tilde{C}_2(w_2, w_3)}{C_1^2(w_3)x'(w_3)}. \end{aligned}$$

In fact, Theorem 2.4.1 is merely a special case of a more general functional relation for the higher genus theory explained in Section 2.2 and Section 2.3. Indeed, we proved in [BCGF$^+$23] a more general functional relation for all genera. In the following, we want to discuss this formula as well. First, the set of trees in Theorem 2.4.1 must be replaced by a more general set of graphs.

Definition 2.4.4.
Let $n \in \mathbb{N}$, then we denote by $\mathcal{G}_n^\bullet$ the set of possibly disconnected bicoloured graphs

i) having n labeled white vertices with labels $1, \dots, n$;

ii) having black vertices of valency $v \geq 2$; and

iii) such that edges only connect vertices of different colours.

If, in addition, we require

iv) the graph is connected,

then we denote the set by $\mathcal{G}_n$.

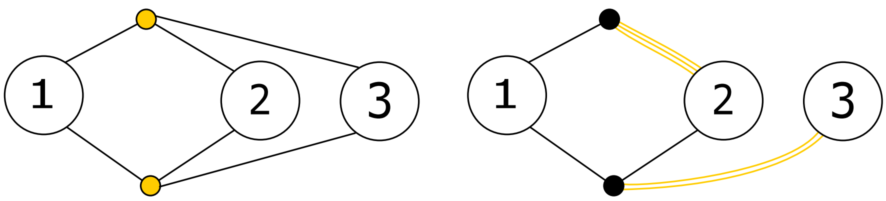
Remark 2.4.5.
We want to think of the black vertices as *hyper-edges*, that is, multisets containing the information of which white vertices are adjacent to it. More precisely, a multiset in $[n]$ is a function $f\colon [n] \to \mathbb{N}$ and we say that i belongs to the multiset with multiplicity $f(i)$ if $f(i) > 0$. We denote by $\mathcal{I}(\Gamma)$ the set of hyper-edges of a graph $\Gamma \in \mathcal{G}_n^{\bullet}$. Moreover, we define the cardinality of the multiset by $\#f = \sum_{i\in[n]} f(i)$. If I is a multiset in $[n]$, we write $I \subset [n]$.

We want to understand the automorphisms of the graphs $\mathcal{G}_n$ in the following way.

Definition 2.4.6.
Let $\Gamma \in \mathcal{G}_n$ be a graph. We define the automorphism group $\mathrm{Aut}(\Gamma)$ of Γ to be the set of permutations, permuting the edges without changing the structure of the graph or labeling of the vertices.

Let us give some examples for the definitions.

Example 2.4.7.
Compared to $\mathcal{G}_{0,n}$, we now allow multiple edges and in particular cycles within the graph. Consider the following two graphs.



$\#\mathrm{Aut} = 2$ $\#\mathrm{Aut} = 12$

In the first one we can interchange the edges adjacent to the yellow vertices. We understand this type of automorphisms as automorphisms of the hyper-edges. In the second graph we can permute the yellow edges, which we understand as an automorphism of multiple edges.

The functional relations for all genera are given by the following theorem.

Theorem 2.4.8.
Let Z^{φ}, Z^{κ} be two topological partition functions and $G^{\varphi}_{g,n}, G^{\kappa}_{g,n}$ the generating series appearing in the topological expansion of their n-point functions, cf. Remark 2.3.4.

Suppose that $\varphi = \zeta_\hbar \circledast \kappa$. Then, under the substitution

$$x_i = \frac{w_i}{G^\kappa_{0,1}(w_i)}, \qquad P^\kappa(w_i) = \frac{\mathrm{d}\ln w_i}{\mathrm{d}\ln x_i}, \tag{2.4.3}$$

we have

$$\begin{aligned} G^\varphi_{0,1}(x_1) &= G^\kappa_{0,1}(w_1), \\ G^\varphi_{0,2}(x_1,x_2) &= P^\kappa(w_1)P^\kappa(w_2)\left(G^\kappa_{0,2}(w_1,w_2) + \frac{w_1w_2}{(w_1-w_2)^2}\right) - \frac{x_1x_2}{(x_1-x_2)^2}, \end{aligned} \tag{2.4.4}$$

and for $2g-2+n>0$:

$$G^\varphi_{g,n}(x_1,\ldots,x_n) = \delta_{n,1}\Delta^\kappa_g(x_1) + [\hbar^{2g-2+n}] \sum_{\Gamma\in\mathcal{G}_n} \frac{1}{\#\mathrm{Aut}(\Gamma)} \prod_{i=1}^n \vec{\mathsf{O}}^\kappa(w_i) \prod_{I\in\mathcal{I}(\Gamma)} \mathsf{c}^\kappa(u_I,w_I). \tag{2.4.5}$$

Here

- the i-th white vertex weight is

$$\begin{aligned} \vec{\mathsf{O}}^\kappa(w_i) = \sum_{m\geq 0} & \big(P^\kappa(w_i)w_i\partial_{w_i}\big)^m P^\kappa(w_i) \\ & \cdot [v_i^m] \sum_{r\geq 0} \left(\partial_y + \frac{v_i}{y}\right)^r \exp\left(v_i\,\frac{\varsigma(\hbar v_i\partial_y)}{\varsigma(\hbar\partial_y)}\ln y - v_i\ln y\right)\Big|_{y=G^\kappa_{0,1}(w_i)} \\ & \cdot [u_i^r]\ \frac{\exp\big(\hbar u_i\varsigma(\hbar u_iw_i\partial_{w_i})(G^\kappa_1(w_i)-\hbar^{-1}) - u_i(G^\kappa_{0,1}(w_i)-1)\big)}{\hbar u_i\,\varsigma(\hbar u_i)}; \end{aligned} \tag{2.4.6}$$

- the black vertex or hyper-edge weight is

$$\mathsf{c}^\kappa(u_I,w_I) = \begin{cases} \big(\hbar u_i\varsigma(\hbar u_iw_i\partial_{w_i})\big)^2 G^\kappa_2(w_i,w_i) & \text{if } \#I=2 \text{ and } I(i)=2, \\ \prod_{i\in I}\hbar u_i\varsigma(\hbar u_iw_i\partial_{w_i})\widetilde{G}^\kappa_{\#I}(w_I) & \text{otherwise;} \end{cases} \tag{2.4.7}$$

- the correction term appearing for $n=1$ is:

$$\begin{aligned} \Delta^\kappa_g(x) = [\hbar^{2g}] \sum_{m\geq 0} & \big(P^\kappa(w)w\partial_w\big)^m [v^{m+1}] \exp\left(v\frac{\varsigma(\hbar v\partial_y)}{\varsigma(\hbar\partial_y)}\ln y - v\ln y\right)\Big|_{y=G^\kappa_{0,1}(w)} \\ & \cdot P^\kappa(w)w\partial_w G^\kappa_{0,1}(w). \end{aligned} \tag{2.4.8}$$

Remark 2.4.9.

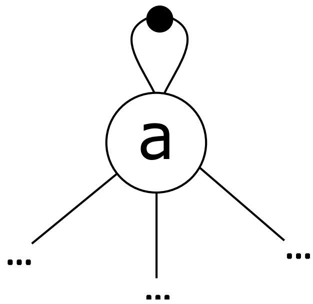
i) Equation (2.4.5) remains valid for $(g,n)=(0,2)$, provided the left-hand side is replaced with $\widetilde{G}_{0,2}(x_1,x_2)$. This recovers the second equation in (2.1.5).

ii) Equation (2.4.5) remains also valid for $(g,n) = (0,1)$, provided we extend the summation over m to $m \geq -1$ in the definition of $\Delta_0^\kappa(x)$ and identify

$$\big(P^\kappa(w)w\partial_w\big)^{-1} P^\kappa(w)w\partial_w G_{0,1}^\kappa(w)$$

with $G_{0,1}^\kappa(w)$. This is the only contribution (the sum over graphs does not contribute, as it contains only nonnegative powers of $\hbar$). This recovers the first equation in (2.4.4).

Example 2.4.10.
Recall the graph with multiple edges from Example 2.4.7. The upper vertex amounts to a factor $G_4(w_1, w_3, w_3, w_3)$ and the lower one to $G_4(w_1, w_2, w_3, w_3)$. Note that the introduction of possible multiple edges can cause problems in the $n = 2$ contributions: The following kind of black vertices connect a white vertex to itself, generating a *loop*.



Contribution $G_2(w_a, w_a)$

This vertex would give an ill-defined expression in the second summand of

$$\tilde{C}_2(w_i, w_j) = C_2(w_i, w_j) + \frac{w_i w_j}{(w_i - w_j)^2},$$

since we would have $i = j$. This problem is dealt with by the correction in (2.4.7).

Finally, let us emphasize the dependencies and connection to existing results.

Remark 2.4.11.
Theorem 2.4.8 is based on [BDBKS23, Theorem 4.14 and Remark 4.15]. Our main achievement in [BCGF$^+$23] is establishing the connection between free probability and the framework of [BDBKS22, BDBKS23] and use this to solve the fundamental open problem in the theory of higher order free probability ([CMSS07]).

2.5 Proof of the main result

In this section, we want to present a step-by-step proof for the functional relations between the moment and cumulant generating series, i.e. of Theorem 2.4.8 and in particular Theorem 2.4.1. In the following we will explain roughly the route of the proof and in the upcoming subsection we give detailed proofs of every step.

The proof heavily relies on techniques developed in a series of papers by B. Bychkov, P. Dunin-Barkowski, M. Kazarian and S. Shadrin [BDBKS22], [BDBKS23]. The latter techniques involve manipulations of operators in the Fock space $\mathscr{B}_\hbar$. We reviewed the description of the generating functions of moments and cumulants in the Section 2.3, but we are still missing the reformulation of the moment-cumulant relations $\varphi = \zeta \circledast \kappa$ in terms of the Fock space language. It turns out it can be described by the following operator.

Definition 2.5.1.
We define a linear operator $\mathsf{D} \in \mathrm{End}(\mathscr{B}_\hbar)$ by the formula

$$\mathsf{D}\, s_\lambda = \prod_{(i,j)\in\lambda} (1 + \hbar(j-i))s_\lambda, \tag{2.5.1}$$

where

$$(i,j) \in \lambda \iff i \leq \ell(\lambda) \text{ and } j \leq \lambda_i$$

and s_λ are the Schur functions from Definition 1.2.8. Note that D has a logarithm, and $\ln \mathsf{D}|\rangle = 0$ as well as $\langle| \ln \mathsf{D} = 0$.

Then the equivalent description of the moment-cumulant relations and the first main step towards the proof of the main result Theorem 2.4.1 (or more precisely to the general result Theorem 2.4.8), are given by the following theorem from our paper [BCGF+23]. We will discuss it in detail in Section 2.5.1.

Theorem 2.5.2.
Consider two topological partition functions Z^φ, Z^κ (or equivalently multiplicative functions φ, κ) and $d \in \mathbb{N}$. The following four properties are equivalent:

i) $Z^\varphi(\lambda) = z_\lambda \sum_{\nu\vdash d} H^<(\lambda,\nu) Z^\kappa(\nu)$ holds for any $\lambda \vdash d$;

ii) $\varphi = \zeta_\hbar \circledast \kappa$ holds as functions on $\mathcal{PS}(d)$;

iii) $Z^\kappa(\nu) = z_\nu \sum_{\lambda\vdash d} H^\leq(\nu,\lambda) Z^\varphi(\lambda)$ holds for any $\nu \vdash d$;

iv) $\kappa = \mu_\hbar \circledast \varphi$ holds as functions on $\mathcal{PS}(d)$.

Besides, the property $Z^\varphi = \mathsf{D} Z^\kappa$ is equivalent to any of these conditions simultaneously for all $d > 0$.

For the second step, recall that by Lemma 2.3.6, we may express the n-point functions by the vacuum expectation value

$$\widetilde{G}^\varphi_n(x_1,\ldots,x_n) = \hbar^{-1}\delta_{n,1} + \Big\langle\Big| \prod_{i=1}^n \widetilde{\mathsf{J}}(x_i) \cdot e^{\mathsf{F}^\varphi} \Big|\Big\rangle^\circ,$$

then by Theorem 2.5.2 we have $e^{\mathsf{F}_\varphi}|\rangle = Z^\varphi = \mathsf{D}Z^\kappa = \mathsf{D}e^{\mathsf{F}^\kappa}|\rangle$. Consequently, we may write

$$\begin{aligned}
\widetilde{G}^\varphi_n(x_1,\dots,x_n) &= \hbar^{-1}\delta_{n,1} + \Big\langle \Big| \prod_{i=1}^n \widetilde{\mathsf{J}}(x_i)\cdot e^{\mathsf{F}^\varphi}\Big|\Big\rangle^\circ \\
&= \hbar^{-1}\delta_{n,1} + \Big\langle \Big| \prod_{i=1}^n \widetilde{\mathsf{J}}(x_i)\cdot \mathsf{D}e^{\mathsf{F}^\kappa}\Big|\Big\rangle^\circ \\
&= \hbar^{-1}\delta_{n,1} + \Big\langle \Big| \prod_{i=1}^n \Big(\mathsf{D}^{-1}\widetilde{\mathsf{J}}(x_i)\mathsf{D}\Big)\cdot e^{\mathsf{F}^\kappa}\Big|\Big\rangle^\circ \\
&=: \hbar^{-1}\delta_{n,1} + \Big\langle \Big| \prod_{i=1}^n \mathbb{J}(x_i)\cdot e^{\mathsf{F}^\kappa}\Big|\Big\rangle^\circ,
\end{aligned} \tag{2.5.2}$$

where we used $\langle|\mathsf{D} = \langle|$. Note that we have expressed the n-point functions of φ in terms of the cumulants κ, i.e. used the moment-cumulant relations. Thus, the goal is to better understand the right-hand side, in particular the operators $\mathbb{J}$. The proof proceeds in using the following formula for the $\mathbb{J}$ operators.

Proposition 2.5.3 ([BDBKS22])**.**
With the definition

$$\mathbb{J}(x) = \mathsf{D}^{-1}\widetilde{\mathsf{J}}(x)\mathsf{D} = \sum_{k\in\mathbb{Z}} \mathsf{D}^{-1}\mathsf{J}_k\mathsf{D}x^k =: \sum_{k\in\mathbb{Z}} \mathbb{J}_k x^k,$$

we have

$$\begin{aligned}
\mathbb{J}_k &= \sum_{i\in\mathbb{Z}+\frac12} \exp\Big(\sum_{j=1}^k \ln\Big(1+\hbar(i+j-k-\frac12)\Big)\Big)\widehat{E}_{i-k,i} \\
&= \sum_{i\in\mathbb{Z}+\frac12}\prod_{j=1}^k \Big(1+\hbar\Big(i+j-k-\frac12\Big)\Big)\widehat{E}_{i-k,i}
\end{aligned}$$

in $\widehat{\mathfrak{gl}}_\infty$ and in terms of differential operators in $\mathrm{End}(\mathscr{B}_\hbar)$, we can write

$$\begin{aligned}
\mathbb{J}_k = \sum_{r\geq 0}\partial_y^r \exp\Big(k\frac{\varsigma(k\hbar\partial_y)}{\varsigma(\hbar\partial_y)}\ln(1+y)\Big)\Big|_{y=0}[w^k u^r]\frac{1}{\hbar u\varsigma(\hbar u)} \\
\cdot\exp\Big(\sum_{m>0}\hbar u\varsigma(\hbar u w\partial_w)\mathsf{J}_{-m}w^{-m}\Big)\exp\Big(\sum_{m>0}\hbar u\varsigma(\hbar u w\partial_w)\mathsf{J}_m w^m\Big),
\end{aligned} \tag{2.5.3}$$

where $\varsigma(u) = \frac{e^{\frac u2}-e^{-\frac u2}}{u}$.

Remark 2.5.4.
When we use more than one copy of $\mathbb{J}(x)$ and write $\mathbb{J}(x_i)$ (eg. in (2.5.2)) it is understood that every copy carries its own variables u_i, w_i, that is

$$\begin{aligned}\mathbb{J}(x_i) = \sum_{k_i\in\mathbb{Z}} x_i^{k_i} \sum_{r_i\geq 0} \partial_y^{r_i} \exp\left(k_i \frac{\varsigma(k_i\hbar\partial_y)}{\varsigma(\hbar\partial_y)} \ln(1+y)\right)\Big|_{y=0} [w_i^k u_i^r] \frac{1}{\hbar u_i \varsigma(\hbar u_i)} \\ \cdot \exp\left(\sum_{m>0} \hbar u_i \varsigma(\hbar u_i w_i \partial_{w_i}) \mathsf{J}_{-m} w_i^{-m}\right) \exp\left(\sum_{m>0} \hbar u_i \varsigma(\hbar u_i w_i \partial_{w_i}) \mathsf{J}_{m} w_i^{m}\right).\end{aligned}$$

We will present the proof of Proposition 2.5.3 of [BDBKS22] in Section 2.5.2. After the proposition is established, we use the commutation relations for the operators J_m to commute J_{-m} to the left and J_m to the right in (2.5.2). Then by the fact that $\langle|\mathsf{J}_{-m} = 0 = \mathsf{J}_m|\rangle$, for $m \in \mathbb{N}$, we will obtain an expression having no J operator contributions and the first combinatorial formula for G_n^φ.

Lemma 2.5.5 (Key combinatorial identity)**.**
Let $n \in \mathbb{N}$, $\Gamma \in \mathcal{G}_n$, I a hyper-edge of Γ and $i \in [n]$. Moreover, we define the i-th white vertex operator by

$$\begin{aligned}\vec{\mathsf{U}}^\kappa(x_i) = \sum_{k\in\mathbb{Z}} x_i^k \cdot [w_i^k] \sum_{r\geq 0} \partial_y^r \exp\left(k \frac{\varsigma(k\hbar\partial_y)}{\varsigma(\hbar\partial_y)} \ln(1+y)\right)\Big|_{y=0} \\ \cdot [u_i^r] \; \frac{\exp\left(\hbar u_i\, \varsigma(\hbar u_i w_i \partial_{w_i})(G_1^\kappa(w_i) - \hbar^{-1})\right)}{\hbar u_i \varsigma(\hbar u_i)}\end{aligned} \tag{2.5.4}$$

and the hyper-edge weight by

$$\mathsf{c}^\kappa(u_I, w_I) = \begin{cases} \left(\hbar u_i \varsigma(\hbar u_i w_i \partial_{w_i})\right)^2 G_2^\kappa(w_i, w_i) & \text{if } \#I = 2 \text{ and } I(i) = 2, \\ \prod_{i\in I} \hbar u_i \varsigma(\hbar u_i w_i \partial_{w_i}) \widetilde{G}^\kappa_{\#I}(w_I) & \text{otherwise.} \end{cases} \tag{2.5.5}$$

Then the relation $Z^\varphi = \mathsf{D} Z^\kappa$ is equivalent to

$$\forall n > 0, \qquad \widetilde{G}_n^\varphi(x_1, \ldots, x_n) = \sum_{\Gamma\in\mathcal{G}_n} \frac{1}{\#\mathrm{Aut}(\Gamma)} \prod_{i=1}^n \vec{\mathsf{U}}^\kappa(x_i) \prod_{I\in\mathcal{I}(\Gamma)} \mathsf{c}^\kappa(u_I, w_I), \tag{2.5.6}$$

where the first product is taken from left to right with i increasing.

The second to last step is to derive Theorem 2.4.8 from the key combinatorial identity. The proof proceeds in three steps, which were suggested by M. Kazarian in an unpublished manuscript (see [Kaz19], [Kaz20]). An analogous exposition can be found in our paper [BCGF+23]. We will discuss the three steps in detail in Section 2.5.2. Finally, the last subject is to extract the genus zero sector in Theorem 2.4.8 in order to get Theorem 2.4.1.

2.5.1 Facets of the moment-cumulant relations

The subject of this section is to discuss the different manifestations of the moment-cumulant relations in Definition 2.2.7. More precisely, we want to discuss the equivalences of Theorem 2.5.2. We start by computing small examples and then proceed to give the proof of Theorem 2.5.2.

Example 2.5.6.
Let us compute some moment-cumulant relations for multiplicative functions φ, κ. Recall that we have to solve

$$\varphi(\mathcal{U},\gamma) = \zeta_\hbar \circledast \kappa(\mathcal{U},\gamma) = \sum_{(\mathbf{0}_\pi,\pi)\odot(\mathcal{V},\sigma)=(\mathcal{U},\gamma)} \zeta_\hbar(\mathbf{0}_\pi,\pi)\kappa(\mathcal{V},\sigma) = \sum_{(\mathbf{0}_\pi,\pi)\odot(\mathcal{V},\sigma)=(\mathcal{U},\gamma)} \hbar^{|\pi|}\kappa(\mathcal{V},\sigma).$$

We have the following factorizations for $d = 1, 2$.

$(\mathcal{U},\gamma)$	$(\mathbf{0}_\pi,\pi)$	$(\mathcal{V},\sigma)$	$\zeta(0,\pi)\kappa(\mathcal{V},\sigma)$
$(\mathbf{1}_1,e)$	$(\mathbf{1}_1,e)$	$(\mathbf{1}_1,e)$	κ_1

Factorizations for $d = 1$.

$(\mathcal{U},\gamma)$	$(\mathbf{0}_\pi,\pi)$	$(\mathcal{V},\sigma)$	$\zeta(0,\pi)\kappa(\mathcal{V},\sigma)$
$(\mathbf{1}_2,(12))$	$(\mathbf{1}_2,(12))$	$(\mathbf{1}_2,e)$	$\hbar\kappa_{1,1}$
	$(\mathbf{1}_2,(12)$	$(\{\{1\},\{2\}\},e)$	$\hbar\kappa_1^2$
	$(\{\{1\},\{2\}\},e)$	$(\mathbf{1}_2,(12))$	κ_2
$(\mathbf{1}_2,e)$	$(\{\{1\},\{2\}\},e)$	$(\mathbf{1}_2,e)$	$\kappa_{1,1}$
	$(\mathbf{1}_2,(12))$	$(\mathbf{1}_2,(12)))$	$\hbar\kappa_2$
$(\{\{1\},\{2\}\},e)$	$(\{\{1\},\{2\}\},e)$	$(\{\{1\},\{2\}\},e)$	κ_1^2

Factorizations for $d = 2$.

In terms of equations we get

$$\varphi_1 = \varphi(1,e) = \kappa(1,e) = \kappa_1,$$

in $d = 1$ and in $d = 2$ we have

$$\varphi_2 = \varphi(\mathbf{1}_2,(12)) = \kappa(\mathbf{1}_2,(12)) + \hbar\kappa(\mathbf{1}_2,e) + \hbar\kappa(\{\{1\},\{2\}\},e) = \kappa_2 + \hbar(\kappa_{1,1} + \kappa_1^2)$$

and

$$\varphi_{1,1} + \varphi_1^2 = \sum_{\mathcal{U}\geq e} \varphi(\mathcal{U},e) = \hbar\kappa(\mathbf{1}_2,(12)) + \kappa(\mathbf{1}_2,e) + \kappa(\{\{1\},\{2\}\},e) = \hbar\kappa_2 + \kappa_{1,1} + \kappa_1^2.$$

Now let us compare the results with the equations from the master relation. Recall that we can encode the functions φ, κ as functions in the Fock space by

$$\begin{aligned} Z^{\varphi} &= \exp\left(\sum_{n\geq 1}\sum_{g\geq 0}\frac{\hbar^{2g-2+n}}{n!}\sum_{\mu_1,\ldots,\mu_n\geq 0}\varphi^{[g]}(\mathbf{1}_{\sum_{i=1}^n \mu_i}, \gamma_{\mu_1,\ldots,\mu_n})\right) \\ &= \exp\left(\sum_{d\geq 1}\sum_{g\geq 0}\sum_{\mu\vdash d}\varphi^{[g]}(\mathbf{1}_d, \gamma_{\mu_1,\ldots,\mu_{l(\mu)}})\hbar^{2g-2+l(\mu)}\right). \end{aligned} \tag{2.5.7}$$

We want to compute the master relation i) of Theorem 2.5.2. First we need the strictly monotone Hurwitz numbers which are involved. In $d = 1$ we have $H_k^<((1),(1)) = \delta_{k,1}$ and for $d = 2$ we have the following table.

k	$H_k^<((2);(2))$	$H_k^<((2);(1,1)))$	$H_k^<((1,1);(2))$	$H_k^<((1,1));(1,1)))$
0	$\frac{1}{2}$	0	0	$\frac{1}{2}$
1	0	$\frac{1}{2}$	$\frac{1}{2}$	0
$k>1$	0	0	0	0

Strictly monotone Hurwitz numbers in S_2.

Furthermore, we have for z_λ (see (1.2.1)) the following values:

$$z_{(1)} = 1, \quad z_{(2)} = z_{(1,1)} = 2.$$

We obtain in the trivial $d = 1$ case

$$\varphi_1 = \varphi(\mathbf{1}_1,(1)) = Z^{\varphi}((1)) = z_{(1)}\big(H^<((1),(1))Z^{\kappa}((1))\big) = Z^{\kappa}((1)) = \kappa(\mathbf{1}_1,(1)) = \kappa_1.$$

For $d = 2$ we use the definition of $Z^{\varphi}(\lambda)$ (2.3.3) associated to the multiplicative functions φ and κ and obtain

$$\begin{aligned} \varphi(\mathbf{1}_2,(12)) = Z^{\varphi}((2)) &= z_{(2)}\big(H^<((2),(2))Z^{\kappa}((2)) + H^<((2),(1,1))Z^{\kappa}((1,1))\big) \\ &= \kappa(\mathbf{1}_2,(12)) + \hbar\kappa(\mathbf{1}_2,(1)(2)) + \hbar\kappa(\mathbf{1}_1,(1))^2 \end{aligned}$$

and

$$\begin{aligned} \varphi(\mathbf{1}_2,(1)(2)) + \varphi(\mathbf{1}_1,(1))^2 &= Z^{\varphi}((1,1)) \\ &= z_{(1,1)}\big(H^<((1,1),(2))Z^{\kappa}((2)) + H^<((1,1),(1,1))Z^{\kappa}((1,1))\big) \\ &= \hbar\kappa(\mathbf{1}_2,(12)) + \kappa(\mathbf{1}_2,(1)(2)) + \kappa(\mathbf{1}_1,(1))^2. \end{aligned}$$

We see that the relations agree with the equations we computed by $\varphi = \zeta_\hbar \circledast \kappa$. Now for the final equivalence, we want to see that these can be reformulated in the Fock space language. Let us also do some examples for small d. Assume we have two partition functions Z^{φ} and Z^{κ} induced by two multiplicative functions φ and κ and that we have

$$Z^{\varphi} = \mathsf{D}Z^{\kappa}.$$

We will now compute both sides of the equation above and see that this will give precisely the master relation involving the Hurwitz numbers. Since we already have seen that the master relation is equivalent to the moment-cumulant formula, we then will have verified (for small d) that the moment-cumulant formula can be written as an operator in the Fock space. Recall that the operator D is a diagonal operator in the basis $(s_\lambda)_\lambda$, given by

$$\mathsf{D}s_\lambda = \prod_{(i,j)\in\lambda}(1+\hbar(j-i))s_\lambda.$$

Thus we will change the basis from the power sum basis $(p_\mu)_\mu$ to the Schur basis $(s_\lambda)_\lambda$ in order to compute both sides of $Z^\varphi = \mathsf{D}Z^\kappa$. For the $d=1$, i.e. the $\mu=(1)$ term of (2.5.7), we have

$$Z^\varphi((1))p_1 = [p_1]Z^\kappa p_1 = [p_1]Z^\kappa s_1,$$

since $p_1 = s_1$. Hence, the operator equation reads

$$Z^\varphi((1))p_1 = \underbrace{\prod_{(i,j)\in(1)}(1+\hbar(j-i))}_{=1} Z^\kappa((1))p_1 = Z^\kappa((1))p_1,$$

i.e. $Z^\varphi(1) = Z^\kappa(1)$, which agrees with the previous computations. We continue with $d=2$, for changing basis (see Lemma 1.2.11) we need the following character values:

$\lambda\backslash\mu$	(2)	$(1,1)$
(2)	1	1
$(1,1)$	-1	1

Values of the characters $\chi_\lambda(\mu)$.

Thus, by the transformation formula we have

$$s_{(2)} = \frac{1}{2}p_{(2)} + \frac{1}{2}p_{(1)}^2, \qquad s_{(1,1)} = -\frac{1}{2}p_{(2)} + \frac{1}{2}p_{(1)}^2,$$
$$p_{(2)} = s_{(2)} - s_{(1,1)}, \qquad p_{(1,1)} = p_{(1)}^2 = s_{(2)} + s_{(1,1)},$$

and hence

$$\begin{aligned} &Z^\kappa((2))p_{(2)} + Z^\kappa((1,1))p_{(1)}^2 \\ &\quad = Z^\kappa((2))(s_{(2)} - s_{(1,1)})) + Z^\kappa((1,1))(s_{(2)} + s_{(1,1)})) \\ &\quad = (Z^\kappa((2)) + Z((1,1)))s_{(2)} + (-Z^\kappa((2)) + Z^\kappa((1,1)))s_{(1,1)}. \end{aligned}$$

Applying the operator D to Z^κ and comparing the coefficients in $Z^\varphi = \mathsf{D}Z^\kappa$ yields the equations

$$Z^\varphi((2)) + Z^\varphi((1,1))) = (1+\hbar)(Z^\kappa((2)) + Z^\kappa((1,1)))$$

and

$$-Z^{\varphi}((2)) + Z^{\varphi}((1,1)) = (1-\hbar)(-Z^{\kappa}((2)) + Z^{\kappa}((1,1))),$$

which we can solve for $Z^{\varphi}((2))$ and $Z^{\varphi}((1,1))$. We find

$$\begin{aligned} Z^{\varphi}((2)) &= \frac{1}{2}(1+\hbar)(Z^{\kappa}((2)) + Z^{\kappa}((1,1))) - (1-\hbar)(-Z^{\kappa}((2)) + Z^{\kappa}((1,1))) \\ &= Z^{\kappa}((2)) + \hbar Z^{\kappa}((1,1)) \end{aligned}$$

and

$$\begin{aligned} Z^{\varphi}((1,1)) &= \frac{1}{2}(1+\hbar)(Z^{\kappa}((2)) + Z^{\kappa}((1,1))) + (1-\hbar)(-Z^{\kappa}((2)) + Z^{\kappa}((1,1))) \\ &= \hbar Z^{\kappa}((2)) + Z^{\kappa}((1,1)). \end{aligned}$$

These are exactly the relations obtained by the master relation for $d = 2$.

Now that we have seen the equivalences in small examples we want to prove the statement in general.

Proof of Theorem 2.5.2.

- First we prove ii) $\iff$ i). Let $\lambda \vdash d$ be a partition of d. Let us assume that φ and κ satisfy the moment-cumulant relations $\varphi = \zeta_{\hbar} \circledast \kappa$. Then by (2.3.3), we have

$$\begin{aligned} Z^{\varphi}(\lambda) &= \sum_{\substack{\mathcal{V} \in \mathcal{P}(d) \\ \mathbf{0}_{\lambda} \leq \mathcal{V}}} (\zeta_{\hbar} \circledast \kappa)(\mathcal{V}, \gamma_{\lambda}) \\ &= \sum_{\substack{\mathcal{V} \in \mathcal{P}(d) \\ \mathbf{0}_{\lambda} \leq \mathcal{V}}} \sum_{(\mathbf{0}_{\pi}, \pi) \odot (\mathcal{W}, \sigma) = (\mathcal{V}, \gamma_{\lambda})} \hbar^{|\pi|} \kappa(\mathcal{W}, \sigma) \\ &= \sum_{\substack{\pi, \sigma \in S(d) \\ \pi \circ \sigma = \gamma_{\lambda}}} \sum_{\substack{\mathcal{W} \in \mathcal{P}(d) \\ \mathbf{0}_{\sigma} \leq \mathcal{W}}} \hbar^{|\pi|} \kappa(\mathcal{W}, \sigma) \\ &= \sum_{\nu \vdash d} \sum_{\pi \in S(d)} \hbar^{|\pi|} \sum_{\substack{\sigma \in C_{\nu} \\ \pi \circ \sigma = \gamma_{\lambda}}} \left(\sum_{\substack{\mathcal{W} \in \mathcal{P}(d) \\ \mathbf{0}_{\sigma} \leq \mathcal{W}}} \kappa(\mathcal{W}, \sigma) \right), \end{aligned}$$

  where we collected all $\sigma = \pi^{-1} \circ \gamma_{\lambda}$ belonging to the same conjugacy class C_{ν}. By multiplicativity of κ, the sum inside the brackets only depends on the cycle structure of σ. In particular, substituting σ with γ_{ν} does not change the sum, and by comparing with (2.3.3) we recognise the value of $Z^{\kappa}(\nu)$. This yields

$$Z^{\varphi}(\lambda) = \sum_{\nu \vdash d} \sum_{\pi \in S(d)} \hbar^{|\pi|} \sum_{\substack{\sigma \in C_{\nu} \\ \pi \circ \sigma = \gamma_{\lambda}}} Z^{\kappa}(\nu) = \sum_{\nu \vdash d} Z^{\kappa}(\nu) \left(\sum_{\substack{\pi \in S(d), \sigma \in C_{\nu} \\ \pi \circ \sigma = \gamma_{\lambda}}} \hbar^{|\pi|} \right).$$

The last constraint can be rewritten as $\gamma_\lambda^{-1} \circ \pi \circ \sigma = \mathrm{id}$. By comparison with Definition 1.3.10, we recognise the free single Hurwitz numbers,

$$Z^\varphi(\lambda) = z_\lambda \sum_{\nu \vdash d} Z^\kappa(\nu) \bigg(\sum_{r=0}^{d-1} \hbar^r H_r^{|}(\lambda, \nu) \bigg).$$

Here, r is the colength of π, and the factor $z_\lambda = \frac{d!}{\# C_\lambda}$ is explained as follows. The numerator compensates the $\frac{1}{d!}$ in the definition of Hurwitz numbers. The denominator comes from the fact that in the definition of free single Hurwitz numbers, we let the leftmost permutation be any element of the conjugacy class C_λ, so we overcount by a factor of $\# C_\lambda$. By Proposition 1.3.11 and by comparison with the definition of the generating series of strictly monotone Hurwitz numbers in Definition 1.3.3, we get the identity

$$Z^\varphi(\lambda) = z_\lambda \sum_{\nu \vdash d} H^{>}(\lambda, \nu) Z^\kappa(\nu).$$

That is, it holds i), and since all the steps are equivalences we actually have i) $\iff$ ii).

- The equivalence ii) $\Rightarrow$ iv) is clear since $\mu_\hbar$ and $\zeta_\hbar$ are inverses w.r.t. the extended convolution.
- The implication i) $\Rightarrow$ iii) is obtained by multiplying i) by $z_\nu H^{\leq}(\nu, \lambda)$, summing over $\lambda \vdash d$ and using the first line of Lemma 1.3.9, while the converse direction is obtained likewise using the second line of Lemma 1.3.9.

This finishes the proof of all equivalences between i), ii), iii), iv). Let us finally prove the equivalence between i) for all $d > 0$ and $Z^\varphi = \mathsf{D} Z^\kappa$. We write

$$Z^\varphi = \mathsf{D} \sum_{d \geq 1} \sum_{\mu \vdash d} ([p_\mu] Z^\kappa) p_\mu$$

and changing the basis yields

$$Z^\varphi = \mathsf{D} \sum_{d \geq 1} \sum_{\mu, \alpha \vdash d} ([p_\mu] Z^\kappa) \chi_\alpha(\mu) s_\alpha = \sum_{d \geq 1} \sum_{\mu, \alpha \vdash d} ([p_\mu] Z^\kappa) \chi_\alpha(\mu) \prod_{(i,j) \in \alpha} (1 - \hbar(j - i)) s_\alpha.$$

We apply the Hall inner product with p_β and find

$$\begin{aligned} \langle p_\beta, Z^\varphi \rangle &= \sum_{d \geq 1} \sum_{\mu, \alpha \vdash d} ([p_\mu] Z^\kappa) \prod_{(i,j) \in \alpha} (1 - \hbar(j - i)) \chi_\alpha(\mu) \langle p_\beta, s_\alpha \rangle \\ &= \sum_{\mu, \alpha \vdash d} ([p_\mu] Z^\kappa) \prod_{(i,j) \in \alpha} (1 - \hbar(j - i)) \chi_\alpha(\mu) \chi_\alpha(\beta) \\ &= z_\beta \sum_{\mu, \alpha \vdash d} z_\mu ([p_\mu] Z^\kappa) \prod_{(i,j) \in \alpha} (1 - \hbar(j - i)) \frac{\chi_\alpha(\mu)}{z_\mu} \frac{\chi_\alpha(\beta)}{z_\beta} \end{aligned}$$

$$
\begin{aligned}
&= z_\beta \sum_{\mu \vdash d} z_\mu([p_\mu]Z^\kappa) \sum_{\alpha \vdash d} \prod_{(i,j)\in\alpha} (1-\hbar(j-i)) \frac{\chi_\alpha(\mu)}{z_\mu} \frac{\chi_\alpha(\beta)}{z_\beta} \\
&= z_\beta \sum_{\mu \vdash d} z_\mu([p_\mu]Z^\kappa) H^<(\mu,\beta) \\
&= z_\beta \sum_{\mu \vdash d} \langle p_\mu, Z^\kappa \rangle H^<(\mu,\beta).
\end{aligned}
$$

By this calculation we know that $Z^\varphi = \mathsf{D} Z^\kappa$ is equivalent to the master relation for the quantities

$$
\langle p_\mu, Z^\varphi \rangle = z_\mu [p_\mu] Z^\varphi, \quad \langle p_\mu, Z^\kappa \rangle = z_\mu [p_\mu] Z^\kappa.
$$

Comparing with (2.3.2) and (2.3.3) yields the desired result. □

2.5.2 Using the Fock space formalism

Now that we have proven that the moment-cumulant relation can be equivalently formulated as an operator equation of partition functions in the Fock space, we want to explain the main tools from [BDBKS22, BDBKS23] and prove our main result. Note that the moment-cumulant relation or more precisely the corresponding operator D is a special case of an operator equation

$$
Z_1 = \mathsf{D}_\psi Z_2
$$

in [BDBKS22, BDBKS23]. In our case we consider $\psi = \ln(1+y)$. We will not bother with the most general choice of D (resp. ψ) as we are mainly concerned with the higher order free probability application. The more general case has similar results, but it does not describe the moment-cumulant relation in free probability.

Lemma 2.5.7.
The operator D belongs to the Lie group of $\widehat{\mathfrak{gl}}_\infty$ and we have

$$
\mathsf{D} = \exp\Bigg(\sum_{k \in \mathbb{Z}+\frac{1}{2}} w_k \hat{E}_{k,k} \Bigg),
$$

where w_k is determined by $w_{k+\frac{1}{2}} - w_{k-\frac{1}{2}} = \ln(1-k\hbar)$.

Proof. Consider an operator

$$
W = \sum_{k \in \mathbb{Z}+\frac{1}{2}} w_k \widehat{E}_{k,k} \in \widehat{\mathfrak{gl}}_\infty,
$$

then it acts on the fermions via

$$
\sum_{k \in \mathbb{Z}+\frac{1}{2}} w_k \widehat{E}_{k,k} v_\lambda = \Bigg(\sum_{i=1}^{\ell(\lambda)} w_{\lambda_i - i + \frac{1}{2}} - w_{-i+\frac{1}{2}} \Bigg) v_\lambda = w_\lambda v_\lambda. \tag{2.5.8}
$$

In other words w_λ is an eigenvalue of W with eigenvector v_λ. Let us write

$$\psi_k = w_{k+\frac{1}{2}} - w_{k-\frac{1}{2}},$$

then we may rearrange the telescopic sum

$$\begin{aligned}
&w_{\lambda_i-i+\frac{1}{2}} - w_{i+\frac{1}{2}}\\
&= w_{\lambda_i-i+\frac{1}{2}} \underbrace{-w_{\lambda_i-i-\frac{1}{2}} + w_{\lambda_i-i-\frac{1}{2}}}_{=0} + \underbrace{w_{\lambda_i-1-i+\frac{1}{2}} - w_{\lambda_i-1-i+\frac{1}{2}}}_{=0} \cdots + w_{1-i+\frac{1}{2}} - w_{-i+\frac{1}{2}}\\
&= (w_{i+1+\frac{1}{2}} - w_{i+1-\frac{1}{2}}) + \cdots + (w_{\lambda_i-i+\frac{1}{2}} - w_{\lambda_i-i-\frac{1}{2}})\\
&= \sum_{j=1}^{\lambda_i} w_{j-i+\frac{1}{2}} - w_{j-i-\frac{1}{2}}\\
&= \sum_{j=1}^{\lambda_i} \psi_{j-i}.
\end{aligned}$$

Thus we have

$$w_\lambda = \sum_{i=1}^{\ell(\lambda)} \sum_{j=1}^{\lambda_i} \psi_{j-i} = \sum_{(i,j)\in\lambda} \psi_{j-i}.$$

Hence the exponential of W has the eigenvalues

$$\exp\Bigg(\sum_{(i,j)\in\lambda} \psi_{j-i} \Bigg),$$

and for the choice $\psi_k = \ln(1 + (j-i)\hbar)$ we find

$$\exp\Bigg(\sum_{(i,j)\in\lambda} \ln(1+(j-i)\hbar) \Bigg) v_\lambda = \prod_{(i,j)\in\lambda} (1+(j-i)\hbar).$$

Under the boson-fermion correspondence the eigenvectors in $\mathscr{B}_\hbar$ are the Schur polynomials with the same eigenvalue, i.e. the operator we defined agrees with D. $\square$

We proceed by proving the representation of $\mathbb{J}$ in Proposition 2.5.3.

Proof of Proposition 2.5.3. First observe that for $k = 0$ we have

$$\sum_{j\in\mathbb{Z}+\frac{1}{2}} \widehat{E}_{j,j} s_\lambda = 0$$

by (2.5.8), with $w_k = 0$ for all k. Thus J_0 and hence $\mathbb{J}_0$ annihilates the whole space $\mathcal{B}_\hbar$. Let us write

$$\mathsf{W} = \sum_{j\in\mathbb{Z}+\frac{1}{2}} w_j \widehat{E}_{j,j},$$

then for $k \neq 0$ we have

$$\begin{aligned}\mathsf{D}^{-1}\mathsf{J}_k\mathsf{D} &= \exp(-\mathsf{W})\mathsf{J}_k\exp(\mathsf{W})\\ &= \sum_{n=0}^{\infty}\frac{[(-\mathsf{W})^n,\mathsf{J}_k]}{n!}\end{aligned}$$

by the Campbell identity (see [Hal15]), where $[(\mathsf{W})^n,\mathsf{J}_k]$ is a shorthand notation for the n-times application of the commutator $[\mathsf{W},[\mathsf{W},\ldots,[\mathsf{W},\mathsf{J}_k]]]$. Using the commutation relations for the $\widehat{E}_{i,j}$, we compute

$$\begin{aligned}[-\mathsf{W},\mathsf{J}_k] &= -\sum_{i,j\in\mathbb{Z}+\frac{1}{2}} w_j[\widehat{E}_{j,j},\widehat{E}_{i-k,i}]\\ &= -\sum_{i\in\mathbb{Z}+\frac{1}{2}} w_{i-k}\widehat{E}_{i-k,j} + w_i(-\widehat{E}_{i-k,i})\\ &= \sum_{i\in\mathbb{Z}+\frac{1}{2}}(w_i - w_{i-k})\widehat{E}_{i-k,i},\end{aligned}$$

and inductively

$$\begin{aligned}[(-\mathsf{W})^n,\mathsf{J}_k] &= [-\mathsf{W},[-\mathsf{W},\ldots,[-\mathsf{W},\mathsf{J}_k]]]\\ &= \sum_{j_1,\ldots,j_n\in\mathbb{Z}+\frac{1}{2}} w_{j_1}\ldots w_{j_n}[\widehat{E}_{j_1,j_1},[\widehat{E}_{j_2,j_2},\ldots,[\widehat{E}_{j_n,j_n},\mathsf{J}_k]]]\\ &= \sum_{i\in\mathbb{Z}+\frac{1}{2}}(w_i - w_{i-k})^n\widehat{E}_{i-k,i}.\end{aligned}$$

Thus we have

$$\begin{aligned}\mathsf{D}^{-1}\mathsf{J}_k\mathsf{D} &= \sum_{n=0}^{\infty}\frac{[(-\mathsf{W})^n,\mathsf{J}_k]}{n!}\\ &= \sum_{n=0}^{\infty}\sum_{i\in\mathbb{Z}+\frac{1}{2}}\frac{(w_i - w_{i-k})^n}{n!}\widehat{E}_{i-k,i}\\ &= \sum_{i\in\mathbb{Z}+\frac{1}{2}}\exp(w_i - w_{i-k})\widehat{E}_{i-k,i}.\end{aligned}$$

Recall the trick in the last proof of expanding via a telescoping sum, and

$$\ln\Big(1+\hbar\Big(m-\frac{1}{2}\Big)\Big) = w_m - w_{m-1} \quad \text{for} \quad m\in\mathbb{Z}+\frac{1}{2}.$$

Then we obtain

$$\sum_{i\in\mathbb{Z}+\frac{1}{2}} \exp(w_i - w_{i-k})\widehat{E}_{i-k,i} = \sum_{i\in\mathbb{Z}+\frac{1}{2}} \exp\left(\sum_{j=0}^{k-1} w_{i-j} - w_{i-j-1}\right)\widehat{E}_{i-k,i}$$
$$= \sum_{i\in\mathbb{Z}+\frac{1}{2}} \exp\left(\sum_{j=0}^{k-1} \ln\left(1+\hbar\Big(i-j-\frac{1}{2}\Big)\right)\right)\widehat{E}_{i-k,i}$$
$$= \sum_{i\in\mathbb{Z}+\frac{1}{2}} \prod_{j=0}^{k-1}\left(1+\hbar\Big(i-j-\frac{1}{2}\Big)\right)\widehat{E}_{i-k,i}.$$

Finally, a change of indices yields the first formula in Proposition 2.5.3. Now let us rewrite the product in the following way

$$\prod_{j=1}^{k}\left(1+\hbar\Big(i+j-k-\frac{1}{2}\Big)\right) = \prod_{j=1}^{k}\left(1+\hbar\Big(\Big(i-\frac{k}{2}\Big)+j-\frac{k+1}{2}\Big)\right)$$
$$= \prod_{j=1}^{k}\left(1+\hbar\Big(y+j-1-\frac{k-1}{2}\Big)\right)\Bigg|_{y=(i-\frac{k}{2})}.$$

We expand the expression using Taylor's formula at $y=0$ and obtain

$$\prod_{j=1}^{k}\left(1+\hbar\Big(y+j-1-\frac{k-1}{2}\Big)\right)\Bigg|_{y=(i-\frac{k}{2})}$$
$$= \sum_{r=0}^{\infty}\partial_y^r\prod_{j=1}^{k}\left(1+\hbar\Big(y+j-1-\frac{k-1}{2}\Big)\right)\Bigg|_{y=0}\frac{(i-\frac{k}{2})^r}{r!}$$
$$=: \sum_{r=0}^{\infty}\partial_y^r\phi_k(y)\Bigg|_{y=0}\frac{(i-\frac{k}{2})^r}{r!}.$$

On the other hand we have

$$\phi_k(y) = \prod_{j=1}^{k}\left(1+\hbar\Big(y+j-1-\frac{k-1}{2}\Big)\right)$$
$$= \exp\left(\sum_{j=1}^{k}\ln\left(1+\hbar\Big(y+j-1-\frac{k-1}{2}\Big)\right)\right)$$
$$= \exp\left(e^{\hbar\frac{k-1}{2}\partial_y}\sum_{j=1}^{k}e^{\hbar(i-1)\partial_y}\ln(1+y)\right).$$

Now we use

$$\frac{x^k-y^k}{x-y} = \sum_{i=1}^{k}x^{i-1}y^{k-i}$$

for $x = e^{\hbar(i-1)\partial_y}, y = 1$ and obtain

$$\begin{aligned}\exp\left(e^{\hbar\frac{k-1}{2}\partial_y}\sum_{j=1}^{k}e^{\hbar(i-1)\partial_y}\ln(1+y)\right) &= \exp\left(e^{\hbar\frac{k-1}{2}\partial_y}\frac{e^{k\hbar\partial_y}-1}{e^{\hbar\partial_y}-1}\ln(1+y)\right)\\ &= \exp\left(e^{\hbar\frac{k-1}{2}\partial_y}\frac{e^{k\hbar\partial_y}-1}{e^{\hbar\partial_y}-1}\ln(1+y)\right)\\ &= \exp\left(\frac{e^{\hbar\frac{m}{2}\partial_y}-e^{-\hbar\frac{m}{2}\partial_y}}{e^{\frac{\hbar}{2}\partial_y}-e^{\frac{-\hbar}{2}\partial_y}}\ln(1+y)\right)\\ &= \exp\left(k\frac{\varsigma(\hbar k\partial_y)}{\varsigma(\hbar\partial_z)}\ln(1+y)\right).\end{aligned}$$

Together with (1.2.4), we obtain the desired result. □

Remark 2.5.8.
The computations in the proof of Proposition 2.5.3 are exactly the computations as in [BDBKS22] in Proposition 3.1 and Lemma 4.6, here for the case

$$\phi_k(y) = \prod_{j=1}^{k}\left(1+\hbar\left(y+j-\frac{k-1}{2}\right)\right).$$

More precisely, Proposition 2.5.3 is merely a special case of the latter two results for our choice of ϕ_k. We presented the proof here for a self-contained reading and because of the importance of understanding the techniques.

Together with Lemma 2.3.6, Proposition 2.5.3 is our starting point in [BCGF$^+$23] for proving Theorem 2.4.8. The next step is to use (2.5.3) and the commutation relations for the operators J, that is

$$[\mathsf{J}_r, \mathsf{J}_\mathsf{s}] = r\delta_{r,-s}$$

and the immediate consequence for the generating series

$$\left[\sum_{r=1}^{\infty}\mathsf{J}_r x_1^r, \sum_{s=1}^{\infty}\mathsf{J}_{-s}x_2^{-s}\right] = \sum_{i=i}^{n} i\left(\frac{x_1}{x_2}\right)^i = \frac{x_1x_2}{(x_1-x_2)^2}.$$

We commute the J_r with $r < 0$ to the left and the ones where $r > 0$ to the right. Then, the latter will be killed by $\langle|$ or by $|\rangle$ respectively. This will result in the *key combinatorial identity*, an expression for the n-point functions as a sum over the graphs.

Lemma 2.5.9 (Key combinatorial identity)**.**
Let $n \in \mathbb{N}$, $\Gamma \in \mathcal{G}_n$, I a hyper-edge of Γ and $i \in [n]$. Moreover, we define the i-th white

vertex operator by

$$\vec{\mathsf{U}}^{\kappa}(x_i) = \sum_{k\in\mathbb{Z}} x_i^k \cdot [w_i^k] \sum_{r\geq 0} \partial_y^r \exp\left(k\frac{\varsigma(k\hbar\partial_y)}{\varsigma(\hbar\partial_y)}\ln(1+y)\right)\Big|_{y=0} \\ \cdot [u_i^r]\ \frac{\exp\left(\hbar u_i\, \varsigma(\hbar u_i w_i \partial_{w_i})(G_1^{\kappa}(w_i) - \hbar^{-1})\right)}{\hbar u_i \varsigma(\hbar u_i)} \tag{2.5.9}$$

and the hyper-edge weight by

$$\mathsf{c}^{\kappa}(u_I, w_I) = \begin{cases} \left(\hbar u_i \varsigma(\hbar u_i w_i \partial_{w_i})\right)^2 G_2^{\kappa}(w_i, w_i) & \text{if } \#I = 2 \text{ and } I(i) = 2, \\ \prod_{i\in I} \hbar u_i \varsigma(\hbar u_i w_i \partial_{w_i}) \widetilde{G}^{\kappa}_{\#I}(w_I) & \text{otherwise.} \end{cases} \tag{2.5.10}$$

Then relation $Z^{\varphi} = \mathsf{D} Z^{\kappa}$ is equivalent to

$$\forall n > 0, \qquad \widetilde{G}_n^{\varphi}(x_1, \dots, x_n) = \sum_{\Gamma\in\mathcal{G}_n} \frac{1}{\#\mathrm{Aut}(\Gamma)} \prod_{i=1}^{n} \vec{\mathsf{U}}^{\kappa}(x_i) \prod_{I\in\mathcal{I}(\Gamma)} \mathsf{c}^{\kappa}(u_I, w_I), \tag{2.5.11}$$

where the first product is taken from left to right with i increasing.

Remark 2.5.10.
Recall that we have added an additional term for the 2-point function

$$\widetilde{G}_2(x_1, x_2) = G_2(x_1, x_2) + \frac{x_1 x_2}{(x_1 - x_2)^2}.$$

Thus a hyper-edge of cardinality two connecting the same vertex to itself would yield an ill-defined expression $\widetilde{G}(w_i, w_i)$. To prevent this, we have introduced a correction in (2.5.10).

Proof of Lemma 2.5.9. Let $Z^{\varphi} = DZ^{\kappa}$. We start by recalling the formula for the n-point function in terms of the operators in the Fock space. More precisely, we will start by considering a *disconnected* expression (cf. Lemma 2.3.5),

$$\widetilde{G}_n^{\varphi,\bullet}(x_1, \dots, x_n) = \langle | \mathbb{J}(x_1) \dots \mathbb{J}(x_n) \exp(\mathsf{F}^{\kappa}) | \rangle, \tag{2.5.12}$$

where $\mathbb{J}(x_1)$ are the generating series of the $\mathbb{J}_k$. Furthermore, from Proposition 2.5.3 we have the expression

$$\mathbb{J}_k = \sum_{r\geq 0} \partial_y^r \exp\left(k\frac{\varsigma(k\hbar\partial_y)}{\varsigma(\hbar\partial_y)}\ln(1+y)\right)\Big|_{y=0} [w^k u^r] \frac{1}{\hbar u \varsigma(\hbar u)} \\ \cdot \exp\left(\sum_{m>0} \hbar u \varsigma(\hbar u w \partial_w) \mathsf{J}_{-m} w^{-m}\right) \exp\left(\sum_{m>0} \hbar u \varsigma(\hbar u w \partial_w) \mathsf{J}_m w^m\right).$$

We focus on the terms

$$\exp\Big(\sum_{i=1}^{\infty} a_i \mathsf{J}_{-m} w_i^{-m}\Big) \exp\Big(\sum_{i=1}^{\infty} a_i \mathsf{J}_m w_i^m\Big).$$

As stated earlier, our goal is to get the J_{-m} annihilated by $\langle|$ and the J_m annihilated by $|\rangle$, i.e. we need to commute the J_{-m} to the left and J_m to the right. We will use the following commutation relation

$$\begin{aligned}&\exp\Big(\sum_{m=1}^{\infty} a_m \mathsf{J}_m\Big)\exp\Big(\sum_{m=1}^{\infty} b_m \mathsf{J}_{-m}\Big)\\ &\qquad = \exp\Big(\sum_{m=1}^{\infty} m a_m b_m\Big)\exp\Big(\sum_{m=1}^{\infty} b_m \mathsf{J}_{-m}\Big)\exp\Big(\sum_{m=1}^{\infty} a_m \mathsf{J}_m\Big).\end{aligned} \tag{2.5.13}$$

It is a special case of the Baker-Campbell-Hausdorff identity and can also be proven by direct computation (see [BDBKS22, Proposition 2.3]). Thus, the product

$$\Big\langle\Big|\prod_{i=1}^{n}\exp\Big(\sum_{m>0}\hbar u_i\varsigma(\hbar u_i w_i\partial_{w_i})w_i^{-m}\mathsf{J}_{-m}\Big)\exp\Big(\sum_{m>0}\hbar u_i\varsigma(\hbar u_i w_i\partial_{w_i})w_i^{m}\mathsf{J}_{m}\Big)\Big|\Big\rangle$$

becomes

$$\begin{aligned}&\prod_{1\le i<j\le n}\exp\Big(\sum_{m\ge 0}\hbar u_i\varsigma(\hbar u_i w_i\partial_{w_i})\hbar u_j\varsigma(\hbar u_j w_j\partial_{w_j})m\Big(\frac{w_i}{w_j}\Big)^m\Big)\\ &\qquad=\prod_{1\le i<j\le n}\exp\Big(\hbar u_i\varsigma(\hbar u_i w_i\partial_{w_i})\hbar u_j\varsigma(\hbar u_j w_j\partial_{w_j})\frac{w_iw_j}{(w_i-w_j)^2}\Big),\\ &\qquad=:\prod_{1\le i<j\le n}\exp\big(\alpha_i\alpha_j B(w_i,w_j)\big),\end{aligned} \tag{2.5.14}$$

where we used

$$a_m^{(i)} = \hbar u_i\varsigma(\hbar u_i w_i\partial_{w_i})w_i^m$$

and

$$b_m^{(i)} = \hbar u_i\varsigma(\hbar u_i w_i\partial_{w_i})w_i^{-m}$$

in (2.5.13). Thus, we have

$$\begin{aligned}&\widetilde{G}^{\varphi}(x_1,\dots,x_n)\\ &\quad=\sum_{\substack{k_1,\dots,k_n\in\mathbb{Z}\\ r_1,\dots,r_n\ge 0}}\prod_{i=1}^{n}x_i^{k_i}\partial_y^{r_i}\exp\Big(k\frac{\varsigma(k\hbar\partial_y)}{\varsigma(\hbar\partial_y)}\ln(1+y)\Big)\Big|_{y=0}\prod_{1\le i<j\le n}\exp\big(\alpha_i\alpha_j B(w_iw_j)\big)\\ &\qquad\Big\langle\Big|\prod_{i=1}^{n}\exp\Big(\sum_{m>0}b_m^{(i)}\mathsf{J}_{-m}\Big)\prod_{i=1}^{n}\exp\Big(\sum_{m>0}a_m^{(i)}\mathsf{J}_m\Big)\exp(F^{\kappa})\Big|\Big\rangle\\ &\quad=\sum_{\substack{k_1,\dots,k_n\in\mathbb{Z}\\ r_1,\dots,r_n\ge 0}}\prod_{i=1}^{n}x_i^{k_i}\partial_y^{r_i}\exp\Big(k\frac{\varsigma(k\hbar\partial_y)}{\varsigma(\hbar\partial_y)}\ln(1+y)\Big)\Big|_{y=0}\prod_{1\le i<j\le n}\exp\big(\alpha_i\alpha_j B(w_iw_j)\big)\\ &\qquad\Big\langle\Big|\prod_{i=1}^{n}\exp\Big(\sum_{m>0}a_m^{(i)}\mathsf{J}_m\Big)\exp(F^{\kappa})\Big|\Big\rangle,\end{aligned}$$

where in the second equation we used the fact that $\langle|$ annihilates all the J_{-m}, and thus leaves only the constant term 1 of the exponential. For the equation, we recall that the exponential of the generating series of J_m operators for positive values m act as a translation operator on any $f = f(p_1, p_2, \dots) \in B_\hbar$, that is

$$\begin{aligned}\exp\Big(\sum_{m=1}^{n} \alpha_m \mathsf{J}_m\Big) f &= \exp\Big(\sum_{m=1}^{n} \alpha_m m \partial_m\Big) f(p_1, p_2, \dots)\\ &= f(p_1 + \alpha_1, p_2 + 2\alpha_2, p_3 + 3\alpha_3, \dots).\end{aligned}$$

Thus we have

$$\begin{aligned}&\Big\langle\Big|\prod_{i=1}^{n} \exp\Big(\sum_{m>0} a_m^{(i)} \mathsf{J}_m\Big) \exp(F^\kappa)\Big|\Big\rangle\\
&= \Big\langle\Big|\prod_{i=1}^{n} \exp\Big(\sum_{m>0} a_m^{(i)} \mathsf{J}_m\Big) \exp\Big(\sum_{\substack{\ell\geq 1\\ g\geq 0}} \frac{\hbar^{2g-2+\ell}}{\ell!} \sum_{s_1,\dots,s_\ell>0} F^\kappa_{g;s_1,\dots,s_\ell} \prod_{j=1}^{\ell} \frac{\mathsf{J}_{-s_j}}{s_j}\Big)\Big|\Big\rangle\\
&= \Big\langle\Big|\prod_{i=1}^{n} \exp\Big(\sum_{m>0} a_m^{(i)} \mathsf{J}_m\Big) \exp\Big(\sum_{\substack{\ell\geq 1\\ g\geq 0}} \frac{\hbar^{2g-2+\ell}}{\ell!} \sum_{s_1,\dots,s_\ell>0} F^\kappa_{g;s_1,\dots,s_\ell} \prod_{j=1}^{\ell} \frac{p_{s_j}}{s_j}\Big)\\
&= \Big\langle\Big| \exp\Big(\sum_{\substack{\ell\geq 1\\ g\geq 0}} \frac{\hbar^{2g-2+\ell}}{\ell!} \sum_{s_1,\dots,s_\ell>0} F^\kappa_{g;s_1,\dots,s_\ell} \prod_{j=1}^{\ell} \Big(\frac{p_{s_j}}{s_j} + \sum_{i=1}^{n} a_{s_j}^{(i)}\Big)\Big)\\
&= \exp\Big(\sum_{\substack{\ell\geq 1\\ g\geq 0}} \frac{\hbar^{2g-2+\ell}}{\ell!} \sum_{s_1,\dots,s_\ell>0} F^\kappa_{g;s_1,\dots,s_\ell} \prod_{j=1}^{\ell} \Big(\sum_{i=1}^{n} a_{s_j}^{(i)}\Big)\Big)\\
&= \exp\Big(\sum_{\substack{\ell\geq 1\\ g\geq 0}} \frac{\hbar^{2g-2+\ell}}{\ell!} \sum_{i_1,\dots,i_l=1}^{n} \sum_{s_1,\dots,s_\ell>0} F^\kappa_{g;s_1,\dots,s_\ell} \prod_{j=1}^{\ell} a_{s_j}^{(i_j)}\Big). \qquad (2.5.15)\end{aligned}$$

First we want to treat the $\ell = 2$ case since we need to match it with terms in (2.5.14) (in this case $\widetilde{G}_2$ differs from G_2), the other terms are more or less in the right form, we will treat them afterward. When $i_1 \neq i_2$, then

$$\begin{aligned}&\exp\Big(\sum_{g\geq 0} \frac{\hbar^{2g}}{2!} \sum_{\substack{i_1,i_2\in[n]\\ i_1\neq i_2}} \sum_{s_1,s_2>0} F^\kappa_{g;s_1,s_2} a_{s_1}^{(i_1)} a_{s_2}^{(i_2)}\Big)\\
&\quad= \exp\Big(\sum_{g\geq 0} \hbar^{2g} \sum_{1\leq i_1<i_2\leq n} \sum_{s_1,s_2>0} F^\kappa_{g;s_1,s_2} a_{s_1}^{(i_1)} a_{s_2}^{(i_2)}\Big)\\
&\quad= \exp\Big(\sum_{g\geq 0} \hbar^{2g} \sum_{1\leq i_1<i_2\leq n} \sum_{s_1,s_2>0} F^\kappa_{g;s_1,s_2} \hbar u_{i_1} \varsigma(\hbar u_{i_1} w_{i_1} \partial_{w_{i_1}}) w_{i_1}^{s_1} \hbar u_{i_2} \varsigma(\hbar u_{i_2} w_{i_2} \partial_{w_{i_2}}) w_{i_2}^{s_2}\Big)\end{aligned}$$

$$
\begin{aligned}
&= \exp\left(\sum_{1\le i_1<i_2\le n} \alpha_{i_1}\alpha_{i_2} \sum_{g\ge 0} \hbar^{2g} \sum_{s_1,s_2>0} F^{\kappa}_{g;s_1,s_2} w_{i_1}^{s_1} w_{i_2}^{s_2} \right) \\
&= \exp\left(\sum_{1\le i_1<i_2\le n} \alpha_{i_1}\alpha_{i_2} G_2^{\kappa}(w_{i_1}, w_{i_2}) \right), \qquad (2.5.16)
\end{aligned}
$$

where the $\frac{1}{\ell!}$ eliminated the symmetry to obtain $G_2^{\kappa}(w_{i_1}, w_{i_1})$. Recall the factor from (2.5.14),

$$
\prod_{1\le i<j\le n} \exp\big(\alpha_i\alpha_j B(w_i, w_j)\big) = \exp\left(\sum_{1\le i<j\le n} \alpha_i\alpha_j B(w_i, w_j) \right),
$$

we can combine it with (2.5.16) to get a term

$$
\exp\left(\sum_{1\le i_1<i_2\le n} \alpha_{i_1}\alpha_{i_2} \big(G_2^{\kappa}(w_{i_1}, w_{i_2}) + B(w_{i_1}, w_{i_1})\big) \right) = \exp\left(\sum_{1\le i_1<i_2\le n} \alpha_{i_1}\alpha_{i_2} \widetilde{G}_2^{\kappa}(w_{i_1}, w_{i_2}) \right).
$$

Now if $i_1 = i_2$ we have a term

$$
\exp\left(\frac{1}{2} \sum_{1\le i_1\le n} \alpha_{i_1}^2 G_2^{\kappa}(w_{i_1}, w_{i_1}) \right) = \exp\left(\frac{1}{2} \sum_{1\le i_1\le n} \alpha_{i_1}^2 \widetilde{G}_2^{\kappa}(w_{i_1}, w_{i_1}) \right).
$$

For $\ell \ge 3$ we have similarly terms of the form

$$
\begin{aligned}
&\exp\left(\sum_{g\ge 0} \frac{\hbar^{2g-2+\ell}}{\ell!} \sum_{i_1,\dots,i_l=1}^{n} \sum_{s_1,\dots,s_\ell>0} F^{\kappa}_{g;s_1,\dots,s_\ell} \prod_{j=1}^{\ell} a_{s_j}^{(i_j)} \right) \\
&\qquad = \exp\left(\frac{1}{\ell!} \sum_{i_1,\dots,i_\ell=1}^{n} \left(\prod_{j=1}^{\ell} \alpha_{i_j} \right) \widetilde{G}_\ell^{\kappa}(w_{i_1}, \dots, w_{i_\ell}) \right).
\end{aligned}
$$

Thus expanding the exponential function

$$
\begin{aligned}
\exp\Bigg(&\frac{1}{2} \sum_{i_1=1}^{n} \alpha_{i_1}^2 G_2^{\kappa}(w_{i_1}, w_{i_1}) + \sum_{i_1<i_2} \alpha_{i_1}\alpha_{i_2} \widetilde{G}_2^{\kappa}(w_{i_1}, w_{i_2}) \\
&+ \sum_{\ell\ge 3} \frac{1}{\ell!} \sum_{i_1,\dots,i_\ell=1}^{n} \left(\prod_{j=1}^{\ell} \alpha_{i_j} \right) \widetilde{G}_\ell^{\kappa}(w_{i_1}, \dots, w_{i_\ell}) \Bigg),
\end{aligned}
$$

we get products of the type $\mathsf{c}^{\kappa}(u_I, w_I)$, i.e. every factor containing $\widetilde{G}_\ell^{\kappa}(w_{i_1}, \dots, w_{i_\ell})$ can be described by a multi-edge connection of white vertices i_j $j = 1, \dots, n$ (possibly multiple times), and every product of such terms is the contribution of a graph in $\mathcal{G}_n$. Note that the factor $\frac{1}{\ell!}$ deals with multiplicity or more precisely the automorphisms of the multisets; see e.g. the $\ell = 2$ case we treated explicitly. The factor from expanding the exponential will deal with the automorphisms of the multi-edges. The only thing left is obtaining the claimed form of the operators U^{κ}. The terms

$$
\sum_{k\in\mathbb{Z}} x_i^k \sum_{r\ge 0} \partial_y^r \exp\left(k \frac{\varsigma(k\hbar\partial_y)}{\varsigma(\hbar\partial_y)} \ln(1+y) \right)\Big|_{y=0} \cdot [u_i^r] \frac{1}{\hbar u_i \varsigma(\hbar u_i)}
$$

are immediate from the representation of the $\mathbb{J}$ operators in Proposition 2.5.3, and the numerator in (2.5.9) is explained by the $\ell = 1$ case of (2.5.15). We have

$$\exp\left(\sum_{g\geq 0}\hbar^{2g-1}\sum_{i_1=1}^{n}\sum_{s_1>0}F^{\kappa}_{g,s_1}a^{i_1}_{s_1}\right) = \exp\left(\sum_{i_1=1}^{n}\alpha_{i_1}(\widetilde{G}^{\kappa}_1(w_{i_1}) - \hbar^{-1})\right),$$

where the extra $\hbar^{-1}$ term is due to the term we added in (2.3.7), the definition of the n-point functions. Note that in the previous computations the graphs we obtain do not need to be connected, i.e. we obtain a sum over graphs in $\mathcal{G}^{\bullet}_n$ for (2.5.12). Thus, applying the inclusion-exclusion principle, we obtain the desired expression for $\widetilde{G}^{\varphi}_n(x_1, \ldots, x_n)$. □

Remark 2.5.11.
In [BCGF+23] Lemma 2.5.9 is cited to be a special case of [BDBKS23, Lemma 2.1] when the quantities are translated carefully. With the proof given here, we aim to fill in more detail from [BDBKS22, Section 3] and [BDBKS23, Section 2] for this special case in order to highlight the method of commuting the operators in the Fock space.

In order to match the expression of the key combinatorial identity to the assertion in Theorem 2.4.8, we proceed in three steps, which were suggested by M. Kazarian in an unpublished manuscript (see [Kaz19],[Kaz20]). An analogous exposition can be found in our paper [BCGF+23].

We have the following three results, the second being merely a simple observation. First, we have the following lemma.

Lemma 2.5.12 (step one)**.**
Let $\Phi(y)$, $\Psi(u)$ and $Y(w)$ be three formal power series and $Y(w) = O(w)$, then

$$\sum_{r\geq 0}(\partial^r_y\Phi)(0)\cdot [u^r]\ \exp(uY(w))\Psi(u) = \sum_{r\geq 0}(\partial^r_y\Phi)(Y(w))\cdot [u^r]\ \Psi(u)\,. \tag{2.5.17}$$

Proof. We have

$$\begin{aligned}\sum_{r\geq 0}(\partial^r_y\Phi)(0)[u^r]\exp(uY(w))\Psi(u) &= \sum_{r\geq 0}(\partial^r_y\Phi)(0)[u^r]\sum_{i=0}^{\infty}\frac{Y(w)^i}{i!}u^i\Psi(u)\\
&= \sum_{r\geq 0}\sum_{i=0}^{\infty}(\partial^r_y\Phi)(0)\frac{Y(w)^i}{i!}[u^r]u^i\Psi(u)\\
&= \sum_{r\geq 0}\sum_{i=0}^{\infty}(\partial^r_y\Phi)(0)\frac{Y(w)^i}{i!}[u^{r-i}]\Psi(u)\\
&= \sum_{r\geq 0}\sum_{i=0}^{\infty}(\partial^{(r-i)+i}_y\Phi)(0)\frac{Y(w)^i}{i!}[u^{r-i}]\Psi(u)\\
&= \sum_{s=0}^{\infty}\sum_{\substack{r,i\geq 0\\ r-i=s}}(\partial^{(r-i)+i}_y\Phi)(0)\frac{Y(w)^i}{i!}[u^{r-i}]\Psi(u)\end{aligned}$$

$$= \sum_{s=0}^{\infty} \sum_{i \geq 0} (\partial_y^{s+i} \Phi)(0) \frac{Y(w)^i}{i!} [u^s] \Psi(u)$$
$$= \sum_{s=0}^{\infty} (\partial_y^s \Phi)(Y(w)) [u^s] \Psi(u),$$

where we used that every pair (s, i) determines r uniquely in the second to last equation and Taylor expansion in the last one. □

Lemma 2.5.13 (step two)**.**
Let $Q(x)$ be a polynomial. Then

$$\sum_{j=0}^{\infty} [v^j] (x\partial_x)^j \sum_{k=0}^{\infty} Q(v) x^k = \sum_{k=0}^{\infty} Q(k) x^k.$$

Proof. We have

$$\sum_{k=0}^{\infty} Q(k) x^k = \sum_{k=0}^{\infty} \left(\sum_{j=0}^{\deg Q} [v^j] Q(v) \cdot k^j \right) x^k$$
$$= \sum_{k=0}^{\infty} \sum_{j=0}^{\deg Q} [v^j] Q(v) \cdot k^j x^k$$
$$= \sum_{k=0}^{\infty} \sum_{j=0}^{\deg Q} [v^j] Q(v) \cdot (x\partial_x)^j x^k$$
$$= \sum_{j=0}^{\deg Q} (x\partial_x)^j [v^j] \sum_{k=0}^{\infty} Q(v) x^k.$$

The formula is still valid if we replace $\deg Q$ by ∞, since then $[v^j]Q(v)$ will yield zero for terms $j \geq \deg Q$. □

For the third and last step, we will use the *Lagrange inversion formula.* Particularly, we will use [Ges16, Theorem 2.1.1] without a proof and only state the special case we need (equation (2.1.5) in op. cit.).

Proposition 2.5.14 (Lagrange inversion formula)**.**
Let $R(t)$ be a formal power series independent of x. Then there is a unique power series $f(x)$ such that $f(x) = xR(f(x))$, and for any Laurent series ψ independent of x and any integer $n \in \mathbb{N}$ we have

$$[x^n] \frac{\psi(f)}{1 - f \frac{R'(f)}{R(f)}} = [t^n] \psi(t) R(t)^n.$$

Remark 2.5.15.
We note that we may see x as a function in f via

$$f(x) = xR(f(x)) \iff x = \frac{f(x)}{R(f(x))} \iff x(f) = \frac{f}{R(f)},$$

and hence

$$\frac{\mathrm{d}}{\mathrm{d}f}x(f) = \frac{R(f) - fR'(f)}{R(f)^2} = \frac{1 - f\frac{R'(f)}{R(f)}}{R(f)}.$$

Consequently we may write

$$\frac{\mathrm{d}\ln f}{\mathrm{d}\ln x} = \frac{x}{f}\frac{\mathrm{d}f}{\mathrm{d}x} = \frac{x}{f}\frac{R(f)}{1 - f\frac{R'(f)}{R(f)}} = \frac{1}{1 - f\frac{R'(f)}{R(f)}},$$

where we used $\frac{R(f)}{f} = \frac{1}{x}$.

Proof of Theorem 2.4.8. First we will prove the formula for $n \geq 2$, since in the case $n = 1$ the graphs in the sum contain an element without a hyper-edge. For $n \geq 2$ the connectedness assures the existence of hyper-edges. We start from Lemma 2.5.9 and use the last three lemmas step by step.

i) Recall the definition of $\vec{\mathsf{U}}^\kappa(x_i)$ from (2.5.9),

$$\vec{\mathsf{U}}^\kappa(x_i) = \sum_{k\in\mathbb{Z}} x_i^k \cdot [w_i^k] \sum_{r\geq 0} \partial_y^r \exp\left(k\frac{\varsigma(k\hbar\partial_y)}{\varsigma(\hbar\partial_y)}\ln(1+y)\right)\Big|_{y=0}$$
$$[u_i^r]\frac{\exp(\hbar u_i\varsigma(\hbar u_i w_i\partial_{w_i})(G_1^\kappa(w_i) - \hbar^{-1}))}{\hbar u_i\varsigma(\hbar u_i)}.$$

Then the second factor has a contribution of the form

$$\exp(\hbar u_i(G_1^\kappa(w_i) - \hbar^{-1})) = \exp(u_i(G_{0,1}^\kappa(w_i) - 1) + u_i\hbar^2 G_{1,1}^\kappa(w_i) + \dots),$$

when we expand $\varsigma(z) = 1 + \frac{x^2}{24} + \dots$ and take the constant coefficient. The consequence is that, with rising order of x_i^k, the operator $[w_i^k]$ takes more and more coefficients of $G_{0,1}^\kappa(w_i)$ into the formula. The contribution in each of x_i and $\hbar$ is finite, but in the equation for the generating series $G_n^\varphi(x_1, \dots, x_n)$, as we sum over the all orders, we get an infinite contribution. Thus, we use Lemma 2.5.12 to get rid of these contributions. If we set $Y(w) = (G^{\kappa}_{0,1}(w) - 1)$ then we may write

$$\begin{aligned}
&\widetilde{G}_n(x_1, \dots, x_n) \\
&= \prod_{i=1}^{n} \sum_{k\in\mathbb{Z}} x_i^k \cdot [w_i^k] \sum_{r\geq 0} \partial_y^r \exp\left(k \frac{\varsigma(k\hbar\partial_y)}{\varsigma(\hbar\partial_y)} \ln(1+y)\right)\Big|_{y=0} [u_i^r] \exp(u_i(G^{\kappa}_{0,1}(w_i) - 1)) \\
&\quad \frac{\exp(\hbar u_i \varsigma(\hbar u_i w_i \partial_{w_i})(G^{\kappa}_1(w_i) - \hbar^{-1}) - u_i(G^{\kappa}_{0,1}(w_i) - 1))}{\hbar u_i \varsigma(\hbar u_i)} \sum_{\Gamma\in\mathcal{G}_n} \prod_{I\in\mathcal{I}(\Gamma)} \frac{\mathsf{c}^{\kappa}(u_I, w_I)}{\#\mathrm{Aut}(\Gamma)} \\
&= \prod_{i=1}^{n} \sum_{k\in\mathbb{Z}} x_i^k \cdot [w_i^k] \sum_{r\geq 0} \partial_y^r \exp\left(k \frac{\varsigma(k\hbar\partial_y)}{\varsigma(\hbar\partial_y)} \ln(1+y)\right)\Big|_{y=G^{\kappa}_{0,1}(w_i)-1} \\
&[u_i^r] \frac{\exp(\hbar u_i \varsigma(\hbar u_i w_i \partial_{w_i})(G^{\kappa}_1(w_i) - \hbar^{-1}) - u_i(G^{\kappa}_{0,1}(w_i) - 1))}{\hbar u_i \varsigma(\hbar u_i)} \sum_{\Gamma\in\mathcal{G}_n} \prod_{I\in\mathcal{I}(\Gamma)} \frac{\mathsf{c}^{\kappa}(u_I, w_I)}{\#\mathrm{Aut}(\Gamma)}.
\end{aligned}$$

ii) In order to use Lemma 2.5.13, we need to get a polynomial dependence in k in the factor

$$\partial_y^r \exp\left(k \frac{\varsigma(k\hbar\partial_y)}{\varsigma(\hbar\partial_y)} \ln(1+y)\right)\Big|_{y=G^{\kappa}_{0,1}(w_i)-1},$$

it has a term $\exp(k\ln(1+y))$ in the expansion of ς. We compute

$$\begin{aligned}
&\partial_y^r \exp\left(k \frac{\varsigma(k\hbar\partial_y)}{\varsigma(\hbar\partial_y)} \ln(1+y)\right)\Big|_{y=G^{\kappa}_{0,1}(w_i)-1} \\
&\quad = \partial_y^r \exp\left(k \frac{\varsigma(k\hbar\partial_y)}{\varsigma(\hbar\partial_y)} \ln(y)\right)\Big|_{y=G^{\kappa}_{0,1}(w_i)} \\
&\quad = (G^{\kappa}_{0,1}(w_i))^k \left(\exp(-k\ln(y)) \cdot \partial_y^r \exp\left(k \frac{\varsigma(k\hbar\partial_y)}{\varsigma(\hbar\partial_y)} \ln(y)\right)\right)\Big|_{y=G^{\kappa}_{0,1}(w_i)}.
\end{aligned} \tag{2.5.18}$$

We claim

$$\exp(-k\ln(y))\partial_y^r \exp\left(k \frac{\varsigma(k\hbar\partial_y)}{\varsigma(\hbar\partial_y)} \ln(y)\right) = \left(\partial_y + \frac{k}{y}\right)^r \exp\left(k\left(\frac{\varsigma(k\hbar\partial_y)}{\varsigma(\hbar\partial_y)} - 1\right)\ln(y)\right). \tag{2.5.19}$$

For $r = 0$ the assertion is true, and for any $r > 0$ we have by induction

$$\begin{aligned}
&\left(\partial_y + \frac{k}{y}\right)^r \exp\left(k\left(\frac{\varsigma(k\hbar\partial_y)}{\varsigma(\hbar\partial_y)} - 1\right)\ln(y)\right) \\
&\quad = \left(\partial_y + \frac{k}{y}\right) \exp(-k\ln(y))\partial_y^{r-1} \exp\left(k \frac{\varsigma(k\hbar\partial_y)}{\varsigma(\hbar\partial_y)} \ln(y)\right) \\
&\quad = \frac{-k}{y} \exp(-k\ln(y))\partial_y^{r-1} \exp\left(k \frac{\varsigma(k\hbar\partial_y)}{\varsigma(\hbar\partial_y)} \ln(y)\right)
\end{aligned}$$

$$
\begin{aligned}
&+\exp(-k\ln(y))\partial_y^r \exp\left(k\frac{\varsigma(k\hbar\partial_y)}{\varsigma(\hbar\partial_y)}\ln(y)\right)\\
&+\frac{k}{y}\exp(-k\ln(y))\partial_y^{r-1}\exp\left(k\frac{\varsigma(k\hbar\partial_y)}{\varsigma(\hbar\partial_y)}\ln(y)\right)\\
=\ &\exp(-k\ln(y))\partial_y^r\exp\left(k\frac{\varsigma(k\hbar\partial_y)}{\varsigma(\hbar\partial_y)}\ln(y)\right).
\end{aligned}
$$

We continue using the formula,

$$
\begin{aligned}
&\partial_y^r \exp\left(k\frac{\varsigma(k\hbar\partial_y)}{\varsigma(\hbar\partial_y)}\ln(1+y)\right)\Big|_{y=G^\kappa_{0,1}(w_i)-1}\\
&=(G^\kappa_{0,1}(w_i))^k\Big(\partial_y+\frac{k}{y}\Big)^r\exp\left(k\Big(\frac{\varsigma(k\hbar\partial_y)}{\varsigma(\hbar\partial_y)}-1\Big)\ln(y)\right)\Big|_{y=G^\kappa_{0,1}(w_i)},
\end{aligned}
$$

where the right-hand side has a polynomial dependence in k at each order of $\hbar$. Thus, we can apply Lemma 2.5.13; we have

$$
\begin{aligned}
&\widetilde{G}_n(x_1,\ldots,x_n)\\
&=\prod_{i=1}^n\sum_{j=0}^\infty(x_i\partial_{x_i})^j[v^j]\sum_{k\in\mathbb{Z}}x_i^k\cdot[w_i^k]\sum_{r\geq 0}(G^\kappa_{0,1}(w_i))^k\\
&\Big(\partial_y+\frac{v}{y}\Big)^r\exp\left(v\Big(\frac{\varsigma(v\hbar\partial_y)}{\varsigma(\hbar\partial_y)}-1\Big)\ln(y)\right)\Big|_{y=G^\kappa_{0,1}(w_i)}\\
&[u_i^r]\frac{\exp(\hbar u_i\varsigma(\hbar u_i w_i\partial_{w_i})(G^\kappa_1(w_i)-\hbar^{-1})-u_i(G^\kappa_{0,1}(w_i)-1))}{\hbar u_i\varsigma(\hbar u_i)}\\
&\sum_{\Gamma\in\mathcal{G}_n}\prod_{I\in\mathcal{I}(\Gamma)}\frac{\mathsf{c}^\kappa(u_I,w_I)}{\#\mathrm{Aut}(\Gamma)}.
\end{aligned}
$$

iii) In the last step we remove $G(w_i)^k$ by Lagrange inversion, i.e. by Proposition 2.5.14 and Remark 2.5.15, with $f=w_i, R=G^\kappa_{0,1}, x_i=\frac{w_i}{G^\kappa_{0,1}(w_i)}$. Then we have

$$
[x_i^k]\frac{\mathrm{d}\ln(w_i)}{\mathrm{d}\ln(x_i)}\psi(w_i)=[x^k]\frac{\psi(w_i)}{1-w_i\frac{G^\kappa_{0,1}{}'(w)}{G^\kappa_{0,1}(w_i)}}=[t^k]\psi(t)G^\kappa_{0,1}(t)^k.
$$

Hence

$$
\sum_{k\in\mathbb{Z}}x_i^k[w_i^k]G^\kappa_{0,1}(w_i)^k\psi(w_i)=\sum_{k\in\mathbb{Z}}x_i^k[x_i^k]\frac{\mathrm{d}\ln(w_i)}{\mathrm{d}\ln(x_i)}\psi(w_i)=\frac{\mathrm{d}\ln(w_i)}{\mathrm{d}\ln(x_i)}\psi(w_i)
$$

for any ψ. With the notation $P^\kappa(w_i)=\frac{\mathrm{d}\ln(w_i)}{\mathrm{d}\ln(x_i)}$ and the observation

$$
x_i\partial_{x_i}=x_i\frac{\mathrm{d}w_i}{\mathrm{d}x_i}\partial_{w_i}=w_i\frac{x_i}{w_i}\frac{\mathrm{d}w_i}{\mathrm{d}x_i}\partial_{w_i}=w_i\frac{\mathrm{d}\ln(w_i)}{\mathrm{d}\ln(x_i)}\partial_{w_i},
$$

we obtain the desired

$$\widetilde{G}_n(x_1,\dots,x_n) = \sum_{\Gamma\in\mathcal{G}_n} \frac{1}{\#\mathrm{Aut}(\Gamma)} \prod_{i=1}^{n} \vec{\mathsf{O}}^\kappa(w_i) \prod_{I\in\mathcal{I}(\Gamma)} \mathsf{c}^\kappa(u_I,w_I)$$

for $n \geq 2$.

Now for $n=1$, there is a graph Γ_0 with no hyper-edges, thus there is a summand with

$$\prod_{I\in\mathcal{I}(\Gamma_0)} \mathsf{c}^\kappa(u_I,w_I) = 1.$$

Hence, when we apply the operator $\vec{\mathsf{U}}^\kappa(w)$, the result and in particular the factor

$$\frac{\exp(\hbar u\varsigma(\hbar uw\partial_w)(G_1^\kappa(w)-\hbar^{-1}))}{\hbar u\varsigma(\hbar u)} = \frac{1}{\hbar u\varsigma(\hbar u)} + \frac{\varsigma(\hbar uw\partial_w)(G_1^\kappa(w)-\hbar^{-1})}{\varsigma(\hbar u)} + \dots$$

will have a pole. To remove this pole, and proceed similar to the first step for $n \geq 2$, we write

$$\begin{aligned}\widetilde{G}^\varphi(x) &= G^\varphi(x) - \hbar^{-1} \qquad (2.5.20)\\ &= \sum_{k\in\mathbb{Z}} x^k\cdot[w^k]\sum_{r\geq 0}\partial_y^r \exp\left(k\frac{\varsigma(k\hbar\partial_y)}{\varsigma(\hbar\partial_y)}\ln(1+y)\right)\Big|_{y=0}\\ &\quad [u^r]\left[\frac{\exp(\hbar u\varsigma(\hbar uw\partial_w)(G_1^\kappa(w)-\hbar^{-1}))}{\hbar u\varsigma(\hbar u)} - \frac{\exp(u(G_{0,1}^\kappa(w)-1))}{u\hbar}\right]\\ &\quad + \sum_{k\in\mathbb{Z}} x^k\cdot[w^k]\sum_{r\geq 0}\partial_y^r \exp\left(k\frac{\varsigma(k\hbar\partial_y)}{\varsigma(\hbar\partial_y)}\ln(1+y)\right)\Big|_{y=0}[u^r]\frac{\exp(u(G_{0,1}^\kappa(w)-1))}{u\hbar}.\end{aligned}$$

(2.5.21)

Then we can apply Lemma 2.5.12 to the term in (2.5.21), we obtain

$$\begin{aligned}&\sum_{k\in\mathbb{Z}} x^k\cdot[w^k]\sum_{r\geq 0}\partial_y^r \exp\left(k\frac{\varsigma(k\hbar\partial_y)}{\varsigma(\hbar\partial_y)}\ln(1+y)\right)\Big|_{y=0}\\ &\qquad [u^r]\left[\frac{\exp(\hbar u\varsigma(\hbar uw\partial_w)(G_1^\kappa(w)-\hbar^{-1}))}{\hbar u\varsigma(\hbar u)} - \frac{\exp(u(G_{0,1}^\kappa(w)-1))}{u\hbar}\right]\\ &= \sum_{k\in\mathbb{Z}} x^k\cdot[w^k]\sum_{r\geq 0}\partial_y^r \exp\left(k\frac{\varsigma(k\hbar\partial_y)}{\varsigma(\hbar\partial_y)}\ln(1+y)\right)\Big|_{y=0}[u^r]\\ &\qquad \exp(u(G_{0,1}^\kappa(w)-1))\left[\frac{\exp(\hbar u\varsigma(\hbar uw\partial_w)(G_1^\kappa(w)-\hbar^{-1})-(u(G_{0,1}^\kappa(w)-1)))}{\hbar u\varsigma(\hbar u)} - \frac{1}{u\hbar}\right]\\ &= \sum_{k\in\mathbb{Z}} x^k\cdot[w^k]\sum_{r\geq 0}\partial_y^r \exp\left(k\frac{\varsigma(k\hbar\partial_y)}{\varsigma(\hbar\partial_y)}\ln(1+y)\right)\Big|_{y=G_{0,1}^\kappa(w)-1}[u^r]\\ &\qquad \left[\frac{\exp(\hbar u\varsigma(\hbar uw\partial_w)(G_1^\kappa(w)-\hbar^{-1})-(u(G_{0,1}^\kappa(w)-1)))}{\hbar u\varsigma(\hbar u)} - \frac{1}{u\hbar}\right].\end{aligned}$$

Since the second sum runs over $r \geq 0$, the term $\frac{1}{u\hbar}$ does not contribute. Next we apply Lemma 2.5.13 and obtain

$$\begin{aligned}\widetilde{G}^{\varphi}(x) =& \sum_{j=0}^{\infty}(x\partial_x)^j[v^j]\sum_{k\in\mathbb{Z}} x^k \cdot [w^k]\sum_{r\geq 0}(G^{\kappa}_{0,1}(w))^k \\ & \Big(\partial_y + \frac{v}{y}\Big)^r \exp\left(v\Big(\frac{\varsigma(v\hbar\partial_y)}{\varsigma(\hbar\partial_y)} - 1\Big)\ln(y)\right)\Big|_{y=G^{\kappa}_{0,1}(w)} \\ & [u^r]\left[\frac{\exp(\hbar u\varsigma(\hbar uw\partial_w)(G^{\kappa}_1(w) - \hbar^{-1}) - u(G^{\kappa}_{0,1}(w) - 1))}{\hbar u\varsigma(\hbar u)}\right]\end{aligned}$$

and finally Lagrange inversion, Proposition 2.5.14, yields

$$\begin{aligned}\widetilde{G}^{\varphi}(x) =& \sum_{j=0}^{\infty}(x\partial_x)^j[v^j]\frac{\mathrm{d}\ln w}{\mathrm{d}\ln x}\Big(\partial_y + \frac{v}{y}\Big)^r \exp\left(v\Big(\frac{\varsigma(v\hbar\partial_y)}{\varsigma(\hbar\partial_y)} - 1\Big)\ln(y)\right)\Big|_{y=G^{\kappa}_{0,1}(w)} \\ & [u^r]\left[\frac{\exp(\hbar u\varsigma(\hbar uw\partial_w)(G^{\kappa}_1(w) - \hbar^{-1}) - u(G^{\kappa}_{0,1}(w) - 1))}{\hbar u\varsigma(\hbar u)}\right] \\ =& \sum_{j=0}^{\infty}(wP^{\kappa}(w)\partial_w)^j[v^j]P^{\kappa}(w)\Big(\partial_y + \frac{v}{y}\Big)^r \exp\left(v\Big(\frac{\varsigma(v\hbar\partial_y)}{\varsigma(\hbar\partial_y)} - 1\Big)\ln(y)\right)\Big|_{y=G^{\kappa}_{0,1}(w)} \\ & [u^r]\left[\frac{\exp(\hbar u\varsigma(\hbar uw\partial_w)(G^{\kappa}_1(w) - \hbar^{-1}) - u(G^{\kappa}_{0,1}(w) - 1))}{\hbar u\varsigma(\hbar u)}\right].\end{aligned}$$

This is exactly the term we get from the graph with no hyper-edges. We still need to match the correction term Δ^{κ}_g in Theorem 2.4.8. We repeat the calculations from our paper [BCGF+23], which are originally from [BDBKS22, Section 6.2]. We have

$$\begin{aligned}& \sum_{k\geq 1} x^k[w^k]\sum_{r\geq 0}\partial_y^r \exp\left(k\frac{\varsigma(k\hbar\partial_y)}{\varsigma(\hbar\partial_y)}\ln(1+y)\right)\Big|_{y=0}[u^r]\frac{\exp\big(u(G^{\kappa}_{0,1}(w) - 1)\big)}{\hbar u} \\ =& \int_0^x \mathrm{d}x\frac{\mathrm{d}}{\mathrm{d}x}\sum_{k\geq 1} x^k[w^k]\sum_{r\geq 0}\partial_y^r \exp\left(k\frac{\varsigma(k\hbar\partial_y)}{\varsigma(\hbar\partial_y)}\ln(1+y)\right)\Big|_{y=0}[u^r]\frac{\exp\big(u(G^{\kappa}_{0,1}(w) - 1)\big)}{\hbar u} \\ =& \int_0^x \frac{\mathrm{d}x}{x}\sum_{k\geq 1} kx^k[w^k]\sum_{r\geq 0}\partial_y^r \exp\left(k\frac{\varsigma(k\hbar\partial_y)}{\varsigma(\hbar\partial_y)}\ln(1+y)\right)\Big|_{y=0}[u^r]\frac{\exp\big(u(G^{\kappa}_{0,1}(w) - 1)\big)}{\hbar u} \\ =& \int_0^x \frac{\mathrm{d}x}{x}\sum_{k\geq 1} x^k[w^k]\sum_{r\geq 0}\partial_y^r \exp\left(k\frac{\varsigma(k\hbar\partial_y)}{\varsigma(\hbar\partial_y)}\ln(1+y)\right)\Big|_{y=0}[u^r]w\partial_w\frac{\exp\big(u(G^{\kappa}_{0,1}(w) - 1)\big)}{\hbar u}.\end{aligned}$$

We have for the derivative

$$[u^r]w\partial_w\frac{\exp\big(u(G^{\kappa}_{0,1}(w) - 1)\big)}{\hbar u} = [u^r]\frac{\exp\big(u\,(G^{\kappa}_{0,1}(w) - 1)\big)}{\hbar}w\partial_w G^{\kappa}_{0,1}(w)$$

and thus we can apply Lemma 2.5.12 to obtain

$$\int_0^x \frac{\mathrm{d}x}{x}\sum_{k\geq 1} x^k[w^k]\sum_{r\geq 0}\partial_y^r \exp\left(k\frac{\varsigma(k\hbar\partial_y)}{\varsigma(\hbar\partial_y)}\ln(1+y)\right)\Big|_{y=0}[u^r]w\partial_w\frac{\exp\big(u(G^{\kappa}_{0,1}(w) - 1)\big)}{\hbar u}$$

$$= \int_0^x \frac{dx}{x} \sum_{k\geq 1} x^k [w^k] \sum_{r\geq 0} \partial_y^r \exp\left(k \frac{\varsigma(k\hbar\partial_y)}{\varsigma(\hbar\partial_y)} \ln y\right)\Big|_{y=G^\kappa_{0,1}(w)} [u^r] \frac{w\partial_w G^\kappa_{0,1}(w)}{\hbar}$$
$$= \int_0^x \frac{dx}{x} \sum_{k\geq 1} x^k [w^k] \exp\left(k \frac{\varsigma(k\hbar\partial_y)}{\varsigma(\hbar\partial_y)} \ln y\right)\Big|_{y=G^\kappa_{0,1}(w)} \frac{w\partial_w G^\kappa_{0,1}(w)}{\hbar},$$

where in the last equality we used that we have no contribution from $[u^r]$ for $r > 0$ in the right most factor. We use the tricks (2.5.18) and (2.5.19) to apply Lemma 2.5.13, we obtain

$$\int_0^x \frac{dx}{x} \sum_{k\geq 1} x^k \cdot [w^k] \exp\left(k \frac{\varsigma(k\hbar\partial_y)}{\varsigma(\hbar\partial_y)} \ln y\right)\Big|_{y=G^\kappa_{0,1}(w)} \frac{w\partial_w G^\kappa_{0,1}(w)}{\hbar}$$
$$= \int_0^x \frac{dx}{x} \sum_{k\geq 1} x^k \cdot [w^k](G^\kappa_{0,1}(w))^k \exp\left(k\left(\frac{\varsigma(k\hbar\partial_y)}{\varsigma(\hbar\partial_y)} - 1\right) \ln y\right)\Big|_{y=G^\kappa_{0,1}(w)} \frac{w\partial_w G^\kappa_{0,1}(w)}{\hbar}$$
$$= \int_0^x \frac{dx}{x} \sum_{j=0}^{\infty} (x\partial_x)^j [v^j] \sum_{k\geq 1} x^k [w^k](G^\kappa_{0,1}(w))^k \exp\left(v\left(\frac{\varsigma(v\hbar\partial_y)}{\varsigma(\hbar\partial_y)} - 1\right) \ln y\right)\Big|_{y=G^\kappa_{0,1}(w)} \frac{w\partial_w G^\kappa_{0,1}(w)}{\hbar}.$$

We can apply Lagrange inversion, Proposition 2.5.14, and get

$$\int_0^x \frac{dx}{x} \sum_{j=0}^{\infty} (x\partial_x)^j [v^j] \sum_{k\geq 1} x^k [w^k](G^\kappa_{0,1}(w))^k \exp\left(v\left(\frac{\varsigma(v\hbar\partial_y)}{\varsigma(\hbar\partial_y)} - 1\right) \ln y\right)\Big|_{y=G^\kappa_{0,1}(w)} \frac{w\partial_w G^\kappa_{0,1}(w)}{\hbar}$$
$$= \frac{1}{\hbar} \int_0^x \frac{dx}{x} \sum_{j=0}^{\infty} (x\partial_x)^j [v^j] \exp\left(v\left(\frac{\varsigma(v\hbar\partial_y)}{\varsigma(\hbar\partial_y)} - 1\right) \ln y\right)\Big|_{y=G^\kappa_{0,1}(w)} P^\kappa(w) w\partial_w G^\kappa_{0,1}(w)$$
$$= \frac{1}{\hbar} \int_0^x \frac{dx}{x} \sum_{j=1}^{\infty} (x\partial_x)^j [v^j] \exp\left(v\left(\frac{\varsigma(v\hbar\partial_y)}{\varsigma(\hbar\partial_y)} - 1\right) \ln y\right)\Big|_{y=G^\kappa_{0,1}(w)} P^\kappa(w) w\partial_w G^\kappa_{0,1}(w)$$
$$+ \frac{1}{\hbar} \int_0^x \frac{dx}{x} P^\kappa(w) w\partial_w G^\kappa_{0,1}(w).$$

For the first summand we integrate against one of the outer derivatives $x\partial_x$

$$\frac{1}{\hbar} \int_0^x \frac{dx}{x} \sum_{j=1}^{\infty} (x\partial_x)^j [v^j] \exp\left(v\left(\frac{\varsigma(v\hbar\partial_y)}{\varsigma(\hbar\partial_y)} - 1\right) \ln y\right)\Big|_{y=G^\kappa_{0,1}(w)} P^\kappa(w) w\partial_w G^\kappa_{0,1}(w)$$
$$= \frac{1}{\hbar} \sum_{j=1}^{\infty} (x\partial_x)^j [v^{j-1}] \exp\left(v\left(\frac{\varsigma(v\hbar\partial_y)}{\varsigma(\hbar\partial_y)} - 1\right) \ln y\right)\Big|_{y=G^\kappa_{0,1}(w)} P^\kappa(w) w\partial_w G^\kappa_{0,1}(w)$$
$$= \sum_{g=1}^{\infty} \hbar^{2g-1} \Delta^\kappa_g(x).$$

and the $j = 0$ term together with $P^\kappa(w) w\partial_w = x\partial_x$ yields

$$\frac{1}{\hbar}\int_0^x \frac{dx}{x} P^\kappa(w) w\partial_w G^\kappa_{0,1}(w) = \frac{1}{\hbar}(G^\kappa_{0,1}(w) - 1).$$

In conclusion, for $2g-2+n > 0$ taking the coefficient $[\hbar^{2g-2+n}]$, we obtain the assertion of Theorem 2.4.8. The special cases $(0,1)$ and $(0,2)$ are the known cases [Voi86, Spe94] and [CMSS07] respectively. □

Remark 2.5.16.
The special cases $(0,1)$ and $(0,2)$ can also be obtained by the techniques in the proof of Theorem 2.4.8; see [BCGF$^+$23, section 4.2].

2.5.3 Extracting the genus zero sector

Finally, we will be extracting the genus zero sector of Theorem 2.4.8 for $n \geq 3$. This solves the problem of finding higher order functional relations for the moment-cumulant formalism posed in [CMSS07].

Proof of Theorem 2.4.1. Let us recall the expression of Theorem 2.4.8 and specialise it to genus 0. We have for $n \geq 3$

$$G^\varphi_{g,n}(x_1,\dots,x_n) = [\hbar^{2g-2+n}] \sum_{\Gamma\in\mathcal{G}_n} \frac{1}{\#\mathrm{Aut}(\Gamma)} \prod_{i=1}^n \vec{\mathsf{O}}^\kappa(w_i) \prod_{I\in\mathcal{I}(\Gamma)} \mathsf{c}^\kappa(u_I, w_I).$$

Thus, for $g = 0$, we obtain

$$G^\varphi_{0,n}(x_1,\dots,x_n) = [\hbar^{n-2}] \sum_{\Gamma\in\mathcal{G}_n} \frac{1}{\#\mathrm{Aut}(\Gamma)} \prod_{i=1}^n \vec{\mathsf{O}}^\kappa(w_i) \prod_{I\in\mathcal{I}(\Gamma)} \mathsf{c}^\kappa(u_I, w_I).$$

The leading order in $\hbar$ of the terms on the right hand side can be read from

$$\mathsf{c}^\kappa(u_i, w_i) = \hbar^{2\#I-2}\Big(\prod_{i\in I} u_i\Big)\widetilde{G}^\kappa_{0,\#I}(w_I) + O(\hbar^{2\#I-1})$$

and

$$\begin{aligned}\vec{\mathsf{O}}^\kappa(w) &= \sum_{m\geq 0}(P^\kappa(w)w\partial_w)^m \cdot [v^m] \sum_{r\geq 0}\Big(\partial_y + \frac{v}{y}\Big)^r 1\Big|_{y=G^\kappa_{0,1}(w)} \cdot [u^r]\ (u\hbar)^{-1} + O(\hbar^0) \\ &= \hbar^{-1}\sum_{r\geq 0} \vec{\mathsf{O}}^\kappa_r(w)\cdot [u^{r+1}] + O(\hbar^0),\end{aligned}$$

where, in the last line, $\vec{\mathsf{O}}^\kappa_r$ is defined in Theorem 2.4.1. Thus, for a graph $\Gamma \in \mathcal{G}_n$, the minimal degree on the right-hand side is given by

$$-n + \sum_{I\in\mathcal{I}(\Gamma)} (2\#I - 2).$$

This number attains its minimal value $n-2$ when Γ is a tree. Hence, the only contribution to the genus 0 part of $\widetilde{G}_n$ is given by the latter. Furthermore, trees have no non-trivial automorphisms. The variable u_i only appears from the hyper-edge contributions, and its power is the valency of the i-th white vertex. Therefore, the extraction of powers of u_i prescribed by the operators at the vertices restricts the sum to the set $\mathcal{G}_{0,n}(\mathbf{r}+1)$ of trees where the i-th white vertex has valency r_i+1. We obtain in genus 0 and for $n \geq 3$

$$G^{\kappa}_{0,n}(x_1,\ldots,x_n) = \sum_{r_1,\ldots,r_n\geq 0} \prod_{i=1}^{n} \vec{\mathsf{O}}^{\kappa}_{r_i}(w_i) \sum_{T\in\mathcal{G}_{0,n}(\mathbf{r}+1)} \prod_{I\in\mathcal{I}(T)} \widetilde{G}^{\kappa}_{0,\#I}(w_I). \tag{2.5.22}$$

With the classical notation of genus zero free probability, that is $G^{\varphi}_{0,n} = M_n$ and $G^{\kappa}_{0,n} = C_n$ as well as the correction for $n=2$, we find precisely Theorem 2.4.1. □

2.6 Surfaced free probability

In this section, we extend higher order free probability to so-called surfaced free probability. Concretely, we will extend the results of [CMSS07, Section 7] to our setting of higher genus. We start this section by recalling higher order free probability of [CMSS07]. Afterward, we introduce an extension of the partitioned permutation, which we call surfaced permutations and show that their theory of multiplicative functions evolves in parallel to partitioned permutations. The extended combinatorial theory allows for a new notion of freeness, we call it (g,n)-freeness. Furthermore, we show that it is a sensible notion by proving important properties: (g,n)-freeness does not depend on generators and constants are free from everything. Moreover, we recover Voiculescu's free independence as well as freeness of all order of [CMSS07]. Surprisingly, we can also recover infinitesimal freeness (cf. Section 1.1.4) by allowing half integer genus. We conclude this section by extending the asymptotic result of Theorem 2.1.4.

2.6.1 Higher order free probability

Recall from Definition 2.1.3, that a second order noncommutative probability space consists of the data $(\mathcal{A},\varphi_1,\varphi_2)$, where $\mathcal{A}$ is a unital algebra, and φ_1 is a unital linear functional and φ_2 a symmetric bilinear form that is tracial in its arguments and vanishes on $1_{\mathcal{A}}\cdot\mathbb{C}\times\mathcal{A}$. Motivated by the random matrix calculations presented in Section 2.1.1, Collins, Mingo, Śniady and Speicher extended the notion of a noncommutative probability space and the moment-cumulant formalism to higher orders. Despite not deriving the higher order functional relations, they already introduced and studied the so-called freeness of all orders.

Definition 2.6.1.

i) A *higher order noncommutative probability space* (HOPS) is the data $(\mathcal{A},\boldsymbol{\varphi})$ consisting of a unital associative (maybe non-commutative) algebra $\mathcal{A}$ over $\mathbb{C}$ and a family $\boldsymbol{\varphi} = (\varphi_n)_{n\geq 1}$ of tracial n-linear forms such that

$$\varphi_1(1) = 1 \quad \text{and} \quad \varphi_n(1,a_2,\ldots,a_n) = 0,$$

for all $n \geq 2$ and $a_2, \dots, a_n \in \mathcal{A}$.

ii) Let $a \in \mathcal{A}$ be a noncommutative random variable in a HOPS $(\mathcal{A}, \boldsymbol{\varphi})$. Then the higher order distribution of a is given by

$$\mu_a^{\mathrm{HO}} = \{\varphi_n(a^{r_1}, \dots, a^{r_n}) \colon n \geq 1, r_1, \dots, r_n \in \mathbb{N}\}.$$

iii) Let $a_1, \dots, a_\ell \in \mathcal{A}$ be noncommutative random variables in a HOPS $(\mathcal{A}, \varphi)$. Then the higher order joint distribution of $a_1, \dots, a_\ell$ is given by

$$\mu_{a_1,\dots,a_\ell}^{\mathrm{HO}} = \{\varphi_n(a_{i_1}^{r_1}, \dots, a_{i_n}^{r_n}) \colon n \geq 1, i_1, \dots, i_n \in [\ell], r_1, \dots, r_n \in \mathbb{N}\}.$$

By the discussions from prior sections, we can encode the higher order distribution of a single element via a multiplicative function $\varphi \colon \mathcal{PS} \to \mathbb{C}$ given by

$$\varphi(1_{r_1+\dots+r_n}, \gamma_{r_1,\dots,r_n}) = \varphi_n(a^{r_1}, \dots, a^{r_n}), n \in \mathbb{N}, r_1, \dots, r_n \in \mathbb{N}. \tag{2.6.1}$$

Then Theorem 2.4.1 gives the functional relations between the moment and cumulant generating functions

$$M_n(x_1, \dots, x_n) = \sum_{r_1,\dots,r_n \geq 1} \varphi_{r_1,\dots,r_n} x_1^{r_1} \dots x_n^{r_n}$$

and

$$C_n(x_1, \dots, x_n) = \sum_{r_1,\dots,r_n \geq 1} \kappa_{r_1,\dots,r_n} x_1^{r_1} \dots x_n^{r_n}.$$

In free probability, we want to study the central notion of freeness, and it only appears if we look at joint distributions of several variables. Thus, we need to make sense of the joint distribution in terms of multiplicative functions. This was already done in [CMSS07] and we briefly explain the set-up.

Definition 2.6.2.
Let $\mathcal{A}$ be an algebra.

i) We define the set of *$\mathcal{A}$-decorated partitioned permutations* by

$$\mathcal{PS}(\mathcal{A}) \coloneqq \bigcup_{d \geq 1} PS(d) \times \mathcal{A}^d.$$

If $f \colon \mathcal{PS}(\mathcal{A}) \to \mathcal{C}$, then we denote the value of f on an element $((\mathcal{V}, \pi), a_1, \dots a_d) \in \mathcal{PS}(d) \times \mathcal{A}^d$ by

$$f(\mathcal{V}, \pi)[a_1, \dots, a_d].$$

ii) Let $f\colon \mathcal{PS} \to \mathbb{C}$ and $g\colon \mathcal{PS}(\mathcal{A}) \to \mathbb{C}$ be two functions, $d \geq 1$, and $(\mathcal{U}, \gamma) \in \mathcal{PS}(d)$. We define the convolution of f and g by $f * g\colon \mathcal{PS}(\mathcal{A}) \to \mathbb{C}$, where

$$(f * g)(\mathcal{U}, \gamma)[a_1, \ldots, a_d] \coloneqq \sum_{(\mathcal{V},\pi)\cdot(\mathcal{W},\sigma)=(\mathcal{U},\gamma)} f(\mathcal{V}, \pi)(g(\mathcal{W}, \sigma)[a_1, \ldots, a_d]).$$

iii) Let $f\colon \mathcal{PS}(\mathcal{A}) \to \mathbb{C}$ be a function, we call f *multiplicative* if the following two properties hold:

a) We have

$$f(\mathbf{1}_d, \tau^{-1} \circ \sigma \circ \tau)[a_1, \ldots, a_d] = f(\mathbf{1}_d, \sigma)[a_{\tau(1)}, \ldots, a_{\tau(d)}]$$

for any $d \geq 1, \pi, \sigma \in S(d)$ and $a_1, \ldots, a_d \in \mathcal{A}$.

b) We have

$$f(\mathcal{A}, \alpha)[a_1, \ldots, a_n] = \prod_{A \in \mathcal{A}} f(\mathbf{1}_{\#A}, \alpha_{|A})[(a_i)_{i\in A}],$$

where bijections $[\#A] \to A$ have been chosen to make sense of the right-hand side, which is independent of this choice due to the first condition.

The latter is used to put higher order moments into the framework of multiplicative functions, in particular it lets us define higher order cumulants via convolution.

Definition 2.6.3.
Let $(\mathcal{A}, \boldsymbol{\varphi})$ be a HOPS.

i) We define the *moment function* to be the multiplicative function $\varphi\colon \mathcal{PS}(\mathcal{A}) \to \mathbb{C}$ given by

$$\varphi(\mathbf{1}_d, \gamma_\lambda)[a_1, \ldots, a_d] = \varphi_n\left(\prod_{j=1}^{\lambda_1} a_j, \prod_{j=1}^{\lambda_2} a_{\lambda_1+j}, \ldots, \prod_{j=1}^{\lambda_n} a_{\lambda_1+\cdots+\lambda_{n-1}+j} \right)$$

for any $d \in \mathbb{N}$ and any partition $\lambda \vdash d$.

ii) We define the *cumulant function* to be the multiplicative function $\kappa\colon \mathcal{PS}(\mathcal{A}) \to \mathbb{C}$ given by

$$\kappa = \mu * \varphi.$$

Moreover, we define the *higher order cumulants* by

$$\kappa_{r_1,\ldots,r_n}(a_1, \ldots, a_n) \coloneqq \kappa(\mathbf{1}_d, \gamma_{r_1,\ldots,r_n})[a_1, \ldots, a_d]$$

for any $n, r_1, \ldots, r_n \in \mathbb{N}$ and $d = r_1 + \cdots + r_n$.

Remark 2.6.4.
Note, with these definitions, we may write

$$M_n(x_1,\ldots,x_n) = \sum_{r_1,\ldots,r_n\geq 1} \varphi(a^{r_1},\ldots,a^{r_n})x_1^{r_1}\ldots x_n^{r_n}$$

and

$$C_n(x_1,\ldots,x_n) = \sum_{r_1,\ldots,r_n\geq 1} \kappa_{r_1,\ldots,r_n}(a,\ldots,a)x_1^{r_1}\ldots x_n^{r_n},$$

as we already mentioned.

Let us recall the main features of higher order free probability of [CMSS07] without a proof. We will later generalize these to higher genus and obtain them as special case.

Definition 2.6.5.
Let $(\mathcal{X}_i)_{i\in I}$ be a family of subsets of $\mathcal{A}$. We call $(\mathcal{X}_i)_{i\in I}$ *free of all orders* if we have the following vanishing of mixed cumulants: For all $d\geq 2$ and all $a_k\in\mathcal{X}_{i(k)}$ $(1\leq k\leq d)$ such that $i(p)\neq i(q)$ for some $1\leq p,q\leq d$ we have

$$\kappa(\mathbf{1}_d,\pi)(a_1,\ldots,a_d)=0$$

for all $\pi\in S(d)$.

Remark 2.6.6.
Since the combinatorics of partitioned permutations is yet to be fully understood it is not clear how to express higher order freeness in terms of the moments, so we have to rely on the vanishing of mixed cumulants. Recall that first order freeness was developed in terms of moments and only later [Spe94] introduced free cumulants and showed that the vanishing of mixed cumulants is equivalent to freeness. In theory, the vanishing of mixed cumulants should explain how the formula for the moments looks like, but up to this day it is still not tangible.

Proposition 2.6.7 ([CMSS07])**.**
Let $(\mathcal{A},\varphi)$ be a HOPS. Then $1_{\mathcal{A}}$ is free of all orders from every set $\mathcal{X}\subseteq\mathcal{A}$, that is

$$\kappa_{r_1,\ldots,r_n}(1_{\mathcal{A}},a_2,\ldots,a_d)=0$$

for any $d\geq 2$ and $r_1,\ldots,r_n$ with $d=r_1+\cdots+r_n$ and any $a_2,\ldots,a_d\in\mathcal{A}$.

Theorem 2.6.8 ([CMSS07])**.**
Let $(\mathcal{A},\varphi)$ be a HOPS and $\mathcal{X}_1,\mathcal{X}_2\subset\mathcal{A}$. We denote $\mathcal{X}_i^+=\mathcal{X}_i\cup\{1\}$. The following statements are equivalent:

i) $\mathcal{X}_1$ and $\mathcal{X}_2$ are free of all orders.

ii) $\mathcal{X}_1^+$ and $\mathcal{X}_2^+$ are free of all orders.

iii) For any $d \in \mathbb{N}$, $(\mathcal{U}, \gamma) \in \mathcal{PS}(d)$,$a_1, \dots, a_d \in \mathcal{X}_1^+$ and any $b_1, \dots, b_d \in \mathcal{X}_2^+$, we have

$$\varphi(\mathcal{U}, \gamma)[a_1 b_1, \dots, a_d b_d] = \sum_{(\mathcal{V},\pi)(\mathcal{W},\sigma)=(\mathcal{U},\gamma)} \kappa(\mathcal{V}, \pi)[a_1, \dots, a_d] \varphi(\mathcal{W}, \sigma)[b_1, \dots, b_d].$$

iv) For any $d \in \mathbb{N}$, any $(\mathcal{U}, \gamma) \in \mathcal{PS}(d)$, $a_1, \dots, a_d \in \mathcal{X}_1^+$ and $b_1, \dots, b_d \in \mathcal{X}_2^+$, we have

$$\kappa(\mathcal{U}, \gamma)[a_1 b_1, \dots, a_d b_d] = \sum_{(\mathcal{V},\pi)(\mathcal{W},\sigma)=(\mathcal{U},\gamma)} \kappa(\mathcal{V}, \pi)[a_1, \dots, a_d] \kappa(\mathcal{W}, \sigma)[b_1, \dots, b_d].$$

Theorem 2.6.9 ([CMSS07])**.**
Let $(\mathcal{X}_i)_{i \in I}$ be a family of subsets of $\mathcal{A}$, and $\mathcal{A}_i$ the unital subalgebra generated by $\mathcal{X}_i$. The freeness of all orders of $(\mathcal{X}_i)_{i \in I}$ is equivalent to the freeness of all orders of $(\mathcal{A}_i)_{i \in I}$.

2.6.2 Surfaced permutations

In this section, we will introduce the combinatorial framework for surfaced free probability. The objects in this theory are called surfaced permutations, motivated by the appendix [CMSS07]. In principle, surfaced permutations are only partitioned permutations with the additional data of a genus on each of its blocks. In order to develop a moment-cumulant formalism using these objects, we need to make this new information compatible with the multiplicative structure. The motivation for these objects comes from the extension in Section 2.2, where the multiplicative functions have values in $\mathbb{C}[\![\hbar]\!]$. The idea is that rather than storing the information of a function in a generating series, we distinguish every coefficient $[h^{|(\mathcal{V},\pi)|+2g}]f(\mathcal{V}, \pi)$ by a value on surfaced permutation with the prescribed genus. With this new point of view, we are able to introduce the notion of (g, n)-freeness and study its properties and applications.

Definition 2.6.10.

i) A *surfaced permutation* of $[d]$ is a triple $(\mathcal{V}, \pi, g)$ where $(\mathcal{V}, \pi) \in \mathcal{PS}(d)$ and $g\colon \mathcal{V} \to \mathbb{N}$ is a function. We denote by $\mathbb{PS}(d)$ the set of surfaced permutations of $[d]$, and set $\mathbb{PS} := \bigcup_{d \geq 1} \mathbb{PS}(d)$.

ii) The colength of $(\mathcal{A}, \alpha, g) \in \mathbb{PS}(d)$ is defined by

$$|(\mathcal{A}, \alpha, g)| := |(\mathcal{A}, \alpha)| + \sum_{A \in \mathcal{A}} 2g(A).$$

Definition 2.6.11.
Let $(\mathcal{V}, \pi, g)$, $(\mathcal{W}, \sigma, h) \in \mathbb{PS}$ be surfaced permutations.

i) We define the extended product of $(\mathcal{V}, \pi, g)$,$(\mathcal{W}, \sigma, h) \in \mathbb{PS}(d)$ by

$$(\mathcal{V}, \pi, g) \odot_{\mathbb{PS}} (\mathcal{W}, \sigma, h) = (\mathcal{V} \vee \mathcal{W}, \pi \circ \sigma, k),$$

in which the genus function on a block $B \in \mathcal{V} \vee \mathcal{W}$ is given by

$$k(B) := \frac{|(\mathcal{V}_{|B}, \pi_{|B}, g|_B)| + |(\mathcal{W}|_B, \sigma|_B, h|_B)| - |(\mathcal{V} \vee \mathcal{W}|_B, \pi \circ \sigma|_B)|}{2} \,. \qquad (2.6.2)$$

Here, if $\mathcal{D} = \{D_1, \ldots, D_l\} \in \mathcal{P}(d)$ and $B \subseteq [d]$, the notation $\mathcal{D}|_C$ stands for $\{D_1 \cap B, \ldots, D_l \cap B\}$ from which one removes the elements which are empty sets.

ii) We define the (planar) product of $(\mathcal{V}, \pi, g)$, $(\mathcal{W}, \sigma, h) \in \mathbb{PS}(d)$ by

$$(\mathcal{V}, \pi, g) \cdot (\mathcal{W}, \sigma, h) = \begin{cases} (\mathcal{V} \vee \mathcal{W}, \pi\sigma, k) & \text{if } |(\mathcal{V}, \pi)| + |(\mathcal{W}, \sigma)| = |(\mathcal{V} \vee \mathcal{W}, \pi\sigma)|, \\ 0 & \text{otherwise,} \end{cases}$$

where k is defined by (2.6.2).

iii) Let $f, g \colon \mathbb{PS}(d) \to \mathbb{C}$ be functions. Then we define the convolution of f and g by $f \circledast_{\mathbb{PS}} g \colon \mathbb{PS}(d) \to \mathbb{C}$,

$$(f \circledast_{\mathbb{PS}} g)(\mathcal{U}, \gamma, k) = \sum_{(\mathcal{V},\pi,g) \odot_{\mathbb{PS}} (\mathcal{W},\sigma,h) = (\mathcal{U},\gamma,k)} f(\mathcal{V}, \pi, g)\; g(\mathcal{W}, \sigma, h)\,. \qquad (2.6.3)$$

Similarly we define the planar convolution where we replace $\odot$ by $\cdot$ in (2.6.3).

iv) We call a function $f \colon \mathbb{PS} \to \mathbb{C}$ multiplicative if the values of f only depend on the conjugacy classes, that is

$$f(\mathbf{1}_d, \pi, g) = f(\mathbf{1}_d, \tau^{-1}\pi\tau, g \circ \tau)$$

for any $d \in \mathbb{N}$ and $\tau \in S(d)$ and we have

$$f(\mathcal{V}, \pi, g) = \prod_{B \in \mathcal{V}} f(\mathbf{1}_B, \pi|_B, g|_B).$$

Remark 2.6.12.
Let us propose another interpretation of the genus (2.6.2): let $(\mathcal{V}, \pi, g)$, $(\mathcal{W}, \sigma, h)$ be surfaced permutations. If we expand the colength we have

$$k(B) = \frac{|(\mathcal{V}|_B, \pi|_B)| + |(\mathcal{W}|_B, \sigma|_B)| - |(\mathcal{V} \vee \mathcal{W}|_B, \pi \circ \sigma)|_B|}{2} + \sum_{\substack{C \in \mathcal{V} \\ C \subseteq B}} g(C) + \sum_{\substack{C \in \mathcal{W} \\ C \subseteq B}} h(C). \qquad (2.6.4)$$

Then the last two sums give the genera of the blocks of $\mathcal{V}$ and $\mathcal{W}$ that contribute towards $B \in \mathcal{V} \vee \mathcal{W}$. Furthermore, having (2.1.7) in mind, the first term is the genus that gets generated by a possible non-planar multiplication. This non-planar contribution only allows integer values. We will explain this idea in Example 2.6.14.

Remark 2.6.13.
The reason for introducing the planar product on $\mathbb{PS}$ is that we later obtain the theory of [CMSS07] as the special case of *planar functions* on $\mathbb{PS}$ together with the planar product. More importantly, we will discuss the relation to infinitesimal freeness, there is is important the genus is only created via the partitions and not via a non-planar product.

Example 2.6.14.
We can reconsider Example 2.1.22: consider the two permutations $\pi_1 = (14)(235)$, $\pi_2 = (15)(2)(34)$ and the product of partitioned permutations

$$(\mathbf{1}_5, \pi_1)(\mathbf{0}_{\pi_2}, \pi_2) = (\mathbf{1}_5, \pi_1, (123)(45)).$$

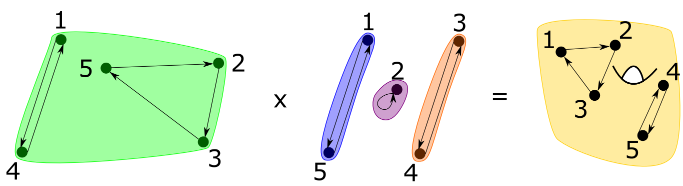
In the context of surfaced permutations we have $(\mathbf{1}_5, \pi_1, g), (\mathbf{0}_{\pi_2}, \pi_2, h)$, with $g \equiv 0, h \equiv 0$ and hence

$$(\mathbf{1}_5, \pi_1, g) \odot_{\mathbb{PS}} (\mathbf{0}_{\pi_2}, \pi_2, h) = (\mathbf{1}_5, \pi_1, (123)(45), k),$$

with

$$\begin{aligned} k(\mathbf{1}_5) &= \underbrace{g(\mathbf{1}_5)}_{=0} + \underbrace{h(\{1,5\})}_{=0} + \underbrace{h(\{2\})}_{=0} + \underbrace{h(\{3,4\})}_{=0} + \frac{|(\mathbf{1}_5, \pi_1)| + |(\mathbf{0}_{\pi_2}, \pi_2)| - |(\mathbf{1}_5, \pi_1\pi_2)|}{2} \\ &= \frac{7-5}{2} \\ &= 1, \end{aligned}$$

i.e. the genus we obtain is indeed coming from taking the product. We recall the picture for a better understanding:



Product of surfaced permutations.

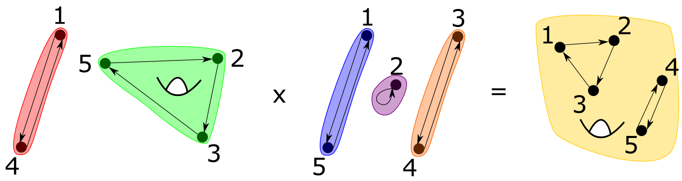
On the other hand, we may alter the first example in Example 2.1.22. We consider

$$(\mathbf{0}_{\pi_1}, \pi_1, \tilde{g})(\mathbf{0}_{\pi_2}, \pi_2, \tilde{h}) = (\mathbf{1}_5, (123)(45), \tilde{h})$$

by taking $\tilde{g}(\{1,4\}) = 0$, $\tilde{g}(\{2,3,5\}) = 1$ and $\tilde{h} \equiv 0$ then

$$\begin{aligned} k(\mathbf{1}_5) &= \underbrace{\tilde{g}(\{1,4\})}_{=0} + \underbrace{\tilde{g}(\{2,3,5\})}_{=1} + \underbrace{h(\{1,5\})}_{=0} + \underbrace{h(\{2\})}_{=0} \underbrace{h(\{3,4\})}_{=0} + \\ &\quad + \underbrace{\frac{|(\mathbf{0}_{\pi_1}, \pi_1)| + |(\mathbf{0}_{\pi_2}, \pi_2)| - |(\mathbf{1}_5, \pi_1\pi_2)|}{2}}_{=0} \\ &= 1 \end{aligned}$$

and the contribution towards the genus of the product is due to the genus we introduced to $(\mathbf{0}_{\pi_1}, \pi_1)$ via $\tilde{g}$. The following diagram visualizes the situation.



Product of surfaced permutations.

Remark 2.6.15.

i) Consider a multiplicative function $f\colon \mathbb{PS} \to \mathbb{C}$, then f induces a multiplicative function $\widehat{f}\colon \mathcal{PS} \to \mathbb{C}[\![\hbar]\!]$ via

$$\widehat{f}(\mathcal{V}, \pi) = \sum_{g\colon \mathcal{V} \to \mathbb{N}} \hbar^{|(\mathcal{V},\pi,g)|} f(\mathcal{V}, \pi, g).$$

ii) There is a natural injection $\iota\colon \mathcal{PS} \to \mathbb{PS}$, $\iota(\mathcal{V}, \pi) = (\mathcal{V}, \pi, 0)$. A function $h\colon \mathcal{PS} \to \mathbb{C}$ can be extended to a function $\iota_* h$ on $\mathbb{PS}$, by $\iota_* h(\mathcal{V}, \pi, 0) = h(\mathcal{V}, \pi)$ and setting it to be zero outside $\iota(\mathcal{PS})$. Furthermore, if h is multiplicative then $\iota_* h$ is also multiplicative.

Lemma 2.6.16.

Let $d \in \mathbb{N}$ and $(\mathcal{V}, \pi, g)$,$(\mathcal{W}, \sigma, h) \in \mathbb{PS}(d)$ be surfaced permutations.

i) If k is given by (2.6.2) then

$$|(\mathcal{V} \vee \mathcal{W}, \pi\sigma, k)| = |(\mathcal{V}, \pi, g)| + |(\mathcal{W}, \sigma, h)|.$$

ii) Let $f, g\colon \mathbb{PS} \to \mathbb{C}$ be functions. Then with the notation from Remark 2.6.15 we have

$$\widehat{f \circledast_{\mathbb{PS}} g} = \widehat{f} \circledast \widehat{g}.$$

Thus we may write $\circledast$ instead $\circledast_{\mathbb{PS}}$.

Proof. For i) we compute

$$\begin{aligned}
|(\mathcal{V} \vee \mathcal{W}, \pi\sigma, k)| &= |(\mathcal{V} \vee \mathcal{W}, \pi\sigma)| + \sum_{B \in \mathcal{V} \vee \mathcal{W}} k(B) \\
&\overset{(2.1.6)}{=} \sum_{B \in \mathcal{V} \vee \mathcal{W}} |(\mathcal{V} \vee \mathcal{W}|_B, \pi\sigma|_B)| + k(B) \\
&\overset{(2.6.4)}{=} \sum_{B \in \mathcal{V} \vee \mathcal{W}} \sum_{\substack{C \subseteq B \\ C \in \mathcal{V}}} g(C) + \sum_{\substack{C \subseteq B \\ C \in \mathcal{W}}} h(C) + |(\mathcal{V}|_B, \pi|_B)| + |(\mathcal{W}|_B, \sigma|_B)|
\end{aligned}$$

$$= \Big(| \sum_{C \in \mathcal{V}} (\mathcal{V}|_C, \pi|_C)| + g(C)) \Big) + \Big(| \sum_{C \in \mathcal{W}} |(\mathcal{W}|_C, \sigma|_C)| + h(C) \Big)$$
$$= |(\mathcal{V}, \pi, g)| + |(\mathcal{W}, \sigma, h)|,$$

where in the second to last equality we used that every $C \in \mathcal{V}$ and $C \in \mathcal{W}$ is contained in a $B \in \mathcal{V} \vee \mathcal{W}$ and for every $(\mathcal{U}, \gamma) \in \mathcal{PS}$ its colength is additive w.r.t. the blocks of $\mathcal{U}$. In particular, we have used

$$|(\mathcal{V}|_B, \pi|_B)| = \sum_{\substack{C \subset B \\ C \in \mathcal{V}}} |(\mathcal{V}|_C, \pi|_C)|.$$

For ii) we use i), we have

$$\begin{aligned}
\widehat{f \circledast_{\mathbb{PS}} g}(\mathcal{U}, \gamma) &= \sum_{k \colon \mathcal{U} \to \mathbb{N}} \hbar^{|(\mathcal{U},\gamma,k)|} f \circledast_{\mathbb{PS}} g(\mathcal{U}, \gamma, k) \\
&= \sum_{k \colon \mathcal{U} \to \mathbb{N}} \hbar^{|(\mathcal{U},\gamma,k)|} \sum_{(\mathcal{V},\pi,h) \odot_{\mathbb{PS}} (\mathcal{W},\sigma,j) = (\mathcal{U},\gamma,k)} f(\mathcal{V}, \pi, h) g(\mathcal{W}, \sigma, j) \\
&\overset{i)}{=} \sum_{k \colon \mathcal{U} \to \mathbb{N}} \sum_{(\mathcal{V},\pi,h) \odot_{\mathbb{PS}} (\mathcal{W},\sigma,j) = (\mathcal{U},\gamma,k)} \hbar^{|(\mathcal{V},\pi,h)|} f(\mathcal{V}, \pi, h) \hbar^{|(\mathcal{W},\sigma,j)|} g(\mathcal{W}, \sigma, j) \\
&= \sum_{(\mathcal{V},\pi) \odot_{\mathbb{PS}} (\mathcal{W},\sigma) = (\mathcal{U},\gamma)} \sum_{h \colon \mathcal{U} \to \mathbb{N}} \hbar^{|(\mathcal{V},\pi,h)|} f(\mathcal{V}, \pi, h) \sum_{j \colon \mathcal{U} \to \mathbb{N}} \hbar^{|(\mathcal{W},\sigma,j)|} g(\mathcal{W}, \sigma, j) \\
&= \sum_{(\mathcal{V},\pi) \odot (\mathcal{W},\sigma) = (\mathcal{U},\gamma)} \widehat{f}(\mathcal{V}, \pi) \widehat{g}(\mathcal{W}, \sigma) \\
&= \widehat{f} \circledast \widehat{g},
\end{aligned}$$

where we used that we may sum over all possible $j \colon \mathcal{V} \to \mathbb{C}$ and $h \colon \mathcal{W} \to \mathbb{C}$ instead of all $k \colon \mathcal{V} \vee \mathcal{W} \to \mathbb{C}$. □

Definition 2.6.17.
We define the following versions of the delta, zeta and Möbius functions:

$$\mathbb{\delta} := \iota_* \delta, \quad \mathbb{\zeta} := \iota_* \zeta$$

and $\mathbb{\mu}$ is the inverse of $\mathbb{\zeta}$ w.r.t. $\circledast_{\mathbb{PS}}$; $\mathbb{\mu}$ is characterized by

$$\widehat{\mathbb{\mu}} = \mu_\hbar.$$

Remark 2.6.18.
We say a function $f \colon \mathbb{PS} \to \mathbb{C}$ is *planar* if for every $d \geq 1$ and $(\mathcal{V}, \pi, g) \in \mathbb{PS}(d)$ we have $f(\mathcal{V}, \pi, g) = 0$ whenever there is a block $B \in \mathcal{V}$ with $g(B) > 0$. Then the planar functions together with the planar product $\cdot$ and planar convolution $*$ recover the theory of [CMSS07].

2.6.3 Relation to infinitesimal freeness

We can recover features from Section 1.1.4 by allowing the genus functions g in $(\mathcal{V}, \pi, g)$ to take values in $\frac{1}{2}\mathbb{Z}_{\geq 0} = \{0, \frac{1}{2}, 1, \frac{3}{2}, 2 \ldots\}$. We denote these extended surfaced permutations by $\mathbb{PS}^{\frac{g}{2}}(d)$. Note, we can repeat the definitions and results from the last section and everything works as in the case of $\mathbb{PS}$. The only thing that changes is that the genus now has possibly positive half integer values. We recover multiplicative functions on $\mathbb{PS}$ from multiplicative functions on $\mathbb{PS}^{\frac{g}{2}}$ similar to recovering functions on $\mathcal{PS}$ from $\mathbb{PS}$.

Remark 2.6.19.
We say a function $f\colon \mathbb{PS}^{\frac{g}{2}} \to \mathbb{C}$ is *even* if for every $d \geq 1$ and $(\mathcal{V}, \pi, g) \in \mathbb{PS}^{\frac{g}{2}}(d)$ we have $f(\mathcal{V}, \pi, g) = 0$ whenever $g(B) \notin \mathbb{N}$. Then the even functions together with the extended product $\odot$ and extended convolution $\circledast$ recover the multiplicative functions on $\mathbb{PS}$.

Recall that in infinitesimal freeness we have two functionals φ, φ'. We can encode the same information in the genus zero part plus the genus $\frac{1}{2}$ part in our theory of surfaced permutations.

Definition 2.6.20.
We say that two multiplicative functions $f_1, f_2\colon \mathbb{PS}^{\frac{g}{2}} \to \mathbb{C}$ agree infinitesimally if their values coincide on $(\mathcal{A}, \alpha, g)$ for any $g\colon \mathcal{A} \to \frac{1}{2}\mathbb{Z}_{\geq 0}$ such that $\sum_{A \in \mathcal{A}} g(A) \leq \frac{1}{2}$. In that case, we write $f_1 \approx f_2$.

The extraction of leading order in Lemma 2.2.8 can be upgraded to include the first sub-leading order (encoded in genus $\frac{1}{2}$):

Lemma 2.6.21.
Let $\phi_1, \phi_2\colon \mathbb{PS}^{\frac{g}{2}} \to \mathbb{C}$ be two multiplicative functions. The relation $\phi_1 = \zeta \circledast \phi_2$ implies the infinitesimal agreement $\phi_1 \approx \zeta * \phi_2$.

Proof. Same as in Lemma 2.2.8, taking into account that the creation of genus occurs by integer units only (see Remark 2.6.12). □

Remark 2.6.22.
This has an equivalent presentation via the ring of dual numbers $\mathbb{C}' = \mathbb{C}[\![\hbar]\!]/(\hbar^2)$. We write for $i = 1, 2$

$$\widehat{f}_i(\mathcal{A}, \alpha) = \hbar^{|(\mathcal{A},\alpha)|}\big(\phi_i(\mathcal{A}, \alpha, 0) + \hbar f_i'(\mathcal{A}, \alpha) + o(\hbar)\big)\,,$$

and define multiplicative functions

$${}^\flat f_i\colon \mathcal{PS} \to \mathbb{C}', \quad {}^\flat f_i(\mathcal{A}, \alpha) = f_i(\mathcal{A}, \alpha, 0) + \hbar f_i'(\mathcal{A}, \alpha)\,.$$

Then, the relation $f_1 = \zeta \circledast f_2$ between $\mathbb{C}$-valued functions on surfaced permutations implies the relation ${}^\flat f_1 = \zeta * ({}^\flat f_2)$ between $\mathbb{C}'$-valued functions on partitioned permutations. Observe that $f_i(-,-,0)$ is a multiplicative function on $\mathcal{PS}$, but f_i' is not. Instead, we have

$$f_i'(\mathcal{A},\alpha) = \sum_{A\in\mathcal{A}} f_i'(\mathbf{1}_{\#A},\alpha_{|A}) \prod_{\substack{A'\in\mathcal{A}\\ A'\neq A}} f_i(\mathbf{1}_{\#A'},\alpha_{|A'},0)\,.$$

Note the similarity to the definition of $\partial\varphi$ in Definition 1.1.30.

With only a little extra work, we can put Theorem 2.4.1 into the context of infinitesimal freeness. Therefore, let us introduce the following set of trees.

Definition 2.6.23.
Let $\mathcal{G}'_{0,n}$ be the set of bicoloured trees as in Theorem 2.4.1, except that they must contain one special black vertex, whose corresponding hyper-edge I' may be univalent. In $\mathcal{G}'_{0,n}(\mathbf{r}+1)$, we require the i-th vertex to have valency r_i+1.

When we again understand multiplicative functions in the sense of (2.6.1) or transition to a decorated version $\mathbb{PS}^{\frac{g}{2}}(\mathcal{A})$ of $\mathbb{PS}^{\frac{g}{2}}$, then we have the following version of Theorem 2.4.1 for infinitesimal freeness.

Theorem 2.6.24.
Let $\varphi,\varphi',\kappa,\kappa'\colon\mathcal{PS}\to\mathbb{C}$ be functions so that

$$^\flat\varphi = \varphi+\hbar\varphi'\colon\mathcal{PS}\to\mathbb{C}',\qquad {}^\flat\kappa=\kappa+\hbar\kappa'\colon\mathcal{PS}\to\mathbb{C}'$$

are multiplicative. Introduce the n-point functions

$$\begin{aligned} G^{\varphi}_{0,n}(x_1,\ldots,x_n) &= \sum_{r_1,\ldots,r_n>0}\varphi(\mathbf{1}_{r_1+\cdots+r_n},\pi_{\lambda(\mathbf{k})})\prod_{i=1}^n x_i^{r_i},\\ G^{\varphi}_{\frac12,n}(x_1,\ldots,x_n) &= \sum_{r_1,\ldots,r_n>0}\varphi'(\mathbf{1}_{r_1+\cdots+r_n},\pi_{\lambda(\mathbf{k})})\prod_{i=1}^n x_i^{r_i}\,, \end{aligned}\tag{2.6.5}$$

and likewise $G^{\kappa}_{\frac12,n}$. Suppose that we have $^\flat\varphi=\zeta*{}^\flat\kappa$. Then, the genus 0 functional relations given in Theorem 2.4.1 hold, and with the same substitution and notations we have for any $n\geq1$

$$G^{\varphi}_{\frac12,n}(x_1,\ldots,x_n) = \sum_{r_1,\ldots,r_n\geq0}\prod_{i=1}^n \vec{\mathsf{O}}^{\kappa}_{r_i}(w_i) \sum_{T\in\mathcal{G}'_{0,n}(\mathbf{r}+1)} G^{\kappa}_{\frac12,\#I'}(w_{I'}) \prod_{I\neq I'}{}' G^{\kappa}_{0,\#I}(w_I)\,.\tag{2.6.6}$$

Proof. First, we put our data into the context of Section 2.2. Thus, we define multiplicative functions $\varphi_\hbar,\kappa_\hbar\colon\mathcal{PS}\to\mathbb{C}[\![\hbar]\!]$ by

$$\kappa_\hbar(\mathbf{1}_d,\pi)=\hbar^{|(\mathbf{1}_d,\pi)|}\big(\kappa(\mathbf{1}_d,\pi)+\hbar\kappa'(\mathbf{1}_d,\pi)\big),\quad \varphi_\hbar=\zeta_\hbar\circledast\kappa_\hbar(\mathbf{1}_d,\pi).\tag{2.6.7}$$

Let $G^{\varphi_\hbar}_n$ and $G^{\kappa_\hbar}_n$ be the n-point functions corresponding to $\varphi_\hbar$ and $\kappa_\hbar$, they have an $\hbar$ expansion of the form in Remark 2.3.4 with half-integer g allowed. Now with that notation and assumptions we know that $G_n^{\varphi_\hbar}$ and $G_n^{\kappa_\hbar}$ satisfy Theorem 2.4.8. To prove the relation, we need to rearrange the formula. First we write

$$\begin{aligned} G_1^{\kappa_\hbar}(w) \sum_{g\in\mathbb{Z}_{\geq 0}} \hbar^{2g-1}\, G_{g,1}^{\kappa_\hbar}(w) + \sum_{g\in\frac{1}{2}+\mathbb{Z}_{\geq 0}} \hbar^{2g-1} G_{g,1}^{\kappa_\hbar}(w) \\ =: G_{\mathrm{even},1}^{\kappa_\hbar}(w) + G_{\mathrm{odd},1}^{\kappa_\hbar}(w). \end{aligned}$$

Then in the definition of the operator O^κ in we may rewrite the last term:

$$\begin{aligned} &\frac{\exp\big(\hbar u_i \varsigma(\hbar u_i w_i \partial_{w_i})(G_1^\kappa(w_i) - \hbar^{-1}) - u_i(G_{0,1}^\kappa(w_i) - 1)\big)}{\hbar u_i\, \varsigma(\hbar u_i)} \\ &= \frac{\exp\big(\hbar u_i \varsigma(\hbar u_i w_i \partial_{w_i})(G_{\mathrm{even},1}^\kappa(w_i) - \hbar^{-1}) - u_i(G_{0,1}^\kappa(w_i) - 1)\big)}{\hbar u_i\, \varsigma(\hbar u_i)} \\ &\quad \exp\big(\hbar u_i \varsigma(\hbar u_i w_i \partial_{w_i})(G_{\mathrm{odd},1}^\kappa(w_i))\big). \end{aligned}$$

The extra factor involving $G_{\mathrm{odd},1}^\kappa(w_i)$ will be incorporated into the weight $\mathsf{c}^\kappa(u_I, w_I)$ by the following changes. We define the set $\mathcal{G}_n'$ of bicoloured graphs like in Definition 2.4.4 but now allowing univalent black vertices and the i-th white vertex weights receive additionally

$$\mathsf{c}^\kappa(u_i, w_i) = \hbar u_i \varsigma(\hbar u_i w_i \partial_{w_i}) G_{\mathrm{odd},1}^\kappa(w_i)$$

for any univalent black vertex connected to the i-th white vertex. With these changes, the relation of Theorem 2.4.8 holds. When extracting the genus $\frac{1}{2}$ part of the formula, only the leading power in $\hbar$ of each of the weights and only the trees in $\mathcal{G}_n'$, in which exactly one factor of $G_{\frac{1}{2},\#I}$ is picked, will contribute. This is because the series ς where $\hbar$ occurs is even, corresponding to the fact that monotone Hurwitz numbers have integer genus. By the definition of the n-point function and (2.6.7) and the fact that the zeta function can only create even genus (cf. proof of Lemma 2.6.21), $G_{0,n}^{\varphi_\hbar}$ and $G_{\frac{1}{2},n}^{\varphi_\hbar}$ must agree with the generating series (2.6.5). Thus, we obtain the claimed formulas. □

We recover the known functional relation in the setting of infinitesimal freeness via the case $n = 1$.

Corollary 2.6.25.
We have

$$G_{\frac{1}{2},1}^\varphi(x) = P^\kappa(w)\, G_{\frac{1}{2},1}^\kappa(w)\,, \quad x = \frac{w}{G_{0,1}^\kappa(w)}\,, \qquad P^\kappa(w) = \frac{\mathrm{d}\ln w}{\mathrm{d}\ln x}\,. \tag{2.6.8}$$

Equivalently we can express the formula in terms of differentials, we have

$$G_{\frac{1}{2},1}^\varphi(x)\frac{\mathrm{d}x}{x} = G_{\frac{1}{2},1}^\kappa(w)\frac{\mathrm{d}w}{w}\,.$$

Proof. We specialise (2.6.24) to $n = 1$. The set $\mathcal{G}_{0,1}'(r+1)$ is non-empty only for $r = 0$ and then contains a single tree, namely the white vertex connected to the special vertex. We already computed $\vec{\mathsf{O}}_0 = P^\kappa(w) = \frac{\mathrm{d}w}{\mathrm{d}x}$ in Example 2.4.2 and we obtain

$$G_{\frac{1}{2},1}(X) = P^\kappa(w) G_{\frac{1}{2},1}^\kappa(w). \qquad \square$$

2.6.4 Freeness in higher genus

In this section, we want to introduce a notion of freeness for our extended combinatorial setting and show that it is a sensible extension of higher order freeness. Since freeness is defined via vanishing of mixed cumulants, we wish to work with moments and cumulants in several variables. Therefore, we introduce the set $\mathbb{PS}(\mathcal{A})$ or $\mathbb{PS}^{\frac{g}{2}}(\mathcal{A})$ of surfaced permutations decorated by elements of an associative algebra $\mathcal{A}$ (cf. Section 2.6.1). Let us start by introducing a central notion, the surfaced probability spaces.

Definition 2.6.26.
A (noncommutative) *surfaced probability space* (SPS) is the data $(\mathcal{A}, \boldsymbol{\varphi})$ consisting of a unital associative algebra $\mathcal{A}$ over $\mathbb{C}$ and a family $\boldsymbol{\varphi} = \big(\varphi_{g,n} \ : \ g \in \frac{1}{2}\mathbb{Z}_{\geq 0}, \ n \in \mathbb{Z}_{>0}\big)$ of symmetric n-linear forms on $\mathcal{A}$ that are tracial in its n arguments such that $\varphi_{0,1}(1) = 1$ and $\varphi_{g,n}(1, a_2, \dots, a_n) = 0$ for any $(g, n) \neq (0, 1)$ and $a_2, \dots, a_n \in \mathcal{A}$.

We can put the setting of surfaced probability spaces into the combinatorial framework of multiplicative functions on the set of surfaced permutations, similar to Definition 2.6.3.

Definition 2.6.27.
Let $(\mathcal{A}, \boldsymbol{\varphi})$ be a SPS.

i) We define the *moment function* to be the multiplicative function $\varphi\colon \mathbb{PS}^{\frac{g}{2}}(\mathcal{A}) \to \mathbb{C}$ given by

$$\varphi(\mathbf{1}_d, \gamma_\lambda, g)[a_1, \dots, a_d] = \varphi_{g(\mathbf{1}_d),n}\Bigg(\prod_{j=1}^{\lambda_1} a_j, \prod_{j=1}^{\lambda_2} a_{\lambda_1+j}, \dots, \prod_{j=1}^{\lambda_n} a_{\lambda_1+\dots+\lambda_{n-1}+j}\Bigg) \tag{2.6.9}$$

for any $d \in \mathbb{N}$ and any partition $\lambda \vdash d$.

ii) We define the *cumulant function* to be the multiplicative function $\kappa\colon \mathbb{PS}^{\frac{g}{2}}(\mathcal{A}) \to \mathbb{C}$ given by

$$\kappa = \mu \circledast \varphi,$$

moreover we define the *higher order cumulants* by

$$\kappa_{g;r_1,\dots,r_n}(a_1, \dots, a_n) \coloneqq \kappa(\mathbf{1}_d, \gamma_{r_1,\dots,r_n}, g)[a_1, \dots, a_d]$$

for any $n, r_1, \dots, r_n \in \mathbb{N}$ and $d = r_1 + \dots + r_n$.

Remark 2.6.28.
By the extension to higher genus, we added new layers to the moment-cumulant relation. In order to understand freeness in higher genus, we need to unwind the dependencies. We overcome this problem by introducing the so-called *type* of a surfaced permutation.

Definition 2.6.29.
Denote $\mathsf{Typ} = \frac{1}{2}\mathbb{Z}_{\geq 0} \times \mathbb{Z}_{>0}$.

i) We define a partial order[2] on $\mathsf{Typ} = \frac{1}{2}\mathbb{Z}_{\geq 0} \times \mathbb{Z}_{>0}$ by declaring $(g,n) \preceq (g',n')$ when $g \leq g'$ and $g + n \leq g' + n'$. We also define
$$\overline{\mathsf{Typ}} = \mathsf{Typ} \cup \Big(\bigcup_{g \in \frac{1}{2}\mathbb{Z}_{\geq 0}} \{(g,\infty)\} \Big) \cup \{(\infty,\infty)\}\,,$$
on which the partial order relation extends naturally, for instance, if $g, g' \in \frac{1}{2}\mathbb{Z}_{\geq 0}$ with $g < g'$ and $n \in \mathbb{Z}_{>0}$
$$(g,n) \prec (g,\infty) \prec (g',\infty)\,, \qquad (g,\infty) \prec (g',n)\,.$$

ii) The *type* of a surfaced permutation $(\mathcal{A},\alpha,g)$ is $(\mathsf{g},\mathsf{n}) \in \mathsf{Typ}$ where n is the number of cycles of α and $\mathsf{g} = \sum_{A \in \mathcal{A}} g(A)$.

Lemma 2.6.30.
Let $(g_0,n_0) \in \overline{\mathsf{Typ}}$. The knowledge of $\varphi_{g,n}$ for all $(g,n) \preceq (g_0,n_0)$ is equivalent to the knowledge of $\kappa_{g;k_1,\ldots,k_n}$ for all $(g,n) \preceq (g_0,n_0)$ and $k_1,\ldots,k_n > 0$.

Proof. This can be extracted by elementary means from the moment-cumulant relations Definition 2.6.27 ii), but we propose here to read it off from Theorem 2.4.8. By multilinearity, it is enough to prove the claim for the evaluations of $\varphi_{g,n}$ and $\kappa_{g;k_1,\ldots,k_n}$ on tuples of the form $(a,\ldots,a)$. The claim clearly holds for $(g,n) = (0,1)$, and for $(\frac{1}{2},1)$ (see Corollary 2.6.25). Now take $a \in \mathcal{A}$ and $(g,n) \in \mathsf{Typ}$ with $2g-2+n > 0$: we consider (2.4.5) expressing the generating functions $G^{\varphi}_{g,n}$ in terms of $G^{\kappa}_{g',n'}$. The summand associated to a graph $\Gamma \in \mathcal{G}_n$ contains contributions from the hyper-edges involving $G^{\vee}_{g_I,\#I}$ for some $g_I \in \frac{1}{2}\mathbb{Z}_{\geq 0}$ with the exception $G^{\kappa}_{0,1}$, and contributions from the i-th vertex which give either a 1 or a product $\prod_{p=1}^{q_i} G^{\kappa}_{g_{i,p},1}$ with $q_i > 0$ and $g_{i,p} \in \frac{1}{2}\mathbb{Z}_{>0}$. The extraction of the correct power of $\hbar$ in (2.4.5) shows that
$$2g - 2 + n = \sum_{i=1}^{n} \Big(-1 + \sum_{p=1}^{q_i} 2g_{i,p} \Big) + \sum_{I \in \mathcal{I}(\Gamma)} 2(g_I - 1 + \#I)\,.$$
In other words:
$$g - 1 + n = \sum_{i=1}^{n} \sum_{p=1}^{q_i} g_{i,p} + \sum_{I \in \mathcal{I}(\Gamma)} (g_I - 1 + \#I)\,. \tag{2.6.10}$$
Besides, as Γ is connected, we must have $\sum_{I \in \mathcal{I}(\Gamma)} (-1 + \#I) \geq n - 1$ and thus
$$g \geq \sum_{i=1}^{n} \sum_{p=1}^{q_i} g_{i,p} + \sum_{I \in \mathcal{I}(\Gamma)} g_I\,. \tag{2.6.11}$$

[2] This is not a total order. For instance, $(1,1)$ and $(0,3)$ are not comparable. More generally, two distinct elements $(g,n),(g',n')$ having $2g-2+n = 2g'-2+n'$ are not comparable.

As in the right-hand side of (2.6.10)-(2.6.11) all terms of the sums are nonnegative, we deduce that only $G^{\kappa}_{g',n'}$ with $g'+n' \leq g+n$ and $g' \leq g$ are involved in the sum over graphs, that is $(g',n') \preceq (g,n)$. Noticing that the correction term $\Delta^{\vee}_{g}$ in (2.4.8) only involves $G^{\kappa}_{g',1}$ for $g' \leq g$, we deduce that $G^{\varphi}_{g,n}$ is expressed as a function of $G^{\kappa}_{g',n'}$ with $(g',n') \preceq (g,n)$. ☐

Example 2.6.31.
We want to express $\varphi_{1;1}(a^2) = \varphi_{1;1}$, i.e. $(g,n) = (1,1)$ in terms of cumulants. We denote $(\mathbf{1}_2, \gamma) = (\mathbf{1}_2, (1,2))$, and for a factorization $(\mathbf{0}_\pi, \pi, g) \circledast (\mathcal{V}, \sigma, h)$ of $(\mathbf{1}_2, \gamma)$ we denote

$$\Omega = \frac{|\pi| + |(\mathcal{V},\sigma)| - |(\mathbf{1}_2,\gamma)|}{2}.$$

By (2.6.4) we must have $\Omega \leq 1$. Recall that Ω describes the genus coming from taking the product, but if the factors itself carry a genus it may contribute as well, thus we obtain the following factorizations:

Ω	$(\mathbf{0}_\pi, \pi)$	$(\mathcal{V}, \sigma)$
0	$(\{\{1\},\{2\}\}, e)$	$(\mathbf{1}_2, (12))$
	$(\mathbf{1}_2, (12)$	$(\{\{1\},\{2\}\}, e)$
1	$(\mathbf{1}_2, (12))$	$(\mathbf{1}_2, e)$

Factorizations of $(\mathbf{1}_2, \gamma)$.

In the $\Omega = 0$ cases, we need to assign a genus to the blocks of the factorizations. In the $\Omega = 1$ case, we have no choice but to assign 0 to all blocks. Thus we have

$$\varphi_{1;1} = \underbrace{\kappa_{1;2} + 2\kappa_{1;1}\kappa_{0;1} + \kappa^2_{\frac{1}{2},1}}_{\Omega=0} + \underbrace{\kappa_{0;1,1}}_{\Omega=1}.$$

Note that we have on the right-hand side the types $(1,1), (0,1), (\frac{1}{2},1), (0,2)$. These are exactly the types that satisfy $(g,n) \preceq (1,1)$.

$\overline{\mathsf{Typ}}$	κ
$(1,1)$	$\kappa_{1;2}, \kappa_{1,1}$
$(0,1)$	$\kappa_{0;1}$
$(\frac{1}{2},1)$	$\kappa_{\frac{1}{2};1}$
$(0,2)$	$\kappa_{0;1,1}$

Type of the κs.

Definition 2.6.32.
Let $(g_0,n_0) \in \overline{\mathsf{Typ}}$. A family $(\mathcal{X}_i)_{i\in I}$ of subsets of $\mathcal{A}$ is called (g_0,n_0)-*free* if for any $(g,n) \preceq (g_0,n_0)$, for any $d \geq 0$, any $(a_1,\ldots,a_d) \in \prod_{p=1}^{d} \mathcal{X}_{i(p)}$ and $r_1,\ldots,r_n > 0$ so that $r_1 + \cdots + r_n = d$, we have $\kappa_{g;r_1,\ldots,r_n}(a_1,\ldots,a_d) = 0$ whenever there exists $p,q \in [d]$ such that $i(p) \neq i(q)$.

Remark 2.6.33.
Voiculescu's freeness is $(0,1)$-freeness, the second-order freeness of [MS06] is $(0,2)$-freeness, the all-order freeness of [CMSS07] is $(0,\infty)$-freeness. Due to Lemma 2.6.21, along with Remark 2.6.22 or looking at the definition of the convolution $*$ on $\mathbb{PS}^{\frac{g}{2}}$, $(\frac{1}{2},1)$-freeness coincides with the notion of infinitesimal freeness of [FN10]. We note that it involves only the free cumulants of type $(g,n)=(0,1)$ and $(\frac{1}{2},1)$. Besides, the order k infinitesimal freeness of [Fév12] corresponds to $(1,0)$-freeness using multiplicative functions valued in the ring R of upper triangular Töplitz matrices of size $(k+1)$ (instead of $\mathbb{C}[\![\hbar]\!]/(\hbar^2)$, which corresponds to $k=1$).

In the following, we show that (g,n)-freeness is a reasonable notion, in particular we prove analogues of the properties of freeness in higher order; see [CMSS07, Section 7].

Lemma 2.6.34.
Let $(\mathcal{A},\varphi)$ be a SPS. Then $1_{\mathcal{A}}$ is $(\infty,1)$-free from any set in $\mathcal{A}$.

Proof. We need to show that

$$0=\kappa_{g;d}(1,a_2,\dots,a_d)=\kappa(\mathbf{1}_d,\gamma_d,g)[1,a_2,\dots,a_d]$$

for any $d>1$ and any $g\geq 0$. Here we abuse notation and write

$$g=g(\mathbf{1}_d),$$

since g is the constant function on the only block $[d]$ of $\mathbf{1}_d$, i.e. we identify it with its value. For $d\geq 1$ and $g=0$, and in particular for $d=2, g=0$ the assertion is immediate. Moreover, $\kappa_{1;1}(1_{\mathcal{A}})=\varphi_{1,1}(1_{\mathcal{A}})=0$. Then for any fixed $d>1$ and g we have

$$\begin{aligned}\varphi(\mathbf{1}_d,\gamma_d,g)[1,a_2,\dots,a_d] &= \zeta \circledast \kappa(\mathbf{1}_d,\gamma_d,g)[1,a_2,\dots,a_d]\\ &= \sum_{(\mathbf{0}_\pi,\pi,0)\odot(\mathcal{V},\sigma,h)=(\mathbf{1}_d,\gamma_d,g)} \kappa(\mathbf{1}_d,\sigma,h)[1,a_2,\dots,a_d]\\ &= \kappa(\mathbf{1}_d,\gamma_{r_1,\dots,r_n},g)[1,a_2,\dots,a_d]\\ &+ \sum_{\substack{(\mathbf{0}_\pi,\pi)\odot(\mathcal{V},\sigma)=(\mathbf{1}_d,\gamma_d,g)\\ |(\mathcal{V},\sigma)|<|(\mathbf{1}_d,\gamma_d)|}} \kappa(\mathbf{1}_d,\sigma,h)[1,a_2,\dots,a_d].\end{aligned}$$

We discuss the sum in the second summand. By the definition of the product of surfaced permutations, particularly (2.6.2),

$$\begin{aligned} g &= \sum_{B\in\mathcal{V}} h(B) + \frac{|\pi|+|(\mathcal{V},\sigma)|-|(\mathbf{1}_d,\gamma_{r_1,\dots,r_n})|}{2}\\ &=: h+\Omega.\end{aligned}$$

If $\Omega>0$ and $1\in B_1, h(B_1)>0$ then all the blocks in the summand on the right-hand side have genus $h(B)<g$ and in particular $h(B_1)<g$, thus these summands vanish by induction hypothesis on g. ☐

Proposition 2.6.35.
Let $(\mathcal{A}, \varphi)$ be a SPS. Then $1_{\mathcal{A}}$ is (∞, ∞) free from every set $\mathcal{X} \subseteq \mathcal{A}$, that is

$$\kappa_{g;r_1,\dots,r_n}(1_{\mathcal{A}}, a_2, \dots, a_d) = 0$$

for any $d \geq 2$, $g \in \frac{1}{2}\mathbb{Z}_{\geq 0}$ and $r_1, \dots, r_n$ with $d = r_1 + \cdots + r_n$ and any $a_2, \dots, a_d \in \mathcal{A}$. In particular, $1_{\mathcal{A}}$ is (g, n) free from every set for any choice of $g \in \mathbb{Z}_{\geq 0}, n \in \mathbb{N}$.

Proof. Let

$$g = g(\mathcal{V}) = \sum_{B \in \mathcal{V}} g(B).$$

We prove the assertion by induction on g. Note, for $g = 0$ and any $d \geq 2$, this is the result of [CMSS07]. Let $d = r_1 + \cdots + r_n$ and $g > 0$. Then we have

$$\begin{aligned}
\varphi_{g;r_1,\dots,r_n}[1_{\mathcal{A}}, a_2, \dots, a_d] &= \varphi(\mathbf{1}_d, \gamma_{r_1,\dots,r_n}, g)[1_{\mathcal{A}}, a_2, \dots, a_d] \\
&= \zeta \circledast \kappa(\mathbf{1}_d, \gamma_{r_1,\dots,r_n}, g)[1, a_2, \dots, a_d] \\
&= \sum_{(\mathbf{0}_\pi, \pi, 0) \odot (\mathcal{V}, \sigma, h) = (\mathbf{1}_d, \gamma_{r_1,\dots,r_n}, g)} \kappa(\mathcal{V}, \sigma, h)[1_{\mathcal{A}}, a_2, \dots, a_d].
\end{aligned} \tag{2.6.12}$$

To simplify the notation of the last summand, let us introduce

$$\mathbb{PS}_\zeta(d) = \{(\mathcal{V}, \sigma, h) \in \mathbb{PS}(r_1, \dots, r_n) \colon (\mathbf{0}_{\gamma\sigma^{-1}}, \gamma\sigma^{-1}, 0) \odot (\mathcal{V}, \sigma, h) = (\mathbf{1}_d, \gamma, g)\},$$

where we abbreviate $\gamma = \gamma_{r_1,\dots,r_n}$. We continue and denote for the factorizations in (2.6.12)

$$g = \sum_{B \in \mathcal{V}} h(B) + \frac{|\pi| + |(\mathcal{V}, \sigma)| - |(1_d, \gamma_{r_1,\dots,r_n})|}{2} \tag{2.6.13}$$

$$=: h + \Omega. \tag{2.6.14}$$

Thus

$$\begin{aligned}
\varphi_{g;r_1,\dots,r_n}[1_{\mathcal{A}}, a_2, \dots, a_d] &= \sum_{(\mathcal{V},\sigma) \in \mathbb{PS}_\zeta(r_1,\dots,r_n)} \sum_{\substack{h \colon \mathcal{V} \to \frac{1}{2}\mathbb{Z}_{\geq 0} \\ g = h + \Omega}} \kappa(\mathcal{V}, \sigma, h)[1_{\mathcal{A}}, a_2, \dots, a_d] \\
&= \sum_{\substack{h \colon \mathcal{V} \to \frac{1}{2}\mathbb{Z}_{\geq 0} \\ g = h + \Omega}} \kappa(\mathbf{1}_d, \gamma_{r_1,\dots,r_n}, h)[1_{\mathcal{A}}, a_2, \dots, a_d] \\
&\quad + \sum_{\substack{(\mathcal{V},\sigma) \in \mathbb{PS}_\zeta(r_1,\dots,r_n) \\ |(\mathcal{V},\pi)| < |(\mathbf{1}_d, \gamma_{r_1,\dots,r_n})|}} \sum_{\substack{h \colon \mathcal{V} \to \frac{1}{2}\mathbb{Z}_{\geq 0} \\ g = h + \Omega}} \kappa(\mathcal{V}, \sigma, h)[1_{\mathcal{A}}, a_2, \dots, a_d] \\
&= \kappa(\mathbf{1}_d, \gamma_{r_1,\dots,r_n}, g)[1_{\mathcal{A}}, a_2, \dots, a_d] \\
&\quad + \sum_{\substack{(\mathcal{V},\sigma) \in \mathbb{PS}_\zeta(r_1,\dots,r_n) \\ |(\mathcal{V},\pi)| < |(\mathbf{1}_d, \gamma_{r_1,\dots,r_n})|}} \sum_{\substack{h \colon \mathcal{V} \to \frac{1}{2}\mathbb{Z}_{\geq 0} \\ g = h + \Omega}} \kappa(\mathcal{V}, \sigma, h)[1_{\mathcal{A}}, a_2 \dots, a_d].
\end{aligned}$$

The last equality is due to the fact that we have no other choice but $h = g$ for $\kappa(1_d, \gamma_{r_1,\ldots,r_n}, h)$ by (2.6.13). We now investigate the remaining sum in the last equality. We denote the block of $\mathcal{V}$ containing 1 by B_1. Then we have the following possibilities for its genus:

- for $g > h(B_1) > 0$ the corresponding κ will vanish by assumption on g,
- for $h(B_1) = 0$ the corresponding κ will vanish unless $\#B_1 = 1$ by Proposition 2.6.7,
- for $h(B_1) = g$ we proceed as follows:

If $h(B_1) = g$ we have by (2.6.13)

$$g = h + \Omega \implies \Omega = 0.$$

Thus, the only contribution in the sum are those $(\mathcal{V}, \sigma)$ that come from planar factorizations of $(\mathbf{1}_d, \gamma_{r_1,\ldots r_n}, g)$, that is

$$0 = \Omega = \frac{|\pi| + |(\mathcal{V}, \sigma)| - |(1_d, \gamma_{r_1,\ldots,r_n})|}{2}.$$

In this case, we have a sum over all planar factorizations that assign g to the block B_1 and zero to all other blocks of $\mathcal{V}$, let us denote the corresponding genus function by g_1. Then our sum reduces to

$$\sum_{(\mathcal{V},\sigma)\in\mathcal{PS}_{NC}(r_1,\ldots,r_n)} \kappa(\mathcal{V}, \sigma, g_1).$$

If $B_1 \neq \mathbf{1}_d$ then the term vanishes by a induction on d and we are left with

$$\sum_{(\mathbf{1}_d,\sigma)\in\mathcal{PS}(r_1,\ldots,r_n)} \kappa(\mathcal{V}, \sigma, g_1). \tag{2.6.15}$$

Note that then we have

$$d + \#\sigma - 2 = |(\mathbf{1}_d, \sigma)| < |(\mathbf{1}_d, \gamma_{r_1,\ldots,r_n})| = d + n - 2 \iff \#\sigma < n.$$

Thus we have to show by induction on n that $\kappa(1_d, \gamma_{s_1,\ldots,s_l}, g) = 0$, for $s_1 + \cdots + s_l = d$ and $l < n$: For $l = 1, s_1 = d$, we have

$$\begin{aligned}\varphi(\mathbf{1}_d, \gamma_d, g)[1, a_2, \ldots, a_d] &= \kappa(\mathbf{1}_d, \gamma_d, g)[1, a_2, \ldots, a_d] \\ &\quad + \sum_{\substack{(\mathcal{V},\sigma)\in\mathbb{PS}_{\mathfrak{C}}(d) \\ |(\mathcal{V},\sigma)|<|(\mathbf{1}_d),\gamma_d)|}} \sum_{h\colon \mathcal{V}\to\frac{1}{2}\mathbb{Z}_{\geq 0}} \kappa(\mathcal{V}, \sigma, h)[1, a_2, \ldots, a_d].\end{aligned}$$

Let us denote the block containing 1 by $\tilde{B}_1$. By the exact same arguments as before we arrive at the possibilities:

- $\#\tilde{B}_1 = 1$ and $h(\tilde{B}_1) = 0$,

- $\mathcal{V} = \mathbf{1}_d$, where $h(\tilde{B}_1) = g$.

The latter implies $\sigma = \gamma_d$, which contradicts $|(\mathbf{1}_d, \sigma)| < |(\mathbf{1}_d, \gamma_d)|$. More precisely, it is the case we already took out of the sum. Thus, we are left with the first case. We can use $\kappa_{0;1}(1_{\mathcal{A}}) = 1$ and arrive at

$$\begin{aligned}\varphi(\mathbf{1}_d, \gamma_d, g)[1, a_2, \dots, a_d] &= \kappa(\mathbf{1}_d, \gamma_d, g)[1, a_2, \dots, a_d] \\ &\quad + \sum_{(\tilde{\mathcal{V}},\tilde{\sigma}))\in\mathbf{PS}_{\mathbb{C}}(d-1)} \sum_{h\colon \tilde{\mathcal{V}}\to\frac{1}{2}\mathbb{Z}_{\geq 0}} \kappa(\tilde{\mathcal{V}}, \tilde{\sigma}, h)[a_2, \dots, a_d] \\ &= \kappa(\mathbf{1}_d, \gamma_d, g)[1, a_2, \dots, a_d] \\ &\quad + \mathbb{C} \circledast \kappa(\mathbf{1}_{d-1}, \gamma_{d-1})[a_2, \dots, a_d] \\ &= \kappa(\mathbf{1}_d, \gamma_d, g)[1, a_2, \dots, a_d] \\ &\quad + \varphi(\mathbf{1}_{d-1}, \gamma_{d-1}, g)[a_2, \dots, a_d],\end{aligned}$$

where the second summand on the right-hand side agrees with the expression on the left-hand side. Thus,

$$\kappa(\mathbf{1}_d, \gamma_d, g)[1, a_2, \dots, a_d] = 0.$$

We need to repeat the steps for $l < n$ again, we arrive at the situation where

$$h(\tilde{B}_1) = 0 \text{ and } \#\tilde{B}_1 = 1$$

or

$$h(\tilde{B}_1) = g \text{ and } \mathcal{V} = 1_d \text{ and } \#\sigma < l.$$

The latter case will vanish by the assumption on l and we obtain, similar as in the $l = 1$ case

$$\begin{aligned}\varphi(\mathbf{1}_d, \gamma_{r_1,\dots,r_l}, g)[1, a_2, \dots, a_d] &= \kappa(\mathbf{1}_d, \gamma_{r_1,\dots,r_l}, g)[1, a_1, \dots, a_d] \\ &\quad + \sum_{(\tilde{\mathcal{V}},\tilde{\sigma})\in\mathbb{PS}_{\mathbb{C}}(r_1-1,r_2,\dots r_l)} \sum_{h\colon \tilde{\mathcal{V}}\to\frac{1}{2}\mathbb{Z}_{\geq 0}} \kappa(\tilde{\mathcal{V}}, \tilde{\sigma}, h)[a_2, \dots, a_d] \\ &= \varphi(\mathbf{1}_{d-1}, \gamma_{r_1-1,r_2\dots,r_l}, g)[a_2, \dots, a_d],\end{aligned}$$

which implies $\kappa(\mathbf{1}_d, \gamma_{r_1,\dots,r_l}, g)[1, a_1, \dots, a_d] = 0$. Note that $r_1 = 1$ cannot contribute since then $\gamma_{1,r_2,\dots,r_l}$ and σ fixed 1 and consequently the same holds true for $\pi = \gamma\sigma^{-1}$. We would end up with $\mathbf{0}_\pi \vee \mathcal{V} \neq \mathbf{1}_d$.
Finally, this means (2.6.15) vanishes and in our original problem only the case $h(B_1) = 0$ with $\#B_1 = 1$ is left. Once again we find

$$\varphi_{g;r_1,\dots,r_n}[1_{\mathcal{A}}, a_2, \dots, a_d] = \kappa(1_d, \gamma_{r_1,\dots,r_n})[1_{\mathcal{A}}, a_2, \dots, a_d] + \varphi(1_{d-1}, \gamma_{r_1-1,\dots,r_n}).$$

Which implies $\kappa(1_d, \gamma_{r_1,\dots,r_n})[1_{\mathcal{A}}, a_2, \dots, a_d] = 0$. $\square$

Lemma 2.6.36.
Let $(\mathcal{A},\varphi)$ be a SPS and $\mathcal{X}_1,\mathcal{X}_2 \subset \mathcal{A}$, and $(g_0,n_0) \in \overline{\mathsf{Typ}}$. We denote $\mathcal{X}_i^+ = \mathcal{X}_i \cup \{1\}$. The following statements are equivalent:

i) $\mathcal{X}_1$ and $\mathcal{X}_2$ are (g_0,n_0)-free.

ii) $\mathcal{X}_1^+$ and $\mathcal{X}_2^+$ are (g_0,n_0)-free.

iii) For any positive integer $d \in \mathbb{Z}_{\geq 0}$, any $(\mathcal{U},\gamma,k) \in \mathbb{PS}(d)$ of type $(\mathsf{g},\mathsf{n}) \preceq (g_0,n_0)$, any $a_1,\ldots,a_d \in \mathcal{X}_1^+$ and $b_1,\ldots,b_d \in \mathcal{X}_2^+$, we have

$$\varphi(\mathcal{U},\gamma,k)[a_1b_1,\ldots,a_db_d] = \sum_{(\mathcal{V},\pi,g)\odot(\mathcal{W},\sigma,h)=(\mathcal{U},\gamma,k)} \kappa(\mathcal{V},\pi,g)[\mathbf{a}_d]\varphi(\mathcal{W},\sigma,h)[\mathbf{b}_d],$$

where we abbreviate $\mathbf{a}_d = (a_1,\ldots,a_d)$ and $\mathbf{b}_d$ respectively.

iv) For any positive integer $d \in \mathbb{Z}_{\geq 0}$, any $(\mathcal{U},\gamma,k) \in \mathbb{PS}(d)$ of type $(\mathsf{g},\mathsf{n}) \preceq (g_0,n_0)$, any $a_1,\ldots,a_d \in \mathcal{X}_1^+$ and $b_1,\ldots,b_d \in \mathcal{X}_2^+$, we have

$$\kappa(\mathcal{U},\gamma,k)[a_1b_1,\ldots,a_db_d] = \sum_{(\mathcal{V},\pi,g)\odot(\mathcal{W},\sigma,h)=(\mathcal{U},\gamma,k)} \kappa(\mathcal{V},\pi,g)[\mathbf{a}_d]\kappa(\mathcal{W},\sigma,h)[\mathbf{b}_d].$$

where we abbreviate $\mathbf{a}_d = (a_1,\ldots,a_d)$ and $\mathbf{b}_d$ respectively.

Proof. This is the higher-genus generalisation of Theorem [CMSS07, Theorem 7.9]. Here we only explain (i) ⇒ (iii). The implication (iii) ⇒ (iv) comes from extended convolution with the Möbius function and (iv) ⇒ (iii) from extended convolution with the zeta function. The equivalence between (i) and (ii) is an immediate consequence of Proposition 2.6.35.

Let $(\mathcal{U},\gamma,k) \in \mathbb{PS}(d)$ of type (g,n). We take a second copy $[\bar{d}]$ of the set $[d]$ and interleave their elements

$$[d,\bar{d}] \coloneqq \{1,\,\bar{1},\,2,\,\bar{2},\,3,\,\bar{3},\ldots,\,d,\,\bar{d}\} \cong [2d]\,.$$

We call $\psi\colon [d] \to [\bar{d}]$ the canonical identification. For a block $B = \{i_1,\ldots,i_\ell\} \subset [d]$, we denote by $\bar{B} = \{\bar{i_1},\ldots,\bar{i_\ell}\} \subset [\bar{d}]$. We define the surfaced permutation $(\hat{\mathcal{U}},\hat{\gamma},\hat{k}) \in \mathbb{PS}(2d)$ as follows:

- the blocks of $\hat{\mathcal{U}}$ are of the form $\hat{B} \coloneqq B \cup \bar{B}$ where $B \in \mathcal{U}$, that is

$$\hat{\mathcal{U}} = \{B \cup \bar{B}\colon B \in \mathcal{U}\};$$

- the permutation $\hat{\gamma}$ is characterised by $\hat{\gamma}|_{[d]} = \psi \circ \gamma$ and $\hat{\gamma}|_{[\bar{d}]} = \gamma \circ \psi^{-1}$, that is

$$\hat{\gamma}(k) = \bar{k} \text{ and } \hat{\gamma}(\bar{k}) = \gamma(k)$$

for all $k \in [d]$; and

- the genus function is inherited via $\hat{k}(C \cup \bar{C}) = k(C)$.

Then we have

$$\begin{aligned}\varphi(\mathcal{U},\gamma,k)[a_1b_1,\dots,a_db_d] &= \varphi(\hat{\mathcal{U}},\hat{\gamma},\hat{k})[a_1,b_1,\dots,a_d,b_d]\\ &= \sum_{(\mathbf{0}_\pi,\pi,0)\odot(\mathcal{V},\sigma,h)=(\hat{\mathcal{U}},\hat{\gamma},\hat{k})} \kappa(\mathcal{V},\sigma,h)[a_1,b_1,\dots,a_d,b_d]. \end{aligned} \tag{2.6.16}$$

Assume that $(\mathcal{X}_1,\mathcal{X}_2)$ is (g_0,n_0)-free. The vanishing of the mixed surfaced free cumulants up to order (g_0,n_0) means that the only terms remaining in the right-hand side of (2.6.16) come from surfaced permutations $(\mathcal{V},\sigma,h)$ where blocks $B\in\mathcal{V}$ are included either in $[d]$ or in $[\bar{d}]$. We denote

$$\begin{aligned}\mathcal{V}_1 &= \{B\colon B\in\mathcal{V}, B\cap[d]\neq\emptyset\}\\ \overline{\mathcal{V}}_2 &= \{B\colon B\in\mathcal{V}, B\cap[d]=\emptyset\}\\ \mathcal{V}_2 &= \psi^{-1}[\overline{\mathcal{V}}_2] = \{\psi^{-1}(B)\colon B\in\mathcal{V}, B\cap[d]=\emptyset\},\end{aligned}$$

i.e. $\mathcal{V}_1$ consists of the blocks containing only non-bar entries of $\mathcal{V}$ and $\overline{\mathcal{V}}_2$ consists of the blocks that contain only bar entries, and $\mathcal{V}_2$ is the identification of $\overline{\mathcal{V}}_2$ in $[d]$ via ψ^{-1}. Since $\mathbf{0}_\sigma \leq \mathcal{V}$, the permutation σ respects this decomposition into $[d]$ and $[\bar{d}]$ and we introduce

$$\hat{\sigma}_1 = \sigma_1 = \sigma|_{[d]}, \quad \hat{\sigma}_2 = \sigma|_{[\hat{d}]} \quad \sigma_2 = \psi^{-1}\circ\hat{\sigma}_2\circ\psi,$$

in order to understand the factorization of σ in $S(d)$. More precisely, we want to use the latter to understand the factorizations on the right-hand side of in (2.6.16) in $S(d)$. Our claim is that

$$(\mathbf{0}_\pi,\pi,0)\odot(\mathcal{V},\sigma,h) = (\hat{\mathcal{U}},\hat{\gamma},\hat{k}) \tag{2.6.17}$$

is equivalent to a factorization

$$(\mathbf{0}_{\tilde{\pi}},\tilde{\pi},0)\odot(\mathcal{V}_1,\sigma_1,h_1)\odot(\mathcal{V}_2,\sigma_2,h_2) = (\mathcal{U},\gamma,k). \tag{2.6.18}$$

It remains to explain $\tilde{\pi},h_1,h_2$ and the fact that the equations (2.6.17) and (2.6.18) are equivalent. First note that if we have an equation like (2.6.17), then with the previous discussion, it holds that

$$\underbrace{\gamma\sigma_2^{-1}\sigma_1^{-1}}_{=:\tilde{\pi}}\sigma_1\sigma_2 = \gamma$$

and vice versa any σ_1,σ_2 determine an element $\sigma\in\mathbb{PS}(2d)$

$$\sigma = \sigma_1\circ(\psi\circ\sigma_2\circ\psi^{-1})$$

and a factorization

$$\underbrace{(\hat{\gamma}\sigma^{-1})}_{=:\pi}\sigma = \hat{\gamma}.$$

Thus on the level of permutations we are settled. Furthermore, by [CMSS07] $\hat{\mathcal{U}} = \mathcal{V} \vee \mathbf{0}_{\hat{\gamma}}$ is equivalent to $\mathcal{U} = \mathcal{V}_1 \vee \mathcal{V}_2 \vee \mathbf{0}_\gamma$. Hence also on the level of partitions the equivalence is clear. In order to make sense of the genus functions, let us observe the following. Since we have $\pi\sigma = \hat{\gamma}$ in $S([d, \bar{d}]) = S(2d)$ and σ decomposes w.r.t. $[d] \cup [\bar{d}]$ we must have that $\pi = \hat{\gamma}\sigma^{-1}$ maps $[d]$ to $[\bar{d}]$. This implies that if $C \cup \bar{C} \in \hat{\mathcal{U}}$ where $C \in \mathcal{U}$ is a block of non-bar elements we have

$$|\pi|_{C\cup\bar{C}}| = |\pi|_C|. \tag{2.6.19}$$

Let us elaborate. First $\mathbf{0}_\pi \le \hat{\mathcal{U}} = \mathbf{0}_\pi \vee \mathcal{V}$, that is by restricting $\pi|_{C\cup\bar{C}}$ we do not break any cycles of π, we only lose some. Hence, $\pi|_{C\cup\bar{C}}$ also maps bar to non-bar elements and vice versa. Then all cycles consist of at least two elements: a bar element and a non-bar element. If we now erase all bar elements we do not change the number of cycles, hence (2.6.19). Let us further note, for any $k \in [d]$, we have

$$\pi = \hat{\gamma}\sigma^{-1}(k) = (\overline{\sigma_1^{-1}(k)}),$$

since σ_1 is the part of σ that operates on non-bar elements in $[d]$. Then we apply once more $\hat{\gamma}\sigma^{-1}$

$$\begin{aligned}\hat{\gamma}\sigma^{-1}(\hat{\gamma}\sigma^{-1}(k)) &= \hat{\gamma}\sigma^{-1}(\overline{\sigma_1^{-1}(k)}) \\ &= \hat{\gamma}(\hat{\sigma}_2^{-1}(\overline{\sigma_1^{-1}(k)})) \\ &= \gamma(\hat{\sigma}_2^{-1}(\overline{\sigma_1^{-1}(k)})),\end{aligned}$$

where we used that only σ_2 operates on bar elements in $[\bar{d}]$ and afterward that $\hat{\gamma}$ maps bar elements $\bar{l}$ to $\gamma(l)$. In particular, restricting π to only non-bar elements yields

$$k \mapsto \gamma(\hat{\sigma}^{-1}(\overline{\sigma_1^{-1}(k)})) = \gamma(\sigma_2^{-1}(\sigma_1^{-1}(k))) = \gamma \circ \sigma_2^{-1} \circ \sigma_1^{-1}(k) = \tilde{\pi},$$

thus

$$|\pi|_C| = |\tilde{\pi}|_C|.$$

Using the latter and the equations of [CMSS07, Lemma 7.10] for the colength, we find by the definition of the genus for $\hat{C} \in \hat{\mathcal{U}}$

$$\begin{aligned}\hat{k}(\hat{C}) &= \frac{|(\mathbf{0}_\pi|_{\hat{C}}, \pi|_{\hat{C}})| + |(\mathcal{V}|_{\hat{C}}, \sigma|_{\hat{C}})| - |(\hat{\mathcal{U}}|_{\hat{C}}, \hat{\gamma}|_{\hat{C}})|}{2} + \sum_{\substack{B\in\mathcal{V}=\mathcal{V}_1\cup\overline{\mathcal{V}}_2 \\ B\subset\hat{C}=C\cup\bar{C}}} h(B) \\ &= \frac{|\pi|_{\hat{C}}| + |(\mathcal{V}|_{\hat{C}}, \sigma|_{\hat{C}})| - |(\hat{\mathcal{U}}|_{\hat{C}}, \hat{\gamma}|_{\hat{C}})|}{2} + \sum_{\substack{B\in\mathcal{V}_1 \\ B\subset C}} h(B) + \sum_{\substack{B\in\overline{\mathcal{V}}_2 \\ B\subset\bar{C}}} h(B) \\ &= \frac{|\tilde{\pi}|_C| + |(\mathcal{V}_1|_C, \sigma_1|_C)| + |(\mathcal{V}_2|_C, \sigma_2|_C)| - |(\mathcal{U}|_C, \gamma|_C)|}{2} \\ &\quad + \sum_{\substack{B\in\mathcal{V}_1 \\ B\subset C}} h(B) + \sum_{\substack{B\in\mathcal{V}_2 \\ \psi(B)\subset\bar{C}}} h(\psi(B)),\end{aligned} \tag{2.6.20}$$

where we used that $\mathcal{V}$ respects the decomposition into bar and non-bar elements. Thus, if we define

$$h_1 \colon \mathcal{V}_1 \to \frac{1}{2}\mathbb{Z}_{\geq 0}, \quad h_1(B) = h(B),$$
$$h_2 \colon \mathcal{V}_2 \to \frac{1}{2}\mathbb{Z}_{\geq 0} \quad h_2(B) = h(\psi(B)),$$

we have achieved the implication we wanted. Note that we could reverse the steps, hence we have an equivalence of factorizations. Finally, this allows us to continue the computation of (2.6.16) by

$$\begin{aligned}
&\sum_{(\mathbf{0}_{\tilde{\pi}},\tilde{\pi},0)\odot(\mathcal{V}_1,\sigma_1,h_1)\odot(\mathcal{V}_2,\sigma_2,h_2)=(\mathcal{U},\gamma,k)} \kappa(\mathcal{V}_1,\sigma_1,h_1)[\mathbf{a}]\kappa(\mathcal{V}_2,\sigma_2,h_2)[\mathbf{b}] \\
&= \sum_{(\mathcal{W},\omega,g)\odot(\mathcal{V}_2,\sigma_2,h_2)=(\mathcal{U},\gamma,k)} \left(\sum_{(\mathbf{0}_{\tilde{\pi}},\tilde{\pi},0)\odot(\mathcal{V}_1,\sigma_1,h_1)=(\mathcal{W},\omega,g)} \zeta(\mathbf{0}_{\tilde{\pi}},\tilde{\pi},0)\kappa(\mathcal{V}_1,\sigma_1,h_1)[\mathbf{a}] \right) \\
&\quad \times \kappa(\mathcal{V}_2,\sigma_2,h_2)[\mathbf{b}] \\
&= \sum_{(\mathcal{W},\omega,g)\odot(\mathcal{V}_2,\sigma_2,h_2)=(\mathcal{U},\gamma,k)} \varphi(\mathcal{W},\omega,g)[\mathbf{a}]\kappa(\mathcal{V}_2,\sigma_2,h_2)[\mathbf{b}],
\end{aligned}$$

where we first used the multiplicativity of κ and then recognized the convolution with ζ. □

Proposition 2.6.37.
Let $(\mathcal{X}_i)_{i\in I}$ be a family of subsets of $\mathcal{A}$, and $\mathcal{A}_i$ the unital subalgebra generated by $\mathcal{X}_i$. Let $(g_0, n_0) \in \overline{\mathsf{Typ}}$. Then (g_0, n_0)-freeness of $(\mathcal{X}_i)_{i\in I}$ is equivalent (g_0, n_0)-freeness of $(\mathcal{A}_i)_{i\in I}$.

Proof of Proposition 2.6.37. By multilinearity, the linear spans of two free sets are free. The only thing that deserves a check is that freeness of $\mathcal{X}_1$ and $\mathcal{X}_2$ implies freeness of $\mathcal{X}_1$ and $\mathcal{X}_2\mathcal{X}_2$. Given Lemma 2.6.36, the proof is identical to [CMSS07, Theorem 7.12]. □

2.6.5 Application in random matrix theory

The formalism of surfaced free cumulants and freeness up to order (g_0, n_0) can be directly applied in random matrix theory, generalising Theorem 1.1.28 of [Voi91] and Theorem 2.1.4 of [CMSS07].

Definition 2.6.38.
If A is a matrix of size N and $\lambda \vdash d$ a partition of length n, with $d \leq N$, we denote

$$p_\lambda(A) = \prod_{i=1}^{n} \mathrm{Tr}(A^{\lambda_i}), \qquad \mathcal{P}_\lambda(A) = \prod_{c=1}^{d} A_{c,\pi_\lambda(c)}. \tag{2.6.21}$$

We recall the following result, which comes from Weingarten calculus. An equivalent formulation can be found in [CMSS07, Theorem 4.4] and [BGF20, Theorem 8.8].

Theorem 2.6.39 ([CMSS07],[BGF20]).
Let A be a random hermitian matrix of size N, whose law is invariant under unitary conjugation. Then for any $\lambda \vdash d$:

$$\mathbb{E}[p_\lambda(A)] = z_\lambda \sum_{\nu \vdash d} N^d \, H^<(\lambda,\nu)\big|_{\hbar=1/N} \, \mathbb{E}[\mathcal{P}_\nu(A)]\,,$$
$$\mathbb{E}[\mathcal{P}_\lambda(A)] = z_\lambda \sum_{\nu \vdash d} N^{-d} H^{\leq}(\lambda,\nu)\big|_{\hbar=1/N} \, \mathbb{E}[p_\nu(A)]\,.$$

Definition 2.6.40.
Let $(A_N)_{N\in\mathbb{N}}, A_N \in M_N(\mathbb{C})$ be a sequence of hermitian random matrices. We say that it admits a limit distribution up to order (g_0,n_0) if there exists $F^A_{g;k_1,\ldots,k_n}$ indexed by $(g,n) \preceq (g_0,n_0)$ and $k_1,\ldots,k_n > 0$, independent of N, such that for any $n \in [\lfloor g_0 \rfloor + n_0]$ and any $k_1,\ldots,k_n > 0$, we have for $N \to \infty$

$$\mathbb{E}^\circ\big[\mathrm{Tr}(A_N^{k_1}),\ldots,\mathrm{Tr}(A_N^{k_n})\big] = \sum_{\substack{g \in \frac{1}{2}\mathbb{Z}_{\geq 0} \\ g \leq g_0 + \min(0,n_0-n)}} N^{2-2g-n}\, F^A_{g;k_1,\ldots,k_n} + o(N^{2-2g_0-n_0+|n_0-n|})\,,$$

where $\mathbb{E}^\circ$ denotes the cumulant expectation value (cf. (2.3.10)). In this expression, the order of the $o(\ldots)$ is adjusted to be the next subleading term compared to the sum. When $g_0 = \infty$, we ask for the existence of such an asymptotic expansion to an arbitrary order $o(N^{-K})$ for all $n \leq n_0$, and in that case we use the notation

$$\mathbb{E}^\circ\big[\mathrm{Tr}(A_N^{k_1}),\ldots,\mathrm{Tr}(A_N^{k_n})\big] = \sum_{g \in \frac{1}{2}\mathbb{Z}_{\geq 0}} N^{2-2g-n}\, F^{\mathrm{A}}_{g;k_1,\ldots,k_n} + o(N^{-\infty})\,. \tag{2.6.22}$$

From Theorem 2.6.39 it can be observed that $(A_N)_N$ has a limit distribution up to order (g_0,n_0), then for any partition $\lambda \vdash d$ of length $n \leq \lfloor g_0 \rfloor + n_0$ we have when $N \to \infty$

$$\mathbb{E}[\mathcal{P}_\lambda(A_N)] = \sum_{\substack{g \in \frac{1}{2}\mathbb{Z}_{\geq 0} \\ g \leq g_0 + \min(0,n_0-n)}} N^{2-2g-n-d}\, \kappa^{\mathrm{A}}_{g;\lambda_1,\ldots,\lambda_n} + o(N^{2-2g_0-n_0+|n_0-n|-d})\,. \tag{2.6.23}$$

We obtain the structure of a SPS on the algebra $\mathcal{A} = \mathbb{C}[a]$ by defining the moments

$$\varphi_{g,n}(a^{k_1},\ldots,a^{k_n}) = F^{\mathrm{A}}_{g;k_1,\ldots,k_n} \qquad \forall (g,n) \preceq (g_0,n_0).$$

Combining Theorem 2.6.39 and the expansion (2.6.23) with Theorem 2.5.2 indicates that $\kappa^{\mathrm{A}}_{g;k_1,\ldots,k_n}$ are the free cumulants at order $(g,n) \preceq (g_0,n_0)$.

Theorem 2.6.41.
Let $(A_N)_N$ and $(B_N)_N$ be two sequences of ensembles of random matrices of size N, at

least one of them being unitarily invariant, and such that for each N, A_N is independent from B_N. Assume that both ensembles have a limit distribution up to order (g_0, n_0) (possibly ∞), and consider the algebra $\mathcal{A} = \mathbb{C}\langle a, b\rangle$ of noncommutative polynomials in two letters. Then, for any $Q \in \mathcal{A}$, $Q(A_N, B_N)$ admits a limit distribution up to order (g_0, n_0), so $\mathcal{A}$ can be upgraded to a SPS. Besides, the subalgebras $\mathbb{C}[a]$ and $\mathbb{C}[b]$ are (g_0, n_0)-free.

Proof. Let $(A_N)_N$ and $(B_N)_N$ be as in the theorem. For any $k, k' > 0$, $(A_N^k)_N$ and $(B_N^{k'})$ clearly have a limit distribution up to order (g_0, n_0). Examining the finite N formula in [CMSS07, Theorem 4.4, (2)], the products $(A_N^k B_N^{k'})_N$ also have a limit distribution up to order (g_0, n_0). Take $\sigma \in S(d)$, $N \geq d$ and a map $\Omega\colon [d] \to \{A_N, B_N\}$. Due to independence of A_N and B_N, and unitary invariance of the law of one of the matrices (say A_N), we have

$$\begin{aligned}
&\mathbb{E}\left[\prod_{c=1}^{d}(\Omega(c))_{c,\sigma(c)}\right] = \mathbb{E}\left[\prod_{c\in\Omega^{-1}(A_N)}(A_N)_{c,\sigma(c)}\right]\mathbb{E}\left[\prod_{c\in\Omega^{-1}(B_N)}(B_N)_{c,\sigma(c)}\right] \\
&= \int_{U(N)}\mathrm{d}U\ \mathbb{E}\left[\prod_{c\in\Omega^{-1}(A_N)}(UA_NU^{-1})_{c,\sigma(c)}\right]\mathbb{E}\left[\prod_{c\in\Omega^{-1}(B_N)}(B_N)_{c,\sigma(c)}\right] \\
&= \sum_{\substack{i_c, j_c\in[N] \\ c\in\Omega^{-1}(A_N)}}\left(\int_{U(N)}\mathrm{d}U \prod_{c\in\Omega^{-1}(A_N)} U_{c,i_c}U^{-1}_{j_c,\sigma(c)}\right)\mathbb{E}\left[\prod_{c\in\Omega^{-1}(A)}(A_N)_{i_c,j_c}\right]\mathbb{E}\left[\prod_{c\in\Omega^{-1}(B)}(B_N)_{c,\sigma(c)}\right].
\end{aligned} \tag{2.6.24}$$

By Weingarten calculus [Col03] the integral over $U(N)$ vanishes unless there exist two permutations $\alpha, \beta \in S(\Omega^{-1}(A_N))$ such that we have $c = j_{\beta(c)}$ and $i_c = \sigma(\alpha(c))$ for all $c \in \Omega^{-1}(A_N)$. This cannot happen when Ω takes at least once the values A_N and B_N, as we can find $c_0 \in [d]$ such that $\Omega(c_0) = A_N$, and $\Omega(\sigma(c_0)) = B_N$ or $\Omega(\sigma^{-1}(c_0)) = B_N$. Since the surfaced free cumulants evaluated on $(\Omega(c))_{c\in[d]}$ are extracted from the asymptotic expansion of (2.6.24) when $N \to \infty$ (recall (2.6.23)), all mixed surfaced free cumulants between $\mathbb{C}[a]$ and $\mathbb{C}[b]$ vanish up to order (g_0, n_0) (which from the assumption is the order up to which the asymptotic expansion exist), i.e. these two algebras are (g_0, n_0)-free in the surfaced probability space $\mathbb{C}\langle a, b\rangle$. ☐

Remark 2.6.42.
The combinatorics underlying infinitesimal free cumulants is the truncation keeping the leading (genus 0) and the first subleading (genus $\frac{1}{2}$) of the master relation involving monotone Hurwitz numbers, whose appearence can be traced back to Weingarten calculus for the unitary group. Typically, for topological expansions in unitarily invariant random hermitian matrices the genus $\frac{1}{2}$ order (corresponding to a term of order N^{-1} compared to the leading term) vanishes. An example of a situation where it does not vanish is the 1-hermitian matrix model with a N-dependent potential of the form $V_0 + N^{-1}V_1$. Although the non-vanishing of the genus $\frac{1}{2}$ order is typical in the topological expansion

for orthogonally invariant random symmetric matrices, the observation of Mingo [Min19] that infinitesimal freeness cannot describe the relations between moments and cumulants in such models is not a surprise, as they should rather be governed by the Weingarten calculus for the orthogonal group.

Remark 2.6.43.
The topological partition function associated with (2.6.22) is in fact the $N \to \infty$ asymptotic series of the formal series in the variables $X_1, X_2, \ldots$

$$\begin{aligned}\mathbb{E}\Big[\prod_i \frac{1}{\det(1 - X_i A_N)}\Big] &= \mathbb{E}\Big[\exp\Big(\sum_{k\geq 1} \frac{\mathrm{Tr}(A_N^k)p_k}{k}\Big)\Big] \\ &= \exp\bigg(\sum_{n\geq 1} \mathbb{E}^\circ\big[\mathrm{Tr}(A^{k_1}), \ldots, \mathrm{Tr}(A^{k_n})\big]\, \frac{p_{k_1}\cdots p_{k_n}}{n!\, k_1 \cdots k_n}\bigg) \\ &= Z\big|_{\hbar=1/N} \cdot \big(1 + O(N^{-\infty})\big)\,,\end{aligned}$$

where $p_k = \sum_i X_i^k$ is the k-th power sum. The first equality is Cauchy's identity, the second one is the definition of the connected expectation value, and the last one comes from comparing (2.6.22) and Section 2.3.

2.7 Relation to topological recursion

Recall that many combinatorial and geometric invariants can be computed by topological recursion. Typically, one recovers the generating series from the TR invariants via

$$\omega_{g,n}(z_1, \ldots, z_n) = \sum_{k_1,\ldots,k_n=1}^{\infty} C_{g,k_1,\ldots,k_n} x(z_1)^{k_1} \ldots x(z_n)^{k_1} \mathrm{d}x(z_1) \ldots \mathrm{d}x(z_n).$$

From our perspective of free probability, the quantities of interest are the moments $\varphi_{g;r_1,\ldots,r_n}$ and the cumulants $\kappa_{g;r_1,\ldots,r_n}$ of random variables in a noncommutative probability space. However, in context of topological recursion we usually are not interested in this setup. In order to be consistent with the notation, we consider two families of n-point functions denoted by G_n^φ, G_n^κ that are related via the master relation of Theorem 2.5.2. But we want to emphasize that despite our notation, G_n^φ, G_n^κ are just formal power series. First, we introduce a change of variable

$$u(w) = \frac{1}{x(w)} = w^{-1} G_{0,1}^\kappa(w). \tag{2.7.1}$$

For $2g-2+n \geq 0$, we define the differential forms of the n-points functions by

$$\begin{aligned} \omega^{\varphi}_{g,n}(u_1,\ldots,u_n) &= \prod_{i=1}^{n} \frac{\mathrm{d}u_i}{u_i}\, G^{\varphi}_{g,n}(u_1^{-1},\ldots,u_n^{-1}) \\ &= \sum_{k_1,\ldots,k_n>0} F^{\varphi}_{g;k_1,\ldots,k_n} \prod_{i=1}^{n} \frac{\mathrm{d}u_i}{u_i^{k_i+1}}\,, \\ \omega^{\kappa}_{g,n}(w_1,\ldots,w_n) &= \prod_{i=1}^{n} \frac{\mathrm{d}w_i}{w_i}\, G^{\kappa}_{g,n}(w_1,\ldots,w_n) \\ &= \sum_{k_1,\ldots,k_n>0} F^{\kappa}_{g;k_1,\ldots,k_n} \prod_{i=1}^{n} w_i^{k_i-1}\mathrm{d}w_i\,, \end{aligned} \tag{2.7.2}$$

and their shifted version for $(g,n)=(0,2)$,

$$\begin{aligned} \widetilde{\omega}^{\varphi}_{0,2}(u_1,u_2) &= \omega^{\varphi}_{0,2}(u_1,u_2) + \frac{\mathrm{d}u_1\mathrm{d}u_2}{(u_1-u_2)^2}\,, \\ \widetilde{\omega}^{\kappa}_{0,2}(w_1,w_2) &= \omega^{\kappa}_{0,2}(w_1,w_2) + \frac{\mathrm{d}w_1\mathrm{d}w_2}{(w_1-w_2)^2}\,. \end{aligned} \tag{2.7.3}$$

Remark 2.7.1.
Note that the change of variables and the formulas (2.7.2) correspond to the Cauchy- and R-transform in Remark 2.1.7.

The relation $G^{\kappa}_{0,1}(w) = G^{\varphi}_{0,1}(u^{-1})$ can be rephrased as the statement on the functional inverse

$$w(u) = \frac{G^{\varphi}_{0,1}(u^{-1})}{u} \qquad \Longleftrightarrow \qquad u(w) = \frac{G^{\kappa}_{0,1}(w)}{w}\,, \tag{2.7.4}$$

again compare to Remark 1.1.21. For $(g,n)=(0,2)$, (2.4.4) becomes

$$\widetilde{\omega}^{\varphi}_{0,2}(u_1,u_2) = \widetilde{\omega}^{\kappa}_{0,2}(w_1,w_2), \tag{2.7.5}$$

which is equivalent to the formula of [CMSS07, Corollary 6.4]. For $n \geq 3$ and under the change of variable (2.7.1) the operator (2.4.2) becomes

$$\vec{\mathsf{O}}^{\kappa}_{r}(w) = \sum_{m\geq 0} (-u\partial_u)^m \frac{-u\mathrm{d}w}{w\mathrm{d}u} \cdot [v^m]\Big(\partial_y + \frac{v}{y}\Big)^r \cdot 1\Big|_{y=uw}\,,$$

and the functional relation in Theorem 2.4.1 becomes

$$\begin{aligned} &\omega^{\varphi}_{0,n}(u_1,\ldots,u_n) = \\ &\quad \sum_{r_1,\ldots,r_n\geq 0} \prod_{i=1}^{n} \frac{\mathrm{d}u_i}{u_i}\, \vec{\mathsf{O}}^{\kappa}_{r_i}(w_i)\Big(\frac{w_i}{\mathrm{d}w_i}\Big)^{r_i+1} \sum_{T\in\mathcal{G}_{0,n}(\mathbf{r}+1)} \prod_{I\in\mathcal{I}(T)}' \omega^{\kappa}_{0,\#I}(w_I)\,, \end{aligned} \tag{2.7.6}$$

where $\prod'$ means that any occurrence of $\omega^{\kappa}_{0,2}(w_i,w_j)$ with $i \neq j$ should be replaced with $\widetilde{\omega}^{\kappa}_{0,2}(w_i,w_j)$.

Consider the example of $G^{\varphi}_{g,n}$ being the generating series of ordinary maps of genus g with n boundaries and $G^{\kappa}_{g,n}$ enumerating fully simple maps of the same topology. It is known that, in this case, the $\omega^{\varphi}_{g,n}$ are computed by the topological recursion [EO07, EO09] applied to the spectral curve

$$\mathcal{S}^{\varphi} = \left(\mathbb{C}P^1, u, w, \tfrac{\mathrm{d}z_1 \mathrm{d}z_2}{(z_1 - z_2)^2}\right),$$

while in [BDBKS23, BCGF21], it was shown that $\omega^{\kappa}_{g,n}$ are computed by the topological recursion applied to

$$\mathcal{S}^{\kappa} = \left(\mathbb{C}P^1, w, u, \tfrac{\mathrm{d}z_1 \mathrm{d}z_2}{(z_1 - z_2)^2}\right).$$

Furthermore, G. Borot and E. Garcia-Failde proved that the generating series are related by the master relation (see [BGF20, BCDGF19]), equivalently they are related by the moment-cumulant relations. This example motivated us in [BCGF$^+$23] to the conjecture that Theorem 2.5.2, or rather the functional relations given in Theorem 2.4.8, when written in terms of differentials, in fact describe the effect of the symplectic exchange $u \leftrightarrow w$ in topological recursion.

Conjecture 2.7.2.
Let $\mathcal{C}$ be a compact Riemann surface and let u, w be meromorphic functions on $\mathcal{C}$ such that $\mathrm{d}u$ and $\mathrm{d}w$ do not have common zeroes. Furthermore, let B be a fundamental bidifferential of the second kind. We denote by $\omega^{\varphi}_{g,n}$ the differentials obtained from the topological recursion with the spectral curve $(\mathcal{C}, u, w, B)$, and by $\omega^{\kappa}_{g,n}$ the ones associated to the spectral curve $(\mathcal{C}, w, u, B)$, and define $\widetilde{\omega}^{\varphi}_{0,2} = \widetilde{\omega}^{\kappa}_{0,2} = B$. Then, these differentials will satisfy the functional relations of Theorem 2.4.8 for all $2g - 2 + n \geq 0$ (after they are converted to relations between meromorphic differentials on $\mathcal{C}$).

The conjecture has in the meantime been proven in [ABDB$^+$22]. Let us state a particularly important result of [ABDB$^+$22] in our language.

Theorem 2.7.3 ([ABDB$^+$22])**.**
Assume that $\omega^{\varphi}_{g,n}$ satisfy topological recursion on the spectral curve (Σ, u, w, B), where u, w are meromorphic such that the ramification points p of w, that is zeroes of $\mathrm{d}w$, are simple and u is regular on p. Then the $\omega^{\kappa}_{g,n}$ given by extending (2.7.6) to higher genus are produced by the spectral curve (Σ, w, u, B).

Let us conclude with the following remark.

Remark 2.7.4.

i) In terms of free probability Theorem 2.7.3 means that given a distribution satisfying the topological recursion on a spectral curve (Σ, u, w, B) such that u satisfies the regularity conditions on the ramification points w, then the cumulants must satisfy topological recursion on the spectral curve given by (Σ, w, u, B).

ii) Note that although we have these results at hand, we still do not know how to phrase the property of satisfying topological recursion in terms of regularity properties in free probability. A probably related notion of free probability is called the *conjugate variables*. Consider a W^*-probability space (M, τ); recall that is M is a von Neumann algebra equipped with a faithful normal tracial state τ. Then a conjugate variable ξ for an element $X \in M$ is an element $\xi \in L^2(M, \tau)$ that has the property that

$$\tau(\xi p(X)) = \tau \otimes \tau(\partial_X p(X))$$

for any noncommutative polynomial $p \in \mathbb{C}\langle X\rangle$ and where

$$\partial_X \colon \mathbb{C}\langle X\rangle \to \mathbb{C}\langle X\rangle \otimes \mathbb{C}\langle X\rangle$$

denotes the noncommutative derivative. It is given by the extension of

$$X^n \mapsto \sum_{k=0}^{n-1} X^k \otimes X^{n-1-k},$$

where $X^0 = 1_M$. In the case where $\xi = (DV)(X)$ is the derivative of a polynomial $V(x)$ evaluated at X, the equation is more or less the first loop equation of a hermitian 1-matrix model with potential V. Furthermore, Mingo and Speicher introduced an extension to second order. A second order conjugate variable satisfies

$$\tau_{0,2}(\xi p(X), q(X)) = \tau_{0,2}(\tau_{0,1} \otimes \mathrm{id} + \mathrm{id} \otimes \tau_{0,1}(\partial_X p(X), q(x)) + \tau_{0,1}(p(X)(Dq)(X));$$

see [MS13]. It also can be easily deduced that the second order equation is equivalent to the second loop equation. Then one may extend these definitions of conjugate variable by requiring that the higher order and higher genus loop equations hold. This procedure could lead to an understanding of how the existence of a more general conjugate variable $\xi \neq (DV)(X)$ might impact the existence of a spectral curve that computes the moments (and hence cumulants) via topological recursion. Furthermore, we propose the study of other regularity properties in regard to topological recursion, e.g. finite *free Fisher information* and finite *free entropy*.

iii) Another still open problem is to understand the joint distribution in terms of the theory of topological recursion and the Fock space formalism. Also, the search for a higher order (and higher genus) formula for cumulants of products is closely related; see [MST09].

3 On quantum curves and topological recursion for various monotone Hurwitz numbers

In this chapter, we discuss the results of [HvIL22] regarding quantum curves and topological recursion. This part is strongly motivated by the works [LMS13] and [DN18].

Topological recursion is a recursive procedure that computes a family of multidifferentials $\omega_{g,n}$ from the initial data of a spectral curve (Σ, x, y, B). A lot of enumerative problems can be computed from this recursion, in particular various types of Hurwitz numbers. Typically the challenge is to find the initial data of a spectral curve that produces these numbers via topological recursion. But, there are techniques that have proved helpful to overcome this problem.

If Σ is a connected compact Riemann surface and $x, y\colon \Sigma \to \mathbb{C}$ meromorphic functions, then x, y satisfy an algebraic equation $P(x, y)$ over $\mathbb{C}$. Thus, the search for the initial data can be formulated as a search for the an algebraic curve P. In many cases, such an equation can be found via the *dequantization* of a so-called quantum curve. A quantum curve for an enumerative problem is a differential equation for the specialized partition function. In terms of Section 2.3, the specialization is the evaluation $\Psi = Z|_{p_i = x^i}$ in a formal variable x. Then a quantum curve is a differential equation $\widehat{\mathsf{P}}(\widehat{\mathsf{y}}, \widehat{\mathsf{x}})\Psi = 0$, involving the operators $\widehat{\mathsf{y}} = \hbar\partial_x$ and $\widehat{\mathsf{x}} = x\cdot$. If $\widehat{\mathsf{P}}$ is a polynomial in $\widehat{\mathsf{y}}$ and $\widehat{\mathsf{x}}$, we can formally replace $\widehat{\mathsf{y}}$ and $\widehat{\mathsf{x}}$ by commuting variables y and x and obtain an algebraic equation $P(x, y)$. In many cases, this procedure yields the spectral curve of the initial enumerative problem. In recent papers, this strategy has been made rigorous; it is called the *quantum curve – topological recursion correspondence*. We start our chapter by explaining this correspondence informally.

X. Liu, M. Mulase and A. Sorkin introduced Hurwitz numbers over a higher genus base curve in [LMS13]. These enumerative quantities count ramified coverings of higher genera Riemann surfaces. In the hope of proving topological recursion, they computed a quantum curve equation for these Hurwitz numbers and studied its semiclassical limit; aka its dequantization. In the same spirit, we introduce (strictly) monotone versions of the numbers of [LMS13] in our paper [HvIL22] and compute quantum curves. When studying the dequantization of strictly monotone Hurwitz numbers over an elliptic curve, we discover a spectral curve for monotone simple Hurwitz numbers, enumerated in an alternating way. These numbers agree with the values of the Möbius function in surfaced free probability. In the last section we prove topological recursion for the Möbius function based on the calculations of [DN18].

3.1 Quantum curves and the QC-TR correspondence

We want to motivate the results of [HvIL22] by giving a short informal introduction to quantum curves and their relation to TR. Let us start from the perspective of Chapter 2, or more precisely Section 2.3. The notion of quantum curves connects the theory of topological recursion with integrable systems. From a perspective of partition functions, the notion of a quantum curve is derived as follows. Consider a partition function

$$Z = \exp\left(\sum_{\substack{g\geq 0\\ n\geq 1}} \frac{\hbar^{2g-2+n}}{n!} \sum_{\mu_1,\dots,\mu_n=1}^{\infty} F_{g,\mu_1,\dots,\mu_n} p_{\mu_1}\cdots p_{\mu_n} \right),$$

then we define its *principle specialization*[1] by

$$\Psi(x,\hbar) \coloneqq Z\Big|_{p_i=x^i\,:\,i\in\mathbb{N}} = \exp\left(\sum_{\substack{g\geq 0\\ n\geq 1}} \frac{\hbar^{2g-2+n}}{n!} \sum_{\mu_1,\dots,\mu_n=1}^{\infty} F_{g,\mu_1,\dots,\mu_n} x^{|\mu|} \right).$$

Assume that the $F_{g,n}$ can be computed by TR of a spectral curve (Σ, x, y, B), such that x, y satisfy a polynomial equation $P(x,y) = 0$[2]. Then a quantization of P is a differential operator $\widehat{\mathsf{P}} = \widehat{\mathsf{P}}(\widehat{\mathsf{x}}, \widehat{\mathsf{y}}, \hbar)$ such that

$$\widehat{\mathsf{P}}(\widehat{\mathsf{x}}, \widehat{\mathsf{y}}, \hbar) = P(\widehat{\mathsf{x}}, \widehat{\mathsf{y}}) + O(\hbar),$$

where $\widehat{\mathsf{x}} = x\cdot$ is the multiplication operator, $\widehat{\mathsf{y}} = \hbar\frac{\mathrm{d}}{\mathrm{d}x}$ and $P(\widehat{\mathsf{x}}, \widehat{\mathsf{y}})$ is the evaluation of the polynomial P in normal ordering. Then a quantum curve is a quantization $\widehat{\mathsf{P}}$ of P that annihilates the principle specialization, i.e.

$$\widehat{\mathsf{P}}\Psi = 0.$$

The perspective from the standpoint of topological recursion is a bit more complicated. Consider a spectral curve (Σ, x, y, B) such that $\Sigma = \mathbb{C}P^1$, then x and y satisfy a polynomial equation, furthermore we denote the topological recursion correlators by $\omega_{g,n}$. Then there is a way to define Ψ without the context of partition functions via

$$\Psi(x(z)) = \exp\left(\sum_{\substack{g\geq 0\\ n\geq 1}} \int_p^{\infty}\cdots\int_p^{\infty} \omega_{g,n} - \delta_{g,0}\delta_{n,2}B(x(z_1)x(z_2)) \right),$$

for a *good* choice of a basepoint p. For higher genus spectral curves there are also ways to define Ψ but they involve some correction. We are ready to formulate the QC-TR correspondence, also called the Gukov-Sulkowski conjecture.

[1]Sometimes Ψ is called the wave function.
[2]This setup is usually referred to as *rational spectral curve*.

Conjecture 3.1.1 ([GS12]).
Let $\mathcal{S} = (\Sigma, x, y, B)$ be an algebraic spectral curve. Then there exists a quantization $\widehat{\mathsf{P}}$ of the polynomial equation $P(x, y) = 0$ for x and y that annihilates the wave function of $\mathcal{S}$.

This conjecture was first proved in a case-by-case manner for various enumerative problems, e.g. [MS12], [DBMN^{+}17] and [DN18]. The first general proof for genus zero curves was given by V. Bouchard and B. Eynard [BE17] for a large class of genus zero spectral curves. Recently the conjecture has been proven for all genera spectral curves with simple ramification behaviour [EGFMO21]. Also there has been an approach beyond algebraic equations [BKW23]. From this perspective it seems sensible to use a reversed approach, that is, given an enumerative problem where the spectral curve is not known, one can try to find a quantum curve and dequantize it to obtain TR. In that spirit we present the following results on the quantum curve for base h (strictly) monotone Hurwitz numbers.

3.2 Stirling numbers

Before we start the discussion of the main results, we introduce the Stirling numbers of the first and second kind. Their relation to Hurwitz numbers and their recursive structure are the main ingredient for proving the quantum curve equations.

Definition 3.2.1.
For $n, k \in \mathbb{N}$, we define *Stirling numbers of the first kind* by the recurrence relation

$$\begin{bmatrix} n+1 \\ k \end{bmatrix} = n \begin{bmatrix} n \\ k \end{bmatrix} + \begin{bmatrix} n \\ k-1 \end{bmatrix} \text{ for } k > 0; \quad \begin{bmatrix} 0 \\ 0 \end{bmatrix} = 1 \text{ and } \begin{bmatrix} n \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ n \end{bmatrix} = 0 \text{ for } n > 0$$

and *Stirling numbers of the second kind* by the recurrence relation

$$\begin{Bmatrix} n+1 \\ k \end{Bmatrix} = k \begin{Bmatrix} n \\ k \end{Bmatrix} + \begin{Bmatrix} n \\ k-1 \end{Bmatrix} \text{ for } k > 0; \quad \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} = 1 \text{ and } \begin{Bmatrix} n \\ 0 \end{Bmatrix} = \begin{Bmatrix} 0 \\ n \end{Bmatrix} = 0 \text{ for } n > 0.$$

We recall a well known fact for the generating series of the Stirling numbers; see e.g. [Cha18].

Lemma 3.2.2.
The generating functions of Stirling numbers satisfy

$$\sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} \frac{x^{n-k}}{n!} = \prod_{r=1}^{n-1} (1 - rx) \quad \text{and} \quad \sum_{n=k}^{\infty} \begin{Bmatrix} n \\ k \end{Bmatrix} x^{n-k} = \prod_{r=1}^{k} \frac{1}{1 - rx}. \tag{3.2.1}$$

3.3 Quantum curve for (strictly) monotone base h Hurwitz numbers

Motivated by the successful study of the base h Hurwitz numbers in [LMS13] and of the monotone Hurwitz numbers in [GGPN14], we introduce base h Hurwitz numbers with additional monotonicity conditions. This section is devoted to deriving a quantum curve for this new enumerative problem.

Recall the connected labeled monotone base h Hurwitz numbers $H^{\leq}_{g,h}(\mu_1, \ldots, \mu_n)$ from Section 1.3.

Definition 3.3.1.
We define

$$F^{\leq}_{g,h}(x_1, \ldots, x_n) = \sum_{\mu \in \mathbb{Z}^n_{\geq 0}} H^{\leq}_{g,h}(\mu_1, \ldots, \mu_n)\, x_1^{\mu_1} \ldots x_n^{\mu_n},$$

and analogously the generating series for the strictly monotone case $F^{<}_{g,h}(x_1, \ldots, x_n)$.

Definition 3.3.2.
We define the specialization of the partition function of the base h monotone Hurwitz numbers as the formal series in variables $x, \hbar$, given by

$$\begin{aligned}\Psi^{\leq}_{\hbar} &= \Psi^{\leq}_{\hbar}(x, \hbar)\\ &= \exp\left[\sum_{g=0}^{\infty}\sum_{n=1}^{\infty} \frac{\hbar^{2g-2+n}}{n!} F^{\leq}_{g,h}(x, x, \ldots, x)\right]\\ &= \exp\left[\sum_{g=0}^{\infty}\sum_{n=1}^{\infty} \frac{\hbar^{2g-2+n}}{n!} \sum_{\mu \in \mathbb{Z}^n_{\geq 0}} H^{\leq}_{g,h}(\mu_1, \ldots, \mu_n)\, x^{|\mu|}\right],\end{aligned}$$

and analogously the partition function $\Psi^{<}_{\hbar} = \Psi^{<}_{\hbar}(x, \hbar)$ of the base h strictly monotone Hurwitz numbers.

Remark 3.3.3.
Throughout the rest of the thesis we refer to Ψ as the partition function instead of its principle specialization for the sake of simplicity.

Proposition 3.3.4.
For the partition functions for the monotone and strictly monotone case, we have

$$\begin{aligned}\Psi^{\leq}_{\hbar} &= 1 + \sum_{d=1}^{\infty}\sum_{r=0}^{\infty} \left\{ \begin{matrix} d+r-1 \\ d-1 \end{matrix} \right\} (d!)^{1-\chi} x^d \hbar^{r+d(1-\chi)}\\ &= 1 + \sum_{d=1}^{\infty} (d!)^{1-\chi} x^d \hbar^{d(1-\chi)} \prod_{j=1}^{d-1} \frac{1}{1 - j\hbar}\end{aligned}$$

and

$$\Psi_{\hbar}^{<} = 1 + \sum_{d=1}^{\infty}\sum_{r=0}^{d-1}\begin{bmatrix} d \\ d-r \end{bmatrix}(d!)^{1-\chi}x^d\hbar^{r+d(1-\chi)}$$
$$= 1 + \sum_{d=1}^{\infty}(d!)^{1-\chi}x^d\hbar^{1-\chi}\prod_{j=1}^{d-1}(1-j\hbar),$$

where $\chi = 2-2h$ and the equalities are understood in the sense of formal power series.

Proof. Since

$$\sum_{g=0}^{\infty}\sum_{n=1}^{\infty}\frac{\hbar^{2g-2+n}}{n!}F_{g,h}^{\leq}(x,x,\ldots,x) = \sum_{g=0}^{\infty}\sum_{n=1}^{\infty}\frac{\hbar^{2g-2+n}}{n!}\sum_{\mu\in\mathbb{Z}_{\geq 0}^{n}} H_{g,h}^{\leq}(\mu_1,\ldots,\mu_n)\,x^{|\mu|}$$

counts transitive (connected) monotone base h factorizations, we can use the exponential formula and find the generating series for the not necessarily transitive factorizations

$$\Psi_{\hbar}^{\leq} = 1 + \sum_{g=0}^{\infty}\sum_{n=1}^{\infty}\frac{\hbar^{2g-2+n}}{n!}\sum_{\mu_1,\ldots,\mu_n=1}^{\infty}\vec{H}_{g,h}^{\bullet,\leq}(\mu_1,\ldots,\mu_n)\,x^{|\mu|}.$$

Collecting all factorizations for given $d = |\mu|$, we get

$$\Psi_{\hbar}^{\leq} = 1 + \sum_{g=0}^{\infty}\sum_{n=1}^{\infty}\frac{\hbar^{2g-2+n}}{n!}\sum_{d=0}^{\infty}\left(\sum_{|\mu|=d}H_{g,h}^{\bullet,\leq}(\mu)\right)x^d.$$

Recall that the number of transpositions is $r = r(g,n,|\mu|) = 2g-2+n-|\mu|(2h-1)$, so we write

$$\sum_{g=0}^{\infty}\sum_{n=1}^{\infty}\frac{\hbar^{2g-2+n}}{n!}\sum_{d=0}^{\infty}\left(\sum_{|\mu|=d}H_{g,h}^{\bullet,\leq}(\mu)\right)x^d = \sum_{g=0}^{\infty}\sum_{n=1}^{\infty}\sum_{d=0}^{\infty}\frac{\sum_{|\mu|=d}H_{g,h}^{\bullet,\leq}(\mu)}{n!}\hbar^r(x\hbar^{2h-1})^d$$
$$= \sum_{g=0}^{\infty}\sum_{n=1}^{\infty}\sum_{d=0}^{\infty}\frac{\sum_{|\mu|=d}H_{g,h}^{\bullet,\leq}(\mu)}{n!}\hbar^r(x\hbar^{1-\chi})^d.$$

Since $H_{g,h}^{\leq}(\mu)$ is non-zero if $r \geq 0$, we can rearrange the series by collecting all possible g, n for a given r and d. Viewing the partition function $\Psi_{\hbar}^{\leq}$ as a series in $\mathbb{Q}[\![\hbar, x\hbar^{1-\chi}]\!]$, we find that the coefficient of $x^d\hbar^{r+d(1-\chi)}$ is precisely the number of monotone base h factorizations of length r in $S(d)$, i.e.

$$[x^d\hbar^{r+d(1-\chi)}]\Psi_{\hbar}^{\leq} = \frac{1}{d!}\#\left\{(\tau_1,\ldots,\tau_r,\sigma,\alpha_1,\beta_1,\ldots,\alpha_h,\beta_h)\,\middle|\,\begin{array}{l}\tau_i \text{ monotone transpositions},\\ \sigma,\alpha_1,\beta_1\ldots\alpha_h,\beta_h\in S_d,\\ \sigma\tau_1\ldots\tau_r = [\alpha_1,\beta_1]\ldots[\alpha_h,\beta_h]\end{array}\right\}.$$

Since the α_i, β_i run over all elements in $S(d)$ and using [DDM17, Lemma 17], we obtain

$$\begin{aligned}[x^d \hbar^{b+d(1-\chi)}]\, \Psi_\hbar^{\leq} &= (d!)^{2g-1} \#\{(\tau_1, \dots, \tau_r) \mid \tau_i \text{ monotone transpositions } \} \\ &= (d!)^{1-\chi} \left\{ \begin{matrix} d+r-1 \\ d-1 \end{matrix} \right\}.\end{aligned}$$

Finally, we have

$$\begin{aligned}\Psi_\hbar^{\leq} &= 1 + \sum_{d=1}^{\infty} \sum_{r=0}^{\infty} \left\{ \begin{matrix} d+r-1 \\ d-1 \end{matrix} \right\} (d!)^{1-\chi} x^d \hbar^{r+d(1-\chi)} \\ &= 1 + \sum_{d=1}^{\infty} (d!)^{1-\chi} x^d \hbar^{d(1-\chi)} \sum_{r=0}^{\infty} \left\{ \begin{matrix} d+r-1 \\ d-1 \end{matrix} \right\} \hbar^r \\ &= 1 + \sum_{d=1}^{\infty} (d!)^{1-\chi} x^d \hbar^{d(1-\chi)} \prod_{j=1}^{d-1} \frac{1}{1-j\hbar},\end{aligned}$$

where we used the well-known identity in (3.2.1).

In the case of the strictly monotone Hurwitz numbers we do a similar calculation and view $\Psi_\hbar^{<}$ as an element of $\mathbb{Q}[\![\hbar, x\hbar^{1-\chi}]\!]$. We find

$$[x^d \hbar^{r+d(1-\chi)}]\, \Psi_\hbar^{<} = (d!)^{1-\chi} \#\{(\tau_1, \dots, \tau_b) \mid \tau_i \text{ strictly monotone transpositions } \}$$

and in particular $[x^d \hbar^{b+d(1-\chi)}]\, \Psi_\hbar^{<} = 0$ for $r \geq d$. These tuples can be expressed by evaluating the elementary symmetric polynomials in the Jucys–Murphy elements (defined in Section 1.3). This also yields an enumeration by evaluating the same polynomials in the number of summands of the i−th Jucys–Murphy element, which yields (see e.g. [KLS16, Equation (4)])

$$\#\{(\tau_1, \dots, \tau_r) \mid \tau_i \text{ strictly monotone transpositions } \} = \left[\begin{matrix} d \\ d-r \end{matrix} \right].$$

for $r \leq d$. Hence we have the assertion

$$\begin{aligned}\Psi_\hbar^{<} &= 1 + \sum_{d=1}^{\infty} \sum_{r=0}^{d-1} \left[\begin{matrix} d \\ d-r \end{matrix} \right] (d!)^{1-\chi} x^d \hbar^{r+d(1-\chi)} \\ &= 1 + \sum_{d=1}^{\infty} (d!)^{1-\chi} x^d \hbar^{1-\chi} \sum_{r=0}^{d-1} \left[\begin{matrix} d \\ d-r \end{matrix} \right] \hbar^d \\ &= 1 + \sum_{d=1}^{\infty} (d!)^{1-\chi} x^d \hbar^{1-\chi} \prod_{j=1}^{d-1} (1-j\hbar),\end{aligned}$$

where the last equality follows by (3.2.1). □

Theorem 3.3.5.
The partition function $\Psi_{\hbar}^{\leq}$ satisfies the differential equation

$$[\hat{\mathsf{x}}\hat{\mathsf{y}}^2 + \hat{\mathsf{y}} + (\hat{\mathsf{y}}\hat{\mathsf{x}})^{2h}]\Psi_{\hbar}^{\leq} = 0,$$

where $\hat{\mathsf{x}} = x\cdot$ and $\hat{\mathsf{y}} = -\hbar\frac{\partial}{\partial x}$.

Proof. The recursion formula for the Stirling numbers of the second kind yields

$$\left\{ {d+r-1 \atop d-1} \right\} = (d-1)\left\{ {d+r-2 \atop d-1} \right\} + \left\{ {d+r-2 \atop d-2} \right\}.$$

We multiply this equation by $\frac{(d!)^{2h}}{(d-1)!}x^d\hbar^{r+d(1-\chi)}$ and sum over $d \geq 1, r \geq 0$. For reasons of clarity, we first do the computations term by term before conflating them. For the term on the left hand side we have

$$\begin{aligned}
\sum_{\substack{d=1\\r=0}}^{\infty} \left\{ {d+r-1 \atop d-1} \right\} \frac{(d!)^{2h}}{(d-1)!} x^d\hbar^{r+d(1-\chi)} &= \sum_{\substack{d=1\\r=0}}^{\infty} \left\{ {d+r-1 \atop d-1} \right\} d(d!)^{1-\chi} x^d\hbar^{r+d(1-\chi)} \\
&= x\frac{\partial}{\partial x} \sum_{\substack{d=1\\r=0}}^{\infty} \left\{ {d+r-1 \atop d-1} \right\} (d!)^{1-\chi} x^d\hbar^{r+d(1-\chi)} \\
&= x\frac{\partial}{\partial x}\left(1 + \sum_{\substack{d=1\\r=0}}^{\infty} \left\{ {d+r-1 \atop d-1} \right\} (d!)^{1-\chi} x^d\hbar^{r+d(1-\chi)}\right) \\
&= x\frac{\partial}{\partial x}\Psi_{\hbar}^{\leq},
\end{aligned}$$

where we used the fact that the derivative of the constant 1 vanishes. For the first expression on the right hand side we get

$$\begin{aligned}
&\sum_{\substack{d=1\\r=0}}^{\infty} (d-1)\left\{ {d+r-2 \atop d-1} \right\} \frac{(d!)^{2h}}{(d-1)!} x^d\hbar^{r+d(1-\chi)} \\
&\quad = \sum_{\substack{d=1\\r=0}}^{\infty} d(d-1)\left\{ {d+r-2 \atop d-1} \right\} (d!)^{1-\chi} x^d\hbar^{r+d(1-\chi)} \\
&\quad = x^2\hbar\frac{\partial^2}{\partial x^2} \sum_{\substack{d=1\\r=0}}^{\infty} \left\{ {d+r-2 \atop d-1} \right\} (d!)^{1-\chi} x^d\hbar^{r-1+d(1-\chi)}.
\end{aligned}$$

Now, note that for $r = 0$ we have $\left\{ {d-2 \atop d-1} \right\} = 0$. Hence

$$\begin{aligned}
&x^2\hbar \frac{\partial^2}{\partial x^2} \sum_{\substack{d=1\\r=0}}^{\infty} \left\{ {d+r-2 \atop d-1} \right\} (d!)^{1-\chi} x^d \hbar^{r-1+d(1-\chi)} \\
&= x^2\hbar \frac{\partial^2}{\partial x^2} \sum_{\substack{d=1\\r=1}}^{\infty} \left\{ {d+r-2 \atop d-1} \right\} (d!)^{1-\chi} x^d \hbar^{r-1+d(1-\chi)} \\
&= x^2\hbar \frac{\partial^2}{\partial x^2} \sum_{\substack{d=1\\r=0}}^{\infty} \left\{ {d+r-1 \atop d-1} \right\} (d!)^{1-\chi} x^d \hbar^{r+d(1-\chi)} \\
&= x^2\hbar \frac{\partial^2}{\partial x^2} \Psi_{\hbar}^{\leq},
\end{aligned}$$

where we performed the shift $r' = r-1$ in the second equality. For the last term we obtain

$$\begin{aligned}
&\sum_{\substack{d=1\\r=0}}^{\infty} \left\{ {d+r-2 \atop d-2} \right\} \frac{(d!)^{2h}}{(d-1)!} x^d \hbar^{r+d(1-\chi)} \\
&= \sum_{\substack{d=1\\r=0}}^{\infty} d^{2h} \left\{ {d+r-2 \atop d-2} \right\} ((d-1)!)^{1-\chi} x^d \hbar^{r+d(1-\chi)} \\
&= \left(x\frac{\partial}{\partial x} \right)^{2h} x\hbar^{1-\chi} \sum_{\substack{d=1\\r=0}}^{\infty} \left\{ {d+r-2 \atop d-2} \right\} ((d-1)!)^{1-\chi} x^{d-1} \hbar^{r+(d-1)(1-\chi)} \\
&= \left(x\frac{\partial}{\partial x} \right)^{2h} x\hbar^{1-\chi} \left(1 + \sum_{\substack{d=1\\r=0}}^{\infty} \left\{ {d+r-1 \atop d-1} \right\} (d!)^{1-\chi} x^d \hbar^{r+d(1-\chi)} \right) \\
&= \frac{x}{\hbar} \left(\hbar\frac{\partial}{\partial x} x \right)^{2h} \left(1 + \sum_{\substack{d=1\\b=0}}^{\infty} \left\{ {d+r-1 \atop d-1} \right\} (d!)^{1-\chi} x^d \hbar^{r+d(1-\chi)} \right) \\
&= \frac{x}{\hbar} \left(\hbar\frac{\partial}{\partial x} x \right)^{2h} \Psi_{\hbar}^{\leq}.
\end{aligned}$$

Putting things together and multiplying by $\frac{\hbar}{x}$ we get

$$\left[x\hbar^2 \frac{\partial}{\partial x} - \hbar\frac{\partial}{\partial x} + \left(\hbar\frac{\partial}{\partial x} x \right)^{2h} \right] \Psi_{\hbar}^{\leq} = 0.$$

Substituting $\hat{x} = x$ and $\hat{y} = -\hbar\frac{\partial}{\partial x}$ we obtain the claim

$$[\hat{x}\hat{y}^2 + \hat{y} + (\hat{y}\hat{x})^{2h}]\Psi_{\hbar}^{\leq} = 0. \qquad \square$$

For the strictly monotone Hurwitz number we have a similar result.

Theorem 3.3.6.
The partition function $\Psi_\hbar^<$ satisfies the differential equation

$$[\hat{\mathsf{y}} + (1 - \widehat{\mathsf{x}\mathsf{y}})(\widehat{\mathsf{y}\mathsf{x}})^{2h}]\Psi_\hbar^< = 0,$$

where $\hat{\mathsf{x}} = x\cdot$ and $\hat{\mathsf{y}} = -\hbar\frac{\partial}{\partial x}$.

Proof. The proof is in the same spirit as in Theorem 3.3.5 but uses the Stirling numbers of the first kind instead. □

Remark 3.3.7.
Note that for $h = 0$, we recover the quantum curve for the usual monotone (strictly monotone) Hurwitz numbers of [DDM17] (resp. [DM14]).

3.4 A mysterious topological recursion

In this section, we consider the quantum curve of the strictly monotone Hurwitz numbers derived in Theorem 3.3.6. We focus on the case with elliptic base curve, i.e. on $h = 1$. Computing the semi-classical limit, we obtain the spectral curve

$$y + (1 - xy)(xy)^2 = 0, \tag{3.4.1}$$

which can be parameterized by

$$x(z) = \frac{(z-1)^2}{z}, \quad y(z) = \frac{z}{(z-1)^3}. \tag{3.4.2}$$

Surprisingly, running topological recursion for this input data yields the monotone single Hurwitz numbers with an additional combinatorial prefactor. More precisely, we obtain the cumulants of the Weingarten function. This points towards an unknown relationship between the combinatorics of strictly monotone Hurwitz numbers with elliptic base curve and monotone Hurwitz numbers with rational base curve.

As already noted in the introduction, topological recursion for monotone single Hurwitz numbers with rational base curves and a different normalisation was proved in [DDM17], however for a different spectral curve and the exclusion of the $(0,2)$-case. In our normalisation, the $(0,2)$-case still encodes the relevant invariants. It is important to note that by (3.4.3) it can be easily deduced that the symplectic transformation

$$(x, y) \mapsto (-\frac{1}{x}, x^2 y)$$

maps the spectral curve (3.4.1) (after cancelling $y = 0$) to the spectral curve

$$xy^2 + y + 1 = 0$$

given in [DDM17]. This has been unnoticed up to now. By [EO07, Theorem 7.1] it follows that the according symplectic invariants agree, moreover in the proof of their Theorem 7.1 they note that the $W_{g,n}$ must be related by a sign[3]. However we still include the proof of TR for the weighted monotone Hurwitz numbers along the lines of [DDM17] for the sake of completeness.

We begin by defining the correct normalisation of monotone single Hurwitz numbers for our purpose, which coincide with the cumulants of the Weingarten function. The latter was motivated by discussions with James Mingo on problems of higher order freeness in free probability [CMSS07]. In particular using the following normalizations, the numbers coincide with certain values of the Möbius function on the set of partitioned permutations, see Section 2.2.1.

Definition 3.4.1.
Let g be a nonnegative integer, d, n be a positive integers and μ a partition d of length l. We denote by $m_{g,n}(\mu)$ the number of connected labeled monotone factorizations of a fixed (but arbitrary) permutation σ with $c(\sigma) = \mu$. Moreover we put

$$C_{g,n}(\mu) = (-1)^{2g-2+n+|\mu|} m_{g,n}(\mu) = (-1)^{n+|\mu|} m_{g,n}(\mu)$$

and denote by $W_{g,n}(x_1, \dots, x_n)$ the corresponding generating series, i.e.

$$W_{g,n}(x_1, \dots, x_n) = \sum_{\mu_1,\dots,\mu_n=1}^{\infty} \frac{C_{g,n}(\mu_1, \dots, \mu_n)}{x_1^{\mu_1+1} \dots x_n^{\mu_n+1}}.$$

Remark 3.4.2.

i) The numbers $C_{g,n}(\mu)$ agree with the monotone Hurwitz numbers up to the combinatorial factor $(-1)^{n+|\mu|} \prod_{i=1}^{l} \mu_i$.

ii) When we drop the connectivity condition in the definition of $m_{g,n}(\mu)$, we obtain the disconnected analogues of $C_{g,n}(\mu)$. These numbers are the coefficients of the asymptotic expansion of the Weingarten function [GGPN14].

iii) In [GGPN13] and [DDM17], the numbers $m_{g,n}(\mu)$ are put into a generating series via

$$M_{g,n}(y_1, \dots, y_n) = \sum_{\mu_1,\dots,\mu_n=1}^{\infty} m_{g,n}(\mu_1, \dots, \mu_n)\, y_1^{\mu_1-1} \cdots y_n^{\mu_n-1}.$$

Thus their generating series relates to $W_{g,n}(x_1, \dots, x_n)$ via

$$W_{g,n}(x_1, \dots, x_n) = \frac{M_{g,n}(\frac{-1}{x_1}, \dots, \frac{-1}{x_n})}{x_1^2 \cdots x_n^2}.$$

[3]The authors of [EO07] actually claim that the $W_{g,n}$ agree, however we think that it might be a small error as we have a sign in our particular example.

Note also that this amounts to a change of variable $y_i = \frac{-1}{x_i}$ if we phrase things in the language of differential forms, i.e.

$$M_{g,n}(y_1, \dots, y_n)\mathrm{d}y_1 \cdots \mathrm{d}y_n = W_{g,n}(x_1, \dots, x_n)\mathrm{d}x_1 \cdots \mathrm{d}x_n. \tag{3.4.3}$$

The following is the main theorem of this section, which we prove in Section 3.4.1.

Theorem 3.4.3.
The numbers $C_{g,n}(\mu)$ satisfy topological recursion with the spectral curve given by

$$x(z) = \frac{(z-1)^2}{z}, \quad y(z) = \frac{z}{(z-1)^3},$$

i.e. the differentials

$$\omega_{g,n}(z_1, \dots, z_n) = \sum_{\mu_1,\dots,\mu_n=1}^{\infty} \frac{C_{g,n}(\mu_1, \dots, \mu_n)}{x(z_1)^{\mu_1+1} \dots x(z_n)^{\mu_n+1}} \mathrm{d}x(z_1) \cdots \mathrm{d}x(z_n) \quad \text{for } (g,n) \neq (0,2)$$

$$\omega_{0,2}(z_1, z_2) = \sum_{\mu_1,\mu_2=1}^{\infty} \frac{C_{0,2}(\mu_1, \mu_2)}{x(z_1)^{\mu_1+1} x(z_2)^{\mu_2+1}} \mathrm{d}x(z_1)\mathrm{d}x(z_2) + \frac{\mathrm{d}x(z_1)\mathrm{d}x(z_2)}{(x(z_1) - x(z_2))^2}$$

satisfy the recursion

$$\begin{aligned} \omega_{g,n}(z_1, \dots, z_n) = \operatorname*{Res}_{z\to\pm1} K(z_1, z) \Bigg[& \omega_{g-1,n+1}(z, \sigma(z), z_2, \dots, z_n) \\ & + \sum_{\substack{g_1+g_2=g \\ I \sqcup J = N\setminus\{1\}}}^{\prime} \omega_{g_1,|I|+1}(z, z_I)\omega_{g_2,|J|+1}(\sigma(z), z_J) \Bigg] \end{aligned} \tag{3.4.4}$$

on $2g - 2 + n > 0$,

$$K(z_1, z) = \frac{\frac{1}{2}\int_{\sigma(z)}^{z} \omega(z_1, \cdot)}{\omega_{0,1}(z) - \omega_{0,1}(\sigma(z))} = \frac{z(z-1)^3 \mathrm{d}z_1}{2(z+1)(z_1 - z)(z_1 z - 1)}, \quad \sigma(z) = \frac{1}{z}$$

and with the initial data given by

$$\omega_{0,1}(z) = y\mathrm{d}x \quad \text{and} \quad \omega_{0,2}(z_1, z_2) = B(z_1, z_2) := \frac{\mathrm{d}z_1 \mathrm{d}z_2}{(z_1 - z_2)^2}.$$

Remark 3.4.4.
Note that $\mathrm{d}x(z) = \frac{z^2-1}{z^2}$ has the zeroes $z = \pm 1$ and since $y(z)$ has a pole of order bigger than 1 at $z = 1$, the spectral curve (x, y) is irregular in the sense of [DN18, section 2.1 item 2(b)]. Hence the invariants $\omega_{g,n}$ agree with the invariants obtained from the local spectral curve obtained by removing the point $z = 1$. In particular the residue at $z = 1$ in (3.4.4) does not contribute and it suffices to compute the residue at $z = -1$.

Moreover, we note that while $\omega_{0,2} \neq W_{0,2}\mathrm{d}x(z_1)\mathrm{d}x(z_2)$, we have

$$\operatorname*{Res}_{z_1,z_2\to\infty} x(z_1)^{\mu_1} x(z_2)^{\mu_2} \omega_{0,2}(z_1, z_2) = C_{0,2}(\mu_1, \mu_2) \tag{3.4.5}$$

since their difference is holomorphic by definition.

The starting point of our proof is the following recursion, which is a direct consequence of [GGPN13, Theorem 2.1].

Proposition 3.4.5.
Let g be a nonnegative integer, $n \in \mathbb{N}$ and $\mu = (\mu_1, \dots, \mu_n)$ a partition of a positive integer. Then we have the recursion

$$
\begin{aligned}
-C_{g,n}(\mu) = &\sum_{j=2}^{n} \mu_j \, C_{g,n-1}(\mu_1 + \mu_j, \mu_{N\setminus\{1\}}) + \sum_{\alpha+\beta=\mu_1} C_{g-1,n+1}(\alpha, \beta, \mu_{N\setminus\{1\}}) \\
&+ \sum_{\alpha+\beta=\mu_1} \sum_{\substack{g_1+g_2=g \\ I \sqcup J = N\setminus\{1\}}} C_{g_1,1+|I|}(\alpha, \mu_I) \, C_{g_2,1+|J|}(\beta, \mu_J),
\end{aligned}
$$

where $\mu_I = (\mu_{i_1}, \dots, \mu_{i_k})$ for $I = \{i_1, \dots, i_k\}$, $N = \{1, \dots, n\}$ and the initial value $C_{0,1}(1) = 1$.

The following proposition reformulates the cut-and-join equation of Proposition 3.4.5 as a differential equation for generating series.

Proposition 3.4.6.
It holds that

$$
\begin{aligned}
&-W_{g,n}(x_1, \dots, x_n) \\
&\quad = \sum_{j=2}^{n} \frac{\partial}{\partial x_j} \frac{x_j}{x_1} \frac{x_1 W_{g,n-1}(X_{\{1,\dots,n\}\setminus\{j\}}) - x_j W_{g,n-1}(X_{\{1,\dots,n\}\setminus\{1\}})}{x_1 - x_j} \\
&\quad\quad + x_1 W_{g-1,n+1}(x_1, X_{\{1,\dots,n\}}) + x_1 \sum_{\substack{g_1+g_2=g \\ I \sqcup J = N\setminus\{1\}}} W_{g_1,|I|+1}(x_1, X_I) W_{g_2,|J|+1}(x_1, X_J) \\
&\quad\quad - \frac{1}{x^2} \delta_{g,0} \delta_{n,1},
\end{aligned}
$$

where $X_I = (x_{i_1}, \dots, x_{i_k})$ for $I = \{i_1, \dots, i_k\} \subset N$.

Proof. We multiply the cut-and-join equations from Proposition 3.4.5 by the variables $x_1^{-(\mu_1+1)} \dots x_n^{-(\mu_n+1)}$ and sum over $\mu_1, \mu_2 \dots \mu_n \geq 1$. We start by dealing with the first term on the right hand side. First observe that for fixed $2 \leq j \leq n$ we have

$$
\begin{aligned}
\sum_{\mu_1, \mu_j = 1}^{\infty} \mu_j \frac{C_{g,n-1}(\mu_1 + \mu_j, \mu_{N\setminus\{\mu_j\}})}{x_1^{\mu_1+1} x_j^{\mu_j+1}} &= -\frac{\partial}{\partial x_j} \frac{1}{x_1^2} \sum_{\mu_1 \geq 1, \mu_j \geq 0} \frac{C_{g,n-1}(\mu_1 + \mu_j, \mu_{N\setminus\{\mu_j\}})}{x_1^{\mu_1-1} x_j^{\mu_j}} \\
&= -\frac{\partial}{\partial x_j} \frac{1}{x_1^2} \sum_{\nu=0}^{\infty} \sum_{\substack{\mu_1+\mu_j=\nu \\ \mu_1, \mu_j \geq 0}} \frac{C_{g,n-1}(\nu + 1, \mu_{N\setminus\{\mu_j\}})}{x_1^{\mu_1} x_j^{\mu_j}}.
\end{aligned}
$$

We note that

$$\sum_{\mu_1+\mu_j=\nu} \frac{1}{x_1^{\mu_1} x_j^{\mu_j}} = -x_1 x_j \frac{\frac{1}{x_1^{\nu+1}} - \frac{1}{x_j^{\nu+1}}}{x_1 - x_j}$$

and hence we find

$$\begin{aligned}
-\frac{\partial}{\partial x_j} \frac{1}{x_1^2} \sum_{\nu=0}^{\infty} \sum_{\substack{\mu_1+\mu_j=\nu \\ \mu_1,\mu_j \geq 0}} & \frac{C_{g,n-1}(\nu+1, \mu_{N\setminus\{\mu_j\}})}{x_1^{\mu_1} x_j^{\mu_j}} \\
&= \frac{\partial}{\partial x_j} \frac{x_j}{x_1} \sum_{\nu=0}^{\infty} \frac{C_{g,n-1}(\nu+1, \mu_{N\setminus\{\mu_j\}})(\frac{x_1}{x_1^{\nu+2}} - \frac{x_j}{x_j^{\nu+2}})}{x_1 - x_j}.
\end{aligned}$$

Further, we can put this in the summation over all $\mu_1, \ldots, \mu_n$ and we obtain

$$\begin{aligned}
\sum_{\mu_1,\ldots,\mu_n=1}^{\infty} & \frac{\mu_j C_{g,n-1}(\mu_1+\mu_j, \mu_{N\setminus\{\mu_j\}})}{x_1^{\mu_1+1} \ldots x_n^{\mu_n+1}} \\
&= \frac{\partial}{\partial x_j} \frac{x_j}{x_1} \frac{x_1 W_{g,n-1}(X_{\{1,\ldots,n\}\setminus\{j\}}) - x_j W_{g,n-1}(X_{\{1,\ldots,n\}\setminus\{1\}})}{x_1 - x_j}.
\end{aligned}$$

We proceed analogously for the second term and observe that

$$\begin{aligned}
\sum_{\mu_1,\ldots,\mu_n=1}^{\infty} \sum_{\alpha+\beta=\mu_1} \frac{C_{g-1,n+1}(\alpha,\beta,\mu_{N\setminus\{1\}})}{x_1^{\mu_1+1} \ldots x_n^{\mu_n+1}} &= \sum_{\mu_2,\ldots\mu_n,\alpha,\beta=1}^{\infty} \frac{C_{g-1,n+1}(\alpha,\beta,\mu_{N\setminus\{1\}})}{x_1^{\alpha+\beta+1} \ldots x_n^{\mu_n+1}} \\
&= x_1 W_{g-1,n+1}(x_1, X_{\{1,\ldots,x_n\}}).
\end{aligned}$$

Finally, for the third term we obtain

$$\begin{aligned}
\sum_{\mu_1,\ldots,\mu_n=1}^{\infty} \sum_{\alpha+\beta=\mu_1} & \sum_{\substack{g_1+g_2=g \\ I\sqcup J=N\setminus\{1\}}} \frac{C_{g_1,1+|I|}(\alpha,\mu_I) C_{g_2,1+|J|}(\beta,\mu_J)}{x_1^{\mu_1+1} \ldots x_n^{\mu_n+1}} \\
&= x_1 \sum_{\substack{g_1+g_2=g \\ I\sqcup J=N\setminus\{1\}}} \sum_{\substack{\alpha,\mu_i=1 \\ i\in I}}^{\infty} \frac{C_{g_1,1+|I|}(\alpha,\mu_I)}{x_1^{\alpha+1} \prod_{i\in I} x_i^{\mu_i+1}} \sum_{\substack{\alpha,\mu_i=1 \\ i\in J}}^{\infty} \frac{C_{g_1,1+|I|}(\beta,\mu_I)}{x_1^{\beta+1} \prod_{i\in J} x_i^{\mu_i+1}} \\
&= x_1 \sum_{\substack{g_1+g_2=g \\ I\sqcup J=N\setminus\{1\}}} W_{g_1,|I|+1}(x_1, X_I) W_{g_2,|J|+1}(x_1, X_J).
\end{aligned}$$

Putting everything together yields the desired equation. □

In the perspective of CEO topological recursion it is handy to rewrite the cut-and-join equation in way that $W_{g,n}(x)$ does not appear on the right hand side.

Corollary 3.4.7.
It holds

$$
\begin{aligned}
-(1+2x_1W_{0,1}(x_1))\,W_{g,n}(x_1,\dots,x_n)&\\
=\sum_{j=2}^{n}\frac{\partial}{\partial x_j}\frac{x_j}{x_1}\frac{x_1W_{g,n-1}(X_{\{1,\dots,n\}\setminus\{j\}})-x_jW_{g,n-1}(X_{\{1,\dots,n\}\setminus\{1\}})}{x_1-x_j}&+\\
+x_1W_{g-1,n+1}(x_1,X_{\{1,\dots,n\}})+x_1\sum_{\substack{g_1+g_2=g\\ I\sqcup J=N\setminus\{1\}}}^{\prime}W_{g_1,|I|+1}(x_1,X_I)W_{g_2,|J|+1}(x_1,X_J)&\\
-\frac{1}{x^2}\delta_{g,0}\delta_{n,1},&
\end{aligned}
$$

where $\sum'$ means that the cases $(g_1,I)=(0,\emptyset)$ or $(g_2,J)=(0,\emptyset)$ are excluded.

We now compute some special cases of $W_{g,n}$, which require special treatment in the CEO topological recursion. We have the following result for the first few values of (g,n).

Corollary 3.4.8 ([GGPN13, Theorem 1.1],[GGPN16, Theorem 6.2])**.**
We have that

$$
\begin{aligned}
C_{0,1}(\mu)&=(-1)^{\mu-1}\frac{1}{\mu}\binom{2\mu-2}{\mu-1},\\
C_{0,2}(\mu_1,\mu_2)&=(-1)^{\mu_1+\mu_2}\frac{2\mu_1\mu_2}{\mu_1+\mu_2}\binom{2\mu_1-1}{\mu_1}\binom{2\mu_2-1}{\mu_2},\\
C_{0,3}(\mu_1,\mu_2,\mu_3)&=(-1)^{\mu_1+\mu_2+\mu_3-1}8\mu_1\binom{2\mu_1-1}{\mu_1}\mu_2\binom{2\mu_2-1}{\mu_2}\mu_3\binom{2\mu_3-1}{\mu_3}.
\end{aligned}
$$

A straightforward calculation shows the following lemma.

Lemma 3.4.9.
The following identities hold:

$$
\begin{aligned}
W_{0,1}(x(z))&=\frac{z}{(z-1)^3},\\
W_{0,2}(x(z_1),x(z_2))&=\frac{z_1^2z_2^2}{(z_1^2-1)(z_2^2-1)(1-z_1z_2)^2},\\
W_{0,3}(x_1,x_2,x_3)&=\frac{8}{x_1^2x_2^2x_3^2(1+\frac{4}{x_1})^{\frac{3}{2}}(1+\frac{4}{x_2})^{\frac{3}{2}}(1+\frac{4}{x_3})^{\frac{3}{2}}}\\
&=\prod_{i=1}^{3}\frac{2}{x_i^2(1+\frac{4}{x_i})^{\frac{3}{2}}}\\
&=\prod_{i=1}^{3}\frac{2}{(z_i+1)^2}\frac{1}{x'(z_i)}.
\end{aligned}
$$

The next lemma is a key step towards proving the topological recursion for the numbers $C_{g,n}(\mu)$, as determining the difference between the Bergman kernel and the $(0,2)$ free energy is important for the input data of the topological recursion.

Lemma 3.4.10.
We have

$$W_{0,2}(x(z_1),x(z_2))\mathrm{d}x(z_1)\mathrm{d}x(z_2) = \frac{\mathrm{d}z_1\mathrm{d}z_2}{(1-z_1z_2)^2} = \frac{\mathrm{d}z_1\mathrm{d}z_2}{(z_1-z_2)^2} - \frac{\mathrm{d}x(z_1)\mathrm{d}x(z_2)}{(x(z_1)-x(z_2))^2}$$

and in particular

$$W_{0,2}(x(z_1),x(z_2))\mathrm{d}x(z_1)\mathrm{d}x(z_2) = -B(\frac{1}{z_1},z_2).$$

Proof. From the last proposition and $x'(z_i) = \frac{z_i^2-1}{z_i^2}$ we obtain

$$W_{0,2}(x(z_1),x(z_2)) = \frac{z_1^2}{(z_1^2-1)}\frac{z_2^2}{(z_2^2-1)}\frac{1}{(1-z_1z_2)^2} = \frac{1}{x'(z_1)}\frac{1}{x'(z_2)}\frac{1}{(1-z_1z_2)^2},$$

from which the first equality follows immediately. For the second one, a straightforward calculation yields

$$(x(z_1)-x(z_2))^2 = \left(\frac{z_1^2+1}{z_1} - \frac{z_2^2+1}{z_2}\right)^2 = \frac{(1-z_1z_2)^2(z_1-z_2)^2}{z_1^2z_2^2},$$

which yields

$$\frac{1}{(z_1-z_2)^2} - \frac{x'(z_1)x'(z_2)}{(x(z_1)-x(z_2))^2} = \frac{1}{(z_1-z_2)^2} - \frac{(z_1^2-1)(z_2^2-1)}{(1-z_1z_2)^2(z_1-z_2)^2} = \frac{1}{(1-z_1z_2)^2}. \square$$

3.4.1 Proof of Theorem 3.4.3

Our proof of Theorem 3.4.3 is inspired by the approach in [DDM17]. We begin by considering the case $(g,n) = (0,3)$, which requires an independent discussion.

Lemma 3.4.11.
The multidifferential $\omega_{0,3}$ satisfies the recursion in (3.4.4).

Proof. Recall that by Lemma 3.4.9, we have

$$\begin{aligned} W_{0,3}(x_1,x_2,x_3) &= \frac{8}{x_1^2x_2^2x_3^2(1+\frac{4}{x_1})^{\frac{3}{2}}(1+\frac{4}{x_2})^{\frac{3}{2}}(1+\frac{4}{x_3})^{\frac{3}{2}}} \\ &= \prod_{i=1}^{3}\frac{2}{x_i^2(1+\frac{4}{x_i})^{\frac{3}{2}}}. \end{aligned}$$

We find

$$\begin{aligned}
W_{0,3}(z_1,z_2,z_3) &= \prod_{i=1}^{3} \frac{2}{x_i(z)^2(1+\frac{4}{x_i(z)})^{\frac{3}{2}}} \\
&= \prod_{i=1}^{3} \frac{2z_i^2(z_i-1)^3}{(z_i-1)^4(z_i+1)^3} \\
&= \prod_{i=1}^{3} \frac{2}{(z_i+1)^2} \frac{z_i^2}{(z_i+1)(z_i-1)} \\
&= \prod_{i=1}^{3} \frac{2}{(z_i+1)^2} \frac{1}{x'(z_i)}.
\end{aligned}$$

The recursion from topological recursion reads

$$\begin{aligned}
&\omega_{0,3}(z_1,z_2,z_3) \\
&= \operatorname*{Res}_{z\to -1} K(z_1,z)\left[w_{0,2}(z,z_2)\, w_{0,2}(\frac{1}{z},z_3) + w_{0,2}(z,z_3)\, w_{0,2}(\frac{1}{z},z_2)\right] \\
&= \operatorname*{Res}_{z\to -1} \frac{z(z-1)^3 \mathrm{d}z_1}{2(z+1)(z_1-z)(z_1 z-1)\mathrm{d}z}\left[\frac{\mathrm{d}z\mathrm{d}z_2}{(z-z_2)^2}\frac{\mathrm{d}\frac{1}{z}\mathrm{d}z_3}{(\frac{1}{z}-z_3)^2} + \frac{\mathrm{d}z\mathrm{d}z_3}{(z-z_3)^2}\frac{\mathrm{d}\frac{1}{z}\mathrm{d}z_2}{(\frac{1}{z}-z_2)^2}\right] \\
&= \operatorname*{Res}_{z\to -1} \frac{1}{z+1}\frac{-(z-1)^3\mathrm{d}z_1}{2z(z_1-z)(z_1 z-1)\mathrm{d}z}\left[\frac{\mathrm{d}z\mathrm{d}z_1\mathrm{d}z_2\mathrm{d}z_3}{(z-z_2)^2(\frac{1}{z}-z_3)^2} + \frac{\mathrm{d}z\mathrm{d}z_1\mathrm{d}z_2\mathrm{d}z_3}{(z-z_3)^2(\frac{1}{z}-z_2)^2}\right],
\end{aligned}$$

which is of the form

$$\frac{f(z,z_1,z_2,z_3)\mathrm{d}z}{(z+1)}\mathrm{d}z_1\mathrm{d}z_2\mathrm{d}z_3$$

where f is holomorphic in z around $z=-1$. Hence we get

$$\omega_{0,3}(z_1,z_2,z_3) = f(-1,z_1,z_2,z_3)\mathrm{d}z_1\mathrm{d}z_2\mathrm{d}z_3 = \frac{8\mathrm{d}z_1\mathrm{d}z_2\mathrm{d}z_3}{(z_1+1)^2(z_2+1)^2(z_3+1)^2},$$

which concludes the proof. □

Recall the polynomiality result for monotone Hurwitz numbers.

Theorem 3.4.12 ([GGPN16]).
There are symmetric rational functions $\vec{P}_{g,h}$ such that

$$\vec{H}_{g,n}(\mu_1,\dots,\mu_n) = \prod_{i=1}^{n}\binom{2\mu_i}{\mu_i}\vec{P}_{g,n}(\mu_1,\mu_2,\dots,\mu_n).$$

Moreover, if $(g,n)\neq(0,1),(0,2)$, then $\vec{P}_{g,n}$ is a polynomial with rational coefficients of degree $3g-3+n$.

Since $C_{g,n}$ agrees with $\vec{H}_{g,h}$ up to the factor $(-1)^r \prod_{i=1}^n \mu_i$, we immediately get

$$C_{g,n} = (-1)^r \prod_{i=1}^{n} \mu_i \binom{2\mu_i}{\mu_i} \vec{P}_{g,n}(\mu_1, \mu_2, \ldots, \mu_n).$$

Thus, for the polynomial

$$\vec{P}_{g,n}(\mu_1, \ldots, \mu_n) = \sum_{\underline{a}=\underline{0}}^{\text{finite}} B_{g,n}(\underline{a}) \prod \mu_i^{a_i} \tag{3.4.6}$$

with coefficients $B_{g,n}(\underline{a})$, we can write our generating function as

$$\begin{aligned} W_{g,n}(x_1, \ldots, x_n) &= \sum_{\underline{a}=\underline{0}}^{\text{finite}} B_{g,n}(\underline{a}) \prod_{i=1}^{n} \sum_{\mu_i=1}^{\infty} \mu_i^{a_i+1} \binom{2\mu_i}{\mu_i} \left(\frac{-1}{x_i}\right)^{\mu_i+1} \\ &= \sum_{\underline{a}=\underline{0}}^{\text{finite}} B_{g,n}(\underline{a}) \prod_{i=1}^{n} f_{a_i}(x_i) \end{aligned}$$

with

$$f_a(x) = \sum_{\mu=1}^{\infty} \mu^{a+1} \binom{2\mu}{\mu} \left(\frac{-1}{x}\right)^{\mu+1}. \tag{3.4.7}$$

A careful analysis of the functions f_a will give us the analytic properties of $W_{g,n}$.

Lemma 3.4.13.
Let $(g, n) \neq (0, 1), (0, 2)$, then the functions $W_{g,n}(z_1, \ldots, z_n)$ satisfy

$$W_{g,n}(z_1, \ldots, z_n) = -W_{g,n}(\sigma(z_1), z_2, \ldots, z_n) = -W_{g,n}(\frac{1}{z_1}, z_2, \ldots, z_n).$$

Moreover they are rational functions in each z_i having poles at $z_i = 1$ and at $z_i = -1$.

Proof. Note that the functions f_a satisfy the recursion

$$f_a(x) = \sum_{\mu=1}^{\infty} \mu^{a+1} \binom{2\mu}{\mu} \left(\frac{-1}{x}\right)^{\mu+1} = -\frac{\partial}{\partial x} x f_{a-1}(x), \tag{3.4.8}$$

i.e.

$$f_a(x) = (-\frac{\partial}{\partial x} x)^a f_0(x)$$

and

$$f_0(x) = \sum_{n=1}^{\infty} \mu \binom{2\mu}{\mu} \left(\frac{-1}{x}\right)^{\mu+1} = \frac{2}{\sqrt{x}(x+4)^{\frac{3}{2}}}.$$

In the variable z we get

$$f_0(z) = \frac{2z^2}{(z-1)(z+1)^3}$$

and

$$f_0(\sigma(z)) = f_0\Big(\frac{1}{z}\Big) = \frac{2\frac{1}{z^2}}{(\frac{1}{z}-1)(\frac{1}{z}+1)^3} = -\frac{2z^2}{(z-1)(z+1)^3} = -f_0(z).$$

We find by induction

$$f_a(z) + f_a(\sigma(z)) = \frac{\partial}{\partial x} x(f_{a-1}(z) + f_{a-1}(\sigma(z))) = 0$$

and hence

$$\begin{aligned} W_{g,n}(z_1, z_2, \dots, z_n) + W_{g,n}(\sigma(z_1), z_2, \dots, z_n) &= \sum_{\underline{a}=\underline{0}}^{\text{finite}} B_{g,n}(\underline{a})[f_{a_1}(z) + f_{a_1}(\sigma(z))] \prod_{i=2}^{n} f_{a_i}(x_i) \\ &= 0. \end{aligned}$$

Moreover note that (3.4.8) reads

$$f_a(z) = \left(\frac{-z^2}{z^2-1} \frac{\partial}{\partial z} \frac{(z-1)^2}{z} \right) f_{a-1}(z)$$

in the variable z. It follows by induction that f_a is rational and has a pole of order 1 at $z = 1$ and a pole of order $2a + 3$ of $z = -1$. □

The last result can be reformulated in terms of the forms $\omega_{g,n}$.

Corollary 3.4.14.
For $(g, n) \neq (0, 1), (0, 2)$, the forms $\omega_{g,n}(z_1, \dots, z_n)$ are antisymmetric w.r.t. σ, i.e.

$$\omega_{g,n}(z_1, \dots, z_n) = -\omega_{g,n}(\sigma(z_1), z_2, \dots, z_n) = -\omega_{g,n}\left(\frac{1}{z_1}, z_2, \dots, z_n\right).$$

They only have poles at $z = \pm 1$, where the pole at $z = 1$ has at most order 1.

Proof. The assertions follow from the Lemma 3.4.13. □

Now we are ready to prove Theorem 3.4.3.

Proof of Theorem 3.4.3. The initial data is given by $(g, n) = (0, 1), (0, 2)$ and the case of $(g, n) = (0, 3)$ was proved in Lemma 3.4.11. Thus, in the following we assume that $(g, n) \neq (0, 1), (0, 2), (0, 3)$.

The idea is to add the recursions for $W_{g,n}(z_1, z_2, \dots, z_n)$ and $W_{g,n}(\sigma(z_1), z_2, \dots, z_n)$ and proceed with a careful combinatorial analysis after substituting the identity in Lemma 3.4.13.

- First we note that the left hand side yields the following

$$
\begin{aligned}
&-(1+2x_1W_{0,1}(z_1))W_{g,n}(z_1,z_2,\dots,z_n)-(1+2x_1W_{0,1}(\sigma(z_1)))W_{g,n}(\sigma(z_1),z_2,\dots,z_n)\\
&=-(1+2x_1W_{0,1}(z_1))W_{g,n}(z_1,z_2,\dots,z_n)+(1+2x_1W_{0,1}(\sigma(z_1)))W_{g,n}(z_1,z_2,\dots,z_n)\\
&=-2x_1[W_{0,1}(z_1)-W_{0,1}(\sigma(z_1))]W_{g,n}(z_1,z_2,\dots,z_n).
\end{aligned}
$$

- The first term on the right hand side is

$$\sum_{j=2}^{n}\frac{\partial}{\partial x_j}\frac{x_j}{x_1}\frac{x_1W_{g,n-1}(Z_{\{1,\dots,n\}\setminus\{j\}})-x_jW_{g,n-1}(Z_{\{1,\dots,n\}\setminus\{1\}})}{x_1-x_j}$$

and its counterpart where z_1 is replaced by $\sigma(z_1)$. First we note that we can rewrite the derivatives by

$$\frac{\partial}{\partial x}=\frac{z^2}{z^2-1}\frac{\partial}{\partial z}.$$

Now we want to focus on the change of z_1 with $\sigma(z_1)$, using $x(\sigma(z_1))=x(z_1)$ we get terms of the form

$$\frac{z_j^2}{z_j^2-1}\frac{\partial}{\partial z_j}\frac{x(z_j)}{x(z_1)}\frac{x(\sigma(z_1))W_{g,n-1}(\sigma(z_1),Z_{\{2,\dots,n\}\setminus\{j\}})-x_jW_{g,n-1}(Z_{\{1,\dots,n\}\setminus\{1\}})}{x_1-x_j}.$$

By our observations the latter is equivalent to

$$\frac{z_j^2}{z_j^2-1}\frac{\partial}{\partial z_j}\frac{x(z_j)}{x(z_1)}\frac{-x_1W_{g,n-1}(Z_{\{1,\dots,n\}\setminus\{j\}})-x_jW_{g,n-1}(Z_{\{1,\dots,n\}\setminus\{1\}})}{x_1-x_j}.$$

Thus $x_1W_{g,n-1}(Z_{\{1,\dots,n\}\setminus\{j\}})$ cancels in the sum and we end up with the term

$$-2\sum_{j=2}^{n}\frac{z_j^2}{z_j^2-1}\frac{\partial}{\partial z_j}\frac{x(z_j)^2}{x(z_1)}\frac{W_{g,n-1}(Z_{\{1,\dots,n\}\setminus\{1\}})}{x_1-x_j}.$$

- The second term on the right hand side is

$$
\begin{aligned}
x_1\Big(W_{g-1,n+1}(z_1,z_1,z_2,\dots,z_n)+W_{g-1,n+1}(\sigma(z_1),\sigma(z_1),z_2,\dots,z_n)\Big)&\\
=-2x_1W_{g-1,n+1}(z_1,\sigma(z_1),z_2,\dots,z_n).&
\end{aligned}
$$

- The third term is

$$
\begin{aligned}
x_1\Big[& \sum_{\substack{g_1+g_2=g \\ I\sqcup J=N\setminus\{1\}}} W_{g_1,|I|+1}(z_1,Z_I)W_{g_2,|J|+1}(z_1,Z_J) \\
&+ \sum_{\substack{g_1+g_2=g \\ I\sqcup J=N\setminus\{1\}}} W_{g_1,|I|+1}(\sigma(z_1),Z_I)W_{g_2,|J|+1}(\sigma(z_1),Z_J) \\
&+ 2\sum_{j=2}^{n}\Big(W_{0,2}(z_1,z_j)W_{g,n-1}(z_1,Z_{\{1,\dots,n\}\setminus\{j\}}) \\
&+ W_{0,2}(\sigma(z_1),z_j)W_{g,n-1}(\sigma(z_1),Z_{\{1,\dots,n\}\setminus\{j\}})\Big)\Big].
\end{aligned}
$$

By Lemma 3.4.10 and Corollary 3.4.14 this yields

$$
\begin{aligned}
-2x_1\Big[& \sum_{\substack{g_1+g_2=g \\ I\sqcup J=N\setminus\{1\}}} W_{g_1,|I|+1}(z_1,Z_I)W_{g_2,|J|+1}(\sigma(z_1),Z_J)+ \\
&+ \sum_{j=2}^{n}\Big(\frac{B(\sigma(z_1),z_j)}{\mathrm{d}x(z_1)\mathrm{d}x(z_j)}W_{g,n-1}(z_1,Z_{\{1,\dots,n\}\setminus\{j\}}) \\
&+ \frac{B(z_1,z_j)}{\mathrm{d}x(\sigma(z_1))\mathrm{d}x(z_j)}W_{g,n-1}(\sigma(z_1),Z_{\{1,\dots,n\}\setminus\{j\}})\Big)\Big].
\end{aligned}
$$

As $(g,n)\neq(0,1),(0,2),(0,3)$ we have $(g,n-1)\neq(0,1),(0,2)$. Thus the differentials $W_{g,n-1}dx(z_1)\cdots dx(z_{n-1})$ satisfies Corollary 3.4.14. Therefore, putting things together, dividing by

$$
-2x_1[W_{0,1}(x(z_1))-W_{0,1}(x(\sigma(z_1)))]=-2x_1(\omega_{0,1}(z_1)-\omega_{0,1}(\sigma(z_1)))\frac{1}{\mathrm{d}x_1}
$$

and multiplying with $\mathrm{d}x_1\dots\mathrm{d}x_n$, we obtain

$$
\begin{aligned}
&\omega_{g,n}(z_1,\dots,z_n) \\
&= \frac{1}{\omega_{0,1}(z_1)-\omega_{0,1}(\sigma(z_1))}\Big[\sum_{j=2}^{n}\frac{\mathrm{d}x_1\mathrm{d}x_1}{x_1^2}\frac{z_j^2}{z_j^2-1}\frac{\partial}{\partial z_j}x_j^2\frac{\omega_{g,n-1}(z_2,\dots,z_n)}{x_1-x_j} \\
&\quad + \omega_{g-1,n+1}(z_1,\sigma(z_1),z_2,\dots,z_n) + \sum_{\substack{g_1+g_2=g \\ I\sqcup J=N\setminus\{1\}}}^{\circ} \omega_{g_1,|I|+1}(z_1,Z_I)\omega_{g_2,|J|+1}(\sigma(z_1),Z_J) \\
&\quad + \sum_{j=2}\omega_{0,2}(\sigma(z_1),z_j)\omega_{g,n-1}(z_1,Z_{\{1,\dots,n\}\setminus\{j\}})+\omega_{0,2}(z_1,z_j)\omega_{g,n-1}(\sigma(z_1),Z_{\{1,\dots,n\}\setminus\{j\}})\Big]
\end{aligned}
$$

$$= \frac{1}{\omega_{0,1}(z_1) - \omega_{0,1}(\sigma(z_1))} \Bigg[\sum_{j=2}^{n} \frac{\mathrm{d}x_1 \mathrm{d}x_1}{x_1^2} \frac{z_j^2}{z_j^2 - 1} \frac{\partial}{\partial z_j} x_j^2 \frac{\omega_{g,n-1}(z_2, \ldots, z_n)}{x_1 - x_j}$$
$$+ \omega_{g-1,n+1}(z_1, \sigma(z_1), z_2, \ldots, z_n) + \sideset{}{'}\sum_{\substack{g_1+g_2=g \\ I \sqcup J = N\setminus\{1\}}} \omega_{g_1,|I|+1}(z_1, Z_I)\omega_{g_2,|J|+1}(\sigma(z_1), Z_J) \Bigg].$$

The next step is to apply Cauchy's formula and use the fact that the $\omega_{g,n}$ are rational forms (in particular in z_1) having only poles at ± 1. We have

$$\begin{aligned} \omega_{g,n}(z_1, \ldots, z_n) &= \operatorname*{Res}_{z \to z_1} \frac{\omega_{g,n}(z, z_2, \ldots, z_n)\mathrm{d}z_1}{z - z_1} \\ &= \operatorname*{Res}_{z \to \pm 1} \frac{\mathrm{d}z_1}{z_1 - z} \omega_{g,n}(z, z_2, \ldots, z_n) \\ &= \operatorname*{Res}_{z \to \pm 1} \frac{\mathrm{d}z_1}{z_1 - \frac{1}{z}} \omega_{g,n}(\frac{1}{z}, z_2, \ldots, z_n) \\ &= \operatorname*{Res}_{z \to \pm 1} \frac{\mathrm{d}z_1}{z_1 - \sigma(z)} \omega_{g,n}(z, z_2, \ldots, z_n), \end{aligned}$$

the second equality is due to the fact that $\omega_{g,n}$ are rational differentials in each z_i, hence the sum over all residue must vanish, i.e.

$$0 = \operatorname*{Res}_{z \to z_1} \frac{\omega_{g,n}(z, z_2, \ldots, z_n)\mathrm{d}z_1}{z - z_1} + \operatorname*{Res}_{z \to \pm z_1} \frac{\omega_{g,n}(z, z_2, \ldots, z_n)\mathrm{d}z_1}{z - z_1},$$

where $\operatorname*{Res}_{z \to \pm 1}$ denotes the sum of the residues at 1 and -1. Thus we get

$$\begin{aligned} \omega_{g,n}(z_1, \ldots, z_n) &= \operatorname*{Res}_{z \to \pm 1} \frac{1}{2} \left[\frac{\mathrm{d}z_1}{z_1 - z} - \frac{\mathrm{d}z_1}{z_1 - \sigma(z)} \right] \omega_{g,n}(z, z_2, \ldots, z_n) \\ &= \operatorname*{Res}_{z \to \pm 1} \left[\frac{1}{2} \int_{\sigma(z)}^{z} \omega_{0,2}(z_1, \cdot) \right] \omega_{g,n}(z, z_2, \ldots, z_n). \end{aligned}$$

Now we want to invoke the recursion for the $\omega_{g,n}$ which we established before. We get

$$\omega_{g,n}(z_1, \ldots, z_n)$$
$$= \operatorname*{Res}_{z \to \pm 1} \frac{\frac{1}{2} \int_{\sigma(z)}^{z} \omega_{0,2}(z_1, \cdot)}{\omega_{0,1}(z) - \omega_{0,1}(\sigma(z))} \Bigg[\sum_{j=2}^{n} \frac{\mathrm{d}x \mathrm{d}x}{x^2} \frac{z_j^2}{z_j^2 - 1} \frac{\partial}{\partial z_j} x_j^2 \frac{\omega_{g,n-1}(z_2, \ldots, z_n)}{x - x_j}$$
$$+ \omega_{g-1,n+1}(z, \sigma(z), z_2, \ldots, z_n) + \sum_{\substack{g_1+g_2=g \\ I \sqcup J = N\setminus\{1\}}} \omega_{g_1,|I|+1}(z, Z_I)\omega_{g_2,|J|+1}(\sigma(z), Z_J) \Bigg].$$

First we need to argue that the residue at $z = 1$ does not contribute. But this is the case since

$$K(z_1, z) = \frac{\frac{1}{2} \int_{\sigma(z)}^{z} \omega(z_1, \cdot)}{\omega_{0,1}(z) - \omega_{0,1}(\sigma(z))} = \frac{z(z-1)^3 \mathrm{d}z_1}{2(z+1)(z_1 - z)(z_1 z - 1)\mathrm{d}z},$$

i.e. K has a zero of order 3 at $z = 1$ which cancels the poles (of order 1) of the differential $\omega_{g,n}(z, z_2, \ldots, z_n)$. Hence the last two terms on the right hand side vanish. For the first one, note that

$$\frac{\mathrm{d}x\mathrm{d}x}{x^2} = \frac{(z+1)^2\mathrm{d}z\mathrm{d}z}{(z-1)^2 z^2}$$

has pole of order 2, so the zero of $K(z_1, z)$ cancels this as well. Lastly we show that the first term on the right hand side vanishes if we take the residue at $z = -1$. But by the last equation we see that the pole of order 1 of K is removed by the zero of order 2. Thus we finally arrive at

$$\begin{aligned}\omega_{g,n}(z_1, \ldots, z_n) = \ & \mathrm{Res}_{z \to -1} \frac{\frac{1}{2}\int_{\sigma(z)}^{z} \omega_{0,2}(z_1, \cdot)}{\omega_{0,1}(z) - \omega_{0,1}(\sigma(z))} \Bigg[\omega_{g-1,n+1}(z, \sigma(z), z_2, \ldots, z_n) \\ & + \sum_{\substack{g_1+g_2=g \\ I \sqcup J = N \setminus \{1\}}} \omega_{g_1,|I|+1}(z, Z_I)\omega_{g_2,|J|+1}(\sigma(z), Z_J)\Bigg].\end{aligned}$$

□

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