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Reading the log canonical threshold of a plane curve singularity from its Newton polyhedron

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Abstract
There is a proposition due to Kollár as reported by Kollár (Proceedings of the summer research institute, Santa Cruz, CA, USA, July 9–29, 1995, American Mathematical Society, Providence, 1997) on computing log canonical thresholds of certain hypersurface germs using weighted blowups, which we extend to weighted blowups with non-negative weights. Using this, we show that the log canonical threshold of a convergent complex power series is at most $1/c$, where $(c, \dots, c)$ is a point on a facet of its Newton polyhedron. Moreover, in the case $n = 2$, if the power series is weakly normalised with respect to this facet or the point (c, c) belongs to two facets, then we have equality. This generalises a theorem of Varchenko 1982 to non-isolated singularities.

Keywords Complex singularity exponent · Complex oscillation index · Newton polygon · Remoteness

Mathematics Subject Classification Primary 14H20 · 14B05 · 32S25 · 14Q05 · 14H50

Contents

1 Introduction . . . 1070
2 Preliminaries . . . 1072
2.1 Power series and Newton polyhedron . . . 1072
2.2 Log canonical threshold and weighted blowups . . . 1073
3 Main . . . 1074
3.1 Computing the log canonical threshold using weights . . . 1074
3.2 Choosing a coordinate system . . . 1076
3.3 Reading the log canonical threshold from the Newton polyhedron . . . 1079
3.4 Example—sum and product of powers . . . 1080
References . . . 1082

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1 Introduction

Let f be a holomorphic function, not identically zero, on a domain of $\mathbb{C}^n$ where n is any positive integer and let P be a zero of f. The log canonical threshold of f at P can be characterised in many ways:

(1) it is the greatest positive rational number λ such that the pair $(\mathbb{C}^2, \lambda V(f))$ is log canonical (Definition 2.5),
(2) it is the supremum of rational numbers λ such that $|f|^{-\lambda}$ is L^2 in a neighbourhood of P, sometimes called the *complex singularity exponent* ([10, Definition 8.4]),
(3) it is the smallest *jumping number* of $V(f)$ ([16, Definition 9.3.22]),
(4) it is the negative of the largest zero of the Bernstein–Sato polynomial of f ([13, Theorem 10.6]),
(5) if f defines an isolated singularity, then the log canonical threshold is equal to $\min(1, \beta_{\mathbb{C}}(f))$ where $\beta_{\mathbb{C}}(f)$ is the *complex singular index* (see [20, Section 4] or [13, Theorem 9.5]), called the *complex oscillation index* in [2, 13.1.5] and the *complex singularity index* in [18],
(6) if f defines an isolated singularity, then the complex singular index minus one coincides with the smallest *spectrum number* ([18, p 558]).

If f does not define an isolated singularity, then it is not clear whether the log canonical threshold and the complex singular index coincide ([13, 9.6]).

Considering f as a power series in $\mathbb{C}\{x_1, \ldots, x_n\}$, the Newton polyhedron (Definition 2.2) is the convex hull of all the points $(i_1, \ldots, i_n) \in \mathbb{R}^n$ such that the monomial $x_1^{i_1} \cdot \ldots \cdot x_n^{i_n}$ has a non-zero coefficient in some power series g in the ideal generated by (f). The *distance to the Newton polyhedron* is defined as the rational number c such that the point $(c, \ldots, c)$ is on the boundary of the Newton polyhedron of f.

In case of isolated plane curve singularities, by [20, Theorem 4.4] there is a coordinate change (described in [19] for isolated real plane curve singularities) such that the log canonical threshold is equal to the reciprocal of the distance. The proof is analytic in nature. See [14, Theorem 6.40] for an algebraic proof. Note that the proof of [14, Theorem 6.40] works only for reduced plane curves since the last sentence of the proof uses finite determinacy of isolated singularities.

The main result of this paper is extending [20, Theorem 4.4] to non-isolated plane curve singularities:

Theorem *(= Theorem 3.10) Let $f \in \mathbb{C}\{x_1, \ldots, x_n\}$ be any non-zero power series satisfying $f(\mathbf{0}) = 0$. Let $c \in \mathbb{Q}_{>0}$ be the unique number such that a (not necessarily compact) facet Δ of the Newton polyhedron of f contains the point $(c, \ldots, c) \in \mathbb{R}^n$. Then, $\operatorname{lct}_{\mathbf{0}}(f) \leq 1/c$. Moreover, in the case $n = 2$, if f is weakly normalised with respect to Δ or the point (c, c) is in the intersection of two facets, then $\operatorname{lct}_{\mathbf{0}}(f) = 1/c$.*

The proof relies on Proposition 3.2, which is a generalisation of [13, Proposition 8.14] to weighted blowups with non-negative weights.

A generalisation of [3, Algorithm 4] shows how to find a coordinate change such that $f \in \mathbb{C}\{x, y\}$ is normalised with respect to every facet of the Newton polyhedron (Proposition 3.6). To read off the log canonical threshold, a weaker condition is enough, which we call *weakly normalised* (Definition 3.5). In case the singularity is non-isolated, Theorem 3.6 only gives a formal coordinate system, which is enough for computing the log canonical threshold (Remark 3.11(b)). In case the principal part of $f \in \mathbb{C}\{x, y\}$ is non-degenerate, by [3, Remark 3.28], all power series that are right equivalent to f and are normalised with respect to every compact facet of their Newton polyhedrons have the same Newton polyhedron. This does not always hold without the non-degeneracy assumption as shown by Examples 3.8 and 3.9.

The log canonical threshold at a point of a reduced plane curve can also be computed using Puiseux expansions. An explicit formula is given in [6, Theorem 2.13], which generalises [15, Theorems 1.2 and 1.3], that depends only on the first two maximal contact values of the branches and their intersection numbers.

Lastly, we mention some similar results that have been proved in higher dimensions, assuming the power series has enough monomials and with general enough coefficients. More precisely, we say that a power series $f \in \mathbb{C}\{x_1, \dots, x_n\}$ is *non-degenerate* with respect to a face σ of its Newton polyhedron if the polynomials

$$\frac{\partial f_\sigma}{\partial x_1}, \dots, \frac{\partial f_\sigma}{\partial x_n}$$

do not have common zeros in $(\mathbb{C} \setminus \{0\})^n$, where f_σ is the sum of the terms of f lying on σ. We say that f has *non-degenerate principal part* if f is non-degenerate with respect to all compact faces ([2, Definition 6.2.2]). As a stronger condition, we say that f is *non-degenerate* if f is non-degenerate with respect to all faces ([9, Definition 5]). In particular, if f is non-degenerate, then f has to be non-singular outside the coordinate hyperplanes.

By [2, Theorem 13.2(1)], if $f \in \mathbb{C}\{x_1, \dots, x_n\}$ defines an isolated singularity and has non-degenerate principal part, then the log canonical threshold of f is at most the reciprocal of the distance of the Newton polyhedron, with equality if the distance is greater than 1. The proof is analytic.

If f is a non-degenerate polynomial, then we can combinatorially compute all the jumping numbers, not only the log canonical threshold, and in fact describe the multiplier ideal of the divisor defined by f. Namely, let $\mathfrak{a}_f$ denote the ideal generated by the monomials appearing with a non-zero coefficient in f. By [8, Main Theorem] (also given in [16, Theorem 9.3.27]), the multiplier ideal $\mathcal{J}(r \cdot \mathfrak{a})$ of a monomial ideal $\mathfrak{a} \subseteq \mathbb{C}[x_1, \dots, x_n]$ coincides with the ideal generated by all monomials $x_1^{i_1} \cdot \ldots \cdot x_n^{i_n}$ such that $(i_1, \dots, i_n) + (1, \dots, 1)$ is in the interior of rP, where rP is the homothety with factor r applied to the the Newton polyhedron P of $\mathfrak{a}$. In the preprint [9, Theorem 12], it is proved that if f is non-degenerate, then a toric log resolution of $\mathfrak{a}_f$ also log-resolves div f. In this case, by [9, Corollary 13] (stated also in [16, Theorem 9.3.37]), for every rational number $0 < r < 1$, we have the equality of multiplier ideals,

$$\mathcal{J}(r \cdot \operatorname{div} f) = \mathcal{J}(r \cdot \mathfrak{a}_f).$$

As detailed in [16, Example 9.3.31], this recovers the result on log canonical thresholds of [2, Theorem 13.2(1)] in the special case of non-degenerate polynomials.

2 Preliminaries

2.1 Power series and Newton polyhedron

Definition 2.1 The field of complex numbers is denoted $\mathbb{C}$. The $\mathbb{C}$-algebra of power series in variables $x_1, \dots, x_n$ that are absolutely convergent in a neighbourhood of $\mathbf{0}$ is denoted by $\mathbb{C}\{x_1, \dots, x_n\}$, where n is a positive integer. The (possibly non-reduced) complex subspace of $\mathbb{C}^n$ defined by a convergent power series $f \in \mathbb{C}\{x_1, \dots, x_n\}$ is denoted by $V(f)$. The **saturation** of a power series $f \in \mathbb{C}[[x_1, \dots, x_n]]$, denoted $\operatorname{sat}(f)$, is the power series $f/(x_1^{a_1} \cdot \dots \cdot x_n^{a_n})$ where $a_1, \dots, a_n \in \mathbb{Z}_{\geq 0}$ are as large as possible.

Two formal power series $f, g \in \mathbb{C}[[x_1, \dots, x_n]]$ are **formally right equivalent** if there exists a $\mathbb{C}$-algebra isomorphism Φ of $\mathbb{C}[[x_1, \dots, x_n]]$ such that $\Phi(f) = g$. Two convergent power series $f, g \in \mathbb{C}\{x_1, \dots, x_n\}$ are **right equivalent** if there exists a $\mathbb{C}$-algebra isomorphism Φ of $\mathbb{C}\{x_1, \dots, x_n\}$ such that $\Phi(f) = g$ ([7, Definition I.2.9]).

Let $\boldsymbol{w} = (w_1, \dots, w_n) \in \mathbb{Q}^n_{\geq 0}$ be non-negative rational numbers, not all zero, and let

$$f := \sum_{i_1, \dots, i_n \in \mathbb{Z}_{\geq 0}} a_{i_1, \dots, i_n} x_1^{i_1} \cdot \dots \cdot x_n^{i_n}$$

be a power series in $\mathbb{C}[[x_1, \dots, x_n]]$, where $a_{i_1, \dots, i_n}$ are complex numbers. The ***w*-weight** of f, denoted $\operatorname{wt}_{\boldsymbol{w}}(f)$, is defined to be

$$\operatorname{wt}_{\boldsymbol{w}}(f) := \min \left\{ i_1 w_1 + \dots + i_n w_n \;\middle|\; \begin{array}{l} i_1, \dots, i_n \in \mathbb{Z}_{\geq 0}, \text{ the coefficient} \\ \text{of } x_1^{i_1} \cdot \dots \cdot x_n^{i_n} \text{ is non-zero in } f \end{array} \right\} \in \mathbb{Q}_{\geq 0} \cup \{\infty\}.$$

If f is not zero, then the ***w*-weighted-homogeneous leading term** of f, denoted $f_{\boldsymbol{w}}$, is defined to be

$$\sum_{\substack{i_1, \dots, i_n \in \mathbb{Z}_{\geq 0} \\ i_1 w_1 + \dots + i_n w_n = \operatorname{wt}_{\boldsymbol{w}}(f)}} a_{i_1, \dots, i_n} x^{i_1} \cdot \dots \cdot x^{i_n}.$$

Definition 2.2 ([1, Section 12.7], *local Newton polytope* in [7, Definition 2.14]) The **Newton polyhedron** of $f \in \mathbb{C}[[x_1, \dots, x_n]]$ is the subset of $\mathbb{R}^n$ given by the convex hull of the union of the subsets $(i_1, \dots, i_n) + (\mathbb{R}_{\geq 0})^n$ taken over non-negative integers $i_1, \dots, i_n$ such that $x_1^{i_1} \cdot \dots \cdot x_n^{i_n}$ has a non-zero coefficient in f.

Definition 2.3 [[7, Definition I.2.15]] Let $f \in \mathbb{C}[[x_1, \dots, x_n]]$ be a power series such that its Newton polyhedron has non-empty intersection with every coordinate axis. We say that f is **Newton non-degenerate** or has **non-degenerate Newton boundary** if for every compact facet Δ of its Newton polyhedron and non-zero normal vector $\boldsymbol{w} \in \mathbb{Z}^2_{\geq 0}$ of Δ, $\operatorname{sat}(f_{\boldsymbol{w}})$ defines an isolated singularity.

2.2 Log canonical threshold and weighted blowups

Definition 2.4 [[12, Notation 0.4]] Let X be a smooth complex space. A **$\mathbb{Q}$-divisor** on X is a formal $\mathbb{Q}$-linear combination $\sum \lambda_i D_i$ of prime divisors D_i where $\lambda_i \in \mathbb{Q}$. A $\mathbb{Z}$-divisor $\sum D_i$ is **snc** if all the prime divisors D_i are smooth and around every point of X, $\sum D_i$ is locally given by $V(x_1 \cdot \ldots \cdot x_k)$ in $\mathbb{C}^n$ where $0 \leq k \leq n$.

Let D be a $\mathbb{Q}$-divisor on a smooth complex space X. A **log resolution of** (X, D) is a proper modification ([4, Definition VII.1.1]) $\pi : X' \to X$ from a smooth complex space X' such that the exceptional locus E of π is of pure codimension 1 and $E \cup \pi^{-1}(\mathrm{Supp}(D))$ is snc.

For any proper bimeromorphic holomorphism $\varphi \colon X' \to X$ from a smooth complex space X', the **relative canonical divisor** of φ, denoted K_φ, is the unique $\mathbb{Q}$-divisor that is linearly equivalent to $\varphi^*(K_X) - K_{X'}$ and supported on the exceptional locus of φ, where K_X and $K_{X'}$ are the canonical classes of respectively X and X'. The **log pullback** of D with respect to φ is the $\mathbb{Q}$-divisor $D' = K_\varphi + \varphi^* D$ on X'. We have the $\mathbb{Q}$-linear equivalence $K_{X'} + D' \sim \varphi^*(K_X + D)$.

Definition 2.5 [[13, Definition 3.5] or [12, Definition 2.34]] Let D be a $\mathbb{Q}$-divisor on a smooth complex space X and let $P \in X$ be a point. The pair (X, D) is **log canonical at** P if we can restrict (X, D) to an open neighbourhood of P such that there exists a log resolution with the coefficient of every prime divisor in the log pullback of D at most 1. The **log canonical threshold of** (X, D) **at** P is

$$\mathrm{lct}_P(X, D) := \sup\big\{\lambda \in \mathbb{Q}_{>0} \;\big|\; (X, \lambda D) \text{ is log canonical at } P\big\}.$$

If $f \in \mathbb{C}\{x_1, \ldots, x_n\}$ is not zero, then $\mathrm{lct}_{\mathbf{0}}(f)$ is the log canonical threshold of $(\mathbb{C}^n, V(f))$ at the origin, where $\mathbb{C}^n$ and $V(f)$ are considered as complex spaces defined around the origin.

Proposition 2.6 *[[13, Proposition 8.19]] Let $f \in \mathbb{C}\{x_1, \ldots, x_n\}$ be non-zero and satisfy $f(\mathbf{0}) = 0$. Let d be a non-negative integer such that the truncation $f_{\leq d}$ of f up to degree d is non-zero. Then,*

$$\big|\mathrm{lct}_{\mathbf{0}}(f) - \mathrm{lct}_{\mathbf{0}}(f_{\leq d})\big| \leq \frac{n}{d+1}.$$

Remark 2.7 See [11, Proposition-Definition 10.3] for a toric description and [12, Definition 4.56] for an algebraic description of weighted blowups of affine space with positive integer weights.

If the variables of an affine space $\mathbb{A}^k$ over $\mathbb{C}$ have non-negative integer weights, not all zero, then write $\mathbb{A}^k$ as a product $\mathbb{A}^m \times \mathbb{A}^n$ where all the weights of $\mathbb{A}^m$ are zero and all the weights of $\mathbb{A}^n$ are positive. The weighted blowup of $\mathbb{A}^k$ is the morphism $\mathrm{id}_{\mathbb{A}^m} \times \varphi$ where φ is the weighted blowup of $\mathbb{A}^n$.

If X is a complex subspace of $\mathbb{C}^k$, then the weighted blowup of X is the induced holomorphism from strict transform of X with respect to the analytification of the weighted blowup of $\mathbb{A}^k$.

Lemma 2.8 *Let $\varphi\colon W \to \mathbb{C}^n$ be a weighted blowup of $\mathbb{C}^n$ with non-negative weights $\boldsymbol{w} = (w_1, \dots, w_n)$ satisfying $\gcd(w_1, \dots, w_n) = 1$. Let E be the exceptional prime divisor. Let $f \in \mathbb{C}\{x_1, \dots, x_n\}$ be any non-zero power series. Let λ be any rational number. Then, the coefficient of E in the log pullback of $\lambda V(f)$ is $1 + \lambda \operatorname{wt}_{\boldsymbol{w}}(f) - \sum w_i$.*

Proof Using toric geometry ([5, Lemma 11.4.10]), we find

$$K_W = \varphi^* K_{\mathbb{C}^n} + \Big(\sum w_i - 1\Big) E.$$

Let v be the discrete valuation of the field of meromorphic functions of $\mathbb{C}^n$ given by the order of vanishing along the exceptional divisor E. Then, $v(f) = \operatorname{wt}_{\mathbf{0}}(f)$. Let $V(f) = \sum \alpha_j C_j$, where C_j are prime divisors and the coefficients α_j are non-negative integers. Then,

$$\sum \alpha_j (\varphi_*^{-1} C_j) = \varphi^* V(f) - \operatorname{wt}_{\mathbf{0}}(f) E.$$

The lemma follows. $\square$

3 Main

3.1 Computing the log canonical threshold using weights

In this Section, we prove Proposition 3.2, which is the main tool used in Sect. 3.3 for computing log canonical thresholds.

Remark 3.1 A version of Proposition 3.2 Is proved in [13, Proposition 8.14] with two differences:

1. in [13, Proposition 8.14], the weights are required to be positive, whereas we allow any non-negative weights, not all zero,
2. the condition that $(\mathbb{C}^n, b \cdot V(f_{\boldsymbol{w}}))$ is log canonical is replaced by the condition that $(\mathbb{P}^{n-1}, V(f_{\boldsymbol{w}}(x_1^{w_1}, \dots, x_n^{w_n})))$ is log canonical, where $\mathbb{P}^{n-1}$ is the exceptional divisor of the blowup of $\mathbb{C}^n$ at $\mathbf{0}$.

The statement in [13, Proposition 8.14] contains an error, it should instead say that $(\mathbb{P}^{n-1}, b \cdot V(f_{\boldsymbol{w}}(x_1^{w_1}, \dots, x_n^{w_n})))$ is log canonical, otherwise Examples 3.3 and 3.4 are counter-examples.

Proposition3.2 is stated, without proof, for positive weights in [15, Proposition 2.1].

Proposition 3.2 *Let n be a positive integer. Let $f \in \mathbb{C}\{x_1, \dots, x_n\}$ be a non-zero power series. Assign non-negative rational weights $\boldsymbol{w} = (w_1, \dots, w_n)$ to the variables, not all zero. Let $f_{\boldsymbol{w}}$ denote the $\boldsymbol{w}$-weighted-homogeneous leading term of f. Define $b := \sum_i w_i / \operatorname{wt}_{\boldsymbol{w}}(f) \in \mathbb{Q}_{>0} \cup \{\infty\}$. Define the subset*

$$C := V_{\mathbb{C}^n}(\{x_i \mid i \in \{1, \dots, n\},\ w_i > 0\}).$$

Then, $\operatorname{lct}_{\mathbf{0}}(f) \le b$. *Moreover, considering* $\mathbb{C}^n$ *and* $V(f_{\boldsymbol{w}})$ *as complex spaces defined in a neighbourhood of* $\mathbf{0}$, *if* b *is finite and* $(\mathbb{C}^n, b \cdot V(f_{\boldsymbol{w}}))$ *is log canonical outside* C, *then* $\operatorname{lct}_{\mathbf{0}}(f) = b$.

Proof After scaling by a suitable positive rational number, the numbers $w_1, \ldots, w_n$ are non-negative integers with $\gcd(w_1, \ldots, w_n) = 1$.

If there is exactly one index $i_0 \in \{1, \ldots, n\}$ such that w_{i_0} is positive, then $\operatorname{wt}(f) = 1/b$ is the coefficient of the prime divisor $V(x_{i_0})$ in the divisor $V(f)$. Therefore, $\operatorname{lct}_{\mathbf{0}}(f) \le b$. Now, let b be finite and let $(\mathbb{C}^n, b \cdot V(f_{\boldsymbol{w}}))$ be log canonical away from $V(x_{i_0})$. The complex space $V_{\mathbb{C}^n}(f_{\boldsymbol{w}}) \setminus V_{\mathbb{C}^n}(x_{i_0})$ is biholomorphic to $(\mathbb{C}^1 \setminus \{0\}) \times V_{\mathbb{C}^{n-1}}(f_{\boldsymbol{w}}/x_{i_0}^{1/b})$. Therefore, $(\mathbb{C}^{n-1}, b \cdot V(f_{\boldsymbol{w}}/x_{i_0}^{1/b}))$ is log canonical. Note that $V(x_{i_0}) + b \cdot V(f/x_{i_0}^{1/b})$ is precisely the divisor $b \cdot V(f)$ and that the restriction of $b \cdot V(f/x_{i_0}^{1/b})$ to $V(x_{i_0})$ is $b \cdot V(f_{\boldsymbol{w}}/x_{i_0}^{1/b})$. By inversion of adjunction ([13, Theorem 7.5]), $(\mathbb{C}^n, b \cdot V(f))$ is log canonical. Therefore, $\operatorname{lct}_{\mathbf{0}}(f) = b$.

Below we consider the case where there are at least two positive weights among $\boldsymbol{w}$. Let $\varphi\colon W \to \mathbb{C}^n$ be the $\boldsymbol{w}$-weighted blowup and E its exceptional divisor. For every $\lambda \in \mathbb{Q}_{>0}$, by Lemma 2.8, the coefficient of E in the log pullback of $\lambda V(f)$ is

$$1 + \lambda \operatorname{wt}_{\boldsymbol{w}}(f) - \sum w_i.$$

Therefore, $\operatorname{lct}_{\mathbf{0}}(f) \le b$.

Now, let b be finite and let $(\mathbb{C}^n, b \cdot V(f_{\boldsymbol{w}}))$ be log canonical outside C. We show that $(E, b \cdot V(f_{\boldsymbol{w}}))$ is log canonical. For every $i \in \{1, \ldots, n\}$ such that w_i is positive, let the group μ_{w_i} of w_i-th roots of unity act on $\mathbb{C}^n$ by $\xi \cdot (x_1, \ldots, x_n) = (\xi^{w_1} x_1, \ldots, \xi^{w_n} x_n)$, where ξ is a primitive w_i-th root of unity. Let

$$p\colon \mathbb{C}^n \setminus V(x_i) \to \frac{\mathbb{C}^n \setminus V(x_i)}{\mu_{w_i}}$$

be the natural holomorphism. By [13, Theorem 8.12],

$$\left(\frac{\mathbb{C}^n \setminus V(x_i)}{\mu_{w_i}}, V(f_{\boldsymbol{w}})\right)$$

is log canonical. For every monomial M in the μ_{w_i}-invariant $\mathbb{C}$-algebra $\mathbb{C}[x_0, \ldots, x_n, x_i^{-1}]^{\mu_{w_i}}$, there exists an integer k_M such that $\operatorname{wt}_{\boldsymbol{w}}(M) = k_M w_i$. The $\mathbb{C}$-algebra isomorphism

$$\begin{aligned} \mathbb{C}[x_0, \ldots, x_n, x_i^{-1}]^{\mu_{w_i}} &\to \mathbb{C}[x_1, \ldots, x_n, x_i^{-1}]_0 \otimes_{\mathbb{C}} \mathbb{C}[y, y^{-1}] \\ x_i &\mapsto y \\ M &\mapsto y^{k_M} x_i^{-k_M} M \end{aligned}$$

induces a biholomorphism

$$\frac{\mathbb{C}^n \setminus V(x_i)}{\mu_{w_i}} \cong (E \setminus V(x_i)) \times (\mathbb{C}^1 \setminus \{0\})$$

which takes $V(f_{\boldsymbol{w}}) \subseteq (\mathbb{C}^n \setminus V(x_i))/(\mu_{w_i})$ to $V_{E\setminus V(x_i)}(f_{\boldsymbol{w}}) \times (\mathbb{C}^1 \setminus \{\mathrm{pt}\})$. Since log canonicity is a local analytic property ([17, Proposition 4.4.4]) and since taking a product with a smooth complex space does not change discrepancies, $(E, b \cdot V(f_{\boldsymbol{w}}))$ is log canonical.

We show that $\mathrm{lct}_{\mathbf{0}}(\mathbb{C}^n, V(f)) = b$. Since the exceptional divisor E of the weighted blowup $\varphi\colon W \to \mathbb{C}^n$ is the product of an affine space and a weighted projective space, E is log terminal. Let $D'_{\boldsymbol{w}}$ be the log pullback of $b \cdot V(f_{\boldsymbol{w}})$. By inversion of adjunction ([13, Theorem 7.5]), since $(E, b \cdot V(f_{\boldsymbol{w}}))$ is log canonical, $(W, D'_{\boldsymbol{w}})$ is log canonical near E. By [13, Lemma 3.10.2], $(\mathbb{C}^n, b \cdot V(f))$ is log canonical at $\mathbf{0}$. Therefore, $\mathrm{lct}_{\mathbf{0}}(f) = b$. □

Examples 3.3 and 3.4 show that the number b in Proposition 3.2 can be greater than 1.

Example 3.3 Let $f := x + y^d$, where d is any positive integer. Since f is smooth at $\mathbf{0}$, $\mathrm{lct}_{\mathbf{0}}(f) = 1$. On the other hand, choosing weights $(1, 1/d)$, we find $b = (d+1)/d > 1$.

Example 3.4 Let $f = x_1^2 + \ldots + x_{n-1}^2 + x_n^{k+1}$, where $n \geq 3$ and $k \geq 1$. If $n = 3$, then f defines a canonical surface singularity, and if $n \geq 4$, then f defines a terminal $(n-1)$-fold singularity. By inversion of adjunction ([13, Theorem 7.5]), $\mathrm{lct}_{\mathbf{0}}(f) = 1$. On the other hand, choosing weights $(1/2, \ldots, 1/2, 1/(k+1))$, we find $b = (n-1)/2 + 1/(k+1) > 1$.

3.2 Choosing a coordinate system

In most coordinate systems, the Newton polyhedron gives very little information about the power series. Namely, for every non-zero non-unit power series, after a general linear coordinate change, its Newton polyhedron has exactly one compact facet, and that compact facet has a normal vector $(1, \ldots, 1)$. Proposition 3.6 describes a natural coordinate change such that the Newton polyhedron is interesting for our purposes. In this case, we can read off the log canonical threshold from the Newton polyhedron by Theorem 3.10.

Definition 3.5 is [3, Definition 3.24] generalised to the situation where the power series is not necessarily right equivalent to a Newton non-degenerate power series. It coincides with [3, Definition 3.24] if the power series is right equivalent to a Newton non-degenerate one. In Theorem 3.10, we only need a weaker condition which we call *weakly normalised with respect to* Δ.

Definition 3.5 Let $f \in \mathbb{C}[[x, y]]$ be non-zero. Let Δ be a (not necessarily compact) facet of the Newton polyhedron of f. Let $\boldsymbol{w} = (w_x, w_y)$ be the unique non-negative integers with $\gcd(w_x, w_y) = 1$ such that $\boldsymbol{w}$ is normal vector of Δ. Let $f_{\boldsymbol{w}}$ be the $\boldsymbol{w}$-weighted homogeneous leading term of f. Let a, b be the greatest non-negative integers such that $f_{\boldsymbol{w}}/(x^a y^b)$ is a power series. Let d be the greatest non-negative integer such that an irreducible power series to the power d divides $\operatorname{sat}(f_{\boldsymbol{w}})$. We say that f is **weakly normalised with respect to** Δ if one of the following holds:

(W1) $w_x = 0$ or $w_y = 0$,
(W2) $d \leq \max(a, b)$, or
(W3) $w_x > 1$ and $w_y > 1$.

We say f is **normalised with respect to** Δ if one of the following holds:

(N1) $w_x = 0$ or $w_y = 0$,
(N2) $w_x = 1$ and $w_y = 1$ and $d \leq \min(a, b)$,
(N3) $w_x = 1$ and $w_y > 1$ and $d \leq b$,
(N4) $w_x > 1$ and $w_y = 1$ and $d \leq a$, or
(N5) $w_x > 1$ and $w_y > 1$.

Proposition 3.6 *Every formal power series $f \in \mathbb{C}[[x, y]]$ is formally right equivalent to a formal power series $g \in \mathbb{C}[[x, y]]$ which is normalised with respect to each compact facet of its Newton polyhedron. If $f \in \{x, y\}$ defines an isolated singularity, then there exists a right equivalence between f and a polynomial $g \in [x, y]$ which is normalised with respect to each compact facet of its Newton polyhedron.*

Proof We get a formal power series by lines 1–28 and 43 of [3, Algorithm 4] (skipping lines 29–42). If f defines an isolated singularity, then f is $(\mu + 1)$-determined ([7, Corollary 2.24]) where μ is its local Milnor number at the origin. Therefore, we have a right equivalence to a polynomial g of degree $\mu + 1$ which is normalised with respect to each compact facet of its Newton polyhedron. □

Remark 3.7 (a) By [3, Algorithm 4] lines 1–28 and 43, if $f \in \mathbb{C}\{x, y\}$ is right equivalent to a Newton non-degenerate power series and f is normalised with respect to each compact facet of its Newton polyhedron, then f is Newton non-degenerate.
(b) By [3, Remarks 3.27 and 3.28], if $f \in \mathbb{C}\{x, y\}$ is right equivalent to a Newton non-degenerate power series, then we can associate a unique, up to reflection, polyhedron to the right equivalence class of f, namely the Newton polyhedron of any power series in the right equivalence class of f which is normalised with respect to each compact facet of its Newton polyhedron.

If $f \in \mathbb{C}\{x, y\}$ is not right equivalent to a Newton non-degenerate power series, then we do not have such uniqueness as in Remark 3.7(b):

Example 3.8 The right equivalent polynomials

$$\begin{aligned} f &:= x^2 y^2 (x + y)^2 + x^9 + y^7, \\ g &:= x^2 y^2 (x + y)^2 + x^9 - (x + y)^7 \end{aligned}$$

are both normalised with respect to each facet of their Newton polyhedrons, but the Newton polyhedron of f is strictly contained in the Newton polyhedron of g. The local Milnor number of f at the origin is 30 and by Theorem 3.10 the log canonical threshold of f at the origin is $1/3$.

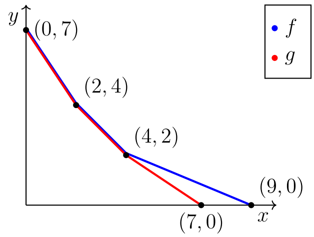

Fig. 1 Compact faces of the Newton polyhedrons in Example 3.8

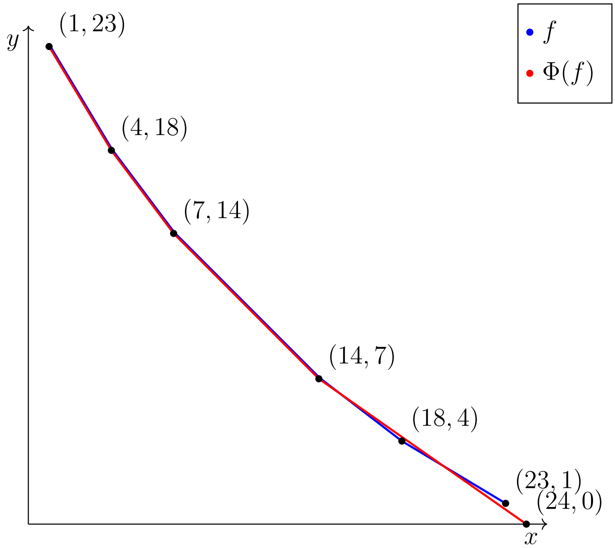

Fig. 2 Compact faces of the Newton polyhedrons in Example 3.9

With more care, we can also construct an example where the Newton polyhedrons are not contained in one another:

Example 3.9 Let the set of polyhedrons in $\mathbb{R}^2$ have a partial order "$\leq$" given by $N_1 \leq N_2$ if and only if $N_1 \subseteq N_2$ or $r(N_1) \subseteq N_2$, where r is the reflection with respect to the line passing through the origin and $(1, 1)$. Define

$$f := \big(xy(x+y)\big)^7 + (xy)^4(x+y)^6(x^8+y^8) + xy(x^{22}+y^{22}).$$

Let Φ be the $\mathbb{C}$-algebra automorphism of $\mathbb{C}\{x, y\}$ given by $\Phi\colon x \mapsto x$, $y \mapsto -x-y$. Then f and $\Phi(f)$ are both normalised with respect to each compact facet of their Newton polyhedrons but the Newton polyhedrons of f and $\Phi(f)$ are incomparable. Moreover, the Newton polyhedrons of f and $\Phi(f)$ are precisely all the minimal Newton polyhedrons, up to reflection, in the formal right equivalence class of f. The local Milnor number of f at the origin is 454 and by Theorem 3.10 the log canonical threshold of f at the origin is $2/21$.

3.3 Reading the log canonical threshold from the Newton polyhedron

Theorem 3.10 *Let $f \in \mathbb{C}\{x_1, \dots, x_n\}$ be any non-zero power series satisfying $f(\mathbf{0}) = 0$. Let $c \in \mathbb{Q}_{>0}$ be the unique number such that a (not necessarily compact) facet Δ of the Newton polyhedron of f contains the point $(c, \dots, c) \in \mathbb{R}^n$. Then, $\operatorname{lct}_{\mathbf{0}}(f) \leq 1/c$. Moreover, in the case $n = 2$, if f is weakly normalised with respect to Δ or the point (c, c) is in the intersection of two facets, then $\operatorname{lct}_{\mathbf{0}}(f) = 1/c$.*

Proof Let Δ be any (not necessarily compact) facet of the Newton polyhedron of f that contains the point $(c, \dots, c)$. Let $\boldsymbol{w} \in \mathbb{Z}^n_{\geq 0}$ be the unique normal vector of Δ such that $\gcd(w_1, \dots, w_n) = 1$. Let $f_{\boldsymbol{w}}$ denote the $\boldsymbol{w}$-weighted-homogeneous leading term of f. Then, $\operatorname{wt}_{\boldsymbol{w}}(f)/\sum w_i = c$. By Proposition 3.2, $\operatorname{lct}_{\mathbf{0}}(f) \leq 1/c$.

Now, let $n = 2$ and let f be weakly normalised with respect to Δ. We use Proposition3.2 to prove that $\operatorname{lct}_{\mathbf{0}}(f) = 1/c$. For this, we need to show that the pair $(\mathbb{C}^2, \frac{1}{c} V(f_{\boldsymbol{w}}))$ is log canonical outside the origin, or equivalently that all the irreducible components of $V(f_{\boldsymbol{w}})$ have multiplicity less than or equal to c.

If $w_2 = 0$, then $f_{\boldsymbol{w}} = u x_1^c x_2^k$ for some unit $u \in \mathbb{C}\{x_2\}$ and non-negative integer $k \leq c$, showing that all the irreducible components of $V(f_{\boldsymbol{w}})$ have multiplicity less than or equal to c. Similarly in the case $w_1 = 0$.

Below, we consider the case where w_1 and w_2 are both positive. Let (P_1, P_2) and (Q_1, Q_2) be the two extreme points of Δ, where $P_1 < Q_1$ and $Q_2 < P_2$. Since Δ contains the point (c, c), we necessarily have $P_1 \leq c$ and $Q_2 \leq c$. Let d be the greatest integer such that an irreducible power series g to the power d divides the saturation $\operatorname{sat}(f_{\boldsymbol{w}})$ of $f_{\boldsymbol{w}}$. To show that all the irreducible components of $V(f_{\boldsymbol{w}})$ have multiplicity less than or equal to c, it suffices to prove that $d \leq c$.

If $w_1 = 1$ or $w_2 = 1$, then since f is weakly normalised with respect to Δ, we have $d \leq \max(P_1, Q_2) \leq c$. Otherwise, after possibly permuting x_1 and x_2, we have $1 < w_1 < w_2$. Therefore, $\deg g(0, x_2) \geq 2$. Since $g^d \mid \operatorname{sat}(f)$, we find that

$$g(0, x_2)^d \mid \operatorname{sat}(f_{\boldsymbol{w}})(0, x_2) = x_2^{P_2 - Q_2}.$$

Therefore, $d \leq (P_2 - Q_2)/2$. Since Δ has slope $-w_1/w_2 > -1$ and Δ contains the point (c, c), the line containing Δ contains the point $(0, \alpha)$ where $\alpha < 2c$. Therefore, $P_2 < 2c$. This proves that $d < c$.

Finally, let $n = 2$ and let (c, c) be contained in the intersection of two facets. Let $(w_1, w_2) \in \mathbb{Z}^2_{>0}$ be any vector such that $f_{\boldsymbol{w}}$ is the monomial $x^c y^c$. Then $\operatorname{wt}_{\boldsymbol{w}}(f)/\sum w_i = c$. By Proposition 3.2, since the pair $(\mathbb{C}^2, \frac{1}{c} V(x^c y^c))$ is log canonical, we find $\operatorname{lct}_{\mathbf{0}}(f) = 1/c$. □

Remark 3.11 (a) By Proposition3.6, Theorem 3.10 gives an algorithm for computing the log canonical threshold for every non-zero non-unit power series $f \in \mathbb{C}\{x, y\}$ that defines an isolated singularity. The algorithm outputs positive integers a, b such that the log canonical threshold is a/b.

(b) By Propositions 2.6and 3.6 and Theorem 3.10, if f is formally right equivalent to a power series $g \in \mathbb{C}[[x, y]]$ such that g is normalised with respect to a facet of the Newton polyhedron that contains the point (c, c), then the log canonical threshold of f is $1/c$. In particular, this gives an algorithm for computing the log canonical threshold for every non-zero power series $f \in \mathbb{C}\{x, y\}$ arbitrarily precisely, but it is not necessarily able to find two positive integers such that the log canonical threshold is their quotient.

(c) The above in (b) is the best possible in the following sense: given two computable convergent power series (meaning where we can compute the terms up to arbitrarily high order), there exists no algorithm to compute two positive integers such that the log canonical threshold is their quotient. This is not surprising, since there is even no algorithm to determine whether a computable power series is equal to zero.

(d) In case f is a polynomial, it would be desirable to know a bound on the denominator of the log canonical threshold, perhaps depending only on the degree. By (b), this would give an algorithm for computing the exact value of the log canonical threshold, as a quotient of two positive integers.

Remark 3.12 The number c in Theorem 3.10 is the minimum of the numbers $\mathrm{wt}_{\boldsymbol{w}'}(f)/\sum w'_i$ taken over all facets of the Newton polyhedron.

More precisely: let Δ' be any (not necessarily compact) facet of the Newton polyhedron of a non-zero power series $f \in \{x_1, \ldots, x_n\}$ satisfying $f(\mathbf{0}) = 0$. Let $\boldsymbol{w}' = (w'_1, \ldots, w'_n) \in \mathbb{Q}^n_{\geq 0}$ be any non-negative numbers such that $\boldsymbol{w}'$ is a non-zero normal vector to Δ'. Then, $\mathrm{wt}_{\boldsymbol{w}'}(f)/\sum w'_i = c'$, where $c' \in \mathbb{Q}_{\geq 0}$ is the unique rational number such that the hyperplane containing Δ' contains the point $(c', \ldots, c')$. In the notation of Theorem 3.10, we have $c' \leq c$, with equality if and only if the ray from the origin through the point $(1, \ldots, 1)$ intersects Δ'.

3.4 Example—sum and product of powers

As an application of Proposition 3.2, we compute the log canonical thresholds of power series of the form $(\prod x_i^{a_i})(\sum x_i^{b_i})$.

Remark 3.13 Proposition 3.14 is the corrected version of [13, Example 8.17] and [15, Proposition 2.2], where we have added that the log canonical threshold is at most 1. Examples 3.3 and 3.4 show that this correction is indeed needed.

Proposition 3.14 *Let $n \geq 2$. Let $f = (\prod x_i^{a_i})(\sum x_i^{b_i})$, where $a_1, \ldots, a_n$ are non-negative integers and $b_1, \ldots, b_n$ are positive integers. Then,*

$$\mathrm{lct}_{\mathbf{0}}(f) = \min\left(\left\{1, \frac{\sum 1/b_i}{1 + \sum a_i/b_i}\right\} \cup \left\{\frac{1}{a_j} \,\middle|\, j \in \{1, \ldots, n\},\ a_j > 0\right\}\right).$$

Proof Denote

$$M := \min\left(\left\{\frac{\sum 1/b_i}{1 + \sum a_i/b_i}\right\} \cup \left\{\frac{1}{a_j} \,\middle|\, j \in \{1, \ldots, n\},\ a_j > 0\right\}\right).$$

If $M > 1$, then replace a_n by 1.

If $M = (\sum 1/b_i)/(1 + \sum a_i/b_i)$, then by Proposition 3.2, it suffices to show that $(\mathbb{C}^n, M \cdot V(f))$ is log canonical outside the origin. For every point $P \in \mathbb{C}^n$, if the j-th coordinate P_j of P is non-zero, then after a suitable biholomorphism locally around P, $V(f)$ becomes either $D_1 := V((x_j - P_j) \cdot \prod_{i \neq j} x_i^{a_i})$ or $D_2 := V(\prod_{i \neq j} x_i^{a_i})$. Since the support of both D_1 and D_2 is snc and since $M \leq 1/a_i$ for all i, we find that $(\mathbb{C}^n, M \cdot V(f))$ is log canonical at P.

If $M = 1/a_1$ and $n = 2$, then $a_1 \leq a_2 + b_2$. Therefore, $(\mathbb{C}^2, M \cdot V(x_1^{a_1} x_2^{a_2+b_2}))$ is log canonical outside $V(x_1)$. By Proposition 3.2, $\mathrm{lct}_{\mathbf{0}}(f) = M$. The case where $M = 1/a_2$ and $n = 2$ is similar.

Lastly, if $M = 1/a_j$ and $n \geq 3$, then let $\boldsymbol{w}$ be the non-negative weights where $w_i > 0$ if and only if $i = j$. We have $f_{\boldsymbol{w}} = (\sum_i x_i^{a_i})(\sum_{i \neq j} x_i^{b_i})$. By Proposition 3.2, it suffices to show that the pair $(\mathbb{C}^n, M \cdot V(f_{\boldsymbol{w}}))$ is log canonical outside $V(x_j)$. For this, it suffices to show that the pair

$$\Big(\mathbb{C}^{n-1},\ M \cdot V\big((\prod_{i \neq j} x_i^{a_i})(\sum_{i \neq j} x_i^{b_i})\big)\Big)$$

is log canonical, where both the product and the sum is over $i \in \{1, \ldots, n\} \setminus \{j\}$. The equality $M = 1/a_j$ implies that

$$M \leq \frac{\sum_{j \neq i} 1/b_i}{c + \sum_{j \neq i} a_i/b_i}.$$

Log canonicity is now proved by induction over n. □

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