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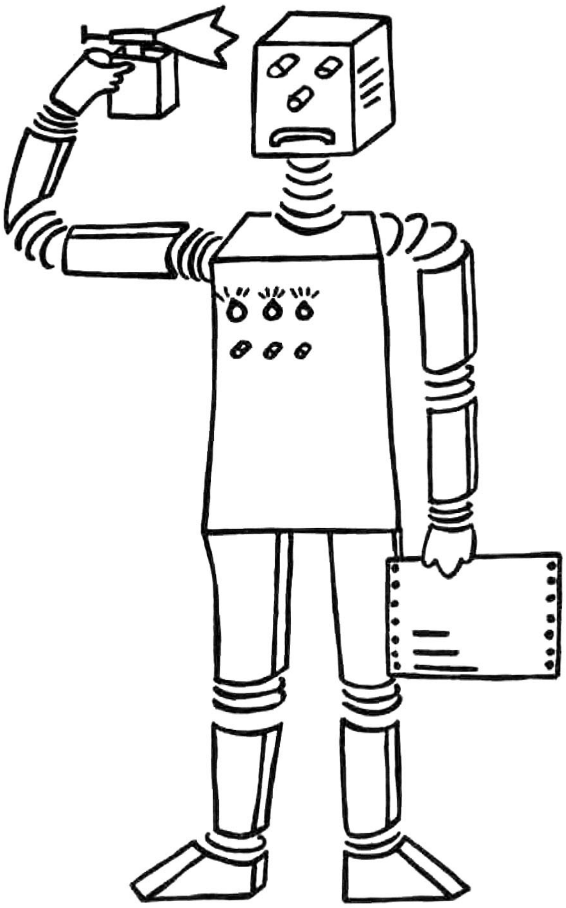

Ground Confluence

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Abstract

In this paper we introduce a test for confluence on ground terms. This test allows us to prove the ground confluence of term rewriting systems where the Knuth-Bendix Algorithm does not terminate.
Ground Confluence of term rewriting systems is sufficient, if one is interested in congruences on ground terms. This is for example the case in the domain of inductive proofs or in the domain of program synthesis.

(1) Introduction

The Knuth-Bendix Algorithm is a procedure for transforming a set of equations into a confluent term rewriting system. It has recently been applied in theories where only the congruence on ground terms of the equational theory is of interest, for example in the domain of inductive proofs (e.g. [MU 80], [HH 82], [KM 83], [JK 85]) or in the domain of program synthesis (e.g. [DE 85]). Although it is sufficient for these applications to generate term rewriting systems which are confluent on ground terms, the classical Knuth-Bendix Algorithm tries to generate a term rewriting system which is confluent on arbitrary terms. This often leads to cases where the classical Knuth-Bendix Algorithm generates an infinite system, even though the infinite system contains a finite ground confluent system. In this paper we will introduce a test for ground confluence which is stronger than the classical confluence test. This test allows us to prove the ground confluence of term rewriting systems where the classical Knuth-Bendix Algorithm does not terminate.

Tests for ground confluence have also been considered in [GO 85b] and [FR 86], where restrictions to the number of critical pairs are introduced if we apply the Knuth-Bendix Algorithm for inductive proofs.

In [GO 85b] a term rewriting system is splitted into three disjoint sets, a confluent set of rules (A) which are axioms for the inductive theory, a set of rules (I) which are inductive consequences of A and a set containing all other rules (O). The critical pairs are computed only from A ∪ O but they are reduced by all rules (A ∪ I ∪ O). If the function symbols can be splitted into constructors and defined functions, then the number of critical pairs can be further reduced by considering only overlappings where variables are replaced by constructor terms.

In [FR 86] the term rewriting system is also splitted into axioms (A) and a set containing all other rules (O). Critical pairs between rules of O are not considered and for a rule in O only critical overlappings at a single position in the left hand side of the rule have to be considered. For this position a critical overlapping with a rule from A has to exist for every (constructor-) ground instance of the rule. If no position in the rule satisfies this condition, then the confluence test can not be applied.

For both methods one needs special informations about the term rewriting system:

- One has to to identify a subset of rules as axioms A
- In [GO 85b] one has to identify also a set of consequences I
- In [FR 86] one has to choose a position in the left hand side of a rule

If one is not able to identify a set of axioms in a set of rules R which is smaller than R, then both methods can not be applied. This is the case for example in program synthesis where one starts with an unstructured set of equations. But also in the area of inductive proofs, one may get problems if the term rewriting systems contains more than one complete set of axioms in a set of rules. We will give an example with two complete sets of axioms, where the Knuth-Bendix Algorithm with the ground confluence test of [GO 85b] or [FR 86] does not terminate if we choose the wrong set of axioms.
In [FR 86], one has to choose also positions in left hand sides of rules. Often, a wrong choice also causes the completion procedure to generate an infinite system.
In this paper we give a ground confluence test for term rewriting systems without the problem of choosing a set of axioms or a position in the left hand side of rules (section 3). This test can be improved for term rewriting systems with convertible function symbols (section 4), where a function symbol is convertible (defined function) if it does not occur in any ground term normal form. Finally we give examples where the ground confluence tests in [GO 85b] and [FR 86] fail and with the test developed in this paper we can prove the ground confluence of these examples.

(2) Notation and Basic Definitions

We assume familiarity of the reader with the basic proofs and results of the Knuth-Bendix Algorithm (e.g. [HU 77], [HO 80], [KB 70]), its extension for inductive proofs (e.g. [MU 80], [HH 82]) and the generalized Newman Lemma [WB 83].
We denote by VA the set of all variables and by FS the set of all function symbols. TE(F,V) is the set of all terms constructed by variables from $V \subseteq VA$ and by function symbols from $F \subseteq FS$. A single term is denoted by t or by α, β, γ or δ if it occurs in a rule or equation. Occurences in terms are denoted by u, v and w. The symbol ε denotes the top level occurence of a term. O(t) is the set of all occurences of the term t and O'(t) is the set of all non variable occurences of t. V(t) returns all variables of the term t. Substitutions

are denoted by σ, τ and φ.

A set of pairs of terms is denoted by P, if we consider this pairs as rules we will denote it by R and if we consider these pairs as equations we will denote it by E. $\alpha \longrightarrow \beta$ and $\gamma \longrightarrow \delta$ denote single rules and $\alpha = \beta$ and $\gamma = \delta$ denote single equations. For every term rewriting system R in this paper we assume a R-compatible term ordering >, therefore we consider only terminating term rewriting systems. A one step derivation with a rule in P is denoted by $t \longrightarrow_P t'$ and $\longmapsto_P$ is the symmetric closure of $\longrightarrow_P$. $\overset{*}{\longrightarrow}_P$ and $\overset{*}{\longmapsto}_P$ are the reflexive and transitive closures of $\longrightarrow_P$ and $\longmapsto_P$. Two terms t_1 and t_2 are connected in one step below a term t ($t_1 \longmapsto_{P,t} t_2$), if $t_1 \longmapsto_P t_2$, $t > t_1$ and $t > t_2$. The reflexive and transitive closure of $\longmapsto_{P,t}$ is denoted by $\overset{*}{\longmapsto}_{P,t}$.

The set IRR(R) contains all terms which are irreducible in R and $IRR_G(R)$ contains all ground terms from IRR(R). Terms from IRR(R) are called to be in R normal form. $t\downarrow R$ is the normal form of the term t in R, if t has a unique normal form.

A term rewriting system is ground confluent, iff for all derivations t_1 and t_2 from a ground term t, there exists a term t' which is derivable from t_1 and t_2.

In this paper we will distinguish between critical overlappings CO(R) and critical triples CT(R) of a term rewriting system R. This two sets are defined as follows:

<u>(2.1) Definition</u>

Let R be a term rewriting system. CO(R) denotes the set of all critical overlappings of R:

$$CO(R) = \{ (\alpha \longrightarrow \beta, u, \gamma \longrightarrow \delta) \mid \alpha \longrightarrow \beta, \gamma \longrightarrow \delta \in R \wedge u \in O'(\alpha) \wedge \exists \sigma, \tau : \sigma(\alpha) / u = \tau(\gamma) \}$$

$CT(\alpha \longrightarrow \beta, u, \gamma \longrightarrow \delta)$ returns a critical triple for a critical overlapping from CO(R):

$$CT(R) := (\sigma(\alpha), \sigma(\alpha) [u \longleftarrow \sigma(\varphi(\delta))], \sigma(\beta))$$

with:
- φ is a renaming substitution of γ
- $V(\varphi(\gamma)) \cap V(\alpha) = \emptyset$
- σ is a most general unifier of α/u and $\varphi(\gamma)$.

We denote the set of all critical triples of R by CTS(R).

The confluence and the connectedness of a critical triple are defined as follows:

(2.2) Definition

Let R be a term rewriting system and (t_1, t_2, t_3) be a triple of terms. Then:

- (t_1, t_2, t_3) is confluent in R iff $t_2{\downarrow}R = t_3{\downarrow}R$
- (t_1, t_2, t_3) is subconnected in R iff $t_2 \vdash\!\dashv_{R,\, t_1} t_3$.

(3) A test for ground confluence

In this section, we introduce a test for ground confluence. We start with the observation that a term rewriting system is ground confluent, iff every ground instance of a critical triple is confluent. Usually there are infinitely many ground instances of critical triples, therefore we have to find a testable criterion. For confluent critical triples, the confluence of every ground instance of the triple is obviously satisfied, but the ground instances of other critical triples may be confluent even though the triple itself is not confluent. In this paper we introduce a test for proving the confluence of all ground instances for critical triples which are not confluent. This test bases on the following ideas:

- We create a finite set M of instances for a critical triple (t_1, t_2, t_3) if the critical triple is not confluent $(t_2{\downarrow}R \neq t_3{\downarrow}R)$
- We prove that every triple (t'_1, t'_2, t'_3) from M is subconnected $(t'_2 \overset{*}{\vdash\!\dashv}_{R,\, t'_1} t'_3)$
- The critical triple (t_1, t_2, t_3) is ground subconnected if every ground instance of (t_1, t_2, t_3) is also a ground instance of a triple in M (M covers $\{(t_1, t_2, t_3)\}$)

Note, that we use the subconnectedness of triples instead of confluence. The confluence of a triple always implies its subconnectedness and for confluent term rewriting systems, these properties are equivalent [WB 83]. For other term rewriting systems the subconnectedness may also hold for triples which are not confluent. This is also true in our case where we may have ground confluence but not full confluence for a term rewriting system. Therefore a critical triple with variables may be subconnected even though only every ground instance of the triple is confluent but not the critical triple itself. We will show examples of this case in the appendix.

We will now formalize these ideas. The proofs for this section and the next section can be found in the appendix.
We will first give some notation:

(3.1) Definition

Let (t_1, t_2, t_3) be a triple of terms and M be a set of triples of terms.

- I(M) denotes the set of all ground instances of triples in M:

$$I(M) := \{ (\sigma(t_1), \sigma(t_2), \sigma(t_3)) \mid (t_1, t_2, t_3) \in M \wedge \forall x \in V(t_1) \cup V(t_2) \cup V(t_3) : \sigma(x) \in TB(FS, \emptyset) \}$$

- (t_1, t_2, t_3) is **ground confluent** if all ground instances of (t_1, t_2, t_3) are confluent:

$$\forall (t'_1, t'_2, t'_3) \in I(\{ (t_1, t_2, t_3) \}) : t'_2{\downarrow}R = t'_3{\downarrow}R$$

- (t_1, t_2, t_3) is **ground subconnected** if all ground instances of (t_1, t_2, t_3) are subconnected:

$$\forall (t'_1, t'_2, t'_3) \in I(\{ (t_1, t_2, t_3) \}) : t'_2 \overset{*}{\longleftrightarrow}_{R, t'_1} t'_3$$

The basis for this paper is the next theorem:

(3.2) Theorem

Let R be a terminating term rewriting system. Then, R is ground confluent iff every critical triple is ground subconnected:

$$\forall (t_1, t_2, t_3) \in I(CTS(R)) : t_2 \overset{*}{\longleftrightarrow}_{R, t_1} t_3$$

Every ground instance of a critical triple is connected if it is covered by a connected set of triples M (defintion 3.3, lemma 3.4):

(3.3) Definition

Let M_1 and M_2 be sets of triples of terms. M_1 covers M_2 iff every ground instance of a triple in in M_2 is also a ground instance of a triple in M_1:

$$I(M_2) \subseteq I(M_1)$$

(3.4) Lemma

Let R be a term rewriting system, (t_1, t_2, t_3) a critical triple from CTS(R) and M be a set of triples of terms with:

$$\forall\, (t'_1, t'_2, t'_3) \in M : t'_2 \overset{*}{\mapsto}_{R,\, t'_1}\, t'_3$$

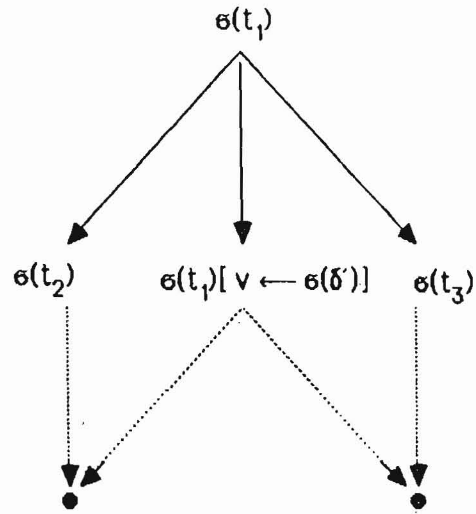
Then, (t_1, t_2, t_3) is ground connected if $\{\ (t_1, t_2, t_3)\ \}$ is covered by M.

Now we need a way to generate instances of critical triples and to prove that these instances are connected. The method in this paper bases on an extension of the confluence test given by [WB 83] and [KU 85].

For a critical triple which is not confluent we try to unify the left hand sides of rules with a subterm of the first component of the triple. Assume $(t_1, t_2, t_3) = CT(\alpha \longrightarrow \beta, u, \gamma \longrightarrow \delta)$ is a critical triple which is not confluent and $\gamma' \longrightarrow \delta'$ is a rule where γ' is unifiable with t_1/u by σ. Then the rules $\alpha \longrightarrow \beta$, $\gamma \longrightarrow \delta$ and $\gamma' \longrightarrow \delta'$ can be applied at $\sigma(t_1)$:

$$\begin{array}{lll} \sigma(t_1) & \longrightarrow_R \sigma(t_2) & \text{with } \gamma \longrightarrow \delta \text{ at } u \\ & \longrightarrow_R \sigma(t_1)\,[\,v \longleftarrow \sigma(\delta')\,] & \text{with } \gamma' \longrightarrow \delta' \text{ at } v \\ & \longrightarrow_R \sigma(t_3) & \text{with } \alpha \longrightarrow \beta \text{ at } \varepsilon \end{array}$$

The triple $(\sigma(t_1), \sigma(t_2), \sigma(t_3))$ is subconnected if the triples $(\sigma(t_1), \sigma(t_2), \sigma(t_1)\,[\,v \longleftarrow \sigma(\delta')\,])$ and $(\sigma(t_1),\ \sigma(t_1)\,[\,v \longleftarrow \sigma(\delta')\,], \sigma(t_3))$ are confluent:



We try to prove the confluence of the triples by considering the positions where the rules $\gamma \longrightarrow \delta$, $\gamma' \longrightarrow \delta'$ are applied (case 1) for the triple $(\sigma(t_1), \sigma(t_2), \sigma(t_1)[\, v \longleftarrow \sigma(\delta')\,])$ and the positions where the rules $\gamma' \longrightarrow \delta'$, $\alpha \longrightarrow \beta$ are applied (case 2) for the triple $(\sigma(t_1),\ \sigma(t_1)[\, v \longleftarrow \sigma(\delta')\,], \sigma(t_3))$.

If two rules are applied at positions which do not critically overlap, then the derived terms can be reduced to a common term, otherwise we have to check the critical overlappings.

We formalize this idea by introducing a new set DCO(C,R) (double critical overlappings) of critical overlappings. DCO(C,R) contains an element of the form $(\alpha \longrightarrow \beta, u, \gamma \longrightarrow \delta, v, \gamma' \longrightarrow \delta')$ if $(\alpha \longrightarrow \beta, u, \gamma \longrightarrow \delta)$ is an element from C and a non variable subterm of the first component from (t_1, t_2, t_3) - $CT(\alpha \longrightarrow \beta, u, \gamma \longrightarrow \delta)$ is unifiable with a rule from R. We denote the instance of the triple $CT(\alpha \longrightarrow \beta, u, \gamma \longrightarrow \delta)$ by $DCT(\alpha \longrightarrow \beta, u, \gamma \longrightarrow \delta, v, \gamma' \longrightarrow \delta')$ (double critical triple). The set DCTS(C,R) contains a double critical triple for every element from DCR(C,R).

(3.5) Definition

Let R be a set of term rewriting rules and C a subset of CO(R). We define the sets DCO, DCT and DCTS on R and C as follows:

$DCO(C, R) := \{ (\alpha \longrightarrow \beta, u, \gamma \longrightarrow \delta, v, \gamma' \longrightarrow \delta') \mid (\alpha \longrightarrow \beta, u, \gamma \longrightarrow \delta) \in C \wedge \gamma' \longrightarrow \delta' \in R$

$\wedge ((t_1, t_2, t_3) = CT(\alpha \longrightarrow \beta, u, \gamma \longrightarrow \delta)$

$\Longrightarrow v \in O'(t_1) \wedge \exists \sigma, \tau : \sigma(t_1)/v = \tau(\gamma')) \}$

$DCT(\alpha \longrightarrow \beta, u, \gamma \longrightarrow \delta, v, \gamma' \longrightarrow \delta') := (\sigma(t_1), \sigma(t_2), \sigma(t_3))$

with: $(t_1, t_2, t_3) = CT(\alpha \longrightarrow \beta, u, \gamma \longrightarrow \delta)$

$\wedge$ φ' renaming substitution of γ' and $V(\varphi'(\gamma')) \cap V(t_1) = \emptyset$

$\wedge$ $\sigma \in mgu(t_1/v, \varphi'(\gamma'))$

$DCTS(C, R) := \{ DCT(\alpha \longrightarrow \beta, u, \gamma \longrightarrow \delta, v, \gamma' \longrightarrow \delta') \mid$

$(\alpha \longrightarrow \beta, u, \gamma \longrightarrow \delta, v, \gamma' \longrightarrow \delta') \in DCO(C, R) \}$

In lemma 3.6 we give the criterion for the ground subconnectedness of a double critical triple $DCT(\alpha \longrightarrow \beta, u, \gamma \longrightarrow \delta, v, \gamma' \longrightarrow \delta')$. If there is no critical overlapping between the rule $\gamma' \longrightarrow \delta'$ and the rules $\alpha \longrightarrow \beta$ and $\gamma \longrightarrow \delta$ (the conditions $v \in O'(\alpha)$, $u = v.u' \wedge u' \in O'(\gamma')$ and $v = u.v' \wedge v' \in O'(\gamma)$ are not satisfied), then the ground subconnectedness is immediately satisfied otherwise we have to check critical overlappings between the rule $\gamma' \longrightarrow \delta'$ and the rules $\alpha \longrightarrow \beta$ and $\gamma \longrightarrow \delta$.

<u>(3.6) Lemma</u>

Let R be a term rewriting system and $(\alpha \longrightarrow \beta, u, \gamma \longrightarrow \delta, v, \gamma' \longrightarrow \delta')$ be from $DCO(CO(R), R)$. The triple $DCT(\alpha \longrightarrow \beta, u, \gamma \longrightarrow \delta, v, \gamma' \longrightarrow \delta')$ is ground subconnected, if:

($v \in O'(\alpha) \Longrightarrow CT(\alpha \longrightarrow \beta, v, \gamma' \longrightarrow \delta')$ is confluent)

$\wedge$ ($u = v.u' \wedge u' \in O'(\gamma') \Longrightarrow CT(\gamma' \longrightarrow \delta', u', \gamma \longrightarrow \delta)$ is confluent)

$\wedge$ ($v = u.v' \wedge v' \in O'(\gamma) \Longrightarrow CT(\gamma \longrightarrow \delta, v', \gamma' \longrightarrow \delta')$ is confluent)

Basing on these definitions and Lemmata, we can give a test for ground confluence:

Confluence Test

(1) $\text{CONFL-CO} := \{ (\alpha \longrightarrow \beta, u, \gamma \longrightarrow \delta) \mid (\alpha \longrightarrow \beta, u, \gamma \longrightarrow \delta) \in CO(R)$ and $CT(\alpha \longrightarrow \beta, u, \gamma \longrightarrow \delta)$ is confluent $\}$

$\text{NOT-CONFL-CO} := \{ (\alpha \longrightarrow \beta, u, \gamma \longrightarrow \delta) \mid (\alpha \longrightarrow \beta, u, \gamma \longrightarrow \delta) \in CO(R)$ and $CT(\alpha \longrightarrow \beta, u, \gamma \longrightarrow \delta)$ is not confluent $\}$

(2) $\text{CONNECTED-DCO} := \{ (\alpha \longrightarrow \beta, u, \gamma \longrightarrow \delta, v, \gamma' \longrightarrow \delta') \mid$

$(\alpha \longrightarrow \beta, u, \gamma \longrightarrow \delta, v, \gamma' \longrightarrow \delta') \in DCO(\text{NOT-CONFL-CO}, R)$

$\wedge (v \in O'(\alpha) \Longrightarrow CT(\alpha \longrightarrow \beta, v, \gamma' \longrightarrow \delta')$ is confluent $)$

$\wedge (u = v.u' \wedge u' \in O'(\gamma')$

$\Longrightarrow CT(\gamma' \longrightarrow \delta', u', \gamma \longrightarrow \delta)$ is confluent $)$

$\wedge (v = u.v' \wedge v' \in O'(\gamma)$

$\Longrightarrow CT(\gamma \longrightarrow \delta, v', \gamma' \longrightarrow \delta')$ is confluent $) \}$

(3) R is ground confluent, if CONNECTED-DCO $\cup$ CONFL-CO covers NOT-CONFL-CO

For the correctness of this test, we have to prove that every critical triple of R is ground connected (Theorem 3.2). If a triple is confluent, then it is also ground connected. Otherwise we consider a set of connected triples CONNECTED-DCO (Lemma 3.6). Then, the critical triple is connected if it is covered by the set of connected triples CONNECTED-DCO $\cup$ CONFL-CO (Lemma 3.4).

To complete this confluence test, we need a method for proving the coveredness property. With the test of Kounalis [KO 85] for example we can prove the coveredness of a set of terms by another set of terms. This test can be extended to triples of terms, by introducing a new ternary operator f and applying it to every triple. Then, we check the coveredness of a set of triples M_1 by a set of triples M_2 as follows:

(1) Transform the triples in M_1 and M_2 into terms:

$M'_1 := \{ f(t_1, t_2, t_3) \mid (t_1, t_2, t_3) \in M_1 \}$

$M'_2 := \{ f(t_1, t_2, t_3) \mid (t_1, t_2, t_3) \in M_2 \}$

(2) M_1 is covered by M_2 iff M'_1 is covered by M'_2

(3) An extension to convertible functions

In [GO 85b] the number of critical pairs has been reduced by introducing a weaker rewrite relation $\twoheadrightarrow_R$ for a term rewriting system R. A term t' is derivable from a term t by $\twoheadrightarrow_R$ if t can be reduced to t' by a rule from R and all variables in the left hand side of the rule are replaced by constructor terms. It has been proved, that the ground confluence of the relation $\twoheadrightarrow_R$ is equivalent to the ground confluence of $\longrightarrow_R$, if every ground term is reducible to a constructor ground term. In this case, it is sufficient to consider only critical pairs where variables are replaced by constructor terms.

Here, we will also consider a weaker rewrite relation denoted by $\twoheadrightarrow_R$, but the definition of this relation differs from the relation given in [GO 85b]. We allow an application of a rule if the variables on the left hand side are replaced by normal forms:

(4.1) Definition

Let R be a term rewriting system. We define the relations $\twoheadrightarrow_R$, $\leftrightarrow_R$ and $\leftrightarrow_{R,t}$ as follows:

- $t_1 \twoheadrightarrow_R t_2 \iff \exists\, \alpha \longrightarrow \beta \in R : \exists\, \sigma\, \exists\, u \in O(t) : t_1/u = \sigma(\alpha)$
  $\wedge\ (\forall\, x \in V(\alpha) : \sigma(x) \in IRR(R))$
  $\wedge\ t_2 = t_1[\, u \leftarrow \sigma(\beta)\,]$

- $t_1 \leftrightarrow_R t_2 \iff t_1 \twoheadrightarrow_R t_2 \vee t_2 \twoheadrightarrow_R t_1$

- $t_1 \leftrightarrow_{R,t} t_2 \iff t_1 \leftrightarrow_R t_2 \wedge t > t_1 \wedge t > t_2$

We denote by $\overset{*}{\twoheadrightarrow}$ the reflexive and transitive closure of $\twoheadrightarrow$ and by $\Vdash\overset{*}{-}\!\!\!\Vdash$ the reflexive and transitive closure of $\Vdash\!\!-\!\!\!\Vdash$.

The relations $\twoheadrightarrow_R$ and $\longrightarrow_R$ have the same normal forms, their symmetric, transitive and reflexive closures are equivalent and $\twoheadrightarrow_R$ is confluent iff $\longrightarrow_R$ is confluent:

(4.2) Theorem

Let R be a term rewriting system. Then:

- $\forall t : t$ irreducible by $\twoheadrightarrow_R \Longleftrightarrow t$ irreducible by $\longrightarrow_R$
- $\Vdash\overset{*}{-}\!\!\!\Vdash_R \Longleftrightarrow \vdash\overset{*}{-}\!\!\dashv_R$
- $\twoheadrightarrow_R$ is confluent iff $\longrightarrow_R$ is confluent.

If we use the relation $\twoheadrightarrow_R$ on ground terms, then the variables of a rule are only replaced by ground terms which contain no convertible function symbols. We can modify the test given in section 2 for the relation $\twoheadrightarrow_R$ by considering only those function symbols F which are not convertible for the computation of critical overlappings and for the coveredness test of a critical triple.

In definition 4.3 we restrict the critical overlappings to cases where variables in rules can be replaced by terms without convertible functions:

(4.3) Definition

Let R be a term rewriting system and $F \subset FS$. We define the set of critical overlappings $CO_F(R)$ and the set of critical triples $CTS_F(R)$ as follows:

$$
\begin{aligned}
CO_F(R) := \{ (\alpha \longrightarrow \beta, u, \gamma \longrightarrow \delta) \mid\ & \alpha \longrightarrow \beta, \gamma \longrightarrow \delta \in R \wedge u \in O'(\alpha) \\
& \wedge \exists \sigma, \tau : \sigma(\alpha)/u = \tau(\gamma) \\
& \wedge \forall x \in V(\alpha) : \sigma(x) \in TE(F,VA) \\
& \wedge \forall x \in V(\gamma) : \tau(x) \in TE(F,VA) \}
\end{aligned}
$$

$$
CTS_F(R) := \{ CT(\alpha \longrightarrow \beta, u, \gamma \longrightarrow \delta) \mid (\alpha \longrightarrow \beta, u, \gamma \longrightarrow \delta) \in CO_F(R) \}
$$

Note, that the CO(R) is equal to $CO_{FS}(R)$ and CTS(R) is equal to $CTS_{FS}(R)$.

Instead of considering all ground instances of triples, we consider only instances where variables are replaced by ground terms without convertible function symbols:

<u>(4.4) Definition</u> (modified version of definition 3.1)

Let (t_1, t_2, t_3) be a triple of terms, $F \subset FS$ a set of function symbols and M be a set of triples of terms.

- $I_F(M)$ denotes the set of all ground instances in F of triples in M:

$$I_F(M) := \{ (\sigma(t_1), \sigma(t_2), \sigma(t_3)) \mid (t_1, t_2, t_3) \in M \wedge \forall x \in V(t_1) \cup V(t_2) \cup V(t_3) : \sigma(x) \in TE(F, \emptyset) \}$$

- (t_1, t_2, t_3) is **ground confluent in F** if all ground instances in F of (t_1, t_2, t_3) are confluent:

$$\forall (t'_1, t'_2, t'_3) \in I_F(\{ (t_1, t_2, t_3) \}) : t'_2{\downarrow}R = t'_3{\downarrow}R$$

- (t_1, t_2, t_3) is **ground connected in F** if all ground instances in F of (t_1, t_2, t_3) are connected:

$$\forall (t'_1, t'_2, t'_3) \in I_F(\{ (t_1, t_2, t_3) \}) : t'_2 \overset{*}{\longmapsto}_{R, t'_1} t'_3$$

A term rewriting system is ground confluent if all ground instances in F of $CPS_F(R)$ are connected and FS\F contains only convertible function symbols:

<u>(4.5) Theorem</u> (modified version of theorem 3.2)

Let R be a terminating term rewriting system, $F \subset FS$ a set of function symbols and FS\F contains only convertible function symbols. Then, R is ground confluent iff:

$$\forall (t_1, t_2, t_3) \in I_F(CTS(R)) : t_2 \overset{*}{\longmapsto}_{R, t_1} t_3$$

The definition of the coveredness property can also be extended for convertible functions:

(4.6) Definition (modified version of definition 3.3)

Let M_1 and M_2 be sets of triples of terms and $F \subset FS$ a set of function symbols. M_1 covers M_2 in F iff:

$$I_F(M_1) \subset I_F(M_2)$$

Every ground instance in F of a critical triple is connected if it is covered in F by a connected set of triples M:

(4.7) Lemma (modified version of lemma 3.4)

Let R be a term rewriting system, $F \subset FS$ a set of function symbols, (t_1, t_2, t_3) a critical triple from $CTS_F(R)$ and M be a set of triples of terms with:

$$\forall (t'_1, t'_2, t'_3) \in M : t_2 \overset{*}{\longmapsto}_{R, t1} t_3$$

Then, (t_1, t_2, t_3) is ground connected in F if $\{ (t'_1, t'_2, t'_3) \}$ is covered in F by M.

Now, we can also give a modified test for ground confluence:

Confluence Test

(1) CONFL-CO := $\{ (\alpha \longrightarrow \beta, u, \gamma \longrightarrow \delta) \mid (\alpha \longrightarrow \beta, u, \gamma \longrightarrow \delta) \in CO_{FS}(R)$ and $CT(\alpha \longrightarrow \beta, u, \gamma \longrightarrow \delta)$ is confluent $\}$

NOT-CONFL-CO := $\{ (\alpha \longrightarrow \beta, u, \gamma \longrightarrow \delta) \mid (\alpha \longrightarrow \beta, u, \gamma \longrightarrow \delta) \in CO_F(R)$ and $CT(\alpha \longrightarrow \beta, u, \gamma \longrightarrow \delta)$ is not confluent $\}$

(2) CONNECTED-DCO := $\{ (\alpha \longrightarrow \beta, u, \gamma \longrightarrow \delta, v, \gamma' \longrightarrow \delta') \mid$
$(\alpha \longrightarrow \beta, u, \gamma \longrightarrow \delta, v, \gamma' \longrightarrow \delta') \in DCO($NOT-CONFL-CO$, R)$
$\wedge (v \in O'(\alpha) \Longrightarrow CT(\alpha \longrightarrow \beta, v, \gamma' \longrightarrow \delta')$ is confluent $)$
$\wedge (u = v.u' \wedge u' \in O'(\gamma') \Longrightarrow CT(\gamma' \longrightarrow \delta', u', \gamma \longrightarrow \delta)$ is confluent$)$
$\wedge (v = u.v' \wedge v' \in O'(\gamma) \Longrightarrow CT(\gamma \longrightarrow \delta, v', \gamma' \longrightarrow \delta')$ is confluent $)$

(3) R is ground confluent, if CONNECTED-DCO $\cup$ CONFL-CO covers NOT-CONFL-CO in F

Note, that we consider all confluent critical triples from $CO_{FS}(R)$ because this may help us to prove more triples in DCO(nred-crit,R) to be confluent than to consider only confluent critical triples from $CO_F(R)$.

The coveredness test of Kounalis [KO 85] has not to be modified, because it allows us to distinguish constructors (F) and defined functions (FS\F), and prove the coveredness of a set of terms only for constructor ground instances.

Conclusion

The test developed in this paper allowed us to prove the ground confluence of many systems where the classical Knuth-Bendix generates an infinite system. The ground confluence could be proven without splitting the term rewriting system into axioms and other rules and without choosing position in rules, which was necessary in [GO 85b] or [FR 86]. In fact, we could prove more systems to be ground confluent than with the methods given in [GO 85b] and [FR 86], but it is not obvious that our method is a stronger test for any term rewriting system.

A major problem for this test are term rewriting systems which are not terminating. In future, we plan to extend the test for globally finite term rewriting systems on the base of [GO 85a]. Also the method of [HR 86] will be considered because it may help in cases where the term rewriting system is not even globally finite.

Appendix 1 Examples

We have implemented a simple completion procedure with the ground confluence test described in this paper. We will give three examples, where the ground conflunce can be proved by our implementation and the classical completion procedure generates an infinite set of rules.

(1) Associativity of Add

In the following term rewriting system rule 1 and 2 axiomize the addition on natural numbers and rule 3 is an inductive consequence of rule 1 and rule 2:

(1) add(0,y) ⟶ y

(2) add(s(x),y) ⟶ s(add(x,y))

(3) add(x,add(y,z)) ⟶ add(add(x,y),z)

This example has been considered in [FR 86] and could be proven to be ground confluent, even though the classical Knuth-Bendix Algorithm generates an infinite set of rules.
For this example we show completely how our ground conflunce test works. For lack of space we will skip details of the test in the other examples.

For this example, we assume that add is convertible. We get the following critical overlappings and critical triples:

(rule 3, 2, rule 2) : (add(x,add(s(y),z)), add(x,s(add(y,z))), add(add(x,s(y)),z))

(rule 3, ε, rule 2) : (add(s(x),add(y,z)), s(add(x,add(y,z))), add(add(s(x),y),z))

(rule 3, 2, rule 1) : (add(x,add(0,y)), add(x,y), add(add(x,0),z))

(rule 3, ε, rule 1) : (add(0,add(x,y)), add(x,y), add(add(0,x),y))

(rule 3, 2, rule 3) :

(add(x,add(y,add(z,u))), add(x,add(add(y,z),u)), add(add(x,y),add(z,u)))

Now, the critical triples of the second critical overlapping (rule 3, ε, rule 2), the fourth critical overlapping (rule 3, ε, rule 1) and the fifth critical overlapping (rule 3, 2, rule 3) are confluent. The other critical triples are not confluent but belong to $CO_P(R)$ therefore we have to generate instances by unifying left hand sides of rules with their first elements:

(rule 3, 2, rule 1, ε, rule 2) :

(add(s(x),add(0,y)), add(s(x),y), add(add(s(x),0),y))

(rule 3, 2, rule 1, ε, rule 1) :

(add(0,add(0,y)), add(0,y), add(add(0,0),y))

(rule 3, 2, rule 2, ε, rule 2) :

(add(s(x),add(s(y),z)), add(s(x),s(add(y,z))), add(add(s(x),s(y)),z))

(rule 3, 2, rule 2, ε, rule 1) :

(add(0,add(s(y),z)), add(0, s(add(y,z))), add(add(0,s(y)),z))

Now, we can prove that all these triples are connected. Consider for example the triple of the overlapping (rule 3, 2, rule 2, ε, rule 1). This triple is confluent because the triple of the overlapping (rule 3, ε, rule 1) is confluent and rule 2 has been applied at a subterm which has been matched by a variable of rule 1 (for details see lemma 3.6).

Note, that all four doble critical triples are not confluent and for every example in this appendix the confluence of triples is not sufficient for proving the ground confluence of the examples.

It remains to proof that the critical triples of the overlappings (rule 3, 2, rule 2) and (rule 3, 2, rule 1) are covered in {s, 0} by their connected instances. This is easy for both triples because the instances have been generated by replacing a single variable by s(x) and 0, and s(x), 0 cover all ground instances in {s, 0}.

The associativity of add can also be proved by the classical Knuth-Bendix Algorithm, if we add the inductive consequences add(x,0) ⟶ x and add(x,s(y)) ⟶ s(add(x,y)) to the system or use the associativity law in the reverse direction. In the next example it does not help to add other rules or to orient rules in the other direction, the classical Knuth-Bendix Algorithm will always generate an infinite system.

(2) Add and Sub on Natural Numbers

In this example we give two equivalent axiomatization of the addition (rules (1), (2) and rules (3), (4)) and one axiomatization of the subtraction (rules (5) - (7)) on natural numbers. Rules (8) and (9) are inductive consequences of the rules (1) - (7):

(1) add(0,y) ⟶ y
(2) add(s(x),y) ⟶ s(add(x,y))
(3) add(x,0) ⟶ x
(4) add(x,s(y)) ⟶ s(add(x,y))

(5) sub(s(x),s(y)) ⟶ sub(x,y)
(6) sub(0,y) ⟶ y
(7) sub(x,0) ⟶ x

(8) sub(add(x,y),y) ⟶ x
(9) sub(add(y,x),y) ⟶ x

With our confluence test we can prove the ground confluence of this system. The methods presented in [GO 85b] and [FR 86] fail for this example. The ground confluence of the subsystem 1-8 and the subsystem 1-7, 9 can be proved by both methods if we choose the right axiomatization of add but the classical Knuth-Bendix Algorithm still diverges for these subsystems.

(3) Greater on Natural Numbers

In the last example, we give an axiomatization of the predicate greater on natural numbers (rules 1, 2, 3), of boolean functions (rules 4 - 9) and three inductive consequences of greater (greater is irreflexive (rule 10), antisymmetric (rule 11) and transitive (rule 12)). These rules have been given to our completion procedure which generates three other rules (rules 13 - 15) and proves the ground confluence of the final system.

(1) gr(0,x) ⟶ false

(2) gr(s(x),0) ⟶ true

(3) gr(s(x),s(y)) ⟶ gr(x,y)

(4) not(true) ⟶ false

(5) not(false) ⟶ true

(6) or(true,y) ⟶ true

(7) or(false,y) ⟶ y

(8) and(true,y) ⟶ y

(9) and(false,y) ⟶ false

(10) gr(x,x) ⟶ false

(11) or(not(gr(x,y)),not(gr(y,x))) ⟶ true

(12) or(not(gr(x,y)),or(not(gr(y,z)),gr(x,z))) ⟶ true

(12),(10) --> (13) or(not(gr(x,y)),true) ⟶ true

(12),(10) --> (14) or(not(gr(x,y)),or(not(gr(y,x)),false)) ⟶ true

(12),(3) --> (15) or(not(gr(x,s(y))),or(not(gr(y,z)),gr(x,s(z)))) ⟶ true

Appendix 2 Proofs

Instead of proving theorem 3.2 and lemma 3.4 we will prove the more general theorem 4.5 and lemma 4.7. Therefore we start with the proof of theorem 4.2 because the proof of theorem 4.5 bases on this proof.

Proof of theorem 4.2

(1) $\forall t : t$ irreducible by $\twoheadrightarrow_R \Longleftrightarrow t$ irreducible by $\longrightarrow_R$

($\Longleftarrow$) obvious because $\twoheadrightarrow_R \subset \longrightarrow_R$

($\Longrightarrow$) t irreducible by $\twoheadrightarrow_R$

The proof follows from the fact that a term t is reducible by $\longrightarrow$ iff there is a subterm of t which is only reducible on top and therefore the subterm is also reducible by $\twoheadrightarrow$:

Assume t is reducible by $\longrightarrow_R$

Let u be the deepest occurence in t, which is reducible :

$\exists \alpha \longrightarrow \beta \in R \; \exists \sigma : t/u = \sigma(\alpha) \wedge \forall v \in O(t) : v > u \Longrightarrow t/v \in IRR(R)$

Then, every variable in α matches an irreducible term :

α is of the form $f(t_1, \ldots, t_n)$ because no left hand side consists of a single variable :

$\Longrightarrow t/u = \sigma(f(t_1, \ldots, t_n)) \qquad t/u = \sigma(\alpha)$

$\Longrightarrow t/u = f(\sigma(t_1'), \ldots, \sigma(t_n))$

$\Longrightarrow \sigma(t_1), \ldots, \sigma(t_n) \in IRR(R)$

$\Longrightarrow \forall x \in V(t_1) \cup \ldots \cup V(t_n) : \sigma(x) \in IRR(R)$

$\Longrightarrow \forall x \in V(f(t_1, \ldots, t_n)) : \sigma(x) \in IRR(R)$

$\Longrightarrow$ t is reducible by $\twoheadrightarrow_R$ at u

$\Longrightarrow$ contradiction : t is irreducible by $\twoheadrightarrow_R$

(2) $\overset{*}{\Vdash\!\dashv}_R \Longleftrightarrow \overset{*}{\vdash\!\dashv}_R$

($\Longrightarrow$) obvious : $\overset{*}{\Vdash\!\dashv}_R \subset \overset{*}{\vdash\!\dashv}_R$

($\Longleftarrow$) We will prove:

$t_1 \vdash\!\dashv_R t_2 \Longrightarrow t_1 \overset{*}{\Vdash\!\dashv}_R t_2$

Then, the case:

$t_1 \overset{*}{\vdash\!\dashv}_R t_2 \Longrightarrow t_1 \overset{*}{\Vdash\!\dashv}_R t_2$

is obvious, because $\overset{*}{\vdash\!\dashv}_R$ is the reflexive and transitive closure of $\vdash\!\dashv_R$.

$t_1 \vdash\!\dashv_R t_2$

$\Longrightarrow$ (i) $\exists\, \alpha \longrightarrow \beta \in R\ \exists\, u, \sigma : t_1/u = \sigma(\alpha) \wedge t_2 = t_1 [\, u \leftarrow \sigma(\beta)\,]$

$\vee$ (ii) $\exists\, \alpha \longrightarrow \beta \in R\ \exists\, u, \sigma : t_2/u = \sigma(\alpha) \wedge t_1 = t_2 [\, u \leftarrow \sigma(\beta)\,]$

(i) We can reduce the subterms of t_1 and t_2 which are matched by variables from α and β ($\sigma(x)$) by $\twoheadrightarrow$ to their normal forms ($\sigma'(x)$) and derive the reduced t_2 from the reduced t_1 by $\twoheadrightarrow$. We define $\sigma'(x)$ as follows :

$x \in V(\alpha) \Longrightarrow \sigma'(x) := \text{normal form of } \sigma(x) \text{ in } \overset{*}{\twoheadrightarrow}_R$

$x \notin V(\alpha) \Longrightarrow \sigma'(x) := x$

Then: $\forall\, x \in V(\alpha) : \sigma'(x) \in IRR(R)$ with (1)

$\Longrightarrow t_1 [\, u \leftarrow \sigma'(\alpha)\,] \twoheadrightarrow_R t_2 [\, u \leftarrow \sigma'(\beta)\,]$

$\Longrightarrow t_1 \overset{*}{\twoheadrightarrow}_R t_1 [\, u \leftarrow \sigma'(\alpha)\,] \twoheadrightarrow_R t_2 [\, u \leftarrow \sigma'(\beta)\,]\ {}_R\overset{*}{\twoheadleftarrow} t_2$

$\Longrightarrow t_1 \overset{*}{\Vdash\!\!\dashv}_R t_2$

(ii) this proof is equivalent to (i)

(3) $\twoheadrightarrow_R$ confluent $\Longleftrightarrow \longrightarrow_R$ is confluent.

($\Longrightarrow$) $\twoheadrightarrow_R$ is confluent

Assume $t \xrightarrow{*}_R t_1 \wedge t \xrightarrow{*}_R t_2$

$\Longrightarrow t_1 \overset{*}{\vdash\!\dashv}_R t_2$

$\Longrightarrow t_1 \overset{*}{\Vdash\!\!\dashv}_R t_2$ with (2)

$\Longrightarrow \exists\, t' : t_1 \overset{*}{\twoheadrightarrow}_R t' \wedge t_2 \overset{*}{\twoheadrightarrow}_R t'$ $\quad \twoheadrightarrow_R$ is Church-Rosser

$\Longrightarrow t_1 \xrightarrow{*}_R t' \wedge t_2 \xrightarrow{*}_R t'$ $\quad \overset{*}{\twoheadrightarrow}_R \subset \xrightarrow{*}_R$

$\Longrightarrow \xrightarrow{*}_R$ is confluent

($\Longleftarrow$) $\longrightarrow_R$ is confluent

Assume $t \overset{*}{\twoheadrightarrow}_R t_1 \wedge t \overset{*}{\twoheadrightarrow}_R t_2$

Then reduce t_1 and t_2 to their normal forms t'_1 and t'_2 by $\overset{*}{\twoheadrightarrow}_R$:

$t_1 \overset{*}{\twoheadrightarrow}_R t'_1 \wedge t_2 \overset{*}{\twoheadrightarrow}_R t'_2$

$\Longrightarrow t'_1 \overset{*}{\Vdash\!\!\dashv}_R t'_2$

$\Longrightarrow t'_1 \overset{*}{\vdash\!\dashv}_R t'_2$ with (2)

$\Longrightarrow \exists\, t' : t'_1 \xrightarrow{*}_R t' \wedge t'_2 \xrightarrow{*}_R t'$ $\quad \xrightarrow{*}_R$ Church-Rosser

with t'_1, t'_2 irreducible by $\overset{*}{\twoheadrightarrow}_R \Longrightarrow t'_1$, t'_2 irreducible by $\xrightarrow{*}_R$ (1) :

$t'_1 = t' \wedge t'_2 = t'$

$\Longrightarrow t'_1 = t'_2$

$\Longrightarrow \longrightarrow\!\!\!\!\rightarrow_R$ is confluent

Lemma A.1 allows us to simplify the proofs of theorem 4.5 and lemma 4.6:

<u>(A.1) Lemma</u>

Let R be a term rewriting system, F a subset of FS, t a ground term and $\alpha \longrightarrow \beta$, $\gamma \longrightarrow \delta$ be rules from R. The rule $\alpha \longrightarrow \beta$ can be applied at u in t, the rule $\gamma \longrightarrow \delta$ can be applied at v in t and all variables in α and β match terms from TE(F,Ø):

- $t/u = \sigma(\alpha) \wedge t/v = \tau(\gamma) \wedge t \in TE(FS, \emptyset)$
- $\forall x \in V(\alpha) : \sigma(x) \in TE(F, \emptyset) \wedge \forall x \in V(\gamma) : \tau(x) \in TE(F, \emptyset)$

Then the triple $(t, t[u \leftarrow \sigma(\beta)], t[v \leftarrow \tau(\delta)])$ is connected if:

(1) u and v are disjoint

or (2) $v = u.v' \wedge v' \notin O'(\alpha)$

or (3) $v = u.v' \wedge v' \in O'(\alpha)$

$$\Longrightarrow \forall (t_1, t_2, t_3) \in I_F((CT(\alpha \longrightarrow \beta, v', \gamma \longrightarrow \delta))) : t_2 \overset{*}{\longmapsto}_{R, t_1} t_3$$

<u>Proof</u>

(1) u, v are disjoint

Then $\gamma \longrightarrow \delta$ can still be applied at $t[u \leftarrow \sigma(\beta)]$

$t[u \leftarrow \sigma(\beta)] / v = t/v = \tau(\gamma)$ with: $t/v = \tau(\gamma) \wedge$ u, v are disjoint

$\Longrightarrow t[u \leftarrow \sigma(\beta)] \longrightarrow_R t[u \leftarrow \sigma(\beta)][v \leftarrow \tau(\delta)]$

and $\alpha \longrightarrow \beta$ can be applied at $t[v \leftarrow \tau(\delta)]$:

$t[v \leftarrow \tau(\delta)] / u = t/u = \sigma(\alpha)$ with: $t/u = \sigma(\alpha) \wedge$ u, v are disjoint

$\Longrightarrow t[v \leftarrow \tau(\delta)] \longrightarrow_R t[v \leftarrow \tau(\delta)][u \leftarrow \sigma(\beta)]$

$t[\,u \leftarrow \sigma(\beta)\,][\,v \leftarrow \tau(\delta)\,] = t[\,v \leftarrow \tau(\delta)\,][\,u \leftarrow \sigma(\beta)\,]$ u, v are disjoint

$\Longrightarrow t[\,u \leftarrow \sigma(\beta)\,] \overset{*}{\mapsto\!\dashv}_{R,t} t[\,v \leftarrow \tau(\delta)\,]$

(2) $v = u.v' \wedge v' \notin O'(\alpha)$

$\Longrightarrow \exists\, v_1, v_2 : v' = v_1.v_2 \wedge \alpha/v_1 \in VA$

Let $\{\, w_1, \ldots, w_n \,\}$ be the positions of this variable in α :

$\{\, w_1, \ldots, w_n \,\} := \{\, w \mid \alpha/w = \alpha/v_1 \,\}$

All subterms t at $u.w_i.v_2$ have to be equal to t/v $(i = 1, \ldots, n)$:

$t/u.w_i.v_2 = \sigma(\alpha)/w_i.v_2$ with $t/u = \sigma(\alpha)$

$\Longrightarrow t/u.w_i.v_2 = \sigma(\alpha/w_i)/v_2$

$\Longrightarrow t/u.w_i.v_2 = \sigma(\alpha/v_1)/v_2$ with : $\alpha/v_1 = \alpha/w_i$ for $i = 1, \ldots, n$

$\Longrightarrow t/u.w_i.v_2 = \sigma(\alpha)/v_1.v_2$

$\Longrightarrow t/u.w_i.v_2 = (t/u)/\, v_1.v_2$ with : $t/u = \sigma(\alpha)$

$\Longrightarrow t/u.w_i.v_2 = t/v$ with : $v' = v_1.v_2 \wedge v = u.v'$

$\gamma \longrightarrow \delta$ can be applied at every position $t/u.w_i.v_2$ and we get :

$t \longrightarrow_R t[\,v \leftarrow \tau(\delta)\,] \overset{*}{\longrightarrow}_R t[\,u.w_1.v_2 \leftarrow \tau(\delta)\,] \ldots [\,u.w_n.v_2 \leftarrow \tau(\delta)\,]$

Now $\alpha \longrightarrow \beta$ can be applied at $t[\,u.w_1.v_2 \leftarrow \tau(\delta)\,] \ldots [\,u.w_n.v_2 \leftarrow \tau(\delta)\,]$.

We define a matcher σ' for α as follows :

- $x = \alpha/v_1 : \sigma'(x) := \sigma(x)[\,u_2 \leftarrow \tau(\delta)\,]$
- $x \neq \alpha/v_1 : \sigma'(x) := \sigma(x)$

Then :

$t[\,u.w_1.v_2 \leftarrow \tau(\delta)\,] \ldots [\,u.w_n.v_2 \leftarrow \tau(\delta)\,]/u$

$= t/u\,[\,w_1.v_2 \leftarrow \tau(\delta)\,] \ldots [\,w_n.v_2 \leftarrow \tau(\delta)\,]$

$= \sigma(\alpha)[\ w_1.u_2 \leftarrow \tau(\delta)\ ] \ldots [\ w_n.u_2 \leftarrow \tau(\delta)\ ]$ with : $t/u = \sigma(\alpha)$

$= \sigma'(\alpha)$ with : $\{\ w_1, \ldots, w_n\ \} := \{\ w \mid \alpha/w = \alpha/v_1\ \}$

$\Longrightarrow t[\ u.w_1.v_2 \leftarrow \tau(\delta)\ ] \ldots [\ u.w_n.v_2 \leftarrow \tau(\delta)\ ]$

$\longrightarrow_R t[\ u.w_1.v_2 \leftarrow \tau(\delta)\ ] \ldots [\ u.w_n.v_2 \leftarrow \tau(\delta)\ ][\ u \leftarrow \sigma'(\beta)\ ]$

$= t[\ u \leftarrow \sigma'(\beta)\ ]$

In the other reduction $t[\ u \leftarrow \sigma(\beta)\ ]$ the subterm which was matched by α/v_1 will occur at any position w in $\sigma(\beta)$, where $\beta/w = \alpha/v_1$:

Let $\{\ w'_1, \ldots, w'_m\ \} := \{\ w' \mid \beta/w' = \alpha/v_1\ \}$

Then, $\gamma \longrightarrow \delta$ can be applied at every position $t[\ u \leftarrow \sigma(\beta)\ ]/\ u.w'_i.v_2$ for $i = 1, \ldots, n$:

$t[\ u \leftarrow \sigma(\beta)\ ]/\ u.w'_i.v_2 \quad = \sigma(\beta)/w'_i.v_2$

$= \sigma(\beta/w'_i)/v_2$

$= \sigma(\alpha/v_1)/v_2$ with : $\beta/w'_i = \alpha/v_1$ for $i = 1, \ldots, n$

$= \sigma(\alpha)/v_1.v_2$

$= t/u.v_1.v_2$ with : $t/u = \sigma(\alpha)$

$= t/v$ with : $v_1.v_2 = v' \wedge v = u.v'$

$= \tau(\gamma)$

$\Longrightarrow t[\ u \leftarrow \sigma(\beta)\ ]$

$\xrightarrow{*}_R t[\ u \leftarrow \sigma(\beta)\ ][\ u.w'_1.v_2 \leftarrow \tau(\delta)\ ] \ldots [\ u.w'_m.v_2 \leftarrow \tau(\delta)\ ]$

$= t[\ u \leftarrow \sigma(\beta)[\ w'_1.v_2 \leftarrow \tau(\delta)\ ] \ldots [\ w'_m.v_2 \leftarrow \tau(\delta)\ ]\ ]$

$= t[\ v \leftarrow \sigma'(\beta)\ ]$

We have derived $t[\ v \leftarrow \sigma'(\beta)\ ]$ from $t[\ u \leftarrow \sigma(\beta)\ ]$ and $t[\ v \leftarrow \tau(\delta)\ ]$ and we get :

$t[\ u \leftarrow \sigma(\beta)\ ] \mapsto^{*}_{R,t} t[\ v \leftarrow \tau(\delta)\ ]$

(3) $v = u.v' \wedge v' \in O'(\alpha)$

with $t/u = \sigma(\alpha)$:

$t/u.v' = \sigma(\alpha)/v'$

$\Longrightarrow t/v = \sigma(\alpha)/v'$ $\quad u.v' = v$

$\Longrightarrow \tau(\gamma) = \sigma(\alpha)/v'$ $\quad t/v = \tau(\gamma)$

$\Longrightarrow (\alpha \longrightarrow \beta, v', \gamma \longrightarrow \delta) \in CO_F(R)$

with: $\forall x \in V(\alpha) : \sigma(x) \in TE(F, \emptyset) \wedge \forall x \in V(\gamma) : \tau(x) \in TE(F, \emptyset)$

We get $CT(\alpha \longrightarrow \beta, v', \gamma \longrightarrow \delta)$ as follows :

Let φ be a renaming substitution of γ with $V(\alpha) \cap V(\varphi(\gamma)) = \emptyset$ and φ' the converse of φ ($\varphi' \circ \varphi = id$). Then :

$$\sigma(\alpha)/v' = \tau(\varphi'(\varphi(\gamma)))$$

α/v' and $\varphi(\gamma)$ are unifiable by the following substitution σ_1:

- $x \in V(\alpha)$: $\sigma_1(x) = \sigma(x)$
- $x \in V(\varphi(\gamma))$: $\sigma_1(x) = \tau(\varphi'(x))$
- $x \notin V(\alpha) \cup V(\varphi(\gamma))$: $\sigma_1(x) = x$

$\Longrightarrow \underline{\sigma_1(\alpha)/u} = \sigma(\alpha)/v' = \tau(\varphi'(\varphi(\gamma))) = \underline{\sigma_1(\varphi(\gamma))}$

and we can find a most general unifier for $\alpha/v', \varphi(\gamma)$:

$\Longrightarrow \exists\, \sigma_2 : \sigma_2 \in mgu(\alpha/v', \varphi(\gamma))$

$\wedge\ (\sigma_2(\alpha), \sigma_2(\alpha)[v' \longleftarrow \sigma_2(\varphi(\delta))], \sigma_2(\beta)) = CT(\alpha \longrightarrow \beta, v', \gamma \longrightarrow \delta)$

$\Longrightarrow \exists\, \sigma_3 : \sigma_1 = \sigma_3 \circ \sigma_2$ $\quad$ with: $\sigma_2 \in mgu(\alpha/v', \varphi(\gamma)) \wedge \sigma_1(\alpha/v') = \sigma_1(\varphi(\gamma))$

$\Longrightarrow (\sigma_3(\sigma_2(\alpha)), \sigma_3(\sigma_2(\alpha))[v' \longleftarrow \sigma_3(\sigma_2(\varphi(\delta)))], \sigma_3(\sigma_2(\beta)))$

$= (\sigma_1(\alpha), \sigma_1(\alpha)[v' \longleftarrow \sigma_1(\varphi(\delta))], \sigma_1(\beta))$ $\quad \sigma_1 = \sigma_3 \circ \sigma_2$.

$= (\sigma(\alpha), \sigma(\alpha)[v' \longleftarrow \tau(\delta)], \sigma(\beta))$ $\quad$ definition of σ_1

If we can prove that:

$\forall x \in V(\sigma_2(\alpha)) \cup V(\sigma_2(\alpha)[v' \longleftarrow \sigma_2(\varphi(\delta))]) \cup V(\sigma_2(\beta)) : \sigma_3(x) \in TE(F, \emptyset)$

then the triple $(t, t[u \longleftarrow \sigma(\beta)], t[v \longleftarrow \tau(\delta)])$ is subconnected:

$$(\sigma(\alpha), \sigma(\alpha) [v' \longleftarrow \tau(\delta)], \sigma(\beta)),$$

$$\in I_F (\{ (\sigma_2(\alpha), \sigma_2(\alpha) [v' \longleftarrow \sigma_2(\varphi(\delta))], \sigma_2(\beta)) \})$$

$$\Longrightarrow (\sigma(\alpha), \sigma(\alpha) [v' \longleftarrow \tau(\delta)], \sigma(\beta)) \in I_F(CT(\alpha \longrightarrow \beta, v', \gamma \longrightarrow \delta))$$

$$\Longrightarrow (\sigma(\alpha), \sigma(\alpha) [v' \longleftarrow \tau(\delta)], \sigma(\beta)) \in I_F(CTS_F(R))$$

$$\Longrightarrow \sigma(\alpha) [v' \longleftarrow \tau(\delta)] \overset{*}{\longmapsto}_{R, \sigma(\alpha)} \sigma(\beta)$$

$$\Longrightarrow t [u \longleftarrow \sigma(\alpha) [v' \longleftarrow \tau(\delta)]] \overset{*}{\longmapsto}_{R, t [u \longleftarrow \sigma(\alpha)]} t [u \longleftarrow \sigma(\beta)]$$

$$\Longrightarrow t [u \longleftarrow \sigma(\alpha)] [u.v' \longleftarrow \tau(\delta)] \overset{*}{\longmapsto}_{R, t [u \longleftarrow \sigma(\alpha)]} t [u \longleftarrow \sigma(\beta)]$$

$$\Longrightarrow t [u.v' \longleftarrow \tau(\delta)] \overset{*}{\longmapsto}_{R, t} t [u \longleftarrow \sigma(\beta)] \qquad t/u = \sigma(\alpha)$$

$$\Longrightarrow t [v \longleftarrow \tau(\delta)] \overset{*}{\longmapsto}_{R, t} t [u \longleftarrow \sigma(\beta)] \qquad u.v' = v$$

Now, it remains to prove:

$$\forall x \in V(\sigma_2(\alpha)) \cup V(\sigma_2(\alpha) [v' \longleftarrow \sigma_2(\varphi(\delta))]) \cup V(\sigma_2(\beta)) : \sigma_3(x) \in TE(F, \emptyset)$$

$$x \in V(\sigma_2(\alpha)) \cup V(\sigma_2(\alpha) [v' \longleftarrow \sigma_2(\varphi(\delta))]) \cup V(\sigma_2(\beta))$$

$$\Longleftrightarrow x \in V(\sigma_2(\alpha)) \cup V(\sigma_2(\alpha) [v' \longleftarrow \sigma_2(\varphi(\delta))]) \qquad V(\beta) \subset V(\alpha)$$

$$\Longleftrightarrow x \in V(\sigma_2(\alpha)) \cup V(\sigma_2(\varphi(\delta)))$$

$$\Longleftrightarrow x \in V(\sigma_2(\alpha)) \cup V(\sigma_2(\varphi(\gamma))) \qquad V(\delta) \subset V(\gamma)$$

We consider two cases, either $x \in V(\sigma_2(\alpha))$ (case i) or $x \in V(\sigma_2(\varphi(\gamma)))$ (case ii) :

(i) $x \in V(\sigma_2(\alpha))$

$$\Longrightarrow \exists\, w : \sigma_2(\alpha)/w = x$$

$$\Longrightarrow \exists\, w_1, w_2 : w_1.w_2 = w \wedge \alpha/w_1 = y \wedge y \in VA$$

$$\Longrightarrow \sigma(y) \in TE(F, \emptyset) \qquad \forall y \in V(\alpha) : \sigma(y) \in TE(F, \emptyset)$$

$$\Longrightarrow \sigma_1(y) \in TE(F, \emptyset) \qquad \sigma_1(y) = \sigma(y) \text{ for } y \in V(\alpha)$$

$$\Longrightarrow \sigma_3(\sigma_2(y)) \in TE(F, \emptyset) \qquad \sigma_1 = \sigma_3 \cdot \sigma_2$$

$$\Longrightarrow \sigma_3(\sigma_2(\alpha/w_1)) \in TE(F, \emptyset) \qquad \alpha/w_1 = y$$

$\Longrightarrow \sigma_3(\sigma_2(\alpha/w_1))/w_2 \in TE(F, \emptyset)$ $\quad w_2 \in O(\alpha/w_1)$

$\Longrightarrow \sigma_3(\sigma_2(\alpha)/w_1.w_2) \in TE(F, \emptyset)$

$\Longrightarrow \sigma_3(\sigma_2(\alpha)/w) \in TE(F, \emptyset)$ $\quad w_1.w_2 = w$

$\Longrightarrow \sigma_3(x) \in TE(F, \emptyset)$ $\quad \sigma_2(\alpha)/w = x$

(ii) $x \in V(\sigma_2(\varphi(\gamma)))$

$\Longrightarrow \exists\, w : \sigma_2(\varphi(\gamma))/w = x$

$\Longrightarrow \exists\, w_1, w_2 : w_1.w_2 = w \wedge \varphi(\gamma)/w_1 = y \wedge y \in VA$

$\Longrightarrow \exists\, z \in V(\gamma) : \varphi(z) = y$

$\tau(z) \in TE(F, \emptyset)$ $\quad \forall\, z \in V(\gamma) : \tau(z) \in TE(F, \emptyset)$

$\Longrightarrow \tau(\varphi'(\varphi(z))) \in TE(F, \emptyset)$

$\Longrightarrow \tau(\varphi'(y)) \in TE(F, \emptyset)$ $\quad \varphi(z) = y$

$\Longrightarrow \sigma_1(y) \in TE(F, \emptyset)$ $\quad \sigma_1(y) = \tau(\varphi'(y))$ for $y \in V(\varphi(\gamma))$

$\Longrightarrow \sigma_3(\sigma_2(y)) \in TE(F, \emptyset)$ $\quad \sigma_1 = \sigma_3 \cdot \sigma_2$

$\Longrightarrow \sigma_3(\sigma_2(\varphi(\gamma)/w_1)) \in TE(F, \emptyset)$ $\quad \varphi(\gamma)/w_1 = y$

$\Longrightarrow \sigma_3(\sigma_2(\varphi(\gamma)/w_1))/w_2 \in TE(F, \emptyset)$ $\quad w_2 \in O(\varphi(\gamma)/w_1)$

$\Longrightarrow \sigma_3(\sigma_2(\varphi(\gamma))/w_1.w_2) \in TE(F, \emptyset)$

$\Longrightarrow \sigma_3(\sigma_2(\varphi(\gamma))/w) \in TE(F, \emptyset)$ $\quad w_1.w_2 = w$

$\Longrightarrow \sigma_3(x) \in TE(F, \emptyset)$ $\quad \sigma_2(\varphi(\gamma))/w = x$

With (i), (ii) :

$\Longrightarrow \forall\, x \in V(\sigma_2(\alpha)) \cup V(\sigma_2(\alpha)[\, u \longleftarrow \sigma_2(\varphi(\delta))\,]) \cup V(\sigma_2(\beta)) : \sigma_3(x) \in TE(F, \emptyset)$

Proof of theorem 4.5 (theorem 3.2)

($\Longrightarrow$)

Let $(t_1, t_2, t_3) \in I_F(CTS_F(R))$

$\Longrightarrow \exists (\alpha \longrightarrow \beta, u, \gamma \longrightarrow \delta) \in CO_F(R) : (t_1, t_2, t_3) \in I_F((CT(\alpha \longrightarrow \beta, u, \gamma \longrightarrow \delta)))$

$\Longrightarrow \exists \sigma, \tau\ \exists v : t_1 = \sigma(\alpha) \wedge t_1/v = \tau(\gamma) \wedge t_2 = t_1 [v \longleftarrow \tau(\gamma)] \wedge t_3 = \sigma(\beta)$

$\Longrightarrow t_1 \longrightarrow_R t_2 \wedge t_1 \longrightarrow_R t_3$

$\Longrightarrow \exists t : t_2 \xrightarrow{*}_R t \wedge t_3 \xrightarrow{*}_R t$ R is ground confluent

$\Longrightarrow t_2 \overset{*}{\longmapsto\!\dashv}_{R, t1}\ t_3$

($\Longleftarrow$)

It is sufficient to prove

$\forall t, t_1, t_2 \in TE(FS, \emptyset) : t \twoheadrightarrow_R t_1 \wedge t \twoheadrightarrow_R t_2 \Longrightarrow t_1 \overset{*}{\longmapsto\!\dashv}_{R, t}\ t_2$:

$\longrightarrow_R$ confluent

$\Longleftrightarrow \twoheadrightarrow_R$ confluent theorem 4.2

$\Longleftrightarrow \forall t, t_1, t_2 \in TE(FS, \emptyset) : t \twoheadrightarrow_R t_1 \wedge t \twoheadrightarrow_R t_2 \Longrightarrow t_1 \overset{*}{\longmapsto\!\dashv}_{R, t}\ t_2$

Generalized Newman Lemma [WB 83]

$\Longleftrightarrow \forall t, t_1, t_2 \in TE(FS, \emptyset) : t \twoheadrightarrow_R t_1 \wedge t \twoheadrightarrow_R t_2 \Longrightarrow t_1 \overset{*}{\longmapsto\!\dashv}_{R, t}\ t_2$

theorem 4.2

For $t \twoheadrightarrow_R t_1 \wedge t \twoheadrightarrow_R t_2$ we get:

$t \twoheadrightarrow_R t_1 \Longrightarrow \exists u \in O(t)\ \exists \alpha \longrightarrow \beta \in R\ \exists \sigma : t/u = \sigma(\alpha) \wedge t [u \longleftarrow \sigma(\beta)] = t_1$

$\wedge \forall x \in V(\alpha) : \sigma(x) \in IRR_G(R)$

$t \twoheadrightarrow_R t_2 \Longrightarrow \exists v \in O(t)\ \exists \gamma \longrightarrow \delta \in R\ \exists \tau : t/v = \tau(\gamma) \wedge t [v \longleftarrow \tau(\delta)] = t_2$

$\wedge \forall x \in V(\gamma) : \tau(x) \in IRR_G(R)$

F contains all function symbols which are not convertible :

$\Longrightarrow IRR_G(R) \subset TE(F, \emptyset)$

$\Longrightarrow \forall x \in V(\alpha) : \sigma(x) \in TE(F, \emptyset) \wedge \forall x \in V(\gamma) : \tau(x) \in TE(F, \emptyset)$

By lemma A.1, the subconnctedness of (t, t_1, t_2) follows immedeately for the case where we have no critical overlapping between the rules (case 1 and 2). For the other cases where we have a critical overlapping ($v = u.v' \wedge v' \in O'(\alpha)$ or $v = u.v' \wedge v' \in O'(\alpha)$), we can prove the subconnectedness by considering critical triples:

Assume w.l.o.g.: $v = u.v' \wedge v' \in O'(\alpha)$

with: $t/u = \sigma(\alpha)$

$\Longrightarrow t/u.v' = \sigma(\alpha)/v'$

$\Longrightarrow t/v = \sigma(\alpha)/v'$ $\qquad v = u.v'$

$\Longrightarrow \tau(\gamma) = \sigma(\alpha)/v'$ $\qquad \tau(\gamma) = t/v$

$\Longrightarrow (\alpha \longrightarrow \beta, v', \gamma \longrightarrow \delta) \in CO_F(R)$

with: $\forall x \in V(\alpha) : \sigma(x) \in TE(F, \emptyset) \wedge \forall x \in V(\gamma) : \tau(x) \in TE(F, \emptyset)$

$\Longrightarrow CT(\alpha \longrightarrow \beta, v', \gamma \longrightarrow \delta) \in CTS_F(R)$

$\Longrightarrow \forall (t'_1, t'_2, t'_3) \in I_F(\{CT(\alpha \longrightarrow \beta, v', \gamma \longrightarrow \delta)\}) : t'_2 \overset{*}{\vdash\!\dashv}_{R, t'_1} t'_3$

with assumption of theorem

$\Longrightarrow (t, t_1, t_2)$ is subconnected $\qquad$ lemma A.1

<u>Proof of lemma 4.7</u> (lemma 3.4)

Assume : $(t'_1, t'_2, t'_3) \in I_F(\{ (t_1, t_2, t_3) \})$

$\Longrightarrow (t'_1, t'_2, t'_3) \in I_F(M)$

with: $\{ (t_1, t_2, t_3) \}$ is well covered by M

$\Longrightarrow t'_2 \overset{*}{\vdash\!\dashv}_{E, t'_1} t'_3$ with assumption of lemma

$\Longrightarrow (t_1, t_2, t_3)$ is ground connected

Proof of lemma 3.6

Assume: $(\alpha \longrightarrow \beta, u, \gamma \longrightarrow \delta, v, \gamma' \longrightarrow \delta') \in DCO(CO(R), R)$

$\wedge\ (t_1, t_2, t_3) \in I_{PS}(\{ DCT(\alpha \longrightarrow \beta, u, \gamma \longrightarrow \delta, v, \gamma' \longrightarrow \delta') \})$

Then $\alpha \longrightarrow \beta$ can be applied at ε, $\gamma \longrightarrow \delta$ can be applied at u and $\gamma' \longrightarrow \delta'$ can be applied at v in t_1. We will consider the pair of terms which can be derived by $\alpha \longrightarrow \beta$ and $\gamma \longrightarrow \delta$ from t_1 (case 1) and the pair which can be derived by $\gamma \longrightarrow \delta$ and $\gamma' \longrightarrow \delta'$ (case 2). We will show that both pairs are connected below t_1 by Lemma A.1, then by the transitivity of $\overset{*}{\longmapsto}_{R,\, t1}$ the terms t_2 and t_3 are connected below t_1.

Let: $(\sigma_1(\alpha), \sigma_1(\alpha)[\, u \longleftarrow \sigma_1(\varphi(\delta))\,], \sigma_1(\beta)) = CT(\alpha \longrightarrow \beta, u, \gamma \longrightarrow \delta)$

$(\sigma_2(\sigma_1(\alpha)), \sigma_2(\sigma_1(\alpha))[\, u \longleftarrow \sigma_2(\sigma_1(\varphi(\delta)))\,], \sigma_2(\sigma_1(\beta)))$

$= DCT(\alpha \longrightarrow \beta, u, \gamma \longrightarrow \delta, v, \gamma' \longrightarrow \delta')$

$\sigma_2(\sigma_1(\alpha))/v = \tau(\gamma')$

Then :

$(t_1, t_2, t_3) \in I_{PS}(\{ DCT(\alpha \longrightarrow \beta, u, \gamma \longrightarrow \delta, v, \gamma' \longrightarrow \delta') \})$

$\Longrightarrow \exists \sigma : t_1 = \sigma(\sigma_2(\sigma_1(\alpha)))$

$\wedge\ t_2 = \sigma(\sigma_2(\sigma_1(\alpha)))[\, u \longleftarrow \sigma(\sigma_2(\sigma_1(\varphi(\delta))))\,]$

$\wedge\ t_3 = \sigma(\sigma_2(\sigma_1(\beta)))$

(1) $\alpha \longrightarrow \beta, \gamma' \longrightarrow \delta'$:

$\sigma(\sigma_2(\sigma_1(\alpha)))/v = \sigma(\tau(\gamma')) \wedge \sigma(\sigma_2(\sigma_1(\alpha)))/\varepsilon = \sigma(\sigma_2(\sigma_1(\alpha)))$

(1.1) $v \in O'(\alpha)$

$\Longrightarrow CT(\alpha \longrightarrow \beta, v, \gamma' \longrightarrow \delta')$ is confluent assumption of Lemma

$\Longrightarrow \forall (t_1, t_2, t_3) \in I_{FS}(\{ CT(\alpha \longrightarrow \beta, v, \gamma' \longrightarrow \delta') \}) : t_2 \overset{*}{\longmapsto}_{R, t1} t_3$

$\Longrightarrow \sigma(\sigma_2(\sigma_1(\alpha)))[v \longleftarrow \sigma(\tau(\delta'))] \overset{*}{\longmapsto}_{R, \sigma(\sigma2(\sigma1(\alpha)))} \sigma(\sigma_2(\sigma_1(\beta)))$

with: Lemma A.1 case 3

(1.2) $v \notin O'(\alpha)$

$\Longrightarrow \sigma(\sigma_2(\sigma_1(\alpha)))[v \longleftarrow \sigma(\tau(\delta'))] \overset{*}{\longmapsto}_{R, \sigma(\sigma2(\sigma1(\alpha)))} \sigma(\sigma_2(\sigma_1(\beta)))$

with: Lemma A.1 case 2

(2) $\gamma \longrightarrow \delta, \gamma' \longrightarrow \delta'$:

$\sigma(\sigma_2(\sigma_1(\alpha)))/v = \sigma(\tau(\gamma')) \wedge \sigma(\sigma_2(\sigma_1(\alpha)))/u = \sigma(\sigma_2(\sigma_1(\varphi(\gamma))))$

(2.1) $u = v.u' \wedge u' \in O'(\gamma')$

$\Longrightarrow CT(\gamma' \longrightarrow \delta', u', \gamma \longrightarrow \delta)$ is confluent assumption of Lemma

$\Longrightarrow \forall (t_1, t_2, t_3) \in I_{FS}(\{ CT(\gamma' \longrightarrow \delta', u', \gamma \longrightarrow \delta) \}) : t_2 \overset{*}{\longmapsto}_{R, t1} t_3$

$\Longrightarrow \sigma(\sigma_2(\sigma_1(\alpha)))[v \longleftarrow \sigma(\tau(\delta'))]$

$\overset{*}{\longmapsto}_{R,\sigma(\sigma2(\sigma1(\alpha)))} \sigma(\sigma_2(\sigma_1(\alpha)))[u \longleftarrow \sigma(\sigma_2(\sigma_1(\varphi(\delta))))]$

with lemma A.1 case 3

(2.2) $v = u.v' \wedge v' \notin O'(\gamma)$

$\Longrightarrow \sigma(\sigma_2(\sigma_1(\alpha)))[v \longleftarrow \sigma(\tau(\delta'))]$

$\overset{*}{\longmapsto}_{R, \sigma(\sigma2(\sigma1(\alpha)))} \sigma(\sigma_2(\sigma_1(\alpha)))[u \longleftarrow \sigma(\sigma_2(\sigma_1(\varphi(\delta))))]$

with lemma A.1 case 2

(2.2) u and v are disjoint

$\Longrightarrow \sigma(\sigma_2(\sigma_1(\alpha)))[v \longleftarrow \sigma(\tau(\delta'))]$

$\overset{*}{\longmapsto}_{R,\, \sigma(\sigma2(\sigma1(\alpha)))}\, \sigma(\sigma_2(\sigma_1(\alpha)))[u \longleftarrow \sigma(\sigma_2(\sigma_1(\varphi(\delta))))]$

with lemma A.1 case 1

(1), (2) $\Longrightarrow \sigma(\sigma_2(\sigma_1(\beta))) \overset{*}{\longmapsto}_{R,\, \sigma(\sigma2(\sigma1(\alpha)))}\, \sigma(\sigma_2(\sigma_1(\alpha)))[u \longleftarrow \sigma(\sigma_2(\sigma_1(\varphi(\delta))))]$

$\Longrightarrow t_3 \overset{*}{\longmapsto}_{R,\, t1}\, t_2$ with : $t_1 = \sigma(\sigma_2(\sigma_1(\alpha)))$

$t_2 = \sigma(\sigma_2(\sigma_1(\alpha)))\,[\, u \longleftarrow \sigma(\sigma_2(\sigma_1(\varphi(\delta))))\,]$

$t_3 = \sigma(\sigma_2(\sigma_1(\beta)))$

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