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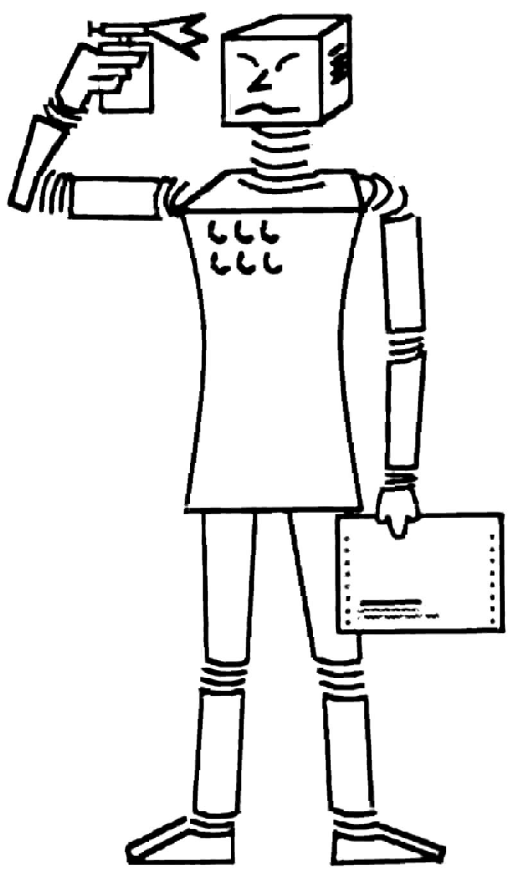

Check your Ordering - Termination Proofs and Open Problems

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Check Your Ordering

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Termination Proofs and Open Problems*

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Abstract

Termination is a decisive property of term rewriting systems that is often needed. Simplification orderings provide a mechanism for proving this property. The power of such an ordering depends on the number of examples (occurring in practice) to which it can successfully be applied.

In addition to a brief introduction to the presented simplification orderings, this report contains a large collection of term rewriting systems all of which were tested for completion by our completion environment COMTES. In a lot of cases simplification orderings can be found that enable the generation of canonical systems. However, a lot of examples exist for which the problem of completion cannot be solved.

1 Preface

The use of term rewriting systems presumes termination. In [Der87] several different methods which guarantee this important property are presented. The strength of these methods, called orderings, should be investigated in practice.

Therefore, we have collected various examples of rewriting systems and will present them together with some orderings proving their termination. The orderings investigated in this report are all taken from [Ste88] except for the ordering on polynomial interpretations ([Lan79], [BL87], [SZ90]). Besides the latter, the list of orderings includes the recursive path ordering with status (RPOS), a new recursive decomposition ordering with status (IRDS) and the Knuth-Bendix ordering with status (KBOS). The following subsection contains a brief description of these methods.

We have investigated 155 exmples. They belong to several different classes including algebraic structures, arithmetic theories, lists and string rewriting systems. We used two criteria for selecting these rule systems: i) We examined systems which often appear in the literature. ii) The remaining systems are interesting from a termination point of view, i.e., the orderings cannot be applied straightforward in order to prove the termination. More details about the examples examined (in particular, their presentation) will be given in subsection 1.2.

1.1 Used Orderings

First of all, we briefly introduce the used orderings. For more details on orderings, see for example [Der87], [BL87], [Ste88] and [SZ90]. [HO80], [JL87], [Klo87], [DJ89] and [AM90] contain general aspects and formal definitions of term rewriting systems.

A (partial) ordering $\succ$ on any set $\mathcal{M}$ is a transitive and irreflexive binary relation on $\mathcal{M}$. The application of an ordering $\succ$ w.r.t. term rewriting requires the well-foundedness to prevent infinite derivations of terms.

Note that $\succ$ is used to compare terms. Since operators have terms as arguments we define an extension of $\succ$, called lexicographically greater ($\succ^{lex}$), on tuples of elements as follows:

$$\begin{array}{lll} (m_1, m_2, ..., m_p) \succ^{lex} (n_1, n_2, ..., n_q) & & \\ \quad if \;\; either & p > 0 & \wedge \;\; q = 0 \\ \quad\quad\quad or & m_1 \succ n_1 & \\ \quad\quad\quad or & m_1 \sim n_1 & \wedge \;\; (m_2, ..., m_p) \succ^{lex} (n_2, ..., n_q). \end{array}$$

Multisets are treated like sets, that allow multiple occurrences of identical elements. The extension of $\succ$ on multisets of elements is defined as follows: A multiset $\mathcal{M}_1$ is greater than a multiset $\mathcal{M}_2$ over $\mathcal{M}$, denoted by

$$\begin{array}{ll} \mathcal{M}_1 \succ_{mul} \mathcal{M}_2 & \\ \quad iff \;\; i) & \mathcal{M}_1 \neq \mathcal{M}_2 \;\; \wedge \\ \quad\quad\quad ii) & (\forall y \in \mathcal{M}_2 \setminus \mathcal{M}_1)(\exists x \in \mathcal{M}_1 \setminus \mathcal{M}_2) \; x \succ y. \end{array}$$

All orderings used in this paper share one essential characteristic: Each operator has a status that determines the order according to which the direct subterms of f are compared ([KL80]). Formally, the status is a function τ which maps the set of operators $\mathcal{F}$ into the set $\{mul, left, right\}$:

$$\tau : \mathcal{F} \rightarrow \{mul, left, right\}.$$

Therefore, the status of a function symbol can have one of the following three values: *mul* (the arguments will be compared as multisets), *left* (lexicographical comparison from left to right) and *right* (the arguments will be compared lexicographically from right to left).

The index '$\tau(f)$' in '$\succ_{ord,\tau(f)}$' denotes the extension of the ordering $\succ_{ord}$ w.r.t. the status of the operator f:

$$\begin{array}{lll} (s_1, ..., s_m) \succ_{ord,\tau(f)} (t_1, ..., t_n) & & \\ \quad iff \quad\quad & \tau(f) = mul & \wedge \;\; \{s_1, ..., s_m\} \succ_{ord,mul} \{t_1, ..., t_n\} \\ \quad\quad\quad or & \tau(f) = left & \wedge \;\; (s_1, ..., s_m) \succ^{lex}_{ord} (t_1, ..., t_n) \\ \quad\quad\quad or & \tau(f) = right & \wedge \;\; (s_m, ..., s_1) \succ^{lex}_{ord} (t_n, ..., t_1) \end{array}$$

All of the following orderings uniquely define a congruence $\sim$ depending on $\mathcal{F}$ and τ via $f(s_1, ..., s_m) \sim g(t_1, ..., t_n)$ iff $f = g$ and $m = n$ and i) $\tau(f) = mul$ and there is a permutation π of the set $\{1, ..., n\}$ such that $s_i \sim t_{\pi(i)}$, for all $i \in [1, n]$ or ii) $\tau(f) \neq mul \wedge s_i \sim t_i$, for all $i \in [1, n]$.

Most of the orderings are based on the principle of root orderings, i.e., the comparison of two terms depends on their leading function symbols. The presentation of the case distinctions which result is of the following form: $s \succ t$ iff i) *condition*$_1$ ii) *condition*$_2$ ... denoting $s \succ t$ iff *condition*$_1$ or *condition*$_2$ or ... holds. If the conditions of the case distinctions are marked by hyphens they are to be evaluated 'lexicographically', e.g., $s \succ t$ iff $-\ s \succ_1 t\ \ -\ s \succ_2 t$ stands for $s \succ t$ iff $s \succ_1 t$ or $(s \sim_1 t \ \wedge\ s \succ_2 t)$. Here, the symbol $\sim_1$ denotes the congruence relation induced by the quasi-ordering $\succeq_1$.

When writing s, t and $\succ$ we will always assume that s and t are terms and $\succ$ is a precedence on the set $\mathcal{F}$ of operators.

Definition 1.1 ([KL80]) *Recursive Path Ordering with Status :* **RPOS**[1]

$s \succ_{RPOS} t \iff$

$i)$ $\mathcal{H}ead(s) \succ \mathcal{H}ead(t) \ \wedge\ \{s\} \succ_{RPOS,mul} \mathcal{A}rgs(t)$

$ii)$ $\mathcal{H}ead(s) = \mathcal{H}ead(t) \ \wedge\ \tau(\mathcal{H}ead(s)) = mul \ \wedge\ \mathcal{A}rgs(s) \succ_{RPOS,mul} \mathcal{A}rgs(t)$

$iii)$ $\mathcal{H}ead(s) = \mathcal{H}ead(t) \ \wedge\ \tau(\mathcal{H}ead(s)) \neq mul \ \wedge\ \mathcal{A}rgs(s) \succ_{RPOS,\tau(\mathcal{H}ead(s))} \mathcal{A}rgs(t)$
$\wedge\ \{s\} \succ_{RPOS,mul} \mathcal{A}rgs(t)$

$iv)$ $\mathcal{A}rgs(s) \succeq_{RPOS,mul} \{t\}$

To define the improved recursive decomposition ordering with status, we need some kind of formalism. A path of a term is a sequence of terms starting with the whole term followed by a path of one of its arguments:

- $path_\varepsilon(\triangle) \ = \ \triangle$ $\quad$ *if* $\triangle$ is a constant or a variable,
- $path_{i.u}(f(t_1,...,t_n)) \ = \ f(t_1,...,t_n); path_u(t_i)$ $\quad$ *if* $u \in \mathcal{P}os^*(t_i)$[2]

For a path $p = [t_1; t_2; ...; t_n]$ we denote the set $\{t_1, ..., t_n\}$ of all terms in p by *set*(p). This set will also be called path-decomposition and its abbreviation is $dec_u(t)$ (and is equal to $set(path_u(t))$).

The decomposition $dec(\{t_1,...,t_n\}) = \{dec_u(t_i) \mid i \in [1,n]\ ,\ u \in \mathcal{P}os^*(t_i)\}$ is the multiset of all path-decompositions of the terms $t_1, ..., t_n$.

The set of subterms of a path-decomposition P relative to a term t is defined as

$$sub(P,t) = \{s \in P \mid (\exists u \neq \varepsilon)\ t|_u = s\}.$$ [3]

Suppose $t = (x * y) + (x * z)$, then $dec_{21}(t) = \{t, x * z, x\}$, $dec(\{t\}) = \{\{t, x * y, x\}, \{t, x * y, y\}, \{t, x * z, x\}, \{t, x * z, z\}\}$ and $sub(dec_{21}(t), x) = \emptyset$.

Definition 1.2 ([Ste88] , [Ste89b]) *Improved Recursive Decomposition Ordering with Status :* **IRDS**[4]

$s \succ_{IRDS} t$

[1] The RPOS is based on the recursive path ordering of [Der82].

[2] where $\mathcal{P}os^*(t_i)$ is the set of all terminal occurrences of t_i.

[3] $t|_u$ denotes the subterm of t at occurrence u.

[4] The IRDS is based on the recursive decomposition ordering developed by Lescanne (see for example [Les83]). The power of the IRDS is identical to that of the path ordering with status of [KNS85].

$$
\begin{array}{ll}
\Longleftrightarrow & dec(\{s\}) \succ_{EL,mul}\succ_{EL,mul} dec(\{t\}) \\
& with\ dec_u(s) \ni s' \succ_{EL} t' \in dec_v(t) \\
& \quad \Longleftrightarrow \quad i) \quad \mathcal{H}ead(s') \succ \mathcal{H}ead(t') \\
& \qquad\qquad ii) \quad \mathcal{H}ead(s') = \mathcal{H}ead(t') \ \wedge\ \tau(\mathcal{H}ead(s')) = mul \ \wedge \\
& \qquad\qquad\qquad - \ sub(dec_u(s), s') \succ_{EL,mul} sub(dec_v(t), t') \\
& \qquad\qquad\qquad - \ dec(\mathcal{A}rgs(s')) \succ_{EL,mul}\succ_{EL,mul} dec(\mathcal{A}rgs(t')) \\
& \qquad\qquad iii) \quad \mathcal{H}ead(s') = \mathcal{H}ead(t') \ \wedge\ \tau(\mathcal{H}ead(s')) \neq mul \ \wedge \\
& \qquad\qquad\qquad \mathcal{A}rgs(s') \succ_{IRDS,\tau(\mathcal{H}ead(s'))} \mathcal{A}rgs(t') \ \wedge\ \{s'\} \succ_{IRDS,mul} \mathcal{A}rgs(t')
\end{array}
$$

To describe the Knuth-Bendix ordering with status and the polynomial ordering, more background information and helpful definitions will be presented.

If x is a variable and t is a term we denote the number of occurrences of x in t by $\#_x(t)$. We assign a non-negative integer $\varphi(f)$ – the weight of f – to each operator and a positive integer φ_0 to each variable such that

$$
\begin{array}{ll}
\varphi(c) \geq \varphi_0 & \text{if } c \text{ is a constant,} \\
\varphi(f) = 0 & \text{for one unary operator } f \text{ at most; } f \text{ has to be maximal w.r.t.} \succ .
\end{array}
$$

We extend this weight function on operators to terms:

$$\varphi(f(t_1, ..., t_n)) = \varphi(f) + \sum_{i=1}^{n} \varphi(t_i).$$

The weight of an operator f can be considered as a special linear function: $x_1 + ... + x_n + \varphi(f)$. A polynomial ordering even allows polynomials as the 'weight' of operators. For each n-ary operator f an interpretation $[f](x_1, ..., x_n)$ defines a polynomial with n variables which satisfies the following conditions:

- $[f]$ must have the same arity as f
- the coefficients of $[f]$ must be integers
- $(\forall x, y)\ [f](..., x, ...) > [f](..., y, ...)\ \ if\ \ x > y.$

The extension of an interpretation of an operator to a term $f(t_1, ..., t_n)$ is defined as follows:

$$[f(t_1, ..., t_n)](x_1, ..., x_m) = [f]([t_1](x_1, ..., x_m), ..., [t_n](x_1, ..., x_m)).$$

Definition 1.3 ([Ste88] , [Ste89b]) *Knuth-Bendix Ordering with Status:* **KBOS**[5]

$$
\begin{array}{lll}
s \succ_{KBOS} t & & \\
\Longleftrightarrow & (\forall x) & \#_x(s) \geq \#_x(t) \ \wedge \\
& i) & s \equiv f(...f(t)) \\
& ii) & - \ \varphi(s) > \varphi(t) \\
& & - \ \mathcal{H}ead(s) \succ \mathcal{H}ead(t) \\
& & - \ \mathcal{A}rgs(s) \succ_{KBOS,\tau(\mathcal{H}ead(s))} \mathcal{A}rgs(t)
\end{array}
$$

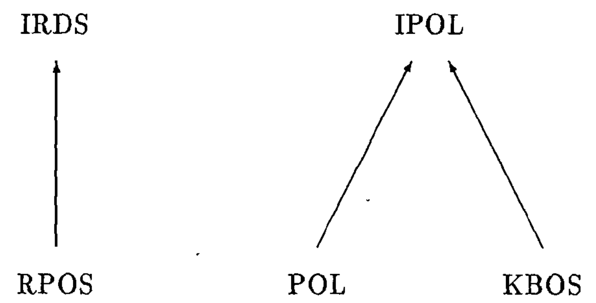
[5]The KBOS is an extension of the original Knuth-Bendix ordering of [KB70].

Definition 1.4 ([Lan79]) *Ordering on Polynomial Interpretations:* **POL**
$s \succ_{POL} t$
$\iff$ $[s] \sqsupset [t]$
where $p \sqsupset q$[6]
$\iff$ $(\forall x_i \in \mathcal{M} \subseteq \mathcal{N}at)\ p(x_1, ..., x_n) > q(x_1, ..., x_n)$

Definition 1.5 ([SZ90]) *Improved Polynomial Ordering:* **IPOL**
For each constant c, $[c]() \geq 2$.
$s \succ_{IPOL} t$
$\iff$
- $s \succ_{POL} t$
- $\mathcal{H}ead(s) \succ \mathcal{H}ead(t)$
- $\mathcal{A}rgs(s) \succ_{IPOL,\tau(\mathcal{H}ead(s))} \mathcal{A}rgs(t)$

Note that, if f is a unary operator interpreted by the identity function (i.e., $[f](x) = x$) there must not exist another operator being greater than f w.r.t. $\succ$. Furthermore, $\succ_{IPOL}$ is a simplification ordering only if the condition '$f(...f(s)) \succ_{IPOL} s$ if $[f](x) = x$' is additionally required.

The above described orderings were applied to prove the termination of rule systems. The following diagram depicts the comparison of their powers. The power of an ordering is described by the set of comparable pairs of terms. Thus, we examine the relation between two such sets. We use a kind of Hasse diagrams to represent these relations, i.e., if $\succ_1 \subset \succ_2$ then we arrange $\succ_2$ above $\succ_1$ joining them with an arrow:



For more details about the comparison of these orderings, see [Ste88], [Ste89b] and [SZ90].

[6]Note that $[s]$ and $[t]$ are polynomials. Thus, $\sqsupset$ is an ordering on polynomials (p,q). $p(x_1, ..., x_n)$ represents the polynomial $\sum_{r_1,...,r_n} c_{r_1...r_n} x_1^{r_1} \cdot ... \cdot x_n^{r_n}$ based on n variables. $\mathcal{N}at$ denotes the set of natural numbers.

1.2 Presentation of the Examples

All examples will be presented in the following way:

$$\begin{array}{lcl} l_1 & \to & r_1 \\ & \vdots & \\ l_n & \to & r_n \end{array}$$

$$\begin{array}{lcl} l_{n+1} & \to & r_{n+1} \\ & \vdots & \\ l_m & \to & r_m \\ IPOL & : & \ldots^7 \\ IRDS & : & \ldots \\ KBOS & : & \ldots \\ POL & : & \ldots \\ RPOS & : & \ldots \end{array}$$

The initial system consists of the rules $l_1 \to r_1 \cdots l_n \to r_n$. Throughout the completion process new rules, $l_{n+1} \to r_{n+1} \cdots l_m \to r_m$, are generated. The canonical system contains all of the presented rules except those marked with '$\diamond$'. Appropriate term orderings are specified by their parameters (such as the precedence, the status function, the weight function, the polynomial interpretations). Note that each set of orderings is minimal w.r.t. the power of the orderings, e.g., if there exists an RPOS we do not provide an appropriate IRDS (since the IRDS is stronger than the RPOS, see subsection 1.1). However, if none of the two strongest orderings of subsection 1.1 (either IRDS or IPOL) is enumerated the termination of the corresponding example cannot be proved (with the help of the presented orderings).

We use x, y, z, u, v, w to represent variables. Function symbols will be denoted by $f, g, h, h_1, h_2, i, j, k, p, s$ and a, b, b', c, d, d', e (constant symbols). Some additional function symbols which occur in the examples require no further elucidation.

The tested examples belong to several domains. The list of these classes consists of *algebraic structures* (groups, rings, etc.), *arithmetic theories* (addition, multiplication, etc.), *boolean theories* (if-then-else, etc.), *lists* (append, reverse, flatten, etc.), *string rewriting systems*[8] and *other systems* (systems that do not fall into one of the five former classes). Each example is characterized by an abbreviation of the class it belongs to (G: algebraic structures, A: arihmetical theories, B: Boolean theories, L: lists, S: string rewriting systems, O: other systems).

For completing the initial rule systems we used our completion environment COMTES (see [AMS89] , [Ste89a]) in which all presented orderings have been integrated. Obviously[9], we did not employ 'special tricks' such as postponing incomparable pairs, introducing new operators, etc. More precisely, the generation of an incomparable pair caused the completion process to break off.

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[7] $\ldots$ represents the parameters of the orderings.

[8] The structure of string rewriting systems is easier than that of lists since they use unary operators, only.

[9] We are only interested in testing the power of the presented orderings.

In the next section we deal with the termination of systems which are already canonical (sometimes the systems have to be normalized). The third section contains rewriting systems where at least one non-trivial critical pair has to be generated in order to complete them. Finally, in the fourth section, we present a lot of systems for which the termination cannot be guaranteed using the methods described in subsection 1.1. However, all of these examples are terminating. We know that there exist some other orderings[10] with which the termination property of some of these examples can be proved. However, this report stands at the end of our research on simplification orderings and represents a beginning of investigations about orderings that can handle homeomorphic embeddings (in particular extremely recursive definitions).

[10] For example, semantic path orderings of [KL80] and transformation orderings of [BD86] and [BL88].

2 Canonical Systems

EXAMPLE 2.1 G: Group Theory ([KB70])

$$
\begin{array}{lcl}
i(0) & \to & 0 \\
0+y & \to & y \\
x+0 & \to & x \\
i(i(x)) & \to & x \\
i(x)+x & \to & 0 \\
x+i(x) & \to & 0 \\
i(x+y) & \to & i(y)+i(x) \\
x+(y+z) & \to & (x+y)+z \\
(x+i(y))+y & \to & x \\
(x+y)+i(y) & \to & x
\end{array}
$$

KBOS: $\varphi(i)=0, \varphi(+)>\varphi(0)$
$i \succ +$
$\tau(+)=right$

POL:
$$
\begin{array}{lcl}
[+](x,y) & = & 2xy+y \\
[i](x) & = & x^2 \\
[0]() & = & 2
\end{array}
$$

RPOS: $i \succ + \succ 0$
$\tau(+)=right$ □

EXAMPLE 2.2 G: Associativity and Endomorphism ([BL88])

$$
\begin{array}{lcl}
(x+y)+z & \to & x+(y+z) \\
f(x)+f(y) & \to & f(x+y) \\
f(x)+(f(y)+z) & \to & f(x+y)+z
\end{array}
$$

KBOS: $\varphi(f)>0$
$\tau(+)=left$

POL:
$$
\begin{array}{lcl}
[+](x,y) & = & xy+x \\
[f](x) & = & 2x
\end{array}
$$
□

EXAMPLE 2.3 G: [Kir87]

$$
\begin{array}{lcl}
--x & \to & x \\
-h(x) & \to & h(-x) \\
-f(x,y) & \to & f(-y,-x)
\end{array}
$$

KBOS: $\varphi(-)=0$
$- \succ h, - \succ f$

POL:
$$
\begin{array}{lcl}
[-](x) & = & 2x \\
[f](x,y) & = & x+y+1 \\
[h](x) & = & x+1
\end{array}
$$

RPOS: $- \succ h, - \succ f$ □

EXAMPLE 2.4 G: [Kir87]

$$\begin{array}{lcl} f(x+0) & \to & f(x) \\ x+(y+z) & \to & (x+y)+z \end{array}$$

KBOS: $\tau(+) = right$

POL:
$$\begin{array}{lcl} [+](x,y) & = & x+2y \\ [f](x) & = & x \\ [0]() & = & 1 \end{array}$$

RPOS: $\tau(+) = right$ □

EXAMPLE 2.5 G: Associativity and Distributivity

$$\begin{array}{lcl} x+(y+z) & \to & (x+y)+z \\ x*(y+z) & \to & (x*y)+(x*z) \\ (x+(y*z))+(y*u) & \to & x+(y*(z+u)) \end{array} \quad \diamond$$

POL:
$$\begin{array}{lcl} [+](x,y) & = & x+2y+1 \\ [*](x,y) & = & xy \end{array}$$
□

EXAMPLE 2.6 G: [Mar87]

$$\begin{array}{lcl} (x+y)+z & \to & x+(y+z) \\ (x*y)+(x*z) & \to & x*(y+z) \\ (x*y)+((x*z)+u) & \to & (x*(y+z))+u \end{array}$$

KBOS: $\varphi(*) > 0$
$\tau(+) = left$

POL:
$$\begin{array}{lcl} [+](x,y) & = & xy+x \\ [*](x,y) & = & x+y \end{array}$$
□

EXAMPLE 2.7 G: [Bel86]

$$\begin{array}{lcl} f(0,y) & \to & y \\ f(x,0) & \to & x \\ f(i(x),y) & \to & i(x) \\ f(f(x,y),z) & \to & f(x,f(y,z)) \\ f(g(x,y),z) & \to & g(f(x,z),f(y,z)) \\ f(1,g(x,y)) & \to & x \\ f(2,g(x,y)) & \to & y \end{array}$$

POL: $[f](x,y) = xy + x$
$[g](x,y) = x + y + 1$
$[i](x) = x$
$[0]() = 1$
$[1]() = 1$
$[2]() = 1$

RPOS: $f \succ g$
$\tau(f) = left$

□

EXAMPLE 2.8 G: [KNZ86]

$$\begin{array}{lcl} a + b & \rightarrow & b + a \\ a + (b + z) & \rightarrow & b + (a + z) \\ (x + y) + z & \rightarrow & x + (y + z) \\ f(a, y) & \rightarrow & a \\ f(b, y) & \rightarrow & b \\ f(x + y, z) & \rightarrow & f(x, z) + f(y, z) \end{array}$$

IPOL: $[+](x,y) = x + y + 1$
$[f](x,y) = xy$
$[a]() = 3$
$[b]() = 2$
$\tau(+) = left$

RPOS: $a \succ b, f \succ +$
$\tau(+) = left$

□

EXAMPLE 2.9 A: [PF86]

$$\begin{array}{lcl} 0 + y & \rightarrow & y \\ s(x) + y & \rightarrow & s(x + y) \\ s(x) + y & \rightarrow & x + s(y) \quad \diamond \end{array}$$

$$x + s(y) \rightarrow s(x + y)$$

KBOS: $+ \succ s$
$\tau(+) = left$

POL: $[+](x,y) = 3x + 2y$
$[s](x) = x + 1$
$[0]() = 1$

RPOS: $+ \succ s$
$\tau(+) = left$

□

EXAMPLE 2.10 A: Unary Integer Addition ([Der85])

$$\begin{array}{lcl} -0 & \to & 0 \\ x+0 & \to & x \\ 0+y & \to & y \\ (-1)+1 & \to & 0 \\ --x & \to & x \\ x+(-y) & \to & -((-x)+y) \\ x+(y+z) & \to & (x+y)+z \\ (-(x+1))+1 & \to & -x \end{array}$$

POL: $$\begin{array}{lcl} [+](x,y) & = & x+3y \\ [-](x) & = & x+1 \\ [0]() & = & 1 \\ [1]() & = & 1 \end{array}$$

RPOS: $+ \succ -, + \succ 0$

$\tau(+) = right$

□

EXAMPLE 2.11 A: Addition and Subtraction

$$\begin{array}{lcl} 0+y & \to & y \\ s(x)+y & \to & s(x+y) \\ 0-y & \to & 0 \\ x-0 & \to & x \\ s(x)-s(y) & \to & x-y \end{array}$$

KBOS: $\varphi(s) > 0$

$+ \succ s$

POL: $$\begin{array}{lcl} [+](x,y) & = & 2x+y \\ [-](x,y) & = & x+y \\ [s](x) & = & x+1 \\ [0]() & = & 1 \end{array}$$

RPOS: $+ \succ s$

□

EXAMPLE 2.12 A: [NRS89]

$$\begin{array}{lcl} 0+y & \to & y \\ s(x)+y & \to & s(x+y) \\ p(x)+y & \to & p(x+y) \\ -0 & \to & 0 \\ -s(x) & \to & p(-x) \\ -p(x) & \to & s(-x) \\ 0*y & \to & 0 \\ s(x)*y & \to & (x*y)+y \\ p(x)*y & \to & (x*y)+(-y) \end{array}$$

IPOL:	$[+](x,y)$	$=$	$x+y$
	$[*](x,y)$	$=$	$xy+1$
	$[-](x)$	$=$	$x+1$
	$[p](x)$	$=$	$x+2$
	$[s](x)$	$=$	$x+2$
	$[0]()$	$=$	1
	$* \succ + \succ p, + \succ s, - \succ p, - \succ s$		
RPOS:	$* \succ - \succ p, - \succ s, * \succ + \succ p, + \succ s$		

□

EXAMPLE 2.13 A[11]:

$double(0)$	$\rightarrow$	0	◇
$double(s(x))$	$\rightarrow$	$s(s(double(x)))$	◇
$x+0$	$\rightarrow$	x	
$x+s(y)$	$\rightarrow$	$s(x+y)$	
$s(x)+y$	$\rightarrow$	$s(x+y)$	
$double(x)$	$\rightarrow$	$x+x$	

POL:	$[+](x,y)$	$=$	$2x+2y$
	$[double](x)$	$=$	$4x+1$
	$[s](x)$	$=$	$x+1$
	$[0]()$	$=$	1
RPOS:	$double \succ + \succ s$		

□

EXAMPLE 2.14 A: [Toy86]

$double(0)$	$\rightarrow$	0
$double(s(x))$	$\rightarrow$	$s(s(double(x)))$
$half(0)$	$\rightarrow$	0
$half(s(0))$	$\rightarrow$	0
$half(s(s(x)))$	$\rightarrow$	$s(half(x))$
$x-0$	$\rightarrow$	x
$s(x)-s(y)$	$\rightarrow$	$x-y$
$if(0,y,z)$	$\rightarrow$	y
$if(s(x),y,z)$	$\rightarrow$	z
$half(double(x))$	$\rightarrow$	x

POL:	$[-](x,y)$	$=$	$x+y$
	$[double](x)$	$=$	$3x$
	$[half](x)$	$=$	$x+1$
	$[if](x,y,z)$	$=$	$x+y+z$
	$[s](x)$	$=$	$x+1$
	$[0]()$	$=$	1
RPOS:	$double \succ s, half \succ s$		

□

[11] The 'inductive theorem' described by the last rule would lead to divergence if the fifth rule were not included.

EXAMPLE 2.15 A: [JSS85]

$$\begin{array}{lll} f(0) & \rightarrow & 1 \\ f(s(x)) & \rightarrow & g(f(x)) \quad \diamond \\ g(x) & \rightarrow & x + s(x) \end{array}$$

$$f(s(x)) \rightarrow f(x) + s(f(x))$$

POL:

$$\begin{array}{lll} [+](x,y) & = & x+y \\ [f](x) & = & x^2 \\ [g](x) & = & 3x+1 \\ [s](x) & = & 2x \\ [0]() & = & 2 \\ [1]() & = & 2 \end{array}$$

RPOS: $f \succ 1, f \succ g \succ +, g \succ s$ □

EXAMPLE 2.16 A: [Toy86]

$$\begin{array}{lll} f(0) & \rightarrow & 0 \\ f(s(x)) & \rightarrow & g(x, s(x)) \\ g(0,y) & \rightarrow & y \\ g(s(x),y) & \rightarrow & g(x, y+s(x)) \quad \diamond \\ x+0 & \rightarrow & x \\ x+s(y) & \rightarrow & s(x+y) \end{array}$$

$$g(s(x),y) \rightarrow g(x, s(y+x))$$

POL:

$$\begin{array}{lll} [+](x,y) & = & x+2y \\ [f](x) & = & x^2 \\ [g](x,y) & = & x^2+y \\ [s](x) & = & x+2 \\ [0]() & = & 2 \end{array}$$

RPOS: $f \succ g \succ + \succ s$
$\tau(g) = left$ □

EXAMPLE 2.17 A: Sum of Natural Numbers

$$\begin{array}{lll} sum(0) & \rightarrow & 0 \\ sum(s(x)) & \rightarrow & sum(x) + s(x) \\ sum1(0) & \rightarrow & 0 \\ sum1(s(x)) & \rightarrow & s(sum1(x) + (x+x)) \end{array}$$

POL:

$$\begin{array}{lll} [+](x,y) & = & x+y \\ [s](x) & = & x+2 \\ [sum](x) & = & x^2 \\ [sum1](x) & = & x^2 \\ [0]() & = & 2 \end{array}$$

RPOS: $sum \succ +, sum1 \succ +, sum1 \succ s$ □

EXAMPLE 2.18 A[12]:

$$\begin{array}{lll} sum(0) & \rightarrow & 0 \\ sum(s(x)) & \rightarrow & s(x) + sum(x) \\ x + 0 & \rightarrow & x \\ x + s(y) & \rightarrow & s(x+y) \end{array}$$

IPOL:

$$\begin{array}{lll} [+](x,y) & = & x + y \\ [s](x) & = & x + 1 \\ [sum](x) & = & x^2 \\ [0]() & = & 1 \\ + \succ s & & \end{array}$$

RPOS: $sum \succ + \succ s$ □

EXAMPLE 2.19 A: Square Numbers ([HL89])

$$\begin{array}{lll} sqr(0) & \rightarrow & 0 \\ sqr(s(x)) & \rightarrow & sqr(x) + s(double(x)) \quad \diamond \\ double(0) & \rightarrow & 0 \\ double(s(x)) & \rightarrow & s(s(double(x))) \\ x + 0 & \rightarrow & x \\ x + s(y) & \rightarrow & s(x+y) \end{array}$$

$$sqr(s(x)) \rightarrow s(sqr(x) + double(x))$$

POL:

$$\begin{array}{lll} [+](x,y) & = & x + 2y \\ [double](x) & = & 3x \\ [s](x) & = & x + 3 \\ [sqr](x) & = & x^2 \\ [0]() & = & 2 \end{array}$$

RPOS: $sqr \succ + \succ s, sqr \succ double \succ s$ □

EXAMPLE 2.20 A: Sum of Square Numbers

$$\begin{array}{lll} sum(0) & \rightarrow & 0 \\ sum(s(x)) & \rightarrow & sqr(s(x)) + sum(x) \quad \diamond \\ sqr(x) & \rightarrow & x * x \end{array}$$

$$sum(s(x)) \rightarrow (s(x) * s(x)) + sum(x)$$

POL:

$$\begin{array}{lll} [+](x,y) & = & x + y \\ [*](x,y) & = & x + y \\ [s](x) & = & x + 2 \\ [sqr](x) & = & 2x + 1 \\ [sum](x) & = & x^2 \\ [0]() & = & 2 \end{array}$$

RPOS: $sum \succ +, sum \succ sqr \succ *$ □

[12]The following exponential functions can also guarantee the termination of the system: $[+](x,y) = x + 2y$, $[s](x) = x + 2$, $[sum](x) = 2^x$, $[0]() = 2$.

EXAMPLE 2.21 A: Binomial Coefficients

$$\begin{array}{lcl} bin(x,0) & \rightarrow & s(0) \\ bin(0,s(y)) & \rightarrow & 0 \\ bin(s(x),s(y)) & \rightarrow & bin(x,s(y)) + bin(x,y) \end{array}$$

POL: $$\begin{array}{lcl} [+](x,y) & = & x+y \\ [bin](x,y) & = & xy+x \\ [s](x) & = & 2x \\ [0]() & = & 2 \end{array}$$

RPOS: $bin \succ s, bin \succ +$ □

EXAMPLE 2.22 A: Arithmetical Theory[13]

$$\begin{array}{lcl} exp(x,0) & \rightarrow & s(0) \\ exp(x,s(y)) & \rightarrow & x * exp(x,y) \\ 0 * y & \rightarrow & 0 \\ s(x) * y & \rightarrow & y + (x * y) \\ 0 - y & \rightarrow & 0 \\ x - 0 & \rightarrow & x \\ s(x) - s(y) & \rightarrow & x - y \end{array}$$

RPOS: $exp \succ * \succ +, exp \succ s$ □

EXAMPLE 2.23 A: Factorial Function[14] ([Pad89]) ◇

$$\begin{array}{lcl} fac(0) & \rightarrow & 1 \\ fac(s(x)) & \rightarrow & s(x) * fac(x) \\ floop(0,y) & \rightarrow & y \\ floop(s(x),y) & \rightarrow & floop(x, s(x) * y) \\ x * 0 & \rightarrow & 0 \\ x * s(y) & \rightarrow & (x * y) + x \\ x + 0 & \rightarrow & x \\ x + s(y) & \rightarrow & s(x+y) \\ 1 & \rightarrow & s(0) \end{array}$$

$fac(0) \rightarrow s(0)$

RPOS: $floop \succ * \succ + \succ s, fac \succ *, fac \succ 1 \succ 0, 1 \succ s$
$\tau(floop) = left$ □

EXAMPLE 2.24 A: Fibonacci Function

$$\begin{array}{lcl} fib(0) & \rightarrow & 0 \\ fib(s(0)) & \rightarrow & s(0) \\ fib(s(s(x))) & \rightarrow & fib(s(x)) + fib(x) \end{array}$$

[13]The system can be oriented by the exponential functions $[+](x,y) = x+y$, $[*](x,y) = xy$, $[-](x,y) = x+y$, $[exp](x,y) = 2^{xy}$, $[s](x) = 2x$, $[0]() = 2$.

[14]There exist exponential interpretations of the operators with which the rules of the system can be oriented except for the fourth rule.

POL:	$[+](x,y) = x+y$	
	$[fib](x) = x+1$	
	$[s](x) = 2x$	
	$[0]() = 2$	
RPOS:	$fib \succ +$	□

EXAMPLE 2.25 A: Fibonacci Function[15] ([Toy86])

$$
\begin{array}{lcl}
fib(0) & \rightarrow & 0 \\
fib(s(0)) & \rightarrow & s(0) \\
fib(s(s(x))) & \rightarrow & fib(s(x)) + fib(x) \\
x + 0 & \rightarrow & x \\
x + s(y) & \rightarrow & s(x+y)
\end{array}
$$

RPOS: $fib \succ + \succ s$ □

EXAMPLE 2.26 A: Fibonacci Function[16] ([Toy86])

$$
\begin{array}{lcll}
f(0) & \rightarrow & s(0) & \\
f(s(0)) & \rightarrow & s(0) & \\
f(s(s(x))) & \rightarrow & p(h(g(x))) & \diamond \\
g(0) & \rightarrow & <s(0), s(0)> & \\
g(s(x)) & \rightarrow & h(g(x)) & \diamond \\
h(x) & \rightarrow & <p(x) + q(x), p(x)> & \\
p(<x,y>) & \rightarrow & x & \\
q(<x,y>) & \rightarrow & y & \\
x + 0 & \rightarrow & x & \\
x + s(y) & \rightarrow & s(x+y) &
\end{array}
$$

$$
\begin{array}{lcl}
f(s(s(x))) & \rightarrow & p(g(x)) + q(g(x)) \\
g(s(x)) & \rightarrow & <p(g(x)) + q(g(x)), p(g(x))>
\end{array}
$$

RPOS: $f \succ g \succ h \succ + \succ s, h \succ <>, h \succ p; h \succ q$ □

[15]The system can be oriented by the exponential functions $[+](x,y) = x+2y$, $[fib](x) = 2^x$, $[s](x) = x+2$, $[0]() = 1$.

[16]The system can be oriented by the following exponential functions: $[+](x,y) = x+y$, $[<>](x,y) = x+y$, $[f](x) = 2^x$, $[g](x) = 2^x$, $[h](x) = 4x+1$, $[p](x) = x$, $[q](x) = x$, $[s](x) = x+3$, $[0]() = 1$.

EXAMPLE 2.27 A: Fibonacci Function[17] ([Red89])

$$\begin{array}{lll} fib(0) & \rightarrow & 0 \\ fib(s(0)) & \rightarrow & s(0) \\ fib(s(s(0))) & \rightarrow & s(0) & \diamond \\ fib(s(s(x))) & \rightarrow & s_p(g(x)) \\ g(0) & \rightarrow & <s(0),0> \\ g(s(0)) & \rightarrow & <s(0),s(0)> & \diamond \\ g(s(x)) & \rightarrow & n_p(g(x)) \\ s_p(<x,y>) & \rightarrow & x+y \\ n_p(<x,y>) & \rightarrow & <x+y,x> \\ x+0 & \rightarrow & x \\ x+s(y) & \rightarrow & s(x+y) \end{array}$$

RPOS: $fib \succ g \succ n_p \succ <> \succ + \succ s, g \succ s_p$ □

EXAMPLE 2.28 A: [Les83]

$$dfib(s(s(x)),y) \rightarrow dfib(s(x), dfib(x,y))$$

POL:

$$\begin{array}{lll} [dfib](x,y) & = & x+y \\ [s](x) & = & 2x \end{array}$$

RPOS: $\tau(dfib) = left$ □

EXAMPLE 2.29 A: Prime Numbers ([BM79])

$$\begin{array}{lll} prime(0) & \rightarrow & false \\ prime(s(0)) & \rightarrow & false \\ prime(s(s(x))) & \rightarrow & prime1(s(s(x)),s(x)) \\ prime1(x,0) & \rightarrow & false \\ prime1(x,s(0)) & \rightarrow & true \\ prime1(x,s(s(y))) & \rightarrow & (\neg divp(s(s(y)),x)) \wedge prime1(x,s(y)) \\ divp(x,y) & \rightarrow & rem(x,y) \equiv 0 \end{array}$$

POL:

$$\begin{array}{lll} [\wedge](x,y) & = & x+y \\ [\equiv](x,y) & = & x+y \\ [\neg](x) & = & x \\ [divp](x,y) & = & x+y+3 \\ [prime](x) & = & x^3 \\ [prime1](x,y) & = & xy^2 \\ [rem](x,y) & = & x+y \\ [s](x) & = & 2x \\ [0]() & = & 2 \\ [false]() & = & 2 \end{array}$$

[17] The system can be oriented by the exponential functions $[+](x,y) = x + 2y$, $[<>](x,y) = x + y$, $[f](x) = 2^x$, $[g](x) = 2^x + 1$, $[n_p](x) = 2x + 1$, $[s](x) = x + 2$, $[s_p](x) = 2x$, $[0]() = 1$.

RPOS: $prime \succ prime1 \succ false, prime1 \succ true, prime1 \succ divp \succ rem, prime1 \succ \wedge, prime1 \succ \neg, divp \succ \equiv, divp \succ 0$ □

EXAMPLE 2.30 B: Boolean Ring ([Hsi85])

$$\begin{array}{lcl} \neg x & \rightarrow & x \oplus true \\ x \supset y & \rightarrow & (x \wedge y) \oplus (x \oplus true) \\ x \vee y & \rightarrow & (x \wedge y) \oplus (x \oplus y) \\ x \equiv y & \rightarrow & x \oplus (y \oplus true) \end{array}$$

POL:

$$\begin{array}{lcl} [\wedge](x,y) & = & x + y \\ [\vee](x,y) & = & 2x + 2y + 1 \\ [\equiv](x,y) & = & x + y + 2 \\ [\supset](x,y) & = & 2x + y + 2 \\ [\oplus](x,y) & = & x + y \\ [\neg](x) & = & x + 2 \\ [true]() & = & 1 \end{array}$$

RPOS: $\neg \succ \oplus, \neg \succ true, \supset \succ \oplus, \supset \succ \wedge, \supset \succ true, \vee \succ \oplus, \vee \succ \wedge, \equiv \succ \oplus, \equiv \succ true$ □

EXAMPLE 2.31 B: [Pue81]

$$\begin{array}{lcl} \neg true & \rightarrow & false \\ \neg false & \rightarrow & true \\ odd(0) & \rightarrow & false \\ odd(s(x)) & \rightarrow & \neg odd(x) \\ x + 0 & \rightarrow & x \\ x + s(y) & \rightarrow & s(x + y) \\ s(x) + y & \rightarrow & s(x + y) \end{array}$$

KBOS: $\varphi(s) > \varphi(\neg) > \varphi(false), \varphi(\neg) > \varphi(true)$
$+ \succ s$

POL:

$$\begin{array}{lcl} [+](x,y) & = & 3x + 2y \\ [\neg](x) & = & x + 1 \\ [odd](x) & = & x + 1 \\ [s](x) & = & x + 2 \\ [0]() & = & 1 \\ [false]() & = & 1 \\ [true]() & = & 1 \end{array}$$

RPOS: $odd \succ \neg \succ false, \neg \succ true, + \succ s$ □

EXAMPLE 2.32 B: [Mus80]

$$\begin{array}{lcll}
\neg x & \to & if(x, false, true) & \\
x \wedge y & \to & if(x, y, false) & \\
x \vee y & \to & if(x, true, y) & \\
x \supset y & \to & if(x, y, true) & \\
x \equiv x & \to & true & \diamond \\
x \equiv y & \to & if(x, y, \neg y) & \diamond \\
if(true, x, y) & \to & x & \\
if(false, x, y) & \to & y &
\end{array}$$

$$\begin{array}{lcl}
if(x, x, if(x, false, true)) & \to & true \\
x \equiv y & \to & if(x, y, if(y, false, true))
\end{array}$$

POL:
$$\begin{array}{lcl}
[\wedge](x,y) & = & x+y+2 \\
[\vee](x,y) & = & x+y+2 \\
[\equiv](x,y) & = & x+2y+3 \\
[\supset](x,y) & = & x+y+2 \\
[\neg](x) & = & x+3 \\
[if](x,y,z) & = & x+y+z \\
[false]() & = & 1 \\
[true]() & = & 1
\end{array}$$

RPOS: $\wedge \succ if, \wedge \succ false, \vee \succ if, \vee \succ true, \supset \succ if, \supset \succ true, \equiv \succ \neg \succ if, \neg \succ false, \neg \succ true$ □

EXAMPLE 2.33 B: Disjunctive Normal Form[18]

$$\begin{array}{lcl}
x \vee x & \to & x \\
x \wedge x & \to & x \\
\neg\neg x & \to & x \\
\neg(x \wedge y) & \to & (\neg x) \vee (\neg y) \\
\neg(x \vee y) & \to & (\neg x) \wedge (\neg y)
\end{array}$$

KBOS: $\varphi(\neg) = 0$
$\neg \succ \vee, \neg \succ \wedge$

POL:
$$\begin{array}{lcl}
[\wedge](x,y) & = & x+y \\
[\vee](x,y) & = & x+y \\
[\neg](x) & = & x^2
\end{array}$$

RPOS: $\neg \succ \vee, \neg \succ \wedge$ □

EXAMPLE 2.34 B: If-then-else

$$\begin{array}{lcl}
if(true, x, y) & \to & x \\
if(false, x, y) & \to & y \\
if(x, y, y) & \to & y \\
if(if(x, y, z), u, v) & \to & if(x, if(y, u, v), if(z, u, v))
\end{array}$$

---

[18] The orientation of the two last rules in reverse order implies divergence.

POL: $[if](x,y,z) = xy + xz + x$
$[false]() = 1$
$[true]() = 1$

RPOS: $\tau(if) = left$ □

EXAMPLE 2.35 B: Implication ([Ste89c])

$$\begin{array}{lcl} x \wedge false & \to & false \\ x \wedge (\neg false) & \to & x \\ \neg\neg x & \to & x \\ false \supset y & \to & \neg false \\ x \supset false & \to & \neg x \\ (\neg x) \supset (\neg y) & \to & y \supset (x \wedge y) \end{array}$$

POL: $[\wedge](x,y) = x + y$
$[\supset](x,y) = 2x + 2y$
$[\neg](x) = 2x$
$[false]() = 1$

IRDS: $\supset \succ \neg \succ \wedge$ □

EXAMPLE 2.36 B: Implication

$$\begin{array}{lcl} (\neg x) \supset y & \to & x \vee y \\ (\neg x) \supset (y \vee z) & \to & y \supset (x \vee z) \quad \diamond \end{array}$$

$$x \supset (y \vee z) \to y \vee (x \supset z)$$

KBOS: $\varphi(\neg) + \varphi(\supset) > \varphi(\vee), \varphi(\neg) > 0$
$\supset \succ \vee$

POL: $[\vee](x,y) = x + y$
$[\supset](x,y) = x + 2y$
$[\neg](x) = 3x$

IRDS: $\neg \succ \supset \succ \vee$ □

EXAMPLE 2.37 B: Ternary $\wedge$-Operator ([Ste88])

$$and(\neg\neg x, y, \neg z) \to and(y, x \mathbin{\bar{\wedge}} z, x)$$

POL: $[and](x,y,z) = x + y + z$
$[\bar{\wedge}](x,y) = x + y$
$[\neg](x) = 2x$

IRDS: $\neg \succ \bar{\wedge}$ □

EXAMPLE 2.38 L: Functional List Append[19] ([DP88])

$$\begin{array}{lcl} nil \circ y & \rightarrow & y \\ x \circ nil & \rightarrow & x \\ (x.y) \circ z & \rightarrow & x.(y \circ z) \\ (x \circ y) \circ z & \rightarrow & x \circ (y \circ z) \end{array}$$

KBOS: $\circ \succ .$
$\tau(\circ) = left$

POL:
$$\begin{array}{lcl} [\circ](x,y) & = & 2x + y \\ [.](x,y) & = & x + y \\ [nil]() & = & 1 \end{array}$$

RPOS: $\circ \succ .$
$\tau(\circ) = left$ □

EXAMPLE 2.39 L: List ([Ait85])

$$\begin{array}{lcl} rev(nil) & \rightarrow & nil \\ rev(x.y) & \rightarrow & rev(y) \circ (x.nil) \\ car(x.y) & \rightarrow & x \\ cdr(x.y) & \rightarrow & y \\ null(nil) & \rightarrow & true \\ null(x.y) & \rightarrow & false \\ nil \circ y & \rightarrow & y \\ (x.y) \circ z & \rightarrow & x.(y \circ z) \end{array}$$

POL:
$$\begin{array}{lcl} [\circ](x,y) & = & xy \\ [.](x,y) & = & xy + y + 1 \\ [car](x) & = & x \\ [cdr](x) & = & x \\ [null](x) & = & x + 1 \\ [rev](x) & = & x^2 \\ [nil]() & = & 2 \\ [false]() & = & 2 \\ [true]() & = & 2 \end{array}$$

RPOS: $rev \succ \circ \succ .,rev \succ nil, null \succ true, null \succ false$ □

EXAMPLE 2.40 L: Member ([HO82])

$$\begin{array}{lcl} true \vee y & \rightarrow & true \\ x \vee true & \rightarrow & true \\ false \vee false & \rightarrow & false \\ mem(x, nil) & \rightarrow & false \\ mem(x, set(y)) & \rightarrow & x \equiv y \\ mem(x, union(y,z)) & \rightarrow & mem(x,y) \vee mem(x,z) \end{array}$$

---

[19] Using the KBOS with $\tau(\circ) = left$ and $. \succ \circ$, the completion process diverges. The completion also diverges if we use the KBOS with $\tau(\circ) = right$.

POL:

$$
\begin{array}{lcl}
[\vee](x,y) & = & x+y \\
[\equiv](x,y) & = & x+y \\
[mem](x,y) & = & xy \\
[set](x) & = & x+2 \\
[union](x,y) & = & x+y+1 \\
[nil]() & = & 2 \\
[false]() & = & 1 \\
[true]() & = & 1
\end{array}
$$

RPOS: $mem \succ false, mem \succ \equiv, mem \succ \vee$ □

EXAMPLE 2.41 L: Finite Sequences of Natural Numbers ([KM87])

$$
\begin{array}{lcl}
\|nil\| & \to & 0 \\
\|g(x,y)\| & \to & s(\|x\|) \\
f(x,nil) & \to & g(nil,x) \\
f(x,g(y,z)) & \to & g(f(x,y),z) \\
rem(nil,y) & \to & nil \\
rem(g(x,y),0) & \to & g(x,y) \\
rem(g(x,y),s(z)) & \to & rem(x,z)
\end{array}
$$

KBOS: $\varphi(f) > \varphi(g) > \varphi(s), \varphi(nil) > \varphi(0)$
$f \succ g$

POL:

$$
\begin{array}{lcl}
[\|\cdot\|](x) & = & x \\
[f](x,y) & = & x+2y \\
[g](x,y) & = & x+y \\
[rem](x,y) & = & x+y \\
[s](x) & = & x \\
[0]() & = & 1 \\
[nil]() & = & 2
\end{array}
$$

RPOS: $\|.\| \succ 0, \|.\| \succ s, f \succ g$ □

EXAMPLE 2.42 L: Flattening and Reverse Operation

$$
\begin{array}{lcl}
flatten(nil) & \to & nil \\
flatten(unit(x)) & \to & flatten(x) \\
flatten(x \circ y) & \to & flatten(x) \circ flatten(y) \\
flatten(unit(x) \circ y) & \to & flatten(x) \circ flatten(y) \quad \diamond \\
flatten(flatten(x)) & \to & flatten(x) \\
rev(nil) & \to & nil \\
rev(unit(x)) & \to & unit(x) \\
rev(x \circ y) & \to & rev(y) \circ rev(x) \\
rev(rev(x)) & \to & x \\
x \circ nil & \to & x \\
nil \circ y & \to & y \\
(x \circ y) \circ z & \to & x \circ (y \circ z)
\end{array}
$$

KBOS:	$\varphi(unit) > 0, \varphi(flatten) = \varphi(rev) = 0$ $rev \sim flatten \succ \circ$[20] $\tau(\circ) = left$	
POL:	$[\circ](x,y) = 3xy + 2x$ $[flatten](x) = x^2$ $[rev](x) = x^2$ $[unit](x) = x + 1$ $[nil]() = 2$	
RPOS:	$flatten \succ \circ, rev \succ \circ$ $\tau(\circ) = left$	□

EXAMPLE 2.43 L: [Nar89]

$$
\begin{array}{lcl}
merge(nil, y) & \rightarrow & y \\
merge(x, nil) & \rightarrow & x \\
merge(x.y, u.v) & \rightarrow & if(x < u, x.merge(y, u.v), u.merge(x.y, v)) \\
nil \circ y & \rightarrow & y \\
(x.y) \circ z & \rightarrow & x.(y \circ z) \\
if(true, x, y) & \rightarrow & x \\
if(false, x, y) & \rightarrow & y
\end{array}
$$

POL:	$[\circ](x,y) = x^2y$ $[.](x,y) = x + 2y$ $[<](x,y) = x + y$ $[if](x,y,z) = x + y + z$ $[merge](x,y) = x^2y^2$ $[nil]() = 2$ $[false]() = 1$ $[true]() = 1$	
RPOS:	$merge \succ if, merge \succ <, merge \succ ., \circ \succ .$	□

EXAMPLE 2.44 L: [MS86]

$$
\begin{array}{lcl}
del(x.(y.z)) & \rightarrow & f(x \equiv y, x, y, z) \\
f(true, x, y, z) & \rightarrow & del(y.z) \\
f(false, x, y, z) & \rightarrow & x.del(y.z) \\
nil \equiv nil & \rightarrow & true \\
(x.y) \equiv nil & \rightarrow & false \\
nil \equiv (y.z) & \rightarrow & false \\
(x.y) \equiv (u.v) & \rightarrow & (x \equiv u) \wedge (y \equiv v)
\end{array}
$$

---

[20] Using a quasi-ordering on the operators, we may admit more than one unary operator with weight zero (see [Ste88]).

POL:

$$\begin{aligned}
[\wedge](x,y) &= x+y \\
[\equiv](x,y) &= xy \\
[.](x,y) &= x+y \\
[del](x) &= x^2 \\
[f](x,y,z,u) &= x+y+z^2+2zu+u^2 \\
[nil]() &= 2 \\
[false]() &= 2 \\
[true]() &= 2
\end{aligned}$$

□

EXAMPLE 2.45 L: [Nar89]

$$\begin{aligned}
admit(x, nil) &\to nil \\
admit(x, u.v.w.z) &\to cond(sum(x,u,v) \equiv w, u.v.w.admit(carry(x,u,v),z)) \\
cond(true, y) &\to y
\end{aligned}$$

POL:

$$\begin{aligned}
[\equiv](x,y) &= x+y \\
[.](x,y) &= xy \\
[admit](x,y) &= xy^3 \\
[carry](x,y,z) &= x+y+z \\
[cond](x,y) &= x+y \\
[sum](x,y,z) &= x+y+z \\
[nil]() &= 2 \\
[true]() &= 2
\end{aligned}$$

RPOS: $admit \succ cond, admit \succ sum, admit \succ \equiv, admit \succ . \succ carry$
$\tau(admit) = right$

□

EXAMPLE 2.46 S:

$$\begin{aligned}
aa &\to bb \\
bba &\to abb
\end{aligned}$$

KBOS: $\varphi(a) > \varphi(b)$
$b \succ a$

POL:

$$\begin{aligned}
[a](x) &= 2x+1 \\
[b](x) &= 2x
\end{aligned}$$

□

EXAMPLE 2.47 S: [HP86][21]

$$\begin{aligned}
ab &\to ba \\
ac &\to \lambda
\end{aligned}$$

KBOS: $a \succ b$

POL:

$$\begin{aligned}
[a](x) &= 2x \\
[b](x) &= x+1 \\
[c](x) &= x
\end{aligned}$$

[21]The process diverges if the first rule is oriented in reverse order. Note that λ denotes the empty word.

RPOS: $a \succ b$ □

EXAMPLE 2.48 S: [HP86][22]

$$\begin{array}{lll} d & \rightarrow & eu \\ du & \rightarrow & c \quad \diamond \\ cu & \rightarrow & b \quad \diamond \\ ve & \rightarrow & \lambda \\ bu & \rightarrow & ae \quad \diamond \end{array}$$

$$\begin{array}{lll} c & \rightarrow & euu \\ b & \rightarrow & euuu \\ ae & \rightarrow & euuuu \end{array}$$

KBOS: $\varphi(e) = \varphi(u) = 1, \varphi(a) = 5, \varphi(b) = \varphi(c) = \varphi(d) = 6$

POL:
$$\begin{array}{lll} [a](x) & = & x+1 \\ [b](x) & = & x+1 \\ [c](x) & = & x+1 \\ [d](x) & = & x+1 \\ [e](x) & = & x \\ [u](x) & = & x \\ [v](x) & = & x+1 \end{array}$$

RPOS: $c \succ b \succ a \succ e \succ d \succ u$ □

EXAMPLE 2.49 S: Monoid

$$\begin{array}{lll} acd & \rightarrow & c \\ ubdd & \rightarrow & b \\ vaa & \rightarrow & uv \\ vac & \rightarrow & ubd \\ vc & \rightarrow & b \\ waa & \rightarrow & uw \\ wac & \rightarrow & ubd \\ wc & \rightarrow & b \end{array}$$

KBOS: $\varphi(b) = \varphi(d) = \varphi(u) = \varphi(w) = 1, \varphi(a) = \varphi(v) = 2, \varphi(c) = 3$

POL:
$$\begin{array}{lll} [a](x) & = & x+1 \\ [b](x) & = & x \\ [c](x) & = & x \\ [d](x) & = & x+1 \\ [u](x) & = & x \\ [v](x) & = & x+1 \\ [w](x) & = & x+1 \end{array}$$

[22]The original system of [HP86] is parameterized, such that $a_{n+1}a_{n-1} \rightarrow \lambda$, $a_1 a_n \rightarrow a_0 a_{n-1}$, $a_i a_n \rightarrow a_{i-1}$ for all $i = n-1, ..., 2 (n \geq 3)$. The termination of this general system can be proved with the help of the following polynomial ordering: $[a_i](x) = x+1$ for all $i = 0, ..., n-2, n+1$ and $[a_i](x) = x$ for $i = n-1, n$.

RPOS: $v \succ u, v \succ b, v \succ d, w \succ u, w \succ b, w \succ d$ □

EXAMPLE 2.50 S:

$$ab \rightarrow bba$$

POL:

$$[a](x) = 3x$$
$$[b](x) = x+1$$

RPOS: $a \succ b$ □

EXAMPLE 2.51 O: Ackermann Function

$$ack(0,y) \rightarrow s(y)$$
$$ack(s(x),0) \rightarrow ack(x,s(0))$$
$$ack(s(x),s(y)) \rightarrow ack(x,ack(s(x),y))$$

RPOS: $ack \succ s$

$\tau(ack) = left$ □

EXAMPLE 2.52 O: [Thi84]

$$f(x,0,0) \rightarrow s(x)$$
$$f(0,y,0) \rightarrow s(y)$$
$$f(0,0,z) \rightarrow s(z)$$
$$f(s(0),y,z) \rightarrow f(0,s(y),s(z))$$
$$f(s(x),s(y),0) \rightarrow f(x,y,s(0))$$
$$f(s(x),0,s(z)) \rightarrow f(x,s(0),z)$$
$$f(0,s(0),s(0)) \rightarrow s(s(0))$$
$$f(s(x),s(y),s(z)) \rightarrow f(x,y,f(s(x),s(y),z))$$
$$f(0,s(s(y)),s(0)) \rightarrow f(0,y,s(0))$$
$$f(0,s(0),s(s(z))) \rightarrow f(0,s(0),z)$$
$$f(0,s(s(y)),s(s(z))) \rightarrow f(0,y,f(0,s(s(y)),s(z)))$$

RPOS: $f \succ s$

$\tau(f) = left$ □

EXAMPLE 2.53 O: [BD86]

$$f(x,y) \rightarrow g(x,y)$$
$$g(h(x),y) \rightarrow h(f(x,y)) \quad \diamond$$

$$g(h(x),y) \rightarrow h(g(x,y))$$

KBOS: $\varphi(f) = \varphi(g)$

$f \succ g \succ h$

POL: $[f](x,y) = 2x+y+1$
$[g](x,y) = 2x+y$
$[h](x) = x+2$ □

EXAMPLE 2.54 O: [Gra88][23]

$$f(x,a) \rightarrow x$$
$$f(x,g(y)) \rightarrow f(g(x),y)$$

KBOS: $\tau(f) = right$

POL: $[f](x,y) = x+2y$
$[g](x) = x+1$
$[a]() = 1$

RPOS: $f \succ g$
$\tau(f) = right$ □

EXAMPLE 2.55 O: [Gra88][24]

$$f(x,g(x)) \rightarrow x$$
$$f(x,h(y)) \rightarrow f(h(x),y)$$

KBOS: $\tau(f) = right$

POL: $[f](x,y) = x+2y$
$[g](x) = x$
$[h](x) = x+1$

RPOS: $f \succ h$
$\tau(f) = right$ □

EXAMPLE 2.56 O: [You88]

$$f(a,x) \rightarrow g(a,x) \quad \diamond$$
$$g(a,x) \rightarrow f(b,x)$$

$$f(a,x) \rightarrow f(b,x)$$

KBOS: $\varphi(f) = \varphi(g), \varphi(a) > \varphi(b)$
$f \succ g$

POL: $[f](x,y) = 2x+y$
$[g](x,y) = x+y$
$[a]() = 3$
$[b]() = 1$ □

EXAMPLE 2.57 O: [RJ81]

$$f(g(h(x,y)),f(a,a)) \rightarrow f(h(x,x),g(f(y,a)))$$

---

[23] If the last rule is oriented from right to left, the completion process will diverge.
[24] The orientation of the second rule in reverse order implies divergence.

POL: $[f](x,y) = x + y$
$[g](x) = 2x$
$[h](x,y) = x + y + 1$
$[a]() = 1$

RPOS: $f \succ g \succ h$
$\tau(f) = left$

□

EXAMPLE 2.58 O: [Mid88]

$$f(x,y) \rightarrow x$$
$$g(a) \rightarrow h(a,b,a)$$
$$i(x) \rightarrow f(x,x) \qquad \diamond$$
$$h(x,x,y) \rightarrow g(x)$$

$$i(x) \rightarrow x$$

POL: $[f](x,y) = x + y$
$[g](x) = x^2$
$[h](x,y,z) = x + y^2 + z$
$[i](x) = 2x + 1$
$[a]() = 3$
$[b]() = 1$

□

EXAMPLE 2.59 O: [Lév86]

$$f(g(x),y,y) \rightarrow g(f(x,x,y))$$

POL: $[f](x,y,z) = x^2 + y + z$
$[g](x) = x + 1$

RPOS: $f \succ g$

□

EXAMPLE 2.60 O:

$$f(g(f(a),h(a,f(a)))) \rightarrow f(h(g(f(a),a),g(f(a),f(a))))$$

POL: $[f](x) = x$
$[g](x,y) = xy$
$[h](x,y) = x + y + 1$
$[a]() = 2$

RPOS: $g \succ h$

□

EXAMPLE 2.61 O: [Lan89]

$$
\begin{array}{lcl}
f(j(x,y),y) & \to & g(f(x,k(y))) \\
f(x,h_1(y,z)) & \to & h_2(0,x,h_1(y,z)) \\
g(h_2(x,y,h_1(z,u))) & \to & h_2(s(x),y,h_1(z,u)) \\
h_2(x,j(y,h_1(z,u)),h_1(z,u)) & \to & h_2(s(x),y,h_1(s(z),u)) \\
i(f(x,h(y))) & \to & y \\
i(h_2(s(x),y,h_1(x,z))) & \to & z \\
k(h(x)) & \to & h_1(0,x) \\
k(h_1(x,y)) & \to & h_1(s(x),y)
\end{array}
$$

KBOS: $\varphi(h)=\varphi(h_1)=\varphi(h_2)=\varphi(s)=\varphi(0)=1, \varphi(g)=\varphi(k)=2, \varphi(f)=3, \varphi(j)=4$

POL:

$$
\begin{array}{lcl}
[f](x,y) & = & x+y+2 \\
[g](x) & = & x+1 \\
[h](x) & = & x+1 \\
[h_1](x,y) & = & x+y \\
[h_2](x,y,z) & = & x+y+z \\
[i](x) & = & x \\
[j](x,y) & = & x+y+2 \\
[k](x) & = & x+1 \\
[s](x) & = & x \\
[0]() & = & 1
\end{array}
$$

□

3 Systems to Complete

EXAMPLE 3.1 G: [Les84]

$$\begin{array}{lcl} 1 * y & \to & y \\ i(x) * x & \to & 1 \\ (x * y) * z & \to & x * (y * z) \\ x/y & \to & x * i(y) \end{array}$$

$$\begin{array}{lcl} x * 1 & \to & x \\ i(1) & \to & 1 \\ i(i(x)) & \to & x \\ x * i(x) & \to & 1 \\ i(x * y) & \to & i(y) * i(x) \\ i(x) * (x * y) & \to & y \\ x * (i(x) * y) & \to & y \end{array}$$

KBOS: $\varphi(*) = \varphi(i) = 0, \varphi(/) = \varphi(1) = 1$
$\tau(*) = left$

POL:
$$\begin{array}{lcl} [*](x,y) & = & 2xy + x \\ [/](x,y) & = & 2xy^2 + x + 1 \\ [i](x) & = & x^2 \\ [1]() & = & 2 \end{array}$$

RPOS: $/ \succ i \succ *, i \succ 1$
$\tau(*) = left$

□

EXAMPLE 3.2 G: [BL88]

$$\begin{array}{lcll} f(f(x)) & \to & x & \\ f(x) + f(y) & \to & f(x+y) & \diamond \\ (x+y)+z & \to & x + (y+z) & \end{array}$$

$$f(x) + y \to f(x + f(y))$$

POL:
$$\begin{array}{lcl} [+](x,y) & = & 3x + y \\ [f](x) & = & x + 1 \end{array}$$

RPOS: $+ \succ f$
$\tau(+) = left$

□

EXAMPLE 3.3 G: Group ([KS83])

$$\begin{array}{lcll} (x/x)/((y/y)/y) & \to & y & \diamond \\ (x/y)/(z/y) & \to & x/z & \diamond \\ x/x & \to & 1 & \diamond \\ 1/y & \to & i(y) & \diamond \\ x/i(y) & \to & x * y & \diamond \end{array}$$

$$
\begin{array}{lll}
i(1) & \rightarrow & 1 \\
i(i(x)) & \rightarrow & x \\
i(x*y) & \rightarrow & i(y)*i(x) \\
i(x)*x & \rightarrow & 1 \\
x*i(x) & \rightarrow & 1 \\
x*1 & \rightarrow & x \\
1*y & \rightarrow & y \\
(x*y)*z & \rightarrow & x*(y*z) \\
x*(i(x)*y) & \rightarrow & y \\
i(x)*(x*y) & \rightarrow & y \\
x/y & \rightarrow & x*i(y)
\end{array}
$$

KBOS: $\varphi(*) = \varphi(/) = \varphi(i) = 0, \varphi(1) = 1$
$/ \succ *, i \succ *$
$\tau(*) = left$

POL:

$$
\begin{array}{lll}
[*](x,y) & = & 2xy + x \\
[/](x,y) & = & 2xy^2 + x + 1 \\
[i](x) & = & x^2 \\
[1]() & = & 2
\end{array}
$$

RPOS: $/ \succ i \succ * \succ 1$
$\tau(*) = left$

□

EXAMPLE 3.4 G: Group ([KB70])

$$
\begin{array}{lll}
(x*y)*z & \rightarrow & x*(y*z) \\
1*y & \rightarrow & y \\
g(x) & \rightarrow & f(x)*x \\
g(x)*y & \rightarrow & y \quad \diamond
\end{array}
$$

$$
\begin{array}{lll}
f(x)*(x*y) & \rightarrow & y \\
f(1)*y & \rightarrow & y \\
f(f(x))*y & \rightarrow & x*y \\
x*(f(x)*y) & \rightarrow & y \\
f(x*y)*z & \rightarrow & f(y)*(f(x)*z)
\end{array}
$$

POL:

$$
\begin{array}{lll}
[*](x,y) & = & 2x + y \\
[f](x) & = & 2x \\
[g](x) & = & 5x + 1 \\
[1]() & = & 1
\end{array}
$$

RPOS: $g \succ f \succ *$
$\tau(*) = left$

□

EXAMPLE 3.5 G: Central Groupoid ([KB70])

$$
\begin{array}{lll}
(x*y)*(y*z) & \rightarrow & y \quad \diamond \\
(x*x)*x & \rightarrow & f(x) \quad \diamond \\
x*(x*x) & \rightarrow & g(x) \quad \diamond \\
g(x)*y & \rightarrow & x*y
\end{array}
$$

$f(f(x))$	$\rightarrow$	$f(x)$
$f(g(x))$	$\rightarrow$	$g(x)$
$g(g(x))$	$\rightarrow$	$g(x)$
$g(f(x))$	$\rightarrow$	$f(x)$
$f(x) * x$	$\rightarrow$	$f(x)$
$f(x) * g(x)$	$\rightarrow$	x
$x * f(y)$	$\rightarrow$	$x * y$
$f(x * y)$	$\rightarrow$	$g(x)$
$x * g(x)$	$\rightarrow$	$g(x)$
$g(x * y)$	$\rightarrow$	$f(y)$
$x * (y * z)$	$\rightarrow$	$x * g(y)$
$(x * y) * z$	$\rightarrow$	$f(y) * z$

KBOS: $\varphi(*) = \varphi(f) = \varphi(g) = 1$
$* \succ g$

POL:
$$[*](x,y) = x + y$$
$$[f](x) = x + 1$$
$$[g](x) = x + 1$$

RPOS: $* \succ f, * \succ g$ □

EXAMPLE 3.6 G: Taussky Group ([KB70])

$x * (y * z)$	$\rightarrow$	$(x * y) * z$	
$1 * 1$	$\rightarrow$	1	$\diamond$
$x * i(x)$	$\rightarrow$	1	
$g(x * y, y)$	$\rightarrow$	$f(x * y, x)$	$\diamond$
$f(1, y)$	$\rightarrow$	y	

$x * 1$	$\rightarrow$	x
$1 * y$	$\rightarrow$	y
$i(i(x))$	$\rightarrow$	x
$i(x * y)$	$\rightarrow$	$i(y) * i(x)$
$i(x) * x$	$\rightarrow$	1
$g(x, y)$	$\rightarrow$	$f(x, x * i(y))$
$(x * y) * i(y)$	$\rightarrow$	x
$(x * i(y)) * y$	$\rightarrow$	x

POL:
$$[*](x,y) = 2xy + y$$
$$[f](x,y) = x + y$$
$$[g](x,y) = x + y^2 + 2xy^2 + 1$$
$$[i](x) = x^2$$
$$[1]() = 2$$

RPOS: $g \succ i \succ * \succ 1, g \succ f \succ *$
$\tau(*) = right$ □

EXAMPLE 3.7 G: [Sak84]

$$
\begin{array}{lll}
(x+y)+z & \to & x+(y+z) \\
0+0 & \to & 0 \quad \diamond \\
x+(-x) & \to & 0 \\
f(0,y,z) & \to & y \\
g(x+y,y) & \to & f(x+y,x,y) \quad \diamond
\end{array}
$$

$$
\begin{array}{lll}
-0 & \to & 0 \\
--x & \to & x \\
-(x+y) & \to & (-y)+(-x) \\
x+0 & \to & x \\
0+y & \to & y \\
(-x)+x & \to & 0 \\
(-x)+(x+y) & \to & y \\
x+((-x)+y) & \to & y \\
g(x,y) & \to & f(x,x+(-y),y) .
\end{array}
$$

POL:

$$
\begin{array}{lll}
[+](x,y) & = & 2xy+x \\
[-](x) & = & x^2 \\
[f](x,y,z) & = & x+y+z \\
[g](x,y) & = & 2xy^2+2x+y+1 \\
[0]() & = & 2
\end{array}
$$

RPOS: $g \succ - \succ + \succ 0, g \succ f$
$\tau(+) = left$ □

EXAMPLE 3.8 G: Quasigroup ([Hul80b])

$$
\begin{array}{lll}
x*(x\backslash y) & \to & y \quad \diamond \\
x\backslash(x*y) & \to & y \quad \diamond \\
(x*y)/y & \to & x \quad \diamond \\
(x/y)*y & \to & x \quad \diamond \\
x*(y*x) & \to & y
\end{array}
$$

$$
\begin{array}{lll}
x\backslash y & \to & y*x \\
y/x & \to & x*y \\
(x*y)*x & \to & y
\end{array}
$$

KBOS: $\varphi(*) = \varphi(\backslash) = \varphi(/) = 1$
$\backslash \succ *, / \succ *$

POL:

$$
\begin{array}{lll}
[*](x,y) & = & x+y \\
[\backslash](x,y) & = & x+y+1 \\
[/](x,y) & = & x+y+1
\end{array}
$$

RPOS: $\backslash \succ *, / \succ *$ □

EXAMPLE 3.9 G: [Chr89][25]

$$
\begin{array}{lll}
f(f(x,y),z) & \rightarrow & f(x,f(y,z)) \\
f(a_k,y) & \rightarrow & y \qquad for\ k=1,...,12 \\
f(x,i_1(x)) & \rightarrow & a_1 \\
f(x,i_k(x)) & \rightarrow & a_k \qquad for\ k=2,...,12 \qquad \diamond
\end{array}
$$

$$
\begin{array}{lll}
i_k(x) & \rightarrow & f(i_1(x),a_k) \qquad for\ k=2,...,12 \\
i_1(a_k) & \rightarrow & a_1 \qquad for\ k=1,...,12 \\
i_1(i_1(x)) & \rightarrow & f(x,a_1) \\
i_1(f(x,y)) & \rightarrow & f(i_1(y),i_1(x)) \\
f(x,f(i_1(x),y)) & \rightarrow & y \\
f(i_1(x),f(x,y)) & \rightarrow & y \\
f(i_1(x),a_1) & \rightarrow & i_1(x)
\end{array}
$$

KBOS: $\varphi(f)=\varphi(i_1)=0$
$\varphi(a_k)=1 \qquad for\ k=1,...,12$
$\varphi(i_k)=2 \qquad for\ k=2,...,12$
$i_1 \succ a_1$
$\tau(f)=left$

POL:

$$
\begin{array}{lll}
[f](x,y) & = & 2xy+x \\
[i_1](x) & = & x^2 \\
[i_k](x) & = & 5x^2+1 \qquad for\ k=2,...,12 \\
[a_k]() & = & 2 \qquad for\ k=1,...,12
\end{array}
$$

RPOS: $i_2 \succ i_1$
$i_3 \succ i_k \succ i_1 \succ f \succ a_k \qquad for\ k=1,...,12$
$\tau(f)=left$

□

EXAMPLE 3.10 G: Cancellation Law ([KB70])

$$
\begin{array}{lll}
f(x,x*y) & \rightarrow & y \\
g(x*y,y) & \rightarrow & x \\
x*1 & \rightarrow & x \\
1*y & \rightarrow & y
\end{array}
$$

$$
\begin{array}{lll}
f(1,y) & \rightarrow & y \\
f(x,x) & \rightarrow & 1 \\
g(x,1) & \rightarrow & x \\
g(x,x) & \rightarrow & 1
\end{array}
$$

KBOS: $\varphi(*)=\varphi(f)=\varphi(g)=\varphi(1)=1$

POL:

$$
\begin{array}{lll}
[*](x,y) & = & x+y \\
[f](x,y) & = & x+y \\
[g](x,y) & = & x+y \\
[1]() & = & 1
\end{array}
$$

[25] Using the KBOS, the canonical system is supplemented by $i_1(i_1(i_1(x))) \rightarrow i_1(x)$ and the equation $i_1(i_1(x)) = f(x,a_1)$ is oriented in reverse order.

RPOS: $f \succ 1, g \succ 1$ □

EXAMPLE 3.11 A: [BBK87]

$p(0)$	$\rightarrow$	0	$\diamond$
$p(s(x))$	$\rightarrow$	x	$\diamond$
$x+0$	$\rightarrow$	x	
$s(x+p(y))$	$\rightarrow$	$x+y$	$\diamond$

$p(x)$	$\rightarrow$	x
$s(x)$	$\rightarrow$	x

KBOS: $\varphi(+) = \varphi(p) = \varphi(s) = \varphi(0) = 1$

POL:
$$[+](x,y) = x+y$$
$$[p](x) = x+1$$
$$[s](x) = x+1$$
$$[0]() = 1$$

RPOS: empty $\succ$ □

EXAMPLE 3.12 A: Division ([Dic87])

x/x	$\rightarrow$	1
$x/1$	$\rightarrow$	x
$i(x/y)$	$\rightarrow$	y/x
$(x/y)/z$	$\rightarrow$	$x/(z/i(y))$

$i(1)$	$\rightarrow$	1
$1/y$	$\rightarrow$	$i(y)$
$i(i(x))$	$\rightarrow$	x
$x/(y/i(x))$	$\rightarrow$	$i(y)$
$i(x)/(y/x)$	$\rightarrow$	$i(y)$
$x/(y/(i(x)/z))$	$\rightarrow$	$i(z)/y$
$i(x)/(y/(x/z))$	$\rightarrow$	$i(z)/y$

KBOS: $\varphi(/) = \varphi(i) = 0, \varphi(1) = 1$
$i \succ /$
$\tau(/) = left$ □

EXAMPLE 3.13 A: [Zeh89]

$(x+y)+z$	$\rightarrow$	$x+(y+z)$
$x+0$	$\rightarrow$	x
$(x*y)*z$	$\rightarrow$	$x*(y*z)$
$x*1$	$\rightarrow$	x
$exp(0)$	$\rightarrow$	1
$exp(x+y)$	$\rightarrow$	$exp(x)*exp(y)$

$x+(0+y)$	$\rightarrow$	$x+y$
$x*(1*y)$	$\rightarrow$	$x*y$

KBOS: $\varphi(exp) = 0, \varphi(+) = \varphi(*) = \varphi(0) = \varphi(1) = 1$
$exp \succ *, exp \succ 1$
$\tau(+) = \tau(*) = left$

POL:

$$\begin{aligned} [+](x,y) &= 3x + y \\ [*](x,y) &= 2x + y \\ [exp](x) &= 2x \\ [0]() &= 1 \\ [1]() &= 1 \end{aligned}$$

RPOS: $exp \succ *, exp \succ 1$
$\tau(+) = \tau(*) = left$

□

EXAMPLE 3.14 A: [KNZ86]

$$\begin{aligned} s(s(x)) &\rightarrow x \\ f(0,y) &\rightarrow y \\ f(s(x),y) &\rightarrow s(f(x,y)) \\ f(f(g(x,y),0),0) &\rightarrow g(x,y) \quad \diamond \\ g(0,y) &\rightarrow y \\ g(s(x),y) &\rightarrow f(g(x,y),0) \\ h(0) &\rightarrow s(0) \end{aligned}$$

$$f(f(x,0),0) \rightarrow x$$

KBOS: $\varphi(f) = 0, \varphi(g) = \varphi(0) = 1, \varphi(s) = 2, \varphi(h) = 3$
$f \succ s$

POL:

$$\begin{aligned} [f](x,y) &= xy + x \\ [g](x,y) &= xy + x \\ [h](x) &= 2x + 2 \\ [s](x) &= 2x + 1 \\ [0]() &= 1 \end{aligned}$$

RPOS: $g \succ f \succ s, g \succ 0, h \succ s$

□

EXAMPLE 3.15 A: Integers With Equality (Boolean Algebra) ([Com86])

$$\begin{aligned} p(s(x)) &\rightarrow x \\ x \equiv x &\rightarrow true \\ s(x) \equiv x &\rightarrow false \quad \diamond \\ x \equiv s(x) &\rightarrow false \quad \diamond \\ s(x) \equiv s(y) &\rightarrow x \equiv y \quad \diamond \\ x \equiv p(x) &\rightarrow false \\ p(x) \equiv x &\rightarrow false \\ p(x) \equiv p(y) &\rightarrow x \equiv y \end{aligned}$$

$$\begin{aligned} x \equiv s(y) &\rightarrow p(x) \equiv y \\ s(x) \equiv y &\rightarrow x \equiv p(y) \end{aligned}$$

KBOS: $\varphi(\equiv) = \varphi(p) = \varphi(false) = \varphi(true) = 1, \varphi(s) = 2$

POL:

$$\begin{aligned} [\equiv](x,y) &= x+y \\ [p](x) &= x+1 \\ [s](x) &= x+2 \\ [false]() &= 1 \\ [true]() &= 1 \end{aligned}$$

□

EXAMPLE 3.16 A: [Bur68]

$$\begin{aligned} car(x.y) &\to x \\ cdr(x.y) &\to y \\ car(x).cdr(x) &\to x \\ atom(x.y) &\to false \quad \diamond \end{aligned}$$

$$atom(x) \to false$$

KBOS: $\varphi(.) = \varphi(atom) = \varphi(car) = \varphi(cdr) = \varphi(false) = 1$

POL:

$$\begin{aligned} [.](x,y) &= x+y \\ [atom](x) &= x+1 \\ [car](x) &= x \\ [cdr](x) &= x \\ [false]() &= 1 \end{aligned}$$

RPOS: $atom \succ false$

□

EXAMPLE 3.17 B:

$$\begin{aligned} (x \wedge y) \vee (z \wedge y) &\to (x \vee z) \wedge y \\ x \wedge x &\to x \\ x \vee x &\to x \end{aligned}$$

$$\begin{aligned} (x \wedge y) \vee y &\to (x \vee y) \wedge y \\ x \vee (y \wedge x) &\to (x \vee y) \wedge x \end{aligned}$$

KBOS: $\varphi(\wedge) = \varphi(\vee) = 1$
$\vee \succ \wedge$

POL:

$$\begin{aligned} [\wedge](x,y) &= x+y+1 \\ [\vee](x,y) &= 2x+2y \end{aligned}$$

□

EXAMPLE 3.18 L: Reverse ([Küc87])

$$\begin{aligned} nil \circ y &\to y \\ (x.y) \circ z &\to x.(y \circ z) \\ rev(nil) &\to nil \\ rev(x.y) &\to rev(y) \circ (x.nil) \\ rev(rev(x)) &\to x \end{aligned}$$

$$rev(x \circ (y.nil)) \to y.rev(x)$$

POL:	$[\circ](x,y) = xy$	
	$[.](x,y) = xy+y+1$	
	$[rev](x) = x^2$	
	$[nil]() = 2$	
RPOS:	$rev \succ \circ \succ . \succ nil$	□

EXAMPLE 3.19 L: [HO80]

$$\begin{array}{lll}
nil \circ y & \rightarrow & y \\
(x.y) \circ z & \rightarrow & x.(y \circ z) \\
(x \circ y) \circ z & \rightarrow & x \circ (y \circ z) \\
reviter(nil, y) & \rightarrow & y \\
reviter(x.y, z) & \rightarrow & reviter(y, x.z) \\
rev(nil) & \rightarrow & nil \quad \diamond \\
rev(x.y) & \rightarrow & rev(y) \circ (x.nil) \quad \diamond \\
rev(x) \circ y & \rightarrow & reviter(x, y) \quad \diamond \\
rev(x) & \rightarrow & reviter(x, nil)
\end{array}$$

$$reviter(x,y) \circ z \rightarrow reviter(x, y \circ z)$$

POL:	$[\circ](x,y) = xy+x$	
	$[.](x,y) = xy+y+1$	
	$[rev](x) = 2x^2$	
	$[reviter](x,y) = xy+x$	
	$[nil]() = 2$	
RPOS:	$rev \succ \circ \succ reviter \succ . \succ nil$	
	$\tau(\circ) = \tau(reviter) = left$	□

EXAMPLE 3.20 L: [Mus80]

$$\begin{array}{lll}
x \equiv x & \rightarrow & true \\
nil \equiv end(y,z) & \rightarrow & false \\
end(x,y) \equiv nil & \rightarrow & false \\
end(x,y) \equiv end(u,v) & \rightarrow & (y \equiv v) \wedge (x \equiv u) \\
f(x, nil) & \rightarrow & end(nil, x) \\
f(x, end(y,z)) & \rightarrow & end(f(x,y), z) \\
nil.y & \rightarrow & y \\
end(x,y).z & \rightarrow & x.f(y,z) \\
null(nil) & \rightarrow & true \\
null(end(x,y)) & \rightarrow & false
\end{array}$$

$$true \wedge true \rightarrow true$$

KBOS: $\varphi(\wedge) = 0, \varphi(\equiv) = \varphi(.) = \varphi(end) = \varphi(f) = \varphi(null) = \varphi(nil) = \varphi(false) = \varphi(true) = 1$
$\tau(.) = left$

POL:

$$\begin{array}{lcl} [\wedge](x,y) & = & xy \\ [\equiv](x,y) & = & xy \\ [.](x,y) & = & x^2y \\ [end](x,y) & = & xy+1 \\ [f](x,y) & = & 2xy \\ [null](x) & = & x+1 \\ [nil]() & = & 2 \\ [false]() & = & 2 \\ [true]() & = & 2 \end{array}$$

RPOS: $. \succ f \succ end, \equiv \succ false, \equiv \succ true, null \succ false, null \succ true$
$\tau(.) = left$ □

EXAMPLE 3.21 L: Finite Sequences of Natural Numbers ([KM87])

$$\begin{array}{lcll} f(x,nil) & \rightarrow & g(nil,x) & \diamond \\ f(x,g(y,z)) & \rightarrow & g(f(x,y),z) & \diamond \\ g(g(x,y),y) & \rightarrow & g(x,y) & \diamond \\ g(cdr(g(x,y)),y) & \rightarrow & cdr(g(x,y)) & \diamond \\ x \circ nil & \rightarrow & x & \\ x \circ g(y,z) & \rightarrow & g(x \circ y, z) & \diamond \\ cdr(nil) & \rightarrow & nil & \\ cdr(g(nil,y)) & \rightarrow & nil & \diamond \\ cdr(g(g(x,y),z)) & \rightarrow & g(cdr(g(x,y)),z) & \diamond \end{array}$$

$$\begin{array}{lcl} f(x,nil) & \rightarrow & nil \\ g(x,y) & \rightarrow & x \end{array}$$

KBOS: $\varphi(\circ) = \varphi(cdr) = \varphi(f) = \varphi(g) = \varphi(nil) = 1$
$f \succ g, \circ \succ g, cdr \succ g$

POL:

$$\begin{array}{lcl} [\circ](x,y) & = & x+2y \\ [cdr](x) & = & 2x \\ [f](x,y) & = & x+2y \\ [g](x,y) & = & x+y \\ [nil]() & = & 2 \end{array}$$

RPOS: $f \succ g, \circ \succ g, cdr \succ g$ □

EXAMPLE 3.22 S: Fibonacci Group

$$\begin{array}{lcll} ae & \rightarrow & d & \diamond \\ ba & \rightarrow & e & \diamond \\ a & \rightarrow & cb & \diamond \\ b & \rightarrow & dc & \diamond \\ c & \rightarrow & ed & \diamond \end{array}$$

$$\begin{array}{lcl} a & \rightarrow & eee \\ b & \rightarrow & eeeeeeeee \\ c & \rightarrow & eeeee \\ d & \rightarrow & eeee \\ eeeeeeeeeee & \rightarrow & e \end{array}$$

KBOS: $\varphi(e) = 1, \varphi(d) = 5, \varphi(c) = 7, \varphi(b) = 13, \varphi(a) = 21$
$d \succ e$

POL:
$$\begin{aligned}[a](x) &= 8x + 41\\ [b](x) &= 4x + 17\\ [c](x) &= 2x + 6\\ [d](x) &= 2x + 4\\ [e](x) &= x + 1\end{aligned}$$

RPOS: $a \succ b \succ c \succ d \succ e$ □

EXAMPLE 3.23 S: [HP86]

$$\begin{aligned} bd &\to \lambda\\ da &\to e\\ ec &\to a\\ ce &\to a \end{aligned}$$

$$\begin{aligned} be &\to a\\ ca &\to ac\\ ea &\to ae\\ ba &\to ac \end{aligned}$$

KBOS: $\varphi(a) = \varphi(b) = \varphi(c) = \varphi(d) = \varphi(e) = 1$
$e \succ a, b \succ c \succ a$

POL:
$$\begin{aligned}[a](x) &= x + 1\\ [b](x) &= 2x\\ [c](x) &= 2x\\ [d](x) &= 2x\\ [e](x) &= 2x\end{aligned}$$

RPOS: $d \succ e \succ a, b \succ c \succ a$ □

EXAMPLE 3.24 S: [Sat88]

$$\begin{aligned} aba &\to ba \quad \diamond\\ c &\to a \quad \diamond\\ c &\to b \quad \diamond \end{aligned}$$

$$\begin{aligned} c &\to b\\ a &\to b\\ bbb &\to bb \end{aligned}$$

KBOS: $\varphi(a) = \varphi(b) = \varphi(c)$
$c \succ a \succ b$

POL:
$$\begin{aligned}[a](x) &= 3x\\ [b](x) &= 2x\\ [c](x) &= 5x\end{aligned}$$

RPOS: $c \succ a \succ b$ □

EXAMPLE 3.25 S: [Mar86]

$abaab \rightarrow a$

$aaab \rightarrow abaa$

KBOS: $\varphi(a) = \varphi(b)$
$a \succ b$

POL: $[a](x) = 2x$
$[b](x) = x + 1$

RPOS: $a \succ b$ □

EXAMPLE 3.26 S: [Mar86]

$ac \rightarrow cbb$ ◇
$ba \rightarrow acc$ ◇
$cb \rightarrow baa$ ◇
$ua \rightarrow \lambda$ ◇
$vb \rightarrow \lambda$ ◇
$wc \rightarrow \lambda$ ◇

$a \rightarrow ccccbccccc$
$u \rightarrow bc$
$v \rightarrow cccccbccccc$
$w \rightarrow cccccc$
$bb \rightarrow cccbcccccc$
$bcb \rightarrow ccbc$
$bccb \rightarrow ccccccbccccc$
$bcccb \rightarrow cbcccc$
$bccccb \rightarrow ccccbcc$
$bcccccb \rightarrow cc$
$bccccccb \rightarrow cccccbccc$
$ccccccc \rightarrow \lambda$

KBOS: $\varphi(c) = 1, \varphi(b) = 10, \varphi(a) = 20, \varphi(u) = \varphi(v) = \varphi(w) = 21$
$v \succ a \succ c, b \succ c$

POL: $[a](x) = 11x + 11$
$[b](x) = 5x + 5$
$[c](x) = x + 1$
$[u](x) = 13x + 13$
$[v](x) = 13x + 13$
$[w](x) = 13x + 13$

RPOS: $a \succ b \succ c, u \succ a, v \succ a, w \succ a$ □

EXAMPLE 3.27 S: [Mar86][26]

a	$\rightarrow$	$cccbcc$	$\diamond$
bcb	$\rightarrow$	$ccbc$	$\diamond$
$bccb$	$\rightarrow$	$ccccccbccccc$	$\diamond$
$bcccb$	$\rightarrow$	$cbcccc$	$\diamond$
$bbcccb$	$\rightarrow$	$ccccbcc$	$\diamond$
$bccccb$	$\rightarrow$	cc	$\diamond$
$bcccccb$	$\rightarrow$	$cccccbccc$	$\diamond$
$ccccccc$	$\rightarrow$	λ	$\diamond$

$a \rightarrow \lambda$
$b \rightarrow \lambda$
$c \rightarrow \lambda$

KBOS: $\varphi(c) = 1, \varphi(b) = 10, \varphi(a) = 16$
$b \succ c$

POL:
$$[a](x) = 2x + 8$$
$$[b](x) = 2x$$
$$[c](x) = x + 1$$

RPOS: $a \succ b \succ c$ □

EXAMPLE 3.28 S: [Mar86][27]

bu	$\rightarrow$	ab	$\diamond$
au	$\rightarrow$	λ	$\diamond$
aw	$\rightarrow$	ca	
cv	$\rightarrow$	bc	$\diamond$
bv	$\rightarrow$	λ	$\diamond$
cw	$\rightarrow$	λ	
ua	$\rightarrow$	λ	$\diamond$
vb	$\rightarrow$	λ	$\diamond$
wc	$\rightarrow$	λ	

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u	$\rightarrow$	$wbcab$
v	$\rightarrow$	wbc
aba	$\rightarrow$	b
abb	$\rightarrow$	bba
$abca$	$\rightarrow$	$wwbc$
ac	$\rightarrow$	wa
$bbcab$	$\rightarrow$	$ccabc$
bcb	$\rightarrow$	c
bcc	$\rightarrow$	ccb
bw	$\rightarrow$	$wwbc$

[26] Using the given orderings, the completion produces only 39 rules and 8 critical pairs while the ordering on the polynomial interpretations $[a](x) = x + 1$, $[b](x) = 23x + 23$ and $[c](x) = 11x + 11$ leads to 530 rules and 1006 critical pairs (before generating the canonical system above).

[27] The above canonical system is generated w.r.t. the given POL. Using the RPOS, the rule $bbcab \rightarrow ccabc$ is redundant, and $abc \rightarrow wwbbcab$ is included instead of $abca \rightarrow wwbc$.

POL:

$$\begin{aligned}
[a](x) &= 7x+7\\
[b](x) &= 5x+5\\
[c](x) &= x+1\\
[u](x) &= 176x+224\\
[v](x) &= 5x+13\\
[w](x) &= x+2
\end{aligned}$$

RPOS: $u \succ a \succ v \succ b \succ w \succ c$ □

EXAMPLE 3.29 S:

$$\begin{aligned}
ac &\to ea\\
ad &\to va\\
bc &\to eb\\
bd &\to vb\\
au &\to bu \quad \diamond\\
u &\to cw\\
dw &\to u \quad \diamond
\end{aligned}$$

$$\begin{aligned}
dw &\to cw\\
eaw &\to vbw\\
vaw &\to vbw\\
ebw &\to vbw
\end{aligned}$$

KBOS: $\varphi(b) = \varphi(v) = \varphi(w) = 1, \varphi(a) = 2, \varphi(e) = 3, \varphi(c) = 4, \varphi(d) = 5, \varphi(u) = 7$

POL:

$$\begin{aligned}
[a](x) &= x+2\\
[b](x) &= x\\
[c](x) &= x+3\\
[d](x) &= x+4\\
[e](x) &= x+2\\
[u](x) &= x+5\\
[v](x) &= x+1\\
[w](x) &= x
\end{aligned}$$

RPOS: $u \succ d \succ c \succ e \succ a \succ v \succ b \succ w$ □

EXAMPLE 3.30 O[28]:

$$\begin{aligned}
f(g(x),x) &\to a \quad \diamond\\
f(g(x),y) &\to h(y) \quad \diamond\\
f(g(x),f(y,z)) &\to k(f(g(x),y),f(g(x),z)) \quad \diamond
\end{aligned}$$

$$\begin{aligned}
f(g(x),y) &\to a\\
h(x) &\to a\\
k(a,a) &\to a
\end{aligned}$$

[28] If the third rule of the initial rule system is normalized to $k(a,a) \to a$ the following KBOS can prove its termination: $\varphi(a) = \varphi(f) = \varphi(g) = \varphi(h) = \varphi(k) = 1$.

POL: $$\begin{aligned}[f](x,y) &= x+y\\ [g](x) &= x+1\\ [h](x) &= x+1\\ [k](x,y) &= x+y\\ [a]() &= 1\end{aligned}$$

RPOS: $f \succ h \succ a$ □

EXAMPLE 3.31 O:

$$\begin{aligned}f(x,h(y)) &\to j(x)\\ f(h(x),y) &\to j(h(x))\\ g(f(x,x)) &\to i(x)\end{aligned}$$

$$g(j(h(x))) \to i(h(x))$$

KBOS: $\varphi(f)=\varphi(g)=\varphi(h)=\varphi(i)=\varphi(j)=1$

POL: $$\begin{aligned}[f](x,y) &= x+y\\ [g](x) &= x+1\\ [h](x) &= x\\ [i](x) &= x\\ [j](x) &= x\end{aligned}$$

RPOS: $g \succ i, f \succ j$ □

EXAMPLE 3.32 O:

$$\begin{aligned}f(x,x) &\to x && \diamond\\ f(g(x),y) &\to g(x) && \diamond\\ g(g(x)) &\to x\end{aligned}$$

$$f(x,y) \to x$$

KBOS: $\varphi(f)=\varphi(g)$

POL: $$\begin{aligned}[f](x,y) &= x+y\\ [g](x) &= x+1\end{aligned}$$

RPOS: empty $\succ$ □

EXAMPLE 3.33 O: [Hul80a]

$$\begin{aligned}f(g(x)) &\to g(x) && \diamond\\ g(a) &\to a\\ g(g(x)) &\to x\end{aligned}$$

$$f(x) \to x$$

KBOS: $\varphi(f)=\varphi(g)=\varphi(a)$

POL: $$\begin{aligned}[f](x) &= x+1\\ [g](x) &= x+1\\ [a]() &= 1\end{aligned}$$

RPOS: empty $\succ$ □

4 Open Problems

EXAMPLE 4.1 G: [JK86]

$$\begin{array}{lcl} --x & \rightarrow & x \\ -(x+y) & \rightarrow & (-y)+(-x) \\ (-x)+(x+y) & \rightarrow & y \\ (x+y)+(-y) & \rightarrow & x \end{array}$$

Problem: Divergence[29] □

EXAMPLE 4.2 G: Commutator ([Mar86])

$$\begin{array}{lcl} x*1 & \rightarrow & x \\ 1*y & \rightarrow & y \\ i(x)*x & \rightarrow & 1 \\ x*i(x) & \rightarrow & 1 \\ x*(y*z) & \rightarrow & (x*y)*z \\ i(1) & \rightarrow & 1 \\ (x*y)*i(y) & \rightarrow & x \\ (x*i(y))*y & \rightarrow & x \\ i(i(x)) & \rightarrow & x \\ i(x*y) & \rightarrow & i(y)*i(x) \\ k(x,1) & \rightarrow & 1 \\ k(x,x) & \rightarrow & 1 \\ k(x,y)*k(y,x) & \rightarrow & 1 \\ (i(x)*k(y,z))*x & \rightarrow & k((i(x)*y)*x,(i(x)*z)*x) \\ k(x*i(y),y*i(x)) & \rightarrow & 1 \end{array}$$

Problem: Divergence □

EXAMPLE 4.3 G: [Pau84]

$$\begin{array}{lcl} x+0 & \rightarrow & x \\ (-x)+x & \rightarrow & 0 \\ -0 & \rightarrow & 0 \\ --x & \rightarrow & x \\ -(x+y) & \rightarrow & (-x)+(-y) \\ x*1 & \rightarrow & x \\ x*0 & \rightarrow & 0 \\ x*(y+z) & \rightarrow & (x*y)+(x*z) \\ x*(-y) & \rightarrow & -(x*y) \end{array}$$

Problem: Divergence[30] □

[29] The completion process will converge if + is used as an AC-operator and a new operator is introduced.

[30] Using + as an AC-operator, the completion converges.

EXAMPLE 4.4 G: [Kir87]

$$\begin{array}{lcl} (x*y)+(a*y) & \to & (x+a)*y \\ (x*y)*z & \to & x*(y*z) \end{array}$$

Problem: Divergence □

EXAMPLE 4.5 G: Distributivity ([Der87])

$$\begin{array}{lcl} x*(y+z) & \to & (x*y)+(x*z) \\ (x+y)*z & \to & (x*z)+(y*z) \\ x*1 & \to & x \\ 1*y & \to & y \end{array}$$

Problem: Divergence & Termination[31] □

EXAMPLE 4.6 G: Associativity and Distributivity ([Der87])

$$\begin{array}{lcl} (x*y)*z & \to & x*(y*z) \\ (x+y)*z & \to & (x*z)+(y*z) \\ x*(y+f(z)) & \to & g(x,z)*(y+y) \end{array}$$

Problem: Termination □

EXAMPLE 4.7 G: [Dic86]

$$\begin{array}{lcl} i(x)*x & \to & 1 \\ 1*y & \to & y \\ x*0 & \to & 0 \\ (x*y)*z & \to & x*(y*z) \end{array}$$

Problem: Termination[32] □

EXAMPLE 4.8 G: Idempotent Semigroup ([Hul80b])

$$\begin{array}{lcl} x*(y*z) & \to & (x*y)*z \\ x*x & \to & x \end{array}$$

Problem: Divergence[33] □

EXAMPLE 4.9 G: Ring ([Hul80b])

$$\begin{array}{lcl} x+0 & \to & x \\ x+i(x) & \to & 0 \\ (x+y)+z & \to & x+(y+z) \\ x*(y+z) & \to & (x*y)+(x*z) \\ (x+y)*z & \to & (x*z)+(y*z) \end{array}$$

Problem: Termination & Divergence □

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[31] The AC-completion (+ is AC) leads to a canonical system.

[32] The pair $x = 1$ will be generated.

[33] AC-completion terminates.

EXAMPLE 4.10 G: A-Modules ([Hul80b])

$$\begin{array}{lcl} x * (y * z) & \rightarrow & (x \otimes y) * z \\ 1 * y & \rightarrow & y \\ (x + y) * z & \rightarrow & (x * z) \oplus (y * z) \\ x * (y \oplus z) & \rightarrow & (x * y) \oplus (x * z) \end{array}$$

Problem: Termination & Divergence □

EXAMPLE 4.11 A: Addition ([JL87])

$$\begin{array}{lcl} x + 0 & \rightarrow & x \\ x + s(y) & \rightarrow & s(x + y) \\ 0 + s(y) & \rightarrow & s(y) \\ s(0 + y) & \rightarrow & s(y) \end{array}$$

Problem: Divergence □

EXAMPLE 4.12 A: Addition

$$\begin{array}{lcl} 0 + y & \rightarrow & y \\ s(x) + 0 & \rightarrow & s(x) \\ s(x) + s(y) & \rightarrow & s(s(x) + (y + 0)) \end{array}$$

Problem: Termination □

EXAMPLE 4.13 A: Difference

$$\begin{array}{lcl} 0 - y & \rightarrow & 0 \\ x - 0 & \rightarrow & x \\ x - s(y) & \rightarrow & if(x > s(y), s(x - p(s(y))), 0) \\ p(0) & \rightarrow & 0 \\ p(s(x)) & \rightarrow & x \end{array}$$

Problem: Termination □

EXAMPLE 4.14 A: [NRS89]

$$\begin{array}{lcl} p(s(x)) & \rightarrow & x \\ s(p(x)) & \rightarrow & x \\ 0 + y & \rightarrow & y \\ s(x) + y & \rightarrow & s(x + y) \\ p(x) + y & \rightarrow & p(x + y) \\ -0 & \rightarrow & 0 \\ -s(x) & \rightarrow & p(-x) \\ -p(x) & \rightarrow & s(-x) \\ 0 * y & \rightarrow & 0 \\ s(x) * y & \rightarrow & (x * y) + y \\ p(x) * y & \rightarrow & (x * y) + (-y) \end{array}$$

Problem: Divergence[34] □

[34] The above system is ground confluent.

EXAMPLE 4.15 A: Addition and Subtraction ([Bac87])

$$\begin{array}{lcl} (x-y)+z & \rightarrow & (x+z)-y \\ (x+y)-y & \rightarrow & x \end{array}$$

Problem: Divergence □

EXAMPLE 4.16 A: [Zeh89]

$$\begin{array}{lcl} f(0) & \rightarrow & s(0) \\ f(s(0)) & \rightarrow & s(s(0)) \\ f(s(0)) & \rightarrow & s(s(0)) * f(0) \\ f(x+s(0)) & \rightarrow & s(s(0)) * f(x) \\ f(x+y) & \rightarrow & f(x) * f(y) \end{array}$$

Problem: Divergence □

EXAMPLE 4.17 A: Factorial Function ([KL80])

$$\begin{array}{lcl} fac(s(x)) & \rightarrow & fac(p(s(x))) * s(x) \\ p(s(0)) & \rightarrow & 0 \\ p(s(s(x))) & \rightarrow & s(p(s(x))) \end{array}$$

Problem: Termination □

EXAMPLE 4.18 A: Greatest Common Divisor ([BM79])

$$\begin{array}{lcl} gcd(x,0) & \rightarrow & x \\ gcd(0,y) & \rightarrow & y \\ gcd(s(x),s(y)) & \rightarrow & if(x < y, gcd(s(x), y-x), gcd(x-y, s(y))) \end{array}$$

Problem: Termination □

EXAMPLE 4.19 A: [Ste88]

$$((-x)-(-x))-((-y)-(-y)) \rightarrow (x-y)-(x-y)$$

Problem: Divergence[35] □

EXAMPLE 4.20 B: Boolean Ring ([Ait85])

$$\begin{array}{lcl} \neg x & \rightarrow & x \oplus true \\ x \vee y & \rightarrow & (x \wedge y) \oplus (x \oplus y) \\ x \supset y & \rightarrow & (x \wedge y) \oplus (x \oplus true) \\ x \wedge true & \rightarrow & x \\ x \wedge false & \rightarrow & false \\ x \wedge x & \rightarrow & x \\ x \oplus false & \rightarrow & x \\ x \oplus x & \rightarrow & false \\ (x \oplus y) \wedge z & \rightarrow & (x \wedge z) \oplus (y \wedge z) \end{array}$$

[35]Note that this rule cannot be oriented with the help of the RPOS. However, it is possible using the IRDS or a polynomial ordering.

Problem: Divergence[36] □

EXAMPLE 4.21 B: Distributive Lattices ([BPW89])

$x \wedge (y \vee z)$	$\rightarrow$	$(x \wedge y) \vee (x \wedge z)$
$x \wedge (y \wedge y)$	$\rightarrow$	$x \wedge y$
$(x \vee y) \vee (y \wedge z)$	$\rightarrow$	$x \vee y$
$x \vee (x \wedge y)$	$\rightarrow$	x
$true \vee y$	$\rightarrow$	$true$
$x \vee false$	$\rightarrow$	x
$x \vee x$	$\rightarrow$	x
$x \vee (y \vee y)$	$\rightarrow$	$x \vee y$
$x \wedge true$	$\rightarrow$	x
$false \wedge y$	$\rightarrow$	$false$
$x \wedge x$	$\rightarrow$	x

Problem: Divergence[37] □

EXAMPLE 4.22 B: Disjunctive Normal Form ([Der83b])

$\neg(x \wedge y)$	$\rightarrow$	$(\neg x) \vee (\neg y)$
$\neg(x \vee y)$	$\rightarrow$	$(\neg x) \wedge (\neg y)$
$x \wedge (y \vee z)$	$\rightarrow$	$(x \wedge y) \vee (x \wedge z)$

Problem: Divergence □

EXAMPLE 4.23 B: If-then-else

$if(true, x, y)$	$\rightarrow$	x
$if(false, x, y)$	$\rightarrow$	y
$if(x, y, y)$	$\rightarrow$	y
$if(if(x, y, z), u, v)$	$\rightarrow$	$if(x, if(y, u, v), if(z, u, v))$
$if(x, if(x, y, z), z)$	$\rightarrow$	$if(x, y, z)$
$if(x, y, if(x, y, z))$	$\rightarrow$	$if(x, y, z)$

Problem: Divergence □

EXAMPLE 4.24 L: Reverse

$rev(nil)$	$\rightarrow$	nil
$rev(x \circ y)$	$\rightarrow$	$rev_1(x, y) \circ rev_2(x, y)$
$rev_1(x, nil)$	$\rightarrow$	x
$rev_1(x, y \circ z)$	$\rightarrow$	$rev_1(y, z)$
$rev_2(x, nil)$	$\rightarrow$	nil
$rev(x, y \circ z)$	$\rightarrow$	$rev(x \circ rev(rev_2(y, z)))$

Problem: Termination □

[36] Using $\wedge$ and $\oplus$ as AC-operators, the completion process terminates.
[37] The AC-completion method will converge, if $\wedge$ and $\vee$ are used as AC-operators.

EXAMPLE 4.25 L: Reverse

$$\begin{array}{lcl} rev(a) & \to & a \\ rev(b) & \to & b \\ rev(x \circ y) & \to & rev(y) \circ rev(x) \\ rev(x \circ x) & \to & rev(x) \end{array}$$

Problem: Divergence □

EXAMPLE 4.26 L: [Par88]

$$\begin{array}{lcl} rev(nil) & \to & nil \\ rev(rev(x)) & \to & x \\ rev(x \circ y) & \to & rev(y) \circ rev(x) \\ nil \circ y & \to & y \\ x \circ nil & \to & x \\ (x.y) \circ z & \to & x.(y \circ z) \\ x \circ (y \circ z) & \to & (x \circ y) \circ z \\ make(x) & \to & x.nil \end{array}$$

Problem: Divergence □

EXAMPLE 4.27 L: List of Integers

$$\begin{array}{lcl} int(0,0) & \to & 0.nil \\ int(0,s(y)) & \to & 0.int(s(0),s(y)) \\ int(s(x),0) & \to & nil \\ int(s(x),s(y)) & \to & int_{list}(int(x,y)) \\ int_{list}(nil) & \to & nil \\ int_{list}(x.y) & \to & s(x).int_{list}(y) \end{array}$$

Problem: Termination □

EXAMPLE 4.28 L: Finite Sequences of Natural Numbers ([KM87])

$$\begin{array}{lcl} f(x,nil) & \to & g(nil,x) \\ f(x,g(y,z)) & \to & g(f(x,y),z) \\ x \circ nil & \to & x \\ x \circ g(y,z) & \to & g(x \circ y,z) \\ null(nil) & \to & true \\ null(g(x,y)) & \to & false \\ mem(nil,y) & \to & false \\ mem(g(x,y),z) & \to & (y \equiv z) \vee mem(x,z) \\ mem(x,max(x)) & \to & \neg null(x) \\ max(g(g(nil,x),y)) & \to & max'(x,y) \\ max(g(g(g(x,y),z),u)) & \to & max'(max(g(g(x,y),z)),u) \end{array}$$

Problem: Divergence □

EXAMPLE 4.29 L: Merging ([Bou83])

$$
\begin{array}{lcl}
merge(x, nil) & \rightarrow & x \\
merge(nil, y) & \rightarrow & y \\
merge(x \circ y, u \circ v) & \rightarrow & x \circ merge(y, u \circ v) \\
merge(x \circ y, u \circ v) & \rightarrow & u \circ merge(x \circ y, v)
\end{array}
$$

Problem: Divergence □

EXAMPLE 4.30 L: Binary Trees

$$
\begin{array}{lcl}
f(nil) & \rightarrow & nil \\
f(nil.y) & \rightarrow & nil.f(y) \\
f((x.y).z) & \rightarrow & f(x.(y.z)) \\
g(nil) & \rightarrow & nil \\
g(x.nil) & \rightarrow & g(x).nil \\
g(x.(y.z)) & \rightarrow & g((x.y).z)
\end{array}
$$

Problem: Termination □

EXAMPLE 4.31 L: Purging ([Wal88])

$$
\begin{array}{lcl}
purge(nil) & \rightarrow & nil \\
purge(x.y) & \rightarrow & x.purge(remove(x, y)) \\
remove(x, nil) & \rightarrow & nil \\
remove(x, y.z) & \rightarrow & if(x \equiv y, remove(x, z), y.remove(x, z))
\end{array}
$$

Problem: Termination □

EXAMPLE 4.32 S: [Mar86]

$$
\begin{array}{lcl}
ab & \rightarrow & baa \\
bc & \rightarrow & cbb \\
ca & \rightarrow & acc \\
ua & \rightarrow & \lambda \\
vb & \rightarrow & \lambda \\
wc & \rightarrow & \lambda \\
au & \rightarrow & \lambda \\
bv & \rightarrow & \lambda \\
cw & \rightarrow & \lambda
\end{array}
$$

Problem: Divergence □

EXAMPLE 4.33 S: [KN85]

$$aba \rightarrow bab$$

Problem: Divergence & Termination □

EXAMPLE 4.34 S: [Der87]

$$ab \rightarrow bbaa$$

Problem: Termination □

EXAMPLE 4.35 S: [BL88]

$$ab \rightarrow acb$$

Problem: Termination □

EXAMPLE 4.36 S: [BL88]

$$aa \rightarrow aba$$

Problem: Termination □

EXAMPLE 4.37 S: [HP86]

$$aba \rightarrow ba$$

Problem: Divergence □

EXAMPLE 4.38 O: Associative Ring ([Hul80b])

$$\begin{array}{lcl} g(f(x,y),z) & \rightarrow & f(x,g(y,z)) \\ g(h(x,y),z) & \rightarrow & g(x,f(y,z)) \\ g(x,h(y,z)) & \rightarrow & h(g(x,y),z) \end{array}$$

Problem: Divergence □

EXAMPLE 4.39 O: [Ste88]

$$x * ((-y) * y) \rightarrow (-(y * y)) * x$$

Problem: Divergence □

EXAMPLE 4.40 O: Combinatory Logic ([Klo87])

$$\begin{array}{lcl} f(f(f(a,x),y),z) & \rightarrow & f(f(x,z),f(y,z)) \\ f(f(b,x),y) & \rightarrow & x \\ f(c,y) & \rightarrow & y \end{array}$$

Problem: Termination □

EXAMPLE 4.41 O: Forward Chains ([Der83a])

$$\begin{array}{lcl} f(a,y) & \rightarrow & f(y,g(y)) \\ g(a) & \rightarrow & b \\ g(b) & \rightarrow & b \end{array}$$

Problem: Termination □

EXAMPLE 4.42 O:

$$\begin{array}{lcl} f(a,g(y)) & \rightarrow & g(g(y)) \\ f(g(x),a) & \rightarrow & f(x,g(a)) \\ f(g(x),g(y)) & \rightarrow & h(g(y),x,g(y)) \\ h(g(x),y,z) & \rightarrow & f(y,h(x,y,z)) \\ h(a,y,z) & \rightarrow & z \end{array}$$

Problem: Termination □

EXAMPLE 4.43 O: [TJ89]

$$\begin{array}{lcl} x+0 & \to & x \\ x+s(y) & \to & s(x+y) \\ 0+y & \to & y \\ s(x)+y & \to & s(x+y) \\ x+(y+z) & \to & (x+y)+z \\ f(g(f(x))) & \to & f(h(s(0),x)) \\ f(g(h(x,y))) & \to & f(h(s(x),y)) \\ f(h(x,h(y,z))) & \to & f(h(x+y,z)) \end{array}$$

Problem: Divergence □

EXAMPLE 4.44 O: [Bac87]

$$\begin{array}{lcl} f(h(x)) & \to & f(i(x)) \\ g(i(x)) & \to & g(h(x)) \\ h(a) & \to & b \\ i(a) & \to & b \end{array}$$

Problem: Termination □

EXAMPLE 4.45 O: [Bau88]

$$\begin{array}{lcl} f(x,x) & \to & a \\ f(g(x),y) & \to & f(x,y) \end{array}$$

Problem: Divergence □

EXAMPLE 4.46 O: [BL88]

$$\begin{array}{lcl} f(a) & \to & f(b) \\ g(b) & \to & g(a) \end{array}$$

Problem: Termination □

EXAMPLE 4.47 O: [Die89]

$$\begin{array}{lcl} f(g(i(a,b,b'),c),d) & \to & if(e,f(b.c,d'),f(b'.c,d')) \\ f(g(h(a,b),c),d) & \to & if(e,f(b.g(h(a,b),c),d),f(c,d')) \end{array}$$

Problem: Termination □

EXAMPLE 4.48 O: Ternary Boolean Algebra ([Wos89])

$$\begin{array}{lcl} f(f(x,y,z),u,f(x,y,v)) & \to & f(x,y,f(z,u,v)) \\ f(x,y,y) & \to & y \\ f(x,y,g(y)) & \to & x \\ f(x,x,y) & \to & x \\ f(g(x),x,y) & \to & y \end{array}$$

Problem: Divergence □

EXAMPLE 4.49 O:

$$f(x,y,f(z,u,v)) \rightarrow f(f(x,y,z),u,f(x,y,v))$$

Problem: Termination □

EXAMPLE 4.50 O: [Toy87]

$$f(0,1,x) \rightarrow f(x,x,x)$$

Problem: Termination □

EXAMPLE 4.51 O: [Pla86]

$$\begin{array}{lcl} f(a) & \rightarrow & g(h(a)) \\ h(g(x)) & \rightarrow & g(h(f(x))) \\ k(x,h(x),a) & \rightarrow & h(x) \\ k(f(x),y,x) & \rightarrow & f(x) \end{array}$$

Problem: Termination □

EXAMPLE 4.52 O: [Nip89]

$$\begin{array}{lcl} s(a) & \rightarrow & a \\ s(s(x)) & \rightarrow & x \\ s(f(x,y)) & \rightarrow & f(s(y),s(x)) \\ s(g(x,y)) & \rightarrow & g(s(x),s(y)) \\ f(x,a) & \rightarrow & x \\ f(a,y) & \rightarrow & y \\ f(g(x,y),g(u,v)) & \rightarrow & g(f(x,u),f(y,v)) \\ g(a,a) & \rightarrow & a \end{array}$$

Problem: Divergence □

EXAMPLE 4.53 O:

$$\begin{array}{lcl} f(a) & \rightarrow & b \\ f(c) & \rightarrow & d \\ f(g(x,y)) & \rightarrow & g(f(x),f(y)) \\ f(h(x,y)) & \rightarrow & g(h(y,f(x)),h(x,f(y))) \\ g(x,x) & \rightarrow & h(e,x) \end{array}$$

Problem: Divergence □

EXAMPLE 4.54 O:

$$g(f(x,y)) \rightarrow f(f(g(g(x)),g(g(y))),f(g(g(x)),g(g(y))))$$

Problem: Termination □

EXAMPLE 4.55 O:

$$\begin{array}{lcl} f(x,x) & \rightarrow & f(a,b) \\ b & \rightarrow & c \end{array}$$

Problem: Termination □

EXAMPLE 4.56 O:

$$\begin{array}{lcl} f(a,b) & \to & f(a,c) \\ f(c,d) & \to & f(b,d) \end{array}$$

Problem: Termination □

EXAMPLE 4.57 O:

$$\begin{array}{lcl} f(x,y,z) & \to & g(x \leq y, x, y, z) \\ g(true,x,y,z) & \to & z \\ g(false,x,y,z) & \to & f(f(p(x),y,z), f(p(y),z,x), f(p(z),x,y)) \\ p(0) & \to & 0 \\ p(s(x)) & \to & x \end{array}$$

Problem: Termination □

EXAMPLE 4.58 O: Battle of Hydra and Hercules ([DJ89])

$$\begin{array}{lcl} f(x,g(y)) & \to & f(h(x), i(x,y)) \\ i(x,j(0,0)) & \to & g(0) \\ i(x,j(y,z)) & \to & j(g(y), i(x,z)) \\ i(h(x), j(j(y,z),0)) & \to & j(i(h(x), j(y,z)), i(x, j(y,z))) \\ j(g(x), g(y)) & \to & g(j(x,y)) \end{array}$$

Problem: Termination □

EXAMPLE 4.59 O: Quick-Sort ([Gra90])

$$\begin{array}{lcl} qsort(nil) & \to & nil \\ qsort(x.y) & \to & qsort(lowers(x,y)) \circ (x.qsort(greaters(x,y))) \\ lowers(x,nil) & \to & nil \\ lowers(x,y.z) & \to & if(y \leq x, y.lowers(x,z), lowers(x,z)) \\ greaters(x,nil) & \to & nil \\ greaters(x,y.z) & \to & if(y \leq x, greaters(x,z), y.greaters(x,z)) \end{array}$$

Problem: Termination □

EXAMPLE 4.60 O: Minimum-Sort ([Gra90])

$$\begin{array}{lcl} msort(nil) & \to & nil \\ msort(x.y) & \to & min(x,y).msort(del(min(x,y), x.y)) \\ min(x,nil) & \to & x \\ min(x,y.z) & \to & if(x \leq y, min(x,z), min(y,z)) \\ del(x,nil) & \to & nil \\ del(x,y.z) & \to & if(x \equiv y, z, y.del(x,z)) \end{array}$$

Problem: Termination □

EXAMPLE 4.61 O: Bubble-Sort ([Gra90])

$$
\begin{array}{lcl}
bsort(nil) & \rightarrow & nil \\
bsort(x.y) & \rightarrow & last(bubble(x.y)).bsort(butlast(bubble(x.y))) \\
bubble(nil) & \rightarrow & nil \\
bubble(x.nil) & \rightarrow & x.nil \\
bubble(x.(y.z)) & \rightarrow & if(x \leq y, y.bubble(x.z), x.bubble(y.z)) \\
last(nil) & \rightarrow & 0 \\
last(x.nil) & \rightarrow & x \\
last(x.(y.z)) & \rightarrow & last(y.z) \\
butlast(nil) & \rightarrow & nil \\
butlast(x.nil) & \rightarrow & nil \\
butlast(x.(y.z)) & \rightarrow & x.butlast(y.z)
\end{array}
$$

Problem: Termination □

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