Fachbereich Informatik
Universität Kaiserslautern
Postfach 3049
D-6750 Kaiserslautern

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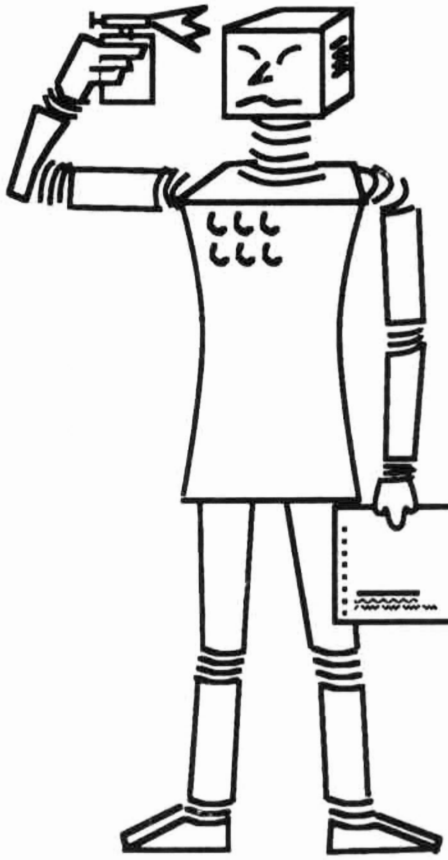

A STRUCTURAL ANALYSIS OF MODULAR TERMINATION OF TERM REWRITING SYSTEMS

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Bernhard Gramlich*
Fachbereich Informatik
Universität Kaiserslautern
Erwin Schrödinger Straße
W-6750 Kaiserslautern
Germany
gramlich@informatik.uni-kl.de

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Abstract

Modular properties of term rewriting systems, i.e. properties which are preserved under disjoint unions, have attracted an increasing attention within the last few years. Whereas confluence is modular this does not hold true in general for termination. By means of a careful analysis of potential counterexamples we prove the following abstract result. Whenever the disjoint union $\mathcal{R}_1 \oplus \mathcal{R}_2$ of two (finite) terminating term rewriting systems $\mathcal{R}_1$, $\mathcal{R}_2$ is non-terminating, then one of the systems, say $\mathcal{R}_1$, enjoys an interesting (undecidable) property, namely it is not termination preserving under non-deterministic collapses, i.e. $\mathcal{R}_1 \oplus \{G(x,y) \to x, G(x,y) \to y\}$ is non-terminating, and the other system $\mathcal{R}_2$ is collapsing, i.e. contains a rule with a variable right hand side. This result generalizes known sufficient syntactical criteria for modular termination of rewriting. Then we develop a specialized version of the 'increasing interpretation method' for proving termination of combinations of term rewriting systems. This method is applied to establish modularity of termination for certain classes of term rewriting systems. In particular, termination turns out to be modular for the class of systems, for which termination can be shown by simplification orderings (this result has recently been obtained by Kurihara & Ohuchi by a similar, but less general proof technique). Moreover, we show that the weaker property of being non-self-embedding which also implies termination is not modular. We prove that the finiteness restrictions in our main results concerning the term rewriting systems involved can be considerably weakened. Furthermore, we prove that the minimal rank of potential counterexamples in disjoint unions may be arbitrarily high. Hence, a general analysis of arbitrarily complicated 'mixed' term seems to be necessary when modularity of termination is investigated. Finally, we show that generalizations of our main results are possible for the cases of conditional term rewriting systems as well as for some restricted form of non-disjoint combinations of term rewriting systems involving common constructors.

Topics: Term Rewriting Systems, Termination, Modularity.

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Contents

1 Introduction

The question whether properties of combinations of term rewriting systems (TRSs for short) are inherited from the corresponding properties of the constituent TRSs is of great importance, e.g. in the field of abstract data type specifications. In principle and also for efficiency reasons it is very useful to know whether a combined TRS has some property whenever this property already holds for the single 'modules'. A simple and natural way of such 'modular' constructions is given by the concept of 'direct sum' ([24]) or 'disjoint union'.[1] Two TRSs $\mathcal{R}_1$ and $\mathcal{R}_2$ over disjoint signatures $\mathcal{F}_1$ and $\mathcal{F}_2$, respectively, are said to be *disjoint* if $\mathcal{F}_1$ and $\mathcal{F}_2$ are disjoint, i.e. $\mathcal{F}_1 \cap \mathcal{F}_2 = \emptyset$ (in that case the rule sets of $\mathcal{R}_1$ and $\mathcal{R}_2$ are necessarily disjoint, too). The (disjoint) union of two disjoint TRSs $\mathcal{R}_1$, $\mathcal{R}_2$ is denoted by $R_1 \oplus R_2$. We shall also speak of the disjoint union of $\mathcal{R}_1$ and $\mathcal{R}_2$ using the implicit convention that $\mathcal{R}_1$ and $\mathcal{R}_2$ are assumed to be disjoint TRSs. A property P of TRSs is said to be *modular* if the following holds for all disjoint TRSs R_1, R_2: $R_1 \oplus R_2$ has property P iff both R_1 and R_2 have property P. Toyama [24] has shown that confluence is modular. The termination property, however, is in general not modular as witnessed by the following counterexample of [24]:

Example 1.1 $\quad \mathcal{R}_1 : \quad f(a,b,x) \to f(x,x,x) \qquad \mathcal{R}_2 : \quad \begin{array}{l} G(x,y) \to x \\ G(x,y) \to y \end{array}$

Clearly, both R_1 and R_2 are terminating, but $\mathcal{R}_1 \oplus \mathcal{R}_2$ admits e.g. the following infinite derivation:

$$\begin{array}{rll} f(a,b,G(a,b)) & \to_{\mathcal{R}_1} & f(G(a,b),G(a,b),G(a,b)) \\ & \to_{\mathcal{R}_2} & f(a,G(a,b),G(a,b)) \\ & \to_{\mathcal{R}_2} & f(a,b,G(a,b)) \\ & \to_{\mathcal{R}_1} & \cdots \end{array}$$

Note, that in this example $\mathcal{R}_2$ is not confluent. Other, more complicated examples by Klop & Barendregt as well as by Toyama gathered in [23] show that $\mathcal{R}_1 \oplus \mathcal{R}_2$ may be non-terminating even if R_1 and R_2 are both terminating, confluent and interreduced. All these counterexamples have some common feature. Namely, one of the systems contains a duplicating rule, i.e. a rule $l \to r$ where some variable occurs strictly more often in r than in l, and the other system contains a collapsing rule $l' \to r'$, i.e. r' is a variable[2]. This observation was exploited by Rusinowitch [22] and Middeldorp [17] (see conditions (a)-(c) below). The counterexamples in [23], involving only two confluent systems R_1 and R_2, contained non-left-linear rules which turned out to be essential as shown by Toyama, Klop and Barendregt [25] (see condition (d) below). These results may be summarized as follows:

Given two disjoint TRSs R_1, R_2, their disjoint union $\mathcal{R}_1 \oplus \mathcal{R}_2$ is terminating, if R_1, R_2 are terminating and one of the following conditions is satisfied:

(a) Neither R_1 nor R_2 contains a duplicating rule [22].

(b) Neither R_1 nor R_2 contains a collapsing rule [22].

(c) One of the system $\mathcal{R}_1$, $\mathcal{R}_2$ contains neither collapsing nor duplicating rules [17].

(d) Both R_1 and R_2 are left-linear and confluent [25].

[1] Roughly spoken, the concept of 'direct sum' as defined in [24] is slightly more general than that of 'disjoint union' because it allows for renaming function symbols in order to obtain disjointness.

[2] A system without collapsing rules is said to be *collapse-free*.

As discussed in [20] conditions (a)-(c) together with example 1.1 provide a complete analysis for the termination of the disjoint union of two terminating TRSs $\mathcal{R}_1$, $\mathcal{R}_2$ in terms of the distribution of collapsing and duplicating rules among $\mathcal{R}_1$ and $\mathcal{R}_2$.

Condition (a) above implies that termination is modular for right-linear TRSs, in particular for string rewriting systems. Unfortunately, duplicating and collapsing rules occur quite often and naturally in many cases. Hence, practical applicability of conditions (a)-(c) is rather limited.

Some other interesting modularity properties of TRSs related to normal forms are investigated by Middeldorp in [16]. These results are generalized to (various versions of) conditional TRSs in [18], [19]. In particular, concerning the termination property it turns out that condition (b) above is still sufficient for ensuring termination of the disjoint union of conditional TRSs, but (a) and (c) are shown to hold only under the additional requirement that $\mathcal{R}_1$ and $\mathcal{R}_2$ are confluent.

Another interesting line of research is pursued by Kurihara & Kaji [13] where modular properties of TRSs w.r.t. a modified reduction relation are investigated. Essentially, this so-called 'modular reduction' requires that, given a disjoint union of several 'module' TRSs, successive reduction steps have to be performed in the same module as long as possible, i.e. until a normal form w.r.t. this module is obtained. Reduction to a normal form in one module is considered to be one step of 'modular reduction'.

Ganzinger & Giegerich [7] consider the termination property in restricted combinations of heterogeneous, i.e. many-sorted TRSs, where the involved signatures do not have to be completely disjoint. The disjointness requirement of combinations of TRSs is also relaxed in recent investigations of Kurihara & Ohuchi [15] and Middeldorp & Toyama [21]. This will be discussed later on.

Before going into details now, let us motivate and sketch our approach for analyzing modularity of termination. Having again a closer look on example 1.1 above and the non-terminating derivation indicated there, it is obvious that the collapsing R_2-steps using the rules $G(x,y) \to x, G(x,y) \to y$ play an essential role for enabling the derivation to be infinite. In fact, this observation can be generalized to arbitrary situations where we have terminating TRSs R_1, R_2 over disjoint signatures $\mathcal{F}_1$ and $\mathcal{F}_2$, respectively, such that the direct sum $\mathcal{R}_1 \oplus \mathcal{R}_2$ is non-terminating. In any infinite $(\mathcal{R}_1 \oplus \mathcal{R}_2)$-derivation $s_0 \to s_1 \to s_2 \to s_3 \to \ldots$, all the s_i's must be 'mixed' terms, i.e. involve function symbols from both signatures $\mathcal{F}_1$ and $\mathcal{F}_2$. We shall show that for any counterexample satisfying a certain minimality property concerning the 'layer structure' of its terms, one can construct from this counterexample an infinite derivation in $R_i \oplus \{G(x,y) \to x, G(x,y) \to y\}$ for $i = 1$ or $i = 2$, let's say for $i = 1$. This is achieved by an appropiate transformation from terms over $\mathcal{F}_1 \uplus \mathcal{F}_2$ into terms over $\mathcal{F}_1 \uplus \{A, G\}$ (here A is a new constant symbol and G is a new binary function symbol) which abstracts from the concrete form of $\mathcal{F}_2$-layers but retains enough relevant information for the translation of the reduction steps. This characteristic property of minimal counterexamples provides the basis for a couple of modularity results derived subsequently. It also corresponds nicely to the intuition that the existence of counterexamples crucially depends on 'non-deterministic collapsing' reduction steps. Hence, example 1.1 above is in a sense the simplest conceivable counterexample.

In the next section we briefly recall the basic notions, definitions and facts for TRSs needed later on. In section 3 the main results and their applications will be presented and discussed. In section 4 possible extensions and generalizations are developed.

2 Preliminaries

2.1 Basic Notations and Definitions

We briefly recall the basic terminology needed for dealing with TRSs (e.g. [12]). Let $\mathcal{V}$ be a countably infinite set of *variables* and $\mathcal{F}$ be a set of *function symbols* with $\mathcal{V} \cap \mathcal{F} = \emptyset$. Associated to every $f \in \mathcal{F}$ is a natural number denoting its arity. Function symbols of arity 0 are called *constants*. The set $\mathcal{T}(\mathcal{F}, \mathcal{V})$ of *terms* over $\mathcal{F}$ and $\mathcal{V}$ is the smallest set with (1) $\mathcal{V} \subseteq \mathcal{T}(\mathcal{F}, \mathcal{V})$ and (2) if $f \in \mathcal{F}$ has arity n and $t_1, \ldots, t_n \in \mathcal{T}(\mathcal{F}, \mathcal{V})$ then $f(t_1, \ldots, t_n) \in \mathcal{T}(\mathcal{F}, \mathcal{V})$. If some function symbols are allowed to be varyadic then the definition of $\mathcal{T}(\mathcal{F}, \mathcal{V})$ is generalized in an obvious way. The set of all *ground terms* (over $\mathcal{F}$), i.e. terms with no variables, is denoted by $\mathcal{T}(\mathcal{F})$. In the following we shall always assume that $\mathcal{T}(\mathcal{F})$ is non-empty, i.e. there is at least one constant in $\mathcal{F}$. Identity of terms is denoted by $\equiv$. The set of variables occurring in a term t is denoted by $V(t)$.

A *context* $C[, \ldots,]$ is a term with 'holes', i.e. a term in $\mathcal{T}(\mathcal{F} \uplus \{\Box\}, \mathcal{V})$ where $\Box$ is a new special constant symbol. If $C[, \ldots,]$ is a context with n occurrences of $\Box$ and $t_1, \ldots, t_n$ are terms then $C[t_1, \ldots, t_n]$ is the term obtained from $C[, \ldots,]$ by replacing from left to right the occurrences of $\Box$ by $t_1, \ldots, t_n$. A context containing precisely one occurrence of $\Box$ is denoted by $C[]$. For the set $\mathcal{T}(\mathcal{F} \uplus \{\Box\}, \mathcal{V})$ we also write $\mathcal{CON}(\mathcal{F}, \mathcal{V})$. A *non-empty* context is a term from $\mathcal{CON}(\mathcal{F}, \mathcal{V}) \setminus \mathcal{T}(\mathcal{F}, \mathcal{V})$ which is different from $\Box$. A term s is a *subterm* of a term t if there exists a context $C[]$ with $t \equiv C[s]$. If in addition $C[] \not\equiv \Box$ then s is a *proper* subterm of t. A substitution σ is a mapping from $\mathcal{V}$ to $\mathcal{T}(\mathcal{F}, \mathcal{V})$ such that its domain $dom(\sigma)$ $\{x \in \mathcal{V} | \sigma x \not\equiv x\}$ is finite. Its homomorphic extension to a mapping from $\mathcal{T}(\mathcal{F}, \mathcal{V})$ to $\mathcal{T}(\mathcal{F}, \mathcal{V})$ is also denoted by σ.

A *term rewriting system (TRS)* is a pair $(\mathcal{R}, \mathcal{F})$ consisting of a signature $\mathcal{F}$ and a set $\mathcal{R} \subseteq \mathcal{T}(\mathcal{F}, \mathcal{V}) \times \mathcal{T}(\mathcal{F}, \mathcal{V})$ of (rewrite) rules (l, r) denoted by $l \to r$ with $l \notin \mathcal{V}$ and $V(r) \subseteq V(l)$. [3]

Instead of $(\mathcal{R}, \mathcal{F})$ we also write $\mathcal{R}^{\mathcal{F}}$ or simply $\mathcal{R}$ when $\mathcal{F}$ is clear from the context or irrelevant. Given a TRS $\mathcal{R}^{\mathcal{F}}$ the rewrite relation $\to_{\mathcal{R}^{\mathcal{F}}}$ for terms $s, t \in \mathcal{T}(\mathcal{F}, \mathcal{V})$ is defined as follows: $s \to_{\mathcal{R}^{\mathcal{F}}} t$ if there exists a rule $l \to r \in \mathcal{R}$, a substitution σ and a context $C[]$ such that $s \equiv C[\sigma l]$ and $t \equiv C[\sigma r]$. We also write $\to_{\mathcal{R}}$ or simply $\to$ when $\mathcal{F}$ or $\mathcal{R}^{\mathcal{F}}$ is clear from the context, respectively. The symmetric, transitive and transitive-reflexive closures of $\to$ are denoted by $\leftrightarrow$, $\to^{+}$ and $\to^{*}$, respectively. Two terms s, t are *joinable in* $\mathcal{R}$, denoted by $s \downarrow_{\mathcal{R}} t$, if there exists a term u with $s \;{}_{\mathcal{R}}^{*}\!\!\leftarrow u \to_{\mathcal{R}}^{*} t$. A term s is *irreducible* or a *normal form* if there is no term t with $s \to t$. A TRS $\mathcal{R}$ is *terminating* or *strongly normalizing* if $\to$ is noetherian, i.e. if there is no infinite reduction sequence $s_1 \to s_2 \to s_3 \to \cdots$.

A *partial ordering* $>$ on a set D is a transitive and irreflexive binary relation on D. A partial ordering $>$ on $\mathcal{T}(\mathcal{F}, \mathcal{V})$ is said to be *monotonic (w.r.t. the term structure)* if it possesses the *replacement property*

$$s > t \quad \Longrightarrow \quad C[s] > C[t]$$

for all $s, t, C[]$. It is *stable (w.r.t. substitutions)* if

$$s > t \quad \Longrightarrow \quad \sigma s > \sigma t$$

for all s, t, σ. A *term ordering on* $\mathcal{T}(\mathcal{F}, \mathcal{V})$ is a monotonic and stable partial ordering on $\mathcal{T}(\mathcal{F}, \mathcal{V})$. A *reduction ordering* is a well-founded term ordering. A term ordering $>$ is said to be a simplification ordering if it additionally enjoys the subterm property

$$C[s] > s$$

[3]This restriction of excluding variable left-hand sides and right-hand side extra-variables is not a severe one. In particular, concerning termination of rewriting it only excludes trivial cases.

for any s and any non-empty context $C[]$.[4] The *homeomorphic embedding relation* $\trianglelefteq$ on terms is recursively defined by $s \equiv f(s_1, \ldots, s_m) \trianglelefteq g(t_1, \ldots, t_n) \equiv t$ if either $s \trianglelefteq t_i$ for some $i \in \{1, \ldots, n\}$ or $f \equiv g$ and $s_j \trianglelefteq t_{i_j}$ for all $j \in \{1, \ldots, m\}$ where $1 \le i_1 < i_2 < \cdots < i_m \le n$. A TRS $\mathcal{R}$ is said to be *self-embedding* if there exists a self-embedding $\mathcal{R}$-derivation, i.e. a reduction sequence $s_1 \to s_2 \to s_3 \to \cdots$ with $s_i \trianglelefteq s_j$ for some i, j with $i < j$.

A TRS is confluent if ${}^*\!\leftarrow \circ \to^* \subseteq \to^* \circ {}^*\!\leftarrow$ and locally confluent if $\leftarrow \circ \to \subseteq \to^* \circ {}^*\!\leftarrow$.[5] A confluent and terminating TRS is said to be *convergent* or *complete*.

2.2 Disjoint Unions

The following notations and definitions for dealing with disjoint unions of TRSs mainly follow [24] and [20].

Let $\mathcal{R}_1^{\mathcal{F}_1}$, $\mathcal{R}_2^{\mathcal{F}_2}$ be TRSs with disjoint signatures $\mathcal{F}_1$, $\mathcal{F}_2$. Their *disjoint union* $\mathcal{R}_1 \oplus \mathcal{R}_2$ is the TRS $(\mathcal{R}_1 \uplus \mathcal{R}_2, \mathcal{F}_1 \uplus \mathcal{F}_2)$.[6] A property $\mathcal{P}$ of TRSs is said to be *modular* if for all disjoint TRSs $\mathcal{R}_1^{\mathcal{F}_1}$, $\mathcal{R}_2^{\mathcal{F}_2}$ the following holds: $\mathcal{R}_1 \oplus \mathcal{R}_2$ has property $\mathcal{P}$ iff both $\mathcal{R}_1^{\mathcal{F}_1}$ and $\mathcal{R}_2^{\mathcal{F}_2}$ have property $\mathcal{P}$.

Let $t \equiv C[t_1, \ldots, t_n]$ with $C[, \ldots,] \not\equiv \Box$. We write $t \equiv C[\![t_1, \ldots, t_n]\!]$ if $C[, \ldots,] \in CON(\mathcal{F}_a, \mathcal{V})$ and $root(t_1), \ldots, root(t_n) \in \mathcal{F}_b$ for some $a, b \in \{1, 2\}$ with $a \neq b$. In this case the t_i's are the *principal subterms* or *principal aliens* of t. Note that every $t \in \mathcal{T}(\mathcal{F}_1 \uplus \mathcal{F}_2, \mathcal{V}) \setminus (\mathcal{T}(\mathcal{F}_1, \mathcal{V}) \cup \mathcal{T}(\mathcal{F}_2, \mathcal{V}))$ has a unique representation of the form $t \equiv C[\![t_1, \ldots, t_n]\!]$. The set of all principal subterms of t is denoted by *PS(t)*. The set *SS(t)* of *special subterms* or *aliens* of t is recursively defined by

$$
\begin{aligned}
SS_1(t) &= \{t\} \\
SS_{n+1}(t) &= \begin{cases} \emptyset & \text{if} \quad rank(t) = 1 \\ SS_n(t_1) \cup \ldots \cup SS_n(t_m) & \text{if} \quad t \equiv C[\![t_1, \ldots, t_m]\!] \end{cases}, \\
SS(t) &= \bigcup_{i \ge 1} SS_i(t)
\end{aligned}
$$

The *rank* of a term $t \in \mathcal{T}(\mathcal{F}_1 \uplus \mathcal{F}_2, \mathcal{V})$ is defined by

$$
rank(t) = \begin{cases} 1 & \text{if} \quad t \in \mathcal{T}(\mathcal{F}_1, \mathcal{V}) \cup \mathcal{T}(\mathcal{F}_2, \mathcal{V}) \\ 1 + max\{rank(t_i) | 1 \le i \le n\} & \text{if} \quad t \equiv C[\![t_1, \ldots, t_n]\!] \end{cases}
$$

An important basic fact about the rank of terms occurring in a $(\mathcal{R}_1 \oplus \mathcal{R}_2)$-derivation is the following ([24]): If $s \to^* t$ then $rank(s) \ge rank(t)$. Moreover, if $s \in \mathcal{T}(\mathcal{F}_1 \uplus \mathcal{F}_2, \mathcal{V})$ with $rank(s) = n$ then there exists a ground instance σs of s with $rank(\sigma s) = n$, too.[7] A (finite or infinite) derivation $D : s_1 \to s_2 \to s_3 \ldots$ is said to have rank n ($rank(D) = n$) if n is the minimal rank of all the s_i's, i.e. $n = \min\{rank(s_i) | 1 \le i\}$.

The topmost homogeneous part of a term $t \in \mathcal{T}(\mathcal{F}_1, \mathcal{V}) \cup \mathcal{T}(\mathcal{F}_2, \mathcal{V})$, denoted by $top(t)$, is obtained from t by replacing all principal subterms by $\Box$, i.e.

$$
top(t) = \begin{cases} t & \text{if} \quad rank(t) = 1 \\ C[, \ldots,] & \text{if} \quad t \equiv C[\![t_1, \ldots, t_n]\!] \end{cases}
$$

Furthermore we shall use the abbreviations $\mathcal{GT}_\oplus$ for $\mathcal{T}(\mathcal{F}_1 \uplus \mathcal{F}_2)$, $\mathcal{GT}_\oplus^n$ for $\{t \in \mathcal{GT}_\oplus | rank(t) = n\}$, $\mathcal{GT}_\oplus^{<n}$ for $\{t \in \mathcal{GT}_\oplus | rank(t) < n\}$ and $\mathcal{GT}_\oplus^{\le n}$ for $\{t \in \mathcal{GT}_\oplus | rank(t) \le n\}$.

[4] For the case that varyadic function symbols are allowed one additionally requires here the so-called 'deletion'-property (cf. Dershowitz [2]).

[5] Here, '$\circ$' denotes relation composition.

[6] The symbol '$\uplus$' denotes union of disjoint sets.

[7] This is easily verified by substituting appropriately $\mathcal{F}_1$- or $\mathcal{F}_2$-ground terms for those variables which occur in the 'deepest layer' of s.

For the sake of better readability the function symbols from $\mathcal{F}_1$ are considered to be black and those of $\mathcal{F}_2$ to be white. Variables have no colour. A *top black (white)* term has a black (white) root symbol.

For $s, t \in \mathcal{GT}_\oplus$ the one-step reduction $s \to t$ is said to be *inner* – denoted by $s \xrightarrow{i} t$ – if the reduction takes place in one of the principal subterms of s. Otherwise, we speak of an *outer* reduction step and write $s \xrightarrow{o} t$. A rewrite step $s \to t$ is ***destructive at level 1*** if the root symbols of s and t have different colours. The step $s \to t$ is ***destructive at level*** $n+1$ (for $n \geq 1$) if $s \equiv C[\![s_1, \ldots, s_j, \ldots, s_n]\!] \xrightarrow{i} C[s_1, \ldots, t_j, \ldots, s_n] \equiv t$ with $s_j \to t_j$ destructive at level n. Clearly, if a rewrite step is destructive (at some level) then the applied rewrite rule is collapsing, i.e. has a variable right-hand side. This is a basic fact which should be kept in mind subsequently.

For coding principal subterms, e.g. by new variables or constants, and for dealing with outer rewrite steps involving non-linear rules the following definitions are useful. For $s_1, \ldots, s_n, t_1, \ldots, t_n \in \mathcal{T}_\oplus$ we write $\langle s_1, \ldots, s_n \rangle \propto \langle t_1, \ldots, t_n \rangle$ if $t_i \equiv t_j$ whenever $s_i \equiv s_j$, for all $1 \leq i < j \leq n$. The conjunction of $\langle s_1, \ldots, s_n \rangle \propto \langle t_1, \ldots, t_n \rangle$ and $\langle t_1, \ldots, t_n \rangle \propto \langle s_1, \ldots, s_n \rangle$ is denoted by $\langle s_1, \ldots, s_n \rangle \infty \langle t_1, \ldots, t_n \rangle$. The following basic properties of outer and inner reduction steps will be freely used in the sequel:

- If $s \xrightarrow{o} t$ then $s \equiv C[\![s_1, \ldots, s_n]\!], t \equiv C'[s_{i_1}, \ldots, s_{i_m}]$ for some contexts $C[, \ldots,]$, $C'[, \ldots,], i_1, \ldots, i_m \in \{1, \ldots, n\}$ and terms $s_1, \ldots, s_n \in \mathcal{T}_\oplus$. If moreover $s \xrightarrow{o} t$ is not destructive at level 1 then $t \equiv C'[\![s_{i_1}, \ldots, s_{i_m}]\!]$.

- If $s \xrightarrow{i} t$ then $s \equiv C[\![s_1, \ldots, s_j, \ldots, s_n]\!]$ and $t \equiv C[s_1, \ldots, t_j, \ldots, s_n]$ for some context $C[, \ldots,], j \in \{1, \ldots, n\}$ and terms $s_1, \ldots s_n, t_j \in \mathcal{T}_\oplus$ with $s_j \to t_j$. If moreover $s \xrightarrow{i} t$ is not destructive at level 2 then $t \equiv C[\![s_1, \ldots, t_j, \ldots, s_n]\!]$.

- If $C[\![s_1, \ldots, s_n]\!] \xrightarrow{o} C'[\![s_{i_1}, \ldots, s_{i_m}]\!], 1 \leq i_j \leq n,\ j \in \{1, \ldots, m\}$, by application of some rule then $C[t_1, \ldots, t_n] \to C'[t_{i_1}, \ldots, t_{i_m}]$ by the same rule for all terms $t_1, \ldots, t_n$ with $\langle s_1, \ldots, s_n \rangle \propto \langle t_1, \ldots, t_n \rangle$.

3 Structural Properties of Minimal Counterexamples

3.1 Characterization of Minimal Counterexamples

Before formally stating and proving the main result we shall now illustrate the essential ideas and construction steps via an example from Drosten [6] which shows that termination need not be modular even for confluent TRSs.

Example 3.1

$$\mathcal{R}_1 : \begin{array}{rcl} f(a, b, x) & \to & f(x, x, x) \\ f(x, y, z) & \to & c \\ a & \to & c \\ b & \to & c \end{array} \qquad \mathcal{R}_2 : \begin{array}{rcl} K(x, y, y) & \to & x \\ K(y, y, x) & \to & x \end{array}$$

Here, both $\mathcal{R}_1$ and $\mathcal{R}_2$ are clearly terminating and confluent, but their disjoint union is non-terminating. For instance, we have the following infinite derivation:

$$\begin{array}{llll} D: & f(a, b, K(a, b, b)) & \xrightarrow{o}_{\mathcal{R}_1} f(K(a, b, b), K(a, b, b), K(a, b, b)) & (1) \\ & & \xrightarrow{i}_{\mathcal{R}_2} f(a, K(a, b, b), K(a, b, b)) & (2) \\ & & \xrightarrow{i}_{\mathcal{R}_1} f(a, K(c, b, b), K(a, b, b)) & (3) \\ & & \xrightarrow{i}_{\mathcal{R}_1} f(a, K(c, c, b), K(a, b, b)) & (4) \\ & & \xrightarrow{i}_{\mathcal{R}_2} f(a, b, K(a, b, b)) & (5) \\ & & \xrightarrow{o}_{\mathcal{R}_1} \ldots & \end{array}$$

Obviously, the crucial steps which enable this derivation to be infinite (and even cyclic) are the inner reductions (2)-(5), in particular the steps (2) and (5) which are destructive at level 2. They modify substantially the topmost homogeneous black [8] layer thereby enabling an outer reduction step previously not possible. The idea now is to abstract from the concrete form of these inner steps but retain the essential information which permits subsequent outer steps. For that purpose it is sufficient to consider the principal top white, i.e. $\mathcal{F}_2$-rooted, aliens and collect those top black, i.e. $\mathcal{F}_1$-rooted, terms to which the former may reduce. In other words, colour changing derivations issued by principal aliens are essential. The coding of the collected top black successors of some principal top white alien will be achieved by some new function symbol(s) which in a sense serve(s) for abstracting from the concrete form of white layers while keeping only the 'layer separating' information. Since in general also top black aliens hidden in deeper layers (cf. subsection 3.4 below) may eventually become principal top black aliens the whole process has to be performed in a recursive fashion in general (which is not necessary in the example). After this abstracting transformation process sequences of inner reduction steps like (2)-(5) above in the original derivation may be simulated by ('deletion' and) 'subterm' steps in the transformed derivation. In order to explain this in more detail let us choose H as a (varyadic) new layer separating function symbol. Then we get the transformed derivation

$$\begin{array}{llll} D': & f(a,b,H(a,b,c)) & \xrightarrow{o}_{\mathcal{R}_1} f(H(a,b,c),H(a,b,c),H(a,b,c)) & (1') \\ & & \xrightarrow{i}_{\mathcal{R}_2'} f(a,H(a,b,c),H(a,b,c)) & (2') \\ & & \xrightarrow{i}_{\mathcal{R}_2'} f(a,H(b,c),H(a,b,c)) & (3') \\ & & \equiv \quad f(a,H(b,c),H(a,b,c)) & (4') \\ & & \xrightarrow{i}_{\mathcal{R}_2'} f(a,b,H(a,b,c)) & (5') \\ & & \xrightarrow{o}_{\mathcal{R}_1} \cdots & \end{array}$$

where $\mathcal{R}_1$ is as above and $\mathcal{R}_2' = \mathcal{R}_{sub}^H \cup \mathcal{R}_{del}^H$ with

$$\begin{array}{lll} \mathcal{R}_{sub}^H & = & \{H(x_1,\ldots,x_j,\ldots,x_n) \to x_j \mid 1 \le j \le n\}\ , \\ \mathcal{R}_{del}^H & = & \{H(x_1,\ldots,x_j,\ldots,x_n) \to H(x_1,\ldots,x_{j-1},x_{j+1},\ldots,x_n) \mid 1 \le j \le n\}. \end{array}$$

The top white principal alien $t := K(a,b,b)$ of the top white starting term $s := f(a,b,K(a,b,b))$ of D can be reduced (in arbitrarily many steps) to the top black successors a, b and c. Hence, the abstracting transformation of t yields $H(a,b,c)$ and the whole starting term s is transformed into $f(a,b,H(a,b,c))$. Furthermore, any outer step in D corresponds to an outerstep in D' using the same rule. Any inner step in D which is not destructive at level 2, e.g. (3) and (4), corresponds in D' to a (possibly empty) sequence of inner $\mathcal{R}_{del}^H$-steps not destructive at level 2 (here (3') and (4'), respectively). Any inner step in D which is destructive at level 2 (hence collapsing), e.g. (2) and (5), corresponds in D' to an $\mathcal{R}_{sub}^H$-step (here (2') and (5'), respectively).

In order to stay within the usual scenario of fixed-arity function symbols we modify the above transformation by taking a new binary function symbol G and a new constant A instead of the varyadic symbol H. With the correspondence

$$H(t_1,\ldots,t_n) = \begin{cases} A & \text{if } \ n = 0 \\ G(t_1, G(t_2, \ldots G(t_{n-1}, G(t_n, A)) \ldots)) & \text{if } \ n > 0 \end{cases}$$

[8] Remember that function symbols from $\mathcal{R}_1$ and $\mathcal{R}_2$ are considered to be black and white, respectively.

the above construction easily carries over and we obtain the derivation

$$
\begin{array}{llr}
D'': & f(a,b,G(a,G(b,G(c,A)))) & \\
\stackrel{o}{\rightarrow}_{\mathcal{R}_1} & f(G(a,G(b,G(c,A))),G(a,G(b,G(c,A))),G(a,G(b,G(c,A)))) & (1'') \\
\stackrel{i}{\rightarrow}_{\mathcal{R}_2''} & f(a,G(a,G(b,G(c,A))),G(a,G(b,G(c,A)))) & (2'') \\
\stackrel{i}{\rightarrow}_{\mathcal{R}_2''} & f(a,G(b,G(c,A)),G(a,G(b,G(c,A)))) & (3'') \\
\equiv & f(a,G(b,G(c,A)),G(a,G(b,G(c,A)))) & (4'') \\
\stackrel{i}{\rightarrow}_{\mathcal{R}_2''} & f(a,b,G(a,G(b,G(c,A)))) & (5'') \\
\stackrel{o}{\rightarrow}_{\mathcal{R}_1} & \ldots &
\end{array}
$$

Here, $\mathcal{R}_2''$ is to be interpreted as $\mathcal{R}_2'' = \mathcal{R}_{sub}^G$ with

$$\mathcal{R}_{sub}^G = \{G(x,y) \to x,\ G(x,y) \to y\},$$

i.e. deletion rules are not necessary any more. In the following formal presentation we shall use the latter transformation.

Definition 3.2 *A TRS $\mathcal{R}$ is said to be* termination preserving under non-deterministic collapses *if termination of $\mathcal{R}$ implies termination of $\mathcal{R} \oplus \{G(x,y) \to x, G(x,y) \to y\}$.*

Lemma 3.3 *Let $\mathcal{R}_1$, $\mathcal{R}_2$ be two ¡terminating disjoint TRSs such that*

$$D : s_1 \to s_2 \to s_3 \to \ldots$$

is an infinite derivation in $\mathcal{R}_1 \oplus \mathcal{R}_2$ (involving only ground terms) of minimal rank, i.e. any derivation in $\mathcal{R}_1 \oplus \mathcal{R}_2$ of smaller rank is finite. Then we have:

(a) $rank(D) \geq 3$.

(b) Infinitely many steps in D are outer steps.

(c) Infinitely many steps in D are inner reductions which are destructive at level 2.

Proof:

(a) Follows from (c) since whenever $s_i \stackrel{i}{\rightarrow} s_{i+1}$ is destructive at level 2 then $rank(s_i) \geq 3$.

(b) Assume for a proof by contradiction that only finitely many steps in D are outer ones. W.l.o.g. we may further assume that no step in D is an outer one. Hence, for $s_1 \equiv C[\![t_1,\ldots,t_n]\!]$ all reductions in D are inner ones and take place below one of the positions of the s_i's. Since D is infinite we conclude by the pigeon hole principle that at least one of the s_i's initiates an infinite derivation whose rank is smaller than $rank(D)$. But this is a contradiction to the minimality assumption concerning $rank(D)$.

(c) For a proof by contradiction assume w.l.o.g. that no inner step in D is destructive at level 2. Then, With $\widetilde{s_i} := top(s_i)$ any outer step $s_i \stackrel{o}{\rightarrow} s_{i+1}$ in D yields $\widetilde{s_i} \to \widetilde{s_{i+1}}$ using the same rule from $\mathcal{R}_1 \oplus \mathcal{R}_2$ and for every inner step $s_i \stackrel{i}{\rightarrow} s_{i+1}$ we have $\widetilde{s_i} \equiv \widetilde{s_{i+1}}$. Assuming w.l.o.g. that all the s_i's are top black, i.e. $\mathcal{F}_1$-rooted, we can conclude by (b) that $\mathcal{R}_1$ is non-terminating which yields a contradiction.

■

Next we formalize the transformation process illustrated above.

Definition 3.4 *Let $\mathcal{R}_1$, $\mathcal{R}_2$ be two terminating disjoint TRSs, $\mathcal{R} := \mathcal{R}_1 \oplus \mathcal{R}_2$ and $n \in \mathbb{N}$ such that for every $s \in \mathcal{T}(\mathcal{F}_1 \uplus \mathcal{F}_2)$ with $rank(s) \leq n$ there is no infinite $\mathcal{R}$-derivation starting with s. Moreover, let $<_{\mathcal{T}(\mathcal{F}_1 \uplus \{A,G\})}$ be some arbitrary, but fixed total ordering on $\mathcal{T}(\mathcal{F}_1 \uplus \{A,G\})$. Then the $\mathcal{F}_2$- (or* white*)* abstraction *is defined to be the mapping*

$$\Phi : \mathcal{GT}_{\oplus}^{\leq n} \uplus \{t \in \mathcal{GT}_{\oplus}^{n+1} | root(t) \in \mathcal{F}_1\} \quad \longrightarrow \quad \mathcal{T}(\mathcal{F}_1 \uplus \{A,G\})$$

given by

$$\Phi(t) := \begin{cases} t & \text{if} \quad t \in \mathcal{T}(\mathcal{F}_1) \\ A & \text{if} \quad t \in \mathcal{T}(\mathcal{F}_2) \\ C[\![\Phi(t_1), \ldots, \Phi(t_m)]\!] & \text{if} \quad t \equiv C[\![t_1, \ldots, t_m]\!], m \geq 1, root(t) \in \mathcal{F}_1 \\ CONS(SORT(\Phi^*(SUCC^{\mathcal{F}_1}(t)))) & \text{if} \quad t \equiv C[\![t_1, \ldots, t_m]\!], m \geq 1, root(t) \in \mathcal{F}_2 \end{cases}$$

with

$$\begin{aligned} SUCC^{\mathcal{F}_1}(t) &:= \{t' \in \mathcal{T}(\mathcal{F}_1 \uplus \mathcal{F}_2) | t \rightarrow_{\mathcal{R}}^* t', root(t') \in \mathcal{F}_1\}, \\ \Phi^*(M) &:= \{\Phi(t) | t \in M\} \quad \text{for} \quad M \subseteq dom(\Phi), \\ CONS(\langle\rangle) &:= A, \\ CONS(\langle s_1, \ldots, s_{k+1}\rangle) &:= G(s_1, CONS(\langle s_2, \ldots, s_{k+1}\rangle)) \quad \text{and} \\ SORT(\{s_1, \ldots, s_k\} &:= \langle s_{\pi(1)}, \ldots, s_{\pi(k)}\rangle, \end{aligned}$$

such that $s_{\pi(j)} \leq_{\mathcal{T}(\mathcal{F}_1 \uplus \{A,G\})} s_{\pi(j+1)}$ for $1 \leq j < k$.

Intuitively, for computing $\Phi(t)$ one proceeds top-down in a recusive fashion. Top black layers are left invariant whereas (for the case of top black t) the principal top white subterms are transformed by computing for every such top white subterm the set of possible top black successors, abstracting the resulting terms recursively, sorting the resulting set of abstracted terms and finally constructing again an ordinary term by means of using the new constant symbol A (for empty arguments sets) and the new binary function symbol G (for non-empty argument sets). The sorting process and the total ordering involved here are due to some proof-technical subtleties which will become clear later on. For illustration let us consider again example 3.1.

Example 3.5 *(example 3.1 continued) Here the white abstraction of the s_i's in the original derivation D yields (using alphabetical sorting)*

$$\begin{aligned} \Phi(s_1) &= \Phi(f(a,b,K(a,b,b))) = f(a,b,\Phi(K(a,b,b))) \\ &= f(a,b,CONS(SORT(\Phi^*(SUCC^{\mathcal{F}_1}(K(a,b,b)))))) \\ &= f(a,b,CONS(SORT(\Phi^*(\{a,b,c\})))) = f(a,b,CONS(SORT(\{a,b,c\}))) \\ &= f(a,b,CONS(\langle a,b,c\rangle\})) = f(a,b,G(a,G(b,G(c,A)))) \quad , \\ \Phi(s_3) &= \Phi(f(a,K(a,b,b),K(a,b,b))) = f(a,\Phi(K(a,b,b),\Phi(K(a,b,b)))) \\ &= f(a,CONS(SORT(\Phi^*(\{a,b,c\}))),CONS(SORT(\Phi^*(\{a,b,c\})))) \\ &= f(a,CONS(SORT(\{a,b,c\})),CONS(SORT(\{a,b,c\}))) \\ &= f(a,CONS(\langle a,b,c\rangle\}),CONS(\langle a,b,c\rangle\})) \\ &= f(a,G(a,G(b,G(c,A))),G(a,G(b,G(c,A)))) \quad \text{and} \\ \Phi(s_4) &= \Phi(f(a,K(c,b,b),K(a,b,b))) = f(a,\Phi(K(c,b,b),\Phi(K(a,b,b)))) \\ &= f(a,CONS(SORT(\Phi^*(\{b,c\}))),CONS(SORT(\Phi^*(\{a,b,c\})))) \\ &= f(a,CONS(SORT(\{b,c\})),CONS(SORT(\{a,b,c\}))) \\ &= f(a,CONS(\langle b,c\rangle\}),CONS(\langle a,b,c\rangle\})) = f(a,G(b,G(c,A)),G(a,G(b,G(c,A)))). \end{aligned}$$

Note that the subterm rewrite step $s_3 \rightarrow s_4$ reducing $G(a,G(b,G(c,A)))$ to $G(b,G(c,A))$ would not have been possible if we had sorted $\{b,c\}$ as $\langle b,c\rangle$ and $\{a,b,c\}$ as $\langle c,b,a\rangle$.

In the following we shall implicitly use the convention that notions like *rank* or *inner* and *outer* reduction steps have to be interpreted w.r.t. some specific disjoint union which is clear from the context.

The next lemmas capture the important properties of the above defined abstracting transformation.

Lemma 3.6 *Let $\mathcal{R}_1$, $\mathcal{R}_2$, $\mathcal{R} := \mathcal{R}_1 \oplus \mathcal{R}_2$, n and Φ be given as in definition 3.4. Then, for any $s, t \in \mathcal{T}(\mathcal{F}_1 \uplus \mathcal{F}_2)$ with $rank(s) \leq n$ and $root(s) \in \mathcal{F}_2$ we have:*

$$s \rightarrow_{\mathcal{R}} t \quad \Longrightarrow \quad \Phi(s) \rightarrow^*_{\mathcal{R}'_2} \Phi(t),$$

where $\mathcal{R}'_2 := \mathcal{R}^G_{sub} := \{G(x,y) \rightarrow x, G(x,y) \rightarrow y\}$.

Proof: Let s, t be given with $rank(s) \leq n$, $root(s) \in \mathcal{F}_2$ and $s \rightarrow t$. We distinguish two cases. For $rank(s) = 1$ we have $s, t \in \mathcal{T}(\mathcal{F}_2)$, hence $\Phi(s) = A = \Phi(t)$ by definition of Φ. If $rank(s) > 1$ then s has the form $s \equiv C[\![s_1, \ldots, s_n]\!], n \geq 1$. By the recursive case of definition 3.4 this implies $\Phi(s) = CONS(SORT(\Phi^*(SUCC^{\mathcal{F}_1}(s))))$. From $s \rightarrow t$ we obtain $SUCC^{\mathcal{F}_1}(s) \supseteq SUCC^{\mathcal{F}_1}(t)$, hence $\Phi^*(SUCC^{\mathcal{F}_1}(s)) \supseteq \Phi^*(SUCC^{\mathcal{F}_1}(t))$. By definition of $SORT$ and $CONS$, finally, we get $\Phi(t) \rightarrow^*_{\mathcal{R}'_2} \Phi(t)$. [9] ■

Lemma 3.7 *Let $\mathcal{R}_1$, $\mathcal{R}_2$ and Φ be given as in definition 3.4. Then, Φ is rank decreasing, i.e. for any $s \in dom(\Phi) := \mathcal{GT}^{\leq n}_{\oplus} \uplus \{t \in \mathcal{GT}^{n+1}_{\oplus} | root(t) \in \mathcal{F}_1\}$ we have $rank(\Phi(s)) \leq rank(s)$.*

Proof: By an easy induction on $rank(s)$ using the definition of Φ. ■

Inner and outer reduction steps in $(\mathcal{R}_1 \oplus \mathcal{R}_2)$-derivations can be translated into corresponding (sequences of) inner and outer steps in the transformed derivation in $\mathcal{R}_1 \oplus \{G(x,y) \rightarrow x, G(x,y) \rightarrow y\}$ as described by

Lemma 3.8 *Let $\mathcal{R}_1$, $\mathcal{R}_2$, $\mathcal{R} = \mathcal{R}_1 \oplus \mathcal{R}_2$, $\mathcal{R}'_2 = \mathcal{R}^G_{sub} = \{G(x,y) \rightarrow x, G(x,y) \rightarrow y\}$, n and the $\mathcal{F}_2$-abstraction Φ be given as above. Then, for any $s, t \in \mathcal{T}(\mathcal{F}_1 \uplus \mathcal{F}_2)$ with $rank(s) \leq n + 1$, $root(s) \in \mathcal{F}_1$ and $s \rightarrow_{\mathcal{R}} t$ we have:*

- *(a) If $s \xrightarrow{o}_{\mathcal{R}_1} t$ is not destructive at level 1 then $\Phi(s) \xrightarrow{o}_{\mathcal{R}_1} \Phi(t)$ using the same $\mathcal{R}_1$-rule, and moreover this step is also not destructive at level 1.*
- *(b) If $s \xrightarrow{o}_{\mathcal{R}_1} t$ is destructive at level 1 then $\Phi(s) \xrightarrow{o}_{\mathcal{R}_1} \Phi(t)$ using the same $\mathcal{R}_1$-rule, and moreover this step is also destructive at level 1.*
- *(c) If $s \xrightarrow{i}_{\mathcal{R}} t$ is not destructive at level 2 then $\Phi(s) \xrightarrow{i}{}^*_{\mathcal{R}'_2} \Phi(t)$ with all steps not destructive at level 2.*
- *(d) If $s \xrightarrow{i}_{\mathcal{R}} t$ is destructive at level 2 then $\Phi(s) \xrightarrow{i}{}^+_{\mathcal{R}'_2} \Phi(t)$ such that exactly one of these steps is destructive at level 2.*

Proof: Under the assumptions of the lemma assume that $s, t \in \mathcal{T}(\mathcal{F}_1 \uplus \mathcal{F}_2)$ are given with $rank(s) \leq n + 1$, $root(s) \in \mathcal{F}_1$ and $s \rightarrow_{\mathcal{R}} t$.

- (a) If $s \xrightarrow{o}_{\mathcal{R}_1} t$ is not destructive at level 1 then we have $s \equiv C[\![s_1, \ldots, s_m]\!]$, $t \equiv C'[\![s_{i_1}, \ldots, s_{i_k}]\!]$, $1 \leq i_j \leq m$, $1 \leq j \leq k$ for some contexts C, C'. By definition of Φ this implies $\Phi(s) = C[\![\Phi(s_1), \ldots, \Phi(s_m)]\!]$ and $\Phi(t) = C'[\![\Phi(s_{i_1}), \ldots, \Phi(s_{i_k})]\!]$, hence also $\Phi(s) \xrightarrow{o}_{\mathcal{R}_1} \Phi(t)$ using the same $\mathcal{R}_1$-rule because of $\langle s_1, \ldots, s_m \rangle \propto \langle \Phi(s_1), \ldots, \Phi(s_m) \rangle$. Clearly, $\Phi(s) \xrightarrow{o}_{\mathcal{R}_1} \Phi(t)$ is not destructive at level 1, too.

[9] The sorting process involved here is needed for ensuring that $\Phi(t)$ is homeomorphically embedded in $\Phi(s)$, or more precisely, that $\Phi(t)$ can be obtained from $\Phi(s)$ by applying subterm rules from $\mathcal{R}^G_{sub}$.

(b) If $s \xrightarrow{o}_{\mathcal{R}_1} t$ is destructive at level 1 then we have $s \equiv C[\![s_1, \ldots, s_m]\!]$, $t \equiv s_j$ for some j with $1 \leq j \leq m$ and some context C. By definition of Φ this implies $\Phi(s) = C[\![\Phi(s_1), \ldots, \Phi(s_m)]\!]$ and $\Phi(t) = \Phi(s_j)$, hence also $\Phi(s) \xrightarrow{o}_{\mathcal{R}_1} \Phi(t)$ using the same $\mathcal{R}_1$-rule because of $\langle s_1, \ldots, s_m \rangle \propto \langle \Phi(s_1), \ldots, \Phi(s_m) \rangle$. Clearly, $\Phi(s) \xrightarrow{o}_{\mathcal{R}_1} \Phi(t)$ is destructive at level 1, too.

(c) If $s \xrightarrow{i}_{\mathcal{R}} t$ is not destructive at level 2 then we have $s \equiv C[\![s_1, \ldots, s_j, \ldots, s_m]\!]$, $t \equiv C[\![s_1, \ldots, s'_j, \ldots, s_m]\!], s_j \rightarrow_{\mathcal{R}} s'_j$ for some j with $1 \leq j \leq m$ and some context C. By definition of Φ this implies $\Phi(s) = C[\![\Phi(s_1), \ldots, \Phi(s_j), \ldots, \Phi(s_m)]\!]$ and $\Phi(t) = C[\![\Phi(s_1), \ldots, \Phi(s'_j), \ldots, \Phi(s_m)]\!]$. Since s_j, s'_j are top white, i.e. $\mathcal{F}_2$-rooted, we get $\Phi(s_j) = A = \Phi(s'_j)$ for the case $s_j \in \mathcal{T}(\mathcal{F}_2)$ and
$\Phi(s_j) = CONS(SORT(\Phi^*(SUCC^{\mathcal{F}_1}(s_j))))$,
$\Phi(s'_j) = CONS(SORT(\Phi^*(SUCC^{\mathcal{F}_1}(s'_j))))$, otherwise. Since $s_j \rightarrow_{\mathcal{R}} s'_j$ this implies $SUCC^{\mathcal{F}_1}(s_j) \supseteq SUCC^{\mathcal{F}_1}(s'_j)$, hence $\Phi(s_j) \rightarrow^*_{\mathcal{R}'_2} \Phi(s'_j)$ and also $\Phi(s) \xrightarrow{i}{}^*_{\mathcal{R}'_2} \Phi(t)$ with no step destructive at level 2.

(d) If $s \xrightarrow{i}_{\mathcal{R}} t$ is destructive at level 2 then we have $s \equiv C[\![s_1, \ldots, s_j, \ldots, s_m]\!]$, $t \equiv C[s_1, \ldots, s'_j, \ldots, s_m]$ with $s_j \rightarrow_{\mathcal{R}} s'_j$ colour changing for some j with $1 \leq j \leq m$ and some context C. By definition of Φ this implies $\Phi(s) = C[\![\Phi(s_1), \ldots, \Phi(s_j), \ldots, \Phi(s_m)]\!]$ and $\Phi(t) = C[\Phi(s_1), \ldots, \Phi(s'_j), \ldots, \Phi(s_m)]$. Moreover, $s'_j \in SUCC^{\mathcal{F}_1}(s_j)$, hence $\Phi(s) \xrightarrow{i}{}^+_{\mathcal{R}'_2} \Phi(t)$. In this derivation there is exactly one (inner) step which is destructive at level 2, namely the last one.

■

Now we are prepared to state and prove the main result of this section.

Theorem 3.9 *Let $\mathcal{R}_1, \mathcal{R}_2$ be two disjoint (finite) TRSs which are both terminating such that their disjoint union $\mathcal{R}_1 \oplus \mathcal{R}_2$ is non-terminating. Then $\mathcal{R}_j$ is not termination preserving under non-deterministic collapses for some $j \in \{1,2\}$ and the other system $\mathcal{R}_k$, $k \in \{1,2\} \setminus \{j\}$ is collapsing. Moreover, the minimal rank of counterexamples in $\mathcal{R}_j \oplus \{G(x,y) \rightarrow x, G(x,y) \rightarrow y\}$ is less than or equal to the minimal rank of counterexamples in $\mathcal{R}_1 \oplus \mathcal{R}_2$.*

Proof: Let $\mathcal{R}_1, \mathcal{R}_2$ with $\mathcal{R} := \mathcal{R}_1 \oplus \mathcal{R}_2$ be given as stated above. We consider a minimal counterexample, i.e. an infinite $\mathcal{R}$-derivation

$$D: \quad s_1 \rightarrow s_2 \rightarrow s_3 \rightarrow \quad \ldots$$

of minimal rank, let's say $n+1$. W.l.o.g. we may assume that all the s_i's are top black, i.e. $\mathcal{F}_1$-rooted ground terms having $rank\ n+1$. Since the preconditions of definition 3.4 are satisfied we may apply the white ($\mathcal{F}_2$-) abstraction function Φ to the s_i's. As it will be shown this yields an infinite $\mathcal{R}'$-derivation

$$D': \quad \Phi(s_1) \rightarrow^* \Phi(s_2) \rightarrow^* \Phi(s_3) \rightarrow^* \quad \ldots$$

where $\mathcal{R}' := \mathcal{R}_1 \oplus \mathcal{R}'_2$ with $\mathcal{R}'_2 := \mathcal{R}^G_{sub} = \{G(x,y) \rightarrow x, G(x,y) \rightarrow y\}$.
Using lemma 3.8 we conclude that for any step $s_j \rightarrow s_{j+1}$ in D we have

$$\begin{aligned} s_j \xrightarrow{o}_{\mathcal{R}_1} s_{j+1} &\implies \Phi(s_j) \xrightarrow{o}_{\mathcal{R}_1} \Phi(s_{j+1}), \\ s_j \xrightarrow{i}_{\mathcal{R}} s_{j+1} &\implies \Phi(s_j) \xrightarrow{i}{}^*_{\mathcal{R}'_2} \Phi(s_{j+1}). \end{aligned}$$

Hence, D' is indeed an $\mathcal{R}'$-derivation. Since according to lemma 3.3 (b) infinitely many steps in D are outer ones, the derivation D' is infinite, too. But this means that $\mathcal{R}_1$ is not termination preserving under non-deterministic collapses. Moreover, lemma 3.3 (c) implies that $\mathcal{R}_2$ is collapsing. This can also be inferred more directly by observing that for non-collapsing $\mathcal{R}_2$ the $\mathcal{F}_2$-abstraction of principal subterms of a minimal counterexample always yields the constant A which implies that the transformed infinite derivation is an $\mathcal{R}_1$-derivation contradicting termination of $\mathcal{R}_1$. Lemma 3.7 finally implies $rank(D') \leq rank(D)$ which finishes the proof. ■

As an immediate consequence of this result we obtain

Corollary 3.10 *Termination (and hence also completeness) is modular for the class of (finite) TRSs which are termination preserving under non-deterministic collapses.*

By observing that $\mathcal{R}^G_{sub}$ is termination preserving under non-deterministic collapses we even get

Corollary 3.11 *The disjoint union of two (finite) terminating TRSs is again terminating whenever one of the systems is termination preserving under non-deterministic collapses and non-collapsing.*

The next result shows that the class of TRSs which are termination preserving under non-deterministic collapses comprises all non-duplicating TRSs.

Lemma 3.12 *Whenever a (finite) TRS is non-duplicating then it is termination preserving under non-deterministic collapses.*

Proof: Let $\mathcal{R}_1$ be a non-duplicating and terminating TRS. Then consider $\mathcal{R} := \mathcal{R}_1 \oplus \mathcal{R}_2$ with $\mathcal{R}_2 := \mathcal{R}^G_{sub} := \{G(x,y) \to x, G(x,y) \to y\}$. We define the term ordering $>$ on $\mathcal{T}(\mathcal{F}_1 \uplus \mathcal{F}_2, \mathcal{V})$ by lexicographically combining $\to^+_{\mathcal{R}_1}$ and the ordering $>_G$ which counts occurrences of G as follows: $s >_G t \,:\Longleftrightarrow\, oc(G,s) >_{nat} oc(G,t)$, $s \approx_G t \,:\Longleftrightarrow\, oc(G,s) =_{nat} oc(G,t)$[10] and $> := lex(>_G, \to^+_{\mathcal{R}_1})$. The form of $\mathcal{R}_2$ and the fact that $\mathcal{R}_1$ is non-duplicating implies $s \to_{\mathcal{R}} t \implies s \geq_G t$ and $s \to_{\mathcal{R}_2} t \implies s >_G$. Since both $\to^+_{\mathcal{R}_1}$ and $>_G$ are well-founded term orderings the lexicographic combination $>$ is well-founded, too. Moreover, $>$ is monotonic w.r.t. replacement and $> \cap \to^+_{\mathcal{R}_1}$ is stable w.r.t. substitutions. Hence it suffices to show $l > r$ for any rule $l \to r \in \mathcal{R}$. The case $l \to_{\mathcal{R}_2} r$ is trivial. For $l \to_{\mathcal{R}_1} r$ we have $l \approx_G r$, $l \to^+_{\mathcal{R}_1} r$ and hence $l > r$. This shows that $\mathcal{R}$ is terminating, i.e. $\mathcal{R}_1$ is termination preserving under non-deterministic collapses. ■

Theorem 3.9, corollaries 3.10, 3.11 and lemma 3.12 constitute a generalization of the main results of [22] and [17].

Theorem 3.9 corresponds nicely to the intuition that the existence of counterexamples crucially depends on 'non-deterministic collapsing' reduction steps. Hence, example 1.1 above is in a sense the simplest conceivable counterexample.

On the one side the general result stated in theorem 3.9 reveals an interesting structural property of potential counterexamples to modularity of termination. On the other side it is still rather abstract. The obvious question arising is which TRSs are indeed termination preserving under non-deterministic collapses. This question will be tackled

[10] Here, $oc(f,s)$ yields the number of occurrences of the symbol f in the term s. By $>_{nat}$ and $=_{nat}$ we mean the usual ordering and equality on natural numbers.

next. In section 4 we shall show how the finiteness condition concerning the TRSs involved can be weakened.

Given an arbitrary TRS $\mathcal{R}$ it would be desirable to have a method for testing whether $\mathcal{R}$ is termination preserving under non-deterministic collapses. But it turns out that this is an undecidable property in general.

Theorem 3.13 *The property of TRSs to be termination preserving under non-deterministic collapses is undecidable.*

Proof sketch: This result is an implicit consequence of the proof of the fact that termination is an undecidable property of disjoint unions of terminating TRSs as shown by Middeldorp and Dershowitz (cf. [20]).[11] Roughly spoken the construction proceeds as follows: Given an arbitrary TRS $\mathcal{R}$, another TRS $\mathcal{R}_1$ is constructed by appropriately combining $\mathcal{R}$ with the system $\mathcal{R}_2 := \{f(a,b,x) \rightarrow f(x,x,x)\}$ of the introductory example 1.1 in such a way that $\mathcal{R}_1$ is terminating notwithstanding the fact that $\mathcal{R}$ may be non-terminating. Moreover, choosing $\mathcal{R}_2 := \{G(x,y) \rightarrow x, G(x,y) \rightarrow y\}$, it can be shown that the disjoint union $\mathcal{R}_1 \oplus \mathcal{R}_2$ is terminating if and only if $\mathcal{R}$ is terminating. Since for arbitrary TRSs termination is known to be undecidable (cf. [8]) it follows that the property of TRSs of being termination preserving under non-deterministic collapses is undecidable, too. ■

3.2 The Increasing Interpretation Method

In order to obtain easily verifiable sufficient conditions for the property of being termination preserving under non-deterministic collapses we shall now use a general method for termination proofs – namely the well-founded mapping method ([9], [3]) – and adapt it to the scenario of disjoint unions.

Let $\mathcal{R}^{\mathcal{F}}$ be a TRS over some signature $\mathcal{F}$. For proving termination of $\mathcal{R}$ it suffices to exhibit a well-founded partial ordering $>$ on $\mathcal{T}(\mathcal{F})$ satisfying

$$(1) \quad \forall s,t \in \mathcal{T}(\mathcal{F}): \quad s \rightarrow_{\mathcal{R}} t \quad \Longrightarrow \quad s > t\,.$$

The well-founded mapping method suggests to take a well-founded partial ordering $>_D$ on some set D and some termination function $\tau : \mathcal{T}(\mathcal{F}) \longrightarrow D$ for defining $>$ by

$$(2) \quad s > t \quad :\Longleftrightarrow \quad \tau(s) >_D \tau(t)\,.$$

This method is specialized to the increasing interpretation method by taking D to be an $\mathcal{F}$-algebra and τ to be the unique $\mathcal{F}$-homomorphism from $\mathcal{T}(\mathcal{F})$ to D. Then (1) is guaranteed by

$$(3) \quad \forall s,t \in \mathcal{T}(\mathcal{F})\, \forall f \in \mathcal{F}: \quad s > t \quad \Longrightarrow \quad f(\ldots,s,\ldots) > f(\ldots,t,\ldots)$$

and

$$(4) \quad \forall l \rightarrow r \in \mathcal{R}\ \forall \sigma, \sigma\ \mathcal{T}(\mathcal{F})\text{–ground substitution}: \quad \sigma(l) > \sigma(r)\,.$$

Let us now consider the scenario where two TRSs $\mathcal{R}_1$ and $\mathcal{R}_2$ over signatures $\mathcal{F}_1$ and $\mathcal{F}_2$, respectively, are given such that $\mathcal{R}_1$ is terminating. For proving termination of $\mathcal{R}_1 \cup \mathcal{R}_2$ we apply the increasing interpretation method as follows: Choose D to be $\mathcal{T}(\mathcal{F}_1)$ considered as $\mathcal{F}$-algebra $\mathcal{D}$ with $\mathcal{F} = \mathcal{F}_1 \cup \mathcal{F}_2$, where $\mathcal{F}_1$-operations are interpreted as in the term algebra $\mathcal{T}(\mathcal{F}_1)$ and every $\mathcal{F}_2$-operation is interpreted in some fixed way in terms of $\mathcal{F}_1$-operations, i.e.

$$f^{\mathcal{D}} := \lambda x_1,\ldots,x_n\,.f(x_1,\ldots,x_n) \quad \text{for} \quad f \in \mathcal{F}_1$$

[11] Middeldorp states in [20] that this result has been independently obtained by Dershowitz.

and

$$f^{\mathcal{D}} := \lambda x_1, \ldots, x_n . t_f, t_f \in \mathcal{T}(\mathcal{F}_1, \{x_1, \ldots, x_n\}) \quad \text{for} \quad f \in \mathcal{F}_2 .$$

Hence, the unique homomorphism $\varphi : \mathcal{T}(\mathcal{F}_1 \cup \mathcal{F}_2) \to \mathcal{D}$ is given by $\varphi(f) = f^{\mathcal{D}}$. As well-founded partial ordering $>_D$ on $D = \mathcal{T}(\mathcal{F}_1)$ we take $>_D := \to^+_{\mathcal{R}_1}$. For this case (3) and (4) specialize to

$$(3') \quad \forall s, t \in \mathcal{T}(\mathcal{F}_1) \forall f \in \mathcal{F}_2 : \quad s \to^+_{\mathcal{R}_1} t \quad \Longrightarrow \quad (\varphi f)(\ldots, s, \ldots) \to^+_{\mathcal{R}_1} (\varphi f)(\ldots, t, \ldots)$$

and

$$(4') \quad \forall l \to r \in \mathcal{R}_2 \, \forall \sigma, \sigma \; \mathcal{T}(\mathcal{F}_1)\text{–ground substitution} : \quad \varphi(\sigma l) \to^+_{\mathcal{R}_1} \varphi(\sigma r) .$$

Now, it is easily verified that (3′) is satisfied whenever φ is a *strict* interpretation for $\mathcal{F}_2$, i.e. for any $f \in \mathcal{F}_2$ we have $V(f(x_1, \ldots, x_n)) \subseteq V(\varphi(f(x_1, \ldots, x_n)))$. For verifying (4′) it suffices to show that $\mathcal{R}_2$-rules can be 'simulated' by $\mathcal{R}_1$-rules. To be more precise, we get

Lemma 3.14 *Let $\mathcal{R}_1^{\mathcal{F}}, \mathcal{R}_2^{\mathcal{F}}$ be TRSs such that $\mathcal{R}_1$ is terminating. Moreover, let φ be an interpretation of $(\mathcal{F}_1 \cup \mathcal{F}_2)$-operations in terms of $\mathcal{F}_1$-operations which is the identity on $\mathcal{F}_1$ and which is strict on $\mathcal{F}_2$. Then the union $(\mathcal{R}_1 \cup \mathcal{R}_2)^{\mathcal{F}_1 \cup \mathcal{F}_2}$ is terminating, too, provided that for every rule $l \to r \in \mathcal{R}_2$ we have $\varphi(l) \to^+_{\mathcal{R}_1} \varphi(r)$.*

A trivial consequence of this result is the following

Corollary 3.15 *Whenever a TRS $\mathcal{R}^{\mathcal{F}}$ is terminating then $\mathcal{R}^{\mathcal{F}'}$ is terminating, too, for any enriched signature $\mathcal{F}' \supseteq \mathcal{F}$.*

Proof: Condition (4′) above is vacuously satisfied, and condition (3′) can also be easily fulfilled by interpreting $\mathcal{F}'$-operations in some arbitrary strict way. This is always possible provided that $\mathcal{F}_1$ contains at least one function symbol of an arity greater 1. For the special case that $\mathcal{F}_2$ contains only constants and unary function symbols an easy direct proof is possible. ∎

Of course, the method for proving termination according to the above lemma is rather restricted, because it requires in a sense that $\mathcal{R}_1 \cup \mathcal{R}_2$ terminates for the same reason as $\mathcal{R}_1$ alone. But in particular for the scenario of disjoint unions it is well-suited as we shall see now.

3.3 Derived Criteria for Modularity of Termination

Concrete sufficient criteria for modularity of termination are now easily obtained by combining the previous considerations with corollary 3.10. Firstly, we need

Definition 3.16 *A TRS $\mathcal{R}^{\mathcal{F}}$ is said to be* non-deterministically collapsing *if there exists a term $s[x,y] \in \mathcal{T}(\mathcal{F}, \mathcal{V})$ with $x, y \in \mathcal{V}$ such that $s[x,y] \to^+ x$ and $s[x,y] \to^+ y$, i.e. if some term can be reduced to two distinct variables.*

Lemma 3.17 *Termination is modular for the class of (finite) TRSs which are non-deterministically collapsing.*

Proof: Let $\mathcal{R}_1^{\mathcal{F}_1}$ be a terminating and non-deterministically collapsing TRS. According to theorem 3.9 it suffices to show that the disjoint union $\mathcal{R}_1 \oplus \mathcal{R}_2$ with $\mathcal{R}_2 = \{G(x,y) \to x,\, G(x,y) \to y\}$ is terminating. Since $\mathcal{R}_1^{\mathcal{F}_1}$ is non-deterministically collapsing there exists some term $s[x,y] \in \mathcal{T}(\mathcal{F}_1, \mathcal{V})$ with $x, y \in \mathcal{V}$ such that $s[x,y] \to^+ x$ and $s[x,y] \to^+ y$. W.l.o.g. we may further assume that x, y are the only variables appearing in $s[x,y]$. Now we interpret the function symbol G by $\varphi f = \lambda x, y \,.\, s[x, y]$ and simply apply lemma 3.14 the preconditions of which are satisfied. ■

Next we consider cases where a terminating TRS $\mathcal{R}$ does not necessarily contain collapsing rules but remains terminating when such rules are added.

Definition 3.18 *Let $\mathcal{R}^{\mathcal{F}}$ be a TRS and $f \in \mathcal{F}, \mathcal{F}' \subseteq \mathcal{F}$. Then, $\mathcal{R}^{\mathcal{F}}$ is said to be* f-simply terminating *if $\mathcal{R}^{\mathcal{F}} \cup \mathcal{R}^{f}_{sub}$ with $\mathcal{R}^{f}_{sub} := \{f(x_1, \ldots, x_j, \ldots, x_n) \to x_j | 1 \leq j \leq n\}$ is terminating. $\mathcal{R}^{\mathcal{F}}$ is* F'-simply terminating *if $\mathcal{R}^{\mathcal{F}} \cup \bigcup_{f \in \mathcal{F}'} \mathcal{R}^{f}_{sub}$ is terminating. $\mathcal{R}^{\mathcal{F}}$ is* simply terminating *(Kurihara & Ohuchi [15]) if $\mathcal{R}^{\mathcal{F}}$ is $\mathcal{F}$-simply terminating, i.e. $\mathcal{R}^{\mathcal{F}} \cup \bigcup_{f \in \mathcal{F}} \mathcal{R}^{f}_{sub}$ is terminating.*

Clearly, if a TRS $\mathcal{R}^{\mathcal{F}}$ is $\mathcal{F}'$-simply terminating for some $\mathcal{F}' \subseteq \mathcal{F}$ then it is terminating, i.e. simple termination implies termination.

Using again the specialized increasing interpretation method we obtain

Lemma 3.19 *Let $\mathcal{R}^{\mathcal{F}}$ be f-simply terminating for some $f \in \mathcal{F}$ of arity greater than 1. Then $\mathcal{R}^{\mathcal{F} \cup \mathcal{F}'}$ is $(\{f\} \cup \mathcal{F}')$-simply terminating for any $\mathcal{F}'$ with $\mathcal{F}' \cap \mathcal{F} = \emptyset$.*

Proof: Let $\mathcal{R}^{\mathcal{F}}$ be f-simply terminating for some $f \in \mathcal{F}$ with $arity(f) > 1$ and let $\mathcal{F}'$ be given with $\mathcal{F}' \cap \mathcal{F} = \emptyset$. W.l.o.g. we may assume that f has arity 2 and that $\mathcal{F}'$ has no constants.[12] We shall apply lemma 3.14 for proving that $\mathcal{R}^{\mathcal{F} \cup \mathcal{F}'}$ is $(\{f\} \cup \mathcal{F}')$-simply terminating. For that purpose we interpret every $G \in \mathcal{F}'$ strictly in terms of $\mathcal{F}$-operations as follows:

$$G(x_1, \ldots, x_n) = \begin{cases} f(x_1, x_1) & \text{if} \quad n = 1 \\ f(x_1, f(x_2, \ldots f(x_{n-1}, x_n) \ldots)) & \text{if} \quad n > 1 \,. \end{cases}$$

Now the assumptions of lemma 3.14 are clearly satisfied and we can conclude that $\mathcal{R}^{\mathcal{F} \cup \mathcal{F}'}$ is $(\{f\} \cup \mathcal{F}')$-simply terminating. ■

Combining this result with lemma 3.17 we obtain

Corollary 3.20 *Let $\mathcal{R}_1^{\mathcal{F}_1}$, $\mathcal{R}_2^{\mathcal{F}_2}$ be two (finite) disjoint TRSs with $f_1 \in \mathcal{F}_1, f_2 \in \mathcal{F}_2$ of arity greater than 1 such that $\mathcal{R}_i$ is f_i-simply terminating for $i = 1, 2$. Then the disjoint union $\mathcal{R}_1 \oplus \mathcal{R}_2$ is (f_1- and f_2-simply) terminating, too.*

The intuition behind the notion of $\mathcal{F}$-simple termination is its close relationship to simplification orderings, an important subclass of reduction orderings which in practice are very often used for termination proofs. Simplification orderings are well-suited for that purpose due to the following result from Dershowitz [1] which we present in a slightly generalized version. [13]

Lemma 3.21 *A (possibly infinite) TRS $\mathcal{R}^{\mathcal{F}}$ over some finite signature $\mathcal{F}$ terminates if there exists a simplification ordering $\succ$ on $\mathcal{T}(\mathcal{F}, \mathcal{V})$ such that $l \succ r$ for every rule $l \to r \in \mathcal{R}^{\mathcal{F}}$.*

---

[12] Enriching the signature of a TRS by new constants does not change the termination behaviour (cf. corollary 3.15).

[13] The proof is based on Kruskal's tree theorem which roughly spoken states that any infinite sequence of terms from $\mathcal{T}(\mathcal{F})$ with $\mathcal{F}$ finite is self-embedding. Hence, the requirement that $\mathcal{R}$ must be finite can be weakened.

Kurihara & Ohuchi [14] have shown that for finite TRSs simple termination can be characterized by means of simplification orderings. This yields in slightly generalized form:

Lemma 3.22 *A (possibly infinite) TRS $\mathcal{R}^{\mathcal{F}}$ over some finite signature $\mathcal{F}$ is ($\mathcal{F}-$) simply terminating if and only if there exists a simplification ordering $\succ$ with $l \succ r$ for every rule $l \to r \in \mathcal{R}^{\mathcal{F}}$.*

By specializing corollary 3.20 we finally obtain the main result from Kurihara & Ohuchi [14]:

Theorem 3.23 *Simple termination is a modular property of (finite) TRSs.*

This fact generalizes the well-known observation that common classes of (precedence[14] based) simplification orderings like recursive path orderings or recursive decomposition orderings exhibit a modular behaviour simply by combining the corresponding disjoint precedences.

In [14] theorem 3.23 is directly proved by means of a construction which has some similarity with our approach presented in the last section. Instead of our black (and white) abstraction function Kurihara & Ohuchi define a mapping called 'alien-replacement' which is tailored to some specific finite reduction sequence. Moreover their construction is in a sense incremental, but not rank-decreasing. To be more precise, consider some finite derivation

$$D: \quad s_0 \to s_1 \to s_2 \to \ldots \to s_m$$

in $\mathcal{R} := (\mathcal{R}_1 \cup \mathcal{R}_{sub}^{\mathcal{F}_1} \cup \mathcal{R}_{del}^{\mathcal{F}_1}) \oplus (\mathcal{R}_2 \cup \mathcal{R}_{sub}^{\mathcal{F}_2} \cup \mathcal{R}_{del}^{\mathcal{F}_2})$ with all s_i's top black and such that every $\mathcal{R}$-derivation starting from any (top white) principal alien of s_0 is finite. Then their 'alien replacement' construction for D essentially consists in (recursively) collecting, for any principal alien occurring in D, all direct descendants occurring in D and abstracting them via a new varyadic (black) function symbol. Using this transformation the $\mathcal{R}$-derivation D can be translated in a one-to-one manner into a corresponding $(\mathcal{R}_1 \cup \mathcal{R}_{sub}^{\mathcal{F}_1} \cup \mathcal{R}_{del}^{\mathcal{F}_1})$-derivation from which one can easily infer the modularity of simple termination using lemma 3.22.[15]

3.4 Minimal Counterexamples of Arbitrary Rank

Besides the features mentioned all counterexamples to modularity of termination presented above and in the literature (cf. [23]) have some more common property. Namely, the rank n of minimal counterexamples always equals 3. According to lemma 3.3 (a) we must have $n \geq 3$. So, the question naturally arises whether this is a general phenomenon saying that, whenever the disjoint union of two terminating TRSs is non-terminating then there is a counterexamples having rank 3. This question is not only interesting by itself but also because many proofs concerning results on modular termination have to consider 'mixed' terms of arbitrary rank. In particular, the extremely complicated analysis performed in [25] for proving that completeness is modular for left-linear TRSs could be considerably simplified if counterexamples of rank 3 were always possible. Surprisingly (at least for the author) this is not the case as illustrated by

Example 3.24 $\quad \mathcal{R}_1 : \quad f(x, g(x), y) \to f(y, y, y) \qquad \mathcal{R}_2 : \quad G(x) \to x$
$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad G(x) \to A$

[14] A precedence is a partial ordering on a set $\mathcal{F}$ of function symbols.

[15] cf. [14], [15] for details; in fact, compared to [14], [15] contains a simplified and clarified version of 'alien replacement'.

Here, both $\mathcal{R}_1$ and $\mathcal{R}_2$ are clearly terminating, but $\mathcal{R} := \mathcal{R}_1 \oplus \mathcal{R}_2$ is non-terminating. For instance, we have the following infinite $\mathcal{R}$-derivation

$$
\begin{array}{llll}
D: & f(G(g(A)),G(g(A)),G(g(A))) & \to_{\mathcal{R}_2} & f(A,G(g(A)),G(g(A))) \\
& & \to_{\mathcal{R}_2} & f(A,g(A),G(g(A))) \\
& & \to_{\mathcal{R}_1} & f(G(g(A)),G(g(A)),G(g(A))) \\
& & \to_{\mathcal{R}_2} \cdots &
\end{array}
$$

of rank 4. By analyzing for which mixed terms s, t it is possible that $s \to_{\mathcal{R}} t$ and $s \to_{\mathcal{R}} g(t)$ one can show that the minimal rank of a non-terminating $\mathcal{R}$-derivation is exactly 4.

Moreover, example 3.24 can be easily generalized in order to show that the rank of minimal counterexamples may be arbitrarily high.

Example 3.25 $\quad \mathcal{R}_1 : \; f(x, g(x), \ldots, g^n(x), y) \to f(y, \ldots, y) \qquad \mathcal{R}_2 : \; \begin{array}{l} G(x) \to x \\ G(x) \to A \end{array}$

Here, f has arity $n+2$ and $g^n(x)$ stands for the n-fold application of g to x. Both $\mathcal{R}_1$ and $\mathcal{R}_2$ are clearly terminating, but $\mathcal{R}_1 \oplus \mathcal{R}_2$ is non-terminating. For instance, we have the following infinite $(\mathcal{R}_1 \oplus \mathcal{R}_2)$-derivation[16]

$$
\begin{array}{llll}
D: & f((Gg)^n A, (Gg)^n A, \ldots (Gg)^n A) & \to_{\mathcal{R}_2} & f(A, (Gg)^n A, \ldots, (Gg)^n A) \\
& & \to_{\mathcal{R}_2} & f(A, g(Gg)^{n-1} A, \ldots, (Gg)^n A) \\
& & \to_{\mathcal{R}_2} & f(A, gA, \ldots, (Gg)^n A) \\
& & \vdots & \\
& & \to_{\mathcal{R}_2} & f(A, gA, g^2 A, \ldots, g^n A, (Gg)^n A) \\
& & \to_{\mathcal{R}_1} & f((Gg)^n A, (Gg)^n A, \ldots, (Gg)^n A) \\
& & \to_{\mathcal{R}_2} & \cdots
\end{array}
$$

of rank $2n+2$. Again a careful analysis of possible reductions shows that for this example $2n+2$ is the minimal rank of any conceivable non-terminating $(\mathcal{R}_1 \oplus \mathcal{R}_2)$-derivation. Moreover, it is straightforward to modify the above examples in such a way that only finite signatures with function symbols of (uniformly) bounded arities are involved. For instance, one may use a binary f' and the encoding $f'(x_1, f'(x_2, \ldots, f'(x_{n-1}, x_n)\ldots))$ for $f(x_1, \ldots, x_n)$.

Hence, we can conclude that for terminating disjoint TRSs with non-terminating disjoint union minimal counterexamples may have an arbitrarily high rank. This shows that the interaction in disjoint unions of TRSs may be very subtle, in particular concerning termination properties.

Having a closer look on examples 3.24 and 3.25 it is obvious that $\mathcal{R}_1$ is non-left-linear and $\mathcal{R}_2$ is non-confluent. On the other side the main result from [25] implies that one of the systems involved must be non-left-linear or non-confluent. Even stronger, we have the following

> **Conjecture:** Whenever $\mathcal{R}_1, \mathcal{R}_2$ are two terminating disjoint TRSs such that their disjoint union $\mathcal{R}_1 \oplus \mathcal{R}_2$ is non-terminating with $rank(D) > 3$ for any infinite $(\mathcal{R}_1 \oplus \mathcal{R}_2)$-derivation D then one of the systems is non-left-linear and duplicating and the other one is non-confluent and collapsing.

[16] The notation used here should be self-explanatory. For example, $(Gg)^2(A)$ stands for $G(g(G(g(A))))$.

Note that there is a close relationship between this conjecture and the main result of [25] which says that termination (and hence completeness) is modular for left-linear and confluent TRSs. If we could prove the above conjecture then the very complicated proof in [25] given for modularity of completeness of left-linear TRSs could be considerably simplified.

4 Extensions and Generalizations

4.1 Non-Self-Embedding Systems

We have seen that – as a consequence of our main result – simple termination is a modular property of (finite) TRSs. In other words, termination of a disjoint union $\mathcal{R}_1 \oplus \mathcal{R}_2$ can be shown by a simplification ordering if and only if this holds already for $\mathcal{R}_1$ and $\mathcal{R}_2$. Now simplification orderings are closely connected to the self-embedding property of TRSs. According to Kruskal's tree theorem the property of being non-self-embedding implies termination (for finite TRSs). Furthermore, simple termination is sufficient for being non-self-embedding. Hence, a natural question is to ask whether termination is also modular for non-self-embedding systems, or in slightly sharpened form: Is the property of being non-self-embedding a modular one? Having again a closer look on example 1.1 with $\mathcal{R}_1 = \{f(a,b,x) \to f(x,x,x)\}$, $\mathcal{R}_2 = \{G(x,y) \to x\ , G(x,y) \to y\}$ it is clear that $\mathcal{R}_1$ is terminating, but cannot be simply terminating because it is self-embedding as witnessed e.g. by the one-step-derivation $f(a,b,f(a,b,b)) \to_{\mathcal{R}_1} f(f(a,b,b),f(a,b,b),f(a,b,b))$. Now consider the following modified version of example 1.1:

Example 4.1 $\quad \mathcal{R}_1 : \quad f(a,b,x) \to h(x,x,x) \qquad \mathcal{R}_2 : \quad G(x,y) \to x$
$\qquad\qquad\qquad\qquad\qquad h(a,b,x) \to f(x,x,x) \qquad\qquad\qquad\quad G(x,y) \to y$

Clearly, both R_1 and R_2 are terminating and even non-self-embedding as can be easily shown, but $\mathcal{R}_1 \oplus \mathcal{R}_2$ admits e.g. the following infinite (and hence self-embedding) derivation:

$$\begin{array}{rll} f(a,b,G(a,b)) & \to_{\mathcal{R}_1} & h(G(a,b),G(a,b),G(a,b)) \\ & \to_{\mathcal{R}_2} & h(a,G(a,b),G(a,b)) \\ & \to_{\mathcal{R}_2} & h(a,b,G(a,b)) \\ & \to_{\mathcal{R}_1} & f(G(a,b),G(a,b),G(a,b)) \\ & \to_{\mathcal{R}_2} & f(a,G(a,b),G(a,b)) \\ & \to_{\mathcal{R}_2} & f(a,b,G(a,b)) \\ & \to_{\mathcal{R}_1} & \cdots \end{array}$$

Thus, we may conclude that termination is not modular in general for non-self-embedding TRSs or – slightly stronger – that the property of being non-self-embedding is not a modular one. Note, that this reveals a gap between simply terminating and non-self-embedding systems. In fact, every simply terminating TRS is non-self-embedding, but not vice-versa because we have e.g. in $\mathcal{R}_1 \cup \mathcal{R}^f_{sub}$ with $\mathcal{R}_1$ as above:

$$\begin{array}{rll} f(a,b,f(a,b,b)) & \to & h(f(a,b,b),f(a,b,b),f(a,b,b)) \\ & \to^+ & h(a,b,f(a,b,b)) \\ & \to & f(f(a,b,b),f(a,b,b),f(a,b,b)) \\ & \to^+ & f(a,b,f(a,b,b)) \\ & \to & \cdots \end{array}$$

Hence, both implications

$$\mathcal{R} \text{ simply terminating} \implies \mathcal{R} \text{ non-self-embedding} \implies \mathcal{R} \text{ terminating}$$

cannot be reversed. This is well-known for the latter one (cf. e.g. Dershowitz [3]) but – as far as we know – it is nowhere mentioned in the literature for the first one. Moreover, the gap between non-self-embedding and simply terminating TRSs exists even for TRSs which contain only unary function symbols, hence for string rewriting systems. To this end consider the system $\mathcal{R} := \{g(g(x)) \to h(f(h(x))), h(h(x)) \to g(f(g(x)))\}$ over $\mathcal{F} := \{f, g, h\}$. Here, $\mathcal{R}$ is easily shown to be non-self-embedding but it is not (f-)simply terminating because we have for instance the following infinite (cyclic), hence self-embedding derivation in $\mathcal{R} \cup \mathcal{R}^f_{sub}$:

$$g(g(x)) \to h(f(h(x))) \to h(h(x)) \to g(f(g(x))) \to g(g(x)) \to \cdots .$$

4.2 Weakening the Finiteness Requirement

Most results presented so far which rely on our main theorem 3.9 required that the involved TRSs have only finitely many rewrite rules. This assumption can be considerably weakened as it will be shown now. In fact, the reason for this finiteness condition was to ensure well-definedness of the white (or black) abstraction function Φ in definition 3.4. A closer look at the definition reveals that the essential property needed is that for any mixed (ground) term s of *rank* less than or equal n with n as in the definition, the set of possible successors of s, i.e. $SUCC(s) := \{s' \in \mathcal{T}(\mathcal{F}_1 \uplus \mathcal{F}_2) | s \to^+_{\mathcal{R}} s'\}$ with $\mathcal{R} := \mathcal{R}_1 \oplus \mathcal{R}_2$, is finite. For that purpose it is sufficient to require that $\mathcal{R}$ is finitely branching, i.e. for any term $s \in \mathcal{T}(\mathcal{F}_1 \uplus \mathcal{F}_2)$ the one-step-successor set $\{s' \in \mathcal{T}(\mathcal{F}_1 \uplus \mathcal{F}_2) | s \to_{\mathcal{R}} s'\}$ is finite. In that case one may simply apply Königs lemma. The following result provides a characterization of the property of TRSs to be finitely branching.

Lemma 4.2 *A (possibly infinite) TRS $\mathcal{R}^{\mathcal{F}}$ is finitely branching if and only if for every rule $l \to r \in \mathcal{R}^{\mathcal{F}}$ there are only finitely many different rules in $\mathcal{R}^{\mathcal{F}}$ with the same left hand side l.* [17]

Proof: Consider an arbitrary ground term s and possible $\mathcal{R}^{\mathcal{F}}$-reductions. Clearly, there are only finitely many different left hand sides of rules in $\mathcal{R}^{\mathcal{F}}$ which can match some subterm of s. Hence, the set of one-step-successors of s can be infinite only in the case that there are infinitely many different rules in $\mathcal{R}^{\mathcal{F}}$ with the same left hand side. The only-if-direction of the lemma is trivial. ■

Corollary 4.3 *The property of (possibly infinite) TRSs to be finitely branching is modular.*

Hence, all our results basing on our main theorem 3.9 can be generalized by requiring the involved TRSs to be only finitely branching instead of finite. Note that the signature may still be infinite. This case is only problematic if simplification orderings are used for trying to prove termination. For infinite signatures the lemmas 3.21 and 3.22 do not hold any more in general because Kruskal's tree theorem is no longer valid. Hence, if a TRS $\mathcal{R}$ can be oriented by some simplification ordering this does not necessarily imply termination of $\mathcal{R}$ any more.

The restriction to finitely branching TRSs is essential as can be seen from the following example involving a non-finitely branching TRS over some infinite signature.

[17] Note that rules which can be obtained from one another by renaming variables are considered to be equal.

Example 4.4 *Let $\mathcal{R}_1^{\mathcal{F}_1}, \mathcal{R}_2^{\mathcal{F}_2}$ be given with $\mathcal{R}_2 = \{H(x,y,y) \to x, H(y,x,y) \to x\}$, $\mathcal{F}_2 = \{H, A\}$, $\mathcal{F}_1 = \{f_0, f_1, f_2, \ldots\} \cup \{0,1,2,\ldots\} \cup \{\omega\}$ and*

$$\begin{array}{llll} \mathcal{R}_1: & f_0(0,1,x) \to f_1(x,x,x) & 0 \to 2 \quad 0 \to 4 \quad 0 \to 6 & \ldots \\ & f_1(2,3,x) \to f_2(x,x,x) & 1 \to 3 \quad 1 \to 5 \quad 1 \to 7 & \ldots \\ & f_2(4,5,x) \to f_3(x,x,x) & 0 \to \omega & \\ & \quad \vdots \quad \to \quad \vdots & 1 \to \omega & \end{array}$$

Here, $\mathcal{R}_1$ and $\mathcal{R}_2$ are terminating but $\mathcal{R} := \mathcal{R}_1 \oplus \mathcal{R}_2$ is non-terminating as can be seen from the infinite $\mathcal{R}$-derivation:

$$\begin{array}{lll} & f_0(H(0,1,1), H(0,1,1), H(0,1,1)) & \to_{\mathcal{R}}^{+} \; f_0(0,1,H(0,1,1)) \\ \to_{\mathcal{R}_1} & f_1(H(0,1,1), H(0,1,1), H(0,1,1)) & \to_{\mathcal{R}}^{+} \; f_1(2,3,H(0,1,1)) \\ \to_{\mathcal{R}_1} & f_2(H(0,1,1), H(0,1,1), H(0,1,1)) & \to_{\mathcal{R}}^{+} \; f_2(4,5,H(0,1,1)) \\ \to_{\mathcal{R}_1} & f_3(H(0,1,1), H(0,1,1), H(0,1,1)) & \to_{\mathcal{R}}^{+} \; f_3(6,7,H(0,1,1)) \ \ldots \end{array}$$

The infinity of $\mathcal{R}_1$ is essential for the existence of this counterexample because for every finite subset $\mathcal{R}_1'$ of $\mathcal{R}_1$ the disjoint union $\mathcal{R}_1' \oplus \mathcal{R}_2$ is again terminating. The abstracting transformation underlying theorem 3.9 is not applicable here since it would yield infinite terms. For instance, $H(0,1,1)$ has the infinitely many different $\mathcal{F}_1$-rooted successors $\{0,2,4,\ldots\} \cup \{1,3,5,\ldots\} \cup \{\omega\}$ in $\mathcal{R}$. Hence, the whole transformation process would yield an infinite derivation consisting of infinite terms. Nevertheless, $\mathcal{R}_1$ is not termination preserving under non-deterministic collapses, because for $\mathcal{R}' := \mathcal{R}_1 \oplus \{G(x,y) \to x, G(x,y) \to y\}$ we have e.g.

$$\begin{array}{lll} & f_0(G(0,1), G(0,1), G(0,1)) & \to_{\mathcal{R}'}^{+} \; f_0(0,1,G(0,1)) \\ \to_{\mathcal{R}_1} & f_1(G(0,1), G(0,1), G(0,1)) & \to_{\mathcal{R}'}^{+} \; f_1(2,3,G(0,1)) \\ \to_{\mathcal{R}_1} & f_2(G(0,1), G(0,1), G(0,1)) & \to_{\mathcal{R}'}^{+} \; f_2(4,5,G(0,1)) \\ \to_{\mathcal{R}_1} & f_3(G(0,1), G(0,1), G(0,1)) & \to_{\mathcal{R}'}^{+} \; f_3(6,7,G(0,1)) \ \ldots \end{array}$$

This means that the conclusion of theorem 3.9 holds for this example although we cannot apply 3.9 due to the required finiteness conditions.

In fact, it is possible to completely drop any finiteness assumption in theorem 3.9 and derived results. But for proving this generalization a substantially different approach has to be taken which will be detailed elsewhere.

4.3 Weakening the Disjointness Requirement

For practical purposes the invariance of properties of TRSs under non-disjoint unions is very important, too. In general, most interesting properties do not exhibit such an invariant behaviour under arbitrary non-disjoint unions. But for certain restricted variants of combinations some results are known (e.g. [7], [21], [15]). We shall now investigate for which cases our results can be generalized.

4.3.1 Hierarchical Combinations

One natural kind of non-disjoint union of TRSs is a hierarchical combination in the following sense. Let some TRS $\mathcal{R}_1 \subseteq \mathcal{T}(\mathcal{F}_1, \mathcal{V}) \times \mathcal{T}(\mathcal{F}_1, \mathcal{V})$ over some signature $\mathcal{F}_1$ and some TRS $\mathcal{R}_2 \subseteq \mathcal{T}(\mathcal{F}_2, \mathcal{V}) \times \mathcal{T}(\mathcal{F}_1 \uplus \mathcal{F}_2, \mathcal{V})$ over the signature $\mathcal{F}_1 \uplus \mathcal{F}_2$ be given. Since the left hand sides of $\mathcal{R}_2$ do not contain $\mathcal{F}_1$-symbols, $\mathcal{R} := \mathcal{R}_1 \cup \mathcal{R}_2$ may be considered as a hierarchical extension of $\mathcal{R}_1$. Now, in general such a hierarchical combination of TRSs clearly does not preserve termination of its constituents but perhaps under some further restrictions.

In fact, for well-known classes of simplification orderings like the recursive path ordering (RPO) such an invariance result for the termination property may be achieved. To be more precise, let us assume that $\mathcal{R}_1, \mathcal{R}_2$ are given as above such that termination of both systems can be shown by appropiate RPOs induced by precedences $>_{\mathcal{F}_1}$ and $>_{\mathcal{F}_2 \cup \mathcal{F}_1}$, respectively. Then it is easy to prove (by structural induction according to the definition of the RPO) that $>_{\mathcal{F}_2 \cup \mathcal{F}_1} \; = \; >_{\mathcal{F}_2 \cup \mathcal{F}_1} |_{\mathcal{F}_2 \times \mathcal{F}_2}$ may be assumed w.l.o.g., i.e. relations of $>_{\mathcal{F}_2 \cup \mathcal{F}_1}$ involving an $\mathcal{F}_1$-symbol are not really necessary for ensuring termination of $\mathcal{R}_2$. Hence, it follows that the union $\mathcal{R}_1 \cup \mathcal{R}_2$ is terminating, too, simply by taking the RPO induced bye the union of the two precedences.

This kind of reasoning should also be possible for other classes of precedence based simplication orderings. An obvious question arising therefrom is whether the termination property is also preserved under such hierarchical combinations under the weaker assumption that termination of $\mathcal{R}_1$ and $\mathcal{R}_2$ can be shown by some simplification ordering. Unfortunately, this is not the case as shown by the following simple

Example 4.5 *Let $\mathcal{R}_1$: $a \to b$ and $\mathcal{R}_2$: $h(x,x) \to h(a,b)$ be given over signatures $\mathcal{F}_1 = \{a,b\}$, $\mathcal{F}_2 = \{h\}$. Then both $\mathcal{R}_1$ and $\mathcal{R}_2$ are simply terminating. This is obvious for $\mathcal{R}_1$ and easy to show for $\mathcal{R}_2$ by considering $\mathcal{R}_2' := \mathcal{R}_2 \cup \mathcal{R}_{sub}^H = \{h(x,x) \to h(a,b), h(x,y) \to x, h(x,y) \to y\}$. But $\mathcal{R} := \mathcal{R}_1 \cup \mathcal{R}_2$ is non-terminating. For instance we have $h(b,b) \to_{\mathcal{R}_2} h(a,b) \to_{\mathcal{R}_1} h(b,b) \to \cdots$.*

4.3.2 Non-Disjoint Unions with Common Constructors

In practice the necessity of considering non-disjoint unions of TRSs often comes from the fact that some class of function symbols naturally occurs in several distinct component TRSs. This is for instance the case with constructors.

Definition 4.6 *([21]) A* constructor system (CS) *is a TRS $\mathcal{R}^{\mathcal{F}}$ with the property that $\mathcal{F}$ can be partitioned into $\mathcal{F} = \mathcal{C} \uplus \mathcal{D}$ such that every left hand side $f(s_1, \ldots, s_n)$ of a rewrite rule from $\mathcal{R}^{\mathcal{F}}$ satisfies $f \in \mathcal{D}$ and $s_1, \ldots, s_n \in \mathcal{T}(\mathcal{C}, \mathcal{V})$.[18] Function symbols in $\mathcal{D}$ are called* defined symbols *and those in $\mathcal{C}$* constructors. *Slightly abusing notation we also write $\mathcal{T}(\mathcal{C}, \mathcal{D}, \mathcal{V})$ instead of $\mathcal{T}(\mathcal{F}, \mathcal{V})$.*

Middeldorp and Toyama have shown in [21] that completeness is preserved under the union of constructor systems with disjoint sets of defined symbols (and common set of constructor symbols). In fact, a slightly more general result is proved in [21].

Kurihara & Ohuchi ([15]) investigate another notion of combining TRSs with common constructors.

Definition 4.7 *([15]) A TRS $\mathcal{R}^{\mathcal{F}}$ with a fixed partition of $\mathcal{F}$ into $\mathcal{F} = \mathcal{C} \uplus \mathcal{D}$ is said to be a* TRS with constructors *provided that for any rule $l \to r \in \mathcal{R}^{\mathcal{F}}$ we have $root(l) \in \mathcal{D}$. Given two TRSs $\mathcal{R}_1$, $\mathcal{R}_2$ with constructors over signatures $\mathcal{F}_1 = \mathcal{C} \uplus \mathcal{D}_1$, $\mathcal{F}_2 = \mathcal{C} \uplus \mathcal{D}_2$, the TRS $\mathcal{R} := \mathcal{R}_1 \cup \mathcal{R}_2$ over the signature $\mathcal{F} := \mathcal{C} \uplus (\mathcal{D}_1 \uplus \mathcal{D}_2)$ is called the* combined system with shared constructors *$\mathcal{C}$.*

Of course, every union of constructor systems with disjoint sets of defined symbols (and common set of constructor symbols) is a combined system with shared constructors, but not vice-versa. For combined systems with shared constructors Kurihara & Ohuchi [15] have generalized in a straightforward manner their main result from [14], namely modularity of simple termination.

[18] This definition of constructor system corresponds to what is usually meant when one speaks of a constructor discipline (for specifying functions).

In an analogous manner our structural analysis of potential counterexamples presented in the last section can also be generalized from the disjoint union case to the case of combined systems with shared constructors. This will be sketched now. Firstly we need some terminology from [15].

Let us assume in the following that $\mathcal{R}_1^{\mathcal{F}_1}$, $\mathcal{R}_2^{\mathcal{F}_2}$ with $\mathcal{F}_1 = \mathcal{C} \uplus \mathcal{D}_1$, $\mathcal{F}_2 = \mathcal{C} \uplus \mathcal{D}_2$, $\mathcal{D}_1 \cap \mathcal{D}_2 = \emptyset$ are finite[19] TRSs with constructors such that $\mathcal{R} = \mathcal{R}_1 \cup \mathcal{R}_2$ is a combined system with shared constructors $\mathcal{C}$. The defined function symbols from $\mathcal{D}_1$ and $\mathcal{D}_2$ are considered to be black and white, respectively. Constructor symbols (and variables) are painted according to the context above the actual position. This means that if a constructor symbol appears at the the root of a term then it is considered to be transparent. Otherwise, its colour is (recursively) the same as the colour of its predecessor (in tree representation). Analogously, the notions of being top white, top black and top transparent are defined.

Definition 4.8 *A term t is said to be a* principal alien *or* principal subterm *of a term s if t is a non-variable proper subterm of s which is maximal w.r.t. the subterm relation such that $root(t)$ and $root(s)$ have different colours. Again we use the notation $s \equiv C[\![s_1, \ldots, s_n]\!]$ where all principal aliens s_i of s are displayed. The sets $PS(s)$ of principal aliens and the set $SS(s)$of all aliens of s are defined analogously as in the disjoint union case. The* rank *of a term s is defined by*

$$rank(s) = \begin{cases} 0 & \text{if} \quad s \in \mathcal{T}(\mathcal{C}) \\ 1 & \text{if} \quad s \in (\mathcal{T}(\mathcal{D}_1 \uplus \mathcal{C}, \mathcal{V}) \cup \mathcal{T}(\mathcal{D}_2 \uplus \mathcal{C}, \mathcal{V})) \setminus \mathcal{T}(\mathcal{C}) \\ max\{rank(s_i) | 1 \leq i \leq n\} & \text{if} \quad s \equiv C[\![s_1, \ldots, s_n]\!] \text{ with } C \in \mathcal{CON}(\mathcal{C}, \mathcal{V}) \\ 1 + max\{rank(s_i) | 1 \leq i \leq n\} & \text{if} \quad s \equiv C[\![s_1, \ldots, s_n]\!] \text{ with } C \notin \mathcal{CON}(\mathcal{C}, \mathcal{V}) \end{cases}$$

The notions of inner, outer and destructive reduction steps are generalized in a straightforward manner. A subterm t of s is an inner *subterm of s if it is a subterm of some alien of s. Otherwise, it is an* outer *subterm of s.*

Now we are prepared for generalizing our structural analysis for the disjoint union case to the scenario of non-disjoint combinations of TRSs with shared constructors.

Lemma 4.9 *Let $\mathcal{R}_1$, $\mathcal{R}_2$ be terminating such that*

$$D : s_1 \to s_2 \to s_3 \to \ldots$$

is an infinite derivation in the combined system $\mathcal{R}$ (involving only ground terms) of minimal rank, i.e. any derivation in $\mathcal{R}$ of smaller rank is finite. Then we have:

(a) $rank(D) \geq 3$.

(b) Infinitely many steps in D are outer steps.

(c) Infinitely many steps in D are inner reductions which are destructive at level 2.

Definition 4.10 *Let $\mathcal{R}_1$, $\mathcal{R}_2$ be terminating TRSs over signatures $\mathcal{F}_1 = \mathcal{C} \uplus \mathcal{D}_1$ and $\mathcal{F}_2 = \mathcal{C} \uplus \mathcal{D}_2$, respectively, with $\mathcal{F} = \mathcal{C} \uplus \mathcal{D}_1 \uplus \mathcal{D}_2$ and $n \in \mathrm{N}$ such that for every $s \in \mathcal{T}(\mathcal{F})$ with $rank(s) \leq n$ there is no infinite derivation in the combined system $\mathcal{R}$ with shared constructors starting with s. Moreover, let $\mathcal{T}(\mathcal{F})^{\leq n} := \{t \in \mathcal{T}(\mathcal{F}) | rank(t) \leq n\}$, $\mathcal{T}(\mathcal{F})^{n+1} := \{t \in \mathcal{T}(\mathcal{F}) | rank(t) = n + 1\}$. Moreover, let $<_{\mathcal{T}(\mathcal{F}_1 \uplus \{A,G\})}$ be some arbitrary,*

[19]The finiteness condition required here can be weakened by the same line of reasoning as presented in subsection 4.2.

but fixed total ordering on $\mathcal{T}(\mathcal{F}_1 \uplus \{A,G\})$. *Then the* $\mathcal{D}_2$- *(or* white*)* abstraction *is defined to be the mapping*

$$\Psi : \mathcal{T}(\mathcal{F})^{\leq n} \uplus \{t \in \mathcal{T}(\mathcal{F})^{n+1} | root(t) \in \mathcal{F}_1 \uplus \mathcal{C}\} \longrightarrow \mathcal{T}(\mathcal{F}_1 \uplus \{A,G\})$$

given by

$$\Psi(t) := \begin{cases} t & \text{if} \quad t \in \mathcal{T}(\mathcal{D}_1 \uplus \mathcal{C}) \\ A & \text{if} \quad t \in \mathcal{T}(\mathcal{D}_2) \\ C[\![\Psi(t_1),\ldots,\Psi(t_m)]\!] & \text{if} \quad t \equiv C[\![t_1,\ldots,t_m]\!], m \geq 1, root(t) \in \mathcal{D}_1 \uplus \mathcal{C} \\ CONS(SORT(\Psi^*(SUCC^{\mathcal{F}_1}(t)))) & \text{if} \quad t \equiv C[\![t_1,\ldots,t_m]\!], m \geq 1, root(t) \in \mathcal{D}_2 \end{cases}$$

with

$$\begin{array}{rcl} SUCC^{\mathcal{F}_1}(t) & := & \{t' \in \mathcal{T}(\mathcal{F}) | t \to^*_{\mathcal{R}} t', root(t') \in \mathcal{F}_1\}, \\ \Psi^*(M) & := & \{\Psi(t) | t \in M\} \quad \text{for} \quad M \subseteq dom(\Psi), \\ CONS(\langle\rangle) & := & A, \\ CONS(\langle s_1,\ldots,s_{k+1}\rangle) & := & G(s_1, CONS(\langle s_2,\ldots,s_{k+1}\rangle)) \quad \text{and} \\ SORT(\{s_1,\ldots,s_k\} & := & \langle s_{\pi(1)},\ldots,s_{\pi(k)}\rangle, \end{array}$$

such that $s_{\pi(j)} \leq_{\mathcal{T}(\mathcal{F}_1 \uplus \{A,G\})} s_{\pi(j+1)}$ *for* $1 \leq j < k$.

Lemma 4.11 *Let* $\mathcal{R}_1$, $\mathcal{R}_2$, $\mathcal{R} := \mathcal{R}_1 \oplus \mathcal{R}_2$, n *and* Ψ *be given as in definition 4.10. Then, for any* $s,t \in \mathcal{T}(\mathcal{F})$ *with* $rank(s) \leq n$ *and* $root(s) \in \mathcal{D}_2$ *we have:*

$$s \to_{\mathcal{R}} t \quad \Longrightarrow \quad \Psi(s) \to^*_{\mathcal{R}'_2} \Psi(t),$$

where $\mathcal{R}'_2 := \mathcal{R}^G_{sub} := \{G(x,y) \to x, G(x,y) \to y\}$.

Lemma 4.12 *Let* $\mathcal{R}_1$, $\mathcal{R}_2$ *and* Ψ *be given as in definition 4.10. Then,* Ψ *is rank decreasing, i.e. for any* $s \in dom(\Psi) := \mathcal{GT}^{\leq n}_{\oplus} \uplus \{t \in \mathcal{GT}^{n+1}_{\oplus} | root(t) \in \mathcal{F}_1\}$ *we have* $rank(\Psi(s)) \leq rank(s)$.

Lemma 4.13 *Let* $\mathcal{R}_1$, $\mathcal{R}_2$, $\mathcal{R} = \mathcal{R}_1 \cup \mathcal{R}_2$, $\mathcal{R}'_2 = \mathcal{R}^G_{sub} = \{G(x,y) \to x, G(x,y) \to y\}$, n *and the* $\mathcal{D}_2$*-abstraction* Ψ *be given as above. Then, for any* $s,t \in \mathcal{T}(\mathcal{F})$ *with* $rank(s) \leq n+1$, $root(s) \in \mathcal{D}_1 \uplus \mathcal{C}$ *and* $s \to_{\mathcal{R}} t$ *we have:*

(a) *If* $s \stackrel{o}{\to}_{\mathcal{R}_1} t$ *is not destructive at level 1 then* $\Psi(s) \stackrel{o}{\to}_{\mathcal{R}_1} \Psi(t)$ *using the same* $\mathcal{R}_1$*-rule, and moreover this step is also not destructive at level 1.*

(b) *If* $s \stackrel{o}{\to}_{\mathcal{R}_1} t$ *is destructive at level 1 then* $\Psi(s) \stackrel{o}{\to}_{\mathcal{R}_1} \Psi(t)$ *using the same* $\mathcal{R}_1$*-rule, and moreover this step is also destructive at level 1.*

(c) *If* $s \stackrel{i}{\to}_{\mathcal{R}} t$ *is not destructive at level 2 then* $\Psi(s) \stackrel{i}{\to}^*_{\mathcal{R}'_2} \Psi(t)$ *with all steps not destructive at level 2.*

(d) *If* $s \stackrel{i}{\to}_{\mathcal{R}} t$ *is destructive at level 2 then* $\Psi(s) \stackrel{i}{\to}^+_{\mathcal{R}'_2} \Psi(t)$ *such that exactly one of these steps is destructive at level 2.*

Theorem 4.14 *Let* $\mathcal{R} = \mathcal{R}_1 \uplus \mathcal{R}_2$ *be a combined TRS with shared constructors such that both systems* $\mathcal{R}_1$ *and* $\mathcal{R}_2$ *are terminating and such that* $\mathcal{R}$ *is non-terminating. Then* $\mathcal{R}_j$ *is not termination preserving under non-deterministic collapses for some* $j \in \{1,2\}$ *and the other system* $\mathcal{R}_k$, $k \in \{1,2\} \setminus \{j\}$ *is collapsing. Moreover, the minimal rank of counterexamples in* $\mathcal{R}_j \oplus \{G(x,y) \to x, G(x,y) \to y\}$ *is less than or equal to the minimal rank of counterexamples in* $\mathcal{R}$.

Corollary 4.15 *If $\mathcal{R} = \mathcal{R}_1 \uplus \mathcal{R}_2$ is a combined system with shared constructors such that $\mathcal{R}_1$, $\mathcal{R}_2$ are terminating TRSs which are termination preserving under non-deterministic collapses then $\mathcal{R}$ is terminating, too.*

Lemma 4.16 *If $\mathcal{R} = \mathcal{R}_1 \uplus \mathcal{R}_2$ is a combined system with shared constructors such that both $\mathcal{R}_1$ and $\mathcal{R}_2$ are non-deterministically collapsing then $\mathcal{R}$ is terminating (and confluent) if and only if both $\mathcal{R}_1$ and $\mathcal{R}_2$ are terminating (and confluent), too.*

Corollary 4.17 *Let $\mathcal{R} = \mathcal{R}_1 \uplus \mathcal{R}_2$ be a combined system with shared constructors with $f_1 \in \mathcal{D}_1, f_2 \in \mathcal{D}_2$ of arity greater than 1 such that $\mathcal{R}_i$ is f_i-simply terminating for $i = 1, 2$. Then $\mathcal{R}$ is (f_1- and f_2-simply) terminating, too.*

By specializing this corollary we finally obtain the main result from [15]:

Theorem 4.18 *A combined system $\mathcal{R} = \mathcal{R}_1 \cup \mathcal{R}_2$ with shared constructors is simply terminating (and confluent) if and only if both $\mathcal{R}_1$ and $\mathcal{R}_2$ are simply terminating (and confluent).*

Note that the invariance of confluence (under the termination assumption) is guaranteed by the critical pair lemma and the fact that for the set $CP(\mathcal{R})$ of critical pairs in a combined system with shared constructors $\mathcal{R}$ we have: $CP(\mathcal{R}) = CP(\mathcal{R}_1 \cup \mathcal{R}_2) = CP(\mathcal{R}_1) \cup CP(\mathcal{R}_2)$.

4.4 Generalization to Conditional Term Rewriting Systems

We show now how to generalize our structural analysis to conditional term rewriting systems (CTRSs for short). Firstly, we need some basic terminology.

Definition 4.19 *A* CTRS *is a pair $(\mathcal{R}, \mathcal{F})$ consisting of a signature $\mathcal{F}$ and a set of* conditional rewrite rules *of the form*

$$s_1 = t_1 \ \wedge \ldots \wedge \ s_n = t_n \implies l \rightarrow r$$

with $s_1, \ldots, s_n, t_1, \ldots, t_n, l, r \in \mathcal{T}(\mathcal{F}, \mathcal{V})$. Moreover, we require $l \notin \mathcal{V}$ and $V(r) \subseteq V(l)$ as for unconditional TRSs, i.e. no variable left hand sides and no extra variables on the right side. For our purposes it will be useful to exclude extra variables in the conditions, too. This means to require additionally $\bigcup_{i=1}^{n} \{V(s_i), V(t_i)\} \subseteq V(l)$.[20] If the condition is empty, i.e. $n = 0$, we simply write $l \rightarrow r$. Instead of $(\mathcal{R}, \mathcal{F})$ we also write $\mathcal{R}^{\mathcal{F}}$ or simply $\mathcal{R}$ when $\mathcal{F}$ is clear from the context or irrelevant.

Depending on the interpretation of the equality sign in the conditions of rewrite rules, different reduction relations may be associated with a given CTRS.

Definition 4.20

(1) In a join CTRS *$\mathcal{R}$ the equality sign in the conditions of rewrite rules is interpreted as joinability. Formally this means: $s \rightarrow_{\mathcal{R}} t$ if there exists a rewrite rule $s_1 = t_1 \ \wedge \ldots \wedge \ s_n = t_n \implies l \rightarrow r \in \mathcal{R}$, a substitution σ and a context $C[]$ such that $s \equiv C[\sigma l]$, $t \equiv C[\sigma r]$ and $\sigma s_i \downarrow_{\mathcal{R}} \sigma t_i$ for all $i \in \{1, \ldots, n\}$. For rewrite rules of a join CTRS we shall use the notation $s_1 \downarrow t_1 \ \wedge \ldots \wedge \ s_n \downarrow t_n \implies l \rightarrow r$.*

[20] Extra variables in the conditions may be quite natural in many situations, in particular from a specification or programming point of view. Later on we will discuss the reason for excluding them here.

(2) Semi-equational CTRSs are obtained by interpreting the equality sign in the conditions as convertibility, i.e. as $\overset{*}{\leftrightarrow}$.

Definition 4.21 *The reduction relation corresponding to a given CTRS $\mathcal{R}$ is inductively defined as follows ($\Box$ denotes $\downarrow$ or $\overset{*}{\leftrightarrow}$, respectively):*

$$\begin{array}{rcl} \mathcal{R}_0 & = & \{l \to r \mid l \to r \in \mathcal{R}\}\,, \\ \mathcal{R}_{i+1} & = & \{\sigma l \to \sigma r \mid s_1 \Box t_1 \ \wedge \ldots \wedge \ s_n \Box t_n \Longrightarrow l \to r \ \in \mathcal{R}, \\ & & \sigma s_j \Box_{\mathcal{R}_i} \sigma t_j \text{ for } j = 1, \ldots, n\}\,, \\ s \to_{\mathcal{R}} t & :\Longleftrightarrow & s \to_{\mathcal{R}_i} t \text{ for some } i \geq 0, \text{i.e.} \to_{\mathcal{R}} = \bigcup_{i \geq 0} \to_{\mathcal{R}_i}\,. \end{array}$$

The depth *of a rewrite step $s \to_{\mathcal{R}} t$ is defined to be the minimal i with $s \to_{\mathcal{R}_i} t$.*

In general, conditional rewriting is much more complicated than unconditional rewriting. For instance, the rewrite relation may be undecidable even for complete CTRSs without extra variables in the conditions (cf. [11]).

Definition 4.22 *([5]) A CTRS $\mathcal{R}$ is* decreasing *if there exists an extension $>$ of the reduction relation induced by $\mathcal{R}$ which satisfies the following properties:*

(1) $>$ is noetherian.

(2) $>$ has the subterm property, i.e. $C[s] > s$ for every term s and every non-empty context $C[]$.

(3) If $s_1 = t_1 \ \wedge \ldots \wedge \ s_n = t_n \ \Longrightarrow \ l \to r$ is a rule in $\mathcal{R}$ and σ is a substitution then $\sigma l > \sigma s_i$ and $\sigma l > \sigma t_i$ for $i = 1, \ldots, n$.

A CTRS $\mathcal{R}$ is simplifying[21] *([11]) if there exists a simplification ordering $>$ with (1)-(3) satisfying additionally*

(4) If $s_1 = t_1 \ \wedge \ldots \wedge \ s_n = t_n \ \Longrightarrow \ l \to r$ is a rule in $\mathcal{R}$ and σ is a substitution then $\sigma l > \sigma r$.

A CTRS $\mathcal{R}$ is reductive *([10]) if there exists a well-founded monotonic extension $>$ of the reduction relation induced by $\mathcal{R}$ satisfying (3).*

Clearly, every decreasing system is terminating. Both simplifying and reductive systems are special cases of decreasing ones. In fact, decreasingness exactly captures the finiteness of recursive evaluation of terms (cf. [4]). For decreasing (join) CTRSs all the basic notions are decidable, e.g. reducibility and joinability. Moreover, fundamental results like the critical pair lemma hold for decreasing (join) CTRSs which is not the case in general for arbitrary (terminating join) CTRSs.

In the following we shall tacitly assume that all CTRSs considered are join CTRSs (which is the most important case in practice), except for cases where another kind of CTRSs is explicitly mentioned.

The notions and terminology for disjoint unions of (unconditional) TRSs are generalized in a straightforward manner to CTRSs.

But for generalizing results concerning modular properties of TRSs to the conditional case a careful analysis is necessary. As mentioned by Middeldorp (cf. [20]), the additional complications mainly arise from the fact that the fundamental property

$$s \to_{\mathcal{R}_1 \oplus \mathcal{R}_2} t \quad \Longrightarrow \quad s \to_{\mathcal{R}_1} t \quad \vee \quad s \to_{\mathcal{R}_2} t \quad (*)$$

[21] Conditions (1) and (2) are satisfied by any simplification ordering (over some finite signature).

only holds for unconditional TRSs but not for CTRSs in general. This is due to the fact that for verifying the applicability of an $\mathcal{R}_1$-rule, i.e. for proving the corresponding instantiated conditions, rules from $\mathcal{R}_2$ may be crucial. Consider for instance

Example 4.23

$$\mathcal{R}_1 = \{ \ x \downarrow b \ \wedge \ x \downarrow c \ \Longrightarrow \ a \to a \ \} \ over \ \mathcal{F}_1 = \{a, b, c\},$$

$$\mathcal{R}_2 = \{G(x, y) \to x, G(x, y) \to y\} \ over \ \mathcal{F}_2 = \{G, A\}.$$

Here, we have $a \to_{\mathcal{R}_1 \oplus \mathcal{R}_2} a$ by applying the $\mathcal{R}_1$-rule (x is substituted by $G(b, c)$), but neither $a \to_{\mathcal{R}_1} a$ nor $a \to_{\mathcal{R}_2} a$. Hence, this is also a very simple counterexample to modularity of termination for CTRSs, because both $\mathcal{R}_1$ and $\mathcal{R}_2$ are terminating (the reduction relation of $\mathcal{R}_1$ is empty). Moreover, the infinite $(\mathcal{R}_1 \oplus \mathcal{R}_2)$-derivation $a \to a \to a \to \ldots$ has rank one, a phenomenon which cannot occur in the unconditional case. Note that the system $\mathcal{R}_1$ above has the extra variable x in the condition part of its rule.[22] Therefore it cannot be decreasing. When we forbid extra variables in the conditions then the minimal rank of potential counterexamples is at least 3 as we shall see. But the fundamental property $(*)$ above may still be violated as shown by

Example 4.24 *(see [20] for similar counterexamples)*

$$\mathcal{R}_1 = \{ \ x \downarrow a \ \wedge \ x \downarrow b \ \Longrightarrow \ f(x) \to f(x) \ \} \ over \ \mathcal{F}_1 = \{a, b, f\},$$

$$\mathcal{R}_2 = \{G(x, y) \to x, G(x, y) \to y\} \ over \ \mathcal{F}_2 = \{G, A\}.$$

Here, both $\mathcal{R}_1$ and $\mathcal{R}_2$ are clearly decreasing (and even reductive), hence terminating, but $\mathcal{R}_1 \oplus \mathcal{R}_2$ is non-terminating. This example shows that – as mentioned in the introduction – the conditions

(a) neither R_1 nor R_2 contains a duplicating rule ([22]), and

(c) one of the system $\mathcal{R}_1$, $\mathcal{R}_2$ contains neither collapsing nor duplicating rules ([17])

are sufficient for ensuring modularity of termination only for unconditional TRSs, but not for CTRSs in general. In [18] it is shown that (a) and (c) are sufficient under the additional assumption that both systems are confluent. Moreover, confluence turns out to be a modular property of CTRSs as shown by Middeldorp in [20].

In the following we shall show that the essential features and results of our structural analysis of modular termination for the unconditional case can be generalized to CTRSs in a rather straightforward manner. The numbers of corresponding definitions or results for the unconditional case are given in parentheses.

Let us start with some basic properties of disjoint unions of CTRSs. It is easy to see that conditional reduction steps are rank decreasing, i.e. $s \to_{\mathcal{R}_1 \oplus \mathcal{R}_2} t$ implies $rank(s) \geq rank(t)$. As shown by Middeldorp ([20]) any non-destructive outer reduction step in a mixed term can be abstracted into a 'pure' step using the same rule provided that there are no collapsing rules. Formally we get

Lemma 4.25 *(see Middeldorp [20], p. 74, Proposition 4.3.2) Let $\mathcal{R}_1^{\mathcal{F}_1}$, $\mathcal{R}_2^{\mathcal{F}_2}$ be two collapse-free disjoint CTRSs and let $s, t \in \mathcal{T}(\mathcal{F}_1 \uplus \mathcal{F}_2, \mathcal{V})$ be given. Then $s \xrightarrow{o}_{\mathcal{R}} t$ implies $top(s) \, (\to_{\mathcal{R}_1} \cup \to_{\mathcal{R}_2}) \, top(t)$.*

[22] Hence, strictly spoken $\mathcal{R}_1$ is no CTRS in the sense of definition 4.19.

Note that this result is a technical key lemma which will be important subsequently. Lemma 3.3 is generalized to

Lemma 4.26 *(3.3) Let $\mathcal{R}_1$, $\mathcal{R}_2$ be two terminating disjoint CTRSs such that*

$$D : s_1 \to s_2 \to s_3 \to \dots$$

is an infinite derivation in $\mathcal{R}_1 \oplus \mathcal{R}_2$ (involving only ground terms) of minimal rank, i.e. any derivation in $\mathcal{R}_1 \oplus \mathcal{R}_2$ of smaller rank is finite. Then we have:

(a) $rank(D) \geq 3$.

(b) Infinitely many steps in D are outer steps.

Proof: The proof of (b) is the same as for lemma 3.3 (b). For proving (a) we first remark that $rank(D) = 1$ is impossible.[23] For showing by contradiction that $rank(D) = 2$ is impossible, too, we consider an infinite $(\mathcal{R}_1 \oplus \mathcal{R}_2)$-derivation

$$D' : t_1 \to t_2 \to t_3 \to \cdots .$$

where w.l.o.g. all t_j's are top black and have *rank* 2. Since there can be no collapsing step in this derivation (this would contradict the case $rank(D') = 1$ above) we may apply lemma 4.25 in slightly strengthened version yielding for all j: $t_j \xrightarrow{o}_{\mathcal{R}_1 \oplus \mathcal{R}_2} t_{j+1}$ using an $\mathcal{R}_1$-rule implies $top(t_j) \xrightarrow{o}_{\mathcal{R}_1} top(t_{j+1})$ using the same $\mathcal{R}_1$-rule, and $t_j \xrightarrow{i}_{\mathcal{R}_1 \oplus \mathcal{R}_2} t_{j+1}$ implies $top(t_j) \equiv top(t_{j+1})$. Thus we get the derivation

$$D'' : top(t_1) \to^*_{\mathcal{R}_1} top(t_2) \to^*_{\mathcal{R}_1} top(t_3) \to^*_{\mathcal{R}_1} \cdots .$$

According to (b) infinitely many steps in D'' are outer $\mathcal{R}_1$-steps, hence D'' is an infinite $\mathcal{R}_1$-derivation contradicting termination of $\mathcal{R}_1$.

■

Note, that lemma 3.3 (c) which says that infinitely many steps in D are inner reductions which are destructive at level 2 does not hold for CTRSs in general. To wit, consider the infinite $(\mathcal{R}_1 \oplus \mathcal{R}_2)$-derivation

$$f(G(a,b)) \to f(G(a,b)) \to f(G(a,b)) \to \dots$$

in example 4.24 above where all reductions are outer $\mathcal{R}_1$-steps.

The property of being termination preserving under non-deterministic collapses (cf. definition 3.4) and the central white (and black) abstraction mapping Ψ (cf. definition 3.4) are defined as for the unconditional case:

Definition 4.27 *(3.2) A CTRS $\mathcal{R}$ is said to be* termination preserving under non-deterministic collapses *if termination of $\mathcal{R}$ implies termination of $\mathcal{R} \oplus \{G(x,y) \to x, G(x,y) \to y\}$.*

Definition 4.28 *(3.4) Let $\mathcal{R}_1$, $\mathcal{R}_2$ be two terminating disjoint CTRSs, $\mathcal{R} := \mathcal{R}_1 \oplus \mathcal{R}_2$ and $n \in$ N such that for every $s \in \mathcal{T}(\mathcal{F}_1 \uplus \mathcal{F}_2)$ with $rank(s) \leq n$ there is no infinite $\mathcal{R}$-derivation starting with s. Moreover, let $<_{\mathcal{T}(\mathcal{F}_1 \uplus \{A,G\})}$ be some arbitrary, but fixed total*

[23] Here our general assumption that extra variables in the conditions of rules are forbidden is crucial! See also example 4.23.

ordering on $\mathcal{T}(\mathcal{F}_1 \uplus \{A,G\})$. Then the $\mathcal{F}_2$- (or white*)* abstraction *is defined to be the mapping*

$$\Phi : \mathcal{GT}_{\oplus}^{\leq n} \uplus \{t \in \mathcal{GT}_{\oplus}^{n+1} | root(t) \in \mathcal{F}_1\} \longrightarrow \mathcal{T}(\mathcal{F}_1 \uplus \{A,G\})$$

given by

$$\Phi(t) := \begin{cases} t & \text{if} \quad t \in \mathcal{T}(\mathcal{F}_1) \\ A & \text{if} \quad t \in \mathcal{T}(\mathcal{F}_2) \\ C[\![\Phi(t_1),\ldots,\Phi(t_m)]\!] & \text{if} \quad t \equiv C[\![t_1,\ldots,t_m]\!], m \geq 1, root(t) \in \mathcal{F}_1 \\ CONS(SORT(\Phi^*(SUCC^{\mathcal{F}_1}(t)))) & \text{if} \quad t \equiv C[\![t_1,\ldots,t_m]\!], m \geq 1, root(t) \in \mathcal{F}_2 \end{cases}$$

with

$$\begin{aligned} SUCC^{\mathcal{F}_1}(t) &:= \{t' \in \mathcal{T}(\mathcal{F}_1 \uplus \mathcal{F}_2) | t \to_{\mathcal{R}}^* t', root(t') \in \mathcal{F}_1\}, \\ \Phi^*(M) &:= \{\Phi(t) | t \in M\} \quad \text{for} \quad M \subseteq dom(\Phi), \\ CONS(\langle\rangle) &:= A, \\ CONS(\langle s_1,\ldots,s_{k+1}\rangle) &:= G(s_1, CONS(\langle s_2,\ldots,s_{k+1}\rangle)) \quad \text{and} \\ SORT(\{s_1,\ldots,s_k\} &:= \langle s_{\pi(1)},\ldots,s_{\pi(k)}\rangle, \end{aligned}$$

such that $s_{\pi(j)} \leq_{\mathcal{T}(\mathcal{F}_1 \uplus \{A,G\})} s_{\pi(j+1)}$ for $1 \leq j < k$.

Now the corresponding results for the unconditional case can be generalized to the conditional one.

Lemma 4.29 *(3.6) Let $\mathcal{R}_1$, $\mathcal{R}_2$, $\mathcal{R} := \mathcal{R}_1 \oplus \mathcal{R}_2$, n and Φ be given as in definition 4.28. Then, for any $s,t \in \mathcal{T}(\mathcal{F}_1 \uplus \mathcal{F}_2)$ with $rank(s) \leq n$ and $root(s) \in \mathcal{F}_2$ we have:*

$$s \to_{\mathcal{R}} t \quad \Longrightarrow \quad \Phi(s) \to_{\mathcal{R}_2'}^* \Phi(t),$$

where $\mathcal{R}_2' := \mathcal{R}_{sub}^G := \{G(x,y) \to x, G(x,y) \to y\}$.

Proof: Analogous to the proof of lemma 3.6. ■

Lemma 4.30 *(3.7) Let $\mathcal{R}_1$, $\mathcal{R}_2$ and Φ be given as in definition 4.28. Then, Φ is rank decreasing, i.e. for any $s \in dom(\Phi) := \mathcal{GT}_{\oplus}^{\leq n} \uplus \{t \in \mathcal{GT}_{\oplus}^{n+1} | root(t) \in \mathcal{F}_1\}$ we have $rank(\Phi(s)) \leq rank(s)$.*

Proof: Analogous to the proof of lemma 3.7. ■

Lemma 4.31 *(3.8) Let $\mathcal{R}_1$, $\mathcal{R}_2$, $\mathcal{R} = \mathcal{R}_1 \oplus \mathcal{R}_2$, $\mathcal{R}_2' = \mathcal{R}_{sub}^G = \{G(x,y) \to x,\ G(x,y) \to y\}$, $\mathcal{R}' = \mathcal{R}_1 \uplus \mathcal{R}_2'$, n and the $\mathcal{F}_2$-abstraction Φ be given as in lemma 4.29. Then, for any $s,t \in \mathcal{T}(\mathcal{F}_1 \uplus \mathcal{F}_2)$ with $rank(s) \leq n+1$, $root(s) \in \mathcal{F}_1$ and $s \to_{\mathcal{R}} t$ we have:*

(a) If $s \xrightarrow{o}_{\mathcal{R}} t$ using an $\mathcal{R}_1$-rule is not destructive at level 1 then $\Phi(s) \xrightarrow{o}_{\mathcal{R}'} \Phi(t)$ using the same $\mathcal{R}_1$-rule, and moreover this step is also not destructive at level 1.

(b) If $s \xrightarrow{o}_{\mathcal{R}} t$ using an $\mathcal{R}_1$-rule is destructive at level 1 then $\Phi(s) \xrightarrow{o}_{\mathcal{R}'} \Phi(t)$ using the same $\mathcal{R}_1$-rule, and moreover this step is also destructive at level 1.

(c) If $s \xrightarrow{i}_{\mathcal{R}} t$ is not destructive at level 2 then $\Phi(s) \xrightarrow{i}{}^*_{\mathcal{R}_2'} \Phi(t)$ with all steps not destructive at level 2.

(d) If $s \xrightarrow{i}_{\mathcal{R}} t$ is destructive at level 2 then $\Phi(s) \xrightarrow{i}{}^+_{\mathcal{R}_2'} \Phi(t)$ such that exactly one of these steps is destructive at level 2.

Proof sketch: The general proof structure is as follows: We show by induction over the depth of rewriting steps the implication

$$s \to_{\mathcal{R}} t \quad \Longrightarrow \quad \Phi(s) \to^*_{\mathcal{R}'} \Phi(s)\,.$$

The case distinction for inner and outer as well as for destructive and non-destructive reduction steps proceeds as in lemma 3.8 yielding a proof of (a)-(d).[24] ■

Finally we obtain the generalized structure theorem for CTRSs.

Theorem 4.32 *(3.9) Let $\mathcal{R}_1, \mathcal{R}_2$ be two disjoint (finite) CTRSs which are both terminating such that their disjoint union $\mathcal{R}_1 \oplus \mathcal{R}_2$ is non-terminating. Then $\mathcal{R}_j$ is not termination preserving under non-deterministic collapses for some $j \in \{1,2\}$ and the other system $\mathcal{R}_k$, $k \in \{1,2\} \setminus \{j\}$, is collapsing. Moreover, the minimal rank of counterexamples in $\mathcal{R}_j \oplus \{G(x,y) \to x,\, G(x,y) \to y\}$ is less than or equal to the minimal rank of counterexamples in $\mathcal{R}_1 \oplus \mathcal{R}_2$.*

Proof: Let $\mathcal{R}_1, \mathcal{R}_2$ with $\mathcal{R} := \mathcal{R}_1 \oplus \mathcal{R}_2$ be given as stated above. We consider a minimal counterexample, i.e. an infinite $\mathcal{R}$-derivation

$$D: \quad s_1 \to s_2 \to s_3 \to \quad \ldots$$

of minimal rank, let's say $n+1$. W.l.o.g. we may assume that all the s_i's are top black, i.e. $\mathcal{F}_1$-rooted ground terms having *rank* $n+1$. Since the preconditions of definition 4.28 are satisfied we may apply the white ($\mathcal{F}_2$-) abstraction function Φ to the s_i's. As it will be shown this yields an infinite $\mathcal{R}'$-derivation

$$D': \quad \Phi(s_1) \to^* \Phi(s_2) \to^* \Phi(s_3) \to^* \quad \ldots$$

where $\mathcal{R}' := \mathcal{R}_1 \oplus \mathcal{R}'_2$ with $\mathcal{R}'_2 := \mathcal{R}^G_{sub} = \{G(x,y) \to x,\, G(x,y) \to y\}$. Using lemma 4.31 we conclude that for any step $s_j \to s_{j+1}$ in D we have

$$s_j \xrightarrow{o}_{\mathcal{R}} s_{j+1} \quad \Longrightarrow \quad \Phi(s_j) \xrightarrow{o}_{\mathcal{R}'} \Phi(s_{j+1})\,,$$
$$s_j \xrightarrow{i}_{\mathcal{R}} s_{j+1} \quad \Longrightarrow \quad \Phi(s_j) \xrightarrow{i\,*}_{\mathcal{R}'} \Phi(s_{j+1}).$$

Hence, D' is indeed an $\mathcal{R}'$-derivation. Since according to lemma 4.26 (b) infinitely many steps in D are outer ones, the derivation D' is infinite, too. But this means that $\mathcal{R}_1$ is not termination preserving under non-deterministic collapses. Moreover, under the assumption that $\mathcal{R}_2$ is non-collapsing the $\mathcal{F}_2$-abstraction of principal subterms of a minimal counterexample always yields the constant A which implies that the transformed infinite derivation is an $\mathcal{R}_1$-derivation contradicting termination of $\mathcal{R}_1$. Thus $\mathcal{R}_2$ must be collapsing. Lemma 4.30 finally implies $rank(D') \leq rank(D)$ which finishes the proof. ■

Corollary 4.33 *(3.10) Termination (and hence also completeness) is modular for the class of (finite) CTRSs which are termination preserving under non-deterministic collapses.*

[24] Note that the assumption of having no extra variables in the conditions is important because this would cause problems with the *rank* of instantiated condition terms. In that case substitution of the extra variables in the condition part which are implicitly existentially quantified might yield terms of arbitrarily high *rank* which in turn might prevent Φ from being well-defined for these instantiated terms.

As in the unconditional case this general result can now be exploited for deriving a couple of sufficient criteria for modular termination of CTRSs.

Definition 4.34 *(3.16) A CTRS $\mathcal{R}^{\mathcal{F}}$ is said to be* non-deterministically collapsing *if there exists a term $s[x,y] \in \mathcal{T}(\mathcal{F},\mathcal{V})$ with $x,y \in \mathcal{V}$ such that $s[x,y] \to_{\mathcal{R}}^{+} x$ and $s[x,y] \to_{\mathcal{R}}^{+} y$, i.e. if some term can be reduced to two distinct variables.*

Lemma 4.35 *(3.17) Termination is modular for the class of (finite) CTRSs which are non-deterministically collapsing.*

Definition 4.36 *(3.18) Let $\mathcal{R}^{\mathcal{F}}$ be a CTRS and $f \in \mathcal{F}, \mathcal{F}' \subseteq \mathcal{F}$. Then, $\mathcal{R}^{\mathcal{F}}$ is said to be* f-simply terminating *if $\mathcal{R}^{\mathcal{F}} \cup \mathcal{R}^{f}_{sub}$ with $\mathcal{R}^{f}_{sub} := \{f(x_1,\ldots,x_j,\ldots,x_n) \to x_j | 1 \le j \le n\}$ is terminating. $\mathcal{R}^{\mathcal{F}}$ is* $\mathcal{F}'$-simply terminating *if $\mathcal{R}^{\mathcal{F}} \cup \bigcup_{f \in \mathcal{F}'} \mathcal{R}^{f}_{sub}$ is terminating. $\mathcal{R}^{\mathcal{F}}$ is* simply terminating *if $\mathcal{R}^{\mathcal{F}}$ is $\mathcal{F}$-simply terminating, i.e. $\mathcal{R}^{\mathcal{F}} \cup \bigcup_{f \in \mathcal{F}} \mathcal{R}^{f}_{sub}$ is terminating.*

Clearly, simple termination again implies termination for CTRSs.

Lemma 4.37 *(3.19) Let $\mathcal{R}^{\mathcal{F}}$ be f-simply terminating CTRSs for some $f \in \mathcal{F}$ of arity greater than 1. Then $\mathcal{R}^{\mathcal{F} \cup \mathcal{F}'}$ is $(\{f\} \cup \mathcal{F}')$-simply terminating for any $\mathcal{F}'$ with $\mathcal{F}' \cap \mathcal{F} = \emptyset$.*

Corollary 4.38 *(3.20) Let $\mathcal{R}_1^{\mathcal{F}_1}, \mathcal{R}_2^{\mathcal{F}_2}$ be two (finite) disjoint CTRSs with $f_1 \in \mathcal{F}_1, f_2 \in \mathcal{F}_2$ of arity greater than 1 such that $\mathcal{R}_i$ is f_i-simply terminating for $i = 1,2$. Then the disjoint union $\mathcal{R}_1 \oplus \mathcal{R}_2$ is (f_1- and f_2-simply) terminating, too.*

A characterization of simple termination of CTRSs analogous to the case of unconditional TRSs (see lemma 3.22) is not possible in a straightforward manner. Obviously, any simply terminating CTRS can be shown to be terminating by some simplification ordering, but not vice-versa in general. To see this, let us have again a look on example 4.24.

Example 4.39 *(example 4.24 continued)*
Consider $\mathcal{R}_1 = \{\ x \downarrow a \ \wedge\ x \downarrow b \implies f(x) \to f(x)\ \}$ over the extended signature $\mathcal{F}_1 = \{a,b,f,G\}$, with G binary. Here the reduction relation induced by $\mathcal{R}_1$ is empty, hence any simplification ordering trivially suffices for ensuring termination of $\mathcal{R}_1$. But, due to the non-termination of $\mathcal{R}_1 \cup \{f(x) \to x, G(x,y) \to x, G(x,y) \to y\}$, $\mathcal{R}_1$ is not simply terminating .

Note moreover that every simplifying CTRS is clearly simply terminating but not vice-versa in general. Simple termination even does not imply decreasingness as shown e.g. by the CTRS consisting of the single rule $a \downarrow b \implies a \to a$.

By specializing corollary 4.38 we finally obtain

Theorem 4.40 *Simple termination is modular for the class of (finite) CTRSs $\mathcal{R}^{\mathcal{F}}$ such that there exists at least one function symbol $f \in \mathcal{F}$ of arity greater than 1.*

Before concluding let us mention some aspects not yet handled. Firstly our results have only been proved for join CTRS. But it should (at least be partially) possible to extend them to the semi-equational case. Note that again new subtle effects may occur in semi-equational CTRSs. For instance, lemma 4.26 (a) does not hold any more. To wit, consider

Example 4.41 $\mathcal{R}_1 = \{\ b \stackrel{*}{\leftrightarrow} c \implies a \to a\ \}, \mathcal{R}_2 = \{G(x,y) \to x, G(x,y) \to y\}$.

Both $\mathcal{R}_1$ and $\mathcal{R}_2$ are terminating (and decreasing) but $\mathcal{R}_1 \oplus \mathcal{R}_2$ is non-terminating. We even have a counterexample of *rank* 1, namely $a \rightarrow_{\mathcal{R}_1 \oplus \mathcal{R}_2} a \rightarrow_{\mathcal{R}_1 \oplus \mathcal{R}_2} a \ldots$ because we have $b \leftarrow G(b,c) \rightarrow c$ in $\mathcal{R}_2$, hence $b \xrightarrow{*}_{\mathcal{R}_2} c$. Considered as join CTRS $\mathcal{R}_1 \oplus \mathcal{R}_2$ is terminating, however!

Secondly, the variable conditions required for CTRSs should be investigated in more detail. It should be possible to allow extra variables in the conditions. Extra variables in right hand sides seem to be much more difficult to handle than extra variables only in the conditions (see the discussion in [20]).

Moreover, the relationship between those results in [20] dealing with sufficient conditions for modular termination of CTRSs and our results should be clarified.

5 Conclusion

We have presented a structural analysis of minimal counterexamples to modular termination of rewriting. It has been shown that the abstract property of TRSs to be termination preserving under non-deterministic collapses is crucial for the invariance of termination under disjoint unions. Although this property turns out to be undecidable in general it provides the basis for a couple of sufficient criteria for ensuring modularity of termination. For that purpose we have developed a specialized version of the increasing interpretation method for proving termination of rewriting. Our general approach and the resulting sufficient conditions for modularity of termination generalize known results of [22], [17], [14] and [15]. In particular, the basic ideas and constructions have been shown to be also applicable to more general situations, namely for (non-disjoint) unions of TRSs with common constructors as well as for conditional TRSs. Moreover, we have given counterexamples for some interesting conceivable conjectures, namely the modularity of the non-self-embedding property as well as the invariance of simple termination under hierarchical combinations of TRSs. And finally, a very simple class of examples has been presented which proves that the minimal rank of non-terminating derivations in disjoint unions of terminating TRSs may be arbitrarily high. This reflects in a sense the very subtle interaction of rewriting in disjoint unions and shows that arbitrarily complicated layer structures may be essential w.r.t. the termination behaviour.

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