Graph Isomorphism: Some Special Cases

Rolf Socher

SEKI-Report SR-87-10 August 1987

Graph Isomorphism: Some special cases

Rolf Socher, Fachbereich Informatik, Universität Kaiserslautern
Postfach 3049, D-6750 Kaiserslautern, W.-Germany
net-address: UUCP ..!mcvax!unido!uklirb!socher

Abstract:
This paper investigates some special cases of graphs with a small number of vertices or edges where a characteristic property of the vertices and edges already determines the graph up to isomorphism. We also present counterexamples that show the limits of this approach to the graph isomorphism problem. The main interest in graph isomorphism for automated deduction lies in the fact that the problem of deciding whether a clause is a variant of another clause is a generalization of the graph isomorphism problem.

1. Introduction

For a deduction system the detection of equivalence or entailment of two clauses is very useful [Ei81]. Since equivalence as well as entailment of two clauses is undecidable in general, other concepts that are stronger than equivalence and implication have been used in automated theorem proving, e.g. the notion of subsumption of two clauses [CL73]. In [So87] the concept of a variant of a clause, a specialization of general equivalence, is investigated. It is easy to see that the decision whether a clause is a variant of another clause is a generalization of the graph isomorphism problem:

The problem to decide whether the clauses

$$C = \{Pxy, Pyz, Pxz\} \quad \text{and } D = \{Pvu, Pwu, Pvw\}$$

are variants of each other, is equivalent to the following isomorphism problem for directed graphs:

In a similar way clauses whose literals all have the same predicate symbol with arity m≥2 correspond to so called m-graphs, that are sets of m-tuples. In order to treat clauses containing different predicate symbols we introduce so called labeled graphs in chapter 4.

The graph isomorphism problem can be solved in the following standard way: possible pairings of vertices (i.e. bijective mappings from the vertices of the first graph to the vertices of the second) are tested, whether they yield a graph isomorphism. The (exponential) number of possible pairings is restricted by the fact that graph isomorphisms must preserve some properties of vertices and edges. Several properties that may be useful to restrict the set of possible bijective mappings have been proposed [Un64], [Kn71]. But not all of these can be generalized to m-graphs. One of those properties, which can be used also for m-graphs, is the characteristic function of a graph [So87] that assigns to each vertex the pair <number of incoming arcs, number of outgoing arcs>. Another advantage of the characteristic is its easily being computable.

In this paper we investigate certain special cases of m-graphs with a small number of vertices or arcs where the agreement of the two m-graphs with respect to their characteristics is already a sufficient condition for isomorphism. Of course those cases are not representative for most of the usual cases where graph isomorphism problems arise. But in the field of automated deduction clauses with only three variables or three literals are not rare.

2. Notation

An **m-graph** G is a pair G = (V,E) where V is a finite set of vertices and E is a set of m-tuples of elements of V (called edges) such that no edge has a multiple occurrence of a vertex. (For m=2 this corresponds to a directed graph). Since no confusion can arise, we simply use the term graph. We call m the **degree** of the m-graph G and write m=deg(G).

Let G=(V,E) and H=(W,F) be m-graphs. A **graph isomorphism** from G onto H is a bijective mapping $\sigma: V \rightarrow W$ such that $(v_1,\ldots,v_m) \in E$ implies $(v_1\sigma,\ldots,v_m\sigma) \in F$. We write G≅H, if there is a graph isomorphism from G onto H.

For each $v \in V$ let v^* be the **characteristic** of v, i.e. the m-tuple $(v_1^*,\ldots,v_m^*)$ where v_i^* is the number

of occurrences of v at coordinate position i in m-tuples of E. We define $V^* := \{v^* \mid v \in V\}$.
For each $e=(v_1,...,v_m) \in E$ let e^* be the m×m-matrix (e_{ij}^*) where e_{ij}^* is the number of occurrences of v_j at coordinate position i in m-tuples of E. We define $E^* := \{e^* \mid e \in E\}$

3. Graph Isomorphism

In the following G=(V,E) and H=(W,F) are connected m-graphs with $|V|=|W|$ and $|E|=|F|$.
The main results of this chapter are:

- in the case $|V| \leq 3$ the characteristics of the vertices determine a graph up to isomorphism, i.e. $V^*=W^*$ implies $G \cong H$.
- in the case $|E| \leq 3$ the characteristics of the vertices together with the characteristics of the edges determine a graph up to isomorphism.

As all m-graphs with m=1 and equal number of vertices are isomorphic we consider only the case $m \geq 2$.
Since no vertex occurs more than once in one edge, we have the following

3.1 Lemma:
If $|V|=3$, then $m \leq 3$ and

$$0 \leq v_i^* \leq 2 \text{ holds for all } v \in V,\ 1 \leq i \leq m$$ ■

3.2 Lemma:
Let $|V|=3$ with $V=\{x,y,z\}$ and $V^*=W^*$. If $x^*=y^*$ and all components of z^* are even then G and H are isomorphic.

Proof:
Let $W=\{u,v,w\}$ with $u^*=v^*=x^*=y^*$ and $w^*=z^*$. First we remark that from 2.1 follows $z_i^*=0$ or $z_i^*=2$ for each $1 \leq i \leq m$; the same holds for w. Let $\sigma: V \to W$ be defined by $x\sigma=u$, $y\sigma=v$, $z\sigma=w$.
a) Let e be an element of E that contains z at position i. Then $z_i^*=2$. Hence there must be another element $f \in E$ that contains z at position i. It is easy to see that if x occurs at a position $j \neq i$ in e then y must occur a the same position in f and vice versa. Analogously there must be e', f' in F containing w at position i, and if u occurs at position $j \neq i$ in e' then v must occur at position j in f' and vice versa. Hence $e\sigma=e'$ or $e\sigma=f'$; in both cases $e\sigma \in F$.
b) Let g be an element of E that does not contain z. From a) and the fact that $|E|=|F|$ follows that there is an element g' in F that does not contain w. This is only possible if m=2. Thus we can assume w.l.o.g. that g=(x,y). Let $E'=E \setminus \{e \in E \mid e \text{ contains } z\}$. Let $G' = (V \setminus \{z\}, E')$. Since according to a) for each element of E that contains z and x there is a corresponding element that contains z and y, the characteristics of x and y with respect to G' must be equal, too. This implies that $h:=(y,x) \in E$. The same argument shows that $(u,v) \in F$ and $(v,u) \in F$. Hence $g\sigma \in F$. ■

3.3 Lemma:
Let $|V|=3$, $V^*=W^*$. If $v_i=1$ for all $v \in V$ and $i \leq m$, then G and H are isomorphic.

Proof:

Let V:={x,y,z}. In the case m=2 one of the edges (x,y) and (x,z) must be in E. W.l.o.g. let (x,y) be in E. It is easy to see that the only possibility for E is E={(x,y),(y,z),(z,x)}. Analogously one of (u,v) and (u,w) must be in F, say (u,v). Then F must be {(u,v),(v,w),(w,u)}. Hence G and H are isomorphic. In the case m=3 one of the edges (x,y,z) or (x,z,y) must be in E, let (x,y,z) be that edge. Then we obtain E={(x,y,z),(y,z,x),(z,x,y)}. Analogously if the edge (u,v,w) is in F, then we obtain F={(u,v,w),(v,w,u),(w,u,v)} and again G and H are isomorphic. ■

3.4 Lemma:

If |V|=3 and |V*|=3, i.e. the v* are pairwise different, then G is isomorphic to H iff V*=W*.

Proof:

Suppose V*=W* and let V={x,y,z} and W={u,v,w} such that x*=u*, y*=v* and z*=w*. The elements x*, y* and z* are pairwise different. Define the mapping φ by $x\varphi=u$, $y\varphi=v$, $z\varphi=w$.

a) Let m=2. We show that $e\in E$ implies $e\varphi\in F$.

Let e=(x,y). We show that $e\varphi=(u,v)\in F$. Suppose $(u,v)\notin F$. We have $x_1^*\geq 1$ and $y_2^*\geq 1$, hence also $u_1^*\geq 1$ and $v_2^*\geq 1$. Since $(u,v)\notin F$, (u,w) and (w,v) must be in F. This implies that $z_1^*\geq 1$ and $z_2^*\geq 1$. If $(x,z)\in E$, then $x_1^*\geq 2$, which implies $u_1^*\geq 2$, and then (u,v) must be in F. If (z,y) is in E, then $y_2^*\geq 2$ which implies $v_2^*\geq 2$ and then (u,v) must be in F, too. If neither (x,z) nor (z,y) is in E, then (y,z) and (z,x) must be in E. If E={(x,y), (y,z), (z,x)} then x*=y*. So there must be at least one more element in E, say (y,x). Then $y_1^*\geq 2$, whence $v_1^*\geq 2$ and $x_2^*\geq 2$, whence $u_2^*\geq 2$. Then (v,w) and (w,u) must be in F. Since z*=w* this shows that (z,y) and (x,z) must be in E. But then E={(x,y), (y,x), (y,z), (z,y), (x,z), (z,x)}and x*=y*=z* which is a contradiction.

b) Let m=3. We show that $e\in E$ implies $e\varphi\in F$.

Let e=(x,y,z). We show that $e\varphi=(u,v,w)\in F$. We have $x_1^*\geq 1$, $y_2^*\geq 1$, $z_3^*\geq 1$, hence $u_1^*\geq 1$, $v_2^*\geq 1$, $w_3^*\geq 1$. Suppose $(u,v,w)\notin F$. Since u must occur at first position in F, v at second and w at third position, (u,w,v), (w,v,u) and (v,u,w) must be in F. Since $u^*\neq v^*$ there must be at least one more element in F, say (w,u,v). Hence $v_3^*\geq 2$ and therefore $y_3^*\geq 2$. But this implies that (z,x,y) and (x,z,y) must be in E, hence also $x_1^*\geq 2$. From this follows $u_1^*\geq 2$, which says that (u,w,v) and (u,v,w) are in F which is a contradiction. Hence $(u,v,w)\in F$.

3.5 Theorem:

If |V|=3 then G is isomorphic to H iff V*=W*.

Proof:

Let V={x,y,z} and e:=|E|. If G is isomorphic to H then clearly V*=W*. Now suppose V*=W*. We have to show that G and H are isomorphic.

It is easy to verify that the following conditions must hold:

(1) $x_i^*+y_i^*+z_i^* = e$ for $1\leq i\leq m$

(2) $x_i^* + y_i^* - 2 \leq z_k^*$ for any pair (i,k) with $i\neq k$

a) Let m=2.

If all the v* for $v\in\{x,y,z\}$ are different, then G and H are isomorphic by 3.4. So we assume that at least two of the v* are equal, say x*=y*. Then (1) yields

(1´) $2x_1^* + z_1^* = 2x_2^* + z_2^*$.

Case 1: $x_1^*=2$. Then $2x_2^*+z_2^*-z_1^*=4 \Rightarrow 2x_2^* \geq 4-z_2^* \geq 2 \Rightarrow x_2^* \geq 1$. If $x_2^*=1$, then $z_2^*-z_1^*=2$ which is only possible for $z_2^*=2$ and $z_1^*=0$. If $x_2^*=2$ then $z_2^*-z_1^*=0$, i.e. $z_1^*=z_2^*$. From (2) we obtain $z_2^*=2$, hence also $z_1^*=2$. In both cases z_1^* and z_2^* are even, thus G and H are isomorphic according to 3.1.

Case 2: $x_1^*=1$. If $x_2^*=2$, then we have a situation analogous to Case 1. So let $x_2^*=1$. Then either $z_1^*=z_2^*=1$ and G and H are isomorphic according to 3.2 or $z_1^*=z_2^*=2$ and then z_1^* and z_2^* are even and again G and H are isomorphic. If $x_2^*=0$, then $z_1^*+2=z_2^*$. This is only possible for $z_2^*=2$, $z_1^*=0$, so z_1^* and z_2^* are even and G and H are isomorphic.

b) Let m=3. As in a) let $x^*=y^*$.

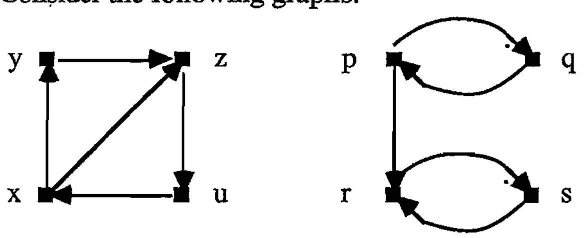
Case 1: $x_1^*=2$. Then from (2) follows $z_2^* = z_3^* = 2$. Furthermore from (1) follows $z_1^*=2x_2^*-2$, hence z_1^* is even. In this case all z_i^* are even and the assertion follows from 3.1.

Case 2: $x_1^*=1$. If $x_2^* = 2$ or $x_3^* = 2$ then an analogous argument to Case 1 proves the assertion. If $x_2^*=0$ then (1) yields $2+z_1^*=z_2^*$ which can only occur if $z_2^*=2$ and $z_1^*=0$. Then $z_3^*=2-x_3^*$ is even. Analogously if $x_3^*=0$ it follows that all z_i^* are even. Now let $x_2^*=x_3^*=1$. Then from (1) follows $z_1^*=z_2^*=z_3^*$. If $z_1^*=1$ then lemma 3.2 proves the assertion, if $z_1^*=2$ then the assertion follows from lemma 3.1 ■

The following example shows that in the case of a graph with four vertices the characteristics of the vertices do not characterize the graph up to isomorphism:

3.6 Example:

Consider the following graphs:



The equalities

$$x^* = p^*,\ y^* = q^*,\ z^* = r^* \text{ and } u^* = s^*$$

hold but nevertheless the two graphs are not isomorphic.

For the case of graphs with less than four *edges* the characteristics of the vertices together with the characteristics of the edges are needed to determine the graph uniquely.

3.7 Theorem:

If $|E|=3$ then G is isomorphic to H iff $V^*=W^*$ and $E^*=F^*$.

Proof:

We show that any bijective mapping $\varphi: E \to F$ with $(e\varphi)^*=e^*$ for all $e \in E$ induces a bijective mapping from V to W.

Let $E=\{e_1,e_2,e_3\}$ and $F=\{f_1,f_2,f_3\}$

Let $x \in V$. If there is no $y \in V$ with $y^*=x^*$, then there must be exactly one $a^* \in W^*$ with $x^*=a^*$. Then φ

induces a mapping φ' with $x\varphi'=a$.

Let there be a $y\in V$ with $y^*=x^*$. W.l.o.g we assume that x occurs in the first 3 positions.

Case 1: $\Sigma x_i^*=3$. Then $\Sigma y_i^*=3$. Since $|E|=3$, $x_i\leq 1$ and $y_i\leq 1$ for all i. This means that there must be distinct elements $e_1=(x,..,..,..)$, $e_2=(.,x,..,..)$ and $e_3=(.,.,x,..)$ in E. The same elements with y instead of x must be in E, too. There are only two possibilities: Either

$E=\{(x,y,u,..),(v,x,y,..),(y,w,x,..)\}$ or

$E=\{(y,x,u,..),(v,y,x,..),(x,w,y,..)\}$.

W.l.o.g. we assume that $E=\{(x,y,u,..),(v,x,y,..),(y,w,x,..)\}$.

Case 1.1: If u=v=w, then all elements in E* have the following form:

$$\begin{bmatrix} 1 & 1 & 1 & . \\ 1 & 1 & 1 & . \\ 1 & 1 & 1 & . \\ . & . & . & . \end{bmatrix}$$

Since E*=F* there must be elements a,b,c in W such that $F=\{(a,b,c,..), (c,a,b,..), (b,c,a,..)\}$. Now it is easy to see that any bijective mapping φ from E to F with $(e\varphi)^*=e^*$ induces a bijective mapping φ' from V to W. (E.g. if $(x,y,u,..)\varphi= (a,b,c,..)$, $(u,x,y,..)\varphi= (c,a,b,..)$, $(y,u,x,..)\varphi= (b,c,a,..)$, then $x\varphi'=a$, $y\varphi'=b$, $u\varphi'=c$.).

Case 1.2: If $u\neq v$, then $E=\{(x,y,u,..), (v,x,y,..), (y,w,x,..)\}$ and the elements in E* have the following form:

$$\begin{bmatrix} 1 & 1 & * & . \\ 1 & 1 & * & . \\ 1 & 1 & * & . \\ . & . & . & . \end{bmatrix}, \begin{bmatrix} * & 1 & 1 & . \\ * & 1 & 1 & . \\ * & 1 & 1 & . \\ . & . & . & . \end{bmatrix}, \begin{bmatrix} 1 & * & 1 & . \\ 1 & * & 1 & . \\ 1 & * & 1 & . \\ . & . & . & . \end{bmatrix}$$

where in each matrix at least one of the asterisks is equal to 0. From this follows that the e_i^* are pairwise different. Since E*=F*, there must be $a,b\in W$ with $F=\{(a,b,..), (.,a,b,..), (b,.,a,..)\}$. Any bijective mapping φ from E to F with $(e\varphi)^*=e^*$ must map (x,y,u,..) to (a,b,..), (v,x,y,..) to (.,a,b,..) and (y,w,x,..) to (b,.,a,..) and hence induces a mapping $\varphi':V\rightarrow W$ with $x\varphi'=a$, $y\varphi'=b$.

Case 2: $\Sigma x_i=2$. Then $\Sigma y_i=2$ and all the x_i and y_i are 1. E must be one of the following sets:

2.1 $\{(x,y,..), (y,x,..),(u,v,..)\}$

2.2 $\{(x,y,..), (u,x,..), (y,v,..)\}$, where $u\neq v$

2.3 $\{(x,y,..), (u,x,..), (y,u,..)\}$

In case 2.1, E* has the following form:

$$\begin{bmatrix} 1 & 1 & . \\ 1 & 1 & . \\ . & . & . \end{bmatrix}, \begin{bmatrix} 1 & 1 & . \\ 1 & 1 & . \\ . & . & . \end{bmatrix}, \begin{bmatrix} 1 & 0 & . \\ 0 & 1 & . \\ . & . & . \end{bmatrix}$$

Since F*=E*, there must be elements $a,b,c,d\in W$ such that $F=\{(a,b,..), (b,a,..), (c,d,..)\}$. Any bijective mapping φ from E to F with $(e\varphi)^*=e^*$ for all $e\in E$ induces a mapping $\varphi':V\rightarrow W$ with $x\varphi'=a$, $y\varphi'=b$ or $x\varphi'=b$, $y\varphi'=a$.

In case 2.2, E* has the following form:

$$\begin{bmatrix} 1 & 1 & . \\ 1 & 1 & . \\ . & . & . \end{bmatrix}, \begin{bmatrix} 1 & 1 & . \\ 0 & 1 & . \\ . & . & . \end{bmatrix}, \begin{bmatrix} 1 & 0 & . \\ 1 & 1 & . \\ . & . & . \end{bmatrix}$$

F* has the same form and this can only be the case, if F={(a,b,..), (c,a,..), (b,d,..)} for some a,b,c,d$\in$W. Any bijective mapping φ from E to F with $(e\varphi)^*=e^*$ for all e$\in$E must map (x,y,..) to (a,b,..), (u,x,..) to (c,a,..) and (y,v,..) to (b,d,..) and hence induces a mapping φ':V$\rightarrow$W with $x\varphi'=a$, $y\varphi'=b$.

In case 2.3, all elements of E* have the following form:

$$\begin{bmatrix} 1 & 1 & . \\ 1 & 1 & . \\ . & . & . \end{bmatrix}$$

All elements of F* have the same form and this can only be the case, if F={(a,b,..), (b,c,..), (c,a,..)} for some a,b,c$\in$W. Now it is easy to see that any bijective mapping φ from E to F with $(e\varphi)^*=e^*$ for all e$\in$E induces a bijective mapping φ' from V to W.

Now we have established for all cases that E*=F* implies that G and H are isomorphic. ■

The following example shows that even for graphs with only two edges the premise E*=F* of theorem 3.7 cannot be dropped.

3.8 Example:

Let G=(V,E) and H=(V,F) with V={x,y,z,u,v,w}, W={x',y',z',u',v',w'}

E = {(x,u,y,v), (w,x,z,y)}, F = {(x',u',v',y'), (w',x',y',z')}.

The two graphs G and H are not isomorphic although V*=W* holds.

4. Labeled Graphs

M-graphs correspond to clauses C={L_1,...,L_n} where all literals L_i have the same predicate symbol. In order to generalize some of the above results to arbitrary clauses we introduce the notion of simultaneous isomorphism of graphs: A clause C whose literals may have different predicate symbols from a set $\mathcal{P}$ corresponds to a family $\mathcal{G}$ = (G(P) | P$\in$ $\mathcal{P}$) of graphs G(P) = (V, E(P)) over the same set V of vertices.

The main result of this chapter are:

- in the case |V|≤3 the characteristics of the vertices determine a labeled graph up to isomorphism, if there is no graph G(P) in which the characteristics of the three vertices are equal.
- in the case where each |E(P)|≤2 the characteristics of the vertices together with the characteristics the edges determine a labeled graph up to isomorphism.

<u>4.1 Definition:</u>

(i) Let $\mathcal{P} := \bigcup_{n\in N} \mathcal{P}_n$ where each $\mathcal{P}_n$ is a finite set of n-place labels and V a finite set of vertices.

A **labeled graph** is a family $\mathcal{G}$ = (G(P) | P$\in$ $\mathcal{P}$) of graphs G(P) = (V, E(P)) where deg(G(P))=m, iff $P \in \mathcal{P}_m$.

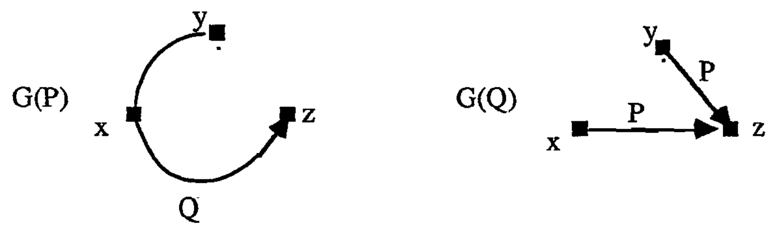
(ii) Let $\mathcal{G} = (V, E)$ and $\mathcal{H} = (W, F)$ be labeled graphs. A graph isomorphism from $\mathcal{G}$ to $\mathcal{H}$ is a bijective mapping $\sigma: V \rightarrow W$ that induces a graph isomorphism from G(P) onto H(P) for each $P \in \mathcal{P}$.

4.2 Definition:

For each $v \in V$ and $P \in \mathcal{P}_m$ let v^P be the m-tuple $(v_1^P,\dots,v_m^P)$ where v_i^P is the number of occurences of v at coordinate position i in all edges of E(P). Let $V^P := \{v^P \mid v \in V\}$. Let $v^* = \{(P,v^P) \mid P \in \mathcal{P}\}$ and $V^* = \{v^* \mid v \in V\}$.

4.3 Example:

The clause {Qyxz, Pxz, Pyz} corresponds to the family (G(Q),G(P)) of graphs:



In the following $\mathcal{G} = (V, E(P) \mid P \in \mathcal{P})$ and $\mathcal{H} = (W, F(P) \mid P \in \mathcal{P})$ are labeled graphs with $|V| = |W|$ and $|E(P)| = |F(P)|$ for each $P \in \mathcal{P}$.

4.4 Lemma:

If $|V|=3$ and $|\{v^P \mid v \in V\}|=3$ for all $P \in \mathcal{P}$, then $\mathcal{G}$ is isomorphic to $\mathcal{H}$, iff $V^* = W^*$.

Proof:

The generalization of the proof of lemma 3.4 to prove 4.4 is tedious but straightforward. ■

4.5 Lemma:

Let $|V|=3$ and $V=\{x,y,z\}$. Suppose $x^P=y^P \neq z^P$ for all $P \in \mathcal{P}$. Then $\mathcal{G}$ is isomorphic to $\mathcal{H}$ iff $V^* = W^*$.

Proof:

For each $m \in \mathbb{N}$ and $P \in \mathcal{P}_m$ the equations

(1) $x_i^P+y_i^P+z_i^P = |E(P)|$ for $1 \leq i \leq m$

(2) $x_i^P + y_i^P - 2 \leq z_k^P$ for any pair (i,k) with $i \neq k$

must hold. The proof of 3.5 shows that all components of z^P must be even. Then an argument analogous to that of the proof of 3.2 proves that $\mathcal{G}$ is isomorphic to $\mathcal{H}$ ■

4.6 Lemma:

Let $|V|=3$, $V=\{x,y,z\}$. Suppose that $x^P=y^P \neq z^P$ for some $P \in \mathcal{P}$, and $x^Q \neq y^Q$ for some $Q \in \mathcal{P}$. Then $\mathcal{G}$ is isomorphic to $\mathcal{H}$ iff $V^P = W^P$ for all $P \in \mathcal{P}$.

Proof:

Since $x^Q \neq y^Q$ we have $x^* \neq y^*$ and since $x^P= y^P \neq z^P$ we have $x^* \neq z^*$ and $y^* \neq z^*$. Let $W=\{u,v,w\}$ with

$x^*=u^*$, $y^*=v^*$ and $z^*=w^*$. We define $\sigma:V\to W$ by $x\sigma=u$, $y\sigma=v$, $z\sigma=w$. Again all components of z^P must be even. The proof of lemma 3.2 now shows that σ is an isomorphism from $\mathcal{G}$ onto $\mathcal{H}$. ■

4.7 Corollary:

Let $|V|=3$, $V=\{x,y,z\}$. Suppose that there is no $P\in\mathcal{P}$ such that $x^P=y^P=z^P$. Then $\mathcal{G}$ is isomorphic to $\mathcal{H}$ iff $V^*=W^*$ ■

The complete analogon of 3.5, however, does not hold for labeled graphs:

4.7 Example:

The two labeled graphs $\mathcal{G}$ and $\mathcal{H}$ belonging to the clauses

$C=\{Pxy, Pyz, Pzx, Qxy, Qyz, Qzx\}$

$D=\{Puv, Pvw, Pwu, Qvu, Qwv, Quw\}$

satisfy $V^*=W^*$ but they are not isomorphic. In this case $x^P=y^P=z^P$ holds.

Since $E(P)^P=F(P)^P$ and $E(Q)^Q=F(Q)^Q$ holds for the above example, it shows also that theorem 3.7 can not be completely generalized to labeled graphs. But there is a weaker version of 3.7 for labeled graphs:

4.8 Lemma:

If $|E(P)|\leq 2$ for all $P\in\mathcal{P}$

then G is isomorphic to H iff $V^*=W^*$ and $E(P)^P=F(P)^P$ for all $P\in\mathcal{P}$.

Proof:

For each $P\in\mathcal{P}$ let φ_P be a bijective mapping $\varphi_P:E(P)\to F(P)$ with $(e\varphi_P)^P=e^P$ for all $e\in E(P)$. Let φ be the composition of all φ_P. We show that we can choose the φ_P in such a way that φ induces a bijective mapping from V to W.

According to theorem 3.7 each bijective mapping $\varphi_P:E(P)\to F(P)$ induces a bijective mapping φ'_P from V to W. We only have to show that the φ_P can be chosen such that $\varphi'_P\equiv\varphi'_Q$ for all $P, Q\in\mathcal{P}$.

If $x\in V$ such that there is no $y\in V$ with $x^*=y^*$, then there is exactly one $a\in W$ such that $x^*=a^*$. Let $P\in\mathcal{P}$. Either there is no $y\in V$ with $x^P=y^P$ and then $x\varphi'_P = a$ must hold. If on the other hand there is an $y\in V$ with $x^P=y^P$ then there are $a,b\in W$ such that $x^P=a^P=b^P$. Then $x^*=a^*$ or $x^*=b^*$. Let $x^*=a^*$. For E(P) only case 2.1 of the proof of 3.6 is possible. But then one can choose the mapping φ_P such that $x\varphi'_P = a$. If there are $x, y\in V$ with $x^*=y^*$ then $x^P=y^P$ must hold for each $P\in\mathcal{P}$ and then there are $a,b\in W$ with $a^*=b^*$ and one can choose the mappings φ_P such that $x\varphi'_P=a$ for all $P\in\mathcal{P}$. This shows that the mappings φ_P can be chosen in such a way that $x\varphi_P = x\varphi_Q$ holds for all $x\in V$ and all $P,Q\in\mathcal{P}$. ■

Acknowledgement

I wish to thank my colleagues Hans-Jürgen Bürckert and Christoph Lingenfelder who read and improved an earlier draft of this paper.

References

[Be73] Berztiss, A.T.: A Backtrack Procedure for Isomorphism of Directed Graphs. JACM, Vol. 20, 3, 365-377 (1973).

[CL73] Chang, C.L.; Lee, R.C.: Symbolic Logic and Mechanical Theorem Proving. Academic Press, New York (1973).

[Ei81] Eisinger, N.: Subsumption and Connection Graphs. Proc. of the 7th IJCAI, Vancouver, 480-486 (1981).

[Kn71] Knödel, W.: Ein Verfahren zur Feststellung der Isomorphie von endlichen, zusammenhängenden Graphen. Computing 8, 329 - 334 (1971).

[So87] Socher, R: Boolean Simplification of First Order Formulae. SEKI-Report SR-87-04, Universität Kaiserslautern (1987).

[Un64] Unger, S.H.: GIT - A Heuristic Program for Testing Pairs of Directed Line Graphs for Isomorphism. Comm. of the ACM 7, 26 - 34 (1964).

Dokument-Berichts-Blatt

Titel: Graph Isomorphism: Some Special Cases

Autor(en): Rolf Socher

Produktart: ~~SR~~ SWP Produktnummer: SR-87-10

Einverstanden: 1) Lingenfelder

2) Fürcher

Doppelveröffentlichung: ~~ja~~ ~~nein~~ (wo:)

Auflage:

Kostenstelle: 8900

Kaiserslautern, den 5-8-87