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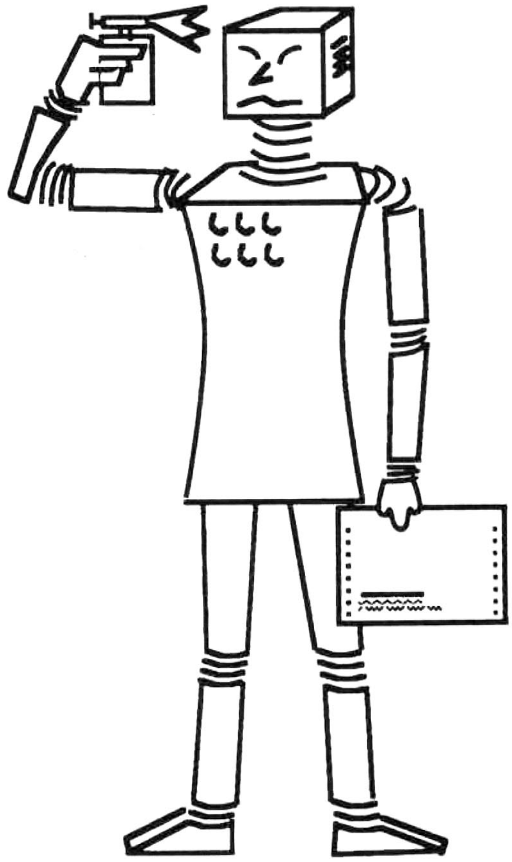

Another Technique for Proving Completeness of Resolution

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Abstract: This paper contains a very short proof of the refutation completeness of ground resolution, that is, it proves that each inconsistent set S of ground clauses admits a resolution derivation of the empty clause. The proof technique works uniformly for pure resolution, hyperresolution and ordered resolution.

Keywords: Resolution Theorem Proving, Completeness Proofs.

A Proof of Completeness of Resolution

Proofs for the refutation completeness of resolution usually consist of two parts: A completeness proof for ground resolution and a lifting lemma, which shows that the proof can be transferred from the ground case to the general (first order) case. Several proofs for the completeness of ground resolution have been given in the literature: The first one by J.A. Robinson [4] as well as Chang & Lee's [2] proof use the technique of *semantic trees*, while Loveland [3] proceeds by induction on the number of *excess literals* (that is the number of literal occurrences in the clause set minus the number of clauses in the set). Recently, Bachmair & Dershowitz [1] presented a technique for proving the completeness of rewrite based proof methods, which is based on the notion of *proof orderings*.

This paper presents a very short proof, which manages with a minimum of semantic notions. The proof technique also applies to some refinements of resolution, like *hyperresolution* [5], or *ordered resolution* [2,3].

In the following, a deduction of a clause C from a clause set S is called trivial, if $C \in S$ holds. The empty clause is denoted by the symbol $\square$.

<u>Completeness Theorem:</u>

Let S be a finite set of ground clauses. If S is unsatisfiable, then there is a resolution deduction of $\square$ from S.

Proof: The proof proceeds by induction on the number n of atoms occurring in the set S. If n=1, then S is of the form $\{\{L\}, \{\neg L\}\}$, and we are done. Assume the theorem holds for n-1. Let S be a clause set with n atoms. Take an arbitrary literal L occurring in S, and let S' be the set obtained from S by deleting each clause containing $\neg L$, and removing the literal L from the remaining clauses. S' is also unsatisfiable[1], and it has n-1 atoms. Thus, by induction hypothesis, there is a (possibly trivial) resolution deduction of □ from S'. By adjoining the literal L back to every clause used in this deduction, we obtain a (possibly trivial) deduction of the unit clause $\{L\}$ or of the empty clause from S. Since L was arbitrary, we can derive either □ or each literal occurring in S. Since S contains two complementary literals, the empty clause is in either case derivable from S. ■

We also sketch an analogous completeness proof for negative hyperresolution:

<u>Completeness Theorem for Hyperresolution:</u>

Let S be any finite set of ground clauses. If S is unsatisfiable, then there is a hyperresolution deduction of the empty clause from S.

Proof: The proof for negative[2] hyperresolution proceeds in the same way as the one for resolution, with the only difference that each *negative* unit clause Lin S can be deduced with hyperresolution from S. This is shown in the same way as before, by observing that the step of adjoining a negative literal preserves the negative hyperresolution property. Since S must contain a positive clause $C = \{L_1,\ldots,L_n\}$, (i.e. all L_i are positive), such that all literals $\neg L_i$ occur in S, a hyperresolution step between C and the unit clauses $\neg L_i$ yields □. ■

In the literature, several concepts for ordered resolution can be found, which, however, do not differ essentially for the ground case. We will adopt a very general notion of ordered resolution. The (ground) literals of the set S are assumed to be partially ordered by an ordering $<$, and the restriction on resolution is that only the literals, which are maximal in a clause are resolved upon. In other words, if a clause C contains literals L and K with $L<K$, then resolution with C on L is excluded.

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[1] Note that this standard argument is the only semantic one in the proof. If S' had a model $\mathcal{M}'$, then $\mathcal{M}'$ could be extended to a model $\mathcal{M}$ for S, by interpreting the literal L as false.

[2] Hyperresolution is called negative, if the electrons in each hyperresolution step are negative clauses.

Completeness Theorem for Ordered Resolution:

Let S be any finite set of ground clauses. If S is unsatisfiable, then there is a ordered resolution deduction of the empty clause from S.

Proof: W.l.o.g. we can assume that S is minimally unsatisfiable. This implies that for each literal L occurring in S also $\neg L$ occurs in S, otherwise $S \setminus \{C \in S \mid L \in C\}$ is also unsatisfiable contradicting the minimality of S. Now the proof proceeds in the same way as the one for resolution, except that we show that each unit clause $\{L\}$, where L is minimal w.r.t. $<$, can be deduced with ordered resolution from S. (Note that the step of adjoining a minimal literal preserves the ordered resolution property.) Analogously the unit clause $\{\neg L\}$ can be derived from S, and an ordered resolution step between $\{L\}$ and $\{\neg L\}$ yields $\square$. ∎

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