# Generalised pulse solutions for a class of quasilinear evolutionary equations with applications to water waves

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# Abstract

In this thesis we present a rigorous existence theory for a class of quasilinear analytic wave equations. This theory is developed by writing both wave equations as infinite-dimensional evolutionary systems via the spatial dynamics approach. Using methods from centre-manifold theory and conserved quantities of the resulting evolutionary equations, we construct a family of generalised pulse solutions on the entire real line. These generalised pulses are exponentially close to a reversible homoclinic solution to either

$$\begin{aligned} \dot{z}_1 &= z_2 + \varepsilon \mathcal{R}_1^\varepsilon(z_1, z_2), \\ \dot{z}_2 &= z_1 - C z_1^2 + \varepsilon \mathcal{R}_2^\varepsilon(z_1, z_2) \end{aligned}$$

or

$$\begin{aligned} \dot{x}_1 &= y_1 + \varepsilon \mathcal{R}_{1,1}^\varepsilon(x_1, y_1, x_2, y_2), \\ \dot{y}_1 &= x_1 - C x_1(x_1^2 + x_2^2) + \varepsilon \mathcal{R}_{1,2}^\varepsilon(x_1, y_1, x_2, y_2), \\ \dot{x}_2 &= y_2 + \varepsilon \mathcal{R}_{2,1}^\varepsilon(x_1, y_1, x_2, y_2), \\ \dot{y}_2 &= x_2 - C x_2(x_1^2 + x_2^2) + \varepsilon \mathcal{R}_{2,2}^\varepsilon(x_1, y_1, x_2, y_2), \end{aligned}$$

for all times, where $\mathcal{R}_i$ are quadratic and $\mathcal{R}_{i,j}$ cubic reversible perturbations and $\varepsilon$ is a small parameter. As applications we consider steady gravity water waves with vorticity and steady three-dimensional gravity capillary water waves.

# Zusammenfassung

In dieser Arbeit entwickeln wir eine rigorose Existenztheorie für eine Klasse quasilinearer, analytischer Wellengleichungen indem wir diese mit Hilfe räumlicher Dynamik als unendlich dimensionaler Evolultionsgleichungen formulieren. Methoden aus dem Bereich der Zentrumsmannigfaltigkeitstheorie kombiniert mit Erhaltungsgrößen der gegebenen Evolutionsgleichungen ermöglichen es uns eine Familie verallgemeinerter Pulslösungen für alle reellen Zeiten zu konstruieren. Diese Lösungen liegen für alle Zeiten exponentiell dicht an einer reversiblen homoklinen Lösung des Systems

$$\begin{aligned} \dot{z}_1 &= z_2 + \varepsilon \mathcal{R}_1^\varepsilon(z_1, z_2), \\ \dot{z}_2 &= z_1 - C z_1^2 + \varepsilon \mathcal{R}_2^\varepsilon(z_1, z_2) \end{aligned}$$

oder

$$\begin{aligned} \dot{x}_1 &= y_1 + \varepsilon \mathcal{R}_{1,1}^\varepsilon(x_1, y_1, x_2, y_2), \\ \dot{y}_1 &= x_1 - C x_1(x_1^2 + x_2^2) + \varepsilon \mathcal{R}_{1,2}^\varepsilon(x_1, y_1, x_2, y_2), \\ \dot{x}_2 &= y_2 + \varepsilon \mathcal{R}_{2,1}^\varepsilon(x_1, y_1, x_2, y_2), \\ \dot{y}_2 &= x_2 - C x_2(x_1^2 + x_2^2) + \varepsilon \mathcal{R}_{2,2}^\varepsilon(x_1, y_1, x_2, y_2), \end{aligned}$$

wobei $\mathcal{R}_i$ quadratische und $\mathcal{R}_{i,j}$ kubische reversible Störterme sind und $\varepsilon$ ein kleiner Parameter. Als Anwendungen betrachten wir permanente Wasserwellen unter Schwerkraft und Vortizität sowie dreidimensionale Wasserwellen unter Schwerkraft und Oberflächenspannung.

# Acknowledgements

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# Contents

# 1 Introduction

## 1.1 Pulses

The system

$$\dot{z}_1 = z_2, \tag{1.1}$$
$$\dot{z}_2 = z_1 - Cz_1^2 \tag{1.2}$$

with $C \neq 0$ and $z_1, z_2 \colon \mathbb{R} \to \mathbb{R}$ is a simple example of a dynamical system admitting a homoclinic solution (see Figure 1.1). This solution is given by the explicit formula

$$\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} h \\ \dot{h} \end{pmatrix},$$

where

$$h(t) = \frac{3}{2C} \operatorname{sech}^2\left(\frac{t}{2}\right).$$

Perturbations of equations (1.1) and (1.2) of the form

$$\dot{z}_1 = z_2 + \varepsilon \mathcal{R}_1^\varepsilon(z_1, z_2), \tag{1.3}$$
$$\dot{z}_2 = z_1 - Cz_1^2 + \varepsilon \mathcal{R}_2^\varepsilon(z_1, z_2), \tag{1.4}$$

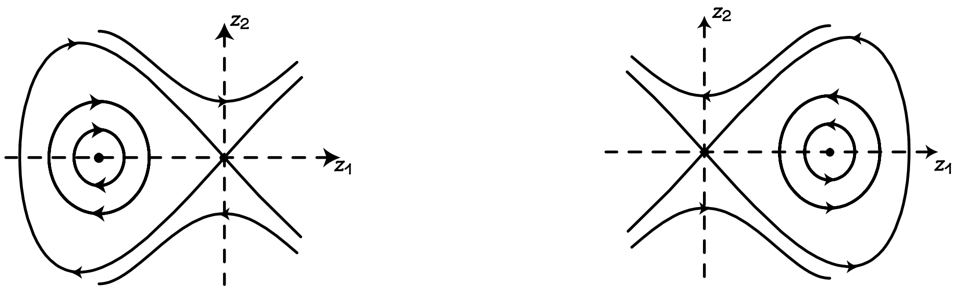
where

$$\mathcal{R}_1^\varepsilon(z_1, z_2) = \mathcal{O}(|(z_1, z_2)|^2)$$

and $\varepsilon$ is small parameter, in general do not have homoclinic solutions. If we additionally assume that equations (1.3) and (1.4) are *reversible*, i.e. equations (1.3) and (1.4) are invariant under $t \mapsto -t$, $(z_1, z_2) \mapsto (z_1, -z_2)$, we still are able to construct symmetric homoclinic solutions by showing that the stable manifold $W_\mathrm{s}^\varepsilon$ consisting of points on orbits which converge to zero as $t \to \infty$ intersects the symmetric section $\{z_2 = 0\}$. Indeed

$$W_\mathrm{s}^0 = \left\{ \begin{pmatrix} h(t) \\ \dot{h}(t) \end{pmatrix} : t \in \mathbb{R} \right\},$$

which intersects $\{z_2 = 0\}$ transversally at $(h(0), 0)^\mathrm{T}$; since transversality is an open phenomenon the intersection persists for small positive values of $\varepsilon$. (See Lemma 4.4 for a functional-analytic proof of this result.)




(a) The case $C < 0$. (b) The case $C > 0$.

Figure 1.1: Phase portrait of equations (1.1) and (1.2).

An example of a system with a cubic nonlinearity and homoclinic solutions is given by

$$\dot{x}_1 = y_1, \tag{1.5}$$
$$\dot{y}_1 = x_1 - Cx_1^3, \tag{1.6}$$

where $C > 0$ (Figure 1.2); they are given by the formula

$$\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \pm \begin{pmatrix} h \\ \dot{h} \end{pmatrix},$$

where

$$h(t) = \Big(\frac{2}{C}\Big)^{\frac{1}{2}} \operatorname{sech}(t).$$

The four dimensional system

$$\dot{x}_1 = y_1, \tag{1.7}$$
$$\dot{y}_1 = x_1 - Cx_1(x_1^2 + x_2^2), \tag{1.8}$$
$$\dot{x}_2 = y_2, \tag{1.9}$$
$$\dot{y}_2 = x_2 - Cx_2(x_1^2 + x_2^2), \tag{1.10}$$

also has homoclinic solutions since it admits the invariant plane $\{(x_1, y_1)\}$, the flow in which is governed by equations (1.5) and (1.6).

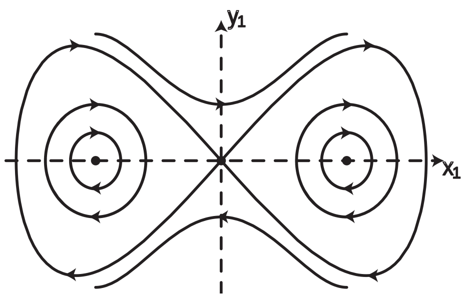



Figure 1.2: Phase portrait of the invariant plane $\{(x_1, y_1)\}$ of equations (1.7) – (1.10).

Now consider the perturbed system

$$\dot{x}_1 = y_1 + \varepsilon \mathcal{R}^{\varepsilon}_{1,1}(x_1, y_1, x_2, y_2), \tag{1.11}$$
$$\dot{y}_1 = x_1 - Cx_1(x_1^2 + x_2^2) + \varepsilon \mathcal{R}^{\varepsilon}_{1,2}(x_1, y_1, x_2, y_2), \tag{1.12}$$
$$\dot{x}_2 = y_2 + \varepsilon \mathcal{R}^{\varepsilon}_{2,1}(x_1, y_1, x_2, y_2), \tag{1.13}$$
$$\dot{y}_2 = x_2 - Cx_2(x_1^2 + x_2^2) + \varepsilon \mathcal{R}^{\varepsilon}_{2,2}(x_1, y_1, x_2, y_2), \tag{1.14}$$

where

$$R^{\varepsilon}_{i,j}(x_1, y_1, x_2, y_2) = \mathcal{O}(|(x_1, y_1, x_2, y_2)|^3),$$

assuming the reversibility $t \mapsto -t$, $(x_1, y_1, x_2, y_2) \mapsto (x_1, y_1, -x_2, -y_2)$. In this case

$$W^0_{\mathrm{c}} = \left\{ \begin{pmatrix} x_1 \\ y_1 \\ x_2 \\ y_2 \end{pmatrix} : \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = R_\theta \begin{pmatrix} h(t) \\ 0 \end{pmatrix}, \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = R_\theta \begin{pmatrix} \dot{h}(t) \\ 0 \end{pmatrix}, t \in \mathbb{R}, \theta \in [0, 2\pi) \right\},$$

where $R_\theta$ is a rotation through $\theta$, and this manifold intersects the symmetric section $\{y_1 = 0, x_2 = 0\}$ transversally in the points $(\pm h(0), 0, 0, 0)^{\mathrm{T}}$. We thus obtain the existence of two symmetric homoclinic solutions for small positive values of $\varepsilon$. (See Lemma 4.12 for a functional-analytic proof of this result.)

Equations (1.3) and (1.4) arise from the system

$$\dot{z}_1 = z_2 + \dots, \tag{1.15}$$
$$\dot{z}_2 = \lambda^{\varepsilon} z_1 - C z_1^2 + \dots, \tag{1.16}$$

where $\lambda^{\varepsilon} = \varepsilon + \mathcal{O}(\varepsilon^2)$, by the scaling

$$\check{t} = \lambda^{\varepsilon} t, \quad \begin{pmatrix} z_1(t) \\ z_2(t) \end{pmatrix} = \begin{pmatrix} (\lambda^{\varepsilon})^2 \check{z}_1(\check{t}) \\ (\lambda^{\varepsilon})^3 \check{z}_2(\check{t}) \end{pmatrix}.$$

This system exhibits *homoclinic bifurcation* (from zero) associated with a $0^2$ *resonance* (the eigenvalues of the linearised system form a purely imaginary pair for $\varepsilon < 0$ which collides at the origin as $\varepsilon \uparrow 0$, forming a Jordan block, and splits into a real pair for $\varepsilon > 0$; see Figure 1.3). Similarly, equations (1.11) – (1.14) arise from the system given in complex coordinates by

$$\dot{z}_1 = \mathrm{i}(\omega + \sigma^{\varepsilon}) z_1 + z_2 + \dots, \tag{1.17}$$
$$\dot{z}_2 = (\lambda^{\varepsilon})^2 z_1 + \mathrm{i}(\omega + \sigma^{\varepsilon}) z_2 - C z_2 |z_1|^2 + \dots, \tag{1.18}$$

where $\lambda^{\varepsilon} = \varepsilon + \mathcal{O}(\varepsilon^2)$, $\sigma^{\varepsilon} = \mathcal{O}(\varepsilon)$, by the scaling

$$\check{t} = \varepsilon t, \quad \begin{pmatrix} z_1(t) \\ z_2(t) \end{pmatrix} = \mathrm{e}^{\mathrm{i}(\omega + \sigma^{\varepsilon})t} \begin{pmatrix} \lambda^{\varepsilon} \big( \check{x}_1(\check{t}) + \mathrm{i} \check{x}_2(\check{t}) \big) \\ (\lambda^{\varepsilon})^2 \big( \check{y}_1(\check{t}) + \mathrm{i} \check{y}_2(\check{t}) \big) \end{pmatrix}.$$

This system exhibits homoclinic bifurcation associated with an $(\mathrm{i}\omega)^2$ *resonance* (the eigenvalues of the linearised system form two purely imaginary pairs for $\varepsilon < 0$ which collide pairwise as $\varepsilon \uparrow 0$, forming two Jordan blocks, and split into a complex quartet for $\varepsilon > 0$; see Figure 1.4).

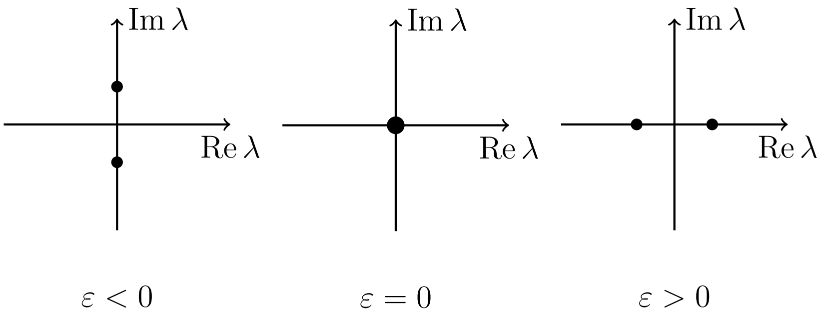



Figure 1.3: An $0^2$ resonance at $\varepsilon = 0$.

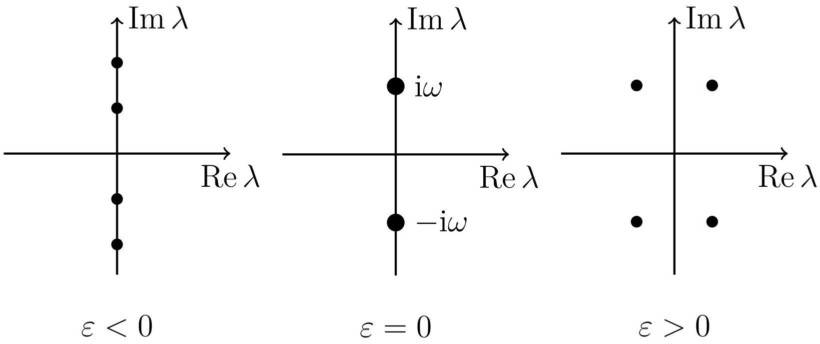



Figure 1.4: An $(\mathrm{i}\omega)^2$ resonance at $\varepsilon = 0$.

It is convenient to write system (1.15) and (1.16) or system (1.17) and (1.18) as

$$\dot{z} = L^\varepsilon_{\mathrm{wh}} z + h^\varepsilon_{\mathrm{wh}}(z), \tag{1.19}$$

where $z \in \mathbb{R}^n$, $n \in \{2, 4\}$, $L^\varepsilon_{\mathrm{wh}}$ is the matrix associated with the $0^2$ or $(\mathrm{i}\omega)^2$ resonance and $h^\varepsilon_{\mathrm{wh}}$ is the (parameter-dependent) nonlinearity. Let us now couple equation (1.19) with a second, possibly infinite-dimensional equation, to arrive at the system

$$\dot{z} = L^\varepsilon_{\mathrm{wh}} z + g^\varepsilon_{\mathrm{wh}}(z, u) + h^\varepsilon_{\mathrm{wh}}(z), \tag{1.20}$$
$$\dot{u} = L^\varepsilon_{\mathrm{sh}} u + g^\varepsilon_{\mathrm{sh}}(z, u) + h^\varepsilon_{\mathrm{sh}}(z), \tag{1.21}$$

where the new nonlinearity $g^\varepsilon_{\mathrm{wh}}$ satisfies $g^\varepsilon_{\mathrm{wh}}(z, 0) = 0$. The variable $u$ belongs to a Banach space $\mathcal{X}_{\mathrm{sh}}$, and $L^\varepsilon_{\mathrm{sh}} \colon \mathcal{D}_{\mathrm{sh}} \subseteq \mathcal{X}_{\mathrm{sh}} \to \mathcal{X}_{\mathrm{sh}}$ is a densely defined, closed linear operator, while $g^\varepsilon_{\mathrm{sh}} \colon \mathbb{R}^n \times \mathcal{D}_{\mathrm{sh}} \to \mathcal{X}_{\mathrm{sh}}$ and $h^\varepsilon_{\mathrm{sh}} \colon \mathbb{R}^n \to \mathcal{X}_{\mathrm{sh}}$ are nonlinearities with $g^\varepsilon_{\mathrm{sh}}(z, 0) = 0$. Under the assumption that the spectrum of $L^\varepsilon_{\mathrm{sh}}$ lies in two wedge-shaped regions (see Figure 1.5), the *centre-manifold theorem* asserts that all small, globally bounded solutions to equations (1.20) and (1.21) lie on a locally invariant manifold

$$M^\varepsilon_{\mathrm{c}} = \{(z, u) : u = r^\varepsilon(z)\}$$

given as the graph of a nonlinear reduction function $r^\varepsilon$, where $\varepsilon$, $z$, $u$ lie in neighbourhoods of the origin in $\mathbb{R}$, $\mathbb{R}^n$ and $\mathcal{D}_{\mathrm{sh}}$. The flow on the centre manifold is determined by the reduced equation obtained by inserting $u = r^\varepsilon(z)$ into equation (1.20). The reduced equation is of the type (1.19), and one can investigate homoclinic bifurcation for this equation, and hence system (1.20) and (1.21), using the above techniques. This method has been particularly successful in studies of travelling gravity-capillary water waves. Writing the hydrodynamic equation as evolutionary system in which the horizontal spatial coordinate plays the role of time ('spatial dynamics'), one finds that the system has the form (1.20), (1.21). The above procedure thus reduces the problem to a $0^2$ resonance (see Kirchgässner [16]) or $(\mathrm{i}\omega)^2$ resonance (see Iooss and Kirchgässner [10]) depending upon the values of the physical parameters. The resulting homoclinic bifurcation generates homoclinic solutions to system (1.20) and (1.21) known as *solitary waves*.

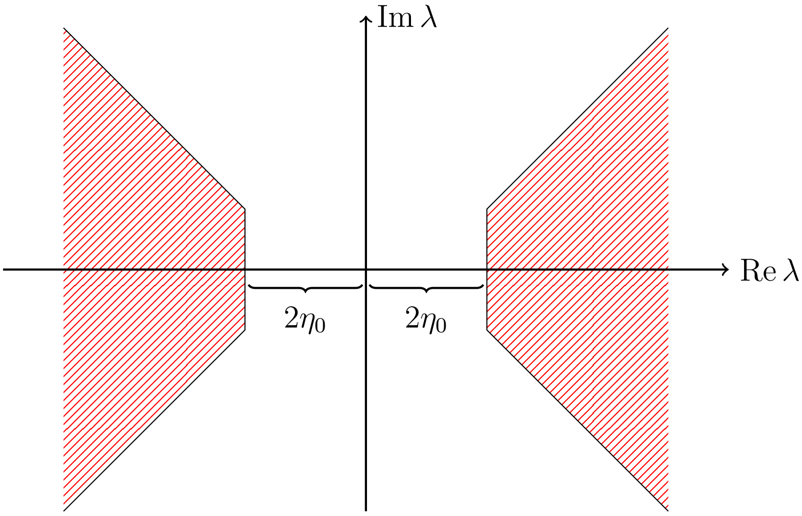



Figure 1.5: The spectrum of $L^0_{\mathrm{sh}}$ is contained in wedges with *spectral gap* of distance $2\eta_0$ to the imaginary axis.

Alternatively, one can consider the coupled system

$$\dot{z} = L^\varepsilon_{\mathrm{wh}} z + g^\varepsilon_{\mathrm{wh}}(z, w) + h^\varepsilon_{\mathrm{wh}}(z), \tag{1.22}$$
$$\dot{w} = L^\varepsilon_{\mathrm{c}} w + g^\varepsilon_{\mathrm{c}}(z, w) + h^\varepsilon_{\mathrm{c}}(z), \tag{1.23}$$

where $w \in \mathbb{R}^{2d}$ and $L^\varepsilon_{\mathrm{c}}$ is a matrix with $d$ pairs of purely imaginary eigenvalues denoted by $\pm\mathrm{i}\omega_1, \dots, \pm\mathrm{i}\omega_d$ (see Figure 1.6), and $g^\varepsilon_{\mathrm{c}} \colon \mathbb{R}^n \times \mathbb{R}^{2d} \to \mathbb{R}^{2d}$, $h^\varepsilon_{\mathrm{c}} \colon \mathbb{R}^n \to \mathbb{R}^{2d}$ are nonlinearities with $g^\varepsilon_{\mathrm{c}}(z, 0) = 0$. In general system (1.22) and (1.23) does not have homoclinic solutions. Each point on a homoclinic solution is a point of intersection of the stable and unstable manifolds $W^\varepsilon_{\mathrm{s}}$ and $W^\varepsilon_{\mathrm{u}}$, which are both $\frac{n}{2}$-dimensional ($\dim W^\varepsilon_{\mathrm{s}}$ and $\dim W^\varepsilon_{\mathrm{u}}$ are the number of eigenvalues of $L^\varepsilon_{\mathrm{wh}}$ with respectively negative and positive real part), and in general two $\frac{n}{2}$-dimensional manifolds do not intersect in $\mathbb{R}^{n+2d}$.

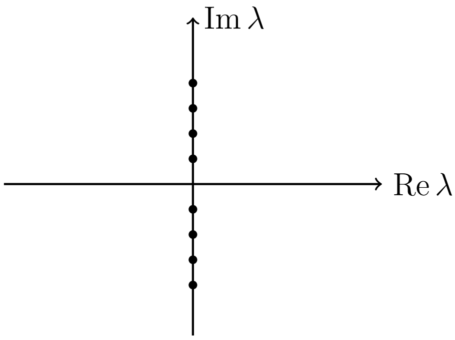



Figure 1.6: The spectrum of $L^\varepsilon_{\mathrm{c}}$ consists of $d$ pairs of purely imaginary eigenvalues.

Equations of the form (1.22), (1.23) were considered in a series of papers by Groves and Schneider [7, 8, 9] (they actually considered an infinite-dimensional version of these equations which arise from nonlinear wave equations, but the issue is the same). In the special case that $h_{\mathrm{c}}^{\varepsilon} = 0$ system (1.22) and (1.23) has the invariant subspace $\{w = 0\}$, the flow in which is determined by equation (1.19), and one can find a homoclinic solution $p^{\varepsilon}$ to equation (1.19), and hence a homoclinic solution $(p^{\varepsilon}, 0)$ to (1.22), (1.23), using the methods described above. Groves and Schneider showed that if $h_{\mathrm{c}}(z) = \mathcal{O}(|z|^N)$ then (1.22), (1.23) has reversible solutions $(z, w)$ which satisfy

$$|\big(z(t) - p^{\varepsilon}(t), w(t)\big)| = \mathcal{O}(\varepsilon^N), \quad t \in [-t^*, t^*],$$

where $t^* = \mathcal{O}(\varepsilon^{-N})$. Furthermore, a sequence of near-identity 'normal-form' transformations can be used to successively remove terms in the Maclaurin expansion of $h_{\mathrm{c}}^{\varepsilon}(z)$ and hence achieve the condition $h_{\mathrm{c}}^{\varepsilon}(z) = \mathcal{O}(|z|^N)$. The result is thus a family of 'generalised pulses' which exist and are approximated algebraically closely by $p^{\varepsilon}$ over an algebraically long time scale (here 'algebraic' means with respect to the amplitude of $p^{\varepsilon}$, which is $\mathcal{O}(\varepsilon^{\frac{n}{2}})$. This result can be considerably strengthened if the nonlinearities are analytic. In this case one can find an optimal value of $N$ so that $h_{\mathrm{c}}^{\varepsilon}(z)$ is exponentially small for values of $(z, \varepsilon)$ in an appropriately chosen neighbourhood of the origin. One then finds that system (1.22) and (1.23) has reversible solutions $(z, w)$ which satisfy

$$|(z(t) - p^{\varepsilon}(t), 0)| = \mathcal{O}(\mathrm{e}^{-\frac{1}{2\sqrt{\varepsilon}}})$$

for $t \in [-t^*, t^*]$, where $t^* = \mathcal{O}(\mathrm{e}^{1/2\sqrt{\varepsilon}})$. These generalised pulses exist and are exponentially close to $p^{\varepsilon}$ over an exponentially long time scale (see Figure 1.7).

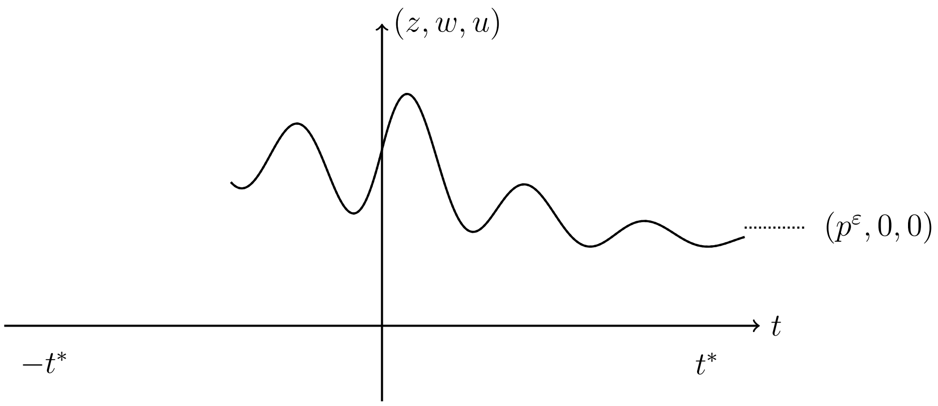



Figure 1.7: A generalised pulse solution to equations (1.24) – (1.26) lies in an exponentially thin tubular neighbourhood of $(p^{\mu}, 0, 0)$ over an exponentially long time scale.

In this thesis we study the general coupled system

$$\dot{z} = L_{\mathrm{wh}}^{\varepsilon} z + g_{\mathrm{wh}}^{\varepsilon}(z, w, u) + h_{\mathrm{wh}}^{\varepsilon}(z), \tag{1.24}$$

$$\dot{w} = L_{\mathrm{c}}^{\varepsilon} w + g_{\mathrm{c}}^{\varepsilon}(z, w, u) + h_{\mathrm{c}}^{\varepsilon}(z), \tag{1.25}$$

$$\dot{u} = L_{\mathrm{sh}}^{\varepsilon} u + g_{\mathrm{sh}}^{\varepsilon}(z, w, u) + h_{\mathrm{sh}}^{\varepsilon}(z) \tag{1.26}$$

obtained by combining equations (1.20) and (1.21) with equations (1.22) and (1.23). Assuming analyticity of the nonlinearities, we show that system (1.24) – (1.26) has generalised pulse solutions which exist and are exponentially close to $p^\varepsilon$ (a homoclinic solution in the invariant subspace $\{(w,u)=(0,0)\}$ over an exponentially long time scale. (Note that this result cannot be achieved by the standard method of constructing a locally invariant centre manifold of the form

$$\{(z,w,u) : (w,u) = r^\varepsilon(z)\}$$

since the reduction function $r^\varepsilon$ typically does not inherit the analyticity of the equations.) We improve this result further for systems with a conserved quantity which is positive definite on the subspace $\{(z,u)=(0,0)\}$, showing that the generalised pulse can be extended to the entire real line. These solutions do not necessarily decay to zero at infinity but they are exponentially close to $p^\varepsilon$ for all times. A similar result was obtained by Groves and Schneider [9] for algebraic approximations to $p^\varepsilon$ in the context of system (1.22) and (1.23).

## 1.2 The main result

In this thesis we consider the system

$$\dot{z} = L^\varepsilon_{\mathrm{wh}} z + g^\varepsilon_{\mathrm{wh}}(z,w,u) + h^\varepsilon_{\mathrm{wh}}(z), \tag{1.27}$$
$$\dot{w} = L^\varepsilon_{\mathrm{c}} w + g^\varepsilon_{\mathrm{c}}(z,w,u) + h^\varepsilon_{\mathrm{c}}(z), \tag{1.28}$$
$$\dot{u} = L^\varepsilon_{\mathrm{sh}} u + g^\varepsilon_{\mathrm{sh}}(z,w,u) + h^\varepsilon_{\mathrm{sh}}(z), \tag{1.29}$$

for $(z,w,u)\colon \mathbb{R} \to \mathbb{R}^n \times \mathbb{R}^{2d} \times \mathcal{D}_{\mathrm{sh}}$, where $n \in \{2,4\}$, $d \in \mathbb{N}$ and $\mathcal{D}_{\mathrm{sh}}$ is a dense subspace of a Banach space $\mathcal{X}_{\mathrm{sh}}$. (We use the terminology 'weakly hyperbolic', 'strongly hyperbolic' and 'centre' and the subscripts 'wh', 'sh' and 'c' in analogy with finite-dimensional dynamical systems.) We abbreviate $\mathbb{R}^n \times \mathbb{R}^{2d} \times \mathcal{X}_{\mathrm{sh}}$, $\mathbb{R}^n \times \mathbb{R}^{2d} \times \mathcal{D}_{\mathrm{sh}}$ to respectively $\mathcal{X}$, $\mathcal{D}$ and $\mathbb{R}^{2d} \times \mathcal{X}_{\mathrm{sh}}$, $\mathbb{R}^{2d} \times \mathcal{D}_{\mathrm{sh}}$ to respectively $\mathcal{X}_{\mathrm{c,sh}}$, $\mathcal{D}_{\mathrm{c,sh}}$. Similarly we write $L^\varepsilon$ instead of $(L^\varepsilon_{\mathrm{wh}}, L^\varepsilon_{\mathrm{c}}, L^\varepsilon_{\mathrm{sh}})$. On the right-hand side we make the following assumptions.

(A1) The bounded linear operators $L^\varepsilon_{\mathrm{wh}}\colon \mathbb{R}^n \to \mathbb{R}^n$, $L^\varepsilon_{\mathrm{c}}\colon \mathbb{R}^{2d} \to \mathbb{R}^{2d}$ and $L^\varepsilon_{\mathrm{sh}}\colon \mathcal{D}_{\mathrm{sh}} \to \mathcal{X}_{\mathrm{sh}}$ depend analytically upon $\varepsilon$.

(A2) The functions $g^{(\cdot)}_{\mathrm{wh}}$, $g^{(\cdot)}_{\mathrm{c}}$, $g^{(\cdot)}_{\mathrm{sh}}$, $h^{(\cdot)}_{\mathrm{wh}}$, $h^{(\cdot)}_{\mathrm{c}}$, $h^{(\cdot)}_{\mathrm{sh}}$ take values in respectively $\mathbb{R}^n$, $\mathbb{R}^{2d}$, $\mathcal{X}_{\mathrm{sh}}$, $\mathbb{R}^n$, $\mathbb{R}^{2d}$, $\mathcal{X}_{\mathrm{sh}}$ and are analytic at the origin in respectively $\mathbb{R} \times \mathcal{D}$ and $\mathbb{R} \times \mathbb{R}^n$. We suppose that

$$g^\varepsilon_{\mathrm{wh}}(z,w,u), g^\varepsilon_{\mathrm{c}}(z,w,u), g^\varepsilon_{\mathrm{sh}}(z,w,u) = \mathcal{O}(\|(z,w,u)\|_{\mathcal{D}} \|(w,u)\|_{\mathcal{D}_{\mathrm{c,sh}}}),$$
$$h^\varepsilon_{\mathrm{wh}}(z), h^\varepsilon_{\mathrm{c}}(z), h^\varepsilon_{\mathrm{sh}}(z) = \mathcal{O}(|z|^2).$$

(A3) The spectrum of the complexified operator $L^\varepsilon_{\mathrm{c}} \in \mathbb{C}^{2d \times 2d}$ consists of finitely many simple purely imaginary eigenvalues $\pm \mathrm{i}\omega^\varepsilon_1, \dots, \pm \mathrm{i}\omega^\varepsilon_d$, where $\omega^\varepsilon_1, \dots, \omega^\varepsilon_d > 0$ (see Figure 1.6).

(A4) The system (1.27) – (1.29) is reversible, i.e. there exist $S_{\mathrm{wh}} \in \mathbb{R}^{n \times n}$, $S_{\mathrm{c}} \in \mathbb{R}^{2d \times 2d}$ and $S_{\mathrm{sh}} \in \mathcal{L}(\mathcal{D}_{\mathrm{sh}}) \cap \mathcal{L}(\mathcal{X}_{\mathrm{sh}})$ such that system (1.27) – (1.29) is invariant under $t \mapsto -t$, $(z,w,u) \mapsto (S_{\mathrm{wh}} z, S_{\mathrm{c}} w, S_{\mathrm{sh}} u)$.

(A5) There exists a real-valued function $\mathcal{I}^{(\cdot)}$ which is analytic at the origin in $\mathbb{R} \times \mathcal{D}$, satisfies

$$\mathcal{I}^\varepsilon(z, w, u) = \mathcal{O}(\|(z, w, u)\|_{\mathcal{D}}^2)$$

and

$$\mathcal{I}^0(0, w, 0) = |w|^2 + \mathcal{O}(|w|^3)$$

and is such that $\mathcal{I}^\varepsilon$ a conserved quantity of system (1.27) – (1.29).

(A6) The linear operator $L_{\mathrm{sh}}^0 \colon \mathcal{D}_{\mathrm{sh}} \subseteq \mathcal{X}_{\mathrm{sh}} \to \mathcal{X}_{\mathrm{sh}}$ is closed and satisfies the estimate

$$\|(\mathrm{i}sI - L_{\mathrm{sh}}^0)^{-1}\|_{\mathcal{L}(\mathcal{X}_{\mathrm{sh}})} \lesssim \frac{1}{1+|s|}$$

for $s \in \mathbb{R}$.

For the weakly hyperbolic component we treat two distinct cases. For $n = 2$ we assume the following properties.

(B1) The spectrum of the linear operator $L_{\mathrm{wh}}^\varepsilon \in \mathbb{R}^{2\times 2}$ exhibits a $0^2$ resonance at $\varepsilon = 0$, meaning that as $\varepsilon \uparrow 0$ a pair of purely imaginary eigenvalues of $L_{\mathrm{wh}}^\varepsilon$ collides at the origin (forming a Jordan block) and splits into a pair of real eigenvalues for $\varepsilon > 0$ (see Figure 1.3). We write $z \in \mathbb{R}^2$ as

$$z = z_1 e + z_2 f,$$

where $L_{\mathrm{wh}}^0 e = 0$, $L_{\mathrm{wh}}^0 f = e$, and assume that

$$L_{\mathrm{wh}}^\varepsilon = \begin{pmatrix} 0 & 1 \\ (\lambda^\varepsilon)^2 & 0 \end{pmatrix},$$

where $\lambda^\varepsilon$ is an analytic function of $\varepsilon$ with

$$\lambda^\varepsilon = \varepsilon + \mathcal{O}(\varepsilon^2).$$

Additionally, we assume that $S_{\mathrm{wh}}(z_1, z_2) = (z_1, -z_2)$.

(B2) The coefficient of $z_1^2 e$ in the Maclaurin expansion of $h_{\mathrm{wh}}^\varepsilon(z_1, z_2)$ denoted by $-C$ does not vanish.

For $n = 4$ we make the following assumptions.

(C1) The right-hand side of equations (1.27) – (1.29) satisfies the estimates

$$h^{\varepsilon}_{\mathrm{wh}}(z) = \mathcal{O}(|z|^3),$$
$$h^{\varepsilon}_{\mathrm{c}}(z), h^{\varepsilon}_{\mathrm{sh}}(z) = \mathcal{O}(|z|^2).$$

(C2) The spectrum of the complexified linear operator $L^{\varepsilon}_{\mathrm{wh}} \in \mathbb{C}^{4\times 4}$ exhibits an $(\mathrm{i}\omega)^2$ resonance at $\varepsilon = 0$, meaning that there exists $\omega \geq 0$ such that as $\varepsilon \uparrow 0$ two pairs of purely imaginary eigenvalues of $L^{\varepsilon}_{\mathrm{wh}}$ collide to form geometrically simple and algebraically double eigenvalues $\pm\mathrm{i}\omega$ and split into a complex eigenvalue quartet for $\varepsilon > 0$ (see Figure 1.4). When working with complex coordinates we write $z \in \mathbb{R}^4$ as

$$z = z_1 e + z_2 f + \bar{z}_1 \bar{e} + \bar{z}_2 \bar{f}, \quad z_1, z_2 \in \mathbb{C},$$

where $(L^0_{\mathrm{wh}} - \mathrm{i}\omega I)e = 0$, $(L^0_{\mathrm{wh}} - \mathrm{i}\omega I)f = e$, and assume that

$$L^{\varepsilon}_{\mathrm{wh}} = \begin{pmatrix} \mathrm{i}(\omega+\sigma^{\varepsilon}) & 1 & 0 & 0 \\ (\lambda^{\varepsilon})^2 & \mathrm{i}(\omega+\sigma^{\varepsilon}) & 0 & 0 \\ 0 & 0 & -\mathrm{i}(\omega+\sigma^{\varepsilon}) & 1 \\ 0 & 0 & (\lambda^{\varepsilon})^2 & -\mathrm{i}(\omega+\sigma^{\varepsilon}) \end{pmatrix},$$

where $\lambda^{\varepsilon}$, $\sigma^{\varepsilon}$ are analytic functions of $\varepsilon$ with

$$\lambda^{\varepsilon} = \varepsilon + \mathcal{O}(\varepsilon^2), \qquad \sigma^{\varepsilon} = \mathcal{O}(\varepsilon).$$

Additionally, we assume that $S_{\mathrm{wh}}(z_1, z_2) = (\bar{z}_1, -\bar{z}_2)$ and $k\omega \neq \omega^0_j$ for all $j \in \{1, \ldots, d\}$ and $k \in \mathbb{N}$.

(C3) The conserved quantity $\mathcal{I}^{\varepsilon}$ satisfies

$$\mathcal{I}^{\varepsilon}((z_1, z_2), w, u) = \mathcal{O}(\|(z_2, w, u)\|_{\mathbb{R}^2\times\mathbb{R}^{2d}\times\mathcal{D}_{\mathrm{sh}}}\|(z_1, z_2, w, u)\|_{\mathcal{D}} + \varepsilon\|(z_1, z_2, w, u)\|^2_{\mathcal{D}})$$

(in complex coordinates).

(C4) The coefficient of $z_1|z_2|^2 f$ in $h^{\varepsilon}_{\mathrm{wh}}(z) - g^{\varepsilon}_{\mathrm{wh}}(z, X_{\mathrm{c}}(z,z), X_{\mathrm{sh}}(z,z))$ denoted by $-C$, where $X(z,z)$ is the unique solution of the equation

$$-L^0_{\mathrm{c,sh}} X(z,z) + X(z, 2L^0_{\mathrm{wh}} z) = -\frac{1}{2}\,\mathrm{d}^2 h^0_{\mathrm{c,sh}}[0](z,z),$$

is positive.

We prove the following results for each fixed $\nu \in (0,1)$.

**Theorem 1.1.** Suppose $n = 2$ and Assumptions (A1) – (A6) and Assumptions (B1) and (B2) hold. There exist positive constants $\varepsilon_0$ and $c^*$ with the property that for each $\varepsilon \in (0, \varepsilon_0)$ equations (1.27) – (1.29) admit a family of generalised pulse solutions satisfying

$$(z(-t), w(-t), u(-t)) = S(z(t), w(t), u(t))$$

and

$$\left| \begin{pmatrix} z_1(t) \\ z_2(t) \end{pmatrix} - \begin{pmatrix} (\lambda^\varepsilon)^2 \breve{p}_1^\varepsilon(\lambda^\varepsilon t) \\ (\lambda^\varepsilon)^3 \breve{p}_2^\varepsilon(\lambda^\varepsilon t) \end{pmatrix} \right| \leq \mathrm{e}^{-\frac{c^*}{2\sqrt{\varepsilon}}},$$
$$\|(w(t), u(t))\|_{\mathcal{D}_{\mathrm{c,sh}}} \leq \mathrm{e}^{-\frac{c^*}{2\sqrt{\varepsilon}}}$$

for all $t \in \mathbb{R}$, where

$$\breve{p}_1^\varepsilon(t) = \frac{3}{2C} \operatorname{sech}^2\left(\frac{t}{2}\right) + \mathcal{O}(\varepsilon \mathrm{e}^{-\nu|t|}),$$
$$\breve{p}_2^\varepsilon(t) = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{3}{2C} \operatorname{sech}^2\left(\frac{t}{2}\right)\right) + \mathcal{O}(\varepsilon \mathrm{e}^{-\nu|t|}).$$

**Theorem 1.2.** Suppose $n = 4$ and Assumptions (A1) – (A6) and Assumptions (C1) – (C4) hold. There exist positive constants $\varepsilon_0$ and $c^*$ with the property that for each $\varepsilon \in (0, \varepsilon_0)$ equations (1.27) – (1.29) admit a family of generalised pulse solutions satisfying

$$(z(-t), w(-t), u(-t)) = S(z(t), w(t), u(t))$$

and

$$\left| \begin{pmatrix} z_1(t) \\ z_2(t) \end{pmatrix} - \mathrm{e}^{\mathrm{i}(\omega+\sigma^\varepsilon)t} \begin{pmatrix} \lambda^\varepsilon\Big(\breve{p}_1^\varepsilon(\lambda^\varepsilon t) + \mathrm{i}\breve{p}_2^\varepsilon(\lambda^\varepsilon t)\Big) \\ (\lambda^\varepsilon)^2\Big(\breve{q}_1^\varepsilon(\lambda^\varepsilon t) + \mathrm{i}\breve{q}_2^\varepsilon(\lambda^\varepsilon t)\Big) \end{pmatrix} \right| \leq \mathrm{e}^{-\frac{c^*}{2\sqrt{\varepsilon}}},$$
$$\|(w(t), u(t))\|_{\mathcal{D}_{\mathrm{c,sh}}} \leq \mathrm{e}^{-\frac{c^*}{2\sqrt{\varepsilon}}}$$

for all $t \in \mathbb{R}$, where

$$\breve{p}_1^\varepsilon(t) = \pm\Big(\frac{2}{C}\Big)^{\frac{1}{2}} \operatorname{sech}(t) + \mathcal{O}(\varepsilon \mathrm{e}^{-\nu|t|}),$$
$$\breve{q}_1^\varepsilon(t) = \pm\frac{\mathrm{d}}{\mathrm{d}t}\Big(\Big(\frac{2}{C}\Big)^{\frac{1}{2}} \operatorname{sech}^2(t)\Big) + \mathcal{O}(\varepsilon \mathrm{e}^{-\nu|t|})$$
$$\breve{p}_2^\varepsilon(t), \breve{q}_2^\varepsilon(t) = \mathcal{O}(\varepsilon \mathrm{e}^{-\nu|t|}).$$

**Remark 1.3.** Theorems 1.1 and 1.2 hold for $t \in [-\mathrm{e}^{-c^*/2\sqrt{\varepsilon}}, \mathrm{e}^{c^*/2\sqrt{\varepsilon}}]$ rather than $t \in \mathbb{R}$ if Assumption (B2) or Assumption (C4) is not satisfied.

The proofs of Theorems 1.1 and 1.2 make heavy use of techniques employed in the proof of the centre-manifold theorem for quasilinear evolution equations, and we therefore begin with a thorough review of this proof in Chapter 2. In the construction of the centre manifold it is necessary to solve equations of the form

$$\dot{u} = L^0_{\mathrm{sh}} u + f(t) \tag{1.30}$$

with a function $f\colon \mathbb{R} \to \mathcal{X}_{\mathrm{sh}}$ (in the above notation). For this purpose we use a *maximal regularity* result by Arendt et al. [1] which states that for each $f \in C^\alpha_{\mathrm{b}}(\mathbb{R}; \mathcal{X}_{\mathrm{sh}})$ equation (1.30) has a unique solution $u \in C^\alpha_{\mathrm{b}}(\mathbb{R}; \mathcal{D}_{\mathrm{sh}}) \cap C^{1,\alpha}_{\mathrm{b}}(\mathbb{R}; \mathcal{X}_{\mathrm{sh}})$ provided that

$$\|(\mathrm{i}sI - L^0_{\mathrm{sh}})^{-1}\|_{\mathcal{L}(\mathcal{X}_{\mathrm{sh}})} \lesssim \frac{1}{1+s}$$

for all $s \in \mathbb{R}$ (here $\alpha \in (0,1)$ is fixed). In general this result does not hold for $\alpha = 0$, which necessitates the use of Hölder spaces in the entire construction. In particular we discuss composition operators in such spaces. Our proof is a slight modification of the proof given by Kirrmann [17]. It is also possible to replace $C^\alpha_{\mathrm{b}}$ with $L^{\mathrm{p}}$ for $p > 1$ (see Arendt et al. [1]), and a proof of the centre-manifold theorem in this setting was given by Mielke [21].

In Chapters 3 and 4 we adapt the normal-form theory given by Groves and Schneider [9] on the basis of earlier work by Iooss and Lombardi [11] to construct a sequence of near-identity changes of variable which systematically remove the $j$th order terms of the Maclaurin expansion of $h^\varepsilon_{\mathrm{c,sh}}$ for $j \in \{1, \ldots, p\}$ while preserving the overall structure of the system. In general it is not possible to remove $h^\varepsilon_{\mathrm{c,sh}}$ completely but we can at least make an optimal choice of $p$ so that the remaining terms are exponentially small in comparison to $\varepsilon$ in a suitable neighbourhood of the origin. This normal-form theory requires that $L^0_{\mathrm{wh}}$ is diagonalisable. Since this condition is evidently not met we use the following change of parameter to 'replace' it with a diagonal matrix.

In the case of a $0^2$ resonance we write $\varepsilon = \mu^2$ and introduce the scaled variables

$$\begin{pmatrix} Z_1 \\ Z_2 \end{pmatrix} = \begin{pmatrix} (\tilde{\lambda}^\mu)^{-1} z_1 \\ (\tilde{\lambda}^\mu)^{-2} z_2 \end{pmatrix}, \qquad \begin{pmatrix} W \\ U \end{pmatrix} = (\tilde{\lambda}^\mu)^{-1} \begin{pmatrix} w \\ u \end{pmatrix}, \qquad \tilde{\lambda}^\mu = \lambda^\varepsilon|_{\varepsilon=\mu^2}$$

to convert equations (1.27) – (1.29) into

$$\dot{Z} = L^\mu_{\mathrm{wh}} Z + G^\mu_{\mathrm{wh}}(Z,W,U) + H^\mu_{\mathrm{wh}}(Z), \tag{1.31}$$
$$\dot{W} = L^\mu_{\mathrm{c}} W + G^\mu_{\mathrm{c}}(Z,W,U) + H^\mu_{\mathrm{c}}(Z), \tag{1.32}$$
$$\dot{U} = L^\mu_{\mathrm{sh}} U + G^\mu_{\mathrm{sh}}(Z,W,U) + H^\mu_{\mathrm{sh}}(Z), \tag{1.33}$$

where

$$L^\mu_{\mathrm{wh}} \begin{pmatrix} Z_1 \\ Z_2 \end{pmatrix} = \begin{pmatrix} 0 & \tilde{\lambda}^\mu \\ \tilde{\lambda}^\mu & 0 \end{pmatrix} \begin{pmatrix} Z_1 \\ Z_2 \end{pmatrix},$$
$$G^\mu_{\mathrm{wh}}(Z,W,U) = \mathcal{O}(\|(W,U)\|_{\mathcal{D}_{\mathrm{c,sh}}} \|(Z,W,U)\|_{\mathcal{D}}),$$
$$G^\mu_{\mathrm{c}}(Z,W,U), G^\mu_{\mathrm{sh}}(Z,W,U) = \mathcal{O}\Big(\mu^2 \|(W,U)\|_{\mathcal{D}_{\mathrm{c,sh}}} \|(Z,W,U)\|_{\mathcal{D}}\Big),$$
$$H^\mu_{\mathrm{wh}}(Z) = \mathcal{O}(|Z|^2),$$

$$H_{\mathrm{c}}^{\mu}(Z), H_{\mathrm{sh}}^{\mu}(Z) = \mathcal{O}(\mu^2|Z|^2),$$

and we have abbreviated $L_{\mathrm{c}}^{\varepsilon}|_{\varepsilon=\mu^2}$, $L_{\mathrm{sh}}^{\varepsilon}|_{\varepsilon=\mu^2}$ to $L_{\mathrm{c}}^{\mu}$, $L_{\mathrm{sh}}^{\mu}$.

In the case of an $(\mathrm{i}\omega)^2$ resonance we again write $\varepsilon = \mu^2$ after an additional preliminary transformation (this step is necessary since the scaling leads to slightly worse estimates for the nonlinearities) and introduce the different scaled variables

$$\begin{aligned}
\tilde{\lambda}^{\mu} &= \lambda^{\varepsilon}|_{\varepsilon=\mu^2}, \\
\tilde{\sigma}^{\mu} &= \sigma^{\varepsilon}|_{\varepsilon=\mu^2}, \\
\begin{pmatrix} Z_1 \\ Z_2 \end{pmatrix} &= \begin{pmatrix} (\tilde{\lambda}^{\mu})^{-\frac{1}{2}} z_1 \\ (\tilde{\lambda}^{\mu})^{-\frac{3}{2}} z_2 \end{pmatrix}, \\
W &= (\tilde{\lambda}^{\mu})^{-1} w, \\
U &= (\tilde{\lambda}^{\mu})^{-\frac{3}{2}} u.
\end{aligned}$$

This transformation converts equations (1.27) – (1.29) into

$$\dot{Z} = L_{\mathrm{wh}}^{\mu} Z + G_{\mathrm{wh}}^{\mu}(Z, W, U) + H_{\mathrm{wh}}^{\mu}(Z), \tag{1.34}$$

$$\dot{W} = L_{\mathrm{c}}^{\mu} W + G_{\mathrm{c}}^{\mu}(Z, W, U) + H_{\mathrm{c}}^{\mu}(Z), \tag{1.35}$$

$$\dot{U} = L_{\mathrm{sh}}^{\mu} U + G_{\mathrm{sh}}^{\mu}(Z, W, U) + H_{\mathrm{sh}}^{\mu}(Z), \tag{1.36}$$

where

$$L_{\mathrm{wh}}^{\mu} = \begin{pmatrix} \mathrm{i}(\omega + \tilde{\sigma}^{\mu}) & \tilde{\lambda}^{\mu} & 0 & 0 \\ \tilde{\lambda}^{\mu} & \mathrm{i}(\omega + \tilde{\sigma}^{\mu}) & 0 & 0 \\ 0 & 0 & -\mathrm{i}(\omega + \tilde{\sigma}^{\mu}) & \tilde{\lambda}^{\mu} \\ 0 & 0 & \tilde{\lambda}^{\mu} & -\mathrm{i}(\omega + \tilde{\sigma}^{\mu}) \end{pmatrix}$$

(in complex coordinates), and

$$\begin{aligned}
G_{\mathrm{wh}}^{\mu}(Z, W, U) &= \mathcal{O}\Big(|W| \|(Z, W, U)\|_{\mathcal{D}} + \mu \|U\|_{\mathcal{D}_{\mathrm{sh}}} \|(Z, W, U)\|_{\mathcal{D}}\Big), \\
G_{\mathrm{c}}^{\mu}(Z, W, U) &= \mathcal{O}\Big(\mu |W| \|(Z, W, U)\|_{\mathcal{D}} + \mu^2 \|U\|_{\mathcal{D}_{\mathrm{sh}}} \|(Z, W, U)\|_{\mathcal{D}}\Big), \\
G_{\mathrm{sh}}^{\mu}(Z, W, U) &= \mathcal{O}\Big(\|(W, U)\|_{\mathcal{D}_{\mathrm{c,sh}}} \|(Z, W, U)\|_{\mathcal{D}}\Big), \\
H_{\mathrm{wh}}^{\mu}(Z) &= \mathcal{O}(|Z|^3), \\
H_{\mathrm{c}}^{\mu}(Z), H_{\mathrm{sh}}^{\mu}(Z) &= \mathcal{O}(|(Z, \mu)||Z|^2)
\end{aligned}$$

and we have abbreviated $L_{\mathrm{c}}^{\varepsilon}|_{\varepsilon=\mu^2}$, $L_{\mathrm{sh}}^{\varepsilon}|_{\varepsilon=\mu^2}$ to $L_{\mathrm{c}}^{\mu}$, $L_{\mathrm{sh}}^{\mu}$.

The result of the normal-form theory is the existence of an optimal value of $p$ and $\delta > 0$, $c^* > 0$ such that the transformed system

$$\dot{Z} = L^\mu_{\mathrm{wh}} Z + \tilde{G}^\mu_{\mathrm{wh}}(Z,W,U) + \tilde{H}^\mu_{\mathrm{wh}}(Z), \tag{1.37}$$
$$\dot{W} = L^\mu_{\mathrm{c}} W + \tilde{G}^\mu_{\mathrm{c}}(Z,W,U) + \tilde{H}^\mu_{\mathrm{c}}(Z), \tag{1.38}$$
$$\dot{U} = L^\mu_{\mathrm{sh}} U + \tilde{G}^\mu_{\mathrm{sh}}(Z,W,U) + \tilde{H}^\mu_{\mathrm{sh}}(Z) \tag{1.39}$$

satisfies

$$\|\tilde{H}^\mu_{\mathrm{c,sh}}(Z)\|_{\mathcal{X}_{\mathrm{c,sh}}} \lesssim \mu^2 \mathrm{e}^{-\frac{c^*}{2\mu}},$$
$$\|\mathrm{d}\tilde{H}^\mu_{\mathrm{c,sh}}(Z)\|_{\mathcal{L}(\mathbb{R}^3;\mathcal{X}_{\mathrm{c,sh}})} \lesssim \mu \mathrm{e}^{-\frac{c^*}{2\mu}}$$

for $|(Z,\mu)| < \delta$.

The normal-form theory is presented in a general context in Chapter 3 and applied to our specific problems in Chapter 4. Chapter 4 also contains the following results for the ‘approximate pulses’ in the invariant subspace $\{(w,u) = (0,0)\}$ obtained by selecting $h^\varepsilon_{\mathrm{c,sh}}(z) = 0$, and estimates for the transformed nonlinearities are given in Remarks 4.8 and 4.16.

**Lemma 1.4.**

(i) Suppose $n = 2$. For fixed $\nu \in (0,1)$ and $\varepsilon > 0$ the system

$$\dot{z} = L^\varepsilon_{\mathrm{wh}} z + h^\varepsilon_{\mathrm{wh}}(z)$$

has a reversible homoclinic solution of the form

$$\begin{pmatrix} p^\varepsilon_1(t) \\ p^\varepsilon_2(t) \end{pmatrix} = \begin{pmatrix} (\lambda^\varepsilon)^2 \breve{p}^\varepsilon_1(t) \\ (\lambda^\varepsilon)^3 \breve{p}^\varepsilon_2(t) \end{pmatrix},$$

where

$$|\breve{p}^\varepsilon_j(t)| \lesssim \mathrm{e}^{-\nu|t|}$$

for all $t \in \mathbb{R}$. The scaled equation

$$\dot{z} = L^\mu_{\mathrm{wh}} Z + H^\mu_{\mathrm{wh}}(Z)$$

has the reversible homoclinic solution

$$P^\mu(t) = \begin{pmatrix} P^\mu_1(t) \\ P^\mu_2(t) \end{pmatrix} = \begin{pmatrix} \tilde{\lambda}^\mu \breve{p}^\varepsilon_1(\tilde{\lambda}^\mu t)|_{\varepsilon=\mu^2} \\ \tilde{\lambda}^\mu \breve{p}^\varepsilon_2(\tilde{\lambda}^\mu t)|_{\varepsilon=\mu^2} \end{pmatrix},$$

which satisfies the estimate

$$|P^\mu(t)| \lesssim \tilde{\lambda}^\mu \mathrm{e}^{-\nu\tilde{\lambda}^\mu|t|}$$

for all $t \in \mathbb{R}$.

(ii) Suppose that $n=4$. For fixed $\nu \in (0,1)$ and $\varepsilon > 0$ the system

$$\dot{z} = L^\varepsilon_{\mathrm{wh}} z + h^\varepsilon_{\mathrm{wh}}(z)$$

has a pair of reversible homoclinic solutions of the form

$$\mathrm{e}^{\mathrm{i}(\omega+\sigma^\varepsilon)t} \begin{pmatrix} \lambda^\varepsilon(\check{p}^\varepsilon_1(\lambda^\varepsilon t) + \mathrm{i}\check{p}^\varepsilon_2(\lambda^\varepsilon t)) \\ (\lambda^\varepsilon)^2(\check{q}^\varepsilon_1(\lambda^\varepsilon t) + \mathrm{i}\check{q}^\varepsilon_2(\lambda^\varepsilon t)) \end{pmatrix},$$

where

$$|\check{p}^\varepsilon_i(t)|, |\check{q}^\varepsilon_i(t)| \lesssim \mathrm{e}^{-\nu|t|}$$

for all $t \in \mathbb{R}$. The scaled equation

$$\dot{z} = L^\mu_{\mathrm{wh}} Z + H^\mu_{\mathrm{wh}}(Z)$$

has the reversible homoclinic solutions

$$P^\mu(t) = \begin{pmatrix} P^\mu_1(t) \\ P^\mu_2(t) \end{pmatrix} = \mathrm{e}^{\mathrm{i}(\omega+\tilde{\sigma}^\mu)} \begin{pmatrix} (\tilde{\lambda}^\mu)^{\frac{1}{2}}(\check{p}^\varepsilon_1(\tilde{\lambda}^\mu t) + \mathrm{i}\check{p}^\varepsilon_2(\tilde{\lambda}^\mu t))|_{\varepsilon=\mu^2} \\ (\tilde{\lambda}^\mu)^{\frac{1}{2}}(\check{q}^\varepsilon_1(\tilde{\lambda}^\mu t) + \mathrm{i}\check{q}^\varepsilon_2(\tilde{\lambda}^\mu t))|_{\varepsilon=\mu^2} \end{pmatrix},$$

which satisfy the estimate

$$|P^\mu(t)| \lesssim (\tilde{\lambda}^\mu)^{\frac{1}{2}} \mathrm{e}^{-\nu\tilde{\lambda}^\mu|t|}$$

for all $t \in \mathbb{R}$.

In Chapter 5 we turn to the construction of generalised pulses. Writing

$$Z = P^\mu + R,$$

we obtain the system

$$\dot{R} = K^\mu R + N^\mu(R,W,U), \tag{1.40}$$
$$\dot{W} = L^\mu_{\mathrm{c}} W + \tilde{G}^\mu_{\mathrm{c}}(P^\mu + R, W, U) + \tilde{H}^\mu_{\mathrm{c}}(P^\mu + R), \tag{1.41}$$
$$\dot{U} = L^\mu_{\mathrm{sh}} U + \tilde{G}^\mu_{\mathrm{sh}}(P^\mu + R, W, U) + \tilde{H}^\mu_{\mathrm{sh}}(P^\mu + R), \tag{1.42}$$

where

$$K^\mu R = L^\mu_{\mathrm{wh}} R + \mathrm{d}\tilde{H}^\mu_{\mathrm{wh}}[P^\mu](R),$$
$$N^\mu(R,W,U) = \tilde{H}^\mu_{\mathrm{wh}}(P^\mu + R) - \tilde{H}^\mu_{\mathrm{wh}}(P^\mu) - \mathrm{d}\tilde{H}^\mu_{\mathrm{wh}}[P^\mu](R) + \tilde{G}^\mu_{\mathrm{wh}}(P^\mu + R, W, U).$$

Since we anticipate the $W$-component to grow linearly (being the centre part of the system), we cannot expect solutions of equations (1.40) – (1.42) to be bounded in their $W$-component. We therefore modify system (1.40) – (1.42) by replacing $W$ in the nonlinearities by $\phi(W) = \psi(\mathrm{e}^{\frac{c^*}{2\mu}}|W|)W$, where $\psi\colon \mathbb{R} \to \mathbb{R}$ is a smooth cut-off function with

$$\psi(t) = \begin{cases} 1, & |t| \le 1, \\ 0, & |t| > 2 \end{cases}$$

and $|\psi^{(l)}(t)| \le 2^l$ for $t \in \mathbb{R}$ and $l \in \mathbb{N}_0$. We write the modified system as

$$\dot{R} = K^\mu R + \underset{\sim}{N}^\mu_{\mathrm{wh}}(R, W, U), \tag{1.43}$$
$$\dot{W} = L^\mu_{\mathrm{c}} W + \underset{\sim}{\tilde{G}}^\mu_{\mathrm{c}}(P^\mu + R, W, U) + \tilde{H}^\mu_{\mathrm{c}}(P^\mu + R), \tag{1.44}$$
$$\dot{U} = L^\mu_{\mathrm{sh}} U + \underset{\sim}{\tilde{G}}^\mu_{\mathrm{sh}}(P^\mu + R, W, U) + \tilde{H}^\mu_{\mathrm{sh}}(P^\mu + R). \tag{1.45}$$

In Section 5.2 we establish the following theorem by formulating equations (1.43) – (1.45) as a fixed-point problem.

**Theorem 1.5.** Fix $\nu \in (0,1)$. For each $W_0 \in \mathbb{R}^{2d}$ with $S_{\mathrm{c}} W_0 = W_0$ and $|W_0| \le \mu \mathrm{e}^{-\frac{c^*}{2\mu}}$ there exists a reversible solution $(R^*_{W_0}, W^*_{W_0}, U^*_{W_0})$ of equations (1.43) – (1.45) with $W(0) = W_0$ in

$$E_{\nu\tilde{\lambda}^\mu} = C^\alpha_{\nu\tilde{\lambda}^\mu}(\mathbb{R}; \mathbb{R}^n) \times C^1_{\nu\tilde{\lambda}^\mu}(\mathbb{R}; \mathbb{R}^{2d}) \times C^\alpha_{\nu\tilde{\lambda}^\mu}(\mathbb{R}; \mathcal{D}_{\mathrm{sh}})$$

(where the subscript refers to the use of the exponential weighting $\mathrm{e}^{-\nu\tilde{\lambda}^\mu|t|}$ in the norms) satisfying

$$\sup_{t\in\mathbb{R}} |R^*_{W_0}(t)| \lesssim \mu \mathrm{e}^{-\frac{c^*}{2\mu}},$$
$$\sup_{t\in[-\mathrm{e}^{\frac{c^*}{2\mu}}, \mathrm{e}^{\frac{c^*}{2\mu}}]} |W^*_{W_0}(t)| \lesssim \mu^\delta \mathrm{e}^{-\frac{c^*}{2\mu}},$$
$$\sup_{t\in\mathbb{R}} \|U^*_{W_0}(t)\|_{\mathcal{D}_{\mathrm{sh}}} \lesssim \mu \mathrm{e}^{-\frac{c^*}{2\mu}}$$

for some $\delta > 0$. These solutions thus in particular solve equations (1.37) – (1.39) for $t \in [-\mathrm{e}^{c^*/2\mu}, \mathrm{e}^{c^*/2\mu}]$. In analogy with familiar dynamical-systems theory we define

$$W^{\mathrm{cs}}_{\mathrm{loc}} = \{(R^*_{W_0}, W^*_{W_0}, U^*_{W_0})(0) : S_{\mathrm{c}} W_0 = W_0, |W_0| \le \mu \mathrm{e}^{-\frac{c^*}{2\mu}}\}$$

and refer to $W^{\mathrm{cs}}_{\mathrm{loc}}$ as the *local centre-stable manifold for reversible solutions to equations (1.37) – (1.39)*. The behaviour of functions with initial values lying on $W^{\mathrm{cs}}_{\mathrm{loc}}$ is summarised in Figures 1.8 – 1.10.

The proof of Theorems 1.1 and 1.2 is completed by showing that in fact

$$\sup_{t\in\mathbb{R}} |W^*_{W_0}(t)| < \mathrm{e}^{-\frac{c^*}{2\mu}},$$

so that $(R^*_{W_0}, W^*_{W_0}, U^*_{W_0})$ solves equations (1.37) – (1.39) for all $t \in \mathbb{R}$. We accomplish this result in several stages in Sections 5.3 and 5.4, by proving the following results. Theorems 1.6 and 1.8 are more precise than usual versions of familiar results in centre-manifold theory, while Lemma 1.7 is proved using the conserved quantity $\mathcal{I}^\varepsilon|_{\varepsilon=\mu^2}$ as a Lyapunov functional.

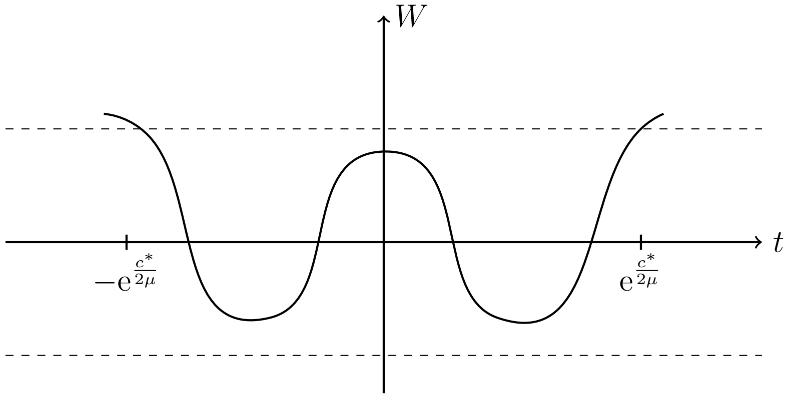



Figure 1.8: The central part of functions with initial values on $W^{\mathrm{cs}}_{\mathrm{loc}}$ satisfies $|W(t)| \leq \mathrm{e}^{-\frac{c^*}{2\mu}}$ for $t \in [-\mathrm{e}^{\frac{c^*}{2\mu}}, \mathrm{e}^{\frac{c^*}{2\mu}}]$. It may leave this neighbourhood of the origin for larger values of $|t|$.

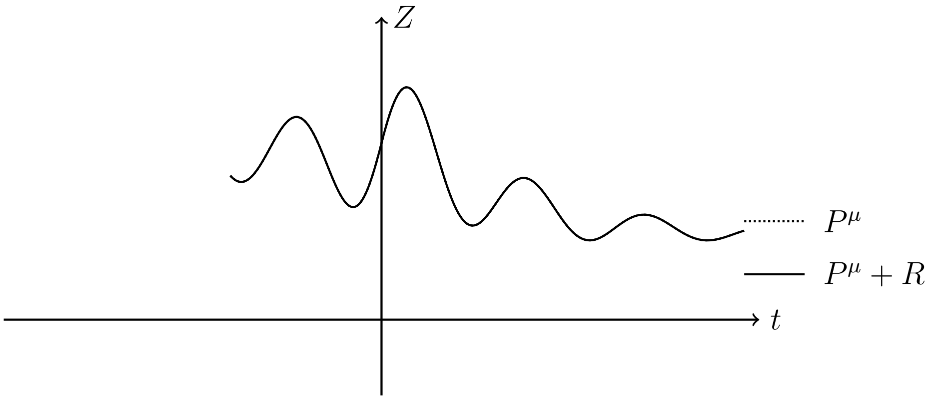



Figure 1.9: The weakly hyperbolic part of functions with initial values on $W^{\mathrm{cs}}_{\mathrm{loc}}$ lies in a tubular neighbourhood of $P^\mu$ such that $|(Z(t) - P^\mu(t))| \leq \mathrm{e}^{-\frac{c^*}{2\mu}}$ for all $t \in \mathbb{R}$.

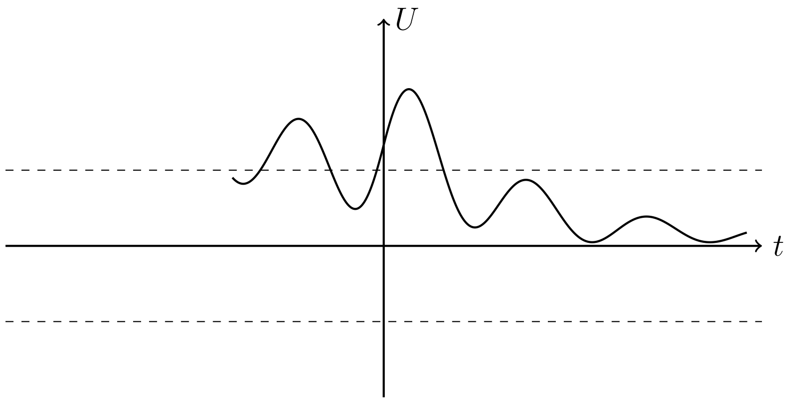



Figure 1.10: The strongly hyperbolic part of functions with initial values on $W^{\mathrm{cs}}_{\mathrm{loc}}$ satisfies $\|U(t)\|_{\mathcal{D}_{\mathrm{sh}}} \leq \mathrm{e}^{-\frac{c^*}{2\mu}}$ for all $t \in \mathbb{R}$.

**Theorem 1.6.** The equations

$$\dot{Z} = L^{\mu}_{\rm wh} Z + \underline{N}^{\mu}_{\rm wh}(Z,W,U) + \tilde{H}^{\mu}_{\rm wh}(Z), \tag{1.46}$$
$$\dot{W} = L^{\mu}_{\rm c} W + \underline{\tilde{G}}^{\mu}_{\rm c}(Z,W,U) + \tilde{H}^{\mu}_{\rm c}(Z), \tag{1.47}$$
$$\dot{U} = L^{\mu}_{\rm sh} U + \underline{\tilde{G}}^{\mu}_{\rm sh}(Z,W,U) + \tilde{H}^{\mu}_{\rm sh}(Z) \tag{1.48}$$

obtained from equations (1.37) – (1.39) using the cut-off function have a *centre manifold*

$$W^{\rm c} = \{(Z,W,U) : (Z,U) = \Psi(W)\},$$

where $\Psi \colon \bar{B}_{{\rm e}^{-c^*/2\mu}}(0) \to \mathbb{R}^n \times \mathcal{D}_{\rm sh}$ satisfies the estimate

$$\|\Psi(W)\|_{\mathbb{R}^n \times \mathcal{D}_{\rm sh}} \lesssim |W|^2.$$

Any solution $(Z,W,U)$ to equations (1.46) – (1.48) satisfying

$$\|Z\|_{C_{\rm b}(\mathbb{R};\mathbb{R}^n)}, \|\dot{W} - L^{\mu}_{\rm c} W\|_{C_{\rm b}(\mathbb{R};\mathbb{R}^{2d})}, \|U\|_{C_{\rm b}(\mathbb{R};\mathcal{D}_{\rm sh})} \le {\rm e}^{-\frac{c^*}{2\mu}}$$

lies completely on $W^{\rm c}$.

Furthermore, any solution to equations (1.37) – (1.39) with

$$\|Z\|_{C_{\rm b}(\mathbb{R};\mathbb{R}^n)}, \|W\|_{C_{\rm b}(\mathbb{R};\mathbb{R}^{2d})}, \|U\|_{C_{\rm b}(\mathbb{R};\mathcal{D}_{\rm sh})} \le {\rm e}^{-\frac{c^*}{2\mu}}$$

lies on

$$W^{\rm c}_{\rm loc} = \{(Z,W,U) : (Z,U) = \Psi(W), |W| \le {\rm e}^{-\frac{c^*}{2\mu}}\},$$

and any solution passing through a point on $W^{\rm c}_{\rm loc}$ remains on $W^{\rm c}_{\rm loc}$ as long as it remains in

$$\{(Z,W,U) \in \mathcal{D} : |Z|, |W|, \|U\|_{\mathcal{D}_{\rm sh}} \le {\rm e}^{-\frac{c^*}{2\mu}}\}.$$

**Lemma 1.7.** Any solution $(Z,W,U)$ of system (1.46) – (1.48) lying on the manifold $W^{\rm c}$ with $|W(t_0)| \le \frac{1}{2}{\rm e}^{-\frac{c^*}{2\mu}}$ (so that $(Z,W,U)(t_0) \in W^{\rm c}_{\rm loc}$) satisfies $|W(t)| \le \frac{3}{4}{\rm e}^{-\frac{c^*}{2\mu}}$ (and hence $(Z,W,U)(t) \in W^{\rm c}_{\rm loc}$) for all $t \ge t_0$.

**Theorem 1.8.** Let $(R^*_{W_0}, W^*_{W_0}, U^*_{W_0})$ be a solution to equations (1.37) – (1.39) with $(R^*_{W_0}, W^*_{W_0}, U^*_{W_0})(0) \in W^{\rm cs}_{\rm loc}$. There exists a solution $(Z,W,U)$ of equations (1.46) – (1.48) on $W^{\rm c}$ such that

$$\sup_{t \in [t^*,\infty)} |W^*_{W_0}(t) - W(t)| \le \mu {\rm e}^{-\frac{c^*}{2\mu}}$$

for $t^* = \frac{c^*}{\mu\nu\tilde{\lambda}^{\mu}}$ (see Figure 1.11).

Noting that $t^* < {\rm e}^{-\frac{c^*}{2\mu}}$, we find from the estimate

$$\sup_{t \in [0,{\rm e}^{-\frac{c^*}{2\mu}}]} |W^*_{W_0}(t)| \lesssim \mu^{\delta} {\rm e}^{-\frac{c^*}{2\mu}}$$

that $|W^*_{W_0}(t^*)| \le \frac{1}{2}{\rm e}^{-\frac{c^*}{2\mu}}$ and hence $|W^*_{W_0}(t)| \le \frac{3}{4}{\rm e}^{-\frac{c^*}{2\mu}}$ for $t \ge t^*$. The fact that $|W^*_{W_0}(t)| \le {\rm e}^{-\frac{c^*}{2\mu}}$ for all $t \in \mathbb{R}$ follows from the symmetry of $W^*_{W_0}$. The generalised pulses in Theorems 1.1 and 1.2 are obtained from $(P^{\mu} + R^*_{W_0}, W^*_{W_0}, U^*_{W_0})$ by reversing the scaling from $\varepsilon$ to $\mu$.

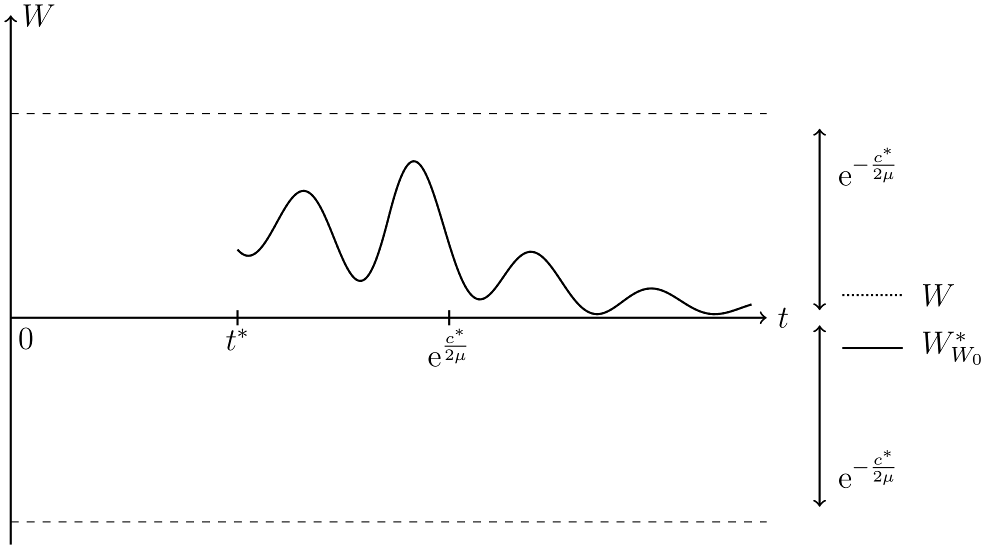



Figure 1.11: The central part $W^*_{W_0}$ of a function with initial values on $W^{\mathrm{cs}}_{\mathrm{loc}}$ converges exponentially to the central part $W$ of a solution to equations (1.46) – (1.48) on $W^{\mathrm{c}}$.

## 1.3 Applications

### 1.3.1 Generalised solitary waves on rotational flows

In Section 6.2 we consider gravity-driven steady waves on the surface of water in a uniform rectangular channel bounded below by a rigid horizontal bottom and above by a free surface. In a Cartesian coordinate system moving with the wave the fluid domain is

$$\{(x, y) : x \in \mathbb{R}, 0 < y < \eta(x)\}$$

for some profile function $\eta\colon \mathbb{R} \to (0, \infty)$. Working in dimensionless coordinates, we seek the velocity field in the form $(\psi_y, -\psi_x)$, where the *stream function* $\psi\colon \mathbb{R}^2 \to \mathbb{R}$ satisfies the boundary-value problem

$$\psi_{xx} + \psi_{yy} + \omega^{\varepsilon}(\psi) = 0, \qquad 0 < y < \eta, \tag{1.49}$$

$$\psi = 0, \qquad y = 0, \tag{1.50}$$

$$\psi = 1, \qquad y = \eta, \tag{1.51}$$

$$\psi_x^2 + \psi_y^2 + 2\eta = 3r, \qquad y = \eta \tag{1.52}$$

(see Keady and Norbury [14]). Here the *vorticity function* $\omega^{(\cdot)}$ is a real-valued function which is analytic at the origin in $\mathbb{R} \times \mathbb{R}$ and $r$ is a parameter referred to as the *Bernoulli constant*. A *solitary wave* is a solution $(\eta, \psi)$ to equations (6.8) – (6.11) such that $\eta$ decays to a constant, while a *generalised solitary wave* instead decays to small ripples far up- and downstream. Solitary waves were found by Kozlov et al. [19] under the assumption that $\omega$ is a large negative constant. In this section we apply the results of Chapter 5 to establish the existence of generalised solitary waves with exponentially small tails for linear vorticity functions (see Figure 1.12).

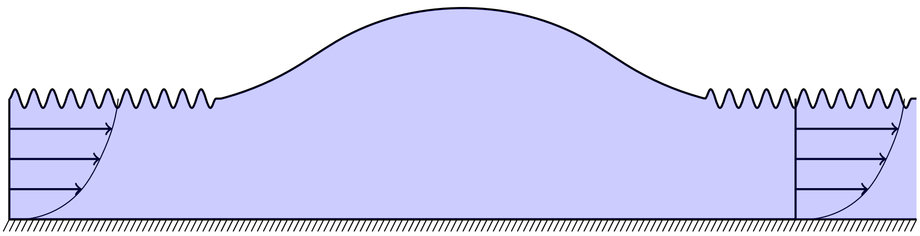



Figure 1.12: A generalised solitary wave on the surface of a stream solution.

We start by interpreting our flow as a perturbation of a *stream solution* $(\Lambda^\varepsilon, h)$ of equations (1.49) – (1.52), that is a solution $(\eta, \psi)$ with $\psi = \Lambda^\varepsilon(\tilde{y})$ and $\eta = h$, so that

$$\begin{aligned}
(\Lambda^\varepsilon)'' + \omega^\varepsilon(\Lambda^\varepsilon) &= 0, && 0 < \tilde{y} < h,\\
\Lambda^\varepsilon &= 0, && y = 0,\\
\Lambda^\varepsilon &= 1, && y = h,\\
((\Lambda^\varepsilon)')^2 + 2h &= 3r, && y = h
\end{aligned}$$

(see Kozlov and Kuznetsov [18] for a complete discussion of stream solutions); we assume that $\Lambda^\varepsilon \in H^2(0,h)$ depends analytically upon $\varepsilon$. Equations (1.49) – (1.52) are equivalent to the spatial evolutionary system

$$\tilde{\Phi}_{\tilde{x}} = \tilde{\Psi} + N_1^\varepsilon(\tilde{\Phi}, \tilde{\Psi}), \qquad 0 < \tilde{y} < h, \tag{1.53}$$

$$\tilde{\Psi}_{\tilde{x}} = -\tilde{\Phi}_{\tilde{y}\tilde{y}} - (\omega^\varepsilon)'(\Lambda^\varepsilon)\tilde{\Phi} + N_2^\varepsilon(\tilde{\Phi}, \tilde{\Psi}), \qquad 0 < \tilde{y} < h, \tag{1.54}$$

$$\tilde{\Phi} = 0, \qquad y = 0, \tag{1.55}$$

$$\tilde{\Psi} = 0, \qquad y = 0, \tag{1.56}$$

$$\tilde{\Phi}_{\tilde{y}} - \kappa^\varepsilon\tilde{\Phi} = N_3^\varepsilon(\tilde{\Phi}, \tilde{\Psi}), \qquad y = h, \tag{1.57}$$

in terms of the new coordinates

$$\tilde{x} = x, \quad \tilde{y} = \frac{h}{\eta(x)}y$$

and variables

$$\begin{aligned}
\tilde{\Phi} &= \Phi - \Lambda^\varepsilon - \tilde{y}\frac{(\Lambda^\varepsilon)'}{h}\zeta,\\
\tilde{\Psi} &= \Psi,\\
\zeta &= \eta - h - \frac{\tilde{\Phi}(\cdot, h)}{(\Lambda^\varepsilon)'(h)},
\end{aligned}$$

where

$$\begin{aligned}
\Phi(\tilde{x}, \tilde{y}) &= \psi(x, y),\\
\Psi(\tilde{x}, \tilde{y}) &= \frac{\eta}{h}(\Phi_{\tilde{x}} - \frac{\eta_{\tilde{x}}}{\eta}\tilde{y}\Phi_{\tilde{y}}).
\end{aligned}$$

The nonlinearities are given by

$$
\begin{aligned}
N_1^\varepsilon(\tilde\Phi,\tilde\Psi) &= -\frac{\zeta}{\zeta+h}\tilde\Psi + \frac{y\zeta_{\tilde x}}{\zeta+h}\Big(\tilde\Phi_{\tilde y} + \frac{y}{h}(\Lambda^\varepsilon)''\zeta\Big),\\
N_2^\varepsilon(\tilde\Phi,\tilde\Psi) &= -\frac{\zeta}{h}(\omega^\varepsilon)'(\Lambda^\varepsilon)\tilde\Phi + \frac{\zeta_{\tilde x}}{\zeta+h}(\tilde y\tilde\Psi)_{\tilde y}\\
&\quad + \frac{\zeta}{\zeta+h}\Big(\tilde\Phi_{\tilde y\tilde y} - \frac{\zeta}{h}\omega^\varepsilon(\Lambda^\varepsilon) - \frac{\zeta+2h}{h}(\omega^\varepsilon)'(\Lambda^\varepsilon)(\Lambda^\varepsilon)'\tilde y\frac{\zeta}{h}\Big)\\
&\quad - \frac{\zeta+h}{h}\Big(\omega^\varepsilon\Big(\tilde\Phi + \Lambda^\varepsilon + \tilde y(\Lambda^\varepsilon)'\frac{\zeta}{h}\Big) - \omega^\varepsilon(\Lambda^\varepsilon) - (\omega^\varepsilon)'(\Lambda^\varepsilon)\Big(\tilde\Phi + \tilde y(\Lambda^\varepsilon)'\frac{\zeta}{h}\Big)\Big),\\
N_3^\varepsilon(\tilde\Phi,\tilde\Psi) &= \frac{1}{2(\Lambda^\varepsilon)'}\Big(-\tilde\Psi^2 - \Big(\tilde\Phi_{\tilde y} - \Big(\frac{1}{h} + \frac{(\Lambda^\varepsilon)''}{(\Lambda^\varepsilon)'}\tilde\Phi\Big)^2 + P\Big(h - \frac{\tilde\Phi}{(\Lambda^\varepsilon)'}\Big) - P(h)\\
&\qquad + P'(h)\frac{\tilde\Phi}{(\Lambda^\varepsilon)'}\Big)\Big)
\end{aligned}
$$

with

$$
\begin{aligned}
\kappa^\varepsilon &= \Big(\frac{1}{(\Lambda^\varepsilon)'}\Big)^2 - \frac{\omega^\varepsilon(1)}{(\Lambda^\varepsilon)'},\\
P(\eta) &= \frac{\eta^2}{h^2}(3r-2\eta).
\end{aligned}
$$

Equations (1.53) – (1.57) have the conserved quantity

$$
\begin{aligned}
\tilde H^\varepsilon(\tilde\Phi,\tilde\Psi) = \int_0^h \Big(&\frac{h}{2(\zeta+h)}\Big(\tilde\Psi^2 - \Big(\tilde\Phi_{\tilde y} + (\Lambda^\varepsilon)' + \frac{(\tilde y(\Lambda^\varepsilon)')'}{h}\zeta\Big)^2\Big)\\
&+ \frac{\zeta+h}{h}\Omega^\varepsilon\Big(\tilde\Phi + \Lambda^\varepsilon + \frac{\tilde y(\Lambda^\varepsilon)'\zeta}{h}\Big)\Big)\,\mathrm{d}\tilde y + \frac{1}{2}(\zeta+h)^2 - \frac{3}{2}r(\zeta+h).
\end{aligned}
$$

Using spatial dynamics we write equations (1.53) – (1.57) as the evolutionary system

$$
\begin{pmatrix}\tilde\Phi\\ \tilde\Psi\end{pmatrix}_{\tilde x} = f^\varepsilon(\tilde\Phi,\tilde\Psi), \tag{1.58}
$$

where

$$
f^\varepsilon(\tilde\Phi,\tilde\Psi) = \begin{pmatrix}\tilde\Psi + N_1^\varepsilon(\tilde\Phi,\tilde\Psi)\\ -\tilde\Phi_{\tilde y\tilde y} - (\omega^\varepsilon)'(\Lambda^\varepsilon)\tilde\Phi + N_2^\varepsilon(\tilde\Phi,\tilde\Psi)\end{pmatrix};
$$

note that $f^\varepsilon$ takes values in $\mathcal{X}$ and is analytic at the origin in $\mathbb{R}\times\mathcal{Y}$, where

$$
\begin{aligned}
\mathcal{Y} &= \Big\{(\tilde\Phi,\tilde\Psi)\in H^2(0,h)\times H^1(0,h) : \tilde\Phi|_{\tilde y=0} = 0,\, \tilde\Psi|_{\tilde y=0} = 0\Big\},\\
\mathcal{X} &= \Big\{(\tilde\Phi,\tilde\Psi)\in H^1(0,h)\times L^2(0,h) : \tilde\Phi|_{\tilde y=0} = 0\Big\}.
\end{aligned}
$$

The domain of the vector field on the right-hand side of equation (1.58) is

$$
\{(\tilde\Phi,\tilde\Psi)\in\mathcal{Y} : \tilde\Phi_{\tilde y} - \kappa^\varepsilon\tilde\Phi = N_3^\varepsilon(\tilde\Phi,\tilde\Psi) \text{ for } y = h\}.
$$

To linearise the nonlinear boundary condition associated with equation (1.58) we introduce new coordinates $\tilde\Gamma, \tilde\xi$ given by

$$\tilde\Gamma = \tilde\Phi + \frac{\tilde y}{h}\int_{\tilde y}^h N_3^\varepsilon(\tilde\Phi,\tilde\Psi)(s)\,\mathrm{d}s,$$
$$\tilde\xi = \tilde\Psi.$$

This change of variable transforms equation (1.58) into the evolutionary system

$$\begin{pmatrix}\tilde\Gamma\\ \tilde\xi\end{pmatrix}_{\tilde x} = k^\varepsilon(\tilde\Gamma,\tilde\xi), \tag{1.59}$$

where $k^\varepsilon(\tilde\Gamma,\tilde\xi)$ takes values in $\mathcal{X}$ and is analytic at the origin in $\mathbb{R}\times\mathcal{Y}$. The domain of the vector field on the right-hand side of equation (1.59) is

$$\mathcal{D}^\varepsilon = \left\{(\tilde\Gamma,\tilde\xi)\in H^2(0,h)\times H^1(0,h) : \tilde\Gamma(0),\tilde\xi(0)=0, \tilde\Gamma_{\tilde y}(h)-\kappa^\varepsilon\tilde\Gamma(h)=0\right\}$$

and $\tilde H^\varepsilon$ is transformed into a conserved quantity for equation (1.59).

We note that the linearisation of $k^\varepsilon$ is given by $\check L^\varepsilon\colon \mathcal{D}^\varepsilon\subseteq\mathcal{X}\to\mathcal{X}$ with

$$\check L^\varepsilon\begin{pmatrix}\tilde\Gamma\\ \tilde\xi\end{pmatrix} = \begin{pmatrix}\tilde\xi\\ -\tilde\Gamma_{\tilde y\tilde y}-(\omega^\varepsilon)'(\Lambda^\varepsilon)\tilde\Gamma\end{pmatrix}$$

and study the spectrum of $\check L^\varepsilon$ for the specific choice $\omega^\varepsilon(\Psi)=(b+\varepsilon)\Psi$. We show that for each $N\in\mathbb{N}$ there exists a choice $b=b_N^*$ for which the spectrum $\sigma(\check L^\varepsilon)$ consists of purely imaginary eigenvalues $\pm\mathrm{i}(-\mu_1^\varepsilon)^{1/2}$, $\pm\mathrm{i}(-\mu_2^\varepsilon)^{1/2}$, $\ldots,\pm\mathrm{i}(-\mu_N^\varepsilon)^{1/2}$, real eigenvalues $\pm(\mu_{N+2}^\varepsilon)^{1/2}$, $\pm(\mu_{N+3}^\varepsilon)^{1/2},\ldots$ and additionally

(i) a pair of purely imaginary eigenvalues $\pm\mathrm{i}(-\mu_{N+1}^\varepsilon)^{1/2}$ for $\varepsilon>0$,

(ii) a pair of real eigenvalues $\pm(\mu_{N+1}^\varepsilon)^{1/2}$ for $\varepsilon<0$,

(iii) a zero eigenvalue for $\varepsilon=0$

(see Figure 1.13).

We proceed by decomposing $\mathcal{X}=\mathcal{X}_{\mathrm{wh}}^\varepsilon\oplus\mathcal{X}_{\mathrm{c}}^\varepsilon\oplus\mathcal{X}_{\mathrm{sh}}^\varepsilon$, where $\mathcal{X}_{\mathrm{wh}}=P_{\mathrm{wh}}^\varepsilon[\mathcal{X}]$, $\mathcal{X}_{\mathrm{c}}=P_{\mathrm{c}}^\varepsilon[\mathcal{X}]$ and $\mathcal{X}_{\mathrm{sh}}=P_{\mathrm{sh}}^\varepsilon[\mathcal{X}]$, $P_{\mathrm{c}}^\varepsilon$ and $P_{\mathrm{sh}}^\varepsilon$ are the spectral projections corresponding to respectively $\{\pm\mathrm{i}(-\mu_1^\varepsilon)^{1/2},\ldots,\pm\mathrm{i}(-\mu_N^\varepsilon)^{1/2}\}$ and $\{\pm(\mu_{N+2}^\varepsilon)^{1/2},\pm(\mu_{N+3}^\varepsilon)^{1/2},\ldots\}$ and $P_{\mathrm{wh}}^\varepsilon=I-P_{\mathrm{c}}^\varepsilon-P_{\mathrm{sh}}^\varepsilon$. A further near-identity change of variable maps $\mathcal{X}_{\mathrm{wh}}^\varepsilon$, $\mathcal{X}_{\mathrm{c}}^\varepsilon$ and $\mathcal{X}_{\mathrm{sh}}^\varepsilon$ into the fixed spaces $\mathcal{X}_{\mathrm{wh}}^0$,$\mathcal{X}_{\mathrm{c}}^0$ and $\mathcal{X}_{\mathrm{sh}}^0$ (see Section 6.1 for full details of the procedure) and thus converts equation (1.59) into a system of the form equations (1.27) – (1.29). The following theorem is finally obtained by verifying Assumptions (A1) – (A6) and Assumptions (B1) and (B2).

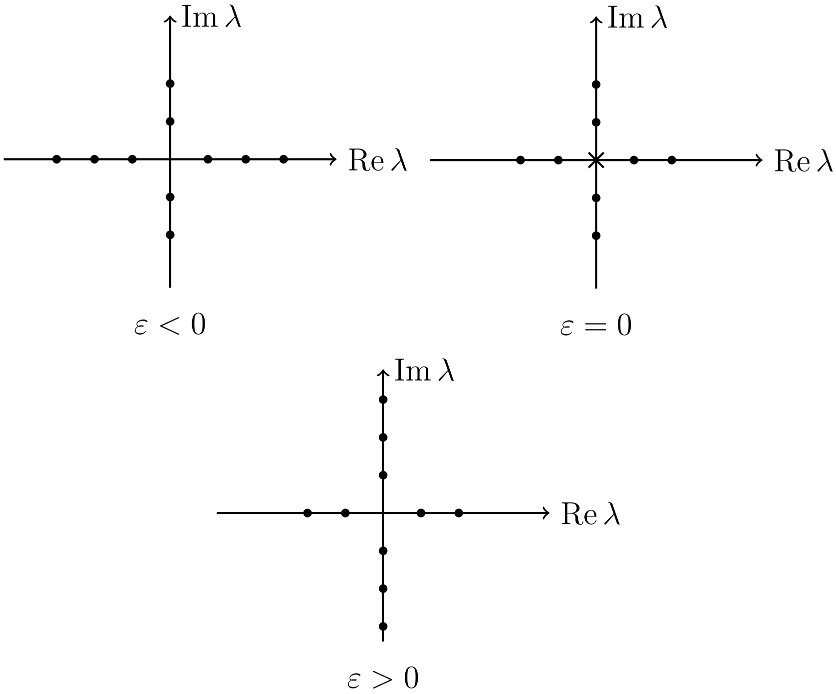



Figure 1.13: The spectrum of $\check{L}^\varepsilon$.

**Theorem 1.9.** For each $n \in \mathbb{N}$ there exist positive constants $b_N^*$, $\varepsilon_0$ and $c^*$ such that for each $\varepsilon \in (0, \varepsilon)$ equations (1.53) – (1.57) admit a family of generalised pulse solutions satisfying in particular $\eta(\tilde{x}) = h + \zeta(\tilde{x})$, where $\zeta(\tilde{x}) = \zeta(-\tilde{x})$ and

$$\left|\zeta(\tilde{x}) + \frac{1}{\sqrt{\gamma b_{N^*}}} \tan(\sqrt{b_{N^*}} h)(\lambda^\varepsilon)^2 \check{p}^\varepsilon(\lambda^\varepsilon \tilde{x})\right| < \mathrm{e}^{-\frac{c^*}{2\sqrt{\varepsilon}}}.$$

Here $\gamma$ and $(-1)^N C$ are positive numbers (given by equation (6.41) with $b = b_N^*$ and equation (6.42)),

$$\check{p}^\varepsilon(t) = \frac{3}{2C} \operatorname{sech}^2\left(\frac{t}{2}\right) + \mathcal{O}(\varepsilon \mathrm{e}^{-\nu|t|})$$

and $\pm\lambda^\varepsilon$ are the two real eigenvalues of the linearised problem with $\lambda^\varepsilon = \mathcal{O}(\varepsilon)$.

### 1.3.2 Periodic steady gravity-capillary water waves with localised transverse profiles

In Section 6.3 we consider gravity-capillary steady waves on the surface of water bounded below by a rigid horizontal bottom and above by a free surface. In a dimensionless Cartesian coordinate system moving with the wave the fluid domain is

$$\{(x, y, z) : x, y, z \in \mathbb{R}, 0 < y < 1 + \eta(x, z)\}$$

for some profile function $\eta \colon \mathbb{R}^2 \to (-1, \infty)$ which is $2\pi/\tau$-periodic in the $x$-direction.

Working in dimensionless variables, we seek the velocity field in the form $(\phi_x, \phi_y, \phi_z)$, where the *velocity potential* $\phi\colon \mathbb{R}^3 \to \mathbb{R}$ satisfies the boundary-value problem

$$\tau^2\phi_{xx} + \phi_{yy} + \phi_{zz} = 0, \qquad 0 < y < 1+\eta, \tag{1.60}$$

$$\phi_y = 0, \qquad y = 0, \tag{1.61}$$

$$\phi_y = \tau^2\eta_x\phi_x - \eta_z\phi_z - \tau\eta_x, \qquad y = 1+\eta \tag{1.62}$$

and

$$\begin{aligned} &-\tau\phi_x + \frac{1}{2}(\tau^2\phi_x^2 + \phi_y^2 + \phi_z^2) + \alpha\eta \\ &\quad - \beta\tau^2\Big[\frac{\eta_x}{\sqrt{1+\tau^2\eta_x^2+\eta_z^2}}\Big]_x - \beta\Big[\frac{\eta_z}{\sqrt{1+\tau^2\eta_x^2+\eta_z^2}}\Big]_z = 0, \quad y = 1+\eta. \end{aligned} \tag{1.63}$$

Here the period in the $x$-direction has been normalised to $2\pi$ and $\alpha, \beta$ are dimensionless parameters which measure respectively the speed of the wave and the strength of surface tension (see Groves [6]). In this section we establish the existence of solutions to equations (1.60) – (1.63) with localised profiles which decay to small ripples in the transversal direction (see Figure 1.14). Waves with localised profiles which decay to zero in the transverse direction were found by Groves [6]. We introduce a bifurcation parameter by writing $(\beta, \alpha) = (\beta_0, \alpha_0 + \varepsilon)$, where the values $(\beta_0, \alpha_0)$ are chosen later.

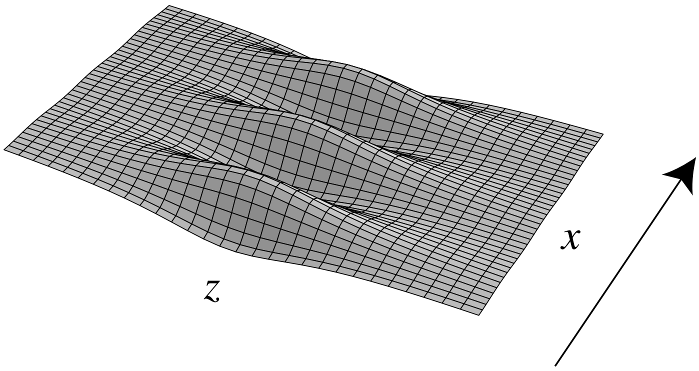



Figure 1.14: A steady wave which is periodic in the direction of travel and spatially localised in the transverse direction.

Equations (1.60) – (1.63) are equivalent to the spatial evolutionary system

$$\eta_{\tilde z} = \left(\frac{1+\tau^2\eta_{\tilde x}^2}{\beta_0^2 - W^2}\right)^{\frac12} W, \tag{1.64}$$

$$\begin{aligned} \omega_{\tilde z} = {} & \frac{W}{(1+\eta)^2}\left(\frac{1+\tau^2\eta_{\tilde x}^2}{\beta_0^2 - W^2}\right)^{\frac12} \int_0^1 \Psi'\tilde y \Phi'_{\tilde y}\,\mathrm{d}\tilde y - \tau^2\left[\eta_{\tilde x}\left(\frac{\beta_0^2 - W^2}{1+\tau^2\eta_{\tilde x}^2}\right)^{\frac12}\right]_{\tilde x} \\ & + \int_0^1 \left(\frac{\Psi'^2 - \Phi_{\tilde y}'^{\,2}}{2(1+\eta)^2} + \frac{\tau^2}{2}\left(\Phi'_{\tilde x} - \frac{\Phi'_{\tilde y}\tilde y \eta_{\tilde x}}{1+\eta}\right)^2 \right. \\ & \qquad + \tau^2\left(\Phi'_{\tilde x} - \frac{\Phi'_{\tilde y}\tilde y\eta_{\tilde x}}{1+\eta}\right)\frac{\Phi'_{\tilde y}\tilde y\eta_{\tilde x}}{1+\eta} \\ & \qquad \left. + \tau^2\left[\left(\Phi'_{\tilde x} - \frac{\Phi'_{\tilde y}\tilde y\eta_{\tilde x}}{1+\eta}\right)\Phi'_{\tilde y}\tilde y\right]_{\tilde x}\right)\mathrm{d}\tilde y \\ & + (\alpha+\varepsilon)\eta - \tau\Phi'_{\tilde x}|_{\tilde y=1}, \end{aligned} \tag{1.65}$$

$$\begin{aligned} \Phi'_{\tilde z} = {} & \frac{\Psi'}{1+\eta} + \left(\frac{1+\tau^2\eta_{\tilde x}^2}{\beta_0^2 - W^2}\right)^{\frac12}\frac{\tilde y\Phi'_{\tilde y}W}{1+\eta} \\ & - \frac{1}{2\pi}\int_0^{2\pi}\int_0^1 \left(\frac{-\eta\Psi'}{1+\eta} + \left(\frac{1+\tau^2\eta_{\tilde x}^2}{\beta_0^2 - W^2}\right)^{\frac12}\frac{\tilde y\Phi'_{\tilde y}W}{1+\eta}\right)\mathrm{d}\tilde y\,\mathrm{d}\tilde x, \end{aligned} \tag{1.66}$$

$$\begin{aligned} \Psi'_{\tilde z} = {} & -\frac{\Phi'_{\tilde y\tilde y}}{1+\eta} - \tau^2\left[(1+\eta)\left(\Phi'_{\tilde x} - \frac{\Phi'_{\tilde y}\tilde y\eta_{\tilde x}}{1+\eta}\right)\right]_{\tilde x} \\ & + \tau^2\left[\tilde y\eta_{\tilde x}\left(\Phi'_{\tilde x} - \frac{\Phi'_{\tilde y}\tilde y\eta_{\tilde x}}{1+\eta}\right)\right]_{\tilde y} + \frac{W(\tilde y\Psi')_{\tilde y}}{1+\eta}\left(\frac{1+\tau^2\eta_{\tilde x}^2}{\beta_0^2 - W^2}\right)^{\frac12}, \end{aligned} \tag{1.67}$$

with boundary conditions

$$\Phi'_{\tilde y} = 0, \qquad \tilde y = 0, \tag{1.68}$$

$$\tau\eta_{\tilde x} + \frac{\Phi'_{\tilde y}}{1+\eta} = \tau^2\eta_{\tilde x}\left(\Phi'_{\tilde x} - \frac{\Phi'_{\tilde y}\eta_{\tilde x}}{1+\eta}\right) + \frac{W\Psi'}{1+\eta}\left(\frac{1+\tau^2\eta_{\tilde x}^2}{\beta_0^2 - W^2}\right)^{\frac12}, \qquad \tilde y = 1, \tag{1.69}$$

where

$$W = \omega + \frac{1}{1+\eta}\int_0^1 \Psi'\tilde y\Phi'_{\tilde y}\,\mathrm{d}\tilde y.$$

The coordinates and variables are

$$\tilde x = x, \quad \tilde y = \frac{y}{1+\eta(x,z)}, \quad \tilde z = z$$

and

$$\Phi' = \Phi - \frac{1}{2\pi}\int_0^{2\pi}\int_0^1 \Phi\,\mathrm{d}\tilde y\,\mathrm{d}x, \quad \Psi' = \Psi - \frac{1}{2\pi}\int_0^{2\pi}\int_0^1 \Psi\,\mathrm{d}\tilde y\,\mathrm{d}x,$$

where

$$\Phi(\tilde{x},\tilde{y},\tilde{z}) = \phi(x,y,z),$$
$$\omega = -\int_0^1 \Big(\Phi_{\tilde{z}} - \tilde{y}\frac{\Phi_{\tilde{y}}\eta_{\tilde{z}}}{1+\eta}\Big)\tilde{y}\Phi_{\tilde{y}}\,\mathrm{d}\tilde{y} + \beta_0 \frac{\eta_{\tilde{z}}}{\sqrt{1+\tau^2\eta_{\tilde{x}}^2+\eta_{\tilde{z}}^2}},$$
$$\Psi = \Big(\Phi_{\tilde{z}} - \tilde{y}\frac{\Phi_{\tilde{z}}\eta_{\tilde{z}}}{1+\eta}\Big)(1+\eta).$$

We define the spaces

$$X_s = H^{s+1}_{\text{per}}(S) \times H^s_{\text{per}}(S) \times \mathring{H}^{s+1}_{\text{per}}(\Sigma) \times \mathring{H}^s_{\text{per}}(\Sigma)$$

for $s \geq 0$, where $S = (0, 2\pi)$ and $\Sigma = (0,1)\times(0,2\pi)$ and

$$\mathring{H}^s_{\text{per}}(\Sigma) = \Big\{u \in H^s_{\text{per}}(\Sigma) : \int_\Sigma u\,\mathrm{d}\tilde{y}\,\mathrm{d}\tilde{x} = 0\Big\},$$

and $\mathcal{X} = X_0$, $\mathcal{Y} = X_1$. We can then formulate equations (1.64) – (1.67) and boundary conditions (1.68) and (1.69) as the evolutionary system

$$\begin{pmatrix}\eta\\ \omega\\ \Phi'\\ \Psi'\end{pmatrix}_{\tilde{z}} = f^\varepsilon(\eta,\omega,\Phi',\Psi'); \tag{1.70}$$

the domain of the vector field on the right-hand side of this equation is given by

$$\mathcal{D} = \{(\eta,\omega,\Phi',\Psi') \in \mathcal{Y} :\ (1.68) \text{ and } (1.69) \text{ are satisfied}\}.$$

To linearise the nonlinear boundary conditions associated with equation (1.70), we note that they are equivalent to

$$\Phi'_{\tilde{y}} + \tilde{y}\tau\eta_x = F(\eta,\omega,\Phi',\Psi'), \quad \tilde{y} \in \{0,1\},$$

where

$$F(\eta,\omega,\Phi',\Psi') = -\tilde{y}\tau\eta\eta_{\tilde{x}} + \tilde{y}\tau^2\eta_{\tilde{x}}(\Phi'_{\tilde{x}}(1+\eta) - \Phi'_{\tilde{y}}\eta_{\tilde{x}}) + \tilde{y}W\Psi'\Big(\frac{1+\tau^2\eta_{\tilde{x}}^2}{\beta^2-W^2}\Big)^{\frac{1}{2}}.$$

The requisite change of variable is given by

$$(\rho,\theta,\Gamma,\xi) = (\eta,\omega,\Phi'-\chi_{\tilde{y}},\Psi')$$

with $\chi = \Delta^{-1}F(\rho,\omega,\Phi',\Psi')$ being the unique solution to the elliptic boundary value problem

$$\begin{aligned} \tau^2\chi_{\tilde{x}\tilde{x}} + \chi_{\tilde{y}\tilde{y}} &= F(\rho,\omega,\Phi',\Psi'), \qquad && (\tilde{x},\tilde{y}) \in \Sigma,\\ \chi &= 0, && \tilde{y} \in \{0,1\}. \end{aligned}$$

This change of variable transforms equation (1.70) into the evolutionary system

$$\begin{pmatrix}\rho\\ \theta\\ \Gamma\\ \xi\end{pmatrix}_{\tilde{z}} = k^\varepsilon(\rho,\theta,\Gamma,\xi), \tag{1.71}$$

where $k^\varepsilon(\rho,\theta,\Gamma,\xi)$ takes values in $\mathcal{X}$ and is analytic at the origin in $\mathbb{R}\times\mathcal{Y}$, and $H^\varepsilon$ is transformed into a conserved quantity for equation (1.71). The domain of the vector field on the right-hand side of equation (1.71) is

$$\mathcal{D}=\Big\{(\rho,\theta,\Gamma,\xi)\in\mathcal{Y}:\Gamma_{\tilde{y}}|_{\tilde{y}=0}=0,\tau\rho_x+\Gamma_{\tilde{y}}|_{\tilde{y}=1}=0\Big\}.$$

We note that the linearisation of $k^\varepsilon$ is given by $\check{L}^\varepsilon\colon\mathcal{D}\subseteq\mathcal{X}\to\mathcal{X}$ with

$$\check{L}^\varepsilon\begin{pmatrix}\rho\\ \theta\\ \Gamma\\ \xi\end{pmatrix}=\begin{pmatrix}\frac{1}{\beta_0}\theta\\ -\tau\Gamma_{\tilde{x}}|_{\tilde{y}=1}+(\alpha_0+\varepsilon)\rho-\tau^2\beta_0\rho_{\tilde{x}\tilde{x}}\\ \xi\\ -\tau^2\Gamma_{\tilde{x}\tilde{x}}-\Gamma_{\tilde{y}\tilde{y}}\end{pmatrix}$$

and study the spectrum of this operator using Fourier series in the $\tilde{x}$-coordinate.

**Lemma 1.10.**

(i) The eigenvalues of $\check{L}^0$ with eigenvector in the 0th Fourier mode are $\pm(\alpha_0/\beta_0)^{1/2}$ and $\pm n\pi$, $n\in\mathbb{N}$.

(ii) Suppose $m\in\mathbb{N}$. A complex number $\lambda$ is an eigenvalue of $\check{L}^0$ with corresponding eigenvectors in the $m$th Fourier mode if and only if

$$(\alpha_0-\beta_0\sigma_m^2)\sigma_m\sin\sigma_m+m^2\tau^2\cos\sigma_m=0, \tag{1.72}$$

where

$$\sigma_m^2=\lambda^2-m^2\tau^2.$$

In particular $\lambda$ is either real or purely imaginary.

For each $m\in\mathbb{N}$ equation (1.72) has at most two purely imaginary solutions $\pm\mathrm{i}s_m$ which correspond to geometrically double eigenvalues of $\check{L}^0$. The eigenvalues $\pm\mathrm{i}s_m$ collide at the origin at points of the line

$$C_m=\{(\beta_0,\alpha_0):(\alpha_0+\beta_0m^2\tau^2)\sinh m\tau=m\tau\cosh m\tau\}$$

in $(\beta_0,\alpha_0)$ parameter space (see Figure 1.15). At these points the two zero eigenvectors each have a Jordan chain of length 2. We proceed by choosing

$$(\beta_0,\alpha_0)\in C_m\setminus\{P_{1,m},\dots,P_{m-1,m},P_{m,m+1},P_{m,m+2}\},$$

defining $\mathcal{X}^\varepsilon_{\mathrm{wh}}$, $\mathcal{X}^\varepsilon_{\mathrm{c}}$ and $\mathcal{X}^\varepsilon_{\mathrm{sh}}$ in the obvious fashion using spectral projections and mapping those spaces to $\mathcal{X}^0_{\mathrm{wh}}$, $\mathcal{X}^0_{\mathrm{c}}$ and $\mathcal{X}^0_{\mathrm{sh}}$. The following theorem is obtained by verifying Assumptions (A1) – (A6) and Assumptions (C1) – (C4) and applying Theorem 1.2 to equation (1.71).

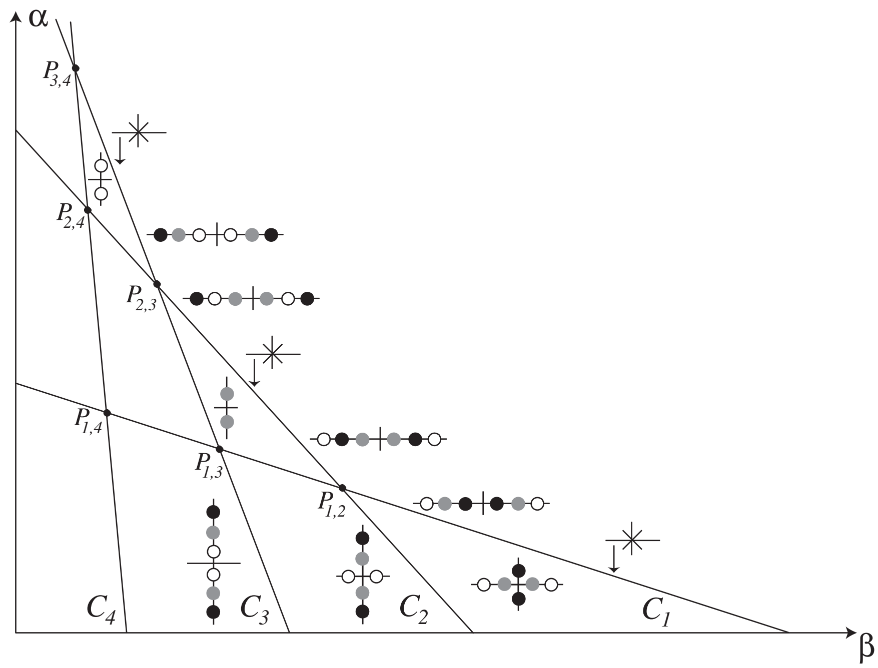



Figure 1.15: The line $C_m$ consists of points in $\beta, \alpha$ parameter space at which two real eigenvalues in the $m$th Fourier mode become purely imaginary by passing through 0. It connects $((m\tau)^{-1}\coth(m\tau), 0)$ with $(0, m\tau\coth(m\tau))$ and crosses $C_{m+1}, C_{m+2}, \ldots$ at the points $P_{m,m+1}, P_{m,m+2}, \ldots$.

**Theorem 1.11.** Suppose that $(\beta_0, \alpha_0) \in C_m \setminus \{P_{1,m}, \ldots, P_{m-1,m}, P_{m,m+1}, P_{m,m+2}, \ldots\}$ and $\beta_0 \in \left(\frac{\tanh(m\tau)}{3m\tau}, \frac{\coth(m\tau)}{m\tau}\right)$ for $m \in \mathbb{N}$. There exist positive constants $\varepsilon_0$ and $c^*$ such that for each $\varepsilon \in (0, \varepsilon_0)$ equations (1.64) – (1.67) and boundary conditions (1.68) and (1.69) admit a family of generalised pulse solutions satisfying in particular $\eta(\tilde{x}, \tilde{z}) = \eta(-\tilde{x}, -\tilde{z})$, $\eta(\tilde{x} + 2\pi, \tilde{z}) = \eta(\tilde{x}, \tilde{z})$ and

$$\left|\eta(\tilde{x}, \tilde{z}) - \frac{2}{\sqrt{\gamma_{0,m}}}\sinh(m\tau)\cos(m\tilde{x})\lambda^\varepsilon \check{p}^\varepsilon(\check{\lambda}^\varepsilon \tilde{z})\right| < \mathrm{e}^{-\frac{c^*}{2\sqrt{\varepsilon}}}.$$

Here $\gamma_{0,m}$ and $C$ are positive numbers (given by equations (6.79) and (6.80)),

$$\check{p}^\varepsilon(t) = \pm\left(\frac{2}{C}\right)^{\frac{1}{2}}\operatorname{sech}(t) + \mathcal{O}(\varepsilon \mathrm{e}^{-\nu|t|})$$

and $\pm\lambda^\varepsilon$ are the two real mode $m$ eigenvalues of the linearised problem with $\lambda^\varepsilon = \mathcal{O}(\varepsilon)$.

# 2 The centre-manifold theorem

The reduction of a (possibly infinite-dimensional) parameter-dependent evolution equation

$$\dot{x} = f(\lambda; x) \tag{2.1}$$

to a low-dimensional dynamical system is an established method to analyse its local bifurcations. The *centre-manifold theorem* states an existence criterion for a locally invariant manifold which contains all small global solutions of equation (2.1). In contrast to abstract methods such as the Lyapunov-Schmidt reduction the centre-manifold theorem has the advantage of preserving the status of equation (2.1) as an evolution equation. There is a huge and growing literature on centre-manifold theory, and here we mention just a few key results. Early versions of the centre-manifold theorem were proved by Kelley [15] and Pliss [22] for finite dimensional systems. Vanderbauwhede [23] wrote a survey about the theorem for semilinear systems and Mielke [21] proved it for quasilinear systems on Hilbert spaces.

In this chapter we present a complete proof of the centre-manifold theorem for quasilinear systems on Banach spaces due to Kirrmann [17], the technical details of which form the basis of the research we present later. While Mielke and Kirrmann both employ maximal regularity results for the 'hyperbolic' part of the linearised equation, Mielke works in $L^p$-based spaces while Kirrmann uses Hölder spaces. Kirrmann's approach has the advantage of being applicable in Banach spaces but involves many technical estimates for composition operators in Hölder spaces which we review in particular detail as they are needed later. We are also able to simplify Kirrmann's proof slightly by using maximal regularity results due to Arendt et al. [1].

## 2.1 Function spaces

First we introduce the function spaces used in our proof of the centre-manifold theorem.

**Definition 2.1.** Let $\omega\colon I \to (0, \infty)$ be a weight function, that is a non-constant, positive, continuous function, $n \in \mathbb{N}$, $\alpha \in (0, 1)$, $I$ an interval and $X, X_1, \ldots, X_n$ Banach spaces.

(i) We denote the space of all bounded functions from $I$ to $X$ by $B(I; X)$.

(ii) We denote the Banach space of all bounded continuous functions from $I$ to $X$ by $C_{\mathrm{b}}(I; X)$ and equip it with the usual norm

$$\|u\|_{C_{\mathrm{b}}(I;X)} := \sup_{t \in I} \|u(t)\|_X.$$

(iii) We denote the Banach space of all $n$-linear mappings from $X_1, \dots, X_n$ to $X$ by $\mathcal{L}^{(n)}(X_1, \dots, X_n; X)$. For notational simplicity we write $\mathcal{L}^{(n)}(X_1; X)$ instead of $\mathcal{L}^{(n)}(X_1, \dots, X_n; X)$ if $X_1 = \dots = X_n$.

(iv) We denote the Banach space of all bounded uniformly continuous functions from $I$ to $X$ by $C_{\mathrm{b,u}}(I; X)$.

(v) We define the Banach space of bounded Hölder continuous functions from $I$ to $X$ with Hölder exponent $\alpha$ as

$$C_{\mathrm{b}}^{\alpha}(I; X) := \left\{ u \in C(I; X) : \|u\|_{C_{\mathrm{b}}^{\alpha}(I;X)} < \infty \right\},$$

where

$$\|u\|_{C_{\mathrm{b}}^{\alpha}(I;X)} := \|u\|_{C_{\mathrm{b}}(I;X)} + \sup_{\substack{t_1, t_2 \in I \\ t_2 < t_1}} \frac{\|u(t_1) - u(t_2)\|_X}{|t_1 - t_2|^{\alpha}}.$$

The second term in the definition of $\|\cdot\|_{C_{\mathrm{b}}^{\alpha}(I;X)}$ is called the *Hölder seminorm.*

(vi) We define the Banach space of Hölder continuous weighted functions from $I$ to $X$ with weight $\omega$ and Hölder exponent $\alpha$ as

$$C_{\omega}^{\alpha}(I; X) := \left\{ u \in C(I; X) : \|u\|_{C_{\omega}^{\alpha}(I;X)} < \infty \right\},$$

where $\|u\|_{C_{\omega}^{\alpha}(I;X)} := \|\omega u\|_{C_{\mathrm{b}}^{\alpha}(I;X)}$.

For notational simplicity we write $C_{\eta}^{\alpha}(I; X)$ instead of $C_{\mathrm{e}^{-\eta|\cdot|}}^{\alpha}(I; X)$ and $C_{\eta,\pm}^{\alpha}(I; X)$ instead of $C_{\mathrm{e}^{\pm\eta(\cdot)}}^{\alpha}(I; X)$.

(vii) The Banach space of $k$ times continuously differentiable weighted functions from $I$ to $X$ with weight $\omega \colon I \to (0, \infty)$ is defined as

$$C_{\mathrm{b}}^{k}(\omega; I, X) = \left\{ u \in C^k(I; X) : \|u\|_{C_{\mathrm{b}}^{k}(\omega;I,X)} < \infty \right\},$$

where

$$\|u\|_{C_{\mathrm{b}}^{k}(\omega;I,X)} := \sup_{t \in I} \omega(t)\|u(t)\|_X + \sum_{j=1}^{k} \sup_{t \in I} \omega(t)\|u^{(j)}(t)\|_X.$$

In the case $\omega = \mathrm{e}^{-\eta|\cdot|}$ for $\eta \in \mathbb{R}$ we write $C_{\eta}^{k}(I; X)$ instead of $C_{\mathrm{b}}^{k}(\mathrm{e}^{-\eta|\cdot|}; I, X)$ and abbreviate the case $\eta = 0$ to $C_{\mathrm{b}}^{k}(I; X)$.

(viii) We denote the Banach space of even continuous bounded functions from $\mathbb{R}$ to $X$ by $C_{\mathrm{e}}(\mathbb{R}; X)$ and the Banach space of odd continuous bounded functions from $\mathbb{R}$ to $X$ by $C_{\mathrm{o}}(\mathbb{R}; X)$.

For some frequently used weight functions such as exponential functions there exists an equivalent norm

$$|||u|||_{C_{\omega}^{\alpha}(I;X)} := \sup_{t \in I} \omega(t)\|u(t)\|_X + \sup_{\substack{t_1, t_2 \in I \\ t_2 < t_1}} \min\{\omega(t_1), \omega(t_2)\} \frac{\|u(t_1) - u(t_2)\|_X}{|t_1 - t_2|^{\alpha}}$$

on $C_{\omega}^{\alpha}(I; X)$ which is often easier to handle in Lipschitz estimates.

**Proposition 2.2.** Suppose that $\omega$ is a weight function satisfying $|||\omega|||_{C^{\alpha}_{\omega^{-1}}(I;\mathbb{R})} < \infty$. The norm $|||\cdot|||_{C^{\alpha}_{\omega}(I,X)}$ is equivalent to $\|\cdot\|_{C^{\alpha}_{\omega}(I;X)}$.

*Proof.* We suppose that $\min\{\omega(t_1), \omega(t_2)\} = \omega(t_1)$ for fixed $-\infty < t_2 < t_1 < \infty$ (the other case is handled similarly). We find that

$$
\begin{aligned}
\|\omega(t_1)u(t_1) - \omega(t_2)u(t_2)\|_X &\le \omega(t_1)\|u(t_1) - u(t_2)\|_X + |\omega(t_1) - \omega(t_2)|\|u(t_2)\|_X \\
&\le |||u|||_{C^{\alpha}_{\omega}(I;X)}|t_1 - t_2|^{\alpha} + \frac{|\omega(t_1) - \omega(t_2)|}{|t_1 - t_2|^{\alpha}\omega(t_2)}|||u|||_{C^{\alpha}_{\omega}(I;X)}|t_1 - t_2|^{\alpha} \\
&\le |||u|||_{C^{\alpha}_{\omega}(I;X)}|t_1 - t_2|^{\alpha} + |||\omega|||_{C^{\alpha}_{\omega^{-1}}(I;\mathbb{R})}|||u|||_{C^{\alpha}_{\omega}(I;X)}|t_1 - t_2|^{\alpha} \\
&= (1 + |||\omega|||_{C^{\alpha}_{\omega^{-1}}(I;\mathbb{R})})|||u|||_{C^{\alpha}_{\omega}(I;X)}|t_1 - t_2|^{\alpha}
\end{aligned}
$$

and

$$
\begin{aligned}
\omega(t_1)\|u(t_1) - u(t_2)\|_X &\le \|\omega(t_1)u(t_1) - \omega(t_2)u(t_2)\|_X + \|\omega(t_2)u(t_2) - \omega(t_1)u(t_2)\|_X \\
&\le \|u\|_{C^{\alpha}_{\omega}(I;X)}|t_1 - t_2|^{\alpha} + \frac{|\omega(t_1) - \omega(t_2)|}{|t_1 - t_2|^{\alpha}\omega(t_2)}\|u\|_{C^{\alpha}_{\omega}(I;X)}|t_1 - t_2|^{\alpha} \\
&\le (1 + |||\omega|||_{C^{\alpha}_{\omega^{-1}}(I;\mathbb{R})})\|u\|_{C^{\alpha}_{\omega}(I;X)}|t_1 - t_2|^{\alpha}
\end{aligned}
$$

for $t_1, t_2 \in I$ satisfying $t_2 < t_1$. □

**Corollary 2.3.**

(i) The norms $\|\cdot\|_{C_{\eta}(I;X)}$ and $|||\cdot|||_{C_{\eta}(I;X)}$ are equivalent.

(ii) The norms $\|\cdot\|_{C_{\eta,\pm}(I;X)}$ and $|||\cdot|||_{C_{\eta,\pm}(I;X)}$ are equivalent.

*Proof.*

(i) Suppose that $-\infty < t_2 < t_1 < \infty$. In the case $\min\{e^{\eta|t_1|}, e^{\eta|t_2|}\} = e^{\eta|t_2|}$ we find that

$$
\begin{aligned}
\frac{|e^{-\eta|t_1|} - e^{-\eta|t_2|}|}{|t_1 - t_2|^{\alpha}}\min\{e^{\eta|t_1|}, e^{\eta|t_2|}\} &= \frac{|e^{-\eta|t_1|} - e^{-\eta|t_2|}|}{|t_1 - t_2|^{\alpha}}e^{\eta|t_2|} \\
&\le \frac{1 - e^{-\eta(|t_1| - |t_2|)}}{|t_1 - t_2|^{\alpha}} \\
&\le \frac{1 - e^{-\eta|t_1 - t_2|}}{|t_1 - t_2|^{\alpha}} \\
&\le \sup_{t>0}\frac{1 - e^{-\eta t}}{t^{\alpha}} \\
&< \infty.
\end{aligned}
\tag{2.2}
$$

The other case is treated in the same manner, so that $|||e^{-\eta|\cdot|}|||_{C_{-\eta}(I;X)} < \infty$. Proposition 2.2 therefore yields the equivalence of both norms.

(ii) For $\mathrm{e}^{\eta(\cdot)}$ we consider that

$$\begin{aligned}
\mathrm{e}^{-\eta t_1}\frac{|\mathrm{e}^{\eta t_1}-\mathrm{e}^{\eta t_2}|}{|t_1-t_2|^\alpha} &= \frac{1-\mathrm{e}^{\eta(t_2-t_1)}}{|t_1-t_2|^\alpha} \\
&= \frac{1-\mathrm{e}^{-\eta|t_1-t_2|}}{|t_1-t_2|^\alpha} \\
&\leq \sup_{t>0}\frac{1-\mathrm{e}^{-\eta t}}{t^\alpha} \\
&< \infty.
\end{aligned}$$

The case $\mathrm{e}^{-\eta(\cdot)}$ is treated in the same manner as $\mathrm{e}^{\eta(\cdot)}$, so that $|||\mathrm{e}^{\pm\eta(\cdot)}|||_{C_{\eta,\mp}(I;X)} < \infty$. Proposition 2.2 therefore yields the equivalence of both norms.

□

## 2.2 Cut-off functions

A *cut-off function* $\chi \in C^\infty(\mathbb{R})$ is a function with the properties

$$\chi(r) = \begin{cases} 1, & |r| \leq 1, \\ 0, & |r| \geq 2, \end{cases}$$

and $|\chi^{(l)}(r)| \leq 2^l$ for all $r \in \mathbb{R}$ and $l \in \mathbb{N}$. Let the function $h_\varepsilon\colon \mathbb{R}^d \to \mathbb{R}^d$ be given by

$$h_\varepsilon(z) = \chi(\varepsilon^{-1}|z|)z$$

for all $z \in \mathbb{R}^d$ and $\varepsilon > 0$. The function $h_\varepsilon$ induces a composition operator $H_\varepsilon\colon C(I;\mathbb{R}^d) \to C(I;\mathbb{R}^d)$ in a natural way with the formula

$$H_\varepsilon(u)(t) = h_\varepsilon(u(t))$$

which satisfies

$$\|H_\varepsilon(u)\|_{C_\mathrm{b}(I;\mathbb{R}^d)} \leq 2\varepsilon$$

by construction for all $u \in C(I;\mathbb{R}^d)$. In this section we study $H_\varepsilon$ as an operator $C^1_\eta(I;\mathbb{R}^d) \to C^1_\eta(I;\mathbb{R}^d)$ and $C^1_\mathrm{b}(I;\mathbb{R}^d) \to C^1_\mathrm{b}(I;\mathbb{R}^d)$.

**Proposition 2.4.** The function $h_\varepsilon\colon \mathbb{R}^d \to \mathbb{R}^d$ satisfies the estimates

$$\begin{gathered}
|h_\varepsilon(z)| \leq 2\varepsilon, \\
\|\mathrm{d}^j h_\varepsilon[z]\|_{\mathcal{L}^{(j)}(\mathbb{R}^d;\mathbb{R}^d)} \lesssim \varepsilon^{-j+1},
\end{gathered}$$

for all $z \in \mathbb{R}^d$ and $j \in \mathbb{N}$.

*Proof.* Clearly

$$|h_\varepsilon(z)| \leq |\chi(\varepsilon^{-1}|z|)||z| \leq 2\varepsilon$$

for all $z \in \mathbb{R}^d$ because $\chi(\varepsilon^{-1}|z|) = 0$ for all $|z| \geq 2\varepsilon$. Similarly

$$\mathrm{d}h_\varepsilon[z] = \varepsilon^{-1}\dot{\chi}(\varepsilon^{-1}|z|)\frac{zz^{\mathrm{T}}}{|z|} + \chi(\varepsilon^{-1}|z|)I$$

and hence

$$\|\mathrm{d}h_\varepsilon[z]\|_{\mathcal{L}(\mathbb{R}^d)} \leq \varepsilon^{-1}|\dot{\chi}(\varepsilon^{-1}|z|)||z| + |\chi(\varepsilon^{-1}|z|)| \lesssim 1,$$

where we have used the facts that $\dot{\chi}(\varepsilon^{-1}|z|) = 0$ for all $|z| \geq 2\varepsilon$ and $|\chi(\varepsilon^{-1}|z|)| \leq 1$ for all $z \in \mathbb{R}^d$.

The higher-order estimates are obtained inductively in the same fashion. □

**Definition 2.5.** Let $\eta, \varepsilon > 0$ and $A \in \mathbb{R}^{d\times d}$. We define

$$E_{\eta,\varepsilon,A}(I;\mathbb{R}^d) = \left\{u \in C^1_\eta(I;\mathbb{R}^d) : |\dot{u}(t) - Au(t)| \leq \varepsilon \text{ for all } t \in I\right\}.$$

In fact $E_{\eta,\varepsilon,A}(I;\mathbb{R}^d)$ is a closed and convex subset of $C^1_\eta(I;\mathbb{R}^d)$.

**Lemma 2.6.** The set $E_{\eta,\varepsilon,A}(I;\mathbb{R}^d)$ is convex and $(E_{\eta,\varepsilon,A}(I;\mathbb{R}^d), \|\cdot\|_{C^1_\eta(I;\mathbb{R}^d)})$ is a complete metric space.

*Proof.* The convexity of $E_{\eta,\varepsilon,A}(I;\mathbb{R}^d)$ follows directly from the calculation

$$\begin{aligned}|r\dot{u}_1(t) + (1-r)\dot{u}_2(t) - rAu_1(t) - (1-r)Au_2(t)| &\leq r|\dot{u}_1(t) - Au_1(t)| \\ &\qquad + (1-r)|\dot{u}_2(t) - Au_2(t)| \\ &\leq \varepsilon\end{aligned}$$

for all $u_1, u_2 \in E_{\eta,\varepsilon,A}(I;\mathbb{R}^d)$, $r \in (0,1)$ and $t \in I$.

The space $E_{\eta,\varepsilon,A}(I;\mathbb{R}^d)$ is clearly closed in $C^1_\eta(I;\mathbb{R}^d)$, so that it is complete. □

**Proposition 2.7.**

(i) The operator $H_\varepsilon$ satisfies the estimate

$$\|H_\varepsilon(u)\|_{C^1_{\mathrm{b}}(I;\mathbb{R}^d)} \lesssim \left(1 + \|A\|_{\mathbb{R}^{d\times d}}\right)\varepsilon$$

for all $u \in E_{\eta,\varepsilon,A}(I;\mathbb{R}^d)$.

(ii) We have that

$$\|H_\varepsilon(u_1) - H_\varepsilon(u_2)\|_{C^1_\eta(I;\mathbb{R}^d)} \lesssim \left(1 + \|A\|_{\mathbb{R}^{d\times d}}\right)\|u_1 - u_2\|_{C^1_\eta(I;\mathbb{R}^d)}$$

for all $u_1, u_2 \in E_{\eta,\varepsilon,A}(I;\mathbb{R}^d)$.

(iii) The operator $H_{2\varepsilon}$ also satisfies the estimates given in (i) and (ii) for all $u_1, u_2 \in E_{\eta,\varepsilon,A}(I;\mathbb{R}^d)$.

*Proof.*

(i) From Proposition 2.4 it follows that $|H_\varepsilon(u)(t)| \leq 2\varepsilon$ for all $u \in E_{\eta,\varepsilon,A}(I;\mathbb{R}^d)$ and $t \in I$. Furthermore, Proposition 2.4 and $\mathrm{d}h_\varepsilon[z] = 0$ for $|z| \geq 2\varepsilon$ yield

$$\begin{aligned}\Big|\frac{\mathrm{d}}{\mathrm{d}t}H_\varepsilon(u)(t)\Big| &= |\mathrm{d}h_\varepsilon[u(t)]\big(\dot{u}(t)\big)| \\ &= |\mathrm{d}h_\varepsilon[u(t)]\big(\dot{u}(t) - Au(t) + Au(t)\big)| \\ &\lesssim \big(1 + \|A\|_{\mathbb{R}^{d\times d}}\big)\varepsilon\end{aligned}$$

so that

$$\|H_\varepsilon(u)\|_{C^1_{\mathrm{b}}(I;\mathbb{R}^d)} \lesssim \big(1 + \|A\|_{\mathbb{R}^{d\times d}}\big)\varepsilon$$

for all $u \in E_{\eta,\varepsilon,A}(I;\mathbb{R}^d)$.

(ii) Next, we find that

$$\begin{aligned}|H_\varepsilon(u_1)(t) - H_\varepsilon(u_2)(t)| &= \Big|\int_0^1 \mathrm{d}h_\varepsilon[\sigma u_1(t) + (1-\sigma)u_2(t)]\big(u_1(t) - u_2(t)\big)\,\mathrm{d}\sigma\Big| \\ &\lesssim \mathrm{e}^{\eta|t|}\|u_1 - u_2\|_{C_\eta(I;\mathbb{R}^d)}\end{aligned}$$

for $u_1, u_2 \in E_{\eta,\varepsilon,A}(I;\mathbb{R}^d)$ and $t \in I$.

Before considering $\frac{\mathrm{d}}{\mathrm{d}t}H_\varepsilon$ we notice that for all $v_1$, $v_2 \in \mathbb{R}^d$ we obtain the estimate

$$\begin{aligned}\|\mathrm{d}h_\varepsilon[z_2] - \mathrm{d}h_\varepsilon[z_2]\|_{\mathcal{L}(\mathbb{R}^d)} &= \Big\|\varepsilon^{-1}\dot{\chi}(\varepsilon^{-1}|z_1|)\frac{z_1 z_1^{\mathrm{T}}}{|z_1|} + \chi(\varepsilon^{-1}|z_1|)I \\ &\qquad - \varepsilon^{-1}\dot{\chi}(\varepsilon^{-1}|z_2|)\frac{z_2 z_2^{\mathrm{T}}}{|z_2|} - \chi(\varepsilon^{-1}|z_2|)I\Big\|_{\mathcal{L}(\mathbb{R}^d)} \\ &\leq \varepsilon^{-1}|\dot{\chi}(\varepsilon^{-1}|z_1|) - \dot{\chi}(\varepsilon^{-1}|z_2|)||z_1| \\ &\qquad + \varepsilon^{-1}|\dot{\chi}(\varepsilon^{-1}|z_2|)|\Big|\frac{z_1 z_1^{\mathrm{T}}}{|z_1|} - \frac{z_2 z_2^{\mathrm{T}}}{|z_2|}\Big| \\ &\qquad + |\chi(\varepsilon^{-1}|z_1|) - \chi(\varepsilon^{-1}|z_2|)| \\ &\leq 2\sup_{\tau\in[0,1]}|\ddot{\chi}((1-\tau)\varepsilon^{-1}|z_1| + \tau\varepsilon^{-1}|z_1|)|\varepsilon^{-1}|z_1 - z_2| \\ &\qquad + 4\varepsilon^{-1}\sup_{\tau\in\mathbb{R}}|\dot{\chi}(\tau)||z_1 - z_2| \\ &\lesssim \varepsilon^{-1}|z_1 - z_2|,\end{aligned}$$

where we have again used the facts that $\chi(\varepsilon^{-1}|z|), \dot{\chi}(\varepsilon^{-1}|z|), \ddot{\chi}(\varepsilon^{-1}|z|) = 0$ for $|z| \geq 2\varepsilon$ combined with the estimate

$$\Big||z_2|z_1 z_1^{\mathrm{T}} - |z_1|z_2 z_2^{\mathrm{T}}\Big| \leq 3|z_1||z_2||z_1 - z_2|.$$

The above estimates imply that

$$\begin{aligned}
&\left|\frac{\mathrm{d}}{\mathrm{d}t}\Big(H_\varepsilon(u_1)(t) - H_\varepsilon(u_2)(t)\Big)\right| \\
&\qquad \le \|\mathrm{d}h_\varepsilon[u_1(t)]\|_{\mathbb{R}^{d\times d}}|\dot{u}_1(t) - \dot{u}_2(t)| \\
&\qquad\qquad + \|\mathrm{d}h_\varepsilon[u_1(t)] - \mathrm{d}h_\varepsilon[u_2(t)]\|_{\mathbb{R}^{d\times d}}|\dot{u}_2(t) - Au_2(t) + Au_2(t)| \\
&\qquad \lesssim |\dot{u}_1(t) - \dot{u}_2(t)| + (1 + \|A\|_{\mathbb{R}^{d\times d}})|u_1(t) - u_2(t)| \\
&\qquad \lesssim \Big(1 + \|A\|_{\mathbb{R}^{d\times d}}\Big)\mathrm{e}^{\eta|t|}\|u_1 - u_2\|_{C^1_\eta(I;\mathbb{R}^d)}
\end{aligned}$$

for all $u \in E_{\eta,\varepsilon,A}(I;\mathbb{R}^d)$.

(iii) The assertion follows by replacing $\varepsilon$ by $2\varepsilon$ except in the estimate $|\dot{u}(t) - Au(t)| < \varepsilon$.

□

**Remark 2.8.** Since the embedding $C^\alpha_\eta(I;\mathbb{R}^d) \hookrightarrow C^1_\eta(I;\mathbb{R}^d)$ is continuous for $\alpha \in [0,1]$ and $\eta \ge 0$ we can change the norm from $\|\cdot\|_{C^1_\eta(\mathbb{R};\mathbb{R}^d)}$ to $\|\cdot\|_{C^\alpha_\eta(\mathbb{R};\mathbb{R}^d)}$ on the left-hand side of the estimates in Proposition 2.7.

## 2.3 Localisation and composition operators

Suppose that $X_1$, $X_2$ are Banach spaces and $U$, $V$ are neighbourhoods of the origin in $\mathbb{R}^d$ and $X_2$. The *localisation* of a continuous function $g\colon U \times V \to X_1$ is the continuous function $g_\varepsilon\colon \mathbb{R}^d \times V \to X_1$ defined by the formula

$$g_\varepsilon(z,y) = g(h_\varepsilon(z), y),$$

where $\varepsilon > 0$ is chosen such that $\overline{B}_{4\varepsilon} \times \overline{B}_{2\varepsilon} \subseteq U \times V$. The localisation of $g$ in turn defines a *composition operator* $G_\varepsilon\colon C(I;\mathbb{R}^d) \times C(I;X_2) \to C(I;X_1)$ by the formula

$$G_\varepsilon(u,v)(t) = g_\varepsilon(u(t), v(t)).$$

For better readability we henceforth use the notation

$$C_\zeta := C^1_\zeta(I;\mathbb{R}^d) \times C^\alpha_\zeta(I;X_2)$$

for $\zeta > 0$. In this section we study $G_\varepsilon$ on the closed subset

$$E^\alpha_{\eta,\varepsilon,A}(I) := E_{\eta,\varepsilon,A}(I;\mathbb{R}^d) \times \overline{B}_\varepsilon = \{(u,v) \in E_{\eta,\varepsilon,A}(I) \times C^\alpha_{\mathrm{b}}(I;X_2) : \|v\|_{C^\alpha_{\mathrm{b}}(I;X_2)} \le \varepsilon\}$$

of $C_\eta$ for fixed $\eta > 0$, $\varepsilon > 0$ and $A \in \mathbb{R}^{n\times n}$. First we show that $G_\varepsilon$ and $G_{2\varepsilon}$ are well-defined as operators into $C^\alpha_{\mathrm{b}}(I;X_1)$ or $C^\alpha_\mu(I;X_1)$ for $\mu > 0$.

**Lemma 2.9.** Suppose that $g \in C^1_{\mathrm{b,u}}(U \times V; X_1)$.

(i) The operators $G_\varepsilon$ and $G_{2\varepsilon}$ map $E^\alpha_{\eta,\varepsilon,A}(I)$ into $C^\alpha_{\mathrm{b}}(I; X_1)$ and satisfy

$$\|G_\varepsilon(u,v)\|_{C^\alpha_{\mathrm{b}}(I;X_1)} \lesssim \sup_{(z,y)\in\overline{B}_{2\varepsilon}\times\overline{B}_\varepsilon} \|g(z,y)\|_{X_1} + \varepsilon \sup_{(z,y)\in\overline{B}_{2\varepsilon}\times\overline{B}_\varepsilon} \|\mathrm{d}g[z,y]\|_{\mathcal{L}(\mathbb{R}^d\times X_2;X_1)}$$

and

$$\|G_{2\varepsilon}(u,v)\|_{C^\alpha_{\mathrm{b}}(I;X_1)} \lesssim \sup_{(z,y)\in\overline{B}_{4\varepsilon}\times\overline{B}_{2\varepsilon}} \|g(z,y)\|_{X_1} + \varepsilon \sup_{(z,y)\in\overline{B}_{4\varepsilon}\times\overline{B}_{2\varepsilon}} \|\mathrm{d}g[z,y]\|_{\mathcal{L}(\mathbb{R}^d\times X_2;X_1)}$$

for all $(u,v) \in E^\alpha_{\eta,\varepsilon,A}(I)$.

(ii) For each $\mu > 0$ the operators $G_\varepsilon$ and $G_{2\varepsilon}$ map $E^\alpha_{\eta,\varepsilon,A}(I)$ uniformly continuously into $C^\alpha_\mu(I; X_1)$.

*Proof.*

(i) We observe that

$$\|G_\varepsilon(u,v)(t)\|_{X_1} = \|g_\varepsilon(u(t),v(t))\|_{X_1} \leq \sup_{(z,y)\in\overline{B}_{2\varepsilon}\times\overline{B}_\varepsilon} \|g(z,y)\|_{X_1}$$

for all $t \in I$, and Proposition 2.7 and Remark 2.8 yield the estimate

$$\begin{aligned}
&\|G_\varepsilon(u,v)(t) - G_\varepsilon(u,v)(s)\|_{X_1}\\
&\quad = \|g_\varepsilon(u(t),v(t)) - g_\varepsilon(u(s),v(s))\|_{X_1}\\
&\quad \leq \sup_{(z,y)\in\overline{B}_{2\varepsilon}\times\overline{B}_\varepsilon} \|\mathrm{d}g[z,y]\|_{\mathcal{L}(\mathbb{R}^d\times X_2;X_1)}|t-s|^\alpha \left(\|H_\varepsilon(u)\|_{C^\alpha_{\mathrm{b}}(\mathbb{R};\mathbb{R}^d)} + \|v\|_{C^\alpha_{\mathrm{b}}(\mathbb{R};X_2)}\right)\\
&\quad \lesssim \varepsilon \sup_{(z,y)\in\overline{B}_{2\varepsilon}\times\overline{B}_\varepsilon} \|\mathrm{d}g[z,y]\|_{\mathcal{L}(\mathbb{R}^d\times X_2;X_1)}|t-s|^\alpha
\end{aligned}$$

for $t, s \in I$ with $s < t$.

The proof for $G_{2\varepsilon}$ is identical.

(ii) First we prove uniform continuity with respect to the weighted supremum norm. To that end let $\rho > 0$ be given and observe that

$$\begin{aligned}
\sup_{t\geq R} \mathrm{e}^{-\mu|t|}\|(G_\varepsilon(u_1,v_1) - G_\varepsilon(u_2,v_2))(t)\|_{X_1} &\leq 2 \sup_{(z,y)\in\overline{B}_{2\varepsilon}\times\overline{B}_\varepsilon} \|g(z,y)\|_{X_1}\mathrm{e}^{-\mu R}\\
&< \frac{1}{4}\rho
\end{aligned}$$

for $R > 0$ sufficiently large. Since the function $g_\varepsilon \colon \mathbb{R}^d \times \overline{B}_\varepsilon \to X_1$ is uniformly continuous, there exists $\delta_1 > 0$ such that

$$\|g_\varepsilon(z_1,y_1) - g_\varepsilon(z_2,y_2)\|_{X_1} < \frac{1}{4}\rho$$

for all $(z_1,y_1),(z_2,y_2) \in \mathbb{R}^d \times \overline{B}_\varepsilon$ with $\|(z_1,y_1)-(z_2,y_2)\|_{\mathbb{R}^d\times X_2} < \delta_1$. From this fact we deduce that $\|(u_1,v_1)-(u_2,v_2)\|_{C_\eta} < \mathrm{e}^{-\eta R}\delta_1$ implies

$$\sup_{|t|\leq R} \mathrm{e}^{-\mu|t|}\|G_\varepsilon(u_1,v_1)(t) - G_\varepsilon(u_2,v_2)(t)\|_{X_1} < \frac{1}{4}\rho.$$

Hence

$$\|G_\varepsilon(u_1,v_1) - G_\varepsilon(u_2,v_2)\|_{C_\mu(I;X_2)} < \frac{1}{2}\rho$$

for all $(u_1,v_1),(u_2,v_2) \in E^\alpha_{\eta,\varepsilon,A}(I)$ with $\|(u_1,v_1)-(u_2,v_2)\|_{C_\eta} < \delta_1$.

It remains to prove uniform continuity with respect to the Hölder seminorm. For $R > 0$ sufficiently large and $s < t$ with $\max\{|s|,|t|\} \geq R$ we obtain the estimate

$$\begin{aligned}
&\|(G_\varepsilon(u_1,v_1) - G_\varepsilon(u_2,v_2))(t) - (G_\varepsilon(u_1,v_1) - G_\varepsilon(u_2,v_2))(s)\|_{X_1} \\
&\quad \leq \|g_\varepsilon(u_1(t),v_1(t)) - g_\varepsilon(u_1(s),v_1(s))\|_{X_1} \\
&\qquad + \|g_\varepsilon(u_2(t),v_2(t)) - g_\varepsilon(u_2(s),v_2(s))\|_{X_1} \\
&\quad \leq \sup_{(z,y)\in \overline{B}_{2\varepsilon}\times \overline{B}_\varepsilon} \|\,\mathrm{d}g[z,y]\|_{\mathcal{L}(\mathbb{R}^d\times X_2;X_1)}|t-s|^\alpha \Big(\|H_\varepsilon(u_1)\|_{C^\alpha_{\mathrm{b}}(\mathbb{R};\mathbb{R}^d)} + \|H_\varepsilon(u_2)\|_{C^\alpha_{\mathrm{b}}(\mathbb{R};\mathbb{R}^d)} + 2\varepsilon\Big) \\
&\quad \lesssim \varepsilon |t-s|^\alpha \max\{\mathrm{e}^{\mu|t|},\mathrm{e}^{\mu|s|}\}\mathrm{e}^{-\mu R} \\
&\quad < \frac{1}{4}\rho.
\end{aligned}$$

For $-R < s < t < R$ we write

$$\begin{aligned}
&G_\varepsilon(u_1,v_1)(t) - G_\varepsilon(u_2,v_2)(t) - \Big(G_\varepsilon(u_1,v_1)(s) - G_\varepsilon(u_2,v_2)(s)\Big) \\
&\quad = g_\varepsilon(u_1(t),v_1(t)) - g_\varepsilon(u_1(s),v_1(s)) - \Big(g_\varepsilon(u_2(t),v_2(t)) - g_\varepsilon(u_2(s),v_2(s))\Big) \\
&\quad = \int_0^1 \mathrm{d}g\Big[r\Big(h_\varepsilon(u_1(t)),v_1(t)\Big) + (1-r)\Big(h_\varepsilon(u_1(s)),v_1(s)\Big)\Big]\Big(h_\varepsilon(u_1(t)) - h_\varepsilon(u_1(s)), \\
&\qquad\qquad v_1(t)-v_1(s)\Big)\,\mathrm{d}r \\
&\qquad - \int_0^1 \mathrm{d}g\Big[r\Big(h_\varepsilon(u_2(t)),v_2(t)\Big) + (1-r)\Big(h_\varepsilon(u_2(s)),v_2(s)\Big)\Big]\Big(h_\varepsilon(u_2(t)) - h_\varepsilon(u_2(s)), \\
&\qquad\qquad v_2(t)-v_2(s)\Big)\,\mathrm{d}r.
\end{aligned}$$

The uniform continuity of the mapping

$$\begin{aligned}
&[0,1]\times\mathbb{R}^d\times\overline{B}_\varepsilon\times\mathbb{R}^d\times\overline{B}_\varepsilon \to \mathcal{L}(\mathbb{R}^d\times\overline{B}_\varepsilon;X_1), \\
&\qquad (r,z_1,y_1,z_2,y_2) \mapsto \mathrm{d}g[r(h_\varepsilon(z_1),y_1) + (1-r)(h_\varepsilon(z_2),y_2)]
\end{aligned}$$

implies the existence of $\delta_2 > 0$ such that

$$\begin{aligned}
&\|\,\mathrm{d}g[r(h_\varepsilon(z_1),y_1) + (1-r)(h_\varepsilon(z_2),y_2)] \\
&\qquad - \mathrm{d}g[r(h_\varepsilon(z_1'),y_1') + (1-r)(h_\varepsilon(z_2'),y_2')]\|_{\mathcal{L}(\mathbb{R}^d\times X_2;X_1)} < \frac{\rho}{4\varepsilon}
\end{aligned}$$

for all $(r,z_1,y_1,z_2,y_2),(r',z_1',y_1',z_2',y_2') \in [0,1]\times\mathbb{R}^d\times\overline{B}_\varepsilon\times\mathbb{R}^d\times\overline{B}_\varepsilon$ with

$$\|(z_1,y_1,z_2,y_2) - (z_1',y_1',z_2',y_2')\|_{[0,1]\times\mathbb{R}^d\times\overline{B}_\varepsilon\times\mathbb{R}^d\times\overline{B}_\varepsilon} < \delta_2.$$

Proposition 2.7 and Remark 2.8 yield the estimates

$$\begin{aligned}
\|h_\varepsilon(u_1(t)) - h_\varepsilon(u_1(s)), v_1(t)-v_1(s)\|_{\mathbb{R}^d\times X_2} &\lesssim \Big(\|H_\varepsilon(u_1)\|_{C^\alpha_{\mathrm{b}}(I;\mathbb{R}^d)} + \|v_1\|_{C^\alpha_{\mathrm{b}}(I;X_2)}\Big)|t-s|^\alpha \\
&\lesssim \varepsilon|t-s|^\alpha
\end{aligned}$$

and

$$\|h_\varepsilon(u_1(t)) - h_\varepsilon(u_2(t)) - \Big(h_\varepsilon(u_1(s)) - h_\varepsilon(u_2(s))\Big), v_1(t) - v_2(t) - \Big(v_1(s) - v_2(s)\Big)\|_{\mathbb{R}^d\times X_2}$$
$$\lesssim |t-s|^\alpha \mathrm{e}^{\eta R}\|(u_1 - u_2, v_1 - v_2)\|_{C^1_\eta(I;\mathbb{R}^d)\times C^\alpha_{\mathrm{b}}(I;X_2)}$$

for all $(u_1, v_1), (u_2, v_2) \in E^\alpha_{\eta,\varepsilon,A}(I)$ with

$$\|(u_1, v_1) - (u_2, v_2)\|_{C_\eta} < \delta_2.$$

For $\delta := \min\{\delta_1, \delta_2\}$ we thus obtain

$$\|G_\varepsilon(u_1, v_1) - G_\varepsilon(u_2, v_2)\|_{C^\alpha_\mu(I;X_1)} < \rho$$

for all $(u_1, v_1), (u_2, v_2) \in E^\alpha_{\eta,\varepsilon,A}(I)$ with

$$\|(u_1, v_1) - (u_2, v_2)\|_{C_\eta} < \delta.$$

The proof for $G_{2\varepsilon}$ is identical. □

In fact $G_\varepsilon$ is Lipschitz continuous under the stronger assumption that $g \in C^2_{\mathrm{b,u}}(U \times V; X_1)$.

**Lemma 2.10.** Suppose that $g \in C^2_{\mathrm{b,u}}(U\times V; X_1)$. The operator $G_\varepsilon\colon E^\alpha_{\eta,\varepsilon,A}(I) \to C_\eta(I; X_1)$ is Lipschitz continuous; more precisely it satisfies the estimate

$$\begin{aligned}
&\|G_\varepsilon(u_1, v_1) - G_\varepsilon(u_2, v_2)\|_{C^\alpha_\eta(I;X_1)}\\
&\lesssim \Bigg( \sup_{(z,y)\in \overline{B}_{2\varepsilon}\times\overline{B}_\varepsilon} \|\,\mathrm{d}g[z, y]\|_{\mathcal{L}(\mathbb{R}^d\times X_2;X_1)}\\
&\qquad + \sup_{(z,y)\in \overline{B}_{2\varepsilon}\times\overline{B}_\varepsilon} \|\,\mathrm{d}^2 g[z, y]\|_{\mathcal{L}^{(2)}(\mathbb{R}^d\times X_2;X_1)}\varepsilon\Bigg) \|(u_1 - u_2, v_1 - v_2)\|_{C_\eta}
\end{aligned}$$

for all $(u_1, v_1), (u_2, v_2) \in E^\alpha_{\eta,\varepsilon,A}(I)$.

*Proof.* We find that

$$\begin{aligned}
\|G_\varepsilon(u_1, v_1) - G_\varepsilon(u_2, v_2)\|_{C_\eta(I;X_1)} &= \sup_{t\in I} \mathrm{e}^{-\eta|t|}\|G_\varepsilon(u_1, v_1)(t) - G_\varepsilon(u_2, v_2)(t)\|_{X_1}\\
&\le \sup_{(z,y)\in \overline{B}_{2\varepsilon}\times\overline{B}_\varepsilon} \|\,\mathrm{d}g[z, y]\|_{\mathcal{L}(\mathbb{R}^d\times X_2;X_1)}\\
&\quad \times \Big(\|H_\varepsilon(u_1) - H_\varepsilon(u_2)\|_{C_\eta(I;\mathbb{R}^d)} + \|v_1 - v_2\|_{C_\eta(I;X_2)}\Big)
\end{aligned}$$

for all $(u_1, v_1), (u_2, v_2) \in E^\alpha_{\eta,\varepsilon,A}(I)$. The estimates from Proposition 2.7 now imply that

$$\begin{aligned}
&\|G_\varepsilon(u_1, v_1) - G_\varepsilon(u_2, v_2)\|_{C_\eta(I;X_1)}\\
&\lesssim \sup_{(z,y)\in \overline{B}_{2\varepsilon}\times\overline{B}_\varepsilon} \|\,\mathrm{d}g[z, y]\|_{\mathcal{L}(\mathbb{R}^d\times X_2;X_1)} \|(u_1 - u_2, v_1 - v_2)\|_{C_\eta}.
\end{aligned}$$

Next we reformulate the Hölder seminorm as

$$
\begin{aligned}
&g_\varepsilon(u_1(t), v_1(t)) - g_\varepsilon(u_1(s), v_1(s)) - (g_\varepsilon(u_2(t), v_2(t)) - g_\varepsilon(u_2(s), v_2(s)))\\
&= \int_0^1 \mathrm{d}g\Big[r\Big(h_\varepsilon(u_1(t)), v_1(t)\Big) + (1-r)\Big(h_\varepsilon(u_1(s)), v_1(s)\Big)\Big]\Big(h_\varepsilon(u_1(t)) - h_\varepsilon(u_1(s)),\\
&\qquad\qquad\qquad v_1(t) - v_1(s)\Big)\,\mathrm{d}r\\
&\quad - \int_0^1 \mathrm{d}g\Big[r\Big(h_\varepsilon(u_2(t)), v_2(t)\Big) + (1-r)\Big(h_\varepsilon(u_2(s)), v_2(s)\Big)\Big]\Big(h_\varepsilon(u_2(t)) - h_\varepsilon(u_2(s)),\\
&\qquad\qquad\qquad v_2(t) - v_2(s)\Big)\,\mathrm{d}r\\
&= \int_0^1\int_0^1 \mathrm{d}^2 g\Big[\sigma\Big(r\Big(h_\varepsilon(u_1(t)), v_1(t)\Big) + (1-r)\Big(h_\varepsilon(u_1(s)), v_1(s)\Big)\Big)\\
&\qquad + (1-\sigma)\Big(r\Big(h_\varepsilon(u_2(t)), v_2(t)\Big) + (1-r)\Big(h_\varepsilon(u_2(s)), v_2(s)\Big)\Big)\Big]\\
&\qquad \Big(r\Big(h_\varepsilon(u_1(t)) - h_\varepsilon(u_1(s)), v_1(t) - v_1(s)\Big)\\
&\qquad + (1-r)\Big(h_\varepsilon(u_2(t)) - h_\varepsilon(u_2(s)), v_2(t) - v_2(s)\Big),\\
&\qquad \Big(h_\varepsilon(u_1(t)) - h_\varepsilon(u_1(s)), v_1(t) - v_1(s)\Big)\Big)\,\mathrm{d}\sigma\,\mathrm{d}r\\
&\quad + \int_0^1 \mathrm{d}g\Big[r\Big(h_\varepsilon(u_2(t)), v_2(t)\Big) + (1-r)\Big(h_\varepsilon(u_2(s)), v_2(s)\Big)\Big]\\
&\qquad \Big(\Big(h_\varepsilon(u_1(t)) - h_\varepsilon(u_1(s)) - \Big(h_\varepsilon(u_2(t)) - h_\varepsilon(u_2(s))\Big),\\
&\qquad v_1(t) - v_1(s) - \Big(v_2(t) - v_2(s)\Big)\Big)\Big)\,\mathrm{d}r.
\end{aligned}
$$

and again use Proposition 2.7 to obtain

$$
\begin{aligned}
&\|G_\varepsilon(u_1, v_1)(t) - G_\varepsilon(u_2, v_2)(t) - (G_\varepsilon(u_1, v_1)(s) - G_\varepsilon(u_2, v_2)(s))\|_{X_1}\\
&\le \sup_{(z,y)\in \overline{B}_{2\varepsilon}\times\overline{B}_\varepsilon} \|\mathrm{d}^2 g[z,y]\|_{\mathcal{L}^{(2)}(\mathbb{R}^d\times X_2;X_1)}\\
&\quad \times \Big\|\Big(h_\varepsilon(u_1(t)) - h_\varepsilon(u_2(t)) - \Big(h_\varepsilon(u_1(s)) - h_\varepsilon(u_2(s))\Big),\\
&\qquad v_1(t) - v_2(t) - \Big(v_1(s) - v_2(s)\Big)\Big)\Big\|_{\mathbb{R}^d\times X_2}\\
&\quad \times \Big\|\Big(h_\varepsilon(u_1(t)) - h_\varepsilon(u_1(s)), v_1(t) - v_1(s)\Big)\Big\|_{\mathbb{R}^d\times X_2}\\
&\quad + \sup_{(z,y)\in \overline{B}_{2\varepsilon}\times\overline{B}_\varepsilon} \|\mathrm{d}g[z,y]\|_{\mathcal{L}(\mathbb{R}^d\times X_2;X_1)}\\
&\qquad \times \Big\|\Big(h_\varepsilon(u_1(t)) - h_\varepsilon(u_2(t)) - \Big(h_\varepsilon(u_1(s)) - h_\varepsilon(u_2(s))\Big),\\
&\qquad\quad v_1(t) - v_2(t) - \Big(v_1(s) - v_2(s)\Big)\Big)\Big\|_{\mathbb{R}^d\times X_2}\\
&\le \sup_{(z,y)\in \overline{B}_{2\varepsilon}\times\overline{B}_\varepsilon} \|\mathrm{d}^2 g[z,y]\|_{\mathcal{L}^{(2)}(\mathbb{R}^d\times X_2;X_1)}\\
&\quad \times \Big(\|H_\varepsilon(u_1) - H_\varepsilon(u_2)\|_{C^\alpha_\eta(I;\mathbb{R}^d)} + \|v_1 - v_2\|_{C^\alpha_\eta(I;X_2)}\Big)\\
&\quad \times \Big(\|H_\varepsilon(u_1) - H_\varepsilon(u_2)\|_{C^\alpha_\eta(I;\mathbb{R}^d)} + \|v_1 - v_2\|_{C^\alpha_\eta(I;X_2)}\Big)\\
&\quad \times \max\{\mathrm{e}^{\eta|s|}, \mathrm{e}^{\eta|t|}\}|t-s|^\alpha\\
&\quad + \sup_{(z,y)\in \overline{B}_{2\varepsilon}\times\overline{B}_\varepsilon} \|\mathrm{d}g[z,y]\|_{\mathcal{L}(\mathbb{R}^d\times X_2;X_1)}\\
&\qquad \times \Big(\|H_\varepsilon(u_1) - H_\varepsilon(u_2)\|_{C^\alpha_\eta(I;\mathbb{R}^d)} + \|v_1 - v_2\|_{C^\alpha_\eta(I;X_2)}\Big)\max\{\mathrm{e}^{\eta|s|}, \mathrm{e}^{\eta|t|}\}|t-s|^\alpha
\end{aligned}
$$

$$\lesssim \Big( \sup_{(z,y)\in \overline{B}_{2\varepsilon}\times \overline{B}_\varepsilon} \|\mathrm{d}^2 g[z,y]\|_{\mathcal{L}^{(2)}(\mathbb{R}^d\times X_2;X_1)}\varepsilon + \sup_{(z,y)\in \overline{B}_{2\varepsilon}\times \overline{B}_\varepsilon} \|\mathrm{d}g[z,y]\|_{\mathcal{L}(\mathbb{R}^d\times X_2;X_1)}\Big)$$
$$\times \|(u_1-u_2, v_1-v_2)\|_{C^1_\eta(I;\mathbb{R}^d)\times C^\alpha_\eta(I;X_2)} \max\{\mathrm{e}^{\eta|s|},\mathrm{e}^{\eta|t|}\}|t-s|^\alpha.$$

□

The reformulation of the Hölder seminorm used in the proof of Lemma 2.10 can be used as a general strategy for estimating the Hölder norm of composition operators on Banach spaces.

**Remark 2.11.** Let $B_1,\dots,B_d,B$ be Banach spaces, $I$ an interval, $\omega$ a weight function satisfying $\omega \in C^\alpha_{\omega^{-1}}(I;(0,\infty))$ and $f\colon U_1\times\ldots\times U_d \to B$ an analytic function, where $U_1,\dots,U_d$ are neighbourhoods of the origin in respectively $B_1,\dots,B_d$. From the identities

$$f(u_1,\dots,u_d)(t) - f(v_1,\dots,v_d)(t) = \sum_{k=1}^d \int_0^1 \mathrm{d}_k f[\{v_j(t)\}_{j=1}^{k-1}, \sigma u_k(t) + (1-\sigma)v_k(t), \{u_j(t)\}_{j=k+1}^d](u_k(t)-v_k(t))\,\mathrm{d}\sigma$$

and

$$\begin{aligned}
&f(u_1,\dots,u_d)(t) - f(v_1,\dots,v_d)(t) \\
&\quad - \Big(f(u_1,\dots,u_d)(s) - f(v_1,\dots,v_d)(s)\Big) \\
&= \sum_{k=1}^d \int_0^1 \mathrm{d}_k f[\{u_j(s)\}_{j=1}^{k-1}, \sigma u_k(t) + (1-\sigma)u_k(s), \{u_j(t)\}_{j=k+1}^d] \\
&\qquad\qquad \Big(u_k(t)-u_k(s)\Big)\,\mathrm{d}\sigma \\
&\quad - \sum_{k=1}^d \int_0^1 \mathrm{d}_k f[\{v_j(s)\}_{j=1}^{k-1}, \sigma v_k(t) + (1-\sigma)v_k(s), \{v_j(t)\}_{j=k+1}^d] \\
&\qquad\qquad \Big(v_k(t)-v_k(s)\Big)\,\mathrm{d}\sigma \\
&= \sum_{k=1}^d \int_0^1 \mathrm{d}_k f[\{u_j(s)\}_{j=1}^{k-1}, \sigma u_k(t) + (1-\sigma)u_k(s), \{u_j(t)\}_{j=k+1}^d] \\
&\qquad\qquad \Big(u_k(t)-u_k(s) - (v_k(t)-v_k(s))\Big)\,\mathrm{d}\sigma \\
&\quad + \sum_{k=1}^d \int_0^1\int_0^1 \Big( \sum_{l=1}^{k-1} \mathrm{d}_k\,\mathrm{d}_l f[\{v_j(s)\}_{j=1}^{l-1}, \tau u_l(s) + (1-\tau)v_l(s), \{u_j(s)\}_{j=l+1}^{k-1}, \\
&\qquad\qquad \sigma u_k(t) + (1-\sigma)u_k(s), \{u_j(t)\}_{j=k+1}^d] \\
&\qquad\qquad \Big(u_l(s)-v_l(s), v_k(t)-v_k(s)\Big) \\
&\qquad + \mathrm{d}_k^2 f[\{v_j(s)\}_{j=1}^{k-1}, \tau\Big(\sigma u_k(t) + (1-\sigma)u_k(s)\Big) \\
&\qquad\qquad + (1-\tau)\Big(\sigma v_k(t) + (1-\sigma)v_k(s)\Big), \{u_j(t)\}_{j=k+1}^d] \\
&\qquad\qquad \Big(\sigma(u_k(t)-v_k(t)) + (1-\sigma, u_k(s)-v_k(s)), v_k(t)-v_k(s)\Big) \\
&\qquad + \sum_{l=k+1}^d \mathrm{d}_k\,\mathrm{d}_l f[\{v_j(s)\}_{j=1}^{k-1}, \tau u_l(s) + (1-\tau)v_l(s), \\
&\qquad\qquad \sigma v_k(t) + (1-\sigma)v_k(s), \{v_j(t)\}_{j=k+1}^{l-1}, \\
&\qquad\qquad \tau u_l(t) + (1-\tau)u_l(s), \{u_j(t)\}_{j=l+1}^d] \\
&\qquad\qquad \Big(u_l(s)-v_l(s), v_k(t)-v_k(s)\Big)\Big)\,\mathrm{d}\tau\,\mathrm{d}\sigma
\end{aligned}$$

we find that the composition operator $F$ induced by $f$ given by

$$F(u_1,\dots,u_d)(t) = f(u_1(t),\dots,u_d(t))$$

satisfies

$$\begin{aligned}
&\|F(u_1,\dots,u_d) - F(v_1,\dots,v_d)\|_{C^\alpha_\omega(I,B)} \\
&\le \sup_{\substack{t,s\in I\\ s<t}} \Big( \sum_{k=1}^{d} \sup_{\sigma\in[0,1]} \|\,\mathrm{d}_k f[\{u_j(s)\}_{j=1}^{k-1}, \sigma u_k(t) + (1-\sigma)u_k(s), \{u_j(t)\}_{j=k+1}^{d}]\|_{\mathcal{L}(B_k;B)} \\
&\qquad\qquad \times \|u_k - v_k\|_{C^\alpha_\omega(I,B_k)} \\
&\quad + \sum_{k=1}^{d} \Big( \sum_{l=1}^{k-1} \sup_{\sigma,\tau\in[0,1]} \|\mathrm{d}_k\,\mathrm{d}_l f[\{v_j(s)\}_{j=1}^{l-1}, \tau u_l(s) + (1-\tau)v_l(s), \{u_j(s)\}_{j=l+1}^{k-1}, \\
&\qquad\qquad \sigma u_k(t) + (1-\sigma)u_k(s), \{u_j(t)\}_{j=k+1}^{d}]\|_{\mathcal{L}(B_k\times B_l;B)} \\
&\qquad\qquad \times \|u_l - v_l\|_{C^\alpha_\omega(I,B_l)} \|v_k\|_{C_{\mathrm{b}}(I;B_k)} \\
&\qquad + \sup_{\sigma,\tau\in[0,1]} \|\,\mathrm{d}_k^2 f[\{v_j(s)\}_{j=1}^{k-1}, \tau\big(\sigma u_k(t) + (1-\sigma)u_k(s)\big) \\
&\qquad\qquad + (1-\tau)\big(\sigma v_k(t) + (1-\sigma)v_k(s)\big), \{u_j(t)\}_{j=k+1}^{d}]\|_{\mathcal{L}(B_k^2;B)} \\
&\qquad\qquad \times \|u_k - v_k\|_{C^\alpha_\omega(I,B_k)} \|v_k\|_{C_{\mathrm{b}}(I;B_k)} \\
&\qquad + \sum_{l=k+1}^{d} \sup_{\sigma,\tau\in[0,1]} \|\mathrm{d}_k\,\mathrm{d}_l f[\{v_j(s)\}_{j=1}^{k-1}, \tau u_l(s) + (1-\tau)v_l(s), \\
&\qquad\qquad \sigma v_k(t) + (1-\sigma)v_k(s), \{v_j(t)\}_{j=k+1}^{l-1}, \\
&\qquad\qquad \tau u_l(t) + (1-\tau)u_l(s), \{u_j(t)\}_{j=l+1}^{d}]\|_{\mathcal{L}(B_k\times B_l;B)} \\
&\qquad\qquad \times \|u_l - v_l\|_{C^\alpha_\omega(I;B_l)} \|v_k\|_{C_{\mathrm{b}}(I;B_k)} \Big)\Big).
\end{aligned}$$

for all $(u_1,\dots,u_d),(v_1,\dots,v_d) \in C^\alpha_\omega(I,B_1)\times\ldots\times C^\alpha_\omega(I,B_d)$.

## 2.4 Composition operators induced by derivatives

In the sense of Section 2.3 the composition operator

$$G^{(j)}_\varepsilon\colon C(I;\mathbb{R}^d)\times C(I;X_2) \to C(I;\mathcal{L}^{(j)}(\mathbb{R}^d\times X_2;X_1))$$

induced by the $j$th derivative of $g_\varepsilon \in C^j_{\mathrm{b,u}}(\mathbb{R}^d\times V;X_1)$ is defined by the formula

$$G^{(j)}_\varepsilon(u,v)(t) = \mathrm{d}^j g_\varepsilon[u(t),v(t)].$$

In this section we study the operator

$$\tilde G^{(j)}_\varepsilon\colon C(I;\mathbb{R}^d)\times C(I;X_2) \to \mathcal{L}^{(j)}\big(C(I;\mathbb{R}^d)\times C(I;X_2);C(I;X_1)\big)$$

given by

$$\tilde G^{(j)}_\varepsilon(u,v)(\{(u_i,v_i)\}_{i=1}^{j})(t) = \mathrm{d}^j g_\varepsilon[u(t),v(t)](\{(u_i(t),v_i(t))\}_{i=1}^{j})$$

and its connection to $G^{(j)}_\varepsilon$, using the same notation as in Section 2.3.

**Lemma 2.12.** Suppose that $g \in C^{k+1}_{\mathrm{b,u}}(\overline{B}_{4\varepsilon} \times \overline{B}_{2\varepsilon}; X_1)$. For all $j \in \{1, \ldots, k\}$ the operator $G^{(j)}_\varepsilon$ has the following properties.

(i) The operator $G^{(j)}_\varepsilon$ maps $E^\alpha_{\eta,\varepsilon,A}(I)$ into $C^\alpha_{\mathrm{b}}(I; \mathcal{L}^{(j)}(\mathbb{R}^d \times X_2; X_1))$ and satisfies

$$\|G^{(j)}_\varepsilon(u,v)\|_{C^\alpha_{\mathrm{b}}(I;\mathcal{L}^{(j)}(\mathbb{R}^d \times X_2;X_1))} \lesssim \sum_{i=1}^{j+1} \varepsilon^{-j+i} \sup_{(z,y)\in\overline{B}_{4\varepsilon}\times\overline{B}_{2\varepsilon}} \| \mathrm{d}^i g[z,y]\|_{\mathcal{L}^{(i)}(\mathbb{R}^d\times X_2;X_1)}$$

for all $(u,v) \in E^\alpha_{\eta,\varepsilon,A}(I)$.

(ii) For each $\mu > 0$ the operator $G^{(j)}_\varepsilon$ maps $E^\alpha_{\eta,\varepsilon,A}(I)$ uniformly continuously into $C^\alpha_\mu(I; \mathcal{L}^{(j)}(\mathbb{R}^d \times X_2; X_1))$.

*Proof.*

(i) Notice that

$$\mathrm{d}^j g_\varepsilon[z,y] = S_j\Big(\sum_{i=1}^{j} \sum_{m\in M_{j,i}} \frac{i!}{m_1! \ldots m_i!} \\ \times \Lambda^{j,m_1,\ldots,m_i}\, \mathrm{d}^i g[\tilde{h}_\varepsilon(z,y)](\mathrm{d}^{m_1}\tilde{h}_\varepsilon[z,y], \ldots, \mathrm{d}^{m_i}\tilde{h}_\varepsilon[z,y])\Big),$$

where

$$\begin{aligned}\tilde{h}_\varepsilon(z,y) &= (h_\varepsilon(z), y),\\ M_{j,i} &= \Big\{(m_1,\ldots,m_i) : m_1 + \ldots + m_i = j\Big\},\\ S_j(L)(x_1,\ldots,x_j) &= \frac{1}{j!}\sum_{\sigma\in S_j} L(x_{\sigma(1)},\ldots,x_{\sigma(j)}),\end{aligned}$$

and

$$\Lambda^{j,m_1,\ldots,m_i}\Big(K(L_1,\ldots,L_i)\Big)(x_1,\ldots,x_j) \\ = K\Big(L_1(x_1,\ldots,x_{m_1}),\ldots,(x_{m_1+\ldots+m_{i-1}+1},\ldots,x_j)\Big).$$

Using Proposition 2.4 yields

$$\|\mathrm{d}^m\tilde{h}_\varepsilon[z,y]\|_{\mathcal{L}^{(j)}(\mathbb{R}^d\times X_2;\mathbb{R}^d\times X_2)} \lesssim \varepsilon^{-m+1},$$

and the calculation

$$\varepsilon^{-m_1+1}\cdot \ldots \cdot \varepsilon^{-m_i+1} = \varepsilon^{-j+i}$$

implies the estimates

$$\sup_{(z,y)\in\overline{B}_{4\varepsilon}\times\overline{B}_{2\varepsilon}} \| \mathrm{d}^j g_\varepsilon[z,y]\|_{\mathcal{L}^{(j)}(\mathbb{R}^d\times X_2;X_1)} \lesssim \sum_{i=1}^{j} \varepsilon^{-j+i} \sup_{(z,y)\in\overline{B}_{4\varepsilon}\times\overline{B}_{2\varepsilon}} \| \mathrm{d}^i g[z,y]\|_{\mathcal{L}^{(i)}(\mathbb{R}^d\times X_2;X_1)}$$

for each fixed $j \in \{1,\ldots,k\}$.

Writing $k = \mathrm{d}^j g_\varepsilon$ and noting that

$$k(z,y) = \mathrm{d}^j g_\varepsilon[z,y] = \mathrm{d}^j g_\varepsilon[h_{2\varepsilon}(z),y] = k_{2\varepsilon}(z,y)$$

(since $\mathrm{d}^j g_\varepsilon[z,y] = 0$ for $|z| > 2\varepsilon$ and $h_{2\varepsilon}(z) = 1$ for $|z| \le 2\varepsilon$), we find from Lemma 2.9 that

$$\begin{aligned}\|G_\varepsilon^{(j)}(u,v)\|_{C_\mathrm{b}^\alpha(I;\mathcal{L}^{(j)}(\mathbb{R}^d\times X_2;X_1))} &= \|K_{2\varepsilon}(u,v)\|_{C_\mathrm{b}^\alpha(I;\mathcal{L}^{(j)}(\mathbb{R}^d\times X_2;X_1))}\\ &\lesssim \sup_{(z,y)\in\overline{B}_{4\varepsilon}\times\overline{B}_{2\varepsilon}} \|\mathrm{d}^j g_\varepsilon[z,y]\|_{\mathcal{L}^{(j)}(\mathbb{R}^d\times X_2;X_1)}\\ &\qquad + \varepsilon \sup_{(z,y)\in\overline{B}_{4\varepsilon}\times\overline{B}_{2\varepsilon}} \|\mathrm{d}^{j+1} g_\varepsilon[z,y]\|_{\mathcal{L}^{(j+1)}(\mathbb{R}^d\times X_2;X_1)}\end{aligned}$$

and hence

$$\|G_\varepsilon^{(j)}(u,v)\|_{C_\mathrm{b}^\alpha(I;\mathcal{L}^{(j)}(\mathbb{R}^d\times X_2;X_1))} \lesssim \sum_{i=1}^{j+1} \varepsilon^{-j+i} \sup_{(z,y)\in\overline{B}_{4\varepsilon}\times\overline{B}_{2\varepsilon}} \|\,\mathrm{d}^i g[z,y]\|_{\mathcal{L}^{(i)}(\mathbb{R}^d\times X_2;X_1)}.$$

(ii) This is a direct consequence of Lemma 2.9(ii).

□

**Corollary 2.13.** Suppose that $g \in C_{\mathrm{b,u}}^{k+1}(U\times V;X_1)$ and $\zeta_1,\dots,\zeta_j \ge \eta$. The operator $\tilde{G}_\varepsilon^{(j)}$ has the following properties.

(i) The operator $\tilde{G}_\varepsilon^{(j)}$ maps $E_{\eta,\varepsilon,A}^\alpha(I)$ into $\mathcal{L}^{(j)}(C_{\zeta_1},\dots,C_{\zeta_j};C_{\zeta_1+\ldots+\zeta_j}^\alpha(I;X_1))$ and satisfies

$$\begin{aligned}&\|\tilde{G}_\varepsilon^{(j)}(u,v)\|_{\mathcal{L}^{(j)}(C_{\zeta_1},\dots,C_{\zeta_j};C_{\zeta_1+\ldots+\zeta_j}^\alpha(I;X_1))}\\ &\qquad\lesssim \sum_{i=1}^{j+1} \varepsilon^{-j+i} \sup_{(z,y)\in\overline{B}_{4\varepsilon}\times\overline{B}_{2\varepsilon}} \|\,\mathrm{d}^i g[z,y]\|_{\mathcal{L}^{(i)}(\mathbb{R}^d\times X_2;X_1)}\end{aligned}$$

for all $(u,v) \in E_{\eta,\varepsilon,A}^\alpha(I)$.

(ii) Suppose that $\zeta > \zeta_1+\ldots+\zeta_j$. The operator $\tilde{G}_\varepsilon^{(j)}$ maps the set $E_{\eta,\varepsilon,A}^\alpha(I)$ uniformly continuously into the space $\mathcal{L}^{(j)}(C_{\zeta_1},\dots,C_{\zeta_j};C_\zeta^\alpha(I;X_1))$.

(iii) The operator $\tilde{G}_\varepsilon^{(j-1)}\colon E_{\eta,\varepsilon,A}^\alpha(I) \to \mathcal{L}^{(j-1)}(C_{\zeta_1},\dots,C_{\zeta_{j-1}};C_\zeta^\alpha(I;X_1))$ is differentiable with

$$\mathrm{d}\tilde{G}_\varepsilon^{(j-1)} = \tilde{G}_\varepsilon^{(j)},$$

where $\zeta_1+\ldots+\zeta_{j-1} < \zeta-\eta$.

*Proof.*

(i) By definition

$$\begin{aligned}
&\|\tilde G_\varepsilon^{(j)}(u,v)\|_{\mathcal{L}^{(j)}(C_{\zeta_1},\dots,C_{\zeta_j};C^\alpha_{\zeta_1+\dots+\zeta_j}(I;X_1))}\\
&\quad= \sup\left\{\|\tilde G_\varepsilon^{(j)}(u,v)(\{(u_i,v_i)\}_{i=1}^j)\|_{C^\alpha_{\zeta_1+\dots+\zeta_j}(I;X_1)} : \|(u_i,v_i)\|_{C_{\zeta_i}}=1, i\in\{1,\dots,j\}\right\}.
\end{aligned}$$

Estimating the weighted supremum norm by

$$\begin{aligned}
&\|\tilde G_\varepsilon^{(j)}(u,v)(\{(u_i,v_i)\}_{i=1}^j)\|_{C_{\zeta_1+\dots+\zeta_j}(I;X_1)}\\
&\qquad= \sup_{t\in I} \mathrm{e}^{-(\zeta_1+\dots+\zeta_j)|t|}\|\,\mathrm{d}^j g_\varepsilon[u(t),v(t)](\{(u_i(t),v_i(t))\}_{i=1}^j)\|_{X_1}\\
&\qquad\le \sup_{t\in I}\|\,\mathrm{d}^j g_\varepsilon[u(t),v(t)]\|_{\mathcal{L}^{(j)}(\mathbb{R}^d\times X_2;X_1)}\prod_{i=1}^j\|(u_i,v_i)\|_{C_{\zeta_i}}\\
&\qquad= \|G_\varepsilon^{(j)}(u,v)\|_{C_\mathrm{b}(I;\mathcal{L}^{(j)}(\mathbb{R}^d\times X_2;X_1))}\prod_{i=1}^j\|(u_i,v_i)\|_{C_{\zeta_i}}
\end{aligned}$$

and the weighted Hölder seminorm by

$$\begin{aligned}
&\|\tilde G_\varepsilon^{(j)}(u,v)(\{(u_i,v_i)\}_{i=1}^j)(t)-\tilde G_\varepsilon^{(j)}(u,v)(\{(u_i,v_i)_{i=1}^j\})(s)\|_{X_1}\\
&\quad= \|\mathrm{d}^j g_\varepsilon[u(t),v(t)](\{(u_i(t),v_i(t))\}_{i=1}^j)-\mathrm{d}^j g_\varepsilon[u(s),v(s)](\{(u_i(s),v_i(s))\}_{i=1}^j)\|_{X_1}\\
&\quad\le \|\mathrm{d}^j g_\varepsilon[u(t),v(t)]-\mathrm{d}^j g_\varepsilon[u(s),v(s)]\|_{\mathcal{L}^{(j)}(\mathbb{R}^d\times X_2;X_1)}\prod_{i=1}^j\|(u_i(t),v_i(t))\|_{\mathbb{R}^d\times X_2}\\
&\qquad+\|\mathrm{d}^j g_\varepsilon[u(s),v(s)]\|_{\mathcal{L}^{(j)}(\mathbb{R}^d\times X_2;X_1)}\prod_{i=1}^j\|(u_i(t)-u_i(s),v_i(t)-v_i(s))\|_{\mathbb{R}^d\times X_2}\\
&\quad\le \max\{\mathrm{e}^{(\zeta_1+\dots+\zeta_j)|t|},\mathrm{e}^{(\zeta_1+\dots+\zeta_j)|s|}\}|t-s|^\alpha\|G_\varepsilon^{(j)}(u,v)\|_{C^\alpha_\mathrm{b}(I;\mathcal{L}^{(j)}(\mathbb{R}^d\times X_2;X_1))}\\
&\qquad\times\prod_{i=1}^j\|(u_i,v_i)\|_{C_{\zeta_i}}
\end{aligned}$$

we therefore find that

$$\|\tilde G_\varepsilon^{(j)}(u,v)\|_{\mathcal{L}^{(j)}(C_{\zeta_1},\dots,C_{\zeta_j};C^\alpha_{\zeta_1+\dots+\zeta_j}(I;X_1))}\lesssim\|G_\varepsilon^{(j)}(u,v)\|_{C^\alpha_\mathrm{b}(I;\mathcal{L}^{(j)}(\mathbb{R}^d\times X_2;X_1))}\tag{2.3}$$

for $(u,v)\in E^\alpha_{\eta,\varepsilon,A}(I)$. Lemma 2.12(i) and estimate (2.3) imply the desired estimate.

(ii) This is a direct consequence of Lemma 2.12(ii) and estimate (2.3).

(iii) For the weighted supremum norm we find that

$$\begin{aligned}
&\left\|G_\varepsilon(u,v)-G_\varepsilon(\tilde u,\tilde v)-\tilde G_\varepsilon^{(1)}(u,v)(u-\tilde u,v-\tilde v)\right\|_{C_\zeta(I;X_1)}\\
&\quad=\sup_{t\in I}\mathrm{e}^{-\zeta|t|}\Big\|g_\varepsilon(u(t),v(t))-g_\varepsilon(\tilde u(t),\tilde v(t))\\
&\qquad\qquad-\mathrm{d}g_\varepsilon[u(t),v(t)]\big(u(t)-\tilde u(t),v(t)-\tilde v(t)\big)\Big\|_{X_1}\\
&\quad\le\sup_{r\in[0,1]}\sup_{t\in I}\mathrm{e}^{-\mu|t|}\|\,\mathrm{d}g_\varepsilon[ru(t)+(1-r)\tilde u(t),rv(t)+(1-r)\tilde v(t)]\\
&\qquad\qquad-\mathrm{d}g_\varepsilon[u(t),v(t)]\|_{\mathcal{L}(\mathbb{R}^d\times X_2;X_1)}\|(u-\tilde u,v-\tilde v)\|_{C_\eta}
\end{aligned}$$

$$\leq \sup_{r\in[0,1]} \|G_\varepsilon^{(1)}(ru+(1-r)\tilde{u}, rv+(1-r)\tilde{v}) - G_\varepsilon^{(1)}(u,v)\|_{C_\mu(I;\mathcal{L}(\mathbb{R}^d\times X_2;X_1))} \times \|(u-\tilde{u}, v-\tilde{v})\|_{C_\eta},$$

where $\mu = \zeta - \eta$. For the weighted Hölder seminorm we find that

$$\begin{aligned}
&\Big\|G_\varepsilon(u,v)(t) - G_\varepsilon(\tilde{u},\tilde{v})(t) - \tilde{G}_\varepsilon^{(1)}(u,v)(u-\tilde{u}, v-\tilde{v})(t) \\
&\quad - \Big(G_\varepsilon(u,v)(s) - G_\varepsilon(\tilde{u},\tilde{v})(s) - \tilde{G}_\varepsilon^{(1)}(u,v)(u-\tilde{u}, v-\tilde{v})(s)\Big)\Big\|_{X_1} \\
&\quad = \Big\|g_\varepsilon(u(t),v(t)) - g_\varepsilon(\tilde{u}(t),\tilde{v}(t)) - \mathrm{d}g_\varepsilon[u(t),v(t)]\Big(u(t)-\tilde{u}(t), v(t)-\tilde{v}(t)\Big) \\
&\qquad - \Big(g_\varepsilon(u(s),v(s)) - g_\varepsilon(\tilde{u}(s),\tilde{v}(s)) \\
&\qquad\quad - \mathrm{d}g_\varepsilon[u(s),v(s)]\Big(u(s)-\tilde{u}(s), v(s)-\tilde{v}(s)\Big)\Big)\Big\|_{X_1} \\
&\quad \leq \Big\|g_\varepsilon(u(t),v(t)) - g_\varepsilon(\tilde{u}(t),\tilde{v}(t)) \\
&\qquad - \mathrm{d}g_\varepsilon[u(t),v(t)]\Big(u(t)-\tilde{u}(t), v(t)-\tilde{v}(t)\Big) \\
&\qquad - g_\varepsilon(u(s),v(s)) - g_\varepsilon(\tilde{u}(s),\tilde{v}(s)) \\
&\qquad - \mathrm{d}g_\varepsilon[u(s),v(s)]\Big(u(s)-\tilde{u}(s), v(s)-\tilde{v}(s)\Big)\Big\|_{X_1} \\
&\qquad + \Big\|g_\varepsilon(u(t),v(t)) - g_\varepsilon(\tilde{u}(t),\tilde{v}(t)) \\
&\qquad\quad - \mathrm{d}g_\varepsilon[u(t),v(t)]\Big(u(t)-\tilde{u}(t), v(t)-\tilde{v}(t)\Big)\Big\|_{X_1} \\
&\quad \leq \sup_{r\in[0,1]} \|G_\varepsilon^{(1)}(ru+(1-r)\tilde{u}, rv+(1-r)\tilde{v}) - G_\varepsilon^{(1)}(u,v)\|_{C_\mu^\alpha(I;\mathcal{L}(\mathbb{R}^d\times X_2;X_1))} \\
&\qquad \times \max\{\mathrm{e}^{\zeta|t|}, \mathrm{e}^{\zeta|s|}\}|t-s|^\alpha \|(u-\tilde{u}, v-\tilde{v})\|_{C_\eta}.
\end{aligned}$$

Lemma 2.12(ii) yields

$$\begin{aligned}
&\|G_\varepsilon^{(1)}(ru+(1-r)\tilde{u}, rv+(1-r)\tilde{v}) - G_\varepsilon^{(1)}(u,v)\|_{C_\mu^\alpha(I;\mathcal{L}(\mathbb{R}^d\times X_2;X_1))}\|(u-\tilde{u}, v-\tilde{v})\|_{C_\eta} \\
&\quad = o(\|(u-\tilde{u}, v-\tilde{v})\|_{C_\eta})
\end{aligned}$$

as $(u,v)\to(\tilde{u},\tilde{v})$, so that

$$\mathrm{d}G_\varepsilon = \tilde{G}_\varepsilon^{(1)}.$$

Now suppose that $j \geq 2$. For the weighted supremum norm of $\tilde{G}_\varepsilon^{(j-1)}$ we find that

$$\begin{aligned}
&\Big\|\Big(\tilde{G}_\varepsilon^{(j-1)}(u,v) - \tilde{G}_\varepsilon^{(j-1)}(\tilde{u},\tilde{v}) - \tilde{G}_\varepsilon^{(j)}(u,v)(u-\tilde{u}, v-\tilde{v})\Big)\Big(\{(u_i,v_i)\}_{i=1}^{j-1}\Big)\Big\|_{C_\zeta(I;X_1)} \\
&\quad = \sup_{t\in I} \mathrm{e}^{-\zeta|t|}\Big\|\Big(\mathrm{d}^{j-1}g_\varepsilon[u(t),v(t)] - \mathrm{d}^{j-1}g_\varepsilon[\tilde{u}(t),\tilde{v}(t)] \\
&\qquad\qquad - \mathrm{d}^j g_\varepsilon[u(t),v(t)]\Big(u(t)-\tilde{u}(t), v(t)-\tilde{v}(t)\Big)\Big)\Big(\{(u_i(t),v_i(t))\}_{i=1}^{j-1}\Big)\Big\|_{X_1} \\
&\quad \leq \sup_{t\in I} \mathrm{e}^{-(\eta+\mu)|t|}\|\mathrm{d}^{j-1}g_\varepsilon[u(t),v(t)] - \mathrm{d}^{j-1}g_\varepsilon[\tilde{u}(t),\tilde{v}(t)] \\
&\qquad\qquad - \mathrm{d}^j g_\varepsilon[u(t),v(t)]\Big(u(t)-\tilde{u}(t), v(t)-\tilde{v}(t)\Big)\Big\|_{\mathcal{L}^{(j-1)}(\mathbb{R}^d\times X_2;X_1)} \\
&\qquad \times \prod_{i=1}^{j-1}\|(u_i,v_i)\|_{C_{\zeta_i}} \\
&\quad \leq \sup_{r\in[0,1]}\sup_{t\in I} \mathrm{e}^{-\mu|t|}\|\,\mathrm{d}^j g_\varepsilon[ru(t)+(1-r)\tilde{u}(t), rv(t)+(1-r)\tilde{v}(t)] \\
&\qquad\qquad - \mathrm{d}^j g_\varepsilon[u(t),v(t)]\|_{\mathcal{L}^{(j)}(\mathbb{R}^d\times X_2;X_1)}\|(u-\tilde{u}, v-\tilde{v})\|_{C_\eta}
\end{aligned}$$

$$\leq \sup_{r\in[0,1]} \|G_\varepsilon^{(j)}(ru+(1-r)\tilde{u}, rv+(1-r)\tilde{v}) - G_\varepsilon^{(j)}(u,v)\|_{C_\mu(I;\mathcal{L}^{(j)}(\mathbb{R}^d\times X_2;X_1))} \times \|(u-\tilde{u}, v-\tilde{v})\|_{C_\eta},$$

where $\mu = \zeta - \zeta_1 - \ldots - \zeta_{j-1} - \eta$. For the weighted Hölder seminorm we find that

$$\begin{aligned}
&\Big\|\Big(\tilde{G}_\varepsilon^{(j-1)}(u,v) - \tilde{G}_\varepsilon^{(j-1)}(\tilde{u},\tilde{v}) - \tilde{G}_\varepsilon^{(j)}(u,v)\big(u-\tilde{u}, v-\tilde{v}\big)\Big)\Big(\{(u_i,v_i)\}_{i=1}^{j-1}\Big)(t) \\
&\quad - \Big(\tilde{G}_\varepsilon^{(j-1)}(u,v) - \tilde{G}_\varepsilon^{(j-1)}(\tilde{u},\tilde{v}) - \tilde{G}_\varepsilon^{(j)}(u,v)\big(u-\tilde{u}, v-\tilde{v}\big)\Big)\Big(\{(u_i,v_i)\}_{i=1}^{j-1}\Big)(s)\Big\|_{X_1} \\
&\quad = \Big\|\Big( \mathrm{d}^{j-1}g_\varepsilon[u(t),v(t)] - \mathrm{d}^{j-1}g_\varepsilon[\tilde{u}(t),\tilde{v}(t)] \\
&\qquad - \mathrm{d}^j g_\varepsilon[u(t),v(t)]\big(u(t)-\tilde{u}(t), v(t)-\tilde{v}(t)\big)\Big)\Big(\{(u_i(t),v_i(t))\}_{i=1}^{j-1}\Big) \\
&\qquad - \Big( \mathrm{d}^{j-1}g_\varepsilon[u(s),v(s)] - \mathrm{d}^{j-1}g_\varepsilon[\tilde{u}(s),\tilde{v}(s)] \\
&\qquad\quad - \mathrm{d}^j g_\varepsilon[u(s),v(s)]\big(u(s)-\tilde{u}(s), v(s)-\tilde{v}(s)\big)\Big)\Big(\{(u_i(s),v_i(s))\}_{i=1}^{j-1}\Big)\Big\|_{X_1} \\
&\quad \leq \Big\| \mathrm{d}^{j-1}g_\varepsilon[u(t),v(t)] - \mathrm{d}^{j-1}g_\varepsilon[\tilde{u}(t),\tilde{v}(t)] \\
&\qquad - \mathrm{d}^j g_\varepsilon[u(t),v(t)]\big(u(t)-\tilde{u}(t), v(t)-\tilde{v}(t)\big) \\
&\qquad - \mathrm{d}^{j-1}g_\varepsilon[u(s),v(s)] - \mathrm{d}^{j-1}g_\varepsilon[\tilde{u}(s),\tilde{v}(s)] \\
&\qquad - \mathrm{d}^j g_\varepsilon[u(s),v(s)]\big(u(s)-\tilde{u}(s), v(s)-\tilde{v}(s)\big)\Big\|_{\mathcal{L}^{(j-1)}(\mathbb{R}^d\times X_2;X_1)} \\
&\qquad \times \prod_{i=1}^{j-1} \|(u_i(t),v_i(t))\|_{\mathbb{R}^d\times X_2} \\
&\qquad + \Big\| \mathrm{d}^{j-1}g_\varepsilon[u(t),v(t)] - \mathrm{d}^{j-1}g_\varepsilon[\tilde{u}(t),\tilde{v}(t)] \\
&\qquad\quad - \mathrm{d}^j g_\varepsilon[u(t),v(t)]\big(u(t)-\tilde{u}(t), v(t)-\tilde{v}(t)\big)\Big\|_{\mathcal{L}^{(j-1)}(\mathbb{R}^d\times X_2;X_1)} \\
&\qquad \times \prod_{i=1}^{j-1} \|(u_i(t)-u_i(s), v_i(t)-v_i(s))\|_{\mathbb{R}^d\times X_2} \\
&\quad \leq \sup_{r\in[0,1]} \|G_\varepsilon^{(j)}(ru+(1-r)\tilde{u}, rv+(1-r)\tilde{v}) - G_\varepsilon^{(j)}(u,v)\|_{C_\mu^\alpha(I;\mathcal{L}^{(j)}(\mathbb{R}^d\times X_2;X_1))} \\
&\qquad \times \max\{\mathrm{e}^{\zeta|t|}, \mathrm{e}^{\zeta|s|}\}|t-s|^\alpha \|(u-\tilde{u}, v-\tilde{v})\|_{C_\eta} \prod_{i=1}^{j-1} \|(u_i,v_i)\|_{C_{\zeta_i}}.
\end{aligned}$$

Lemma 2.12(ii) yields

$$\begin{aligned}
&\|G_\varepsilon^{(j)}(ru+(1-r)\tilde{u}, rv+(1-r)\tilde{v}) - G_\varepsilon^{(j)}(u,v)\|_{C_\mu^\alpha(I;\mathcal{L}^{(j)}(\mathbb{R}^d\times X_2;X_1))} \\
&\qquad \times \|(u-\tilde{u}, v-\tilde{v})\|_{C_\eta} \\
&\quad = o(\|(u-\tilde{u}, v-\tilde{v})\|_{C_\eta})
\end{aligned}$$

as $(u,v) \to (\tilde{u},\tilde{v})$, so that

$$\mathrm{d}\tilde{G}_\varepsilon^{(j-1)} = \tilde{G}_\varepsilon^{(j)}.$$

□

## 2.5 Maximal regularity

The linear equation

$$\dot{u}(t) = Lu(t) + f(t) \tag{2.4}$$

has *maximal* $C^\alpha_{\mathrm{b}}$*-regularity* in a Banach space $B$ if for all $f \in C^\alpha_{\mathrm{b}}(\mathbb{R}; B)$ it has a unique solution $u \in C^{1,\alpha}_{\mathrm{b}}(\mathbb{R}; B) \cap C^\alpha_{\mathrm{b}}(\mathbb{R}; \mathcal{D}(L))$. It follows from the closed-graph theorem that this solution satisfies

$$\|u\|_{C^{1,\alpha}_{\mathrm{b}}(\mathbb{R};B)} + \|u\|_{C^\alpha_{\mathrm{b}}(\mathbb{R};\mathcal{D}(L))} \lesssim \|f\|_{C^\alpha_{\mathrm{b}}(\mathbb{R};B)}.$$

The following maximal regularity result by Arendt et al. [1, Theorem 6.1 and Remark 6.3(a)] implies a similar result for weighted spaces.

**Lemma 2.14.** Let $\alpha \in (0, 1)$ and $L$ be a closed, densely defined linear operator on a Banach space $B$. The equation

$$\dot{u}(t) = Lu(t) + f(t)$$

has a unique solution $u \in C^{1,\alpha}_{\mathrm{b}}(\mathbb{R}; B) \cap C^\alpha_{\mathrm{b}}(\mathbb{R}; \mathcal{D}(L))$ with

$$\|u\|_{C^{1,\alpha}_{\mathrm{b}}(\mathbb{R};B)} + \|u\|_{C^\alpha_{\mathrm{b}}(\mathbb{R};\mathcal{D}(L))} \lesssim \|f\|_{C^\alpha_{\mathrm{b}}(\mathbb{R};B)}$$

for each $f \in C^\alpha_{\mathrm{b}}(\mathbb{R}; B)$ if and only if the operator $L$ satisfies $\mathrm{i}\mathbb{R} \subseteq \rho(L)$ and

$$\|(\mathrm{i}sI - L)^{-1}\|_{\mathcal{L}(B)} \lesssim \frac{1}{1 + |s|}$$

for all $s \in \mathbb{R}$.

**Remark 2.15.** It was shown by Baillon [4] that when $B$ is a reflexive Banach space equation (2.4) has maximal $C_{\mathrm{b}}$-regularity if and only if $L$ is bounded.

**Corollary 2.16.** Let $\alpha \in (0, 1)$, $\omega_\eta \in \{\mathrm{e}^{-\eta|\cdot|}, \mathrm{e}^{\eta(\cdot)}, \mathrm{e}^{-\eta(\cdot)}\}$ and $L$ be a closed, densely defined linear operator on a Banach space $B$ which satisfies $\mathrm{i}\mathbb{R} \subseteq \rho(L)$ and the estimate

$$\|(\mathrm{i}sI - L)^{-1}\|_{\mathcal{L}(B)} \lesssim \frac{1}{1 + |s|} \tag{2.5}$$

for all $s \in \mathbb{R}$. There exists $\eta_0 > 0$ such that the equation

$$\dot{u}(t) = Lu(t) + f(t) \tag{2.6}$$

has a unique solution

$$u \in C^{1,\alpha}_{\omega_\eta}(\mathbb{R}; B) \cap C^\alpha_{\omega_\eta}(\mathbb{R}; \mathcal{D}(L))$$

for every $f \in C^\alpha_{\omega_\eta}(\mathbb{R}; B)$ and $\eta \in [0, \eta_0]$. Furthermore,

$$\|u\|_{C^{1,\alpha}_{\omega_\eta}(\mathbb{R},B)} + \|u\|_{C^\alpha_{\omega_\eta}(\mathbb{R};\mathcal{D}(L))} \lesssim \|f\|_{C^\alpha_{\omega_\eta}(\mathbb{R};B)} \tag{2.7}$$

uniformly over $\eta \in [0, \eta_0]$.

*Proof.* Estimate (2.5) implies the existence of positive numbers $\gamma, s_0$ such that

$$\|(\lambda I - L)^{-1}\|_{\mathcal{L}(B;B)} \leq \frac{\gamma}{|\lambda|} \tag{2.8}$$

for all $\lambda \in \mathbb{C}$ with $\lambda = \mathrm{i}s$ and $|s| \geq s_0$. We find that estimate (2.8) is extendable to a cone $|\mathrm{Re}\,\lambda| \leq q_0|\mathrm{Im}\,\lambda|$ and $|s| = |\mathrm{Im}\,\lambda| \geq s_0$, where $0 < q_0 < \gamma^{-1}$, by considering

$$\begin{aligned}\|(\lambda I - L)^{-1}\|_{\mathcal{L}(B;B)} &= \|\Big(I - (\lambda - \mathrm{i}s)(L - \mathrm{i}sI)^{-1}\Big)^{-1}(L - \mathrm{i}sI)^{-1}\|_{\mathcal{L}(B;B)} \\ &\leq (1 - \|(\lambda - \mathrm{i}s)(L - \mathrm{i}sI)\|_{\mathcal{L}(B;B)})^{-1}\|(L - \mathrm{i}sI)^{-1}\|_{\mathcal{L}(B;B)} \\ &\leq \frac{(1+q_0)\gamma}{(1-\gamma q_0)|\lambda|}.\end{aligned}$$

Choosing $\eta_0 = \frac{1}{2}\,\mathrm{dist}(\sigma(L), \mathrm{i}\mathbb{R})$ therefore implies that $L \pm \eta I$ also satisfies estimate (2.5), uniformly over $\eta \in [0, \eta_0]$.

Suppose that $\omega_\eta = \mathrm{e}^{-\eta|\cdot|}$. We define

$$\begin{aligned} f_+(t) &= \mathrm{e}^{-\eta t}\chi(t)f(t), \\ f_-(t) &= \mathrm{e}^{\eta t}(1 - \chi(t))f(t),\end{aligned}$$

where $\chi \in C^\infty(\mathbb{R})$ is a smooth cut-off function such that

$$\chi(t) = \begin{cases} 0, & t \leq -1, \\ 1, & t \geq 1,\end{cases}$$

so that $f_\pm \in C^\alpha_{\mathrm{b}}(\mathbb{R}; B)$. From Lemma 2.14 we find that the problems

$$\dot{u}(t) = (L \mp \eta I)u(t) + f_\pm(t)$$

have unique solutions $u_\pm \in C^{1,\alpha}_{\mathrm{b}}(\mathbb{R}; B) \cap C^\alpha_{\mathrm{b}}(\mathbb{R}; \mathcal{D}(L))$ satisfying the estimates

$$\|u_\pm\|_{C^{1,\alpha}_{\mathrm{b}}(\mathbb{R};B)} + \|u_\pm\|_{C^\alpha_{\mathrm{b}}(\mathbb{R};\mathcal{D}(L))} \lesssim \|f\|_{C^\alpha_{\mathrm{b}}(\mathbb{R};B)} \tag{2.9}$$

uniformly over $\eta \in [0, \eta_0]$. The function $u\colon \mathbb{R} \to \mathcal{D}(L)$ defined by

$$u(t) = \mathrm{e}^{\eta t}u_+(t) + \mathrm{e}^{-\eta t}u_-(t)$$

for $t \in \mathbb{R}$ solves equation (2.6) and satisfies

$$\begin{aligned}\|u\|_{C^{1,\alpha}_\eta(\mathbb{R};B)} &\lesssim \|\mathrm{e}^{\eta(\cdot)}u_+\|_{C^{1,\alpha}_\eta(\mathbb{R};B)} + \|\mathrm{e}^{-\eta(\cdot)}u_-\|_{C^{1,\alpha}_\eta(\mathbb{R};B)} \\ &\lesssim \|u_+\|_{C^{1,\alpha}_{\mathrm{b}}(\mathbb{R};B)} + \|u_-\|_{C^{1,\alpha}_{\mathrm{b}}(\mathbb{R};B)} \\ &\lesssim \|f_+\|_{C^\alpha_{\mathrm{b}}(\mathbb{R};B)} + \|f_-\|_{C^\alpha_{\mathrm{b}}(\mathbb{R};B)} \\ &\lesssim \|f\|_{C^\alpha_\eta(\mathbb{R};B)}.\end{aligned}$$

uniformly over $\eta \in [0, \eta_0]$.

The cases $\omega_\eta = \mathrm{e}^{\eta(\cdot)}$ and $\omega_\eta = \mathrm{e}^{-\eta(\cdot)}$ are treated analogously by applying Lemma 2.14 to

$$g(t) = \mathrm{e}^{\eta t}\chi(t)f(t)$$

and

$$g(t) = \mathrm{e}^{-\eta t}\chi(t)f(t).$$

□

## 2.6 Proof of the centre-manifold theorem

In this section we give a complete proof of the centre-manifold theorem for autonomous quasilinear evolutionary equations. Let $(X, \|\cdot\|_X$ and $(Y, \|\cdot\|_Y)$ be Banach spaces with $Y$ continuously and densely embedded in $X$. We consider the nonlinear differential equations

$$\dot{u}(t) = Au(t) + g_1(u(t), v(t)), \tag{2.10}$$
$$\dot{v}(t) = Lv(t) + g_2(u(t), v(t)) \tag{2.11}$$

for $t \in \mathbb{R}$, where $A \in \mathbb{R}^{d\times d}$ has purely imaginary spectrum. Furthermore the linear operator $L\colon Y \subseteq X \to X$ and the functions $g_1\colon \mathbb{R}^d \times Y \to \mathbb{R}^d$, $g_2\colon \mathbb{R}^d \times Y \to X$ satisfy the following assumptions.

**Hypothesis 2.17.**

(i) The linear operator $L\colon Y \subseteq X \to X$ is closed.

(ii) The spectrum of $L$ satisfies $\sigma(L) \cap \mathrm{i}\mathbb{R} = \varnothing$ and

$$\|(\mathrm{i}sI - L)^{-1}\|_{\mathcal{L}(X)} \lesssim \frac{1}{1 + |s|}$$

for all $s \in \mathbb{R}$.

(iii) There exist $k \in \mathbb{N}$ and neighbourhoods $U$ and $V$ of the origin in respectively $\mathbb{R}^d$ and $Y$ such that $g_1 \in C^{k+1}_{\mathrm{b,u}}(U \times V; \mathbb{R}^d)$ and $g_2 \in C^{k+1}_{\mathrm{b,u}}(U \times V; X)$. Additionally we assume that

$$g_1(0,0), g_2(0,0) = 0,$$
$$\mathrm{d}g_1[0,0], \mathrm{d}g_2[0,0] = 0.$$

Choose $\eta > 0$ and $\varepsilon > 0$ such that $\overline{B}_{4\varepsilon} \times \overline{B}_{2\varepsilon} \subseteq U \times V$. The following generalized contraction result by Vanderbauwhede [23, pp. 105-106] is one of the main ingredients in the proof of the centre-manifold theorem.

**Lemma 2.18.** Let $k$ be a natural number and $X_0, \dots, X_k$ complete metric spaces. Additionally, let $F\colon X_1 \times \ldots \times X_k \to X_1 \times \ldots \times X_k$ be a function of the form

$$F(x_0, \dots, x_k) = (F_0(x_0), F_1(x_0, x_1), \dots, F_k(x_0, \dots, x_k)),$$

where $F_j\colon X_0 \times \ldots \times X_j \to X_j$ is a contraction in its $j$th argument which is uniform in its remaining arguments.

(i) The function $F$ has a unique fixed point $(\bar{x}_0, \dots, \bar{x}_k) \in X_0 \times \ldots \times X_k$.

(ii) Suppose that the functions $F_j(\cdot, \bar{x}_j)\colon X_0 \times \ldots \times X_{j-1} \to X_j$ are continuous for all $j \in \{0, \dots, k\}$. The fixed point of $F$ is *attractive*, i.e.

$$\lim_{n\to\infty} F^n(x_0, \dots, x_k) = F(\bar{x}_0, \dots, \bar{x}_k)$$

for all $(x_0, \dots, x_k) \in X_0 \times \ldots \times X_k$.

In the proof of the centre-manifold theorem we consider the linear equations

$$\dot{u}(t) = Au(t) + k_1(t), \quad u(0) = w, \tag{2.12}$$
$$\dot{v}(t) = Lv(t) + k_2(t), \tag{2.13}$$

where $k_1 \in C^\alpha_\mathrm{b}(\mathbb{R};\mathbb{R}^d)$ and $k_2 \in C^\alpha_\mathrm{b}(\mathbb{R};X)$ for $\alpha \in (0,1)$. The solution to equation (2.12) is

$$u(t) = (\mathcal{S}w)(t) + (\mathcal{S}_0 k_1)(t),$$

where $\mathcal{S} \in \mathcal{L}(\mathbb{R}^d; C^1_\mathrm{b}(\mathbb{R};\mathbb{R}^d))$ is given by

$$(\mathcal{S}w)(t) = \mathrm{e}^{At}w,$$

and

$$\mathcal{S}_0 \in \mathcal{L}(C^\alpha_\mathrm{b}(\mathbb{R};\mathbb{R}^d); C^{1,\alpha}_\mathrm{b}(\mathbb{R};\mathbb{R}^d)) \cap \mathcal{L}(C^\alpha_\eta(\mathbb{R};\mathbb{R}^d); C^{1,\alpha}_\eta(\mathbb{R};\mathbb{R}^d))$$

is given by

$$(\mathcal{S}_0 k_1)(t) = \int_0^t \mathrm{e}^{A(t-s)} k_1(s)\,\mathrm{d}s.$$

Equation (2.13) is solved by maximal regularity methods (see Section 2.5). We denote its solution by $\mathcal{M}k_2$ and note that

$$\mathcal{M} \in \mathcal{L}\Big(C^\alpha(\mathbb{R};X); C^{1,\alpha}_\mathrm{b}(\mathbb{R};Y)\Big) \cap \mathcal{L}\Big(C^\alpha_\eta(\mathbb{R};X); C^{1,\alpha}_\eta(\mathbb{R};Y)\Big)$$

with $\eta \in (0,\eta_0]$ for some $\eta_0 > 0$ (see Lemma 2.14 and Corollary 2.16).

**Theorem 2.19** (Centre-manifold theorem)**.** Under Hypothesis 2.17 there exist neighbourhoods $\tilde{U}$ and $\tilde{V}$ of the origin in respectively $\mathbb{R}^d$ and $Y$ with $\tilde{U} \subseteq U$ and $\tilde{V} \subseteq V$, and a function $\Psi \in C^k(\tilde{U};\tilde{V})$ with $\Psi(0) = 0$ and $\mathrm{d}\Psi[0] = 0$ such that the *centre manifold*

$$M_\mathrm{c} = \Big\{(u_0, \Psi(u_0)) : u_0 \in \tilde{U}\Big\} \subseteq \tilde{U} \times \tilde{V}$$

has the following properties.

(i) The manifold $M_\mathrm{c}$ is locally invariant, i.e. if $(u,v)\colon [0,T] \to X$ for $T > 0$ is a solution of equations (2.10) and (2.11) with $(u,v)(0) \in M_\mathrm{c}$ and $(u,v)(t) \in \tilde{U} \times \tilde{V}$ for all $t \in [0,T]$ then $(u,v)(t) \in M_\mathrm{c}$ for all $t \in [0,T]$.

(ii) Any solution of equations (2.10) and (2.11) with $(u,v)(t) \in \tilde{U} \times \tilde{V}$ for all $t \in \mathbb{R}$ satisfies $(u,v)(t) \in M_\mathrm{c}$ for all $t \in \mathbb{R}$.

*Proof.* Let $W \subseteq \mathbb{R}^d$ be a bounded neighbourhood of the origin. For $w \in W$ we consider the system

$$\dot{u}(t) = Au(t) + g_{1,\varepsilon}(u(t), v(t)), \quad u(0) = w, \tag{2.14}$$
$$\dot{v}(t) = Lv(t) + g_{2,\varepsilon}(u(t), v(t)), \tag{2.15}$$

where $g_{1,\varepsilon}$ and $g_{2,\varepsilon}$ are the localisations of $g_1$ and $g_2$ in the sense of Section 2.3. First we reformulate the above system as the fixed-point problem

$$\begin{aligned}\begin{pmatrix} u \\ v \end{pmatrix} &= \begin{pmatrix} \mathcal{S}w + \mathcal{S}_0 G_{1,\varepsilon}(u,v) \\ \mathcal{M}G_{2,\varepsilon}(u,v) \end{pmatrix} \\ &=: F_0(u,v), \end{aligned} \tag{2.16}$$

where $G_{1,\varepsilon}$ and $G_{2,\varepsilon}$ are the localised composition operators corresponding to $g_1$ and $g_2$ in the sense of Section 2.3.

We construct a $k$ times (with respect to $w$) differentiable solution to equation (2.16) by formally differentiating equation (2.16) with respect to $w$ to obtain a fixed-point problem suitable for Lemma 2.18. Denoting $(\mathrm{d}^j u[w], \mathrm{d}^j v[w])$ by $(u^{(j)}, v^{(j)})$ we find that

$$\begin{aligned}\dot{u}^{(j)}(t) &= Au^{(j)}(t) + S_j\Big(\sum_{l=1}^{j} \sum_{m\in M_{j,l}} \frac{l!}{m_1! \cdot \ldots \cdot m_l!} \\ &\qquad \Lambda^{j,m_1,\ldots,m_l}\, \mathrm{d}^{(l)} g_{1,\varepsilon}[u(t),v(t)](\{(u^{(m_i)}(t), v^{(m_i)}(t))\}_{i=1}^{l})\Big), \\ \dot{v}^{(j)}(t) &= Lv^{(j)}(t) + S_j\Big(\sum_{l=1}^{j} \sum_{m\in M_{j,l}} \frac{l!}{m_1! \cdot \ldots \cdot m_l!} \\ &\qquad \Lambda^{j,m_1,\ldots,m_l}\, \mathrm{d}^{(l)} g_{2,\varepsilon}[u(t),v(t)](\{(u^{(m_i)}(t), v^{(m_i)}(t))\}_{i=1}^{l})\Big)\end{aligned}$$

with $u^{(1)}(0) = I$, $v^{(j)}(0) = 0$ for $j \in \{2, \ldots, k\}$, which leads to

$$\begin{aligned}\begin{pmatrix} u^{(1)} \\ v^{(1)} \end{pmatrix} &= \begin{pmatrix} \mathcal{S}I + \mathcal{S}_0 \tilde{G}^{(1)}_{1,\varepsilon}(u,v)(u^{(1)}, v^{(1)}) \\ \mathcal{M}\tilde{G}^{(1)}_{2,\varepsilon}(u,v)(u^{(1)}, v^{(1)}) \end{pmatrix} \\ &=: F_1(u,v)(u^{(1)}, v^{(1)}), \end{aligned} \tag{2.17}$$

and

$$\begin{aligned}\begin{pmatrix} u^{(j)} \\ v^{(j)} \end{pmatrix} &= \sum_{l=1}^{j} \sum_{m\in M_{j,l}} \frac{l!}{m_1! \cdot \ldots \cdot m_l!} \begin{pmatrix} S_j\Big(\Lambda^{j,m_1,\ldots,m_l} \mathcal{S}_0 \tilde{G}^{(l)}_{1,\varepsilon}(u,v)(\{(u^{(m_i)}, v^{(m_i)})\}_{i=1}^{l})\Big) \\ S_j\Big(\Lambda^{j,m_1,\ldots,m_l} \mathcal{M} \tilde{G}^{(l)}_{2,\varepsilon}(u,v)(\{(u^{(m_i)}, v^{(m_i)})\}_{i=1}^{l})\Big) \end{pmatrix} \\ &=: F_j(u,v)(\{(u^{(i)}, v^{(i)})\}_{i=1}^{j}), \end{aligned} \tag{2.18}$$

where $\tilde{G}^{(l)}_{1,\varepsilon}$ and $\tilde{G}^{(l)}_{2,\varepsilon}$ are defined as in Section 2.4, and

$$\begin{aligned} M_{j,i} &= \Big\{(m_1, \ldots, m_i) : m_1 + \ldots + m_i = j\Big\}, \\ S_j(L)(x_1, \ldots, x_j) &= \frac{1}{j!} \sum_{\sigma \in S_j} L(x_{\sigma(1)}, \ldots, x_{\sigma(j)}) \end{aligned}$$

and

$$\begin{aligned} &\Lambda^{j,m_1,\ldots,m_i}\Big(K(L_1, \ldots, L_i)\Big)(x_1, \ldots, x_j) \\ &\quad = K\Big(L_1(x_1, \ldots, x_{m_1}), \ldots, (x_{m_1+\ldots+m_{i-1}+1}, \ldots, x_j)\Big). \end{aligned}$$

Define

$$F = (F_0, \dots, F_j),$$

and

$$\begin{aligned} X_0 &= C_{\mathrm{b}}(W; E^{\alpha}_{\eta,\varepsilon,A}),\\ X_j^{\mu} &= B(W; \mathcal{L}^{(j)}(\mathbb{R}^d; C_{j\eta+(2j-1)\mu})) \end{aligned}$$

for $j \in \{1, \dots, k\}$, $\eta > 0$ and $\mu \geq 0$, where we use the notation

$$\begin{aligned} C_{\zeta} &= C^1_{\zeta}(\mathbb{R}; \mathbb{R}^d) \times C^{\alpha}_{\zeta}(\mathbb{R}; Y),\\ D_{\zeta} &= C^{\alpha}_{\zeta}(\mathbb{R}; \mathbb{R}^d) \times C^{\alpha}_{\zeta}(\mathbb{R}; X) \end{aligned}$$

for $\zeta > 0$ and

$$E^{\alpha}_{\eta,\varepsilon,A} = \{(u, v) \in E_{\eta,\varepsilon,A}(\mathbb{R}) \times C^{\alpha}_{\mathrm{b}}(\mathbb{R}; Y) : \|v\|_{C^{\alpha}_{\mathrm{b}}(\mathbb{R};Y)} \leq \varepsilon\},$$

which we consider as a closed subset of $C_{\eta}$. The calculations

$$\begin{aligned} \Big\|\frac{\mathrm{d}}{\mathrm{d}t}\Big(\mathcal{S}w + \mathcal{S}_0 G_{1,\varepsilon}(u, v)\Big) - A\Big(\mathcal{S}w + \mathcal{S}_0 G_{1,\varepsilon}(u, v)\Big)\Big\|_{C_{\mathrm{b}}(\mathbb{R};\mathbb{R}^d)} &\lesssim \|G_{1,\varepsilon}(u, v)\|_{C^{\alpha}_{\mathrm{b}}(\mathbb{R};X)}\\ &\lesssim \varepsilon^2 \end{aligned}$$

(which is obtained from Lemma 2.9 and the facts $g(0) = 0$ and $\mathrm{d}g[0] = 0$) and

$$\begin{aligned} \|\mathcal{M}G_{2,\varepsilon}(u, v)\|_{C^{\alpha}_{\mathrm{b}}(\mathbb{R};Y)} &\lesssim \|G_{2,\varepsilon}(u, v)\|_{C^{\alpha}_{\mathrm{b}}(\mathbb{R};X)}\\ &\lesssim \varepsilon^2 \end{aligned}$$

(which is obtained from Lemma 2.9 and again the facts $g(0) = 0$ and $\mathrm{d}g[0] = 0$) imply that $F_0$ maps $X_0$ into itself. Corollary 2.16 yields the existence of $\eta_0 > 0$ such that

$$\|F_0(u, v) - F_0(\tilde{u}, \tilde{v})\|_{X_0} \lesssim \|G_{\varepsilon}(u, v) - G_{\varepsilon}(\tilde{u}, \tilde{v})\|_{B(W;D_{\eta})}$$

for $\eta \in (0, \eta_0)$. Lemma 2.9 therefore implies that

$$\|F_0(u, v) - F_0(\tilde{u}, \tilde{v})\|_{X_0} \lesssim \varepsilon \|(u - \tilde{u}, v - \tilde{v})\|_{X_0},$$

so that $F_0 \colon X_0 \to X_0$ is a contraction.

Turning to $F_j$ we assume that $\mu$ satisfies

$$k\eta + (2k - 1)\mu < \eta_0$$

in order that Corollaries 2.13 and 2.16 remain applicable. We notice that

$$\tilde{G}^{(l)}_{\varepsilon} := \begin{pmatrix} \tilde{G}^{(l)}_{1,\varepsilon} \\ \tilde{G}^{(l)}_{2,\varepsilon} \end{pmatrix}$$

maps $E^{\alpha}_{\eta,\varepsilon,A}$ boundedly into $\mathcal{L}^{(j)}(C_{m_1\eta+(2m_1-1)\mu}, \dots, C_{m_l\eta+(2m_l-1)\mu}; D^{\alpha}_{j\eta+(2j-l)\mu})$ by Corollary 2.13(i), since $(m_1, \dots, m_l) \in M_{j,l}$ satisfies

$$\sum_{i=1}^{l} m_i\eta + (2m_i - 1)\mu = j\eta + (2j - l)\mu$$

by definition. Furthermore $(u^{(m_i)}, v^{(m_i)}) \in B(W; \mathcal{L}^{(m_i)}(\mathbb{R}^d; C_{m_i\eta+(2m_i-1)\mu}))$. Thus

$$\big((u,v), \{(u^{(m_i)}, v^{(m_i)})\}_{i=1}^l\big) \mapsto \tilde{G}_\varepsilon^{(l)}(u,v)(\{(u^{(m_i)}, v^{(m_i)})\}_{i=1}^l)$$

maps $X_0 \times X_{m_1}^\mu \times \ldots \times X_{m_l}^\mu$ into $B(W; \mathcal{L}^{(j)}(\mathbb{R}^d; D_{j\eta+(2j-l)\mu}))$. The fact that

$$D_{j\eta+(2j-l)\mu} \hookrightarrow D_{j\eta+(2j-1)\mu}$$

shows that $F_j$ maps $X_0 \times X_1^\mu \times \ldots \times X_j^\mu$ into $X_j^\mu$. Corollary 2.13 implies that

$$\begin{aligned}
&\|F_j\big((u,v), \{(u^{(i)}, v^{(i)})\}_{i=1}^j\big) - F_j\big((u,v), \{(\tilde{u}^{(i)}, \tilde{v}^{(i)})\}_{i=1}^j\big)\|_{X_j^\mu} \\
&\quad = \Big\| \frac{1}{j!} \begin{pmatrix} \mathcal{S}_0 \tilde{G}_{1,\varepsilon}^{(1)}(u,v)(u^{(j)} - \tilde{u}^{(j)}, v^{(j)} - \tilde{v}^{(j)}) \\ \mathcal{M} \tilde{G}_{2,\varepsilon}^{(1)}(u,v)(u^{(j)} - \tilde{u}^{(j)}, v^{(j)} - \tilde{v}^{(j)}) \end{pmatrix} \Big\|_{X_j^\mu} \\
&\quad \lesssim \|\tilde{G}_\varepsilon^{(1)}(u,v)(u^{(j)} - \tilde{u}^{(j)}, v^{(j)} - \tilde{v}^{(j)})\|_{B(W; \mathcal{L}^{(j)}(\mathbb{R}^d; D_{j\eta+(2j-1)\mu}))} \\
&\quad \lesssim \varepsilon \|(u^{(j)} - \tilde{u}^{(j)}, v^{(j)} - \tilde{v}^{(j)})\|_{X_j^\mu},
\end{aligned}$$

so that $F_j((u,v), \{(u^{(i)}, v^{(i)})\}_{i=1}^{j-1}, \cdot)$ is a uniform contraction.

It follows from Lemma 2.18 that $F\colon X_0 \times X_1^\mu \times \ldots \times X_j^\mu \to X_0 \times X_1^\mu \times \ldots \times X_j^\mu$ has a unique fixed point $\big((\bar{u}, \bar{v}), (\bar{u}^{(1)}, \bar{v}^{(1)}), \ldots, (\bar{u}^{(j)}, \bar{v}^{(j)})\big)$. Since $(\bar{u}^{(i)}, \bar{v}^{(i)})$ is unique and $X_i^0 \subseteq X_i^\mu$, we conclude that $(\bar{u}^{(i)}, \bar{v}^{(i)}) \in X_i^0$ for all $i \in \{1, \ldots, j\}$.

Now we suppose that $\mu > 0$. It remains to prove the differentiability of $(\bar{u}, \bar{v})$. To that end we first prove that $F_j(\cdot, (\bar{u}^{(j)}, \bar{v}^{(j)}))$ is continuous for all $j \in \{1, \ldots, k\}$. Corollary 2.13(ii) implies that

$$\begin{aligned}
&\|\tilde{G}_\varepsilon^{(1)}(u,v)(\bar{u}^{(j)}, \bar{v}^{(j)}) - \tilde{G}_\varepsilon^{(1)}(\tilde{u},\tilde{v})(\bar{u}^{(j)}, \bar{v}^{(j)})\|_{B(W; \mathcal{L}^{(j)}(\mathbb{R}^d; D_{j\eta+(2j-1)\mu}))} \\
&\quad \leq \|\tilde{G}_\varepsilon^{(1)}(u,v) - \tilde{G}_\varepsilon^{(1)}(\tilde{u},\tilde{v})\|_{B(W; \mathcal{L}(C_{j\eta}; D_{j\eta+(2j-1)\mu}))} \|(\bar{u}^{(j)}, \bar{v}^{(j)})\|_{X_j^0} \\
&\quad \to 0
\end{aligned}$$

as $\|(u,v) - (\tilde{u},\tilde{v})\|_{X_0} \to 0$. Similarly Corollaries 2.13(i) and 2.13(ii) yield that

$$\begin{aligned}
&\|\tilde{G}_\varepsilon^{(l)}(u,v)(\{(u^{(m_i)}, v^{(m_i)})\}_{i=1}^l) - \tilde{G}_\varepsilon^{(l)}(\tilde{u},\tilde{v})(\{(\tilde{u}^{(m_i)}, \tilde{v}^{(m_i)})\}_{i=1}^l)\|_{B(W; \mathcal{L}^{(j)}(\mathbb{R}^d; D_{j\eta+(2j-1)\mu}))} \\
&\quad \leq \|\big(\tilde{G}_\varepsilon^{(l)}(u,v) - \tilde{G}_\varepsilon^{(l)}(\tilde{u},\tilde{v})\big)(\{(\tilde{u}^{(m_i)}, \tilde{v}^{(m_i)})\}_{i=1}^l)\|_{B(W; \mathcal{L}^{(j)}(\mathbb{R}^d; D_{j\eta+(2j-1)\mu}))} \\
&\qquad + \|\tilde{G}_\varepsilon^{(l)}(\tilde{u},\tilde{v})(\{(u^{(m_i)} - \tilde{u}^{(m_i)}, v^{(m_i)} - \tilde{v}^{(m_i)}))\}_{i=1}^l)\|_{B(W; \mathcal{L}^{(j)}(\mathbb{R}^d; D_{j\eta+(2j-1)\mu}))} \\
&\quad \leq \|\tilde{G}_\varepsilon^{(l)}(u,v)\|_{B(W; \mathcal{L}^{(l)}(\{C_{m_i\eta+(2m_i-1)\mu}\}_{i=1}^l; D_{j\eta+(2j-l)\mu}))} \\
&\qquad \times \prod_{i=1}^l \|(u^{(m_i)} - \tilde{u}^{(m_i)}, v^{(m_i)} - \tilde{v}^{(m_i)})\|_{X_{m_i}^\mu} \\
&\qquad + \|\tilde{G}_\varepsilon^{(l)}(u,v) - \tilde{G}_\varepsilon^{(l)}(\tilde{u},\tilde{v})\|_{B(W; \mathcal{L}^{(l)}(\{C_{m_i\eta+(2m_i-1)\mu}\}_{i=1}^l; D_{j\eta+(2j-1)\mu}))} \\
&\qquad\quad \times \prod_{i=1}^l \|(u^{(m_i)}, v^{(m_i)})\|_{X_{m_i}^\mu} \\
&\quad \to 0
\end{aligned}$$

as $\|(u,v) - (\tilde{u},\tilde{v})\|_{X_0} \to 0$ for $l \in \{2, \ldots, j\}$, where we used the fact that

$$\sum_{i=1}^l m_i\eta + (2m_i - 1)\mu = j\eta + (2j - l)\mu < j\eta + (2j - 1)\mu,$$

so that

$$D_{j\eta+(2j-l)\mu} \subseteq D_{j\eta+(2j-1)\mu},$$

to apply Corollary 2.13(ii). Lemma 2.18 now yields that $\big((\bar{u},\bar{v}),(\bar{u}^{(1)},\bar{v}^{(1)}),\dots,(\bar{u}^{(j)},\bar{v}^{(j)})\big)$ is an attractive fixed point of $F$.

Now set

$$(u_0,v_0),(u_0^{(1)},v_0^{(1)}),\dots,(u_0^{(j)},v_0^{(j)}) = (0,0)$$

and define

$$\big((u_{n+1},v_{n+1}),(u_{n+1}^{(1)},v_{n+1}^{(1)}),\dots,(u_{n+1}^{(j)},v_{n+1}^{(j)})\big) = F\big((u_n,v_n),(u_n^{(1)},v_n^{(1)}),\dots,(u_n^{(j)},v_n^{(j)})\big)$$

for $n \in \mathbb{N}_0$. By construction $(u_n,v_n) \in C^j(W;C_{j\eta+(2j-1)\mu})$ with

$$(\mathrm{d}^j u_n, \mathrm{d}^j v_n) = (u_n^{(j)}, v_n^{(j)})$$

(see Corollary 2.13(iii)). Uniform convergence yields that $(\bar{u},\bar{v}) \in C^j(W;C_{j\eta+(2j-1)\mu})$ with $(\mathrm{d}^j\bar{u},\mathrm{d}^j\bar{v}) = (\bar{u}^{(j)},\bar{v}^{(j)})$ for all $j \in \{1,\dots,k\}$.

Set $\tilde{U} = B_\varepsilon \subseteq \mathbb{R}^d$, $\tilde{V} = B_\varepsilon \subseteq Y$ and

$$\Psi(w) = \bar{v}(w)|_{t=0}.$$

The facts that

$$(\bar{u}(0),\bar{v}(0)) = (0,0), \quad (\bar{u}^{(1)}(0),\bar{v}^{(1)}(0)) = (0,0)$$

imply $\Psi(0) = 0$, $\mathrm{d}\Psi[0] = 0$, and assertions (i) and (ii) are obtained as follows.

(i) We first show that $M_c$ is a globally invariant manifold for equations (2.14) and (2.15). Suppose that $(u^\varepsilon,v^\varepsilon) \in E^\alpha_{\eta,\varepsilon,A}$ is a solution to equations (2.14) and (2.15) with

$$v^\varepsilon(0) = \Psi(u^\varepsilon(0)).$$

We notice that

$$(\tilde{u}^\varepsilon,\tilde{v}^\varepsilon)(t) = (u^\varepsilon(t+s), v^\varepsilon(t+s))$$

is also a solution to equations (2.14) and (2.15) in $E^\alpha_{\eta,\varepsilon,A}$ for each $s \in \mathbb{R}$, so that

$$\tilde{v}^\varepsilon(0) = \Psi(\tilde{u}^\varepsilon(0)),$$

which is equivalent to

$$v^\varepsilon(s) = \Psi(u^\varepsilon(s))$$

for all $s \in \mathbb{R}$.

Now we take a solution $(u,v)$ of equations (2.10) and (2.11) with

$$v(0) = \Psi(u(0))$$

and

$$(u,v)(t) \in \tilde{U} \times \tilde{V}$$

for all $t \in [0, T]$. Let $(u^\varepsilon, v^\varepsilon) \in E^\alpha_{\eta,\varepsilon,A}$ be the solution of equations (2.14) and (2.15) with $u^\varepsilon(0) = u(0)$. By construction

$$(u, v)(t) = (u^\varepsilon, v^\varepsilon)(t)$$

and

$$v^\varepsilon(t) = \Psi(u^\varepsilon(t))$$

for all $t \in [0, T]$, so that

$$v(t) = \Psi(u(t))$$

for all $t \in [0, T]$.

(ii) Next let $(u, v) \colon \mathbb{R} \to \mathbb{R}^d \times Y$ be a solution to equations (2.10) and (2.11) with $(u, v)(t) \in \tilde{U} \times \tilde{V}$ for all $t \in \mathbb{R}$, so that $(u, v)$ is a solution of equation (2.16) with $(u, v) \in E^\alpha_{\eta,\varepsilon,A}$. This fact implies that

$$(u, v) = (\bar{u}, \bar{v})(u(0)),$$

and the result follows from (i).

□

Next we consider the parameter dependent system

$$\dot{u}(t) = Au(t) + g_1(\lambda; u(t), v(t)), \tag{2.19}$$
$$\dot{v}(t) = Lv(t) + g_2(\lambda; u(t), v(t)) \tag{2.20}$$

for $t \in \mathbb{R}$. Instead of Hypothesis 2.17(iii) we now assume the following hypothesis.

**Hypothesis 2.20.** There exist $k \in \mathbb{N}$ and neighbourhoods $\Lambda$, $U$ and $V$ of the origin in respectively $\mathbb{R}^p$, $\mathbb{R}^d$ and $Y$ such that $g_1 \in C^{k+1}_{\mathrm{b,u}}(\Lambda \times U \times V; \mathbb{R}^d)$ and $g_2 \in C^{k+1}_{\mathrm{b,u}}(\Lambda \times U \times V; X)$. Additionally we assume that

$$g_1(0,0,0), g_2(0,0,0) = 0,$$
$$\mathrm{d}_2 g_1[0,0,0], \mathrm{d}_2 g_2[0,0,0] = 0,$$

where $\mathrm{d}_2$ denotes the derivative with respect to $(u, v)$.

**Corollary 2.21.** There exist neighbourhoods $\tilde{\Lambda}$, $\tilde{U}$ and $\tilde{V}$ of the origin in respectively $\mathbb{R}^p$, $\mathbb{R}^d$ and $Y$ with $\tilde{\Lambda} \subseteq \Lambda$, $\tilde{U} \subseteq U$ and $\tilde{V} \subseteq V$, and a function $\Psi \in C^k(\tilde{\Lambda} \times \tilde{U}; \tilde{V})$ with $\Psi(0,0) = 0$ and $\mathrm{d}_2\Psi[0,0] = 0$ such that the *centre manifold*

$$M^\lambda_{\mathrm{c}} = \left\{ (u_0, \Psi(\lambda; u_0)) : u_0 \in \tilde{U} \right\} \subseteq \tilde{U} \times \tilde{V}$$

has the following properties for each $\lambda \in \tilde{\Lambda}$.

(i) The manifold $M_c^\lambda$ is locally invariant, i.e. if $(u,v)\colon [0,T] \to X$ for $T>0$ is a solution of equations (2.19) and (2.20) with $(u,v)(0) \in M_c^\lambda$ and $(u,v)(t) \in \tilde{U} \times \tilde{V}$ for all $t \in [0,T]$ then $(u,v)(t) \in M_c^\lambda$ for all $t \in [0,T]$.

(ii) Any solution of equations (2.19) and (2.20) with $(u,v)(t) \in \tilde{U} \times \tilde{V}$ for all $t \in \mathbb{R}$ satisfies $(u,v)(t) \in M_c^\lambda$ for all $t \in \mathbb{R}$.

*Proof.* We extend equations (2.19) and (2.20) to the system

$$\begin{aligned} \dot{\lambda} &= 0, \\ \dot{u}(t) &= Au(t) + g_1(\lambda; u(t), v(t)), \\ \dot{v}(t) &= Lv(t) + g_2(\lambda; u(t), v(t)). \end{aligned}$$

Set

$$\hat{A} = \begin{pmatrix} 0 \\ A \end{pmatrix} \in \mathbb{R}^{(p+d)\times(p+d)},$$

define $\hat{g}_1\colon \Lambda \times U \times V \to \mathbb{R}^{p+d}$ by

$$\tilde{g}_1 = \begin{pmatrix} 0 \\ g_1 \end{pmatrix}$$

and apply Theorem 2.19 with $A$, $g$ and $U$ replaced by respectively $\hat{A}$, $\hat{g}$ and $\hat{U} = \Lambda \times U$. Without loss of generality we may assume that the reduction function $\Psi$ is defined on a 'rectangular' neighbourhood of the origin $\tilde{\Lambda} \times \tilde{U}$ in $\mathbb{R}^{p+d}$. The properties of $\Psi$ are deduced by noting that all solutions $(\lambda(t), u(t), v(t))$ satisfy

$$\lambda = \text{const};$$

in particular setting $\lambda = 0$ returns us to the 'standard' parameter-independent setting and yields $\Psi(0,0)$, $\mathrm{d}_2\Psi[0,0] = 0$. □

# 3 Normal-form theory

Suppose that $\mathcal{D}$ and $\mathcal{X}$ are Banach spaces with $\mathcal{D}$ continuously and densely embedded in $\mathcal{X}$. We consider the parameter-dependent evolutionary system

$$\dot{Z} = F^\mu(Z, Q), \tag{3.1}$$
$$\dot{Q} = L^\mu Q + G^\mu(Z, Q) + H^\mu(Z), \tag{3.2}$$

for $(Z, Q)\colon \mathbb{R} \to \mathbb{R}^n \times \mathcal{D}$. Here $L^\mu\colon \mathcal{D} \subseteq \mathcal{X} \to \mathcal{X}$ is a closed linear operator such that $L^\mu \in \mathcal{L}(\mathcal{D}; \mathcal{X})$ depends analytically upon $\mu$ and $F^{(\cdot)}\colon \mathbb{R} \times \mathbb{R}^n \times \mathcal{D} \to \mathbb{R}^n$, $G^{(\cdot)}\colon \mathbb{R} \times \mathbb{R}^n \times \mathcal{D} \to \mathcal{X}$, $H^{(\cdot)}\colon \mathbb{R} \times \mathbb{R}^n \to \mathcal{X}$ are functions analytic at the origin with

$$\begin{aligned} F^\mu(Z, Q) &= \mathcal{O}(|Z| + \|(Z, Q)\|^2_{\mathbb{R}^n \times \mathcal{D}}), \\ G^\mu(Z, Q) &= \mathcal{O}(\|Q\|_{\mathcal{D}} \|(Z, Q)\|_{\mathbb{R}^n \times \mathcal{D}}), \\ H^\mu(Z) &= \mathcal{O}(\mu^a |Z|^b) \end{aligned}$$

for some $a \geq 0$ and $b \geq 2$. Systems of this kind arise from scalings of an evolutionary system (for $Q$) coupled to a dynamical system (for $Z$) which undergoes a change in the number of its purely imaginary eigenvalues as a bifurcation parameter $\varepsilon$ is varied through zero; the spectrum of the linear operator for $Q$ is on the other hand non-critical in the sense that it is either purely imaginary or bounded away from the imaginary axis for all values of $\varepsilon$.

This chapter is motivated by the observation that if the $Q$-independent term $H^\mu$ in equation (3.2) is not present the set $\{Q = 0\}$ is an invariant subspace of equations (3.1) and (3.2), so that the solutions of the *approximate system*

$$\dot{Z} = F^\mu(Z, 0) \tag{3.3}$$

also solve equations (3.1) and (3.2). We construct a normal-form theory consisting of a sequence of changes of variable which systematically remove the $j$th order terms of the Maclaurin expansion of $H^\mu$ with respect to $(Z, \mu)$ for all $j \in \{2, \ldots, p\}$ while preserving the overall structure of the system. In general it is not possible to remove $H^\mu$ completely but we can at least make an optimal choice of $p$ so that the remaining terms are exponentially small in comparison to $(Z, \mu)$ in a neighbourhood of the origin. In Chapter 4 we use this fact to prove that under certain circumstances homoclinic solutions of equation (3.3) approximate solutions of equations (3.1) and (3.2) with exponentially small remainder. Our analysis is based upon a theory for finite-dimensional dynamical systems given by Iooss and Lombardi [11], and we use their notation and refer to several of their combinatorial results here.

## 3.1 Banach-space valued polynomials

Before we begin the construction of our normal form we restate some useful facts about Banach-space valued polynomials of $n$ real variables. Suppose that $B$ is a Banach space.

**Definition 3.1.** Let $k \in \mathbb{N}_0$.

(i) We call an expression

$$\Psi_k(x) = \sum_{|\alpha|=k} v_\alpha x_1^{\alpha_1} \cdot \ldots \cdot x_{n+1}^{\alpha_{n+1}},$$

where $v_\alpha \in B$, a homogeneous polynomial from $\mathbb{R}^{n+1}$ to $B$ of degree $k$ and denote the space of all such polynomial functions by $\mathcal{P}_k(\mathbb{R}^{n+1}; B)$.

(ii) We call $\Psi = \sum_{j=0}^{k} \Psi_j$, where $\Psi_j \in \mathcal{P}_j(\mathbb{R}^{n+1}; B)$ and $\Psi_k \neq 0$, a polynomial from $\mathbb{R}^{n+1}$ to $B$ of degree $k$.

Additionally to the definition of homogeneous Banach-space valued polynomials given above there is an equivalent one in terms of bounded symmetric operators. Since it is quite easy to compute an abstract formula of the derivative of a given polynomial in this alternative definition it is often more convenient to use in proofs.

**Remark 3.2.** Let $k \in \mathbb{N}$.

(i) A given function $\Psi_k \colon \mathbb{R}^{n+1} \to B$ is a homogeneous polynomial of degree $k$ if and only if there exists a bounded symmetric $k$-linear operator $A_k \colon \mathbb{R}^{n+1} \to B$ satisfying

$$\Psi_k(x_1, \ldots, x_{n+1}) = A_k\Big(\{(x_1, \ldots, x_{n+1})\}^{(k)}\Big)$$

for all $(x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+1}$, so that

$$v_\alpha = \binom{k}{\alpha} A_k\Big(\{e_1\}^{(\alpha_1)}, \ldots, \{e_{n+1}\}^{(\alpha_{n+1})}\Big)$$

in the notation of Definition 3.1, where $\{e_1, \ldots, e_{n+1}\}$ is the standard basis of $\mathbb{R}^{n+1}$.

(ii) The expression

$$|\Psi_k|_2 \coloneqq \sqrt{\sum_{|\alpha|=k} \|v_\alpha\|_B^2 \alpha_1! \cdot \ldots \cdot \alpha_{n+1}!}$$

defines a norm on $\mathcal{P}_k(\mathbb{R}^{n+1}; B)$.

(iii) The derivative of the polynomial function $\Psi_k \colon \mathbb{R}^{n+1} \to B$ is given by

$$\mathrm{d}\Psi_k[x](v) = k A_k\Big(\{x\}^{(k-1)}, v\Big)$$

where $v \in \mathbb{R}^{n+1}$ and $A_k$ is defined as in (i), so that $\mathrm{d}\Psi_k \in \mathcal{P}_{k-1}(\mathbb{R}^{n+1}; \mathcal{L}(\mathbb{R}^{n+1}; B))$.

Next we introduce two additional norms for $\mathcal{P}_k(\mathbb{R}^{n+1}; B)$. In a context where we consider the polynomial function induced by $\Psi$ we use the operator norm

$$|\Psi|_{0,k} = \sup_{Y \in \mathbb{R}^{n+1}} \frac{\|\Psi(Y)\|_B}{|Y|^k}$$

and if we treat $\Psi$ like a traditional polynomial we use

$$|\Psi|_{2,k} = \frac{1}{\sqrt{k!}} |\Psi|_2.$$

The relations between the two norms for $\mathcal{P}_k(\mathbb{R}^{n+1}, B)$ are given in the following proposition (see Iooss and Lombardi [11, Lemma 2.10 and 2.11]).

**Proposition 3.3.**

(i) The estimate

$$|\Psi_k|_{0,k} \leq |\Psi_k|_{2,k} \leq \sqrt{\binom{k+2}{2}} |\Psi_k|_{0,k} \leq \sqrt{n+1} k |\Psi_k|_{0,k}$$

holds for every $\Psi_k \in \mathcal{P}_k(\mathbb{R}^{n+1}; B)$.

(ii) Let $q \in \mathbb{N}$, $i \in \{0, \dots, q\}$ and $p_1, \dots, p_{q-i} \in \mathbb{N}$ be given; additionally let the operator $R_q \in \mathcal{L}^{(q)}(\mathbb{R}^{(n+1)i} \times B^{q-i}; \tilde{B})$ be $q$-linear and $\tilde{B} \in \{B, \mathbb{R}^{n+1}\}$.

For every polynomial function $\Psi_{p_l} \in \mathcal{P}_{p_l}(\mathbb{R}^{n+1}; \tilde{B})$ and $l \in \{1, \dots, q-i\}$ the function $Y \mapsto R_q(\{Y\}^i, \Psi_{p_1}(Y), \dots, \Psi_{p_{q-i}}(Y))$ lies in $\mathcal{P}_{i+p}(\mathbb{R}^{n+1}; \tilde{B})$ with $p = p_1 + \dots + p_{q-i}$ and

$$\begin{aligned} &|R_q(\{Y\}^i, \Psi_{p_1}(Y), \dots, \Psi_{p_{q-i}}(Y))|_{2,i+p} \\ &\qquad \leq |||R_q|||_{\mathcal{L}(\mathbb{R}^{(n+1)i} \times B^{q-i}; \tilde{B})} (\sqrt{n+1})^i |\Psi_{p_1}(Y)|_{2,p_1} \cdot \ldots \cdot |\Psi_{p_{q-i}}(Y)|_{2,p_{q-i}}. \end{aligned}$$

(iii) Let $k \in \mathbb{N}$, $p \in \mathbb{N}_0$ and $\Psi_k \in \mathcal{P}_k(\mathbb{R}^{n+1}; B)$, $N_p \in \mathcal{P}_p(\mathbb{R}^{n+1}, \mathbb{R}^{n+1})$.

The polynomial function $Y \mapsto \mathrm{d}\Psi_k[Y](N_p(Y))$ lies in $\mathcal{P}_{k-1+p}(\mathbb{R}^{n+1}, B)$ and

$$|\mathrm{d}\Psi_k[Y](N_p(Y))|_{2,k-1+p} \leq \sqrt{k^2 + nk} |\Psi_k(Y)|_{2,k} |N_p(Y)|_{2,p}.$$

## 3.2 Construction of the normal-form transformation

Writing $Y = (Z, \mu)$ and appending the new equation

$$\dot{\mu} = 0,$$

we reformulate equations (3.1) and (3.2) as

$$\begin{aligned} \dot{Y} &= F(Y, Q), \\ \dot{Q} &= L^\mu Q + G(Y, Q) + H(Y), \end{aligned}$$

where

$$F(Y,Q) = F^\mu(Z,Q),$$
$$G(Y,Q) = G^\mu(Z,Q),$$
$$H(Y) = H^\mu(Z).$$

To construct our normal form we opt for an near-identity transformation

$$\tilde{Y} = Y, \quad \tilde{Q} = Q + \Phi(Y).$$

This transformation leads to the equations

$$\dot{Y} = \tilde{F}(Y,Q), \tag{3.4}$$
$$\dot{Q} = L^\mu Q + \tilde{G}(Y,Q) + \tilde{H}(Y), \tag{3.5}$$

where $\Phi$ maps $\mathbb{R}^{n+1}$ to $\mathcal{D}$ and

$$\tilde{F}(Y,Q) = \Big(F\big(Y, Q - \Phi(Y)\big) - F\big(Y, -\Phi(Y)\big), 0\Big), \tag{3.6}$$
$$\begin{aligned}\tilde{G}(Y,Q) = {} & \mathrm{d}\Phi[Y]\Big(F\big(Y, Q - \Phi(Y)\big) - F\big(Y, -\Phi(Y)\big)\Big) \\ & + G\big(Y, Q - \Phi(Y)\big) - G\big(Y, -\Phi(Y)\big),\end{aligned} \tag{3.7}$$
$$\begin{aligned}\tilde{H}(Y) = {} & -L^\mu\Phi(Y) + \mathrm{d}\Phi[Y]\Big(F\big(Y, -\Phi(Y)\big)\Big) \\ & + G\big(Y, -\Phi(Y)\big) + H(Y).\end{aligned} \tag{3.8}$$

Here we have dropped the tildes on the variables for notational simplicity.

Restricting ourselves to polynomial transformations

$$\Phi(Y) = \sum_{k=2}^{p} \Phi_k(Y)$$

preserves the analytic structure of our system and also enables us to explicitly compute the normal form up to order $p$. To calculate a suitable polynomial transformation that removes the $m$th order terms of $\tilde{H}$ we consider its linear and nonlinear (with respect to $\Phi$) parts which are given by

$$(\mathcal{L}\Phi)(Y) = L^0\Phi(Y) - \mathrm{d}\Phi[Y]\Big(\mathrm{d}_1F[0,0](Y)\Big) \tag{3.9}$$

and

$$\begin{aligned}N(Y) = {} & -(L^\mu - L^0)\Phi(Y) + G\big(Y, -\Phi(Y)\big) + H(Y) \\ & + \mathrm{d}\Phi[Y]\Big(F(Y, -\Phi) - \mathrm{d}_1F[0,0](Y)\Big).\end{aligned}$$

We construct $\Phi_k$ by successively solving the equation

$$\mathcal{L}\Phi_k = N_k \tag{3.10}$$

for $k \in \{2, \dots, p\}$, where $N_k$ denotes the part of $N$ that is homogeneous of degree $k$. This choice of $\Phi$ ensures that the Maclaurin expansion of $\tilde{H}$ does not contain any terms of order less than $p+1$ and that

$$\Phi(Y) = \mathcal{O}(\mu^a|Z|^b).$$

**Hypothesis 3.4.** We assume that $\mathcal{L}\colon \mathcal{P}_m(\mathbb{R}^{n+1};\mathcal{D}) \to \mathcal{P}_m(\mathbb{R}^{n+1};\mathcal{X})$ is invertible with operator norm $a := |||\mathcal{L}^{-1}|||$ independent of $m$.

**Remark 3.5.** Hypothesis 3.4 typically requires that $\mathrm{d}_1 F[0,0]$ is semi-simple (see Iooss and Lombardi [11]).

## 3.3 Estimates for the transformation

Before tackling the estimates of $\tilde{H}$ and $\mathrm{d}\tilde{H}$ it is necessary to derive precise estimates for $\Phi$ and the transformed nonlinearities defined in equations (3.7) and (3.8). We calculate the constants in order of magnitude explicitly and use the notation

$$F_{i,j}(Y_1,\dots,Y_i,Q_1,\dots,Q_j) = \frac{1}{(i+j)!}\,\mathrm{d}_1^i\,\mathrm{d}_2^j F[0,0](Y_1,\dots,Y_i,Q_1,\dots,Q_j)$$
$$G_{i,j}(Y_1,\dots,Y_i,Q_1,\dots,Q_j) = \frac{1}{(i+j)!}\,\mathrm{d}_1^i\,\mathrm{d}_2^j G[0,0](Y_1,\dots,Y_i,Q_1,\dots,Q_j)$$

and

$$L_i(Y_1,\dots,Y_i) = \frac{1}{i!}\,\mathrm{d}^i L[0](Y_1,\dots,Y_i),$$
$$H_i(Y_1,\dots,Y_i) = \frac{1}{i!}\,\mathrm{d}^i H[0](Y_1,\dots,Y_i).$$

We also introduce the notation

$$\Phi_{\boldsymbol{p}} := (\Phi_{p_1},\dots,\Phi_{p_q})$$

and

$$\phi_{\boldsymbol{p}} = \prod_{i=1}^{q} |\Phi_{p_i}|_{2,p_i}$$

for a polynomial $\Phi$ of degree of at most $p$ and $\boldsymbol{p} \in \mathbb{N}^q$ or $\boldsymbol{p} \in \mathbb{N}_2^q$ with $q \in \{0,\dots,p\}$, where $\mathbb{N}_2$ is the set of all natural numbers greater or equal than 2.

Using equation (3.10) and Hypothesis 3.4 yields

$$|\Phi_m|_{2,m} \le a|N_m|_{2,m}$$

for $m \in \{2,\dots,p\}$. A straightforward calculation shows

$$\begin{aligned} N_m(Y) = {}& H_m(Y) - \sum_{2\le k\le m-1} L_{m-k}(Y)\Phi_k(Y) \\ &+ \sum_{2\le q\le m}\sum_{i=0}^{q}\sum_{\substack{\boldsymbol{p}\in\mathbb{N}_2^{q-i}\\ |\boldsymbol{p}|=m-i}} G_{i,q-i}(\{Y\}^i, -\Phi_{\boldsymbol{p}}(Y)) \\ &+ \sum_{2\le k\le m-1} \mathrm{d}\Phi_k[Y]\Bigg(\sum_{2\le q\le m-k+1}\sum_{i=0}^{q}\sum_{\substack{\boldsymbol{p}\in\mathbb{N}_2^{q-i}\\ |\boldsymbol{p}|=m-k+1-i}} F_{i,q-i}(\{Y\}^i, -\Phi_{\boldsymbol{p}}(Y))\Bigg). \end{aligned} \tag{3.11}$$

Since $F$, $G$ and $H$ are analytic, there exist constants $c, \rho > 0$ such that

$$\begin{aligned}
|F_{i,j}(Y_1, \ldots, Y_i, Q_1, \ldots, Q_j)| &\leq \frac{c}{2\rho^{i+j}} |Y_1| \cdot \ldots \cdot |Y_i| \cdot \|Q_1\|_{\mathcal{D}} \cdot \ldots \cdot \|Q_j\|_{\mathcal{D}}, \\
\|G_{i,j}(Y_1, \ldots, Y_i, Q_1, \ldots, Q_j)\|_{\mathcal{X}} &\leq \frac{c}{2\rho^{i+j}} |Y_1| \cdot \ldots \cdot |Y_i| \cdot \|Q_1\|_{\mathcal{D}} \cdot \ldots \cdot \|Q_j\|_{\mathcal{D}}, \\
\|L_i(Y)\|_{\mathcal{X}_{\mathrm{c,sh}}} &\leq \frac{c}{2\rho^i} |\mu|^i, \\
\|H_i(Y_1, \ldots, Y_i)\|_{\mathcal{X}} &\leq \frac{c}{2\rho^i} |Y_1| \cdot \ldots \cdot |Y_i|.
\end{aligned}$$

Using Proposition 3.3 yields

$$\begin{aligned}
|N_m|_{2,m} \leq{}& \frac{c}{\rho^m} (\sqrt{n+1})^m + \frac{1}{2} \sum_{2 \leq k \leq m-1} \frac{c}{\rho^{m-k}} (\sqrt{n+1})^{m-k} \phi_k \\
&+ \frac{1}{2} \sum_{2 \leq q \leq m} \sum_{i=0}^{q} \sum_{\substack{\boldsymbol{p} \in \mathbb{N}_2^{q-i} \\ |\boldsymbol{p}| = m-i}} \frac{c}{\rho^q} (\sqrt{n+1})^i \phi_{\boldsymbol{p}} \\
&+ \sum_{2 \leq k \leq m-1} \sqrt{k^2 + nk} \phi_k \sum_{2 \leq q \leq m-k+1} \sum_{i=0}^{q} \sum_{\substack{\boldsymbol{p} \in \mathbb{N}_2^{q-i} \\ |\boldsymbol{p}| = m-k+1-i}} \frac{c}{\rho^q} (\sqrt{n+1})^i \phi_{\boldsymbol{p}} \\
\leq{}& \frac{c}{\rho^m} (\sqrt{n+1})^m + \sum_{2 \leq q \leq m} \sum_{i=0}^{q} \sum_{\substack{\boldsymbol{p} \in \mathbb{N}_2^{q-i} \\ |\boldsymbol{p}| = m-i}} \frac{c}{\rho^q} (\sqrt{n+1})^i \phi_{\boldsymbol{p}} \\
&+ \sum_{2 \leq k \leq m-1} \sqrt{k^2 + nk} \phi_k \sum_{2 \leq q \leq m-k+1} \sum_{i=0}^{q} \sum_{\substack{\boldsymbol{p} \in \mathbb{N}_2^{q-i} \\ |\boldsymbol{p}| = m-k+1-i}} \frac{c}{\rho^q} (\sqrt{n+1})^i \phi_{\boldsymbol{p}},
\end{aligned}$$

from which we construct a recursively defined sequence $\{\beta_m\}_{m=1}^{\infty} \subseteq \mathbb{R}$ bounding $\phi_m$.

**Proposition 3.6.** Consider the sequence $\{\beta_m\}_{m=1}^{\infty} \subseteq \mathbb{R}$ defined recursively by

$$\begin{aligned}
\beta_1 &= 1, \\
\beta_m &= \left(\frac{\rho}{ac}\right)^{m-2} \sum_{2 \leq q \leq m} \sum_{i=0}^{q} \sum_{\substack{\boldsymbol{p} \in \mathbb{N}_2^{q-i} \\ |\boldsymbol{p}| = m-i}} \left(\frac{\rho}{ac}\right)^{q-2} \beta_{\boldsymbol{p}} \\
&\quad + n \sum_{2 \leq k \leq m-1} k \beta_k \sum_{2 \leq q \leq m-k+1} \sum_{i=0}^{q} \sum_{\substack{\boldsymbol{p} \in \mathbb{N}_2^{q-i} \\ |\boldsymbol{p}| = m-k+1-i}} \left(\frac{\rho}{ac}\right)^{q-2} \beta_{\boldsymbol{p}}
\end{aligned}$$

for $m \geq 2$, where we have used the notation

$$\beta_{\boldsymbol{p}} = \prod_{j=1}^{q} \beta_{p_j}.$$

The estimate

$$\phi_m \leq \sqrt{n+1} \left(\frac{\sqrt{n+1}ac}{\rho^2}\right)^{m-1} \beta_m \tag{3.12}$$

holds for all $m \geq 2$.

*Proof.* We establish this result by mathematical induction. For $m = 2$ we have that $\beta_2 = 2$ and

$$\begin{aligned}
\phi_2 &\le a|N_2|_{2,2} \\
&\le a\frac{c}{\rho^2}(\sqrt{n+1})^2 + a\sum_{i=0}^{2}\sum_{\substack{\boldsymbol{p}\in\mathbb{N}_2^{q-i}\\|\boldsymbol{p}|=2-i}}\frac{c}{\rho^2}(\sqrt{n+1})^i\phi_{\boldsymbol{p}} \\
&= 2\sqrt{n+1}\Big(\frac{ac}{\rho^2}\sqrt{n+1}\Big) \\
&= \sqrt{n+1}\Big(\frac{ac}{\rho^2}\sqrt{n+1}\Big)\beta_2.
\end{aligned}$$

Now we assume that the result holds for all $k < m$ with $m \ge 3$. We find

$$\begin{aligned}
\phi_m \le{}& \frac{ac}{\rho^m}(\sqrt{n+1})^m + \sum_{2\le q\le m}\sum_{i=0}^{q}\sum_{\substack{\boldsymbol{p}\in\mathbb{N}_2^{q-i}\\|\boldsymbol{p}|=m-i}}\frac{ac}{\rho^q}(\sqrt{n+1})^q\Big(\frac{ac\sqrt{n+1}}{\rho^q}\Big)^{m-q}\beta_{\boldsymbol{p}} \\
&+ \sum_{2\le k\le m-1}\sqrt{k^2+nk}\sqrt{n+1}\Big(\frac{ac\sqrt{n+1}}{\rho^2}\Big)^{k-1}\beta_k \\
&\times \sum_{2\le q\le m-k+1}\sum_{i=0}^{q}\sum_{\substack{\boldsymbol{p}\in\mathbb{N}_2^{q-i}\\|\boldsymbol{p}|=m-k+1-i}}\Big(\frac{ac}{\rho^q}(\sqrt{n+1})^q\Big(\frac{ac\sqrt{n+1}}{\rho^q}\Big)^{m-k-q+1}\beta_{\boldsymbol{p}}\Big) \\
\le{}& \sqrt{n+1}\Big(\frac{ac\sqrt{n+1}}{\rho^2}\Big)^{m-1}\Big(\frac{ac}{\rho^m}\Big(\frac{\rho^2}{ac}\Big)^{m-1} + \sum_{2\le q\le m}\sum_{i=0}^{q}\sum_{\substack{\boldsymbol{p}\in\mathbb{N}_2^{q-i}\\|\boldsymbol{p}|=m-i}}\frac{ac}{\rho^q}\Big(\frac{ac}{\rho^2}\Big)^{1-q}\beta_{\boldsymbol{p}} \\
&+ \sum_{2\le k\le m-1}nk\beta_k\sum_{2\le q\le m-k+1}\sum_{i=0}^{q}\sum_{\substack{\boldsymbol{p}\in\mathbb{N}_2^{q-i}\\|\boldsymbol{p}|=m-k+1-i}}\frac{ac}{\rho^q}\Big(\frac{ac}{\rho^2}\Big)^{1-q}\beta_{\boldsymbol{p}}\Big) \\
={}& \sqrt{n+1}\Big(\frac{ac\sqrt{n+1}}{\rho^2}\Big)^{m-1}\beta_m
\end{aligned}$$

using the facts

$$\frac{ac}{\rho^q}\Big(\frac{ac}{\rho^2}\Big)^{1-q} = \Big(\frac{\rho}{ac}\Big)^{q-2}, \quad \frac{ac}{\rho^m}\Big(\frac{\rho^2}{ac}\Big)^{m-1} = \Big(\frac{\rho}{ac}\Big)^{m-2}.$$

□

**Remark 3.7.** Because of $\beta_1 = 1$ and

$$\sum_{i=0}^{q}\sum_{\substack{\boldsymbol{p}\in\mathbb{N}_2^{q-i}\\|\boldsymbol{p}|=m-i}}\beta_{\boldsymbol{p}} = \sum_{\substack{\boldsymbol{p}\in\mathbb{N}^q\\|\boldsymbol{p}|=m}}\beta_{\boldsymbol{p}}$$

we can write

$$\begin{aligned}
\beta_m ={}& \Big(\frac{\rho}{ac}\Big)^{m-2} + \sum_{2\le q\le m}\sum_{\substack{\boldsymbol{p}\in\mathbb{N}^q\\|\boldsymbol{p}|=m}}\Big(\frac{\rho}{ac}\Big)^{q-2}\beta_{\boldsymbol{p}} \\
&+ n\sum_{2\le k\le m-1}k\beta_k\sum_{2\le q\le m-k+1}\sum_{\substack{\boldsymbol{p}\in\mathbb{N}^q\\|\boldsymbol{p}|=m-k+1}}\Big(\frac{\rho}{ac}\Big)^{q-2}\beta_{\boldsymbol{p}} \qquad (3.13)
\end{aligned}$$

for all $m \geq 2$.

Finally, we construct a non recursively defined sequence $\{\alpha_m\}_{m=1}^{\infty} \subseteq \mathbb{R}$ to estimate $\{\beta_m\}_{m=1}^{\infty}$.

**Proposition 3.8.** We define

$$\alpha_m = \theta^{m-2}(m-2)!$$

for $m \geq 2$ and $\alpha_1 = 1$, where $\theta$ is chosen large enough to satisfy $\rho/(ac\theta) < 1/n$ and

$$\frac{1}{2} + \frac{5(n+1)}{2\theta} + \frac{2}{\theta} + \frac{10(n+1)}{\theta} + \frac{1}{8(n+1)} \leq 1.$$

The estimate

$$\beta_m \leq 2^m \alpha_m \tag{3.14}$$

is satisfied for all $m \in \mathbb{N}$.

*Proof.* We prove the statement by mathematical induction. First we note that

$$\beta_1 = 1 = \alpha_1 = \alpha_2$$

and

$$\beta_2 = 2 \leq 4\alpha_2.$$

Now we assume the estimate $\beta_k \leq 2^k \alpha_k$ to hold true for all $k < m$, where $m \geq 3$. Combining the inductive hypothesis with equation (3.13) we find that

$$\beta_m \leq 2^m \left(\Delta_m^1 + \Delta_m^2 + \Delta_m^3 + \Delta_m^4\right) + \left(\frac{\rho}{ac}\right)^{m-2},$$

where

$$\begin{aligned}
\Delta_m^1 &= \sum_{3 \leq q \leq m} \sum_{\substack{\boldsymbol{p} \in \mathbb{N}^q \\ |\boldsymbol{p}| = m}} \left(\frac{\rho}{ac}\right)^{q-2} \alpha_{\boldsymbol{p}}, \\
\Delta_m^2 &= 2(n+1) \sum_{2 \leq k \leq m-1} k\alpha_k \sum_{3 \leq q \leq m-k+1} \sum_{\substack{\boldsymbol{p} \in \mathbb{N}^q \\ |\boldsymbol{p}| = m-k+1}} \left(\frac{\rho}{ac}\right)^{q-2} \alpha_{\boldsymbol{p}}, \\
\Delta_m^3 &= \sum_{1 \leq k \leq m-1} \alpha_k \alpha_{m-k}, \\
\Delta_m^4 &= 2(n+1) \sum_{2 \leq k \leq m-1} k\alpha_k \sum_{1 \leq j \leq m-k} \alpha_j \alpha_{m-k+1-j},
\end{aligned}$$

where we have used the notation

$$\alpha_{\boldsymbol{p}} = \prod_{j=1}^{q} \alpha_{p_j}$$

for $\boldsymbol{p} \in \mathbb{N}^q$.

Using the same combinatorial arguments as Iooss and Lombardi [11, Steps 4 and 5 in the proof of lEmma 2.13 ] we obtain that

$$\Delta_m^1 \leq \sum_{3\leq q\leq m} 2\Big(\frac{\rho}{ac\theta}\Big)^{q-2}\alpha_m \leq \sum_{3\leq q\leq m} 2\Big(\frac{1}{n+1}\Big)^{q-2}\alpha_m \leq \frac{\alpha_m}{2}.$$

The above estimate in conjunction with Iooss and Lombardi [11, Lemma 2.13 step 2] yields

$$\begin{aligned}\Delta_m^2 &\leq n \sum_{2\leq k\leq m-1} k\alpha_k\alpha_{m-k+1} \\ &= n \sum_{2\leq k\leq m-1} k\theta^{m-3}(k-2)!(m-k-1)! \\ &= \frac{n+1}{\theta}\alpha_m \sum_{2\leq k\leq m-1} \frac{k(k-2)!(m-k-1)!}{(m-2)!} \\ &\leq \frac{5(n+1)}{2\theta}\alpha_m.\end{aligned}$$

In the same fashion we estimate

$$\Delta_m^3 \leq \frac{2}{\theta}\alpha_m$$

and

$$\Delta_m^4 \leq \alpha_m \frac{4(n+1)}{\theta} \sum_{2\leq k\leq m-1} \frac{k(k-2)!(m-k-1)!}{(m-2)!} \leq \frac{10(n+1)}{\theta}\alpha_m.$$

Finally we find

$$\Big(\frac{\rho}{ac}\Big)^{m-2} = \Big(\frac{\rho}{ac\theta}\Big)^{m-2}\theta^{m-2} \leq \frac{1}{n+1}\theta^{m-2} \leq \frac{2^m}{8(n+1)}\alpha_m,$$

since $n \cdot 2^m \geq 8(n+1)$ for $m \geq 3$.

Altogether we find

$$\beta_m \leq \Big(\frac{1}{2} + \frac{5(n+1)}{2\theta} + \frac{2}{\theta} + \frac{10(n+1)}{\theta} + \frac{1}{8(n+1)}\Big)2^m\alpha_m \leq 2^m\alpha_m.$$

□

The preceding proposition implies that

$$\phi_m \leq 2\sqrt{n+1}\Big(\frac{2\sqrt{n+1}ac}{\rho^2}\Big)^{m-1}\theta^{m-2}(m-2)! = \frac{4(n+1)ac}{\rho^2}\Big(\frac{2\sqrt{n+1}ac\theta}{\rho^2}\Big)^{m-2}(m-2)! \tag{3.15}$$

holds for $m \geq 2$ and by imposing a mutual constraint on the order $p$ of the normal-form and $\delta$, we can use estimate (3.15) to obtain an estimate of the supremum norm of $\Phi$ and of the operator norms of $\mathrm{d}\Phi$ and $\mathrm{d}^2\Phi$.

**Proposition 3.9.** Let $B$ be a Banach space and $\Phi_k\colon \mathbb{R}^{n+1} \to B$ a homogeneous polynomial of degree $k$. The derivative of $\Phi_k$ satisfies the estimate

$$|\mathrm{d}\Phi_k|_2 \leq \sqrt{(n+1)k}|\Phi_k|_2.$$

*Proof.* By Remark 3.2 there exists a symmetric $k$-linear operator $A_k\colon \mathbb{R}^{(n+1)k} \to B$ such that

$$\Phi_k(Y) = A_k(\{Y\}^{(k)})$$

and the derivative of $\Phi_k$ is given by

$$\mathrm{d}\Phi_k[Y] = kA_k(\{Y\}^{(k-1)}, \cdot),$$

so that

$$\begin{aligned}
|\mathrm{d}\Phi_k|_2^2 &= k^2 \sum_{|\alpha|=k-1} \alpha_1! \cdot \ldots \cdot \alpha_{n+1}! \binom{k-1}{\alpha}^2 \|A_k(\{e_1\}^{(\alpha_1)}, \ldots, \{e_{n+1}\}^{(\alpha_{n+1})}, \cdot)\|^2_{\mathcal{L}(\mathbb{R}^{n+1};B)} \\
&= k^2 \sum_{|\alpha|=k-1} \alpha_1! \cdot \ldots \cdot \alpha_{n+1}! \binom{k-1}{\alpha}^2 \sup_{|\tilde{Y}|=1} \|A_k(\{e_1\}^{(\alpha_1)}, \ldots, \{e_{n+1}\}^{(\alpha_{n+1})}, \tilde{Y})\|^2_B \\
&\leq (n+1)k \sum_{|\alpha|=k} \alpha_1! \cdot \ldots \cdot \alpha_{n+1}! \binom{k}{\alpha}^2 \|A_k(\{e_1\}^{(\alpha_1)}, \ldots, \{e_{n+1}\}^{(\alpha_{n+1})})\|^2_B \\
&= (n+1)k|\Phi_k|_2^2.
\end{aligned}$$

□

**Proposition 3.10.** We suppose $\delta > 0$ and $p$ to satisfy

$$\delta p \leq \frac{\rho^2}{4\sqrt{n+1}ac\theta} \tag{3.16}$$

and find the estimates

$$\begin{aligned}
\Big\| \sum_{2\leq k\leq p} \Phi_k(Y) \Big\|_{\mathcal{D}} &\leq \frac{\sqrt{n+1}\delta}{\theta}, \\
\Big\| \sum_{2\leq k\leq p} \mathrm{d}_2\Phi_k[Y] \Big\|_{\mathcal{L}(\mathbb{R}^{n+1};\mathcal{D})} &\leq \frac{2(n+1)}{\theta}, \\
\Big\| \sum_{2\leq k\leq p} \mathrm{d}_2^2\Phi_k[Y] \Big\|_{\mathcal{L}^{(2)}(\mathbb{R}^{2(n+1)};\mathcal{D})} &\leq \frac{2(n+1)^{\frac{3}{2}}}{\theta\delta},
\end{aligned}$$

to hold for $|Y| \leq \delta$.

*Proof.* Using Proposition 3.3 and estimate (3.15), we find that

$$\begin{aligned}
\Big\| \sum_{2\leq k\leq p} \Phi_k(Y) \Big\|_{\mathcal{D}} &\leq \sum_{2\leq k\leq p} |\Phi_k|_{0,k} |Y|^k \\
&\leq \sum_{2\leq k\leq p} \phi_k \delta^k \\
&\leq \sum_{2\leq k\leq p} \frac{4(n+1)ac\delta^2}{\rho^2} \Big(\frac{2\sqrt{n+1}ac\theta\delta}{\rho^2}\Big)^{k-2} (k-2)! \\
&= \frac{2\sqrt{n+1}\delta}{\theta} \sum_{2\leq k\leq p} \Big(\frac{2\sqrt{n+1}ac\theta\delta}{\rho^2}\Big)^{k-1} (k-2)!
\end{aligned}$$

$$\leq \frac{2\sqrt{n+1}\delta}{\theta} \sum_{2\leq k\leq p} \left(\frac{1}{2p}\right)^{k-1}(k-2)!$$
$$\leq \frac{\sqrt{n+1}\delta}{\theta},$$

where we have used $(k-2)!/p^{k-1} \leq 1/p$ for $2 \leq k \leq p$.

The remaining estimates are obtained by similar calculations combined with Proposition 3.9. □

## 3.4 Estimates for the transformed coupling term

Now we are in a position to construct an estimate for $\tilde{H}(Y)$ by computing an explicit expression for it. Assuming the normal form has been constructed up to order $p$ we find that

$$\begin{aligned}
\tilde{H}(Y) = &\sum_{p+1\leq q} H_q(\{Y\}^{(q)}) - \sum_{\substack{2\leq k\leq p\\ p+1\leq q}} L_{q-k}(\Phi_k(Y)) \qquad (3.17)\\
&+ \sum_{2\leq q}\sum_{i=0}^{q} \sum_{\substack{\boldsymbol{p}\in\mathbb{N}_2^{q-i}\\ |\boldsymbol{p}|\geq p+m-i}} G_{i,q-i}(\{Y\}^{(i)}, -\Phi_{\boldsymbol{p}}(Y))\\
&+ \sum_{k_1+k_2\geq p+2} \mathrm{d}\Phi_{k_1}[Y]\Bigg(\sum_{2\leq q\leq k_2}\sum_{i=0}^{q}\sum_{\substack{\boldsymbol{p}\in\mathbb{N}_2^{q-i}\\ |\boldsymbol{p}|=k_2-i}} F_{i,q-i}(\{Y\}^{(i)}, -\Phi_{\boldsymbol{p}}(Y))\Bigg)\\
= &\sum_{p+1\leq q} H_q(\{Y\}^{(q)}) - \sum_{\substack{2\leq q\leq p\\ p+1\leq m}} L_q(\Phi_{m-q}(Y)) - \sum_{p+1\leq q} L_q(Y)\Big(\sum_{2\leq k\leq p}\Phi_k(Y)\Big)\\
&+ \sum_{2\leq q\leq p}\sum_{i=0}^{q}\sum_{\substack{\boldsymbol{p}\in\mathbb{N}_2^{q-i}\\ |\boldsymbol{p}|\geq p-i}} G_{i,q-i}(\{Y\}^{(i)}, -\Phi_{\boldsymbol{p}}(Y))\\
&+ \sum_{p+1\leq q}\sum_{i=0}^{q} G_{i,q-i}\Bigg(\{Y\}^{(i)}, \Big\{\sum_{2\leq k\leq p}\Phi_k(Y)\Big\}^{(q-i)}\Bigg)\\
&+ \Big(\sum_{k=2}^{p}\mathrm{d}\Phi_k[Y]\Big)\Bigg(\sum_{2\leq q\leq p}\sum_{i=0}^{q}\sum_{\substack{\boldsymbol{p}\in\mathbb{N}_2^{q-i}\\ |\boldsymbol{p}|\geq p-i}} F_{i,q-i}(\{Y\}^{(i)}, -\Phi_{\boldsymbol{p}}(Y))\\
&\qquad + \sum_{p+1\leq q}\sum_{i=0}^{q} F_{i,q-i}\Bigg(\{Y\}^{(i)}, \Big\{\sum_{2\leq k\leq p}\Phi_k(Y)\Big\}^{(q-i)}\Bigg)\Bigg)\\
&+ \sum_{\substack{2\leq k\leq p\\ p+1\leq m\leq p+k-1}} \mathrm{d}\Phi_k[Y]\Bigg(\sum_{2\leq q\leq m-k+1}\sum_{i=0}^{q}\sum_{\substack{\boldsymbol{p}\in\mathbb{N}_2^{q-i}\\ |\boldsymbol{p}|=m-k+1-i}} F_{i,q-i}(\{Y\}^{(i)}, -\Phi_{\boldsymbol{p}}(Y))\Bigg). \qquad (3.18)
\end{aligned}$$

In the neighbourhood $B_\delta$ of the origin, where $p$ and $\delta$ satisfy

$$\delta p \le \frac{\rho^2}{4\sqrt{n+1}ac\theta},$$

Proposition 3.10 yields

$$\begin{aligned}
\|\tilde{H}(Y)\|_{\mathcal{X}} \le{}& \frac{1}{2}\sum_{p+1\le q}\frac{c}{\rho^q}(\sqrt{n+1})^q\delta^q + \frac{1}{2}\sum_{\substack{2\le q\le p\\ p+1\le m}}\frac{c}{\rho^q}(\sqrt{n+1})^q\delta^m\phi_{m-q}\\
&+\frac{1}{2}\sum_{p+1\le q}\frac{c}{\rho^q}\frac{\sqrt{n+1}}{\theta}\delta^{q+1}\\
&+\Big(\frac{1}{2}+\frac{2(n+1)}{\theta}\Big)\sum_{\substack{2\le q\le p\\ p+1\le m}}\sum_{i=0}^{q}\sum_{\substack{\boldsymbol{p}\in\mathbb{N}_2^{q-i}\\ |\boldsymbol{p}|=m-i}}\frac{c\delta^m}{\rho^q}(\sqrt{n+1})^i\phi_{\boldsymbol{p}}\\
&+\Big(1+\frac{2(n+1)}{\theta}\Big)\sum_{p+1\le q}\sum_{i=0}^{q}\frac{c}{\rho^q}(\sqrt{n+1}\delta)^i\Big(\frac{\sqrt{n+1}\delta}{\theta}\Big)^{q-i}\\
&+\sum_{\substack{2\le k\le p\\ p+1\le m\le p+k-1}}\sqrt{n+1}k\phi_k\sum_{2\le q\le m-k+1}\sum_{i=0}^{q}\sum_{\substack{\boldsymbol{p}\in\mathbb{N}_2^{q-i}\\ |\boldsymbol{p}|=m-k+1-i}}\frac{c}{\rho^q}(\sqrt{n+1})^i\phi_{\boldsymbol{p}}\delta^m\\
\le{}& \Big(2+\frac{2(n+1)}{\theta}\Big)\sum_{p+1\le q}\frac{c}{\rho^q}(\sqrt{n+1})^q\delta^q(q+1)\\
&+\Big(1+\frac{2(n+1)}{\theta}\Big)\sum_{\substack{2\le q\le p\\ p+1\le m}}\sum_{\substack{\boldsymbol{p}\in\{1,\dots,p\}^q\\ |\boldsymbol{p}|=m}}\frac{c\delta^m}{\rho^q}\phi_{\boldsymbol{p}}\\
&+\sum_{\substack{2\le k\le p\\ p+1\le m\le p+k-1}}\sqrt{n+1}k\phi_k\sum_{2\le q\le m-k+1}\sum_{\substack{\boldsymbol{p}\in\mathbb{N}^q\\ |\boldsymbol{p}|=m-k+1}}\frac{c}{\rho^q}\phi_{\boldsymbol{p}}\delta^m,
\end{aligned}$$

where we have chosen $\theta \ge 1$ and defined $\phi_1 = \sqrt{n+1}$. From the above inequalities we now conclude an estimate independent of $\Phi$.

**Proposition 3.11.** Suppose that

$$\delta p \le \frac{\rho^2}{4\sqrt{n+1}eac\theta},$$

together with $\theta \ge 1$ and

$$r = \frac{\rho}{ac\theta} < \frac{1}{n+1}.$$

The estimate

$$\|\tilde{H}(Y)\|_{\mathcal{X}} \le \frac{2(n+1)+1}{n+1}c\Big((C\delta)^{p+1}p! + \frac{1}{e^{p+1}p^2}\Big),$$

holds for all $Y \in \mathbb{R}^{n+1}$ with $|Y| \le \delta$, where

$$C = \frac{4\sqrt{n+1}ac\theta}{\rho^2}.$$

*Proof.* We first note that the estimate

$$\phi_m \le \frac{4(n+1)ac}{\rho^2}\Big(\frac{2\sqrt{n+1}ac}{\rho^2}\Big)^{m-2}\alpha_m$$

continues to hold for $m=1$ since

$$\frac{4(n+1)ac}{\rho^2}\Big(\frac{2\sqrt{n+1}ac}{\rho^2}\Big)^{-1}\alpha_1 = 2\sqrt{n+1}.$$

Furthermore, since $\theta \ge 1$ we have that

$$\alpha_m \le \theta^{m-1}(m-2)!,$$

for $m \ge 2$, and the estimate also continues to hold for $m=1$, since $(-1)!=1$ and $\alpha_1 = 1$.

Now we define

$$\Delta_p^1 = \Big(2+\frac{2(n+1)}{\theta}\Big)\sum_{p+1\le q}\frac{c}{\rho^q}(\sqrt{n+1})^q\delta^q(q+1),$$

$$\Delta_p^2 = \Big(1+\frac{2(n+1)}{\theta}\Big)\sum_{\substack{2\le q\le p\\ p+1\le n}}\sum_{\substack{\boldsymbol{p}\in\{1,\dots,p\}^q\\ |\boldsymbol{p}|=n+1}}\frac{c\delta^{n+1}}{\rho^q}\phi_{\boldsymbol{p}}\delta^{n+1},$$

$$\Delta_p^3 = \sum_{\substack{2\le k\le p\\ p\le n\le p+k-2}}\sqrt{n+1}k\phi_k\sum_{2\le q\le n-k+2}\sum_{\substack{\boldsymbol{p}\in\{1,\dots,p\}^q\\ |\boldsymbol{p}|=n-k+2}}\frac{c}{\rho^q}\phi_{\boldsymbol{p}}\delta^{n+1}.$$

We note that

$$\frac{2\sqrt{n+1}\delta}{\rho} \le \frac{\rho}{2acp\theta} \le \frac{1}{4(n+1)}$$

for $p \ge 2$, so that

$$\begin{aligned}\sum_{p+1\le q}\frac{c}{\rho^q}(q+1)(\sqrt{n+1})^q\delta^q &\le c\Big(\frac{2\sqrt{n+1}\delta}{\rho}\Big)^{p+1}\sum_{q=0}^{\infty}\Big(\frac{1}{4(n+1)}\Big)^q\\ &\le \frac{4(n+1)c}{4(n+1)-1}\Big(\frac{2\sqrt{n+1}\delta}{\rho}\Big)^{p+1},\end{aligned}$$

and hence

$$\Delta_p^1 \le \frac{(2+2(n+1))4(n+1)c}{4(n+1)-1}\Big(\frac{2\sqrt{n+1}\delta}{\rho}\Big)^{p+1}.$$

Next we find

$$\begin{aligned}&\sum_{2\le q\le p}\sum_{p+1\le m}\sum_{\substack{\boldsymbol{p}\in\{1,\dots,p\}^q\\ |\boldsymbol{p}|=m}}\frac{c\delta^m}{\rho^q}\phi_{\boldsymbol{p}}\\ &\quad\le \sum_{2\le q\le p}\sum_{p+1\le m}\sum_{\substack{\boldsymbol{p}\in\{1,\dots,p\}^q\\ |\boldsymbol{p}|=m}}\frac{c\delta^m}{\rho^q}\Big(\frac{4(n+1)ac}{\rho^2}\Big)^q\Big(\frac{2\sqrt{n+1}ac}{\rho^2}\Big)^{m-2q}\alpha_{\boldsymbol{p}}\\ &\quad\le c\sum_{2\le q\le p}r^q\sum_{p+1\le m}\sum_{\substack{\boldsymbol{p}\in\{1,\dots,p\}^q\\ |\boldsymbol{p}|=m}}(c\delta)^m(p_1-2)!\cdot\ldots\cdot(p_q-2)!\end{aligned}$$

$$
\begin{aligned}
&\leq c\Big(\frac{1}{\mathrm{e}}\Big)^{p+1}\sum_{2\leq q\leq p} r^q \sum_{p+1\leq m}\sum_{\substack{\boldsymbol{p}\in\{1,\dots,p\}^q\\ |\boldsymbol{p}|=m}} \Big(\frac{1}{p}\Big)^m (p_1-2)!\cdot\ldots\cdot(p_q-2)!\\
&\leq c\Big(\frac{1}{\mathrm{e}}\Big)^{p+1}\sum_{2\leq q\leq p} r^q \Big(\sum_{j=1}^{p}\Big(\frac{1}{p}\Big)^j (j-2)!\Big)^q\\
&\leq c\Big(\frac{1}{\mathrm{e}}\Big)^{p+1}\sum_{2\leq q\leq p} r^q \Big(\frac{1}{p}+\frac{p-1}{p^2}\Big)^q\\
&\leq c\Big(\frac{1}{\mathrm{e}}\Big)^{p+1}\sum_{2\leq q\leq p} \Big(\frac{2r}{p}\Big)^q\\
&\leq c\Big(\frac{1}{\mathrm{e}}\Big)^{p+1}\frac{4r^2}{p^2}\frac{1}{1-\frac{2r}{p}}
\end{aligned}
$$

since $2/p \leq 1$. Hence

$$
\Delta_p^2 \leq \frac{(1+2(n+1)\theta)4cr^2}{e^{p+1}}\cdot\frac{1}{p^2}\cdot\frac{1}{1-r}.
$$

Finally we obtain

$$
\begin{aligned}
\Delta_p^3 &\leq \frac{1}{a}\sum_{\substack{2\leq k\leq p\\ p+1\leq n\leq p+k-1}} \sqrt{n+1}k\frac{4(n+1)ac}{\rho^2}\Big(\frac{2\sqrt{n+1}ac}{\rho^2}\Big)^{k-2}\alpha_k\\
&\quad\times\sum_{2\leq q\leq n-k+2}\sum_{\substack{\boldsymbol{p}\in\mathbb{N}^q\\ |\boldsymbol{p}|=n-k+2}}\frac{ac}{\rho^q}\bigg(\Big(\frac{4(n+1)ac}{\rho^2}\Big)^q\Big(\frac{2\sqrt{n+1}ac}{\rho^2}\Big)^{n-k+2-2q}\alpha_{\boldsymbol{p}}\delta^{n+1}\bigg)\\
&=\frac{1}{a}\sum_{\substack{2\leq k\leq p\\ p+1\leq n\leq p+k-1}} \sqrt{n+1}k\frac{4(n+1)ac}{\rho^2}\Big(\frac{2\sqrt{n+1}ac}{\rho^2}\Big)^{n-2}\alpha_k\\
&\quad\times\sum_{2\leq q\leq n-k+2}\sum_{\substack{\boldsymbol{p}\in\mathbb{N}^q\\ |\boldsymbol{p}|=n-k+2}}\frac{ac}{\rho^q}\bigg(\Big(\frac{4(n+1)ac}{\rho^2}\Big)^q\Big(\frac{2\sqrt{n+1}ac}{\rho^2}\Big)^{1-2q}\alpha_{\boldsymbol{p}}\delta^{n+1}\bigg)\\
&=\frac{1}{a}\sum_{\substack{2\leq k\leq p\\ p+1\leq n\leq p+k-1}}\bigg(4(n+1)\sqrt{n+1}\Big(\frac{2\sqrt{n+1}ac}{\rho^2}\Big)^{n-1}k\alpha_k\\
&\quad\times\underbrace{\sum_{2\leq q\leq n-k+2}\sum_{\substack{\boldsymbol{p}\in\mathbb{N}^q\\ |\boldsymbol{p}|=n-k+2}}\Big(\frac{\rho}{ac}\Big)^{q-2}\alpha_{\boldsymbol{p}}\delta^{n+1}\bigg)}_{\leq\frac{\frac{2\rho}{ac\theta}}{1-\frac{\rho}{ac\theta}}\alpha_{n-k+2}}\\
&\leq\frac{2(n+1)^{\frac{3}{2}}}{a}\sum_{\substack{2\leq k\leq p\\ p+1\leq n\leq p+k-1}}\delta^{n+1}\Big(\frac{2\sqrt{n+1}ac}{\rho^2}\Big)^{n-1}\theta^{n-3}k(k-2)!(n-k-1)!\\
&\leq\frac{3\rho^2}{a^2c\theta^3}\sum_{2\leq k\leq p}k(k-2)!(C\delta)^{p+1}\sum_{p\leq n\leq p+k-2}(C\delta)^{n-p-1}(n-k-1)!\\
&\leq\frac{3\rho^2}{a^2c\theta^3}\sum_{2\leq k\leq p}k(k-2)!(C\delta)^{p+1}\sum_{p\leq n\leq p+k-2}\Big(\frac{1}{2}\Big)^{n-p-1}\frac{(n-k-1)!}{p^{n-p-1}},
\end{aligned}
$$

where we have used the fact $C\delta \le 1/(\mathrm{e}p) \le 1/(2p)$. Now we observe that

$$\frac{(m-k-1)!}{p^{m-p-1}} \le (p-k)!$$

holds for $p+1 \le m \le p+k-1$, and thus we obtain

$$\begin{aligned}
\Delta_p^3 &\le \frac{n\rho^2}{a^2 c\theta^3} \sum_{2\le k\le p} k(k-2)!(C\delta)^{p+1} 2(p-k)! \\
&\le \frac{2(n+1)\rho^2}{a^2 c\theta^3}(C\delta)^{p+1} p! \sum_{2\le k\le p} \frac{1}{\binom{p}{k}(k-1)} \\
&\le \frac{2(n+1)\rho^2}{a^2 c\theta^3}(C\delta)^{p+1} p! \sum_{2\le k\le p} \frac{1}{p-1} \\
&= \frac{2(n+1)\rho^2}{a^2 c\theta^3}(C\delta)^{p+1} p!.
\end{aligned}$$

Altogether we have

$$\begin{aligned}
\|\tilde{H}(Y)\|_{\mathcal{X}} &\le \frac{(2+2(n+1))4(n+1)c}{4(n+1)-1}\Big(\frac{2\sqrt{n+1}\delta}{\rho}\Big)^{p+1} + \frac{(1+2(n+1))4cr^2}{\mathrm{e}^{p+1}} \\
&\quad + \frac{2(n+1)\rho^2}{a^2 c\theta^3}(C\delta)^{p+1} p! \\
&= \frac{(2+2(n+1))4(n+1)c}{4(n+1)-1} C(rC\delta)^{p+1} + \frac{(1+2(n+1))4cr^2}{\mathrm{e}^{p+1}} \cdot \frac{1}{p^2} \cdot \frac{1}{1-r} \\
&\quad + \frac{2(n+1)cr^2}{\theta}(C\delta)^{p+1} p! \\
&\le \frac{2(n+1)+1}{n+1} c\Big((C\delta)^{p+1} p! + \frac{1}{\mathrm{e}^{p+1} p^2}\Big).
\end{aligned}$$

□

Next we use Iooss and Lombardi [11, Lemma 2.19] to determine a sufficient order up to which the normal form of $H$ has to be constructed so that it is exponentially small.

**Lemma 3.12.** Let $\varepsilon > 0$ be given. We define $f_\varepsilon \colon \mathbb{Z} \to \mathbb{R}$ by $f_\varepsilon(p) = \varepsilon^{p+1} p!$ and extend $f_\varepsilon$ to $\tilde{f}_\varepsilon \colon \mathbb{R} \to \mathbb{R}$ by defining

$$\tilde{f}_\varepsilon(x) = f_\varepsilon(\lfloor x \rfloor).$$

Choosing

$$p_{\mathrm{opt}} = \Big\lfloor \frac{1}{\varepsilon \mathrm{e}} \Big\rfloor$$

we find

$$\tilde{f}_\varepsilon\Big(\frac{1}{\varepsilon \mathrm{e}}\Big) \le m \sqrt{\frac{\varepsilon}{\mathrm{e}}}\, \mathrm{e}^{-\frac{2}{\varepsilon \mathrm{e}}},$$

where

$$m = \sup_{p\in\mathbb{N}} \frac{\mathrm{e}^2 p!}{p^{p+\frac{1}{2}} \mathrm{e}^{-p}} < \infty$$

(by Stirling's formula).

Setting $\varepsilon = C\delta$ in the preceding lemma we obtain

$$p_{\text{opt}} = \left\lfloor \frac{1}{\mathrm{e}C\delta} \right\rfloor$$

and note that $p_{\text{opt}}$ satisfies

$$\delta p_{\text{opt}} \leq \frac{1}{\mathrm{e}C}.$$

Clearly

$$2 \leq p_{\text{opt}} \leq \frac{1}{\mathrm{e}C\delta} \leq p_{\text{opt}} + 1$$

and

$$\frac{1}{p_{\text{opt}}} \leq 2\mathrm{e}C\delta.$$

Hence for $|Y| \leq \delta$ we find

$$\begin{aligned}
\|\tilde{H}(Y)\|_{\mathcal{X}} &\leq \frac{2(n+1)+1}{n+1} c\left((C\delta)^{p+1}p! + \frac{1}{\mathrm{e}^{p+1}p^2}\right) \\
&= \frac{2(n+1)+1}{n+1} c\left(\tilde{f}_{C\delta}(p) + \frac{1}{\mathrm{e}^{p+1}p^2}\right) \\
&\leq \frac{2(n+1)+1}{n+1} c\left(m\sqrt{\frac{C\delta}{\mathrm{e}}}\mathrm{e}^{-\frac{2}{C\delta\mathrm{e}}} + \frac{1}{p^2\mathrm{e}^{p+1}}\right) \\
&\leq \frac{2(n+1)+1}{n+1} c\left(m\sqrt{\frac{C\delta}{\mathrm{e}}}\mathrm{e}^{-\frac{2}{C\delta\mathrm{e}}} + (2\mathrm{e}C\delta)^2\mathrm{e}^{-\frac{1}{\mathrm{e}C\delta}}\right) \\
&\leq \frac{2(n+1)+1}{n+1} c(\mathrm{e}C\delta)^2\mathrm{e}^{-\frac{1}{\mathrm{e}C\delta}}\left(\frac{m}{\mathrm{e}}\underbrace{(\mathrm{e}^{C\delta})^{-\frac{n+1}{2}}\mathrm{e}^{-\frac{1}{C\delta\mathrm{e}}}}_{\leq 1} + 4\right) \\
&\leq \frac{2(n+1)+1}{n+1} c(\mathrm{e}C\delta)^2\mathrm{e}^{-\frac{1}{\mathrm{e}C\delta}}\left(\frac{m}{\mathrm{e}} + 4\right) \\
&= \frac{2(n+1)+1}{n+1} cC^2(m\mathrm{e} + 4\mathrm{e}^2)\delta^2\mathrm{e}^{-\frac{1}{\mathrm{e}C\delta}}.
\end{aligned}$$

## 3.5 Estimates for the derivatives of the transformed coupling term

Before we derive the corresponding estimates for $\mathrm{d}\tilde{H}$ we introduce the following notation. For a polynomial $\Phi$ and $\boldsymbol{p} \in \mathbb{N}^q$ or $\boldsymbol{p} \in \mathbb{N}_2^q$ and $\boldsymbol{l} \in \mathbb{N}_0^q$ with $q \in \mathbb{N}$ we introduce the notation

$$\Phi_{\boldsymbol{p}}^{(\boldsymbol{l})}(Y;\tilde{Y}) \coloneqq \left(\mathrm{d}^{l_1}\Phi_{p_1}[Y](\tilde{Y}_1), \ldots, \mathrm{d}^{l_q}\Phi_{p_q}[Y](\tilde{Y}_q)\right)$$

for $\tilde{Y} = (\tilde{Y}_1, \ldots, \tilde{Y}_q)$ with $\tilde{Y}_i \in (\mathbb{R}^{n+1})^{l_i}$ for $i \in \{1, \ldots, q\}$. In the case $\boldsymbol{l} = \boldsymbol{0}$ we just write

$$\Phi_{\boldsymbol{p}}^{(\boldsymbol{0})}(Y;\tilde{Y}) = \Phi_{\boldsymbol{p}}(Y)$$

for notational simplicity.

**Remark 3.13.** For a polynomial $\Phi$ and $\boldsymbol{p} \in \mathbb{N}^q$, $\boldsymbol{l} \in \mathbb{N}_0^q$ we obtain from Proposition 3.9 that

$$\phi_{\boldsymbol{p}}^{(\boldsymbol{l})} = \prod_{i=1}^{q} |\mathrm{d}^{l_i}\phi_{p_i}|_{2,p_i-l_i} \leq (\sqrt{n}+1)^{\boldsymbol{l}} \prod_{i=1}^{q} \frac{p_i!}{(p_i-l_i)!}\phi_{\boldsymbol{p}}.$$

Differentiating equation (3.17) yields

$$\begin{aligned}
&\mathrm{d}\tilde{H}_{\mathrm{c,sh}}[Y](\tilde{Y}) \\
&= \sum_{p+1\leq q} qH_q(\{Y\}^{(q-1)}, \tilde{Y}) \\
&\quad - \sum_{\substack{2\leq q\leq p\\ p+1\leq m}} \Big(qL_q(\{Y\}^{(q-1)}, \tilde{Y})\big(\Phi_{m-q}(Y)\big) + L_q(Y)\,\mathrm{d}\Phi_{m-q}[Y](\tilde{Y})\Big) \\
&\quad - \sum_{p+1\leq q} \Big(qL_q(\{Y\}^{(q-1)}, \tilde{Y})\big(\sum_{2\leq k\leq p} \Phi_k(Y)\big) + L_q(Y) \sum_{2\leq k\leq p} \mathrm{d}\Phi_k[Y](\tilde{Y})\Big) \\
&\quad + \sum_{\substack{2\leq k\leq p\\ p+1\leq m\leq p+k-1}} \mathrm{d}^2\Phi_k[Y]\Big( \sum_{2\leq q\leq m-k+1} \sum_{i=0}^{q} \sum_{\substack{\boldsymbol{p}\in\mathbb{N}_2^{q-i}\\ |\boldsymbol{p}|=m-k+1-i}} F_{i,q-i}(\{Y\}^{(i)}, -\Phi_{\boldsymbol{p}}(Y)), \tilde{Y}\Big) \\
&\quad + \sum_{\substack{2\leq k\leq p\\ p+1\leq m\leq p+k-1}} \mathrm{d}\Phi_k[Y]\Big( \sum_{2\leq q\leq m-k+1} \sum_{i=0}^{q} \sum_{\substack{\boldsymbol{p}\in\mathbb{N}_2^{q-i}\\ |\boldsymbol{p}|=m-k+1-i}} \Big(iF_{i,q-i}(\{Y\}^{(i-1)}, \tilde{Y}, -\Phi_{\boldsymbol{p}}(Y)) \\
&\qquad\qquad + \sum_{\substack{\boldsymbol{l}\in\mathbb{N}_0^{(q-i)}\\ |\boldsymbol{l}|=1}} F_{i,q-i}(\{Y\}^{(i)}, -\Phi_{\boldsymbol{p}}^{(\boldsymbol{l})}(Y;\tilde{Y}))\Big)\Big) \\
&\quad + \sum_{2\leq q\leq p} \sum_{i=0}^{q} \sum_{\substack{\boldsymbol{p}\in\mathbb{N}_2^{q-i}\\ |\boldsymbol{p}|\geq p+1-i}} \Big(iG_{i,q-i}(\{Y\}^{(i-1)}, \tilde{Y}, -\Phi_{\boldsymbol{p}}(Y)) \\
&\qquad\qquad + \sum_{\substack{\boldsymbol{l}\in\mathbb{N}_0^{q-i}\\ |\boldsymbol{l}|=1}} G_{i,q-i}(\{Y\}^{(i)}, -\Phi_{\boldsymbol{p}}^{(\boldsymbol{l})}(Y;\tilde{Y}))\Big) \\
&\quad + \sum_{p+1\leq q} \sum_{i=0}^{q} \Big(iG_{i,q-i}\Big(\{Y\}^{(i)}, \tilde{Y}, \Big\{\sum_{2\leq k\leq p} \Phi_k(Y)\Big\}^{(q-i)}\Big) \\
&\qquad\qquad + (q-i)G_{i,q-i}\Big(\{Y\}^{(i)}, \Big\{\sum_{2\leq k\leq p} \Phi_k(Y)\Big\}^{(q-i-1)}, \sum_{2\leq k\leq p} \mathrm{d}\Phi_k(Y)(\tilde{Y})\Big)\Big) \\
&\quad + \Big(\sum_{k=2}^{p} \mathrm{d}^2\Phi_k[Y]\Big)\Big( \sum_{2\leq q\leq p} \sum_{i=0}^{q} \sum_{\substack{\boldsymbol{p}\in\mathbb{N}_2^{q-i}\\ |\boldsymbol{p}|\geq p+1-i}} F_{i,q-i}(\{Y\}^{(i)}, -\Phi_{\boldsymbol{p}}(Y)) \\
&\qquad\qquad + \sum_{p+1\leq q} \sum_{i=0}^{q} F_{i,q-i}\Big(\{Y\}^{(i)}, \Big\{\sum_{2\leq k\leq p} \Phi_k(Y)\Big\}^{(q-i)}\Big), \tilde{Y}\Big)
\end{aligned}$$

$$+\Big(\sum_{k=2}^{p}\mathrm{d}\Phi_k[Y]\Big)\Bigg(\sum_{2\le q\le p}\sum_{i=0}^{q}\sum_{\substack{\boldsymbol{p}\in\mathbb{N}_2^{q-i}\\|\boldsymbol{p}|\ge p+1-i}}\Bigg(iF_{i,q-i}\Big(\{Y\}^{(q-i)},\tilde{Y},-\Phi_{\boldsymbol{p}}(Y)\Big)$$

$$+\sum_{\substack{\boldsymbol{l}\in\mathbb{N}_0^q\\|\boldsymbol{l}|=1}}F_{i,q-i}\Big(\{Y\}^{(i)},-\Phi_{\boldsymbol{p}}^{(\boldsymbol{l})}(Y;\tilde{Y})\Big)\Bigg)$$

$$+\sum_{p+1\le q}\sum_{i=0}^{q}\Bigg(iF_{i,q-i}\Big(\{Y\}^{(i-1)},\tilde{Y},\Big\{\sum_{2\le k\le p}\Phi_k(Y)\Big\}^{(q-i)},\sum_{2\le k\le p}\mathrm{d}\Phi_k(Y)(\tilde{Y})\Big)\Bigg)\Bigg).\tag{3.19}$$

By using Proposition 3.10 and Remark 3.13 we therefore have that

$$\begin{aligned}
&\|\mathrm{d}\tilde{H}[Y]\|_{\mathcal{L}(\mathbb{R}^{n+1};\mathcal{X})}\\
&\le\frac{1}{2}\sum_{p+1\le q}\frac{c}{\rho^q}q(\sqrt{n+1})^q\delta^{q-1}\\
&\quad+\frac{1}{2}\sum_{\substack{2\le q\le p\\p+1\le m}}\frac{c}{\rho^q}\Big(q(\sqrt{n+1})^{q-1}+\frac{\sqrt{n+1}}{\theta}\Big)\delta^{m-1}\phi_k\\
&\quad+\frac{1}{2}\sum_{p+1\le q}\frac{c}{\rho^q}\delta^q\Big(q(\sqrt{n+1})^{q-1}+\frac{\sqrt{n+1}}{\theta}\Big)\\
&\quad+\frac{2(n+1)}{\theta}\sum_{2\le q\le p}\sum_{\substack{\boldsymbol{p}\in\mathbb{N}^q\\|\boldsymbol{p}|\ge p+1}}\frac{c}{\rho^q}\phi_{\boldsymbol{p}}\delta^{|\boldsymbol{p}|-1}\\
&\quad+\frac{2(n+1)}{\theta}\sum_{p+1\le q}\frac{c}{\rho^q}(\sqrt{n+1})^q\delta^{q-i}(q+1)\\
&\quad+\Big(1+\frac{2(n+1)}{\theta}\Big)\sum_{2\le q\le p}\sum_{\substack{\boldsymbol{p}\in\mathbb{N}^q\\|\boldsymbol{p}|\ge p+1}}|\boldsymbol{p}|\frac{c\delta^{|\boldsymbol{p}|-1}}{\rho^q}\phi_{\boldsymbol{p}}\\
&\quad+\Big(1+\frac{2(n+1)}{\theta}\Big)\sum_{p+1\le q}\frac{c}{\rho^q}q(\sqrt{n+1})^q\delta^{q-1}(q+1)\\
&\quad+\sum_{\substack{2\le k\le p\\p+1\le m\le p+k-1}}\sqrt{n+1}k\phi_k\sum_{2\le q\le m-k+1}\sum_{\substack{\boldsymbol{p}\in\mathbb{N}^q\\|\boldsymbol{p}|=m-k+1}}m\frac{c}{\rho^q}\phi_{\boldsymbol{p}}\delta^{m-1}\\
&\le 2\Big(1+\frac{2(n+1)}{\theta}\Big)\sum_{p+1\le q}\frac{c}{\rho^q}q(\sqrt{n+1})^q\delta^{q-1}(q+1)\\
&\quad+\Big(1+\frac{2(n+1)+2}{\theta}\Big)\sum_{2\le q\le p}\sum_{\substack{\boldsymbol{p}\in\mathbb{N}^q\\|\boldsymbol{p}|\ge p+1}}|\boldsymbol{p}|\frac{c\delta^{|\boldsymbol{p}|-1}}{\rho^q}\phi_{\boldsymbol{p}}\\
&\quad+\sum_{\substack{2\le k\le p\\p+1\le k\le p+k-1}}\sqrt{n+1}k\phi_k\sum_{2\le q\le m-k+1}\sum_{\substack{\boldsymbol{p}\in\mathbb{N}^q\\|\boldsymbol{p}|=m-k+1}}m\frac{c}{\rho^q}\phi_{\boldsymbol{p}}\delta^{m-1},
\end{aligned}$$

where we have chosen $\theta\ge 1$ and defined $\phi_1=\sqrt{n+1}$.

An estimate for $\|\mathrm{d}\tilde{H}\|_{\mathcal{L}(\mathbb{R}^{n+1};\mathcal{D})}$ is obtained from the above calculations and repeating the proof of Proposition 3.11.

**Proposition 3.14.** Suppose that

$$\delta p \leq \frac{\rho^2}{4\sqrt{n+1}\mathrm{e}ac\theta},$$

that $\theta \geq 1$ and that

$$r = \frac{\rho}{ac\theta} < \frac{1}{n+1}.$$

For every $Y \in \mathbb{R}^{n+1}$ with $|Y| \leq \delta$ we find

$$\|\mathrm{d}\tilde{H}[Y]\|_{\mathcal{L}(\mathbb{R}^{n+1};\mathcal{X})} \leq \frac{(2(n+1)+1)c}{(n+1)\delta}\Big((C\delta)^{p+1}p! + \frac{1}{\mathrm{e}^{p+1}p^2}\Big),$$

where

$$C = \frac{4\sqrt{n+1}ac\theta}{\rho^2}.$$

*Proof.* Define

$$\Delta_p^1 = 2\Big(1 + \frac{2(n+1)}{\theta}\Big) \sum_{p+1\leq q} \frac{c}{\rho^q} q(\sqrt{n+1})^q \delta^{q-1}(q+1),$$

$$\Delta_p^2 = \Big(1 + \frac{4(n+1)}{\theta}\Big) \sum_{2\leq q\leq p} \sum_{\substack{\boldsymbol{p}\in\mathbb{N}^q \\ |\boldsymbol{p}|\geq p+1}} |\boldsymbol{p}| \frac{c\delta^{|\boldsymbol{p}|-1}}{\rho^q}\phi_{\boldsymbol{p}},$$

$$\Delta_p^3 = \sum_{\substack{2\leq k\leq p \\ p+1\leq m\leq p+k-1}} \sqrt{n+1}k\phi_k \sum_{2\leq q\leq m-k+1} \sum_{\substack{\boldsymbol{p}\in\mathbb{N}^q \\ |\boldsymbol{p}|=m-k+1}} \frac{cm}{\rho^q}\phi_{\boldsymbol{p}}\delta^m.$$

We note that

$$(q+1)q(\sqrt{n+1})^q \leq (2\sqrt{n+1})^q$$

for $q \geq n$ and

$$\frac{2\sqrt{n+1}\delta}{\rho} \leq \frac{\rho}{4ac\theta p} \leq \frac{1}{4(n+1)p} \leq \frac{1}{8(n+1)}$$

for $p \geq 2$ so that

$$\begin{aligned}\sum_{p+1\leq q} \frac{c}{\rho^q} q(q+1)(\sqrt{n+1})^q \delta^{q-1} &\leq \frac{c}{\delta}\Big(\frac{2\sqrt{n+1}\delta}{\rho}\Big)^{p+1} \sum_{q=0}^{\infty}\Big(\frac{1}{8(n+1)}\Big)^q \\ &\leq \frac{8(n+1)c}{(8(n+1)-1)\delta}\Big(\frac{2\sqrt{n+1}\delta}{\rho}\Big)^{p+1}\end{aligned}$$

and hence

$$\Delta_p^1 \leq \frac{16(n+1)c}{(8(n+1)-1)\delta}\Big(1 + \frac{2(n+1)}{\theta}\Big)\Big(\frac{2\sqrt{n+1}\delta}{\rho}\Big)^{p+1},$$

where we have assumed $p \geq 4$.

Turning to $\Delta_p^2$, we have that

$$\sum_{2\leq q\leq p} \sum_{m\geq p+1} \sum_{\substack{\boldsymbol{p}\in\mathbb{N}^q \\ |\boldsymbol{p}|=m}} \frac{cm}{\rho^q}\delta^m \phi_{\boldsymbol{p}}$$

$$\leq \sum_{2\leq q\leq p} \sum_{m\geq p+1} \sum_{\substack{\boldsymbol{p}\in\mathbb{N}^q \\ |\boldsymbol{p}|=m}} c\Big(\frac{4\sqrt{n+1}ac\theta\delta}{\rho^2}\Big)^m \Big(\frac{\rho}{ac\theta}\Big)^q (p_1-2)!\cdot\ldots\cdot(p_q-2)!$$

$$\leq \frac{1}{\delta}\frac{4cr^2}{\mathrm{e}^{p+1}}\cdot\frac{1}{p^2}\cdot\frac{1}{1-r}$$

by requiring $C < \frac{1}{\mathrm{e}}$. Hence

$$\Delta_p^2 \leq \frac{(1+4(n+1))4cr^2}{\mathrm{e}^{p+1}\delta}\cdot\frac{1}{p^2}\cdot\frac{1}{1-r}.$$

Finally we have that

$$\Delta_p^3 \leq \frac{2(n+1)\sqrt{n+1}}{a\delta} \sum_{\substack{2\leq k\leq p \\ p+1\leq m\leq p+k-1}} \delta^m m\Big(\frac{2\sqrt{n+1}ac}{\rho^2}\Big)^{m-1}\theta^{m-3}k(k-2)!(m-k-1)!$$

$$\leq \frac{3\rho^2}{a^2c\theta^3\delta} \sum_{\substack{2\leq k\leq p \\ p+1\leq m\leq p+k-1}} \Big(\frac{4\sqrt{n+1}ac\theta\delta}{\rho^2}\Big)^m k(k-2)!(m-k-1)!$$

$$\leq \frac{2(n+1)\rho^2}{a^2c\theta^3\delta}(C\delta)^{p+1}p!$$

by our previous method.

Altogether we find

$$\begin{aligned}
\|\mathrm{d}\tilde{H}[Y]\|_{\mathcal{L}(\mathbb{R}^{n+1};\mathcal{X})} &\leq \frac{48(n+1)^2c}{(8(n+1)-1)\delta}\Big(\frac{2\sqrt{n+1}\delta}{\rho}\Big)^{p+1} + \frac{20(n+1)cr^2}{\mathrm{e}^{p+1}\delta}\cdot\frac{1}{p^2}\cdot\frac{1}{1-r} \\
&\quad + \frac{2(n+1)\rho^2}{a^2c\delta\theta^3}(C\delta)^{p+1}p! \\
&\leq \frac{48(n+1)^2c}{(8(n+1)-1)\delta}\Big(\frac{rC\delta}{2}\Big)^{p+1} + \frac{20(n+1)cr^2}{\mathrm{e}^{p+1}\delta}\cdot\frac{1}{p^2}\cdot\frac{1}{1-r} \\
&\quad + \frac{2(n+1)c}{(n+1)^2\delta}(C\delta)^{p+1}p! \\
&\leq \frac{48(n+1)^2c}{(8(n+1)-1)(2(n+1))^{p+1}\delta}(C\delta)^{p+1} + \frac{20(n+1)c}{(n+1)^2\mathrm{e}^{p+1}p^2\delta} \\
&\quad + \frac{2c}{(n+1)\delta}(C\delta)^{p+1}p! \\
&\leq \frac{23}{(n+1)\delta}c\Big((C\delta)^{p+1}p! + \frac{1}{\mathrm{e}^{p+1}p^2}\Big).
\end{aligned}$$

□

Arguing as before, we find that

$$\|\mathrm{d}\tilde{H}[Y]\|_{\mathcal{L}(\mathbb{R}^{n+1};\mathcal{X})} \leq \frac{23c}{n+1}C^2(m\mathrm{e}+4\mathrm{e}^2)\delta\mathrm{e}^{-\frac{1}{eC\delta}}$$

for all $Y\in\mathbb{R}^{n+1}$ with $|Y|\leq\delta$.

# 4 Approximate pulses

We consider the evolutionary equation system

$$\dot{z} = L^\varepsilon_{\mathrm{wh}} z + g^\varepsilon_{\mathrm{wh}}(z, w, u) + h^\varepsilon_{\mathrm{wh}}(z), \tag{4.1}$$
$$\dot{w} = L^\varepsilon_{\mathrm{c}} w + g^\varepsilon_{\mathrm{c}}(z, w, u) + h^\varepsilon_{\mathrm{c}}(z), \tag{4.2}$$
$$\dot{u} = L^\varepsilon_{\mathrm{sh}} u + g^\varepsilon_{\mathrm{sh}}(z, w, u) + h^\varepsilon_{\mathrm{sh}}(z) \tag{4.3}$$

for $(z, w, u)\colon \mathbb{R} \to \mathbb{R}^n \times \mathbb{R}^{2d} \times \mathcal{D}_{\mathrm{sh}}$, where $n, d \in \mathbb{N}$ and $\mathcal{D}_{\mathrm{sh}}$ is a dense subspace of a Banach space $\mathcal{X}_{\mathrm{sh}}$. We abbreviate $\mathbb{R}^n \times \mathbb{R}^{2d} \times \mathcal{X}_{\mathrm{sh}}$, $\mathbb{R}^n \times \mathbb{R}^{2d} \times \mathcal{D}_{\mathrm{sh}}$ to respectively $\mathcal{X}$, $\mathcal{D}$ and $\mathbb{R}^{2d} \times \mathcal{X}_{\mathrm{sh}}$, $\mathbb{R}^{2d} \times \mathcal{D}_{\mathrm{sh}}$ to respectively $\mathcal{X}_{\mathrm{c,sh}}$, $\mathcal{D}_{\mathrm{c,sh}}$. On the right-hand side we make the following assumptions.

(A1) The bounded linear operators $L^\varepsilon_{\mathrm{wh}}\colon \mathbb{R}^n \to \mathbb{R}^n$, $L^\varepsilon_{\mathrm{c}}\colon \mathbb{R}^{2d} \to \mathbb{R}^{2d}$ and $L^\varepsilon_{\mathrm{sh}}\colon \mathcal{D}_{\mathrm{sh}} \to \mathcal{X}_{\mathrm{sh}}$ depend analytically upon $\varepsilon$.

(A2) The functions $g^{(\cdot)}_{\mathrm{wh}}$, $g^{(\cdot)}_{\mathrm{c}}$, $g^{(\cdot)}_{\mathrm{sh}}$, $h^{(\cdot)}_{\mathrm{wh}}$, $h^{(\cdot)}_{\mathrm{c}}$, $h^{(\cdot)}_{\mathrm{sh}}$ take values in respectively $\mathbb{R}^n$, $\mathbb{R}^{2d}$, $\mathcal{X}_{\mathrm{sh}}$, $\mathbb{R}^n$, $\mathbb{R}^{2d}$, $\mathcal{X}_{\mathrm{sh}}$ and are analytic at the origin in respectively $\mathbb{R} \times \mathcal{D}$ and $\mathbb{R} \times \mathbb{R}^n$. We suppose that

$$g^\varepsilon_{\mathrm{wh}}(z, w, u), g^\varepsilon_{\mathrm{c}}(z, w, u), g^\varepsilon_{\mathrm{sh}}(z, w, u) = \mathcal{O}(\|(z, w, u)\|_{\mathcal{D}} \|(w, u)\|_{\mathcal{D}_{\mathrm{c,sh}}}),$$
$$h^\varepsilon_{\mathrm{wh}}(z), h^\varepsilon_{\mathrm{c}}(z), h^\varepsilon_{\mathrm{sh}}(z) = \mathcal{O}(|z|^2).$$

(A3) The spectrum of the complexified operator $L^\varepsilon_{\mathrm{c}} \in \mathbb{C}^{2d \times 2d}$ consists of finitely many simple purely imaginary eigenvalues $\pm \mathrm{i}\omega^\varepsilon_1, \ldots, \pm \mathrm{i}\omega^\varepsilon_d$, where $\omega^\varepsilon_1, \ldots, \omega^\varepsilon_d > 0$ (see Figure 4.1). For later use we denote the corresponding eigenvectors by $e^\varepsilon_1, \ldots, e^\varepsilon_d$ and $\bar{e}^\varepsilon_1, \ldots, \bar{e}^\varepsilon_d$.

(A4) The system (4.1) – (4.3) is reversible, i.e. there exist $S_{\mathrm{wh}} \in \mathbb{R}^{n \times n}$, $S_{\mathrm{c}} \in \mathbb{R}^{2d \times 2d}$ and $S_{\mathrm{sh}} \in \mathcal{L}(\mathcal{D}_{\mathrm{sh}}) \cap \mathcal{L}(\mathcal{X}_{\mathrm{sh}})$ such that system (4.1) – (4.3) is invariant under $t \mapsto -t$, $(z, w, u) \mapsto (S_{\mathrm{wh}} z, S_{\mathrm{c}} w, S_{\mathrm{sh}} u)$.

(A5) There exists a real-valued function $\mathcal{I}^{(\cdot)}$ which is analytic at the origin in $\mathbb{R} \times \mathcal{D}$, satisfies

$$\mathcal{I}^\varepsilon(z, w, u) = \mathcal{O}(\|(z, w, u)\|^2_{\mathcal{D}})$$

and

$$\mathcal{I}^0(0, w, 0) = |w|^2 + \mathcal{O}(|w|^3)$$

and is such that $\mathcal{I}^\varepsilon$ is a conserved quantity of system (4.1) – (4.3).

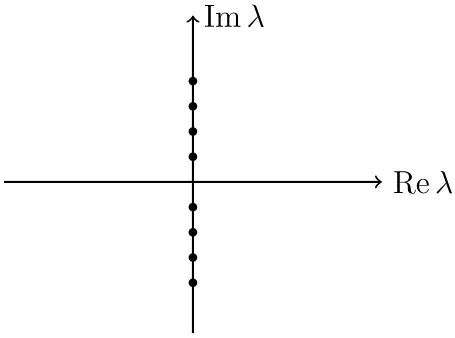



Figure 4.1: The spectrum of $L_c^\varepsilon$ consists of $d$ pairs of purely imaginary eigenvalues.

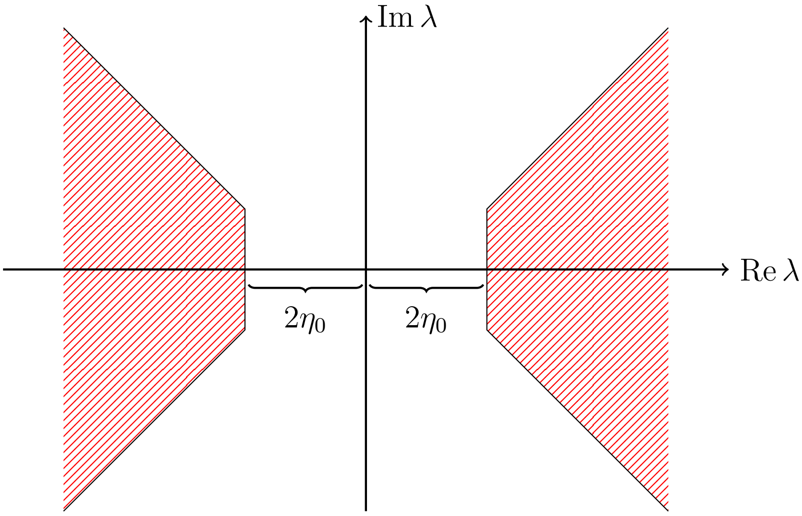



Figure 4.2: The spectrum of $L_{sh}^0$ is contained in wedges. We call the distance $2\eta_0$ of the wedges to the imaginary axis the *spectral gap*.

(A6) The linear operator $L_{sh}^0 \colon \mathcal{D}_{sh} \subseteq \mathcal{X}_{sh} \to \mathcal{X}_{sh}$ is closed and satisfies the estimate

$$\|(\mathrm{i}sI - L_{sh}^0)^{-1}\|_{\mathcal{L}(\mathcal{X}_{sh})} \lesssim \frac{1}{1+|s|}$$

for $s \in \mathbb{R}$. This estimate implies the existence of $\gamma, \eta_0 > 0$ such that $\sigma(L_{sh}^\varepsilon)$ lies in the region

$$\{\lambda \in \mathbb{C} : |\mathrm{Re}\,\lambda| \geq 2\eta_0, |\mathrm{Re}\,\lambda| \geq \gamma |\mathrm{Im}\,\lambda|\}$$

(see the proof of Corollary 2.16 for details).

**Remark 4.1.** Assumptions (A5) and (A6) are not needed to prove the existence of approximate pulses in this chapter but will be heavily used in Chapter 5.

In this chapter we construct homoclinic solutions to the (reversible) *approximate system*

$$\dot{z} = L_{wh}^\varepsilon z + h_{wh}^\varepsilon(z) \tag{4.4}$$

and use a scaling and the normal-form theory from Chapter 3 to transform equations (4.1) – (4.3) into a system with the same structure but for which the coupling terms in equations (4.2) and (4.3) are exponentially small in a neighbourhood of the origin. We treat the cases $n = 2$ and $n = 4$ in detail, assuming that equation (4.4) undergoes respectively a $0^2$ and an $(\mathrm{i}\omega)^2$ resonance at $\varepsilon = 0$.

## 4.1 The $0^2$ resonance

In the case $n = 2$ we assume that equations (4.1) – (4.3) satisfy the following additional assumptions.

(B1) The spectrum of the linear operator $L^\varepsilon_{\mathrm{wh}} \in \mathbb{R}^{2\times 2}$ exhibits a $0^2$ resonance at $\varepsilon = 0$, meaning that as $\varepsilon \uparrow 0$ a pair of purely imaginary eigenvalues of $L^\varepsilon_{\mathrm{wh}}$ collides at the origin (forming a Jordan block) and splits into a pair of real eigenvalues for $\varepsilon > 0$ (see Figure 4.3). We write $z \in \mathbb{R}^2$ as

$$z = z_1 e + z_2 f,$$

where $L^0_{\mathrm{wh}} e = 0$, $L^0_{\mathrm{wh}} f = e$, and assume that

$$L^\varepsilon_{\mathrm{wh}} = \begin{pmatrix} 0 & 1 \\ (\lambda^\varepsilon)^2 & 0 \end{pmatrix},$$

where $\lambda^\varepsilon$ is an analytic function of $\varepsilon$ with

$$\lambda^\varepsilon = \varepsilon + \mathcal{O}(\varepsilon^2).$$

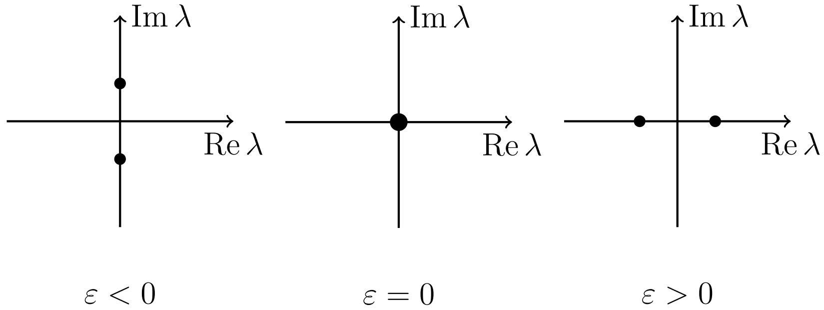



Figure 4.3: The $0^2$ resonance at $\varepsilon = 0$.

(B2) The term

$$h^\varepsilon_{\mathrm{wh}}(z_1, z_2) = \begin{pmatrix} h^\varepsilon_{\mathrm{wh},1}(z_1, z_2) \\ h^\varepsilon_{\mathrm{wh},2}(z_1, z_2) \end{pmatrix}$$

satisfies

$$C := -\frac{1}{2}\, \mathrm{d}_1^2 h^\varepsilon_{\mathrm{wh},2}[0](1,1) \neq 0.$$

We show that the approximate system (4.4) has homoclinic solutions by a method due to Kirchgässner [16, Proposition 5.1].

**Proposition 4.2.** The linear operator $L\colon C^2(\mathbb{R}) \to C(\mathbb{R})$ given by

$$Ly = \ddot{y} - y$$

has the following properties.

(i) The restrictions $L\colon C^2_{-\nu}(\mathbb{R}) \to C_{-\nu}(\mathbb{R})$ and $L\colon C^2_{-\nu}(\mathbb{R}) \cap C_\mathrm{e}(\mathbb{R}) \to C_{-\nu}(\mathbb{R}) \cap C_\mathrm{e}(\mathbb{R})$ of $L$ are well defined for all $\nu \geq 0$.

(ii) For $0 \leq \nu < 1$ the operator $L\colon C^2_{-\nu}(\mathbb{R}) \to C_{-\nu}(\mathbb{R})$ is invertible with

$$L^{-1}f(t) = -\frac{1}{2}\int_{-\infty}^{\infty} \mathrm{e}^{-|t-s|} f(s)\,\mathrm{d}s.$$

**Proposition 4.3.** The operator $A_h\colon C(\mathbb{R}) \to C(\mathbb{R})$ given by

$$A_h y = L^{-1}(-2Chy),$$

where $h \in C_{-1}(\mathbb{R})$ is even, $C \in \mathbb{R}$ and $L$ is defined as in Proposition 4.2, has the following properties.

(i) The restrictions $A_h\colon C_\mathrm{b}(\mathbb{R}) \to C^2_{-\nu}(\mathbb{R})$ and $A_h\colon C_\mathrm{e}(\mathbb{R}) \to C^2_{-\nu}(\mathbb{R})$ of $A_h$ are bounded linear operators for $0 \leq \nu < 1$.

(ii) The restriction $A_h\colon C_\mathrm{b}(\mathbb{R}) \to C_{-\nu}(\mathbb{R})$ of $A_h$ is compact for $0 \leq \nu < 1$.

(iii) The restriction $A_h\colon C_{-\nu}(\mathbb{R}) \to C_{-\nu}(\mathbb{R})$ of $A_h$ is compact for $0 \leq \nu < 1$.

**Lemma 4.4.** For fixed $\nu \in (0,1)$ and $\varepsilon > 0$ the system (4.4) has a reversible homoclinic solution, that is invariant under the transformation $(z_1, z_2)(t) \mapsto (z_1, -z_2)(-t)$, of the form

$$\begin{pmatrix} p_1^\varepsilon(t) \\ p_2^\varepsilon(t) \end{pmatrix} = \begin{pmatrix} (\lambda^\varepsilon)^2 \check{p}_1^\varepsilon(\lambda^\varepsilon t) \\ (\lambda^\varepsilon)^3 \check{p}_2^\varepsilon(\lambda^\varepsilon t) \end{pmatrix},$$

where

$$|\check{p}_j^\varepsilon(t)| \lesssim \mathrm{e}^{-\nu|t|}.$$

*Proof.* The scaling

$$\check{t} = \lambda^\varepsilon t, \qquad \begin{pmatrix} z_1(t) \\ z_2(t) \end{pmatrix} = \begin{pmatrix} (\lambda^\varepsilon)^2 \check{z}_1(\check{t}) \\ (\lambda^\varepsilon)^3 \check{z}_2(\check{t}) \end{pmatrix}$$

converts equation (4.4) to

$$\dot{\check{z}}_1 = \check{z}_2 + \lambda^\varepsilon \mathcal{R}_1^\varepsilon(\check{z}_1, \check{z}_2), \tag{4.5}$$
$$\dot{\check{z}}_2 = \check{z}_1 - C\check{z}_1^2 + \lambda^\varepsilon \mathcal{R}_2^\varepsilon(\check{z}_1, \check{z}_2), \tag{4.6}$$

where $\mathcal{R}_1^\varepsilon$, $\mathcal{R}_2^\varepsilon$ are respectively odd and even in $\check{z}_2$ with

$$|\mathcal{R}_1^\varepsilon(\check{z}_1, \check{z}_2)| = \mathcal{O}(|\check{z}|^2).$$

For $\varepsilon = 0$ this system has the explicit reversible homoclinic solution

$$\begin{pmatrix} \check{z}_1 \\ \check{z}_2 \end{pmatrix} = \begin{pmatrix} h \\ \dot{h} \end{pmatrix},$$

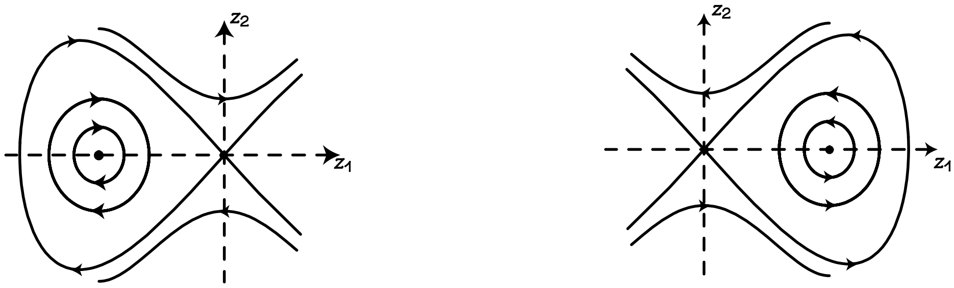
where

$$h(t) = \frac{3}{2C} \operatorname{sech}^2 \left(\frac{t}{2}\right).$$

In fact we have the family

$$\left\{(\check{z}_1, \check{z}_2) = \left(h(t_0 + \cdot), \dot{h}(t_0 + \cdot)\right)\right\}_{t_0 \in \mathbb{R}}$$

of homoclinic solutions. Note that the solution in the case $t_0 = 0$ is reversible (see Figure 4.4).




(a) The case $C < 0$. (b) The case $C > 0$.

Figure 4.4: Phase portrait of equations (4.5) and (4.6) for $\varepsilon = 0$.

Using the implicit function theorem we can solve equation (4.5) for $\check{z}_2 = \check{z}_2(\check{z}_1, \dot{\check{z}}_1)$, to find that

$$\check{z}_2 = \dot{\check{z}}_1 + \varepsilon v^\varepsilon(\check{z}_1, \dot{\check{z}}_1),$$

where $v^\varepsilon$ is analytic and odd in $\dot{\check{z}}_1$ and $|v^\varepsilon(\check{z}_1, \dot{\check{z}}_1)| = \mathcal{O}(|(\check{z}_1, \dot{\check{z}}_1)||(\check{z}_1, \dot{\check{z}}_1, \varepsilon)|)$. Hence we can rewrite equations (4.5) and (4.6) as

$$\ddot{\check{z}}_1 = \check{z}_1 - C\check{z}_1^2 + \varepsilon \mathcal{S}^\varepsilon(\check{z}_1, \dot{\check{z}}_1),$$

where $\mathcal{S}^\varepsilon$ is analytic and even in $\dot{\check{z}}_1$ and

$$|\mathcal{S}^\varepsilon(\check{z}_1, \dot{\check{z}}_1)| = \mathcal{O}(|(\check{z}_1, \dot{\check{z}}_1)||(\check{z}_1, \dot{\check{z}}_1, \varepsilon)|).$$

Now we write $\check{z}_1$ as perturbation

$$\check{z}_1 = h + y$$

of $h$, so that

$$\ddot{y} - y = -2Chy + r^\varepsilon(y, \dot{y}, t),$$

with the obvious definition of $r^\varepsilon$. We study this equation in the space $C^2_{-\nu}(\mathbb{R})$ with fixed $\nu \in (0, 1)$ and consider the nonlinearity $r^\varepsilon$ with a slight abuse of notation as an analytic mapping $C^2_{-\nu}(\mathbb{R}) \to C_{-\nu}(\mathbb{R})$ and $C^2_{-\nu}(\mathbb{R}) \cap C_{\mathrm{e}}(\mathbb{R}) \to C_{-\nu}(\mathbb{R}) \cap C_{\mathrm{e}}(\mathbb{R})$ with

$$\|r^\varepsilon(y)\|_{C_{-\nu}(\mathbb{R})} \lesssim \varepsilon + \|y\|^2_{C^2_{-\nu}(\mathbb{R})}.$$

Next we write the above equation as

$$y = A_h y + L^{-1} r^{\varepsilon}(y), \tag{4.7}$$

where $A_h$ and $L$ are defined in Propositions 4.2 and 4.3. To solve equation (4.7) by means of the implicit function theorem we prove $I - A_h\colon C_{-\nu}(\mathbb{R}) \cap C_{\mathrm{e}}(\mathbb{R}) \to C^2_{-\nu}(\mathbb{R}) \cap C_{\mathrm{e}}(\mathbb{R})$ to be invertible. To this end it suffices to check that the eigenspace of $A_h$ to the eigenvalue 1 lies in $C_{-\nu}(\mathbb{R}) \cap C_{\mathrm{o}}(\mathbb{R})$. We know that

$$A_h y = y$$

is equivalent to

$$\ddot{y} - y + 2Chy = 0$$

which has the fundamental solution set $\{y_1, y_2\}$, where

$$y_1(t) = \frac{1}{2}\operatorname{sech}^2\left(\frac{t}{2}\right)\tanh\left(\frac{t}{2}\right) = -\frac{C}{3}\dot{h}(t) \tag{4.8}$$

and

$$y_2(t) = \cosh(t) + \frac{3}{2}\operatorname{sech}^2\left(\frac{t}{2}\right)\Big(-8 + 2\cosh(t) + 5t\tanh\left(\frac{t}{2}\right)\Big). \tag{4.9}$$

We note that $y_1$ is odd and $y_2$ is even. Hence all bounded solutions of

$$A_h y = y$$

are multiples of $\dot{h}$. The eigenspace of $A_h$ to the eigenvalue 1 is therefore $\langle \dot{h} \rangle$ and lies in $C_{-\nu}(\mathbb{R}) \cap C_{\mathrm{o}}(\mathbb{R})$. The implicit function theorem now yields a solution $h^*$ to equation (4.7) which satisfies

$$\|h^*\|_{C^2_{-\nu}(\mathbb{R})} = \mathcal{O}(\varepsilon).$$

The result now follows with

$$\begin{aligned}
\check{p}_1^{\varepsilon}(\check{t}) &= h(\check{t}) + h^*(\check{t}),\\
\check{p}_2^{\varepsilon}(\check{t}) &= h(\check{t}) + h^*(\check{t}) + \varepsilon v^{\varepsilon}\Big(h(\check{t}) + y^*(\check{t}), \dot{h}(\check{t}) + \dot{y}^*(\check{t})\Big).
\end{aligned}$$

□

To apply the normal-form theory in Chapter 3 the parameter-independent part of $L^{\varepsilon}_{\mathrm{wh}}$ should be diagonalisable as discussed in Remark 3.5. Since $L^0_{\mathrm{wh}}$ is evidently not diagonalisable, we use a change of parameter to 'replace' it by the zero matrix. Writing $\varepsilon = \mu^2$ and introducing the scaled variables

$$\begin{pmatrix} Z_1 \\ Z_2 \end{pmatrix} = \begin{pmatrix} (\tilde{\lambda}^{\mu})^{-1} z_1 \\ (\tilde{\lambda}^{\mu})^{-2} z_2 \end{pmatrix}, \quad \begin{pmatrix} W \\ U \end{pmatrix} = (\tilde{\lambda}^{\mu})^{-1}\begin{pmatrix} w \\ u \end{pmatrix}, \quad \tilde{\lambda}^{\mu} = \lambda^{\varepsilon}|_{\varepsilon=\mu^2}$$

converts equations (4.1) – (4.3) into

$$\dot{Z} = L^{\mu}_{\rm wh} Z + G^{\mu}_{\rm wh}(Z, W, U) + H^{\mu}_{\rm wh}(Z), \tag{4.10}$$
$$\dot{W} = L^{\mu}_{\rm c} W + G^{\mu}_{\rm c}(Z, W, U) + H^{\mu}_{\rm c}(Z), \tag{4.11}$$
$$\dot{U} = L^{\mu}_{\rm sh} U + G^{\mu}_{\rm sh}(Z, W, U) + H^{\mu}_{\rm sh}(Z), \tag{4.12}$$

where

$$\begin{aligned}
L^{\mu}_{\rm wh}\begin{pmatrix} Z_1 \\ Z_2 \end{pmatrix} &= \begin{pmatrix} 0 & \tilde{\lambda}^{\mu} \\ \tilde{\lambda}^{\mu} & 0 \end{pmatrix}\begin{pmatrix} Z_1 \\ Z_2 \end{pmatrix}, \\
G^{\mu}_{\rm wh}(Z, W, U) &= \mathcal{O}(\|(W, U)\|_{\mathcal{D}_{\rm c,sh}} \|(Z, W, U)\|_{\mathcal{D}}), \\
G^{\mu}_{\rm c}(Z, W, U), G^{\mu}_{\rm sh}(Z, W, U) &= \mathcal{O}\big(\mu^2 \|(W, U)\|_{\mathcal{D}_{\rm c,sh}} \|(Z, W, U)\|_{\mathcal{D}}\big), \\
H^{\mu}_{\rm wh}(Z) &= \mathcal{O}(|Z|^2), \\
H^{\mu}_{\rm c}(Z), H^{\mu}_{\rm sh}(Z) &= \mathcal{O}(\mu^2 |Z|^2),
\end{aligned}$$

and we have abbreviated $L^{\varepsilon}_{\rm c}|_{\varepsilon=\mu^2}$, $L^{\varepsilon}_{\rm sh}|_{\varepsilon=\mu^2}$ to $L^{\mu}_{\rm c}$, $L^{\mu}_{\rm sh}$.

**Remark 4.5.** The formula

$$\mathcal{I}^{\mu}(Z, W, U) := \mathcal{I}^{\varepsilon}(z, w, u)|_{\varepsilon=\mu^2}$$

defines a conserved quantity of equations (4.10) – (4.12) and satisfies

$$\mathcal{I}^{\mu}(Z, W, U) = \mathcal{O}(\mu^4 \|(Z, W, U)\|^2_{\mathcal{D}}).$$

The equation

$$\dot{Z} = L^{\mu}_{\rm wh} Z + H^{\mu}_{\rm wh}(Z)$$

has the reversible homoclinic solution

$$P^{\mu}(t) = \begin{pmatrix} P^{\mu}_1(t) \\ P^{\mu}_2(t) \end{pmatrix} = \begin{pmatrix} \tilde{\lambda}^{\mu} \check{p}^{\varepsilon}_1(\tilde{\lambda}^{\mu} t)|_{\varepsilon=\mu^2} \\ \tilde{\lambda}^{\mu} \check{p}^{\varepsilon}_2(\tilde{\lambda}^{\mu} t)|_{\varepsilon=\mu^2} \end{pmatrix}, \tag{4.13}$$

which satisfies the estimate

$$|P^{\mu}(t)| \lesssim \tilde{\lambda}^{\mu} \mathrm{e}^{-\nu \tilde{\lambda}^{\mu} |t|}$$

(see Lemma 4.4).

The estimates gathered above are not sufficient to construct a contractive iteration scheme which will be our main tool in the existence proof of generalised pulse solutions in Chapter 5. The following preliminary transformation improves these estimates by removing those terms of $G^{\mu}_{\rm wh}$ which are linear or quadratic in $(Z, \mu)$ and linear in $(W, U)$ at the expense of modifying higher-order terms.

**Lemma 4.6.** There exists a near-identity, finite-dimensional change of variables which transforms equations (4.10) – (4.12) into

$$\dot{Z} = L^{\mu}_{\mathrm{wh}} Z + \hat{G}^{\mu}_{\mathrm{wh}}(Z, W, U) + \hat{H}^{\mu}_{\mathrm{wh}}(Z), \tag{4.14}$$

$$\dot{W} = L^{\mu}_{\mathrm{c}} W + \hat{G}^{\mu}_{\mathrm{c}}(Z, W, U) + \hat{H}^{\mu}_{\mathrm{c}}(Z), \tag{4.15}$$

$$\dot{U} = L^{\mu}_{\mathrm{sh}} U + \hat{G}^{\mu}_{\mathrm{sh}}(Z, W, U) + \hat{H}^{\mu}_{\mathrm{sh}}(Z) \tag{4.16}$$

and preserves the reversibility. The transformed nonlinearities $\hat{G}^{\mu}_{\mathrm{c}}$, $\hat{G}^{\mu}_{\mathrm{sh}}$ and $\hat{H}^{\mu}_{\mathrm{c}}$, $\hat{H}^{\mu}_{\mathrm{sh}}$ satisfy the same estimates as respectively $G^{\mu}_{\mathrm{c}}$, $G^{\mu}_{\mathrm{sh}}$ and $H^{\mu}_{\mathrm{c}}$, $H^{\mu}_{\mathrm{sh}}$, while

$$\hat{H}^{\varepsilon}_{\mathrm{wh}}(Z) - H^{\mu}_{\mathrm{wh}}(Z) = \mathcal{O}(|(Z, \mu)||Z|^2),$$

so that the $\mu$-independent quadratic terms of $H^{\mu}_{\mathrm{wh}}$ are untouched by the transformation, and

$$\hat{G}^{\varepsilon}_{\mathrm{wh}}(Z, W, U) = \mathcal{O}(|Z||(Z, \mu)|\|(W, U)\|_{\mathcal{D}_{\mathrm{c,sh}}} + \mu^2|W|^2 + \|U\|_{\mathcal{D}_{\mathrm{sh}}}\|(W, U)\|_{\mathcal{D}_{\mathrm{c,sh}}}).$$

*Proof.* We consider the near-identity transformation

$$\hat{Z} = Z + d(Z, W), \quad (\hat{W}, \hat{U}) = (W, U),$$

where $d\colon \mathbb{R}^2 \times \mathbb{R}^{2d} \to \mathbb{R}^{2d}$ is bilinear; its inverse is given by

$$Z = \hat{Z} - d(\hat{Z}, \hat{W}) + \mathcal{R}(\hat{Z}, \hat{W}),$$

where $\mathcal{R}$ is analytic and satisfies

$$\mathcal{R}(\hat{Z}, \hat{W}) = \mathcal{O}(|\hat{Z}||\hat{W}|^2).$$

We find that

$$\dot{\hat{Z}} = L^{\mu}_{\mathrm{wh}} \hat{Z} + F^{\mu}(\hat{Z}, \hat{W}, \hat{U}),$$

where

$$\begin{aligned}
&F^{\mu}(\hat{Z}, \hat{W}, \hat{U}) \\
&= L^{\mu}_{\mathrm{wh}}(\mathcal{R}(\hat{Z}, \hat{W}) - d(\hat{Z}, \hat{W})) + H^{\mu}_{\mathrm{wh}}\big(\hat{Z} - d(\hat{Z}, \hat{W}) + \mathcal{R}(\hat{Z}, \hat{W})\big) \\
&\qquad + G^{\mu}_{\mathrm{wh}}\big(\hat{Z} - d(\hat{Z}, \hat{W}) + \mathcal{R}(\hat{Z}, \hat{W}), \hat{W}, \hat{U}\big) \\
&\qquad + d\big(L^{\mu}_{\mathrm{wh}}(\hat{Z} - d(\hat{Z}, \hat{W}) + \mathcal{R}(\hat{Z}, \hat{W})) + H^{\mu}_{\mathrm{wh}}(\hat{Z} - d(\hat{Z}, \hat{W}) \\
&\qquad\qquad + \mathcal{R}(\hat{Z}, \hat{W})) + G^{\mu}_{\mathrm{wh}}(\hat{Z} - d(\hat{Z}, \hat{W}) + \mathcal{R}(\hat{Z}, \hat{W}), \hat{W}, \hat{U}), \hat{W}\big) \\
&\qquad + d\Big(\hat{Z}, L^{\mu}_{\mathrm{c}}\hat{W} + G^{\mu}_{\mathrm{c}}\big(\hat{Z} - d(\hat{Z}, \hat{W}) + \mathcal{R}(\hat{Z}, \hat{W}), \hat{W}, \hat{U}\big) \\
&\qquad\qquad + H^{\mu}_{\mathrm{c}}\big(\hat{Z} - d(\hat{Z}, \hat{W}) + \mathcal{R}(\hat{Z}, \hat{W})\big)\Big) \\
&= H^{\mu}_{\mathrm{wh}}(\hat{Z}) + G^{\mu}_{\mathrm{wh}}(\hat{Z}, \hat{W}, \hat{U}) + d(\hat{Z}, L^0_{\mathrm{c}}\hat{W}) + \mathcal{O}(|\hat{Z}|\|(\hat{W}, \hat{U})\|_{\mathcal{D}_{\mathrm{c,sh}}}|(\hat{Z}, \mu)| + \|(\hat{W}, \hat{U})\|^2_{\mathcal{D}_{\mathrm{c,sh}}}),
\end{aligned}$$

so that

$$F_{0,1,1,0}(\hat{Z}, \hat{W}) = \hat{f}_{0,1,1,0}(\hat{Z}, \hat{W}) + d(\hat{Z}, L^0_{\mathrm{c}}\hat{W}),$$

where $F_{0,1,1,0}$ and $\hat{f}_{0,1,1,0}$ are the parts of $F$ and $\tilde{f}_2^\mu$ that are linear in $\hat{Z}$ and $\hat{W}$. Choosing

$$d(\hat{Z},\hat{W}) = -\hat{f}_{0,1,1,0}(\hat{Z},(L_{\mathrm{c}}^0)^{-1}\hat{W})$$

therefore leads to

$$F_{0,1,1,0} = 0.$$

By also applying a similar near-identity transformation

$$\hat{Z} = Z - \hat{f}_{0,1,0,1}(Z,(L_{\mathrm{sh}}^0)^{-1}U)$$

we achieve $F_{0,1,0,1} = 0$, where $F_{0,1,0,1}$ and $\hat{f}_{0,1,0,1}$ are the parts of $F$ and $\tilde{f}_2^\mu$ that are linear in $Z$ and $U$.

To eliminate terms that are homogeneous of degree 2 in $\hat{W}$ and homogeneous of degree $i \in \{0,1\}$ in $\mu$ we use the near-identity transformations

$$\hat{Z} = Z - \frac{1}{2}\hat{f}_{0,0,2,0}((L_{\mathrm{c}}^0)^{-1}\hat{W},\hat{W})$$

and

$$\hat{Z} = Z - \frac{1}{2\mu}\hat{f}_{1,0,2,0}((L_{\mathrm{c}}^\mu)_1^{-1}\hat{W},\hat{W}),$$

where $\hat{f}_{i,0,2,0}$ is the part of $\tilde{f}_2^\mu$ that is homogeneous of degree $i \in \{0,1\}$ in $\mu$ and quadratic in $\hat{W}$, and $(L_{\mathrm{c}}^\mu)_1$ is the part of $L_{\mathrm{c}}^\mu$ that is linear in $\mu$.

Defining

$$\hat{H}_{\mathrm{wh}}^\mu(\hat{Z}) = F^\mu(\hat{Z},0,0)$$

and

$$\hat{G}_{\mathrm{wh}}^\mu(\hat{Z},\hat{W},\hat{U}) = F^\mu(\hat{Z},\hat{W},\hat{U}) - \hat{H}_{\mathrm{wh}}^\mu(\hat{Z})$$

yields the desired estimates. □

The next step is to apply the normal-form theory presented in Chapter 3 to equations (4.14) – (4.16). Defining

$$\begin{aligned}
\mathcal{X} &= \mathcal{X}_{\mathrm{c,sh}} := \mathcal{X}_{\mathrm{c}} \times \mathcal{X}_{\mathrm{sh}},\\
\mathcal{D} &= \mathcal{D}_{\mathrm{c,sh}} := \mathcal{D}_{\mathrm{c}} \times \mathcal{D}_{\mathrm{sh}},\\
Q &= (W,U)
\end{aligned}$$

and

$$\begin{aligned}
F^\mu(Z,Q) &= L_{\mathrm{wh}}^\mu Z + \hat{G}_{\mathrm{wh}}^\mu(Z,W,U) + \hat{H}_{\mathrm{wh}}^\mu(Z),\\
L^\mu Q &= L_{\mathrm{c,sh}}^\mu(W,U) := (L_{\mathrm{c}}^\mu W, L_{\mathrm{sh}}^\mu U),\\
G^\mu(Z,Q) &= \hat{G}_{\mathrm{c,sh}}^\mu(Z,W,U) := \big(\hat{G}_{\mathrm{c}}^\mu(Z,W,U),\hat{G}_{\mathrm{sh}}^\mu(Z,W,U)\big)
\end{aligned}$$

we are now in the setting described in Chapter 3 and note that

$$\mathrm{d}_1 F^0[0,0] = 0.$$

Hypothesis 3.4 is verified in the following result.

**Lemma 4.7.** The operator $\mathcal{L}\colon \mathcal{P}_k(\mathbb{R}^3;\mathcal{D}_{\mathrm{c,sh}}) \to \mathcal{P}_k(\mathbb{R}^3;\mathcal{D}_{\mathrm{c,sh}})$ defined by

$$(\mathcal{L}\Psi_k)(Y) = L^0_{\mathrm{c,sh}}\Psi_k(Y)$$

is invertible on $\mathcal{P}_k(\mathbb{R}^3;\mathcal{D}_{\mathrm{c,sh}})$ and the operator norm of its inverse

$$\sup_{|\Psi_k|_2=1} |(\mathcal{L})^{-1}\Psi_k|_2$$

is independent of $k$.

*Proof.* The fact that $0 \in \rho(L^0_{\mathrm{c,sh}})$ yields the existence of $(L^0_{\mathrm{c,sh}})^{-1}$ and we find that

$$\Big(\mathcal{L}^{-1}\Psi_k\Big)(Y) = (L^0_{\mathrm{c,sh}})^{-1}\Big(\Psi_k(Y)\Big)$$

for every $\Psi_k \in \mathcal{P}_k(\mathbb{R}^3;\mathcal{D}_{\mathrm{c,sh}})$. From this fact we obtain for an arbitrary polynomial function $\Psi_k \in \mathcal{P}_k(\mathbb{R}^3;\mathcal{D}_{\mathrm{c,sh}})$ given by

$$\Psi_k(Y) = \sum_{|\alpha|=k} v_\alpha y_1^{\alpha_1} y_2^{\alpha_2} y_3^{\alpha_3},$$

where $v_\alpha \in \mathcal{D}_{\mathrm{c,sh}}$, that

$$|\mathcal{L}^{-1}\Psi_k|_2^2 \le \|(L^0_{\mathrm{c,sh}})^{-1}\|_{\mathcal{L}(\mathcal{D}_{\mathrm{c,sh}})} \sum_{|\alpha|=k} \|v_\alpha\|_{\mathcal{D}_{\mathrm{c,sh}}} \alpha_1!\alpha_2!\alpha_3! = \|(L^0_{\mathrm{c,sh}})^{-1}\|_{\mathcal{L}(\mathcal{D}_{\mathrm{c,sh}})} |\Psi_k|_2^2$$

holds independently of $\mu$ and $k$. □

Chapter 3 therefore yields the existence of a near-identity, finite-dimensional change of variable

$$\tilde{Y} = Y := (Z,\mu), \quad \tilde{W} = W + \Phi_{\mathrm{c}}(Y), \quad \tilde{U} = U + \Phi_{\mathrm{sh}}(Y)$$

satisfying

$$\Phi(Z,\mu) = \mathcal{O}(|(Z,\mu)||Z|^2) \tag{4.17}$$

which transforms equations (4.14) – (4.16) into

$$\begin{aligned}
\dot{Z} &= L^\mu_{\mathrm{wh}} Z + \tilde{G}^\mu_{\mathrm{wh}}(Z,W,U) + \tilde{H}^\mu_{\mathrm{wh}}(Z),\\
\dot{W} &= L^\mu_{\mathrm{c}} W + \tilde{G}^\mu_{\mathrm{c}}(Z,W,U) + \tilde{H}^\mu_{\mathrm{c}}(Z),\\
\dot{U} &= L^\mu_{\mathrm{sh}} U + \tilde{G}^\mu_{\mathrm{sh}}(Z,W,U) + \tilde{H}^\mu_{\mathrm{sh}}(Z)
\end{aligned}$$

and $\delta > 0$ such that

$$\begin{aligned}
\|\tilde{H}_{\mathrm{c,sh}}(Y)\|_{\mathcal{X}_{\mathrm{c,sh}}} &\lesssim \mu^2 \mathrm{e}^{-\frac{c^*}{2\mu}},\\
\|\mathrm{d}\tilde{H}_{\mathrm{c,sh}}(Y)\|_{\mathcal{L}(\mathbb{R}^3;\mathcal{X}_{\mathrm{c,sh}})} &\lesssim \mu \mathrm{e}^{-\frac{c^*}{2\mu}}
\end{aligned}$$

for $|Y| \le \delta$. Equations (3.6) – (3.8) and estimate (4.17) imply the following estimates for the transformed nonlinearities.

**Remark 4.8.** The transformed nonlinearities satisfy

(i) $\tilde{H}^{\mu}_{\mathrm{wh}}(Z) - \hat{H}^{\mu}_{\mathrm{wh}}(Z) = \mathcal{O}\big(|(Z,\mu)||Z|^2\big)$, so that the $\mu$-independent quadratic terms of $H^{\mu}_{\mathrm{wh}}$ are untouched by the transformation,

(ii) $\tilde{G}^{\mu}_{\mathrm{wh}}(Z,W,U) = \mathcal{O}\Big(|Z||(Z,\mu)|\|(W,U)\|_{\mathcal{D}_{\mathrm{c,sh}}} + \mu^2|W|^2 + \|U\|_{\mathcal{D}_{\mathrm{sh}}}\|(W,U)\|_{\mathcal{D}_{\mathrm{c,sh}}}\Big)$,

(iii) $\tilde{G}^{\mu}_{\mathrm{c,sh}}(Z,W,U) = \mathcal{O}\big(|Z||(Z,\mu)|^3\|(W,U)\|_{\mathcal{D}_{\mathrm{c,sh}}} + |(Z,\mu)|^2\|(W,U)\|^2_{\mathcal{D}_{\mathrm{c,sh}}}\big)$,

(iv) $\mathrm{d}\tilde{H}^{\mu}_{\mathrm{wh}}[Z] = \mathcal{O}\big(|Z|\big)$,

(v) $\mathrm{d}_1\tilde{G}^{\mu}_{\mathrm{wh}}[Z,W,U] = \mathcal{O}\Big(|(Z,\mu)|\|(W,U)\|_{\mathcal{D}_{\mathrm{c,sh}}} + \|(W,U)\|^2_{\mathcal{D}_{\mathrm{c,sh}}}\Big)$,

(vi) $\mathrm{d}_2\tilde{G}^{\mu}_{\mathrm{wh}}[Z,W,U], \mathrm{d}_3\tilde{G}^{\mu}_{\mathrm{wh}}[Z,W,U] = \mathcal{O}\big(|Z|^2 + \mu|Z| + \|(W,U)\|_{\mathcal{D}_{\mathrm{c,sh}}}\big)$,

(vii) $\mathrm{d}_1\tilde{G}^{\mu}_{\mathrm{c,sh}}[Z,W,U] = \mathcal{O}\big(|(Z,\mu)|^3\|(W,U)\|_{\mathcal{D}_{\mathrm{c,sh}}} + |(Z,\mu)|\|(W,U)\|^2_{\mathcal{D}_{\mathrm{c,sh}}}\big)$,

(viii) $\mathrm{d}_2\tilde{G}^{\mu}_{\mathrm{c,sh}}[Z,W,U], \mathrm{d}_3\tilde{G}^{\mu}_{\mathrm{c,sh}}[Z,W,U] = \mathcal{O}\big(|Z||(Z,\mu)|^3 + |(Z,\mu)|^2\|(W,U)\|_{\mathcal{D}_{\mathrm{c,sh}}}\big)$.

## 4.2 The $(\mathrm{i}\omega)^2$ resonance

In the case $n = 4$ we assume that equations (4.1) – (4.3) satisfy the following additional assumptions.

(C1) The right-hand side of equations (4.1) – (4.3) satisfies the estimates

$$h^{\varepsilon}_{\mathrm{wh}}(z) = \mathcal{O}(|z|^3),$$
$$h^{\varepsilon}_{\mathrm{c}}(z), h^{\varepsilon}_{\mathrm{sh}}(z) = \mathcal{O}(|z|^2).$$

(C2) The spectrum of the complexified linear operator $L^{\varepsilon}_{\mathrm{wh}} \in \mathbb{C}^{4\times 4}$ exhibits an $(\mathrm{i}\omega)^2$ resonance at $\varepsilon = 0$, meaning that there exists $\omega \geq 0$ such that as $\varepsilon \uparrow 0$ two pairs of purely imaginary eigenvalues of $L^{\varepsilon}_{\mathrm{wh}}$ collide to form geometrically simple and algebraically double eigenvalues $\pm\mathrm{i}\omega$ and split into a complex eigenvalue quartet for $\varepsilon > 0$ (see Figure 4.5). When working with complex coordinates we write $z \in \mathbb{R}^4$ as

$$z = z_1 e + z_2 f + \bar{z}_1\bar{e} + \bar{z}_2\bar{f}, \quad z_1, z_2 \in \mathbb{C},$$

where $(L^0_{\mathrm{wh}} - \mathrm{i}\omega I)e = 0$, $(L^0_{\mathrm{wh}} - \mathrm{i}\omega I)f = e$, and assume that

$$L^{\varepsilon}_{\mathrm{wh}} = \begin{pmatrix} \mathrm{i}(\omega+\sigma^{\varepsilon}) & 1 & 0 & 0 \\ (\lambda^{\varepsilon})^2 & \mathrm{i}(\omega+\sigma^{\varepsilon}) & 0 & 0 \\ 0 & 0 & -\mathrm{i}(\omega+\sigma^{\varepsilon}) & 1 \\ 0 & 0 & (\lambda^{\varepsilon})^2 & -\mathrm{i}(\omega+\sigma^{\varepsilon}) \end{pmatrix},$$

where $\lambda^{\varepsilon}$, $\sigma^{\varepsilon}$ are analytic functions of $\varepsilon$ with

$$\lambda^{\varepsilon} = \varepsilon + \mathcal{O}(\varepsilon^2), \qquad \sigma^{\varepsilon} = \mathcal{O}(\varepsilon).$$

Additionally, we assume that $S_{\mathrm{wh}}(z_1, z_2) = (\bar{z}_1, -\bar{z}_2)$ and $k\omega \neq \omega_j^0$ for all $j \in \{1, \dots, d\}$ and $k \in \mathbb{N}$.

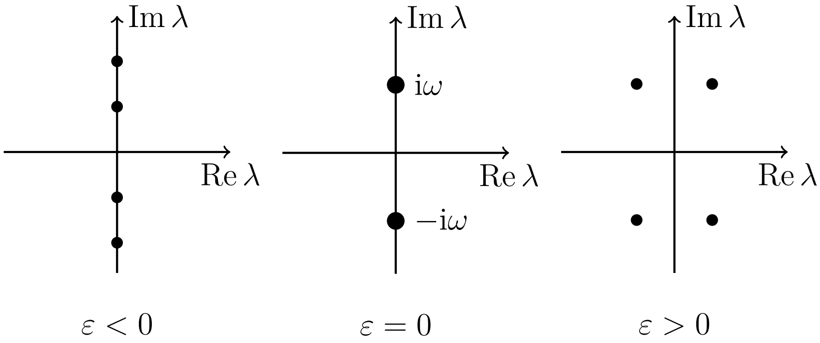



Figure 4.5: The $(\mathrm{i}\omega)^2$ resonance at $\varepsilon = 0$.

(C3) The conserved quantity $\mathcal{I}^\varepsilon$ satisfies

$$\mathcal{I}^\varepsilon((z_1, z_2), w, u) = \mathcal{O}(\|(z_2, w, u)\|_{\mathbb{R}^2 \times \mathbb{R}^{2d} \times \mathcal{D}_{\mathrm{sh}}} \|(z_1, z_2, w, u)\|_{\mathcal{D}} + \varepsilon \|(z_1, z_2, w, u)\|_{\mathcal{D}}^2)$$

(in complex coordinates).

Since the scaling in this case leads to slightly worse estimates for the nonlinearities we need to eliminate the $\varepsilon$-independent term of $h^\varepsilon_{\mathrm{c,sh}}$ which is homogeneous of degree two in $z$. This change of variable affects the coefficient in $h^\varepsilon_{\mathrm{wh}}$ (part of the cubic terms of $h^0_{\mathrm{wh}}$) which determines whether homoclinic bifurcation takes place. We therefore eliminate this term as a separate preliminary step by writing

$$\tilde{z} = z, \quad \tilde{w} = w + X_{\mathrm{c}}(z, z), \quad \tilde{u} = u + X_{\mathrm{sh}}(z, z),$$

where $X \coloneqq (X_{\mathrm{c}}, X_{\mathrm{sh}}) \colon \mathbb{R}^4 \times \mathbb{R}^4 \to \mathcal{D}_{\mathrm{c,sh}}$ is a bounded symmetric bilinear operator. Applying a transformation of this kind leads to the equations

$$\dot{\tilde{z}} = L^\varepsilon_{\mathrm{wh}} \tilde{z} + \tilde{g}^\varepsilon_{\mathrm{wh}}(\tilde{z}, \tilde{w}, \tilde{u}) + \tilde{h}^\varepsilon_{\mathrm{wh}}(\tilde{z}), \tag{4.18}$$

$$\dot{\tilde{w}} = L^\varepsilon_{\mathrm{c}} \tilde{w} + \tilde{g}^\varepsilon_{\mathrm{c}}(\tilde{z}, \tilde{w}, \tilde{u}) + \tilde{h}^\varepsilon_{\mathrm{c}}(\tilde{z}), \tag{4.19}$$

$$\dot{\tilde{u}} = L^\varepsilon_{\mathrm{sh}} \tilde{u} + \tilde{g}^\varepsilon_{\mathrm{sh}}(\tilde{z}, \tilde{w}, \tilde{u}) + \tilde{h}^\varepsilon_{\mathrm{sh}}(\tilde{z}), \tag{4.20}$$

where

$$\begin{aligned}
\tilde{g}^\varepsilon_{\mathrm{wh}}(\tilde{z}, \tilde{w}, \tilde{u}) &= g^\varepsilon_{\mathrm{wh}}\Big(\tilde{z}, \tilde{w} - X_{\mathrm{c}}(\tilde{z}, \tilde{z}), \tilde{u} - X_{\mathrm{sh}}(\tilde{z}, \tilde{z})\Big) - g^\varepsilon_{\mathrm{wh}}\Big(\tilde{z}, -X_{\mathrm{c}}(\tilde{z}, \tilde{z}), -X_{\mathrm{sh}}(\tilde{z}, \tilde{z})\Big), \\
\tilde{g}^\varepsilon_{\mathrm{c}}(\tilde{z}, \tilde{w}, \tilde{u}) &= 2X_{\mathrm{c}}\Big(\tilde{z}, (f^\varepsilon\Big(\tilde{z}, \tilde{w} - X_{\mathrm{c}}(\tilde{z}, \tilde{z}), \tilde{u} - X_{\mathrm{sh}}(\tilde{z}, \tilde{z})\Big) - f^\varepsilon\Big(\tilde{z}, -X_{\mathrm{c}}(\tilde{z}, \tilde{z}), -X_{\mathrm{sh}}(\tilde{z}, \tilde{z})\Big)\Big) \\
&\quad + g^\varepsilon_{\mathrm{c}}\Big(\tilde{z}, \tilde{w} - X_{\mathrm{c}}(\tilde{z}, \tilde{z}), \tilde{u} - X_{\mathrm{sh}}(\tilde{z}, \tilde{z})\Big) - g^\varepsilon_{\mathrm{c}}\Big(\tilde{z}, -X_{\mathrm{c}}(\tilde{z}, \tilde{z}), -X_{\mathrm{sh}}(\tilde{z}, \tilde{z})\Big), \\
\tilde{g}^\varepsilon_{\mathrm{sh}}(\tilde{z}, \tilde{w}, \tilde{u}) &= 2X_{\mathrm{sh}}\Big(\tilde{z}, (f^\varepsilon\Big(\tilde{z}, \tilde{w} - X_{\mathrm{c}}(\tilde{z}, \tilde{z}), \tilde{u} - X_{\mathrm{sh}}(\tilde{z}, \tilde{z})\Big) \\
&\qquad - f^\varepsilon\Big(\tilde{z}, -X_{\mathrm{c}}(\tilde{z}, \tilde{z}), -X_{\mathrm{sh}}(\tilde{z}, \tilde{z})\Big)\Big) \\
&\quad + g^\varepsilon_{\mathrm{sh}}\Big(\tilde{z}, \tilde{w} - X_{\mathrm{c}}(\tilde{z}, \tilde{z}), \tilde{u} - X_{\mathrm{sh}}(\tilde{z}, \tilde{z})\Big) - g^\varepsilon_{\mathrm{sh}}\Big(\tilde{z}, -X_{\mathrm{c}}(\tilde{z}, \tilde{z}), -X_{\mathrm{sh}}(\tilde{z}, \tilde{z})\Big), \\
\tilde{h}^\varepsilon_{\mathrm{wh}}(\tilde{z}) &= h^\varepsilon_{\mathrm{wh}}(\tilde{z}) + g^\varepsilon_{\mathrm{wh}}\Big(\tilde{z}, -X_{\mathrm{c}}(\tilde{z}, \tilde{z}), -X_{\mathrm{sh}}(\tilde{z}, \tilde{z})\Big),
\end{aligned}$$

$$\tilde{h}_{\mathrm{c}}^{\varepsilon}(\tilde{z}) = -L_{\mathrm{c}}^{\varepsilon} X_{\mathrm{c}}(\tilde{z},\tilde{z}) + 2X_{\mathrm{c}}\Big(\tilde{z}, f^{\varepsilon}\big(\tilde{z}, -X_{\mathrm{c}}(\tilde{z},\tilde{z}), -X_{\mathrm{sh}}(\tilde{z},\tilde{z})\big)\Big) + g_{\mathrm{c}}^{\varepsilon}\big(\tilde{z}, X_{\mathrm{c}}(\tilde{z},\tilde{z}), -X_{\mathrm{sh}}(\tilde{z},\tilde{z})\big) + h_{\mathrm{c}}^{\varepsilon}(\tilde{z}),$$

$$\tilde{h}_{\mathrm{sh}}^{\varepsilon}(\tilde{z}) = -L_{\mathrm{sh}}^{\varepsilon} X_{\mathrm{sh}}(\tilde{z},\tilde{z}) + 2X_{\mathrm{sh}}\Big(\tilde{z}, f^{\varepsilon}\big(\tilde{z}, 0, -X_{\mathrm{sh}}(\tilde{z},\tilde{z})\big)\Big) + g_{\mathrm{sh}}^{\varepsilon}\big(\tilde{z}, -X_{\mathrm{c}}(\tilde{z},\tilde{z}), -X_{\mathrm{sh}}(\tilde{z},\tilde{z})\big) + h_{\mathrm{sh}}^{\varepsilon}(\tilde{z})$$

and

$$f^{\varepsilon}(\tilde{z},\tilde{w},\tilde{u}) = L_{\mathrm{wh}}^{\varepsilon}\tilde{z} + g_{\mathrm{wh}}^{\varepsilon}(\tilde{z},\tilde{w},\tilde{u}) + h_{\mathrm{wh}}^{\varepsilon}(\tilde{z}),$$

so that the quadratic terms of $\tilde{h}_{\mathrm{c,sh}}^{0}$ are given by

$$-L_{\mathrm{c,sh}}^{0} X(\tilde{z},\tilde{z}) + X(\tilde{z}, 2L_{\mathrm{wh}}^{0}\tilde{z}) + \frac{1}{2}\,\mathrm{d}^2 h_{\mathrm{c,sh}}^{0}[0](\tilde{z},\tilde{z}),$$

while the cubic terms of $\tilde{h}_{\mathrm{wh}}^{0}$ are

$$\frac{1}{6}\,\mathrm{d}^3 h_{\mathrm{wh}}^{0}[0](\tilde{z},\tilde{z},\tilde{z}) - \mathrm{d}^2 g_{\mathrm{wh}}^{0}[0]\Big((\tilde{z},0,0),(0, X_{\mathrm{c}}(\tilde{z},\tilde{z}), X_{\mathrm{sh}}(\tilde{z},\tilde{z}))\Big). \tag{4.21}$$

We therefore have to solve the operator equation

$$-L_{\mathrm{c,sh}}^{0} X(\tilde{z},\tilde{z}) + X(\tilde{z}, 2L_{\mathrm{wh}}^{0}\tilde{z}) = -\frac{1}{2}\,\mathrm{d}^2 h_{\mathrm{c,sh}}^{0}[0](\tilde{z},\tilde{z}). \tag{4.22}$$

The following result was proved by Arendt et al. [2].

**Lemma 4.9.** Suppose that $E$, $F$ are Banach spaces, $A\colon \mathcal{D}(A) \subseteq E \to E$ is a closed linear operator, $B \in \mathcal{L}(F;F)$ and the spectrum of $-B$ is separated from the spectrum of $A$ by a finite number of simple closed curves $\gamma$. The Sylvester equation

$$AX + XB = Y \tag{4.23}$$

has a unique solution $X \in \mathcal{L}(F;E)$ for each $Y \in \mathcal{L}(F;E)$, and this solution is given by the formula

$$X = \frac{1}{2\pi\mathrm{i}} \int_{\gamma} (\lambda I - A)^{-1} Y (\lambda I + B)^{-1}\,\mathrm{d}\lambda. \tag{4.24}$$

**Proposition 4.10.** Suppose that the hypotheses of Lemma 4.9 hold. The solution $X$ to equation (4.23) lies in $\mathcal{L}(F;\mathcal{D}(A))$ for each $Y \in \mathcal{L}(F;E)$. Furthermore $X$ depends continuously on $Y$ in this sense with

$$\|X\|_{\mathcal{L}(F;\mathcal{D}(A))} \lesssim \sup_{\lambda\in\operatorname{Im}\gamma} \|(\lambda I - A)^{-1}\|_{\mathcal{L}(E;\mathcal{D}(A))} \|(\lambda I + B)^{-1}\|_{\mathcal{L}(F;F)} \|Y\|_{\mathcal{L}(F;E)}.$$

*Proof.* We observe that the resolvent of $A$ is a holomorphic function $\rho(A) \to \mathcal{L}(E;\mathcal{D}(A))$, so that $(\lambda I - A)^{-1} Y (\lambda I + B)^{-1} \in \mathcal{L}(F;\mathcal{D}(A))$ depends continuously on $\lambda \in \operatorname{Im}\gamma$; the integral on the right-hand side of equation (4.24) therefore converges in this sense. Furthermore $X \mapsto AX + XB$ is a bounded bijective mapping $\mathcal{L}(F;\mathcal{D}(A)) \to \mathcal{L}(F;E)$, so that by the inverse mapping theorem its inverse is also continuous; the estimate follows directly from equation (4.24). □

Applying Proposition 4.10 with $A = -L^0_{\mathrm{c,sh}}$ and $B = 2L^0_{\mathrm{wh}}$ to the equation

$$-L^0_{\mathrm{c,sh}} X(\tilde{z},\cdot) + X(\tilde{z}, 2L^0_{\mathrm{wh}}) = -\frac{1}{2}\,\mathrm{d}^2 h^0_{\mathrm{c,sh}}[0](\tilde{z},\cdot)$$

yields a unique solution $X(\tilde{z},\cdot) \in \mathcal{L}(\mathbb{R}^4; \mathcal{D}_{\mathrm{c,sh}})$ which depends linearly and continuously on $\tilde{z} \in \mathbb{R}^4$. It follows that $X$ is a bounded symmetric bilinear operator $\mathbb{R}^4 \times \mathbb{R}^4 \to \mathcal{D}_{\mathrm{c,sh}}$ which satisfies equation (4.22). The transformed nonlinearities satisfy

$$\begin{aligned} g^\varepsilon_{\mathrm{wh}}(z,w,u), g^\varepsilon_{\mathrm{c}}(z,w,u), g^\varepsilon_{\mathrm{sh}}(z,w,u) &= \mathcal{O}(\|(z,w,u)\|_{\mathcal{D}} \|(w,u)\|_{\mathcal{D}_{\mathrm{c,sh}}}), \\ h^\varepsilon_{\mathrm{wh}}(z) &= \mathcal{O}(|z|^3), \\ h^\varepsilon_{\mathrm{c}}(z), h^\varepsilon_{\mathrm{sh}}(z) &= \mathcal{O}(|(\varepsilon,z)||z|^2), \end{aligned}$$

where we have dropped the tildes for notational simplicity.

Now we turn to the approximate equation

$$\dot{z} = L^\varepsilon_{\mathrm{wh}} z + h^\varepsilon_{\mathrm{wh}}(z). \tag{4.25}$$

The following result is a straightforward application of the normal-form theory given by Iooss and Pérouème [12].

**Proposition 4.11.** There is a polynomial change of variable

$$\tilde{z} = z + \sum_{\substack{i+j=3 \\ j\geq 3}}^{5} \varepsilon^i \Phi^i_j(z), \tag{4.26}$$

where $\Phi^i_j$ is homogeneous of degree $j$ in $z$, which converts equation (4.25) into

$$\dot{\tilde{z}} = L_{\mathrm{wh}}\tilde{z} + \tilde{h}_{\mathrm{wh}}(\tilde{z}),$$

where in the same notation

$$R_\theta \tilde{h}^i_{\mathrm{wh},j}(\tilde{z}) = \tilde{h}^i_{\mathrm{wh},j}(R_\theta \tilde{z}), \quad i+j \in \{3,4,5\}, \quad j \geq 3$$

for all rotations

$$R_\theta(z_1, z_2, \bar{z}_1, \bar{z}_2) = (\mathrm{e}^{\mathrm{i}\theta} z_1, \mathrm{e}^{\mathrm{i}\theta} z_2, \mathrm{e}^{-\mathrm{i}\theta}\bar{z}_1, \mathrm{e}^{-\mathrm{i}\theta}\bar{z}_2), \quad \theta \in [0, 2\pi].$$

In particular $\tilde{h}^0_{\mathrm{wh},3}$ is obtained from $h^0_{\mathrm{wh},3}$ by the removal of all monomials which are not equivalent under rotations.

Applying the change of variable (4.26) to the complete system

$$\begin{aligned} \dot{z} &= L^\varepsilon_{\mathrm{wh}} z + g^\varepsilon_{\mathrm{wh}}(z,w,u) + h^\varepsilon_{\mathrm{wh}}(z), \\ \dot{w} &= L^\varepsilon_{\mathrm{c}} z + g^\varepsilon_{\mathrm{c}}(z,w,u) + h^\varepsilon_{\mathrm{c}}(z), \\ \dot{u} &= L^\varepsilon_{\mathrm{sh}} z + g^\varepsilon_{\mathrm{sh}}(z,w,u) + h^\varepsilon_{\mathrm{sh}}(z), \end{aligned}$$

transforms $h^\varepsilon_{\mathrm{wh}}$ in the same way, while the estimates for the other nonlinearities remain unchanged. Without loss of generality we therefore assume that

$$R_\theta h^i_{\mathrm{wh},j}(z) = h^i_{\mathrm{wh},j}(R_\theta z), \quad i+j \in \{3,4,5\}, \quad j \geq 3$$

for all $\theta \in [0, 2\pi]$, and we write equation (4.25) as

$$\dot{z}_1 = \mathrm{i}(\omega + \sigma^\varepsilon)z_1 + z_2 + \mathcal{P}_1^\varepsilon(z_1, z_2, \bar{z}_1, \bar{z}_2) + \mathcal{R}_1^\varepsilon(z_1, z_2, \bar{z}_1, \bar{z}_2), \quad (4.27)$$
$$\dot{z}_2 = (\lambda^\varepsilon)^2 z_1 + \mathrm{i}(\omega + \sigma^\varepsilon)z_2 + \mathcal{P}_2^\varepsilon(z_1, z_2, \bar{z}_1, \bar{z}_2) + \mathcal{R}_2^\varepsilon(z_1, z_2, \bar{z}_1, \bar{z}_2), \quad (4.28)$$

where

$$\overline{\mathcal{P}_1^\varepsilon(\bar{z}_1, \bar{z}_2, \bar{z}_1, \bar{z}_2)} = -\mathcal{P}_1^\varepsilon(z_1, -z_2, \bar{z}_1, -\bar{z}_2), \quad \overline{\mathcal{P}_2^\varepsilon(\bar{z}_1, \bar{z}_2, \bar{z}_1, \bar{z}_2)} = \mathcal{P}_2^\varepsilon(z_1, -z_2, \bar{z}_1, -\bar{z}_2),$$
$$\overline{\mathcal{R}_1^\varepsilon(\bar{z}_1, \bar{z}_2, \bar{z}_1, \bar{z}_2)} = -\mathcal{R}_1^\varepsilon(z_1, -z_2, \bar{z}_1, -\bar{z}_2), \quad \overline{\mathcal{R}_2^\varepsilon(\bar{z}_1, \bar{z}_2, \bar{z}_1, \bar{z}_2)} = \mathcal{R}_2^\varepsilon(z_1, -z_2, \bar{z}_1, -\bar{z}_2),$$

as well as

$$\mathcal{P}_j^\varepsilon(\mathrm{e}^{\mathrm{i}\theta} z_1, \mathrm{e}^{\mathrm{i}\theta} z_2, \mathrm{e}^{-\mathrm{i}\theta}\bar{z}_1, \mathrm{e}^{-\mathrm{i}\theta}\bar{z}_2) = \mathrm{e}^{\mathrm{i}\theta}\mathcal{P}_j^\varepsilon(z_1, z_2, \bar{z}_1, \bar{z}_2),$$
$$|\mathcal{R}_j^\varepsilon(z_1, z_2, \bar{z}_1, \bar{z}_2)| = \mathcal{O}(|(\varepsilon, z)|^3|z|^3)$$

for $\theta \in [0, 2\pi]$ and $j \in \{1, 2\}$. At this point we make an additional hypothesis concerning the term of $h_{\mathrm{wh}}^\varepsilon$ which determines whether equation (4.25) has a homoclinic solution.

(C4) We assume that the coefficient of $z_1|z_1|^2$ in $\mathcal{P}_2^\varepsilon(z_1, z_2, \bar{z}_1, \bar{z}_2)$ is negative and can therefore be written as $-C$ for some $C > 0$.

Changing to real coordinates by writing

$$z_1 = x_1 + \mathrm{i}x_2, \qquad z_2 = y_1 + \mathrm{i}y_2,$$

we find that (with a slight abuse of notation)

$$\dot{x} = (\omega + \sigma^\varepsilon)R_{\frac{\pi}{2}}x + y + \mathcal{P}_1^\varepsilon(x, y) + \mathcal{R}_1^\varepsilon(x, y), \quad (4.29)$$
$$\dot{y} = (\omega + \sigma^\varepsilon)R_{\frac{\pi}{2}}y + (\lambda^\varepsilon)^2 x + \mathcal{P}_2^\varepsilon(x, y) + \mathcal{R}_2^\varepsilon(x, y), \quad (4.30)$$

where $x = (x_1, x_2)^{\mathrm{T}}$, $y = (y_1, y_2)^{\mathrm{T}}$.

**Lemma 4.12.** For fixed $\nu \in (0, 1)$ and $\varepsilon > 0$ the system (4.29) and (4.30) has a pair of reversible homoclinic solutions, that is invariant under the transformation $(x_1, x_2, y_1, y_2)(t) \mapsto (x_1, -x_2, -y_1, y_2)(-t)$, of the form

$$p^{\varepsilon\pm}(t) = R_{(\omega+\sigma^\varepsilon)t}\begin{pmatrix} \lambda^\varepsilon(\check{p}_1^\pm(\lambda^\varepsilon t), \check{p}_2^\pm(\lambda^\varepsilon t))^{\mathrm{T}} \\ (\lambda^\varepsilon)^2(\check{q}_1^\pm(\lambda^\varepsilon t), \check{q}_2^\pm(\lambda^\varepsilon t))^{\mathrm{T}} \end{pmatrix}$$

which satisfy the estimate

$$|\check{p}_j^\pm(t)|, |\check{q}_j^\pm(t)| \lesssim \mathrm{e}^{-\nu|t|}.$$

*Proof.* Writing

$$y = \dot{x} - (\omega + \sigma^\varepsilon)R_{\frac{\pi}{2}}x + v,$$

we find from equation (4.29) that

$$v + \mathcal{P}_1^\varepsilon(x, \dot{x} - (\omega + \sigma^\varepsilon)R_{\frac{\pi}{2}}x + v) + \mathcal{R}_1^\varepsilon(x, \dot{x} - (\omega + \sigma^\varepsilon)R_{\frac{\pi}{2}}x + v) = 0.$$

We seek solutions to this equation of the form

$$v = v_1^\varepsilon(x, \dot{x} - (\omega + \sigma^\varepsilon)R_{\frac{\pi}{2}}x) + v_2^\varepsilon(x, \dot{x} - (\omega + \sigma^\varepsilon)R_{\frac{\pi}{2}}x),$$

where

$$v_1^\varepsilon + \mathcal{P}_1^\varepsilon(x, \dot{x} - (\omega + \sigma^\varepsilon)R_{\frac{\pi}{2}}x + v_1^\varepsilon) = 0 \tag{4.31}$$

and

$$\begin{aligned} v_2^\varepsilon &+ \mathcal{P}_1^\varepsilon(x, \dot{x} - (\omega + \sigma^\varepsilon)R_{\frac{\pi}{2}}x + v_1^\varepsilon(x, \dot{x} - (\omega + \sigma^\varepsilon)R_{\frac{\pi}{2}}x) + v_2^\varepsilon) \\ &- \mathcal{P}_1^\varepsilon(x, \dot{x} - (\omega + \sigma^\varepsilon)R_{\frac{\pi}{2}}x + v_1^\varepsilon(x, \dot{x} - (\omega + \sigma^\varepsilon)R_{\frac{\pi}{2}}x)) \\ &+ \mathcal{R}_1^\varepsilon(x, \dot{x} - (\omega + \sigma^\varepsilon)R_{\frac{\pi}{2}}x + v_1^\varepsilon(x, \dot{x} - (\omega + \sigma^\varepsilon)R_{\frac{\pi}{2}}x) + v_2^\varepsilon) = 0. \end{aligned} \tag{4.32}$$

Using the implicit function theorem we obtain unique solutions $v_1^\varepsilon = v_1^\varepsilon(x, \dot{x} - (\omega + \sigma^\varepsilon)R_{\frac{\pi}{2}}x)$ and $v_2^\varepsilon = v_2^\varepsilon(x, \dot{x} - (\omega + \sigma^\varepsilon)R_{\frac{\pi}{2}}x)$ to equations (4.31) and (4.32) respectively depending analytically upon $\varepsilon$, $x$ and $\dot{x} - (\omega + \sigma^\varepsilon R_{\frac{\pi}{2}}x)$ and satisfying

$$\begin{aligned} v_1^\varepsilon(x, \dot{x} - (\omega + \sigma^\varepsilon)R_{\frac{\pi}{2}}x) &= \mathcal{O}(|(x, \dot{x} - (\omega + \sigma^\varepsilon)R_{\frac{\pi}{2}}x)|^3), \\ R_\theta v_1^\varepsilon(x, \dot{x} - (\omega + \sigma^\varepsilon)R_{\frac{\pi}{2}}x) &= v_1^\varepsilon(R_\theta x, R_\theta(\dot{x} - (\omega + \sigma^\varepsilon)R_{\frac{\pi}{2}}x)) \end{aligned}$$

for all $\theta \in [0, 2\pi)$ and

$$v_2^\varepsilon(x, \dot{x} - (\omega + \sigma^\varepsilon)R_{\frac{\pi}{2}}x) = \mathcal{O}(|(x, \dot{x} - (\omega + \sigma^\varepsilon)R_{\frac{\pi}{2}}x)|^3 |(\varepsilon, x, \dot{x} - (\omega + \sigma^\varepsilon)R_{\frac{\pi}{2}}x)|).$$

Substituting

$$y = \dot{x} - (\omega + \sigma^\varepsilon)R_{\frac{\pi}{2}}x + v_1^\varepsilon + v_2^\varepsilon$$

into equation (4.30) we obtain that

$$\begin{aligned} (\partial_t - (\omega + \sigma^\varepsilon)R_{\frac{\pi}{2}})^2 x = &-(\partial_t - (\omega + \sigma^\varepsilon)R_{\frac{\pi}{2}})(v_1^\varepsilon + v_2^\varepsilon) \\ &+ (\lambda^\varepsilon)^2 x + \hat{\mathcal{P}}^\varepsilon(x, \dot{x} - (\omega + \sigma^\varepsilon)R_{\frac{\pi}{2}}x) + \hat{\mathcal{R}}^\varepsilon(x, \dot{x} - (\omega + \sigma^\varepsilon)R_{\frac{\pi}{2}}x), \end{aligned}$$

where

$$\begin{aligned} \hat{\mathcal{P}}^\varepsilon(x, \dot{x} - (\omega + \sigma^\varepsilon)R_{\frac{\pi}{2}}x) &= \mathcal{P}_2^\varepsilon(x, \dot{x} - (\omega + \sigma^\varepsilon)R_{\frac{\pi}{2}}x + v_1^\varepsilon), \\ \hat{\mathcal{R}}^\varepsilon(x, \dot{x} - (\omega + \sigma^\varepsilon)R_{\frac{\pi}{2}}x) &= \mathcal{P}_2^\varepsilon(x, \dot{x} - (\omega + \sigma^\varepsilon)R_{\frac{\pi}{2}}x + v_1^\varepsilon + v_2^\varepsilon) - \mathcal{P}_2^\varepsilon(x, \dot{x} - (\omega + \sigma^\varepsilon)R_{\frac{\pi}{2}}x + v_1^\varepsilon) \\ &\quad + \mathcal{R}_2^\varepsilon(x, \dot{x} - (\omega + \sigma^\varepsilon)R_{\frac{\pi}{2}}x + v_1^\varepsilon + v_2^\varepsilon). \end{aligned}$$

Therefore, we find that

$$\begin{aligned} (\partial_t - (\omega + \sigma^\varepsilon)R_{\frac{\pi}{2}})^2 x = &-\partial_1 v_1^\varepsilon(\dot{x} - (\omega + \sigma^\varepsilon)R_{\frac{\pi}{2}}x) - \partial_2 v_1^\varepsilon(\partial_t - (\omega + \sigma^\varepsilon)R_{\frac{\pi}{2}})^2 x \\ &+ (\lambda^\varepsilon)^2 x + \hat{\mathcal{P}}^\varepsilon(x, \dot{x} - (\omega + \sigma^\varepsilon)R_{\frac{\pi}{2}}x) \\ &- \partial_1 v_2^\varepsilon(\dot{x} - (\omega + \sigma^\varepsilon)R_{\frac{\pi}{2}}x) - \partial_2 v_2^\varepsilon(\partial_t - (\omega + \sigma^\varepsilon)R_{\frac{\pi}{2}})^2 x \\ &- \partial_1 v_2^\varepsilon(\omega + \sigma^\varepsilon)R_{\frac{\pi}{2}}x - \partial_2 v_2^\varepsilon(\omega + \sigma^\varepsilon)R_{\frac{\pi}{2}}(\dot{x} - (\omega + \sigma^\varepsilon)R_{\frac{\pi}{2}}x) \\ &+ (\omega + \sigma^\varepsilon)R_{\frac{\pi}{2}}v_2^\varepsilon + \hat{\mathcal{R}}^\varepsilon(x, \dot{x} - (\omega + \sigma^\varepsilon)R_{\frac{\pi}{2}}x), \end{aligned} \tag{4.33}$$

where we have used the notation $\partial_j v_k^\varepsilon = \mathrm{d}_j v_k^\varepsilon[x, \dot{x} - (\omega + \sigma^\varepsilon)R_{\frac{\pi}{2}}x]$ and the calculation

$$\begin{aligned} &(\partial_t - sR_{\frac{\pi}{2}})v_1^\varepsilon(x, \dot{x} - (\omega + \sigma^\varepsilon)R_{\frac{\pi}{2}}x) \\ &\quad = \partial_1 v_1^\varepsilon(x, \dot{x} - (\omega + \sigma^\varepsilon)R_{\frac{\pi}{2}}x)(\dot{x} - (\omega + \sigma^\varepsilon)R_{\frac{\pi}{2}}x) \\ &\qquad + \partial_2 v_1^\varepsilon(x, \dot{x} - (\omega + \sigma^\varepsilon)R_{\frac{\pi}{2}}x)(\partial_t - (\omega + \sigma^\varepsilon)R_{\frac{\pi}{2}})^2 x \end{aligned}$$

based on the rotational invariance of $v_1^\varepsilon$.

Introducing the scaled variables

$$x(t) = \lambda^\varepsilon R_{(\omega+\sigma^\varepsilon)t}\check{x}(\check{t}), \quad \check{t} = \lambda^\varepsilon t,$$

equation (4.33) transforms into

$$\ddot{\check{x}} = \check{x} - C\check{x}|\check{x}|^2 + \mathcal{P}^\varepsilon(\check{x}, \dot{\check{x}}) + R_{-(\omega+\sigma^\varepsilon)t/\lambda^\varepsilon}\mathcal{Q}^\varepsilon(R_{(\omega+\sigma^\varepsilon)t/\lambda^\varepsilon}\check{x}, R_{(\omega+\sigma^\varepsilon)t/\lambda^\varepsilon}\dot{\check{x}}, R_{(\omega+\sigma^\varepsilon)t/\lambda^\varepsilon}\ddot{\check{x}}), \tag{4.34}$$

where

$$\mathcal{P}^\varepsilon(\check{x}, \dot{\check{x}}) = \mathcal{O}(\varepsilon|(\check{x}, \dot{\check{x}})|), \quad \mathcal{Q}^\varepsilon(\check{x}, \dot{\check{x}}, \ddot{\check{x}}) = \mathcal{O}(\varepsilon^2|(\check{x}, \dot{\check{x}}, \ddot{\check{x}})|),$$

since

$$\dot{x} - (\omega + \sigma^\varepsilon)R_{\frac{\pi}{2}}x = (\lambda^\varepsilon)^2 R_{(\omega+\sigma^\varepsilon)t}\dot{\check{x}}(\check{t}), \quad (\partial_t - (\omega + \sigma^\varepsilon)R_{\frac{\pi}{2}})^2 x = (\lambda^\varepsilon)^2 R_{(\omega+\sigma^\varepsilon)t}\ddot{\check{x}}(\check{t}).$$

We can rewrite equation (4.34) as the second order system

$$\ddot{\check{x}}_1 = \check{x}_1 - C\check{x}_1(\check{x}_1^2 + \check{x}_2^2) + S_1^\varepsilon(\check{x}_1, \check{x}_2, \dot{\check{x}}_1, \dot{\check{x}}_2, \ddot{\check{x}}_1, \ddot{\check{x}}_2), \tag{4.35}$$
$$\ddot{\check{x}}_2 = \check{x}_2 - C\check{x}_2(\check{x}_1^2 + \check{x}_2^2) + S_2^\varepsilon(\check{x}_1, \check{x}_2, \dot{\check{x}}_1, \dot{\check{x}}_2, \ddot{\check{x}}_1, \ddot{\check{x}}_2), \tag{4.36}$$

where $S_1^\varepsilon$ and $S_2^\varepsilon$ are continuously differentiable functions of $\varepsilon$ and their arguments in a neighbourhood of the origin, and $S_1^\varepsilon$ is even and $S_2^\varepsilon$ odd in $(\dot{\check{x}}_1, \check{x}_2, \ddot{\check{x}}_2)$ with

$$S_j^\varepsilon(\check{x}_1, \check{x}_2, \dot{\check{x}}_1, \dot{\check{x}}_2, \ddot{\check{x}}_1, \ddot{\check{x}}_2) = \mathcal{O}(\varepsilon)$$

for $j \in \{1, 2\}$. For $\varepsilon = 0$ this system has the explicit reversible homoclinic solution

$$(\check{x}_1, \check{x}_2) = (h, 0),$$

where

$$h(\check{t}) = \left(\frac{2}{C}\right)^{\frac{1}{2}} \operatorname{sech}(\check{t}).$$

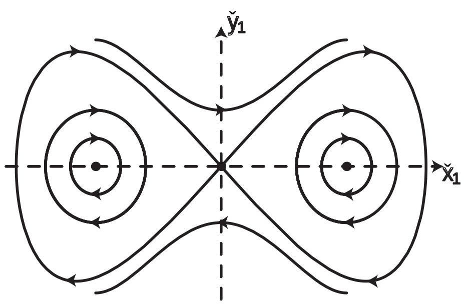



Figure 4.6: Phase portrait of the invariant plane $\{(\check{x}_1, \dot{\check{x}}_1)\}$ of equations (4.35) and (4.36) for $\varepsilon = 0$.

In fact, we have the family

$$\left\{(\check{x}_1,\check{x}_2) = R_a(h(\check{t}_0+\cdot),0), a\in[0,2\pi), \check{t}_0\in\mathbb{R}\right\}$$

of homoclinic solutions, where the solutions $\check{p}^+$ for $a=0$ and $\check{t}_0=0$ as well as $\check{p}^-$ for $a=\pi$ and $\check{t}_0=0$ are reversible (see Figure 4.6).

Now we write $\check{x}$ as perturbation

$$\check{x}_1 = h + p_1, \quad \check{x}_2 = p_2$$

of the homoclinic solution $(h,0)$ so that equations (4.35) and (4.36) become

$$\ddot{p}_1 - p_1 = -3Ch^2p_1 + r_1^\varepsilon(p_1,p_2,\dot{p}_1,\dot{p}_2,\ddot{p}_1,\ddot{p}_2,h), \tag{4.37}$$
$$\ddot{p}_2 - p_2 = -Ch^2p_2 + r_2^\varepsilon(p_1,p_2,\dot{p}_1,\dot{p}_2,\ddot{p}_1,\ddot{p}_2,h), \tag{4.38}$$

with the obvious definitions of $r_1^\varepsilon$ and $r_2^\varepsilon$. We study this system in the space $C^2_{-\nu}(\mathbb{R})^2$ and consider the nonlinearities $r_1^\varepsilon$ and $r_2^\varepsilon$, with a slight abuse of notation, as continuously differentiable mappings $C^2_{-\nu}(\mathbb{R})^2 \to C_{-\nu}(\mathbb{R})^2$ with

$$r_j^\varepsilon(p_1,p_2) = \mathcal{O}(\varepsilon + \|(p_1,p_2)\|^2_{C^2_{-\nu}(\mathbb{R})}).$$

In terms of the (vector-valued versions of the) operators $L$ and $A_h$ defined in Propositions 4.2 and 4.3 we write equations (4.37) and (4.38) as

$$p = A_hp + L^{-1}r^\varepsilon(p), \tag{4.39}$$

where $p=(p_1,p_2)$ and $r^\varepsilon = (r_1^\varepsilon, r_2^\varepsilon)$. In order to solve equation (4.39), we show that the operator

$$I - A_h\colon \Big(C_{-\nu}(\mathbb{R})\cap C_{\mathrm{e}}(\mathbb{R})\Big)\times\Big(C_{-\nu}(\mathbb{R})\cap C_{\mathrm{o}}(\mathbb{R})\Big) \to \Big(C^2_{-\nu}(\mathbb{R})\cap C_{\mathrm{e}}(\mathbb{R})\Big)\times\Big(C^2_{-\nu}(\mathbb{R})\cap C_{\mathrm{o}}(\mathbb{R})\Big)$$

is invertible.

As a first step we show that 1 is a geometrically double eigenvalue of

$$A_h\colon C_{-\nu}(\mathbb{R})^2 \to C_{-\nu}(\mathbb{R})^2,$$

noting that the eigenvalue problem

$$A_hp = p$$

is equivalent to the decoupled system

$$\ddot{p}_1 = p_1 - 3Ch^2p_1, \tag{4.40}$$
$$\ddot{p}_2 = p_2 - Chp_2 \tag{4.41}$$

of ordinary differential equations. Now let

$$\begin{aligned}
v_1(\check{t}) &= \operatorname{sech}(\check{t})\tanh(\check{t}),\\
v_2(\check{t}) &= \operatorname{sech}(\check{t})(-3+\cosh^2(\check{t}) + 3\check{t}\tanh\check{t}),\\
w_1(\check{t}) &= \operatorname{sech}(\check{t}),\\
w_2(\check{t}) &= \operatorname{sech}(\check{t})(2\check{t} + \operatorname{sech}(2\check{t})),
\end{aligned}$$

and note that $\{v_1, v_2\}$ is a fundamental solution set for equation (4.40) and $\{w_1, w_2\}$ is a fundamental solution set for equation (4.41). Since $v_1, w_1$ are bounded and $v_2, w_2$ are unbounded, we conclude that all bounded solutions of equation (4.40) are multiples of $v_1 = -\dot{h}$ and all bounded solution of equation (4.41) are multiples of $w_1 = (2/C)^{-\frac{1}{2}}h$. The eigenspace of $A_h\colon C_{-\nu}(\mathbb{R})^2 \to C_{-\nu}(\mathbb{R})^2$ corresponding to the eigenvalue 1 is therefore

$$\left\langle \begin{pmatrix} \dot{h} \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ h \end{pmatrix} \right\rangle$$

and lies in $\Big(C_{-\nu}(\mathbb{R}) \cap C_{\mathrm{o}}(\mathbb{R})\Big) \times \Big(C_{-\nu}(\mathbb{R}) \cap C_{\mathrm{e}}(\mathbb{R})\Big)$.

In particular 1 is not an eigenvalue of $A_h|_{C_{-\nu}(\mathbb{R})\cap C_{\mathrm{e}}(\mathbb{R})\times C_{-\nu,\mathrm{o}}(\mathbb{R})}$ and since the operator $A_h$ is a compact operator $C_{-\nu}(\mathbb{R})^2 \to C_{-\nu}(\mathbb{R})^2$ by Proposition 4.3, it follows that the spectrum of $A_h|_{C_{-\nu}(\mathbb{R})\cap C_{\mathrm{e}}(\mathbb{R})\times C_{-\nu,\mathrm{o}}(\mathbb{R})}$ consists only of eigenvalues, so that

$$I - A_h\colon \Big(C_{-\nu}(\mathbb{R}) \cap C_{\mathrm{e}}(\mathbb{R})\Big) \times \Big(C_{-\nu}(\mathbb{R}) \cap C_{\mathrm{o}}(\mathbb{R})\Big) \to \Big(C^2_{-\nu}(\mathbb{R}) \cap C_{\mathrm{e}}(\mathbb{R})\Big) \times \Big(C^2_{-\nu}(\mathbb{R}) \cap C_{\mathrm{o}}(\mathbb{R})\Big)$$

is invertible. We can therefore solve equation (4.39) for sufficiently small values of $\varepsilon > 0$ using the implicit function theorem. The solution $h^*$ satisfies $\|h^*\|_{C^2_{-\nu}(\mathbb{R})} = \mathcal{O}(\varepsilon)$. The lemma now follows with

$$\begin{aligned}
\check{p}(\check{t}) &= h(\check{t}) + h^*(\check{t}), \\
\check{q}(\check{t}) &= \dot{h}(\check{t}) + h^*(\check{t}) \\
&\quad + (\lambda^\varepsilon)^{-2} v^\varepsilon\Big((\lambda^\varepsilon)^2 \dot{\check{x}}(t), \lambda^\varepsilon \check{x}(t)\Big) \\
&\quad + (\lambda^\varepsilon)^{-2} R_{-(\omega+\sigma^\varepsilon)t} v^\varepsilon \Big(R_{(\omega+\sigma^\varepsilon)\check{t}}(\lambda^\varepsilon)^2 \dot{\check{x}}(\check{t}), R_{(\omega+\sigma^\varepsilon)\check{t}} \lambda^\varepsilon \check{x}(\check{t})\Big),
\end{aligned}$$

where $v^\varepsilon = v_1^\varepsilon + v_2^\varepsilon$ and $t = \check{t}/\lambda^\varepsilon$.

The second homoclinic solution $\check{p}^{\varepsilon -}$ satisfying the above estimates is obtained analogously to $\check{p}^-$. □

Since the matrix $L^0_{\mathrm{wh}}$ is evidently not diagonalisable, we need to use a change of variable to 'replace' it by a diagonalisable matrix to apply the normal-form theory in Chapter 3, as in the $0^2$ resonance case (see Remark 3.5 for further details). Writing $\varepsilon = \mu^2$ and introducing the scaled variables

$$\begin{aligned}
\tilde{\lambda}^\mu &= \lambda^\varepsilon|_{\varepsilon=\mu^2}, \\
\tilde{\sigma}^\mu &= \sigma^\varepsilon|_{\varepsilon=\mu^2}, \\
\begin{pmatrix} Z_1 \\ Z_2 \end{pmatrix} &= \begin{pmatrix} (\tilde{\lambda}^\mu)^{-\frac{1}{2}} z_1 \\ (\tilde{\lambda}^\mu)^{-\frac{3}{2}} z_2 \end{pmatrix}, \\
W &= (\tilde{\lambda}^\mu)^{-1} w, \\
U &= (\tilde{\lambda}^\mu)^{-\frac{3}{2}} u,
\end{aligned}$$

converts equations (4.18) – (4.20) into

$$\dot{Z} = L^\mu_{\mathrm{wh}} Z + G^\mu_{\mathrm{wh}}(Z, W, U) + H^\mu_{\mathrm{wh}}(Z), \tag{4.42}$$

$$\dot{W} = L_{\mathrm{c}}^{\mu} W + G_{\mathrm{c}}^{\mu}(Z,W,U) + H_{\mathrm{c}}^{\mu}(Z), \tag{4.43}$$
$$\dot{U} = L_{\mathrm{sh}}^{\mu} U + G_{\mathrm{sh}}^{\mu}(Z,W,U) + H_{\mathrm{sh}}^{\mu}(Z), \tag{4.44}$$

where

$$L_{\mathrm{wh}}^{\mu} = \begin{pmatrix} \mathrm{i}(\omega+\tilde{\sigma}^{\mu}) & \tilde{\lambda}^{\mu} & 0 & 0 \\ \tilde{\lambda}^{\mu} & \mathrm{i}(\omega+\tilde{\sigma}^{\mu}) & 0 & 0 \\ 0 & 0 & -\mathrm{i}(\omega+\tilde{\sigma}^{\mu}) & \tilde{\lambda}^{\mu} \\ 0 & 0 & \tilde{\lambda}^{\mu} & -\mathrm{i}(\omega+\tilde{\sigma}^{\mu}) \end{pmatrix}$$

(in complex coordinates), and

$$\begin{aligned} G_{\mathrm{wh}}^{\mu}(Z,W,U) &= \mathcal{O}\Big(|W|\|(Z,W,U)\|_{\mathcal{D}} + \mu\|U\|_{\mathcal{D}_{\mathrm{sh}}}\|(Z,W,U)\|_{\mathcal{D}}\Big), \\ G_{\mathrm{c}}^{\mu}(Z,W,U) &= \mathcal{O}\Big(\mu|W|\|(Z,W,U)\|_{\mathcal{D}} + \mu^2\|U\|_{\mathcal{D}_{\mathrm{sh}}}\|(Z,W,U)\|_{\mathcal{D}}\Big), \\ G_{\mathrm{sh}}^{\mu}(Z,W,U) &= \mathcal{O}\Big(\|(W,U)\|_{\mathcal{D}_{\mathrm{c,sh}}}\|(Z,W,U)\|_{\mathcal{D}}\Big), \\ H_{\mathrm{wh}}^{\mu}(Z) &= \mathcal{O}(|Z|^3), \\ H_{\mathrm{c}}^{\mu}(Z), H_{\mathrm{sh}}^{\mu}(Z) &= \mathcal{O}(|(Z,\mu)||Z|^2) \end{aligned}$$

and we have abbreviated $L_{\mathrm{c}}^{\varepsilon}|_{\varepsilon=\mu^2}$, $L_{\mathrm{sh}}^{\varepsilon}|_{\varepsilon=\mu^2}$ to $L_{\mathrm{c}}^{\mu}$, $L_{\mathrm{sh}}^{\mu}$.

**Remark 4.13.** The formula

$$\mathcal{I}^{\mu}(Z,W,U) \coloneqq \mathcal{I}^{\varepsilon}(z,w,u)|_{\varepsilon=\mu^2}$$

defines a conserved quantity of equations (4.42) – (4.44) and satisfies

$$\mathcal{I}^{\mu}(Z,W,U) = \mathcal{O}(\mu^4\|(Z,W,U)\|_{\mathcal{D}}^2).$$

The equation

$$\dot{Z} = L_{\mathrm{wh}}^{\mu} Z + H_{\mathrm{wh}}^{\mu}(Z)$$

has the reversible homoclinic solution

$$P^{\mu} = \mathrm{e}^{\mathrm{i}(\omega+\tilde{\sigma}^{\mu})t} \begin{pmatrix} (\tilde{\lambda}^{\mu})^{\frac{1}{2}}(\check{p}_1^{\varepsilon\pm}(\tilde{\lambda}^{\mu}t) + \mathrm{i}\check{p}_2^{\varepsilon\pm}(\tilde{\lambda}^{\mu}t))|_{\varepsilon=\mu^2} \\ (\tilde{\lambda}^{\mu})^{\frac{1}{2}}(\check{q}_1^{\varepsilon\pm}(\tilde{\lambda}^{\mu}t) + \mathrm{i}\check{q}_2^{\varepsilon\pm}(\tilde{\lambda}^{\mu}t))|_{\varepsilon=\mu^2} \end{pmatrix},$$

which satisfies the estimate

$$|P^{\mu}(t)| \lesssim (\tilde{\lambda}^{\mu})^{\frac{1}{2}}\mathrm{e}^{-\nu\tilde{\lambda}^{\mu}|t|}$$

(see Lemma 4.12).

The estimates gathered above are not sufficient to construct a contractive iteration scheme which will be our main tool in the existence proof of generalised pulse solutions in Chapter 5. The following preliminary transformation improves these estimates by removing those terms of $G_{\mathrm{wh}}^{\mu}$ which are linear or quadratic in $(Z,\mu)$ and linear in $(W,U)$ at the expense of modifying higher-order terms.

**Lemma 4.14.** There exists a near-identity, finite-dimensional change of variables which transforms equations (4.42) – (4.44) into

$$\dot{Z} = L^\mu_{\mathrm{wh}} Z + \hat{G}^\mu_{\mathrm{wh}}(Z,W,U) + \hat{H}^\mu_{\mathrm{wh}}(Z), \tag{4.45}$$

$$\dot{W} = L^\mu_{\mathrm{c}} W + \hat{G}^\mu_{\mathrm{c}}(Z,W,U) + \hat{H}^\mu_{\mathrm{c}}(Z), \tag{4.46}$$

$$\dot{U} = L^\mu_{\mathrm{sh}} U + \hat{G}^\mu_{\mathrm{sh}}(Z,W,U) + \hat{H}^\mu_{\mathrm{sh}}(Z) \tag{4.47}$$

and preserves the reversibility. The transformed nonlinearities $\hat{G}^\mu_{\mathrm{c}}, \hat{G}^\mu_{\mathrm{sh}}$ and $\hat{H}^\mu_{\mathrm{c}}, \hat{H}^\mu_{\mathrm{sh}}$ satisfy the same estimates as respectively $G^\mu_{\mathrm{c}}$, $G^\mu_{\mathrm{sh}}$ and $H^\mu_{\mathrm{c}}$, $H^\mu_{\mathrm{sh}}$, while

$$\hat{H}^\mu_{\mathrm{wh}}(Z) - H^\mu_{\mathrm{wh}}(Z) = \mathcal{O}(|(Z,\mu)||Z|^3),$$

so that the $\mu$-independent cubic terms of $H^\mu_{\mathrm{wh}}$ are untouched by the transformation and

$$\hat{G}^\mu_{\mathrm{wh}}(Z,W,U) = \mathcal{O}(|(Z,\mu)|^2|Z|\|(W,U)\|_{\mathcal{D}_{\mathrm{c,sh}}} + \|(W,U)\|^2_{\mathcal{D}_{\mathrm{c,sh}}}).$$

*Proof.* We consider the near-identity transformation

$$\hat{Z} = Z + \mu^i d_j(Z,W), \quad \hat{W} = W, \quad \hat{U} = U,$$

where $d_j \colon \mathbb{R}^4 \times \mathbb{R}^{2d} \to \mathbb{R}^4$ is homogeneous of degree $j$ in the first argument and linear in the second; its inverse is given by

$$Z = \hat{Z} - \mu^i d_j(\hat{Z},\hat{W}) + \mathcal{R}(\hat{Z},\hat{W},\mu),$$

where $\mathcal{R}$ is analytic at the origin and satisfies

$$\mathcal{R}(\hat{Z},\hat{W},\mu) = \mathcal{O}(\mu^{(j+1)i}|\hat{Z}|^j|\hat{W}|^2).$$

We find that

$$\dot{\hat{Z}} = \tilde{F}^\mu(\hat{Z},\hat{W},\hat{U}),$$

where

$$\begin{aligned}
&\tilde{F}^\mu(\hat{Z},\hat{W},\hat{U}) \\
&= L^\mu_{\mathrm{wh}}\hat{Z} + L^\mu_{\mathrm{wh}}(\mathcal{R}(\hat{Z},\hat{W}) - \mu^i d_j(\hat{Z},\hat{W})) + H^\mu_{\mathrm{wh}}\big(\hat{Z} - \mu^i d_j(\hat{Z},\hat{W}) + \mathcal{R}(\hat{Z},\hat{W},\mu)\big) \\
&\quad + G^\mu_{\mathrm{wh}}\big(\hat{Z} - \mu^i d_j(\hat{Z},\hat{W}) + \mathcal{R}(\hat{Z},\hat{W},\mu),\hat{W},\hat{U}\big) \\
&\quad + \mu^i d_j\Big(L^\mu_{\mathrm{wh}}(\hat{Z} - \mu^i d_j(\hat{Z},\hat{W}) + \mathcal{R}(\hat{Z},\hat{W},\mu)) \\
&\qquad\qquad + H^\mu_{\mathrm{wh}}(\hat{Z} - \mu^i d_j(\hat{Z},\hat{W}) + \mathcal{R}(\hat{Z},\hat{W},\mu)) \\
&\qquad\qquad + G^\mu_{\mathrm{wh}}(\hat{Z} - \mu^i d_j(\hat{Z},\hat{W}) + \mathcal{R}(\hat{Z},\hat{W},\mu),\hat{W},\hat{U}),\hat{W}\Big) \\
&\quad + \mu^i d_j\Big(\hat{Z}, L^\mu_{\mathrm{c}}\hat{W} + G^\mu_{\mathrm{c}}\big(\hat{Z} - d(\hat{Z},\hat{W}) + \mathcal{R}(\hat{Z},\hat{W}),\hat{W},\hat{U}\big) \\
&\qquad\qquad + H^\mu_{\mathrm{c}}\big(\hat{Z} - d(\hat{Z},\hat{W}) + \mathcal{R}(\hat{Z},\hat{W})\big)\Big) \\
&= H^\mu_{\mathrm{wh}}(\hat{Z}) + G^\mu_{\mathrm{wh}}(\hat{Z},\hat{W},\hat{U}) - \mu^i L^0_{\mathrm{wh}} d_j(\hat{Z},\hat{W}) + \mu^i d_j(L^0_{\mathrm{wh}}\hat{Z},\hat{W}) + \mu^i d_j(\hat{Z},L^0_{\mathrm{c}}\hat{W}) \\
&\quad + \mathcal{O}\Big((\mu^{i+1}|\hat{Z}|^j + \mu^i|\hat{Z}|^{j+1})\|(\hat{W},\hat{U})\|_{\mathcal{D}_{\mathrm{c,sh}}} + \|(\hat{Z},\hat{W},\hat{U})\|_{\mathcal{D}}\|(\hat{W},\hat{U})\|^2_{\mathcal{D}_{\mathrm{c,sh}}} \\
&\qquad\qquad + \mu^i|(\hat{Z},\mu)||\hat{Z}|^{j+2}\Big),
\end{aligned}$$

so that

$$\begin{aligned}\hat{F}_{i,j,1,0}(\{\mu\}^{(i)},\{\hat{Z}\}^{(j)},\hat{W}) = & -\mu^i L^0_{\mathrm{wh}} d_j(\hat{Z},\hat{W}) + \mu^i d_j(L^0_{\mathrm{wh}}\hat{Z},\hat{W}) \\ & + \mu^i d_j(\hat{Z}, L^0_{\mathrm{c}}\hat{W}) + (G_{\mathrm{wh}})_{i,j,1,0}(\{\mu\}^{(i)},\{\hat{Z}\}^{(j)},\hat{W}),\end{aligned}$$

where $\hat{F}_{i,j,1,0}$ and $(G_{\mathrm{wh}})_{i,j,1,0}$ are the parts of $\hat{F}$ and $G^{\mu}_{\mathrm{wh}}$ that are homogeneous of degree $i$ in $\mu$, homogeneous of degree $j$ in $\hat{Z}$ and linear in $\hat{W}$. Therefore, to achieve $\hat{F}_{i,j,1,0} = 0$ we have to solve the equation

$$-\mu^i L^0_{\mathrm{wh}} d_j(\hat{Z},\hat{W}) + \mu^i d_j(L^0_{\mathrm{wh}}\hat{Z},\hat{W}) + \mu^i d_j(\hat{Z}, L^0_{\mathrm{c}}\hat{W}) = (G_{\mathrm{wh}})_{i,j,1,0}(\{\mu\}^{(i)},\{\hat{Z}\}^{(j)},\hat{W}),$$

which is equivalent to the system

$$(\mathrm{i}\omega^0_m I - L^0_{\mathrm{wh}}) d_j(\hat{Z}, e_m) + d_j(L_{\mathrm{wh}}\hat{Z}, e_m) = (G_{\mathrm{wh}})_{i,j,1,0}(\{1\}^{(i)},\{\hat{Z}\}^{(j)}, e_m), \tag{4.48}$$

$$-(\mathrm{i}\omega^0_m I + L^0_{\mathrm{wh}}) d_j(\hat{Z}, \bar{e}_m) + d_j(L_{\mathrm{wh}}\hat{Z}, \bar{e}_m) = (G_{\mathrm{wh}})_{i,j,1,0}(\{1\}^{(i)},\{\hat{Z}\}^{(j)}, \bar{e}_m) \tag{4.49}$$

for $m \in \{1,\ldots,d\}$, where $d_j$ and $(G_{\mathrm{wh}})_{i,j,1,0}$ have been extended linearly to complex-valued second and third arguments. Applying Lemma 4.9 with $A = \mathrm{i}\omega^0_m I - L^0_{\mathrm{wh}}$ and $B = L^0_{\mathrm{wh}}$ to equation (4.48) and with $A = -(\mathrm{i}\omega^0_m I + L^0_{\mathrm{wh}})$ and $B = L^0_{\mathrm{wh}}$ to equation (4.49) we find unique solutions $d_j(\cdot, e_m) \in \mathcal{L}(\mathbb{R}^4;\mathbb{C}^4)$ and $d_j(\cdot,\bar{e}_m) \in \mathcal{L}(\mathbb{R}^4;\mathbb{C}^4)$ for $m \in \{1,\ldots,d\}$. (Note that $d_j(\cdot,\bar{e}_m) = \overline{d_j(\cdot, e_m)}$.) Constructing $d_j$ for $(i,j) \in \{(1,0),(2,0),(1,1)\}$ yields

$$G^{\mu}_{\mathrm{wh}}(Z,W,U) = \mathcal{O}(|(Z,\mu)|^2|Z|\|(W,U)\|_{\mathcal{D}_{\mathrm{c,sh}}} + \mu|Z|\|U\|_{\mathcal{D}_{\mathrm{sh}}} + \|(W,U)\|^2_{\mathcal{D}_{\mathrm{c,sh}}})$$

in the new variables, where we have dropped the hats.

To obtain the desired estimates we consider the near-identity transformation

$$\hat{Z} = Z + \mu d(Z,U), \quad \hat{W} = W, \quad \hat{U} = U,$$

where $d\colon \mathbb{R}^4 \times \mathcal{D}_{\mathrm{sh}} \to \mathbb{R}^4$ bilinear; its inverse is given by

$$Z = \hat{Z} - \mu d(\hat{Z},\hat{U}) + \hat{\mathcal{R}}(\hat{Z},\hat{U},\mu),$$

where $\hat{\mathcal{R}}$ is analytic at the origin and satisfies

$$\hat{\mathcal{R}}(\hat{Z},\hat{U},\mu) = \mathcal{O}(\mu^2|\hat{Z}||\hat{U}|^2).$$

Repeating the same arguments, we find that

$$\dot{\hat{Z}} = \tilde{F}^{\mu}(\hat{Z},\hat{W},\hat{U}),$$

where

$$\begin{aligned}\tilde{F}^{\mu}(\hat{Z},\hat{W},\hat{U}) = H^{\mu}_{\mathrm{wh}}(\hat{Z}) + G^{\mu}_{\mathrm{wh}}(\hat{Z},\hat{W},\hat{U}) - \mu L^0_{\mathrm{wh}} d(\hat{Z},\hat{U}) + \mu d(L^0_{\mathrm{wh}}\hat{Z},\hat{U}) + \mu d(\hat{Z}, L^0_{\mathrm{sh}}\hat{U}) \\ + \mathcal{O}\Big((\mu^2|\hat{Z}| + \mu|\hat{Z}|^2)\|(\hat{W},\hat{U})\|_{\mathcal{D}_{\mathrm{c,sh}}} + \|(\hat{Z},\hat{W},\hat{U})\|_{\mathcal{D}}\|(\hat{W},\hat{U})\|^2_{\mathcal{D}_{\mathrm{c,sh}}} \\ + \mu|(\hat{Z},\mu)||\hat{Z}|^3\Big),\end{aligned}$$

so that

$$\begin{aligned}\hat{F}_{1,1,0,1}(\hat{Z},\hat{U},\mu) = & -\mu L^0_{\mathrm{wh}} d(\hat{Z},\hat{U}) + \mu d(L^0_{\mathrm{wh}}\hat{Z},\hat{U}) \\ & + \mu d(\hat{Z}, L^0_{\mathrm{sh}}\hat{U}) + (G_{\mathrm{wh}})_{i,j,0,1}(\mu,\hat{Z},\hat{U}),\end{aligned}$$

where $\hat{F}_{i,j,0,1}$ and $(G_{\mathrm{wh}})_{i,j,0,1}$ are the parts of $\hat{F}$ and $G^{\mu}_{\mathrm{wh}}$ that are homogeneous of degree $i$ in $\mu$, homogeneous of degree $j$ in $\hat{Z}$ and linear in $\hat{U}$. To achieve $\hat{F}_{1,1,0,1} = 0$ we have to solve the equation

$$-\mu L^0_{\mathrm{wh}} d(\hat{Z}, \hat{U}) + \mu d(L^0_{\mathrm{wh}}\hat{Z}, \hat{U}) + \mu d(\hat{Z}, L^0_{\mathrm{sh}}) = (G_{\mathrm{wh}})_{1,1,0,1}(\mu, \hat{Z}, \hat{U}),$$

which is equivalent to the system

$$(L^0_{\mathrm{wh}} + \mathrm{i}\omega) d(e, \hat{U}) + d(e, L^0_{\mathrm{sh}}\hat{U}) = (G_{\mathrm{wh}})_{1,1,0,1}(1, e, \hat{U}) \tag{4.50}$$

$$(L^0_{\mathrm{wh}} + \mathrm{i}\omega) d(f, \hat{U}) + d(f, L^0_{\mathrm{sh}}\hat{U}) = (G_{\mathrm{wh}})_{1,1,0,1}(1, f, \hat{U}) \tag{4.51}$$

$$(L^0_{\mathrm{wh}} - \mathrm{i}\omega) d(\bar{e}, \hat{U}) + d(\bar{e}, L^0_{\mathrm{sh}}\hat{U}) = (G_{\mathrm{wh}})_{1,1,0,1}(1, \bar{e}, \hat{U}), \tag{4.52}$$

$$(L^0_{\mathrm{wh}} - \mathrm{i}\omega) d(\bar{f}, \hat{U}) + d(\bar{f}, L^0_{\mathrm{sh}}\hat{U}) = (G_{\mathrm{wh}})_{1,1,0,1}(1, \bar{f}, \hat{U}), \tag{4.53}$$

where $d$ and $(G_{\mathrm{wh}})_{i,j,0,1}$ have been extended linearly to complex-valued first and second arguments. Applying Lemma 4.9 with $A = L^0_{\mathrm{wh}} + \mathrm{i}\omega I$ and $B = L^0_{\mathrm{sh}}$ to equations (4.50) and (4.51) and with $A = L^0_{\mathrm{wh}} - \mathrm{i}\omega I$ and $B = L^0_{\mathrm{sh}}$ to equations (4.52) and (4.53) we find unique solutions $d(e, \cdot), d(f, \cdot), d(\bar{e}, \cdot), d(\bar{f}, \cdot) \in \mathcal{L}(\mathcal{D}_{\mathrm{sh}}; \mathbb{C}^4)$. (Note that $d(\bar{e}, \cdot) = \overline{d(\cdot, e)}$ and $d(\bar{f}, \cdot) = \overline{d(\cdot, f)}$.) □

Finally, define

$$\begin{aligned} \mathcal{X} &= \mathcal{X}_{\mathrm{c,sh}} := \mathcal{X}_{\mathrm{c}} \times \mathcal{X}_{\mathrm{sh}}, \\ \mathcal{D} &= \mathcal{D}_{\mathrm{c,sh}} := \mathcal{D}_{\mathrm{c}} \times \mathcal{D}_{\mathrm{sh}}, \\ Q &= (W, U) \end{aligned}$$

and

$$\begin{aligned} F^{\mu}(Z, Q) &= L^{\mu}_{\mathrm{wh}} Z + \hat{G}^{\mu}_{\mathrm{wh}}(Z, W, U) + \hat{H}^{\mu}_{\mathrm{wh}}(Z), \\ L^{\mu} Q &= L^{\mu}_{\mathrm{c,sh}}(W, U) := (L^{\mu}_{\mathrm{c}} W, L^{\mu}_{\mathrm{sh}} U), \\ G^{\mu}(Z, Q) &= \hat{G}^{\mu}_{\mathrm{c,sh}}(Z, W, U) := \big(\hat{G}^{\mu}_{\mathrm{c}}(Z, W, U), \hat{G}^{\mu}_{\mathrm{sh}}(Z, W, U)\big), \end{aligned}$$

so that we are now in the setting described in Chapter 3 and note that

$$\mathrm{d}_1 F^0[0, 0] = K_{\mathrm{wh}} := \begin{pmatrix} L^0_{\mathrm{wh}} \\ 0 \end{pmatrix}.$$

Hypothesis 3.4 is verified in the following result.

**Lemma 4.15.** The operator $\mathcal{L}\colon \mathcal{P}_k(\mathbb{R}^{2n}; \mathcal{D}_{\mathrm{c,sh}}) \to \mathcal{P}_k(\mathbb{R}^{2n}; \mathcal{X}_{\mathrm{c,sh}})$ defined in equation (3.9) is invertible and the operator norm of its inverse

$$\sup_{|\Psi_k|_{2,k}=1} |(\mathcal{L})^{-1}\Psi_k|_{2,k}$$

is independent of $k$.

*Proof.* In order to invert $\mathcal{L}$ we have to find a $k$-linear operator $A_k\colon \mathbb{R}^{2n} \to \mathcal{D}_{\mathrm{c,sh}}$ for a given $k$-linear operator $B_k\colon \mathbb{R}^{2n} \to \mathcal{X}_{\mathrm{c,sh}}$ solving

$$L^0_{\mathrm{c,sh}} A_k(\{Y\}^{(k)}) - k A_k(\{Y\}^{(k-1)}, K_{\mathrm{wh}} Y) = B_k(\{Y\}^{(k)}) \tag{4.54}$$

for all $Y \in \mathbb{R}^{2n}$. Instead of equation (4.54) it is sufficient to solve the operator equation

$$L^0_{\mathrm{c,sh}} A_k(\{Y\}^{(k-1)}, \cdot) - A_k(\{Y\}^{(k-1)}, kK_{\mathrm{wh}}) = B_k(\{Y\}^{(k-1)}, \cdot) \tag{4.55}$$

for fixed $Y \in \mathbb{R}^{2n}$. Using Proposition 4.10 with $A = L^0_{\mathrm{c,sh}}$, $B = -kK_{\mathrm{wh}}$ and $\gamma$ as the union of circles around $\pm ik\omega$ and 0 with sufficiently small radius to separate $\sigma(L^0_{\mathrm{c,sh}})$ and $\sigma(K_{\mathrm{wh}})$ yields a unique solution $A_k(\{Y\}^{(k-1)}, \cdot) \in \mathcal{L}(\mathbb{R}^{2n}; \mathcal{D}_{\mathrm{c,sh}})$ which depends $(k-1)$-linearly and continuously upon $Y \in \mathbb{R}^{2n}$. From the calculation

$$\|(\lambda I - kK_{\mathrm{wh}})^{-1}\|_{\mathcal{L}(\mathbb{R}^{2n})} \lesssim \frac{1}{1 + k\omega} + \frac{1}{k}$$

for all $\lambda \in \gamma$ we obtain

$$\|A_k(\{Y\}^{(k-1)}, \cdot)\|_{\mathcal{L}(\mathbb{R}^{2n};\mathcal{D}_{\mathrm{c,sh}})} \lesssim \frac{1}{k} \|B_k(\{Y\}^{(k-1)}, \cdot)\|_{\mathcal{L}(\mathbb{R}^{2n};\mathcal{X}_{\mathrm{c,sh}})},$$

so that

$$|\mathcal{L}^{-1} B_k|_{2,k} \leq \sqrt{5} k |\mathcal{L}^{-1} B_k|_{0,k} \leq \sqrt{5} |B_k|_{0,k} \leq |B_k|_{2,k}$$

holds independently of $k$, where we have slightly abused the notation by using $B_k$ interchangeably with its induced polynomial. □

Chapter 3 therefore yields the existence of a near-identity, finite-dimensional change of variable

$$\tilde{Y} = Y := (Z, \mu), \quad \tilde{W} = W + \Phi_{\mathrm{c}}(Y), \quad \tilde{U} = U + \Phi_{\mathrm{sh}}(Y)$$

satisfying

$$\Phi(Z, \mu) = \mathcal{O}(|(Z, \mu)||Z|^2) \tag{4.56}$$

which transforms equations (4.10) – (4.12) into

$$\begin{aligned} \dot{Z} &= L^\mu_{\mathrm{wh}} Z + \tilde{G}^\mu_{\mathrm{wh}}(Z,W,U) + \tilde{H}^\mu_{\mathrm{wh}}(Z), \\ \dot{W} &= L^\mu_{\mathrm{c}} W + \tilde{G}^\mu_{\mathrm{c}}(Z,W,U) + \tilde{H}^\mu_{\mathrm{c}}(Z), \\ \dot{U} &= L^\mu_{\mathrm{sh}} U + \tilde{G}^\mu_{\mathrm{sh}}(Z,W,U) + \tilde{H}^\mu_{\mathrm{sh}}(Z) \end{aligned}$$

and $\delta > 0$ such that

$$\begin{aligned} \|\tilde{H}_{\mathrm{c,sh}}(Y)\|_{\mathcal{X}_{\mathrm{c,sh}}} &\lesssim \mu^2 \mathrm{e}^{-\frac{c^*}{2\mu}}, \\ \|\mathrm{d}\tilde{H}_{\mathrm{c,sh}}(Y)\|_{\mathcal{L}(\mathbb{R}^5;\mathcal{X}_{\mathrm{c,sh}})} &\lesssim \mu \mathrm{e}^{-\frac{c^*}{2\mu}} \end{aligned}$$

for $|Y| \leq \delta$. Equations (3.6) – (3.8) and estimate (4.17) imply the following estimates for the transformed nonlinearities.

**Remark 4.16.** The transformed nonlinearities satisfy

(i) $\tilde{H}^\mu_{\mathrm{wh}}(Z) - \hat{H}^\mu_{\mathrm{wh}}(Z) = \mathcal{O}\Big(|(Z,\mu)||Z|^3\Big)$, so that the $\mu$-independent cubic terms of $H^\mu_{\mathrm{wh}}$ are untouched by the transformation,

(ii) $\tilde{G}^\mu_{\mathrm{wh}}(Z,W,U) = \mathcal{O}\Big(|(Z,\mu)|^2|Z|\|(W,U)\|_{\mathcal{D}_{\mathrm{c,sh}}} + \|(W,U)\|^2_{\mathcal{D}_{\mathrm{c,sh}}}\Big)$,

(iii) $\tilde{G}^\mu_{\mathrm{c}}(Z,W,U) = \mathcal{O}\Big(\mu|W|\|(Z,W,U)\|_{\mathcal{D}} + \mu^2\|U\|_{\mathcal{D}_{\mathrm{sh}}}\|(Z,W,U)\|_{\mathcal{D}}\Big)$,

(iv) $\tilde{G}^\mu_{\mathrm{sh}}(Z,W,U) = \mathcal{O}\Big(\|(W,U)\|_{\mathcal{D}_{\mathrm{c,sh}}}\|(Z,W,U)\|_{\mathcal{D}}\Big)$,

(v) $\mathrm{d}\tilde{H}^\mu_{\mathrm{wh}}[Z] = \mathcal{O}\Big(|Z|^2\Big)$,

(vi) $\mathrm{d}_1\tilde{G}^\mu_{\mathrm{wh}}[Z,W,U] = \mathcal{O}\Big(|(Z,\mu)|^2\|(W,U)\|_{\mathcal{D}_{\mathrm{c,sh}}} + \|(W,U)\|^2_{\mathcal{D}_{\mathrm{c,sh}}}\Big)$,

(vii) $\mathrm{d}_2\tilde{G}^\mu_{\mathrm{wh}}[Z,W,U], \mathrm{d}_3\tilde{G}^\mu_{\mathrm{wh}}[Z,W,U] = \mathcal{O}\Big(|(Z,\mu)|^2|Z| + \|(W,U)\|_{\mathcal{D}_{\mathrm{c,sh}}}\Big)$,

(viii) $\mathrm{d}_1\tilde{G}^\mu_{\mathrm{c}}[Z,W,U] = \mathcal{O}\Big(\mu\|(W,U)\|_{\mathcal{D}_{\mathrm{c,sh}}}\Big)$,

(ix) $\mathrm{d}_2\tilde{G}^\mu_{\mathrm{c}}[Z,W,U] = \mathcal{O}\Big(\mu\|(Z,W,U)\|_{\mathcal{D}}\Big)$,

(x) $\mathrm{d}_3\tilde{G}^\mu_{\mathrm{c}}[Z,W,U] = \mathcal{O}\Big(\mu|W| + \mu^2\|(Z,W,U)\|_{\mathcal{D}}\Big)$,

(xi) $\mathrm{d}_1\tilde{G}^\mu_{\mathrm{sh}}[Z,W,U] = \mathcal{O}\Big(\|(W,U)\|_{\mathcal{D}_{\mathrm{c,sh}}}\Big)$,

(xii) $\mathrm{d}_2\tilde{G}^\mu_{\mathrm{sh}}[Z,W,U], \mathrm{d}_3\tilde{G}^\mu_{\mathrm{sh}}[Z,W,U] = \mathcal{O}\Big(\|(Z,W,U)\|_{\mathcal{D}}\Big)$.

# 5 Construction of generalised pulses

In this chapter we consider the evolutionary system

$$\dot{Z} = L^\mu_{\mathrm{wh}} Z + \tilde{G}^\mu_{\mathrm{wh}}(Z,W,U) + \tilde{H}^\mu_{\mathrm{wh}}(Z), \tag{5.1}$$

$$\dot{W} = L^\mu_{\mathrm{c}} W + \tilde{G}^\mu_{\mathrm{c}}(Z,W,U) + \tilde{H}^\mu_{\mathrm{c}}(Z), \tag{5.2}$$

$$\dot{U} = L^\mu_{\mathrm{sh}} U + \tilde{G}^\mu_{\mathrm{sh}}(Z,W,U) + \tilde{H}^\mu_{\mathrm{sh}}(Z) \tag{5.3}$$

for $(Z,W,U)\colon \mathbb{R} \to \mathbb{R}^n \times \mathbb{R}^{2d} \times \mathcal{D}_{\mathrm{sh}}$ derived in Chapter 4. Recall that these equations have the following properties.

(D1) The bounded linear operators $L^\mu_{\mathrm{wh}}\colon \mathbb{R}^n \to \mathbb{R}^n$, $L^\mu_{\mathrm{c}}\colon \mathbb{R}^{2d} \to \mathbb{R}^{2d}$, $L^\mu_{\mathrm{sh}}\colon \mathcal{D}_{\mathrm{sh}} \to \mathcal{X}_{\mathrm{sh}}$ depend analytically upon $\mu$.

(D2) We assume that

$$L^\mu_{\mathrm{wh}} = \begin{pmatrix} 0 & \tilde{\lambda}^\mu \\ \tilde{\lambda}^\mu & 0 \end{pmatrix}$$

for $n=2$ and

$$L^\mu_{\mathrm{wh}} = \begin{pmatrix} \mathrm{i}(\omega+\tilde{\sigma}^\mu) & \tilde{\lambda}^\mu & 0 & 0 \\ \tilde{\lambda}^\mu & \mathrm{i}(\omega+\tilde{\sigma}^\mu) & 0 & 0 \\ 0 & 0 & -\mathrm{i}(\omega+\tilde{\sigma}^\mu) & \tilde{\lambda}^\mu \\ 0 & 0 & \tilde{\lambda}^\mu & -\mathrm{i}(\omega+\tilde{\sigma}^\mu) \end{pmatrix}$$

(in complex coordinates) for $n=4$, where $\tilde{\lambda}^\mu$ and $\tilde{\sigma}^\mu$ are analytic functions of $\mu$ with

$$\tilde{\lambda}^\mu = \mu^2 + \mathcal{O}(\mu^4), \quad \tilde{\sigma}^\mu = \mathcal{O}(\mu^2).$$

(D3) The functions $\tilde{G}^{(\cdot)}_{\mathrm{wh}}$, $\tilde{G}^{(\cdot)}_{\mathrm{c}}$, $\tilde{G}^{(\cdot)}_{\mathrm{sh}}$, $\tilde{H}^{(\cdot)}_{\mathrm{wh}}$, $\tilde{H}^{(\cdot)}_{\mathrm{c}}$, $\tilde{H}^{(\cdot)}_{\mathrm{sh}}$ take values in respectively $\mathbb{R}^n$, $\mathbb{R}^{2d}$, $\mathcal{X}_{\mathrm{sh}}$, $\mathbb{R}^n$, $\mathbb{R}^{2d}$, $\mathcal{X}_{\mathrm{sh}}$, are analytic at the origin in respectively $\mathbb{R}\times\mathcal{D}$ and $\mathbb{R}\times\mathbb{R}^n$, and satisfy the estimates given in Remarks 4.8 and 4.16 for $n=2$ and $n=4$ respectively.

(D4) The spectrum of the complexified operator $L^\mu_{\mathrm{c}} \in \mathbb{C}^{2d\times 2d}$ consists of finitely many simple purely imaginary eigenvalues $\pm\mathrm{i}\omega^\mu_1,\ldots,\pm\mathrm{i}\omega^\mu_d$, where $\omega^\mu_1,\ldots,\omega^\mu_d > 0$.

(D5) The system (5.1) – (5.3) is reversible, i.e. there exist $S_{\mathrm{wh}} \in \mathbb{R}^{n\times n}$, $S_{\mathrm{c}} \in \mathbb{R}^{2d\times 2d}$ and $S_{\mathrm{sh}} \in \mathcal{L}(\mathcal{D}_{\mathrm{sh}}) \cap \mathcal{L}(\mathcal{X}_{\mathrm{sh}})$ such that system (4.1) – (4.3) is invariant under $t \mapsto -t$, $(Z,W,U) \mapsto (S_{\mathrm{wh}}Z, S_{\mathrm{c}}W, S_{\mathrm{sh}}U)$.

(D6) There exists a conserved quantity $\mathcal{I}^{(\cdot)}$ of system (5.1) – (5.3) which is analytic at the origin in $\mathbb{R}\times\mathcal{D}$ and satisfies

$$\mathcal{I}^\mu(Z,W,U) = \mathcal{O}(\mu^4\|(Z,W,U)\|^2_{\mathcal{D}}).$$

(D7) The linear operator $L^0_{\mathrm{sh}} \colon \mathcal{D}_{\mathrm{sh}} \subseteq \mathcal{X}_{\mathrm{sh}} \to \mathcal{X}_{\mathrm{sh}}$ is closed and satisfies the estimate

$$\|(\mathrm{i}sI - L^0_{\mathrm{sh}})^{-1}\|_{\mathcal{L}(\mathcal{X}_{\mathrm{sh}})} \lesssim \frac{1}{1+|s|}$$

for $s \in \mathbb{R}$.

(D8) The approximate system

$$\dot{Z} = L^\mu_{\mathrm{wh}} Z + \tilde{H}^\mu_{\mathrm{wh}}(Z) \tag{5.4}$$

has a homoclinic solution $P^\mu$ satisfying the estimate

$$|P^\mu(t)| \leq c_{\mathrm{h},\nu} \mu^2 \mathrm{e}^{-\nu \tilde{\lambda}^\mu |t|} \tag{5.5}$$

in the case $n = 2$, and

$$|P^\mu(t)| \leq c_{\mathrm{h},\nu} \mu \mathrm{e}^{-\nu \tilde{\lambda}^\mu |t|} \tag{5.6}$$

in the case $n = 4$ for all $t \in \mathbb{R}$.

Our goal in this chapter is to complete the proofs of Theorems 1.1 and 1.2 by constructing generalised pulse solutions to equations (5.1) – (5.3), meaning solutions of the form $(P^\mu + R, W, U)$, where $(R, W, U)$ is an exponentially small remainder which does not necessarily vanish as $t \to \pm\infty$ (see Figure 5.1). To that end we reformulate equations (5.1) – (5.3) again by interpreting $Z$ as a perturbation of the homoclinic solution $P^\mu$. To guarantee that $|Y| = |(Z, \mu)| \leq \delta$, so that

$$\|\tilde{H}(Y)\|_{\mathcal{X}_{\mathrm{c,sh}}} \lesssim \mu^2 \mathrm{e}^{-\frac{c^*}{\mu}}, \quad \|\mathrm{d}\tilde{H}[Y]\|_{\mathcal{L}(\mathbb{R}^{n+1};\mathcal{X}_{\mathrm{c,sh}})} \lesssim \mu \mathrm{e}^{-\frac{c^*}{\mu}} \tag{5.7}$$

for some constant $c^* > 0$, we set

$$\delta = (2c_{\mathrm{h},\nu} + 1)\mu_0$$

and restrict the perturbation $R = Z - P^\mu$ to $\{|R| \leq c_{\mathrm{h},\nu}\mu_0\}$, i.e. $(Z, \mu)$ to $\{|Z| \leq 2c_{\mathrm{h},\nu}\mu_0, 0 < \mu \leq \mu_0\}$ and define

$$c^* = \frac{1}{\mathrm{e}C(2c_{\mathrm{h},\nu} + 1)}.$$

Writing

$$Z = P^\mu + R,$$

we obtain the system

$$\dot{R} = K^\mu R + N^\mu(R, W, U), \tag{5.8}$$

$$\dot{W} = L^\mu_{\mathrm{c}} W + \tilde{G}^\mu_{\mathrm{c}}(P^\mu + R, W, U) + \tilde{H}^\mu_{\mathrm{c}}(P^\mu + R), \tag{5.9}$$

$$\dot{U} = L^\mu_{\mathrm{sh}} U + \tilde{G}^\mu_{\mathrm{sh}}(P^\mu + R, W, U) + \tilde{H}^\mu_{\mathrm{sh}}(P^\mu + R), \tag{5.10}$$

where

$$K^\mu R = L^\mu_{\mathrm{wh}} R + \mathrm{d}\tilde{H}^\mu_{\mathrm{wh}}[P^\mu](R),$$
$$N^\mu(R, W, U) = \tilde{H}^\mu_{\mathrm{wh}}(P^\mu + R) - \tilde{H}^\mu_{\mathrm{wh}}(P^\mu) - \mathrm{d}\tilde{H}^\mu_{\mathrm{wh}}[P^\mu](R) + \tilde{G}^\mu_{\mathrm{wh}}(P^\mu + R, W, U).$$

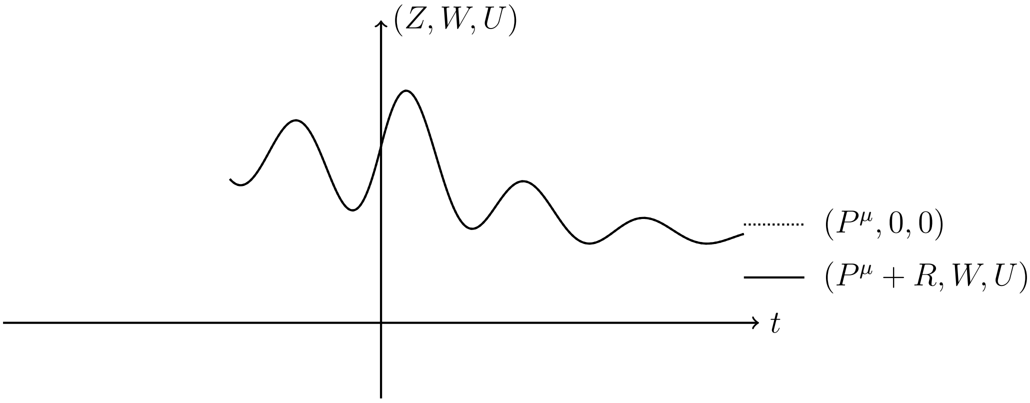



Figure 5.1: A generalised pulse solution to equations (5.1) – (5.3) lies in an exponentially thin tubular neighbourhood of $(P^\mu, 0, 0)$.

**Remark 5.1.** Using the analyticity of $\tilde{H}^\mu_{\mathrm{wh}}$ and Remarks 4.8 and 4.16 we find the following estimates.

(i) In the case $n = 2$ we have that

$$N^\mu(R, W, U) = \mathcal{O}\Big(|R|^2 + \big(\mu^3 + |R||(R,\mu)|\big)\|(W,U)\|_{\mathcal{D}_{\mathrm{c,sh}}} + \|(W,U)\|^2_{\mathcal{D}_{\mathrm{c,sh}}}\Big)$$

and

$$\begin{aligned} \mathrm{d}_1 N^\mu[R, W, U] &= \mathcal{O}\Big(|(R,\mu)|\|(W,U)\|_{\mathcal{D}_{\mathrm{c,sh}}} + \|(W,U)\|^2_{\mathcal{D}_{\mathrm{c,sh}}}\Big), \\ \mathrm{d}_2 N^\mu[R, W, U], \mathrm{d}_3 N^\mu[R, W, U] &= \mathcal{O}\Big(\mu^3 + |R||(R,\mu)| + \|(W,U)\|_{\mathcal{D}_{\mathrm{c,sh}}}\Big). \end{aligned}$$

(ii) In the case $n = 4$ we have that

$$N^\mu(R, W, U) = \mathcal{O}\Big(|R|^2 + |(R,\mu)|^3\|(W,U)\|_{\mathcal{D}_{\mathrm{c,sh}}} + \|(W,U)\|^2_{\mathcal{D}_{\mathrm{c,sh}}}\Big)$$

and

$$\begin{aligned} \mathrm{d}_1 N^\mu[R, W, U] &= \mathcal{O}\Big(|R| + |(R,\mu)|^2\|(W,U)\|_{\mathcal{D}_{\mathrm{c,sh}}} + \|(W,U)\|^2_{\mathcal{D}_{\mathrm{c,sh}}}\Big), \\ \mathrm{d}_2 N^\mu[R, W, U], \mathrm{d}_3 N^\mu[R, W, U] &= \mathcal{O}\Big(|(R,\mu)|^3 + \|(W,U)\|_{\mathcal{D}_{\mathrm{c,sh}}}\Big). \end{aligned}$$

We first construct a family of reversible solutions to equations (5.8) – (5.10) which remain exponentially small in comparison to $\mu$ over exponentially long timescales and thus define, in analogy with familiar dynamical-systems theory, a local centre-stable manifold for equations (5.8) – (5.10) consisting of their initial data. Finally, by showing that solutions with initial data on our constructed local centre-stable manifold converge to solutions on a local centre manifold of equations (5.1) – (5.3) and using a Lyapunov stability argument, we obtain the existence of a global centre-stable manifold for equations (5.1) – (5.3) consisting of initial data of generalised pulses.

## 5.1 Formulation as a fixed-point problem

Since we anticipate the $W$-component to grow linearly (being the centre part of the system), we cannot expect solutions to equations (5.8) – (5.10) to be bounded in their $W$-component. We therefore modify system (5.8) – (5.10) by artificially bounding the $W$-component. Let $\phi\colon \mathbb{R}^{2d} \to \mathbb{R}^{2d}$ be defined by

$$\phi(v) = \psi\Big(\mathrm{e}^{\frac{c^*}{2\mu}}|v|\Big)v, \tag{5.11}$$

where $\psi\colon \mathbb{R} \to \mathbb{R}$ is a smooth cut-off function with

$$\psi(t) = \begin{cases} 1, & |t| \leq 1, \\ 0, & |t| \geq 2 \end{cases}$$

and $|\psi^{(l)}(t)| \leq 2^l$ for $t \in \mathbb{R}$ and $l \in \mathbb{N}_0$. Consider the modified equations

$$\dot{R} = K^\mu R + \underset{\sim}{N}^\mu(R, W, U), \tag{5.12}$$

$$\dot{W} = L^\mu_{\mathrm{c}} W + \underset{\sim}{\tilde{G}}{}^\mu_{\mathrm{c}}(P^\mu + R, W, U) + \tilde{H}^\mu_{\mathrm{c}}(P^\mu + R), \tag{5.13}$$

$$\dot{U} = L^\mu_{\mathrm{sh}} U + \underset{\sim}{\tilde{G}}{}^\mu_{\mathrm{sh}}(P^\mu + R, W, U) + \tilde{H}^\mu_{\mathrm{sh}}(P^\mu + R), \tag{5.14}$$

where

$$\begin{aligned} \underset{\sim}{N}^\mu(R, W, U) &= N^\mu(R, \phi(W), U), \\ \underset{\sim}{\tilde{G}}{}^\mu_{\mathrm{c}}(Z, W, U) &= \tilde{G}^\mu_{\mathrm{c}}(Z, \phi(W), U), \\ \underset{\sim}{\tilde{G}}{}^\mu_{\mathrm{sh}}(Z, W, U) &= \tilde{G}^\mu_{\mathrm{sh}}(Z, \phi(W), U). \end{aligned}$$

Observe that the system (5.12) – (5.14) is equivalent to the fixed-point problem

$$\begin{aligned} R &= \mathcal{F}_{\mathrm{wh}}(R, W, U), \\ W &= \mathcal{F}_{\mathrm{c}}(R, W, U), \\ U &= \mathcal{F}_{\mathrm{sh}}(R, W, U), \end{aligned}$$

where

$$\begin{aligned} \mathcal{F}_{\mathrm{wh}}(R, W, U)(t) = &\int_0^t \langle \underset{\sim}{N}^\mu(R, W, U)(\tau); (s^\mu)^*(\tau)\rangle \,\mathrm{d}\tau\, s^\mu(t) \\ &- \int_t^\infty \langle \underset{\sim}{N}^\mu(R, W, U)(\tau); (u^\mu)^*(\tau)\rangle \,\mathrm{d}\tau\, u^\mu(t), \end{aligned} \tag{5.15}$$

in the case $n = 2$,

$$\begin{aligned} \mathcal{F}_{\mathrm{wh}}(R, W, U)(t) = &\sum_{j=1}^{2} \int_0^t \langle \underset{\sim}{N}^\mu(R, W, U)(\tau); (s^\mu_j)^*(\tau)\rangle \,\mathrm{d}\tau\, s^\mu_j(t) \\ &- \sum_{j=1}^{2} \int_t^\infty \langle \underset{\sim}{N}^\mu(R, W, U)(\tau); (u^\mu_j)^*(\tau)\rangle \,\mathrm{d}\tau\, u^\mu_j(t), \end{aligned} \tag{5.16}$$

in the case $n = 4$,

$$\begin{aligned} \mathcal{F}_{\mathrm{c}}(R, W, U)(t) = \mathrm{e}^{L^\mu_{\mathrm{c}} t} W_0 + \int_0^t \mathrm{e}^{L^\mu_{\mathrm{c}}(t-\tau)} \Big(&\underset{\sim}{\tilde{G}}{}^\mu_{\mathrm{c}}(P^\mu + R, W, U)(\tau) \\ &+ \tilde{H}^\mu_{\mathrm{c}}(P^\mu + R)(\tau)\Big) \,\mathrm{d}\tau, \end{aligned} \tag{5.17}$$

and finally $\mathcal{F}_{\mathrm{sh}}(R, W, U)$ is the unique solution of the equation

$$\dot{M} = L^{\mu}_{\mathrm{sh}} M + \tilde{G}^{\mu}_{\mathrm{sh}}(P^{\mu} + \mathcal{E}_{\mathrm{wh}} R, \mathcal{E}_{\mathrm{c}} W, U) + \tilde{H}^{\mu}_{\mathrm{sh}}(P^{\mu} + \mathcal{E}_{\mathrm{wh}} R) \tag{5.18}$$

in a suitable class of functions (see below). The initial value $W_0 \in \mathbb{R}^{2d}$ is chosen such that $S_{\mathrm{c}} W_0 = W_0$. Here $\mathcal{E}_{\mathrm{wh}}$ and $\mathcal{E}_{\mathrm{c}}$ denote the extensions to $\mathbb{R}$ by reversibility given by

$$(\mathcal{E}_{\mathrm{wh}} R)(t) = \begin{cases} R(t), & t \geq 0, \\ (S_{\mathrm{wh}} R)(-t), & t < 0 \end{cases}$$

and

$$(\mathcal{E}_{\mathrm{c}} W)(t) = \begin{cases} W(t), & t \geq 0, \\ (S_{\mathrm{c}} W)(-t), & t < 0. \end{cases}$$

In equations (5.15) and (5.16) $\{s^{\mu}, u^{\mu}\}$ and $\{s^{\mu}_1, s^{\mu}_2, u^{\mu}_1, u^{\mu}_2\}$ are fundamental solution sets for the linear problem

$$\dot{R} = K^{\mu} R$$

(with dual bases $\{(s^{\mu})^*, (u^{\mu})^*\}$ and $\{(s^{\mu}_1)^*, (s^{\mu}_2)^*, (u^{\mu}_1)^*, (u^{\mu}_2)^*\}$) constructed in the following results.

**Lemma 5.2.** Suppose $n = 2$. The linear equation

$$\dot{R} = K^{\mu} R$$

has solutions $s^{\mu}, u^{\mu} \colon [0, \infty) \to \mathbb{R}^2$ such that the estimates

$$|s^{\mu}(t)| \lesssim \mathrm{e}^{-\tilde{\lambda}^{\mu} t}, \qquad |u^{\mu}(t)| \lesssim \mathrm{e}^{\tilde{\lambda}^{\mu} t}$$

hold for $t \in [0, \infty)$. The dual basis $\{(s^{\mu})^*(t), (u^{\mu})^*(t)\}$ to $\{s^{\mu}(t), u^{\mu}(t)\}$ in $\mathbb{R}^2$ satisfies the estimates

$$|(s^{\mu})^*(t)| \lesssim \mathrm{e}^{\tilde{\lambda}^{\mu} t}, \qquad |(u^{\mu})^*(t)| \lesssim \mathrm{e}^{-\tilde{\lambda}^{\mu} t}$$

for $t \in [0, \infty)$. Furthermore $u^{\mu}$ is reversible and $s^{\mu}$ antireversible, i.e. they satisfy

$$u^{\mu}(-t) = (S_{\mathrm{wh}} u^{\mu})(t), \quad s^{\mu}(-t) = -(S_{\mathrm{wh}} s^{\mu})(t)$$

for $t \in [0, \infty)$.

*Proof.* Observe that

$$\tilde{H}^{\mu}_{\mathrm{wh}}(Z) = \begin{pmatrix} 0 \\ -C Z_1^2 \end{pmatrix} + \mathcal{O}(|(Z, \mu)||Z|^2),$$

and since by construction

$$P^{\mu}(t) = \tilde{\lambda}^{\mu} \begin{pmatrix} h(\tilde{\lambda}^{\mu} t) \\ \dot{h}(\tilde{\lambda}^{\mu} t) \end{pmatrix} + \mathcal{R}^{\mu}(t)$$

with

$$|\mathcal{R}^{\mu}(t)| \lesssim \mu^3 \mathrm{e}^{-\nu \tilde{\lambda}^{\mu} t}$$

we conclude that

$$K^{\mu} R = L^{\mu}_{\mathrm{wh}} R + T^{\mu}_1(R) + T^{\mu}_2(R), \tag{5.19}$$

where

$$T_1^\mu(Z) = \begin{pmatrix} 0 \\ -2Ch(\tilde{\lambda}^\mu t)\tilde{\lambda}^\mu Z_1 \end{pmatrix}$$

and

$$|T_2^\mu(t)| \lesssim \mu^3 \mathrm{e}^{-\nu\tilde{\lambda}^\mu |t|}$$

for $t \in [0, \infty)$.

We first consider the system

$$\dot{Z} = L_{\mathrm{wh}}^\mu Z + T_1^\mu(Z),$$

which has the explicit solutions

$$s(t) = \begin{pmatrix} y_1(\tilde{\lambda}^\mu t) \\ \dot{y}_1(\tilde{\lambda}^\mu t) \end{pmatrix}, \tag{5.20}$$

$$u(t) = \begin{pmatrix} y_2(\tilde{\lambda}^\mu t) \\ \dot{y}_2(\tilde{\lambda}^\mu t) \end{pmatrix}, \tag{5.21}$$

where $y_1$, $y_2$ are given by equations (4.8) and (4.9), and these solutions satisfy

$$|s(t)| \lesssim \mathrm{e}^{-\tilde{\lambda}^\mu t}, \quad |u(t)| \lesssim \mathrm{e}^{\tilde{\lambda}^\mu t}$$

for $t \in [0, \infty)$. The dual basis $\{s^*(t), u^*(t)\}$ to $\{s(t), u(t)\}$ is given by

$$s^*(t) = \frac{1}{2}\begin{pmatrix} \dot{y}_2(\tilde{\lambda}^\mu t) \\ -y_2(\tilde{\lambda}^\mu t) \end{pmatrix}, \tag{5.22}$$

$$u^*(t) = \frac{1}{2}\begin{pmatrix} -\dot{y}_1(\tilde{\lambda}^\mu t) \\ y_1(\tilde{\lambda}^\mu t) \end{pmatrix} \tag{5.23}$$

and satisfies

$$|s^*(t)| \lesssim \mathrm{e}^{\tilde{\lambda}^\mu t}, \quad |u^*(t)| \lesssim \mathrm{e}^{-\tilde{\lambda}^\mu t}$$

for $t \in [0, \infty)$.

Now we turn to equation (5.19). We note that any solution of either

$$\begin{aligned} s^\mu(t) = s(t) &- s(t)\int_t^\infty \langle T_2^\mu(\tau)s^\mu(\tau), s^*(\tau)\rangle \,\mathrm{d}\tau \\ &- u(t)\int_t^\infty \langle T_2^\mu(\tau)s^\mu(\tau), u^*(\tau)\rangle \,\mathrm{d}\tau, \end{aligned} \tag{5.24}$$

or

$$\begin{aligned} u^\mu(t) = u(t) &+ s(t)\int_0^t \langle T_2^\mu(\tau)u^\mu(\tau), s^*(\tau)\rangle \,\mathrm{d}\tau \\ &- u(t)\int_t^\infty \langle T_2^\mu(\tau)u^\mu(\tau), u^*(\tau)\rangle \,\mathrm{d}\tau, \end{aligned} \tag{5.25}$$

is also a solution of equation (5.19). To construct a solution of equation (5.24) we denote its right-hand side by $\mathcal{G}(s^\mu)$ and use the estimates for $s$ and $u$ to obtain that

$$|\mathcal{G}(s^\mu)(t) - s(t)| \lesssim \int_t^\infty |T_2^\mu(\tau)| \,\mathrm{d}\tau \|s^\mu\|_{C_{-\tilde{\lambda}^\mu}([0,\infty);\mathbb{R}^2)} \mathrm{e}^{-\tilde{\lambda}^\mu t}.$$

This estimate now implies

$$\begin{aligned}\|\mathcal{G}(s^\mu)-s\|_{C_{-\tilde\lambda^\mu}([0,\infty);\mathbb{R}^2)} &\lesssim \|s^\mu\|_{C_{-\tilde\lambda^\mu}([0,\infty);\mathbb{R}^2)}\int_t^\infty |T_2^\mu(\tau)|\,\mathrm{d}\tau\\ &\lesssim \|s^\mu\|_{C_{-\tilde\lambda^\mu}([0,\infty);\mathbb{R}^2)}\int_t^\infty \mu^3\mathrm{e}^{-\nu\tilde\lambda^\mu\tau}\,\mathrm{d}\tau\\ &\lesssim \mu\|s^\mu\|_{C_{-\tilde\lambda^\mu}([0,\infty);\mathbb{R}^2)},\end{aligned}$$

so that

$$\|\mathcal{G}(s)-\mathcal{G}(s')\|_{C_{-\tilde\lambda^\mu}([0,\infty);\mathbb{R}^2)} = \|\mathcal{G}(s-s')-s\|_{C_{-\tilde\lambda^\mu}([0,\infty);\mathbb{R}^2)} \lesssim \mu\|s_1-s_2\|_{C_{-\tilde\lambda^\mu}([0,\infty);\mathbb{R}^2)}.$$

Hence the operator $\mathcal{G}\colon C_{-\tilde\lambda^\mu}([0,\infty);\mathbb{R}^2)\to C_{-\tilde\lambda^\mu}([0,\infty);\mathbb{R}^2)$ is a contraction on $\bar B_1(s)\subseteq C_{-\tilde\lambda^\mu}([0,\infty);\mathbb{R}^2)$. Thus $\mathcal{G}$ has a fixed point $s^\mu$ satisfying

$$|s^\mu(t)-s(t)|\mathrm{e}^{\tilde\lambda^\mu t}\le 1$$

and in particular $|s^\mu(t)|\lesssim \mathrm{e}^{-\tilde\lambda^\mu t}$ for $t\in[0,\infty)$. We also observe that

$$|s^\mu(t)-s(t)|\mathrm{e}^{\tilde\lambda^\mu t}\le \int_t^\infty |T_2^\mu(\tau)|\,\mathrm{d}\tau\|s^\mu\|_{C_{-\tilde\lambda^\mu}([0,\infty);\mathbb{R}^2)}\to 0$$

as $t\to\infty$ and

$$\lim_{t\to\infty} s^\mu(t)\mathrm{e}^{\tilde\lambda^\mu t} = \lim_{t\to\infty} s(t)\mathrm{e}^{\tilde\lambda^\mu t} = \begin{pmatrix}2\\-2\end{pmatrix}.$$

In a similar fashion we construct a solution $u^\mu$ of equation (5.25) satisfying

$$|u^\mu(t)|\lesssim \mathrm{e}^{\tilde\lambda^\mu t}$$

for $t\in[0,\infty)$ and

$$\lim_{t\to\infty} u^\mu(t)\mathrm{e}^{-\tilde\lambda^\mu t} = \lim_{t\to\infty} u(t)\mathrm{e}^{-\tilde\lambda^\mu t} = \begin{pmatrix}\frac12\\ \frac12\end{pmatrix}.$$

Next we compute the dual basis $\{(s^\mu)^*(t),(u^\mu)^*(t)\}$ to $\{s(t),u(t)\}$. To this end we define

$$M(t)=\big(s^\mu(t)|u^\mu(t)\big)$$

and

$$M^*(t)=\big((s^\mu)^*(t)|(u^\mu)^*(t)\big)$$

so that $M^*=(M^{\mathrm{T}})^{-1}$ or $(M^*)^{\mathrm{T}}=M^{-1}$. Hence we find that

$$\begin{aligned}\det M(t) &= \lim_{t\to\infty}\det M(t)\\ &= \lim_{t\to\infty}\det\Big(s^\mu(t)\mathrm{e}^{\tilde\lambda^\mu t}|u^\mu(t)e^{-\tilde\lambda^\mu t}\Big)\\ &= \lim_{t\to\infty}\det\Big(s(t)\mathrm{e}^{\tilde\lambda^\mu t}|u(t)e^{-\tilde\lambda^\mu t}\Big)\\ &= \lim_{t\to\infty}\det\Big(s(t)|u(t)\Big)\\ &= 2\end{aligned}$$

since

$$\frac{\mathrm{d}}{\mathrm{d}t}\det M = \operatorname{tr} K^\mu \det M = 0.$$

From this computation and the identities

$$
\begin{aligned}
(s^\mu)^*(t) &= \frac{1}{\det M(t)} \begin{pmatrix} \det\left(e_1|u^\mu(t)\right) \\ \det\left(e_2|u^\mu(t)\right) \end{pmatrix}, \\
(u^\mu)^*(t) &= \frac{1}{\det M(t)} \begin{pmatrix} \det\left(s^\mu(t)|e_1\right) \\ \det\left(s^\mu(t)|e_2\right) \end{pmatrix}
\end{aligned}
$$

we obtain

$$|(s^\mu)^*(t)| \lesssim |u^\mu(t)| \lesssim \mathrm{e}^{\tilde{\lambda}^\mu t}$$

and

$$|(u^\mu)^*(t)| \lesssim |s^\mu(t)| \lesssim \mathrm{e}^{-\tilde{\lambda}^\mu t}$$

for all $t \geq 0$.

Finally we note that $u^\mu$ is reversible by construction since $u^\mu(0) \in \langle u(0) \rangle$. Furthermore $P^\mu(\tilde{\lambda}^\mu t)$ is an antireversible solution of equation (5.19). Since $\{\dot{P}^\mu(\tilde{\lambda}^\mu t), u^\mu(t)\}$ is a fundamental solution set for equation (5.19) we find every bounded solution of equation (5.19) to be a multiple of $\dot{P}^\mu(\tilde{\lambda}^\mu t)$. This fact is in particular true of $s^\mu$, which is therefore antireversible. □

**Lemma 5.3.** Suppose that $n = 4$. The linear equation

$$\dot{R} = K^\mu R \tag{5.26}$$

has solutions $s_1^\mu, s_2^\mu, u_1^\mu, u_2^\mu \colon [0, \infty) \to \mathbb{R}^4$ such that the estimates

$$|s_j^\mu(t)| \lesssim \mathrm{e}^{-\tilde{\lambda}^\mu t}, \quad |u_j^\mu(t)| \lesssim \mathrm{e}^{\tilde{\lambda}^\mu t}$$

hold for all $t \in [0, \infty)$ and $j \in \{1, 2\}$. The dual basis $\{(s_1^\mu)^*(t), (s_2^\mu)^*(t), (u_1^\mu)^*(t), (u_2^\mu)^*(t)\}$ to $\{s_1^\mu(t), s_2^\mu(t), u_1^\mu(t), u_2^\mu(t)\}$ in $\mathbb{R}^4$ satisfies the estimates

$$|(s_j^\mu)^*(t)| \lesssim \mathrm{e}^{\tilde{\lambda}^\mu t}, \quad |(u_j^\mu)^*(t)| \lesssim \mathrm{e}^{-\tilde{\lambda}^\mu t}$$

for all $t \in [0, \infty)$ and $j \in \{1, 2\}$. Furthermore $u_j^\mu$ are reversible and $s_j^\mu$ antireversible, i.e. they satisfy

$$S_{\mathrm{wh}} u_j^\mu(0) = u_j^\mu(0), \quad S_{\mathrm{wh}} s_j^\mu(0) = -s_j^\mu(0).$$

*Proof.* We use the transformation

$$r_1(t) = \mathrm{e}^{\mathrm{i}(\omega + \tilde{\sigma}^\mu)t} z_1(t), \quad r_2(t) = \mathrm{e}^{\mathrm{i}(\omega + \tilde{\sigma}^\mu)t} z_2(t),$$

to find that it suffices to prove this result for the special case $\omega + \tilde{\sigma}^\mu = 0$. Using real coordinates

$$z_1(t) = x_1 + \mathrm{i} x_2, \quad z_2(t) = y_1 + \mathrm{i} y_2,$$

we observe that

$$L_{\mathrm{wh}}^\mu Z + H_{\mathrm{wh}}^\mu(Z) = \tilde{\lambda}^\mu T_1(Z) + \begin{pmatrix} 0 \\ -C x_1 (x_1^2 + x_2^2) \\ 0 \\ -C x_2 (x_1^2 + x_2^2) \end{pmatrix} + \mathcal{O}(|(Z, \mu)||Z|^3),$$

where

$$T_1 \begin{pmatrix} x_1 \\ y_1 \\ x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} y_1 \\ x_1 \\ y_2 \\ x_2 \end{pmatrix},$$

and since by construction

$$p^\mu(t) = (\tilde\lambda^\mu)^{\frac{1}{2}} \begin{pmatrix} h(\tilde\lambda^\mu t) \\ \dot h(\tilde\lambda^\mu t) \\ 0 \\ 0 \end{pmatrix} + \mathcal{R}^\mu(t)$$

with

$$|\mathcal{R}^\mu(t)| \lesssim \mu^2 \mathrm{e}^{-\nu\tilde\lambda^\mu t},$$

we conclude that

$$K^\mu R = \tilde\lambda^\mu T_1(R) + T_2^\mu(R) + T_3^\mu(t),$$

where

$$T_2^\mu \begin{pmatrix} x_1 \\ y_1 \\ x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 \\ -3Ch^2(\tilde\lambda^\mu t)\tilde\lambda^\mu x_1 \\ 0 \\ -Ch^2(\tilde\lambda^\mu t)\tilde\lambda^\mu x_2 \end{pmatrix},$$

and

$$|T_3^\mu(t)| \lesssim \mu^3 \mathrm{e}^{-\nu\tilde\lambda^\mu t}.$$

We first consider the system

$$\dot Z = \tilde\lambda^\mu T_1(Z) + T_2^\mu(Z),$$

which has the explicit solutions

$$s_1(t) = \begin{pmatrix} v_1(\tilde\lambda^\mu t) \\ \dot v_1(\tilde\lambda^\mu t) \\ 0 \\ 0 \end{pmatrix}, \quad s_2(t) = \begin{pmatrix} 0 \\ 0 \\ w_1(\tilde\lambda^\mu t) \\ \dot w_1(\tilde\lambda^\mu t) \end{pmatrix}, \tag{5.27}$$

$$u_1(t) = \begin{pmatrix} v_2(\tilde\lambda^\mu t) \\ \dot v_2(\tilde\lambda^\mu t) \\ 0 \\ 0 \end{pmatrix}, \quad u_2(t) = \begin{pmatrix} 0 \\ 0 \\ w_2(\tilde\lambda^\mu t) \\ \dot w_2(\tilde\lambda^\mu t) \end{pmatrix}, \tag{5.28}$$

where

$$\begin{aligned} v_1(t) &= \operatorname{sech}(t)\tanh(t), \\ v_2(t) &= \operatorname{sech}(t)(-3 + \cosh^2(t) + 3t\tanh(t)), \\ w_1(t) &= \operatorname{sech}(t), \\ w_2(t) &= \operatorname{sech}(t)(2t + \sinh(2t)), \end{aligned}$$

which satisfy

$$|s_1(t)|, |s_2(t)| \lesssim \mathrm{e}^{-\tilde\lambda^\mu t}, \quad |u_1(t)|, |u_2(t)| \lesssim \mathrm{e}^{\tilde\lambda^\mu t}$$

for $t \in [0,\infty)$. The dual basis $\{s_1^*(t), s_2^*(t), u_1^*(t), u_2^*(t)\}$ to $\{s_1(t), s_2(t), u_1(t), u_2(t)\}$ is given by

$$s_1^*(t) = \frac{1}{2}\begin{pmatrix} \dot{v}_2(\tilde{\lambda}^\mu t) \\ -v_2(\tilde{\lambda}^\mu t) \\ 0 \\ 0 \end{pmatrix}, \quad s_2^*(t) = \frac{1}{4}\begin{pmatrix} 0 \\ 0 \\ \dot{w}_2(\tilde{\lambda}^\mu t) \\ -w_2(\tilde{\lambda}^\mu t) \end{pmatrix}, \tag{5.29}$$

$$u_1^*(t) = \frac{1}{2}\begin{pmatrix} -\dot{v}_1(\tilde{\lambda}^\mu t) \\ v_1(\tilde{\lambda}^\mu t) \\ 0 \\ 0 \end{pmatrix}, \quad u_2^*(t) = \frac{1}{4}\begin{pmatrix} 0 \\ 0 \\ -\dot{w}_1(\tilde{\lambda}^\mu t) \\ w_1(\tilde{\lambda}^\mu t) \end{pmatrix}, \tag{5.30}$$

and satisfies

$$|s_j^*(t)| \lesssim \mathrm{e}^{\tilde{\lambda}^\mu t}, \quad |u_j^*(t)| \lesssim \mathrm{e}^{-\tilde{\lambda}^\mu t}$$

for $t \in [0,\infty)$ and $j \in \{1,2\}$.

Now we turn to (5.26). Consider the integral equations

$$s_j^\mu(t) = s_j(t) - \sum_{i=1}^{2} \int_t^\infty \langle T_3^\mu(\tau) s_j^\mu(\tau), s_i^*(\tau)\rangle \ \mathrm{d}\tau \ s_i(t) - \sum_{i=1}^{2} \int_t^\infty \langle T_3^\mu(\tau) s_j^\mu(\tau), u_i^*(\tau)\rangle \ \mathrm{d}\tau \ u_i(t), \tag{5.31}$$

$$u_j^\mu(t) = u_j(t) + \sum_{i=1}^{2} \int_0^t \langle T_3^\mu(\tau) u_j^\mu(\tau), s_i^*(\tau)\rangle \ \mathrm{d}\tau \ s_i(t) - \sum_{i=1}^{2} \int_t^\infty \langle T_3^\mu(\tau) u_j^\mu(\tau), u_i^*(\tau)\rangle \ \mathrm{d}\tau \ u_i(t), \tag{5.32}$$

for $j \in \{1,2\}$ and note that any solution of equation (5.31) or (5.32) is a solution of equation (5.26). Our task is therefore to find solutions to equations (5.31) and (5.32) which satisfy the desired estimates. To this end denote the right-hand side of equation (5.31) by $\mathcal{G}_j(s_j^\mu)$. Arguing as in the proof of Lemma 5.2, we find that the mapping $\mathcal{G}_j$ is a contraction on $\overline{B}_1(s_j) \subseteq C_{-\tilde{\lambda}^\mu}([0,\infty);\mathbb{R}^4)$ and thus has a fixed point $s_j^\mu$ satisfying $|s_j^\mu(t)| \lesssim \mathrm{e}^{-\tilde{\lambda}^\mu t}$ for $t \in [0,\infty)$. We also observe that

$$|s_j^\mu(t) - s_j(t)|\mathrm{e}^{\tilde{\lambda}^\mu t} \leq \int_t^\infty |T_3^\mu(\tau)| \ \mathrm{d}\tau \ \|s_j^\mu\|_{C_{-\tilde{\lambda}^\mu}([0,\infty);\mathbb{R}^4)} \to 0$$

as $t \to \infty$ and conclude that

$$\lim_{t\to\infty} s_j^\mu(t) e^{\tilde{\lambda}^\mu t} = \lim_{t\to\infty} s_j(t) e^{\tilde{\lambda}^\mu t} = \begin{cases} (2,-2,0,0)^\top, & j=1, \\ (0,0,2,-2)^\top, & j=2. \end{cases} \tag{5.33}$$

In a similar fashion we construct a solution $u_j^\mu$ to equation (5.32) with

$$|u_j^\mu(t)| \lesssim \mathrm{e}^{\tilde{\lambda}^\mu t}$$

and

$$\lim_{t\to\infty} u_j^\mu(t) e^{-\tilde{\lambda}^\mu t} = \lim_{t\to\infty} u_j(t) e^{-\tilde{\lambda}^\mu t} = \begin{cases} (\frac{1}{2},\frac{1}{2},0,0)^\top, & j=1, \\ (0,0,1,1)^\top, & j=2. \end{cases}$$

The next step is to compute the dual basis $\{(s_1^\mu)^*(t), (s_2^\mu)^*(t), (u_1^\mu)^*(t), (u_2^\mu)^*(t)\}$ in $\mathbb{R}^4$. To this end define the matrices

$$M(t) = (s_1^\mu(t)|s_2^\mu(t)|u_1^\mu(t)|u_2^\mu(t)), \quad M^*(t) = ((s_1^\mu)^*(t)|(s_2^\mu)^*(t)|(u_1^\mu)^*(t)|(u_2^\mu)^*(t))$$

and note that $M^* = (M^\top)^{-1}$ by definition of the dual basis. In order to compute $M^*$, we note that

$$\frac{\mathrm{d}}{\mathrm{d}t} \det M = \operatorname{tr} K^\mu \det M = 0.$$

Using the multilinearity and continuity of the determinant we compute

$$\begin{aligned}
\det M &= \lim_{t\to\infty} \det M \\
&= \lim_{t\to\infty} \det(s_1^\mu(t)\mathrm{e}^{\tilde{\lambda}^\mu t}|s_2^\mu(t)\mathrm{e}^{\tilde{\lambda}^\mu t}|u_1^\mu(t)\mathrm{e}^{-\tilde{\lambda}^\mu t}|u_2^\mu(t)\mathrm{e}^{-\tilde{\lambda}^\mu t}) \\
&= \lim_{t\to\infty} \det(s_1(t)\mathrm{e}^{\tilde{\lambda}^\mu t}|s_2(t)\mathrm{e}^{\tilde{\lambda}^\mu t}|u_1(t)\mathrm{e}^{-\tilde{\lambda}^\mu t}|u_2(t)\mathrm{e}^{-\tilde{\lambda}^\mu t}) \\
&= \lim_{t\to\infty} \det(s_1(t)|s_2(t)|u_1(t)|u_2(t)) \\
&= 8.
\end{aligned}$$

In terms of the adjunct matrix $M^\#$ the entries of $M^*$ can now be computed via

$$(M^*)^\top = \frac{1}{\det M^\top} M^\#$$

with

$$M^\#_{ij} = (-1)^{i+j} \det(c_1^M|\cdots|c_{i-1}^M|e_j|c_{i+1}^M|\cdots|c_4^M),$$

in which $c_j^M$ denotes the $j$-th column of $M$ and $\{e_1, \ldots, e_4\}$ is the usual basis for $\mathbb{R}^4$. We thus obtain the explicit formulae

$$(s_1^\mu)^*(t) = \frac{1}{8}\begin{pmatrix} \det(e_1|s_2^\mu(t)|u_1^\mu(t)|u_2^\mu(t)) \\ -\det(e_2|s_2^\mu(t)|u_1^\mu(t)|u_2^\mu(t)) \\ \det(e_3|s_2^\mu(t)|u_1^\mu(t)|u_2^\mu(t)) \\ -\det(e_4|s_2^\mu(t)|u_1^\mu(t)|u_2^\mu(t)) \end{pmatrix},$$

$$(u_1^\mu)^*(t) = \frac{1}{8}\begin{pmatrix} \det(s_1^\mu(t)|s_2^\mu(t)|e_1|u_2^\mu(t)) \\ -\det(s_1^\mu(t)|s_2^\mu(t)|e_2|u_2^\mu(t)) \\ \det(s_1^\mu(t)|s_2^\mu(t)|e_3|u_2^\mu(t)) \\ -\det(s_1^\mu(t)|s_2^\mu(t)|e_4|u_2^\mu(t)) \end{pmatrix},$$

$$(s_2^\mu)^*(t) = \frac{1}{8}\begin{pmatrix} -\det(s_1^\mu(t)|e_1|u_1^\mu(t)|u_2^\mu(t)) \\ \det(s_1^\mu(t)|e_2|u_1^\mu(t)|u_2^\mu(t)) \\ -\det(s_1^\mu(t)|e_3|u_1^\mu(t)|u_2^\mu(t)) \\ \det(s_1^\mu(t)|e_4|u_1^\mu(t)|u_2^\mu(t)) \end{pmatrix},$$

$$(u_1^\mu)^*(t) = \frac{1}{8}\begin{pmatrix} -\det(s_1^\mu(t)|s_2^\mu(t)|u_1^\mu(t)|e_1) \\ \det(s_1^\mu(t)|s_2^\mu(t)|u_1^\mu(t)|e_2) \\ -\det(s_1^\mu(t)|s_2^\mu(t)|u_1^\mu(t)|e_3) \\ \det(s_1^\mu(t)|s_2^\mu(t)|u_1^\mu(t)|e_4) \end{pmatrix}$$

and again using the multilinearity of the determinant, we find that

$$\begin{aligned}
|(s_1^\mu)^*(t)| &\lesssim |s_2^\mu(t)||u_1^\mu(t)||u_2^\mu(t)| \lesssim \mathrm{e}^{\tilde{\lambda}^\mu t}, \\
|(s_2^\mu)^*(t)| &\lesssim |s_1^\mu(t)||u_1^\mu(t)||u_2^\mu(t)| \lesssim \mathrm{e}^{\tilde{\lambda}^\mu t}, \\
|(u_1^\mu)^*(t)| &\lesssim |s_1^\mu(t)||s_2^\mu(t)||u_2^\mu(t)| \lesssim \mathrm{e}^{-\tilde{\lambda}^\mu t}, \\
|(u_2^\mu)^*(t)| &\lesssim |s_1^\mu(t)||s_2^\mu(t)||u_1^\mu(t)| \lesssim \mathrm{e}^{-\tilde{\lambda}^\mu t}
\end{aligned}$$

for $t \in [0,\infty)$.

It remains to demonstrate the reversibility of $u_j^\mu$ and the antireversibility of $s_j^\mu$. It follows from equation (5.32) that $u_1^\mu(0), u_2^\mu(0) \in \langle u_1^\mu(0), u_2^\mu(0)\rangle$, and since the functions $u_1^\mu$ and $u_2^\mu$ are reversible, we conclude that $u_1^\mu$ and $u_2^\mu$ are reversible. Moreover, recall that equation (4.4), written in the new variables as

$$z_t = L_{\mathrm{wh}}^\mu Z + H_{\mathrm{wh}}^\mu(z),$$

has the family

$$\{R_a p^\mu(t_0+\cdot)\colon a\in[0,2\pi),\ t_0\in\mathbb{R}\}$$

of homoclinic solutions. Hence the functions

$$\tilde{S}_1 = \frac{\mathrm{d}}{\mathrm{d}t_0} R_a p^\mu(t_0+\cdot)\Big|_{a,t_0=0} = (p^\mu)', \quad \tilde{S}_2 = \frac{\mathrm{d}}{\mathrm{d}a} R_a p^\mu(t_0+\cdot)\Big|_{a,t_0=0} = R_{\pi/2}p^\mu$$

are linearly independent, antireversible, homoclinic solutions of equation (5.26), and the following argument shows that the set $\{\tilde{S}_1, \tilde{S}_2, u_1^\mu, u_2^\mu\}$ is a fundamental solution set to equation (5.26). Suppose that

$$\alpha_1\tilde{S}_1(t) + \alpha_2\tilde{S}_2(t) = -\beta_1 u_1^\mu(t) - \beta_2 u_2^\mu(t), \quad t\in[0,\infty),$$

for some scalars $\alpha_j$ and $\beta_j$. The function on the left-hand side is antireversible, while the function on the right-hand side is reversible, so that

$$\alpha_1\tilde{S}_1(t) + \alpha_2\tilde{S}_2(t) = 0, \quad \beta_1 u_1^\mu(t) + \beta_2 u_2^\mu(t) = 0, \qquad t\in[0,\infty),$$

and thus $\alpha_1=\alpha_2=\beta_1=\beta_2=0$, since $\tilde{S}_1, \tilde{S}_2$ as well as $u_1^\mu, u_2^\mu$ are linearly independent. Now by construction $\{s_1^\mu, s_2^\mu, u_1^\mu, u_2^\mu\}$ is also a fundamental solution set to equation (5.26), so that each of the bounded solutions $s_1^\mu$ and $s_2^\mu$ is a linear combination of $\tilde{S}_1$ and $\tilde{S}_2$. In particular $s_1^\mu$ and $s_2^\mu$ are antireversible (note that we cannot simply take $s_1^\mu=\tilde{S}_1,\ s_2^\mu=\tilde{S}_2$ since we *a priori* know only that $|\tilde{S}_1(t)|, |\tilde{S}_2(t)| \lesssim \mathrm{e}^{-\nu\tilde{\lambda}^\mu t}$ for $t\in[0,\infty)$). □

## 5.2 A local centre-stable manifold

In this section we show that the function

$$\mathcal{F}(R,W,U) = \Big(\mathcal{F}_{\mathrm{wh}}(R,W,U), \mathcal{F}_{\mathrm{c}}(R,W,U), \mathcal{F}_{\mathrm{sh}}(R,W,U)\Big) \tag{5.34}$$

is a contraction on the closed, convex subset

$$\begin{aligned} B_{\nu\tilde{\lambda}^\mu}^+ = \Big\{ &(R,W,U)\in C_{\mathrm{b}}^\alpha([0,\infty);\mathbb{R}^n)\times C_{\nu\tilde{\lambda}^\mu}^1([0,\infty);\mathbb{R}^{2d})\times C_{\mathrm{b}}^\alpha(\mathbb{R};\mathcal{D}_{\mathrm{sh}}) : \\ &\|R\|_{C_{\mathrm{b}}^\alpha([0,\infty);\mathbb{R}^n)}, \|U\|_{C_{\mathrm{b}}^\alpha(\mathbb{R};\mathcal{D}_{\mathrm{sh}})}, \|\dot{W} - L_{\mathrm{c}}^\mu W\|_{C_{\mathrm{b}}([0,\infty);\mathbb{R}^{2d})} \le \mathrm{e}^{-\frac{c^*}{2\mu}}, \\ &(S_{\mathrm{wh}}R)(0) = R(0), (S_{\mathrm{c}}W)(0) = W(0)\Big\} \end{aligned}$$

of

$$E_{\nu\tilde{\lambda}^\mu}^+ = C_{\nu\tilde{\lambda}^\mu}^\alpha([0,\infty);\mathbb{R}^n)\times C_{\nu\tilde{\lambda}^\mu}^1([0,\infty);\mathbb{R}^{2d})\times C_{\nu\tilde{\lambda}^\mu}^\alpha(\mathbb{R};\mathcal{D}_{\mathrm{sh}})$$

provided that $S_c W_0 = W_0$. An auxiliary argument shows that its fixed point denoted by $(R^*_{W_0}, W^*_{W_0}, U^*_{W_0})$ satisfies $S_{\mathrm{sh}} U^*_{W_0}(0) = U^*_{W_0}(0)$, so that $(R^*_{W_0}, W^*_{W_0}, U^*_{W_0})$ is a reversible, bounded solution of equations (5.12) – (5.14).

Our results show that the fixed points $(R^*_{W_0}, W^*_{W_0}, U^*_{W_0})$ induce a family of reversible solutions to equations (5.12) – (5.14) in

$$E_{\nu\tilde{\lambda}^\mu} = C^\alpha_{\nu\tilde{\lambda}^\mu}(\mathbb{R};\mathbb{R}^n) \times C^1_{\nu\tilde{\lambda}^\mu}(\mathbb{R};\mathbb{R}^{2d}) \times C^\alpha_{\nu\tilde{\lambda}^\mu}(\mathbb{R};\mathcal{D}_{\mathrm{sh}})$$

for $|W_0| \le \mu \mathrm{e}^{-\frac{c^*}{2\mu}}$ satisfying

$$\sup_{t\in\mathbb{R}} |R^*_{W_0}(t)| \lesssim \mu \mathrm{e}^{-\frac{c^*}{2\mu}},$$

$$\sup_{t\in[-\mathrm{e}^{c^*/2\mu}, \mathrm{e}^{c^*/2\mu}]} |W^*_{W_0}(t)| \lesssim \mu^\delta \mathrm{e}^{-\frac{c^*}{2\mu}},$$

$$\sup_{t\in\mathbb{R}} \|U^*_{W_0}(t)\|_{\mathcal{D}_{\mathrm{sh}}} \lesssim \mu \mathrm{e}^{-\frac{c^*}{2\mu}}$$

for some $\delta > 0$ (in the case $n = 2$ we have $\delta = 1$). These solutions thus in particular solve equations (5.8) – (5.10) for $t \in [-\mathrm{e}^{c^*/2\mu}, \mathrm{e}^{c^*/2\mu}]$. In analogy with familiar dynamical-systems theory (see Kelley [15]) we define

$$W^{\mathrm{cs}}_{\mathrm{loc}} = \left\{ (R^*_{W_0}, W^*_{W_0}, U^*_{W_0})(0) : S_c W_0 = W_0, |W_0| \le \mu \mathrm{e}^{-\frac{c^*}{2\mu}} \right\}$$

and refer to $W^{\mathrm{cs}}_{\mathrm{loc}}$ as the *local centre-stable manifold for reversible solutions to equations* (5.8) – (5.10). Observe that $W^{\mathrm{cs}}_{\mathrm{loc}}$ is a graph over $\{W_0 \in \mathbb{R}^{2d} : S_c W_0 = W_0, |W_0| \le \mu \mathrm{e}^{-\frac{c^*}{2\mu}}\}$ since $W^*_{W_0}(0) = W_0$. The behaviour of functions with initial values lying on $W^{\mathrm{cs}}_{\mathrm{loc}}$ is summarised in Figures 5.2 – 5.4. In this section and Section 5.4 below we show that in fact $|W^*_{W_0}(t)| \le \mathrm{e}^{-\frac{c^*}{2\mu}}$ for all $t \in \mathbb{R}$ (cf. Figures 5.2 – 5.4), so that $W^{\mathrm{cs}}_{\mathrm{loc}}$ is actually a global centre-stable manifold for equations (5.8) – (5.10). We treat the cases $n = 2$ and $n = 4$ separately. In the proofs of our theorems we use the fact that the norms $\|\cdot\|_{C^\alpha_{\nu\tilde{\lambda}^\mu}(I;B)}$ and $|||\cdot|||_{C^\alpha_{\nu\tilde{\lambda}^\mu}(I;B)}$ are equivalent, uniformly in $\mu$, for $I \in \{[0,\infty), \mathbb{R}\}$ and $B \in \{\mathbb{R}^2, \mathbb{R}^{2d}, \mathcal{D}_{\mathrm{sh}}\}$, so that changing between them does not introduce any new dependency upon $\mu$.

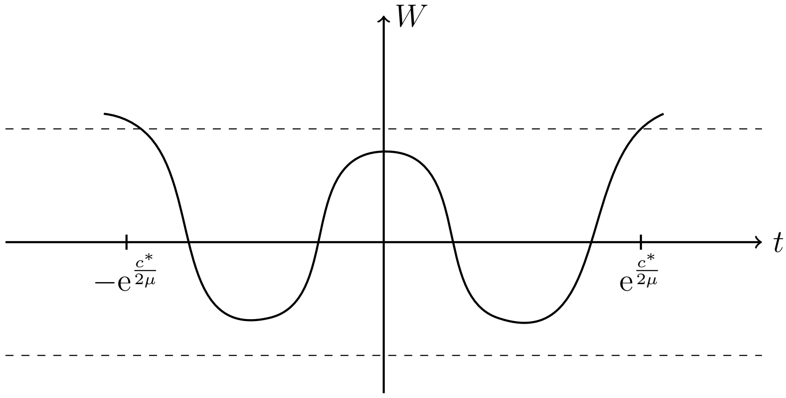



Figure 5.2: The central part of functions with initial values on $W^{cs}_{loc}$ satisfies $|W(t)| \leq e^{-\frac{c^*}{2\mu}}$ for $t \in [-e^{\frac{c^*}{2\mu}}, e^{\frac{c^*}{2\mu}}]$. It may leave this neighbourhood of the origin for larger values of $|t|$.

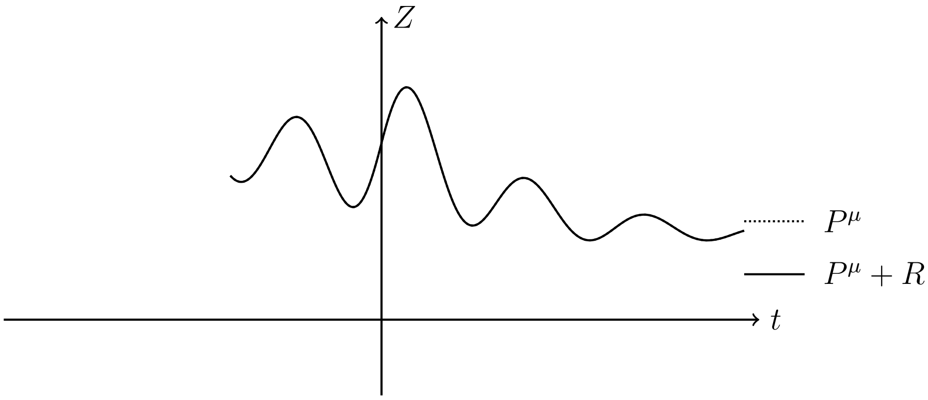



Figure 5.3: The weakly hyperbolic part of functions with initial values on $W^{cs}_{loc}$ lies in a tubular neighbourhood of $P^\mu$ such that $|(Z(t) - P^\mu(t))| \leq e^{-\frac{c^*}{2\mu}}$ for all $t \in \mathbb{R}$.

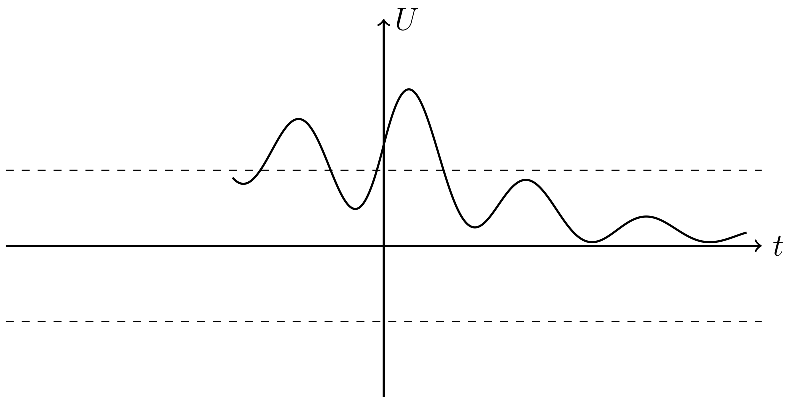



Figure 5.4: The strongly hyperbolic part of functions with initial values on $W^{cs}_{loc}$ satisfies $\|U(t)\|_{\mathcal{D}_{sh}} \leq e^{-\frac{c^*}{2\mu}}$ for all $t \in \mathbb{R}$.

### 5.2.1 The case $n = 2$

First we show that $\mathcal{F}$ is a contraction.

**Theorem 5.4.** The operator $\mathcal{F}$ given in equation (5.34) maps $B^+_{\nu\tilde{\lambda}^\mu}$ into itself. Furthermore,

$$\|\mathcal{F}_{\mathrm{wh}}(R,W,U)\|_{C^\alpha_{\mathrm{b}}([0,\infty);\mathbb{R}^2)} \lesssim \mu \mathrm{e}^{-\frac{c^*}{2\mu}}, \tag{5.35}$$

$$\Big\|\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{F}_{\mathrm{c}}(R,W,U) - L^\mu_{\mathrm{c}}\mathcal{F}_{\mathrm{c}}(R,W,U)\Big\|_{C_{\mathrm{b}}(\mathbb{R};\mathbb{R}^{2d})} \lesssim \mu \mathrm{e}^{-\frac{c^*}{2\mu}}, \tag{5.36}$$

$$\|\mathcal{F}_{\mathrm{sh}}(R,W,U)\|_{C^\alpha_{\mathrm{b}}(\mathbb{R};\mathcal{D}_{\mathrm{sh}})} \lesssim \mu \mathrm{e}^{-\frac{c^*}{2\mu}} \tag{5.37}$$

and

$$\sup_{t\in[0,\mathrm{e}^{\frac{c^*}{2\mu}}]} |\mathcal{F}_{\mathrm{c}}(R,W,U)(t)| \lesssim \mu \mathrm{e}^{-\frac{c^*}{2\mu}} \tag{5.38}$$

for all $(R,W,U) \in B^+_{\nu\tilde{\lambda}^\mu}$ provided that $|W_0| \le \mu \mathrm{e}^{-\frac{c^*}{2\mu}}$.

*Proof.* Suppose that $(R,W,U) \in B^+_{\nu\tilde{\lambda}^\mu}$. Inspecting equations (5.15) and (5.17) shows that $(S_{\mathrm{wh}}\mathcal{F}_{\mathrm{wh}}R)(0) = R(0)$ and $(S_{\mathrm{c}}\mathcal{F}_{\mathrm{c}}W)(0) = W(0)$ (recall that $S_{\mathrm{c}}W_0 = W_0$ and $(S_{\mathrm{wh}}(u^\mu)(0) = u^\mu(0)$.)

To estimate $\mathcal{F}_{\mathrm{wh}}(R,W,U)$ we observe that

$$\begin{aligned}
&|N^\mu(R,W,U)(t)| \\
&\lesssim \|R\|^2_{C^\alpha_{\mathrm{b}}([0,\infty);\mathbb{R}^2)} + \Big(\mu^3 + \mu\|R\|_{C^\alpha_{\mathrm{b}}([0,\infty);\mathbb{R})} + \|R\|^2_{C^\alpha_{\mathrm{b}}([0,\infty);\mathbb{R}^2)}\Big)\|(\phi(W),U)\|_{C^\alpha_{\mathrm{b}}([0,\infty);\mathcal{D}_{\mathrm{c,sh}})} \\
&\qquad + \|(\phi(W),U)\|^2_{C^\alpha_{\mathrm{b}}([0,\infty);\mathcal{D}_{\mathrm{c,sh}})} \\
&\lesssim \mu^3 \mathrm{e}^{-\frac{c^*}{2\mu}},
\end{aligned} \tag{5.39}$$

where we have used Remark 5.1. The above estimate implies that

$$\begin{aligned}
\Big|\int_0^t \langle \underline{N}^\mu(R,W,U)(\tau); (s^\mu)^*(\tau)\rangle \,\mathrm{d}\tau\, s^\mu(t)\Big| &\lesssim \int_0^t \mu^3 \mathrm{e}^{-\frac{c^*}{2\mu}} \mathrm{e}^{\tilde{\lambda}^\mu\tau}\,\mathrm{d}\tau\, \mathrm{e}^{-\tilde{\lambda}^\mu t} \\
&= \mu(1-\mathrm{e}^{-\tilde{\lambda}^\mu t})\mathrm{e}^{-\frac{c^*}{2\mu}} \\
&\le \mu \mathrm{e}^{-\frac{c^*}{2\mu}}
\end{aligned}$$

and

$$\begin{aligned}
\Big|\int_0^t \langle \underline{N}^\mu(R,W,U)(\tau); (u^\mu)^*(\tau)\rangle \,\mathrm{d}\tau\, u^\mu(t)\Big| &\lesssim \mu^3 \mathrm{e}^{-\frac{c^*}{2\mu}} \int_t^\infty \mathrm{e}^{-\tilde{\lambda}^\mu\tau}\,\mathrm{d}\tau\, \mathrm{e}^{\tilde{\lambda}^\mu t} \\
&= \mu \mathrm{e}^{-\frac{c^*}{2\mu}}
\end{aligned}$$

for all $t \in [0,\infty)$, so that

$$\sup_{t\in[0,\infty)} |\mathcal{F}_{\mathrm{wh}}(R,W,U)(t)| \lesssim \mu \mathrm{e}^{-\frac{c^*}{2\mu}}.$$

Next we observe that

$$\|\mathrm{e}^{\tilde{\lambda}^\mu(\cdot)}\|_{C^\alpha_{\tilde{\lambda}^\mu}([0,\infty);\mathbb{R})} \lesssim \|\mathrm{e}^{\tilde{\lambda}^\mu(\cdot)}\|_{C^1_{\tilde{\lambda}^\mu}([0,\infty))} \lesssim 1$$

and

$$\begin{aligned}\|s^\mu\|_{C^\alpha_{-\tilde{\lambda}^\mu}([0,\infty);\mathbb{R}^2)} &\lesssim \|s^\mu\|_{C^1_{-\tilde{\lambda}^\mu}([0,\infty);\mathbb{R}^2)} \\ &\lesssim 1 + \sup_{t\in[0,\infty)} \mathrm{e}^{\tilde{\lambda}^\mu t}|K^\mu(t)s^\mu(t)| \\ &\lesssim 1,\end{aligned}$$

where we have used the fact that $\|\cdot\|_{C^\alpha_{\tilde{\lambda}^\mu}([0,\infty);\mathbb{R})} \leq \|\cdot\|_{C^1_{\tilde{\lambda}^\mu}([0,\infty);\mathbb{R})}$. Similarly we obtain

$$\|u^\mu\|_{C^\alpha_{\tilde{\lambda}^\mu}([0,\infty);\mathbb{R}^2)} \lesssim 1.$$

Using Lemma 5.2 and estimate (5.39) we find that

$$\begin{aligned}&\left|\int_0^{t_1} \langle \underline{N}^\mu(R,W,U)(\tau);(s^\mu)^*(\tau)\rangle\,\mathrm{d}\tau\, s^\mu(t_1) - \int_0^{t_2} \langle \underline{N}^\mu(R,W,U)(\tau);(s^\mu)^*(\tau)\rangle\,\mathrm{d}\tau\, s^\mu(t_2)\right| \\ &= \Bigg|\int_{t_2}^{t_1} \langle \underline{N}^\mu(R,W,U)(\tau);(s^\mu)^*(\tau)\rangle\,\mathrm{d}\tau\, s^\mu(t_1) \\ &\qquad - \int_0^{t_2} \langle \underline{N}^\mu(R,W,U)(\tau);(s^\mu)^*(\tau)\rangle\,\mathrm{d}\tau\Big(s^\mu(t_1)-s^\mu(t_2)\Big)\Bigg| \\ &\lesssim \mu\mathrm{e}^{-\frac{c^*}{2\mu}}(\mathrm{e}^{\tilde{\lambda}^\mu t_1}-\mathrm{e}^{\tilde{\lambda}^\mu t_2})\mathrm{e}^{-\tilde{\lambda}^\mu t_1} + \mu\mathrm{e}^{-\frac{c^*}{2\mu}}(\mathrm{e}^{\tilde{\lambda}^\mu t_2}-1)|s^\mu(t_1)-s^\mu(t_2)| \\ &\leq \mu\mathrm{e}^{-\frac{c^*}{2\mu}}\|\mathrm{e}^{\tilde{\lambda}^\mu(\cdot)}\|_{C^\alpha_{\tilde{\lambda}^\mu}([0,\infty))}|t_1-t_2|^\alpha + \mu\mathrm{e}^{-\frac{c^*}{2\mu}}\|s^\mu\|_{C^\alpha_{-\tilde{\lambda}^\mu}([0,\infty);\mathbb{R}^2)}|t_1-t_2|^\alpha \\ &\lesssim \mu\mathrm{e}^{-\frac{c^*}{2\mu}}|t_1-t_2|^\alpha\end{aligned}$$

and in a similar fashion

$$\begin{aligned}&\left|\int_0^{t_1} \langle \underline{N}^\mu(R,W,U)(\tau);(u^\mu)^*(\tau)\rangle\,\mathrm{d}\tau u^\mu(t_1) - \int_0^{t_2} \langle \underline{N}^\mu(R,W,U)(\tau);(u^\mu)^*(\tau)\rangle\,\mathrm{d}\tau u^\mu(t_2)\right| \\ &\leq \mu\mathrm{e}^{-\frac{c^*}{2\mu}}|t_1-t_2|^\alpha\end{aligned}$$

for $0 \leq t_2 < t_1 < \infty$, so that

$$|\mathcal{F}_{\mathrm{wh}}(R,W,U)(t_1) - \mathcal{F}_{\mathrm{wh}}(R,W,U)(t_2)| \lesssim \mu\mathrm{e}^{-\frac{c^*}{2\mu}}|t_1-t_2|^\alpha.$$

Altogether we obtain

$$\|\mathcal{F}_{\mathrm{wh}}(R,W,U)\|_{C^\alpha_{\mathrm{b}}([0,\infty);\mathbb{R}^2)} \leq \mu\mathrm{e}^{-\frac{c^*}{2\mu}}. \tag{5.40}$$

For $\mathcal{F}_{\mathrm{c}}(R,W,U)$ we use Remark 4.8 and estimate (5.7) to find that

$$\begin{aligned}&|\mathcal{F}_{\mathrm{c}}(R,W,U)(t)| \\ &\qquad\leq |W_0| + \left|\int_0^t \mathrm{e}^{L^\mu_{\mathrm{c}}(t-\tau)}\Big(\underline{\tilde{G}}^\mu_{\mathrm{c}}(P^\mu+R,W,U)(\tau) + \tilde{H}^\mu_{\mathrm{c}}(P^\mu+R)(\tau)\Big)\,\mathrm{d}\tau\right| \\ &\qquad\lesssim |W_0| + \int_0^t \Big(|(P^\mu+R)(\tau)|\mu^3\mathrm{e}^{-\frac{c^*}{2\mu}} + \mu^2\mathrm{e}^{-\frac{c^*}{\mu}}\Big)\,\mathrm{d}\tau \\ &\qquad\lesssim |W_0| + \int_0^t \Big(\mu^5\mathrm{e}^{-\nu\tilde{\lambda}^\mu\tau}\mathrm{e}^{-\frac{c^*}{2\mu}} + \mu^2\mathrm{e}^{-\frac{c^*}{\mu}}\Big)\,\mathrm{d}\tau \\ &\qquad\leq |W_0| + \mu^3\mathrm{e}^{-\frac{c^*}{2\mu}} + t\mu^2\mathrm{e}^{-\frac{c^*}{\mu}}\end{aligned} \tag{5.41}$$

and

$$\Big|\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{F}_{\mathrm{c}}(R,W,U)(t)\Big| \lesssim \|\mathcal{F}_{\mathrm{c}}(R,W,U)\|_{C_{\mathrm{b}}^{\alpha}([0,\infty);\mathbb{R}^{2d})} + \|\tilde{G}_{\mathrm{c}}^{\mu}(P^{\mu}+R,W,U)\|_{C_{\mathrm{b}}^{\alpha}([0,\infty);\mathbb{R}^{2d})} + \|\tilde{H}_{\mathrm{c}}^{\mu}(P^{\mu}+R)\|_{C_{\mathrm{b}}^{\alpha}([0,\infty);\mathbb{R}^{2d})}$$
$$\lesssim |W_0| + (t+1)\mu^3 \mathrm{e}^{-\frac{c^*}{2\mu}}$$

for $t \in [0,\infty)$. Altogether we obtain

$$\|\mathcal{F}_{\mathrm{c}}(R,W,U)\|_{C^1_{\nu\tilde{\lambda}^{\mu}}([0,\infty);\mathbb{R}^{2d})} < \infty$$

and

$$\begin{aligned}\|\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{F}_{\mathrm{c}}(R,W,U) - L_{\mathrm{c}}^{\mu}\mathcal{F}_{\mathrm{c}}(R,W,U)\|_{C_{\mathrm{b}}(\mathbb{R};\mathbb{R}^{2d})} &= \|\tilde{G}_{\mathrm{c}}^{\mu}(P^{\mu}+R,W,U) + \tilde{H}_{\mathrm{c}}^{\mu}(P^{\mu}+R)\|_{C_{\mathrm{b}}(\mathbb{R};\mathbb{R}^{2d})} \\ &\lesssim \mu^5 \mathrm{e}^{-\frac{c^*}{2\mu}}. \qquad (5.42)\end{aligned}$$

For $\mathcal{F}_{\mathrm{sh}}(R,W,U)$ we first consider

$$\begin{aligned}
&\|\tilde{G}_{\mathrm{sh}}^{\mu}(P^{\mu}+\mathcal{E}_{\mathrm{wh}}R,\mathcal{E}_{\mathrm{c}}W,U)(t_1) - \tilde{G}_{\mathrm{sh}}^{\mu}(P^{\mu}+\mathcal{E}_{\mathrm{wh}}R,\mathcal{E}_{\mathrm{c}}W,U)(t_2)\|_{\mathcal{X}_{\mathrm{sh}}} \\
&\quad = \Big\|\int_0^1 \mathrm{d}_1\tilde{G}_{\mathrm{sh}}^{\mu}\Big[\sigma(P^{\mu}+\mathcal{E}_{\mathrm{wh}}R)(t_1) + (1-\sigma)(P^{\mu}+\mathcal{E}_{\mathrm{wh}}R)(t_2), \phi(\mathcal{E}_{\mathrm{c}}W)(t_1), U(t_1)\Big] \\
&\qquad\qquad \Big((P^{\mu}+\mathcal{E}_{\mathrm{wh}}R)(t_1) - (P^{\mu}+\mathcal{E}_{\mathrm{wh}}R)(t_2)\Big)\,\mathrm{d}\sigma \\
&\qquad + \int_0^1 \mathrm{d}_2\tilde{G}_{\mathrm{sh}}^{\mu}\Big[(P^{\mu}+\mathcal{E}_{\mathrm{wh}}R)(t_2), \sigma\phi(\mathcal{E}_{\mathrm{c}}W)(t_1) + (1-\sigma)\phi(\mathcal{E}_{\mathrm{c}}W)(t_2), U(t_1)\Big] \\
&\qquad\qquad \Big(\phi(\mathcal{E}_{\mathrm{c}}W)(t_1) - \phi(\mathcal{E}_{\mathrm{c}}W)(t_2)\Big)\,\mathrm{d}\sigma \\
&\qquad + \int_0^1 \mathrm{d}_3\tilde{G}_{\mathrm{sh}}^{\mu}\Big[(P^{\mu}+\mathcal{E}_{\mathrm{wh}}R)(t_2), \phi(\mathcal{E}_{\mathrm{c}}W)(t_2), \sigma U(t_1) + (1-\sigma)U(t_2)\Big] \\
&\qquad\qquad \Big(U(t_1) - U(t_2)\Big)\,\mathrm{d}\sigma\Big\|_{\mathcal{X}_{\mathrm{sh}}} \\
&\quad \lesssim \mu^3 \mathrm{e}^{-\frac{c^*}{2\mu}}|(P^{\mu}+\mathcal{E}_{\mathrm{wh}}R)(t_1) - (P^{\mu}+\mathcal{E}_{\mathrm{wh}}R)(t_2)| \\
&\qquad + \mu^5\Big(|\phi(\mathcal{E}_{\mathrm{c}}W)(t_1) - \phi(\mathcal{E}_{\mathrm{c}}W)(t_2)| + \|U(t_1)-U(t_2)\|_{\mathcal{D}_{\mathrm{sh}}}\Big) \\
&\quad \le |t_1-t_2|^{\alpha}\mu^3 \mathrm{e}^{-\frac{c^*}{2\mu}}\Big(\|P^{\mu}\|_{C_{\mathrm{b}}^{\alpha}(\mathbb{R};\mathbb{R}^2)} + \|\mathcal{E}_{\mathrm{wh}}R\|_{C_{\mathrm{b}}^{\alpha}(\mathbb{R};\mathbb{R}^2)}\Big) \\
&\qquad + |t_1-t_2|^{\alpha}\mu^5\Big(\|\phi(\mathcal{E}_{\mathrm{c}}W)\|_{C_{\mathrm{b}}^{\alpha}(\mathbb{R};\mathbb{R}^{2d})} + \|U\|_{C_{\mathrm{b}}^{\alpha}(\mathbb{R};\mathcal{D}_{\mathrm{sh}})}\Big) \\
&\quad \le \mu^3 \mathrm{e}^{-\frac{c^*}{2\mu}}|t_1-t_2|^{\alpha} \qquad (5.43)
\end{aligned}$$

and

$$\begin{aligned}
&\|\tilde{H}_{\mathrm{sh}}^{\mu}(P^{\mu}+\mathcal{E}_{\mathrm{wh}}R)(t_1) - \tilde{H}_{\mathrm{sh}}^{\mu}(P^{\mu}+\mathcal{E}_{\mathrm{wh}}R)(t_2)\|_{\mathcal{X}_{\mathrm{sh}}} \\
&\quad = \int_0^1 \mathrm{d}\tilde{H}_{\mathrm{sh}}^{\mu}[\sigma(P^{\mu}+\mathcal{E}_{\mathrm{wh}}R)(t_1) \\
&\qquad\qquad + (1-\sigma)(P^{\mu}+\mathcal{E}_{\mathrm{wh}}R)(t_2)]\Big((P^{\mu}+\mathcal{E}_{\mathrm{wh}}R)(t_1) - (P^{\mu}+\mathcal{E}_{\mathrm{wh}}R)(t_2)\Big)\,\mathrm{d}\sigma \\
&\quad \lesssim \mu \mathrm{e}^{-\frac{c^*}{\mu}}\|P^{\mu}+\mathcal{E}_{\mathrm{wh}}R\|_{C_{\mathrm{b}}^{\alpha}(\mathbb{R};\mathbb{R}^2)}|t_1-t_2|^{\alpha}, \qquad (5.44)
\end{aligned}$$

where we have used Proposition 2.7, Remark 4.8, and estimate (5.7). Using the maximal regularity result (Lemma 2.14) we find that

$$\begin{aligned}\|\mathcal{F}_{\mathrm{sh}}(R,W,U)\|_{C^\alpha_{\mathrm{b}}(\mathbb{R};\mathcal{D}_{\mathrm{sh}})} &\lesssim \|(L^\mu_{\mathrm{sh}}-L^0_{\mathrm{sh}})U+\tilde{\underline{G}}^\mu_{\mathrm{sh}}(P^\mu+R,W,U)+\tilde{H}^\mu_{\mathrm{sh}}(P^\mu+R)\|_{C^\alpha_{\mathrm{b}}(\mathbb{R};\mathcal{X}_{\mathrm{sh}})}\\ &\lesssim \mu\mathrm{e}^{-\frac{c^*}{2\mu}}. \end{aligned} \tag{5.45}$$

Finally, we note that estimates (5.35) – (5.38) follow directly from estimates (5.40) – (5.42) and (5.45). □

To show that $\mathcal{F}\colon B^+_{\nu\tilde\lambda^\mu}\to B^+_{\nu\tilde\lambda^\mu}$ is a contraction we use Remark 2.11 to prove that the composition operators induced by the nonlinearities of the right-hand side of equation (5.18) are Lipschitz-continuous.

**Theorem 5.5.** The operator $\mathcal{F}\colon B^+_{\nu\tilde\lambda^\mu}\to B^+_{\nu\tilde\lambda^\mu}$ is a contraction provided that $S_{\mathrm{c}}W_0=W_0$ and $|W_0|\le\mu\mathrm{e}^{-\frac{c^*}{2\mu}}$.

*Proof.* Suppose that $(R_1,W_1,U_1),(R_2,W_2,U_2)\in B^+_{\nu\tilde\lambda^\mu}$. For $\mathcal{F}_{\mathrm{wh}}$ we first consider

$$\begin{aligned}&|\underline{N}^\mu(R_1,W_1,U_1)-\underline{N}^\mu(R_2,W_2,U_2)(t)|\\ &\quad=\Big|\int_0^1 \mathrm{d}_1N^\mu[\sigma R_1+(1-\sigma)R_2,\phi(W_1),U_1](R_1-R_2)(t)\,\mathrm{d}\sigma\\ &\qquad\quad+\int_0^1 \mathrm{d}_2N^\mu[R_2,\sigma\phi(W_1)+(1-\sigma)\phi(W_2),U_2]\Big(\phi(W_1)-\phi(W_2)\Big)(t)\,\mathrm{d}\sigma\\ &\qquad\quad+\int_0^1 \mathrm{d}_3N^\mu[R_2,\phi(W_2),\sigma U_1+(1-\sigma)U_2](U_1-U_2)(t)\,\mathrm{d}\sigma\Big|\\ &\quad\lesssim\mu^3\|(R_1,W_1,U_1)(t)-(R_2,W_2,U_2)(t)\|_{\mathcal{D}},\end{aligned}$$

where we have used Remark 5.1. The above estimate now implies that

$$\begin{aligned}&\Big|\int_0^t\Big\langle\Big(\underline{N}^\mu(R_1,W_1,U_1)-\underline{N}^\mu(R_2,W_2,U_2)\Big)(\tau);(s^\mu)^*(\tau)\Big\rangle\,\mathrm{d}\tau\, s^\mu(t)\Big|\\ &\quad\lesssim\int_0^t\Big|\Big(\underline{N}^\mu(R_1,W_1,U_1)-\underline{N}^\mu(R_2,W_2,U_2)\Big)(\tau)\Big|\mathrm{e}^{\tilde\lambda^\mu\tau}\,\mathrm{d}\tau\mathrm{e}^{-\tilde\lambda^\mu t}\\ &\quad\le\int_0^t\mu^3\Big(|(R_1-R_2)(\tau)|\\ &\qquad\qquad+|(\phi(W_1)-\phi(W_2))(\tau)|+\|(U_1-U_2)(\tau)\|_{\mathcal{D}_{\mathrm{sh}}}\Big)\mathrm{e}^{\tilde\lambda^\mu\tau}\,\mathrm{d}\tau\mathrm{e}^{-\tilde\lambda^\mu t}\\ &\quad\le\mu^3\mathrm{e}^{-\tilde\lambda^\mu t}\|(R_1,W_1,U_1)-(R_2,W_2,U_2)\|_{E^+_{\nu\tilde\lambda^\mu}}\int_0^t\mathrm{e}^{(1+\nu)\tilde\lambda^\mu\tau}\,\mathrm{d}\tau\\ &\quad\le\frac{\mu^3}{(1+\nu)\tilde\lambda^\mu}\mathrm{e}^{\nu\tilde\lambda^\mu t}\|(R_1,W_1,U_1)-(R_2,W_2,U_2)\|_{E^+_{\nu\tilde\lambda^\mu}}\\ &\quad\le\mu\mathrm{e}^{\nu\tilde\lambda^\mu t}\|(R_1,W_1,U_1)-(R_2,W_2,U_2)\|_{E^+_{\nu\tilde\lambda^\mu}}\end{aligned}$$

and

$$
\begin{aligned}
&\Big| \int_t^\infty \langle \underline{N}^\mu(R_1,W_1,U_1)(\tau) - \underline{N}^\mu(R_2,W_2,U_2)(\tau); (u^\mu)^*(\tau)\rangle \,\mathrm{d}\tau\, u^\mu(t)\Big| \\
&\quad \lesssim \mu^3 \int_t^\infty \|\underline{N}^\mu(R_1,W_1,U_1)(\tau) - \underline{N}^\mu(R_2,W_2,U_2)(\tau)\| \mathrm{e}^{-\tilde{\lambda}^\mu \tau}\,\mathrm{d}\tau \mathrm{e}^{\tilde{\lambda}^\mu t} \\
&\quad \lesssim \mu^3 \|(R_1,W_1,U_1) - (R_2,W_2,U_2)\|_{E^+_{\nu\tilde{\lambda}^\mu}} \int_t^\infty \mathrm{e}^{(\nu-1)\tilde{\lambda}^\mu \tau}\,\mathrm{d}\tau \mathrm{e}^{\tilde{\lambda}^\mu t} \\
&\quad \lesssim \frac{\mu^3}{(1-\nu)\tilde{\lambda}^\mu} \mathrm{e}^{\nu\tilde{\lambda}^\mu t} \|(R_1,W_1,U_1) - (R_2,W_2,U_2)\|_{E^+_{\nu\tilde{\lambda}^\mu}} \\
&\quad \lesssim \mu \mathrm{e}^{\nu\tilde{\lambda}^\mu t} \|(R_1,W_1,U_1) - (R_2,W_2,U_2)\|_{E^+_{\nu\tilde{\lambda}^\mu}}.
\end{aligned}
$$

We also find that

$$
\begin{aligned}
&\Big| \int_0^{t_1} \langle \underline{N}^\mu(R_1,W_1,U_1)(\tau) - \underline{N}^\mu(R_2,W_2,U_2)(\tau); (s^\mu)^*(\tau)\rangle \,\mathrm{d}\tau\, s^\mu(t_1) \\
&\qquad - \Big( \int_0^{t_2} \langle \underline{N}^\mu(R_1,W_1,U_1)(\tau) - \underline{N}^\mu(R_2,W_2,U_2)(\tau); (s^\mu)^*(\tau)\rangle \,\mathrm{d}\tau\, s^\mu(t_2)\Big)\Big| \\
&\quad \leq \int_{t_2}^{t_1} \Big| \int_0^1 \mathrm{d}_1 N^\mu[\sigma R_1 + (1-\sigma)R_2, \phi(W_1), U_1](R_1 - R_2)(\tau)\,\mathrm{d}\sigma \\
&\qquad\qquad + \int_0^1 \mathrm{d}_2 N^\mu[R_2, \sigma\phi(W_1) + (1-\sigma)\phi(W_2), U_1]\Big(\phi(W_1) - \phi(W_2)\Big)(\tau)\,\mathrm{d}\sigma \\
&\qquad\qquad + \int_0^1 \mathrm{d}_3 N^\mu[R_2, \phi(W_2), \sigma U_1 + (1-\sigma)U_2](U_1 - U_2)(\tau)\,\mathrm{d}\sigma\Big| \mathrm{e}^{\tilde{\lambda}^\mu \tau}\,\mathrm{d}\tau \mathrm{e}^{-\tilde{\lambda}^\mu t_1} \\
&\qquad + \int_0^{t_2} \Big| \int_0^1 \mathrm{d}_1 N^\mu[\sigma R_1 + (1-\sigma)R_2, \phi(W_1), U_1](R_1 - R_2)(\tau)\,\mathrm{d}\sigma \\
&\qquad\quad + \int_0^1 \mathrm{d}_2 N^\mu[R_2, \sigma\phi(W_1) + (1-\sigma)\phi(W_2), U_1]\Big(\phi(W_1) - \phi(W_2)\Big)(\tau)\,\mathrm{d}\sigma \\
&\qquad\quad + \int_0^1 \mathrm{d}_3 N^\mu[R_2, \phi(W_2), \sigma U_1 + (1-\sigma)U_2](U_1 - U_2)(\tau)\,\mathrm{d}\sigma\Big| \mathrm{e}^{\tilde{\lambda}^\mu \tau}\,\mathrm{d}\tau |s^\mu(t_1) - s^\mu(t_2)| \\
&\quad \lesssim \mu \mathrm{e}^{\nu\tilde{\lambda}^\mu t_1} |t_1 - t_2|^\alpha \|(R_1,W_1,U_1) - (R_2,W_2,U_2)\|_{E^+_{\nu\tilde{\lambda}^\mu}}
\end{aligned}
$$

and

$$
\begin{aligned}
&\Big| \int_0^{t_1} \langle \underline{N}^\mu(R_1,W_1,U_1)(\tau) - \underline{N}^\mu(R_2,W_2,U_2)(\tau); (u^\mu)^*\rangle \,\mathrm{d}\tau\, u^\mu(t_1) \\
&\qquad - \Big( \int_0^{t_2} \langle \underline{N}^\mu(R_1,W_1,U_1)(\tau) - \underline{N}^\mu(R_2,W_2,U_2)(\tau); (u^\mu)^*\rangle \,\mathrm{d}\tau\, u^\mu(t_2)\Big)\Big| \\
&\quad \lesssim \mu \mathrm{e}^{\nu\tilde{\lambda}^\mu t_1} |t_1 - t_2|^\alpha \|(R_1,W_1,U_1) - (R_2,W_2,U_2)\|_{E^+_{\nu\tilde{\lambda}^\mu}}
\end{aligned}
$$

for $0 \leq t_2 < t_1 < \infty$, so that

$$
\|\mathcal{F}_{\mathrm{wh}}(R_1,W_1,U_1) - \mathcal{F}_{\mathrm{wh}}(R_2,W_2,U_2)\|_{C^\alpha_{\nu\tilde{\lambda}^\mu}([0,\infty);\mathbb{R}^2)} \lesssim \mu \|(R_1,W_1,U_1)\|_{E^+_{\nu\tilde{\lambda}^\mu}}.
$$

Considering $\mathcal{F}_{\mathrm{c}}(R_1,W_1,U_1) - \mathcal{F}_{\mathrm{c}}(R_2,W_2,U_2)$, we obtain the estimates

$$
\begin{aligned}
&\Big| \mathrm{e}^{L_{\mathrm{c}}^\mu t} W_0 + \int_0^t \mathrm{e}^{L_{\mathrm{c}}^\mu (t-\tau)} \Big( \underline{\tilde{G}}_{\mathrm{c}}^\mu(P^\mu + R_1, W_1, U_1)(\tau) + \tilde{H}_{\mathrm{c}}^\mu(P^\mu + R_1)(\tau)\Big)\,\mathrm{d}\tau \\
&\quad - \Big( \mathrm{e}^{L_{\mathrm{c}}^\mu t} W_0 + \int_0^t \mathrm{e}^{L_{\mathrm{c}}^\mu (t-\tau)} \Big( \underline{\tilde{G}}_{\mathrm{c}}^\mu(P^\mu + R_2, W_2, U_2)(\tau) + \tilde{H}_{\mathrm{c}}^\mu(P^\mu + R_2)(\tau)\Big)\,\mathrm{d}\tau\Big)\Big|
\end{aligned}
$$

$$
\begin{aligned}
&\lesssim \int_0^t \Big| \Big( \int_0^1 \mathrm{d}_1 \tilde{G}_{\mathrm{c}}^\mu [\sigma(P^\mu + R_1) + (1-\sigma)(P^\mu + R_2), \phi(W_1), U_1](R_1 - R_2)(\tau)\, \mathrm{d}\sigma \\
&\qquad + \int_0^1 \mathrm{d}_2 \tilde{G}_{\mathrm{c}}^\mu [P^\mu + R_2, \sigma\phi(W_1) + (1-\sigma)\phi(W_2), U_1]\big(\phi(W_1) - \phi(W_2)\big)(\tau)\, \mathrm{d}\sigma \\
&\qquad + \int_0^1 \mathrm{d}_3 \tilde{G}_{\mathrm{c}}^\mu [P^\mu + R_2, \phi(W_2), \sigma U_1 + (1-\sigma)U_2](U_1 - U_2)(\tau)\, \mathrm{d}\sigma \\
&\qquad + \int_0^1 \mathrm{d}\tilde{H}_{\mathrm{c}}^\mu [\sigma(P^\mu + R_1) + (1-\sigma)(P^\mu + R_2)](R_1 - R_2)(\tau)\, \mathrm{d}\sigma \Big) \Big|\, \mathrm{d}\tau \\
&\lesssim \sup_{\sigma\in[0,1]} \int_0^t \mathrm{e}^{\nu\tilde{\lambda}^\mu \tau} \Big( \mu^3 \|(\phi(W_1), U_1)(\tau)\|_{\mathcal{D}_{\mathrm{c,sh}}} \|R_1 - R_2\|_{C^\alpha_{\nu\tilde{\lambda}^\mu}([0,\infty);\mathbb{R}^2)} \\
&\qquad + \Big(\mu^5 + \mu^2 \|(\sigma\phi(W_1) + (1-\sigma)\phi(W_2), U_2)(\tau)\|_{\mathcal{D}_{\mathrm{c,sh}}}\Big) \|W_1 - W_2\|_{C^\alpha_{\nu\tilde{\lambda}^\mu}([0,\infty);\mathbb{R}^{2d})} \\
&\qquad + \Big(\mu^5 + \mu^2 \|(\phi(W_2), \sigma U_1 + (1-\sigma)U_2)(\tau)\|_{\mathcal{D}_{\mathrm{c,sh}}}\Big) \|U_1 - U_2\|_{C^\alpha_{\nu\tilde{\lambda}^\mu}(\mathbb{R};\mathcal{D}_{\mathrm{sh}})} \\
&\qquad + \mu \mathrm{e}^{-\frac{c^*}{\mu}} \|R_1 - R_2\|_{C^\alpha_{\nu\tilde{\lambda}^\mu}([0,\infty);\mathbb{R}^2)} \Big)\, \mathrm{d}\tau \\
&\lesssim \mu \mathrm{e}^{\nu\tilde{\lambda}^\mu t} \|(R_1, W_1, U_1) - (R_2, W_2, U_2)\|_{E^+_{\nu\tilde{\lambda}^\mu}},
\end{aligned}
$$

where we again have used Remark 4.8 and Proposition 2.7. We also find that

$$
\begin{aligned}
&\Big| \frac{\mathrm{d}}{\mathrm{d}t} \Big( \mathcal{F}_{\mathrm{c}}(R_1, W_1, U_1) - \mathcal{F}_{\mathrm{c}}(R_2, W_2, U_2) \Big)(t) \Big| \\
&\qquad \lesssim |\mathcal{F}_{\mathrm{c}}(R_1, W_1, U_1)(t) - \mathcal{F}_{\mathrm{c}}(R_2, W_2, U_2)(t)| \\
&\qquad\quad + |\tilde{G}_{\mathrm{c}}^\mu(P^\mu + R_1, W_1, U_1)(t) - \tilde{G}_{\mathrm{c}}^\mu(P^\mu + R_2, W_2, U_2)(t)| \\
&\qquad\quad + |\tilde{H}_{\mathrm{c}}^\mu(P^\mu + R_1)(t) - \tilde{H}_{\mathrm{c}}^\mu(P^\mu + R_2)(t)| \\
&\qquad \lesssim \mu \mathrm{e}^{\nu\tilde{\lambda}^\mu t} \|(R_1, W_1, U_1) - (R_2, W_2, U_2)\|_{E^+_{\nu\tilde{\lambda}^\mu}},
\end{aligned}
$$

so that

$$
\|\mathcal{F}_{\mathrm{c}}(R_1, W_1, U_1) - \mathcal{F}_{\mathrm{c}}(R_2, W_2, U_2)\|_{C^1_{\nu\tilde{\lambda}^\mu}([0,\infty);\mathbb{R}^{2d})} \lesssim \mu \|(R_1, W_1, U_1)\|_{E^+_{\nu\tilde{\lambda}^\mu}}.
$$

For $\mathcal{F}_{\mathrm{sh}}(R_1, W_1, U_1) - \mathcal{F}_{\mathrm{sh}}(R_2, W_2, U_2)$ we obtain from maximal regularity (see Corollary 2.16) that

$$
\begin{aligned}
&\|\mathcal{F}_{\mathrm{sh}}(R_1, W_1, U_1) - \mathcal{F}_{\mathrm{sh}}(R_2, W_2, U_2)\|_{C^\alpha_{\nu\tilde{\lambda}^\mu}(\mathbb{R};\mathcal{D}_{\mathrm{sh}})} \\
&\quad \lesssim \|(L^\mu_{\mathrm{sh}} - L^0_{\mathrm{sh}})(U_1 - U_2)\|_{C^\alpha_{\nu\tilde{\lambda}^\mu}([0,\infty);\mathcal{X}_{\mathrm{sh}})} \\
&\qquad + \|\tilde{G}^\mu_{\mathrm{sh}}(P^\mu + \mathcal{E}_{\mathrm{wh}} R_1, \mathcal{E}_{\mathrm{c}} W_1, U_1) - \tilde{G}^\mu_{\mathrm{sh}}(P^\mu + \mathcal{E}_{\mathrm{wh}} R_2, \mathcal{E}_{\mathrm{c}} W_2, U_2)\|_{C^\alpha_{\nu\tilde{\lambda}^\mu}([0,\infty);\mathcal{X}_{\mathrm{sh}})} \\
&\qquad + \|\tilde{H}^\mu_{\mathrm{sh}}(P^\mu + \mathcal{E}_{\mathrm{wh}} R_1) - \tilde{H}^\mu_{\mathrm{sh}}(P^\mu + \mathcal{E}_{\mathrm{wh}} R_2)\|_{C^\alpha_{\nu\tilde{\lambda}^\mu}([0,\infty);\mathcal{X}_{\mathrm{sh}})} \\
&\quad \lesssim \mu \|U_1 - U_2\|_{C^\alpha_{\nu\tilde{\lambda}^\mu}([0,\infty);\mathcal{D}_{\mathrm{sh}})} \\
&\qquad + \|\tilde{G}^\mu_{\mathrm{sh}}(P^\mu + \mathcal{E}_{\mathrm{wh}} R_1, \mathcal{E}_{\mathrm{c}} W_1, U_1) - \tilde{G}^\mu_{\mathrm{sh}}(P^\mu + \mathcal{E}_{\mathrm{wh}} R_2, \mathcal{E}_{\mathrm{c}} W_2, U_2)\|_{C^\alpha_{\nu\tilde{\lambda}^\mu}([0,\infty);\mathcal{X}_{\mathrm{sh}})} \\
&\qquad + \|\tilde{H}^\mu_{\mathrm{sh}}(P^\mu + \mathcal{E}_{\mathrm{wh}} R_1) - \tilde{H}^\mu_{\mathrm{sh}}(P^\mu + \mathcal{E}_{\mathrm{wh}} R_2)\|_{C^\alpha_{\nu\tilde{\lambda}^\mu}([0,\infty);\mathcal{X}_{\mathrm{sh}})}
\end{aligned}
$$

and it follows from Remarks 2.11 and 4.8, estimate (5.7), and Proposition 2.7 combined with the facts that all second derivatives of $\tilde{G}^\mu_{\mathrm{sh}}$ and $\tilde{H}^\mu_{\mathrm{sh}}$ are $\mathcal{O}(1)$ that

$$
\begin{aligned}
&\|\tilde{G}^\mu_{\mathrm{sh}}(P^\mu + \mathcal{E}_{\mathrm{wh}} R_1, \mathcal{E}_{\mathrm{c}} W_1, U_1) - \tilde{G}^\mu_{\mathrm{sh}}(P^\mu + \mathcal{E}_{\mathrm{wh}} R_2, \mathcal{E}_{\mathrm{c}} W_2, U_2)\|_{C^\alpha_{\nu\tilde{\lambda}^\mu}([0,\infty);\mathcal{X}_{\mathrm{sh}})} \\
&\qquad \lesssim \mu \|(R_1, W_1, U_1) - (R_2, W_2, U_2)\|_{E^+_{\nu\tilde{\lambda}^\mu}}
\end{aligned}
$$

and

$$\|\tilde{H}^{\mu}_{\mathrm{sh}}(P^{\mu}+\mathcal{E}_{\mathrm{wh}}R_1)-\tilde{H}^{\mu}_{\mathrm{sh}}(P^{\mu}+\mathcal{E}_{\mathrm{wh}}R_2)\|_{C^{\alpha}_{\nu\tilde{\lambda}^{\mu}}([0,\infty);\mathcal{X}_{\mathrm{sh}})}$$
$$\lesssim \mu\|(R_1,W_1,U_1)-(R_2,W_2,U_2)\|_{E^{+}_{\nu\tilde{\lambda}^{\mu}}}. \qquad \square$$

**Corollary 5.6.** The operator $\mathcal{F}$ has a unique fixed point $(R^*_{W_0},W^*_{W_0},U^*_{W_0})\in B^{+}_{\nu\tilde{\lambda}^{\mu}}$ for each $W_0\in\mathbb{R}^{2d}$ with $S_{\mathrm{c}}W_0=W_0$ and $|W_0|\le\mu\mathrm{e}^{-\frac{c^*}{2\mu}}$ in the case $n=2$. This fixed point satisfies $(S_{\mathrm{sh}}U^*_{W_0})(0)=U^*_{W_0}(0)$.

*Proof.* The existence and uniqueness of $R^*_{W_0}$, $W^*_{W_0}$ and $U^*_{W_0}$ follows directly from Theorems 5.4 and 5.5. Observing that $(R^*_{W_0},W^*_{W_0},S_{\mathrm{sh}}U^*_{W_0})\in B^{+}_{\nu\tilde{\lambda}^{\mu}}$ is also a fixed point of $\mathcal{F}$, we conclude that $S_{\mathrm{sh}}U^*_{W_0}=U^*_{W_0}$, i.e. $(S_{\mathrm{sh}}U^*_{W_0})(0)=U^*_{W_0}(0)$. $\square$

### 5.2.2 The case $n=4$

The estimates for $\mathcal{F}_{\mathrm{c}}$ are more involved in the case $n=4$ because of the weaker estimate

$$|P^{\mu}(t)|\lesssim\mu\mathrm{e}^{-\nu\tilde{\lambda}^{\mu}t}.$$

It does not suffice to estimate $|P^{\mu}(t)|\lesssim\mu$ for all $t\ge0$. Instead we use a two-step approach, first estimating $|P^{\mu}(t)|\lesssim\mu^{\gamma}$ over the interval $[0,t^*]$, where $\mathrm{e}^{-\nu\tilde{\lambda}^{\mu}t^*}=\mu^{\gamma}$ for an appropriately chosen constant $\gamma$, and then using this result to obtain the final estimate over the exponentially long interval $[0,\mathrm{e}^{c^*/2\mu}]$.

**Theorem 5.7.** The operator $\mathcal{F}$ given in equation (5.34) maps $B^{+}_{\nu\tilde{\lambda}^{\mu}}$ into itself. Furthermore,

$$\|\mathcal{F}_{\mathrm{wh}}(R,W,U)\|_{C^{\alpha}_{\mathrm{b}}([0,\infty);\mathbb{R}^4)}\lesssim\mu\mathrm{e}^{-\frac{c^*}{2\mu}}, \tag{5.46}$$
$$\left\|\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{F}_{\mathrm{c}}(R,W,U)-L^{\mu}_{\mathrm{c}}\mathcal{F}_{\mathrm{c}}(R,W,U)\right\|_{C_{\mathrm{b}}(\mathbb{R};\mathbb{R}^{2d})}\lesssim\mu\mathrm{e}^{-\frac{c^*}{2\mu}}, \tag{5.47}$$
$$\|\mathcal{F}_{\mathrm{sh}}(R,W,U)\|_{C^{\alpha}_{\mathrm{b}}(\mathbb{R};\mathcal{D}_{\mathrm{sh}})}\lesssim\mu\mathrm{e}^{-\frac{c^*}{2\mu}} \tag{5.48}$$

for all $(R,W,U)\in B^{+}_{\nu\tilde{\lambda}^{\mu}}$ provided that $|W_0|\le\mu\mathrm{e}^{-\frac{c^*}{2\mu}}$.

*Proof.* Suppose that $(R,W,U)\in B^{+}_{\nu\tilde{\lambda}^{\mu}}$. Inspecting equations (5.16) and (5.17) shows that $(S_{\mathrm{wh}}\mathcal{F}_{\mathrm{wh}}R)(0)=R(0)$ and $(S_{\mathrm{c}}\mathcal{F}_{\mathrm{c}}W)(0)=W(0)$ (recall that $S_{\mathrm{c}}W_0=W_0$ and $(S_{\mathrm{wh}}(u^{\mu})(0)=u^{\mu}(0)$.)

To estimate $\mathcal{F}_{\mathrm{wh}}(R,W,U)$ we observe that

$$\begin{aligned}&|N^{\mu}(R,W,U)(t)|\\&\lesssim\|R\|^2_{C^{\alpha}_{\mathrm{b}}([0,\infty);\mathbb{R}^4)}+\Big(\mu^3+\mu\|R\|_{C^{\alpha}_{\mathrm{b}}([0,\infty);\mathbb{R})}+\|R\|^2_{C^{\alpha}_{\mathrm{b}}([0,\infty);\mathbb{R}^4)}\Big)\|(\phi(W),U)\|_{C^{\alpha}_{\mathrm{b}}([0,\infty);\mathcal{D}_{\mathrm{c,sh}})}\\&\qquad+\|(\phi(W),U)\|^2_{C^{\alpha}_{\mathrm{b}}([0,\infty);\mathcal{D}_{\mathrm{c,sh}})}\\&\lesssim\mu^3\mathrm{e}^{-\frac{c^*}{2\mu}},\end{aligned} \tag{5.49}$$

where we have used Remark 5.1. The above estimate implies that

$$\left|\int_0^t \langle \underline{N}^\mu(R,W,U)(\tau);(s_j^\mu)^*(\tau)\rangle \,\mathrm{d}\tau\, s_j^\mu(t)\right| \lesssim \int_0^t \mu^3 \mathrm{e}^{-\frac{c^*}{2\mu}} \mathrm{e}^{\tilde{\lambda}^\mu \tau}\,\mathrm{d}\tau\, \mathrm{e}^{-\tilde{\lambda}^\mu t}$$
$$= \mu(1-\mathrm{e}^{-\tilde{\lambda}^\mu t})\mathrm{e}^{-\frac{c^*}{2\mu}}$$
$$\leq \mu \mathrm{e}^{-\frac{c^*}{2\mu}}$$

and

$$\left|\int_0^t \langle \underline{N}^\mu(R,W,U)(\tau);(u_j^\mu)^*(\tau)\rangle \,\mathrm{d}\tau\, u_j^\mu(t)\right| \lesssim \mu^3 \mathrm{e}^{-\frac{c^*}{2\mu}} \int_t^\infty \mathrm{e}^{-\tilde{\lambda}^\mu \tau}\,\mathrm{d}\tau\, \mathrm{e}^{\tilde{\lambda}^\mu t} = \mu \mathrm{e}^{-\frac{c^*}{2\mu}}$$

for all $t \in [0,\infty)$, so that

$$\sup_{t\in[0,\infty)} |\mathcal{F}_{\mathrm{wh}}(R,W,U)(t)| \lesssim \mu \mathrm{e}^{-\frac{c^*}{2\mu}}.$$

Next we observe that

$$\|\mathrm{e}^{\tilde{\lambda}^\mu(\cdot)}\|_{C^\alpha_{\tilde{\lambda}^\mu}([0,\infty);\mathbb{R})} \lesssim \|\mathrm{e}^{\tilde{\lambda}^\mu(\cdot)}\|_{C^1_{\tilde{\lambda}^\mu}([0,\infty))} \lesssim 1$$

and

$$\|s_j^\mu\|_{C^\alpha_{-\tilde{\lambda}^\mu}([0,\infty);\mathbb{R}^4)} \lesssim \|s_j^\mu\|_{C^1_{-\tilde{\lambda}^\mu}([0,\infty);\mathbb{R}^4)}$$
$$\lesssim 1 + \sup_{t\in[0,\infty)} \mathrm{e}^{\tilde{\lambda}^\mu t}|K^\mu(t) s_j^\mu(t)|$$
$$\lesssim 1,$$

where we have used the fact that $\|\cdot\|_{C^\alpha_{\tilde{\lambda}^\mu}([0,\infty);\mathbb{R})} \leq \|\cdot\|_{C^1_{\tilde{\lambda}^\mu}([0,\infty);\mathbb{R})}$. Similarly we obtain

$$\|u^\mu\|_{C^\alpha_{\tilde{\lambda}^\mu}([0,\infty);\mathbb{R}^4)} \lesssim 1.$$

Using estimate (5.49) and Lemma 5.3 we find that

$$\left|\int_0^{t_1} \langle \underline{N}^\mu(R,W,U)(\tau);(s_j^\mu)^*(\tau)\rangle \,\mathrm{d}\tau\, s_j^\mu(t_1) - \int_0^{t_2} \langle \underline{N}^\mu(R,W,U)(\tau);(s_j^\mu)^*(\tau)\rangle \,\mathrm{d}\tau\, s_j^\mu(t_2)\right|$$
$$= \left|\int_{t_2}^{t_1} \langle \underline{N}^\mu(R,W,U)(\tau);(s_j^\mu)^*(\tau)\rangle \,\mathrm{d}\tau\, s_j^\mu(t_1) - \int_0^{t_2} \langle \underline{N}^\mu(R,W,U)(\tau);(s_j^\mu)^*(\tau)\rangle \,\mathrm{d}\tau\Big(s_j^\mu(t_1) - s_j^\mu(t_2)\Big)\right|$$
$$\lesssim \mu \mathrm{e}^{-\frac{c^*}{2\mu}}(\mathrm{e}^{\tilde{\lambda}^\mu t_1} - \mathrm{e}^{\tilde{\lambda}^\mu t_2})\mathrm{e}^{-\tilde{\lambda}^\mu t_1} + \mu \mathrm{e}^{-\frac{c^*}{2\mu}}(\mathrm{e}^{\tilde{\lambda}^\mu t_2} - 1)|s_j^\mu(t_1) - s_j^\mu(t_2)|$$
$$\leq \mu \mathrm{e}^{-\frac{c^*}{2\mu}}\|\mathrm{e}^{\tilde{\lambda}^\mu(\cdot)}\|_{C^\alpha_{\tilde{\lambda}^\mu}([0,\infty))}|t_1-t_2|^\alpha + \mu \mathrm{e}^{-\frac{c^*}{2\mu}}\|s_j^\mu\|_{C^\alpha_{-\tilde{\lambda}^\mu}([0,\infty);\mathbb{R}^4)}|t_1-t_2|^\alpha$$
$$\lesssim \mu \mathrm{e}^{-\frac{c^*}{2\mu}}|t_1-t_2|^\alpha$$

and in a similar fashion

$$\left|\int_0^{t_1} \langle \underline{N}^\mu(R,W,U)(\tau);(u_j^\mu)^*(\tau)\rangle \,\mathrm{d}\tau u_j^\mu(t_1) - \int_0^{t_2} \langle \underline{N}^\mu(R,W,U)(\tau);(u_j^\mu)^*(\tau)\rangle \,\mathrm{d}\tau u_j^\mu(t_2)\right|$$
$$\leq \mu \mathrm{e}^{-\frac{c^*}{2\mu}}|t_1-t_2|^\alpha$$

for $0 \le t_2 < t_1 < \infty$, so that

$$|\mathcal{F}_{\mathrm{wh}}(R,W,U)(t_1) - \mathcal{F}_{\mathrm{wh}}(R,W,U)(t_2)| \lesssim \mu \mathrm{e}^{-\frac{c^*}{2\mu}} |t_1 - t_2|^{\alpha}.$$

Altogether we obtain

$$\|\mathcal{F}_{\mathrm{wh}}(R,W,U)\|_{C^{\alpha}_{\mathrm{b}}([0,\infty);\mathbb{R}^4)} \le \mu \mathrm{e}^{-\frac{c^*}{2\mu}}. \tag{5.50}$$

Using Remark 4.16, estimate (5.7), and Proposition 2.7 we find that

$$\begin{aligned}
&|\mathcal{F}_{\mathrm{c}}(R,W,U)(t)| \\
&\qquad \le |W_0| + \left| \int_0^t \mathrm{e}^{L^{\mu}_{\mathrm{c}}(t-\tau)} \Big( \tilde{\underline{G}}^{\mu}_{\mathrm{c}}(P^{\mu}+R,W,U)(\tau) + \tilde{H}^{\mu}_{\mathrm{c}}(P^{\mu}+R)(\tau) \Big) \,\mathrm{d}\tau \right| \\
&\qquad \lesssim |W_0| + \mu^2 \mathrm{e}^{\nu \tilde{\lambda}^{\mu} t} \\
&\qquad \lesssim \mathrm{e}^{\nu \tilde{\lambda}^{\mu} t}
\end{aligned} \tag{5.51}$$

and

$$\begin{aligned}
\left| \frac{\mathrm{d}}{\mathrm{d}t} \mathcal{F}_{\mathrm{c}}(R,W,U)(t) \right| &\lesssim \|\mathcal{F}_{\mathrm{c}}(R,W,U)\|_{C^{\alpha}_{\mathrm{b}}([0,\infty);\mathbb{R}^{2d})} + \|\tilde{\underline{G}}^{\mu}_{\mathrm{c}}(R,W,U)\|_{C^{\alpha}_{\mathrm{b}}([0,\infty);\mathbb{R}^{2d})} \\
&\qquad + \|\tilde{H}^{\mu}_{\mathrm{c}}(P^{\mu}+R)\|_{C^{\alpha}_{\mathrm{b}}([0,\infty);\mathbb{R}^{2d})} \\
&\lesssim |W_0| + \mu \mathrm{e}^{\nu \tilde{\lambda}^{\mu} t} \\
&\lesssim \mathrm{e}^{\nu \tilde{\lambda}^{\mu} t}.
\end{aligned}$$

for $t \in [0,\infty)$. Altogether we obtain

$$\|\mathcal{F}_{\mathrm{c}}(R,W,U)\|_{C^1_{\nu\tilde{\lambda}^{\mu}}([0,\infty);\mathbb{R}^{2d})} < \infty$$

and

$$\begin{aligned}
&\|\frac{\mathrm{d}}{\mathrm{d}t} \mathcal{F}_{\mathrm{c}}(R,W,U) - L^{\mu}_{\mathrm{c}} \mathcal{F}_{\mathrm{c}}(R,W,U)\|_{C_{\mathrm{b}}(\mathbb{R};\mathbb{R}^{2d})} \\
&\qquad = \|\tilde{\underline{G}}^{\mu}_{\mathrm{c}}(P^{\mu}+R,W,U) + \tilde{H}^{\mu}_{\mathrm{c}}(P^{\mu}+R)\|_{C_{\mathrm{b}}(\mathbb{R};\mathbb{R}^{2d})} \\
&\qquad \lesssim \mu^3 \mathrm{e}^{-\frac{c^*}{2\mu}}.
\end{aligned} \tag{5.52}$$

For $\mathcal{F}_{\mathrm{sh}}(R,W,U)$ we first consider

$$\begin{aligned}
&\|\tilde{\underline{G}}^{\mu}_{\mathrm{sh}}(P^{\mu}+\mathcal{E}_{\mathrm{wh}}R,\mathcal{E}_{\mathrm{c}}W,U)(t_1) - \tilde{\underline{G}}^{\mu}_{\mathrm{sh}}(P^{\mu}+\mathcal{E}_{\mathrm{wh}}R,\mathcal{E}_{\mathrm{c}}W,U)(t_2)\|_{\mathcal{X}_{\mathrm{sh}}} \\
&\quad = \Bigg\| \int_0^1 \mathrm{d}_1 \tilde{G}^{\mu}_{\mathrm{sh}} \Big[ \sigma (P^{\mu}+\mathcal{E}_{\mathrm{wh}}R)(t_1) + (1-\sigma)(P^{\mu}+\mathcal{E}_{\mathrm{wh}}R)(t_2), \phi(\mathcal{E}_{\mathrm{c}}W)(t_1), U(t_1) \Big] \\
&\qquad\qquad \Big( (P^{\mu}+\mathcal{E}_{\mathrm{wh}}R)(t_1) - (P^{\mu}+\mathcal{E}_{\mathrm{wh}}R)(t_2) \Big) \,\mathrm{d}\sigma \\
&\qquad + \int_0^1 \mathrm{d}_2 \tilde{G}^{\mu}_{\mathrm{sh}} \Big[ (P^{\mu}+\mathcal{E}_{\mathrm{wh}}R)(t_2), \sigma\phi(\mathcal{E}_{\mathrm{c}}W)(t_1) + (1-\sigma)\phi(\mathcal{E}_{\mathrm{c}}W)(t_2), U(t_1) \Big] \\
&\qquad\qquad \Big( \phi(\mathcal{E}_{\mathrm{c}}W)(t_1) - \phi(\mathcal{E}_{\mathrm{c}}W)(t_2) \Big) \,\mathrm{d}\sigma \\
&\qquad + \int_0^1 \mathrm{d}_3 \tilde{G}^{\mu}_{\mathrm{sh}} \Big[ (P^{\mu}+\mathcal{E}_{\mathrm{wh}}R)(t_2), \phi(\mathcal{E}_{\mathrm{c}}W)(t_2), \sigma U(t_1) + (1-\sigma)U(t_2) \Big] \\
&\qquad\qquad \Big( U(t_1) - U(t_2) \Big) \,\mathrm{d}\sigma \Bigg\|_{\mathcal{X}_{\mathrm{sh}}}
\end{aligned}$$

$$
\begin{aligned}
&\lesssim \|(\phi(W),U)\|_{C_{\mathrm{b}}(\mathbb{R};\mathcal{D}_{\mathrm{c,sh}})}|(P^\mu+\mathcal{E}_{\mathrm{wh}}R)(t_1)-(P^\mu+\mathcal{E}_{\mathrm{wh}}R)(t_2)| \\
&\quad + \|(P^\mu+\mathcal{E}_{\mathrm{wh}}R,\phi(W),U)\|_{C_{\mathrm{b}}(\mathbb{R};\mathcal{D})}\Big(|\phi(\mathcal{E}_{\mathrm{c}}W)(t_1)-\phi(\mathcal{E}_{\mathrm{c}}W)(t_2)| \\
&\qquad\qquad + \|U(t_1)-U(t_2)\|_{\mathcal{D}_{\mathrm{sh}}}\Big) \\
&\le |t_1-t_2|^\alpha\Big(\mathrm{e}^{-\frac{c^*}{2\mu}}\Big(\|P^\mu\|_{C^\alpha_{\mathrm{b}}(\mathbb{R};\mathbb{R}^4)}+\|\mathcal{E}_{\mathrm{wh}}R\|_{C^\alpha_{\mathrm{b}}(\mathbb{R};\mathbb{R}^4)}\Big) \\
&\quad + \|(P^\mu+\mathcal{E}_{\mathrm{wh}}R,\phi(W),U)\|_{C_{\mathrm{b}}(\mathbb{R};\mathcal{D})}\Big(\|\phi(\mathcal{E}_{\mathrm{c}}W)\|_{C^\alpha_{\mathrm{b}}(\mathbb{R};\mathbb{R}^{2d})}+\|U\|_{C^\alpha_{\mathrm{b}}(\mathbb{R};\mathcal{D}_{\mathrm{sh}})}\Big)\Big) \qquad (5.53)\\
&\le \mu\mathrm{e}^{-\frac{c^*}{2\mu}}|t_1-t_2|^\alpha
\end{aligned}
$$

and

$$
\begin{aligned}
&\|\tilde{H}^\mu_{\mathrm{sh}}(P^\mu+\mathcal{E}_{\mathrm{wh}}R)(t_1)-\tilde{H}^\mu_{\mathrm{sh}}(P^\mu+\mathcal{E}_{\mathrm{wh}}R)(t_2)\|_{\mathcal{X}_{\mathrm{sh}}} \\
&\quad = \int_0^1 \mathrm{d}\tilde{H}^\mu_{\mathrm{sh}}[\sigma(P^\mu+\mathcal{E}_{\mathrm{wh}}R)(t_1) \\
&\qquad\qquad + (1-\sigma)(P^\mu+\mathcal{E}_{\mathrm{wh}}R)(t_2)]\Big((P^\mu+\mathcal{E}_{\mathrm{wh}}R)(t_1)-(P^\mu+\mathcal{E}_{\mathrm{wh}}R)(t_2)\Big)\,\mathrm{d}\sigma \\
&\quad \lesssim \mu\mathrm{e}^{-\frac{c^*}{\mu}}\|P^\mu+\mathcal{E}_{\mathrm{wh}}R\|_{C^\alpha_{\mathrm{b}}(\mathbb{R};\mathbb{R}^4)}|t_1-t_2|^\alpha, \qquad (5.54)
\end{aligned}
$$

where we have used Remark 4.16, estimate (5.7), and Proposition 2.7. Using the maximal regularity result (Lemma 2.14) we find that

$$
\begin{aligned}
\|\mathcal{F}_{\mathrm{sh}}(R,W,U)\|_{C^\alpha_{\mathrm{b}}(\mathbb{R};\mathcal{D}_{\mathrm{sh}})} &\lesssim \|(L^\mu_{\mathrm{sh}}-L^0_{\mathrm{sh}})U+\tilde{G}^\mu_{\mathrm{sh}}(P^\mu+R,W,U) \\
&\qquad + \tilde{H}^\mu_{\mathrm{sh}}(P^\mu+R)\|_{C^\alpha_{\mathrm{b}}(\mathbb{R};\mathcal{X}_{\mathrm{sh}})} \\
&\lesssim \mu\mathrm{e}^{-\frac{c^*}{2\mu}}. \qquad (5.55)
\end{aligned}
$$

Finally, we note that estimates (5.46) – (5.48) and (5.61) follow directly from estimates (5.50) – (5.52) and (5.55). □

**Theorem 5.8.** The operator $\mathcal{F}\colon B^+_{\nu\tilde{\lambda}^\mu}\to B^+_{\nu\tilde{\lambda}^\mu}$ is a contraction provided that $S_{\mathrm{c}}W_0=W_0$ and $|W_0|\le\mu\mathrm{e}^{-\frac{c^*}{2\mu}}$.

*Proof.* Suppose that $(R_1,W_1,U_1),(R_2,W_2,U_2)\in B^+_{\nu\tilde{\lambda}^\mu}$. For $\mathcal{F}_{\mathrm{wh}}$ we first consider

$$
\begin{aligned}
&|\underline{N}^\mu(R_1,W_1,U_1)-\underline{N}^\mu(R_2,W_2,U_2)(t)| \\
&\quad = \Big|\int_0^1 \mathrm{d}_1N^\mu[\sigma R_1+(1-\sigma)R_2,\phi(W_1),U_1](R_1-R_2)(t)\,\mathrm{d}\sigma \\
&\qquad + \int_0^1 \mathrm{d}_2N^\mu[R_2,\sigma\phi(W_1)+(1-\sigma)\phi(W_2),U_2]\Big(\phi(W_1)-\phi(W_2)\Big)(t)\,\mathrm{d}\sigma \\
&\qquad + \int_0^1 \mathrm{d}_3N^\mu[R_2,\phi(W_2),\sigma U_1+(1-\sigma)U_2](U_1-U_2)(t)\,\mathrm{d}\sigma\Big| \\
&\quad \lesssim \mu^3\|(R_1,W_1,U_1)(t)-(R_2,W_2,U_2)(t)\|_{\mathcal{D}},
\end{aligned}
$$

where we have used Remark 5.1. The above estimate now implies that

$$
\Big|\int_0^t\Big\langle\Big(\underline{N}^\mu(R_1,W_1,U_1)-\underline{N}^\mu(R_2,W_2,U_2)\Big)(\tau);(s^\mu_j)^*(\tau)\Big\rangle\,\mathrm{d}\tau\, s^\mu_j(t)\Big|
$$

$$
\begin{aligned}
&\lesssim \int_0^t \Big|\Big(\underline{N}^\mu(R_1,W_1,U_1) - \underline{N}^\mu(R_2,W_2,U_2)\Big)(\tau)\Big| \mathrm{e}^{\tilde\lambda^\mu \tau}\,\mathrm{d}\tau \mathrm{e}^{-\tilde\lambda^\mu t}\\
&\le \int_0^t \mu^3 \Big( |(R_1-R_2)(\tau)| \\
&\qquad\qquad + |(\phi(W_1)-\phi(W_2))(\tau)| + \|(U_1-U_2)(\tau)\|_{\mathcal{D}_{\mathrm{sh}}}\Big) \mathrm{e}^{\tilde\lambda^\mu \tau}\,\mathrm{d}\tau \mathrm{e}^{-\tilde\lambda^\mu t}\\
&\le \mu^3 \mathrm{e}^{-\tilde\lambda^\mu t} \|(R_1,W_1,U_1)-(R_2,W_2,U_2)\|_{E^+_{\nu\tilde\lambda^\mu}} \int_0^t \mathrm{e}^{(1+\nu)\tilde\lambda^\mu \tau}\,\mathrm{d}\tau\\
&\le \frac{\mu^3}{(1+\nu)\tilde\lambda^\mu} \mathrm{e}^{\nu\tilde\lambda^\mu t} \|(R_1,W_1,U_1)-(R_2,W_2,U_2)\|_{E^+_{\nu\tilde\lambda^\mu}}\\
&\le \mu \mathrm{e}^{\nu\tilde\lambda^\mu t} \|(R_1,W_1,U_1)-(R_2,W_2,U_2)\|_{E^+_{\nu\tilde\lambda^\mu}}
\end{aligned}
$$

and

$$
\begin{aligned}
&\Big| \int_t^\infty \langle \underline{N}^\mu(R_1,W_1,U_1)(\tau) - \underline{N}^\mu(R_2,W_2,U_2)(\tau); (u_j^\mu)^*(\tau)\rangle\,\mathrm{d}\tau\, u_j^\mu(t)\Big|\\
&\quad\lesssim \mu^3 \int_t^\infty \|\underline{N}^\mu(R_1,W_1,U_1)(\tau) - \underline{N}^\mu(R_2,W_2,U_2)(\tau)\| \mathrm{e}^{-\tilde\lambda^\mu \tau}\,\mathrm{d}\tau \mathrm{e}^{\tilde\lambda^\mu t}\\
&\quad\lesssim \mu^3 \|(R_1,W_1,U_1)-(R_2,W_2,U_2)\|_{E^+_{\nu\tilde\lambda^\mu}} \int_t^\infty \mathrm{e}^{(\nu-1)\tilde\lambda^\mu \tau}\,\mathrm{d}\tau \mathrm{e}^{\tilde\lambda^\mu t}\\
&\quad\lesssim \frac{\mu^3}{(1-\nu)\tilde\lambda^\mu} \mathrm{e}^{\nu\tilde\lambda^\mu t} \|(R_1,W_1,U_1)-(R_2,W_2,U_2)\|_{E^+_{\nu\tilde\lambda^\mu}}\\
&\quad\lesssim \mu \mathrm{e}^{\nu\tilde\lambda^\mu t} \|(R_1,W_1,U_1)-(R_2,W_2,U_2)\|_{E^+_{\nu\tilde\lambda^\mu}}.
\end{aligned}
$$

We also find that

$$
\begin{aligned}
&\Big| \int_0^{t_1} \langle \underline{N}^\mu(R_1,W_1,U_1)(\tau) - \underline{N}^\mu(R_2,W_2,U_2)(\tau); (s_j^\mu)^*(\tau)\rangle\,\mathrm{d}\tau\, s_j^\mu(t_1)\\
&\quad - \Big( \int_0^{t_2} \langle \underline{N}^\mu(R_1,W_1,U_1)(\tau) - \underline{N}^\mu(R_2,W_2,U_2)(\tau); (s_j^\mu)^*(\tau)\rangle\,\mathrm{d}\tau\, s_j^\mu(t_2)\Big)\Big|\\
&\le \int_{t_2}^{t_1} \Big| \int_0^1 \mathrm{d}_1 N^\mu[\sigma R_1 + (1-\sigma)R_2, \phi(W_1), U_1](R_1-R_2)(\tau)\,\mathrm{d}\sigma\\
&\qquad + \int_0^1 \mathrm{d}_2 N^\mu[R_2, \sigma\phi(W_1) + (1-\sigma)\phi(W_2), U_1]\Big(\phi(W_1)-\phi(W_2)\Big)(\tau)\,\mathrm{d}\sigma\\
&\qquad + \int_0^1 \mathrm{d}_3 N^\mu[R_2, \phi(W_2), \sigma U_1 + (1-\sigma)U_2](U_1-U_2)(\tau)\,\mathrm{d}\sigma\Big| \mathrm{e}^{\tilde\lambda^\mu \tau}\,\mathrm{d}\tau \mathrm{e}^{-\tilde\lambda^\mu t_1}\\
&\quad + \int_0^{t_2} \Big| \int_0^1 \mathrm{d}_1 N^\mu[\sigma R_1 + (1-\sigma)R_2, \phi(W_1), U_1](R_1-R_2)(\tau)\,\mathrm{d}\sigma\\
&\qquad + \int_0^1 \mathrm{d}_2 N^\mu[R_2, \sigma\phi(W_1) + (1-\sigma)\phi(W_2), U_1]\Big(\phi(W_1)-\phi(W_2)\Big)(\tau)\,\mathrm{d}\sigma\\
&\qquad + \int_0^1 \mathrm{d}_3 N^\mu[R_2, \phi(W_2), \sigma U_1 + (1-\sigma)U_2](U_1-U_2)(\tau)\,\mathrm{d}\sigma\Big| \mathrm{e}^{\tilde\lambda^\mu \tau}\,\mathrm{d}\tau |s_j^\mu(t_1) - s_j^\mu(t_2)|\\
&\lesssim \mu \mathrm{e}^{\nu\tilde\lambda^\mu t_1} |t_1-t_2|^\alpha \|(R_1,W_1,U_1)-(R_2,W_2,U_2)\|_{E^+_{\nu\tilde\lambda^\mu}}
\end{aligned}
$$

and

$$
\begin{aligned}
&\Big| \int_0^{t_1} \langle \underline{N}^\mu(R_1,W_1,U_1)(\tau) - \underline{N}^\mu(R_2,W_2,U_2)(\tau); (u^\mu)^*\rangle\,\mathrm{d}\tau\, u_j^\mu(t_1)\\
&\quad - \Big( \int_0^{t_2} \langle \underline{N}^\mu(R_1,W_1,U_1)(\tau) - \underline{N}^\mu(R_2,W_2,U_2)(\tau); (u^\mu)^*\rangle\,\mathrm{d}\tau\, u_j^\mu(t_2)\Big)\Big|
\end{aligned}
$$

$$\lesssim \mu e^{\nu\tilde{\lambda}^\mu t_1}|t_1-t_2|^\alpha \|(R_1,W_1,U_1)-(R_2,W_2,U_2)\|_{E^+_{\nu\tilde{\lambda}^\mu}}$$

for $0 \le t_2 < t_1 < \infty$, so that

$$\|\mathcal{F}_{\mathrm{wh}}(R_1,W_1,U_1)-\mathcal{F}_{\mathrm{wh}}(R_2,W_2,U_2)\|_{C^\alpha_{\nu\tilde{\lambda}^\mu}([0,\infty);\mathbb{R}^4)} \lesssim \mu\|(R_1,W_1,U_1)\|_{E^+_{\nu\tilde{\lambda}^\mu}}.$$

For $\mathcal{F}_{\mathrm{c}}$ we find that

$$\begin{aligned}
&\Big|e^{L^\mu_{\mathrm{c}}t}W_0+\int_0^t e^{L^\mu_{\mathrm{c}}(t-\tau)}\Big(\tilde{G}^\mu_{\mathrm{c}}(P^\mu+R_1,W_1,U_1)(\tau)+\tilde{H}^\mu_{\mathrm{c}}(P^\mu+R_1)(\tau)\Big)\,\mathrm{d}\tau\\
&\quad-\Big(e^{L^\mu_{\mathrm{c}}t}W_0+\int_0^t e^{L^\mu_{\mathrm{c}}(t-\tau)}\Big(\tilde{G}^\mu_{\mathrm{c}}(P^\mu+R_2,W_2,U_2)(\tau)+\tilde{H}^\mu_{\mathrm{c}}(P^\mu+R_2)(\tau)\Big)\,\mathrm{d}\tau\Big)\Big|\\
&\lesssim \int_0^t\Big|\Big(\int_0^1 \mathrm{d}_1\tilde{G}^\mu_{\mathrm{c}}[\sigma(P^\mu+R_1)+(1-\sigma)(P^\mu+R_2),\phi(W_1),U_1](R_1-R_2)(\tau)\,\mathrm{d}\sigma\\
&\qquad\quad+\int_0^1 \mathrm{d}_2\tilde{G}^\mu_{\mathrm{c}}[P^\mu+R_2,\sigma\phi(W_1)+(1-\sigma)\phi(W_2),U_1]\\
&\qquad\qquad\qquad\Big(\phi(W_1)-\phi(W_2)\Big)(\tau)\,\mathrm{d}\sigma\\
&\qquad\quad+\int_0^1 \mathrm{d}_3\tilde{G}^\mu_{\mathrm{c}}[P^\mu+R_2,\phi(W_2),\sigma U_1+(1-\sigma)U_2](U_1-U_2)(\tau)\,\mathrm{d}\sigma\\
&\qquad\quad+\int_0^1 \mathrm{d}\tilde{H}^\mu_{\mathrm{c}}[\sigma(P^\mu+R_1)+(1-\sigma)(P^\mu+R_2)](R_1-R_2)(\tau)\,\mathrm{d}\sigma\Big)\Big|\,\mathrm{d}\tau\\
&\lesssim \sup_{\sigma\in[0,1]}\int_0^t\Big(\mu\|(\phi(W_1),U_1)(\tau)\|_{\mathcal{D}_{\mathrm{c,sh}}}|(R_1-R_2)(\tau)|\\
&\qquad+\Big(\mu^3+\mu\|(\sigma\phi(W_1)+(1-\sigma)\phi(W_2),U_2)(\tau)\|_{\mathcal{D}_{\mathrm{c,sh}}}\Big)|\phi(W_1)(\tau)-\phi(W_2)(\tau)|\\
&\qquad+\Big(\mu^3+\mu\|(\phi(W_2),\sigma U_1+(1-\sigma)U_2)(\tau)\|_{\mathcal{D}_{\mathrm{c,sh}}}\Big)\|U_1(\tau)-U_2(\tau)\|_{\mathcal{D}_{\mathrm{sh}}}\\
&\qquad+\mu e^{-\frac{c^*}{\mu}}|R_1(\tau)-R_2(\tau)|\Big)\,\mathrm{d}\tau\\
&\qquad+\int_0^t \mu|(P^\mu+R_2)(\tau)||W_1(\tau)-W_2(\tau)|\,\mathrm{d}\tau\\
&\lesssim \sup_{\sigma\in[0,1]}\int_0^t e^{\nu\tilde{\lambda}^\mu\tau}\Big(\mu\|(\phi(W_1),U_1)(\tau)\|_{\mathcal{D}_{\mathrm{c,sh}}}\|R_1-R_2\|_{C^\alpha_{\nu\tilde{\lambda}^\mu}([0,\infty);\mathbb{R}^4)}\\
&\qquad+\Big(\mu^3+\mu\|(\sigma\phi(W_1)+(1-\sigma)\phi(W_2),U_2)(\tau)\|_{\mathcal{D}_{\mathrm{c,sh}}}\Big)\|\phi(W_1)-\phi(W_2)\|_{C^\alpha_{\nu\tilde{\lambda}^\mu}(\mathbb{R};\mathbb{R}^{2d})}\\
&\qquad+\Big(\mu^3+\mu\|(\phi(W_2),\sigma U_1+(1-\sigma)U_2)(\tau)\|_{\mathcal{D}_{\mathrm{c,sh}}}\Big)\|U_1-U_2\|_{C^\alpha_{\nu\tilde{\lambda}^\mu}(\mathbb{R};\mathcal{D}_{\mathrm{sh}})}\\
&\qquad+\mu e^{-\frac{c^*}{\mu}}\|R_1-R_2\|_{C^\alpha_{\nu\tilde{\lambda}^\mu}([0,\infty);\mathbb{R}^4)}\Big)\,\mathrm{d}\tau\\
&\qquad+\mu^2 t\|W_1-W_2\|_{C^\alpha_{\nu\tilde{\lambda}^\mu}([0,\infty);\mathbb{R}^{2d})}\\
&\lesssim \mu e^{\nu\tilde{\lambda}^\mu t}\|(R_1,W_1,U_1)-(R_2,W_2,U_2)\|_{E^+_{\nu\tilde{\lambda}^\mu}},
\end{aligned}$$

where we again have used Remark 4.16 and Proposition 2.7. We also find that

$$\begin{aligned}
&\Big|\frac{\mathrm{d}}{\mathrm{d}t}\Big(\mathcal{F}_{\mathrm{c}}(R_1,W_1,U_1)-\mathcal{F}_{\mathrm{c}}(R_2,W_2,U_2)\Big)(t)\Big|\\
&\qquad\lesssim |\mathcal{F}_{\mathrm{c}}(R_1,W_1,U_1)(t)-\mathcal{F}_{\mathrm{c}}(R_2,W_2,U_2)(t)|\\
&\qquad\qquad+|\tilde{G}^\mu_{\mathrm{c}}(P^\mu+R_1,W_1,U_1)(t)-\tilde{G}^\mu_{\mathrm{c}}(P^\mu+R_2,W_2,U_2)(t)|\\
&\qquad\qquad+|\tilde{H}^\mu_{\mathrm{c}}(P^\mu+R_1)(t)-\tilde{H}^\mu_{\mathrm{c}}(P^\mu+R_2)(t)|
\end{aligned}$$

$$\lesssim \mu e^{\nu\tilde{\lambda}^\mu t}\|(R_1,W_1,U_1)-(R_2,W_2,U_2)\|_{E^+_{\nu\tilde{\lambda}^\mu}},$$

so that

$$\begin{aligned}&\|\mathcal{F}_{\mathrm{c}}(R_1,W_1,U_1)-\mathcal{F}_{\mathrm{c}}(R_2,W_2,U_2)\|_{C^1_{\nu\tilde{\lambda}^\mu}([0,\infty);\mathbb{R}^{2d})}\\ &\qquad\lesssim \mu\|(R_1,W_1,U_1)-(R_2,W_2,U_2)\|_{E^+_{\nu\tilde{\lambda}^\mu}}.\end{aligned}$$

For $\mathcal{F}_{\mathrm{sh}}(R_1,W_1,U_1)-\mathcal{F}_{\mathrm{sh}}(R_2,W_2,U_2)$ we obtain from maximal regularity (see Corollary 2.16) that

$$\begin{aligned}&\|\mathcal{F}_{\mathrm{sh}}(R_1,W_1,U_1)-\mathcal{F}_{\mathrm{sh}}(R_2,W_2,U_2)\|_{C^\alpha_{\nu\tilde{\lambda}^\mu}(\mathbb{R};\mathcal{D}_{\mathrm{sh}})}\\
&\quad\lesssim \|(L^\mu_{\mathrm{sh}}-L^0_{\mathrm{sh}})(U_1-U_2)\|_{C^\alpha_{\nu\tilde{\lambda}^\mu}([0,\infty);\mathcal{X}_{\mathrm{sh}})}\\
&\qquad+\|\tilde{G}^\mu_{\mathrm{sh}}(P^\mu+\mathcal{E}_{\mathrm{wh}}R_1,\mathcal{E}_{\mathrm{c}}W_1,U_1)-\tilde{G}^\mu_{\mathrm{sh}}(P^\mu+\mathcal{E}_{\mathrm{wh}}R_2,\mathcal{E}_{\mathrm{c}}W_2,U_2)\|_{C^\alpha_{\nu\tilde{\lambda}^\mu}([0,\infty);\mathcal{X}_{\mathrm{sh}})}\\
&\qquad+\|\tilde{H}^\mu_{\mathrm{sh}}(P^\mu+\mathcal{E}_{\mathrm{wh}}R_1)-\tilde{H}^\mu_{\mathrm{sh}}(P^\mu+\mathcal{E}_{\mathrm{wh}}R_2)\|_{C^\alpha_{\nu\tilde{\lambda}^\mu}([0,\infty);\mathcal{X}_{\mathrm{sh}})}\\
&\quad\lesssim \mu\|U_1-U_2\|_{C^\alpha_{\nu\tilde{\lambda}^\mu}([0,\infty);\mathcal{X}_{\mathrm{sh}})}\\
&\qquad+\|\tilde{G}^\mu_{\mathrm{sh}}(P^\mu+\mathcal{E}_{\mathrm{wh}}R_1,\mathcal{E}_{\mathrm{c}}W_1,U_1)-\tilde{G}^\mu_{\mathrm{sh}}(P^\mu+\mathcal{E}_{\mathrm{wh}}R_2,\mathcal{E}_{\mathrm{c}}W_2,U_2)\|_{C^\alpha_{\nu\tilde{\lambda}^\mu}([0,\infty);\mathcal{X}_{\mathrm{sh}})}\\
&\qquad+\|\tilde{H}^\mu_{\mathrm{sh}}(P^\mu+\mathcal{E}_{\mathrm{wh}}R_1)-\tilde{H}^\mu_{\mathrm{sh}}(P^\mu+\mathcal{E}_{\mathrm{wh}}R_2)\|_{C^\alpha_{\nu\tilde{\lambda}^\mu}([0,\infty);\mathcal{X}_{\mathrm{sh}})}\end{aligned}$$

and it follows from Proposition 2.7, Remarks 2.11 and 4.16, and estimate (5.7) combined with the facts that all second derivatives of $\tilde{G}^\mu_{\mathrm{sh}}$ and $\tilde{H}^\mu_{\mathrm{sh}}$ are $\mathcal{O}(1)$ that

$$\begin{aligned}&\|\tilde{G}^\mu_{\mathrm{sh}}(P^\mu+\mathcal{E}_{\mathrm{wh}}R_1,\mathcal{E}_{\mathrm{c}}W_1,U_1)-\tilde{G}^\mu_{\mathrm{sh}}(P^\mu+\mathcal{E}_{\mathrm{wh}}R_2,\mathcal{E}_{\mathrm{c}}W_2,U_2)\|_{C^\alpha_{\nu\tilde{\lambda}^\mu}([0,\infty);\mathcal{X}_{\mathrm{sh}})}\\ &\qquad\lesssim \mu\|(R_1,W_1,U_1)-(R_2,W_2,U_2)\|_{E^+_{\nu\tilde{\lambda}^\mu}}\end{aligned}$$

and

$$\begin{aligned}&\|\tilde{H}^\mu_{\mathrm{sh}}(P^\mu+\mathcal{E}_{\mathrm{wh}}R_1)-\tilde{H}^\mu_{\mathrm{sh}}(P^\mu+\mathcal{E}_{\mathrm{wh}}R_2)\|_{C^\alpha_{\nu\tilde{\lambda}^\mu}([0,\infty);\mathcal{X}_{\mathrm{sh}})}\\ &\qquad\lesssim \mu\|(R_1,W_1,U_1)-(R_2,W_2,U_2)\|_{E^+_{\nu\tilde{\lambda}^\mu}}.\end{aligned}$$

□

**Corollary 5.9.** The operator $\mathcal{F}$ has a unique fixed point $(R^*_{W_0},W^*_{W_0},U^*_{W_0})\in B^+_{\nu\tilde{\lambda}^\mu}$ for each $W_0\in\mathbb{R}^{2d}$ with $S_{\mathrm{c}}W_0=W_0$ and $|W_0|\le \mu e^{-\frac{c^*}{2\mu}}$ in the case $n=4$. This fixed point satisfies $(S_{\mathrm{sh}}U^*_{W_0})(0)=U^*_{W_0}(0)$.

## 5.3 The local centre manifold

The *local centre manifold* $W^{\mathrm{c}}_{\mathrm{loc}}$ for solutions to equations (5.1) – (5.3) is constructed in a similar fashion to $W^{\mathrm{cs}}_{\mathrm{loc}}$. We formulate the modified equations

$$\dot{Z}=L^\mu_{\mathrm{wh}}Z+\tilde{G}^\mu_{\mathrm{wh}}(Z,W,U)+\tilde{H}^\mu_{\mathrm{wh}}(Z), \tag{5.56}$$

$$\dot{W}=L^\mu_{\mathrm{c}}W+\tilde{G}^\mu_{\mathrm{c}}(Z,W,U)+\tilde{H}^\mu_{\mathrm{c}}(Z), \tag{5.57}$$

$$\dot{U} = L^{\mu}_{\mathrm{sh}}U + \tilde{\underline{G}}^{\mu}_{\mathrm{sh}}(Z, W, U) + \tilde{H}^{\mu}_{\mathrm{sh}}(Z), \tag{5.58}$$

where

$$\tilde{\underline{G}}^{\mu}_{\mathrm{wh}}(Z, W, U) = \tilde{G}^{\mu}_{\mathrm{wh}}(Z, \phi(W), U),$$

as the fixed-point problem

$$(Z, W, U) = \Big(\mathcal{G}_{\mathrm{wh}}(Z, W, U), \mathcal{G}_{\mathrm{c}}(Z, W, U), \mathcal{G}_{\mathrm{sh}}(Z, W, U)\Big), \tag{5.59}$$

where

$$\begin{aligned}\mathcal{G}_{\mathrm{wh}}(Z, W, U)(t) = &\int_{-\infty}^{t} \Big\langle \Big(\tilde{H}^{\mu}_{\mathrm{wh}}(Z) + \tilde{\underline{G}}^{\mu}_{\mathrm{wh}}(Z, W, U)\Big)(\tau); s^{*}\mathrm{e}^{\tilde{\lambda}^{\mu}\tau}\Big\rangle \,\mathrm{d}\tau\, s\,\mathrm{e}^{-\tilde{\lambda}^{\mu}t} \\ &- \int_{t}^{\infty} \Big\langle \Big(\tilde{H}^{\mu}_{\mathrm{wh}}(Z) + \tilde{\underline{G}}^{\mu}_{\mathrm{wh}}(Z, W, U)\Big)(\tau); u^{*}\mathrm{e}^{-\tilde{\lambda}^{\mu}\tau}\Big\rangle \,\mathrm{d}\tau\, u\,\mathrm{e}^{\tilde{\lambda}^{\mu}t},\end{aligned}$$

in the case $n = 2$,

$$\begin{aligned}\mathcal{G}_{\mathrm{wh}}(Z, W, U)(t) = &\sum_{j=1}^{2}\int_{-\infty}^{t} \Big\langle \Big(\tilde{H}^{\mu}_{\mathrm{wh}}(Z) + \tilde{\underline{G}}^{\mu}_{\mathrm{wh}}(Z, W, U)\Big)(\tau); s_j^{*}\mathrm{e}^{\tilde{\lambda}^{\mu}\tau}\Big\rangle \,\mathrm{d}\tau\, s_j\mathrm{e}^{-\tilde{\lambda}^{\mu}t} \\ &- \sum_{j=1}^{2}\int_{t}^{\infty} \Big\langle \Big(\tilde{H}^{\mu}_{\mathrm{wh}}(Z) + \tilde{\underline{G}}^{\mu}_{\mathrm{wh}}(Z, W, U)\Big)(\tau); u_j^{*}\mathrm{e}^{-\tilde{\lambda}^{\mu}\tau}\Big\rangle \,\mathrm{d}\tau\, u_j\mathrm{e}^{\tilde{\lambda}^{\mu}t}\end{aligned}$$

in the case $n = 4$,

$$\mathcal{G}_{\mathrm{c}}(Z, W, U)(t) = \mathrm{e}^{L^{\mu}_{\mathrm{c}}t}W_0 + \int_0^t \mathrm{e}^{L^{\mu}_{\mathrm{c}}(t-\tau)}\Big(\tilde{\underline{G}}^{\mu}_{\mathrm{c}}(Z, W, U)(\tau) + \tilde{H}^{\mu}_{\mathrm{c}}(Z)(\tau)\Big)\,\mathrm{d}\tau,$$

where $s, u, s^*, u^*$ are defined in equations (5.20) – (5.23) and $s_j, u_j, s_j^*, u_j^*$ are defined in equations (5.27) – (5.30) (with the obvious modifications if $\omega + \tilde{\sigma}^{\mu} \neq 0$), and $\mathcal{G}_{\mathrm{sh}}(Z, W, U)$ is the unique solution of

$$\dot{M} = L^{\mu}_{\mathrm{sh}}M + \tilde{\underline{G}}^{\mu}_{\mathrm{sh}}(Z, W, U) + \tilde{H}^{\mu}_{\mathrm{sh}}(Z)$$

(in the appropriate sense). Repeating the previous arguments, we find that $\mathcal{G}$ is a contraction on the closed, convex subset

$$B_{\nu\tilde{\lambda}^{\mu}} = \Big\{(Z, W, U) \in E_{\nu\tilde{\lambda}^{\mu}} : \|Z\|_{C^{\alpha}_{\mathrm{b}}(\mathbb{R};\mathbb{R}^n)}, \|U\|_{C^{\alpha}_{\mathrm{b}}(\mathbb{R};\mathcal{D}_{\mathrm{sh}})}, \|\dot{W} - L^{\mu}_{\mathrm{c}}W\|_{C_{\mathrm{b}}(\mathbb{R};\mathbb{R}^{2d})} \leq \mathrm{e}^{-\frac{c^*}{2\mu}}\Big\}$$

of

$$E_{\nu\tilde{\lambda}^{\mu}} = C^{\alpha}_{\nu\tilde{\lambda}^{\mu}}(\mathbb{R};\mathbb{R}^n) \times C^{1}_{\nu\tilde{\lambda}^{\mu}}(\mathbb{R};\mathbb{R}^{2d}) \times C^{\alpha}_{\nu\tilde{\lambda}^{\mu}}(\mathbb{R};\mathcal{D}_{\mathrm{sh}}).$$

Its unique fixed point $(Z^{**}_{W_0}, W^{**}_{W_0}, U^{**}_{W_0})$ satisfies

$$\sup_{t\in\mathbb{R}}|Z^{**}_{W_0}(t)| \lesssim \mu\mathrm{e}^{-\frac{c^*}{2\mu}},$$

$$\sup_{t\in\mathbb{R}}\|U^{**}_{W_0}(t)\|_{\mathcal{D}_{\mathrm{sh}}} \lesssim \mu\mathrm{e}^{-\frac{c^*}{2\mu}}.$$

Any solution $(Z, W, U)$ to equations (5.56) – (5.58) satisfying

$$\|Z\|_{C^{\alpha}_{\mathrm{b}}(\mathbb{R};\mathbb{R}^n)}, \|U\|_{C^{\alpha}_{\mathrm{b}}(\mathbb{R};\mathcal{D}_{\mathrm{sh}})}, \|\dot{W} - L^{\mu}_{\mathrm{c}}W\|_{C_{\mathrm{b}}(\mathbb{R};\mathbb{R}^{2d})} \leq \mathrm{e}^{-\frac{c^*}{2\mu}}$$

lies in $E_{\nu\tilde{\lambda}^{\mu}}$ and is a fixed point of $\mathcal{G}$ with $W(0) = W_0$; its initial value $(Z, W, U)(0)$ therefore lies on

$$\begin{aligned} W^{\mathrm{c}} &= \left\{ (Z^{**}_{W_0}, W^{**}_{W_0}, U^{**}_{W_0})(0) : W_0 \in \mathbb{R}^{2d} \right\} \\ &= \left\{ (Z, W_0, U) \in \mathcal{D} : (Z, U) = \Psi(W_0), W_0 \in \mathbb{R}^{2d} \right\} \end{aligned}$$

(by uniqueness of the fixed point of $\mathcal{G}$), where

$$\Psi(W_0) = (\Psi_{\mathrm{wh}}(W_0), \Psi_{\mathrm{sh}}(W_0)) = (Z^{**}_{W_0}, U^{**}_{W_0})(0).$$

Using the autonomy of equations (5.56) – (5.58) we further conclude that in fact each point $(Z, W, U)(t)$ of $(Z, W, U)$ lies on $W^{\mathrm{c}}$ for all $t \in \mathbb{R}$. Restricting the domain of $\Psi$ to $B_{\mathrm{e}^{-c^*/2\mu}}(0)$ leads to the local centre manifold

$$W^{\mathrm{c}}_{\mathrm{loc}} = \left\{ (Z, W, U) : (Z, U) = \Psi(W), |W| \leq \mathrm{e}^{-\frac{c^*}{2\mu}} \right\}$$

for equations (5.1) – (5.3); all solutions with

$$\|Z\|_{C_{\mathrm{b}}(\mathbb{R};\mathbb{R}^n)}, \|W\|_{C_{\mathrm{b}}(\mathbb{R};\mathbb{R}^{2d})}, \|U\|_{C_{\mathrm{b}}(\mathbb{R};\mathcal{D}_{\mathrm{sh}})} \leq \mathrm{e}^{-\frac{c^*}{2\mu}}$$

lie on $W^{\mathrm{c}}_{\mathrm{loc}}$ and any solution passing through a point on $W^{\mathrm{c}}_{\mathrm{loc}}$ remains on $W^{\mathrm{c}}_{\mathrm{loc}}$ as long as it remains in

$$\left\{ (Z, W, U) \in \mathcal{D} : |Z|, |W|, \|U\|_{\mathcal{D}_{\mathrm{sh}}} \leq \mathrm{e}^{-\frac{c^*}{2\mu}} \right\}.$$

The following results show that in fact any solution $(Z, W, U)$ with $(Z, W, U)(t_0) \in W^{\mathrm{c}}_{\mathrm{loc}}$ and $|W(t_0)| \leq \frac{1}{2}\mathrm{e}^{-\frac{c^*}{2\mu}}$ satisfies $(Z, W, U)(t) \in W^{\mathrm{c}}_{\mathrm{loc}}$ for all $t \geq t_0$.

**Proposition 5.10.** The function $\Psi\colon \bar{B}_{\mathrm{e}^{-c^*/2\mu}}(0) \to \mathbb{R}^n \times \mathcal{D}_{\mathrm{sh}}$ satisfies the estimate

$$\|\Psi(W_0)\|_{\mathbb{R}^n \times \mathcal{D}_{\mathrm{sh}}} \lesssim |W_0|^2.$$

*Proof.* Since $\mathcal{G}\colon B_{\nu\tilde{\lambda}^{\mu}} \to B_{\nu\tilde{\lambda}^{\mu}}$ is a contraction with Lipschitz constant less than or equal to $1/2$, we know that

$$\|(Z, W, U) - (Z^{**}_{W_0}, W^{**}_{W_0}, U^{**}_{W_0})\|_{E_{\nu\tilde{\lambda}^{\mu}}} \leq 2\|(Z, W, U) - \mathcal{G}(Z, W, U)\|_{E_{\nu\tilde{\lambda}^{\mu}}}$$

for all $(Z, W, U) \in B_{\nu\tilde{\lambda}^{\mu}}$. Using this result with $(Z, W, U) = (0, \mathrm{e}^{L^{\mu}_{\mathrm{c}}(\cdot)}W_0, 0)$, we find that

$$\begin{aligned} \|\Psi(W_0)\|_{\mathbb{R}^n \times \mathcal{D}_{\mathrm{sh}}} &= \|(\Psi_{\mathrm{wh}}(W_0), W_0, \Psi_{\mathrm{sh}}(W_0)) - (0, W_0, 0)\|_{\mathcal{D}} \\ &\leq \|(0, \mathrm{e}^{L^{\mu}_{\mathrm{c}}(\cdot)}W_0, 0) - (Z^{**}_{W_0}, W^{**}_{W_0}, U^{**}_{W_0})\|_{E_{\nu\tilde{\lambda}^{\mu}}} \\ &\leq 2\|(0, \mathrm{e}^{L^{\mu}_{\mathrm{c}}(\cdot)}W_0, 0) - \mathcal{G}(0, \mathrm{e}^{L^{\mu}_{\mathrm{c}}(\cdot)}W_0, 0)\|_{E_{\nu\tilde{\lambda}^{\mu}}} \\ &\lesssim \|\mathcal{G}_{\mathrm{wh}}(0, \mathrm{e}^{L^{\mu}_{\mathrm{c}}(\cdot)}W_0, 0)\|_{C^{\alpha}_{\nu\tilde{\lambda}^{\mu}}(\mathbb{R};\mathbb{R}^n)} \\ &\qquad + \|\mathrm{e}^{L^{\mu}_{\mathrm{c}}(\cdot)}W_0 - \mathcal{G}_{\mathrm{c}}(0, \mathrm{e}^{L^{\mu}_{\mathrm{c}}(\cdot)}W_0, 0)\|_{C^{1}_{\nu\tilde{\lambda}^{\mu}}(\mathbb{R};\mathbb{R}^{2d})} \\ &\qquad + \|\mathcal{G}_{\mathrm{sh}}(0, \mathrm{e}^{L^{\mu}_{\mathrm{c}}(\cdot)}W_0, 0)\|_{C^{\alpha}_{\nu\tilde{\lambda}^{\mu}}(\mathbb{R};\mathcal{D}_{\mathrm{sh}})}. \end{aligned}$$

Observing that $|\mathrm{e}^{L_\mathrm{c}^\mu(\cdot)}W_0| = |W_0|$ and $\phi(W_0) = W_0$, we find that

$$\begin{aligned}\|\mathcal{G}_\mathrm{wh}(0, \mathrm{e}^{L_\mathrm{c}^\mu(\cdot)}W_0, 0)\|_{C^\alpha_{\nu\tilde{\lambda}^\mu}(\mathbb{R};\mathbb{R}^n)} &\lesssim \int_{-\infty}^t \mu^2 |\mathrm{e}^{L_\mathrm{c}^\mu \tau}W_0|^2 \mathrm{e}^{\tilde{\lambda}^\mu \tau}\,\mathrm{d}\tau\, \mathrm{e}^{-\tilde{\lambda}^\mu t} \\ &\qquad + \int_t^\infty \mu^2 |\mathrm{e}^{L_\mathrm{c}^\mu \tau}W_0|^2 \mathrm{e}^{-\tilde{\lambda}^\mu \tau}\,\mathrm{d}\tau\, \mathrm{e}^{\tilde{\lambda}^\mu t} \\ &\lesssim |W_0|^2.\end{aligned}$$

From the fact that $\mathcal{G}_\mathrm{sh}(0, \mathrm{e}^{L_\mathrm{c}^\mu(\cdot)}W_0, 0)$ is the unique solution to

$$\dot{M} = L_\mathrm{sh}^\mu M + \tilde{G}_\mathrm{sh}^\mu(0, \mathrm{e}^{L_\mathrm{c}^\mu(\cdot)}W_0, 0)$$

in $B_{\nu\tilde{\lambda}^\mu}$ and Corollary 2.16 we obtain that

$$\|\mathcal{G}_\mathrm{sh}(0, \mathrm{e}^{L_\mathrm{c}^\mu(\cdot)}W_0, 0)\|_{C^\alpha_{\nu\tilde{\lambda}^\mu}(\mathbb{R};\mathcal{D}_\mathrm{sh})} \lesssim \|\mathcal{G}_\mathrm{sh}(0, \mathrm{e}^{L_\mathrm{c}^\mu(\cdot)}W_0, 0)\|_{C^\alpha_\mathrm{b}(\mathbb{R};\mathcal{D}_\mathrm{sh})} \lesssim |W_0|^2.$$

Finally we find that

$$|\mathrm{e}^{L_\mathrm{c}^\mu t}W_0 - \mathcal{G}_\mathrm{c}(0, \mathrm{e}^{L_\mathrm{c}^\mu t}W_0, 0)| = \Big|\int_0^t \mathrm{e}^{L_\mathrm{c}^\mu(t-\tau)}\tilde{G}_\mathrm{c}^\mu(0, \mathrm{e}^{L_\mathrm{c}^\mu \tau}W_0, 0)\,\mathrm{d}\tau\Big| \lesssim \mu^2|t||W_0|^2$$

and

$$\begin{aligned}\Big|\frac{\mathrm{d}}{\mathrm{d}t}(\mathrm{e}^{L_\mathrm{c}^\mu t}W_0 - \mathcal{G}_\mathrm{c}(0, \mathrm{e}^{L_\mathrm{c}^\mu t}W_0, 0))\Big| &= \Big|L_\mathrm{c}^\mu \int_0^t \mathrm{e}^{L_\mathrm{c}^\mu(t-\tau)}\tilde{G}_\mathrm{c}^\mu(0, \mathrm{e}^{L_\mathrm{c}^\mu \tau}W_0, 0)\,\mathrm{d}\tau + \tilde{G}_\mathrm{c}^\mu(0, \mathrm{e}^{L_\mathrm{c}^\mu t}W_0, 0)\Big| \\ &\lesssim \mu^2(|t|+1)|W_0|^2,\end{aligned}$$

so that

$$\|\mathcal{G}_\mathrm{c}(0, \mathrm{e}^{L_\mathrm{c}^\mu(\cdot)}W_0, 0)\|_{C^1_{\nu\tilde{\lambda}^\mu}(\mathbb{R};\mathbb{R}^{2d})} \lesssim |W_0|^2$$

(since $\mu^2|t|\mathrm{e}^{-\nu\tilde{\lambda}^\mu|t|} \lesssim 1$). □

**Lemma 5.11.** Any solution $(Z, W, U)$ of system (5.56) – (5.58) lying on $W^\mathrm{c}$ with $|W(t_0)| \leq \frac{1}{2}\mathrm{e}^{-\frac{c^*}{2\mu}}$ (so that $(Z, W, U)(t_0) \in W^\mathrm{c}_\mathrm{loc}$) satisfies $|W(t)| \leq \frac{3}{4}\mathrm{e}^{-\frac{c^*}{2\mu}}$ (and hence $(Z, W, U)(t) \in W^\mathrm{c}_\mathrm{loc}$) for all $t \geq t_0$.

*Proof.* Using Assumption (D6) we find that

$$\tilde{\mathcal{I}}^\mu(Z, W, U) = \tilde{\mathcal{I}}^\mu(\Psi_\mathrm{wh}(W), W, \Psi_\mathrm{sh}(W)) \geq \frac{1}{2}\mu^4|W|^2$$

and

$$\tilde{\mathcal{I}}^\mu(Z, W, U) = \tilde{\mathcal{I}}^\mu(\Psi_\mathrm{wh}(W), W, \Psi_\mathrm{sh}(W)) \leq \frac{3}{2}\mu^4|W|^2$$

for $(Z, W, U) \in W^\mathrm{c}_\mathrm{loc}$.

Now we suppose that $(Z, W, U)$ is a global solution of equations (5.56) – (5.58) on $W^\mathrm{c}$ with $|W(t_0)| \leq \frac{1}{2}\mathrm{e}^{-\frac{c^*}{2\mu}}$. Assume there exists a time $t > t_0$ such that $|W(t)| > \frac{3}{4}\mathrm{e}^{-\frac{c^*}{2\mu}}$. The continuity of $W$ yields the existence of a time $t^* \in (t_0, t)$ such that $|W(t^*)| = \frac{3}{4}\mathrm{e}^{-\frac{c^*}{2\mu}}$. Since $(Z, W, U)$ solves equations (5.1) – (5.3) for $t \in [t_0, t^*]$ we find that

$$\begin{aligned}\frac{9}{32}\mathrm{e}^{-\frac{c^*}{\mu}} &= \frac{1}{2}|W(t^*)|^2 \\ &\leq \mu^{-4}\tilde{\mathcal{I}}^\mu(Z(t^*), W(t^*), U(t^*)) \\ &= \mu^{-4}\tilde{\mathcal{I}}^\mu(Z(t_0), W(t_0), U(t_0)) \\ &\leq \frac{3}{16}\mathrm{e}^{-\frac{c^*}{\mu}},\end{aligned}$$

which is a contradiction. □

## 5.4 The global centre-stable manifold

Suppose that $(R^*_{W_0}, W^*_{W_0}, U^*_{W_0})$ is the fixed point of the operator $\mathcal{F}$. We observe that $(P^\mu + R^*_{W_0}, W^*_{W_0}, U^*_{W_0})$ yields a generalised pulse solution to equations (5.1) – (5.3) if

$$|W^*_{W_0}(t)| \leq \mathrm{e}^{-\frac{c^*}{2\mu}} \tag{5.60}$$

for all $t \geq 0$ (and by the symmetry of $W^*_{W_0}$ for all $t \in \mathbb{R}$), since $(P^\mu + R^*_{W_0}, W^*_{W_0}, U^*_{W_0})$ then solves equations (5.1) – (5.3), in which no cut-off function is used. Since we have already proved it for $t \in [0, \mathrm{e}^{\frac{c^*}{2\mu}}]$ it remains to show that estimate (5.60) remains true for $t \geq \mathrm{e}^{\frac{c^*}{2\mu}}$. To this end we show that $(P^\mu + R^*_{W_0}, W^*_{W_0}, U^*_{W_0})$ converges to a solution $(Z, W, U)$ of equations (5.56) – (5.58) on $W^{\mathrm{c}}$, so that in particular

$$\sup_{t\in[t^*,\infty)} |W^*_{W_0}(t) - W(t)| \leq \mu \mathrm{e}^{-\frac{c^*}{2\mu}}$$

for some $t^*$ (see Figure 5.5).

**Proposition 5.12.** There exists $\delta > 0$ such that

$$\sup_{t\in[0,\mathrm{e}^{\frac{c^*}{2\mu}}]} |W^*_{W_0}(t)| \lesssim \mu^\delta \mathrm{e}^{-\frac{c^*}{2\mu}}. \tag{5.61}$$

*Proof.* The central component $W^*_{W_0}$ of the fixed point of $\mathcal{F}$ satisfies equation (5.13) so that Remark 4.16 and estimate (5.7) yield an a priori estimate for $|W^*_{W_0}|$ over the interval $[0, t^*]$, where

$$t^* = \frac{\gamma|\log\mu|}{\nu\tilde{\lambda}^\mu},$$

so that $\mathrm{e}^{-\nu\tilde{\lambda}^\mu t^*} = \mu^\gamma$ for an appropriately chosen constant $\gamma$. Applying $\mathrm{e}^{-L^\mu_{\mathrm{c}} t}$ to equation (5.13) and taking the inner product with $\mathrm{e}^{-L^\mu_{\mathrm{c}} t} W$ we obtain

$$\begin{aligned}&\langle \mathrm{e}^{-L^\mu_{\mathrm{c}} t}\dot{W}^*_{W_0}(t) - L^\mu_{\mathrm{c}}\mathrm{e}^{-L^\mu_{\mathrm{c}} t}W^*_{W_0}(t); \mathrm{e}^{-L^\mu_{\mathrm{c}} t}W^*_{W_0}(t)\rangle \\ &\quad = \langle \mathrm{e}^{-L^\mu_{\mathrm{c}} t}\Big(\tilde{G}^\mu_{\mathrm{c}}(P^\mu + R^*_{W_0}, W^*_{W_0}, U^*_{W_0})(t) + \tilde{H}_{\mathrm{c}}(P^\mu + R^*_{W_0})(t)\Big); \mathrm{e}^{-L^\mu_{\mathrm{c}} t}W^*_{W_0}(t)\rangle,\end{aligned}$$

so that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|\mathrm{e}^{-L_c^\mu t}W^*_{W_0}(t)|^2 \lesssim (\mu^2|W^*_{W_0}(t)| + \mu^3\mathrm{e}^{-\frac{c^*}{2\mu}})|\mathrm{e}^{-L_c^\mu t}W^*_{W_0}(t)|$$

and thus

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|\mathrm{e}^{-L_c^\mu t}W^*_{W_0}(t)|^2 \leq c_1\tilde{\lambda}^\mu|W^*_{W_0}|^2 + c_2\mu^4\mathrm{e}^{-\frac{c^*}{\mu}}.$$

Gronwall's inequality now yields

$$|W^*_{W_0}(t)|^2 \leq (|W_0|^2 + c_2\mu^4\mathrm{e}^{-\frac{c^*}{\mu}}t)\mathrm{e}^{c_1\tilde{\lambda}^\mu t}$$

(since $|\mathrm{e}^{-L_c^\mu t}(\cdot)|$ is equivalent to the usual norm for $\mathbb{R}^{2d}$, uniformly in $\mu$). By choosing $\gamma = \nu/c_1$, so that $\mathrm{e}^{c_1\tilde{\lambda}^\mu t^*} = 1/\mu$, we find

$$\begin{aligned}|W^*_{W_0}(t)|^2 &\leq \frac{1}{\mu}(|W_0|^2 + c_2\mu^4\mathrm{e}^{-\frac{c^*}{\mu}}t^*)\\ &\leq \frac{1}{\mu}(|W_0|^2 + c_2\mu^2\mathrm{e}^{-\frac{c^*}{\mu}}\frac{\nu}{c_1}|\log\mu|)\\ &\lesssim \mu|\log\mu|\mathrm{e}^{-\frac{c^*}{\mu}}\end{aligned}$$

for all $t \in [0, t^*]$. Using the above estimate yields

$$\begin{aligned}|W^*_{W_0}(t) &= |\mathcal{F}_c(R^*_{W_0}, W^*_{W_0}, U^*_{W_0})(t)|\\ &\leq |W_0| + \left|\int_0^t \mathrm{e}^{L_c^\mu(t-\tau)}\Big(\underline{\tilde{G}}^\mu_c(P^\mu + R^*_{W_0}, W^*_{W_0}, U^*_{W_0})(\tau) + \tilde{H}^\mu_c(P^\mu + R^*_{W_0})(\tau)\Big)\,\mathrm{d}\tau\right|\\ &\lesssim |W_0| + \int_0^{t^*}(\mu^2|W^*_{W_0}(\tau)| + \mu^3\mathrm{e}^{-\frac{c^*}{2\mu}})\,\mathrm{d}\tau + \int_{t^*}^t(\mu^2\mathrm{e}^{-\frac{c^*}{2\mu}}\mathrm{e}^{-\nu\tilde{\lambda}^\mu\tau} + \mu\mathrm{e}^{-\frac{c^*}{\mu}})\,\mathrm{d}\tau\\ &\lesssim |W_0| + \mu^{\frac{5}{2}}|\log\mu|^{\frac{1}{2}}\mathrm{e}^{-\frac{c^*}{2\mu}}t^* + \mathrm{e}^{-\frac{c^*}{2\mu}}\mathrm{e}^{-\nu\tilde{\lambda}^\mu t^*} + t\mu\mathrm{e}^{-\frac{c^*}{\mu}}\\ &\lesssim |W_0| + (\mu^{\frac{1}{2}}|\log\mu|^{\frac{3}{2}} + \mu^\gamma)\mathrm{e}^{-\frac{c^*}{2\mu}} + t\mu\mathrm{e}^{-\frac{c^*}{\mu}}\end{aligned}$$

□

In view of the estimate

$$\sup_{t\in[0,\mathrm{e}^{-\frac{c^*}{2\mu}}]}|W^*_{W_0}(t)| \lesssim \mu^\delta\mathrm{e}^{-\frac{c^*}{2\mu}},$$

showing that $t^* < \mathrm{e}^{\frac{c^*}{2\mu}}$ yields $|W(t^*)| \leq \frac{1}{2}\mathrm{e}^{-\frac{c^*}{2\mu}}$ and hence

$$\sup_{t\in[t^*,\infty)}|W^*_{W_0}(t)| \leq \frac{3}{4}\mathrm{e}^{-\frac{c^*}{2\mu}}$$

(by Lemma 5.11).

Setting

$$t^* = \frac{c^*}{\mu\nu\tilde{\lambda}^\mu} \in [0, \mathrm{e}^{\frac{c^*}{2\mu}}],$$

we find that

$$\mathrm{e}^{-\nu\tilde{\lambda}^\mu\frac{t^*}{2}} = \mathrm{e}^{-\frac{c^*}{2\mu}},$$

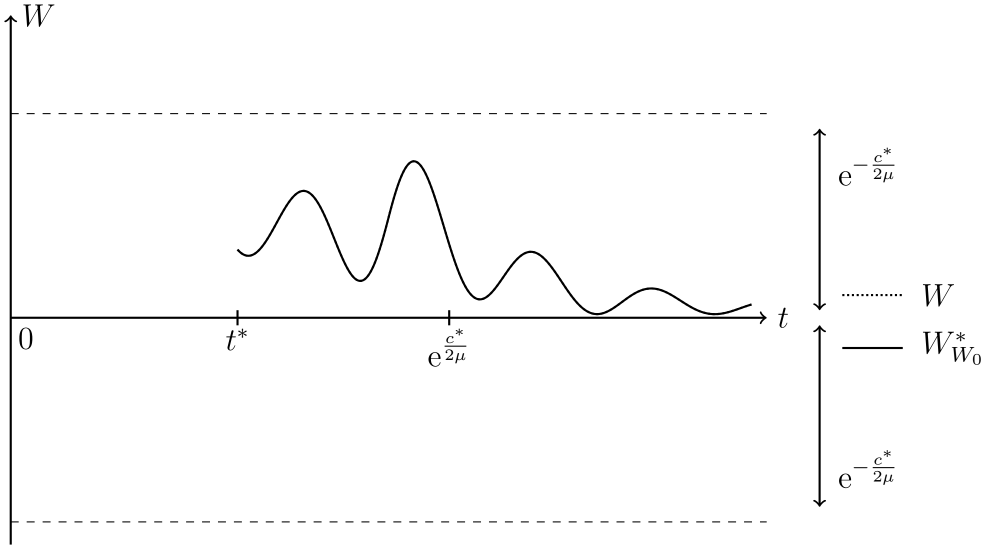



Figure 5.5: The central part $W^*_{W_0}$ of a function with initial values on $W^{\mathrm{cs}}_{\mathrm{loc}}$ converges exponentially to the central part $W$ of a solution to equations (5.56) – (5.58) on $W^{\mathrm{c}}$.

so that

$$|P^\mu(t)| \le \begin{cases} \mu^2 \mathrm{e}^{-\frac{c^*}{2\mu}}, & n = 2, \\ \mu \mathrm{e}^{-\frac{c^*}{2\mu}}, & n = 4, \end{cases}$$

for $t \ge \frac{t^*}{2}$. Let $\chi\colon \mathbb{R} \to \mathbb{R}$ be a smooth cut-off function with

$$\chi(t) = \begin{cases} 0, & t \le \frac{t^*}{2}, \\ 1, & t \ge t^*, \end{cases}$$

and $|\chi^{(l)}(t)| \le 2^l$ for $t \in \mathbb{R}$ and $l \in \mathbb{N}_0$. The function $(\hat{Z}, \hat{W}, \hat{U})$ given by

$$(\hat{Z}, \hat{W}, \hat{U}) = \chi(P^\mu + R^*_{W_0}, W^*_{W_0}, U^*_{W_0})$$

satisfies the equations

$$\begin{aligned} \dot{\hat{Z}} &= L^\mu_{\mathrm{wh}} \hat{Z} + \underline{\tilde{G}}^\mu_{\mathrm{wh}}(\hat{Z}, \hat{W}, \hat{U}) + \tilde{H}^\mu_{\mathrm{wh}}(\hat{Z}) + Q_{\mathrm{wh}}, \\ \dot{\hat{W}} &= L^\mu_{\mathrm{c}} \hat{W} + \underline{\tilde{G}}^\mu_{\mathrm{c}}(\hat{Z}, \hat{W}, \hat{U}) + \tilde{H}^\mu_{\mathrm{c}}(\hat{Z}) + Q_{\mathrm{c}}, \\ \dot{\hat{U}} &= L^\mu_{\mathrm{sh}} \hat{U} + \underline{\tilde{G}}^\mu_{\mathrm{sh}}(\hat{Z}, \hat{W}, \hat{U}) + \tilde{H}^\mu_{\mathrm{sh}}(\hat{Z}) + Q_{\mathrm{sh}}, \end{aligned}$$

where

$$\begin{aligned} Q_{\mathrm{wh}} &= \chi \tilde{H}^\mu_{\mathrm{wh}}(P^\mu + R^*_{W_0}) + \chi \underline{\tilde{G}}^\mu_{\mathrm{wh}}(P^\mu + R^*_{W_0}, W^*_{W_0}, U^*_{W_0}) + \dot{\chi}(P^\mu + R^*_{W_0}) \\ &\quad - \tilde{H}^\mu_{\mathrm{wh}}(\chi(P^\mu + R^*_{W_0})) - \underline{\tilde{G}}^\mu_{\mathrm{wh}}(\chi(P^\mu + R^*_{W_0}, W^*_{W_0}, U^*_{W_0})), \\ Q_{\mathrm{c}} &= \chi \underline{\tilde{G}}^\mu_{\mathrm{c}}(P^\mu + R^*_{W_0}, W^*_{W_0}, U^*_{W_0}) + \chi \tilde{H}^\mu_{\mathrm{c}}(P^\mu + R^*_{W_0}) + \dot{\chi} W^*_{W_0} \\ &\quad - \underline{\tilde{G}}^\mu_{\mathrm{c}}(\chi(P^\mu + R^*_{W_0}, W^*_{W_0}, U^*_{W_0})) - \tilde{H}^\mu_{\mathrm{c}}(\chi(P^\mu + R^*_{W_0})), \\ Q_{\mathrm{sh}} &= \chi \underline{\tilde{G}}^\mu_{\mathrm{sh}}(P^\mu + R^*_{W_0}, W^*_{W_0}, U^*_{W_0}) + \chi \tilde{H}^\mu_{\mathrm{sh}}(P^\mu + R^*_{W_0}) + \dot{\chi} U^*_{W_0} \\ &\quad - \underline{\tilde{G}}^\mu_{\mathrm{sh}}(\chi(P^\mu + R^*_{W_0}, W^*_{W_0}, U^*_{W_0})) - \tilde{H}^\mu_{\mathrm{sh}}(\chi(P^\mu + R^*_{W_0})). \end{aligned}$$

To show that $(\hat{Z},\hat{W},\hat{U})$ converges exponentially to a solution of equations (5.56) – (5.58) on $W^{\mathrm{c}}$ we have to find a solution $(\Delta_{\mathrm{wh}},\Delta_{\mathrm{c}},\Delta_{\mathrm{sh}})$ to

$$\begin{aligned}\dot{\Delta}_{\mathrm{wh}} &= L^{\mu}_{\mathrm{wh}}\Delta_{\mathrm{wh}} + \tilde{H}^{\mu}_{\mathrm{wh}}(\hat{Z}) - \tilde{H}^{\mu}_{\mathrm{wh}}(\hat{Z}-\Delta_{\mathrm{wh}}) + \tilde{G}^{\mu}_{\mathrm{wh}}(\hat{Z},\hat{W},\hat{U}) \\ &\quad - \tilde{G}^{\mu}_{\mathrm{wh}}(\hat{Z}-\Delta_{\mathrm{wh}},\hat{W}-\Delta_{\mathrm{c}},\hat{U}-\Delta_{\mathrm{sh}}) + Q_{\mathrm{wh}}, \qquad (5.62)\\ \dot{\Delta}_{\mathrm{c}} &= L^{\mu}_{\mathrm{c}}\Delta_{\mathrm{c}} + \tilde{G}^{\mu}_{\mathrm{c}}(\hat{Z},\hat{W},\hat{U}) + \tilde{H}^{\mu}_{\mathrm{c}}(\hat{Z}) \\ &\quad - \tilde{G}^{\mu}_{\mathrm{c}}(\hat{Z}-\Delta_{\mathrm{wh}},\hat{W}-\Delta_{\mathrm{c}},\hat{U}-\Delta_{\mathrm{sh}}) - \tilde{H}^{\mu}_{\mathrm{c}}(\hat{Z}-\Delta_{\mathrm{wh}}) + Q_{\mathrm{c}}, \qquad (5.63)\\ \dot{\Delta}_{\mathrm{sh}} &= L^{\mu}_{\mathrm{sh}}\Delta_{\mathrm{sh}} + \tilde{G}^{\mu}_{\mathrm{sh}}(\hat{Z},\hat{W},\hat{U}) + \tilde{H}^{\mu}_{\mathrm{sh}}(\hat{Z}) \\ &\quad - \tilde{G}^{\mu}_{\mathrm{sh}}(\hat{Z}-\Delta_{\mathrm{wh}},\hat{W}-\Delta_{\mathrm{c}},\hat{U}-\Delta_{\mathrm{sh}}) - \tilde{H}^{\mu}_{\mathrm{sh}}(\hat{Z}-\Delta_{\mathrm{wh}}) + Q_{\mathrm{sh}}, \qquad (5.64)\end{aligned}$$

which decays exponentially to zero as $t\to\infty$ and has the property that

$$(Z,W,U) = (\hat{Z},\hat{W},\hat{U}) - (\Delta_{\mathrm{wh}},\Delta_{\mathrm{c}},\Delta_{\mathrm{sh}})$$

lies on $W^{\mathrm{c}}$, i.e. is a fixed point of the operator $\mathcal{G}$ defined in Section 5.3. To this end we formulate equations (5.62) – (5.64) as the fixed-point problem

$$\begin{aligned}\Delta_{\mathrm{wh}} &= \mathcal{K}_{\mathrm{wh}}(\Delta_{\mathrm{wh}},\Delta_{\mathrm{c}},\Delta_{\mathrm{sh}}),\\ \Delta_{\mathrm{c}} &= \mathcal{K}_{\mathrm{c}}(\Delta_{\mathrm{wh}},\Delta_{\mathrm{c}},\Delta_{\mathrm{sh}}),\\ \Delta_{\mathrm{sh}} &= \mathcal{K}_{\mathrm{sh}}(\Delta_{\mathrm{wh}},\Delta_{\mathrm{c}},\Delta_{\mathrm{sh}}).\end{aligned}$$

Here

$$\begin{aligned}\mathcal{K}_{\mathrm{wh}}(\Delta_{\mathrm{wh}},\Delta_{\mathrm{c}},\Delta_{\mathrm{sh}})(t) &= \int_{-\infty}^{t}\Big\langle \tilde{H}^{\mu}_{\mathrm{wh}}(\hat{Z})(\tau) + \tilde{G}^{\mu}_{\mathrm{wh}}(\hat{Z},\hat{W},\hat{U})(\tau) - \tilde{H}^{\mu}_{\mathrm{wh}}(\hat{Z}-\Delta_{\mathrm{wh}})(\tau) \\ &\qquad - \tilde{G}^{\mu}_{\mathrm{wh}}(\hat{Z}-\Delta_{\mathrm{wh}},\hat{W}-\Delta_{\mathrm{c}},\hat{U}-\Delta_{\mathrm{sh}})(\tau) \\ &\qquad + Q_{\mathrm{wh}}(\tau); s^{*}\mathrm{e}^{\tilde{\lambda}^{\mu}\tau}\Big\rangle\,\mathrm{d}\tau\, s\,\mathrm{e}^{-\tilde{\lambda}^{\mu}t} \\ &\quad - \int_{t}^{\infty}\Big\langle \tilde{H}^{\mu}_{\mathrm{wh}}(\hat{Z})(\tau) + \tilde{G}^{\mu}_{\mathrm{wh}}(\hat{Z},\hat{W},\hat{U})(\tau) - \tilde{H}^{\mu}_{\mathrm{wh}}(\hat{Z}-\Delta_{\mathrm{wh}})(\tau) \\ &\qquad - \tilde{G}^{\mu}_{\mathrm{wh}}(\hat{Z}-\Delta_{\mathrm{wh}},\hat{W}-\Delta_{\mathrm{c}},\hat{U}-\Delta_{\mathrm{sh}})(\tau) \\ &\qquad + Q_{\mathrm{wh}}(\tau); u^{*}\mathrm{e}^{-\tilde{\lambda}^{\mu}\tau}\Big\rangle\,\mathrm{d}\tau\, u\,\mathrm{e}^{\tilde{\lambda}^{\mu}t} \qquad (5.65)\end{aligned}$$

in the case $n=2$,

$$\begin{aligned}\mathcal{K}_{\mathrm{wh}}(\Delta_{\mathrm{wh}},\Delta_{\mathrm{c}},\Delta_{\mathrm{sh}})(t) &= \sum_{j=1}^{2}\int_{-\infty}^{t}\Big\langle \tilde{H}^{\mu}_{\mathrm{wh}}(\hat{Z})(\tau) + \tilde{G}^{\mu}_{\mathrm{wh}}(\hat{Z},\hat{W},\hat{U})(\tau) - \tilde{H}^{\mu}_{\mathrm{wh}}(\hat{Z}-\Delta_{\mathrm{wh}})(\tau) \\ &\qquad - \tilde{G}^{\mu}_{\mathrm{wh}}(\hat{Z}-\Delta_{\mathrm{wh}},\hat{W}-\Delta_{\mathrm{c}},\hat{U}-\Delta_{\mathrm{sh}})(\tau) \\ &\qquad + Q_{\mathrm{wh}}(\tau); s_j^{*}\mathrm{e}^{\tilde{\lambda}^{\mu}\tau}\Big\rangle\,\mathrm{d}\tau\, s_j\mathrm{e}^{-\tilde{\lambda}^{\mu}t} \\ &\quad - \sum_{j=1}^{2}\int_{t}^{\infty}\Big\langle \tilde{H}^{\mu}_{\mathrm{wh}}(\hat{Z})(\tau) + \tilde{G}^{\mu}_{\mathrm{wh}}(\hat{Z},\hat{W},\hat{U})(\tau) - \tilde{H}^{\mu}_{\mathrm{wh}}(\hat{Z}-\Delta_{\mathrm{wh}})(\tau) \\ &\qquad - \tilde{G}^{\mu}_{\mathrm{wh}}(\hat{Z}-\Delta_{\mathrm{wh}},\hat{W}-\Delta_{\mathrm{c}},\hat{U}-\Delta_{\mathrm{sh}})(\tau) \\ &\qquad + Q_{\mathrm{wh}}(\tau); u_j^{*}\mathrm{e}^{-\tilde{\lambda}^{\mu}\tau}\Big\rangle\,\mathrm{d}\tau\, u_j\mathrm{e}^{\tilde{\lambda}^{\mu}t} \qquad (5.66)\end{aligned}$$

in the case $n=4$,

$$\begin{aligned}\mathcal{K}_{\mathrm{c}}(\Delta_{\mathrm{wh}},\Delta_{\mathrm{c}},\Delta_{\mathrm{sh}})(t) &= -\int_{t}^{\infty}\mathrm{e}^{L^{\mu}_{\mathrm{c}}(t-\tau)}\Big(\tilde{G}^{\mu}_{\mathrm{c}}(\hat{Z},\hat{W},\hat{U}) + \tilde{H}^{\mu}_{\mathrm{c}}(\hat{Z}) \\ &\qquad - \tilde{G}^{\mu}_{\mathrm{c}}(\hat{Z}-\Delta_{\mathrm{wh}},\hat{W}-\Delta_{\mathrm{c}},\hat{U}-\Delta_{\mathrm{sh}}) \\ &\qquad - \tilde{H}^{\mu}_{\mathrm{c}}(\hat{Z}-\Delta_{\mathrm{wh}}) + Q_{\mathrm{c}}\Big)(\tau)\,\mathrm{d}\tau, \qquad (5.67)\end{aligned}$$

where $s, u, s^*, u^*$ are defined in equations (5.20) – (5.23), $s_j, u_j, s_j^*, u_j^*$ are defined in equations (5.27) – (5.30), and $\mathcal{K}_{\mathrm{sh}}(\Delta_{\mathrm{wh}}, \Delta_{\mathrm{c}}, \Delta_{\mathrm{sh}})$ is the unique solution to

$$\begin{aligned}\dot{M} =& L^\mu_{\mathrm{sh}} M + \tilde{\underline{G}}^\mu_{\mathrm{sh}}(\hat{Z}, \hat{W}, \hat{U}) + \tilde{H}^\mu_{\mathrm{sh}}(\hat{Z}) - \tilde{\underline{G}}^\mu_{\mathrm{sh}}(\hat{Z} - \Delta_{\mathrm{wh}}, \hat{W} - \Delta_{\mathrm{c}}, \hat{U} - \Delta_{\mathrm{sh}}) \\ & - \tilde{H}^\mu_{\mathrm{sh}}(\hat{Z} - \Delta_{\mathrm{wh}}) + Q_{\mathrm{sh}} \end{aligned} \tag{5.68}$$

(in the sense specified below). We now prove that $\mathcal{K} = (\mathcal{K}_{\mathrm{wh}}, \mathcal{K}_{\mathrm{c}}, \mathcal{K}_{\mathrm{sh}})$ is a contraction on the closed, convex subset

$$B_{\nu\tilde{\lambda}^\mu,+} = \left\{(Z, W, U) \in E_{\nu\tilde{\lambda}^\mu,+} : \|Z\|_{C^\alpha_{\mathrm{b}}(\mathbb{R};\mathbb{R}^n)}, \|U\|_{C^\alpha_{\mathrm{b}}(\mathbb{R};\mathcal{D}_{\mathrm{sh}})}, \|\dot{W} - L^\mu_{\mathrm{c}} W\|_{C_{\mathrm{b}}(\mathbb{R};\mathbb{R}^{2d})} \leq \mathrm{e}^{-\frac{c^*}{2\mu}}\right\}$$

of

$$E_{\nu\tilde{\lambda}^\mu,+} = C^\alpha_{\nu\tilde{\lambda}^\mu,+}(\mathbb{R}, \mathbb{R}^n) \times C^1_{\nu\tilde{\lambda}^\mu,+}(\mathbb{R}, \mathbb{R}^{2d}) \times C^\alpha_{\nu\tilde{\lambda}^\mu,+}(\mathbb{R}, \mathcal{D}_{\mathrm{sh}}).$$

As in Section 5.2 we use the fact that $|||\cdot|||_{C^\alpha_{\nu\tilde{\lambda}^\mu,+}(\mathbb{R};B)}$ and $\|\cdot\|_{C^\alpha_{\nu\tilde{\lambda}^\mu,+}(\mathbb{R};B)}$ are equivalent norms for $C^\alpha_{\nu\tilde{\lambda}^\mu,+}(\mathbb{R}; B)$, uniformly in $\mu$, where $B \in \{\mathbb{R}^n, \mathbb{R}^{2d}, \mathcal{D}_{\mathrm{sh}}\}$.

**Proposition 5.13.** The operators $Q_{\mathrm{wh}}$, $Q_{\mathrm{c}}$ and $Q_{\mathrm{sh}}$ satisfy the estimates

$$\|Q_{\mathrm{wh}}\|_{C_{\mathrm{b}}(\mathbb{R};\mathbb{R}^n)}, \|Q_{\mathrm{c}}\|_{C_{\mathrm{b}}(\mathbb{R};\mathbb{R}^{2d})}, \|Q_{\mathrm{sh}}\|_{C^\alpha_{\mathrm{b}}(\mathbb{R};\mathcal{X}_{\mathrm{sh}})} \lesssim \mu^3 \mathrm{e}^{-\frac{c^*}{2\mu}}.$$

Furthermore,

$$\|Q_{\mathrm{wh}}\|_{C_{\nu\tilde{\lambda}^\mu,+}(\mathbb{R};\mathbb{R}^n)}, \|Q_{\mathrm{c}}\|_{C_{\nu\tilde{\lambda}^\mu,+}(\mathbb{R};\mathbb{R}^{2d})}, \|Q_{\mathrm{sh}}\|_{C^\alpha_{\nu\tilde{\lambda}^\mu,+}(\mathbb{R};\mathcal{X}_{\mathrm{sh}})} < \infty.$$

*Proof.* Observing that $t^* < \mathrm{e}^{-\frac{c^*}{2\mu}}$, so that

$$|W^*_{W_0}(t)| \lesssim \mu \mathrm{e}^{-\frac{c^*}{2\mu}}$$

and thus $\phi(W^*_{W_0}(t)) = W^*_{W_0}(t)$ for $t \in [\frac{t^*}{2}, t^*]$, we find that

$$\begin{aligned}|Q_{\mathrm{wh}}(t)|, |Q_{\mathrm{c}}(t)|, \|Q_{\mathrm{sh}}(t)\|_{\mathcal{X}_{\mathrm{sh}}} & \\ &\lesssim \|(P^\mu + R^*_{W_0}, W^*_{W_0}, U^*_{W_0})(t)\|^2_{\mathcal{D}} + \|\dot{\chi}(P^\mu + R^*_{W_0}, W^*_{W_0}, U^*_{W_0})(t)\|_{\mathcal{X}} \\ &\lesssim \|(P^\mu + R^*_{W_0}, W^*_{W_0}, U^*_{W_0})(t)\|^2_{\mathcal{D}} + |\dot{\chi}(t)| \|(P^\mu + R^*_{W_0}, W^*_{W_0}, U^*_{W_0})(t)\|_{\mathcal{D}} \\ &\lesssim \mu^3 \mathrm{e}^{-\frac{c^*}{2\mu}} \end{aligned} \tag{5.69}$$

because $|(P^\mu + R^*_{W_0})(t)| \lesssim \mu \mathrm{e}^{-\frac{c^*}{2\mu}}$ and $|\dot{\chi}(t)| \lesssim \frac{1}{t^*} \lesssim \mu^2$ for $t \in [\frac{t^*}{2}, t^*]$. Next we note that $|\chi(P^\mu + R^*_{W_0})(t)| \lesssim \mu \mathrm{e}^{-\frac{c^*}{2\mu}}$ for all $t \in \mathbb{R}$. As in the proof of Theorem 5.4 (see estimates (5.43) and (5.44)) we find that

$$\begin{aligned}&\|\tilde{\underline{G}}^\mu_{\mathrm{sh}}\Big(\chi(P^\mu + R^*_{W_0}, W^*_{W_0}, U^*_{W_0})\Big)(t_1) - \tilde{\underline{G}}^\mu_{\mathrm{sh}}\Big(\chi(P^\mu + R^*_{W_0}, W^*_{W_0}, U^*_{W_0})\Big)(t_2)\|_{\mathcal{X}_{\mathrm{sh}}} \\ &\quad \lesssim \mu^3 \mathrm{e}^{-\frac{c^*}{2\mu}} |t_1 - t_2|^\alpha \end{aligned} \tag{5.70}$$

and

$$\begin{aligned}&\|\chi\tilde{\underline{G}}^\mu_{\mathrm{sh}}(P^\mu + R^*_{W_0}, W^*_{W_0}, U^*_{W_0})(t_1) - \chi\tilde{\underline{G}}^\mu_{\mathrm{sh}}(P^\mu + R^*_{W_0}, W^*_{W_0}, U^*_{W_0})(t_2)\|_{\mathcal{X}_{\mathrm{sh}}} \\ &\quad \lesssim \|\tilde{\underline{G}}^\mu_{\mathrm{sh}}(P^\mu + R^*_{W_0}, W^*_{W_0}, U^*_{W_0})(t_1) - \tilde{\underline{G}}^\mu_{\mathrm{sh}}(P^\mu + R^*_{W_0}, W^*_{W_0}, U^*_{W_0})(t_2)\|_{\mathcal{X}_{\mathrm{sh}}} \\ &\quad \lesssim \mu^3 \mathrm{e}^{-\frac{c^*}{2\mu}} |t_1 - t_2|^\alpha \end{aligned} \tag{5.71}$$

and similarly

$$\|H(\chi(P^\mu + R_{W_0}))(t_1) - H(\chi(P^\mu + R_{W_0}))(t_2)\|_{\mathcal{X}_{\rm sh}} \lesssim \mu \mathrm{e}^{-\frac{c^*}{\mu}} |t_1 - t_2|^\alpha \tag{5.72}$$

and

$$\|\chi H(P^\mu + R_{W_0}^*)(t_1) - \chi H(P^\mu + R_{W_0}^*)(t_2)\|_{\mathcal{X}_{\rm sh}} \lesssim \mu \mathrm{e}^{-\frac{c^*}{\mu}} |t_1 - t_2|^\alpha \tag{5.73}$$

for $-\infty < t_2 < t_1 < \infty$, where we have used the estimate

$$\|\chi\|_{C_{\rm b}^\alpha(\mathbb{R})} \leq \|\chi\|_{C_{\rm b}^1(\mathbb{R})} \lesssim 1.$$

Combining estimates (5.69) – (5.73) we obtain

$$\|Q_{\rm sh}\|_{C_{\rm b}^\alpha(\mathbb{R})} \lesssim \mu^3 \mathrm{e}^{-\frac{c^*}{2\mu}}.$$

For the second assertion we first consider the calculation

$$\begin{aligned}
|Q_{\rm wh}(t)|, |Q_{\rm c}(t)| &\lesssim \mu^3 \mathrm{e}^{-\frac{c^*}{2\mu}} \\
&= \mathrm{e}^{-\nu\tilde{\lambda}^\mu \frac{t^*}{2}} \\
&= \mathrm{e}^{\nu\tilde{\lambda}^\mu \frac{t^*}{2}} \mathrm{e}^{-\nu\tilde{\lambda}^\mu t^*} \\
&\leq \mathrm{e}^{\nu\tilde{\lambda}^\mu \frac{t^*}{2}} \mathrm{e}^{-\nu\tilde{\lambda}^\mu t} \\
&= \mathrm{e}^{\frac{c^*}{2\mu}} \mathrm{e}^{-\nu\tilde{\lambda}^\mu t},
\end{aligned}$$

where we have used the fact that $Q_{\rm wh}(t), Q_{\rm c}(t) = 0$ for all $t > t^*$. Next we observe that $\hat{\chi}(t) Q_{\rm sh}(t) = Q_{\rm sh}(t)$ for all $t \in \mathbb{R}$, where $\hat{\chi} \in C^\infty(\mathbb{R})$ is a smooth cut-off function satisfying

$$\hat{\chi}(t) = \begin{cases} 1, & |t| \leq t^* \\ 0, & |t| \geq 2t^*. \end{cases}$$

We find that

$$\begin{aligned}
\|Q_{\rm sh}\|_{C^\alpha_{\nu\tilde{\lambda}^\mu,+}(\mathbb{R};\mathcal{X}_{\rm sh})} &= \|\mathrm{e}^{\nu\tilde{\lambda}^\mu(\cdot)} Q_{\rm sh}\|_{C_{\rm b}^\alpha(\mathbb{R};\mathcal{X}_{\rm sh})} \\
&= \|\hat{\chi}\mathrm{e}^{\nu\tilde{\lambda}^\mu(\cdot)} Q_{\rm sh}\|_{C_{\rm b}^\alpha(\mathbb{R};\mathcal{X}_{\rm sh})} \\
&\lesssim \|\hat{\chi}\mathrm{e}^{\nu\tilde{\lambda}^\mu(\cdot)}\|_{C_{\rm b}^\alpha(\mathbb{R})} \|Q_{\rm sh}\|_{C_{\rm b}^\alpha(\mathbb{R};\mathcal{X}_{\rm sh})} \\
&\lesssim \|\hat{\chi}\mathrm{e}^{\nu\tilde{\lambda}^\mu(\cdot)}\|_{C_{\rm b}^1(\mathbb{R})} \mu^3 \mathrm{e}^{-\frac{c^*}{2\mu}} \\
&< \infty,
\end{aligned}$$

where we again have used the fact that $Q_{\rm sh}(t) = 0$ for all $t > t^*$. □

**Lemma 5.14.** The operator $\mathcal{K}$ maps $B_{\nu\tilde{\lambda}^\mu,+}$ contractively into itself. Furthermore,

$$\begin{aligned}
\|\mathcal{K}_{\rm wh}(\Delta_{\rm wh}, \Delta_{\rm c}, \Delta_{\rm sh})\|_{C_{\rm b}^\alpha(\mathbb{R};\mathbb{R}^n)} &\lesssim \mu \mathrm{e}^{-\frac{c^*}{2\mu}}, \\
\left\|\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{K}_{\rm c}(\Delta_{\rm wh}, \Delta_{\rm c}, \Delta_{\rm sh}) - L_{\rm c}^\mu \mathcal{K}_{\rm c}(\Delta_{\rm wh}, \Delta_{\rm c}, \Delta_{\rm sh})\right\|_{C_{\rm b}(\mathbb{R};\mathbb{R}^{2d})} &\lesssim \mu \mathrm{e}^{-\frac{c^*}{2\mu}}, \\
\|\mathcal{K}_{\rm sh}(\Delta_{\rm wh}, \Delta_{\rm c}, \Delta_{\rm sh})\|_{C_{\rm b}^\alpha(\mathbb{R};\mathcal{D}_{\rm sh})} &\lesssim \mu \mathrm{e}^{-\frac{c^*}{2\mu}}
\end{aligned}$$

for all $(\Delta_{\rm wh}, \Delta_{\rm c}, \Delta_{\rm sh}) \in B_{\nu\tilde{\lambda}^\mu,+}$.

*Proof.* Suppose that $v$ is one of $s$, $u$, $s_j$, $u_j$, $s^*$, $u^*$, $s_j^*$, or $u_j^*$ and $(\Delta_{\mathrm{wh}}, \Delta_{\mathrm{c}}, \Delta_{\mathrm{sh}}) \in B_{\nu\tilde{\lambda}^\mu,+}$.

Using Remarks 4.8 and 4.16 we find for $\mathcal{K}_{\mathrm{wh}}(\Delta_{\mathrm{wh}}, \Delta_{\mathrm{c}}, \Delta_{\mathrm{sh}})$ that

$$\begin{aligned}
&\Big| \int_{-\infty}^{t} \Big\langle \tilde{H}^\mu_{\mathrm{wh}}(\hat{Z})(\tau) + \tilde{\underline{G}}^\mu_{\mathrm{wh}}(\hat{Z}, \hat{W}, \hat{U})(\tau) - \tilde{H}^\mu_{\mathrm{wh}}(\hat{Z} - \Delta_{\mathrm{wh}})(\tau) \\
&\qquad - \tilde{\underline{G}}^\mu_{\mathrm{wh}}(\hat{Z} - \Delta_{\mathrm{wh}}, \hat{W} - \Delta_{\mathrm{c}}, \hat{U} - \Delta_{\mathrm{sh}})(\tau); v(\tau)\mathrm{e}^{\tilde{\lambda}^\mu \tau} \Big\rangle \,\mathrm{d}\tau \, v(t)\mathrm{e}^{-\tilde{\lambda}^\mu t} \\
&\quad + \int_{t}^{\infty} \Big\langle \tilde{H}^\mu_{\mathrm{wh}}(\hat{Z})(\tau) + \tilde{\underline{G}}^\mu_{\mathrm{wh}}(\hat{Z}, \hat{W}, \hat{U})(\tau) - \tilde{H}^\mu_{\mathrm{wh}}(\hat{Z} - \Delta_{\mathrm{wh}})(\tau) \\
&\qquad - \tilde{\underline{G}}^\mu_{\mathrm{wh}}(\hat{Z} - \Delta_{\mathrm{wh}}, \hat{W} - \Delta_{\mathrm{c}}, \hat{U} - \Delta_{\mathrm{sh}})(\tau); v(\tau)\mathrm{e}^{-\tilde{\lambda}^\mu \tau} \Big\rangle \,\mathrm{d}\tau \, v(t)\mathrm{e}^{\tilde{\lambda}^\mu t} \Big| \\
&\lesssim \int_{-\infty}^{t} \mu^3 \mathrm{e}^{\tilde{\lambda}^\mu \tau} \mathrm{e}^{-\nu\tilde{\lambda}^\mu \tau} \|(\Delta_{\mathrm{wh}}, \phi(\hat{W}) - \phi(\hat{W} - \Delta_{\mathrm{c}}), \Delta_{\mathrm{sh}})\|_{E_{\nu\tilde{\lambda}^\mu,+}} \,\mathrm{d}\tau \, \mathrm{e}^{-\tilde{\lambda}^\mu t} \\
&\quad + \int_{t}^{\infty} \mu^3 \mathrm{e}^{-\tilde{\lambda}^\mu \tau} \mathrm{e}^{-\nu\tilde{\lambda}^\mu \tau} \|(\Delta_{\mathrm{wh}}, \phi(\hat{W}) - \phi(\hat{W} - \Delta_{\mathrm{c}}), \Delta_{\mathrm{sh}})\|_{E_{\nu\tilde{\lambda}^\mu,+}} \,\mathrm{d}\tau \, \mathrm{e}^{\tilde{\lambda}^\mu t} \\
&\lesssim \frac{\mu^3}{\nu\tilde{\lambda}^\mu} \|(\Delta_{\mathrm{wh}}, \phi(\hat{W}) - \phi(\hat{W} - \Delta_{\mathrm{c}}), \Delta_{\mathrm{sh}})\|_{E_{\nu\tilde{\lambda}^\mu,+}} \mathrm{e}^{-\nu\tilde{\lambda}^\mu t} \\
&\quad + \frac{\mu^3}{\nu\tilde{\lambda}^\mu} \|(\Delta_{\mathrm{wh}}, \phi(\hat{W}) - \phi(\hat{W} - \Delta_{\mathrm{c}}), \Delta_{\mathrm{sh}})\|_{E_{\nu\tilde{\lambda}^\mu,+}} \mathrm{e}^{-\nu\tilde{\lambda}^\mu t} \\
&\lesssim \mu \|(\Delta_{\mathrm{wh}}, \phi(\hat{W}) - \phi(\hat{W} - \Delta_{\mathrm{c}}), \Delta_{\mathrm{sh}})\|_{E_{\nu\tilde{\lambda}^\mu,+}} \mathrm{e}^{-\nu\tilde{\lambda}^\mu t}
\end{aligned} \tag{5.74}$$

and

$$\begin{aligned}
&\Big| \int_{-\infty}^{t} \Big\langle Q_{\mathrm{wh}}(\tau); v(\tau)\mathrm{e}^{-\tilde{\lambda}^\mu \tau} \Big\rangle \,\mathrm{d}\tau \, v(t)\mathrm{e}^{\tilde{\lambda}^\mu t} \\
&\quad + \int_{t}^{\infty} \Big\langle Q_{\mathrm{wh}}(\tau); v(\tau)\mathrm{e}^{\tilde{\lambda}^\mu \tau} \Big\rangle \,\mathrm{d}\tau \, v(t)\mathrm{e}^{-\tilde{\lambda}^\mu t} \Big| \\
&\lesssim \int_{-\infty}^{t} |Q_{\mathrm{wh}}(\tau)|\mathrm{e}^{\tilde{\lambda}^\mu \tau} \,\mathrm{d}\tau \, \mathrm{e}^{-\tilde{\lambda}^\mu t} + \int_{t}^{\infty} |Q_{\mathrm{wh}}(\tau)|\mathrm{e}^{-\tilde{\lambda}^\mu \tau} \,\mathrm{d}\tau \, \mathrm{e}^{\tilde{\lambda}^\mu t} \\
&\le \|Q_{\mathrm{wh}}\|_{C_{\nu\tilde{\lambda}^\mu,+}(\mathbb{R};\mathbb{R}^n)} \int_{-\infty}^{t} \mathrm{e}^{(1-\nu)\tilde{\lambda}^\mu \tau} \,\mathrm{d}\tau \, \mathrm{e}^{-\tilde{\lambda}^\mu t} + \|Q_{\mathrm{wh}}\|_{C_{\nu\tilde{\lambda}^\mu,+}(\mathbb{R};\mathbb{R}^n)} \int_{t}^{\infty} \mathrm{e}^{-(1+\nu)\tilde{\lambda}^\mu \tau} \,\mathrm{d}\tau \, \mathrm{e}^{\tilde{\lambda}^\mu t} \\
&\lesssim \frac{1}{(1-\nu)\tilde{\lambda}^\mu} \|Q_{\mathrm{wh}}\|_{C_{\nu\tilde{\lambda}^\mu,+}(\mathbb{R};\mathbb{R}^n)} \mathrm{e}^{-\nu\tilde{\lambda}^\mu t}
\end{aligned} \tag{5.75}$$

for $t \in \mathbb{R}$. Remark 4.8 and Proposition 5.13 yield that

$$\begin{aligned}
&\Big| \int_{-\infty}^{t_1} \Big\langle \tilde{H}^\mu_{\mathrm{wh}}(\hat{Z})(\tau) + \tilde{\underline{G}}^\mu_{\mathrm{wh}}(\hat{Z}, \hat{W}, \hat{U})(\tau) - \tilde{H}^\mu_{\mathrm{wh}}(\hat{Z} - \Delta_{\mathrm{wh}})(\tau) \\
&\qquad - \tilde{\underline{G}}^\mu_{\mathrm{wh}}(\hat{Z} - \Delta_{\mathrm{wh}}, \hat{W} - \Delta_{\mathrm{c}}, \hat{U} - \Delta_{\mathrm{sh}})(\tau); v(\tau)\mathrm{e}^{\tilde{\lambda}^\mu \tau} \Big\rangle \,\mathrm{d}\tau \, v(t_1)\mathrm{e}^{-\tilde{\lambda}^\mu t_1} \\
&\quad + \int_{t_1}^{\infty} \Big\langle \tilde{H}^\mu_{\mathrm{wh}}(\hat{Z})(\tau) + \tilde{\underline{G}}^\mu_{\mathrm{wh}}(\hat{Z}, \hat{W}, \hat{U})(\tau) - \tilde{H}^\mu_{\mathrm{wh}}(\hat{Z} - \Delta_{\mathrm{wh}})(\tau) \\
&\qquad - \tilde{\underline{G}}^\mu_{\mathrm{wh}}(\hat{Z} - \Delta_{\mathrm{wh}}, \hat{W} - \Delta_{\mathrm{c}}, \hat{U} - \Delta_{\mathrm{sh}})(\tau); v(\tau)\mathrm{e}^{-\tilde{\lambda}^\mu \tau} \Big\rangle \,\mathrm{d}\tau \, v(t_1)\mathrm{e}^{\tilde{\lambda}^\mu t_1} \\
&\quad - \int_{-\infty}^{t_2} \Big\langle \tilde{H}^\mu_{\mathrm{wh}}(\hat{Z})(\tau) + \tilde{\underline{G}}^\mu_{\mathrm{wh}}(\hat{Z}, \hat{W}, \hat{U})(\tau) - \tilde{H}^\mu_{\mathrm{wh}}(\hat{Z} - \Delta_{\mathrm{wh}})(\tau) \\
&\qquad - \tilde{\underline{G}}^\mu_{\mathrm{wh}}(\hat{Z} - \Delta_{\mathrm{wh}}, \hat{W} - \Delta_{\mathrm{c}}, \hat{U} - \Delta_{\mathrm{sh}})(\tau); v(\tau)\mathrm{e}^{\tilde{\lambda}^\mu \tau} \Big\rangle \,\mathrm{d}\tau \, v(t_2)\mathrm{e}^{-\tilde{\lambda}^\mu t_2} \\
&\quad - \int_{t_2}^{\infty} \Big\langle \tilde{H}^\mu_{\mathrm{wh}}(\hat{Z})(\tau) + \tilde{\underline{G}}^\mu_{\mathrm{wh}}(\hat{Z}, \hat{W}, \hat{U})(\tau) - \tilde{H}^\mu_{\mathrm{wh}}(\hat{Z} - \Delta_{\mathrm{wh}})(\tau) \\
&\qquad - \tilde{\underline{G}}^\mu_{\mathrm{wh}}(\hat{Z} - \Delta_{\mathrm{wh}}, \hat{W} - \Delta_{\mathrm{c}}, \hat{U} - \Delta_{\mathrm{sh}})(\tau); v(\tau)\mathrm{e}^{-\tilde{\lambda}^\mu \tau} \Big\rangle \,\mathrm{d}\tau \, v(t_2)\mathrm{e}^{\tilde{\lambda}^\mu t_2} \Big|
\end{aligned}$$

$$
\begin{aligned}
\lesssim \int_{t_2}^{t_1}\int_0^1 \Big| & \mathrm{d}\tilde{H}^\mu_{\mathrm{wh}}[\sigma\hat{Z}+(1-\sigma)(\hat{Z}-\Delta_{\mathrm{wh}})](\Delta_{\mathrm{wh}})(\tau) \\
& + \mathrm{d}_1\tilde{G}^\mu_{\mathrm{wh}}[\sigma\hat{Z}+(1-\sigma)(\hat{Z}-\Delta_{\mathrm{wh}}),\phi(\hat{W}),\hat{U}](\Delta_{\mathrm{wh}})(\tau) \\
& + \mathrm{d}_2\tilde{G}^\mu_{\mathrm{wh}}[\hat{Z}-\Delta_{\mathrm{wh}},\sigma\phi(\hat{W})+(1-\sigma)(\phi(\hat{W})-\phi(\Delta_{\mathrm{c}})),\hat{U}](\phi(\Delta_{\mathrm{c}}))(\tau) \\
& + \mathrm{d}_3\tilde{G}^\mu_{\mathrm{wh}}[\hat{Z}-\Delta_{\mathrm{wh}},\phi(\hat{W})-\phi(\Delta_{\mathrm{c}}), \\
& \qquad \sigma\hat{U}+(1-\sigma)(\hat{U}-\Delta_{\mathrm{sh}})](\Delta_{\mathrm{sh}})(\tau)\Big| \,\mathrm{d}\sigma\, \mathrm{e}^{\tilde{\lambda}^\mu\tau}\,\mathrm{d}\tau\, \mathrm{e}^{-\tilde{\lambda}^\mu t_1} \\
+ \int_{-\infty}^{t_2}\int_0^1 \Big| & \mathrm{d}\tilde{H}^\mu_{\mathrm{wh}}[\sigma\hat{Z}+(1-\sigma)(\hat{Z}-\Delta_{\mathrm{wh}})](\Delta_{\mathrm{wh}})(\tau) \\
& + \mathrm{d}_1\tilde{G}^\mu_{\mathrm{wh}}[\sigma\hat{Z}+(1-\sigma)(\hat{Z}-\Delta_{\mathrm{wh}}),\phi(\hat{W}),\hat{U}](\Delta_{\mathrm{wh}})(\tau) \\
& + \mathrm{d}_2\tilde{G}^\mu_{\mathrm{wh}}[\hat{Z}-\Delta_{\mathrm{wh}},\sigma\phi(\hat{W})+(1-\sigma)(\phi(\hat{W})-\phi(\Delta_{\mathrm{c}})),\hat{U}](\phi(\Delta_{\mathrm{c}}))(\tau) \\
& + \mathrm{d}_3\tilde{G}^\mu_{\mathrm{wh}}[\hat{Z}-\Delta_{\mathrm{wh}},\phi(\hat{W})-\phi(\Delta_{\mathrm{c}}), \\
& \qquad \sigma\hat{U}+(1-\sigma)(\hat{U}-\Delta_{\mathrm{sh}})](\Delta_{\mathrm{sh}})(\tau)\Big| \,\mathrm{d}\sigma\, \mathrm{e}^{\tilde{\lambda}^\mu\tau}\,\mathrm{d}\tau\, |\mathrm{e}^{-\tilde{\lambda}^\mu t_1}-\mathrm{e}^{-\tilde{\lambda}^\mu t_2}| \\
+ \int_{t_1}^{\infty}\int_0^1 \Big| & \mathrm{d}\tilde{H}^\mu_{\mathrm{wh}}[\sigma\hat{Z}+(1-\sigma)(\hat{Z}-\Delta_{\mathrm{wh}})](\Delta_{\mathrm{wh}})(\tau) \\
& + \mathrm{d}_1\tilde{G}^\mu_{\mathrm{wh}}[\sigma\hat{Z}+(1-\sigma)(\hat{Z}-\Delta_{\mathrm{wh}}),\phi(\hat{W}),\hat{U}](\Delta_{\mathrm{wh}})(\tau) \\
& + \mathrm{d}_2\tilde{G}^\mu_{\mathrm{wh}}[\hat{Z}-\Delta_{\mathrm{wh}},\sigma\phi(\hat{W})+(1-\sigma)(\phi(\hat{W})-\phi(\Delta_{\mathrm{c}})),\hat{U}](\phi(\Delta_{\mathrm{c}}))(\tau) \\
& + \mathrm{d}_3\tilde{G}^\mu_{\mathrm{wh}}[\hat{Z}-\Delta_{\mathrm{wh}},\phi(\hat{W})-\phi(\Delta_{\mathrm{c}}), \\
& \qquad \sigma\hat{U}+(1-\sigma)(\hat{U}-\Delta_{\mathrm{sh}})](\Delta_{\mathrm{sh}})(\tau)\Big| \,\mathrm{d}\sigma\, \mathrm{e}^{-\tilde{\lambda}^\mu\tau}\,\mathrm{d}\tau\, |\mathrm{e}^{-\tilde{\lambda}^\mu t_1}-\mathrm{e}^{-\tilde{\lambda}^\mu t_2}| \\
\lesssim \mu\|(\Delta_{\mathrm{wh}}, & \phi(\hat{W})-\phi(\hat{W}-\Delta_{\mathrm{c}}),\Delta_{\mathrm{sh}})\|_{E_{\nu\tilde{\lambda}^\mu,+}}|t_1-t_2|^\alpha \mathrm{e}^{-\nu\tilde{\lambda}^\mu t_2}
\end{aligned} \tag{5.76}
$$

and

$$
\begin{aligned}
\Big| & \int_{-\infty}^{t_1}\Big\langle Q_{\mathrm{wh}}(\tau); v(\tau)\mathrm{e}^{\tilde{\lambda}^\mu\tau}\Big\rangle \mathrm{d}\tau\, v(t_1)\mathrm{e}^{-\tilde{\lambda}^\mu t_1} + \int_{t_1}^{\infty}\Big\langle Q_{\mathrm{wh}}(\tau); v(\tau)\mathrm{e}^{-\tilde{\lambda}^\mu\tau}\Big\rangle \mathrm{d}\tau\, v(t_1)\mathrm{e}^{\tilde{\lambda}^\mu t_1} \\
& - \int_{-\infty}^{t_2}\Big\langle Q_{\mathrm{wh}}(\tau); v(\tau)\mathrm{e}^{\tilde{\lambda}^\mu\tau}\Big\rangle \mathrm{d}\tau\, v(t_2)\mathrm{e}^{-\tilde{\lambda}^\mu t_2} - \int_{t_2}^{\infty}\Big\langle Q_{\mathrm{wh}}(\tau); v(\tau)\mathrm{e}^{-\tilde{\lambda}^\mu\tau}\Big\rangle \mathrm{d}\tau\, v(t_2)\mathrm{e}^{\tilde{\lambda}^\mu t_2}\Big| \\
\lesssim & \int_{t_2}^{t_1}|Q_{\mathrm{wh}}(\tau)|\mathrm{e}^{\tilde{\lambda}^\mu\tau}\,\mathrm{d}\tau\,\mathrm{e}^{-\tilde{\lambda}^\mu t_1} + \int_{-\infty}^{t_2}|Q_{\mathrm{wh}}(\tau)|\mathrm{e}^{\tilde{\lambda}^\mu\tau}\,\mathrm{d}\tau\,(\mathrm{e}^{-\tilde{\lambda}^\mu t_2}-\mathrm{e}^{-\tilde{\lambda}^\mu t_1}) \\
& + \int_{t_1}^{\infty}|Q_{\mathrm{wh}}(\tau)|\mathrm{e}^{-\tilde{\lambda}^\mu\tau}\,\mathrm{d}\tau\,(\mathrm{e}^{-\tilde{\lambda}^\mu t_2}-\mathrm{e}^{-\tilde{\lambda}^\mu t_1}) \\
\lesssim & \frac{1}{(1-\nu)\tilde{\lambda}^\mu}\|Q_{\mathrm{wh}}\|_{C_{\nu\tilde{\lambda}^\mu,+}(\mathbb{R};\mathbb{R}^n)}(\mathrm{e}^{(1-\nu)\tilde{\lambda}^\mu t_1}-\mathrm{e}^{(1-\nu)\tilde{\lambda}^\mu t_2})\mathrm{e}^{-\tilde{\lambda}^\mu t_1} \\
& + \frac{1}{(1-\nu)\tilde{\lambda}^\mu}\|Q_{\mathrm{wh}}\|_{C_{\nu\tilde{\lambda}^\mu,+}(\mathbb{R};\mathbb{R}^n)}\mathrm{e}^{(1-\nu)\tilde{\lambda}^\mu t_2}(\mathrm{e}^{-\tilde{\lambda}^\mu t_2}-\mathrm{e}^{-\tilde{\lambda}^\mu t_1}) \\
& + \frac{1}{(1+\nu)\tilde{\lambda}^\mu}\|Q_{\mathrm{wh}}\|_{C_{\nu\tilde{\lambda}^\mu,+}(\mathbb{R};\mathbb{R}^n)}\mathrm{e}^{-(1+\nu)\tilde{\lambda}^\mu t_2}(\mathrm{e}^{\tilde{\lambda}^\mu t_1}-\mathrm{e}^{\tilde{\lambda}^\mu t_2}) \\
\lesssim & \frac{1}{(1-\nu)\tilde{\lambda}^\mu}\|Q_{\mathrm{wh}}\|_{C_{\nu\tilde{\lambda}^\mu,+}(\mathbb{R};\mathbb{R}^n)}\|\mathrm{e}^{(1-\nu)\tilde{\lambda}^\mu(\cdot)}\|_{C^\alpha_{(1-\nu)\tilde{\lambda}^\mu,-}(\mathbb{R})}|t_1-t_2|^\alpha\mathrm{e}^{(1-\nu)\tilde{\lambda}^\mu t_2}\mathrm{e}^{-\tilde{\lambda}^\mu t_2} \\
& + \frac{1}{(1-\nu)\tilde{\lambda}^\mu}\|Q_{\mathrm{wh}}\|_{C_{\nu\tilde{\lambda}^\mu,+}(\mathbb{R};\mathbb{R}^n)}\frac{1-\mathrm{e}^{\tilde{\lambda}^\mu(t_2-t_1)}}{|t_1-t_2|^\alpha}|t_1-t_2|^\alpha\mathrm{e}^{-\nu\tilde{\lambda}^\mu t_2} \\
& + \frac{1}{(1+\nu)\tilde{\lambda}^\mu}\|Q_{\mathrm{wh}}\|_{C_{\nu\tilde{\lambda}^\mu,+}(\mathbb{R};\mathbb{R}^n)}\frac{1-\mathrm{e}^{\tilde{\lambda}^\mu(t_2-t_1)}}{|t_1-t_2|^\alpha}|t_1-t_2|^\alpha\mathrm{e}^{-\nu\tilde{\lambda}^\mu t_2}
\end{aligned}
$$

$$\lesssim \frac{1}{(1-\nu)\tilde{\lambda}^\mu} \|Q_{\mathrm{wh}}\|_{C_{\nu\tilde{\lambda}^\mu,+}(\mathbb{R};\mathbb{R}^n)} |t_1 - t_2|^\alpha \mathrm{e}^{-\nu\tilde{\lambda}^\mu t_2} \tag{5.77}$$

for $-\infty < t_2 < t_1 < \infty$, where we have estimated $\mathrm{e}^{-\tilde{\lambda}^\mu t_1} \leq \mathrm{e}^{-\tilde{\lambda}^\mu t_2}$ between the third and fourth line and

$$\frac{1 - \mathrm{e}^{-\tilde{\lambda}^\mu |t_1 - t_2|}}{|t_1 - t_2|^\alpha} \lesssim 1$$

between the fourth and fifth line. From Proposition 2.7 and estimates (5.74) – (5.77) we obtain $\mathcal{K}_{\mathrm{wh}}(\Delta_{\mathrm{c}}, \Delta_{\mathrm{c}}, \Delta_{\mathrm{sh}}) \in C^\alpha_{\nu\tilde{\lambda}^\mu,+}(\mathbb{R}, \mathbb{R}^n)$.

For $\mathcal{K}_{\mathrm{c}}(\Delta_{\mathrm{wh}}, \Delta_{\mathrm{c}}, \Delta_{\mathrm{sh}})$ we use Proposition 5.13 and similar estimates as for $\mathcal{F}_{\mathrm{c}}$ in the proofs of Theorems 5.5 and 5.8 to find that

$$\begin{aligned}
|\mathcal{K}_{\mathrm{c}}(\Delta_{\mathrm{wh}}, \Delta_{\mathrm{c}}, \Delta_{\mathrm{sh}})(t)| &\lesssim \mu \mathrm{e}^{-\nu\tilde{\lambda}^\mu t} \|(\Delta_{\mathrm{wh}}, \phi(\hat{W}) - \phi(\hat{W} - \Delta_{\mathrm{c}}), \Delta_{\mathrm{sh}})\|_{E_{\nu\tilde{\lambda}^\mu,+}} \\
&\quad + \int_t^\infty |Q_{\mathrm{c}}(\tau)| \,\mathrm{d}\tau \\
&\lesssim \mu \mathrm{e}^{-\nu\tilde{\lambda}^\mu t} \|(\Delta_{\mathrm{wh}}, \phi(\hat{W}) - \phi(\hat{W} - \Delta_{\mathrm{c}}), \Delta_{\mathrm{sh}})\|_{E_{\nu\tilde{\lambda}^\mu,+}} \\
&\quad + \frac{1}{\nu\tilde{\lambda}^\mu} \|Q_{\mathrm{c}}\|_{C_{\nu\tilde{\lambda}^\mu,+}(\mathbb{R};\mathbb{R}^{2d})} \mathrm{e}^{-\nu\tilde{\lambda}^\mu t}
\end{aligned} \tag{5.78}$$

and

$$\begin{aligned}
\Big| \frac{\mathrm{d}}{\mathrm{d}t} \mathcal{K}_{\mathrm{c}}(\Delta_{\mathrm{wh}}, \Delta_{\mathrm{c}}, \Delta_{\mathrm{sh}})(t) \Big| \leq &\mu \mathrm{e}^{-\nu\tilde{\lambda}^\mu t} \|(\Delta_{\mathrm{wh}}, \phi(\hat{W}) - \phi(\hat{W} - \Delta_{\mathrm{c}}), \Delta_{\mathrm{sh}})\|_{E_{\nu\tilde{\lambda}^\mu,+}} \\
&+ \|Q_{\mathrm{c}}\|_{C_{\nu\tilde{\lambda}^\mu,+}(\mathbb{R};\mathbb{R}^{2d})} \mathrm{e}^{-\nu\tilde{\lambda}^\mu t}
\end{aligned} \tag{5.79}$$

for $t \in \mathbb{R}$, so that $\mathcal{K}_{\mathrm{c}}(\Delta_{\mathrm{wh}}, \Delta_{\mathrm{c}}, \Delta_{\mathrm{sh}}) \in C^1_{-\nu\tilde{\lambda}^\mu,+}(\mathbb{R}; \mathbb{R}^{2d})$.

For $\mathcal{K}_{\mathrm{sh}}(\Delta_{\mathrm{wh}}, \Delta_{\mathrm{c}}, \Delta_{\mathrm{sh}})$ we first note that it follows from Proposition 2.7, Remarks 2.11, 4.8 and 4.16, and estimate (5.7) combined with the facts that all second derivatives of $\tilde{G}^\mu_{\mathrm{sh}}$ and $\tilde{H}^\mu_{\mathrm{sh}}$ are $\mathcal{O}(1)$ that

$$\|\tilde{G}^\mu_{\mathrm{sh}}(Z_1, W_1, U_1) - \tilde{G}^\mu_{\mathrm{sh}}(Z_2, W_2, U_2)\|_{C^\alpha_{\nu\tilde{\lambda}^\mu,+}(\mathbb{R}, \mathcal{D}_{\mathrm{sh}})} \lesssim \mu \|(Z_1, W_1, U_1) - (Z_2, W_2, U_2)\|_{E_{\nu\tilde{\lambda}^\mu,+}}$$

and

$$\|\tilde{H}^\mu_{\mathrm{sh}}(Z_1) - \tilde{H}^\mu_{\mathrm{sh}}(Z_2)\|_{C^\alpha_{\nu\tilde{\lambda}^\mu,+}(\mathbb{R}, \mathcal{X}_{\mathrm{c,sh_{c,sh}}})} \lesssim \mu \|Z_1 - Z_2\|_{C^\alpha_{\nu\tilde{\lambda}^\mu,+}(\mathbb{R}, \mathbb{R}^n)}$$

for all $(Z_1, W_1, U_1), (Z_2, W_2, U_2) \in B_{\nu\tilde{\lambda}^\mu,+}$. Using these estimates together with Corollary 2.16 and Proposition 5.13 we obtain that

$$\begin{aligned}
\|\mathcal{K}_{\mathrm{sh}}(\Delta_{\mathrm{wh}}, \Delta_{\mathrm{c}}, \Delta_{\mathrm{sh}})\|_{C^\alpha_{\nu\tilde{\lambda}^\mu,+}(\mathbb{R}, \mathcal{D}_{\mathrm{sh}})} &\lesssim \|\tilde{G}^\mu_{\mathrm{sh}}(\hat{Z}, \hat{W}, \hat{U}) - \tilde{G}^\mu_{\mathrm{sh}}(\hat{Z} - \Delta_{\mathrm{wh}}, \hat{W} - \Delta_{\mathrm{c}}, \hat{U} - \Delta_{\mathrm{sh}}) \\
&\qquad + \tilde{H}^\mu_{\mathrm{sh}}(\hat{Z}) - \tilde{H}^\mu_{\mathrm{sh}}(\hat{Z} - \Delta_{\mathrm{wh}}) + Q_{\mathrm{sh}}\|_{C^\alpha_{\nu\tilde{\lambda}^\mu,+}(\mathbb{R}, \mathcal{X}_{\mathrm{sh}})} \\
&\lesssim \mu \|(\Delta_{\mathrm{wh}}, \phi(\hat{W}) - \phi(\hat{W} - \Delta_{\mathrm{c}}), \Delta_{\mathrm{sh}})\|_{E_{\nu\tilde{\lambda}^\mu,+}} \\
&\qquad + \|Q_{\mathrm{sh}}\|_{C^\alpha_{\nu\tilde{\lambda}^\mu,+}(\mathbb{R}, \mathcal{X}_{\mathrm{sh}})} \\
&\lesssim \mu \|(\Delta_{\mathrm{wh}}, \phi(\hat{W}) - \phi(\hat{W} - \Delta_{\mathrm{c}}), \Delta_{\mathrm{sh}})\|_{E_{\nu\tilde{\lambda}^\mu,+}} \\
&\qquad + \|Q_{\mathrm{sh}}\|_{C^\alpha_{\nu\tilde{\lambda}^\mu,+}(\mathbb{R}, \mathcal{X}_{\mathrm{sh}})},
\end{aligned} \tag{5.80}$$

so that $\mathcal{K}_{\mathrm{sh}}(\Delta_{\mathrm{wh}}, \Delta_{\mathrm{c}}, \Delta_{\mathrm{sh}}) \in C^{\alpha}_{\nu\tilde{\lambda}^{\mu},+}(\mathbb{R}, \mathcal{D}_{\mathrm{sh}})$, where we also have used Proposition 2.7.

Now we show that $\mathcal{K}$ maps $B_{\nu\tilde{\lambda}^{\mu},+}$ into itself. Repeating the arguments used in estimates (5.74) and (5.75) with $\nu = 0$ yields that

$$|\mathcal{K}_{\mathrm{wh}}(\Delta_{\mathrm{wh}}, \Delta_{\mathrm{c}}, \Delta_{\mathrm{sh}})(t)| \lesssim \mu\|(\Delta_{\mathrm{wh}}, \phi(\hat{W}) - \phi(\hat{W} - \Delta_{\mathrm{c}}), \Delta_{\mathrm{sh}})\|_{C^{\alpha}_{\mathrm{b}}(\mathbb{R};\mathcal{D})} + \mu \mathrm{e}^{-\frac{c^*}{2\mu}}$$

for all $t \in \mathbb{R}$, and from estimates (5.76) and (5.77) we obtain

$$\begin{aligned} &|\mathcal{K}_{\mathrm{wh}}(\Delta_{\mathrm{wh}}, \Delta_{\mathrm{c}}, \Delta_{\mathrm{sh}})(t_1) - \mathcal{K}_{\mathrm{wh}}(\Delta_{\mathrm{wh}}, \Delta_{\mathrm{c}}, \Delta_{\mathrm{sh}})(t_2)| \\ &\qquad \lesssim \mu\|(\Delta_{\mathrm{wh}}, \phi(\hat{W}) - \phi(\hat{W} - \Delta_{\mathrm{c}}), \Delta_{\mathrm{sh}})\|_{C^{\alpha}_{\mathrm{b}}(\mathbb{R};\mathcal{D})}|t_1 - t_2|^{\alpha} + \mu \mathrm{e}^{-\frac{c^*}{2\mu}}|t_1 - t_2|^{\alpha} \end{aligned}$$

for all $-\infty < t_2 < t_1 < \infty$. Combining the two previous estimates yields

$$\|\mathcal{K}_{\mathrm{wh}}(\Delta_{\mathrm{wh}}, \Delta_{\mathrm{c}}, \Delta_{\mathrm{sh}})\|_{C^{\alpha}_{\mathrm{b}}(\mathbb{R};\mathbb{R}^n)} \lesssim \mu \mathrm{e}^{-\frac{c^*}{2\mu}}. \tag{5.81}$$

Using Remarks 4.8 and 4.16 and Proposition 5.13 again we find that

$$\begin{aligned} &\Big|\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{K}_{\mathrm{c}}(\Delta_{\mathrm{wh}}, \Delta_{\mathrm{c}}, \Delta_{\mathrm{sh}})(t) - L^{\mu}_{\mathrm{c}}\mathcal{K}_{\mathrm{c}}(\Delta_{\mathrm{wh}}, \Delta_{\mathrm{c}}, \Delta_{\mathrm{sh}})(t)\Big| \\ &\quad = |\tilde{G}^{\mu}_{\mathrm{c}}(\hat{Z}, \hat{W}, \hat{U})(t) + \tilde{H}^{\mu}_{\mathrm{c}}(\hat{Z})(t) \\ &\qquad - \tilde{G}^{\mu}_{\mathrm{c}}(\hat{Z} - \Delta_{\mathrm{wh}}, \hat{W} - \Delta_{\mathrm{c}}, \hat{U} - \Delta_{\mathrm{sh}})(t) - \tilde{H}^{\mu}_{\mathrm{c}}(\hat{Z} - \Delta_{\mathrm{wh}})(t) + Q_{\mathrm{c}}(t)| \\ &\quad \lesssim \mu\|(\Delta_{\mathrm{wh}}, \phi(\hat{W}) - \phi(\hat{W} - \Delta_{\mathrm{c}}), \Delta_{\mathrm{sh}})\|_{E_{\nu\tilde{\lambda}^{\mu},+}} + \mu^2 \mathrm{e}^{-\frac{c^*}{2\mu}} \end{aligned}$$

for all $t \in \mathbb{R}$, so that

$$\Big\|\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{K}_{\mathrm{c}}(\Delta_{\mathrm{wh}}, \Delta_{\mathrm{c}}, \Delta_{\mathrm{sh}}) - L^{\mu}_{\mathrm{c}}\mathcal{K}_{\mathrm{c}}(\Delta_{\mathrm{wh}}, \Delta_{\mathrm{c}}, \Delta_{\mathrm{sh}})\Big\|_{C_{\mathrm{b}}(\mathbb{R};\mathbb{R}^{2d})} \lesssim \mu \mathrm{e}^{-\frac{c^*}{2\mu}}.$$

Finally, using Proposition 5.13 and the same arguments as in estimate (5.80) we obtain

$$\begin{aligned} \|\mathcal{K}_{\mathrm{sh}}(\Delta_{\mathrm{wh}}, \Delta_{\mathrm{c}}, \Delta_{\mathrm{sh}})\|_{C^{\alpha}_{\mathrm{b}}(\mathbb{R};\mathcal{D}_{\mathrm{sh}})} &\lesssim \mu\|(\Delta_{\mathrm{wh}}, \phi(\hat{W}) - \phi(\hat{W} - \Delta_{\mathrm{c}}), \Delta_{\mathrm{sh}})\|_{C^{\alpha}_{\mathrm{b}}(\mathbb{R};\mathcal{D})} + \mu^2 \mathrm{e}^{-\frac{c^*}{2\mu}} \\ &\lesssim \mu \mathrm{e}^{-\frac{c^*}{2\mu}}, \end{aligned}$$

so that $\mathcal{K}(\Delta_{\mathrm{wh}}, \Delta_{\mathrm{c}}, \Delta_{\mathrm{sh}}) \in B_{\nu\tilde{\lambda}^{\mu},+}$.

The fact that $\mathcal{K}\colon B_{\nu\tilde{\lambda}^{\mu},+} \to B_{\nu\tilde{\lambda}^{\mu},+}$ is a contraction is established by repeating the above arguments with $(\hat{Z}, \hat{W}, \hat{U})$ and $(\hat{Z} - \Delta_{\mathrm{wh}}, \hat{W} - \Delta_{\mathrm{c}}, \hat{U} - \Delta_{\mathrm{sh}})$ by respectively $(\Delta^{(1)}_{\mathrm{wh}}, \Delta^{(1)}_{\mathrm{c}}, \Delta^{(1)}_{\mathrm{sh}})$ and $(\Delta^{(2)}_{\mathrm{wh}}, \Delta^{(2)}_{\mathrm{c}}, \Delta^{(2)}_{\mathrm{sh}})$. □

**Corollary 5.15.** The operator $\mathcal{K}\colon B_{\nu\tilde{\lambda}^{\mu},+} \to B_{\nu\tilde{\lambda}^{\mu},+}$ has a unique fixed point denoted by $(\Delta_{\mathrm{wh}}, \Delta_{\mathrm{c}}, \Delta_{\mathrm{sh}})$ which satisfies the estimates

$$\begin{aligned} \|\Delta_{\mathrm{wh}}\|_{C^{\alpha}_{\mathrm{b}}([0,\infty);\mathbb{R}^n)} &\lesssim \mu \mathrm{e}^{-\frac{c^*}{2\mu}}, \\ \|\Delta_{\mathrm{sh}}\|_{C^{\alpha}_{\mathrm{b}}(\mathbb{R};\mathcal{D}_{\mathrm{sh}})} &\lesssim \mu \mathrm{e}^{-\frac{c^*}{2\mu}}, \end{aligned}$$

and

$$\|(\Delta_{\mathrm{wh}}, \Delta_{\mathrm{c}}, \Delta_{\mathrm{sh}})\|_{E_{\nu\tilde{\lambda}^{\mu},+}} \lesssim \mu \mathrm{e}^{\nu\tilde{\lambda}^{\mu}\frac{t^*}{2}}.$$

Furthermore

$$(Z, W, U) = (\hat{Z}, \hat{W}, \hat{U}) - (\Delta_{\mathrm{wh}}, \Delta_{\mathrm{c}}, \Delta_{\mathrm{sh}})$$

lies on $W^{\mathrm{c}}$.

*Proof.* The first assertion follows from Lemma 5.14 and the fact that

$$\|\mathcal{K}(0)\|_{E_{\nu\tilde{\lambda}^\mu,+}} \lesssim \mu e^{\nu\tilde{\lambda}^\mu \frac{t^*}{2}}, \quad \mathrm{Lip}_{B_{\nu\tilde{\lambda}^\mu,+}} \mathcal{K} < \frac{1}{2}$$

(see estimates (5.78) – (5.81)).

The function

$$(Z, W, U) = (\hat{Z}, \hat{W}, \hat{U}) - (\Delta_{\mathrm{wh}}, \Delta_{\mathrm{c}}, \Delta_{\mathrm{sh}})$$

is by construction a solution of equations (5.56) – (5.58) and because of

$$\|Z\|_{C_{\mathrm{b}}^\alpha(\mathbb{R};\mathbb{R}^n)} \leq \|\hat{Z}\|_{C_{\mathrm{b}}^\alpha(\mathbb{R};\mathbb{R}^n)} + \|\Delta_{\mathrm{wh}}\|_{C_{\mathrm{b}}^\alpha(\mathbb{R};\mathbb{R}^n)} \lesssim \mu e^{-\frac{c^*}{2\mu}},$$
$$\|U\|_{C_{\mathrm{b}}^\alpha(\mathbb{R};\mathcal{D}_{\mathrm{sh}})} \leq \|\hat{U}\|_{C_{\mathrm{b}}^\alpha(\mathbb{R};\mathbb{R}^n)} + \|\Delta_{\mathrm{sh}}\|_{C_{\mathrm{b}}^\alpha(\mathbb{R};\mathcal{D}_{\mathrm{sh}})} \lesssim \mu e^{-\frac{c^*}{2\mu}},$$
$$\|\dot{W} - L_{\mathrm{c}}^\mu W\|_{C_{\mathrm{b}}(\mathbb{R};\mathbb{R}^{2d})} \leq \|\dot{\hat{W}} - L_{\mathrm{c}}^\mu \hat{W}\|_{C_{\mathrm{b}}(\mathbb{R};\mathbb{R}^{2d})} + \|\dot{\Delta}_{\mathrm{c}} - L_{\mathrm{c}}^\mu \Delta_{\mathrm{c}}\|_{C_{\mathrm{b}}(\mathbb{R};\mathbb{R}^{2d})} \lesssim \mu e^{-\frac{c^*}{2\mu}}$$

it lies on $W^{\mathrm{c}}$. □

It remains to deduce the assertion given at the beginning of this section.

**Theorem 5.16.** The function $W_{W_0}^*$ satisfies

$$|W_{W_0}^*(t)| \leq e^{-\frac{c^*}{2\mu}}$$

for all $t \in \mathbb{R}$.

*Proof.* In the notation of Corollary 5.15, we observe that

$$\begin{aligned} |W_{W_0}^*(t) - W(t)| &= |\hat{W}(t) - W(t)| \\ &\leq \|(\Delta_{\mathrm{wh}}, \Delta_{\mathrm{c}}, \Delta_{\mathrm{sh}})(t)\|_{\mathcal{D}} \\ &\leq \|(\Delta_{\mathrm{wh}}, \Delta_{\mathrm{c}}, \Delta_{\mathrm{sh}})\|_{E_{\nu\tilde{\lambda}^\mu,+}} e^{-\nu\tilde{\lambda}^\mu t} \\ &\lesssim \mu e^{\nu\tilde{\lambda}^\mu \frac{t^*}{2}} e^{-\nu\tilde{\lambda}^\mu t} \\ &\leq \mu e^{-\nu\tilde{\lambda}^\mu \frac{t^*}{2}} \\ &= \mu e^{-\frac{c^*}{2\mu}} \end{aligned} \tag{5.82}$$

for $t \geq t^*$. On the other hand, by construction

$$|W_{W_0}^*(t)| \lesssim \mu e^{-\frac{c^*}{2\mu}}$$

for $t \in [0, e^{\frac{c^*}{2\mu}}] \supseteq [0, t^*]$, so that $|W(t^*)| \lesssim \frac{1}{2} e^{-\frac{c^*}{2\mu}}$ and hence by Lemma 5.11

$$|W(t)| \leq \frac{3}{4} e^{-\frac{c^*}{2\mu}} \tag{5.83}$$

for $t \geq t^*$. Combining estimates (5.82) and (5.83) we find that

$$|W_{W_0}^*(t)| \leq e^{-\frac{c^*}{2\mu}}$$

for $t \geq t^*$. Since $W_{W_0}^*$ is symmetric, we have that

$$|W_{W_0}^*(t)| \leq e^{-\frac{c^*}{2\mu}}$$

for all $t \in \mathbb{R}$. □

# 6 Applications

## 6.1 Generalities

In this chapter we apply Theorems 1.1 and 1.2 to the evolutionary equation

$$\dot{v} = \check{L}^\varepsilon v + \check{N}^\varepsilon(v) \tag{6.1}$$

for $v\colon \mathbb{R} \to \mathcal{Y}$, where $\mathcal{X}, \mathcal{Y}$ are Banach spaces with $\mathcal{Y}$ continuously and densely embedded in $\mathcal{X}$. Let $\{\mathcal{D}^\varepsilon\}_{\varepsilon\in\mathbb{R}}$ be a family of closed subspaces of $\mathcal{Y}$ and suppose that

(i) the linear and nonlinear functions $\check{L}^{(\cdot)}$ and $\check{N}^{(\cdot)}$ take values in $\mathcal{X}$ and are analytic at the origin in $\mathbb{R} \times \mathcal{Y}$ with

$$\|\check{N}^\varepsilon(v)\|_{\mathcal{X}} = \mathcal{O}(\|v\|_{\mathcal{Y}}^2),$$

(ii) the linear operator $\check{L}^\varepsilon\colon \mathcal{D}^\varepsilon \subseteq \mathcal{X} \to \mathcal{X}$ is closed, densely defined and depends analytically upon $\varepsilon$ in the sense of Kato [13, VII-§2]: there exists $\check{T}^\varepsilon \in \mathcal{L}(\mathcal{D}^0; \mathcal{Y})$ such that $\check{T}^\varepsilon\colon \mathcal{D}^0 \to \mathcal{D}^\varepsilon$ is a bijection and $\check{T}^\varepsilon \in \mathcal{L}(\mathcal{D}^0; \mathcal{Y})$ (and hence $\check{L}^\varepsilon \check{T}^\varepsilon \in \mathcal{L}(\mathcal{D}^0; \mathcal{X})$) depends analytically upon $\varepsilon$.

The spectral hypotheses on $\check{L}^\varepsilon$ are that

(i) it has finitely many simple purely imaginary eigenvalues $\pm\mathrm{i}\omega_1^\varepsilon, \ldots, \pm\omega_d^\varepsilon$, where $\omega_1^\varepsilon, \ldots, \omega_d^\varepsilon > 0$,

(ii) it exhibits a $0^2$ resonance (a pair of imaginary eigenvalues become real by colliding at the origin) or an $(\mathrm{i}\omega)^2$ resonance (two pairs of imaginary eigenvalues become complex by colliding on the imaginary axis) at $\varepsilon = 0$,

(iii) the rest of $\sigma(\check{L}^\varepsilon)$ is bounded away from the imaginary axis, uniformly in $\varepsilon$.

Let $\gamma_\mathrm{c}$ and $\gamma_\mathrm{wh}$ be simple closed curves enclosing respectively $\pm\mathrm{i}\omega_1^\varepsilon, \ldots, \pm\mathrm{i}\omega_d^\varepsilon$ and the colliding eigenvalues in the $0^2$ or $(\mathrm{i}\omega)^2$ resonance and no other points of $\sigma(\check{L}^\varepsilon)$. Define the corresponding spectral projections by the formulae

$$\begin{aligned} P_\mathrm{wh}^\varepsilon v &= \frac{1}{2\pi\mathrm{i}} \int_{\gamma_\mathrm{wh}} (\lambda I - \check{L}^\varepsilon)^{-1} v \, \mathrm{d}\lambda, \\ P_\mathrm{c}^\varepsilon v &= \frac{1}{2\pi\mathrm{i}} \int_{\gamma_\mathrm{c}} (\lambda I - \check{L}^\varepsilon)^{-1} v \, \mathrm{d}\lambda, \\ P_\mathrm{sh}^\varepsilon v &= (I - P_\mathrm{c}^\varepsilon - P_\mathrm{wh}^\varepsilon) v. \end{aligned}$$

Defining $\mathcal{X}^\varepsilon_{\mathrm{wh}} = P^\varepsilon_{\mathrm{wh}}\mathcal{X}$, $\mathcal{X}^\varepsilon_{\mathrm{c}} = P^\varepsilon_{\mathrm{c}}\mathcal{X}$, $\mathcal{X}^\varepsilon_{\mathrm{sh}} = P^\varepsilon_{\mathrm{sh}}\mathcal{X}$ (with similar notation for $\mathcal{Y}$ and $\mathcal{D}^\varepsilon$), we can rewrite equation (6.1) as the system

$$\dot{v}_{\mathrm{wh}} = \check{L}^\varepsilon_{\mathrm{wh}} v_{\mathrm{wh}} + \check{N}^\varepsilon_{\mathrm{wh}}(v_{\mathrm{wh}}, v_{\mathrm{c}}, v_{\mathrm{sh}}), \tag{6.2}$$

$$\dot{v}_{\mathrm{c}} = \check{L}^\varepsilon_{\mathrm{c}} v_{\mathrm{c}} + \check{N}^\varepsilon_{\mathrm{c}}(v_{\mathrm{wh}}, v_{\mathrm{c}}, v_{\mathrm{sh}}), \tag{6.3}$$

$$\dot{v}_{\mathrm{sh}} = \check{L}^\varepsilon_{\mathrm{sh}} v_{\mathrm{sh}} + \check{N}^\varepsilon_{\mathrm{sh}}(v_{\mathrm{wh}}, v_{\mathrm{c}}, v_{\mathrm{sh}}) \tag{6.4}$$

for $v_{\mathrm{wh}} = P^\varepsilon_{\mathrm{wh}} v$, $v_{\mathrm{c}} = P^\varepsilon_{\mathrm{c}} v$, $v_{\mathrm{sh}} = P^\varepsilon_{\mathrm{sh}} v$ (with the obvious definitions of $\check{L}^\varepsilon_{\mathrm{wh}}$, $\check{L}^\varepsilon_{\mathrm{c}}$, $\check{L}^\varepsilon_{\mathrm{sh}}$ and $\check{N}^\varepsilon_{\mathrm{wh}}$, $\check{N}^\varepsilon_{\mathrm{c}}$, $\check{N}^\varepsilon_{\mathrm{sh}}$). The dependence of the function spaces upon $\varepsilon$ can be removed using the following result (Kato [13, II-§4]).

**Proposition 6.1.** Let $B$ be a Banach space and $P^\varepsilon_1, \ldots, P^\varepsilon_l \in \mathcal{B}(B)$ be projections which depend analytically upon $\varepsilon$ in a neighbourhood of the origin and satisfy

$$P^\varepsilon_i P^\varepsilon_j = \delta_{ij} P^\varepsilon_i$$

for $i, j \in \{1, \ldots, l\}$ and

$$\sum_{i=1}^{l} P^\varepsilon_i = I.$$

There exists $Q^\varepsilon \in \mathcal{L}(B)$ such that

$$Q^\varepsilon P^0_i (Q^\varepsilon)^{-1} = P^\varepsilon_i$$

for $i \in \{1, \ldots, l\}$. Furthermore $Q^\varepsilon$ and $(Q^\varepsilon)^{-1}$ depend analytically upon $\varepsilon$.

Applying Proposition 6.1 with $B = \mathcal{L}(\mathcal{X}) \cap \mathcal{L}(\mathcal{Y})$ and $P^\varepsilon_1 = P^\varepsilon_{\mathrm{wh}}$, $P^\varepsilon_2 = P^\varepsilon_{\mathrm{c}}$, $P^\varepsilon_3 = P^\varepsilon_{\mathrm{sh}}$ yields $Q^\varepsilon \in \mathcal{L}(\mathcal{X}) \cap \mathcal{L}(\mathcal{Y})$ such that

$$\begin{aligned} Q^\varepsilon P^0_{\mathrm{wh}} (Q^\varepsilon)^{-1} &= P^\varepsilon_{\mathrm{wh}}, \\ Q^\varepsilon P^0_{\mathrm{c}} (Q^\varepsilon)^{-1} &= P^\varepsilon_{\mathrm{c}}, \\ Q^\varepsilon P^0_{\mathrm{sh}} (Q^\varepsilon)^{-1} &= P^\varepsilon_{\mathrm{sh}}. \end{aligned}$$

Applying the equations $P^0_{\mathrm{wh}}(Q^\varepsilon)^{-1} = (Q^\varepsilon)^{-1} P^\varepsilon_{\mathrm{wh}}$ and $P^\varepsilon_{\mathrm{wh}}(Q^\varepsilon) = Q^\varepsilon P^0_{\mathrm{wh}}$ to $\mathcal{X}$ and $\mathcal{Y}$, we find that $Q^\varepsilon[\mathcal{X}^0_{\mathrm{wh}}] = \mathcal{X}^\varepsilon_{\mathrm{wh}}$ and $Q^\varepsilon[\mathcal{D}^0_{\mathrm{wh}}] = \mathcal{D}^\varepsilon_{\mathrm{wh}}$ and similarly $Q^\varepsilon[\mathcal{X}^0_{\mathrm{c}}] = \mathcal{X}^\varepsilon_{\mathrm{c}}$, $Q^\varepsilon[\mathcal{X}^0_{\mathrm{sh}}] = \mathcal{X}^\varepsilon_{\mathrm{sh}}$ and $Q^\varepsilon[\mathcal{D}^0_{\mathrm{c}}] = \mathcal{D}^\varepsilon_{\mathrm{c}}$, $Q^\varepsilon[\mathcal{D}^0_{\mathrm{sh}}] = \mathcal{D}^\varepsilon_{\mathrm{sh}}$. The change of variable

$$v_{\mathrm{wh}} = Q^\varepsilon z, \quad v_{\mathrm{c}} = Q^\varepsilon w, \quad v_{\mathrm{sh}} = Q^\varepsilon u,$$

transforms equations (6.2) – (6.4) into

$$\dot{z} = L^\varepsilon_{\mathrm{wh}} z + N^\varepsilon_{\mathrm{wh}}(z, w, u), \tag{6.5}$$

$$\dot{w} = L^\varepsilon_{\mathrm{c}} w + N^\varepsilon_{\mathrm{c}}(z, w, u), \tag{6.6}$$

$$\dot{u} = L^\varepsilon_{\mathrm{sh}} u + N^\varepsilon_{\mathrm{sh}}(z, w, u), \tag{6.7}$$

where

$$\begin{aligned} L^\varepsilon_{\mathrm{wh}} &= (Q^\varepsilon)^{-1} \check{L}^\varepsilon_{\mathrm{wh}} Q^\varepsilon, \\ N^\varepsilon_{\mathrm{wh}} &= (Q^\varepsilon)^{-1} \check{N}^\varepsilon_{\mathrm{wh}}(Q^\varepsilon z, Q^\varepsilon w, Q^\varepsilon u) \end{aligned}$$

with corresponding definitions for $L^\varepsilon_{\mathrm{c}}$, $L^\varepsilon_{\mathrm{sh}}$ and $N^\varepsilon_{\mathrm{c}}$, $N^\varepsilon_{\mathrm{sh}}$. We can also write

$$N^\varepsilon_{\mathrm{wh}}(z,w,u) = g^\varepsilon_{\mathrm{wh}}(z,w,u) + h^\varepsilon_{\mathrm{wh}}(z),$$

where $h_{\mathrm{wh}}(z) = N^\varepsilon(z,0,0)$, with corresponding definitions for $g^\varepsilon_{\mathrm{c}}$, $g^\varepsilon_{\mathrm{sh}}$ and $h^\varepsilon_{\mathrm{c}}$, $h^\varepsilon_{\mathrm{sh}}$. Equations (6.5) – (6.7) are of the form (1.27) – (1.29) with $\mathcal{X}^0_{\mathrm{wh}}, \mathcal{D}^0_{\mathrm{wh}} \cong \mathbb{R}^n$, $\mathcal{X}^0_{\mathrm{c}}, \mathcal{D}^0_{\mathrm{c}} \cong \mathbb{R}^{2d}$ and $\mathcal{X}_{\mathrm{sh}} = \mathcal{X}^0_{\mathrm{sh}}$, $\mathcal{D}_{\mathrm{sh}} = \mathcal{D}^0_{\mathrm{sh}}$. Note that the spectra of $\check{L}^\varepsilon_{\mathrm{wh}}$ and $L^\varepsilon_{\mathrm{wh}}$, $\check{L}^\varepsilon_{\mathrm{c}}$ and $L^\varepsilon_{\mathrm{c}}$, $\check{L}^\varepsilon_{\mathrm{sh}}$ and $L^\varepsilon_{\mathrm{sh}}$ are identical, and furthermore (6.2) – (6.4) and (6.5) – (6.7) coincide for $\varepsilon = 0$.

We now turn to the special case that the linearised equation

$$\dot{v} = \check{L}^0 v$$

represents Hamilton's equations for a linear Hamiltonian system $(\mathcal{X}, \Omega, H^0)$. Here the symplectic 2-form $\Omega$ is a bounded, bilinear, skew-symmetric mapping $\mathcal{X} \times \mathcal{X} \to \mathbb{R}$ and the Hamiltonian $H^0 \colon \mathcal{X} \to \mathbb{R}$ is a functional which is homogeneous of degree 2; they have the property that

$$\Omega(\check{L}^0 v, w) = \mathrm{d}H^0[v](w)$$

for all $w \in \mathcal{X}$. The following result gives a simple representation of the spectral projections $P^0_{\mathrm{c}}$ and $P^0_{\mathrm{wh}}$ (see Mielke [21, §3.1]).

**Proposition 6.2.**

(i) Let $e_1, \bar{e}_1, \ldots, e_n, \bar{e}_n$ be eigenvectors corresponding to the eigenvalues $\pm \mathrm{i}\omega^0_1, \ldots, \pm \mathrm{i}\omega^0_n$, normalised such that $\Omega(e_i, \bar{e}_i) = \pm \mathrm{i}$, where $\Omega$ is extended bilinearly to the complexification of $\mathcal{X}$. The spectral projection $P^0_{\mathrm{c}}$ is given by

$$P^0_{\mathrm{c}} v = \sum_{i=1}^n \Big( s_i \Omega(v, \bar{e}_i) e_i - s_i \Omega(v, e_i) \bar{e}_i \Big),$$

where $s_i = -\Omega(e_i, \bar{e}_i)$.

(ii) Suppose that $\check{L}^\varepsilon$ exhibits a $0^2$ resonance at $\varepsilon = 0$. Let $f_1$, $f_2$ satisfy $\check{L}^0 f_1 = 0$, $\check{L}^0 f_2 = f_1$ and $\Omega(f_1, f_2) = 1$. The spectral projection $P^0_{\mathrm{wh}}$ is given by

$$P^0_{\mathrm{wh}} v = \Omega(v, f_2) f_1 - \Omega(v, f_1) f_2.$$

(iii) Suppose that $\check{L}^\varepsilon$ exhibits an $(\mathrm{i}\omega)^2$ resonance at $\varepsilon = 0$. Let $e$, $f$ satisfy $\check{L}^0 e = \mathrm{i}\omega e$, $(\check{L}^0 - \mathrm{i}\omega I) f = e$ and $\Omega(e, \bar{f}) = 1$, $\Omega(\bar{e}, f) = -1$, all other symplectic products being zero (and $\Omega$ is extended bilinearly to the complexification of $\mathcal{X}$). The spectral projection $P^0_{\mathrm{wh}}$ is given by

$$\begin{aligned} P^0_{\mathrm{wh}} v = \Omega(v, \bar{f}) e - \Omega(v, \bar{e}) f \\ + \Omega(v, f) \bar{e} - \Omega(v, e) \bar{f}. \end{aligned}$$

## 6.2 Generalised solitary waves on rotational flows

In this section we consider gravity-driven steady waves on the surface of water in a uniform rectangular channel bounded below by a rigid horizontal bottom and above by a free surface. In a Cartesian coordinate system moving with the wave the fluid domain is

$$\{(x, y) : x \in \mathbb{R}, 0 < y < \eta(x)\}$$

for some profile function $\eta\colon \mathbb{R} \to (0, \infty)$. Working in dimensionless coordinates, we seek the velocity field in the form $(\psi_y, -\psi_x)$, where the *stream function* $\psi\colon \mathbb{R}^2 \to \mathbb{R}$ satisfies the boundary-value problem

$$\psi_{xx} + \psi_{yy} + \omega^\varepsilon(\psi) = 0, \qquad 0 < y < \eta, \tag{6.8}$$

$$\psi = 0, \qquad y = 0, \tag{6.9}$$

$$\psi = 1, \qquad y = \eta, \tag{6.10}$$

$$\psi_x^2 + \psi_y^2 + 2\eta = 3r, \qquad y = \eta \tag{6.11}$$

(see Keady and Norbury [14]). Here the *vorticity function* $\omega^{(\cdot)}$ is a real-valued function which is analytic at the origin in $\mathbb{R} \times \mathbb{R}$ and $r$ is a parameter referred to as the *Bernoulli constant.* A *solitary wave* is a solution $(\eta, \psi)$ to equations (6.8) – (6.11) such that $\eta$ decays to a constant, while a *generalised solitary wave* instead decays to small ripples far up- and downstream. Solitary waves were found by Kozlov et al. [19] under the assumption that $\omega$ is a large negative constant. In this section we apply the results of Chapter 5 to establish the existence of generalised solitary waves with exponentially small tails for linear vorticity functions (see Figure 6.1).

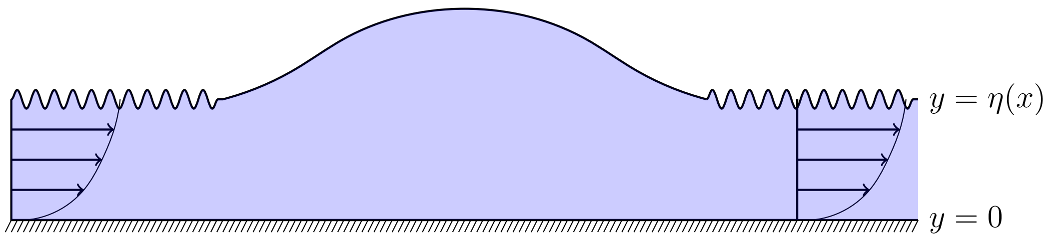



Figure 6.1: A generalised solitary wave on the surface of a stream solution.

### 6.2.1 Formulation as an evolutionary system

We begin by transforming equations (6.8) – (6.11) to a boundary value problem on the fixed strip $\mathbb{R} \times [0, h]$. We achieve this aim by introducing the new coordinates

$$\tilde{x} = x, \quad \tilde{y} = \frac{h}{\eta(x)} y$$

and variable

$$\Phi(\tilde{x}, \tilde{y}) = \psi(x, y),$$

to transform equations (6.8) – (6.11) into

$$\Phi_{\tilde{x}\tilde{x}} + \frac{\eta^2}{h^2}\Phi_{\tilde{y}\tilde{y}} + \omega^\varepsilon(\Phi) = \frac{\eta_{\tilde{x}}}{\eta}\Big(\tilde{y}\Phi_{\tilde{x}} - \frac{\eta_{\tilde{x}}}{\eta}\tilde{y}^2\Phi_{\tilde{y}}\Big)_{\tilde{y}} + (\Phi_{\tilde{x}} - \frac{\eta_{\tilde{x}}}{\eta}\tilde{y}\Phi_{\tilde{y}})_{\tilde{x}}, \quad 0 < \tilde{y} < h, \tag{6.12}$$

$$\Phi = 0, \qquad \tilde{y} = 0, \tag{6.13}$$

$$\Phi = 1, \qquad \tilde{y} = h, \tag{6.14}$$

$$(\Phi_{\tilde{x}} - \frac{\eta_{\tilde{x}}}{\eta}\tilde{y}\Phi_{\tilde{y}})^2 + \frac{\eta^2}{h^2}\Phi_{\tilde{y}}^2 + 2\eta = 3r, \qquad \tilde{y} = h. \tag{6.15}$$

Introducing the additional variable

$$\Psi(\tilde{x}, \tilde{y}) = \frac{\eta}{h}(\Phi_{\tilde{x}} - \frac{\eta_{\tilde{x}}}{\eta}\tilde{y}\Phi_{\tilde{y}})$$

we find equations (6.12) – (6.15) to be equivalent to

$$\Phi_{\tilde{x}} = \frac{h}{\eta}\Psi + \frac{\eta_{\tilde{x}}}{\eta}\tilde{y}\Phi_{\tilde{y}}, \qquad 0 < \tilde{y} < h, \tag{6.16}$$

$$\Psi_{\tilde{x}} = \frac{\eta_{\tilde{x}}}{\eta}(\tilde{y}\Psi)_{\tilde{y}} - \frac{h}{\eta}\Phi_{\tilde{y}\tilde{y}} - \frac{\eta}{h}\omega^\varepsilon(\Phi), \qquad 0 < \tilde{y} < h, \tag{6.17}$$

$$\Phi = 0, \qquad \tilde{y} = 0, \tag{6.18}$$

$$\Psi = 0, \qquad \tilde{y} = 0, \tag{6.19}$$

$$\Phi = 1, \qquad \tilde{y} = h, \tag{6.20}$$

$$\Psi^2 + \Phi_{\tilde{y}}^2 = (3r - 2\eta)\Big(\frac{\eta}{h}\Big)^2, \qquad \tilde{y} = h. \tag{6.21}$$

From equation (6.20) we find that $\Phi_{\tilde{x}}(\cdot, h) = 0$, so that the definition of $\Psi$ yields

$$\eta_{\tilde{x}} = \frac{\Psi(\cdot, h)}{\Phi_{\tilde{y}}(\cdot, h)}. \tag{6.22}$$

Regarding equations (6.16), (6.17) and (6.22) as an evolutionary system in which $\tilde{x}$ is the time-like variable (spatial dynamics) and equations (6.18) – (6.21) are boundary conditions, we can show it has a conserved quantity.

**Proposition 6.3.** The system of equations (6.16) – (6.22) has the conserved quantity

$$H^\varepsilon(\eta, \Phi, \Psi) = \int_0^h \Big(\frac{h}{2\eta}(\Psi^2 + \Phi_{\tilde{y}}^2) + \frac{\eta}{h}\Omega^\varepsilon(\Phi)\Big)\,\mathrm{d}\tilde{y} + \frac{\eta^2}{2} - \frac{3}{2}r\eta,$$

where

$$\Omega^\varepsilon(t) = -\int_t^1 \omega^\varepsilon(s)\,\mathrm{d}s.$$

*Proof.* We find that

$$\begin{aligned} H^\varepsilon_{\tilde{x}}(\eta, \Phi, \Psi) = &\int_0^h \Big(\frac{h}{\eta}(\Psi\Psi_{\tilde{x}} - \Phi_{\tilde{y}}\Phi_{\tilde{x}\tilde{y}}) + \frac{\eta}{h}\omega^\varepsilon(\Phi)\Phi_{\tilde{x}}\Big)\,\mathrm{d}\tilde{y} \\ &+ \Big(\int_0^h \Big(\frac{-h}{2\eta^2}(\Psi^2 - \Phi_{\tilde{y}}^2) + \frac{1}{h}\Omega^\varepsilon(\Phi)\Big)\,\mathrm{d}\tilde{y} + \eta - \frac{3r}{2}\Big)\eta_{\tilde{x}} \end{aligned}$$

$$
\begin{aligned}
&= \int_0^h \Big( \frac{h\eta_{\tilde{x}}}{\eta^2}\Psi(\tilde{y}\Psi)_{\tilde{y}} - \frac{h^2}{\eta^2}\Psi\Phi_{\tilde{y}\tilde{y}} - \Psi\omega^\varepsilon(\Phi) - \frac{h^2}{\eta^2}\Phi_{\tilde{y}}\Psi_{\tilde{y}} - \frac{h}{\eta^2}\eta_{\tilde{x}}\Phi_{\tilde{y}}(\tilde{y}\Phi_{\tilde{y}})_{\tilde{y}} \\
&\qquad\qquad + \omega^\varepsilon(\Phi)\Psi + \frac{\tilde{y}}{h}\eta_{\tilde{x}}\omega^\varepsilon(\Phi)\Phi_{\tilde{y}}\Big) \\
&\qquad + \Big( \int_0^h \Big(\frac{-h}{2\eta^2}(\Psi^2 - \Phi_{\tilde{y}}^2) + \frac{1}{h}\Omega^\varepsilon(\Phi)\Big)\, \mathrm{d}\tilde{y} + \eta - \frac{3}{2}r\Big)\eta_{\tilde{x}} \\
&= \int_0^h \Big( \frac{h\eta_{\tilde{x}}}{\eta^2}\Big(\frac{1}{2}\tilde{y}\Psi^2\Big)_{\tilde{y}} - \frac{h\eta_{\tilde{x}}}{\eta^2}\Big(\frac{1}{2}\tilde{y}\Phi_{\tilde{y}}^2\Big)_{\tilde{y}} - \frac{h^2}{\eta^2}(\Psi\Phi_{\tilde{y}})_{\tilde{y}} + \frac{\eta_{\tilde{x}}}{h}\Big(\tilde{y}\Omega^\varepsilon(\Phi)\Big)_{\tilde{y}}\Big)\, \mathrm{d}\tilde{y} \\
&\qquad + \Big(\eta - \frac{3}{2}r\Big)\eta_{\tilde{x}} \\
&= \frac{h^2\eta_{\tilde{x}}}{\eta^2}\frac{1}{2}\Big(\Psi^2(\cdot,h) - \Phi_{\tilde{y}}^2(\cdot,h)\Big) - \frac{h^2}{\eta^2}\Psi(\cdot,h)\Phi_{\tilde{y}}(\cdot,h) \\
&\qquad + \eta_{\tilde{x}}\Omega^\varepsilon\Big(\Phi(\cdot,h)\Big) + \frac{h^2}{\eta^2}\Psi(\cdot,h)\Phi_{\tilde{y}}(\cdot,h) + \Big(\eta - \frac{3}{2}r\Big)\eta_{\tilde{x}} \\
&= \frac{h^2}{2\eta^2}\Big(\Psi^2(\cdot,h) + \Psi_{\tilde{y}}^2(\cdot,h)\Big)\eta_{\tilde{x}} + \Big(\eta - \frac{3}{2}r\Big)\eta_{\tilde{x}} \\
&= 0,
\end{aligned}
$$

where we have used the boundary conditions in the last two steps. □

We proceed by interpreting our flow as a perturbation of a *stream solution* $(\Lambda^\varepsilon, h)$ of equations (6.8) – (6.11), that is a solution $(\eta, \psi)$ with $\psi = \Lambda^\varepsilon(\tilde{y})$ and $\eta = h$, so that

$$
\begin{aligned}
(\Lambda^\varepsilon)'' + \omega^\varepsilon(\Lambda^\varepsilon) &= 0, && 0 < \tilde{y} < h, \\
\Lambda^\varepsilon &= 0, && y = 0, \\
\Lambda^\varepsilon &= 1, && y = h, \\
((\Lambda^\varepsilon)')^2 + 2h &= 3r, && y = h
\end{aligned}
$$

(see Kozlov and Kuznetsov [18] for a complete discussion of stream solutions); we assume that $\Lambda^\varepsilon \in H^2(0,h)$ depends analytically upon $\varepsilon$. Note that $(\eta, \Phi, \Psi) = (h, \Lambda^\varepsilon, 0)$ solves equations (6.16) – (6.22). We thus introduce the new variables

$$
\tilde{\Phi} = \Phi - \Lambda^\varepsilon - \tilde{y}\frac{(\Lambda^\varepsilon)'}{h}\zeta, \quad \tilde{\Psi} = \Psi, \quad \zeta = \eta - h.
$$

Since $(h, \Lambda^\varepsilon)$ satisfies equations (6.13) – (6.15) we find that

$$
\zeta = \frac{-\tilde{\Phi}(\cdot,h)}{(\Lambda^\varepsilon)'(h)}
$$

and

$$
\zeta_{\tilde{x}} = -\frac{h\tilde{\Psi}(\cdot,h)}{h\Big(\tilde{\Phi}_{\tilde{y}}(\cdot,h) + (\Lambda^\varepsilon)'(h)\Big) - \tilde{\Phi}(\cdot,h)\Big(1 - h\frac{\omega^\varepsilon(1)}{(\Lambda^\varepsilon)'(h)}\Big)}.
$$

Substituting $\Phi$ and $\Psi$ in equations (6.16) – (6.21), we obtain

$$
\tilde{\Phi}_{\tilde{x}} = \tilde{\Psi} + N_1^\varepsilon(\tilde{\Phi}, \tilde{\Psi}), \qquad 0 < \tilde{y} < h, \tag{6.23}
$$

$$
\tilde{\Psi}_{\tilde{x}} = -\tilde{\Phi}_{\tilde{y}\tilde{y}} - (\omega^\varepsilon)'(\Lambda^\varepsilon)\tilde{\Phi} + N_2^\varepsilon(\tilde{\Phi}, \tilde{\Psi}), \qquad 0 < \tilde{y} < h, \tag{6.24}
$$

$$\tilde{\Phi} = 0, \qquad y = 0, \tag{6.25}$$

$$\tilde{\Psi} = 0, \qquad y = 0, \tag{6.26}$$

$$\tilde{\Phi}_{\tilde{y}} - \kappa^\varepsilon \tilde{\Phi} = N_3^\varepsilon(\tilde{\Phi}, \tilde{\Psi}), \qquad y = h, \tag{6.27}$$

where the nonlinearities are given by

$$\begin{aligned}
N_1^\varepsilon(\tilde{\Phi}, \tilde{\Psi}) &= -\frac{\zeta}{\zeta + h}\tilde{\Psi} + \frac{y\zeta_{\tilde{x}}}{\zeta + h}\Big(\tilde{\Phi}_{\tilde{y}} + \frac{y}{h}(\Lambda^\varepsilon)''\zeta\Big),\\
N_2^\varepsilon(\tilde{\Phi}, \tilde{\Psi}) &= -\frac{\zeta}{h}(\omega^\varepsilon)'(\Lambda^\varepsilon)\tilde{\Phi} + \frac{\zeta_{\tilde{x}}}{\zeta + h}(\tilde{y}\tilde{\Psi})_{\tilde{y}}\\
&\quad + \frac{\zeta}{\zeta + h}\Big(\tilde{\Phi}_{\tilde{y}\tilde{y}} - \frac{\zeta}{h}\omega^\varepsilon(\Lambda^\varepsilon) - \frac{\zeta + 2h}{h}(\omega^\varepsilon)'(\Lambda^\varepsilon)(\Lambda^\varepsilon)'\tilde{y}\frac{\zeta}{h}\Big)\\
&\quad - \frac{\zeta + h}{h}\Big(\omega^\varepsilon\Big(\tilde{\Phi} + \Lambda^\varepsilon + \tilde{y}(\Lambda^\varepsilon)'\frac{\zeta}{h}\Big) - \omega^\varepsilon(\Lambda^\varepsilon) - (\omega^\varepsilon)'(\Lambda^\varepsilon)\Big(\tilde{\Phi} + \tilde{y}(\Lambda^\varepsilon)'\frac{\zeta}{h}\Big)\Big),\\
N_3^\varepsilon(\tilde{\Phi}, \tilde{\Psi}) &= \frac{1}{2(\Lambda^\varepsilon)'}\Big( - \tilde{\Psi}^2 - \Big(\tilde{\Phi}_{\tilde{y}} - \Big(\frac{1}{h} + \frac{(\Lambda^\varepsilon)''}{(\Lambda^\varepsilon)'}\tilde{\Phi}\Big)^2 + P\Big(h - \frac{\tilde{\Phi}}{(\Lambda^\varepsilon)'}\Big) - P(h)\\
&\qquad + P'(h)\frac{\tilde{\Phi}}{(\Lambda^\varepsilon)'}\Big)\Big)
\end{aligned}$$

with

$$\begin{aligned}
\kappa^\varepsilon &= \Big(\frac{1}{(\Lambda^\varepsilon)'}\Big)^2 - \frac{\omega^\varepsilon(1)}{(\Lambda^\varepsilon)'},\\
P(\eta) &= \frac{\eta^2}{h^2}(3r - 2\eta).
\end{aligned}$$

Now we can write equations (6.23) – (6.27) as the evolutionary system

$$\begin{pmatrix} \tilde{\Phi} \\ \tilde{\Psi} \end{pmatrix}_{\tilde{x}} = f^\varepsilon(\tilde{\Phi}, \tilde{\Psi}), \tag{6.28}$$

where

$$f^\varepsilon(\tilde{\Phi}, \tilde{\Psi}) = \begin{pmatrix} \tilde{\Psi} + N_1^\varepsilon(\tilde{\Phi}, \tilde{\Psi}) \\ -\tilde{\Phi}_{\tilde{y}\tilde{y}} - (\omega^\varepsilon)'(\Lambda^\varepsilon)\tilde{\Phi} + N_2^\varepsilon(\tilde{\Phi}, \tilde{\Psi}) \end{pmatrix};$$

note that $f^\varepsilon$ takes values in $\mathcal{X}$ and is analytic at the origin in $\mathbb{R} \times \mathcal{Y}$, where

$$\begin{aligned}
\mathcal{Y} &= \Big\{(\tilde{\Phi}, \tilde{\Psi}) \in H^2(0, h) \times H^1(0, h) : \tilde{\Phi}|_{\tilde{y}=0} = 0, \tilde{\Psi}|_{\tilde{y}=0} = 0\Big\},\\
\mathcal{X} &= \Big\{(\tilde{\Phi}, \tilde{\Psi}) \in H^1(0, h) \times L^2(0, h) : \tilde{\Phi}|_{\tilde{y}=0} = 0\Big\}.
\end{aligned}$$

The domain of the vector field on the right-hand side of equation (6.28) is

$$\{(\tilde{\Phi}, \tilde{\Psi}) \in \mathcal{Y} : \tilde{\Phi}_{\tilde{y}} - \kappa^\varepsilon\tilde{\Phi} = N_3^\varepsilon(\tilde{\Phi}, \tilde{\Psi}) \text{ for } y = h\}.$$

The conserved quantity of equations (6.16) – (6.22) evidently transforms into a conserved quantity of equations (6.23) – (6.27).

**Proposition 6.4.** The functional $\tilde{H}^\varepsilon$ defined by

$$\begin{aligned}\tilde{H}^\varepsilon(\tilde{\Phi},\tilde{\Psi}) = \int_0^h \bigg(&\frac{h}{2(\zeta+h)}\Big(\tilde{\Psi}^2 - \Big(\tilde{\Phi}_{\tilde{y}} + (\Lambda^\varepsilon)' + \frac{(\tilde{y}(\Lambda^\varepsilon)')'}{h}\zeta\Big)^2\Big)\\ &+ \frac{\zeta+h}{h}\Omega^\varepsilon\Big(\tilde{\Phi} + \Lambda^\varepsilon + \frac{\tilde{y}(\Lambda^\varepsilon)'\zeta}{h}\Big)\bigg)\,\mathrm{d}\tilde{y} + \frac{1}{2}(\zeta+h)^2 - \frac{3}{2}r(\zeta+h)\end{aligned} \tag{6.29}$$

is a conserved quantity of equations (6.23) – (6.27) and satisfies

$$\begin{aligned}\tilde{H}^\varepsilon(0) &= -\int_0^h ((\Lambda^\varepsilon)')^2\,\mathrm{d}\tilde{y} - \frac{1}{2}h^2,\\ \mathrm{d}\tilde{H}^\varepsilon[0] &= 0,\\ \frac{1}{2}\,\mathrm{d}^2\tilde{H}^\varepsilon[0](\tilde{\Phi},\tilde{\Psi}) &= \frac{1}{2}\int_0^h \Big(\tilde{\Psi}^2 - \tilde{\Phi}_{\tilde{y}}^2 + (\omega^\varepsilon)'(\Lambda^\varepsilon)\tilde{\Phi}^2\Big)\,\mathrm{d}\tilde{y} + \frac{1}{2}\kappa^\varepsilon\tilde{\Phi}^2(h).\end{aligned}$$

*Proof.* We find that

$$\begin{aligned}\tilde{H}^\varepsilon(0) &= \int_0^h \Big(-\frac{1}{2}((\Lambda^\varepsilon)')^2 + \Omega^\varepsilon(\Lambda^\varepsilon)\Big)\,\mathrm{d}\tilde{y} + \frac{1}{2}h^2 - \frac{3}{2}rh\\ &= \int_0^h \Big(-\frac{1}{2}((\Lambda^\varepsilon)')^2 + \Big(\tilde{y}\Omega^\varepsilon(\Lambda^\varepsilon)\Big)_{\tilde{y}} - \tilde{y}\omega^\varepsilon(\Lambda^\varepsilon)(\Lambda^\varepsilon)'\Big)\,\mathrm{d}\tilde{y} + \frac{1}{2}h^2 - \frac{3}{2}rh\\ &= \int_0^h \Big(-\frac{1}{2}((\Lambda^\varepsilon)')^2 + \frac{1}{2}\tilde{y}\Big(((\Lambda^\varepsilon)')^2\Big)_{\tilde{y}} + \Big(\tilde{y}\Omega^\varepsilon(\Lambda^\varepsilon)\Big)_{\tilde{y}}\Big)\,\mathrm{d}\tilde{y} + \frac{1}{2}h^2 - \frac{3}{2}rh\\ &= \int_0^h \Big(-((\Lambda^\varepsilon)')^2 + \Big(\frac{1}{2}\tilde{y}((\Lambda^\varepsilon)')^2 + \tilde{y}\Omega^\varepsilon(\Lambda^\varepsilon)\Big)_{\tilde{y}}\Big)\,\mathrm{d}\tilde{y} + \frac{1}{2}h^2 - \frac{3}{2}rh\\ &= -\int_0^h ((\Lambda^\varepsilon)')^2\,\mathrm{d}\tilde{y} + \frac{1}{2}\Big((\Lambda^\varepsilon)'(h)^2 + 2h - 3r\Big)h - \frac{1}{2}h^2\\ &= -\int_0^h ((\Lambda^\varepsilon)')^2\,\mathrm{d}\tilde{y} - \frac{1}{2}h^2,\end{aligned}$$

$$\begin{aligned}\mathrm{d}\tilde{H}^\varepsilon[0](\tilde{\Phi},\tilde{\Psi}) &= \int_0^h \bigg(\frac{((\Lambda^\varepsilon)')^2\zeta}{2h} - (\Lambda^\varepsilon)'\tilde{\Phi}_{\tilde{y}} - (\tilde{y}(\Lambda^\varepsilon)')'\frac{u\zeta}{h} + \frac{\zeta}{h}\Omega^\varepsilon(\Lambda^\varepsilon)\\ &\qquad + \omega^\varepsilon(\Lambda^\varepsilon)\Big(\tilde{\Phi} + \tilde{y}\frac{(\Lambda^\varepsilon)'\zeta}{h}\Big)\bigg)\,\mathrm{d}\tilde{y} + \zeta h - \frac{3}{2}r\zeta\\ &= \int_0^h \Big((\Lambda^\varepsilon)'' + \omega^\varepsilon(\Lambda^\varepsilon)\Big)\tilde{\Phi}\,\mathrm{d}\tilde{y} - (\Lambda^\varepsilon)'(h)\tilde{\Phi} - \frac{\zeta}{2}(\Lambda^\varepsilon)'(h)^2 + \zeta h - \frac{3}{2}r\zeta\\ &= \frac{1}{2}\zeta\Big((\Lambda^\varepsilon)'(h)^2 + 2h - 3r\Big)\\ &= 0,\end{aligned}$$

and

$$\begin{aligned}\mathrm{d}^2\tilde{H}^\varepsilon[0](\tilde{\Phi},\tilde{\Psi}) = \int_0^h \bigg(&\frac{1}{2}\Big(\tilde{\Psi}^2 - \tilde{\Phi}_{\tilde{y}}^2 - \Big((\tilde{y}(\Lambda^\varepsilon)')'\frac{\zeta}{h}\Big)^2 - 2\tilde{\Phi}_{\tilde{y}}(\tilde{y}(\Lambda^\varepsilon)')'\frac{\zeta}{h}\Big) + \frac{\zeta}{2h}\Big(2\tilde{\Phi}_{\tilde{y}}(\Lambda^\varepsilon)'\\ &+ 2(\Lambda^\varepsilon)'(\tilde{y}(\Lambda^\varepsilon)')'\frac{\zeta}{h}\Big) - \frac{\zeta^2}{2h^2}((\Lambda^\varepsilon)')^2 + \frac{\zeta}{h}\omega^\varepsilon(\Lambda^\varepsilon)\Big(\tilde{\Phi} + \tilde{y}(\Lambda^\varepsilon)'\frac{\zeta}{h}\Big)\\ &+ \frac{1}{2}(\omega^\varepsilon)'(\Lambda^\varepsilon)\Big(\tilde{\Phi} + \tilde{y}(\Lambda^\varepsilon)'\frac{\zeta}{h}\Big)^2\bigg)\,\mathrm{d}\tilde{y} + \frac{1}{2}\zeta^2\end{aligned}$$

$$
\begin{aligned}
&= \int_0^h \frac{1}{2}(\tilde{\Psi}^2 - \tilde{\Phi}_{\tilde{y}}^2 + (\omega^\varepsilon)'(\Lambda^\varepsilon)\tilde{\Phi}^2)\,\mathrm{d}\tilde{y} \\
&\qquad + \frac{\zeta}{h}\int_0^h \Big( - \tilde{\Phi}_{\tilde{y}}(\tilde{y}(\Lambda^\varepsilon)')' + \tilde{\Phi}_{\tilde{y}}(\Lambda^\varepsilon)' + \omega^\varepsilon(\Lambda^\varepsilon)\tilde{\Phi} + (\omega^\varepsilon)'(\Lambda^\varepsilon)\tilde{y}(\Lambda^\varepsilon)'\tilde{\Phi}\Big)\,\mathrm{d}\tilde{y} \\
&\qquad + \frac{\zeta^2}{h^2}\int_0^h \Big( - \frac{1}{2}\Big((\tilde{y}(\Lambda^\varepsilon)')'\Big)^2 + (\Lambda^\varepsilon)'(\tilde{y}(\Lambda^\varepsilon)')' \\
&\qquad\qquad - \frac{1}{2}((\Lambda^\varepsilon)')^2 + \omega^\varepsilon(\Lambda^\varepsilon)\tilde{y}(\Lambda^\varepsilon)' + \frac{1}{2}(\omega^\varepsilon)'(\tilde{y}(\Lambda^\varepsilon)')^2\Big)\,\mathrm{d}\tilde{y} + \frac{1}{2}\zeta^2 \\
&= \int_0^h \frac{1}{2}(\tilde{\Psi}^2 - \tilde{\Phi}_{\tilde{y}}^2 + (\omega^\varepsilon)'(\Lambda^\varepsilon)\tilde{\Phi}^2)\,\mathrm{d}\tilde{y} + \zeta\tilde{\Phi}(h)\omega^\varepsilon(u(h)) \\
&\qquad + \frac{\zeta^2}{h^2}\Big( - \frac{1}{2}(\tilde{y}(\Lambda^\varepsilon)')'(h)h(\Lambda^\varepsilon)'(h) + \frac{1}{2}(\tilde{y}(\Lambda^\varepsilon)')^2(h)\Big) + \frac{1}{2}\zeta^2 \\
&= \int_0^h \frac{1}{2}(\tilde{\Psi}^2 - \tilde{\Phi}_{\tilde{y}}^2 + (\omega^\varepsilon)'(\Lambda^\varepsilon)\tilde{\Phi}^2)\,\mathrm{d}\tilde{y} \\
&\qquad + \zeta^2\Big( - \omega^\varepsilon(\Lambda^\varepsilon)(\Lambda^\varepsilon)' - \frac{1}{2}(\Lambda^\varepsilon)'(\Lambda^\varepsilon)''\Big)(h) + \frac{1}{2}\zeta^2 \\
&= \int_0^h \frac{1}{2}(\tilde{\Psi}^2 - \tilde{\Phi}_{\tilde{y}}^2 + (\omega^\varepsilon)'(\Lambda^\varepsilon)\tilde{\Phi}^2)\,\mathrm{d}\tilde{y} + \zeta^2\Big(1 - \frac{1}{2}\omega^\varepsilon(\Lambda^\varepsilon)(\Lambda^\varepsilon)'\Big)(h) \\
&= \int_0^h \frac{1}{2}(\tilde{\Psi}^2 - \tilde{\Phi}_{\tilde{y}}^2 + (\omega^\varepsilon)'(\Lambda^\varepsilon)\tilde{\Phi}^2)\,\mathrm{d}\tilde{y} + \frac{1}{2}\tilde{\Phi}^2(h)\Big(\frac{1}{(\Lambda^\varepsilon)'(h)^2} - \frac{\omega^\varepsilon(1)}{(\Lambda^\varepsilon)'(h)}\Big).
\end{aligned}
$$

□

To linearise the nonlinear boundary condition associated with equation (6.28) we introduce new coordinates $\tilde{\Gamma}, \tilde{\xi}$ given by

$$(\tilde{\Gamma}, \tilde{\xi}) = G^\varepsilon(\tilde{\Phi}, \tilde{\Psi}),$$

where

$$
\begin{aligned}
\tilde{\Gamma} &= \tilde{\Phi} + \frac{\tilde{y}}{h}\int_{\tilde{y}}^h N_3^\varepsilon(\tilde{\Phi}, \tilde{\Psi})(s)\,\mathrm{d}s, \\
\tilde{\xi} &= \tilde{\Psi}.
\end{aligned}
$$

The following result shows that $G^\varepsilon$ defines a valid change of variables.

**Proposition 6.5.** For each $\varepsilon$ in a neighbourhood of the origin in $\mathbb{R}$ the mapping $G^\varepsilon$ is an analytic diffeomorphism of a neighbourhood of the origin in $\mathcal{Y}$ onto a neighbourhood of the origin in $\mathcal{Y}$ which, together with its inverse, depends analytically upon $\varepsilon \in \mathbb{R}$.

Furthermore the operator $\mathrm{d}G^\varepsilon[\tilde{\Phi}, \tilde{\Psi}] \in \mathcal{L}(\mathcal{Y})$ extends to an isomorphism $\widehat{\mathrm{d}G^\varepsilon}[\tilde{\Phi}, \tilde{\Psi}] \in \mathcal{L}(\mathcal{X})$ which, together with its inverse, depends analytically upon $\varepsilon \in \mathbb{R}$ and $(\tilde{\Phi}, \tilde{\Psi}) \in \mathcal{Y}$.

*Proof.* Observe that $G^{(\cdot)}$ takes values in $\mathcal{Y}$ and is analytic at the origin in $\mathbb{R} \times \mathcal{Y}$ and

$$\mathrm{d}G^\varepsilon[\tilde{\Phi}, \tilde{\Psi}] = \begin{pmatrix} I + \dfrac{\tilde{y}}{h}\displaystyle\int_{\tilde{y}}^h \mathrm{d}N_3^\varepsilon[\tilde{\Phi}, \tilde{\Psi}](\cdot)\,\mathrm{d}s \\ I \end{pmatrix}.$$

Furthermore, for each $\varepsilon$ in a neighbourhood of the origin in $\mathbb{R}$ and $(\tilde{\Phi}, \tilde{\Psi})$ in a neighbourhood of the origin in $\mathcal{Y}$ the operator $\mathrm{d}N_3^\varepsilon[\tilde{\Phi}, \tilde{\Psi}] \in \mathcal{L}(Y)$ extends to an operator $\widehat{\mathrm{d}N_3^\varepsilon}[\tilde{\Phi}, \tilde{\Psi}] \in \mathcal{L}(X)$ which depends analytically upon $\varepsilon$ and $(\tilde{\Phi}, \tilde{\Psi})$; the above formula shows that the same is true of $\mathrm{d}G^\varepsilon[\tilde{\Phi}, \tilde{\Psi}]$. Finally, observe that $\mathrm{d}G^0[0]$ and $\widehat{\mathrm{d}G^0}[0]$ are the identity operators in respectively $\mathcal{L}(\mathcal{Y})$ and $\mathcal{L}(\mathcal{X})$.

The assertions now follow by applying the implicit-function theorem to the functions $F_1 \colon \mathcal{Y} \times \mathcal{Y} \times \mathbb{R} \to \mathcal{Y}$ and $F_2 \colon \mathcal{L}(\mathcal{X}) \times \mathcal{X} \times \mathbb{R} \to \mathcal{L}(\mathcal{X})$ given by

$$\begin{aligned} F_1((\tilde{\Gamma}, \tilde{\xi}), (\tilde{\Phi}, \tilde{\Psi}), \varepsilon) &= G^\varepsilon(\tilde{\Phi}, \tilde{\Psi}) - (\tilde{\Gamma}, \tilde{\xi}), \\ F_2(T, (\tilde{\Phi}, \tilde{\Psi}), \varepsilon) &= \widehat{\mathrm{d}G^\varepsilon}[\tilde{\Phi}, \tilde{\Psi}]T - I. \end{aligned}$$

□

The change of variable given by $G^\varepsilon$ transforms equation (6.28) into the evolutionary system

$$\begin{pmatrix} \tilde{\Gamma} \\ \tilde{\xi} \end{pmatrix}_{\tilde{x}} = k^\varepsilon(\tilde{\Gamma}, \tilde{\xi}) = \begin{pmatrix} k_1^\varepsilon(\tilde{\Gamma}, \tilde{\xi}) \\ k_2^\varepsilon(\tilde{\Gamma}, \tilde{\xi}) \end{pmatrix}, \tag{6.30}$$

where

$$k^\varepsilon(\tilde{\Gamma}, \tilde{\xi}) = \widehat{\mathrm{d}G}^\varepsilon[(G^\varepsilon)^{-1}(\tilde{\Gamma}, \tilde{\xi})]\Big(f^\varepsilon\big((G^\varepsilon)^{-1}(\tilde{\Gamma}, \tilde{\xi})\big)\Big)$$

takes values in $\mathcal{X}$ and is analytic at the origin in $\mathbb{R} \times \mathcal{Y}$. The domain of the vector field on the right-hand side of equation (6.30) is

$$\mathcal{D}^\varepsilon = \Big\{(\tilde{\Gamma}, \tilde{\xi}) \in H^2(0,h) \times H^1(0,h) : \tilde{\Gamma}(0), \tilde{\xi}(0) = 0, \tilde{\Gamma}_{\tilde{y}}(h) - \kappa^\varepsilon \tilde{\Gamma}(h) = 0\Big\}.$$

We note that the linearisation of $k^\varepsilon$ is given by $\check{L}^\varepsilon \colon \mathcal{D}^\varepsilon \subseteq \mathcal{X} \to \mathcal{X}$ with

$$\check{L}^\varepsilon \begin{pmatrix} \tilde{\Gamma} \\ \tilde{\xi} \end{pmatrix} = \begin{pmatrix} \tilde{\xi} \\ -\tilde{\Gamma}_{\tilde{y}\tilde{y}} - (\omega^\varepsilon)'(\Lambda^\varepsilon)\tilde{\Gamma} \end{pmatrix} \tag{6.31}$$

(since $\mathrm{d}G^\varepsilon[0] = I$) and this operator depends analytically upon $\varepsilon$ since

$$T^\varepsilon \begin{pmatrix} \tilde{\Gamma} \\ \tilde{\xi} \end{pmatrix} = \begin{pmatrix} \tilde{\Gamma} - \dfrac{\tilde{y}}{h}(\kappa^\varepsilon - \kappa^0) \displaystyle\int_{\tilde{y}}^h \tilde{\Gamma}(s)\,\mathrm{d}s \\ \tilde{\xi} \end{pmatrix}$$

defines an isomorphism $T^\varepsilon \in \mathcal{L}(\mathcal{Y})$ which depends analytically upon $\varepsilon$ and maps $\mathcal{D}^0$ onto $\mathcal{D}^\varepsilon$. The quadratic terms are given by

$$\frac{1}{2}\,\mathrm{d}^2 k_1^\varepsilon[0](\tilde{\Gamma}, \tilde{\xi}) = M_1^\varepsilon(\tilde{\Gamma}, \tilde{\xi}) + \frac{\tilde{y}}{h}\int_{\tilde{y}}^h \mathrm{d}M_2^\varepsilon[\tilde{\Gamma}, \tilde{\xi}](\tilde{\xi}, -\tilde{\Gamma}_{\tilde{y}\tilde{y}} - (\omega^\varepsilon)'(\Lambda^\varepsilon)\tilde{\Gamma}),$$

where

$$\begin{aligned} M_1^\varepsilon(\tilde{\Gamma}, \tilde{\xi}) &= \frac{\tilde{\Gamma}(h)}{h(\Lambda^\varepsilon)'(h)}\tilde{\xi} - y\frac{\tilde{\xi}(h)}{h(\Lambda^\varepsilon)'(h)}\Big(\tilde{\Gamma}_{\tilde{y}} - \frac{\tilde{\Gamma}(h)}{(\Lambda^\varepsilon)'(h)}\frac{\tilde{y}}{h}(\Lambda^\varepsilon)''\Big), \\ M_2^\varepsilon(\tilde{\Gamma}, \tilde{\xi}) &= \frac{1}{2(\Lambda^\varepsilon)'(h)}\Big( - \tilde{\xi}^2 - \Big(\tilde{\Gamma}_{\tilde{y}} - \Big(\frac{1}{h} + \frac{(\Lambda^\varepsilon)''(h)}{(\Lambda^\varepsilon)'(h)}\Big)\tilde{\Gamma}\Big)^2 + \Big(\frac{3r}{h^2} - \frac{6}{h}\Big)\frac{\tilde{\Gamma}^2}{(\Lambda^\varepsilon)''(h)^2}\Big), \end{aligned}$$

and

$$\frac{1}{2}\,\mathrm{d}^2k_2^\varepsilon[0](\tilde\Gamma,\tilde\xi) = \frac{2}{h}M_2^\varepsilon(\tilde\Gamma,\tilde\xi) + \frac{\tilde y}{h}\partial_{\tilde y}M_2^\varepsilon(\tilde\Gamma,\tilde\xi) - (\omega^\varepsilon)'(\Lambda^\varepsilon)\frac{\tilde y}{h}\int_{\tilde y}^h M_2^\varepsilon(\tilde\Gamma,\tilde\xi) + M_3^\varepsilon(\tilde\Gamma,\tilde\xi),$$

where

$$\begin{aligned} M_3^\varepsilon(\tilde\Gamma,\tilde\xi) = &-\frac{\tilde\xi(h)}{(\Lambda^\varepsilon)'(h)}\frac{(\tilde y\tilde\xi)_{\tilde y}}{h} - \frac{\tilde\Gamma(h)^2}{h^2(\Lambda^\varepsilon)'(h)^2}\omega^\varepsilon(\Lambda^\varepsilon) + \frac{\tilde\Gamma(h)}{h(\Lambda^\varepsilon)'(h)}(\omega^\varepsilon)'(\Lambda^\varepsilon)\tilde\Gamma \\ &- \frac{2\tilde\Gamma(h)^2}{h(\Lambda^\varepsilon)'(h)^2}(\omega^\varepsilon)^2(\Lambda^\varepsilon)\frac{\tilde y(\Lambda^\varepsilon)'}{h} - \frac{1}{2}(\omega^\varepsilon)''(\Lambda^\varepsilon)\Big(\tilde\Gamma - \frac{\tilde\Gamma(h)}{(\Lambda^\varepsilon)'(h)}\frac{\tilde y(\Lambda^\varepsilon)'}{h}\Big)^2. \end{aligned}$$

### 6.2.2 The linearised system

In this section we study the linear operator $\check L^\varepsilon\colon \mathcal{D}^\varepsilon \subseteq \mathcal{X} \to \mathcal{X}$ given by equation (6.31). We begin with two general results which hold for all choices of vorticity functions $\omega^\varepsilon$.

**Proposition 6.6.** The linear equation

$$\begin{pmatrix}\tilde\Gamma\\ \tilde\xi\end{pmatrix}_{\tilde x} = \check L^\varepsilon\begin{pmatrix}\tilde\Gamma\\ \tilde\xi\end{pmatrix}$$

represents Hamilton's equations for the linear Hamiltonian system $(\mathcal{X},\Omega,H_2^\varepsilon)$, where

$$\Omega\Big((\tilde\Gamma_1,\tilde\xi_1),(\tilde\Gamma_2,\tilde\xi_2)\Big) = \int_0^h (\tilde\xi_2\tilde\Gamma_1 - \tilde\xi_1\tilde\Gamma_2)\,\mathrm{d}\tilde y$$

and

$$H_2^\varepsilon(\tilde\Gamma,\tilde\xi) = \frac{1}{2}\int_0^h (\tilde\xi^2 - \tilde\Gamma_{\tilde y}^2 + (\omega^\varepsilon)'(\Lambda^\varepsilon)\tilde\Gamma^2)\,\mathrm{d}\tilde y + \frac{1}{2}\kappa^\varepsilon\tilde\Phi(h)^2.$$

*Proof.* A direct calculation shows that

$$\Omega(\check L^\varepsilon(\tilde\Gamma,\tilde\xi),(\check\Gamma,\check\xi)) = \mathrm{d}H_2^\varepsilon[\tilde\Gamma,\tilde\xi](\check\Gamma,\check\xi)$$

for all $(\check\Gamma,\check\xi)\in\mathcal{X}$ and $(\tilde\Gamma,\tilde\xi)\in\mathcal{D}^\varepsilon$. □

**Lemma 6.7.** The operator $\check L^\varepsilon$ satisfies the estimates

$$\begin{aligned} \|(\check L^\varepsilon - \mathrm{i}sI)^{-1}\|_{\mathcal{L}(\mathcal{X})} &\lesssim |s|^{-1}, \\ \|(\check L^\varepsilon - \mathrm{i}sI)^{-1}\|_{\mathcal{L}(\mathcal{X};\mathcal{D}^\varepsilon)} &\lesssim 1 \end{aligned}$$

as $|s|\to\infty$.

*Proof.* Consider the self-adjoint operator $\check A^\varepsilon\colon \tilde{\mathcal{D}}^\varepsilon \subseteq \tilde{\mathcal{X}} \to \tilde{\mathcal{X}}$ given by

$$\check A^\varepsilon(\tilde\Gamma) = -\tilde\Gamma_{\tilde y\tilde y} - (\omega^\varepsilon)'(\Lambda^\varepsilon)\tilde\Gamma,$$

where

$$\tilde{\mathcal{X}} = L^2(0,h), \qquad \tilde{\mathcal{D}}^\varepsilon = \{\tilde{\Gamma} \in H^2(0,h) : \tilde{\Gamma}(0) = 0, \tilde{\Gamma}_{\tilde{y}}(h) = \kappa^\varepsilon \tilde{\Gamma}(h)\}.$$

Observing that

$$\tilde{\xi} - \mathrm{i}s\tilde{\Gamma} = \tilde{\Gamma}^*, \tag{6.32}$$

$$-\tilde{\Gamma}_{\tilde{y}\tilde{y}} - (\omega^\varepsilon)'(\Lambda^\varepsilon)\tilde{\Gamma} - \mathrm{i}s\tilde{\xi} = \tilde{\xi}^* \tag{6.33}$$

if and only if

$$-\tilde{\Gamma}_{\tilde{y}\tilde{y}} - (\omega^\varepsilon)'(\Lambda^\varepsilon)\tilde{\Gamma} + s^2\tilde{\Gamma} = \tilde{\xi}^* + \mathrm{i}s\tilde{\Gamma}^*,$$

we find that $\pm\mathrm{i}s \in \rho(\check{L}^\varepsilon)$ if and only if $-s^2 \in \rho(\check{A}^\varepsilon)$. The calculation

$$\begin{aligned}
&\int_0^h \Big( -\tilde{\Gamma}_{\tilde{y}\tilde{y}} - (\omega^\varepsilon)'(\Lambda^\varepsilon)\tilde{\Gamma}\Big)\tilde{\Gamma}\,\mathrm{d}\tilde{y} \\
&\quad = \int_0^h \Big( -(\omega^\varepsilon)'(\Lambda^\varepsilon)|\tilde{\Gamma}|^2 + |\tilde{\Gamma}_{\tilde{y}}|^2\Big)\,\mathrm{d}\tilde{y} - \kappa^\varepsilon|\Gamma(h)|^2 \\
&\quad = \int_0^h \Big( -(\omega^\varepsilon)'(\Lambda^\varepsilon)|\tilde{\Gamma}|^2 + |\tilde{\Gamma}_{\tilde{y}}|^2\Big)\,\mathrm{d}\tilde{y} - 2\kappa^\varepsilon\int_0^h \tilde{\Gamma}\tilde{\Gamma}_{\tilde{y}}\,\mathrm{d}\tilde{y} \\
&\quad \geq -\int_0^h (\omega^\varepsilon)'(\Lambda^\varepsilon)|\tilde{\Gamma}|^2\,\mathrm{d}\tilde{y} + (1-\kappa^\varepsilon\delta)\int_0^h |\tilde{\Gamma}_{\tilde{y}}|^2\,\mathrm{d}\tilde{y} - \frac{\kappa^\varepsilon}{\delta}\int_0^h |\tilde{\Gamma}|^2\,\mathrm{d}\tilde{y} \\
&\quad \geq -\Big(\frac{\kappa^\varepsilon}{\delta} + \sup_{\tilde{y}\in[0,h]}|(\omega^\varepsilon)'(\Lambda^\varepsilon)|\Big)\int_0^h |\tilde{\Gamma}|^2\,\mathrm{d}\tilde{y}
\end{aligned}$$

for $\tilde{\Gamma} \in \tilde{\mathcal{D}}^\varepsilon$ and sufficiently small $\delta$ shows that $\sigma(\check{A}^\varepsilon)$ is bounded from below and in particular that $-s^2 \in \rho(\check{A}^\varepsilon)$ and hence $\pm\mathrm{i}s \in \rho(\check{L}^\varepsilon)$ for sufficiently large values of $|s|$.

The estimates

$$\|(\check{L}^\varepsilon - \mathrm{i}sI)^{-1}\|_{\mathcal{L}(\mathcal{X})} \lesssim |s|^{-1},$$

and

$$\|(\check{L}^\varepsilon - \mathrm{i}sI)^{-1}\|_{\mathcal{L}(\mathcal{X},\mathcal{D}^\varepsilon)} \lesssim 1$$

for sufficiently large values of $|s|$ follow from equations (6.32) and (6.33) by the calculation

$$\begin{aligned}
&\int_0^h \Big(|\tilde{\Gamma}^*|^2 + |\tilde{\Gamma}^*_{\tilde{y}}|^2 + |\tilde{\xi}^*|^2\Big)\,\mathrm{d}\tilde{y} \\
&\quad = \int_0^h \Big(|\tilde{\xi} - \mathrm{i}s\tilde{\Gamma}|^2 + |\tilde{\xi}_{\tilde{y}} - \mathrm{i}s\tilde{\Gamma}_{\tilde{y}}|^2 + |\tilde{\Gamma}_{\tilde{y}\tilde{y}} + (\omega^\varepsilon)'(\Lambda^\varepsilon)\tilde{\Gamma} + \mathrm{i}s\tilde{\xi}|^2\Big)\,\mathrm{d}\tilde{y} \\
&\quad = \int_0^h \Big(|\tilde{\xi}|^2 + |\tilde{\xi}_{\tilde{y}}|^2 + |\tilde{\Gamma}_{\tilde{y}\tilde{y}}|^2 + s^2(|\tilde{\Gamma}|^2 + |\tilde{\Gamma}_{\tilde{y}}|^2 + |\tilde{\xi}|^2) + (\omega^\varepsilon)'(\Lambda^\varepsilon)^2|\tilde{\Gamma}|^2\Big)\,\mathrm{d}\tilde{y} \\
&\qquad\quad - 2s\,\mathrm{Im}\int_0^h \Big(\tilde{\xi}\bar{\tilde{\Gamma}} + \tilde{\xi}\bar{\tilde{\Gamma}}_{\tilde{y}} + \tilde{\xi}\bar{\tilde{\Gamma}}_{\tilde{y}\tilde{y}} + \tilde{\xi}(\omega^\varepsilon)'(\Lambda^\varepsilon)\bar{\tilde{\Gamma}}\Big)\,\mathrm{d}\tilde{y} \\
&\qquad\quad + 2\,\mathrm{Re}\int_0^h (\omega^\varepsilon)'(\Lambda^\varepsilon)\tilde{\Gamma}\bar{\tilde{\Gamma}}_{\tilde{y}\tilde{y}}\,\mathrm{d}\tilde{y}
\end{aligned}$$

and the inequalities

$$\left|2s\int_0^h |\tilde{\xi}||\tilde{\Gamma}|\,\mathrm{d}\tilde{y}\right| \leq |s|\int_0^h \Big(|\tilde{\xi}|^2 + |\tilde{\Gamma}|^2\Big)\,\mathrm{d}\tilde{y},$$

$$
\begin{aligned}
|2s\,\mathrm{Im}\,\tilde{\xi}(h)\bar{\tilde{\Gamma}}(h)| &\leq |s|\delta|\tilde{\xi}(h)|^2 + \frac{|s|}{\delta}|\tilde{\Gamma}(h)|^2 \\
&= |s|\delta\int_0^h \frac{\mathrm{d}}{\mathrm{d}\tilde{y}}|\tilde{\xi}|^2\,\mathrm{d}\tilde{y} + \frac{|s|}{\delta}\int_0^h \frac{\mathrm{d}}{\mathrm{d}\tilde{y}}|\tilde{\Gamma}|^2\,\mathrm{d}\tilde{y} \\
&\leq \delta\int_0^h \left(|\tilde{\xi}_{\tilde{y}}|^2 + s^2|\tilde{\xi}|^2\right)\mathrm{d}\tilde{y} + \frac{|s|}{\delta}\int_0^h \left(|\tilde{\Gamma}_{\tilde{y}}|^2 + |\tilde{\Gamma}|^2\right)\mathrm{d}\tilde{y}, \\
\left|2\int_0^h |\tilde{\Gamma}||\tilde{\Gamma}_{\tilde{y}\tilde{y}}|\,\mathrm{d}\tilde{y}\right| &\leq \delta\int_0^h |\tilde{\Gamma}_{\tilde{y}\tilde{y}}|^2\,\mathrm{d}\tilde{y} + \frac{1}{\delta}\int_0^h |\tilde{\Gamma}|^2\,\mathrm{d}\tilde{y}
\end{aligned}
$$

(for sufficiently small values of $\delta$). □

**Corollary 6.8.** The spectrum of $\check{L}^\varepsilon$ consists only of isolated eigenvalues with finite algebraic multiplicity.

*Proof.* Since $\mathcal{D}^\varepsilon$ is compactly embedded in $\mathcal{X}$ we know that $(\check{L}^\varepsilon - \mathrm{i}sI)^{-1} \in \mathcal{L}(\mathcal{X})$ is compact for sufficiently large values of $|s|$. The result now follows from Kato [13, Theorem III.6.29]. □

Now we specialise to the case $\omega^\varepsilon(\Psi) = (b+\varepsilon)\Psi$, where $b$ is a positive constant, so that

$$
\begin{aligned}
\Lambda^\varepsilon(\tilde{y}) &= \sin(\sqrt{b+\varepsilon}\tilde{y}), \\
r &= \frac{\sin^2(\sqrt{b+\varepsilon}h) + 2h}{3}, \\
\kappa^\varepsilon &= \kappa(b+\varepsilon),
\end{aligned}
$$

where

$$
\kappa(b) = \frac{1}{b}\tan^2(\sqrt{b}h) - \sqrt{b}\tan(\sqrt{b}h).
$$

To calculate the spectrum of $\check{L}^\varepsilon$ we consider the eigenvalue problem

$$
\begin{aligned}
\tilde{\xi} &= \lambda\tilde{\Gamma}, \\
-\tilde{\Gamma}_{\tilde{y}\tilde{y}} - (b+\varepsilon)\tilde{\Gamma} &= \lambda\tilde{\xi},
\end{aligned}
$$

which is equivalent to the eigenvalue problem

$$
-\tilde{\Gamma}_{\tilde{y}\tilde{y}} - (b+\varepsilon)\tilde{\Gamma} = \mu\tilde{\Gamma}
$$

for the self-adjoint operator $A^\varepsilon\colon \tilde{\mathcal{D}}^\varepsilon \subseteq \tilde{\mathcal{X}} \to \tilde{\mathcal{X}}$ given by

$$
A^\varepsilon\tilde{\Gamma} = -\tilde{\Gamma}_{\tilde{y}\tilde{y}} - (b+\varepsilon)\tilde{\Gamma},
$$

where

$$
\tilde{\mathcal{X}} = L^2(0,h), \quad \tilde{\mathcal{D}}^\varepsilon = \{\tilde{\Gamma} \in H^2(0,h) : \tilde{\Gamma}(0) = 0, \tilde{\Gamma}_{\tilde{y}}(h) = \kappa(b+\varepsilon)\tilde{\Gamma}(h)\}.
$$

The following auxiliary result is a combination of Zettl [24, Theorem 3.5.1, Theorem 3.8.2 and Theorem 4.4.3] (simplified for our needs) concerning the continuous dependence of the spectrum upon the boundary conditions.

**Lemma 6.9.** The self-adjoint Sturm-Liouville problem

$$-\tilde{\Phi}_{\tilde{y}\tilde{y}} - q\tilde{\Phi} = \mu\tilde{\Phi}, \qquad \tilde{y} \in (0,h), \tag{6.34}$$

$$\tilde{\Phi}(0) = 0, \tag{6.35}$$

$$A\tilde{\Phi}(h) + B\tilde{\Phi}_{\tilde{y}}(h) = 0, \tag{6.36}$$

where $q, A \in \mathbb{R}$ and $B \in \mathbb{R} \setminus \{0\}$, has a countable infinite family $\{\mu_n\}_{n=1}^{\infty}$ of simple eigenvalues with

$$\mu_1 < \mu_2 < \mu_3 < \dots$$

and $\mu_n \to \infty$ as $n \to \infty$. Each eigenvalue of problem (6.34) – (6.36) depends continuously on $(q, A, B)$.

Furthermore in the case $A = 1$ the eigenvalue $\mu_n$ depends continuously upon $B$ for $B \neq 0$ but has a jump discontinuity at $B = 0$; more precisely we have

(i) $\mu_n(B) \to \mu_n(0)$ as $B \downarrow 0$ for each $n \in \mathbb{N}$,

(ii) $\mu_{n+1}(B) \to \mu_n(0)$ as $B \uparrow 0$ for each $n \in \mathbb{N}$.

**Lemma 6.10.** For any $N \geq 1$ and $h > 0$ there exists $b_N^* > 0$ such that the eigenvalues $\{\mu_n^\varepsilon\}_{n=1}^{\infty}$ of the Sturm-Liouville problem

$$-\tilde{\Gamma}_{\tilde{y}\tilde{y}} - b\tilde{\Gamma} = \mu\tilde{\Gamma}, \qquad \tilde{y} \in (0,h), \tag{6.37}$$

$$\tilde{\Gamma}(0) = 0, \tag{6.38}$$

$$\tilde{\Gamma}_{\tilde{y}}(h) = \kappa(b)\tilde{\Gamma}(h) \tag{6.39}$$

with $b = b_N^* + \varepsilon$ satisfy $\mu_1^\varepsilon, \dots, \mu_N^\varepsilon < 0$ and $0 < \mu_n^\varepsilon$ for $n > N+1$ for sufficiently small $\varepsilon \in \mathbb{R}$. Furthermore, $\mu_{N+1}^0 = 0$, $\mu_{N+1}^\varepsilon > 0$ for $\varepsilon < 0$ and $\mu_{N+1}^\varepsilon < 0$ for $\varepsilon > 0$.

*Proof.* The solution of (6.37) and (6.38) is given (up to a multiplicative factor) by $\tilde{\Gamma}(\tilde{y}) = \sin(\sqrt{b+\varepsilon+\mu}\,\tilde{y})$. With $b = b_N := \left(\frac{N\pi}{h}\right)^2$, equation (6.39) yields the eigenvalues

$$\mu_n^N = \left((n - \frac{1}{2})^2 - N^2\right)\left(\frac{\pi}{h}\right)^2, \quad n \in \mathbb{N},$$

so that $\mu_n^N < 0$ for $n \leq N$ and $\mu_n^N > 0$ for $n > N$. We proceed by showing that there is precisely one value of $b$ in $(b_N, b_{N+1})$, which lies in the interval $(b_N, b_{N+1/2})$, for which 0 is an eigenvalue of problem (6.37) – (6.39).

For $\mu = 0$ to be an eigenvalue of problem (6.37) – (6.39) we need to find $b$ such that $\tilde{\Gamma}(\tilde{y}) = \sin(\sqrt{b}\tilde{y})$ satisfies equation (6.39), so that

$$\frac{1}{b}\frac{\sin^3(\sqrt{b}h)}{\cos^2(\sqrt{b}h)} - \sqrt{b}\frac{\sin^2(\sqrt{b}h)}{\cos(\sqrt{b}h)} - \sqrt{b}\cos(\sqrt{b}h) = 0, \tag{6.40}$$

or equivalently

$$g(b) := \tan(\sqrt{b}h) - b^{\frac{3}{2}} - \frac{1}{2}\sin(2\sqrt{b}h) = 0.$$

A lengthy calculation shows that

$$g'(b) = \frac{h}{2\sqrt{b}}\Big(\frac{1}{\cos^2(\sqrt{b}h)} - \cos(2\sqrt{b}h)\Big) - \frac{3}{2}\sqrt{b}$$

and

$$\begin{aligned} g^{(3)}(b) &= \frac{-\sin^2(2\sqrt{b}h)(2+\cos(2\sqrt{b}h)) + 6}{8\cos^4(\sqrt{b}h)b^{\frac{3}{2}}}h^3 \\ &\quad - \frac{3\sin(2\sqrt{b}h)(1+2\cos^4(\sqrt{b}h))}{8\cos^4(\sqrt{b}h)b^2}h^2 \\ &\quad + \frac{3\sin^2(2\sqrt{b}h)(2+\cos(2\sqrt{b}h))}{32\cos^4(\sqrt{b}h)b^{\frac{5}{2}}}h + \frac{3}{8b^{\frac{3}{2}}} \\ &\geq \frac{3h^3\sqrt{b} - 3h^2}{8\cos^4(\sqrt{b}h)b^2} \\ &> 0 \end{aligned}$$

for $b \geq b_1$, so that in particular $g'$ is strictly convex in $(b_N, b_{N+\frac{1}{2}})$ and thus has at most one minimum and no other stationary points in this interval. From $g'(b_N) < 0$ and $g'(b) \to \infty$ as $b \uparrow b_{N+\frac{1}{2}}$ we find that $g'$ has exactly one zero in $(b_N, b_{N+\frac{1}{2}})$. Combined with the facts $g(b_N) < 0$ and $g(b) \to \infty$ as $b \uparrow b_{N+\frac{1}{2}}$ we conclude that $g$ has exactly one zero in $(b_N, b_{N+1/2})$. Furthermore $g$ is negative in $(b_{N+1/2}, b_{N+1})$ and therefore has no zeros in this interval.

Denote the unique zero of $g$ in $(b_N, b_{N+1/2})$ by $b_N^*$ (and note for later use that $b_N^* \in (b_{N+1/4}, b_{N+1/2})$ since $g(b_{N+1/4}) < 0$ and $g(b) \to \infty$ as $b \uparrow b_{N+1/2}$). Since the eigenvalues $\{\mu_n\}_{n=1}^{\infty}$ of problem (6.37) – (6.39) depend continuously on $b \in [b_N, b_{N+1/2})$ (see Lemma 6.9), we conclude that either

(i) $\mu_N < 0$ for $b_N < b < b_N^*$, $\mu_N = 0$ for $b = b_N^*$, $\mu > 0$ for $b_N^* < b < b_{N+1/2}$ (with $\mu_1, \ldots, \mu_{N-1} < 0$ and $\mu_{N+1}, \mu_{N+2}, \ldots > 0$ for all $b \in (b_N, b_{N+1/2})$;

(ii) $\mu_{N+1} > 0$ for $b_N < b < b_N^*$, $\mu_{N+1} = 0$ for $b = b_N^*$, $\mu_{N+1} < 0$ for $b_N^* < b < b_{N+1/2}$ (with $\mu_1, \ldots, \mu_N < 0$ and $\mu_{N+2}, \mu_{N+3}, \ldots > 0$ for all $b \in (b_N, b_{N+1/2})$.

By reformulating equation (6.39) as

$$\tilde{\Gamma}(h) - \frac{1}{\kappa(b)}\tilde{\Gamma}_{\tilde{y}}(h) = 0$$

and noting that $-\kappa(b)^{-1} \uparrow 0$ as $b \uparrow b_{N+1/2}$, we find from Lemma 6.9 that $\mu_{n+1} \to \mu_n^0$ as $b \uparrow b_{N+1/2}$, where

$$\mu_n^0 = \Big(n^2 - \big(N + \frac{1}{2}\big)^2\Big)\Big(\frac{\pi}{h}\Big)^2$$

are the Dirichlet eigenvalues. It follows that (ii) holds. □

Set $b = b_N^*$. Since $\lambda$ is an eigenvalue of $\check{L}^\varepsilon$ if and only if $\lambda^2$ is an eigenvalue of $A^\varepsilon$, we conclude that the spectrum of $\check{L}^\varepsilon$ consists of purely imaginary eigenvalues $\pm\mathrm{i}(-\mu_1^\varepsilon)^{1/2}$, $\pm\mathrm{i}(-\mu_2^\varepsilon)^{1/2}$, $\ldots, \pm\mathrm{i}(-\mu_N^\varepsilon)^{1/2}$, real eigenvalues $\pm(\mu_{N+2}^\varepsilon)^{1/2}$, $\pm(\mu_{N+3}^\varepsilon)^{1/2}, \ldots$ and additionally

(i) a pair of purely imaginary eigenvalues $\pm\mathrm{i}(\mu^\varepsilon_{N+1})^{1/2}$ for $\varepsilon > 0$,

(ii) a pair of real eigenvalues $\pm(\mu^\varepsilon_{N+1})^{1/2}$ for $\varepsilon < 0$,

(iii) a zero eigenvalue for $\varepsilon = 0$

(see Figure 6.2). The eigenvectors corresponding to $\pm\mathrm{i}\mu^\varepsilon_i$ are given by $e^\varepsilon_i$ and $e^{-\varepsilon}_i$, where

$$e^\varepsilon_i = \frac{1}{\sqrt{\gamma^\varepsilon_i}} \begin{pmatrix} \sin(\sqrt{b-(\mu^\varepsilon_i)^2}\tilde{y}) \\ \mathrm{i}\mu^\varepsilon_i \sin(\sqrt{b-(\mu^\varepsilon_i)^2}\tilde{y}) \end{pmatrix}, \quad i \in \{1,\ldots,N\},$$

and

$$\gamma^\varepsilon_i = \Big(1 - \frac{\sin(2h\sqrt{b-(\mu^\varepsilon_i)^2})}{2h\sqrt{b-(\mu^\varepsilon_i)^2}}\Big) h\mu^\varepsilon_i,$$

so that

$$\Omega(e^\varepsilon_i, \bar{e}^\varepsilon_i) = -\mathrm{i}, \quad i \in \{1,\ldots,n\}.$$

For $\varepsilon = 0$ the vectors

$$e = \frac{1}{\sqrt{\gamma}} \begin{pmatrix} \sin(\sqrt{b}\tilde{y}) \\ 0 \end{pmatrix}, \quad f = \frac{1}{\sqrt{\gamma}} \begin{pmatrix} 0 \\ \sin(\sqrt{b}\tilde{y}) \end{pmatrix}$$

are the generalised eigenvectors for the zero eigenvalue with

$$L^0 e = 0, \quad L^0 f = e.$$

Here

$$\gamma = \Big(1 - \frac{\sin(2\sqrt{b}h)}{2\sqrt{b}h}\Big)\frac{h}{2}, \tag{6.41}$$

so that

$$\Omega(e,f) = 1.$$

We can therefore write

$$\begin{aligned} w &= C_1 e^0_1 + \ldots + C_N e^0_N + \bar{C}_1 \bar{e}^0_1 + \ldots + \bar{C}_N \bar{e}^0_N, \\ z &= z_1 e + z_2 f, \end{aligned}$$

so that

$$H^0_2(0,w,0) = -\frac{1}{2}(\mu^0_1)^2 |C_1|^2 - \ldots - \frac{1}{2}(\mu^0_N)^2 |C_N|^2$$

and

$$P^0_{\mathrm{wh}}(\cdot) = \Omega(\cdot, f)e - \Omega(\cdot, e)f.$$

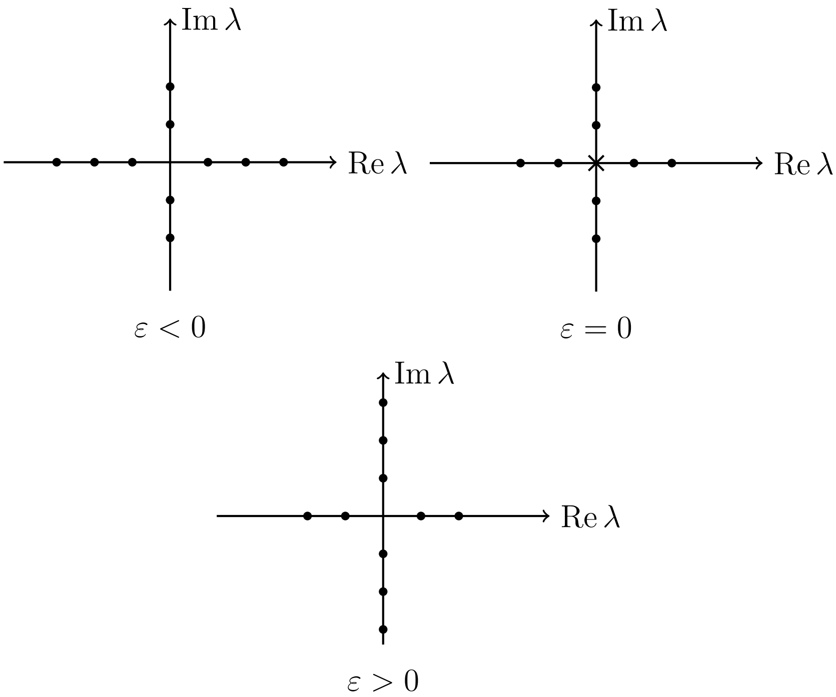



Figure 6.2: The spectrum of $\check{L}^\varepsilon$.

### 6.2.3 Existence theory

We now set

$$\omega^\varepsilon(\Psi) = (b_N^* + \varepsilon)\Psi,$$

derive equations (6.5) – (6.7) from equation (6.30) as explained in Section 6.1 and prove Theorem 1.9 by applying Theorem 1.1. Assumptions (A1) – (A3) and (B1) are obviously satisfied, while Assumption (A6) follows from Lemma 6.7. It therefore remains to verify Assumptions (A4), (A5) and (B2). To this end we note that equation (6.30) is reversible with reverser

$$S(\tilde{\Gamma}, \tilde{\xi}) = (\tilde{\Gamma}, -\tilde{\xi})$$

and has the conserved quantity

$$\begin{aligned} I^\varepsilon(\tilde{\Gamma}, \tilde{\xi}) &= -\tilde{H}^\varepsilon\Big((G^\varepsilon)^{-1}(\tilde{\Gamma}, \tilde{\xi})\Big) + \tilde{H}^\varepsilon(0) \\ &= \mathcal{O}(\|(\tilde{\Gamma}, \tilde{\xi})\|_{\mathcal{D}}^2), \end{aligned}$$

which satisfies, with a change of notation

$$\begin{aligned} I^0(0, w, 0) &= -\tilde{H}_2^0(w) + \mathcal{O}(|w|^3) \\ &= \frac{1}{2}(\mu_1^0)^2|C_1|^2 + \ldots + \frac{1}{2}(\mu_N^0)^2|C_N|^2 + \mathcal{O}(|w|^3). \end{aligned}$$

Finally, note that the coefficient of $z_1^2$ in the Maclaurin expansion of $h_{\mathrm{wh}}^\varepsilon$ is given by

$$\begin{pmatrix} \Omega(\frac{1}{2}\,\mathrm{d}^2k^0[0](e, e), f) \\ -\Omega(\frac{1}{2}\,\mathrm{d}^2k^0[0](e, e), e) \end{pmatrix},$$

so that

$$C = \Omega\Big(\tfrac{1}{2}\,\mathrm{d}^2 k^0[0](e,e), e\Big).$$

A lengthy calculation shows that

$$\begin{aligned} C = -\frac{1}{2\,h^2}\sec(\sqrt{b_N^*}h)\Big(&-4h - b_N^*\,(4\,{b_N^*}^2 h^2 + b_N^* - 3) + 4\,({b_N^*}^3 h^2 + b_N^*)\cos(2\,\sqrt{b_N^*}h) \\ &+ ({b_N^*}^2 + b_N^* + 4\,h)\cos(4\,\sqrt{b_N^*}h) \\ &+ 8\,{b_N^*}^{5/2} h\,(\cos(\sqrt{b_N^*}h))^3 \sin(\sqrt{b_N^*}h)\Big) \\ \times\,(\tan(\sqrt{b_N^*}h))^2 {b_N^*}^{-\frac{5}{2}}&\Big(2\,h - \sin(2\,\sqrt{b_N^*}h)\frac{1}{\sqrt{b_N^*}}\Big)^{-\frac{3}{2}}. \end{aligned} \tag{6.42}$$

**Proposition 6.11.** The coefficient $C$ is negative if $N$ is odd and positive if $N$ is even.

*Proof.* We note that

$$C = -\frac{1}{2h^6 (b_N^*)^{\frac{5}{2}}}\sec(\tilde{b})\tan^2(\tilde{b})\Big(\frac{2\tilde{b} - \sin(2\tilde{b})}{\sqrt{b_N^*}}\Big)^{-\frac{3}{2}} D,$$

where

$$\begin{aligned} D = 8\tilde{b}^2\cos^4(\tilde{b})h^2 - 8\sin^2(2\tilde{b})h^5 \\ + \tilde{b}^4(-1 - 4\tilde{b}^2\cos(2\tilde{b}) + \cos(4\tilde{b}) + 8\tilde{b}\cos^3(\tilde{b})\sin(\tilde{b})) \end{aligned}$$

and $\tilde{b} = \sqrt{b_N^*}h$. For fixed $\tilde{b}$ we find that

$$\frac{\mathrm{d}D}{\mathrm{d}h} = 16\tilde{b}\cos^4(\tilde{b})h - 40\sin^2(2\tilde{b})h^4,$$

so that $D$ has a unique global maximum at

$$h = \Big(\frac{2}{5}\Big)^{\frac{1}{3}}\tilde{b}^{\frac{2}{3}}\cos^{\frac{4}{3}}(\tilde{b})\operatorname{cosec}^{\frac{2}{3}}(2\tilde{b});$$

its value is

$$D_{\max} = \tilde{b}^4(-1 + \cos(4\tilde{b})) + \frac{24}{5}\Big(\frac{2}{5}\Big)^{\frac{1}{3}}\tilde{b}^{\frac{10}{3}}\cos^{\frac{20}{3}}(\tilde{b})\operatorname{cosec}^{\frac{4}{3}}(2\tilde{b}) + 8\tilde{b}^5\cos^3(\tilde{b}) - 8b^6\sin^2(\tilde{b}).$$

Noting that

$$\begin{aligned} -1 + \cos(4\tilde{b}) &\le 0, \\ \cos^{\frac{20}{3}}(\tilde{b})\operatorname{cosec}^{\frac{4}{3}}(2\tilde{b}) &\le \frac{1}{8.2^{\frac{1}{3}}}, \\ \cos^3(\tilde{b})\sin(\tilde{b}) &\le \frac{1}{4}, \\ -\sin^2(\tilde{b}) &\le -\frac{1}{2} \end{aligned}$$

for $\tilde{b} \in ((N + 1/4)\pi, (N + 1/2)\pi)$, we conclude that

$$D_{\max} \le 2\tilde{b}^5 + \frac{3}{5}\frac{2^{\frac{1}{3}}}{5^{\frac{2}{3}}}\tilde{b}^{\frac{10}{3}} - \tilde{b}^6$$

for $\tilde{b} > 5\pi/4$. The sign of $C$ therefore agrees with the sign of $\sec(\tilde{b})$ in the interval $((N + 1/4)\pi, (N + 1/2)\pi)$. □

## 6.3 Periodic steady gravity-capillary water waves with localised transverse profiles

In this section we consider gravity-capillary steady waves on the surface of water bounded below by a rigid horizontal bottom and above by a free surface. In a dimensionless Cartesian coordinate system moving with the wave the fluid domain is

$$\{(x,y,z) : x,y,z \in \mathbb{R}, 0 < y < 1+\eta(x,z)\}$$

for some profile function $\eta\colon \mathbb{R}^2 \to (-1,\infty)$ which is $2\pi/\tau$-periodic in the $x$-direction. Working in dimensionless variables, we seek the velocity field in the form $(\phi_x, \phi_y, \phi_z)$, where the *velocity potential* $\phi\colon \mathbb{R}^3 \to \mathbb{R}$ satisfies the boundary-value problem

$$\tau^2\phi_{xx} + \phi_{yy} + \phi_{zz} = 0, \qquad 0 < y < 1+\eta, \tag{6.43}$$
$$\phi_y = 0, \qquad y = 0, \tag{6.44}$$
$$\phi_y = \tau^2\eta_x\phi_x - \eta_z\phi_z - \tau\eta_x, \qquad y = 1+\eta \tag{6.45}$$

and

$$\begin{aligned} &-\tau\phi_x + \frac{1}{2}(\tau^2\phi_x^2 + \phi_y^2 + \phi_z^2) + \alpha\eta \\ &\quad - \beta\tau^2\left[\frac{\eta_x}{\sqrt{1+\tau^2\eta_x^2+\eta_z^2}}\right]_x - \beta\left[\frac{\eta_z}{\sqrt{1+\tau^2\eta_x^2+\eta_z^2}}\right]_z = 0, \quad y = 1+\eta. \end{aligned} \tag{6.46}$$

Here the period in the $x$-direction has been normalised to $2\pi$ and $\alpha, \beta$ are dimensionless parameters which measure respectively the speed of the wave and the strength of surface tension (see Groves [6]). In this section we apply the results of Chapter 5 to establish the existence of solutions to equations (6.43) – (6.46) with localised profiles which decay to small ripples in the transversal direction (see Figure 6.3). We introduce a bifurcation parameter by writing $(\beta, \alpha) = (\beta_0, \alpha_0 + \varepsilon)$, where the values $(\beta_0, \alpha_0)$ are chosen later.

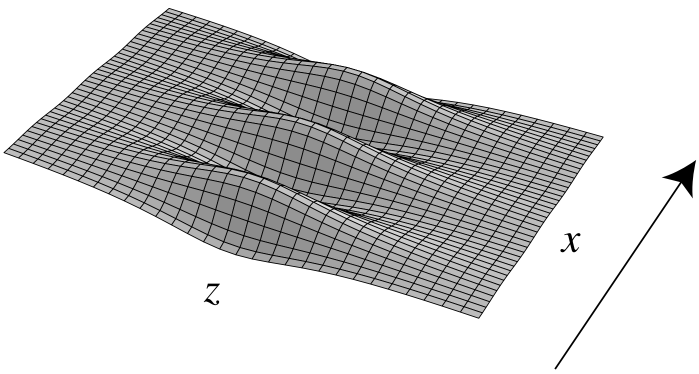



Figure 6.3: A steady wave which is periodic in the direction of travel and spatially localised in the transverse direction.

### 6.3.1 Formulation as an evolutionary system

We begin by transforming equations (6.43) – (6.46) to a boundary-value problem in the fixed strip $\mathbb{R} \times (0,1) \times (0,2\pi)$. In terms of the new coordinates

$$\tilde{x} = x, \quad \tilde{y} = \frac{y}{1+\eta(x,z)}, \quad \tilde{z} = z$$

and variable

$$\Phi(\tilde{x},\tilde{y},\tilde{z}) = \phi(x,y,z),$$

we find that

$$\begin{aligned}
\tau^2\Phi_{\tilde{x}\tilde{x}} + \frac{1}{(1+\eta)^2}\Phi_{\tilde{y}\tilde{y}} + \Phi_{\tilde{z}\tilde{z}} = \tau^2\Big(&2\frac{\eta_{\tilde{x}}}{1+\eta}\tilde{y}\Phi_{\tilde{x}\tilde{y}} + \big(\frac{\eta_{\tilde{x}\tilde{x}}}{1+\eta} - 2\frac{\eta_{\tilde{x}}^2}{(1+\eta)^2}\big)\tilde{y}\Phi_{\tilde{y}} \\
&- \big(\frac{\eta_{\tilde{x}}}{1+\eta}\tilde{y}\big)^2\Phi_{\tilde{y}\tilde{y}}\Big) \\
&+ 2\frac{\eta_{\tilde{z}}}{1+\eta}\tilde{y}\Phi_{\tilde{z}\tilde{y}} + \big(\frac{\eta_{\tilde{z}\tilde{z}}}{1+\eta} - 2\frac{\eta_{\tilde{z}}^2}{(1+\eta)^2}\big)\tilde{y}\Phi_{\tilde{y}} \\
&- \big(\frac{\eta_{\tilde{z}}}{1+\eta}\tilde{y}\big)^2\Phi_{\tilde{y}\tilde{y}}, \qquad 0 < \tilde{y} < 1, \qquad (6.47) \\
\Phi_{\tilde{y}} = 0, \qquad & \qquad \tilde{y} = 0, \qquad (6.48) \\
\frac{1}{1+\eta}\Phi_{\tilde{y}} = \tau^2\eta_{\tilde{x}}\big(&\Phi_{\tilde{x}} - \frac{\eta_{\tilde{x}}}{1+\eta}\tilde{y}\Phi_{\tilde{y}}\big) \\
&- \eta_{\tilde{z}}\big(\Phi_{\tilde{z}} - \frac{\eta_{\tilde{z}}}{1+\eta}\tilde{y}\Phi_{\tilde{y}}\big) - \tau\eta_{\tilde{x}}, \qquad \tilde{y} = 1 \qquad (6.49)
\end{aligned}$$

and

$$\begin{aligned}
&-\tau\big(\Phi_{\tilde{x}} - \frac{\eta_{\tilde{x}}}{1+\eta}\tilde{y}\Phi_{\tilde{y}}\big) + \frac{1}{2}\Big(\tau^2\big(\Phi_{\tilde{x}} - \frac{\eta_{\tilde{x}}}{1+\eta}\tilde{y}\Phi_{\tilde{y}}\big)^2 \\
&+ \frac{1}{(1+\eta)^2}\Phi_{\tilde{y}}^2 + \big(\Phi_{\tilde{z}} - \frac{\eta_{\tilde{z}}}{1+\eta}\tilde{y}\Phi_{\tilde{y}}\big)^2\Big) \\
&+ (\alpha_0+\varepsilon)\eta - \beta_0\tau^2\Big[\frac{\eta_{\tilde{x}}}{\sqrt{1+\tau^2\eta_{\tilde{x}}^2+\eta_{\tilde{z}}^2}}\Big]_{\tilde{x}} - \beta_0\Big[\frac{\eta_{\tilde{z}}}{\sqrt{1+\tau^2\eta_{\tilde{x}}^2+\eta_{\tilde{z}}^2}}\Big]_{\tilde{z}} = 0, \quad \tilde{y} = 1. \quad (6.50)
\end{aligned}$$

Introducing additional variables

$$\begin{aligned}
\omega &= -\int_0^1 \Big(\Phi_{\tilde{z}} - \tilde{y}\frac{\Phi_{\tilde{y}}\eta_{\tilde{z}}}{1+\eta}\Big)\tilde{y}\Phi_{\tilde{y}}\,\mathrm{d}\tilde{y} + \beta_0\frac{\eta_{\tilde{z}}}{\sqrt{1+\tau^2\eta_{\tilde{x}}^2+\eta_{\tilde{z}}^2}}, \\
\Psi &= \Big(\Phi_{\tilde{z}} - \tilde{y}\frac{\Phi_{\tilde{z}}\eta_{\tilde{z}}}{1+\eta}\Big)(1+\eta)
\end{aligned}$$

and setting

$$W = \omega + \frac{1}{1+\eta}\int_0^1 \Psi\tilde{y}\Phi_{\tilde{y}}\,\mathrm{d}\tilde{y},$$

we find equations (6.47) – (6.50) to be equivalent to the spatial evolutionary system

$$\eta_{\tilde{z}} = \Big(\frac{1+\tau^2\eta_{\tilde{x}}^2}{\beta_0^2 - W^2}\Big)^{\frac{1}{2}}W, \qquad (6.51)$$

$$
\begin{aligned}
\omega_{\tilde z} = {} & \frac{W}{(1+\eta)^2}\Big(\frac{1+\tau^2\eta_{\tilde x}^2}{\beta_0^2-W^2}\Big)^{\frac12}\int_0^1 \Psi\tilde y\Phi_{\tilde y}\,\mathrm{d}\tilde y - \tau^2\Big[\eta_{\tilde x}\Big(\frac{\beta_0^2-W^2}{1+\tau^2\eta_{\tilde x}^2}\Big)^{\frac12}\Big]_{\tilde x} \\
& + \int_0^1 \Bigg(\frac{\Psi^2-\Phi_{\tilde y}^2}{2(1+\eta)^2} + \frac{\tau^2}{2}\Big(\Phi_{\tilde x} - \frac{\Phi_{\tilde y}\tilde y\eta_{\tilde x}}{1+\eta}\Big)^2 \\
& \qquad + \tau^2\Big(\Phi_{\tilde x} - \frac{\Phi_{\tilde y}\tilde y\eta_{\tilde x}}{1+\eta}\Big)\frac{\Phi_{\tilde y}\tilde y\eta_{\tilde x}}{1+\eta} + \tau^2\Big[\Big(\Phi_{\tilde x} - \frac{\Phi_{\tilde y}\tilde y\eta_{\tilde x}}{1+\eta}\Big)\Phi_{\tilde y}\tilde y\Big]_{\tilde x}\Bigg)\,\mathrm{d}\tilde y \\
& + (\alpha_0+\varepsilon)\eta - \tau\Phi_{\tilde x}|_{\tilde y=1}, \qquad (6.52)\\
\Phi_{\tilde z} = {} & \frac{\Psi}{1+\eta} + \Big(\frac{1+\tau^2\eta_{\tilde x}^2}{\beta_0^2-W^2}\Big)^{\frac12}\frac{\tilde y\Phi_{\tilde y}W}{1+\eta}, \qquad (6.53)\\
\Psi_{\tilde z} = {} & -\frac{\Phi_{\tilde y\tilde y}}{1+\eta} - \tau^2\Big[(1+\eta)\Big(\Phi_{\tilde x} - \frac{\Phi_{\tilde y}\tilde y\eta_{\tilde x}}{1+\eta}\Big)\Big]_{\tilde x} + \tau^2\Big[\tilde y\eta_{\tilde x}\Big(\Phi_{\tilde x} - \frac{\Phi_{\tilde y}\tilde y\eta_{\tilde x}}{1+\eta}\Big)\Big]_{\tilde y} \\
& + \frac{W(\tilde y\Psi)_{\tilde y}}{1+\eta}\Big(\frac{1+\tau^2\eta_{\tilde x}^2}{\beta_0^2-W^2}\Big)^{\frac12} \qquad (6.54)
\end{aligned}
$$

with boundary conditions

$$
\begin{aligned}
\Phi_{\tilde y} &= 0, && \tilde y = 0, \qquad (6.55)\\
\tau\eta_{\tilde x} + \frac{\Phi_{\tilde y}}{1+\eta} &= \tau^2\eta_{\tilde x}\Big(\Phi_{\tilde x} - \frac{\Phi_{\tilde y}\eta_{\tilde x}}{1+\eta}\Big) + \frac{W\Psi}{1+\eta}\Big(\frac{1+\tau^2\eta_{\tilde x}^2}{\beta_0^2-W^2}\Big)^{\frac12}, && \tilde y = 1. \qquad (6.56)
\end{aligned}
$$

Equations (6.51) – (6.54) and boundary conditions (6.55) and (6.56) are invariant under the transformation $\Phi \mapsto \Phi + c$ for any constant $c$. To eliminate this symmetry it is convenient to replace $\Phi$, $\Psi$ with new variables $\Psi'$, $\Psi'$, $\Phi_0$, $\Psi_0$, where

$$
\begin{aligned}
\Phi_0 &= \frac{1}{2\pi}\int_\Sigma \Phi\,\mathrm{d}\tilde y\,\mathrm{d}\tilde x,\\
\Psi_0 &= \frac{1}{2\pi}\int_\Sigma \Psi\,\mathrm{d}\tilde y\,\mathrm{d}\tilde x,\\
\Phi' &= \Phi - \Phi_0,\\
\Psi' &= \Psi - \Psi_0
\end{aligned}
$$

and $\Sigma = (0,1)\times(0,2\pi)$. We note that

$$
\begin{aligned}
\int_\Sigma \Phi'\,\mathrm{d}\tilde y\,\mathrm{d}\tilde x &= 0,\\
\int_\Sigma \Psi'\,\mathrm{d}\tilde y\,\mathrm{d}\tilde x &= 0.
\end{aligned}
$$

The new variables satisfy the equations

$$
\eta_{\tilde z} = \Big(\frac{1+\tau^2\eta_{\tilde x}^2}{\beta_0^2-W^2}\Big)^{\frac12} W, \qquad (6.57)
$$

$$
\begin{aligned}
\omega_{\tilde{z}} = {} & \frac{W}{(1+\eta)^2}\left(\frac{1+\tau^2\eta_{\tilde{x}}^2}{\beta_0^2-W^2}\right)^{\frac{1}{2}}\int_0^1(\Psi'+\Psi_0)\tilde{y}\Phi'_{\tilde{y}}\,\mathrm{d}\tilde{y} - \tau^2\left[\eta_{\tilde{x}}\left(\frac{\beta_0^2-W^2}{1+\tau^2\eta_{\tilde{x}}^2}\right)^{\frac{1}{2}}\right]_{\tilde{x}} \\
& + \int_0^1\left(\frac{(\Psi'+\Psi_0)^2-{\Phi'_{\tilde{y}}}^2}{2(1+\eta)^2} + \frac{\tau^2}{2}\left(\Phi'_{\tilde{x}} - \frac{\Phi'_{\tilde{y}}\tilde{y}\eta_{\tilde{x}}}{1+\eta}\right)^2\right. \\
& \qquad + \tau^2\left(\Phi'_{\tilde{x}} - \frac{\Phi'_{\tilde{y}}\tilde{y}\eta_{\tilde{x}}}{1+\eta}\right)\frac{\Phi'_{\tilde{y}}\tilde{y}\eta_{\tilde{x}}}{1+\eta} \\
& \qquad \left. + \tau^2\left[\left(\Phi'_{\tilde{x}} - \frac{\Phi'_{\tilde{y}}\tilde{y}\eta_{\tilde{x}}}{1+\eta}\right)\Phi'_{\tilde{y}}\tilde{y}\right]_{\tilde{x}}\right)\mathrm{d}\tilde{y} \\
& + (\alpha_0+\varepsilon)\eta - \tau\Phi'_{\tilde{x}}|_{\tilde{y}=1}, \qquad (6.58)\\
\Phi'_{\tilde{z}} = {} & \frac{\Psi'+\Psi_0}{1+\eta} + \left(\frac{1+\tau^2\eta_{\tilde{x}}^2}{\beta_0^2-W^2}\right)^{\frac{1}{2}}\frac{\tilde{y}\Phi'_{\tilde{y}}W}{1+\eta} \\
& - \frac{1}{2\pi}\int_\Sigma\left(\frac{\Psi'+\Psi_0}{1+\eta} + \left(\frac{1+\tau^2\eta_{\tilde{x}}^2}{\beta_0^2-W^2}\right)^{\frac{1}{2}}\frac{\tilde{y}\Phi'_{\tilde{y}}W}{1+\eta}\right)\mathrm{d}\tilde{y}\,\mathrm{d}\tilde{x}, \qquad (6.59)\\
\Psi'_{\tilde{z}} = {} & -\frac{\Phi'_{\tilde{y}\tilde{y}}}{1+\eta} - \tau^2\left[(1+\eta)\left(\Phi'_{\tilde{x}} - \frac{\Phi'_{\tilde{y}}\tilde{y}\eta_{\tilde{x}}}{1+\eta}\right)\right]_{\tilde{x}} \\
& + \tau^2\left[\tilde{y}\eta_{\tilde{x}}\left(\Phi'_{\tilde{x}} - \frac{\Phi'_{\tilde{y}}\tilde{y}\eta_{\tilde{x}}}{1+\eta}\right)\right]_{\tilde{y}} + \frac{W\big(\tilde{y}(\Psi'+\Phi_0)\big)_{\tilde{y}}}{1+\eta}\left(\frac{1+\tau^2\eta_{\tilde{x}}^2}{\beta_0^2-W^2}\right)^{\frac{1}{2}}, \qquad (6.60)\\
\Phi_{0\tilde{z}} = {} & \frac{1}{2\pi}\int_\Sigma\left(\frac{\Psi'+\Psi_0}{1+\eta} - \left(\frac{1+\tau^2\eta_{\tilde{x}}^2}{\beta_0^2-W^2}\right)^{\frac{1}{2}}\frac{\tilde{y}\Phi'_{\tilde{y}}W}{1+\eta}\right)\mathrm{d}\tilde{y}\,\mathrm{d}\tilde{x}, \qquad (6.61)\\
\Psi_{0\tilde{z}} = {} & 0, \qquad (6.62)
\end{aligned}
$$

with boundary conditions

$$
\begin{aligned}
\Phi'_{\tilde{y}} &= 0, && \tilde{y}=0, \qquad (6.63)\\
\tau\eta_{\tilde{x}} + \frac{\Phi'_{\tilde{y}}}{1+\eta} &= \tau^2\eta_{\tilde{x}}\left(\Phi'_{\tilde{x}} - \frac{\Phi'_{\tilde{y}}\eta_{\tilde{x}}}{1+\eta}\right) + \frac{W(\Psi'+\Psi_0)}{1+\eta}\left(\frac{1+\tau^2\eta_{\tilde{x}}^2}{\beta_0^2-W^2}\right)^{\frac{1}{2}}, && \tilde{y}=1, \qquad (6.64)
\end{aligned}
$$

where

$$
W = \omega + \frac{1}{1+\eta}\int_0^1(\Psi'+\Psi_0)\tilde{y}\Phi'_{\tilde{y}}\,\mathrm{d}y.
$$

We note that $\Psi_0$ is a conserved quantity and $\Phi_0$ does not appear in equations (6.57) – (6.60), (6.63) and (6.64). We can therefore eliminate these variables by setting $\Psi_0 = 0$, solving equations (6.57) – (6.60), (6.63) and (6.64) for $(\eta, \omega, \Phi, \Psi)$ and recovering $\Phi_0$ by quadrature from equation (6.61).

Our final system of equations is thus given by

$$
\eta_{\tilde{z}} = \left(\frac{1+\tau^2\eta_{\tilde{x}}^2}{\beta_0^2-W^2}\right)^{\frac{1}{2}}W, \qquad (6.65)
$$

$$
\begin{aligned}
\omega_{\tilde z} = \frac{W}{(1+\eta)^2}\Big(\frac{1+\tau^2\eta_{\tilde x}^2}{\beta_0^2-W^2}\Big)^{\frac12}\int_0^1 \Psi'\tilde y\Phi'_{\tilde y}\,\mathrm{d}\tilde y - \tau^2\Big[\eta_{\tilde x}\Big(\frac{\beta_0^2-W^2}{1+\tau^2\eta_{\tilde x}^2}\Big)^{\frac12}\Big]_{\tilde x} \\
+ \int_0^1 \Big(\frac{\Psi'^2-\Phi_{\tilde y}'^{\,2}}{2(1+\eta)^2} + \frac{\tau^2}{2}\Big(\Phi'_{\tilde x} - \frac{\Phi'_{\tilde y}\tilde y\eta_{\tilde x}}{1+\eta}\Big)^2 \\
+ \tau^2\Big(\Phi'_{\tilde x} - \frac{\Phi'_{\tilde y}\tilde y\eta_{\tilde x}}{1+\eta}\Big)\frac{\Phi'_{\tilde y}\tilde y\eta_{\tilde x}}{1+\eta} \\
+ \tau^2\Big[\Big(\Phi'_{\tilde x} - \frac{\Phi'_{\tilde y}\tilde y\eta_{\tilde x}}{1+\eta}\Big)\Phi'_{\tilde y}\tilde y\Big]_{\tilde x}\Big)\,\mathrm{d}\tilde y \\
+ (\alpha_0+\varepsilon)\eta - \tau\Phi'_{\tilde x}|_{\tilde y=1}, \qquad (6.66)
\end{aligned}
$$

$$
\begin{aligned}
\Phi'_{\tilde z} = \frac{\Psi'}{1+\eta} + \Big(\frac{1+\tau^2\eta_{\tilde x}^2}{\beta_0^2-W^2}\Big)^{\frac12}\frac{\tilde y\Phi'_{\tilde y}W}{1+\eta} \\
- \frac{1}{2\pi}\int_\Sigma \Big(\frac{-\eta\Psi'}{1+\eta} + \Big(\frac{1+\tau^2\eta_{\tilde x}^2}{\beta_0^2-W^2}\Big)^{\frac12}\frac{\tilde y\Phi'_{\tilde y}W}{1+\eta}\Big)\,\mathrm{d}\tilde y\,\mathrm{d}\tilde x, \qquad (6.67)
\end{aligned}
$$

$$
\begin{aligned}
\Psi'_{\tilde z} = -\frac{\Phi'_{\tilde y\tilde y}}{1+\eta} - \tau^2\Big[(1+\eta)\Big(\Phi'_{\tilde x} - \frac{\Phi'_{\tilde y}\tilde y\eta_{\tilde x}}{1+\eta}\Big)\Big]_{\tilde x} \\
+ \tau^2\Big[\tilde y\eta_{\tilde x}\Big(\Phi'_{\tilde x} - \frac{\Phi'_{\tilde y}\tilde y\eta_{\tilde x}}{1+\eta}\Big)\Big]_{\tilde y} + \frac{W(\tilde y\Psi')_{\tilde y}}{1+\eta}\Big(\frac{1+\tau^2\eta_{\tilde x}^2}{\beta_0^2-W^2}\Big)^{\frac12}, \qquad (6.68)
\end{aligned}
$$

with boundary conditions

$$
\Phi'_{\tilde y} = 0, \qquad \tilde y = 0, \qquad (6.69)
$$

$$
\tau\eta_{\tilde x} + \frac{\Phi'_{\tilde y}}{1+\eta} = \tau^2\eta_{\tilde x}\Big(\Phi'_{\tilde x} - \frac{\Phi'_{\tilde y}\eta_{\tilde x}}{1+\eta}\Big) + \frac{W\Psi'}{1+\eta}\Big(\frac{1+\tau^2\eta_{\tilde x}^2}{\beta_0^2-W^2}\Big)^{\frac12}, \qquad \tilde y = 1, \qquad (6.70)
$$

where

$$
W = \omega + \frac{1}{1+\eta}\int_0^1 \Psi'\tilde y\Phi'_{\tilde y}\,\mathrm{d}\tilde y.
$$

**Proposition 6.12.** The system of equations (6.65) – (6.68) and boundary conditions (6.69) and (6.70) has the conserved quantity

$$
\begin{aligned}
H^\varepsilon(\eta,\omega,\Phi',\Psi') = \int_\Sigma \Big(\tau(1+\eta)\Phi'_{\tilde x} - \tau\Phi'_{\tilde y}\tilde y\eta_{\tilde x} + \frac{(\Psi')^2-\Phi_{\tilde y}'^{\,2}}{2(1+\eta)} \\
- \frac{1+\eta}{2}\tau^2\Big(\Phi'_{\tilde x} - \frac{\Phi'_{\tilde y}\tilde y\eta_{\tilde x}}{1+\eta}\Big)^2\Big)\,\mathrm{d}\tilde y\,\mathrm{d}\tilde x \\
+ \int_S \Big(-\frac12(\alpha_0+\varepsilon)\eta^2 + \beta_0 - (\beta_0^2-W^2)^{\frac12}(1+\tau^2\eta_{\tilde x}^2)^{\frac12}\Big)\,\mathrm{d}\tilde x,
\end{aligned}
$$

where $S = (0,2\pi)$ and $\Sigma = (0,1)\times(0,2\pi)$.

*Proof.* We find that

$$
\begin{aligned}
\frac{\mathrm{d}H}{\mathrm{d}\tilde z} = \int_S \omega_{\tilde z} W\Big(\frac{1+\tau^2\eta_{\tilde x}^2}{\beta_0^2-W^2}\Big)^{\frac12}\,\mathrm{d}\tilde x + \int_\Sigma \eta_{\tilde x\tilde z}\Big(\tau\Phi'_{\tilde x} - \tau\frac{\tilde y\Phi'_{\tilde y}\eta_{\tilde x}}{1+\eta}\Big)\tilde y\Phi'_{\tilde y}\,\mathrm{d}\tilde y\,\mathrm{d}\tilde x \\
&- \int_\Sigma \eta_{\tilde z}\Big(\frac{(\Psi')^2-{\Phi'_{\tilde y}}^2}{2(1+\eta)^2} + \frac12\Big(\tau\Phi'_{\tilde x} - \frac{\tau\tilde y\Phi'_{\tilde y}\eta_{\tilde x}}{1+\eta}\Big)^2 \\
&\qquad + \Big(\tau\Phi'_{\tilde x} - \tau\frac{\tilde y\Phi'_{\tilde y}\eta_{\tilde x}}{1+\eta}\Big)\tau\frac{\tilde y\Phi'_{\tilde y}\eta_{\tilde x}}{1+\eta}\Big)\,\mathrm{d}\tilde y\,\mathrm{d}\tilde x \\
&- \int_S \tau^2\eta_{\tilde x\tilde z}\eta_{\tilde x}\Big(\frac{\beta_0^2-W^2}{1+\tau^2\eta_{\tilde x}^2}\Big)^{\frac12}\,\mathrm{d}\tilde x - \int_\Sigma \eta_{\tilde z}\frac{W\Psi'\tilde y\Phi'_{\tilde y}}{(1+\eta)^2}\Big(\frac{1+\tau^2\eta_{\tilde x}^2}{\beta_0^2-W^2}\Big)^{\frac12}\,\mathrm{d}\tilde y\,\mathrm{d}\tilde x \\
&- \int_S(\alpha_0+\varepsilon)\eta\eta_{\tilde z}\,\mathrm{d}\tilde x + \tau\int_\Sigma(\eta_{\tilde z}\Phi'_{\tilde x} + \tilde y\Phi'_{\tilde y}\eta_{\tilde x\tilde z})\,\mathrm{d}\tilde y\,\mathrm{d}\tilde x \\
&+ \int_\Sigma \Phi'_{\tilde y\tilde z}\Big(\tau^2\Big(\Phi'_{\tilde x} - \frac{\tilde y\Phi'_{\tilde y}\eta_{\tilde x}}{1+\eta}\Big)\tilde y\eta_{\tilde x} + \frac{W\tilde y\Psi'}{1+\eta}\Big(\frac{1+\tau^2\eta_{\tilde x}^2}{\beta_0^2-W^2}\Big)^{\frac12} - \frac{\Phi'_{\tilde y}}{1+\eta}\Big)\,\mathrm{d}\tilde y\,\mathrm{d}\tilde x \\
&- \int_\Sigma \tau^2\Phi'_{\tilde x\tilde z}\Big(\Phi'_{\tilde x} - \frac{\tilde y\Phi'_{\tilde y}\eta_{\tilde x}}{1+\eta}\Big)(1+\eta)\,\mathrm{d}\tilde y\,\mathrm{d}\tilde x \\
&+ \int_\Sigma \tau\Big((1+\eta)\Phi'_{\tilde x\tilde z} - \Phi'_{\tilde y\tilde z}\tilde y\eta_{\tilde x}\Big)\,\mathrm{d}\tilde y\,\mathrm{d}\tilde x \\
&+ \int_\Sigma \Psi'_{\tilde z}\Big(\frac{\Psi'}{1+\eta} + \frac{W\tilde y\Phi'_{\tilde y}}{1+\eta}\Big(\frac{1+\tau^2\eta_{\tilde x}^2}{\beta_0^2-W^2}\Big)^{\frac12}\Big)\,\mathrm{d}\tilde y\,\mathrm{d}\tilde x,
\end{aligned}
$$

and, by integrating by parts in $\tilde y$ and $\tilde x$, therefore

$$
\begin{aligned}
\frac{\mathrm{d}H}{\mathrm{d}\tilde z} &= \int_S \omega_{\tilde z} W\Big(\frac{1+\tau^2\eta_{\tilde x}^2}{\beta_0^2-W^2}\Big)^{\frac12}\,\mathrm{d}\tilde x \\
&+ \int_S \eta_{\tilde z}\Big(-(\alpha_0+\varepsilon)\eta + \Big(\tau\eta_{\tilde x}\Big(\frac{\beta_0^2-W^2}{1+\tau^2\eta_{\tilde x}^2}\Big)^{\frac12}\Big)_{\tilde x} \\
&\qquad - \int_0^1 \Big(\frac{(\Psi')^2-{\Phi'_{\tilde y}}^2}{2(1+\eta)^2} + \frac12\tau^2\Big(\Phi'_{\tilde x} - \frac{\tilde y\Phi'_{\tilde y}\eta_{\tilde x}}{1+\eta}\Big)^2 \\
&\qquad\qquad + \tau^2\Big(\Phi'_{\tilde x} - \frac{\tilde y\Phi'_{\tilde y}\eta_{\tilde x}}{1+\eta}\Big)\frac{\tilde y\Phi'_{\tilde y}\eta_{\tilde x}}{1+\eta} + \frac{W\Psi'\tilde y\Phi'_{\tilde y}}{(1+\eta)^2}\Big(\frac{1+\tau^2\eta_{\tilde x}^2}{\beta_0^2-W^2}\Big)^{\frac12}\Big)\,\mathrm{d}\tilde y \\
&\qquad - \int_0^1 \Big(\tau^2\Big(\Phi'_{\tilde x} - \frac{\tilde y\Phi'_{\tilde y}\eta_{\tilde x}}{1+\eta}\Big)\tilde y\Phi'_{\tilde y}\Big)_{\tilde x}\,\mathrm{d}\tilde y + \int_0^1 \tau(\Phi'_{\tilde x} - \tilde y\Phi'_{\tilde x\tilde y})\,\mathrm{d}\tilde y\Big)\,\mathrm{d}\tilde x \\
&+ \int_\Sigma \Psi'_{\tilde z}\Big(\frac{\Psi'}{1+\eta} - \frac{W\tilde y\Phi'_{\tilde y}}{1+\eta}\Big(\frac{1+\tau^2\eta_{\tilde x}^2}{\beta_0^2-W^2}\Big)^{\frac12}\Big)\,\mathrm{d}\tilde y\,\mathrm{d}\tilde x \\
&- \int_S \Big(\int_0^1 \Phi'_{\tilde z}\Big(\tau^2\Big(\Phi'_{\tilde x} - \frac{\tilde y\Phi'_{\tilde y}\eta_{\tilde x}}{1+\eta}\Big)\tilde y\eta_{\tilde x} + \frac{W\tilde y\Psi'}{1+\eta}\Big(\frac{1+\tau^2\eta_{\tilde x}^2}{\beta_0^2-W^2}\Big)^{\frac12} - \frac{\Phi'_{\tilde y}}{1+\eta}\Big)_{\tilde y}\,\mathrm{d}\tilde y \\
&\qquad + \Big[\Phi'_{\tilde z}\Big(\tau^2\Big(\Phi'_{\tilde x} - \frac{\tilde y\Phi'_{\tilde y}\eta_{\tilde x}}{1+\tau\eta_{\tilde x}}\Big)\tilde y\eta_{\tilde x} + \frac{W\tilde y\Psi'}{1+\eta}\Big(\frac{1+\tau^2\eta_{\tilde x}^2}{\beta_0^2-W^2}\Big)^{\frac12} - \frac{\Phi'_{\tilde y}}{1+\eta}\Big)\Big]_0^1\Big)\,\mathrm{d}\tilde x \\
&+ \int_\Sigma \Phi'_{\tilde z}\Big(\tau\Big(\Phi'_{\tilde x} - \frac{\tilde y\Phi'_{\tilde y}\eta_{\tilde x}}{1+\eta}\Big)(1+\eta)\Big)_{\tilde x}\,\mathrm{d}\tilde y\,\mathrm{d}\tilde x \\
&+ \int_S \Big(\int_0^1 \tau(-\eta_{\tilde x}\Phi'_{\tilde z} + \eta_{\tilde x}\Phi'_{\tilde z})\,\mathrm{d}\tilde y - [\tilde y\tau\eta_{\tilde x}\Phi'_{\tilde z}]_0^1\Big)\,\mathrm{d}\tilde x \\
&= \int_S(\omega_{\tilde z}\eta_{\tilde z} - \eta_{\tilde z}\omega_{\tilde z})\,\mathrm{d}\tilde x + \int_\Sigma(\Psi'_{\tilde z}\Phi'_{\tilde z} - \Phi'_{\tilde z}\Psi'_{\tilde z})\,\mathrm{d}\tilde y\,\mathrm{d}\tilde x \\
&+ \int_S \Big[\Big(\tau^2\Big(\Phi'_{\tilde x} - \frac{\tilde y\Phi'_{\tilde y}\eta_{\tilde x}}{1+\eta}\Big)\tilde y\eta_{\tilde x} + \frac{W\tilde y\Psi'}{1+\eta}\Big(\frac{1+\tau^2\eta_{\tilde x}^2}{\beta_0^2-W^2}\Big)^{\frac12} - \frac{\Phi'_{\tilde y}}{1+\eta} - \tilde y\tau\eta_{\tilde x}\Big)\Phi'_{\tilde z}\Big]_0^1\,\mathrm{d}\tilde x
\end{aligned}
$$

$$= 0,$$

where we have used equations (6.65) – (6.68) and boundary conditions (6.69), (6.70) in combination with the calculation

$$\begin{aligned}
&\int_\Sigma \Psi'_{\tilde{z}}\Big(\frac{\Psi'}{1+\eta}+\frac{W\tilde{y}\Phi'_{\tilde{y}}}{1+\eta}\Big(\frac{1+\tau^2\eta_{\tilde{x}}^2}{\beta_0^2-W^2}\Big)^{\frac{1}{2}}\Big)\,\mathrm{d}\tilde{y}\,\mathrm{d}\tilde{x}\\
&\quad=\int_\Sigma \Psi'_{\tilde{z}}\Big(\frac{\Psi'}{1+\eta}+\frac{W\tilde{y}\Phi'_{\tilde{y}}}{1+\eta}\Big(\frac{1+\tau^2\eta_{\tilde{x}}^2}{\beta_0^2-W^2}\Big)^{\frac{1}{2}}\Big)\,\mathrm{d}\tilde{y}\,\mathrm{d}\tilde{x}\\
&\qquad\quad-\frac{1}{2\pi}\int_\Sigma \Psi'_{\tilde{z}}\,\mathrm{d}\tilde{y}\,\mathrm{d}\tilde{x}\int_\Sigma\Big(\frac{\Psi'}{1+\eta}+\frac{W\tilde{y}\Phi'_{\tilde{y}}}{1+\eta}\Big(\frac{1+\tau^2\eta_{\tilde{x}}^2}{\beta_0^2-W^2}\Big)^{\frac{1}{2}}\Big)\,\mathrm{d}\tilde{y}\,\mathrm{d}\tilde{x}\\
&\quad=\int_\Sigma \Psi'_{\tilde{z}}\Big(\frac{\Psi'}{1+\eta}+\frac{W\tilde{y}\Phi'_{\tilde{y}}}{1+\eta}\Big(\frac{1+\tau^2\eta_{\tilde{x}}^2}{\beta_0^2-W^2}\Big)^{\frac{1}{2}}\\
&\qquad\quad-\frac{1}{2\pi}\int_\Sigma \frac{\Psi'}{1+\eta}+\frac{W\tilde{y}\Phi'_{\tilde{y}}}{1+\eta}\Big(\frac{1+\tau^2\eta_{\tilde{x}}^2}{\beta_0^2-W^2}\Big)^{\frac{1}{2}}\,\mathrm{d}\tilde{y}\,\mathrm{d}\tilde{x}\Big)\,\mathrm{d}\tilde{y}\,\mathrm{d}\tilde{x}\\
&\quad=\int_\Sigma \Psi'_{\tilde{z}}\Phi'_{\tilde{z}}\,\mathrm{d}\tilde{y}\,\mathrm{d}\tilde{x},
\end{aligned}$$

noting that

$$\frac{1}{2\pi}\int_\Sigma \Psi'_{\tilde{z}}\,\mathrm{d}\tilde{y}\,\mathrm{d}\tilde{x} = 0.$$

□

We define the spaces

$$X_s = H^{s+1}_{\mathrm{per}}(S)\times H^s_{\mathrm{per}}(S)\times \mathring{H}^{s+1}_{\mathrm{per}}(\Sigma)\times \mathring{H}^s_{\mathrm{per}}(\Sigma)$$

for $s \geq 0$, where

$$\mathring{H}^s_{\mathrm{per}}(\Sigma) = \{u \in H^s_{\mathrm{per}}(\Sigma) : \int_\Sigma u\,\mathrm{d}\tilde{y}\,\mathrm{d}\tilde{x} = 0\},$$

and $\mathcal{X} = X_0$, $\mathcal{Y} = X_1$. We can then write equations (6.51) – (6.54) and boundary conditions (6.55) and (6.56) as the evolutionary system

$$\begin{pmatrix}\eta\\ \omega\\ \Phi'\\ \Psi'\end{pmatrix}_{\tilde{z}} = f^\varepsilon(\eta,\omega,\Phi',\Psi'); \tag{6.71}$$

the domain of the vector field on the right-hand side of this equation is given by

$$\mathcal{D} = \{(\eta,\omega,\Phi',\Psi') \in \mathcal{Y} : \text{(6.69) and (6.70) are satisfied}\}.$$

The following proposition shows that $f^\varepsilon$ and $H^\varepsilon$ take values in respectively $\mathcal{X}$ and $\mathbb{R}$ and are analytic at the origin in $\mathcal{Y}$ (see Lions et al. [20, Theorems 9.4 and 9.8], Bagri and Groves [3, Proposition 2.1] and Buffoni and Toland [5]).

**Proposition 6.13.**

(i) The spaces $H^{s_1}_{\mathrm{per}}(S)$ and $H^{s_2}_{\mathrm{per}}(\Sigma)$ are Banach algebras for $s_1 > 1/2$ and $s_2 > 1$.

(ii) The formulae $w \mapsto w|_{\tilde{y}=0}$, $w \mapsto w|_{\tilde{y}=1}$ define continuous linear mappings $H^s_{\mathrm{per}}(\Sigma) \to H^{s-1/2}_{\mathrm{per}}(S)$ for each $s > 0$.

(iii) The formula
$$(w_1, w_2) \mapsto w_1 w_2$$
defines continuous bilinear mappings $L^2_{\mathrm{per}}(\Sigma) \times H^1_{\mathrm{per}}(S) \to L^2_{\mathrm{per}}(\Sigma)$, $H^1_{\mathrm{per}}(\Sigma) \times L^2_{\mathrm{per}}(S) \to L^2_{\mathrm{per}}(\Sigma)$ and $H^1_{\mathrm{per}}(\Sigma) \times H^1_{\mathrm{per}}(S) \to H^1_{\mathrm{per}}(\Sigma)$.

(iv) The formula
$$(w_1, w_2) \mapsto \int_0^1 w_1(\cdot, \tilde{y}) w_2(\cdot, \tilde{y}) \,\mathrm{d}\tilde{y}$$
defines continuous bilinear mappings $L^2_{\mathrm{per}}(\Sigma) \times H^1_{\mathrm{per}}(S) \to L^2_{\mathrm{per}}(S)$, $H^1_{\mathrm{per}}(\Sigma) \times L^2_{\mathrm{per}}(S) \to L^2_{\mathrm{per}}(S)$ and $H^1_{\mathrm{per}}(\Sigma) \times H^1_{\mathrm{per}}(S) \to H^1_{\mathrm{per}}(S)$.

(v) Let $f\colon \mathbb{R}^n \to \mathbb{R}$ be analytic at the origin and satisfy $f(0) = 0$ and let $s > 1/2$. The induced composition operator $f\colon H^s(\mathbb{R}) \to \mathbb{R}$ is analytic at the origin.

The next step is to linearise the nonlinear boundary conditions associated with equation (6.71). To this end note that the boundary conditions are equivalent to
$$\Phi'_{\tilde{y}} + \tilde{y}\tau\eta_x = F(\eta, \omega, \Phi', \Psi'), \quad \tilde{y} \in \{0, 1\},$$
where
$$F(\eta, \omega, \Phi', \Psi') = -\tilde{y}\tau\eta\eta_{\tilde{x}} + \tilde{y}\tau^2\eta_{\tilde{x}}(\Phi'_{\tilde{x}}(1+\eta) - \Phi'_{\tilde{y}}\eta_{\tilde{x}}) + \tilde{y}W\Psi'\Big(\frac{1+\tau^2\eta^2_{\tilde{x}}}{\beta^2 - W^2}\Big)^{\frac{1}{2}}.$$
The requisite change of variable is given by
$$(\rho, \theta, \Gamma, \xi) = (\eta, \omega, \Phi' - \chi_{\tilde{y}}, \Psi') =: G(\eta, \omega, \Phi', \Psi')$$
with $\chi = \Delta^{-1}F(\rho, \omega, \Phi', \Psi')$ being the unique solution to the elliptic boundary value problem
$$\begin{aligned} \tau^2\chi_{\tilde{x}\tilde{x}} + \chi_{\tilde{y}\tilde{y}} &= F(\rho, \omega, \Phi', \Psi'), && (\tilde{x}, \tilde{y}) \in \Sigma, \\ \chi &= 0, && \tilde{y} \in \{0, 1\}. \end{aligned}$$
We note that
$$\begin{aligned} \int_\Sigma \Gamma \,\mathrm{d}x\,\mathrm{d}y &= \int_\Sigma \Phi' \,\mathrm{d}\tilde{x}\,\mathrm{d}\tilde{y} - \int_\Sigma \chi_{\tilde{y}} \,\mathrm{d}\tilde{y}\,\mathrm{d}\tilde{x} \\ &= \int_\Sigma \Phi' \,\mathrm{d}\tilde{x}\,\mathrm{d}\tilde{y} - \int_S [\chi]_0^1 \,\mathrm{d}\tilde{x} \\ &= \int_\Sigma \Phi' \,\mathrm{d}\tilde{x}\,\mathrm{d}\tilde{y}. \end{aligned}$$

**Lemma 6.14.**

(i) The mapping $G$ is an analytic diffeomorphism from the neighbourhood $V$ of 0 in $\mathcal{Y}$ onto a neighbourhood $\tilde{V}$ of the origin in $\mathcal{Y}$.

(ii) The operator $\mathrm{d}G[v] \colon \mathcal{Y} \to \mathcal{Y}$ extends to an isomorphism $\widehat{\mathrm{d}G}[v] \colon \mathcal{X} \to \mathcal{X}$ for each $v \in V$. The operators $\widehat{\mathrm{d}G}[v]$, $\widehat{\mathrm{d}G}[v]^{-1} \in \mathcal{L}(\mathcal{X})$ depend analytically upon $v \in V$.

*Proof.*

(i) This result follows by the analytic inverse function theorem since $G$ is a near identity transformation.

(ii) Using Proposition 6.13 we note that the operator $\mathrm{d}F[v] \in \mathcal{L}(\mathcal{Y}; H^1_{\mathrm{per}}(\Sigma))$ given by the formula

$$\begin{aligned}
&\mathrm{d}F[\eta,\omega,\Phi',\Psi'](\tilde{\eta},\tilde{\omega},\tilde{\Phi}',\tilde{\Psi}') \\
&\quad = -\tilde{y}\tau\tilde{\eta}\eta_{\tilde{x}} - \tilde{y}\tau\eta\tilde{\eta}_{\tilde{x}} + \tilde{y}\tau^2\tilde{\eta}_{\tilde{x}}\Big(\Phi'_{\tilde{x}}(1+\eta) - \Phi'_{\tilde{y}}\eta_{\tilde{x}}\Big) \\
&\qquad + \tilde{y}\tau^2\eta_{\tilde{x}}\Big(\tilde{\Phi}'_{\tilde{x}}(1+\eta) - \tilde{\Phi}'_{\tilde{y}}\eta_{\tilde{x}}\Big) + \tilde{y}\tau^2\eta_{\tilde{x}}(\Phi'_{\tilde{x}}\tilde{\eta} - \Phi'_{\tilde{y}}\tilde{\eta}_{\tilde{x}}) \\
&\qquad + \tilde{y}W\tilde{\Psi}'\left(\frac{1+\tau^2\eta_{\tilde{x}}^2}{\beta^2 - W^2}\right)^{\frac{1}{2}} + \frac{\tilde{y}W\Psi'}{(\beta_0^2 - W^2)^{\frac{1}{2}}}\frac{\tau^2\eta_{\tilde{x}}\tilde{\eta}_{\tilde{x}}}{(1+\tau^2\eta_{\tilde{x}})^{\frac{1}{2}}} \\
&\qquad + \beta_0^2\tilde{y}W^2\Psi'\frac{(1+\tau^2\eta_{\tilde{x}}^2)^{\frac{1}{2}}}{(\beta_0^2 - W^2)^{\frac{3}{2}}}\Bigg(\tilde{\omega} - \frac{\tilde{\eta}}{(1+\eta)^2}\int_0^1 \Psi'\tilde{y}\Phi'_{\tilde{y}}\,\mathrm{d}\tilde{y} \\
&\qquad\qquad + \frac{1}{1+\eta}\int_0^1 (\tilde{\Psi}'\tilde{y}\Phi'_{\tilde{y}} + \Psi'\tilde{y}\tilde{\Phi}'_{\tilde{y}})\,\mathrm{d}\tilde{y}\Bigg)
\end{aligned}$$

extends to $\widehat{\mathrm{d}F}[v] \in \mathcal{L}(\mathcal{X}; L^2_{\mathrm{per}}(\Sigma))$ which depends analytically upon $v \in V$. Hence $\widehat{\mathrm{d}G}[v] \in \mathcal{L}(\mathcal{Y})$ extends to $\widehat{\mathrm{d}G}[v] \in \mathcal{L}(\mathcal{X})$ which depends analytically upon $v \in V$. Furthermore,

$$\widehat{\mathrm{d}G}[0] = I$$

is an isomorphism, and the result follows from the analytic implicit and inverse function theorems.

□

The change of variable given by $G$ transforms equation (6.71) into the evolutionary system

$$\begin{pmatrix}\rho\\ \theta\\ \Gamma\\ \xi\end{pmatrix}_{\tilde{z}} = k^{\varepsilon}(\rho,\theta,\Gamma,\xi), \tag{6.72}$$

where

$$k^{\varepsilon}(\rho,\theta,\Gamma,\xi) = \widehat{\mathrm{d}G}[G^{-1}(\rho,\theta,\Gamma,\xi)]\Big(f^{\varepsilon}(G^{-1}(\rho,\theta,\Gamma,\xi))\Big)$$

takes values in $\mathcal{X}$ and is analytic at the origin in $\mathbb{R} \times \mathcal{Y}$. The domain of the vector field on the right-hand side of equation (6.72) is

$$\mathcal{D} = \Big\{(\rho, \theta, \Gamma, \xi) \in \mathcal{Y} : \Gamma_{\tilde{y}}|_{\tilde{y}=0} = 0, \tau\rho_x + \Gamma_{\tilde{y}}|_{\tilde{y}=1} = 0\Big\}$$

because

$$\begin{aligned}\Gamma_{\tilde{y}}|_{\tilde{y}\in\{0,1\}} + \tilde{y}\tau\rho_{\tilde{x}} &= \Phi_{\tilde{y}} + \tilde{y}\tau\eta_{\tilde{x}} - \chi_{\tilde{y}\tilde{y}}|_{\tilde{y}\in\{0,1\}} \\ &= \Phi_{\tilde{y}} + \tilde{y}\tau\eta_{\tilde{x}} + \tau^2\chi_{\tilde{x}\tilde{x}} - F(\eta, \omega, \Phi, \Psi)|_{\tilde{y}\in\{0,1\}} \\ &= \Phi_{\tilde{y}} + \tilde{y}\tau\eta_{\tilde{x}} - F(\eta, \omega, \Phi, \Psi)|_{\tilde{y}\in\{0,1\}}.\end{aligned}$$

We note that the linearisation of $k^\varepsilon$ is given by $\check{L}^\varepsilon \colon \mathcal{D} \subseteq \mathcal{X} \to \mathcal{X}$ with

$$\check{L}^\varepsilon \begin{pmatrix}\rho\\ \theta\\ \Gamma\\ \xi\end{pmatrix} = \begin{pmatrix}\frac{1}{\beta_0}\theta \\ -\tau\Gamma_{\tilde{x}}|_{\tilde{y}=1} + (\alpha_0+\varepsilon)\rho - \tau^2\beta_0\rho_{\tilde{x}\tilde{x}} \\ \xi \\ -\tau^2\Gamma_{\tilde{x}\tilde{x}} - \Gamma_{\tilde{y}\tilde{y}}\end{pmatrix} \tag{6.73}$$

(since $\mathrm{d}G[0] = I$) and this operator depends analytically upon $\varepsilon$. The $\varepsilon$-independent quadratic and cubic terms are given by

$$\begin{aligned}\frac{1}{2}\mathrm{d}^2k^0[0](\rho, \theta, \Gamma, \xi) = \frac{1}{\beta_0}\int_0^1 \tilde{y}\xi\Gamma_{\tilde{y}}\,\mathrm{d}\tilde{y} + \frac{1}{2}\int_0^1 \Big(\xi^2 - \Gamma_{\tilde{y}}^2\Big)\,\mathrm{d}\tilde{y}& \\ + \frac{\tau^2}{2}\int_0^1 \Big(\Gamma_{\tilde{x}}^2 + \tilde{y}\Gamma_{\tilde{x}}\Gamma_{\tilde{y}\tilde{x}} + \tilde{y}\Gamma_{\tilde{x}\tilde{x}}\Gamma_{\tilde{y}}\Big)\,\mathrm{d}\tilde{y}& \\ - \tau\Phi_{2\tilde{x}}|_{\tilde{y}=1} + \tau^2\tilde{y}\rho_{\tilde{x}}\Gamma_{\tilde{y}\tilde{x}} + \tau^2\rho_{\tilde{x}\tilde{x}}\tilde{y}\Gamma_{\tilde{y}} - \tau^2\rho\Gamma_{\tilde{x}\tilde{x}} - \tau^2\rho_{\tilde{x}}\Gamma_{\tilde{x}}& \\ + \tau\rho\rho_{\tilde{x}} + \rho\Gamma_{\tilde{y}\tilde{y}} - \rho\xi + \frac{\theta}{\beta_0}\tilde{y}\Gamma_{\tilde{y}}& \\ - \frac{1}{2\pi}\int_\Sigma \Big(-\theta\xi + \frac{\theta}{\beta_0}\tilde{y}\Gamma_{\tilde{y}}\Big)\,\mathrm{d}\tilde{y}\,\mathrm{d}\tilde{x}& \\ - \partial_{\tilde{y}}\Delta^{-1}\Big(\frac{\tilde{y}}{\beta_0}\xi(\alpha_0\rho - \beta_0\rho_{\tilde{x}\tilde{x}} - \tau\Gamma_{\tilde{x}}|_{\tilde{y}=1}) + \frac{\tilde{y}}{\beta_0}\theta(-\Gamma_{\tilde{y}\tilde{y}} - \tau^2\Gamma_{\tilde{x}\tilde{x}})& \\ - \frac{\tilde{y}\tau}{\beta_0}\theta\rho_{\tilde{x}} - \frac{\tilde{y}\tau}{\beta_0}\theta_{\tilde{x}}\rho + \frac{\tilde{y}}{\beta_0}\tau^2\theta_{\tilde{x}}\Gamma_{\tilde{x}} + \tilde{y}\tau^2\rho_{\tilde{x}}\xi_{\tilde{x}}\Big)&\end{aligned}$$

and

$$\begin{aligned}&\frac{1}{6}\mathrm{d}^3k[0](\rho, \theta, \Gamma, \xi) \\ &= \frac{1}{\beta}\int_0^1 \tilde{y}\xi\Phi_{2\tilde{y}}\,\mathrm{d}\tilde{y} - \frac{\rho}{\beta}\int_0^1 \tilde{y}\xi\Gamma_{\tilde{y}}\,\mathrm{d}\tilde{y} + \frac{\theta^3}{2\beta^3} + \frac{\tau^2\rho_{\tilde{x}}^2\theta}{4\beta}\int_0^1 \Big(\xi^2 - \Gamma_{\tilde{y}}^2\Big)\,\mathrm{d}\tilde{y} \\ &\quad + \frac{\tau^2}{2}\Big(\int_0^1 \Big(\Gamma_{\tilde{x}}^2 + \tilde{y}\Gamma_{\tilde{x}}\Gamma_{\tilde{y}\tilde{x}} + \tilde{y}\Gamma_{\tilde{x}\tilde{x}}\Gamma_{\tilde{y}}\Big)\,\mathrm{d}\tilde{y}\Big) - \tau\Phi_{2\tilde{x}}|_{\tilde{y}=1} - \frac{\theta\rho\xi}{\beta} \\ &\quad + \tau^2\rho_{\tilde{x}}^2\Gamma_{\tilde{y}} - 2\tau^2\rho_{\tilde{x}}^2\tilde{y}\Gamma_{\tilde{y}} + \tau^2\rho_{\tilde{x}\tilde{x}}\tilde{y}\Phi_{2\tilde{y}} - \frac{\theta\rho\tilde{y}\xi_{\tilde{y}}}{\beta} \\ &\quad - \rho^2\Gamma_{\tilde{y}\tilde{y}} + \tau^2\rho_{\tilde{x}}^2\tilde{y}\Gamma_{\tilde{y}\tilde{y}} - \tau^2\rho_{\tilde{x}}^2\tilde{y}^2\Gamma_{\tilde{y}\tilde{y}} + \rho\Phi_{2\tilde{y}\tilde{y}} - \tau^2\rho\rho_{\tilde{x}}\Gamma_{\tilde{x}} - \tau^2\rho_{\tilde{x}}\Phi_{2\tilde{x}} \\ &\quad - \tau^2\rho\rho_{\tilde{x}}\tilde{y}\Gamma_{\tilde{y}\tilde{x}} + \tau^2\rho_{\tilde{x}}\tilde{y}\Phi_{2\tilde{y}\tilde{x}} - \tau^2\rho\Phi_{2\tilde{x}\tilde{x}} + \rho^2\xi + \frac{\tilde{y}}{\beta}\Gamma_{\tilde{y}}\int_0^1 \tilde{y}\xi\Gamma_{\tilde{y}}\,\mathrm{d}\tilde{y}\end{aligned}$$

$$
\begin{aligned}
&- \frac{\tilde{y}}{\beta}\rho\theta\Gamma_{\tilde{y}} + \frac{\tilde{y}\theta}{\beta}\Phi_{2\tilde{y}} - \frac{1}{2\pi}\int_\Sigma \Big(\rho^2\xi\int_0^1 \tilde{y}\xi\Gamma_{\tilde{y}}\,\mathrm{d}\tilde{y} - \frac{\tilde{y}}{\beta}\rho\theta\Gamma_{\tilde{y}} + \frac{\tilde{y}\theta}{\beta}\Phi_{2\tilde{y}}\Big)\,\mathrm{d}\tilde{x}\,\mathrm{d}\tilde{y} \\
&- \partial_{\tilde{y}}\Delta^{-1}\Big(\frac{\tilde{y}^2}{\beta^2}\tau^2\theta_{\tilde{x}}\Phi_{2\tilde{x}} + \frac{\tilde{y}}{\beta}(-\Gamma_{\tilde{y}\tilde{y}} - \tau^2\Gamma_{\tilde{x}\tilde{x}})\int_0^1 \tilde{y}\xi\Gamma_{\tilde{y}}\,\mathrm{d}\tilde{y} \\
&\qquad + \frac{\tilde{y}\xi}{\beta}\int_0^1 \tilde{y}(-\Gamma_{\tilde{y}\tilde{y}} - \tau^2\Gamma_{\tilde{x}\tilde{x}})\,\mathrm{d}\tilde{y} + \frac{\tilde{y}\xi}{\beta}\int_0^1 \tilde{y}\xi\xi_{\tilde{y}}\,\mathrm{d}\tilde{y} \\
&\qquad - \frac{2\tau^2}{\beta}\rho_{\tilde{x}}\theta_{\tilde{x}}\tilde{y}\Gamma_{\tilde{y}} - \tau^2\rho_{\tilde{x}}^2\tilde{y}\xi_{\tilde{y}} + \frac{\tau^2}{\beta}\theta\rho_{\tilde{x}}\tilde{y}\Gamma_{\tilde{x}} \\
&\qquad + \frac{\tau^2}{\beta}\rho\theta_{\tilde{x}}\tilde{y}\Gamma_{\tilde{x}} + \tau^2\rho\rho_{\tilde{x}}\tilde{y}\xi_{\tilde{x}} \\
&\qquad + \frac{\tilde{y}}{\beta}\xi\Big(\frac{1}{2}\int_0^1 (\xi^2 - \Gamma_{\tilde{y}}^2)\,\mathrm{d}\tilde{y} + \frac{\tau^2}{2}\int_0^1 (\Gamma_{\tilde{x}}^2 + \tilde{y}\Gamma_{\tilde{x}}\Gamma_{\tilde{x}\tilde{y}} + \tilde{y}\Gamma_{\tilde{x}\tilde{x}}\Gamma_{\tilde{y}})\,\mathrm{d}\tilde{y} \\
&\qquad\qquad - \tau\Phi_{2\tilde{x}}|_{\tilde{y}=1}\Big) \\
&\qquad + \frac{\tilde{y}}{\beta}\theta(\tau\rho\rho_{\tilde{x}} + \tau^2\rho_{\tilde{x}\tilde{x}}\tilde{y}\Gamma_{\tilde{y}} + \rho\Gamma_{\tilde{y}\tilde{y}} - \tau^2\rho_{\tilde{x}}\Gamma_{\tilde{x}} + \tau^2\tilde{y}\rho_{\tilde{x}}\Gamma_{\tilde{x}\tilde{y}} - \tau^2\rho\Gamma_{\tilde{x}\tilde{x}}) \\
&\qquad - \frac{\tilde{y}\tau}{\beta}\rho_{\tilde{x}}\int_0^1 \tilde{y}\xi\Gamma_{\tilde{y}}\,\mathrm{d}\tilde{y} - \frac{\tilde{y}}{\beta}\tau\rho\int_0^1 \tilde{y}(\xi\Gamma_{\tilde{y}})_{\tilde{x}}\,\mathrm{d}\tilde{y} + \frac{\tilde{y}}{\beta}\tau^2\Gamma_{\tilde{x}}\int_0^1 \tilde{y}(\xi\Gamma_{\tilde{y}})_{\tilde{x}}\,\mathrm{d}\tilde{y} \\
&\qquad + \tilde{y}\tau^2\rho_{\tilde{x}}\Big(-\rho\xi + \frac{\theta}{\beta}\tilde{y}\Gamma_{\tilde{y}}\Big)_{\tilde{x}}\Big).
\end{aligned}
$$

### 6.3.2 The linearised system

In this section we study the linear operator $\check{L}^\varepsilon\colon \mathcal{D} \subseteq \mathcal{X} \to \mathcal{X}$ given by equation (6.73).

**Proposition 6.15.** The linear equation

$$
\begin{pmatrix}\rho\\ \theta\\ \Gamma\\ \xi\end{pmatrix}_{\tilde{z}} = \check{L}^\varepsilon \begin{pmatrix}\rho\\ \theta\\ \Gamma\\ \xi\end{pmatrix}
$$

represents Hamilton's equations for the linear Hamiltonian system $(\mathcal{X}, \Omega, H_2^\varepsilon)$, where

$$
\begin{aligned}
\Omega\Big((\rho_1,\theta_1,\Gamma_1,\xi_1),(\rho_2,\theta_2,\Gamma_2,\xi_2)\Big) &= \int_S (\theta_2\rho_1 - \rho_2\theta_1)\,\mathrm{d}\tilde{x} + \int_\Sigma (\xi_2\Gamma_1 - \Gamma_2\xi_1)\,\mathrm{d}\tilde{y}\,\mathrm{d}\tilde{x}, \\
H_2^\varepsilon(\rho,\theta,\Gamma,\xi) &= \int_\Sigma \Big(\frac{1}{2}(\xi^2 - \tau^2\Gamma_{\tilde{x}}^2 - \Gamma_{\tilde{y}}^2) + \tau\rho\Gamma_{\tilde{x}} - \tau\rho_{\tilde{x}}\tilde{y}\Gamma_{\tilde{y}}\Big)\,\mathrm{d}\tilde{y}\,\mathrm{d}\tilde{x} \\
&\quad + \int_\Sigma \Big(-\frac{1}{2}(\alpha_0+\varepsilon)\rho^2 - \frac{1}{2}\beta_0\tau^2\rho_{\tilde{x}}^2 + \frac{1}{2\beta_0}\theta^2\Big)\,\mathrm{d}\tilde{x}.
\end{aligned}
$$

*Proof.* A direct calculation shows that

$$
\Omega\Big(\check{L}^\varepsilon(\rho,\theta,\Gamma,\xi),(\hat{\rho},\hat{\theta},\hat{\Gamma},\hat{\xi})\Big) = \mathrm{d}H_2^\varepsilon[\rho,\theta,\Gamma,\xi](\hat{\rho},\hat{\theta},\hat{\Gamma},\hat{\xi})
$$

for all $(\hat{\rho},\hat{\theta},\hat{\Gamma},\hat{\xi}) \in \mathcal{X}$ and $(\rho,\theta,\Gamma,\xi) \in \mathcal{D}$. □

**Lemma 6.16.** The operator $\check{L}^\varepsilon$ satisfies the estimates

$$\begin{aligned}\|(\check{L}^\varepsilon - \mathrm{i}sI)^{-1}\|_{\mathcal{L}(\mathcal{X})} &\lesssim |s|^{-1},\\ \|(\check{L}^\varepsilon - \mathrm{i}sI)^{-1}\|_{\mathcal{L}(\mathcal{X};\mathcal{D})} &\lesssim 1\end{aligned}$$

as $|s| \to \infty$.

*Proof.* Consider the self-adjoint operator $\check{A}^\varepsilon \colon \tilde{\mathcal{D}} \subseteq \tilde{\mathcal{X}} \to \tilde{\mathcal{X}}$ given by

$$\check{A}^\varepsilon \begin{pmatrix}\rho\\ \Gamma\end{pmatrix} = \begin{pmatrix} -\frac{\tau}{\beta_0}\Gamma_{\tilde{x}}|_{\tilde{y}=1} + \frac{1}{\beta_0}(\alpha_0+\varepsilon)\rho - \tau^2\rho_{\tilde{x}\tilde{x}} \\ -\tau^2\Gamma_{\tilde{x}\tilde{x}} - \Gamma_{\tilde{y}\tilde{y}}\end{pmatrix},$$

where

$$\begin{aligned}\tilde{\mathcal{X}} &= L^2_{\mathrm{per}}(S) \times L^2_{\mathrm{per}}(\Sigma),\\ \tilde{\mathcal{D}} &= \{(\rho,\Gamma) \in H^2_{\mathrm{per}}(S) \times H^2_{\mathrm{per}}(\Sigma) : \Gamma_{\tilde{y}}(0) = 0, \tau\rho_{\tilde{x}} + \Gamma_{\tilde{y}}(1) = 0\},\end{aligned}$$

and $\tilde{\mathcal{X}}$ is equipped with the inner product

$$\langle(\rho_1,\Gamma_1),(\rho_2,\Gamma_2)\rangle = \beta_0 \int_S \rho_1\rho_2 \,\mathrm{d}\tilde{x} + \int_\Sigma \Gamma_1\Gamma_2 \,\mathrm{d}\tilde{y}\,\mathrm{d}\tilde{x}.$$

Observing that

$$\frac{1}{\beta_0}\theta - \mathrm{i}s\rho = \rho^*, \tag{6.74}$$

$$-\tau\Gamma_{\tilde{x}}|_{\tilde{y}=1} + (\alpha_0+\varepsilon)\rho - \tau^2\beta_0\rho_{\tilde{x}\tilde{x}} - \mathrm{i}s\theta = \theta^*, \tag{6.75}$$

$$\xi - \mathrm{i}s\Gamma = \Gamma^*, \tag{6.76}$$

$$-\tau^2\Gamma_{\tilde{x}\tilde{x}} - \Gamma_{\tilde{y}\tilde{y}} - \mathrm{i}s\xi = \xi^* \tag{6.77}$$

if and only if

$$\begin{aligned}-\frac{\tau}{\beta_0}\Gamma_{\tilde{x}}|_{\tilde{y}=1} + \frac{1}{\beta_0}(\alpha_0+\varepsilon)\rho - \tau^2\rho_{\tilde{x}\tilde{x}} + s^2\rho &= \frac{1}{\beta_0}\theta^* + \mathrm{i}s\rho^*,\\ -\tau^2\Gamma_{\tilde{x}\tilde{x}} - \Gamma_{\tilde{y}\tilde{y}} + s^2\Gamma &= \xi^* + \mathrm{i}s\Gamma^*,\end{aligned}$$

we find that $\pm\mathrm{i}s \in \rho(\check{L}^\varepsilon)$ if and only if $-s^2 \in \rho(\check{A}^\varepsilon)$.

The calculations

$$\begin{aligned}&\beta_0 \int_S \Big(-\frac{\tau}{\beta_0}\Gamma_{\tilde{x}}|_{\tilde{y}=1} + \frac{1}{\beta_0}(\alpha_0+\varepsilon)\rho - \tau^2\rho_{\tilde{x}\tilde{x}}\Big)\rho\,\mathrm{d}\tilde{x} + \int_\Sigma (-\tau^2\Gamma_{\tilde{x}\tilde{x}} - \Gamma_{\tilde{y}\tilde{y}})\Gamma\,\mathrm{d}\tilde{y}\,\mathrm{d}\tilde{x}\\ &\quad= \int_S (-2\tau\Gamma_{\tilde{x}}|_{\tilde{y}=1}\rho + (\alpha_0+\varepsilon)|\rho|^2 + \tau^2|\rho_{\tilde{x}}|^2)\,\mathrm{d}\tilde{x} + \int_\Sigma (\tau^2|\Gamma_{\tilde{x}}|^2 + |\Gamma_{\tilde{y}}|^2)\,\mathrm{d}\tilde{y}\,\mathrm{d}\tilde{x}\\ &\quad= \int_S (2\tau\Gamma|_{\tilde{y}=1}\rho_{\tilde{x}} + (\alpha_0+\varepsilon)|\rho|^2 + \tau^2|\rho_{\tilde{x}}|^2)\,\mathrm{d}\tilde{x} + \int_\Sigma (\tau^2|\Gamma_{\tilde{x}}|^2 + |\Gamma_{\tilde{y}}|^2)\,\mathrm{d}\tilde{y}\,\mathrm{d}\tilde{x}\end{aligned}$$

and estimate

$$\left|2\tau \int_S \Gamma|_{\tilde{y}=1}\rho_{\tilde{x}}\,\mathrm{d}\tilde{x}\right| \leq \frac{1}{\delta}\int_S |\Gamma|_{\tilde{y}=1}|^2\,\mathrm{d}\tilde{x} + \delta\int_\Sigma \tau^2|\rho_{\tilde{x}}|^2\,\mathrm{d}\tilde{y}\,\mathrm{d}\tilde{x}$$

$$\leq \delta \int_\Sigma |\Gamma_{\tilde{y}}|^2 \,\mathrm{d}\tilde{y}\,\mathrm{d}\tilde{x} + \Big(\frac{1}{\delta} + \frac{1}{\delta^2}\Big) \int_\Sigma |\Gamma|^2 \,\mathrm{d}\tilde{y}\,\mathrm{d}\tilde{x} + \delta \int_S \tau^2 |\rho_{\tilde{x}}|^2 \,\mathrm{d}\tilde{x}$$

for $\Gamma \in \mathcal{D}$ and sufficiently small $\delta$ show that

$$\begin{aligned}
&\beta_0 \int_S \Big( -\frac{\tau}{\beta_0}\Gamma_{\tilde{x}}|_{\tilde{y}=1} + \frac{1}{\beta_0}(\alpha_0+\varepsilon)\rho - \tau^2 \rho_{\tilde{x}\tilde{x}} \Big)\rho \,\mathrm{d}\tilde{x} + \int_\Sigma (-\tau^2 \Gamma_{\tilde{x}\tilde{x}} - \Gamma_{\tilde{y}\tilde{y}})\Gamma \,\mathrm{d}\tilde{y}\,\mathrm{d}\tilde{x} \\
&\quad \geq (\alpha_0+\varepsilon) \int_S |\rho|^2 \,\mathrm{d}\tilde{x} - \Big(\frac{1}{\delta} + \frac{1}{\delta^2}\Big) \int_\Sigma |\Gamma|^2 \,\mathrm{d}\tilde{y}\,\mathrm{d}\tilde{x} \\
&\quad \geq -\Big(\frac{1}{\delta} + \frac{1}{\delta^2}\Big)\Big(\beta_0 \int_S |\rho|^2 \,\mathrm{d}\tilde{x} + \int_\Sigma |\Gamma|^2 \,\mathrm{d}\tilde{y}\,\mathrm{d}\tilde{x}\Big).
\end{aligned}$$

Thus $\sigma(\check{A}^\varepsilon)$ is bounded from below and in particular $-s^2 \in \rho(\check{A}^\varepsilon)$ and hence $\pm \mathrm{i}s \in \rho(\check{L}^\varepsilon)$ for sufficiently large values of $|s|$.

The estimates

$$\|(\check{L}^\varepsilon - \mathrm{i}sI)^{-1}\|_{\mathcal{L}(\mathcal{X})} \lesssim |s|^{-1}, \quad \|(\check{L}^\varepsilon - \mathrm{i}sI)^{-1}\|_{\mathcal{L}(\mathcal{X};\mathcal{D})} \lesssim 1$$

for sufficiently large values of $|s|$ follow from equations (6.74) – (6.77) by the calculation

$$\begin{aligned}
&\int_S (\beta_0^2 |\rho^*|^2 + \tau^2 \beta_0^2 |\rho^*_{\tilde{x}}|^2 + |\theta^*|^2) \,\mathrm{d}\tilde{x} + \int_\Sigma (|\Gamma^*|^2 + \tau^2 |\Gamma^*_{\tilde{x}}|^2 + |\Gamma^*_{\tilde{y}}|^2 + |\xi^*|^2) \,\mathrm{d}\tilde{y}\,\mathrm{d}\tilde{x} \\
&\quad = \int_S (|\theta - \mathrm{i}s\beta_0\rho|^2 + \tau^2 |\theta_{\tilde{x}} - \mathrm{i}s\beta_0\rho_{\tilde{x}}|^2 \\
&\qquad\qquad + |-\tau\Gamma_{\tilde{x}}|_{\tilde{y}=1} + (\alpha_0+\varepsilon)\rho - \tau^2\beta_0\rho_{\tilde{x}\tilde{x}} - \mathrm{i}s\theta|^2) \,\mathrm{d}\tilde{x} \\
&\qquad + \int_\Sigma (|\xi - \mathrm{i}s\Gamma|^2 + \tau^2 |\xi_{\tilde{x}} - \mathrm{i}s\Gamma_{\tilde{x}}|^2 + |\xi_{\tilde{y}} - \mathrm{i}s\Gamma_{\tilde{y}}|^2 \\
&\qquad\qquad + |\tau^2\Gamma_{\tilde{x}\tilde{x}} + \Gamma_{\tilde{y}\tilde{y}} + \mathrm{i}s\xi|^2) \,\mathrm{d}\tilde{y}\,\mathrm{d}\tilde{x} \\
&\quad = \int_S \Big( |\theta|^2 + \tau^2 |\theta_{\tilde{x}}|^2 + \tau^2 |\Gamma_{\tilde{x}}|_{\tilde{y}=1}|^2 + (\alpha_0+\varepsilon)^2 |\rho|^2 + \tau^4 \beta_0^2 |\rho_{\tilde{x}\tilde{x}}|^2 \\
&\qquad\qquad + s^2(\beta_0^2 |\rho|^2 + \tau^2 \beta_0^2 |\rho_{\tilde{x}}|^2 + |\theta|^2) \Big) \,\mathrm{d}\tilde{x} \\
&\qquad - 2s \,\mathrm{Im} \int_S (\beta_0\theta\bar{\rho} + \tau^2\beta_0\theta_{\tilde{x}}\bar{\rho}_{\tilde{x}} - \tau\Gamma_{\tilde{x}}|_{\tilde{y}=1}\bar{\theta} + (\alpha_0+\varepsilon)\rho\bar{\theta} + \tau^2\beta_0\theta\bar{\rho}_{\tilde{x}\tilde{x}}) \,\mathrm{d}\tilde{x} \\
&\qquad + 2 \,\mathrm{Re} \int_S (-\tau(\alpha_0+\varepsilon)\Gamma_{\tilde{x}}|_{\tilde{y}=1}\bar{\rho} + \beta_0\tau^3\Gamma_{\tilde{x}}|_{\tilde{y}=1}\bar{\rho}_{\tilde{x}\tilde{x}} - (\alpha_0+\varepsilon)\tau^2\beta_0\rho\bar{\rho}_{\tilde{x}\tilde{x}}) \,\mathrm{d}\tilde{x} \\
&\qquad + \int_\Sigma \Big( |\xi|^2 + \tau^2 |\xi_{\tilde{x}}|^2 + |\xi_{\tilde{y}}|^2 + \tau^2 |\Gamma_{\tilde{x}\tilde{x}}|^2 + |\Gamma_{\tilde{y}\tilde{y}}|^2 \\
&\qquad\qquad + s^2(|\Gamma|^2 + \tau^2 |\Gamma_{\tilde{x}}|^2 + |\Gamma_{\tilde{y}}|^2 + |\xi|^2) \Big) \,\mathrm{d}\tilde{y}\,\mathrm{d}\tilde{x} \\
&\qquad - 2s \,\mathrm{Im} \int_\Sigma (\xi\bar{\Gamma} + \tau^2\xi_{\tilde{x}}\bar{\Gamma}_{\tilde{x}} + \xi_{\tilde{y}}\bar{\Gamma}_{\tilde{y}} + \tau^2\xi\bar{\Gamma}_{\tilde{x}\tilde{x}} + \xi\bar{\Gamma}_{\tilde{y}\tilde{y}}) \,\mathrm{d}\tilde{y}\,\mathrm{d}\tilde{x} \\
&\qquad + 2 \,\mathrm{Re} \int_\Sigma \tau^2 \Gamma_{\tilde{x}\tilde{x}} \bar{\Gamma}_{\tilde{y}\tilde{y}} \,\mathrm{d}\tilde{y}\,\mathrm{d}\tilde{x}
\end{aligned}$$

$$
\begin{aligned}
\geq \int_S \Big( & |\theta|^2 + \tau^2|\theta_{\tilde{x}}|^2 + (\alpha_0+\varepsilon)^2|\rho|^2 + \tau^4\beta_0^2|\rho_{\tilde{x}\tilde{x}}|^2 + 2(\alpha_0+\varepsilon)\tau^2\beta_0|\rho_{\tilde{x}}|^2 \\
& + s^2(\beta_0^2|\rho|^2 + \tau^2\beta_0^2|\rho_{\tilde{x}}|^2 + |\theta|^2)\Big)\,\mathrm{d}\tilde{x} \\
& - 2s\,\mathrm{Im}\int_S (\beta_0\theta\bar{\rho} - \tau\Gamma_{\tilde{x}}|_{\tilde{y}=1}\bar{\theta} + (\alpha_0+\varepsilon)\rho\bar{\theta})\,\mathrm{d}\tilde{x} \\
& + 2\,\mathrm{Re}\int_S (-\tau(\alpha_0+\varepsilon)\Gamma_{\tilde{x}}|_{\tilde{y}=1}\bar{\rho} + \beta_0\tau^3\Gamma_{\tilde{x}}|_{\tilde{y}=1}\bar{\rho}_{\tilde{x}\tilde{x}})\,\mathrm{d}\tilde{x} \\
& + \int_\Sigma \Big(|\xi|^2 + \tau^2|\xi_{\tilde{x}}|^2 + |\xi_{\tilde{y}}|^2 + \tau^2|\Gamma_{\tilde{x}\tilde{x}}|^2 + |\Gamma_{\tilde{y}\tilde{y}}|^2 + 2|\Gamma_{\tilde{x}\tilde{y}}|^2 \\
& \qquad + s^2(|\Gamma|^2 + \tau^2|\Gamma_{\tilde{x}}|^2 + |\Gamma_{\tilde{y}}|^2 + |\xi|^2)\Big)\,\mathrm{d}\tilde{y}\,\mathrm{d}\tilde{x} \\
& - 2s\,\mathrm{Im}\left(\int_\Sigma \xi\bar{\Gamma}\,\mathrm{d}\tilde{y}\,\mathrm{d}\tilde{x} - \int_S \tau\xi|_{\tilde{y}=1}\bar{\rho}_{\tilde{x}}\right) \\
& + \mathrm{Re}\int_S \tau^3\Gamma_{\tilde{x}}|_{\tilde{y}=1}\bar{\rho}_{\tilde{x}\tilde{x}}\,\mathrm{d}\tilde{x}
\end{aligned}
$$

and the inequalities

$$
\begin{aligned}
\left|2s\,\mathrm{Im}\int_S \theta\bar{\rho}\,\mathrm{d}\tilde{x}\right| &\leq |s|\int_S (|\theta|^2 + |\rho|^2)\,\mathrm{d}\tilde{x}, \\
\left|2s\,\mathrm{Im}\int_\Sigma \xi\bar{\Gamma}\,\mathrm{d}\tilde{y}\,\mathrm{d}\tilde{x}\right| &\leq |s|\int_\Sigma (|\xi|^2 + |\Gamma|^2)\,\mathrm{d}\tilde{y}\,\mathrm{d}\tilde{x}, \\
\left|2s\,\mathrm{Im}\int_S \Gamma_{\tilde{x}}|_{\tilde{y}=1}\bar{\theta}\,\mathrm{d}\tilde{x}\right| &\leq \delta|s|\int_S |\Gamma_{\tilde{x}}|_{\tilde{y}=1}|^2\,\mathrm{d}\tilde{x} + \frac{|s|}{\delta}\int_S |\theta|^2\,\mathrm{d}\tilde{x} \\
&= \delta|s|\int_\Sigma \frac{\mathrm{d}}{\mathrm{d}\tilde{y}}(\tilde{y}|\Gamma_{\tilde{x}}|^2)\,\mathrm{d}\tilde{y}\,\mathrm{d}\tilde{x} + \frac{|s|}{\delta}\int_S |\theta|^2\,\mathrm{d}\tilde{y} \\
&\leq \delta\int_\Sigma (|\Gamma_{\tilde{x}\tilde{y}}|^2 + s^2|\Gamma_{\tilde{x}}|^2)\,\mathrm{d}\tilde{y}\,\mathrm{d}\tilde{x} + \delta|s|\int_\Sigma |\Gamma_{\tilde{x}}|^2\,\mathrm{d}\tilde{y}\,\mathrm{d}\tilde{x} + \frac{|s|}{\delta}\int_S |\theta|^2\,\mathrm{d}\tilde{x}, \\
\left|2s\,\mathrm{Im}\int_S \xi|_{\tilde{y}=1}\bar{\rho}_{\tilde{x}}\,\mathrm{d}\tilde{x}\right| &\leq \delta\int_\Sigma (|\xi_{\tilde{y}}|^2 + s^2|\xi|^2)\,\mathrm{d}\tilde{y}\,\mathrm{d}\tilde{x} + \delta|s|\int_\Sigma |\xi|^2\,\mathrm{d}\tilde{y}\,\mathrm{d}\tilde{x} + \frac{|s|}{\delta}\int_S |\rho_{\tilde{x}}|^2\,\mathrm{d}\tilde{x}, \\
\left|2\,\mathrm{Re}\int_S \Gamma_{\tilde{x}}|_{\tilde{y}=1}\bar{\rho}\right| &\leq \delta\int_S |\Gamma_{\tilde{x}}|_{\tilde{y}=1}|^2 + \frac{1}{\delta}\int_S |\bar{\rho}|^2\,\mathrm{d}\tilde{x} \\
&\leq \delta\int_\Sigma (2|\Gamma_{\tilde{x}}|^2 + |\Gamma_{\tilde{x}\tilde{y}}|^2)\,\mathrm{d}\tilde{y}\,\mathrm{d}\tilde{x} + \frac{1}{\delta}\int_S |\bar{\rho}|^2\,\mathrm{d}\tilde{x}, \\
\left|2\,\mathrm{Re}\int_S \Gamma_{\tilde{x}}|_{\tilde{y}=1}\bar{\rho}_{\tilde{x}\tilde{x}}\,\mathrm{d}\tilde{x}\right| &\leq \frac{1}{\delta}\int_S |\Gamma_{\tilde{x}}|_{\tilde{y}=1}|^2\,\mathrm{d}\tilde{x} + \delta\int_S |\rho_{\tilde{x}\tilde{x}}|^2\,\mathrm{d}\tilde{x} \\
&\leq \delta\int_\Sigma |\Gamma_{\tilde{x}\tilde{y}}|^2\,\mathrm{d}\tilde{y}\,\mathrm{d}\tilde{x} + \left(\frac{1}{\delta} + \frac{1}{\delta^2}\right)\int_\Sigma |\Gamma_{\tilde{x}}|^2\,\mathrm{d}\tilde{y}\,\mathrm{d}\tilde{x} + \delta\int_S |\rho_{\tilde{x}\tilde{x}}|^2\,\mathrm{d}\tilde{x}.
\end{aligned}
$$

□

**Corollary 6.17.** The spectrum of $\check{L}^\varepsilon$ consists only of isolated eigenvalues with finite algebraic multiplicity.

*Proof.* Since $\mathcal{D}$ is compactly embedded in $\mathcal{X}$ we know that $(\check{L}^\varepsilon - \mathrm{i}sI)^{-1} \in \mathcal{L}(\mathcal{X})$ is compact for sufficiently large values of $|s|$. The result now follows from Kato [13, Theorem III.6.29]. □

Our next result is proved by a direct calculation.

**Lemma 6.18.**

(i) The eigenvalues of $\check{L}^0$ with eigenvalue in the 0th Fourier mode are $\pm(\alpha_0/\beta_0)^{1/2}$ and $\pm n\pi$, $n \in \mathbb{N}$.

(ii) Suppose $m \in \mathbb{N}$. A complex number $\lambda$ is an eigenvalue of $\check{L}^0$ with corresponding eigenvectors in the $m$th Fourier mode if and only if

$$(\alpha_0 - \beta_0\sigma_m^2)\sigma_m \sin\sigma_m + m^2\tau^2 \cos\sigma_m = 0, \tag{6.78}$$

where

$$\sigma_m^2 = \lambda^2 - m^2\tau^2.$$

In particular $\lambda$ is either real or purely imaginary.

For each $m \in \mathbb{N}$ equation (6.78) has at most two purely imaginary solutions $\pm \mathrm{i}s_m$ with eigenvectors $e_m^+$, $\bar{e}_m^+$ and $e_m^-$, $\bar{e}_m^-$, where

$$e_m^+ = \frac{1}{\sqrt{\gamma}_m}\begin{pmatrix} \frac{2}{m\tau}(s_m^2+m^2\tau^2)^{\frac12}\sinh(s_m^2+m^2\tau^2)^{\frac12}\cos m\tilde{x} \\ \frac{-2\mathrm{i}s_m\beta_0}{m\tau}(s_m^2+m^2\tau^2)^{\frac12}\sinh(s_m^2+m^2\tau^2)^{\frac12}\cos m\tilde{x} \\ 2\cosh(s_m^2+m^2\tau^2)^{\frac12}\tilde{y}\sin m\tilde{x} \\ -2\mathrm{i}s_m\cosh(s_m^2+m^2\tau^2)^{\frac12}\tilde{y}\sin m\tilde{x} \end{pmatrix},$$

$$e_m^- = \frac{1}{\sqrt{\gamma_m}}\begin{pmatrix} -\frac{2}{m\tau}(s_m^2+m^2\tau^2)^{\frac12}\sinh(s_m^2+m^2\tau^2)^{\frac12}\sin m\tilde{x} \\ \frac{2\mathrm{i}s_m\beta_0}{m\tau}(s_m^2+m^2\tau^2)^{\frac12}\sinh(s_m^2+m^2\tau^2)^{\frac12}\sin m\tilde{x} \\ 2\cosh(s_m^2+m^2\tau^2)^{\frac12}\tilde{y}\cos m\tilde{x} \\ -2\mathrm{i}s_m\cosh(s_m^2+m^2\tau^2)^{\frac12}\tilde{y}\cos m\tilde{x} \end{pmatrix}$$

and

$$\gamma_m = 2\pi s_m\Big(2 + 4\beta_0\Big(1+\frac{s_m^2}{m^2\tau^2}\Big)\sinh^2(s_m^2+m^2\tau^2)^{\frac12} + \frac{\sinh 2(s_m^2+m^2\tau^2)^{\frac12}}{(s_m^2+m^2\tau^2)^{\frac12}}\Big).$$

The eigenvectors have been normalised such that

$$\Omega(e_m^+, \bar{e}_m^+) = \Omega(e_m^-, \bar{e}_m^-) = -\mathrm{i}$$

and the symplectic product of any other combination is zero. The eigenvalues $\pm\mathrm{i}s_m$ collide at the origin at points of the line

$$C_m = \{(\beta_0, \alpha_0) : (\alpha_0 + \beta_0 m^2\tau^2)\sinh m\tau = m\tau\cosh m\tau\}$$

in $(\beta_0, \alpha_0)$ parameter space (see Figure 6.4). At these points the two zero eigenvectors each have a Jordan chain of length 2: the vectors

$$e_1 = \frac{1}{\sqrt{\gamma_{0,m}}}\begin{pmatrix} 2\sinh(m\tau)\cos(m\tilde{x}) \\ 0 \\ 2\cosh(m\tau\tilde{y})\sin(m\tilde{x}) \\ 0 \end{pmatrix}, \qquad e_2 = \frac{1}{\sqrt{\gamma_{0,m}}}\begin{pmatrix} -2\sinh(m\tau)\sin(m\tilde{x}) \\ 0 \\ 2\cosh(m\tau\tilde{y})\cos(m\tilde{x}) \\ 0 \end{pmatrix},$$

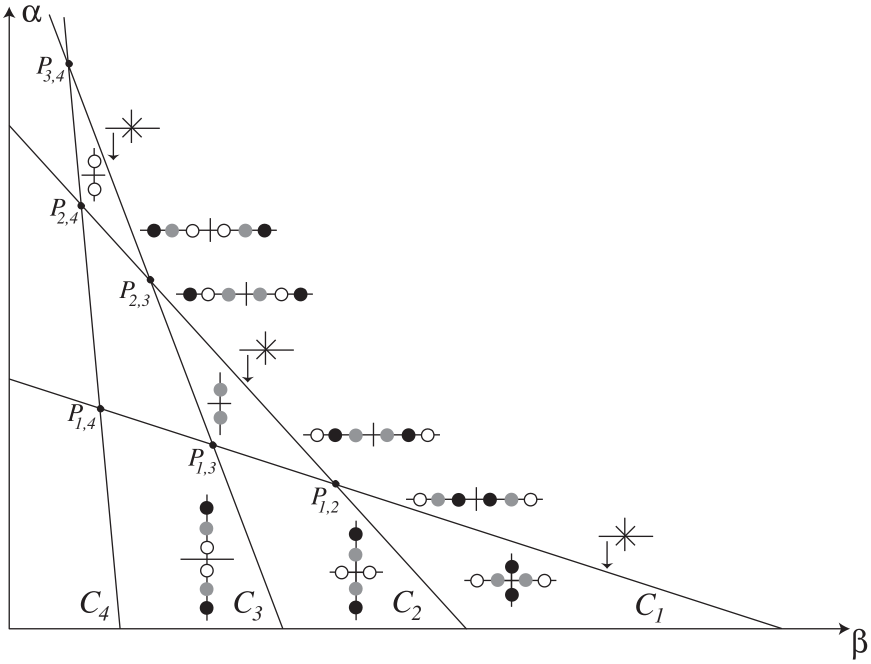



Figure 6.4: The line $C_m$ consists of points in $\beta, \alpha$ parameter space at which two real eigenvaules in the $m$th Fourier mode become purely imaginary by passing through 0. It connects $((\tau m)^{-1}\coth(\tau m), 0)$ with $(0, \tau m \coth(\tau m))$ and crosses $C_{m+1}, C_{m+2}, \ldots$ at the points $P_{m,m+1}, P_{m,m+2}, \ldots$.

$$f_1 = \frac{1}{\sqrt{\gamma_{0,m}}} \begin{pmatrix} 0 \\ 2\beta_0 \sinh(m\tau)\cos(m\tilde{x}) \\ 0 \\ 2\cosh(m\tau\tilde{y})\sin(m\tilde{x}) \end{pmatrix}, \qquad f_2 = \begin{pmatrix} 0 \\ 2\beta_0 \sinh(m\tau)\sin(m\tilde{x}) \\ 0 \\ 2\cosh(m\tau\tilde{y})\cos(m\tilde{x}) \end{pmatrix},$$

where

$$\gamma_{0,m} = \pi\Big(2 + 4\beta\sinh^2(m\tau) + \frac{\sinh(2m\tau)}{m\tau}\Big), \tag{6.79}$$

satisfy $Le^{\pm}_{0,m} = 0$, $Lf^{\pm}_{0,m} = e^{\pm}_{0,m}$; moreover $\Omega(e^{+}_{0,m}, f^{+}_{0,m}) = 1$, $\Omega(e^{-}_{0,m}, f^{-}_{0,m}) = 1$ and the symplectic product of any other combination is zero.

We choose $(\beta_0, \alpha_0) \in C_m \setminus \{P_{1,m}, \ldots, P_{m-1,m}, P_{m,m+1}, P_{m,m+2}, \ldots\}$ and write

$$w = \sum (C^+_j e^+_j + \bar{C}^-_j \bar{e}^-_j + \bar{C}^+_j \bar{e}^+_j + \bar{C}^-_j \bar{e}^-_j),$$
$$z = z_1 e + z_2 f + \bar{z}_1 \bar{e} + \bar{z}_2 \bar{f},$$

where

$$e = \frac{1}{2}(e_1 - \mathrm{i}e_2), \qquad f = \frac{1}{2}(f_1 - \mathrm{i}f_2),$$

so that

$$H^0_2(0, w, 0) = -\sum \frac{1}{2} s_j (|C^+_j|^2 + |C^-_j|^2)$$

and

$$P^0_{\mathrm{wh}}(\cdot) = 2\Omega(\cdot, \bar{f})e - 2\Omega(\cdot, \bar{e})f + 2\Omega(\cdot, f)\bar{e} - 2\Omega(\cdot, e)\bar{f}.$$

In these formulae the sums are taken over those values of $j$ for which $\check{L}^0$ has two mode $j$ eigenvalues $\pm \mathrm{i}s_j$.

### 6.3.3 Existence theory

We now derive equations (6.5) – (6.7) from equation (6.72) as explained in Section 6.1 and prove Theorem 1.11 by applying Theorem 1.2. Assumptions (A1) – (A3) are obviously satisfied, and Assumption (A6) follows from Lemma 6.16. It therefore remains to verify Assumptions (A4), (A5) and (C1), (C3), (C4). To this end we note that equation (6.72) is reversible with reverser

$$S\Big(\rho(\tilde{x}), \theta(\tilde{x}), \Gamma(\tilde{x}, \tilde{y}), \xi(\tilde{x}, \tilde{y})\Big) = \Big(\rho(-\tilde{x}), -\theta(-\tilde{x}), -\Gamma(-\tilde{x}, \tilde{y}), \xi(-\tilde{x}, \tilde{y})\Big)$$

and has the conserved quantity

$$\begin{aligned}\mathcal{I}^\varepsilon(\rho, \theta, \Gamma, \xi) &= -H^\varepsilon(G^{-1}(\rho, \theta, \Gamma, \xi)) \\ &= \mathcal{O}(\|(\rho, \theta, \Gamma, \xi)\|_{\mathcal{D}}^2),\end{aligned}$$

which satisfies, with a change of notation,

$$\begin{aligned}\mathcal{I}^0(0, w, 0) &= -H_2^0(w) + \mathcal{O}(|w|^3) \\ &= -\sum_{j=1}^{\infty} \frac{1}{2} s_j (|C_j^+|^2 + |C_j^{-1}|^2), \\ \mathcal{I}^0\Big((z_1, 0), 0, 0\Big) &= -H_2^0(z_1 e + \bar{z}_1 \bar{e}, 0, 0) \\ &= 0,\end{aligned}$$

so that

$$\mathcal{I}^\varepsilon\Big((z_1, z_2), w, u\Big) = \mathcal{O}(\|(z_2, w, u)\|_{\mathbb{R}^2 \times \mathbb{R}^{2d} \times \mathcal{D}_{\mathrm{sh}}} \|(z_1, z_2, w, u)\|_{\mathcal{D}} + \varepsilon \|(z_1, z_2, w, u)\|_{\mathcal{D}}^2).$$

To verify Assumptions (C1) and (C4) we change to real coordinates by writing

$$z_1 = \tilde{x}_1 + \mathrm{i}\tilde{x}_2, \qquad z_2 = \tilde{y}_1 + \mathrm{i}\tilde{y}_2,$$

so that

$$z = \tilde{x}_1 e_1 + \tilde{y}_1 f_1 + \tilde{x}_2 e_2 + \tilde{y}_2 f_2$$

and

$$P_{\mathrm{wh}} = \Omega(\cdot, f_1) e_1 - \Omega(\cdot, e_1) f_1 + \Omega(\cdot, f_2) e_2 - \Omega(\cdot, e_2) f_2.$$

Writing

$$\frac{1}{2}\,\mathrm{d}^2 h_{\mathrm{wh}}^0[0](z) = \sum_{i+j+k+l=2} h_{\mathrm{wh},ijkl}^0 \tilde{x}_1^i \tilde{x}_2^j \tilde{y}_1^k \tilde{y}_2^l$$

with similar notation for the other nonlinearities, we find that

$$\begin{aligned}h_{\mathrm{wh},2000}^0 &= P_{\mathrm{wh}} h_{2000}^0 \\ &= \Omega(\tfrac{1}{2}\,\mathrm{d}^2 k^0[0](e_1, e_1), f_1) e_1 - \Omega(\tfrac{1}{2}\,\mathrm{d}^2 k^0[0](e_1, e_1), e_1) f_1 \\ &\qquad + \Omega(\tfrac{1}{2}\,\mathrm{d}^2 k^0[0](e_1, e_1), f_2) e_2 - \Omega(\tfrac{1}{2}\,\mathrm{d}^2 k^0[0](e_1, e_1), e_2) f_2 \\ &= 0\end{aligned}$$

since

$$\int_0^{2\pi} \begin{Bmatrix}\cos(m\tilde{x}) \\ \sin(m\tilde{x})\end{Bmatrix} \begin{Bmatrix}\cos(m\tilde{x}) \\ \sin(m\tilde{x})\end{Bmatrix} \begin{Bmatrix}\cos(m\tilde{x}) \\ \sin(m\tilde{x})\end{Bmatrix} \mathrm{d}\tilde{x} = 0.$$

Similar calculations show that the remaining components of $h^0_{\mathrm{wh}}$ also vanish.

Similarly, writing

$$X(z,z) = \sum_{i+j+k+l=2} X_{ijkl}\tilde{x}_1^i \tilde{x}_2^j \tilde{y}_1^k \tilde{y}_2^l,$$

we find from equation (4.21) that

$$L_{\mathrm{c,sh}} X_{2000} = P_{\mathrm{c,sh}}(\tfrac{1}{2}\,\mathrm{d}^2 k^0[0](e_1,e_1)),$$

which can also be written as

$$L^0 X_{2000} = \frac{1}{2}\,\mathrm{d}^2 k^0[0](e_1,e_1)$$

because $\mathrm{d}^2 h^0_{\mathrm{wh}}[0] = 0$. We find that

$$X_{2000} = (L^0)^{-1}\Big(\frac{1}{2}\,\mathrm{d}^2 k^0[0](e_1,e_1)\Big) = \begin{pmatrix} X^{(1)}_{2000} \\ X^{(2)}_{2000} \\ X^{(3)}_{2000} \\ X^{(4)}_{2000} \end{pmatrix},$$

where

$$\begin{aligned}
X^{(1)}_{2000} &= \frac{m^2\tau^2}{\alpha_0} + \frac{1}{2}m^2\tau^2 \frac{\cos(2m\tilde{x})\big(4\sinh(2m\tau)+\sinh(4m\tau)\big)}{(\alpha_0+4m^2\tau^2\beta_0)\sinh(2m\tau) - 2m\tau\cosh(2m\tau)},\\
X^{(2)}_{2000} &= 0,\\
X^{(3)}_{2000} &= \frac{1}{18}\sin(2m\tilde{x})\Big(20\cosh(m\tau\tilde{y})\sinh(m\tau) - \frac{9\sinh^2(m\tau)}{m\tau}\\
&\qquad - 24m\tau\tilde{y}\sinh(m\tau)\sinh(m\tau\tilde{y}) + \cosh(2m\tau\tilde{y})\tanh(m\tau)\\
&\qquad + \cosh(2m\tau\tilde{y})\frac{6m^2\tau^2(6+\cosh(2m\tau))}{(\alpha_0+4m\tau^2\beta_0)\sinh(2m\tau) - 2m\tau\cosh(2m\tau)}\\
&\qquad + \cosh(2m\tau\tilde{y})\frac{6m\tau(\alpha_0+4m^2\tau^2\beta_0)\sinh(2m\tau)}{(\alpha_0+4m\tau^2\beta_0)\sinh(2m\tau) - 2m\tau\cosh(2m\tau)}\Big),\\
X^{(4)}_{2000} &= 0.
\end{aligned}$$

The coefficient $C$ is given by

$$\begin{aligned}
C &= \Omega(\tfrac{1}{6}\,\mathrm{d}^3 h^0_{\mathrm{wh}}[0](e_1,e_1,e_1),e_1) - 2\Omega(\tfrac{1}{2}\,\mathrm{d}^2 g^0_{\mathrm{wh}}[0](e_1,X_{2000}),e_1)\\
&= \Omega(\tfrac{1}{6}\,\mathrm{d}^3 k^0[0](e_1,e_1,e_1),e_1) - 2\Omega(\tfrac{1}{2}\,\mathrm{d}^2 k^0[0](e_1,X_{2000}),e_1)\\
&= m^5\tau^5\Big(-8(13+3m^2\tau^2\beta_0^2)\cosh(2m\tau)\\
&\qquad - 4(40+15m^2\tau^2\beta_0^2 + 4\cosh(4m\tau) + 2\cosh(6m\tau))\\
&\qquad + m\tau\beta_0\Big(12m\tau\beta_0(9\cosh(4m\tau) - 2\cosh(6m\tau)\\
&\qquad\qquad + 24m\tau\beta_0\cosh(m\tau)\sinh^5(m\tau))\\
&\qquad\qquad - 93\sinh(2m\tau) - 60\sinh(4m\tau) + 23\sinh(6m\tau)\Big)\Big)\\
&\quad \times \frac{1}{16\pi}(3m\tau\beta_0\cosh(m\tau) - \sinh(m\tau))^{-1}(m\tau\beta_0\sinh(m\tau) - \cosh(m\tau))^{-1}\\
&\quad \times (2m\tau + 4m\tau\beta_0\sinh^2(m\tau) + \sinh(2m\tau))^{-2},
\end{aligned} \tag{6.80}$$

where we have eliminated $\alpha_0$ using the relation

$$\alpha_0 = -\beta_0 m^2\tau^2 + m\tau\coth(m\tau).$$

**Proposition 6.19.** The coefficient $C$ is negative for $\beta_0 \in (0, \frac{\tanh(m\tau)}{3m\tau})$ and positive for $\beta_0 \in (\frac{\tanh(m\tau)}{3m\tau}, \frac{\coth(m\tau)}{m\tau})$.

*Proof.* It suffices to show that the numerator in the above formula for $C$ is negative for $\beta_0 \in (0, \frac{\coth(m\tau)}{m\tau})$. We can write the numerator as the polynomial

$$\begin{aligned} n(\beta_0) = &-8s^5(20 + 13\cosh(2s) + 2\cosh(4s) + \cosh(6s)) \\ &+ s^6(-93\sinh(2s) - 60\sinh(4s) + 23\sinh(6s))\beta_0 \\ &+ 48s^7(5 + 5\cosh(2s) - 2\cosh(4s))\sinh^2(s)\beta_0^2 \\ &+ 288s^8\cosh s\sinh^5(s)\beta_0^3, \end{aligned}$$

where $s = m\tau$. Observe that the coefficient of $\beta_0^3$ is positive, while the constant term is negative.

Define

$$c_2(s) = 5 + 5\cosh(2s) - 2\cosh(4s)$$

and note that $n$ has precisely one positive root if $c_2(s) > 0$ (by Decartes's rule of signs). Since

$$\begin{aligned} n(0) &= -8s^5(20 + 13\cosh(2s) + 2\cosh(4s) + \cosh(6s)) < 0, \\ n\Big(\frac{\coth(s)}{s}\Big) &= -64s^4(2 + \cosh(2s)) < 0 \end{aligned}$$

and $n(\beta_0) \to \infty$ as $\beta_0 \to \infty$ we conclude that this root is larger than $\frac{\coth s}{s}$ and $n(s) < 0$ for $\beta_0 \in (0, \frac{\coth s}{s})$. We note that

$$c_2'(s) = 10\sinh(2s) - 8\sinh(4s).$$

Since

$$\frac{5}{8}\operatorname{sech}(s) < 1$$

is true for all $s$ and equivalent to

$$c_2'(s) < 0$$

we find that $c_2(s)$ is a strictly decreasing function of $s$, and since $c_2(0.68) > 0$ we conclude that $c_2(s) > 0$ for all $s \in (0, 0.68]$, so that $n(\beta_0) < 0$ for $\beta_0 \in (0, \frac{\coth s}{s})$ and $s \in (0, 0.68]$.

Next we note that

$$\begin{aligned} n'(\beta_0) = &\, s^6(-93\sinh(2s) - 60\sinh(4s) + 23\sinh(6s)) \\ &+ 96s^7(5 + 5\cosh(2s) - 2\cosh(4s))\sinh^2(s)\beta_0 \\ &+ 864s^8\cosh s\sinh^5(s)\beta_0^2 \end{aligned}$$

is a quadratic polynomial in $\beta_0$. Its discriminant is given by

$$\begin{aligned}\Delta(s) = \Big(&96s^7(5+5\cosh(2s)-2\cosh(4s))\sinh^2(s)\Big)^2\\ &-3456s^6(-93\sinh(2s)-60\sinh(4s)+23\sinh(6s))s^8\cosh(s)\sinh^5(s)\\ =\ &288s^{14}\sinh^4 s\tilde{\Delta}(s),\end{aligned}$$

where

$$\tilde{\Delta}(s) = 1100\cosh(2s)+108\cosh(4s)-140\cosh(6s)-5\cosh(8s).$$

We note that

$$\tilde{\Delta}'(s) = 8(275\sinh(2s)+54\sinh(4s))-40(21\sinh(6s)+\sinh(8s))$$

and that

$$\tilde{\Delta}'(s) < 0$$

is equivalent to

$$d(s) := \frac{1110\sinh(2s)+27\sinh(4s)}{20(21\sinh(6s)+\sinh(8s))} < 1.$$

Furthermore

$$\begin{aligned}d'(s) &= \frac{-(6101+92724\cosh(2s)+7734\cosh(4s)+108\cosh(6s))\sinh^3(2s)}{5(51\sinh(6s)+\sinh(8s))^2}\\ &< 0,\end{aligned}$$

so that $d(s)$ is a strictly decreasing function of $s$ and therefore

$$d(s) \leq \lim_{s\to 0} d(s) = \frac{577}{670} < 1.$$

Hence $\tilde{\Delta}(s)$ is also a strictly decreasing function of $s$ and since $\tilde{\Delta}(s) < 0$ we conclude that $\tilde{\Delta}(s) < 0$ for all $s \in [0.67, \infty)$. The fact that $n'$ has no roots for these values of $s$ implies that $n$ has no critical points among these values. Because $n(\beta_0) \to \pm\infty$ as $\beta_0 \to \pm\infty$ we conclude that $n$ is strictly increasing. Since $n(\frac{\coth s}{s}) < 0$ we deduce that $n(\beta_0) < 0$ for $\beta_0 \in (0, \frac{\coth s}{s})$, so that $n(\beta_0) < 0$ for $\beta_0 \in (0, \frac{\coth s}{s})$ and $s \in [0.67, \infty)$.

Altogether we have shown that $n(\beta_0) < 0$ for all $\beta_0 \in (0, \frac{\coth s}{s})$ and all $s > 0$.

□

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