Unification, Weak Unification, Upper Bound, Lower Bound, and Generalization Problems

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We define E-unification, weak E-unification, E-upper bound, E-lower bound and E-generalization problems and the corresponding notions of unification, weak unification, upper bound, lower bound and generalization type of an equational theory. Most general unifiers, most general weak unifiers, suprema, infima and most specific generalizers correspond to "weak versions" of well-known categorical concepts.

The problems are first studied for the empty theory using the restricted instantiation ordering (i.e., substitutions are compared w.r.t. their behaviour on finite sets of variables) and the unrestricted instantiation ordering (i.e., substitutions are compared w.r.t. their behaviour on all variables). This shows that the unrestricted instantiation ordering should only be used for unification. For the other problems the restricted ordering yields much better results. We shall also show that there exists an equational theory where unification problems always have most general unifiers w.r.t. the restricted instantiation ordering but not w.r.t. the unrestricted instantiation ordering. This accounts for the fact that equational unification is mostly done with restricted instantiation.

Most general unifiers (i.e., weak coequalizers) modulo commutative theories cannot always be chosen as coequalizers. But we can give algebraic conditions under which this is possible. For the class of commutative theories there always exist least specific generalizers. That means that all commutative theories have generalization type "unitary".

1. Introduction

Unification of terms plays an important rôle in automated theorem proving, term rewriting and logic programming. A unification problem is a term equation $\Gamma = < s = t >$ and a solution or unifier of Γ is a substitution θ such that $s\theta = t\theta$. A substitution θ is an endomorphism of the term algebra such that $x\theta = x$ for almost all variables x. In their seminal papers for automated theorem proving and term rewriting, Robinson (1965) and Knuth-Bendix (1967) independently showed that a solvable unification problem Γ always has a most general unifier, i.e., a unifier from which all unifiers may be generated by instantiation. Terms as well as substitutions are ordered w.r.t. instantiation preorderings (see e.g. Huet (1980) and Eder (1985)). In Robinson's paper and in many subsequent papers on unification (e.g. Eder (1985)) the instantiation preorder $\leq$ on substitutions is defined by $\sigma \leq \theta$ iff there exists a substitution λ such that $\theta = \sigma\lambda$. This preorder will be called unrestricted instantiation ordering in the sequel. In other papers (e.g. Rydeheard-Burstall (1986)) the ordering is restricted to the variables occurring in the unification problem, i.e., they just require that $x\theta = x\sigma\lambda$ for all variables x occurring in some term of the unification problem. This preorder will be called restricted instantiation ordering. The fact that there always exists a most general unifier does not depend on the chosen instantiation ordering.

In some applications – for example, if we want to compute critical pairs of rewrite rules – we do not directly have a unification problem, but a weak unification problem: for given

1) This research was done while the author was still at the IMMD 1, University Erlangen.

variables, this can be made to a unification problem. In order to avoid variable renaming, Eder (1985) has introduced the notion of weak unification. The term $s\sigma = t\tau$ is an upper bound of s, t in the instantiation lattice of first order terms (see Huet (1980)). Two weakly unifiable terms always have a single most general upper bound u, i.e., a term u which is the supremum of s, t in the instantiation lattice. But this does not mean that the pair σ, τ of weak unifiers with $s\sigma = u = t\tau$ is most general. If we take the unrestricted instantiation ordering, there exist terms s, t which are weakly unifiable but do not have a most general pair of weak unifiers (Eder (1985), see also Section 5).

A concept closely related to weak unification is generalization of terms: for given terms s, t we want to find a term g and substitutions σ, τ with $s = g\sigma$ and $t = g\tau$ (see e.g. Plotkin (1970), Huet (1980)). The term g is a lower bound of s, t in the instantiation lattice of first order terms. In this sense weak unification and generalization are duals of each other. Two terms s, t always have a single least general lower bound g, i.e., an infimum in the instantiation lattice. Again, this does not imply that the corresponding substitutions σ, τ with $s = g\sigma$ and $t = g\tau$ are least general w.r.t. the unrestricted instantiation ordering.

If unification is generalized to equational unification, most authors (see e.g. Plotkin (1972) and Siekmann (1989)) use the restricted instantiation ordering. In Section 6 we shall give an example of an equational theory – namly the theory of commutative, idempotent monoids – where unification problems always have most general unifiers w.r.t. the restricted instantiation ordering but not w.r.t. the unrestricted instantiation ordering.

Eder (1985) has generalized the notion of supremum (w.r.t. the unrestricted instantiation ordering) and weak unification from terms to substitutions. The same can be done for unification, generalization and infimum.

Now we can consider categories which have term algebras as objects and substitutions as morphisms. The choice of the appropriate category depends on the instantiation ordering. Unification, weak unification, etc, can thus be expressed in a categorical way. Rydeheard-Burstall (1986) used this categorical reformulation of the unification problem to obtain a categorical unification algorithm.

In this paper we shall also use the categorical framework to clarify the connection between unification, weak unification and generalization and to show the influence of the different instantiation orderings. Unification has something to do with weak coequalizers and weak unification with weak pushouts, but generalization does not correspond to the categorical dual concepts of weak equalizers or pullbacks (see Mac Lane (1971) or Section 3 below for the definition of weak limits and colimits). In order to formulate weak unification and generalization as duals in the categorical sense, we have to use a different category (see Section 3). This construction also clarifies the difference between finding most general pairs of weak unifiers and finding suprema w.r.t. instantiation.

For the unrestricted instantiation ordering, non-trivial most general unifiers are never coequalizers in the corresponding category, because they do not satisfy the uniqueness condition which is required for coequalizers but not for weak coequalizers and most general unifiers. Moreover, this category does not have binary (weak) coproducts, which accounts for the problems that arise when weak unification is considered w.r.t. the unrestricted instantiation ordering.

If we take the restricted instantiation ordering we can always find most general unifiers which are coequalizers in the corresponding category. Since this category also has binary coproducts, pushouts and hence most general weak unifiers can be obtain using a well-known categorical construction.

The categorical reformulation of equational unification (with the restricted instantiation ordering) was used in Baader (1989a) to derive general results on unification in the

class of commutative theories. In Section 7 we shall show under which conditions most general unifiers modulo a commutative theory can be chosen as coequalizers.

Generalization of terms and substitutions can also be done modulo an equational theory. But then terms may have more than one least general lower bound (see e.g. Pottier (1989)). We shall show that in commutative theories a single least general lower bound always exists.

2. Basic Definitions and Notations

Let Ω be a signature, i.e., a set of function symbols with fixed arity, and let V be a countable set of variables. For any subset X of V we denote the set of all *Ω-terms* with variables in X by F(X). This set is the carrier of the *free Ω-algebra* with generators X, which will also be denoted by F(X). Any mapping of X into an Ω-algebra $\mathcal{A}$ can be uniquely extended to a homomorphism of F(X) into $\mathcal{A}$. We write homomorphisms in suffix notation, i.e., $s\theta$ instead of $\theta(s)$. Consequently, composition is written from left to right, i.e. $\sigma\theta$ means first σ and then θ. An endomorphism θ of F(V) is called *substitution* iff it has finite domain, where the domain of θ is defined as $D(\theta) := \{ x; x\theta \neq x \}$.

Let s be a term, θ be a substitution and X be a subset of V. The set of all variables occurring in s is denoted by V(s). The set $\{ y;$ There is $x \in X$ with $y \in V(x\theta) \}$ is denoted by $V(X\theta)$.

Let E be a set of identities (equational theory) and let $=_E$ be the equality of terms, induced by E. The equational theory E defines a variety V(E), i.e. the class of all algebras (over the given signature Ω), which satisfy each identity of E. For any subset X of V the quotient algebra $F(X)/{=_E}$ is the *E-free Ω-algebra* with generators X, which is an element of V(E) and which will be denoted by $F_E(X)$.

The relation $=_E$ can be extended to substitutions in the obvious way, namely $\sigma =_E \tau$ iff $x\sigma =_E x\tau$ for all variables $x \in V$. Terms and substitutions may be ordered by E-instantiation orderings. We shall define these orderings on n-tuples ($n \geq 1$) of terms (resp. substitutions). For an n-tuple of terms $\underline{s} = (s_1,...,s_n)$, an n-tuple of substitutions $\underline{\sigma} = (\sigma_1,...,\sigma_n)$ and a substitution λ, let $\underline{s}\lambda := (s_1\lambda,...,s_n\lambda)$, $\underline{\sigma}\lambda := (\sigma_1\lambda,...,\sigma_n\lambda)$ and $\lambda\underline{\sigma} = (\lambda\sigma_1,...,\lambda\sigma_n)$.

DEFINITION 2.1. (E-instantiation preorder on n-tuples)

(1) Let $\underline{s} = (s_1,...,s_n)$ and $\underline{t} = (t_1,...,t_n)$ be n-tuples of terms. Then we define

$\underline{s} \leq_E \underline{t}$:⇔ There exists a substitution λ such that $\underline{s}\lambda =_E \underline{t}$.

(2) Let $\underline{\sigma} = (\sigma_1,...,\sigma_n)$ and $\underline{\tau} = (\tau_1,...,\tau_n)$ be n-tuples of substitutions and let $\underline{X} = (X_1,..., X_n)$ be an n-tuple of finite subsets of V. We define the *restricted E-instantiation preorder* $\leq_E \langle\underline{X}\rangle$ by

$\underline{\sigma} \leq_E \underline{\tau} \langle\underline{X}\rangle$:⇔ There exists a substitution λ such that for all i, $1 \leq i \leq n$, we have

$$x\sigma_i\lambda =_E x\tau_i \text{ for all } x \in X_i.$$

(3) Let $\underline{\sigma} = (\sigma_1,...,\sigma_n)$ and $\underline{\tau} = (\tau_1,...,\tau_n)$ be n-tuples of substitutions. We define the *unrestricted instantiation preorder* $\leq_E$ by

$\underline{\sigma} \leq_E \underline{\tau}$:⇔ There exists an endomorphism λ such that $\underline{\sigma}\lambda =_E \underline{\tau}$.

We shall omit the index "$_E$" if E is the empty set. For n = 1 we have the usual instantiation preorders (see e.g. Huet (1980), Eder (1985), Siekmann (1989)). But n-tuples are not ordered componentwise w.r.t. the usual instantiation preorder, because we require the same λ for all components. The substitutions which are smaller w.r.t. $\leq_E$ are left factors of the greater ones. For generalization we shall also consider preorders which are defined by using right factors.

Let ≤ be a *preorder*, i.e., a reflexive, transitive relation, on a set Q. This preorder defines an equivalence relation ≡ in the usual way: a ≡ b iff a ≤ b and b ≤ a. Now ≤ induces a partial order on the equivalence classes [a] = { b; a ≡ b } of ≡ by [a] ≤ [b] iff a ≤ b.

A non-empty subset A of Q is a *lower set* (*upper set*) iff a ∈ A and b ≤ a implies b ∈ A (a ∈ A and a ≤ b implies b ∈ A). The lower set (upper set) A is generated by B ⊆ A iff A = { a ∈ Q; There is b ∈ B such that a ≤ b } (A = { a ∈ Q; There is b ∈ B such that b ≤ a }). Let A be a lower set (upper set) which is generated by B. Then B is called a *basis* of A iff two different elements of B are not comparable w.r.t. ≤.

LEMMA 2.2. Let ≤ be a preorder on the set Q and let [Q] be the set of all ≡-classes. Moreover, let A be an upper set (lower set) in Q and let M be the set of all minimal (maximal) elements of [A] = { [a]; a ∈ A }.

(1) A has a basis (w.r.t. ≤ on Q) iff M generates [A] (w.r.t. ≤ on [Q]).

(2) If B is a basis of A then M = { [b]; b ∈ B }.

(3) If M generates [A] then any set of representatives for M is a basis of A.

PROOF. See Baader (1989), Lemma 2.2 and Proposition 2.3. ❑

Evidently, the lower set (upper set) A may have four possible types:

(1) M generates [A] and is a singleton (type unitary).

(2) M generates [A] and is finite (type finitary).

(3) M generates [A] and is infinite (type infinitary).

(4) A does not have a basis (type zero).

These types are ordered as follows: unitary < finitary < infinitary < zero. This will be used to define unification types, weak unification types, and so on. But first, we have to define the notions unification, weak unification, generalization, infimum and supremum for n-tuples of terms. If we want to consider infinite problems then n-tuple must be replaced by ω-tuples (here ω denotes order type of the non-negative integers).

DEFINITION 2.3. All problems are of the form $\Gamma = < \underline{s}, \underline{t} >_E$, where $\underline{s} = (s_1,...,s_n)$ and $\underline{t} = (t_1,...,t_n)$ are n-tuples of terms.

(1) Let $\Gamma = < \underline{s}, \underline{t} >_E$ be an *E-unification problem.* An E-unifier of Γ is a substitution σ such that $\underline{s}\sigma =_E \underline{t}\sigma$. The set of all E-unifiers of Γ is denoted by $U_E(\Gamma)$. This set is the set of solutions of the unification problem.

(2) Let $\Gamma = < \underline{s}, \underline{t} >_E$ be a *weak E-unification problem.* A weak E-unifier of Γ is a pair of substitution $\underline{\sigma} = (\sigma_1,\sigma_2)$ such that $\underline{s}\sigma_1 =_E \underline{t}\sigma_2$. The set of all weak E-unifiers of Γ is denoted by $W_E(\Gamma)$. This set is the set of solutions of the weak unification problem.

(3) Let $\Gamma = < \underline{s}, \underline{t} >_E$ be an *E-upper bound problem.* An E-upper bound of Γ is an n-tuple $\underline{u}$ of terms such that $\underline{s} \leq_E \underline{u}$ and $\underline{t} \leq_E \underline{u}$. The set of all upper bounds of Γ is denoted by $UB_E(\Gamma)$. This set is the set of solutions of the upper bound problem.

(4) Let $\Gamma = < \underline{s}, \underline{t} >_E$ be an *E-lower bound problem.* An E-lower bound of Γ is an n-tuple $\underline{g}$ of terms such that $\underline{g} \leq_E \underline{s}$ and $\underline{g} \leq_E \underline{t}$. The set of all E-lower bounds of Γ is denoted by $LB_E(\Gamma)$. This set is the set of solutions of the lower bound problem.

(5) Let $\Gamma = < \underline{s}, \underline{t} >_E$ be an *E-generalization problem.* An E-generalizer of Γ is a pair of substitution $\underline{\sigma} = (\sigma_1,\sigma_2)$ and an n-tuple $\underline{g}$ of terms such that $\underline{g}\sigma_1 =_E \underline{s}$ and $\underline{g}\sigma_2 =_E \underline{t}$. The set of all E-generalizer of Γ is denoted by $G_E(\Gamma)$. This set is the set of solutions of the generalization problem.

A problem is solvable iff the corresponding set of solutions is not empty. The possible solutions of the problems may be preordered by restricted or unrestricted E-instantiation preorders.

(1) $U_E(\Gamma)$ is a set of substitutions. The set of all substitutions can be preordered by $\leq_E$ (Definition 2.1.3 for n = 1) or by $\leq_E <X>$ (Definition 2.1.2 for n = 1), where X is the set of all variables occurring in some s_i or t_i (i = 1, ..., n).

(2) $W_E(\Gamma)$ is a set of pairs of substitutions. The set of all pairs of substitutions can be ordered by $\leq_E$ (Definition 2.1.3 for n = 2) or by $\leq_E <X_1,X_2>$ (Definition 2.1.2 for n = 2), where X_1 is the set of all variables occurring in some s_i (i = 1, ..., n) and X_2 is the set of all variables occurring in some t_i (i = 1, ..., n).

(3) $UB_E(\Gamma)$ is a set of n-tuples of terms. The set of all n-tuples of terms can be ordered by $\leq_E$ (Definition 2.1.1).

(4) $LB_E(\Gamma)$ is a set of n-tuples of terms. The set of all n-tuples of terms can be ordered by $\leq_E$ (Definition 2.1.1).

(5) The elements of $G_E(\Gamma)$ are of the form $(\underline{\sigma},\underline{g})$, where $\underline{\sigma}$ is a pair of substitutions and $\underline{g}$ is an n-tuple of terms. Let Q be the set { $(\underline{\sigma},\underline{g})$; $\underline{\sigma}$ is a pair of substitutions and $\underline{g}$ is an n-tuple of terms }.

(5.1) The unrestricted preorder on Q is defined by

$(\underline{\sigma},\underline{g}) \leq_E (\underline{\sigma}',\underline{g}')$:⇔ There exists a substitution λ such that $\underline{\sigma} =_E \lambda\underline{\sigma}'$ and $\underline{g}\lambda =_E \underline{g}'$.

(5.2) The restricted preorder on Q is defined by

$(\underline{\sigma},\underline{g}) \leq_E (\underline{\sigma}',\underline{g}')$:⇔ There exists a substitution λ such that $\underline{g}\lambda =_E \underline{g}'$ and $x\underline{\sigma} =_E x\lambda\underline{\sigma}'$ for all variables x occurring in $\underline{g}$.

Obviously, the relations $\leq_E$ defined in (1) – (4) and (5.1) of the definition are preorders. In (5.1), the greater pair of substitutions $\underline{\sigma}'$ is a right factor of the smaller pair $\underline{\sigma}$. The set X which is used in (5.2) for the restriction depends on the term part g of the smaller tupel $(\underline{\sigma},g)$ and not on the problem Γ. Nevertheless, it can be easily shown that the defined relation is a preorder.

LEMMA 2.4. (1) The sets $U_E(\Gamma)$, $W_E(\Gamma)$ and $UB_E(\Gamma)$ are upper sets w.r.t. the corresponding preorders.
(2) The sets $LB_E(\Gamma)$ and $G_E(\Gamma)$ are lower sets w.r.t. the corresponding preorders.

Please note that $W_E(\Gamma)$ would not be an upper set w.r.t. the componentwise instantiation ordering which is used in Eder (1985) to compare weak unifiers.

DEFINITION 2.5. (Types of problems and equational theories)
(1) Let Γ be a solvable E-unification (weak E-unification, E-upper bound, E-lower bound, E-generalization) problem and let A be the set of solutions of Γ. Then A is an upper set or lower set w.r.t. the restricted (unrestricted) E-instantiation ordering. The restricted (unrestricted) type of Γ is defined to be the smallest type of A:

$$\text{type}(\Gamma) := \min\{ T;\ A \text{ has type } T \}.$$

(2) Let E be an equational theory. Then the restricted (unrestricted) unification (weak unification, upper bound, lower bound, generalization) type of E is defined as

max{ T; T is the restricted (unrestricted) type of a solvable E-unification (weak E-unification, E-upper bound, E-lower bound, E-generalization) problem }.

In the present paper we shall only be interested in the question whether a given problem or theory is unitary or not. Let Γ be a unitary E-unification (weak E-unification, E-upper bound, E-lower bound, E-generalization) problem. Then all solutions of Γ can be generated from a single solution. This solution is unique up to equivalence and is called most general E-unifier (most general weak E-unifier, E-supremum, E-infimum, most specific E-generalizer) of Γ.
Finite E-unification (weak E-unification, E-upper bound, E-lower bound, E-generalization) problems can be easily formulated in a categorical way, if we use the restricted E-instantiation ordering. The notions most general E-unifier (most general weak E-unifier, E-supremum, E-infimum, most specific E-generalizer) correspond to well-known categorical concepts (see Section 3 and 4). For the unrestricted E-instantiation ordering we shall also have to consider infinite problems (see Section 5).

If we work with the restricted E-instantiation ordering we do not distinguish between $=_E$-equal substitutions and we are only interested in their behaviour on finite sets of variables. Hence substitutions can be regarded as morphisms in the following category:

DEFINITION 2.6. The category $C_r(E)$ is defined as follows:

(1) The objects of $C_r(E)$ are the algebras $F_E(X)$ for finite subsets X of V.

(2) The morphisms of $C_r(E)$ are the homomorphisms between these objects.

(3) The composition of morphisms is the usual composition of mappings.

For the unrestricted E-instantiation ordering we still do not distinguish between $=_E$-equal substitutions but we are interested in their behaviour on the whole set of variables V. This yields the category $C_u(E)$:

DEFINITION 2.7. The category $C_u(E)$ is defined as follows:

(1) The only object of $C_u(E)$ is the algebra $F_E(V)$.

(2) The morphisms of $C_u(E)$ are all substitutions (which can be considered as endomorphisms of $F_E(V)$).

(3) The composition of morphisms is the usual composition of mappings.

3. Categories

Let C be a category and A, B be objects of C. We denote by hom(A,B) the set of morphisms with domain A and codomain B. Note that composition of morphisms is also written from left to right. The identity morphism in hom(A,A) is denoted by 1_A or just 1. A morphism f is called epimorphism iff for any two morphisms g, h the equality fg = fh implies g = h. An isomorphism is an invertible morphism.
We say that the object P is a *product* of A, B iff there exist morphisms p_1: P $\to$ A, p_2: P $\to$ B such that for every pair of morphisms f: X $\to$ A, g: X $\to$ B there is a unique morphism h: X $\to$ P such that the product diagram of Figure 3.1 commutes.

FIGURE 3.1

$$\begin{tikzcd}
& A \\
X \arrow[ur, "f"] \arrow[r, "h"] \arrow[dr, "g"'] & P \arrow[u, "p_1"'] \arrow[d, "p_2"] \\
& B
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
A \arrow[d, "u_1"'] \arrow[dr, "f"] & \\
S \arrow[r, "h"] & X \\
B \arrow[u, "u_2"] \arrow[ur, "g"'] &
\end{tikzcd}$$

product diagram | coproduct diagram

A product of two objects may not exist, but if it exists it is unique up to isomorphism. We denote the product of A and B by A×B and call the corresponding morphisms projections. The dual of the product is the coproduct. An object S is a *coproduct* of A, B iff there exist morphisms u_1: A $\to$ S, u_2: B $\to$ S such that for every pair of morphisms f: A $\to$ X, g: B $\to$ X there is a unique morphism h: S $\to$ X such that the coproduct diagram of Figure 4.1 commutes. We denote the coproduct of A and B (if it exists) by A+B and call the corresponding morphisms injections. If we do not have uniqueness of the morphism h in the

above definitions we say that we have a *weak product* (*weak coproduct*). Weak products and coproducts need not be unique up to isomorphism (see Mac Lane (1971), Chapter 10, for the definition of weak limits and colimits). Please note that this notion of "weak" has nothing to do with the "weak" in "weak unifier".
Let g, h be morphisms with common domain and codomain. A *coequalizer* of the parallel pair g, h is a morphism f such that (1) gf = hf and (2) for any f' with gf' = hf' there is a unique morphism k such that f' = fk (see Figure 3.2). Obviously, any coequalizer is an epimorphism.

FIGURE 3.2

$$\begin{tikzcd} A \arrow[r, shift left, "g"] \arrow[r, shift right, "h"'] & B \arrow[r, "f"] \arrow[dr, "f'"'] & C \arrow[d, "k"] \\ & & C' \end{tikzcd}$$

coequalizer

$$\begin{tikzcd} & B \arrow[dr, "f_1"] \arrow[drr, "f'_1"] & & \\ A \arrow[ur, "g"] \arrow[dr, "h"'] & & D \arrow[r, "k"] & E \\ & C \arrow[ur, "f_2"] \arrow[urr, "f'_2"'] & & \end{tikzcd}$$

pushout

A *pushout* of two morphisms g, h with common domain is given by a pair of morphisms f_1, f_2 such that (1) $gf_1 = hf_2$ and (2) for any pair f'_1, f'_2 with $gf'_1 = hf'_2$ there is a unique morphism k such that $f'_1 = f_1k$ and $f'_2 = f_2k$.
The dual concepts are called equalizers and pullbacks. If we do not have uniqueness of the morphism k in the above definitions we say that we have a *weak coequalizer* (*weak pushout*).
Pushouts (weak pushouts) can be constructed using (weak) coproducts and (weak) coequalizers (see e.g. Burstall-Rydeheard (1988) and Proposition 3.8 below).

Let **C** be a category. We define the following two derived categories:
(1) The *morphism-category* $\mathbf{C_m}$ has as objects the morphisms of **C**. For two objects f: A → B and g: A' → B' we define $\hom_{\mathbf{C_m}}(f,g) = \varnothing$ if A ≠ A'. Otherwise $\hom_{\mathbf{C_m}}(f,g)$ contains all morphisms h: B → B' of **C** which satisfy g = fh (see Figure 3.3). The composition of morphisms is the composition in **C**.

FIGURE 3.3

$$\begin{tikzcd} & B \arrow[dd, "h"] \\ A \arrow[ur, "f"] \arrow[dr, "g"'] & \\ & B' \end{tikzcd}$$

a morphism of $\mathbf{C_m}$

$$\begin{tikzcd} & B \arrow[dd, "!_{f,g}"] \\ A \arrow[ur, "f"] \arrow[dr, "g"'] & \\ & B' \end{tikzcd}$$

a morphism of $\mathbf{C_p}$

(2) The *preorder-category* $\mathbf{C_p}$ has as objects the morphisms of **C**. For two objects f: A → B and g: A' → B' we define $\hom_{\mathbf{C_p}}(f,g) = \varnothing$ iff $\hom_{\mathbf{C_m}}(f,g) = \varnothing$. Otherwise $\hom_{\mathbf{C_p}}(f,g)$ contains a unique morphism $!_{f,g}$. These morphisms are composed in the obvious way, namely, $!_{f,g}!_{g,h} = !_{f,h}$.

The following proposition states that (weak) pushouts in **C** correspond to *(weak) coproducts in C_m*.

PROPOSITION 3.4. (1) Let f: $A \to D$ with the injections u_1: $B \to D$, u_2: $C \to D$ be an (weak) coproduct of f_1: $A \to B$ and f_2: $A \to C$ in $\mathbf{C_m}$. Then u_1, u_2 is a (weak) pushout of f_1, f_2 in **C**.

(2) Let u_1: $B \to D$, u_2: $C \to D$ be a (weak) pushout of f_1: $A \to B$ and f_2: $A \to C$ in **C**. Then $f := f_1u_1 = f_2u_2$: $A \to D$ with the injections u_1, u_2 is a (weak) coproduct of f_1, f_2 in $\mathbf{C_m}$.

Please note that (weak) pullbacks in **C** have nothing to do with (weak) products in $\mathbf{C_m}$. Weak products in $\mathbf{C_m}$ can be used for the categorical description of generalization (this was first mentioned in Plotkin (1970), p. 155).

In the category $\mathbf{C_p}$, weak coproducts (products, ...) are already coproducts, (products, ...). We can define a *functor* F: $\mathbf{C_m} \to \mathbf{C_p}$ as follows: F is the identity on the objects of $\mathbf{C_m}$. Let h be a morphism of $\mathbf{C_m}$ with domain f: $A \to B$ and codomain g: $A \to C$. Then F(h) $:= !_{f,g}$. It is easy to see that F preserves products and coproducts.

PROPOSITION 3.5. Let f: $A \to D$ with the injections u_1: $B \to D$, u_2: $C \to D$ (projections p_1: $D \to B$, p_2: $D \to C$) be a weak coproduct (weak product) of f_1: $A \to B$ and f_2: $A \to C$ in $\mathbf{C_m}$. Then f: $A \to D$ with the injections $F(u_1) = !_{f1,f}$, $F(u_2) = !_{f2,f}$ (projections $F(p_1) = !_{f,f1}$, $F(p_2) = !_{f,f2}$) is a coproduct (product) of f_1: $A \to B$ and f_2: $A \to C$ in $\mathbf{C_p}$.

REMARK. Let us keep the notations of Proposition 3.5. Taking a *coproduct (product) f in C_p* instead of $\mathbf{C_m}$ has the following meaning in **C** :

We are not interested in the morphisms u_1 and u_2 (p_1 and p_2), but only in the morphism f. Let g, v_1 and v_2 (h, q_1 and q_2) be morphisms such that $f_1v_1 = g = f_2v_2$ ($hq_1 = f_1$, $hq_2 = f_2$). Then we only require that there is a morphism k such that g = fk (hk = f), but we do not require $u_1k = v_1$ and $u_2k = v_2$ ($kp_1 = q_1$ and $kp_2 = q_2$).

If, however, f (resp. h) is an epimorphism then $u_1k = v_1$ and $u_2k = v_2$ (resp. $kp_1 = q_1$ and $kp_2 = q_2$) is a consequence of g = fk (hk = f).

Unification, weak unification, upper bound, lower bound and generalization problems can also be defined for categories:

DEFINITION 3.6. Let **C** be a category and let A, B, C, D be objects of **C**.

(1) A unification problem in **C** is a parallel pair of morphisms f, g: $A \to B$ and a unifier of the pair $< f, g >$ is a morphism h: $B \to C$ such that fh = gh.

(2) A weak unification problem in $\mathbf{C}$ is a pair of morphisms $f: A \to B$, $g: A \to C$ and a weak unifier of $< f, g >$ is a pair of morphisms $h: B \to D$, $k: C \to D$ such that $fh = gk$.

(3) An upper bound problem in $\mathbf{C}$ is a pair of morphisms $f: A \to B$, $g: A \to C$ and an upper bound of $< f, g >$ is a morphism $h: A \to D$ such that there exist morphisms k_1, k_2 with $fk_1 = h$ and $gk_2 = h$.

(4) A lower bound problem in $\mathbf{C}$ is a pair of morphisms $f: A \to B$, $g: A \to C$ and a lower bound of $< f, g >$ is a morphism $h: A \to D$ such that there exist morphisms k_1, k_2 with $hk_1 = f$ and $hk_2 = g$.

(5) A generalization problem in $\mathbf{C}$ is a pair of morphisms $f: A \to B$, $g: A \to C$ and a generalizer of $< f, g >$ is a morphism $h: A \to D$ and a pair of morphisms $k_1: D \to B$, $k_2: D \to C$ such that $hk_1 = f$ and $hk_2 = g$.

The solutions of the problems (1) – (4) are morphisms or pairs of morphisms. We order morphisms and pairs of morphisms with the following instantiation orderings:

$f \leq f'$ iff there is a morphism h such that $fh = f'$,

$(f,g) \leq (f',g')$ iff there is a morphism h such that $fh = f'$ and $gh = g'$.

Generalizers are ordered as follows:

$(k_1,k_2,h) \leq (k_1',k_2',h')$ iff there is a morphism m such that $hm = h'$, $k_1 = mk_1'$ and $k_2 = mk_2'$.

As in Section 2, we can now define the type of a problem and the notions *most general unifier* (*most general weak unifier, supremum, infimum, most specific generalizer*) of pairs of morphisms in $\mathbf{C}$.

PROPOSITION 3.7. Let $\mathbf{C}$ be a category.

(1) The morphism h is a most general unifier of the parallel pair $f, g: A \to B$ iff h is a weak coequalizer of f, g.

(2) The pair of morphisms $h: B \to D$, $k: C \to D$ is a most general weak unifier of $f: A \to B$, $g: A \to C$ iff h, k is a weak pushout of f, g. By Proposition 3.4, this means that $fh = gk$ with the injections h, k is a weak coproduct of f, g in $\mathbf{C}_{\mathbf{m}}$.

(3) The morphism h is a supremum of $f: A \to B$, $g: A \to C$ iff h with the injections $!_{f,h}$, $!_{g,h}$ is a coproduct of f, g in $\mathbf{C}_{\mathbf{p}}$.

(4) The morphism h is an infimum of $f: A \to B$, $g: A \to C$ iff h with the projections $!_{h,f}$, $!_{h,g}$ is a product of f, g in $\mathbf{C}_{\mathbf{p}}$.

(5) $h: A \to D$ and the pair of morphisms $k_1: D \to B$, $k_2: D \to C$ is a most specific generalizer of $f: A \to B$, $g: A \to C$ iff h with the projections k_1, k_2 is a weak product of f, g in $\mathbf{C}_{\mathbf{m}}$.

Since weak pushouts can be constructed using weak coequalizers and weak coproducts, weak unification in $\mathbf{C}$ can be reduced to unification in $\mathbf{C}$, provided that $\mathbf{C}$ has all weak binary coproducts.

PROPOSITION 3.8. Let **C** be a category such that every pair of objects has a weak coproduct. If all solvable unification problems are unitary then all solvable weak unification problems are also unitary.

PROOF. Let $< f, g >$ with $f: A \rightarrow B$, $g: A \rightarrow C$ be a solvable weak unification problem and let B+C with the injections u_1, u_2 be a weak coproduct of B, C. We consider the unification problem $< fu_1, fu_2 >$. Let the pair (h,k) be a weak unifier of $< f, g >$. Then there exists a morphism m such that $u_1m = h$ and $u_2m = k$ (by the definition of weak coproduct). Obviously, m is a unifier of $< fu_1, fu_2 >$. This shows that $< fu_1, fu_2 >$ is solvable. Let u be a most general unifier of $< fu_1, fu_2 >$ (i.e., weak coequalizer of fu_1, fu_2). It is easy to see that (u_1u, u_2u) is a most general weak unifier of $< f, g >$ (i.e., weak pushout of f, g). ❑

4. Substitutions with the Restricted ∅-Instantiation Ordering

Let $\Gamma = < (\underline{s},\underline{t}) >$ be a ∅-unification problem and X be the (finite) set of variables occurring in some s_i or t_i. Evidently, we can consider the s_i and t_i as elements of F(X). Since we use the restricted instantiation ordering, any ∅-unifier of Γ can be regarded as a homomorphism of F(X) into F(Y) for some finite set Y (of variables). Let $I = \{ x_1, ..., x_n \}$ be a set of cardinality n. We define homomorphisms

$$\sigma, \tau: F(I) \rightarrow F(X) \text{ by } x_i\sigma := s_i \text{ and } x_i\tau := t_i \ (i = 1, ..., n).$$

Now $\delta: F(X) \rightarrow F(Y)$ is a unifier of Γ iff $x_i\sigma\delta = s_i\delta = t_i\delta = x_i\tau\delta$ for $i = 1, ..., n$, i.e. iff $\sigma\delta = \tau\delta$. Thus a finite term unification problem can be written as a unification problem $< \sigma, \tau >$ in the category $C_r(\emptyset)$.

The same holds for the other problems introduced in Definition 2.3 and 3.6. That means that we can restrict our attention to problems which are given as pairs of morphisms in $C_r(\emptyset)$.

In this section, let **C** denote the category $C_r(\emptyset)$. It is easy to see that a morphism $\sigma: F(X) \rightarrow F(Y)$ is an epimorphism of **C** iff $V(X\sigma) = Y$. Hence any morphism $\sigma: F(X) \rightarrow F(Y)$ can be considered as an epimorphism with domain F(X) and codomain $F(V(X\sigma))$. In general, this nice property does not hold if we consider categories $C_r(E)$ for $E \neq \emptyset$. The coproduct of two objects F(X), F(Y) of **C** is given by $F(X \dot{\cup} Y)$, where $\dot{\cup}$ denotes disjoint union.

4.1 Unification

It is well-known that a finite, sovable ∅-unification problem always has a most general unifier (even w.r.t. unrestricted instantiation and thus, all the more, w.r.t. restricted instantiation). That means that any unifiable parallel pair $\sigma, \tau: F(I) \rightarrow F(X)$ of morphisms in $C_r(\emptyset)$ has a weak coequalizer (i.e., most general unifier) $\gamma: F(X) \rightarrow F(Y)$.

The morphism γ is a coequalizer of σ, τ iff $V(X\gamma) = Y$, i.e., iff γ is an epimorphism in **C**. This shows that not all most general unifiers are coequalizers. But we can always find a

most general unifiers which is a coequalizer: we just consider γ as morphism from F(X) into F(V(Xγ)).

4.2 Weak Unification

The category $\mathbf{C} = \mathbf{C}_r(\emptyset)$ has all binary coproducts. Hence, Section 4.1 and Proposition 3.8 imply that a finite, solvable weak $\emptyset$-unification problem σ: F(I) $\rightarrow$ F(X), τ: F(I) $\rightarrow$ F(Y) always has a most general weak unifier. This most general weak unifier can even be chosen as pushout of σ, τ, since we have coproducts and coequalizers and not just weak coproducts and coequalizers.

4.3 Upper Bound Problems

Proposition 3.5, 3.7 and Section 4.2 imply that a solvable upper bound problem $< \sigma, \tau >$ (where σ: F(I) $\rightarrow$ F(X), τ: F(I) $\rightarrow$ F(Y)) always has a supremum in **C**, i.e., a coproduct in $\mathbf{C}_p$. Hence, any pair $\underline{s}$, $\underline{t}$ of n-tuples of terms has a supremum, if it has an upper bound.

4.4 Lower Bound Problems

It is well-known that a pair of terms always has an infimum (see e.g. Huet (1980)). Let $\underline{s} = (s_1,\ldots,s_n)$, $\underline{t} = (t_1,\ldots,t_n)$ be n-tuples of terms and let f be a binary function symbol. We define $s := f(s_1,f(s_2,\ldots f(s_{n-1},s_n)\ldots))$ and $t := f(t_1,f(t_2,\ldots f(t_{n-1},t_n)\ldots))$. Then $\underline{g} = (g_1,\ldots,g_n)$ is an infimum of $\underline{s}$, $\underline{t}$ iff $g = f(g_1,f(g_2,\ldots f(g_{n-1},g_n)\ldots))$ is an infimum of s, t.

This shows that a pair of morphisms σ: F(I) $\rightarrow$ F(X), τ: F(I) $\rightarrow$ F(Y) always has an infimum in **C**, i.e., a product in $\mathbf{C}_p$.

4.5 Generalization

Obviously, any generalization problem is solvable in **C**. We shall first show that there exist objects σ, τ in $\mathbf{C}_m$ which do not have a weak product in $\mathbf{C}_m$.

EXAMPLE 4.1. Let the signature consist of the two unary function symbols f and g. We define σ: F(x) $\rightarrow$ F(u), τ: F(x) $\rightarrow$ F(v) by

$$x\sigma := f(u) \text{ and } x\tau := g(v).$$

Assume that γ: F(x) $\rightarrow$ F(Z) with the projections π_1: F(Z) $\rightarrow$ F(u), π_2: F(Z) $\rightarrow$ F(v) is a weak product of σ, τ in $\mathbf{C}_m$.

Obviously, $x\gamma\pi_1 = f(u)$ and $x\gamma\pi_2 = g(v)$ implies that $x\gamma$ is a variable. Let $x\gamma$ be the variable z and let Z be the set { z, z_1, ..., z_m }. We have $z\pi_1 = f(u)$, $z\pi_2 = g(v)$ and $z_i\pi_1 = s_i(u)$, $z_i\pi_2 = t_i(v)$ for terms $s_i(u)$ and $t_i(v)$.

Let $k > 1$ be a positive integer such that $f^k(u) \neq s_i(u)$ for all i, $1 \leq i \leq m$. We consider the morphisms δ: F(x) $\rightarrow$ F({ y, y_1 }), δ_1: F({ y, y_1 }) $\rightarrow$ F(u) and δ_2: F({ y, y_1 }) $\rightarrow$ F(v) which are defined by

$$x\delta := y,\ y\delta_1 := f(u),\ y\delta_2 := g(v) \text{ and } y_1\delta_1 := f^k(u) \text{ and } y_1\delta_2 := g^k(u).$$

Now $\delta\delta_1 = \sigma$ ($\delta\delta_2 = \tau$) implies that δ_1 (δ_2) can be considered as a morphism of $\mathbf{C}_m$

with domain δ and codomain σ (codomain τ). Since γ is a weak product of σ, τ in $\mathbf{C_m}$, there is a morphism λ such that $\delta\lambda = \gamma$, $\lambda\pi_1 = \delta_1$ and $\lambda\pi_2 = \delta_2$.

Evidently, $y_1\lambda\pi_1 = y_1\delta_1 = f^k(u)$ and $y_1\lambda\pi_2 = y_1\delta_2 = g^k(u)$ imply that $y_1\lambda$ is a variable. Since $z\pi_1 = f(u)$ and $k > 1$, we get $y_1\lambda \neq z$, i.e., $y_1\lambda = z_i$ for some i, $1 \leq i \leq m$. That means that $z_i\pi_1 = f^k(u)$, which is a contradiction.

This does not mean that there exist term generalization problems which do not have most specific generalizers w.r.t. the restricted instantiation ordering. It only means that $\mathbf{C} = \mathbf{C}_r(\emptyset)$ is not the appropriate category.

Recall that the restricted preorder on Q = { ($\underline{\sigma}$,$\underline{g}$); $\underline{\sigma}$ is a pair of substitutions and $\underline{g}$ is an n-tuple of terms } was defined by ($\underline{\sigma}$,g) $\leq_E$ ($\underline{\sigma}$',g') :$\Leftrightarrow$ There exists a substitution λ such that $\underline{g}\lambda =_E g'$ and $x\underline{\sigma} =_E x\lambda\underline{\sigma}'$ *for all variables x occurring in* $\underline{g}$ (see Definition 2.3)

In the example, this means that we only require $y\lambda\pi_1 = y\delta_1$ and $y\lambda\pi_2 = y\delta_2$, since $V(x\delta) = \{ y \}$.

In order to express most specific $\emptyset$-generalizers (w.r.t. the restricted instantiation ordering) as product in a morphism category we have to take the following subcategory of $\mathbf{C} = \mathbf{C}_r(\emptyset)$: The category $\mathbf{C}_e(\emptyset)$ has the same objects as $\mathbf{C}_r(\emptyset)$ but only the epimorphisms of $\mathbf{C}_r(\emptyset)$ as morphisms.

Let $\mathbf{C'}$ denote the category $\mathbf{C}_e(\emptyset)$. The results of Section 4.1 – 4.4 also hold with $\mathbf{C'}$ in place of $\mathbf{C}$, because we can consider any morphism γ: F(X) $\rightarrow$ F(Y) as morphism from F(X) into F(V(Xγ)). But note that most general unifiers are now automaticly coequalizers.

Section 4.4 and the remark after Proposition 3.4 imply that products always exist in $\mathbf{C'_m}$. This shows that we always have most specific $\emptyset$-generalizers w.r.t. the restricted instantiation ordering.

5. Substitutions with the Unrestricted Instantiation Ordering

We shall now consider the problems of Definition 2.3 for E = $\emptyset$ and unrestricted instantiation ordering. In this section let $\mathbf{C}$ denote the category $\mathbf{C}_u(\emptyset)$.

It is easy to see that a morphism σ of $\mathbf{C}$ is an epimorphism (in the categorical sense, as defined in Section 3) iff $V(V\sigma) = V$. Please note that for an epimorphism σ of $\mathbf{C}$ the mapping σ from F(V) into F(V) need not be surjective.

4.1 Unification

Unification of a pair of substitutions corresponds to a finite unification problem for terms. Let σ and τ be two morphisms of $\mathbf{C}$. We consider the unification problem

$$\Gamma(\sigma,\tau) := < (x\sigma)_{x\in D(\sigma)\cup D(\tau)}, (x\tau)_{x\in D(\sigma)\cup D(\tau)} >.$$

Obviously, any term unification problem Γ can be obtained as $\Gamma = \Gamma(\sigma,\tau)$ for suitable sub-

stitutions σ, τ. A substitution δ (i.e., a morphism of C) is a unifier of $\Gamma(\sigma,\tau)$ iff $x\sigma\delta = x\tau\delta$ for all $x \in D(\sigma) \cup D(\tau)$. For $y \notin D(\sigma) \cup D(\tau)$ we have $y\sigma\delta = y\delta = y\tau\delta$. This yields

LEMMA 5.1. δ is a unifier of $\Gamma(\sigma,\tau)$ iff δ is a unifier of σ and τ in **C**.

It is well known (e.g. Robinson (1965), Eder (1985)) that any solvable unification problem $\Gamma = < \underline{s}, \underline{t} >$ has a most general unifier δ (w.r.t. unrestricted instantiation) which satisfies the following properties:
(P1) $D(\delta) \cup V(D(\delta)\delta) \subseteq V_0$, where V_0 is the set of all variables occurring in some s_i or t_i ($i = 1, ..., n$).
(P2) δ is idempotent, i.e., $D(\delta) \cap V(D(\delta)\delta) = \emptyset$.
Most general unifiers are unique up to $\equiv$-equivalence (where $\equiv$ denotes the equivalence induced by the unrestricted instantiation preorder). The equivalence relation $\equiv$ can be described as follows (see Eder (1985)): $\sigma \equiv \tau <V>$ iff there exists a substitution π which is a permutation of variables and which satisfies $\sigma = \tau\pi$. Obviously, π is an isomorphism of C.
The next proposition states the connection between most general unifiers and (weak) coequalizers.

PROPOSITION 5.2. Let $\Gamma = \Gamma(\sigma,\tau)$ be a solvable unification problem and let γ be a most general unifier of Γ.
(1) γ is a weak coequalizer (i.e., most general unifier) of the parallel pair σ, τ in C and any weak coequalizer of σ, τ is a most general unifier of Γ.
(2) γ is a coequalizer of σ, τ if and only if $\sigma = \tau$.
PROOF. (1) The first part of the proposition is an immediate consequence of Lemma 5.1 and Proposition 3.7.
(2) If $\sigma = \tau$ then γ is an isomorphism and hence a coequalizer. If $\sigma \neq \tau$ then any unifier of Γ has non-empty domain. Let δ be a most general unifier of Γ which satisfies the Properties P1 and P2. Then δ is not an epimorphism because the variables of $D(\delta)$ are not contained in $V(V\delta)$. Since $\gamma = \delta\pi$ for an isomorphism π, the morphism γ is also not an epimorphism. But coequalizers are always epimorphisms. ❑

The fact that only trivial parallel pairs have coequalizers in **C** is the first unpleasant feature of the unrestricted instantiation ordering. It gets even worse if we consider weak unification.

5.2 Weak Unification and Upper Bound Problems

Since the definition of suprema for n-tuples of terms has nothing to do with restricted or unrestricted instantiation, Section 4.3 can be used for n-tuples of terms.
But solvable upper bound problems in **C** need not have suprema.

LEMMA 5.3. Let γ_1, γ_2 be the substitutions defined by $D(\gamma_1) := D(\gamma_2) := \{ x, y, z \}$ and $x\gamma_1 := y\gamma_1 := z\gamma_1 := f(f(x,y),z)$, $x\gamma_2 := y\gamma_2 := z\gamma_2 := f(x,f(y,z))$. The upper bound problem $< \gamma_1, \gamma_2 >$ is solvable, but there does not exist a supremum of γ_1, γ_2 in **C**.

PROOF. See Eder (1985), Example 2.7. ❑

Eder shows that if he restricts himself to idempotent substitutions, i.e., substitutions σ such that $\sigma\sigma = \sigma$, then every set of such substitutions which has an upper bound has a supremum. This restriction cannot be used in our categorical framework, since the composition of idempotent substitutions need not be idempotent (Eder (1985), p.38).
If weak unifiers are compared using the unrestricted instantiation ordering then the following lemma holds.

LEMMA 5.4. A weak unification problem for n-tuples of terms does not have a most general pair of weak unifiers (w.r.t. the unrestricted instantiation ordering).

PROOF. Let $\Gamma = < \underline{s}, \underline{t} >$ with $\underline{s} = (s_1,...,s_n)$ and $\underline{t} = (t_1,...,t_n)$ be a weak unification problem and let X be the set of variables occurring in some s_i or t_i. Assume that γ, δ is a most general pair of weak unifiers of Γ. For $z \notin X \cup D(\gamma) \cup D(\delta)$ we have $z\gamma = z = z\delta$ and hence $z\gamma\lambda = z\delta\lambda$ for all substitutions λ. Now any pair γ', δ' with $x\gamma' = x\gamma$ and $x\delta' = x\delta$ for $x \neq z$ and $z\gamma' \neq x\delta'$ is a pair of weak unifiers of Γ which is not an instance of γ, δ.
❑

This proof depends on the fact that we require a common right factor λ to obtain γ' and δ'. Nevertheless, we shall use this ordering instead of the componentwise instantiation ordering for the following reasons:
(1) Instances of weak unifiers should also be weak unifiers.
(2) Most general pairs of weak unifiers would not correspond to pushouts if we used the componentwise instantiation ordering.
(3) Even with the componentwise instantiation ordering, the terms $s = x$ and $t = f(x,y)$ have weak unifiers but they do not have a most general pair of weak unifiers (see Eder (1985), Example 5.5).

To express weak unification of morphisms in $\mathbf{C} = \mathbf{C}_u(\emptyset)$, we shall in general need infinite weak unification problems for terms.
Let σ, τ be a pair of morphisms of $\mathbf{C}$. First, we consider $\Gamma(\sigma,\tau) = < (x\sigma)_{x\in D(\sigma)\cup D(\tau)}, (x\tau)_{x\in D(\sigma)\cup D(\tau)} >$ as weak unification problem. Obviously, a pair of substitutions γ, δ with $\sigma\gamma = \tau\delta$ is a pair of weak unifiers of $\Gamma(\sigma,\tau)$, but it is easy to see that the opposite need not be true. Moreover, a weak unification problem $\Gamma(\sigma,\tau)$ may be solvable, even if there are no substitutions γ, δ with $\sigma\gamma = \tau\delta$ (consider σ, τ defined by $D(\sigma) := \{ x \} =: D(\tau)$ and $x\sigma := f(y)$, $x\tau := f(f(y))$). Eder (1985) considers the infinite weak unification problem

$$\Lambda(\sigma,\tau) := < (x\sigma)_{x\in V}, (x\tau)_{x\in V} >.$$

Obviously, the pair of substitution γ, δ is a weak unifier of $\Lambda(\sigma,\tau)$ iff the pair of morphisms γ, δ is a weak unifier (i.e., weak pushout) of σ, τ in $\mathbf{C}$.
In general, weak unification problems for n-tuples of terms cannot be expressed as weak unification problems in $\mathbf{C}$, since they need not be of the form $\Lambda(\sigma,\tau)$ for substitutions σ, τ.
In Section 3 we have seen that weak pushouts in $\mathbf{C}$ (i.e., weak unifiers in $\mathbf{C}$) correspond to weak coproducts in $\mathbf{C}_m$. If we are only interested in suprema of substitutions we have

to consider coproducts in C_p. We have already seen that a solvable upper bound problem in C need not have a supremum. But even if a supremum in C (i.e., coproducts in C_p) exists, we need not have a weak coproduct in C_m.

PROPOSITION 5.5. There exist substitutions σ, τ which have a coproduct in C_p, but which do not have a weak coproduct in C_m.

PROOF. Consider the substitutions σ, τ defined by $D(\sigma) := \{ x \}$, $x\sigma := y$ and $D(\tau) := \{ x, y \}$, $x\tau := f(x,y)$, $y\tau := f(x,y)$. Eder (1985) shows: τ is a supremum of $\{ \sigma, \tau \}$ in the set of substitutions, but $\Lambda(\sigma,\tau)$ does not have a most general pair of weak unifiers w.r.t. the unrestricted componentwise instantiation ordering. Hence $\Lambda(\sigma,\tau)$ does not have a most general pair of weak unifiers w.r.t. our unrestricted instantiation ordering. Now Proposition 3.7 yields that the pair σ, τ has a coproduct in C_p, but does not have a weak coproduct in C_m. ❑

This subsection shows that the unrestricted instantiation ordering is not well-suited for handling weak unification. In Section 4.2 we have seen, that the restricted instantiation ordering yields much better results.

5.3 Generalization and Lower Bound Problems

Since the definition of infima for n-tuples of terms has nothing to do with restricted or unrestricted instantiation, Section 4.5 can be used for n-tuples of terms.
But lower bound problems in C (which are always solvable since the identity is a lower bound for all substitutions) need not have infima. Before we can show this we have to prove a technical lemma.

LEMMA 5.6. Let σ, τ, γ be substitutions such that $\gamma \leq \sigma$ and $\gamma \leq \tau$. We define $V_0 := D(\sigma) \cup D(\tau)$ and $V_1 := V(V_0\gamma)$. Assume that for any λ, ρ such that $\gamma\lambda = \sigma$ and $\gamma\rho = \tau$ and any $x \in V_1$ we have $x\lambda \neq x\rho$. Then $|V_0| \geq |V_1|$.

PROOF. Assume that $|V_0| < |V_1|$. Without loss of generality we may even assume that $V_0 \subset V_1$. Otherwise, let W be a subset of V_1 of cardinality $|V_0|$ and let π be a substitution such that π is a permutation of variables with $W\pi = V_0$. Then $\gamma\pi$ satisfies the assumptions of the lemma and $V_0 \subseteq V(V_0\gamma\pi)$.

Let z_0 be an element of $V_1 \setminus V_0$. Then $z_0\lambda \neq z_0\rho$, $z_0 = z_0\sigma = z_0\gamma\lambda$ and $z_0 = z_0\tau = z_0\gamma\rho$. Hence $z_1 := z_0\gamma$ is a variable and $z_1\lambda = z_0 = z_1\rho$. This implies $z_1 \notin V_1$ and thus $z_1 \notin V_0$ and $z_1 \neq z_0$. Now assume that we have already defined n+1 different variables $z_0, z_1, ..., z_n$ ($n \geq 1$) such that $z_1, ..., z_n \notin V_1$, $z_i\gamma = z_{i+1}$ and $z_{i+1}\lambda = z_i = z_{i+1}\rho$. Since $z_n \notin V_0$ we have $z_n\gamma\lambda = z_n\sigma = z_n = z_n\tau = z_n\gamma\rho$. Thus $z_{n+1} := z_n\gamma$ is a variable and $z_{n+1}\lambda = z_n = z_{n+1}\rho$. This implies $z_{n+1} \notin V_1$ and thus $z_{n+1} \neq z_0$. For $1 \leq i \leq n$, $z_i\lambda = z_{i-1} \neq z_n$ implies $z_{n+1} \neq z_i$.

By induction, we thus get that $D(\gamma)$ cannot be finite, which is a contradiction. ❑

PROPOSITION 5.7. There exist substitutions σ, τ such that $\{ \sigma, \tau \}$ does not have an infimum in the set of substitutions (i.e., σ, τ does not have a product in $\mathbf{C}_p$)

PROOF. Let $\sigma, \tau, \gamma_1, \gamma_2$ be the substitutions defined by

$D(\sigma) := D(\tau) := D(\gamma_1) := D(\gamma_2) := \{ x, y, z \}$ and

$x\sigma := y\sigma := z\sigma := f(f(a,b),f(c,d))$, $x\tau := y\tau := z\tau := f(f(b,a),f(d,c))$,

$x\gamma_1 := y\gamma_1 := z\gamma_1 := f(f(x,y),z)$, $x\gamma_2 := y\gamma_2 := z\gamma_2 := f(x,f(y,z))$,

where x, y, z are variables, a, b, c, d are constants and f is a binary function symbol.
It is easy to see that γ_1, γ_2 are lower bounds of $\{ \sigma, \tau \}$ in the set of substitutions. Assume that the substitution γ is an infimum of $\{ \sigma, \tau \}$. Then γ is an upper bound of $\{ \gamma_1, \gamma_2 \}$, which yields $\{ x, y, z \} \subseteq D(\gamma)$ and $x\gamma = y\gamma = z\gamma = f(f(q,r),f(s,t))$ for terms q, r, s, t (see Eder (1985), Example 2.7). Since γ is also a lower bound of $\{ \sigma, \tau \}$, the terms q, r, s, t are pairwise different variables. It can be easily shown that σ, τ, γ satisfy the assumptions of Lemma 4.7. Thus $|\{ x, y, z \}| < |V(\{ x, y, z \}\gamma)|$ is a contradiction. ❑

Eder (1985) has shown that two idempotent substitutions always have an infimum in the set of idempotent substitutions. The substitutions σ, τ in the proof of Part (2) of the proposition are idempotent, but this does not contradict Eder's result, because the substitutions γ_1, γ_2 are not equivalent to idempotent substitutions.

As in the case of weak unification, the generalization problem for substitutions σ, τ can in general only be expressed by the infinite term generalization problem $\Lambda(\sigma,\tau) = < (x\sigma)_{x\in V},(x\tau)_{x\in V} >$. But even finite term generalization problems need not have a most specific generalizer w.r.t. the unrestricted instantiation ordering.

EXAMPLE 5.8. Let the signature consist of the two unary function symbols f and g. We define $s := f(x)$ and $t := g(x)$. Assume that the term h with the substitutions π_1, π_2 is a most specific generalizer of $< s, t >$. Obviously, $h\pi_1 = f(x)$ and $h\pi_2 = g(x)$ implies that h is a variable z. Let $Z = \{ z, z_1, ..., z_n \}$ be the set $D(\pi_1) \cup D(\pi_2)$. We can now continue as in Example 4.1 to get a contradiction.

We have already seen that lower bound problems in $\mathbf{C}$ are always solvable but need not have infima. But even if an infimum in $\mathbf{C}$ (i.e., product in $\mathbf{C}_p$) exists, we need not have a most specific generalizer (i.e., weak product in $\mathbf{C}_m$).

PROPOSITION 5.9. There exist substitutions σ, τ which have a product in $\mathbf{C}_p$, but which do not have a weak product in $\mathbf{C}_m$.

PROOF. Let the signature consist of the two unary function symbols f and g. We define σ, τ by $D(\sigma) := \{ x \}$, $x\sigma := y$, $D(\tau) = \{ x, y \}$, $x\tau := f(u)$ and $y\tau := f(u)$. Evidently, $\sigma\tau = \tau$

and thus σ is an infimum of { σ, τ }. This shows that σ, τ have a product in C_p.
Assume that γ with the projections π_1, π_2 is a weak product of σ, τ in C_m.
Let k be a positive integer such that $f^k(u) \neq v\pi_1$ for all $v \in V$ (such an integer k exists since $D(\pi_1)$ is finite). We consider the substitutions δ, δ_1 and δ_2 which are defined by $\delta := \sigma$, $D(\delta_1) := \{ x \}$, $x\delta_1 := f^k(u)$, $D(\delta_2) := \{ x, y \}$, $y\delta_2 := f(u)$ and $x\delta_2 := g^k(u)$.
Now $\delta\delta_1 = \sigma$ ($\delta\delta_2 = \tau$) implies that δ_1 (δ_2) can be regarded as a morphism of C_m with domain δ and codomain σ (codomain τ). Since γ is a weak product of σ, τ in C_m , there is a morphism λ such that $\delta\lambda = \gamma$, $\lambda\pi_1 = \delta_1$ and $\lambda\pi_2 = \delta_2$.
Evidently, $x\lambda\pi_1 = x\delta_1 = f^k(u)$ and $x\lambda\pi_2 = x\delta_2 = g^k(u)$ implies that $x\lambda$ is a variable v. But now $v\pi_1 = x\lambda\pi_1 = x\delta_1 = f^k(u)$ is a contradiction. ❑

6. The Unification Type of a Theory Depends on the Instantiation Ordering

Until now we have seen that the weak unification (generalization) type of a theory depends on the chosen instantiation ordering: the empty theory has weak unification (generalization) type "unitary" if we use the restricted $\emptyset$-instantiation ordering; with respect to the unrestricted $\emptyset$-instantiation ordering, the empty theory does not have weak unification (generalization) type "unitary".
In this section we give an example of an equational theory E which has unification type "unitary" w.r.t. the restricted E-instantiation ordering, but not w.r.t. the unrestricted E-instantiation ordering.
Let CIM be the theory of commutative idempotent monoids, i.e., the signature consists of a binary function symbol "+" and a constant symbol "0" and the equational theory is

$$CIM := \{ x + 0 = x, x + y = y + x, x + (y + z) = (x + y) + z, x + x = x \}.$$

Terms s, t are equal w.r.t. CIM iff $V(s) = V(t)$ and $s =_{CIM} 0$ iff $V(s) = \emptyset$.
Since CIM is a commutative theory and the finitely generated CIM-free objects are finite, CIM has unification type "unitary" w.r.t. the restricted CIM-instantiation ordering (Baader (1988), see Section 7 of the present paper for the definition of commutative theories). A unification algorithm can be found in Baader-Büttner (1988).
Let s, t be terms, $\Gamma = < s, t >_{CIM}$ be a CIM-unification problem and $V_0 := V(s) \cup V(t)$ be the set of all variables occurring in s or t. Assume that the substitution σ is a most general CIM-unifier of Γ, where "most general" is meant w.r.t. the *unrestricted instantiation ordering*. We define $W_0 := V(V_0\sigma)$.
From Baader-Büttner (1988) one can easyly derive that there exist CIM-unification problems such that $|W_0| > |V_0|$ holds. Assume that Γ is such a unification problem. Without loss of generality we may also assume that $V_0 \subseteq W_0$. Otherwise, let W be a subset of W_0 of cardinality V_0 and let π be a substitution such that π is a permutation of variables with $W\pi = V_0$. Then $\sigma\pi$ is a most general CIM-unifier of Γ which satisfies $V_0 \subseteq V(V_0\sigma\pi)$.

Let x_0 be an element of W_0 which is not contained in V_0.

LEMMA 6.1. We have $x_0 \in D(\sigma)$ and $x_0\sigma \neq_{CIM} 0$.

PROOF. (1) Assume $x_0\sigma =_{CIM} x_0$. We define a substitution τ by $x\tau := x\sigma$ for $x \neq x_0$ and $x_0\tau := 0$. Since σ and τ coincide on V_0, τ is also a CIM-unifier of Γ. Hence there exists a substitution λ such that $\tau =_{CIM} \sigma\lambda$. Now $0 = x_0\tau =_{CIM} x_0\sigma\lambda =_{CIM} x_0\lambda$ shows that $x_0 \notin V(V\sigma\lambda)$. But $x_0 \in V(V_0\sigma) = V(V_0\tau) \subseteq V(V\tau)$ and $\tau =_{CIM} \sigma\lambda$ implies $V(V\tau) = V(V\sigma\lambda)$.

(2) Assume $x_0\sigma =_{CIM} 0$. We define a substitution τ by $x\tau := x\sigma$ for $x \neq x_0$ and $x_0\tau := z$ for some variable z. Since τ is a CIM-unifier of Γ, there exists a substitution λ such that $\tau =_{CIM} \sigma\lambda$. But $x_0\sigma =_{CIM} 0$ implies $x_0\sigma\lambda =_{CIM} 0 \neq_{CIM} x_0\tau$. ❑

Let x_1 be an element of $V(x_0\sigma)$. The variable x_1 exists, since $x_0\sigma \neq_{CIM} 0$ implies $V(x_0\sigma) \neq \emptyset$.

LEMMA 6.2. We have $x_1 \notin W_0$ and hence $x_1 \notin V_0$ and $x_0 \neq x_1$.

PROOF. Assume $x_1 \in W_0$. We define τ by $x\tau := x\sigma$ for $x \neq x_0$ and $x_0\tau := 0$. Since τ is a CIM-unifier of Γ, there exists a substitution λ such that $\tau =_{CIM} \sigma\lambda$. Now $0 = x_0\tau =_{CIM} (x_0\sigma)\lambda$ and $x_1 \in V(x_0\sigma)$ implies that $x_1\lambda =_{CIM} 0$. Hence $x_1 \notin V(V\sigma\lambda)$, but $x_1 \in W_0 = V(V_0\sigma) = V(V_0\tau) \subseteq V(V\tau)$. ❑

Now assume that we have already defined n+1 different variables $x_0, x_1, ..., x_n$ ($n \geq 1$) which satisfy the following conditions:

(1) $x_0 \in W_0 \setminus V_0$,

(2) $x_{i+1} \in V(x_i\sigma)$ for all i, $0 \leq i \leq n-1$,

(3) $x_0, x_1, ..., x_{n-1} \in D(\sigma)$ and

(4) $x_1, ..., x_n \notin W_0$ and thus $x_1, ..., x_n \notin V_0$.

LEMMA 6.3. We have $x_n \in D(\sigma)$ and $x_n\sigma \neq_{CIM} 0$.

PROOF. Assume $x_n\sigma =_{CIM} x_n$. We define a substitution τ by $x\tau := x\sigma$ for $x \neq x_n$ and $x_n\tau := 0$. As in Lemma 6.1 we get $\tau =_{CIM} \sigma\lambda$ and $x_n\lambda =_{CIM} 0$. Hence $x_n \notin V(V\sigma\lambda)$, but $x_n \in V(x_{n-1}\sigma) = V(x_{n-1}\tau)$.

(2) The proof of the second assertion is same as for Lemma 6.1. ❑

Let x_{n+1} be an element of $V(x_n\sigma)$.

LEMMA 6.4. We have $x_{n+1} \notin W_0 \cup \{ x_1, ..., x_n \}$. Hence x_{n+1} is different from $x_0, x_1, ...,$

x_n and $x_{n+1} \notin V_0$.

PROOF. Assume that $x_{n+1} \in W_0 \cup \{ x_1, ..., x_n \}$. We define $x\tau := x\sigma$ for $x \neq x_n$ and $x_n\tau := 0$. Since τ is a CIM-unifier of Γ, there exists a substitution λ such that $\tau =_{CIM} \sigma\lambda$. As in Lemma 6.2 we can deduce that $x_{n+1}\lambda =_{CIM} 0$. Hence $x_{n+1} \notin V(V\sigma\lambda)$. For $x_{n+1} \in W_0$ we have $x_{n+1} \in V(V_0\sigma) = V(V_0\tau) \subseteq V(V\tau)$ and for $x_{n+1} = x_i \in \{ x_1, ..., x_n \}$ we have $x_{n+1} \in V(x_{i-1}\sigma) = V(x_{i-1}\tau) \subseteq V(V\tau)$. ❑

By induction we thus get an infinite chain $x_0, x_1, x_2, ...$ of different variables such that $x_i \in D(\sigma)$. Hence $D(\sigma)$ can not be finite, which is a contradiction.

Thus there does not exist a most general CIM-unifier of Γ (where "most general" is meant w.r.t. the unrestricted instantiation ordering), which shows that CIM does not have unification type "unitary" w.r.t. the unrestricted CIM-instantiation ordering.

This accounts for the fact that unification modulo equational theories is mostly done with restricted instantiation.

7. Unification and Generalization in Commutative Theories

Let $E \neq \emptyset$ be an equational theory. From now on we shall only use the restricted E-instantiation ordering. That means that we work with the category $C_r(E)$.

DEFINITION 7.1. (1) A catgory **C** is *semiadditive* iff **C** has a zero object and every pair of objects has a coproduct which is also a product of these objects (see Baader (1989a) or Herrlich-Strecker (1973) for more information about semiadditive categories).

(2) The theory E is called *commutative* iff $C_r(E)$ is a semiadditive category (see Baader (1989a) for more information about commutative theories).

It has been pointed out to me at the Summer Conference on Category Theory and Computer Science 1989 that these theories should be called semiadditive, since the notion "commutative theory" is already used otherwise. In order to be consistent with Baader (1989a,1989b) I shall keep the name commutative in this paper. In the following, commutative theories are what we have defined in Definitition 7.1.

Examples of commutative theories are the theory CM of commutative monoids, the theory CIM of commutative idempotent monoids, the theory AB of abelian groups or the theory CMH of commutative monoids with a homomorphism (see Baader (1989a,1989b)).

Let E be a commutative theory. The morphism σ: $F_E(X) \to F_E(Y)$ of $C_r(E)$ is given by an $|X|\times|Y|$-matrix M_σ with entries from a semiring S(E). The composition of morphisms corresponds to multiplication of matrices (see Nutt (1988), Baader (1989b)).

For the above examples we have $S(CM) \cong \mathbb{N}$, S(CIM) is isomorphic to the 2-element boolean semiring, $S(AB) \cong \mathbb{Z}$ and $S(CMH) \cong \mathbb{N}[X]$.

7.1. Unification

The unification type of a commutative theory is either unitary or zero. The theories CM, CIM and AB are unitary (Baader (1989a)) and CMH has type zero (see Baader (1989b)). Since $C_r(E)$ has all binary coproducts, weak E-unification can be reduced to E-unification (see Proposition 3.8).

Let E be a commutative theory and let σ, τ: $F_E(X) \to F_E(Y)$ be morphisms of $C_r(E)$. The morphism δ: $F_E(Y) \to F_E(Z)$ is a most general E-unifier (i.e., weak coequalizer) of the parallel pair σ, τ iff the columns of M_δ generate the right S(E)-semimodule $U(M_\sigma,M_\tau) :=$ $\{ \underline{x} \in S(E)^{n\times 1};\ M_\sigma \underline{x} = M_\tau \underline{x} \}$ (see Nutt (1988) and Baader (1988,1989)).

DEFINITION 7.2. Let S be a semiring and U be a right S-semimodule.
The multiset $B = \{ b_1, ..., b_k \}$ is a *base* of U if and only if

(1) $U = \{ b_1s_1 + ... + b_ks_k;\ s_1, ..., s_k \in S \}$ and

(2) $b_1s_1 + ... + b_ks_k = b_1s_1' + ... + b_ks_k'$ implies $s_1 = s_1', ..., s_k = s_k'$.

PROPOSITION 7.3. Let E be a commutative theory and let δ: $F_E(Y) \to F_E(Z)$ be a most general E-unifier (i.e., weak coequalizer) of σ, τ: $F_E(X) \to F_E(Y)$ in $C_r(E)$. Then δ is a coequalizer of σ, τ iff the columns of M_δ are a base of $U(M_\sigma,M_\tau)$.

PROOF. This is an easy consequence of the definitions of weak coequalizer, coequalizer and base. ❑

Finitely generated right S(E)-semimodules need not have a base.

EXAMPLE 7.3. We consider the theory CM of commutative monoids. Since $S(CM) \cong \mathbb{N}$, morphisms of $C_r(CM)$ can be written as matrices with entries in $\mathbb{N}$. Let σ, τ be morphisms such that $M_\sigma = (\ 2\ \ 3\ \ 0\ )$ and $M_\tau = (\ 0\ \ 0\ \ 5\ )$. The elements of $U := U(M_\sigma,M_\tau)$ can be orderd by the componentwise $\leq$-ordering on natural numbers. The semimodule U is generated by the minimal elements of $U \setminus \{ \underline{0} \}$ and any set that generates U must contain these minimal elements. It is easy to see that $(\ 5\ \ 0\ \ 2\ )^T$, $(\ 0\ \ 5\ \ 3\ )^T$ and $(\ 1\ \ 1\ \ 1\ )^T$ are minimal elements of $U \setminus \{ \underline{0} \}$. Since $(\ 5\ \ 0\ \ 2\ )^T\cdot 1 + (\ 0\ \ 5\ \ 3\ )^T\cdot 1 = (\ 1\ \ 1\ \ 1\ )^T\cdot 5$, the semimodule U does not have a base. This shows

PROPOSITION 7.4. There exist morphisms σ, τ in $C_r(CM)$ which have a weak coequalizer, but which do not have a coequalizer.

However, if S is a principal ideal domain, then any finitely generated S-module has a base (see e.g. Oeljeklaus-Remmert (1974)). As a consequence we get

PROPOSITION 7.5. Any solvable unification problem $< \sigma, \tau >$ in $C_r(AB)$ has a coequalizer.

7.2. Generalization

Let E be a commutative theory and let **C** denote the category $\mathbf{C}_r(E)$. We are now interested in weak products in $\mathbf{C}_m$. The objects of $\mathbf{C}_m$ are matrices with entries in S(E). Let A $\in S(E)^{k\times n}$, B $\in S(E)^{k\times m}$ be two objects of $\mathbf{C}_m$. A morphism of $\mathbf{C}_m$ with domain A and codomain B is a matrix $C \in S(E)^{n\times m}$ such that AC = B.

PROPOSITION 7.6. Any pair of objects $A \in S(E)^{k\times n}$, $B \in S(E)^{k\times m}$ of $\mathbf{C}_m$ has a weak product.

PROOF. Let $a_1, ..., a_n$ be the columns of A and $b_1, ..., b_m$ be the columns of B. Let C = (A B) be the k×(n+m)-matrix with columns $a_1, ..., a_n, b_1, ..., b_m$. We shall show that C is a weak product of A, B in $\mathbf{C}_m$. The corresponding projections P_1, P_2 are defined as follows:

$$P_1 := \begin{pmatrix} E_{n\times n} \\ Z_{m\times m} \end{pmatrix} \quad \text{and} \quad P_2 := \begin{pmatrix} Z_{n\times n} \\ E_{m\times m} \end{pmatrix},$$

where the matrix $Z_{m\times m}$ ($Z_{n\times n}$) is the m×m (n×n) zero matrix and the matrix $E_{n\times n}$ ($E_{m\times m}$) is the n×n (m×m) identity matrix. Obviously, $CP_1 = A$ and $CP_2 = B$.

Assume that there are matrices $D \in S(E)^{k\times s}$, $Q_1 \in S(E)^{s\times n}$, $Q_2 \in S(E)^{s\times m}$ such that $DQ_1 = A$ and $DQ_2 = B$. We have to find a matrix $L \in S(E)^{s\times(n+m)}$ such that DL = C, $LP_1 = Q_1$ and $LP_2 = Q_2$. Let $L := (Q_1\ Q_2)$ be the s×(n+m) matrix which consists of the columns of Q_1 followed by the columns of Q_2. Now $DL = D\cdot(Q_1\ Q_2) = (DQ_1\ DQ_2) = (A\ B) = C$,

$$LP_1 = (Q_1\ Q_2)\cdot \begin{pmatrix} E_{n\times n} \\ Z_{m\times m} \end{pmatrix} = Q_1\cdot E_{n\times n} + Q_2\cdot Z_{m\times m} = Q_1 \text{ and analogously } LP_2 = Q_2.$$

This shows that C with the projections P_1, P_2 is a weak product of A, B in $\mathbf{C}_m$. ❑

Proposition 3.7 and 3.5 together with the above proposition yield

THEOREM 7.7. Any commutative theory E has lower bound and generalization type "unitary" (w.r.t. the restricted E-instantiation ordering).

8. Conclusion

We have seen that Eder's (Eder (1985)) negative results for weak ∅-unification can be avoided by using the restricted instantiation ordering. This is so because the category $C_r(\emptyset)$ – which corresponds to the restricted ∅-instantiation ordering – has all binary coproducts while the category $C_u(\emptyset)$ – which corresponds to the unrestricted ∅-instantiation ordering – does not have binary coproducts.

Another possibility to avoid this problem would be to use arbitrary endomorphisms in-

stead of substitutions. The corresponding category – which has F(V) as only object and all endomorphisms as morphisms – also has binary coproducts. For this category one gets results which are similar to those of Section 4. But this seems to be only of theoretical interest because the morphisms need not have finite descriptions.
Section 6 also shows that it is better to use restricted instantiation orderings for unification modulo equational theories.
For the empty theory a most specific generalizer of two terms yields a shorter description of these terms (see Ohlbach (1989)). In Section 7 we have seen that a commutative theory E has generalization type "unitary". But in this case a most specific E-generalizer of two terms does not give a shorter description of the terms (see the proof of Proposition 7.6).
In this paper categories were used to find the correct definitions (e.g. of the instantiation ordering on generalizers) and to clarify the connection between different notions (such as unification and weak unification or unification and generalization), a method which was also proposed in Goguen (1989). We have seen that most general unifiers correspond to *weak* coequalizers and not to coequalizers. This is an observation which seems to have escaped attention until now.

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