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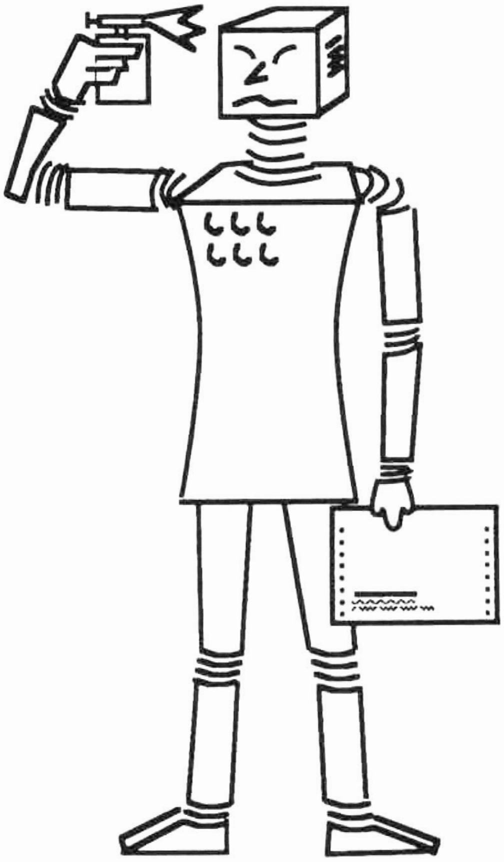

Dynamic Features of Topographical Multiset Orderings for Terms

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Abstract

Multiset orderings are usually used to prove the termination of production systems in comparing elements directly with respect to a given precedence ordering. Topographical multiset orderings are based on the position of elements in the graph induced by the precedence. This concept results in more flexible and stronger multiset orderings. To support the dynamic aspect of incremental refinement of a multiset ordering the notion of Depth Graphs is introduced. This concept leads to the use of a graph of which the nodes are terms (instead of constants and function symbols). It replaces the standard precedence graph. Moreover, it can be used to define a new recursive decomposition ordering on terms which is stronger than the original one.

1 Motivation

"Suppose you have a big box filled with red, green and blue balls", said the fox. "And suppose further that you are allowed to throw away any red ball you can find in the box, but you have to put in thousand green balls for each blue one you remove and each green one has to be replaced by a million of reds. Do you think that you can ever succeed in emptying the box?" "Yes, I do", replied the owl. "Are you sure?" "Of course, I can prove it", answered the owl, with a twinkle in her eyes.

The sophisticated owl knows about the concept of multiset orderings. Beside solving puzzles, multiset orderings are used to prove termination of programs and processes ([DM79]) and they serve as a basis for many recursive term orderings which in turn are used in proofs for the well-foundedness of term rewriting systems ([AM89], [De87], [HO80], [Ru87], [St89]). Their properties have been studied in [JL82], [Ma89], [St86], [MS86], [Fe88]. Especially, for the improvement of term orderings they are very helpful since the term ordering will be stronger if the underlying multiset ordering gets stronger.

"Well", said the fox, "but what happens if you are in addition allowed to replace balls by boxes of balls. Say, blue balls might be replaced by boxes with any number of green and red balls. Green ones may be replaced by two boxes filled with red balls. You may also handle the balls as before and any empty box will be thrown away. So now you will have boxes in boxes which contain boxes and balls, etc. Do you think that you will end up eventually with one empty box?" "Sure" said the owl, "it's the same story."

If complex objects O_1, O_2 must be compared, we usually have a partial ordering on the simple objects. Thus, complex objects are decomposed into (multi-) sets of less complex objects and the task of comparing O_1, O_2 is reduced to comparing these sets. By decreasing the complexity of the objects stepwise we eventually generate sets of simple objects which are compared with respect to the given ordering. If terms, treated as complex objects, have to be compared, we usually have a precedence ordering on the function symbols. For example, we would like to compare h(f(a),b) and h(g(a),b) with respect to (w.r.t.) the well-known recursive path ordering (see [De87]). The given objects are incomparable since the precedence is empty. We may refine the precedence by $f \triangleright g$ and will get $f(a) > g(a)$. This relation implies $h(f(a),b) > h(g(a),b)$ to be valid since it is equivalent to $\{f(a),b\} \gg \{g(a),b\}$. But, for the sake of receiving $f(a) > g(a)$ we have fixed the value of other comparisons, as well. For instance, $f(g(a)) > g(f(a))$ inducing $h(f(g(a)),b) > h(g(f(a)),b)$ is derived. Thus, it is impossible to have $h(g(f(a)),b) > h(f(g(a)),b)$ under the precondition that $f(g(a)) > g(f(a))$.

This example illustrates that a precedence determines the comparison of a class of object pairs. In order to weaken this inflexible approach of extending a precedence we will refine the ordering in a more moderate way. The original ordering compares two objects by comparing parts of these objects as multisets. We will generalize a precedence to a graph of complex objects (not only function symbols) by simultaneously using a stronger ordering on multisets of objects called dynamic depth ordering. The dynamic depth ordering is a topographical multiset ordering which compares two objects by using their depths (natural numbers) w.r.t. the precedence graph.

After a brief description of the classical multiset ordering we will explain the technique of topographical multiset orderings by presenting some examples of orderings. Section 2.3 will deal with the definition of the dynamic depth ordering which can be used as part of a term ordering. The incorporation of this multiset ordering in a recursive decomposition ordering (by using a graph instead of a precedence) will be the main part of chapter 3.

2 Multiset Orderings

Intuitively, a multiset is a collection of elements of one set. In contrast to subsets every element of a multiset can possibly occur more than once. More formally, a multiset M on S is a mapping from S to the natural numbers. Each element of S is associated with the number of times it appears in the multiset M.

Definition Let S be any set.
A mapping $M : S \mapsto \mathbb{N}$ is called a **multiset** on S. ∎

However, we will use the informal "bracket" notation to describe the contents of a multiset, e.g. $M = \{a,b,b,b,c\}$ instead of $M[a] = M[c] = 1$, $M[b] = 3$.

Note that multisets on S, with elements occurring once at the most (i.e.: $M[x] \leq 1$, $\forall x \in S$), can be identified with the subsets of S. Based on operations with natural numbers, the common operations known of sets, such as union, intersection, difference and inclusion, can also be defined on multisets. We expect the reader to be familiar with these operations (an exact definition is included in [HO80], [Fe88] and [St86], for example). In order to preserve computability, we are exclusively interested in the class of all multisets containing a finite number of elements.

Definition Let S be any set.
Mult(S) $:= \{ M : S \mapsto \mathbb{N} \mid M \text{ is finite} \}$
is called the set of all finite multisets on S. ∎

Note that Mult(S) is closed under multiset union, intersection and difference.

2.1 Standard Multiset Ordering

We focus our interest on comparing finite multisets by defining well-founded partial orderings on Mult(S). A partial ordering on a set S is a relation $> \subseteq S \times S$ which is irreflexive and transitive. If $x,y \in S$ are incomparable (w.r.t. >) we write $x \# y$. A partial ordering > on S is well-founded if there exists no infinite decreasing sequence $x_1 > x_2 > \dots$ of elements of S.

One way to define a multiset ordering over S (a partial ordering on Mult(S)) is to "lift" a given partial ordering > on S to Mult(S). Reasonably, the resulting multiset ordering » should be an extension of > to multisets (i.e. $x > y$ implies $\{x\} \gg \{y\}$, $\forall x,y \in S$).

Following this idea, Dershowitz and Manna have presented a multiset ordering that is induced by a partial ordering > on the underlying set S.

Definition [Multiset ordering of Dershowitz-Manna, [DM79]]

Let > be a partial ordering on S and M,N $\in$ Mult(S).

$M \gg_{DM} N$ iff $\exists X,Y \in \text{Mult}(S)$ such that

i) $\emptyset \neq X \subseteq M$

ii) $(M \setminus X) \cup Y = N$

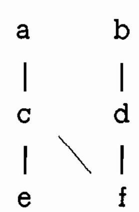
iii) $\forall y \in Y \;\; \exists x \in X : \quad x > y$ ■

A lot of definitions of multiset orderings equivalent to the one of Dershowitz and Manna exist (see for example [St86]). Therefore, we call $\gg_{DM}$ the **Standard Multiset Ordering** and will simply refer to it by $\gg$.

We present an example to illustrate the definiton of the Standard Multiset Ordering. This example also indicates the disadvantages of $\gg$. Therefore, it will be used throughout this paper to demonstrate the differences between the presented orderings.

Example Let S = {a,b,c,d,e,f} and $a > c > e$, $c > f$, $b > d > f$

Hasse diagram:



Further,

$M = \{a,c\}$ $\quad N = \{c,c,e,f,f,f\}$

$M' = \{b,e\}$ $\quad N' = \{b,d\}$

$M'' = \{a,c,d\}$ $\quad N'' = \{b,d,e,f\}$

$M \gg N$ since $(M \setminus \{a\}) \cup \{c,e,f,f,f\} = N$ and $a > c$, $a > e$, $a > f$

$M' \# N'$ since $(M' \setminus \{e\}) \cup \{d\} = N'$ and $e \# d$
or $(N' \setminus \{d\}) \cup \{e\} = M'$ and $d \# e$

$M'' \# N''$ since $(M'' \setminus \{a,c\}) \cup \{b,e,f\} = N''$ and $a \# b$, $c \# b$
or $(N'' \setminus \{b,e,f\}) \cup \{a,c\} = M''$ and $b \# a$, $e < a$, $f < a$ ■

Intuitively, one can imagine that M′ is smaller than N′ and M″ dominates N″. Therefore, the result of comparing these multisets with $\gg$ requires the search for more powerful multiset orderings.

2.2 Topographical Multiset Orderings

Our new multiset orderings are based on the topographical aspects of the graphical representation of a given ordering on S. The basic idea is: "The higher an element is situated, the bigger it is"! More precisely, we introduce the *depth* of an element which characterizes its position in the underlying partial ordering on S.

Definition [Depth of an element]

Let > be a partial ordering on S. $D^S : S \mapsto \mathbb{N}$ with
$D^S[x] := \max \{ D^S[y] \mid x < y \in S \} + 1$ [where $\max[\emptyset] = 0$]
is called the **depth** of x in S. ∎

In general, it should be noted that the existence of the depth is not guaranteed for all elements of S. In fact, most of the interesting orderings [especially those presented in the second part of this paper] do not have the desired property, i.e. there are some elements with an infinite depth. A partial ordering > with all elements of S possessing finite depth is called **co-bounded** [i.e. < is bounded]. Now, we connect the notion of the depth of elements of S with multisets on S resulting in definitions which are used to construct some topographical multiset orderings.

Definition [Optimum, hierarchy level and depth multiset]

Let > be a co-bounded partial ordering on S, $M \in Mult[S]$ and $n \in \mathbb{N}$.
$Opt^S : Mult[S] \mapsto \mathbb{N} \cup \{\infty\}$ with
$Opt^S[M] := \min \{ D^S[x] \mid x \in M \}$ [where $Opt^S[\emptyset] = \infty$]
is called the **optimum** of M.

$L^S : Mult[S] \times \mathbb{N} \mapsto Mult[S]$ with
$L^S[M,n] := \{ x \in M \mid D^S[x] = n \}$
[exactly: $L^S[M,n][x] := M[x]$ iff $D^S[x] = n$]
is called the n-th **hierarchy level** of M.

$D^S : Mult[S] \mapsto Mult[\mathbb{N}]$ with
$D^S[M] := \{ D^S[x] \mid x \in M \}$ [exactly: $D^S[M][n] := |L^S[M,n]|$]
is called the **depth multiset** of M. ∎

We will write D[x] [and Opt[M], L[M,n], D[M]], if it is obvious which set S is referred to.

The fundamental idea of the following multiset ordering [see [St86]] is, that a multiset M is bigger than a multiset N if its biggest element [w.r.t. >] is "higher" than that of N. Therefore, this multiset ordering follows from the idea given above.

Definition [Optimum Ordering]

Let > be a co-bounded partial ordering on S.

$M \gg_O N$ iff $Opt^S[M\backslash N] < Opt^S[N\backslash M]$ ∎

The presented example will illustrate how the Optimum Ordering works.

Example Let S, > and six multisets be as in the example above.

$M \gg_O N$	since	$Opt[M\backslash N]$	$= Opt[\{a\}]$	$= D[a]$		$= 1$	and
		$Opt[N\backslash M]$	$= Opt[\{c,e,f,f,f\}]$	$= \min\{2,3,3,3,3\}$		$= 2$	
$M' \ll_O N'$	since	$Opt[M'\backslash N']$	$= Opt[\{e\}]$	$= D[e]$		$= 3$	and
		$Opt[N'\backslash M']$	$= Opt[\{d\}]$	$= D[d]$		$= 2$	
$M'' \#_O N''$	since	$Opt[M''\backslash N'']$	$= Opt[\{a,c\}]$	$= \min\{1,2\}$		$= 1$	and
		$Opt[N''\backslash M'']$	$= Opt[\{b,e,f\}]$	$= \min\{1,3,3\}$		$= 1$	∎

In addition to M and N, M′ and N′ are now comparable in the desired manner. In fact, $\gg_O$ is stronger than the Standard Multiset Ordering, but M″ and N″ remain incomparable. Moreover, the Optimum Ordering demands the ordering on S to be co-bounded, i.e. the depths of all elements must be finite. To overcome this restriction, the definition of $\gg_O$ may be altered in such a way that the depths of the elements are computed w.r.t. a finite subset of S. For a comparison, this subset must contain all the elements needed. The first attempt in generalizing the Optimum Ordering in this manner is based on the following concept [see [St86]]: When comparing two multisets M and N the required depths are computed w.r.t. *set[M∪N]* which denotes the set of elements contained in the union of M and N.

Definition [General Optimum Ordering]

Let [S,>] be a partially ordered set and M,N ∈ Mult[S].

$M \gg_O^{\cup} N$ iff $Opt^{set(M\cup N)}[M\backslash N] < Opt^{set(M\cup N)}[N\backslash M]$ ∎

As our example will prove, the General Optimum Ordering differs from the original one [see the second comparison].

Example Let S, > and six multisets be as in the example above.

$M \gg_{O}^{\cup} N$	since	$Opt^{set(M \cup N)}[M \backslash N]$	$= Opt^{\{a,c,e,f\}}[\{a\}]$	$= D^{\{a,c,e,f\}}[a]$	$= 1$
	and	$Opt^{set(M \cup N)}[N \backslash M]$	$= Opt^{\{a,c,e,f\}}[\{c,e,f,f,f\}]$	$= \min\{2,3,3,3,3\}$	$= 2$
$M' \gg_{O}^{\cup} N'$	since	$Opt^{set(M' \cup N')}[M' \backslash N']$	$= Opt^{\{b,d,e\}}[\{e\}]$	$= D^{\{b,d,e\}}[e]$	$= 1$
	and	$Opt^{set(M' \cup N')}[N' \backslash M']$	$= Opt^{\{b,d,e\}}[\{d\}]$	$= D^{\{b,d,e\}}[d]$	$= 2$
$M'' \#_{O}^{\cup} N''$	since	$Opt^{set(M'' \cup N'')}[M'' \backslash N'']$	$= Opt^{S}[\{a,c\}]$	$= \min\{1,2\}$	$= 1$
	and	$Opt^{set(M'' \cup N'')}[N'' \backslash M'']$	$= Opt^{S}[\{b,e,f\}]$	$= \min\{1,3,3\}$	$= 1$

■

Note that $\gg_{O}^{\cup}$ is still stronger than the Standard Multiset Ordering. Moreover, the General Optimum Ordering is equivalent to the multiset ordering based on disjunctive partitions ($\gg_M$) of Jouannaud and Lescanne ([JL82]). Thus, the definition of $\gg_{O}^{\cup}$ gives useful hints for an efficient implementation of the Disjunctive Partition Based Ordering.

A closer look at the definition of $\gg_{O}^{\cup}$ shows that, in general, set[M∪N] contains elements whose depths are never needed in the comparison process. These are elements appearing equally in number in both multisets. Therefore, another version of the Optimum Ordering can be constructed, where the depths are computed w.r.t. *set(M⊕N)* that denotes the set of elements occurring in either M or N but not in equal quantities in both multisets.

Definition [Basic Optimum Ordering]

Let [S,>] be a partially ordered set and M,N ∈ Mult[S].

$M \gg_{O}^{\oplus} N$ iff $Opt^{set(M \oplus N)}[M \backslash N] < Opt^{set(M \oplus N)}[N \backslash M]$ ■

It is easy to see that the comparison of the multisets of our example with the Basic Optimum Ordering provides the same results as with the Standard Multiset Ordering. Moreover, $\gg_{O}^{\oplus}$ and » are equivalent, i.e. the Basic Optimum Ordering is a topographical definition of ». Like $\gg_{O}^{\cup}$, the definition of $\gg_{O}^{\oplus}$ is very useful to efficiently implement the well-known Standard Multiset Ordering.

Now, we will concentrate on the problem that the example still contains two incomparable multisets (M'' and N''). All Optimum Orderings only use the topmost elements to decide which of the two compared multisets is the greater one. If the optima are equal, the two multisets are incomparable, no matter what depths the smaller elements possess. The following multiset ordering ([St86]) solves this problem. It compares lexicographically the number of elements on each hierarchy level.

Definition [Level Ordering]

Let > be a co-bounded partial ordering on S.

$M \gg_L N$ iff $\exists k \in \mathbb{N}$ such that

i] $|L^S[M,i]| = |L^S[N,i]| \quad \forall i < k$

ii] $|L^S[M,k]| > |L^S[N,k]|$ ∎

If the topmost elements are of the same depth, they are neglected. The comparison process will proceed by recursively comparing the remaining multisets until a decision can be made or the multisets are empty. Thus, we may call $\gg_L$ a "recursive version" of $\gg_O$. From this point of view, it seems natural that the Level Ordering is in fact stronger than the Optimum Ordering. To illustrate the definition of $\gg_L$, we again use the example given above.

Example Let S, > and six multisets be as in the example above.

$M \gg_L N$ since $|L[M,1]| = |\{a\}| = 1 > 0 = |\emptyset| = |L[N,1]|$

$M' \ll_L N'$ since $|L[M',1]| = |\{b\}| = 1 = |\{b\}| = |L[N',1]|$ and $|L[M',2]| = |\emptyset| = 0 < 1 = |\{d\}| = |L[N',2]|$

$M'' \gg_L N''$ since $|L[M'',1]| = |\{a\}| = 1 = |\{b\}| = |L[N'',1]|$ and $|L[M'',2]| = |\{c,d\}| = 2 > 1 = |\{d\}| = |L[N'',2]|$ ∎

In contrast to all the Optimum Orderings, M'' and N'' are comparable. Since D[a] = D[b] = 1, both elements are removed from the two multisets. M'' is at least greater than N'' since it contains more elements than N'' on the second hierarchy level.

We deduce another definition of a multiset ordering straight from the extension of the depth function to multisets (i.e. the depth multisets).

Definition [Depth Ordering]

Let > be a co-bounded partial ordering on S.
Further, let $\gg^<$ be the Standard Multiset Ordering on Mult[$\mathbb{N}$] that respects the ordering < on $\mathbb{N}$.

$M \gg_D N$ iff $D^S[M] \gg^< D^S[N]$ ∎

The ordering $\gg^<$ is the multiset extension of the reverse ordering on $\mathbb{N}$, i.e. $1 > 2 > \ldots$. Note that $\gg^<$ differs from $\ll$, e.g. $\{1,3\} \gg^< \{2\}$ but also $\{1,3\} \gg \{2\}$. Since the natural numbers are totally ordered, a comparison of two multisets with $\gg_D$ can be done by sorting the corresponding depth multisets (w.r.t. <) and comparing them lexicographically (w.r.t. <). This process reveals a certain similarity between $\gg_L$ and $\gg_D$. Later on, we will state that they are not only similar but equal.

The definition of the technical term *hierarchy level* (resp. *depth multiset*) demands the same restrictions on $\gg_L$ (resp. $\gg_D$) as on $\gg_O$. But, if we try to overcome these restrictions in the same fashion as we did with the Optimum Ordering, we lose the transitivity property. Therefore, $\gg_L^{\cup}$ (resp. $\gg_D^{\cup}$) would not be an ordering at all. This fact is shown by the following example.

Example Let S = {a,b,c,d} and b > c, d > a.

Hasse diagram:

```
b   d
|   |
c   a
```

Further, let A,B,C ∈ Mult(S) with A = {a,a,a} , B = {b,b} , C = {c,d}. Assume the "General Level Ordering" ($\gg_L^{\cup}$) is developed from $\gg_L$ as $\gg_O^{\cup}$ from $\gg_O$. When computing the depths of the elements w.r.t. set(A∪B) (resp. set(B∪C) and set(A∪C)), it is easy to see that A $\gg_L^{\cup}$ B $\gg_L^{\cup}$ C $\gg_L^{\cup}$ A. This contradicts transitivity, because A $\gg_L^{\cup}$ B and B $\gg_L^{\cup}$ C must imply A $\gg_L^{\cup}$ C. ∎

2.3 Dynamic Depth Ordering

The loss of transitivity by generalizing the Depth (Level) Ordering is caused by the fact that several elements appear on the same level one time and on different levels at another time. Each comparison generates its own environment of depths. The decision which of two multisets is the greater one strongly depends on this environment. Furthermore, the comparison process of the Depth Ordering does not respect any other environment than the current one. A possible way out of this awkward situation is given by dynamically generating a singleton depth function. It has to be constructed by "freezing" each environment once generated. Following this idea we introduce the notion of a *Depth Graph*. It simply relates depths with the elements of an appropriate subset of S.

Definition (Graph)

Let > be a partial ordering on S, P ⊆ S and D : P ↦ ℕ.
G := (P,D) is called a **graph** from S
iff $\forall x,y \in P: \quad x > y \implies D(x) < D(y)$ ∎

Later on, we will discuss how to expand a graph dynamically without destroying the present depth relations. The notions of *hierarchy level* and *depth multiset* can easily be adapted to graphs. They are denoted by $L^G(M,n)$ (resp. $D^G(M)$) with the corresponding depth graph G as index.

Now, we are able to redefine the Depth (Level) Ordering by exchanging the originally used depth function with the one given by an underlying depth graph.

Definition (Dynamic Depth Ordering)

Let (S,>) be a partially ordered set, G = (P,D) a graph from S, M,N $\in$ Mult(P) and $\gg^<$ the Standard Multiset Ordering over $\mathbb{N}$ that respects the ordering < on $\mathbb{N}$.

$M \gg^G_D N$ iff $D^G(M) \gg^< D^G(N)$ ∎

Only multisets over $P \subseteq S$ can be compared with $\gg^G_D$. The definition says nothing about the elements of Mult(S), in general. This doesn't satisfy our goal since we are not only interested in a multiset ordering on subsets of S but also on the whole set S. At this point the dynamic extendability of the depth graph takes effect. When comparing two multisets over S w.r.t. an underlying depth graph G, the first thing to do is to extend G, such that it contains all the elements needed for the comparison. The extension of a depth graph has to be done carefully, in order to preserve the (frozen) results of all previous comparisons. This can be guaranteed if the extension does not destroy existing hierarchy levels. Also, one can show the existence of such an extension independent of the actual depth graph and the underlying ordering > on S. Proofs of these two statements are included in [Fe88].

Example Let S = {a,b,c,d} and b > c, d > a.

Hasse diagram:

```
b   d
|   |
c   a
```

Further, let A,B,C $\in$ Mult(S) with A = {a,a,a}, B = {b,b}, C = {c,d}.

Let G = ({a,b},D) with D(a) = D(b) = 1. G is a depth graph from S.

$A \gg^G_D B$ since $D^G(A) = \{1,1,1\} \gg^< \{1,1\} = D^G(B)$

Let G' = ({a,b,c,d},D') with D'(a) = D'(b) = 2, D'(c) = 3 and D'(d) = 1.

G' is an extension of G that preserves $A \gg^G_D B$.

$C \gg^{G'}_D A$ since $D^{G'}(C) = \{1,3\} \gg^< \{2,2,2\} = D^{G'}(A)$

$C \gg^{G'}_D B$ since $D^{G'}(C) = \{1,3\} \gg^< \{2,2\} = D^{G'}(B)$ ∎

2.4 Comparing the introduced orderings

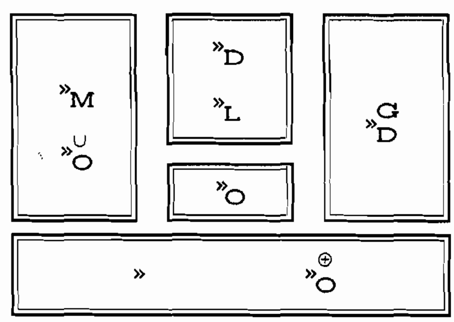
In this section we summarize the comparisons between the presented orderings. There are three possible relations: Two orderings can be **equivalent** [$\gg = \gg$] , one ordering can be **properly included** in the other [$\gg \subset \gg$] or they **overlap** each other [$\gg \# \gg$]. Two orderings overlap each other if there exist multisets M, N, M′, N′ such that $M \gg N \wedge M \not\gg N$ and $M' \gg N' \wedge M' \not\gg N'$.

The proofs of the following lemmata can be found in [Fe88].

Lemma Let > be a partial ordering on S. Then the following holds:

$$\gg \;=\; \gg_O^{\oplus}$$
$$\gg \;\subset\; \gg_O^{\cup} \;=\; \gg_M$$
$$\gg \;\subset\; \gg_O \;\subset\; \gg_L \;=\; \gg_D \qquad \text{, if > is co-bounded}$$
$$\gg_O \;\#\; \gg_O^{\cup} \;\#\; \gg_L$$
$$\gg \;\subset\; \gg_D^{G} \;\#\; \gg_O^{\cup} \qquad \text{, restricted to P}$$
$$\gg_O \;\#\; \gg_D^{G} \;\#\; \gg_L$$

Graphical representation:



■

Lemma Let > be a total ordering on S.
Then, all the orderings presented are equivalent:

$$\gg \;=\; \gg_O^{\oplus} \;=\; \gg_M \;=\; \gg_O^{\cup} \;=$$
$$\gg_O \;=\; \gg_L \;=\; \gg_D \;= \qquad \text{[if > is co-bounded]}$$
$$\gg_D^{G} \qquad \text{[restricted to P]}$$

■

2.5 Properties of the introduced orderings

All the presented orderings are really partial orderings [irreflexive and transitive relations] on Mult[S], of course. We want to point out a few other characteristics of the topographical multiset orderings. The proofs of all statements are included in [Fe88].

It is very useful for implementation purposes to have multiset orderings which are *additive* ([Ma89]) and *closed under difference* ([St86]). If a comparison result never changes by adding (deleting) equal elements to (from) both multisets, the used ordering is called additive (closed under difference). Only the General Optimum Ordering ($\gg_O^{\cup}$) is neither additive nor closed under difference. All of the other orderings possess both properties.

For termination proofs, it is important to know under which conditions a multiset ordering is well-founded. The Basic ($\gg_O^{\oplus}$) and General ($\gg_O^{\cup}$) Optimum Ordering (just like the Standard Multiset Ordering) are well-founded if and only if the ordering > on S is well-founded. The Optimum Ordering ($\gg_O$) itself as well as the Depth (Level) Ordering ($\gg_D = \gg_L$) and its dynamic version ($\gg_D^G$) are well-founded if and only if the values of the corresponding depth functions have an upper bound.

The Standard Multiset Ordering (») is a monotonous extension of > on S to multisets, i.e. a stronger ordering on S implies a stronger multiset ordering. As Jouannaud and Lescanne ([JL82]) have shown, » is the maximal multiset ordering possessing this property (called incrementality). Therefore, none of the presented topographical multiset orderings has this property (except the Basic Optimum Ordering since it is equivalent to »).

The Dynamic Depth Ordering ($\gg_D^G$) features monotony w.r.t. depth graphs which is similar to the monotony of the Standard Multiset Ordering. This property leads to the concept of dynamical extensions of $\gg_D^G$ stated above. It also reveals the great difference to all other topographical multiset orderings. A comparison result not only reflects a fixed relation. The relation itself is constructed during the comparison process. Therefore, it is possible to specify desirable results in order to dynamically adapt $\gg_D^G$ to the multisets to be compared.

It seems possible to simulate the adaptation process of the Dynamic Depth Ordering ($\gg_D^G$) with the Standard Multiset Ordering using appropriate extensions of > on S. The simulation requires that > can be extended in any desired direction, but this does not hold for each partially ordered set (see the second part of this paper).

We now present an application of the Dynamic Depth Ordering.

3 Term Orderings

Term rewriting provides a simple mechanism that can be applied to reasoning in structures defined by equations. The effective calculation using term rewriting systems presumes termination. Orderings on terms are able to guarantee this property. Most of the published term orderings are recursively constructed by applying the definition to the multisets of the subterms. The multiset ordering needed is the standard one. This chapter deals with the substitution of this weaker multiset ordering by the Dynamic Depth Ordering.

First of all, we briefly recapitulate the most important notions concerning term rewriting systems and their termination. A detailed description is presented in [HO80] and [AM89].

A term rewriting system (TRS) is a set of rules $\mathfrak{R}$, each of the form $l \Rightarrow r$. l and r are terms built from a set of function symbols $\mathfrak{F}$ and a set of variables $\mathfrak{V}$. A TRS $\mathfrak{R}$ defines a binary relation $\Longrightarrow_{\mathfrak{R}}$ on the set of terms which is called reduction relation. A term s can be reduced to another term t under the TRS $\mathfrak{R}$ ($s \Longrightarrow_{\mathfrak{R}} t$) if and only if there exists a rule $l \Rightarrow r \in \mathfrak{R}$ and a match from l into s. By replacing the matched subterm of s with an instance of r, t is derived from s. A more formal introduction to TRS theory is contained in [HO80] or [AM89], for example.

A TRS terminates if and only if each reduction sequence starting with any term ends after a finite number of steps in an irreducible term. Proving the termination of an arbitrary TRS $\mathfrak{R}$ is an important but generally undecidable problem. Nevertheless, some methods have been developed that can prove the termination of a large number of TRSs. A very successful method is to search for a well-founded ordering on terms which includes the reduction relation. If such a reduction ordering exists, the TRS must terminate. Moreover, the existence of a simplification ordering (a special kind of reduction ordering) is sufficient to guarantee the termination of a TRS ([De87]). An ordering on terms is a *simplification ordering* if and only if it possesses the subterm and the replacement property. The *subterm property* guarantees that a term is bigger than any of its proper subterms. The replacement property ensures that the value of a term will be decreased if any one of its subterms is decreased.

To prove the inclusion of a given reduction relation in a simplification ordering $>$, it is sufficient to show that $\sigma(l) > \sigma(r)$ for all ground substitutions σ of each rule $l \Rightarrow r \in \mathfrak{R}$. To obtain a finite termination proof (of a finite TRS) the chosen simplification ordering is required to be stable w.r.t. substitutions (i.e. $s > t \Longrightarrow \sigma(s) > \sigma(t)$, for all σ).

A large class of simplification orderings is known as path orderings ([De87], [St89]). Each definition of a path ordering contains recursive calls to a multiset ordering. Traditionally, the Standard Multiset Ordering is used in these definitions. We will demonstrate the possible usage of the Dynamic Depth Ordering in the definition of the Improved Recursive Decomposition Ordering IRD ([Ru87], [St89]). During the first attempt we restrict the definition to ground terms. But first of all, we need some notation.

The leading function symbol of a term t is referred to by top(t). Terms are labelled with sequences of natural numbers to identify the positions of their subterms. The set of all labels of a term t is called set of all occurrences of t, O(t). Ot(t) denotes the set of all terminal occurrences of the term t, i.e. the labels of its leaves. A specific subterm of a term t is determined by $t|u$ with $u \in O(t)$.

Definition (Occurrences and Subterm)

$$O(f(t_1,\dots,t_n)) = \{\varepsilon\} \cup \{iu \mid u \in O(t_i),\ 1 \le i \le n\}$$
$$Ot(t) = \{u \in O(t) \mid \forall v \neq \varepsilon\ \forall w \in O(t){:}\ w \neq uv\}$$
$$t|\varepsilon = t \qquad f(t_1,\dots,t_n)|iu = t_i|u$$
■

We now introduce the notion of the decomposition of a term which is used to define the IRD. Our notation is influenced by [St89] but it is not exactly the same. This notation (hopefully) provides a somewhat easier definition of the IRD.

Definition (Decomposition)

$$dec_\varepsilon(t) = \Phi \qquad dec_{iu}(f(t_1,\dots,t_n)) = \{t_i\} \cup dec_u(t_i)$$
$$dec(t) = \{\ dec_u(t) \mid u \in Ot(t)\ \}$$
■

The decomposition of a term t is a multiset consisting of multisets of elementary decompositions. In our notation, an elementary decomposition looks like a (proper) subterm of t. Note that additional information is needed for an exact characterization of an elementary decomposition, i.e. an elementary decomposition is closely related to its multiset. A term s is greater that a term t (w.r.t. the IRD) if the decomposition of s is greater than the decomposition of t. The ordering on these multisets (» ») is an extension of the basic ordering on terms (>) to multisets of multisets.

Definition (Improved Recursive Decomposition Ordering, [Ru87])

Let $\triangleright$ be a partial ordering on the set of function symbols and $\boldsymbol{f}$ an additional unary function symbol.

$$s >_{IRD} t \quad \text{iff} \quad dec[\boldsymbol{f}(s)] \gg\gg dec[\boldsymbol{f}(t)]$$

$$dec_{pu}(s) \quad s|p = s' > t' = t|q \in dec_{qv}(t)$$

iff

- $top(s') \triangleright top(t')$
- $dec_u(s') \gg dec_v(t')$
- $dec(s') \gg\gg dec(t')$ ■

The evaluation of the conditions is marked by hyphens: s > t iff - $s >_1 t$, - $s >_2 t$ stands for s > t iff $s >_1 t$ or $(s =_1 t \wedge s >_2 t)$. Here, the equality sign $=_1$ is the congruence relation induced by the quasi-ordering $\geq_1$. Two terms are equivalent under $=_{IRD}$ if they are the same up to equivalent function symbols and permutations of subterms.

Before modifying the IRD we explain its definition with a simple example.

Example Let $\mathfrak{F} = \{a,f,g\}$ and $f \triangleright g$.

- $f(a) >_{IRD} g(a)$ since $dec[\boldsymbol{f}(f(a))] = \{\{f(a),a\}\}$, $dec[\boldsymbol{f}(g(a))] = \{\{g(a),a\}\}$ and $dec_{11}(\boldsymbol{f}(f(a)))$ $f(a) > g(a) \in dec_{11}(\boldsymbol{f}(g(a)))$

- $f(g(a)) >_{IRD} g(f(a))$ since $dec[\boldsymbol{f}(f(g(a)))] = \{\{f(g(a)),g(a),a\}\}$, $dec[\boldsymbol{f}(g(f(a)))] = \{\{g(f(a)),f(a),a\}\}$ and $dec_{111}(\boldsymbol{f}(f(g(a))))$ $f(g(a)) > f(a) \in dec_{111}(\boldsymbol{f}(g(f(a))))$, $dec_{111}(\boldsymbol{f}(f(g(a))))$ $f(g(a)) > g(f(a)) \in dec_{111}(\boldsymbol{f}(g(f(a))))$ ■

Further, we are going to improve this decomposition ordering by using the Dynamic Depth Ordering instead of the Standard Multiset Ordering, i.e. the precedence of the IRD will be replaced by the more flexible structure of a depth graph. We change the definition of the IRD at the "inner" one of its two multiset comparisons. In the dynamic version the IRD will compare multisets of elementary decompositions with the Dynamic Depth Ordering ($\gg_D^G$) instead of the Standard Multiset Ordering. Therefore, we need a depth graph consisting of elementary decompositions. The depth graph has to respect an ordering on its elements. We choose a modified version of > on elementary decompositions to construct our depth graph. The modification is the same as with the IRD: multisets of elementary decompositions are compared with $\gg_D^G$.

Definition (Dynamic IRD)

Let $\triangleright$ be a partial ordering on the set of function symbols, $\boldsymbol{f}$ an additional unary function symbol and G a depth graph containing all elementary decompositions appearing in dec($\boldsymbol{f}$ (s)) $\cup$ dec($\boldsymbol{f}$ (t)).

$$s >_{DIRD} t \quad \text{iff} \quad dec(\boldsymbol{f}(s)) \gg\gg_D^G dec(\boldsymbol{f}(t))$$

with

$$dec_{pu}(s) \ni s|p = s' >_{DEL} t' = t|q \in dec_{qv}(t)$$

iff

- $top(s') \triangleright top(t')$
- $dec_u(s') \gg_D^G dec_v(t')$
- $dec(s') \gg\gg_D^G dec(t')$ ∎

It is easy to see that two elementary decompositions s and t can only be compared with DEL if the corresponding depth graph contains all elementary decompositions occurring in dec(s) $\cup$ dec(t). This reveals the indirect recursion of DIRD: before inserting a term into a depth graph, all of its arguments have to be inserted recursively. In contrast to the originals, DEL and DIRD themselves are not recursive. We decided on using the (strongest) decomposition ordering since the needed partition into subterms represents a simple and efficient method for determining the order in which the terms are integrated in the depth graph.

Example Let $\mathfrak{F}$ = {a,f,g}.

Let G = ({a,f(a),g(a)},D) with D(a) = 3, D(f(a)) = 2 and D(g(a)) = 1

$g(a) >_{DIRD} f(a)$ since $D^G(dec_{11}(\boldsymbol{f}(g(a)))) = \{1,3\}$, $D^G(dec_{11}(\boldsymbol{f}(f(a)))) = \{2,3\}$ and $\{1,3\} \gg^< \{2,3\}$

Let G′ = ({a,f(a),g(a),f(g(a)),g(f(a))},D′) with

D′(a) = 5, D′(f(a)) = 4, D′(g(a)) = 3, D′(g(f(a))) = 2 and D′(f(g(a))) = 1.

G′ is an extension of G which respects $g(a) >_{DIRD} f(a)$.

$f(g(a)) >_{DIRD} g(f(a))$ since $D^G(dec_{111}(\boldsymbol{f}(f(g(a))))) = \{1,3,5\}$, $D^G(dec_{111}(\boldsymbol{f}(g(f(a))))) = \{2,4,5\}$ and $\{1,3,5\} \gg^< \{2,4,5\}$ ∎

Note that this result cannot be obtained with the original IRD, irrespective of the chosen precedence: $g(a) >_{IRD} f(a)$ requires $g \triangleright f$, but $f(g(a)) >_{IRD} g(f(a))$ requires $f \triangleright g$ which contradicts the former choice of $\triangleright$.

The IRD is monotonous w.r.t. the precedence which permits an incremental generation of the precedence. This property demands the multiset ordering to be a monotonous extension function. Therefore, we cannot hope to transfer the precedence monotony to the DIRD. But we have replaced it by a similar property. Owing to the characteristics of $\gg_D^G$, the Dynamic IRD is monotonous w.r.t. the depth graph. As you can see in the example above, this concept allows a very flexible adaptation of the DIRD to the set of terms to compare. It is more flexible than a dynamic generation of the precedence.

To be usable as a tool for proving the termination of a TRS, the DIRD has to possess certain characteristics. Of course, it has to be a partial ordering. Furthermore, the condition of being a simplification ordering (subterm and replacement property) has to be fulfilled. In addition, the Dynamic IRD should be an extension of the original one in order to justify its definition.

We formulate lemmata about the characteristics of DIRD and will prove them.

Lemma DIRD is a simplification ordering.

Proof Let $G = (P,D)$ be a depth graph which contains all elementary decompositions needed for the proof.

a) DIRD is irreflexive and transitive since the Standard Multiset Ordering ($\gg$) as well as the Dynamic Depth Ordering ($\gg_D^G$) is a partial ordering.

b) DIRD has the subterm property:
Let be $\varepsilon \neq u \in O(t)$
$\Longrightarrow t >_{IRD} t|u$, since the IRD has the subterm property
$\Longrightarrow t >_{DIRD} t|u$, since IRD $\subseteq$ DIRD (see below)

c) DIRD has the replacement property:
Let be $s >_{DIRD} s'$. We have to show that

$$t = f(t_1,\dots,s,\dots,t_n) >_{DIRD} f(t_1,\dots,s',\dots,t_n) = t'$$
$$\Longleftrightarrow \quad dec(\boldsymbol{f}(t)) \gg \gg_D^G dec(\boldsymbol{f}(t'))$$
$$\Longleftrightarrow \quad \forall v \in Ot(\boldsymbol{f}(t')) \ \exists u \in Ot(\boldsymbol{f}(t)) : \quad dec_u(\boldsymbol{f}(t)) \gg_D^G dec_v(\boldsymbol{f}(t'))$$
since $dec(\boldsymbol{f}(t)) \cap dec(\boldsymbol{f}(t')) = \emptyset$

This is true because

- $\forall v \in Ot[f[t']]\ \exists u \in Ot[f[t]]$:

  $dec_u[f[t]] \quad t >_{DEL} t' \in dec_v[f[t']]$

  (case distinction whether v determines a position in s' or not)
- $t >_{DEL} t' \implies D[t] < D[t']$ since G respects DEL
- $dec[f[s]] \gg \gg_D^G dec[f[s']]$ by precondition $(s >_{DIRD} s')$
- $\gg_D^G$ is additive ∎

The property of being a simplification ordering authorizes the DIRD to guarantee the termination of a ground term rewriting system. The following lemma gives preference to the DIRD since it is stronger than the IRD.

Lemma IRD ⊆ DIRD

Proof
- $> \subseteq >_{DEL}$: By induction on the structure of terms and owing to the construction of the depth graph $(s >_{DEL} t \implies D[s] < D[t])$

- $s >_{IRD} t$

$\iff$	$dec[f[s]]$	$\gg \gg$	$dec[f[t]]$	(Def. of the IRD)
$\implies$	$dec[f[s]]$	$\gg \gg_{DEL}$	$dec[f[t]]$	(since $> \subseteq >_{DEL}$)
$\implies$	$dec[f[s]]$	$\gg \gg_D^G$	$dec[f[t]]$	(G respects DEL)
$\iff$	$s >_{DIRD} t$			(Def. of the DIRD) ∎

Neither in the definition nor in the proofs of the properties of the DIRD, the restriction to ground terms is needed. The question arises why we have demanded it at the beginning? Well, the utilization of terms containing variables causes some problems which we wanted to neglect. Firstly, the treatment of terms which are identical except for the names of their variables cannot yet be specified. This class of terms raises both practical and theoretical questions. For reasons of efficiency, it is desirable to avoid redundant information in the depth graph. From a theoretical point of view, this problem is connected with the general question of how to construct a depth graph correctly? A correctly constructed depth graph guarantees the stability w.r.t. substitutions of the DIRD.

Up to now, we have no solution to this problem. It is, however, obvious that the definition of the depth graph must be altered. With the presented version it is probably impossible to guarantee the stability w.r.t. substitutions. Suppose $a \triangleright b$ to be constants and x to be a variable. x is incomparable with both a and b with all simplification orderings stable w.r.t. substitutions (because x is unifiable with both constants]. To receive the same results with DIRD we have to construct a depth graph with the restriction D[a] = D[x] = D[b] concerning the depth function. But the precedence [a $\triangleright$ b] demands D[a] < D[b] since a $>_{DEL}$ b. Obviously, it is impossible to satisfy both constraints.

4 Conclusion

We have developed new multiset orderings which are classified to be topographical. All presented orderings are equal to or stronger than the Standard Multiset Ordering (on similar conditions). The Dynamic Depth Ordering ($\gg^{G}_{D}$) is incrementally adaptable to the multisets to be compared. It is not a fixed relation, but is generated during the comparison process. This unique flexibility also causes some problems when $\gg^{G}_{D}$ is used in a term ordering environment. We have altered the improved recursive decomposition ordering (IRD) by replacing the Standard Multiset Ordering with $\gg^{G}_{D}$. The resulting DIRD is a simplification ordering but it is not stable w.r.t. substitutions. We hope to find a modification of the depth graph to gain back this property. Our future work is influenced by the idea of not only characterizing the depth of an element by a single natural number, but by an interval.

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