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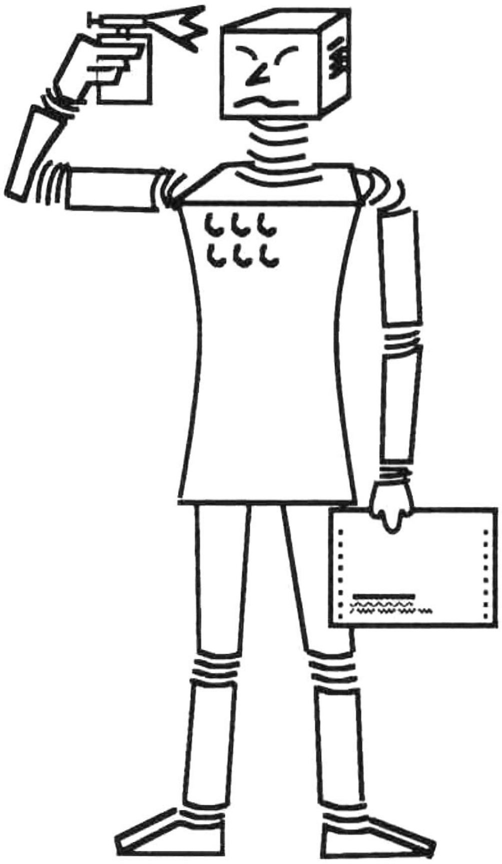

Unification Algebras: An Axiomatic Approach to Unification, Equation Solving, and Constraint Solving

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Unification Algebras: An Axiomatic Approach to Unification, Equation Solving and Constraint Solving

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Abstract.

Traditionally unification is viewed as solving an equation in an algebra given an explicit construction method for terms and substitutions. We abstract from this explicit term construction methods and give a set of axioms describing unification algebras that consist of objects and mappings, where objects abstract terms and mappings abstract substitutions. A unification problem in a given unification algebra is the problem to find mappings for a system of equations $\langle s_i = t_i \mid i \in I\rangle$, where s_i and t_i are objects, such that s_i and t_i are mapped onto the same term. Typical instances of unification algebras and unification problems are: Term unification with respect to equational theories and sorts, standard equation solving in mathematics, unification in the λ-calculus, constraint solving, disunification, and unification of rational terms.

Within this framework we give general purpose unification rules that can be used in every unification algorithm in unification algebras. Furthermore we demonstrate the use of this framework by investigating the analogue of syntactic unification and unification of rational terms.

Keywords: Unification algebra, universal algebra, equation solving, constraint solving, equational theories

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1 Introduction.

What are the common features of unification of first-order terms with respect to an equational theory, unification in λ-calculus, equation solving in mathematics and answering a query with respect to a logic program? In order to approximate the answer let us look more closely to the different problems.

– Unification is the task to make two terms equal, i.e., given two terms s,t, which contain variables (in some sense), find replacements for these variables, such that s and t are equal after the replacement. This task arises in several settings, for free first order terms [Her30, MM82 , Hu76], for terms together with an equational theory [Pl72,Si87] and for terms in sorted signatures [Wa88,Sch88, SNMG87].

– Unification in λ-calculus [Hu75, SG88] is the task given two λ-expressions with free variables, find λ-expressions that substituted for these variables make the two expressions equal after application of reduction rules.
– Solving an equation s = t over a fixed (universal) algebra A is to find assignments from variables in s and t to elements of A, such that s and t are mapped to the same element of the algebra by the assignment.
– Solving equations in mathematics, for example solving Diophantine equations over the naturals, means given a polynomial p with several variables, to find natural numbers for every variable in p such that the p becomes zero after replacing the variables by those naturals.
– Solving constraints [Co82b, JL86, DSV87] is the task to find solutions, i.e., substitutions into variables, such that a given constraint is satisfied. It is also common in practice only to require a test for solvability of a constraint rather than explicitely computing solutions.
– Answering a query with respect to a logic program is given a query including variables, findone or all answers, i.e., all instantiation of variables in the query, such that the instantiated query follows from the logic program.

To summarize, some common features of all these problems are:

i) there are objects having variables,
ii) the names of the variables do not matter
iii) there exists an operation like substituting objects into variables
iv) there is a domain where the valid solutions come from.

The common problems are that some or all instantiations are wanted that solve some equation or makes some formulae true.

A further common problem is that methods are needed to represent infinite sets of solutions in a finite way. For example the equation $x^2 = y^2$ has an infinite number of solutions over the integers, but every solution can be represented using variables, and in this case either by $x = y$ or by $x = -y$.

This paper is an attempt to give an axiomatic framework for unification in terms of unification algebras, such that all the problems above can be seen as unification problems within this framework. We develop this unification theory to the extent that a set of lemmas and theorems can be derived including some nondeterministic unification rules that are valid in general.

A particular advantage of this approach is that many of the familiar lemmata and theorems of standard unification theory can be shown for unification algebras and hence can be used for every problem domain, which satisfies the axioms of a unification algebra.

This work was inspired by a recent excellent paper of J. Goguen [Go88], which advocates a categorical approach to unification, viewing unifiers as equalizers in some category. His attempt to provide a framework for unification theory clearly advanced the field and had strong influence in that it showed a new line of development. However, category theory appears to have too strong an emphasis on the substitutions (as arrows) and underrates the inner structure of the objects, for example the role of the variables in objects. Furthermore his approach has a built-in renaming every time a unifier or a unifying step is executed. This makes completeness proofs for unification algorithms that rename variables only if necessary overly complicated. Though we were inspired by J. Goguen, the first authors to consider unification in categories were R. M. Burstall and D.E.Rydeheard [RB85, RS87].

Our approach allows a natural treatment of unification algorithms based on transforming systems of equations or multi-equations as for example used in [Her30, MM82, GS87, Hu76, Sch87]. This approach allows easy proofs of completeness of such nonrenaming algorithms, wich are of high practical interest. A further advantage is that our appeoach allows a unified treatment of sorts and of equational theories.

The paper is structured as follows: In sections 2-4 we give the basic axioms for a unification algebra, define the notions of unification type and provide some consequences in order to show that the substitutions behave as expected. Section 5 on homomorphisms and congruences shows that unification algebras form a category where the usual homomorphism theorem holds. In section 6 we give complete transformation steps for equation systems that can be used in every unification algorithm. Section 7 presents the notion of dimension and redundant equations in unification algebras. In section 8 we discuss the notion of minimal representations of solutions. Section 9 presents a complete unification algorithm for the equivalent of the Robinson-case and also for the case of rational infinite terms.

2. Unification Algebras

Basically, unification algebras consist of a set **OBJ** and a set **MAP** of mappings from **OBJ** into **OBJ**. Intuitively, the set **OBJ** can be viewed as the set of all well-formed expressions with respect to some language modulo a congruence and **MAP** as the set of well-formed variable-replacements modulo the same congruence. We will distinguish some elements in **OBJ** as variables **V**. This gives a level of abstraction, in which the properties of the defined structure can be investigated without regarding the language which is used to define this structure.

The design decisions we have made are for example that variables are explicitly available in contrast to the categorical approach, where variables are considered modulo renamings. Furthermore it is not possible to prohibit the application of a substitution to some term, for example it is impossible to say that it is disallowed to subtitute 0 for x in the term y/x if substituting 0 for x is allowed in other terms.

As a preliminary for the definition of unification algebra we define a **unification quasi-algebra** $\mathcal{A}$ as a triple (**V**, **OBJ**, **MAP**), where **OBJ** $\neq \emptyset$, **MAP** is a set of total mappings σ: **OBJ** $\rightarrow$ **OBJ** and **V** $\subseteq$ **OBJ**. We tacitly assume that equal mappings on **OBJ** are equal elements of **MAP**.

Usually we will refer to elements of **OBJ** as **objects**, to **MAP** as **mappings** and to **V** as

variables.

We try to provide a minimal axiomatization, such that it is easy to check that a given structure is a unification algebra, and that the machinery for unification algebras can be used.

Now we give the axioms which a unification algebra $\mathcal{A}$ should obey.
In the following we assume that $\mathcal{A} = (\mathbf{V}, \mathbf{OBJ}, \mathbf{MAP})$ is a unification quasi-algebra.

MON) **MAP** is a monoid with respect to composition of mappings with identity Id.

We use the usual notation for substitutions als for mappings $\sigma \in \mathbf{MAP}$. By $DOM(\sigma)$ we denote $\{x \in \mathbf{V} \mid \sigma x \neq x\}$ and by $COD(\sigma) = \sigma DOM(\sigma)$. The notation $\sigma = \tau\ [W]$ for mappings σ,τ means that $\sigma x = \tau x$ for all variables x in W.

The next axiom states that every mapping can be characterized by its values on **V**:

V1) (Basis axiom)
$\forall \sigma,\tau \in \mathbf{MAP}: \sigma = \tau\ [\mathbf{V}] \Rightarrow \sigma = \tau.$

We represent a mapping $\sigma \in \mathbf{MAP}$ by $\sigma = \{x_i \leftarrow \sigma x_i \mid i \in I\}$, where $DOM(\sigma) = \{x_i \mid i \in I\}$.

The axiom (V2) captures the intuition that one can independently choose instantiations for variables.

V2) (Restriction axiom)
$\forall \sigma \in \mathbf{MAP}\ \forall W \subseteq \mathbf{V}\ \exists \tau \in \mathbf{MAP}\quad \sigma = \tau\ [W]$ and $\tau y = y$ for all $y \notin W$.
We denote τ as $\sigma_{|W}$.

2.1 Definition. We say ξ is a **variable permutation** iff

i) $\xi \mathbf{V} \subseteq \mathbf{V}$, and
ii) There exists a mapping $\xi^- \in \mathbf{MAP}$ such that $\xi^-\xi = Id$.

We say two variables x,y are **equivalent**, if $\xi x = y$ for some variable permutation ξ.

Axiom V3 has the task to axiomatize that the name of variables is irrelevant.

V3) (Renaming axiom)
For all finite sets W,W' of variables there exists a variable permutation $\xi \in \mathbf{MAP}$ such that $\xi W \cap W' = \emptyset$.

The following axiom is the first of the finiteness axioms, and together with the second (finiteness of variables occurring in an object) we are enabled consider the set of variables as coinfinite, i.e., we can always assume that new variables can be introduced. This is a slight restriction, since this makes it impossible to solve problems including an infinite number of variables. A solution for this more general case would be to choose **V** such that the cardinality of **V** is greater than the cardinality of the set of variables in an object and the set of variables in the domain of a mapping. Note that in the infinite case, the axioms may be insufficient, since we have tried to make the axioms as small as possible.

V4) (Finiteness of domains of mappings)
$DOM(\sigma)$ is finite for all $\sigma \in$ **MAP**.

2.2 Definition. The set of variables of an object $t \in$ **OBJ** is defined as the following set:

$$V(t) := \{x \in V \mid \exists \sigma \ \sigma_{|\{x\}} t \neq t \}.$$

We denote the set of variables introduced by a mapping σ by $I(\sigma) := V(COD(\sigma))$.

V5) (Finiteness axiom for objects)
$\forall t \in$ **OBJ** : $V(t)$ is finite. ■

2.3 Definition. (**V**,**OBJ**, **MAP**) is a **unification algebra**, iff it is a unification quasi-algebra and the axioms (MON) and V1) - V5) are satisfied. ■

We denote the set of the set of reachable objects as **TERM** := **MAP**(**V**) = $\{\sigma x \mid \sigma \in$ **MAP**, $x \in$ **V**$\}$ and refer to objects in **TERM** as **terms**.

2.4 Definition. A unification algebra (**V**,**OBJ**, **MAP**) is called **unsorted**, iff all variables are equivalent and **MAP**(**OBJ** – **TERM**) $\subseteq$ **OBJ** – **TERM**. Otherwise it is called **sorted.** ■

The condition **MAP(OBJ – TERM)** $\subseteq$ **OBJ – TERM** can be interpreted as: instances of literals are again literals, and literals and terms are different. If for some object t that is not a term some instance is a term, then t can be considered as a nonwell-sorted term, which has become well-sorted after instantiation.
The definition characterizes the set of unsorted terms and literals with respect to some signature as unsorted unification algebra.

2.5 Example.

1) Let Σ be a signature, $\mathcal{V}$ be a set of variables, $\mathcal{T}(\Sigma,\mathcal{V})$ be the set of first order terms and let SUB be the set of substitutions over $\mathcal{T}(\Sigma,\mathcal{V})$. It can easily be verified that ($\mathcal{V}$, $\mathcal{T}(\Sigma,\mathcal{V})$, SUB) is an (unsorted) unification algebra.
2) Let Σ, $\mathcal{V}$, $\mathcal{T}(\Sigma,\mathcal{V})$, and SUB be as above and let ~ be a congruence on $\mathcal{T}(\Sigma,\mathcal{V})$, such that s ~ t implies $\sigma s \sim \sigma t$ for all terms s,t and all substitutions $\sigma \in$ **SUB**.
Let $\mathcal{V}/\sim$, $\mathcal{T}(\Sigma,\mathcal{V})/\sim$, SUB/~ be the quotients of variables, terms and substitutions modulo ~. Again it can easily be verified that ($\mathcal{V}/\sim$, $\mathcal{T}(\Sigma,\mathcal{V})/\sim$, SUB/~) is an (unsorted) unification algebra.
If the congruence comes from an equational theory $\mathcal{E}$, then $\mathcal{E}$-equality transforms into identity

of objects and substitutions in the unification algebra.

It should be noted that the set of variables in a term defined here is not the syntactic one as in term algebras. For example the theory $\mathcal{E}$ axiomatized by {f(x) = f(y)} causes f(x) to contain no variables with respect to the unification algebra, since it cannot be changed (modulo $\mathcal{E}$) by instantiating the variable x.

3) Let $\mathcal{F}$ be the set of set of first-order-expressions with respect to some signature, i.e., the set consisting of variables, terms, and first order formulae.

As equivalence ≡ we use the change of bound variables in formulae. Since the names of bound variables should not conflict with free variables, we assume that the set of free variables is disjoint from the set of bound variables. Then let **OBJ** = $\mathcal{F}/\equiv$, and **MAP** be the set of first-order substitutions with respect to the terms over free variables. This constitutes a unification algebra.

4) Let $\mathcal{F}$ be the set of set of first-order-expressions with respect to some signature, i.e., the set consisting of variables, terms, and first order formulae. We include also the constants *true* and *false*. We assume that for every ground literal (i.e., without variables), we know whether it is true or false. Furthermore we assume that every literal has a ground instance.

We choose an equivalence different to 3): A formulae L is equivalent to *true*, iff it contains no free variables and is interpreted as *true* with respect to the given semantics. A formulae L is equivalent to *false*, iff it contains no free variables and is interpreted as *true* with respect to the given semantics. For arbitrary formulae we assume that two formulae are equivalent (≡), iff they always evaluate to the same truth-value under every interpretation. As above, we can assume that the set of free variables is disjoint from the set of bound variables.

Then let **OBJ** := $\mathcal{F}/\equiv$, and let **MAP** be the set of first-order substitutions over the set of terms with respect to free variables. This constitutes a unification algebra.

We have not allowed that free variables are captured, for example it is not allowed to replace y by x in the formula ∀x P(x,y), since then there exists no variable permutation that renames y.

5) A slight variation of the example in 4) is that the semantics is defined via a logic program, and formulae are only the queries, i.e., clauses with negative literals, where the variables in the query are considered as free. As equivalence we may use the following: two queries are equivalent (≡), iff they are equal under associativity, commutativity and idempotence of ∨.

Then $\mathcal{F}/\equiv$ together with the set of first-order substitutions over the set of terms with respect to free variables constitute a unification algebra.

It is also possible to have stronger equivalences, for example an equational theory on the term-algebra, and a theory on literals, such as symmetry of predicates.

6) The well-sorted terms of a sorted termalgebra [Wa83, Sch88] together with the well-sorted substitutions form a sorted unification algebra, as is easily verified.

This unification algebras are sorted in the sense of Definition 2.4, since in general not all variables are equivalent.

7) The set of all polynomials over the integers together with substitutions that substitute polynomials into variables is a unification algebra.

8) Let $\mathcal{F}$ be the set of λ-expressions (including free variables) over some signature modulo some equivalence ($\beta\eta$-reduction, denoted by $\equiv$). We assume that the set of bound and free variables are disjoint. Then $\mathcal{F}/\equiv$ together with the set of substitutions that substitute λ-expressions into variables are a unification algebra.

9) Solving equations over fields:
Let K be a field. We add an error-element *error*. We take the set of first order formulae as **OBJ**, where we assume that = is a built-in binary predicate, and use an appropriate congruence on first order-formulae. For example p/0 = *error*. Two formulae are equivalent, if they can be made equal by renaming of bound variables. The mappings **MAP** are all assignments of rational polynomials (including *error*) to variables (modulo the congruence).
This constitutes a unification algebra.
Solving an equation p/q = 0 means to solve the problem $\langle p/q = 0, q \neq 0\rangle$. ■

2.6 Example.

1) Matching as defined in [BHS87] can be seen as solving equations in a unification algebra. The unification algebra for matching is constructed from the term-algebra by considering some variables as constants, i.e., by restricting the set of substitutions.

2) Matching as defined in [FH83] cannot be seen as solving equations in a unification algebra in the sense that a matching problem is replaced by its solution. In our framework this type of matching means to add equations to a to-be-solved system, where these additonal equations come from a substitution.

3) Disunification [Co84, Com88, Bü88] can also be seen as solving equations in a unification algebra, where the encoding as equations may be via formulae as in Example 2.5.
Disunification with parameters can be interpreted as solving an infinite system of equations containing only a finite set of variables.

3. Semantics and Unification Problems.

In this section we develop the notion of solving equations and systems of equations adopting the notion of solutions as ground solutions. This is according to our intuition of equation solving in mathematics, but seems not to capture term-unification. However, considering the variables in a problem and the variables in terms, which occur in solutions, as different things, we can view the variables in terms as free constants, whereas the variables in a problem to be solved are variables in the sense of unification algebras.

In sections 5 and 6 we show more explicitly the relation between our notion of the solution of an equation and with the unification in term-algebras.

Rather than to provide the semantics of expressions with respect to some external models, we prefer to use a similar notion as the Herbrand-model, which uses ground terms and atoms for providing a semantics. This (internal) semantics can be seen as 'definite' semantics, and captures also the case where semantics is defined via a class of models.

Let $\mathbf{OBJ}_{gr} := \{t \in \mathbf{OBJ} \mid V(t) = \emptyset\}$ be the set of **ground objects**, and let the set of **ground mappings** on a set of variables W be $\mathbf{MAP}_{gr,W} = \{\sigma \in \mathbf{MAP} \mid W \subseteq DOM(\sigma) \text{ and } I(\sigma) = \emptyset\}$. If $\sigma \in \mathbf{MAP}_{gr,W}$ we say also **σ is ground on W.**

3.1 Definition. A unification algebra $\mathcal{A}$ is called **inhabited,** iff for every $x \in \mathbf{V}$, there exists an object t with $V(t) = \emptyset$, such that $\{x \leftarrow t\} \in \mathbf{MAP}$. ■

In the following we assume that $\mathcal{A}$ is inhabited.

3.2 Definition. A **unification problem** is a set Γ of pairs of objects, also denoted by $\Gamma = \langle s_i = t_i \mid i \in I\rangle$, such that $V(\Gamma)$ is finite.
A **solution** σ of $\Gamma = \langle s_i = t_i \mid i \in I\rangle$ is a mapping σ ground on $V(\Gamma)$, such that $\sigma s_i = \sigma t_i$ for all $i \in I$.
The set of solutions of Γ is also denoted $SOL(\Gamma)$. We say Γ is **solvable**, if $SOL(\Gamma) \neq \emptyset$, otherwise it is called **unsolvable.** ■

For a mapping $\sigma = \{x_1 \leftarrow t_1,\ldots,x_n \leftarrow t_n\}$, we denote the unification problem $\langle x_1 = t_1,\ldots,x_n = t_n\rangle$ by $\langle\sigma\rangle$.
For a set of mappings U we define the set of restrictions on a set of variables W as $U_{|W} = \{\sigma_{|W} \mid \sigma \in U\}$. For two sets of mappings U and U', we say U and U' are equal modulo a set of variables W, denoted by $U = U'\ [W]$, iff $U_{|W} = U'_{|W}$.

We define systems of solved forms as an extension of idempotent substitutions:
3.3 Definition. Let $\mathcal{S}$ be a set of unification problems.
We say $\mathcal{S}$ is a system of **solved problems,** iff

i) $SOL(\Delta) \neq \emptyset$ for every $\Delta \in \mathcal{S}$.
ii) For every unification problem Γ, there exists a subset $\mathcal{D} \subseteq \mathcal{S}$ such that $SOL(\Gamma) = \cup\{SOL(\Delta) \mid \Delta \in \mathcal{D}\}\ [V(\Gamma)]$.
Such a set $\mathcal{D}$ is also called an **$\mathcal{S}$-representation** of $SOL(\Gamma)$.
iii) For every $\sigma \in \mathbf{MAP}$ with $DOM(\sigma) \cap I(\sigma) = \emptyset$: $\langle\sigma\rangle \in \mathcal{S}$. ■

In section 4 (Lemma 4.15) we show that this definition is consistent, since for mappings $\sigma \in \mathbf{MAP}$ with $DOM(\sigma) \cap I(\sigma) = \emptyset$, the unification problem $\langle\sigma\rangle$ is solvable.

3.4 Definition. Let $\mathcal{S}$ be a system of solved problem. The **unification type** of unification problems and algebras is defined with respect to a system $\mathcal{S}$ of solved problems.
Let Γ be a solvable unification proble and let $\mathcal{A}$ be a unification algebra.

i) We say Γ is **$\mathcal{S}$-unitary,** iff there exists a $\mathcal{S}$-representation of $SOL(\Gamma)$ that is a singleton.

ii) We say Γ is $\mathcal{S}$-**finitary**, iff there exists a finite $\mathcal{S}$–representation of SOL(Γ).

iii) We say Γ is $\mathcal{S}$-**infinitary**, iff there exists no finite $\mathcal{S}$–representation.

iv) We say $\mathcal{A}$ is $\mathcal{S}$-**unitary**, iff every solvable Γ is $\mathcal{S}$-unitary.

v) We say $\mathcal{A}$ is $\mathcal{S}$-**finitary**, iff every solvable Γ is $\mathcal{S}$-finitary

vi) We say $\mathcal{A}$ is $\mathcal{S}$-**infinitary**, if there exists some solvable Γ that is $\mathcal{S}$-**infinitary**. ■

It is common in unification theory to use as system $\mathcal{S}$ of solved problems only the set of all $\langle\sigma\rangle$ for all mappings σ with DOM(σ} $\cap$I(σ) = $\emptyset$, i.e., all idempotent substitutions. However, there are also examples where the system of solved problems is larger. For rational terms, it is accepted that cyclic problems are also solved forms, and for unification in λ-calculus there are also flexible-flexible term pairs allowed in a solved form [Hu75, SG88].

We compare systems of equations (unification problems) with an ordering.

3.5 Definition. Let W be a set of variables and let Γ, Δ be unification problems. Then

$\Gamma \subseteq_W \Delta$, iff $SOL(\Gamma)_{|W} \subseteq SOL(\Delta)_{|W}$. ■

We do not define minimal problems as in [Si88], since for the general framework given here, this notion seems to be unimportant, see our discussion in section 8 on optimal representations.

Let $\mathcal{S}_{UNI}$ be the standard (and minimal) set of solved forms consisting of all equational systems that correspond to idempotent substitutions, i.e., $\mathcal{S}_{UNI} := \{\langle\sigma\rangle \mid \sigma \in$ **MAP** and DOM(σ)$\cap$I(σ) = $\emptyset\}$. We give the definition of unifiers and correct and complete sets of unifiers:

3.6 Definition. Let Γ be a unification problem.

i) A mapping σ is a **unifier of** Γ, if whenever $\lambda\sigma$ is ground on V(Γ) it is also a solution of Γ.

ii) A set cU is a **correct set of unifiers of** Γ, if every mapping in cU is a unifier.

iii) A set cU is a **complete set of unifiers of** Γ, if for every $\sigma \in$ SOL(Γ), there exists a $\tau \in$ cU and a mapping λ, such that $\lambda\tau = \sigma$ [V(Γ)].

iv) A correct and complete set of unifiers is also called a **unifier-representation** of SOL(Γ). ■

In the rest of this paper we will only consider solved forms that correspond to unifiers, if not stated otherwise.

4. Properties of Unification Algebras

In this section we explore some consequences of our axioms and show that the behaviour of mappings is as expected. The proofs are in general simple, but some are rather tedious to our surprise.

Throughout this section we assume that **V** $\neq \emptyset$.

4.1 Lemma. (Nontriviality of variables)

i) For all $x \in \mathbf{V}$ there exists a $\sigma \in \mathbf{MAP}$ with $\sigma x \neq x$.

ii) $\mathbf{V}$ is an infinite set

Proof.

i) Follows from Axiom V6).

ii) Axiom V3) allows the introduction of infinitely many variables, hence $\mathbf{V}$ is infinite. ■

As noted above, every mapping σ can be represented in a finite way as $\{x_1 \leftarrow t_1,\dots,x_n \leftarrow t_n\}$ where $DOM(\sigma) = \{x_1,\dots,x_n\}$ and $COD(\sigma) = \{t_1,\dots,t_n\}$. The mappings $\{x_i \leftarrow t_i\}$ are called **components** of σ. We show, how to compute the representation of the composition of mappings $\sigma,\tau \in \mathbf{MAP}$:

4.2 Lemma. If $\sigma = \{x_1 \leftarrow s_1 ,\dots, x_n \leftarrow s_n\}$ and $\tau = \{y_1 \leftarrow t_1 ,\dots, y_m \leftarrow t_m\}$, then $\sigma\tau = \{y_1 \leftarrow \sigma t_1 ,\dots, y_m \leftarrow \sigma t_m\} \cup \{x_i \leftarrow s_i \mid x_i \notin DOM(\tau)\}$.

Proof. Using V1) we can compute $\sigma\tau$ by testing $\sigma\tau$ on variables: For $x \notin DOM(\sigma) \cup DOM(\tau)$, we have $\sigma\tau x = \sigma x = x$. If $x \in DOM(\sigma) - DOM(\tau)$, then $\sigma\tau x = \sigma x$. If $x \in DOM(\tau)$, then $\sigma\tau x = \sigma(\tau x)$.■

4.3 Proposition. For every $x \in \mathbf{V}$ there exist infinitely many variables x' equivalent to x.

Proof. Let x be a variable and assume there are only finitely many variables W equivalent to x. Then application of axiom V3 yields for the set W a variable permutation ξ, such that $W \cap \xi W = \emptyset$. Since ξW is a set of variables equivalent to x, this is a contradiction to the assumption thate W is the largest set of variables equivalent to x. ■

This proposition justfies the notion of **new variable**: if we have already used a finite set of variables W and we have a variable x, then it is always possible to select a variable x' equivalent to x, such that $x' \notin W$.

Let s,t be two objects. We say s is **more general than** t (or t is an instance of s), iff there exists a mapping σ with $\sigma s = t$. This is denoted by $s \leq t$. We say s and t are **equivalent**, iff $s \leq t$ and $t \leq s$, and denote this by $s \equiv t$. The next lemma shows, that this is consistent with the notion of equivalent variables.

4.4 Lemma. The following statements are equivalent:

i) The variables x and y are equivalent,

ii) $\{y \leftarrow x\} \in \mathbf{MAP}$ and $\{x \leftarrow y\} \in \mathbf{MAP}$

iii) There exist $\sigma,\tau \in \mathbf{MAP}$ with $\sigma x = y$ and $\sigma y = x$.

Proof. i) ⇒ii) follows from the definition of variable permutation and V2).

ii) ⇒ iii) trivial.

iii) ⇒ ii) follows from the restriction axiom V2).

ii) ⇒ i) Let $\{x \leftarrow y\} \in \mathbf{MAP}$ and $\{y \leftarrow x\} \in \mathbf{MAP}$. Let y' be a new variable equivalent to x. Then $\{x \leftarrow y'\}$, $\{y' \leftarrow x\} \in \mathbf{MAP}$, since i) implies ii). Hence also $(\{x \leftarrow y\}\{y' \leftarrow x\})_{|\{y'\}} = \{y' \leftarrow y\} \in \mathbf{MAP}$.

Obviously we have $(\{y' \leftarrow y\}\{y \leftarrow x\}\{x \leftarrow y'\})_{|\{x,y\}} = \{x \leftarrow y, y \leftarrow x\}$.

Since $\{x \leftarrow y, y \leftarrow x\}\ \{x \leftarrow y, y \leftarrow x\} = Id$, we have constructed a variable permutation $\xi := \{x \leftarrow y, y \leftarrow x\}$ with $\xi x = y$. ■

4.5 Lemma. Let $\xi = \{x_1 \leftarrow y_1,\ldots,x_n \leftarrow y_n\}$ be a variable permutation. Then

i) the left-inverse ξ^- is also a right-inverse

ii) the inverse ξ^- is unique.

iii) the inverse ξ^- is a variable permutation

iii) ξ is a bijection on **V**.

iv) $\xi^- = \{y_1 \leftarrow x_1,\ldots,y_n \leftarrow x_n\}$

Proof. Let ξ^- be a mapping with $\xi^-\xi = Id$.

For $x_i \in DOM(\xi)$ we have $\xi^-\xi x_i = \xi^- y_i = x_i$, since ξ^- is a left inverse of ξ.

For $x \notin DOM(\xi)$ we have $\xi^-\xi x = \xi^- x = x$, since ξ^- is a left inverse of ξ.

Let $y_i \in COD(\xi) - DOM(\xi)$, then on the one hand, we have $\xi^- y_i = y_i$, since $y_i \notin DOM(\xi)$, on the other hand, we have $\xi^- y_i = x_i$. This implies $x_i = y_i$, hence we have the contradiction that $x_i \notin DOM(\xi)$. We have proved that $COD(\xi) \subseteq DOM(\xi)$.

ξ is injective on $DOM(\xi)$, since $\xi x_j = \xi x_k$ implies $\sigma\xi x_j = \sigma\xi x_k$ which is equivalent $x_j = x_k$.

Since $DOM(\xi)$ is finite, we have $DOM(\xi) = COD(\xi)$.

Summarizing, we have shown that $\xi^- = \{y_1 \leftarrow x_1,\ldots,y_n \leftarrow x_n\}$, which also shows that the inverse is unique.

Now ξ^- is also a right inverse of ξ:

For $x \notin DOM(\xi)$, we have $\xi\xi^- x = x$.

For $x \in DOM(\xi)$, we can assume that $x = y_k$ for some k. Then $\xi\xi^- y_k = \xi x_k = y_k$.

Hence ξ^- is a variable permutation.

That ξ is a bijection on **V** follows, since ξ is surjective as $DOM(\xi) = COD(\xi)$, and since ξ is injective, which is implied by the fact that ξ has a left inverse. ■

A **renaming** $\rho \in$ **MAP** is a restriction of a variable permutation ξ, such that $DOM(\rho) \cap I(\rho) = \emptyset$. If $\rho = \{x_1 \leftarrow y_1,\ldots,x_n \leftarrow y_n\}$, then the **converse** ρ^- is defined as $\rho^- = \{y_1 \leftarrow x_1,\ldots,y_n \leftarrow x_n\}$. If the domain of a renaming ρ is W and the codomain of ρ consists of new variables, we will call ρ a **renaming** of W.

4.6 Lemma. Let ρ be a renaming.

i) A renaming is the product of its components.

ii) ρ is idempotent, i.e., $\rho\rho = \rho$.

iii) The converse ρ^- of a renaming exists and is a renaming.

iv) $\rho^-\rho = \rho^-$, $\rho\rho^- = \rho$. ■

In the following we analyse the notion of variables in an object, and show that it behaves as expected.

4.7 Lemma. Let t be an object, let y be a variable and let y' be a new variable equivalent to y. Then $y \in \mathbf{V}(t) \Leftrightarrow \{y \leftarrow y'\}t \neq t$,

Proof. "$\Leftarrow$" is trivial.

"$\Rightarrow$ Let $y \in \mathbf{V}(t)$. Then there exists an object s such that $\{y \leftarrow s\}t \neq t$. Let y' be a new variable equivalent to y. Since y' is new we have $y' \notin \mathbf{V}(t) \cup \mathbf{V}(\{y \leftarrow s\}t) \cup \mathbf{V}(s)$. Now consider the product: $\{y \leftarrow s\}\ \{y' \leftarrow y\} = \{y' \leftarrow s, y \leftarrow s\}$, hence by V2 the mapping $\{y' \leftarrow s\}$ is in **MAP**. Now $\{y' \leftarrow s\}\ \{y \leftarrow y'\} = \{y' \leftarrow s, y \leftarrow s\} = \{y' \leftarrow s\}\ \{y \leftarrow s\}$, hence $\{y' \leftarrow s\}\ \{y \leftarrow y'\}t = \{y' \leftarrow s\}\ \{y \leftarrow s\}t = \{y \leftarrow s\}t \neq t$. Furthermore $\{y' \leftarrow y\}\ \{y \leftarrow y'\}t = \{y' \leftarrow y\}t = t$. This means $y' \in \mathbf{V}(\ \{y \leftarrow y'\}t\ )$, hence $\{y \leftarrow y'\}t \neq t$.■

4.7 Lemma. Let t be a nonvariable object with $y \in \mathbf{V}(t)$ and let y' be a new variable equivalent to y. Then $\{y \leftarrow y'\}t \notin \mathbf{V}$.

Proof. Assume for contradiction that $\{y \leftarrow y'\}t = z \in \mathbf{V}$. Then $\{y' \leftarrow y\}\{y \leftarrow y'\}t = t$, hence $\{y' \leftarrow y\}z = t$. This means that $t = z$ or $t = y'$, which is a contradiction. ■

Now the notion of mappings and objects seem to be understandable, however, one problem remains: we have not proved that $\sigma t = t$, if $DOM(\sigma) \cap \mathbf{V}(t) = \emptyset$. Surprisingly, the proof is tedious:

4.8 Lemma. Let t be an object and let $\rho \in$ **MAP** be a renaming with $I(\rho) \cap \mathbf{V}(t) = \emptyset$. Then $\mathbf{V}(\rho t) = \rho\mathbf{V}(t)$.

Proof.

i) It suffices to consider a single component of ρ, since $\rho = \rho_1\rho_2 \ldots \rho_n$:
Assume as base case for the induction on the number of components, that the lemma is true for a single component.
Then we can prove the induction step:
We have $I(\rho_1) \cap \mathbf{V}(\rho_2 \ldots \rho_n t) = I(\rho_1) \cap \rho_2 \ldots \rho_n \mathbf{V}(t) = \emptyset$ (by induction hypothesis and assumption.
Then we can conclude $\mathbf{V}(\rho_1\rho_2 \ldots \rho_n t) = \rho_1\mathbf{V}(\rho_2 \ldots \rho_n t) = \rho_1\rho_2 \ldots \rho_n \mathbf{V}(t)$, since the lemma holds for a single component.

ii) The lemma holds for a single component, i.e. for different (but equivalent) $x,x' \in \mathbf{V}$ with $x' \notin \mathbf{V}(t)$, we have $\mathbf{V}(\{x \leftarrow x'\}t) = \{x \leftarrow x'\}\mathbf{V}(t)$:
The case that $x \notin \mathbf{V}(t)$ is trivial, since then by definition of $\mathbf{V}(t)$ we have $\{x \leftarrow x'\}t = t$, hence $\mathbf{V}(\{x \leftarrow x'\}t) = \mathbf{V}(t) = \{x \leftarrow x'\}\mathbf{V}(t)$. Thus we can assume that $x \in \mathbf{V}(t)$.
Now we can compute $\{x \leftarrow x'\}\mathbf{V}(t) = \{x'\} \cup (\mathbf{V}(t) - \{x\})$.
Since for all mappings $\{x \leftarrow s\}$, we have $\{x \leftarrow s\}\ (\{x \leftarrow x'\}t) = (\{x \leftarrow s\}\ \{x \leftarrow x'\})\ t = \{x \leftarrow x'\}t$, we have that x is not a variable of the object $\{x \leftarrow x'\}t$, hence $\{x \leftarrow x'\}t \neq t$.
Since x and x' are equivalent, there exists a mapping $\{x' \leftarrow x\}$, hence $\{x' \leftarrow x\}\ \{x \leftarrow x'\}t = \{x' \leftarrow x\}t = t$ implies that $x' \in \mathbf{V}(\{x \leftarrow x'\}t)$.
We show that $\mathbf{V}(\{x \leftarrow x'\}t) - \{x'\} = \{x \leftarrow x'\}\mathbf{V}(t) - \{x'\}$:

1) $\mathbf{V}(\{x \leftarrow x'\}t) - \{x'\} \subseteq \{x \leftarrow x'\}\mathbf{V}(t) - \{x'\}$:
Assume by contradiction that for some $y \in \mathbf{V}(\{x \leftarrow x'\}t)$ with $y \neq x,x'$ we have $y \notin \{x \leftarrow x'\}\mathbf{V}(t)$. Then $y \notin \mathbf{V}(t)$. Let $y' \equiv y$ be a variable such that y' is new.
We have $\{y \leftarrow y'\}\ \{x \leftarrow x'\}t \neq \{x \leftarrow x'\}t$, since $y \in \mathbf{V}(\{x \leftarrow x'\}t)$. However,

$\{y \leftarrow y'\}\ \{x \leftarrow x'\} = \{x \leftarrow x'\}\ \{y \leftarrow y'\}$, since all variables are different, hence $\{y \leftarrow y'\}\ \{x \leftarrow x'\}t = \{x \leftarrow x'\}\ \{y \leftarrow y'\}t = \{x \leftarrow x'\}t$. This is a contradiction.

2) $\{x \leftarrow x'\}\mathbf{V}(t) - \{x'\} \subseteq \mathbf{V}(\{x \leftarrow x'\}t) - \{x'\}$:

Assume there exists a variable $y \neq x,x'$ with $y \in \mathbf{V}(t)$, but $y \notin \mathbf{V}(\{x \leftarrow x'\}t)$.
Let $y' \equiv y$ be a new variable Then $\{y \leftarrow y'\}\ \{x \leftarrow x'\}t = \{x \leftarrow x'\}t$. Application of $\{x' \leftarrow x\}$ gives $\{x' \leftarrow x\}\ \{y \leftarrow y'\}\ \{x \leftarrow x'\}t = \{x' \leftarrow x\}\ \{x \leftarrow x'\}t = t$, but $\{x' \leftarrow x\}\ \{y \leftarrow y'\}\ \{x \leftarrow x'\}t = \{y \leftarrow y'\}\ \{x' \leftarrow x\}\ \{x \leftarrow x'\}t = \{y \leftarrow y'\}t$. This is a contradiction, since $t \neq \{y \leftarrow y'\}t$. ■

4.9 Lemma. For $\sigma \in$ **MAP** and $t \in$ **OBJ** we have $\sigma t = (\sigma_{|V(t)})t$.

Proof. Let $\sigma \in$ **MAP** and let t be an object. Let ρ be a renaming of $DOM(\sigma) \cap I(\sigma)$, such that $I(\rho)$ consists of new variables. Assume that $DOM(\sigma) = \{x_1,\dots,x_n\}$ and that $DOM(\sigma) \cap \mathbf{V}(t) = \{x_1,\dots,x_k\}$. From $\sigma = \{x_1 \leftarrow t_1,\dots, x_n \leftarrow t_n\}$ we get $\rho\sigma = \{x_1 \leftarrow \rho t_1,\dots, x_n \leftarrow \rho t_n\}$.
Furthermore by Lemma 4.8 we get $\mathbf{V}(\rho t_i) \cap DOM(\sigma) = \emptyset$.
Hence $\rho\sigma = \{x_1 \leftarrow \rho t_1\}\ \{x_2 \leftarrow \rho t_2\}\ \dots\ \{x_n \leftarrow \rho t_n\}$. This factorization implies $\rho\sigma t = (\rho\sigma)_{|V(t)}t$.
Obviously the mapping $(\rho\sigma)_{|V(t)}$ has the representation $\{x_1 \leftarrow \rho t_1,\dots, x_k \leftarrow \rho t_k\}$
Applying ρ^- to the equation $\rho\sigma t = (\rho\sigma)_{|V(t)}t$ gives $\rho^-\rho\sigma t = \rho^-(\rho\sigma)_{|V(t)}t$ and thus $\rho^-\sigma t = \rho^-\rho\sigma t = \rho^-(\rho\sigma)_{|V(t)}t = \rho^-\{x_1 \leftarrow \rho t_1,\dots, x_k \leftarrow \rho t_k\}t = \rho^-\{x_1 \leftarrow t_1,\dots, x_k \leftarrow t_k\}t = \rho^-(\sigma_{|V(t)})t$.
Since $DOM(\rho^-)$ is disjoint from $V(\sigma t)$ and $V((\sigma_{|V(t)})t)$, we get $\sigma t = \rho^-\sigma t = \rho^-(\sigma_{|V(t)})t = (\sigma_{|V(t)})t$.
■

4.10 Corollary. Let $\sigma \in$ **MAP** and let t be an object such that $\mathbf{V}(t) \cap DOM(\sigma) = \emptyset$. Then $\sigma t = t$.

4.11 Corollary. Let $\sigma,\tau \in$ **MAP** and let $t \in$ **T**. Then
$\sigma = \tau[V(t)] \Rightarrow \sigma t = \tau t$.

Proof. Since $\sigma t = (\sigma_{|V(t)})t$, $\tau t = (\tau_{|V(t)})t$ and $\sigma_{|V(t)} = \tau_{|V(t)}$, we can conclude $\sigma t = \tau t$. ■

4.12 Lemma $\mathbf{V}(\sigma t) \subseteq \cup\{\mathbf{V}(\sigma x) | x \in \mathbf{V}(t)\}$.

Proof. We can assume $DOM(\sigma) = \mathbf{V}(t)$, since $\sigma t = \sigma_{|V(t)}t$ by Lemma 4.9.
Note that $\cup\{\mathbf{V}(\sigma x) | x \in \mathbf{V}(t)\} = I(\sigma) \cup (V(t) - DOM(\sigma))$.
Let y be a variable with $y \notin I(\sigma) \cup (\mathbf{V}(t) - DOM(\sigma))$. That means $y \notin I(\sigma)$. Furthermore either $y \notin V(t)$ or $y \in DOM(\sigma)$. Let y' be a new variable that is equivalent to y'.
If $y \notin \mathbf{V}(t)$, then $\{y \leftarrow y'\}\ \sigma t = \sigma\ \{y \leftarrow y'\}\ t = \sigma t$, hence $y \notin \mathbf{V}(\sigma t)$.
If $y \in DOM(\sigma)$, then $\{y \leftarrow y'\}\ \sigma = \sigma$, hence $y \notin \mathbf{V}(\sigma t)$. This proves the lemma. ■

4.13 Lemma. The union of mappings exists:

Let $\sigma,\tau \in$ **MAP** such that $\sigma = \tau\ [DOM(\sigma) \cap DOM(\tau)]$.
Then there exists a mapping μ, such that $\mu = \sigma\ [DOM(\sigma)]$ and $\mu = \tau\ [DOM(\tau)]$.

Proof. Let ρ be a renaming of $I(\tau)$ by new variables. Then let $\tau' := (\rho^-\tau)_{|DOM(\tau)}$ and $\mu := \rho\sigma\tau'$.
Let $x \in DOM(\tau)$. Then $\mu x = \rho\sigma\tau' x = \rho\sigma\rho^-\tau x = \rho\rho^-\tau x = \rho\tau x = \tau x$ using Lemma 4.9.
If $x \in DOM(\sigma) \cap DOM(\tau)$, then $\mu x = \tau x = \sigma x$ by the assumption of the lemma.

If $x \in DOM(\sigma) - DOM(\tau)$, then $\mu x = \rho\sigma\tau' x = \rho\sigma x = \sigma x$. ■

For two mappings σ,τ with $\sigma = \tau$ [DOM(σ) $\cap$ DOM(τ)] we define $\sigma \cup \tau$ as the mapping given in the lemma above restricted to DOM(σ) $\cup$ DOM(τ). This union can be seen as a union of the representations of σ and τ.

Now we can prove some required lemmas concerning solvability of equation systems.
We can characterize $\mathbf{OBJ}_{gr}$ as the set of fixed points under **MAP**:

4.14 Lemma. $\mathbf{OBJ}_{gr} = \{t \in \mathbf{OBJ} \mid \forall\sigma \in \mathbf{MAP}: \sigma t = t\}$.

Proof. If $t \in \mathbf{OBJ}_{gr}$, then $V(t) = \emptyset$, hence $\sigma t = t$ for all $\sigma \in \mathbf{MAP}$ by Corollary 4.10. On the other hand, if for some $t \in \mathbf{OBJ}$ is fixed under all $\sigma \in \mathbf{MAP}$, it is fixed under all mappings $\{x \leftarrow s\}$, hence $\mathbf{V}(t) = \emptyset$. ■

We show that the equation systems in solved forme are solvable:

4.15 Lemma. Let $\mathcal{A}$ be an inhabited unification algebra and let σ be a mapping with DOM(σ) $\cap$ I(σ) = $\emptyset$. Then $\langle\sigma\rangle$ is solvable.

Proof. For every $x \in I(\sigma)$ there exists a ground term s_x with $\{x \leftarrow s_x\} \in \mathbf{MAP}$. The union τ of all these mappings exists by Lemma 4.13. Then $\tau\sigma$ is a solution of $\langle\sigma\rangle$. For a component $\langle x = \sigma x\rangle$, we have $\tau\sigma x = \tau\sigma\sigma x$, since σ is idempotent. Furthermore $\tau\sigma$ is ground on $\mathbf{V}(\Gamma)$. ■

According to Definition 3.5, we define two subsumption relations on mappings.

4.16 Definition. Let σ,τ be mappings and W be a set of finite variables,

i) $\sigma \sqsupseteq_W \tau$, iff for all $\lambda \in \mathbf{MAP}$, such that $\lambda\tau$ is ground on W, there exists a λ', such that $\lambda'\sigma = \lambda\tau$ [W].
Intuitively, $\sigma \sqsupseteq_W \tau$ means that σ represents more ground substitutions than τ.

ii) $\sigma \le \tau$ [W], iff there exists a mapping λ such that $\lambda\sigma = \tau$ [W]. ■

Obviously $\le$[W] and $\sqsupseteq_W$ are quasi-orderings on **MAP** for a fixed W.

4.17 Lemma. $\sigma \le \tau$ [W] implies $\sigma \sqsupseteq_W \tau$, but the converse may be false.

Proof. Let μ be such that $\mu\sigma = \tau$ [W]. Let $\lambda \in \mathbf{MAP}$, such that $\lambda\tau$ is ground on W. Then $\lambda\mu\sigma$ is also ground on W and $\lambda\mu\sigma = \lambda\tau$ [W]. Hence $\sigma \sqsupseteq_W \tau$,
We give an example that the converse is false:
Consider substitutions over the integers.
Let $\sigma = \{x \leftarrow y\}$ and let $\tau := \{x \leftarrow x_1^2 + x_2^2 + x_3^2 + x_4^2 - (y_1^2 + y_2^2 + y_3^2 + y_4^2)\}$
Obviously we have $\sigma \sqsupseteq_W \tau$ and $\sigma \sqsupseteq_W \tau$, since both substitutions range over the whole set of integers as is well-known from the theory of numbers. The relation $\sigma \le \tau$ [W] holds, but not $\sigma \ge \tau$ [W], since it is not possible to obtain y by substituting polynomials into the polynomial τx.
■

We show in paragraph 5 that for unification in free term algebras the two relations $\leq [W]$ and $\supseteq_W$ are the same.

4.18 Example. Consider Pythagoras´ equation $x^2+y^2 = z^2$ over the naturals (including zero). This equations has infinitely many solutions. A minimal representation consists of the two unifiers $\{x \leftarrow 2p(p+q),\ y \leftarrow q^2+2pq,\ z \leftarrow p^2+(p+q)^2\}$ and $\{x \leftarrow q^2+2pq,\ y \leftarrow 2p(p+q),\ z \leftarrow p^2+(p+q)^2\}$.

5. Homorphisms and Congruences

In this paragraph we develop the algebraic tools that correspond to unification algebras like homomorphisms and quotients and show that the isomorphism theorem holds. This makes the unification algebras to be a category with some additional properties.

A the end of this paragraph we argue that every unification algebra is isomorphic to the quotient of of an order-sorted term-algebra [Sch88, SNMG87] (including ill-sorted terms), where the quotient is made with respect to a stable congruence. At the first glance this appears to be a drawback, since we have arrived at what was to be generalized. However, there are several merits. Usually it is easier to prove that some problem can be formulated using unification algebras than to give a signature, a congruence and a sort-structure that describes the problem domain, since there may be infinitely many symbols, equations and term-sort declarations. A further advantage of unification algebras is the notion of isomorphism, which is superior to isomorphisms of universal algebras, since it includes weak isomorphism of universal algebras (definitional equivalence, polynomial equivalence), [Gr79, BS81] isomorphisms or symmetries of signatures, and symmetries of the sort-structure.

5.1 Definition. Let $\mathcal{A}_1 = (\mathbf{V}_1,\mathbf{OBJ}_1,\mathbf{MAP}_1)$ and $\mathcal{A}_2 = (\mathbf{V}_2,\mathbf{OBJ}_2,\mathbf{MAP}_2)$ be unification algebras. A mapping $\psi: \mathcal{A}_1 \to \mathcal{A}_2$ is a **homomorphism**, iff

i) ψ is a mapping $\psi:\mathbf{OBJ}_1 \to \mathbf{OBJ}_2$ and $\psi:\mathbf{MAP} \to \mathbf{MAP}_2$
ii) $\psi(\sigma\tau) = \psi(\sigma)\psi(\tau)$
iii) $\forall\sigma \in \mathbf{MAP}_1\ \forall t \in \mathbf{OBJ}_1$: $\psi(\sigma t) = (\psi\sigma)(\psi t)$.
iv) $\mathbf{V}_2 \subseteq \psi(\mathbf{V}_1)$.■

Let $\psi: \mathcal{A}_1 \to \mathcal{A}_2$ be a **homomorphism**. We say ψ is an **isomorphism**, iff it is a homomorphism that is a bijection on the set objects and mappings, such that the inverse is an homomorphism.
Let range(ψ) := $(\mathbf{V}_\psi, \psi\mathbf{OBJ}_1, \psi\mathbf{MAP}_1)$, where $\mathbf{V}_\psi := \{x \in \mathbf{V}_2 \mid$ there exists exactly one variable $y \in \mathbf{V}_1$ with $\psi y = x\}$.

The unification algebras form a category with this notion of homomorphism. There is a final object, which is the trivial unification algebra $(\emptyset, \{a\}, \{Id\})$

We say $\mathcal{A}_1$ is **embedded** in $\mathcal{A}_2$ via ψ, iff $\psi:\mathcal{A}_1 \to \mathcal{A}_2$ is an injective homomorphism.

Let $\mathcal{A}$, $\mathcal{B}$ be unification algebras, such that $V_{\mathcal{A}} = V_{\mathcal{B}}$, $\mathbf{OBJ}_{\mathcal{A}} \subseteq \mathbf{OBJ}_{\mathcal{B}}$, $\mathbf{MAP}_{\mathcal{A}} \subseteq \mathbf{MAP}_{\mathcal{B}}$. Then $\mathcal{A}$ is called **strongly embedded** in $\mathcal{B}$.

5.2 Lemma. Let $\psi: \mathcal{A}_1 \to \mathcal{A}_2$ be a homomorphism. Then:
$DOM(\psi\sigma) \subseteq \psi DOM(\sigma)$ for all $\sigma \in \mathbf{MAP}$.

Proof. Let $x_2 \in DOM(\psi\sigma)$. There exists a variable $x_1 \in \mathbf{V}_1$ with $\psi x_1 = x_2$. Assume $x_1 \notin DOM(\sigma)$. Then $\sigma x_1 = x_1$, which implies $x_2 = \psi(x_1) = \psi(\sigma x_1) = \psi(\sigma)\psi(x_1) = \psi(\sigma)x_2$, which contradicts $x_2 \in DOM(\psi\sigma)$. Hence $x_1 \in DOM(\sigma)$. ■

5.3 Lemma. A homomorphism $\psi: \mathcal{A}_1 \to \mathcal{A}_2$ that is bijective on objects and mappings, is an isomorphism.

Proof. Let ψ^- be the inverse mapping of ψ. Condition i) of Definition 5.1 is trivially satisfied. In order to prove ii), let σ_2,τ_2 be mappings in MAP_2. There are mappings σ_1,τ_1 in MAP_1 with $\psi\sigma_1 = \sigma_2$ and $\psi\tau_1 = \tau_2$. We have $\psi^-(\sigma_2\tau_2) = \psi^-((\psi\sigma_1)\psi(\tau_1)) = \psi^-(\psi(\sigma_1\tau_1)) = \sigma_1\tau_1 = \psi^-(\sigma_2)\ \psi^-(\tau_2)$. A similar computation shows that iii) also holds.
We have already $\mathbf{V}_2 \subseteq \psi(\mathbf{V}_1)$, since ψ is a homomorphism. We have to show that $\mathbf{V}_1 \subseteq \psi^-(\mathbf{V}_2)$. From $\mathbf{V}_2 \subseteq \psi(\mathbf{V}_1)$ we get $\psi^-(\mathbf{V}_2) \subseteq \mathbf{V}_1$. Assume for contradiction, that there is a variable $x \in \mathbf{V}_1 - \psi^-(\mathbf{V}_2)$. That means $\psi x \notin \mathbf{V}_2$. Let y be a variable equivalent to x. The mapping $\sigma := \{x \leftarrow y\}$ must be mapped to Id_2, since $DOM(\psi\sigma) = \mathbf{V}_2 \cap \{\psi x\} = \emptyset$. Since ψ is a bijection on **MAP**, this implies that $\sigma = Id_2$, which is a contradiction. Hence $\mathbf{V}_2 = \psi(\mathbf{V}_1)$. ■

The next lemma clarifies the effect of homomorphisms on variables.

5.4 Lemma. Let $\psi: \mathcal{A}_1 \to \mathcal{A}_2$ be a homomorphism, and let $x \in \mathbf{V}_1$. Then

i) $\psi x \notin \mathbf{V}_2$ implies that there exists a variable $y \neq x$ with $\psi x = \psi y$.

ii) If there exists a variable $y \neq x$ with $\psi x = \psi y$, then

   a) ψ is constant on the set $\{s \mid \{x \leftarrow s\} \in \mathbf{SUB}$ or $\{y \leftarrow s\} \in \mathbf{SUB}\}$

   b) ψx is fixed by every substitution in range(ψ).

Proof.

i) Let $\psi x \notin \mathbf{V}_2$. There exists a variable $y \neq x$ that is equivalent to x. Consider the mapping $\{x \leftarrow y\}$. Since $\psi x \notin \mathbf{V}_2$, we have $DOM(\psi(\{x \leftarrow y\}) = \emptyset$, hence $\psi x = \psi y$.

ii) Let y be a variable with $y \neq x$ and $\psi x = \psi y$. Let s be a term such that $\{x \leftarrow s\} \in \mathbf{MAP}$ or $\{y \leftarrow s\} \in \mathbf{MAP}$. Without loss of generality we can assume that $\{x \leftarrow s\} \in \mathbf{MAP}$. Then $\psi s = \psi(\{x \leftarrow s\}x) = \psi(\{x \leftarrow s\})\ \psi x = \psi(\{x \leftarrow s\})\psi y = \psi(\{x \leftarrow s\}y) = \psi y$.
This proves part a). Let $\psi\sigma$ be a mapping in range(ψ). Then $\{x \leftarrow \sigma x\} \in \mathbf{MAP}$, hence $\psi\sigma\ \psi x = \psi\sigma x = \psi x$ due to a). This proves part b).■

We give an example, that $\psi x = \psi y$ for different variables x,y does not preclude that ψx is a variable in $\mathbf{V}_2$. Let $\mathcal{A}_1$ be a unification algebra consisting only of variables. The variables are partitioned in classes S_i, i = 1,2,..., such that x is equivalent to y, iff they belong to the same class. The mappings in $\mathbf{MAP}_1$ are the possible substitutions. Let $\mathcal{A}_2$ consist only of variables z_i, i= 1,2,... that are all equivalent, and of all possible substitutions on these variables.

Let the homomorphism $\psi:\mathcal{A}_1 \to \mathcal{A}_2$ be such that $\psi x = z_i$, iff $x \in S_i$, and all mappings in $\mathbf{MAP}_1$ are mapped to the identity. Then ψ is a homomorphism and maps different variables in $\mathcal{A}_1$ to the same variable in $\mathcal{A}_2$.

5.5 Lemma. Let DOM_ψ (.) and $\mathbf{V}_\psi$(.) be that domain and variable operator, respectively, with respect to range(ψ). Then

i) $DOM_\psi(\psi\sigma) \subseteq \psi DOM(\sigma) \cap V_\psi$, where $V_\psi = \{x \in \mathbf{V}_2 \mid$ there exists exactly one variable $y \in \mathbf{V}_1$ with $\psi y = x\}$.

ii) $\mathbf{V}_\psi(\psi t) \subseteq \psi\mathbf{V}(t)$

Proof.

i) Let $\sigma \in \mathbf{MAP}$. Lemma 5.2 shows that $DOM(\psi\sigma) \subseteq \psi DOM(\sigma)$. Let $z_2 \in \mathbf{V}_2$. Then there exists a variable z_1 with $\psi z_1 = z_2$. If there is another variable z_1' with $\psi z_1' = z_2$, then Lemma 5.4.ii.b) shows that $\psi\sigma$ does not change z_2, hence $z_2 \notin DOM(\psi\sigma)$. This means $DOM(\psi\sigma) \subseteq \psi DOM(\sigma) \cap \mathbf{V}_\psi$.

ii) Let $x_2 \in \mathbf{V}_\psi(\psi t)$. Then there exists a mapping $\{x_2 \leftarrow s_2\} \in \psi\mathbf{MAP}_1$ with $\{x_2 \leftarrow s_2\}\psi t \neq \psi t$. Hence there exists a mapping $\{x_1 \leftarrow s_1\}$ with $\psi\{x_1 \leftarrow s_1\} = \{x_2 \leftarrow s_2\}$ and $\psi x_1 = x_2$.. This means $\psi\{x_1 \leftarrow s_1\}\ \psi t = \psi(\{x_1 \leftarrow s_1\}t) \neq \psi t$, hence $\{x_1 \leftarrow s_1\}t \neq t$. Thus $x_1 \in \mathbf{V}(t)$. ■

5.6 Proposition. range(ψ) is a unification algebra.

Proof. MON) follows from the definition of a homomorphism and since $\mathcal{A}_1$ is a unification algebra.

V1) Let $\psi\sigma = \psi\tau\ [\mathbf{V}_\psi]$. Lemma 5.4 shows that $DOM(\psi\sigma) \subseteq \mathbf{V}_\psi$ and $DOM(\psi\tau) \subseteq \mathbf{V}_\psi$. Hence $\psi\sigma = \psi\tau\ [\mathbf{V}_2]$ which implies $\psi\sigma = \psi\tau$, since $(\mathbf{V}_2,\mathbf{T}_2,\mathbf{SUB}_2)$ is a unification algebra.

V2) Trivial.

V3) Let $W_2 \subseteq \mathbf{V}_\psi$ be a finite set of variables. Since every variable in W_2 is the image of a (unique) variable in $\mathbf{V}_1$, there exists a finite set of variables $W_1 \subseteq \mathbf{V}_1$ with $\psi W_1 = W_2$. Let ξ be a variable permutation that renames W_1. The image of ξ is also variable-permutation: If $DOM(\psi\xi) \neq \psi DOM(\xi)$, then there exists a component $\{x \leftarrow y\}$ of ξ such that $\psi x = \psi y$, which contradicts $W_2 \subseteq \mathbf{V}_\psi$. Hence $DOM(\psi\xi) = \psi DOM(\xi)$. Since $\psi\xi$ has as inverse $\psi\xi^-$, it is a variable permutation. Now the set $\psi(\xi)W_2$ is a set of variables that is disjoint with W_2. Hence V3) holds.

V4) follows from Lemma 5.2.

V5) follows, since $\mathcal{A}_2$ is a unification algebra. ■

5.6 Corollary. An embedding $\psi:\mathcal{A}_1 \to \mathcal{A}_2$ is an isomorphism $\psi:\mathcal{A}_1 \to$ range(ψ)

Proof. Holds, since range(ψ) is a unification algebra by Proposition 5.5 and by Lemma 5.3. ■

5.7 Definition. Let $\mathcal{A} = (\mathbf{V},\mathbf{OBJ},\mathbf{MAP})$ be a unification algebra.

i) A equivalence relation ~ on **OBJ** is a **congruence**, iff $s \sim t$ and $\sigma x \sim \tau x$ for all $x \in \mathbf{V}$ implies $\sigma s \sim \tau t$.

ii) For a congruence ~ we define the **quotient** $\mathcal{A}/\sim$ as $(\mathbf{V}_\sim,\mathbf{T}/\sim,\mathbf{SUB}/\sim)$, where $\mathbf{V}_\sim := \{x/\sim \mid x \in \mathbf{V}$ and $x/\sim \cap \mathbf{V} = \{x\}\}$. The relation ~ is extended to mappings by $\sigma \sim \tau$

iff $\sigma x \sim \tau x$ for all variables x. The operations are defined as $(\sigma/\sim)(\tau/\sim) := (\sigma\tau)/\sim$ and $(\sigma/\sim)(t/\sim) := (\sigma t)/\sim$.

5.8 Lemma. quotients are unification algebras.

Proof. Operations are well-defined: Let $\sigma_1 \sim \sigma_2$ and $\tau_1 \sim \tau_2$. Then we have $\sigma_1\tau_1 x \sim \sigma_2\tau_2 x$ for all variables x, hence $\sigma_1\tau_1 \sim \sigma_2\tau_2$. The other follows from the definition of congruence.

The proofs are tedious, but exactly analogous to the proofs that show that the range of a homomorphism is a unification algebra, hence we omit them. ■

5.9 Proposition. Let $\mathcal{A}_1 = (\mathbf{V}_1, \mathbf{T}_1, \mathbf{SUB}_1)$ and $\mathcal{A}_2 = (\mathbf{V}_2, \mathbf{T}_2, \mathbf{SUB}_2)$ be unification algebras and let $\psi: \mathcal{A}_1 \to \mathcal{A}_2$ be a homomorphism.

Then the relation $\sim$ on $\mathbf{T}_1$ with $s \sim t :\Leftrightarrow \psi s = \psi t$ is a congruence.

Furthermore $\mathcal{A}_1/\sim$ is isomorphic to range(ψ).

Proof. Let $s,t \in \mathbf{T}$ with $\psi s = \psi t$ and let $\sigma,\tau \in \mathbf{SUB}$ such that $\psi(\sigma x) = \psi(\tau x)$ for all $x \in \mathbf{V}$.

Then $\psi(\sigma)(\psi x) = \psi(\tau)(\psi x)$ for all $x \in \mathbf{V}$. Since $\psi\mathbf{V} \subseteq \mathbf{V}'$, we have $\psi(\sigma) = \psi(\tau)$, hence $\psi(\sigma)(\psi s) = \psi(\tau)(\psi t)$.

In order to prove the isomorphism between $\mathcal{A}_1/\sim$ and range(ψ), we have to check that ψ is a bijection, which is obvious. Then we can apply Lemma 5.3. ■

There are some natural isomorphisms on unification algebras:

5.10 Lemma. Variable permutations provide isomorphisms on unification algebras.

For a variable permutation ξ, the corresponding isomorphism φ_ξ on the unification algebra operates as follows: $\varphi_\xi\, t := \xi t$ for $t \in \mathbf{OBJ}$ and $\varphi_\xi\, \sigma := \xi\sigma\xi^-$ for $\sigma \in \mathbf{MAP}$.

5.11 Lemma. Let $\mathcal{A} := (\mathrm{V}, \mathbf{OBJ}, \mathbf{MAP})$ be a unification algebra, and let $\mathrm{V} = \mathrm{V}_1 \cup \mathrm{V}_2$ be a partition of V, such that there is a bijection $\varphi: \mathrm{V} \to \mathrm{V}_1$ such that φx is equivalent to x for all x. Let $\mathcal{A}_0$ be the following unification algebra: $\mathcal{A}_0 = (\mathrm{V}_1, \mathrm{T}, \{\sigma \in \mathbf{MAP} \mid \mathrm{DOM}(\sigma) \subseteq \mathrm{V}_1\}$,

Then $\mathcal{A}$ is embeddable in $\mathcal{A}_0$.

Proof. The injective homomorphism $\psi: \mathcal{A} \to \mathcal{A}_0$ is defined as follows: Let $\psi t = \varphi t$ and let $\psi\sigma = \varphi\sigma\varphi^-$, where φ^- is the inverse of φ. These definition are not quite correct as they stand. φt for example can be seen as the result of applying the substitution $\varphi_{|V(t)}$ to t. Note that the mapping $\varphi\sigma\varphi^-$ has a finite domain and hence is a mapping in **MAP**. Now all conditions can be easily verified. ■

If in the construction above the partition $\mathrm{V} = \mathrm{V}_1 \cup \mathrm{V}_2$ is such that there is also a bijection $\varphi: \mathrm{V} \to \mathrm{V}_2$, then we will call the unification algebra $\mathcal{A}_0 = (\mathrm{V}_1, \mathrm{T}, \{\sigma \in \mathrm{SUB} \mid \mathrm{DOM}(\sigma) \subseteq \mathrm{V}_1\}$ the **unification algebra extended by constants**. For these distinguished constants it makes sense to speak of constants occurring in a term. Let $\mathrm{CONST}(t) := \mathrm{V}_{\mathcal{A}}(t) - \mathrm{V}_1$.

The usual notion of unification in the free term algebra as considered in [Si86, Si88] and for sorted signatures [Sch88] is a specialization of unification in a unifcation algebra extended by free constants.

In order to show that minimal representations for usual term unification are exactly minimal sets of unifiers in the usual sense, we use the unification algebra extended by constants. This is the same as viewing the variables in solutions as ground and only the new variables in unifiers as variables in which something can be substituted. The same effect can be achieved by adding an infinite set of free constants to a term-algebra.

5.12 Lemma. For a unification algebra extended by constants: $\sigma \leq \tau\ [W]$ is equivalent to $\sigma \sqsupseteq_W \tau$ for finite W .

Proof. Let $\mathcal{A}$ = (**V**,**T**,**SUB**) be a unification algebra, let $V = V_0 \cup V_1$ such that $\mathcal{A}_0$ = (V_0,T_0,SUB_0) has variables V_0 and is the unification algebra extended by constants. Now we refer elements of V_1 as constants and to elements of V_0 as variables.

Let σ,τ be substitutions with $\sigma \sqsupseteq_W \tau$. Without loss of generality we can assume that $DOM(\sigma) = DOM(\tau) \subseteq W$ and that $I(\sigma) \cap I(\tau) = \emptyset$. Let $x_1,\ldots,x_n$ be the variables in $V(\tau W)$ and let $a_1,\ldots,a_n$ be constants not occurring in the terms of $COD(\sigma) \cup COD(\tau)$, such that x_i is equivalent to a_i in $\mathcal{A}$. Then $\tau_{gr} := \{x_i \leftarrow a_i\}\tau$ is an instance of τ ground on W. There exists a substitution λ, such that $\lambda\sigma$ is ground on W and $\lambda\sigma = \tau_{gr}\ [W]$. Now we switch to $\mathcal{A}$, the larger unification algebra: Then $\{a_i \leftarrow x_i\}$ is a substitution in **SUB**. Applying it to the equation above gives $\{a_i \leftarrow x_i\}\lambda\sigma = \{a_i \leftarrow x_i\}\ \tau_{gr} = \{a_i \leftarrow x_i\}\ \{x_i \leftarrow a_i\}\tau = \tau\ [W]$. Furthermore $\{a_i \leftarrow x_i\}\lambda\sigma = (\{a_i \leftarrow x_i\}\lambda)_{|I(\sigma)}\sigma\ [W]$. The substitution $(\{a_i \leftarrow x_i\}\lambda)_{|I(\sigma)}$ is in SUB_0, hence $\sigma \leq \tau\ [W]$ in $\mathcal{A}_0$.
■

5.13 Corollary. For unification in a unification algebra extended by constants: $\sigma \leq \tau\ [W]$ is equivalent to $\sigma \sqsupseteq_W \tau$, for every finite set of variables W .

Proof. Follows from 5.12.

The following structure theorem holds:

5.14 Theorem. Every unification algebra is isomorphic to $\mathcal{T}/\!\sim$, where $\mathcal{T}$ is a set of terms with respect to some order-sorted signature, such that $\mathcal{T}$ is closed under well-sorted instantiation, $\mathcal{T}$ contains all well-sorted terms, but may contain also ill-sorted ones, and ~ is a stable congruence on $\mathcal{T}$.

Proof. Let $\mathcal{A}$ = (**V**,**OBJ**, **MAP**). Then we construct the signature Σ as follows:

The set of sorts is the set of equivalence classes of variables and the ordering on sorts is $[x] \sqsubseteq [y]$, iff $\{y \leftarrow x\} \in$ **MAP**. For every object $t \in$ **OBJ** – **V** there is a function symbol $f_t \in \Sigma$. We can assume that the operator V(.) gives a vector of variables instead of a set. Then if $V(t) = (x_1,\ldots,x_n)$, the function symbol f_t has arity n. For every variable y such that $\{y \leftarrow t\} \in$ **MAP**, there is a function declaration $f_t\colon [x_1]\times\ldots\times[x_n] \rightarrow [y]$. Every variable y is considered to have sort [y] and all greater sorts.

Let $\mathcal{B}_0 := (\mathbf{V}, \mathcal{T}_+(\Sigma,\mathbf{V}), SUB(\Sigma,V))$, where $SUB(\Sigma,V)$ is the set of well-sorted substitutions on (the set of well-sorted terms) $\mathcal{T}(\Sigma,\mathbf{V})$. $\mathcal{T}_+(\Sigma,\mathbf{V})$ is $\mathbf{V} \cup \{\sigma f_t(x_1,\ldots,x_{n(t)}) \mid$

$\sigma \in SUB(\Sigma,V)$, $t \in \mathbf{OBJ}$, $V(t) = \{x_1,\ldots,x_{n(t)}\}\}$. The set $\mathcal{T}_+(\Sigma,V)$ contains $\mathcal{T}(\Sigma,V)$, but is in general not equal to the set of all syntactic possible terms over Σ.

Let $\varphi: \mathcal{B} \rightarrow \mathcal{A}$ be defined as follows:

i) $\varphi(x) := x$ for $x \in \mathbf{V}$.

ii) $\varphi\, f_t(x_1,\ldots,x_n) := t$, where t is an object, and $V(t) = (x_1,\ldots,x_n)$.

iii) $\varphi(\{x_1 \leftarrow y_1,\ldots, x_n \leftarrow y_n\}) := \{x_1 \leftarrow y_1,\ldots, x_n \leftarrow y_n\}$ for variables x_i, y_i

iv) $\varphi\, f_t(t_1,\ldots,t_n) := \{x_1 \leftarrow \varphi t_1,\ldots, x_n \leftarrow \varphi t_n\}t$, where t is an object, and $V(t) = (x_1,\ldots,x_n)$.

v) $\varphi\{x_1 \leftarrow t_1,\ldots, x_n \leftarrow t_n\} := \{x_1 \leftarrow \varphi t_1,\ldots, x_n \leftarrow \varphi t_n\}$.

We have to show that the definition of φ makes sense and that φ is a homomorphism.

1) if s is of sort [x], then $\{x \leftarrow \varphi s\}$ is a mapping in **MAP**:

Proof. By induction on the term depth.

If y is of sort [x], then $\{x \leftarrow y\}$ is in **MAP** by definition.

If s is of sort [x], then $s = f_t(s_1,\ldots,s_n)$, and $\{x \leftarrow t\}$ is in **MAP**. By the definition of $\mathcal{T}_+(\Sigma,V)$, s_i is a term of sort $[x_i]$ for all $i = 1,\ldots,n$. By induction hypothesis, $\{x_i \leftarrow \varphi s_i\}$ is a mapping in **MAP** for all $i = 1,\ldots,n$. Since the union of mappings exists, we have also that $\sigma := \{x_1 \leftarrow \varphi s_1,\ldots, x_n \leftarrow \varphi s_n\}$ is a mapping. Composition and restriction gives that $\{x \leftarrow \sigma t\}$ is in **MAP**. By definition $\{x \leftarrow \sigma t\} = \{x \leftarrow \varphi f_t(s_1,\ldots,s_n)\} = \{x \leftarrow \varphi s\}$.

2) φ is surjective on **V**, **OBJ**, **MAP**:

φ is obviously surjective on **V**. φ is also surjective on **OBJ**, since for every $t \in \mathbf{OBJ}$, there exists a term $f_t(x_1,\ldots,x_{n(t)})$ with $\varphi\, f_t(x_1,\ldots,x_{n(t)}) = t$. It is surjective on **MAP**, since for every mapping $\{x_1 \leftarrow t_1,\ldots, x_n \leftarrow t_n\}$ there exists terms $\varphi^{-1}t_i$ of sort $[x_i]$, such that $\{x_i \leftarrow \varphi^{-1}t_i\}$ is a substitution.

3) φ is a homomorphism:

Definition 5.1.iii holds by the definition of φ. That 5.1 iii) can be shown by an easy computation using $\varphi(\sigma t) = \varphi(\sigma)\varphi t$. Definition of φ implies that 5.1 iv) holds. ❑

Proposition 5.9 now implies that $\mathcal{A}$ is isomorphic to a quotient of $\mathcal{B}$. ■

Note that this structure theorem allows some ill-sorted terms without well-sorted instances. Such terms could have been also considered as literals. For the sake of simplicity, we have treated them as ill-sorted terms.

For unsorted unification algebras, from Definition 2.4 and Theorem 5.14 it is easy to deduce that 5.14 can be sharpened to:

5.15 Proposition. Every unsorted unification algebra is isomorphic to $\mathcal{T}\cup\mathcal{AT}/\sim$, where $\mathcal{T}$ is a set of terms with respect to some unsorted signature, $\mathcal{AT}$ is the set of atoms, and $\sim$ is a stable congruence on $\mathcal{T}\cup\mathcal{AT}$, such that terms and atoms are never congruent.

6. Nondeterministic Transformations

In this paragraph we give transformation rules for constructing unification algorithms and show that they apply universally.

As basic datastructure we use **systems of multi-equations** Γ, where Γ is a multiset $\{M_i \mid i \in I\}$, M_i are multisets of objects in **OBJ** also called multi-equations, such that $V(\Gamma)$ is finite. The set of solutions of Γ is denoted by $SOL(\Gamma)$ is the set of substitutions ground on $V(\Gamma)$, such that for every $\sigma \in SOL(\Gamma)$, $M \in \Gamma$ and for all $s,t \in M$ we have $\sigma s = \sigma t$. Every unification problem Γ can be considered as a system of multi-equations. Let $VAR(\Gamma)$ be the set of variables that occur as elements in some multi-equation in Γ and let $OBJ(\Gamma)$ be the set of nonvariable objects that occur in some multi-equation from Γ.

Multisets [DM79] are like sets but allow multiple occurrences of the same element. We use the set-theoretic operators $\cup, \cap, -, \in$ for multisets in their obvious meaning.

We compare two sets of solutions as follows: $S_1 \subseteq_W S_2$, iff $S_{1|W} \subseteq S_{2|W}$ and $S_1 =_W S_2$ iff $S_1 \subseteq_W S_2$ and $S_1 \supseteq_W S_2$.

Assume given a unification problem Γ_0 for which we want to know all solutions. Since in general there is an infinite number of such solutions it is useful either to have a finite representation for all solutions or at least to compute further constraints on the solutions. We use the method of applying transformations to multi-equation systems. Such a transformation is denoted as $\Gamma \Rightarrow_W \Delta$, where Γ, Δ are systems of multi-equations and W is a finite set of variables. In the following we give rules that specify classes of transformations. We will define solved forms of multi-equations. As solution for a system Γwe will accept a transformation $\Gamma \Rightarrow_{V(\Gamma)} \Delta$, where Δ is a system in solved form with $SOL(\Gamma) =_W SOL(\Delta)$. Usually, a set of solved forms may be necessary to represent the solutions of Γ adequately.

6.1 Definition. We say a specific transformation $\Gamma \Rightarrow_W \Delta$ is **complete**, iff $SOL(\Gamma) =_W SOL(\Delta)$.
We say a rule is complete, iff every application provides a complete transformation. ■

Obviously, the application of transformations is transitive: If $\Gamma_1 \Rightarrow_W \Gamma_2$ is complete and $\Gamma_2 \Rightarrow_W \Gamma_3$ is complete, then $\Gamma_1 \Rightarrow_W \Gamma_3$ is complete.

We use the following conventions for denoting the rules:
$\Rightarrow_W$ denotes the transformation relation with respect to W.
M denotes a multi-equation and Γ denotes a system of multi-equations.

6.2 Definition. The basic rule set $\mathcal{BRS}$ is defined as follows:
Rule: Equal objects. $\{M\} \cup \Gamma \Rightarrow_W \{M-\{s\}\} \cup \Gamma$
if M contains s more than once.

Rule: Trivial multi-equation. $\Gamma \Rightarrow_W \Gamma - \{M\}$,
if M is a singleton.

Rule: Merging. $\{M_1\} \cup \{M_2\} \cup \Gamma \Rightarrow_W \{M_1 \cup M_2\} \cup \Gamma$,
if $M_1 \cap M_2 \neq \emptyset$

Rule: Auxiliary variables $\{M\} \cup \Gamma \Rightarrow_W \{M - \{x\}\} \cup \Gamma$,
if x is a variable with $x \notin W$ and x does nor occur elsewhere in Γ and M and there exists a term $t \in M - \{x\}$, such that $\{x \leftarrow t\} \in$ **MAP**.

Rule: Unfolding. $\{\{t\} \cup M\} \cup \Gamma \Rightarrow_W \{\{s\} \cup M\} \cup \Gamma \cup \langle \tau \rangle$
if $\tau s = t$ and $V(s)$ consists of new variables, $DOM(\tau) = V(s)$, and $DOM(\tau) \cap I(\tau) = \emptyset$.

Rule: Replacement. $\{\{s_2\} \cup M_2\} \cup \{\{s_1, t_1\} \cup M_1\} \Rightarrow_W \{\{\{x \leftarrow t_1\}s\} \cup M_2\} \cup \{\{s_1, t_1\} \cup M_1\}$
if s is an object, x a variable, $\{x \leftarrow s_1\}$ a mapping, such that $\{x \leftarrow s_1\}s = s_2$

Rule: Application of substitution: $\{\{x,t\} \cup M\} \cup \Gamma \Rightarrow_W \{\{x,t\} \cup M\} \cup \{x \leftarrow t\}\Gamma$,
if $\{x \leftarrow t\} \in$ **MAP**.

Rule: Partial solution. $\Gamma_1 \cup \Delta \Rightarrow_W \Gamma_2 \cup \Delta$
if $\Gamma_1 \Rightarrow_{V(\Gamma_1)} \Gamma_2$ is complete and all variables in $V(\Gamma_2) - V(\Gamma_1)$ are new variables.

6.3 Theorem. All the rules in $\mathcal{BSR}$ are complete.

Proof. The completeness of the equal-objects rules, the trivial multi-equations rule and the merging rule is trivial.

Auxiliary variables :

Obviously $SOL(\{M\} \cup \Gamma, W) \subseteq_W SOL(\{M - \{x\}\} \cup \Gamma)$. Let $\sigma \in SOL(\{M - \{x\}\} \cup \Gamma)$. Without loss of generality we can assume that $x \notin DOM(\sigma)$. Let t be the term in $M - \{x\}$ such that $\{x \leftarrow t\} \in$ **MAP**. Construct σ' such that $DOM(\sigma') = DOM(\sigma) \cup \{x\}$, $\sigma' = \sigma\ [DOM(\sigma)]$ and $\sigma'x := \sigma t$. Then σ' is a mapping, since $\{x \leftarrow \sigma t\} = (\sigma\{x \leftarrow t\})_{|\{x\}}$. σ' is also a solution of $\{M\} \cup \Gamma$ with $\sigma = \sigma'[W]$, since $x \notin W$.

Unfolding:

Let σ be a solution of $\{\{t\} \cup M\} \cup \Gamma$ with $DOM(\sigma) \subseteq V(\{\{t\} \cup M\} \cup \Gamma)$. Then $\sigma\tau$ is a solution of $\{\{s\} \cup M\} \cup \Gamma \cup \langle \tau \rangle$: we have $\sigma\tau s = \sigma t$, and $\sigma = \sigma\tau\ [V(\{\{t\} \cup M\} \cup \Gamma)]$, hence $\sigma\tau$ solves $\{\{s\} \cup M\} \cup \Gamma$. For $x \in DOM(\tau)$ we have $\sigma\tau(\tau x) = \sigma(\tau\tau)x = \sigma\tau x$, hence $\sigma\tau$ solves $\langle \tau \rangle$.

Let σ be a solution of $\{\{s\} \cup M\} \cup \Gamma \cup \langle \tau \rangle$. Then $\sigma x = \sigma\tau x$ for all $x \in DOM(\tau)$, hence $\sigma s = \sigma\tau s = \sigma t$.

Replacement:

We can assume that $x \in V(s)$. Let σ be a solution of $\{\{s_2\} \cup M_2\} \cup \{\{s_1, t_1\} \cup M_1\}$. Then $\sigma s_1 = \sigma t_1$, hence $\sigma\{x \leftarrow t_1\}s = \sigma\{x \leftarrow s_1\}s = \sigma s_2$. This shows one direction, the other direction is a symmetric case.

Application of substitution:

Follows from repeated application of the replacement rule.

Partial solution:

If $\Gamma_1 \Rightarrow_{V(\Gamma 1)} \Gamma_2$ is complete and all variables in $V(\Gamma_2) - V(\Gamma_1)$ are new variables.

Let σ be a solution of $\Gamma_1 \cup \Delta$. Since $\Gamma_1 \Rightarrow_{V(\Gamma 1)} \Gamma_2$ is complete, there exists a solution τ of Γ_2 with $\sigma = \tau\ [V(\Gamma_1)]$. Since $DOM(\sigma) \cap DOM(\tau) = V(\Gamma_1)$, we can define $\theta := \tau \cup \sigma$. Obviously, this is a solution of $\Gamma_2 \cup \Delta$.

Let σ be a solution of $\Gamma_2 \cup \Delta$. Since $\Gamma_1 \Rightarrow_{V(\Gamma 1)} \Gamma_2$ is complete, there exists a solution τ of Γ_1 with $\sigma = \tau\ [V(\Gamma_1)]$. Since $DOM(\sigma) \cap DOM(\tau) = V(\Gamma_1)$, we can define $\theta := \tau \cup \sigma$. Obviously, this is a solution of $\Gamma_1 \cup \Delta$. ■

6.4 Proposition. The equal terms rule, the trivial multi-equations rule and the merging rule can be applied only a finite number of times.

Proof. Obvious, since the number of multi-equations and the number of terms in Γ is decreased.■

A system of multi-equations, where none of the rules 'equal terms', 'Trivial multi-equations' and merging can be applied, is called **merged.**

This generally applicable rules have the practical advantage that a unification algorithm can solve partially a unification problem and that the variables introduced by unifiers may contain some "old" variables.

To obtain a similar proof for this fact within the framework of [Go88] would be hard, since his substitution systems behave as if all codomains are renamed away.

6.5 Definition. Let Γ be a system of multi-equations.

A set of pairs $\{(x_i,t_i) \mid i = 1,\ldots,n\}$ is called a **cycle,** iff x_i,t_i are in the same multi-equation of Γ, where the x_i's are variables and the t_i's are nonvariable objects, and $x_i \in V(t_{i+1})$ for $i = 1,\ldots,n-1$ and $x_n \in V(t_1)$.

6.6 Definition. Let Γ be a merged system of multi-equations.

i) a multi-equation M is in **solved form,** iff M contains a term t, such that $M - \{t\}$ is a set of variables $\{x_1,\ldots,x_n\}$ (t maybe a variable), such that $\tau_M := \{x_1 \leftarrow t,\ldots,x_n \leftarrow t\}$ is a mapping and $V(t) \cap V(M-\{t\}) = \emptyset$.

ii) Γ is called **solved,** if every multi-equation M in Γ is in solved form and Γ contains no cycles.

6.7 Proposition. Every solved system is solvable and has a unitary representation.

Proof. Let Γ be sequentially solved system. We partition every multi-equation M_i into $M_i := M_{i0} \cup \{t_i\}$, where M_{i0} is a multiset of variables, such that $\{x \leftarrow t\}$ is a substitution for every $x \in M_{i0}$.

We can assume that in the case that t_i is a variable, ihe variables M_{i0} do not occur elsewhere in Γ

(by applying the appropriate substitution.

We introduce a transitive ordering < on multi-sets generated by the pairs $M_i < M_j$, iff $M_{i0} \cap V(t_j) \neq \emptyset$.

This ordering is cycle-free, since Γ contains no cycles.

Let M_i be minimal with respect to this ordering and let M_j be direct succesor of M_i, then by applying the substitution $\{x_1 \leftarrow t_i, \ldots, x_n \leftarrow t\}$ for $M_{i0} = \{x_1, \ldots, x_n\}$, we obtain a system Γ' that is smaller in the sense that there is a smaller number of generating pairs for the ordering. Note that the application has an effect only on the terms t_k and that Γ' is a solved system.

Hence Γ can be brought into a form such that it is solved and the ordering < is empty.

Now the substitution σ that represents all solutions is the union of all substitiuions for every multi-equation.

It remains to show that this substitution is indeed a representation.

Assume there is a solution θ of (Γ, W). Then $\theta\sigma = \theta$: Consider an $M_i = M_{i0} \cup \{t_i\}$ and $x \in M_{i0}$. Then $\theta\sigma x = \theta t_i$ by definition of σ and $\theta t_i = \theta x$ since $\theta$is a solution. For variables $y \in V(t_i)$ we have $\theta\sigma y = \theta y$, since $y \notin DOM(\sigma)$.

Let λ be such that $\lambda\sigma$ is ground on $V(\Gamma) \cup W$. Then $\lambda\sigma$ is a solution, since for $x,y \in M_{i0}$ we have $\lambda\sigma x = \lambda\sigma y$, since $\sigma x = \sigma y$. Furthermore we have $\lambda\sigma x = \lambda t_i$ by definition of σ and $\lambda\sigma t_i = \lambda t_i$, since $DOM(\sigma) \cap V(t_i) = \emptyset$. ■

A unification algorithm can now be described as a set of rules that describe transformations of equation systems. These rules are in general considered as nondeterministic, where the nondeterminism has two instances: "don´t-know" and "don´t care" nondeterminism. "Don´t know" means that we have to choose between several alternatives and that for a complete algorithm, all alternatives have to be explored, whereas "don´t care" means that we can choose one alternative and forget the other without loosing completeness.

We haven´t said what completeness means:

A unification algorithm is **complete**, iff

for every Γ and every solution σ of Γ, there exists a system of multi-equations Δ that can be reached from Γ using correct transformations specified by the algorithm, the transformations are with respect to $V(\Gamma)$, Δ is in solved form, and σ is a solution of Δ.

Given a system of multi-equations Γ, every complete unification algorithm can be used to enumerate a set of representatives for Γ, if a breadth-first-like method is exploited to search for all reachable solved systems.

The advantage of describing an algorithm by nondeterministic rules over a description using a disjunction of systems of multi-equations is that for the nondeterministic approach it is easier to handle cases, where an infinite set of alternatives has to be explored.

In order to handle such sets of alternatives, we introduce the notion of a complete sets of alternatives:

6.8 Definition. Let Γ, Γ_i, $i \in I$ be systems of multi-equations and let W be a set of variables, such that $\Gamma \Rightarrow_W \Gamma_i$ is a correct transformation for all $i \in I$.
Then $\{\Gamma \Rightarrow_W \Gamma_i \mid i \in I\}$ is a **complete set of alternatives**, iff
$\mathrm{SOL}(\Gamma) =_W \bigcup\{\mathrm{SOL}(\Gamma_i) \mid i \in I\}$. ■

Complete sets of alternatives can be combined:
6.9 Lemma. If $\{\Gamma \Rightarrow_W \Gamma_i \mid i \in I\}$ and $\{\Gamma_{i0} \Rightarrow_W \Delta_j \mid j \in J\}$ are complete sets of alternatives, where $i_0 \in I$, then also $\{\Gamma \Rightarrow_W \Gamma_i \mid i \in I-\{i_0\}\} \cup \{\Gamma \Rightarrow_W \Delta_j \mid j \in J\}$ is a complete set of alternatives.

On the basis of a complete (nondeterministic) algorithm $\mathcal{S}_{sing}$ for solving the equation s = t, i.e., for solving sytems {{s,t}} it is easy to construct a complete algorithm for arbitrary systems of equations Γ. For this purpose we can assume (w.l.o.g.) that Γ is a system of multi-equations, where every multi-equation contains exactly two terms. Furthermore we can assume that $\mathcal{S}_{sing}$ has as result a system $\langle\tau\rangle$ that comes from a unifier τ with $\mathrm{DOM}(\tau) \subseteq \mathrm{V}(s,t)$ and $\mathrm{VCOD}(\tau)$ consists only of new variables.

The algorithm $\mathcal{S}_{sys}$ works as follows:

Input: Γ
$\Delta := \emptyset$
while $\Gamma \neq \emptyset$ *do*
  Let {s,t} be a multi-equation in Γ.
  Let $\sigma_{s,t}$ be some output of $\mathcal{S}_{sing}(s,t)$
  Let $\Delta := \Delta \cup \langle\sigma_{s,t}\rangle$ and let let $\Gamma := \sigma_{s,t}(\Gamma - \{s,t\})$
endwhile
Output: Δ

6.10 Proposition. Given a complete algorithm $\mathcal{S}_{sing}$ for single equations, the algorithm $\mathcal{S}_{sys}$ is a complete unification algorithm for systems of equations.
Proof. Obviously, $\mathcal{S}_{sys}$ terminates, since Γ is reduced in every step. Furthermore, Δ is solved, since it has no cycles due to the condition that $\sigma_{s,t}$ introduces only new variables.
The completeness is shown by induction.
Let θ be a solution of Γ. Since $\mathcal{S}_{sing}(s,t)$ is complete, there is a nondeterminstic execution of $\mathcal{S}_{sing}$, such that $\sigma_{s,t}$ is the output, such that there is a solution θ' of $\langle\sigma_{s,t}\rangle$ and $\theta = \theta\ [\mathrm{V}(s,t)]$. Theorem 6.3 shows that application of $\sigma_{s,t}$ is a complete step. Hence no solution is lost. ■

Now we consider solution methods for strongly embedded unification algebras. Note that the notion of strong embedding can be applied to the embedding of the theory of AC into AC1 [HS87], for restricted unification and matching [BHS87, Bü86] and for unification in sorted equational theories as considered in [Sch88, Sch86b].

6.11 Lemma. If $\mathcal{A}$ is strongly embedded in $\mathcal{B}$, then for all equation systems Γ containing only objects from $\mathcal{A}$: $\mathrm{SOL}_{\mathcal{A}}(\Gamma) = \mathrm{SOL}_{\mathcal{B}}(\Gamma) \cap \mathbf{MAP}_{\mathcal{A}}$. ■

Let $\mathcal{A}$ be strongly embedded in $\mathcal{B}$, and let Γ be an equation systems containing only objects from $\mathcal{A}$. Suppose, there is a complete algorithm $\mathcal{S}_{\mathcal{B}}$ for solving equation systems in $\mathcal{B}$ and a complete algorithm $\mathcal{W}_{\mathcal{B}\to\mathcal{A}}$ that takes a mapping in $\mathbf{MAP}_{\mathcal{B}}$ and generates an instance in $\mathbf{MAP}_{\mathcal{A}}$. Then we can use the combined algorithm $\mathcal{W}_{\mathcal{B}\to\mathcal{A}}\circ\mathcal{S}_{\mathcal{B}}$ as unification algorithm for $\mathcal{A}$. $\mathcal{W}_{\mathcal{B}\to\mathcal{A}}\circ\mathcal{S}_{\mathcal{B}}$ works as follows: first it computes a unifier with respect to $\mathcal{B}$ and afterwards instantiates it such that the instance is an $\mathcal{A}$-mapping. We give a condition for completeness of $\mathcal{W}_{\mathcal{B}\to\mathcal{A}}\circ\mathcal{S}_{\mathcal{B}}$..
We say the "weakening-algorithm" $\mathcal{W}_{\mathcal{B}\to\mathcal{A}}$ is **$\mathcal{A}$-$\mathcal{B}$-complete**, iff the following holds: for input σ, and every $\mathcal{B}$-instance $\theta \in \mathbf{MAP}_{\mathcal{A}}$, it can generate a $\mathcal{B}$-instance τ of σ, such that $\tau \in \mathbf{MAP}_{\mathcal{A}}$ and θ is a $\mathcal{A}$-instance of τ.

6.12 Theorem. Under the conditions above, and if $\mathcal{W}_{\mathcal{B}\to\mathcal{A}}$ is $\mathcal{A}$-$\mathcal{B}$-complete, then $\mathcal{W}_{\mathcal{B}\to\mathcal{A}}\circ\mathcal{S}_{\mathcal{B}}$ is a complete unification algorithm for $\mathcal{A}$.

Proof. Let Γ be a system of equations with respect to $\mathcal{A}$, and let $\theta \in \mathrm{SOL}_{\mathcal{A}}(\Gamma)$. Then $\theta \in \mathrm{SOL}_{\mathcal{B}}(\Gamma)$ and $\mathcal{S}_{\mathcal{B}}$ gives a $\mathcal{B}$-unifier σ, such that θ is a $\mathcal{B}$-instance of σ. $\mathcal{W}_{\mathcal{B}\to\mathcal{A}}$ now generates a $\mathcal{B}$-instance τ of σ with $\tau \in \mathbf{MAP}_{\mathcal{A}}$, such that θ is a $\mathcal{A}$-instance of τ. Hence $\mathcal{W}_{\mathcal{B}\to\mathcal{A}}\circ\mathcal{S}_{\mathcal{B}}$ is complete. ■

The hard part for such a strong embedding is to show that there exists such an $\mathcal{A}$-$\mathcal{B}$-complete weakening algorithm. Once this is shown, we can apply Theorem 6.12. Such $\mathcal{A}$-$\mathcal{B}$-complete weakening algorithms exist for the embedding of AC into AC1, for the case of restricted unification, and for unification in some special sorted equational theories.

7. Unification Algebras with Dimension

In this section we assume that all equation systems have only multi-equations with exactly two elements.

We define the **rank** of a system of equations Γ as the number of equations in it and denote it by $|\Gamma|$ [LMM87]. The number of variables in Γ is denoted by $|V(\Gamma)|$.
A single equation $s = t$ in Γ is **redundant**, iff $\mathrm{SOL}(\Gamma) = \mathrm{SOL}(\Gamma - \{s = t\})\ [V(\Gamma)]$. A system of equations Γ is called **redundant**, iff it contains a redundant equation, otherwise it is called **irredundant**.

7.1 Definition. We say a unification algebra $\mathcal{A}$ **has a (linear) dimension**, iff for every solvable system of equations Γ : $|\Gamma| > |V(\Gamma)|$ implies that Γ is redundant.
As extension we say a unification algebra $\mathcal{A}$ **has a f(n)-dimension**, iff for every solvable system of equations Γ : $|\Gamma| > f(|V(\Gamma)|)$ implies that Γ is redundant. ■

Abelian groups [LBB84], the empty theory [Ro65, LMM87] and vector spaces have a dimension (as defined above). Below we will show that Abelian semigroups and Abelian monoids also have a dimension.

The theory of associativity with only axiom $\{f(x, f(y, z)) = f(f(x, y), z)\}$ does not have a linear dimension: Consider the system $\langle abx=xba, abax=xaba\rangle$, which has only $\{x \leftarrow aba\}$ as solution, but the first equation has also $\{x \leftarrow a\}$ as solution, which is not a solution to the second one, and the second one has $\{x \leftarrow abaaba\}$ as solution, which in turn is not a solution to the first one.
The theory of commutativity with only axiom $\{f(x, y) = f(y, x)\}$ does not have a linear dimension: Consider the system $\langle f(f(x,x), f(a, b)) = f(f(x, a), f(x, b)), f(f(x,x), f(b, c)) = f(f(x, b), f(x, c))\rangle$, which has only $x = b$ as solution, but the first equation has as solutions: $\{\{x \leftarrow a\}, \{x \leftarrow b\}\}$ and the second equation has as solutions: $\{\{x \leftarrow b\}, \{x \leftarrow c\}\}$.
The theory of Boolean rings does not have an n-dimension for the function n: the proposition below shows, that f(n) is exactly $2^{kn} - 1$, where k is the number of free constants in the signature, since unification in Boolean algebras is a special case of solving equations in a term algebra generated by a primal, finite algebra.

More generally, if we consider unification in primal algebras [BS81, Ni88] we have the following:

7.2 Proposition. If $\mathcal{A}$ is the term algebra generated by a finite algebra A with $|A| \geq 2$, then the corresponding dimension function f(n) is $|A|^n - 1$, which can easily be verified. If A is in addition primal, then the dimension function is exactly $|A|^n - 1$.

Proof. Let Γ be an irredundant, solvable system of equations with $|V(\Gamma)| = n$. Then there are at most $|A|^n$ possible different solutions to $|\Gamma|$. The possible solutions can be seen as vectors of length n over A. The difference $A^n - SOL(\Gamma)$ is the same as $\bigcup\{A^n - SOL(s_i = t_i) \mid \{s_i, t_i\} \in \Gamma\}$. Furthermore, the set $A^n - SOL(\Gamma)$ is not empty. Were there more than $|A|^n - 1$ different equations in Γ, then one set in the union above is redundant, hence one equation is redundant.
If A is primal, then every function is a term. This means that for two elements 0,1 in A we can construct the following terms: a (ground) term t_0 that is equal to 0 (and contains at most one variable), and terms t_v, where v is a vector in A^n, such that $t_v(w) = 0$ for $v \neq w$ and $t_v(v) = 1$.
The system $\Gamma := \{\{t_0, t_v\} \mid v \in A^n - \{(0,\ldots,0)\}\}$ contains no redundant equation, has at most n variables in $V(\Gamma)$ and Γ contains $|A|^n - 1$ equations. ■

7.3 Proposition. Let $\mathcal{A}$ be a unification algebra with dimension. Let Γ be a solvable system of equations, such that $|\Gamma| > |V(\Gamma)|$. Then there are at least $|\Gamma| - |V(\Gamma)|$ redundant equations in Γ.

Recall that $\mathcal{A}$ is called strongly embedded in $\mathcal{B}$, iff $V_{\mathcal{A}} = V_{\mathcal{B}}$, $\mathbf{OBJ}_{\mathcal{A}} \subseteq \mathbf{OBJ}_{\mathcal{B}}$ and $\mathbf{MAP}_{\mathcal{A}} \subseteq \mathbf{MAP}_{\mathcal{B}}$.

7.4 Lemma. If $\mathcal{A}$ is strongly embedded in $\mathcal{B}$, then for all equation systems Γ:

$$SOL_{\mathcal{A}}(\Gamma) = SOL_{\mathcal{B}}(\Gamma) \cap \mathbf{MAP}_{\mathcal{A}}.$$

7.5 Lemma. Let $\mathcal{A}$, $\mathcal{B}$ be unification algebras, such that $\mathcal{A}$ is strongly embedded in $\mathcal{B}$. Then: If $\mathcal{B}$ has a dimension, then $\mathcal{A}$ has a dimension.

Proof. Let Γ be an equation system containing only objects from $\mathbf{OBJ}_{\mathcal{A}}$, such that Γ is solvable with respect to $\mathcal{A}$ and let $|\Gamma| > |V(\Gamma)|$. Then Γ is also solvable with respect to $\mathcal{B}$. Hence there exists a proper subsystem Γ' of Γ, such that $SOL_{\mathcal{B}}(\Gamma') = SOL_{\mathcal{B}}(\Gamma)\ [V(\Gamma)]$. By Lemma 7.3 we have

$SOL_{\mathcal{A}}(\Gamma') = SOL_{\mathcal{B}}(\Gamma') \cap \mathbf{MAP}_{\mathcal{A}}$ and $SOL_{\mathcal{A}}(\Gamma) = SOL_{\mathcal{B}}(\Gamma) \cap \mathbf{MAP}_{\mathcal{A}}$. Hence $SOL_{\mathcal{A}}(\Gamma') = SOL_{\mathcal{A}}(\Gamma)\ [V(\Gamma)]$. This means that Γ is redundant. ■

We can give a sufficient criterion for $\mathcal{A}$ has a dimension:

7.6 Theorem. Let $\mathcal{A}$ be unitary. If for every equation $s = t$ there exists a most general unifier σ with $|VCOD(\sigma)| < |V(s,t)|$, then $\mathcal{A}$ has a dimension.

Proof. Let $\Gamma := \langle s_1 = t_1,\ldots,s_n = t_n\rangle$ be a solvable system of equations with $|V(s_1,t_1,\ldots,s_n,t_n)| < n$. If $n = 1$, then $|V(s_1,t_1)| = 0$, hence $s_1 = t_1$ holds and $\langle s_1 = t_1\rangle$ is redundant.
In order to prove the induction step, first assume that $s_1 = t_1$ is not redundant, otherwise we are ready. Let σ be a most general unifier of $s_1 = t_1$ that introduces only new variables and less than $|V(s_1,t_1)|$. Then the system $\langle\sigma\rangle \cup \langle\sigma s_2 = \sigma t_2,\ldots,\sigma s_n = \sigma t_n\rangle$ is equivalent to Γ due to Theorem 6.3 on the variables $V(\Gamma)$. The induction hypothesis applies to $\langle\sigma s_2 = \sigma t_2,\ldots,\sigma s_n = \sigma t_n\rangle$, since $|\langle\sigma s_2 = \sigma t_2,\ldots,\sigma s_n = \sigma t_n\rangle| < n\text{-}1$. This yields that one equation is redundant, say $\sigma s_j = \sigma t_j$. But then $s_j = t_j$ is redundant in Γ. ■

7.7 Theorem. Let $\mathcal{A}$ be unitary such that for every equation $s = t$ there exists a most general unifier σ with $|VCOD(\sigma)| < |V(s,t)|$. Let Γ be a solvable, irredundant system of equations.
Then there exists a most general unifier σ of Γ such that $|VCOD(\sigma)| < |V(\Gamma)| - |\Gamma| - 1$.

Proof. Let $\Gamma := \langle s_1 = t_1,\ldots,s_n = t_n\rangle$ be a solvable, irredundant system of equations.
Consider one step of solving this system sequentially. A general situation is that $\langle\tau\rangle \cup \langle s_1 = t_1,\ldots,s_n = t_n\rangle$ has to be solved.
Let σ be a most general unifier of $s_1 = t_1$ that introduces only new variables and less than $|V(s_1,t_1)|$. Then the system $\langle\sigma\tau\rangle \cup \langle\sigma s_2 = \sigma t_2,\ldots,\sigma s_n = \sigma t_n\rangle$ is equivalent to Γ due to Theorem 6.3 on the variables $V(\Gamma)$. Obviously, $|V(\langle\sigma s_2 = \sigma t_2,\ldots,\sigma s_n = \sigma t_n\rangle \cup VCOD(\sigma\tau)| < |V(\Gamma)|$.
Since no equation in Γ is redundant, the same holds for the derived system, and hence every step reduces the number of variables by 1. Thus the equation $|VCOD(\sigma)| < |V(\Gamma)| - |\Gamma| - 1$ holds for the finally constructed most general unifier σ of Γ. ■

7.8 Proposition. Abelian monoids and Abelian semigroups have a dimension.

Proof. Follows, since Abelian monoids and Abelien semigroups can be strongly embedded into Abelian groups, and Abelian groups have a dimension [LBB84]. ■

As mentioned above, the theories of associativity and commutativity don't have a linear dimension. However, a defect lemma holds for both theories, stating that for every Γ there exists a complete set of unifiers that use at most $|V(\Gamma)| - 1$ variables in their codomains. Similarly, for Boolean rings it is well-known that the most general unifier requires at most $|V(\Gamma)|$ variables in its codomain. Nevertheless, the impact of such a defect to the redundancy of systems of equations is unclear and should be investigated in the future.

8. Minimal and Optimal Representations

Unification theory has as an important notion the definition of what it means for a complete set to be minimal, and a unification hierarchy for equational theories depending on the existence and cardinality of minimal complete sets. We show that for unification algebras extended by constants such a usual notion is available, i.e. in particular for term algebras. Furthermore some counterexamples are given that in the general case there is no satisfactory definition of minimality.

8.1 Definition. Let cU be a representation of solutions of the unification problem Γ.

i) We say cU is in addition **minimal**, iff no proper subset of cU is complete.

ii) Let U_1 and U_2 be minimal representations.
Then U_1 is **more general** than U_2 (modulo $V(\Gamma)$), iff $\forall \sigma \in U_2\ \exists \tau \in U_1$ and $\tau \supseteq_{V(\Gamma)} \sigma$.
We say U_1 is **properly more general** than U_2, if U_2 is not more general than U_1.
We say U_1 and U_2 are **equivalent** representations, iff U_1 is more general than U_2 and vice versa.

iii) A minimal representation cU is in addition an **optimal** representation, iff there is no properly more general minimal representation.

iv) A unifier σ is **maximal**, if for every unifier τ with $\tau \supseteq_{V(\Gamma)} \sigma$ we have also $\sigma \supseteq_{V(\Gamma)} \tau$ ■

In several theories and fields one has a measure for the set of solutions. For example for linear systems of equations of a field the set of solution is a vectorspace and has a dimension. A translation of this in terms of unification theory would be the number of variables (or parameters) in the codomain of a a complete representation. There are a lot of interesting theories where the number of parameters depends in a fixed way from the number of variables in the problem. For Boolean rings, Abelian groups, the empty theory and linear equations the number of parameters is not greater than the number of variables in the original problem. In [LMM87] such a notion of dimension is considered for the free term algebra.

8.2 Lemma. Let U be an minimal representation of the solutions of Γ. Then U is optimal iff there is no other properly (not necessarily minimal) more general representation.

Proof. "$\Leftarrow$" is trivial.

"$\Rightarrow$": Let U be an optimal representation and let U_0 be a representation that is properly more general than U. Since U_0 is properly more general, there exists a substitution $\tau \in U_0$, such that $\sigma \supseteq_{V(\Gamma)} \tau$ is false for all $\sigma \in U$. On the other hand there exists a $\sigma_0 \in U$, such that $\sigma_0 \subseteq_{V(\Gamma)} \tau$. Let $U' := (U - \{\sigma_0\}) \cup \{\tau\}$. This is a representation that is properly more general than U. Let $U'' := U' - \{\lambda \in U \mid \lambda \subseteq_{V(\Gamma)} \tau\}$. This is a minimal representation that is properly more general than U, which is a contradiction. ■

8.3 Lemma. Let U be an minimal representation of the solutions of Γ. Then U is optimal iff it consists only of maximal unifiers.

Proof. "⇒": Let U be optimal and let σ be a unifier in U that is not maximal. Then there exists a unifier τ such that $\tau \sqsupseteq_{V(\Gamma)} \sigma$ but not $\sigma \sqsupseteq_{V(\Gamma)} \tau$. Let U' be $(U - \{\sigma\}) \cup \{\tau\}$. Then U' is a properly more general representation, a contradiction tio Lemma 8.2.
"⇐": trivial. ■

We show below that for sets of unifiers in term algebras modulo an equational theory the two notions collapse to the notion of minimality with respect to unification in termalgebras. Hence we can use the usual examples to show that minimal and optimal sets may not exist [FH83].
In general, minimal sets of unifiers are not elementwise equivalent nor have a fixed cardinality, since for example the set of all ground solutions is minimal in this sense. Unfortunately, the same holds for optimal sets of unifiers, as the next example shows.

8.4 Example.

The following part of the constructions is always the same in the four parts of this lemma.
We construct a term-algebra, such that the initial algebra consists of the set of naturals $\mathbb{N}$ including zero. There is a unary function symbol f with $f(0) = 0$ and $f(n) = 1$ for all $n \geq 1$. Let $\mathbb{P} := \{n \in \mathbb{N} \mid n \geq 1\}$.
For every example we select a fixed set S of subsets of $\mathbb{P}$, such that for every $A \in S$ there is a unary function symbol g_A, such that $g_A(\mathbb{N}) = A$ and such that $\cup S = \mathbb{P}$. We assume that an appropriate set of ground equations is given.
The unification problem is $\langle f(x) = 1 \rangle$, which has as solution the set $\{ \{x \leftarrow n \mid n \in \mathbb{P}\}$.
It is obvious that every maximal unifier has the form $\{x \leftarrow g_A(y)\}$. Now the problem to find optimal sets is equivalent to find "optimal" coverings of $\mathbb{P}$ using sets in S. So in the following we give only the set-theoretic part of the arguments

i) Optimal representations of the same size may be uncomparable.
Let $S := \{\mathbb{P}-\{1\}, \mathbb{P}-\{2\}, \mathbb{P}-\{3\}\}$.
Then $\mathbb{P}-\{1\} \cup \mathbb{P}-\{2\} = \mathbb{P}-\{1\} \cup \mathbb{P}-\{3\} = \mathbb{P}$, but the coverings are not comparable.

ii) There exists a unification problem such that for every $n = 2,3,\ldots$ and even for $n = \infty$ there exist optimal representations.
Let the following sets be in S:
a) Every set $\{2n\text{-}1,2n\}$ for $n \geq 1$.
b) Every set $\{2n\} \cup \{k \mid k \geq 2n+1 \text{ and } k \text{ is odd}\}$ for $n \geq 1$.
c) Every set $\{2n\text{-}1\} \cup \{k \mid k \geq 2n+2 \text{ and } k \text{ is even}\}$ for $n \geq 1$.
These sets are all maximal in the sense that they cannot be compared by $\subseteq$.
The finite coverings of $\mathbb{P}$ are:
For every $m \geq 2$ we have the covering of $\mathbb{P}$ with m elements from S
The m-2 smallest sets of a), $\{2m\text{-}2)\} \cup \{k \mid k \geq 2m\text{-}1) \text{ and } k \text{ is odd}\}$ and $\{2m\text{-}3\} \cup \{k \mid k \geq 2m \text{ and } k \text{ is even}\}$.
For the infinite covering take all sets of type a).

iii) There may exist a Γ with a minimal representation consisting of two elements. But no optimal representation exists.

Let S consist of the following sets:

a) The set of positive odd numbers

b) for $n \geq 1$ the sets $A_n := \{k \mid 1 \leq k \leq 2n-1\} \cup \{k \in \mathbb{P} \mid k \text{ is even}\}$.

Then the sets A_n are an ascending chain with respect to $\subseteq$ without maximal element in S. There is no optimal covering, since the sets A_n are necessary, but not maximal. A minimal complete covering of cardinality 2 is A_1 and the odd numbers. ■

8.5 Lemma. For a unitary unification problem, all optimal representations are equivalent.

Proof. Trivial. ■

In the following we show that for unification in unification algebras extended by free constants (in particular in free term algebras) the notion of minimal representation and optimal representation are the same and that all minimal sets are equivalent [FH83].

8.6 Theorem. Let A be a unification algebra extended by constants and let Γ be a unification problem

i) Every minimal representation is optimal.

ii) All minimal representations are equivalent and of the same cardinality.

Proof. Follows by standard arguments [FH83]. ■

Now we can define a special unification type (extended by constants), which corresponds exactly to the usual one: [Si75, Si88]

8.7 Definition. Let A be a unification algebra extended by constants.

i) Let Γ be a solvable unification problem .
   Γ is called **unitary**, if an optimal representation exists that is a singleton.
   Γ is called **finitary**, if a finite optimal representation exists.
   Γ is called **infinitary**, if an infinite optimal representation exists.
   Γ is called **nullary**, if no optimal representation exists.

ii) A is called **unification based**, iff no solvable Γ is nullary
   A is called **unitary**, if all solvable Γ are unitary.
   A is called **finitary**, if all solvable Γ are finitary.
   A is called **infinitary**, if A is unification based and some Γ isf infinitary.
   A is called **nullary**, if some Γ is nullary. ■

9. Classes of Unification Algebras: How to Obtain the Martelli-Montanari Algorithm

In this section we investigate some classes of unification algebras and give unification procedures for these classes. In particular we show how an algorithm in the Martelli-Montanari style can be used for

solving unification problems in unification algebras corresponding to free term algebras and the term algebra of rational terms.
The first part is a preliminary and provides the required notions and some connections between them. Throughout the whole section we assume that unification algebras are unsorted and nontrivial and that **TERM** = **OBJ**, hence we can speak of terms instead of objects.

9.1 Properties of Unification Algebras.

In the following we define some typical properties of free term algebras and show how to use them to construct complete unification steps.

9.1.1 Definition. Let $\mathcal{A}$ = (**V**,**T**,**SUB**) be a unification algebra.

i) A term $t \in$ **T** is **Ω-free**, iff for all $\sigma,\tau \in$ **SUB**: $\sigma t = \tau t \Rightarrow \sigma = \tau$ [V(t)].
ii) A unification algebra $\mathcal{A}$ is **Ω-free**, iff all terms are **Ω-free**.
iii) A unification algebra $\mathcal{A}$ is **decomposable,** if for all nonvariable objects s,t: if $\langle s = t \rangle$ is solvable, then there exists a nonvariable object r and $\sigma,\tau \in$ **SUB**, such that $\sigma r = s$ and $\tau r = t$. ■

An Ω-free term algebra [Sz82] (see also [BHS87]) and in particular the free term-algebra are Ω-free in this sense.

We need a notion of subterms in order to characterize properties of unification algebras.
9.1.2 Definition. A term s is a **subterm** of t, if there exists a nonvariable term r, such that $x \in V(r)$ and $\{x \leftarrow s\}r = t$.
We denote this by s *sub* t. ■

9.1.3 Definition. Let $\mathcal{A}$ be a unification algebra.

i) $\mathcal{A}$ is called **subterm-cycle-free**, if *sub* does not contain cycles.
ii) $\mathcal{A}$ is called **subterm-finite**, if every term contains at most finitely many subterms. Accordingly we say a term t is **subterm-finite**, if t has a finite number of subterms.
iii) $\mathcal{A}$ is called **collapsing**, iff there is a nonvariable term t with $x \in V(t)$ and a term s such that $\{x \leftarrow s\}t$ is a variable. Otherwise $\mathcal{A}$ is called **collapse-free**.
iv) $\mathcal{A}$ is called **regular**, iff for all $\sigma \in$ SUB and all $t \in$ **T**: $V(\sigma t) = \bigcup\{V(\sigma x) \mid x \in V(t)\}$. ■

If we consider usual term algebras modulo an equational theory, then we have the following analogies. A simple theory has a subterm-cycle-free term-algebra as unification algebra. However, simplicity depends not only on the 'structure' of the equational theory, but also on the signature. An almost collapse-free theory has a collapse-free term-algebra (modulo theory) as unification algebra, and vice versa. Regular equational theories provide regular unification algebras, but since regularity of equational theory depends also on syntax, there are examples of nonregular equational theories, which provide regular unification algebras. An example is the theory axiomatized by $E := \{f(x,y) = f(x,z)\}$. This theory is not regular. However, the provided unification algebra is regular, since y does not count as variable in f(x y) due to our definition.

We investigate some properties of the subterm relation.

9.1.4 Lemma.

i) In Definition 9.1.2, we can assume that x is a new variable.

ii) $x \in V(t) \Rightarrow x$ *sub* t

Proof. i) Let $\{x \leftarrow s\}r = t$. Then with a new variable x', we have $\{x \leftarrow s\}r = \{x \leftarrow s\}\{x' \leftarrow x\}\ \{x \leftarrow x'\}\ r = t$. The term $r' = \{x \leftarrow x'\}r$ is not a variable, furthermore $x' \in V(r')$. Hence $\{x' \leftarrow s\}r' = t$

ii) If $x \in V(t)$, we have $\{x' \leftarrow x\}\{x' \leftarrow x\}t = t$, where x' is a new variable and $t' := \{x' \leftarrow x\}t$ is a nonvariable term with $x' \in V(t')$. ■

The converse of Lemma 9.1.4 ii) may be false:

Consider the theory $E := \{f(x, x) = a\}$. Then $V(f(x,y)) = \{x,y\}$. Thus $\{x \leftarrow y\}f(x, y) = a$. This means y *sub* a, but $y \notin V(a)$.

9.1.5 Lemma. Let $\mathcal{A}$ be a regular, collapse-free unification algebra. Then

i) s *sub* t $\Rightarrow V(s) \subseteq V(t)$

ii) x *sub* t iff $x \in V(t)$.

iii) the relation *sub* is transitive.

iv) s *sub* t $\Rightarrow \sigma s$ *sub* σt

Proof. i) Holds, since $\mathcal{A}$ is regular.

ii) If $x \in V(t)$, then $\{x' \leftarrow x\}(\{x \leftarrow x'\}t) = t$, where x' is a new variable, hence x *sub* t. The other direction follows from i)

iii) Let r *sub* s *sub* t. There exist terms r', s', with $x \in V(r')$ and $y \in V(s')$ such that $\{x \leftarrow r\}r' = s$ and $\{y \leftarrow s\}s' = t$. We can assume that x and y are new variables. We have $\{x \leftarrow r\}\{y \leftarrow r'\}s' = \{y \leftarrow \{x \leftarrow r\}r'\}s' = \{y \leftarrow s\}s' = t$. Since $\mathcal{A}$ is regular, $x \in V(\{y \leftarrow r'\}s')$, and since $\mathcal{A}$ is collapse-free, $\{y \leftarrow r'\}s'$ is not a variable. Hence r *sub* t.

iv) s *sub* t means $\{x \leftarrow s\}s' = t$ for some s' with $x \in V(s')$. We can assume that x is a new variable. We can assume that $DOM(\sigma) \subseteq V(t)$, due to i). Applying σ gives $\sigma t = \sigma\{x \leftarrow s\}s' = (\sigma \cup \{x \leftarrow \sigma s\})\ s' = \{x \leftarrow \sigma s\}\ (\sigma s')$. Since $\mathcal{A}$ is regular $x \in V(\sigma s')$, and since A is collapse-free $\sigma s'$ is not a variable, thus σs is a subterm of σt. ■

9.1.6 Example. i) The relation *sub* may be transitive for a nonregular unification algebra: An example is the theory $E := \{f(x\ a) = f(a\ x) = b\}$.

ii) Consider the theory $\mathcal{E}$ axiomatized by $E = \{f(f(x)) = x\}$. The theory $\mathcal{E}$ is regular, Ω-free, decomposable, subterm-finite, and collapsing. In this theory, terms are equal, iff they contain the same variable and the same number of f's modulo 2. Ω-freeness holds since $f(s) =_{\mathcal{E}} f(t)$ implies that the number of f's in s and t is equal modulo 2. $\mathcal{E}$ is decomposable, since every term is an instance of f(x). The relation *sub* is not transitive: We have f(x) *sub* x *sub* f(x), but not f(x) *sub* f(x), since $\{y \leftarrow f(x)\}t = f(x)$ with $y \in V(t)$ implies t = y.

9.1.7 Lemma. $\mathcal{A}$ is regular, iff for all $\{x \leftarrow s\} \in SUB$, $t \in T$ with $x \in V(t)$:

$V(\{x \leftarrow s\}t) = (V(t)-\{x\}) \cup V(s)$.

Proof. Follows by a renaming technique. Rename all variables in I(σ) by new variables. Then the renamed σ is a product of its components. Use induction on the number of components. ■

9.1.8 Lemma. Every Ω-free theory is also regular.

Proof. Suppose there is a $\sigma \in SUB$ and a $t \in \mathbf{T}$ such that $V(\sigma t) \subset \cup\{V(\sigma x) | x \in V(t)\}$.

Let y be a variable in $\cup\{V(\sigma x) | x \in V(t)\} - V(\sigma t)$. Let x and z be two new different variables, such that $\{y \leftarrow x\}$ and $\{y \leftarrow z\}$ are substitutions. Then $\{y \leftarrow x\}\sigma t = \{y \leftarrow z\}\sigma t$, since y is not a variable in σt. Since $\mathcal{A}$ is Ω-free, we get $\{y \leftarrow x\}\sigma = \{y \leftarrow z\}\sigma\ [V(t)]$. This is not possible due to Lemma 2.13. We have reached a contradiction. ■

9.1.9 Lemma. Let $\mathcal{A}$ be an Ω-free unification algebra.

Then for variables x,y, nonvariable terms s,t with $x \in V(t)$: the collapse-equation $\{x \leftarrow s\}t = y$ implies that $V(t) = \{x\}$, $V(s) = y$ and $s \neq y$.

Proof. Assume $V(t) \supset \{x\}$. Since $\mathcal{A}$ is regular by Lemma 9.1.8, $V(t) = \{x,y\}$.

Now $\{y \leftarrow t\}\{x \leftarrow s\}t = ID_T\ t$, hence by Ω-freeness, we have $\{y \leftarrow t\}\{x \leftarrow s\} = ID_T\ [\{x,y\}]$, and finally the contradiction t = y. We conclude $V(t) = \{x\}$.

$V(s) = \{y\}$ holds, since $\mathcal{A}$ is regular.

Assume for contradiction that s = y. Then $\{x \leftarrow y\}t = y$. If $y \notin V(t)$, then $x = \{y \leftarrow x\}\ \{x \leftarrow y\}t = t$, which is not possible. If $y \in V(t)$, we get the same contradiction as above. ■

9.1.10 Lemma. If $\mathcal{A}$ is Ω-free and subterm-cycle-free, then $\mathcal{A}$ is collapse-free.

Proof. Assume, $\mathcal{A}$ is not collapse-free. Then there exists a term s and a nonvariable term t with $x \in V(t)$, such that $\{x \leftarrow s\}t = y$ for some variable y. If $y \notin V(\{x \leftarrow s\}t)$, then all terms in $\mathcal{A}$ are equal, which contradicts our assumption that $\mathcal{A}$ is nontrivial. Hence $y \in V(\{x \leftarrow s\}t)$. Lemma 9.1.9 shows that $V(t) = \{x\}$, $V(s) = \{y\}$ and $s \neq y$. Now $\{y \leftarrow t\}\{x \leftarrow s\}t = \{y \leftarrow t\}y = t$ and $\{y \leftarrow t\}\{x \leftarrow s\}t = \{x \leftarrow \{y \leftarrow t\}s\}t$ imply that $\{y \leftarrow t\}s$ is a subterm of t. Since s is not a variable, we have t *sub* $\{y \leftarrow t\}s$ *sub* t, which is a cycle in the subterm relation. This contradicts our assumption. ■

9.1.11 Lemma. Let $\mathcal{A}$ be Ω-free and subterm-cycle-free.

Then for equivalent terms s,t there always exists a renaming ρ with ρs = t.

Proof. There exists σ,τ with DOM(σ) = V(s), DOM(τ) = V(t) and σs = t and τt = s. Hence στt = t and τσs = s. Ω-freeness implies $\sigma\tau = ID_{\mathcal{A}}\ [V(t)]$ and $\tau\sigma = ID_{\mathcal{A}}\ [V(s)]$. Since $\mathcal{A}$ is collapse-free by Lemma 9.1.10, COD(σ) and COD(τ) consist of variables. The substitution σ is invertable, hence it is a renaming. ■

We denote by $\langle\sigma = \tau\rangle$ the equation system $\langle s_1 = t_1,\ldots,s_n = t_n\rangle$, for substitutions $\sigma = \{x_i \leftarrow s_i \mid i = 1,\ldots,n\}$ and $\tau = \{x_i \leftarrow t_i \mid i = 1,\ldots,n\}$, where $DOM(\sigma) = DOM(\tau) = \{x_i \mid i = 1,\ldots,n\}$.

Now we can give rules that are sufficient for the empty theory:

9.1.12 Definition.

Rule: Decomposition. $\{\{s,t\}\cup M\} \Rightarrow_W \{\{s\}\cup M\} \cup \langle\sigma = \tau\rangle$,
iff there is a nonvariable term r such that $\sigma r = s$ and $\tau r = t$ and $DOM(\sigma) = DOM(\tau) = V(r)$.

Rule: Occur-check. $\Gamma \Rightarrow_W$ FAIL,
if Γ contains a cycle.

Rule: Clash $\{s,t\}\cup M \Rightarrow_W$ FAIL,
if s and t are nonvariable terms and decomposition is not applicable.

9.1.13 Proposition. Let $\mathcal{A}$ be a unification algebra extended by constants.

i) If $\mathcal{A}$ is Ω-free, then decomposition is complete.
ii) If $\mathcal{A}$ is subterm-cycle-free, then the occur-check is complete.
iii) If $\mathcal{A}$ is decomposable, then the clash-rule is complete.

Proof. i) Let $\mathcal{A}$ be Ω-free.

"$\Leftarrow_W$": Let θ be a solution of $\{\{s\}\cup M\} \cup \langle\sigma = \tau\rangle$. Then θ is also a unifier of $s = t$, as $\theta\sigma r = \theta\tau r$. Hence θ is a solution of $\{s,t\}\cup M$.

"$\Rightarrow_W$": Let θ be a solution of $\{s,t\}\cup M$, let $\sigma,\tau \in SUB$ with $\sigma r = s$ and $\tau r = t$ and $DOM(\sigma) = DOM(\tau) = V(r)$. Since $\mathcal{A}$ is Ω-free, we have $\theta\sigma = \theta\tau\ [V(r)]$, hence θ is a solution of $\langle\sigma = \tau\rangle$.

ii) Let $\mathcal{A}$ be subterm-cycle-free. It is sufficient to show that a Γ with a cycle has no solution. Assume there is a solution θ of Γ. There exists a cycle (x_i,t_i) in Γ. Since θ unifies the cycle and t_1 is not a variable, we have that θt_1 is a subterm of itself, which is a contradiction.

iii) Let $\mathcal{A}$ be decomposable. It is sufficient to show that $\{s,t\}\cup M$ has no solution. Assume there is a solution θ of $s = t$. Since $\mathcal{A}$ is decomposable, there exists substitutions σ,τ and a nonvariable term r, such that $\sigma r = s$ and $\tau r = t$. But then decomposition would be applicable. ■

If a unification algebra is regular, collapse-free, subterm-cycle-free, then we can define the **depth** of terms for all terms that have a finite number of subterms as follows $depth(x) := 0$ for $x \in V$, and $depth(t) := 1 + \max\{depth(s) \mid s\ \mathit{sub}\ t\}$

9.1.14 Lemma. If $\mathcal{A}$ is regular, collapse-free, and subterm-cycle-free, then the definition of depths of terms is sensible for subterm-finite terms t for a subterm-finite term t we have *sub* t implies $depth(s) < depth(t)$.

Proof. If $\mathcal{A}$ is regular and collapse-free, variables have no subterms, hence $depth(x) := 0$ is compatible with the definition of depth. Transitivity of *sub* yields that for *sub* t, the $\{r \mid r\ \mathit{sub}\ s\}$ is a subset of $\{r \mid r\ \mathit{sub}\ t\}$. Subterm-cycle-freeness implies that the subset relation is proper. For subterm-finite t, these subsets are finite by assumption, hence depth is a natural nuber and *sub* t implies depth(s) < depth(t). ■

Note that since $\mathcal{A}$ is unsorted, solved systems of equations are exactly those, which are merged, have no cylces and in every multi-equation there is at most one nonvariable term.

Now we can show termination of decomposition-merge for a class of unification algebras:

9.1.15 Theorem. Let $\mathcal{A}$ be Ω-free, subterm-cycle-free and decomposable and let Γ be a unification problem, such that all all terms in Γ have only a finite number of subterms.
Then decomposition, merge, occur-check and clash provide a terminating, complete unification algorithm for Γ.

Proof. Proposition 9.1.13 and Theorem 6.3 show that the rules preserve the solution space.
We show that the the application of rules terminates. Therefore we need a slight variation of the decomposition rule: In the multi-equation $\{s,t\} \cup M$, we delete the term with greater depth and keep the one with a smaller depth.
The measure for showing termination is $\mu(\Gamma) = (\mu_1, \mu_2, \mu_3)$, ordered lexicographically, where μ_1 is the multiset $\{\text{depth}(t) \mid t \in OBJ(\Gamma)\}$ and the ordering on these multisets is inherited from the ordering on natural numbers, and μ_2 is the number of multi-equations in Γ.
Decomposition strictly decreases μ_1, since the terms in $\langle\sigma = \tau\rangle$ are subterms of either s or t, hence the depth of all terms in $\langle\sigma = \tau\rangle$ is strictly smaller than $\max\{\text{depth(s)}, \text{depth(t)}\}$ by Lemma 9.1.14.
The merge rule leaves μ_1 invariant and strictly decreases μ_2.
Since μ is well-founded, the procedure terminates.
It remains to be shown that the returned system is in solved form, if Γ is unifiable. If no rule is applicable, then the system has no cycles. Furthermore, every multi-equation contains at most one nonvariable term, since otherwise either decompositon or clash is applicable. This means that Γ is solved. (Note that we have assumed that $\mathcal{A}$ is unsorted) ■

9.1.16 Corollary. Let $\mathcal{A}$ be Ω-free, subterm-cycle-free, decomposable and subterm-finite.
Then decomposition, merge, occur-check and clash provide a terminating, complete unification algorithm for Γ. Morover, $\mathcal{A}$ is unitary and has a dimension in the sense of 7.1 ■

9.2 Unification of Free Terms.

9.2.1 Definition.

i) A substitution σ is **Noetherian** (modulo W), if there is no properly decreasing infinite chain $\sigma_0 >_W \sigma_1 >_W \sigma_2 >_W \ldots$

ii) An term t is **Noetherian** (modulo W), if there is no properly decreasing infinite chain $t_0 > t_1 > t_2 > \ldots$

iii) A unification algebra is **Noetherian**, if every term is Noetherian and for every finite set of variables W every substititution is Noetherian. ■

9.2.2 Theorem. Let $\mathcal{A}$ be a unification algebra, such that $\mathcal{A}$ is Ω-free, subterm-cycle-free, decomposable, subterm-finite and Noetherian.

Then $\mathcal{A}$ is isomorphic to the terms over a free signature.

Proof.

1) For every term t there is a term t_0, with $t_0 \leq t$, t_0 has only variables as subterms, and t_0 is minimal with respect to $\leq$. This term is unique up to $\equiv$:

Since t is Noetherian, there exists a $\leq$-minimal nonvariable term t_0 with $t_0 \leq t$. Assume, t_0 has a nonvariable subterm s. Then there exists a nonvariable term s' such that $\{x \leftarrow s\}s' = t_0$. This means $s' \leq t_0$. Since t_0 is minimal, there exists a σ such that $\sigma t_0 = s'$. Thus $\sigma\{x \leftarrow s\}s' = s'$, and hence by Ω-freeness we obtain $\sigma\{x \leftarrow s\} = ID_{\mathcal{A}}$. Now $\sigma s = x$ implies that $\mathcal{A}$ is not collapse-free, which contradicts Lemma 9.1.10.

Assume, there is another minimal term t_1 with the described properties. Without loss of generality we can assume that t_0 and t_1 are variable disjoint, by applying a variable permutation if necessary. Then t_0 and t_1 are unifiable, and hence we can use decomposition. There exists a term r and substitutions σ,τ with $\sigma r = t_0$ and $\tau r = t_1$. By minimality, we have $r \equiv t_0$ and $r \equiv t_1$, hence $t_0 \equiv t_1$.

2) Now we can define the signature Σ:

We can assume that the set of variables is ordered by a total partial ordering. For every equivalence class EC of $\leq$-minimal nonvariable terms, we select a representative t_{EC}. Due to Lemma 9.1.8, all terms in EC have the same number of variables, say n_{EC}. We select a n_{EC}-ary function symbol f_{EC}. The signature Σ then exactly consists of all function symbols f_{EC} for all such equivalence classes. We assume that the variables are exactly the set **V**.

We define the generating terms $t_{\Sigma,EC} := f_{EC}(x_1,\ldots,x_{nEC})$, where $\{x_1,\ldots,x_{nEC}\} = V(t_{EC})$ and the variables are ascending with respect to the ordering on variables.

The set of all terms $\mathcal{T}(\Sigma,V)$ can be constructed from the generating terms $t_{\Sigma,EC}$.

Every term in $\mathcal{T}(\Sigma,V)$ is either a variable or of the form $\{x_1 \leftarrow t_1,\ldots,x_{nEC} \leftarrow t_{nEC}\}\, t_{\Sigma,EC}$ for some terms t_i. This representation is unique for terms in $\mathcal{T}(\Sigma,V)$.

3) $\mathcal{T}(\Sigma,V)$ and $\mathcal{A}$ are isomorphic:

We define a mapping φ: $\mathcal{T}(\Sigma,V) \rightarrow \mathcal{A}$, and show that φ is an isomorphism of unification algebras.

$\varphi(x) := x$ for variables $x \in$ **V**.

$\varphi(\{x_1 \leftarrow t_1,\ldots,x_{nEC} \leftarrow t_{nEC}\}\, t_{\Sigma,EC}) := \{x_1 \leftarrow \varphi t_1,\ldots,x_{nEC} \leftarrow \varphi t_{nEC}\}\, t_{EC}$.

For substitutions $\sigma = \{x_1 \leftarrow t_1,\ldots,x_n \leftarrow t_n\}$ we define $\varphi(\sigma) := \{x_1 \leftarrow \varphi t_1,\ldots,x_n \leftarrow \varphi t_n\}$

i) φ: $\mathcal{T}(\Sigma,V) \rightarrow T_{\mathcal{A}}$ is injective:

$\varphi(\{x_1 \leftarrow t_1,\ldots,x_{nEC} \leftarrow t_{nEC}\}t_{\Sigma,EC}) = \varphi(\{x_1 \leftarrow s_1,\ldots,x_{nEC'} \leftarrow s_{nEC'}\}\, t_{\Sigma,EC'})$ implies $\{x_1 \leftarrow \varphi t_1,\ldots,x_{nEC} \leftarrow \varphi t_{nEC}\}t_{EC} = \{x_1 \leftarrow \varphi t_1,\ldots,x_{nEC} \leftarrow \varphi t_{nEC}\}\, t_{EC'}$. Hence t_{EC} and $t_{EC'}$ are unifiable in $\mathcal{A}$, which is only possible if $t_{EC} = t_{EC'}$ by (1). Now Ω-freeness of $\mathcal{A}$ implies $\varphi t_i = \varphi s_i$ for all i, and by induction on the depth of terms in $\mathcal{A}$ we conclude that φ injective.

ii) φ:$\mathcal{T}(\Sigma,V) \rightarrow T_{\mathcal{A}}$ is a surjective:

Let s be a term in $\mathcal{A}$. If s is a variable, then σ is in the image of φ. Let s be a nonvariable term. Then $s = \{x_1 \leftarrow s_1,\ldots,x_n \leftarrow s_n\}\, s_{0,EC}$, where $s_{0,EC}$ is the minimal nonvariable term that is more general than s. Ω-freeness of $\mathcal{A}$ implies that s_i are unique, and subterm-cycle-freeness of $\mathcal{A}$ implies that the $\mathcal{A}$-depths of the s_i's are strictly smaller than the depth of s. Now induction on the $\mathcal{A}$-depth shows that there are terms $s_{i,T}$ with $\varphi s_{i,T} = s_i$. Hence $\varphi(\{x_1 \leftarrow s_{1,T},\ldots,x_n \leftarrow s_{n,T}\}\, s_{\Sigma,0,EC}) = s$.

iii) φ is also a bijection $\varphi := SUB_\Sigma \to SUB_{\mathcal{A}}$.

iv) $\varphi(\sigma t) = \varphi(\sigma)\varphi(t)$:

If t is a variable, then the equations holds by definition.

If t is not a variable, then t can be represented as $t = \{x_1 \leftarrow t_1,\ldots,x_{nEC} \leftarrow t_{nEC}\}\, t_{\Sigma,EC}$, where the t_i's have smaller term depths than t. By definition, we have $\varphi(\sigma t) = \{x_1 \leftarrow \varphi(\sigma t_1),\ldots,x_{nEC} \leftarrow \varphi(\sigma t_{nEC})\}t_{EC}$. Induction on the term depth shows that this expression is equal to $\{x_1 \leftarrow \varphi(\sigma)\varphi(t_1),\ldots,x_{nEC} \leftarrow \varphi(\sigma)\varphi(t_{nEC})\}t_{EC}$. Since $V(t_{EC}) = \{x_1,\ldots,x_{nEC}\}$, this expression is equal to $\varphi(\sigma)(\{x_1 \leftarrow \varphi(t_1),\ldots,x_{nEC} \leftarrow \varphi(t_{nEC})\}t_{EC})$. This is exactly $\varphi(\sigma)\varphi(t)$.

v) $\varphi(\sigma\tau) = \varphi(\sigma)\varphi(\tau)$:

Follows easy from part iv).

vi) φ is an isomorphism:

Using Definition 5.1, we have shown that φ is a bijection on terms and substitutions and is a homomorphism of unification algebras. That φ is an isomorphism follows from Lemma 5.3.

■

The following example gives a theory that is unitary due to Corollary 9.1.16 but cannot be equivalent to a free term algebra, since it is not Noetherian.

9.2.3 Example.

Consider the following theory. The theory can be viewed as a theory of infinite sequences s, such that for every sequence s, there exists a number n such that s_m becomes constant for for all $m \geq n$. The signature has for every $n > 0$ an n-ary function symbol f_n. The theory $\mathcal{E}$ is defined by the (infinite) canonical term rewriting system:

$R := \{f_n(x_1,\ldots,x_{n-2},x,x) \to f_{n-1}(x_1,\ldots,x_{n-2},x) \mid n \geq 2\}$.

<u>$\mathcal{E}$ is Ω-free</u>: Assume $f_n(s_1,\ldots,s_n) =_{\mathcal{E}} f_n(t_1,\ldots,t_n)$ for some f_n, s_i and t_i and assume that for some $j \in \{1,\ldots,n\}$, we have $s_j \neq_{\mathcal{E}} t_j$. We can assume that all s_i and t_i are in normal form. Furthermore, we can assume that n is the smallest number, which violates Ω-freeness. Since $f_n(s_1,\ldots,s_n) =_{\mathcal{E}} f_n(t_1,\ldots,t_n)$, one of them must be reducible, say $f_n(s_1,\ldots,s_n)$, hence $n \geq 2$. Reducability of $f_n(s_1,\ldots,s_n)$ implies $s_n = s_{n-1}$. Now $f_n(t_1,\ldots,t_n)$ must be reducible, too, hence $t_n = t_{n-1}$. Thus we have $f_{n-1}(s_1,\ldots,s_{n-1}) =_{\mathcal{E}} f_{n-1}(t_1,\ldots,t_{n-1})$, which contradicts the minimal choice of n.

<u>$\mathcal{E}$ is subterm-cycle-free</u>: Holds, since the (usual) term depth is invariant in equivalence classes with respect to $\mathcal{E}$.

<u>$\mathcal{E}$ is subterm-finite</u>: The equations show, that application of rewrite rules does not change equivalence classes of subterms, hence a term has only a finite number of subterms modulo $\mathcal{E}$.

<u>$\mathcal{E}$ is decomposable:</u> Let s,t be nonvariable, unifiable terms. By applying rewrite rules backwards, we can assume that $s = f_n(s_1,\ldots,s_n)$ and $t = f_n(t_1,\ldots,t_n)$ for some n. Then we can choose r =

$f(x_1,\ldots,x_n)$, where $x_1,\ldots,x_n$ are new variables, and $\sigma := \{x_i \leftarrow s_i \mid i = 1,\ldots,n\}$ and $\tau := \{x_i \leftarrow t_i \mid i = 1,\ldots,n\}$.

<u>$\mathcal{E}$ is not Noetherian:</u> the following chain $f_1(x_1) > f_2(x_1,x_2) > f_3(x_1,x_2,x_3) > \ldots$ shows this: We have $\{x_n \leftarrow x_{n-1}\}f_n(x_1,\ldots,x_n) = f_{n-1}(x_1,\ldots,x_{n-1})$. In order to show that $f_{n-1}(x_1,\ldots,x_{n-1})$ is a proper $\mathcal{E}$-instance of $f_n(x_1,\ldots,x_n)$, assume $f_{n-1}(t_1,\ldots,t_{n-1}) =_{\mathcal{E}} f_n(x_1,\ldots,x_n)$. The term $f_n(x_1,\ldots,x_n)$ is in normal form, thus $f_{n-1}(t_1,\ldots,t_{n-1})$ must be reducible. However, it is not possible to reduce it to a term starting with f_n. We have reached a contradiction.

Now Corollary 9.1.16 shows that for this theory the algorithm consisting of the rules decomposition, merge is a unification algorithm and that it is of unification type unitary. ■

9.3 Unification of Rational Terms.

In order to give unification rules that are able to deal with the algebra of rational terms [Co82], we need more properties of unification algebras.

9.3.1 Definition. Let $\mathcal{A}$ be a unification algebra.

$\mathcal{A}$ **solves cycles uniquely**, iff for every nonvariable term r with $x \in V(r)$, and two terms s, t: $\{x \leftarrow s\}r = s$ and $\{x \leftarrow t\}r = t$ implies $s = t$. ■

9.3.2 Definition.

Rule: **Rational-Unfolding**:

$t = M \Rightarrow_W x = t' = M$,

if t contains a nonvariable subterm s, $\{x \leftarrow s\}t' = t$, where t' is a nonvariable term with $x \in V(t')$, and x is a new variable. ■

9.3.3 Proposition. If $\mathcal{A}$ is Ω-free, collapse-free, and solves cycles uniquely, then rational-unfolding is a complete step.

Proof.

1) The tranformation is complete.

"$\Rightarrow$": Let θ be a solution of $t = M$. Then define θ' such that $\theta' = \theta\ [V(t,M)]$, and $\theta'x := \theta t$. Obviously $\theta' = \theta'\{x \leftarrow t\}$. This implies $\theta't' = \theta'\{x \leftarrow t\}t' = \theta't = \theta t$.

"$\Leftarrow$": Let θ be a solution of $x = t' = M$. Without loss of generality we can assume that $DOM(\theta) \cap I(\theta) = \emptyset$. With $\theta' := \theta_{|V(t')-\{x\}}$, we can partitition θ as follows: $\theta = \theta' \cup \{x \leftarrow \theta t'\}$. Then we have $\theta x = \theta t' = \{x \leftarrow \theta t'\}(\theta' t')$.

$\theta't'$ is not a variable, since $\mathcal{A}$ is collapse-free. We have $\theta t = (\theta' \cup \{x \leftarrow \theta t'\})\{x \leftarrow t\}t' = \{x \leftarrow \theta t'\}\theta'\{x \leftarrow \theta't\}t' = \{x \leftarrow \theta't\}\theta't' = \{x \leftarrow \theta t\}\theta't'$. $\mathcal{A}$ solves cycles uniquely, hence $\theta t = \theta t'$. This means, θ is a unifier of $t = M$.

9.3.4 Proposition. If $\mathcal{A}$ is Ω-free, collapse-free, and Noetherian, then the rational-unfolding rule terminates with a Γ, such that all terms in OBJ(Γ) have only variables as subterms.

Proof. We show that the transformation terminates:

Let the measure be the multiset of all nonvariable terms that have nonvariable subterms, ordered by the ordering that comes from the instance relation. Since $\mathcal{A}$ is Noetherian, the multi-set ordering is well-founded. We have to show that $t' < t$. Obviously $t' \leq t$. Assume $\sigma t = t'$ for some substitution σ with $DOM(\sigma) = V(t)$. Then $\{x \leftarrow s\}\sigma t = t$. Furthermore $t \neq t'$, which implies $V(t) \neq \emptyset$. Now Ω-freeness implies $\{x \leftarrow s\}\sigma = ID_T\ [V(t)]$, which is impossible, since $\mathcal{A}$ is regular and collapse-free.

We have shown that rational-unfolding terminates.

If rational-unfolding stops, then there is no term with a nonvariable subterm, hence the last claim holds. ■

Cyclically solved systems of equations are exactly those, which are merged and in every multi-equation there is at most one nonvariable term.

9.3.5 Theorem. If $\mathcal{A}$ is Ω-free, collapse-free, decomposable, Noetherian, and solves cycles uniquely, then the following procedure is complete and terminates.

1) first use rational-unfolding until this is no longer possible,
2) use decomposition, merge and clash.

If it terminates, then Γ is in cyclically solved form.

Proof. Lemma 9.3.4 shows that step 1 yields a Γ, which contains only variables or terms that have no nonvariable subterms.

Let $s = t$ be in a multi-equation and nonvariable terms. If they are unifiable, then decomposition is applicable. Let $\sigma r = s$, $\tau r = t$ with $DOM(\sigma) = DOM(\tau) = V(r)$. We have that $COD(\sigma)$ abnd $COD(\tau)$ consist of variables, since otherwise s or t have nonvariable subterms as $\mathcal{A}$ is regular and collapse-free. Hence every decomposition removes a nonvariable term. This means, the merge-decomposition process terminates. Furthermore it is complete due to Lemma 9.1.13. ■

All the above properties hold for rational terms [Co 82], hence our algorithm can be applied to them.

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