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Unification of Term Schemes
- Theory and Applications

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Abstract

We present a new approach for solving certain infinite sets of first order unification problems represented by term schemes. Within the framework of second order equational logic solving such scheme unification problems amounts exactly to solving (variable-) restricted unification problems. A method known for solving first order restricted unification problems is generalized to the second order case. Essentially this is achieved by transforming a restricted unification problem into an unrestricted one, solving the latter and retransforming the solutions obtained. The results on second order restricted unification are then used to solve the original problem, namely to decide the solvability of scheme unification problems and - in the positive case - to compute the corresponding most general unifiers. Finally the results are applied to provide sufficient conditions for a property of "repeated unifiability" which in turn is crucial for the analysis of divergence of completion procedures for term rewriting systems. Although the study of divergent completion behaviour was the starting point for the work presented, the results obtained are not only applicable to divergence analysis but may be useful for other applications, too.

Contents

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1 Introduction

Knuth-Bendix like completion procedures for term rewriting systems provide a very powerful means for automating equational reasoning. But one of the main drawbacks of such methods up to now is their potential nontermination (divergence) on certain inputs, i.e. the completion procedure when given an inital set of equations and a term reduction ordering runs forever producing infinitely many rewrite rules. Let us give a simple example to illustrate some of the problems encountered in the analysis of divergent completion behaviour. For A: f(h(u),v) = f(u,h(v)), B_0: f(w,a) = w and C_0: f(w,g(w)) = w completion with input {A, B_0} and {A, C_0} respectively and an appropiate reduction ordering produces the following infinite sequence of rewrite rules:

A	$f(h(u),v) \to f(u,h(v))$	A	$f(h(u),v) \to f(u,h(v))$
B_0	$f(w,a) \to w$	C_0	$f(w,g(w)) \to w$
B_1	$f(w,h(a)) \to h(w)$	C_1	$f(w,h(g(h(w)))) \to h(w)$
B_2	$f(w,h(h(a))) \to h(h(w))$	C_2	$f(w,h(h(g(h(h(w)))))) \to h(h(w))$
....		....	
B_n	$f(w,h^n(a)) \to h^n(w)$	C_n	$f(w,h^n(g(h^n(w)))) \to h^n(w)$
....		....	

The reason for divergence obviously comes from the fact that any left hand side l_n of B_n and C_n respectively is unifiable with the left hand side l of A yielding via critical pair construction $l_{n+1} \to r_{n+1}$. Now, why are all those (infinitely many) l_n indeed unifiable with l? One might conjecture that this "repeated unifiablity" of l with every l_n is due to the fact that l is unifiable with any instance of the "term scheme" f(w,X) resp. f(w,Y(w)). In order to give a formal definition of this kind of unification problems and to provide solutions for them we have to make precise the notions "term scheme" and "unification of term schemes". This will be done in the framework of second order equational logic.

In chapter 2 basic notions and definitions for second order term languages are given. Unification problems as mentioned and solution methods are dealt with in chapter 3. The link between the theoretical results and and the application to divergence analysis including some examples are sketched in chapter 4.

2 Second Order Term Languages

The basic definitions, notions and lemmas about second order term languages are essentially taken from [Hu76] and [SnGa88]. Moreover we assume familiarity with the basic notions and results of λ-calculus (cf.[Hu76]).

Definition 2.1 (second order terms)
For all $i \geq 0$ let $\mathcal{V}_i$ be a denumerable set of distinct (function) variables of arity i with $\mathcal{V}_i \cap \mathcal{V}_j = \emptyset$ for $i \neq j$. The set of all variables is defined as $\mathcal{V} = \cup_{i\geq 0}\mathcal{V}_i$. Let $\mathcal{C}$ with $\mathcal{C} \cap \mathcal{V} = \emptyset$ be a finite or denumerable set of distinct (function) constants together with their respective arity (denoted ar(...)). We use X,Y,Z,... for denoting variables, x,y,z,... for variables of arity 0, f,g,h,... for function constants and A,B,C,... for denoting variables or constants. The set of restricted second order terms $\mathcal{T} = \mathcal{T}(\mathcal{C},\mathcal{V})$ is the smallest set satisfying

(i) $A \in \mathcal{V} \cup \mathcal{C}$, $ar(A) = 0 \Rightarrow A \in \mathcal{T}$, and

(ii) $A \in \mathcal{V} \cup \mathcal{C}$, $ar(A) > 0$, $t_1,\ldots,t_{ar(A)} \in \mathcal{T} \Rightarrow A(t_1,\ldots,t_{ar(A)}) \in \mathcal{T}$.

$\mathcal{T}_0 := \{t \in \mathcal{T} \mid V(t) \subseteq \mathcal{V}_0\}$ is the subset of first order terms of $\mathcal{T}$, where the set V(t) of variables of a term t is defined by $V(A(t_1,\ldots,t_{ar(A)})) := \cup_{i=1,\ldots,ar(A)} V(t_i) \cup (\{A\} \cap \mathcal{V})$. In order to get general second order terms (i.e. functional objects, too) we complete $\mathcal{T}$ into $\underline{\mathcal{T}}$ by introducing bound variables as follows. $\underline{\mathcal{T}}$ is the smallest set containing $\mathcal{T}$ that satisfies the closure property $\forall t \in \underline{\mathcal{T}}\ \forall x \in \mathcal{V}_0 : \lambda x.t \in \underline{\mathcal{T}}$. As usual we abbreviate a term of the form $\lambda x_1.\ldots\lambda x_n.t$ as $\lambda x_1\ldots x_n.t$. In fact, as pointed out in [Hu76], $\underline{\mathcal{T}}$ may be regarded as the restriction of the language of extensional normal forms of typed λ-calculus (with one base type) to second order terms. The definition of V(t) is recursively extended for $\underline{\mathcal{T}}$ by defining $V(\lambda x.t) := V(t) \setminus \{x\}$, which describes the set of **free variables** of t. By V(M) with $M \subseteq \mathcal{T}$ we mean $\cup_{t \in M} V(t)$. In the following terms are always compared modulo α-conversion (e.g. renaming of bound variables) which greatly simplifies the presentation. Furthermore w.l.o.g. we assume that for any term under consideration the set of free variables is distinct from the set of all bound variables. By $\eta[A] := \lambda x_1\ldots x_{ar(A)}.A(x_1\ldots x_{ar(A)})$ we denote the extensional normal form of $A \in \mathcal{V} \cup \mathcal{C}$ (i.e. the normal form of A under the converse of the η-reduction rule in λ-calculus). The top symbol of a term $t \in \mathcal{T}$ is denoted by top(t).

Definition 2.2 (second order substitutions, cf. [Hu76], [SnGa88])
A (second order) substitution σ is a finite set of (type respecting) variable assignments, i.e. pairs (X,t) with $t \in \underline{\mathcal{T}}$ also denoted $X \leftarrow t$. It may be considered to be a total function $\sigma : \mathcal{V} \rightarrow \underline{\mathcal{T}}$ by defining $\sigma(X) := \eta[X]$, if there is no pair $X \leftarrow t$ in σ. The **domain** of a substitution σ is $\mathbf{D}(\sigma) := \{X \in \mathcal{V} \mid \sigma(X) \neq \eta[X]\}$, the set of **introduced variables** is $\mathbf{I}(\sigma) := \cup_{X \in D(\sigma)} I(\sigma,X)$, where $I(\sigma,X) := V(t)$ for $(X \leftarrow t) \in \sigma$, $X \in D(\sigma)$. The application of substitutions to terms yielding again terms is recursively defined by (denoting the extension also by σ)

(i) $\sigma(A(t_1,\ldots,t_{ar(A)})) := A(\sigma(t_1),\ldots,\sigma(t_{ar(A)}))$ for $A \in \mathcal{C} \cup \mathcal{V} \setminus D(\sigma)$,

(ii) $\sigma(X(t_1,\ldots,t_{ar(X)})) := \varphi(t)$ for $X \in \mathcal{V} \cap D(\sigma)$, $X \leftarrow \lambda x_1\ldots x_{ar(X)}.t \in \sigma$ and the substitution

$\varphi := \{x_i \leftarrow \sigma(t_i) | 1 \le i \le ar(X)\}$ and

(iii) $\sigma(\lambda x.t) := \lambda x.\sigma(t)$, where the bound variable x is supposed not to be in $D(\sigma)$, which is implicitely satisfied since we work modulo renaming of bound variables (note that $\sigma(X) = \sigma(\eta[X])$ iff $X \notin D(\sigma)$).

By **SUB:=SUB($\mathcal{T}$)** we denote the set of all (second order) substitutions, by **SUB($\mathcal{T}_0$)** the subset of all first order substitutions, i.e. substitutions σ with $D(\sigma), I(\sigma) \subseteq \mathcal{V}_0$. **Equality for substitutions** is defined by $\sigma = \sigma'$ iff $D(\sigma) = D(\sigma')$ and $\forall X \in D(\sigma)$: $\sigma(X) = \sigma'(X)$. By **id** we denote the identity substitution ($D(id) = \emptyset$). The **composition of substitutions** is defined by $\varphi\sigma := \{X \leftarrow \varphi(\sigma(X)) | X \in D(\sigma), \varphi(\sigma(X)) \neq \eta[X]\} \cup \{X \leftarrow \varphi(X) | X \in D(\varphi) \setminus D(\sigma)\}$. The **restriction of a substitution σ to a (finite) set V of variables** is defined as $\sigma|_V := \{(X \leftarrow t) \in \sigma | X \in V \cap D(\sigma)\}$. **Equality of substitutions restricted to a (finite) set of variables V** is defined by $\sigma =_V \varphi$ iff $\sigma|_V = \varphi|_V$.

A substitution σ is **idempotent** iff $\sigma\sigma = \sigma$.It It is a **renaming substitution away from** $V \subseteq \mathcal{V}$ iff (1) $\forall X \in D(\sigma)$: $\sigma(X) = \eta[Y]$ for some $Y \in \mathcal{V} \setminus V$ and (2) $\forall X,Y \in D(\sigma)$: $\sigma(X) = \sigma(Y) \Rightarrow X = Y$. It is a **W-renaming for W** $\subseteq \mathcal{V}$ iff (1') $\forall X \in W$: $\sigma(X) = \eta[Y]$ for some $Y \in \mathcal{V}$ and (2') $\forall X,Y \in W$: $\sigma(X) = \sigma(Y) \Rightarrow X = Y$. And σ is said to be **strict** iff $\forall X \in D(\sigma)$, $\sigma(X) = \lambda u_1,\ldots,u_{ar(X)}.t$: $\{u_1,\ldots,u_{ar(X)}\} \subseteq V(t)$. For $M \subseteq \mathcal{T}$ we denote the set $\{\varphi(t) | t \in M\}$ by $\varphi(M)$.

Definition 2.3 (preorderings on terms and substitutions)

The preordering $\le$ on terms is defined by $s \le t$ iff $\exists\, \sigma \in$ SUB: $t = \sigma(s)$. For $s \le t$ we also say that t is an instance of s or s is more general than t. Analogously (and abusing notation) $\le$ for substitutions is defined by $\sigma \le \varphi$ iff $\exists\, \tau \in$ SUB: $\varphi = \tau\sigma$. We say that σ is more general than φ on $V \subseteq \mathcal{V}$ (written $\sigma \le_V \varphi$) iff $\exists\, \tau \in$ SUB: $\varphi =_V \tau\sigma$. The preordering $\le_V$ induces an equivalence relation $\equiv_V$ on SUB via $\sigma \equiv_V \varphi$ iff $\sigma \le_V \varphi$ and $\varphi \le_V \sigma$.

Lemma 2.4 (some elementary properties of substitutions, cf. [Hu76])

The following properties hold for all $t \in \mathcal{T}$, $\sigma, \varphi, \varrho \in$ SUB, $V, W \subseteq \mathcal{V}$:

(1) $\forall X \in \mathcal{V}: \sigma(X) = \varphi(X) \Leftrightarrow \forall s \in \mathcal{T}: \sigma(s) = \varphi(s)$

(2) $V(t) \subseteq W \Rightarrow \sigma(t) = \sigma|_W(t)$

(3) $D(\sigma) \cap V(t) = \emptyset \Rightarrow \sigma(t) = t$

(4) $(\varphi\sigma)(t) = \varphi(\sigma(t))$

(5) $(\varrho\varphi)\sigma = \varrho(\varphi\sigma)$

(6) $V \subseteq W \Rightarrow [\sigma \le_W \varphi \Rightarrow \sigma \le_V \varphi]$

(7) $\sigma =_V \varphi \Rightarrow \varrho\sigma =_V \varrho\varphi$.

Proof: see [Hu76].

■

The next lemma will be useful in subsequent proofs.

Lemma 2.5
For all $\sigma, \varphi \in SUB$ we have

(a) $I(\sigma) \cap D(\sigma) = \emptyset \Rightarrow \sigma$ is idempotent.
(b) $D(\sigma) \cap D(\varphi) = I(\varphi) \cap (D(\sigma) \cup D(\varphi)) = \emptyset \Rightarrow \varphi\sigma = (\varphi\sigma)\varphi$.
(c) $D(\sigma) \cap D(\varphi) = D(\sigma) \cap I(\varphi) = D(\varphi) \cap I(\sigma) = \emptyset \Rightarrow \varphi\sigma = \sigma\varphi$.
(d) $D(\sigma) \cap D(\varphi) = D(\sigma) \cap I(\varphi) = \emptyset \Rightarrow \varphi\sigma = (\varphi\sigma)|_{D(\sigma)}\varphi$.

Proof:

(a) Let σ with $I(\sigma) \cap D(\sigma) = \emptyset$ be given. If $X \notin D(\sigma)$ then we have $\sigma(X) = \eta[X]$ implying $\sigma(\sigma(X)) = \sigma(\eta[X]) = \sigma(X)$ else $\sigma(\sigma(X)) = \sigma(X)$ because of $I(\sigma) \cap D(\sigma) = \emptyset$.

(b) Let $\sigma, \varphi \in SUB$ satisfy the assumptions. If $X \in D(\sigma)$ then $\varphi(X) = \eta[X]$ which implies $(\varphi\sigma)(\varphi(X)) = (\varphi\sigma)(X)$. If $X \in D(\varphi)$ then $X \notin D(\sigma)$ implies $\varphi(\sigma(X)) = \varphi(X) = \varphi(\varphi(X))$ since φ is idempotent by (a). Moreover $I(\varphi) \cap D(\sigma) = \emptyset$ implies $\varphi(X) = \sigma(\varphi(X))$ yielding together $(\varphi\sigma)(X) = \varphi(\sigma(X)) = \varphi(X) = \varphi(\varphi(X)) = \varphi(\sigma(\varphi(X))) = ((\varphi\sigma)\varphi)(X)$. The case $X \notin D(\sigma) \cup D(\varphi)$ is trivial.

(c) and (d): Again by an easy case analysis for $X \in \mathcal{V}$.

■

Note that the condition $I(\sigma) \cap D(\sigma) = \emptyset$ is equivalent to idempotency of σ in the first order case but is only sufficient in general here (example: $\sigma = \{X \leftarrow \lambda u.X(a)\}$).

3 Unification of Term Schemes and Restricted Unification

We now come back to our original problem and make precise the notions of term schemes and scheme unification. In order to solve scheme unification problems we then generalize the notion of (variable-) restricted unification (cf. [Sz82], [Bü86]) to the second order case, develop solution methods for them and finally show that scheme unification problems may be solved as a special case in this more general framework.

Definition 3.1 (term scheme, scheme unification)
A **term scheme** is any term $s \in \mathcal{T}$ with its variables partitioned into a set of ordinary first order variables and a set of (possibly second order) scheme variables. A **scheme unification problem (SUP)** is represented by a finite set E of (unordered) pairs of term schemes where the role of any variable occurring in E is fixed, either as an ordinary variable or as a scheme variable. Thus, we may denote the SUP by $\langle E/W \rangle$ where W is the set of ordinary variables of E and $W^c := V(E)\setminus W$ the set of scheme variables of E. Solving the SUP $\langle E/W \rangle$ consists in deciding whether every first order unification problem $\langle \psi(E) \rangle$ with $\psi \in SUB$, $D(\psi) \subseteq W^c$, $I(\psi) \cap W = \emptyset$ is solvable, and if so, in computing the unifiers (see Def. 3.2) for all these problems.

In order to be able to handle scheme unification problems, we first deal now with so-called restricted unification problems. Let us start with a generalized definition of (second order) unification problems following [GaSn88] but forbidding certain variables to be instantiated.

Definition 3.2 (variable-restricted unification)
A **(variable-) restricted unification problem (RUP)** is a finite set E of (unordered) term pairs (s_i,t_i) together with a set of variables $W \subseteq V(E) := \bigcup_i(V(s_i) \cup V(t_i))$, also denoted by $\langle E/W \rangle = \langle s_1=t_1,\ldots,s_n=t_n/W \rangle$. A **solution** or **(variable-) restricted unifier** of $\langle E/W \rangle$ is any substition σ with $D(\sigma) \cap V(E) \subseteq W$ and $\sigma(s_i) = \sigma(t_i)$ for $i=1,\ldots,n$. The set of all solutions of $\langle E/W \rangle$ is denoted by **U(E/W)**. Since $\sigma \in U(E/W)$ iff $\sigma|_W \in U(E/W)$ we are in most cases interested in $\mathbf{U_W(E/W)} := \{\sigma \in U(E/W) | D(\sigma) \subseteq W\}$, the set of all solutions of U(E/W) with their domain restricted to the relevant variables. We say $\langle E/W \rangle$ is **solvable** iff $U(E/W) \neq \emptyset$. Two terms s and t are said to be **W-unifiable** for $W \subseteq V(s) \cup V(t)$ iff $\langle s=t/W \rangle$ is solvable.

Choosing $W=V(E)$ we get the ordinary unification problem (UP) $\langle E/V(E) \rangle$ for (second order) terms, simply denoted $\langle E \rangle$. Intuitively W describes the set of variables substitution is restricted to whereas $W^c := V(E) \setminus W$ contains the protected variables where substitution is not allowed.
Second order unification is in general undecidable as shown in [Go81], but of course the set of unifiers for a given problem is recursively enumerable. Next we give some basic definitions that are important for finite descriptions of unifier sets. These definitions are a straightforward generalization for RUP's of the ones given in [SnGa88] for ordinary UP's.

Definition 3.3 (complete/minimal set of unifiers, most general unifier)
Given a RUP $\langle E/W \rangle$, a set S of substitutions and a finite set of additional "protected" variables V with $V \cap W^c = \emptyset$, we say that
S is a **complete** set of unifiers (csu) for $\langle E/W \rangle$ away from V iff

(i) $\forall\, \sigma \in S$: $D(\sigma) \subseteq W$ and $I(\sigma) \cap (D(\sigma) \cup V)=\emptyset$,

(ii) $S \subseteq U(E/W)$, and

(iii) $\forall\, \varphi \in U(E/W)\ \exists\, \sigma \in S$: $\sigma \leq_{V(E)} \varphi$.

S is a **complete minimal** set of unifiers **(cmsu)** for $\langle E/W \rangle$ away from V iff additionally

(iv) $\forall\, \sigma,\sigma' \in S, \sigma \neq \sigma'$: $\sigma \not\leq_{V(E)} \sigma'$ holds.

σ is a **most general unifier (mgu)** of $\langle E/W \rangle$ away from V iff $\{\sigma\}$ is a csu of $\langle E/W \rangle$ away from V.

The set V of "protected" variables may be used to separate new variables which are introduced by a csu of a given problem from the variables of the context where the problem was extracted from. Indeed, in contrast to first order unification, it is in general necessary for higher order unification to introduce new (free) variables in order to describe csu's. When V is not significant we drop it.

For any solvable RUP $\langle E/W \rangle$ a csu clearly exists but may not be finite. Mgu's do not always exist for unifiable second order terms as well as cmsu's for unifiable third order terms whereas the existence of cmsu's for unifiable second order terms has been conjectured but not yet been proved. If a cmsu exists it is unique up to an isomorphism (see [Hu76]).

Condition (i) for csu's above is of technical nature only. This is shown by the following

Lemma 3.4 (cf. [GaSn88])

Let a RUP $\langle E/W \rangle$, $\varphi \in SUB$ and a set V of "protected" variables with $V \cap W^c = \emptyset$ be given. If $\varphi \in U(E/W)$ then there exists $\sigma \in SUB$ such that

(i) $D(\sigma) \subseteq W$ and $I(\sigma) \cap (D(\sigma) \cup V) = \emptyset$,

(ii) $\sigma \in U(E/W)$ and

(iii) $\sigma \equiv_{V(E)} \varphi$.

Proof: (analogous to the proof of Lemma 4.4 in [SnGa88])

If $\sigma := \varphi|_W$ satisfies condition (i) we are done. Otherwise, if $I(\varphi)\backslash W^c = \{X_1,\ldots,X_n\}$ then let $\{Y_1,\ldots,Y_n\}$ be a set of new variables disjoint from the variables in V, $\{X_1,\ldots,X_n\}$ and V(E) with corresponding type. Now define the renaming substitutions $\varrho_1 := \{X_i \leftarrow \eta[Y_i] | 1 \le i \le n\}$, $\varrho_2 := \{Y_i \leftarrow \eta[X_i] \mid 1 \le i \le n\}$. With $\sigma := (\varrho_1\varphi)|_{V(E)}$ we have $D(\sigma) \subseteq W$, $I(\sigma) \cap D(\sigma) = \emptyset$ and $I(\sigma) \cap V = \emptyset$ since $V \cap W^c = \emptyset$. Thus conditions (i) and (ii) of Def.3.3 are satisfied for σ and we have $\varphi \le_{V(E)} \sigma$. Now $\varrho_2\varrho_1 =_{V(E)\cup I(\varphi)} id$ and $\sigma = (\varrho_1\varphi)|_{V(E)}$ imply $\varphi =_{V(E)\cup I(\varphi)} (\varrho_2(\varrho_1\varphi)) = (\varrho_2\varrho_1)\varphi =_{V(E)} \varrho_2\sigma$ and thus $\sigma \le_{V(E)} \varphi$. Therefore we have $\sigma \le_{V(E)} \varphi$ and $\varphi \le_{V(E)} \sigma$.

■

This lemma shows that w.l.o.g. we can restrict ourselves to idempotent unifiers σ with $I(\sigma) \cap D(\sigma) = \emptyset$ when regarding csu's for a given RUP $\langle E/W \rangle$. The following monotonicity lemma is straightforward from the definition of solution sets.

Lemma 3.5

For any set E of equations and sets W,W' of variables with $W \subseteq W' \subseteq V(E)$ the inclusion $U(E/W) \subseteq U(E/W')$ holds.

Proof: Obvious.

■

Definition 3.6 (solved form)

A RUP $\langle E/W \rangle$ is in solved form iff for all s=t in E the term s is of the form $\eta[X]$ with $X \in W \subseteq \mathcal{V}$ and X does not occur anywhere else in E. For any RUP $\langle E/W \rangle$ in solved form we define $\sigma_E := \{X \leftarrow t \mid \eta[X] = t \in E\}$ and conversely for $\sigma \in SUB$ let E_σ be defined by $E_\sigma := \{\eta[X] = t \mid (X \leftarrow t) \in \sigma\}$.

The relevance of solved forms is explained by

Lemma 3.7

If $\langle E/W \rangle$ is a RUP in solved form then the substitution σ_E is a mgu for $\langle E/W \rangle$ away from V for any $V \subseteq \mathcal{V}$ with $V \cap V(E) = \emptyset$.

Proof: (analogous to the proof of lemma 4.5 in [SnGa88])

By definition of σ_E condition (i) and (ii) of Def. 3.3 are satisfied, since $D(\sigma_E) \subseteq W$, $I(\sigma_E) \cap D(\sigma_E) = \emptyset$, $I(\sigma_E) \cap V = (V(E) \setminus D(\sigma_E)) \cap V \subseteq V(E) \cap V = \emptyset$ and $\forall\, \eta[X] = t \in E$: $\sigma_E(\eta[X]) = \sigma_E(X) = t = \sigma_E(t)$. Now, if $\varphi \in U(E/W)$ (w.l.o.g. φ is assumed to be idempotent with $D(\varphi) \cap I(\varphi) = \emptyset$) then we have $\varphi(\eta[X]) = \varphi(X) = \varphi(\varphi(X)) = \varphi(t) = \varphi(\sigma_E(X))$ for any $X \in D(\sigma_E)$ with $X = t \in E$ and $\varphi(X) = \varphi(\eta[X]) = \varphi(\sigma_E(X))$ otherwise. Thus $\sigma_E \leq \varphi$ and of course also $\sigma_E \leq_{V(E)} \varphi$.

■

We will show now that an RUP $\langle E/W \rangle$ may be transformed into an ordinary (unrestricted) UP $\langle E' \rangle$ such that the solutions of $\langle E/W \rangle$ can be obtained from the solutions of $\langle E' \rangle$ by an inverse transformation. The transformation function involved interprets the forbidden variables in W^c as distinct new constants of corresponding arity. To be precise, for $W^c = \{X_1,\ldots,X_n\}$ let C_i be a new constant with $ar(X_i) = ar(C_i)$ for every i, $1 \leq i \leq n$. Defining $\mathcal{C}^* := \mathcal{C} \cup \{C_i \mid 1 \leq i \leq n\}$, $\mathcal{V}^* := \mathcal{V} \setminus W^c$, the set of transformed terms becomes $\mathcal{T}^* := \mathcal{T}(\mathcal{C}^*,\mathcal{V}^*)$ and $\underline{\mathcal{T}^*} := \underline{\mathcal{T}(\mathcal{C}^*,\mathcal{V}^*)}$ respectively. The transformation function $\Phi: \underline{\mathcal{T}} \rightarrow \underline{\mathcal{T}^*}$ is defined by homomorphic extension of $\Phi: \mathcal{V} \rightarrow \underline{\mathcal{T}^*}$, $\Phi(X_i) := \eta[C_i]$ for $X_i \in W^c$ and $\Phi(X) := \eta[X]$ else. It may also be extended to substitutions that leave the variables of W^c unchanged: With $SUB^*(\underline{\mathcal{T}}) := \{\sigma \in SUB(\underline{\mathcal{T}}) \mid D(\sigma) \cap W^c = \emptyset\}$ we get Φ_S: $SUB^*(\underline{\mathcal{T}}) \rightarrow SUB(\underline{\mathcal{T}^*})$ defined by $\Phi_S(\sigma) := \{X \leftarrow \Phi(\sigma(X)) \mid X \in D(\sigma)\} = (\Phi\sigma)|_{D(\sigma)}$. It is easily verified that Φ, Φ_S are bijections with Φ^{-1}, Φ_S^{-1} defined by $\Phi^{-1}(\lambda u_1,\ldots u_m.A(t_1,\ldots t_n)) := \lambda u_1,\ldots u_m.A'(\Phi^{-1}(t_1),\ldots,\Phi^{-1}(t_n))$ with $A' := X_i$ if $A = C_i$ and $A' := A$ else, and $\Phi_S^{-1}(\sigma^*) := \{X \leftarrow \Phi^{-1}(\sigma^*(X)) \mid X \in D(\sigma^*)\}$ (note that Φ^{-1} is not a substitution!). Moreover we have the following simple properties :

Lemma 3.8

For all $\sigma, \sigma_1, \sigma_2 \in SUB^*(\underline{\mathcal{T}})$, $\sigma^*, \sigma_1^*, \sigma_2^* \in SUB(\underline{\mathcal{T}^*})$, $t \in \underline{\mathcal{T}}$ and $t^* \in \underline{\mathcal{T}^*}$ it holds:

(a) $\Phi(\sigma(t)) = \Phi_S(\sigma)(\Phi(t))$

(b) $\Phi^{-1}(\sigma^*(t^*)) = \Phi_S^{-1}(\sigma^*)(\Phi^{-1}(t^*))$

(c) $\Phi_S(\sigma_2\sigma_1) = \Phi_S(\sigma_2)\Phi_S(\sigma_1)$

(d) $\Phi_S^{-1}(\sigma_2^*\sigma_1^*) = \Phi_S^{-1}(\sigma_2^*)\Phi_S^{-1}(\sigma_1^*)$.

Proof:

(a) Follows from Lemma 2.5 (b), which is applicable because of $D(\Phi) \cap D(\sigma) = \emptyset$, $I(\Phi) = \emptyset = I(\Phi) \cap (D(\sigma) \cup D(\Phi))$.

(b) Let $\sigma^* \in SUB(\underline{\mathcal{T}^*})$, $t^* \in \underline{\mathcal{T}^*}$ be given. Since Φ and Φ_S are bijections it suffices to show for $t^* = \Phi(t)$ that $\Phi(\Phi^{-1}(\sigma^*(\Phi(t))) = \Phi(\Phi_S^{-1}(\sigma^*)(\Phi^{-1}(\Phi(t))))$ which is equivalent to $\sigma^*(\Phi(t) =$

$\Phi(\Phi_S^{-1}(\sigma^*)(t))$. Using (a) and again Lemma 2.5 (b) we get $\Phi(\Phi_S^{-1}(\sigma^*)(t)) = \Phi_S(\Phi_S^{-1}(\sigma^*)(\Phi(t))) = \sigma^*(\Phi(t))$.

(c) Let $\sigma_1, \sigma_2 \in$ SUB*($\mathcal{T}$), $t^* \in \mathcal{T}^*$ with $t^* = \Phi(t)$ be given. Then using (a) we get $(\Phi_S(\sigma_2\sigma_1))(t^*) = (\Phi_S(\sigma_2\sigma_1))(\Phi(t)) = \Phi((\sigma_2\sigma_1)(t)) = \Phi(\sigma_2(\sigma_1(t))) = \Phi_S(\sigma_2)(\Phi(\sigma_1(t))) = \Phi_S(\sigma_2)(\Phi_S(\sigma_1)(\Phi(t))) = (\Phi_S(\sigma_2)\Phi_S(\sigma_1))(t^*)$.

(d) Follows easily from (c).

■

With $\Phi(E) := \{\Phi(s)=\Phi(t) | s=t \in E\}$ the connection between the solution sets of the RUP ‹E/W› and the transformed (unrestricted) UP ‹Φ(E)› is captured by

Lemma 3.9 (correspondence between solution sets)

The properties of substitutions to be a (most general) solution and of solution sets to be complete (and minimal) are preserved under the transformation Φ, i.e. for all $\sigma \in$ SUB*($\mathcal{T}$), Q $\subseteq$ SUB*($\mathcal{T}$) the following properties hold:

(a) $\sigma \in U(E/W) \Leftrightarrow \Phi_S(\sigma) \in U(\Phi(E))$

(b) Q is a csu for ‹E/W› $\Leftrightarrow$ $\Phi_S(Q)$ is a csu for ‹Φ(E)›

(c) Q is a cmsu for ‹E/W› $\Leftrightarrow$ $\Phi_S(Q)$ is a cmsu for ‹Φ(E)›

(d) σ is a mgu for ‹E/W› $\Leftrightarrow$ $\Phi_S(\sigma)$ is a mgu for ‹Φ(E)›.

Proof:

(a) Let $\sigma \in U(E/W)$ be given. Then, for all $t = t' \in E$, $\sigma(t) = \sigma(t')$ implies $\Phi(\sigma(t)) = \Phi(\sigma(t'))$ which is equivalent to $\Phi_S(\sigma)(\Phi(t)) = \Phi_S(\sigma)(\Phi(t'))$ yielding $\Phi_S(\sigma) \in U(\Phi(E))$. Conversely, for $\sigma^* \in U(\Phi(E))$ we have for all $\Phi(t) = \Phi(t') \in \Phi(E)$ that $\sigma^*(\Phi(t)) = \sigma^*(\Phi(t'))$ implying $\Phi^{-1}(\sigma^*(\Phi(t))) = \Phi^{-1}(\sigma^*(\Phi(t')))$ which is equivalent to $\Phi_S^{-1}(\sigma^*)(\Phi^{-1}(\Phi(t))) = \Phi_S^{-1}(\sigma^*)(\Phi^{-1}(\Phi(t')))$. Thus we get $\Phi_S^{-1}(\sigma^*) \in U(E/W)$.

(b) Let Q $\subseteq$ SUB*($\mathcal{T}$) be a csu for ‹E/W›, $\sigma^* \in U(\Phi(E))$. For $\sigma := \Phi_S^{-1}(\sigma^*) \in U(E/W)$ we know that there exists $\psi \in Q$, $\tau \in$ SUB*($\mathcal{T}$) with $\tau\psi \equiv_{V(E)} \sigma$ which implies $\Phi_S(\tau\psi) \equiv_W \Phi_S(\sigma) = \sigma^*$. Using Lemma 3.8 (c) we get $\Phi_S(\tau)\Phi_S(\psi) \equiv_W \sigma^*$ with $\Phi_S(\psi) \in \Phi_S(Q)$. The inverse direction is analogous.

(c) follows from the property $\varphi_1 \leq_{V(E)} \varphi_2 \Leftrightarrow \Phi_S(\varphi_1) \leq_W \Phi_S(\varphi_2)$ ($\varphi_1, \varphi_2 \in$ SUB*($\mathcal{T}$)) and

(d) is a consequence of (b).

■

In the following we are mainly interested in RUP's, where only the forbidden variables may be of second order. In this case the transformed problem clearly is a first order one. Solving an unrestricted UP ‹E› with $E \subseteq \mathcal{T}^2$, $V(E) \subseteq \mathcal{V}_0$ over $\mathcal{T}$ is essentially the same as solving it over $\mathcal{T}_0$, i.e. as an ordinary first order UP over first order terms. This is obvious considering the following solution preserving and terminating set $\mathcal{R}$ of transformation rules for such UP's over $\mathcal{T}$ where the symbol FAIL is to denote unsolvability (cf. [GaSn87], [GaSn88], [MaMo82]):

(1) Deletion of trivial pairs

$\{ s = s \} \cup E \rightarrow E$

(2) Term decomposition

$\{ f(s_1,\ldots,s_n) = f(t_1,\ldots,t_n) \} \cup E \rightarrow \{ s_i = t_i \mid 1 \le i \le n \} \cup E$

(3) Variable elimination

$\{ x = t \} \cup E \rightarrow \{ x = t \} \cup \sigma(E),$ if $x \in V(E), x \notin V(t), \sigma = \{x \leftarrow t\}$

(4) Clash

$\{ f(s_1,\ldots,s_m) = g(t_1,\ldots,t_n) \} \cup E \rightarrow FAIL,$ if $f \neq g$

(5) Occur check

$\{x = t\} \cup E \rightarrow FAIL,$ if $x \neq t, x \in V(t)$

Using the rules (1)-(3) any solvable UP ‹E› can be transformed into an equivalent UP ‹E'› which is in solved form and thus describes an mgu for ‹E'› as well as for ‹E› (see [GaSn87]). Rules (4) and (5) are needed to detect unsolvability. The rule system $\mathcal{R}$ may be looked upon as an abstract formulation of a big class of unification algorithms since it is terminating but does not fix the control structure for rule application (note that in the above presentation there is an additional source of nondeterminism because pairs s=t are considered as unordered multisets). Moreover it provides a very useful computational proof technique as we will see later on.

Using Lemma 3.9 (d) we can deduce that any RUP ‹E/W›, $E \subseteq \mathcal{T}^2$, with $W \subseteq \mathcal{V}_0$ is either unsolvable or possesses a mgu (which is unique up to $\equiv_{V(E)}$-equivalence) and may be computed by any of the well-known algorithms for first order unification (see figure).

```
                        Φ
‹E/W› -----------------> ‹Φ(E)›
  |                        |
  |                        | first order unification
  |                Φ⁻¹     |
Φ⁻¹(E') <-------------------  E' in solved form
Φ_S⁻¹(σ_E') mgu for          σ_E' mgu for
‹Φ⁻¹(E')/W› and ‹E/W›        ‹E'› and ‹Φ(E)›
```

The explicit translation steps using Φ, Φ^{-1} may be avoided by freezing the forbidden variables from W^c within the first order unification process.

We have seen that, if an RUP ‹E/W› with $W \subseteq \mathcal{V}_0$ is solvable, then it has an mgu which is unique (up to $\equiv_{V(E)}$-equivalence). It is possible that ‹E› is solvable, but not ‹E/W›. The next result shows that if ‹E/W› with $V(E) \subseteq \mathcal{V}_0$ is solvable the resulting mgu is also most general for the unrestricted UP ‹E› whereas this property cannot be generalized to arbitrary second order RUP's.

Lemma 3.10

Let $\langle E/W \rangle$, $E \subseteq \mathcal{T}^2$ be given.

(a) If $V(E) \subseteq \mathcal{V}_0$ and σ is a mgu for $\langle E/W \rangle$ then σ is also a mgu for $\langle E \rangle$.

(b) If $V(E) \nsubseteq \mathcal{V}_0$ and S is a csu for $\langle E/W \rangle$ then S is not necessarily a csu for $\langle E \rangle$.

Proof:

(a) Let σ be a mgu for $\langle E/W \rangle$ obtained by transformation of $\langle \Phi(E) \rangle$ into solved form $\langle E' \rangle$ using (rules (1)-(3) of) $\mathcal{R}$: $\langle \Phi(E) \rangle \rightarrow_{\mathcal{R}}^{*} \langle E' \rangle$, $\sigma = \Phi_S^{-1}(\sigma_{E'})$. Any step $E_1 \rightarrow_{(i)} E_2$ using rule (i), $1 \le i \le 3$, in this derivation may be translated into a step $\Phi^{-1}(E_1) \rightarrow_{(i)} \Phi^{-1}(E_2)$. For i = 1,2 this is obvious. For the variable elimination rule

$$\{ x = t^* \} \cup E^* \rightarrow_{(3)} \{ x = t^* \} \cup \sigma^*(E^*), \quad \text{if } x \in V(E^*), x \notin V(t^*), \sigma^* = \{x \leftarrow t^*\}$$

we have

$$\Phi^{-1}(x) = x \notin V(\Phi^{-1}(t^*)) \subseteq V(t^*), \quad x \in V(\Phi^{-1}(E^*))$$

implying

$$\{ x = \Phi^{-1}(t^*) \} \cup \Phi^{-1}(E^*) \rightarrow_{(3)} \{ x = \Phi^{-1}(t^*) \} \cup \sigma(\Phi^{-1}(E^*))$$

with $\sigma := \{x \leftarrow \Phi^{-1}(t^*)\}$. Using $\sigma = \Phi_S^{-1}(\sigma^*)$, $t^* = \Phi(t)$ and Lemma 3.8 (b) we get

$$\sigma(\Phi^{-1}(E^*)) = \Phi_S^{-1}(\sigma^*)(\Phi^{-1}(E^*) = \Phi^{-1}(\sigma^*(E^*))$$

and thus

$$\Phi^{-1}(\{x = t^* \} \cup E^*) \rightarrow_{(3)} \Phi^{-1}(\{ x = t^* \} \cup \sigma^*(E^*))$$

as desired. From $\langle \Phi(E) \rangle \rightarrow_{\mathcal{R}}^{*} \langle E' \rangle$ and $\langle \Phi^{-1}(\Phi(E)) \rangle \rightarrow_{\mathcal{R}}^{*} \langle \Phi^{-1}(E') \rangle$ we finally deduce that $\langle \Phi^{-1}(E') \rangle$ is also in solved form yielding the mgu $\Phi_S^{-1}(\sigma_{E'}) = \sigma$ for the unrestricted UP $\langle E \rangle$ (note that this result could also be proved without the rule system $\mathcal{R}$ using only reasoning about first order substitutions and renamings).

(b) Consider the following counterexample: For E: X(y) = X(a), $X \in \mathcal{V}_1$. It may easily be verified that $\{\sigma_1, \sigma_2\}$ with $\sigma_1 := \{y \leftarrow a\}$, $\sigma_2 := \{X \leftarrow \lambda u.z\}$ is a csu for $\langle E \rangle$ such that σ_1 and σ_2 are incomparable w.r.t. $\le_{V(E)}$. For W = {y}, $W^c = \{X\}$, σ_1 is a mgu for $\langle E/W \rangle$. For W = {X}, $W^c = \{y\}$, σ_2 is again a mgu for $\langle E/W \rangle$, but neither $\{\sigma_1\}$ nor $\{\sigma_2\}$ is a csu for the unrestricted UP $\langle E \rangle$.

■

If the RUP $\langle E/W \rangle$ with $E \subseteq \mathcal{T}^2$, $W \subseteq \mathcal{V}_0$ is solvable, say with mgu σ, then the instantiated problem $\langle \psi(E)/W \rangle$ is solvable, too, for every substitution ψ that leaves the variables from W untouched and does not introduce any such variable. Moreover, in the case of a strict ψ the resulting mgu may be computed from σ and ψ without explicitely unifying again. This is detailed by

Theorem 3.11 (solutions for instantiated RUP's)

Let $\langle E/W \rangle$ with $E \subseteq \mathcal{T}^2$, $W \subseteq V(E)$, $W \subseteq \mathcal{V}_0$ be a solvable RUP and σ be a mgu for it (w.l.o.g. let $I(\sigma) \subseteq V(E)$). Assume further that ψ is a substitution with $D(\psi) \subseteq V(E)\backslash W$, $I(\psi) \cap W = \emptyset$. Then $(\psi\sigma)|_{D(\sigma)}$ is a solution for the RUP $\langle \psi(E)/W \rangle$ which is most general for strict ψ.

Proof: Let $\langle E/W \rangle$, σ and ψ satisfy the assumptions. By $\rightarrow_{\mathcal{R}V}$, $V \subseteq \mathcal{V}$ we denote the reduction relation defined by the transformation system $\mathcal{R}$, where the variables from V are frozen, i.e.

considered as distinct new constants. Analogously $\rightarrow_{(i),V}$ denotes one $(\mathcal{R},V)$-step using rule (i), $1 \le i \le 3$. W.l.o.g. we assume further that the mgu σ for $\langle E/W \rangle$ has been constructed by transformation of $\langle E/W \rangle$ into solved form E' using $\mathcal{R}$: $E \rightarrow_{\mathcal{R},P}^{*} E'$, $P := V(E)\backslash W$, $\sigma = \sigma_{E'}$. We will show now that for strict ψ this derivation can be translated into a derivation $\psi(E) \rightarrow_{\mathcal{R},Q}^{*} \psi(E')$ with $Q := V(\psi(E))\backslash W$. The conditions $D(\psi) \subseteq V(E)\backslash W$ and $I(\psi) \cap W = \emptyset$ imply then that $\langle \psi(E') \rangle$ is in solved form, too. Thus, $\sigma_{\psi(E')} = (\psi\sigma)|_{D(\sigma)}$ is a mgu for $\langle \psi(E)/W \rangle$ as required. So, let us consider an arbitrary step $E_1 \rightarrow_{\mathcal{R},P} E_2$ within the derivation $E \rightarrow_{\mathcal{R},P} E'$. Translation will yield $\psi(E_1) \rightarrow_{\mathcal{R},Q}^{+} \psi(E_2)$ according to the following cases.

(1) $\{ s = s \} \cup E_1 \rightarrow_{(1),P} E_1$ obviously implies $\{ \psi(s) = \psi(s) \} \cup \psi(E_1) \rightarrow_{(1),Q} \psi(E_1)$.

(2) $\{ f(s_1,\ldots,s_n) = f(t_1,\ldots,t_n) \} \cup E_1 \rightarrow_{(2),P} \{ s_i = t_i \mid 1 \le i \le n \} \cup E_1$:
If f is not among the frozen variables from $P = V(E)\backslash W$ interpreted as new constants then we have $\psi(\{ f(s_1,\ldots,s_n) = f(t_1,\ldots,t_n) \} \cup E_1) \rightarrow_{(2),Q} \psi(\{ s_i = t_i \mid 1 \le i \le n \} \cup E_1)$. Otherwise $\psi(f(s_1,\ldots,s_n)) = \psi(f(t_1,\ldots,t_n))$ can be simplified using $\rightarrow_{\mathcal{R},Q}$ by repeated application of decomposition (2) and deletion of trivial problems (1) until only the pairs $\psi(s_i) = \psi(t_i)$, $1 \le i \le n$ are left. Note that the s_i, t_i do not disappear because ψ is strict. Thus we get
$\psi(\{ f(s_1,\ldots,s_n) = f(t_1,\ldots,t_n) \} \cup E_1) \rightarrow_{\mathcal{R},Q}^{+} \psi(\{ s_i = t_i \mid 1 \le i \le n \} \cup E_1)$.

(3) $\{ x = t \} \cup E_1 \rightarrow_{(3),P} \{ x = t \} \cup \varphi(E_1)$ with $x \in V(E_1)\backslash P$, $x \notin V(t)$, $\varphi = \{x \leftarrow t\}$:
From the fact that no new variables are introduced by rule application and the assumptions about W and ψ we deduce $x \in V(E_1)\backslash P \subseteq V(E)\backslash P = W$, $x \in V(\psi(E_1))\backslash Q$, $x = \psi(x) \notin V(\psi(t))$. This implies $\psi(\{x = t\} \cup E_1) \rightarrow_{(3),Q} \psi(\{x = t\}) \cup \varphi'(\psi(E_1))$ with $x \in V(\psi(E_1))\backslash Q$, $x \notin V(\psi(t))$, $\varphi' := \{x \leftarrow \psi(t)\}$. Since $\varphi'(\psi(E_1)) = (\psi\varphi)|_{D(\varphi)}(\psi(E_1)) = \psi(\varphi(E_1))$ using Lemma 2.5 (d) we can conclude $\psi(\{x = t\} \cup E_1) \rightarrow_{(3),Q} \psi(\{x = t\} \cup \varphi(E_1))$ as desired.

Note that $(\psi\sigma)|_{D(\sigma)} \in U(\psi(E)/W)$ also holds for every (not necessarily strict) ψ with $D(\psi) \subseteq V(E)\backslash W$, $I(\psi) \cap W = \emptyset$ because of Lemma 2.5 (d).

■

The strictness assumption for ψ in the above proof is necessary to ensure that the solution $(\psi\sigma)|_{D(\sigma)}$ of $\langle \psi(E)/W \rangle$ is most general (cf. case (2) in the proof). Take for example E: $f(u,X(u)) = f(v,X(a))$, $W = \{u,v\}$ and $\psi = \{X \leftarrow \lambda z.w\}$. Here $\sigma := \{u \leftarrow a, v \leftarrow a\}$ is a mgu for $\langle E/W \rangle$ but $(\psi\sigma)|_{D(\sigma)} = \sigma \in U(\psi(E)/W)$ is not most general. Indeed we get $\psi(E)$: $f(u,w) = f(v,w)$, which has as mgu $\tau := \{u \leftarrow v\}$ such that $\tau < \sigma$, i.e. $\tau \le \sigma$, but $\sigma \not\le \tau$.

Theorem 3.11 essentially provides a characterization of a whole class of RUP's as described by

Corollary 3.12

For $E \subseteq \mathcal{T}^2$, $W \subseteq V(E)$, $W \subseteq \mathcal{V}_0$ the RUP $\langle E/W \rangle$ is solvable iff $\langle \psi(E)/W \rangle$ is solvable for every ψ with $D(\psi) \subseteq V(E)\backslash W$ and $I(\psi) \cap W = \emptyset$, and if so, for any such ψ which is strict the mgu for $\langle \psi(E)/W \rangle$ may be computed from the mgu for $\langle E/W \rangle$ as described in Theorem 3.11.

Proof: The only-if-direction is provided by theorem 3.11 and for the if-direction we simply choose the identitiy substitution for ψ.

■

We now come back to our original problem of solving scheme unification problems (SUP's, see Def. 3.1), i.e. to decide for given $\langle E/W \rangle$ with $W \subseteq V(E)$, $W \subseteq \mathcal{V}_0$, whether all first order (unrestricted) UP's $\langle \psi(E) \rangle$ with $D(\psi) \subseteq V(E)\backslash W \cap \mathcal{V}_0$ and $I(\psi) \cap W = \emptyset$ are solvable, and if so, to compute the corresponding mgu's. The following result shows how to do that. Essentially it is a stronger version of corollary 3.12 stating that under slightly stronger preconditions concerning the underlying signature the "if-direction" remains true, when ψ is restricted to first order substitutions (i.e., $I(\psi) \subseteq \mathcal{V}_0$).

Theorem 3.13 (solving scheme unification problems)
Let $E \subseteq \mathcal{T}(C,\mathcal{V})^2$, $W \subseteq V(E)$, $W \subseteq \mathcal{V}_0$ be given. Assume further that the underlying signature contains at least one function constant of arity ≥ 1 and another one of arity ≥ 2. Then the RUP $\langle E/W \rangle$ is solvable iff the SUP $\langle E/W \rangle$ is solvable, i.e. $\langle \psi(E) \rangle$ is solvable for every ψ with $D(\psi) \subseteq V(E)\backslash W$, $I(\psi) \cap W = \emptyset$ and $I(\psi) \subseteq \mathcal{V}_0$, and if so, for any such ψ which is strict the mgu for $\langle \psi(E) \rangle$ may be computed from the mgu σ for $\langle E/W \rangle$ as $(\psi\sigma)|_{D(\sigma)}$.

Proof: The interesting if-direction is again proved by considering $\mathcal{R}$-derivations to construct mgu's. Assume $\langle E/W \rangle$ is unsolvable but $\langle \psi(E) \rangle$ is solvable for every ψ with $D(\psi) \subseteq V(E)\backslash W$, $I(\psi) \cap W = \emptyset$ and $I(\psi) \subseteq \mathcal{V}_0$. Using the same notations as in the proof of Theorem 3.11 we construct from a derivation $E \rightarrow_{\mathcal{R},P}^* E' \rightarrow_{\mathcal{R},P}$ FAIL with $P := V(E)\backslash W$ another derivation $\psi(E) \rightarrow_{\mathcal{R},Q}^* \psi(E') \rightarrow_{\mathcal{R},Q}$ FAIL for some strict ψ with $D(\psi) \subseteq V(E)\backslash W$, $I(\psi) \cap W = \emptyset$ and $I(\psi) \subseteq \mathcal{V}_0$ contradicting the assumption that $\langle \psi(E) \rangle$ is solvable for every such ψ. Obviously the last step of the first derivation must be an application of clash (4) or occur ckeck (5) rule. Thus we have the following two cases:

(a) $E \rightarrow_{\mathcal{R},P}^* E' \rightarrow_{(4),P}$ **FAIL with** $E' = \{ f(s_1,\ldots,s_m) = g(t_1,\ldots,t_n) \} \cup E''$, $f \neq g$ **:**
Now, depending on whether $f,g \in C$, we choose any strict ψ with $D(\psi) \subseteq V(E)\backslash W$, $I(\psi) \cap W = \emptyset$ and $I(\psi) \subseteq \mathcal{V}_0$ and construct a derivation $\psi(E) \rightarrow_{\mathcal{R},Q}^* \psi(E') \rightarrow_{(4),Q}$ FAIL with $\psi(E') = \{f(\psi(s_1),\ldots,\psi(s_m)) = g(\psi(t_1),\ldots,\psi(t_n))\} \cup \psi(E'')$, $f \neq g$, $Q := V(\psi(E))\backslash W$ as in theorem 3.11
Again we have four subcases:

(i) For $f,g \in C$ we can choose any such ψ.

(ii) For $f \in C$, $g \notin C$, g a frozen variable corresponding to $X \in Q$, we choose a strict ψ with $\psi(X) = \lambda u_1,\ldots,u_{ar(X)}.t$, $f \neq top(t) \in C$.

(iii) The case $g \in C$, $f \notin C$ is symmetric to (ii).

(iv) For $f \notin C$, $g \notin C$, f and g corresponding to frozen variables $X \in Q$ and $Y \in Q$ respectively , we choose a strict ψ with $\psi(X) = \lambda u_1,\ldots,u_{ar(X)}.s$, $\psi(Y) = \lambda u_1,\ldots,u_{ar(Y)}.t$, $top(s) \neq top(t)$, $top(s),top(t) \in C$.

Note that for the cases (ii)-(iv) the assumption that there exist at least one function constant of arity ≥ 1 and another one of arity ≥ 2 is essential for constructing a strict ψ with the required properties.

(b) $E \rightarrow_{\mathcal{R},P}^* E' \rightarrow_{(4),P}$ **FAIL with** $E' = \{x = t\} \cup E$, $x \neq t$, $x \in V(t)$ **:**
In this case any strict ψ with $D(\psi) \subseteq V(E)\backslash W$, $I(\psi) \cap W = \emptyset$ and $I(\psi) \subseteq \mathcal{V}_0$ is sufficient to construct a derivation $\psi(E) \rightarrow_{\mathcal{R},Q}^* \psi(E') \rightarrow_{(4),Q}$ FAIL, since $x \neq t$, $x \in V(t)$ implies $x \neq \psi(t)$, $x \in V(\psi(t))$ for

strict ψ.

∎

In order to illustrate this result we consider the second example of the introduction. Here we have the SUP $\langle E/W \rangle$, $E := \{f(h(u),v)=f(w,Y(w))\}$, $W := \{u,v,w\}$ with only one (unary) scheme variable, namely Y. Solving this problem with the techniques described amounts to solving the RUP $\langle E/W \rangle$ which yields a mgu $\sigma := \{w \leftarrow h(u), v \leftarrow Y(h(u))\}$. Now, take for instance the strict substitution $\psi := \{Y \leftarrow \lambda x.f(y,h(x))\}$. Then $(\psi\sigma)|_{D(\sigma)} = \{w \leftarrow h(u), v \leftarrow f(y,h(h(u)))\}$ is a mgu for $\langle \psi(E) \rangle = \langle f(h(u),v)=f(w,f(y,h(w))) \rangle$.

The preconditions concerning the signature in the above theorem are really necessary in general for the validity of the "if-direction". Consider the following counterexamples:

(a) **$\mathcal{C}$ contains only function constants of arity 0 or 1:**
For E: $X(X(v,w),X(w,v)) = X(u,u)$, $W := \{u\}$, the RUP $\langle E/W \rangle$ is unsolvable, because we have $\{\Phi(E)\} \to_{\mathcal{RP}}^* \{u = X(v,w),\ v = w\} \to_{(4),P}$ FAIL. But for every ψ with $D(\psi) \subseteq V(E)\backslash W$, $I(\psi) \cap W = \emptyset$, $I(\psi) \subseteq \mathcal{V}_0$ the UP $\langle \psi(E) \rangle$ is solvable. For $\psi(X) = \lambda yz.t$ we know that t cannot contain both y and z because of the restricted signature. By an easy case analysis for t we can prove now that $\langle \psi(E) \rangle$ is always solvable. For example, if $y \in V(t)$, then y is the only variable occurring in t (moreover it occurs only once). Thus, with the notation $t = t[y]$ we get $\psi(E)$: $t[t[\psi(v)]] = t[u]$, which, transformed into solved form, yields E': $u = t[\psi(v)]$.

(b) **$\mathcal{C}$ contains one function constant f of arity greater or equal 2 and all others have arity 0:**
For X with $ar(X) = ar(f)$ and E: $X(X(y,\ldots,y),\ldots,X(y,\ldots,y)) = X(f(z_1,\ldots,z_{ar(f)}),\ldots,f(z_1,\ldots,z_{ar(f)}))$, $W := \{y, z_1,\ldots,z_{ar(f)}\}$ one may prove by case analysis analogous to (a) that the RUP $\langle E/W \rangle$ again is unsolvable but for every ψ with the corresponding conditions $\langle \psi(E) \rangle$ is solvable.

Let us conclude this chapter with some historical remarks. The idea of freezing certain variables in unification problems, i.e. considering them as distinct new constants, has already been used in the pioneering work of Huet ([Hu76]) in order to compute complete independent sets of preunifiers for semi-unification problems in typed λ-calculus. From a logical point of view restricted unification problems may be represented by formulae of the form

$$(*)\quad \forall\, \underline{x}\, \exists\, \underline{y} : s_1 = t_1 \wedge \ldots \wedge s_n = t_n ,$$

where the variables of the s_i, t_i are partitioned into the disjunct variable sets $\underline{x}$ and $\underline{y}$. Solving (*) as a restricted unification problem amounts to looking for values for the existentially quantified variables $\underline{y}$ that make the formulae valid, whereas the universally quantified variables have to be left untouched (cf. [Bü86], [Co88]). Within this logical framework it becomes obvious that considering the universally quantified variables as distinct new constants essentially is an application of the "Theorem on Constants" (cf. [Sh73]). This fundamental result, roughly spoken, states that universally quantified variables in logical formulae may be equivalently replaced by distinct new constants. In our context we do not only have the equivalence between (*) and

$$(**)\ \exists\ \underline{y} : \Phi(s_1) = \Phi(t_1) \ \wedge\ \dots\ \wedge\ \Phi(s_n) = \Phi(t_n)\ ,$$

i.e. equivalence concerning solvability, but also the correspondence between solutions as follows:

σ solves $\forall\ \underline{x} \exists\ \underline{y} : \bigwedge_{i=1,\dots,n} s_i = t_i$

$\Leftrightarrow\ \forall\ \underline{x} : \bigwedge_{i=1,\dots,n} \sigma(s_i) = \sigma(t_i)$

$\Leftrightarrow\ \bigwedge_{i=1,\dots,n} \Phi(\sigma(s_i)) = \Phi(\sigma(t_i))$

$\Leftrightarrow\ \bigwedge_{i=1,\dots,n} \Phi_S(\sigma)(\Phi(s_i)) = \Phi_S(\sigma)(\Phi(t_i))$ (using lemma 2.5 (d))

$\Leftrightarrow\ \Phi_S(\sigma)$ solves $\exists\ \underline{y} : \bigwedge_{i=1,\dots,n} \Phi(s_i) = \Phi(t_i)$.

4. Applications to Divergence of Completion

The previous results are applied now to provide a method for establishing the solvability of an infinite sequence of UP's (constructed recursively) and for computing the corresponding mgu's. Essentially this is achieved by identifying all these UP's as special instances of a certain SUP, which is known to be solvable. For a better understanding of the following theorem which is a little bit technical because of its generality, we will first explain the scenario it is aimed for and the notations used. We are interested in repeated critical pair constructions where one of the rules involved is always fixed. In this rule A with variable set W_1 the terms s and t are (sub)terms of the left and right hand side respectively. If for a second rule B_0: $l_0 \to r_0$ the left hand side l_0 (or a subterm of it) may be represented as $\psi_0(M)$, i.e. as a certain instance of a term scheme M with scheme variable set W_{22}, then theorem 4.1 states sufficient conditions for a property of repeated unifiability, i.e. for repeatedly constructing critical pairs starting from A and B_0.

Theorem 4.1 (a criterion for repeated unifiability)
Let $s,t \in \mathcal{T}_0$, $M \in \mathcal{T}$ be given with $V(s) \cup V(t) \subseteq W_1 \subseteq \mathcal{V}_0$, $V(M) =: W_{21} \cup W_{22} \subseteq W_2 \subseteq \mathcal{V}$, $W_{21} \subseteq \mathcal{V}_0$, $W_1 \cap W_2 = \emptyset$ and $W := V(s) \cup W_{21}$. Assume further that the RUP $\langle s=M/W \rangle$ is solvable, say with mgu σ, such that $I(\sigma) \subseteq V(s) \cup V(M)$ and $\sigma(t) \equiv \pi(\sigma(t)) = \psi(M)$ where

(i) π is a renaming substitution for $V(\sigma(t))\backslash W_{22}$ away from W_{22}, and

(ii) ψ is a strict substitution with $D(\psi) \subseteq W_{22}$ and $I(\psi) \cap (W_1 \cup W_2) \subseteq W_{22}$.

Then for every strict ψ_0 satisfying $P(\psi_0)$ with

$$P(\tau) := [\, D(\tau) \subseteq W_{22} \ \wedge\ V(\tau(M)) \subseteq \mathcal{V}_0 \ \wedge\ I(\tau) \cap (W_1 \cup W_2 \cup I(\psi)) \subseteq W_{22} \,]$$

the UP $\langle s=\psi_0(M) \rangle$ is solvable with mgu $(\psi_0\sigma)|_{D(\sigma)}$ such that $(\psi_0\sigma)|_{D(\sigma)}(t) \equiv \psi_1(M)$ for some strict ψ_1 again satisfying $P(\psi_1)$.

Proof: Let s, t, M, W_1, W_2, W_{21}, W_{22}, σ, π and ψ satisfy the assumptions and let ψ_0 be any strict substitution satisfying $P(\psi_0)$. Since the RUP $\langle s=M/W \rangle$ is solvable with mgu σ, Theorem 3.13 implies that the UP $\langle s=\psi_0(M) \rangle$ is solvable with mgu $(\psi_0\sigma)|_{D(\sigma)}$. From $V(t) \cap D(\psi_0) = \emptyset$ we get $(\psi_0\sigma)|_{D(\sigma)}(t) = (\psi_0\sigma)(t)$. And from $\sigma(t) \equiv (\pi\sigma)(t) = \psi(M)$ and $P(\psi_0)$ we deduce $(\psi_0\sigma)(t) \equiv (\psi_0\pi\sigma)(t) = (\psi_0\psi)(M)$. Choosing $\psi_1' := \psi_0\psi$ now, we are almost done, because the strictness of

ψ_1' and the properties $D(\psi_1') \subseteq W_{22}$, $V(\psi_1'(M)) \subseteq \mathcal{V}_0$ and $I(\psi_1') \cap (W_1 \cup W_2) \subseteq W_{22}$ are inherited from the corresponding properties of ψ_0 and ψ. In order to establish $I(\psi_1) \cap I(\psi) \subseteq W_{22}$ we have to replace the (first order) variables from $(I(\psi_1') \cap I(\psi))\backslash W_{22}$ by new ones, say with the renaming substitution π_0, thus yielding $(\psi_0\sigma)|_{D(\sigma)}(t) \equiv \psi_1'(M) = (\pi_0\psi_0\psi)(M) \equiv (\psi_0\pi_0\psi)(M) = (\psi_0(\pi_0\psi)|_{D(\psi)})(M)$. For $\psi_1 := \psi_0(\pi_0\psi)|_{D(\psi)}$ all required conditions are satisfied now, as it can be easily verified.

■

Note that for given s, t and M in the above theorem the existence of π and ψ with the required properties is decidable because second order matching is decidable (cf. [Hu76]).

Before concluding let us give an idea of how to apply the above criterion to divergence analysis. A more detailed and extended investigation of divergence phenomena of completion procedures is the subject of a forthcoming paper ([Gr88]).

We will take up again the examples of the introduction to illustrate the methodology. In the first example completion with input A: $f(h(u),v) = f(u,h(v))$ and B_0: $f(w,a) = w$ runs forever producing an infinite sequence of rules B_n: $f(w,h^n(a)) \to h^n(w)$. Using the notations of theorem 4.1 and choosing

$$M:= f(w,X),\ s = f(h(u),v),\ t = f(u,h(v)),$$

$$W_1 := V(s) = V(t),\ W_{21} := \{w\},\ W_{22} := \{X\},\ W_2 := V(M) \text{ and } W := \{u,v,w\}$$

we get

$$\sigma := mgu(s=M/W) = \{w \leftarrow h(u), v \leftarrow X\}.$$

This yields

$$\sigma(t) = f(u,h(X)) \equiv (\pi\sigma)(t) = \psi(M)$$

with

$$\pi := \{u \leftarrow w\},\ \psi := \{X \leftarrow h(X)\}.$$

Furthermore we have

$$l_0 = \psi_0(M) \text{ with } \psi_0 := \{X \leftarrow a\}$$

and, using theorem 4.1,

$$\sigma_0 := mgu(s=\psi_0(M)) = (\psi_0\sigma)|_{D(\sigma)} = \{w \leftarrow h(u), v \leftarrow a\}$$

yielding the critical pair

$$(\sigma_0(t),\sigma_0(r_0)),\ \sigma_0(t) = f(u,h(a)),\ \sigma_0(r_0) = h(u).$$

Renaming with π and orientation leads to B_1: $l_1 \to r_1$ with

$$l_1 = f(w,h(a)) = (\pi\sigma_0)(t) = (\pi\sigma_0)|_{V(t)}(t) = (\psi_0\psi)(M) = \psi_1(M),\ \psi_1 := \psi_0\psi = \{X \leftarrow h(a)\}$$

and

$$r_1 = h(w) = (\pi\sigma_0)(r_0) = (\pi\sigma_0)|_{V(r0)}(r_0).$$

Renaming again in order to establish $I(\psi_1) \cap I(\psi) \subseteq W_{22}$ is not necessary for the next step, i.e. a new application of theorem 4.1 using ψ_1 instead of ψ_0. Indeed, since there are no new variables introduced here in the unification process, we can easily verify by induction that B_n has the general form

$$(\psi_0\psi^n)(M) \to (\pi\sigma_0)|_{V(r0)}{}^n(r_0)$$

with

$$\psi_0\psi^n = \{X\leftarrow a\}\ \{X\leftarrow h(X)\}^n = \{X\leftarrow a\}\ \{X\leftarrow h^n(X)\} = \{X\leftarrow h^n(a)\}$$

and

$$(\pi\sigma_0)|_{V(r0)}{}^n = (\ \{u\leftarrow w\}\ \{w\leftarrow h(u),v\leftarrow a\}\ )|_{\{w\}}{}^n = \{w\leftarrow h(w)\}^n = \{w\leftarrow h^n(w)\}$$

yielding

$$B_n:\ \{X\leftarrow h^n(a)\}\ f(w,X)\ \rightarrow\ \{w\leftarrow h^n(w)\}\ w.$$

In the second example of the introduction completion with input A: f(h(u),v) = f(u,h(v)) and C_0: f(w,g(w)) = w diverges producing C_n: $f(w,h^n(g(h^n(w)))) \rightarrow h^n(w)$, $n\geq 0$. Choosing s, t, W_1, W_{21} as above and M := f(w,Y(w)), W_{22} :={Y}, W_2 := V(M) = {w,Y} we get σ := mgu(s=M/W) = {w← h(u),v←Y(h(u))}. Thus we have $\sigma(t) = f(u,h(Y(h(u)))) \equiv (\pi\sigma)(t) = f(w,h(Y(h(w)))) = \psi(M)$ with $\pi := \{u\leftarrow w\}$, $\psi := \{Y\leftarrow\lambda z.h(Y(h(z)))\}$. Since $l_0 = \psi_0(M)$ with $\psi_0 := \{Y\leftarrow\lambda z.g(z)\}$ repeated application of theorem 4.1 as above results in the general form for C_n

$$C_n:\ \{Y\leftarrow\lambda z.h^n(g(h^n(z)))\}\ f(w,Y(w))\ \rightarrow\ \{w\rightarrow h^n(w)\}\ w.$$

Finally, in order to demonstrate the generality of the framework presented, let us give a more complicated famous example of divergent completion behaviour, namely associativity and idempotency. Starting from A: f(f(u,v),w) = f(u,f(v,w)) and B_0: f(x,x) = x completion diverges producing

A	: $f(f(u,v),w) \rightarrow f(u,f(v,w))$	
B_0	: $f(x,x) \rightarrow x$	
B_1	: $f(x,f(x_0,f(x,x_0))) \rightarrow f(x,x_0)$	$CP(A,\varepsilon,B_0)$
B_1'	: $f(x,f(x,x_0)) \rightarrow f(x,x_0)$	$CP(B_0',1,A)$
B_2	: $f(x,f(x_1,f(x_0,f(x,f(x_1,x_0))))) \rightarrow f(x,f(x_1,x_0))$	$CP(A,\varepsilon,B_1)$
B_2'	: $f(x,f(x_1,f(x,f(x_1,x_0)))) \rightarrow f(x,f(x_1,x_0))$	$CP(B_1',1,A)$
....		
B_{n+1}	: $f(x,f(x_n,\ldots f(x_0,f(x,f(x_n,\ldots,f(x_1,x_0)\ldots)))\ldots)) \rightarrow f(x,f(x_n,\ldots,f(x_1,x_0)\ldots))$	$CP(A,\varepsilon,B_n)$
B_{n+1}'	: $f(x,f(x_n,\ldots,f(x_1,f(x,f(x_n,\ldots,f(x_1,x_0)\ldots)))\ldots)) \rightarrow f(x,f(x_n,\ldots,f(x_1,x_0)\ldots))$	$CP(B_n',1,A)$
....		

The notion CP(D,o,E) in the right column indicates that the corresponding rule stems from the critical pair obtained by superposition of rule D into rule E at position o. Here we have (among others) two infinite sequences of rules generated and moreover the example is still more complicated since there are new variables introduced in any generated rule. Identifying the left hand side $l_0 := l_0' :=$ f(x,x) of $B_0 =: B_0'$ as $\psi_0(M)$ with

$$M := f(x,Y(x)),\ \psi_0 := \{Y\leftarrow\lambda y.y\},\ W := \{u,v,w,x\},\ W_{22} := \{Y\},$$

we can proceed as above and get

$$\sigma := mgu(s=M/W) = \{x\leftarrow f(u,v),w\leftarrow Y(f(u,v))\},$$

$$\sigma(t) = f(u,f(v,Y(f(u,v))) \equiv (\pi\sigma)(t) = f(x,f(z,Y(f(x,z)))) = \psi(M)$$

with

$$\psi := \{Y \leftarrow \lambda y.f(z,Y(f(y,z)))\}$$

and

$$\pi := \{u \leftarrow x, v \leftarrow z\}.$$

This yields

$$\sigma_0 := mgu(s{=}l_0) = (\psi_0\sigma)|_{D(\sigma)} = \{x \leftarrow f(u,v), w \leftarrow f(u,v)\}$$

with

$$\sigma_0(t) = f(u,f(v,f(u,v))),\ \sigma_0(r_0) = f(u,v).$$

After renaming with π we get

$$\pi(\sigma_0(t)) = f(x,f(z,f(x,z))) = \psi_1'(M),\ \psi_1' := \psi_0\psi,$$
$$\pi(\sigma_0(r_0)) = f(x,z).$$

Renaming again with $\pi_0 := \{z \leftarrow x_0\}$ in order to establish $I(\psi_1) \cap I(\psi) \subseteq W_{22} = \{Y\}$ leads to $B_1{:}l_1 \rightarrow r_1$,

$$l_1 := f(x,f(x_0,f(x,x_0))) = \psi_0(\pi_0\psi)|_{D(\psi)}(M) = \psi_1(M),\ \psi_1 := \psi_0(\pi_0\psi)|_{D(\psi)}$$

and

$$r_1 := f(x,x_0) = (\pi_0\pi\sigma_0)|_{V(r0)}(r_0).$$

By induction and taking into account reduction of critical pairs to normal forms (here: using (A)) we can see that $B_{n+1}{:}\ l_{n+1} \rightarrow r_{n+1}$ has the general form

$$l_{n+1} = \psi_{n+1}(M),\ r_{n+1} = \tau_{n+1}(r_0)$$

with

$$\psi_{n+1} = \{Y \leftarrow \lambda y.f(x_n,\ldots,f(x_0,f(y,f(x_n,\ldots,f(x_1,x_0)\ldots)))\ldots)\}$$

and

$$\tau_{n+1} = \{x \leftarrow f(x,f(x_n,f(x_{n-1},\ldots f(x_1,x_0)\ldots)))\}.$$

In an analogous manner we may infer for $CP(B_n',1,A)$ the general form $B_{n+1}'{:}\ l_{n+1}' \rightarrow r_{n+1}'$,

$$l_{n+1}' = \varphi(M),\ r_{n+1}' = \varrho(r_0)$$

with

$$\varphi := \{Y \leftarrow \lambda y.f(y,f(x_n,\ldots,f(x_1,f(y,f(x_n,\ldots f(x_1,x_0)\ldots)))\ldots))\}$$

and

$$\varrho := \{x \leftarrow f(x,f(x_n,\ldots,f(x_1,x_0)\ldots))\}.$$

Note that the above considerations do not only explain divergent completion behaviour for associativity (A) and idempotency (B_0), but cover a whole class of cases, namely any starting set $\{A,B^*\}$ of equations, where B^* may be represented as $\psi^*(f(x,Y(x))) \rightarrow x$ for some ψ^* having the same properties as ψ_0, e.g. $\psi^* := \{Y \leftarrow \lambda u.f(u,u)\}$. Moreover in the process of establishing the repeated possibility of constructing new critical pairs we get as a by-product the formation rule describing their regularity. This information essentially is represented by the scheme M and the corresponding (second order) substitution ψ.

A straightforward extension of the ideas presented is to schematize the right hand side of rules too, i.e. to take into account rule schemes like

$$f(x,Y(x)) \rightarrow Y(x)$$

or

$$f(x,Y(x)) \rightarrow Z(x)$$

(in the latter example one has to be careful about the introduction of extra-variables on the right hand side via scheme instanciation).

The approach may also be used to develop techniques for avoiding divergence of completion or, alternatively, deduce finite representations of resulting infinite rewrite systems. For instance, using string notation for unary function symbols, the infinite system of the second example of the introduction may be represented by

$$A: \ f(hu,v) \rightarrow f(u,hv)$$

and

$$C_S : f(w,SgSw) \rightarrow Sw, \ S \in \{h\}^*.$$

All these topics as well as a comparison with related work on divergence of completion procedures (e.g. [Ki85], [Ki87], [HePr86], [He88]) will be treated in [Gr88].

5. Conclusion

We have presented a new approach for solving certain infinite sets of first order unification problems represented by term schemes. Within the framework of second order equational logic we have shown how to solve such problems via (variable-) restricted second order unification. For that purpose a transformational solution technique known for first order restricted unification has been generalized to the second order case. We conjecture that most of the results obtained can be extended to general higher order logic, too.

As a first application of the theoretical results it has been demonstrated how to use them for analyzing divergence phenomena of completion procedures. We think that the approach presented provides a well-suited and genêral basis for attacking the divergence problem.

Moreover, it might be interesting to investigate a generalization of the approach to solving infinite systems of equations modulo an underlying theory.

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