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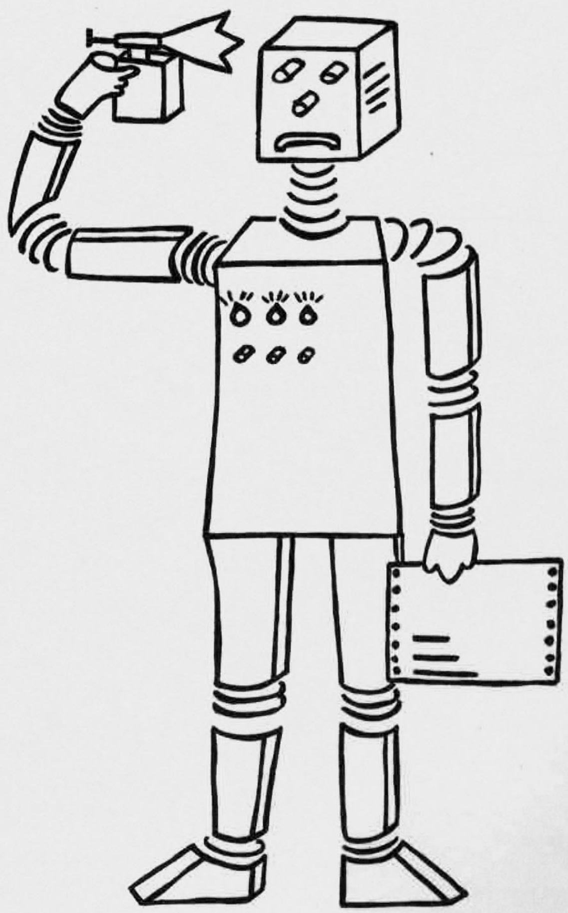

Did Gödel Prove that We Are Not Machines? (On Philosophical Consequences of Gödel's Theorem)

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Did Gödel prove that we are not machines ?
(On philosophical consequences of Gödel's theorem.)

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Gödel's incompleteness theorem has been the most famous example of a mathematical theorem from which deep philosophical consequences follow. They are said to give an insight, first, into the nature of mathematics, and more generally of human knowledge, and second, into the nature of the mind. The limitations of logicist or formalist programmes of mathematics have had a clear significance against the background of the foundational schools of the early decades of this century. The limitations of mechanism, or of the vision underlying research in the field of Artificial Inteligence, gain significance only now. Yet,while the limitations imposed by Gödel's theorem upon the extent of formal methods seem unquestionable they seem to have very little to say about the restrictions concerning mathematical or computer practice. And the alleged consequences concerning the non-mechanical character of human mind are questionable. The standard reasoning, known as Lucas' argument, begs the question, and actually implies that Lucas is inconsistent!

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0. Introduction

The old mechanist thesis stated by La Mettrie says:
(M) Man is a machine.
This can be given many interpretations.[1] Of special interest is the modern development provided by the work in the area of artificial intelligence. Its vision can be summarized in the following thesis:
(AI) Human mind can be simulated by a computer.
In science fiction it is often assumed that even the human body can be simulated. Of course, to investigate mechanization of intelligent behaviour, one does not have to believe (AI). But it seems that it expresses the natural horizon of the research in AI. In fact, an essential distinction should be made: the strong version of (AI) is that the structure of mind and the process of thinking can be simulated, while according to the weaker version only the results of thinking can be reproduced by a suitably programmed computer. As is well known, there is a world of difference between the weaker and the stronger claim. A restricted variant would be that it is possible to simulate mathematical thinking, or - in the weaker version - mathematical results. Now, if we could refute the weakest version that computers can achieve in principle all mathematical theorems that we can prove, then of course all stronger versions of the (AI) thesis would be refuted, and a fortiori the mechanist claim (M) as well.

It has been repeatedly said that Gödel´s theorem can be used to achieve this refutation.The theorem is a strict mathematical result which seems to have a mysterious, even mystical significance concerning the nature of our mind.[2] As a matter of fact it was proved by Gödel some 57 years ago to settle the major problems arising from the formalist and the logicist programmes in the foundations of mathematics.He did end the ambitions of those programmes. But did he prove our superiority over machines?
Before answering the question let us formulate a standard version of the theorem.

1. Gödel´s incompleteness theorem

Let S be an arbitrary axiomatic theory formalized in the first order logic. We assume that
(a) the axioms of the theory S are given in an effective way (i.e. are recursively enumerable),
(b) the basic elementary arithmetic Ar is contained in S (i.e. Ar is interpretable in S).
The assumption (a) is not really a limitation; every axiomatic theory which one encounters in practice can be axiomatized effectively, in most cases by a finite number of axioms or axiom schemes. The condition (b) eliminates weak theories but all usual theories of natural numbers and all set theories satisfy the assumptions of the Gödel´s theorem. The theory Ar is in the language in which there are symbols for addition and multiplication, for zero and one, and the variables range over the natural numbers 0,1,2,3,4,...[3] For every such theory S we have

The first incompleteness theorem
There exists a sentence G_S of the theory S such that
(*) if S is consistent then G_S is not provable in S, even though G_S is true.
Analysing the proof we can get, as Gödel himself did in his famous paper[4]

The second incompleteness theorem
The property (*) is enjoyed by the sentence $Cons_S$, a natural formalization of the statement of the consistency of the theory S.

For the theorem to be true the sentence $Cons_S$ must arise as a natural formalization of consistency, and this requirement can be given a precise technical sense.[5] The construction of G_S or $Cons_S$ is possible because of the Gödel´s method of arithmetization: we assign a distinct number to each symbol of the language of the theory S, so that metamathematical properties, i.e. properties of expressions, can be translated into the language of number theory as properties of numbers. Then it is possible to express these arithmetical properties as formulae of the theory Ar, so - due to (b) - as formulae of S. The condition (a) is needed to express the property of "provability-in-S" which makes possible to make Gödel´s famous construction resulting in the sentence "I am not provable", i.e. the sentence G_S expressing the fact that "G_S is not provable-in-S" so that G_S is indeed unprovable in S and true.[6]
An interesting strenghtening known as J.Robinson-Davis-Putnam-Matiasievich theorem was obtained in 1970.[7] The sentence satisfying (*) may be of the shape
"there is no solution for the diophantine equation $p_S=0$",
where p_S is a polynomial in many variables with integer coefficients (p_S depends on S) and the solutions are sought in integers only (this is the meaning of "diophantine").

2. Unprovability of consistency

Gödel´s theorem showed that the original Hilbert´s programme could not work. The programme resulted from earlier develpments, both their successes and troubles. The main success was the creation of a universal language in which all existing mathematical results could be expressed. This was possible because, first, in the 19th century the calculus was arithmetized and numbers were expressed as set theoretical constructions, and secondly, a formal theory of logical concepts as they are used in mathematics was created. This made it possible to create an axiomatic system which had as its primitive concepts both logical and set theoretical ones (both e.g. "or", "if...then", "for all", and "a class","is a member of"). Such a system, as e.g. *Principia Mathematica* of Russell and Whitehead who used the achievments of Frege, Cantor, Peano and their predecessors, was sufficient for the reconstruction of all classical mathematics. I would like to stress that what was achieved is nothing more than a reconstruction : to develop mathematics, many languages with their specific approaches are necessary. It is true though that some philosophers thought that this reconstruction is really a reduction which reveals the essence of mathematical concepts and their essentialy logical nature. In any case it seemed that it was possible to materialize the Leibniz´ dream about the possibility of *mathesis universalis*, a calculus that would mechanically solve all problems, at least all mathematical problems.

The trouble that lay in the background of Hilbert´s programme was the emergence of logical antinomies. Systems of type theory and of axiomatic set theory came into being as a response to these problems which beset the naive set theory. They made possible an elimination of antinomies while still serving as foundational theories capable of expressing the whole of mathematics. This success created a new problem. In a relatively short time abstract infinite sets became a major method or rather the natural universe, the element, of the modern mathematics. Yet it was not clear whether the antinomies could not reappear. Some of the results of the theory of abstract, arbitrary sets seemed so bizarre that their generated suspicions if some contradicton was not concealed in the whole approach. The most famous of those paradoxical results were Zermelo´s well ordering of the reals and the paradoxical decomposition of the sphere.[8]

Hilbert planned an absolutely sure proof showing the consistency of mathematics. To achieve this he proposed to consider a foundational theory sufficient to express mathematics, both mathematical concepts and proofs (or rather logical reconstructions of proofs), from a purely formal point of view. If we consider only the form of expressions ignoring their meaning we can analyze them using only the simplest means, which are absolutely safe. If such a finitistic combinatorial analysis showed that no sentence of the shape 0=1 is provable in the theory then we would have an absolute consistency proof for the theory, hence for all of mathematics as well.[9]

The second theorem of Gödel shows the failure of Hilbert´s hopes if we assume that the finitistic methods that Hilbert wanted to use in the study of the form of linguistic expressions are all expressible in the theory Ar. May we assume that? There is evidence which allows us to say "yes". But this cannot be proved because the concept of the simplest method based on the immediate intuition of symbols is necessarily vague. It is only the practical experience of specialists which shows that the theory Ar contains very rich possibilities of encoding finite combinatorics. Thus an absolute proof of the consistency of S, or Ar itself, could be reproduced in Ar, which would mean that in S the sentence $Cons_S$ would be formally provable, which contradicts the second Gödel´s theorem. Even more can be said. If S is a consistent theory which is sufficiently strong, i.e. interpreting Ar, its consistency has to be postulated as an act of faith, because the proof of it requires a stronger theory. Yet in practice the danger of fundamental contradictions in mathematics is really slight. Mathematicians are not afraid of them in the least.
Another aspect of Hilbert´s program was the question of conservativity. It could be expected that the introduction of new ideal elements, like infinite complicated structures, cannot add new simple provable properties of natural numbers. However Gödel´s theorem shows that if a theory S is strenghtened to a theory T in which the consistency of S is provable than a new theorem on numbers becomes provable, namely Con_S, and moreover a statement saying that there is no solution of a certain diophantine equation is provable in T but not in S. Yet, even though the sentence $Cons_S$ is about the natural numbers it has no understandable purely arithmetical meaning. Its sense is derived from the fact that it was constructed as a code of some metamathematical properties. Also the statement about the diofantine equation could not arise in normal number theoretical considerations despite its , apparently transparent, purely arithmetical character. The simplicity is an illusion: the coefficients of the equation must be so huge and complicated as to encode the whole logical reasoning leading to them. Thus there is no hope that one could grasp, let alone write down, this particular equation. In fact, it was very difficult to find a mathematically meaningful, as opposed to logically meaningful, arithmetical sentence independent of the Peano Arithmetic.[10] (Some people argue that it was because it needed some essentially infinite construction codable as a relation among natural numbers.)

3.Incompleteness

The first incompleteness theorem showed that the whole of mathematics cannot be reduced to a single axiomatic system, even if everything can be expressed in a universal language. Even some simple arithmetical truths can be found outside every effectively given axiomatic theory. (Since "simple" is a misleading term in our context, I add that what is meant are the truths expressible as formulas of low logical complexity). Thus the logicist programme failed: no single system is sufficient for a reconstruction of all possible mathematics (as opposed to all existing mathematics). We might say that provability "in general" goes beyond "provability-in-S" for any S. But one might ask: doesn´t it depend on the particular formalization of logic we used to formalize our theories S ? Perhaps a more powerful universal language would do? (Cf. with the insufficiency of rationals to express the square root of 2).

The answer to the questions is an emphatic "no". Limitations revealed by Gödel are extremely general and no extension of methods can help, provided that they remain effective. The intiuitive notion of being effective, or mechanical, or computable, turns out to be a very rigid, objective notion captured by the mathematical concept of recursive functions or that of Turing machines or other kinds of idealized computers. Any system of mechanical, i.e. recursive, generation of elementary arithmetical theorems, if consistent, is incomplete. It is subject to Gödel´s theorem because recursivity is closely connected to the theory Ar via representability: recursivity of a set or a function is equivalent to provability in Ar (or S) of appropriate formulae. Thus the general notion of effectivity is in a sense equivalent to the usual logic with the standard formalization.[11]

An interesting method of establishing incompletness is based on information theory. Chaitin observed that in every formalism there is an upper limit for the complexity of the information theoretic contents of provable formulae. Thus some learning from the outside environment seems to be as necessary for any system or machine with high ambitions as it is for us.[12]

More philosophically ambitious reformulations of the consequences of the incompleteness theorem are possible. For example, no machine producing truths can produce all truths, even if it runs infinitely long. Or we might say that truth is beyond formalisms.[13] Yet while truth cannot be reduced to provability in a single formal system it is not excluded that truth can be given by provability in an informal collection of formal systems[14] or else by an intuitive unformalizable provability, as the intuitionists maintain.

The exciting grand conclusions look now less relevant than they did 50 or 30 years ago. It could seem then that the undecidability theorems (like that of Church on the predicate logic) which are closely connected to Gödel´s theorem set a limit for computer applications.The practical attempts to formalize notions and reasoning with view of computer implementations have revealed the fact that even those branches of mathematics which admit a complete and decidable formalization can be as far from any practical mechanization as undecidable branches. For example Pressburger arithmetic, i.e. the theory of the natural numbers with addition alone (ignoring multiplication) admits a simple complete axiomatization and is decidable, but it is known that every decision procedure is superexponential. Even simpler algorithms like that detecting propositional tautologies cannot be directly applied since an exponential growth of time or space makes the procedure practically unfeasible. Efficient decidability requires special methods which have been being developed in the last decades.

Yet the philosophical impact of Gödel´s results has not diminished. For example they seem to support the idea that an irreducible intuition is unavoidable at some level. Speaking more precisely: on the one hand, it would seem that a finite definition of the natural numbers is possible. Then all truths would follow from this definition. On the other hand, it is not possible to have one system or machine encompassing the whole universe of the natural numbers. So no finite definition is possible. What are then the definitions given in textbooks? Well, they are definitions but only when an intuitive background knowledge is assumed. This intuition is necessary. We know that the Gödelian sentence is true because we can look at the system, machine or definition from outside and see its truth (that it is not provable) using the consistency of our concept of natural numbers. If this intuition can never be completely exhausted, does it mean that it is innate? Is then artificial inteligence impossible?

This conclusion is not warranted, as Gödel himself stressed. It may be possible that we would not be able to look at a system from outside because it couldn´t be given to us ! Perhaps we are a machine, only we do not know which one? A superhuman mind could see this but we cannot.
So let us look more closely at the argument that Gödel´s theorem implies that we are not machines. Lucas´ 1961 article is commonly quoted as a standard exposition of this view.[15]

4. Is there a mathematical proof that we are not machines?

If Mechanist says that a machine M is equivalent to Lucas, or rather to mathematical powers of Lucas, then Lucas produces the Gödelian formula G_M corresponding to the theory T(M) consisting of arithmetical statements provable by M. How we define T(M) is not essential as long as we assume that M is equivalent in principle to a Turing machine or other standard notion. If not then the concept of a machine becomes so unclear that no rigorous argument is possible. For all ordinary machines T(M) is always semidecidable (i.e. recursively enumerable).
Now there are two possibilities:
Case 1: the theory T(M) is consistent,
Case 2: the theory T(M) is inconsistent.
In the first case G_M is unprovable by M (i.e. unprovable in the theory T(M)) but it is seen to be true, i.e. provable, by Lucas - according to the Gödel´s theorem together with the assumption of consistency. And in Case 2 every sentence, also e.g. 2+2=5, is provable by M, i.e. in the theory T(M), which certainly is not true about Lucas. In any case Lucas can show that he is different from M, or as we´ll say he can "out-Gödel" M.

In principle, the above procedure can be mechanized. Let us assume that all machines are listed in a sequence M1,M2,M3,.... With a little experience in logic it is easy to see that there is a recursive function g such that for every n:

(+) if T(Mn) is consistent then g(n) is the Gödel number of a (Gödelian) sentence which is true and unprovable in T(Mn).

An objection to the significance of the whole procedure can be made on the basis of (+). If a machine (corresponding to the function g) can simulate the procedure of out-Gödeling then machines are not really worse than we. The reply is that this machine can be out-Gödeled too. In general, the aim is not to dominate all machines at once but but rather each machine proposed by the Mechanist. Lucas describes the procedure as dialectical, or as a game in which Lucas wins in every move.

It has been argued that the procedure does not work if the machine´s program is not known. Thus Lucas´ mathematical powers might be equal to a machine but he would not be able to find n in order to produce g(n). This possibility was mentioned above: perhaps we are a machine without knowing which one (Benceraff)? The reply to this would be that theoretically it is possible to do the out-Gödeling. The appropriate Gödelian sentence is somewhere there, waiting for us. Also, we can use again the "dialectical" nature of the argument: if the Mechanist presents a number n then Lucas can produce g(n). The sentence corresponding to g(n) is true provided Mn is consistent (i.e. the theory T(Mn) is consistent). But how do we know that it is consistent ?

This question is more serious than it could seem at the first glance. It could be impossible to establish that a given machine is consistent even if it is. I do not refer to practical limitations. They are ignored in our context anyway. For example, computing g(n) can be infeasible for many n. Yet the algorithm exists. In contrast the problem "is a given machine consistent?" is recursively undecidable. The set

$C = \{n: T(Mn) \text{ is consistent}\}$

is not recursive.To see this, let us note that the halting problem can be reduced to the complement of C. Namely given a machine M we make another machine M´ which imitates M but when M stops M´ prints 0=1. Then: M stops iff M´ does not belong to C.

We see that it is theoretically impossible to distinguish Case 1 from Case 2! (Of course, in many instances it can be easily done but Lucas´ argument requires us to worry about all possible machines.) To distinguish the cases requires nonrecursive skills, that is the procedure of out-Gödeling assumes a non-mechanical nature of Lucas, the thesis it was supposed to prove. Lucas could still maintain that either the argument of Case 1 or that of Case 2 applies. But then the "dialectical" nature of the argument becomes doubtful, and moreover Lucas is not allowed to commit even a single mistake. Namely the Gödelian sentence for an inconsistent theory is false and in fact contradicts the most elementary arithmetical truths (those with limited quantifiers). And we saw that out-Gödeling in Case 2 depended entirely on Lucas´ consistency.

5. Are we consistent ?

Everyday experience shows that humans are rather inconsistent. Lucas remarks: "certainly women are, and politicians." Putnam maintained that it is conceivable that we are inconsistent machines. Inconsistent machines seem to prove everything. Strictly speaking, every sentence follows logically but it does not mean that an inconsistent machine actually produces all formulae. In fact it may happen that the proofs of contradictions would be too long to matter in practice, and it would be possible to remain in safe contradictionfree areas all the time. This is actually the case with large programs which contain bugs in marginal areas. Also, more to the point, the infinitesimal calculus was developed on inconsistent foundations for centuries. In fact the lethal contradictions were avoided and it was assumed (rightly !) that the theory is fundamentally consistent. It seems that in order to do mathematics such assumption is necessary. Behind our mistakes and contradictions a solid, consistent framework exists. Our mind´s consistency functions as a regulative idea in Kant´s sense.

Incidentally, the second Gödel´s theorem seems to imply that we cannot prove our consistency in a mathematical way, "moro geometrico", even if we are fundamentally consistent. Otherwise, the proof could be simulated by a machine including the appropriate part of our mathematical capacities. The machine would prove a fortiori its own consistency, which contradicts the second incompleteness theorem.

Of course we believe that our mistakes are corrigible. Lucas argues that our inconsistencies are rather like machine malfunctionings. We remove reasons of inconsistencies, so that we are in principle consistent. To "fallible but self correcting machines" out-Gödeling applies.Yet it is not the end of the analysis.

In his 1982 paper G. Lee Bowie remarked that whatever Lucas may claim about his consistency, we know enough to prove that he is actually inconsistent. As we know the systematic use of of the Lucas´ procedure can be expressed as the recursivity of the appropriate function g satisfying the condition (+). Now the point is that the range of g, i.e. the set

A = {sentence with Gödel number g(n) : n = 1,2,3,...}

is inconsistent! To prove this let us assume that it is generated by a machine Mk (such a k exists since A is recursively enumerable). If A were consistent then by (+) the sentence with Gödel number g(k) would not be provable in T(Mk). Yet as an element of A it would belong to T(Mk). This is a contradiction.[16]

To consider the whole of the range of g is adequate: Lucas must be able to reply, whatever machine the Mechanist presents. Lucas cannot assume that he would do it only for consistent machines that is that g(n) is considered only for n belonging to C: as C is not recursive (not even recursively enumerable) the whole Lucas´ procedure would become actually nonrecursive. To apply it, Lucas would have to be nonmechanical to begin with.

Is it possible to modify the argument to avoid the above criticism? No. The proof of the contradiction of the procedure of out-Gödeling is valid as long as the attempted procedure is effective, and more precisely as long as

1° g is partial recursive,

2° the domain of g includes the set C (of the numbers of consistent machines),

3° for each n in C the sentence with Gödel number g(n) is unprovable in T(Mn).

It seems that these conditions are precisely what is necessary for the out-Gödeling to take place. We do not even assume in 3° that the sentence g(n) is true or provable by anyone. In 1° any procedure of ignoring machines is allowed if only it is effective, while in 2° it is assured that no consistent machine may be ignored.[17]

It seems that it has been proved that the class of inconsistent humans is not empty: it contains the philosophers believing in Gödel based mathemathical proof of their own superiority over machines.

6. What was the opinion of Gödel ?

Gödel himself was looking for arguments that "laws of thought are not mechanical".[18] In fact, according to Wang, he believed that mind can exist separately from matter, and according to Kreisel, he was interested in demonology throughout his life. Yet he did not think that his theorem implied the nonmechanical nature of the mind. His results do not exclude the existence of a machine which could achieve exactly the same mathematical results as we do. To maintain this it is not necessary to show such a machine. It is enough to show that it could possibly come into being. How? It was von Neumann who first showed that it is possible in principle to have machines producing other machines, and even copies of themselves. It is possible to apply to machines the concept of evolution. Von Rucker gave a colourful description of a civilization of robots on the moon. Random factors could cause mutations so that a natural selection process would occur. As a result a computer might come into existence whose mathematical capacities would be exactly the same as those of Lucas. However, noone would be probably able to detect this. Now, if the Mechanist introduced this lovely robot to Lucas he would not be able to do any nasty out-Gödeling trick. Neither would he find the robot´s number in the sequence of the machines, nor would he be able to find out if the robot is consistent or not, so that the embarrasing situation would last indefinitely.

According to Gödel we cannot detect the equivalence of our mind with a given machine because of the second incompleteness theorem. It is so if we assume our consistency, which - it seems - was for Gödel beyond doubt. If we could prove the equivalence, the argument runs, the machine would be consistent (being equivalent to us), so it could prove this (being equivalent to us), which is impossible.

Finally, it was stressed by Gödel that his theorems do imply the thesis that mind is not mechanical, under an additional assumption. Namely it is sufficient to believe that we can solve every diophantine equation. No machine can do this. Thus Gödel´s theorem can be invoked to minimize the extent of faith needed to give a precise cogent argument that we are not machines. It was Hilbert who stated the belief that every mathematical problem can be solved. "In mathematics there is no ignorabimus", he wrote. Gödel shared this belief.

Notes

1. See Webb [27] for an extensive analysis.

2. The most attractive, if not necessarily conclusive, treatment of the problem and its mystical dimension is available in Hofstadter's bestseller [10].

3. More about the theory Ar and the conditions (a), (b) can be found e.g. in Tarski et al. [24] and Barwise [2].

4. Godel [8]. He wanted to publish a second part of the paper with a detailed proof of the second theorem but he abandoned this because the results were immediately accepted by experts. First full proof written by Bernays appeared in [11].

5. See Feferman [7].

6. The construction was extended by Rosser [22]. It can be found in Smorynski's chapter in Barwise [2] together with a review of later developments. A noteworthy popular presentation is given in [19].

7. Matiasievich [17]. For a fuller account see e.g. [3].

8. Exhaustive history of the axiom of choice, Zermelo's proof and paradoxical decomposition, and of controversies they provoked, can be found in Moore [18].

9. This project emerged in the beginning of this century. It is described in Hilbert, Bernays [11] and in some sections of the anthology [4].

10. It was done by Paris and Harrington in Barwise [2].

11. Rogers [21] and Barwise [2] are among good expositions of these matters.

12. A representative sample of Chaitin's work is included in Tymoczko [25].

13. Rucker [23] dwells more on such statements.

14. Cf. Wang [26], p.322.

15. Lucas [15], reprinted in [1] and partly in [10]. See also Benacerraf [5].

16. Inconsistency of the set A is mentioned in Webb [27] but the application against Lucas is not made. Undecidability of the set C had been known before, and it was mentioned in connection with Godel's theorems by Webb and by Wang [26], p.317.

17. The conditions 1^{o}-3^{o} are implicit in Lee Bowie [14]. They were formulated explicitely in [12]. Reinhardt [20] considers the possibility that the Mechanist playing against Lucas must present only consistent machines. Using this fact Lucas would be able to "out-Godel" him. Yet this proves nothing: if the Mechanist is able to decide the set C he is assumed to be nonmechanical - despite his own conviction - and Lucas can take advantage of this to acquire nonmechanical skills.

18. Main sources for Godel's own views are: Wang [26] pp.324-326, and Kreisel [13] pp.216-218.

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