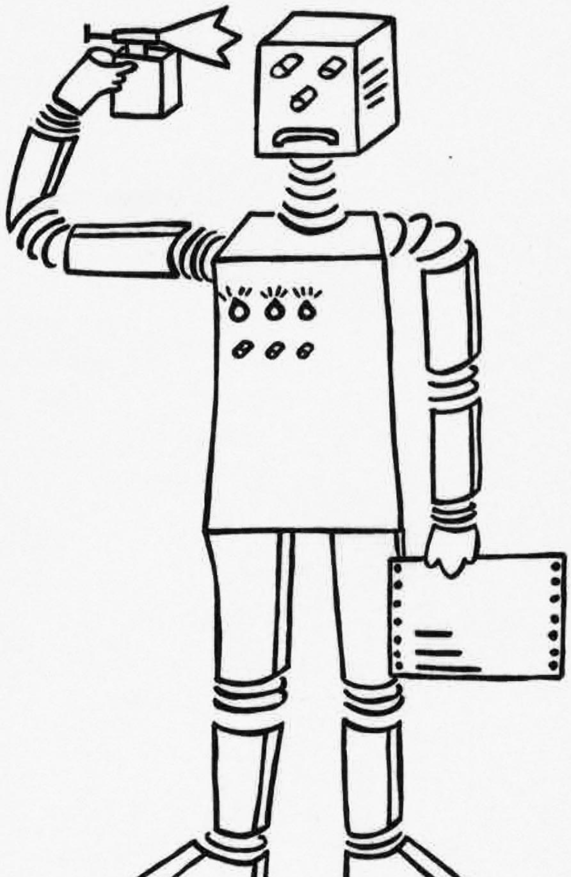
SEKI-REPORT

Artificial
Intelligence
Laboratories

Fachbereich Informatik
Universität Kaiserslautern
Postfach 3049
D-6750 Kaiserslautern 1, W. Germany



A Representation for the Non-Instances of Linear Terms

Horst Gerlach

SEKI Report SR-88-05

A representation for the non-instances of linear terms

Horst Gerlach
FB Informatik, Universität Kaiserslautern
Postfach 3049
D-6750 Kaiserslautern
FR Germany
(E-mail : gerlach@uklirb.uucp)

SEKI-Report SR-88-05

Abstract

Given a signature (S,Σ) and a Σ-term $\tau \in T_\Sigma(X)^s$ of sort $s \in S$, $X = (X_s)_{s \in S}$ being a family of countably infinite sets of variables, we call a set $\mathcal{B} \subseteq T_\Sigma(X)^s$ of Σ-terms a *representation for the non-instances of* τ *(w.r.t.* (S,Σ)*)* iff the Σ-groundinstances of the elements of $\mathcal{B}$ are precisely those Σ-groundterms of sort s that are not a Σ-instance of τ. We recursively define a family $(\mathcal{B}_\tau)_{\tau \in T_\Sigma(X)^s, s \in S}$ of sets of Σ-terms (w.r.t. an implicit well ordering of the sets X_s of variables) in such a way, that for each linear Σ-term τ, $\mathcal{B}_\tau$ is a representation for the non-instances of τ (w.r.t. (S,Σ)) that satisfies both a minimality and an embedding property. In particular, $\mathcal{B}_\tau$ is computable and finite if, in addition, Σ contains but a finite number of operation symbols.

Acknowledgements

I am indebted to Hans-Jürgen Bürckert and Manfred Schmidt-Schauß who kindly supported this work with their expertise. In particular, Hans-Jürgen Bürckert read a first draft of this paper and contributed a lot of valuable hints to the final presentation of the subject.

Contents

0. Introduction

$\mathbb{P}$: *Given a signature* (S,Σ) *and a* Σ*-term* $\tau \in T_\Sigma(X)^s$ *of sort* $s \in S$. *Is there a suitable representation for the set of all* Σ*-groundterms* τ_1 *of sort* s *that are not a* Σ*-instance of* τ ?

This problem was encountered by the author in the course of his work on the application of term rewriting techniques to the simulation and analysis of High Level Petri Nets over algebraic specifications with constructors that is part of the ESPRIT project **GRASPIN** (see [Ge 88]). In such a net, the enabling of a transition t in a given situation (essentially) depends on the availability of some groundinstance (w.r.t. a common substitution σ) of the terms labelling the input-arcs of t in the corresponding input-places, and the firing of t results in removing these groundinstances and in adding the σ-instances of the terms labelling the output-arcs of t to the corresponding output-places. Assuming that the net has exactly n places $p_1,\ldots,p_n$ and that the current state of the system, i.e. the current distribution of groundterms over the places, is represented by the groundterm

$$\tau_1 \;{}_{def}\!= \;\text{make}\,(a_1,\ldots,a_n)$$

where a_i denotes the groundterm residing in place p_i resp. a_i = EMPTY if p_i is empty, this availability-condition holds iff τ_1 is an instance of the term

$$\tau \;{}_{def}\!= \;\text{make}\,(u_1,\ldots,u_n)$$

where u_i is the term labelling the input-arc joining p_i and t if p_i is an input-place of transition t, u_i = EMPTY if p_i is an output-place of t, but no input place of t and all other u_j´s are pairwise different variables. Consequently, by choosing appropriate terms $v_1,\ldots,v_n$ derived from the labellings of the output-arcs of transition t, the effect of the firing of t in situations that satisfy the above availability-condition can be correctly described by a rewrite rule of the form

$$\tau \;=\; \text{make}\,(u_1,\ldots,u_n) \;\rightarrow\; \text{make}\,(v_1,\ldots,v_n).$$

Thus, for instance, the firing of the transition t in the environment as represented in Figure 1 is described by the rewrite rule

$$\tau \;=\; \text{make (s(s(x)),true,cons(y,l),Empty,Empty)}$$
$$\rightarrow\; \text{make (Empty,Empty,Empty,+(*(x,x),s(0)),cons(x,cons(y,l)))}$$

(here, s denotes the successor operation on naturals, true one of the boolean constants and cons the cons operation on lists). Using this rewrite rule, the term

$$\tau_1 \;{}_{def}\!= \;\text{make (s(s(s(0))),true,cons(0,NIL),Empty,Empty)}$$

representing the state illustrated in Figure 1 is rewritten to the term

$$\tau_2 \;{}_{def}\!= \;\text{make (Empty,Empty,Empty,+(*(s(0),s(0)),s(0)),cons(s(0),cons(0,NIL)))}$$

that, on his part, represents the state ensuing from the firing of the transition t.

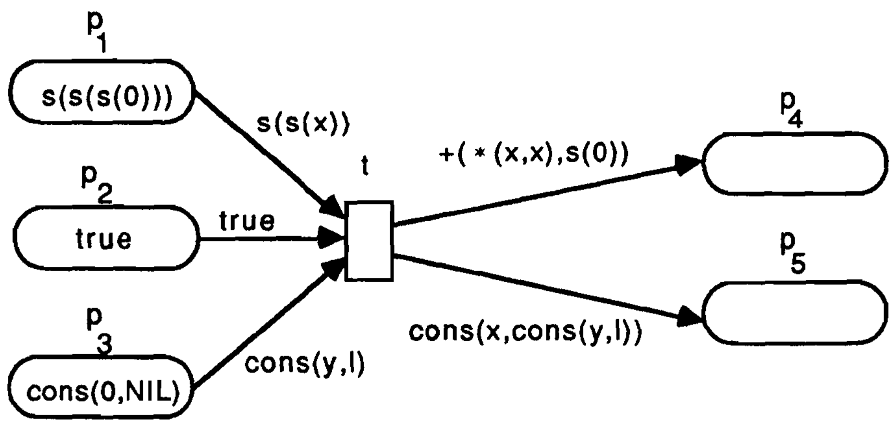


Figure 1 : A transition t with input-places p_1, p_2 and p_3 carrying the data s(s(s(0))), true and cons(0,NIL) and empty output-places p_4 and p_5.

However, in order to represent the "no-action" behavior of t in all remaining situations (by corresponding rewrite rules), it is desirable to have a suitable representation of all these situations, i.e. a suitable representation of all groundterms that are not an instance of τ. Thus, we arrive at the problem $\mathbb{P}$.

The problem $\mathbb{P}$ is easily seen to be a specialization of the following, more general representation problem :

$\mathbb{P}'$: *Given a signature* (S,Σ) *and* Σ*-terms* $\tau, \tau^1,\ldots,\tau^r \in T_\Sigma(X)^s$ *of sort* $s \in S$. *Is there a suitable representation for the set* $\tau/\{\tau^1 \vee\ldots\vee \tau^r\}$ *of all* Σ*-groundterms* τ_1 *of sort* s *that are a* Σ*-instance of* τ *but not a* Σ*-instance of any of the* τ^i (i=1,...,r) ?

Clearly, $\mathbb{P}$ corresponds to finding a suitable representation for $x/\{\tau\}$, where x is any variable of the same sort as τ. Apart from the interest in $\mathbb{P}$, and hence also in $\mathbb{P}'$, stemming from the sketched relation between the rewrite world and the Petri Net scenario, an even more lively interest in $\mathbb{P}'$ is due to the fact that solving $\mathbb{P}'$ corresponds to learning concepts from examples and counter examples, as is pointed out in the work of J.-L. Lassez and K. Marriott [La/Ma 87] (see also [Mic 83], [Mit 78] and [Ver 80]). Restricting to the case that S = {s} is single-sorted and the set $\bigcup \{\Sigma_{w,s'} \mid w \in S^*, s' \in S\}$ of Σ-operation symbols is finite, they provide an algorithm that decides whether or not an "implicit representation" $\tau/\{\tau^1 \vee\ldots\vee \tau^r\}$ has an "explicit representation" $\zeta^1 \vee\ldots\vee \zeta^m$ (here,

$\zeta^1 \vee ... \vee \zeta^m$ denotes the set of all Σ-groundinstances of some of the terms ζ^i) satisfying that $\tau/\{\tau^1 \vee ... \vee \tau^r\} = \zeta^1 \vee ... \vee \zeta^m$ and, if so, calculates $\zeta^1,...,\zeta^m$. Their results (see [La/Ma 87], Proposition 4.5 and Proposition 4.6) also give evidence that nonlinear terms cause principal obstacles.

Our work is related to that of J.-L. Lassez and K. Marriott [La/Ma 87] in that we consider the restricted problem of finding a suitable representation for the implicit representation $x/\{\tau\}$ in case of an arbitrary set S of sorts. Furthermore, apart from the results presented in [La/Ma 87], we prove that the kind of representation proposed in the work at hand enjoys some nice properties of minimality (both w.r.t. set inclusion and cardinality) and practicability.

In Paragraph 1, we briefly resume our notations and some basic concepts and results of (sorted) universal algebra needed in the sequel.
Paragraph 2 introduces the notion of a *representation for the non-instances of a* Σ*-term* τ *(w.r.t. a signature* (S,Σ) *with variables* X*)* as a set $\mathcal{B} \subseteq T_\Sigma(X)^s$ (s being the sort of τ) satisfying that every Σ-groundterm τ_1 of sort s is not a Σ-instance of τ iff it is a Σ-instance of some element of $\mathcal{B}$ (Definition 2.1). It is noted that, as a matter of triviality, a representation for the non-instances of τ (w.r.t. (S,Σ)) always exists. Subsequently, two technical lemmata are anticipated (Lemma 2.2 and Lemma 2.3).
In Paragraph 3, we recursively define, w.r.t. a signature (S,Σ), the family $(\mathcal{B}_\tau)_{\tau \in T_\Sigma(X)^s, s \in S}$ of sets of Σ-terms that constitute the subject of the main results contained in this work (Definition 3.1). In particular, we prove that $\mathcal{B}_\tau$ is a linear (i.e. containing only linear Σ-terms) representation for the non-instances of τ (w.r.t. (S,Σ)) provided τ is linear (Proposition 3.2). Moreover, $\mathcal{B}_\tau$ is distinguished by the facts that it is a minimal representation (w.r.t. set inclusion) for the non-instances of τ (w.r.t. (S,Σ)) (Proposition 3.3) and that it can be embedded in any other representation for the non-instances of τ (w.r.t. (S,Σ)) all of whose Σ-terms having the same top-level symbol as τ are linear (Proposition 3.4). Here again, τ is assumed to be linear. As a corollary, no finite/countable representation of this type exists if $\mathcal{B}_\tau$ is infinite/uncountable. Finally, we prove that all the sets $\mathcal{B}_\tau$ are finite and can be effectively computed provided that the set $\bigcup \{\Sigma_{w,s'} \mid w \in S^*, s' \in S\}$ of Σ-operation symbols is finite (Proposition 3.5).

1. Notations

We assume that the reader is familiar with the basic concepts and results of (sorted) universal algebra as they can be found in the common literature (see, for instance, [Grä 79] or [Lug 76]).

Essentially, we use the following notations and facts : A **signature** is a pair (S,Σ) where S is a set, $\Sigma = (\Sigma_{w,s})_{w\in S^*,s\in S}$ is a $S^*\times S$-indexed family of indexwise disjoint sets that are also disjoint from S.[1] The elements of S are called **sorts** and, for each $(w,s) \in S^*\times S$, the elements $f \in \Sigma_{w,s}$ are called **Σ-operation symbols** of **arity** w and **co-arity** s, or, in short terms, of **functionality** (w,s). Usually, together with a signature (S,Σ), we are given a family of sets of **variables**, i.e. a S-indexed family $X = (X_s)_{s\in S}$ of indexwise disjoint sets that are also disjoint from S and any of the sets $\Sigma_{w,s}$ with $(w,s) \in S^*\times S$.
Given a signature (S,Σ), a **(S,Σ)-algebra** (or, alternatively, an algebra over (S,Σ)) is a pair $\mathcal{A} = ((\mathcal{A}^s)_{s\in S},(f^{\mathcal{A}})_{w\in S^*,s\in S,f\in\Sigma_{w,s}})$, where each $\mathcal{A}^s$ is a set and, for all $w \in S^*$, $s \in S$ and $f \in \Sigma_{w,s}$, $f^{\mathcal{A}} : \mathcal{A}^{w_1} \times ... \times \mathcal{A}^{w_{|w|}} \rightarrow \mathcal{A}^s$ is a mapping. Given two (S,Σ)-algebras $\mathcal{A}$ and $\mathcal{B}$, say $\mathcal{A} = ((\mathcal{A}^s)_{s\in S},(f^{\mathcal{A}})_{w\in S^*,s\in S,f\in\Sigma_{w,s}})$ and $\mathcal{B} = ((\mathcal{B}^s)_{s\in S},(f^{\mathcal{B}})_{w\in S^*,s\in S,f\in\Sigma_{w,s}})$, a **homomorphism** σ from $\mathcal{A}$ into $\mathcal{B}$ is a S-indexed family $\sigma = (\sigma_s)_{s\in S}$ of mappings $\sigma_s : \mathcal{A}^s \rightarrow \mathcal{B}^s$ satisfying, for all $w \in S^*$, $s \in S$, $f \in \Sigma_{w,s}$ and $(a_1,...,a_{|w|}) \in \mathcal{A}^{w_1} \times ... \times \mathcal{A}^{w_{|w|}}$, the equation $\sigma_s(f^{\mathcal{A}}(a_1,...,a_{|w|})) = f^{\mathcal{B}}(\sigma_{w_1}(a_1),...,\sigma_{w_{|w|}}(a_{|w|}))$. If there is no danger of confusion, we usually drop the sort-index "$_s$" in σ_s.
Among all (S,Σ)-algebras, we are particularly interested in the (S,Σ)-algebra of terms, i.e. the (S,Σ)-algebra $T_\Sigma(X) = ((T_\Sigma(X)^s)_{s\in S},(f^{T_\Sigma(X)})_{w\in S^*,s\in S,f\in\Sigma_{w,s}})$, where $(T_\Sigma(X)^s)_{s\in S}$ is the least family of sets of strings s.t. for all $s \in S$, $T_\Sigma(X)^s$ contains the variables and constants of sort s, i.e. the set $X_s \cup \bigcup \{\Sigma_{\Lambda,s} \mid s \in S\}$[2], and, if $w \in S^*$, $s \in S$, $(\tau_1,...,\tau_{|w|}) \in T_\Sigma(X)^{w_1} \times ... \times T_\Sigma(X)^{w_{|w|}}$ and $f \in \Sigma_{w,s}$, also $f(\tau_1,...,\tau_{|w|}) \in T_\Sigma(X)^s$; accordingly, the mapping $f^{T_\Sigma(X)} : T_\Sigma(X)^{w_1} \times ... \times T_\Sigma(X)^{w_{|w|}} \rightarrow T_\Sigma(X)^s$ is defined by $f^{T_\Sigma(X)}(\tau_1,...,\tau_{|w|}) =_{def} f(\tau_1,...,\tau_{|w|})$. We let $T_\Sigma =_{def} T_\Sigma((\emptyset)_{s\in S})$. The elements of $T_\Sigma(X)^s$ (resp. $(T_\Sigma)^s$) are called **Σ-terms of sort s** (resp. **Σ-groundterms of sort s**). For a Σ-term τ, **Var(τ)** denotes the set of variables $x \in \bigcup\{X_s \mid s \in S\}$ occurring in τ. τ is said to be **linear** iff no variable $x \in \bigcup\{X_s \mid s \in S\}$ occurs more than once in the string τ. The (S,Σ)-algebra $T_\Sigma(X)$ is distinguished by the fact that any family $\sigma = (\sigma_s)_{s\in S}$ of

---

1 Note that, unless stated otherwise, no assumption is made on the cardinality of any of the sets S or $\Sigma_{w,s}$ (for $w \in S^*$, $s \in S$);

2 Λ denotes the empty word in S^*;

mappings $\sigma_s : X_s \to \mathcal{B}^s$, where $\mathcal{B} = ((\mathcal{B}^s)_{s \in S}, (f^{\mathcal{B}})_{w \in S^*, s \in S, f \in \Sigma_{w,s}})$ is an arbitrary (S,Σ)-algebra, can be uniquely extended to a homomorphism $\hat{\sigma}$ from $T_\Sigma(X)$ to $\mathcal{B}$.
A homomorphism σ from $T_\Sigma(X)$ to $T_\Sigma(X)$ that moves only a finite number of variables $x \in \bigcup \{X_s \mid s \in S\}$ is called a **substitution**. It is noted that $\sigma(\tau) = \varrho(\tau)$ for a Σ-terms τ and all substitutions σ, ϱ satisfying $\sigma(x) = \varrho(x)$ for all $x \in Var(\tau)$, i.e. $\sigma(\tau)$ only depends on the effect of σ on the variables of τ. Given Σ-terms τ and τ' of equal sort, τ' is called a **Σ-instance** of τ iff $\tau' = \sigma(\tau)$ for some substitution σ, and τ' is called a **Σ-groundinstance** of τ iff τ' is a Σ-instance of τ and τ' is ground.

2. Representation of non-instances of terms

Recalling that, if (S,Σ) is a signature with corresponding variables $X = (X_s)_{s \in S}$, the sets of operation symbols and variables are sortwise disjoint and that substitutions preserve sorts (see Paragraph 1), it is easily recognized that if τ is a Σ-term of sort $s \in S$, no Σ-groundterm τ_1 of some sort $s' \in S \setminus \{s\}$ can ever be a Σ-instance of τ. Therefore, the problem of finding a suitable representation for those Σ-groundterms τ_1 that are not a Σ-instance of the Σ-term τ may be restricted to Σ-groundterms of the sort s of τ; for all other sorts $s' \in S \setminus \{s\}$, this problem is, trivially enough by the above remark, settled by saying that the "non-instances" of sort s' are precisely the Σ-groundinstances of some element of the set $\mathcal{B} =_{def} \{x\}$, x being any variable of sort s'. Sets $\mathcal{B}$ that enjoy this property in the nontrivial case $s' = s$ are captured by the following definition.

2.1 Definition :

Let 1) (S,Σ) be a signature with variables X;

2) $\tau \in T_\Sigma(X)^s$ with $s \in S$.

A set $\mathcal{B} \subseteq T_\Sigma(X)^s$ is a *representation for the non-instances of* τ *(w.r.t.* (S,Σ)*)* iff, for every Σ-groundterm $\tau_1 \in (T_\Sigma)^s$,

τ_1 is not a Σ-instance of τ <u>iff</u> τ_1 is a Σ-instance of some element of $\mathcal{B}$. ■

Note that, if (S,Σ) is a signature with variables X and τ is a Σ-term of sort $s \in S$, the set

$$\mathcal{B} =_{def} \{\tau_1 \in (T_\Sigma)^s \mid \tau_1 \text{ is not a } \Sigma\text{-instance of } \tau\}$$

is immediately seen to be a representation for the non-instances of τ (w.r.t. (S,Σ)). Thus a representation for the non-instances of τ (w.r.t. (S,Σ)) always exists!

However, the above set $\mathcal{B}$ may be infinite even if Σ contains but a finite number of operation symbols (and therefore fail to be of any practical use). This can be easily seen by choosing a signature (S^{fin},Σ^{fin}) with exactly one sort $s \in S^{fin}$, two constant symbols a and b, and an unary function symbol f. Under these assumptions and letting $\tau =_{def} b$, we have

$$\mathcal{B} = \{a,f(a),f(f(a)),f(f(f(a))), \ldots ,f(b),f(f(b)),f(f(f(b))), \ldots \}.$$

In Paragraph 3 we will define a set $\mathcal{B}_\tau$ of Σ-terms (in fact a family $(\mathcal{B}_\tau)_{\tau \in T_\Sigma(X)^s, s \in S}$) that evades this shortcoming (see Proposition 3.5) and also constitutes a representation for the non-instances of τ (w.r.t. (S,Σ)) provided τ is <u>linear</u> (see Proposition 3.2). Moreover, $\mathcal{B}_\tau$ is distinguished by some additional properties (see Proposition 3.3 and Proposition 3.4). These properties will be proven by the use of the following two lemmata, the first of which states that the g-component $\mathcal{B}|_g$ of any representation $\mathcal{B}$ for the non-instances of τ (w.r.t. (S,Σ)) is not empty provided g is an operation symbol different from the top-level symbol of τ and all argument sorts of g are not empty (w.r.t. (S,Σ)).[3]

2.2 Lemma :

Let 1) (S,Σ) be a signature with variables X;

2) $\tau = f(\tau^{(1)},\ldots,\tau^{(|v|)})$ with $s \in S$, $v \in S^*$, $f \in \Sigma_{v,s}$ and $\tau^{(\kappa)} \in T_\Sigma(X)^{v_\kappa}$ for all $\kappa \in \{1,\ldots,|v|\}$;

3) $\mathcal{B} \subseteq T_\Sigma(X)^s$ a representation for the non-instances of τ (w.r.t. (S,Σ)).

Then, for every $w \in S^*$ s.t. $w_1,\ldots,w_{|w|}$ are not empty (w.r.t. (S,Σ)) and every operation symbol $g \in \Sigma_{w,s} \setminus \{f\}$, the set

$$\mathcal{B}|_g \;{}_{\mathrm{def}}{=}\; \{\tau' \in \mathcal{B} \mid \tau' \text{ is not a variable; g is the top-level symbol of } \tau'\}$$

is not empty. ■

Proof :

Assume that $w \in S^*$ and $w_1,\ldots,w_{|w|}$ are not empty (w.r.t. (S,Σ)) and $g \in \Sigma_{w,s} \setminus \{f\}$. Since w_i is not empty (w.r.t. (S,Σ)), there exists a Σ-groundterm $u_i \in (T_\Sigma)^{w_i}$ $(i = 1,\ldots,|w|)$. Now, since we have $g \neq f$, the Σ-groundterm $g(u_1,\ldots,u_{|w|}) \in (T_\Sigma)^s$ is not a Σ-instance of τ, and hence, due to assumption 3) of Lemma 2.2, is a Σ-instance of some element of $\mathcal{B}$, i.e. $g(u_1,\ldots,u_{|w|}) = \sigma(\tau')$ for some $\tau' \in \mathcal{B}$ and some substitution $\sigma : T_\Sigma(X) \to T_\Sigma(X)$. Consequently, g must be the top-level symbol of τ', and therefore, $\mathcal{B}|_g \neq \emptyset$. ■

The next lemma now explains how certain representations for the non-instances of a Σ-term $\tau = f(\tau^{(1)},\ldots,\tau^{(|v|)})$ induce representations for the non-instances of all of its argument terms $\tau^{(i)}$ provided τ is linear and at least one Σ-instance of τ is ground. Since these assumptions on τ are handed down to all of its non-variable subterms, this lemma may be used as a basis for a top-down strategy to construct representations for the non-instances of all non-variable subterms of τ starting with a suitable representations for the non-instances of the Σ-term τ itself.

3 A sort $s \in S$ is empty (w.r.t. (S,Σ)) iff $(T_\Sigma)^s = \emptyset$;

2.3 Lemma :

Let 1) (S,Σ) be a signature with variables X;

2) $\tau = f(\tau^{(1)},...,\tau^{(|v|)})$ be linear with $v \in S^*$, $s \in S$, $f \in \Sigma_{v,s}$, $\tau^{(\kappa)} \in T_\Sigma(X)^{v_\kappa}$ for all $\kappa \in \{1,...,|v|\}$ s.t. some Σ-instance of τ is ground;

3) $\mathcal{B} \subseteq T_\Sigma(X)^s$ be an f-linear[4] representation for the non-instances of τ (w.r.t. (S,Σ)).

Then, for every $i \in \{1,...,|v|\}$, the set

$$\mathcal{B}_i \ {}_{def}{=} \ \{b \mid \exists u_1,...,u_{i-1},u_{i+1},...,u_{|v|}: f(u_1,...,u_{i-1},b,u_{i+1},...,u_{|v|}) \in \mathcal{B}; \text{ no } \Sigma\text{-groundinstance of } b \text{ is a } \Sigma\text{-instance of } \tau^{(i)}; \text{ some } \Sigma\text{-groundinstance of } u_\kappa \text{ is a } \Sigma\text{-instance of } \tau^{(\kappa)} \ (\kappa=1,...,|v|, \kappa\neq i)\}$$

is a linear[5] representation for the non-instances of $\tau^{(i)}$ (w.r.t. (S,Σ)). ■

Proof :

Clearly, $\mathcal{B}_i$ is linear since $\mathcal{B}$ is f-linear (assumption 3) of Lemma 2.3). Now let $\tau_i' \in (T_\Sigma)^{v_i}$. If we assume that τ_i' is a Σ-instance of some element $b \in \mathcal{B}_i$, there exist $u_1,...,u_{i-1},u_{i+1},...,u_{|v|}$ s.t.

$$f(u_1,...,u_{i-1},b,u_{i+1},...,u_{|v|}) \in \mathcal{B};$$
$$\text{no } \Sigma\text{-groundinstance of } b \text{ is a } \Sigma\text{-instance of } \tau^{(i)}; \tag{1}$$
$$\text{some } \Sigma\text{-groundinstance of } u_\kappa \text{ is a } \Sigma\text{-instance of } \tau^{(\kappa)} \ (\kappa=1,...,|v|, \kappa\neq i)$$

and a substitution $\sigma_i : T_\Sigma(X) \to T_\Sigma(X)$ s.t.

$$\tau_i' = \sigma_i(b). \tag{2}$$

Hence, due to (1) and (2), τ_i' is not a Σ-instance of $\tau^{(i)}$. Conversely assume that τ_i' is not a Σ-instance of $\tau^{(i)}$. We have to show that τ_i' is a Σ-instance of some element $b \in \mathcal{B}_i$. According to assumption 2) of Lemma 2.3 there exists a substitution $\varrho : T_\Sigma(X) \to T_\Sigma(X)$ s.t.

$$f(\varrho(\tau^{(1)}),...,\varrho(\tau^{(|v|)})) = \varrho(f(\tau^{(1)},...,\tau^{(|v|)})) = \varrho(\tau) \in (T_\Sigma)^s. \tag{3}$$

We consider

$$\tau_1 \ {}_{def}{=} \ f(\varrho(\tau^{(1)}),...,\varrho(\tau^{(i-1)}),\tau_i',\varrho(\tau^{(i+1)}),...,\varrho(\tau^{(|v|)})) \in (T_\Sigma)^s. \tag{4}$$

Since τ_i' is not a Σ-instance of $\tau^{(i)}$, τ_1 is not a Σ-instance of τ $(= f(\tau^{(1)},...,\tau^{(|v|)}))$. Consequently, due to assumption 3) of Lemma 2.3 and Definition 2.1, τ_1 is a Σ-instance of some element

$$f(u_1,...,u_{i-1},b,u_{i+1},...,u_{|v|}) \in \mathcal{B}. \tag{5}$$

Therefore, there exists a substitution $\sigma : T_\Sigma(X) \to T_\Sigma(X)$ s.t.

---

4 I.e. all Σ-terms $\tau' \in \mathcal{B}$ with top-level symbol f are linear;

5 I.e. all Σ-terms $\tau' \in \mathcal{B}_i$ are linear;

$$
\begin{aligned}
& f(\varrho(\tau^{(1)}),\ldots,\varrho(\tau^{(i-1)}),\tau_i',\varrho(\tau^{(i+1)}),\ldots,\varrho(\tau^{(|v|)})) && = \\
& \tau_1 && = \\
& \text{(see (4))} && && (6)\\
& \sigma(f(u_1,\ldots,u_{i-1},b,u_{i+1},\ldots,u_{|v|})) && = \\
& f(\sigma(u_1),\ldots,\sigma(u_{i-1}),\sigma(b),\sigma(u_{i+1}),\ldots,\sigma(u_{|v|})).
\end{aligned}
$$

In particular,

$$\tau_i' = \sigma(b). \tag{7}$$

Due to (6), $\sigma(u_\kappa) = \varrho(\tau^{(\kappa)})$, i.e. some Σ-groundinstance of u_κ is a Σ-instance of $\tau^{(\kappa)}$ ($\kappa=1,\ldots,|v|$, $\kappa \neq i$). Thus, in order to prove that

$$b \in \mathcal{B}_i, \tag{8}$$

we only have to show that no Σ-groundinstance of b is a Σ-instance of $\tau^{(i)}$. Assume the contrary, i.e. that there exist substitutions $\sigma', \varrho' : T_\Sigma(X) \to T_\Sigma(X)$ s.t.

$$\sigma'(b) = \varrho'(\tau^{(i)}) \in (T_\Sigma)^{v_i} \tag{9}$$

is a Σ-groundterm. Now both $f(u_1,\ldots,u_{i-1},b,u_{i+1},\ldots,u_{|v|})$ and $f(\tau^{(1)},\ldots,\tau^{(|v|)})$ are linear due to assumption 3) and assumption 2) of Lemma 2.3 resp.. Hence, there exist substitutions $\hat{\sigma}, \hat{\varrho} : T_\Sigma(X) \to T_\Sigma(X)$ satisfying

$$\hat{\sigma}(x) = \begin{cases} \sigma(x) & x \in \bigcup_{\kappa=1,\ldots,|v|,\kappa\neq i} Var(u_\kappa) \\ \sigma'(x) & x \in Var(b) \\ x & \text{in all remaining cases} \end{cases} \tag{10}$$

$$\hat{\varrho}(x) = \begin{cases} \varrho(x) & x \in \bigcup_{\kappa=1,\ldots,|v|,\kappa\neq i} Var(\tau^{(\kappa)}) \\ \varrho'(x) & x \in Var(\tau^{(i)}) \\ x & \text{in all remaining cases.} \end{cases} \tag{11}$$

From (10) and (11) we conclude that

$$
\begin{aligned}
& \hat{\sigma}(f(u_1,\ldots,u_{i-1},b,u_{i+1},\ldots,u_{|v|})) && = \\
& f(\hat{\sigma}(u_1),\ldots,\hat{\sigma}(u_{i-1}),\hat{\sigma}(b),\hat{\sigma}(u_{i+1}),\ldots,\hat{\sigma}(u_{|v|})) && = \\
& f(\sigma(u_1),\ldots,\sigma(u_{i-1}),\sigma'(b),\sigma(u_{i+1}),\ldots,\sigma(u_{|v|})) && = \\
& \text{(due to (10))} \\
& f(\varrho(\tau^{(1)}),\ldots,\varrho(\tau^{(i-1)}),\varrho'(\tau^{(i)}),\varrho(\tau^{(i+1)}),\ldots,\varrho(\tau^{(|v|)})) && = && (12)\\
& \text{(due to (6) and (9))} \\
& f(\hat{\varrho}(\tau^{(1)}),\ldots,\hat{\varrho}(\tau^{(i-1)}),\hat{\varrho}(\tau^{(i)}),\hat{\varrho}(\tau^{(i+1)}),\ldots,\hat{\varrho}(\tau^{(|v|)})) && = \\
& \text{(due to (11))} \\
& \hat{\varrho}(f(\tau^{(1)},\ldots,\tau^{(i-1)},\tau^{(i)},\tau^{(i+1)},\ldots,\tau^{(|v|)})) && = \\
& \hat{\varrho}(\tau) \\
& \text{(assumption 2) of Lemma 2.3).}
\end{aligned}
$$

Thus, due to (12), (3), (9) and (5), $\hat{\varrho}(\tau) \in (T_\Sigma)^s$ is a Σ-groundterm that is a Σ-instance of τ and, at the same time, a Σ-instance of some element of $\mathcal{B}$. This, however, is a contradiction to assumption 3) of Lemma 2.3 and Definition 2.1. ■

3. The family $(\mathcal{B}_\tau)_{\tau \in T_\Sigma(X)^s, s \in S}$ of sets of terms and its properties

We now turn to the definition of the family $(\mathcal{B}_\tau)_{\tau \in T_\Sigma(X)^s, s \in S}$ of sets $\mathcal{B}_\tau$ of Σ-terms that are intended to be a suitable representation for the non-instances of τ (w.r.t. (S,Σ)), at least for linear Σ-terms τ. This definition requires the following additional assumptions on the family $X = (X_s)_{s \in S}$ of the sets of variables that will be tacitly assumed throughout this paragraph :

for each sort $s \in S$, the set X_s of variables of sort s is

1. infinite
2. well ordered (without making explicit reference to the well ordering) in such a way that for each finite subset $Y \subseteq X_s$, the minimum $\min(X_s \setminus Y) \in X_s$ is effectively computable.

Clearly, we may chose $\mathcal{B}_\tau \;_{def}\!= \emptyset$ if the sort of τ is empty (w.r.t. (S,Σ)), for in that case there are no Σ-groundterms of the same sort as τ at all. If, however, the sort of τ is not empty (w.r.t. (S,Σ)) but none of the Σ-instances of τ is ground, obviously $\mathcal{B}_\tau \;_{def}\!= \{x\}$ (x a variables of the sort of τ) serves as a representation for the non-instances of τ (w.r.t. (S,Σ)). In all remaining cases, except when τ itself is a variable, $\mathcal{B}_\tau$ is defined by recursion.

3.1 Definition :

Let (S,Σ) be a signature with variables X.

The family $(\mathcal{B}_\tau)_{\tau \in T_\Sigma(X)^s, s \in S}$ of sets of Σ-terms is recursively defined by :

a) (i) if $\tau \in T_\Sigma(X)^s$ (with $s \in S$) and s is empty (w.r.t. (S,Σ)), let

$$\mathcal{B}_\tau \;_{def}\!= \emptyset$$

(ii) if $\tau \in T_\Sigma(X)^s$ (with $s \in S$), s is not empty (w.r.t. (S,Σ)) and no Σ-instance of τ is ground, let

$$\mathcal{B}_\tau \;_{def}\!= \{x\}$$

where x is the least element in the set X_s

b) if $\tau \in T_\Sigma(X)^s$ (with $s \in S$), s is not empty (w.r.t. (S,Σ)) and some Σ-instance of τ is ground, let

$$\mathcal{B}_\tau \;_{def}\!= \emptyset$$

if τ is a variable of sort s, and let

$\mathcal{B}_\tau \;\; {}_{def}{=} \;\; D \cup \bigcup_{i=1,\dots,|v|} D_i$

where

$D \;\; {}_{def}{=} \;\; \{ g(x_1,\dots,x_{|w|}) \mid$
$w \in S^*;\; g \in \Sigma_{w,s}\setminus\{f\};\; w_1,\dots,w_{|w|}$ not empty (w.r.t. (S,Σ));
for every $\kappa \in \{1,\dots,|w|\}$: x_κ is the least element in the set $X_{w_\kappa} \setminus \{x_\mu \mid \mu=1,\dots,\kappa-1\}\}$

$D_i \;\; {}_{def}{=} \;\; \{ f(x_1,\dots,x_{i-1},b,x_{i+1},\dots,x_{|v|}) \mid$
$b \in \mathcal{B}_{\tau(i)}$;
for every $\kappa \in \{1,\dots,|v|\}\setminus\{i\}$: x_κ is the least element in the set $X_{v_\kappa} \setminus (Var(b) \cup \{x_\mu \mid \mu=1,\dots,\kappa-1,\, \mu\neq i\})\}$

if $\tau = f(\tau^{(1)},\dots,\tau^{(|v|)})$ with $v \in S^*$, $f \in \Sigma_{v,s}$, $\tau^{(i)} \in T_\Sigma(X)^{v_i}$ $(i = 1,\dots,|v|)$. ■

The following proposition now states that the sets $\mathcal{B}_\tau$ in fact meet our primary intention, namely that $\mathcal{B}_\tau$ is a (linear) representation for the non-instances of τ (w.r.t. (S,Σ)) provided τ is linear.

3.2 Proposition : *(basic property of $\mathcal{B}_\tau$)*

Let 1) (S,Σ) be a signature with variables X;
2) $\tau \in T_\Sigma(X)^s$ linear with $s \in S$.

Then, the set $\mathcal{B}_\tau \subseteq T_\Sigma(X)^s$ is a linear representation for the non-instances of τ (w.r.t. (S,Σ)). ■

Proof :

If the sort s is empty (w.r.t. (S,Σ)), we have

$\mathcal{B}_\tau \;\; {}_{def}{=} \;\; \emptyset$

and obviously, $\mathcal{B}_\tau$ is linear and a representation for the non-instances of τ (w.r.t. (S,Σ)).So, let us assume that

s is not empty (w.r.t. (S,Σ)). **(1)**

We consider the following two cases :

case 1 : *No Σ-instance of τ is ground.*

Then, according to Definition 3.1,

$\mathcal{B}_\tau \;\; {}_{def}{=} \;\; \{x\}$

where x is the least element in the set X_s. Clearly, $\mathcal{B}_\tau$ is linear. Furthermore, due to the assumption made in case 1, every Σ-groundterm $\tau_1 \in (T_\Sigma)^s$ is not a Σ-instance of

τ and a Σ-instance of an element of $\mathcal{B}_\tau$, i.e. $\mathcal{B}_\tau$ is a representation for the non-instances of τ (w.r.t. (S,Σ)).

case 2 : *Some Σ-instance of τ is ground.*

To prove the assertion of Proposition 3.2 under this assumption, we use induction on $d(\tau)$, the maximum depth of the Σ-term τ. To start with, we assume

$$d(\tau) \;=\; 0 \qquad (2)$$

i.e. τ is either a variable or a constant of sort s. If τ is a variable of sort s, every Σ-groundterm $\tau_1 \in (T_\Sigma)^s$ is a Σ-instance of τ, and consequently,

$$\mathcal{B}_\tau \;{}_{def}\!\!= \;\emptyset$$

is a linear representation for the non-instances of τ (w.r.t. (S,Σ)). The case that τ is a constant of sort s is included (by setting $v \;{}_{def}\!\!= \Lambda$) in the subsequent proof for the case "$d(\tau) > 0$". So let us assume that τ is linear with

$$d(\tau) \;>\; 0 \qquad (3)$$

and that the assertion of Proposition 3.2 holds for every sort $s' \in S$ and every Σ-term $\tau' \in T_\Sigma(X)^{s'}$ with $d(\tau') < d(\tau)$. Now, referring to (3), we have

$$\tau \;=\; f\,(\tau^{(1)},\ldots,\tau^{(|v|)}) \qquad (4)$$

$$\text{with } v \in S^*\backslash\{\Lambda\},\, f \in \Sigma_{v,s}\,,\, \tau^{(i)} \in T_\Sigma(X)^{v_i}\ (i=1,\ldots,|v|).$$

Furthermore, for every $i \in \{1,\ldots,|v|\}$, we have $d(\tau^{(i)}) < d(\tau)$ and $\tau^{(i)}$ is linear, since τ is, such that $\mathcal{B}_{\tau^{(i)}}$ is a linear representation for the non-instances of $\tau^{(i)}$ (w.r.t. (S,Σ)) according to the induction hypothesis. According to Definition 3.1,

$$\mathcal{B}_\tau \;{}_{def}\!\!= \;D \;\cup\; \bigcup_{i=1,\ldots,|v|} D_i$$

where

$$D \;{}_{def}\!\!= \;\{\, g\,(x_1,\ldots,x_{|w|}) \mid$$
$$w \in S^*;\ g \in \Sigma_{w,s}\backslash\{f\};\ w_1,\ldots,w_{|w|} \text{ not empty (w.r.t. } (S,\Sigma));$$
$$\text{for every } \kappa \in \{1,\ldots,|w|\} : x_\kappa \text{ is the least element in the set}$$
$$X_{w_\kappa} \backslash \{x_\mu \mid \mu=1,\ldots,\kappa-1\}\,\}$$

$$D_i \;{}_{def}\!\!= \;\{\, f\,(x_1,\ldots,x_{i-1},b,x_{i+1},\ldots,x_{|v|}) \mid$$
$$b \in \mathcal{B}_{\tau^{(i)}};$$
$$\text{for every } \kappa \in \{1,\ldots,|v|\}\backslash\{i\} : x_\kappa \text{ is the least element}$$
$$\text{in the set } X_{v_\kappa} \backslash (Var(b) \cup \{x_\mu \mid \mu=1,\ldots,\kappa-1,\ \mu\neq i\})\,\}.$$

We have to to establish the following three assertions :

Assertion 1 : *If $\tau_1 \in (T_\Sigma)^s$ is a Σ-instance of an element of $\mathcal{B}_\tau$, then τ_1 is not a Σ-instance of τ.*

Assume that $\tau_1 \in (T_\Sigma)^s$ is a Σ-instance of an element of $\mathcal{B}_\tau$, i.e. there is a substitution $\sigma : T_\Sigma(X) \rightarrow T_\Sigma(X)$ s.t.

either

$$\tau_1 = \sigma\,(g\,(x_1,\ldots,x_{|w|})) \tag{5}$$
$$= g\,(\sigma\,(x_1),\ldots,\sigma\,(x_{|w|}))$$

where : $w \in S^*$; $g \in \Sigma_{w,s} \setminus \{f\}$;
$w_1,\ldots,w_{|w|}$ not empty (w.r.t. (S,Σ));
for every $\kappa \in \{1,\ldots,|w|\}$:
x_κ is the least element in the set
$X_{w_\kappa} \setminus \{x_\mu \mid \mu=1,\ldots,\kappa-1\}$

or

$$\tau_1 = \sigma\,(f\,(x_1,\ldots,x_{i-1},b,x_{i+1},\ldots,x_{|v|})) \tag{6}$$
$$f\,(\sigma\,(x_1),\ldots,\sigma\,(x_{i-1}),\sigma\,(b),\sigma\,(x_{i+1}),\ldots,\sigma\,(x_{|v|}))$$

where : $b \in \mathcal{B}_{\tau^{(i)}}$;
for every $\kappa \in \{1,\ldots,|v|\} \setminus \{i\}$:
x_κ is the least element in the set
$X_{v_\kappa} \setminus (Var(b) \cup \{x_\mu \mid \mu=1,\ldots,\kappa-1, \mu \neq i\})$.

In the first case, τ_1 is obviously not a Σ-instance of τ, since τ satisfies (4) and we have $g \neq f$ in (5). However, if (6) holds and τ_1 were a Σ-instance of τ, there would be a substitution $\sigma' : T_\Sigma(X) \to T_\Sigma(X)$ satisfying

$$f\,(\sigma\,(x_1),\ldots,\sigma\,(x_{i-1}),\sigma\,(b),\sigma\,(x_{i+1}),\ldots,\sigma\,(x_{|v|})) =$$
$$\tau_1 =$$
(due to (6))
$$\sigma'\,(\tau) = \tag{7}$$
$$\sigma'\,(f\,(\tau^{(1)},\ldots,\tau^{(i-1)},\tau^{(i)},\tau^{(i+1)},\ldots,\tau^{(|v|)})) =$$
(due to (4))
$$f\,(\sigma'\,(\tau^{(1)}),\ldots,\sigma'\,(\tau^{(i-1)}),\sigma'\,(\tau^{(i)}),\sigma'\,(\tau^{(i+1)}),\ldots,\sigma'\,(\tau^{(|v|)}))$$

leading us to

$$\sigma\,(b) = \sigma'\,(\tau^{(i)})$$
$$\in (T_\Sigma)^{v_i}$$
(due to (7) and because $\tau_1 \in (T_\Sigma)^s$)

and contradicting the fact that $\sigma\,(b)$ is a Σ-instance of an element of $\mathcal{B}_{\tau^{(i)}}$ and hence not a Σ-instance of $\tau^{(i)}$ since $\mathcal{B}_{\tau^{(i)}}$ is a representation for the non-instances of $\tau^{(i)}$ (w.r.t. (S,Σ)). Therefore in either of the two cases, τ_1 is not a Σ-instance of τ.

Assertion 2 : *If $\tau_1 \in (T_\Sigma)^s$ is not a Σ-instance of τ, then τ_1 is a Σ-instance of some element of $\mathcal{B}$.*

Let $\tau_1 \in (T_\Sigma)^s$ and τ_1 not be a Σ-instance of τ. We have to show that τ_1 is a Σ-instance of some element of $\mathcal{B}_\tau$. To this aim, we first assume that

$$\tau_1 \quad = \quad g\,(\tau_1{}^{(1)},\ldots,\tau_1{}^{(|w|)}) \tag{8}$$

with $w \in S^*$, $g \in \Sigma_{w,s} \backslash \{f\}$, $\tau_1{}^{(\kappa)} \in (T_\Sigma)^{w_\kappa}$ ($\kappa=1,\ldots,|w|$).

Since $\tau^{(\kappa)} \in (T_\Sigma)^{w_\kappa}$, w_κ is not empty (w.r.t. (S,Σ)) ($\kappa=1,\ldots,|w|$) and consequently,

$$g\,(x_1,\ldots,x_{|w|}) \;\in\; \mathcal{B}_\tau. \tag{9}$$

where x_κ is the least element in the set $X_{w_\kappa} \setminus \{x_\mu \mid \mu=1,\ldots,\kappa-1\}$ ($\kappa = 1,\ldots,|w|$).

Now, defining the substitution σ be by

$\sigma\,(x) \;{}_{def}\!= \;\tau_1{}^{(\kappa)}$ if $\kappa \in \{1,\ldots,|w|\}$ and $x = x_\kappa$

$\sigma\,(x) \;{}_{def}\!= \;x$ if $x \in \bigcup\{X_{s'} \mid s' \in S\} \setminus \{x_1,\ldots,x_{|w|}\}$,

we arrive at

$$\begin{aligned} \tau_1 &= g\,(\tau_1{}^{(1)},\ldots,\tau_1{}^{(|w|)}) \\ &\quad \text{(cf. (8))} \\ &= g\,(\sigma\,(x_1),\ldots,\sigma\,(x_{|w|})) \\ &= \sigma\,(g\,(x_1,\ldots,x_{|w|})). \end{aligned} \tag{10}$$

Putting (9) and (10) together, we see that τ_1 is a Σ-instance of an element of $\mathcal{B}_\tau$ provided that (8) holds. So we still have to consider the case

$$\tau_1 \quad = \quad f\,(\tau_1{}^{(1)},\ldots,\tau_1{}^{(|v|)}) \tag{11}$$

with $\tau_1{}^{(\kappa)} \in (T_\Sigma)^{v_\kappa}$ ($\kappa=1,\ldots,|v|$).

If, for every $\kappa \in \{1,\ldots,|v|\}$, there would be a substitution $\sigma_\kappa : T_\Sigma(X) \to T_\Sigma(X)$ with $\tau_1{}^{(\kappa)} = \sigma_\kappa\,(\tau^{(\kappa)})$, the substitution σ defined by

$\sigma\,(x) \;{}_{def}\!= \;\sigma_\kappa\,(x)$ if $\kappa \in \{1,\ldots,|v|\}$ and $x \in Var(\tau^{(\kappa)})$

$\sigma\,(x) \;{}_{def}\!= \;x$ if $x \in \bigcup\{X_{s'} \mid s' \in S\} \setminus \bigcup\{Var(\tau^{(\kappa)}) \mid \kappa=1,\ldots,|v|\}$

would turn τ_1 into a Σ-instance of τ, since

$$\begin{aligned} \tau_1 &= f\,(\tau_1{}^{(1)},\ldots,\tau_1{}^{(|v|)}) \\ &\quad \text{(cf. (11))} \\ &= f\,(\sigma_1\,(\tau^{(1)}),\ldots,\sigma_{|v|}\,(\tau^{(|v|)})) \\ &= f\,(\sigma\,(\tau^{(1)}),\ldots,\sigma\,(\tau^{(|v|)})) \\ &\quad \text{(since } \sigma_{|Var(\tau^{(\kappa)})} = \sigma_{\kappa|Var(\tau^{(\kappa)})} \;(\kappa=1,\ldots,|v|)) \\ &= \sigma\,(f\,(\tau^{(1)},\ldots,\tau^{(|v|)})) \\ &= \sigma\,(\tau) \\ &\quad \text{(cf. (4)),} \end{aligned}$$

and thus confront us with a contradiction (note that due to the assumption that τ is linear, $Var(\tau^{(\kappa)}) \cap Var(\tau^{(\mu)}) = \emptyset$ for $\kappa, \mu \in \{1,\ldots,|v|\}$, $\kappa \neq \mu$, so that σ is well-defined). As a consequence, there exists an index $i \in \{1,\ldots,|v|\}$ s.t. $\tau_1{}^{(i)}$ is not a Σ-instance of $\tau^{(i)}$ and therefore a Σ-instance of some element of $\mathcal{B}_{\tau^{(i)}}$, say

$$\tau_1{}^{(i)} = \quad \sigma_i\,(b) \tag{12}$$

with $b \in \mathcal{B}_{\tau^{(i)}}$ and a substitution $\sigma_i : T_\Sigma(X) \to T_\Sigma(X)$,

because $\mathcal{B}_{\tau^{(i)}}$ is a representation for the non-instances of $\tau^{(i)}$ (w.r.t. (S,Σ)).

Consequently,

$$f\ (x_1,\dots,x_{i-1},b,x_{i+1},\dots,x_{|v|})\ \in\ \mathcal{B}_\tau \tag{13}$$

where, for every $\kappa \in \{1,\dots,|v|\}\backslash\{i\}$, the variable x_κ is the least element in the set $X_{v_\kappa} \setminus (\mathrm{Var}(b) \cup \{x_\mu \mid \mu=1,\dots,\kappa-1,\ \mu\neq i\})$. Since we have that the intersection $\{x_1,\dots,x_{i-1},x_{i+1},\dots,x_{|v|}\} \cap \mathrm{Var}(b)$ is empty and that $x_1,\dots,x_{i-1},x_{i+1},\dots,x_{|v|}$ are pairwise different, the substitution $\sigma : T_\Sigma(X) \to T_\Sigma(X)$ with

$$\begin{array}{lll} \sigma\ (x) & _{def}= \sigma_i\ (x) & \text{if } x \in \mathrm{Var}(b) \\ \sigma\ (x) & _{def}= \tau_1^{(\kappa)} & \text{if } \kappa \in \{1,\dots,|v|\}\backslash\{i\} \text{ and } x = x_\kappa \\ \sigma\ (x) & _{def}= x & \text{if } x \in \bigcup\{X_{s'} \mid s' \in S\} \setminus \\ & & \quad (\mathrm{Var}(b) \cup \{x_1,\dots,x_{i-1},x_{i+1},\dots,x_{|v|}\}) \end{array}$$

is well-defined and gives us

$$\begin{array}{rll} \tau_1 & = & f\ (\tau_1^{(1)},\dots,\tau_1^{(|v|)}) \\ & & (\text{cf. } (11)) \\ & = & f\ (\sigma\ (x_1),\dots,\sigma\ (x_{i-1}),\tau_1^{(i)},\sigma\ (x_{i+1}),\dots,\sigma\ (x_{|v|})) \\ & = & f\ (\sigma\ (x_1),\dots,\sigma\ (x_{i-1}),\sigma_i\ (b),\sigma\ (x_{i+1}),\dots,\sigma\ (x_{|v|})) \\ & & (\text{cf. } (12)) \\ & = & f\ (\sigma\ (x_1),\dots,\sigma\ (x_{i-1}),\sigma\ (b),\sigma\ (x_{i+1}),\dots,\sigma\ (x_{|v|})) \\ & & (\text{since } \sigma_{|\mathrm{Var}(b)} = \sigma_{i|\mathrm{Var}(b)}) \\ & = & \sigma\ (f\ (x_1,\dots,x_{i-1},b,x_{i+1},\dots,x_{|v|})). \end{array} \tag{14}$$

Thus, by (13) and (14), τ_1 is a Σ-instance of an element of $\mathcal{B}_\tau$.

<u>Assertion 3 :</u> $\mathcal{B}_\tau$ *is linear.*

This assertion is immediate from the fact that $\mathcal{B}_{\tau(i)}$ is linear ($i=1,\dots,|v|$) and the definition of $\mathcal{B}_\tau$ in the present case.

Thus, summarizing Assertion 1, Assertion 2 and Assertion 3, we have shown that $\mathcal{B}_\tau$ is in fact a linear representation for the non-instances of τ (w.r.t. (S,Σ)). ■

Having established now the basic property of being a (linear) representation for the non-instances of τ (w.r.t. (S,Σ)) if τ is a linear Σ-term, the remainder of this paragraph is to point out some distinguishing features of the sets $\mathcal{B}_\tau$. I particular we are going to prove the following three assertions, where τ is assumed to be linear in 1. and 2., but may be an arbitrary Σ-term in 3. :

1. No proper subset of $\mathcal{B}_\tau$ is a representation for the non-instances of τ (w.r.t. (S,Σ))
   (*minimality property of* $\mathcal{B}_\tau$).

2. $\mathcal{B}_\tau$ can be embedded in any other representation $\mathcal{B}$ for the non-instances of τ (w.r.t. (S,Σ)) that is f-linear if τ is not a variable and f is the top-level symbol of τ
(*embedding property of* $\mathcal{B}_\tau$).
3. If Σ contains but a finite number of operation symbols, also $\mathcal{B}_\tau$ is finite and can be effectively computed
(*computability of* $\mathcal{B}_\tau$).

The minimality property of $\mathcal{B}_\tau$ is tackled in the following Proposition 3.3 :

3.3 Proposition : (*minimality property of* $\mathcal{B}_\tau$)

Let 1) (S,Σ) be a signature with variables X;
2) $\tau \in T_\Sigma(X)^s$ linear with $s \in S$.

Then, no proper subset of $\mathcal{B}_\tau$ is a representation for the non-instances of τ (w.r.t. (S,Σ)). ■

Proof :

If the sort s is empty (w.r.t. (S,Σ)), then

$$\mathcal{B}_\tau \ {}_{\text{def}}\!= \ \emptyset$$

and hence there is no proper subset of $\mathcal{B}_\tau$ at all. Therefore let us assume that

s is not empty (w.r.t. (S,Σ)). **(1)**

As in the proof of Proposition 3.2, we consider the following two cases :

case 1 : *No Σ-instance of τ is ground.*

Then, according to Definition 3.1,

$$\mathcal{B}_\tau \ {}_{\text{def}}\!= \ \{x\}$$

where x is the least element in the set X_s. As we have assumed in (1), there exists a Σ-groundterm $\tau_1 \in (T_\Sigma)^s$, and consequently, according to the assumption made in case 1, τ_1 is not a Σ-instance of τ. But τ_1 cannot be a Σ-instance of an element of a proper subset of $\mathcal{B}_\tau$, since $\emptyset$ is the only proper subset of $\mathcal{B}_\tau$. Therefore, no proper subset of $\mathcal{B}_\tau$ is a representation for the non-instances of τ (w.r.t. (S,Σ)).

case 2 : *Some Σ-instance of τ is ground.*

To prove the assertion of Proposition 3.3 under this assumption, we use induction on $d(\tau)$, the maximum depth of the Σ-term τ. To start with, let us assume that

$$d(\tau) = 0 \qquad \textbf{(2)}$$

i.e. τ is either a variable or a constant of sort s. If τ is a variable of sort s,

$$\mathcal{B}_\tau \ {}_{\text{def}}\!= \ \emptyset$$

and again, there is no proper subset of $\mathcal{B}_\tau$ at all. The case that τ is a constant of sort s is included (by setting $v \;_{def}\!\!= \Lambda$) in the subsequent proof for the case "$d(\tau) > 0$". So let us assume that τ is linear with

$$d(\tau) \;>\; 0 \tag{3}$$

and that the assertion of the Proposition 3.3 holds for every sort $s' \in S$ and every linear Σ-term $\tau' \in T_\Sigma(X)^{s'}$ with $d(\tau') < d(\tau)$. Now, referring to (3), we have

$$\tau \;=\; f\,(\tau^{(1)},\ldots,\tau^{(|v|)}) \tag{4}$$

$$\text{with } v \in S^*\backslash\{\Lambda\},\; f \in \Sigma_{v,s},\; \tau^{(i)} \in T_\Sigma(X)^{v_i}\; (i=1,\ldots,|v|).$$

Furthermore, for every $i \in \{1,\ldots,|v|\}$, we have $d(\tau^{(i)}) < d(\tau)$ and $\tau^{(i)}$ is linear, because τ is, so that no proper subset of $\mathcal{B}_{\tau^{(i)}}$ is a linear representation for the non-instances of $\tau^{(i)}$ (w.r.t. (S,Σ)) according to the induction hypothesis. According to Definition 3.1 we have

$$\mathcal{B}_\tau \;\;_{def}\!\!=\;\; D \cup \bigcup_{i=1,\ldots,|v|} D_i$$

where

$$D \;\;_{def}\!\!=\;\; \{\, g\,(x_1,\ldots,x_{|w|}) \mid$$
$$w \in S^*;\; g \in \Sigma_{w,s}\backslash\{f\};\; w_1,\ldots,w_{|w|} \text{ not empty (w.r.t. } (S,\Sigma));$$
$$\text{for every } \kappa \in \{1,\ldots,|w|\} : x_\kappa \text{ is the least element in the set}$$
$$X_{w_\kappa} \setminus \{x_\mu \mid \mu=1,\ldots,\kappa-1\}\,\}$$

$$D_i \;\;_{def}\!\!=\;\; \{\, f\,(x_1,\ldots,x_{i-1},b,x_{i+1},\ldots,x_{|v|}) \mid$$
$$b \in \mathcal{B}_{\tau^{(i)}};$$
$$\text{for every } \kappa \in \{1,\ldots,|v|\}\backslash\{i\} : x_\kappa \text{ is the least element}$$
$$\text{in the set } X_{v_\kappa} \setminus (Var(b) \cup \{x_\mu \mid \mu=1,\ldots,\kappa-1,\; \mu\neq i\})\,\}.$$

We have to to establish the following assertion :

Assertion 1 : *If $\mathcal{B} \subset \mathcal{B}_\tau$ is a proper subset of $\mathcal{B}_\tau$,*
then there exist a Σ-groundterm $\tau_1 \in (T_\Sigma)^s$ s.t.
either

τ_1 is not a Σ-instance of τ and
τ_1 is not a Σ-instance of some element of $\mathcal{B}$

or

τ_1 is a Σ-instance of τ and
τ_1 is a Σ-instance of some element of $\mathcal{B}$.

Let $\mathcal{B} \subset \mathcal{B}_\tau$ be a proper subset of $\mathcal{B}_\tau$ and assume first that

$$\mathcal{B} \;\subseteq\; \mathcal{B}_\tau \setminus \{g\,(x_1,\ldots,x_{|w|})\} \tag{5}$$
$$\text{for some } w \in S^*;\; g \in \Sigma_{w,s}\backslash\{f\} \text{ s.t.}$$
$$w_1,\ldots,w_{|w|} \text{ are not empty (w.r.t. } (S,\Sigma)) \text{ and,}$$
$$\text{for every } \kappa \in \{1,\ldots,|w|\},\; x_\kappa \text{ is the least element}$$
$$\text{in the set } X_{w_\kappa} \setminus \{x_\mu \mid \mu=1,\ldots,\kappa-1\}.$$

Now, since the sorts $w_1,\dots,w_{|w|}$ are not empty (w.r.t. (S,Σ)) (cf. (5)), there exist Σ-groundterms $\tau_1^{(i)} \in (T_\Sigma)^{w_i}$ (i=1,...,|w|). As a consequence we have that clearly $\tau_1 \;_{def}\!\!= g\,(\tau_1^{(1)},\dots,\tau_1^{(|w|)}) \in (T_\Sigma)^s$ and that τ_1 is not a Σ-instance of τ since τ is of the form (4) and $g \in \Sigma_{w,s}\backslash\{f\}$. However, τ_1 cannot be a Σ-instance of some of the elements of $\mathcal{B}$, since, due to (5), they are of the form $g'\,(\tau_1'^{(1)},\dots,\tau_1'^{(|w'|)})$ with Σ-groundterms $\tau_1'^{(i)} \in (T_\Sigma)^{w'_i}$ (i=1,...,|w'|), where $g' \neq g$. Therefore τ_1 satisfies the first alternative in Assertion 1. Now, assume that

$$\mathcal{B} \subseteq \mathcal{B}_\tau \setminus \{f\,(x_1,\dots,x_{i-1},b,x_{i+1},\dots,x_{|v|})\} \quad (6)$$

for some $i \in \{1,\dots,|v|\}$ and some $b \in \mathcal{B}_{\tau^{(i)}}$,
where, for every $\kappa \in \{1,\dots,|v|\}\backslash\{i\}$,
x_κ is the least element in the set
$X_{v_\kappa} \setminus (Var(b) \cup \{x_\mu \mid \mu=1,\dots,\kappa-1,\ \mu\neq i\})$

and let

$$\mathcal{C}_{(i)} \;_{def}\!\!= \{b' \in \mathcal{B}_{\tau^{(i)}} \mid f\,(x_1,\dots,x_{i-1},b',x_{i+1},\dots,x_{|v|}) \in \mathcal{B}; \quad (7)$$

for every $\kappa \in \{1,\dots,|v|\}\backslash\{i\}$,
x_κ is the least element in the set
$X_{v_\kappa} \setminus (Var(b') \cup \{x_\mu \mid \mu=1,\dots,\kappa-1,\ \mu\neq i\})\}$.

(6) implies $\mathcal{C}_{(i)} \subseteq \mathcal{B}_{\tau^{(i)}}\backslash\{b\}$, i.e. $\mathcal{C}_{(i)}$ is a proper subset of $\mathcal{B}_{\tau^{(i)}}$, and since $\mathcal{B}_{\tau^{(i)}}$ satisfies the assertion of Proposition 3.3 (due to the induction hypothesis), $\mathcal{C}_{(i)}$ is not a representation for the non-instances of $\tau^{(i)}$ (w.r.t. (S,Σ)), i.e. there exists a Σ-groundterm $\tau_1^{(i)} \in (T_\Sigma)^{v_i}$ s.t.

either

$\tau_1^{(i)}$ is not a Σ-instance of $\tau^{(i)}$ and
$\tau_1^{(i)}$ is not a Σ-instance of some element of $\mathcal{C}_{(i)}$ **(8)**

or

$\tau_1^{(i)}$ is a Σ-instance of $\tau^{(i)}$ and
$\tau_1^{(i)}$ is a Σ-instance of some element of $\mathcal{C}_{(i)}$. **(9)**

Furthermore, there is a Σ-groundinstance of $\tau = f\,(\tau^{(1)},\dots,\tau^{(|v|)})$ (cf. (4)) due to assumption of case 2. This implies that, for every $\kappa \in \{1,\dots,|v|\}\backslash\{i\}$, there exists a Σ-groundterm $\tau_1^{(\kappa)} \in (T_\Sigma)^{v_\kappa}$ that is a Σ-instance of $\tau^{(\kappa)}$, say

$$\tau_1^{(\kappa)} = \sigma_\kappa\,(\tau^{(\kappa)}) \quad (10)$$

for all $\kappa \in \{1,\dots,|v|\}\backslash\{i\}$ and some substitution σ_κ.

Particularly,

$$\tau_1 \;_{def}\!\!= f\,(\tau_1^{(1)},\dots,\tau_1^{(i-1)},\tau_1^{(i)},\tau_1^{(i+1)},\dots,\tau_1^{(|v|)}) \in (T_\Sigma)^s. \quad (11)$$

Now assume (8). Then, due to (4) and (11), τ_1 is not a Σ-instance of τ. However, if τ_1 were a Σ-instance of some element of $\mathcal{B}$ ($\subseteq \mathcal{B}_\tau$; see (6)), there would be a pair (i′,b′) with $i' \in \{1,\dots,|v|\}$ and $b' \in \mathcal{B}_{\tau^{(i')}}$ s.t. $(i',b') \neq (i,b)$ and

τ_1 is a Σ-instance of

$f(x_1,...,x_{i'-1},b',x_{i'+1},...,x_{|v|}) \in \mathcal{B}$ **(12)**

where, for every $\kappa \in \{1,...,|v|\}\backslash\{i'\}$, x_κ is the least element in the set $X_{v_\kappa} \backslash (Var(b') \cup \{x_\mu \mid \mu=1,...,\kappa-1, \mu \neq i'\}$.

Hence for some substitution $\sigma : T_\Sigma(X) \to T_\Sigma(X)$, we would have

$$f(\tau_1^{(1)},...,\tau_1^{(i-1)},\tau_1^{(i)},\tau_1^{(i+1)},...,\tau_1^{(|v|)}) = \tag{13}$$

$$\tau_1 =$$

(cf. (11))

$$\sigma\,(f(x_1,...,x_{i'-1},b',x_{i'+1},...,x_{|v|})) =$$

$$f(\sigma(x_1),...,\sigma(x_{i'-1}),\sigma(b'),\sigma(x_{i'+1}),...,\sigma(x_{|v|})),$$

showing that $\tau_1^{(i')}$ ($= \sigma(b')$) is a Σ-instance of some element of $\mathcal{B}_{\tau(i')}$ and consequently, since $\mathcal{B}_{\tau(i')}$ is a representation for the non-instances of $\tau^{(i')}$ (w.r.t. (S,Σ)) (see Proposition 3.2), that $\tau_1^{(i')}$ is not a Σ-instance of $\tau^{(i')}$. Because of (10), this would imply

$$i' = i. \tag{14}$$

Finally, looking at (13) in the light of (14), we could infer

$$\tau_1^{(i)} = \sigma(b')$$

with $f(x_1,...,x_{i-1},b',x_{i+1},...,x_{|v|}) = f(x_1,...,x_{i'-1},b',x_{i'+1},...,x_{|v|}) \in \mathcal{B}$ (cf. (14) and (12)) and therefore $b' \in \mathcal{C}_{(i)}$ (cf. (7)), in contradiction to (8). Therefore τ_1 is not a Σ-instance of τ and also not a Σ-instance of some element of $\mathcal{B}$ if (8) holds, i.e. satisfies the first alternative in Assertion 1. Thus, to complete the proof of Assertion 1, we still have to look at the situation as given by (9). Now, assuming (9), $\kappa = i$ is no longer an exception in (10), i.e. we have

$$\tau_1^{(\kappa)} = \sigma_\kappa(\tau^{(\kappa)}) \tag{15}$$

for all $\kappa \in \{1,...,|v|\}$and some substitution σ_κ.

Since τ is linear, $Var(\tau^{(\kappa)}) \cap Var(\tau^{(\mu)}) = \emptyset$ for $\kappa, \mu \in \{1,...,|v|\}$, $\kappa \neq \mu$, so that the substitution $\sigma : T_\Sigma(X) \to T_\Sigma(X)$ with

$\sigma(x) \;_{def}\!= \sigma_\kappa(x)$ if $\kappa \in \{1,...,|v|\}$ and $x \in Var(\tau^{(\kappa)})$

$\sigma(x) \;_{def}\!= x$ if $x \in \bigcup\{X_{s'} \mid s' \in S\} \setminus \bigcup\{Var(\tau^{(\kappa)}) \mid \kappa=1,...,|v|\}$

is well-defined and satisfies

$$\tau_1 = f(\tau_1^{(1)},...,\tau_1^{(|v|)})$$

(cf. (11))

$$= f(\sigma_1(\tau^{(1)}),...,\sigma_{|v|}(\tau^{(|v|)}))$$

(cf. (15))

$$= f(\sigma(\tau^{(1)}),...,\sigma(\tau^{(|v|)}))$$

(since $\sigma_{|Var(\tau^{(\kappa)})} = \sigma_{\kappa|Var(\tau^{(\kappa)})}$ ($\kappa=1,...,|v|$))

$$= \sigma(f(\tau^{(1)},...,\tau^{(|v|)}))$$

$$= \sigma(\tau)$$

(cf. (4)).

Hence τ_1 is a Σ-instance of τ. Furthermore, due to (9), $\tau_1^{(i)}$ is a Σ-instance of some element $b' \in \mathcal{C}_{(i)}$, that is

$$\tau_1^{(i)} = \hat{\sigma}_i(b') \quad \text{for some substitution } \hat{\sigma}_i : T_\Sigma(X) \to T_\Sigma(X). \tag{16}$$

From $b' \in \mathcal{C}_{(i)}$ and (7) we conclude that

$$f(x_1,\ldots,x_{i-1},b',x_{i+1},\ldots,x_{|v|}) \in \mathcal{B} \tag{17}$$

where, for every $\kappa \in \{1,\ldots,|v|\}\setminus\{i\}$, x_κ is the least element in the set $X_{v_\kappa} \setminus (Var(b') \cup \{x_\mu \mid \mu=1,\ldots,\kappa-1, \mu \neq i\})$.

Since $\{x_1,\ldots,x_{i-1},x_{i+1},\ldots,x_{|v|}\} \cap Var(b') = \emptyset$ and $x_1,\ldots,x_{i-1},x_{i+1},\ldots,x_{|v|}$ are pairwise different, the substitution $\sigma : T_\Sigma(X) \to T_\Sigma(X)$ with

$$\begin{aligned}
\sigma(x) &=_{def} \hat{\sigma}_i(x) && \text{if } x \in Var(b') \\
\sigma(x) &=_{def} \tau_1^{(\kappa)} && \text{if } \kappa \in \{1,\ldots,|v|\}\setminus\{i\} \text{ and } x = x_\kappa \\
\sigma(x) &=_{def} x && \text{if } x \in \bigcup\{X_{s'} \mid s' \in S\} \setminus \\
& && \quad (Var(b') \cup \{x_1,\ldots,x_{i-1},x_{i+1},\ldots,x_{|v|}\})
\end{aligned}$$

is well-defined and gives us

$$\begin{aligned}
\tau_1 &= f(\tau_1^{(1)},\ldots,\tau_1^{(i-1)},\tau_1^{(i)},\tau_1^{(i+1)},\ldots,\tau_1^{(|v|)}) \quad \text{(cf. (11))} \\
&= f(\sigma(x_1),\ldots,\sigma(x_{i-1}),\tau_1^{(i)},\sigma(x_{i+1}),\ldots,\sigma(x_{|v|})) \\
&= f(\sigma(x_1),\ldots,\sigma(x_{i-1}),\hat{\sigma}_i(b'),\sigma(x_{i+1}),\ldots,\sigma(x_{|v|})) \quad \text{(cf. (16))} \\
&= f(\sigma(x_1),\ldots,\sigma(x_{i-1}),\sigma(b'),\sigma(x_{i+1}),\ldots,\sigma(x_{|v|})) \quad \text{(since } \sigma|Var(b') = \hat{\sigma}_i|Var(b')\text{)} \\
&= \sigma(f(x_1,\ldots,x_{i-1},b',x_{i+1},\ldots,x_{|v|})).
\end{aligned} \tag{18}$$

Thus, by (17) and (18), τ_1 is also a Σ-instance of an element of $\mathcal{B}$ and therefore satisfies the second alternative in Assertion 1.

This completes the proof of Proposition 3.3. ∎

At this point we would like to point out that, given a signature (S,Σ) with variables X and a Σ-term τ of sort $s \in S$, also the minimality property stated in Proposition 3.3 is shared by the set

$$\mathcal{B} =_{def} \{\tau_1 \in (T_\Sigma)^s \mid \tau_1 \text{ is not a } \Sigma\text{-instance of } \tau\}$$

that we have already considered in the discussion immediately following Definition 2.1. However, this is no longer true (in general) for the embedding property that is established for $\mathcal{B}_\tau$ in Proposition 3.4 below. Reusing for instance the signature (S^{fin},Σ^{fin}) specified in the above-mentioned discussion and again letting $\tau =_{def} b$, we have seen that

$$\mathcal{B} = \{a, f(a), f(f(a)), f(f(f(a))), \ldots, f(b), f(f(b)), f(f(f(b))), \ldots\}.$$

Furthermore, by Definition 3.1,

$$\mathcal{B}_\tau \;=\; \{a, f(x)\}$$

where x is a variable of sort s. Both $\mathcal{B}$ and $\mathcal{B}_\tau$ are representation for the non-instances of τ (w.r.t. (S^{fin}, Σ^{fin})) (see Proposition 3.2) that are even linear; however, $\mathcal{B}$ cannot be embedded in $\mathcal{B}_\tau$.

3.4 Proposition : *(embedding property of* $\mathcal{B}_\tau$*)*

Let 1) (S,Σ) be a signature with variables X

2) $\tau \in T_\Sigma(X)^s$ linear with $s \in S$.

Then, if $\mathcal{B} \subseteq T_\Sigma(X)^s$ is a representation for the non-instances of τ (w.r.t. (S,Σ)) that is f-linear if τ is not a variable and f is the top-level symbol of τ, there exists an injection I : $\mathcal{B}_\tau \to \mathcal{B}$. ■

Proof :

Let $\mathcal{B} \subseteq T_\Sigma(X)^s$ be an f-linear representation for the non-instances of τ (w.r.t. (S,Σ)). Due to Definition 3.1, if the sort s is empty (w.r.t. (S,Σ)), then

$$\mathcal{B}_\tau \;{}_{def}\!\!= \emptyset$$

and hence the empty mapping $I \;{}_{def}\!\!= \emptyset : \mathcal{B}_\tau \to \mathcal{B}$ is an injection. Therefore assume that

s is not empty (w.r.t. (S,Σ)). **(1)**

As in the proofs of the preceding Propositions, we consider the following two cases :

case 1 : *No Σ-instance of τ is ground.*

Then, according to Definition 3.1,

$$\mathcal{B}_\tau \;{}_{def}\!\!= \{x\}$$

where x is the least element in the set X_s. Obviously, in the present situation, it suffices to show that $\mathcal{B}$ is not empty. Now, s is not empty (w.r.t. (S,Σ)) (see (1)), i.e. there exists a Σ-groundterm $\tau_1 \in (T_\Sigma)^s$, and, due to the assumption made in case 1, τ_1 is not a Σ-instance of τ. Since $\mathcal{B}$ is a representation for the non-instances of τ (w.r.t. (S,Σ)), τ_1 is a Σ-instance of some element of $\mathcal{B}$ (see Definition 2.1). In particular, $\mathcal{B}$ is not empty.

case 2 : *Some Σ-instance of τ is ground.*

Now we use induction on $d(\tau)$, the maximum depth of the Σ-term τ. Let us first assume that

$d(\tau) \;=\; 0.$ **(2)**

Then τ is either a variable or a constant of sort s. If τ is a variable of sort s,

$$\mathcal{B}_\tau \;{}_{def}\!\!= \emptyset$$

by Definition 3.1 and again, the empty mapping $I \;{}_{def}\!\!= \emptyset : \mathcal{B}_\tau \to \mathcal{B}$ is an injection.

The case that τ is a constant of sort s is again included (by setting $v \;_{def}\!= \Lambda$) in the subsequent proof for the case "$d(\tau) > 0$". Now assume that τ is linear with

$$d(\tau) \;>\; 0 \tag{3}$$

and that the assertion of Proposition 3.4 holds for every sort $s' \in S$ and every Σ-term $\tau' \in T_\Sigma(X)^{s'}$ with $d(\tau') < d(\tau)$. Because of (3), we have

$$\tau \;=\; f\,(\tau^{(1)},\ldots,\tau^{(|v|)}) \tag{4}$$
$$\text{with } v \in S^*\backslash\{\Lambda\},\ f \in \Sigma_{v,s},\ \tau^{(i)} \in T_\Sigma(X)^{v_i}\ (i=1,\ldots,|v|)$$

and, according to Definition 3.1,

$$\mathcal{B}_\tau \;_{def}\!= \; D \cup \bigcup_{i=1,\ldots,|v|} D_i \tag{5}$$

where

$$D \;_{def}\!= \; \{\, g\,(x_1,\ldots,x_{|w|}) \mid \tag{6}$$
$$w \in S^*;\ g \in \Sigma_{w,s}\backslash\{f\};\ w_1,\ldots,w_{|w|} \text{ not empty (w.r.t. } (S,\Sigma));$$
$$\text{for every } \kappa \in \{1,\ldots,|w|\} : x_\kappa \text{ is the least element in the set}$$
$$X_{w_\kappa} \setminus \{x_\mu \mid \mu=1,\ldots,\kappa-1\}\,\}$$

$$D_i \;_{def}\!= \; \{\, f\,(x_1,\ldots,x_{i-1},b,x_{i+1},\ldots,x_{|v|}) \mid \tag{7}$$
$$b \in \mathcal{B}_{\tau^{(i)}};$$
$$\text{for every } \kappa \in \{1,\ldots,|v|\}\backslash\{i\} : x_\kappa \text{ is the least element}$$
$$\text{in the set } X_{v_\kappa} \setminus (Var(b) \cup \{x_\mu \mid \mu=1,\ldots,\kappa-1,\ \mu\neq i\})\,\}.$$

We claim that $\mathcal{B}_\tau$ is even the disjoint union of the sets $D, D_1,\ldots, D_{|v|}$:

<u>Assertion 1 :</u> *The sets* $D, D_1,\ldots, D_{|v|}$ *are pairwise disjoint.*

Clearly $D \cap D_i = \emptyset$ for every $i \in \{1,\ldots,|v|\}$ since all Σ-terms $\tau' \in D$ have some top-level symbol $g \neq f$, whereas all Σ-terms $\tau' \in D_i$ have top-level symbol f. Now let $i, j \in \{1,\ldots,|v|\}$, $i \neq j$, and assume that $D_i \cap D_j \neq \emptyset$. By the definition of D_i (see (7)) there exists $b \in \mathcal{B}_{\tau^{(i)}}$ s.t.

$$f\,(x_1,\ldots,x_{i-1},b,x_{i+1},\ldots,x_{|v|}) \;\in\; D_j \tag{8}$$

where, for every $\kappa \in \{1,\ldots,|v|\}\backslash\{i\}$, the variable x_κ is the least element in the set $X_{v_\kappa} \setminus (Var(b) \cup \{x_\mu \mid \mu=1,\ldots,\kappa-1,\ \mu\neq i\})$. However, since $i \neq j$, the i-th argument term of any Σ-term in D_j is a variable. Consequently, by (8), b is a variable and therefore, every Σ-groundterm $\tau_1^{(i)} \in (T_\Sigma)^{v_i}$ is a Σ-instance of b. Since $b \in \mathcal{B}_{\tau^{(i)}}$ and $\mathcal{B}_{\tau^{(i)}}$ is a representation for the non-instances of $\tau^{(i)}$ (w.r.t. (S,Σ)) (see Proposition 3.2), no Σ-groundterm $\tau_1^{(i)} \in (T_\Sigma)^{v_i}$ is a Σ-instance of $\tau^{(i)}$ or, in other words, no Σ-instance of $\tau^{(i)}$ is ground. Because of (4), this implies that also no Σ-instance of τ is ground, a contradiction to our assumption in case 2 ! Therefore, we must have $D_i \cap D_j = \emptyset$.

We are now going to prove the existence of injections

$$J \;:\; D \;\rightarrow\; \mathcal{B}$$

$$J_i \;:\; D_i \;\rightarrow\; \mathcal{B} \qquad (i=1,\ldots,|v|)$$

whose ranges Im (J), Im (J_1),..., Im ($J_{|v|}$) are pairwise disjoint. Once having established this assertion, we may define

$$I \;_{def}{=}\; J \cup J_1 \cup \ldots \cup J_{|v|}$$

and obtain an injection $I : \mathcal{B}_\tau \rightarrow \mathcal{B}$ (cf. (5) and Assertion 1). Now, due to the definition of D (see (6)) and due to Lemma 2.2, for every Σ-term $g\,(x_1,\ldots,x_{|w|}) \in D$, the set

$$\mathcal{B}|_g \;_{def}{=}\; \{\tau' \in \mathcal{B} \mid \tau' \text{ is not a variable; g is the top-level symbol of } \tau'\}$$

is contained in $\mathcal{B}$ and not empty. Thus, the Axiom of Choice provides us with a mapping

$$J \;:\; D \;\rightarrow\; \mathcal{B} \tag{9}$$

satisfying

$$J\,(g\,(x_1,\ldots,x_{|w|})) \;\in\; \mathcal{B}|_g \quad \text{for every } g\,(x_1,\ldots,x_{|w|}) \in D. \tag{10}$$

From (10) and (6) we infer that

$$J \text{ is injective.} \tag{11}$$

Now let $i \in \{1,\ldots,|v|\}$. Then, the mapping

$$\begin{array}{lccc} J_i' : & D_i & \rightarrow & \mathcal{B}_{\tau^{(i)}} \\ & f\,(x_1,\ldots,x_{i-1},b,x_{i+1},\ldots,x_{|v|}) & \mapsto & b \end{array} \tag{12}$$

is an injection (cf. (7)). Furthermore, by Lemma 2.3, the set

$$\begin{aligned} \mathcal{B}_i \;_{def}{=}\; \{b \mid \exists u_1,\ldots,u_{i-1},u_{i+1},\ldots,u_{|v|}:\; & f\,(u_1,\ldots,u_{i-1},b,u_{i+1},\ldots,u_{|v|}) \in \mathcal{B}; \\ & \text{no } \Sigma\text{-groundinstance of b is a } \Sigma\text{-instance of } \tau^{(i)}; \\ & \text{some } \Sigma\text{-groundinstance of } u_\kappa \text{ is a } \\ & \Sigma\text{-instance of } \tau^{(\kappa)}\ (\kappa=1,\ldots,|v|,\ \kappa\neq i)\} \end{aligned} \tag{13}$$

is a linear representation for the non-instances of $\tau^{(i)}$ (w.r.t. (S,Σ)). Consequently, due to the induction hypothesis, there exists an injection

$$J_i'' \;:\; \mathcal{B}_{\tau^{(i)}} \;\rightarrow\; \mathcal{B}_i. \tag{14}$$

Finally, for every $b \in \mathcal{B}_i$, the set

$$\begin{aligned} \mathcal{C}_{(b)} \;_{def}{=}\; \{\, & f\,(u_1,\ldots,u_{i-1},b,u_{i+1},\ldots,u_{|v|}) \in \mathcal{B} \mid \\ & \text{no } \Sigma\text{-groundinstance of b is a } \Sigma\text{-instance of } \tau^{(i)}\text{; some } \Sigma\text{-} \\ & \text{groundinstance of } u_\kappa \text{ is a } \Sigma\text{-instance of } \tau^{(\kappa)}\ (\kappa=1,\ldots,|v|,\ \kappa\neq i)\} \end{aligned}$$

is contained in $\mathcal{B}$ and not empty. Hence, again referring to the Axiom of Choice, there is a mapping

$$J_i''' \;:\; \mathcal{B}_i \;\rightarrow\; \mathcal{B}. \tag{15}$$

satisfying

$$J_i'''(b) \;\in\; \mathcal{C}_{(b)} \quad \text{for every } b \in \mathcal{B}_i. \tag{16}$$

(16) immediately implies that J_i''' is injective. Now letting

$$J_i \;_{def}{=}\; J_i''' \circ J_i'' \circ J_i' \;:\; D_i \;\rightarrow\; \mathcal{B}, \tag{17}$$

the injectivity of J_i''', J_i'' and J_i' implies that also

J_i is injective. **(18)**

Note that, due to (12), (16) and (17), for every $f(x_1,...,x_{i-1},b,x_{i+1},...,x_{|v|}) \in D_i$ we have

$$J_i(f(x_1,...,x_{i-1},b,x_{i+1},...,x_{|v|})) = f(u_1,...,u_{i-1},J_i''(b),u_{i+1},...,u_{|v|}) \quad \textbf{(19)}$$

where $J_i''(b) \in T_\Sigma(X)^{v_i}$, no Σ-groundinstance of $J_i''(b)$ is a Σ-instance of $\tau^{(i)}$, and, for every $\kappa \in \{1,...,|v|\}\setminus\{i\}$, $u_\kappa \in T_\Sigma(X)^{v_\kappa}$, some Σ-groundinstance of u_κ is a Σ-instance of $\tau^{(\kappa)}$.

According to the foregoing remark, we still have to establish the following assertion :

<u>Assertion 2 :</u> *The sets* Im (J), Im (J_1),..., Im ($J_{|v|}$) *are pairwise disjoint.*

Since all values of J are Σ-terms with some top-level symbol $g \neq f$ (cf. (9) and (10)) whereas the values of J_i are Σ-terms with top-level symbol f (cf. (17) and (19)), Im (J) $\cap$ Im (J_i) = $\emptyset$ for every $i \in \{1,...,|v|\}$. Now let $i, j \in \{1,...,|v|\}$ with $i \neq j$, and assume that Im (J_i)$\cap$Im (J_j) $\neq \emptyset$. Referring to the representation (19) of the values of the mappings J_i and J_j resp. we infer that there exist Σ-terms $u_1,...,u_{i-1},U_i,u_{i+1},...,u_{|v|}$ as well as Σ-terms $r_1,...,r_{j-1},R_j,r_{j+1},...,r_{|v|}$ s.t.

$$f(u_1,...,u_{i-1},U_i,u_{i+1},...,u_{|v|}) = f(r_1,...,r_{j-1},R_j,r_{j+1},...,r_{|v|}) \quad \textbf{(20)}$$

where

$U_i \in T_\Sigma(X)^{v_i}$, no Σ-groundinstance of U_i is a Σ-instance of $\tau^{(i)}$, **(21)**
and, for every $\kappa \in \{1,...,|v|\}\setminus\{i\}$, $u_\kappa \in T_\Sigma(X)^{v_\kappa}$, some Σ-groundinstance of u_κ is a Σ-instance of $\tau^{(\kappa)}$

and

$R_j \in T_\Sigma(X)^{v_j}$, no Σ-groundinstance of R_j is a Σ-instance of $\tau^{(j)}$, **(22)**
and, for every $\mu \in \{1,...,|v|\}\setminus\{j\}$, $r_\mu \in T_\Sigma(X)^{v_\mu}$, some Σ-groundinstance of r_μ is a Σ-instance of $\tau^{(\mu)}$.

From (20) and $i \neq j$ we further conclude that

$$U_i = r_i.$$

Hence, by (22), some Σ-groundinstance of U_i is a Σ-instance of $\tau^{(i)}$, in contradiction to (21). Consequently, Im (J_i)$\cap$Im (J_j) $\neq \emptyset$ cannot hold, i.e. we have Im (J_i)$\cap$Im (J_j) = $\emptyset$. ■

Note that, as a corollary to Proposition 3.4, if $\mathcal{B}_\tau$ is infinite/uncountable, there is no finite /countable representation $\mathcal{B}$ for the non-instances of τ (w.r.t. (S,Σ)) that is f-linear if τ is not a variable and f is the top-level symbol of τ (hypothesis of Proposition 3.4 assumed).

We conclude this paragraph with the following proposition that ensures the (finiteness and) computability of the sets $\mathcal{B}_\tau$ in the situation that the signature (S,Σ) under consideration contains but a finite number of operation symbols. Although this is a rigorous restriction of generality, this situation is most common in practice. Note that Proposition 3.5 does not assume that τ is linear.

3.5 Proposition : *(computability of $\mathcal{B}_\tau$)*

Let (S,Σ) be a signature with variables X satisfying that the set $\bigcup \{\Sigma_{w,s'} \mid w \in S^*, s' \in S\}$ of Σ-operation symbols is finite.

Then, for every Σ-term $\tau \in T_\Sigma(X)^s$ of sort $s \in S$, the set $\mathcal{B}_\tau$ (is finite and) can be effectively computed. ■

Proof :

Since the set $\bigcup \{\Sigma_{w,s'} \mid w \in S^*, s' \in S\}$ of Σ-operation symbols is assumed to be finite, it is obvious from Definition 3.1 that also $\mathcal{B}_\tau$ is finite for every sort $s \in S$ and every Σ-term $\tau \in T_\Sigma(X)^s$. In order to prove that $\mathcal{B}_\tau$ can be effectively computed, we first establish the following assertion :

Assertion 1 : *The set*

$$S_1 \;{}_{def}{=}\; \{s' \in S \mid s' \text{ is not empty (w.r.t. } (S,\Sigma))\}$$

can be effectively computed.

Consider the following algorithm NONEMPTYSORTS:

Input : $\Sigma_1 \;{}_{def}{=}\; \{(g,w,s') \mid w \in S^*, s' \in S, g \in \Sigma_{w,s'}\}$

Output : $S_1 \;{}_{def}{=}\; \{s' \in S \mid s' \text{ is not empty (w.r.t. } (S,\Sigma))\}$

Algorithm : 1) $S_2 := \emptyset$

2) $N := \{s' \in S \setminus S_2 \mid \exists w \in S_2^*, g : (g,w,s') \in \Sigma_1\}$

3) **If** $N = \emptyset$

then Output S_2;
STOP.

else $S_2 := S_2 \cup N$;
goto 2).

Note that Σ_1 is finite and consequently, that only a finite number of sorts $s' \in S$ may occur as the third component of a triple in Σ_1. Therefore NONEMPTYSORTS will eventually stop. Now, at any state of the computation of NONEMPTYSORTS, the set S_2 constructed so far satisfies

$$S_2 \subseteq \{s' \in S \mid s' \text{ is not empty (w.r.t. } (S,\Sigma))\} \qquad (1)$$

since S_2 is initialized by the empty set and, at each iteration, supplemented only by sorts s´ which possess an operation symbol whose argument sorts belong to S_2 and hence are not empty (w.r.t. (S,Σ)). Now let us assume that the inclusion (1) is proper, i.e.

$$\{s' \in S \backslash S_2 \mid s' \text{ is not empty (w.r.t. } (S,\Sigma))\} \neq \emptyset .$$

Then the set of natural numbers

$$\{d(\tau') \mid s' \in S\backslash S_2, \tau' \in (T_\Sigma)^{s'}\},$$

where $d(\tau')$ denotes the maximum depth of the Σ-term τ', contains a least element, say

$$n^\circ \quad = \quad d(\tau')$$
$$\text{with } \tau' \in (T_\Sigma)^{s'} \text{ for some } s' \in S\backslash S_2.$$

Since τ' is a Σ-groundterm, we have

$$\tau' \quad = \quad g\,(\tau^{(1)\prime},\dots,\tau^{(|w|)\prime})$$
$$\text{with } w \in S^*, g \in \Sigma_{w,s'}, \tau^{(i)\prime} \in (T_\Sigma)^{w_i}\ (i=1,\dots,|w|).$$

Furthermore, for every $i \in \{1,\dots,|w|\}$, we know $d(\tau^{(i)\prime}) < d(\tau') = n^\circ$ and therefore, due to the minimality property of n°, $w_i \in S_2$. Resuming these fact, we have found a sort $s' \in S\backslash S_2$, a word $w \in S_2{}^*$ and an operation symbol g s.t. $(g,w,s') \in \Sigma_1$. As a consequence, if (1) is a proper inclusion, the set N constructed in step 2) of NONEMPTYSORTS is not empty so that NONEMPTYSORTS cannot stop unless (1) turns out to be an equality. This proves the correctness of NONEMPTYSORTS.

Now, referring to Definition 3.1, the following algorithm NONINSTANCES actually computes the set $\mathcal{B}_\tau$ for every sort $s \in S$ and every Σ-term $\tau \in T_\Sigma(X)^s$:

Input: $\Sigma_1 \,_{def}\!= \{(g,w,s') \mid w \in S^*, s' \in S, g \in \Sigma_{w,s'}\}$,
a Σ-term $\tau \in T_\Sigma(X)^s$ with $s \in S$

Output: $\mathcal{B}_\tau$

Algorithm: 1) a) **If** s is empty (w.r.t. (S,Σ))
then $\mathcal{B}_\tau := \emptyset$;
Output $\mathcal{B}_\tau$;
STOP.

b) **If** for at least one variable $y \in Var(\tau)$,
the sort s´of y is empty (w.r.t. (S,Σ))
then $\mathcal{B}_\tau := \{x\}$
where x is the least element in the set X_s;
Output $\mathcal{B}_\tau$;
STOP.

2) a) **If** τ is a variable
then $\mathcal{B}_\tau := \emptyset$;
Output $\mathcal{B}_\tau$;
STOP.

b) **If** $\tau = f(\tau^{(1)},...,\tau^{(|v|)})$
with $v \in S^*$, $f \in \Sigma_{v,s}$, $\tau^{(i)} \in T_\Sigma(X)^{v_i}$ $(i=1,...,|v|)$
then **for every**
$i \in \{1,...,|v|\}$
do
compute $\mathcal{B}_{\tau^{(i)}}$ by a recursive call of algorithm NONINSTANCES
$\mathcal{B}_\tau := \emptyset$;
for every
$(g,w,s') \in \Sigma_1$ s.t. $g \neq f$, $w_1,...,w_{|w|}$ not empty (w.r.t. (S,Σ)), $s' = s$
do
$\mathcal{B}_\tau := \mathcal{B}_\tau \cup \{g(x_1,...,x_{|w|})\}$
where, for every $\kappa \in \{1,...,|w|\}$,
x_κ is the least element in the set
$X_{w_\kappa} \setminus \{x_\mu \mid \mu=1,...,\kappa-1\}$;
for every
$i \in \{1,...,|v|\}$, $b \in \mathcal{B}_{\tau^{(i)}}$
do
$\mathcal{B}_\tau := \mathcal{B}_\tau \cup$
$\{f(x_1,...,x_{i-1},b,x_{i+1},...,x_{|v|})\}$
where, for every $\kappa \in \{1,...,|v|\} \setminus \{i\}$,
x_κ is the least element in the set
$X_{v_\kappa} \setminus (Var(b) \cup \{x_\mu \mid \mu=1,...,\kappa-1, \mu \neq i\})$;
Output $\mathcal{B}_\tau$;
STOP.

Note that, due to Assertion 1, it can be decided whether or not a sort $s \in S$ is not empty (w.r.t. (S,Σ)). Furthermore, NONINSTANCES will eventually stop upon input Σ_1 and τ, since all recursive calls of NONINSTANCES use a proper subterm $\tau^{(i)}$ of the original input Σ-term τ as input and since all loop statements in NONINSTANCES vary over a finite index set (see step 2)b) in NONINSTANCES). This establishes Proposition 3.5. ■

4. Conclusion

Motivated by the problem of finding a set of rewrite rules that is appropriate to model the "no-action" behavior of disabled transitions in High Level Petri Nets over algebraic specifications with constructors, we have abstracted the general problem of finding a suitable representation for the set of all ground terms τ_1 that are not an instance of a given term τ over a sorted signature (S,Σ). Restricting to the case that S = {s} is a singleton, a more general version of this problem has already been considered by J.-L. Lassez and K. Marriott [La/Ma 87]. The solution we propose consists in a recursively defined family $(\mathcal{B}_\tau)_{\tau \in T_\Sigma(X)^s, s \in S}$ of sets of terms s.t. for every linear term τ, the non-instances of τ are precisely the ground instances of the elements of $\mathcal{B}_\tau$ and, moreover, $\mathcal{B}_\tau$ is minimal (both w.r.t. set inclusion and w.r.t. cardinality) among all sets of linear terms enjoying this property. As to the practicability of our approach, we prove that the sets $\mathcal{B}_\tau$ are finite and can be effectively computed whenever the number of operation symbols presented by the signature (S,Σ) is finite, an assumption which is usually satisfied in practice.

Three lines along which the above results may be generalized are now offering for future work : extending J.-L. Lassez´ and K. Marriotts results to arbitrary signatures (S,Σ), incorporating equational theories and, last but not least, considering the case of nonlinear terms. This case has been excluded so far, not at least because of some results in [La/Ma 87] (see Proposition 4.5 and Proposition 4.6) that show that the non-instances of nonlinear terms cannot be finitely represented in the above described way (unless, of course, the underlying Herbrand universe is finite). Hence, tackling nonlinear terms seems to require new ideas as to the representation of their non-instances.

Bibliography

[Ge 88] H. Gerlach : *Rewriting Applied to Pr/E-nets over Algebraic Specifications with Constructors*
GRASPIN Technical Paper KAI 29/1 , Universität Kaiserslautern , (1988)

[Grä 79] G. Grätzer : *Universal Algebra*
Springer-Verlag New York Heidelberg Berlin (2nd ed.) , (1979)

[La/Ma 87] J.-L. Lassez, K. Marriott : *Explicit Representation of Terms defined by Counter Examples*
Journal of Automated Reasoning 3 , pp 301 - 317 , (1987)

[Lug 76] H. Lugowski : *Grundzüge der Universellen Algebra*
BSB B.G. Teubner Verlagsgesellschaft Leipzig , (1976)

[Mic 83] R. S. Michalski : *A Theory and Methodology of Inductive Learning*
Artificial Intelligence 20 , pp 111 - 161 , (1983)

[Mit 78] T. M. Mitchell : *Version Spaces: An Approach to Concept Learning*
Ph.D. Thesis , STAN-CS-78-711 , Comp. Science Dept. , Stanford University , (1978)

[Ver 80] S. A. Vere : *Multilevel Counterfactuals for Generalization of Relational Concepts and Productions*
Artificial Intelligence 14 , pp 139 - 164 , (1980)