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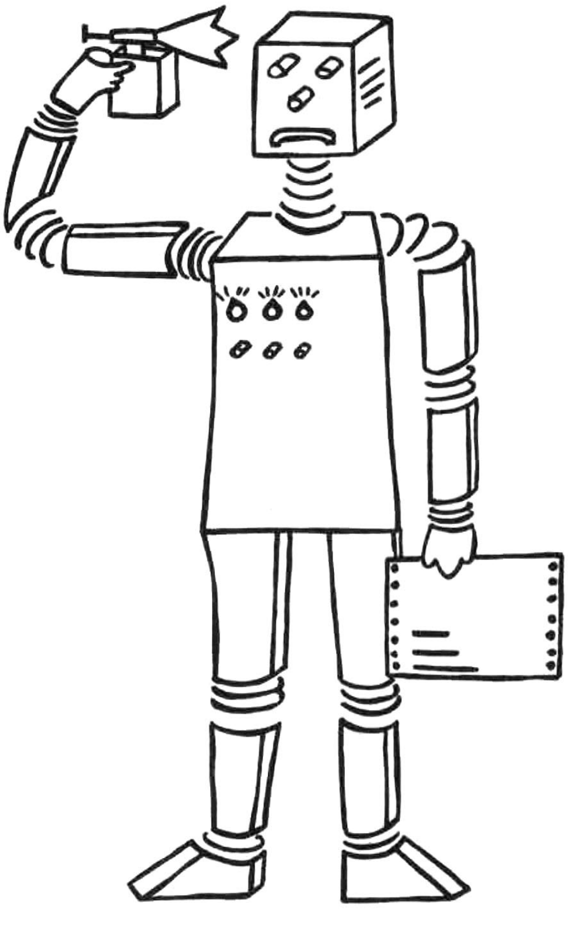

UNIFICATION IN A COMBINATION OF ARBITRARY DISJOINT EQUATIONAL THEORIES

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Unification in a Combination of Arbitrary Disjoint Equational Theories.

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Abstract. The unification problem of terms in a disjoint combination $\mathcal{E}_1+\ldots+\mathcal{E}_n$ of arbitrary theories is reduced to a combination of pure unification problems in $\mathcal{E}_j$, where free constants may occur in terms, and to constant elimination problems like: find all substitutions σ such that c_i is not a constant in the term σt_i, $i = 1,\ldots,n$, where t_i are terms in the theory $\mathcal{E}_j$.

The algorithm consists of the following basic steps: First of all the terms to be unified are transformed via variable abstraction into terms belonging to one particular theory. Terms belonging to the same theory can now be unified with the algorithm for this theory. For terms in some multi-equation belonging to different theories it is sufficient to select some theory and collapse all terms not belonging to this particular theory into a common constant. Finally constant elimination must be applied in order to solve cyclic unification problems like $\langle x = f(x)\rangle$.

The algorithm shows that a combination of finitary unifying regular theories, of Boolean rings, of Abelian groups or of theories of Hullot-type is of unification-type finitary, since these theories have finitary constant-elimination problems. As a special case, unification in a combination of a free Boolean ring with free function symbols is decidable and finitary; the same holds for Abelian groups. Remarkably, it can be shown that unification problems can be solved in the general case $\mathcal{E}_1+\ldots+\mathcal{E}_n$ if for every i there is a method to solve unification problems in a combination of $\mathcal{E}_i$ with free function symbols. Thus, unification in a combination with free function symbols is the really hard case.

This paper presents solutions to the important open questions of combining unification algorithms in a disjoint combination of theories. As a special case it provides a solution to the unification of general terms (i.e. terms, where free function symbols are permitted) in Abelian groups and Boolean rings. It extends the known results on unification in a combination of regular and collapse-free theories in two aspects: Arbitrary theories are admissable and we can use complete unification procedures (including universal unification procedures such as narrowing) that may produce an infinite complete set of unifiers for a special theory.

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Key words: Unification, Equational theories, Decidability of Unification, Combination of equational theories, Boolean rings, Abelian groups

1. Introduction.

Unification of terms with respect to an equational theory $\mathcal{E}$ [Pl72, Si75, Si86] is the problem given a set of equations $\Gamma = \langle s_1 = t_1, \ldots, s_n = t_n \rangle_{\mathcal{E}}$ to find some or all substitutions σ such that all equations are solved, i.e. $\mathcal{E}$ implies $\sigma s_i = \sigma t_i$ for all i. There are several theories $\mathcal{E}$ for which a unification algorithm or a unification procedure is known, for example commutativity (C), associtivity and commutativity or Abelian semigroups (AC), Boolean rings (BR) and Abelian groups (AG). For a survey see [Si86, Si87].

In general the terms that are allowed as input for a unification algorithm for a theory $\mathcal{E}$ are restricted to consist of variables and function symbols that belong to $\mathcal{E}$, for example a BR-unification algorithm allows only terms built with +,*,0,1 and free constants. All useful unification algorithms known so far accept terms built with a fixed set of theory function symbols and arbitrary free constants. However, it is an open problem how to construct from an $\mathcal{E}$-unification algorithm for $\mathcal{E}$-pure terms (i.e. without free constants) a unification algorithm that also accepts terms including free constants [Bü86, Bü87]. H.-J. Bürckert and the author have given an example where unification becomes undecidable after the addition of free constants [Bü86, Sch87a]. We exclude this problem by assuming that free constants are permitted in terms.

The application of unification algorithms would be rather restricted if only terms containing the function symbols belonging to $\mathcal{E}$ and free constants are possible as input. For example in Automated Deduction systems Skolem-function (i.e., free function symbols) occur frequently and usually terms containing the theory symbols and free function symbols have to be unified. A similar situation arises in completion procedures modulo a congruence [LB77, Hu80, JK84], where in general a unification algorithm for an equational theory plus free function symbols is required.

The combination of unification algorithms for theories with disjoint sets of function symbols has been considered first by M. Stickel [St75, St81] , M. Livesey and J. Siekmann [LS78] and F. Fages [Fa84] for the associative-commutative case. The algorithms accept terms built with several AC-function symbols and free function symbols. A more general combination problem was tackled by K. Yellick, C. Kirchner, E. Tidén and A. Herold [Ye87, Ki85, Ti86a, He86]. They came up with algorithms for a combination of equational theories that obey some restrictions. C. Kirchner [Ki85] requires the theories to be simple. K.Yellick and A. Herold [Ye87, He86] require the theories to be regular and collapse-free and E. Tidén [Ti86a, Ti86b] considered the more general case of collapse-free theories. Recently, P. Jouannaud and A. Boudet [BJ87] announced a unification procedure for a combination of arbitrary and a simple theory.

We loosen these restrictions to allow arbitrary theories in a disjoint combination. The presented algorithm can be seen as an extension of C. Kirchner's method to transform systems of multi-equations. The idea of constant-abstraction [LS78, He86] is indispensible and used heavily in our algorithm. We show that in order to solve unification problems in a combination, it is sufficient to have a unification algorithm for terms with free constants for all theories and a solution method for constant elimination problems in every theory. Alternatively we can also use a unification procedure for a combination of every theory with free function symbols. A complete solution is presented for a combination of theories, where every theory $\mathcal{E}$ in the combination satisfies one of the following cases:

i) $\mathcal{E}$ is regular

ii) $\mathcal{E}$ is a free Abelian group

iii) $\mathcal{E}$ is a free Boolean ring

iv) $\mathcal{E}$ admits a canonical TRS and basic narrowing terminates (is of Hullot-type)

Note that in this paper we will always assume that an $\mathcal{E}$-unification algorithm also accepts free constants and that all notions and definitions refer to a signature that includes infinitely many free constants.

The paper is organised as follows. In the paragraphs 2-4 we give the basic definitions and an analysis of properties of a combination and paragraph 5 deals with the basic transformation rules used for unification algorithms. In sections 6 the algorithm for the general combinations is presented and in section 7 we prove its completeness. Paragraph 8 presents methods to solve constant-elimination problems. Paragraphs 9 and 10 deal with the special cases of combining an arbitrary and a simple theory and combining collapse-free and regular theories. In paragraph 11 a decidability result for unification in a combination is given.

2. Equational Theories.

We assume that the reader is familiar with terms, substitutions and algebras [HO80, Hu80, Si87]. The set of terms $\mathcal{T}(\Sigma,V)$ is defined over a signature Σ consisting of fixed-arity function symbols and a countably infinite set of variables **V**. The set $\mathcal{T}(\Sigma,V)$ is a free algebra over Σ. Nullary function symbols are also called constants. We shall use hd(t) to denote the top level function symbol of t or hd(t) = t, if t is a variable. The set of variables in a term t is denoted by V(t). A **substitution** σ is an endomorphism on $\mathcal{T}(\Sigma,V)$ that moves at most finitely many variables. Substitutions can be represented by a set of variable-term pairs $\sigma = \{x_1 \leftarrow t_1, \ldots, x_n \leftarrow t_n\}$ with $x_i \neq t_i$. The set $\{x_1,\ldots,x_n\}$ is the **domain** $DOM(\sigma)$ of σ and the set $\{t_1,\ldots,t_n\}$ is the **codomain** $COD(\sigma)$, the set of variables introduced by σ is denoted as $I(\sigma) := V(COD(\sigma))$. The **union** $\sigma\cup\tau$ of two substitutions σ and τ with $DOM(\sigma) \cap DOM(\tau) = \emptyset$ is defined as $(\sigma\cup\tau)x = \sigma x$, if

$x \in DOM(\sigma)$, $(\sigma \cup \tau)x = \tau x$, if $x \in DOM(\sigma)$ and $(\sigma \cup \tau)x = x$, otherwise. The **restriction** of a substitution σ to a set of variables W is the substitution $\sigma_{|W}$ with $\sigma_{|W}x = \sigma x$ for $x \in W$ and $\sigma_{|W}x = x$, otherwise. We also use $U_{|W}$ for the set $\{\sigma_{|W} \in U \mid \sigma \in U\}$. A substitution σ is called **idempotent**, iff $\sigma\sigma = \sigma$. Note that σ is idempotent, iff $DOM(\sigma) \cap I(\sigma) = \emptyset$.

In order to have access to subterms of a term t, we use **occurrences** [Hu80]. The subterm of t at occurrence π is denoted by $t\backslash\pi$ and the term constructed from t by replacing the subterm at occurrence π by term s is denoted as $t[\pi \leftarrow s]$.

In the following we sometimes use the phrase 'new variable', which always means that a variable is now used, which is never used before dependent on the context. Such a choice is always possible, since there are countably many variables.

An **equational theory** $\mathcal{E}$ is a pair (Σ, E), where Σ is a signature and E is a set of equations. The equations in E are also called **axioms.** If a function symbol occurs in an equation in E it is called **interpreted** function symbols, otherwise a **free** function symbol. The set of free constants in a term t is denoted by FRC(t).

An algebra A over Σ **satisfies** E or is a **model** of $\mathcal{E}$ ($A \vDash E$), if for every assignment of elements in A to variables in an equation $l = r$ in E, the corresponding equation holds in A. We say an equation $s = t$ is a **consequence** of E ($E \vDash s = t$), if every model A of E also satisfies $s = t$. We will also use $s =_{\mathcal{E}} t$ instead of $\mathcal{E} \vDash s = t$. Note that the relation $=_{\mathcal{E}}$ is a congruence relation on $\mathcal{T}(\Sigma,V)$ and that $\mathcal{T}(\Sigma,V)/=_{\mathcal{E}}$ is a free model of $\mathcal{E}$. The equivalence class of t with respect to $=_{\mathcal{E}}$ is denoted as $[t]_{\mathcal{E}}$.

It is well-known [Bi35], that there are derivation systems that produce all consequences of a set of axioms E: The following rules are sufficient:

i) $\vdash \{t = t\}$ (reflexivity)
ii) $\{s = t\} \vdash \{t = s\}$ (symmetry)
iii) $\{r = s\}$ & $\{s = t\} \vdash \{r = t\}$ (transitivity)
iv) $\vdash \{s = t\}$ for $s = t$ in E.
v) $\{s = t\} \vdash \{\sigma s = \sigma t\}$ for all substitutions σ.
vi) $\{s_i = t_i \mid i = 1,\ldots,n\} \vdash \{f(s_1,\ldots,s_n) = f(t_1,\ldots,t_n)\}$ for an n-ary function symbol f.

It is also well-known that iv) and v) can be replaced by
iv)' $\vdash \{\sigma s = \sigma t\}$ for all substitutions σ and for all $s = t$ in E.

A further complete derivation system is demodulation (also called rewriting) [WRC67, Mc67]. In order to prove $s =_{\mathcal{E}} t$, the deduction starts with s and uses only one rule:

$s \longrightarrow s[\pi \leftarrow \sigma r]$,

if there is an occurrence π in s, an instance $\sigma l = \sigma r$ of an axiom $l = r$ or $r = l$ such that σl is syntactically equal to $s\backslash\pi$.

This derivation system is complete in the sense, that for every valid equation $s =_{\mathcal{E}} t$, there is a demodulation proof starting with s and arriving at t.

A **term rewriting system** R is a set of directed equations $R = \{l_i \to r_i\}$, where $V(r_i) \subseteq V(l_i)$. The corresponding derivation relation $\xrightarrow{*}_R$ is rewriting where $l_i \to r_i$ is used only in the given direction. A TRS is called **terminating**, if there are no infinite derivation. A TRS is called **confluent**, iff for all terms s, s_1, s_2 with $s \xrightarrow{*}_R s_1$ and $s \xrightarrow{*}_R s_2$ there exists a term s_3 with $s_1 \xrightarrow{*}_R s_3$ and $s_2 \xrightarrow{*}_R s_3$. A terminating and confluent TRS is called **canonical.** A term is in **normalform**, if no reductions are possible. In an equational theory admitting a canonical TRS every term t can be reduced a unique normalform, denoted by $t\!\downarrow$.

An equational theory $\mathcal{E}$ is called **consistent**, iff there is a nontrivial model of $\mathcal{E}$, equivalently if the equation $x = y$ for different variables x and y is not deducable from E. An equational theory $\mathcal{E}$ is called **collapse-free**, iff there is no valid equation $x =_{\mathcal{E}} t$, where t is not the variable x. An equational theory is called **regular**, iff for every valid equation $s =_{\mathcal{E}} t$ we have $V(s) = V(t)$. Equational theories are regular or collapse-free, iff the corresponding sets of axioms have this property. A theory $\mathcal{E}$ is called **simple**, iff $s =_{\mathcal{E}} t$ does not hold for a proper subterm s of t. Note that a simple theory is regular and collapse-free, but that the converse is false [BHS87].

We extend $\mathcal{E}$-equality of terms to substitutions: Two substitutions σ and τ are equal modulo $\mathcal{E}$ over a set of variables W $(\sigma =_{\mathcal{E}} \tau\ [W])$, if $\sigma x =_{\mathcal{E}} \tau x$ for all variable $x \in W$.

We say σ is an **instance** of τ or τ is **more general** than σ over a set of variables W $(\tau \leq_{\mathcal{E}} \sigma\ [W])$, if there exists a substitution λ with $(\lambda\tau =_{\mathcal{E}} \sigma\ [W])$. Furthermore we say σ is **equivalent** to τ over W $(\tau \equiv_{\mathcal{E}} \sigma\ [W])$, iff $\tau \leq_{\mathcal{E}} \sigma\ [W]$ and $\sigma \leq_{\mathcal{E}} \tau\ [W]$.

Let $\Gamma := \langle s_i = t_i \mid i = 1,\ldots,n\rangle$ be a system of equations. (We will also use the letter Δ for denoting a system of equations.

A substitution σ **$\mathcal{E}$-unifies** Γ if for every equation $s_j = t_j$ in Γ we have $\sigma s_j =_{\mathcal{E}} \sigma t_j$. In this case we say σ is an $\mathcal{E}$-unifier of Γ. The set of all $\mathcal{E}$-unifiers is denoted by $U_{\mathcal{E}}(\Gamma)$.

A **complete** set $cU_{\mathcal{E}}(\Gamma)$ of unifiers of Γ is a set satisfying

i) $cU_{\mathcal{E}}(\Gamma) \subseteq U_{\mathcal{E}}(\Gamma)$ (correctness)

ii) $\forall \sigma \in U_{\mathcal{E}}(\Gamma)\ \exists \tau \in cU_{\mathcal{E}}(\Gamma)$: $\tau \leq_{\mathcal{E}} \sigma\ [V(\Gamma)]$ (completeness)

A complete set is called **minimal** or a set of most general unifiers (mgus), iff additionally

iii) $\forall \sigma,\tau \in cU_{\mathcal{E}}(\Gamma)$ $\tau \leq_{\mathcal{E}} \sigma\ [V(\Gamma)] \Rightarrow \tau = \sigma$ (minimality)

Minimal sets are also designated as $\mu U_{\mathcal{E}}(\Gamma)$. Note that for fixed Γ all the sets $\mu U_{\mathcal{E}}(\Gamma)$ are equivalent [FH83].

An equational theory is called **unification based**, iff $\mu U_{\mathcal{E}}(\Gamma)$ exists for all Γ.

A unification based equational theory $\mathcal{E}$ is called **unitary, finitary or infinitary** depending on the maximal cardinality of $\mu U_{\mathcal{E}}(\Gamma)$ for all Γ. Theories that are not unification based are also called **nullary**.

There are theories of interest in every class [Sz82,Si86, FH83, Ba86, Sch86].

In the unification procedure described later we need constant-elimination problems to unify cyclic unification problems where several theories are involved. E. Tidén [Ti86] used a similar method for this urpose, which he called 'variable elimination'.

A **constant elimination** problem $\mathcal{C}$ in the theory $\mathcal{E}$ is of the form $\mathcal{C} := \langle c_i \notin t_{ij} \mid i = 1,\ldots,n,\ j=1,\ldots,m\rangle$, where c_i are different free constants and t_{ij} are $\mathcal{E}$-terms. The set of solutions of $\mathcal{C}$ is the following set: $U_{\mathcal{E}}(\mathcal{C}) := \{\sigma \mid \exists t_{ij}'\ \ t_{ij}' =_{\mathcal{E}} \sigma t_{ij}$ and $c_i \notin FRC(t_{ij}')$ for $i = 1,\ldots,n,\ j=1,\ldots,m\ \}$.

A **complete set of constant eliminators** $cU_{\mathcal{E}}(\mathcal{C})$ is a set of substitutions, such that

i) $cU_{\mathcal{E}}(\mathcal{C}) \subseteq U_{\mathcal{E}}(\mathcal{C})$ and

ii) For every $\theta \in U_{\mathcal{E}}(\mathcal{C})$ there exists a $\sigma \in cU_{\mathcal{E}}(\mathcal{C})$, such that $\sigma \leq_{\mathcal{E}} \theta\ [V(\mathcal{C})]$.

Note that instances of constant eliminators of $\mathcal{C}$ may not be constant eliminators for $\mathcal{C}$. However, in special theories like Boolean rings, for which we determine a set of constant eliminators below, it is always clear how to obtain every constant eliminator by instantiating the most general ones.

3. Combination of Equational Theories.

The following notions and definitions are adapted or generalized notions from [He86, Ye87, Ti86a].

In the following we investigate equations and unification in a combination of two equational theories $\mathcal{E}_1 = (\Sigma_{\mathcal{E}1},E_1)$ and $\mathcal{E}_2 = (\Sigma_{\mathcal{E}2},E_2)$, where the only symbols common to $\Sigma_{\mathcal{E}1}$ and $\Sigma_{\mathcal{E}2}$ are free constants, i.e., constants that do not occur E_1 and E_2. Furthermore we assume that $\Sigma_{\mathcal{E}1}$ and $\Sigma_{\mathcal{E}2}$ contain the same set of countably infinitely many free constants $\mathcal{FRC}$. We denote the disjoint combination as $\mathcal{E}_1+\mathcal{E}_2 := (\Sigma_{\mathcal{E}1}\cup\Sigma_{\mathcal{E}2}, E_1\cup E_2)$. As abbreviation we sometimes write $\mathcal{E}+$ instead of $\mathcal{E}_1+\mathcal{E}_2$. For the purposes in this paragraph it is no loss of generality to consider the case of a combination of two equational theories, since all theorems can easily be generalized to the case of $N > 2$ theories..

3.1 Assumption. In the following we assume that all $\mathcal{E}_j$ are consistent theories. ■

From now on we assume that $\mathcal{E}_1$ and $\mathcal{E}_2$ are given and that terms are from $\mathcal{T}(\Sigma_{\mathcal{E}1}\cup\Sigma_{\mathcal{E}2}, V)$. We use the convention that $\mathcal{E}$ denotes a general theory that can be specialized to either $\mathcal{E}_1$ or $\mathcal{E}_2$.

Let s be a term, then $TH(s) = \mathcal{E}$, if the top-level function symbol of s belongs to $\mathcal{E}$; in this case we say s is an $\mathcal{E}$-term or has (syntactical) **theory** $\mathcal{E}$. We also say that variables and free constants are $\mathcal{E}$-terms. A term s is called **pure** if s is a term from $\mathcal{T}(\Sigma_{\mathcal{E}}, V)$ for some theory $\mathcal{E}$, otherwise a term is called a **mixed** or **general** term. In order to emphasize the theory of its top symbol, we sometimes say s is $\mathcal{E}$-pure if $TH(t) = \mathcal{E}$. We also say a term s is a **proper $\mathcal{E}$-term**, iff s is an $\mathcal{E}$-term but not a variable or free constant. A subterm s of a term t is called **$\mathcal{E}$-alien**, if every proper superterm of s in t is an $\mathcal{E}$-term, but s is not an $\mathcal{E}$-term. A subterm s of an $\mathcal{E}$-term t is an **alien** subterm, iff every proper superterm of s in t is an $\mathcal{E}$-term and s and t have different theories. The set of all alien subterms of a term t is denoted as ALIEN(t) and the set of all $\mathcal{E}$-alien subterms is denoted as $ALIEN_{\mathcal{E}}(t)$. Sometimes we need also the set of equivalence-classes of $\mathcal{E}$-alien terms denoted by $ALIEN_{\mathcal{E}}(t, \mathcal{E}+) := \{[r]_{\mathcal{E}+} \mid r \in ALIEN_{\mathcal{E}}(t)\}$. Note that free constants or variables do not count as alien subterms.

We define the **syntactical theory height** of a term t as the maximal number of theory changes of a term [He87, Ti86a]:

i) $THT(t) := 0$, if t is a variable or free constant.

ii) $THT(t) := 1$, if t is a pure term.

iii) $THT(t) := 1 + \max \{THT(s) \mid s \in ALIEN(t)\}$.

The following well-known lemma of G. Gentzen [Ge35] states that free constants can be replaced by new variables.

3.2 Lemma. Let $\mathcal{E}$ be an equational theory, let s,t be terms, let $c \in FRC(s,t)$ and let y be a new variable. The terms s' and t' are constructed from s and t, respectively, by replacing every occurrence of c by y.

Then $s =_{\mathcal{E}} t \Leftrightarrow s' =_{\mathcal{E}} t'$.

Proof. "$\Leftarrow$" is trivial by applying the substitution $\{y \leftarrow c\}$ to $s' =_{\mathcal{E}} t'$.

"$\Rightarrow$" If A be a model of $\mathcal{E}$ and let γ be a assignment of values to variables in s' and t'. If we construct a A' from A by changing the interpretation of c to be $c_{A'} := \gamma(y)$, then A' is a $\mathcal{E}$-model, hence $\gamma(s) = \gamma(t)$. Furthermore $\gamma(s') = \gamma(s) = \gamma(t) = \gamma(t')$, hence $\gamma(s') = \gamma(t')$ in the model A. This arguments hold for every $\mathcal{E}$-model and interpretation, hence $s' =_{\mathcal{E}} t'$. ■

The addition of free constants is a conservative extension:

3.3 Lemma. Let $\mathcal{E} = (\Sigma_{\mathcal{E}}, E)$ be an equational theory, let C be a set of free constants and let $\mathcal{F} = (\Sigma_{\mathcal{E}} \cup C, E)$ be the theory where free constants are added.

Then for all $\mathcal{E}$-terms s,t: $s =_{\mathcal{E}} t \Leftrightarrow s =_{\mathcal{F}} t$.

Proof. "$\Rightarrow$" is trivial, since $\mathcal{E}$-deductions are also $\mathcal{F}$-deductions.

"$\Leftarrow$" Let s,t be terms with $s \neq_{\mathcal{E}} t$ and let A be an $\mathcal{E}$-model such that $\gamma(s) \neq \gamma(t)$ for some interpretation γ. Then A can be made an $\mathcal{F}$-model by assigning arbitrary values to free constants from C. Since we have $\gamma(s) \neq \gamma(t)$ in the $\mathcal{F}$-model A, we have also $s \neq_{\mathcal{F}} t$. ■

3.4 Corollary. Let $\mathcal{E}$ be a consistent theory.

Then $a \neq_{\mathcal{E}} b$ for different free constants. ■

In the following we construct a model A of $\mathcal{E}+$, which turns out to be isomorphic to the free term-model. As a consequence of this construction we get the same results as E. Tidén [Ti86a], that the combination of several theories does not influence equality of pure terms, i.e. that the combination of disjoint equational theories is a consaervative extension of every theory, and that complete sets of unifiers of pure terms can be computed locally in the corresponding theory, however, the construction given below appears to be simpler than the proofs in [Ti86a]. Weaker versions of these results are proved in [Ye87, He86].

We proceed by defining a chain of sets $A_{j,n}$ and $B_{j,n}$ with partially defined operations.
Let $A_0 := A_{1,0} := A_{2,0} := \mathcal{FRC} \cup V$ and let $A_{j,1} := \mathcal{T}(\Sigma_{\mathcal{E}j}, V)/=_{\mathcal{E}j}$ be the free term model of $\mathcal{E}_j$, $j=1,2$. Since $\mathcal{E}_1$ and $\mathcal{E}_2$ are consistent, we can assume that A_0 is embedded in $A_{j,1}$ by considering a and $[a]_{=\mathcal{E}j}$ as the same element where 'a' is a variable or free constant.
Let $B_{j,1} := A_{j,1} - A_0$ for $j = 1,2$. For convenience we assume that $B_{1,0} = B_{2,0} = \emptyset$.
(Note that we use the term 3-j in order to switch from 1 to 2 or from 2 to 1.)
For $j = 1,2$ we define $A_{j,n}$ recursively as $A_{j,n} := \mathcal{T}(\Sigma_{\mathcal{E}j} \cup B_{3-j,n-1}, V)/=_{\mathcal{E}j}$ where the elements of $B_{3-j,n-1}$ are considered as free constants. It is assumed that free constants and variables are identified with their congruence-class, i.e. a and $[a]_{=\mathcal{E}j}$ are considered as the same element where $a \in B_{3-j,n-1} \cup A_0$. Let $B_{j,n} := A_{j,n} - (A_0 \cup B_{3-j,n-1})$.
Intuitively, the sets $B_{j,n}$ contain all terms that cannot be collapsed to elements in $A_0 \cup B_{3-j,n-1}$.
Due to Lemma 3.3 we can assume that $A_{j,n}$ is embedded in $A_{j,n+1}$ in the case $B_{3-j,n-1} \subseteq B_{3-j,n}$.
The following holds:

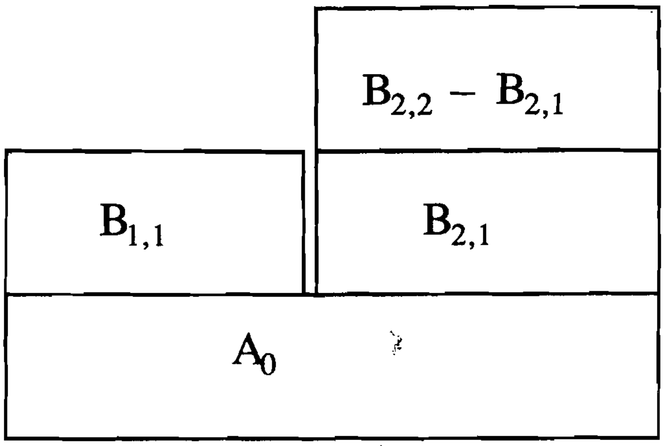
i) $A_{j,n} \subseteq A_{j,n+1}$ for $n \geq 1$ and $j = 1,2$.

ii) $B_{j,n} \subseteq B_{j,n+1}$ for $n \geq 1$ and $j = 1,2$.

Proof: The base cases for induction are $A_{j,0} \subseteq A_{j,1}$ and $B_{j,0} \subseteq B_{j,1}$, which hold trivially.
For some $n \geq 1$ let the induction hypothesis be $A_{j,n-1} \subseteq A_{j,n}$ and $B_{j,n-1} \subseteq B_{j,n}$ for $j = 1,2$.
From $B_{3-j,n-1} \subseteq B_{3-j,n}$ we derive $A_{j,n} \subseteq A_{j,n+1}$ by the definition of $A_{j,n}$.
An element $b \in B_{j,n}$ is an element in $\mathcal{T}(\Sigma_{\mathcal{E}j} \cup B_{3-j,n-1}, V)/=_{\mathcal{E}j}$ that is not equal to a constant or a variable (in this algebra). If we extend $B_{3-j,n-1}$ by $B_{3-j,n}$, then Lemma 3.3 yields that $A_{j,n}$ can be embedded into $A_{j,n+1}$, hence $b \in A_{j,n+1} - (A_0 \cup B_{3-j,n})$. We conclude that $B_{j,n} \subseteq B_{j,n+1}$. □

Note that $B_{1,n}$, $B_{2,n}$ and A_0 are pairwise disjoint. The following diagram shows as an illustrating example the set-diagram of $A_{2,2}$:

$A_{2,2}$:



The algebra A is defined as the union of all sets $A_{j,n}$, i.e., $A := \cup\{A_{j,n} \mid n = 1,2,\ldots, j=1,2\}$. We define also $B_{j,\infty} := \cup\{B_{j,n} \mid n = 1,2,\ldots,\}$. Note that $A = A_0 \cup B_{1,\infty} \cup B_{2,\infty}$, where the union is disjoint.

The set A is an algebra for $\Sigma_{\mathcal{E}1} \cup \Sigma_{\mathcal{E}1}$ if we use the following interpretation:
Free constants in $\Sigma_{\mathcal{E}j}$ are interpreted by themselves. Let $f \in \Sigma_{\mathcal{E}j}$ be a function symbol and let $a_1,\ldots,a_n \in A$. There is an m such that $a_i \in A_0 \cup B_{1,m} \cup B_{2,m}$ for all $i = 1,\ldots,n$. Then all a_i are elements in $A_{j,m+1}$. We define $f_A(a_1,\ldots,a_n) := f(a_1,\ldots,a_n)/=_{\mathcal{E}j}$. This is an element in $A_{j,m+1}$.

Lemma 3.5 A is an $\mathcal{E}_1+\mathcal{E}_2$ -model:

Proof. Let $l = r$ be an axiom in $\mathcal{E}_j$ and let γ be an assignment of values from A to variables in $V(l,r)$. There is an m such that $\gamma x \in A_0 \cup B_{1,m} \cup B_{2,m}$ for all $x \in V(l,r)$. We can view the assignment γ as a mapping with values in $A_{j,m+1}$. Since $A_{j,m+1}$ is a model of $l = r$, we have that γl denotes the same element as γr. ■

Every term t in $\mathcal{T}(\Sigma_{\mathcal{E}1} \cup \Sigma_{\mathcal{E}2}, V)$ has a unique interpretation $\iota_A(t)$ in A if variables are interpreted by themselves and function symbols f as f_A.

The combination $\mathcal{E}_1+\mathcal{E}_2$ is a conservative extension of $\mathcal{E}_1$ and $\mathcal{E}_2$:

3.6 Lemma. Let s,t be $\mathcal{E}_j$-pure terms.

Then $s =_{\mathcal{E}j} t \Leftrightarrow s =_{\mathcal{E}+} t$.

Proof. "$\Rightarrow$" is trivial.

"$\Leftarrow$": Let $s \neq_{\mathcal{E}j} t$. Consider the model A of $\mathcal{E}_1+\mathcal{E}_2$. Obviously we have $\iota_A(s) \neq \iota_A(t)$, hence also $s \neq_{\mathcal{E}+} t$. ■

3.7 Lemma. A is isomorphic to the free term model $\mathcal{T}(\Sigma_{\mathcal{E}1} \cup \Sigma_{\mathcal{E}2}, V)/=_{\mathcal{E}+}$

Proof. We show that $\iota_A/=_{\mathcal{E}+} : \mathcal{T}(\Sigma_{\mathcal{E}1} \cup \Sigma_{\mathcal{E}2}, V)/=_{\mathcal{E}+} \rightarrow A$ is an isomorphism:

Since A is an $\mathcal{E}_1+\mathcal{E}_2$-model, we have that $s =_{\mathcal{E}+} t$ implies that $\iota_A(s) = \iota_A(t)$.
In order to show the converse let $\iota_A(s) = \iota_A(t)$. By the construction of A it is obvious that the relation $\equiv$ on $\mathcal{T}(\Sigma_{\mathcal{E}1} \cup \Sigma_{\mathcal{E}2}, V)$ induced by ι_A via $s \equiv t$ iff $\iota_A(s) = \iota_A(t)$ is a congruence-relation with $s \equiv t$ implies $s =_{\mathcal{E}+} t$. ■

The above construction also yields as a corollary that the decidability of the word-problem of the involved theories is inherited to the combination. This result is also implicitly contained in [Ti86a]. Y.Toyama [To86, To87] investigates the disjoint combination of theories admitting a canonical term rewriting system and shows that confluence is inherited, but not termination. This corollary can be seen as a supplement to Toyama's results.

3.8 Corollary. If the word-problem in $\mathcal{E}_1$ and $\mathcal{E}_2$ is decidable, then the word-problem in $\mathcal{E}_1+\mathcal{E}_2$ is decidable.
Proof. The embedding ι_A is computable, if the word-problem in $\mathcal{E}_1$ and $\mathcal{E}_2$ is decidable.■

The following notions of semantical theory and semantical theory height are needed in order to deal with unification problems in combinations of arbitrary theories. In the case where all theories in a combination are collapse-free and regular, those semantical notions coincide with the syntactical notions.

3.9 Definition.

i) We say a term t has **semantical theory** $\mathcal{E}_j$ iff $\iota_A(t) \in B_{j,\infty}$ or semantical theory $\mathcal{E}_0$, iff $\iota_A(t) \in A_0$. We denote this by S-THT(t) = $\mathcal{E}_j$ or S-THT(t) = $\mathcal{E}_0$, respectively.
ii) The **semantical theory height** is defined as follows:
If $t \in A_0$, then S-THT(t) = 0,
otherwise it is the smallest number m, such that $\iota_A(t) \in B_{j,m}$ for some j.
iii) A term t is called **$\mathcal{E}$+-normalized,** iff for every subterm s of t : S-THT(s) = THT(s).
iv) A term t is called **$\mathcal{E}_j$-normalized,** iff every $\mathcal{E}_j$-alien subterm is $\mathcal{E}$+-normalized.
iv) A substitution σ is $\mathcal{E}_j$-normalized, iff every term in COD(σ) is $\mathcal{E}_j$-normalized. ■

The construction of the model A shows that the notions of semantical theory and semantical theory height are uniquely defined.
Note that an $\mathcal{E}_j$-normalized term t may not be $\mathcal{E}$+-normalized, in particular it may have another semantical theory than $\mathcal{E}_j$. Furthermore an $\mathcal{E}$+-normalized term may not be the smallest possible representative of a term. For example if $E_1 = \{f(x\ y) = f(x'\ y)\}$ and E_2 is empty, then f(g(a), g(a)) is $\mathcal{E}$+-normalized, but $\mathcal{E}$+-equal to f(a, g(a)).

In equations between $\mathcal{E}_j$-normalized terms we can replace alien terms by new free constants:

3.10 Lemma. Let s and t be $\mathcal{E}_j$-normalized terms and let s' and t' be constructed from s and t by replacing all $\mathcal{E}_j$-alien terms consistently by new variables, i.e. $\mathcal{E}$+-equal $\mathcal{E}_j$-alien terms are replaced by the same variable and $\mathcal{E}$+-unequal terms by different variables.

Then $s =_{\mathcal{E}+} t \Leftrightarrow s' =_{\mathcal{E}j} t'$.

Proof. "$\Leftarrow$": Obvious.

"$\Rightarrow$": Let s,t be terms with $s =_{\mathcal{E}+} t$. In the $\mathcal{E}$+-model A we have $\iota_A(s) = \iota_A(t)$. Since $\mathcal{E}_j$-alien terms have a different semantical theory than $\mathcal{E}_j$, this means precisely, that s and t are equal, if $\mathcal{E}_j$-alien subterms are considered as constants. With Lemma 3.2 we can replace the free constants by new variables. This proves the claim. ■

If s and t are not $\mathcal{E}$-normalized, then Lemma 3.10 may be false:

3.11 Example. Let $E_1 := \{f(x, x) = g(x, x)\}$ and $E_2 := \{h(x) = x\}$. Then $f(h(x), x) =_{\mathcal{E}+} g(h(x), h(x))$. We have $\mathrm{ALIEN}_{\mathcal{E}1}(f(h(x), x)) = \{h(x)\}$ and $\mathrm{ALIEN}_{\mathcal{E}1}(g(h(x), h(x))) = \{h(x)\}$. If we replace h(x) by y, then we would have the equation $f(y, x) =_{\mathcal{E}+} g(y, y)$, which does not hold. ■

3.12 Lemma. If the substitution σ is $\mathcal{E}_j$-normalized and the term t is $\mathcal{E}_j$-pure, then σt is $\mathcal{E}_j$-normalized. ■

The following lemma is Lemma 3.2.1 in [Ti86a] . We give a proof as a corollary of Lemma 3.6.

3.13 Lemma. Let $\Gamma = \{s_i = t_i \mid i = 1,\dots,n\}$ be an $\mathcal{E}_j$-pure system of equations.

Then $U_{\mathcal{E}j}(\Gamma)$ is a complete set of $\mathcal{E}$+-unifiers for $U_{\mathcal{E}+}(\Gamma)$.

Proof. Let σ be an $\mathcal{E}$+-unifier of Γ. It suffices to show that there exists an $\mathcal{E}_j$-unifier σ_{mg} of Γ, which is more general than σ over $V(\Gamma)$. We can assume without loss of generality that σ is $\mathcal{E}_j$-normalized. Then all terms σs_i and σt_i are also $\mathcal{E}_j$-normalized by Lemma 3.12. Lemma 3.10 shows that the equations in $\sigma\Gamma$ remain solved, if σ is changed such that the $\mathcal{E}_j$-alien subterms of codomain terms of σ are replaced by new variables. This gives an $\mathcal{E}_j$-pure substitution σ_{mg}, which solves Γ with respect to $\mathcal{E}$+ and hence by Lemma 3.6 also with respect to $\mathcal{E}_j$. Furthermore σ_{mg} is more general than σ over $V(\Gamma)$. ■

4. Properties of Essential Terms.

For $\mathcal{E}$-normalized terms t we say $\alpha_{\mathcal{E}}(t)$ is an **$\mathcal{E}$-variable-abstraction** of t, iff every $\mathcal{E}$-alien subterm r of t is replaced by a new variable $y_{[r]}$, such that for alien subterms r and r' we have $r =_{\mathcal{E}+} r'$ if and only if $y_{[r]} = y_{[r']}$. Similarily an **$\mathcal{E}$-constant-abstractions** $\beta_{\mathcal{E}}$ is defined, where new free constants are used instead of new variables. Obviously the terms $\alpha_{\mathcal{E}}(t)$ and $\beta_{\mathcal{E}}(t)$ are pure $\mathcal{E}$-terms. It is obvious that abstractions can be simultaneously defined for finite sets of terms.

We will also abstract $\mathcal{E}$-normalized substitutions σ: the $\mathcal{E}$-variable abstraction $\alpha_{\mathcal{E}}(\sigma)$ of σ is defined by $\alpha_{\mathcal{E}}(\sigma)x := \alpha_{\mathcal{E}}(\sigma x)$ for all x. In the same way we define the constant-abstraction of an $\mathcal{E}$-normalized substitution.

4.1 Lemma. Let s, t be $\mathcal{E}$-normalized terms and let $\alpha_{\mathcal{E}}$ be a variable abstraction and $\beta_{\mathcal{E}}$ be a constant-abstraction.

Then $s =_{\mathcal{E}+} t \Leftrightarrow \alpha_{\mathcal{E}}(s) =_{\mathcal{E}} \alpha_{\mathcal{E}}(t) \Leftrightarrow \beta_{\mathcal{E}}(s) =_{\mathcal{E}} \beta_{\mathcal{E}}(t)$.

Proof. Follows from Lemmas 3.2 and 3.10. ■

In nonregular theories $\mathcal{E}$ there are usually terms s,t with $s =_{\mathcal{E}} t$, but $V(s) \neq V(t)$. The following definitions and investigations are in order to deal with this problem.
The set ESS–FRC(t) of **essential free constants** of an $\mathcal{E}$-pure term t is the intersection of all sets FRC(t') for all terms t' with $t' =_{\mathcal{E}} t$. The same can be defined for variables in a pure term t and the corresponding set of **essential variables** is denoted as ESS-V(t). Variables or constants in FRC(t) – ESS-FRC(t) or V(t) – ESS-V(t) are called **inessential.**
If t is an $\mathcal{E}$-normalized term, then we define the set of equivalence classes of **essential $\mathcal{E}$-alien subterms** of t, denoted as ESS-ALIEN$_{\mathcal{E}}$(t, $\mathcal{E}$+) as the $\mathcal{E}$+-equivalence-class of $\mathcal{E}$-alien terms, that are replaced by an essential variable in the $\mathcal{E}$-variable-abstraction.
The set ESS-ALIEN$_{\mathcal{E}}$(t) is the set of all $\mathcal{E}$-alien subterms r of t with $[r]_{\mathcal{E}+} \in$ ESS-ALIEN$_{\mathcal{E}}$(t, $\mathcal{E}$+).
A term from ALIEN$_{\mathcal{E}}$(t) – ESS-ALIEN$_{\mathcal{E}}$(t) is called **inessential.**
An obvious fact is that for $\mathcal{E}$-pure terms s,t with $s =_{\mathcal{E}} t$ we have ESS-FRC(s) = ESS-FRC(t) and ESS-V(s) = ESS-V(t) and that for $\mathcal{E}$-normalized terms s,t with $s =_{\mathcal{E}+} t$ we have ESS-ALIEN$_{\mathcal{E}}$(s, $\mathcal{E}$+) = ESS-ALIEN$_{\mathcal{E}}$(t, $\mathcal{E}$+),

4.2 Lemma. Let t be an $\mathcal{E}$-pure term. Then the following statements are equivalent:

i) $c \notin$ ESS-FRC(t) for a free constant c.

ii) $t =_{\mathcal{E}} t'$, where t' is obtained from t by replacing all occurrences of c by a new variable x_c.

iii) $t =_{\mathcal{E}} t''$, where t'' is obtained from t by replacing all occurrences of c by a new constant d.

Proof. (ii) $\Leftrightarrow$ (iii) follows from Lemma 3.2, since d and x_c do not occur in t.

ii) $\Rightarrow$ i) follows, since $c \notin$ ESS-FRC(t') and $t =_{\mathcal{E}} t'$.

i) $\Rightarrow$ ii): There exists a term t_0 with $t =_{\mathcal{E}} t_0$, such that $c \notin$ FRC(t_0).

With Lemma 3.2 we obtain $t' =_{\mathcal{E}} t_0$, hence by transitivity we obtain $t' =_{\mathcal{E}} t$. ■

4.3 Lemma. Let t be an $\mathcal{E}$-normalized term and let $r \in$ ALIEN$_{\mathcal{E}}$(t). The following statements are equivalent:

i) $r \notin$ ESS-ALIEN$_{\mathcal{E}}$(t).

ii) $t =_{\mathcal{E}+} t'$, where t' is obtained from t by replacing all $\mathcal{E}$-alien subterms $\mathcal{E}$+–equal to r by a new variable x_r.

iii) $t =_{\mathcal{E}+} t''$, where t'' is obtained from t by replacing all $\mathcal{E}$-alien subterms $\mathcal{E}$+-equal to r by a new constant c_r.

Proof. (ii) $\Leftrightarrow$ (iii) follows from Lemma 3.2, since c_r and x_r do not occur in t.

ii) $\Rightarrow$ i) follows from the definition of essential $\mathcal{E}$-alien subterms and Lemma 4.1.

i) $\Rightarrow$ ii): From t we construct $\beta_{\mathcal{E}}(t)$ by a constant-abstraction. By definition we have that $\beta_{\mathcal{E}}(r)$ is an inessential constant in $\beta_{\mathcal{E}}(t)$, hence we can use Lemma 4.2 to construct a term t' with $\beta_{\mathcal{E}}(t) =_{\mathcal{E}} \beta_{\mathcal{E}}(t')$. Lemma 3.2 and Lemma 4.1 show that $t =_{\mathcal{E}+} t'$. ■

4.4 Lemma.

i) For every $\mathcal{E}$-pure term t, there exists a term s with $s =_{\mathcal{E}+} t$ and ESS-FRC(t) = FRC(s).

ii) For every $\mathcal{E}$-normalized term t, there exists an $\mathcal{E}$-normalized term s with $s =_{\mathcal{E}+} t$ and $\text{ESS-ALIEN}_{\mathcal{E}}(t, \mathcal{E}+) = \text{ALIEN}_{\mathcal{E}}(s, \mathcal{E}+)$.

Proof. i) follows from Lemma 4.2 and ii) follows from Lemma 4.3 by repeated application.■

4.5 Proposition. Let t be an $\mathcal{E}$-normalized term and let s be an $\mathcal{E}$-alien subterm of t with S-THT(s) $\geq$ S-THT(t).

Then s is an inessential $\mathcal{E}$-alien subterm of t.

Proof. Assume s is an essential $\mathcal{E}$-alien subterm of t and let $n = \max\{\text{S-THT}(r) \mid r \in \text{ALIEN}_{\mathcal{E}}(t)\}$. Consider the construction of the $\mathcal{E}$+-model A. If we consider all alien terms as constants, then all terms equal to t in the model $A_{j,n}$ contain the constant corresponding to s. Hence there is no m less than S-THT(s) such that $\iota_A(t)$ is in $A_{j,m}$. This means that S-THT(t) > S-THT(s), which contradicts our assumption. ■

If $\mathcal{E}$ is regular, then there are no inessential terms:

4.6 Lemma. If $\mathcal{E}$ is regular, then

i) For every $\mathcal{E}$-pure term t: ESS-FRC(t) = FRC(t).

ii) For all $\mathcal{E}$-pure terms s, t with $s =_{\mathcal{E}} t$: FRC(s) = FRC(t).

iii) For every $\mathcal{E}$-normalized t: $\text{ESS-ALIEN}_{\mathcal{E}}(t, \mathcal{E}+) = \text{ALIEN}_{\mathcal{E}}(t, \mathcal{E}+)$.

iv) For all $\mathcal{E}$-normalized terms s, t with $s =_{\mathcal{E}+} t$: $\text{ALIEN}_{\mathcal{E}}(s, \mathcal{E}+) = \text{ALIEN}_{\mathcal{E}}(t, \mathcal{E}+)$.

Proof. Due to Lemma 4.1, 4.2 and 4.3 it is sufficient to prove i), which in turn follows immediately from Lemma 4.2, since $\mathcal{E}$ is regular. ■

5. Unification as Transformations of Systems of Equations

We consider the process of unification (or solving equations) as a sequence of (maybe nondeterministic) transformations that starts with a system of equations and stops with one in

solved form. This follows the ideas of J. Herbrand [Her30], A. Martelli, U. Montanari [MM82] and C. Kirchner [Ki85]. We shall also use multi-equations instead of equations, since they are more appropriate. We assume that Γ is a set of multi-equations and that each multi-equation M_i is a set of terms $\{t_1,\ldots,t_n\}$, also denoted as $t_1 = t_2 = \ldots. = t_n$. Obviously every system of equations can be considered to have this form. We use $s = t \in \Gamma$ synonymously with $s,t \in M$, where $M \in \Gamma$. Furthermore we assume that merging is built-in, i.e. if $r = s$ and $s = t$ are in Γ, then also $r = t$ is in Γ, or equivalently that all multi-equations are disjoint. Note that 'merge' usually means to consider only variables common to some multi-equations and that we consider also common terms. As abbreviation we shall also use equations of the form $S = T$, where the uppercase letters denote sets of terms and $S = T$ means the conjunction of all equations $s_i = t_j$ for $s_i \in S$ and $t_j \in T$. With $VAR(\Gamma)$ and $TER(\Gamma)$ we denote the set of variables and terms, respectively, that occur as arguments of equations in Γ.
In the following we consider transformations of an system of equations Γ_1 to a system Γ_2 with respect to a set of variables W, denoted by $\Gamma_1 \Rightarrow_W \Gamma_2$. This set W is usually the set of variables of an orginal system of equations Γ_0 to be solved. We will sometimes call the set W the set of **significant variables**, and the other variables **auxiliary variables**. Usually we abbreviate $\Gamma \Rightarrow_{V(\Gamma)} \Gamma'$ as $\Gamma \Rightarrow \Gamma'$.
We say a transformation $\Gamma \Rightarrow_W \Gamma'$ is **correct**, iff $U_{\mathcal{E}}(\Gamma)_{|W} \supseteq U_{\mathcal{E}}(\Gamma')_{|W}$, and that $\Gamma \Rightarrow_W \Gamma'$ is **complete** (or preserves solutions), iff $U_{\mathcal{E}}(\Gamma)_{|W} = U_{\mathcal{E}}(\Gamma')_{|W}$. We say a set of correct transformations $\{\Gamma \Rightarrow_W \Gamma_i \mid i \in I\}$ is a **complete set of alternatives**, iff $U_{\mathcal{E}}(\Gamma)_{|W} = \cup\{U_{\mathcal{E}}(\Gamma_i)_{|W} \mid i \in I\}$. This is of particular interest if the set of transformations comes from a rule. In this case we say this rule provides a complete set of alternatives.

The proofs of the following three lemmas are straightforward.

5.1 Lemma.

i) For all $\sigma \in U_{\mathcal{E}}(\Gamma)$, $\tau \in SUB_\Sigma$ and $\sigma \leq_{\mathcal{E}} \tau\ [V(\Gamma)] \Rightarrow \tau \in U_{\mathcal{E}}(\Gamma)$.

ii) For all $\sigma \in U_{\mathcal{E}}(\Gamma)$, $\tau \in SUB_\Sigma$ and $\sigma \equiv_{\mathcal{E}} \tau\ [V(\Gamma)] \Rightarrow \tau \in U_{\mathcal{E}}(\Gamma)$.

iii) For every $\sigma \in U_{\mathcal{E}}(\Gamma)$, there exists an idempotent substitution $\tau \in U_{\mathcal{E}}(\Gamma)$ such that $\sigma \equiv \tau\ [V(\Gamma)]$, $DOM(\tau) = V(\Gamma)$ and $I(\tau)$ consists of new variables. ■

We have the following criteria to recognize the completeness of transformations:

5.2 Lemma.

i) $\{\Gamma \Rightarrow_W \Gamma_i \mid i \in I\}$ is a complete set of alternatives, iff
for every $\sigma \in U_{\mathcal{E}}(\Gamma)$ there exists a $\tau \in \cup\{U_{\mathcal{E}}(\Gamma_i) \mid i \in I\}$ with $\sigma =_{\mathcal{E}} \tau\ [W]$ and
for every $\tau \in \cup\{U_{\mathcal{E}}(\Gamma_i) \mid i \in I\}$ there exists a $\sigma \in U_{\mathcal{E}}(\Gamma)$ with $\sigma =_{\mathcal{E}} \tau\ [W]$.

ii) $\{\Gamma \Rightarrow_W \Gamma_i \mid i \in I\}$ is a complete set of alternatives, iff
for every $\sigma \in U_{\mathcal{E}}(\Gamma)$ there exists a $\tau \in \cup\{U_{\mathcal{E}}(\Gamma_i) \mid i \in I\}$ with $\sigma \equiv_{\mathcal{E}} \tau\ [W]$ and

for every $\tau \in \cup\{U_{\mathcal{E}}(\Gamma_i) \mid i \in I\}$ there exists a $\sigma \in U_{\mathcal{E}}(\Gamma)$ with $\sigma \equiv_{\mathcal{E}} \tau\ [W]$. ■

5.3 Lemma.

i) If $\Gamma_1 \Rightarrow_W \Gamma_2$ and $\Gamma_2 \Rightarrow_W \Gamma_3$ are correct, then $\Gamma_1 \Rightarrow_W \Gamma_3$ is correct.

ii) If $\Gamma_1 \Rightarrow_W \Gamma_2$ is complete and $\Gamma_2 \Rightarrow_W \Gamma_3$ is complete, then $\Gamma_1 \Rightarrow_W \Gamma_3$ is complete.

iii) If $V \subseteq W$ and $\Gamma_1 \Rightarrow_W \Gamma_2$ is correct, then $\Gamma_1 \Rightarrow_V \Gamma_2$ is correct

iv) If $V \subseteq W$ and $\Gamma_1 \Rightarrow_W \Gamma_2$ is complete, then $\Gamma_1 \Rightarrow_V \Gamma_2$ is complete

v) If $V \subseteq W$ and $\{\Gamma \Rightarrow_W \Gamma_i \mid i \in I\}$ is a complete set of alternatives , then $\{\Gamma \Rightarrow_V \Gamma_i \mid i \in I\}$ is a complete set of alternatives.

vi) If If $V \subseteq W$ and $\{\Gamma \Rightarrow_V \Gamma_i \mid i \in I\}$ is a complete set of alternatives and $\{\Gamma_1 \Rightarrow_W \Gamma_{1,j} \mid j \in J\}$ is a complete set of alternatives then $\{\Gamma \Rightarrow_V \Gamma_i \mid i \in I-\{1\}\} \cup \{\Gamma \Rightarrow_W \Gamma_{1,j} \mid j \in J\}$ is a complete set of alternatives.

vii) If $V(\Gamma_1) \subseteq W$ and $V(\Gamma_2) \subseteq W$ then:
$\Gamma_1 \Rightarrow_W \Gamma_2$ is complete, iff $U_{\mathcal{E}}(\Gamma_1) = U_{\mathcal{E}}(\Gamma_2)$.

viii) $\Gamma_1 \Rightarrow_W \Gamma_2$ is complete, iff for every idempotent substitution $\sigma \in U_{\mathcal{E}}(\Gamma_1)$ with $DOM(\sigma) = V(\Gamma_1)$ there exists a λ with $DOM(\lambda) \subseteq V(\Gamma_2)-V(\Gamma_1)$, such that $\sigma_{|W} \cup \lambda \in U_{\mathcal{E}}(\Gamma_2)$. ■

In the case where an infinite number of free constants is in the signature, it makes no difference to use ground substitutions instead of arbitrary substitutions for testing completeness. If σ is an idempotent unifier of Γ with $DOM(\sigma) = V(\Gamma)$, then let σ_c be the constantified unifier, where every variable in $I(\sigma)$ is replaced by a free constant not in Γ. Obviously σ_c is a unifier of Γ. Conversely, if σ_c is a unifier of Γ, then σ is also a unifier due to Lemma 3.2.

In the following we denote the conjunction of two problems Γ_1 and Γ_2 as $\Gamma_1 \& \Gamma_2$. Obviously we have $U_{\mathcal{E}}(\Gamma_1 \& \Gamma_2) = U_{\mathcal{E}}(\Gamma_1) \cap U_{\mathcal{E}}(\Gamma_2)$. The following lemma shows that complete tranformations made on one conjunct can be lifted provided the transformation introduces only new variables.

5.4 Lemma. Let Γ_i and Δ be systems of equations.

i) Let W be a set of variables and let $\Gamma_1 \Rightarrow_{V(\Gamma 1)} \Gamma_2$ be complete such that all variables in $V(\Gamma_2) - V(\Gamma_1)$ are new ones.
Then $\Gamma_1 \& \Delta \Rightarrow_W \Gamma_2 \& \Delta$ is complete.

ii) Let W be a set of variables and let $\{\Gamma \Rightarrow_{V(\Gamma)} \Gamma_i \mid i \in I\}$ be a complete set of alternatives such that all variables in $\cup\{V(\Gamma_i) \mid i \in I\} - V(\Gamma)$ are new ones.
Then $\{\Gamma \& \Delta \Rightarrow_W \Gamma_i \& \Delta \mid i \in I\}$ is a complete set of alternatives .

Proof. It is sufficient to prove ii):

Let σ be a solution of $\Gamma \& \Delta$ with $DOM(\sigma) \subseteq V(\Gamma \& \Delta)$. Since $\{\Gamma \Rightarrow_{V(\Gamma)} \Gamma_i \mid i \in I\}$ is a complete set of alternatives, there exists an index j and a substitution $\tau \in U_{\mathcal{E}}(\Gamma_j)$ with

$DOM(\tau) \subseteq V(\Gamma_j)$ and $\sigma_{|V(\Gamma)} = \tau_{|V(\Gamma)}$. Now $\theta := \tau \cup \sigma_{|V(\Delta)-V(\Gamma)}$ is a unifier of Γ_j & Δ, since $\theta = \sigma\ [V(\Delta)]$.

In order to prove the converse, let $j \in I$ and let τ be a solution of Γ_j & Δ with $DOM(\tau) \subseteq V(\Gamma_j)$. There exists a substitution $\sigma \in U_{\mathcal{E}}(\Gamma)$ with $DOM(\sigma) \subseteq V(\Gamma)$ and $\sigma_{|V(\Gamma)} = \tau_{|V(\Gamma)}$. Now $\sigma \cup \tau_{|V(\Delta)-V(\Gamma)}$ is a unifier of Γ & Δ. ■

A special complete transformation is to replace a system Γ by a complete set of $\mathcal{E}$-unifiers of Γ. We use $\langle\sigma\rangle$ to denote the system of equations that come from a substitution σ, i.e., if $\sigma = \{x_1 \leftarrow t_1,\ldots, x_n \leftarrow t_n\}$, then $\langle\sigma\rangle = \langle x_1 = t_1,\ldots,x_n = t_n\rangle$.

5.5 Proposition. Let Γ be a unification problem and let U be a complete set of idempotent $\mathcal{E}$-unifiers of Γ, such that $DOM(\sigma) \subseteq V(\Gamma_1)$.

Then $\{\Gamma \Rightarrow_{V(\Gamma)} \langle\sigma\rangle \mid \sigma \in U\}$ is a complete set of alternatives.

Proof.

i) Correctness: Let $\tau \in U_{\mathcal{E}}(\langle\sigma\rangle)$ for some $\sigma \in U$. Then we have $\tau x =_{\mathcal{E}} \tau\sigma x$ for all $x \in V(\Gamma)$. Hence $\sigma \leq_{\mathcal{E}} \tau\ [V(\Gamma)]$, which implies $\tau \in U_{\mathcal{E}}(\Gamma)$ by Lemma 5.1.

ii) Completeness: Let $\tau \in U_{\mathcal{E}}(\Gamma)$. Then there exists a $\sigma \in U$, such that $\sigma \leq_{\mathcal{E}} \tau\ [V(\Gamma)]$, hence there exists a substitution λ such that $\lambda\sigma =_{\mathcal{E}} \tau\ [V(\Gamma)]$. For a component $x = \sigma x$ in $\langle\sigma\rangle$, we have $x \in V(\Gamma)$, hence $\lambda\sigma x = \lambda\sigma\sigma x$ and thus $\lambda\sigma \in U_{\mathcal{E}}(\langle\sigma\rangle)$. Since $\lambda\sigma =_{\mathcal{E}} \tau\ [V(\Gamma)]$, we are ready by Lemma 5.2. ■

The idempotency of unifiers is necessary in Proposition 5.5:

Consider the system of equations $\langle x = f(y)\rangle$. Then $\{x \leftarrow f(x), y \leftarrow x\}$ is a most general unifier for Γ, but the system $\{x = f(x), y = x\}$ is unsolvable.

A **cycle** in Γ is defined as follows:

Let $x_i = t_i$, $i = 1,\ldots,n$ be equations in Γ, where x_i is a variable and t_i is a nonvariable term, such that $x_{i+1} \in V(t_i)$ for $i = 1,\ldots,n-1$ and $x_1 \in V(t_n)$. Then x_i, t_i, $i=1,\ldots,n$ is a cycle in Γ. A system of multi-equations Γ is in **sequentially solved form**, iff in every multiequation there is at most one nonvariable term and Γ contains no cycle. It is in **solved form**, iff no variables in $VAR(\Gamma)$ occur in some term from $TER(\Gamma)$. Note that every sequentially solved system can be transformed into a solved one by the replacement rule defined below without loosing solutions. From every Γ in solved form, we can immediately construct an idempotent substitution σ_Γ as follows: If M is a multi-equation in Γ with a nonvariable term t, then let $\sigma_\Gamma x := t$ for all $x \in M$. If M is a multi-equation in Γ consisting only of variables, then choose a variable $y \in M$ and let $\sigma_\Gamma x := y$ for all $x \in M$. Obviously σ_Γ is idempotent for systems Γ in solved form.

Solved equation systems have the right solution and are unitary solvable:

5.6 Lemma. Let Γ be a solved equational system. Then

σ_Γ is a most general $\mathcal{E}$-unifier of Γ.

Proof. Let σ be a $\mathcal{E}$-unifier of $\Gamma = \{x_1 = t_1,\ldots,x_n = t_n\}$. Then $\sigma x_i =_{\mathcal{E}} \sigma t_i$ for $i = 1,\ldots,n$. We have to show $\sigma =_{\mathcal{E}} \sigma\,\sigma_\Gamma\,[V(\Gamma)]$. For $x \in \{x_1,\ldots,x_n\}$, this follows from $\sigma x_i =_{\mathcal{E}} \sigma t_i = \sigma\sigma_\Gamma\, x_i$. For $x \in V(t_1,\ldots,t_n)$, we have $\sigma x =_{\mathcal{E}} \sigma\sigma_\Gamma\, x$, since $\sigma_\Gamma\, x = x$. ■

In the following we give some transformation rules that are useful for all equational theories.
First we describe some **general don´t care rules** that can be applied to systems of equations, i.e., these rules are complete. The first four rules are also referred to as **reduction rules.** Note that we do not mention the usual merge-rule, since we assume that it is built-in.
In the following rules we mean by $s = t$ that s and t are different terms of the same multi-equation M in Γ

5.7 Definition.

Rule: Trivial Multi-equations. $M \,\&\, \Gamma \Rightarrow \Gamma$,
if M contains only one element.

Rule: Auxiliary Variables. $\Gamma \,\&\, M \Rightarrow_W \Gamma \,\&\, M\text{-}\{z\}$,
if $z \notin W$ and z does not occur elsewhere in Γ.

Rule: Theory-Merge. $M_1 \,\&\, M_2 \Rightarrow M_1 \cup M_2$,
if there are terms $t_1 \in M_1$ and $t_2 \in M_2$ with $t_1 =_{\mathcal{E}+} t_2$.

Rule: Equal terms. $M \Rightarrow M-\{s\}$,
if M contains two different terms s, t with $s =_{\mathcal{E}} t$.

Rule: Demodulation. $s = t \Rightarrow s' = t$
if $s =_{\mathcal{E}} s'$.

Rule: Replacement. $s = t \,\&\, r = l \Rightarrow s = t \,\&\, r[\pi \leftarrow t] = l$,
if $r\backslash\pi =_{\mathcal{E}} s$.

Rule: Variable elimination $x = t \,\&\, \Gamma \Rightarrow x = t \,\&\, \{x \leftarrow t\}\Gamma$
if $x \notin V(t)$.

Rule: Renaming. $\Gamma \Rightarrow \{x \leftarrow x'\}\Gamma \,\&\, x = x'$,
where $x \in V(t)$ for some $t \in TER(\Gamma)$ and x' is a new variable.

Rule: Unfold. $s = t \Rightarrow s[\pi \leftarrow x] = t \& x = r$,
if r is an alien subterm of s at occurrence π and x is a new variable. ■

Note that the variable-elimination rule can be simulated by the replacement rule.

5.8 Proposition. The rules in Definition 5.7 are complete transformation rules. ■

This proposition together with Lemma 5.4 and Proposition 5.5 shows also that the computation of minimal sets of unifiers can be sequentialized. In order to solve $\Gamma_1 \& \Gamma_2$ first compute a minimal set of unifiers for Γ_1, apply the obtained unifiers to Γ_2 and solve the obtained system. The only requirement for this method to be complete is that unifiers of Γ_1 should not introduce variables that occur only in Γ_2.

We emphasize that the deletion of auxiliary variables is not just for the sake of efficiency, but is an important rule that ensures termination of the general unification algorithm in a combination. Such a rule also appears in [NRS87].

6. A Unification Procedure for Mixed Terms.

We present in the following the basic steps, the nondeterministic rules and a strategy for unification in a combination of disjoint theories $\mathcal{E}_j$. The procedure is described in a way suitable for proving completeness and termination. We do not consider all possible failure rules. In this paragraph we only prove termination, completeness is more complicated and proved in a separate paragraph.

In order to design such an nondeterministic algorithm one should have in mind that a solution σ of the original system of equations Γ_0 is given and that it must be possible to direct the solution process such that a solution σ_{mg} is returned that is more general than σ over $V(\Gamma_0)$. We design the steps and rules in such a way that for every nondeterministic step in this process, the number of different possibilities is finite unless an involved theory povides an infinite set of unifiers or constant-eliminators.

The procedure is described for a combination of N theories, since we have found no way to solve constant-elimination problems in a combination if there are algorithms for every theory, which would be required by an induction argument.

We denote the actual system of multi-equations with Γ and assume that it consists of multi-equations M_i, i.e., $\Gamma = \{M_i \mid i = 1,\ldots,M\}$.

We will use $\mathcal{E}_0$ standing for the 'theory' of free constant.

We assume in the following that the reduction rules are performed whenever possible and do not

explicitely mention them. However, we do not assume that all possibilities for the rule theory-merge and equal terms are peformed, since we may run into trouble if the word-problem in $\mathcal{E}+$ is undecidable. However, we assume that after the application of a unifier σ to terms s and t the resulting terms σs and σt are $\mathcal{E}+$-equal and that the reduction rules can use this fact.
We describe the procedure as a sequence of steps that use some specified rules.

GU-Step 1. Transform Γ into unfolded normalform.

6.1 Definition. A system Γ in **unfolded normalform** (UNF) has the following form and properties:

Γ consists of the multi-equations M_i, $i = 1,\ldots,M$. Every multi-equation M_i has the form $X_i = T_{i0} = T_{i1} = \ldots = T_{iN}$, where some constituents may be empty.

The set $\cup\{X_i \mid i = 1,\ldots,M\}$ contains exactly the significant variables and significant variables do not occur elsewhere in Γ. The sets T_{ij}, $j=1,\ldots,N$ contain all proper $\mathcal{E}_j$-pure terms and all variables of $VAR(M_i) \cap V(T_{ij} \mid i = 1,\ldots,M\}$. , and the sets T_{i0} contain all free constants in $TER(M_i)$.

Proper $\mathcal{E}_j$-terms in T_{ij}, $j=1,\ldots,N$ do not contain free constants.

The $\mathcal{E}_0$-part does not contain variables.

We have $V(T_{ij} \mid i = 1,\ldots,M\} \cap V(T_{ik} \mid i = 1,\ldots,M\} = \emptyset$ for $j \neq k$.

The variables in $V(T_{ij} \mid i = 1,\ldots,M\}$ are called the **$\mathcal{E}_j$-related variables** and the terms in $\cup\{T_{ij} \mid i = 1,\ldots,M\}$ are called the **$\mathcal{E}_j$-related terms.**

The part Γ_j is the set of multi-equations $\{T_{ij} \mid j = 1,\ldots,M\}$ and is called the **$\mathcal{E}_j$-part** of Γ. ■

For example the unfolded normalform of $\langle x = f(x*y)\rangle$ is:

significant variables		Boolean ring		free functions
$\langle$ x	=	x'	=	f(z)
y	=	y'		
		x'*y'	=	z $\rangle$.

6.2 Lemma. Every system of equations can be transformed into UNF.

Proof. First one can use unfolding to make all terms pure. As a second step it is possible to satisfy the disjointness conditions for variables by application of renaming . ■

6.3 Lemma. Let Γ be an unfolded normalform, let $\mathcal{E}_j$ be a theory and let σ be an idempotent substitution such that $DOM(\sigma) \subseteq V(\Gamma_j)$, $I(\sigma)$ consists of new variables, and all terms in $COD(\sigma)$ are variables, free constants or pure $\mathcal{E}_j$-terms.Then the following holds:

i) $\Gamma \,\&\, \langle\sigma\rangle \Rightarrow_W \sigma\Gamma$ is a complete transformation.

ii) $\sigma\Gamma$ is in unfolded normalform.

iii) the number of multi-equations in $\sigma\Gamma$ is less or equal the number of multi-equations in Γ,

iv) the number of $\mathcal{E}_j$-related terms in $\sigma\Gamma$ is less or equal the number of $\mathcal{E}_j$-related terms in Γ.

Proof. Note that the application of σ to Γ has only an effect on the $\mathcal{E}_j$-part of Γ.

i) Obviously, $\Gamma \,\&\, \langle\sigma\rangle \Rightarrow_W \sigma\Gamma \,\&\, \langle\sigma\rangle$ is a complete transformation by Proposition 5.8. After the application of σ, the variables in $DOM(\sigma)$ do not occur elsewhere in Γ, and hence we can delete by Proposition 5.8 the variables z that are in $VAR(\Gamma) \cap DOM(\sigma)$, since σ is idempotent and all variables in $DOM(\sigma)$ are auxiliary. For a term $t \in COD(\sigma)$ there are two posibilities: either t is a term in $\sigma\Gamma$, or t forms a multi-equation consisting of one element and can then be deleted. Hence the reduction rules reduce $\sigma\Gamma \,\&\, \langle\sigma\rangle$ to $\sigma\Gamma$.

ii) $\sigma\Gamma$ is in UNF, since all terms in $COD(\sigma)$ are $\mathcal{E}_j$-terms and all variables in $I(\sigma)$ are new ones.

iii) and iv) are obvious. ■

GU-Step 2. Transform Γ by several GU-unifications and by one GU-identification until there is at most one term per theory-part and multi-equation.

Rule: GU-Unification. $\quad\quad \Gamma \Rightarrow \sigma\Gamma$,

where σ is a (most general) unifier of the $\mathcal{E}_j$-related part Γ_j.

This rule is nondeterministic, since a complete set of unifiers may contain more than one substitution.

6.4 Remark. In the following we assume that all substitutions or unifiers introduced by some operation introduce only new variables and hence are idempotent.

6.5 Lemma. Every nontrivial GU-unification application properly decreases the number of terms in Γ.

Proof. Follows from Lemma 6.3 and since at least two terms are made $\mathcal{E}_j$-equal in $\sigma\Gamma$.

6.6 Proposition. Every system of multi-equations is transformable by a finite number of applications of GU-unification into a system in unfolded normalform, such that in every multi-equation there is at most one $\mathcal{E}_j$-term.

Proof. By Lemma 6.5 the application of the GU-unification rule terminates. As long as there are nontrivial multi-equations in some $\mathcal{E}_j$-part of Γ we can apply unification. Hence the claim of this proposition holds. ■

The following rule is a nondeterministic one and is intended to partition the solution space into substitutions with a different identification pattern on the multi-equations.

Rule: GU-Identification.

$\Gamma \Rightarrow \Gamma'$,

where $\sim$ is some equivalence relation on multi-equations and

Γ' is constructed from Γ by joining multi-equations M_i and M_j, iff $M_i \sim M_j$.

After an identification, it is sufficient to consider only those substitutions in the solution space that do not further identify multi-equations. So it is sufficient to perform GU-identification step only once. If we write U for GU-unification and I for GU-identification, then the application sequence is like U^*IU^*. The GU-identification rule is a proviso for the application of the collapsing-rule by constan-abstraction, since we can then abstract different multi-equations by different constants.

6.7 Lemma. GU-identification and GU-unification together terminate. Furthermore all resulting systems are in unfolded normalform. ■

In the following we can assume that the system of equations is in unfolded normalform and that every multi-equation M_i contains at most one $\mathcal{E}_j$-term (sometimes denoted t_{ij}). For correctness, we consider the set of solutions of Γ as the full set $U_{\mathcal{E}}(\Gamma)$, but for completeness, we consider the following set of solutions: $\{\theta \in U_{\mathcal{E}}(\Gamma) \mid \theta M_i \neq_{\mathcal{E}} \theta M_j$ for $i \neq j$ and $M_i, M_j \in \Gamma\}$.

Step 3: Labeling multi-equations.

We label every multi-equation in Γ with exactly one theory ranging from $\mathcal{E}_0$ to $\mathcal{E}_N$ and add to every multi-equation M_i a new extra variable y_i, that does not belong to any theory-part. ❑

This extra variables y_i shall play the role of constant abstractions in the following rules.

The rule is nondeterministic in nature and after applying it, the system stands for the following set of substitutions: $\{\theta \in U_{\mathcal{E}+}(\Gamma) \mid \theta M_i \neq_{\mathcal{E}+} \theta M_k$ for $i \neq k$ and $M_i, M_k \in \Gamma$, S-TH$(\theta M_i) = \mathcal{E}_j$, if M_i has label $\mathcal{E}_j\}$.

In general not all possible labelings have to be considered, for example if a multi-equation contains a free constant, then the only sensible theory is $\mathcal{E}_0$. For $j = 1,\ldots,N$ we define the sets I_{-j} and I_{+j} as follows:

Let $I_{-j} := \{i \mid M_i \in \Gamma, M_i$ is not labeled with $\mathcal{E}_j$ and M_i contains an $\mathcal{E}_j$-term $t_{ij}\}$

and let $I_{+j} := \{i \mid M_i \in \Gamma, M_i$ is labeled with $\mathcal{E}_j\}$

The following step should ensure that the system of equations is consistent with the labeling, i.e. that in multi-equations labeled $\mathcal{E}_j$ there do not occur terms with the wrong syntactical theory.

Gu-Step 4. Apply GU-collapsing once for every theory $\mathcal{E}_j$, $j \in \{1,\ldots,N\}$.

Rule: GU-Collapsing.

$\Gamma \Rightarrow \sigma\Gamma$,

where $\mathcal{E}_j$ ($j \in \{1,\ldots,N\}$ is a theory and σ is a (most general) unifier of the problem $\langle t_{ij} = y_i \mid i \in I_{-j}\rangle$, where the y_i's are considered as free constants.

We apply the rule 'collapsing' exactly once for every proper theory. Note that the substitutions σ generated by this rule may have y_i's in the codomain terms, but only such y_i's that are considered as constants, hence after application of σ the terms t_{ij} may contain some y_k's, however, all this y_k are from a multi-equations with a different theory and can be considered as constants in this term.

After all applications and simplifications the system of equations has a special form. It consists of multi-equations of the form:

$X_i = y_i = t_{ij}$ or $X_i = y_i$, where X_i may be empty. For multi-equations M_i labeled $\mathcal{E}_0$ there is only the possibility $X_i = y_i$.

Since there is at most one proper theory-term in every multi-equation, in the following we write t_i instead of t_{ij}. We say also y_k or t_i is labeled with a theory $\mathcal{E}$, if the corresponding multi-equation is labeled with this theory.

The remaining problem now is that the resulting system may have cycles where more than one theory is involved. If there are cycles, we use constant elimination to resolve the cycles. Otherwise, the system is in sequentially solved form and we are ready.

GU-Step 5: Choose Constant-elimination problem.

Choose nondeterministically a constant-elimination problem $\mathcal{C}$ consisting of pairs $y_k \notin t_i$, where y_k and t_i are labeled with different theories and t_i and y_k is not labeled with $\mathcal{E}_0$.

GU-Step 6: Resolve Constant-elimination problem.

$\Gamma \Rightarrow \sigma\Gamma$,

where σ is the union $\sigma_1 \cup \ldots \cup \sigma_N$, where σ_i are (most general) solution to the $\mathcal{E}_i$-parts of the constant elimination problem $\mathcal{C}$.

If the system $\sigma\Gamma$ is in sequentially solved form after $\mathcal{E}_j$-normalizing terms, the system is returned as solution.

Failure-Rules: The following criteria are used to terminate the procedure with failure:

i) If the $\mathcal{E}_j$-part is nontrivial, but not unifiable, then stop with failure.

ii) If a collapse-problem is nontrivial but unsolvable, then stop with failure.

iii) If a theory-merge of different multi-equations becomes possible after identification, then stop with failure.

6.8 Theorem. The above nondeterminisitc procedure always terminates. Furthermore if every theory is finitary unifying and for every theory the constant-elimination problems are finitary solvable, then the procedure returns finitely many solutions. ■

6.9 Proposition. The procedure above provides correct transformations of systems of equations, if the systems are considered as pure systems of equations, i.e. without labeling and without the restriction that solutions do not unify different multi-equations.

Proof. The effect of the application of every rule is either to join some multi-equations or to instantiate Γ, which is always sound.■

This procedure requires branching at several points. We can always choose among several most general unifiers, there are several possibilities for identification, there are several possibilities for labeling, and in addition several solutions for every collapse problem, in the last step there are several possible sensible constant-elimination problems and in addition several possible solutions to every constant-elimination problem. Thus this procedure is not very efficient

Efficiency could eventually be improved by avoiding the rigorous renaming of variables and the abstraction of constants. However, this requires a more complicated measure for termination and a new rule for the handling of free constants that appear in Γ. For every multi-equation labeled $\mathcal{E}_0$ we must try not only the collapsing to y_i, but also to all free constants that are in Γ. For example, let f and g be two idempotent function symbols, i.e., $f(x, x) = x$ and $g(x, x) = x$ holds. If the multi-equation $y_1 = f(x, a) = g(z, a)$ is labeled with $\mathcal{E}_0$, then it is not sufficient to collapse $f(x, a)$ to y_1 (considered as free constant), since this problöem is unsolvable. However, a solution is $\{x \leftarrow a, y \leftarrow a\}$.

Furthermore a weaker unification rule may be very useful in practice. The idea is not to solve the whole $\mathcal{E}_j$-part of Γ but only a subsystem, for example a single equation. For example if two terms x,t belonging to the same theory , where x is a variable and t a term not containing x, are contained in the same multi-equation, then we can make progress by applying $\{x \leftarrow t\}$. Similarly, it can be an improvement to make decomposition for decomposable function symbols (cf. [Ki85]), but it is not clear whether the procedure with decomposition terminates.

7. The Algorithm is Complete for General Terms.

In this paragraph we show the completeness of the combination algorithm presented in the previous chapter 6. Due to Proposition 6.9 all operations are correct if we ignore the restrictions on

GU-identification and theory-labeling. The completeness proof , however, makes heavy use of these restrictions.

By Lemmas 5.1-5.5 unification provides complete sets of alternatives. We can assume that after the application of identification, every resulting system stands only for solutions θ with the additional property that $\theta y_i \neq_{\mathcal{E}+} \theta y_j$ for $i \neq j$, i.e. θ does not further identify multi-equations.

7.1 Lemma. For every solution θ of Γ_0 we can reach by the above algorithm a system of equations Γ such that there exists a solution θ' of $\Gamma$with $\theta =_{\mathcal{E}+} \theta'$ $[\mathbf{V}(\Gamma_0)]$ and θ' does not identify multi-equations in Γ and for every multi-equation M we have that S-TH(θt) = $\mathcal{E}_j$ where $t \in M$ and $\mathcal{E}_j$ is the label of M.

Proof. Obvious. ■

We say a solution θ **respects multi-equations and theory-labeling,** if $\theta M_i \neq_{\mathcal{E}+} \theta M_j$ for $i \neq j$ and if θM_j has as semantical theory equal to the theory-label of M_i.

7.2 Lemma. Collapsing is a complete step for solutions that respect multi-equations and theory-labeling.

Proof. Let $\theta \in U_{\mathcal{E}+}(\Gamma)$ be a substitution that respects multi-equations and their labeling. We can assume that θ is ground (hence idempotent), that DOM(θ) = $\mathbf{V}(\Gamma)$ and that θ is $\mathcal{E}+$–normalized.

Let $\mathcal{E} := \mathcal{E}_k$ be a theory such that I_{-k} is not empty and hence an application of the collapsing rule is possible. Let $\Delta := \langle t_{ik} = y_i \mid i \in I_{-k}\rangle$ be the collapsing problem to be solved. We have to show that the collapsing rule provides a unifier σ, such that there is a unifier θ' of $\Gamma \,\&\langle\sigma\rangle$ with $\theta = \theta'$ $[\mathbf{V}(\Gamma)]$.

We construct an abstraction θ_{abs} from θ with DOM(θ_{abs}) = $\mathbf{V}(t_{ik} \mid i \in I_{-k})$ by replacing every $\mathcal{E}$-alien subterm in θx by a variable, such that θy_i is abstracted by y_i for $i \in I_{-k}$ and other $\mathcal{E}$-alien subterms are abstracted by new variables z_i (collected in a set Z). Note that $i \in I_{-k}$ implies that θy_i has not semantical theory $\mathcal{E}_k$, i.e. has either another theory or is a free constant. Since FRC($t_{ik} \mid i \in I_{-k}$) = $\emptyset$, we can also replace the free constants θy_i that correspond to $\mathcal{E}_0$, by the variable y_i. Since $\theta y_i \neq_{\mathcal{E}+} \theta y_j$ for $i \neq j$, this can be done in a consistent way, i.e., the abstracted subterms are abstracted by the same variable, iff they are $\mathcal{E}+$-equal. Let $Y := \{y_i \mid i \in I_{-k}\}$ and let λ_θ be the substitution that reverses the abstraction, i.e. $\lambda_\theta y_i = \theta y_i$ for $i \in I_{-k}$ and DOM(λ_θ) = $Y \cup Z$ and $\theta = \lambda_\theta \theta_{abs}$ $[\mathbf{V}(\Delta)]$.

Due to Lemma 3.6 and Lemma 3.2 θ_{abs} is a solution of the collapse-problem $\langle t_{ik} = y_i \mid i \in I_{-k}\rangle$ with respect to $\mathcal{E}$. Let σ be an $\mathcal{E}$-unifier of Δ, where variables in Y are treated as constants, such that σ is more general than θ_{abs}, i.e., $\theta_{abs} \geq_{\mathcal{E}} \sigma$ $[\mathbf{V}(\Delta)]$, DOM(σ) = $\mathbf{V}(\Delta)$ – Y and such that I(σ) -Y consists of new variables .

We show that there is a solution θ' of Γ & $\langle\sigma\rangle$ such that $\theta = \theta'$ $[V(\Gamma)]$:

Let λ_σ be such that $\theta_{abs} =_{\mathcal{E}} \lambda_\sigma\sigma$ $[V(\Delta)]$ and hence $\theta =_{\mathcal{E}} \lambda_\theta\theta_{abs} =_{\mathcal{E}} \lambda_\theta\lambda_\sigma\sigma$ $[V(\Delta)]$, where $DOM(\lambda_\sigma) = I(\sigma) - (Y \cup Z)$

Let $\theta' := \theta\lambda_\theta\lambda_\sigma$.

We show that $\theta' = \theta$ $[V(\Gamma)]$:

For variables $y_i \in Y$ we have $\theta\lambda_\theta\lambda_\sigma y_i = \theta\lambda_\theta y_i = \theta\theta y_i = \theta y_i$.

For $x \in V(\Delta) - Y$ we have $\theta\lambda_\theta\lambda_\sigma x = \theta\lambda_\theta x = \theta x$.

For $x \in V(\Gamma) - V(\Delta)$ we have $\theta\lambda_\theta\lambda_\sigma x = \theta\lambda_\theta x = \theta x$.

Furthermore θ' is a solution of Γ & $\langle\sigma\rangle$:

Let $\langle\sigma\rangle = \langle x_j = s_j \mid j = 1,\ldots,m\rangle$. By the above computations we have $\theta' = \theta$ $[V(\Gamma)]$, hence θ' solves Γ. We have to show that $\theta' x_j =_{\mathcal{E}+} \theta' s_j$ for all $j = 1,\ldots,m$.

We obtain $\theta' x_j = \theta x_j$ since $\theta' = \theta$ $[V(\Gamma)]$ and $x_j \in DOM(\sigma) = V(\Delta) - Y$. Furthermore $\theta' s_j = \theta\lambda_\theta\lambda_\sigma s_j =_{\mathcal{E}} \theta\lambda_\theta\lambda_\sigma\sigma x_j =_{\mathcal{E}} \theta\lambda_\theta\theta_{abs} x_j =_{\mathcal{E}} \theta\theta x_j =_{\mathcal{E}} \theta x_j$. Hence θ' is a solution of Γ & $\langle\sigma\rangle$ with $\theta = \theta'$ $[V(\Gamma)]$. ■

In the following lemma we use union of substitutions in the following sense:
$\cup\{\sigma_k \mid k \in K\}$ is a substitution defined by $\cup\{\sigma_k \mid k \in K\}x =_{\mathcal{E}+} \sigma_k x$, if $x \in DOM(\sigma_k)$. In order to ensure well-definedness it is required, that for all $y \in DOM(\sigma_i) \cap DOM(\sigma_k)$ we have $\sigma_i y =_{\mathcal{E}+} \sigma_k y$.

7.3 Lemma. Selecting and solving constant elimination problems is a complete step for solutions that respect multi-equations and their theory-labeling.

Proof.

Let θ be a ground solution of Γ that respects multi-equations and theory-labeling. We can assume that all terms in $COD(\theta)$ are $\mathcal{E}$+-normalized.

Now choose a constant elimination problem as follows: Let all pairs $y_k \notin t_i$ be in $\mathcal{C}$, where $\mathcal{E}_j$ is a theory not equal to $\mathcal{E}_0$, t_i is a theory-term in Γ labeled by $\mathcal{E}_j$ and y_k is labeled by another theory than $\mathcal{E}_j$ and $\mathcal{E}_0$, and θy_k is not an essential $\mathcal{E}_j$-alien subterm in θt_i.

Let $\mathcal{C}_j$ be the $\mathcal{E}_j$-part of $\mathcal{C}$.

We construct the abstractions $\theta_{abs,j}$ from θ by restricting θ to $V(t_i \mid i \in I_{+j}\}$ and then by replacing every $\mathcal{E}_j$-alien subterm in θx for $x \in V(t_i \mid i \in I_{+j}\}$ by a variable, such that terms $\mathcal{E}$+-equal to θy_i are abstracted by y_i for $i \in I_{-j}$ and other $\mathcal{E}_j$-alien subterms are abstracted by new variables collected in Z_j. Since $\theta y_i \neq_{\mathcal{E}+} \theta y_j$ for $i \neq j$, this can be done in a consistent way, i.e., two $\mathcal{E}_j$-alien subterms are abstracted by the same variable, iff they are $\mathcal{E}$+-equal. Let $Y_{-j} := \{y_k \mid M_k$ is not labeled by $\mathcal{E}_j\}$ and let $Y := \{y_i \mid i=1,\ldots,M\}$.

Let $\lambda_{\theta,j}$ be such that it reverses the abstraction, i.e. $DOM(\lambda_{\theta,j}) \subseteq Y_{-j} \cup Z_j$ and $\lambda_{\theta,j}\, y_k = \theta y_k$ for $y_k \in Y_j \cap I(\theta_{abs,j})$. Hence $\theta =_{\mathcal{E}+} \lambda_{\theta,j}\, \theta_{abs,j}$ $[V(t_i \mid i \in I_{+j})]$.

Let σ_j be a solution of $\mathcal{C}_j$ that is more general than $\theta_{abs,j}$. We can assume that $DOM(\sigma_j) =$

$V(t_i \mid i \in I_{+j})$ and that σ_j introduces no new free constants and only new variables (besides $y_i \in I_{-j}$).

Then there exists a substitution $\lambda_{\sigma,j}$ with $DOM(\lambda_{\sigma,j}) = I(\sigma_j) - Y_j$ and $\theta_{abs,j} =_{\mathcal{E}+} \lambda_{\sigma,j}\sigma_j\,[V(t_i \mid i \in I_{+j})]$.

Let $\sigma := \bigcup\{\sigma_j \mid j = 1,\ldots,N\}$ be the substitution applied in the constant-elimination rule.

Let $\theta' := \bigcup\{\theta\lambda_{\theta,j}\lambda_{\sigma,j} \mid j = 1,\ldots,N\}$.

i) θ' is well-defined:

The only variables common to some domains of $\lambda_{\theta,j}\lambda_{\sigma,j}$ may be the variables y_i.

We have $\theta\lambda_{\theta,j}\lambda_{\sigma,j}\, y_i = \theta\lambda_{\theta,j}y_i$ and since $\lambda_{\theta,j}y_i$ is either y_i or θy_i, we obtain $\theta\lambda_{\theta,j}y_i = \theta y_i$.

ii) $\theta' =_{\mathcal{E}+} \theta\,[V(\Gamma)]$:

For y_i we have already shown that $\theta y_i =_{\mathcal{E}+} \theta' y_i$.

For variables $x \in V(\Gamma) - Y$ we have $\lambda_{\theta,j}\lambda_{\sigma,j}x = x$, hence $\theta x =_{\mathcal{E}+} \theta' x$.

iii) θ' is a solution of $\Gamma\ \&\ \langle\sigma\rangle$, i.e. $\theta' =_{\mathcal{E}+} \theta'\sigma\,[V(\Gamma)]$

Since $\theta' =_{\mathcal{E}+} \theta\,[V(\Gamma)]$, it is sufficient to show that θ' solves $\langle\sigma\rangle$.

For a variable x in $DOM(\sigma)$ we have to show $\theta' x =_{\mathcal{E}+} \theta'\sigma x$. Let $\mathcal{E}_j$ be the theory corresponding to x. Then

$$\begin{aligned}
\theta'\sigma x &=_{\mathcal{E}+} \theta' \circ \bigcup\{\sigma_k \mid k = 1,\ldots,N\}x \\
&=_{\mathcal{E}+} \theta'\sigma_j x \\
&=_{\mathcal{E}+} (\bigcup\{\theta\lambda_{\theta,k}\lambda_{\sigma,k} \mid k = 1,\ldots,N\})\circ\sigma_j x \\
&=_{\mathcal{E}+} \theta\lambda_{\theta,j}\lambda_{\sigma,j}\circ\sigma_j x \qquad (\text{since } \bigcup\{\theta\lambda_{\theta,k}\lambda_{\sigma,k} \mid k = 1,\ldots,n\} =_{\mathcal{E}+} \theta\lambda_{\theta,j}\lambda_{\sigma,j}\,[V(\sigma_j x)]\,) \\
&=_{\mathcal{E}+} \theta\lambda_{\theta,j}\theta_{abs,j}x \\
&=_{\mathcal{E}+} \theta\theta x \quad =_{\mathcal{E}+} \theta x\,.
\end{aligned}$$

iv) $\Gamma\ \&\ \langle\sigma\rangle$ can be transformed by a complete step into sequentially solved form:

The transformation $\Gamma\ \&\ \langle\sigma\rangle \Rightarrow \sigma\Gamma$ is complete due to Lemma 6.3.

We can assume that every y_k that occurs in some σt_i is essential, since otherwise we can choose an equal term that contains no inessential variables due to Lemma 4.4 and Proposition 5.7.

Assume there is a cycle in $\sigma\Gamma$. Then the cycle is of the form $y_{i1} = \sigma t_{i1}$, $y_{i2} = \sigma t_{i2}$, $y_{i3} = \sigma t_{i3}, \ldots$, $y_{ik} = \sigma t_{ik}$ with $y_{ij} \in V(\sigma t_{i,j-1})$, and $y_{i1} \in V(\sigma t_{ik})$, where $y_{ij} \in \{y_i \mid i = 1,\ldots,M\}$ and $t_{ij} \in \{t_i \mid i=1,..,M\}$.

We have that θ' solves this cycle: $\theta'\sigma t_{ij} = \theta' t_{ij}$ by iii) and $\theta' t_{ij} =_{\mathcal{E}+} \theta y_{ij}$, since θ' is a unifier of Γ. Hence $\theta'\sigma t_{ij} =_{\mathcal{E}+} \theta' t_{ij} = \theta t_{ij} =_{\mathcal{E}+} \theta y_{ij} = \theta' y_{ij}$

Without loss of generality we can assume that $\theta'\sigma t_{i1} = \theta t_{i1}$ has the smallest semantical theory height in this chain. Since we have assumed that the semantical theory of θt_{i2} is different from the semantical theory of θt_{i1}, the term θt_{i2} is an inessential $\mathcal{E}_j$-alien subterm of θt_{i1} by Proposition 4.5. But then we have that $y_{i2} \notin t_{i1}$ is a pair in C, since in addition y_{i2} and t_{i1} are not labeled by the theory $\mathcal{E}_0$. But then y_{i2} is eliminated by σ and hence y_{i2} is not a variable in σt_{i1}. This contradicts our assumption that we have a cycle. ■

7.4 Theorem. If there exist complete $\mathcal{E}_j$-unification procedures for every system of equations including free constants and an algorithms for every theory $\mathcal{E}_j$ that provides complete sets of constant eliminators for every constant elimination problem, then our procedure is a correct and complete procedure for solving unification problems in systems of equations in the combination of the theories $\mathcal{E}_j$. ■

7.5 Corollary. If all $\mathcal{E}_j$ are finitary and there always exists a finite complete set of constant eliminators for $\mathcal{E}_j$, then unification in the combination is also finitary.

Proof. The above procedure returns only finitely many solved systems of equations since at every choice-point there exist only a finite number of possible choices .■

7.6 Corollary. If all $\mathcal{E}_j$ are finitary and regular, then unification in the combination is also finitary.

Proof. In regular theories every nontrivial constant elimination problems is unsolvable. Hence we can use Corollary 7.5.

8. Solving Constant Elimination Problems.

Besides regular theories, where all nontrivial constant-elimination problems are unsolvable, there are nonregular theories for which we can describe an algorithm for solving constant-elimination problems. Note that in general it is obvious that a complete set of constant eliminators is recursive enumerable.

8.1 Constant-Elimination in Boolean Rings.

The unification problem in Boolean rings is known to be decidable and unitary [MN86, BS86]. In is well-known that terms in Boolean rings can be transformed into normalform as a sum of products (cf. [HD83]). Note that a term in normalform has no inessential variables and constants.

We give a method how to solve constant elimination problems $\mathcal{C}$ in Boolean rings.

Let $\mathcal{C} = \{c_i \notin t_{ij} \mid i = 1,\ldots,n, j = 1,\ldots,m\}$ be a constant elimination problem. Let $C_0 := \{c_i \mid i = 1,\ldots,n\}$ and let $V_0 := \mathbf{V}\{t_{ij} \mid i = 1,\ldots,n, j = 1,\ldots,m\} = \{z_k \mid k = 1,\ldots,K\}$. Let D be the set of all possible products of elements in C_0, i.e., $D := \{c_{i1} * c_{i2} * \ldots * c_{ig} \mid \{i_1,\ldots,i_g\} \subseteq \{1,\ldots,n\}\}$. Note that D contains the element 1 as an empty product and hence the set D generated by C_0 has 2^n elements.

We try a 'general' substitution σ with $\mathrm{DOM}(\sigma) = V_0$. A general representation is $\sigma z_k =$

$\Sigma\{y_{k,d}*d \mid d \in D\}$, where $y_{k,d}$ are different new variables and stand for terms not containing constants from C_0. If we apply σ to $\mathcal{C}$ we get the representation $\sigma t_{ij} = \Sigma\{t_{i,j,d}*d \mid d \in D\}$, where $t_{i,j,d}$ is a term not containing constants from C_0. The unification problem $\Gamma_{\mathcal{C}}$ corresponding to $\mathcal{C}$ is as follows: $\Gamma_{\mathcal{C}} := \{t_{i,j,d} = 0 \mid d \in D$ where c_i is a factor of d, i = 1,...,n, j=1,...,m }. This unification problem does not contain constants from C_0 and is to be solved without these constants. The obtained mgu can be transformed into a solution of the constant-elimination problem $\mathcal{C}$. Since Boolean rings are unitary unifying, there is at most one most general constant-eliminator necessary in Boolean rings. ■

Thus we have the Theorem:

8.2 Theorem. Constant-elimination problems in Boolean rings are unitary solvable. ■

8.3 Constant-Elimination in Abelian Groups.

Unification in free Abelian groups is considered in [LBB84] and it is shown there that it is of type finitary and that a set of most general unifiers can be computed by solving linear Diophantine equations. We use the operators +,-,0 in Abelian groups. It is well-known that terms in Abelian groups can be transformed into normalform as a sum of the form Σn_i*a_i, where n_i*a_i represents a sum of n_i elements a_i if n_i is positive and a sum of $-n_i$ elements $(-a_i)$ if n_i is negative. Note that a term in normalform contains no inessential variables and constants.

We show how to solve constant elimination problems $\mathcal{C}$ in Abelian groups.

Let $\mathcal{C} = \{c_i \notin t_{ij} \mid i = 1,\ldots,n,\ j = 1,\ldots,m\}$ be a constant elimination problem. Let $C_0 := \{c_i \mid i = 1,\ldots,n\}$ and let $V_0 := V\{t_{ij} \mid i = 1,\ldots,n,\ j = 1,\ldots,m\} = \{z_k \mid j = 1,\ldots,K\}$.

A general solution σ of $\mathcal{C}$ has the form $\sigma z_k = \Sigma\{z_{k,c} \mid c \in C_0\} + e_k$, where $z_{k,c}$ is a variable standing for a term n_1*a, where n_1 is an integer and e_k does not contain constants from C_0. Substituting this sum into the variables of t_{ij} we obtain a representation $t_{ij} = t_{ij,0} + t_{ij,R}$, where $t_{ij,0}$ contains all c_i-terms, i.e., all c_i's and all variables standing for a sum of c_i's. The condition that $c_i \notin t_{i,j}$ is now equivalent to the condition $t_{i,j,0} = 0$ due to independency. Thus the solution of the whole problem $\mathcal{C}$ can be solved by considering the unification problem $\Gamma_{\mathcal{C}} := \{t_{i,j,0} = 0 \mid i = 1,\ldots,n,\ j = 1,\ldots,m\}$. Since unification in Abelian groups is finitary there are at most finitely many constant eliminators necessary.

Thus the following holds.

8.3 Theorem. Constant-elimination problems in Abelian groups are finitary solvable. ■

8.4 Constant-Elimination in Canonical Theories.

Let $\mathcal{E}$ be a theory with a canonical term rewriting system $R_{\mathcal{E}}$. Then the first observation is that every term t in normalform does not contain inessential free variables or constants, since the rewriting relation removes variables from terms, but does not add new variables. Hence a solution θ to a constant-elimination problem $C = \{c_i \notin t_{ij} \mid i = 1,\ldots,n, j = 1,\ldots,m\}$ has the property that $c_i \notin FRC((\theta t_{ij})\downarrow)$. Since we assume that an infinite number of free constants is in the signature, it is sufficient for an investigation of completeness to assume that θ is ground and normalized, eventually replacing variables by new free constants. If we know the solution θ then consider the unification problem $\langle t_{ij} = (\theta t_{ij})\downarrow\rangle$. Let's try to solve this problem by basic narrowing [Hul80, NRS87]. Narrowing steps have to be performed only on the left hand side, (i.e. on t_{ij}) until there is a derived term t_{ij}' that is syntactically more general than $(\theta t_{ij})\downarrow$. Obviously $FRC(t_{ij}')$ does not contain c_i. Now J.-M. Hullot [Hul80] has shown that basic narrowing is a complete unification procedure for theories admitting a canonical TRS. An application of this result shows that we get a complete set of constant-eliminators, if basic narrowing is performed on the terms t_{ij} and all narrowing substitutions are returned that correspond to a set of derived terms that satisfy the elimination conditions.

For a special case of theories for which basic narrowing always terminates we obtain always finite complete sets of constant-eliminators. A criterion for termination of basic narrowing given in [Hul80] is that basic narrowing terminates on the right hand sides of the rules in a TRS.

8.5 Theorem. Basic narrowing is a complete procedure for solving constant-elimination problems for theories admitting a canonical TRS. Furthermore, if basic narrowing always terminates, then constant-elimination problems are finitary solvable. ■

8.6 Remark. In order to have an approximation of the solutions of a constant-elimination problem, it is possible to use an idea of E. Tidén [Ti86a]. Instead of solving $c \notin t$ solve the unification problem $t = t'$, where t' is obtained from t by replacing the constant c by a new constant c' and by renaming all variables in t. A complete set of unifiers to this problem is complete for the constant-elimination problem, but it may contain unifiers that are not eliminators, hence for the exact solution a search for the right instances is necessary. Since the application of a substitution is always correct in the algorithm below, such an approximation (E. Tidén called it a total complete set of eliminators) may be of practical use. This idea works for a variety of theories.

Note that in general a set of constant-elimination problems cannot be encoded this way, a counterexample can be constructed from Example 11.4. ■

9. Combining Disjointly an Arbitrary and a Simple Theory.

The above procedure can be improved for the special case of a disjoint combination $\mathcal{E}+\mathcal{F}$, where $\mathcal{E}$ is arbitrary and $\mathcal{F}$ is a simple theory. The improvements over the general procedure of paragraph 7 originate in some nice properties of simple theories. For example a proper syntactical $\mathcal{F}$-term is also a semantical $\mathcal{F}$-term and cyclic systems of equations in simple theories are not solvable. Note that this algorithm is not a specialization of the general algorithm.

The improvements are that identification is only necessary for multi-equations containing proper $\mathcal{F}$-terms. A further improvement is that no labelling of multi-equations is necessary and that free constants remain in the terms and are not abstracted by variables.

In this procedure we do not introduce the theory $\mathcal{E}_0$ and we assume in contrast to the general procedure that free constants are allowed in the terms in an unfolded normalform. Thus an ASU-UNF is like an UNF, but free constants are not abstracted.

ASU-step 1. Transform Γ into a system in ASU-UNF.

ASU-step 2. Transform Γ into a system in separated UNF by the rules GU-unification and ASU-Identification

The ASU-identification rule used for step 2 is more restricted than GU-identification:

Rule: ASU-Identification. $\Gamma \Rightarrow \Gamma'$,

where in Γ' some multi-equations containing proper $\mathcal{F}$-terms from Γ are joined together.

If we write 'U' for unification and 'I' for identification, then the application sequence is like $U^*(IU^*)^*$ in contrast to U^*IU^* for the general case.

Every multi-equation in the system has now at most one term for every theory. Furthermore the system is intended to represent only solutions θ with $\theta s \neq_{\mathcal{E}+\mathcal{F}} \theta t$ for different proper $\mathcal{F}$-terms s and t in Γ.

The next step is like collapsing. The goal is to solve equations $s = t$, where s is a proper pure $\mathcal{F}$-term and t is a pure $\mathcal{E}$-term. The method used is constant-abstraction [LS78, He87]:

ASU-step 3. Abstract proper $\mathcal{F}$-terms by different constants and solve the system with respect to $\mathcal{E}$.

We add to every multi-equation M_i that contains a proper $\mathcal{F}$-term a new variable y_i, which is used as constant-abstraction. The set of such indices is denoted by I.

Rule: ASU-Collapsing. $\Gamma \Rightarrow \sigma\Gamma$,

where σ is a (most general) unifier of Δ, where the y_i are treated as constants during unification. The system Δ is the $\mathcal{E}$-part of Γ including the y_i's.

The system has now the following properties: Every multi-equation that contains a proper $\mathcal{F}$-term does not contain an $\mathcal{E}$-term. The variables y_i may occur in proper $\mathcal{E}$-terms. Note that there may be multi-equations containing an $\mathcal{F}$-variable and an $\mathcal{E}$-term.
Note that after ASU-collapsing there may be some theory-merges possible for multi-equations M_i and M_j where $i \notin I$ or $j \notin I$. Thus there may occur new problems for the theory $\mathcal{F}$ of the type $x = t$. If $x \in \mathbf{V}(t)$, then we have failure, if not, we can apply $\{x \leftarrow t\}$ to Γ and then delete x from Γ. This is a complete transformation. Hence we can assume that multi-equations have one of the following forms :i) $X = y_{iu} = t_{\mathcal{F}}$, ii) $X = t_{\mathcal{E}}$, iii) $X = x = t_{\mathcal{E}}$.

ASU-step 4. Select a constant-elimination problem corresponding to Γ and $\mathcal{E}$.

This is performed by choosing nondeterministically a constant-elimination problem $\mathcal{C}$ consisting of pairs $y_i \notin t_j$, where $y_i \in M_i$, t_j is the $\mathcal{E}$-term in M_j, and M_i has a nontrivial $\mathcal{F}$-part.

ASU-Step 5. Transform Γ into $\sigma\Gamma$,

where σ is a (most general) constant-eliminator of $\mathcal{C}$.
A solution to the original system is obtained, if the system $\sigma\Gamma$ is in sequentially solved form after deletion of inessential variables.

Failure-Rules: The following criteria are used to terminate the procedure with failure:

i) If the $\mathcal{E}$-part ($\mathcal{F}$-part) is nontrivial, but not unifiable, then stop with failure.
ii) If the collapse-problem is nontrivial, but unsolvable, then stop with failure.
iii) If in step 3, 4 or 5 a theory-merge of different multi-equations containing proper $\mathcal{F}$-terms becomes possible, then stop with failure.

First we investigate termination of the above procedure.

9.1 Lemma. GU-unification and ASU-identification terminate.

Proof. Follows since unification decreases properly the number of terms in Γ and decreases the number of multi-equations in Γ and ASU-identification properly decreases the number of multi-equations.

9.2 Proposition. The procedure ASU terminates. ■

The completeness of the procedure is shown similarily to the completeness of the general procedure. Since unification provides a complete set of alternatives, we have:

9.3 Lemma. For every solution θ of Γ_0 we can reach by the procedure ASU in a finite number of steps a system of multi-equations Γ such that there exists a solution θ' of Γ with $\theta =_{\mathcal{E}+} \theta'$ $[V(\Gamma_0)]$ and θ' does not identify multi-equations in Γ that contain a proper $\mathcal{F}$-term.

Proof. Obvious. ■

9.4 Lemma. ASU-Collapsing is a complete step for solutions that respect multi-equations with proper $\mathcal{F}$-terms.

Proof. We adapt the proof of Lemma 7.2.

Let $\theta \in U_{\mathcal{E}+}(\Gamma)$ be a substitution that respects multi-equations with proper $\mathcal{F}$-terms. We can assume that θ is ground (hence idempotent), that $DOM(\theta) = V(\Gamma)$ and that θ is $\mathcal{E}$+–normalized.

Let Δ be the collapsing problem to be solved. Without loss of generality we can assume that Δ is nontrivial. We have to show that the collapsing rule provides a unifier σ, such that there is a unifier θ' of Γ &$\langle\sigma\rangle$ with $\theta = \theta'$ $[V(\Gamma)]$.

We construct an abstraction θ_{abs} from θ with $DOM(\theta_{abs}) = V(\Delta)$ by replacing every $\mathcal{E}$-alien subterm in θx by a variable, such that θy_i is abstracted by y_i and other $\mathcal{E}$-alien subterms are abstracted by other new variables z_i (collected in a set Z). Note that θy_i has semantical theory $\mathcal{F}$. This can be done in a consistent way, i.e., the abstracted subterms are abstracted by the same variable, iff they are $\mathcal{E}$+-equal. Let $Y := \{y_i \mid i \in I\}$ and let λ_θ be the substitution that reverses the abstraction, i.e. $\lambda_\theta y_i = \theta y_i$ for $i \in I$ and $DOM(\lambda_\theta) = Y \cup Z$ and $\theta =_{\mathcal{E}} \lambda_\theta \theta_{abs}$ $[V(\Delta)]$.

Due to Lemma 3.6 θ_{abs} is a solution of the collapse-problem Δ with respect to $\mathcal{E}$. Let σ be an $\mathcal{E}$-unifier of Δ, where variables in Y are treated as constants, such that σ is more general than θ_{abs}, i.e., $\theta_{abs} \geq_{\mathcal{E}} \sigma$ $[V(\Delta)]$, $DOM(\sigma) = V(\Delta) - Y$ and such that $I(\sigma)$ -Y consists of new variables.

We show that there is a solution θ' of Γ & $\langle\sigma\rangle$ such that $\theta = \theta'$ $[V(\Gamma)]$:

Let λ_σ be such that $\theta_{abs} =_{\mathcal{E}} \lambda_\sigma \sigma$ $[V(\Delta)]$ and $\theta =_{\mathcal{E}} \lambda_\theta \theta_{abs} =_{\mathcal{E}} \lambda_\theta \lambda_\sigma \sigma$ $[V(\Delta)]$, where $DOM(\lambda_\sigma) = I(\sigma) - (Y \cup Z)$

Let $\theta' := \theta\lambda_\theta\lambda_\sigma$.

We show that $\theta' = \theta$ $[V(\Gamma)]$:

For variables $y_i \in Y$ we have $\theta\lambda_\theta\lambda_\sigma y_i = \theta\lambda_\theta y_i = \theta\theta y_i = \theta y_i$.

For $x \in V(\Delta) - Y$ we have $\theta\lambda_\theta\lambda_\sigma x = \theta\lambda_\theta x = \theta x$.

For $x \in (V(\Gamma) - V(\Delta))$ we have $\theta\lambda_\theta\lambda_\sigma x = \theta\lambda_\theta x = \theta x$.

Furthermore θ' is a solution of Γ & $\langle\sigma\rangle$:

Let $\langle\sigma\rangle = \langle x_j = s_j \mid j = 1,\ldots,m\rangle$. By the above computations we have $\theta' = \theta$ $[V(\Gamma)]$, hence θ'

solves Γ. We have to show that $\theta'x_j =_{\mathcal{E}+} \theta's_j$ for all $j = 1,\ldots,m$.

We obtain $\theta'x_j = \theta x_j$ since $\theta' = \theta\ [V(\Gamma)]$ and $x_j \in DOM(\sigma) = V(\Delta) - Y$. Furthermore $\theta's_j = \theta\lambda_\theta\lambda_\sigma s_j =_{\mathcal{E}} \theta\lambda_\theta\lambda_\sigma\sigma x_j =_{\mathcal{E}} \theta\lambda_\theta\theta_{abs}x_j =_{\mathcal{E}} \theta\theta x_j =_{\mathcal{E}} \theta x_j$. Hence θ' is a solution of $\Gamma\ \&\ \langle\sigma\rangle$ with $\theta = \theta'\ [V(\Gamma)]$. ■

9.5 Lemma. Selecting and solving constant elimination problems is a complete step for solutions that respect multi-equations and their labeling.

Proof. We adapt the proof of Lemma 7.3.

Let θ be a ground solution of Γ that respects multi-equations with proper $\mathcal{F}$-terms. We can assume that all terms in $COD(\theta)$ are $\mathcal{E}$+-normalized.

Now choose a constant elimination problem as follows: Let all pairs $y_i \notin t_j$ be in $\mathcal{C}$, where t_j is a $\mathcal{E}$-term in Γ and M_j contains an $\mathcal{F}$-term and θy_i is not an essential $\mathcal{E}$-alien subterm in θt_j. Let $V_{\mathcal{E}}$ be the set of variables occurring in $\mathcal{E}$-terms.

We construct an abstractions θ_{abs} from θ by restricting θ to $V_{\mathcal{E}}$ and then by replacing every $\mathcal{E}$-alien subterm in θx for $x \in V_{\mathcal{E}}$ by a variable, such that terms $\mathcal{E}$+-equal to θy_i are abstracted by y_i for $i \in I$ and other $\mathcal{E}$-alien subterms are abstracted by new variables collected in Z. Since $\theta y_i \neq_{\mathcal{E}+} \theta y_j$ for $i \neq j$, this can be done in a consistent way, i.e., two $\mathcal{E}$-alien subterms are abstracted by the same variable, iff they are $\mathcal{E}$+-equal. Let $Y := \{y_i \mid i \in I\}$.

Let λ_θ be such that it reverses the abstraction, i.e. $DOM(\lambda_\theta) \subseteq Y \cup Z$ and $\lambda_\theta y_k = \theta y_k$ for $y_k \in Y \cap I(\theta_{abs})$. Hence $\theta =_{\mathcal{E}+} \lambda_\theta\, \theta_{abs}\ [V_{\mathcal{E}}]$.

Let σ be a solution of $\mathcal{C}$ that is more general than θ_{abs}. We can assume that $DOM(\sigma) = V_{\mathcal{E}}$ and that σ introduces no new free constants and only new variables (besides $y_i \in Y$).

Then there exists a substitution λ_σ with $DOM(\lambda_\sigma) = I(\sigma) - Y$ and $\theta_{abs} =_{\mathcal{E}+} \lambda_\sigma\sigma\ \ [V_{\mathcal{E}}]$.

Let $\theta' := \theta\lambda_\theta\lambda_\sigma$.

i) $\theta' =_{\mathcal{E}+} \theta\ [V(\Gamma)]$:

We have $\theta\lambda_\theta\lambda_\sigma\, y_i = \theta\lambda_\theta y_i$ and since $\lambda_\theta y_i$ is either y_i or θy_i, we obtain $\theta\lambda_\theta y_i = \theta y_i$.

Hence $\theta y_i =_{\mathcal{E}+} \theta' y_i$.

For variables $x \in V(\Gamma)$ - Y we have $\lambda_{\theta,j}\lambda_{\sigma,j}x = x$, hence $\theta x =_{\mathcal{E}+} \theta'x$.

ii) θ' is a solution of $\Gamma\ \&\ \langle\sigma\rangle$, i.e. $\theta' =_{\mathcal{E}+} \theta'\sigma\ [V(\Gamma)]$

Since $\theta' =_{\mathcal{E}+} \theta\ [V(\Gamma)]$, it is sufficient to show that θ' solves $\langle\sigma\rangle$.

For a variable x in $DOM(\sigma)$ we have to show $\theta'x =_{\mathcal{E}+} \theta'\sigma x$.We have

$$\begin{aligned}
&\theta'\sigma x \\
=_{\mathcal{E}+}\ &\theta\lambda_\theta\lambda_\sigma\sigma x \\
=_{\mathcal{E}+}\ &\theta\lambda_\theta\theta_{abs}x && \text{since } x \in V_{\mathcal{E}} \\
=_{\mathcal{E}+}\ &\theta\circ\theta x \quad =_{\mathcal{E}+} \theta x\ .
\end{aligned}$$

iii) $\Gamma\ \&\ \langle\sigma\rangle$ can be transformed by a complete step into sequentially solved form:

The transformation $\Gamma\ \&\ \langle\sigma\rangle \Rightarrow \sigma\Gamma$ is complete due to Lemma 6.3.

We can assume that every variable that occurs in some σt_j is essential, since otherwise we can choose an $\mathcal{E}$+-equal term that contains no inessential variables due to Lemma 4.4 and Proposition 5.7.

Assume there is a cycle in $\sigma\Gamma$. Then the cycle is of the form $x_{i1} = \sigma t_{i1}$, $y_{i1} = s_{i1}$, $x_{i2} = \sigma t_{i2}$, $y_{i2} = s_{i2}, \ldots, x_{ik} = \sigma t_{ik}$, $y_{ik} = s_{ik}$ with $y_{ij} \in V(\sigma t_{i,j})$, $x_{ij} \in V(s_{i,j-1})$, and $x_{i1} \in V(s_{ik})$, where $y_{ij} \in \{y_i \mid i \in I\}$ and t_{ij} is a $\mathcal{E}$-term from Γ and s_{ij} is a proper $\mathcal{F}$-term from Γ.

We have that θ' solves this cycle: $\theta'\sigma t_{ij} = \theta' t_{ij}$ by ii) and $\theta' t_{ij} =_{\mathcal{E}+} \theta x_{ij}$, since θ' is a unifier of Γ. Furthermore $\theta y_{ij} =_{\mathcal{E}+} \theta s_{ij}$, since θ' is a unifier of Γ. Hence $\theta'\sigma t_{ij} =_{\mathcal{E}+} \theta' t_{ij} = \theta t_{ij} =_{\mathcal{E}+} \theta x_{ij} = \theta' x_{ij}$ and $\theta' s_{ij} = \theta s_{ij} =_{\mathcal{E}+} \theta y_{ij} = \theta' y_{ij}$.

The condition that θy_{ij} is an essential $\mathcal{E}$-alien subterm of $\theta'\sigma t_{ij} = \theta t_{ij}$ and that $\mathcal{F}$ is regular yields that the semantical theory-height must decrease along the cycle, hence all terms have the same semantical theory height. Since θs_{i1} has semantical theory $\mathcal{F}$ all terms θx_{ij} and θy_{ij} have semantical theory $\mathcal{F}$. Since $\mathcal{F}$ is regular and y_{ij} is an essential variable in σt_{ij}, we have $\theta y_{ij} =_{\mathcal{E}+} \theta'\sigma t_{ij} =_{\mathcal{E}+} \theta x_{ij}$ by Lemma 4.6. This gives a contradiction to the simplicity of $\mathcal{F}$, since then θ' is a unifier of the cyclic pure $\mathcal{F}$-problem $x_{i1} = s_{i1}, x_{i2} = s_{i2}, \ldots, x_{ik} = s_{ik}$.

■

9.6 Theorem. If there exist complete $\mathcal{E}$- and $\mathcal{F}$-unification procedures for every system of equations including free constants and an algorithm for the theory $\mathcal{E}$ that provides a complete set of constant eliminators for every constant elimination problem, then the ASU procedure is a correct and complete procedure for solving systems of equations in the combination of the theories $\mathcal{E}+\mathcal{F}$.■

9.7 Corollary. If $\mathcal{E}$ and $\mathcal{F}$ are finitary unifying and there always exists a finite complete set of constant eliminators for $\mathcal{E}$, then ASU returns a finite, complete set of unifiers .■

9.8 Example. Solving x = f(x*y).

We consider the unification problem $\langle x = f(x*y)\rangle$ in a combination of a Boolean ring with operators *,+,0,1 and a free function symbol f.

This problem was posed by U.Martin at the first unification workshop in Val d'Ajol as a test-example [Ki87b]. We use the algorithms for a combination of an arbitrary and a simple theory.

The unfolded normal form of this problem is:

$\langle x = x' = f(z), y = y', x'*y' = z\rangle$. Unification or ASU-identification is not applicable. ASU-step 3 means to transform this system into $\langle x = y_1 = x' = f(z), y = y', x'*y' = z\rangle$ and then to solve $\langle x' = y_1\rangle$. Application of the solution $\{x' \leftarrow y_1\}$ yields the multi-equation system $\langle x = y_1 = f(z), y = y', y_1*y' = z\rangle$. This system has a cycle and hence is not in solved form.

The only possible constant-elimination problems is: $C := \{y_1 \notin y_1*y'\}$, where y_1 is to be considered as a constant. A constant-eliminator of $C = \{y_1 \notin y_1*y'\}$ is computed as follows:

Let $y' := y_a + y_b*y_1$. The problem to be solved is $y_1 \notin y_1*(y_a + y_b*y_1)$, which is equivalent to the condition $y_a+y_b = 0$, since y_1 is treated as a constant and y_1 is not allowed in terms that are substituted for y_a or y_b. The unique solution is $y_a = y_b$, hence the constant-eliminator is $\{y' \leftarrow y''(1+y_1)\}$. The application to the original problem gives the solution $\{x \leftarrow f(0), y \leftarrow y''*(1+f(0))\}$.■

Note that unification in a combination of a Boolean ring and free function symbols is not unitary, since the unification problem $\langle f(x)*f(y) = f(a)*f(b)\rangle$ (cf. [MN86]) has as a minimal complete set of unifiers consisting of two substitutions: $\{\{x \leftarrow a, y \leftarrow b\}, \{y \leftarrow a, x \leftarrow b\}\}$.

10. Combining Collapse-free, Regular Theories.

In this paragraph we show how to obtain an algorithm for two theories $\mathcal{E}_1$ and $\mathcal{E}_2$ that are collapse-free and regular. Unification algorithms for this special case have been given by K. Yellick, A. Herold, E. Tidén and C. Kirchner [Ye87, Ki85, He86, Ti86a]. Our aim is to give a very simple algorithm for this case that can be compared to theirs. The described algorithm appears to be closest to the algorithm of C. Kirchner´s, which uses variable abstraction, but I believe that also the algorithm of A. Herold [He86] that uses the constant-abstraction method can be reformulated with the tools developed in this paper.

Two important facts for a combination of two collapse-free and regular theories are (cf. [Ye87, Ki85, He86, Ti86a]:

i) if s is a proper $\mathcal{E}_1$-term and t is a proper $\mathcal{E}_2$-term, then s and t are not unifiable.
ii) if Γ has a cycle that contains proper terms from both theories, then Γ is not unifiable.

The algorithm has the following basic steps:

i) Transform Γ into UNF, but do not abstract constants by variables.
ii) Perform (nondeterministically) unification steps on the $\mathcal{E}_1$ and $\mathcal{E}_2$-part until the $\mathcal{E}_1$-part and $\mathcal{E}_2$-part is solved.
   Note that constants are both in the $\mathcal{E}_1$-part and $\mathcal{E}_2$-part.
   Now every multi-equation of the system that has a nontricial $\mathcal{E}_1$-part and a nontricial $\mathcal{E}_2$-part has the form x = t.
iii) Check whether the resulting Γ has a cycle. If there is none, return Γas solution.

It is straightforward to proof the correctness and completeness of this procedure using the methods in this paper. It is obvious that it nondeterministically always terminates, but the number of alternatives may be infinite if the number of mgu´s for some intermediate unification problem is infinite.

An improvement of the algorithm can be obtained, if not the whole E_j-part is unified, but only a subsystem for which solutions are easily computable. For example if there are two variables x_1, x_2 in the $\mathcal{E}_j$-part and in the same multi-equation, then apply the unifier $\{x_1 \leftarrow x', x_2 \leftarrow x'\}$, where x' is a new variable. This is in effect the variable-canonization described in [Ki85].

11. Decidability of Unification in $\mathcal{E}$+.

The following variation of the general procedure shows that unification in the general combination $\mathcal{E}_1+\ldots+\mathcal{E}_N$ is decidable if unification in the combination of every theory with free function symbols is decidable. The main difference to the general procedure is that unification and collapsing are delayed to the end, which is necessary, since complete sets of unifiers may be infinite.

The nondeterminisitic test-procedure has the following steps:

Step 1. Transform Γ into UNF.

Step 2. Apply the general identification step.

Step 3. Label every multi-equations by a theory $\mathcal{E}_j$ where $j \in \{0,1,\ldots,N\}$ and add a new variable y_i to every multi-equation not labeled $\mathcal{E}_0$.

Step 4. Select a minimal cycle-free constant-elimination problem $\mathcal{C}$ consisting of pairs $y_i \not\in y_k$, where y_i and y_k are labeled with different theories and neither y_i nor y_k are labeled with $\mathcal{E}_0$.

Step 5. Check unifiability of all $(\Gamma_j,\mathcal{C}_j)$ (defined below) and return 'unifiable', if this is the case for all $j \in \{1,\ldots,N\}$.

Let Y be the set of the new variables y_i.

This constant-elimination problems are denoted a bit different from the ones defined in paragraph 2. The meaning is similar, namely that for a solution θ and a pair $y_i \not\in y_k$ in $\mathcal{C}$, θy_i is not an essential alien subterm of θy_k. By **cycle-free** we mean that for every cycle $y_{i1}, y_{i2},\ldots,y_{ik}, y_{ik+1}$ $(= y_{i1})$ for $y_{ij} \in$ Y, where y_{ij} and $y_{i,j+1}$ have different theory label, there is an index j such that $y_{ij} \not\in y_{ij+1}$ is in $\mathcal{C}$. **Minimal** means that every subset of $\mathcal{C}$ is not cycle-free. Let $<_c$ be the transitive relation defined by the pairs $y_i <_c y_k$, iff $y_i \not\in y_k$ is not a pair in $\mathcal{C}$. That $\mathcal{C}$ is minimal cycle-free means that the relation $<_c$ is a transitive, assymetric and irreflexive relation. that is maximal in the following sense. Whenever a relation $y_i <_c y_k$ is added for y_i, y_k labeled differently, then $<_c$ has a cycle.

11.1 Lemma. Let $\mathcal{C}$ be minimal cycle-free . Then for all y_i, y_k labeled differently, either $y_i <_c y_k$ or $y_k <_c y_i$ holds.

Proof. If neither $y_i <_c y_k$ nor $y_k <_c y_i$ holds, then we can add one of them without creating a cycle in the relation $<_c$. This contradicts the maximality of $<_c$.■

11.2 Corollary. $\mathcal{C}$ is minimal cycle-free iff for all y_i, y_k labeled differently, exactly one of either $y_i \notin y_k$ or $y_k \notin y_i$ are in $\mathcal{C}$. ■

The problem $(\Gamma_j,\mathcal{C}_j)$ for the theory $\mathcal{E}_j$ is constructed as follows:
Γ_j consists of the $\mathcal{E}_j$-parts of the multi-equations and the variables y_i. $\mathcal{C}_j$ consists of the pairs $y_i \notin y_k$ in $\mathcal{C}$, where y_i is not labeled with $\mathcal{E}_j$ and y_k is labeled $\mathcal{E}_j$. The variables y_i that are not labeled with $\mathcal{E}_j$ are considered as constants in this problem. A substitution σ is a solution of $(\Gamma_j,\mathcal{C}_j)$, iff σ solves Γ_j, $\sigma y_i = y_i$ for y_i not labeled $\mathcal{E}_j$ and if $y_i \notin y_k$ is in $\mathcal{C}_j$, then y_i is not an essential constant in σy_k.

11.3 Proposition. Let $\mathcal{C}$ be a minimal cycle-free constant-elimination problem and let $(\Gamma_j,\mathcal{C}_j)$ be the $\mathcal{E}_j$-part. Then there exists an equivalent unification problem Δ in a combination of $\mathcal{E}_j$ with free function symbols, such that Γ_j is solvable, iff $\mathcal{C}_j$ is solvable.

Proof. First we construct the unification problem Δ from $(\Gamma_j,\mathcal{C}_j)$. Δ is derived from Γ_j by replacing the y_i that are not labeled $\mathcal{E}_j$ by $q_i := f_i(\ldots)$, where f_i is a new free function symbol and q_i has as arguments the variables y_k with $y_i \notin y_k$ in $\mathcal{C}_j$.

i) Let Δ be solvable.
Then there is a ground substitution θ that solves Δ. We can assume that θ is $\mathcal{E}_j$-normalized. If we construct an abstraction θ_{abs} of θ by consistently replacing all $\mathcal{E}_j$-alien terms by variables such that θq_i is replaced by y_i, we obtain a solution of Γ_j that additionally satisfies $\mathcal{C}_j$, since for $y_i \notin y_k$ in $\mathcal{C}_j$ the term θq_i is not essential in θy_k, since y_k occurs in q_i.

ii) Let $(\Gamma_j,\mathcal{C}_j)$ be solvable.
Then there is a substitution θ that solves $(\Gamma_j,\mathcal{C}_j)$. We can assume that θ is ground and $\mathcal{E}_j$-normalized. We construct an equation system $\langle\mu\rangle$ from θ be replacing all subterms y_i (not labeled $\mathcal{E}_j$) by the term q_i. Now it is obvious (by abstraction) that every solution of $\langle\mu\rangle$ is also a solution of Δ. It suffices to show that $\langle\mu\rangle$ has a solution which is the case if and only if $\langle\mu\rangle$ is cycle-free. Every cycle in $\langle\mu\rangle$ immediately yields a cycle in $<_{\mathcal{C}}$ Hence $\langle\mu\rangle$ is cycle-free, since $<_c$ is assumed to be cycle-free.■

This shows that all the constant-elimination problems in a specific theory, which are relevant for a combination algorithm, can be encoded as unification problems of the form $x = t(\ldots f_c(\ldots,x,\ldots) \ldots)$ by replacing the constants c by terms $f_c(\ldots,x,\ldots)$. The following example shows that this methods fails in general .

11.4 Example. Let $E := \{ g(x, x, y) = h(y)\}$ and consider the constant elimination problem

$C := \{a \not\in g(x, a, y),\ b \not\in g(y, b, x)\}$. The only solution is $\{x \leftarrow a, y \leftarrow b\}$. The intuitive encoding would be $\langle x_1 = g(x, f_a(x_1), y), x_2 = g(y, f_b(x_2), x)\rangle$, where f_a and f_b are free unary function symbols. However, to remove the x_1-cycle and x_2-cycle requires the substitution $\{x \leftarrow f_a(x_1), x \leftarrow f_b(x_2)\}$. This results in the equation system
$\langle x_1 = h(f_b(x_2)), x_2 = h(f_a(x_1))\rangle$, which has no solution. ■

We have to show that the above nondeterminisitic procedure can be used as decision algorithm for unifiability of a system Γ, if decision algorithms for the problems (Γ_j, C_j) for every theory $\mathcal{E}_j$, $j \in \{1,\ldots,N\}$ exist. Since the branching rate of the procedure is finite, the unifiability problem of the originial system is then equivalent to a finite disjunction of a conjunction of problems (Γ_j, C_j).

11.5 Lemma. If the original system Γ_0 is unifiable, then there is an execution of the procedure such that all the final problems (Γ_j, C_j) are solvable.

Proof. Let θ be a unifier of Γ_0. Without loss of generality we can assume that θ is ground and $\mathcal{E}$-normalized. It is obvious that up to step 3 there is a path, i.e. we can assume that we have a labeled system $\Gamma = \{M_1,\ldots,M_M\}$ and a substitution θ such that $\theta M_i \neq \theta M_j$ for $i \neq j$ and the semantical theory of θM_i corresponds to the label of M_i.
Let C_0 be the following set of pairs: $y_i \not\in y_k$ is in C_0, iff θy_i and θy_k are not equal to free constants, θy_i and θy_k have different semantical theory and θy_i is not an essential alien subterm of θy_k. It follows from Proposition 4.5 that C_0 is cycle-free. As C we choose a minimal cycle-free subset of C_0. For every theory $\mathcal{E}_j$ we abstract θ, such that θy_i is abstracted by y_i, iff y_i is labeled different from $\mathcal{E}_j$. The obtained substitution is a solution to the problem (Γ_j, C_j). ■

11.6 Lemma. If the procedure says unifiable, then the original system has a solution.

Proof. The only nontrivial step is to show that whenever all problems (Γ_j, C_j) are solvable, then the system obtained in step 3 has a solution. Therefore it is sufficient to show that the solutions of the systems (Γ_j, C_j) can be combined.
Let σ_j be solutions of (Γ_j, C_j), $j \in \{1,\ldots,N\}$ restricted to $V(\Gamma_1,\ldots,\Gamma_N)$.
Application of all σ_j's to Γ_3, the system obtained in step 3, has the following effect: Every multi-equations labeled $\mathcal{E}_0$ can be transformed into one of the form $X = y_i$ or $X = y_i = a$ and every other multi-equation labeled $\mathcal{E}_j$ into one of the form $X = y_i$ or $X = y_i = t_i$, where t_i is an $\mathcal{E}_j$-term and if $y_k \in Y$ is an essential variable in t_i, then $y_k <_C y_i$. Since $<_C$ is cycle-free, the obtained system is in sequentially solved form and hence has a solution. ■

Finally, we have the following result on decidability of unification in the combined theory $\mathcal{E}+$:

11.7 Theorem. Unification in $\mathcal{E}+$ is decidable, if for every $\mathcal{E}_i$, unification in a combination of $\mathcal{E}_i$ with free function symbols is decidable. ■

The following open problem remains to be solved.

11.8 Open Problem. Is unification in a combination of $\mathcal{E}$ with free functions decidable, if unification in $\mathcal{E}$ is decidable?

Conclusion.

This paper gives a unification procedure for mixed terms in a combination of arbitrary disjoint theories. This algorithm is constructed on the base of an $\mathcal{E}$-unification algorithm for every involved theory and a method to solve constant-elimination problems in every theory $\mathcal{E}$.

It is not clear whether there exists a general method to construct an algorithm for constant elimination from a unification algorithm as it is possible for Boolean rings and Abelian groups or whether a theory with decidable $\mathcal{E}$-unification also has decidable constant elimination problems.

Unfortunately, the described general combination procedure has a high complexity, for example for every significant variable in the problem we have to guess its semantical labeling. So some research is needed to recognize possible redundant steps in this algorithm and to find more efficient versions of our algorithm. Some possibilities to enhance efficiency are i) to make partial unification, i.e. to solve only parts of an $\mathcal{E}_j$-part instead of the whole $\mathcal{E}_j$-part, ii) to loosen the rigid sequence of steps and rules in order to support a lazy unification method. iii) to avoid the renaming of variables and the abstraction of constants.

This paper also provides a basis for future research in the combinatin of nondisjoint equational theories. I conjecture that theories can be combined if the combination can be described as a disjoint combination over some equational theory.

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