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Group and Lie algebra filtrations and homotopy groups of spheres

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Abstract

We establish a bridge between homotopy groups of spheres and commutator calculus in groups, and solve in this manner the "dimension problem" by providing a converse to Sjogren's theorem: every abelian group of bounded exponent can be embedded in the dimension quotient of a group. This is proven by embedding for arbitrary s, d the torsion of the homotopy group $\pi_s(S^d)$ into a dimension quotient, via a result of Wu. In particular, this invalidates some long-standing results in the literature, as for every prime p, there is some p-torsion in $\pi_{2p}(S^2)$ by a result of Serre. We explain in this manner Rips's famous counterexample to the dimension conjecture in terms of the homotopy group $\pi_4(S^2) = \mathbb{Z}/2\mathbb{Z}$. We finally obtain analogous results in the context of Lie rings: for every prime p there exists a Lie ring with p-torsion in some dimension quotient.

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1 | INTRODUCTION

The fundamental problem of combinatorial group theory can be phrased as: "*Given a group G presented as a quotient of a free group, what can be said of quotients of G itself?*". Of main importance are those quotients produced by universal constructions; prominently the maximal nilpotent

In memoriam John R. Stallings, 1935–2008.

quotients $G/\gamma_n(G)$, and more generally G/N for $N \lhd G$ obtained from G using the elementary operations of product, intersection, commutation and power.

An altogether different family of quotients arise from associative algebra. Every group G naturally embeds in its *group ring*$\mathbb{Z}G$, leading to images of G in the quotients $\mathbb{Z}G/\varpi^n$ by powers of the augmentation ideal; and more generally $G/G \cap (1+\mathfrak{N})$ for ideals $\mathfrak{N} \lhd \mathbb{Z}G$ obtained from ϖ using the elementary operations of product, intersection, sum, and scalar multiple. These quotients play a crucial role as the receptacle for numerous topological invariants, such as Milnor's link invariants [16, 17, 36].

A key insight of Magnus [39] was that the filtrations $\gamma_n(G)$ of G and ϖ^n of $\mathbb{Z}G$ are deeply related: defining $\delta_n(G) := G \cap (1+\varpi^n) = \ker(G \to \mathbb{Z}G/\varpi^n)$ the nth *dimension subgroup*, one has $\gamma_n(G) \leqslant \delta_n(G)$, and the *dimension problem* asks to understand when $\delta_n(G) = \gamma_n(G)$. In that same paper, Magnus showed that if F is a free group then $\delta_n(F) = \gamma_n(F)$ for all n.

It was claimed on numerous occasions [12, 37, 41] that $\delta_n(G) = \gamma_n(G)$ holds for all n and all groups G. It is relatively easy to prove $\delta_n(G) = \gamma_n(G)$ for $n \leqslant 3$, but a counterexample was found by Rips [51], with $\delta_4(G)/\gamma_4(G) = \mathbb{Z}/2\mathbb{Z}$. Nevertheless, the quotient $\delta_n(G)/\gamma_n(G)$ is always abelian, and Sjogren bounded its exponent by a function of n only [57]; the following claim even stood in the literature:

"Theorem" (Gupta [27, section 4; 28, Theorem 2.2]). *For all n and all groups G, one has $\delta_n(G)^2 \leqslant \gamma_n(G) \leqslant \delta_n(G)$.*

Our main result is that this cannot hold, and Sjogren's result alluded to above is essentially optimal:

Theorem A (See Theorem 1.3). *For every abelian group H of bounded exponent and every n large enough, there exists a group G whose quotient $\delta_n(G)/\gamma_n(G)$ contains H as a subgroup. In particular p-torsion may appear in $\delta_n(G)/\gamma_n(G)$ for all primes p.*

Consider, for example, H cyclic. It is known that every finite cyclic group appears as a subgroup of $\pi_s(S^d)$ for some s, d. For these parameters, we construct a group G, integer n and monomorphism $\mathsf{torsion}(\pi_s(S^d)) \hookrightarrow \delta_n(G)/\gamma_n(G)$. We even construct explicitly, for $p = 2$ and $p = 3$, an element of order p in $\delta_n(G)/\gamma_n(G)$, based on the element of order p in $\pi_{2p}(S^2)$ discovered by Serre [55].

Theorem A is thus a converse to Sjogren's theorem: the best general constraint on the quotients $\delta_n(G)/\gamma_n(G)$ is precisely that they are abelian of bounded exponent.

Our method is in principle applicable in a broad setting, producing for every $K(\pi, 1)$ space X and integer s a group G, an integer n and a homomorphism

$$\mathsf{torsion}\big(\pi_{s+1}(\Sigma X)\big) \to \delta_n(G)/\gamma_n(G),$$

which we expect to be injective under mild finiteness conditions on X. Our main result comes from the space $X = S^2 \vee S^2$.

Homotopy groups of spheres are so fundamental objects that they pervade topology, with applications ranging from Brouwer's fixed-point theorem to Rokhlin's theorem on signatures of spin 4-manifolds. Their nontriviality and finiteness (apart from the $\pi_n(S^n)$ and $\pi_{4n-1}(S^{2n})$) are among the most profound results of mathematics. Theorem A shows that they are also tightly linked to a question in pure algebra.

Cohen, Wu and their coauthors revealed deep links between combinatorial group theory and homotopy [9–11, 30, 34, 62, 63]. At the heart of our method is a formula by Wu, expressing homotopy groups of spheres as quotients between two subgroups of a finitely generated free group. It is based on simplicial sets and corresponding simplicial groups, see May's fundamental reference [42].

Topological methods are inherent to the modern study of group theory, as witnessed by the monumental treatises by Gromov [21], Bridson and Haefliger [4], and Geoghegan [19]. Stallings [58, p. 117], in a program carried out by Sjogren [57], already recognized the value of homological arguments toward studying dimension quotients.

Nonetheless, Theorem A is the first instance of a classical problem in algebra that is solved using higher algebraic topology, in effect harnessing the powerful instruments of Steenrod algebra and spectral sequences, notably the Adams, Curtis, and May spectral sequences.

1.1 | History of the dimension problem

The dimension problem has a long history, starting with Magnus's investigation of the lower central series of a free group, and its associated Lie algebra [39]; he showed with Witt that the dimension property holds for free groups, see [40, 61], attributing the first proof to Grün [22]. For a small subset of the literature, we refer to [24, 25, 43, 49, 59], and for further historical remarks to [52]. Remarkably, incorrect proofs of the dimension problem appeared more than once, by Cohn [12], Losey [37] (Lyndon remarks dryly, in his MathSciNet review, "*The main content of this paper is another incomplete proof that the (integral) dimension subgroups of an arbitrary group are the terms of its lower central series*"), and even Magnus himself [41, Theorem 5.15(i)]!

We note that if one replaces the ring $\mathbb{Z}$ by a field, then there is an elegant and elementary description of the corresponding dimension subgroup, depending only on the field's characteristic; see [31, 32].

The dimension problem is quantitatively studied in terms of the quotients $\delta_n(G)/\gamma_n(G)$ called *dimension quotients*. Gupta and Kuzmin proved in [23] that they are all abelian, and Sjogren proved in [57] that they have finite exponent, bounded by a function of n only: there exists a minimal $s(n) \in \mathbb{N}$ (at most $(n!)^n$) such that $\delta_n(G)^{s(n)} \subseteq \gamma_n(G) \leqslant \delta_n(G)$ holds for all groups G.

In that terminology, one has $s(1) = s(2) = s(3) = 1$, and Rips's example implies $2 \mid s(4)$, which generalizes to $2 \mid s(n)$ for all $n \geqslant 4$. Passi [48] gave $s(4) = 2$, and Tahara [60] gave $s(5) \in \{2, 6\}$.

This can be improved in the case of metabelian groups: Gupta proved in [26] that $s(n)$ is a power of 2. He then claimed that $s(n)$ is a power of 2 for all groups, a proof is published in [28]; and even that it may be improved to $s(n) = 2$ for all $n \geqslant 4$, see [27]. However, many parts of his arguments were never fully understood.

Now our main result, stated above, shows that the function $s(n)$ is unbounded, and its values cannot even be constrained to a finite collection of primes.

1.2 | Lie rings

An variant of the dimension problem may be asked for Lie rings; namely, Lie algebras over $\mathbb{Z}$. Every Lie ring A embeds in its universal enveloping algebra $U(A)$, which also admits an augmentation ideal. The dimension subrings are defined analogously by $\delta_n(A) = A \cap \varpi^n$, see [2]. Again $\delta_n(A) = \gamma_n(A)$ when $n \leqslant 3$, and there is a Lie ring A with $\delta_n(A)/\gamma_n(A) = \mathbb{Z}/2$. Sjogren's bound also holds for Lie rings [56], and many details are simpler in the category of Lie rings.

Even though we are not aware of any direct construction of a group from a Lie ring or vice versa that preserves dimension quotients, it often happens that a presentation involving only powers and commutators, which may therefore be interpreted either as group or Lie algebra presentation, yields isomorphic dimension quotients.

To give a quick taste of dimension quotients in Lie rings, we reproduce first an example due to Pierre Cartier of a Lie algebra over a commutative ring $\Bbbk$ *not* embedding in its universal envelope [8]: consider $\Bbbk = \mathbb{F}_2[x_0, x_1, x_2]/(x_0^2, x_1^2, x_2^2)$, and

$$A = \langle e_0, e_1, e_2 \mid x_0 e_0 + x_1 e_1 + x_2 e_2 = 0\rangle \text{ qua } \Bbbk\text{-Lie algebra.}$$

Then $\alpha := x_0x_1[e_0, e_1] + x_0x_2[e_0, e_2] + x_1x_2[e_1, e_2]$ is nontrivial in A, but in any associative algebra it maps to $(x_0e_0 + x_1e_1 + x_2e_2)^2 = 0$.

Rips' example, or rather its Lie algebra variant [2, Theorem 4.7], is of a similar spirit. In $\Bbbk = \mathbb{Z}$ one can of course not choose x_i nilpotent; but one may choose x_i a large power of 2 and impose relations that guarantee that elements with large 2-valuation are mapped far in the lower central series: set $x_i = 2^{2+i}$ and consider

$$\begin{aligned} A = \langle e_0, e_1, e_2, \cdots \mid\ & 2^{2i+2} e_i \in \gamma_2 \text{ for all } i \in \{0,1,2\}, \\ & x_j x_k e_i \pm x_i x_k e_j \in 2^{2k+2}\gamma_2 + \gamma_3 \text{ for all } \{i,j,k\} = \{0,1,2\}\rangle \end{aligned} \tag{1.1}$$

with the element $\alpha = \sum_{0 \leqslant i < j \leqslant 2} x_i x_j [e_i, e_j]$. Then the relations imply $\alpha \in A \cap (\gamma_2(A) \cdot \gamma_2(A) + A \cdot \gamma_3(A)) \subseteq \delta_4(A)$, while it is easy to make choices of elements in γ_2 and $2^{2+2k}\gamma_2 + \gamma_3$ that yield, by direct computation, that α has a nontrivial image in the quotient $A/\gamma_4(A)$.

It is even possible to write a 3-related Lie algebra, based on [43, Example 2.3], that satisfies (1.1) and $\alpha \in \delta_4(A) \setminus \gamma_4(A)$:

$$A = \langle e_0, e_1, e_2, z \mid 2^2 e_0 = [z, e_1 + 2e_2], 2^4 e_1 = [z, -e_0 + 4e_2], 2^6 e_2 = [z, -2e_0 - 4e_1]\rangle$$

with as before $\alpha = 2^5[e_0, e_1] + 2^6[e_0, e_2] + 2^7[e_1, e_2]$.

This Lie algebra presentation may also be interpreted as a group presentation,

$$G = \langle e_0, e_1, e_2, z \mid e_0^4 = [z, e_1] \cdot [z, e_2]^2, e_1^{16} = [z, e_0]^{-1} \cdot [z, e_2]^4, e_2^{64} = [z, e_0]^{-2} \cdot [z, e_1]^{-4}\rangle,$$

in which the element $\alpha = [e_0, e_1]^{32}[e_0, e_2]^{64}[e_1, e_2]^{128}$ belongs to $\delta_4(G) \setminus \gamma_4(G)$.

1.3 | Homotopy groups of the two-sphere: Main statement and sketch of proof

According to a result of Wu [15, 62], we may express the homotopy groups of spheres $\pi_{s+1}(S^2)$ as a quotient of two normal subgroups in a free group. More precisely, write $F_s = \langle x_0, \dots, x_s \mid x_0 \cdots x_s = 1\rangle$ a free group of rank s with one redundant generator, and for $i = 0, \dots, s$ let R_i denote the normal closure of x_i in F_s. We write iterated commutators as left-normed: $[x_1, x_2, \dots, x_d] =$

$[[\cdots[x_1,x_2],\dots],x_d]$, and denote by Σ_{s+1} the symmetric group on $\{0,\dots,s\}$. Then

$$\frac{R_0\cap\cdots\cap R_s}{\prod_{\rho\in\Sigma_{s+1}}[R_{\rho(0)},\dots,R_{\rho(s)}]}\simeq\pi_{s+1}(S^2). \tag{1.2}$$

We can now state more precisely the main step toward our result:

Theorem 1.1′. *Given an integer $s\geqslant 3$, there is for all n large enough a group G and a set-wise map $F_s\to G$ inducing via (1.2) an injective homomorphism*

$$\pi_{s+1}(S^2)\hookrightarrow\delta_n(G)/\gamma_n(G).$$

In particular, the exponent of $\delta_n(G)/\gamma_n(G)$ is divisible by that of $\pi_{s+1}(S^2)$.

We note that for $s=3$ this result produces a variant of Rips's example [51], "explaining" the 2-torsion in $\delta_4(G)/\gamma_4(G)$ as that of $\pi_4(S^2)$; see Subsection 9.1.

An analogous result holds in the realm of Lie algebras. There, starting with a free Lie ring $L_s=\langle x_0,\dots,x_s\mid x_0+\cdots+x_s=0\rangle$, define analogously ideals $I_i=\langle x_i\rangle^{L_s}$; then

$$\frac{I_0\cap\cdots\cap I_s}{\prod_{\rho\in\Sigma_{s+1}}[I_{\rho(0)},\dots,I_{\rho(s)}]}\simeq\bigoplus_{i\geqslant 1}E^1_{i,s}, \tag{1.3}$$

the sth column of the lower central series spectral sequence for S^2. Our result, for Lie algebras, states:

Theorem 1.2′. *Given an integer $s\geqslant 3$, there is for all n large enough a Lie ring A and a linear map $L_s\to A$ inducing via (1.3) an injective homomorphism*

$$\bigoplus_{i\geqslant 1}E^1_{i,s}\hookrightarrow\delta_n(A)/\gamma_n(A).$$

In particular, the exponent of $\delta_n(A)/\gamma_n(A)$ is divisible by all primes appearing in the order of $\bigoplus_{i\geqslant 1}E^1_{i,s}$.

The spectral sequence $E^*_{*,s}$ converges to $\pi_{s+1}(S^2)$, so for $s=2p-1$ there is an order-p term $\alpha_p\in E^1_{s+1,s}$ that survives as the Serre element in $\pi_{s+1}(S^2)$. We are able to write it explicitly in terms of a shuffle product.

In fact, the groups G and Lie algebras A appearing in Theorems 1.1′, 1.2′ can be written quite concretely. In a group or Lie algebra presentation, we introduce the following notation: for $d\in\mathbb{N}$, when we write a generator $x^{(d)}$ of degree d we mean a list of generators $x_1,\dots,x_d$; when x appears in a relator, it is a shorthand for the left-normed iterated commutator $x:=[x_1,\dots,x_d]$ of the generators $x_1,\dots,x_d$. Thus, "$\langle x_1^{(2)},y^{(3)}\mid[x_1,y]\rangle$" is shorthand for "$\langle x_{1,1},x_{1,2},y_1,y_2,y_3\mid[[x_{1,1},x_{1,2}],[y_1,y_2,y_3]]\rangle$". The following results, which precise Theorems 1.1′ and 1.2′, will, respectively, be proven in Sections 5 and 4.

Theorem 1.1. *Given an integer $s \geqslant 3$, there are integers $e, c_0, \dots, c_s$ and $n = c_0 + \dots + c_s$ such that, in the group*

$$G = \left\langle \begin{matrix} x_0, \dots, x_s, y_0^{(c_0)}, \dots, y_s^{(c_s)}, \\ (r_w)_{w \in \langle x_0, \dots, x_s \rangle} \end{matrix} \middle| \begin{matrix} x_0 \cdots x_s = 1, x_i^{e^{nc_i}} = y_i \text{ for } i = 0, \dots, s, \\ r_w^{e^n} = w \text{ for all } w \in \langle x_0, \dots, x_s \rangle \end{matrix} \right\rangle$$

the map $\iota\colon\ w(x_0, \dots, x_s) \in F_s \mapsto w(x_0, \dots, x_s)^{e^{n^2}}$ induces via (1.2) an injective homomorphism

$$\bar{\iota}\colon\ \pi_{s+1}(S^2) \hookrightarrow \delta_n(G)/\gamma_n(G).$$

The c_i must only satisfy some linear inequalities, and every n large enough may be obtained.

Note that only a finite number of roots r_w of group elements are required, though it seems messy to specify exactly which ones.

Theorem 1.2. *Given an integer $s \geqslant 3$, there are integers $e, c_0, \dots, c_s$ and $n = c_0 + \dots + c_s$ such that, in the Lie ring*

$$A = \langle x_0 \dots, x_s, y_0^{(c_0)}, \dots, y_s^{(c_s)} \mid x_0 + \dots + x_s = 0, e^{c_i} x_i = y_i \text{ for } i = 0, \dots, s \rangle$$

the linear map $\iota\colon\ w(x_0, \dots, x_s) \in L_s \mapsto e^n w(x_0, \dots, x_s) \in A$ induces via (1.3) an injective homomorphism

$$\bar{\iota}\colon\ \bigoplus_i E^1_{i,s} \hookrightarrow \delta_n(A)/\gamma_n(A).$$

The c_i must only satisfy some linear inequalities, and every n large enough may be obtained.

The constants e and c_i are somewhat explicit, based on the exponent of $\pi_{s+1}(S^2)$ and the connectivity of certain simplicial groups. We have determined tighter values for $p = 2$ and $p = 3$, see Section 9.

Here is a sketch of the proof in the group case; the Lie algebra case is essentially the same, and slightly simpler. The first claim follows from the fact that G is a free product with amalgamation. The second claim splits in three parts: $\iota(\prod_\rho [R_{\rho(0)}, \dots, R_{\rho(s)}]) \leqslant \gamma_n(G)$, which follows from standard commutator calculus; $\iota(R_0 \cap \dots \cap R_s) \leqslant \delta_n(G)$, which boils down to two ingredients: the Hurewicz homomorphism, see Proposition 3.1, connecting the group and associative algebra universes, and the presence of roots r_w of elements $w \in \langle x_0, \dots, x_s \rangle$ in G; and the last part, $\iota^{-1}(\gamma_n(G)) \cap R_0 \cap \dots \cap R_s \leqslant \prod_\rho [R_{\rho(0)}, \dots, R_{\rho(s)}]$.

For this last part, we first invoke Curtis' connectedness theorem [13], from which there is an integer k such that $\gamma_k(F_s) \cap R_0 \cap \dots \cap R_s \leqslant \prod_\rho [R_{\rho(0)}, \dots, R_{\rho(s)}]$. It therefore suffices to prove $\iota^{-1}(\gamma_n(G)) \cap R_0 \cap \dots \cap R_s \leqslant \prod_\rho [R_{\rho(0)}, \dots, R_{\rho(s)}] \gamma_k(F_s)$.

We then use the particular form of the presentation of G: consider an element of F_s, identified with its image in G, and write it in terms of commutators $g = [z_1, \dots, z_j]$ with each $z_i \in \{x_0, \dots, x_s\}$, for some $j < k$. This element g seemingly defines an element of $\gamma_j(G)$, but in G it may be rewritten, in the presence of sufficiently high powers of e, into a commutator of higher weight by replacing some x_i by the corresponding y_i. If the c_i are chosen such that $c_0 \geqslant k$ and $c_i / c_{i-1} \geqslant k$

for all i, then in order for g to belong to $\gamma_n(G)$ either each x_i must have been replaced at least once by a y_i, so the commutator had to belong to some $[R_{\rho(0)}, \dots, R_{\rho(s)}]$, or a larger power of e is required, and therefore the original term in F_s itself was a power of e. In this manner we obtain $g \in \prod_\rho [R_{\rho(0)}, \dots, R_{\rho(s)}] F_s^e \gamma_k(F_s)$.

In effect, we use two filtrations on G, by the lower central series and by powers of e. Only the substitution $x_s \rightsquigarrow y_s$ increases much the degree in the first filtration; but it consumes a high degree in the second. All the other substitutions $x_i \rightsquigarrow y_i$ for $i < s$ involve a trade-off between how much they increase either degree, and each one requires the previous one. Finally, the substitutions $x_i \rightsquigarrow v_i$ or $y_i \rightsquigarrow w_i$ decrease the first degree too much to be of any use in attaining $\gamma_n(G)$.

We derive in Theorem 8.1 an expression for the p-torsion of $\pi_{2p}(S^2)$, first at the level of Lie algebras, namely on the first page of the Curtis spectral sequence, and deduce in Proposition 8.2 some properties of its representation $\widetilde{\alpha}_p$ as an element of the free group F_{2p-1}. We make use of an explicit form for $p = 2$ and $p = 3$ to obtain smaller examples, in particular for $p = 2$ we obtain straightforward constructions, for arbitrary $n \geqslant 4$, of Lie algebras in which δ_n/γ_n contains 2-torsion, and for $p = 3$ we obtain a Lie algebra and a group in which δ_7/γ_7 contains 3-torsion. These examples have also been checked using computer algebra software.

1.4 | Wedges of spheres

An analogous statement to (1.2) holds for wedges of two-spheres, and even more generally for suspensions of spaces with contractible universal cover. We restrict ourselves here to the space $S^2 \vee S^2$, which is sufficient to prove Theorem A.

Consider the group $\overline{F}_s = F_s * F_s$, and identify the generators of its factors as $x_{i,j}$ for $i = 0, \dots, s$ and $j \in \{1, 2\}$; set $\overline{R}_i = \langle x_{i,1}, x_{i,2} \rangle^{\overline{F}_s}$. Then

$$\frac{\overline{R}_0 \cap \cdots \cap \overline{R}_s}{\prod_{\rho \in \Sigma_{s+1}} [\overline{R}_{\rho(0)}, \dots, \overline{R}_{\rho(s)}]} \simeq \pi_{s+1}(S^2 \vee S^2). \tag{1.4}$$

Analogously to Theorem 1.1, we have

Theorem 1.3. *Given an integer $s \geqslant 3$, there are integers $e, c_0, \dots, c_s$ and $n = c_0 + \cdots + c_s$ such that, in the group*

$$G = \left\langle \begin{matrix} x_{0,1}, x_{0,2}, \dots, x_{s,1}, x_{s,2}, \\ y_{0,1}^{(c_0)}, y_{0,2}^{(c_0)}, \dots, y_{s,1}^{(c_s)}, y_{s,2}^{(c_s)}, \\ (r_w)_{w \in \langle x_{0,1}, x_{0,2}, \dots, x_{s,1}, x_{s,2} \rangle} \end{matrix} \middle| \begin{matrix} x_{0,1} \cdots x_{s,1} = x_{0,2} \cdots x_{s,2} = 1, \\ x_{i,j}^{e^{nc_i}} = y_{i,j} \text{ for } i = 0, \dots, s, i \in \{1, 2\}, \\ r_w^{e^n} = w \text{ for all } w \in \langle x_{0,1}, x_{0,2}, \dots, x_{s,1}, x_{s,2} \rangle \end{matrix} \right\rangle$$

the map $\iota : w(x_{i,j}) \in F_s * F_s \mapsto w(x_{i,j})^{e^{n^2}}$ induces via (1.4) an injective homomorphism

$$\bar{\iota} : \mathit{torsion}\big(\pi_{s+1}(S^2 \vee S^2)\big) \hookrightarrow \delta_n(G)/\gamma_n(G).$$

Note that only a finite number of roots r_w of group elements are required, though it seems messy to specify exactly which ones. Note also that the group constructed in Theorem 1.3 is, apart from the adjunction of roots r_w, the free product of two copies of the group constructed in Theorem 1.1.

There is a Lie algebra analogue to Theorem 1.3, which we do not state because it does not seem to have any applications. In particular, we do not know whether there exists a Lie algebra such that one of its dimension quotients contains $\mathbb{Z}/p^2\mathbb{Z}$-torsion for some prime p. Indeed the torsion in the first page of the spectral sequence converging to $\pi_*(S^2 \vee S^2)$ has only prime orders. There seems to be a fundamental difference, here, between groups and Lie algebras.

2 | DIMENSION QUOTIENTS

We recall in this section some classical facts about dimension quotients, and their Lie algebra equivalents:

Proposition 2.1 [23, 57]. *For arbitrary group G or Lie algebra A, the quotient $\delta_n(G)/\gamma_n(G)$, respectively, $\delta_n(A)/\gamma_n(A)$, is abelian of bounded exponent.*

The "bounded exponent" statement is due to Sjogren. In his notation, let b_m denote the least common multiple of $\{1, \ldots, m\}$, and define integers a_i^j and c_n recursively by

$$a_1^n = 1, \qquad a_{k+1}^n = \prod_{i=1}^{k} a_i^{n-k+i} b_{n-k}, \qquad c_n = \prod_{k=1}^{n-1} a_k^k.$$

Then for any group G we have $\delta_n(G)^{c_n} \subseteq \gamma_n(G)$. By [56], the same result holds in Lie algebras: for any Lie algebra A we have $c_n\delta_n(A) \subseteq \gamma_n(A)$.

Gupta and Kuzmin prove even that the quotient $\delta_n(G)/\gamma_{n+1}(G)$ is abelian; following [43] the same result is easily seen to hold in Lie algebras. We reproduce the argument, in the Lie algebra case, because of its simplicity:

Proof (that $\delta_n(A)/\gamma_{n+1}(A)$ is abelian). Let A be nilpotent of class n; we are to show that $\delta_n(A)$ is abelian. Let M be maximal abelian normal in A; so M is an A-module via adjunction. For any $m \in M, x \in \delta_k(A)$ we have $[m, x] \in \gamma_{k+1}(M)$, so $\delta_n(A)$ centralizes M. Now as M is maximal it is self-centralizing, so $\delta_n(A) \leqslant M$ and therefore is abelian. □

3 | HOMOTOPY GROUPS OF SPHERES

We describe in this section the group-theoretic and Lie-algebra-theoretic formulations of homotopy groups of spheres. They will be essential in the proofs of Theorems 1.2 and 1.1. We use "n" in this section for what is written "s" in the rest of the text, to avoid confusion with the degeneracies s_i in simplicial objects.

3.1 | Groups

Fix an integer $n \geqslant 1$ and let $F = \langle x_0, \ldots, x_n \mid x_0 \cdots x_n = 1\rangle$ be a free group of rank n. Consider its normal subgroups

$$R_i := \langle x_i \rangle^F \text{ for } i = 0, \ldots, n.$$

Note that F is the fundamental group of a 2-sphere with $n+1$ punctures, and R_i contains the conjugacy class of a loop around the ith puncture; the operation of filling-in the ith puncture induces the map $F \to F/R_i$ on fundamental groups.

Denote by Σ_{n+1} the symmetric group on $\{0, \dots, n\}$, and define the symmetric commutator product of the above subgroups by

$$[R_0, \dots, R_n]_\Sigma := \prod_{\rho \in \Sigma_{n+1}} [R_{\rho(0)}, \dots, R_{\rho(n)}].$$

Here and below the iterated commutators are assumed to be left-normalized, namely, $[R_0, R_1, R_2] = [[R_0, R_1], R_2]$, and so on.

We view the circle S^1 as a simplicial set. Milnor's F construction produces a group complex, having in degree n a free group on the degree-n objects of S^1 subject to a single relation ($s_0^n(*) = 1$) and the same boundaries and degeneracies as S^1. According to a formula due to Jie Wu [15, 62], considered in the standard basis of Milnor's $F[S^1]$-construction, homotopy groups of the sphere S^2 can be presented in the following way:

$$\pi_{n+1}(S^2) \simeq \frac{R_0 \cap \dots \cap R_n}{[R_0, \dots, R_n]_\Sigma}.$$

Consider now for $i = 0, \dots, n$ the ideals $\mathfrak{r}_i := (x_i - 1)\mathbb{Z}[F]$ in the free group ring $\mathbb{Z}[F]$, and their symmetric product

$$(\mathfrak{r}_0, \dots, \mathfrak{r}_n)_\Sigma := \sum_{\rho \in \Sigma_{n+1}} \mathfrak{r}_{\rho(0)} \cdots \mathfrak{r}_{\rho(n)},$$

which is also an ideal in $\mathbb{Z}[F]$.

Proposition 3.1. *For $n \geqslant 3$ we have $R_0 \cap \dots \cap R_n \leqslant F \cap (1 + (\mathfrak{r}_0, \dots, \mathfrak{r}_n)_\Sigma)$ when considered in $\mathbb{Z}[F]$.*

Proof. It is shown in [44] that the quotient $\frac{\mathfrak{r}_0 \cap \dots \cap \mathfrak{r}_n}{(\mathfrak{r}_0, \dots, \mathfrak{r}_n)_\Sigma}$ can be viewed as the nth homotopy group of the simplicial abelian group $\mathbb{Z}[F[S^1]]$, and the map $F \to \mathbb{Z}[F]$ given by $f \mapsto f - 1$ induces the following commutative diagram

$$\begin{array}{ccc} \dfrac{R_0 \cap \dots \cap R_n}{[R_0, \dots, R_n]_\Sigma} & \longrightarrow & \dfrac{\mathfrak{r}_0 \cap \dots \cap \mathfrak{r}_n}{(\mathfrak{r}_0, \dots, \mathfrak{r}_n)_\Sigma} \\ \| & & \| \\ \pi_{n+1}(S^2) = \pi_n(\Omega S^2) & \longrightarrow & H_n(\Omega S^2). \end{array}$$

The lower map is the nth Hurewicz homomorphism for the loop space ΩS^2. As all homotopy groups $\pi_n(\Omega S^2)$ are finite for $n \geqslant 3$, but all homology groups $H_n(\Omega S^2)$ are infinite cyclic ($H_*(\Omega S^2)$ is the tensor algebra generated by the homology of S^1 in dimension one [3]), we conclude that, for $n \geqslant 3$, the map in the above diagram is zero. □

3.2 | Lie algebras

One obtains an analogous picture in the case of Lie algebras over $\mathbb{Z}$. The homotopy groups of the simplicial Lie algebra

$$L[S^1] := \bigoplus_i \gamma_i(F[S^1])/\gamma_{i+1}(F[S^1])$$

are equal to the direct sum of terms in rows of the E^1-term of the Curtis spectral sequence

$$E^1_{i,j} := \pi_j\big(\gamma_i(F[S^1])/\gamma_{i+1}(F[S^1])\big) \Longrightarrow \pi_{j+1}(S^2).$$

The mod-p-lower central series spectral sequence is well-studied, see, for example, the foundational paper [7]. The integral case that we consider here has similar properties, see [6, 33]. Here we will only need elementary properties of this spectral sequence.

Observe that the E^1-page of the above spectral sequence consists of derived functors $\mathbb{L}_j$ in the sense of Dold–Puppe, applied to Lie functors: if $\mathscr{L}^i$ denotes the ith Lie functor in the category of abelian groups, then

$$\pi_j\big(\gamma_i(F[S^1])/\gamma_{i+1}(F[S^1])\big) = \mathbb{L}_j\mathscr{L}^i(\mathbb{Z}, 1).$$

Recall the definition of derived functors. Let B be an abelian group, and let F be an endofunctor on the category of abelian groups. For every $i, n \geqslant 0$ the derived functors of F in the sense of Dold–Puppe [14] are defined by

$$\mathbb{L}_iF(B, n) = \pi_i(FKP_*[n]),$$

where $P_* \to B$ is a projective resolution of B, and K is the Dold–Kan transform, inverse to the Moore normalization functor from simplicial abelian groups to chain complexes. We denote by $\mathbb{L}F(B, n)$ the object $FK(P_*[n])$ in the homotopy category of simplicial abelian groups determined by $FK(P_*[n])$, so that $\mathbb{L}_iF(B, n) = \pi_i(\mathbb{L}F(B, n))$.

Consider a free Lie algebra L over $\mathbb{Z}$ with generators $x_0, \dots, x_n$ and relation $x_0 + \cdots + x_n = 0$, and the Lie ideals

$$I_i := \langle x_i \rangle^L \text{ for } i = 0, \dots, n.$$

Define their symmetric product by

$$[I_0, \dots, I_n]_\Sigma = \sum_{\rho \in \Sigma_{n+1}} [I_{\rho(0)}, \dots, I_{\rho(n)}].$$

The same arguments as in the group case imply the Lie analog of Wu's formula:

$$\frac{I_0 \cap \cdots \cap I_n}{[I_0, \dots, I_n]_\Sigma} \simeq \bigoplus_{i \geqslant 1} E^1_{i,n} = \bigoplus_{i \geqslant 1} \mathbb{L}_n\mathscr{L}^i(\mathbb{Z}, 1).$$

In fact, but we shall not need this, each term $E^1_{i,n}$ may be singled out by filtering L via its lower central series: one has

$$\frac{I_0 \cap \cdots \cap I_n \cap \gamma_i(L)}{([I_0, \dots, I_n]_\Sigma \cap \gamma_i(L)) + (I_0 \cap \cdots \cap I_n \cap \gamma_{i+1}(L))} \simeq E^1_{i,n}.$$

Consider the universal enveloping algebra $U(L)$, the corresponding ideals $\mathfrak{i}_i := x_i U(L)$ in $U(L)$, and their symmetric product:

$$(\mathfrak{i}_0, \dots, \mathfrak{i}_n)_\Sigma = \sum_{\rho \in \Sigma_{n+1}} \mathfrak{i}_{\rho(0)} \cdots \mathfrak{i}_{\rho(n)}.$$

Proposition 3.2. *For $n \geqslant 3$, we have $I_0 \cap \cdots \cap I_n \leqslant L \cap (\mathfrak{i}_0, \dots, \mathfrak{i}_n)_\Sigma$ when considered in the universal enveloping algebra.*

Proof. Similarly to the group case, the natural map $L \to U(L)$ induces

$$\begin{array}{ccc} \dfrac{I_0 \cap \cdots \cap I_n}{[I_0, \dots, I_n]_\Sigma} & \longrightarrow & \dfrac{\mathfrak{i}_0 \cap \cdots \cap \mathfrak{i}_n}{(\mathfrak{i}_0, \dots, \mathfrak{i}_n)_\Sigma} \\ \| & & \| \\ \bigoplus_{i \geq 1} E^1_{i,n} & \longrightarrow & H_n\big(U(L[S^1])\big). \end{array}$$

By [53], the $E^1_{i,j}$-terms of the lower central series spectral sequence for S^2 are finite for all $j \geqslant 3$, while the universal enveloping simplicial algebra $U(L[S^1])$ has infinite cyclic homology groups in all dimensions. It follows that the map is 0. □

4 | PROOF OF THEOREM 1.2

We begin with the proof of Theorem 1.2 on Lie algebras, as it is slightly simpler, while conceptually similar, to the corresponding statement for groups. Let an integer $s \geqslant 3$ be fixed throughout this section.

First claim: ι exists and is injective

The assignment $w(x_0, \dots, x_s) \mapsto e^n w(x_0, \dots, x_s)$ naturally defines a linear map $\iota : L_s \to A$, as L_s's only relator holds in A. Furthermore, A is an iterated amalgamated free product, to wit start with $\langle x_i (0 \leqslant i \leqslant s) \mid x_0 + \cdots + x_s = 0 \rangle$ and repeatedly amalgamate, for $i = 0, \dots, s$, with $\langle y_{i,j} (1 \leqslant j \leqslant \ell + c_i) \rangle$ along a 1-dimensional subalgebra, so ι is injective by the normal form of amalgamated free products, see [5, Theorem 4.4.2].

Second claim: $\iota([I_0, \dots, I_s]_\Sigma) \leqslant \gamma_n(A)$

Recall $n = \deg(y_0) + \cdots + \deg(y_s)$. Then

$$\begin{aligned} \iota([I_0, \dots, I_s]_\Sigma) &\leqslant e^n [\langle x_0 \rangle^A, \dots, \langle x_s \rangle^A]_\Sigma = [\langle e^{c_0} x_0 \rangle^A, \dots, \langle e^{c_s} x_s \rangle^A]_\Sigma \\ &= [\langle y_0 \rangle^A, \dots, \langle y_s \rangle^A]_\Sigma \leqslant [\gamma_{c_0}(A), \dots, \gamma_{c_s}(A)]_\Sigma \leqslant \gamma_n(A). \end{aligned}$$

Third claim: $\iota(I_0 \cap \cdots \cap I_s) \leqslant \delta_n(A)$

$$\begin{aligned}\iota(I_0 \cap \cdots \cap I_s) &= \iota(L_s \cap (\mathfrak{i}_0, \dots, \mathfrak{i}_s)_\Sigma) \text{ by Proposition 3.2}\\ &\leqslant A \cap e^n(\langle x_0\rangle^{U(A)}, \dots, \langle x_s\rangle^{U(A)})_\Sigma = A \cap (\langle e^{c_0}x_0\rangle^{U(A)}, \dots, \langle e^{c_s}x_s\rangle^{U(A)})_\Sigma\\ &= A \cap (\langle y_0\rangle^{U(A)}, \dots, \langle y_s\rangle^{U(A)})_\Sigma \leqslant A \cap (\varpi^{c_0}, \dots, \varpi^{c_s})_\Sigma \leqslant \delta_n(A).\end{aligned}$$

It follows that ι induces a map $\bar{\iota}: (I_0 \cap \cdots \cap I_s)/[I_0, \dots, I_s]_\Sigma \to \delta_n(A)/\gamma_n(A)$, which is a homomorphism because its domain (and range) are abelian.

Fourth claim: $\bar{\iota}$ is injective

It is time to specify more precisely the admissible parameters in the construction of A. The parameter e is the exponent of $\bigoplus_i E^1_{i,s}$, or any multiple thereof. By the Curtis connectivity theorem [13], or more precisely its variant for Lie algebras, there is an integer k such that $\gamma_k(F_s) \cap I_0 \cap \cdots \cap I_s \leqslant [I_0, \dots, I_s]_\Sigma$: let us quickly sketch the argument. For any connected free simplicial group F, consider the associated Lie algebra $L(F) = \bigoplus_i L^i(F_{\mathrm{ab}})$. Now $\gamma_n(L(F)) = \bigoplus_{i \geqslant n} L^i(F_{\mathrm{ab}})$. By Curtis' theorem, $L^i(F_{\mathrm{ab}})$ is $\log_2 i$-connected, namely $\pi_s L^i(F_{\mathrm{ab}}) = 0$ for all $s \leqslant \log_2 i$. Therefore, fixing s, we get $\gamma_k(L) \cap I_0 \cap \cdots \cap I_s \cap \leqslant [I_0, \dots, I_s]_\Sigma$ for all $k \geqslant 2^s$. We apply this to $F = F[S^1]$ and its Lie algebra $L[S^1]$.

We may choose the parameters $c_0, \dots, c_s$ arbitrarily so long as $c_0 \geqslant k$ and $c_i \geqslant kc_{i-1}$ for all $i = 1, \dots, s$. To fix matters, let us choose $c_i = k^{i+1}$.

As ι is injective, we are to prove $\iota^{-1}(\gamma_n(A)) \cap I_0 \cap \cdots \cap I_s \leqslant [I_0, \dots, I_s]_\Sigma$. We note that $I_i \cap eL_s = eI_i$ for all i, so $I_0 \cap \cdots \cap I_s \cap eL_s = e(I_0 \cap \cdots \cap I_s)$. By the choice of k and e, it therefore suffices to prove

$$\iota^{-1}(\gamma_n(A) \cap I_0 \cap \cdots \cap I_s) \leqslant [I_0, \dots, I_s]_\Sigma + eL_s + \gamma_k(L_s).$$

Consider the free graded Lie algebra $M = \langle x_0, \dots, x_s, y_0, \dots, y_s\rangle$, in which y_i has degree c_i: it admits a natural surjection $\pi : M \twoheadrightarrow A$. Given $v \in \langle x_0, \dots, x_s\rangle \leqslant M$, consider the collection of expressions in $\pi^{-1}(\pi(v))$ that are obtained by replacing, in an expression of v, some terms x_i by the corresponding y_i, adjusting appropriately the power of e. We call v *in x-form*, and the corresponding expressions obtained by replacing some x_i by y_i are called in *xy-form*. We also denote by ρ the natural map $e^n\langle x_0, \dots, x_s\rangle \subseteq M \twoheadrightarrow L_s$; we have $\iota \circ \rho = \pi$.

Let us consider $w \in I_0 \cap \cdots \cap I_s$, and assume $\iota(w) \in \gamma_n(A)$. We shall write $w = w_0 + w_1 + w_2$, with $w_0 \in [I_0, \dots, I_s]_\Sigma$ and $w_1 \in eL_s$ and $w_2 \in \gamma_k(L_s)$. Now by assumption $\iota(w)$ may be written as an element $\tilde{v} \in M$, all of whose terms have degree at least n; we express this in two steps: first, $\iota(w)$ gives rise to an x-form $v \in M$ by application of the relation $x_0 + \cdots + x_s = 0$; and then v gives rise to an xy-form $\tilde{v}$ of v by application of the other relations, namely, replacement of x_i by y_i with absorption of e^{c_i} in the coefficient. Indeed it follows from the form of A as an amalgamated free product that the xy-form $\tilde{v}$ may be obtained from w first by selecting an appropriate x-form using the relation $x_0 + \cdots + x_s = 0$, and then converting it to $\tilde{v}$; thus there is a natural bijection between the summands of v and $\tilde{v}$.

As M is graded and free, we may write $\tilde{v}$ in a standard basis of free Lie algebras, such as a selection of left-normed commutators. Let us consider in turn all summands of $\tilde{v}$, a typical one being of the form $\tilde{\theta} := [z_1, \dots, z_\ell]$ with all $z_i \in \{x_0, \dots, x_s, y_0, \dots, y_s\}$; let θ be the corresponding monomial in v.

If $\ell \geqslant k$, we put $\rho(\theta)$ in w_2. We may therefore from now on suppose $\ell < k$. On the other hand, say for $i = 0, \dots, s$ that n_i of the z_j's are the generator y_i, and that n_∞ of the z_j's are in $\{x_0, \dots, x_s\}$; then the weight of $\tilde{\theta}$ is

$$n_\infty + n_0 c_0 + \cdots + n_s c_s \geqslant n = c_0 + \cdots + c_s. \tag{4.1}$$

Combining $0 \leqslant n_i < k$ for all i with $c_i \geqslant k c_{i-1}$ and $c_0 \geqslant k$, we see by unicity of the base-k representation of an integer that either $n_0 = \cdots = n_s = 1$ or $n_0 c_0 + \cdots + n_s c_s > n$.

In the former case, each of the $y_0, \dots, y_s$ in $\tilde{\theta}$ may be replaced by the corresponding $x_0, \dots, x_s$ to produce a monomial θ in v with coefficient multiplied by $e^{c_0 + \cdots + c_s}$; and then this summand belongs to $\iota([I_0, \dots, I_s]_\Sigma)$. Add $\rho(\theta)$ to w_0.

In the latter case, replace again all y_i in $\tilde{\theta}$ by the corresponding x_i to produce the monomial θ in v with coefficient multiplied by $e^{n_0 c_0 + \cdots + n_s c_s}$. Remembering that its coefficient is divisible by e^{n+1}, add $\rho(\theta)$ to w_1.

We have in this manner expressed w in the required form $w_0 + w_1 + w_2$, concluding the proof that $\bar{\iota}$ is injective.

5 | PROOF OF THEOREM 1.1

The proof of Theorem 1.1 follows closely that of the previous section; the main difference is that the connection between a group and its group ring is not quite at tight as that between an algebra and its universal enveloping algebra. We remedy this issue by adding roots of elements of $\langle x_0, \dots, x_s \rangle$. Note that only finitely many elements need a root, but we added all for simplicity of the argument.

Let an integer $s \geqslant 3$ be fixed throughout this section. We begin by specifying more precisely the admissible parameters in the construction of G. The parameter e is the exponent of $\pi_{s+1}(S^2)$, or any multiple thereof. By [13], there is an integer k such that $\gamma_k(F_s) \cap R_0 \cap \cdots \cap R_s \leqslant [R_0, \dots, R_s]_\Sigma$. We then choose $c_0, \dots, c_s$ as before subject to $c_0, c_i/c_{i-1} \geqslant k$, for example, $c_i = k^{i+1}$.

First claim: ι exists and is injective

The assignment $w(x_0, \dots, x_s) \mapsto w(x_0, \dots, x_s)^{e^n}$ naturally defines a map $\iota\colon F_s \to G$, as F_s's only relator holds in G. Furthermore, G is an iterated amalgamated free product, to wit start with $\langle x_0, \dots, x_s \mid x_0 \cdots x_s = 1 \rangle$ and repeatedly amalgamate, for $i = 0, \dots, s$, with $\langle y_{i,j} (1 \leqslant j \leqslant c_i) \rangle$ along a cyclic subgroup. Then amalgamate, for $w \in \langle x_0, \dots, x_s \rangle$, with $\langle r_w \rangle$ along a cyclic subgroup. By the standard normal form theorem for free products with amalgamation (see [38, Theorem IV.2.6]), the map ι is injective.

Second claim: $\iota([R_0, \dots, R_s]_\Sigma) \leqslant \gamma_n(G)$

The Lie algebra identities $e[x, y] = [ex, y] = [x, ey]$ do not quite hold in groups, but in the presence of sufficient roots a close analogue exists. For g in a group H we denote by g^H the normal closure of $\langle g \rangle$ in H, and by $[g, (c)h]$ the iterated commutator $[g, h, \dots, h]$ with c copies of "h":

Lemma 5.1. *Let H be a group, let c, e be integers, and assume there are elements $g \in \gamma_m(H)$ and $h, v \in H$ with $v^{e^c} = h$. Then for all $d \in \mathbb{N}$ we have*

$$[g, h^{e^d}] \in [g, h]^{e^d H}[g, (c+1)v]^H, \tag{5.1}$$

$$[g, h]^{e^d} \in [g, h^{e^d}]^H[g, (c+1)v]^H. \tag{5.2}$$

Proof. It suffices to prove the statement for $d = 1$, as it may be applied repeatedly to $[g, h^{e^i}]$, respectively, $[g, h]^{e^i}$, for $i = 1, \dots, d$. We thus restrict ourselves to $d = 1$.

The claims are proven by induction on c, the case $c = 0$ being covered by the first line below. For (5.1), write $v_1 = v^{e^{c-1}}$ and note

$$\begin{aligned}[g, h^e] &= [g, h] \cdot [g, h]^h \cdots [g, h]^{h^{e-1}} \in [g, h]^e[g, h, h]^H;\\ \text{and } [g, h, h] &= [g, h, v_1^e] \in [g, h, v_1]^{eH}[g, h, (c)v]^H \text{ by induction}\\ &\leqslant [g, h]^{eH}[g, (c+1)v]^H\end{aligned}$$

as $[g, h, (c)v] \in [g, (c+1)v]^H$. For (5.2), note

$$\begin{aligned}[g, h]^e &\in [g, h^e] \cdot [g, h, h]^H \text{ as before;}\\ \text{and } [g, h, h] &= [g, h, v_1^e] \in [g, h, v_1]^{eH}[g, h, (c)v]^H \text{ by (5.1)}\\ &\leqslant [g, v_1, h]^{eH}[g, (c+1)v]^H \text{ as } v_1, h \text{ commute}\\ &\leqslant [g, v_1, h^e]^H[g, (c+1)v]^H \text{ by induction}\\ &\leqslant [g, h^e]^H[g, (c+1)v]^H.\end{aligned}$$

□

Now as in the Lie ring case we have, recalling $n = \deg(y_0) + \cdots + \deg(y_s)$,

$$\begin{aligned}\iota([R_0, \dots, R_s]_\Sigma) &\leqslant ([x_0^G, \dots, x_s^G]_\Sigma)^{e^{n^2}} = [x_0^{e^{nc_0}G}, \dots, x_s^{e^{nc_s}G}]_\Sigma \cdot \gamma_n(G) \text{ by (5.2)}\\ &= [y_0^G, \dots, y_s^G]_\Sigma \cdot \gamma_n(G) \leqslant [\gamma_{c_0}(G), \dots, \gamma_{c_s}(G)]_\Sigma \cdot \gamma_n(G) \leqslant \gamma_n(G).\end{aligned}$$

Third claim: $\iota(R_0 \cap \cdots \cap R_s) \leqslant \delta_n(R)$

Recall that in any group H, if m is an integer and $h \in H$ then

$$h^m - 1 = (h - 1 + 1)^m - 1 = \sum_{i \geqslant 1} \binom{m}{i} (h-1)^i \in m(h-1) + (h-1)^2\mathbb{Z}H.$$

We extend this identity, in the presence of roots, as follows:

Lemma 5.2. *Let H be a group, let c, e be integers, and assume H contains elements h, v with $v^{e^c} = h$. Then for all $d \in \mathbb{N}$ we have*

$$h^{e^d} - 1 \in e^d(h-1)\mathbb{Z}H + (v-1)^{c+1}\mathbb{Z}H, \tag{5.3}$$

$$e^d(h-1) \in (h^{e^d} - 1)\mathbb{Z}H + (v-1)^{c+1}\mathbb{Z}H. \tag{5.4}$$

Proof. By applying repeatedly the lemma with (h^{e^i}, v^{e^i}) for $i = 0, \dots, d-1$, it is enough to consider $d = 1$.

For (5.3), we proceed by induction on c, noting that the case $c = 0$ follows from the first line. Setting $v_1 = v^{e^{c-1}}$,

$$\begin{aligned} h^e - 1 = (h-1+1)^e - 1 = \sum_{1\leqslant i\leqslant e} \binom{e}{i}(h-1)^i &\in e(h-1) + (h-1)^2\mathbb{Z}H \\ &= e(h-1) + (h-1)(v_1^e - 1)\mathbb{Z}H \\ &\leqslant e(h-1) + (h-1)(e(v_1-1) + (v-1)^c\mathbb{Z}H) \text{ by induction} \\ &\leqslant e(h-1)\mathbb{Z}H + (v-1)^{c+1}\mathbb{Z}H \end{aligned}$$

as $v-1$ divides $v_1 - 1$ and $h-1$ and commutes with them. For (5.4), we also proceed by induction on c:

$$\begin{aligned} e(h-1) = (h^e - 1) - \sum_{2\leqslant i\leqslant e} \binom{e}{i}(h-1)^i &\in (h^e-1) + (h-1)^2\mathbb{Z}H \\ &= (h^e - 1) + (h-1)(v_1^e - 1)\mathbb{Z}H \\ &\leqslant (h^e-1) + (h-1)(e(v_1-1) + (v-1)^c)\mathbb{Z}H \text{ by (5.3)} \\ &\leqslant (h^e-1) + e(h-1)(v_1-1) + (v-1)^{c+1}\mathbb{Z}H \\ &\leqslant (h^e-1)\mathbb{Z}H + (v-1)^{c+1}\mathbb{Z}H \text{ by induction.} \end{aligned}$$

□

We are now ready to prove $\iota(R_0 \cap \cdots \cap R_s) \leqslant \delta_n(G)$. We have

$$\begin{aligned} \iota(R_0 \cap \cdots \cap R_s) - 1 &= \iota(F_s \cap (1 + (\mathfrak{r}_0, \dots, \mathfrak{r}_s)_\Sigma)) - 1 \text{ by Proposition 3.1} \\ &\leqslant (1 + (\langle x_0 - 1\rangle^{\mathbb{Z}G}, \dots, \langle x_s - 1\rangle^{\mathbb{Z}G})_\Sigma)^{e^{n^2}} - 1 \\ &\leqslant e^{n^2}(\langle x_0 - 1\rangle^{\mathbb{Z}G}, \dots, \langle x_s - 1\rangle^{\mathbb{Z}G})_\Sigma + \varpi^n \text{ by (5.3)} \\ &= (\langle e^{nc_0}(x_0 - 1)\rangle^{\mathbb{Z}G}, \dots, \langle e^{nc_s}(x_s - 1)\rangle^{\mathbb{Z}G})_\Sigma + \varpi^n \\ &= (\langle x_0^{e^{nc_0}} - 1\rangle^{\mathbb{Z}G}, \dots, \langle x_s^{e^{nc_s}} - 1)\rangle^{\mathbb{Z}G})_\Sigma + \varpi^n \text{ by (5.4)} \\ &= (\langle y_0 - 1\rangle^{\mathbb{Z}G}, \dots, \langle y_s - 1\rangle^{\mathbb{Z}G})_\Sigma + \varpi^n \\ &\leqslant (\varpi^{c_0}, \dots, \varpi^{c_s})_\Sigma + \varpi^n = \varpi^n. \end{aligned}$$

It follows that ι induces a map $\bar{\iota} \colon (R_0 \cap \cdots \cap R_s)/[R_0, \dots, R_s]_\Sigma \to \delta_n(G)/\gamma_n(G)$, which is a homomorphism because its domain (and range) are abelian.

Fourth claim: $\bar{\iota}$ is injective

The argument is essentially the same as in the Lie algebra case, so we only indicate the differences. We are to prove

$$\iota^{-1}(\gamma_n(G) \cap R_0 \cap \cdots \cap R_s) \leqslant [R_0, \ldots, R_s]_\Sigma \cdot F_s^e \cdot \gamma_k(F_s).$$

We again start with $w \in R_0 \cap \cdots \cap R_s$ and assume $\iota(w) \in \gamma_n(G)$, and write it as $w = w_0 \cdot w_1 \cdot w_2$ with $w_0 \in [R_0, \ldots, R_s]_\Sigma$ and $w_1 \in F_s^e$ and $w_2 \in \gamma_k(F_s)$. Again we write $\iota(w)$ as $\tilde{v}$ in the free group with generators $x_0, \ldots, x_s, y_0, \ldots, y_s$, and let v be the corresponding x-form of $\tilde{v}$. These elements are written as left-normed commutators, more precisely as left-normed commutators of generators if their length is $< k$, and as arbitrary commutators of length $\geqslant k$. As before, those commutators of length $\geqslant k$ are gathered in w_0. For each of the remaining ones in $\tilde{v}$, again the number of generators y_i in them is denoted by n_i, and the number of other generators is denoted by n_∞; and (4.1) still holds.

If all $n_i \geqslant 1$, then we get a term in w_0, while if some $n_i = 0$ then (4.1) is strict, and we consider the equation coming from the exponents. Now the other generators are either x_i or their roots r_{x_i}; let us write $n_\infty = n'_\infty + n''_\infty$ with n'_∞ the number of generators in $\{x_0, \ldots, x_s\}$. Then, in converting a term into its x-form, the exponent gets multiplied by

$$e^{n_0(nc_0)+\cdots+n_s(nc_s)-n''_\infty n} = e^{n(n_0c_0+\cdots+n_sc_s-n''_\infty)} > e^{n^2},$$

as all c_i are divisible by k and $n''_\infty < k$. Therefore, we again get a term in w_1.

6 | PROOF OF THEOREM 1.3

We begin by an analogue of Proposition 3.1. Dropping the “overlines” from our notation, consider the group

$$F_s = \langle x_{0,1}, x_{0,2}, \ldots, x_{s,1}, x_{s,2} \mid x_{0,1} \cdots x_{s,1} = x_{0,2} \cdots x_{s,2} = 1 = \rangle$$

and its normal subgroups $R_i = \langle x_{i,1}, x_{i,2} \rangle^{F_s}$ for $i = 0, \ldots, s$. Consider also the ideal $\mathfrak{r}_i = (R_i - 1)\mathbb{Z}F_s$.

Proposition 6.1. *For $s \geqslant 2$, we have*

$$\text{torsion}\big(\pi_{s+1}(S^2 \vee S^2)\big) \cong \frac{F_s \cap (1 + (\mathfrak{r}_0, \ldots, \mathfrak{r}_n)_\Sigma)}{[R_0, \ldots, R_s]_\Sigma}.$$

Proof. Following [44], the quotient $\frac{\mathfrak{r}_0 \cap \cdots \cap \mathfrak{r}_s}{(\mathfrak{r}_0, \ldots, \mathfrak{r}_s)_\Sigma}$ can be viewed as the sth homotopy group of the simplicial abelian group $\mathbb{Z}[F[S^1 \vee S^1]]$, and the map $F_s \to \mathbb{Z}[F_s]$ given by $f \mapsto f - 1$ induces the following commutative diagram

$$\begin{array}{ccc}
\dfrac{R_0 \cap \cdots \cap R_s}{[R_0, \ldots, R_s]_\Sigma} & \longrightarrow & \dfrac{\mathfrak{r}_0 \cap \cdots \cap \mathfrak{r}_s}{(\mathfrak{r}_0, \ldots, \mathfrak{r}_s)_\Sigma} \\
\Big\| & & \Big\| \\
\pi_{s+1}(S^2 \vee S^2) = \pi_s\big(\Omega(S^2 \vee S^2)\big) & \longrightarrow & H_s\big(\Omega(S^2 \vee S^2)\big).
\end{array}$$

The lower map is the sth Hurewicz homomorphism for the loop space $\Omega(S^2 \vee S^2)$. By [3], the lower right term is the degree-s part of the free associative algebra on two generators (the homology of $S^1 \vee S^1$). The homotopy group $\pi_{s+1}(S^2 \vee S^2)$ is the sum of its torsion and torsion-free part, and the torsion-free part is the degree-s part of the free Lie algebra on two generators (also the homology of $S^1 \vee S^1$). The lower map, on the torsion-free part, is the natural inclusion of the free Lie algebra into the free associative algebra, so the kernel of the lower map coincides with the torsion subgroup of $\pi_{s+1}(S^2 \vee S^2)$. □

Combining Serre's finiteness theorem and Hilton's theorem [29], the torsion of $\pi_{s+1}(S^2 \vee S^2)$ is finite, and in particular has bounded exponent e. We may also apply Curtis's theorem: there is an integer k such that $\gamma_k(F_s) \cap R_0 \cap \cdots \cap R_s = 1$. The proof of Theorem 1.3 then proceeds exactly as that of Theorem 1.1, with Proposition 6.1 used as a replacement of Proposition 3.1.

7 | PROOF OF THEOREM A

Let H be an abelian group of bounded exponent. We begin by recalling Prüfer's "first" theorem [50]: every abelian group of bounded exponent is a direct product of cyclic groups. Now clearly

$$\delta_n\left(\prod_\alpha G_\alpha\right) = \prod_\alpha \delta_n(G_\alpha), \quad \gamma_n\left(\prod_\alpha G_\alpha\right) = \prod_\alpha \gamma_n(G_\alpha),$$

so it suffices to prove Theorem A for cyclic H.

We recall next Hilton's theorem [29]:

$$\pi_{s+1}(S^2 \vee S^2) = \bigoplus_d \pi_{s+1}(S^d) \otimes \mathscr{L}_d(\mathbb{Z}^2).$$

Therefore, in particular every $\pi_{s+1}(S^d)$ is a direct summand of $\pi_{s+1}(S^2 \vee S^2)$.

We finally recall Gray's theorem [20], proving that the exponent bound of Cohen–Moore–Neisendorfer is optimal: arbitrary cyclic groups appear as subgroups of some $\pi_{s+1}(S^d)$.

It follows that for every cyclic group H there is an integer s such that $\pi_{s+1}(S^2 \vee S^2)$ contains a copy of H. We then conclude by Theorem 1.3.

8 | THE SERRE ELEMENT IN $\pi_{2p}(S^2)$

Let p be a prime. In this subsection, we describe explicitly a copy of $\mathbb{Z}/p$ in $\pi_{2p}(S^2)$ due to Serre [55], by computing its (pre)image α_p in the E^1-term of the lower central spectral sequence associated to $F[S^1]$. There is a single $(\mathbb{Z}/p)$-term in dimension $2p-1$ of the spectral sequence

$$\text{p-torsion}\left(\frac{I_0 \cap \cdots \cap I_{2p-1}}{[I_0, \ldots, I_{2p-1}]_\Sigma}\right) = \mathbb{L}_{2p-1}\mathscr{L}^{2p}(\mathbb{Z}, 1) = \mathbb{Z}/p,$$

and α_p will be a generator of this subgroup.

Theorem 8.1. *Let x_i for $i = 0, \dots, 2p-2$ be free generators of a free Lie algebra, and consider the following element*

$$\alpha_p = \sum_{\substack{\rho\in\Sigma_{2p-2} \text{ a } 2^{p-1}\text{-shuffle} \\ \rho(1)<\rho(3)<\cdots<\rho(2p-5)}} (-1)^\rho[[x_{\rho(0)}, x_{2p-2}], [x_{\rho(1)}, x_{2p-2}], [x_{\rho(2)}, x_{\rho(3)}], \dots, [x_{\rho(2p-4)}, x_{\rho(2p-3)}]];$$

the sum is taken over all permutations $(\rho(0), \dots, \rho(2p-3)) \in \Sigma_{2p-2}$ satisfying $\rho(0) < \rho(1), \dots, \rho(2p-4) < \rho(2p-3)$ as well as $\rho(1) < \rho(3) < \cdots < \rho(2p-5)$. Then α_p represents a generator of the p-torsion in $\mathbb{L}_{2p-1}\mathscr{L}^{2p}(\mathbb{Z}, 1)$.

Proof. Consider the free abelian simplicial group $K(\mathbb{Z}, 2)$: it has a single generator σ in degree 2, and its other generators may be chosen to be all iterated degeneracies of σ. We will use the dual notation for generators: for $k > 2$ the free abelian group $K(\mathbb{Z}, 2)_k$ is generated by ordered sequences of two elements

$$(i_1\, i_2) := s_{k-1} \cdots \widehat{s_{i_2}} \cdots \widehat{s_{i_1}} \cdots s_0(\sigma)$$

with $0 \leqslant i_1 < i_2 < k$. For example, $K(\mathbb{Z}, 2)_5$ has generators

$$(0\,1) := s_4s_3s_2(\sigma), (0\,2) := s_4s_3s_1(\sigma), (0\,3) := s_4s_2s_1(\sigma), (0\,4) := s_3s_2s_1(\sigma),$$
$$(1\,2) := s_4s_3s_0(\sigma), (1\,3) := s_4s_2s_0(\sigma), (1\,4) := s_3s_2s_0(\sigma), (2\,3) := s_4s_1s_0(\sigma),$$
$$(2\,4) := s_3s_1s_0(\sigma), (3\,4) := s_2s_1s_0(\sigma).$$

For $n \geqslant 1$, define the functor J^n as the *metabelianization* of the nth Lie functor $\mathscr{L}^n$. For a group A, there is a natural epimorphism

$$\mathscr{L}^p(A) \twoheadrightarrow J^p(A)$$

with kernel generated by Lie brackets of the form $[[*, *], [*, *]]$. The elements of J^p can also be written as linear combinations of Lie brackets, namely as elements of the Lie functor $\mathscr{L}^p$, but there is additional rule that holds in J^p but not hold in $\mathscr{L}^p$ in general:

$$[a_1, a_2, \dots, a_p] = [a_1, a_2, a_{\rho(3)}, \dots, a_{\rho(p)}]$$

for arbitrary a_i and permutation $(\rho(3), \dots, \rho(p))$ of $\{3, \dots, p\}$. For $p = 3$, the functors $\mathscr{L}^3$ and J^3 are equal.

For $n \geqslant 1$, denote by S^n the nth symmetric power functor

$$S^n : \text{Abelian groups} \to \text{Abelian groups}.$$

For a free abelian group A, there is a natural short exact sequence [53, Proposition 3.2]

$$0 \to J^n(A) \to S^{n-1}(A) \otimes A \to S^n(A) \to 0, \tag{8.1}$$

where the left-hand map is given by

$$[b_1, \dots, b_n] \mapsto b_2b_3 \cdots b_n \otimes b_1 - b_1b_3 \cdots b_n \otimes b_2 \text{ for } b_i \in A. \tag{8.2}$$

Applying the functors $J^p \hookrightarrow S^{p-1} \otimes id \twoheadrightarrow S^p$ to the simplicial abelian group $K(\mathbb{Z}, 2n)$, and taking the homotopy groups, we get the long exact sequence

$$\pi_{2pn}\big(S^{p-1}K(\mathbb{Z},2n)\otimes K(\mathbb{Z},2n)\big) \to \mathbb{L}_{2pn}S^p(\mathbb{Z},2n) \to \mathbb{L}_{2pn-1}J^p(\mathbb{Z},2n) \to$$
$$\to \pi_{2pn-1}\big(S^{p-1}K(\mathbb{Z},2n)\otimes K(\mathbb{Z},2n)\big).$$

It follows from [14, p. 307] that the above sequence has the following form:

$$\begin{array}{ccccc} \pi_{2pn}\big(S^{p-1}K(\mathbb{Z},2n)\otimes K(\mathbb{Z},2n)\big) & \longrightarrow & \mathbb{L}_{2pn}S^p(\mathbb{Z},2n) & \longrightarrow & \mathbb{L}_{2pn-1}J^p(\mathbb{Z},2n) \\ \| & & \| & & \| \\ \mathbb{Z} & \xrightarrow{\quad p \quad} & \mathbb{Z} & \twoheadrightarrow & \mathbb{Z}/p. \end{array}$$

By [53, Proposition 4.7], the natural epimorphism $\mathscr{L}^p \twoheadrightarrow J^p$ gives a natural isomorphism of derived functors

$$\mathbb{L}_{2pn-1}\mathscr{L}^p(\mathbb{Z},2n) \xrightarrow{\simeq} \mathbb{L}_{2pn-1}J^p(\mathbb{Z},2n) \simeq \mathbb{Z}/p.$$

Let us first find a simplicial generator of $\mathbb{L}_{2p}S^p(\mathbb{Z},2)$. For this, we observe that the inclusion of the symmetric power into the tensor power $S^p \hookrightarrow \otimes^p$ induces an isomorphism of derived functors

$$\mathbb{L}_{2p}S^p(\mathbb{Z},2) \to \mathbb{L}_{2p}\otimes^p(\mathbb{Z},2).$$

A simplicial generator of $\mathbb{L}_{2p}\otimes^p(\mathbb{Z},2)$ can be given by the Eilenberg–Zilber shuffle-product theorem. Using interchangeably the notation $\rho(i)$ and ρ_i, this is the element

$$\sum_{\rho\in\Sigma_{2p}\text{ a }2^p\text{-shuffle}} (-1)^\rho(\rho_0\,\rho_1)\otimes(\rho_2\,\rho_3)\otimes\cdots\otimes(\rho_{2p-2}\,\rho_{2p-1}).$$

It follows immediately from the definition of 2^p-shuffles that the symmetric group Σ_p, acting by permutation on blocks $\{2i, 2i+1\}$ of size 2, acts on 2^p-shuffles. A generator of $\mathbb{L}_{2p}S^p(\mathbb{Z},2)$ can be chosen by keeping only a single element per Σ_p-orbit, and replacing tensor products by symmetric products:

$$\beta := \sum_{\substack{\rho\in\Sigma_{2p}\text{ a }2^p\text{-shuffle}\\ \rho(1)<\rho(3)<\cdots<\rho(2p-1)}} (-1)^\rho(\rho_0\,\rho_1)\cdot(\rho_2\,\rho_3)\cdots(\rho_{2p-2}\,\rho_{2p-1}).$$

The conditions imply $\rho(2p-1) = 2p-1$. For example, for $p = 3$ we get the element

$$(0\,1)(2\,3)(4\,5) - (0\,1)(2\,4)(3\,5) + (0\,1)(3\,4)(2\,5) - (0\,2)(3\,4)(1\,5) - (0\,2)(1\,3)(4\,5)$$
$$+ (0\,3)(2\,4)(1\,5) - (0\,3)(1\,4)(2\,5) + (1\,2)(0\,3)(4\,5) - (2\,3)(0\,4)(1\,5) - (1\,2)(0\,4)(3\,5)$$
$$+ (1\,2)(3\,4)(0\,5) - (1\,3)(2\,4)(0\,5) + (2\,3)(1\,4)(0\,5) + (0\,2)(1\,4)(3\,5) + (1\,3)(0\,4)(2\,5).$$

Now we lift the element from $S^pK(\mathbb{Z},2)_{2p}$ to $(S^{p-1}K(\mathbb{Z},2)\otimes K(\mathbb{Z},2))_{2p}$ in a standard way:

$$\tilde\beta := \sum_{\substack{\rho\in\Sigma_{2p}\text{ a }2^p\text{-shuffle}\\ \rho(1)<\rho(3)<\cdots<\rho(2p-1)}} (-1)^\rho(\rho_0\,\rho_1)\cdots(\rho_{2p-4}\,\rho_{2p-3})\otimes(\rho_{2p-2}\,\rho_{2p-1}).$$

Observe that we have

$$d_j(i_1\,i_2) = \begin{cases} (i_1\,i_2) & \text{if } i_2 < j, \\ (i_1\,i_2-1) & \text{if } i_1 < j \leqslant i_2, \\ (i_1-1\,i_2-1) & \text{if } j \leqslant i_1 \end{cases}$$

with the understanding that $(i\,i) = 0$, that we use the same notation $(i_1\,i_2)$ for elements of varying degree, and that $d_0(0\,i_2) = 0$ and $d_j(i_1\,i_2) = 0$ if $\deg(i_1\,i_2) = j = i_2 - 1$. Thus, for example, $d_0(0\,4) = d_5(0\,4) = 0$ and $d_1(0\,4) = d_2(0\,4) = d_3(0\,4) = d_4(0\,4) = (0\,3)$ while $d_0(2\,3) = d_1(2\,3) = d_2(2\,3) = (1\,2)$ and $d_3(2\,3) = 0$ and $d_4(2\,3) = d_5(2\,3) = (2\,3)$.

Clearly, $d_0(\tilde\beta) = d_{2p-1}(\tilde\beta) = 0$. If $j < 2p-2$, then we express $\tilde\beta$ as a sum over all possible values of $r := \rho(2p-2)$ (remembering $\rho(2p-1) = 2p-1$) and obtain

$$d_j(\tilde\beta) = \sum_{r=0}^{2p-2} (-1)^r \big(d_j(\cdots) \otimes (r\,2p-1) + (\cdots) \otimes d_j(r\,2p-1)\big).$$

Now the sum in $(\cdots)$ is a symmetric product similar to β, but with $p-1$ instead of p factors, so $(\cdots)$ is exact. The second terms telescope, so we get $d_j(\tilde\beta) = 0$ when $j < 2p-2$. However, $\tilde\beta$ is not a cycle in $S^{p-1}(\mathbb{Z},2) \otimes K(\mathbb{Z},2)$, because $d_{2p-2}(\tilde\beta)$ is not zero: we compute

$$d_{2p-2}(\tilde\beta) = \sum_{\substack{\rho\in\Sigma_{2p} \text{ a } 2^p\text{-shuffle} \\ \rho(1)<\cdots<\rho(2p-3)=2p-2>\rho(2p-2)}} (-1)^\rho (\rho_0\,\rho_1)\ldots(\rho_{2p-4}\,\rho_{2p-3}) \otimes (\rho_{2p-2}\,2p-2).$$

We use the long exact sequence associated with (8.1) to obtain a cycle in $J^p(\mathbb{Z},2)_{2p-1}$. The ascending 2^p-shuffles $(\rho(0),\ldots,\rho(2p-1))$ appearing in the sum can in fact be viewed as 2^{p-1}-shuffles $(\rho(0),\rho(1),\ldots,\rho(2p-6),\rho(2p-5),\rho(2p-4),\rho(2p-2))$ or $(\rho(0),\rho(1),\ldots,\rho(2p-6),\rho(2p-5),\rho(2p-2),\rho(2p-4))$, depending on whether $\rho(2p-2) < \rho(2p-4)$ or not, and in all cases completed by the values $(2p-2, 2p-1)$. Furthermore, these two shuffles come with opposite signs, and can be combined, via (8.2), into

$$\sum_{\substack{\rho\in\Sigma_{2p-2} \text{ a } 2^{p-1}\text{-shuffle} \\ \rho(1)<\cdots<\rho(2p-5)}} (-1)^\rho [(\rho_{2p-3}\,2p-2),(\rho_{2p-4}\,2p-2),(\rho_0\,\rho_1),\ldots,(\rho_{2p-6}\,\rho_{2p-5})]. \tag{8.3}$$

We now consider the simplicial map $K(\mathbb{Z},2) \to \mathscr{L}^2 K(\mathbb{Z},1)$, given by $\sigma \mapsto [s_0(\sigma'), s_1(\sigma')]$, where σ' is the generator of $K(\mathbb{Z},1)_1$; it is a homotopy equivalence of complexes. The abelian group $K(\mathbb{Z},1)_k$ is k-dimensional, with generators

$$x_i := s_k \cdots \widehat{s_i} \cdots s_0(\sigma')$$

for all $0 \leqslant i < k$, and we have $(i_1\,i_2) \mapsto [x_{i_1}, x_{i_2}]$ under this homotopy equivalence. Thus, $\mathscr{L}^*(\mathbb{Z},2)_{2p-1}$ is a free Lie algebra on $2p-1$ generators. There is an induced map

$$\mathscr{L}^p K(\mathbb{Z},2) \to \mathscr{L}^p \circ \mathscr{L}^2 K(\mathbb{Z},1) \to \mathscr{L}^{2p} K(\mathbb{Z},1),$$

which also is a homotopy equivalence of complexes. The image of the element (8.3) is

$$\sum_{\substack{\rho\in\Sigma_{2p-2}\text{ a }2^{p-1}\text{-shuffle}\\ \rho(1)<\cdots<\rho(2p-5)}} (-1)^{\rho}[[x_{\rho(2p-3)},x_{2p-2}],[x_{\rho(2p-4)},x_{2p-2}],[x_{\rho(0)},x_{\rho(1)}],\dots,[x_{\rho(2p-6)},x_{\rho(2p-5)}]].$$

Up to sign and renumbering, this is exactly our element α_p. □

Note that we considered, in the beginning of this section, a free Lie algebra of rank $2p-1$ with $2p$ generators $x_0,\dots,x_{2p-1}$ subject to the relation $\sum x_i = 0$. Any choice of $2p-1$ out of these $2p$ generators yields a free Lie algebra on $2p-1$ generators, and an expression α_p. The point being made is that every such expression involves one of the generators (here x_{2p-2}) twice, and omits another (here x_{2p-1}).

We summarize the properties of the element α_p that will be useful to us as follows.

Proposition 8.2. *For every prime p there is an element $\widetilde{\alpha}_p$ in the free group $\langle x_0,\dots,x_{2p-1} \mid x_0\cdots x_{2p-1}=1\rangle$ with the properties:*

- $\widetilde{\alpha}_p - 1 \in (\mathfrak{r}_0,\dots,\mathfrak{r}_{2p-1})_\Sigma$;
- $\widetilde{\alpha}_p \notin [R_0,\dots,R_{2p-1}]_\Sigma$;
- $\widetilde{\alpha}_p^p \in [R_0,\dots,R_{2p-1}]_\Sigma$.

Furthermore, $\widetilde{\alpha}_p - 1 \in ([\mathfrak{r}_0,\mathfrak{r}_1],\dots,[\mathfrak{r}_{2p-2},\mathfrak{r}_{2p-1}])_\Sigma$, namely in the sum of all p-fold associative products of brackets of $\mathfrak{r}_i$ in any of the $(2p)!$ orderings.

Proof. The first claim follows from Proposition 3.1, as α_p represents an element of $\pi_{2p}(S^2)$. The second claim holds because this element is nontrivial in $\pi_{2p}(S^2)$. The third claim holds because it has order p in $\pi_{2p}(S^2)$. The last claim follows from general facts: $\mathbb{L}_i\mathscr{L}^n(\mathbb{Z},1)=0$ for odd n, and $\mathbb{L}_i\mathscr{L}^{2n}(\mathbb{Z},1)=\mathbb{L}_i\mathscr{L}^n(\mathbb{Z},2)$. □

The same statement holds for Lie algebras; we omit the proof.

Proposition 8.3. *For every prime p there is an element α_p in the free Lie algebra $\langle x_0,\dots,x_{2p-1} \mid x_0+\cdots+x_{2p-1}=0\rangle$ with the properties:*

- $\alpha_p \in (I_0,\dots,I_{2p-1})_\Sigma$;
- $\alpha_p \notin [I_0,\dots,I_{2p-1}]_\Sigma$;
- $p\alpha_p \in [I_0,\dots,I_{2p-1}]_\Sigma$.

Furthermore, $\alpha_p \in ([R_0,R_1],\dots,[R_{2p-2},R_{2p-1}])_\Sigma$.

Example 8.4. Here is an explicit generator of $\pi_4(S^2)=\mathbb{Z}/2$. If we consider $p=2$ in Theorem 8.1, we have only one [2]-shuffle and the element α_2 is $[[x_0,x_2],[x_1,x_2]]$. Reintroducing $x_3=-x_0-x_1-x_2$, we can easily check that $\alpha_2\in(I_0,I_1,I_2,I_3)_\Sigma$:

$$\begin{aligned}\alpha_2 &:= [[x_0,x_2],[x_1,x_2]]\\ &= [x_2,x_3]\cdot[x_0,x_1]+[x_1,x_2]\cdot[x_0,x_3]-[x_0,x_2]\cdot[x_1,x_3].\end{aligned} \tag{8.4}$$

Applying to it the Dynkin idempotent $u\cdot v\mapsto \frac{1}{2}[u,v]$ gives then $2\alpha_2\in[I_0,I_1,I_2,I_3]_\Sigma$.

It is only slightly harder to write a generator of $\pi_4(S^2)$ in the language of groups. We may lift α_2 to $\widetilde{\alpha}_2 \in F$, the free group $\langle x_0, x_1, x_2, x_3 \mid x_0 \cdots x_3\rangle$, as

$$\widetilde{\alpha}_2 = [[x_0, x_2], [x_0 x_1, x_2]],$$

since then the Hall–Witt identities give $\widetilde{\alpha}_2 = [[x_0, x_2], [x_3^{-1}, x_2]^{x_2^{-1}}] = [[x_0, x_2], [x_0, x_2]^{x_1}[x_1, x_2]] = [[x_0, x_2], [x_1, x_2]] \cdot [[x_0, x_2], [[x_0, x_2], x_1^{x_2}]]$ so $\widetilde{\alpha}_2 \in R_0 \cap \cdots \cap R_3$. We have thus produced a nontrivial cycle $\widetilde{\alpha}_2 \in (R_0 \cap R_1 \cap R_2 \cap R_3)/[R_0, R_1, R_2, R_3]_\Sigma$.

Example 8.5. Here is a generator of the 3-torsion in $\pi_6(S^2)$. For $p = 3$, we have six [2,2]-shuffles in Theorem 8.1:

$$\begin{aligned} &(0,1,2,3) \text{ with sign} = 1, && (0,2,1,3) \text{ with sign} = -1,\\ &(0,3,1,2) \text{ with sign} = 1 && (2,3,0,1) \text{ with sign} = 1,\\ &(1,3,0,2) \text{ with sign} = -1, && (1,2,0,3) \text{ with sign} = 1. \end{aligned}$$

The element α_3 representing 3-torsion in $\pi_6(S^2)$ is

$$\begin{aligned} \alpha_3 :=& [[x_0, x_4], [x_1, x_4], [x_2, x_3]] - [[x_0, x_4], [x_2, x_4], [x_1, x_3]]\\ &+ [[x_0, x_4], [x_3, x_4], [x_1, x_2]] + [[x_1, x_4], [x_2, x_4], [x_0, x_3]]\\ &- [[x_1, x_4], [x_3, x_4], [x_0, x_2]] + [[x_2, x_4], [x_3, x_4], [x_0, x_1]]. \end{aligned}$$

It may be expressed as a sum of 30 associative products of the form $\pm[x_a, x_b] \cdot [x_c, x_d] \cdot [x_e, x_f]$ with $\{a, b, c, d, e, f\} = \{0, 1, 2, 3, 4, 5\}$.

Again it is possible (but now with considerably more effort) to lift α_3 to a generator of $\pi_6(S^2)$ in terms of free groups. We return to the notation of simplicial free groups: we consider the free group $F = \langle z_0, \ldots, z_4\rangle$ and normal subgroups $R_0 = \langle z_0\rangle^F$, $R_i = \langle z_{i-1}^{-1} z_i\rangle^F$ for $i \in \{1, \ldots, 4\}$ and $R_5 = \langle z_4\rangle^F$. In other words, we set $z_i := x_0 \cdots x_i$. Here is a lift of α_3 to F that defines a simplicial cycle, that is, which lies in the intersection $R_0 \cap \cdots \cap R_5$: it is the product of the following fourteen elements

$$\begin{aligned} \widetilde{\alpha}_3 =& [[z_0, z_4], [z_2, z_4], [z_1, z_3]^{[z_0,z_4]}]^{-1} \cdot [[z_1, z_4], [z_2, z_4], [z_0, z_3]^{[z_1,z_4]}]\\ &\cdot [[z_1, z_4], [z_2, z_3], [z_0, z_4]^{[z_1,z_4]}]^{-1} \cdot [[z_0, z_4], [z_2, z_3], [z_1, z_4]^{[z_0,z_4]}]\\ &\cdot [[z_2, z_4], [z_0, z_4], [z_1, z_3]^{[z_2,z_4]}] \cdot [[z_2, z_4], [z_1, z_4], [z_0, z_3]^{[z_2,z_4]}]^{-1}\\ &\cdot [[z_2, z_3], [z_1, z_4], [z_0, z_4]^{[z_2,z_3]}] \cdot [[z_2, z_3], [z_0, z_4], [z_1, z_4]^{[z_2,z_3]}]^{-1}\\ &\cdot [[z_3, z_4], [z_1, z_4], [z_0, z_2]^{[z_3,z_4]}] \cdot [[z_3, z_4], [z_0, z_4], [z_1, z_2]^{[z_3,z_4]}]^{-1}\\ &\cdot [[z_3, z_4], [z_2, z_4], [z_0, z_1]^{[z_3,z_4]}]^{-1} \cdot [[z_1, z_4], [z_3, z_4], [z_0, z_2]^{[z_1,z_4]}]^{-1}\\ &\cdot [[z_0, z_4], [z_3, z_4], [z_1, z_2]^{[z_0,z_4]}] \cdot [[z_2, z_4], [z_3, z_4], [z_0, z_1]^{[z_2,z_4]}]. \end{aligned}$$

One can directly check that $\widetilde{\alpha}_3$ defines a simplicial cycle and that modulo the seventh term of the lower central series it represents exactly the element α_3.

Remark 8.6. We have $\pi_6(S^2) = \mathbb{Z}/3 \times \mathbb{Z}/4$, and it is also possible to give an explicit generator of the 4-torsion. In the same notation as above, it is

$$\begin{aligned}
\widetilde{\alpha}_4 = &[[[z_3, z_1], [z_3, z_2]], [[z_4, z_0], [z_4, z_2]]] \\
&\cdot [[[z_4, z_1], [z_4, z_2]], [[z_3, z_0], [z_3, z_2]]] \\
&\cdot [[[[z_4, z_1], [z_4, z_2]], [[z_4, z_0], [z_4, z_2]]], [[z_3, z_2], [z_3, z_1]]] \\
&\cdot [[[z_3, z_1], [z_3, z_2]], [[[z_4, z_2], [z_4, z_0]], [[z_4, z_2], [z_4, z_1]]]] \\
&\cdot [[[z_3, z_2], [z_3, z_1]], [[z_4, z_2], [z_4, z_0]]] \\
&\cdot [[[z_4, z_2], [z_4, z_1]], [[z_3, z_2], [z_3, z_0]]] \\
&\cdot [[[z_4, z_2], [z_3, z_0]], [[z_4, z_2], [z_3, z_1]]] \\
&\cdot [[[z_4, z_1], [z_3, z_2]], [[z_4, z_2], [z_3, z_0]]] \\
&\cdot [[[z_4, z_2], [z_3, z_1]], [[z_4, z_0], [z_3, z_2]]] \\
&\cdot [[[z_4, z_0], [z_3, z_2]], [[z_4, z_1], [z_3, z_2]]] \\
&\cdot [[[z_4, z_2], [z_3, z_1]], [[z_4, z_3], [z_2, z_0]]] \\
&\cdot [[[z_4, z_3], [z_2, z_1]], [[z_4, z_2], [z_3, z_0]]] \\
&\cdot [[[z_4, z_3], [z_2, z_0]], [[z_4, z_1], [z_3, z_2]]] \\
&\cdot [[[z_4, z_0], [z_3, z_2]], [[z_4, z_3], [z_2, z_1]]] \\
&\cdot [[[z_4, z_3], [z_2, z_0]], [[z_4, z_3], [z_2, z_1]]].
\end{aligned}$$

Note that $\widetilde{\alpha}_4^2$ is, up to the symmetric commutator $[R_0, \dots, R_5]_\Sigma$, equal to

$$[[[[z_0, z_1], [z_0, z_2]], [[z_0, z_1], [z_0, z_3]]], [[[z_0, z_1], [z_0, z_2]], [[z_0, z_1], [z_0, z_4]]]]. \tag{8.5}$$

Here is a brief explanation of the origin of $\widetilde{\alpha}_4$. The elements of the E^1-page of the spectral sequence can be coded by generators of lambda-algebra. Serre elements, which we study, correspond to the elements λ_1. The element $\widetilde{\alpha}_4$ corresponds to $\lambda_2\lambda_1$ of the lambda-algebra. The $E^\infty_{*,5}$ column of S^2 has the following nontrivial terms: $E^\infty_{8,5} = \mathbb{Z}/2$ (generator $\lambda_2\lambda_1$), $E^\infty_{6,5} = \mathbb{Z}/3$ (generator λ_1 for $p = 3$), $E^\infty_{16,5} = \mathbb{Z}/2$ (generator λ_1^3). The 4-torsion in $\pi_6(S^2)$ is glued from two terms in E^∞: λ_1^3 and $\lambda_2\lambda_1$. A representative of λ_1^3 is the bracket (8.5), see, for example, [15]. More generally, each λ_i corresponds to an operation on a simplicial group, with λ_1 corresponding (for $p = 2$) to a simple bracketing $u \mapsto [s_0 u, s_1 u]$. Iterating it three times gives (8.5); see [46] for details.

To show that $\widetilde{\alpha}_4$ represents the 4-torsion, we observe first that it is a cycle, namely that it lies in $R_0 \cap \cdots \cap R_5$, and second we show that, modulo $\gamma_9\gamma_8^2$, it represents the element $\lambda_2\lambda_1$ of the simplicial Lie algebra, given as a sum

$$\begin{aligned}
&[[[z_0, z_3], [z_2, z_4]] + [[z_0, z_4], [z_2, z_3]] + [[z_3, z_4], [z_0, z_2]], \\
&\qquad [[z_1, z_3], [z_2, z_4]] + [[z_1, z_4], [z_2, z_3]] + [[z_3, z_4], [z_1, z_2]]] \\
&= -[[[z_3, z_1], [z_3, z_2]], [[z_4, z_0], [z_4, z_2]]]
\end{aligned}$$

$$
\begin{aligned}
&- [[[z_4, z_1], [z_4, z_2]], [[z_3, z_0], [z_3, z_2]]] \\
&+ [[[z_3, z_2], [z_3, z_1]], [[z_4, z_2], [z_4, z_0]]] \\
&+ [[[z_4, z_2], [z_4, z_1]], [[z_3, z_2], [z_3, z_0]]] \\
&+ [[[z_4, z_2], [z_3, z_0]], [[z_4, z_2], [z_3, z_1]]] \\
&+ [[[z_4, z_1], [z_3, z_2]], [[z_4, z_2], [z_3, z_0]]] \\
&+ [[[z_4, z_2], [z_3, z_1]], [[z_4, z_0], [z_3, z_2]]] \\
&+ [[[z_4, z_0], [z_3, z_2]], [[z_4, z_1], [z_3, z_2]]] \\
&+ [[[z_4, z_2], [z_3, z_1]], [[z_4, z_3], [z_2, z_0]]] \\
&+ [[[z_4, z_3], [z_2, z_1]], [[z_4, z_2], [z_3, z_0]]] \\
&+ [[[z_4, z_3], [z_2, z_0]], [[z_4, z_1], [z_3, z_2]]] \\
&+ [[[z_4, z_0], [z_3, z_2]], [[z_4, z_3], [z_2, z_1]]] \\
&+ [[[z_4, z_3], [z_2, z_0]], [[z_4, z_3], [z_2, z_1]]].
\end{aligned}
$$

9 | EXAMPLES

The homotopy classes presented above yielded with relatively little computational effort Lie algebras and groups with p-torsion in some high-degree dimension quotient. Using more computational resources, we were able to find p-torsion in lower degree for $p = 2$ and $p = 3$.

A general simplification (see Propositions 8.2 and 8.3) is that we can start by an element α_p of degree p and not $2p$, by writing generators x_{ij} in place of $[x_i, x_j]$. Indeed all the computations that express α_p as an symmetrized associative product actually take place in $\mathscr{L}_p\mathscr{L}_2(\mathbb{Z}^{2p}) \subset \mathscr{L}_{2p}(\mathbb{Z}^{2p})$. In fact, this amounts to working in Milnor's simplicial construction $F[S^2]$, whose geometric realization is ΩS^3, and in its Lie analog $L[S^2]$. Observe that, for spheres S^d of dimension $d > 3$, as well as of Moore spaces, there is a description of homotopy groups as centers of explicitly defined finitely generated groups [45]. However, these groups are not as easily defined as in the case of S^2, when we quotient by the symmetric commutator. This is why we concentrated on $S^2 \vee S^2$ in this article.

9.1 | $p = 2$

The construction given in the proof of Theorem 1.2 has generators x_0, x_1, x_2 and $x_3 := -x_0 - x_1 - x_2$. The element ω belongs to $\delta_{14}(A) \setminus \gamma_{14}(A)$. It is possible to be a little bit more economical, by keeping the nilpotency degrees of the y_i more under control: the best we could achieve is

$$
\begin{aligned}
A = \langle & x_0, x_1, x_2, x_3, y_0^{(1)}, y_1^{(2)}, y_2^{(2)}, y_3^{(2)} \mid \\
& x_0 + x_1 + x_2 + x_3 = 0,\ x_0 = 2^6 y_0,\ 2^6 x_1 = 2^5 y_1,\ 2^5 x_2 = 2^3 y_2,\ 2^3 x_3 = y_3 \rangle
\end{aligned}
$$

and the element $\omega = [[x_0, x_2], [x_1, x_2]]$. In that Lie algebra, we have $\omega \in \delta_7(A) \setminus \gamma_7(A)$ and $2\omega \in \gamma_7(A)$. This can be checked by hand, or computer using the program `lienq` by Csaba Schneider [54], or its improvement `anq` [1]; see the Appendix.

Rewriting $[x_i, x_j]$ as x_{ij} allows more simplifications; we may consider general presentations of the form

$$A = \langle x_{01}, x_{02}, x_{03}, x_{12}, x_{13}, x_{23}, y_{01}^{(2)}, y_{02}^{(2)}, y_{03}^{(2)}, y_{12}^{(2)}, y_{13}^{(2)}, y_{23}^{(2)} \mid$$
$$x_{01} + x_{02} + x_{03} = -x_{01} + x_{12} + x_{13} = -x_{02} - x_{12} + x_{23} = 0,$$
$$2^{a_{ij}} x_{ij} = 2^{b_{ij}} y_{ij} \text{ for all } i, j\rangle.$$

Suppose that the element ω is $2^n[x_{02}x_{12}]$ and we want to show that it belongs to $\delta_4(A) \setminus \gamma_4(A)$. Using the associative rewriting $[x_{02}, x_{12}] = x_{23}x_{01} + x_{12}x_{03} - x_{02}x_{13}$ from (8.4), we will have $\omega \in \delta_4(A)$ as soon as A has relations of the form $2^n x_{23} = 2^{a_{23}} y_{23}$ and $2^{a_{23}} x_{01} = y_{01}$ and similarly for the other generators. The condition $\omega \notin \gamma_4(A)$ can be checked by a direct calculation, for example, using `anq`.

We may also replace the variables x_{ij} by $2^{b_{ij}} x_{ij}$ for well-chosen b_{ij}. This amounts, essentially, to letting the x_{ij} have degree less than 1. For instance, replacing x_{ij} by $2^{i+j} x_{ij}$ in the example above and simplifying somewhat, we get

$$A = \langle x_{01}, x_{02}, x_{12}, y_{01}^{(2)}, y_{02}^{(2)}, y_{03}^{(2)}, y_{12}^{(2)}, y_{13}^{(2)}, y_{23}^{(2)} \mid$$
$$2^2 x_{01} = y_{01},\ 2^4 x_{02} = y_{02},\ 2^6 x_{12} = y_{12},$$
$$2^6(-x_{01} - 2x_{02}) = 2^6 y_{03},\ 2^5(x_{01} - 4x_{12}) = 2^4 y_{13},\ 2^5(x_{02} + 2x_{12}) = 2^2 y_{23}\rangle$$

with $\omega := 2^5[x_{02}, x_{12}] \in \delta_4(A) \setminus \gamma_4(A)$. We have $\omega \equiv 2^5[x_{01}, x_{02}] + 2^6[x_{01}, x_{12}] + 2^7[x_{02}, x_{12}]$ modulo $\gamma_4(A)$. We may also choose $n \geqslant 4$ and let y_{0j} have degree $n - 2$ for all j, obtaining in this manner examples with 2-torsion in $\delta_n(A)/\gamma_n(A)$.

It is straightforward to convert the example above into a group: it will be

$$G = \langle x_{01}, x_{02}, x_{12}, y_{01}^{(2)}, y_{02}^{(2)}, y_{03}^{(2)}, y_{12}^{(2)}, y_{13}^{(2)}, y_{23}^{(2)} \mid$$
$$x_{01}^4 = y_{01},\ x_{02}^{16} = y_{02},\ x_{12}^{64} = y_{12},$$
$$x_{01}^{-64} x_{02}^{-128} = y_{03}^{64},\ x_{01}^{32} x_{12}^{-128} = y_{13}^{16},\ x_{02}^{32} x_{12}^{64} = y_{23}^{4}\rangle$$

and the element $\omega = [e_0, e_1]^{32}[e_0, e_2]^{64}[e_1, e_2]^{128}$ belongs to $\delta_4(G) \setminus \gamma_4(G)$. Increasing the degree of the y_{0j} leads, for every $n \geqslant 4$, to a group G with 2-torsion in $\delta_n(G)/\gamma_n(G)$. This is essentially Rips's original example (1.1), except that his example contains more relations that make the group finite.

9.2 | $p = 3$

As in the $p = 2$ case, we may construct a Lie algebra with generators x_{ij} as follows:

$$A = \langle x_{ij}, y_{ij}^{(i+j+1)} \text{ for } 0 \leqslant i < j \leqslant 5,$$
$$x_{01} + x_{02} + x_{03} + x_{04} + x_{05} = 0,$$

$$-x_{01}+x_{12}+x_{13}+x_{14}+x_{15}=0,$$
$$-x_{02}-x_{12}+x_{23}+x_{24}+x_{25}=0,$$
$$-x_{03}-x_{13}-x_{23}+x_{34}+x_{35}=0,$$
$$-x_{04}-x_{14}-x_{24}-x_{34}+x_{45}=0,$$
$$3^{i+j}x_{ij}=y_{ij} \text{ for } 0\leqslant i<j\leqslant 5\rangle$$

and the element $\omega = 3^{15}([x_{04},x_{14},x_{23}]-[x_{04},x_{24},x_{13}]+[x_{04},x_{34},x_{12}]+[x_{14},x_{24},x_{03}]-[x_{14},x_{34},x_{02}]+[x_{24},x_{34},x_{01}])$ that belongs to $\delta_{18}(A)\setminus\gamma_{18}(A)$.

Again there is substantial flexibility in this example: the degrees of the y_{ij} may be adjusted, and the last relations may be changed to $3^{a_{ij}}x_{ij}=3^{c_{ij}}y_{ij}$ for well-chosen a_{ij},c_{ij}. The variables x_{ij} themselves may be replaced by $3^{b_{ij}}x_{ij}$ for well-chosen b_{ij}. Finally, some extra linear conditions may be imposed on the variables, such as $x_{02}=x_{13}=x_{15}=x_{24}=x_{34}=0$. After some experimentation, we arrived at the following reasonably small example:

$$\begin{aligned} A = \langle e_0,e_1,e_2,e_3,y_i^{(2)} \text{ for } i\in\{0,\dots,3\}, y_{ij}^{(3)} \text{ for } 0\leqslant i<j\leqslant 3 \mid \\ 3^{2i}e_i=y_i,\ 3^{12-i}e_j+3^{12-j}e_i=3^{12-2i-2j}y_{ij} \text{ for } (i,j)\in\{(0,1),(0,2),(1,3),(2,3)\}\rangle \end{aligned} \tag{9.1}$$

with $\omega=3^9[e_2,e_1,e_0]$.

Proposition 9.1. *For the Lie ring A defined in (9.1) we have $\omega\in\delta_7(A)\setminus\gamma_7(A)$ and $3\omega\in\gamma_7(A)$.*

Proof. Expanding ω associatively, we get

$$\omega = 3^9(e_0e_1e_2-e_0e_2e_1-e_1e_2e_0+e_2e_1e_0).$$

We may rewrite it as

$$\begin{aligned}\omega = &-e_0(3^9e_2+3^{10}e_3)e_1-e_1(3^9e_2+3^{10}e_3)e_0\\ &+e_0(3^9e_1+3^{11}e_3)e_2+e_2(3^9e_1+3^{11}e_3)e_0\\ &+(3^{10}e_0+3^{12}e_2)e_3e_1+e_1e_3(3^{10}e_0+3^{12}e_2)\\ &-(3^{11}e_0+3^{12}e_1)e_3e_2-e_2e_3(3^{11}e_0+3^{12}e_1).\end{aligned}$$

Each of the summands belongs to $\varpi^7(A)$: they are all products of e_k, e_ℓ and $3^{12-i}e_j+3^{12-j}e_i$ for some $\{i,j,k,\ell\}=\{0,1,2,3\}$. The binomial term equals $3^{12-2i-2j}y_{ij}=3^{2k+2\ell}y_{ij}$, so the summand is the product of $3^{2k}e_k$, $3^{2\ell}e_\ell$ and y_{ij}, namely the product of y_k,y_ℓ,y_{ij}, of respective degrees 2,2,3.

To check that ω does not belong to $\gamma_7(A)$ but that 3ω does, we compute nilpotent quotients of A. We did the calculation using two different programs: `lienq` by Csaba Schneider and `LieRing` [35] for `GAP` [18] by Willem de Graaf and Serena Cicalò. □

In the next subsection, we give a direct proof that the associated group has 3-torsion in δ_7/γ_7.

9.3 | A small, finite 3-group G with $\delta_7(G) \neq \gamma_7(G)$

We consider the group G given by the presentation (9.1), namely,

$$\begin{aligned} G = \langle e_0, e_1, e_2, e_3, y_i^{(2)} \text{ for } i \in \{0, \dots, 3\}, y_{ij}^{(3)} \text{ for } 0 \leqslant i < j \leqslant 3 \mid \\ e_i^{3^{2i}} = y_i,\ e_j^{3^{12-i}} e_i^{3^{12-j}} = y_{ij}^{3^{12-2i-2j}} \text{ for } (i,j) \in \{(0,1),(0,2),(1,3),(2,3)\}\rangle \end{aligned} \tag{9.2}$$

with $\omega = [e_2, e_1, e_0]^{3^9}$.

Proposition 9.2. *In the group defined by (9.2), we have $\omega \in \delta_7(G)$.*

Proof. We will use the following well-known identity, which holds for any element $x \in G$ and $d \geqslant 2$:

$$x^d - 1 = \sum_{k=1}^{d} \binom{d}{k} (x-1)^k. \tag{9.3}$$

We compute modulo $\varpi^7(G)$, and from now on write $\equiv$ to mean equivalence modulo $\varpi^7(G)$. We get

$$\begin{aligned} \omega - 1 &\equiv 3^9([e_2,e_1,e_0]-1) \text{ as } [e_2,e_1,e_0] \in \gamma_4(G) \\ &= 3^9[e_1,e_2]e_0^{-1}(([e_2,e_1]-1)(e_0-1) - (e_0-1)([e_2,e_1]-1)) \\ &= 3^9[e_1,e_2]e_0^{-1}\big(e_2^{-1}e_1^{-1}((e_2-1)(e_1-1)-(e_1-1)(e_2-1))(e_0-1) \\ &\quad -(e_0-1)e_2^{-1}e_1^{-1}((e_2-1)(e_1-1)-(e_1-1)(e_2-1))\big); \end{aligned}$$

and as 3^9 is divisible by the product of exponent of e_0, e_1, e_2 modulo $\gamma_2(G)$,

$$\begin{aligned} &\equiv 3^9((e_0-1)(e_1-1)(e_2-1) - (e_0-1)(e_2-1)(e_1-1) \\ &\quad - (e_1-1)(e_2-1)(e_0-1) + (e_2-1)(e_1-1)(e_0-1)). \end{aligned}$$

Let us write $f_i := e_i - 1$ for $i \in \{0,1,2,3\}$. Then, as in the proof of Proposition 9.1, we have

$$\begin{aligned} \omega - 1 \equiv\, & 3^9(f_0f_1f_2 - f_0f_2f_1 - f_1f_2f_0 + f_2f_1f_0) \\ =\, & -f_0(3^9f_2 + 3^{10}f_3)f_1 - f_1(3^9f_2 + 3^{10}f_3)f_0 \\ & + f_0(3^9f_1 + 3^{11}f_3)f_2 + f_2(3^9f_1 + 3^{11}f_3)f_0 \\ & + (3^{10}f_0 + 3^{12}f_2)f_3f_1 + f_1f_3(3^{10}f_0 + 3^{12}f_2) \\ & - (3^{11}f_0 + 3^{12}f_1)f_3f_2 - f_2f_3(3^{11}f_0 + 3^{12}f_1). \end{aligned}$$

Next, using (9.3) we have

$$\begin{aligned} e_i^{3^{12-j}} e_j^{3^{12-i}} - 1 &= (e_i^{3^{12-j}} - 1) + (e_j^{3^{12-i}} - 1) + (e_i^{3^{12-j}} - 1)(e_j^{3^{12-i}} - 1) \\ &= 3^{12-j} f_i + \binom{3^{12-j}}{2} f_i^2 + \binom{3^{12-j}}{3} f_i^3 + \cdots \end{aligned}$$

$$+ 3^{12-i} f_j + \binom{3^{12-i}}{2} f_j^2 + \binom{3^{12-i}}{3} f_j^3 + \cdots$$

$$+ 3^{24-i-j} f_i f_j + \cdots$$

$$= y_{ij}^{3^{12-2i-2j}} - 1 = 3^{12-2i-2j}(y_{ij} - 1) + \binom{3^{12-2i-2j}}{2}(y_{ij} - 1)^2 + \cdots$$

Again using (9.3), the relations $e_i^{3^{2i}} = y_i$ imply $3^{2i} f_i \in \varpi^2$. Now $3^{12-2i-2j}$ divides $\binom{3^{12-j}}{3}/3^{2i}$, so $\binom{3^{12-j}}{2} f_i^2 \in 3^{12-2i-2j}\varpi^3$. Similarly, $3^{12-2i-2j}$ divides $\binom{3^{12-j}}{3}$ and $\binom{3^{12-j}}{4}$ so all terms with a binomial coëfficient belong to $3^{12-2i-2j}\varpi^3 + \varpi^5$. The same holds for all terms in the last two rows. We therefore have

$$3^{12-j} f_i + 3^{12-i} f_j \in 3^{12-2i-2j}\varpi^3 + \varpi^5.$$

We note $3^{12-2i-2j} = 3^{2k+2\ell}$ whenever $\{i, j, k, \ell\} = \{0, 1, 2, 3\}$. Returning to computations modulo ϖ^7, we consider a typical summand $f_k(3^{12-j} f_i + 3^{12-i} f_j) f_\ell$ in our decomposition of $\omega - 1$. We write $3^{12-j} f_i + 3^{12-i} f_j = 3^{12-2i-2j} u + v$ with $u \in \varpi^3, v \in \varpi^5$ to get

$$f_k(3^{12-j} f_i + 3^{12-i} f_j) f_\ell = f_k(3^{2k+2\ell} u + v) f_\ell = (3^{2k} f_k) u (3^{2\ell} f_\ell) + f_k v f_\ell,$$

where each summand belongs to ϖ^7. □

Proposition 9.3. *In the group defined by (9.2), the element ω defined above does not belong to $\gamma_7(G)$, but its cube does.*

Proof. The proof is computer-assisted. It suffices to exhibit a quotient $\overline{G}$ of G in which the image of ω does not belong to $\gamma_7(\overline{G})$ but its cube does, and we shall exhibit a finite 3-group as quotient.

To make the computations more manageable, we replace the generators y_i and y_{ij} by generators $z_0, \dots, z_3$, and impose the choices

$$y_0 = [z_0, z_1], \quad y_1 = [z_0, z_2], \quad y_2 = [z_0, z_3], \quad y_3 = [z_1, z_2],$$

$$y_{01} = 1, \quad y_{02} = [z_1, z_3, z_3], \quad y_{13} = [z_1, z_3, z_1], \quad y_{23} = [z_1, z_3, z_0].$$

In this manner, we obtain an 8-generated group $\langle e_0, \dots, e_3, z_0, \dots, z_3 \rangle$. We next impose extra commutation relations: $[e_2, z_2], [e_3, z_2], [e_1, z_3], [e_2, z_3], [e_3, z_3]$.

We compute a basis of left-normed commutators of length at most 6 in that group; notice that ω may be expressed as $[z_3, z_2, z_3, z_1, z_1, e_3]^{3^5}$, and impose extra relations making γ_6 cyclic and central.

The resulting finite presentation may be fed to the program pq by Eamonn O'Brien [47], to compute the maximal quotient of 3-class 17. This is a group of order 3^{3996}, and can (barely) be loaded in the computer algebra system GAP [18] so as to check (for safety) that the relations of G hold, and that the element ω has a nontrivial image in it.

Finally, the order of the group may be reduced by iteratively quotienting by maximal subgroups of the center that do not contain ω. □

The resulting group, which is the minimal-order 3-group with nontrivial dimension quotient that we could obtain, has order 3^{494}.

It may be loaded in any GAP distribution by downloading the ancillary file 3group.gap to the current directory and running Read("3group.gap"); in a GAP session.

APPENDIX: THE PROGRAM ANQ

We have developed a power computer program to explore nilpotent quotients of finitely presented Lie rings. It is freely available at https://github.com/laurentbartholdi/anq. The first example in Subsection 9.1 is entered by putting the following in a file:

```
< x0, x1, x2, x3, omega, omega2, y0; y1; y2; y3 | x0+x1+x2+x3,
  x0 = 2^7*y0, 2^1*x1 = y1, 2^2*x2 = y2, 2^4*x3 = y3,
  omega := [[x0,x2],[x1,x2]], omega2 := 2*omega >
```

Without going into details: generators are listed, separated by commas (,) or semicolons (;); the degree of a generator is one more than the number of preceding semicolons. Relations are then listed after the |. The last two relations are *aliases*: they define a generator in terms of previously-listed generators. (Being an alias merely speeds up the program.)

anq supports a variety of rings; in particular, finite rings $\mathbb{Z}/p^n\mathbb{Z}$ in which arithmetic is very fast; fixed-precision integers (which abort under overflow); and arbitrary-precision integers (which tend to be quite slow). As for the example above we are interested in 2-torsion, we compile an executable for $p = 2$ and n large:

```
% make nq_l_2_64
```

The "l" means "Lie algebra," and p and n are given separated by underscores (_). Replacing "l" by "g" would compile a group quotient program. If the presentation above was saved in file twotorsion, we could then invoke

```
% ./nq_l_2_64 -N4 -W9 ./twotorsion | grep omega
#      omega |-> 256*a149 + 4*a150 + 252*a152 + 1*a156 + 2*a157
                 + 256*a159 + 1*a161
#     omega2 |->
```

This tells us that, in the maximal quotient of the given Lie algebra of nilpotency class 4 and maximal degree 9, the element ω is nontrivial but its double is trivial. Note that the nilpotency class option "-N4" forces all five-fold iterated commutators to vanish, and also serves as a speedup; as the generators y1,y2,y3 have degree 2,3,4, respectively, the effective nilpotency class of the quotient is at most $1 + 2 + 3 + 4 = 10$, if each yi is written as an $(i + 1)$-fold iterated commutator of degree-1 generators.

We have in this manner verified that ω does not belong to γ_{10}, but that 2ω does belong to γ_{10}. It remains to check, by hand, that ω belongs to δ_{10} to conclude that indeed the example above has 2-torsion in δ_{10}/γ_{10}.

Note that the same verification could have been made by computing with coefficients $\mathbb{Z}/2^{15}$; but the answer with coefficients $\mathbb{Z}/2^{14}$ would have been inconclusive.

The Lie algebra example (9.1) was also checked using anq, as follows:

```
% make nq_l_3_38
% echo '< e0, e1, e2, e3, omega, omega3; y0, y1, y2, y3; y00,
        y01, y02, y03, y12, y13, y23 |
```

```
  3^0*e0 = y0, 3^2*e1 = y1, 3^4*e2 = y2, 3^6*e3 = y3,
  3^12*e1 + 3^11*e0 = 3^10*y01,
  3^12*e2 + 3^10*e0 = 3^8*y02,
  3^11*e3 + 3^9*e1 = 3^4*y13,
  3^10*e3 + 3^9*e2 = 3^2*y23,
  omega := 3^9*[e2,e1,e0], omega3 := 3*omega >' | ./nq_l_3_38 -W6 -
N3 | grep omega
#      omega |-> 2*a323
#     omega3 |->
```

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REFERENCES

1. L. Bartholdi, *ANQ : all nilpotent quotients*, version 5.0.0, 2021. https://github.com/laurentbartholdi/anq
2. L. Bartholdi and I. B. S. Passi, *Lie dimension subrings*, Internat. J. Algebra Comput. **25** (2015), no. 8, 1301–1325. https://doi.org/10.1142/S0218196715500423. MR 3438163
3. R. H. Bott and H. Samelson, *On the Pontryagin product in spaces of paths*, Comment. Math. Helv. **27** (1953), 320–337 (1954). https://doi.org/10.1007/BF02564566. MR 60233
4. M. R. Bridson and A. Haefliger, *Metric spaces of non-positive curvature*, Springer, Berlin, 1999. MR 2000k:53038
5. L. A. Bokut' and G. P. Kukin, *Algorithmic and combinatorial algebra*, Mathematics and its Applications, vol. 255, Kluwer Academic Publishers Group, Dordrecht, 1994. MR 1292459
6. L. Breen and R. Mikhailov, *Derived functors of nonadditive functors and homotopy theory*, Algebr. Geom. Topol. **11** (2011), no. 1, 327–415. https://doi.org/10.2140/agt.2011.11.327. MR 2764044
7. A. K. Bousfield, E. B. Curtis, D. M. Kan, D. G. Quillen, D. L. Rector, and J. W. Schlesinger, *The* mod$-p$ *lower central series and the Adams spectral sequence*, Topology **5** (1966), 331–342. https://doi.org/10.1016/0040-9383(66)90024-3. MR 0199862
8. P. Cartier, *Remarques sur le théorème de Birkhoff-Witt*, Ann. Scuola Norm. Sup. Pisa (3) **12** (1958), 1–4. (French.) MR 0098121 (20 #4583)
9. F. R. Cohen, *On combinatorial group theory in homotopy*, Homotopy theory and its applications (Cocoyoc, 1993), Contemp. Math., vol. 188, American Mathematical Society, Providence, RI, 1995, pp. 57–63. https://doi.org/10.1090/conm/188/02233. MR 1349129
10. F. R. Cohen and J. Wu, *On braid groups, free groups, and the loop space of the 2-sphere*, Categorical decomposition techniques in algebraic topology (Isle of Skye, 2001), Progr. Math., vol. 215, Birkhäuser, Basel, 2004, pp. 93–105. MR 2039761.

11. F. R. Cohen and J. Wu, *Artin's braid groups, free groups, and the loop space of the 2-sphere*, Q. J. Math. **62** (2011), no. 4, 891–921. https://doi.org/10.1093/qmath/haq010. MR 2853222
12. P. M. Cohn, *Generalization of a theorem of Magnus*, Proc. Lond. Math. Soc. (3) **2** (1952), 297–310. https://doi.org/10.1112/plms/s3-2.1.297. MR 0051834
13. E. B. Curtis, *Some relations between homotopy and homology*, Ann. of Math. (2) **82** (1965), 386–413. https://doi.org/10.2307/1970703. MR 184231
14. A. Dold and D. Puppe, *Homologie nicht-additiver Funktoren. Anwendungen*, Ann. Inst. Fourier Grenoble **11** (1961), 201–312. (German, with French summary.) MR 0150183
15. G. Ellis and R. Mikhailov, *A colimit of classifying spaces*, Adv. Math. **223** (2010), no. 6, 2097–2113. https://doi.org/10.1016/j.aim.2009.11.003. MR 2601009
16. R. H. Fox, *Free differential calculus. I. Derivation in the free group ring*, Ann. of Math. (2) **57** (1953), 547–560. MR 0053938 (14,843d)
17. K. T. Chen, R. H. Fox, and R. C. Lyndon, *Free differential calculus. IV. The quotient groups of the lower central series*, Ann. of Math. (2) **68** (1958), 81–95. MR 0102539 (21 #1330)
18. The GAP Group, *GAP : Groups, Algorithms, and Programming*, Version 4.8.10, 2018. https://www.gap-system.org/
19. R. Geoghegan, *Topological methods in group theory*, Graduate Texts in Mathematics, vol. 243, Springer, New York, 2008. MR 2365352
20. B. I. Gray, *On the sphere of origin of infinite families in the homotopy groups of spheres*, Topology **8** (1969), 219–232. https://doi.org/10.1016/0040-9383(69)90012-3. MR 245008
21. M. Gromov, *Asymptotic invariants of infinite groups*, Geometric group theory, vol. 2 (Sussex, 1991), London Math. Soc. Lecture Note Ser., vol. 182, Cambridge University Press, Cambridge, 1993, pp. 1–295. MR 1253544
22. O. Grün, *Über eine Faktorgruppe freier Gruppen I*, Deutsche Mathematik **1** (1936), 772–782.
23. N. Gupta and Y. Kuz'min, *On varietal quotients defined by ideals generated by Fox derivatives*, J. Pure Appl. Algebra **78** (1992), no. 2, 165–172. https://doi.org/10.1016/0022-4049(92)90094-V. MR 1161340
24. N. D. Gupta, *Free group rings*, Contemporary Mathematics, vol. 66, American Mathematical Society, Providence, RI, 1987. MR 895359
25. N. D. Gupta, *Dimension subgroups of metabelian p-groups*, J. Pure Appl. Algebra **51** (1988), no. 3, 241–249. https://doi.org/10.1016/0022-4049(88)90063-1. MR 946575
26. N. D. Gupta, *A solution of the dimension subgroup problem*, J. Algebra **138** (1991), no. 2, 479–490. https://doi.org/10.1016/0021-8693(91)90182-8. MR 1102820
27. N. D. Gupta, *Lectures on dimension subgroups*, Resenhas **2** (1996), no. 3, 263–273. Group rings and related topics (Portuguese) (São Paulo, 1995). MR 1410298
28. N. D. Gupta, *The dimension subgroup conjecture holds for odd order groups*, J. Group Theory **5** (2002), no. 4, 481–491. MR 1931371 (2003m:20019)
29. P. J. Hilton, *On the homotopy groups of the union of spheres*, J. Lond. Math. Soc. **30** (1955), 154–172. https://doi.org/10.1112/jlms/s1-30.2.154. MR 68218
30. R. Huang and J. Wu, *Combinatorics of double loop suspensions, evaluation maps and Cohen groups*, J. Math. Soc. Japan **72** (2020), no. 3, 847–889. https://doi.org/10.2969/jmsj/81678167. MR 4125848
31. S. A. Jennings, *The group ring of a class of infinite nilpotent groups*, Canad. J. Math. **7** (1955), 169–187.
32. M. Lazard, *Sur les groupes nilpotents et les anneaux de Lie*, Ann. École Norm. Sup. (3) **71** (1953), 101–190.
33. D. Leibowitz, *The E^1 term of the lower central series spectral sequence for the homotopy of spaces*, Ph.D. thesis, Brandeis University, 1972.
34. J. Y. Li and J. Wu, *On symmetric commutator subgroups, braids, links and homotopy groups*, Trans. Amer. Math. Soc. **363** (2011), no. 7, 3829–3852. https://doi.org/10.1090/S0002-9947-2011-05339-0. MR 2775829
35. W. de Graaf and S. Cicalò, *LieRing: a GAP package*, version 2.3, 2016. https://gap-packages.github.io/liering/
36. X.-S. Lin, *Power series expansions and invariants of links*, Geometric topology (Athens, GA, 1993), AMS/IP Stud. Adv. Math., vol. 2, American Mathematical Society, Providence, RI, 1997, pp. 184–202. MR 1470727
37. G. Losey, *On dimension subgroups*, Trans. Amer. Math. Soc. **97** (1960), 474–486. https://doi.org/10.2307/1993383. MR 0148754
38. R. C. Lyndon and P. E. Schupp, *Combinatorial group theory*, Springer, Berlin, 1970.
39. W. Magnus, *Beziehungen zwischen Gruppen und Idealen in einem speziellen Ring*, Math. Ann. **111** (1935), no. 1, 259–280. (German.) https://doi.org/10.1007/BF01472217. MR 1512992

40. W. Magnus, *Über Beziehungen zwischen höheren Kommutatoren*, J. Reine Angew. Math. **177** (1937), 105–115. (German.) https://doi.org/10.1515/crll.1937.177.105. MR 1581549
41. W. Magnus, A. Karrass, and D. Solitar, *Combinatorial group theory: presentations of groups in terms of generators and relations*, Interscience Publishers [John Wiley & Sons, Inc.], New York-London-Sydney, 1966. MR 0207802.
42. J. P. May, *Simplicial objects in algebraic topology*, Van Nostrand Mathematical Studies, No. 11, D. Van Nostrand Co., Inc., Princeton, NJ-Toronto, Ont.-London, 1967. MR 0222892.
43. R. Mikhailov and I. B. S. Passi, *Lower central and dimension series of groups*, Lecture Notes in Mathematics, vol. 1952, Springer, Berlin, 2009. MR 2460089
44. R. Mikhailov, I. B. S. Passi, and J. Wu, *Symmetric ideals in group rings and simplicial homotopy*, J. Pure Appl. Algebra **215** (2011), no. 5, 1085–1092. https://doi.org/10.1016/j.jpaa.2010.07.013. MR 2747240
45. R. Mikhailov and J. Wu, *Combinatorial group theory and the homotopy groups of finite complexes*, Geom. Topol. **17** (2013), no. 1, 235–272. https://doi.org/10.2140/gt.2013.17.235. MR 3035327
46. R. Mikhailov, *Homotopy theory of Lie functors*, arXiv:1808.00681, 2018.
47. E. A. O'Brien, *The p-group generation algorithm*, J. Symbolic Comput. **9** (1990), no. 5–6, 677–698. (Computational group theory, Part 1.) https://doi.org/10.1016/S0747-7171(08)80082-X. MR 1075431
48. I. B. S. Passi, *Dimension subgroups*, J. Algebra **9** (1968), 152–182. https://doi.org/10.1016/0021-8693(68)90018-5. MR 0231916
49. I. B. S. Passi, *Group rings and their augmentation ideals*, Lecture Notes in Mathematics, vol. 715, Springer, Berlin, 1979. MR 537126.
50. H. Prüfer, *Untersuchungen über die Zerlegbarkeit der abzählbaren primären Abelschen Gruppen*, Math. Z. **17** (1923), no. 1, 35–61. (German.) https://doi.org/10.1007/BF01504333. MR 1544601
51. E. Rips, *On the fourth integer dimension subgroup*, Israel J. Math. **12** (1972), 342–346. MR 0314988 (47 #3537).
52. F. Röhl, *Review and some critical comments on a paper of Grün concerning the dimension subgroup conjecture*, Bol. Soc. Brasil. Mat. **16** (1985), 17. https://doi.org/10.1007/BF02584798. MR 847113
53. J. W. Schlesinger, *The semi-simplicial free Lie ring*, Trans. Amer. Math. Soc. **122** (1966), 436–442. https://doi.org/10.2307/1994559. MR 0199861
54. C. Schneider, *Computing nilpotent quotients in finitely presented Lie rings*, Discrete Math. Theor. Comput. Sci. **1** (1997), no. 1, 1–16. MR 1471347
55. J.-P. Serre, *Homologie singulière des espaces fibrés. Applications*, Ann. of Math. (2) **54** (1951), 425–505. (French.) https://doi.org/10.2307/1969485. MR 0045386
56. T. Sicking, *Dimension subrings of Lie rings*, Ph.D. thesis, Göttingen University, 2020.
57. J. A. Sjogren, *Dimension and lower central subgroups*, J. Pure Appl. Algebra **14** (1979), no. 2, 175–194. https://doi.org/10.1016/0022-4049(79)90006-9. MR 524186
58. J. R. Stallings, *Quotients of the powers of the augmentation ideal in a group ring*, Knots, groups, and 3-manifolds, Ann. of Math. Studies, No. 84 (Papers dedicated to the memory of R. H. Fox), Princeton University Press, Princeton, NJ, 1975, pp. 101–118. https://doi.org/10.1515/9781400881512-010. MR 0379685
59. K.-I. Tahara, *The fourth dimension subgroups and polynomial maps*, J. Algebra **45** (1977), no. 1, 102–131. https://doi.org/10.1016/0021-8693(77)90365-9. MR 0432763
60. K.-I. Tahara, *The augmentation quotients of group rings and the fifth dimension subgroups*, J. Algebra **71** (1981), no. 1, 141–173. https://doi.org/10.1016/0021-8693(81)90111-3. MR 627430
61. E. Witt, *Treue Darstellung Liescher Ringe*, J. Reine Angew. Math. **177** (1937), 152–160. (German.) https://doi.org/10.1515/crll.1937.177.152. MR 1581553
62. J. Wu, *Combinatorial descriptions of homotopy groups of certain spaces*, Math. Proc. Cambridge Philos. Soc. **130** (2001), no. 3, 489–513. https://doi.org/10.1017/S030500410100487X. MR 1816806
63. J. Wu, *Simplicial objects and homotopy groups*, Braids, Lect. Notes Ser. Inst. Math. Sci. Natl. Univ. Singap., vol. 19, World Sci. Publ., Hackensack, NJ, 2010, pp. 31–181. https://doi.org/10.1142/9789814291415_0002. MR 2605306