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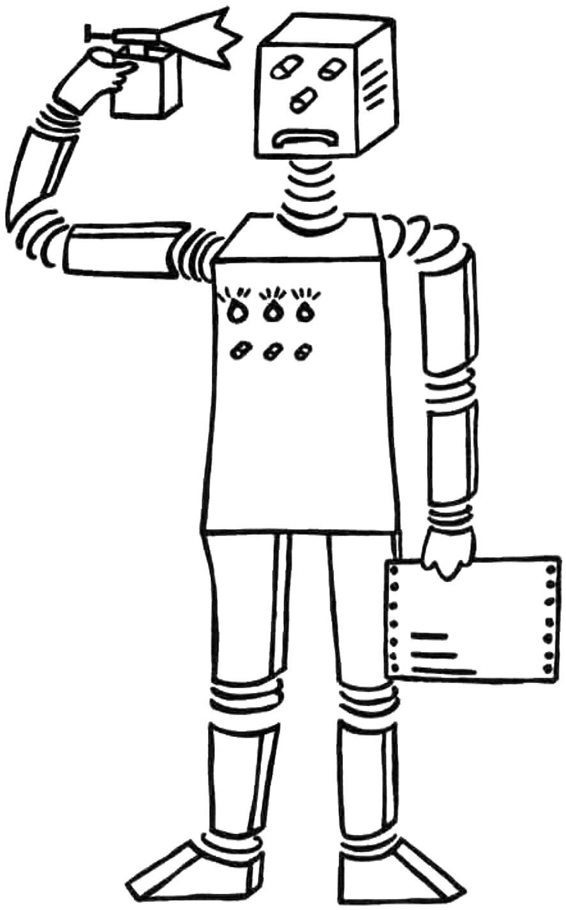

Some Relationships between Unification Restricted Unification and Matching

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Some Relationships between Unification, Restricted Unification, and Matching

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Abstract:
We present restricted **T**-unification that is unification of terms under a given equational theory **T** with the restriction that not all variables are allowed to be substituted. Some relationships between restricted **T**-unification, unrestricted **T**-unification and **T**-matching (one-sided **T**-unification) are established. Our main result is that, in the case of an almost collapse free equational theory the most general restricted unifiers and for certain termpairs the most general matchers are also most general unrestricted unifiers, this does not hold for more general theories. Almost collapse free theories are theories, where only terms starting with projection symbols may collapse (i.e to be **T**-equal) to variables.

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1 Introduction

Restricted **T**-unification is unification of terms under a given equational theory **T**, where only some of their variables are allowed to be substituted. Szabo mentioned this notion to be necessary for writing down some of his proofs more exactly (he used the notation 'partial **T**-unification' [Szabo82]). However, he gave no definitions and he did not investigate the relationships of this notion with common ones. He described how to get unrestricted unification and matching as special cases. But this will only hold for a restricted form of matching, also called semi-unification in [Huet76]. We present the necessary definitions and show some of the relationships between restricted **T**-unification, unrestricted **T**-unification, semi-**T**-unification and **T**-matching. For that purpose we introduce the class of 'almost collapse free' equational theories - a slight generalization of the wellknown collapse free theories [Szabo82, Yelick85, Tiden85, Herold86]. Collapse free theories have the property that no non-variable term is **T**-equal ('collapses') to a variable. In almost collapse free theories special terms starting with a projection symbol - a function symbol f for which the **T**-equation $f(v_1,\ldots,v_i,\ldots,v_n) =_T v_i$ holds - may be **T**-equal to variables. For those theories we show that every most general restricted **T**-unifier, and for termpairs, where one term is ground, also every most general **T**-matcher, is a most general unrestricted **T**-unifier. Some examples demonstrate that this is not the case, if we admit arbitrary collapsing terms.

This result is of high significance in the area of theory classification known under the notion 'unification hierarchy' [Siekmann84]. Equational theories are classified by the cardinality of the most general unifier/matcher sets into unitary, finitary, infinitary and nullary unifying/matching theories. We extend this classification to restricted unification. Then our result above shows that in the case of almost collapse free theories every infinitary (nullary) restricted unifying theory is also infinitary (nullary) unrestricted unifying and every finitary (unitary) unrestricted unifying theory is also finitary (unitary) restricted unifying. These results will especially hold for semi-unification as a special kind of restricted unification. Some examples demonstrate that the classification in the restricted case cannot be obtained from the classification in the unrestricted case by replacing the blocked variables of a unification problem by free constants. There might exist no free constants in the signature and addition of free constants will destroy decidability of unification (see appendix).

There is some closer kinship between almost collapse free and collapse free theories: Every almost collapse free theory can be transformed into a collapse free theory without losing any information about the unification problems. This is an immediate consequence of a result holding for arbitrary theories: All projection symbols can be removed by recursively replacing terms

starting with such function symbols by the argument they are projecting to. In this way we can map every unification problem of a given theory to a problem of a theory containing no projections with essentially the same solution set as the original problem.
These theoretical investigations were triggered by some applications in automatic theorem proving [Loveland78]: Computation of semi-unifier sets from unifier sets are used in several reduction mechanisms of clause graph theorem provers (for example subsumption and replacement factoring in the MKRP-system [Raph84]).

The next section collects the definitions of terms, substitutions and equations. In section3 we define V-restricted unification and matching problems and we show the relationships between the solution sets of these problems. Almost collapse free theories and their kinship to collapse free theories are investigated in section4. The main theorem of this paper, which relates the minimal solution sets in the case of almost collapse free theories, is presented in section5. In the last section we classify the unification problems by the unification hierarchy and we show the relationships between the hierarchy classes. Finally we give an algorithm to compute the restricted unifiers from unrestricted unifiers and point out some applications of this algorithm in automatic theorem proving. In an appendix we demonstrate by an example that in some theories restricted unification is undecidable, while unrestricted unification is decidable.

2 Terms, Substitutions, Equations

In this section we summarize the common algebraic terminology [Grätzer79, Burris&Sankappanavar83] used in unification theory [Siekmann84].

Given a signature $\mathbf{F}=\bigcup \mathbf{F}_n$ of finite sets $\mathbf{F}_n$ of *n-ary function* symbols ($n \geq 0$), and a countable set $\mathbf{V}$ of *variable* symbols, we define the *term algebra* $\mathbf{T}:=\mathbf{T}(\mathbf{F},\mathbf{V})$ to be the least set with

(i) $\mathbf{V}, \mathbf{F}_0 \subseteq \mathbf{T}$ and (ii) $f \in \mathbf{F}_n$ and $t_1,\ldots,t_n \in \mathbf{T} \Rightarrow f(t_1,\ldots,t_n) \in \mathbf{T}$,

together with the usual operations induced by the function symbols.
For any object O consisting of or containing terms, we use the following abbreviations to denote the symbols occurring in this object: $\mathbf{V}(O)$ is the set of its variable symbols, $\mathbf{F}(O)$ is the set of its function symbols.

The *substitution monoid* $\mathbf{\Sigma}:=\mathbf{\Sigma}(\mathbf{F},\mathbf{V})$ of a term algebra is the set of finitely representable endomorphisms on the term algebra being a monoid by the composition operation and the identity:

(i) $\varepsilon\in\mathbf{\Sigma}$ (identity)

(ii) $\sigma,\tau\in\mathbf{\Sigma} \Rightarrow \sigma\tau\in\mathbf{\Sigma}$ (composition)

(iii) $c\in\mathbf{F}_0, f(t_1,\dots,t_n)\in\mathbf{T} \Rightarrow \sigma c=c, \sigma f(t_1,\dots,t_n)=f(\sigma t_1,\dots,\sigma t_n)$ (homomorphism)

(iv) $\sigma\in\mathbf{\Sigma} \Rightarrow \text{card}(\{v\in\mathbf{V}: \sigma v\neq v\})<\infty$. (finite domain)

The set $\text{Dom}\sigma:=\{v\in\mathbf{V}: \sigma v\neq v\}$ is the *domain*, the set $\text{Cod}\sigma:=\{\sigma v\in\mathbf{T}: v\in\text{Dom}\sigma\}$ is the *codomain* of the substitution σ, and $\text{VCod}\sigma:=\mathbf{V}(\text{Cod}\sigma)$ is the set of variables introduced by σ. Every substitution σ can be finitely represented by the set $\{v\leftarrow\sigma v: v\in\text{Dom}\sigma\}$. For $V\subseteq\mathbf{V}$ the *V-restriction* $\sigma|_V$ of $\sigma\in\mathbf{\Sigma}$ is defined by $\sigma|_V v:=\sigma v$ $(v\in V)$ and $\sigma|_V v:=v$ $(v\notin V)$, that is $\text{Dom}\sigma|_V\subseteq V$, $\mathbf{\Sigma}|_V$ denotes the set of all V-restrictions with fixed $V\subseteq\mathbf{V}$. A substitution $\varrho\in\mathbf{\Sigma}$ is called a *V-renaming*, iff $\text{Dom}\varrho=V$, $\text{Cod}\varrho\subseteq\mathbf{V}\setminus V$ and $\varrho v=\varrho w \Rightarrow v=w$ $(\forall v,w\in V)$. The *converse* ϱ^c of a V-renaming ϱ is defined by: $\varrho^c v=w$, if $v=\varrho w$ and $w\in\text{Dom}\varrho$, and $\varrho^c v=v$ otherwise. The set of all V-renamings is written Ren(V).

2.Proposition1: (see [Herold83])

(i) $\varrho\in\text{Ren}(V) \Rightarrow \varrho^c\in\text{Ren}(\varrho V)$

(ii) $\varrho\in\text{Ren}(V) \Rightarrow \varrho\varrho^c=\varrho$ and $\varrho^c\varrho=\varrho^c$.

A finite set $\mathbf{T}:=\{(s , t): s,t\in\mathbf{T}\}$ of term pairs is called an *axiomatization* of an (equational) theory, the elements are called *axioms* The *equational theory* is the least $\mathbf{\Sigma}$-invariant congruence relation $=_T$ on $\mathbf{T}$ containing the set $\mathbf{T}$:

(i) $=_T$ is an equivalence relation with: $(s , t)\in\mathbf{T} \Rightarrow s=_T t$

(ii) $s_1=_T t_1 ,\dots, s_n=_T t_n , f\in\mathbf{F}_n \Rightarrow f(s_1,\dots,s_n)=_T f(t_1,\dots,t_n)$ (congruence)

(iii) $s =_T t , \sigma\in\mathbf{\Sigma} \Rightarrow \sigma s=_T\sigma t$ ($\mathbf{\Sigma}$-invariance)

We only consider *consistent* theories, that are theories that do not collapse into a single equivalence class: $\forall v,w\in\mathbf{V}: v=_T w \Rightarrow v=w$. We frequently call the axiomatization $\mathbf{T}$ itself a theory, and we write $s = t$ instead of (s , t) for the axioms, if no confusion is possible.

This algebraic notion of equational theories is equivalent to the common logical one: the fact that $s=_T t$ holds is known in first order (equational) logics as $s=t$ is *deducible* from $\mathbf{T}$ and abbreviated by $\mathbf{T}\vdash s=t$.

It is wellknown that the relation $=_T$ can be constructed from $\mathbf{T}$ by a closure operator. For arbitrary sets of term pairs $\mathbf{P}$ we define the following operators:

$\mathbf{REFL}(\mathbf{P}):=\mathbf{P}\cup\{(p , p): p\in\mathbf{T}\}$

$\mathbf{SYM}(\mathbf{P}):=\{(q , p): (p , q)\in\mathbf{P}\}$

$\mathbf{TRANS}(\mathbf{P}):=\{(p , q): (p , r),(r , q)\in\mathbf{P}$ for some $r\in\mathbf{T}\}$

$\mathbf{CON}(\mathbf{P}):=\{(f(p_1,\dots,p_n) , f(q_1,\dots,q_n)): (p_i , q_i)\in\mathbf{P}\ (1\le i\le n)$ and $f\in\mathbf{F}_n\ (n\ge 1)\}$

$\mathbf{SUBST}(\mathbf{P}):=\{(\sigma p , \sigma q): (p , q)\in\mathbf{P}$ and $\sigma\in\mathbf{\Sigma}\}$

$\mathfrak{C}(\mathbf{P}) := \mathbf{REFL}(\mathbf{P}) \cup \mathbf{SUM}(\mathbf{P}) \cup \mathbf{TRANS}(\mathbf{P}) \cup \mathbf{CON}(\mathbf{P}) \cup \mathbf{SUBST}(\mathbf{P})$.

With $\mathfrak{C}^0(\mathbf{P}) := \mathbf{P}$ and $\mathfrak{C}^{n+1}(\mathbf{P}) := \mathfrak{C}(\mathfrak{C}^n(\mathbf{P}))$ for each $n \geq 0$, we obtain a representation of the equational theory $=_T$, being useful for induction proofs: The relation $=_T$ is just the union set $\bigcup\{\mathfrak{C}^n(\mathbf{T}): n \geq 0\}$.

2.Proposition2: (see [Burris&Sankappanavar83])

$\forall s,t \in \mathbf{T}:\ s =_T t \Leftrightarrow \exists n \geq 0: (s, t) \in \mathfrak{C}^n(\mathbf{T})$.

Proof: It is easy to see, that $\mathbf{U} := \bigcup\{\mathfrak{C}^n(\mathbf{T}): n \geq 0\}$ is a $\mathbf{\Sigma}$-invariant congruence relation $=_T$ on $\mathbf{T}$ containing the set $\mathbf{T}$. Hence $=_T$ is a subset of $\mathbf{U}$. Conversely, by induction one can easily see, that for each $(s, t) \in \mathfrak{C}^n(\mathbf{T})$, the relation $s =_T t$ holds.

An *equation* $s =_T t$ is *regular*, iff $\mathbf{V}(s) = \mathbf{V}(t)$. Equations $t =_T v$ with $v \in \mathbf{V}$, $t \notin \mathbf{V}$ are called *collapse* equations. A theory is called *regular*, iff all equations are regular. A theory without collapse equations is called *collapse free*. Both properties are inherited from the axiomatization to the whole equational theory [Plonka69, Yelick85, Tiden85, Herold86]:

2.Proposition3:

(i) A theory is regular, iff all axioms are regular.

(ii) A theory is collapse free, iff no axiom is a collapse axiom.

For any $s,t \in \mathbf{T}$, $\sigma,\tau \in \mathbf{\Sigma}$ and $V \subseteq \mathbf{V}$ we define the following relations on terms and substitutions:

(i) σ is $\mathbf{T}$-*equal on V* to τ ($\sigma =_T \tau\ [V]$), iff $\sigma v =_T \tau v\ \forall v \in V$.

(ii) s is a $\mathbf{T}$-*instance* of t ($s \leq_T t$), iff $\exists \lambda \in \mathbf{\Sigma}$ with $s =_T \lambda t$.

(iii) s is $\mathbf{T}$-*equivalent* to t ($s \equiv_T t$), iff $s \leq_T t$ and $s \geq_T t$.

(iv) σ is a $\mathbf{T}$-*instance* of τ *on V* ($\sigma \leq_T \tau\ [V]$), iff $\exists \lambda \in \mathbf{\Sigma}$ with $\sigma =_T \lambda\tau\ [V]$.

(v) σ is $\mathbf{T}$-*equivalent* to τ *on V* ($\sigma \equiv_T \tau\ [V]$), iff $\sigma \leq_T \tau\ [V]$ and $\sigma \geq_T \tau\ [V]$.

If $V = \mathbf{V}$, we drop the suffix $[V]$. The 'instance' relations are reflexive and transitive, and the 'equivalent' relations are in addition symmetric (that is they are equivalences).

2.Proposition4:

Let $\mathbf{T}$ be a theory.

(i) Let $t \in \mathbf{T}$ and $\sigma,\tau \in \mathbf{\Sigma}$ with $\sigma =_T \tau\ [\mathbf{V}(t)]$, then $\sigma t =_T \tau t$.

(ii) Let $\sigma,\tau \in \mathbf{\Sigma}$ and $V \subseteq W \subseteq \mathbf{V}$ with $\mathrm{Dom}\sigma \subseteq V$ and $V\mathrm{Cod}\sigma \cap W = \emptyset$.
Then $\tau \leq_T \sigma\ [V]$ implies $\tau \leq_T \sigma\ [W]$. (Fortsetzungslemma)

Proof: (i) can easily be seen by induction on the term structure. A proof of (ii) is given in [Herold86].

3 Unification, Restricted Unification, Matching

Unification theory is the general theory of solving equations [Huet76, Szabo82, Siekmann84] and as usual we are interested in computing a base of the solution space. Therefore we introduce the notion of the base of a substitution set as the set of its most general elements, that is, we admit only the $\leq_T$-maximal elements.

Let $\Sigma \subseteq \mathbf{\Sigma}$ be any set of substitutions and $W \subseteq \mathbf{V}$. Then $\mu\Sigma$ is a set of ***most general substitutions on W*** of Σ (abbreviated to ***base*** or ***μ-set on W***), iff the following conditions hold:

(B1) $\mu\Sigma \subseteq \Sigma$ (correctness)

(B2) $\forall \delta \in \Sigma\ \exists \sigma \in \mu\Sigma$ with $\delta \leq_T \sigma$ **[W]** (completeness)

(B3) $\forall \sigma, \tau \in \mu\Sigma$: $\sigma \leq_T \tau$ **[W]** $\Rightarrow \sigma = \tau$ (minimality)

The set $\mu\Sigma$ may not exist and if it exists, it is not unique. However, it is unique modulo the equivalence relation $\equiv_T$**[W]** [Huet76, Fages&Huet83].

The solution set of a given term equation is represented by the set of substitutions equalizing both terms.

3.Definition1: (Unification)

For a pair of terms $s,t \in \mathbf{T}$ and a theory $\mathbf{T}$, the set $U\Sigma[s =_T t] := \{\sigma \in \mathbf{\Sigma} : \sigma s =_T \sigma t\}$ is called the set of $\mathbf{T}$-***unifiers*** of s and t or the ***solution set*** of the ***unrestricted unification problem*** $\langle s =_T t \rangle$. The set of ***most general*** $\mathbf{T}$-unifiers $\mu U\Sigma[s =_T t]$ is defined as a base on $W := \mathbf{V}(s,t)$ of $U\Sigma[s =_T t]$.

We always choose a base $\mu U\Sigma[s =_T t]$ with the additional property:

(*) $\forall \sigma \in \mu U\Sigma[s =_T t]$: $Dom\sigma = \mathbf{V}(s,t)$ and $VCod\sigma \cap \mathbf{V}(s,t) = \emptyset$.

This property is only technical and it is always fulfilled by an appropriate μ-set [Huet76, Fage&Huet83]. Some applications require, that the variable disjointness holds for some superset Z of $\mathbf{V}(s,t)$, the set of *protected* variables; this property is known as 'away from Z' in unification theory.

Sometimes one is interested in substituting only into the variables of one side of an equation in order to solve it; for example, to find out, whether a term is an instance of another one. This is called matching.

3.Definition2: (Matching)

For $s,t \in \mathbf{T}$ and a theory $\mathbf{T}$ let $M\Sigma[s \leq_T t] := \{\sigma \in \mathbf{\Sigma} : s =_T \sigma t\}$ be the set of $\mathbf{T}$-***matchers*** of the ***matching problem*** $\langle s \leq_T t \rangle$. Its base on $W := \mathbf{V}(t)$, the set of ***most general*** $\mathbf{T}$-matchers, is denoted by $\mu M\Sigma[s \leq_T t]$.

The technical restriction will be (protected variables should not contain $\mathbf{V}(s)$):

(*) $\forall \mu \in \mu M\Sigma[s \leq_T t]$: $Dom\mu \subseteq \mathbf{V}(t)$ and $VCod\mu \cap (\mathbf{V}(t) \setminus \mathbf{V}(s)) = \emptyset$.

Remark:

A matcher is not a special unifier in general. For example, the substitution $\mu=\{x\leftarrow f(x)\}$ is a matcher of the problem $\langle f(x) \leq_{\emptyset} x\rangle$, but not a unifier of the corresponding unification problem $\langle f(x) =_{\emptyset} x\rangle$: $\mu\in M\Sigma[f(x)\leq_{\emptyset}x]$ but $\mu\notin U\Sigma[f(x)=_{\emptyset}x]$. Here $\boldsymbol{\emptyset}$ denotes the empty theory.

Matching is closely related to semi-unification [Huet76], where the domain of a unifier is not allowed to contain the variables of one of the two terms. This kinship is particularly close for equations, where both sides have disjoint variable sets - especially if one side contains no variables (see 3.Proposition4(iii)). In several applications it is necessary to generalize this concept by restricting the domain of the unifiers to arbitrary subsets V of the variables of the unification problem [Szabo82]. We call this V-restricted unification. This is somewhat similar as if we regard the blocked variables of the problem as constants, a view often proposed in applications. However, this amounts to changes of the signature, which is not convenient and - as our definition shows - not necessary from a theoretical point of view. Moreover, such an extension of the signature might destroy the decidability of the solution problem of **T**-unification (see appendix).

3.Definition3: (Restricted unification)

For a *V-restricted unification problem* $\langle s =_T t\ ,\ V\rangle$ with $V\subseteq\mathbf{V}(s,t)$ the set $U\Sigma|_V[s=_Tt]:=\{\sigma\in\boldsymbol{\Sigma}|_V: \sigma s=_T\sigma t\}=\{\sigma\in\boldsymbol{\Sigma}: \sigma s=_T\sigma t,\ Dom\sigma\subseteq V\}$ is the *V-restricted solution set*, also called the set of *V-restricted* **T**-*unifiers* of $s,t\in\mathbf{T}$. The corresponding base on $W:=\mathbf{V}(s,t)$ is denoted by $\mu U\Sigma|_V[s=_Tt]$, its elements are the *most general* V-restricted **T**-unifiers of s and t. As a special case of V-restriction the problem $\langle s =_T t\ ,\ \mathbf{V}(t)\setminus\mathbf{V}(s)\rangle$ is called a *semi-unification problem*.

Again we require some technical property of the chosen µ-sets:

(*) $\forall\sigma\in\ \mu U\Sigma|_V[s=_Tt]$: $Dom\sigma=V$ and $VCod\sigma\cap V=\emptyset$.

As for unification problems this property might be 'away from Z', but for restricted unification Z must not contain any of the blocked variables.

Remarks:

1. $U\Sigma|_V[s=_Tt]\neq\emptyset$ is the *solvability problem* of V-restricted **T**-unification and $U\Sigma|_{\emptyset}[s=_Tt]\neq\emptyset$ is just the *word problem* of the equational theory **T**.

2. Every semi-unifier is a matcher, but not conversely (see the above example).

3. Sometimes semi-unification is called matching and the more general definition of matching is not used. This seems to be allright for practical applications, because usually the instance problem will arise only for variable disjoint terms.

4. Certain theoretical aspects require the use of another instance relation on the set $U\Sigma|_V[s=_Tt]$ of restricted unifiers, namely

$$\sigma <_T \tau [V] \Leftrightarrow \exists \lambda \in \mathbf{\Sigma} \text{ with } Dom\lambda \cap V^c = \emptyset : \sigma =_T \lambda\tau [V], \text{ where } V^c = \mathbf{V}(s,t).$$

Notice, that there are two differences to the above relation: the first is, that only certain substitutions, not substituting into the blocked variables, are allowed; and the second is, that instantiation is retricted onto the set V. However, it is easy to see that both relations are the same on the set of restricted unifiers: $\forall \sigma,\tau \in U\Sigma|_V[s=_Tt]: \sigma <_T \tau [V] \Leftrightarrow \sigma \le_T \tau [V(s,t)]$.

5. Szabo uses still another instance relation in his matching definition, which is in fact semi-unification: For instantiation only substitutions $\lambda \in \mathbf{\Sigma}$ with $Dom\lambda \cap VCod\lambda = \emptyset$ are allowed. This differs on the set $U\Sigma|_V[s=_Tt]$ from our instance relation, but it will generate the same bases with the technical requirement (*) of 3.Definition3.

The following proposition shows some relationships between these three kinds of solving equations, especially that $\mathbf{V}(s,t)$-restricted unification and unrestricted unification are essentially the same and that semi-unification and matching (for variable disjoint terms) are closely related.

3.Proposition4:

(i) For $V \subseteq W \subseteq \mathbf{V}(s,t)$ we have $U\Sigma|_V[s=_Tt] \subseteq U\Sigma|_W[s=_Tt] \subseteq U\Sigma[s=_Tt]$.

(ii) $U\Sigma|_{\mathbf{V}(s,t)}[s=_Tt]$ is a complete subset of $U\Sigma[s=_Tt]$, thus $\mu U\Sigma|_{\mathbf{V}(s,t)}[s=_Tt]$ is a base of the unrestricted unification problem.

(iii) If $\mathbf{V}(s) \cap \mathbf{V}(t) = \emptyset$, then $\mu U\Sigma|_{\mathbf{V}(t)\setminus\mathbf{V}(s)}[s=_Tt]$ is a complete subset of $M\Sigma[s \le_T t]$.

(iv) If $\mathbf{V}(s) = \emptyset$, then $\mu U\Sigma[s=_Tt]$ is a base of $M\Sigma[s \le_T t]$ on $\mathbf{V}(t) = \mathbf{V}(s,t)$.

Note, that (iii) is less trivial than (ii), because of the different variable sets, the instance relations for the μ-sets are based on.

4 Almost Collapse Free Theories

We introduce the notion of projection equations and of almost collapse free theories, where projection equations are essentially the only collapse equations. We give some useful technical characterization of these theories, and we show that projections are superfluous in unification theory.

4.Definition1: (projections, almost collapse free)

(i) Equations $p^{(i)}(v_1,\dots,v_i,\dots,v_n) =_T v_i$ (for some i with $1 \le i \le n$) with pairwise different $v_1,\dots,v_n \in \mathbf{V}$ are called *projection* equations; in this case $p^{(i)} \in \mathbf{F}_n$ is called a *projection* symbol, *projecting* to the i-th argument. The set of projection symbols induced by **T** is denoted $\mathbf{P}_T$.

(ii) A theory is called ***almost collapse free***, iff the leading function symbol of every collapse equation is a projection symbol.

Remarks:
1. The consistency requirement for theories enforces that a projection symbol can project only to one of its arguments.
2. Every collapse free theory is almost collapse free.

We give an example for an axiomatization of an almost collapse free theory.

4.Example2:
Let $\mathbf{T}:=\{f(g(x)) = g(x)\ ,\ g(x) = x\ ,\ h(f(x)) = x\}$. Then all function symbols are projection symbols, since the equations $f(x)=_T x$, $g(x)=_T x$ and $h(x)=_T x$ can be deduced. Obviously the theory is almost collapse free.

Almost collapse free theories can be characterized by some useful properties of terms and substitutions: Every term that is **T**-equivalent to a variable is **T**-equal to a variable, and every substitution that is **T**-equivalent to a renaming is **T**-equal to a renaming (note, that on their domain renamings are always **T**-equivalent to the identity).

4.Lemma3:
Let **T** be a theory. The following three statements are equivalent:
(i) **T** is almost collapse free.
(ii) $\forall t\in\mathbf{T}$: $t\equiv_T v$ for some $v\in\mathbf{V}$ $\Rightarrow$ $t=_T v'$ for some $v'\in\mathbf{V}$
(iii) $\forall \sigma\in\mathbf{\Sigma}$ with $\mathrm{Dom}\sigma\cap\mathrm{VCod}\sigma=\emptyset$ and $\forall V\subseteq\mathbf{V}$:

$$\sigma\equiv_T\varepsilon\ [V]\ \Rightarrow\ \sigma=_T\varrho\ [W]\ \text{ for some } \varrho\in\mathrm{Ren}(W),\ W:=V\cap\mathrm{Dom}\sigma.$$

Proof: (i) $\Rightarrow$ (ii): Let **T** be an almost collapse free theory. Let $t\equiv_T v$ and $v\in\mathbf{V}$, then there is some $\delta\in\mathbf{\Sigma}$ with $\delta t=_T v$. Either t is a variable, or t has the form $f(t_1,...,t_n)$ and hence $\delta f(t_1,...,t_n)=f(\delta t_1,...\delta t_n)=_T v$. Since **T** is almost collapse free, f must be a projection symbol: $f(v_1,...,v_i,...,v_n)=_T v_i$. We can deduce $f(\delta t_1,...\delta t_i,...\delta t_n)=_T\delta t_i=_T v$, that is $t_i\equiv_T v$. By induction on the nesting depth of the terms we obtain $t=_T v'$ for some $v'\in\mathbf{V}$.

(ii) $\Rightarrow$ (iii): Let **T** have property (ii). Let $\sigma\in\mathbf{\Sigma}$, $\mathrm{Dom}\sigma\cap\mathrm{VCod}\sigma=\emptyset$ and $\sigma\equiv_T\varepsilon\ [V]$, then $\varepsilon\leq_T\sigma\ [V]$ and there is a $\delta\in\mathbf{\Sigma}$ with: $\forall x\in V\ \delta\sigma x=_T x$ and hence $\sigma x=_T v_x$ with some $v_x\in\mathbf{V}\backslash V$, by property (ii). Thus $\forall y\in W:=V\cap\mathrm{Dom}\sigma$: $\sigma y=_T v_y$. We define $\varrho\in\Sigma$ by $\varrho y:=v_y$ for $y\in W$ and $\varrho y:=y$ otherwise. Then $\varrho\in\mathrm{Ren}(W)$:

I. $\mathrm{Cod}\varrho\subseteq\mathbf{V}\backslash V$ by definition.

II. $\mathrm{Dom}\varrho=W$: By the definition of ϱ holds $\mathrm{Dom}\varrho\subseteq W$. Now let $y\in W$. Assume $\varrho y=y$. Then $v_y=y$ and hence $\sigma y=_T y$. The consistency of the theory enforces $y\in\mathbf{V}(\sigma y)\subseteq\mathrm{VCod}\sigma$ or $\sigma y=y$. Both is impossible, since $y\in W\subseteq\mathrm{Dom}\sigma$ and

$Dom\sigma \cap VCod\sigma = \emptyset$. Therefore the assumption was wrong and hence $y \in Dom\varrho$.
III. Let $\varrho x = \varrho y$ for $x, y \in W \Rightarrow v_x = v_y \Rightarrow \sigma x =_T \sigma y \Rightarrow \delta\sigma x =_T \delta\sigma y \Rightarrow x =_T y \Rightarrow x = y$.
By definition $\sigma =_T \varrho$ **[**W**]** holds.

(iii) $\Rightarrow$ (i): Let **T** have property (iii). Let $f(t_1,...,t_n) =_T v$ be any collapse equation. Let $v_1,...,v_n$ be pairwise different variables, not occurring in the collapse equation. Then $f(v_1,...,v_n) \geq_T f(t_1,...,t_n) =_T v$ (take $\lambda := \{v_1 \leftarrow t_1,...,v_n \leftarrow t_n\}$) . Let $\sigma \in \mathbf{\Sigma}$ be represented by $\sigma := \{v \leftarrow f(v_1,...,v_n)\}$. Then $Dom\sigma \cap VCod\sigma = \emptyset$ and $\sigma \equiv_T \varepsilon$ **[**$\{v\}$**]**, since $\lambda\sigma v =_T v$ with the above $\lambda \in \mathbf{\Sigma}$. Property (iii) implies $\sigma =_T \varrho$ **[**$\{v\}$**]**, that is $f(v_1,...,v_n)$ is **T**-equal to a variable. Therefore f is a projection symbol.

We want to show, that these theories are also in some unification theoretical sense 'almost' collapse free: Every almost collapse free theory can be transformed (by a computable mapping on the terms) into a collapse free theory, such that the transformed unification problems have essentially the same bases as the original problems.

Given a theory **T** with the set $\mathbf{P}_T$ of its projection symbols. We define a mapping $': \mathbf{T} \to \mathbf{T}$, $t \mapsto t'$ recursively by

(i) $\forall v \in \mathbf{V}: v' := v$ and $\forall c \in \mathbf{F}_0: c' := c$

(ii) $\forall f \in \mathbf{F}_n \setminus \mathbf{P}_T, \forall t_1,...,t_n \in \mathbf{T}: (f(t_1,...,t_n))' := f(t_1',...,t_n')$ and
$\forall p^{(i)} \in \mathbf{F}_n \cap \mathbf{P}_T, \forall t_1,...,t_n \in \mathbf{T}: (p^{(i)}(t_1,...,t_n))' := t_i'$ $(n \geq 1)$

and we extend this mapping to substitutions by

(iii) $\forall \sigma \in \mathbf{\Sigma}, x \in Dom\sigma: \sigma' x := (\sigma x)'$.

The images of **T** and $\mathbf{\Sigma}$ are denoted **T'** and $\mathbf{\Sigma}'$. Now we apply this mapping to the axiomatization **T** and we obtain the axiomatization

$\mathbf{T}' := \{(l', r') : (l, r) \in \mathbf{T}\}$.

Thus this mapping removes all projection symbols and we may regard the image **T'** of the term algebra **T** under this mapping as a term algebra with the reduced signature $\mathbf{F} \setminus \mathbf{P}_T$. **T'** is then an axiomatization of an equational theory on this reduced term algebra.

The following technical lemma shows more about the relationship between the axiomatizations **T** and **T'**, especially that they are equivalent on the reduced term algebra **T'**, that is, they induce the same equational theory on **T'**.

<u>4.Lemma4:</u>

(i) $\forall t \in \mathbf{T}: t =_T t'$.

(ii) $\forall \sigma \in \mathbf{\Sigma}\ \forall t \in \mathbf{T}: (\sigma t)' = \sigma' t'$.

(iii) $\forall (l', r') \in \mathbf{T}': l' =_T r'$ (i.e. $\mathbf{T} \vdash \mathbf{T}'$), and more general:
$\forall s', t' \in \mathbf{T}': s' =_{T'} t' \Rightarrow s' =_T t'$ (i.e. $\mathbf{T}' \vdash s' = t' \Rightarrow \mathbf{T} \vdash s' = t'$).

(iv) $\forall s, t \in \mathbf{T}: s =_T t \Rightarrow s' =_{T'} t'$ (i.e. $\mathbf{T} \vdash s = t \Rightarrow \mathbf{T}' \vdash s' = t'$).

Proof : (i) and (ii) can easily be derived from the above definition by structural induction on the terms.
(iii) The first part is an immediate consequence of the definition of $\mathbf{T}'$. The second part follows by the closure property of the equational theories: $=_{T'}$ is the least $\mathbf{\Sigma}$-invariant congruence containing $\mathbf{T}'$. The assertion holds also, if we consider $=_{T'}$ as $\mathbf{\Sigma}'$-invariant congruence on the reduced term algebra $\mathbf{T}'$.
(iv) We use induction on the closure construction of an equational theory.
Let $s=_T t$. Then $(s , t)\in\mathbf{C}^n(\mathbf{T})$ for some $n\geq 1$ (by 2.Proposition2).
$n=0$: $(s , t)\in\mathbf{C}^0(\mathbf{T})=\mathbf{T} \Rightarrow (s' , t')\in\mathbf{T}' \Rightarrow s'=_{T'}t'$.
$n\to n+1$: $(s , t)\in\mathbf{C}^{n+1}(\mathbf{T})=\mathbf{C}(\mathbf{C}^n(\mathbf{T}))$.
In order to prove that $s'=_{T'}t'$ holds, we distinguish between the different cases of generating the pair (s , t) from term pairs in $\mathbf{C}^n(\mathbf{T})$ by the five suboperators defining the operator $\mathbf{C}$.
REFL: obvious
SYM: $(t , s)\in\mathbf{C}^n(\mathbf{T}) \Rightarrow t'=_{T'}s' \Rightarrow s'=_{T'}t'$
TRANS: $\exists r\in\mathbf{T}$ with $(s , r),(r , t)\in\mathbf{C}^n(\mathbf{T}) \Rightarrow s'=_{T'}r', r'=_{T'}t' \Rightarrow s'=_{T'}t$
CON: $\exists f\in\mathbf{F}_m$ with $s=f(s_1,\dots,s_m)$, $t=f(t_1,\dots,t_m)$, $(s_i,t_i)\in\mathbf{C}^n(\mathbf{T})$ $(1\leq i\leq m) \Rightarrow s_i'=_{T'}t_i'$ $(1\leq i\leq m)$

Case1: $f\notin\mathbf{P}_T \Rightarrow s'=f(s_1',\dots,s_m')$ and $t'=f(t_1',\dots,t_m') \Rightarrow s'=_{T'}t'$

Case2: $f=p^{(i)}\in\mathbf{P}_T$ for some j $(1\leq j\leq m) \Rightarrow s'=s_j'$ and $t'=t_j' \Rightarrow s'=_{T'}t'$

SUBST: $\exists\sigma\in\mathbf{\Sigma}$ with $s=\sigma s_0$, $t=\sigma t_0$, $(s_0 , t_0)\in\mathbf{C}^n(\mathbf{T}) \Rightarrow s_0=_{T'}t_0 \Rightarrow s'=_{T'}t'$
Hence by induction $s'=_{T'}t'$.
Note, that this proof also holds, if we regard $=_{T'}$ as an equational theory on the term algebra $\mathbf{T}'$ over the reduced signature $\mathbf{F}\backslash\mathbf{P}_T$.

Now we can show that deletion of all projection symbols in this way will not affect the bases of the unifier sets of the unification problems, in other words, projection symbols are superfluous in a unification theoretical sense.
<u>4.Theorem5</u>:
With the above notations we obtain:
(i) $U\Sigma'[s'=_{T'}t']:=\{\sigma'\in\mathbf{\Sigma}' : \sigma's'=_{T'}\sigma't'\}$ is a complete subset of $U\Sigma[s=_T t]$.
(ii) The base $\mu U\Sigma'[s'=_{T'}t']$ of $U\Sigma'[s'=_{T'}t']$ is also a base of $U\Sigma[s=_T t]$.
Proof: (i) Correctness: $\delta'\in U\Sigma'[s'=_{T'}t'] \Rightarrow \delta's'=_{T'}\delta't' \Rightarrow \delta's'=_T\delta't'$ (4.Lemma4(ii)+(iii)) $\Rightarrow$ $\delta's=_T\delta't$ (4.Lemma4(i)) $\Rightarrow \delta'\in U\Sigma[s=_T t]$.
Completeness: $\delta\in U\Sigma[s=_T t] \Rightarrow \delta s=_T\delta t \Rightarrow \delta's'=(\delta s)'=_{T'}(\delta t)'=\delta't'$ (4.Lemma4(ii)+(iv)) $\Rightarrow$ $\delta'\in U\Sigma'[s'=_{T'}t']$. By 4.Lemma4(i)+(ii) we obtain $\delta'x=\delta'x'=(\delta x)'=_T\delta x$ $\forall x\in\mathbf{V}(s,t)$ and hence $\delta\leq_T\delta'$ $[\mathbf{V}(s,t)]$.

(ii) Correctness: $\mu U\Sigma'[s' =_{T'} t'] \subseteq U\Sigma'[s' =_{T'} t'] \subseteq U\Sigma[s =_T t]$.

Completeness: We must show that $\forall \delta \in U\Sigma[s =_T t]\ \exists \sigma' \in \mu U\Sigma'[s' =_{T'} t']: \delta \leq_T \sigma' [\mathbf{V}(s,t)]$.

Let $\delta \in U\Sigma[s =_T t] \Rightarrow \delta' \in U\Sigma'[s' =_{T'} t']$ and $\delta \leq_T \delta' [\mathbf{V}(s,t)]$ (by part (i)) $\Rightarrow \exists \sigma' \in \mu U\Sigma'[s' =_{T'} t']$: $\delta' \leq_{T'} \sigma' [\mathbf{V}(s',t')]$. By the definition of μ sets $Dom\,\sigma' = \mathbf{V}(s',t')$ and $VCod\,\sigma' \cap \mathbf{V}(s,t) = \emptyset$ and by 2.Proposition4(ii) we obtain that $\delta' \leq_{T'} \sigma' [\mathbf{V}(s,t)]$. 4.Lemma4(iii) then implies $\delta' \leq_T \sigma' [\mathbf{V}(s,t)]$. Hence $\delta \leq_T \sigma' [\mathbf{V}(s,t)]$ by the transitivity of $\leq_T [\mathbf{V}(s,t)]$.

Minimality: Let $\sigma', \tau' \in \mu U\Sigma'[s' =_{T'} t']$ and let $\sigma' \leq_T \tau' [\mathbf{V}(s,t)]$. Then $\sigma' \leq_T \tau' [\mathbf{V}(s',t')]$, and with 4.Lemma4(iv) $\sigma' \leq_{T'} \tau' [\mathbf{V}(s',t')]$. By minimality of $\mu U\Sigma'[s' =_{T'} t']$ with respect to $\leq_{T'} [\mathbf{V}(s',t')]$, we obtain $\sigma' = \tau'$. Thus we have shown also minimality with respect to $\leq_T [\mathbf{V}(s,t)]$.

This theorem is a generalization of a result of [Szabo82]. A collapse equation of the form $f(v) =_T v$ with $f \in \mathbf{F}_1$ is called ***monadic***. Szabo shows that monadic collapse equations are superfluous in a model theoretic sense: For every theory $\mathbf{T}$ there is a theory $\mathbf{T}'$ without monadic collapse equations, but with essentially the same models as $\mathbf{T}$, that is they are 'definition-equivalent' (see also [Taylor79] for a more detailed definition of and some literature about these notions). It is easy to see that 4.Theorem5 holds for definition-equivalent theories $\mathbf{T}$ and $\mathbf{T}'$.

Applying 4.Theorem5 to almost collapse free theories we obtain that unification in almost collapse free theories is the same as unification in collapse free theories.

4.Corollary6:

For each almost collapse free theory $\mathbf{T}$ there is a collapse free theory $\mathbf{T}'$, such that $\mathbf{T}$-unification and $\mathbf{T}'$-unification are related as above, that is (i) and (ii) of 4.Theorem5 hold.

The theory of 4.Example2 is a regular theory with monadic collapse equations and it is definition-equivalent to the theory induced by the empty axiomatization. Thus unification in this theory is the same as syntactic unification.

These results can be prescribed with the notions of term rewriting systems: The set of projection equations of a theory can be regarded as a canonical term rewriting system. Then t' denotes just the normal form of a term t with respect to this term rewriting system (see [Huet&Oppen80]).

5 Relationhips Between μ-Sets

In this section we show the main result: In almost collapse free theories the most general V-restricted unifiers are most general unrestricted unifiers. Throughout this section we denote the blocked variables of a V-restricted unification problem by $V^c := \mathbf{V}(s,t)\setminus V$.

First we show, that a solvable unification problem has V-restricted $\mathbf{T}$-unifiers, iff there are some most general unrestricted unifiers being $\mathbf{T}$-equivalent on the blocked variables to the identity, in other words, the substitution of the blocked variables is not essential to solve the problem. We collect these unifiers in the set

$$U_V := \{\sigma \in \mu U\Sigma[s =_T t] : \sigma \equiv_T \varepsilon\ [V^c]\},$$

for $s,t \in \mathbf{T}$ with existing $\mu U\Sigma[s =_T t]$ and for $V \subseteq \mathbf{V}(s,t)$.

5.Lemma1:

Let $s,t \in \mathbf{T}$ with existing $\mu U\Sigma[s =_T t]$ and let $V \subseteq \mathbf{V}(s,t)$. Then:

$$U_V \neq \emptyset \Leftrightarrow U\Sigma|_V[s =_T t] \neq \emptyset.$$

Proof: "⇒" Let $\sigma \in U_V \Rightarrow \sigma \equiv_T \varepsilon\ [V^c] \Rightarrow \exists \lambda \in \mathbf{\Sigma}: \lambda\sigma =_T \varepsilon\ [V^c]$, that is $\lambda\sigma w =_T w\ \forall w \in V^c$ $\Rightarrow (\lambda\sigma)|_V v = \lambda\sigma v\ \ \forall v \in V$ and $(\lambda\sigma)|_V w = w =_T \lambda\sigma w\ \ \forall w \in V^c \Rightarrow (\lambda\sigma)|_V =_T \lambda\sigma\ [\mathbf{V}(s,t)]$. By $\sigma \in \mu U\Sigma[s =_T t]$ and with 2.Proposition4 we get $(\lambda\sigma)|_V s =_T \lambda\sigma s =_T \lambda\sigma t =_T (\lambda\sigma)|_V t$. Hence $(\lambda\sigma)|_V \in U\Sigma|_V[s =_T t]$.

"⇐" Let $\mu \in U\Sigma|_V[s =_T t] \Rightarrow \mu \in U\Sigma[s =_T t] \Rightarrow \exists \sigma \in \mu U\Sigma[s =_T t]: \mu \leq_T \sigma\ [\mathbf{V}(s,t)]$. Hence $\exists \lambda \in \mathbf{\Sigma}$ with $\mu =_T \lambda\sigma\ [\mathbf{V}(s,t)]$. Since $\mu w = w\ \forall w \in V^c$ (remember that $\mathrm{Dom}\mu \subseteq V$), hence $\lambda\sigma w =_T \mu w = w\ \forall w \in V^c$ and hence $\sigma \equiv_T \varepsilon\ [V^c]$. This is $\sigma \in U_V$.

In almost collapse free theories the elements of U_V are $\mathbf{T}$-equal to renamings (4.Lemma3), hence

$$U_V = \{\sigma \in \mu U\Sigma[s =_T t] : \sigma =_T \varrho_\sigma\ [V^c],\ \varrho_\sigma \in \mathrm{Ren}(V^c)\}.$$

In this case we obtain a base of V-restricted unifiers by composing the converses of these renamings with the corresponding unifiers, hence the set

$$U^c := \{(\varrho^c\sigma)|_V : \sigma \in U_V,\ \varrho := \varrho_\sigma\}$$

is a base. This means that in almost collapse free theories the most general unrestricted unifiers differ from the restricted ones by a renaming of the blocked variables.

5.Theorem2:

Let $\mathbf{T}$ be almost collapse free. Let $s,t \in \mathbf{T}$ with existing $\mu U\Sigma[s =_T t]$.

Then $U_V \neq \emptyset$ for $V \subseteq \mathbf{V}(s,t)$ implies U^c is a base of $U\Sigma|_V[s =_T t]$.

Proof: First we have to show the correctness: $U^c \subseteq U\Sigma|_V[s=_T t]$.
Let $(\varrho^c\sigma)|_V \in U^c$. Since σ is a **T**-unifier, hence also $\varrho^c\sigma s =_T \varrho^c\sigma t$. Let $w \in V^c$, then $\varrho^c\sigma w =_T \varrho^c\varrho w = \varrho^c w = w$ (the first equation holds with $\sigma =_T \varrho$ **[**V^c**]**, the second one by 2.Proposition1, and the last one by the definition of ϱ^c: $Dom\varrho^c \cap V^c = \emptyset$). This implies $(\varrho^c\sigma)|_V =_T \varrho^c\sigma$ **[V**(s,t)**]**, and with 2.Proposition4 we get $(\varrho^c\sigma)|_V s =_T (\varrho^c\sigma)|_V t$. Hence $(\varrho^c\sigma)|_V \in U\Sigma|_V[s=_T t]$.
Next we show the completeness: $\forall \mu \in U\Sigma|_V[s=_T t]\ \exists \nu \in U^c$: $\mu \leq_T \nu$ **[V**(s,t)**]**.
Let $\mu \in U\Sigma|_V[s=_T t]$. As in the second proof part of 5.Lemma1 there is some $\sigma \in U_V$, that is $\sigma =_T \varrho$ **[**V^c**]**, with $\mu \leq_T \sigma$ **[V**(s,t)**]**.
With $VCod\sigma \cap Dom\varrho = \emptyset$ and with 2.Proposition1 we get $\sigma = \varrho\sigma = \varrho\varrho^c\sigma$. Obviously $\varrho\varrho^c\sigma \leq_T \varrho^c\sigma$ **[V**(s,t)**]** and as in the correctness part $(\varrho^c\sigma)|_V =_T \varrho^c\sigma$ **[V**(s,t)**]**. Thus with transitivity of the **T**-instance relation $\mu \leq_T (\varrho^c\sigma)|_V$ **[V**(s,t)**]** holds.
Finally we show the minimality of U^c:
Given $(\varrho^c\sigma)|_V, (\varrho'^c\sigma')|_V \in U^c$ with $(\varrho^c\sigma)|_V \leq_T (\varrho'^c\sigma')|_V$ **[V**(s,t)**]**. This implies $\varrho^c\sigma \leq_T \varrho'^c\sigma'$ **[V**(s,t)**]**, i.e. $\exists \lambda \in \mathbf{\Sigma}$: $\varrho^c\sigma =_T \lambda\varrho'^c\sigma'$ **[V**(s,t)**]**, and hence we obtain $\varrho\varrho^c\sigma =_T \varrho\lambda\varrho'^c\sigma'$ **[V**(s,t)**]**. By $\sigma = \varrho\sigma = \varrho\varrho^c\sigma$: $\sigma =_T \varrho\lambda\varrho'^c\sigma'$ **[V**(s,t)**]**, i.e. $\sigma \leq_T \sigma'$ **[V**(s,t)**]**. The minimality of $\mu U\Sigma[s=_T t]$ implies $\sigma = \sigma'$, and hence $(\varrho^c\sigma)|_V = (\varrho'^c\sigma')|_V$.

With the same techniques we can prove that most general V-restricted unifiers are most general W-restricted unifiers, if $V \subseteq W \subseteq$ **V**(s,t): In the above lemma and theorem we can replace $\mu U\Sigma[s=_T t]$ by $\mu U\Sigma|_W[s=_T t]$.

The next corollary is an immediate consequence of 3.Proposition4(iii). It shows that the results of the above lemma and the theorem also hold for matching problems with variable disjoint terms.
5.Corollary3:
Let $s,t \in \mathbf{T}$ with existing $\mu U\Sigma[s=_T t]$. If **V**(s)$\cap$**V**(t)$=\emptyset$, then we obtain (with V=**V**(t) and V^c=**V**(s)):
(i) $U_V \neq \emptyset \Leftrightarrow M\Sigma[s \leq_T t] \neq \emptyset$.
If **T** is in addition almost collapse free, then:
(ii) $U_V \neq \emptyset \Rightarrow U^c := \{(\varrho^c\sigma)|_{\mathbf{V}(t)} : \sigma \in U_V\}$ is a complete subset of $M\Sigma[s \leq_T t]$.

Remarks:
1. In the correctness part of the theorem, we prove $(\varrho^c\sigma)|_V =_T \varrho^c\sigma$ **[V**(s,t)**]**; this implies $(\varrho^c\sigma)|_V \equiv_T \sigma$ **[V**(s,t)**]**. Hence every most general restricted unifier is a most general unifier and we can abbreviate the above results by the following notation ('$\subseteq_T$' means 'subset modulo $\equiv_T$'):

(i) $V \subseteq W \subseteq \mathbf{V}(s,t) \Rightarrow \mu U\Sigma|_V[s=_T t] \subseteq_T \mu U\Sigma|_W[s=_T t] \subseteq_T \mu U\Sigma[s=_T t]$ (5.Theorem2)

(ii) $\mathbf{V}(s) \cap \mathbf{V}(t) = \emptyset \Rightarrow \mu M\Sigma[s \leq_T t] \subseteq_T \mu U\Sigma[s=_T t]$ (5.Corollary3(ii)).

2. If $\mathbf{T}$ is collapse free, then $U_V = \{\sigma \in \mu U\Sigma[s=_T t]: \sigma = \varrho_\sigma\ [V^c],\ \varrho_\sigma \in Ren(V^c)\}$, since being $\mathbf{T}$-equal to a renaming enforces being identical to a renaming.

Let us demonstrate these results by an example:

5.Example5:

Let $\mathbf{T} := \{fgx = fx\}$ with $f,g \in \mathbf{F}_1$; the theory is almost collapse free (for ease of notation we drop the parantheses for unary function symbols and abbreviate multiple nestings of the same function symbol by exponents: $f^0x := x$, $f^{n+1}x := f(f^n x)$ for $f \in \mathbf{F}_1$). Then consider the terms $s := fx$ and $t := fy$.

1. The unification problem $\langle s =_T t \rangle$ has a base $\mu U\Sigma[s=_T t] = \{\sigma_{00}\} \cup \{\sigma_{nm}: n,m>0, n \neq m\}$ with $\sigma_{nm} := \{x \leftarrow g^n v_{nm}\ ,\ y \leftarrow g^m v_{nm}\}$, where $v_{nm} \in \mathbf{V} \setminus \{x,y\}$ are pairwise different variables ($n,m \geq 0$ and $n \neq m$ for $n,m>0$).

2. For the semi-unification problem $\langle s =_T t\ ,\{y\}\rangle$ we obtain $U_{\{y\}} = \{\sigma_{0m}: m \geq 0\}$, and a base is $U^c = \{\tau_m: m \geq 0\}$ with $\tau_m = (\varrho_m{}^c\sigma_{0m})|_{\{y\}} = \{y \leftarrow g^m x\}$, where $\varrho_m = \{x \leftarrow v_{0m}\}$ ($m \geq 0$).

3. The matching problem $\langle s \leq_T t \rangle$ has the same sets $U_{\{y\}}$ and U^c, but it has a base with a single matcher: $\mu M\Sigma[s \leq_T t] = \{\mu\}$ with $\mu = \tau_0 = \{y \leftarrow x\} \in U^c$. Every other element of U^c is a $\mathbf{T}$-instance of μ on $\mathbf{V}(t) = \{y\}$: $\tau_m =_T \lambda\mu\ [\{y\}]$ with $\lambda = \{x \leftarrow g^m x\}$ ($m \geq 1$).

This also demonstrates that we cannot get minimality for U^c in 5.Corollary3.

We give some counterexamples for the above results, if the preconditions are weakened.

5.Example6:

If we drop the 'almost collapse free' requirement, the examples (i)-(iii) are semi-unification problems contradicting both 5.Corollary3 and 5.Theorem2; that is the most general unrestricted unifiers differ (in general) from the restricted ones not only by a renaming of the blocked variables.

Note, that the axiomatizations of these examples just represent the three main possibilities of violating the almost collapse free property with regular theories.

(i) $\mathbf{T} := \{f(x,x) = x\}$ (idempotence)

Consider the following terms $s := x$ and $t := f(y,z)$ and the substitutions $\sigma := \{x \leftarrow f(u,v)\ ,\ y \leftarrow u\ ,\ z \leftarrow v\}$ and $\mu := \{y \leftarrow x\ ,\ z \leftarrow x\}$. Then $\sigma \in \mu U\Sigma[s =_T t]$, and $\mu \in \mu M\Sigma[s \leq_T t]$, but $\mu \leq_T \sigma\ [\mathbf{V}(s,t)]$ with $\lambda = \{u \leftarrow x\ ,\ v \leftarrow x\}$ and $\mu \not\equiv_T \sigma\ [\mathbf{V}(s,t)]$, i.e. the most general matcher (semi-unifier) is a proper instance of the most general unifier. Note, that 5.Lemma1 holds, since $\sigma \in U_{\{y,z\}}$. However, σ is not $\mathbf{T}$-equal to a renaming on the bolcked variables $V^c := \{x\}$.

(ii) $T := \{f(1,x) = x\}$ (unit element)

Let $s := x$ and $t := f(y,z)$, let $\sigma = \{x \leftarrow f(u,v), y \leftarrow u, z \leftarrow v\}$ and $\mu = \{y \leftarrow 1, z \leftarrow x\}$. Then σ is a most general unifier and μ a most general matcher. With the substitution $\lambda = \{u \leftarrow 1, v \leftarrow x\}$ we have again $\mu \leq_T \sigma\ [\mathbf{V}(s,t)]$, but $\mu \neq_T \sigma$.

(iii) $T := \{f(g(x)) = x\}$.

Let $s := x$ and $t := f(y)$. Let $\sigma := \{x \leftarrow f(u), y \leftarrow u\}$ and $\mu := \{y \leftarrow g(x)\}$. Analogous to the former examples $\mu \leq_T \sigma\ [\mathbf{V}(s,t)]$ with $\lambda = \{u \leftarrow g(x)\}$ and $\mu \neq_T \sigma$. Again σ is a most general unifier and μ a most general matcher.

5.Example7:

This example shows, that 5.Corollary3 does not hold for arbitrary matching problems, that is, if we drop the variable disjointness requirement. There may exist more most general matchers than most general unifiers and in addition the latter may be proper instances of the former, if the terms have common variables. Note that the theory is collapse free.

$T := \{f(x,y) = f(y,x)\}$ (commutativity)

With the terms $s := f(g(x),y)$ and $t := f(x,z)$ and with the substitutions $\sigma := \{x \leftarrow u, y \leftarrow u, z \leftarrow g(u)\}$, $\mu_1 := \{x \leftarrow g(x), z \leftarrow y\}$ and $\mu_2 := \{x \leftarrow y, z \leftarrow g(x)\}$ we obtain: $\sigma \in \mu U\Sigma[s =_T t]$ and $\mu_1, \mu_2 \in \mu M\Sigma[s \leq_T t]$ and $\sigma \leq_T \mu_2\ [\mathbf{V}(s,t)]$ with $\lambda = \{x \leftarrow u, y \leftarrow u\}$, but not conversely. On the other hand σ and μ_1 are not comparable in $\leq_T$ and σ is the only most general unifier. Note, that the corresponding semi-unification problem $\langle s =_T t, \mathbf{V}(t) \setminus \mathbf{V}(s) \rangle$ has no solution.

6 Consequences and Applications

Depending on the cardinality of the μ-sets we classify the unification problems and the theories. This is known as *unification hierarchy* [Siekmann84].

6.Definition1:

(i) A solvable unification problem is called *nullary*, iff the μ-set does not exist (the cardinality is null). It is called *unitary/finitary/infinitary*, iff the μ-set exists and its cardinality is one/finite/infinite.

(ii) A theory is *unitary/finitary unifying*, iff every solvable unification problem is unitary/finitary. It is called *nullary/infinitary unifying*, iff at least one solvable unification problem is nullary/infinitary.

Analogously we define this for matching and restricted unification.

The following theorem describes the relationship between the hierarchy classes of restricted and unrestricted unification problems.

6.Theorem2:

Let **T** be an almost collapse free theory.

(i) If **T** is nullary restricted unifying, then **T** is nullary unifying.

If in addition the μ-set exists for each problem (that is **T** is not nullary), then we have the following hierarchy results:

(ii) If **T** is unitary/finitary unifying, then **T** is unitary/finitary restricted unifying.

(iii) If **T** is infinitary restricted unifying, then **T** is infinitary unifying.

Proof: (i) If **T** is not nullary unifying, every base exists and by 5.Theorem2 the bases of the restricted unification problems exist also.

(ii)+(iii) The bases of the restricted unification problems are subsets of the bases of the unrestricted unification problems modulo **T**-equivalence (5.Theorem2 and remarks in section5) and hence they have less or equal cardinalities.

Since semi-unification is a special case of restricted unification, we at once have the corresponding hierarchy results for semi-unification:

6.Corollary3:

Let **T** be an almost collapse free theory.

(i) If **T** is nullary semi-unifying, then **T** is nullary unifying.

If in addition all μ-sets exist, then:

(ii) If **T** is unitary/finitary unifying, then **T** is unitary/finitary semi-unifying.

(iii) If **T** is infinitary semi-unifying, then **T** is infinitary unifying.

Remarks:

1. The converses of the implications in 6.Corollary3 do not hold in general. Szabo gives an infinitary unifying, but unitary semi-unifying theory [Szabo82].
2. In 6.Theorem2 we of course have equivalence, since by 3.Proposition4(ii) unification is a special form of restricted unification.
3. The results for semi-unification are covered by some results in [Szabo82].
   - Every unitary unifying theory is unitary semi-unifying.
   - Every nullary semi-unifying theory is nullary unifying.

   Here the theories need not to be almost collapse free! However, Szabo's proof of the nullary case is incomplete, and moreover it is based upon the idea of replacing the blocked variables by ground terms, which is not possible in general (see 5. below and the appendix example).
4. We cannot get the nullary hierarchy result for matching, since we obtain only completeness of the matcher sets constructed by our method. But of course we have the other results provided all the bases exist:
   - If **T** is unitary/finitary unifying, then every **T**-matching problem for variable disjoint terms is unitary/finitary.
   - If there is an infinitary **T**-matching problem with variable disjoint terms,

then **T** is infinitary unifying.

Remember, that semi-unification is needed in applications instead of the general matching definition, so the results of 6.Corollary3 are sufficient.

5. Occasionally there has been the suggestion in the literature to simply replace the blocked variables by some ground terms in order to prove such hierarchy results. That this does not work in general is demonstrated by the following example (which was an important motivation for this paper):

Let c be the only constant, let f,g,h be unary functions, and let us assume, that there are no further functions at all. We again drop the parantheses.

Let the theory be defined by

$\mathbf{T}:=\{gfx = gx\ ,\ fc = c\ ,\ gc = c\ ,\ hc = c\}$

and consider the following variable disjoint terms:

$s:=ghy$ and $t:=gx$.

Then the semi-unification problem $\langle s =_T t\ ,\ \{x\}\rangle$ has an infinite base

$\mu U\Sigma|_{\{x\}}[s=_T t]:=\{\sigma_n : n\geq 0\}$ with $\sigma_n:=\{x\leftarrow f^n hy\}\ (n\geq 0)$,

that is, it is an infinitary problem, and hence the theory will be infinitary semi-unifying.

In order to show with the above idea that **T** is also infinitary unifying, we have to replace the variable of s by a ground term to get an infinitary unification problem. But with the only existing ground substitution $\gamma=\{y\leftarrow c\}$ (all ground terms are **T**-equal to c) we get:

$\mu U\Sigma[\gamma s=_T t]=\mu U\Sigma[ghc=_T gx]=\{\{x\leftarrow c\}\}$.

Hence the corresponding unification problem is unitary. However, the theory is of course infinitary unifying by 6.Theorem2(iii) (**T** is collapse free). An infinitary unification problem will be given by original terms s and t.

From the main theorem of the last section we can infer an algorithm to compute most general restricted unifiers from most general unifiers. This implies, that for every almost collapse free theory with an existing minimal unification algorithm (that is an algorithm computing a base for every solvable unification problem) there is also a minimal restricted unification algorithm.

6.Algorithm2:

Unifier_to_Restricted_Unifier

Input: - a (finite) base of unifiers of a unification problem $\langle s =_T t\rangle$
- a subset V of the variables of s and t

Output: - a base of V-restricted unifiers of the problem $\langle s =_T t\ ,\ V\rangle$, if this problem is solvable
- FAILURE, if the problem is not solvable

- If there is no most general unifier with a V-renaming part,
then return FAILURE.

- Else for each most general unifier with a V-renaming part do:
  remove the V-renaming part
  apply the converse of the V-renaming to the codomain of the rest.
- Return all changed unifiers.

This algorithm is particularily useful for clause graph theorem proving procedures [Kowalski75] like the MKRP-system [Raph84] at Kaiserslautern:
In clause graph procedures the clause sets are transformed into graphs with
- nodes labelled with the literals of the clauses
- arcs between nodes labelled by unifiable literals (with opposite sign) of different clauses (*resolution links*)
- arcs between unifiable literals (with same sign) of different clauses (*subsumption links*)

The resolution links are labelled by μUΣ-sets and characterize resolution possibilities. The subsumption links support the application of the subsumption rule [Loveland78]: If there are two clauses C,D and a substitution μ with $\mu C \subseteq D$, then the clause D can be removed. This can be extended to clause graphs using the above subsumption links [Eisinger81]. Therefore these links should be labelled by semi-unification bases, but since the direction of the semi-unification problem is not known in advance, the links are also labelled by μUΣ-sets and the semi-unification bases are computed dynamically by the above algorithm.

7 Conclusions

We have seen that for almost collapse free theories the most general restricted unifiers can be computed from a set of most general unrestricted unifiers. An open question is, whether this can be done in the general case. By 5.Lemma1 we can decide the restricted unification problem, if the minimal solution set of the unrestricted unification problem exists and is finite (if the unification problem is infinitary, we obtain at least a semi-decision procedure for the restricted problem). The proof of this lemma gives some hints for computing restricted unifiers from those most general unifiers that are equivalent to the identity on the blocked variables: we have to instantiate them and restrict the instances on the unblocked variables. Finding out the appropriate instantiations - for almost collapse free theories, these are the renamings - might also lead to a minimal solution set. Solving this problem then would also yield the still missing hierarchy results for theories with arbitrary collapse equations.
Our result on the unification theoretical relationship between almost collapse free and collapse free theories (4.Corollary6) affects also the problem of combining unification algorithms of theories with disjoint function sets

[Yelick85, Tiden85, Herold86]. This is still only solved for collapse free theories and our result yields, that the collapse free requirement can be weakened by admitting projection equations.

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Appendix

We want to consider unification under distributivity and associativity and some extension to obtain an interesting undecidability result.

Let + and × be binary function symbols written in infix notation: $\mathbf{F'}_2=\{+,\times\}$. Let us assume that there also is at least one constant symbol: $\mathbf{F'}_0=\{c\}$. Then our signature is $\mathbf{F'}=\mathbf{F'}_0\cup\mathbf{F'}_2$. We define the distributivity laws and the associativity law to be:

Dl :={ x×(y+z) = (x×y)+(x×z) } (left distributivity)
Dr:={ (x+y)×z = (x×z)+(y×z) } (right distributivity)
A :={ x+(y+z) = (x+y)+z } (associativity)

The solvability problem of unification of **F'**-terms in the theory **DA**:=**Dl**∪**Dr**∪**A** of distributivity and associativity is known to be undecidable [Szabo82].

Now given another signature $\mathbf{F}=\mathbf{F}_0\cup\mathbf{F}_3$ with $\mathbf{F}_0=\{a\}$ and $\mathbf{F}_3=\{f,g\}$, i.e. we have one constant and two ternary function symbols, we define some generalizations of the left and right distributivity and of the associativity axioms for binary function symbols to ternary function symbols:

Dl3 := { f(g(x,y,v),z,v) = g(f(x,z,v),f(y,z,v),v) }
Dr3:= { f(x,g(y,z,v),v) = g(f(x,y,v),f(x,z,v),v) }
A3 := { g(g(x,y,v),z,v) = g(x,g(y,z,v),v) }

Then we consider the collapse free theory **T** := **Dl3**∪**Dr3**∪**A3**∪**Ta** with

Ta:= { f(x,y,a) = a , f(x,y,f(u,v,w)) = a , f(x,y,g(u,v,w)) = a ,
g(x,y,a) = a , g(x,y,f(u,v,w)) = a , g(x,y,g(u,v,w)) = a }.

In this theory every term starting with a ternary function symbol (a *complex* term) is **T**-equal to the constant a, if its third top argument is a non-variable. Every **T**-unification problem constructed with the signature **F** is solvable: We

substitute all variables of the problem by the constant a. Then both terms will become $\mathbf{T}$-equal to the constant a by the subtheory $\mathbf{T_a}$. Hence unification under this theory is decidable within the given signature.
But, if we introduce a new free constant b, unification in this theory will become undecidable. We can reduce a subset of the unification problems to the unfication under distributivity and associativity of binary function symbols introduced above. Therefore we consider a subset $\mathbf{T}_b$ of the term algebra $\mathbf{T}(\mathbf{F}\cup\{b\},\mathbf{V})$ with the extended signature $\mathbf{F}\cup\{b\}$:

(i) $a,b\in\mathbf{T}_b$ and $\mathbf{V}\subseteq\mathbf{T}_b$

(ii) $t_1,t_2\in\mathbf{T}_b \Rightarrow f(t_1,t_2,b),g(t_1,t_2,b)\in\mathbf{T}_b$.

That is, $\mathbf{T}_b$ is the subset of $\mathbf{F}\cup\{b\}$-terms, where the third argument of every complex term is only allowed to be the constant b. Every $\mathbf{T}$-unification problem $\langle s =_T t\rangle$ built up by those terms is solvable, iff the $\mathbf{DA}$-unification problem $\langle s' =_{DA} t'\rangle$ is solvable. Here we obtain s' and t' from s and t by the following mapping:

(i) $a \mapsto c$, $b \mapsto c$ and $v \mapsto v$ $\forall v\in\mathbf{V}$

(ii) $f(t_1,t_2,b) \mapsto t_1 \times t_2$, $g(t_1,t_2,b) \mapsto t_1+t_2$ $\forall t_1,t_2\in\mathbf{T}_b$.

Thus in this theory unification will become undecidable, if we introduce new constants.

An analogous reduction will demonstrate that restricted unification is not necessarily decidable, when unrestricted unification is. Take again the above theory $\mathbf{T}$ and the signature $\mathbf{F}$. As we have seen, unification is decidable. However considering restricted unification problems $\langle s =_T t\ ,\ \mathbf{V}(s,t)\setminus\{x\}\rangle$, where the variable x occurs at the third argument of each complex term, we get an undecidable set of problems.