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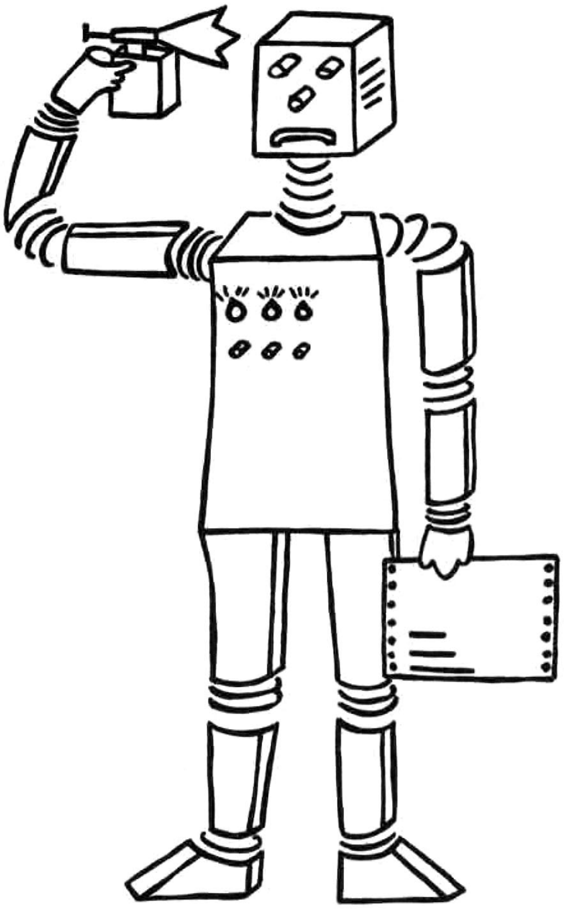

UNIFICATION PROPERTIES OF IDEMPOTENT SEMIGROUPS

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August 1986 SEKI-REPORT SR-86-07

Unification Properties of Idempotent Semigroups.

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(UUCP: seismo! mcvax!unido! uklirb! schauss)

Abstract.

Unification in free idempotent semigroups is of unification type zero, i.e. there are unifiable terms s,t but there is no minimal, complete set of unifiers for these two terms. Unification in free idempotent semigroups is strongly complete, i.e. the unification problem $<x =_{AI} t>$ is always solvable with unifier $\{x \leftarrow t\}$, even if x occurs in t.

We give a generalization of the usual unification hierarchy and demonstrate that the number of independent unifiers in A+I-unifier sets is not bounded.

It is known that there is a conditional, canonical term rewriting system for idempotent semigroups. To strengthen this result, we show that there can be no unconditioned and finite rewriting system.

Keywords: Unification, Equational Theories, Idempotent Semigroups, Rewriting systems

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This work is supported by the Deutsche Forschungsgemeinschaft, SFB 314

1. Introduction.

Unification theory is concerned with the problem to find solutions for an equation $<s = t>$, where s and t are terms. Solutions of $<s = t>$ are substitutions σ with $\sigma s = \sigma t$. The substitution σ is called a unifier for s and t.
An extension of this problem is the following: Given a set of equations T we say two terms t_1 and t_2 are equal w.r.t. T, denoted as $t_1 =_T t_2$, iff $t_1 = t_2$ logically follows from T. A T-unification problem $<s = t>_T$ is the problem to find solutions σ such that $\sigma s =_T \sigma t$.
The set of all unifying substitutions (i.e. of all solutions) of $<s = t>_T$ is denoted as $U\Sigma_T(s,t)$. In many cases, the set of all solutions $U\Sigma_T(s,t)$ can be generated from a minimal subset of solutions, the set of most general unifiers $\mu U\Sigma_T(s,t)$, which is defined as follows:
We say the substitution σ is more general than τ on the set of variables W ($\tau \leq_T \sigma$ [W]) iff there exists a substitution λ such that $\tau x =_T \lambda\sigma x$ for all $x \in W$. Note that the set $U\Sigma_T(s,t)$ is ordered by the quasi ordering $\leq_T[V(s,t)]$.
The set $\mu U\Sigma_T(s,t)$ is characterized by three conditions:

i) correctness: $\mu U\Sigma_T(s,t) \subseteq U\Sigma_T(s,t)$
ii) completeness: $\forall\theta \in U\Sigma_T(s,t)\ \exists\, \sigma \in \mu U\Sigma_T(s,t)\ \ \theta \leq_T \sigma\,[W]$ where $W = V(s,t)$
iii) minimality: $\forall\sigma,\tau \in \mu U\Sigma_T(s,t)\ \ \sigma \leq_T \tau\,[W] \Rightarrow \sigma = \tau.$ where $W = V(s,t)$

Unification theory classifies equational theories by the cardinality of the set $\mu U\Sigma_T(s,t)$

i) A theory T is unitary, iff $\mu U\Sigma_T(s,t)$ always exists and has at most one element.
ii) A theory T is finitary, iff $\mu U\Sigma_T(s,t)$ always exists and is finite.
iii) A theory T is infinitary, iff $\mu U\Sigma_T(s,t)$ always exists and there exists a pair of terms s,t such that $\mu U\Sigma_T(s,t)$ is infinite.
iv) A theory T is nullary, iff $\mu U\Sigma_T(s,t)$ does not exist for some terms s and t.

In [Sz82,Si84] unification theories of type nullary are not subclassified.
In order to give a finer classification of equational theories (including theories of type nullary) by the maximal width of the sets $U\Sigma_T(s,t)$ we introduce some notions to handle enumerable, quasi ordered sets.

Let U be a set ordered by the quasi ordering $\leq$. (we are interested in the case $U = U\Sigma(s,t)$ and $\leq$ is the quasi-ordering $\leq_T[V(s,t)]$.)
Let $\sim$ be the equivalence relation corresponding to $\leq$, i.e. $a \sim b$ iff $a \leq b$ and $b \leq a$. We say a subset V of U is complete, iff $\forall u \in U\ \exists v \in V$: $u \leq v$. An element u is maximal in U, iff $\forall v \in V$: $u \leq v \Rightarrow u \sim v$. The set of all maximal elements of U is denoted as max(U). $\mu(U)$ denotes a set of $\sim$-representatives of max(U).
A subset V of U is minimal, iff $\forall u,v \in V$: $u \leq v \Rightarrow u = v$.

We have: If max(U) is complete, then $\mu(U)$ is a complete and minimal subset of U.

We define $\eta(U)$ as the "nullary" part of U: $\eta(U) := \{u \in U \mid \forall v \in \mu(U) : u \nleq v\}$. I.e. $\eta(U)$ is the set of all elements of U that are not smaller than a maximal element. Obviously the set $\eta(U)$ does not contain maximal elements.
A subset B of U is called independent, iff for all $b_1,b_2 \in B$ with $b_1 \neq b_2$ there does not exist a common $u \in U$ with $b_1 \leq u$ and $b_2 \leq u$. A subset B is called maximal independent, if every set C with $B \subsetneq C \subseteq U$ is dependent.

An easy application of the Zorn's Lemma shows that every independent set is contained in a maximal independent set. Note that for countable sets, Zorn's Lemma can be proved from the other axioms of set theory.

Let $N_\infty := N \cup \{\infty\}$.
We can measure quasi-ordered sets in the following way :

The width of U (width(U)) is a pair $(a,b) \in N_\infty \times N_\infty$ where:

i) a is the cardinality of a set $\mu(U)$.

ii) b is the maximal cardinality of a (maximal) independent subset of $\eta(U)$. ■

In the appendix it is shown that it is not possible that every independent subset of U is finite but the maximal cardinality of independent subsets has no upper bound.

In the following we use the set $N_{\omega,\infty} := N \cup \{\omega,\infty\}$ with ordering $n < \omega < \infty$ where ω denotes as usual the supremum of all natural numbers. Pairs are ordered with respect to the product ordering: $(a,b) \le (c,d)$, iff $a \le c$ and $b \le d$. The set $N_{\omega,\infty}$ is well-ordered. We define the supremum of a set M in $N_{\omega,\infty} \times N_{\omega,\infty}$ as follows:
Construct the closure M* of M:

i) $M^* \supseteq M$

ii) for every ascending chain (a_i,b_i) in M* add the element $(\sup(a_i), \sup(b_i))$ to M*

Now we define sup(M) := max(M*).

For example if $M = \{(4,2),(3,5),(2,5),(2,6),\ldots,(2,i),\ldots\}$, then $\sup(M) = \{(4,2),(3,5),(2,\omega)\}$.

We extend the usual classification of unification to theories of type nullary:

1.1 Definition. Let T be an equational theory. The unification type of T is a set of pairs defined as follows:

$\text{type}(T) := \sup\{\text{width}(U\Sigma_T(s,t)) \mid s,t \in T\}$.

I.e. type(T) is a set of pairs that describes the maximal possible widths of sets $U\Sigma_T(s,t)$.

The maximal unification type is a single pair (m_1,m_2), where
$m_1 := \sup\{a \mid (a,b) \in \text{type}(T)\}$ and $m_2 := \sup\{b \mid (a,b) \in \text{type}(T)\}$.

For example if the set $\{\text{width}(U\Sigma_T(s,t)) \mid s,t \in T\} = M = \{(4,2),(3,5),(2,5),(2,6),\ldots,(2,i),\ldots\}$, then the type of T is $\sup(M) = \{(4,2),(3,5),(2,\omega)\}$ and the maximal unification type of T is $(4,\omega)$.
This classification is a generalization of the usual one: A unitary theory has type $\{(1,0)\}$ and maximal unification type (1,0). Finitary theories have maximal unification type (n,0) or $(\omega,0)$ and infinitary theories have maximal unification type $(\infty,0)$. Theories of type nullary have a maximal unification type with nonzero second argument.
In the appendix it is shown that for a countable quasi-ordered set U with width(U) = (0,1). (i.e. the set U has no maximal elements and the maximal cardinality of an independent subset is 1) there exists one increasing chain C of elements of U such that C is complete in U. Similarily, if the width is a finite number n, then n chains are sufficient to construct a complete subset.
In the case $\text{width}(U\Sigma(s,t)) = (0,n)$ it is never necessary in a theorem prover to consider more than n unifiers for this problem in parallel, since in this case for every set of unifiers of s,t with more than n elements there exists a set of more general unifiers with at most n unifiers.

2 Idempotent Semigroups.

The equational theory A+I is generated by the two axioms:

A: $f(f(x, y), z) = f(x, f(y, z))$

I: $f(x, x) = x$

These two equations define free idempotent semigroups (bands). It is a known fact, that finitely generated bands only have a finite number of elements [Ho76]. An immediate consequence is that A+I is finitary matching, i.e. a correct, complete and minimal subset of the set $M\Sigma_{AI}(s,t) := \{\sigma \mid \sigma s =_{AI} t \text{ and } DOM(\sigma) \cap V(t) = \emptyset\}$ is finite for all terms s and t, since there is only a finite number of substitutions with a fixed domain and with fixed symbols in the codomain. Furthermore it is decidable, whether two terms are A+I-unifiable [Sz82]. A+I-unification is of type nullary [Sch86,Ba86] in the usual sense.

2.1 Basic Notions

We use the standard notation of unification-theory [Si84], the main notions are listed below. For convenience, associative terms are denoted as strings.

$s = t$	the terms s and t are symbolwise equal.
$s =_{AI} t$	the terms are equal under the theory A+I, i.e. s and t are congruent w.r.t. the congruence relation generated by A and I.
$\sigma =_{AI} \tau\ [W]$	$\sigma x =_{AI} \tau x$ for all variables in the set W.
$\sigma \leq_{AI} \tau\ [W]$	There exists a substitution λ with $\sigma =_{AI} \lambda\tau\ [W]$.
$\sigma_{\mid W}$	the restriction of σ to the set W, i.e. $\sigma_{\mid W} x$ is σx for $x \in W$ and the identity otherwise.
V(o)	the set of variables occurring in the object o.
$DOM(\sigma)$	the domain of σ, i.e. the set $\{x \mid \sigma x \neq x\}$
$COD(\sigma)$	the codomain of σ, i.e. $\{\sigma x \mid \sigma x \neq x\}$
$VCOD(\sigma)$	$V(COD(\sigma))$
Sy(o)	the set of symbols occurring in the object o.
C(o)	the set of constants occurring in the object o.
C#(o)	the number of constant occurrences in the object o.
$s\downarrow$	denotes the A+I-normalform of s, which exists (see below).

We assume that the reader is familiar with rewrite rules on terms (cf. [HO80]).
The following two conditional rewrite rules form a terminating, confluent rewriting system for idempotent semigroups (see [SS83,Sz82]), hence a unique normal form $s\downarrow$ exists for every term s.
Let x be a variable and let s,t,u be A+I-terms.

Rule 1) $xx \to x$
Rule 2) $stu \to su$, provided $Sy(s) = Sy(u)$ and $Sy(u) \supseteq Sy(t)$

Note that it suffices to use the simpler rule 2' :

2') $stu \to su$, provided $Sy(s) = Sy(u)$ and t is a symbol with $t \in Sy(u)$.

The following lemmata that are either well-known (cf. [Ho76]) or obvious are helpful in later proofs:

2.1.1 Lemma [Ho76] Let a and b be symbols, let s,t be terms

i) $sa =_{AI} tb \Rightarrow a = b$.
ii) $as =_{AI} bt \Rightarrow a = b$.

Proof. [Ho76] .

2.1.2 Lemma. Let s,t be terms with $s =_{AI} t$. Then

i) $C(s) = C(s\downarrow)$ ii) $Sy(s) = Sy(s\downarrow)$

iii) $C\#(s) \geq C\#(s\downarrow)$ iv) $C(s) = C(t)$

v) $Sy(s) = Sy(t)$

Proof. Obvious.

We say a substitution λ is constant-free on W, iff $\forall x \in W$: $C\#(\lambda x) = 0$

2.1.3 Lemma Let s,t be terms and let λ be a substitution that is constant-free on V(s,t).

i) Then $C\#(s) \geq C\#((\lambda s)\downarrow)$.

ii) If t is in normal form and $C\#(s) < C\#(t)$ then $\lambda s \neq_{AI} t$.

Proof. Obvious.

2.1.4 Lemma. [Ho76] Let a be a symbol and let $s_1 a s_2 =_{AI} t_1 a t_2$. Then:

i) $a \notin Sy(s_1)$ and $a \notin Sy(t_1)$ $\Rightarrow$ $s_1 =_{AI} t_1$.

ii) $a \notin Sy(s_2)$ and $a \notin Sy(t_2)$ $\Rightarrow$ $s_2 =_{AI} t_2$.

Proof. [Ho76] . ■

For a string t we say p is the full prefix of t provided p is the shortest prefix of t with $Sy(p) = Sy(t)$; s is the full suffix of t if s is the shortest suffix of t with $Sy(s) = Sy(t)$

2.1.5 Lemma. [Ho76] Let s,t be terms and p_s, p_t, q_s, q_t be the full prefixes and full suffixes of s and t, respectively. Then:

$s =_{AI} t \Leftrightarrow p_s =_{AI} p_t$ and $q_s =_{AI} q_t$. ■

We show that t can be reduced to its normalform $t\downarrow$ in the following way: First reduce t to its normalform t_1 with respect to rule 2' by checking applicability of rule 2' from left to right and then reduce t_1 to its normalform t_2 with respect to rule 1.

The next proposition shows that the deletion of symbols by rule 2' can be made in parallel. Furthermore it shows that it is sufficient to check every symbol from left to right.

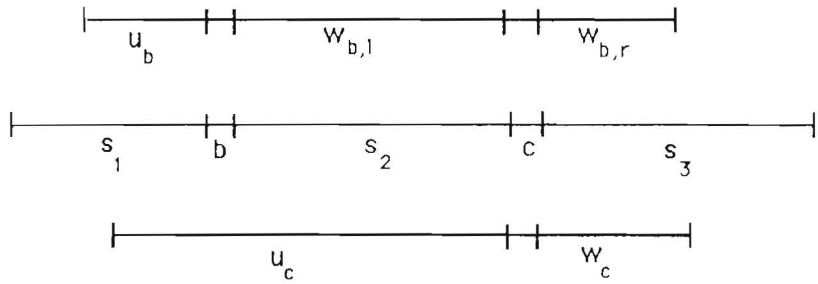
2.1.6 Proposition. The set of symbols in a string t that are deletable with rule 2' does not change after application of rule 2'.

Proof. Let b and c be symbols in the string s.

i) If b,c are deletable by rule 2', then c remains deletable by rule 2' after deleting b:

Assume by contradiction that c is not deletable by rule 2' after deletion of b.

Then we have the case $s = s_1 b s_2 c s_3$. s_1 and s_3 are not empty and b is not contained in s_2. Since b and c are deletable by rule 2' there exist substrings of s of the following form: $u_b b w_b$ with $Sy(u_b) = Sy(w_b)$ and $u_c c w_c$ with $Sy(u_c) = Sy(w_c)$. Since b is not contained in s_2, w_b overlaps c and we can split w_b in the right and left part $w_b = w_{b,l} c w_{b,r}$. Now we can construct new strings u_n and w_n that show that c is deletable by rule 2': Let u_n be the string that contains exactly $u_b, w_{b,l}$ and u_c', where u_c' is u_c after deleting the symbol b. Let w_n be the string containing exactly $w_{b,r}$ and w_c.



We show: $Sy(u_b) \cup Sy(w_{b,l}) \cup Sy(u_c') = Sy(w_{b,r}) \cup Sy(w_c)$:

"$\subseteq$": $Sy(w_{b,l}) \subseteq Sy(u_c) = Sy(w_c)$.

$Sy(u_c') \subseteq Sy(u_c) = Sy(w_c)$.

$Sy(u_b) \subseteq Sy(w_{b,r}) \cup \{c\} \cup Sy(w_{b,l}) \subseteq Sy(u_c) \cup Sy(w_c) \cup Sy(w_{b,r})$

$\subseteq Sy(w_{b,r}) \cup Sy(w_c)$.

"$\supseteq$": $Sy(w_c) \subseteq Sy(u_c') \cup \{b\} \subseteq Sy(u_c') \cup Sy(u_b)$

$Sy(w_{b,r}) \subseteq Sy(u_b)$. □

ii) If b is deletable by rule 2' and c is not deletable then c remains undeletable by rule 2' after deleting b:

Assume by contradiction that c is deletable by rule 2' after deletion of b.

Then we have the case $s = s_1bs_2cs_3$. s_1 and s_3 are not empty and b is not contained in s_2. Since b is deletable by rule 2' there exists a substring of s of the form u_bbw_b with $Sy(u_b) = Sy(w_b)$. Since c is deletable by rule 2' in $s' := s \setminus b$, there exist substrings of s' of the form u_ccw_c with $Sy(u_c) = Sy(w_c)$. Note that u_c covers s_2. Since b is not contained in s_2, w_b overlaps c and we can split w_b in the right and left part $w_b = w_{b,l}cw_{b,r}$. Now we can construct new strings u_n and w_n that show that c is deletable by rule 2' in the string s: Let u_n be the string that contains exactly $u_b, w_{b,l}$, b and u_c. Let w_n be the string containing exactly $w_{b,r}$ and w_c.

We show: $Sy(u_b) \cup Sy(w_{b,l}) \cup Sy(u_c) \cup \{b\} = Sy(w_{b,r}) \cup Sy(w_c)$:

"$\subseteq$": $Sy(w_{b,l}) \subseteq Sy(u_c) = Sy(w_c)$.

$Sy(u_c) = Sy(w_c)$.

$Sy(u_b) \subseteq Sy(w_{b,l}) \cup \{c\} \cup Sy(w_{b,r}) \subseteq Sy(u_c) \cup Sy(w_c) \cup Sy(w_{b,r})$

$\subseteq Sy(w_{b,r}) \cup Sy(w_c)$.

$c \in Sy(w_c)$

"$\supseteq$": $Sy(w_c) \subseteq Sy(u_c)$

$Sy(w_{b,r}) \subseteq Sy(u_b)$. ■

<u>2.1.7 Proposition.</u> Let t be a term that is not reducible by rule 2. Let t' be obtained by a reduction using rule1. Then t' is not reducable with rule 2.

<u>Proof.</u> It suffices to show this for a one-step reduction.

Let $t = t_1t_2t_2t_3$ be a term, which is not reducible by rule 2.

Assume there is an element e in $t' = t_1t_2t_3$ that is reducible by rule 2'.

If e is in t_1 or t_3 , then e is reducible by rule 2 in t. Hence e is in t_2.

Let w_l be the word on the left side and w_r be the word on the right side with $e \in w_l$, $e \in w_r$ and $Sy(w_l) = Sy(w_r)$.

If one of them is a substring of t_2 then e is reducible in t. Hence they are not substrings of t_2. This means $t_2 \subseteq w_l \cup w_r = Sy(w_r)$. So e is reducible with Rule 2 in t. ■

Using these observations the reduction algorithm based on rule 1 and rule 2 can be improved: First apply rule 2' to every element in the string to be reduced. Then apply rule 1 in all possible ways until the term is not further reducible.

2.2. A+I is of Type Nullary

We assume that there is one free constant a in the signature.
Consider the unification problem <zaxaz $=_{AI}$ zaz> , where x,z are variables and a is a constant. In the following we fix the unifier $\theta := \{x \leftarrow z_1az_2;\ z \leftarrow z_1z_2\ \}$ of zaxaz and zaz, where the z_i are variables.

2.2.1 Lemma For all $\sigma \in U\Sigma_{AI}$(zaxaz, zaz) with $\theta \leq_{AI} \sigma$ [x,z]:

i) σz consists only of variables.

ii) $C(\sigma x) = \{a\}$.

iii) The last element of σx is a variable

iv) $V(\sigma x) \subseteq V(\sigma z)$.

Proof. Let $\sigma \in U\Sigma_{AI}$(zaxaz, zaz) and let $\lambda \in \Sigma$ with $\theta =_{AI} \lambda\sigma$ [x,z].
$\theta z = z_1z_2$ and σz is more general than hz, hence i) holds. iv) holds since $(\sigma z)a(\sigma x)a(\sigma z) =_{AI} (\sigma z)a(\sigma z)$.
The substitution λ is constant-free on $V(\sigma x) \subseteq V(\sigma z)$ and $\lambda\sigma x =_{AI} z_1az_2$, hence $C(\sigma x) = \{a\}$ and ii) holds.
The last element of σx is a variable since σx is more general than z_1az_2. ■

2.2.2 Lemma For all $\sigma \in U\Sigma_{AI}$(zaxaz, zaz) with $\theta \leq_{AI} \sigma$ [x,z] there exists a $\sigma' \in U\Sigma_{AI}$(zaxaz, zaz) with $\sigma \leq_{AI} \sigma'$ [x,z] and $\sigma' \not\leq_{AI} \sigma$ [x,z] .

Proof.: Without loss of generality we can assume that σx is in normalform.
Lemma 2.2.1 iv) shows that $V(\sigma x) \subseteq V(\sigma z)$. We define a unifier σ' that is more general than σ as follows:
$\sigma'x := (\sigma x)au$, $\sigma'z := (\sigma z)u(\sigma z)$, where u is a new variable. $\sigma'x$ is in normalform, since σx is in normalform.

1) σ' is a unifier of zaxaz and zaz:
$(\sigma z)u(\sigma z)a(\sigma x)aua(\sigma z)u(\sigma z) =_{AI} (\sigma z)u(\sigma z)a(\sigma z)u(\sigma z)$ since $V(\sigma x) \subseteq V(\sigma z)$

2) $\sigma \leq_{AI} \sigma'$ [x,z]:
Let $\sigma x = s_1as_2$, where s_2 is a nonempty string of variables and s_1 is a nonempty string.
We have $\sigma =_{AI} \mu\sigma'$ [x,z] for $\mu := \{u \leftarrow s_2\}$, since $(\sigma z)s_2(\sigma z) =_{AI} \sigma z$ by rule 2 and $(\sigma x)as_2 =_{AI} \sigma x$.

3) $\sigma' \not\leq_{AI} \sigma$ [x,z]:
Assume there exists a substitution μ with $\sigma' =_{AI} \mu\sigma$ [x,z] . Then μ is constant-free on all variables in $V(\sigma z)$.
By Lemma 2.1.3 we have $\mu\sigma x \neq_{AI} \sigma'x$, since $\sigma'x$ is in normalform and $C\#(\sigma'x) = C\#(\sigma x) + 1$. ■

Using Lemma 2.2.2 we can construct for every unifier σ of zaxaz and zaz that is more general than θ another unifier σ' that is more general than σ, hence we have shown:

2.2.3 Theorem. $\mu U\Sigma_{AI}$(zaxaz, zaz) does not exist.

This immediately implies:
2.2.4 Corollary: The equational theory A+I (idempotent semigroups) is of type nullary.

In [Ba86] it is shown that there are unifiers in $U\Sigma_{AI}$(zxz, zyz) that are not instances of a most general unifier.

2.3. A+I is strongly complete.

The equational theory A+I is an example for strongly complete theories [Ki85], that is theories in which the unification problem <x $=_T$ t> with $x \in V(t)$ is either not solvable or solvable with a unifier σ with $DOM(\sigma) = \{x\}$.

2.3.1 Proposition. The unification problem <x $=_{AI}$ t> has the most general unifier $\sigma = \{x \leftarrow t\}$.

I.e. $\mu U\Sigma_{AI}(x,t)$ exists and is a singleton for all possibilities of x and t.

Proof. It is well-known that in the case $x \notin V(t)$ the most general unifier is $\{x \leftarrow t\}$.

In the case $x \in V(t)$ the most general unifier is $\sigma = \{x \leftarrow t\}$:

i) σ is a unifier:

Let $t = s_1 x s_2 x s_3$, where x does not occur in s_1 and s_3.

Then $\sigma t = s_1 t (\sigma s_2) t s_3$

$s_1 t (\sigma s_2) t s_3 =_{AI} t(\sigma s_2) t$ since t starts with s_1 and stops with s_3

$t(\sigma s_2) t =_{AI} t$ since $Sy(t) \supseteq Sy(\sigma s_2)$.

ii) σ is most general:

Let θ be a unifier of x and t.

We show $\theta =_{AI} \theta\sigma [V(x,t)]$: $\theta\sigma x = \theta t =_{AI} \theta x$ and $\theta\sigma y = \theta y$ for variables $y \neq x$. ■

As a nontrivial example for unification in idempotent semigroups we analyze the structure of the set of unifiers of the unification problem <xa $=_{AI}$ ya> where a is a constant and show that this problem has 6 mgu's.

2.3.2 Lemma. $\sigma_1 := \{x \leftarrow z, y \leftarrow z\}$ is a most general unifier of <xa $=_{AI}$ ya>

Proof. Let σ be a unifier of xa and ya that is more general than $\{x \leftarrow z, y \leftarrow z\}$. Then σx and σy are strings of variables. Lemma 2.1.4 shows that $\sigma x =_{AI} \sigma y$. ■

2.3.3 Lemma $\sigma_2 := \{x \leftarrow za, y \leftarrow z\}$ and $\sigma_3 := \{x \leftarrow z, y \leftarrow za\}$ are most general unifiers of <xa $=_{AI}$ ya>

Proof. It suffices to show that $\{x \leftarrow za, y \leftarrow z\}$ is most general.

Let σ be a unifier of xa and ya that is more general than $\{x \leftarrow za, y \leftarrow z\}$. Then σy is a string of variables and the rightmost symbol of σx is the constant a. We have $\sigma =_{AI} \{z \leftarrow \sigma y\} \circ \{x \leftarrow za, y \leftarrow z\}$ [x,y]:

$\{z \leftarrow \sigma y\} \circ \{x \leftarrow za, y \leftarrow z\} x =_{AI} (\sigma y)a =_{AI} (\sigma x)a =_{AI} \sigma x$.

$\{z \leftarrow \sigma y\} \circ \{x \leftarrow za, y \leftarrow z\} y = \sigma y$. ■

The above lemmas show:

2.3.4 Lemma. Every unifier σ of xa and ya that is not an instance of $\sigma_1, \sigma_2, \sigma_3$ has the following properties:

i) $\sigma x \neq_{AI} \sigma y$.

ii) the last symbol of σx and σy is not the constant a.

iii) The constant a is either contained in σx or σy. ■

2.3.5 Lemma. Let s,t be irreducible strings that start and stop with variables and let a be a symbol with $a \notin Sy(s)$ and $a \in Sy(t)$. Furthermore let $sa =_{AI} ta$.

Then $t = t_1 a t_2$ with $a \notin Sy(t_1)$ and $a \notin Sy(t_2)$.

Proof. Assume for contradiction that the lemma is false. Then $t = t_1 a t_2 a \ldots a t_n$ with $a \notin Sy(t_i)$ and $n \geq 3$.

We can assume that the sum of the lengths of s and t is minimal.

Obviously we have $s =_{AI} t_1$ and $Sy(s) \supseteq Sy(t_i)$. Let u_1 be the first variable of s and t. Let $s = u_1 s'$ and $t = u_1 t'$.

1) u_1 occurs in s' or t':

Otherwise it is $s'a =_{AI} t'a$. If the first element of s' is a variable, then s', t' is a smaller pair than s,t which is a contradiction. The other case is $s = u_1$ and $t = u_1 a u_1$, which is a contradiction, too. □

Let s_l, t_l be the full prefices of s and t, respectively and let s_r, t_r be the full suffices of s and t, respectively.

2) t_r covers $t_2 a \ldots a t_n$:

Otherwise t is reducible by rule 2, since $Sy(t_1 a) = Sy(t) = Sy(t_r)$.

3) t_r covers $a t_2 a \ldots a t_n$:

If $t_r = t_2 a \ldots a t_n$, then $t_r a$ is a full suffix of ta. Since $s_r a$ is a full suffix of sa, we have $t_r a =_{AI} s_r a$. Minimality of s,t implies $t_r = t_2 a t_3$. Hence $t_2 a t_3 a =_{AI} s_r a$, hence $t_2 =_{AI} s_r$. Multiplying $t_1 a$ from left we obtain: $t_1 a =_{AI} t_1 a t_2 a t_3 a =_{AI} t_1 a t_2 a$. But then t contains the reducible substring $t_1 a t_2 a$, a contradiction. □

4) Final contradiction:

We have proved that $t_r = t_1' a t_2 a \ldots a t_n$. Since $t_r a$ is a full suffix of ta and $s_r a$ is a full suffix of sa, we have $t_r a =_{AI} s_r a$. From 1) it follows that $t_r \neq t$ or $s_r \neq s$. Hence n = 2. ■

2.3.6 Lemma. Every unifier σ of xa and ya that is not an instance of $\sigma_1, \sigma_2, \sigma_3$ is an instance of $\sigma_4 := \{x \leftarrow z_1 z_2, y \leftarrow z_1 z_2 a z_2\}$ or $\sigma_5 := \{x \leftarrow z_1 z_2 a z_2, y \leftarrow z_1 z_2\}$ or
$\sigma_6 := \{x \leftarrow z_1 z_2\, z_3 z_4\, z_2 a\, z_3 z_2\, z_3 z_4\, z_2, y \leftarrow z_1 z_2\, z_3 z_4\, z_2 a\, z_4 z_2\, z_3 z_4\, z_2\}$

Proof. Assume by contradiction that there exists a unifier σ of xa and ya that is not an instance of a σ_i for i = 1,...,6. We can assume that the sum of the lengths of the strings σx and σy is minimal. Furthermore we can assume that σx and σy are in normalform. We use as abbreviation $\sigma x = s$ and $\sigma y = t$.

1) s and t contain occurrences of a:

It follows from Lemmas 2.3.4 that one of them contains an occurrence of a.

Assume t contains an occurrence of a and s is a-free. Then by Lemma 2.3.5 $t = t_1 a t_2$.

Now σ is an instance of σ_4: $\sigma =_{AI} \{z_1 \leftarrow s, z_2 \leftarrow t_2\} \sigma_4 [x,y]$. Note that $t_1 t_2 =_{AI} t_1$. □

2) We can assume that the first symbol of s and t is a variable.

Otherwise we can replace the a at the start by a new variable and obtain a more general unifier with the same number of symbols.

Let u_1 be the first variable of s and t. Let $s = u_1 s'$ and $t = u_1 t'$

3) u_1 occurs in s' or t':

Assume s' and t' do not contain u_1. Then $s'a =_{AI} t'a$. Let $\sigma' := \{x \leftarrow s', y \leftarrow t'\}$ be the corresponding unifier of xa and ya. Since σ is minimal, σ' is an instance of some σ_i. The structure of the σ_i implies that σ is also an instance of the same σ_i. □

Let s_l, t_l be the full prefices of s and t, respectively and let $s_r a$, $t_r a$ be the full suffices of sa and ta, respectively. $\sigma' := \{x \leftarrow s_r, y \leftarrow t_r\}$ is a unifier of xa and ya that is an instance of some σ_j. Hence we have $\sigma' =_{AI} \lambda\sigma_j [x,y]$ for some λ. Let $s_r' := s_{r1} s_r$ and $t_r' := t_{r1} t_r$ be the full suffices of s and t, respectively.

4) Either s_r or t_r is a-free.

If both contain an occurrence of a, then $\sigma =_{AI} \lambda'\sigma_j [x,y]$ with $\lambda' z_1 := s_l \lambda z_1$.

$\lambda'\sigma_j x = s_l s_r =_{AI} s_l s_{r1} s_r =_{AI} s$, since $Sy(s_l) = Sy(s_r)$.

$\lambda'\sigma_j y = s_l t_r =_{AI} t_l t_{r1} t_r =_{AI} t$, since $Sy(t_l) = Sy(t_r)$. □

Assume that s_r is a-free.

5) t_r is a-free:

If t_r contains an a, then $\sigma =_{AI} \lambda'\sigma_j [x,y]$ with $\lambda' z_1 := s_l s_{r1} \lambda z_1$.

$\lambda'\sigma_j x = s_l s_{r1} s_r =_{AI} s$ and $\lambda'\sigma_j y = s_l s_{r1} t_r = t_l s_{r1} t_r =_{AI} t_l t_r =_{AI} t$,

since $Sy(t_l) = Sy(t_r)$. □

6) Final contradiction.

s_r and t_r are both a-free. Then it is $s_r =_{AI} t_r$.

Let $s_r' := a s_{r1} s_r$ and $t_r' := a t_{r1} t_r$ be the full suffices of s and t, respectively. Now σ is an instance of σ_6:

$\theta\sigma_6\, x = s_l s_r\, s_{r1}\, t_{r1} s_r\, a\, s_{r1} s_r\, s_{r1}\, t_{r1} s_r$

$=_{AI}\ s_l\ a\ s_{r1} s_r\, s_{r1}\, t_{r1} s_r$ since $Sy(s_l) = Sy(s_r a) \supseteq Sy(s_r s_{r1}\, t_{r1})$.

$=_{AI}\ s_l\ a\ s_{r1} s_r$ since $Sy(s_r) \supseteq Sy(s_{r1}\, t_{r1})$.

$=_{AI}\ s$.

Analogously, we obtain $\theta\sigma_6\, y =_{AI} t$.

The case where s_{r1} or t_{r1} are empty can be treated in the same way by using the component $z_3 \leftarrow s_r$ or $z_4 \leftarrow t_r$ instead of the components $z_3 \leftarrow s_{r1}$, $z_4 \leftarrow t_{r1}$. ■

2.3.7 Proposition. $\mu U\Sigma(xa, ya) = \{\sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5, \sigma_6\}$.

Now we analyze the example in §2.2 more thoroughly and show that $U\Sigma(zaxaz, zaz)$ has width (1,1):

2.3.8 Lemma. $\langle zaxaz = zaz\rangle_{AI}$ has the most general unifier $\tau := \{z \leftarrow zxz\}$ and the chain $\mu_i := \{x \leftarrow u_1 a u_2 a \ldots a u_n,\ z \leftarrow u_{n+1} u_1 u_2 \ldots u_n u_{n+1}\}$. The set $\{\tau, \mu_i\}$ is a complete set of unifiers.

Proof.

1) If θ is a unifier with $a \in Sy(\theta z)$ or $a \notin Sy(\theta x)$ then $\theta \le \tau[x,z]$:

We have $\theta = \theta\tau\,[x,z]$: $\theta z\ \theta x\ \theta z =_{AI} \theta z$ under the conditions above.

2) If θ is a unifier with $a \notin Sy(\theta z)$ and $a \in Sy(\theta x)$ then there exists a n with $\theta \le \mu_n\,[x,z]$:

We have $\theta x = s_1 a \ldots a s_m$ with $a \notin Sy(s_i)$. Choose $n = m$ and λ as follows:

$\lambda u_i := s_i$, $\lambda u_{n+1} := \sigma z$. Then $\sigma =_{AI} \lambda\mu_n\,[x,z]$.

The case were s_1 or s_m is empty can be treated with $\lambda u_1 := a$,

3) Obviously μ_i is an ascending chain of substitutions. Furthermore τ and μ_i are independent. ■

Theorem 2.2.3 and Lemma 2.3.8 show that the following holds:

2.3.9 Proposition. $U\Sigma(zaxaz, zaz)$ has width (1,1):

2.4 A Lower Bound for the Maximal Unification Type of A+I.

2.4.1 Theorem. If one constants a is available, then the maximal unification type of A+I is at least (ω,ω).

Proof. We show that for every n there exist terms s,t such that $U\Sigma_{AI}(s,t)$ has type $\ge (n,n)$

Let s_i, t_i, i=1,...,n-1 be variants of the problem $\langle xa =_{AI} ya\rangle$ with most general unifiers σ_1 and σ_2 given as in Lemma 2.3.2 and 2.3.3. Let s_n and t_n be variants of the problem $\langle zxz =_{AI} zyz\rangle$. This problem has a most general unifier $\{x \leftarrow y\}$ and a nonempty $\eta(U\Sigma_{AI}(zxz, zyz))$ [Ba86].

Now let $s = s_1 s_2 \ldots s_{n-1} s_n$ and $t = t_1 t_2 \ldots t_{n-1} t_n$ and let τ_i be unifiers of s_i and t_i, such that the codomains are pairwise disjoint. Consider all possible combinations of these unifiers. Every such combination unifies s and t.

Let μ be a unifier of s and t with $\tau_1 \circ \ldots \circ \tau_n \le \mu\ [V(s,t)]$. Since all codomains of the τ_i's are disjoint, the variable sets $V(\mu(s_i), \mu(t_i))$ are disjoint. Furthermore the first symbol of μx is a variable, since every $\tau_i x$ starts with a variable.

Lemmas 2.1.1 and 2.1.4 show that $\mu s_i = \mu t_i$ for all i, hence different combinations $\tau_1 \circ \ldots \circ \tau_n$ are independent. Hence there are at least 2^{n-1} elements in $\mu(U\Sigma_{AI}(s,t))$ and at least 2^{n-1} independent elements in $\eta(U\Sigma_{AI}(s,t))$. ■

2.5 Rewrite Systems on Free Idempotent Semigroups.

In [SS83] a conditional rewrite system for idempotent semigroups is presented. However as unconditional rewrite rule systems are much more preferable in practice, there remain the open problem, if a construction as in [SS83] is really necessary.

We now show that there exists no unconditional canonical rewriting system for idempotent semigroups.:

In the following we denote the A+I-normalform of a term t with $t\downarrow$ and the normalform with respect to a rewrite rule system R with $t\downarrow_R$.

We can assume that there are no constants. Hence every string in the following is a string of variables.

2.5.1 Theorem. There does not exist a finite unconditional canonical string rewriting system for the equation $xx \to x$.

Proof.

i) Suppose there exists a canonical rewrite rule system on strings $R = \{l_i \to r_i \mid i = 1,\ldots,n\}$.

1) We can assume that all r_i are in R-normalform.

2) If $s \to s'$, then $|s| > |s'|$. Particularly, $|l_i| > |r_i|$ for all rewrite rules.

Assume by contradiction that $s \to s'$ and $|s| \le |s'|$. We have $x^{|s|} \to x^{|s'|}$, since $x^{|s|}$ is an instance of s. Then for the term $x^{|s|}$ there exists an infinite reduction, since $x^{|s|}$ is reduced to $x^{|s'|}$ and $x^{|s|}$ is a substring of $x^{|s'|}$. □

Let m be the maximal length of all l_i. Consider the term $t = z_m z_1 z_2 \ldots z_{m-2}\, z_{m-1}\, z_1\, z_m z_{m-1} \ldots z_2\, z_1$.

3) Obviously the A+I-normalform of t is $t\downarrow = z_m z_1 \ldots z_{m-1} z_m z_{m-1} \ldots z_1$

4) All substrings of length ≥ 2 of the string t are different and all substrings of length ≥ 2 of $t\downarrow$ are different:

It suffices to consider substrings of length 2. By construction these substrings are all different.

5) All proper substrings of t and $t\downarrow$ are in normalform:

Rule 1 is never applicable due to 4). Obviously Rule 2 is not applicable to proper substrings of t.

6) There exists no rule $l \to r$ in R, which reduces t.

Assume $l \to r$ reduces t.

Then l must reduce the term t at toplevel, since all substrings are in normalform.

Let $l = y_1 \ldots y_k$,where y_i are variables. Let σ be a sbstitution with $\sigma l = t$, then $\sigma r = t\downarrow$, since $|t| > |\sigma r|$ and the only possibility for σr is $t\downarrow$. It follows from 4) that for all variables x that occur at least twice in l we have σx is a variable.

There exists a variable y in t such that σy is not a variable, since $|l| < |t|$. It follows from 4) that y occurs exactly once in l. Since $r =_{AI} l$, $y \in V(r)$ and y occurs exactly once in r due to 4). Hence we have the representation $l = l_S\, y l_E$, $r = r_S y r_E$ and $y \in Sy(l_S, l_E, r_S, r_E)$.

Lemma 2.1.4 yields $l_S =_{AI} r_S$ and $l_E =_{AI} r_E$. Furthermore either $|l_S| > |r_S|$ or $|l_E| > |r_E|$. Repeating this argument we obtain nonempty substrings s_l of l and s_r of r with the property: $s_l =_{AI} s_r$, $|\sigma s_l| = |s_l|$, $|\sigma s_r| = |s_r|$ and $|s_l| > |s_r|$.

This means σs_l and σs_r are proper substrings of t and $t\downarrow$ respectively that are equal under idempotence and have a different number of symbols. Such substring do not exist due to 5) ■

2.5.2 Theorem

There does not exist a finte unconditional canonical rewrite rule system for idempotent semigroups.

Proof.

i) Suppose there exists a canonical rewrite rule system $R = \{l_i \to r_i \mid i = 1,\ldots,n\}$.

1) We can assume without loss of generality that $f(f(x\,y)\,z) \to f(x\,f(y\,z))$ is in R and that normalforms are of the form $f(x_1\, f(x_2\, f(\ldots)))$:

The terms x and f(x y) are in normalform, since otherwise R is nonterminating.
We have: f(x f(y z)) is equal to f(f(x y) z). Hence they can be reduced to the same normalform. If none of them is in normalform, then reduction cannot terminate, since we always can move brackets around. Assume f(f(x y) z) is irreducible. Then there exists a reduction from f(x f(y z)) to f(f(x y) z)
Hence we can add the rule f(x f(y z)) $\to$ f(f(x y) z) to R without changing canonicity or normalforms.

For convenience we call a term fully reduced by f(f(x y) z) $\to$ f(x f(y z)) and containing only variables in standard-form and denote them as a string of their variables. The set of all terms in standardform is denoted as T_S. We denote with $t\downarrow$ the A+I-normalform of t in standardform

2) For every term t in standardform: If $t \to t'$, t has more symbols than t':
Assume for contradiction $|t| \le |t'|$. Obviously we can reduce t' to a term t'' in standardform with $|t'| = |t''|$. This means that a term t_x obtained from t by making all variables equal reduces to a term t''_x. Both t_x and t''_x are in standardform, hence t_x is a subterm of t''_x. This is a contradiction to the termination of R. □
3) We can assume that all r_i are in standardform.
4) The subsystem R_S of rules with left side in standardform is a canonical rewrite rule system on the set T_S of terms in standardform:
Termination follows from 1).
If a rule $l \to r$ reduces a term in standardform, then l is in standardform. Assume R_S is not confluent. Then there exists an R_S-irreducible term t in standardform such that $t \ne t\downarrow$. Since R is confluent, and $|t| > |t\downarrow|$, R reduces t. But every rule that reduces t is in R_S. This contradiction shows that R_S is confluent.
5) R_S reduces every term t to its normalform $t\downarrow$.
6) Let $R_{S,A}$ be the associative version of R_S. Then $R_{S,A}$ is a canonical rewrite rule system for idempotency on strings:
Obviuosly $R_{S,A}$ reduces strings t to their normalform $t\downarrow$. Furthermore $R_{S,A}$ satisfies 2), i.e. shortens every string during reduction, since R_S does so. Hence $R_{S,A}$ is canonical.
This is a contradiction to the Theorem 2.5.1 above ■

Acknowledgements.

I would like to thank: R. Göbel for his support concerning rewriting systems and J.Siekmann, F. Baader, A.Herold and H.J. Bürckert and for reading and improving some drafts of this paper.

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Appendix

Let U be a countable quasi-ordered set. In the following we denote with $[b,\infty]$ the subset $\{a \mid b \le a\}$ of U.

A.1 Lemma Let B be an independent subset of U.

Then the sets $[b,\infty]$ are mutually disjoint, where $b \in B$.

Furthermore if B is maximal independent then $\cup\{ [b,\infty] \mid b \in B\}$ is complete in U.

Proof. The disjointness of $[b_1,\infty]$ and $[b_2,\infty]$ follows from the independence of B.

Let B be a maximal independent subset of U and let $u \in U$ be an arbitrary element in U. Then $B \cup \{u\}$ is dependent, hence there exists $v \in U$ and $b \in B$ with $u \le v$ and $b \le v$. Thus $v \in [b,\infty]$ and we have shown that $\cup \{ [b,\infty] \mid b \in B\}$ is complete in U.

A.2 Lemma Let B, C be maximal independent subsets of U with $|C| > |B|$.

Then there exists a $b \in B$ and $c_1 \neq c_2 \in C$ with $[b,\infty] \cap [c_1,\infty] \neq \emptyset$ and $[b,\infty] \cap [c_2,\infty]$.

Proof. Assume by contradiction that the lemma is false.

Then for all $b \in B$ there exists at most one $c \in C$ with $[b,\infty] \cap [c,\infty] \neq \emptyset$.

Since $|C| > |B|$ there exists a $c_0 \in C$ such that $[b,\infty] \cap [c_0,\infty] = \emptyset$ for all $b \in B$.

This means the set $B \cup \{c_0\}$ is independent, a contradiction to the maximality of B.

A.3 Lemma. Let B be an independent subset of U and let C_b be independent subset of U contained in $[b,\infty]$.

i) Then $\cup \{C_b \mid b \in B\}$ is independent.

ii) If B is maximal independent and the sets C_b are maximal independent, then $\cup \{C_b \mid b \in B\}$ is maximal independent.

Proof. i) Obvious.

ii) Assume $\cup\{C_b \mid b \in B\}$ is not maximal independent. Then there exists a $c \in U$ such that $\cup\{C_b \mid b \in B\} \cup \{c\}$ is independent. By Lemma A.1 there exists an element $d \in \cup\{[b,\infty] \mid b \in B\}$ such that $d \ge c$. Let $d \in [b',\infty]$. Application of Lemma A.1 to $C_{b'}$ yields an element $d' \in \cup\{[c',\infty] \mid c' \in C_{b'}\}$ with $d' \ge d \ge c$. That is a contradiction to the independence of $\cup \{C_b \mid b \in B\} \cup \{c\}$. ■

A.4 Theorem. Let U be a quasi-ordered set without maximal elements.

Then either the cardinality of independent subsets B is bounded by a natural number n_0 or there exists an infinite, independent subset B of U.

Proof. By contradiction.

Assume the theorem is false.

Then there exists a sequence of finite, maximal independent subsets B_i of U with $|B_{i+1}| > |B_i|$.

Our aim is to construct an infinite independent subset B of U:

1) For every chain B_i of maximal independent subsets of U and for every i there exists a $b \in B_i$ such that the set $[b,\infty]$ contains a sequence of finite, maximal independent subsets C_j with $|C_{j+1}| > |C_j|$:

Assume the assertion is false.

Then for every $b \in B_i$ the number of elements in a maximal independent subset of $[b,\infty]$ is bound.

For every $b \in B_i$ let D_b be an independent set in $[b,\infty]$ of maximal cardinality.

Then the set $D := \cup D_b$ is a maximal independent subset of U due to Lemma A.3

There exists a maximal independent set B_k with $|B_k| > |D|$, since the cardinality of maximal independent subsets is not bound.

Lemma A.2 shows that there exist elements $b_{1,k}, b_{2,k} \in B_k$ and an element $d \in D$ such that there exist elements $b_{1,k}' \in [b_{1,k},\infty] \cap [d,\infty]$ and $b_{2,k}' \in [b_{2,k},\infty] \cap [d,\infty]$. The element d is in some $D_{b'}$. The replacement of the element d in $D_{b'}$ with the elements $b_{1,k}', b_{2,k}'$ yields an independent subset in $[b',\infty]$ of greater cardinality than $D_{b'}$. This is a contradiction.

iii) There exists an infinite , independent subset of U:

Let b_2 be the element of B_2 that satisfies ii), i.e. there exists a chain C_j of maximal independent subsets in $[b_2,\infty]$ with $|C_j| < |C_{j+1}|$. We define $D_1 := B_2 \setminus \{b_2\}$. Not that $D_1 \neq \emptyset$.

The same construction yields a nonempty set $D_2 \subsetneq [b_2,\infty]$ and an element b_3 such that $[b_3,\infty]$ contains an chain according to ii).

Repeating the construction we obtain an infinite sequence of independent subsets D_i of U with the additional property that their union is independent. The set $\cup\{D_i | i = 1,2,\ldots\}$ is an infinite independent subset of U.

We have reached a contradiction. ■

A.5 Lemma . Let U be a countable quasi-ordered set with width(U) = (0,1). I.e. the set U has no maximal elements and the maximal cardinality of an independent subset is 1.

Then there exists an increasing chain C of elements of U such that C is complete in U.

Proof. Let $u_1, u_2, \ldots$ be the elements of U. Let $c_1 := u_1$ and define c_i recursively such that c_{i+1} is an element greater than c_i and u_i.

Then obviously $C := \{c_1, c_2 \ldots\}$ is an increasing chain and C is a complete subset of U. ■

Note that the lemma is false for noncountable quasi-ordered sets:

The set of all finite subsets of a noncountable set S ordered by the subset ordering has width (0,1), but every increasing chain C covers only a countable subset of the set S.

A.6 Lemma . Let U be a countable quasi-ordered set with width(U) = (0,n). I.e. the set U has no maximal elements and the maximal cardinality of an independent subset is n.

Then there exist n increasing chains C_i of elements of U such that $\cup C_i$ is complete in U.

Proof follows from A.1 and A.5 ■

If width($U\Sigma(s,t)$) = ∞ , then the number of increasing chains, that form a complete subset may be not countable. Consider for example an infinite binary tree .