CLASSIFICATION AND HOMOLOGICAL INVARIANTS OF COMPACT QUANTUM GROUPS OF COMBINATORIAL TYPE

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Abstract

Compact quantum groups can be found by solving certain combinatorics problems, as first shown by Banica and Speicher. Any system of partitions of finite sets which is closed under reflection and two kinds of concatenation gives rise to a quantum subgroup of the free orthogonal quantum group. Later Freslon, Tarrago and Weber extended this construction to colored partitions. Only recently, Mančinska and Roberson generalized this from finite sets to finite graphs. The present thesis contributes to the classification programs for quantum groups induced by two-colored partitions in Chapter 1 and those induced by uncolored graphs in Chapter 2. While these constructions produce numerous quantum groups, little is known about which of those are actually new and not isomorphic to others. In an effort to elucidate this, Chapter 3 shows that any such quantum group interpolating the unitary group and the free unitary quantum group can be written as a quotient of a wreath graph product of one of the two. Another way of making distinctions between such quantum groups of combinatorial type is to study quantum group invariants, such as cohomology. Chapter 4 computes the first order with trivial coefficients for the discrete duals of all of Tarrago and Weber's quantum groups. For a handful of those Chapter 5 computes the L^2-Betti numbers following Bichon, Kyed and Raum's method. Chapter 6 proposes a common categorial framework covering all the aforementioned constructions for the first time.

Zusammenfassung

Durch das Lösen gewisser Kombinatorikrätsel lassen sich kompakte Quantengruppen finden, wie von Banica und Speicher gezeigt. Jede Sammlung von Partitionen endlicher Mengen, die unter Spiegelung und zwei Arten Konkatenierung abgeschlossen ist, ergibt eine Unterquantengruppe der freien orthogonalen Quantengruppe. Freslon, Tarrago und Weber erweiterten dies auf "gefärbte Partionen". Erst kürzlich ersetzten Mancinska und Roberson die endlichen Mengen durch endliche Graphen. Die Dissertation trägt zu zwei entsprechenden Klassifikationsvorhaben bei: zweifarbige Partitionen in Kapitel 1, ungefärbte Graphen in Kapitel 2. Zwar ergeben sich viele Quantengruppen. Doch ist nur wenig darüber bekannt, welche davon tatsächlich neu sind. Um dieser Frage nachzugehen, wird in Kapitel 3 bewiesen, dass jede solche Quantengruppe zwischen der unitären Gruppe und der freien unitären Quantengruppe Quotient eines Kranzgraphprodukts einer dieser beiden ist. Eine andere Möglichkeit, solche Quantengruppen kombinatorischen Typs von einander zu unterscheiden bieten Invarianten wie Kohomologie. Von letzterer, mit trivialen Koeffizienten, wird in Kapitel 4 die erste Ordnung berechnet, und zwar für die diskreten Dualen aller von Tarrago und Webers Quantengruppen. Für eine handvoll davon werden in Kapitel 5 noch nach der Methode von Bichon, Kyed und Raum die L^2-Betti-Zahlen bestimmt. Kapitel 6 enthält den Vorschlag eines gemeinsamen Rahmens für erstmals alle zuvor genannten Konstruktionen von Quantengruppen.

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Contents

Introduction

At the latest with Banica and Speicher's seminal work in [BS09] it became clear that large numbers of compact quantum groups in the sense of [Wor87; Wor91; Wor98] can be constructed by combinatorial means. Namely, any set of "partitions" of finite sets which is closed under horizontal and vertical concatenation and under reflection gives rise to an entire family of compact quantum groups via the Tannaka-Krein theorem proved in [Wor88]. More precisely, each quantum group of this kind, called "easy" by Banica and Speicher, can be understood as a quantum subgroup of Wang's free orthogonal quantum group defined in [Wan95a].

Not long after their discovery, all easy quantum groups were classified in [BCS10; Web13; RW14; RW16a; RW16b]. However, in [TW18], Tarrago and Weber found a way of modifying Banica and Speicher's approach to construct even more quantum groups. By considering "two-colored partitions" rather than uncolored ones they provided a framework for finding new quantum subgroups of Wang's free unitary quantum group, also defined in [Wan95a]. While many of these so-called "unitary easy", in distinction from Banica and Speicher's "orthogonal easy", quantum groups have been classified, an unknown number remain undiscovered at the time of writing.

Tarrago and Weber's work is not the end of the story, though. In [Fre17], Freslon expanded their definitions once more to "partitions" with an arbitrary number of colors. The resulting, one might call them "general easy", quantum groups can be interpreted as quantum subgroups of free products of free orthogonal and free unitary quantum groups.

However, only recently, Mančinska and Roberson demonstrated impressively, that there is actually no need to confine oneself to "partitions" of finite sets. In [MR19], they generalized the known constructions yet again by replacing the finite sets by finite graphs. By finding sets of what they call "bi-labeled graphs" (in a reference to Lovász's language in [Lov12]) which are again invariant under certain concatenation and reflection operations, they construct "graph-theoretic" quantum groups. These include notably the quantum automorphism groups of finite simple graphs in the sense of [Ban05]. But, as Mančinska and Roberson explain in [MR19], that is really only the tip of the iceberg. They note that their construction also works for multi-graphs, directed graphs, graphs with colored vertices, graphs with colored edges and more. Thus, in truth, it seems there are few bounds on which combinatorial entities can*not* be turned in quantum groups.

It is this wealth of quantum groups, constructed from un-, two- or multi-colored partitions or all kinds of graphs, that forms the subject matter of the present thesis, gathered under the provisional umbrella term "compact quantum groups of combinatorial type". Those quantum groups are studied from three different angles, corresponding to the three parts of the thesis.

Part 1 The first part is a contribution to the programs initiated by Tarrago and Weber in [TW18] respectively Mančinska and Roberson in [MR19] to classify all unitary easy compact quantum groups and all graph-theoretic compact quantum groups.

Part 2 In the second part, some small results are offered to address the difficult question of deciding which of all the compact quantum groups of combinatorial type are isomorphic to each other and to already known quantum groups.

Part 3 Finally, building on work by Deligne in [Del07b] and Knop in [Kno07], the third part investigates a general construction, formulated in the language of enriched category theory, that unifies all the aforementioned approaches by Banica, Speicher, Tarrago, Weber, Freslon, Mančinska and Roberson at the same time.

The three parts are thematically related but logically independent. In order to make the contents of this thesis as widely and conveniently accessible as possible, great care has been taken to keep not only each part but also each chapter self-contained. In particular, every chapter is endowed with tailor-made preliminaries which address exactly the priors of the chapter in question and nothing more.

Part 1

Classification results

CHAPTER 1

Hyperoctahedral categories of two-colored partitions

Via Woronowicz's Tannaka-Krein theorem monoidal involutive categories with finite sets of two-colored points as objects and partitions of two such sets as morphisms correspond to certain quantum subgroups of Wang's free unitary quantum group, the non-commutative generalization of the unitary group. An infinite sublattice of the lattice of all such categories is classified, both in terms of partitions and generators. The categories in question, all very close to the free case, generalize the classical hyperoctahedral group. While quantum-algebraic in its implication, the chapter is purely combinatorial in its concepts, techniques and results.

1. Introduction

1.1. Background and Context. In [TW18], Tarrago and Weber initiated a classification program to determine all *categories of two-colored partitions*. Finite totally ordered sets of two sorts of points form the objects. The morphisms between two such sets are given by partitions of their disjoint union. Composition of morphisms is defined via vertical concatenation. Horizontal concatenation yields a tensor product. Swapping the two sets produces adjoint morphisms. Thus, categories of two-colored partitions are small concrete involutive monoidal categories with natural combinatorial operations.

Via a Tannaka-Krein type result of Woronowicz's (see [Wor88]) each such category induces for every dimension $N \in \mathbb{N}$ a C^*-algebraic compact (matrix) quantum group (see [Wor87], [Wor98]), namely a quantum subgroup of Wang's (see [Wan95a]) free unitary quantum group U_N^*, a non-commutative analogue of the unitary group U_N. This was shown by Freslon and Weber in [FW16], generalizing a seminal result of Banica and Speicher's from [BS09]. The second objective of the classification program from [TW18] is to find all the quantum groups arising in this way, the so-called *unitary easy quantum groups*.

Multiple partial classification results for categories of two-colored partitions have been obtained already. In [TW18], Tarrago and Weber started the program by determining all *non-crossing* categories and all categories belonging to the *group case*, i.e., all categories $\mathcal{C} \subseteq \mathcal{P}^{\circ\bullet}$ with $\mathcal{C} \subseteq \langle \sqcap_{\circ\circ}, \sqcap\sqcap\sqcap_{\circ\bullet\circ\bullet}, \uparrow_{\circ} \rangle$ or $\times_{\circ\circ}^{\circ\circ} \in \mathcal{C}$. Gromada then classified all *globally colorized* categories in [Gro18], which is to say all categories $\mathcal{C} \subseteq \mathcal{P}^{\circ\bullet}$ with $\sqcap_{\circ\circ} \otimes \sqcap_{\bullet\bullet} \in \mathcal{C}$. All *unitary half-liberations*, categories $\mathcal{C} \subseteq \mathcal{P}^{\circ\bullet}$ with $\mathcal{C} \subseteq \langle \times_{\circ\circ}^{\circ\circ} \rangle$, were foundin [MW20] and [MW21a] and all non-hyperoctahedral categories, i.e., categories $\mathcal{C} \subseteq \mathcal{P}^{\circ\bullet}$ with $\uparrow_{\circ} \otimes \uparrow_{\bullet} \in \mathcal{C}$ or $\sqcap\sqcap\sqcap_{\circ\bullet\circ\bullet} \notin \mathcal{C}$, in [MW21b], [MW21c], [MW22a], [MW22b] and [MW22c]. Hence, the remaining unknown categories are at this point in time the hyperoctahedral locally colorized non-group-case categories with crossings, categories $\mathcal{C} \subseteq \mathcal{P}^{\circ\bullet}$ with {[crossing ○○], [pair ○○]⊗[pair ●●], [block ○●○●], [singleton ○]⊗[singleton ●]} $\cap\, \mathcal{C}$ = {[block ○●○●]} and $\mathcal{C} \notin \langle$[pair ○○], [block ○●○●], [singleton ○]$\rangle$.

In the following a first partial classification result is provided for such categories (see the next subsection). We define explicitly an infinite combinatorial index set $\mathcal{R}$ (of certain sets of binary words) and a mapping $R \mapsto \mathcal{W}_R$ assigning to every $R \in \mathcal{R}$ a set $\mathcal{W}_R$ of partitions. Then, we prove that this mapping is an isomorphism of complete lattices from $(\mathcal{R}, \subseteq)$ onto the set of all hyperoctahedral subcategories of a certain category $\mathcal{W}$. All categories in this lattice except for the minimal one, $\langle$[block ○●○●]$\rangle$, are new. Moreover, we determine a natural generator $\{\pi_c \,|\, c \in R\}$ of $\mathcal{W}_R$ for every $R \in \mathcal{R}$, consisting of explicitly known partitions.

The results obtained below in a sense generalize those obtained by Raum and Weber in [RW16b] for categories of uncolored partitions in the original sense of Banica and Speicher's. More precisely, the categories $\mathcal{W}_R$ for $R \in \mathcal{R}$ bear close resemblance to the *non-group-theoretical hyperoctahedral categories of partitions* $\langle \pi_k \,|\, k \leq \ell \rangle$ for $\ell \in \mathbb{N} \cup \{\infty\}$ classified there. For a discussion of this relationship see Section 10.1.

1.2. Main Result. To obtain the *normalized color*, invert the colors of upper (i.e., domain) points (and leave the colors of lower (co-domain) points unchanged). The *cyclic order* is the one where the lower row is traversed left-to-right and the upper row right-to-left and where the leftmost lower point succeeds the leftmost upper one and the rightmost upper point the rightmost lower one.

Theorem. *If $\mathcal{R}$ is the set of all non-empty sets R of words over the alphabet $\{\circ, \bullet\}$ which are closed under the three operations of*

- □ *reflection and simultaneous color inversion,*
- □ *forgetting the first letter and*
- □ *forgetting any two neighboring letters which are different,*

then the following are true:

(a) A hyperoctahedral category $\mathcal{W}$ of two-colored partitions is given by the set of all two-colored partitions satisfying the following conditions (Theorem 4.7):
  - *(i) There are equally many points of each normalized color $\circ$ and $\bullet$ overall.*
  - *(ii) Any block has equal numbers of legs bearing normalized colors $\circ$ and $\bullet$.*
  - *(iii) Any two cyclically subsequent legs of the same block have opposite normalized colors.*
  - *(iv) Between any two cyclically subsequent points of the same block there are as many points of normalized color $\circ$ as of normalized color $\bullet$.*
  - *(v) For any two distinct blocks B and B' evenly many legs of B' lie between any two distinct legs of B, in the cyclical sense.*

(b) For every $R \in \mathcal{R}$ a hyperoctahedral category $\mathcal{W}_R$ of two-colored partitions is given by the set of all elements of $\mathcal{W}$ with the following property (Theorem 4.42):

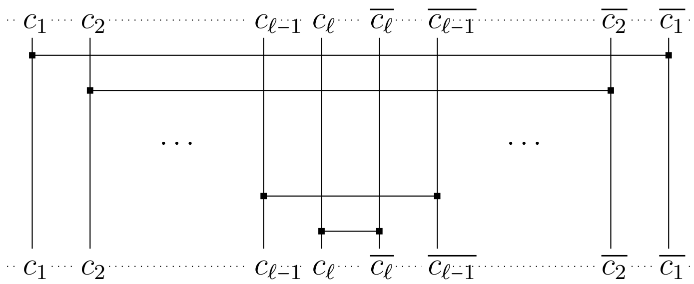
Whenever two distinct blocks A and C cross each other, the following holds for all legs α of A and γ of C:

If k is the number of blocks B with $A \neq B \neq C$ and with the property that oddly many legs of B lie between α and γ, viewed cyclically, and if these k blocks are enumerated $B_1, B_2, \dots B_k$ in the order their respective first legs appear in the space between α and γ, then an element of R is given by the following word of length $k+2$:

- □ *The leftmost letter is given by the normalized color of α if there are evenly many legs of A located properly in between α and γ, in the cyclical sense, and the opposite of that color, otherwise .*
- □ *For $i = 1, 2, \dots, k$ the i-th following letter is given by the normalized color of the first leg of B_i in between α and γ, in the cyclical sense.*
- □ *The last letter is given by the normalized color of γ if there are evenly many legs of C located properly in between α and γ, in the cyclical sense, and the opposite of that color, otherwise.*

(c) *Distinct elements R of $\mathcal{R}$ yield distinct categories $\mathcal{W}_R$ (Theorem 7.1).*

(d) *If for any word $c = (c_1, c_2, \dots, c_\ell)$ of length ℓ over $\{\circ, \bullet\}$ the partition*



is denoted by π_c, where $\overline{\circ} = \bullet$ and $\overline{\bullet} = \circ$, then for any $R \in \mathcal{R}$ the category $\mathcal{W}_R$ is generated by the set of all π_c with $c \in R$ (Theorem 6.24). The category $\mathcal{W}$ is generated by the set of all conceivable π_c (Theorem 6.19).

(e) *The mapping $R \mapsto \mathcal{W}_R$ is an isomorphism of complete lattices with respect to $\subseteq$ from $\mathcal{R}$ to the set of all hyperoctahedral subcategories of two-colored partitions of $\mathcal{W}$ (Theorem 9.4).*

In fact, the condition (iii) in (a) that subsequent legs of the same block alternate in normalized color is redundant. Moreover, there is a more easily digestible formulation of (b) which makes use of, in particular, a kind of "$\star$-betweenness" relation on the set of blocks and induced order relations (see Section 3). However, this formulation requires preparation.

1.3. Structure of the chapter. The prerequisites from the theory of two-colored partitions and their categories are briefly recapitulated in Section 2. Following that, Section 3 defines the index set $\mathcal{R}$ and the sets $\mathcal{W}_R$ for $R \in \mathcal{R}$, which include $\mathcal{W}$ as a special (maximal) case. The proof of the main theorem then proceeds as follows:

Step 1: $\mathcal{W}_R$ is category for every $R \in \mathcal{R}$ (Section 4).
Step 2: $\mathcal{W}_R$ is generated by $\{\pi_c \,|\, c \in R\}$ for every $R \in \mathcal{R}$ (Section 6).

Step 3: $(\mathcal{W}_R)_{R\in\mathcal{R}}$ are pairwise distinct (Section 7).
Step 4: For any hyperoctahedral $\mathcal{C}\subseteq\mathcal{W}$ there is $R\in\mathcal{R}$ with $\mathcal{C}=\mathcal{W}_R$ (Section 8).
Step 5: $R\mapsto\mathcal{W}_R$ is an isomorphism of complete lattices (Section 9).

The proof of the second step is assisted by an auxiliary section which introduces a set $\mathscr{R}\subseteq\mathcal{P}^{\circ\bullet}$ and gives a way to reconstruct any category $\mathcal{C}\subseteq\mathcal{P}^{\circ\bullet}$ from its set $\mathcal{C}\cap\mathscr{R}$ (Section 5). Finally, Section 10 discusses the results.

1.4. Notation. Throughout, $0\notin\mathbb{N}$. Instead, $\mathbb{N}_0=\{0\}\cup\mathbb{N}$. Moreover, $[\![k]\!]:=\{1,2,\ldots,k\}$ for any $k\in\mathbb{N}$ and $[\![0]\!]:=\varnothing$. Finally, $x\equiv_m y$ stands for $x-y\in m\mathbb{Z}$, where $m,x,y\in\mathbb{Z}$.

2. Preliminaries: Two-Colored Partitions and Their Categories

A detailed exposition of the basic definitions of and results about two-colored partitions can be found in [TW18, Section 1]. Moreover, certain concepts and theorems from [MW21b, Sections 3 and 4], from [MW21c, Sections 3 and 4] and from [MW22b, Sections 3–5] will be used. For the convenience of the reader those are repeated here, however without proofs.

2.1. Two-Colored Partitions. By a *two-colored partition*, in symbols: an element of $\mathcal{P}^{\circ\bullet}$, we mean an exhaustive division of the disjoint union of two distinguishable totally ordered (not necessarily non-empty) sets, the *upper row* (consisting of *upper points*) and the *lower row* (of *lower points*), into pairwise disjoint subsets, the *blocks* (whose elements are called *legs*), together with a binary-valued ($\circ$ or $\bullet$, which are called *inverse* or *opposite* to each other, in symbols: $\overline{\circ}:=\bullet$ and $\overline{\bullet}:=\circ$) mapping on the points, the *(native) coloring*.

Lower and upper points are addressed as ${}_{\blacksquare}x$ and ${}^{\blacksquare}x$, respectively, where x is their rank in the total order of the respective row. We say that lower ranks are to the *left* and higher ranks to the *right* of any point. We write $p\in\mathcal{P}^{\circ\bullet}(k,\ell)$ if p has k upper and ℓ lower points and then define $\|p\|:=k+\ell$. To say that B is a block of p, we write $B\in p$. Unions of blocks are called *subpartitions*. Blocks are said to be *through blocks* if they have both lower and upper legs and *lower/upper non-through blocks* if they are confined to the lower/upper row. Given partitions p and q, we say that p refines q, in symbols: $p\leq q$, if for any $B\in p$ there exists $C\in q$ with $B\subseteq C$.

The *normalized color* is given by the native color for lower points but the opposite of the native color for upper points. Assigning densities $\sigma(\circ):=1$ and $\sigma(\bullet):=-1$ with respect to normalized color defines a signed measure σ_p on the points of any $p\in\mathcal{P}^{\circ\bullet}$. The *total color sum* $\Sigma(p)$ is the color sum of the set of all points of p. Point sets with color sum 0 are called *neutral*.

On the points a *cyclic order* is imposed by extending the total order of the lower and the exact opposite total order of the upper row and by defining the least/greatest lower point the successor/predecessor of the least/greatest upper point. *Intervals* $[\alpha,\beta]_p$, $]\alpha,\beta[_p$, etc., between points α and β of $p\in\mathcal{P}^{\circ\bullet}$ with, importantly, always

$\alpha \neq \beta$ refer to this cyclic order. *Consecutive* sets are empty, singleton or interval sets of points. A *turn* is a neutral consecutive set of size 2.

The *color distance* $\delta_p(\alpha,\beta)$ from a point α of $p \in \mathcal{P}^{\circ\bullet}$ to a point β of p is defined as $\Sigma(p)$ if $\alpha = \beta$ and otherwise as $\delta_p(\alpha,\beta) = \sigma_p(]\alpha,\beta]_p) + \frac{1}{2}(\sigma_p(\{\alpha\}) - \sigma_p(\{\beta\}))$. It has properties of a "signed distance".

LEMMA 2.1. [MW20, Lemma 3.1] *For any $p \in \mathcal{P}^{\circ\bullet}$ and any points α, β, γ of p,*
(a) $\delta_p(\alpha,\alpha) \equiv 0 \mod \Sigma(p)$,
(b) $\delta_p(\alpha,\beta) \equiv -\delta_p(\beta,\alpha) \mod \Sigma(p)$,
(c) $\delta_p(\alpha,\gamma) \equiv \delta_p(\alpha,\beta) + \delta_p(\beta,\gamma) \mod \Sigma(p)$.

2.2. Operations for Two-Colored Partitions. We call a pairing (p,p') of $p,p' \in \mathcal{P}^{\circ\bullet}$ *composable* if the upper row of p and the lower row of p' are of equal size and if points of equal rank concur in native color. If for every $q \in \{p,p'\}$ the lower row of q is denoted by $R_{q,L}$ and the upper row by $R_{q,U}$, then the *composition* pp' inherits $R_{p,L}$ as lower and $R_{p',U}$ as upper row, colors included, and, if s is the join $(\vee)$ of $\{B \cap R_{p,U} \,|\, B \in p\}\backslash\{\varnothing\}$ and $\{B' \cap R_{p',L} \,|\, B' \in p'\}\backslash\{\varnothing\}$, where we identity $R_{p,U} \cong R_{p',L}$, then the blocks of pp' are precisely the lower non-through blocks of p, the upper non-through blocks of p' and the non-empty ones of the sets $\bigcup\{B \cap R_{p,L} \,|\, B \in p,\, B \cap D \neq \varnothing\} \uplus \bigcup\{B' \cap R_{p',U} \,|\, B' \in p',\, B' \cap D \neq \varnothing\}$ for blocks $D \in s$.

If for every $i \in \{1,2\}$ the lower and upper rows of $p_i \in \mathcal{P}^{\circ\bullet}$ are given by $R_{p_i,L}$ and $R_{p_i,U}$, respectively, then the *tensor product* $p_1 \otimes p_2$ of $p_1, p_2 \in \mathcal{P}^{\circ\bullet}$ has as lower and upper rows the disjoint unions $R_{p_1,L} \sqcup R_{p_2,L}$ and $R_{p_1,U} \sqcup R_{p_2,U}$, respectively, (points and colors) and the new total orders are the unique extensions of the ones of p_1 and p_2 with $\gamma_1 \leq \gamma_2$ for all $\gamma_1 \in R_{p_1,X}$, $\gamma_2 \in R_{p_2,X}$ and $X \in \{L,U\}$.

The *involution* or *adjoint* p^* of $p \in \mathcal{P}^{\circ\bullet}$ is obtained from p by declaring the lower row the new upper row and the upper row the new lower row.

To produce the *color inversion* $\overline{p}$ of $p \in \mathcal{P}^{\circ\bullet}$ all native colors are turned into their opposites. The *reflection* $\hat{p}$ results from reversing the total orders of the rows in p. And the *verticolor reflection* $\tilde{p}$ is the reflection of the color inversion of p.

If $p \in \mathcal{P}^{\circ\bullet}(k,\ell)$ with $1 \leq k$, then the *rotation* $p^{\Lsh} \in \mathcal{P}^{\circ\bullet}(k-1,\ell+1)$ is obtained by transferring ${}^{\blacksquare}1$ from the upper to the lower row, declaring it the new ${}_{\blacksquare}1$ and inverting its native color, blocks unaffected. Similarly, $p^{\Rsh} \in \mathcal{P}^{\circ\bullet}(k-1,\ell+1)$ results from considering ${}^{\blacksquare}k$ to be ${}_{\blacksquare}(\ell{+}1)$ and inverting its color. Moreover, let $p^{\curvearrowleft} := ((p^*)^{\Lsh})^*$ and $p^{\curvearrowright} := ((p^*)^{\Rsh})^*$ as well as $p^{\circlearrowright} := (p^{\curvearrowleft})^{\Rsh}$ and $p^{\circlearrowleft} := (p^{\Lsh})^{\curvearrowright}$. In the latter case we speak of (*clockwise* and *counter-clockwise*) *cyclic rotations.*

Disconnecting a set S in $p \in \mathcal{P}^{\circ\bullet}$ replaces all $B \in p$ with $B \cap S \neq \varnothing$ by the two sets $B \cap S$ and $B\backslash S$ provided they are non-empty. Somewhat conversely, if $\{B_1, B_2\} \subseteq p$, then *combining* B_1 and B_2 switches them out for the new block $B_1 \cup B_2$.

Finally, if T is a turn in p, then the *erasing* $E(p,T)$ of T from p is obtained by removing T from the set of points of p and by combining all the blocks of p which have a non-empty intersection with T into one.

2.3. Categories of Two-Colored Partitions. A *category of two-colored partitions* is any set which includes $\{\varnothing, |_{\circ}^{\circ}, |_{\bullet}^{\bullet}, \sqcap_{\circ\bullet}, \sqcap_{\bullet\circ}\}$ and which is closed under composition, tensor products and involution. There is an equivalent set of conditions which is often easier to verify.

Lemma 2.2. [MW21b, Lemma 4.3] *Any set of partitions is a category if and only if it contains* $|_{\circ}^{\circ}$ *and is closed under rotation, tensor products, verticolor reflection and erasing turns.*

We say that a category $\mathcal{C} \subseteq \mathcal{P}^{\circ\bullet}$ is *hyperoctahedral* if $\uparrow_{\circ}\otimes\uparrow_{\bullet} \notin \mathcal{C}$ and $\sqcap\!\sqcap\!\sqcap_{\circ\bullet\circ\bullet} \in \mathcal{C}$. Otherwise, it is called *non-hyperoctahedral*. These conditions have implications for the allowed block sizes:

Lemma 2.3. [TW18, Lemmata 1.3 (b), 2.1 (a)] *Let* $\mathcal{C} \subseteq \mathcal{P}^{\circ\bullet}$ *be any category.*

(a) $\langle \uparrow_{\circ}\otimes\uparrow_{\bullet} \rangle = \langle \uparrow_{\bullet}\otimes\uparrow_{\circ} \rangle = \langle {}^{\circ}_{\circ}\!\!\nwarrow \rangle = \langle {}^{\bullet}_{\bullet}\!\!\nwarrow \rangle$.

(b) $\uparrow_{\circ}\otimes\uparrow_{\bullet} \in \mathcal{C}$ *if and only if there exist* $p \in \mathcal{C}$ *and* $B \in p$ *with* $|B| < 2$.

(c) If $\uparrow_{\circ}\otimes\uparrow_{\bullet} \in \mathcal{C}$, *then* $\mathcal{C}$ *is closed under disconnecting points from their blocks.*

Lemma 2.4. [TW18, Lemmata 1.3 (d), 2.1 (b)] *Let* $\mathcal{C} \subseteq \mathcal{P}^{\circ\bullet}$ *be any category.*

(a) $\langle \sqcap\!\sqcap\!\sqcap_{\circ\bullet\circ\bullet} \rangle = \langle \sqcap\!\sqcap\!\sqcap_{\bullet\circ\bullet\circ} \rangle = \langle {}^{\circ\bullet}_{\circ\bullet} \rangle = \langle {}^{\bullet\circ}_{\bullet\circ} \rangle$.

(b) $\sqcap\!\sqcap\!\sqcap_{\circ\bullet\circ\bullet} \in \mathcal{C}$ *if and only if there exist* $p \in \mathcal{C}$ *and* $B \in p$ *with* $|B| > 2$.

(c) If $\sqcap\!\sqcap\!\sqcap_{\circ\bullet\circ\bullet} \in \mathcal{C}$, *then* $\mathcal{C}$ *is closed under combining all blocks intersecting a turn.*

2.4. Special Kinds of Partitions. A partition $p \in \mathcal{P}^{\circ\bullet}$ is called *verticolor-reflexive* if $\tilde{p} = p$. If so, then p has an even number of points in each of its rows. We say that a partition $p \in \mathcal{P}^{\circ\bullet}$ is *projective* if $p = p^* = pp$. If so and if p has $\ell \in \mathbb{N}$ with $2 \leq \ell$ lower points, then p is called a *bracket* if there exists $B \in p$ with $\{_{\blacksquare}1, _{\blacksquare}\ell\} \subseteq B \subseteq \{_{\blacksquare}1, _{\blacksquare}2, \ldots, _{\blacksquare}\ell\}$, and a *co-bracket* if there exists $B \in p$ with $\{_{\blacksquare}1, _{\blacksquare}2, \ldots, _{\blacksquare}\ell\} \cap B = \{_{\blacksquare}1, _{\blacksquare}\ell\} \neq B$.

2.5. Equivalence and Projection. For all $i \in \{1, 2\}$, let P_{p_i} denote the set of all points of $p_i \in \mathcal{P}^{\circ\bullet}$ and let $S_i \subseteq P_{p_i}$ be consecutive. We call (p_1, S_1) and (p_2, S_2) *equivalent* if $S_1 = S_2 = \varnothing$ or if the following is true: There exist $n \in \mathbb{N}$ and for each $i \in [\![2]\!]$ pairwise distinct points $\gamma_{i,1}, \ldots, \gamma_{i,n}$ in p_i such that $(\gamma_{i,1}, \ldots, \gamma_{i,n})$ is ordered in p_i and $S_i = \{\gamma_{i,1}, \ldots, \gamma_{i,n}\}$ and such that for all $\{j, j'\} \subseteq\in [\![n]\!]$ (possibly $j = j'$) the following are true:

(i) $\sigma_p(\{\gamma_{1,j}\}) = \sigma_p(\{\gamma_{2,j}\})$.

(ii) $\exists B_1 \in p_1\colon \{\gamma_{1,j}, \gamma_{1,j'}\} \subseteq B_1 \subseteq S_1$ if and only if $\exists B_2 \in p_2\colon \{\gamma_{2,j}, \gamma_{2,j'}\} \subseteq B_2 \subseteq S_2$.

(iii) $\exists B_1 \in p_1\colon \{\gamma_{1,j}, \gamma_{1,j'}\} \subseteq B_1 \nsubseteq S_1$ if and only if $\exists B_2 \in p_2\colon \{\gamma_{2,j}, \gamma_{2,j'}\} \subseteq B_2 \nsubseteq S_2$.

For any consecutive set S in any $p \in \mathcal{P}^{\circ\bullet}$ we call the unique projective partition q with lower row M such that (q, M) and (p, S) are equivalent the *projection* $P(p, S)$ of (p, S). Categories are closed under arbitrary projections.

Lemma 2.5. [MW21c, Lemma 3.3] $P(p, S) \in \langle p \rangle$ *for any consecutive set* S *of points in any partition* $p \in \mathcal{P}^{\circ\bullet}$.

3. Definition of $\mathcal{W}_R$

We define an index set $\mathcal{R}$ (Definition 3.1) and for every $R \in \mathcal{R}$ a set $\mathcal{W}_R \subseteq \mathcal{P}^{\circ\bullet}$ of two-colored partitions (Definition 3.8), which we ultimately claim to be a category.

Definition 3.1. (a) We call R a *$\mathcal{W}$-parameter set* if $R \subseteq \bigcup_{n\in\mathbb{N}}([\![n]\!] \to \{\circ,\bullet\})$ and if for all $n \in \mathbb{N}$ and all $(c_1,\dots,c_n) \in R$ the following are true:

(i) $\circ \in R$.
(ii) $(\overline{c_n}, \overline{c_{n-1}}, \dots, \overline{c_1}) \in R$.
(iii) $(c_2, c_3, \dots, c_n) \in R$ if $2 \leq n$.
(iv) $(c_1, c_2 \dots, c_{i-1}, c_{i+2}, c_{i+3}, \dots, c_n) \in R$ if $4 \leq n$, if $1 < i < n-1$ and if $c_i \neq c_{i+1}$.

(b) Denote the set of all $\mathcal{W}$-parameter sets by $\mathcal{R}$.

Note that, trivially, $\{\circ,\bullet\}$ and $\bigcup \mathcal{R} = \bigcup_{n\in\mathbb{N}}([\![n]\!] \to \{\circ,\bullet\})$ are, respectively, the smallest and largest $\mathcal{W}$-parameter sets possible.

The next six auxiliary definitions enable that of the sets $(\mathcal{W}_R)_{R\in\mathcal{R}}$. Lemma 2.5 shows that the following is well-defined.

Definition 3.2. Let $m \in \mathbb{N}_0$, let $p \in \mathcal{P}^{\circ\bullet}$ satisfy $\Sigma(p) \equiv_m 0$ and let P_p be the set of all points of p. Define the equivalence relation $\sim_{p,m}$ on P_p by

$$\alpha \sim_{p,m} \beta :\Longleftrightarrow \delta_p(\alpha,\beta) \equiv_m 0$$

for any $\{\alpha,\beta\} \subseteq P_p$, and let $\Delta_m p$ be the partition of P_p given by the set of equivalence classes of $\sim_{p,m}$. A block of $\Delta_m p$ is called an *m-part* of p.

Definition 3.3. Let P_p be the set of all points of $p \in \mathcal{P}^{\circ\bullet}$, let $\Sigma(p) = 0$, let $p \leq \Delta_0 p$, let $\varnothing \subsetneq A \subseteq P_p$ and $\varnothing \subsetneq B \subseteq P_p$, let $A \neq B$ and let $\sigma_p(A) = \sigma_p(B) = 0$. We call B *non-interferent* with A in p if $\sigma_p([\alpha,\gamma]_p \cap B) = 0$ for any $\{\alpha,\gamma\} \subseteq A$ with $\alpha \neq \gamma$.

Definition 3.4. Let $\mathcal{W}$ be the set of all $p \in \mathcal{P}^{\circ\bullet}$ such that

(i) $\sigma_p(A) = 0$ for all $A \in p$,
(ii) $p \leq \Delta_0 p$,
(iii) B is non-interferent with A for all $\{A,B\} \subseteq p$ with $A \neq B$.

Definition 3.5. For each $p \in \mathcal{W}$ define on the blocks of p the ternary relation

$$\chi_p \coloneqq \{(A,B,C) \mid \{A,B,C\} \subseteq p \colon \neg(A = B = C) \text{ and } \exists(\alpha,\gamma) \in A \times C\colon \\ \alpha \neq \gamma \text{ and } \sigma_p([\alpha,\gamma]_p \cap B) \neq 0\}.$$

Definition 3.6. For each $p \in \mathcal{W}$ between χ_p and $\mathbb{Z}$ define the binary relation

$$\lambda_p \coloneqq \{((A,B,C), \sigma_p([\alpha,\gamma]_p \cap B)) \mid (A,B,C) \in \chi_p, (\alpha,\gamma) \in A \times C\colon \\ \alpha \neq \gamma \text{ and } \sigma_p([\alpha,\gamma]_p \cap B) \neq 0\}.$$

Definition 3.7. Let $p \in \mathcal{W}$, let $\{A,C\} \subseteq p$ and let $A \neq C$. On the set

$$\langle A,C\rangle_p \coloneqq \{B \mid B \in p\colon (A,B,C) \in \chi_p\}$$

define the binary relation

$$\leq_{p,A,C} := \{(B_1, B_2) \mid \{B_1, B_2\} \subseteq \langle A, C \rangle_p \colon B_1 = B_2 \text{ or } (A, B_1, B_2) \in \chi_p\}.$$

DEFINITION 3.8. For each $R \in \mathcal{R}$ let $\mathcal{W}_R$ be the set of all $p \in \mathcal{W}$ with

$$c \in R$$

for all $n \in \mathbb{N}$, all $\{B_1, B_2, \dots, B_n\} \subseteq p$ and all $c : [\![n]\!] \to \{\circ, \bullet\}$ such that

(i) $2 \leq n$,
(ii) $B_1, B_2, \dots, B_n$ are pairwise distinct,
(iii) B_1 and B_n cross in p,
(iv) $\langle B_1, B_n \rangle_p = \{B_1, B_2, \dots, B_n\}$,
(v) $B_1 \leq B_2 \leq \dots \leq B_n$ with respect to $\leq_{p,B_1,B_n}$, and
(vi) $((B_1, B_i, B_n), \sigma(c_i)) \in \lambda_p$ for every $i \in [\![n]\!]$.

Because $\{\{\circ, \bullet\}, \bigcup \mathcal{R}\} \subseteq \mathcal{R}$ the sets $\mathcal{W}_{\{\circ,\bullet\}}$ and $\mathcal{W}_{\bigcup \mathcal{R}}$ are well-defined. And, of course, $\mathcal{W}_{\bigcup \mathcal{R}} = \mathcal{W}$. The definition also immediately implies $\mathcal{W}_{\{\circ,\bullet\}} \subseteq \mathcal{W}_{R_1} \subseteq \mathcal{W}_{R_2} \subseteq \mathcal{W}$ for all $\{R_1, R_2\} \subseteq \mathcal{R}$ with $R_1 \subseteq R_2$.

4. Invariance

In this longest and most difficult section of the chapter we show that $\mathcal{W}_R$ is a hyperoctahedral category of two-colored partitions for every $R \in \mathcal{R}$ (Theorem 4.42). It will be convenient to prove this for the special case $\mathcal{W} = \mathcal{W}_{\bigcup \mathcal{R}}$ first (Theorem 4.7).

4.1. Invariance of $\mathcal{W}$. Instead of checking the definition, we will use Lemma 2.2. Even with that, some preparatory remarks, definitions and results are in order.

NOTATION 4.1. If P_p is the set of all points of $p \in \mathcal{P}^{\circ\bullet}$ and if $B \subseteq P_p$, let

$$\delta_p^B : (P_p \backslash B) \times (P_p \backslash B) \to \mathbb{Z},\ (\alpha, \gamma) \mapsto \begin{cases} \sigma_p(B) & \text{if } \alpha = \gamma, \\ \sigma_p([\alpha, \gamma]_p \cap B) & \text{otherwise.} \end{cases}$$

LEMMA 4.2. *Let $m \in \mathbb{N}_0$, let $p \in \mathcal{P}^{\circ\bullet}$, let S be a set of points of p and let $\sigma_p(S) \equiv_m 0$. Then, for all points α, β and γ of p with $\{\alpha, \beta, \gamma\} \cap S = \emptyset$,*

(a) $\delta_p^S(\alpha, \alpha) \equiv_m 0$,
(b) $\delta_p^S(\alpha, \beta) \equiv_m -\delta_p^S(\beta, \alpha)$,
(c) $\delta_p^S(\alpha, \gamma) \equiv_m \delta_p^S(\alpha, \beta) + \delta_p^S(\beta, \gamma)$.

PROOF. (a) By definition, $\delta_p^S(\alpha, \alpha) = \sigma_p(S) \equiv_m 0$.

(b) If $\alpha = \beta$, then (b) holds by (a). If $\alpha \neq \beta$, then, as $S \backslash \{\alpha, \beta\} = S$ and $\sigma_p(S) \equiv_m 0$,

$$\begin{aligned} \delta_p^S(\alpha, \beta) &= \sigma_p([\alpha, \beta]_p \cap S) = \sigma_p(S \backslash]\beta, \alpha[_p) = \sigma_p(S) - \sigma_p(]\beta, \alpha[_p \cap S) \\ &\equiv_m -\sigma_p([\beta, \alpha]_p \cap S), \end{aligned}$$

which proves the claim.

(c) If α, β and γ are not pairwise distinct, then (c) is true by (a) and (b) together. For pairwise distinct α, β and γ, if (α,β,γ) is ordered in p, then, because $\beta \notin S$,

$$\begin{aligned}\delta_p^S(\alpha,\gamma) &= \sigma_p([\alpha,\beta]_p \cap S) + \sigma_p(]\beta,\gamma]_p \cap S)\\ &= \sigma_p([\alpha,\beta]_p \cap S) + \sigma_p([\beta,\gamma]_p \cap S)\\ &= \delta_p^S(\alpha,\beta) + \delta_p^S(\beta,\gamma)\end{aligned}$$

and, if (α,γ,β) is ordered in p instead, then, because $\gamma \notin S$,

$$\begin{aligned}\delta_p^S(\alpha,\gamma) &= \sigma_p([\alpha,\beta]_p \cap S) - \sigma_p(]\gamma,\beta]_p \cap S)\\ &= \sigma_p([\alpha,\beta]_p \cap S) - \sigma_p([\gamma,\beta]_p \cap S)\\ &= \delta_p^S(\alpha,\beta) - \delta_p^S(\gamma,\beta)\\ &\equiv_m \delta_p^S(\eta,\epsilon) + \delta_p^S(\epsilon,\theta),\end{aligned}$$

where we have used (b) in the last step. □

The invariance proof demands being aware of how the concepts featuring in the definition of $\mathcal{W}$ transform under category operations.

Remark 4.3. The definitions of the operations of rotation, verticolor reflection and tensor product have the following implications.

(a) Let $r \in \{⤸, ⤺, ⤻, ⤹\}$ and let ρ be the map rotating the points of p^r to their original positions in $p \in \mathcal{P}^{\circ\bullet}$.
  (i) In terms of blocks, $p^r = \{\rho^{-1}(B) \mid B \in p\}$.
  (ii) $\sigma_{p^r}(S) = \sigma_p(\rho(S))$ for every set S of points of p^r.
  (iii) $\rho([\gamma_1,\gamma_2]_{p^r}) = [\rho(\gamma_1),\rho(\gamma_2)]_p$ for all points γ_1 and γ_2 of p^r with $\gamma_1 \neq \gamma_2$.

(b) Let ϱ be the map reflecting the points of $\tilde{p}$ back to their positions in $p \in \mathcal{P}^{\circ\bullet}$.
  (i) In terms of blocks, $\tilde{p} = \{\varrho^{-1}(B) \mid B \in p\}$.
  (ii) $\sigma_{\tilde{p}}(S) = -\sigma_p(\varrho(S))$ for every set S of points of $\tilde{p}$.
  (iii) $\rho([\gamma_1,\gamma_2]_{\tilde{p}}) = [\varrho(\gamma_2),\varrho(\gamma_1)]_p$ for all points γ_1 and γ_2 of $\tilde{p}$ with $\gamma_1 \neq \gamma_2$.

(c) Let $p_1, p_2 \in \mathcal{P}^{\circ\bullet}$ and for every $i \in [\![2]\!]$ let S_i be the set of points of $p_1 \otimes p_2$ coming from p_i and let τ_i be the map sending the points of S_i to their original positions in p_i.
  (i) In terms of blocks, $p_1 \otimes p_2 = \{\tau_i^{-1}(B) \mid i \in [\![2]\!],\, B \in p_i\}$.
  (ii) $\sigma_{p_1\otimes p_2}(S) = \sum_{i=1}^2 \sigma_{p_i}(\tau_i(S \cap S_i))$ for every set S of points of $p_1 \otimes p_2$.
  (iii) For any $\{\neg i, i\} \subseteq [\![2]\!]$ with $\neg i \neq i$ and any $\{\gamma_1,\gamma_2\} \subseteq S_i$ with $\gamma_1 \neq \gamma_2$ the set $[\gamma_1,\gamma_2]_{p_1\otimes p_2}$ is either $\tau_i^{-1}([\tau_i(\gamma_1),\tau_i(\gamma_2)]_{p_i})$ or $\tau_i^{-1}([\tau_i(\gamma_1),\tau_i(\gamma_2)]_{p_i}) \cup S_{\neg i}$.

That leaves the transformation rules of the erasing operation to address.

Notation 4.4. Let $p \in \mathcal{P}^{\circ\bullet}$ be arbitrary, let T be a turn in p, let π be the map which sends the blocks (not the points) of p not contained in T to the blocks they become in $E(p,T)$, let ϵ be the map sending the points of $E(p,T)$ to their former positions in p and let $B \in E(p,T)$.

(a) *Parents* of B with respect to (p,T) are any two (not necessarily distinct) blocks $\{B_1,B_2\}\subseteq p$ with $\pi(B_1)=\pi(B_2)=B$ and $\epsilon(B)=(B_1\cup B_2)\backslash T$.
(b) With respect to (p,T) we say that B is
   (i) … *case* E_{I} if $B_1=B_2$ and $(B_1\cup B_2)\cap T=\varnothing$ for any parents $\{B_1,B_2\}$,
   (ii) … *case* E_{II} if $B_1=B_2$ and $T\subsetneq B_1\cup B_2$ for any parents $\{B_1,B_2\}$,
   (iii) … *case* $\mathrm{E}_{\mathrm{III}}$ if $B_1\neq B_2$ and $T\subsetneq B_1\uplus B_2$ and $\varnothing\neq B_1\cap T$ and $\varnothing\neq B_2\cap T$ for any parents $\{B_1,B_2\}$.

It is important to note that in any partition $p\in\mathcal{P}^{\circ\bullet}$ per erased turn T there can be *at most one* block of $E(p,T)$ which is not case E_{I}.

Lemma 4.5. *Let $p\in\mathcal{P}^{\circ\bullet}$, let T be a turn in p, let ϵ be the map which sends the points of $E(p,T)$ to their original positions in p, let $\{A,B,C\}\subseteq p$ and let $\{B_1,B_2\}$ be parents of B with respect to (p,T). Then, for all $\alpha\in A$ and all $\gamma\in C$,*

$$\sigma_{E(p,T)}([\alpha,\gamma]_{E(p,T)}\cap B)=\begin{cases}\sum_{j=1}^{2}\sigma_p([\epsilon(\alpha),\epsilon(\gamma)]_p\cap B_j) & \text{if } B \text{ is case } \mathrm{E}_{\mathrm{III}},\\ \sigma_p([\epsilon(\alpha),\epsilon(\gamma)]_p\cap B_1) & \text{otherwise.}\end{cases}$$

Proof. The definition of $E(p,T)$ implies $\sigma_{E(p,T)}(S)=\sigma_p(\epsilon(S))$ for all sets S of points of $E(p,T)$ and $\epsilon([\gamma_1,\gamma_2]_{E(p,T)})=[\epsilon(\gamma_1),\epsilon(\gamma_2)]_p\backslash T$ for all points γ_1 and γ_2 of $E(p,T)$ with $\gamma_1\neq\gamma_2$. Thus, we infer, because ϵ is injective,

$$\begin{aligned}\sigma_{E(p,T)}([\alpha,\gamma]_{E(p,T)}\cap B)&=\sigma_p(\epsilon([\alpha,\gamma]_{E(p,T)}\cap B))\\&=\sigma_p(\epsilon([\alpha,\gamma]_{E(p,T)})\cap\epsilon(B))\\&=\sigma_p(([\epsilon(\alpha),\epsilon(\gamma)]_p\backslash T)\cap((B_1\cup B_2)\backslash T))\\&=\sigma_p(([\epsilon(\alpha),\epsilon(\gamma)]_p\cap(B_1\cup B_2))\backslash T).\end{aligned}$$

Moreover, as $\{\epsilon(\alpha),\epsilon(\gamma)\}\cap T\subseteq\operatorname{ran}(\pi)\cap T=\varnothing$ and because T is consecutive, either $[\epsilon(\alpha),\epsilon(\gamma)]_p\cap T=\varnothing$ or $T\subseteq]\epsilon(\alpha),\epsilon(\gamma)[_p$. On the other hand, always $T\subseteq B_1\cup B_2$. In consequence,

$$\begin{aligned}&\sigma_p(([\epsilon(\alpha),\epsilon(\gamma)]_p\cap(B_1\cup B_2))\backslash T)\\&\quad=\begin{cases}\sigma_p([\epsilon(\alpha),\epsilon(\gamma)]_p\cap(B_1\cup B_2)) & \text{if } [\epsilon(\alpha),\epsilon(\gamma)]_p\cap T=\varnothing,\\ \sigma_p([\epsilon(\alpha),\epsilon(\gamma)]_p\cap(B_1\cup B_2))-\sigma_p(T) & \text{if } T\subseteq]\epsilon(\alpha),\epsilon(\gamma)[_p\end{cases}\\&\quad=\sigma_p([\epsilon(\alpha),\epsilon(\gamma)]_p\cap(B_1\cup B_2)),\end{aligned}$$

where in the last step we have used that T is a turn and thus $\sigma_p(T)=0$. In regard of the definitions of cases E_{I}–$\mathrm{E}_{\mathrm{III}}$, that proves the claim. □

We can employ two results from [MW21b] to reduce the set of conditions whose stability under category operations we have to prove.

Proposition 4.6. [MW21b, Lemmata 6.13 (a) and 7.1] *A category of two-colored partitions is given by the set $\{p\,|\,p\in\mathcal{P}^{\circ\bullet}\colon p\leq\Delta_0 p,\ \forall A\in p\colon \sigma_p(A)=0\}$.*

Theorem 4.7. *$\mathcal{W}$ is a hyperoctahedral category of partitions.*

Proof. Because $\uparrow_\circ\otimes\uparrow_\bullet$ has non-neutral blocks, $\uparrow_\circ\otimes\uparrow_\bullet \notin \mathcal{W}$. And $\sqcap_{\circ\bullet\circ\bullet} \in \mathcal{W}$ is also clear by definition. Thus, once we show that $\mathcal{W}$ is indeed a category, we can be certain that it is hyperoctahedral.

In order to prove that $\mathcal{W}$ is a category we use Lemma 2.2. Clearly, $\sqcap_{\circ}^{\circ} \in \mathcal{W}$. By Proposition 4.6 it suffices to verify that the property of all blocks being mutually non-interferent is preserved by rotations, tensor product, verticolor reflection and erasing turns. Let $p, p_1, p_2 \in \mathcal{W}$, let $r \in \{\curvearrowleft, \curvearrowright, \curvearrowleft, \curvearrowright\}$, let T be a turn in p, let $p' \in \{p^r, \tilde{p}, p_1 \otimes p_2, E(p,T)\}$, let $\{A, B\} \subseteq p'$, let $A \neq B$, let $\{\alpha, \gamma\} \subseteq A$ and let $\alpha \neq \gamma$. We show $\sigma_{p'}([\alpha,\gamma]_{p'} \cap B) = 0$.

Case 1: *Rotations.* First, let $p' = p^r$ and let ρ be the map rotating the points of p^r to their original positions in p. By Remark 4.3 (a), both $\{\rho(A), \rho(B)\} \subseteq p$ and $\rho(A) \neq \rho(B)$. Hence, B is non-interferent with A by $p \in \mathcal{W}$. As $\{\rho(\alpha), \rho(\gamma)\} \subseteq \rho(A)$ and $\rho(\alpha) \neq \rho(\gamma)$ we can therefore infer

$$0 = \sigma_p([\rho(\alpha), \rho(\gamma)]_p \cap \rho(B)) = \sigma_p(\rho([\alpha,\gamma]_{p^r} \cap B)) = \sigma_{p^r}([\alpha,\gamma]_{p^r} \cap B),$$

where we have used Remark 4.3 (a). That proves $p^r \in \mathcal{W}$.

Case 2: *Verticolor reflection.* Next, let $p' = \tilde{p}$, let ϱ be the map reflecting the points of $\tilde{p}$ back to their places in p. Again, $\{\varrho(A), \varrho(B)\} \subseteq p$ and $\varrho(A) \neq \varrho(B)$, this time by Remark 4.3 (b). Because $\varrho(B)$ is thus non-interferent with $\varrho(A)$ and because $\{\varrho(\gamma), \varrho(\alpha)\} \subseteq \varrho(A)$ and $\varrho(\gamma) \neq \varrho(\alpha)$, we conclude, using the same remark,

$$0 = \sigma_p([\varrho(\gamma), \varrho(\alpha)]_p \cap \varrho(B)) = \sigma_p(\varrho([\alpha,\gamma]_{\tilde{p}} \cap B)) = -\sigma_{\tilde{p}}([\alpha,\gamma]_{\tilde{p}} \cap B),$$

which confirms $\tilde{p} \in \mathcal{W}$.

Case 3: *Tensor products.* Now, let $p' = p_1 \otimes p_2$ and for every $i \in [\![2]\!]$ let S_i be the set of points of $p_1 \otimes p_2$ stemming from p_i and let τ_i be the map sending the points of S_i to their former positions in p_i. By Remark 4.3 (c) there exist $\{i, j\} \subseteq [\![2]\!]$ such that $A \subseteq S_i$ and $B \subseteq S_j$. In particular, $\{\alpha, \gamma\} \subseteq S_i$. We distinguish two cases.

Case 3.1: *A and B came from the same partition.* If $i = j$, then $\{\tau_i(A), \tau_i(B)\} \subseteq p_i$ and $\tau_i(A) \neq \tau_i(B)$ by Remark 4.3 (c). Thus, $\tau_i(B)$ is non-interferent with $\tau_i(A)$ by $p_i \in \mathcal{W}$. As $\{\tau_i(\alpha), \tau_i(\gamma)\} \subseteq \tau_i(A)$ and $\tau_i(\alpha) \neq \tau_i(\gamma)$ it hence follows by the same remark

$$\begin{aligned} 0 &= \sigma_{p_i}([\tau_i(\alpha), \tau_i(\gamma)]_{p_i} \cap \tau_i(B)) = \sigma_{p_i}(\tau_i(([\alpha,\gamma]_{p_1\otimes p_2} \cap B) \cap S_i)) \\ &= \textstyle\sum_{\ell=1}^{2} \sigma_{p_\ell}(\tau_\ell(([\alpha,\gamma]_{p_1\otimes p_2} \cap B) \cap S_\ell)) = \sigma_{p_1\otimes p_2}([\alpha,\gamma]_{p_1\otimes p_2} \cap B) \end{aligned}$$

because $S_1 \cap S_2 = \varnothing$ and $[\alpha,\gamma]_{p_1\otimes p_2} \cap B \subseteq S_i$ per the assumptions $B \subseteq S_j$ and $i = j$.

Case 3.2: *A and B came from different partitions.* Alternatively, let $i \neq j$. By Remark 4.3 (c), either $[\alpha,\gamma]_{p_1\otimes p_2} = \tau_i^{-1}([\tau_i(\alpha), \tau_i(\gamma)]_{p_i}) \subseteq S_i$ or $[\alpha,\gamma]_{p_1\otimes p_2} = \tau_i^{-1}([\tau_i(\alpha), \tau_i(\gamma)]_{p_i}) \cup S_j$. The assumptions $B \subseteq S_j$ and $i \neq j$ then imply $[\alpha,\gamma]_{p_1\otimes p_2} \cap B \in \{\varnothing, B\}$. In the first case,

$$\sigma_{p_1\otimes p_2}([\alpha,\gamma]_{p_1\otimes p_2} \cap B) = \sigma_{p_1\otimes p_2}(\varnothing) = 0.$$

In the second, the same remark and the inclusion $B \subseteq S_j$ yield

$$\sigma_{p_1\otimes p_2}([\alpha,\gamma]_{p_1\otimes p_2} \cap B) = \sigma_{p_1\otimes p_2}(B) = \textstyle\sum_{\ell=1}^{2} \sigma_{p_\ell}(\tau_\ell(B \cap S_\ell)) = \sigma_{p_j}(\tau_j(B)) = 0$$

by $p_j \in \mathcal{W}$ since $\tau_j(B) \in p_j$. Thus, $p_1 \otimes p_2 \in \mathcal{W}$ has been proven.

Case 4: *Erasing turns.* Finally, let $p' = E(p,T)$, let ϵ be the map sending the points of $E(p,T)$ to their former positions in p, let $\{A_1, A_2\}$ be parents of A with respect to (p,T), let $\epsilon(\alpha) \in A_1$ and let $\{B_1, B_2\}$ be parents of B. Since $A \neq B$, also $\{A_1, A_2\} \cap \{B_1, B_2\} = \varnothing$. Lemma 4.5 therefore yields

$$\sigma_{E(p,T)}([\alpha,\gamma]_{E(p,T)} \cap B) = \begin{cases} \sum_{j=1}^{2} \sigma_p([\epsilon(\alpha),\epsilon(\gamma)]_p \cap B_j) & \text{if } B \text{ is case } \mathrm{E_{III}}, \\ \sigma_p([\epsilon(\alpha),\epsilon(\gamma)]_p \cap B_1) & \text{otherwise.} \end{cases} \tag{1}$$

Two cases must be distinguished.

Case 4.1: *α and γ came from the same block of p.* First, suppose $\epsilon(\gamma) \in A_1$. As $A_1 \notin \{B_1, B_2\}$, the assumption $p \in \mathcal{W}$ then guarantees

$$\sigma_p([\epsilon(\alpha),\epsilon(\gamma)]_p \cap B_j) = 0$$

for every $j \in [\![2]\!]$. That proves the claim by (1), regardless of which case B is.

Case 4.2: *α and γ came from two different blocks of p.* Now, let $\epsilon(\gamma) \notin A_1$ instead, implying $\epsilon(\gamma) \in A_2$ and $A_1 \neq A_2$. Then, B is case $\mathrm{E_I}$ and thus $B_1 = B_2$. Because A is case $\mathrm{E_{III}}$, for every $i \in [\![2]\!]$ there exists $\tau_i \in A_i \cap T$. Then, $\{\epsilon(\alpha), \epsilon(\gamma)\} \cap \{\tau_1, \tau_2\} = \varnothing$. Therefore and because $p \in \mathcal{W}$ and $B_1 \notin \{A_1, A_2\}$ we can infer

$$\sigma_p([\epsilon(\alpha),\tau_1]_p \cap B_1) = 0 = \sigma_p([\tau_2,\epsilon(\gamma)]_p \cap B_1).$$

Furthermore, T being a turn and the identity $B_1 \cap T = \varnothing$ ensure

$$\sigma_p([\tau_1,\tau_2]_p \cap B_1) = \left\{ \begin{array}{ll} \sigma_p(B_1 \cap T) & \text{if } \tau_2 \text{ succeeds } \tau_1, \\ \sigma_p(B_1) & \text{if } \tau_2 \text{ precedes } \tau_1 \end{array} \right\} = 0$$

because $\sigma_p(B_1) = 0$ by $p \in \mathcal{W}$. In consequence, by Lemma 4.2 (c),

$$\begin{aligned} &\sigma_p([\epsilon(\alpha),\epsilon(\gamma)]_p \cap B_1) \\ &\quad = \sigma_p([\epsilon(\alpha),\tau_1]_p \cap B_1) + \sigma_p([\tau_1,\tau_2]_p \cap B_1) + \sigma_p([\tau_2,\epsilon(\gamma)]_p \cap B_1) = 0. \end{aligned}$$

Again, this proves the claim by (1). That concludes the proof overall. □

4.2. Properties of Partitions in $\mathcal{W}$. The proof that $\mathcal{W}_R$ is a category for arbitrary $R \in \mathcal{R}$ requires numerous auxiliary result about the relationship between the block structure and the coloring in partitions of $\mathcal{W}$ which go beyond what is already asked for in the definition.

4.2.1. *Colors of Subsequent Legs.* We begin by proving that in any partition of $\mathcal{W}$ the legs of any block alternate in normalized color (Proposition 4.13). That actually holds true in a larger set than $\mathcal{W}$ and the proofs are the same, which is why we show it in the more general version.

In the next crucial Lemma 4.9 we will use twice the following discrete analogue of the intermediate value theorem for functions resembling Dyck paths.

LEMMA 4.8 (Discrete Intermediate Value Theorem). *Let $a, b \in \mathbb{Z}$, let $a < b$ and let $f : \{i \in \mathbb{Z} \mid a \leq i \leq b\} \to \mathbb{Z}$. If $\partial f : \{i \in \mathbb{Z} \mid a < i \leq b\} \to \mathbb{Z}$, $j \mapsto f(j) - f(j-1)$ satisfies $\mathrm{ran}(\partial f) \subseteq \{-1, 0, 1\}$ and if $f(a) \cdot f(b) < 0$, then there exists $x \in \mathbb{Z}$ with $a < x < b$ such that $f(x) = 0$.*

Proof. The induction proof is elementary and omitted. □

We prove that the points of any 0-part alternate in normalized color, and any two subsequent points of distinct parts have identical normalized colors.

Lemma 4.9. *Let* $p \in \mathcal{P}^{\circ\bullet}$*, let* $\Sigma(p) = 0$*, let* $\{A, C\} \subseteq \Delta_0 p$*, let* $\{\alpha, \gamma\} \subseteq A \cup C$*, let* $\alpha \neq \gamma$ *and let* $]\alpha, \gamma[_p \cap (A \cup C) = \varnothing$*.*

(a) If $A = C$*, then* $\sigma_p(\{\alpha, \gamma\}) = 0$*.*

(b) If $A \neq C$*, then* $\sigma_p(\{\alpha, \gamma\}) \neq 0$*.*

Proof. We prove both claims simultaneously and distinguish two cases.

Case 1: *α and γ are neighbors.* If $]\alpha, \gamma[_p = \varnothing$, then, per definition, $\delta_p(\alpha, \gamma) = \sigma_p(\varnothing) = 0$ if and only if $\sigma_p(\{\alpha, \gamma\}) = 0$ and $\delta_p(\alpha, \gamma) = \sigma_p(\{\gamma\}) \neq 0$ if and only if $\sigma_p(\{\alpha, \gamma\}) \neq 0$. The assumption $A = C$ demands $\delta_p(\alpha, \gamma) = 0$, while the premise $A \neq C$ requires $\delta_p(\alpha, \gamma) \neq 0$. Hence, the assertion holds if $]\alpha, \gamma[_p = \varnothing$.

Case 2: *α and γ are not neighbors.* Now, let $]\alpha, \gamma[_p \neq \varnothing$ instead. Then, there exist $\ell \in \mathbb{N}$ with $1 < \ell$ and pairwise distinct points $\theta_0, \theta_1, \ldots, \theta_\ell$ such that $(\theta_0, \theta_1, \ldots, \theta_\ell)$ is ordered in p and such that $\{\theta_0, \theta_1, \ldots, \theta_\ell\} = [\alpha, \gamma]_p$, in particular, $\theta_0 = \alpha$ and $\theta_\ell = \gamma$. For every $j \in \{0\} \cup [\![\ell]\!]$ let c_j be the normalized color of θ_j. Then, we must show $c_0 \neq c_\ell$ if $A = C$ and $c_0 = c_\ell$ if $A \neq C$. Define the function $f : \{0\} \cup [\![\ell]\!] \to \mathbb{Z}, j \mapsto \delta_p(\theta_j, \theta_\ell)$.

Step 2.1: *$f^{-1}(\{0\})$ is $\{0, \ell\}$ if $A = C$ and $\{\ell\}$ if $A \neq C$.* Per definition, for any $j \in \{0\} \cup [\![\ell]\!]$, the statement $f(j) = 0$ is equivalent to $\delta_p(\theta_j, \theta_\ell) = 0$, which in turn is the same as saying $\theta_j \in C$ because $\gamma = \theta_\ell \in C$. Because $]\theta_0, \theta_\ell[_p \cap C =]\alpha, \gamma[_p \cap C = \varnothing$ per assumption, $f^{-1}(\{0\}) \subseteq \{0, \ell\}$ has thus been shown. Moreover, we can conclude that $\ell \in f^{-1}(\{0\})$ is always true and that $0 \in f^{-1}(\{0\})$ holds if and only if $A = C$ because $\alpha = \theta_0 \in A$.

Step 2.2: *$f^{-1}(\{f(0)\})$ is $\{0, \ell\}$ if $A = C$ and $\{0\}$ if $A \neq C$.* For all $j \in \{0\} \cup [\![\ell]\!]$ the statement $f(j) = f(0)$ is equivalent to $\delta_p(\theta_j, \theta_\ell) = \delta_p(\theta_0, \theta_\ell)$; and this in turn is true if and only if $0 = \delta_p(\theta_0, \theta_\ell) - \delta_p(\theta_j, \theta_\ell) = \delta_p(\theta_0, \theta_\ell) + \delta_p(\theta_\ell, \theta_j) = \delta_p(\theta_0, \theta_j)$ by Lemma 2.1. As $\alpha = \theta_0 \in A$ we have thus checked for any $j \in [\![\ell]\!]$ that $f(j) = f(0)$ if and only if $\theta_j \in A$. It follows, on the one hand, $f^{-1}(\{0\}) \subseteq \{0, \ell\}$ because $]\theta_0, \theta_\ell[_p \cap A =]\alpha, \gamma[_p \cap A = \varnothing$, and, on the other hand, $\ell \in f^{-1}(\{f(0)\})$ if and only if $A = C$ because $\gamma = \theta_\ell \in C$.

Step 2.3: *Proving* $\operatorname{ran}(\partial f) \subseteq \{-1, 0, 1\}$ *and a formula for* ∂f*.* We show that $\partial f : [\![\ell]\!] \to \mathbb{Z}, j \mapsto f(j) - f(j-1)$ satisfies $\operatorname{ran}(\partial f) \subseteq \{-1, 0, 1\}$ and find a formula for

∂f. Per definition, for all $j \in [\![\ell]\!]$,

$$\begin{aligned} f(j) - f(j-1) &= \delta_p(\theta_j, \theta_\ell) - \delta_p(\theta_{j-1}, \theta_\ell) \\ &= \sigma_p(]\theta_j, \theta_\ell]_p) + \tfrac{1}{2}(\sigma_p(\{\theta_j\}) - \sigma_p(\{\theta_\ell\})) \\ &\quad - (\sigma_p(]\theta_{j-1}, \theta_\ell]_p) + \tfrac{1}{2}(\sigma_p(\{\theta_{j-1}\}) - \sigma_p(\{\theta_\ell\}))) \\ &= -\sigma_p(]\theta_{j-1}, \theta_\ell]_p \backslash]\theta_j, \theta_\ell]_p) - \tfrac{1}{2}\sigma_p(\{\theta_{j-1}\}) + \tfrac{1}{2}\sigma_p(\{\theta_j\}) \\ &= -\tfrac{1}{2}\sigma_p(\{\theta_{j-1}, \theta_j\}) \\ &= \begin{cases} -1 & \text{if } c_{j-1} = c_j = \circ, \\ 1 & \text{if } c_{j-1} = c_j = \bullet, \\ 0 & \text{if } c_{j-1} = c_j, \end{cases} \end{aligned}$$

which in particular proves $\operatorname{ran}(\partial f) \subseteq \{-1, 0, 1\}$.

Step 2.4: *Proving* $\partial f(1) \neq 0 \neq \partial f(\ell)$. From $\{\ell\} \subseteq f^{-1}(\{0\}) \subseteq \{0, \ell\}$ by Step 2.1 it follows $\partial f(\ell) = f(\ell) - f(\ell - 1) = -f(\ell - 1) \neq 0$ because $1 < \ell$. Likewise, $\{0\} \subseteq f^{-1}(\{f(0)\}) \subseteq \{0, \ell\}$ by Step 2.2 proves $\partial f(1) = f(1) - f(0) \neq 0$, also since $1 < \ell$.

Step 2.5: *Sign of* f *and definition of* ε. We show by contradiction that there exists $\varepsilon \in \{-1, 1\}$ such that $\varepsilon f(j) > 0$ for all $j \in [\![\ell - 1]\!]$.

Suppose that there exist $\{j, j'\} \subseteq [\![\ell - 1]\!]$ with $j < j'$ such that $f(j) \cdot f(j') < 0$. Because $\operatorname{ran}(\partial f) \subseteq \{-1, 0, 1\}$, Lemma 4.8 (discrete intermediate value theorem) then ensures the existence of $i \in [\![\ell]\!]$ with $0 < j < i < j' < \ell$ and $f(i) = 0$. That contradicts the result $f^{-1}(\{0\}) \subseteq \{0, \ell\}$ of Step 2.1. Hence, an $\varepsilon \in \{-1, 1\}$ such that $\varepsilon f(j) > 0$ for all $j \in [\![\ell - 1]\!]$ exists.

Step 2.6 *Sign of* $f(0) - f$. Now, we prove that, if $f(0) \neq 0$, then $\varepsilon(f(0) - f(j)) > 0$ for all $j \in [\![\ell - 1]\!]$.

Again, assume that $f(0) \neq 0$ and that there exist $\{j, j'\} \subseteq [\![\ell - 1]\!]$ with $j < j'$ such that $(f(0) - f(j)) \cdot (f(0) - f(j')) < 0$. Because $\operatorname{ran}(\partial(f(0) - f)) = -\operatorname{ran}(\partial f) = -\{-1, 0, 1\} = \{-1, 0, 1\}$, by Lemma 4.8 there is $i \in [\![\ell]\!]$ such that $0 < j < i < j' < \ell$ and $f(0) - f(i) = 0$. This is the contradiction we sought because $f^{-1}(\{f(0)\}) \subseteq \{0, \ell\}$ by Step 2.2.

Step 2.7: $\partial f(1) \neq \partial f(\ell)$ *if and only if* $A = C$. According to Step 2.1 the assumption $A = C$ is equivalent to $f(0) = 0$. Because $\partial f(1) \neq 0 \neq \partial f(\ell)$ by Step 2.4 and because $\operatorname{ran}(\partial f) \subseteq \{-1, 0, 1\}$ by Step 2.3, it suffices to prove that $\partial f(1)$ and $\partial f(\ell)$ have different signs if and only if $f(0) = 0$ in order to see that $\partial f(1) \neq \partial f(\ell)$ if and only if $A = C$.

First, suppose $f(0) = 0$. Then, $\partial f(1) = f(1) - f(0) = f(1)$ and $\partial f(\ell) = f(\ell) - f(\ell - 1) = -f(\ell - 1)$ by Step 2.1 and thus $\varepsilon \partial f(1) = \varepsilon f(1)$ and $\varepsilon \partial f(\ell) = -\varepsilon f(\ell - 1)$. Because $\varepsilon f(1) > 0$ and $\varepsilon f(\ell - 1) > 0$ by Step 2.5, it follows $\varepsilon \partial f(1) > 0$ and $\varepsilon \partial f(\ell) < 0$. In conclusion, $\partial f(1) \neq \partial f(\ell)$.

Alternatively, let $f(0) \neq 0$. Then, $\varepsilon \partial f(1) = -\varepsilon(f(0) - f(1))$ and, still, $\varepsilon \partial f(\ell) = -\varepsilon f(\ell - 1)$ because $f(\ell) = 0$ by Step 2.1. Now, $\varepsilon f(\ell - 1) > 0$ by Step 2.5 and $\varepsilon(f(0) - f(1)) > 0$ by Step 2.6 imply $\varepsilon \partial f(1) < 0$ and $\varepsilon \partial f(\ell) < 0$. Thus, in this case, $\partial f(1) = \partial f(\ell)$.

Step 2.8: *$c_0 \neq c_\ell$ if and only if $A = C$.* From the statement $0 \neq \partial f(1) \neq \partial f(\ell) \neq 0$ which holds in case $A = C$ by Steps 2.4 and 2.7 and from our formula for ∂f found in Step 2.3 it follows $0 \neq -\frac{1}{2}\sigma_p(\{\theta_0,\theta_1\}) \neq -\frac{1}{2}\sigma_p(\{\theta_{\ell-1},\theta_\ell\}) \neq 0$ and thus $0 \neq \sigma_p(\{\theta_0,\theta_1\}) \neq \sigma_p(\{\theta_{\ell-1},\theta_\ell\}) \neq 0$. That is only possible if $c_0 = c_1 \neq c_{\ell-1} = c_\ell$.

Likewise, if $A \neq C$, then $\partial f(1) = \partial f(\ell) \neq 0$, as shown in Steps 2.4 and 2.7, then $\sigma_p(\{\theta_0,\theta_1\}) = \sigma_p(\{\theta_{\ell-1},\theta_\ell\}) \neq 0$ by Step 2.3 requires $c_0 = c_1 = c_{\ell-1} = c_\ell$. That is what we needed to see. □

Lemma 4.9 can be immediately extended to points of the same 0-part separated by an even number of points of that part.

LEMMA 4.10. *Let $p \in \mathcal{P}^{\circ\bullet}$ satisfy $\Sigma(p) = 0$. Then, $\sigma_p(\{\alpha,\gamma\}) = 0$ for all $A \in \Delta_0 p$ and all $\{\alpha,\gamma\} \subseteq A$ with $\alpha \neq \gamma$ and $|[\alpha,\gamma]_p \cap A| \equiv_2 0$.*

PROOF. For the case $]\alpha,\gamma[_p \cap A = \varnothing$ the assertion was shown in Lemma 4.9 (a). Hence, we can assume $]\alpha,\gamma[_p \cap A \neq \varnothing$. We find pairwise distinct points $\theta_0,\theta_1,\ldots,\theta_\ell$ such that $(\theta_0,\theta_1,\ldots,\theta_\ell)$ is ordered in p and $\{\theta_0,\theta_1,\ldots,\theta_\ell\} = [\alpha,\gamma]_p \cap A$, in particular $\theta_0 = \alpha$ and $\theta_\ell = \gamma$. For every $i \in \{0\} \cup [\![\ell]\!]$ let c_i be the normalized color of θ_i in p. Since $]\theta_{i-1},\theta_i[_p \cap A = \varnothing$ we find $c_i = \overline{c_{i-1}}$ by Lemma 4.9 (a) for every $i \in [\![\ell]\!]$. For every such i we infer by induction, $c_i = \overline{c_0}$ if i is odd and $c_i = c_0$ if i is even. Our assumption $|]\alpha,\gamma[_p \cap A| \equiv_2 0$ makes ℓ an odd number. It follows $c_0 = \overline{c_\ell}$, which is what we needed to see. □

Although not required for the proof of the main result, it provides a good motivation for the notion of non-interference to see that all 0-parts have this property.

REMARK 4.11. Let $p \in \mathcal{P}^{\circ\bullet}$ satisfy $\Sigma(p) = 0$. Then, $\sigma_p(A) = 0$ for all $A \in \Delta_0 p$.

PROOF. Let $k \in \mathbb{N}$ and let the pairwise distinct points $\theta_1,\theta_2,\ldots,\theta_k$ be such that $(\theta_1,\theta_2,\ldots,\theta_k)$ is ordered in p and that $\{\theta_1,\theta_2,\ldots,\theta_k\} = C$. Let $\nu : [\![k]\!] \to [\![k]\!]$ be the permutation with $i \mapsto i+1$ for all $i \in [\![k-1]\!]$ and with $k \mapsto 1$. Let P_p denote the set of all points of p. Then, the decomposition $P_p = \{\theta_1,\ldots,\theta_k\} \cup \bigcup_{i\in[\![k]\!]}]\theta_i,\theta_{\nu(i)}[_p$ and the assumption $\Sigma(p) = 0$ together imply

$$0 = \Sigma(p) = \sigma_p(P_p) = \sigma_p(\{\theta_1,\ldots,\theta_k\}) + \textstyle\sum_{i\in[\![k]\!]} \sigma_p(]\theta_i,\theta_{\nu(i)}[_p)$$

and thus the formula $\sigma_p(C) = -\sum_{i\in[\![k]\!]} \sigma_p(]\theta_i,\theta_{\nu(i)}[_p)$.

Since $]\theta_i,\theta_{\nu(i)}[_p \cap A = \varnothing$ for all $i \in [\![k]\!]$, Lemma 4.9 (a) guarantees $\sigma_p(\{\theta_i,\theta_{\nu(i)}\}) = 0$ for all $i \in [\![k]\!]$. The definition of δ_p consequently implies $\delta_p(\theta_i,\theta_{\nu(i)}) = \sigma_p(]\theta_i,\theta_{\nu(i)}[_p)$ for all $i \in [\![k]\!]$. As $\{\theta_1,\ldots,\theta_k\} \subseteq A$ means $\delta_p(\theta_i,\theta_{\nu(i)}) = 0$ for every $i \in [\![k]\!]$, we conclude $\sigma_p(]\theta_i,\theta_{\nu(i)}[_p) = 0$ for every $i \in [\![k]\!]$. Thus, $\sigma_p(C) = 0$ by the above formula. □

REMARK 4.12. If $p \in \mathcal{P}^{\circ\bullet}$ satisfies $\Sigma(p) = 0$, then any two 0-parts of p are mutually non-interferent.

PROOF. Let $\{A,B\} \subseteq \Delta_0 p$, let $A \neq B$, let $\{\alpha,\gamma\} \subseteq A$ and let $\alpha \neq \gamma$.

Case 1: *Subsequent points of A.* For simplicity let us first suppose in addition that $]\alpha,\gamma[_p \cap A = \varnothing$. If $[\alpha,\gamma]_p \cap B = \varnothing$, the claim is obviously true. Hence, we can assume $[\alpha,\gamma]_p \cap B \neq \varnothing$. Now, there exist $k \in \mathbb{N}$ and pairwise distinct $\theta_0, \theta_1, \ldots, \theta_k$ such that $(\theta_0, \theta_1, \ldots, \theta_k)$ is ordered in p and $\{\theta_0, \theta_1, \ldots, \theta_k\} = [\alpha,\gamma]_p \cap B$. Then, we have to prove $\sigma_p(\{\theta_0, \theta_1, \ldots, \theta_k\}) = 0$.

Applying Lemma 4.9 (a) to B on its own shows that $\sigma_p(\{\theta_{i-1}\}) = -\sigma_p(\{\theta_i\})$ for all $i \in [\![k]\!]$. In conclusion, $\sigma_p(\{\theta_0, \theta_1, \ldots, \theta_k\}) = 0$ holds if and only if k is odd. Moreover, k being even is equivalent to $\sigma_p(\{\theta_0, \theta_1, \ldots, \theta_k\}) = \sigma_p(\{\theta_0\}) = \sigma_p(\{\theta_\ell\})$ holding.

Lemma 4.9 (b) shows $\sigma_p(\{\alpha\}) = \sigma_p(\{\theta_0\})$ and $\sigma_p(\{\theta_k\}) = \sigma_p(\{\gamma\})$. If we apply Lemma 4.9 (a) to A, we can infer $\sigma_p(\{\alpha\}) = -\sigma_p(\{\gamma\})$. Combining both results yields $\sigma_p(\{\theta_0\}) = -\sigma_p(\{\theta_\ell\})$. That excludes $\sigma_p(\{\theta_0\}) = \sigma_p(\{\theta_\ell\})$. Hence, k must be odd. That concludes the proof in this case.

Case 2: *Not necessarily subsequent α and γ.* If $]\alpha,\gamma[_p \cap A \neq \varnothing$, then there exists $\ell \in \mathbb{N}$ with $1 < \ell$ and pairwise distinct points $\eta_0, \eta_1, \ldots, \eta_\ell$ such that $(\eta_0, \eta_1, \ldots, \eta_\ell)$ is ordered in p and such that $\{\eta_0, \eta_1, \ldots, \eta_\ell\} = [\alpha,\gamma]_p \cap A$, in particular $\eta_0 = \alpha$ and $\eta_\ell = \gamma$. For all $j \in [\![\ell]\!]$, because $]\eta_{j-1}, \eta_j[_p \cap A = \varnothing$, we can infer $\sigma_p([\eta_{j-1}, \eta_j]_p \cap B) = 0$ by Case 1. Consequently, $\{\eta_0, \eta_1, \ldots, \eta_\ell\} \cap B = \varnothing$ allows us to conclude $\sigma_p([\alpha,\gamma]_p \cap B) = \sum_{i=1}^{\ell} \sigma_p([\eta_{j-1}, \eta_j]_p \cap B) = 0$, which is what we wanted to see. □

The ensuing lemma is the general version of the claim that in any partition of $\mathcal{W}$ any two subsequent legs of the same block alternate in normalized color.

Proposition 4.13. *Let $p \in \mathcal{P}^{\circ\bullet}$, let $\Sigma(p) = 0$, let $p \leq \Delta_0 p$ and let B be non-interferent with A for all $P \in \Delta_0 p$ and all $\{A, B\} \subseteq p$ with $A \neq B$ and $A \cup B \subseteq P$.*

Then, $\sigma_p(\{\alpha,\gamma\}) = 0$ for all $A \in p$ and $\{\alpha,\gamma\} \subseteq A$ with $\alpha \neq \gamma$ and $|[\alpha,\gamma]_p \cap A| \equiv_2 0$.

Proof. Let $P \in \Delta_0 p$ be such that $A \subseteq P$ and let P be the set of all $B \in p$ with $A \neq B$ and $B \subseteq P$. Then,

$$[\alpha,\gamma]_p \cap P = [\alpha,\gamma]_p \cap (A \cup \textstyle\bigcup_{B \in \mathsf{P}} B) = ([\alpha,\gamma]_p \cap A) \uplus \biguplus_{B \in \mathsf{P}} ([\alpha,\gamma]_p \cap B)$$

For every $B \in \mathsf{P}$ the premise that B is non-interferent with A guarantees $\sigma_p([\alpha,\gamma]_p \cap B) = 0$ and thus, in particular, $|[\alpha,\gamma]_p \cap B| \equiv_2 0$. Thus, the assumption $|[\alpha,\gamma]_p \cap A| \equiv_2 0$ implies $|[\alpha,\gamma]_p \cap P| \equiv_2 0$ by the above decomposition. Now, Lemma 4.10 proves $\sigma_p(\{\alpha,\gamma\}) = 0$ because $\{\alpha,\gamma\} \subseteq P$. □

4.2.2. *Revisiting the Definition of $\mathcal{W}_R$.* The next step is to better understand the relations χ_p and λ_p for $p \in \mathcal{W}$ appearing in the definition of the sets $(\mathcal{W}_R)_{R \in \mathcal{R}}$. We show that the $\exists$-quantor in Definition 3.5 can be replaced with a $\forall$-quantor in the case of pairwise distinct blocks (Lemma 4.15). Moreover, we prove that the binary relation from Definition 3.6 is a actually a mapping to $\{-1, 1\}$ (Lemma 4.17). Finally, we characterize both relations in terms of parity conditions with the help of the result of the preceding subsection (Lemma 4.18).

Lemma 4.14. *Let $p \in \mathcal{W}$, let $\{A, B, C\} \subseteq p$ and let $A \neq B \neq C$. If $\{\alpha, \alpha'\} \subseteq A$ and $\{\gamma, \gamma'\} \subseteq C$ and if $\alpha \neq \gamma$ and $\alpha' \neq \gamma'$, then*

$$\sigma_p([\alpha,\gamma]_p \cap B) = \sigma_p([\alpha',\gamma']_p \cap B).$$

PROOF. It is crucial that $(A \cup C) \cap B = \varnothing$ and that $\sigma_p(B) = 0$ by assumption. The definition of $\mathcal{W}$ further implies $\delta_p^B(\alpha, \alpha') = 0$ and $\delta_p^B(\gamma', \gamma) = 0$, where we have used Lemma 4.2 (a). Applying Lemma 4.2 (c) twice hence yields

$$\begin{aligned}\sigma_p([\alpha,\gamma]_p \cap B) &= \delta_p^B(\alpha,\gamma)\\ &= \delta_p^B(\alpha,\alpha') + \delta_p^B(\alpha',\gamma') + \delta_p^B(\gamma',\gamma) = \delta_p^B(\alpha',\gamma') = \sigma_p([\alpha',\gamma']_p \cap B),\end{aligned}$$

which proves the claim. □

LEMMA 4.15. *Let $p \in \mathcal{W}$, let $\{A, B, C\} \subseteq p$ and let A, B and C be* pairwise distinct. *Then, the following are equivalent:*

(a) $(A, B, C) \in \chi_p$.
(b) There exist $\alpha \in A$ and $\gamma \in C$ such that $\alpha \neq \gamma$ and $\sigma_p([\alpha,\gamma]_p \cap B) \neq 0$.
(c) $\sigma_p([\alpha,\gamma]_p \cap B) \neq 0$ for all $\alpha \in A$ and $\gamma \in C$ with $\alpha \neq \gamma$.

PROOF. Follows immediately by Lemma 4.14. □

The corresponding statement is *false* if some of the three blocks are identical. Rather, if so, the extra assumptions in the following lemma are essential.

LEMMA 4.16. *Let $p \in \mathcal{W}$, let $\{A, C\} \subseteq p$ and let $A \neq C$. Then, for all $\{\alpha, \alpha'\} \subseteq A$ and $\{\gamma, \gamma'\} \subseteq C$ such that $\sigma_p([\alpha,\gamma]_p \cap A) \neq 0$ and $\sigma_p([\alpha',\gamma']_p \cap A) \neq 0$,*

$$\begin{aligned}\sigma_p([\alpha,\gamma]_p \cap A) &= \sigma_p([\alpha',\gamma']_p \cap A),\\ \sigma_p([\alpha,\gamma]_p \cap C) &= \sigma_p([\alpha',\gamma']_p \cap C).\end{aligned}$$

PROOF. We only prove the first identity. The second can be shown analogously.

Step 1: *Moving $\gamma \to \gamma'$.* First, we verify that $\sigma_p([\alpha,\gamma]_p \cap A) = \sigma_p([\alpha,\gamma']_p \cap A)$. If $\gamma = \gamma'$, this is vacuously true. Otherwise, $p \in \mathcal{W}$ and $\{\gamma,\gamma'\} \subseteq C \neq A$ imply $\sigma_p([\gamma,\gamma']_p \cap A) = \sigma_p([\gamma',\gamma]_p \cap A) = 0$. As $\{\gamma,\gamma'\} \cap A = \varnothing$, in particular $\sigma_p(]\gamma,\gamma']_p \cap A) = \sigma_p(]\gamma',\gamma]_p \cap A) = 0$. It thus follows, if (α,γ,γ') is ordered,

$$\sigma_p([\alpha,\gamma']_p \cap A) = \sigma_p([\alpha,\gamma]_p \cap A) + \sigma_p(]\gamma,\gamma']_p \cap A) = \sigma_p([\alpha,\gamma]_p \cap A),$$

and, if (α,γ',γ) is ordered

$$\sigma_p([\alpha,\gamma']_p \cap A) = \sigma_p([\alpha,\gamma]_p \cap A) - \sigma_p(]\gamma',\gamma]_p \cap A) = \sigma_p([\alpha,\gamma]_p \cap A).$$

Step 2: *Moving $\alpha \to \alpha'$.* It remains to show $\sigma_p([\alpha,\gamma']_p \cap A) = \sigma_p([\alpha',\gamma']_p \cap A)$. For $\alpha = \alpha'$ this is trivial. Hence, suppose $\alpha \neq \alpha'$. Because $p \in \mathcal{W}$, the legs of A alternate in color by Proposition 4.13. This fact can be equivalently expressed by saying that for all consecutive sets S in p, if $|A \cap S| \equiv_2 0$, then $\sigma_p(A \cap S) = 0$. Actually, the converse is true as well.

The assumptions $\sigma_p([\alpha,\gamma']_p \cap A) = \sigma_p([\alpha,\gamma]_p \cap A) \neq 0$ and $\sigma_p([\alpha',\gamma']_p \cap A) \neq 0$ therefore mean that $|[\alpha,\gamma']_p \cap A| \equiv_2 |[\alpha',\gamma']_p \cap A| \equiv_2 1$. We distinguish two cases

Case 2.1: Let (α',α,γ') be ordered in p. Then,

$$|[\alpha',\alpha[_p \cap A| = |[\alpha',\gamma']_p \cap A| - |[\alpha,\gamma']_p \cap A| \equiv_2 1 - 1 \equiv_2 0$$

and consequently, $\sigma_p([\alpha',\alpha[_p\cap A) = 0$. It thus follows

$$\sigma_p([\alpha',\gamma']_p\cap A) = \sigma_p([\alpha',\alpha[_p\cap A) + \sigma_p([\alpha,\gamma']_p\cap A) = \sigma_p([\alpha,\gamma']_p\cap A)$$

as claimed.

Case 2.2: If (α,α',γ') is ordered instead, then we deduce

$$|[\alpha,\alpha'[_p\cap A| = |[\alpha,\gamma']_p\cap A| - |[\alpha',\gamma']_p\cap A| \equiv_2 1-1 \equiv_2 0$$

and thus $\sigma_p([\alpha,\alpha'[_p\cap A) = 0$, implying

$$\sigma_p([\alpha',\gamma']_p\cap A) = \sigma_p([\alpha,\gamma']_p\cap A) - \sigma_p([\alpha,\alpha'[_p\cap A) = \sigma_p([\alpha,\gamma']_p\cap A)$$

as well. That concludes the proof. □

Lemma 4.17. *λ_p is a mapping $\chi_p\to\{-1,1\}$ for every $p\in\mathcal{W}$.*

Proof. That λ_p is a mapping is the combined result of Lemmata 4.14 and 4.16. Because the legs of p alternative in normalized color by Proposition 4.13 any color sum $\sigma_p([\alpha,\gamma]_p\cap B)$ for $B\in p$ and points α and γ of p with $\alpha\neq\gamma$ can only have the values $\{-1,0,1\}$. The definition of λ_p hence ensures $\operatorname{ran}(\lambda_p)\subseteq\{-1,1\}$. □

Lemma 4.18. *Let $p\in\mathcal{W}$, let $\{A,B,C\}\subseteq p$ and let $A\neq C$.*

(a) The following are equivalent:
  (i) $(A,B,C)\in\chi_p$.
  (ii) There exist $\alpha\in A$ and $\gamma\in C$ such that $\alpha\neq\gamma$ and $|[\alpha,\gamma]_p\cap B|\equiv_2 1$.

(b) If $A\neq B\neq C$, then the following are equivalent:
  (i) $(A,B,C)\in\chi_p$.
  (ii) There exist $\alpha\in A$ and $\gamma\in C$ such that $\alpha\neq\gamma$ and $|[\alpha,\gamma]_p\cap B|\equiv_2 1$.
  (iii) $|[\alpha,\gamma]_p\cap B|\equiv_2 1$ for all $\alpha\in A$ and $\gamma\in C$ with $\alpha\neq\gamma$.

(c) If $(A,B,C)\in\chi_p$, then $\lambda_p(A,B,C)=\sigma_p([\alpha,\gamma]_p\cap B)$ for all $\alpha\in A$ and $\gamma\in C$ with $\alpha\neq\gamma$ and $|[\alpha,\gamma]_p\cap B|\equiv_2 1$.

Proof. Because $p\in\mathcal{W}$ the legs of B alternate in color by Proposition 4.13. As a consequence, $\sigma_p([\alpha,\gamma]_p\cap B)\neq 0$ if and only if $|[\alpha,\gamma]_p\cap B|\equiv_2 1$. Now, the claims are clear by Lemmata 4.14 and 4.16. □

4.2.3. *Crossing Proliferation.* With Definitions 3.5 and 3.6 suitably understood, it is time to note a special link between the ternary relation and the binary crossing relation.

Lemma 4.19. *Let $p\in\mathcal{W}$, let $\{A,B,C\}\subseteq p$ and $A\neq C$. If A and C cross in p and if $(A,B,C)\in\chi_p$, then X crosses Y in p for any $\{X,Y\}\subseteq\{A,B,C\}$ with $X\neq Y$.*

Proof. If $A=B$ or $B=C$, there is nothing to show. Hence, we can assume that A, B and C are pairwise distinct and have to prove that B crosses both A and C.

Because A and C cross we find $\{\alpha_1,\alpha_2\}\subseteq A$ and $\{\gamma_1,\gamma_2\}\subseteq C$ such that $\alpha_1\neq\alpha_2$ and $\gamma_1\neq\gamma_2$ and such that $(\alpha_1,\gamma_1,\alpha_2,\gamma_2)$ is ordered in p. Since A, B and C are pairwise disjoint the assumption $(A,B,C)\in\chi_p$ ensures $\sigma_p([\alpha_i,\gamma_i]_p\cap B)\neq 0$ for every

$i \in [\![2]\!]$ by Lemma 4.15. In particular $[\alpha_i, \gamma_i]_p \cap B \neq \varnothing$ for each $i \in [\![2]\!]$. Hence, also $]\alpha_i, \gamma_i[_p \cap B \neq \varnothing$ since $A \neq B \neq C$. For every $i \in [\![2]\!]$ let $\beta_i \in]\alpha_i, \gamma_i[_p \cap B$ be arbitrary. It follows that $(\alpha_i, \beta_i, \gamma_i)$ is ordered for every $i \in [\![2]\!]$. Because $(\alpha_1, \gamma_1, \alpha_2, \gamma_2)$ is ordered in p, so is then $(\alpha_1, \beta_1, \gamma_1, \alpha_2, \beta_2, \gamma_2)$. Now, the ordered tuple $(\alpha_1, \beta_1, \alpha_2, \beta_2)$ gives a crossing between A and B and, likewise, $(\beta_1, \gamma_1, \beta_2, \gamma_2)$ one between B and C. □

The converse of Lemma 4.19 is false: There do exist $p \in \mathcal{W}$ and therein pairwise distinct blocks A, B and C which pairwise cross each other and still $(A, B, C) \notin \chi_p$. Otherwise, χ_p and λ_p would be of little interest.

4.2.4. *$\ast$-Betweenness.* One ingredient of the definition of the sets $(\mathcal{W}_R)_{R \in \mathcal{R}}$ has yet been ignored. The next step is to prove that the binary relation $\leq_{p,A,C}$ from Definition 3.7 for $p \in \mathcal{W}$ and $\{A, C\} \subseteq p$ with $A \neq C$ is indeed a total order on $\langle A, C \rangle_p$ (Lemma 4.28). Proving this requires us to learn many properties of the ternary relation χ_p for $p \in \mathcal{W}$ (Lemmata 4.20–4.27), which are also relevant to the invariance proof in themselves. These results will reveal that χ_p for $p \in \mathcal{W}$ fits an intuitive notion of "betweenness". We also see that λ_p for $p \in \mathcal{W}$ attaches colors these relations in an involutive way (Lemma 4.23).

The first axiom our ternary relation satisfies is a kind of *reflexivity.*

LEMMA 4.20. *Let $p \in \mathcal{W}$ and let $\{A, C\} \subseteq p$. If $A \neq C$, then both $(A, C, C) \in \chi_p$ and $(A, A, C) \in \chi_p$.*

PROOF. Let P_p denote the set of all points of p and let $\alpha \in A$ be arbitrary. The cyclic order of p induces a total order $\leq$ on $P_p \backslash \{\alpha\}$. Since $A \neq C$ the point $\gamma \coloneqq \min_{\leq}((P_p \backslash \{\alpha\}) \cap C) = \min_{\leq}(C)$ is well-defined. Then, $[\alpha, \gamma]_p \cap C = \{\gamma\}$ by definition of γ. In particular, $\sigma_p([\alpha, \gamma]_p \cap C|) \neq 0 \equiv_2 1$, implying $(A, C, C) \in \chi_p$.

Now, let $\gamma \in C$ and let $\leq$ be the total order on $P_p \backslash \{\gamma\}$ induced by the cyclic order of p. If we let $\alpha \coloneqq \max_{\leq}((P_p \backslash \{\gamma\}) \cap A) = \max_{\leq}(A)$, then $[\alpha, \gamma]_p \cap A = \{\alpha\}$ implies $\sigma_p([\alpha, \gamma]_p \cap C) \neq 0$ and thus $(A, A, C) \in \chi_p$. □

The next result shows that actually only two out of three entries of the ternary relation determine the associated colors. However, it is beneficial to index the information in this way.

LEMMA 4.21. *Let $p \in \mathcal{W}$, let $\{A, B, C\} \subseteq p$ and let A, B and C be pairwise distinct. If $(A, B, C) \in \chi_p$, then $\lambda_p(A, B, C) = \lambda_p(B, B, C) = \lambda_p(A, B, B)$.*

PROOF. By definition of λ_p there exist $(\alpha, \gamma) \in A \times C$ with $\alpha \neq \gamma$ and $\lambda_p(A, B, C) = \sigma_p([\alpha, \gamma]_p \cap B) \neq 0$. That requires in particular $[\alpha, \gamma]_p \cap B \neq \varnothing$. Because $\{\alpha, \gamma\} \subseteq A \cup C$ and $(A \cup C) \cap B = \varnothing$ we can refine this to $]\alpha, \gamma[_p \cap B = \varnothing$. Let $\leq$ be the total order induced on $]\alpha, \gamma[_p$. Then, the legs $\beta^- \coloneqq \min_{\leq}(]\alpha, \gamma[_p \cap B)$ and $\beta^+ \coloneqq \max_{\leq}(]\alpha, \gamma[_p \cap B)$ are well-defined.

The definitions of β^- and β^+ ensure that $[\alpha, \gamma]_p \cap B = [\beta^-, \gamma]_p \cap B = [\alpha, \beta^+]_p \cap B$. We conclude $0 \neq \lambda_p(A, B, C) = \sigma_p([\beta^-, \gamma]_p \cap B) = \sigma_p([\alpha, \beta^+]_p \cap B)$. That proves $\lambda_p(A, B, C) = \lambda_p(B, B, C) = \lambda_p(A, B, B)$ by Lemma 4.16. □

The property in the following lemma is a kind of *minimality.*

LEMMA 4.22. *If $p \in \mathcal{W}$, if $\{A, B, C\} \subseteq p$ and if $(A, B, C) \in \chi_p$, then $A \neq C$.*

PROOF. We show the contraposition by proving $(A, B, A) \notin \chi_p$. If $A = B$, then this is trivially true by definition. If $A \neq B$, then B is non-interferent with A by $p \in \mathcal{W}$. Hence, $\sigma_p([\alpha, \alpha']_p \cap B) = 0$ for all $\{\alpha, \alpha'\} \subseteq A$ with $\alpha \neq \alpha'$. By definition this statement is equivalent to the claim $(A, B, A) \notin \chi_p$. □

Our colored ternary relation is what could be called $*$-*symmetric.*

LEMMA 4.23. *Let $p \in \mathcal{W}$ and let $\{A, B, C\} \subseteq p$.*
(a) $(A, B, C) \in \chi_p$ if and only if $(C, B, A) \in \chi_p$.
(b) If so, then $\lambda_p(C, B, A) = -\lambda_p(A, B, C)$.

PROOF. It suffices to prove one implication of (a). Hence, let $(A, B, C) \in \chi_p$. Then, $A \neq C$ according to Lemma 4.22. We distinguish three cases.

Case 1: First, let $A \neq B \neq C$. Because $(A, B, C) \in \chi_p$ we find $(\alpha, \gamma) \in A \times C$ such that $\alpha \neq \gamma$ and $\lambda_p(A, B, C) = \sigma_p([\alpha, \gamma]_p \cap B) \neq 0$. The assumption $A \neq B \neq C$ implies $\{\alpha, \gamma\} \cap B = \varnothing$. And $p \in \mathcal{W}$ guarantees $\sigma_p(B) = 0$. Hence, we infer $\sigma_p([\gamma, \alpha]_p \cap B) = \sigma_p(]\gamma, \alpha[_p \cap B) = \sigma_p(B) - \sigma_p([\alpha, \gamma]_p \cap B) = -\sigma_p([\alpha, \gamma]_p \cap B) = -\lambda_p(A, B, C) \neq 0$. It thus follows $(C, B, A) \in \chi_p$ and $\lambda_p(C, B, A) = -\lambda_p(A, B, C)$ by Lemma 4.17.

Case 2: Next, suppose $A = B \neq C$. Then $(C, B, A) = (C, A, A) \in \chi_p$ is already clear by Lemma 4.20. We only have to prove $\lambda_p(C, A, A) = -\lambda_p(A, A, C)$. Let $\gamma \in C$, let P_p denote the set of all points of p and let $\leq$ be the total order induced on $P_p \backslash \{\gamma\}$ by the cyclic order of p. Then $\alpha^- \coloneqq \min_\leq((P_p \backslash \{\alpha\}) \cap A) = \min_\leq(A)$ and $\alpha^+ \coloneqq \max_\leq((P_p \backslash \{\alpha\}) \cap A) = \max_\leq(A)$ both exist because $A \neq C$. Moreover, $p \in \mathcal{W}$ requires $\sigma_p(A) = 0$ and thus in particular $|A| > 1$. In conclusion, $\alpha^- \neq \alpha^+$. Since $[\alpha^+, \alpha^-]_p \cap A = \{\alpha^+, \alpha^-\}$ we can deduce $\sigma_p(\{\alpha^-\}) = -\sigma_p(\{\alpha^+\})$ by $p \in \mathcal{W}$ and Proposition 4.13.

By construction, $[\gamma, \alpha^-]_p \cap A = \{\alpha^-\}$ and $[\alpha^+, \gamma]_p \cap A = \{\alpha^+\}$ and thus in particular, $\sigma_p([\alpha^+, \gamma]_p \cap A) = \sigma_p(\{\alpha^+\}) \neq 0$ and $\sigma_p([\gamma, \alpha^-]_p \cap A) = \sigma_p(\{\alpha^-\}) \neq 0$. Hence, Lemma 4.17 guarantees $\lambda_p(C, A, A) = \sigma_p([\gamma, \alpha^-]_p \cap A)$ and $\lambda_p(A, A, C) = \sigma_p([\alpha^+, \gamma]_p \cap A)$. We have thus shown $\lambda_p(C, A, A) = \sigma_p([\gamma, \alpha^-]_p \cap A) = \sigma_p(\{\alpha^-\}) = -\sigma_p(\{\alpha^+\}) = -\sigma_p([\alpha^+, \gamma]_p \cap A) = -\lambda_p(A, A, C)$, which is what we claimed.

Case 3: Finally, if $A \neq B = C$, then $(C, B, A) = (C, C, A) \in \chi_p$ also by Lemma 4.20, proving (a) fully. Part (b) is then also true by Case 2. □

The ternary relation satisfies a kind of *anti-symmetry* axiom.

LEMMA 4.24. *Let $p \in \mathcal{W}$ and let A, B and C be blocks of p.*
(a) If $(A, B, C) \in \chi_p$ and $(B, A, C) \in \chi_p$, then $A = B$.
(b) If $(A, B, C) \in \chi_p$ and $(A, C, B) \in \chi_p$, then $B = C$.

PROOF. (a) We suppose $A \neq B$ and derive a contradiction. By Lemma 4.22, the assumptions $(A, B, C) \in \chi_p$ and $(B, A, C) \in \chi_p$ demand in particular $A \neq C \neq B$. Hence, A, B and C are pairwise distinct. Let P_p denote the set of all points of p and let $\gamma \in C$ be arbitrary. The cyclic order of p induces a total order $\leq$ on the set $P_p \backslash \{\gamma\}$. Because $A \neq C \neq B$ both $\alpha \coloneqq \max_{\leq}((P_p \backslash \{\gamma\}) \cap A) = \max_{\leq}(A)$ and $\beta \coloneqq \max_{\leq}((P_p \backslash \{\gamma\}) \cap B) = \max_{\leq}(B)$ exist. Then, either $\alpha \prec \beta$ or $\beta \prec \alpha$ because $A \neq B$. In the former case, the definition of α ensures $[\beta, \gamma]_p \cap A = \varnothing$. Likewise, in the latter case, $[\alpha, \gamma]_p \cap B = \varnothing$ by the definition of β. In particular, $\sigma_p([\beta, \gamma]_p \cap A) = 0$ or $\sigma_p([\alpha, \gamma]_p \cap B) = 0$. Because $\{(B, A, C), (A, B, C)\} \subseteq \chi_p$ and because A, B and C are pairwise distinct, Lemma 4.15 tells us that $\sigma_p([\beta, \gamma]_p \cap A) \neq 0 \neq \sigma_p([\alpha, \gamma]_p \cap B)$, which is the contradiction we sought.

(b) If $(A, B, C) \in \chi_p$ and $(A, C, B) \in \chi_p$, then $(C, B, A) \in \chi_p$ and $(B, C, A) \in \chi_p$ by Lemma 4.23 (a). Now, Part (a) implies $B = C$. □

The next two results show two *transitivity* properties of our relation.

LEMMA 4.25. *Let $p \in \mathcal{W}$, let $\{A, B, C, D\} \subseteq p$ and let $\neg(B = C = D)$. Whenever $(A, B, C) \in \chi_p$ and $(A, C, D) \in \chi_p$, then $(B, C, D) \in \chi_p$.*

PROOF. We prove the claim in four steps.

Step 1: *We can assume $A \notin \{B, C, D\}$.* The assumptions $(A, B, C) \in \chi_p$ and $(A, C, D) \in \chi_p$ ensure $C \neq A \neq D$ by Lemma 4.22. If $A = B$, then the assumption $(A, C, D) \in \chi_p$ trivially proves the claim $(B, C, D) \in \chi_p$. Hence, no generality is lost in assuming $A \notin \{B, C, D\}$.

Step 2: *We have $B \neq D$.* If $B = D$ were true, then our assumptions $(A, D, C) = (A, B, C) \in \chi_p$ and $(A, C, D) \in \chi_p$ would imply $C = D$ by Lemma 4.24 (b). But the consequence $B = C = D$ contradicts our assumption $|\{B, C, D\}| \geq 2$.

Step 3: *We can assume $B \neq C \neq D$.* If we suppose $B = C$, then $|\{B, C, D\}| \geq 2$ demands $C \neq D$ and we have to show $(C, C, D) \in \chi_p$. And this is then indeed true by Lemma 4.20. The claim is also true if $C = D$: If so, then $B \neq C$ by $|\{B, C, D\}| \geq 2$. It follows $(B, C, D) = (B, C, C) \in \chi_p$ by Lemma 4.20. Therefore, we can assume $B \neq C \neq D$ in the following.

Step 4: *Proving $(B, C, D) \in \chi_p$.* We construct $(\beta, \delta) \in B \times D$ such that $\beta \neq \delta$ and $\sigma_p([\beta, \delta]_p \cap B) \neq 0$. By Steps 1–3 the blocks A, B, C and D are pairwise distinct.

Step 4.1: *Leg δ and auxiliary leg α.* Let $\delta \in D$ and let $\leq$ be the total order induced on $P_p \backslash \{\delta\}$ by the cyclic order of p, where P_p denotes the set of all points of p. Because $A \neq D$, the point $\alpha \coloneqq \min_{\leq}((P_p \backslash \{\delta\}) \cap A) = \min_{\leq}(A)$ exists.

Step 4.2: *Auxiliary leg γ.* Because A, C and D are pairwise distinct and because $(A, C, D) \in \chi_p$ per assumption Lemma 4.15 ensures $\sigma_p([\alpha, \delta]_p \cap C) \neq 0$. In particular, $[\alpha, \delta]_p \cap C \neq \varnothing$. Because $A \neq C \neq D$ it follows $]\alpha, \delta[_p \cap C \neq \varnothing$. Thus $\gamma \coloneqq \min_{\leq}((P_p \backslash \{\delta\}) \cap (]\alpha, \delta[_p \cap C)) = \min_{\leq}(]\alpha, \delta[_p \cap C)$ is well-defined.

Step 4.3: *Leg β.* Since $(A, B, C) \in \chi_p$ and since A, B and C are pairwise distinct, Lemma 4.15 guarantees $\sigma_p([\alpha, \gamma]_p \cap B) \neq 0$, which implies $[\alpha, \gamma]_p \cap B \neq \varnothing$ in particular. We infer $]\alpha, \gamma[_p \cap B \neq \varnothing$ because $A \neq B \neq C$. Hence and because $D \neq B$, we can define $\beta \coloneqq \min_{\leq}((P_p \backslash \{\delta\}) \cap (]\alpha, \gamma[_p \cap B)) = \min_{\leq}(]\alpha, \gamma[_p \cap B)$.

Step 4.4: *Proving $\sigma_p([\beta, \delta]_p \cap C) \neq 0$.* By construction then, the tuple $(\alpha, \beta, \gamma, \delta)$ is ordered in p. The definition of γ therefore ensures $[\alpha, \beta[_p \cap C = \varnothing$ and thus in particular $\sigma_p([\alpha,\beta[_p \cap C) = 0$. Consequently,

$$\sigma_p([\beta,\delta]_p \cap C) = \sigma_p([\alpha,\delta]_p \cap C) - \sigma_p([\alpha,\beta[_p \cap C) = \sigma_p([\alpha,\delta]_p \cap C) \neq 0.$$

That concludes the proof. □

LEMMA 4.26. *Let $p \in \mathcal{W}$ and $\{A,B,C,D\} \subseteq p$. If $(A,B,C) \in \chi_p$ and $(A,C,D) \in \chi_p$, then $(A,B,D) \in \chi_p$.*

PROOF. We prove the claim in three steps.

Step 1: *We can assume $C \notin \{A,B,D\}$.* From $(A,B,C) \in \chi_p$ it follows $A \neq C$ by Lemma 4.22. If $B = C$, then $(A,B,D) \in \chi_p$ is trivially true since we assume $(A,C,D) \in \chi_p$. Likewise, $C = D$ lets us infer $(A,B,D) \in \chi_p$ immediately because we have supposed $(A,B,C) \in \chi_p$. In conclusion, we can let $C \notin \{A,B,D\}$.

Step 2: *We can assume $|\{A,B,D\}| = 3$.* Our premise $(A,C,D) \in \chi_p$ implies $A \neq D$ by Lemma 4.22. Therefore, if $B = D$, then $(A,B,D) = (A,D,D) \in \chi_p$ follows by Lemma 4.20. Likewise, if $A = B$, then $(A,B,D) \in (A,A,D) \in \chi_p$ by Lemma 4.20. Thus, we can suppose that A, B and D are pairwise distinct from now on.

Step 3: *Proving $(A,B,D) \in \chi_p$.* We construct $(\alpha,\delta) \in A \times D$ with $\sigma_p([\alpha,\delta]_p \cap B) \neq 0$. This is enough to verify our claim $(A,B,D) \in \chi_p$ since $A \neq D$. By Steps 1 and 2 all the blocks A, B, C and D are pairwise distinct.

Step 3.1: *Defining α, δ, γ^-, γ^+, β^- and β^+.* By our assumption $(A,C,D) \in \chi_p$ there exist $(\alpha,\delta) \in A \times D$ such that $\alpha \neq \delta$ and $\sigma_p([\alpha,\delta]_p \cap C) \neq 0$, in particular such that $[\alpha,\delta]_p \cap C \neq \emptyset$. Because $A \neq C \neq D$ this actually means $]\alpha,\delta[_p \cap C \neq \emptyset$. Let $\leq$ be the total order on $]\alpha,\delta[_p$ induced by the total order of p. Then, the legs $\gamma^- \coloneqq \min_\leq(]\alpha,\delta[_p \cap C)$ and $\gamma^+ \coloneqq \max_\leq(]\alpha,\delta[_p \cap C)$ are well-defined (and not necessarily distinct). Since we also assume $(A,B,C) \in \chi_p$ and since A, B and C are pairwise distinct, $\sigma_p([\alpha,\gamma^-]_p \cap B) \neq 0$ by Lemma 4.15. Because $A \neq B \neq C$ that ensures $]\alpha,\gamma^-[_p \cap B \neq \emptyset$, guaranteeing the existence of $\beta^- \coloneqq \min_\leq(]\alpha,\gamma^-[_p \cap B)$. On the other hand, because $]\alpha,\gamma^-[_p \cap B \subseteq]\alpha,\delta[_p \cap B$ we also know that $]\alpha,\delta[_p \cap B \neq \emptyset$, which allows us to define $\beta^+ \coloneqq \max_\leq(]\alpha,\delta[_p \cap B)$.

Step 3.2: *Proving $\beta^+ < \gamma^+$.* As $B \neq C$, we know $\gamma^+ \neq \beta^+$. Suppose $\gamma^+ < \beta^+$. We derive a contradiction. Since $\beta^- < \gamma^- \leq \gamma^+$ by definition, our assumption $\gamma^+ < \beta^+$ ensures $\beta^- \neq \beta^+$ and turns $(\alpha,\beta^-,\gamma^-,\gamma^+,\beta^+,\delta)$ into an ordered tuple in p. Hence, the definitions of γ^- and γ^+ imply $[\alpha,\delta]_p \cap C = [\beta^-,\beta^+]_p \cap C$. Because $\beta^- \neq \beta^+$ and $\{\beta^-,\beta^+\} \subseteq B \neq C$, the premise $p \in \mathcal{W}$ forces $\sigma_p([\beta^-,\beta^+]_p \cap C) = 0$. That contradicts the assumption $\sigma_p([\alpha,\delta]_p \cap C) \neq 0$. Hence, $\beta^+ < \gamma^+$ must have been true instead.

Step 3.3: *Verifying $\sigma_p([\alpha,\delta]_p \cap B) \neq 0$.* If $\beta^+ < \gamma^-$, then the tuple $(\alpha,\beta^-,\beta^+,\gamma^-,\delta)$ is ordered and the definitions of β^- and β^+ therefore guarantee $[\alpha,\delta]_p \cap B = [\alpha,\gamma^-]_p \cap B$, which ensures $\sigma_p([\alpha,\delta]_p \cap B) \neq 0$ since $\sigma_p([\alpha,\gamma^-]_p \cap B) \neq 0$ per assumption. As $B \neq C$, we can then assume $\gamma^- < \beta^+$. Since we already know $\beta^+ < \gamma^+$ from Step 3.2, it follows that $\gamma^- \neq \gamma^+$ and that the tuple $(\alpha,\beta^-,\gamma^-,\beta^+,\gamma^+,\delta)$ is ordered in p. The definitions of β^- and β^+ let us infer $[\alpha,\delta]_p \cap B = [\alpha,\gamma^+]_p \cap B = ([\alpha,\gamma^-]_p \cap B) \uplus ([\gamma^-,\gamma^+]_p \cap B)$ since $B \neq C$. Because we assume $p \in \mathcal{W}$, the facts that $\{\gamma^-,\gamma^+\} \subseteq C \neq$

B and $\gamma^- \neq \gamma^+$ necessitate $\sigma_p([\gamma^-, \gamma^+]_p \cap B) = 0$. We conclude

$$\sigma_p([\alpha, \delta]_p \cap B) = \sigma_p([\alpha, \gamma^-]_p \cap B) + \sigma_p([\gamma^-, \gamma^+]_p \cap B) = \sigma_p([\alpha, \gamma^-]_p \cap B) \neq 0$$

by Step 3.1. That completes the proof. □

The next very strong property could be considered a kind of *local totality*.

LEMMA 4.27. *Let $p \in \mathcal{W}$, let $\{A, B, C, D\} \subseteq p$ and $\neg(A = B = C)$. If $(A, B, D) \in \chi_p$ and $(A, C, D) \in \chi_p$, then $(A, B, C) \in \chi_p$ or $(A, C, B) \in \chi_p$.*

PROOF. We show the claim in six steps.

Step 1: *We can assume $D \notin \{A, B, C\}$.* Since we suppose $(A, B, D) \in \chi_p$, the inequality $A \neq D$ is clear by Lemma 4.22. If $B = D$, then $(A, C, B) \in \chi_p$, and thus our claim, is trivially true by our assumption $(A, C, D) \in \chi_p$. Likewise, if $C = D$, then the assertion holds because the premise $(A, B, D) \in \chi_p$ implies $(A, B, C) \in \chi_p$.

Step 2: *We can assume $|\{A, B, C\}| = 3$.* If $A = B$, then $\neg(A = B = C)$ requires $A \neq C$, which implies $(A, B, C) = (A, A, C) \in \chi_p$ by Lemma 4.20, and thus our claim. Likewise, $B = C$ necessitates $A \neq B$ because $\neg(A = B = C)$, implying $(A, B, C) = (A, B, B) \in \chi_p$, and thus the assertion, by Lemma 4.20. Finally, supposing $A = C$ lets us conclude $A \neq B$ by $\neg(A = B = C)$, from which the claim follows as $(A, C, B) = (A, A, B) \in \chi_p$ by Lemma 4.20. Thus, A, B and C are pairwise distinct henceforth.

Step 3: *Parity condition.* It suffices to find $(\alpha, \beta^-, \gamma^-) \in A \times B \times C$ such that $\sigma_p([\alpha, \gamma^-]_p \cap B) \neq 0$ or $\sigma_p([\alpha, \beta^-]_p \cap C) \neq 0$. Actually, since A, B, C and D are pairwise distinct by Steps 1 and 2 and since the claim is invariant under exchanging $B \leftrightarrow C$, we will be able to restrict to the former of these goals at a convenient time.

Step 3.1: *Definition of α, δ, β^-, β^+, γ^- and γ^+.* By the assumption $(A, B, D) \in \chi_p$ there exist $(\alpha, \delta) \in A \times D$ with $\alpha \neq \delta$ and $\sigma_p([\alpha, \delta]_p \cap B) \neq 0$. Because A, C and D are pairwise distinct and because $(A, C, D) \in \chi_p$ Lemma 4.15 then guarantees $\sigma_p([\alpha, \delta]_p \cap C) \neq 0$. It follows $[\alpha, \delta]_p \cap B \neq \varnothing$ and $[\alpha, \delta]_p \cap C \neq \varnothing$ in particular. Because $A \neq B \neq D$ and $A \neq C \neq D$ we can conclude $]\alpha, \delta[_p \cap C \neq \varnothing$ and $]\alpha, \delta[_p \cap B \neq \varnothing$ from that. Because $]\alpha, \delta[_p$ is consecutive and not the entire set of points of p, the cyclic order of p induces a total order $\leq$ on $]\alpha, \delta[_p$. Hence, $\beta^- := \min_\leq(]\alpha, \delta[_p \cap B)$ and $\beta^+ := \max_\leq(]\alpha, \delta[_p \cap B)$ as well as $\gamma^- := \min_\leq(]\alpha, \delta[_p \cap C)$ and $\gamma^+ := \max_\leq(]\alpha, \delta[_p \cap C)$ are well-defined.

Step 3.2: *We can assume $\beta^- < \gamma^-$ and only need to prove $\sigma_p([\alpha, \gamma^-]_p \cap B) \neq 0$.* Because $B \neq C$ we find $\beta^- \neq \gamma^-$ by definition. Since $\leq$ is total, either $\beta^- < \gamma^-$ or $\gamma^- < \beta^-$. Because, as mentioned initially, the claim is symmetric under exchanging the roles of $B \leftrightarrow C$, we can ensure $\beta^- < \gamma^-$ by renaming $B \leftrightarrow C$ if necessary. It then suffices to show $\sigma_p([\alpha, \gamma^-]_p \cap B) \neq 0$.

Step 3.3: *Auxiliary claim $\beta^+ < \gamma^+$.* We show that assuming $\beta^+ \geq \gamma^+$ produces a contradiction. Indeed, since $B \neq C$, this is the same as supposing $\beta^+ > \gamma^+$ and thus $\beta^- < \gamma^- \leq \gamma^+ < \beta^+$ by Step 3.2. It follows, on the one hand, $\beta^- \neq \beta^+$ and, on the other hand, $[\beta^-, \beta^+]_p \cap C = [\alpha, \delta]_p \cap C$ per definition of γ^- and γ^+. In consequence, $\sigma_p([\beta^-, \beta^+]_p \cap C) \neq 0$ since $\sigma_p([\alpha, \delta]_p \cap C) \neq 0$. However, $p \in \mathcal{W}$ and $\{\beta^-, \beta^+\} \subseteq B \neq C$

and $\beta^- \neq \beta^+$ require $\sigma_p([\beta^-, \beta^+]_p \cap C) = 0$. Since this is a contradiction, we must have $\beta^+ \prec \gamma^+$ instead.

Step 6.4: *Auxiliary claim $\sigma_p(]\gamma^-, \delta]_p \cap B) = 0$.* Since $(\alpha, \beta^+, \gamma^+, \delta)$ is ordered in p by Steps 3.1 and 3.3, the definition of β^+ and the assumption $B \neq C$ ensure $]\gamma^-, \delta]_p \cap B = \emptyset$ if $\gamma^- = \gamma^+$ and $]\gamma^-, \delta]_p \cap B = [\gamma^-, \gamma^+]_p \cap B$ if $\gamma^- \neq \gamma^+$. In the former case $\sigma_p(]\gamma^-, \delta]_p \cap B) = 0$ is thus clear. And in the latter case this is true as well since $p \in \mathcal{W}$ and $\{\gamma^-, \gamma^+\} \subseteq C \neq B$ and $\gamma^- \neq \gamma^+$ demand $\sigma_p([\gamma^-, \gamma^+]_p \cap B) = 0$. Hence, $\sigma_p(]\gamma^-, \delta]_p \cap B) = 0$ always.

Step 6.5: *Proving $\sigma_p([\alpha, \gamma^-]_p \cap B) \neq 0$.* Since $A \neq B \neq D$, since $\sigma_p(]\alpha, \delta[_p \cap B) \neq 0$ by Step 3.1 and since $\sigma_p(]\gamma^-, \delta]_p \cap B) = 0$ by Step 3.4, we can decompose

$$\sigma_p([\alpha, \gamma^-]_p \cap B) = \sigma_p(]\alpha, \delta[_p \cap B) - \sigma_p(]\gamma^-, \delta]_p \cap B) = \sigma_p(]\alpha, \delta[_p \cap B) \neq 0.$$

That concludes the proof, according to Step 3.2. □

With these properties of our colored ternary relation gathered we are able to show the main result of this subsection.

LEMMA 4.28. *Let $p \in \mathcal{W}$, let $\{A, C\} \subseteq p$ and let $A \neq C$. Then, $\leq_p^{A,C}$ is a total order on $\langle A, C \rangle_p$.*

PROOF. In this proof, abbreviate $\leq_{p,A,C}$ by $\leq$. It is clear that $\leq$ is reflexive. We check that $\leq$ is anti-symmetric, transitive and total.

Step 1: *Anti-Symmetry.* Let $\{B_1, B_2\} \subseteq p$ with $(A, B_1, C) \in \chi_p$ and $(A, B_2, C) \in \chi$ and let $B_1 \leq B_2$ and $B_2 \leq B_1$ simultaneously. We suppose $B_1 \neq B_2$ and derive a contradiction. If so, then, by definition, $(A, B_1, B_2) \in \chi_p$ and $(A, B_2, B_1) \in \chi_p$. It follows $B_1 = B_2$ by Lemma 4.24 (b), in contradiction to our assumption.

Step 2: *Transitivity.* Let $\{B_1, B_2, B_3\} \subseteq p$, let $(A, B_i, C) \in \chi_p$ for every $i \in [\![3]\!]$ and suppose $B_1 \leq B_2$ and $B_2 \leq B_3$. If $B_1 = B_2$ or $B_2 = B_3$, there is nothing to show. Hence, assume the opposite. Per definition then, $(A, B_i, B_{i+1}) \in \chi_p$ for all $i \in [\![2]\!]$. In consequence, $(A, B_1, B_3) \in \chi_p$ per Lemma 4.26.

Step 3: *Totality.* Let $\{B_1, B_2\} \subseteq p$ and let $(A, B_1, C) \in \chi_p$ and $(A, B_2, C) \in \chi_p$. We prove that $B_1 \leq B_2$ or $B_2 \leq B_1$. We can assume $B_1 \neq B_2$.

As then $\neg(A = B_1 = B_2)$ is ensured, Lemma 4.27 implies that $(A, B_1, B_2) \in \chi_p$ or $(A, B_2, B_1) \in \chi_p$, i.e., that $B_1 \leq B_2$ or $B_2 \leq B_1$. That concludes the proof. □

Very helpful for the invariance proof is the following result showing that we could have just as well have used the right boundary instead of the left one in the definition of the order.

LEMMA 4.29. *Let $p \in \mathcal{W}$, let $\{A, C\} \subseteq p$ and let $A \neq C$. Then, $\langle C, A \rangle_p = \langle A, C \rangle_p$ and $\leq_{p,C,A}$ is the opposite total order of $\leq_{p,A,C}$.*

PROOF. The first part of the claim was shown in Lemma 4.23 (a). Let $\{B_1, B_2\} \subseteq p$, let $(C, B_1, A) \in \chi_p$ and $(C, B_2, A) \in \chi_p$ and let $B_1 \leq_{p,C,A} B_2$. If $B_1 = B_2$, then $B_2 \leq_{p,A,C} B_1$ is clear. Hence, let $B_1 \neq B_2$. Then, by definition, $(C, B_1, B_2) \in \chi_p$. From $\neg(B_1 = B_2 = A)$, from $(C, B_1, B_2) \in \chi_p$ and from $(C, B_2, A) \in \chi_p$ it follows

$(B_1, B_2, A) \in \chi_p$ by Lemma 4.25. According to Lemma 4.23 (a) this is equivalent to $(A, B_2, B_1) \in \chi_p$, which is to say $B_2 \leq_{p,A,C} B_1$. Exchanging the roles of A and C proves the other implication. □

4.3. Invariance of $\mathcal{W}_R$. Equipped with the results of the preceding subsection we can show that $\mathcal{W}_R$ is a category of two-colored partitions for general $R \in \mathcal{R}$. Again, we will employ Lemma 2.2 instead of checking the definition. This time, we treat each of the operations individually.

4.3.1. *Rotation.* Since $\mathcal{W}$ is closed under rotations by Theorem 4.7, the following is a well-defined assertion.

LEMMA 4.30. *Let $p \in \mathcal{W}$, let $r \in \{↷, ⤵, ⤷, ↶\}$ and let ρ be the map rotating the points of p^r to their former positions in p. Then,*

$$\chi_{p^r} = \{(\rho^{-1}(A), \rho^{-1}(B), \rho^{-1}(C)) \mid (A, B, C) \in \chi_p\},$$

and

$$\lambda_{p^r} = \{((\rho^{-1}(A), \rho^{-1}(B), \rho^{-1}(C)), \lambda_p(A, B, C)) \mid (A, B, C) \in \chi_p\}.$$

PROOF. By Remark 4.3 (a), for all $\{A, B, C\} \subseteq p$ and all $(\alpha, \gamma) \in A \times C$ with $\alpha \neq \gamma$,

$$\sigma_{p^r}([\rho^{-1}(\alpha), \rho^{-1}(\gamma)]_{p^r} \cap \rho^{-1}(B)) = \sigma_{p^r}(\rho^{-1}([\alpha, \gamma]_p \cap B)) = \sigma_p([\alpha, \gamma]_p \cap B).$$

Now, the claim follows by Remark 4.3 (a). □

LEMMA 4.31. *$p^r \in \mathcal{W}_R$ for all $r \in \{↷, ⤵, ⤷, ↶\}$, all $p \in \mathcal{W}_R$ and all $R \in \mathcal{R}$.*

PROOF. Follows immediately from Lemma 4.30. □

4.3.2. *Verticolor Reflection.* Because $\mathcal{W}$ is closed under verticolor reflection by Theorem 4.7, the following claim makes sense.

LEMMA 4.32. *Let $p \in \mathcal{W}$ and let ϱ be the map sending the points of $\tilde{p}$ to their former positions in p. Then,*

$$\chi_{\tilde{p}} = \{(\varrho^{-1}(A), \varrho^{-1}(B), \varrho^{-1}(C)) \mid (A, B, C) \in \chi_p\},$$

and

$$\lambda_{\tilde{p}} = \{((\varrho^{-1}(A), \varrho^{-1}(B), \varrho^{-1}(C)), \lambda_p(A, B, C)) \mid (A, B, C) \in \chi_p\}.$$

PROOF. By definition, the statement $(A, B, C) \in \chi_p$ is equivalent to the existence of $(\alpha, \gamma) \in A \times C$ with $\alpha \neq \gamma$ and $\sigma_p([\alpha, \gamma]_p \cap B) \neq 0$. Since

$$\sigma_{\tilde{p}}([\varrho^{-1}(\gamma), \varrho^{-1}(\alpha)]_{\tilde{p}} \cap \varrho^{-1}(B)) = \sigma_{\tilde{p}}(\varrho^{-1}([\alpha, \gamma]_p \cap B)) = -\sigma_p([\alpha, \gamma]_p \cap B)$$

by Remark 4.3 (b), this is true precisely if $(\varrho^{-1}(C), \varrho^{-1}(B), \varrho^{-1}(A)) \in \chi_p$. Moreover, this identity proves that, if so, then $\lambda_{\tilde{p}}(\varrho^{-1}(C), \varrho^{-1}(B), \varrho^{-1}(A)) = -\lambda_p(A, B, C)$ by Lemma 4.17. Now, the claim follows by Parts (a) and (b) of Lemma 4.23. □

LEMMA 4.33. *$\tilde{p} \in \mathcal{W}_R$ for all $p \in \mathcal{W}_R$ and all $R \in \mathcal{R}$.*

PROOF. Follows immediately from Lemma 4.32. □

4.3.3. *Tensor Products.* As $\mathcal{W}$ is closed under tensor products by Theorem 4.7, we can formulate the next result.

LEMMA 4.34. *Let $p_1, p_2 \in \mathcal{W}$ and for every $i \in [\![2]\!]$ let S_i be the set of all points of $p_1 \otimes p_2$ coming from p_i and let τ_i be the map sending the points of S_i to their original positions in p_i. Then, for all $i \in [\![2]\!]$,*

$$\{(D,E,F) \in \chi_{p_1\otimes p_2} \,|\, D,E,F \subseteq S_i\} = \{(\tau_i^{-1}(A), \tau_i^{-1}(B), \tau_i^{-1}(C)) \,|\, (A,B,C) \in \chi_{p_i}\}$$

and

$$\lambda_{p_1\otimes p_2} \supseteq \{((\tau_i^{-1}(A), \tau_i^{-1}(B), \tau_i^{-1}(C)), \lambda_{p_i}(A,B,C)) \,|\, (A,B,C) \in \chi_{p_i}\}.$$

PROOF. Let $i \in [\![2]\!]$ and let $\{A,B,C\} \subseteq p_i$. By Remark 4.3 (c) the sets $\tau_i^{-1}(A)$, $\tau_i^{-1}(B)$ and $\tau_i^{-1}(C)$ are blocks of $p_1 \otimes p_2$ contained in S_i. Moreover, all blocks of $p_1 \otimes p_2$ contained in S_i arise in this way. Hence, if we show that $(A,B,C) \in \chi_{p_i}$ holds if and only if $(\tau_i^{-1}(A), \tau_i^{-1}(A), \tau_i^{-1}(A)) \in \chi_{p_1\otimes p_2}$ is true, the identity in the first claim will have been proven.

According to Remark 4.3 (c) we can infer for all $\alpha \in A$ and $\gamma \in C$ with $\alpha \neq \gamma$, because $S_1 \cap S_2 = \varnothing$ and because τ_i is injective,

$$\begin{aligned}
&\sigma_{p_1\otimes p_2}([\tau_i^{-1}(\alpha), \tau_i^{-1}(\gamma)]_{p_1\otimes p_2} \cap \tau_i^{-1}(B)) \\
&\qquad = \textstyle\sum_{\ell=1}^{2} \sigma_{p_\ell}(\tau_\ell(([\tau_i^{-1}(\alpha), \tau_i^{-1}(\gamma)]_{p_1\otimes p_2} \cap \tau_i^{-1}(B)) \cap S_\ell)) \\
&\qquad = \sigma_{p_i}(\tau_i([\tau_i^{-1}(\alpha), \tau_i^{-1}(\gamma)]_{p_1\otimes p_2} \cap \tau_i^{-1}(B) \cap S_i)) \\
&\qquad = \sigma_{p_i}([\alpha,\gamma]_{p_i} \cap B_i).
\end{aligned}$$

This identity not only shows that the statements $(A,B,C) \in \chi_{p_i}$ and $(\tau_i^{-1}(A), \tau_i^{-1}(B), \tau_i^{-1}(C)) \in \chi_{p_1\otimes p_2}$ are equivalent but also that, if one of them, and thus also the other, is true, then $\lambda_{p_1\otimes p_2}(\tau_i^{-1}(A), \tau_i^{-1}(B), \tau_i^{-1}(C)) = \lambda_{p_i}(A,B,C)$. That is what we needed to see. □

LEMMA 4.35. *$p_1 \otimes p_2 \in \mathcal{W}_R$ for all $\{p_1, p_2\} \subseteq \mathcal{W}_R$ and all $R \in \mathcal{R}$.*

PROOF. For every $i \in [\![2]\!]$ let S_i be the set of all points of $p_1 \otimes p_2$ coming from p_i and let τ_i be the map sending the points of S_i to their original places in p_i. Moreover, let $n \in \mathbb{N}$, let $2 \leq n$, let $\{B_1, B_2, \ldots, B_n\} \subseteq p_1 \otimes p_2$, let $B_1, B_2, \ldots, B_n$ be pairwise distinct, let B_1 and B_n cross in $p_1 \otimes p_2$, let $\langle B_1, B_n\rangle_{E(p,T)} = \{B_1, B_2, \ldots, B_n\}$, let $B_1 \leq B_2 \leq \ldots \leq B_n$ with respect to $\leq_{p_1\otimes p_2, B_1, B_n}$ and for every $i \in [\![n]\!]$ let $c_i \in \{\circ, \bullet\}$ be such that $\lambda_{p_1\otimes p_2}(B_1, B_i, B_n) = \sigma(c_i)$. We prove $(c_1, c_2, \ldots, c_n) \in R$ by finding $k \in [\![2]\!]$ and $\{B_1', B_2', \ldots, B_n'\} \subseteq p_k$ such that $B_1', B_2', \ldots, B_n'$ are pairwise distinct, such that B_1' and B_n' cross in p_k, such that $\langle B_1', B_n'\rangle_{p_k} = \{B_1', B_2', \ldots, B_n'\}$, such that $B_1' \leq B_2' \leq \ldots \leq B_n'$ with respect to $\leq_{p_k, B_1', B_n'}$ and such that $\lambda_{p_k}(B_1', B_i', B_n') = \lambda_{p_1\otimes p_2}(B_1, B_i, B_n)$ for every $i \in [\![n]\!]$.

Step 1: *Defining k and $B_1', B_2', \ldots, B_n'$.* By Remark 4.3 (c) there exists for every $j \in [\![n]\!]$ an $i_j \in [\![2]\!]$ such that $B_j \subseteq S_{i_j}$. The same remark shows that the assumption of B_1 and B_n crossing in $p_1 \otimes p_2$ requires $i_1 = i_n$.

By Lemma 4.19, the crossing between B_1 and B_n in $p_1 \otimes p_2$ also lets us conclude that B_j crosses B_1 and B_n in $p_1 \otimes p_2$ for every $j \in [\![n]\!]$ with $1 < j < n$. Hence, applying Remark 4.3 (c) a second time shows that $i_1 = i_2 = \ldots = i_n$ and that, consequently, if $k \coloneqq i_1$, then $B'_j \coloneqq \tau_k(B_j)$ is a block of p_k for every $j \in [\![n]\!]$.

Step 2: *Crossing.* Because the blocks B_1 and B_n cross in $p_1 \otimes p_2$ so do the blocks $B'_1 = \tau_k(B_1)$ and $B'_n \coloneqq \tau_k(B_n)$ in p_k by Remark 4.3 (c).

Step 3: *Determining* $\langle B'_1, B'_n \rangle_{p_k}$. By Lemma 4.34 for every $j \in [\![n]\!]$, because $B_1 \cup B_j \cup B_n \subseteq S_k$, the assumption $(\tau_k^{-1}(B'_1), \tau_k^{-1}(B'_j), \tau_k^{-1}(B'_n)) = (B_1, B_j, B_n) \in \chi_{p_1 \otimes p_2}$ implies $(B'_1, B'_j, B'_n) \in \chi_{p_k}$. Hence, $\langle B'_1, B'_n \rangle_{p_k} \supseteq \{B'_1, B'_2, \ldots, B'_n\}$.

To see the converse inclusion we let $F' \in p_k$ and $F' \notin \{B'_1, B'_2, \ldots, B'_n\}$ and prove $(B'_1, F', B'_n) \notin \chi_{p_k}$. Then, $F \coloneqq \tau_{p_k}^{-1}(F') \in p_1 \otimes p_2$ and $F \subseteq S_k$ and $F \notin \{B_1, B_2, \ldots, B_n\}$. From $\langle B_1, B_n \rangle_{p_1 \otimes p_2} \subseteq \{B_1, B_2, \ldots, B_n\}$ it hence follows $(\tau_k^{-1}(B'_1), \tau_k^{-1}(F'), \tau_k^{-1}(B'_n)) = (B_1, F, B_n) \notin \chi_{p_1 \otimes p_2}$. Therefore and because $B_1 \cup F \cup B_n \subseteq S_k$ Lemma 4.34 allows us to conclude $(B'_1, F', B'_n) \notin \chi_{p_k}$. Thus, $\langle B'_1, B'_n \rangle_{p_k} \subseteq \{B'_1, B'_2, \ldots, B'_n\}$.

Step 4: *The ordering of* $B'_1, B'_2, \ldots, B'_n$. For every $j \in [\![n]\!]$ with $j < n$ the premise that $B_j \leq B_{j+1}$ with respect to $\leq_{p_1 \otimes p_2, B_1, B_n}$, i.e., that $(\tau_k^{-1}(B'_1), \tau_k^{-1}(B'_j), \tau_k^{-1}(B'_{j+1})) = (B_1, B_j, B_{j+1}) \in \chi_{p_1 \otimes p_2}$, by Lemma 4.34 ensures that $(B'_1, B'_j, B'_{j+1}) \in \chi_{p_k}$ because $B_1 \cup B_j \cup B_{j+1} \subseteq S_k$. Hence, indeed, $B'_1 \leq B'_2 \leq \ldots \leq B'_n$ with respect to $\leq_{p_k, B'_1, B'_n}$.

Step 5: *The colors.* Finally, $\lambda_{p_k}(B'_1, B'_j, B'_n) = \lambda_{p_1 \otimes p_2}(\tau_k^{-1}(B'_1), \tau_k^{-1}(B'_j), \tau_k^{-1}(B'_k)) = \lambda_{p_1 \otimes p_2}(B_1, B_j, B_n) = c_j$ for every $j \in [\![n]\!]$ by Lemma 4.34, concluding the proof. □

4.3.4. *Erasing Turns.* The proof that $\mathcal{W}_R$ for arbitrary $R \in \mathcal{R}$ is closed under erasing turns is the most complicated part. Several preparatory results are required.

The first of these proves that there is nothing "between" two blocks which intersect the same turn.

Lemma 4.36. *Let $p \in \mathcal{W}$, let $\{B_1, B_2\} \subseteq p$, let $B_1 \neq B_2$ and let there exist a turn T in p with $\varnothing \neq B_1 \cap T$ and $\varnothing \neq B_2 \cap T$. Then, there exist no $B' \in p$ with $B \notin \{B_1, B_2\}$ and $(B_1, B', B_2) \in \chi_p$.*

Proof. Let T be as in the premise, let $T = [\tau_1, \tau_2]_p$, let $B' \in p$ and let $B' \notin \{B_1, B_2\}$. Since $T \subseteq B_1 \cup B_2$ it follows $T \cap B' = \varnothing$ and thus $\sigma_p([\tau_1, \tau_2]_p \cap B') = \sigma_p(T \cap B') = \sigma_p(\varnothing) = 0$. Because B_1, B' and B_2 are pairwise distinct this is all we needed to show according to Lemma 4.15. □

The next lemma shows that blocks intersecting the same turn have identical colored relations to any other block.

Lemma 4.37. *Let $p \in \mathcal{W}$ be arbitrary.*

(a) If $\{A_1, A_2, B, C\} \subseteq p$, if $A_1 \neq A_2$, if $\{A_1, A_2\} \cap \{B, C\} = \varnothing$ and if there exists a turn T in p with $\varnothing \neq A_1 \cap T$ and $\varnothing \neq A_2 \cap T$, then the following hold:
  (i) $(A_1, B, C) \in \chi_p$ if and only if $(A_2, B, C) \in \chi_p$.
  (ii) If so, then $\lambda_p(A_1, B, C) = \lambda_p(A_2, B, C)$.

(b) If $\{A, B, C_1, C_2\} \subseteq p$, if $C_1 \neq C_2$, if $\{A, B\} \cap \{C_1, C_2\} = \varnothing$ and if there exists a turn T in p with $\varnothing \neq C_1 \cap T$ and $\varnothing \neq C_2 \cap T$, then the following hold:

(i) $(A, B, C_1) \in \chi_p$ *if and only if* $(A, B, C_2) \in \chi_p$.
(ii) *If so, then* $\lambda_p(A, B, C_1) = \lambda_p(A, B, C_2)$.

PROOF. (a) Thanks to Lemmata 4.14 and 4.16 we can prove both claims simultaneously by showing $\delta_p^B(\alpha_1, \gamma) = \delta_p^B(\alpha_2, \gamma)$ for all $(\alpha_1, \alpha_2, \gamma) \in A_1 \times A_2 \times C$. Let T be a turn in p with $\varnothing \neq A_1 \cap T$ and $\varnothing \neq A_2 \cap T$ and for every $i \in [\![2]\!]$ let $\tau_i \in A_i \cap T$.

If $\{\neg i, i\} = [\![2]\!]$ are such that τ_i is the predecessor of $\tau_{\neg i}$ in the cyclic order of p, then, since $B \notin \{A_1, A_2\}$ and $T \subseteq A_1 \cup A_2$ ensure $T \cap B = \varnothing$,

$$\delta_p^B(\tau_i, \tau_{\neg i}) = \sigma_p([\tau_i, \tau_{\neg i}]_p \cap B) = \sigma_p(T \cap B) = \sigma_p(\varnothing) = 0.$$

Moreover, $p \in \mathcal{W}$ and $B \notin \{A_1, A_2\}$ imply $\delta_p^B(\alpha_i, \tau_i) = \delta_p^B(\tau_{\neg i}, \alpha_{\neg i}) = 0$ because $\{\alpha_i, \tau_i\} \subseteq A_i$ and $\{\tau_{\neg i}, \alpha_{\neg i}\} \subseteq A_{\neg i}$. It follows by Lemma 4.2 (c),

$$\delta_p^B(\alpha_i, \gamma) = \delta_p^B(\alpha_i, \tau_i) + \delta_p^B(\tau_i, \tau_{\neg i}) + \delta_p^B(\tau_{\neg i}, \alpha_{\neg i}) + \delta_p^B(\alpha_{\neg i}, \gamma) = \delta_p^B(\alpha_{\neg i}, \gamma),$$

which is what we needed to see.

(b) Follows by Part (a) and Lemmata 4.23. □

But intersecting the same turn also has implications for the colored relations between the two blocks themselves.

LEMMA 4.38. *Let* $p \in \mathcal{W}$ *be arbitrary.*

(a) *If* $\{A_1, A_2, C\} \subseteq p$, *if* A_1, A_2 *and* C *are pairwise distinct and if there exists a turn* T *in* p *with* $A_1 \cap T \neq \varnothing$ *and* $A_2 \cap T \neq \varnothing$, *then the following hold:*
  (i) *If* $\{(A_1, A_2, C), (A_2, A_1, C)\} \cap \chi_p = \varnothing$, *then* $\lambda_p(A_2, A_2, C) = \lambda_p(A_1, A_1, C)$.
  (ii) *Otherwise,* $\lambda_p(A_2, A_2, C) = -\lambda_p(A_1, A_1, C)$.

(b) *If* $\{A, C_1, C_2\} \subseteq p$, *if* A, C_1 *and* C_2 *are pairwise distinct and if there exists a turn* T *in* p *with* $C_1 \cap T \neq \varnothing$ *and* $C_2 \cap T \neq \varnothing$, *then the following hold:*
  (i) *If* $\{(A, C_2, C_1), (A, C_1, C_2)\} \cap \chi_p = \varnothing$, *then* $\lambda_p(A, C_1, C_1) = \lambda_p(A, C_2, C_2)$.
  (ii) *Otherwise,* $\lambda_p(A, C_1, C_1) = -\lambda_p(A, C_2, C_2)$.

PROOF. (a) We show claims (i) and (ii) simultaneously in three steps.

Step 1: *Rewording the claim.* Because λ_p can only take two values $\{-1, 1\}$ by Lemma 4.17, the claim is equivalently expressed as

$$\lambda_p(A_2, A_2, C) = \lambda_p(A_1, A_1, C) \overset{!}{\iff} (A_1, A_2, C) \notin \chi_p \text{ and } (A_2, A_1, C) \notin \chi_p.$$

By Lemma 4.24 (a) the premise $A_1 \neq A_2$ excludes the possibility that $(A_1, A_2, C) \in \chi_p$ and $(A_2, A_1, C) \in \chi_p$ at the same time. Hence, our assertion is equivalent to claiming

$$\lambda_p(A_2, A_2, C) = \lambda_p(A_1, A_1, C) \overset{!}{\iff} [(A_1, A_2, C) \in \chi_p \Leftrightarrow (A_2, A_1, C) \in \chi_p].$$

That is the version we prove.

Step 2: *Defining* τ_1, τ_2 *and* γ. Let $\gamma \in C$ be arbitrary, let P_p denote the set of all points of p and let $\leq$ be the total order induced on $P_p \backslash \{\gamma\}$ by the cyclic order of p. Since $C \notin \{A_1, A_2\}$ the point $\alpha_i^+ \coloneqq \max_\leq((P_p \backslash \{\gamma\}) \cap A_i) = \max_\leq(A_i)$ is well-defined for each $i \in [\![2]\!]$. Per construction, $[\alpha_i^+, \gamma]_p \cap A_i = \{\alpha_i^+\}$ and therefore $\sigma_p([\alpha_i^+, \gamma]_p \cap A_i) \neq 0$ for every $i \in [\![2]\!]$. Lemma 4.17 hence assures us that $\lambda_p(A_i, A_i, C) = \sigma_p([\alpha_i^+, \gamma]_p \cap A_i) = \sigma_p(\{\alpha_i^+\})$ for every $i \in [\![2]\!]$.

Step 3: *Relating* $\lambda_p(A_1, A_2, C)$ *and* $\lambda_p(A_2, A_1, C)$. By Proposition 4.13 the legs of A_1 and A_2 alternate in normalized color. One way of expressing this fact is to say that for all $i \in [\![2]\!]$ and all $\alpha_i \in A_i$ with $\alpha_i \neq \alpha_i^+$,

$$\sigma_p(\{\alpha_i^+\}) = (-1)^{|[\alpha_i, \alpha_i^+[_p \cap A_i|} \sigma_p(\{\alpha_i\}).$$

Per assumption there exist $\tau_1 \in A_1 \cap T$ and $\tau_2 \in A_2 \cap T$ such that $T = \{\tau_1, \tau_2\}$. In conclusion, for every $i \in [\![2]\!]$,

$$\lambda_p(A_i, A_i, C) = \begin{cases} \sigma_p(\{\tau_i\}) & \text{if } \tau_i = \alpha_i^+, \\ (-1)^{|[\tau_i, \alpha_i^+[_p \cap A_i|} \sigma_p(\{\tau_i\}) & \text{otherwise.} \end{cases}$$

For each $i \in [\![2]\!]$, if $\tau_i \neq \alpha_i^+$, then $(\tau_i, \alpha_i^+, \gamma)$ is ordered in p by the definition of α_i^+, implying $|[\tau_i, \alpha_i^+[_p \cap A_i| = |([\tau_i, \gamma]_p \cap A_i) \backslash \{\alpha_i^+\}| \equiv_2 1 + |[\tau_i, \gamma]_p \cap A_i|$. And, of course, for any $i \in [\![2]\!]$, if $\tau_i = \alpha_i^+$, then also $1 + |[\tau_i, \gamma]_p \cap A_i| \equiv_2 0$. Hence, for any $i \in [\![2]\!]$,

$$\lambda_p(A_i, A_i, C) = (-1)^{1 + |[\tau_i, \gamma]_p \cap A_i|} \sigma_p(\{\tau_i\}).$$

Because T is a turn, we find $\{\neg i, i\} = [\![2]\!]$ such that $T = [\tau_i, \tau_{\neg i}]_p$. It follows that $1 + |[\tau_i, \gamma]_p \cap A_i| = 1 + |\{\tau_i\} \cap A_i| + |[\tau_{\neg i}, \gamma]_p \cap A_i| \equiv_2 |[\tau_{\neg i}, \gamma]_p \cap A_i|$ and thus

$$\lambda_p(A_i, A_i, C) = (-1)^{|[\tau_{\neg i}, \gamma] \cap A_i|} \sigma_p(\{\tau_i\}).$$

On the other hand, $1 + |[\tau_{\neg i}, \gamma]_p \cap A_{\neg i}| = 1 + |\{\tau_i\} \cap A_{\neg i}| + |[\tau_i, \gamma]_p \cap A_{\neg i}| = 1 + |[\tau_i, \gamma]_p \cap A_{\neg i}|$. If we also take into account that T being a turn implies $\sigma_p(\{\tau_{\neg i}\}) = -\sigma_p(\{\tau_i\})$, we can thus conclude

$$\lambda_p(A_{\neg i}, A_{\neg i}, C) = (-1)^{|[\tau_i, \gamma] \cap A_{\neg i}|} \sigma_p(\{\tau_i\}).$$

In summary, we have shown

$$\lambda_p(A_2, A_2, C) = \lambda_p(A_1, A_1, C) \iff |[\tau_2, \gamma] \cap A_1| \equiv_2 |[\tau_1, \gamma] \cap A_2|.$$

Since A_1, A_2 and C are pairwise distinct, Lemma 4.18 (b) tells us that the right hand side of this equivalence is nothing but the statement that $(A_1, A_2, C) \in \chi_p$ if and only if $(A_2, A_1, C) \in \chi_p$, which is what we needed to see.

(b) Follows by Part (a) and Lemma 4.23. □

The following auxiliary result recapitulates the essence of the proof of the well-known fact that the set of all non-crossing two-colored partitions is a category.

Lemma 4.39. *Let $p \in \mathcal{P}^{\circ\bullet}$, let T be a turn in p, let $\{A, C\} \subseteq E(p, T)$, let $A \neq C$ and for each $X \in \{A, C\}$ let $\{X_1, X_2\}$ be parents of X with respect to (p, T). If A and C cross in $E(p, T)$, then there exist $\{i, k\} \subseteq [\![2]\!]$ such that A_i and C_k cross in p.*

Proof. Because the claim is symmetric under renaming $A \leftrightarrow C$ and because at most one of A and C can be anything other than case E_{I}, we can assume that C is not case E_{III}.

Let ϵ be map sending the points of $E(p, T)$ to their former positions in p. The crossing between A and C in $E(p, T)$ implies the existence of $\{\alpha_1, \alpha_2\} \subseteq A$ and $\{\gamma_1, \gamma_2\} \subseteq C$ such that $(\alpha_1, \gamma_1, \alpha_2, \gamma_2)$ is an ordered tuple of pairwise distinct points in $E(p, T)$. Per definition of $E(p, T)$ the injection ϵ preserves the cyclic order.

Hence, if $\alpha'_i \coloneqq \epsilon(\alpha_i)$ and $\gamma'_k \coloneqq \epsilon(\gamma_k)$ for all $\{i,k\} \subseteq [\![2]\!]$, then $(\alpha'_1, \gamma'_1, \alpha'_2, \gamma'_2)$ is ordered in p. Now we distinguish two cases.

Case 1: First, suppose that there exist $\{i,k\} \subseteq [\![2]\!]$ such that $\{\alpha'_1, \alpha'_2\} \subseteq A_i$ and $\{\gamma'_1, \gamma'_2\} \subseteq C_k$. The tuple $(\alpha'_1, \gamma'_1, \alpha'_2, \gamma'_2)$ then provides a crossing between A_i and C_k in p, proving our claim.

Case 2: Assuming the opposite requires in particular $A_1 \neq A_2$ or $C_1 \neq C_2$. Since we have excluded that C is case $\mathrm{E_{III}}$, this only leaves the conclusion that A is case $\mathrm{E_{III}}$. By renaming $A_1 \leftrightarrow A_2$ we can suppose that $\alpha'_1 \in A_1$ and $\alpha'_2 \in A_2$.

If we let $\tau_1 \in A_1 \cap T$ and $\tau_2 \in A_2 \cap T$, then $T = \{\tau_1, \tau_2\}$ and $\{\tau_1, \tau_2\} \cap \{\alpha'_1, \gamma'_1, \alpha'_2, \gamma'_2\} = \varnothing$. Moreover, T being consecutive in p ensures that exactly one of the four tuples $(T, \alpha'_1, \gamma'_1, \alpha'_2, \gamma'_2)$, $(\alpha'_1, T, \gamma'_1, \alpha'_2, \gamma'_2)$, $(\alpha'_1, \gamma'_1, T, \alpha'_2, \gamma'_2)$ and $(\alpha'_1, \gamma'_1, \alpha'_2, T, \gamma'_2)$ is ordered in p. In particular, we can distinguish two alternative subcases:

Case 2.1: If $(T, \alpha'_1, \gamma'_1, \alpha'_2, \gamma'_2)$ or $(\alpha'_1, T, \gamma'_1, \alpha'_2, \gamma'_2)$ is ordered in p, then the ordered tuple $(\tau_2, \gamma'_1, \alpha'_2, \gamma'_2)$ in p gives a crossing in p between A_2 and $C_1 = C_2$.

Case 2.2: If, instead, $(\alpha'_1, \gamma'_1, T, \alpha'_2, \gamma'_2)$ or $(\alpha'_1, \gamma'_1, \alpha'_2, T, \gamma'_2)$ is ordered in p, then A_1 and $C_1 = C_2$ cross in p because $(\alpha'_1, \gamma'_1, \tau_1, \gamma'_2)$ is ordered in p. □

The last auxiliary result shows how the $\star$-betweenness relation transforms under erasing turns.

LEMMA 4.40. *Let $p \in \mathcal{W}$, let T be a turn in p, let $\{A, B, C\} \subseteq E(p,T)$, let $A \neq C$ and for all $X \in \{A, B, C\}$ let $\{X_1, X_2\}$ be parents of X with respect to (p,T).*

(a) Whenever B is not case $\mathrm{E_{III}}$, then for all $\{i,j,k\} \subseteq [\![2]\!]$:
  - *(i) $(A,B,C) \in \chi_{E(p,T)}$ if and only if $(A_i, B_j, C_k) \in \chi_p$.*
  - *(ii) If so, then $\lambda_p(A_i, B_j, C_k) = \lambda_{E(p,T)}(A,B,C)$.*

(b) If A is case $\mathrm{E_{III}}$, then for all $k \in [\![2]\!]$:
  - *(i) If both $(A_2, A_1, C_k) \notin \chi_p$ and $(A_1, A_2, C_k) \notin \chi_p$, then $\lambda_p(A_1, A_1, C_k) = \lambda_p(A_2, A_2, C_k) = \lambda_{E(p,T)}(A, A, C)$.*
  - *(ii) If $(A_{\neg i}, A_i, C_k) \in \chi_p$ for some $\neg i, i \in [\![2]\!]$ with $\neg i \neq i$, then $\lambda_p(A_i, A_i, C_k) = -\lambda_p(A_{\neg i}, A_{\neg i}, C_k) = \lambda_{E(p,T)}(A, A, C)$.*
  - *(iii) $\lambda_p(A_1, C_k, C_k) = \lambda_p(A_2, C_k, C_k) = \lambda_{E(p,T)}(A, C, C)$.*

(c) If $B \notin \{A, C\}$ and if B is case $\mathrm{E_{III}}$, then for all $\{i,k\} \subseteq [\![2]\!]$:
  - *(i) $(A,B,C) \in \chi_{E(p,T)}$ if and only if there exists* exactly one *$j \in [\![2]\!]$ such that $(A_i, B_j, C_k) \in \chi_p$.*
  - *(ii) If so, then $\lambda_p(A_i, B_j, C_k) = \lambda_{E(p,T)}(A,B,C)$ for this j.*

(d) If C is case $\mathrm{E_{III}}$, then for all $i \in [\![2]\!]$:
  - *(i) If both $(A_i, C_2, C_1) \notin \chi_p$ and $(A_i, C_1, C_2) \notin \chi_p$, then $\lambda_p(A_i, C_1, C_1) = \lambda_p(A_i, C_2, C_2) = \lambda_{E(p,T)}(A, C, C)$.*
  - *(ii) If $(A_i, C_k, C_{\neg k}) \in \chi_p$ for some $\neg k, k \in [\![2]\!]$ with $\neg k \neq k$, then $\lambda_p(A_i, C_k, C_k) = -\lambda_p(A_i, C_{\neg k}, C_{\neg k}) = \lambda_{E(p,T)}(A, C, C)$.*
  - *(iii) $\lambda_p(A_i, A_i, C_1) = \lambda_p(A_i, A_i, C_2) = \lambda_{E(p,T)}(A, A, C)$.*

PROOF. Let ϵ be the map sending the points of $E(p,T)$ to their original positions in p. We address the individual assertions (a)–(d) and prove them using Lemma 4.5.

Step 1 *Proof of (a).* We show the two claims of (a) (i) separately and simultaneously prove the identity (a) (ii). Hence, suppose that B is not case E_{III}.

Step 1.1: *If $\chi_{E(p,T)}$, then also χ_p.* First, assume $(A,B,C) \in \chi_{E(p,T)}$. For all $\{i,j,k\} \subseteq [\![2]\!]$ we show $(A_i,B_j,C_k) \in \chi_{E(p,T)}$ and $\lambda_p(A_i,B_j,C_k) = \lambda_{E(p,T)}(A,B,C)$.

By $(A,B,C) \in \chi_{E(p,T)}$ there are $(\alpha,\gamma) \in A{\times}C$ with $\alpha \neq \gamma$ and $0 \neq \lambda_{E(p,T)}(A,B,C) = \sigma_{E(p,T)}([\alpha,\gamma]_{E(p,T)} \cap B)$. Let $\{i,k\} \subseteq [\![2]\!]$ be such that $\alpha' \coloneqq \epsilon(\alpha) \in A_i$ and $\gamma' \coloneqq \epsilon(\gamma) \in C_k$ and let $j \in [\![2]\!]$ be arbitrary. Lemma 4.5 and the assumption $B_1 = B_2$ tell us that

$$0 \neq \lambda_{E(p,T)}(A,B,C) = \sigma_{E(p,T)}([\alpha,\gamma]_{E(p,T)} \cap B) = \sigma_p([\alpha',\gamma']_p \cap B_j).$$

By the definition of χ_p and by Lemma 4.17 this conclusion proves $(A_i,B_j,C_k) \in \chi_p$ and $\lambda_p(A_i,B_j,C_k) = \lambda_{E(p,T)}(A,B,C)$. We now distinguish three cases.

Case 1.1.1: If neither A nor C is case E_{III}, and consequently $A_1 = A_2$ and $C_1 = C_2$, we have thus already shown our claim.

Case 1.1.2: Suppose that A is case E_{III}, let $\neg i \in [\![2]\!]$ and let $\neg i \neq i$. Then, because $C_1 = C_2$, what is left to show is that $(A_{\neg i},B_j,C_k) \in \chi_p$ and that $\lambda_p(A_{\neg i},B_j,C_k) = \lambda_p(A_i,B_j,C_k)$. However, this is precisely the implication of Lemma 4.37 (a) since A being case E_{III} requires $A_1 \neq A_2$ and $\emptyset \neq A_1 \cap T$ and $\emptyset \neq A_2 \cap T$.

Case 1.1.3: Likewise, if C is case E_{III} and if $\neg k \in [\![2]\!]$ is such that $\neg k \neq k$, then Lemma 4.37 (b) proves $(A_i,B_j,C_{\neg k}) \in \chi_p$ and $\lambda_p(A_i,B_j,C_{\neg k}) = \lambda_p(A_i,B_j,C_k)$, which, since $A_1 = A_2$, is what we needed to show.

Step 1.2: *If χ_p, then also $\chi_{E(p,T)}$.* Conversely, let $\{i,j,k\} \subseteq [\![2]\!]$ be arbitrary and suppose $(A_i,B_j,C_k) \in \chi_p$. We show $(A,B,C) \in \chi_{E(p,T)}$ and $\lambda_{E(p,T)}(A,B,C) = \lambda_p(A_i,B_j,C_k)$. Two alternatives must be considered.

Case 1.2.1: First, assume $B \notin \{A,C\}$. Since $\{A_1,A_2\}$ are parents of A and $\{C_1,C_2\}$ parents of C, we find $\alpha' \in A_i \backslash T$ and $\gamma' \in C_k \backslash T$. Our premises $A \neq C$ and $B \notin \{A,C\}$ imply that the sets $\{A_1,A_2\}$, $\{B_1,B_2\}$ and $\{C_1,C_2\}$ are pairwise disjoint. In particular, A_i, B_j and C_k are pairwise distinct. Consequently, the assumption (A_i,B_j,C_k) ensures $\sigma_p([\alpha',\gamma']_p \cap B_j) = \lambda_p(A_i,B_j,C_k) \neq 0$ according to Lemmata 4.15 and 4.17. Because ϵ is injective and because $\{\alpha',\gamma'\} \subseteq \mathrm{ran}(\epsilon)$, we can define $\alpha \coloneqq \epsilon^{-1}(\alpha')$ and $\gamma \coloneqq \epsilon^{-1}(\gamma')$. Then, Lemma 4.5 and $B_1 = B_2$ imply

$$\sigma_{E(p,T)}([\alpha,\gamma]_{E(p,T)} \cap B) = \sigma_p([\alpha',\gamma']_p \cap B_j) = \lambda_p(A_i,B_j,C_k) \neq 0.$$

It follows $(A,B,C) \in \chi_{E(p,T)}$ by definition and $\lambda_{E(p,T)}(A,B,C) = \lambda_p(A_i,B_j,C_k)$ by Lemma 4.17. That proves the claim in this case.

Case 1.2.2: Now, let $B \in \{A,C\}$ instead. Then, Lemma 4.20 assures us that $(A,B,C) \in \{(A,C,C),(A,A,C)\} \subseteq \chi_p$. For that reason, we find $(\alpha,\gamma) \in A \times C$ such that $\sigma_{E(p,T)}([\alpha,\gamma]_{E(p,T)} \cap B) = \lambda_{E(p,T)}(A,B,C) \neq 0$. If we define $\alpha' \coloneqq \epsilon(\alpha)$ and $\gamma' \coloneqq \epsilon(\gamma)$, we can apply Lemma 4.5 and the identity $B_1 = B_2$ to deduce

$$0 \neq \lambda_{E(p,T)}(A,B,C) = \sigma_{E(p,T)}([\alpha,\gamma]_{E(p,T)} \cap B) = \sigma_p([\alpha',\gamma']_p \cap B_j). \tag{2}$$

Now, we must further distinguish four cases.

Case 1.2.2.1: First, assume that $A = B$ and that C is not case E_{III}. Because B is not case E_{III} and $A = B$, block A is not case E_{III} either. Then, $A_1 = A_2$

and $C_1 = C_2$ imply $(\alpha', \gamma') \in A_i \times C_j$. Consequently, (2) already proves our claim $\lambda_p(A_i, B_j, C_k) = \lambda_{E(p,T)}(A, B, C)$ according to Lemma 4.17.

Case 1.2.2.2: If, analogously, $B = C$ and A is not case E_{III}, then as C is not case E_{III}, we infer $\alpha' \in A_1 = A_2 = A_i$ and $\gamma' \in C_1 = C_2 = C_j$. Hence, (2) and Lemma 4.17 verify $\lambda_p(A_i, B_j, C_k) = \lambda_{E(p,T)}(A, B, C)$.

Case 1.2.2.3: Next, let $A = B$, let C be case E_{III} and let $\neg k \in [\![2]\!]$ be such that $\neg k \neq k$. Then, $\alpha' \in A_1 = A_2$ and either $\gamma' \in C_k$ or $\gamma' \in C_{\neg k}$. By Lemma 4.17, if $\gamma' \in C_k$, then (2) yields $\lambda_p(A_i, B_j, C_k) = \lambda_{E(p,T)}(A, B, C)$, proving our claim for this case.

Hence, let $\gamma' \in C_{\neg k}$ instead. Then, (2) and Lemma 4.17 only give $(A_i, B_j, C_{\neg k}) \in \chi_p$ and $\lambda_p(A_i, B_j, C_{\neg k}) = \lambda_{E(p,T)}(A, B, C)$. However, this conclusion is equivalent to our assertion $\lambda_p(A_i, B_j, C_k) = \lambda_{E(p,T)}(A, B, C)$ by Lemma 4.37 (b) because $C_1 \neq C_2$ and $\varnothing \neq C_1 \cap T$ and $\varnothing \neq C_2 \cap T$.

Case 1.2.2.4: Likewise, if $B = C$, if A is case E_{III} and if $\neg i \in [\![2]\!]$ and $\neg i \neq i$, then $\gamma' \in C_k$ and either $\alpha' \in A_i$ or $\alpha' \in A_{\neg i}$. In the former case, (2) proves the claim $\lambda_p(A_i, B_j, C_k) = \lambda_{E(p,T)}(A, B, C)$. In the latter, it only shows $\lambda_p(A_{\neg i}, B_j, C_k) = \lambda_{E(p,T)}(A, B, C)$, which, however, in turn implies the assertion by Lemma 4.37 (a).

Step 2: *Proof of (c).* Due to similarity, it is best to treat (c) right after (a). Again, separate proofs are given for each implication. Let $B \notin \{A, C\}$ and let B be case E_{III}.

Step 2.1: *If $\chi_{E(p,T)}$, then also χ_p.* First, let $\{i, k\} \subseteq [\![2]\!]$ be arbitrary and suppose $(A, B, C) \in \chi_p$. We must show that there is exactly one $j \in [\![2]\!]$ such that $(A_i, B_j, C_k) \in \chi_p$ and that $\lambda_p(A_i, B_j, C_k) = \lambda_{E(p,T)}(A, B, C)$ for this j.

Let $(\alpha, \gamma) \in A \times C$ be such that $0 \neq \lambda_{E(p,T)}(A, B, C) = \sigma_{E(p,T)}([\alpha, \gamma]_{E(p,T)} \cap B)$ and define $\alpha' := \epsilon(\alpha)$ and $\gamma' := \epsilon(\gamma)$. Lemma 4.5 lets us infer

$$0 \neq \lambda_{E(p,T)}(A, B, C) = \sigma_{E(p,T)}([\alpha, \gamma]_{E(p,T)} \cap B) = \textstyle\sum_{j=1}^{2} \sigma_p([\alpha', \gamma']_p \cap B_j). \tag{3}$$

Because for any $j \in [\![2]\!]$ the color sum $\sigma_p([\alpha', \gamma']_p \cap B_j)$ can only take the values -1, 0 or 1, we can deduce from (3) the following: There exist $\{\neg j, j\} = [\![2]\!]$ such that

$$\sigma_p([\alpha', \gamma']_p \cap B_j) = \lambda_{E(p,T)}(A, B, C) \neq 0 \quad \text{and} \quad \sigma_p([\alpha', \gamma']_p \cap B_{\neg j}) = 0. \tag{4}$$

Because $B \notin \{A, C\}$ and because B is case E_{III} we know that both A and C are case E_I. In particular, $\alpha' \in A_i = A_1 = A_2$ and $\gamma' \in C_k = C_1 = C_2$. Hence, the first statement in (4) implies $(A_i, B_j, C_k) \in \lambda_{E(p,T)}$ and $\lambda_p(A_i, B_j, C_k) = \lambda_{E(p,T)}(A, B, C)$ by Lemma 4.17.

Moreover, since $A \neq C$ and $B \notin \{A, C\}$, the blocks A, B and C are actually pairwise distinct, which makes Lemma 4.15 applicable: Since $\alpha' \in A_i$ and $\gamma' \in C_k$, we can infer $(A_i, B_{\neg j}, C_k) \notin \chi_p$ from the second statement in (4). That completes the proof of the first implication.

Step 2.2: *If χ_p, then also $\chi_{E(p,T)}$.* To see the converse, let $\{i, \neg j, j, k\} \subseteq [\![2]\!]$, let $(A_i, B_j, C_k) \in \chi_p$ and let $(A_i, B_{\neg j}, C_k) \notin \chi_p$. We prove that $(A, B, C) \in \chi_{E(p,T)}$ and that $\lambda_{E(p,T)}(A, B, C) = \lambda_p(A_i, B_j, C_k)$.

Recall that the assumption that $B \notin \{A, C\}$ and that B is case E_{III} ensures that $\{A_1, A_2\}$, $\{B_1, B_2\}$ and $\{C_1, C_2\}$ are pairwise disjoint and that $A_1 = A_2$ and

$C_1 = C_2$. Because A_i and C_k are parents of A and C, respectively, there exist $\alpha' \in A_i \backslash T \subseteq \operatorname{ran}(\epsilon)$ and $\gamma' \in C_k \backslash T \subseteq \operatorname{ran}(\epsilon)$. Let $\alpha \coloneqq \epsilon^{-1}(\alpha')$ and $\gamma \coloneqq \epsilon^{-1}(\gamma')$.

Per definition of χ_p, from $(A_i, B_{\neg j}, C_k) \notin \chi_p$ we can infer $\sigma_p([\alpha', \gamma']_p \cap B_{\neg j}) = 0$. On the other hand, the premise $(A_i, B_j, C_k) \in \chi_p$ and Lemmata 4.15 and 4.17 guarantee $\sigma_p([\alpha', \gamma']_p \cap B_j) = \lambda_p(A_i, B_j, C_k) \neq 0$ because A_i, B_j and C_k are pairwise distinct. Hence, with Lemma 4.5,

$$\begin{aligned}&\sigma_{E(p,T)}([\alpha,\gamma]_{E(p,T)} \cap B)\\ &= \sigma_p([\alpha', \gamma']_p \cap B_j) + \sigma_p([\alpha', \gamma']_p \cap B_{\neg j}) = \sigma_p([\alpha', \gamma']_p \cap B_j) = \lambda_{E(p,T)}(A, B, C) \neq 0.\end{aligned}$$

The definition of $\chi_{E(p,T)}$ then implies $(A, B, C) \in \chi_{E(p,T)}$ and Lemma 4.17 gives $\lambda_{E(p,T)}(A, B, C) = \lambda_{E(p,T)}(A, B, C)$. That was our claim.

Step 3: *Proof of (b).* Lemma 4.24 (a) shows that claims (i) and (ii) of (b) are mutually exclusive. We treat them simultaneously. The proof of (iii) is given first, though. Let A be case E_{III} and let $k \in [\![2]\!]$ be arbitrary.

Step 3.1: *Proof of (iii).* By our assumption $A \neq C$ and by Lemma 4.20 it is clear that $(A_1, C_k, C_k) \in \chi_p$ as well as $(A_2, C_k, C_k) \in \chi_p$ and that $(A, C, C) \in \chi_{E(p,T)}$. Hence, the assertion $\lambda_p(A_1, C_k, C_k) = \lambda_p(A_2, C_k, C_k) = \lambda_{E(p,T)}(A, C, C)$ makes sense.

We find $(\alpha, \gamma) \in A \times C$ with $\sigma_{E(p,T)}([\alpha, \gamma]_{E(p,T)} \cap C) = \lambda_{E(p,T)}(A, C, C) \neq 0$. Let $\alpha' \coloneqq \epsilon(\alpha)$ and $\gamma' \coloneqq \epsilon(\gamma)$. Because $A \neq C$, the premise that A is case E_{III} means that C is case E_{I}. In particular, applying Lemma 4.5 to the special case $B = C$, yields,

$$0 \neq \lambda_{E(p,T)}(A, C, C) = \sigma_{E(p,T)}([\alpha, \gamma]_{E(p,T)} \cap C) = \sigma_p([\alpha', \gamma']_p \cap C_k) \tag{5}$$

as $C_k = C_1$. Let $\{\neg i, i\} = [\![2]\!]$ and $\alpha' \in A_i$. Lemma 4.17 lets us infer $\lambda_p(A_i, C_k, C_k) = \lambda_{E(p,T)}(A, C, C)$ from (5). Since $A_1 \neq A_2$ and $\varnothing \neq A_1 \cap T$ and $\varnothing \neq A_2 \cap T$, we can conclude $\lambda_p(A_{\neg i}, C_k, C_k) = \lambda_p(A_i, C_k, C_k)$ by Lemma 4.37 (a). Hence, $\lambda_p(A_1, C_k, C_k) = \lambda_p(A_2, C_k, C_k) = \lambda_{E(p,T)}(A, C, C)$ as claimed.

Step 3.2: *Proof of (i) and (ii).* We prove that, if there exist $\{\neg i, i\} \subseteq [\![2]\!]$ with $\neg i \neq i$ such that $(A_{\neg i}, A_i, C_k) \in \chi_p$, then $\lambda_p(A_i, A_i, C_k) = -\lambda_p(A_{\neg i}, A_{\neg i}, C_k) = \lambda_{E(p,T)}(A, A, C)$ and, otherwise, $\lambda_p(A_1, A_1, C_k) = \lambda_p(A_2, A_2, C_k) = \lambda_{E(p,T)}(A, A, C)$.

Again, the definitions of $\chi_{E(p,T)}$ and $\lambda_{E(p,T)}$ allow us to find $(\alpha, \gamma) \in A \times C$ such that $\sigma_{E(p,T)}([\alpha, \gamma]_{E(p,T)} \cap A) = \lambda_{E(p,T)}(A, A, C) \neq 0$. Let $\alpha' \coloneqq \epsilon(\alpha)$ and $\gamma' \coloneqq \epsilon(\gamma)$. Lemma 4.5 applied to the special case $A = B$ then gives

$$0 \neq \lambda_{E(p,T)}(A, A, C) = \sigma_{E(p,T)}([\alpha, \gamma]_{E(p,T)} \cap A) = \textstyle\sum_{i=1}^{2} \sigma_p([\alpha', \gamma']_p \cap A_i).$$

since A is case E_{III}. Let $\{i_1, i_2\} = [\![2]\!]$ and $\alpha' \in A_{i_1}$. Since $\sigma_p([\alpha', \gamma']_p \cap A_i) \in \{-1, 0, 1\}$ for every $i \in [\![2]\!]$, we conclude that

$$\begin{aligned}\text{either}\quad & \sigma_p([\alpha', \gamma']_p \cap A_{i_1}) = 0 \neq \lambda_{E(p,T)}(A, A, C) = \sigma_p([\alpha', \gamma']_p \cap A_{i_2})\\ \text{or}\quad & \sigma_p([\alpha', \gamma']_p \cap A_{i_2}) = 0 \neq \lambda_{E(p,T)}(A, A, C) = \sigma_p([\alpha', \gamma']_p \cap A_{i_1}).\end{aligned} \tag{6}$$

Three cases must be considered. Recall that Lemma 4.24 (a) guarantees that $(A_1, A_2, C_k) \notin \chi_p$ or $(A_2, A_1, C_k) \notin \chi_p$.

Case 3.2.1: First, suppose that $(A_1, A_2, C_k) \notin \chi_p$ and $(A_2, A_1, C_k) \notin \chi_p$. Then, in particular, $(A_{i_1}, A_{i_2}, C_k) \notin \chi_p$. The definition of χ_p hence implies $\sigma_p([\alpha', \gamma']_p \cap A_{i_2}) =$

0 because $\alpha' \in A_{i_1}$. It follows $\sigma_p([\alpha',\gamma']_p \cap A_{i_1}) = \lambda_{E(p,T)}(A,A,C) \neq 0$ by (6). Thus, $\lambda_p(A_{i_1},A_{i_1},C_k) = \lambda_{E(p,T)}(A,A,C)$ by Lemma 4.17. Finally, Lemma 4.38 (a) (i) proves $\lambda_p(A_{i_2},A_{i_2},C_k) = \lambda_p(A_{i_1},A_{i_1},C_k)$. In conclusion, we have shown our claim for this case, namely that $\lambda_p(A_1,A_1,C_k) = \lambda_p(A_2,A_2,C_k) = \lambda_{E(p,T)}(A,A,C)$.

Case 3.2.2: Next, let $(A_{i_1},A_{i_2},C_k) \in \chi_p$ and $(A_{i_2},A_{i_1},C_k) \notin \chi_p$. Because A_{i_1}, A_{i_2} and C_k are pairwise distinct, $(A_{i_1},A_{i_2},C_k) \in \chi_p$ implies $\sigma_p([\alpha',\gamma']_p \cap A_{i_2}) \neq 0$ by Lemma 4.15. Accordingly, $\sigma_p([\alpha',\gamma']_p \cap A_{i_2}) = \lambda_{E(p,T)}(A,A,C) \neq 0$ by (6). It follows $\lambda_p(A_{i_1},A_{i_2},C_k) = \lambda_{E(p,T)}(A,A,C)$ by Lemma 4.17. By Lemma 4.21 we can thus conclude $\lambda_p(A_{i_2},A_{i_2},C_k) = \lambda_{E(p,T)}(A,A,C)$. And Lemma 4.38 (a) (ii) lets us know that $\lambda_p(A_{i_2},A_{i_2},C_k) = -\lambda_p(A_{i_1},A_{i_1},C_k)$. Thus, we have verified our assertion $\lambda_p(A_{i_2},A_{i_2},C_k) = -\lambda_p(A_{i_1},A_{i_1},C_k) = \lambda_{E(p,T)}(A,A,C)$ for this case as well.

Case 3.2.3: Finally, assume $(A_{i_2},A_{i_1},C_k) \in \chi_p$ and $(A_{i_1},A_{i_2},C_k) \notin \chi_p$. From $\alpha' \in A_{i_1}$ and $(A_{i_1},A_{i_2},C_k) \notin \chi_p$ it follows $\sigma_p([\alpha',\gamma']_p \cap A_{i_2}) = 0$ by definition of χ_p. Hence, (6) lets us conclude $\sigma_p([\alpha',\gamma']_p \cap A_{i_1}) = \lambda_{E(p,T)}(A,A,C) \neq 0$. Lemma 4.17 thus proves $\lambda_p(A_{i_1},A_{i_1},C_k) = \lambda_{E(p,T)}(A,A,C)$. And, $\lambda_p(A_{i_2},A_{i_2},C_k) = -\lambda_p(A_{i_1},A_{i_1},C_k)$ by Lemma 4.38 (a) (ii). Altogether, that proves our claim.

Step 4: *Proof of (d).* Follows by (b) and Lemma 4.23. □

LEMMA 4.41. *$E(p,T) \in \mathcal{W}_R$ for all turns T in p, all $p \in \mathcal{W}_R$ and all $R \in \mathcal{R}$.*

PROOF. Let $n \in \mathbb{N}$, let $2 \leq n$, let $\{B_1,B_2,\dots,B_n\} \subseteq E(p,T)$, let $B_1,B_2,\dots,B_n$ be pairwise distinct, let B_1 and B_n cross in $E(p,T)$, let $\langle B_1,B_n\rangle_{E(p,T)} = \{B_1,B_2,\dots,B_n\}$, let $B_1 \leq B_2 \leq \dots \leq B_n$ with respect to $\leq_{E(p,T),B_1,B_n}$ and for every $i \in [\![n]\!]$ let $c_i \in \{\circ,\bullet\}$ be such that $\lambda_{E(p,T)}(B_1,B_i,B_n) = \sigma(c_i)$. We have to prove $(c_1,c_2,\dots,c_n) \in R$.

For every $i \in [\![n]\!]$ let $\{B_{i,1},B_{i,2}\}$ be parents of B_i with respect to (p,T). Note that $\{B_{i_1,1},B_{i_1,2}\} \cap \{B_{i_2,1},B_{i_2,2}\} = \emptyset$ for all $\{i_1,i_2\} \subseteq [\![n]\!]$ with $i_1 \neq i_2$ because $B_1,B_2,\dots,B_n$ are pairwise distinct. We distinguish which cases E_{I}–E_{III} which of $B_1,B_2,\dots,B_n$ are.

Case 1: *None of the blocks resulted from merging.* First, let B_i be case E_{I} or E_{II} for every $i \in [\![n]\!]$. By definition of $E(p,T)$ then, either T is a subpartition of $E(p,T)$ or that there exists $D \in E(p,T)$ with $D \notin \{B_1,B_2,\dots,B_n\}$ which is case E_{II} or E_{III} with respect to (p,T). If such D exists let $\{D_1,D_2\}$ be parents of D with respect to (p,T). Note that $B_{1,1} = B_{1,2}$ and $B_{n,1} = B_{n,2}$ cross in p by Lemma 4.39 because B_1 and B_n cross in $E(p,T)$. A further case distinction is required

Case 1.1: *T was a block of p, was contained in one or intersected two irrelevant ones.* Suppose that T is a subpartition of p or that D exists and is such that, if D is case E_{III}, then $(B_{1,1},D_j,B_{1,n}) \notin \chi_p$ for both $j \in [\![2]\!]$. We show that $\langle B_{1,1},B_{n,1}\rangle_p = \{B_{1,1},B_{2,1},\dots,B_{n,1}\}$, that $B_{1,1} \leq B_{2,1} \leq \dots \leq B_{n,1}$ with respect to $\leq_{p,B_{1,1},B_{n,1}}$ and that $\lambda_p(B_{1,1},B_{i,1},B_{n,1}) = \sigma(c_i)$ for every $i \in [\![n]\!]$. Since we assume $p \in \mathcal{W}_R$, this is sufficient to prove $(c_1,c_2,\dots,c_n) \in R$.

Step 1.1.1: *The blocks.* We begin by showing $\langle B_{1,1},B_{n,1}\rangle_p = \{B_{1,1},B_{2,1},\dots,B_{n,1}\}$. Each inclusion is treated separately.

Step 1.1.1.1: *Inclusion $\supseteq$.* We let $i \in [\![n]\!]$ be arbitrary and prove $B_{i,1} \in \langle B_{1,1},B_{n,1}\rangle_p$, i.e., $(B_{1,1},B_{i,1},B_{n,1}) \in \chi_p$. And, indeed, because B_i is not case E_{III}, the assumption $(B_1,B_i,B_n) \in \chi_{E(p,T)}$ lets us infer $(B_{1,1},B_{i,1},B_{n,1}) \in \chi_p$ by Lemma 4.40 (a) (i).

Step 1.1.1.2: *Inclusion* $\subseteq$. To see the converse inclusion we let $F_1 \in p$ and $F_1 \notin \{B_{1,1}, B_{2,1}, \ldots, B_{n,1}\} = \{B_{1,2}, B_{2,2}, \ldots, B_{n,2}\}$ and prove that $F_1 \notin \langle B_{1,1}, B_{n,1}\rangle_p$, which is to say $(B_{1,1}, F_1, B_{1,n}) \notin \chi_p$.

If $F_1 = T$, then F_1, as a pair block, must be a connected component of $p \in \mathcal{W}$ by Lemma 6.3. According to Lemma 4.19 this excludes $(B_{1,1}, F_1, B_{1,n}) \in \chi_p$ because $B_{1,1}$ and $B_{1,n}$ cross in p. Hence, we can assume $F_1 \neq T$.

Then, there exist $F \in E(p,T)$ and $F_2 \in p$ such that $\{F_1, F_2\}$ are parents of F with respect to (p,T). The assumption $F_1 \notin \{B_{1,1}, B_{2,1}, \ldots, B_{n,1}\}$ assures us that $F \notin \{B_1, B_2, \ldots, B_n\}$. Since $\langle B_1, B_n\rangle_{E(p,T)} = \{B_1, B_2, \ldots, B_n\}$ it follows $(B_1, F, B_n) \notin \chi_{E(p,T)}$.

If F is not case $\mathrm{E_{III}}$, then Lemma 4.40 (a) (i) allows us to conclude $(B_{1,1}, F_1, B_{1,n}) \notin \chi_p$ from $(B_1, F, B_n) \notin \chi_{E(p,T)}$ immediately.

Should F be case $\mathrm{E_{III}}$, then we must have $F = D$ because there can be at most one block of $E(p,T)$ which belongs to a case other than $\mathrm{E_I}$. If so, then $F_1 \in \{D_1, D_2\}$ since those are parents of D. Hence, our assumption that $(B_{1,1}, D_j, B_{1,n}) \notin \chi_p$ for both $j \in [\![2]\!]$ proves $(B_{1,1}, F, B_{1,n}) \notin \chi_p$, as claimed.

Step 1.1.2: *Their ordering.* Next, we verify that $B_{1,1} \leq B_{1,2} \leq \ldots \leq B_{1,n}$ with respect to $\leq_{p,B_{1,1},B_{1,n}}$. Equivalently, we have to prove $(B_{1,1}, B_{i,1}, B_{i+1,1}) \in \chi_p$ for all $i \in [\![n-1]\!]$. Because $B_{1,1} \neq B_{i+1,1}$ and because $B_{i,1}$ is not case $\mathrm{E_{III}}$, the statement $(B_{1,1}, B_{i,1}, B_{i+1,1}) \in \chi_p$ is equivalent to $(B_1, B_i, B_{i+1}) \in \chi_{E(p,T)}$ by Lemma 4.40 (a) (i). And this latter relation is of course true by the assumption that $B_1 \leq B_2 \leq \ldots \leq B_n$ with respect to $\leq_{E(p,T),B_1,B_n}$. Hence, as asserted, $B_{1,1} \leq B_{1,2} \leq \ldots \leq B_{1,n}$ with respect to $\leq_{p,B_{1,1},B_{1,n}}$.

Step 1.1.3: *The colors.* The last part of the above claim to be confirmed is that $\lambda_p(B_{1,1}, B_{i,1}, B_{n,1}) = \lambda_{E(p,T)}(B_1, B_i, B_n)$ for every $i \in [\![n]\!]$. But this is clear by Lemma 4.40 (a) (ii) because B_i is assumed to not be case $\mathrm{E_{III}}$ for any $i \in [\![n]\!]$.

Case 1.2: T *intersected two blocks of* p*, at least one of them relevant.* Alternatively, assume that D exists, that D is case $\mathrm{E_{III}}$ and that $(B_{1,1}, D_j, B_{1,n}) \in \chi_p$ for at least one $j \in [\![2]\!]$. We prove that then there exist $i_0 \in [\![n-1]\!]$ and $\{j_1, j_2\} \subseteq [\![2]\!]$ with $j_1 \neq j_2$ such that $\langle B_{1,1}, B_{n,1}\rangle_p = \{B_{1,1}, B_{2,1}, \ldots, B_{n,1}\} \cup \{D_1, D_2\}$, such that $B_{1,1} \leq B_{2,1} \leq \ldots \leq B_{i_0,1} \leq D_{j_1} \leq D_{j_2} \leq B_{i_0+1,1} \leq B_{i_0+2,1} \leq \ldots \leq B_{n,1}$ with respect to $\leq_{p,B_{1,1},B_{n,1}}$, such that $\lambda_p(B_{1,1}, B_{i,1}, B_{n,1}) = \sigma(c_i)$ for every $i \in [\![n]\!]$ and such that, if $c \in \{\circ, \bullet\}$ is such that $\lambda_p(B_{1,1}, D_{j_1}, B_{n,1}) = \sigma(c)$, then $\lambda_p(B_{1,1}, D_{j_2}, B_{n,1}) = \sigma(\overline{c})$. Our assumption $p \in \mathcal{W}_R$ then implies $(c_1, c_2, \ldots, c_{i_0}, c, \overline{c}, c_{i_0+1}, c_{i_0+2}, \ldots, c_n) \in R$ and thus our claim $(c_1, c_2 \ldots, c_n) \in R$ because R is a $\mathcal{W}$-parameter set.

Step 1.2.1: *The blocks.* Again, we begin by proving $\langle B_{1,1}, B_{n,1}\rangle_p = \{B_{1,1}, B_{2,1}, \ldots, B_{n,1}\} \cup \{D_1, D_2\}$ and treat each inclusion individually.

Step 1.2.1.1: *Inclusion* $\supseteq$. For any $i \in [\![n]\!]$ the assumptions that B_i is case $\mathrm{E_I}$ and that $B_i \in \langle B_1, B_n\rangle_{E(p,T)}$ ensure $B_{i,1} \in \langle B_{1,1}, B_{n,1}\rangle_p$ by Lemma 4.40 (a) (i). We have explicitly assumed that there exists $j \in [\![2]\!]$ such that $(B_{1,1}, D_j, B_{1,n}) \in \chi_p$, which is to say $D_j \in \langle B_{1,1}, B_{n,1}\rangle_p$. Let $\neg j \in [\![2]\!]$ be such that $\neg j \neq j$. It remains to prove $(B_{1,1}, D_{\neg j}, B_{1,n}) \in \chi_p$. Per the assumption $D \notin \{B_1, B_2, \ldots B_n\} =$

$\langle B_1, B_n\rangle_{E(p,T)}$ we know $(B_1, D, B_n) \notin \chi_{E(p,T)}$. Because D is case $\mathrm{E_{III}}$ the two statements $(B_1, D, B_n) \notin \chi_{E(p,T)}$ and $(B_{1,1}, D_j, B_{1,n}) \in \chi_p$ then require $(B_{1,1}, D_{\neg j}, B_{1,n}) \in \chi_p$ by Lemma 4.40 (c) (i). Hence, $\langle B_{1,1}, B_{n,1}\rangle_p \supseteq \{B_{1,1}, B_{2,1}, \ldots, B_{n,1}\} \cup \{D_1, D_2\}$.

Step 1.2.1.2: *Inclusion* $\subseteq$. Let $F_1 \in p$ and let $F_1 \neq B_{i,1}$ and $F_1 \neq D_j$ for all $i \in [\![n]\!]$ and $j \in [\![2]\!]$. We prove $(B_{1,1}, F_1, B_{n,1}) \notin \chi_p$.

Since D is case $\mathrm{E_{III}}$ we are guaranteed $T \subseteq D_1 \cup D_2$ and thus $F_1 \cap T = \varnothing$ by $F_1 \notin \{D_1, D_2\}$. Hence, by definition of $E(p,T)$ there exist $F \in E(p,T)$ and $F_2 \in p$ such that $\{F_1, F_2\}$ are parents of F with respect to (p,T). Then, $D \neq F$ because $F_1 \notin \{D_1, D_2\}$. In consequence, F is not case $\mathrm{E_{III}}$, as D already is.

And the assumption $F_1 \notin \{B_{1,1}, B_{2,1}, \ldots, B_{n,1}\}$ further ensures $F \notin \{B_1, B_2, \ldots, B_n\}$. As $\langle B_1, B_n\rangle_{E(p,T)} = \{B_1, B_2, \ldots, B_n\}$, consequently, $(B_1, F, B_n) \notin \chi_{E(p,T)}$.

From $(B_1, F, B_n) \notin \chi_{E(p,T)}$ it follows by Lemma 4.40 (a) (i) that $(B_{1,1}, F_1, B_{n,1}) \notin \chi_p$ because F is not case $\mathrm{E_{III}}$. Thus, $\langle B_{1,1}, B_{n,1}\rangle_p \subseteq \{B_{1,1}, B_{2,1}, \ldots, B_{n,1}\} \cup \{D_1, D_2\}$ has been proven.

Step 1.2.2: *Their ordering.* The next step is to prove the existence of $i_0 \in [\![n{-}1]\!]$ and $\{j_1, j_2\} \subseteq [\![n]\!]$ such that $B_{1,1} \leq B_{2,1} \leq \ldots \leq B_{i_0,1} \leq D_{j_1} \leq D_{j_2} \leq B_{i_0+1,1} \leq B_{i_0+2,1} \leq \ldots \leq B_{n,1}$ with respect to $\leq_{p,B_{1,1},B_{n,1}}$.

By Lemma 4.40 (a) (i) for all $i \in [\![n{-}1]\!]$, because B_i is not case $\mathrm{E_{III}}$, the assumption that $(B_1, B_i, B_{i+1}) \in \chi_{E(p,T)}$ ensures that $(B_{1,1}, B_{i,1}, B_{i+1,1}) \in \chi_p$. Hence, $B_{1,1} \leq B_{2,1} \leq \ldots \leq B_{n,1}$ with respect to $\leq_{p,B_{1,1},B_{n,1}}$.

Because $\leq_{p,B_{1,1},B_{n,1}}$ is a total order on $\langle B_{1,1}, B_{n,1}\rangle_p$ by Lemma 4.28 and because $\{D_1, D_2\} \subseteq \langle B_{1,1}, B_{n,1}\rangle_p$ by Step 1.2.1 there must indeed exist $\{j_1, j_2\} \subseteq [\![2]\!]$ such that $j_1 \neq j_2$ and $D_{j_1} \leq D_{j_2}$ with respect to $\leq_{p,B_{1,1},B_{n,1}}$.

Moreover, because $B_{1,1} \leq D_{j_1}$ with respect to $\leq_{p,B_{1,1},B_{n,1}}$ by $B_{1,1} \neq D_{j_1}$ and Lemma 4.20, there exists $i \in [\![n{-}1]\!]$ such that $B_{i,1} \leq D_{j_1}$ with respect to $\leq_{p,B_{1,1},B_{n,1}}$. Hence, and by Lemma 4.28, we can let i_0 be $\leq_{p,B_{1,1},B_{n,1}}$-maximal with this property.

We prove by contradiction that D_{j_1} and D_{j_2} are neighbors with respect to the order $\leq_{p,B_{1,1},B_{n,1}}$ on $\langle B_{1,1}, B_{n,1}\rangle_p$. Let $i_1 \in [\![n{-}1]\!]$ be such that $D_{j_1} \leq B_{i_1,1} \leq D_{j_2}$, i.e., such that $(B_{1,1}, D_{j_1}, B_{i_1,1})$ and $(B_{1,1}, B_{i_1,1}, D_{j_2})$. Lemma 4.25 then allows us to conclude $(D_{j_1}, B_{i_1,1}, D_{j_2}) \in \chi_p$. However, because $\{D_1, D_2\}$ are parents of D with respect to (p,T) and because $B_{i_1,1} \notin \{D_1, D_2\}$, Lemma 4.36 assures us $(D_{j_1}, B_{i_1,1}, D_{j_2}) \notin \chi_p$. That is the contradiction we sought.

Now, the definitions of j_1, j_2 and i_0 and the fact that $B_{1,1} \leq B_{2,1} \leq \ldots \leq B_{n,1}$ with respect to $\leq_{p,B_{1,1},B_{n,1}}$ together imply $B_{1,1} \leq B_{2,1} \leq \ldots \leq B_{i_0,1} \leq D_{j_1} \leq D_{j_2} \leq B_{i_0+1,1} \leq B_{i_0+2,1} \leq \ldots \leq B_{n,1}$ with respect to $\leq_{p,B_{1,1},B_{n,1}}$, which is what we needed to see.

Step 1.2.3: *The colors.* For every $i \in [\![n]\!]$ Lemma 4.40 (a) (ii) lets us infer $\lambda_p(B_{1,1}, B_{i,1}, B_{n,1}) = \lambda_{E(p,T)}(B_1, B_i, B_n)$ because B_i is not case $\mathrm{E_{III}}$. We only need to prove $\lambda_p(B_{1,1}, D_1, B_{n,1}) = -\lambda_p(B_{1,1}, D_2, B_{n,1})$.

In Step 1.2.2 we showed that there exist $\{j_1, j_2\} \subseteq [\![2]\!]$ with $j_1 \neq j_2$ and $D_{j_1} \leq D_{j_2}$ with respect to $\leq_{p,B_{1,1},B_{n,1}}$, which is to say $(B_{1,1}, D_{j_1}, D_{j_2}) \in \chi_p$. Because $B_{1,1}$, D_{j_1} and D_{j_2} are pairwise distinct, because D is case $\mathrm{E_{III}}$ and because $\{D_1, D_2\}$ are parents blocks of D with respect to (p,T) Lemma 4.38 (b) (ii) then proves our claim.

Case 2: *A middle block resulted from merging.* Next, let there exist $i_0 \in [\![n]\!]$ with $1 < i_0 < n$ such that B_{i_0} is case $\mathrm{E_{III}}$. Lemma 4.39 assures us that $B_{1,1} = B_{1,2}$ and $B_{n,1} = B_{n,2}$ cross in p since B_1 and B_n cross in $E(p,T)$. We prove that there exists $j_0 \in [\![2]\!]$ such that $\langle B_{1,1}, B_{n,1}\rangle_p = \{B_{1,1}, B_{2,1}, \ldots, B_{i_0-1,1}, B_{i_0,j_0}, B_{i_0+1,1}, B_{i_0+2,1}, \ldots, B_{n,1}\}$, such that $B_{1,1} \leq B_{2,1} \leq \ldots \leq B_{i_0-1,1} \leq B_{i_0,j_0} \leq B_{i_0+1,1} \leq B_{i_0+2,1} \leq \ldots \leq B_{n,1}$ with respect to $\leq_{p,B_{1,1},B_{n,1}}$, such that $\lambda_p(B_{1,1}, B_{i,1}, B_{n,1}) = \sigma(c_i)$ for every $i \in [\![n]\!]\backslash\{i_0\}$ and such that $\lambda_p(B_{1,1}, B_{i_0,j_0}, B_{n,1}) = \sigma(c_{i_0})$. Again, because $p \in \mathcal{W}_R$, this will then prove $(c_1, c_2, \ldots, c_n) \in R$, as claimed.

Step 2.1: *The blocks.* We prove that $\langle B_{1,1}, B_{n,1}\rangle_p = \{B_{1,1}, B_{2,1}, \ldots, B_{i_0-1,1}, B_{i_0,j_0}, B_{i_0+1,1}, B_{i_0+2,1}, \ldots, B_{n,1}\}$ for an appropriate $j_0 \in [\![2]\!]$.

Step 2.1.1: *Inclusion* $\supseteq$. Because $B_{i_0} \notin \{B_1, B_n\}$, because B_{i_0} is case $\mathrm{E_{III}}$ and because $(B_1, B_{i_0}, B_n) \in \chi_{E(p,T)}$ Lemma 4.40 (c) (i) tells us that there exists exactly one $j_0 \in [\![2]\!]$ such that $(B_{1,1}, B_{i_0,j_0}, B_{n,1}) \in \chi_p$, i.e., such that $B_{i_0,j_0} \in \langle B_{1,1}, B_{1,n}\rangle_p$.

On the other hand, for every $i \in [\![n]\!]\backslash\{i_0\}$ the assumption that B_{i_0} is case $\mathrm{E_{III}}$ implies that B_i is not. Hence, $(B_{1,1}, B_{i,1}, B_{n,1}) \in \chi_p$ by Lemma 4.40 (a) (i) because we have assumed $(B_1, B_i, B_n) \in \chi_{E(p,T)}$. That proves the first inclusion.

Step 2.1.2: *Inclusion* $\subseteq$. Conversely, let $F_1 \in p$, let $F_1 \neq B_{i,1} = B_{i,2}$ for all $i \in [\![n]\!]\backslash\{i_0\}$ and let $F_1 \neq B_{i_0,j_0}$. We prove $(B_{1,1}, F_1, B_{n,1}) \notin \chi_p$.

If $F_1 = B_{i_0,\neg j_0}$ for $\neg j_0 \in [\![2]\!]$ with $\neg j_0 \neq j_0$, then $(B_{1,1}, F_1, B_{n,1}) \notin \chi_p$ is guaranteed since j_0 is unique with the property $(B_{1,1}, B_{i_0,j_0}, B_{n,1}) \in \chi_p$. Hence, we can assume $F_1 \notin \{B_{i_0,1}, B_{i_0,2}\}$, and thus $F_1 \notin \{B_{1,1}, B_{2,1}, \ldots, B_{n,1}\}$, in the following.

Then, there exist $F \in E(p,T)$ and $F_2 \in p$ such that $\{F_1, F_2\}$ are parents of F with respect to (p,T). Moreover, F is case $\mathrm{E_I}$ and $F \notin \{B_1, B_2, \ldots, B_n\}$ by $F_1 \notin \{B_{1,1}, B_{2,1}, \ldots, B_{n,1}\}$. Because $\langle B_1, B_n\rangle_{E(p,T)} = \{B_1, B_2, \ldots, B_n\}$, therefore, $(B_1, F, B_n) \notin \chi_{E(p,T)}$. It follows $(B_{1,1}, F_1, B_{n,1}) \notin \chi_p$ by Lemma 4.40 (a) (i) since F is not case $\mathrm{E_{III}}$. In other words, $F_1 \notin \langle B_{1,1}, B_{n,1}\rangle_p$, proving the other inclusion.

Step 2.2: *Their ordering.* Next, we prove that $B_{1,1} \leq B_{2,1} \leq \ldots \leq B_{i_0-1,1} \leq B_{i_0,j_0} \leq B_{i_0+1,1} \leq B_{i_0+2,1} \leq \ldots \leq B_{n,1}$ with respect to $\leq_{p,B_{1,1},B_{n,1}}$.

For every $i \in [\![n]\!]$ with $i \notin \{i_0 - 1, i_0, n\}$ the block B_i is case $\mathrm{E_I}$. Hence, the assumption that $B_i \leq B_{i+1}$ with respect to $\leq_{E(p,T)}$, i.e., that $(B_1, B_i, B_{i+1}) \in \chi_{E(p,T)}$, implies $(B_{1,1}, B_{i,1}, B_{i+1,i}) \in \chi_p$ by Lemma 4.40 (a) (i), which is to say $B_{i,1} \leq B_{i+1,1}$ with respect to $\leq_{p,B_{1,1},B_{n,1}}$.

The block B_{i_0-1} is case $\mathrm{E_I}$ as well. That is why our premise $(B_1, B_{i_0-1}, B_{i_0}) \in \chi_{E(p,T)}$ ensures $(B_{1,1}, B_{i_0-1,1}, B_{i_0,j_0}) \in \chi_p$ by Lemma 4.40 (a) (i). Hence, $B_{i_0-1,1} \leq B_{i_0,j_0}$ with respect to $\leq_{p,B_{1,1},B_{n,1}}$.

According to Lemma 4.29 the assumption that $B_{i_0} \leq B_{i_0+1}$ with respect to $\leq_{E(p,T),B_1,B_n}$ is equivalent to the relation $B_{i_0+1} \leq B_{i_0}$ with respect to $\leq_{E(p,T),B_n,B_1}$, i.e., to the statement $(B_n, B_{i_0+1}, B_{i_0}) \in \chi_{E(p,T)}$. Because B_{i_0+1} is not case $\mathrm{E_{III}}$ we can apply Lemma 4.40 (a) (i) to infer $(B_{n,1}, B_{i_0+1,1}, B_{i_0,j_0}) \in \chi_p$ or, equivalently, $B_{i_0+1,1} \leq B_{i_0,j_0}$ with respect to $\leq_{p,B_{n,1},B_{1,1}}$. Now, Lemma 4.29 employed a second time implies $B_{i_0,j_0} \leq B_{i_0+1,1}$ with respect to $\leq_{p,B_{1,1},B_{n,1}}$.

That is all which was left to show in order to prove that $B_{1,1} \leq B_{2,1} \leq \ldots \leq B_{i_0-1,1} \leq B_{i_0,j_0} \leq B_{i_0+1,1} \leq B_{i_0+2,1} \leq \ldots \leq B_{n,1}$ with respect to $\leq_{p,B_{1,1},B_{n,1}}$.

Step 2.3: *The colors.* Finally, we verify that $\lambda_p(B_{1,1}, B_{i,1}, B_{n,1}) = \sigma(c_i)$ for every $i \in [\![n]\!] \backslash \{i_0\}$ and that $\lambda_p(B_{1,1}, B_{i_0,j_0}, B_{n,1}) = \sigma(c_{i_0})$. Indeed, for every $i \in [\![n]\!] \backslash \{i_0\}$ the fact that B_i is not case $\mathrm{E}_{\mathrm{III}}$ allows us to infer $\lambda_p(B_{1,1}, B_{i,1}, B_{n,1}) = \lambda_{E(p,T)}(B_1, B_i, B_n)$ by Lemma 4.40 (a) (ii). And the definition of j_0 as the unique index with $(B_{1,1}, B_{i_0,j_0}, B_{n,1}) \in \chi_p$ implies $\lambda_p(B_{1,1}, B_{i_0,j_0}, B_{n,1}) = \lambda_{E(p,T)}(B_1, B_{i_0}, B_n)$ by Lemma 4.40 (c) (ii) since B_{i_0} is case $\mathrm{E}_{\mathrm{III}}$. That completes the proof for this case.

Case 3: *The first block resulted from merging.* Next, let B_1 be case $\mathrm{E}_{\mathrm{III}}$. Since B_1 and B_n cross in p, there exists $j_1 \in [\![2]\!]$ such that B_{1,j_1} and $B_{n,1} = B_{n,2}$ cross in p by Lemma 4.39. Let $j_2 \in [\![2]\!]$ be such that $j_1 \neq j_2$. Now, there are two possibilities.

Case 3.1: *Crossing with the inner one.* First, suppose $(B_{1,j_1}, B_{1,j_2}, B_{n,1}) \notin \chi_p$. We prove that then $\langle B_{1,j_1}, B_{n,1} \rangle_p = \{B_{1,j_1}, B_{2,1}, B_{3,1}, \ldots, B_{n,1}\}$, that $B_{1,j_1} \leq B_{2,1} \leq B_{3,1} \leq \ldots \leq B_{n,1}$ with respect to $\leq_{p,B_{1,j_1},B_{n,1}}$, that $\lambda_p(B_{1,j_1}, B_{1,j_1}, B_{n,1}) = \sigma(c_1)$ and that $\lambda_p(B_{1,j_1}, B_{i,1}, B_{n,1}) = \sigma(c_i)$ for every $i \in [\![n]\!]$ with $2 \leq i$. The assumption $p \in \mathcal{W}_R$ will then prove our assertion $(c_1, c_2, \ldots, c_n) \in R$.

Step 3.1.1: *The blocks.* As usual we begin by proving $\langle B_{1,j_1}, B_{n,1} \rangle_p = \{B_{1,j_1}, B_{2,1}, B_{3,1}, \ldots, B_{n,1}\}$ and give separate proofs for each inclusion.

Step 3.1.1.1: *Inclusion* $\supseteq$. By Lemma 4.20 it is clear that $(B_{1,j_1}, B_{1,j_1}, B_{n,1}) \in \chi_p$, i.e., that $B_{1,j_1} \in \langle B_{1,j_1}, B_{n,1} \rangle_p$. We only need to prove $(B_{1,j_1}, B_{i,1}, B_{n,1}) \in \chi_p$ for every $i \in [\![n]\!]$ with $2 \leq i$. And by Lemma 4.40 (a) (i) for such i this follows from our assumption $(B_1, B_i, B_n) \in \chi_{E(p,T)}$ because B_i is not case $\mathrm{E}_{\mathrm{III}}$. Hence, indeed, $\langle B_{1,j_1}, B_{n,1} \rangle_p \supseteq \{B_{1,j_1}, B_{2,1}, B_{3,1}, \ldots, B_{n,1}\}$.

Step 3.1.1.2: *Inclusion* $\subseteq$. In order to see the converse inclusion, we let $F_1 \in p$ and $F_1 \notin \{B_{1,j_1}, B_{2,1}, B_{3,1}, \ldots, B_{n,1}\}$ and show $(B_{1,j_1}, F_1, B_{n,1}) \notin \chi_p$.

By our assumption $(B_{1,j_1}, B_{1,j_2}, B_{n,1}) \notin \chi_p$ we can assume that $F_1 \neq B_{1,j_2}$. Then, $F_1 \notin \{B_{i,j} \,|\, i \in [\![n]\!], j \in [\![2]\!]\}$. In particular, since $F \notin T$, there must exist $F \in E(p,T)$ and $F_2 \in p$ such that $\{F_1, F_2\}$ are parents of F with respect to (p,T). And F must be case E_{I}. Moreover, $F \notin \{B_1, B_2, \ldots, B_n\}$.

The premise $\langle B_1, B_n \rangle_{E(p,T)} = \{B_1, B_2, \ldots, B_n\}$ then requires $(B_1, F, B_n) \notin \chi_{E(p,T)}$. By Lemma 4.40 (a) (i) we can conclude from this $(B_{1,j_1}, F_1, B_{n,1}) \notin \chi_p$ because F is not case $\mathrm{E}_{\mathrm{III}}$. And that is what we needed to see.

Step 3.1.2: *Their ordering.* Let us prove next that $B_{1,j_1} \leq B_{2,1} \leq B_{3,1} \leq \ldots \leq B_{n,1}$ with respect to $\leq_{p,B_{1,j_1},B_{n,1}}$.

Lemma 4.20 assures us that $(B_{1,j_1}, B_{1,j_1}, B_{2,1}) \in \chi_p$, which is equivalent to the relation $B_{1,j_1} \leq B_{2,1}$ with respect to $\leq_{p,B_{1,j_1},B_{n,1}}$.

Since for every $i \in [\![n]\!]$ with $2 \leq i < n$ the block B_i is not case $\mathrm{E}_{\mathrm{III}}$ and since we assume $B_i \leq B_{i+1}$ with respect to $\leq_{E(p,T),B_1,B_n}$, i.e., $(B_1, B_i, B_{i+1}) \in \chi_{E(p,T)}$, we can apply Lemma 4.40 (a) (i) and infer $(B_{1,j_1}, B_{i,1}, B_{i+1,1}) \in \chi_p$. That proves $B_{2,1} \leq B_{3,1} \leq \ldots \leq B_{n,1}$ with respect to $\leq_{p,B_{1,j_1},B_{n,1}}$, which is what was left to prove.

Step 3.1.3: *The colors.* The last step consists in showing $\lambda_p(B_{1,j_1}, B_{1,j_1}, B_{n,1}) = \sigma(c_1)$ and $\lambda_p(B_{1,j_1}, B_{i,1}, B_{n,1}) = \sigma(c_i)$ for every $i \in [\![n]\!]$ with $2 \leq i$.

There are two possibilities: The first is that $(B_{1,j_2}, B_{1,j_1}, B_{n,1}) \notin \chi_p$, which, because we have assumed $(B_{1,j_1}, B_{1,j_2}, B_{n,1}) \notin \chi_p$, is to say $(B_{1,2}, B_{1,1}, B_{n,1}) \notin \chi_p$ and $(B_{1,1}, B_{1,2}, B_{n,1}) \notin \chi_p$. In this case we can conclude $\lambda_p(B_{1,j_1}, B_{1,j_1}, B_{n,1}) = \lambda_p(B_{1,j_2}, B_{1,j_2}, B_{n,1}) = \lambda_{E(p,T)}(B_1, B_1, B_n)$ by Lemma 4.40 (b) (i). The alternative is that $(B_{1,j_2}, B_{1,j_1}, B_{n,1}) \in \chi_p$. If so, Lemma 4.40 (b) (ii) guarantees $\lambda_p(B_{1,j_1}, B_{1,j_1}, B_{n,1}) = -\lambda_p(B_{1,j_2}, B_{1,j_2}, B_{n,1}) = \lambda_{E(p,T)}(B_1, B_1, B_n)$ as well. Hence, $\lambda_p(B_{1,j_1}, B_{1,j_1}, B_{n,1}) = \sigma(c_1)$ always holds.

On the other hand, for any $i \in [\![n]\!]$ with $i \le 2$ the block B_i not being case $\mathrm{E_{III}}$ allows us to conclude $\lambda_p(B_{1,j_1}, B_{i,1}, B_{n,1}) = \lambda_{E(p,T)}(B_1, B_i, B_n) = \sigma(c_i)$ by Lemma 4.40 (a) (ii), just as claimed.

Case 3.2: *Crossing with the outer one.* Alternatively, let $(B_{1,j_1}, B_{1,j_2}, B_{n,1}) \in \chi_p$. We show that $\langle B_{1,j_1}, B_{n,1}\rangle_p = \{B_{1,j_1}, B_{1,j_2}, B_{2,1}, B_{3,1}, \ldots, B_{n,1}\}$, that $B_{1,j_1} \le B_{1,j_2} \le B_{2,1} \le B_{3,1} \le \ldots \le B_{n,1}$ with respect to $\le_{p,B_{1,j_1},B_{n,1}}$, that $\lambda_p(B_{1,j_1}, B_{1,j_1}, B_{n,1}) = \sigma(\overline{c_1})$, that $\lambda_p(B_{1,j_1}, B_{1,j_2}, B_{n,1}) = \sigma(c_1)$ and that $\lambda_p(B_{1,j_1}, B_{i,1}, B_{n,1}) = \sigma(c_i)$ for every $i \in [\![n]\!]$ with $2 \le i$. From $p \in \mathcal{W}_R$ it will then follow that $(\overline{c_1}, c_1, c_2, \ldots, c_n) \in R$. As R is a $\mathcal{W}$-parameter set, this is enough to prove $(c_1, c_2, \ldots, c_n) \in R$.

Step 3.2.1: *The blocks.* We show $\langle B_{1,j_1}, B_{n,1}\rangle_p = \{B_{1,j_1}, B_{1,j_2}, B_{2,1}, B_{3,1}, \ldots, B_{n,1}\}$, as always one inclusion at a time.

Step 3.2.1.1: *Inclusion* $\supseteq$. By Lemma 4.20, the part $B_{1,j_1} \in \langle B_{1,j_1}, B_{n,1}\rangle_p$ is clear. And we have explicitly assumed $(B_{1,j_1}, B_{1,j_2}, B_{n,1}) \in \chi_p$, i.e., $B_{1,j_2} \in \langle B_{1,j_1}, B_{n,1}\rangle_p$.

For every $i \in [\![n]\!]$ with $2 \le i$ Lemma 4.40 (a) (i) assures us that $(B_{1,j_1}, B_{i,1}, B_{n,1}) \in \chi_p$ because B_i is not case $\mathrm{E_{III}}$ and because $B_i \in \langle B_1, B_n\rangle_{E(p,T)}$ per assumption. Hence, also $\{B_{2,1}, B_{3,1}, \ldots, B_{n,1}\} \subseteq \langle B_{1,j_1}, B_{n,1}\rangle_p$.

Step 3.2.1.2: *Inclusion* $\subseteq$. Conversely, let $F_1 \in p$ and $F_1 \notin \{B_{1,j_1}, B_{1,j_2}, B_{2,1}, B_{3,1}, \ldots, B_{n,1}\}$. We must prove $(B_{1,j_1}, F_1, B_{n,1}) \notin \chi_p$. Because $F_1 \nsubseteq T \subseteq B_{1,1} \cup B_{1,2}$ there exist $F \in E(p,T)$ and $F_2 \in p$ such that $\{F_1, F_2\}$ are parents of F with respect to (p,T), such that $F \notin \{B_1, B_2, \ldots, B_n\}$ and such that F is case $\mathrm{E_I}$. As $\langle B_1, B_n\rangle_{E(p,T)} = \{B_1, B_2, \ldots, B_n\}$ it thus follows $(B_1, F, B_n) \notin \chi_{E(p,T)}$. Block F not being case $\mathrm{E_{III}}$ therefore implies $(B_{1,j_1}, F_1, B_{n,1}) \notin \chi_p$ by Lemma 4.40 (a) (i). And that is what we needed to see.

Step 3.2.2: *Their ordering.* Next, we prove $B_{1,j_1} \le B_{1,j_2} \le B_{2,1} \le B_{3,1} \le \ldots \le B_{n,1}$ with respect to $\le_{p,B_{1,j_1},B_{n,1}}$.

As $B_{1,1} \ne B_{1,2}$ it is clear by Lemma 4.20 that $(B_{1,j_1}, B_{1,j_1}, B_{1,j_2}) \in \chi_p$, proving $B_{1,j_1} \le B_{1,j_2}$ with respect to $\le_{p,B_{1,j_1},B_{n,1}}$.

Because B_1 is case $\mathrm{E_{III}}$ Lemma 4.36 guarantees $(B_{1,j_1}, B_{2,1}, B_{1,j_2}) \notin \chi_p$. In other words, $B_{2,1} \nleq B_{1,j_2}$ with respect to $\le_{p,B_{1,j_1},B_{n,1}}$ since $B_{2,1} \ne B_{1,j_2}$. As $\le_{p,B_{1,j_1},B_{n,1}}$ is a total order by Lemma 4.28, that necessitates $B_{1,j_2} \le B_{2,1}$ with respect to $\le_{p,B_{1,j_1},B_{n,1}}$.

Finally, for every $i \in [\![n]\!]$ with $2 \le i$ the assumptions that $B_i \le B_{i+1}$ with respect to $\le_{E(p,T),B_1,B_n}$ and that B_i is not case $\mathrm{E_{III}}$ ensure $B_{i,1} \le B_{i+1,1}$ with respect to $\le_{p,B_{1,j_1},B_{n,1}}$ by Lemma 4.40 (a) (i).

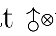
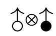
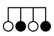
Step 3.2.3: *The colors.* Lastly, it must be verified that $\lambda_p(B_{1,j_1}, B_{1,j_1}, B_{n,1}) = \sigma(\overline{c_1})$, that $\lambda_p(B_{1,j_1}, B_{1,j_2}, B_{n,1}) = \sigma(c_1)$ and that $\lambda_p(B_{1,j_1}, B_{i,1}, B_{n,1}) = \sigma(c_i)$ for every $i \in [\![n]\!]$ with $2 \leq i$.

Because B_1 is case $\mathrm{E_{III}}$ and because $(B_{1,j_1}, B_{1,j_2}, B_{n,1}) \in \chi_p$ per assumption, Lemma 4.40 (b) (ii) proves $\lambda_p(B_{1,j_2}, B_{1,j_2}, B_{n,1}) = -\lambda_p(B_{1,j_1}, B_{1,j_1}, B_{n,1}) = \lambda_{E(p,T)}(B_1, B_1, B_n)$, which proves $\lambda_p(B_{1,j_1}, B_{1,j_1}, B_{n,1}) = \sigma(\overline{c_1})$. But since $\lambda_p(B_{1,j_1}, B_{1,j_2}, B_{n,1}) = \lambda_p(B_{1,j_2}, B_{1,j_2}, B_{n,1})$ by Lemma 4.21 it also shows $\lambda_p(B_{1,j_1}, B_{1,j_2}, B_{n,1}) = \sigma(c_1)$.

And, for every $i \in [\![n]\!]$ with $2 \leq i$, we deduce by Lemma 4.40 (a) (ii) that $\lambda_p(B_{1,j_1}, B_{i,1}, B_{n,1}) = \lambda_{E(p,T)}(B_1, B_i, B_n) = c_i$ because B_i is not case $\mathrm{E_{III}}$. Now the proof is complete for this case.

Case 4: *The last block resulted from merging.* Finally, assume that B_n is case $\mathrm{E_{III}}$. Define $B'_i \coloneqq B_{n-i+1}$ and $B'_{i,j} \coloneqq B_{n-i+1,j}$ for all $i \in [\![n]\!]$ and $j \in [\![2]\!]$. Then, for each $i \in [\![n]\!]$ the blocks $\{B'_{i,1}, B'_{i,2}\}$ of p are parents of the block B'_i of $E(p,T) \in \mathcal{W}_R$ with respect to (p,T). Lemma 4.23 (a) proves that $\langle B'_1, B'_n \rangle_{E(p,T)} = \langle B'_n, B'_1 \rangle_{E(p,T)} = \langle B_1, B_n \rangle_{E(p,T)} = \{B_1, B_2, \ldots, B_n\} = \{B'_1, B'_2, \ldots, B'_n\}$. And Lemma 4.29 ensures $B'_1 \leq B'_2 \leq \ldots \leq B'_n$ with respect to $\leq_{E(p,T),B'_1,B'_n}$. Finally, $\lambda_{E(p,T)}(B'_1, B'_i, B'_n) = \lambda_{E(p,T)}(B_n, B_{n-i+1}, B_1) = -\lambda_{E(p,T)}(B_1, B_{n-i+1}, B_n) = \sigma(\overline{c_{n-i+1}})$ for every $i \in [\![n]\!]$ by Lemma 4.23 (b). As B'_1 is case $\mathrm{E_{III}}$ per assumption, $B'_1, B'_2, \ldots, B'_n$ meet the requirements of Case 3. It follows $(\overline{c_n}, \overline{c_{n-1}}, \ldots, \overline{c_1}) \in R$ by what we have already seen. Because R is a $\mathcal{W}$-parameter set, that concludes the proof overall. □

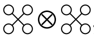
THEOREM 4.42. *$\mathcal{W}_R$ is a hyperoctahedral category of partitions for every $R \in \mathcal{R}$.*

PROOF. That $\mathcal{W}_R$ is a category is the combined implication of Lemmata 4.31, 4.33, 4.35, 4.41 and 2.2. That  $\notin \mathcal{W}_R$ holds by $\mathcal{W}_R \subseteq \mathcal{W}$ because  $\notin \mathcal{W}$. And  $\in \mathcal{W}_R$ is clear by definition. Hence, $\mathcal{W}_R$ is hyperoctahedral. □

5. Tool: Erasing-Minimality

In this auxiliary section we define a set $\mathscr{R} \subseteq \mathcal{P}^{\circ\bullet}$ (see Definition 5.12) and prove $\mathcal{C} = \langle \mathcal{C} \cap \mathscr{R} \rangle$ for every hyperoctahedral category $\mathcal{C} \subseteq \mathcal{P}^{\circ\bullet}$ (see Proposition 5.14). The crucial ingredient in the definition of $\mathscr{R}$ is being *erasing-minimal.*

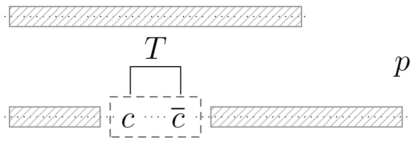
5.1. The Definition of Erasing-Minimality.

Suppose we are trying to understand which category $\langle p \rangle$ a given arbitrary partition $p \in \mathcal{P}^{\circ\bullet}$ generates. A lot of the features of p might be irrelevant to that question. E.g., if $p =$ , then $\langle p \rangle = \langle$$\rangle$. So, what are the crucial parts of p? What of the structure of p is redundant and can be ignored? More formally, we are looking for smaller versions of p which still generate the same category, i.e., numbers $k \in \mathbb{N}$ and partitions $\{p_1, \ldots, p_k\} \subseteq \mathcal{P}^{\circ\bullet}$, each of which has *fewer* points than p, with $\langle p \rangle = \langle p_1, \ldots, p_k \rangle$. An ultimately equivalent question is to ask: Which are the $p \in \mathcal{P}^{\circ\bullet}$ which *cannot* be reduced to smaller versions $\{p_1, \ldots, p_k\} \subseteq \langle p \rangle$ without losing $p \in \langle p_1, \ldots, p_k \rangle$. In Section 5 we will see among other things that being *erasing-minimal* is a necessary condition for such p.

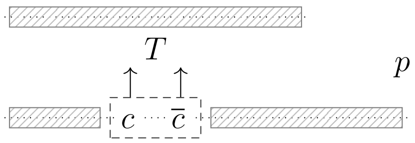
In this section we define the concept of *erasing-minimal* partitions (Definition 5.6), which requires a certain amount of preparations.

REMARK 5.1. There are exactly the following mutually exclusive possibilities for how the blocks of a partition $p \in \mathcal{P}^{\circ\bullet}$ intersecting a turn T in p might be structured:

(a) The turn T itself might be a block of p, necessarily a pair block.

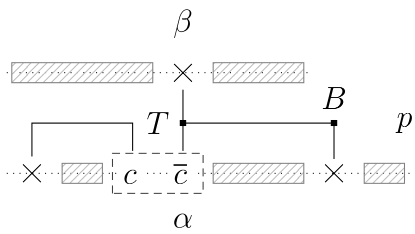


(b) It might be that both of the points of T each form singleton blocks of p.

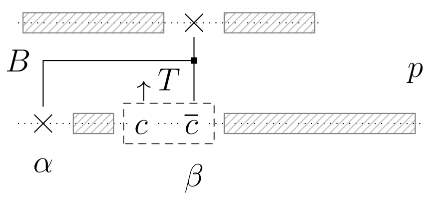


(c) Neither is T a block itself, nor is it a union of blocks of p, and exactly one of the following is true:

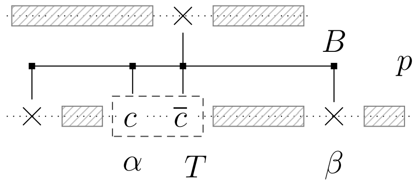
(i) There exist two distinct blocks of p which each supply precisely one point to T and which are not singleton blocks. In this case we can certainly find at least one block B of p and legs $\alpha, \beta \in B$ such that $\alpha \neq \beta$ and $\varnothing \neq T \cap \{\alpha, \beta\} \neq \{\alpha, \beta\}$, i.e., such that either α or β belongs to T, but not both.



(ii) One of the points of T might form a singleton block of p while the other belongs to a non-singleton block. Another way of saying this is that there exists a block B of p (the non-singleton one) and legs $\alpha, \beta \in B$ with $\alpha \neq \beta$ and $\varnothing \neq T \cap \{\alpha, \beta\} \neq \{\alpha, \beta\}$ such that the (necessarily singleton) set $T \backslash \{\alpha, \beta\}$ is a block of p.



(iii) Lastly, it might be that both points of T belong to the same block which is not confined to T, though. Equivalently, we find a block B of p and legs $\alpha, \beta \in B$ with $\alpha \neq \beta$ and $\varnothing \neq T \cap \{\alpha, \beta\} \neq \{\alpha, \beta\}$ such that $T \subseteq B$.



In these last three cases, by choosing the legs α and β appropriately, we can in particular achieve that $]\beta,\alpha[_p \cap B = \varnothing$. Likewise, we can achieve $]\alpha,\beta[_p \cap B = \varnothing$ if we want to.

The following definition allows for a compact definition of erasing-minimality and is a shorthand for the conditions to the later Lemma 5.7.

Definition 5.2. Given $p \in \mathcal{P}^{\circ\bullet}$, an *action* in p is any tuple (T, B, α, β) such that T is a turn in p, such that $B \in p$, such that $\alpha, \beta \in B$ satisfy $\alpha \neq \beta$ and $\varnothing \neq T \cap \{\alpha, \beta\} \neq \{\alpha, \beta\}$ and such that one of the following is true:
(a) $]\beta,\alpha[_p \cap B = \varnothing$.
(b) $T \backslash \{\alpha, \beta\} \in p$.
(c) $T \subseteq B$.
Then, we also say that (T, B, α, β) is an action *for* T in p.

Lemma 5.3. *Given any $p \in \mathcal{P}^{\circ\bullet}$ and any turn T in p, either T is a subpartition of p or there exists an action for T in p.*

Proof. The claim summarizes Remark 5.1 and hence follows from the case distinction made there. □

Definition 5.4. An action (T, B, α, β) in $p \in \mathcal{P}^{\circ\bullet}$ is said to be
(a) …*size-increasing* in p if $|[\alpha,\beta]_p| > \frac{1}{2}\|p\|$.
(b) …*size-preserving* in p if $|[\alpha,\beta]_p| = \frac{1}{2}\|p\|$.
(c) …*size-decreasing* in p if $|[\alpha,\beta]_p| < \frac{1}{2}\|p\|$.

Definition 5.5. A turn T in a $p \in \mathcal{P}^{\circ\bullet}$ is called
(a) …*essential* in p if T is not a subpartition of p and if every action for T in p is size-increasing.
(b) …*redundant* in p if T is a subpartition of p or if there exists a size-decreasing action for T in p.
(c) …*indefinite* in p if T is neither essential nor redundant in p.

In other words, a turn is indefinite if it is not a subpartition and there exists at least one size-preserving action but no size-reducing actions for it.

Definition 5.6. (a) Partitions without redundant turns are *erasing-minimal.*
(b) The set of all erasing-minimal partitions is denoted by $\mathcal{M}$.

5.2. De-Erasing. The following technical lemma is the key that unlocks all other results of this section and shows why erasing-minimality is a useful concept.

Lemma 5.7 (De-Erasing). *Let T be a turn in $p \in \mathcal{P}^{\circ\bullet}$, let $p_1 \coloneqq E(p,T)$ and $p_3 \coloneqq \uparrow^{\circ}\otimes\uparrow^{\bullet}$.*

(a) If T is a pair block of p, then $\langle p \rangle = \langle p_1 \rangle$.

(b) If T is the union of two singleton blocks of p, then $\langle p \rangle = \langle p_1, p_3 \rangle$.

(c) Let $B \in p$ and $\alpha, \beta \in B$ be such that $\alpha \neq \beta$ and $\varnothing \neq T \cap \{\alpha,\beta\} \neq \{\alpha,\beta\}$, and let $p_2 \coloneqq P(p,[\alpha,\beta]_p)$.

(i) If $T\backslash\{\alpha,\beta\}$ is not a block of p and if $]\beta,\alpha[_p \cap B = \varnothing$, then $\langle p \rangle = \langle p_1, p_2 \rangle$.

(ii) If $T\backslash\{\alpha,\beta\}$ is a block of p, then $\langle p \rangle = \langle p_1, p_2, p_3 \rangle$.

(iii) If $T \subseteq B$, then $\langle p \rangle = \langle p_1, p_2 \rangle$.

Especially: If T is a subpartition of p, then $\langle p \rangle = \langle E(p,T), \langle p \rangle \cap \{\uparrow^{\circ}\otimes\uparrow^{\bullet}\}\rangle$; and, if (T,B,α,β) is an action in p, then $\langle p \rangle = \langle E(p,T), P(p,[\alpha,\beta]_p), \langle p \rangle \cap \{\uparrow^{\circ}\otimes\uparrow^{\bullet}\}\rangle$.

Proof. By Lemma 2.3 (b) the addendum follows from Parts (a)–(c).

(a) By Lemma 2.2 it suffices to show $p \in \langle E(p,T)\rangle$ and we can assume $T = \{{}_{\blacksquare}1, {}_{\blacksquare}2\}$. The assumption $T \in p$ then means $p = q \otimes E(p,T)$ for some $q \in \{\sqcap^{\circ\bullet}, \sqcap^{\bullet\circ}\}$. Because $\{\sqcap^{\circ\bullet}, \sqcap^{\bullet\circ}\} \subseteq \langle E(p,T)\rangle$ by definition, this identity proves $p \in \langle E(p,T)\rangle$.

(b) By Lemma 2.3 (b) the presence of singleton blocks in p guarantees $\uparrow^{\circ}\otimes\uparrow^{\bullet} \in \langle p \rangle$. Hence, we only need to show $p \in \langle E(p,T), \uparrow^{\circ}\otimes\uparrow^{\bullet}\rangle$. Once more assume $T = \{{}_{\blacksquare}1, {}_{\blacksquare}2\}$. Supposing T is the union of two singleton blocks then translates to $p = q \otimes E(p,T)$, where, this time, $q \in \{\uparrow^{\circ}\otimes\uparrow^{\bullet}, \uparrow^{\bullet}\otimes\uparrow^{\circ}\}$. It follows $p \in \langle E(p,T), \uparrow^{\circ}\otimes\uparrow^{\bullet}\rangle$ by Lemma 2.3 (a).

(c) The proof of (c) is very similar to that of [MW22b, Lemma 3.1]. In fact, this latter result is implied (a)–(c). We treat cases (i)–(iii) simultaneously. If $T\backslash\{\alpha,\beta\} \notin p$, let $\mathcal{C} \coloneqq \langle E(p,T), P(p,[\alpha,\beta]_p)\rangle$, and let $\mathcal{C} \coloneqq \langle E(p,T), P(p,[\alpha,\beta]_p), \uparrow^{\circ}\otimes\uparrow^{\bullet}\rangle$ otherwise. We suppose (i) or (ii) or (iii) and then have to show $\langle p \rangle = \mathcal{C}$. Again, thanks to Lemma 2.3 (b) and this time also Lemma 2.5, it suffices to show $p \in \mathcal{C}$.

Step 1: *Reducing to the case $\alpha \in T$.* The assumption $\varnothing \neq T \cap \{\alpha,\beta\} \neq \{\alpha,\beta\}$ means that either α or β lies in T, but not both. Suppose we have already shown (c) for the case $\alpha \in T$ and assume now $\beta \in T$ instead. If ϱ is the bijection which reflects the points of p, then, $T' \coloneqq \varrho(T)$ is a turn in the verticolor reflection $p' \coloneqq \tilde{p}$, the points $\alpha' \coloneqq \varrho(\beta)$ and $\beta' \coloneqq \varrho(\alpha)$ in p' with $\alpha' \neq \beta'$ belong to the same block $B' \coloneqq \varrho(B)$ of p' and satisfy $\varrho(\varnothing) = \varnothing \neq \varrho(T \cap \{\alpha,\beta\}) = T' \cap \{\alpha',\beta'\} \neq \varrho(\{\alpha,\beta\}) = \{\alpha',\beta'\}$. So the common requirements of (c) are satisfied for p', T', B', α' and β'.

Now, $\varrho(]\beta,\alpha[_p \cap B) =]\varrho(\alpha),\varrho(\beta)[_{\tilde{p}} \cap \varrho(B) =]\beta',\alpha'[_{p'} \cap B'$ proves that $]\beta,\alpha[_p \cap B = \varnothing$ if and only if $]\beta',\alpha'[_{p'} \cap B' = \varnothing$. The identity $\varrho(T\backslash\{\alpha,\beta\}) = T'\backslash\{\alpha',\beta'\}$ shows that $T\backslash\{\alpha,\beta\} \in p$ if and only if $T'\backslash\{\alpha',\beta'\} \in p'$. And, lastly, $T \subseteq B$ holds if and only if $T' = \varrho(T) \subseteq \varrho(B) = B'$. Because our assumption $\beta \in T$ means $\alpha' = \varrho(\beta) \in \varrho(T) = T'$, the half of (c) which we assume verified thus shows that $\langle p' \rangle = \langle E(p',T'), P(p',[\alpha',\beta']_{p'})\rangle$ if $]\beta,\alpha[_p \cap B = \varnothing$ and $T\backslash\{\alpha,\beta\} \notin p$, that $\langle p' \rangle = \langle E(p',T'), P(p',[\alpha',\beta']_{p'}), \uparrow^{\circ}\otimes\uparrow^{\bullet}\rangle$ if $T\backslash\{\alpha,\beta\} \in p$, and that $\langle p' \rangle = \langle E(p',T'), P(p',[\alpha',\beta']_{p'})\rangle$ if $T \subseteq B$.

Because $E(p',T') = E(\tilde{p},\varrho(T))$ coincides with the verticolor reflection of $E(p,T)$ and $P(p',[\alpha',\beta']_{p'}) = P(\tilde{p},\varrho([\alpha,\beta]_p))$ with that of $P(p,[\alpha,\beta]_p)$ this then proves (i)–(iii) also for the case $\beta \in T$. Hence, we can always assume $\alpha \in T$ in the following.

Step 2: *Defining sets B_1 and B_2 and partitions q, p' and p''.* By Lemma 2.2 we can assume that p has no upper points and that $T = \{{}_{\blacksquare}1, {}_{\blacksquare}2\}$. Now, $\alpha = {}_{\blacksquare}1$ or $\alpha = {}_{\blacksquare}2$ and $\beta = {}_{\blacksquare}j$ for some $j \in \mathbb{N}$ with $j \geq 3$. Let B_1 and B_2 be the blocks of ${}_{\blacksquare}1$ and ${}_{\blacksquare}2$, respectively, in p. Then, $B = B_1$ or $B = B_2$. (Note that possibly $B_1 = B_2$.)

Define $p' \coloneqq \sqcap_{\circ\bullet} \otimes E(p,T)$ if ${}_{\blacksquare}1$ is $\circ$-colored and $p' \coloneqq \sqcap_{\bullet\circ} \otimes E(p,T)$ otherwise. By construction, the lower rows of p and p' agree in length and coloring. Because $\alpha \in \{{}_{\blacksquare}1, {}_{\blacksquare}2\} \not\ni \beta$ there exist unique partitions $u_1 \in \{\varnothing, \wr^{\circ}_{\circ}, \wr^{\bullet}_{\bullet}, |^{\circ}_{\circ}, |^{\bullet}_{\bullet}\}$ and u_2, either $\varnothing$ or a tensor product of suitable partitions from $\{|^{\circ}_{\circ}, |^{\bullet}_{\bullet}\}$, such that, writing $q \coloneqq u_1 \otimes P(p, [\alpha,\beta]_p) \otimes u_2$, the pairing (q, p') is composable and such that $u_1 \in \{\wr^{\circ}_{\circ}, \wr^{\bullet}_{\bullet}\}$ if and only if $B_1 = \{{}_{\blacksquare}1\}$. Finally, let $p'' \coloneqq qp'$.

Step 3: *Recognizing $p'' \in \mathcal{C}$.* Because $p'' = qp'$ it suffices to prove $q \in \mathcal{C}$ and $p' \in \mathcal{C}$ in order to show $p'' \in \mathcal{C}$. Since $E(p,T) \in \mathcal{C}$ by definition of $\mathcal{C}$ and since, naturally, $\{\sqcap_{\circ\bullet}, \sqcap_{\bullet\circ}\} \subseteq \mathcal{C}$, the partition p' is an element of $\mathcal{C}$. To see $q \in \mathcal{C}$ we show that each of the three factors u_1, $P(p, [\alpha,\beta]_p)$ and u_2 in q is element of $\mathcal{C}$. Already by definition, $P(p, [\alpha,\beta]_p) \in \mathcal{C}$. The partition u_2 is included in $\mathcal{C}$ simply because $\mathcal{C}$ is a category. It remains to treat u_1. If $B_1 \neq \{{}_{\blacksquare}1\}$, then u_1 is $\varnothing$ or an identity and thus $u_1 \in \mathcal{C}$ is once again clear. If indeed $B_1 = \{{}_{\blacksquare}1\}$, then $\{\alpha,\beta\} \subseteq B$ and $\alpha \neq \beta$ force $B \neq \{{}_{\blacksquare}1\}$ and thus $\alpha \neq {}_{\blacksquare}1$, which is to say $\alpha = {}_{\blacksquare}2$. Hence, under this assumption $\{{}_{\blacksquare}1\} = T\backslash\{\alpha,\beta\} \in p$, implying $\uparrow_{\circ} \otimes \uparrow_{\bullet} \in \mathcal{C}$ by definition of $\mathcal{C}$. As $u_1 \in \langle \uparrow_{\circ} \otimes \uparrow_{\bullet} \rangle$ by Lemma 2.3 (a), we have thus shown $u_1 \in \mathcal{C}$ in any case. In conclusion, $q \in \mathcal{C}$ and thus $p'' \in \mathcal{C}$.

Step 4: *Defining q' and proving $p'' = p$.* In regard of $p'' \in \mathcal{C}$, we only need to verify $p'' = p$ in order to prove Part (c). Let P_p be the set of all points of p, i.e., the shared lower row of p, p', q and p''. Let $q' \coloneqq \{B \cap P_p \mid B \in q\}\backslash\{\varnothing\}$ be the partition q induces on P_p. Because q is projective, q' is also the partition q induces on its upper row if the latter is identified with P_p. Finally, let $N \coloneqq \bigcup\{B \mid B \in q,\, B \subseteq P_p\}$ be the set of all points of P_p which belong to non-through blocks in q. By definition of the composition operation, because p' has no upper points and because q is projective, N and $P_p\backslash N$ are subpartitions of $p'' = qp'$ and, in terms of blocks,

$$p'' = \{B' \mid B' \in q',\, B' \subseteq N\} \uplus \{B'\backslash N \mid B' \in p' \vee q'\}. \tag{7}$$

We show that every block of p is also one of p''. That requires a case distinction and is most efficiently addressed by first proving two auxiliary statement.

Step 4.1: *Determining the blocks of $p' \vee q'$.* It is convenient to see at this point in two steps that, in terms of blocks,

$$p' \vee q' \overset{!}{=} \{B_1 \cup B_2\} \uplus p\backslash\{B_1, B_2\}. \tag{8}$$

Step 4.1.1: *$B_1 \cup B_2$ is a block of $p' \vee q'$.* By definition of the erasing operation and by construction of p', in terms of blocks,

$$p' = \{T, (B_1 \cup B_2)\backslash T\} \uplus p\backslash\{B_1, B_2\}.$$

Since both $T \in p'$ and $(B_1 \cup B_2)\backslash T \in p'$, in order to see that $B_1 \cup B_2 = T \uplus ((B_1 \cup B_2)\backslash T)$ is a block of $p' \vee q'$ it suffices to show that $B_1 \cup B_2$ is a subpartition of q' and that q' contains a block intersecting both T and $(B_1 \cup B_2)\backslash T$.

By definition of the projection operation and of u_1 and u_2, in terms of blocks,

$$q' = \{[\alpha,\beta]_p \cap B_0 \mid B_0 \in p,\ [\alpha,\beta]_p \cap B_0 \neq \varnothing\} \\ \cup \{\{\gamma\} \mid \gamma \in P_p \backslash [{}_\blacksquare 1,\beta]_p\} \cup \begin{cases} \varnothing & \text{if } \alpha = {}_\blacksquare 1, \\ \{\{{}_\blacksquare 1\}\} & \text{if } \alpha = {}_\blacksquare 2 \end{cases} \tag{9}$$

Because $\alpha \in T = \{{}_\blacksquare 1, {}_\blacksquare 2\}$ we can write

$$B_1 \cup B_2 = ((B_1 \cap B_2) \cap \{{}_\blacksquare 1\}) \cup ((B_1 \cap B_2) \cap [\alpha,\beta]_p) \cup \bigcup_{\gamma \in (B_1 \cup B_2) \backslash [{}_\blacksquare 1,\beta]_p} \{\gamma\}.$$

While the sets on the right hand side need not be non-empty nor pairwise distinct, the non-empty ones are all blocks of q'. Thus, $B_1 \cup B_2$ is indeed a subpartition of q'.

And from our assumptions $T \cap \{\alpha,\beta\} \neq \{\alpha,\beta\}$ and $\alpha \in T$ it follows $\beta \in B \backslash T \subseteq (B_1 \cup B_2) \backslash T$. Thus the block $B \cap [\alpha,\beta]_p$ of q' intersects both T (namely in α) and $(B_1 \cup B_2) \backslash T$ (namely in β). In conclusion, $B_1 \cup B_2 \in p' \vee q'$.

Step 4.1.2: *Blocks of p other than B_1 and B_2 are blocks of $p' \vee q'$.* Given $B_0 \in p$ with $B_0 \notin \{B_1, B_2\}$ we already know $B_0 \in p'$, which is why, in order to see $B_0 \in p' \vee q'$, we only need to see that B_0 is a subpartition of q'. But that is clear, since $\alpha \in T = \{{}_\blacksquare 1, {}_\blacksquare 2\}$ once more allows for the decomposition

$$B_0 = (B_0 \cap \{{}_\blacksquare 1\}) \cup (B_0 \cap [\alpha,\beta]_p) \cup \bigcup_{\gamma \in B_0 \backslash [{}_\blacksquare 1,\beta]_p} \{\gamma\}.$$

With no guarantees whatsoever that all the appearing sets in the decomposition are non-empty or pairwise distinct, they are still all blocks of q', though. And that is all we needed to see.

Step 4.2: *Recognizing the non-through blocks of q.* Let us also recognize that the definition of the projection operation, the definitions of u_1 and u_2 and the fact that p has no upper points imply that the set of non-through blocks of q is given by

$$\{B \mid B \in q,\ B \subseteq P_p\} = \{B_0 \mid B_0 \in p,\ B_0 \subseteq [\alpha,\beta]_p\} \cup \begin{cases} \{{}_\blacksquare 1\} & \text{if } B_1 = \{{}_\blacksquare 1\} \\ \varnothing & \text{otherwise.} \end{cases} \tag{10}$$

Note in particular that $N \subseteq [{}_\blacksquare 1,\beta]_p$ because $\alpha \in T = \{{}_\blacksquare 1, {}_\blacksquare 2\}$.

Step 4.3: *p'' and p have the same blocks.* Let $B_0 \in p$ be arbitrary. We must now make the announced case distinctions and prove $B_0 \in p''$.

Case 4.3.1: If $B_0 \subseteq [\alpha,\beta]_p$, then $B_0 \in q'$ by (9) and $B_0 \subseteq N$ by (10), implying $B_0 \in p''$ by (7).

Case 4.3.2: Next, assume $B_0 \nsubseteq [\alpha,\beta]_p$ and $B_0 \notin \{B_1, B_2\}$. Then, $B_0 \in p' \vee q'$ by (8). Moreover, $B_0 \cap T = \varnothing$ since $T = \{{}_\blacksquare 1, {}_\blacksquare 2\} \subseteq B_1 \cup B_2$. Thus, in particular ${}_\blacksquare 1 \notin B_0$. Consequently, $B_0 \backslash N = B_0$ by (10), no matter whether $B_1 = \{{}_\blacksquare 1\}$ or not. It follows $B_0 \in p''$ by (7).

Case 4.3.3: Let $B_0 \in \{B_1, B_2\}$ and $B_0 \cap [\alpha,\beta]_p = \varnothing$. That only leaves the option that $B_1 \neq B_2 = B$ and $\alpha = {}_\blacksquare 2$ and $B_0 = B_1$: Assuming $B_1 = B_2$ would require $\alpha \in B_0$; if $\alpha = {}_\blacksquare 1$ were true, this would necessitate $(B_1 \cup B_2) \cap [\alpha,\beta]_p \neq \varnothing$; and supposing

$\alpha = {}_{\blacksquare}2$ and $B_0 = B_2$ would imply $\alpha \in B_0$. In particular, $B_1 \neq B_2$ excludes condition (iii). Hence, (i) or (ii) must be true. There are now two subcases.

Case 4.3.3.1: First, suppose $T\backslash\{\alpha,\beta\} \in p$. Then, $\alpha = {}_{\blacksquare}2$ implies $B_0 = B_1 = \{{}_{\blacksquare}1\} = T\backslash\{\alpha,\beta\} \in q'$ by (9). Moreover, $B_1 = \{{}_{\blacksquare}1\}$ yields $B_0 = \{{}_{\blacksquare}1\} \subseteq N$ by (10). Hence, in this case $B_0 \in p''$ by (7).

Case 4.3.3.2: Alternatively, let $T\backslash\{\alpha,\beta\} \notin p$. It follows $B_0 = B_1 \neq \{{}_{\blacksquare}1\} = T\backslash\{\alpha,\beta\}$. Since we have assumed $B_0 \cap [\alpha,\beta]_p = \varnothing$, this means $B_0 \subseteq P_p\backslash N$ by (10). Moreover, the assumption $T\backslash\{\alpha,\beta\} \in p$ denies (ii). As we have already excluded (iii), this forces condition (i) to be true. It follows $]\beta,\alpha[_p \cap B_2 =]\beta,\alpha[_p \cap B = \varnothing$ and thus in particular $B_2 \subseteq [\alpha,\beta]_p$. By (10) this last statement ensures $B_2 \subseteq N$. Since $B_1 \cup B_2 \in p' \vee q'$ by (8) and since $B_0 \subseteq P_p\backslash N$ and $B_2 \subseteq N$, it follows $B_0 = B_1 = (B_1 \cup B_2)\backslash B_2 = (B_1 \cup B_2)\backslash N \in p''$ by (7).

Case 4.3.4: Lastly, let $B_0 \in \{B_1, B_2\}$ and $\varnothing \neq B_0 \cap [\alpha,\beta]_p \neq B_0$. This excludes the possibility that B_0 is a singleton block because $B_0 = \{{}_{\blacksquare}1\}$ would force $\alpha = {}_{\blacksquare}2$ and thus $B_0 \cap [\alpha,\beta]_p = \varnothing$, and because $B_0 = \{{}_{\blacksquare}2\}$ would require $\alpha = {}_{\blacksquare}1$ and thus $B_0 \cap [\alpha,\beta]_p = B_0$. Hence, (10) shows $B_0 \subseteq P_p\backslash N$ because $B_0 \neq \{{}_{\blacksquare}1\}$ and $B_0 \nsubseteq [\alpha,\beta]_p$. We have to distinguish three subcases.

Case 4.3.4.1: If $B_1 = B_2$, then, $B_0 = B_1 \cup B_2 \in p' \vee q'$ by (8), implying $B_0 = B_0\backslash N \in p''$ by (7).

Case 4.3.4.2: Next, let $B_1 \neq B_2$ and $T\backslash\{\alpha,\beta\} \in p$. Because B_0 is not a singleton block, that implies $B_0 = B \neq T\backslash\{\alpha,\beta\}$. Moreover, (10) shows $T\backslash\{\alpha,\beta\} \subseteq N$, either because $\alpha = {}_{\blacksquare}1$ and thus $B_2 = T\backslash\{\alpha,\beta\} = \{{}_{\blacksquare}2\} \subseteq [\alpha,\beta]_p$ or because $\alpha = {}_{\blacksquare}2$ and thus $B_1 = T\backslash\{\alpha,\beta\} = \{{}_{\blacksquare}1\}$. As $B_1 \cup B_2 \in p' \vee q'$ by (8) and because $B_0 = B_0\backslash N$, it follows $B_0 = B = (B_1 \cup B_2)\backslash(T\backslash\{\alpha,\beta\}) = (B_1 \cup B_2)\backslash N \in p''$ by (7).

Case 4.3.4.3: Finally, let $B_1 \neq B_2$ and $T\backslash\{\alpha,\beta\} \notin p$. Because this excludes (ii) and because $B_1 \neq B_2$ keeps (iii) from holding, (i) must be satisfied. This means $]\beta,\alpha[_p \cap B = \varnothing$ and thus in particular $B \subseteq [\alpha,\beta]_p$. Hence, $B \subseteq N$ by (10). Furthermore, because we have assumed $B_0 \cap [\alpha,\beta]_p \neq B_0$, we can infer $B_0 \neq B$. As $B_1 \cup B_2 \in p' \vee q'$ by (8), it follows $B_0 = (B_1 \cup B_2)\backslash B = (B_1 \cup B_2)\backslash N \in p''$, concluding the proof. □

One derives immediately from Lemma 5.7 that any partition which cannot be reduced to smaller versions of itself is necessarily erasing-minimal. While the converse is not true, every category is generated by its erasing-minimal partitions as the next result shows.

Proposition 5.8. *Let $\mathcal{C} \subseteq \mathcal{P}^{\circ\bullet}$ be an arbitrary category. For every $n \in \mathbb{N}_0$,*

$$\{p \,|\, p \in \mathcal{C}\backslash\mathscr{M},\ \|p\| \leq n+1\} \subseteq \langle \{p \,|\, p \in \mathcal{C} \cap \mathscr{M},\ \|p\| \leq n\} \cup (\mathcal{C} \cap \{\uparrow\circ \otimes \uparrow\bullet\}) \rangle.$$

In particular, $\mathcal{C} = \langle \mathcal{C} \cap (\mathscr{M} \cup \{\uparrow\circ \otimes \uparrow\bullet\}) \rangle$.

Proof. Let $\mathcal{C}'_n := \langle \{p \,|\, p \in \mathcal{C} \cap \mathscr{M},\ \|p\| \leq n\} \cup (\mathcal{C} \cap \{\uparrow\circ \otimes \uparrow\bullet\}) \rangle$ for every $n \in \mathbb{N}_0$. All partitions of sizes 0 or 1 are trivially erasing-minimal. Hence, $\mathcal{C}'_0$ contains all partitions of $\mathcal{C}$ with those sizes. Let $n \in \mathbb{N}$ be such that all partitions of $\mathcal{C}\backslash\mathscr{M}$ of size

n or less are elements of $\mathcal{C}'_{n-1}$ and let $p \in \mathcal{C}\backslash\mathcal{M}$ have $n+1$ many points. We prove $p \in \mathcal{C}'_n$.

Note that $\mathcal{C}'_{n-1} \subseteq \mathcal{C}'_n$ by definition. Hence, the induction hypothesis means that $\mathcal{C}'_n$ in fact contains *all* partitions of $\mathcal{C}$ with at most n many points.

By assumption, we find a turn T in p which is either a subpartition of p or for which there exists an action (T, B, α, β) in p with $|[\alpha, \beta]_p| < \frac{n+1}{2}$. By Lemma 5.7 we infer $p \in \langle E(p,T), \langle p\rangle \cap \{\uparrow_{\circ}\otimes\uparrow_{\bullet}\}\rangle$ in the former and $p \in \langle E(p,T), P(p,[\alpha,\beta]_p), \langle p\rangle \cap \{\uparrow_{\circ}\otimes\uparrow_{\bullet}\}\rangle$ in the latter case.

Naturally, $p \in \mathcal{C}$ implies $\langle p\rangle \cap \{\uparrow_{\circ}\otimes\uparrow_{\bullet}\} \subseteq \mathcal{C} \cap \{\uparrow_{\circ}\otimes\uparrow_{\bullet}\} \subseteq \mathcal{C}'_n$. Moreover, the partition $E(p,T) \in \mathcal{C}$ has $\|E(p,T)\| = \|p\| - |T| = (n+1) - 2 = n-1$ points and is thus an element of $\mathcal{C}'_n$. Lastly, by definition of the projection operation, $P(p,[\alpha,\beta]_p)$ has size $2 \cdot |[\alpha,\beta]_p|$. Per assumption, it thus has fewer than $2 \cdot \frac{n+1}{2} = n+1$ many points, i.e., at most n many. The induction hypothesis therefore ensures $P(p,[\alpha,\beta]_p) \in \mathcal{C}'_n$. In conclusion, $p \in \mathcal{C}'_n$. That completes the induction and thus the proof. □

The next result reveals helpful general properties of erasing-minimal partitions.

LEMMA 5.9. *Let (T, B, α, β) be an action in $p \in \mathcal{M}$.*

(a) If $T\backslash\{\alpha,\beta\} \in p$, then the following are true:
- *(i) If $\|p\|$ is odd, then $2 \leq |B| \leq 3$.*
- *(ii) If $\|p\|$ is even, then $2 \leq |B| \leq 4$.*
- *(iii) If $\|p\|$ is odd or if ($\|p\|$ is even and $|B| \neq 3$), then $B\backslash T$ is consecutive.*

(b) If $T \subseteq B$, then the following are true:
- *(i) If $|B| = 3$, then $\uparrow_{\circ}\otimes\uparrow_{\bullet} \in \langle p\rangle$.*
- *(ii) If $\|p\|$ is odd, then $|B| = 3$.*
- *(iii) If $\|p\|$ is even, then $3 \leq |B| \leq 4$.*
- *(iv) $B\backslash T$ is consecutive.*

PROOF. Let P_p denote the set of all points of p, let $n \coloneqq |P_p|$ and let $\gamma_0 \in T \cap \{\alpha,\beta\}$.

(a) If $T\backslash\{\alpha,\beta\} \in p$, then for every $\gamma \in B\backslash\{\gamma_0\}$ both (T, B, γ_0, γ) and (T, B, γ, γ_0) are actions for T. Since $p \in \mathcal{M}$ they are both either size-preserving or size-increasing for every such γ, meaning $|[\gamma,\gamma_0]_p| \geq \frac{n}{2}$ and $|[\gamma_0,\gamma]_p| \geq \frac{n}{2}$. It follows

$$\tfrac{n}{2} \leq |[\gamma_0,\gamma]_p| \leq \tfrac{n}{2} + 2 \tag{11}$$

because $|[\gamma_0,\gamma]_p| = |P_p \backslash]\gamma,\gamma_0[_p| = |P_p| - |]\gamma,\gamma_0[_p| = |P_p| - |[\gamma,\gamma_0]_p| + |\{\gamma,\gamma_0\}| = n + 2 - |[\gamma,\gamma_0]_p| \leq n + 2 - \frac{n}{2} = \frac{n}{2} + 2$. Moreover, recognize that each point $\theta \in P_p\backslash\{\gamma_0\}$ is uniquely determined by $|[\gamma_0,\theta]_p|$ because $P_p\backslash\{\gamma_0\}$ is totally ordered and because $|[\gamma_0,\theta]_p|$ gives the rank of θ in this total order.

Case 1: $\|p\|$ *is odd.* First, let there be $m \in \mathbb{N}$ such that $n = 2m+1$. From (11) it then follows $m + 1 = \lceil\frac{n}{2}\rceil \leq |[\gamma_0,\gamma]_p| \leq \lfloor\frac{n}{2}\rfloor + 2 = m+2$ for all $\gamma \in B\backslash\{\gamma_0\}$. Hence, $|[\gamma_0,\gamma]_p| \in \{m+1, m+2\}$ for any $\gamma \in B\backslash\{\gamma_0\}$. We infer $|B| \leq 3$. Moreover, if $|B| = 3$, then all $B\backslash T$ is consecutive. Of course, $B\backslash T$ is trivially consecutive if $|B| = 2$.

Case 2: $\|p\|$ *is even.* Now, assume instead that we find $m \in \mathbb{N}$ such that $n = 2m$. Then, $m = \frac{n}{2} \leq |[\gamma_0,\gamma]_p| \leq \frac{n}{2} + 2 = m+2$ for any $\gamma \in B_0\backslash\{\gamma_0\}$ by (11). Hence,

$|[\gamma_0,\gamma]_p| \in \{m, m+1, m+2\}$ for such γ. It follows that $|B| \le 4$ and that, if $|B| = 4$, then $B\backslash T$ is consecutive. The same is trivially true if $|B| = 2$.

(b) Now suppose $T \subseteq B$. If $|B| = 3$, then, as T is a turn, $E(p,T) \in \langle p \rangle$ has a singleton block, implying $\uparrow_{\circ} \otimes \uparrow_{\bullet} \in \langle p \rangle$ by Lemma 2.3 (b).

If we let $\gamma_1 \in (T \cap B)\backslash\{\gamma_0\}$, then γ_0 and γ_1 are neighbors. Moreover, for every $\gamma \in B\backslash T$ and every $i \in \{0,1\}$ both (T,B,γ,γ_i) and (T,B,γ_i,γ) are actions for T in p. Because $p \in \mathscr{M}$ we can thus infer $|[\gamma,\gamma_i]_p| \ge \frac{n}{2}$ and $|[\gamma_i,\gamma]_p| \ge \frac{n}{2}$ for all such i and γ. In the same way as in the proof of (a) it hence follows $\frac{n}{2} \le |[\gamma_i,\gamma]_p| \le \frac{n}{2}+2$ for all $i \in \{0,1\}$ and $\gamma \in B\backslash T$.

Again, every $\theta \in P_p\backslash T$ is uniquely determined by $|[\gamma_i,\theta]_p|$ for already any *one* $i \in \{0,1\}$. Here, of course, we have for $\gamma \in B\backslash T$ bounds for $|[\gamma_i,\theta]_p|$ for *both* $i \in \{0,1\}$. Let $i_1, i_2 \in \{0,1\}$ be such that $\{i_1,i_2\} = \{0,1\}$ and such that γ_{i_2} is the successor of γ_{i_1} in p. Then, for all $\gamma \in B\backslash T$, because $(\gamma_{i_1},\gamma_{i_2},\gamma)$ is ordered in p,

$$|[\gamma_{i_1},\gamma]_p| = |[\gamma_{i_2},\gamma]_p| + 1 \tag{12}$$

because $|[\gamma_{i_1},\gamma]_p| = |[\gamma_{i_1},\gamma_{i_2}[_p \cup [\gamma_{i_2},\gamma]_p| = |[\gamma_{i_1},\gamma_{i_2}[_p| + |[\gamma_{i_2},\gamma]_p| = |\{\gamma_{i_1}\}| + |[\gamma_{i_2},\gamma]_p| = |[\gamma_{i_2},\gamma]_p| + 1$.

Case 1: $\|p\|$ *is odd.* First, let $n = 2m+1$ for $m \in \mathbb{N}$. Then, we know $|[\gamma_{i_j},\gamma]_p| \in \{m+1, m+2\}$ for all $j \in \{1,2\}$ and $\gamma \in B\backslash T$ from the proof of (a). By (12) the only way to satisfy these conditions is that $|[\gamma_{i_1},\gamma]_p| = m+2$ and $|[\gamma_{i_2},\gamma]_p| = m+1$. That means $B\backslash T$ is a singleton set and thus in particular consecutive.

Case 2: $\|p\|$ *is even.* If $n = 2m$ for some $m \in \mathbb{N}$ instead, then $|[\gamma_{i_j},\gamma]_p| \in \{m, m+1, m+2\}$ for all $j \in \{1,2\}$ and $\gamma \in B\backslash T$ as seen in the proof of (a). Considering (12), we can conclude from this that $(|[\gamma_{i_1},\gamma]_p|, |[\gamma_{i_2},\gamma]_p|) \in \{(m+2,m+1),(m+1,m)\}$. Hence, $|B\backslash T| \le 2$ and $B\backslash T$ is consecutive (trivially so if $|B| = 1$). □

5.3. Improving Erasing-Minimality with Symmetry. By definition, $p \in \mathcal{P}^{\circ\bullet}$ is erasing-minimal if it has no redundant turns. But, if p is erasing-minimal, p is still allowed to have essential and indefinite ones. The essential turns cannot be helped. But many indefinite turns are actually "unimportant" to the generated category $\langle p \rangle$ in the following sense: While we are not able to reduce p to a smaller version by performing an action on an indefinite turn – we can find an equally-sized version of p which has special symmetry properties and thus contains "less information". That is the idea behind Proposition 5.14.

Being projective or *bi-projective*, which we now define, are such nice symmetry conditions. If $p \in \mathcal{P}^{\circ\bullet}$ is projective, then $\|p\| \in 2\mathbb{N}_0$. Consequently, if p is additionally verticolor-reflexive, then even $\|p\| \in 4\mathbb{N}_0$.

Definition 5.10. (a) We call $p \in \mathcal{P}^{\circ\bullet}$ *bi-projective* if p is projective and verticolor-reflexive and if every block crossing the vertical middle axis of p is symmetric with respect to this axis.

(b) If so, then $p^{\dagger} \coloneqq p^{\circlearrowleft \frac{1}{4}\|p\|} = p^{\circlearrowright \frac{1}{4}\|p\|}$ is called the *dual* of p.

The next lemma motivates the upcoming definition of the improved erasing-minimal partitions. If $p \in \mathcal{P}^{\circ\bullet}$ with $n \coloneqq \|p\|$ is projective, then $\{{}^{\blacksquare}1, {}_{\blacksquare}1\}$ and $\{{}_{\blacksquare}\frac{n}{2}, {}^{\blacksquare}\frac{n}{2}\}$

are turns in p. If p is even bi-projective, then the same is true of $\{{}_{\blacksquare}\frac{n}{4}, {}_{\blacksquare}(\frac{n}{4}+1)\}$ and $\{{}^{\blacksquare}(\frac{n}{4}+1), {}^{\blacksquare}\frac{n}{4}\}$.

LEMMA 5.11. *If $p \in \mathscr{M}\backslash\{\varnothing\}$ is projective, then, $4 \leq \|p\|$ and both $\{{}^{\blacksquare}1, {}_{\blacksquare}1\}$ and $\{{}_{\blacksquare}\frac{1}{2}\|p\|, {}^{\blacksquare}\frac{1}{2}\|p\|\}$ are indefinite turns in p. Moreover, p is a bracket or a co-bracket.*

PROOF. **Step 1:** *Size bound.* If $n \coloneqq \|p\|$, then $0 < n$ since $p \neq \varnothing$. Because p is projective, $n \in 2\mathbb{N}_0$. If $n = 2$ were true, then $p \in \{\substack{\circ\\\circ}, \substack{\bullet\\\bullet}, \substack{\circ\\|\\\circ}, \substack{\bullet\\|\\\bullet}\}$ would contain a turn which is a subpartition, contradicting $p \in \mathscr{M}$. Hence, $4 \leq n$.

Step 2: $\{{}^{\blacksquare}1, {}_{\blacksquare}1\}$ *is indefinite.* Since p is projective, $T \coloneqq \{{}^{\blacksquare}1, {}_{\blacksquare}1\}$ is a turn in p. We prove that T is indefinite and that p is a bracket or co-bracket. Let $B \in p$ satisfy $\alpha \coloneqq {}_{\blacksquare}1 \in B$.

Step 2.1: *B is not confined to T.* We show $B \notin T$ by contradiction. Assuming the opposite would imply $B \in \{T, \{{}_{\blacksquare}1\}\}$. If $B = T$ were true, then T would be both a subpartition and a turn in p, contradicting $p \in \mathscr{M}$. Similarly, if $B = \{{}_{\blacksquare}1\}$ were the case, then, as p is projective, $\{{}^{\blacksquare}1\}$ would be a block of p as well, again making T a subpartition. Hence, $B \notin T$.

Step 2.2: *Definition of i and β.* In conclusion, as p is projective, there is $i \in [\![\frac{n}{2}]\!]$ with $1 < i$ and $\beta \coloneqq {}_{\blacksquare}i \in B$. We thus have a turn T in p, a block $B \in p$ and legs $\{\alpha, \beta\} \subseteq B$ with $\alpha \neq \beta$ and $\varnothing \neq \{{}_{\blacksquare}1\} \subseteq T \cap \{\alpha, \beta\} \neq \{\alpha, \beta\}$.

Step 2.3: *Case distinctions.* We now distinguish whether B is a through block or a (lower) non-through block.

Case 2.3.1: *B is non-through.* If, first, $B \subseteq [{}_{\blacksquare}1, {}_{\blacksquare}\frac{n}{2}]_p$, then we can choose i to be maximal in $[\![\frac{n}{2}]\!]$ with $1 < i$ and ${}_{\blacksquare}i \in B$. This choice of i (and thus β) then ensures $B \subseteq [{}_{\blacksquare}1, {}_{\blacksquare}i]_p = [\alpha, \beta]_p$, which is to say $]\beta, \alpha[_p \cap B = \varnothing$. In other words, (T, B, α, β) is an action in p. As $p \in \mathscr{M}$, that requires $|[{}_{\blacksquare}1, {}_{\blacksquare}i]_p| = |[\alpha, \beta]_p| \geq \frac{n}{2}$. Naturally, this is only possible if $i = \frac{n}{2}$. But then, actually, $|[{}_{\blacksquare}1, {}_{\blacksquare}i]_p| = \frac{n}{2}$, making (T, B, α, β) a size-preserving action and thus $\{{}^{\blacksquare}1, {}_{\blacksquare}1\}$ an indefinite turn of p. And $B \subseteq [{}_{\blacksquare}1, {}_{\blacksquare}\frac{n}{2}]_p$ and $\{{}_{\blacksquare}1, {}_{\blacksquare}\frac{n}{2}\} \subseteq B$ means that p is a bracket.

Case 2.3.2: *B is through.* Now, let $B \nsubseteq [{}_{\blacksquare}1, {}_{\blacksquare}\frac{n}{2}]_p$ instead. We do not need to make any particular choice for i (and β) in this case. Rather, i can be arbitrary in $[\![\frac{n}{2}]\!]$ with $1 < i$ and ${}_{\blacksquare}i \in B$. Because p is projective the assumption of B being a through block entails $T \subseteq B$. Thus, (T, B, α, β) is an action in p. It follows $|[{}_{\blacksquare}1, {}_{\blacksquare}i]_p| = |[\alpha, \beta]_p| \geq \frac{n}{2}$ from $p \in \mathscr{M}$. That implies $i = \frac{n}{2}$ and, in particular, $|[\alpha, \beta]_p| = \frac{n}{2}$, which makes (T, B, α, β) a size-preserving action and thus T an indefinite turn in p. Moreover, as i was arbitrary, we can actually deduce $B \cap [{}_{\blacksquare}1, {}_{\blacksquare}\frac{n}{2}]_p = \{{}_{\blacksquare}1, {}_{\blacksquare}\frac{n}{2}\}$. That is precisely what it means for p to be a co-bracket.

Step 3: $\{{}_{\blacksquare}\frac{n}{2}, {}_{\blacksquare}\frac{n}{2}\}$ *is indefinite.* We have shown the claim about $T = \{{}^{\blacksquare}1, {}_{\blacksquare}1\}$. To see that also $\{{}_{\blacksquare}\frac{n}{2}, {}^{\blacksquare}\frac{n}{2}\}$ is an indefinite turn, apply the already shown result to $\tilde{p}$ and note that $\{{}_{\blacksquare}\frac{n}{2}, {}^{\blacksquare}\frac{n}{2}\}$ is an indefinite turn in p if and only if $\{{}^{\blacksquare}1, {}_{\blacksquare}1\}$ is one in $\tilde{p}$. $\square$

We now define four classes of turn-minimal partitions with special symmetry properties. In one case we make the non-existence of indefinite turns other than the ones shown to exist in Lemma 5.11 a criterion.

Definition 5.12. Let $p \in \mathscr{M}$ be arbitrary.

(a) We say $p \in \mathscr{R}_1$ if p has no turns and no upper points.

(b) We say $p \in \mathscr{R}_2$ if p has turns but none which are indefinite and if p has no upper points.

(c) We say $p \in \mathscr{R}_3$ if p is projective and if $T \cap \{{}^{\blacksquare}1, {}_{\blacksquare}1, {}_{\blacksquare}\frac{1}{2}\|p\|, {}^{\blacksquare}\frac{n}{2}\|p\|\} \neq \varnothing$ for every indefinite turn T of p.

(d) We say $p \in \mathscr{R}_4$ if p is bi-projective.

(e) Write $\mathscr{R} := \bigcup_{i=1}^{4} \mathscr{R}_i$.

The following result is crucial to reducing partitions $\mathcal{C} \subseteq \mathcal{P}^{\circ\bullet}$ to their sets $\mathcal{C} \cap \mathscr{R}$ in the ensuing Proposition 5.14.

Lemma 5.13. *If $p \in \mathscr{M}\backslash\{\varnothing\}$ is projective and $p \notin \mathscr{R}_3$, then $\|p\| \in 4\mathbb{N}$ and there is an action (T, B, α, β) in p such that (α, β) is $({}^{\blacksquare}\frac{1}{4}\|p\|, {}_{\blacksquare}\frac{1}{4}\|p\|)$ or $({}_{\blacksquare}(\frac{1}{4}\|p\|+1), {}^{\blacksquare}(\frac{1}{4}\|p\|+1))$.*

Proof. Since p is projective, $n := \|p\| \in 2\mathbb{N}$ and p has at least two indefinite turns by Lemma 5.11. Per assumption $p \notin \mathscr{R}_3$ we hence find an indefinite turn T with $T \cap \{{}^{\blacksquare}1, {}_{\blacksquare}1, {}_{\blacksquare}\frac{n}{2}, {}^{\blacksquare}\frac{n}{2}\} = \varnothing$. By definition there exists a size-preserving action (T, B, α, β). In particular, $|[\alpha, \beta]_p| = \frac{n}{2}$. We show in four steps that (α, β) is as claimed.

Step 1: *$[\alpha, \beta]_p$ is not contained in one row.* Suppose there existed $i, j \in [\![\frac{n}{2}]\!]$ with $i < j$ such that $(\alpha, \beta) = ({}_{\blacksquare}i, {}_{\blacksquare}j)$ or $(\alpha, \beta) = ({}^{\blacksquare}j, {}^{\blacksquare}i)$. If so, then $j - i + 1 = |[\alpha, \beta]_p| = \frac{n}{2}$ would require $j = i + \frac{n}{2} - 1 \geq 1 + \frac{n}{2} - 1 = \frac{n}{2}$, which is to say $j = \frac{n}{2}$, and thus $i = j - \frac{n}{2} + 1 = \frac{n}{2} - \frac{n}{2} + 1 = 1$. We would conclude $(\alpha, \beta) = ({}_{\blacksquare}1, {}_{\blacksquare}\frac{n}{2})$ or $(\alpha, \beta) = ({}^{\blacksquare}\frac{n}{2}, {}^{\blacksquare}1)$. That would contradict $T \cap \{{}^{\blacksquare}1, {}_{\blacksquare}1, {}_{\blacksquare}\frac{n}{2}, {}^{\blacksquare}\frac{n}{2}\} = \varnothing$ because $T \cap \{\alpha, \beta\} \neq \varnothing$.

Step 2: *α and β lie on different rows.* Next, assume that we could find $i, j \in [\![\frac{n}{2}]\!]$ with $i < j$ such that $(\alpha, \beta) = ({}_{\blacksquare}j, {}_{\blacksquare}i)$ or $(\alpha, \beta) = ({}^{\blacksquare}i, {}^{\blacksquare}j)$. It would follow $\frac{n}{2} + (\frac{n}{2} - i + 1) + j = |[\alpha, \beta]_p| = \frac{n}{2}$, which is to say $n + j - i + 1 = \frac{n}{2}$, i.e., $j = i - \frac{n}{2} - 1$. As $i \leq \frac{n}{2}$ and $1 \leq j$, we would infer $1 \leq j \leq \frac{n}{2} - \frac{n}{2} - 1 = -1$, a contradiction.

Step 3: *α and β are counterparts.* We hence find $i, j \in [\![\frac{n}{2}]\!]$ such that (α, β) is $({}_{\blacksquare}i, {}^{\blacksquare}j)$ or $({}^{\blacksquare}i, {}_{\blacksquare}j)$. If $(\alpha, \beta) = ({}_{\blacksquare}i, {}^{\blacksquare}j)$, then $\frac{n}{2} = |[\alpha, \beta]_p| = (\frac{n}{2} - i + 1) + (\frac{n}{2} - j + 1)$, which is to say $i + j = \frac{n}{2} + 2$. On the other hand, if $(\alpha, \beta) = ({}^{\blacksquare}i, {}_{\blacksquare}j)$, then $\frac{n}{2} = |[\alpha, \beta]_p| = i + j$. In particular, $i + j \in \{\frac{n}{2}, \frac{n}{2} + 2\}$ always. We prove $i = j$ by assuming $i \neq j$ and deriving a contradiction in seven steps.

Step 3.1: *B would have evenly many legs.* Because α and β are legs of B located on different rows, B is a through block. In projective partitions, through blocks are symmetrical with respect to the horizontal axis. In particular, they have even numbers of legs. It follows that $|B|$ is even.

Step 3.2: *The counterparts of α and β would also be legs of B.* The symmetry of B with respect to the horizontal axis and our assumption $\{\alpha, \beta\} \subseteq B$ moreover imply that also the counterparts of α and β are legs of B, i.e., that $\{{}_{\blacksquare}i, {}^{\blacksquare}i, {}_{\blacksquare}j\,{}^{\blacksquare}j\} \subseteq B$. In particular, $|B| \geq 4$ because $i \neq j$.

Step 3.3: *α would not be the subsequent leg of B after β.* If $]\beta, \alpha[_p \cap B = \varnothing$ were true, then because $\{\alpha, \beta\} \subseteq B$ and $]\beta, \alpha[_p \cap B = \varnothing$ and because $\{{}_{\blacksquare}i, {}^{\blacksquare}i, {}_{\blacksquare}j\,{}^{\blacksquare}j\} \subseteq B$ by Step 3.2, (α, β) would be given by $({}_{\blacksquare}\min\{i, j\}, {}^{\blacksquare}\min\{i, j\})$ or $({}^{\blacksquare}\max\{i, j\}, {}_{\blacksquare}\max\{i, j\})$.

However, that would imply $i = j$, contradicting our assumption $i \neq j$. Thus, we must have $]\beta, \alpha[\cap B \neq \emptyset$ instead.

Step 3.4: *B would consist only of α and β and their counterparts and $B \backslash T$ would be consecutive.* Our assumption that (T, B, α, β) is an action in p requires that $]\beta, \alpha[_p \cap B = \emptyset$ or $T \backslash \{\alpha, \beta\} \in p$ or $T \subseteq B$. As we have already excluded the first of the three, one of the latter two must be true. Since n is even by Step 3.1 and since $4 \leq |B|$ by Step 3.2, we can conclude that $|B| = 4$ and that $B \backslash T$ is consecutive by Lemma 5.9. As $\{{}_{\blacksquare}i, {}^{\blacksquare}i, {}_{\blacksquare}j, {}^{\blacksquare}j\} \subseteq B$ by Step 3.2 and as $i \neq j$, the conclusion $|B| = 4$ in fact ensures $B = \{{}_{\blacksquare}i, {}^{\blacksquare}i, {}_{\blacksquare}j, {}^{\blacksquare}j\}$.

Step 3.5: *T would be contained in B.* We show $T \subseteq B$ by contradiction. Because we assume $T \cap \{{}^{\blacksquare}1, {}_{\blacksquare}1, {}_{\blacksquare}\frac{n}{2}, {}^{\blacksquare}\frac{n}{2}\} = \emptyset$, in particular, $T \subseteq]{}_{\blacksquare}1, {}_{\blacksquare}\frac{n}{2}[_p$ or $T \subseteq]{}^{\blacksquare}\frac{n}{2}, {}^{\blacksquare}1[_p$. Therefore, supposing $T \nsubseteq B$ requires, because $T \cap B \neq \emptyset$ and because $B = \{{}_{\blacksquare}i, {}^{\blacksquare}i, {}_{\blacksquare}j, {}^{\blacksquare}j\}$ by Step 3.4, that the set $B \backslash T$ is one of the four $\{{}^{\blacksquare}i, {}_{\blacksquare}j, {}^{\blacksquare}j\}$, $\{{}^{\blacksquare}j, {}_{\blacksquare}j, {}_{\blacksquare}i\}$, $\{{}_{\blacksquare}i, {}^{\blacksquare}i, {}^{\blacksquare}j\}$ and $\{{}_{\blacksquare}j, {}_{\blacksquare}i, {}^{\blacksquare}i\}$. If so, then $B \backslash T$ by Step 3.4 being consecutive demands that $\{i, j\}$ is $\{1, 2\}$ or $\{\frac{n}{2}, \frac{n}{2} - 1\}$. However, then $i + j \in \{\frac{n}{2}, \frac{n}{2} + 2\}$ implies $n \in \{2, 6\}$. As $n = 2$ contradicts $p \in \mathscr{M}$ because p is projective, we must have $n = 6$. Though, then, $T \cap \{{}^{\blacksquare}1, {}_{\blacksquare}1, {}_{\blacksquare}\frac{n}{2}, {}^{\blacksquare}\frac{n}{2}\} = \emptyset$ implies $T \subseteq \{{}_{\blacksquare}2, {}^{\blacksquare}2\}$, contradicting that T is a turn.

Step 3.6: *α and the counterpart of β would be neighbors.* From $T \subseteq]{}_{\blacksquare}1, {}_{\blacksquare}\frac{n}{2}[_p$ or $T \subseteq]{}^{\blacksquare}\frac{n}{2}, {}^{\blacksquare}1[_p$ and from $T \subseteq B = \{{}_{\blacksquare}i, {}^{\blacksquare}i, {}_{\blacksquare}j, {}^{\blacksquare}j\}$ by Steps 3.4 and 3.5 it follows that $\{T, B \backslash T\} = \{\{{}_{\blacksquare}i, {}_{\blacksquare}j\}, \{{}^{\blacksquare}i, {}^{\blacksquare}j\}\}$. Our knowledge from Step 3.4 that $B \backslash T$ is consecutive thus allows us to infer that $j = i + 1$ or $i = j + 1$.

Step 3.7: *p would not be erasing-minimal.* Because $i + j \in \{\frac{n}{2}, \frac{n}{2} + 2\}$ we can conclude $2i + 1 \in \{\frac{n}{2}, \frac{n}{2} + 2\}$ or $2j + 1 \in \{\frac{n}{2}, \frac{n}{2} + 2\}$ from $j = i + 1$ or $i = j + 1$. This shows that $n \in 4\{i, j\} + \{2, 6\}$ and that $\{i, j\}$ is given by $\{\frac{n}{4} - \frac{1}{2}, \frac{n}{4} + \frac{1}{2}\}$ or $\{\frac{n}{4} + \frac{1}{2}, \frac{n}{4} + \frac{3}{2}\}$.

Since p is projective and since $T \in \{\{{}_{\blacksquare}i, {}_{\blacksquare}j\}, \{{}^{\blacksquare}i, {}^{\blacksquare}j\}\}$ by Step 3.6, $T_1 \coloneqq \{{}_{\blacksquare}i, {}_{\blacksquare}j\}$ is a turn in p. Hence, $(T_1, B, \alpha_1, \beta_1)$ is an action in p for $(\alpha_1, \beta_1) = ({}^{\blacksquare}(\frac{n}{4} - \frac{1}{2}), {}_{\blacksquare}(\frac{n}{4} - \frac{1}{2}))$ or $(\alpha_1, \beta_1) = ({}_{\blacksquare}(\frac{n}{4} + \frac{3}{2}), {}^{\blacksquare}(\frac{n}{4} + \frac{3}{2}))$. The assumption $p \in \mathscr{M}$ demands $\frac{n}{2} \leq |[\alpha_1, \beta_1]_p|$, meaning $\frac{n}{2} \leq 2(\frac{n}{2} - (\frac{n}{4} + \frac{3}{2}) + 1) = 2(\frac{n}{4} - \frac{1}{2}) = \frac{n}{2} - 1$ or $\frac{n}{2} \leq 2(\frac{n}{4} - \frac{1}{2}) = \frac{n}{2} - 1$, which is the contradiction we sought.

Step 4: *α and β have the right position.* As $i = j$ by Step 3, we infer $2i \in \{\frac{n}{2}, \frac{n}{2} + 2\}$, implying $n \in \{4i, 4i + 4\}$. That proves $n \in 4\mathbb{N}$ and $i = j \in \{\frac{n}{4}, \frac{n}{4} + 1\}$, as claimed. □

The following proposition is the central result of this section, allowing us to reconstruct any category from its set of special erasing-minimal partitions.

PROPOSITION 5.14. *$\mathcal{C} = \langle \mathcal{C} \cap (\mathscr{R} \cup \{\uparrow_{\circ} \otimes \uparrow_{\bullet}\}) \rangle$ for every category $\mathcal{C} \subseteq \mathcal{P}^{\circ\bullet}$.*

PROOF. Denote by $\mathcal{C}'$ the right-hand side of the claimed identity. By Proposition 5.8 it suffices to show $\mathcal{C} \cap \mathscr{M} \subseteq \mathcal{C}'$, which we prove by induction over partition size. The empty partition $\emptyset$ is the only element of $\mathcal{C} \cap \mathscr{M}$ with size 0 and naturally included in $\mathcal{C}'$. All elements of $\mathcal{C} \cap \mathscr{M}$ of size 1 have no turns and are thus elements of $\langle \mathcal{C} \cap \mathscr{R}_1 \rangle \subseteq \mathcal{C}'$ by Lemma 2.2. Let $n \in \mathbb{N}$, let $2 \leq n$, let $\mathcal{C}'$ include all elements of $\mathcal{C} \cap \mathscr{M}$ with sizes up to $n - 1$, let $p_0 \in \mathcal{C} \cap \mathscr{M}$ and let $\|p_0\| = n$. We show $p_0 \in \mathcal{C}'$.

Note that, by Proposition 5.8, the induction hypothesis means that $\mathcal{C}'$ actually contains all elements of $\mathcal{C}\backslash\mathscr{M}$ with up to n many points. It follows that, in particular, $\mathcal{C}'$ includes *all* partitions of $\mathcal{C}$ of size $n-1$ or less.

Step 1: *Devolving the problem $p_0 \in \mathcal{C}'$ to the problem $p_1 \in \mathcal{C}'$.* If p_0 has no turns, then p_0 is once again a rotation of an element of $\mathcal{C}\cap\mathscr{R}_1$ and thus included in $\mathcal{C}'$ by Lemma 2.2. Hence, let p_0 have at least one turn.

If none of the turns of p_0 are indefinite, then $p_0 \in \langle \mathcal{C}\cap\mathscr{R}_2\rangle \subseteq \mathcal{C}'$ by Lemma 2.2. Hence, let p_0 have at least one indifferent turn.

By assumption, we find an action $(T_0, B_0, \alpha_0, \beta_0)$ for T_0 in p with $|[\alpha_0,\beta_0]_p| = \frac{n}{2}$. Lemma 5.7 tells us $p_0 \in \langle E(p_0,T_0), P(p_0,[\alpha_0,\beta_0]_{p_0}), \langle p_0\rangle \cap \{\uparrow_{\circ}\otimes\uparrow_{\bullet}\}\rangle$. Hence, in order to prove $p_0 \in \mathcal{C}'$ it suffices to show $\{E(p_0,T_0), P(p_0,[\alpha_0,\beta_0]_{p_0})\} \subseteq \mathcal{C}'$. The partition $E(p_0,T_0)\in\mathcal{C}$ has $\|p_0\|-|T_0| = n-2$ points and is thus an element of $\mathcal{C}'$ by the induction hypothesis. Hence, if we can show $p_1 \coloneqq P(p_0,[\alpha_0,\beta_0]_{p_0}) \in \mathcal{C}'$, then also $p_0\in\mathcal{C}'$.

Step 2: *Devolving the problem $p_1 \in \mathcal{C}'$ to the problem $p_2 \in \mathcal{C}'$.* We have $p_1 \in \mathcal{C}$ by Lemma 2.5. Moreover, by definition of the projection operation, $\|p_1\| = 2\cdot|[\alpha_0,\beta_0]_p| = 2\cdot\frac{n}{2} = n$. For that reason, if $p_1 \notin \mathscr{M}$, then $p_1\in\mathcal{C}'$ by the induction hypothesis. Hence, we can assume $p_1 \in \mathscr{M}$. Because p_1 is projective, it has turns and some of its turns are indefinite by Lemma 5.11.

If every indefinite turn of p_1 intersects $\{{}^{\blacksquare}1, {}_{\blacksquare}1, {}_{\blacksquare}\frac{n}{2}, {}^{\blacksquare}\frac{n}{2}\}$, then $p_1 \in \mathcal{C}\cap\mathscr{R}_3 \subseteq \mathcal{C}'$. Therefore, we can assume the opposite.

By Lemma 5.13 then, $n \in 4\mathbb{N}$ and there exists an action $(T_1,B_1,\alpha_1,\beta_1)$ in p_1 such that (α_1,β_1) is given by $({}^{\blacksquare}\frac{n}{4}, {}_{\blacksquare}\frac{n}{4})$ or $({}_{\blacksquare}(\frac{n}{4}+1), {}^{\blacksquare}(\frac{n}{4}+1))$. Lemma 5.7 hence shows $p_1 \in \langle E(p_1,T_1), P(p_1,[\alpha_1,\beta_1]_{p_1}), \langle p_1\rangle\cap\{\uparrow_{\circ}\otimes\uparrow_{\bullet}\}\rangle$. Once again, $E(p_1,T_1)\in\mathcal{C}$ has $\|p_1\| - |T_1| = n-2$ points and is thus an element of $\mathcal{C}'$ by the induction hypothesis. Hence, in order to prove $p_1\in\mathcal{C}'$, it remains to prove that $p_2 \coloneqq P(p_1,[\alpha_1,\beta_1]_{p_1}) \in \mathcal{C}'$.

Step 3: *Proving $p_2 \in \mathcal{C}'$.* Of course, $p_2\in\mathcal{C}$ by Lemma 2.5. And, once more, $\|p_2\| = 2\cdot|[\alpha_1,\beta_1]_{p_1}|$ points. If $(\alpha_1,\beta_1) = ({}^{\blacksquare}\frac{n}{4}, {}_{\blacksquare}\frac{n}{4})$, then $|[\alpha_1,\beta_1]_{p_1}| = 2\cdot\frac{n}{4} = \frac{n}{2}$. Likewise, if $(\alpha_1,\beta_1) = ({}_{\blacksquare}(\frac{n}{4}+1), {}^{\blacksquare}(\frac{n}{4}+1))$, then $|[\alpha_1,\beta_1]_{p_1}| = 2(\frac{n}{2} - (\frac{n}{4}+1)+1) = \frac{n}{2}$. Thus, $\|p_2\| = n$ in any case. Consequently, if $p_2\notin\mathscr{M}$, then $p_2\in\mathcal{C}'$ by the induction hypothesis. Hence, we can assume $p_2\in\mathscr{M}$.

Because $p_1 = P(p_0,[\alpha_0,\beta_0]_{p_0})$ is projective and because α_1 and β_1 are counterparts of each other, $p_2 = P(p_1,[\alpha_1,\beta_1]_{p_1})$ is bi-projective. Being thus both erasing-minimal and bi-projective, p_2 is an element of $\mathcal{C}\cap\mathscr{R}_4\subseteq\mathcal{C}'$. That is what we needed to see. ☐

In the ensuing final lemma of this section we draw some particular conclusions about the sets $\mathcal{C}\cap\mathscr{R}$ for hyperoctahedral categories $\mathcal{C}\subseteq\mathcal{P}^{\circ\bullet}$ which we will need later.

LEMMA 5.15. *Let $\mathcal{C}\subseteq\mathcal{P}^{\circ\bullet}$ be a hyperoctahedral category and let $p\in\mathcal{C}\cap\mathscr{M}$ have no indefinite turns. Then, there are no turns T in p such that $T\subseteq B$ for some $B\in p$.*

PROOF. We prove the claim by contradiction. Let $B\in p$ and $\{\eta,\theta\}\subseteq B$ be such that $T\coloneqq[\eta,\theta]_p$ is a turn in p. We show $|B| = 3$. As $E(p,T)$ then has a singleton block, contradicting the assumption of $\mathcal{C}$ being hyperoctahedral by Lemma 2.3 (b), that is sufficient to prove the claim.

Because $p \in \mathscr{M}$ the turn T is not a subpartition of p, i.e., $B \backslash T \neq \varnothing$. Let $\gamma \in B \backslash T$ be arbitrary. Both (T, B, θ, γ) and (T, B, γ, η) are actions in p. Since $p \in \mathscr{M}$ has no indefinite turns, T must be essential. It follows that (T, B, θ, γ) and (T, B, γ, η) are size-increasing. Let P_p be the set of all points of p and let $n \coloneqq |P_p|$. Then, $|[\theta, \gamma]_p| > \frac{n}{2}$ and $|[\gamma, \eta]_p| > \frac{n}{2}$. That θ is the successor of η hence implies

$$|[\theta,\gamma]_p| = |P_p \backslash]\gamma,\theta[_p| = |P_p| - |]\gamma,\theta[_p| = n - |]\gamma,\eta]_p| = n + 1 - |[\gamma,\eta]_p| < \frac{n}{2} + 1.$$

By $\frac{n}{2} < |[\theta, \gamma]_p| < \frac{n}{2} + 1$ we infer that n is odd and that $|[\theta, \gamma]_p| = \lceil \frac{n}{2} \rceil$. In particular, as $P_p \backslash T$ is totally ordered the condition $|[\theta, \gamma]_p| = \lceil \frac{n}{2} \rceil$ determines $\gamma \in P_p \backslash T$ uniquely. Because γ was arbitrary, B has three legs. □

6. Generators

Employing the auxiliary results from Section 5, we will next show $\mathcal{W}_R = \langle \pi_c \,|\, c \in R \rangle$ for every $R \in \mathcal{R}$ (Theorem 6.24), where π_c are defined in Definition 6.9. As in Section 4 it is best to first prove this claim for the special case $\mathcal{W} = \mathcal{W}_{\bigcup \mathcal{R}}$ (Theorem 6.19).

6.1. Generators of $\mathcal{W}$. We determine the set $\mathcal{W} \cap \mathscr{R}$ (see Proposition 6.18). By Proposition 5.14 this yields a generator of $\mathcal{W}$ (see Theorem 6.19).

6.1.1. *Existence of Turns.* As a first preparatory step the next two results are concerned with the existence of certain actions of the third kind in the sense of Definition 5.2 (c) in partitions of $\mathcal{W}$.

Lemma 6.1. *For all $p \in \mathcal{W}$, all $A \in p$ and all $\{\alpha, \gamma\} \subseteq A$ with $\alpha \neq \gamma$, with $]\alpha, \gamma[_p \neq \varnothing$ and with $]\alpha, \gamma[_p \cap A = \varnothing$ there exist $B \in p$ and a turn T in p such that $T \subseteq]\alpha, \gamma[_p \cap B$.*

Proof. By assumption $]\alpha, \gamma[_p \neq \varnothing$ there exists a block B_1 of p with $]\alpha, \gamma[_p \cap B_1 \neq \varnothing$. Because $]\alpha, \gamma[_p \cap A = \varnothing$, we are assured that $A \neq B_1$. As $p \in \mathcal{W}$ the block B_1 is non-interferent with A in p. Hence, $\sigma_p([\alpha, \gamma]_p \cap B_1) = 0$ and, consequently, in particular, $|[\alpha, \gamma]_p \cap B_1| \equiv_2 0$. Therefore, $]\alpha, \gamma[_p \cap B_1 \neq \varnothing$ actually means $|[\alpha, \gamma]_p \cap B_1| \geq 2$ or, in fact, $|]\alpha, \gamma[_p \cap B_1| \geq 2$ since $A \neq B_1$. We hence find $\{\alpha_1, \gamma_1\} \subseteq]\alpha, \gamma[_p \cap B_1$ with $\alpha_1 \neq \gamma_1$ and, necessarily, $|[\alpha_1, \gamma_1]_p| < |[\alpha, \gamma]_p|$. If $|[\alpha_1, \gamma_1]_p| = 2$, then Proposition 4.13 proves $\sigma_p(\{\alpha_1, \gamma_1\}) = 0$ as $]\alpha_1, \gamma_1[_p \cap B_1 = \varnothing$, meaning we have already discovered our turn $T \coloneqq [\alpha_1, \gamma_1]_p$ and block $B \coloneqq B_1$.

Should $]\alpha_1, \gamma_1[_p \neq \varnothing$ instead, we find ourselves in the initial situation with B_1 playing the role of A. By repeating the same argument we find $B_2 \in p$ and $\{\alpha_2, \gamma_2\} \subseteq]\alpha_1, \gamma_1[_p \cap B_1$ with $\alpha_2 \neq \gamma_2$ and $|[\alpha_2, \gamma_2]_p| < |[\alpha_1, \gamma_1]_p| < |[\alpha, \gamma]_p|$. After a finite number $n \in \mathbb{N}$ of steps we necessarily arrive at a block $B \coloneqq B_n$ and legs $\{\alpha_n, \gamma_n\} \subseteq]\alpha, \gamma[_p \cap B_n$ where $T \coloneqq [\alpha_n, \gamma_n]_p$ has exactly two elements. Once more Proposition 4.13 ensures $\sigma_p(\{\alpha_n, \gamma_n\}) = 0$, making $T \subseteq]\alpha, \gamma[_p \cap B$ the turn we sought. □

Lemma 6.2. *If $p \in \mathcal{W} \backslash \{\varnothing\}$, then there exist $B \in p$ and a turn T in p with $T \subseteq B$.*

Proof. The assumption $p \neq \varnothing$ ensures that p has at least one block A. Moreover, $\sigma_p(A) = 0$ because $p \in \mathcal{W}$. In particular, $|A| \geq 2$. Consequently, there exist $\{\alpha, \gamma\} \subseteq A$ with $\alpha \neq \gamma$ and, especially, $]\alpha, \gamma[_p \cap A = \varnothing$. Two situations are conceivable.

Case 1: If $]\alpha,\gamma[_p = \emptyset$ for all such α and γ, then A is the only block of p and, according to Proposition 4.13, its legs alternate in color. Hence, $B \coloneqq A$ and $T \coloneqq \{\alpha,\gamma\}$ for any such α and γ fit the bill.

Case 2: If instead there do exist $\{\alpha,\gamma\} \subseteq A$ with $\alpha \neq \gamma$, with $]\alpha,\gamma[_p \neq \emptyset$ and with $]\alpha,\gamma[_p \cap A = \emptyset$, then we can find $B \in p$ and a turn T in p with $T \subseteq]\alpha,\beta[_p \cap B$ by Lemma 6.1. That concludes the proof. ☐

6.1.2. *Projective Erasing-Minimal Partitions of $\mathcal{W}$.* As an intermediate step we need to see which restrictions being a member of $\mathcal{W}$ imposes on an erasing-minimal and projective partition.

LEMMA 6.3. *If $p \in \mathcal{W}$, if $B \in p$ and $|B| = 2$, then B is a connected component of p.*

PROOF. Let $A \in p$, let $A \neq B$ and let A and B cross in p. We derive a contradiction. We find $\{\alpha_1,\alpha_2\} \subseteq A$ and $\{\beta_1,\beta_2\} \subseteq B$ such that $(\alpha_1,\beta_1,\alpha_2,\beta_2)$ is an ordered tuple of pairwise distinct points of p. Since $|B| = 2$, actually, $B = \{\beta_1,\beta_2\}$. Hence, $[\alpha_1,\alpha_2]_p \cap B = \{\beta_1\}$ and thus, in particular, $\sigma_p([\alpha_1,\alpha_2]_p \cap B) \neq 0$. Because $\{\alpha_1,\alpha_2\} \subseteq A \neq B$ and $\alpha_1 \neq \alpha_2$ the assumption $p \in \mathcal{W}$ however requires $\sigma_p([\alpha_1,\alpha_2]_p \cap B) = 0$. That is the contradiction we sought. ☐

LEMMA 6.4. *If $p \in (\mathcal{W} \cap \mathcal{M})\backslash\{\emptyset\}$ is projective and $B \in p$ has two neighboring legs, $\|p\| \in 4\mathbb{N}$ and $B = \{{}^{\blacksquare}1, {}_{\blacksquare}1, {}_{\blacksquare}\tfrac{1}{2}\|p\|, {}^{\blacksquare}\tfrac{1}{2}\|p\|\}$ or $B = \{{}^{\blacksquare}\tfrac{1}{4}\|p\|, {}_{\blacksquare}\tfrac{1}{4}\|p\|, {}_{\blacksquare}(\tfrac{1}{4}\|p\|+1), {}^{\blacksquare}(\tfrac{1}{4}\|p\|+1)\}$.*

PROOF. Let $n \coloneqq \|p\|$. We suppose $B \neq \{{}^{\blacksquare}1, {}_{\blacksquare}1, {}_{\blacksquare}\tfrac{n}{2}, {}^{\blacksquare}\tfrac{n}{2}\}$ and show in three steps that $n \in 4\mathbb{N}$ and that $B = \{{}^{\blacksquare}\tfrac{n}{4}, {}_{\blacksquare}\tfrac{n}{4}, {}_{\blacksquare}(\tfrac{n}{4}+1), {}^{\blacksquare}(\tfrac{n}{4}+1)\}$.

Step 1: *B consists of two disjoint turns.* Let β_2 be the successor of β_1 in p and let $T \coloneqq \{\beta_1,\beta_2\} \subseteq B$. Because $p \in \mathcal{W}$ and $\{\beta_1,\beta_2\} \subseteq B$ and $]\beta_1,\beta_2[_p \cap B = \emptyset$, Proposition 4.13 assures us that $\sigma_p(T) = \sigma_p(\{\beta_1,\beta_2\}) = 0$. In other words, T is a turn in p. Since $p \in \mathcal{W}$ there are no singleton blocks in p. Lemma 2.3 (b) therefore guarantees ${\uparrow_{\circ}} \otimes {\uparrow_{\bullet}} \notin \langle p \rangle$. For that reason, by Lemma 5.9 (b) the fact $T \subseteq B$ allows us to conclude that $|B| = 4$ and that $B\backslash T$ is consecutive. A second application of Proposition 4.13 lets us infer $\sigma_p(B\backslash T) = 0$. Hence, B consists of two disjoint turns T and $B\backslash T$.

Step 2: *B is a through block.* We assume that there is a row R of p with $B \subseteq R$ and obtain a contradiction. Let $\leq$ be the total order of R, let $T_1, T_2 \in \{T, B\backslash T\}$ be such that $T_1 < T_2$, let $\gamma_1 \coloneqq \max_{\leq}(T_1)$ and let $\gamma_2 \in T_2$ be arbitrary. This definition ensures $[\gamma_1,\gamma_2]_p \subsetneq R$ because $\min_{\leq}(T_1) \in R\backslash[\gamma_1,\gamma_2]_p$. Consequently, $|[\gamma_1,\gamma_2]_p| < |[{}_{\blacksquare}1, {}_{\blacksquare}\tfrac{n}{2}]_p| = |[{}^{\blacksquare}\tfrac{n}{2}, {}^{\blacksquare}1]_p| = \tfrac{n}{2}$. However, per construction, $(T_1, B, \gamma_1, \gamma_2)$ is an action in p. The assumption $p \in \mathcal{M}$ therefore implies $|[\gamma_1,\gamma_2]_p| \geq \tfrac{n}{2}$, a contradiction. Hence, B is indeed a through block.

Step 3: *B is of the claimed form.* Because p is projective and because B is a through block by Step 2, the block B is symmetric with respect to the horizontal axis. Since B consists of two disjoint turns by Step 1 and since we assume $B \neq \{{}^{\blacksquare}1, {}_{\blacksquare}1, {}_{\blacksquare}\tfrac{n}{2}, {}^{\blacksquare}\tfrac{n}{2}\}$ this allows us to conclude $B = T_L \cup T_U$, where $T_L \coloneqq \{{}_{\blacksquare}i, {}_{\blacksquare}(i+1)\}$ and $T_U \coloneqq \{{}^{\blacksquare}(i+1), {}^{\blacksquare}i\}$ for some $i \in [\![\tfrac{n}{2}-1]\!]$. It remains to proven that $n \in 4\mathbb{N}$ and $i = \tfrac{n}{4}$.

Because T and $B\backslash T$ are consecutive, $\{T, B\backslash T\} = \{T_L, T_U\}$. It follows in particular that T_L is a turn with $T_L \subseteq B$. Consequently, both $(T_L, B, {}^{\blacksquare}i, {}_{\blacksquare}i)$ and $(T_L, B, {}_{\blacksquare}(i{+}1), {}^{\blacksquare}(i{+}1))$ are actions in p. The assumption $p \in \mathcal{M}$ hence implies $2i = |[{}^{\blacksquare}i, {}_{\blacksquare}i]_p| \geq \frac{n}{2}$ and $n - 2i = 2(\frac{n}{2} - (i+1) + 1) = |[{}_{\blacksquare}(i{+}1), {}^{\blacksquare}(i{+}1)]_p| \geq \frac{n}{2}$. We infer $i \geq \frac{n}{4}$ from the first and $i \leq \frac{n}{4}$ from the second inequality. Thus, we have shown $n \in 4\mathbb{N}$ and $i = \frac{n}{4}$, which is what we needed to see. □

Lemma 6.5. *If $p \in \mathcal{W} \cap \mathcal{M}$ is projective, then all blocks of p are through blocks.*

Proof. We can assume $p \neq \emptyset$. Because p is projective, the claim that all blocks of p are through blocks is equivalent to p having no lower non-through blocks. We assume that there exists a lower non-through block B_1 of p and derive a contradiction. Let $n \coloneqq \|p\|$, let $i_1 \in [\![\frac{n}{2}]\!]$ be such that ${}_{\blacksquare}i_1$ is the leftmost leg of B_1. The assumption $p \in \mathcal{W}$ guarantees $\sigma_p(B_1) = 0$ and thus $|B_1| \equiv_2 0$, ensuring $B_1 \backslash \{{}_{\blacksquare}i_1\} \neq \emptyset$. Let $i_2 \in [\![\frac{n}{2}]\!]$ be minimal with the properties $i_1 < i_2$ and ${}_{\blacksquare}i_2 \in B_1$.

Step 1: *${}_{\blacksquare}i_1$ and ${}_{\blacksquare}i_2$ would not be neighbors.* If $i_2 = i_1 + 1$ were true, then B_1 would contain two neighboring legs. It would follow by Lemma 6.4 that $n \in 4\mathbb{N}$ and $B_1 = \{{}^{\blacksquare}1, {}_{\blacksquare}1, {}_{\blacksquare}\frac{n}{2}, {}^{\blacksquare}\frac{n}{2}\}$ or $B_1 = \{{}^{\blacksquare}\frac{n}{4}, {}_{\blacksquare}\frac{n}{4}, {}_{\blacksquare}(\frac{n}{4}{+}1), {}^{\blacksquare}(\frac{n}{4}{+}1)\}$. In particular, B_1 would be a through block, in contradiction to the assumption that B_1 is lower non-through. Hence, $i_1 + 1 < i_2$.

Step 2: *$]{}_{\blacksquare}i_1, {}_{\blacksquare}i_2[_p$ would include a turn T properly contained in a block B.* Because $\{{}_{\blacksquare}i_1, {}_{\blacksquare}i_2\} \subseteq B_1$ and $]{}_{\blacksquare}i_1, {}_{\blacksquare}i_2[_p \cap B_1 = \emptyset$ and because $]{}_{\blacksquare}i_1, {}_{\blacksquare}i_2[_p \neq \emptyset$ by Step 1, Lemma 6.1 ensures that there exist $B \in p$ and a turn T in p with $T = T \cap B \subseteq]{}_{\blacksquare}i_1, {}_{\blacksquare}i_2[_p$. Moreover, $B \neq T$ because the premise $p \in \mathcal{M}$ forbids $T \in p$.

Step 3: *B would be a through block consisting of the central turns on both rows.* By Step 2, the block B has two neighboring legs (in T). Hence, $n \in 4\mathbb{N}$ and $B = \{{}^{\blacksquare}1, {}_{\blacksquare}1, {}_{\blacksquare}\frac{n}{2}, {}^{\blacksquare}\frac{n}{2}\}$ or $B = \{{}^{\blacksquare}\frac{n}{4}, {}_{\blacksquare}\frac{n}{4}, {}_{\blacksquare}(\frac{n}{4}{+}1), {}^{\blacksquare}(\frac{n}{4}{+}1)\}$ by Lemma 6.4. Since $T \subseteq]{}_{\blacksquare}i_1, {}_{\blacksquare}i_2[_p \subseteq B$ excludes the first possibility, $B = \{{}^{\blacksquare}\frac{n}{4}, {}_{\blacksquare}\frac{n}{4}, {}_{\blacksquare}(\frac{n}{4}{+}1), {}^{\blacksquare}(\frac{n}{4}{+}1)\}$ must be true.

Step 4: *p would not be an element of $\mathcal{W}$.* As B_1 is assumed lower non-through, as $B = \{{}^{\blacksquare}\frac{n}{4}, {}_{\blacksquare}\frac{n}{4}, {}_{\blacksquare}(\frac{n}{4}{+}1), {}^{\blacksquare}(\frac{n}{4}{+}1)\}$ by Step 3, as $T = \{{}_{\blacksquare}(\frac{n}{4}{+}1), {}^{\blacksquare}(\frac{n}{4}{+}1)\} \subseteq]{}_{\blacksquare}i_1, {}_{\blacksquare}i_2[_p$ by Step 2 and as $]{}_{\blacksquare}i_1, {}_{\blacksquare}i_2[_p \cap B_1 = \emptyset$ by definition we can infer $[{}^{\blacksquare}\frac{n}{4}, {}_{\blacksquare}\frac{n}{4}]_p \cap B_1 = \{{}_{\blacksquare}i_1\}$. In particular, $\sigma_p([{}^{\blacksquare}\frac{n}{4}, {}_{\blacksquare}\frac{n}{4}]_p \cap B_1) \neq 0$. However, $p \in \mathcal{W}$ and $\{{}^{\blacksquare}\frac{n}{4}, {}_{\blacksquare}\frac{n}{4}\} \subseteq B \neq B_1$ and ${}^{\blacksquare}\frac{n}{4} \neq {}_{\blacksquare}\frac{n}{4}$ demand $\sigma_p([{}^{\blacksquare}\frac{n}{4}, {}_{\blacksquare}\frac{n}{4}]_p \cap B_1) = 0$. That is the contradiction we sought. □

Lemma 6.6. *If $p \in \mathcal{W} \cap \mathcal{M}$ is projective, then $\|p\| \in 4\mathbb{N}_0$ and, provided $p \neq \emptyset$, both $\{{}^{\blacksquare}1, {}_{\blacksquare}1, {}_{\blacksquare}\frac{1}{2}\|p\|, {}^{\blacksquare}\frac{1}{2}\|p\|\} \in p$ and $\{{}^{\blacksquare}\frac{1}{4}\|p\|, {}_{\blacksquare}\frac{1}{4}\|p\|, {}_{\blacksquare}(\frac{1}{4}\|p\|{+}1), {}^{\blacksquare}(\frac{1}{4}\|p\|{+}1)\} \in p$.*

Proof. Since p is erasing-minimal and projective, p is a co-bracket or a bracket by Lemma 5.11. In other words, if $n \coloneqq \|p\| > 0$ and $B_1 \in p$ and ${}_{\blacksquare}1 \in B_1$, then ${}_{\blacksquare}\frac{n}{2} \in B_1$, plus $]{}_{\blacksquare}1, {}_{\blacksquare}\frac{n}{2}[_p \cap B_1 = \emptyset$ or $B_1 \subseteq [{}_{\blacksquare}1, {}_{\blacksquare}\frac{n}{2}]_p$. As p only has through blocks by Lemma 6.5, the latter cannot be true. Hence, $]{}_{\blacksquare}1, {}_{\blacksquare}\frac{n}{2}[_p \cap B_1 = \emptyset$ must hold. As p is projective, only the two options $B_1 = \{{}_{\blacksquare}1, {}_{\blacksquare}\frac{n}{2}\}$ and $B_1 = \{{}^{\blacksquare}1, {}_{\blacksquare}1, {}_{\blacksquare}\frac{n}{2}, {}^{\blacksquare}\frac{n}{2}\}$ remain. Of these, the first is excluded by B_1 being a through block by Lemma 6.5. We have thus shown the first half of the claim.

If $n = 4$, then the two blocks in the assertion coincide and there is nothing left to show. Hence, let $n > 4$ from now on. It follows $]{}_{\blacksquare}1, {}_{\blacksquare}\frac{n}{2}[_p \neq \varnothing$. Since also $\{{}_{\blacksquare}1, {}_{\blacksquare}\frac{n}{2}\} \subseteq B_1$ and $]{}_{\blacksquare}1, {}_{\blacksquare}\frac{n}{2}[_p \cap B_1 \neq \varnothing$, Lemma 6.1 guarantees the existence of $B \in p$ and turn T in p such that $B \neq B_1$ and $T \subseteq]{}_{\blacksquare}1, {}_{\blacksquare}\frac{n}{2}[_p \cap B$. In particular, B has two neighboring legs. Lemma 6.4 thus allows us to conclude $B = \{{}^{\blacksquare}\frac{n}{4}, {}_{\blacksquare}\frac{n}{4}, {}_{\blacksquare}(\frac{n}{4}+1), {}^{\blacksquare}(\frac{n}{4}+1)\}$ because $B \neq B_1 = \{{}^{\blacksquare}1, {}_{\blacksquare}1, {}_{\blacksquare}\frac{n}{2}, {}^{\blacksquare}\frac{n}{2}\}$. That concludes the proof. $\square$

Lemma 6.7. *If $p \in \mathcal{W} \cap \mathcal{M}$ is projective, then p has no pair blocks.*

Proof. We let $B \in p$ and prove $|B| > 2$ by contradiction. Because $p \in \mathcal{W}$ is erasing-minimal and projective, B is a through block by Lemma 6.5. Because p is projective and because we assume $|B| = 2$, there exists $i \in [\![\frac{n}{2}]\!]$ such that $B = \{{}^{\blacksquare}i, {}_{\blacksquare}i\}$.

Because $p \in \mathcal{W}$ is erasing-minimal and projective, $1 \neq \frac{n}{2}$ and $B_1 \coloneqq \{{}^{\blacksquare}1, {}_{\blacksquare}1, {}_{\blacksquare}\frac{n}{2}, {}^{\blacksquare}\frac{n}{2}\} \in p$ by Lemmata 5.11 and 6.5. We infer $B \neq B_1$ because $|B_1| = 4 \neq 2 = |B|$. In consequence, $1 < i < \frac{n}{2}$ because $B \cap B_1 = \varnothing$. Thus, B and B_1 cross in p. However, B is a connected component of $p \in \mathcal{W}$ by Lemma 6.3 because $|B| = 2$. $\square$

Lemma 6.8. *If $p \in \mathcal{W} \cap \mathcal{M}$ is projective, any $B \in p$ other than $\{{}^{\blacksquare}1, {}_{\blacksquare}1, {}_{\blacksquare}\frac{1}{2}\|p\|, {}^{\blacksquare}\frac{1}{2}\|p\|\}$ and $\{{}^{\blacksquare}\frac{1}{4}\|p\|, {}_{\blacksquare}\frac{1}{4}\|p\|, {}_{\blacksquare}(\frac{1}{4}\|p\|+1), {}^{\blacksquare}(\frac{1}{4}\|p\|+1)\}$ crosses these two and has four legs.*

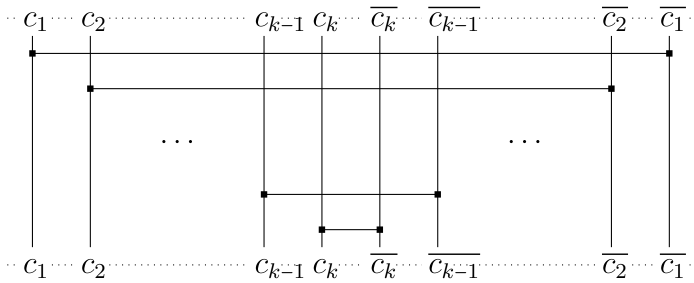
Proof. We can assume $n \coloneqq \|p\| > 0$. As $p \in \mathcal{W} \cap \mathcal{M}$ is projective, $n \in 4\mathbb{N}$ and $B_1 \coloneqq \{{}^{\blacksquare}1, {}_{\blacksquare}1, {}_{\blacksquare}\frac{n}{2}, {}^{\blacksquare}\frac{n}{2}\}$ and $B_2 \coloneqq \{{}^{\blacksquare}\frac{n}{4}, {}_{\blacksquare}\frac{n}{4}, {}_{\blacksquare}(\frac{n}{4}+1), {}^{\blacksquare}(\frac{n}{4}+1)\}$ are blocks of p by Lemma 6.6. For the same reason, B is through by Lemma 6.5. As such, B crosses B_1.

Moreover, $p \in \mathcal{W} \cap \mathcal{M}$ being projective ensures $|B| > 2$ by Lemma 6.7. Since p is projective, it follows that there exist $i_1, i_2 \in [\![\frac{n}{2}]\!]$ such that $i_1 \neq i_2$ and $\{{}^{\blacksquare}i_1, {}_{\blacksquare}i_1, {}_{\blacksquare}i_2, {}^{\blacksquare}i_2\} \subseteq B$. Let i_1 and i_2 be arbitrary with these properties and with $]{}_{\blacksquare}i_1, {}_{\blacksquare}i_2[_p \cap B = \varnothing$. Because we assume $B \neq B_1$ and $B \neq B_2$, Lemma 6.4 guarantees $]{}_{\blacksquare}i_1, {}_{\blacksquare}i_2[_p \neq \varnothing$.

By Lemma 6.1 we find $B' \in p$ and a turn T in p such that $B \neq B'$ and $T \subseteq]{}_{\blacksquare}i_1, {}_{\blacksquare}i_2[_p \cap B'$. Because B' has two neighboring legs, Lemma 6.4 assures us that $B' \in \{B_1, B_2\}$. And $T \subseteq]{}_{\blacksquare}i_1, {}_{\blacksquare}i_2[_p \cap B'$ narrows this down to $B' = B_2$. It follows in particular $i_1 < \frac{n}{4} < \frac{n}{4} + 1 < i_2$. Thus, we have shown that any two subsequent lower legs of B must lie on opposite sides of $T = \{{}_{\blacksquare}\frac{n}{4}, {}_{\blacksquare}(\frac{n}{4}+1)\}$. But this can be true for at most one pair of subsequent legs. Thus, $B = \{{}^{\blacksquare}i_1, {}_{\blacksquare}i_1, {}_{\blacksquare}i_2, {}^{\blacksquare}i_2\}$ since i_1 and i_2 were arbitrary. In particular, B crosses B_2. $\square$

6.1.3. *The Partitions π_c.* We introduce a family of partitions π_c and study some of their properties in preparation for determining $\mathcal{W} \cap \mathcal{R}$.

Definition 6.9. For all $k \in \mathbb{N}$ and $c : [\![k]\!] \to \{\circ, \bullet\}$ define π_c to be the element of $\mathcal{P}^{\circ\bullet}$ which has $2k$ lower and $2k$ upper points, each of whose rows is colored left-to-right $c_1, c_2, \ldots, c_{k-1}, c_k, \overline{c_k}, \overline{c_{k-1}}, \ldots, \overline{c_2}, \overline{c_1}$, and whose blocks are $\{\{{}_{\blacksquare}j, {}^{\blacksquare}j, {}_{\blacksquare}\varrho(j), {}^{\blacksquare}\varrho(j)\} \mid j \in [\![k]\!]\}$, where $\varrho : [\![2k]\!] \to [\![2k]\!], i \mapsto 2k - i + 1$.



LEMMA 6.10. $\{\pi_c \,|\, c \in \bigcup \mathcal{R}\} \subseteq \mathscr{M}$.

PROOF. Let $k \in \mathbb{N}$ and $c : [\![k]\!] \to \{\circ, \bullet\}$, let $\varrho : [\![2k]\!] \to [\![2k]\!], i \mapsto 2k - i + 1$ and for every $j \in [\![k]\!]$ let $B_j \coloneqq \{{}_{\blacksquare}j, {}^{\blacksquare}j, {}_{\blacksquare}\varrho(j), {}^{\blacksquare}\varrho(j)\}$. Let (T, B, α, β) be an action in π_c. We prove $|[\alpha, \beta]_{\pi_c}| \geq 2k$. Because π_c is projective we can assume $T \cap [{}_{\blacksquare}1, {}_{\blacksquare}2k]_{\pi_c} \neq \emptyset$. In fact, since π_c is even bi-projective, no generality is lost in assuming $T \cap [{}_{\blacksquare}1, {}_{\blacksquare}k]_{\pi_c} \neq \emptyset$. Let $j \in [\![k]\!]$ be such that $B = B_j$. We distinguish three cases.

Case 1: Suppose $T = \{{}^{\blacksquare}1, {}_{\blacksquare}1\}$. Since $T \subseteq B_1$ and $T \cap B \neq \emptyset$ this requires $j = 1$. Consequently, $|[\alpha, \beta]_{\pi_c}|$ is one of the numbers $|[{}^{\blacksquare}1, {}^{\blacksquare}2k]_{\pi_c}| = |[{}_{\blacksquare}2k, {}_{\blacksquare}1]_{\pi_c}| = 2k + 2$ or $|[{}^{\blacksquare}1, {}_{\blacksquare}2k]_{\pi_c}| = |[{}_{\blacksquare}1, {}^{\blacksquare}2k]_{\pi_c}| = |[{}^{\blacksquare}2k, {}_{\blacksquare}1]_{\pi_c}| = |[{}_{\blacksquare}2k, {}^{\blacksquare}1]_{\pi_c}| = 2k + 1$ or $|[{}_{\blacksquare}1, {}_{\blacksquare}2k]_{\pi_c}| = |[{}^{\blacksquare}2k, {}^{\blacksquare}1]_{\pi_c}| = 2k$. Thus we have shown $|[\alpha, \beta]_{\pi_c}| \geq 2k$ in this case.

Case 2: If $T = \{{}_{\blacksquare}k, {}_{\blacksquare}(k{+}1)\}$, then $T \subseteq B_k$ requires $j = k$ and a computation analogous to the one in Case 1 yields $|[\alpha, \beta]_{\pi_c}| \geq 2k$.

Case 3: Now, let $T \subseteq [{}_{\blacksquare}1, {}_{\blacksquare}k]_{\pi_c}$. Because $|B_j \cap [{}_{\blacksquare}1, {}_{\blacksquare}k]_{\pi_c}| = 1$ we know $T \nsubseteq B$. And since π_c has no singleton blocks, we must then have $]\beta, \alpha[_p \cap B_j = \emptyset$. Moreover, $\{\alpha, \beta\} \cap T \neq \emptyset$ ensures ${}_{\blacksquare}j \in \{\alpha, \beta\}$. In consequence, $|[\alpha, \beta]_{\pi_c}|$ is one of the numbers $|[{}_{\blacksquare}j, {}^{\blacksquare}j]_{\pi_c}| = 2{\cdot}(2k{-}j{+}1) \geq 2{\cdot}(2k{-}k{+}1) = 2k{+}2$ or $|[{}_{\blacksquare}\varrho(j), {}_{\blacksquare}j]_{\pi_c}| = (2k{-}\varrho(j){+}1){+}2k{+}j = 2k + 2j \geq 2k + 2$. Thus, $|[\alpha, \beta]_{\pi_c}| \geq 2k$ always. That concludes the proof. □

LEMMA 6.11. $\{\pi_{\circ\bullet}, \pi_{\bullet\circ}\} \subseteq \mathscr{R}_3$.

PROOF. For every $c \in \{\circ\bullet, \bullet\circ\}$ the partition π_c is projective. Because the total set of points of π_c is given by $\{{}^{\blacksquare}1, {}_{\blacksquare}1, {}^{\blacksquare}2, {}^{\blacksquare}2\}$ any indefinite turn T of p trivially satisfies $T \cap \{{}^{\blacksquare}1, {}_{\blacksquare}1, {}^{\blacksquare}2, {}^{\blacksquare}2\} \neq \emptyset$. Hence, $\pi_c \in \mathscr{R}_3$. □

LEMMA 6.12. $\{\pi_c \,|\, c \in \bigcup \mathcal{R}\} \subseteq \mathscr{R}_4$.

PROOF. Per definition, π is biprojective. Thus, Lemma 6.10 proves the claim. □

LEMMA 6.13. $\{\pi_c \,|\, c \in \bigcup \mathcal{R}\} \subseteq \mathcal{W}$.

PROOF. Let $k \in \mathbb{N}$ and $c : [\![k]\!] \to \{\circ, \bullet\}$ be arbitrary. For every $j \in [\![k]\!]$ let $B_j \coloneqq \{{}^{\blacksquare}j, {}_{\blacksquare}j, {}_{\blacksquare}\varrho(j), {}^{\blacksquare}\varrho(j)\}$, where $\varrho : [\![2k]\!] \to [\![2k]\!], i \mapsto 2k - i + 1$.

Step 1: *Total color sum.* Because π_c is projective, $\Sigma(\pi_c) = 0$.

Step 2: *Blocks respect* 0*-parts.* By [MW21b, Lemma 3.1], in order to prove $\pi_c \leq \Delta_0 \pi_c$ it suffices to prove for every $j \in [\![k]\!]$ that $\delta_{\pi_c}(\alpha, \beta) = 0$ for all $\alpha, \beta \in B_j$ with $\alpha \neq \beta$ and $]\alpha, \beta[_{\pi_c} \cap B_j = \emptyset$. Since π_c is projective, counterparts always have color distance 0 from each other. For the same reason, $\delta_{\pi_c}({}_{\blacksquare}j, {}_{\blacksquare}\varrho(j)) = -\delta_{\pi_c}({}^{\blacksquare}\varrho(j), {}^{\blacksquare}j)$ for every $j \in [\![k]\!]$. Hence, we can confine ourselves to proving $\delta_{\pi_c}({}_{\blacksquare}j, {}_{\blacksquare}\varrho(j)) = 0$ for every $j \in [\![k]\!]$. Since $\sigma_{\pi_c}(\{{}_{\blacksquare}k, {}_{\blacksquare}(k+1)\}) = 0$ per definition of π_c, trivially, $\delta_{\pi_c}({}_{\blacksquare}k, {}_{\blacksquare}(k+1)) = \sigma_{\pi_c}(\varnothing) = 0$. And for all $j \in [\![k-1]\!]$, because π_c being verticolor-reflexive ensures $\sigma_{\pi_c}([{}_{\blacksquare}\varrho(k), {}_{\blacksquare}\varrho(j)[_{\pi_c}) = -\sigma_{\pi_c}(]{}_{\blacksquare}j, {}_{\blacksquare}k]_p)$, we find

$$\delta_{\pi_c}({}_{\blacksquare}j, {}_{\blacksquare}\varrho(j)) = \sigma_{\pi_c}(]{}_{\blacksquare}j, {}_{\blacksquare}\varrho(j)[_{\pi_c}) = \sigma_{\pi_c}(]{}_{\blacksquare}j, {}_{\blacksquare}k]_p) + \sigma_{\pi_c}([{}_{\blacksquare}\varrho(k), {}_{\blacksquare}\varrho(j)[_{\pi_c}) = 0.$$

Hence, $\pi_c \leq \Delta_0\pi_c$.

Step 3: *Maximal non-interference.* Again, it is enough to show $\sigma_{\pi_c}([\alpha,\gamma]_{\pi_c} \cap B) = 0$ for all $\{A, B\} \subseteq p$ with $A \neq B$ and all $\{\alpha, \gamma\} \subseteq A$ with $\alpha \neq \beta$ and $]\alpha,\gamma[_p \cap A = \varnothing$. For every $i \in [\![k]\!]$ let B_i be the block of ${}_{\blacksquare}i$. Let $\{i, j\} \subseteq [\![k]\!]$ be arbitrary with $i \neq j$. Then, $[{}_{\blacksquare}i, {}_{\blacksquare}\varrho(i)]_{\pi_c} \cap B_j = \varnothing$ if $j < i$ and $[{}_{\blacksquare}i, {}_{\blacksquare}\varrho(i)]_{\pi_c} \cap B_j = \{{}_{\blacksquare}j, {}_{\blacksquare}\varrho(j)\}$ if $i < j$. Similarly, $[{}^{\blacksquare}i, {}_{\blacksquare}i]_{\pi_c} \cap B_j = \{{}^{\blacksquare}j, {}_{\blacksquare}j\}$ if $j < i$ and $[{}^{\blacksquare}i, {}_{\blacksquare}i]_{\pi_c} \cap B_j = \varnothing$ if $i < j$. Hence, $\sigma_{\pi_c}([{}_{\blacksquare}i, {}_{\blacksquare}\varrho(i)]_{\pi_c} \cap B_j) = \sigma_{\pi_c}([{}^{\blacksquare}i, {}_{\blacksquare}i]_{\pi_c} \cap B_j) = 0$. Because p is biprojective, $\sigma_{\pi_c}([{}_{\blacksquare}i, {}_{\blacksquare}\varrho(i)]_{\pi_c} \cap B_j) = \sigma_{\pi_c}([{}^{\blacksquare}\varrho(i), {}^{\blacksquare}i]_{\pi_c} \cap B_j)$ and $\sigma_{\pi_c}([{}^{\blacksquare}i, {}_{\blacksquare}i]_{\pi_c} \cap B_j) = \sigma_{\pi_c}([{}_{\blacksquare}\varrho(i), {}^{\blacksquare}\varrho(i)]_{\pi_c} \cap B_j)$. Thus, B_i and B_j are mutually non-interferent. That concludes the proof. □

6.1.4. *Determining* $\mathcal{W} \cap \mathcal{R}$. With our preparatory results at hand we are now in a position to find a generator of $\mathcal{W}$.

Lemma 6.14. $\mathcal{W} \cap \mathcal{R}_1 = \varnothing$.

Proof. Per definition, $\varnothing \notin \mathcal{R}_1$. By Lemma 6.2 any $p \in \mathcal{W}\backslash\{\varnothing\}$ has turns. However, $p \in \mathcal{R}_1$ would demand that p had none. Hence, $\mathcal{W} \cap \mathcal{R}_1 = \varnothing$. □

Lemma 6.15. $\mathcal{W} \cap \mathcal{R}_2 = \varnothing$.

Proof. We assume that there exists $p \in \mathcal{W} \cap \mathcal{R}_2$ and deduce a contradiction. Per definition, $p \in \mathcal{R}_2$ requires that $p \in \mathcal{M}\backslash\{\varnothing\}$ and that p has no indefinite turns. Since $p \neq \varnothing$, the assumption $p \in \mathcal{W}$ implies by Lemma 6.2 that there exist $B \in p$ and a turn T in p with $T \subseteq B$. However, because $\mathcal{W}$ is a hyperoctahedral category by Theorem 4.7 and because $p \in \mathcal{M}$, Lemma 5.15 guarantees exactly that such B and T may not exist – a contradiction. In conclusion, $p \notin \mathcal{R}_2$ as claimed. □

Lemma 6.16. $\mathcal{W} \cap \mathcal{R}_3 \subseteq \{\pi_{\circ\bullet}, \pi_{\bullet\circ}\}$.

Proof. Recall $\varnothing \notin \mathcal{R}_3$. Let $p \in \mathcal{W} \cap \mathcal{R}_3$ and $n \coloneqq \|p\|$. Since $p \in (\mathcal{W} \cap \mathcal{M})\backslash\varnothing$ is then projective, $n \in 4\mathbb{N}$ and both $B_1 \coloneqq \{{}^{\blacksquare}1, {}_{\blacksquare}1, {}_{\blacksquare}\frac{n}{2}, {}^{\blacksquare}\frac{n}{2}\} \in p$ and $B_2 \coloneqq \{{}^{\blacksquare}\frac{n}{4}, {}_{\blacksquare}\frac{n}{4}, {}_{\blacksquare}(\frac{n}{4}+1), {}^{\blacksquare}(\frac{n}{4}+1)\} \in p$ by Lemma 6.6.

Step 1: *If* $n = 4$*, then* $p = \pi_{\circ\bullet}$ *or* $p = \pi_{\bullet\circ}$. If $n = 4$, then $B_1 = \{{}^{\blacksquare}1, {}_{\blacksquare}1, {}_{\blacksquare}2, {}^{\blacksquare}2\}$ is the only block of p. Proposition 4.13 ensures that ${}_{\blacksquare}1$ and ${}_{\blacksquare}2$ have opposite normalized colors because $p \in \mathcal{W}$ and $]{}_{\blacksquare}1, {}_{\blacksquare}2[_p \cap B_1 = \varnothing$. That is the same as saying $p \in \{\pi_{\circ\bullet}, \pi_{\bullet\circ}\}$ since p is projective.

Step 2: *Proving* $n = 4$. We show $B_1 = B_2$, which is the same as $n = 4$. By $p \in \mathcal{W}$, by $]{}_{\blacksquare}\frac{n}{4}, {}_{\blacksquare}(\frac{n}{4}+1)[_p \cap B_2 = \varnothing$ and by Proposition 4.13 the set $T \coloneqq \{{}_{\blacksquare}\frac{n}{4}, {}_{\blacksquare}(\frac{n}{4}+1)\}$ is a turn in p. Because $T \subseteq B_2$ an action in p is given by $(T, B_2, {}^{\blacksquare}\frac{n}{4}, {}_{\blacksquare}\frac{n}{4})$. The assumption

$p \in \mathscr{M}$ therefore implies $\frac{n}{2} = 2 \cdot \frac{n}{4} = |[{}^{\blacksquare}\frac{n}{4}, {}_{\blacksquare}\frac{n}{4}]_p| \geq \frac{n}{2}$ and thus $|[{}^{\blacksquare}\frac{n}{4}, {}_{\blacksquare}\frac{n}{4}]_p| = \frac{n}{2}$. It follows that $(T, B, {}^{\blacksquare}\frac{n}{4}, {}_{\blacksquare}\frac{n}{4})$ is a size-preserving action and thus T an indefinite turn. Per assumption $p \in \mathscr{R}_3$, this forces $T \cap B_1 \neq \emptyset$. Because $T \subseteq B_2$ this only leaves the conclusion $B_1 = B_2$. That is what we needed to see. □

LEMMA 6.17. $\mathcal{W} \cap \mathscr{R}_4 \subseteq \{\pi_c \,|\, c \in \bigcup \mathcal{R}\}$.

PROOF. Again, $\emptyset \notin \mathscr{R}_4$. Let $p \in \mathcal{W} \cap \mathscr{R}_4$ be arbitrary and $n \coloneqq \|p\|$. Because $p \in (\mathcal{W} \cap \mathscr{M}) \backslash \{\emptyset\}$ is in particular projective, $n \in 4\mathbb{N}$ by Lemma 6.6. Let $c : [\![\frac{n}{4}]\!] \to \{\circ, \bullet\}$ be such that for every $i \in [\![\frac{n}{4}]\!]$ the point ${}_{\blacksquare}i$ has color c_i in p. We prove $p = \pi_c$.

Step 1: *Coloring of* p. Since p is projective, the colorings of the lower and of the upper row agree. And because $p \in \mathscr{R}_4$ is per assumption bi-projective, the lower row of p is colored $c_1, c_2, \ldots, c_{\frac{n}{4}}, \overline{c_{\frac{n}{4}}}, \overline{c_{\frac{n}{4}-1}}, \ldots, \overline{c_1}$ left to right. Thus we have shown that p and π_c have identical colorings.

Step 2: *Blocks of* p. Let $\varrho : [\![\frac{n}{2}]\!] \to [\![\frac{n}{2}]\!], i \mapsto \frac{n}{2} - i + 1$ and for every $j \in [\![\frac{n}{4}]\!]$ let $B_j \in p$ and ${}_{\blacksquare}j \in B_j$. We let $j \in [\![\frac{n}{4}]\!]$ be arbitrary and prove $B_j = \{{}^{\blacksquare}j, {}_{\blacksquare}j, {}_{\blacksquare}\varrho(j), {}^{\blacksquare}\varrho(j)\}$.

Case 2.1: According to Lemma 6.6 both the sets $\{{}^{\blacksquare}1, {}_{\blacksquare}1, {}_{\blacksquare}\frac{n}{2}, {}^{\blacksquare}\frac{n}{2}\}$ and $\{{}^{\blacksquare}\frac{n}{4}, {}_{\blacksquare}\frac{n}{4}, {}_{\blacksquare}(\frac{n}{4}+1), {}^{\blacksquare}(\frac{n}{4}+1)\}$ are blocks of p. That proves the claim about B_j for $j \in \{1, \frac{n}{4}\}$.

Case 2.2: Now let $1 < j < \frac{n}{4}$. Because $p \in \mathcal{W} \cap \mathscr{M}$ is projective, B_j is a through block by Lemma 6.5. Hence, in particular, $\{{}^{\blacksquare}j, {}_{\blacksquare}j\} \subseteq B_j$. Because $1 < j < \frac{n}{4}$ ensures $B_1 \neq B_j \neq B_{\frac{n}{4}}$ by the preceding case, Lemma 6.8 tells us that $|B_j| = 4$. In combination with p being projective, this implies that there exists $i \in [\![\frac{n}{2}]\!]$ such that $i \neq j$ and $B = \{{}^{\blacksquare}j, {}_{\blacksquare}j, {}_{\blacksquare}i, {}^{\blacksquare}i\}$. Lemma 6.8 also guarantees that B_j crosses B_1 and $B_{\frac{n}{4}}$. In particular, B_j crosses the vertical middle axis. Since p is bi-projective this latter fact forces B_j to be symmetric with respect to the vertical axis. Especially, from ${}_{\blacksquare}j \in B_j$ we are allowed to conclude ${}_{\blacksquare}\varrho(j) \in B_j$. Because $j \neq \varrho(j)$ and $i \neq j$ this requires $i = \varrho(j)$. And that is what we needed to show. □

PROPOSITION 6.18. $\mathcal{W} \cap \mathscr{R} = \{\pi_c \,|\, c \in \bigcup \mathcal{R}\}$.

PROOF. That is the combined result of Lemmata 6.10–6.17. □

THEOREM 6.19. $\mathcal{W} = \langle \pi_c \,|\, c \in \bigcup \mathcal{R} \rangle$.

PROOF. Follows by Proposition 5.14. □

6.2. Generators of $\mathcal{W}_R$. Knowing $\mathcal{W} = \langle \pi_c \,|\, c \in \bigcup \mathcal{R} \rangle$, we determine which subsets of $\{\pi_c \,|\, c \in \bigcup \mathcal{R}\}$ generate $\mathcal{W}_R \subseteq \mathcal{W}$ for each $R \in \mathcal{R}$.

DEFINITION 6.20. For all $p \in \mathcal{W}$ let $w(p)$ be the set of all c such that there exist $n \in \mathbb{N}$ and $\{B_1, B_2, \ldots, B_n\} \subseteq p$ such that $c : [\![n]\!] \to \{\circ, \bullet\}$, such that $2 \leq n$, such that $B_1, B_2, \ldots, B_n$ are pairwise distinct, such that B_1 and B_n cross in p, such that $\langle B_1, B_n \rangle_p = \{B_1, B_2, \ldots, B_n\}$, such that $B_1 \leq B_2 \leq \ldots \leq B_n$ with respect to $\leq_{p, B_1, B_n}$ and such that $\lambda_p(B_1, B_i, B_n) = \sigma(c_i)$ for all $i \in [\![n]\!]$.

LEMMA 6.21. *Let $R \in \mathcal{R}$, let $k \in \mathbb{N}$, let $2 \leq k$ and let $c : [\![k]\!] \to \{\circ, \bullet\}$. Then,* $w(\pi_c) = \{(c_i, c_{i+1}, \ldots, c_j), (\overline{c_j}, \overline{c_{j-1}}, \ldots, \overline{c_i}) \,|\, \{i, j\} \subseteq [\![n]\!], \, i < j\}$.

Proof. Let $\varrho : [\![2k]\!] \to [\![2k]\!]$, $i \mapsto 2k - i + 1$ be the reflection and for every $i \in [\![k]\!]$ let $F_k = \{{}^{\blacksquare}i, {}_{\blacksquare}i, {}_{\blacksquare}\varrho(i), {}^{\blacksquare}\varrho(i)\}$ be the block of ${}_{\blacksquare}i$ in π_c. Abbreviate $S_c \coloneqq \{(c_i, c_{i+1}, \ldots, c_j), (\overline{c_j}, \overline{c_{j-1}}, \ldots, \overline{c_i}) \mid i, j \in [\![n]\!],\ i < j\}$. We prove $w(\pi_c) = S_c$.

Step 1: *Showing* $S_c \subseteq w(\pi_c)$. Let $i, j \in [\![n]\!]$ satisfy $i < j$. We show $(c_i, c_{i+1}, \ldots, c_j) \in w(\pi_c)$ and $(\overline{c_j}, \overline{c_{j-1}}, \ldots, \overline{c_i}) \in w(\pi_c)$.

Step 1.1: *Shortenings.* We begin with the claim $(c_i, c_{i+1}, \ldots, c_j) \in w(\pi_c)$. The blocks F_i and F_j cross each other in π_c since $({}_{\blacksquare}i, {}_{\blacksquare}j, {}^{\blacksquare}\varrho(i), {}^{\blacksquare}\varrho(j))$ is ordered there.

For every $\ell \in [\![k]\!]$ the intersection $[{}_{\blacksquare}i, {}_{\blacksquare}j]_p \cap F_\ell$ is empty if $\ell < i$ or $j < \ell$ and it is $\{{}_{\blacksquare}\ell\}$ otherwise. By Lemma 4.15 we have thus shown for every $\ell \in [\![k]\!] \backslash \{i, j\}$ that $(F_i, F_\ell, F_j) \in \chi_{\pi_c}$ if and only if $i < \ell < j$. Lemma 4.20 then allows us to conclude $\langle F_i, F_j \rangle_p = \{F_i, F_{i+1}, \ldots, F_j\}$.

Moreover, for all $\ell \in [\![k]\!]$ with $i \leq \ell < j$ we can infer $F_\ell < F_{\ell+1}$ with respect to $\leq_{\pi_c, F_i, F_j}$ from $[{}_{\blacksquare}i, {}_{\blacksquare}(\ell+1)]_{\pi_c} \cap F_\ell = \{{}_{\blacksquare}\ell\}$. Hence, $F_i \leq F_{i+1} \leq \ldots \leq F_j$ in the order $\leq_{\pi_c, F_i, F_j}$.

Finally, $\lambda_{\pi_c}(F_i, F_\ell, F_j) = \sigma_{\pi_c}([{}_{\blacksquare}i, {}_{\blacksquare}j]_{\pi_c} \cap F_\ell) = \sigma_{\pi_c}(\{{}_{\blacksquare}\ell\}) = \sigma(c_\ell)$ for every $i \in [\![n]\!]$ with $i \leq \ell \leq j$. Thus, we have established $(c_i, c_{i+1}, \ldots, c_j) \in w(\pi_c)$.

Step 1.2: *Verticolor reflections.* Lemmata 4.29 and 4.23 (b) allow us to draw the following conclusions from Step 1.1: The crossing blocks F_j and F_i of π_c satisfy $\langle F_j, F_i \rangle_{\pi_c} = \{F_j, F_{j-1}, \ldots, F_i\}$ and $F_j \leq F_{j-1} \leq \ldots \leq F_i$ with respect to $\leq_{p, F_j, F_i}$ and $\lambda_{\pi_c}(F_j, F_\ell, F_i) = -\sigma(c_\ell) = \sigma(\overline{c_\ell})$ for every $\ell \in [\![k]\!]$ with $i \leq k \leq j$. That proves $(\overline{c_j}, \overline{c_{j-1}}, \ldots, \overline{c_i}) \in w(\pi_c)$, which is what was left to be proven.

Step 2: *Proving* $w(\pi_c) \subseteq S_c$. To see the converse inclusion let $n \in \mathbb{N}$, let $2 \leq n$, let $\{B_1, B_2, \ldots, B_n\} \subseteq \pi_c$, let $B_1, B_2, \ldots, B_n$ be pairwise distinct, let B_1 and B_n cross in π_c, let $\langle B_1, B_n \rangle_{\pi_c} = \{B_1, B_2, \ldots, B_n\}$, let $B_1 \leq B_2 \leq \ldots \leq B_n$ with respect to $\leq_{\pi_c, B_1, B_n}$ and for every $j \in [\![n]\!]$ let $d_j \in \{\circ, \bullet\}$ be such that $\lambda_{\pi_c}(B_1, B_j, B_n) = \sigma(d_j)$. We prove $(d_1, d_2, \ldots, d_n) \in S_c$.

Step 2.1: *Defining* $i_1, i_2, \ldots, i_n$. Because π_c only has k many blocks, $n \leq k$. For every $j \in [\![n]\!]$ there exists a unique $i_j \in [\![k]\!]$ such that $B_j = F_{i_j}$. In particular, $i_1, i_2, \ldots, i_n$ are pairwise distinct and $j \mapsto i_j$ is bijective.

Step 2.2: *We can assume* $i_1 < i_n$. If $i_n < i_1$, we rename $B_j \leftrightarrow B_{n-j+1}$ for every $j \in [\![n]\!]$; which, by Lemmata 4.29 and 4.23 (b), corresponds to replacing $(d_1, d_2, \ldots, d_n)$ by $(\overline{d_n}, \overline{d_{n-1}}, \ldots, \overline{d_1})$. Because, by definition of S_c, we have $(d_1, d_2, \ldots, d_n) \in S_c$ if and only if $(\overline{d_n}, \overline{d_{n-1}}, \ldots, \overline{d_1}) \in S_c$, no generality is lost. Hence, let $i_1 < i_n$ henceforth.

Step 2.3: *Proving* $i_1 < i_2, \ldots, i_{n-1} < i_n$. Let $j \in [\![n]\!]$ with $1 < j < n$ be arbitrary. We prove $i_1 < i_j < i_n$. Because the blocks F_{i_1}, F_{i_j} and F_{i_n} are pairwise distinct, Lemma 4.15 allows us to conclude from $(F_{i_1}, F_{i_j}, F_{i_n}) \in \chi_{\pi_c}$ that $\sigma_{\pi_c}([{}_{\blacksquare}i_1, {}_{\blacksquare}i_n]_{\pi_c} \cap F_{i_j}) \neq 0$, implying in particular $[{}_{\blacksquare}i_1, {}_{\blacksquare}i_n]_{\pi_c} \cap F_{i_j} \neq \varnothing$. By definition of π_c this latter condition can only be satisfied if $i_1 < i_j < i_n$.

Step 2.4: *Proving* $i_1 < i_2 < \ldots < i_n$. We let $\{j, j'\} \subseteq [\![n]\!]$ with $j < j'$ be arbitrary and show $i_j < i_{j'}$. By Steps 2.2 and 2.3 we can assume $1 < j$ and $j' < n$. Then, in particular, F_{i_1}, F_{i_j} and $F_{i_{j'}}$ are pairwise distinct. Once more we can therefore apply Lemma 4.15 to deduce $\sigma_{\pi_c}([{}_{\blacksquare}i_1, {}_{\blacksquare}i_{j'}]_{\pi_c} \cap F_{i_j}) \neq 0$ from the assumption $(F_{i_1}, F_{i_j}, F_{i_{j'}}) \in$

χ_{π_c}. Again, $[{}_{\blacksquare}i_1, {}_{\blacksquare}i_{j'}]_{\pi_c} \cap F_{i_j} \neq \varnothing$ then demands $i_1 < i_j < i_{j'}$, which is what we needed to see.

Step 2.5: *$i_1, i_2, \ldots, i_n$ are consecutive.* Next, we prove $i_j = i_1 + j - 1$ for every $j \in [\![n]\!]$. Let $j \in [\![n-1]\!]$ be arbitrary. Then, $i_j < i_{j+1}$ by Step 2.4. We prove by contradiction that there exists no $i \in [\![k]\!]$ such that $i_j < i < i_{j+1}$. For such an i it would hold that $[{}_{\blacksquare}i_1, {}_{\blacksquare}i_n]_{\pi_c} \cap F_i = \{{}_{\blacksquare}i\}$, that $[{}_{\blacksquare}i_1, {}_{\blacksquare}i]_{\pi_c} \cap F_{i_j} = \{{}_{\blacksquare}i_j\}$ and that $[{}_{\blacksquare}i_1, {}_{\blacksquare}i_{j+1}]_{\pi_c} = \{{}_{\blacksquare}i\}$. That would imply $(F_{i_1}, F_i, F_{i_n}) \in \chi_{\pi_c}$ and $(F_{i_1}, F_{i_j}, F_i) \in \chi_{\pi_c}$ and $(F_{i_1}, F_i, F_{i_{j+1}}) \in \chi_{\pi_c}$. In other words, $F_i \in \langle B_1, B_n \rangle_{\pi_c}$ and $B_j \leq F_i \leq B_{j+1}$ with respect to $\leq_{\pi_c, B_1, B_n}$. And this would contradict the assumptions that $\langle B_1, B_n \rangle_p = \{B_1, B_2, \ldots, B_n\}$ and that $B_1 \leq B_2 \leq \ldots \leq B_n$ in $\leq_{\pi_c, B_1, B_n}$. Thus, indeed, $i_1, \ldots, i_n$ are consecutive.

Step 2.6: *Determining $d_1, \ldots, d_n$.* Finally, we conclude this Step 2 by showing $(d_1, d_2, \ldots, d_n) = (c_{i_1}, c_{i_1+1}, \ldots, c_{i_1+n-1})$. Because $[{}_{\blacksquare}i_1, {}_{\blacksquare}i_n]_{\pi_c} \cap F_{i_1} = \{{}_{\blacksquare}i_1\}$ and, likewise, $[{}_{\blacksquare}i_1, {}_{\blacksquare}i_n]_{\pi_c} \cap F_{i_n} = \{{}_{\blacksquare}i_n\}$, we can conclude $d_1 = c_{i_1}$ and $d_n = c_{i_n}$. And for every $j \in [\![n]\!] \backslash \{1, n\}$, since $i_1 < i_j < i_n$ by Step 2.3, we can infer $\sigma(d_j) = \sigma_{\pi_c}([{}_{\blacksquare}i_1, {}_{\blacksquare}i_n]_{\pi_c} \cap F_{i_j}) = \sigma_{\pi_c}(\{{}_{\blacksquare}i_j\}) = \sigma(c_{i_j})$. It follows $(d_1, d_2, \ldots, d_n) = (c_{i_1}, c_{i_2}, \ldots, c_{i_n})$. Now, Step 2.5 ensures the claim $(d_1, d_2, \ldots, d_n) = (c_{i_1}, c_{i_1+1}, \ldots, c_{i_1+n-1})$, implying $(d_1, d_2, \ldots, d_n) \in S_c$. That concludes the proof. □

LEMMA 6.22. *Let $R \in \mathcal{R}$ and $c \in \bigcup \mathcal{R}$. Then, $\pi_c \in \mathcal{W}_R$ if and only if $c \in R$.*

PROOF. Let $k \in \mathbb{N}$ be such that $c : [\![k]\!] \to \{\circ, \bullet\}$. If $k = 1$, and thus $c \in \{\circ, \bullet\}$, then $c \in R$ and $\pi_c \in \mathcal{W}_R$ are both true by definition. Hence, we can assume $k > 1$.

If we define $S_c := \{(c_i, c_{i+1}, \ldots, c_j), (\overline{c_j}, \overline{c_{j-1}}, \ldots, \overline{c_i}) \mid i, j \in [\![n]\!],\ i < j\}$, then $w(\pi_c) = S_c$ by Lemma 6.21 because $2 \leq k$.

By definition of $w(\pi_c)$ and $\mathcal{W}_R$ the statements $\pi_c \in \mathcal{W}_R$ and $w(\pi_c) \subseteq R$ are equivalent. Because R is a $\mathcal{W}$-parameter, whenever $c \in R$, then also $S_c \subseteq R$. Conversely, $c \in R$ is implied by $S_c \subseteq R$ since $c \in S_c$. Hence, the statements $c \in R$ and $S_c \subseteq R$ are equivalent.

In conclusion, what we claim is that $w(\pi_c) \subseteq R$ if and only if $S_c \subseteq R$. Since $w(\pi_c) = S_c$ that is all we needed to see. □

PROPOSITION 6.23. *$\mathcal{W}_R \cap \mathscr{R} = \{\pi_c \mid c \in R\}$ for every $\mathcal{W}$-parameter set R.*

PROOF. Proposition 6.18 showed $\mathcal{W} \cap \mathscr{R} = \{\pi_c \mid c \in \bigcup \mathcal{R}\}$. As, by definition, $\mathcal{W}_R \subseteq \mathcal{W} = \mathcal{W}_{\bigcup \mathcal{R}}$, it follows $\mathcal{W}_R \cap \mathscr{R} = (\mathcal{W} \cap \mathscr{R}) \cap \mathcal{W}_R = \{\pi_c \mid c \in \bigcup \mathcal{R}\} \cap \mathcal{W}_R = \{\pi_c \mid c \in R\}$, where the last identity is due to Lemma 6.22. □

THEOREM 6.24. *$\mathcal{W}_R = \langle \pi_c \mid c \in R \rangle$ for any $\mathcal{W}$-parameter set R.*

PROOF. Follows immediately from Propositions 5.14 and 6.23. □

7. Distinctness

We can immediately deduce from the results of the preceding Section 6 that the categories $(\mathcal{W}_R)_{R \in \mathcal{R}}$ are pairwise distinct.

THEOREM 7.1. *$\mathcal{W}_{R_1} \neq \mathcal{W}_{R_2}$ for any two $\mathcal{W}$-parameter sets R_1 and R_2 with $R_1 \neq R_2$.*

Proof. The assumption $R_1 \neq R_2$ implies $\{\pi_c \,|\, c \in R_1\} \neq \{\pi_c \,|\, c \in R_2\}$ since the mapping $c \mapsto \pi_c$ is a bijection $\bigcup \mathcal{R} \to \{\pi_c \,|\, c \in \bigcup \mathcal{R}\}$. Proposition 6.23 therefore allows us to infer $\mathcal{W}_{R_1} \cap \mathscr{R} \neq \mathcal{W}_{R_2} \cap \mathscr{R}$. And this requires $\mathcal{W}_{R_1} \neq \mathcal{W}_{R_2}$. □

8. Uniqueness

In this section we prove that every hyperoctahedral subcategory of $\mathcal{W}$ is of the form $\mathcal{W}_R$ for some $R \in \mathcal{R}$ (Theorem 8.5).

Lemma 8.1. *Let $n \in \mathbb{N}$, let $c : [\![n]\!] \to \{\circ, \bullet\}$ and let $p \in \mathcal{W}$. If $c \in w(p)$ and $4n < \|p\|$, then there exists $p' \in \langle p \rangle$ with $c \in w(p')$ and $\|p'\| < \|p\|$.*

Proof. Per assumption there exist $\{B_1, B_2, \ldots, B_n\} \subseteq p$ such that $B_1, B_2, \ldots, B_n$ are pairwise disjoint, such that B_1 and B_n cross in p, such that $\langle B_1, B_n \rangle_p = \{B_1, B_2, \ldots, B_n\}$, such that $B_1 \leq B_2 \leq \ldots \leq B_n$ with respect to $\leq_{p,B_1,B_n}$ and such that $\sigma(c_i) = \lambda_p(B_1, B_i, B_n)$ for every $i \in [\![n]\!]$. We prove in three steps that there exists a turn T in p such that $c \in w(E(p,T))$. As $E(p,T) \in \langle p \rangle$ and $\|E(p,T)\| = \|p\| - |T|$, that is enough to prove the claim.

Step 1: *Sizes of the blocks.* We prove for every $i \in [\![n]\!]$ that $4 \leq |B_i|$ and that, whenever $4 < |B_i|$, then $6 \leq |B_i|$. Since B_1 and B_n cross in p and since $\langle B_1, B_n \rangle_p = \{B_1, B_2, \ldots, B_n\}$, Lemma 4.19 informs us that $B_1, B_2, \ldots, B_n$ all pairwise cross each other. In particular, none of them are connected components. For that reason, by Lemma 6.3, none are pair blocks either. Moreover, each of $B_1, B_2, \ldots, B_n$, being a neutral set by $p \in \mathcal{W}$, has an even number of legs.

Step 2: *Auxiliary claim:* Next, we show that, if T is a turn in p with $T \subseteq B$ for some block B of p and if the sets $B_1 \backslash T$ and $B_n \backslash T$ cross each other in p, then $c \in w(E(p,T))$.

Since $4 \leq |B_i|$ for every $i \in [\![4]\!]$ by Step 1, none of $B_1, B_2, \ldots, B_n$ are contained in T. Hence, for every $i \in [\![n]\!]$ there exists $B_i' \in E(p,T)$ which is not case $\mathrm{E}_{\mathrm{III}}$ such that B_i is a parent of B_i' with respect to (p,T). The assumption that $B_1 \backslash T$ and $B_n \backslash T$ cross each other in p ensures that B_1' and B_n' cross in $E(p,T)$. Hence, if we show that $\langle B_1', B_n' \rangle_{E(p,T)} = \{B_1', \ldots, B_n'\}$, that $B_1' \leq B_2' \leq \ldots \leq B_n'$ with respect to $\leq_{E(p,T),B_1',B_n'}$ and that $\lambda_{E(p,T)}(B_1', B_i', B_n') = \sigma(c_i)$ for every $i \in [\![n]\!]$, then $c \in w(E(p,T))$ will have been proven.

Step 1.1: *The blocks.* We begin by showing $\langle B_1', B_n' \rangle_{E(p,T)} = \{B_1', B_2', \ldots, B_n'\}$. Let $F \in E(p,T)$ and let $\{F_1, F_2\}$ be parents of F with respect to $E(p,T)$. Per assumption on T, the block F is not case $\mathrm{E}_{\mathrm{III}}$. In particular, $F_1 = F_2$. Hence, $F \in \langle B_1', B_n' \rangle_{E(p,T)}$ if and only if $F_1 \in \langle B_1, B_n \rangle_p$ by Lemma 4.40 (a) (i). Consequently, the assumption $\langle B_1, B_n \rangle_p = \{B_1, B_2, \ldots, B_n\}$ ensures that $F \in \langle B_1', B_n' \rangle_{E(p,T)}$ if and only if $F_1 \in \{B_1, B_2, \ldots, B_n\}$. And the latter statement is equivalent to $F \in \{B_1', B_2', \ldots, B_n'\}$, which proves the claim.

Step 1.2: *Their ordering.* For every $i \in [\![n-1]\!]$ the assumption that $B_i \leq B_{i+1}$ with respect to $\leq_{p,B_1,B_n}$, i.e., that $(B_1, B_i, B_{i+1}) \in \chi_p$ by Lemma 4.40 (a) (i) ensures $(B_1', B_i', B_{i+1}') \in \chi_{E(p,T)}$ because B_i' is not case $\mathrm{E}_{\mathrm{III}}$. Hence, $B_1' \leq B_2' \leq \ldots \leq B_n'$ with respect to $\leq_{E(p,T),B_1',B_n'}$.

Step 1.3: *The colors.* Finally, for every $i \in [\![n]\!]$, since B'_i is not case $\mathrm{E}_{\mathrm{III}}$, we can conclude $\lambda_{E(p,T)}(B'_1, B'_i, B'_n) = \lambda_p(B_1, B_i, B_n) = \sigma(c_i)$ by Lemma 4.40 (a) (ii). That concludes the proof of the auxiliary claim.

Step 2: *Existence of T.* By Step 2 it suffices to find a turn T in p and a block B of p with $T \subseteq B$ such that $B_1 \backslash T$ and $B_2 \backslash T$ cross each other in p.

Per assumption, B_1 and B_n cross in p. Hence, there exist legs $\{\beta_{1,1}, \beta_{1,2}\} \subseteq B_1$ and $\{\beta_{n,1}, \beta_{n,2}\} \subseteq B_n$ such that $(\beta_{1,1}, \beta_{n,1}, \beta_{1,2}, \beta_{n,2})$ is an ordered tuple of pairwise distinct points of p. Let $(\alpha_1, \gamma_1) \coloneqq (\beta_{1,1}, \beta_{1,n})$, let $(\alpha_2, \gamma_2) \coloneqq (\beta_{1,n}, \beta_{1,2})$, let $(\alpha_3, \gamma_3) \coloneqq (\beta_{1,2}, \beta_{n,2})$, let $(\alpha_4, \gamma_4) \coloneqq (\beta_{n,2}, \beta_{1,1})$ and for every $j \in [\![4]\!]$ let $S_j \coloneqq]\alpha_j, \gamma_j[_p$.

Step 2.1: *Sufficient criterion for existence of T and B.* We prove that it suffices to show that there exist $B_0 \in p$ and $j \in [\![4]\!]$ such that $2 \leq |B_0 \cap S_j|$.

Indeed, if so, let $\leq$ be the total order induced on S_j by the cyclic order of p. Then, the premise $2 \leq |B_0 \cap S_j|$ ensures that $\theta_1 \coloneqq \min_{\leq}(B_0 \cap S_j)$ and $\theta_2 \coloneqq \min_{\leq}(B_0 \cap]\theta_1, \gamma_j[_p)$ are well-defined legs of B_0 with $\theta_1 \neq \theta_2$.

If $]\theta_1, \theta_2[_p = \varnothing$, then $T \coloneqq \{\theta_1, \theta_2\}$ is a turn in p with $T \subseteq B \coloneqq B_0$ by Proposition 4.13 since $p \in \mathcal{W}$. If, alternatively, $]\theta_1, \theta_2[_p \neq \varnothing$, then, since $]\theta_1, \theta_2[_p \cap B_0 = \varnothing$, Lemma 6.1 guarantees the existence of $B \in p$ and a turn T in p with $T \subseteq]\theta_1, \theta_2[_p \cap B$.

And because $T \subseteq S_j$ we can rest assured that $T \cap \{\beta_{1,1}, \beta_{1,2}, \beta_{n,1}, \beta_{n,2}\} = \varnothing$, implying that $B_1 \backslash T$ and $B_2 \backslash T$ cross in p. That proves the sufficient criterion.

Step 2.2: *Case distinctions.* Now, we consider separately three situations we might encounter. In most of them we can apply the criterion of Step 2.1 to show the existence of T and B.

Case 2.2.1: *Other blocks exist.* First, suppose that there exists $B_0 \in p$ with $B_0 \notin \{B_1, B_2, \ldots, B_n\}$. Then, there is at least one $j \in [\![4]\!]$ such that $B_0 \cap S_j \neq \varnothing$. Since $\langle B_1, B_n \rangle_p = \{B_1, \ldots, B_n\}$ and $B_0 \notin \{B_1, \ldots, B_n\}$ we know $(B_1, B_0, B_n) \notin \chi_p$. Lemma 4.23 (a) further guarantees $(B_n, B_0, B_1) \notin \chi_p$. It follows $|B_0 \cap S_j| = |[\alpha_j, \gamma_j]_p \cap B_0| \equiv_2 0$ according to Lemma 4.18 (a) as $(\alpha_j, \gamma_j) \in (B_1 \times B_n) \cup (B_n \times B_1)$. Because of our assumption $B_0 \cap S_j \neq \varnothing$ we can conclude $|B_0 \cap S_j| \geq 2$, which, by Step 2.1, is what we needed to see.

Case 2.2.2: *One of the middle blocks has extra legs.* Next, let there exist $i_0 \in [\![n]\!]$ with $1 < i_0 < n$ such that $4 < |B_{i_0}|$. Per assumption, $(B_1, B_{i_0}, B_n) \in \chi_p$. Hence, also $(B_n, B_{i_0}, B_1) \in \chi_p$ by Lemma 4.23 (a). Therefore and because B_1, B_{i_0} and B_n are pairwise distinct, Lemma 4.18 (b) lets us conclude $|B_{i_0} \cap S_j| = |[\alpha_j, \gamma_j]_p \cap B_0| \equiv_2 1$ for all $j \in [\![4]\!]$. In particular, $B_{i_0} \cap S_j \neq \varnothing$ for all $j \in [\![4]\!]$. Now, the decomposition $B_{i_0} = \bigcup_{j=1}^{4}(B_0 \cap S_j)$ and the assumption $4 < |B_{i_0}|$ demand the existence of at least one $j \in [\![4]\!]$ with $2 \leq |B_0 \cap S_j|$. That proves the claim for this case according to Step 2.1.

Case 2.2.3: *No other blocks and no extra legs of middle blocks.* The third and final case is that $B_1, B_2, \ldots, B_n$ already constitute all blocks of p and that $|B_i| \leq 4$ for all $i \in [\![n]\!]$ with $1 < i < n$.

By Step 1 we infer that $|B_i| = 4$ for all $i \in [\![n]\!]$ with $1 < i < n$ and that $6 \leq |B_1|$ or $6 \leq |B_n|$. Let $B_0 \in \{B_1, B_n\}$ be arbitrary with $6 \leq |B_0|$. Then, $4 \leq |\bigcup_{j=1}^{4}(B_0 \cap S_j)|$. If there is $j \in [\![4]\!]$ with $2 \leq |B_0 \cap S_j|$, then Step 2.1 proves the claim. Hence, we can assume $|B_0 \cap S_j| = 1$ for all $j \in [\![4]\!]$. Let $\theta \in B_0 \cap S_1$ and define $(\alpha,\gamma) \coloneqq (\beta_{1,1},\theta)$ if $B_0 = B_1$ and $(\alpha,\gamma) \coloneqq (\theta,\beta_{n,1})$ if $B_0 = B_n$. Then, $\{\alpha,\gamma\} \subseteq B_0$ and $\alpha \neq \gamma$ and $]\alpha,\gamma[_p \cap B_0 = \emptyset$ per construction. Two eventualities must be considered.

Case 2.2.3.1: If $]\alpha,\gamma[_p \neq \emptyset$, then, because $]\alpha,\gamma[_p \cap B_0 = \emptyset$, there exists a turn T in p with $T \subseteq]\alpha,\gamma[_p \cap B$ by Lemma 6.1. And since then $T \subseteq S_1$, the blocks $B_1\backslash T$ and $B_2\backslash T$ still have the crossing $(\beta_{1,1},\beta_{n,1},\beta_{1,2},\beta_{1,n})$, proving the claim.

Case 2.2.3.2: If $]\alpha,\gamma[_p = \emptyset$, then $T \coloneqq \{\alpha,\gamma\}$ is a turn in p with $T \subseteq B \coloneqq B_0$ by Proposition 4.13. However $T \nsubseteq S_1$. Nonetheless, $B_1\backslash T$ and $B_n\backslash T$ cross in p: Because $|B_0 \cap S_4| = 1$ there exists $\eta \in B_0 \cap S_4$. Because $\eta \notin T$ it follows that $(\eta,\beta_{n,1},\beta_{1,2},\beta_{n,2})$ is a crossing between $B_1\backslash T$ and $B_n\backslash T$. That concludes the proof. □

LEMMA 8.2. *For any $p \in \mathcal{W}$ and $c \in \bigcup\mathcal{R}$, whenever $c \in w(p)$, then $\pi_c \in \langle p\rangle$.*

PROOF. Let $n \in \mathbb{N}$ and $c : [\![n]\!] \to \{\circ,\bullet\}$. By Lemma 8.2 we can assume $\|p\| \leq 4n$. Because $c \in w(p)$ necessitates $4n \leq \|p\|$ by Lemmata 4.19 and 6.3, that means $\|p\| = 4n$. We show that p is a rotation of π_c. More precisely, we prove the existence of $4n$ pairwise distinct points $(\beta_{i,j})_{(i,j)\in[\![n]\!]\times[\![4]\!]}$ such that $(\beta_{1,1},\beta_{2,1},\ldots,\beta_{n,1},\beta_{n,2},\beta_{n-1,2},\ldots,\beta_{1,2},\beta_{1,3},\beta_{2,3},\ldots,\beta_{n,3},\beta_{n,4},\beta_{n-1,4},\ldots,\beta_{1,4})$ is ordered in p, such that for each $i \in [\![n]\!]$ the set $\{\beta_{i,j}\}_{j=1}^4$ is a block of p and such that for every $i \in [\![n]\!]$ and every $j \in [\![4]\!]$ the point $\beta_{i,j}$ has normalized color c_i in p.

Step 1: *Definitions of $B_1, B_2, \ldots, B_n$.* By the assumption $c \in w(p)$ there exist $\{B_1,B_2,\ldots,B_n\} \subseteq p$ such that $B_1,B_2,\ldots,B_n$ are pairwise distinct, such that B_1 and B_n cross in p, such that $\langle B_1,B_n\rangle_p = \{B_1,B_2,\ldots,B_n\}$, such that $B_1 \leq B_2 \leq \ldots \leq B_n$ with respect to $\leq_{p,B_1,B_n}$ and such that $\lambda_p(B_1,B_i,B_n) = \sigma(c_i)$ for every $i \in [\![n]\!]$. Then, the constraint $\|p\| = 4n$ implies that $B_1,B_2,\ldots,B_n$ are the only blocks of p and that $|B_i| = 4$ for every $i \in [\![n]\!]$.

Step 2: *Definition of ϱ and $\beta_{i,j}$.* Since B_1 and B_n cross in p, since each of them only has four legs and since between any two distinct legs of one can only be an even number (0 or 2) of legs of the other, there must exist for every $i \in \{1,n\}$ legs $\{\beta_{i,j}\}_{j=1}^4 \subseteq B_i$ such that $(\beta_{1,4},\beta_{1,1},\ \beta_{n,1},\beta_{n,2},\ \beta_{1,2},\beta_{1,3},\ \beta_{n,3},\beta_{n,4})$ is ordered in p.

Let $\varrho : [\![n]\!] \to [\![n]\!], i \mapsto n-i+1$, let $i \in [\![n]\!]$, let $1 < i < n$ and let $j \in [\![4]\!]$ be arbitrary. Then, since $(B_1,B_i,B_n) \in \chi_p$ by assumption and thus $(B_n,B_i,B_1) \in \chi_p$ by Lemma 4.23 (a), we can rest assured that $(B_{\varrho^{j-1}(1)},B_i,B_{\varrho^{j-1}(n)}) \in \chi_p$. Because B_1, B_i and B_n are pairwise distinct Lemma 4.15 then guarantees $\sigma_p([\beta_{\varrho^{j-1}(1),j},\beta_{\varrho^{j-1}(n),j}]_p \cap B_i) \neq 0$ and thus in particular $]\beta_{\varrho^{j-1}(1),j},\beta_{\varrho^{j-1}(n),j}[_p \cap B_i = [\beta_{\varrho^{j-1}(1),j},\beta_{\varrho^{j-1}(n),j}]_p \cap B_i \neq \emptyset$. Hence, $\beta_{i,j} \coloneqq \min(]\beta_{\varrho^{j-1}(1),j},\beta_{\varrho^{j-1}(n),j}[_p \cap B_i)$ is well-defined.

Step 3: *Their ordering.* For every $i \in [\![n]\!]$ the definitions ensure that the tuple $(\beta_{1,1},\beta_{i,1},\beta_{n,1},\ \beta_{n,2},\beta_{i,2},\beta_{1,2},\ \beta_{1,3},\beta_{i,3},\beta_{n,3},\ \beta_{n,4},\beta_{i,4},\beta_{1,4})$ is ordered in p. To verify that the points $(\beta_{i,j})_{(i,j)\in[\![n]\!]\times[\![4]\!]}$ have the asserted ordering all that is left to prove is that for all $i \in [\![n]\!]$ with $1 < i < n-1$ and $j \in [\![4]\!]$ the tuple $(\beta_{\varrho^{j-1}(1),j},\beta_{\varrho^{j-1}(i),j},\beta_{\varrho^{j-1}(i+1),j})$ is ordered in p.

For such i the assumptions that $B_i \leq B_{i+1}$ and $B_{n-i} \leq B_{n-i+1}$ with respect to $\leq_{p,B_1,B_n}$, i.e., that $(B_1,B_i,B_{i+1}) \in \chi_p$ and $(B_{n-i},B_{n-i+1},B_n) \in \chi_p$, by Lemma 4.23 (a) imply $(B_{\varrho^{j-1}(1)},B_{\varrho^{j-1}(i)},B_{\varrho^{j-1}(i+1)}) \in \chi_p$ for every $j \in [\![4]\!]$. Because the blocks

$B_{\varrho^{j-1}(1)}, B_{\varrho^{j-1}(i)}, B_{\varrho^{j-1}(i+1)}$ are pairwise distinct, we infer $\sigma_p([\beta_{\varrho^{j-1}(1),j}, \beta_{\varrho^{j-1}(i+1),j}]_p \cap B_{\varrho^{j-1}(i)}) \neq 0$ by Lemma 4.15. The consequence $[\beta_{\varrho^{j-1}(1),j}, \beta_{\varrho^{j-1}(i+1),j}]_p \cap B_{\varrho^{j-1}(i)} \neq \varnothing$ demands $\beta_{\varrho^{j-1}(i),j} \in [\beta_{\varrho^{j-1}(1),j}, \beta_{\varrho^{j-1}(i+1),j}]_p$ because $[\beta_{\varrho^{j-1}(1),j}, \beta_{\varrho^{j-1}(i+1),j}]_p \cap B_{\varrho^{j-1}(i)} \subseteq [\beta_{\varrho^{j-1}(1),j}, \beta_{\varrho^{j-1}(n),j}]_p \cap B_{\varrho^{j-1}(i)} = \{\beta_{\varrho^{j-1}(i+1),j}\}$ by $|B_{\varrho^{j-1}(i)}| = 4$. That means $(\beta_{\varrho^{j-1}(1),j}, \beta_{\varrho^{j-1}(i),j}, \beta_{\varrho^{j-1}(i+1),j})$ is ordered in p, as was claimed.

Step 4: *Their colors.* For every $i \in [\![n]\!]$ and every $j \in [\![4]\!]$ because $|B_i| = 4$ and because $(\beta_{\varrho^{j-1}(1),j}, \beta_{\varrho^{j-1}(i),j}, \beta_{\varrho^{j-1}(n),j})$ is ordered in p, we know $[\beta_{\varrho^{j-1}(1),j}, \beta_{\varrho^{j-1}(n),j}]_p \cap B_{\varrho^{j-1}(i)} = \{\beta_{\varrho^{j-1}(i),j}\}$. The premises $\lambda_p(B_1, B_i, B_n) = \sigma(c_i)$ and $\lambda_p(B_n, B_{n-i+1}, B_1) = \sigma(c_{n-i+1})$ imply $\lambda_p(B_{\varrho^{j-1}(1)}, B_{\varrho^{j-1}(i)}, B_{\varrho^{j-1}(n)}) = c_{\varrho^{j-1}(i)}$. Lemma 4.17 hence allows us to conclude $\sigma_p(\{\beta_{\varrho^{j-1}(i),j}\}) = \sigma_p([\beta_{\varrho^{j-1}(1),j}, \beta_{\varrho^{j-1}(n),j}]_p \cap B_{\varrho^{j-1}(i)}) = \lambda_p(B_{\varrho^{j-1}(1)}, B_{\varrho^{j-1}(i)}, B_{\varrho^{j-1}(n)}) = \sigma(c_{\varrho^{j-1}(i)})$, which concludes the proof. □

LEMMA 8.3. *For all $n \in \mathbb{N}$ and $c : [\![n]\!] \to \{\circ, \bullet\}$ the following are true:*

(a) $\langle \pi_\circ \rangle = \langle \pi_\bullet \rangle = \langle \sqcap\!\sqcap\!\sqcap_{\circ\bullet\circ\bullet} \rangle$.

(b) $\pi_{(\overline{c_n}, \overline{c_{n-1}}, \ldots, \overline{c_1})} \in \langle \pi_c \rangle$.

(c) $\pi_{(c_2, c_3, \ldots, c_n)} \in \langle \pi_c \rangle$ *if* $2 \leq n$.

(d) $\pi_{(c_1, c_2, \ldots, c_{i-1}, c_{i+2}, c_{i+3}, \ldots, c_n)} \in \langle \pi_c \rangle$ *if* $4 \leq n$, *if* $1 < i < n-1$ *and if* $c_i \neq c_{i+1}$.

PROOF. (a) The identities $\pi_\circ = (\sqcap\!\sqcap\!\sqcap_{\circ\bullet\circ\bullet})^{\curvearrowright 2}$ and $\pi_\bullet = \pi_\circ{}^{\circlearrowright}$ prove the claim.

(b) The partition π_c is biprojective and $\pi_{(\overline{c_n}, \overline{c_{n-1}}, \ldots, \overline{c_1})} = \pi_c{}^{\dagger} \in \langle \pi_c \rangle$.

(c) Erase from π_c first the turn $\{{}_{\blacksquare}2n, {}^{\blacksquare}2n\}$ and then from the resulting partition the turn $\{{}^{\blacksquare}1, {}_{\blacksquare}1\}$ to obtain the partition $\pi_{(c_2, c_3, \ldots, c_n)}$.

(d) The partition $\pi_{(c_1, c_2, \ldots, c_{i-1}, c_{i+2}, c_{i+3}, \ldots, c_n)}$ results from π_c by successively erasing, in this order, the four turns $\{{}^{\blacksquare}\varrho(i), {}^{\blacksquare}\varrho(i+1)\}$, $\{{}_{\blacksquare}\varrho(i+1), {}_{\blacksquare}\varrho(i)\}$, $\{{}^{\blacksquare}(i+1), {}^{\blacksquare}i\}$ and $\{{}_{\blacksquare}i, {}_{\blacksquare}(i+1)\}$, where $\varrho : [\![2n]\!] \to [\![2n]\!], i \mapsto 2n - i + 1$ is the reflection. □

LEMMA 8.4. $\{\circ, \bullet\} \cup \bigcup w(\mathcal{C}) \in \mathcal{R}$ *for every hyperoctahedral category* $\mathcal{C} \subseteq \mathcal{W}$.

PROOF. Abbreviate $R_{\mathcal{C}} := \{\circ, \bullet\} \cup \bigcup w(\mathcal{C})$. The requirement $\circ \in R_{\mathcal{C}}$ for $R_{\mathcal{C}}$ being a $\mathcal{W}$-parameter set is satisfied by definition. Let $c \in R_{\mathcal{C}}$ be arbitray. We need to show that $(\overline{c_n}, \overline{c_{n-1}}, \ldots, \overline{c_1}) \in R_{\mathcal{C}}$, that $(c_2, c_3, \ldots, c_n) \in R_{\mathcal{C}}$ if $2 \leq n$ and that $(c_1, c_2, \ldots, c_{i-1}, c_{i+2}, c_{i+3}, \ldots, c_n) \in R_{\mathcal{C}}$ if $4 \leq n$ and $i \in \{2, 3, \ldots, n-2\}$ and $c_i \neq c_{i+1}$. The three claims can be proven simultaneously. Suppose the conditions on n and i are satisfied for the second and third assertion and let d be the tuple for which we need to show $d \in R_{\mathcal{C}}$.

If $n = 1$, then $d = \overline{c} \in R_{\mathcal{C}}$ is all that we claim. And this is guaranteed by $\{\circ, \bullet\} \subseteq R_{\mathcal{C}}$. Hence, we can assume $2 \leq n$ from now on. Then, by definition of $R_{\mathcal{C}}$ there exists $p \in \mathcal{C}$ with $c \in w(p)$. Lemma 8.2 allows us to infer $\pi_c \in \langle p \rangle \subseteq \mathcal{C}$. It follows that $\pi_d \in \langle \pi_c \rangle \subseteq \mathcal{C}$ by Parts (b)–(d) of Lemma 8.3. Because $d \in w(\pi_d)$ by Lemma 6.21, we conclude $d \in \bigcup w(\mathcal{C}) \subseteq R_{\mathcal{C}}$, completing the proof. □

THEOREM 8.5. *For any hyperoctahedral category $\mathcal{C} \subseteq \mathcal{W}$ there is $R \in \mathcal{R}$ with $\mathcal{C} = \mathcal{W}_R$.*

PROOF. Lemma 8.4 informs us that $R_{\mathcal{C}} := \{\circ, \bullet\} \cup \bigcup w(\mathcal{C}) \in \mathcal{R}$. Hence, $\mathcal{W}_{R_{\mathcal{C}}}$ is a well-defined category. We show $\mathcal{C} \subseteq \mathcal{W}_{R_{\mathcal{C}}}$ and $\mathcal{C} \supseteq \mathcal{W}_{R_{\mathcal{C}}}$ separately.

Step 1: *Inclusion* $\subseteq$. By the definitions of the mappings $R \mapsto \mathcal{W}_R$ and $p \mapsto w(p)$, for every $p \in \mathcal{W}$ the statements $p \in \mathcal{W}_{R_\mathcal{C}}$ and $w(p) \subseteq R_\mathcal{C}$ are equivalent. Because $\mathcal{C} \subseteq \mathcal{W}$ by assumption and because for all $p \in \mathcal{C}$ the definition of $R_\mathcal{C}$ ensures $w(p) \subseteq \bigcup w(\mathcal{C}) \subseteq R_\mathcal{C}$, it follows $\mathcal{C} \subseteq \mathcal{W}_{R_\mathcal{C}}$.

Step 2: *Inclusion* $\supseteq$. Since $\mathcal{W}_{R_\mathcal{C}} = \langle \pi_c \,|\, c \in R_\mathcal{C} \rangle$ by Theorem 6.24, if we want to show $\mathcal{W}_{R_\mathcal{C}} \subseteq \mathcal{C}$, it suffices to prove $\{\pi_c \,|\, c \in R_\mathcal{C}\} \subseteq \mathcal{C}$. Because $\mathcal{C}$ is per assumption a hyperoctahedral category, $\underset{\circ\bullet\circ\bullet}{\sqcap\sqcap\sqcap} \in \mathcal{C}$. As $\{\pi_\circ, \pi_\bullet\} \subseteq \langle \underset{\circ\bullet\circ\bullet}{\sqcap\sqcap\sqcap} \rangle$ by Lemma 8.3 (a), only $\pi_c \in \mathcal{C}$ for every $c \in \bigcup w(\mathcal{C})$ remains to be shown. For every $c \in \bigcup w(\mathcal{C})$ there exists $p \in \mathcal{C}$ with $c \in w(p)$. It follows $\pi_c \in \langle p \rangle$ by Lemma 8.2. Hence, $\langle p \rangle \subseteq \mathcal{C}$ concludes the proof. $\square$

9. Lattice Structure

The last step is to show that the mapping $R \mapsto \mathcal{W}_R$ is an isomorphism of complete lattices from $\mathcal{R}$ to the hyperoctahedral subcategories of $\mathcal{W}$ (Theorem 9.4).

Notation 9.1. Denote by $\mathsf{PCat}^{\circ\bullet}_{\mathrm{HO},\subseteq\mathcal{W}}$ the set of all hyperoctahedral categories $\mathcal{C}$ of two-colored partitions with $\mathcal{C} \subseteq \mathcal{W}$.

Note that $\mathcal{W}$ has at least one non-hyperoctahedral subcategory in the form of the set $\langle \varnothing \rangle$ of all non-crossing two-colored pair partitions with neutral blocks.

Lemma 9.2. *The partially ordered set* $(\mathcal{R}, \subseteq)$ *is a complete lattice. For all* $\mathcal{R}' \subseteq \mathcal{R}$ *meet and join of* $\mathcal{R}'$ *are given by* $\{\circ, \bullet\} \cup \bigcap \mathcal{R}'$ *and* $\{\circ, \bullet\} \cup \bigcup \mathcal{R}'$*, respectively.*

Proof. Because $\mathcal{W}$-parameter sets are defined as subsets of $\bigcup \mathcal{R} = \bigcup_{n \in \mathbb{N}} (\llbracket n \rrbracket \to \{\circ, \bullet\})$ which contain $\circ$ and are invariant under certain operations, it is clear that $\mathcal{R}$ is closed under intersections. Because the operations in questions are unary, $\mathcal{R}$ is also closed under unions. Since intersections and unions are meets and joins in the power set of $\bigcup \mathcal{R}$, they must also be meets and joins in $\mathcal{R}$. $\square$

Lemma 9.3. *The partially ordered set* $(\mathsf{PCat}^{\circ\bullet}_{\mathrm{HO},\subseteq\mathcal{W}}, \subseteq)$ *is a complete lattice. For all* $\mathsf{C} \subseteq \mathsf{PCat}^{\circ\bullet}_{\mathrm{HO},\subseteq\mathcal{W}}$ *meet and join of* C *are given by* $\mathcal{W}_{\{\circ,\bullet\}} \cup \bigcap \mathsf{C}$ *and* $\mathcal{W}_{\{\circ,\bullet\}} \cup \langle \bigcup \mathsf{C} \rangle$*, respectively.*

Proof. We treat meets and joins separately. But first, we note the following: Since $\mathcal{W}_{\{\circ,\bullet\}} = \langle \pi_\circ, \pi_\bullet \rangle$ by Theorem 6.24, since $\langle \pi_\circ, \pi_\bullet \rangle = \langle \underset{\circ\bullet\circ\bullet}{\sqcap\sqcap\sqcap} \rangle$ by Lemma 8.3 (a) and since $\underset{\circ\bullet\circ\bullet}{\sqcap\sqcap\sqcap} \in \mathcal{C}$ for every $\mathcal{C} \in \mathsf{C}$ by the definition of what it means for a category to be hyperoctahedral, $\mathcal{W}_{\{\circ,\bullet\}} \subseteq \mathcal{C}$ for every $\mathcal{C} \in \mathsf{C}$.

Step 1: *Meets.* We prove that $\mathcal{W}_{\{\circ,\bullet\}} \cup \bigcap \mathsf{C} \in \mathsf{PCat}^{\circ\bullet}_{\mathrm{HO},\subseteq\mathcal{W}}$ and that this category is indeed the meet of C in $\mathsf{PCat}^{\circ\bullet}_{\mathrm{HO},\subseteq\mathcal{W}}$.

Step 1.1: *Element of* $\mathsf{PCat}^{\circ\bullet}_{\mathrm{HO},\subseteq\mathcal{W}}$. By the initial remark, $\mathcal{W}_{\{\circ,\bullet\}} \subseteq \bigcap \mathsf{C}$ whenever $\mathsf{C} \neq \varnothing$. Consequently, $\mathcal{W}_{\{\circ,\bullet\}} \cup \bigcap \mathsf{C} = \bigcap \mathsf{C}$ if $\mathsf{C} \neq \varnothing$. And, of course, $\mathcal{W}_{\{\circ,\bullet\}} \cup \bigcap \mathsf{C} = \mathcal{W}_{\{\circ,\bullet\}}$ if $\mathsf{C} = \varnothing$. According to Theorem 4.42 the set $\mathcal{W}_{\{\circ,\bullet\}}$ is a hyperoctahedral category. And, if $\mathsf{C} \neq \varnothing$, then the category $\bigcap \mathsf{C}$ is hyperoctahedral because $\underset{\circ\ \bullet}{\updownarrow\otimes\updownarrow} \notin \mathcal{C}$ and $\underset{\circ\bullet\circ\bullet}{\sqcap\sqcap\sqcap} \in \mathcal{C}$ for every $\mathcal{C} \in \mathsf{C}$. Hence, $\mathcal{W}_{\{\circ,\bullet\}} \cup \bigcap \mathsf{C} \in \mathsf{PCat}^{\circ\bullet}_{\mathrm{HO},\subseteq\mathcal{W}}$.

Step 1.2: *Greatest lower bound.* The relation $\mathcal{W}_{\{\circ,\bullet\}} \cup \bigcap \mathsf{C} \subseteq \mathcal{C}$ is clear for every $\mathcal{C} \in \mathsf{C}$ by what was remarked initially. And of course for any $\mathcal{C}_\wedge \in \mathsf{PCat}^{\circ\bullet}_{\mathrm{HO},\subseteq\mathcal{W}}$ with $\mathcal{C}_\wedge \subseteq \mathcal{C}$ for every $\mathcal{C} \in \mathsf{C}$ we have $\mathcal{C}_\wedge \subseteq \bigcap \mathsf{C} \subseteq \mathcal{W}_{\{\circ,\bullet\}} \cup \bigcap \mathsf{C}$. Hence, $\mathcal{W}_{\{\circ,\bullet\}} \cup \bigcap \mathsf{C}$ is indeed the meet of C in $\mathsf{PCat}^{\circ\bullet}_{\mathrm{HO},\subseteq\mathcal{W}}$.

Step 2: *Joins.* As before, we show that $\mathcal{W}_{\{\circ,\bullet\}} \cup \langle \bigcup \mathsf{C}\rangle$ belongs to $\mathsf{PCat}^{\circ\bullet}_{\mathrm{HO},\subseteq\mathcal{W}}$ and is the join of C there.

Step 2.1: *Element of* $\mathsf{PCat}^{\circ\bullet}_{\mathrm{HO},\subseteq\mathcal{W}}$. As seen in Step 1.1, if $\mathsf{C} = \varnothing$, then $\mathcal{W}_{\{\circ,\bullet\}} \cup \langle \bigcup \mathsf{C}\rangle = \mathcal{W}_{\{\circ,\bullet\}} \in \mathsf{PCat}^{\circ\bullet}_{\mathrm{HO},\subseteq\mathcal{W}}$. If $\mathsf{C} \neq \varnothing$, then $\mathcal{W}_{\{\circ,\bullet\}} \cup \langle \bigcup \mathsf{C}\rangle = \langle \bigcup \mathsf{C}\rangle \in \mathsf{PCat}^{\circ\bullet}_{\mathrm{HO},\subseteq\mathcal{W}}$ as well: Indeed, because $\mathcal{C} \subseteq \mathcal{W}$ for every $\mathcal{C} \in \mathsf{C}$, first $\bigcup \mathcal{C} \subseteq \mathcal{W}$ and thus $\langle \bigcup \mathsf{C}\rangle \subseteq \mathcal{W}$ by Theorem 4.7; and because $\updownarrow\!\circ\!\otimes\!\bullet\!\updownarrow \notin \mathcal{W}$ also $\updownarrow\!\circ\!\otimes\!\bullet\!\updownarrow \notin \langle \bigcup \mathsf{C}\rangle \subseteq \mathcal{W}$, and $\sqcap\!\sqcap\!\sqcap_{\circ\bullet\circ\bullet} \in \mathcal{W}_{\{\circ,\bullet\}} \subseteq \langle \bigcup \mathsf{C}\rangle$, making the category $\langle \bigcup \mathsf{C}\rangle$ hyperoctahedral.

Step 2.2: *Least upper bound.* Again, $\mathcal{C} \subseteq \mathcal{W}_{\{\circ,\bullet\}} \cup \langle \bigcup \mathsf{C}\rangle$ is clear for every $\mathcal{C} \subseteq \mathcal{P}^{\circ\bullet}$ because $\mathcal{C} \subseteq \bigcup \mathsf{C} \subseteq \langle \bigcup \mathsf{C}\rangle$. And if $\mathcal{C}_\vee \in \mathsf{PCat}^{\circ\bullet}_{\mathrm{HO},\subseteq\mathcal{W}}$ satisfies $\mathcal{C} \subseteq \mathcal{C}_\vee$ for every $\mathcal{C} \in \mathsf{C}$, then $\sqcap\!\sqcap\!\sqcap_{\circ\bullet\circ\bullet} \in \mathcal{C}_\vee$ demands $\mathcal{W}_{\{\circ,\bullet\}} = \langle \sqcap\!\sqcap\!\sqcap_{\circ\bullet\circ\bullet} \rangle \subseteq \mathcal{C}_\vee$ as well as $\bigcup \mathsf{C} \subseteq \mathcal{C}_\vee$ and thus $\langle \bigcup \mathsf{C}\rangle \subseteq \mathcal{C}_\vee$, proving $\mathcal{W}_{\{\circ,\bullet\}} \cup \langle \bigcup \mathsf{C}\rangle \subseteq \mathcal{C}_\vee$. In other words, $\mathcal{W}_{\{\circ,\bullet\}} \cup \langle \bigcup \mathsf{C}\rangle$ is indeed the join of C in $\mathsf{PCat}^{\circ\bullet}_{\mathrm{HO},\subseteq\mathcal{W}}$. That is all we needed to see. □

THEOREM 9.4. *The mapping* $\Phi : \mathcal{R} \to \mathsf{PCat}^{\circ\bullet}_{\mathrm{HO},\subseteq\mathcal{W}}, R \mapsto \mathcal{W}_R$ *is an isomorphism of complete lattices from* $(\mathcal{R}, \subseteq)$ *to* $(\mathsf{PCat}^{\circ\bullet}_{\mathrm{HO},\subseteq\mathcal{W}}, \subseteq)$.

PROOF. The mapping Φ is well-defined by Theorem 4.42, injective by Theorem 7.1 and surjective by Theorem 8.5. The partially ordered sets $(\mathcal{R}, \subseteq)$ and $(\mathsf{PCat}^{\circ\bullet}_{\mathrm{HO},\subseteq\mathcal{W}}, \subseteq)$ are complete lattices by Lemmata 9.2 and 9.3, respectively. All we have to prove is that Φ preserves meets and joins, the concrete forms of which are known by Lemmata 9.2 and 9.3. Let $\mathcal{R}' \subseteq \mathcal{R}$ be arbitrary.

Step 1: *Meets.* First, we prove that Φ maps the meet of $\mathcal{R}'$ to the meet of $\{\Phi(R) \mid R \in \mathcal{R}'\}$, which is to say that $\mathcal{W}_{\{\circ,\bullet\}\cup\bigcap\mathcal{R}'} = \mathcal{W}_{\{\circ,\bullet\}} \cup \bigcap\{\mathcal{W}_R \mid R \in \mathcal{R}'\}$.

If $\mathcal{R}' = \varnothing$, this is certainly true because then $\mathcal{W}_{\{\circ,\bullet\}\cup\bigcap\mathcal{R}'} = \mathcal{W}_{\{\circ,\bullet\}}$ and $\mathcal{W}_{\{\circ,\bullet\}} \cup \bigcap\{\mathcal{W}_R \mid R \in \mathcal{R}'\} = \mathcal{W}_{\{\circ,\bullet\}}$. Hence, we can assume $\mathcal{R}' \neq \varnothing$.

Since $\{\circ, \bullet\} \subseteq R$ for every $R \in \mathcal{R}'$ by definition, then $\{\circ, \bullet\} \cup \bigcap\mathcal{R}' = \bigcap\mathcal{R}'$. Likewise, because $\mathcal{W}_{\{\circ,\bullet\}} = \langle \sqcap\!\sqcap\!\sqcap_{\circ\bullet\circ\bullet} \rangle \subseteq \mathcal{W}_R$, then $\mathcal{W}_{\{\circ,\bullet\}} \cup \bigcap\{\mathcal{W}_R \mid R \in \mathcal{R}'\} = \bigcap\{\mathcal{W}_R \mid R \in \mathcal{R}'\}$. Hence, what we claim is that then $\mathcal{W}_{\bigcap\mathcal{R}'} = \bigcap\{\mathcal{W}_R \mid R \in \mathcal{R}'\}$. Equivalently, we need to show for every $p \in \mathcal{W}$ that $w(p) \subseteq \bigcap\mathcal{R}'$ if and only if $w(p) \subseteq R$ for every $R \in \mathcal{R}'$. And this is obviously true.

Step 2: *Joins.* In order to see that Φ preserves joins we need to prove $\mathcal{W}_{\{\circ,\bullet\}\cup\bigcup\mathcal{R}'} = \mathcal{W}_{\{\circ,\bullet\}} \cup \langle \bigcup\{\mathcal{W}_R \mid R \in \mathcal{R}'\}\rangle$. In the case $\mathcal{R}' = \varnothing$, this claim reduces to the identity $\mathcal{W}_{\{\circ,\bullet\}} = \mathcal{W}_{\{\circ,\bullet\}} \cup \langle\varnothing\rangle$, which is certainly true because, naturally, $\langle\varnothing\rangle \subseteq \mathcal{W}_{\{\circ,\bullet\}}$.

If $\mathcal{R}' \neq \varnothing$, then $\{\circ, \bullet\} \subseteq \bigcup\mathcal{R}'$ and $\mathcal{W}_{\{\circ,\bullet\}} \subseteq \mathcal{W}_R$ for every $R \in \mathcal{R}'$ imply that what we actually claim is the identity $\mathcal{W}_{\bigcup\mathcal{R}'} = \langle \bigcup\{\mathcal{W}_R \mid R \in \mathcal{R}'\}\rangle$. That is what we now show. Each inclusion is treated separately.

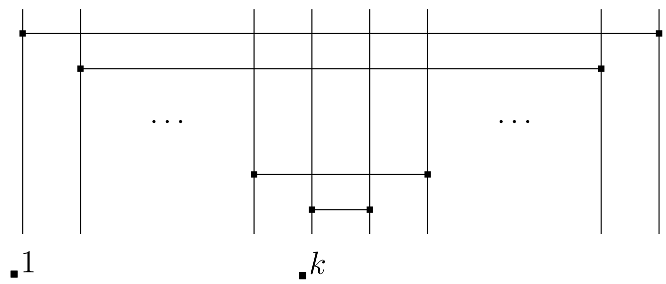
Step 2.1: *Inclusion* $\supseteq$. For every $R \in \mathcal{R}'$, because $R \subseteq \bigcup\mathcal{R}'$, by definition, $\mathcal{W}_R \subseteq \mathcal{W}_{\bigcup\mathcal{R}'}$. Hence, also $\bigcup\{\mathcal{W}_R \mid R \in \mathcal{R}'\} \subseteq \mathcal{W}_{\bigcup\mathcal{R}'}$. That proves $\mathcal{W}_{\bigcup\mathcal{R}'} \supseteq \langle \bigcup\{\mathcal{W}_R \mid R \in \mathcal{R}'\}\rangle$.

Step 2.2: *Inclusion* $\subseteq$. To show the converse relation we employ Theorem 6.24 to see that what we claim is the inclusion $\langle \pi_c \mid c \in \bigcup\mathcal{R}'\rangle \subseteq \langle \bigcup\{\langle \pi_c \mid c \in R\rangle \mid R \in \mathcal{R}'\}\rangle$.

For every $c_0 \in \bigcup \mathcal{R}'$ there exists $R_0 \in \mathcal{R}'$ such that $c_0 \in R_0$. It follows $\pi_0 \in \{\pi_c \,|\, c \in R_0\}$ and thus $\pi_0 \in \langle \pi_c \,|\, c \in R_0 \rangle$, which implies $\pi_0 \in \bigcup\{\langle \pi_c \,|\, c \in R\rangle \,|\, R \in \mathcal{R}'\}$ and thus $\pi_0 \in \langle \bigcup\{\langle \pi_c \,|\, c \in R\rangle \,|\, R \in \mathcal{R}'\}\rangle$. As c_0 was arbitrary, we have shown $\{\pi_c \,|\, c \in \bigcup \mathcal{R}'\} \subseteq \langle \bigcup\{\langle \pi_c \,|\, c \in R\rangle \,|\, R \in \mathcal{R}'\}\rangle$, which then yields the claimed inclusion and concludes the proof overall. □

10. Concluding Remarks

10.1. The Uncolored Case. In [RW16b] Raum and Weber determined all *non-group-theoretical hyperoctahedral* categories of uncolored partitions, i.e., all categories $\mathcal{C} \subseteq \mathcal{P}$ with $\mathcal{C} \cap \{\uparrow\otimes\uparrow, \sqcap\sqcap\sqcap, \times\} = \{\sqcap\sqcap\sqcap\}$. These are given by the category $\langle \pi_\ell \,|\, \ell \in \mathbb{N}\rangle$ and the categories $\langle \pi_k \rangle$ for $k \in \mathbb{N}$, where π_k is the partition obtained by rotating all upper points down in the same (arbitrary) direction in the partition



Raum and Weber proved that there is an isomorphism of partially ordered sets between the set of non-group-theoretical hyperoctahedral categories of uncolored partitions equipped with $\subseteq$ to $(\mathbb{N} \cup \{\infty\}, \leq)$. For any $\ell \in \mathbb{N} \cup \{\infty\}$ the corresponding category is given by the set of all uncolored partitions $p \in \mathcal{P}$ meeting the following requirements if p is considered in its word representation:

(a) Any letter of p appears an even number of times.
(b) If $p = X_1 a X_2 a X_3$ for a letter a and subwords X_1, X_2, X_3, then every letter appearing in X_2 occurs there an even number of times.
(c) p satisfies $\operatorname{wdepth}(p) \leq \ell$, which means that p contains no $\mathcal{W}$ of depth larger than ℓ if $\ell \in \mathbb{N}$ and which is a vacuous condition if $\ell = \infty$. For every $k \in \mathbb{N}$ the partition p is said to *contain a $\mathcal{W}$ of depth k* if there exist letters $a_1, \ldots, a_k$ and words $X_1^\alpha, \ldots, X_k^\alpha$, $X_1^\beta, \ldots, X_{k-1}^\beta$, $X_1^\gamma, \ldots, X_k^\gamma$, $X_1^\delta, \ldots, X_{k-1}^\delta$ and Y_1, Y_2, Y_3 such that
   (i) $p = Y_1 S_\alpha X_k^\alpha S_\beta Y_2 S_\gamma X_k^\gamma S_\delta Y_3$,
   (ii) where $S_\alpha = a_1 X_1^\alpha a_2 X_2^\alpha \ldots a_{k-1} X_{k-1}^\alpha a_k$,
   (iii) where $S_\beta = a_k X_{k-1}^\beta a_{k-1} X_{k-2}^\beta \ldots a_2 X_1^\beta a_1$,
   (iv) where $S_\gamma = a_1 X_1^\gamma a_2 X_2^\gamma \ldots a_{k-1} X_{k-1}^\gamma a_k$,
   (v) where $S_\delta = a_k X_{k-1}^\delta a_{k-1} X_{k-2}^\delta \ldots a_2 X_1^\delta a_1$ and
   (vi) where for every $i \in [\![k]\!]$ the letter a_i appears an *odd* number of times in each word S_α, S_β, S_γ and S_δ and
   (vii) where Y_1, Y_2 and Y_3 contain none of the letters $a_1, \ldots, a_k$.

In the language used here one could formulate:

Theorem 10.1. *If for any uncolored partition* $p \in \mathcal{P}$

- ◻ $B \in p$ *is called* non-interferent *with* $A \in p$ *if* $A \neq B$ *and* $|[\alpha,\gamma]_p \cap B| \equiv_2 0$ *for any* $\{\alpha,\gamma\} \subseteq A$ *with* $\alpha \neq \gamma$,
- ◻ χ_p *is the set of all* (A,B,C) *such that* $\{A,B,C\} \subseteq p$, *such that* $\neg(A = B = C)$ *and such that* $|[\alpha,\gamma]_p \cap B| \equiv_2 1$ *for some* $(\alpha,\gamma) \in A \times C$ *with* $\alpha \neq \gamma$,

then the following are true:

(a) *A category of uncolored partitions is given by the set* $\mathcal{W}$ *of all* $p \in \mathcal{P}$ *such that* $|A| \equiv_2 0$ *for any* $A \in p$ *and such that* B *is non-interferent with* A *for any* $\{A,B\} \subseteq p$ *with* $A \neq B$.

(b) *For any* $\ell \in \mathbb{N} \cup \{\infty\}$ *a category of uncolored partitions is given by the set* $\mathcal{W}_\ell$ *of all* $p \in \mathcal{W}$ *with the property that for any* $\{A,C\} \subseteq p$ *such that* A *and* C *cross in* p *there exist at most* ℓ *many* $B \in p$ *with* $(A,B,C) \in \chi_p$.

(c) $\mathcal{W}_\ell = \langle \pi_k \,|\, k \leq \ell \rangle$ *for every* $\ell \in \mathbb{N} \cup \{\infty\}$.

(d) *The rule* $\ell \mapsto \mathcal{W}_\ell$ *is an isomorphism of complete lattices from* $(\mathbb{N} \cup \{\infty\}, \leq)$ *to the hyperoctahedral subcategories of* $\mathcal{W}$ *equipped with the partial order* $\subseteq$.

Proof. Replace the implicit assumption $\circ \neq \bullet$ underlying all definitions, theorems and proofs with its negation $\circ = \bullet$, for all $p \in \mathcal{P}^{\circ\bullet}$ consider the color sum σ_p not a $\mathbb{Z}$-valued but a $\mathbb{Z}_2$-valued measure (densities $\sigma(\circ) \coloneqq 1$ and $\sigma(\bullet) \coloneqq -1 = 1$ unchanged) and replace the color distance δ_p with the constant map 0 (implying in particular that $p \leq \Delta_0 p$ is always true). ◻

10.2. The Quantum Groups. For any $\mathcal{W}$-parameter set $R \in \mathcal{R}$ and any $N \in \mathbb{N}$ the unitary easy quantum group associated with $\mathcal{W}_R$ and N is the compact matrix quantum group whose algebra is the universal unital C^*-algebra

$$\begin{aligned} C^*\langle \{u_{i,j}\}_{i,j=1}^N \,|\, &\forall i,j \in [\![N]\!]\colon \textstyle\sum_{k=1}^N u_{k,i}^* u_{k,l} = \sum_{k=1}^N u_{i,k} u_{j,k}^* = 1 \text{ and} \\ &\forall \ell \in \mathbb{N}\colon \forall c : [\![\ell]\!] \to \{\circ,\bullet\}\colon c \in R \Rightarrow \forall w,x,y,z : [\![\ell]\!] \to [\![N]\!]\colon \\ &\delta_{x,z}\; u_{w_1,x_1}^{c_1} u_{w_2,x_2}^{c_2} \ldots u_{w_\ell,x_\ell}^{c_\ell} u_{y_\ell,z_\ell}^{\overline{c_\ell}} u_{y_{\ell-1},z_{\ell-1}}^{\overline{c_{\ell-1}}} \ldots u_{y_1,z_1}^{\overline{c_1}} \\ &= \delta_{w,y}\; u_{w_1,x_1}^{c_1} u_{w_2,x_2}^{c_2} \ldots u_{w_\ell,x_\ell}^{c_\ell} u_{y_\ell,z_\ell}^{\overline{c_\ell}} u_{y_{\ell-1},z_{\ell-1}}^{\overline{c_{\ell-1}}} \ldots u_{y_1,z_1}^{\overline{c_1}} \rangle, \end{aligned}$$

where 1 is the unit of the C^*-algebra and where $u_{i,j}^\circ \coloneqq u_{i,j}$ and $u_{i,j}^\bullet \coloneqq u_{i,j}^*$ for all $\{i,j\} \subseteq [\![N]\!]$, and whose fundamental co-representation matrix is $u \coloneqq (u_{i,j})_{i,j=1}^N$.

10.3. Parity Violation. All non-hyperoctahedral categories of two-colored partitions are invariant under reflection or, equivalently, color inversion. This is not true for all hyperoctahedral categories investigated here.

Proposition 10.2. *There exist categories of two-colored partitions which have chirality, i.e., are not closed under reflection.*

Proof. Each of $R_\circ \coloneqq \{\circ\bullet, \circ, \bullet\}$ and $R_\bullet \coloneqq \{\bullet\circ, \circ, \bullet\}$ is a $\mathcal{W}$-parameter set because for any $c \in \{\circ,\bullet\}$ the string $c\overline{c}$ is mapped to itself under reflection and simultaneous color inversion. Because $R_\circ \neq R_\bullet$, also $\mathcal{W}_{R_\circ} \neq \mathcal{W}_{R_\bullet}$ by Theorem 7.1. In particular, while the category $R_\circ$ contains the partition $\pi_{\circ\bullet}$, the reflection $\pi_{\bullet\circ}$ of $\pi_{\circ\bullet}$ is *not* an element of $R_\circ$, both by Lemma 6.21. Hence, $\mathcal{W}_{R_\circ}$ is not closed under reflection. □

Of course, this does *not* yet exclude that the rule $u \mapsto u^t$ defines an automorphism of an associated compact matrix quantum group (A, u). In fact, at least for dimension 1 this would certainly be the case.

10.4. Further Questions. 1. There is no reason to think that $(\mathcal{W}_R)_{R\in\mathcal{R}, R\neq\{\circ,\bullet\}}$ already represent all conceivable locally colorized non-group-case hyperoctahedral categories of two-colored partitions with crossings. A reasonable next step in the classification of those would appear to be considering all possible categories $\mathcal{C} \subseteq \mathcal{V}$, where

$$\begin{aligned}\mathcal{V} := \{p \,|\, p \in \mathcal{P}^{\circ\bullet},\ p \leq \Delta_0 p,\ \forall A \in p\colon \sigma_p(A) = 0,\\ \forall \{A, B\} \subseteq p\colon \exists P \in \Delta_0 p\colon A \cup B \subseteq P \Rightarrow B \text{ non-interferent with } A \text{ in } p\}.\end{aligned}$$

In other words, one could relax the defining condition of $\mathcal{W}$ that any two blocks be mutually non-interferent to that of only any two blocks belonging to the same part being so. By Proposition 4.13 the legs of any block then still alternate in normalized color. That might persuade one to hope that the conditions determining categories $\mathcal{C} \subseteq \mathcal{V}$ do not yet become too complicated. On the other hand, Lemmata 6.1 and 6.2 no longer hold in $\mathcal{V}$. For that reason, while the diagonal subgroup

$$\{a_{f(p(_\blacksquare\ell))}^{\sigma(c(_\blacksquare\ell))} \cdot \ldots \cdot a_{f(p(_\blacksquare1))}^{\sigma(c(_\blacksquare1))} \cdot a_{f(p(^\blacksquare1))}^{\sigma(\bar{c}(^\blacksquare1))} \cdot \ldots \cdot a_{f(p(^\blacksquare k))}^{\sigma(\bar{c}(^\blacksquare k))} \mid \{k, \ell\} \subseteq \mathbb{N}_0,\ (p, c) \in \mathcal{C}(k, \ell),\ f : p \to \mathbb{N}\}$$

for generators $(a_i)_{i\in\mathbb{N}}$ of $(\mathbb{Z}^{*\infty}, \cdot)$ is trivial for $\mathcal{C} = \mathcal{W}$, this need not be true for other subcategories $\mathcal{C} \subseteq \mathcal{V}$. Thus, when attempting to classify such categories one will have to deal with two problems at the same time:

(1) recognizing *relational* conditions similar to those induced by $\star$-betweeness in the presence of (possibly asymmetrically) interferent blocks,
(2) integrating these relational conditions with the *algebraic* ones coming from possibly non-trivial diagonal subgroups.

Determining the set $\mathcal{V} \cap \mathscr{R}$ might be a good starting point. The elements π_c of $\mathcal{W} \cap \mathscr{R}$ give a strong clue to what the conditions of $(\mathcal{W}_R)_{R\in\mathcal{R}}$ are. Maybe one can induct from the knowledge of $\mathcal{V} \cap \mathscr{R}$ those of the categories $\mathcal{C} \subseteq \mathcal{V}$ as well.

2. Of course, nothing at all is known yet about the unitary quantum groups associated with the categories $(\mathcal{W}_R)_{R\in\mathcal{R}}$. Maybe those can be constructed via known procedures like semi-direct products from the orthogonal quantum groups associated with the categories $\langle \pi_k \,|\, k \leq \ell \rangle$ for $\ell \in \mathbb{N} \cup \{\infty\}$.

CHAPTER 2

Categories of bi-labeled graphs

1. Introduction

Several families of new categories of bi-labeled graphs and thus graph-theoretical quantum groups in Mančinska and Roberson's sense are discovered. In particular, two ways are given of producing from any category of partitions in Banica and Speicher's sense a category of bi-labeled graphs. Also, the categories of bi-labeled graphs generated by arbitrary powers of the adjacency bi-labeled graph are determined. Those results are then used to give two categories of bi-labeled graphs which are distinct but yield isomorphic quantum groups under every one of Mančinska and Roberson's fiber functors.

1.1. Background and context. In their preprint [MR19], an excerpt of which has since appeared as [MR20], Mančinska and Roberson use concepts and results from the theory of operator-algebraic quantum groups to address a graph-theoretical question. They show that, in analogy to a classical result by Lováscz [Lov67], two graphs are *quantum-isomorphic* if and only if they admit the same number of homomorphisms from any planar graph.

Beforehand, it was known that proving two graphs quantum-isomorphic is the same as giving a quantum group isomorphism between their *quantum automorphism groups*. Despite its name the quantum automorphism group is not a group but a *compact quantum group* in Woronowicz's sense [Wor87; Wor91; Wor98] defined by Banica in [Ban05] as a liberation of the classical automorphism group. (A slightly different notion of quantum automorphism group had before been proposed by Bichon [Bic03].) According to a Tannaka-Krein type result by Woronowicz [Wor88] compact quantum groups are dual to complete *concrete monoidal W^*-categories with conjugates*. Moreover, any concrete monoidal W^*-category can be completed in an essentially unique way.

What Mančinska and Roberson achieve in [MR19] is to construct for any graph a concrete monoidal W^*-category with conjugates whose completion is the Tannaka-Krein dual of the quantum automorphism group of the graph. Crucially, the entries of the coordinate matrices of the bounded operators making up the morphism spaces of such a category are related to the counts of homorphisms from planar graphs to the graph in question in such a way that two categories coincide if and only if the two graphs admit the same number of homomorphisms from planar graphs.

In passing, Mančinska and Roberson provide a way of obtaining concrete monoidal W^*-categories with conjugates which may represent other compact quantum groups than the quantum automorphism groups. They call them *graph categories* respectively *graph-theoreticcal quantum groups*. Their definitions closely resembles the *categories of partitions* respectively *easy quantum groups* introduced by Banica and Speicher in [BS09]. In both cases, finding new quantum groups is tantamount to solving combinatorics problems.

Namely, one needs to find sets of combinatorial morphisms, *bi-labeled graphs* in Mančinska and Roberson's case, *partitions* in Banica and Speicher's, which are closed under certain *operations*. These operations, known as composition, tensor product, forming adjoints and dualization also have combinatorial definitions, corresponding in the partition case to vertical concatenation, horizontal concatenation, horizontal reflection and vertical reflection, respectively. For bi-labeled graphs the operations are defined in a similar manner.

While all such categories of partitions are known [BS09; BCS10; Web13; RW14; RW16a; RW16a], this is not true for graph categories. Mančinska and Roberson determine one category in order to prove their main result about quantum isomorphisms, namely the category of *planar bi-labeled graphs*, giving rise to the quantum automorphism group via Tannaka-Krein duality. Moreover, they note that, of course, the set of *all bi-labeled graphs* trivially forms a category of bi-labeled graphs, corresponding to the clasical automorphism group. Since then, Gromada has found further examples in [Gro22b], a whole family of so-called *group-theoretical* graph categories. In [Gro22a], he moreover gives an example of a "graph category" in a more general sense than the one defined by Mančinska and Roberson and uses it to define a quantum group liberating the demihyperoctahedral group D_4.

These efforts by Mančinska, Roberson and Gromada will now be continued here. Several new examples of graph categories and thus graph-theoretical quantum groups are provided. One family will moreover enable certain conclusions about how many new compact quantum groups one may expect from Mančinska and Roberson's construction. Furthermore, a number of invariants of graph categories are introduced which might help with classifying them in the future.

1.2. Main results. The main results of the present chapter can be summarized as follows. This was joint work with Moritz Weber and Daniel Gromada.

MAIN RESULT. *(a) Out of any bi-labeled graph* $\mathbf{K}$ *a partition of its labels can be obtained by declaring two labels to be in the same block if and only they are attached to*

- *(i) the same vertex, giving the* vertex partition $\mathbb{P}_{\mathbf{K}}$.
- *(ii) vertices belonging to the same connected component, giving the* component partition $\mathrm{P}_{\mathbf{K}}$.

For any $F \in \{\mathbb{P}, \mathrm{P}\}$ and any category $\mathcal{C}$ of partitions the set of all bi-labeled graphs $\mathbf{K}$ with $F_{\mathbf{K}} \in \mathcal{C}$ is a graph category (Proposition 3.6 resp. 4.6). Conversely, for any graph category $\mathcal{F}$ the set of all $F_{\mathbf{K}}$ where $\mathbf{K} \in \mathcal{F}$ forms a category of partitions (Proposition 3.9 resp. 4.9).

(b) *For any additive subsemigroup S of $\mathbb{N}$ a graph category is given by the set of all bi-labeled graphs with the property that the distances of any two distinct connected labeled vertices is an element of S (Proposition 5.6). Conversely, for any graph category $\mathcal{F}$ the set of distances between any two distinct connected labeled vertices of any bi-labeled graph of $\mathcal{F}$ is an additive subsemigroup of $\mathbb{N}$ (Proposition 5.7).*

(c) *For any additive subsemigroup S of $\mathbb{N}$ and any generator $E \subseteq \mathbb{N}$ of S the category of bi-labeled graphs generated by $\{\mathbf{A}^{\circ k} \mid k \in E\}$, where $\mathbf{A}$ is the adjacency bi-labeled graph, is the set of all bi-labeled graphs $\mathbf{K}$ such that*
   - (i) *$\mathrm{P}_{\mathbf{K}}$ is a non-crossing pair partition,*
   - (ii) *each connected component of $\mathbf{K}$ is either*
     - (1) *a labeled or unlabeled single vertex without a loop,*
     - (2) *an unlabeled single vertex with loop – except if $1 \notin S$,*
     - (3) *an entirely unlabeled cycle graph or*
     - (4) *a path graph which is either entirely unlabeled or only whose degree-one vertices are labeled*
   - (iii) *if $1 \notin S$, then $\mathbf{K}$ has no connected components which are single vertices with a loop*
   - (iv) *for any $m \in \mathbb{N}$, if $m \notin S$, then $\mathbf{K}$ has no connected components which are cycle graphs on m vertices or path graphs on $m+1$ vertices*

   (Propositions 6.4, 6.6 and 6.7).

(d) *There exist distinct graph categories such that for any graph G the compact quantum supergroups of $\mathrm{Aut}(G)$ induced by them are identical (Proposition 7.10).*

1.3. Structure of the chapter. Section 2 provides all the necessary *definitions* as well as certain basic *auxiliaries* about both graph categories (Section 2.2) and categories of partitions (Section 2.3) needed for stating and proving the main results, in particular certain links between the two kinds of categories (Sections 2.1 and 2.4). It is entirely self-contained. The presentation includes a complete list of all categories of partitions in Section 2.3.3.

Sections 3, 4 and 5 all have the same substructure. Sections 3.1, 4.1 and 5.1 define for any bi-labeled graph its *vertex partition*, *component partition* and *label distances*, respectively. This is also where crucial properties of the respective construction are recognized. Sections 3.2, 4.2 and 5.2 then show how placing constraints on the allowed vertex partitions, component partitions or label distances gives rise to new graph categories. Lastly, Sections 3.3, 4.3 and 5.3 show how the respective construction can also become an invariant of any given graph category.

Section 6 answers the question which graph categories are generated by (sets of) powers of the adjacency bi-labeled graph.

Section 7 is the only place where quantum group aspects are considered. Section 7.1 recalls the construction of graph-theoretical quantum groups. In Section 7.2 it is demonstrated that an unfortunate lack of injectivity is inherent to this procedure.

Finally, Section 8 concludes with a few remarks about future research directions concerning graph categories and graph-theoretical quantum groups.

2. Graph categories and categories of partitions

In this section the definitions of both graph categories and categories of partitions are recalled. Morever, all the definitions and results are presented which are not specific to bi-labeled graphs but will be used to show the main results in later sections. The reminder about categories of partitions includes an overview of their classification.

2.1. Common fundamentals. In the following the notation for mappings, products and co-products and equivalence relations if fixed.

NOTATION 2.1. (a) In the following, let $0 \notin \mathbb{N}$ and $\mathbb{N}_0 \coloneqq \{0\} \cup \mathbb{N}$ as well as $[\![0]\!] \coloneqq \varnothing$ and $[\![n]\!] \coloneqq \{1, \ldots, n\}$ for all $n \in \mathbb{N}$.

(b) For any set X let $\wp X = \{S \subseteq X\}$ be its power set.

(c) For any sets X and Y and any mapping $f : X \to Y$ let $f_\to(A) \coloneqq \{f(a) \,|\, a \in A\}$ and $f^\leftarrow(B) \coloneqq \{a \in X \,|\, f(a) \in B\}$.

(d) Moreover, let $Y_1 \boxtimes \ldots \boxtimes Y_k \coloneqq \{(y_1, \ldots, y_k) \,|\, \forall_{i=1}^k\colon y_i \in Y_i\}$ for any sets $Y_1, \ldots, Y_k$ and any $k \in \mathbb{N}$. If $f_i\colon Y_i \to Z_i$ is a mapping for each $i \in [\![k]\!]$, then $f_1 \boxtimes \ldots \boxtimes f_k$ is the mapping $Y_1 \boxtimes \ldots \boxtimes Y_k \to Z_1 \boxtimes \ldots \boxtimes Z_k$ with $(y_1, \ldots, y_k) \mapsto (f_1(y_1), \ldots, f_k(y_k))$ for any $(y_1, \ldots, y_k) \in Y_1 \boxtimes \ldots \boxtimes Y_k$.

(e) For any pair (A_1, A_2) of sets we fix a set $A_1 \boxdot A_2$ and for each $m \in [\![2]\!]$ an injection $\iota^m_{A_1,A_2}\colon A_m \to A_1 \boxdot A_2$ such that $\bigcap_{m=1}^2 (\iota^m_{A_1,A_2})_\to(A_m) = \varnothing$ and $\bigcup_{m=1}^2 (\iota^m_{A_1,A_2})_\to(A_m) = A_1 \boxdot A_2$.

Given any set B and mappings $f_1\colon A_1 \to B$ and $f_2\colon A_2 \to B$ with common co-domain, we write $f_1 \sqcup f_2$ for the unique mapping $f\colon A_1 \boxdot A_2 \to B$ with $f \circ \iota^m_{A_1,A_2} = f_m$ for any $m \in [\![2]\!]$.

(f) Given any equivalence relation $r \subseteq X \boxtimes X$ on any set X and any map $f\colon X \to Y$ with $f(x) = f(x')$ for any $(x, x') \in r$, let X/r be the set of equivalence classes of r, let $\pi_r\colon X \to X/r$ be the mapping which sends each element to its equivalence class and let f/r be the unique mapping $X/r \to Y$ with $f = (f/r) \circ \pi_r$ (i.e., with graph $\{(\pi_r(x), f(x)) \,|\, x \in X\}$).

The ensuing results about equivalence relations are well-known. It will be helpful to be able to refer to them easily later.

REMARK 2.2. (a) If V is any set, if $s \subseteq V \boxtimes V$ is any binary relation on V, if r is the equivalence relation on V generated by s and if $\{v, v'\} \subseteq V$, then $(v, v') \in r$ if and only if there exist $k \in \mathbb{N}_0$ and $\{v_1, \ldots, v_{k+1}\}$ such that $v_1 = v$ and $v_{k+1} = v'$ and $(v_i, v_{i+1}) \in s$ or $(v_{i+1}, v_i) \in s$ for each $i \in [\![k]\!]$.

(b) For any sets V and W, any bijection f from V to W and any equivalence relation r on V, if $r' = (f \boxtimes f)_{\rightarrow}(r)$, then r' is an equivalence relation on W and the mappings $(\pi_{r'} \circ f)/r \colon V/r \to W/r'$ and $(\pi_r \circ f^{-1})/r' \colon W/r' \to V/r$ are well-defined mutually inverse bijections.

(c) Given any two sets V and W, any mapping f from V to W and any binary relation $s \subseteq V \boxtimes V$ on V, if r is the equivalence relation on V generated by s, then the equivalence relations on W generated by $(f \boxtimes f)_{\rightarrow}(r)$ and by $(f \boxtimes f)_{\rightarrow}(s)$ coincide.

Bi-labeled graphs and "partitions" have some common building blocks.

DEFINITION 2.3. Let $\{k, \ell\} \subseteq \mathbb{N}_0$ be arbitrary.

(a) Fix any two injections $i \mapsto {}^{\blacksquare}i$ and $j \mapsto {}_{\blacksquare}j$ defined on $\mathbb{N}$ with ${}^{\blacksquare}i \neq {}_{\blacksquare}j$ for any $\{i, j\} \subseteq \mathbb{N}$.

(b) We call $\Pi^k_\ell := \{{}^{\blacksquare}i, {}_{\blacksquare}j \mid i \in [\![k]\!] \wedge j \in [\![\ell]\!]\}$ the *total set of* k upper and ℓ lower *points*.

(c) Given any set Y, any $\{k, \ell\} \subseteq \mathbb{N}_0$ and any mappings $f : [\![k]\!] \to Y$ and $g : [\![\ell]\!] \to Y$ we write $f\,{}^{\blacksquare}{}_{\blacksquare}\,g$ for the mapping $\Pi^k_\ell \to Y$ defined by ${}^{\blacksquare}i \mapsto f(i)$ and ${}_{\blacksquare}j \mapsto g(j)$ for any $i \in [\![k]\!]$ and $j \in [\![\ell]\!]$.

(d) The *successor function* for k upper and ℓ lower points is the permutation $\nu^k_\ell()$ of Π^k_ℓ defined by ${}^{\blacksquare}i \mapsto {}^{\blacksquare}(i-1)$ and ${}_{\blacksquare}j \mapsto {}_{\blacksquare}(j+1)$ for all $\{i, j\} \subseteq \mathbb{N}$ with $1 < i \leq k$ and $1 \leq j < \ell$, by ${}_{\blacksquare}\ell \mapsto {}^{\blacksquare}k$ if $k \neq 0 \neq \ell$ and ${}_{\blacksquare}\ell \mapsto {}_{\blacksquare}1$ if $k = 0 < \ell$ and by ${}^{\blacksquare}1 \mapsto {}_{\blacksquare}1$ if $k \neq 0 \neq \ell$ and ${}^{\blacksquare}1 \mapsto {}^{\blacksquare}k$ if $\ell = 0 < k$.

(e) By definition, the *cyclic order* $\Gamma^k_\ell \equiv (\cdot \mid \cdot \mid \cdot)^k_\ell$ for k upper and ℓ lower points is the ternary relation on Π^k_ℓ which for any $\{\mathtt{a}, \mathtt{b}, \mathtt{c}\} \subseteq \Pi^k_\ell$ satisfies $(\mathtt{a} \mid \mathtt{b} \mid \mathtt{c})^k_\ell$ if and only if $|\{\mathtt{a}, \mathtt{b}, \mathtt{c}\}| = 3$ and there exist $\{u, v\} \subseteq \mathbb{N}$ such that $u + v < k + \ell$ and $(\nu^k_\ell)^{\circ u}(\mathtt{a}) = \mathtt{b}$ and $(\nu^k_\ell)^{\circ v}(\mathtt{b}) = \mathtt{c}$.

(f) For any two sets $\mathsf{S} \subseteq \Pi^k_\ell$ and $\mathsf{T} \subseteq \Pi^k_\ell$ with $\mathsf{S} \cap \mathsf{T} = \varnothing$ we say that S and T *cross* with respect to Γ^k_ℓ, in symbols: $\mathsf{S} \mathrel{\rlap{$+$}\diagdown}^k_\ell \mathsf{T}$, if there exist $\{\mathtt{a}, \mathtt{c}\} \subseteq \mathsf{S}$ and $\{\mathtt{b}, \mathtt{d}\} \subseteq \mathsf{T}$ with $(\mathtt{a} \mid \mathtt{b} \mid \mathtt{c})^k_\ell$ and $(\mathtt{b} \mid \mathtt{c} \mid \mathtt{d})^k_\ell$ and $(\mathtt{c} \mid \mathtt{d} \mid \mathtt{a})^k_\ell$. Otherwise we call S and T *non-crossing* with respect to Γ^k_ℓ, written as $\mathsf{S} \mathrel{\diagdown\!\!\diagdown}^k_\ell \mathsf{T}$.

(g) Given any $\{\mathtt{a}, \mathtt{b}\} \subseteq \Pi^k_\ell$ with, importantly, $\mathtt{a} \neq \mathtt{b}$ the *(open) cyclic interval* with respect to Γ^k_ℓ from $\mathtt{a}$ to $\mathtt{b}$ is the set $]\mathtt{a}, \mathtt{b}[^k_\ell := \{(\nu^k_\ell)^{\circ i}(\mathtt{a}) \mid i \in [\![u-1]\!]\}$, where $u = \min\{j \in \mathbb{N} \mid (\nu^k_\ell)^{\circ j}(\mathtt{a}) = \mathtt{b}\}$.

Moreover, we let $[\mathtt{a}, \mathtt{b}[^k_\ell := \{\mathtt{a}\} \cup]\mathtt{a}, \mathtt{b}[^k_\ell$ and $]\mathtt{a}, \mathtt{b}]^k_\ell :=]\mathtt{a}, \mathtt{b}[^k_\ell \cup \{\mathtt{b}\}$ as well as $[\mathtt{a}, \mathtt{b}]^k_\ell := \{\mathtt{a}\} \cup]\mathtt{a}, \mathtt{b}[^k_\ell \cup \{\mathtt{b}\}$, speaking of *right-open*, *left-open* and *closed intervals*, respectively.

All these sets are also referred to as *convex* with respect to Γ^k_ℓ.

(h) The *strict linear order* $<^k_\ell$ for k upper and ℓ lower points is the binary relation on Π^k_ℓ which for any $\{\mathtt{a}, \mathtt{b}\} \subseteq \Pi^k_\ell$ satisfies $\mathtt{a} <^k_\ell \mathtt{b}$ if and only if

$({}^{\blacksquare}k \mid \mathtt{a} \mid \mathtt{b})^k_\ell$ or $\mathtt{a} = {}^{\blacksquare}k \neq \mathtt{b}$ in case $0 < k$ and $(\mathtt{a} \mid \mathtt{b} \mid {}_{\blacksquare}\ell)^k_\ell$ or $\mathtt{a} \neq {}_{\blacksquare}\ell = \mathtt{b}$ in case $0 < \ell$ (which is equivalent if both $0 < k$ and $0 < \ell$).

Moreover, $\leq^k_\ell := <^k_\ell \cup \{(\mathtt{a},\mathtt{a}) \mid \mathtt{a} \in \Pi^k_\ell\}$ is called the *linear order* for k upper and ℓ lower points.

(i) The *horizontal reflection* onto k upper and ℓ lower points is the bijection $\kappa^k_\ell \colon \Pi^k_\ell \to \Pi^\ell_k$ with ${}^{\blacksquare}i \mapsto {}_{\blacksquare}i$ and ${}_{\blacksquare}j \mapsto {}^{\blacksquare}j$ for all $i \in [\![k]\!]$ and $j \in [\![\ell]\!]$.

(j) In contrast, the *vertical reflection* of k upper and ℓ lower points is the bijection $\rho^k_\ell \colon \Pi^k_\ell \to \Pi^k_\ell$ defined by ${}^{\blacksquare}i \mapsto {}^{\blacksquare}(k-i+1)$ and ${}_{\blacksquare}j \mapsto {}_{\blacksquare}(\ell-j+1)$ for any $i \in [\![k]\!]$ and $j \in [\![\ell]\!]$.

(k) For any $r \in \{\curvearrowleft, \curvearrowright, \curvearrowleft, \curvearrowright\}$ the *r-rotation* ${\omega^{r,k}_\ell}$ onto k upper and ℓ lower points is defined,

   (i) if $0 < k$, as the bijection $\omega^{\curvearrowleft,k}_\ell \colon \Pi^{k-1}_{\ell+1} \to \Pi^k_\ell$ which satisfies ${}^{\blacksquare}i \mapsto {}^{\blacksquare}(i+1)$ for any $i \in [\![k]\!] \backslash [\![1]\!]$ and ${}_{\blacksquare}1 \mapsto {}^{\blacksquare}1$ and ${}_{\blacksquare}j \mapsto {}_{\blacksquare}(j-1)$ for any $j \in [\![\ell]\!]$.

   (ii) if $0 < \ell$, as the bijection $\omega^{\curvearrowright,k}_\ell \colon \Pi^{k+1}_{\ell-1} \to \Pi^k_\ell$ which satisfies ${}^{\blacksquare}i \mapsto {}^{\blacksquare}i$ for any $i \in [\![k]\!]$ and ${}_{\blacksquare}j \mapsto {}_{\blacksquare}j$ for any $j \in [\![\ell-1]\!]$ and ${}^{\blacksquare}(k+1) \mapsto {}_{\blacksquare}\ell$.

   (iii) if $0 < \ell$, as the bijection $\omega^{\curvearrowleft,k}_\ell \colon \Pi^{k+1}_{\ell-1} \to \Pi^k_\ell$ which satisfies ${}^{\blacksquare}i \mapsto {}^{\blacksquare}(i-1)$ for any $i \in [\![k]\!]$ and ${}^{\blacksquare}1 \mapsto {}_{\blacksquare}1$ and ${}_{\blacksquare}j \mapsto {}_{\blacksquare}(j+1)$ for any $j \in [\![\ell]\!] \backslash [\![1]\!]$.

   (iv) if $0 < k$, as the bijection $\omega^{\curvearrowright,k}_\ell \colon \Pi^{k-1}_{\ell+1} \to \Pi^k_\ell$ which satisfies ${}_{\blacksquare}(\ell+1) \mapsto {}^{\blacksquare}k$ and ${}^{\blacksquare}i \mapsto {}^{\blacksquare}i$ for any $i \in [\![k-1]\!]$ and ${}_{\blacksquare}j \mapsto {}_{\blacksquare}j$ for any $j \in [\![\ell]\!]$.

(l) For any set $\mathsf{S} \subseteq \Pi^k_\ell$, firstly, let $\alpha(\mathsf{S}) = |\Pi^k_0 \cap \mathsf{S}|$ be the *upper point count* of S and $\beta(\mathsf{S}) = |\Pi^0_\ell \cap \mathsf{S}|$ the *lower point count* of S, secondly, let the *upper enumeration* of S be the injection ${\eta_{\mathsf{S}}}^k_\ell : [\![\alpha(\mathsf{S})]\!] \to \Pi^k_\ell$ with the graph

$$\{(|\Pi^i_0 \cap \mathsf{S}|, {}^{\blacksquare}i) \mid i \in [\![k]\!] \wedge {}^{\blacksquare}i \in \mathsf{S}\}$$

and the *lower enumeration* of S the injection ${\theta_{\mathsf{S}}}^k_\ell : [\![\beta(\mathsf{S})]\!] \to \Pi^k_\ell$ with the graph

$$\{(|\Pi^0_j \cap \mathsf{S}|, {}_{\blacksquare}j) \mid j \in [\![\ell]\!] \wedge {}_{\blacksquare}j \in \mathsf{S}\},$$

and, thirdly, let the *insertion* onto S for k upper and ℓ lower points be the injection $\Pi^{\alpha(\mathsf{S})}_{\beta(\mathsf{S})} \to \Pi^k_\ell$ defined by ${\gamma_{\mathsf{S}}}^k_\ell := {\eta_{\mathsf{S}}}^k_\ell \,{}^{\blacksquare}{}_{\blacksquare}\, {\theta_{\mathsf{S}}}^k_\ell$.

2.2. Bi-labeled graphs and graph categories. After fixing which notion of graphs will be employed, this section gives the definition of bi-labeled graphs, their operations and categories. Additionally, further operations for bi-labeled graphs are introduced and an alternative criterion for a set to be a graph categoryd is derived.

2.2.1. *Graphs.* Throughout, all "graphs" will be finite, simple and undirected and may or may not have loops (also known as self-edges).

DEFINITION 2.4. (a) We call G a *graph* if and only if there exist sets V and E with $|V| < \infty$, with $E \subseteq \wp V$, with $|e| \in [\![2]\!]$ for any $e \in E$ and with $G = (V, E)$.

(b) Given any graph $H = (W, F)$ we say that any graph (V, E) is a *subgraph* of H if $V \subseteq W$ and $E \subseteq F$ and a *full subgraph* of H if, additionally, $E = F \cap \wp V$.

(c) For any graphs $G = (V, E)$ and $G' = (V', E')$ any mapping $f\colon V \to V'$ is called a
   (i) *graph homomorphism* from G to G' if $f_{\to}(e) \in E'$ for any $e \in E$,
   (ii) *graph embedding* of G into G' if f is an injective graph homomorphism from G to G' such that $f^{\leftarrow}(e') \in E$ for any $e' \in E'$ with $e' \subseteq \mathrm{ran}(f)$,
   (iii) *graph isomorphism* if f is a surjective graph embedding of G into G'.

(d) Given any graph $G = (V, E)$ and any set $U \subseteq V$ we let $G|_U \coloneqq (U, E \cap \wp U)$.

(e) For any graphs G_1 and G_2 with $G_m = (V_m, E_m)$ for each $m \in [\![2]\!]$ let $G_1 \boxdot G_2 \coloneqq (V_1 \boxdot V_2, \dot\bigcup_{m=1}^{2} \{(\iota^m_{V_1,V_2})_{\to}(e) \,|\, e \in E_m\})$.

(f) Given any graph $G = (V, E)$ and any equivalence relation r on V, define $G/r \coloneqq (V/r, \{(\pi_r)_{\to}(e) \,|\, e \in E\})$.

Of course, for any graphs G_1 and G_2 the pair $G_1 \boxdot G_2$ is a graph as well and $\iota^m_{G_1,G_2}$ is a graph embedding from G_m to $G_1 \boxdot G_2$ for any $m \in [\![2]\!]$. Moreover, for any graph $G = (V, E)$ and any equivalence relation r on V the pair G/r is graph and π_r a graph homomorphism from G to G/r.

DEFINITION 2.5. Let $G = (V, E)$ be any graph and let $\{v, v'\} \subseteq V$.

(a) We say that v and v' are *adjacent* in G, in symbols $v \backsim_G v'$, if $\{v, v'\} \in E$.

(b) We call v and v' *connected* in G if they belong to the same class with respect to the equivalence relation $\simeq_G$ on V generated by $\backsim_G$

(c) The classes of $\simeq_G$ are called the *connected components* of G.

Graph homomorphisms preserve both adjacacency and connectedness.

REMARK 2.6. (a) For any graphs G_1 and G_2 with vertex sets V_1 and V_2, respectively, and for any $\{u, u'\} \subseteq V_1 \boxdot V_2$ the relation $u \backsim_{G_1 \boxdot G_2} u'$ holds if and only if there exist $m \in [\![2]\!]$ and $\{v, v'\} \in V_m$ such that $u = \iota^m_{V_1,V_2}(v)$ and $u' = \iota^m_{V_1,V_2}(v')$ and $v \backsim_{G_m} v'$. Likewise, $u \simeq_{G_1 \boxdot G_2} u'$ holds if and only if there exist $m \in [\![2]\!]$ and $\{v, v'\} \subseteq V_m$ such that $u = \iota^m_{V_1,V_2}(v)$ and $u' = \iota^m_{V_1,V_2}(v')$ and $v \simeq_{G_m} v'$.

(b) For any equivalence relation r on the vertex set V of any graph G and for any $\{u, u'\} \subseteq V/r$ the relation $u \backsim_{G/r} u'$ holds if and only if there exist $\{v, v'\} \subseteq V$ with $u = \pi_r(v)$ and $u' = \pi_r(v')$ and $v \backsim_G v'$.

DEFINITION 2.7. Let $G = (V, E)$ be any graph.

(a) We say that v is (the *vertex sequence of) a walk* in G if there exists $k \in \mathbb{N}$ such that $v\colon [\![k+1]\!] \to V$ and $\{v_i, v_{i+1}\} \in E$ for all $i \in [\![k]\!]$. If so, we call k the *length* of v and we say that v is a walk *from* v_1 *to* v_{k+1}.

(b) Given any $\{x, x'\} \subseteq V$, the *distance* $d_G(x, x')$ between x and x' in G is defined as 0 if $x = x'$, as ∞ if $x \neq x'$ and there is no walk from x to x' in G, and otherwise as the minimal $k \in \mathbb{N}$ such that there exists a walk of length k from x to x'.

(c) Let $\{k, k'\} \subseteq \mathbb{N}$, let v and v' be walks in G of lengths k and k' respectively. We say that v is a *subwalk* of v' if there exists a strictly increasing function $f\colon [\![k+1]\!] \to [\![k'+1]\!]$ such that $v = v' \circ f$ and $v_1 = v'_1$ and $v_{k+1} = v'_{k'+1}$.

(d) For any $k \in \mathbb{N}$ any walk v of length k in G is called a
 (i) *trail* if the mapping $[\![k]\!] \to E$, $i \mapsto \{v_i, v_{i+1}\}$ is injective.
 (ii) *circuit* if v is a trail and if $v_1 = v_{k+1}$
 (iii) *path* if v is a trail and if v is injective.
 (iv) *cycle* if v is a circuit and if the restrictions of v to $[\![k+1]\!]\backslash\{k+1\}$ and to $[\![k+1]\!]\backslash\{1\}$ are both injective.

(e) For any walk v in G of any length k we call the map $v^{\diamondsuit}\colon [\![k+1]\!] \to V$, $i \mapsto v_{k-i+2}$ the *reverse walk* of v

(f) Given any $x \in V$ and any walks v and w in G of length k respectively ℓ to respectively from x the *concatenated walk of* (w, v) is the map $w \oplus v\colon [\![k+\ell+1]\!] \to V$ with $i \mapsto v_i$ if $i \leq k+1$ and $i \mapsto w_{i-k}$ if $k+1 < i$.

Remark 2.8. Let $G = (V, E)$ be any graph.

(a) For any walk v in G from any vertex $x \in V$ to any vertex $x' \in V$ the reverse walk $v^{\diamondsuit}$ of v is a walk in G from x' to x of the same length. In particular, $d_G(x', x) = d_G(x, x')$.

(b) Given any $\{x, x', x''\} \subseteq V$, any walk v in G from x to x' and any walk w in G of from x' to x'' the concatenated walk $w \oplus v$ of (w, v) is a walk in G from x to x'', whose length is the sum of the lengths of v and w. In particular, $d_G(x, x'') \leq d_G(x, x') + d_G(x', x'')$.

(c) For any walk v in G there exists a subwalk of v that is a path or cycle. In fact, for any $\{x, x'\} \subseteq V$ with $x \neq x'$ such that there exists a walk in G from x to x' we can find a path in G from x to x' of length $d_G(x, x')$.

(d) If v is any path in G of length $k = d_G(v_1, v_{k+1})$, then for any $\{j, j'\} \subseteq [\![k]\!]$ with $j < j'$ the mapping $[\![j'-j+1]\!] \to V$, $i \mapsto v_{j+i-1}$ is a path in G from v_j to $v_{j'}$ of length $j' - j = d_G(v_j, v_{j'})$.

(e) For any graph H with vertex set W, for any subgraph G of H with vertex set V and for any $\{x, x'\} \subseteq V$, always, $d_H(x, x') \leq d_G(x, x')$, with equality not guaranteed, even if G is a full subgraph.

(f) Let H be any graph, f any graph homomorphism from G to H and $\{x, x'\} \subseteq V$ arbitrary.
 (i) For any walk v in G from x to x' the map $f \circ v$ is a walk in H from $f(x)$ to $f(x')$. In particular, $d_H(f(x), f(x')) \leq d_G(x, x')$.
 (ii) If f is a graph embedding of G in H and w any walk in H from $f(x)$ to $f(x')$ with $\operatorname{ran}(w) \subseteq \operatorname{ran}(f)$, then the well-defined map $f^{-1} \circ w$ is a walk in G from x to x'.
 (iii) If f is even a graph isomorphism from G to H, then $d_H(f(x), f(x')) = d_G(x, x')$.

(g) For any graphs G_1 and G_2 with vertex sets V_1 and V_2, respectively, and for any $\{u, u'\} \subseteq V_1 \boxdot V_2$, whenever $d_{G_1 \boxdot G_2}(u, u') < \infty$ there exist $m \in [\![2]\!]$ and $\{v, v'\} \subset V_m$ such that $u = \iota^m_{V_1, V_2}(v)$ and $u' = \iota^m_{V_1, V_2}(v')$ and $d_{G_1 \boxdot G_2}(u, u') = d_{G_m}(v, v')$.

DEFINITION 2.9. For any graph $G = (V, E)$ and any $v \in V$ the number $\deg_G(v) \coloneqq |\{v' \in V \wedge \{v, v'\} \in E\}| \in \mathbb{N}_0$ is called the *degree* of v in G.

2.2.2. *Bi-labeled graphs.* The definition of "bi-labeled graph" employed here diverges slightly from the the one given originally by Mančinska and Roberson in [MR20]. However, the difference is purely formal.

DEFINITION 2.10. (a) For any $\{k, \ell\} \subseteq \mathbb{N}_0$ a (k, ℓ)-*bi-labeled graph* is any equivalence class of the following equivalence relation: Of the class of all pairs (G, g) such that $G = (V, E)$ is a graph and $g\colon \Pi_\ell^k \to V$ a mapping it calls any members (G, g) and (G', g') equivalent if and only if there exists a graph isomorphism u from G to G' with $g' = u \circ g$.
(b) Let $\mathcal{G}(k, \ell)$ be the class of all (k, ℓ)-bi-labeled graphs for any $\{k, \ell\} \subseteq \mathbb{N}_0$.
(c) Moreover, let $\mathcal{G} \coloneqq \bigcup_{k,\ell=0}^{\infty} \mathcal{G}(k, \ell)$.
(d) Conversely, for any class $\mathcal{C} \subseteq \mathcal{G}$ let $\mathcal{C}(k, \ell) \coloneqq \mathcal{C} \cap \mathcal{G}(k, \ell)$ for any $\{k, \ell\} \subseteq \mathbb{N}_0$.

The collection $\mathcal{G}(k, \ell)$ is a set for each $\{k, \ell\} \subseteq \mathbb{N}_0$ and, thus, so is $\mathcal{G}$. The following bi-labeled graphs will occur frequently.

DEFINITION 2.11. (a) For any $\{k, \ell\} \subseteq \mathbb{N}_0$ let $\mathbf{M}_{k,\ell} \in \mathcal{G}(k, \ell)$ be such that $(M, m) \in \mathbf{M}_{k,\ell}$, where $M = (\llbracket 1 \rrbracket, \varnothing)$ and where m is the unique mapping $\Pi_\ell^k \to \llbracket 1 \rrbracket$.
(b) In particular, let $\mathbf{I} \coloneqq \mathbf{M}_{1,1}$.
(c) Next, let $\mathbf{S} \in \mathcal{G}(2, 2)$ be such that $(S, s) \in \mathbf{S}$, where $S = (\llbracket 2 \rrbracket, \varnothing)$ and where $s\colon \Pi_2^2 \to \llbracket 2 \rrbracket$ is determined by ${}^{\blacksquare}1, {}_{\blacksquare}2 \mapsto 1$ and ${}^{\blacksquare}2, {}_{\blacksquare}1 \mapsto 2$.
(d) Let $\mathbf{A} \in \mathcal{G}(1, 1)$ be such that $(A, a) \in \mathbf{A}$, where $A = (\llbracket 2 \rrbracket, \{\llbracket 2 \rrbracket\})$ and where $a\colon \Pi_1^1 \to \llbracket 2 \rrbracket$ satisfies ${}^{\blacksquare}1 \mapsto 1$ and ${}_{\blacksquare}1 \mapsto 2$.
(e) The unique element of $\mathcal{G}(0, 0)$ is denoted by $\varnothing$.

2.2.3. *Graph categories.* One can introduce the following (partial) two binary and one unary operations on the set $\mathcal{G}$.

DEFINITION 2.12. (a) For any $\{k, \ell, m\} \subseteq \mathbb{N}_0$ and any $\mathbf{G} \in \mathcal{G}(k, \ell)$ and $\mathbf{H} \in \mathcal{G}(\ell, m)$ the *composition* $\mathbf{H} \circ \mathbf{G} \in \mathcal{G}(k, m)$ is defined by the condition that for any (or, equivalently, some) representatives $(G, g) \in \mathbf{G}$ and $(H, h) \in \mathbf{H}$, if $G = (X, A)$ and $H = (Y, B)$, then $(P, p) \in \mathbf{H} \circ \mathbf{G}$, where $P = (G \boxdot H)/r$ for the equivalence relation r on $X \boxdot Y$ generated by

$$\left\{ \left((\iota_{X,Y}^1 \circ g)({}_{\blacksquare}i), (\iota_{X,Y}^2 \circ h)({}^{\blacksquare}i) \right) \mid i \in \llbracket \ell \rrbracket \right\}$$

and where

$$p = \pi_r \circ \left((\iota_{X,Y}^1 \circ g|_{\Pi_0^k}) \uplus (\iota_{X,Y}^2 \circ h|_{\Pi_m^0}) \right).$$

(b) Given any $\{k_m, \ell_m\} \subseteq \mathbb{N}_0$ and $\mathbf{G}_m \in \mathcal{G}(k_m, \ell_m)$ for any $m \in \llbracket 2 \rrbracket$, the *tensor product* $\mathbf{G}_1 \otimes \mathbf{G}_2$ is the element of $\mathcal{G}(k_1 + k_2, \ell_1 + \ell_2)$ with the property that, if $(G_m, g_m) \in \mathbf{G}_m$ and $G_m = (V_m, E_m)$ for each $m \in \llbracket 2 \rrbracket$, then $(T, t) \in \mathbf{G}_1 \otimes \mathbf{G}_2$, where $T = G_1 \boxdot G_2$ and where

$$t = (\iota_{V_1,V_2}^1 \circ g_1) \uplus (\iota_{V_1,V_2}^2 \circ g_2 \circ \tau_{\ell_1,\ell_2}^{k_1,k_2}),$$

where $\tau^{k_1,k_2}_{\ell_1,\ell_2}\colon \Pi^{k_1+k_2}_{\ell_1+\ell_2}\backslash\Pi^{k_1}_{\ell_1} \to \Pi^{k_2}_{\ell_2}$ is defined by ${}^{\blacksquare}i \mapsto {}^{\blacksquare}(i-k_1)$ and ${}_{\blacksquare}j \mapsto {}_{\blacksquare}(j-\ell_1)$ for any $i \in [\![k_2]\!]$ and $j \in [\![\ell_2]\!]$.

(c) For any $\{k,\ell\} \subseteq \mathbb{N}_0$ and any $\mathbf{G} \in \mathcal{G}(k,\ell)$ the *adjoint* $\mathbf{G}^* \in \mathcal{G}(\ell,k)$ is determined by the demand that, if $(G,g) \in \mathbf{G}$, then $(G, g\circ \kappa^{\ell}_{k}) \in \mathbf{G}^*$, where $\kappa^{\ell}_{k}\colon \Pi^{\ell}_{k} \to \Pi^{k}_{\ell}$ is defined by ${}^{\blacksquare}i \mapsto {}_{\blacksquare}i$ and ${}_{\blacksquare}j \mapsto {}^{\blacksquare}j$ for all $i \in [\![\ell]\!]$ and $j \in [\![k]\!]$.

Both composition and tensor product are associative operations.

Definition 2.13. (a) A *graph category* is any $\mathcal{C} \subseteq \mathcal{G}$ with $\{\varnothing, \mathbf{I}, \mathbf{M}_{0,2}\} \subseteq \mathcal{C}$ which is closed under composition, tensor products and involution.

(b) Given any subset $\mathcal{S}$ of $\mathcal{G}$, the graph category generated by $\mathcal{S}$, i.e., the intersection of all graph categories containing $\mathcal{S}$, is denoted by $\langle \mathcal{S} \rangle$.

Evidently, $\mathcal{G}$ is a graph category. It is often helpful to consider addtional (partial) operations under which graph categories are invariant.

Definition 2.14. For any $\{k,\ell\} \subseteq \mathbb{N}_0$, any $r \in \{⤹, ⤻, ⤺, ⤸\}$ such that $\omega^{r,k}_{\ell}$ is defined and any $\mathbf{G} \in \mathcal{G}(k,\ell)$ the *r-rotation* of $\mathbf{G}$ is the element $\mathbf{G}^r \in \mathcal{G}$ with the property that, if $(G,g) \in \mathbf{G}$, then $(G, g \circ \omega^{r,k}_{\ell}) \in \mathbf{G}^r$.

We also define inductively $\mathbf{G}^{r\,m} = (\mathbf{G}^{r\,(m-1)})^r$ for any $m \in \mathbb{N}$ wherever that makes sense (meaning, e.g., $\mathbf{G}^{⤸ 2} = (\mathbf{G}^{⤸})^{⤸}$ if $r = ⤸$ and $2 \le k$), and we let $\mathbf{G}^{r\,0} = \mathbf{G}$.

Lemma 2.15. *Any graph category is closed under rotations.*

Proof. For any $\{k,\ell\} \subseteq \mathbb{N}_0$ and any $\mathbf{G} \in \mathcal{G}(k,\ell)$, if $0<k$, then $\{\mathbf{G}^{⤹}, \mathbf{G}^{⤸}\} \subseteq \langle \mathbf{G} \rangle$ because $\mathbf{G}^{⤹} = (\mathbf{I}\otimes\mathbf{G})\circ(\mathbf{M}_{0,2}\otimes\mathbf{I}^{\otimes(k-1)})$ and $\mathbf{G}^{⤸} = (\mathbf{G}\otimes\mathbf{I})\circ(\mathbf{I}^{\otimes(k-1)}\otimes\mathbf{M}_{0,2})$. If $0<\ell$, then the identities $\mathbf{G}^{⤺} = ((\mathbf{G}^*)^{⤹})^*$ and $\mathbf{G}^{⤻} = ((\mathbf{G}^*)^{⤸})^*$ hence prove the claim. ☐

Definition 2.16. For any $\{k,\ell\} \subseteq \mathbb{N}_0$ and any $\mathbf{G} \in \mathcal{G}(k,\ell)$ the *reflection* of $\mathbf{G}$ is the unique $\mathbf{G}^{\wedge} \in \mathcal{G}(k,\ell)$ such that, if $(G,g) \in \mathbf{G}$, then $(G, g \circ \rho^{k}_{\ell}) \in \mathbf{G}^{\wedge}$.

Lemma 2.17. *Any graph category is closed under reflection.*

Proof. For any $\{k,\ell\} \subseteq \mathbb{N}_0$ and $\mathbf{G} \in \mathcal{G}(k,\ell)$ the reflection $\mathbf{G}^{\wedge} = ((\mathbf{G}^*)^{⤹\ell})^{⤻k}$ is an element of $\langle \mathbf{G} \rangle$ by Lemma 2.15. That is true even in the cases where one or both of k and ℓ is 0. ☐

Definition 2.18. For any $\{k,\ell\} \subseteq \mathbb{N}_0$, any subset $\mathsf{T} \subseteq \Pi^k_\ell$ with $|\mathsf{T}| = 2$ which is convex with respect to Γ^k_ℓ, and any $\mathbf{G} \in \mathcal{G}(k,\ell)$ the *erasing* of T from $\mathbf{G}$ is the element $E(\mathbf{G},\mathsf{T}) \in \mathcal{G}(\alpha(\mathsf{M}), \beta(\mathsf{M}))$ such that, if $(G,g) \in \mathbf{G}$ and $G = (V,E)$, then $(P,p) \in E(\mathbf{G},\mathsf{T})$, where $P = (W,F)$, where, if $T = g_{\to}(\mathsf{T})$ and $\mathsf{M} = \Pi^k_\ell \backslash \mathsf{T}$, then

$$W = \{\{v\} \mid v \in V\backslash T\} \cup \{T\}$$

and

$$\begin{aligned} F = {} & \{\{\{v\} \mid v \in e\} \mid e \in E \wedge e \cap T = \varnothing\} \\ & \cup \{\{\{v\}, T \mid v \in e\backslash T\} \mid e \in E \wedge \varnothing \neq e \cap T \neq e\} \\ & \cup \{\{T\} \mid \exists e \in E\colon e \subseteq T\} \end{aligned}$$

and

$$p\colon \Pi_{\beta(\mathsf{M})}^{\alpha(\mathsf{M})} \to W,\ \mathsf{a} \mapsto \begin{cases} \{(g \circ \gamma_{\mathsf{M}})(\mathsf{a})\} & \text{if } (g \circ \gamma_{\mathsf{M}})(\mathsf{a}) \notin T \\ T & \text{otherwise.} \end{cases}$$

Remark 2.19. With the same definitions as in the preceding Definition 2.18 and

$$\pi\colon V \to W,\ v \mapsto \begin{cases} \{v\} & \text{if } v \notin T, \\ T & \text{otherwise,} \end{cases}$$

the following hold:

(a) $p = \pi \circ g \circ \gamma_{\mathsf{M}}$.
(b) π is a graph homomorphism from G to P.
(c) $\pi|_{V\setminus T}$ is a graph embedding of $G|_{V\setminus T}$ into P.

Lemma 2.20. *Any graph category is closed under erasing*

Proof. For any $\{k,\ell\} \subseteq \mathbb{N}_0$, any $\mathbf{G} \in \mathcal{G}(k,\ell)$ and any $\mathsf{T} \subseteq \Pi_\ell^k$ with $|\mathsf{T}| = 2$ which is consecutive with respect to Γ_ℓ^k the erasing $E(\mathbf{G},\mathsf{T})$ can be expressed as $(\mathbf{I}^{\otimes(j-1)} \otimes \mathbf{M}_{2,0} \otimes \mathbf{I}^{\otimes(\ell-j-1)}) \circ \mathbf{G}$ if $\mathsf{T} = \{{}_{\blacksquare}j, {}_{\blacksquare}(j+1)\}$ for some $j \in [\![\ell-1]\!]$, as $(\mathbf{I}^{\otimes(i-1)} \otimes \mathbf{M}_{0,2} \otimes \mathbf{I}^{\otimes(k-i-1)}) \circ \mathbf{G}$ if $\mathsf{T} = \{{}^{\blacksquare}i, {}^{\blacksquare}(i+1)\}$ for some $i \in [\![k-1]\!]$, as $(\mathbf{M}_{2,0} \otimes \mathbf{I}^{\otimes(\ell-1)}) \circ (\mathbf{I} \otimes \mathbf{G}) \circ (\mathbf{M}_{0,2} \otimes \mathbf{I}^{\otimes(k-1)})$ if $\mathsf{T} = \{{}^{\blacksquare}1, {}_{\blacksquare}1\}$ and as $(\mathbf{I}^{\otimes(\ell-1)} \otimes \mathbf{M}_{2,0}) \circ (\mathbf{G} \otimes \mathbf{I}) \circ (\mathbf{I}^{\otimes(k-1)} \otimes \mathbf{M}_{0,2})$ if $\mathsf{T} = \{{}^{\blacksquare}k, {}_{\blacksquare}\ell\}$. And that implies $E(\mathbf{G},\mathsf{T}) \in \langle \mathbf{G} \rangle$. □

Invariance under these new operations is usually simpler to check, which is why the following is a useful criterion.

Proposition 2.21. *Any subset $\mathcal{C}$ of $\mathcal{G}$ is a graph category if only if $\mathbf{I} \in \mathcal{C}$ and $\mathcal{C}$ is closed under rotation, tensor products, reflection and erasing.*

Proof. That graph categories are closed under rotation, reflection and erasing was shown in Lemmata 2.15, 2.17 and 2.20, respectively. Hence, let $\mathcal{C} \subseteq \mathcal{G}$ with $\mathbf{I} \in \mathcal{C}$ be closed under rotation, tensor products, reflection and erasing. The identities $\mathbf{M}_{0,2} = \mathbf{I}^{\curvearrowright}$ and $\varnothing = E(\mathbf{I}, \{{}^{\blacksquare}1, {}_{\blacksquare}1\})$ prove $\{\varnothing, \mathbf{I}, \mathbf{M}_{0,2}\} \subseteq \mathcal{C}$. Thus, we only need to show that $\mathcal{C}$ is closed under forming adjoints and under composition. The former follows from the fact that for any $\{k,\ell\} \subseteq \mathbb{N}_0$ and $\mathbf{G} \in \mathcal{C}(k,\ell)$ the adjoint $\mathbf{G}^*$ can be written as $((\mathbf{G}^{\wedge})^{\curvearrowleft \ell})^{\curvearrowright k} \in \mathcal{C}(\ell,k)$. And $\mathcal{C}$ is invariant under composition because for any $\{k,\ell,m\} \subseteq \mathbb{N}_0$ and $\mathbf{H} \in \mathcal{C}(k,\ell)$ and $\mathbf{K} \in \mathcal{C}(\ell,m)$ the composition $\mathbf{K} \circ \mathbf{H}$ is identical to $\mathbf{S}_\ell$, where, first, $\mathbf{R}_0 = \mathbf{K} \otimes \mathbf{I}^{\otimes \ell} \otimes \mathbf{H} \in \mathcal{C}(2\ell+k, m+2\ell)$ and $\mathbf{R}_i = E(\mathbf{R}_{i-1}, \{{}_{\blacksquare}(m+\ell-i+1), {}_{\blacksquare}(m+\ell-i+2)\}) \in \mathcal{C}(2\ell+k, m+2(\ell-i))$ for any $i \in [\![\ell]\!]$ and, then, $\mathbf{S}_0 = \mathbf{R}_\ell \in \mathcal{C}(2\ell+k, m)$ and $\mathbf{S}_i = E(\{{}^{\blacksquare}(\ell-i+1), {}^{\blacksquare}(\ell-i+2)\}, \mathbf{S}_{i-1}) \in \mathcal{C}(2(\ell-i)+k, m)$ for any $i \in [\![\ell]\!]$. □

2.3. Partitions and categories of partitions. The present section recalls the definition of "partitions", their operations and categories. Moreover, a list of all categories of partitions is presented. The section closes with some abstract results about links between graph categories and categories of partitions.

2.3.1. *Partitions.* In [BS09] Banica and Speicher used the following notion of "partitions" for their easy quantum groups.

DEFINITION 2.22. For any $\{k,\ell\} \subseteq \mathbb{N}_0$ a *partition of k upper and ℓ outer points* is any partition of Π^k_ℓ

And it is this notion and this notion only that will be used in the present work.

REMARK 2.23. This definition is decidedly different from the one used in [MR19, p. 11]. For Banica and Speicher the elements of any partition of k upper and ℓ lower points, the so-called *blocks*, are always non-empty. In contrast, Mančinska and Roberson allow "empty blocks" in the following sense: For them a partition of k upper and ℓ lower points is effectively any equivalence class of the following equivalence relation: Of the class of all pairs (V,g) such that V is a finite set and $g\colon \Pi^k_\ell \to V$ a mapping it calls any members (V,g) and (V',g') equivalent if and only if there exists a bijection $u\colon V \to V'$ with $g' = u \circ g$. If one added the condition that in any pair (V,g) the mapping g be surjective, this definition would become equivalent to Banica and Speicher's.

In particular, the results in the literature following [BS09] do generally *not* apply to partitions in Mančinska and Roberson's sense.

DEFINITION 2.24. (a) The trivial partition $\{\Pi^1_1\}$ of Π^1_1 is denoted by $|$.
(b) Similarly, the symbols $\sqcap$ and $\sqcup$ are used for the trivial partitions $\{\Pi^0_2\}$ of Π^0_2 and $\{\Pi^2_0\}$ of Π^2_0, respectively.

The only partition of $\Pi^0_0 = \varnothing$ is of course $\varnothing$.

2.3.2. *Categories of partitions.* There are three basic operations for partitions, composition, tensor products and forming adjoints, just like for bi-labeld graphs. Analogs of the other operations of Section 2.2.3 exist as well but are not needed here.

DEFINITION 2.25. For any $\{k,\ell,m\} \subseteq \mathbb{N}_0$ and any partitions q of Π^k_ℓ and p of Π^ℓ_m, the *composition* of (p,q) is defined as the partition

$$\begin{aligned} pq :=& \\ &\{\mathsf{A} \in q \wedge \mathsf{A} \subseteq \Pi^k_0\} \\ &\cup \Big\{\bigcup\{\mathsf{A} \cap \Pi^k_0 \,|\, \mathsf{A} \in q \wedge \mathsf{A} \cap (\kappa^\ell_0)_{\rightarrow}(\mathsf{B}) \neq \varnothing\} \cup \bigcup\{\mathsf{C} \cap \Pi^0_m \,|\, \mathsf{C} \in p \wedge \mathsf{C} \cap \mathsf{B} \neq \varnothing\}\Big\}_{\mathsf{B} \in s} \backslash \{\varnothing\} \\ &\cup \{\mathsf{C} \in p \wedge \mathsf{C} \subseteq \Pi^0_m\} \end{aligned}$$

of Π^k_m, where $s \equiv ((\kappa^\ell_0)^{\leftarrow}(q|_{\Pi^0_\ell})) \vee (p|_{\Pi^\ell_0})$.

DEFINITION 2.26. For any $\{k,\ell\} \subseteq \mathbb{N}_0$ and any partition p of Π^k_ℓ the *adjoint* of p is the partition $p^* := (\kappa^k_\ell)_{\rightarrow}(p)$ of Π^ℓ_k.

DEFINITION 2.27. For any $\{k_1,k_2,\ell_1,\ell_2\} \subseteq \mathbb{N}_0$ and any partitions p_1 of $\Pi^{k_1}_{\ell_1}$ and p_2 of $\Pi^{k_2}_{\ell_2}$ let

$$p_1 \otimes p_2 := (\gamma_{\mathsf{H}_1})_{\rightarrow}(p_1) \cup (\gamma_{\mathsf{H}_2})_{\rightarrow}(p_2),$$

where $\mathsf{H}_1 \equiv \Pi_{\ell_1}^{k_1}$ and $\mathsf{H}_2 \equiv \Pi_{\ell_1+\ell_2}^{k_1+k_2} \backslash \Pi_{\ell_2}^{k_1}$, be the *tensor product* of (p_1, p_2).

DEFINITION 2.28. A *category of partitions* is any set of partitions in the sense of Definition 2.22 including $\{\shortmid, \sqcap\}$ and closed under composition, tensor products and involution.

For later results it is helpful to see that the operations for partitions can also be expressed in a way that more closely resembles the respective definition for bi-labeled graphs.

LEMMA 2.29. *For any $\{k, \ell, m\} \subseteq \mathbb{N}_0$ and any partitions p of Π_ℓ^k and q of Π_m^ℓ the composition qp can be computed as $\ker(f)$ for any finite sets X and Y and any mappings $g\colon \Pi_\ell^k \to X$ and $h\colon \Pi_m^\ell \to Y$ with $p = \ker(g)$ and $q = \ker(h)$, where r is the equivalence relation on $X \boxdot Y$ generated by*

$$\{((\iota_{X,Y}^1 \circ g)(_\blacksquare i), (\iota_{X,Y}^2 \circ h)(^\blacksquare i)) \mid i \in [\![\ell]\!]\}$$

and where

$$f = \pi_r \circ ((\iota_{X,Y}^1 \circ g|_{\Pi_0^k}) \uplus (\iota_{X,Y}^2 \circ h|_{\Pi_m^0})).$$

PROOF. If $u = ((\kappa_0^\ell)^{\leftarrow}(p)) \vee (q|_{\Pi_0^\ell})$, then we have to show that $\ker(f)$ equals

$$\begin{aligned} qp = \{&\mathsf{A} \in p \wedge \mathsf{A} \subseteq \Pi_0^k\} \\ &\uplus \Big\{\bigcup\{\mathsf{A} \cap \Pi_0^k \mid \mathsf{A} \in p \wedge \mathsf{A} \cap (\kappa_0^\ell)_{\rightarrow}(\mathsf{B}) \neq \varnothing\} \\ &\qquad\qquad \uplus \bigcup\{\mathsf{C} \cap \Pi_m^0 \mid \mathsf{C} \in q \wedge \mathsf{C} \cap \mathsf{B} \neq \varnothing\} \mid \mathsf{B} \in u\Big\} \backslash \{\varnothing\} \\ &\uplus \{\mathsf{C} \in q \wedge \mathsf{C} \subseteq \Pi_m^0\}. \end{aligned}$$

We abbreviate

$$a \equiv \iota_{X,Y}^1 \circ g|_{\Pi_0^k}, \quad b \equiv \iota_{X,Y}^1 \circ g \circ \kappa_0^\ell, \quad c \equiv \iota_{X,Y}^2 \circ h|_{\Pi_0^\ell} \quad \text{and} \quad d \equiv \iota_{X,Y}^2 \circ h|_{\Pi_m^0}.$$

The proof is divided into two steps, relating r and u and then using this intermediate result to prove the claim.

Step 1: We first prove that

$$w := \{\{z\} \mid z \in (X \boxdot Y) \backslash (\operatorname{ran}(b) \cup \operatorname{ran}(c))\} \cup \{b_{\rightarrow}(\mathsf{B}) \cup c_{\rightarrow}(\mathsf{B}) \mid \mathsf{B} \in u\}.$$

is the same as $(X \boxdot Y)/r$. Doing so requires five steps in itself. We have to 1) rewrite u in terms of b and c and then use that to check that 2) the union of w is all of $X \boxdot Y$, that 3) the elements of $X \boxdot Y$ are pairwise disjoint (i.e., that w is actually a partition of $X \boxdot Y$), that 4) the generating set of r is contained in $\sim_w$ and that 5) the partition w is the finest one with this property.

Step 1.1: First, we recognize that

$$u = \ker(b) \vee \ker(c).$$

Indeed, as $\iota_{X,Y}^1$ is injective and as $p = \ker(g)$ the partitions $\ker(\iota_{X,Y}^1 \circ g \circ \kappa_0^\ell)$ and $\ker(g \circ \kappa_0^\ell) = (\kappa_0^\ell)^{\leftarrow}(\ker(g)) = (\kappa_0^\ell)^{\leftarrow}(p)$ coincide and, since $\iota_{X,Y}^2$ is injective and $q = \ker(h)$, so do $\ker(\iota_{X,Y}^2 \circ h|_{\Pi_0^\ell})$ and $q|_{\Pi_0^\ell}$.

Step 1.2: Because u is a partition of Π_0^ℓ the union $\bigcup u$ is all of Π_0^ℓ. For that reason,

$$\bigcup w = (X \boxdot Y)\backslash(\operatorname{ran}(b) \cup \operatorname{ran}(c)) \cup (\operatorname{ran}(b) \cup \operatorname{ran}(c)) = X \boxdot Y.$$

Step 1.3: Next, we need to show that any two elements of w are either identical or disjoint. Hence, let $\{Z, Z'\} \subseteq w$ be arbitrary with $Z \neq Z'$. We prove $Z \cap Z' = \varnothing$. Given that w is defined as the union of two sets, we need to distinguish cases.

Case 1.3.1: If there are $\{z, z'\} \subseteq (X \boxdot Y)\backslash(\operatorname{ran}(b) \cup \operatorname{ran}(c))$ with $Z = \{z\}$ and $Z' = \{z'\}$, then $Z \neq Z'$ requires $z \neq z'$. It follows $Z \cap Z' = \{z\} \cap \{z'\} = \varnothing$.

Case 1.3.2: If there exist $\{\mathsf{B}, \mathsf{B}'\} \subseteq u$ with $Z = b_\rightarrow(\mathsf{B}) \cup c_\rightarrow(\mathsf{B})$ and $Z' = b_\rightarrow(\mathsf{B}') \cup c_\rightarrow(\mathsf{B}')$, then the assumption $Z \neq Z'$ necessitates $\mathsf{B} \neq \mathsf{B}'$. The fact that $\operatorname{ran}(b) \cap \operatorname{ran}(c) \subseteq \operatorname{ran}(\iota^1_{X,Y}) \cap \operatorname{ran}(\iota^2_{X,Y}) = \varnothing$ implies that the intersection

$$Z \cap Z' = (b_\rightarrow(\mathsf{B}) \cup c_\rightarrow(\mathsf{B})) \cap (b_\rightarrow(\mathsf{B}') \cup c_\rightarrow(\mathsf{B}'))$$

collapses to

$$(b_\rightarrow(\mathsf{B}) \cap b_\rightarrow(\mathsf{B}')) \cup (c_\rightarrow(\mathsf{B}) \cap c_\rightarrow(\mathsf{B}')).$$

We conclude from this that Z and Z' are disjoint if and only if both $b_\rightarrow(\mathsf{B}) \cap b_\rightarrow(\mathsf{B}')$ and $c_\rightarrow(\mathsf{B}) \cap c_\rightarrow(\mathsf{B}')$ are empty. And this is indeed true: If there existed points $\mathsf{c} \in \mathsf{B}$ and $\mathsf{c}' \in \mathsf{B}'$ with $b(\mathsf{c}) = b(\mathsf{c}')$, that would imply $\mathsf{c} \sim_{\ker(b)} \mathsf{c}'$ and thus $\mathsf{c} \sim_u \mathsf{c}'$ by $\ker(b) \leq u$ by Step 1.1, yielding the contradiction $\mathsf{B} = \mathsf{B}'$. An analogous argument can be given for $c_\rightarrow(\mathsf{B}) \cap c_\rightarrow(\mathsf{B}') = \varnothing$ due to $\ker(c) \leq u$.

Case 1.3.3: If there exist $\mathsf{B} \in u$ and $z \in (X \boxdot Y)\backslash(\operatorname{ran}(b) \cup \operatorname{ran}(c))$ such that $Z = \{z\}$ and $Z' = b_\rightarrow(\mathsf{B}) \cup c_\rightarrow(\mathsf{B})$, then $Z \cap Z' = \varnothing$ by $z \notin b_\rightarrow(\mathsf{B}) \cup c_\rightarrow(\mathsf{B}) \subseteq \operatorname{ran}(b) \cup \operatorname{ran}(c)$.

And these are all the cases we need to consider. In conclusion, w is indeed a partition of $X \boxdot Y$.

Step 1.4: For any $\mathsf{c} \in \Pi_0^\ell$, if $\mathsf{B} \in u$ is such that $\mathsf{c} \in \mathsf{B}$, then $b(\mathsf{c}) \sim_w c(\mathsf{c})$ because both $b(\mathsf{c}) \in Z$ and $c(\mathsf{c}) \in Z$ for $Z = b_\rightarrow(\mathsf{B}) \cup c_\rightarrow(\mathsf{B}) \in w$. Hence,

$$s = \{(b(\mathsf{c}), c(\mathsf{c})) \mid \mathsf{c} \in \Pi_0^\ell\} \subseteq \sim_w.$$

Step 1.5: It remains to prove that $\sim_w$ is the finest equivalence relation on $X \boxdot Y$ extending s. To do so we let f be any partition of $X \boxdot Y$ such that $s \subseteq \sim_f$ and prove $w \leq f$.

For any $z \in (X \boxdot Y)\backslash(\operatorname{ran}(b) \cup \operatorname{ran}(c))$ it is clear that there exists $F \in f$ with $z \in F$, i.e., $\{z\} \subseteq F$, because f is a partition. Hence, we let $\mathsf{B} \in u$ be arbitrary and must now find a block $F \in f$ with $Z := b_\rightarrow(\mathsf{B}) \cup c_\rightarrow(\mathsf{B}) \subseteq F$.

To that end we define

$$v := \{b^\leftarrow(F) \cup c^\leftarrow(F) \mid F \in f\}$$

and show that this is a partition of Π_0^ℓ coarser than u. By the minimality of u this requires proving that 1) the union of v is Π_0^ℓ, that 2) any two distinct elements of v are disjoint, 3) that u is coarser than $\ker(h|_{\Pi_0^\ell})$ and 4) than $\ker(h|_{\Pi_0^\ell})$.

Step 1.5.1: Because f is a partition of $X \boxdot Y$ the union $\bigcup f$ is all of Π_0^ℓ. That is why

$$\bigcup v = b^{\leftarrow}(X \boxdot Y) \cup c^{\leftarrow}(X \boxdot Y) = \Pi_0^\ell.$$

Step 1.5.2: For any $\{F, F'\} \subseteq f$ such that $\mathsf{V} = b^{\leftarrow}(F) \cup c^{\leftarrow}(F)$ and $\mathsf{V}' = b^{\leftarrow}(F') \cup c^{\leftarrow}(F')$ are distinct it must hold that $F \neq F'$. Since f is a partition of $X \boxdot Y$ we infer that $F \cap F' = \varnothing$ and thus that in the decomposition

$$\begin{aligned}\mathsf{V} \cap \mathsf{V}' = b^{\leftarrow}(F \cap F') \cup c^{\leftarrow}(F \cap F') \\ \cup (b^{\leftarrow}(F) \cap c^{\leftarrow}(F')) \cup (b^{\leftarrow}(F') \cap c^{\leftarrow}(F))\end{aligned}$$

the sets on the right-hand side of the first line are empty. The same is true about the sets in the second line. Indeed, because $s \subseteq \sim_f$, for any $\mathsf{c} \in \Pi_0^\ell$ the vertices $(b)(\mathsf{c})$ and $(c)(\mathsf{c})$ must belong to the same block of f. Due to $\mathsf{V} \neq \mathsf{V}'$ that forbids $\mathsf{c} \in b^{\leftarrow}(F) \cap c^{\leftarrow}(F')$ and, likewise, $\mathsf{c} \in b^{\leftarrow}(F') \cap c^{\leftarrow}(F)$. In conclusion, v is a partition of Π_0^ℓ.

Step 1.5.3: For any $\mathsf{S} \in \ker(b)$ the definition of the kernel implies that $b_{\rightarrow}(\mathsf{S})$ is a singleton set. Because f is a partition of $X \boxdot Y$ there must then exist $F \in f$ with $b_{\rightarrow}(\mathsf{S}) \subseteq F$. Since, trivially, the self-map $b^{\leftarrow} \circ b_{\rightarrow}$ of $\wp(\Pi_0^\ell)$ is increasing with respect to $\subseteq$ it follows

$$\mathsf{S} \subseteq b^{\leftarrow}(b_{\rightarrow}(\mathsf{S})) \subseteq b^{\leftarrow}(F) \subseteq b^{\leftarrow}(F) \cup c^{\leftarrow}(F).$$

Because the set on the right-hand side of the last inclusion is a block of v and since S was arbitrary that proves $\ker(b) \leq f$.

Step 1.5.4: The inequality $\ker(c) \leq f$ can be inferred by an analogous deduction: For any $\mathsf{S}' \in \ker(c)$ there exists $F' \in f$ with $c_{\rightarrow}(\mathsf{S}') \subseteq F'$ because $c_{\rightarrow}(\mathsf{S}')$ is a singleton set. Hence, $\mathsf{S}' \subseteq ((c)^{\leftarrow} \circ c_{\rightarrow})(\mathsf{S}')$ implies that S' is contained in $c^{\leftarrow}(c_{\rightarrow}(\mathsf{S}'))$, which is included in $c^{\leftarrow}(F')$ and thus in the block $b^{\leftarrow}(F') \cup c^{\leftarrow}(F')$ of f.

In conclusion we have shown $u \leq v$. Therefore we can infer the existence of some $\mathsf{V} \in v$ with $\mathsf{B} \subseteq \mathsf{V}$. If $F \in f$ is such that $\mathsf{V} = b^{\leftarrow}(F) \cup c^{\leftarrow}(F)$, then the inequalities $b_{\rightarrow}(\mathsf{B}) \subseteq b_{\rightarrow}(\mathsf{V})$ and $c_{\rightarrow}(\mathsf{B}) \subseteq c_{\rightarrow}(F)$ together imply that the set $Z = b_{\rightarrow}(\mathsf{B}) \cup c_{\rightarrow}(\mathsf{B})$ is contained in

$$b_{\rightarrow}(b^{\leftarrow}(F) \cup c^{\leftarrow}(F)) \cup c_{\rightarrow}(b^{\leftarrow}(F) \cup c^{\leftarrow}(F)),$$

which is the same as

$$(b_{\rightarrow} \circ b^{\leftarrow})(F) \cup (c_{\rightarrow} \circ c^{\leftarrow})(F) \cup c_{\rightarrow}(b^{\leftarrow}(F)) \cup b_{\rightarrow}(c^{\leftarrow}(F)).$$

Because, trivially, the self-maps $b_{\rightarrow} \circ b^{\leftarrow}$ and $c_{\rightarrow} \circ c^{\leftarrow}$ of $\wp(X \boxdot Y)$ are increasing,

$$(b_{\rightarrow} \circ b^{\leftarrow})(F) \cup (c_{\rightarrow} \circ c^{\leftarrow})(F)$$

is a subset of F. But also

$$c_{\rightarrow}(b^{\leftarrow}(F)) \cup b_{\rightarrow}(c^{\leftarrow}(F))$$

is contained in F as well because that is precisely what the assumption $s \subseteq \sim_f$ means: For any $\mathsf{c} \in \Pi_0^\ell$, if $b(\mathsf{c}) \in F$, then also $c(\mathsf{c}) \in F$, and vice versa. Thus, the desired inequality $Z \subseteq F$ follows. That concludes the proof of the identity $w = (X \boxdot Y)/r$.

Step 2: We will now infer our main claim that qp is given by $\ker(f) = \ker(\pi_r \circ (a \uplus d)) = (a \uplus d)^{\leftarrow}(\ker(\pi_r))$. Because $\ker(\pi_r) = (X \boxdot Y)/r = w$ by Step 1 this is the same as the partition

$$\begin{aligned}&\{(a \uplus d)^{\leftarrow}(\{z\}) \mid z \in (X \boxdot Y)\backslash(\operatorname{ran}(b) \cup \operatorname{ran}(c))\}\backslash\{\varnothing\}\\ &\cup \{(a \uplus d)^{\leftarrow}(b_{\rightarrow}(\mathsf{B})) \uplus (a \uplus d)^{\leftarrow}(c_{\rightarrow}(\mathsf{B})) \mid \mathsf{B} \in u\}\backslash\{\varnothing\}\end{aligned}$$

by definition of w. Because $\operatorname{ran} b \cap \operatorname{ran} d = \varnothing$ and $\operatorname{ran} a \cap \operatorname{ran} c = \varnothing$ we can rewrite that as

$$\begin{aligned}&\{a^{\leftarrow}(\{z\}) \mid z \in \operatorname{ran}(a)\backslash\operatorname{ran}(b)\}\\ &\cup \{a^{\leftarrow}(b_{\rightarrow}(\mathsf{B})) \uplus d^{\leftarrow}(c_{\rightarrow}(\mathsf{B})) \mid \mathsf{B} \in u\}\backslash\{\varnothing\}\\ &\cup \{d^{\leftarrow}(\{z\}) \mid z \in \operatorname{ran}(d)\backslash\operatorname{ran}(c)\}\end{aligned}$$

Since $\iota^1_{X,Y}$ and $\iota^2_{X,Y}$ are injective $\ker(f)$ is thus the partition

$$\begin{aligned}&\{(g|_{\Pi_0^k})^{\leftarrow}(\{x\}) \mid x \in \operatorname{ran}(g|_{\Pi_0^k})\backslash\operatorname{ran}(g \circ \kappa_0^\ell)\}\\ &\cup \{(g|_{\Pi_0^k})^{\leftarrow}((g \circ \kappa_0^\ell)_{\rightarrow}(\mathsf{B})) \uplus (h|_{\Pi_m^0})^{\leftarrow}((h|_{\Pi_0^\ell})_{\rightarrow}(\mathsf{B})) \mid \mathsf{B} \in u\}\backslash\{\varnothing\}\\ &\cup \{(h|_{\Pi_m^0})^{\leftarrow}(\{y\}) \mid y \in \operatorname{ran}(h|_{\Pi_m^0})\backslash\operatorname{ran}(h|_{\Pi_0^\ell})\}\end{aligned}$$

which is just another way of writing qp. □

Lemma 2.30. *Given any $\{k_m, \ell_m\} \subseteq \mathbb{N}_0$ and any partition p_m of $\Pi_{\ell_m}^{k_m}$ for each $m \in [\![2]\!]$, the tensor product $p_1 \otimes p_2$ can be computed as $\ker(t)$ for any finite set V_m and any mapping $g_m \colon \Pi_{\ell_m}^{k_m} \to V_m$ with $\ker(g_m) = p_m$ for each $m \in [\![2]\!]$, where*

$$t = (\iota^1_{V_1,V_2} \circ g_1) \uplus (\iota^2_{V_1,V_2} \circ g_2 \circ \tau^{k_1,k_2}_{\ell_1,\ell_2}).$$

Proof. If we abbreviate $x \equiv \iota^1_{V_1,V_2} \circ g_1$ and $y \equiv \iota^2_{V_1,V_2} \circ g_2 \circ \tau^{k_1,k_2}_{\ell_1,\ell_2}$, then, because $\iota^1_{V_1,V_2}$ and $\iota^2_{V_1,V_2}$ and thus also x and y have disjoint ranges, for each $z \in V_1 \boxdot V_2$ at most one of the two sets $x^{\leftarrow}(\{z\})$ and $y^{\leftarrow}(\{z\})$ is non-empty. Hence, the partition

$$\begin{aligned}\ker(t) &= \{(x \cup y)^{\leftarrow}(\{z\}) \mid z \in V_1 \boxdot V_2\}\backslash\{\varnothing\}\\ &= \{x^{\leftarrow}(\{z\}) \cup y^{\leftarrow}(\{z\}) \mid z \in V_1 \boxdot V_2\}\backslash\{\varnothing\}\end{aligned}$$

can be rewritten as

$$\{x^{\leftarrow}(\{z\}) \mid z \in V_1 \boxdot V_2\}\backslash\{\varnothing\} \cup \{y^{\leftarrow}(\{z\}) \mid z \in V_1 \boxdot V_2\}\backslash\{\varnothing\} = \ker(x) \cup \ker(y).$$

Since $\iota^1_{V_1,V_2}$ and $\iota^2_{V_1,V_2}$ are injective, $\ker(x) = \ker(g_1)$ and $\ker(y) = \ker(g_2 \circ \tau^{k_1,k_2}_{\ell_1,\ell_2}) = (\tau^{k_1,k_2}_{\ell_1,\ell_2})^{\leftarrow}(\ker(g_2))$. In conclusion,

$$\ker(t) = \ker(g_1) \cup (\tau^{k_1,k_2}_{\ell_1,\ell_2})^{\leftarrow}(\ker(g_2)) = p_1 \otimes p_2$$

by $\ker(g_1) = p_1$ and $\ker(g_2) = p_2$. □

2.3.3. *List of all categories of partitions.* In combination, the articles [BS09], [BCS10], [Web13], [RW14], [RW16a] and [RW16b] provide a full classification of all categories of partitions. For the convenience of the reader, this section recalls those results.

NOTATION 2.31. (a) Let $\mathbb{Z}_2^{*\infty}$ be a free product of $|\mathbb{N}|$ many copies of the cyclic group $\mathbb{Z}_2$ of order 2, for each $i \in \mathbb{N}$ let a_i be the element of $\mathbb{Z}_2^{*\infty}$ corresponding to the generator of the i-th copy of $\mathbb{Z}_2$ and let $\varnothing$ be the neutral element of $\mathbb{Z}_2^{*\infty}$.

(b) Denote by sS_∞ the strong symmetric semigroup, i.e., the subsemigroup of the semigroup $\mathrm{End}(\mathbb{Z}_2^{*\infty})$ of group endomorphisms of $\mathbb{Z}_2^{*\infty}$ generated by the endomorphisms defined by the rule $a_i \mapsto a_{f(i)}$ for any $i \in \mathbb{N}$ for any mappings $f\colon \mathbb{N} \to \mathbb{N}$ such that $|\mathbb{N}\backslash \mathrm{ran}(f)| < \infty$.

(c) For any set $M \subseteq \mathbb{Z}_2^{*\infty}$ we write $\langle M\rangle_{\mathrm{sS}_\infty}$ for the smallest normal subgroup of $\mathbb{Z}_2^{*\infty}$ containing M and invariant under the action of sS_∞.

DEFINITION 2.32. For any $\{k,\ell\} \subseteq \mathbb{N}_0$ and any partition p of Π_ℓ^k the *word representation* of p is the element

$$F_\infty(p) \coloneqq \left(\overleftarrow{\prod}_{j=1}^{\ell} a_{(r\circ\pi_{\sim p})(\centerdot j)}\right)\left(\overrightarrow{\prod}_{i=1}^{k} a_{(r\circ\pi_{\sim p})(^{\centerdot} i)}\right)$$

of $\mathbb{Z}_2^{*\infty}$, where the operation is that of $\mathbb{Z}_2^{*\infty}$ and where the mapping $r\colon p \to \mathbb{N}$ is defined by the rule $\mathsf{B} \mapsto |\{\mathsf{A} \in p \wedge \min_{\leq_\ell^k}(\mathsf{A}) \leq_\ell^k \min_{\leq_\ell^k}(\mathsf{B})\}|$ for any $\mathsf{B} \in p$.

DEFINITION 2.33. For any $\{k,\ell\} \subseteq \mathbb{N}_0$ any partition p of Π_ℓ^k is said to ...

(a) ... be *even* if $k + \ell \in 2\mathbb{N}_0$.
(b) ... have *small blocks* if $|\mathsf{B}| \leq 2$ for any $\mathsf{B} \in p$.
(c) ... have *even blocks* if $|\mathsf{B}| \in 2\mathbb{N}$ for any $\mathsf{B} \in p$.
(d) ... have *even distances* if $|[\mathsf{a},\mathsf{b}]_\ell^k| \in 2\mathbb{N}$ for any $\mathsf{B} \in p$ and any $\{\mathsf{a},\mathsf{b}\} \subseteq \mathsf{B}$ with $\mathsf{a} \neq \mathsf{b}$.
(e) ... be *non-crossing* if $\mathsf{A} \mathbin{\not\!\!\diagdown}_\ell^k \mathsf{B}$ for any $\{\mathsf{A},\mathsf{B}\} \subseteq p$ with $\mathsf{A} \neq \mathsf{B}$.
(f) ... have *parity-balanced legs* if p has even blocks and $|\{\mathsf{b} \in \mathsf{B} \wedge |]\mathsf{a},\mathsf{b}]_\ell^k| \in 2\mathbb{N}_0\}| = |\{\mathsf{b} \in \mathsf{B} \wedge |]\mathsf{a},\mathsf{b}]_\ell^k| \in 2\mathbb{N}_0 + 1\}|$ for any $\mathsf{a} \in \Pi_\ell^k \backslash \mathsf{B}$ and any $\mathsf{B} \in p$.
(g) ... *contain a $\mathcal{W}$ of depth $m \in \mathbb{N}$* if there exist $\{b_1,\ldots,b_m\} \subseteq \{a_i \,|\, i \in \mathbb{N}\}$ and words $X_1^\alpha,\ldots,X_m^\alpha$, $X_1^\beta,\ldots,X_{m-1}^\beta$, $X_1^\gamma,\ldots,X_m^\gamma$, $X_1^\delta,\ldots,X_{m-1}^\delta$ and Y_1, Y_2, Y_3 over the alphabet $\{a_i \,|\, i \in \mathbb{N}\}$ such that

$$F_\infty(p) = Y_1 S_\alpha X_m^\alpha S_\beta Y_2 S_\gamma X_m^\gamma S_\delta Y_3$$

(i) where $S_\alpha = b_1 X_1^\alpha b_2 X_2^\alpha \ldots b_{m-1} X_{m-1}^\alpha b_m$,
(ii) where $S_\beta = b_m X_{m-1}^\beta b_{k-1} X_{k-2}^\beta \ldots b_2 X_1^\beta b_1$,
(iii) where $S_\gamma = b_1 X_1^\gamma b_2 X_2^\gamma \ldots b_{m-1} X_{m-1}^\gamma b_m$,
(iv) where $S_\delta = b_m X_{k-1}^\delta b_{m-1} X_{m-2}^\delta \ldots b_2 X_1^\delta b_1$ and
(v) where for every $i \in [\![m]\!]$ the letter b_i appears an *odd* number of times in each word S_α, S_β, S_γ and S_δ and
(vi) where Y_1, Y_2 and Y_3 contain none of the letters $b_1,\ldots,b_m$.

The classification can now be stated as follows.

THEOREM 2.34. *If $\mathcal{C}$ is a category of partitions, there exist $\ell \in \mathbb{N} \cup \{\infty\}$ and an sS_∞-invariant normal subgroup A of $\mathbb{Z}_2^{*\infty}$ other than $\langle a_1 a_2\rangle_{\mathrm{sS}_\infty}$ and $\langle a_1\rangle_{\mathrm{sS}_\infty}$ as well as a row in the below table such that $\mathcal{C}$ is given by the set of all partitions p meeting the following conditions:*

(i) p has all the properties listed as "×" in that row (and may or may not have any of the others)

(ii) any $\mathcal{W}$ contained in p must have a depth less than or equal to the value given in the next-to-last cell of that row

(iii) $F_\infty(p)$ is an element of the set given in the last cell of the row.

Moreover, any two categories belonging to different rows in the table are distinct. And so are any two belonging to the row labeled $\mathrm{H}^{<A>}$ but for different values of A or to the row labeled $\mathrm{H}^{\{\ell\}}$ but for different values of ℓ.

	even	*small blocks*	*even blocks*	*even dist.*	*non-cross.*	*par.-bal. legs*	*max.* wdepth	*word representations*
O	×	×	×				0	$\langle a_1a_2a_1a_2\rangle_{\mathrm{sS}_\infty}$
O*	×	×	×	×			0	$\langle a_1a_2a_3a_1a_2a_3\rangle_{\mathrm{sS}_\infty}$
O+	×	×	×	×	×	×	0	$\{\varnothing\}$
B		×					0	$\langle a_1\rangle_{\mathrm{sS}_\infty}$
B′	×	×					0	$\langle a_1a_2\rangle_{\mathrm{sS}_\infty}$
B#*	×	×		×			0	$\langle a_1a_2\rangle_{\mathrm{sS}_\infty}$
B+		×			×	×	0	$\langle a_1\rangle_{\mathrm{sS}_\infty}$
B′+	×	×			×	×	0	$\langle a_1a_2a_1a_3\rangle_{\mathrm{sS}_\infty}$
B#+	×	×		×	×	×	0	$\langle a_1a_2\rangle_{\mathrm{sS}_\infty}$
H	×		×				∞	$\langle a_1a_2a_1a_2\rangle_{\mathrm{sS}_\infty}$
H*	×		×	×			∞	$\langle a_1a_2a_3a_1a_2a_3\rangle_{\mathrm{sS}_\infty}$
$\mathrm{H}^{<A>}$							∞	A
$\mathrm{H}^{\{\ell\}}$	×		×	×		×	ℓ	$\{\varnothing\}$
H+	×		×	×	×	×	0	$\{\varnothing\}$
S							∞	$\langle a_1\rangle_{\mathrm{sS}_\infty}$
S′	×						∞	$\langle a_1a_2\rangle_{\mathrm{sS}_\infty}$
S+					×		0	$\langle a_1\rangle_{\mathrm{sS}_\infty}$
S′+	×				×		0	$\langle a_1a_2\rangle_{\mathrm{sS}_\infty}$

The first column gives the common name of the corresponding category. Redundant constraints usually omitted from the definitions are printed in gray.

2.4. Links between graph categories and categories of partitions. Finally, we will also use twice the following simple fact relating graph categories and categories of partitions.

LEMMA 2.35. *Let P be any mapping from $\mathcal{G}$ to the set of all (uncolored) partitions with the following properties.*

(i) For any $\{\mathbf{G},\mathbf{H}\} \subseteq \mathcal{G}$ the composition $\mathbf{H}\circ\mathbf{G}$ is defined if and only if the composition $P(\mathbf{H})P(\mathbf{G})$ is.

(ii) $P(\mathbf{H}\circ\mathbf{G}) = P(\mathbf{H})P(\mathbf{G})$ for any $\{\mathbf{G},\mathbf{H}\}\subseteq\mathcal{G}$ such that $\mathbf{H}\circ\mathbf{G}$ is defined.

(iii) $P(\mathbf{G}_1\otimes\mathbf{G}_2) = P(\mathbf{G}_1)\otimes P(\mathbf{G}_2)$ for any $\{\mathbf{G}_1,\mathbf{G}_2\}\subseteq\mathcal{G}$.

(iv) $P(\mathbf{G}^*) = P(\mathbf{G})^*$ for any $\mathbf{G}\in\mathcal{G}$.

(v) $P(\mathbf{I}) = |$.

(vi) $P(\mathbf{M}_{0,2}) = \sqcap$.

Then, the following are true.

(a) $P^{\leftarrow}(\mathcal{C})$ is a graph category for each category $\mathcal{C}$ of (uncolored) partitions.

(b) $P_{\rightarrow}(\mathcal{F})$ is a category of (uncolored) partitions for each graph category $\mathcal{F}$.

PROOF. Follows immediately from the definitions of graph categories and categories of partitions. ☐

3. Vertex partitions

3.1. Definition and properties of the vertex partition. In [MR19, Definition 6.2] Mančinska and Roberson associate with each bi-labeled graph a partition. The following definition modifies their construction.

DEFINITION 3.1. For any $\{k,\ell\}\subseteq\mathbb{N}_0$ and any $\mathbf{G}\in\mathcal{G}(k,\ell)$ the *vertex partition* of $\mathbf{G}$ is the partition $\mathbb{P}_{\mathbf{G}} \coloneqq \ker(g)$ of Π^k_ℓ, where $(G,g)\in\mathbf{G}$ can be any representative.

REMARK 3.2. The difference between the mappings $\mathbb{P}$ from [MR19, Definition 6.2] and from Definition 3.1 is of course that the latter forgets the "empty blocks" preserved by the former (see Remark 2.23).

The map $\mathbb{P}$ assigning the vertex partition to a bi-labeled graph is functorial.

LEMMA 3.3. *(a) For any $\{\mathbf{G},\mathbf{H}\}\subseteq\mathcal{G}$ the composition $\mathbf{H}\circ\mathbf{G}$ is defined if and only if the composition $\mathbb{P}_{\mathbf{H}}\mathbb{P}_{\mathbf{G}}$ is.*

(b) $\mathbb{P}_{\mathbf{H}\circ\mathbf{G}} = \mathbb{P}_{\mathbf{H}}\mathbb{P}_{\mathbf{G}}$ for any $\{\mathbf{G},\mathbf{H}\}\subseteq\mathcal{G}$ such that $\mathbf{H}\circ\mathbf{G}$ is defined.

(c) $\mathbb{P}_{\mathbf{G}_1\otimes\mathbf{G}_2} = \mathbb{P}_{\mathbf{G}_1}\otimes\mathbb{P}_{\mathbf{G}_2}$ for any $\{\mathbf{G}_1,\mathbf{G}_2\}\subseteq\mathcal{G}$.

(d) $\mathbb{P}_{\mathbf{G}^*} = (\mathbb{P}_{\mathbf{G}})^*$ for any $\mathbf{G}\in\mathcal{G}$.

(e) $\mathbb{P}_{\mathbf{I}} = |$.

(f) $\mathbb{P}_{\mathbf{M}_{0,2}} = \sqcap$.

PROOF. (a) is clear because for any $\{k,\ell\}\subseteq\mathbb{N}_0$ the operation $\mathbb{P}$ sends (k,ℓ)-bilabeled graphs to partitions of Π^k_ℓ.

(b) If $\{k,\ell,m\}\subseteq\mathbb{N}_0$ are such that $\mathbf{G}\in\mathcal{G}(k,\ell)$ and $\mathbf{H}\in\mathcal{G}(\ell,m)$, and if $(G,g)\in\mathbf{G}$ and $(H,h)\in\mathbf{H}$ and if G and H have the vertex sets X and Y, respectively, then, by definition, $(P,p)\in\mathbf{H}\circ\mathbf{G}$, where $P=(G\boxdot H)/r$, where r is the equivalence relation on $X\boxdot Y$ generated by $\{((\iota^1_{X,Y}\circ g)({}_{\blacksquare}i),(\iota^2_{X,Y}\circ h)({}^{\blacksquare}i))\,|\,i\in[\![\ell]\!]\}$ and where $p=\pi_r\circ((\iota^1_{X,Y}\circ g|_{\Pi^k_0})\cup(\iota^2_{X,Y}\circ h|_{\Pi^0_m}))$. Thus, then $\mathbb{P}_{\mathbf{H}\circ\mathbf{G}}=\ker(p)$. At the same time, since $\mathbb{P}_{\mathbf{H}}=\ker(h)$ and $\mathbb{P}_{\mathbf{G}}=\ker(g)$, also $\mathbb{P}_{\mathbf{H}}\mathbb{P}_{\mathbf{G}}=\ker(p)$ by Lemma 2.29.

(c) If $\{k_m, \ell_m\} \subseteq \mathbb{N}_0$ are such that $\mathbf{G}_m \in \mathcal{G}(k_m, \ell_m)$ and if $(G_m, g_m) \in \mathbf{G}_m$ and if G_m has vertex set V_m for any $m \in [\![2]\!]$, then $(T, t) \in \mathbf{G}_1 \otimes \mathbf{G}_2$, where $T = G_1 \boxdot G_2$ and where $t = (\iota^1_{V_1,V_2} \circ g_1) \cup (\iota^2_{V_1,V_2} \circ g_2 \circ \tau^{k_1,k_2}_{\ell_1,\ell_2})$. Thus, then $\mathbb{P}_{\mathbf{G}_1 \otimes \mathbf{G}_2} = \ker(t)$. On the other hand, $\mathbb{P}_{\mathbf{G}_1} \otimes \mathbb{P}_{\mathbf{G}_2} = \ker(t)$ by $\mathbb{P}_{\mathbf{G}_1} = \ker(g_1)$ and $\mathbb{P}_{\mathbf{G}_2} = \ker(g_2)$ and Lemma 2.30.

(d) If $\{k, \ell\} \subseteq \mathbb{N}_0$ are such that $\mathbf{G} \in \mathcal{G}(k, \ell)$ and if $(G, g) \in \mathbf{G}$, then $(G, g \circ \kappa^\ell_k) \in \mathbf{G}^*$ and thus

$$\mathbb{P}_{\mathbf{G}^*} = \ker(g \circ \kappa^\ell_k) = (\kappa^\ell_k)^{\leftarrow}(\ker(g)) = (\kappa^k_\ell)_{\rightarrow}(\ker(g)) = (\mathbb{P}_{\mathbf{G}})^*$$

since κ^k_ℓ and κ^ℓ_k are inverse to each other.

(e) and (f) are clear because the graph underlying both $\mathbf{I}$ and $\mathbf{M}_{0,2}$ only has a single vertex. □

Remark 3.4. If the definition of "partition" from [MR19, p. 11] is employed instead, Lemma 3.3 remains true.

3.2. Graph categories arising from vertex partition constraints. The preceding lemma allows us to recognize a wealth of new graph categories.

Definition 3.5. Given any category $\mathcal{C}$ of (uncolored) partitions, the set $\mathbb{P}^{\leftarrow}(\mathcal{C})$ is called the *bi-labeled graphs with $\mathcal{C}$-partitioned vertices.*

Proposition 3.6. *For each category $\mathcal{C}$ of partitions the bi-labeled graphs with $\mathcal{C}$-partitioned vertices form a graph category.*

Proof. Follows from Lemma 2.35 (a). □

Remark 3.7.
(a) In [MR19, p. 47 and Theorem 8.3], Mančinska and Roberson show that their mapping $\mathbb{P}$ restricts to a bijection between *edgeless bi-labeled graphs*, i.e., bi-labeled graphs whose underlying graphs have no edges, and "partitions". In fact, they can prove a one-to-one correspondence between graph categories of edgeless bi-labeled graphs and categories of partitions. However, as explained in Remark 2.23, Mančinska and Roberson there employ a different notion of "partition" than the one used by Banica and Speicher and used here. The analog of [MR19, Theorem 8.3] is *false* for the latter sense of "partition". Rather the following is true.
(b) Still, for any $\{k, \ell\} \subseteq \mathbb{N}_0$ any partition p of Π^k_ℓ defines an edgeless bi-labeled graph $\mathbb{E}_p \in \mathcal{G}(k, \ell)$ with $((p, \varnothing), \pi_{\sim_p}) \in \mathbb{E}_p$.
(c) However, this construction $\mathbb{E}$ is *not* inverse to the mapping $\mathbb{P}$ from Definition 3.1. E.g., $(\mathbb{E} \circ \mathbb{P})(\mathbf{M}_{0,0}) = \mathbb{E}_\varnothing = \varnothing \neq \mathbf{M}_{0,0}$.
(d) Neither does $\mathbb{E}$ map categories of partitions to graph categories of edgeless bi-labeled graphs. For example, if $p = \sqcap$ and $q = \sqcup$, then $\mathbb{E}_q \circ \mathbb{E}_p = \mathbf{M}_{2,0} \circ \mathbf{M}_{0,2} = \mathbf{M}_{0,0} \notin \mathrm{ran}(\mathbb{E})$ since $\mathbf{M}_{0,0}$ has unlabeled vertices.
(e) In particular, given a category $\mathcal{C}$ of partitions, if $\mathcal{E} \subseteq \mathcal{C}$ is a generating set of $\mathcal{C}$, the graph categories $\langle \mathbb{E}_p \,|\, p \in \mathcal{E} \rangle$ and $\mathbb{P}^{\leftarrow}(\mathcal{C})$ are generally *not* the same. There is no easy way of translating the results about generating partitions into ones about generating edgeless bi-labeled graphs.

3.3. Vertex partition invariant. We can also use the classification of all categories of partitions to tell graph categories apart under certain circumstances.

DEFINITION 3.8. Given any graph category $\mathcal{F}$, we call the set $\mathbb{P}_{\rightarrow}(\mathcal{F})$ the *vertex partitions of* $\mathcal{F}$.

PROPOSITION 3.9. *For any graph category* $\mathcal{F}$ *the vertex partitions of* $\mathcal{F}$ *form a category of partitions.*

PROOF. Immediate consequence of Lemma 2.35 (b). □

REMARK 3.10. Proposition 3.9 also holds if the definition of "partition" from [BS09, p. 11] is used instead. If categories of partitions in that sense were classified, it would yield a useful invariant.

4. Component partitions

4.1. Definition and properties of the component partition. Definition 3.1 is not the only way of associating with any bi-labeled graph a partition functorially.

DEFINITION 4.1. For any $\{k,\ell\} \subseteq \mathbb{N}_0$ and any $\mathbf{G} \in \mathcal{G}(k,\ell)$ the *component partition* of $\mathbf{G}$ is the partition $\mathrm{P}_{\mathbf{G}} \coloneqq \ker(\pi_{\simeq_G} \circ g)$ of Π^k_ℓ, where $(G,g) \in \mathbf{G}$ can be any representative.

REMARK 4.2. If one adopts the notion of "partition" from [MR19, p. 11] instead, one can define an analogous mapping P with even better properties than the one from Definition 4.1.

In order to see why th assignment P respects the category operations the following bijections are useful.

LEMMA 4.3. *(a) For any graphs* G_1 *and* G_2 *with vertex sets* V_1 *and* V_2*, respectively, the sets*

$$(V_1 \boxdot V_2)/{\simeq_{G_1 \boxdot G_2}} \quad \text{and} \quad (V_1/{\simeq_{G_1}}) \boxdot (V_2/{\simeq_{G_2}})$$

are equinumerous. More precisely, the mappings

$$((\iota^1_{V_1/\simeq_{G_1}, V_2/\simeq_{G_2}} \circ \pi_{\simeq_{G_1}}) \sqcup (\iota^2_{V_1/\simeq_{G_1}, V_2/\simeq_{G_2}} \circ \pi_{\simeq_{G_2}}))/{\simeq_{G_1 \boxdot G_2}}$$

and

$$((\pi_{\simeq_{G_1 \boxdot G_2}} \circ \iota^1_{V_1,V_2})/{\simeq_{G_1}}) \sqcup ((\pi_{\simeq_{G_1 \boxdot G_2}} \circ \iota^2_{V_1,V_2})/{\simeq_{G_2}})$$

are mutually inverse bijections.

(b) For any graph G *and any equivalence relation* r *on the vertex set* V *of* G*, if* s *is the equivalence relation on* $V/{\simeq_G}$ *generated by*

$$\{(\pi_{\simeq_G}(v), \pi_{\simeq_G}(v')) \mid (v,v') \in r\},$$

then the sets

$$(V/r)/{\simeq_{G/r}} \quad \text{and} \quad (V/{\simeq_G})/s$$

are equinumerous. More precisely, the mappings

$$((\pi_s \circ \pi_{\simeq_G})/r)/{\simeq_{G/r}} \quad \text{and} \quad ((\pi_{\simeq_{G/r}} \circ \pi_r)/{\simeq_G})/s$$

are well-defined mutually inverse bijections.

Proof. (a) Proving the claim is the same as showing for any $\{i, i'\} \subseteq [\![2]\!]$ and any $v \in V_i$ and $v' \in V_{i'}$ that

$$(\iota^i_{V_1/\simeq_{G_1}, V_2/\simeq_{G_2}} \circ \pi_{\simeq_{G_i}})(v) = (\iota^{i'}_{V_1/\simeq_{G_1}, V_2/\simeq_{G_2}} \circ \pi_{\simeq_{G_{i'}}})(v')$$

if and only if

$$(\pi_{\simeq_{G_1 \boxdot G_2}} \circ \iota^i_{V_1,V_2})(v) = (\pi_{\simeq_{G_1 \boxdot G_2}} \circ \iota^{i'}_{V_1,V_2})(v').$$

If $(\iota^i_{V_1/\simeq_{G_1}, V_2/\simeq_{G_2}} \circ \pi_{\simeq_{G_i}})(v) = (\iota^{i'}_{V_1/\simeq_{G_1}, V_2/\simeq_{G_2}} \circ \pi_{\simeq_{G_{i'}}})(v')$, then the disjoint ranges of $\iota^1_{V_1/\simeq_{G_1}, V_2/\simeq_{G_2}}$ and $\iota^2_{V_1/\simeq_{G_1}, V_2/\simeq_{G_2}}$ necessitate $i = i'$. Since $\iota^i_{V_1/\simeq_{G_1}, V_2/\simeq_{G_2}}$ is injective we can thus conclude $\pi_{\simeq_{G_i}}(v) = \pi_{\simeq_{G_i}}(v')$ or, equivalently, $v \simeq_{G_i} v'$. Because $\iota^i_{V_1,V_2}$ is a graph homomorphism from G_i to $G_1 \boxdot G_2$ it follows $\iota^i_{V_1,V_2}(v) \simeq_{G_1 \boxdot G_2} \iota^i_{V_1,V_2}(v')$, which is to say $(\pi_{\simeq_{G_1 \boxdot G_2}} \circ \iota^i_{V_1,V_2})(v) = (\pi_{\simeq_{G_1 \boxdot G_2}} \circ \iota^{i'}_{V_1,V_2})(v')$.

To see the converse, assume $(\pi_{\simeq_{G_1 \boxdot G_2}} \circ \iota^i_{V_1,V_2})(v) = (\pi_{\simeq_{G_1 \boxdot G_2}} \circ \iota^{i'}_{V_1,V_2})(v')$ or, equivalently, $\iota^i_{V_1,V_2}(v) \simeq_{G_1 \boxdot G_2} \iota^{i'}_{V_1,V_2}(v')$. By Remark 2.6 (a) and the injectivity of $\iota^1_{V_1,V_2}$ and $\iota^2_{V_1,V_2}$ that requires $i = i'$ and $v \simeq_{G_i} v'$, i.e., $\pi_{\simeq_{G_i}}(v) = \pi_{\simeq_{G_i}}(v')$. Thus also, $(\iota^i_{V_1/\simeq_{G_1}, V_2/\simeq_{G_2}} \circ \pi_{\simeq_{G_i}})(v) = (\iota^{i'}_{V_1/\simeq_{G_1}, V_2/\simeq_{G_2}} \circ \pi_{\simeq_{G_{i'}}})(v')$.

(b) By definition of the quotient maps the claim is equivalent to the statement that for any $\{v, v'\} \subseteq V$,

$$(\pi_s \circ \pi_{\simeq_G})(v) = (\pi_s \circ \pi_{\simeq_G})(v') \iff (\pi_{\simeq_{G/r}} \circ \pi_r)(v) = (\pi_{\simeq_{G/r}} \circ \pi_r)(v').$$

Abbreviate $t \equiv \{(\pi_{\simeq_G}(v), \pi_{\simeq_G}(v')) \mid (v, v') \in r\}$.

First, let $(\pi_s \circ \pi_{\simeq_G})(v) = (\pi_s \circ \pi_{\simeq_G})(v')$ or, equivalently, $(\pi_{\simeq_G}(v), \pi_{\simeq_G}(v')) \in s$. Then, by definition of s and Remark 2.2 (a) there exist $k \in \mathbb{N}_0$ and $\{u_1, \ldots, u_{k+1}\} \subseteq V/{\simeq_G}$ such that $\pi_{\simeq_G}(v) = u_1$ and $u_{k+1} = \pi_{\simeq_G}(v')$ and $(u_i, u_{i+1}) \in t$ or $(u_{i+1}, u_i) \in t$ for any $i \in [\![k]\!]$. Because r is an equivalence relation, and thus in particular symmetric, t is symmetric as well. Hence, actually, $(u_i, u_{i+1}) \in t$ for any $i \in [\![k]\!]$. By definition of t we thus find for any $i \in [\![k]\!]$ some $(x_i, y_i) \in r$ with $\pi_{\simeq_G}(x_i) = u_i$ and $\pi_{\simeq_G}(y_i) = u_{i+1}$. In summary, $v \simeq_G x_1$ and $y_k \simeq_G v'$ and $y_i \simeq_G x_{i+1}$ for each $i \in [\![k-1]\!]$ and $(x_i, y_i) \in r$ for each $i \in [\![k]\!]$. Since π_r is a graph homomorphism from G to G/r it follows that also $\pi_r(v) \simeq_{G/r} \pi_r(x_1)$ and $\pi_r(y_k) \simeq_{G/r} \pi_r(v')$ and $\pi_r(y_i) \simeq_{G/r} \pi_r(x_{i+1})$ for each $i \in [\![k-1]\!]$, which is to say $(\pi_{\simeq_{G/r}} \circ \pi_r)(v) = (\pi_{\simeq_{G/r}} \circ \pi_r)(x_1)$ and $(\pi_{\simeq_{G/r}} \circ \pi_r)(y_k) = (\pi_{\simeq_{G/r}} \circ \pi_r)(v')$ and $(\pi_{\simeq_{G/r}} \circ \pi_r)(y_i) = (\pi_{\simeq_{G/r}} \circ \pi_r)(x_{i+1})$ for each $i \in [\![k-1]\!]$. And, of course, $\pi_r(x_i) = \pi_r(y_i)$ implies $(\pi_{\simeq_{G/r}} \circ \pi_r)(x_i) = (\pi_{\simeq_{G/r}} \circ \pi_r)(y_i)$ for each $i \in [\![k]\!]$. Hence, $(\pi_{\simeq_{G/r}} \circ \pi_r)(v) = (\pi_{\simeq_{G/r}} \circ \pi_r)(v')$ by induction.

For the converse, suppose $(\pi_{\simeq_{G/r}} \circ \pi_r)(v) = (\pi_{\simeq_{G/r}} \circ \pi_r)(v')$, which is to say $\pi_r(v) \simeq_{G/r} \pi_r(v')$. Then, by Remark 2.2 (a) there exist $k \in \mathbb{N}$ and $\{u_1, \ldots, u_{k+1}\} \subseteq V/r$ such that $\pi_r(v) = u_1$ and $u_{k+1} = \pi_r(v')$ and $u_i \backsim_{G/r} u_{i+1}$ or $u_{i+1} \backsim_{G/r} u_i$ for any $i \in [\![k]\!]$.

Since $\backsim_{G/r}$ is symmetric, actually, $u_i \backsim_{G/r} u_{i+1}$ for any $i \in [\![k]\!]$. By Remark 2.6 (b) for each $i \in [\![k]\!]$ we then find $\{x_i, y_i\} \subseteq V$ such that $u_i = \pi_r(x_i)$ and $u_{i+1} = \pi_r(y_i)$ and $x_i \backsim_G y_i$. In other words, $(v, x_1) \in r$ and $(y_k, v') \in r$ and $(y_i, x_{i+1}) \in r$ for each $i \in [\![k-1]\!]$ and $x_i \backsim_G y_i$ for each $i \in [\![k]\!]$. The definitions of t and s thus imply in particular $(\pi_{\simeq_G}(v), \pi_{\simeq_G}(x_1)) \in s$ and $(\pi_{\simeq_G}(y_k), \pi_{\simeq_G}(v')) \in s$ and $(\pi_{\simeq_G}(y_i), \pi_{\simeq_G}(x_{i+1})) \in s$ for each $i \in [\![k-1]\!]$, which is to say $(\pi_s \circ \pi_{\simeq_G})(v) = (\pi_s \circ \pi_{\simeq_G})(x_1)$ and $(\pi_s \circ \pi_{\simeq_G})(y_k) = (\pi_s \circ \pi_{\simeq_G})(v')$ and $(\pi_s \circ \pi_{\simeq_G})(y_i) = (\pi_s \circ \pi_{\simeq_G})(x_{i+1})$ for any $i \in [\![k-1]\!]$. And because $\backsim_G \subseteq \simeq_G$ the relation $x_i \backsim_G y_i$ also implies $x_i \simeq_G y_i$, i.e., $\pi_{\simeq_G}(x_i) = \pi_{\simeq_G}(y_i)$, and thus $(\pi_s \circ \pi_{\simeq_G})(x_i) = (\pi_s \circ \pi_{\simeq_G})(y_i)$ for each $i \in [\![k]\!]$. Hence, $(\pi_s \circ \pi_{\simeq_G})(v) = (\pi_s \circ \pi_{\simeq_G})(v')$ by induction. □

LEMMA 4.4. *(a) For any $\{\mathbf{G}, \mathbf{H}\} \subseteq \mathcal{G}$ the composition $\mathbf{H} \circ \mathbf{G}$ is defined if and only if the composition $\mathrm{P}_{\mathbf{H}}\mathrm{P}_{\mathbf{G}}$ is.*

(b) $\mathrm{P}_{\mathbf{H}\circ\mathbf{G}} = \mathrm{P}_{\mathbf{H}}\mathrm{P}_{\mathbf{G}}$ for any $\{\mathbf{G}, \mathbf{H}\} \subseteq \mathcal{G}$ such that $\mathbf{H} \circ \mathbf{G}$ is defined.

(c) $\mathrm{P}_{\mathbf{G}_1\otimes\mathbf{G}_2} = \mathrm{P}_{\mathbf{G}_1} \otimes \mathrm{P}_{\mathbf{G}_2}$ for any $\{\mathbf{G}_1, \mathbf{G}_2\} \subseteq \mathcal{G}$.

(d) $\mathrm{P}_{\mathbf{G}^*} = (\mathrm{P}_{\mathbf{G}})^*$ for any $\mathbf{G} \in \mathcal{G}$.

(e) $\mathrm{P}_{\mathbf{I}} = |$.

(f) $\mathrm{P}_{\mathbf{M}_{0,2}} = \sqcap$.

PROOF. (a) is clear because for any $\{k, \ell\} \subseteq \mathbb{N}_0$ the operation P sends (k, ℓ)-bilabeled graphs to partitions of Π_ℓ^k.

(b) If $\{k, \ell, m\} \subseteq \mathbb{N}_0$ are such that $\mathbf{G} \in \mathcal{G}(k, \ell)$ and $\mathbf{H} \in \mathcal{G}(\ell, m)$, and if $(G, g) \in \mathbf{G}$ and $(H, h) \in \mathbf{H}$ and if G and H have the vertex sets X and Y, respectively, then, by definition, $(P, p) \in \mathbf{H} \circ \mathbf{G}$ and thus $\mathrm{P}_{\mathbf{H}\circ\mathbf{G}} = \ker(\pi_{\simeq_P} \circ p)$, where $P = (G \boxdot H)/r$, where r is the equivalence relation on $X \boxdot Y$ generated by $\{((\iota^1_{X,Y} \circ g)(_\blacksquare i), (\iota^2_{X,Y} \circ h)(^\blacksquare i)) \,|\, i \in [\![\ell]\!]\}$ and where $p = \pi_r \circ ((\iota^1_{X,Y} \circ g|_{\Pi_0^k}) \cup (\iota^2_{X,Y} \circ h|_{\Pi_m^0}))$. At the same time, since $\mathrm{P}_{\mathbf{H}} = \ker(\pi_{\simeq_H} \circ h)$ and $\mathrm{P}_{\mathbf{G}} = \ker(\pi_{\simeq_G} \circ g)$, if t is the equivalence relation on $(X/\simeq_G) \boxdot (Y/\simeq_H)$ generated by $\{((\iota^1_{X/\simeq_G, Y/\simeq_H} \circ \pi_{\simeq_G} \circ g)(_\blacksquare i), (\iota^2_{X/\simeq_G, Y/\simeq_H} \circ \pi_{\simeq_H} \circ h)(^\blacksquare i)) \,|\, i \in [\![\ell]\!]\}$ and if $q = \pi_t \circ ((\iota^1_{X/\simeq_G, Y/\simeq_H} \circ \pi_{\simeq_G} \circ g|_{\Pi_0^k}) \cup (\iota^2_{X/\simeq_G, Y/\simeq_H} \circ \pi_{\simeq_H} \circ h|_{\Pi_m^0}))$, then $\mathrm{P}_{\mathbf{H}}\mathrm{P}_{\mathbf{G}} = \ker(q)$ by Lemma 2.29. Hence, in order to show $\mathrm{P}_{\mathbf{H}\circ\mathbf{G}} = \mathrm{P}_{\mathbf{H}}\mathrm{P}_{\mathbf{G}}$ we have to prove $\ker(q) = \ker(\pi_{\simeq_P} \circ p)$. We exhibit a bijection f from $((X/\simeq_G) \boxdot (Y/\simeq_H))/t$ to $((X \boxdot Y)/r)/\simeq_P$ with $f \circ q = \pi_{\simeq_P} \circ p$, which then proves the claim. We construct f from auxiliary bijections u and w and v and an auxiliary equivalence s.

Step 1: Construction of s and u. If s denotes the equivalence relation on $(X \boxdot Y)/\simeq_{G\boxdot H}$ generated by $(\pi_{\simeq_{G\boxdot H}} \boxtimes \pi_{\simeq_{G\boxdot H}})_\to(r)$, then by Lemma 4.3 (b) the mapping $u \coloneqq ((\pi_{\simeq_P} \circ \pi_r)/\simeq_{G\boxdot H})/s$ is a well-defined bijection from $((X \boxdot Y)/\simeq_{G\boxdot H})/s$ to $((X \boxdot Y)/r)/\simeq_P$.

Step 2: Construction of w. Moreover, by Lemma 4.3 (a) the map $w \coloneqq ((\pi_{\simeq_{G\boxdot H}} \circ \iota^1_{X,Y})/\simeq_G) \sqcup ((\pi_{\simeq_{G\boxdot H}} \circ \iota^2_{X,Y})/\simeq_H)$ is a well-defined bijection from $(X/\simeq_G) \boxdot (Y/\simeq_H)$ to $(X \boxdot Y)/\simeq_{G\boxdot H}$ with $w^{-1} = ((\iota^1_{X/\simeq_G, Y/\simeq_H} \circ \pi_{\simeq_G}) \sqcup (\iota^2_{X/\simeq_G, Y/\simeq_H} \circ \pi_{\simeq_H}))/\simeq_{G\boxdot H}$.

Step 3: Construction of v. In order to define v we first need to relate t to s and w. More precisely, we prove $t = (w^{-1} \boxtimes w^{-1})_\to(s)$.

According to Remark 2.2 (b) the relation $(w^{-1} \boxtimes w^{-1})_{\to}(s)$ on $(X/{\simeq_G}) \boxdot (Y/{\simeq_H})$ is an equivalence because w^{-1} is invertible. Since s is generated by $(\pi_{\simeq_{G \boxdot H}} \boxtimes \pi_{\simeq_{G \boxdot H}})_{\to}(r)$ the equivalence $(w^{-1} \boxtimes w^{-1})_{\to}(s)$ is generated by the relation $(w^{-1} \boxtimes w^{-1})_{\to}((\pi_{\simeq_{G \boxdot H}} \boxtimes \pi_{\simeq_{G \boxdot H}})_{\to}(r)) = ((w^{-1} \circ \pi_{\simeq_{G \boxdot H}}) \boxtimes (w^{-1} \circ \pi_{\simeq_{G \boxdot H}}))_{\to}(r)$ by Remark 2.2 (c). Since r is in turn generated by $\{((\iota^1_{X,Y} \circ g)(_\blacksquare i), (\iota^2_{X,Y} \circ h)(^\blacksquare i)) \mid i \in [\![\ell]\!]\}$ a second application of Remark 2.2 (c) tells us that $(w^{-1} \boxtimes w^{-1})_{\to}(s)$ is generated by $\{((w^{-1} \circ \pi_{\simeq_{G \boxdot H}} \circ \iota^1_{X,Y} \circ g)(_\blacksquare i), (w^{-1} \circ \pi_{\simeq_{G \boxdot H}} \circ \iota^2_{X,Y} \circ h)(^\blacksquare i)) \mid i \in [\![\ell]\!]\}$. Because, first,

$$\begin{aligned}
&w^{-1} \circ \pi_{\simeq_{G \boxdot H}} \circ \iota^1_{X,Y} \circ g \circ \kappa^\ell_0 \\
&\quad = \Big(\Big((\iota^1_{X/\simeq_G, Y/\simeq_H} \circ \pi_{\simeq_G}) \sqcup (\iota^2_{X/\simeq_G, Y/\simeq_H} \circ \pi_{\simeq_H})\Big)/{\simeq_{G \boxdot H}}\Big) \circ \pi_{\simeq_{G \boxdot H}} \circ \iota^1_{X,Y} \circ g \circ \kappa^\ell_0 \\
&\quad = \Big((\iota^1_{X/\simeq_G, Y/\simeq_H} \circ \pi_{\simeq_G}) \sqcup (\iota^2_{X/\simeq_G, Y/\simeq_H} \circ \pi_{\simeq_H})\Big) \circ \iota^1_{X,Y} \circ g \circ \kappa^\ell_0 \\
&\quad = (\iota^1_{X/\simeq_G, Y/\simeq_H} \circ \pi_{\simeq_G}) \circ g \circ \kappa^\ell_0
\end{aligned}$$

and, second, analogously,

$$w^{-1} \circ \pi_{\simeq_{G \boxdot H}} \circ \iota^2_{X,Y} \circ h|_{\Pi^\ell_0} = (\iota^2_{X/\simeq_G, Y/\simeq_H} \circ \pi_{\simeq_H}) \circ h|_{\Pi^\ell_0}$$

and because, by definition, t is generated by $\{((\iota^1_{X/\simeq_G, Y/\simeq_H} \circ \pi_{\simeq_G} \circ g)(_\blacksquare i), (\iota^2_{X/\simeq_G, Y/\simeq_H} \circ \pi_{\simeq_H} \circ h)(^\blacksquare i)) \mid i \in [\![\ell]\!]\}$ we thus see that $(w^{-1} \boxtimes w^{-1})_{\to}(s)$ coincides with t.

Because $t = (w^{-1} \boxtimes w^{-1})_{\to}(s)$, Remark 2.2 (b) allows us to infer that the mapping $v \coloneqq (\pi_s \circ w)/t$ is a well-defined bijection from $((X/{\simeq_G}) \boxdot (Y/{\simeq_H}))/t$ to $((X \boxdot Y)/{\simeq_{G \boxdot H}})/s$.

Step 4: Construction of f. Finally, we can let $f \coloneqq u \circ v$ and know that is invertible because both u and v are. It remains to show $f \circ q = \pi_{\simeq_P} \circ p$.

Since $((\pi_s \circ w)/t) \circ \pi_t = \pi_s \circ w$,

$$v \circ q = \pi_s \circ w \circ ((\iota^1_{X/\simeq_G, Y/\simeq_H} \circ \pi_{\simeq_G} \circ g|_{\Pi^k_0}) \cup (\iota^2_{X/\simeq_G, Y/\simeq_H} \circ \pi_{\simeq_H} \circ h|_{\Pi^0_m}))$$

and thus, since $((\pi_{\simeq_P} \circ \pi_r)/{\simeq_{G \boxdot H}})/s \circ \pi_s = (\pi_{\simeq_P} \circ \pi_r)/{\simeq_{G \boxdot H}}$,

$$\begin{aligned}
&f \circ q \\
&\quad = ((\pi_{\simeq_P} \circ \pi_r)/{\simeq_{G \boxdot H}}) \circ w \circ ((\iota^1_{X/\simeq_G, Y/\simeq_H} \circ \pi_{\simeq_G} \circ g|_{\Pi^k_0}) \cup (\iota^2_{X/\simeq_G, Y/\simeq_H} \circ \pi_{\simeq_H} \circ h|_{\Pi^0_m})).
\end{aligned}$$

Given that $(((\pi_{\simeq_{G \boxdot H}} \circ \iota^1_{X,Y})/{\simeq_G}) \sqcup ((\pi_{\simeq_{G \boxdot H}} \circ \iota^2_{X,Y})/{\simeq_H})) \circ \iota^1_{X/\simeq_G, Y/\simeq_H} = (\pi_{\simeq_{G \boxdot H}} \circ \iota^1_{X,Y})/{\simeq_G}$,

$$\begin{aligned}
w \circ \iota^1_{X/\simeq_G, Y/\simeq_H} \circ \pi_{\simeq_G} \circ g|_{\Pi^k_0} &= ((\pi_{\simeq_{G \boxdot H}} \circ \iota^1_{X,Y})/{\simeq_G}) \circ \pi_{\simeq_G} \circ g|_{\Pi^k_0} \\
&= \pi_{\simeq_{G \boxdot H}} \circ \iota^1_{X,Y} \circ g|_{\Pi^k_0}.
\end{aligned}$$

Likewise, $(((\pi_{\simeq_{G \boxdot H}} \circ \iota^1_{X,Y})/{\simeq_G}) \sqcup ((\pi_{\simeq_{G \boxdot H}} \circ \iota^2_{X,Y})/{\simeq_H})) \circ \iota^2_{X/\simeq_G, Y/\simeq_H} = (\pi_{\simeq_{G \boxdot H}} \circ \iota^2_{X,Y})/{\simeq_H}$ implies

$$w \circ \iota^2_{X/\simeq_G, Y/\simeq_H} \circ \pi_{\simeq_H} \circ h|_{\Pi^0_m} = \pi_{\simeq_{G \boxdot H}} \circ \iota^2_{X,Y} \circ h|_{\Pi^0_m}.$$

It follows

$$\begin{aligned}
&w \circ ((\iota^1_{X/\simeq_G, Y/\simeq_H} \circ \pi_{\simeq_G} \circ g|_{\Pi^k_0}) \cup (\iota^2_{X/\simeq_G, Y/\simeq_H} \circ \pi_{\simeq_H} \circ h|_{\Pi^0_m})) \\
&\qquad\qquad = \pi_{\simeq_{G \boxdot H}} \circ ((\iota^1_{X,Y} \circ g|_{\Pi^k_0}) \cup (\iota^2_{X,Y} \circ h|_{\Pi^0_m}))
\end{aligned}$$

and thus, by $((\pi_{\simeq_P}\circ\pi_r)/{\simeq_{G\boxdot H}})\circ\pi_{\simeq_{G\boxdot H}}=\pi_{\simeq_P}\circ\pi_r$,

$$f\circ q=\pi_{\simeq_P}\circ\pi_r\circ((\iota^1_{X,Y}\circ g|_{\Pi^k_0})\cup(\iota^2_{X,Y}\circ h|_{\Pi^0_m}))=\pi_{\simeq_P}\circ p.$$

That completes the proof of (b).

(c) If $\{k_m,\ell_m\}\subseteq\mathbb{N}_0$ are such that $\mathbf{G}_m\in\mathcal{G}(k_m,\ell_m)$ and if $(G_m,g_m)\in\mathbf{G}_m$ and if G_m has vertex set V_m for any $m\in[\![2]\!]$, then $(T,t)\in\mathbf{G}_1\otimes\mathbf{G}_2$, where $T=G_1\boxdot G_2$ and where $t=(\iota^1_{V_1,V_2}\circ g_1)\cup(\iota^2_{V_1,V_2}\circ g_2\circ\tau^{k_1,k_2}_{\ell_1,\ell_2})$. Thus, then $\mathrm{P}_{\mathbf{G}_1\otimes\mathbf{G}_2}=\ker(\pi_{\simeq_T}\circ t)$. On the other hand, if $q=(\iota^1_{V_1/\simeq_{G_1},V_2/\simeq_{G_2}}\circ\pi_{\simeq_{G_1}}\circ g_1)\cup(\iota^2_{V_1/\simeq_{G_1},V_2/\simeq_{G_2}}\circ\pi_{\simeq_{G_2}}\circ g_2\circ\tau^{k_1,k_2}_{\ell_1,\ell_2})$, then $\mathrm{P}_{\mathbf{G}_1}\otimes\mathrm{P}_{\mathbf{G}_2}=\ker(q)$ by $\mathrm{P}_{\mathbf{G}_1}=\ker(\pi_{\simeq_{G_1}}\circ g_1)$ and $\mathrm{P}_{\mathbf{G}_2}=\ker(\pi_{\simeq_{G_2}}\circ g_2)$ and Lemma 2.30. According to Lemma 4.3 (a) the mapping $w:=((\pi_{\simeq_{G_1\boxdot G_2}}\circ\iota^1_{V_1,V_2})/{\simeq_{G_1}})\sqcup((\pi_{\simeq_{G_1\boxdot G_2}}\circ\iota^2_{V_1,V_2})/{\simeq_{G_2}})$ is a well-defined bijection from $(V_1/{\simeq_{G_1}})\boxdot(V_2/{\simeq_{G_2}})$ to $(V_1\boxdot V_2)/{\simeq_{G_1\boxdot G_2}}$. Moreover,

$$\begin{aligned}w\circ\iota^1_{V_1/\simeq_{G_1},V_2/\simeq_{G_2}}\circ\pi_{\simeq_{G_1}}\circ g_1&=((\pi_{\simeq_{G_1\boxdot G_2}}\circ\iota^1_{V_1,V_2})/{\simeq_{G_1}})\circ\pi_{\simeq_{G_1}}\circ g_1\\&=\pi_{\simeq_{G_1\boxdot G_2}}\circ\iota^1_{V_1,V_2}\circ g_1\end{aligned}$$

and, likewise,

$$w\circ\iota^2_{V_1/\simeq_{G_1},V_2/\simeq_{G_2}}\circ\pi_{\simeq_{G_2}}\circ g_2\circ\tau^{k_1,k_2}_{\ell_1,\ell_2}=\pi_{\simeq_{G_1\boxdot G_2}}\circ\iota^2_{V_1,V_2}\circ g_2\circ\tau^{k_1,k_2}_{\ell_1,\ell_2}$$

and thus

$$w\circ q=\pi_{\simeq_{G_1\boxdot G_2}}\circ((\iota^1_{V_1,V_2}\circ g_1)\cup(\iota^2_{V_1,V_2}\circ g_2\circ\tau^{k_1,k_2}_{\ell_1,\ell_2}))=\pi_{\simeq_T}\circ t.$$

It follows $\ker(q)=\ker(\pi_{\simeq_T}\circ t)$, which is what we needed to show.

(d) If $\{k,\ell\}\subseteq\mathbb{N}_0$ are such that $\mathbf{G}\in\mathcal{G}(k,\ell)$ and if $(G,g)\in\mathbf{G}$, then $(G,g\circ\kappa^\ell_k)\in\mathbf{G}^*$ and thus

$$\mathrm{P}_{\mathbf{G}^*}=\ker(\pi_{\simeq_G}\circ g\circ\kappa^\ell_k)=(\kappa^\ell_k)^{\leftarrow}(\ker(\pi_{\simeq_G}\circ g))=(\kappa^k_\ell)_{\rightarrow}(\ker(\pi_{\simeq_G}\circ g))=(\mathrm{P}_{\mathbf{G}})^*.$$

(e) and (f) are again clear because the graph underlying both $\mathbf{I}$ and $\mathbf{M}_{0,2}$ only has a single vertex and no edges. □

4.2. Graph categories arising from component partition constraints. Again, the classification of all categories of partitions gives a large number of new examples of graph categories.

Definition 4.5. Given any category $\mathcal{C}$ of (uncolored) partitions, the set $\mathrm{P}^{\leftarrow}(\mathcal{C})$ is called the *bi-labeled graphs with $\mathcal{C}$-partitioned components.*

Proposition 4.6. *For each category $\mathcal{C}$ of partitions the bi-labeled graphs with $\mathcal{C}$-partitioned components form a graph category.*

Proof. Follows from Lemma 2.35 (a). □

Remark 4.7. Proposition 4.6 also holds if one uses the definition of "partition" from [MR19, p. 11] and the corresponding version of P from Remark 4.2.

4.3. Component partition invariant. And, like before, the classification can also be used to give an invariant for arbitrary graph categories.

Definition 4.8. Given any graph category $\mathcal{F}$, we call the set $\mathbb{P}_{\to}(\mathcal{F})$ the *component partitions of* $\mathcal{F}$.

Proposition 4.9. *For any graph category* $\mathcal{F}$ *the component partitions of* $\mathcal{F}$ *form a category of partitions.*

Proof. Immediate consequence of Lemma 2.35 (b). □

Remark 4.10. What was said about Proposition 4.6 in Remark 4.7 is true about Proposition 4.9 as well.

5. Distances of labeled vertices

5.1. Definition and properties of label distances. The object of the results of the present section is the set of distances between any two distinct connected labeled vertices of a bi-labeled graph.

Definition 5.1. For any $\mathbf{G} \in \mathcal{G}$ define the *label distances* of $\mathbf{G}$ as

$$D(\mathbf{G}) := \{d_G(v, v') \mid \{v, v'\} \subseteq \operatorname{ran}(g) \wedge v \neq v' \wedge v \simeq_G v'\},$$

where $(G, g) \in \mathbf{G}$ can be any representative.

Notation 5.2. For any $M \subseteq \mathbb{N}$ let $\langle M \rangle_+$ denote the additive subsemigroup of $\mathbb{N}$ generated by M.

It is important to understand how the label distances of a bi-labeled graph are affected by the varous operations.

Lemma 5.3. *(a)* $D(\mathbf{I}) = \varnothing$.

(b) $D(\mathbf{G}^r) = D(\mathbf{G})$ *for any* $\mathbf{G} \in \mathcal{G}$ *and any* $r \in \{⤹, ⤸, ⤺, ⤻\}$ *such that* $\mathbf{G}^r$ *is defined.*

(c) $D(\mathbf{G}^\wedge) = D(\mathbf{G})$ *for any* $\mathbf{G} \in \mathcal{G}$.

(d) $D(\mathbf{G}_1 \otimes \mathbf{G}_2) \subseteq \bigcup_{m=1}^{2} D(\mathbf{G}_m)$

(e) $D(E(\mathbf{G}, \mathsf{T})) \subseteq \langle D(\mathbf{G}) \rangle_+$ *for any* $\mathbf{G} \in \mathcal{G}$ *and any consecutive set* T *of two points such that* $E(\mathbf{G}, \mathsf{T})$ *is defined.*

Proof. (a) $D(\mathbf{I}) = \varnothing$ because G only has single vertex for any $(G, g) \in \mathbf{I}$.

(b) If $\{k, \ell\} \subseteq \mathbb{N}_0$ are such that $\mathbf{G} \in \mathcal{G}(k, \ell)$ and if $(G, g) \in G$, then $(G, g \circ \omega^{r,k}_{\ell}) \in \mathbf{G}^r$. As $\omega^{r,k}_{\ell}$ is surjective, $\operatorname{ran}(g \circ \omega^{r,k}_{\ell}) = \operatorname{ran}(g)$. Hence, $D(\mathbf{G}^r) = D(\mathbf{G})$.

(c) Similarly, if $\{k, \ell\} \subseteq \mathbb{N}_0$ and $\mathbf{G} \in \mathcal{G}(k, \ell)$ and $(G, g) \in G$, then $(G, g \circ \rho^k_\ell) \in \mathbf{G}^\wedge$, where ρ^k_ℓ is surjective as well. Hence, $\operatorname{ran}(g \circ \rho^k_\ell) = \operatorname{ran}(g)$ and thus $D(\mathbf{G}^\wedge) = D(\mathbf{G})$.

(d) If $\{k_m, \ell_m\} \subseteq \mathbb{N}_0$ are such that $\mathbf{G}_m \in \mathcal{G}(k_m, \ell_m)$ and $(G_m, g_m) \in \mathbf{G}_m$ and if G_m has vertex set V_m for each $m \in [\![2]\!]$, then $(G_1 \boxdot G_2, t) \in \mathbf{G}_1 \otimes \mathbf{G}_2$, where $t = (\iota^1_{V_1,V_2} \circ g_1) \cup (\iota^2_{V_1,V_2} \circ g_2 \circ \tau^{k_1,k_2}_{\ell_1,\ell_2})$. Let $\{u, u'\} \subseteq \operatorname{ran}(t)$ and $u \neq u'$ and $u \simeq_{G_1 \boxdot G_2} u'$. By Remark 2.8 (g) that warrants the existence of $j \in [\![2]\!]$ and $\{v, v'\} \subseteq V_j$ such that $u = \iota^j_{V_1,V_2}(v)$ and $u' = \iota^j_{V_1,V_2}(v')$ and $d_{G_1 \boxdot G_2}(u,u') = d_{G_j}(v,v')$. Because $\iota^1_{V_1,V_2}$ and $\iota^2_{V_1,V_2}$ have disjoint ranges, $\mathrm{ran}(t) = \mathrm{ran}(\iota^1_{V_1,V_2} \circ g_1) \cup \mathrm{ran}(\iota^2_{V_1,V_2} \circ g_2 \circ \tau^{k_1,k_2}_{\ell_1,\ell_2}) = \dot\bigcup_{m=1}^2 (\iota^m_{V_1,V_2})_\to(\mathrm{ran}(g_m))$, where we have also used that $\tau^{k_1,k_2}_{\ell_1,\ell_2}$ is surjective. Hence, the assumption $\{u,u'\} \subseteq \mathrm{ran}(t)$ and the injectivity of $\iota^j_{V_1,V_2}$ demand $\{v,v'\} \subseteq \mathrm{ran}(g_j)$. It follows $d_{G_1 \boxdot G_2}(u,u') = d_{G_j}(v,v') \in D(\mathbf{G}_j)$. Because u and u' were arbitrary we have thus shown $D(\mathbf{G}_1 \otimes \mathbf{G}_2) \subseteq \bigcup_{m=1}^2 D(\mathbf{G}_m)$.

(e) Let $\{k,\ell\} \subseteq \mathbb{N}_0$ be such that $\mathbf{G} \in \mathcal{G}(k,\ell)$, let $(G,g) \in \mathbf{G}$ and let $G = (V,E)$. Then $(P,p) \in E(\mathbf{G},\mathsf{T})$, where $P = (W,F)$, where, if $T = g_\to(\mathsf{T})$ and $\mathsf{M} = \Pi^k_\ell \backslash \mathsf{T}$, then $W = \{\{v\} \,|\, v \in V \backslash T\} \cup \{T\}$ and

$$\begin{aligned} F = &\{\{\{v\} \,|\, v \in e\} \,|\, e \in E \wedge e \cap T = \varnothing\} \\ &\cup \{\{\{v\}, T \,|\, v \in e \backslash T\} \,|\, e \in E \wedge \varnothing \neq e \cap T \neq e\} \\ &\cup \{\{T\} \,|\, \exists e \in E \colon e \subseteq T\} \end{aligned}$$

and where $p = \pi \circ g \circ \gamma_{\mathsf{M}}$ for

$$\pi\colon V \to W,\, v \mapsto \begin{cases} \{v\} & \text{if } v \notin T \\ T & \text{otherwise.} \end{cases}$$

Let $\{y,y'\} \subseteq \mathrm{ran}(p)$ be arbitrary with $y \neq y'$ and $y \simeq_P y'$. Because $\mathrm{ran}(p) \subseteq \pi_\to(\mathrm{ran}(g))$ we find $\{x,x'\} \subseteq \mathrm{ran}(g)$ such that $y = \pi(x)$ and $y' = \pi(x')$ and, necessarily, $x \neq x'$. By Remark 2.8 (c) there exists a path u of length $k \coloneqq d_P(y,y')$ from y to y' in P. We prove $k \in \langle D(\mathbf{G})\rangle_+$ by distinguishing two cases.

Case 1: u does not involve identified vertices. First, suppose $T \notin \mathrm{ran}(u)$. On the one hand, since π is a graph homomorphism from G to P by Remark 2.19 (b) we can infer $k = d_P(y,y') = d_P(\pi(x),\pi(x')) \leq d_G(x,x')$ by Remark 2.8 (f) (i). On the other hand, the asssumption $\mathrm{ran}(u) \subseteq W \backslash \{T\} = \mathrm{ran}(\pi|_{V\backslash T})$ and the fact that $\pi|_{V\backslash T}$ is a graph embedding of $G|_{V\backslash T}$ into P by Remark 2.19 (c) imply that the map $(\pi|_{V\backslash T})^{-1} \circ u$ is a well-defined walk of length k in $G|_{V\backslash T}$ by Remark 2.8 (f) (ii), and thus $d_{G|_{V\backslash T}}(x,x') \leq k$ by definition. Since also $d_G(x,x') \leq d_{G|_{V\backslash T}}(x,x')$ according to Remark 2.8 (e) we have thus shown $d_G(x,x') \leq k$. In conclusion, $k = d_G(x,x') \in D(\mathbf{G})$ by $\{x,x'\} \subseteq \mathrm{ran}(g)$.

Case 2: u involves identified vertices. Now, suppose instead that there exists $j \in [\![k+1]\!]$ such that $u_j = T$. In order to prove $k \in \langle D(\mathbf{G})\rangle_+$ it suffices to show that $j-1 \in D(\mathbf{G})$ if $1 < j$ and that $k-j+1 \in D(\mathbf{G})$ if $j < k+1$. Indeed, if so and if $1 = j$, then $k = k-1+1 = k-j+1 \in D(\mathbf{G})$; likewise, if $j = k+1$, then $k = (k+1)-1 = j-1 \in D(\mathbf{G})$; and, lastly, if $1 < j < k+1$, then both $j-1 \in D(\mathbf{G})$ and $k-j+1 \in D(\mathbf{G})$ and thus $k = (j-1)+(k-j+1) \in \langle D(\mathbf{G})\rangle_+$.

Step 2.1: Definition of b and d. By Remark 2.8 (d), if $1 < j$, then $b\colon [\![j]\!] \to W,\, i \mapsto u_i$ is a path in P from $u_1 = y$ to $u_j = T$ of length $j-1 = d_P(y,T)$. Likewise, if $j < k+1$, then $d\colon [\![k-j+2]\!] \to W,\, i \mapsto u_{j+i-1}$ is a path in P from $u_j = T$ to $u_{k+1} = y' = d_P(T,y')$.

Step 2.2: Definition of $a_1,\ldots,a_{j-1}$ and $c_2,\ldots,c_{k-j+2}$. Because u is a path and $u_j = T$ we know $u_i \neq T$ for any $i \in [\![k+1]\!] \backslash \{j\}$. Hence, by definition of W, if $1 < j$, then for each $i \in [\![j]\!]$ with $i < j$ we find $a_i \in V \backslash T$ with $b_i = \pi(a_i) = \{a_i\}$ by $b_i = u_i$. Likewise, if $j < k+1$, then for each $i \in [\![k-k+2]\!]$ with $1 < i$ there exists $c_i \in V \backslash T$ with $d_i = \pi(c_i) = \{c_i\}$ by $d_i = u_{j+i-1}$.

Step 2.3: Definition of a_j and c_1 and properties of a and c. For any $i \in [\![j-1]\!]$, because b is a walk in P, we know $\{b_i, b_{i+1}\} \in F$. Moreover, if $i < j-1$, then $\{a_i, a_{i+1}\} \cap T = \varnothing$ and the definition of F thus imply $\{a_i, a_{i+1}\} \in E$. In contrast, if $i = j-1$, then $\{b_i, b_{i+1}\} = \{\{a_{j-1}\}, T\}$ and the definition of F thus requires the existence of some $a_j \in T$ such that $\{a_{j-1}, a_j\} \in E$. Altogether we have constructed $a\colon [\![j]\!] \to V$, $i \mapsto a_i$ with $b = \pi \circ a$ and shown it to be a walk in G from $a_1 = x$ to a_j of length $j-1$. Because the restriction of π to $(V \backslash T) \cup \{a_j\}$ is injective and because b is a path in P we even know that a is a path in G.

By an analogous argument, if $j < k+1$, we find $c_1 \in V \backslash T$ such that $c\colon [\![k-j+2]\!] \to V$, $i \mapsto c_i$ is a path in G from c_1 to $c_{k+1} = x'$ of length $k-j+1$ and such that $d = \pi \circ c$.

Step 2.4: Lengths of a and c. Since $\pi(a_1) = y$ and $\pi(a_j) = T$ Remark 2.8 (f) shows $d_P(y, T) \leq d_G(a_1, a_j)$ if $1 < j$. Likewise, $\pi(c_1) = T$ and $\pi(c_{k-j+2}) = y'$ yield $d_P(T, y') \leq d_G(c_1, c_{k-j+2})$ if $j < k+1$.

Because we already know $d_P(y, T) = j-1$ if $1 < j$ and $k-j+1 = d_P(T, y')$ if $j < k+1$ it follows $j-1 \leq d_G(a_1, a_j)$ and $k-j+1 \leq d_G(c_1, c_{k-j+2})$ if $j < k+1$.

On the other hand, $d_G(a_1, a_j) \leq j-1$ if $1 < j$ because a is a walk in G from a_1 to a_j of length $j-1$ then. In the same way, $d_G(c_1, c_{k-j+2}) \leq k-j+1$ if $j < k+1$ because c is a walk in G from c_1 to c_{k-j+2} of length $k-j+1$ in that case.

Hence, $d_G(a_1, a_j) = j-1$ if $1 < j$ and $d_G(c_1, c_{k-j+2}) = k-j+1$ if $j < k+1$.

Step 2.5: Synthesis. If $1 < j$, then $a_1 = x \in \operatorname{ran}(g)$ and $a_j \in T \subseteq \operatorname{ran}(g)$ imply $j-1 = d_G(a_1, a_j) \in D(\mathbf{G})$. Likewise, provided $j < k+1$, we can conclude $k-j+1 = d_G(c_1, c_{k-j+2}) \in D(\mathbf{G})$ from $c_1 = T \subseteq \operatorname{ran}(g)$ and $c_{k-j+2} = x' \in \operatorname{ran}(g)$. That is what we needed to prove. □

5.2. Graph categories arising from label distance constraints. By placing constraints on label distances a set of graph categories can be ontained.

Definition 5.4. For any subsemigroup S of $(\mathbb{N}, +)$ let $\mathcal{D}_S$ be the set of all $\mathbf{G} \in \mathcal{G}$ such that $D(\mathbf{G}) \subseteq S$.

Remark 5.5. (a) $\mathcal{D}_{\mathbb{N}} = \mathcal{G}$ is the graph category of all bi-labeled graphs.

(b) $\mathcal{D}_{\varnothing} = \mathcal{G}$ is given by all bi-labeled graphs whose underlying graphs have at most one labeled vertex per connected component.

(c) For any subsemigroups S and S' of $(\mathbb{N}, +)$, whenever $S \subseteq S'$, then $\mathcal{D}_S \subseteq \mathcal{D}_{S'}$.

Proposition 5.6. *$\mathcal{D}_S$ is a graph category for any subsemigroup S of $(\mathbb{N}, +)$.*

Proof. Follows immediately from Proposition 2.21 and Lemma 5.3. □

5.3. Label distance invariant. The distances of labeled vertices form an invariant of graph categories.

Proposition 5.7. *$\bigcup_{\mathbf{G} \in \mathcal{F}} D(\mathbf{G})$ is an additive subsemigroup of $\mathbb{N}$ for any graph category $\mathcal{F}$.*

Proof. Given any $\{j_1, j_2\} \subseteq \bigcup_{\mathbf{G}\in\mathcal{F}} D(\mathbf{G})$, we exhibit $\mathbf{H} \in \mathcal{F}$ with $j_1 + j_2 \in D(\mathbf{H})$. More precisely, we give a representative $(H, h) \in \mathbf{H}$ and $\{w, w'\} \subseteq \operatorname{ran}(h)$ with $j \coloneqq d_H(w, w') = j_1 + j_2$.

Step 1: Definition of $\mathbf{H}$, H, h, w *and* w'. By assumption, for each $m \in [\![2]\!]$ there exist $\mathbf{G}_m \in \mathcal{F}$ and $(G_m, g_m) \in \mathbf{G}_m$ with $G_m = (V_m, E_m)$ and $\{x_m, x'_m\} \subseteq \operatorname{ran}(g_m)$ with $x_m \neq x'_m$ and $x_m \simeq_{G_m} x'_m$ such that $d_{G_m}(x_m, x'_m) = j_m$. By Lemma 5.3 (b) we can assume for each $m \in [\![2]\!]$ that $\mathbf{G}_m \in \mathcal{G}(1, \ell_m)$ for some $\ell_m \in \mathbb{N}$ and that $x'_m = g_m({}^\blacksquare 1)$. By Remark 2.8 (c), for each $m \in [\![2]\!]$ we find a path a^m in G_m from x_m to x'_m of length j_m. Because $\mathcal{F}$ is a graph category, $\mathbf{G}_1 \otimes \mathbf{G}_2 \in \mathcal{F}(2, \ell_1 + \ell_2)$ and thus $\mathbf{H} \coloneqq E(\mathbf{G}_1 \otimes \mathbf{G}_2, \mathsf{T}) \in \mathcal{F}(0, \ell_1 + \ell_2)$ for $\mathsf{T} = \{{}^\blacksquare 2, {}^\blacksquare 1\}$.

If we abbreviate $\iota_m \equiv \iota^m_{V_1,V_2}$ for each $m \in [\![2]\!]$ and $T = \{\iota_1(x'_1), \iota_2(x'_2)\}$ and $\mathsf{M} = \Pi^0_{\ell_1+\ell_2}$, then by definition $(H, h) \in \mathbf{H}$, where $H = (W, F)$, where

$$W = \dot\bigcup_{m=1}^2 \{\{\iota_m(v)\} \mid v \in V_m \backslash \{x'_m\}\} \cup \{T\}$$

and

$$\begin{aligned} F = &\dot\bigcup_{m=1}^2 \{\{\{\iota_m(v)\} \mid v \in e\} \mid e \in E_m \wedge x'_m \notin e\} \\ &\dot\cup \dot\bigcup_{m=1}^2 \{\{\{\iota_m(v)\}, T \mid v \in e \backslash \{x'_m\}\} \mid e \in E_m \wedge \varnothing \neq e \cap \{x'_m\} \neq e\} \\ &\dot\cup \{\{T\} \mid \exists_{m=1}^2 \colon \{x'_m\} \in E_m\} \end{aligned}$$

and where $h = \pi \circ ((\iota_1 \circ g_1) \dot\cup (\iota_2 \circ g_2 \circ \tau^{1,1}_{\ell_1,\ell_2})) \circ \gamma_{\mathsf{M}}$, where

$$\pi \colon V_1 \boxdot V_2 \to W,\ \iota_m(v) \mapsto \begin{cases} \{\iota_m(v)\} & \text{if } v \neq x'_m, \\ T & \text{otherwise.} \end{cases}$$

Finally, if we let $w \coloneqq (\pi \circ \iota_1)(x_1)$ and $w' \coloneqq (\pi \circ \iota_2)(x_2)$, then $\{w, w'\} \subseteq \operatorname{ran}(h)$ because $x_m \in (g_m)_\to(\Pi^0_{\ell_m})$ and $x_m \neq x'_m$ for each $m \in [\![2]\!]$.

Step 2: Proving $j \leq j_1 + j_2$. Since ι_m is a graph homomorphism from G_m to $G_1 \boxdot G_2$ for each $m \in [\![2]\!]$ and since π is a graph homomorphism from $G_1 \boxdot G_2$ to H by Remark 2.19 (b) the map $\pi \circ \iota_1 \circ a^1$ is a walk in H of length j_1 from w to $(\pi \circ \iota_1)(x'_1) = T$ and $\pi \circ \iota_2 \circ a^2$ is a walk in H of length j_2 from w' to $(\pi \circ \iota_2)(x'_2) = T$ by Remark 2.8 (f) (i). Hence, by Remark 2.8 (a) and (b) the concatenation $(\pi \circ \iota_2 \circ a^2)^{\Diamond} \oplus (\pi \circ \iota_1 \circ a^1)$ of $\pi \circ \iota_1 \circ a^1$ and the reverse of $\pi \circ \iota_2 \circ a^2$ is a walk in H of length $j_1 + j_2$ from w to w'. Thus, $j = d_H(w, w') \leq j_1 + j_2$ by definition.

Step 3: Proving $j_1 + j_2 \leq j$. In order to show $j_1 + j_2 \leq j$ it suffices to give a number $k \in \mathbb{N}$, a walk in G_1 of length $k - 1$ from x_1 to x'_1 and a walk in G_2 of length $j - k + 1$ from x'_2 to x_2. Indeed, adding the two inequalities $j_1 = d_{G_1}(x_1, x'_1) \leq k - 1$ and $j_2 = d_{G_2}(x_2, x'_2) = d_{G_2}(x'_2, x_2) \leq j - k + 1$ valid then implies $j_1 + j_2 \leq (k-1) + (j-k+1) = j$.

Since $w \simeq_H w'$ by the previous step, Remark 2.8 (c) guarantees the existence of a path z in H from w to w' of length $j = d_H(w, w')$.

Step 3.1: z *passes through* T *and stays in the images of* G_1 *before and* G_2 *after.* First, we prove that there exists $k \in \mathbb{N}$ with $1 < k < j + 1$ such that $z_k = T$, such that $\{z_1, \ldots, z_{k-1}\} \subseteq \operatorname{ran}(\pi \circ \iota_1) \backslash \{T\}$ and such that $\{z_{k+1}, \ldots, z_{j+1}\} \subseteq \operatorname{ran}(\pi \circ \iota_2) \backslash \{T\}$.

Step 3.1.1: Reduction to auxiliary claim. This we show by proving that for any $\{s, s'\} \subseteq [\![2]\!]$ with $s \neq s'$ and any $\{p, r\} \subseteq [\![j+1]\!]$ with $p < r$ such that $z_p \in \operatorname{ran}(\pi \circ \iota_s)\backslash\{T\}$ and $z_r \in \operatorname{ran}(\pi \circ \iota_{s'})\backslash\{T\}$ there exists $q \in [\![j+1]\!]$ with $p < q < r$ such that $z_q = T$. Once that is known to be true, the facts $z_1 = w = (\pi\circ\iota_1)(x_1) \in \operatorname{ran}(\pi\circ\iota_1)\backslash\{T\}$ and $z_{j+1} = w' = (\pi \circ \iota_2)(x_2) \in \operatorname{ran}(\pi \circ \iota_2)\backslash\{T\}$ will first of all require the existence of $k \in \mathbb{N}$ with $1 < k < j+1$ such that $z_k = T$. We justify that, in addition though, it will also prove $\{z_1, \ldots, z_{k-1}\} \subseteq \operatorname{ran}(\pi \circ \iota_1)\backslash\{T\}$ and $\{z_{k+1}, \ldots, z_{j+1}\} \subseteq \operatorname{ran}(\pi \circ \iota_2)\backslash\{T\}$.

Indeed, if $z_k = T$, then $z_i \neq T$ for any $i \in [\![j+1]\!]\backslash\{k\}$ because z is a path. Thus, by definition of W, for each $i \in [\![j+1]\!]$ with $i \neq k$ there must exist $t \in [\![2]\!]$ such that $z_i \in \operatorname{ran}(\pi \circ \iota_t)\backslash\{T\}$. If there were $\{s, s'\} \subseteq [\![2]\!]$ with $s \neq s'$ and $\{p, r\} \subseteq [\![j+1]\!]$ such that $1 \leq p < r < k$ or $k < p < r \leq j+1$ and such that $z_p \in \operatorname{ran}(\pi \circ \iota_s)\backslash\{T\}$ and $z_r \in \operatorname{ran}(\pi \circ \iota_{s'})\backslash\{T\}$, we would be able to infer the existence of $q \in [\![j+1]\!]$ with $p < q < r < k$ or $k < p < q < r$ and $z_q = T$, contradicting the uniquness of k. Hence, we must have $\{z_1, \ldots, z_{k-1}\} \subseteq \operatorname{ran}(\pi \circ \iota_1)\backslash\{T\}$ and $\{z_{k+1}, \ldots, z_{j+1}\} \subseteq \operatorname{ran}(\pi \circ \iota_2)\backslash\{T\}$ because $z_1 = w \in \operatorname{ran}(\pi \circ \iota_1)\backslash\{T\}$ and $z_{j+1} = w' \in \operatorname{ran}(\pi \circ \iota_2)\backslash\{T\}$.

Step 3.1.2: Proof of auxiliary claim. Thus, let $\{s, s'\} \subseteq [\![2]\!]$ and $s \neq s'$ and let $\{p, r\} \subseteq [\![j+1]\!]$ be such that $z_p \in \operatorname{ran}(\pi \circ \iota_s)\backslash\{T\}$ and $z_r \in \operatorname{ran}(\pi \circ \iota_{s'})\backslash\{T\}$. We prove by contradiction the existence of $q \in [\![j+1]\!]$ with $p < q < r$ and $z_q = T$. Thus suppose, $T \notin \{z_p, z_{p+1} \ldots, z_r\}$. We show by induction that $z_i \in \operatorname{ran}(\pi \circ \iota_s)$ for each $i \in [\![j+1]\!]$ with $p \leq i \leq r$. That then contradicts $z_r \in \operatorname{ran}(\pi \circ \iota_{s'})\backslash\{T\}$ because $s \neq s'$ and $\bigcap_{t=1}^{2} \operatorname{ran}(\pi \circ \iota_t) = \{T\}$.

As the base case, $z_p \in \operatorname{ran}(\pi \circ \iota_s)$ is true by assumption. Now, let $i \in [\![j+1]\!]$ with $p \leq i < r$ be such that $z_i \in \operatorname{ran}(\pi\circ\iota_s)$. Because z is a path in H and thus $\{z_i, z_{i+1}\} \in F$ and $z_i \neq z_{i+1}$ and because $z_i \neq T \neq z_{i+1}$ by assumption the definition of F implies the existence of $m \in [\![2]\!]$ and $e \in E_m$ with $x'_m \notin e$ such that $\{z_i, z_{i+1}\} = \{\iota_m(v) \,|\, v \in e\}$, which is to say $\{z_i, z_{i+1}\} = (\pi\circ\iota_m)_{\to}(e)$. Hence, $z_i \in \operatorname{ran}(\pi\circ\iota_s)$ demands $m = s$. Thus also $z_{i+1} \in \operatorname{ran}(\pi \circ \iota_s)$, as we needed to see.

Step 3.2: z stems from a path in G_1 and a path in G_2 fused together. The two mappings $y^1\colon [\![k-1]\!] \to W$, $i \mapsto z_i$ and $y^2\colon [\![j-k+1]\!] \to W$, $i \mapsto z_{k+i}$ are subwalks of z and thus walks in H with $\operatorname{ran}(y^1) \subseteq \operatorname{ran}(\pi \circ \iota_1)\backslash\{T\} = \operatorname{ran}(\pi|_{(V_1 \boxdot V_2)\backslash T} \circ \iota_1|_{V_1\backslash\{x'_1\}})$ and $\operatorname{ran}(y^2) \subseteq \operatorname{ran}(\pi \circ \iota_2)\backslash\{T\} = \operatorname{ran}(\pi|_{(V_1\boxdot V_2)\backslash T} \circ \iota_2|_{V_2\backslash\{x'_2\}})$ by Step 3.1. Since ι_m is a graph embeddding of G_m into $G_1 \boxdot G_2$ and thus $\iota_m|_{V_m\backslash\{x'_m\}}$ a graph embedding of $G_m|_{V_m\backslash\{x'_m\}}$ into $G_1 \boxdot G_2|_{(V_1\boxdot V_2)\backslash T}$ and since $\pi|_{(V_1\boxdot V_2)\backslash T}$ is a graph embedding of $(G_1 \boxdot G_2)|_{(V_1\boxdot V_2)\backslash T}$ into H by Remark 2.19 (c) the map $u^m \coloneqq (\pi|_{(V_1\boxdot V_2)\backslash T} \circ \iota_m|_{V_m\backslash\{x'_m\}})^{-1} \circ y^m = (\pi \circ \iota_m)^{-1} \circ y^m$ is a well-defined walk in $G_m|_{V_m\backslash\{x'_m\}}$ by Remark 2.8 (f) (ii) for each $m \in [\![2]\!]$. More precisely, u^1 is a walk of length $k-2$ from $(\pi \circ \iota_1)^{-1}(z_1) = (\pi \circ \iota_1)^{-1}(w) = x_1$ to $(\pi \circ \iota_1)^{-1}(z_{k-1})$ and, similarly, u^2 one of length $j-k$ from $(\pi \circ \iota_1)^{-1}(z_{k+1})$ to $(\pi \circ \iota_2)^{-1}(z_{j+1}) = (\pi \circ \iota_2)^{-1}(w') = x_2$.

Since z is a walk in H and since $z_{k-1} \neq T = z_k$ by Step 3.1 we know $\{z_{k-1}, T\} = \{z_{k-1}, z_k\} \in F$. Hence, by definition of F we can infer that there exists $e \in E_1$ with $\varnothing \neq e \cap \{x'_1\} \neq e$ and $\{z_{k-1}, T\} = \{\{\iota_1(v)\}, T \,|\, v \in e\backslash\{x'_1\}\}$, which is to say $\{(\pi \circ \iota_1)^{-1}(z_{k-1}), x'_1\} = e \in E_1$. In other words, the map $h^1\colon [\![2]\!] \to V_1$ with $1 \mapsto z_{k-1}$

and $2 \mapsto x_1'$ is a walk of length 1 in G_1. Analogously, $z_k = T \neq z_{k+1}$ and $\{z_k, z_{k+1}\} \in F$ demand $\{x_2', (\pi \circ \iota_2)(z_{k+1})\} \in E_2$, thus making $h^2\colon [\![2]\!] \to V_2$ with $1 \mapsto x_2'$ and $2 \mapsto z_{k+1}$ is a walk of length 1 in G_2.

In conclusion, the concatenation $h^1 \oplus u^1$ is a well-defined walk in G_1 of length $(k-2)+1 = k-1$ from x_1 to x_1' and, likewise, $u^2 \oplus h^2$ a walk in G_2 of length $1+(j-k) = j-k+1$ from x_2' to x_2. That concludes the proof. □

6. Further graph categories

There exist graph categories which cannot be described solely in terms of restrictions on the vertex or component partitions or the distances of labeled vertices. Mančinska and Roberson exhibited two examples: the category $\mathcal{P}$ of all planar bi-labeled graphs [MR20, Section II A] and, implicitly, the category of all bi-labeled graphs whose underlying graphs have no edges in Theorem 8.3 of the arXiv version of [MR20]. We add further examples.

Definition 6.1. For any $m \in \mathbb{N}_0$ we call any graph G a
(a) *free graph* on m vertices if $G \cong ([\![m]\!], \varnothing)$,
(b) *co-free graph* on m vertices if $1 \leq m$ and $G \cong ([\![m]\!], \{\{i,j\}\}_{i,j=1}^m)$,
(c) *path graph* on m vertices if $2 \leq m$ and $G \cong ([\![m]\!], \{\{i,i+1\}\}_{i=1}^{m-1})$,
(d) *cycle graph* on m vertices if $3 \leq m$ and $G \cong ([\![m]\!], \{\{i,i+1\}\}_{i=1}^{m-1} \cup \{\{m,1\}\})$.

In particular, the empty graph $(\varnothing, \varnothing)$ is a free graph.

Definition 6.2. Let $\mathcal{A}$ denote the set of all $\mathbf{G} \in \mathcal{G}$ such that for any $(G,g) \in \mathbf{G}$, if V is the vertex set of G, then
(a) $\mathrm{P}_{\mathbf{G}} \in \mathrm{O}^+$
(b) and for any $C \in V/\simeq_G$
  (i) the graph $G|_C$ is a
    (1) free graph on a single vertex,
    (2) co-free graph on a single vertex,
    (3) path graph on two or more vertices or
    (4) cycle graph on three or more vertices
  (ii) and the set $C \cap \operatorname{ran}(g)$ is
    (1) empty if $G|_C$ is a co-free or cycle graph,
    (2) either empty or equal to C if $G|_C$ is a path graph on two vertices,
    (3) equal to $\{v \in C \wedge \deg_G(v) = 1\}$ if $G|_C$ is a path graph on three or more vertices.

Remark 6.3. (a) For any $\mathbf{G} \in \mathcal{G}$, any $(G,g) \in \mathbf{G}$ and any connected component C of G, when considering whether $\mathbf{G}$ satisfies (b) of Definition 6.2 with respect to C, it is useful to recognize that the set $\{v \in C \wedge \deg_G(v) = 1\} = \deg_{G|_C}{}^{\leftarrow}(\{1\})$ only depends on $G|_C$ and not on all of G.

In consequence, if $\mathbf{H} \in \mathcal{G}$ and $(H,h) \in \mathbf{H}$ and if D is any connected component of H, provided $\mathbf{H} \in \mathcal{A}$ is known, it is sufficient to give a graph isomorphism f from $H|_D$ to $G|_C$ with $f_{\to}(D \cap \mathrm{ran}(h)) = C \cap \mathrm{ran}(g)$ in order to verify that $\mathbf{G}$ satisfies (b) of Definition 6.2 with respect to C.

(b) For any graphs G and P with vertex sets V and W, respectively, for any $C \in W/\simeq_P$ and any graph embedding f of G into P, if $C \subseteq \mathrm{ran}(f)$, then $f^{\leftarrow}(C) \in V/\simeq_G$.

PROPOSITION 6.4. *$\mathcal{A}$ is a graph category.*

PROOF. By Proposition 2.21 it is sufficient to show that $\mathcal{A}$ contains $\mathbf{I}$ and is closed under rotation, reflection, tensor products and erasing.

Necessary element: Since a free graph on a single vertex underlies $\mathbf{I}$ and since $\mathrm{P}_{\mathbf{I}} = | \in \mathrm{O}^+$ by Lemmata 3.3 (e) and 4.4 (e) we can infer $\mathbf{I} \in \mathcal{A}$.

Rotation: Given any $\{k,\ell\} \subseteq \mathbb{N}_0$, any $\mathbf{G} \in \mathcal{A}(k,\ell)$ and any $r \in \{\circlearrowleft, \circlearrowright, \circlearrowleft, \circlearrowright\}$ such that $\mathbf{G}^r$ is defined, we already know $\mathrm{P}_{\mathbf{G}^r} \in \mathrm{O}^+$ by Lemma 2.20 and Proposition 4.6. Moreover, for any $(G,g) \in \mathbf{G}$, if $H \coloneqq G$ and $h \coloneqq g \circ \omega^{r,k}_{\ell}$, then $(H,h) \in \mathbf{G}^r$ by definition. For any connected component C of H, of course, C is also a connected component of G and id_C is a graph isomorphism from $G|_C$ to $H|_C$ with $\mathrm{id}_{C\to}(C \cap \mathrm{ran}(g)) = \mathrm{ran}(g \circ \omega^{r,k}_{\ell}) = \mathrm{ran}(g)$ by the surjectivity of $\omega^{r,k}_{\ell}$. Hence, $\mathbf{G}^r$ satisfies (b) of Definition 6.2 with respect to C by Remark 6.3 (a).

Reflection: Similarly, for any $\{k,\ell\} \subseteq \mathbb{N}_0$ and any $\mathbf{G} \in \mathcal{A}(k,\ell)$ it is clear that $\mathrm{P}_{\mathbf{G}^\wedge} \in \mathrm{O}^+$ by Lemma 2.17 and Proposition 4.6. And, if again $(G,g) \in \mathbf{G}$, then $(G, g \circ \rho^k_\ell) \in \mathbf{G}^\wedge$, where once more $\mathrm{ran}(g \circ \rho^k_\ell) = \mathrm{ran}(g)$ because ρ^k_ℓ is surjective. Hence, in the same way as before, Remark 6.3 (a) tells us that $\mathbf{G} \in \mathcal{A}$ implies $\mathbf{G}^\wedge \in \mathcal{A}$.

Tensor product: If $\{k_m,\ell_m\} \subseteq \mathbb{N}_0$ and $\mathbf{G}_m \in \mathcal{A}(k_m,\ell_m)$ and $(G_m,g_m) \in \mathbf{G}_m$ and if G_m has vertex set V_m for any $m \in [\![2]\!]$, then $(G_1 \boxdot G_2, t) \in \mathbf{G}_1 \otimes \mathbf{G}_2$, where $t = (\iota^1_{V_1,V_2} \circ g_1) \cup (\iota^2_{V_1,V_2} \circ g_2 \circ \tau^{k_1,k_2}_{\ell_1,\ell_2})$. Of course, Proposition 4.6 guarantees $\mathrm{P}_{\mathbf{G}_1 \otimes \mathbf{G}_2} \in \mathrm{O}^+$.

Let $C \in (V_1 \boxdot V_2)/\simeq_{G_1 \boxdot G_2}$ be arbitrary. By Lemma 4.3 (a) the mapping $((\pi_{\simeq_{G_1 \boxdot G_2}} \circ \iota^1_{V_1,V_2})/\simeq_{G_1}) \sqcup ((\pi_{\simeq_{G_1 \boxdot G_2}} \circ \iota^2_{V_1,V_2})/\simeq_{G_2})$ is a well-defined bijection from $(V_1/\simeq_{G_1}) \boxdot (V_2/\simeq_{G_2})$ to $(V_1 \boxdot V_2)/\simeq_{G_1 \boxdot G_2}$. Hence, there exists $j \in [\![2]\!]$ and $C_j \in V_j/\simeq_{G_j}$ such that $(\iota^j_{V_1,V_2})_{\to}(C_j) = C$. Because $\iota^j_{V_1,V_2}$ is a graph embedding of G_j into $G_1 \boxdot G_2$ it thus follows that $\iota^j_{V_1,V_2}|_{C_j}$ is a graph isomorphism from $G_j|_{C_j}$ to $(G_1 \boxdot G_2)|_C$. Moreover, the surjectivity of $\tau^{k_1,k_2}_{\ell_1,\ell_2}$ ensures $\mathrm{ran}(t) = \mathrm{ran}(\iota^1_{V_1,V_2} \circ g_1) \cup \mathrm{ran}(\iota^2_{V_1,V_2} \circ g_2 \circ \tau^{k_1,k_2}_{\ell_1,\ell_2}) = \bigcup_{m=1}^2 (\iota^m_{V_1,V_2})_{\to}(\mathrm{ran}(g_m))$. For that reason and because $\iota^j_{V_1,V_2}$ is injective and $\bigcap_{m=1}^2 \mathrm{ran}(\iota^m_{V_1,V_2}) = \varnothing$ it follows $C \cap \mathrm{ran}(t) = (\iota^j_{V_1,V_2})_{\to}(C_j \cap \mathrm{ran}(g_j))$. Hence, $\mathbf{G}_1 \otimes \mathbf{G}_2$ satisfies (b) of Definition 6.2 with respect to C by Remark 6.3 (a).

Erasing: Let $\{k,\ell\} \subseteq \mathbb{N}_0$, let $\mathbf{G} \in \mathcal{A}(k,\ell)$, let $(G,g) \in \mathbf{G}$ and let $G = (V,E)$. Then $(P,p) \in E(\mathbf{G},\mathsf{T})$, where $P = (W,F)$, where, if $T = g_{\to}(\mathsf{T})$ and $\mathsf{M} = \Pi^k_\ell \backslash \mathsf{T}$, then

$W = \{\{v\} \mid v \in V\backslash T\} \uplus \{T\}$ and

$$\begin{aligned} F = &\{\{\{v\} \mid v \in e\} \mid e \in E \wedge e \cap T = \varnothing\} \\ &\uplus \{\{\{v\}, T \mid v \in e\backslash T\} \mid e \in E \wedge \varnothing \neq e \cap T \neq e\} \\ &\uplus \{\{T\} \mid \exists e \in E\colon e \subseteq T\} \end{aligned}$$

and where $p = \pi \circ g \circ \gamma_{\mathsf{M}}$ according to Remark 2.19 (a) for

$$\pi\colon V \to W,\, v \mapsto \begin{cases} \{v\} & \text{if } v \notin T \\ T & \text{otherwise.} \end{cases}$$

Immediately, $\mathrm{P}_{E(\mathbf{G},\mathsf{T})} \in \mathrm{O}^+$ by Lemma 2.20, Proposition 4.6 and $\mathbf{G} \in \mathcal{A}$. Let $C \in W/{\simeq_P}$ be arbitrary. By distinguishing cases, we check that $E(\mathbf{G},\mathsf{T})$ satisfies (b) of Definition 6.2 with respect to C.

Case 1: T is not a vertex of C. If $T \notin C$, then $C \subseteq W\backslash\{T\}$ implies $B \coloneqq \pi^{\leftarrow}(C) \subseteq \pi^{\leftarrow}(W\backslash\{T\}) = \pi^{\leftarrow}(W)\backslash\pi^{\leftarrow}(\{T\}) = V\backslash T$. By Remark 2.19 (c) the restriction $\pi|_{V\backslash T}$ is a graph embedding of $G|_{V\backslash T}$ into P. Hence, in particular, $\pi|_B$ is a graph isomorphism from $G|_B$ to $P|_C$.

Also, then, $B \in (V\backslash T)/{\simeq_{G|_{V\backslash T}}}$ by Remark 6.3 (b) because $C \in W/{\simeq_P}$ and because $\pi|_{V\backslash T}$ is a graph embedding of $G|_{V\backslash T}$ into P. In fact, $B \in V/{\simeq_G}$ because if there were $v \in T$ and $v' \in B$ such that $v \simeq_G v'$ then, since π is a graph homomorphism from G to P by Remark 2.19 (b), that would imply $\pi(v) = T \simeq_P \pi(v')$, contradicting $C \in W/{\simeq_P}$ and $T \notin C$.

Furthermore, since $\mathsf{T} \subseteq g^{\leftarrow}(g_{\rightarrow}(\mathsf{T})) = g^{\leftarrow}(T)$ and thus $\Pi^k_\ell\backslash g^{\leftarrow}(T) \subseteq \Pi^k_\ell\backslash\mathsf{T} = \mathsf{M}$ it follows from $B \subseteq V\backslash T$ that $g^{\leftarrow}(B) \subseteq g^{\leftarrow}(V\backslash T) = g^{\leftarrow}(V)\backslash g^{\leftarrow}(T) = \Pi^k_\ell\backslash g^{\leftarrow}(T) \subseteq \mathsf{M}$. Because the injectivity of γ_{M} ensures $\gamma_{\mathsf{M}\rightarrow} \circ \gamma_{\mathsf{M}}{}^{\leftarrow} = \mathrm{id}_{\wp(\mathsf{M})}$ we have thus shown $g^{\leftarrow}(B) = (\gamma_{\mathsf{M}\rightarrow} \circ \gamma_{\mathsf{M}}{}^{\leftarrow} \circ g^{\leftarrow})(B)$. Hence, $\pi_{\rightarrow}(B \cap \operatorname{ran}(g)) = (\pi_{\rightarrow} \circ g_{\rightarrow} \circ g^{\leftarrow})(B) = (\pi_{\rightarrow} \circ g_{\rightarrow} \circ \gamma_{\mathsf{M}\rightarrow} \circ \gamma_{\mathsf{M}}{}^{\leftarrow} \circ g^{\leftarrow})(B) = ((\pi \circ g \circ \gamma_{\mathsf{M}})_{\rightarrow} \circ (\pi \circ g \circ \gamma_{\mathsf{M}})^{\leftarrow})(C) = (p_{\rightarrow} \circ p^{\leftarrow})(C) = C \cap \operatorname{ran}(p)$.

Hence, $E(\mathbf{G},\mathsf{T})$ satisfies (b) of Definition 6.2 with respect to C by Remark 6.3 (a).

Case 2: T is a vertex of C. If $T \in C$, then by $|T| \leq 2$ there exist $\{B_1, B_2\} \subseteq V/{\simeq_G}$ such that $B_1 \cap T \neq \varnothing \neq B_2 \cap T$, such that $T \subseteq B_1 \cup B_2$ and such that $C = \pi_{\rightarrow}(B_1 \cup B_2)$ and $B_1 \cup B_2 = \pi^{\leftarrow}(C)$. By $\mathbf{G} \in \mathcal{A}$ for any $i \in [\![2]\!]$ the graph $G|_{B_i}$ is a free or co-free graph on a single vertex or a path or cycle graph. Moreover, since $B_i \cap T \neq \varnothing$ and $T = g_{\rightarrow}(\mathsf{T})$ require $B_i \cap \operatorname{ran}(g) \neq \varnothing$ we can exclude by the same assumption that $G|_{B_i}$ is a co-free or cycle graph for any $i \in [\![2]\!]$. In addition, $G|_{B_i}$ is not the empty graph for any $i \in [\![2]\!]$ because $B_i \neq \varnothing$. In summary, for each $i \in [\![2]\!]$ the graph $G|_{B_i}$ must be either a free graph on a single vertex or a path graph.

Moreover, since $(\gamma_{\mathsf{M}\rightarrow} \circ \gamma_{\mathsf{M}}{}^{\leftarrow})(\mathsf{S}) = \mathsf{S}\backslash\mathsf{T}$ for any $\mathsf{S} \subseteq \Pi^k_\ell$ by definition of γ_{M} we infer $C \cap \operatorname{ran}(p) = (p_{\rightarrow} \circ p^{\leftarrow})(C) = ((\pi \circ g \circ \gamma_{\mathsf{M}})_{\rightarrow} \circ (\pi \circ g \circ \gamma_{\mathsf{M}})^{\leftarrow})(C) = (\pi \circ g)_{\rightarrow}((\gamma_{\mathsf{M}\rightarrow} \circ \gamma_{\mathsf{M}}{}^{\leftarrow} \circ g^{\leftarrow})(B_1 \cup B_2)) = (\pi \circ g)_{\rightarrow}(g^{\leftarrow}(B_1 \cup B_2)\backslash\mathsf{T}) = \bigcup_{i=1}^2 (\pi \circ g)_{\rightarrow}(g^{\leftarrow}(B_i)\backslash\mathsf{T})$.

In particular, $\mathsf{T} \subseteq g^{\leftarrow}(g_{\rightarrow}(\mathsf{T})) = g^{\leftarrow}(T)$ implies $g^{\leftarrow}(B_i\backslash T) = g^{\leftarrow}(B_i)\backslash g^{\leftarrow}(T) \subseteq g^{\leftarrow}(B_i)\backslash\mathsf{T}$, thus $(B_i \cap \operatorname{ran}(g))\backslash T = (B_i\backslash T) \cap \operatorname{ran}(g) = g_{\rightarrow}(g^{\leftarrow}(B_i\backslash T)) \subseteq g_{\rightarrow}(g^{\leftarrow}(B_i)\backslash\mathsf{T})$ and thus, ultimately, $\pi_{\rightarrow}((B_i \cap \operatorname{ran}(g))\backslash T) \subseteq (\pi \circ g)_{\rightarrow}(g^{\leftarrow}(B_i)\backslash\mathsf{T})$ for each $i \in [\![2]\!]$.

Case 2.1: T is comprised of only one vertex. If $|T| = 1$, then $B_1 \cap T \neq \varnothing \neq B_2 \cap T$ demands $B_1 = B_2$ because $V/\simeq_G$ is a partition of V. Were $G|_{B_1} = G|_{B_2}$ a path graph, then $|T| = 1 < 2 \leq |B_1|$ would require $T \subsetneq B_1$ and thus $2 = |\mathsf{T}| \leq |g^{\leftarrow}(\mathsf{T})| < |g^{\leftarrow}(B_1)|$ by $\mathsf{T} \subseteq g^{\leftarrow}(T) \subsetneq g^{\leftarrow}(B_1)$, contradicting $g^{\leftarrow}(B_1) \in \mathrm{P}_{\mathbf{G}} \in \mathrm{O}^+$.

In conclusion, $G|_{B_1} = G|_{B_2}$ must be a free graph on a single vertex. Because our assumption $|T| = 1$ makes π an isomorphism from G to P and because $C = \pi_{\rightarrow}(B_1) = \pi_{\rightarrow}(B_2)$ it then follows that $P|_C \cong G|_{B_1}$ is a free graph on a single vertex as well. Since thus no requirement on $C \cap \operatorname{ran}(p)$ is imposed by the definition of $\mathcal{A}$ that is all we needed to show in this case.

Case 2.2: T is comprised of two vertices but B_1 and B_2 still coincide. Next, let $|T| = 2$ and $B_1 = B_2$. Because $g^{\leftarrow}(B_1) \in \mathrm{P}_{\mathbf{G}} \in \mathrm{O}^+$ demands $|g^{\leftarrow}(B_1)| = 2$, because $|\mathsf{T}| = 2$ and because $\mathsf{T} \subseteq g^{\leftarrow}(T) \subseteq g^{\leftarrow}(B_1 \cup B_2) = g^{\leftarrow}(B_1)$ we can conclude $\mathsf{T} = g^{\leftarrow}(B_1)$. Moreover, we infer $C \cap \operatorname{ran}(p) = (\pi \circ g)_{\rightarrow}(g^{\leftarrow}(B_1) \backslash \mathsf{T}) = \varnothing$ from $\mathsf{T} = g^{\leftarrow}(B_1)$. Consequently, it suffices to show that $P|_C$ is a co-free graph on a single vertex, a cycle graph or a path graph on two vertices. We distinguish three subcases depending on $|B_1|$.

First, though, note that because $|T| = 2$ and $T \subseteq B_1 = B_2$ the graph $G|_{B_1} = G|_{B_2}$ cannot be a free graph on a single vertex because $T \subseteq B_1 \cup B_2 = B_1$ demands $2 \leq |B_1|$. Thus, $G|_{B_1}$ must be a path graph instead. By $\mathbf{G} \in \mathcal{A}$ we infer $B_1 \cap \operatorname{ran}(g) = \{v \in B_1 \wedge \deg_G(v) = 1\}$, regardless of the value of $|B_1|$ since $\deg_G(v) = 1$ for any $v \in B_1$ if $|B_1| = 2$. Thus, $T = g_{\rightarrow}(\mathsf{T}) = g_{\rightarrow}(g^{\leftarrow}(B_1)) = B_1 \cap \operatorname{ran}(g) = \{v \in B_1 \wedge \deg_G(v) = 1\}$.

Case 2.2.1: One path graph on two vertices. If $|B_1| = 2$, then $B_1 = T$ by $T \subseteq B_1$ and $|T| = 2$. Thus, $C = \pi_{\rightarrow}(B_1) = \{T\}$ consists of only a single vertex. Moreover, $G|_{B_1} = G|_T$ being a path graph on 2 vertices ensures $T \in E$ and thus $\{T\} \in F$ by definition of F. In other words, $P|_C$ is a co-free graph on a single vertex. Thus, by $C \cap \operatorname{ran}(g) = \varnothing$ the bilabeled graph $E(\mathbf{G}, \mathsf{T})$ satisfies (b) of Definition 6.2 with respect to C.

Case 2.2.2: One path graph on exactly three vertices. Now, let $|B_1| = 3$. Then, $G|_{B_1}$ is a path graph on three vertices. We thus find a graph isomorphism u from $([\![3]\!], \{\{1,2\},\{2,3\}\})$ to $G|_{B_1}$. In particular, $T = \deg_{G|_{B_1}}{}^{\leftarrow}(\{1\}) = \{u_1, u_3\}$ and $C = \pi_{\rightarrow}(B_1) = \{T, \{u_2\}\}$, where $\{u_2\} \cap T = \varnothing$. As a consequence, the mapping $f\colon [\![2]\!] \to C$ with $1 \mapsto T$ and $2 \mapsto \{u_2\}$ is bijective. It is a graph homomorphism from $([\![2]\!], \{\{1,2\}\})$ to $P|_C$ because $\{u_1, u_2\} \in E$ implies $\{T, \{u_2\}\} \in F$ by definition of F. We justify that f is a graph embedding, i.e., $\{T\} \notin F$ and $\{\{u_2\}\} \notin F$. Because u is a graph isomorphism, $\{u_i\} \notin E$ for each $i \in [\![3]\!]$. That has two consequences. First, since $\pi|_{V \backslash T}$ is a graph embedding of $G|_{V \backslash T}$ into P and since $u_2 \in V \backslash T$ that requires $\{\{u_2\}\} \notin F$. Second, by definition of F the only way $\{T\} \in F$ could arise was if $\{u_1, u_3\} = T \in E$, which is not the case since u is a graph embedding of $([\![3]\!], \{\{1,2\},\{2,3\}\})$ into G. Thus, we have shown that $P|_C$ is a path graph on two vertices. Hence, $E(\mathbf{G}, \mathsf{T})$ satisfies (b) of Definition 6.2 with respect to C in this case as well.

Case 2.2.3: One path graph on more than three vertices. Next, consider the case where $m \coloneqq |B_1| > 3$, i.e., where $G|_{B_1}$ is a path graph on more than three vertices. Then, there exists a graph isomorphism u from $(\llbracket m \rrbracket, \{\{i, i+1\}\}_{i=1}^{m})$ to $G|_{B_1}$. It follows $T = \deg_{G|_{B_1}}{}^{\leftarrow}(\{1\}) = \{u_1, u_m\}$ and $C = \pi_{\rightarrow}(B_1) = \{T\} \cup \{\{u_i\}\}_{i=2}^{m-1}$. Hence and since u is injective, the rule $1 \mapsto T$ and $i \mapsto \{u_i\}$ for each $i \in \mathbb{N}$ with $2 \le i \le m-1$ defines a bijection $f \colon \llbracket m-1 \rrbracket \to C$. Moreover, f is graph homomorphism from $(\llbracket m-1 \rrbracket, \{\{i, i+1\}\}_{i=1}^{m-2} \cup \{\{m-1, 1\}\})$ to $P|_C$ because, by definition of F, from $\{u_1, u_2\} \in E$ it follows that $\{T, \{u_2\}\} \in F$, from $\{u_i, u_{i+1}\} \in E$ that $\{\{u_i\}, \{u_{i+1}\}\} \in E$ for any $i \in \mathbb{N}$ with $2 \le i \le m-2$ and from $\{u_{m-1}, u_m\} \in E$ that $\{\{u_{m-1}\}, T\} \in F$.

We show that f is actually a graph embedding. Since u is an embedding into G, since $\pi|_{V \backslash T}$ embeds $G|_{V \backslash T}$ into P and since $\{u_i\}_{i=2}^{m-1} \subseteq V \backslash T$ we can conclude for all $\{i, j\} \subseteq \{2, \dots, m-1\}$ that $\{\{u_i\}, \{u_j\}\} \in E$ if and only if $i = j+1$ or $j = i+1$. Furthermore, u being a graph embedding ensures $\{u_1, u_m\} \notin E$ and $\{u_1\} \notin E$ and $\{u_m\} \notin E$ and thus $\{T\} \notin F$ by definition of F. Finally, if there was $i \in \mathbb{N}$ with $3 \le i \le m-2$ and $\{T, \{u_i\}\} \in F$ by the definition of F that would demand $\{u_1, u_i\} \in E$ or $\{u_m, u_i\} \in E$, which is not true because u is an embedding of $(\llbracket m \rrbracket, \{\{i, i+1\}\}_{i=1}^{m})$ into G. Thus, f is indeed a graph embedding.

In conclusion, $P|_C$ is a cycle graph on $m-1$ vertices, which means that $E(\mathbf{G}, \mathsf{T})$ satisfies (b) of Definition 6.2 with respect to C.

Case 2.3: T is comprised of two vertices and B_1 and B_2 are distinct. Finally, let $|T| = 2$ and $B_1 \neq B_2$. If there was $i \in \llbracket 2 \rrbracket$ with $\mathsf{T} \subseteq g^{\leftarrow}(B_i)$, it would follow $T = g_{\rightarrow}(\mathsf{T}) \subseteq g_{\rightarrow}(g^{\leftarrow}(B_i)) \subseteq B_i$, which would contradict $B_1 \cap T \neq \varnothing \neq B_2 \cap T$ and $B_1 \neq B_2$. Hence, instead, $\mathsf{T} \nsubseteq g^{\leftarrow}(B_i)$ for each $i \in \llbracket 2 \rrbracket$. Since, on the other hand, $T \subseteq B_1 \cup B_2$ implies $\mathsf{T} \subseteq g^{\leftarrow}(g_{\rightarrow}(\mathsf{T})) = g^{\leftarrow}(T) \subseteq g^{\leftarrow}(B_1) \cup g^{\leftarrow}(B_1)$ it must hold $\mathsf{T} \cap g^{\leftarrow}(B_i) \neq \varnothing$ for each $i \in \llbracket 2 \rrbracket$. Because $|\mathsf{T}| = 2$ and $B_1 \neq B_2$ that means $|\mathsf{T} \cap g^{\leftarrow}(B_i)| = 1$ for each $i \in \llbracket 2 \rrbracket$. Correspondingly, $|g^{\leftarrow}(B_i) \backslash \mathsf{T}| = 1$ for each $i \in \llbracket 2 \rrbracket$ since $|g^{\leftarrow}(B_i)| = 2$ by virtue of $g^{\leftarrow}(B_i) \in \mathrm{P}_{\mathbf{G}} \in \mathrm{O}^{+}$. Again we need to consider three subcases separately.

Case 2.3.1: Two free graphs. First, let each of $G|_{B_1}$ and $G|_{B_2}$ be a free graph on a single vertex. That requires $T = B_1 \cup B_2$ by $B_1 \cap T \neq \varnothing \neq B_2 \cap T$, implying $C = \pi_{\rightarrow}(B_1 \cup B_2) = \pi_{\rightarrow}(T) = \{T\}$. Moreover, $G|_{B_1}$ and $G|_{B_2}$ both being free ensures $B_1 \notin E$ and $B_2 \notin E$. Also, $T = B_1 \cup B_2 \notin E$ because B_1 and B_2 are distinct connected components of G. Hence, $\{T\} \notin F$ by definition of F. We have thus proved that $P|_C$ is a free graph on a single vertex as well. Consequently, in this case, $E(\mathbf{G}, \mathsf{T})$ satisfies (b) of Definition 6.2 with respect to C.

Case 2.3.2: One free and one path graph. Next, let one of $G|_{B_1}$ and $G|_{B_2}$ be a free graph on a single vertex and the other any path graph. Without loss of generality we can assume that $G|_{B_1}$ is the free graph and that $G|_{B_2}$ is a path graph on $m \coloneqq |B_2|$ many vertices. Then, we find a graph isomorphism u from $(\llbracket m \rrbracket, \{\{i, i+1\}\}_{i=1}^{m-1})$ to $G|_{B_2}$. Because $\mathbf{G} \in \mathcal{A}$ and $B_2 \cap \mathrm{ran}(g) \neq \varnothing$ we know $B_2 \cap \mathrm{ran}(g) = \{v \in B_2 \wedge \deg_G(v) = 1\} = \{u_1, u_m\}$, irrespective of the value of m because $\deg_G(v) = 1$ for any $v \in B_2$ if $m = 2$. Hence and because $\varnothing \neq T \cap B_2 \subseteq B_2 \cap \mathrm{ran}(g)$, by replacing the walk u in G with its reverse $u^{\Leftrightarrow}$ if necessary we can ensure that $u_1 \in T$, which is to say

$T = B_1 \cup \{u_1\}$ and $C = \{T\} \cup \{\{u_i\}\}_{i=2}^m$. For that reason and since u is bijective, the mapping $f\colon [\![m]\!] \to C$ with $1 \mapsto T$ and $i \mapsto \{u_i\}$ for any $i \in \mathbb{N}$ with $2 \leq i \leq m$ is bijective. Because u is a graph homomorphism from $([\![m]\!], \{\{i, i+1\}\}_{i=1}^{m-1})$ to G and because π is one from G to P the mapping $f = \pi \circ u$ is a graph homomorphism from $([\![m]\!], \{\{i, i+1\}\}_{i=1}^{m-1})$ to P.

We prove that f is in fact a graph embedding. Since u is a graph embedding into G, since $\pi|_{V\backslash T}$ embeds $G|_{V\backslash T}$ into G and since $\{u_i\}_{i=2}^m \subseteq V\backslash T$ by the injectivity of u, we know for any $\{i,j\} \subseteq \{2, \ldots, m\}$ that $\{\{u_i\}, \{u_j\}\} \in F$ if and only if $|i-j| = 1$. For the same reasons, $\{\{u_i\}\} \notin E$ for any $i \in \mathbb{N}$ with $2 \leq i \leq m$. Furthermore, as $\{u_1\} \notin E$ because u is a graph embedding, as $B_1 \notin E$ since $G|_{B_1}$ is free and since $T \notin E$ since B_1 and B_2 are distinct connected components of G, the definition of F implies $\{T\} \notin F$. Lastly, $\{T, \{u_i\}\} \notin E$ for any $i \in \mathbb{N}$ with $3 \leq i \leq m$ because by definition of F that would require $\{u_1, u_i\} \in E$ or $B_1 \cup \{u_i\} \in E$, which is not the case since u is a graph embedding of $([\![m]\!], \{\{i, i+1\}\}_{i=1}^{m-1})$ into G respectively because $u_i \in B_2$ and because B_1 and B_2 are distinct connected components of G. Overall, f is indeed a graph isomorphism from $([\![m]\!], \{\{i, i+1\}\}_{i=1}^{m-1})$ to $P|_C$, which makes the latter a path graph on m vertices.

Hence, $\{x \in C \wedge \deg_P(x) = 1\} = \{f_1, f_m\} = \{T, \{u_m\}\}$ and we have to show $C \cap \operatorname{ran}(p) = \{T, \{u_m\}\}$. Since $C \cap \operatorname{ran}(p) = \biguplus_{i=1}^2 (\pi \circ g)_\to(g^\leftarrow(B_i)\backslash \mathsf{T})$ it suffices to prove $(\pi \circ g)_\to(g^\leftarrow(B_1)\backslash \mathsf{T}) = \{T\}$ and $(\pi \circ g)_\to(g^\leftarrow(B_2)\backslash \mathsf{T}) = \{\{u_m\}\}$. We conclude from $|g^\leftarrow(B_1)\backslash \mathsf{T}| = 1$ in particular $\varnothing \neq (\pi \circ g)_\to(g^\leftarrow(B_1)\backslash \mathsf{T})$ and thus $(\pi \circ g)_\to(g^\leftarrow(B_1)\backslash \mathsf{T}) = \{T\}$ since $(\pi \circ g)_\to(g^\leftarrow(B_1)\backslash \mathsf{T}) \subseteq \pi_\to((g_\to \circ g^\leftarrow)(B_1)) = \pi_\to(B_1) = \{T\}$ by $(g_\to \circ g^\leftarrow)(B_1) = B_1$. Moreover, $u_m \in (B_2 \cap \operatorname{ran}(g))\backslash T$ implies $\{\{u_m\}\} \subseteq \pi_\to((B_i \cap \operatorname{ran}(g))\backslash T) \subseteq (\pi \circ g)_\to(g^\leftarrow(B_i)\backslash \mathsf{T})$ and thus $(\pi \circ g)_\to(g^\leftarrow(B_i)\backslash \mathsf{T}) = \{\{u_m\}\}$ since $|g^\leftarrow(B_2)\backslash \mathsf{T}| = 1$ demands $|(\pi \circ g)_\to(g^\leftarrow(B_2)\backslash \mathsf{T})| = 1$. In conclusion, $E(\mathbf{G}, \mathsf{T})$ satisfies (b) of Definition 6.2 with respect to C.

Case 2.3.3: Two path graphs. Finally, suppose that $G|_{B_1}$ and $G|_{B_2}$ are both path graphs on any numbers $m_1 \coloneqq |B_1|$ respectively $m_2 \coloneqq |B_2|$ of vertices. For each $i \in [\![2]\!]$ we then find a graph isomorphism u^i from $([\![m_i]\!], \{\{r, r+1\}\}_{r=1}^{m_i-1})$ to $G|_{B_i}$. Once more, no matter the value of m_i, we can deduce $B_i \cap \operatorname{ran}(g) = \{v \in B_i \wedge \deg_G(v) = 1\} = \{u_1^i, u_{m_i}^i\}$ for each $i \in [\![2]\!]$. Hence, by replacing one or both of u^1 and u^2 with their respective reverse walks $u^{1\Leftrightarrow}$ and $u^{2\Leftrightarrow}$ we can achieve that $T = \{u_{m_1}^1, u_1^2\}$. Then, because $(\pi \circ u^1)(m_1) = T = (\pi \circ u^2)(1)$ the mapping $f \coloneqq (\pi \circ u^1) \oplus (\pi \circ u^2)$ is a well-defined walk in P, i.e., a graph homomorphism from $([\![m]\!], \{\{r, r+1\}\}_{r=1}^{m-1})$ to P, where $m \coloneqq m_1 + m_2 - 1$.

We show that f is actually a graph embedding. For any $\{i,j\} \subseteq [\![m]\!]\backslash\{m_1\}$ it holds $\{f_i, f_j\} \in F$ if and only if $|i-j| = 1$. Indeed, if $i < m_1$ and $j < m_1$, then $\{f_i, f_j\} = \pi_\to(\{u_i^1, u_j^1\}) \in F$ if and only if $|i-j| = 1$ because u^1 is a graph embedding of $([\![m_1]\!], \{\{r, r+1\}\}_{r=1}^{m_1-1})$ into G, because $\{u_i^1, u_j^1\} \subseteq V\backslash T$ and because $\pi|_{V\backslash T}$ embeds $G|_{V\backslash T}$ into P. Likewise, u^2 being a graph embedding of $([\![m_2]\!], \{\{r, r+1\}\}_{r=1}^{m_2-1})$ into G and $\pi|_{V\backslash T}$ one of $G|_{V\backslash T}$ into P ensure $\{f_i, f_j\} = \pi_\to(\{u_{i-m_1+1}^2, u_{j-m_1+1}^2\}) \in F$ if and only if $|i-j| = 1$ whenever $m_1 < i$ and $m_1 < j$ because $\{u_{i-m_1+1}^2, u_{j-m_1+1}^2\} \subseteq V\backslash T$. And in the case where $i < m_1 < j$ the fact that $u_i^1 \in B_1$ and $u_{j-m_1+1}^2 \in B_2$ lie in distinct connected components of G implies $\{f_i, f_j\} = \pi_{\rightarrow}(\{u_i^1, u_{j-m_1+1}^2\}) \notin F$ in accordance with $|r-t| \neq 1$. If $j < m_1 + 1 < i$, an analogous argument can be given to the same effect. Moreover, u^1 and u^2 being graph embeddings require $\{u_{m_1}^1\} \notin E$ and $\{u_1^2\} \notin E$. Since also $\{u_{m_1}^1, u_1^2\} \notin E$ because $u_{m_1}^1 \in B_1$ and $u_1^2 \in B_2$ that proves $\{T\} \notin F$ by definition of F. Finally, if there was $i \in \mathbb{N}$ with $i < m_1 - 1$ or $m_1 + 1 < i$ such that $\{T, \{f_i\}\} \in F$, then by definition of F that would require $\{u_{m_1}^1, u_i^1\} \in E$ or $\{u_1^2, u_i^1\} \in E$ if $i < m_1 - 1$ and $\{u_{m_1}^1, u_{i-m_1+1}^2\} \in E$ or $\{u_1^2, u_{i-m_1+1}^2\} \in E$ if $m_1 + 1 < i$. However, if $i < m_1 - 1$, then $\{u_{m_1}^1, u_i^1\} \notin E$ because u^1 is a graph embedding and $1 < |i - m_1|$, and $\{u_1^2, u_i^1\} \notin E$ because $u_i^1 \in B_1$ and $u_1^2 \in B_2$. Likewise, if $m_1 + 1 < i$, then $\{u_{m_1}^1, u_{i-m_1+1}^2\} \notin E$ because $u_{m_1}^1 \in B_1$ and $u_{i-m_1+1}^2 \in B_2$, and $\{u_1^2, u_{i-m_1+1}^2\} \notin E$ because u^2 is a graph embedding and $1 < |(i - m_1 + 1) - 1|$. In conclusion, f is a graph isomorphism from $([\![m]\!], \{\{r, r+1\}\}_{r=1}^{m-1})$ to $P|_C$.

It thus remains to show $C \cap \operatorname{ran}(p) = \{f_1, f_m\}$. This is clear once we prove $(\pi \circ g)_{\rightarrow}(g^{\leftarrow}(B_1) \backslash \mathsf{T}) = \{\{u_1^1\}\}$ and $(\pi \circ g)_{\rightarrow}(g^{\leftarrow}(B_2) \backslash \mathsf{T}) = \{\{u_{m_2+1}^2\}\}$. First, $u_1^1 \in (B_1 \cap \operatorname{ran}(g)) \backslash T$ implies $\{\{u_1^1\}\} \subseteq \pi_{\rightarrow}((B_1 \cap \operatorname{ran}(g)) \backslash T) \subseteq (\pi \circ g)_{\rightarrow}(g^{\leftarrow}(B_1) \backslash \mathsf{T})$ and thus $(\pi \circ g)_{\rightarrow}(g^{\leftarrow}(B_1) \backslash \mathsf{T}) = \{\{u_1^1\}\}$ since $|g^{\leftarrow}(B_1) \backslash \mathsf{T}| = 1$ demands $|(\pi \circ g)_{\rightarrow}(g^{\leftarrow}(B_1) \backslash \mathsf{T})| = 1$. Likewise, from $u_{m_2+1}^2 \in (B_1 \cap \operatorname{ran}(g)) \backslash T$ it follows $\{\{u_{m_2+1}^2\}\} \subseteq \pi_{\rightarrow}((B_2 \cap \operatorname{ran}(g)) \backslash T) \subseteq (\pi \circ g)_{\rightarrow}(g^{\leftarrow}(B_2) \backslash \mathsf{T})$ and thus $(\pi \circ g)_{\rightarrow}(g^{\leftarrow}(B_2) \backslash \mathsf{T}) = \{\{u_{m_2+1}^2\}\}$ because $|g^{\leftarrow}(B_2) \backslash \mathsf{T}| = 1$ requires $|(\pi \circ g)_{\rightarrow}(g^{\leftarrow}(B_2) \backslash \mathsf{T})| = 1$. In conclusion, $E(\mathbf{G}, \mathsf{T})$ satisfies (b) of Definition 6.2 with respect to C. And that completes the proof. □

Definition 6.5. For any additive subsemigroup S of $\mathbb{N}$ denote by $\mathcal{A}_S$ the set of all $\mathbf{G} \in \mathcal{A}$ such that for any $(G, g) \in \mathbf{G}$, if V is the vertex set of G, then for any $C \in V/{\simeq_G}$ and any $m \in \mathbb{N}$,

(a) if $1 \notin S$, then $G|_C$ is not co-free,
(b) if $m \notin S$, then $G|_C$ is neither a cycle graph on m vertices nor a path graph on $m+1$ vertices.

Proposition 6.6. *$\mathcal{A}_S$ is a graph category for any additive subsemigroup S of $\mathbb{N}$.*

Proof. Again we use Proposition 2.21. Since $\mathbf{I} \in \mathcal{A}$ and since a free graph on a single vertex underlies $\mathbf{I}$ the bi-labeled graph S is an element of $\mathcal{A}_S$, irrespective of S. As far as invariance of $\mathcal{A}_S$ under rotations, reflection, tensor products and erasing is concerned, we extend the proof of Proposition 6.4. For each operation we make the same assumptions as were made there.

Rotation: If, in adddition, $\mathbf{G} \in \mathcal{A}_S$, then $H|_C$ obeys the restrictions of Definition 6.5 because $G|_C$ does and because $G|_C$ and $H|_C$ are isomorphic. Thus, also $\mathbf{G}^r \in \mathcal{A}_S$.

Reflection: The same argument as for rotation proves $\mathbf{G}^\wedge \in \mathcal{A}_S$ whenever $\mathbf{G} \in \mathcal{A}_S$.

Tensor product: If $\{\mathbf{G}_1, \mathbf{G}_2\} \subseteq \mathcal{A}_S$, then $G_j|_{C_j}$ complies with the exclusion rules of Definition 6.5. Consequently, so does $(G_1 \boxdot G_2)|_C$ since it is isomorphic to $G_j|_{C_j}$. Hence, indeed $\mathbf{G}_1 \otimes \mathbf{G}_2 \in \mathcal{A}_S$.

Erasing: We extend the arguments in the individual cases distinguished in the proof of Proposition 6.4.

Case 1: T is not a vertex of C. If $T \notin C$, then $G|_B$ and $P|_C$ are isomorphic. Hence, if , $\mathbf{G} \in \mathcal{A}_S$ then not only does $G|_B$ respect the rules of Definition 6.5 but so does $P|_C$.

Case 2: T is a vertex of C. If $T \in C$, then we further distinguish based on $|T|$ and on whether B_1 and B_2 coincide or not.

Case 2.1: T is comprised of only one vertex. In the case that $|T| = 1$, and thus $B_1 = B_2$, the graph $P|_C$ is a free graph on a single vertex and thus trivially satisfies the conditions of Definition 6.5 in any case.

Case 2.2: T is comprised of two vertices but B_1 and B_2 still coincide. If $|T| = 2$ and $B_1 = B_2$, then $G|_{B_1}$ is necesarily a path graph.

Case 2.2.1: One path graph on two vertices. If $G|_{B_1}$ is path graph on two vertices, then $P|_C$ is a co-free graph on a single vertex. Thus we need to show that, if $\mathbf{G} \in \mathcal{A}_S$, then $1 \in S$. And, indeed, by the contraposition of Definition 6.5 (b), the fact that $G|_{B_1}$ is path graph on two vertices then requires $1 \in S$.

Case 2.2.2: One path graph on exactly three vertices. In the instance that $G|_{B_1}$ is path graph on three vertices, $P|_C$ is a path graph on two vertices. If we assume $\mathbf{G} \in \mathcal{A}_S$, we hence have to prove $2 \in S$. Fortunately, $G|_{B_1}$ being a path graph on three vertices implies just that by the contraposition of Definition 6.5 (b).

Case 2.2.3: One path graph on more than three vertices. In case $G|_{B_1}$ is a path graph on $m = |B_1| > 3$ vertices, $P|_C$ is a cycle graph on $m - 1$ vertices. Hence, if $\mathbf{G} \in \mathcal{A}_S$, in order to show $E(\mathbf{G}, \mathsf{T}) \in \mathcal{A}_S$ we need to check that $m - 1 \in S$. However, otherwise $G|_{B_1}$ would not be able to be a path graph on m vertices by Definition 6.5 (b). Thus, $m - 1 \in S$.

Case 2.3: T is comprised of two vertices and B_1 and B_2 are distinct. If $B_1 \neq B_2$, then each of $G|_{B_1}$ and $G|_{B_2}$ is a free graph on single vertex or a path graph.

Case 2.3.1: Two free graphs. If $G|_{B_i}$ is a free graph on a single vertex for each $i \in [\![2]\!]$, then so is $P|_C$. Hence, in this case, $P|_C$ complies with the demands of Definition 6.5, no matter what S is.

Case 2.3.2: One free and one path graph. In the case where, without loss of generality, $G|_{B_1}$ is a free graph on a single vertex and $G|_{B_2}$ a path graph on m vertices, $P|_C$ is a path graph on m vertices as well. If now $\mathbf{G} \in \mathcal{A}_S$, then we can infer $m - 1 \in S$ from the fact that $G|_{B_2}$ a path graph on m vertices. And, of course, this is exactly what guarantees that also $P|_C$ meets the conditions of Definition 6.5.

Case 2.3.3: Two path graphs. The final case is that $G|_{B_i}$ is a path graph on $m_i = |B_i|$ vertices for each $i \in [\![2]\!]$, which implies that $P|_C$ is a path graph on $m = m_1 + m_2 - 1$ vertices. If we suppose $\mathbf{G} \in \mathcal{A}_S$, then Definition 6.5 (b) lets us infer $m_i - 1 \in S$ for each $i \in [\![2]\!]$ because $G|_{B_i}$ is a path graph on m_i vertices. Since S is a subsemigroup of $(\mathbb{N}, +)$ it follows $m - 1 = (m_1 - 1) + (m_2 - 1) \in S$. And this is exactly what is required in order for the path graph $P|_C$ on m vertices to comply with Definition 6.5.

In summary, we have shown in all cases that $\mathbf{G} \in \mathcal{A}_S$ also ensures $E(\mathbf{G}, \mathsf{T}) \in \mathcal{A}_S$, which concludes the proof. □

PROPOSITION 6.7. *$\mathcal{A}_S = \langle \mathbf{A}^{\circ k} \,|\, k \in M \rangle$ for any $M \subseteq \mathbb{N}$ with $S = \langle M \rangle_+$.*

PROOF. It suffices to consider the case $M = S$. Because $\mathrm{P}_{\mathbf{A}} = | \in \mathrm{O}^+$ and because a path graph on $k+1$ vertices, of which only the ones of degree one are labeled, underlies $\mathbf{A}^{\circ i}$ for any $i \in \mathbb{N}$, it is clear that $\mathcal{A} \supseteq \langle \mathbf{A}^{\circ i} \,|\, i \in S \rangle$. By Lemma 2.15, in order to show $\mathcal{A} \subseteq \langle \mathbf{A}^{\circ i} \,|\, i \in S \rangle$, it suffices to verify $\mathcal{A}_S(0, k+\ell) \subseteq \langle \mathbf{A}^{\circ i} \,|\, i \in S \rangle$ for any $\{k, \ell\} \subseteq \mathbb{N}_0$. This we prove by induction over the number of connected components of the underlying graph of the bi-labeled graph. The empty bi-labeled graph $\varnothing \in \mathcal{A}(0,0)$, the only one whose underlying graph has zero connected components, is an element of $\langle \mathbf{A}^{\circ i} \,|\, i \in S \rangle$ by definition. Suppose that $\langle \mathbf{A}^{\circ i} \,|\, i \in S \rangle$ contains any bi-labeled graph of $\bigcup_{k,\ell \in \mathbb{N}_0} \mathcal{A}_S(0, k+\ell)$ whose underlying graph has at most $n \in \mathbb{N}_0$ many connected components, and let $\{k, \ell\} \subseteq \mathbb{N}_0$, let $\mathbf{G} \in \mathcal{A}_S(k, \ell)$, let $(G, g) \in \mathbf{G}$, let V be the vertex set of G and let $|V/{\simeq_G}| = n+1$. We prove $\mathbf{G} \in \langle \mathbf{A}^{\circ i} \,|\, i \in S \rangle$ by constructing $\mathbf{H} \in \langle \mathbf{A}^{\circ i} \,|\, i \in S \rangle$ and $\mathbf{T} \in \langle \mathbf{A}^{\circ i} \,|\, i \in S \rangle$ with $\mathbf{G} = \mathbf{H} \otimes \mathbf{T}$.

We can make one simplifying assumption: By the characteristic property of non-crossing partitions, if $0 < k+\ell$, then we find a block $\mathsf{B} \in \mathrm{P}_{\mathbf{G}} \in \mathrm{O}^+(0, k+\ell)$ which is convex with respect to $\Gamma^0_{k+\ell}$, and we can assume $\mathsf{B} = \{{}_{\blacksquare}(k+\ell-1), {}_{\blacksquare}(k+\ell)\}$ by $|\mathsf{B}| = 2$ and Lemma 2.15.

Step 1: Defining $\mathbf{H}$ *and* $\mathbf{T}$. Let $\mathsf{M} \coloneqq \Pi^0_{k+\ell}$ if $k+\ell = 0$ and $\mathsf{M} \coloneqq \Pi^0_{k+\ell} \backslash \mathsf{B} = \Pi^0_{k+\ell-2}$ otherwise. Because $0 < n+1 = |V/{\simeq_G}|$, if $k+\ell = 0$, we can find and fix an arbitrary $B \in V/{\simeq_G}$. In case $0 < k+\ell$ we pick $B \in V/{\simeq_G}$ specifically to satisfy $\mathsf{B} = g^{\leftarrow}(B)$. Now, let $W \coloneqq V \backslash B$, let $H \coloneqq G|_W$ and let $T \coloneqq G|_B$, let h be $g|_{\mathsf{M}}$ seen as a map into W and let t be $g \circ \gamma_{\mathsf{B}}$ seen as a map into B and finally let $\mathbf{H} \in \mathcal{G}(0, k+\ell-2)$ and $\mathbf{T} \in \mathcal{G}(0,2)$ be such that $(H, h) \in \mathbf{H}$ and $(T, t) \in \mathbf{T}$.

Step 2: Proving that $\mathbf{H}$ *belongs to* $\langle \mathbf{A}^{\circ i} \,|\, i \in S \rangle$. Because B is a convex block of $\mathrm{P}_{\mathbf{G}}$, the partition $\mathrm{P}_{\mathbf{H}} = (\mathrm{P}_{\mathbf{G}})|_{\mathsf{M}}$ of Π^0_{n-2} still belongs to O^+. And the assumption that B is a connected component of G implies $W/{\simeq_H} = (V/{\simeq_G}) \backslash \{B\}$. Moreover, $\mathrm{ran}(h) = g_{\rightarrow}(\mathsf{M}) \subseteq W$ ensures $C \cap \mathrm{ran}(h) = C \cap \mathrm{ran}(g)$ for any $C \in W/{\simeq_H}$. Hence, $\mathbf{H}$ is an element of $\mathcal{A}$. Since $|W/{\simeq_H}| = |(V/{\simeq_G}) \backslash \{B\}| = (n+1) - 1 = n$ the induction hypothesis thus yields $\mathbf{H} \in \langle \mathbf{A}^{\circ i} \,|\, i \in S \rangle$.

Step 3: Proving that $\mathbf{T}$ *belongs to* $\langle \mathbf{A}^{\circ i} \,|\, i \in S \rangle$. In order to show $\mathbf{T} \in \langle \mathbf{A}^{\circ i} \,|\, i \in S \rangle$ we need to distinguish cases based on the value of $k+\ell$ and the isomorphism class of $G|_B$. Recall that the assumption $\mathbf{G} \in \mathcal{A}_S \subseteq \mathcal{A}$ requires $G|_B$ to be a free or co-free graph on a single vertex or a path or cycle graph.

Case 3.1: No labeled components. If $k+\ell = 0$, then $g = \varnothing$ and thus in particular $B \cap \mathrm{ran}(g) = \varnothing$. By $\mathbf{G} \in \mathcal{A}_S \subseteq \mathcal{A}$ that requires $G|_B$ to be a free or co-free graph on a single vertex, a cycle graph or a path graph on two vertices.

Case 3.1.1: Free graph. If $G|_B$ is a free graph on a single vertex, then the identity $\mathbf{H} = E(\mathbf{I}, \{{}^{\blacksquare}1, {}_{\blacksquare}1\})$ proves $\mathbf{H} \in \langle \mathbf{A}^{\circ i} \,|\, i \in S \rangle$ by Lemma 2.20 since $\mathbf{I} \in \langle \mathbf{A}^{\circ i} \,|\, i \in S \rangle$ by the definition of graph category.

Case 3.1.1: Co-free graph. In the case that $G|_B$ is a co-free graph on a single vertex, the assumption $\mathbf{G} \in \mathcal{A}_S$ demands $1 \in S$ by Definition 6.5 (a). Because $\mathbf{H} = E(\mathbf{A}, \{{}^{\bullet}1, {}_{\bullet}1\})$, Lemma 2.20 thus shows $\mathbf{H} \in \langle \mathbf{A} \rangle \subseteq \langle \mathbf{A}^{\circ i} \,|\, i \in S \rangle$.

Case 3.1.2: Cycle graph. If $G|_B$ is a cycle graph on $m \coloneqq |B| \geq 3$ many vertices, then, by Definition 6.5 (b), we can infer $m \in S$ from $\mathbf{G} \in \mathcal{A}_S$. Since $\mathbf{H} = E(\mathbf{A}^{\circ m}, \{{}^{\bullet}1, {}_{\bullet}1\})$ it follows $\mathbf{H} \in \langle \mathbf{A}^{\circ m} \rangle \subseteq \langle \mathbf{A}^{\circ i} \,|\, i \in S \rangle$ by Lemma 2.20.

Case 3.1.3: Path on two vertices. Finally, if $G|_B$ is a path graph on two vertices, then the assumption $\mathbf{G} \in \mathcal{A}_S$ forces $1 \in S$ by Definition 6.5 (a). As $\mathbf{H} = E(\mathbf{A}^{\circ 2}, \{{}^{\bullet}1, {}_{\bullet}1\})$, Lemma 2.20 thus proves $\mathbf{H} \in \langle \mathbf{A} \rangle \subseteq \langle \mathbf{A}^{\circ i} \,|\, i \in S \rangle$.

Case 3.2: At least one labeled component. In contrast, if $0 < k+\ell$, then $B \cap \operatorname{ran}(g) \neq \varnothing$ by $\mathsf{B} = g^{\leftarrow}(B)$. Hence, $\mathbf{G} \in \mathcal{A}_S \subseteq \mathcal{A}$ demands that $G|_B$ be a free graph on a single vertex or a path graph.

Case 3.2.1: Free graph. If $G|_B$ is a free graph on a single vertex, then $|\mathsf{B}| = 2$ implies $\mathbf{H} = \mathbf{M}_{0,2}$ and thus $\mathbf{H} \in \langle \mathbf{A}^{\circ i} \,|\, i \in S \rangle$ by the definition of graph category.

Case 3.2.2: Path graph. Lastly, in case $G|_B$ is a path graph on $m \coloneqq |B| \geq 2$ many vertices, then the premise $\mathbf{G} \in \mathcal{A}_S$ allows us to deduce $m-1 \in S$ by Definition 6.5 (b). Hence $\mathbf{H} = \mathbf{A}^{\circ (m-1)}$ shows $\mathbf{H} \in \langle \mathbf{A}^{\circ i} \,|\, i \in S \rangle$, which concludes the induction step and thus the proof. □

Remark 6.8. Though seemingly natural, it is generally *not* true that for an additive subsemigroup S of $\mathbb{N}$ the category $\mathcal{A}_S$ is simply $\mathcal{A} \cap \mathcal{D}_S$. In fact, the two agree if and only if $S = \mathbb{N}$. If $1 \notin S$, then the $(0,0)$-bilabeled graph on a co-free graph on a single vertex, i.e., the $\mathbf{C} \in \mathcal{G}(0,0)$ with $((\{1\}, \{\{1\}\}), \varnothing) \in \mathbf{C}$, is an element of $\mathcal{A} \cap \mathcal{D}_S$ but not of $\mathcal{A}_S$ by Definition 6.5 (a).

7. Graph-theoretical quantum groups

7.1. From graph categories to compact quantum groups. In [MR19] Mančinska and Roberson associate with any pair of a graph and a graph category a compact matrix quantum group. Their construction works as follows.

Notation 7.1. Let B be any $*$-algebra.

(a) For each $n \in \mathbb{N}_0$ the algebra of $n \times n$-matrices with entries from B is denoted by $M_n(B) \cong M_n(\mathbb{C}) \otimes B$.

(b) Given any $n \in \mathbb{N}_0$ and $u = (u_{j,i})_{(j,i) \in [\![n]\!] \boxtimes [\![n]\!]} \in M_n(B)$ let

  (i) $\overline{u} = (u^*_{j,i})_{(j,i) \in [\![n]\!] \boxtimes [\![n]\!]}$ be the conjugate and

  (ii) $u^{\mathrm{t}} = (u_{i,j})_{(j,i) \in [\![n]\!] \boxtimes [\![n]\!]}$ the transpose of u and let

  (iii) $u^{\oplus m} = (u_{j_1,i_1} \ldots u_{j_m,i_m})_{(j,i) \in [\![n]\!]^{\boxtimes m} \boxtimes [\![n]\!]^{\boxtimes m}}$ for any $m \in \mathbb{N}$.

Definition 7.2. For any $n \in \mathbb{N}$, any graph G with vertex set $[\![n]\!]$ and any $\{k, \ell\} \subseteq \mathbb{N}_0$ and any $\mathbf{H} \in \mathcal{G}(k,\ell)$ let $T^{\mathbf{H} \to G}_{k,\ell}$ be the complex $n^\ell \times n^k$-matrix which for any $i \in [\![n]\!]^{\boxtimes k}$ and $j \in [\![n]\!]^{\boxtimes \ell}$ has the (j,i)-entry

$$\left(T^{\mathbf{H} \to G}_{k,\ell}\right)_{j,i} = |\{f \text{ graph homomorphism } H \to G \wedge f \circ h = i \,{}^{\bullet}_{\bullet}\, j\}|,$$

where $(H, h) \in \mathbf{H}$ can be any representative.

Definition 7.3. For any $n \in \mathbb{N}$, any graph G with vertex set $[\![n]\!]$ and any graph category $\mathcal{F}$ let $\mathcal{C}_{\mathcal{F}}^{G} \coloneqq (\mathcal{C}_{\mathcal{F}}^{G}(k,\ell))_{k,\ell \in \mathbb{N}_0}$, where $\mathcal{C}_{\mathcal{F}}^{G}(k,\ell)$ is the vector subspace generated by $\{T_{k,\ell}^{\mathbf{H}\to G} \,|\, \mathbf{H} \in \mathcal{F}(k,\ell)\}$ in the complex vector space of complex $n^\ell \times n^k$-matrices for any $\{k,\ell\} \subseteq \mathbb{N}_0$.

Definition 7.4. For any $n \in \mathbb{N}$, any graph G with vertex set $[\![n]\!]$ and any graph category $\mathcal{F}$ the *graph-theoretical orthogonal compact matrix quantum group* $\mathbb{G}_{\mathcal{F}}^{G}$ associated with G and $\mathcal{F}$ is the pair (B,u), where B is the universal C^*-algebra with n^2 generators $\{u_{j,i}\}_{i,j=1}^{n}$ subject to the relations

$$u = \overline{u} \ \wedge \ \forall \{k,\ell\} \subseteq \mathbb{N}_0\colon \forall T \in \mathcal{C}_{\mathcal{F}}^{G}(k,\ell)\colon u^{\oplus \ell}\,(T \otimes 1) = (T \otimes 1)\, u^{\oplus k}$$

and where $u = (u_{j,i})_{(j,i) \in [\![n]\!] \boxtimes [\![n]\!]}$.

Proposition 7.5. *[MR19, Theorem 8.2] For any $n \in \mathbb{N}$, any graph G with vertex set $[\![n]\!]$ and any graph category $\mathcal{F}$ the pair $\mathbb{G}_{\mathcal{F}}^{G}$ is a compact $n \times n$-matrix quantum group.*

The main reason for that is contained in the following collection of results.

Proposition 7.6. [MR19, Example 3.9, Lemmata 3.21, 3.23 and 3.24] *Let $n \in \mathbb{N}$ and let G be any graph with vertex set $[\![n]\!]$.*

(a) $T_{\ell,m}^{\mathbf{K}\to G} \circ T_{k,\ell}^{\mathbf{H}\to G} = T_{k,m}^{\mathbf{K}\circ\mathbf{H}\to G}$ *for any* $\{k,\ell,m\} \subseteq \mathbb{N}_0$, $\mathbf{H} \in \mathcal{G}(k,\ell)$ *and* $\mathbf{K} \in \mathcal{G}(\ell,m)$.
(b) $T_{k_1,\ell_1}^{\mathbf{H}_1\to G} \otimes T_{k_2,\ell_2}^{\mathbf{H}_2\to G} = T_{k_1+k_2,\ell_1+\ell_2}^{\mathbf{H}_1\otimes\mathbf{H}_2\to G}$ *for any* $\{k_i,\ell_i\}_{i=1}^{2} \subseteq \mathbb{N}_0$, $(\mathbf{H}_i)_{i=1}^{2} \in \boxtimes_{i=1}^{2}\mathcal{G}(k_i,\ell_i)$.
(c) $(T_{k,\ell}^{\mathbf{H}\to G})^* = T_{\ell,k}^{\mathbf{H}^*\to G}$ *for any* $\{k,\ell\} \subseteq \mathbb{N}_0$ *and* $\mathbf{H} \in \mathcal{G}(k,\ell)$.
(d) $T_{1,1}^{\mathbf{I}\to G} = I_n$, *where I_n is the identity $n \times n$-matrix.*
(e) $T_{0,2}^{\mathbf{M}_{0,2}\to G} = (1 \mapsto \sum_{i=1}^{n} e_i \otimes e_i)$, *where $(e_i)_{i=1}^{n}$ is any orthonormal basis of $\mathbb{C}^{\boxtimes n}$.*

The below auxiliary result is implicit in [MR19].

Lemma 7.7. *For any $n \in \mathbb{N}$, any graph G with vertex set $[\![n]\!]$ and any graph category $\mathcal{F}$, if $\mathbb{G}_{\mathcal{F}}^{G} = (B,u)$ and $u = (u_{j,i})_{(j,i)\in[\![n]\!]\boxtimes[\![n]\!]}$, then for any set $\mathcal{E} \subseteq \mathcal{F}$ with $\langle \mathcal{E} \rangle = \mathcal{F}$, the algebra B is also is the universal C^*-algebra with generators $\{u_{j,i}\}_{i,j=1}^{n}$ subject to the relations*

$$u = \overline{u} \ \wedge \ u^{\mathrm{t}} u = I_n \otimes 1 = u u^{\mathrm{t}}$$

and

$$\forall \{k,\ell\} \subseteq \mathbb{N}_0\colon \forall \mathbf{H} \in \mathcal{E}(k,\ell)\colon u^{\oplus \ell}\,(T_{k,\ell}^{\mathbf{H}\to G} \otimes 1) = (T_{k,\ell}^{\mathbf{H}\to G} \otimes 1)\, u^{\oplus k}.$$

Proof. Proven immediately by Proposition 7.6 and [Wor88, Theorem 1.3]. □

7.2. Fiber functor degeneracy. By using the results of Section 6 and basic linear algebra it is possible to see that the construction of graph-theoretical quantum groups regrettably has some unfavorable properties.

Remark 7.8. (a) For any $n \in \mathbb{N}$ and any graph G with vertex set $[\![n]\!]$ the *adjacency matrix* A_G of G is the $n \times n$-matrix which for any $i \in [\![n]\!]$ and $j \in [\![n]\!]$ has the (j,i)-entry 1 if $i \backsim_G j$ and 0 otherwise.

(b) $T_{1,1}^{\mathbf{A}\to G} = A_G$ for any $n \in \mathbb{N}$ and any graph G with vertex set $[\![n]\!]$.

LEMMA 7.9. *For any $n \in \mathbb{N}$ any self-adjoint $A \in M_n(\mathbb{C})$ is an element of the complex linear subspace of $M_n(\mathbb{C})$ spanned by $\{A^k \,|\, k \in \mathbb{N}_0 \backslash \{1\}\}$.*

PROOF. *Regular case:* If A is invertible, then in the characteristic polynomial $\chi_A = \sum_{k=0}^{n} a_k X^k$ of A the coefficient $a_0 = \det(A)$ of the constant monomial is non-zero. Hence, $A = -\frac{1}{a_0}\sum_{k=1}^{n} a_k A^{k+1}$ because $\chi_A(A) = 0$ by the Cayley-Hamilton theorem.

Singluar case: Now, drop the invertibility assumption. Because $A = A^*$ and because $\mathbb{C}^n$ is finite-dimensional, $\ker(A)^\perp = \overline{\mathrm{ran}(A)} = \mathrm{ran}(A)$. In particular, $\ker(A)^\perp$ is an A-invariant subspace. Hence, if V is the linear map $\mathbb{C}^n \cong \ker(A) \oplus \ker(A)^\perp \to \ker(A)^\perp \cong \mathbb{C}^m$, $(u, v) \mapsto v$, then $B \coloneqq VAV^* \in M_m(\mathbb{C})$ is injective and thus invertible. If thus $B = \sum_{k\in\mathbb{N}_0\backslash\{0\}} b_k B^k$ for some $\{b_k\}_{k\in\mathbb{N}_0\backslash\{1\}} \subseteq \mathbb{C}$ with only finitely many non-zero, then $A = V^*BV = \sum_{k\in\mathbb{N}_0\backslash\{0\}} b_k V^*B^kV = \sum_{k\in\mathbb{N}_0\backslash\{0\}} b_k (V^*BV)^k = \sum_{k\in\mathbb{N}_0\backslash\{0\}} b_k A^k$ because $A = AV^*V = V^*VA$ and $VV^* = \mathrm{id}$. □

There are distinct graph categories giving rise to the same graph-theoretical quantum groups, no matter the target graph.

PROPOSITION 7.10. *There exist graph categories $\mathcal{F}_1$ and $\mathcal{F}_2$ such that $\mathbb{G}_{\mathcal{F}_1}^G = \mathbb{G}_{\mathcal{F}_2}^G$ for any graph G even though $\mathcal{F}_1 \neq \mathcal{F}_2$.*

PROOF. If we let $\mathcal{F}_1 \coloneqq \mathcal{A} = \mathcal{A}_{\mathbb{N}}$ and $\mathcal{F}_2 \coloneqq \mathcal{A}_{\mathbb{N}\backslash\{1\}}$, then $\mathcal{F}_1$ and $\mathcal{F}_2$ are both graph categories by Proposition 6.6. Moreover, $\mathcal{F}_1 \neq \mathcal{F}_2$ because $\mathbf{A} \in \mathcal{F}_1 \backslash \mathcal{F}_2$. Let $n \in \mathbb{N}$ and let G be any graph with vertex set $[\![n]\!]$. Then, $\{\mathbf{A}^{\circ k} \,|\, k \in \mathbb{N}\backslash\{1\}\} \subseteq \mathcal{F}_2$ implies $\{{A_G}^k \,|\, k \in \mathbb{N}_0\backslash\{1\}\} \subseteq \mathcal{C}_{\mathcal{F}_2}^G(1,1)$ because $T_{1,1}^{\mathbf{A}\to G} = A_G$ by Remark 7.8 (b) and thus $T_{1,1}^{\mathbf{A}^{\circ k}\to G} = {A_G}^k$ for any $k \in \mathbb{N}_0$ by (a) and (d) of Proposition 7.6. Because A_G is self-adjoint Lemma 7.9 thus proves $A_G \in \mathcal{C}_{\mathcal{F}_2}^G(1,1)$. In conclusion, $\mathbb{G}_{\mathcal{F}_1}^G = \mathbb{G}_{\mathcal{F}_2}^G$ by Lemma 7.7 and $\mathcal{F}_1 = \langle \mathbf{A} \rangle$. □

8. Concluding Remarks

The present work had four main results. It introduced the *vertex* and *component partitions* as well as *label distances* of any given bi-labeled graph and showed (1) how graph categories arise by placing constraints on these quantities and (2) that each of these quantities in turn induces a combinatorial invariant of any given graph category. (3) It described the categories generated by powers of the adjacency bi-labeled graph. (4) It demonstrated an inherent lack of injectivity of the construction of graph-theoretical quantum groups.

8.1. Graph categories from categories of partitions. Besides $\mathbb{P}$ and P of Definitions 3.1 and 4.1, do there exist other mappings P as in Lemma 2.35, thus giving rise to further families of graph categories and graph category invariants?

It can be shown that such a mapping P can be constructed from, in the language of category theory, any right-exact functor from the category of graphs (with graphs as objects and graph homomorphisms as morphisms) to the category of finite sets (with finite sets as objects and mappings as morphisms): Given such a functor F the associated mapping P in the sense of Lemma 2.35 is obtained by sending any bi-labeled graph $\mathbf{K}$ to the partition $\ker(F(g))$, where $(G, g) \in \mathbf{K}$ can be any representative.

A functor is right-exact if and only if it preserves initial objects, co-products and co-equalizers. In particular, any left-adjoint functor is right-exact. The functor which sends any graph to its set of vertices and (which operates as the identity on morphisms) has a left adjoint. It is this left adjoint, namely the functor which sends any finite set V to the free graph $(V, \varnothing)$ over it (and which leaves morphisms unchanged), that induces the mapping $\mathbb{P}$ from Definition 3.1. But also the functor which sends any graph G with vertex set V to its set $V/\simeq_G$ of connected components (and any graph homomorphism f from G to any H to the mapping $(\pi_{\simeq_H} \circ f)/\simeq_G)$ is right-exact. It gives rise to the mapping P from Definition 4.1.

Perhaps it is even possible to classify all right-exact functors from the category of graphs to the category of finite sets.

8.2. Categories of partitions with "empty blocks". As explained in Remark 2.23 the definition of "partition" used in the present work is different from the one employed by Mančinska and Roberson in [MR20, p. 11], in that "empty blocks" are not allowed. The motivation behind this choice was that results were desired which allowed capitalizing on the classifcation of categories of partitions (see Section 2.3.3). However, as mentioned in Remark 3.7 this comes at the cost that theorems about generators of categories of partitions do not translate to such about generators of graph categories consisting solely of edgeless bi-labeled graphs. However, one can give analogous definitions of the mappings $\mathbb{P}$ and P which give partitions which do have "empty blocks". Moreover, Propositions 3.6, 3.9, 4.6 and 4.9 all remain true for those versions of $\mathbb{P}$ and P. Hence, a classification of all categories of partitions with *empty blocks* would immediately yield a classification of all graph categories consisting only of edgeless bi-labeled graphs and new graph category invariants. Thus, such a theorem might be a good place to start in order to continue with the classification of all graph categories.

8.3. Unitary graph categories. Banica and Speicher's categories of partitions from [BS09] and their so-called "easy" quantum groups obtained from those categories always form compact ($n \times n$-matrix) quantum subgroups of the free orthogonal quantum group O_n^+ defined by Wang in [Wan95a]. In [TW18], Tarrago and Weber showed how to modify Banica and Speicher's definitions in order to produce quantum subgroups of Wang's free unitary quantum group U_n^+ instead. Analogous modifications can be made to Mančinska and Robersons graph-theoretical quantum groups.

One fixes a set $\{\circ, \bullet\}$ of two distinct *colors*. For any $\{k, \ell\} \subseteq \mathbb{N}_0$ a *two-colored (k, ℓ)-bi-labeled graph* is any triple $(c, \mathbf{K}, d)$ such that $c\colon [\![k]\!] \to \{\circ, \bullet\}$ and $d\colon [\![\ell]\!] \to \{\circ, \bullet\}$ and such that $\mathbf{K}$ is a (k, ℓ)-bi-labeled graph in the sense of Definition 2.10

(for the same notion of "graph"). Note that it is *not* the underlying graph, neither vertices nor edges, that is endowed with a coloring but the labels themselves. Namely, c is interpreted as the *coloring* of the k upper points (*not* vertices) and d as that of the ℓ lower points. Given any such triples $(b, \mathbf{K}, c)$ and $(a, \mathbf{H}, b)$ their *composition* $(b, \mathbf{K}, c) \circ (a, \mathbf{H}, b)$ is simply defined as $(a, \mathbf{K} \circ \mathbf{H}, c)$. The *adjoint* of $(c, \mathbf{K}, d)$, unsurprisingly, is given by $(d, \mathbf{K}^*, c)$. When forming the tensor product $(c_1, \mathbf{K}_1, d_1) \otimes (c_2, \mathbf{K}_2, d_2) = (c_1 \otimes c_2, \mathbf{K}_1 \otimes \mathbf{K}_2, d_1 \otimes d_2)$ of two triples $(c_1, \mathbf{K}_1, d_1)$ and $(c_2, \mathbf{K}_2, d_2)$ one needs to form the concatenatations $c_1 \otimes c_2$ and $d_1 \otimes d_2$ of the colorings of the upper and lower points, respectively. A *category of two-colored bi-labeled graphs* is then any set closed under composition, tensor products and forming adjoints and containing the four triples $(\circ, \mathbf{I}, \circ)$, $(\bullet, \mathbf{I}, \bullet)$, $(\varnothing, \mathbf{M}_{0,2}, \circ\bullet)$ and $(\varnothing, \mathbf{M}_{0,2}, \bullet\circ)$.

To obtain the *graph-theoretical unitary compact matrix quantum group* associated with such a category $\mathcal{F}$ the definitions of Section 7.1 would have to be adapted in the following way: No modifications whatsoever need to be made to Definition 7.2. The only change to Definition 7.3 is that now, just like $\mathcal{F}$ itself, $\mathcal{C}_{\mathcal{F}}^{G}$ is a family not indexed by pairs (k, ℓ) of numbers k and ℓ but by pairs (c, d) of colorings c and d (of any lengths). Finally, two changes are required in Definition 7.4. First, the relation $u = \overline{u}$ is dropped. Second, given any $\{k, \ell\} \subseteq \mathbb{N}_0$ as well as $c\colon [\![k]\!] \to \{\circ, \bullet\}$ and $d\colon [\![\ell]\!] \to \{\circ, \bullet\}$, for any $T \in \mathcal{C}_{\mathcal{F}}^{G}(c, d)$ rather than $u^{\oplus \ell}\,(T \otimes 1) = (T \otimes 1)\, u^{\oplus k}$ the associated relation imposed has to read

$$(u^{c_1} \oplus u^{c_2} \oplus \ldots \oplus u^{c_k})\,(T \otimes 1) = (T \otimes 1)\,(u^{d_1} \oplus u^{d_2} \oplus \ldots \oplus u^{d_\ell}),$$

where $u^{\circ} = u$ and $u^{\bullet} = \overline{u}$ and where for any $\{v, w\} \subseteq M_n(\mathbb{C})$, if $v = (v_{j,i})_{(j,i) \in [\![n]\!] \boxtimes [\![n]\!]}$ and $w = (w_{j,i})_{(j,i) \in [\![n]\!] \boxtimes [\![n]\!]}$, then $v \oplus w = (v_{j_1,i_1} w_{j_2,i_2})_{(j,i) \in [\![n]\!]^{\boxtimes 2} \boxtimes [\![n]\!]^{\boxtimes 2}}$.

CHAPTER 3

Half-liberated unitary easy quantum groups

1. Introduction

The unitary group is generalized by the free unitary quantum group, the algebra of continuous functions on the unitary group – but liberated from the commutativity constraint. Quantum groups interpolating the unitary group and the free unitary quantum group are known as half-liberations of the unitary group. Three families of such half-liberations arise as so-called easy quantum groups, via Tannaka-Krein duality from categories of two-colored partitions. The present chapter shows that these half-liberations can all be understood as quotients of wreath graph products of either the unitary group or the free unitary quantum group. More precisely, it is shown that their representation categories are full subcategories of those of the said wreath graph products.

1.1. Background and context. Operator-algebraic compact quantum groups as defined by Woronowicz in [Wor87; Wor91; Wor98] can be specified with the help of Woronowicz's Tannaka-Krein duality theorem in [Wor98] by providing their to-be representation functors rather than the otherwise required C^*-algebras. A particularly prolific construction pioneered by Banica and Speicher in [BS09] and later extended by Tarrago and Weber in [TW18] and by Freslon in [Fre17] applies Tannaka-Krein duality to combinatorial functors living on categories of colored partitions. The resulting so-called easy compact quantum groups are quantum subgroups of free products of copies of Wang's free orthogonal and free unitary quantum groups O_n^+ and U_n^+ from [Wan95a], where each copy can live in any dimension n. In particular, both U_n^+ and its classical counterpart the unitary group U_n can be obtained in this way. A full classification of all easy quantum groups interpolating U_n and U_n^+ was given in [MW20; MW21a]. The present chapter is dedicated to the closer study of the three families of those quantum groups, the so-called unitary half-liberations $U_{w,n}^*$, $U_{D,n}^\times$ and $U_{D,n}^{\times+}$.

With the notation $\otimes$ for the cartesian product of sets, the convention that $0 \notin \mathbb{N}$ and the abbreviations that $[\![0]\!] \coloneqq \varnothing$ and that for any $k \in \mathbb{N}$, on the one hand, $[\![k]\!] \coloneqq \{1, 2, \ldots, k\}$ and, on the other hand, in any algebra for any elements $a_1, a_2, \ldots, a_k$,

$$\overrightarrow{\prod}_{i=1}^k a_i = a_1 a_2 \ldots a_k \quad \text{and} \quad \overleftarrow{\prod}_{i=1}^k a_i = a_k a_{k-1} \ldots a_1,$$

the description of the unitary half-liberations coming out of Tannaka-Krein duality reads as follows.

Definition. For any $n \in \mathbb{N}$, any $w \in \mathbb{N}$, any additive subsemigroup of $\mathbb{N}$ and any $G \in \{U^*_{w,n}, U^\times_{D,n}, U^{\times+}_{D,n},\}$ let the compact quantum group G be defined as the formal dual of the universal C^*-algebra $C_{\max}(G)$ generated by n^2-generators $\{u_{j,i}\}_{i,j=1}^n$ subject to the relations $R_{U_n^+} \cup R_G$ and equipped with the unique morphism $\Delta\colon C_{\max}(G) \to C_{\max}(G) \otimes_{\min} C_{\max}(G)$ such that $u_{k,i} \mapsto \sum_{j=1}^n u_{k,j} \otimes u_{j,i}$ for any $\{i,k\} \subseteq [\![n]\!]$, where $R_{U_n^+}$ corresponds to the equations that for any $\{i,k\} \subseteq [\![n]\!]$,

$$\sum_{j=1}^n u_{k,j}u^*_{i,j} = \sum_{j=1}^n u_{j,k}u^*_{j,i} = \sum_{j=1}^n u^*_{j,k}u_{j,i} = \sum_{j=1}^n u^*_{k,j}u_{i,j} = \delta_{k,i}1$$

and where

(a) $R_{U^{\times+}_{D,n}}$ corresponds to the equations that for any $m \in \mathbb{N}$ with $m \notin D$ and any elements $(g^\triangleleft_a)_{a=1}^{m+1}$, $(g^\triangleright_a)_{a=1}^{m+1}$, $(j^\triangleleft_b)_{b=1}^{m+1}$, and $(j^\triangleright_b)_{b=1}^{m+1}$ of $[\![n]\!]^{\otimes m+1}$,

$$\sum_{i^\triangleleft \in [\![n]\!]^{\otimes m+1}} \sum_{i^\triangleright \in [\![n]\!]^{\otimes m+1}} \left(\prod_{\substack{b\in[\![m]\!]\\ \wedge\, b\notin D}} \delta_{i^\triangleleft_{b+1}, i^\triangleright_{b+1}} \delta_{g^\triangleleft_{b+1}, g^\triangleright_{b+1}}\right)\left(\prod_{\substack{b\in[\![m]\!]\cap D\\ \vee\, b=0}} \delta_{i^\triangleleft_{b+1}, g^\triangleleft_{b+1}} \delta_{i^\triangleright_{b+1}, g^\triangleright_{b+1}}\right) \left(\overleftarrow{\prod_{b=0}^{m}} u^*_{j^\triangleleft_{b+1}, i^\triangleleft_{b+1}}\right)\left(\overrightarrow{\prod_{b=0}^{m}} u_{j^\triangleright_{b+1}, i^\triangleright_{b+1}}\right)$$

$$= \sum_{h^\triangleleft \in [\![n]\!]^{\otimes m+1}} \sum_{h^\triangleright \in [\![n]\!]^{\otimes m+1}} \left(\prod_{\substack{a\in[\![m]\!]\\ \wedge\, a\notin D}} \delta_{h^\triangleleft_{a+1}, h^\triangleright_{a+1}} \delta_{j^\triangleleft_{a+1}, j^\triangleright_{a+1}}\right)\left(\prod_{\substack{a\in[\![m]\!]\cap D\\ \vee\, a=0}} \delta_{j^\triangleleft_{a+1}, h^\triangleleft_{a+1}} \delta_{j^\triangleright_{a+1}, h^\triangleright_{a+1}}\right) \left(\overleftarrow{\prod_{a=0}^{m}} u^*_{h^\triangleleft_{a+1}, g^\triangleleft_{a+1}}\right)\left(\overrightarrow{\prod_{a=0}^{m}} u_{h^\triangleright_{a+1}, g^\triangleright_{a+1}}\right).$$

(b) $R_{U^\times_{D,n}}$ corresponds to the union of $R_{U^{\times+}_{D,n}}$ and the relations corresponding to the equations that for any $\{i,j,k,\ell,r,s\} \subseteq [\![n]\!]$,

$$u_{s,r}u^*_{\ell,k}u_{j,i} = u_{j,i}u^*_{\ell,k}u_{s,r}.$$

(c) $R_{U^*_{w,n}}$ corresponds to the equations that for any $\{i,j,r,s\} \subseteq [\![n]\!]$ and any elements $(k_c)_{c=1}^{w-1}$ and $(\ell_c)_{c=1}^{w-1}$ of $[\![n]\!]^{\otimes w-1}$,

$$u_{s,r}\left(\overrightarrow{\prod}_{c=1}^{w-1} u_{\ell_c,k_c}\right)u_{j,i} = u_{j,i}\left(\overrightarrow{\prod}_{c=1}^{w-1} u_{\ell_c,k_c}\right)u_{s,r}.$$

The definition implies that, in particular, $U^*_{1,n} = U_n$ and $U^{\times+}_{\mathbb{N},n} = U_n^+$ for any $n \in \mathbb{N}$. Of course, the definition given above is that of the universal C^*-versions of these quantum groups. In order to obtain the underlying Hopf $*$-algebras one has to consider not the universal C^*-algebra but merely the universal $*$-algebra with the same relations.

For many quantum-algebraic questions it is helpful to present the quantum group to be studied as a quotient of an already well-understood quantum group. For example, by using certain quotient relationships between U_n^+, O_n^+, PO_n^+ and $O_n^+ \hat{*} \widehat{\mathbb{Z}}$ Kyed and Raum were able to compute the first L^2-Betti number of the discrete dual of U_n^+ in [KR17]. Exhibting the half-liberations as quotients of well-known quantum groups was the objective of the present work. In the case of $U^*_{w,n}$ this had already been achieved by Banica and Bichon in [BB17]. It is that article which guided all the considerations in this chapter.

More precisely, the authors of [BB17] present $U^*_{w,n}$ as a quotient quantum group of the wreath product $U_n \mathbin{\hat{\wr}} \widehat{\mathbb{Z}_w}$ of the unitary group U_n and the compact dual of the cyclic group $\mathbb{Z}_w$ of order w, i.e., the crossed product $(U_n)^{\hat{\times}\, w} \hat{\rtimes} \widehat{\mathbb{Z}_w}$ of the direct product $(U_n)^{\hat{\times}\, w}$ of w many copies of U_n with the dual of $\mathbb{Z}_w$, where the action of the latter is by cyclically permuting the copies. Whereas it is relatively straightforward to establish the existence of the C^*-algebra morphism from $C_{\max}(U^*_{w,n})$ to $C(U_n \mathbin{\hat{\wr}} \widehat{\mathbb{Z}_w})$, proving that this morphism is injective is anything but. To do so, Banica and Bichon provide an entire commutative diagram

$$\begin{array}{ccc} C((U_n)^{\hat{\times}\, w}/\otimes) & \hookrightarrow & C((U_n)^{\hat{\times}\, w}) \\ \big\downarrow & & \big\downarrow \\ C_{\max}(U^*_{w,n}) & \hookrightarrow & C(U_n \mathbin{\hat{\wr}} \widehat{\mathbb{Z}_w}) \end{array}$$

of C^*-algebra morphisms. (The three known corners, $U_n \mathbin{\hat{\wr}} \widehat{\mathbb{Z}_w}$, $(U_n)^{\hat{\times}\, w}$ and a certain quotient group $(U_n)^{\hat{\times}\, w}/\otimes$ of the latter, are all co-amenable due to U_n being a compact group and $\mathbb{Z}$ being amenable. Hence, for them, differentiating between different C^*-versions is unnecessary.) The right vertical arrow is the dual of the projection of $(U_n)^{\hat{\times}\, w} \hat{\rtimes} \widehat{\mathbb{Z}_w}$ onto $(U_n)^{\hat{\times}\, w}$. The remaining two arrows can be found as the pull-back of the C^*-morphisms into $C(U_n \mathbin{\hat{\wr}} \widehat{\mathbb{Z}_w})$. By a clever argument involving a $\mathbb{Z}_w$-grading Banica and Bichon show that the lower horizontal arrow is injective if the upper one is. That allows them to infer the sought injectivity by means of Gelfand duality from the fact that the continous group homomorphism from $(U_n)^{\hat{\times}\, w}$ to $(U_n)^{\hat{\times}\, w}/\otimes$ is surjective. By proving $U^*_{w,n}$ a quotient of $U_n \mathbin{\hat{\wr}} \widehat{\mathbb{Z}_w}$, they can moreover conclude that also $U^*_{w,n}$ is co-amenable.

The intent behind the present chapter was to give a characterization of the two remaining families $U^{\times}_{D,n}$ and $U^{\times+}_{D,n}$ of half-liberations which is analogous to that [BB17] gives of $U^*_{w,n}$. And, indeed, a bit of experimentation with the same rule for defining the horizontal arrow in the diagram above quickly suggests a conjecture. Namely that $U^{\times}_{D,n}$ and $U^{\times+}_{D,n}$ are quotient quantum groups of wreath graph products $U_n \mathbin{\hat{\wr}}_{r_D} \widehat{\mathbb{Z}}$ respectively $U^+_n \mathbin{\hat{\wr}}_{r_D} \widehat{\mathbb{Z}}$ with respect to a certain partial commutation relation r_D on $\mathbb{Z}$. Like ordinary wreath products these are crossed products $(U_n)^{\hat{*}(\mathbb{Z},r_D)} \hat{\rtimes} \widehat{\mathbb{Z}}$ respectively $(U^+_n)^{\hat{*}(\mathbb{Z},r_D)} \hat{\rtimes} \widehat{\mathbb{Z}}$, where the action of the compact dual $\widehat{\mathbb{Z}}$ of $\mathbb{Z}$ is again to permute copies. However, rather than the direct products used in the definition of the wreath product in [Wan95b], here $(U_n)^{\hat{*}(\mathbb{Z},r_D)}$ and $(U^+_n)^{\hat{*}(\mathbb{Z},r_D)}$ are graph products, as defined in [CF17] in the quantum group and in [Mło04] and [SW16] in the free probability context.

In fact, it is again relatively easy to confirm that the corresponding C^*-morphisms from $C_{\max}(U^{\times}_{D,n})$ to $C_{\max}(U_n \wr_{r_D} \widehat{\mathbb{Z}})$ respectively from $C_{\max}(U^{\times+}_{D,n})$ to $C_{\max}(U_n \wr_{r_D} \widehat{\mathbb{Z}})$ exist. (Presumably, these wreath graph products are generally not co-amenable, which is why distinctions must be made between different C^*-versions.) As in [BB17], the real challenge is to show that these morphisms are injective. Here, though, Banica and Bichon's proof strategy is unfortunately not applicable: As before the projections of $(U_n)^{\hat{*}(\mathbb{Z},r_D)} \hat{\rtimes} \widehat{\mathbb{Z}}$ onto $(U_n)^{\hat{*}(\mathbb{Z},r_D)}$ respectively of $(U_n^+)^{\hat{*}(\mathbb{Z},r_D)} \hat{\rtimes} \widehat{\mathbb{Z}}$ onto $(U_n^+)^{\hat{*}(\mathbb{Z},r_D)}$ yield right vertical arrows for a diagram analogous to the one above for $U^*_{w,n}$. However, in contrast to the situation [BB17], the compact quantum group $(U_n)^{\hat{*}(\mathbb{Z},r_D)}$ is only a group if $n = 1$ or r_D allows all copies to commute and $(U_n^+)^{\hat{*}(\mathbb{Z},r_D)}$ is even only a group if both are the case. While it is still possible to find the pull-back and to show that the lower horizontal arrow is injective if the upper one is, Gelfand duality is thus generally not available for proving the injectivity of the upper arrow.

It must be emphasized that this chapter is *not* able to remedy this issue. Instead, it has to confine itself to proving that the induced C^*-morphisms from $C_{\mathrm{red}}(U^{\times}_{D,n})$ to $C_{\mathrm{red}}(U_n \wr_{r_D} \widehat{\mathbb{Z}})$ respectively from $C_{\mathrm{red}}(U^{\times+}_{D,n})$ to $C_{\mathrm{red}}(U_n \wr_{r_D} \widehat{\mathbb{Z}})$ between the reduced C^*-versions are injective. More precisely, the combinatorics of partitions (with more than two colors) are employed to prove that the representation theory $\mathrm{Rep}(U^{\times}_{D,n})$ is a full tensor C^*-subcategory of $\mathrm{Rep}(U_n \wr_{r_D} \widehat{\mathbb{Z}})$ respectively that $\mathrm{Rep}(U^{\times+}_{D,n})$ is one of $\mathrm{Rep}(U_n \wr_{r_D} \widehat{\mathbb{Z}})$, which then implies the claim about the reduced C^*-algebras. Whether these results can be extended to the universal C^*-level remains an open question (except in the case of $U^{\times}_{\varnothing,n}$, where an affirmative answer can be given).

1.2. Main results. In the following theorem the statement about the compact quantum groups $U^*_{w,n}$ for $w \in \mathbb{N}$ and $n \in \mathbb{N}$ is implied by the even stronger one shown by Banica and Bichon in [BB17]. Let $\mathbb{Z}_0 = \mathbb{Z}$ and $\mathbb{Z}_w = \{0, 1, \ldots, w-1\}$ for any $w \in \mathbb{N}$ and mind $0 \notin \mathbb{N}$.

MAIN RESULT. *For any $n \in \mathbb{N}$, any $w \in \mathbb{N}$ and any additive subsemigroup D of $\mathbb{N}$, if*

$$m_{U^*_{w,n}} \coloneqq w \quad \wedge \quad m_{U^{\times}_{D,n}} \coloneqq m_{U^{\times+}_{D,n}} = 0$$

and if

$$r_{U^*_{w,n}} \coloneqq \mathbb{Z}_w^{\otimes 2} \quad \wedge \quad r_{U^{\times}_{D,n}} \coloneqq r_{U^{\times+}_{D,n}} \coloneqq \{(s,t) \in \mathbb{Z}^{\otimes 2} \wedge |t-s| \notin D \cup \{0\}\},$$

then for any

$$(G, K_G) \in \{(U^*_{w,n}, U_n), (U^{\times}_{D,n}, U_n), (U^{\times}_{D,n}, O^*_n), (U^{\times+}_{D,n}, U^+_n), (U^{\times+}_{D,n}, O^+_n)\},$$

if $K_G \wr_{r_G} \widehat{\mathbb{Z}_{m_G}}$ denotes the wreath graph product of K_G with $\widehat{\mathbb{Z}_{m_G}}$ with respect to r_G on $\mathbb{Z}_{m_G}$, i.e., the crossed product $K_G{}^{\hat{*}(\mathbb{Z}_{m_G},r_G)} \hat{\rtimes} \widehat{\mathbb{Z}_{m_G}}$, where $\widehat{\mathbb{Z}_{m_G}}$ acts by permuting the copies of K_G in the $\mathbb{Z}_{m_G}$-fold graph product (also known as λ-free or ϵ-independent product) $K_G{}^{\hat{*}(\mathbb{Z}_{m_G},r_G)}$ of K_G with itself according to the partial commutation relation

r_G, then the rule

$$G \twoheadleftarrow K_G \,\hat{\wr}_{r_G}\, \widehat{\mathbb{Z}_{m_G}}\ , \quad u \mapsto v^{(0)} \cdot z,$$

which assigns to the fundamental representation $u = (u_{j,i})_{(j,i)\in[\![n]\!]^{\otimes 2}}$ of the unitary half-liberation G the product representation $u \cdot z = (v^{(0)}_{j,i} z)_{(j,i)\in[\![n]\!]^{\otimes 2}}$ of the fundamental representations $v^{(0)} = (v^{(0)}_{j,i} z)_{(j,i)\in[\![n]\!]^{\otimes 2}}$ of the 0-th copy of K_G and z of the compact dual of $\mathbb{Z}_{m_G}$, induces

(a) a fiber-functor-preserving full tensor C^*-subcategory inclusion of the representation theory of G in the representation theory of $K_G \,\hat{\wr}_{r_G}\, \widehat{\mathbb{Z}_{m_G}}$,
(b) an injective co-multiplication-preserving C^*-algebra morphism from the reduced C^*-version of the continuous functions on G to the reduced C^*-version of the continuous functions on $K_G \,\hat{\wr}_{r_G}\, \widehat{\mathbb{Z}_{m_G}}$
(c) an injective Hopf $*$-algebra morphism from the regular functions of G to the regular functions of $K_G \,\hat{\wr}_{r_G}\, \widehat{\mathbb{Z}_{m_G}}$, and
(d) a C^*-algebra morphism from the universal C^*-version of the continuous functions on G to the universal C^*-version of the continuous functions on $K_G \,\hat{\wr}_{r_G}\, \widehat{\mathbb{Z}_{m_G}}$.

No claim is made about the C^*-algebraic morphism on the universal level being injective in general. The role of the 0-th copy can also be played by any other copy if so desired. The main result implies that $U^{\times}_{\varnothing,n}$ is co-amenable.

1.3. Structure of the chapter. A reminder on compact quantum groups and Tannaka-Krein duality is provided in Section 2.

The main tool for the proof of the main result is the theory of labeled partitions, their categories, fiber functors and associated general easy quantum groups. These are explained in detail in Section 3.

Section 4 introduces wreath graph co-products of categories of labeled partitions. Actually, two such co-products are defined, which are generally distinct, but happen to coincide for the categories relevant to the proof of the main result. As intermediate steps to defining wreath graph co-products of categories, direct, free, graph and crossed co-products are introduced as well.

In Section 5, it is shown that wreath graph co-products of categories of partitions are generated by a very small set of certain labeled "crossing" partitions. That requires proving analogous results about graph products and crossed products incidentally.

The results of the preceding section are then the key to proving in Section 6 that the easy quantum group associated with a wreath graph co-product category is the wreath graph product quantum group of the easy quantum groups associated with the original categories.

Section 7 recalls the definitions of the particular quantum groups which appear in the main result and the categories of partitions they result from.

In Section 8, the categories of the wreath graph product quantum groups from the main result, which by Sections 6–8 are known to be categorical wreath graph co-products of the categories from Section 7, are expressed in a form which is more convenient for the proof of the main result.

Namely, this reformulation simplifies the construction in Section 9 of a strict monoidal full $*$-subcategory inclusion functor from the categories of the unitary half-liberations to the respective wreath graph co-product categories.

Section 10 then combines all the results gathered in the preceding sections to deduce the main result. General results about compact quantum groups and Tannaka-Krein duality recalled and proved in Section 2 play a role here for the last few steps of the proof.

Lastly, Section 11 offers a few observations about the main results, its implications and potential extensions.

2. Compact quantum groups and Tannaka-Krein duality

The objectives of Section 2 are to

- □ recall the definitions of compact quantum groups, in the purely algebraic sense and the analytic sense (however excluding von Neumann algebras) (Section 2.1),
- □ recapitulate the relationship between the different definitions, in particular the universal and regular C^*-envelopes of algebraic compact quantum groups (Section 2.2,
- □ recall the definitions of direct, free, graph and crossed products of compact quantum groups (Section 2.3),
- □ provide a functorial account of Tannaka-Krein duality for not necessarily complete categories (Section 2.5),
- □ explain why full subcategory inclusions induce injective $*$-algebra morphisms for the algebraic and reduced C^*-algebraic duals (Section 2.6).

The reader who is already well-acquainted with quantum groups and familiar with the properties of Tannka-Krein duals of full subcategory inclusions may safely skip this section entirely.

In the following, largely the same notation and conventions as in [Wor88] and [Wan97] are used. Those are recalled below, as are the few but important differences here to be kept in mind.

Notation 2.1. (a) As in [Wor88], all vector spaces and linear maps considered (as well as all vector spaces with additional structures like Banach spaces, algebras, $*$-algebras, C^*-algebras, etc.) are *complex*, i.e., over the ground field $\mathbb{C}$.

(b) All algebras, including $*$- and C^*-algebras, appearing are meant to be *unital* and so are all their morphisms.

(c) We fix a *set* (i.e., not a proper class) containing “all” finite-dimensional Hilbert spaces.

(d) Furthermore, the monoidal category of vector spaces is *strict*, meaning that $\mathbb{C}\otimes V = V = V\otimes\mathbb{C}$ and $(V_1\otimes V_2)\otimes V_3 = V_1\otimes(V_2\otimes V_3)$ for any vector spaces V, V_1, V_2 and V_3. This statement is then also true about the monoidal category of finite-dimensional Hilbert spaces.

(e) For any vector spaces V and W the vector space formed by the linear maps $V\to W$ with pointwise operations is denoted by $[V,W]$. In addition to the simplifications which the convention in the preceding remark provides, for any two vector spaces V_1 and V_2, the natural isomorphism between $[V_1,V_1]\otimes[V_2,V_2]$ and $[V_1\otimes V_2, V_1\otimes V_2]$ is notationally suppressed.

(f) The $*$-algebra formed by $[V,V]$ for any finite-dimensional Hilbert space $H = (V,\langle\cdot\mid\cdot\rangle)$ when equipped with composition as multiplication and forming adjoints as $*$-operation is denoted by $B(H)$.

(g) Identically with the notation from [Wor88], for any algebra with underlying vector space A and any vector spaces V, V_1 and V_2 the symbol $\ominus$ denotes the linear map

$$\begin{aligned}[V,V]\otimes A\otimes[V,V]\otimes A &\to [V,V]\otimes A\otimes A\\ t\otimes a\otimes t'\otimes a' &\longmapsto tt'\otimes a\otimes a',\end{aligned}$$

while the symbol $\oplus$ stands for the linear map

$$\begin{aligned}[V_1,V_1]\otimes A\otimes[V_2,V_2]\otimes A &\to [V_1,V_1]\otimes[V_2,V_2]\otimes A\\ t_1\otimes a_1\otimes t_2\otimes a_2 &\longmapsto t_1\otimes t_2\otimes a_1a_2.\end{aligned}$$

(h) Slightly deviating from the notation in [Wor88] in the interest of clarity, we will distinguish between any vector space and its conjugate, thus avoiding the concept of an "anti-linear map". More precisely, for any vector space V let $V^{\mathrm{cj}}\equiv\mathrm{cj}(V)$ be the same abelian group as V but with the "conjugate action" $(\lambda,x)\mapsto\overline{\lambda}x$ for any scalar λ and vector x of V. Likewise, given any vector spaces V and W and any linear map t from V to W write t^{cj} as well as $\mathrm{cj}(t)$ for the linear map from V^{cj} to W^{cj} with the same underlying morphism of abelian groups as t.

Note that this (for us) strict monoidal involutive endofunctor of the category of vector spaces extends to the category of Hilbert spaces by replacing the scalar product $\langle\cdot\mid\cdot\rangle : V^{\mathrm{cj}}\otimes V\to\mathbb{C}$ of any Hilbert space $H=(V,\langle\cdot\mid\cdot\rangle)$ with the scalar product $\langle\cdot\mid\cdot\rangle^{\mathrm{cj}} : V\otimes V^{\mathrm{cj}}\to\mathbb{C}$ to obtain that of H^{cj}.

(i) For any finite-dimensional Hilbert spaces H and K with underlying vector spaces V respectively W and any linear isomorphism j from V to W^{cj} abbreviate

$$\bar{t}_j \coloneqq \langle\cdot\mid\cdot\rangle\circ(\mathrm{cj}(j^{-1})\otimes\mathrm{id}_V) : W\otimes V\to\mathbb{C},$$

where $\langle\cdot\mid\cdot\rangle$ is the scalar product of H, and abbreviate

$$t_j \coloneqq (\mathrm{id}_H\otimes\mathrm{cj}(j))\circ c : \mathbb{C}\to V\otimes W,$$

where $c : \mathbb{C} \to V \otimes V^{\mathrm{cj}}$ is the unique linear map with

$$1 \mapsto \sum_{i \in I} e_i \otimes e_i$$

for any orthnormal basis $(e_i)_{i \in I}$, which is independent of the particular choice.

(j) Lastly, for any vector spaces J, H, K and L and for any linear maps $f\colon J \to H$, $g\colon H \to K$ and $h\colon K \to L$ let $[f,h](g) \coloneqq hgf$. A particular consequence of this definition is that, given any algebra A and any $v \in [H,K] \otimes A$, we can also write $([f,h] \otimes \mathrm{id})(v)$ for $(h \otimes 1)v(f \otimes 1)$.

(k) Given any basis $(e_i)_{i \in I}$ of any finite-dimensional vector space V, the *matrix units* of $(e_i)_{i \in I}$ are the family $(E_{j,i})_{(i,j) \in I^{\otimes 2}}$ of vectors of $[V,V]$ with $E_{j,i} e_k = \delta_{k,i} e_j$ for any $\{i,j,k\} \subseteq I$.

2.1. The definitions of compact quantum groups. Since von-Neumann-algebraic considerations are omitted in this chapter, all remaining categories commonly referred to as "compact quantum groups" are equivalent to either the C^*-algebraic compact quantum groups defined in the next section or the algebraic compact quantum groups defined in the section thereafter. This includes rigid tensor C^*-categories.

2.1.1. *C^*-algebraic compact quantum groups.* In the C^*-algebraic context compact quantum groups were defined by Woronowicz in [Wor87; Wor91; Wor98].

Definition 2.2. (a) A *compact quantum group C^*-algebra* or *CQG C^*-algebra*, for short, is any pair (A, Δ) such that

(i) A is a C^*-algebra

(ii) Δ is a morphism $A \to A \otimes_{\min} A$ of C^*-algebras, where $\otimes_{\min}$ is the minimal (also known as spatial) tensor product of C^*-algebras,

(iii) $(\Delta \otimes_{\min} \mathrm{id}_A) \circ \Delta = (\mathrm{id}_A \otimes_{\min} \Delta) \circ \Delta$, where the spaces $(A \otimes_{\min} A) \otimes_{\min} A$ and $A \otimes_{\min} (A \otimes_{\min} A)$ have been identified,

(iv) each of $\{(a \otimes 1) \cdot \Delta(b) \,|\, \{a,b\} \subseteq A\}$ and $\{(1 \otimes a) \cdot \Delta(b) \,|\, \{a,b\} \subseteq A\}$ generates $A \otimes_{\min} A$ as a Banach space, where $\cdot$ is the multiplication of $A \otimes_{\min} A$.

Equivalently, we say that the formal dual of (A, Δ) is a *compact quantum group*.

(b) A *morphism of CQG C^*-algebras* from any CQG C^*-algebra (A, Δ) to any CQG C^*-algebra (B, Φ) is any morphism $\psi\colon A \to B$ of C^*-algebras such that $\Phi \circ \psi = (\psi \otimes_{\min} \psi) \circ \Delta$. Equivalently, if H and G are the formal duals of (A, Δ) and (B, Φ), respectively, we speak of the formal dual ω of ψ as a *morphism $G \to H$ of compact quantum group*.

If so and if ψ is surjective, we say that ω exhibits G as a compact quantum *sub*group of H. And, if ψ is injective, ω is said to exhibit H as a compact quantum *quotient* group of G.

2.1.2. *Algebraic quantum groups.* Woronowicz's original definition of compact (matrix) quantum groups, then still called "pseudogroups", from [Wor87] or Wang's generalization from [Wan97] feature a dense Hopf $*$-algebra. It is possible to characterize all the algebras arising in this way (see [KS12, p. 417, Proposition 28]).

DEFINITION 2.3. (a) A *Hopf $*$-algebra* is any $(A, m, 1, \Delta, *, \epsilon, S)$ such that

(i) $(A, m, 1, *)$ is a $*$-algebra (with underlying vector space A, multiplication $m\colon A \otimes A \to A$, unit $1\colon \mathbb{C} \to A$ and star $*\colon A \to A^{\mathrm{cj}}$),

(ii) Δ, the *comultiplication*, is a linear map $A \to A \otimes A$ and a $*$-algebra morphism from $(A, m, 1, *)$ to the tensor product $*$-algebra of $(A, m, 1, *)$ with itself,

(iii) ϵ, the *counit*, is a linear functional on A and a $*$-algebra morphism from $(A, m, 1, *)$ to $\mathbb{C}$,

(iv) S is a linear map $A \to A$, the *antipode* or *coinverse*,

(v) $(\mathrm{id}_A \otimes \Delta) \circ \Delta = (\Delta \otimes \mathrm{id}_A) \circ \Delta$,

(vi) $\mathrm{id}_A = (\epsilon \otimes \mathrm{id}_A) \circ \Delta$,

(vii) $\mathrm{id}_A = (\mathrm{id}_A \otimes \epsilon) \circ \Delta$,

(viii) $m \circ (S \otimes \mathrm{id}_A) \circ \Delta = 1 \circ \epsilon$,

(ix) $m \circ (\mathrm{id}_A \otimes S) \circ \Delta = 1 \circ \epsilon$,

(b) A *morphism of Hopf-$*$-algebras* from any Hopf $*$-algebra $(A, m, 1, *, \Delta, \epsilon, S)$ to any Hopf $*$-algebra $(A', m', 1', *', \Delta', \epsilon', S')$ is any f such that

(i) $f\colon A \to A'$ is a $*$-algebra morphism from $(A, m, 1, *)$ to $(A', m', 1', *')$,

(ii) $(f \otimes f) \circ \Delta = \Delta' \circ f$,

(iii) $\epsilon' \circ f = \epsilon$,

(iv) $f \circ S = S' \circ f$,

(c) For any Hopf-$*$-algebra with underlying vector space A, co-multiplication Δ and unit 1 an *integral* of this Hopf $*$-algebra is any faithful positive unital linear functional h such that

(i) $1 \circ h = (h \otimes \mathrm{id}_A) \circ \Delta$,

(ii) $1 \circ h = (\mathrm{id}_A \otimes h) \circ \Delta$.

The antipode of any Hopf $*$-algebra is antimultiplicative and bijective. Its inverse is given by the conjugation of the antipode by the star. The co-unit and antipode of a Hopf $*$-algebra are uniquely determined by the rest of the structure. On any Hopf $*$-algebra there can exist at most one integral.

DEFINITION 2.4. (a) A *CQG Hopf $*$-algebra* is any Hopf $*$-algebra which admits an integral. Equivalently, we speak of its formal dual as an *algebraic compact quantum group.*

(b) A *morphism of CQG Hopf $*$-algebras* from any CQG Hopf $*$-algebra H to any CQG Hopf $*$-algebra H' is any morphism $u\colon H \to H'$ of Hopf $*$-algebras. Equivalently, if G and G' are the formal duals of H and H', respectively, then we say that the formal dual of u is a *morphism $G' \to G$ of algebraic compact quantum groups.*

2.2. Relationship between the definitions. As mentioned, the categories of compact quantum groups and of algebraic compact quantum groups are not equivalent. Nonetheless, they are closely related. Section 2.2 explains how.

2.2.1. *Auxiliary definitions.* To state the relationship between the categories of compact quantum groups and algebraic compact quantum groups the following definitions will be required. Note that a Hopf $*$-algebra structure is *not* assumed there.

DEFINITION 2.5. (a) Let A be any vector space and Δ any linear map $A \to A \otimes A$ with $(\mathrm{id}_A \otimes \Delta)\Delta = (\Delta \otimes \mathrm{id}_A)\Delta$.

(i) A finite-dimensional *corepresentation* of (A, Δ) is any (V, u) such that
  (1) V is a finite-dimensional vector space, the *carrier space* of u,
  (2) u is a vector of $[V, V] \otimes A$,
  (3) $(\mathrm{id} \otimes \Delta)(u) = u \ominus u$.

  If clear from context or unimportant, we may also speak of just u as the co-representation, suppressing V. Also, the term co-representation will always mean a finite-dimensional one from now on.

(ii) For any two co-representations u and v of (A, Δ) with carrier spaces V respectively W a *corepresentation intertwiner* $u \to v$ of (A, Δ) is any t such that
  (1) t is a vector of $[V, W]$,
  (2) $([\mathrm{id}_V, t] \otimes \mathrm{id}_A)(u) = ([t, \mathrm{id}_W] \otimes \mathrm{id}_A)(v)$.

(iii) Any co-representation (V, u) of (A, Δ) is called *irreducible* if any intertwiner $u \to u$ is a multiple of the identity on V.

(iv) Any two co-representations u and v of (A, Δ) with carrier spaces V respectively W are said to be *equivalent* if there exists an intertwiner $t\colon u \to v$ which is a linear isomorphism $V \to W$.

(v) For any co-representation u on any vector space V and any basis $(e_i)_{i\in I}$ of V the *matrix of* u with respect to $(e_i)_{i\in I}$ is the family $(u_{j,i})_{(j,i)\in I^{\otimes 2}}$ such that, if $(E_{j,i})_{(j,i)\in I^{\otimes 2}}$ are the matrix units of $(e_i)_{i\in I}$, then $u = \sum_{(j,i)\in I^{\otimes 2}} E_{j,i} \otimes u_{j,i}$. If so, we speak of $\{u_{j,i} \mid \{i, j\} \subseteq I\}$ as the set of *matrix coefficients of* u with respect to $(e_i)_{i\in I}$.

(vi) Given any co-representation u_t of (A, Δ) with any carrier space V_t for each $t \in [\![2]\!]$, the *direct sum co-represenation* $u_1 \oplus u_2$ of (A, Δ) of (u_1, u_2) is given by $\sum_{t=1}^{2}([i_t, p_t] \otimes \mathrm{id}_A)(u_t)$ on the carrier space $V_1 \oplus V_2$, where $p_t\colon V_1 \oplus V_2 \to V_t$ is the projection and $i_t\colon V_t \to V_1 \oplus V_2$ the co-projection for the t-th summand in the direct sum $V_1 \oplus V_2$ and where the addition is the one of $[V_1 \oplus V_2, V_1 \oplus V_2]$.

(b) For any algebra $(A, m, 1)$, any algebra morphism Δ from $(A, m, 1)$ to the tensor product algebra $(A, m, 1)$ with itself which satisfies $(\mathrm{id}_A \otimes \Delta)\Delta = (\Delta \otimes \mathrm{id}_A)\Delta$ and any co-representation u_t of (A, Δ) on any carrier space V_t for each $t \in [\![2]\!]$ the *tensor product co-representation* $u_1 \cdot u_2$ of $(A, m, 1, \Delta)$, also known as *exterior product*, of (u_1, u_2) is defined as $u_1 \ominus u_2$ with carrier space $V_1 \otimes V_2$, where we have identified $[V_1, V_1] \otimes [V_2, V_2]$ and $[V_1 \otimes V_2, V_1 \otimes V_2]$.

(c) Let $(A, m, 1, *)$ be any $*$-algebra and Δ any $*$-algebra morphism from $(A, m, 1, *)$ to the tensor product $*$-algebra of $(A, m, 1, *)$ with itself which satisfies $(\mathrm{id}_A \otimes \Delta)\Delta = (\Delta \otimes \mathrm{id}_A)\Delta$.

(i) A finite-dimensional *unitary co-representation* of $(A, m, 1, *, \Delta)$ is any triple $(V, \langle\cdot \mid \cdot\rangle, u)$ such that $(V, \langle\cdot \mid \cdot\rangle)$ is a finite-dimensional Hilbert space, (V, u) a co-representation of (A, Δ) and u a unitary element of the $*$-algebra tensor product of $B(V, \langle\cdot \mid \cdot\rangle)$ and $(A, m, 1, *)$. In that case we refer to $(V, \langle\cdot \mid \cdot\rangle)$ as the *carrier Hilbert space* of $(V, \langle\cdot \mid \cdot\rangle, u)$. And, here too, we may suppress $(V, \langle\cdot \mid \cdot\rangle)$ if there is no risk of misunderstandings as well as drop the qualifier of being finite-dimensional.

(ii) Any co-representation u of (A, Δ) on any carrier space V is called a *unitarizable* co-representation of $(A, m, 1, *, \Delta)$ if there exists an inner product $\langle\cdot \mid \cdot\rangle$ on V turning $(V, \langle\cdot \mid \cdot\rangle)$ into a finite-dimensional Hilbert space in such a way that u becomes a unitary co-representation of $(A, m, 1, *, \Delta)$.

LEMMA 2.6. *Let $(A, m, 1, *)$ be any $*$-algebra, let Δ be any $*$-algebra morphism from $(A, m, 1, *)$ to the tensor product $*$-algebra of $(A, m, 1, *)$ with itself and let $(\mathrm{id}_A \otimes \Delta)\Delta = (\Delta \otimes \mathrm{id}_A)\Delta$.*

(a) For any set $R_{\mathrm{irr}} = \{u^r\}_{r\in R_{\mathrm{irr}}}$ of pairwise inequivalent irreducible unitary co-representations of $(A, m, 1, *, \Delta)$, if M_r is the set of matrix coefficients of u^r with respect to any basis of the carrier space of M_r for each $r \in R_{\mathrm{irr}}$, then $\bigcup_{r\in R_{\mathrm{irr}}} M_r$ is linearly independent in A.

(b) Any unitary co-representation of $(A, m, 1, *)$ is equivalent to a direct sum of pairwise inequivalent irreducible unitary co-representations of $(A, m, 1, *)$.

PROOF. (a) The proof of [KS12, p. 401, Corollary 10] applies, even though no co-unit for Δ is assumed to exist, since also the definitions of co-representations and irreducibility have been adjusted appropriately.

(b) Let u be any unitary co-representation of $(A, m, 1, *)$ and let any finite-dimensional Hilbert space $H = (V, \langle\cdot \mid \cdot\rangle)$ be its carrier. Then the set M of all co-representation intertwiners $u \to u$ of (A, Δ) forms a von Neumann subalgebra of $B(H)$. Indeed, it is a vector subspace and closed under composition as is easily seen from the definition. The assumption that u is unitary ensures that it is also closed under forming adjoints for the following reasons. For any intertwiner $t\colon u \to u$ starring the identity $([t, \mathrm{id}_V] \otimes \mathrm{id}_A)(u) = ([\mathrm{id}_V, t] \otimes \mathrm{id}_A)(u)$ in the $*$-algebra $B(H) \otimes (A, m, 1, *)$ yields the identity $([\mathrm{id}_V, t^*] \otimes \mathrm{id}_A)(u^*) = ([t^*, \mathrm{id}_V] \otimes \mathrm{id}_A)(u^*)$. Since $uu^* = \mathrm{id}_V \otimes 1 = u^*u$ per assumption, multiplying with u from the right transforms this into $t^* \otimes 1 = ([t^*, \mathrm{id}_V] \otimes \mathrm{id}_A)(u^*)u$ and a second multiplication with u, this time from the left, thus produces $([t^*, \mathrm{id}_V] \otimes \mathrm{id}_A)(u) = ([\mathrm{id}_V, t^*] \otimes \mathrm{id}_A)(u)$, proving $t^* \in M$. Because H is finite-dimensional that makes M a von Neumann subalgebra.

As such it permits a decomposition of id_V into a sum $\sum_{i=1}^m P_i$ of finitely many pairwise inequivalent minimal projections $(P_i)_{i=1}^m$ of M. For any $i \in [\![m]\!]$ the image space V_i equipped with the restriction $\langle\cdot\mid\cdot\rangle_i$ of $\langle\cdot\mid\cdot\rangle$ to $V_i^{\mathrm{cj}}\otimes V_i$ forms a finite-dimensional Hilbert space H_i and we can consider P_i a linear map $p_i\colon V\to V_i$. Then, $u^i\coloneqq([p_i^*,p_i]\otimes\mathrm{id}_A)(u)$ is a unitary co-representation of $(A,m,1,*,\Delta)$ on H_i because

$$\begin{aligned}(\mathrm{id}\otimes\Delta)(u^i)&=([p_i^*,p_i]\otimes\Delta)(u)\\&=([p_i^*,p_i]\otimes\mathrm{id}_{A\otimes A})(u\mathbin{\ominus}u)\\&=([p_i^*,p_i]\otimes\mathrm{id}_A)(u)\mathbin{\ominus}([p_i^*,p_i]\otimes\mathrm{id}_A)(u)\\&=u^i\mathbin{\ominus}u^i,\end{aligned}$$

because, by $p_ip_i^*=\mathrm{id}_{V_i}$ and $p_i^*p_i=P_i$ and by $u(P_i\otimes 1)=(P_i\otimes 1)u$,

$$\begin{aligned}u^i(u^i)^*&=(p_i\otimes 1)u(p_i^*\otimes 1)(p_i\otimes 1)u^*(p_i^*\otimes 1)\\&=(p_i\otimes 1)u(P_i\otimes 1)u^*(p_i\otimes 1)\\&=(p_i\otimes 1)(P_i\otimes 1)uu^*(p_i\otimes 1)\\&=p_iP_ip_i^*\otimes 1\\&=\mathrm{id}_{V_i}\otimes 1\end{aligned}$$

and because also $(u^i)^*u^i=\mathrm{id}_{V_i}\otimes 1$ by a similar argument. It is also irreducible for the following reason: Because u^i is a unitary co-representation the set M_i of intertwiners $u^i\to u^i$ forms a von Neumann subalgebra of $B(H_i)$. As such M_i is generated by its projections. Given any projection q of M_i the element $Q\coloneqq p_i^*qp_i$ is a projection of M. Moreover, $Q\leq P_i$ because $QP_i=p_i^*qp_ip_i^*p_i=Q$ and, likewise, $P_iQ=p_i^*p_ip_i^*qp_i=Q$. In particular, Q is subordinated by P_i in M. By the minimality of P_i that forces $Q=P_i$ and thus $q=p_iQp_i^*=p_iP_ip_i^*=\mathrm{id}_{V_i}$. It follows that the von Neumann algebra M_i is trivial, which means that u^i is irreducible.

We show that for any $\{i,j\}\subseteq[\![m]\!]$ with $i\neq j$ the two representations u^i and u^j are inequivalent. In fact, we let s be any intertwiner $u^i\to u^j$ and prove that $s=0$. First of all, s^*, the adjoint of s with respect to $\langle\cdot\mid\cdot\rangle_i$ and $\langle\cdot\mid\cdot\rangle_j$, is an intertwiner $u^j\to u^i$ because

$$\begin{aligned}(s^*\otimes 1)u^j&=u^i(u^i)^*(s^*\otimes 1)u^j\\&=u^i((s\otimes 1)u^i)^*u^j\\&=u^i(u^j(s\otimes 1))^*u^j\\&=u^i(s^*\otimes 1)(u^j)^*u^j\\&=u^i(s^*\otimes 1).\end{aligned}$$

Hence, s^*s is an intertwiner $u^i\to u^i$ and s^*s one $u^j\to u^j$. Because u^i and u^j are irreducible that requires the existence of scalars λ and μ with $s^*s=\lambda\,\mathrm{id}_{V_i}$ and $ss^*=\mu\,\mathrm{id}_{V_j}$. Because M_i and M_j are von Neumann algebras then in particular $0\leq\lambda$ and $0\leq\mu$. Moreover, we infer $\mu s=ss^*s=\lambda s$, which is to say $s=0$ or $\lambda=\mu$. We suppose $s\neq 0$ and derive a contradiction. The assumption forces $\lambda=\mu$. If $\lambda=\mu=0$, then $s^*s=0$ and $ss^*=0$ and thus $s=0$, a contradiction. If $\lambda=\mu\neq 0$, i.e., $\lambda=\mu>0$, then $w\coloneqq\frac{1}{\sqrt{\lambda}}s$ is a unitary $H_i\to H_j$. That makes

$W \coloneqq p_j^* w p_i$ a partial isometry in $B(H)$ because $WW^*W = p_j^* w p_i p_i^* w^* p_j p_j^* w p_i = W$ and $W^*WW^* = p_i^* w^* p_j p_j^* w p_i p_i^* w^* p_j = W^*$. In fact, $W \coloneqq p_j^* w p_i$ is a partial isometry in M since composition of intertwiners produces intertwiners and because p_i is an intertwiner $u \to u^i$ by

$$\begin{aligned}(p_i \otimes 1)u &= (p_i \otimes 1)(P_i \otimes 1)u \\ &= (p_i \otimes 1)u(P_i \otimes 1) \\ &= (p_i \otimes 1)u(p_i^* \otimes 1)(p_i \otimes 1) \\ &= u^i(p_i \otimes 1)\end{aligned}$$

and because the same computation for j in place of i shows that p_j is an intertwiner $u \to u^j$, which then implies that p_j^* is one $u^j \to u$ by the same argument as given for s^*. Since $W^*W = p_i^* w^* p_j p_j^* w p_i = P_i$ and $WW^* = p_j^* w p_i p_i^* w^* p_j = P_j$ it follows that P_i and P_j are equivalent projections of M, which is the contradiction we sought. In conclusion, u^i and u^j are inequivalent co-representations.

Moreover, if $t \coloneqq p_1 \times \ldots \times p_m$ is the linear map $V \to V_1 \oplus \ldots \oplus V_m$ with i-th component p_i for each $i \in [\![m]\!]$, then t is invertible with t^{-1} given by the map $p_1^* \sqcup \ldots \sqcup p_m^*$, the linear map $V_1 \oplus \ldots \oplus V_m \to V$ with i-th co-component p_i^* for each $i \in [\![m]\!]$. And, finally, via t the co-representation u is equivalent to $v \coloneqq u^1 \oplus \ldots \oplus u^m$ because, if π_i and ι_i are the i-th projection respectively co-projection of $V_1 \oplus \ldots \oplus V_m$ for each $i \in [\![m]\!]$, then

$$\begin{aligned}v(t \otimes 1) &= \textstyle\sum_{i=1}^m (\iota_i p_i \otimes 1)u(p_i^* \pi_i t \otimes 1) \\ &= \textstyle\sum_{i=1}^m (\iota_i p_i \otimes 1)u(p_i^* p_i \otimes 1) \\ &= \textstyle\sum_{i=1}^m (\iota_i p_i \otimes 1)u(P_i \otimes 1) \\ &= \textstyle\sum_{i=1}^m (\iota_i p_i \otimes 1)(P_i \otimes 1)u \\ &= \textstyle\sum_{i=1}^m (\iota_i p_i \otimes 1)u \\ &= \textstyle\sum_{i=1}^m (tP_i \otimes 1)u \\ &= (t \otimes 1)u,\end{aligned}$$

where we have used that $\pi_j t P_i = p_j P_i = p_j P_j P_i = \delta_{j,i} p_i P_i = \delta_{j,i} p_i = \pi_j \iota_i p_i$ for any $j \in [\![m]\!]$ and thus $tP_i = \iota_i p_i$ for any $i \subseteq [\![m]\!]$ by the orthogonality of $(P_i)_{i=1}^m$. Hence, u is equivalent to a direct sum of irreducible unitary co-representations. □

2.2.2. *Underlying algebraic compact quantum groups of compact quantum groups.* By considering appropriate representations of compact quantum groups it is possible to assign to any compact quantum group an algebraic compact quantum group which retains much of the information (compare also the original definition of [Wor87] and [Wan97]).

Definition 2.7. For any CQG C^*-algebra (A, Δ) the same definitions as in Definition 2.5 can be given, with the sole difference that in condition (a) (i) (3) of the definition of a co-representation the identity must read $(\mathrm{id} \otimes \Delta)(u) = \iota(u \ominus u)$ instead, where ι is the inclusion of $A \otimes A$ in $A \otimes_{\min} A$.

Moreover, then the results of Lemma 2.6 remain true for co-representations of CQG C^*-algebras. Those enlighten – in part – why the next definition makes sense (see, e.g., [Tim08, Theorem 5.4.1]).

DEFINITION 2.8. (a) For any CQG C^*-algebra (A,Δ), if R_{irr} is any maximal family of pairwise inequivalent irreducible unitary co-representations of (A,Δ) and if $(u^r_{j,i})_{(j,i)\in[\![n_r]\!]^{\otimes 2}}$ is the matrix of u^r with respect to any orthonormal basis of the carrier space for each $r\in R_{\mathrm{irr}}$, then we call the tuple $(R,m,1,\Phi,\epsilon,S,*)$, where

- (i) $(R,m,1,*)$ is the $*$-algebra given by the vector subspace of A generated by $\bigcup_{r\in R_{\mathrm{irr}}}\{u^r_{j,i}\}_{i,j=1}^{n_r}$ with the inherited operations of A (i.e., m, 1 and $*$ are the restrictions to R of, respectively, the multiplication, the unit and the star of A),
- (ii) Φ is the unique linear map $R\to R\otimes R$ with $u^r_{j,i}\mapsto\sum_{s=1}^{n_r}u^r_{j,s}\otimes u^r_{s,i}$ for any $\{i,j\}\subseteq[\![n_r]\!]$ and any $r\in R_{\mathrm{irr}}$,
- (iii) ϵ is the unique linear functional on R such that $u^r_{j,i}\mapsto\delta_{j,i}$ for any $\{i,j\}\subseteq[\![n_r]\!]$ and any $r\in R_{\mathrm{irr}}$,
- (iv) S is the unique linear endomorphism of R with $u^r_{j,i}\mapsto(u^r_{i,j})^*$ for any $\{i,j\}\subseteq[\![n_r]\!]$ and any $r\in R_{\mathrm{irr}}$,

the *underlying CQG Hopf $*$-algebra* $\mathsf{A}(A,\Delta)$ of (A,Δ) and the identity map from R to A the *associated inclusion.* Moreover, we speak of its formal dual as the *underlying algebraic compact quantum group* of the formal dual of (A,Δ).

(b) Given any two CQG C^*-algebras (A,Δ) and (B,Φ) with formal duals G respectively H and any morphism ψ of CQG C^*-algbras from (A,Δ) to (B,Φ) the restriction of ψ to $\mathsf{A}(A,\Delta)$ is called the *underlying CQG Hopf $*$-algebra morphism* $\mathsf{A}(\psi)$ of ψ. Equivalently, we say that the formal dual of $\mathsf{A}(\psi)$ is the *underlying morphism of algebraic comapct quantum groups.*

The underlying CQG Hopf $*$-algebra of any CQG C^*-algebra is the maximal $*$-subcoalgebra which can be turned into a Hopf $*$-algebra (by [BMT01]). In the literature, sometimes, all compact quantum groups with the same underlying algebraic quantum group are identified. If so, the underlying CQG Hopf $*$-algebra of a CQG C^*-algebra with formal dual G is often denoted by $\mathrm{Pol}(G)$ or $\mathcal{O}(G)$. A is a functor from the category of CQG C^*-algebras to the category of CQG Hopf $*$-algebras.

PROPOSITION 2.9. *(a)* $\mathsf{A}(C)$ *is a CQG Hopf $*$-algebra for any CQG C^*-algebra C.*

(b) $\mathsf{A}(\varphi)$ *is a morphism* $\mathsf{A}(C)\to\mathsf{A}(C')$ *of CQG Hopf $*$-algebras for any CQG C^*-algebra morphism $\varphi\colon C\to C'$ and any CQG C^*-algebras C and C'.*

(c) $\mathsf{A}(\varphi'\varphi)=\mathsf{A}(\varphi')\mathsf{A}(\varphi)$ *for any morphisms $\varphi\colon C\to C'$ and $\varphi'\colon C'\to C''$ of CQG C^*-algebras and any CQG C^*-algebras C, C' and C''.*

(d) $\mathsf{A}(\mathrm{id}_C)=\mathrm{id}_{\mathsf{A}(C)}$ *for any CQG C^*-algebra C.*

However, A is not an equivalence. Correspondingly, there are at least two non-isomorphic ways of, conversely, turning algebraic compact quantum groups into C^*-algebraic ones.

2.2.3. *Universal C^*-version of algebraic compact quantum groups.* The first way of producing a CQG C^*-algebra from a CQG Hopf $*$-algebra is the one employed in [Wor88] for Tannaka reconstruction (see, e.g., [Tim08, Theorem 5.4.3]).

DEFINITION 2.10. (a) For any CQG Hopf $*$-algebra $(A, m, 1, *, \Delta, \epsilon, S)$, we call $(A', m', 1', *', \|\cdot\|', \Delta')$, where

(i) A' is the underlying vector space of the co-domain of the inclusion j of the normed space $(A, \|\cdot\|)$ into its Banach space completion $(A', \|\cdot\|')$, where $\|\cdot\|$ is the norm on A with for any vector a of A,

$$a \mapsto \sup\{p(a) \,|\, p \ C^*\text{-seminorm on } (A, m, 1, *)\},$$

(ii) m' is the unique linear map $A' \otimes A' \to A'$ with $m'(j \otimes j) = jm$,

(iii) $1'$ is given by $j(1)$,

(iv) $*'$ is the unique linear map $A' \to A'^{\mathrm{cj}}$ with $*'j = j^{\mathrm{cj}}*$,

(v) Δ' is the unique linear from A' to the vector space underlying the minimal tensor product C^*-algebra of $(A', m', 1', *', \|\cdot\|')$ with itself with $\Delta' j = \iota(j \otimes j)\Delta$, where ι is the inclusion of $A' \otimes A'$ into the tensor product C^* algebra,

the *universal CQG C^*-algebra* $\mathsf{U}(A, m, 1, *, \Delta, \epsilon, S)$ *induced by* $(A, m, 1, *, \Delta, \epsilon, S)$ and j the *associated inclusion.* Furthermore, we say that the formal dual of the CQG C^*-algebra $\mathsf{U}(A, m, 1, *, \Delta, \epsilon, S)$ is the *universal compact quantum group induced by* the formal dual of $(A, m, 1, *, \Delta, \epsilon, S)$.

(b) Given any two CQG Hopf $*$-algebras X and Y and any CQG Hopf $*$-algebra morphism $\psi\colon X \to Y$, if A and B are the underlying vector spaces of X respectively Y, if i and j are the associated inclusions of X into $\mathsf{U}(X)$ respectively Y into $\mathsf{U}(Y)$ and if A' and B' are the underlying vector spaces of $\mathsf{U}(X)$ respectively, $\mathsf{U}(Y)$, then we call the unique linear map $\psi'\colon A' \to B'$ with $\psi' i = j\psi$ the *universal CQG C^*-algebra morphism* $\mathsf{U}(\psi)\colon \mathsf{U}(X) \to \mathsf{U}(Y)$ *induced by* ψ. Equivalently, we say that the formal dual of ψ' is the *universal compact quantum group morphism induced by* the formal dual of ψ.

In the literature, if all CQG C^*-algebra completions of a CQG Hopf $*$-algebra are identified, then the universal CQG C^*-algebra of such an equivalence class with formal dual G is often denoted by $C_{\max}(G)$ or $C_u(G)$. U is a functor from the category of CQG Hopf $*$-algebras to the category of CQG C^*-algebras.

PROPOSITION 2.11. *(a)* $\mathsf{U}(H)$ *is a CQG C^*-algebra for any CQG Hopf $*$-algebra H.*

(b) $\mathsf{U}(\psi)$ *is a morphism* $\mathsf{U}(H) \to \mathsf{U}(H')$ *of CQG C^*-algebras for any morphism $\psi\colon H \to H'$ of CQG Hopf $*$-algebras and CQG Hopf $*$-algebras H and H'.*

(c) $\mathsf{U}(\psi'\psi) = \mathsf{U}(\psi')\mathsf{U}(\psi)$ *for any morphisms $\psi\colon H \to H'$ and $\psi'\colon H' \to H''$ of CQG Hopf $*$-algebras and any CQG Hopf $*$-algebras H, H' and H''.*

(d) $\mathsf{U}(\mathrm{id}_H) = \mathrm{id}_{\mathsf{U}(H)}$ *for any CQG Hopf* $*$*-algebra* H*.*

2.2.4. *Reduced* C^**-version of algebraic compact quantum groups.* The second way of assigning a natural CQG C^*-algebra to a CQG Hopf $*$-algebra is often given in a somewhat roundabout way. Namely, only its factorization through the functor U is provided (see, e.g., [Tim08, Theorem 5.4.5] or [BMT01]). That is likely due to the fact that the definition is usually given in terms of the GNS construction for C^*-algebras. However, the GNS construction can also be performed on any (per our convention unital) $*$-algebra with a state.

Definition 2.12. (a) For any CQG Hopf $*$-algebra $(A, m, 1, *, \Delta, \epsilon, S)$ we call $(A', m', 1', *', \|\cdot\|', \Delta')$, where

(i) A' is the vector space underlying the co-domain of the inclusion j of $(A, \|\cdot\|)$ into its Banach space completion $(A', \|\cdot\|')$, where $\|\cdot\|$ is the norm on A with for any vector a of A,
$$a \mapsto \sup\left\{\sqrt{h(b^*a^*ab)}\,\middle|\, b \in A \wedge h(b^*b) = 1\right\},$$
where h is the integral of $(A, m, 1, *, \Delta, \epsilon, S)$,

(ii) m' is the unique linear map $A' \otimes A' \to A'$ with $m'(j \otimes j) = jm$,

(iii) $1'$ is given by $j(1)$,

(iv) $*'$ is the unique linear map $A' \to A'^{\mathrm{cj}}$ with $*'j = j^{\mathrm{cj}}*$,

(v) Δ' is the unique linear from A' to the vector space underlying the minimal tensor product C^*-algebra of $(A', m', 1', *', \|\cdot\|')$ with itself with $\Delta' j = \iota(j \otimes j)\Delta$, where ι is the inclusion of $A' \otimes A'$ into the tensor product C^* algebra,

the *reduced CQG* C^**-algebra* $\mathsf{R}(A, m, 1, *, \Delta, \epsilon, S)$ *induced by* $(A, m, 1, *, \Delta, \epsilon, S)$ and j the *associated inclusion.* Furthermore, we say that the formal dual of the CQG C^*-algebra $\mathsf{R}(A, m, 1, *, \Delta, \epsilon, S)$ is the *reduced compact quantum group induced by* the formal dual of $(A, m, 1, *, \Delta, \epsilon, S)$.

(b) Given any two CQG Hopf $*$-algebras X and Y and any CQG Hopf $*$-algebra morphism $\psi\colon X \to Y$, if A and B are the underlying vector spaces of X respectively Y, if i and j are the associated inclusions of X into $\mathsf{R}(X)$ respectively Y into $\mathsf{R}(Y)$ and if A' and B' are the underlying vector spaces of $\mathsf{R}(X)$ respectively, $\mathsf{R}(Y)$, then we call the unique linear map $\psi'\colon A' \to B'$ with $\psi' i = j\psi$ the *reduced CQG* C^**-algebra morphism* $\mathsf{R}(\psi)\colon \mathsf{R}(X) \to \mathsf{R}(Y)$ *induced by* ψ. Equivalently, we say that the formal dual of ψ' is the *reduced compact quantum group morphism induced by* the formal dual of ψ.

Where all CQG C^*-algebra completions of any CQG Hopf $*$-algebra are seen as equivalent, the symbols $C_{\mathrm{red}}(G)$ or $C_{\mathrm{r}}(G)$ are often used to refer to the reduced CQG C^*-algebra of the formal dual of such an equivalence class G. R is a functor from the category of CQG Hopf $*$-algebras to the category of CQG C^*-algebras.

Proposition 2.13. *(a)* $\mathsf{R}(H)$ *is a CQG* C^**-algebra for any CQG Hopf* $*$*-algebra* H*.*

(b) $\mathsf{R}(\psi)$ *is a morphism* $\mathsf{R}(H) \to \mathsf{R}(H')$ *of CQG* C^**-algebras for any CQG Hopf* $*$*-algebra morphism* $\psi\colon H \to H'$ *and any CQG Hopf* $*$*-algebras* H *and* H'*.*
(c) $\mathsf{R}(\psi'\psi) = \mathsf{R}(\psi')\mathsf{R}(\psi)$ *for any morphisms* $\psi\colon H \to H'$ *and* $\psi'\colon H' \to H''$ *of CQG Hopf* $*$*-algebras and any CQG Hopf* $*$*-algebras* H*,* H' *and* H''*.*
(d) $\mathsf{R}(\mathrm{id}_H) = \mathrm{id}_{\mathsf{R}(H)}$ *for any CQG Hopf* $*$*-algebra* H*.*

As alluded to, R factorizes through U. The relationships between R, U and A are the topic of the next section.

2.2.5. *Relationship between the three constructions.* In the below definition the existence of the morphism in (a) is guaranteed by the co-universal property of the Banach completion involved in the definition of U. In order to see that the morphism in (b) exists, more effort is needed (see [Tim08, Theorem 5.4.3]).

DEFINITION 2.14. (a) For any CQG C^*-algebra C, if i is the associated inclusion of $\mathsf{A}(C)$ in C, and if j is the associated inclusion of $\mathsf{A}(C)$ in $\mathsf{U}(\mathsf{A}(C))$, we call the unique linear map φ from the underlying vector space of $\mathsf{U}(\mathsf{A}(C))$ to the underlying vector space of C with $\varphi j = i$ the *co-unit morphism* $\mathrm{cu}_C^{\mathsf{U}\dashv\mathsf{A}}$ *of* $\mathsf{U} \dashv \mathsf{A}$ at C.
(b) Given any CQG Hopf $*$-algebra H, if ℓ is the associated inclusion of H in $\mathsf{U}(H)$, and if k is the associated of $\mathsf{A}(\mathsf{U}(H))$ in $\mathsf{U}(H)$, then we call the unique linear map ψ from the underlying vector space of H to the underlying vector space of $\mathsf{A}(\mathsf{U}(H))$ with $k\psi = \ell$ the *unit morphism* $\mathrm{un}_H^{\mathsf{U}\dashv\mathsf{A}}$ *of* $\mathsf{U} \dashv \mathsf{A}$ at H.

$\mathrm{cu}^{\mathsf{U}\dashv\mathsf{A}}$ is the co-unit and $\mathrm{un}^{\mathsf{U}\dashv\mathsf{A}}$ the unit of an adjunction $\mathsf{U} \dashv \mathsf{A}$. In particular, the functor U is left-adjoint and the functor A right-adjoint.

PROPOSITION 2.15. *(a)* $\mathrm{cu}_C^{\mathsf{U}\dashv\mathsf{A}}$ *is a morphism* $\mathsf{U}(\mathsf{A}(C)) \to C$ *of CQG* C^**-algebras for any CQG* C^**-algebra* C*.*
(b) $\mathrm{un}_H^{\mathsf{U}\dashv\mathsf{A}}$ *is a morphism* $H \to \mathsf{A}(\mathsf{U}(H))$ *of CQG Hopf* $*$*-algebras for any CQG Hopf* $*$*-algebra* H*.*
(c) $\mathrm{cu}_{C'}^{\mathsf{U}\dashv\mathsf{A}} \circ \mathsf{U}(\mathsf{A}(\varphi)) = \varphi \circ \mathrm{cu}_C^{\mathsf{U}\dashv\mathsf{A}}$ *for any morphism* $\varphi\colon C \to C'$ *of CQG* C^**-algebras and any CQG* C^**-algebras* C *and* C'*.*
(d) $\mathrm{un}_{H'}^{\mathsf{U}\dashv\mathsf{A}} \circ \psi = \mathsf{A}(\mathsf{U}(\psi)) \circ \mathrm{un}_H^{\mathsf{U}\dashv\mathsf{A}}$ *for any morphism* $\psi\colon H \to H'$ *of CQG Hopf* $*$*-algebras and any CQG Hopf* $*$*-algebras* H *and* H'*.*
(e) $\mathrm{id}_{\mathsf{U}(H)} = \mathrm{cu}_{\mathsf{U}(H)}^{\mathsf{U}\dashv\mathsf{A}} \circ \mathsf{U}(\mathrm{un}_H^{\mathsf{U}\dashv\mathsf{A}})$ *for any CQG Hopf* $*$*-algebra* H*.*
(f) $\mathrm{id}_{\mathsf{A}(C)} = \mathsf{A}(\mathrm{cu}_C^{\mathsf{U}\dashv\mathsf{A}}) \circ \mathrm{un}_{\mathsf{A}(C)}^{\mathsf{U}\dashv\mathsf{A}}$ *for any CQG* C^**-algebra* C*.*

The existence of either of the two morphisms in the next definition is beyond the scope of this overview. See [Tim08, Theorem 5.4.5] for (a) and [Tim08, Proposition 5.4.8] for (b).

DEFINITION 2.16. (a) For any CQG Hopf $*$-algebra H, if i is the associated inclusion of $\mathsf{A}(\mathsf{R}(H))$ in $\mathsf{R}(H)$, and if j is the associated inclusion of H in $\mathsf{R}(H)$, then we call the unique linear map ψ from the underlying vector space of $\mathsf{A}(\mathsf{R}(H))$ to the underlying vector space of H with $j\psi = i$ the *co-unit morphism* $\mathrm{cu}_H^{\mathsf{A}\dashv\mathsf{R}}$ *of* $\mathsf{A} \dashv \mathsf{R}$ at H.

(b) Given any CQG C^*-algebra C we call the unique linear map φ, if k is the associated inclusion of $\mathsf{A}(C)$ in C, and if ℓ is the associated inclusion of $\mathsf{A}(C)$ in $\mathsf{R}(\mathsf{A}(C))$, then we call the unique linear map from the underlying vector space of C to the underlying vector space of $\mathsf{R}(\mathsf{A}(C))$ with $\varphi k = \ell$ the *unit morphism* $\mathrm{un}_C^{\mathsf{A}\dashv\mathsf{R}}$ *of* $\mathsf{A} \dashv \mathsf{R}$ at C.

$\mathrm{cu}^{\mathsf{A}\dashv\mathsf{R}}$ is the co-unit and $\mathrm{un}^{\mathsf{A}\dashv\mathsf{R}}$ the unit of an adjunction $\mathsf{A} \dashv \mathsf{R}$.

Proposition 2.17. *(a)* $\mathrm{cu}_H^{\mathsf{A}\dashv\mathsf{R}}$ *is a morphism* $\mathsf{A}(\mathsf{R}(H)) \to H$ *of CQG Hopf $*$-algebras for any CQG Hopf $*$-algebra H.*

(b) $\mathrm{un}_C^{\mathsf{A}\dashv\mathsf{R}}$ *is a morphism* $C \to \mathsf{R}(\mathsf{A}(C))$ *of CQG C^*-algebras for any CQG C^*-algebra C.*

(c) $\mathrm{cu}_{H'}^{\mathsf{A}\dashv\mathsf{R}} \circ \mathsf{A}(\mathsf{R}(\psi)) = \psi \circ \mathrm{cu}_H^{\mathsf{A}\dashv\mathsf{R}}$ *for any morphism* $\psi\colon H \to H'$ *of CQG Hopf $*$-algebras and any CQG Hopf $*$-algebras H and H'.*

(d) $\mathrm{un}_{C'}^{\mathsf{A}\dashv\mathsf{R}} \circ \varphi = \mathsf{R}(\mathsf{A}(\varphi)) \circ \mathrm{un}_C^{\mathsf{A}\dashv\mathsf{R}}$ *for any morphism* $\varphi\colon C \to C'$ *of CQG C^*-algebras and any CQG C^*-algebras C and C'.*

(e) $\mathrm{id}_{\mathsf{A}(C)} = \mathrm{cu}_{\mathsf{A}(C)}^{\mathsf{A}\dashv\mathsf{R}} \circ \mathsf{A}(\mathrm{un}_C^{\mathsf{A}\dashv\mathsf{R}})$ *for any CQG C^*-algebra C.*

(f) $\mathrm{id}_{\mathsf{R}(H)} = \mathsf{R}(\mathrm{cu}_H^{\mathsf{A}\dashv\mathsf{R}}) \circ \mathrm{un}_{\mathsf{R}(H)}^{\mathsf{A}\dashv\mathsf{R}}$ *for any CQG Hopf $*$-algebra H.*

In combination, Propositions 2.15 and 2.17 show that A is both left- and right-adjoint (although to different functors). It follows, especially, that A preserves co-limits and limits. In contrast, of U and R we only know that U preserves co-limits and that R preserves limits. That is the reason why it is so difficult to say anything about compact quantum quotient groups on the universal and compact quantum subgroups on the reduced level for algebraic compact quantum groups defined via Tannaka-Krein duality.

2.3. Products of compact quantum groups. For the purposes of this chapter it is enough to confine ourselves to products of algebraic compact quantum groups. However, versions of these constructions can also be carried out in the C^*-context.

There are multiple ways one can define the various products considered in this section. The most convenient for our goals is to introduce them by means of the following construction (see [KS12, Section 1.2.7]).

Definition 2.18. Let $(A, m, 1, *, \Delta, \epsilon, S)$ be any Hopf $*$-algebra.

(a) A *Hopf $*$-ideal* is any I vector subspace of A such that
  - (i) I is a $*$-ideal of $(A, m, 1, *)$,
  - (ii) Δ maps I to the vector subspace of $A \otimes A$ generated by the union of the sets $\{a \otimes i \,|\, a \in A \wedge i \in I\}$ and $\{i \otimes a \,|\, i \in I \wedge a \in A\}$,
  - (iii) ϵ maps I to $\{0\}$.
  - (iv) S maps I to itself.

(b) For any Hopf $*$-ideal I we call the Hopf $*$-algebra $(A/I, m', 1', *', \Delta', \epsilon', S')$, where
   (i) $(A/I, m', 1', *')$ is the quotient $*$-algebra of $(A, m, 1, *)$ with repect to the $*$-ideal I (with associated projection $p\colon A \to A/I$),
   (ii) Δ' is the unique linear map $A/I \to A/I \otimes A/I$ with $\Delta' p = (p \otimes p)\Delta'$,
   (iii) ϵ' is the unique linear functional on A/I with $\epsilon' p = \epsilon$,
   (iv) S' is the unique linear endomorphism of A/I with $S'p = pS$,

   the *quotient Hopf $*$-algebra* of $(A, m, 1, *, \Delta, \epsilon, S)$ with respect to I.

The following was recognized by Wang in [Wan95a, Theorem 2.11] for the analogous construction for CQG C^*-algebras.

Proposition 2.19. *Quotient Hopf $*$-algebras of CQG Hopf $*$-algebras by Hopf $*$-ideals are CQG Hopf $*$-algebras.*

2.3.1. *Graph products.* In the free probability context, the graph product was defined independently by Młotkowski in [Mło04] under the name of λ-free product and by Speicher and Wysoczański in [SW16] as ϵ-independence. For compact quantum groups, graph products were introduced by Caspers and Fima in [CF17]. The special cases of free and direct products had already been discussed by Wang in [Wan95a] and [Wan95b], respectively.

Definition 2.20. For any countable set I and any family $(H_i)_{i\in I}$ of CQG Hopf $*$-algebras, if $H_i = (A_i, m_i, 1_i, *_i, \Delta_i, \epsilon_i, S_i)$ for each $i \in I$, then we call the tuple $(A', m', 1', *', \Delta', \epsilon', S')$ together with $(\iota_i)_{i\in I}$, where

(a) $(A', m', 1', *')$ is the co-product $*$-algebra of $(A_i, m_i, 1_i, *_i)_{i\in I}$ (with asssociated co-projections $(\iota_i)_{i\in I}$),
(b) Δ' is the unique morphism of $*$-algebras from $(A', m', 1', *')$ to the tensor product of $(A', m', 1', *')$ with itself such that $(\iota_i \otimes \iota_i)\Delta_i = \Delta' \iota_i$ for any $i \in I$,
(c) ϵ' is the unique morphism of $*$-algebras from $(A', m', 1', *')$ to $\mathbb{C}$ with $\epsilon' \iota_i = \epsilon_i$ for any $i \in I$,
(d) S' is the unique algebra morphism from $(A', m', 1')$ to the opposite algebra of $(A', m', 1')$ with $S' \iota_i = \iota_i S_i$ for any $i \in I$,

the *co-product CQG Hopf $*$-algebra* $*_{i\in I} H_i$ of $(H_i)_{i\in I}$. We also speak of the formal dual of $*_{i\in I} H_i$ as the *free product* of the formal duals of $(H_i)_{i\in I}$.

Definition 2.21. A partial commutation relation on any given set I is any anti-reflexive symmetric binary relation on I.

Definition 2.22. For any countable set I, any partial commutation relation r on I and any family $(H_i)_{i\in I}$ of CQG Hopf $*$-algebras, if H' is the quotient Hopf $*$-algebra of the co-product CQG Hopf $*$-algebra P of $(H_i)_{i\in I}$ with respect to the Hopf $*$-ideal of P given by the $*$-ideal of the underlying $*$-algebra of P generated by the set

$$\{\iota_{i_1}(a_1)\iota_{i_2}(a_2) - \iota_{i_2}(a_2)\iota_{i_1}(a_1) \,|\, (i_1, i_2) \in r \wedge a_1 \in H_{i_1} \wedge a_2 \in H_{i_2}\},$$

where $(\iota_i)_{i\in I}$ are the co-projections associated with the co-product P, and where p is the projection associated with H', then we call H' the *graph co-product CQG Hopf $\ast$-algebra* $\ast^r_{i\in I} H_i$ of $(H_i)_{i\in I}$ with respect to r and we call $(p\iota_i)_{i\in I}$ the *associated co-projections*. Analogously, we say that the formal dual of $\ast^r_{i\in I} H_i$ is the *graph product* of the formal duals of $(H_i)_{i\in I}$ with respect to r.

2.3.2. *Crossed products.* Crossed products of compact quantum groups were also introduced by Wang in [Wan95b] in the C^*-context.

Definition 2.23. For any discrete group Γ with underlying set M, law μ, neutral element e and inversion mapping $(\cdot)^{-1}$ the *group CQG Hopf $\ast$-algebra* $\mathbb{C}[\Gamma]$ of Γ is the CQG Hopf $\ast$-algebra given by the tuple $(A, m, 1, \Delta, \epsilon, S, \ast)$, where

(a) A is the free vector space over M,
(b) m is the linear map $A\otimes A\to A$ with $g\otimes h\mapsto \mu(g,h)$ for any $\{g,h\}\subseteq M$,
(c) $1 = e$,
(d) $\ast$ is the linear map $A\to A^{\mathrm{cj}}$ with $g\mapsto g^{-1}$,
(e) Δ is the linear map $A\to A\otimes A$ with $g\mapsto g\otimes g$ for any $g\in M$,
(f) ϵ is the linear functional on A with $g\mapsto 1$ for any $g\in M$,
(g) S is the linear endomorphism of A with $g\mapsto g^{-1}$ for any $g\in M$.

Definition 2.24. For any CQG Hopf $\ast$-algebra H, any discrete group Γ and any group homomorphism α of from Γ to the group of Hopf $\ast$-automorphisms of H, if H' is the quotient CQG Hopf $\ast$-algebra of the co-product CQG Hopf $\ast$-algebra P of $(H, \mathbb{C}[\Gamma])$ with respect to the Hopf $\ast$-ideal given by the $\ast$-ideal of the underlying $\ast$-algebra of P generated by the set

$$\{\iota_\Gamma(g)\iota_H(a) - \iota_H(\alpha_g(a))\iota_\Gamma(g) \,|\, a\in H \wedge g\in\Gamma\},$$

where (ι_H, ι_Γ) are the co-projections associated with P, and if p is the projection associated with H', then we call H' the *crossed co-product CQG Hopf $\ast$-algebra* $H\rtimes_\alpha \mathbb{C}[\Gamma]$ of H and $\mathbb{C}[\Gamma]$ with respect to α and we call $(p\iota_H, p\iota_\Gamma)$ the *associated co-projections*. Likewise, we say that the formal dual of $H\rtimes_\alpha \mathbb{C}[\Gamma]$ is the *crossed product* of the formal duals of H and $\mathbb{C}[\Gamma]$ with respect to α.

2.3.3. *Wreath graph products.* The special case of finite or infinite wreath graph products with cyclic groups with respect to the empty partial commutation relation was discussed in [Wan95b, Example 4.4 (2), (3)] by Wang for compact quantum groups. The wreath graph product is a combination of graph product and crossed product.

Definition 2.25. For any subgroup S of permutations of any set I any partial commutation relation r on I is called *S-invariant* if $(\sigma(i_1), \sigma(i_2))\in r$ for any $\sigma\in S$ and any $(i_1, i_2)\in r$.

For the next definition we fix a set $\aleph$ with the property that $\aleph\notin I$ for all the index sets I we want to consider.

DEFINITION 2.26. For any countable set I, any discrete subgroup S of permutations of I, any S-invariant partial commutation relation r, and any CQG Hopf $*$-algebra H, if H' is the quotient Hopf $*$-algebra of the co-product CQG Hopf $*$-algebra P of $(H_j)_{j\in I\cup\{\aleph\}}$, where $H_i = H$ for any $i \in I$ and where $H_\aleph = \mathbb{C}[S]$, with respect to the Hopf $*$-ideal given by the $*$-ideal of the underlying $*$-algebra of P generated by the set

$$\{\iota_\aleph(\sigma)\iota_i(a) - \iota_{\sigma(i)}(a)\iota_\aleph(\sigma) \,|\, \sigma \in S \wedge a \in H\} \\ \cup \{\iota_{i_1}(a_1)\iota_{i_2}(a_2) - \iota_{i_2}(a_2)\iota_{i_1}(a_1) \,|\, (i_1, i_2) \in r \wedge \{a_1, a_2\} \subseteq H\},$$

where $(\iota_j)_{j\in I}$ are the co-projections associated with P, and if p is the projection associated with H', then we call H' the *wreath graph co-product* $H \wr_r \mathbb{C}[S]$ *of* H *and* $\mathbb{C}[S]$ *with respect to* r. We speak of the formal dual of $H \wr_r \mathbb{C}[S]$ as the *wreath graph product* of the formal duals of H and $\widehat{S}$.

2.4. Compact (multi-)matrix quantum groups. It follows a side remark unrelated to the proof of the main result. The next definition is obviously inspired by that of a *compact matrix quantum group* from [Wor87, Definition 1.1] if the simplifications from [Wor91] are taken into account. It is however easier to work with because it does not require a C^*-algebra. It is also easier to work with than the definition of a *CMQG algebra* from [KS12, p. 415, Definition 9 and 10] or the equivalent characterizations of [KS12, p. 417, Proposition 28] because it does not call for a Hopf $*$-algebra.

DEFINITION 2.27. (a) For any set N and family $(n_f)_{f\in N}$ of non-negative integers a *CMMQG $*$-algebra* of *profile* $(n_f)_{f\in N}$ is any $(A, m, 1, *, (u^f)_{f\in N})$ such that
  - (i) $(A, m, 1, *)$ is a $*$-algebra,
  - (ii) $u^f = (u^f_{j,i})_{(j,i)\in[\![n_f]\!]^{\otimes 2}}$, the *fundamental co-representation* f, is a unitary element of the $*$-algebra tensor product of $M_{n_f}(\mathbb{C})$ and $(A, m, 1, *)$ for each $f \in N$,
  - (iii) $([(t^f)^{-1}, t^f]\otimes *)(u^f)$ is a unitary unitary element of the $*$-algebra tensor product of $M_{n_f}(\mathbb{C})$ and $(A, m, 1, *)$ for some invertible element t^f of $M_{n_f}(\mathbb{C})$ for each $f \in N$,
  - (iv) $\{u^f_{j,i} \,|\, f \in N \wedge \{i, j\} \subseteq [\![n_f]\!]\}$ generates A as a $*$-algebra,
  - (v) there exists a morphism Δ of $*$-algebras from $(A, m, 1, *)$ to the tensor product $*$-algebra of $(A, m, 1, *)$ with itself such that $\Delta(u^f_{k,i}) = \sum_{j=1}^{n_f} u^f_{k,j} \otimes u^f_{j,i}$ for any $f \in N$ and $\{i, k\} \subseteq [\![n_f]\!]$.

  Equivalently, the formal dual of $(A, m, 1, *, (u^f)_{f\in N})$ is called an *algebraic compact multi-matrix quantum group* of profile $(n_f)_{f\in N}$. If N is a singleton set, we also speak of *CMQG $*$-algebras* respectively *algebraic compact matrix quantum groups.*

(b) Given any profile $(N, (n_f)_{f\in N})$ and any CMMQG $*$-algebras $(A, (u^f)_{f\in N})$ and $(B, (v^f)_{f\in N})$ a *morphism of CMMQG $*$-algebras* from $(A, (u^f)_{f\in N})$ to

$(B,(v^f)_{f\in N})$ is any $*$-algebras morphism φ from A to B with $(\mathrm{id}\otimes\varphi)(u^f) = v^f$ for each $f\in N$. Of course, the formal dual of φ is said to be a *morphism of compact multi-matrix quantum groups* of profile $(N,(n_f)_{f\in N})$ from the formal dual of $(B,(v^f)_{f\in N})$ to the formal dual of $(A,(u^f)_{f\in N})$. Again, if N is a singleton, we speak of *morphisms of CMQG $*$-algebras* respectively *morphisms of algebraic compact matrix quantum groups.*

The next proposition shows that this definition accomplishes what it is intended to do. Arguably, though, the usefulness of morphisms of CMMQG $*$-algebras is limited. It is mostly the objects themselves that are handy.

PROPOSITION 2.28. *(a) For any CMMQG $*$-algebra $(A,m,1,*,(u^f)_{f\in N})$ of any profile $(N,(n_f)_{f\in N})$ there exists a unique (Δ,ϵ,S) such that,*

(i) $(A,m,1,*,\Delta,\epsilon,S)$ is a CQG Hopf $*$-algebra,

(ii) for each $f\in N$, if w^f is the inverse of $(\mathrm{id}\otimes *)(u^f)$ in the $*$-algebra tensor product of $M_{n_f}(\mathbb{C})$ and $(A,m,1,*)$, then

(1) $\Delta(u^f_{k,i}) = \sum_{j=1}^{n_f} u^f_{k,j}\otimes u^f_{j,i}$ for any $\{i,k\}\subseteq [\![n_f]\!]$,

(2) $\epsilon(u^f_{j,i}) = \delta_{j,i}$ for any $\{i,j\}\subseteq [\![n_f]\!]$,

(3) $S(u^f_{j,i}) = (u^f_{i,j})^*$ and $S((u^f_{j,i})^*) = w^f_{j,i}$ for any $\{i,j\}\subseteq [\![n_f]\!]$.

(b) Any CMMQG $*$-algebra morphism φ of from any CMMQG $*$-algebra X to any CMMQG $*$-algebra Y, all of any common profile, is also a Hopf $*$-algebra morphism from the unique Hopf $*$-algebra of X in the sense of (a) to that of Y.

PROOF. (a) *Step 1: Construction of* (Δ,ϵ,S). Let Δ be the $*$-algebra morphism from $(A,m,1,*)$ to the tensor product $*$-algebra of $(A,m,1,*)$ with itself satisfying $\Delta(u^f_{k,i}) = \sum_{j=1}^{n_f} u^f_{k,j}\otimes u^f_{j,i}$ for any $\{i,k\}\subseteq [\![n_f]\!]$ and any $f\in N$, which is guaranteed to exist by condition (v) of Definition 2.27 (a) and which is already known to be unique with that property because of condition (iv) of that definition. For any $f\in N$ and any $\{i,\ell\}\subseteq [\![n_f]\!]$ the two vectors $((\Delta\otimes\mathrm{id}_A)\Delta)(u^f_{\ell,i}) = \sum_{j=1}^{n_f}(\Delta\otimes\mathrm{id}_A)(u^f_{\ell,j}\otimes u^f_{j,i}) = \sum_{k,j=1}^{n_f} u^f_{\ell,k}\otimes u^f_{k,j}\otimes u^f_{j,i}$ and $((\mathrm{id}_A\otimes\Delta)\Delta)(u^f_{\ell,i}) = \sum_{k=1}^{n_f}(\Delta\otimes\mathrm{id}_A)(u^f_{\ell,k}\otimes u^f_{k,i}) = \sum_{k,j=1}^{n_f} u^f_{\ell,k}\otimes u^f_{k,j}\otimes u^f_{j,i}$ coincide. Hence, $(\Delta\otimes\mathrm{id}_A)\Delta = (\mathrm{id}_A\otimes\Delta)\Delta$ by condition (iv) of Definition 2.27 (a). Hence, Definition 2.5 is applicable.

For any $f\in N$ condition (v) of Definition 2.27 (a) says that $(\mathrm{id}\otimes\Delta)(u^f) = u^f \oplus u^f$ and thus that u^f is a co-reprsentation of (A,Δ). Condition (ii) of that definition ensures that, in fact, u^f is a unitary co-representation of $(A,m,1,*,\Delta)$ for any $f\in N$.

For each $f\in N$ let t_f be any invertible element of $M_{n_f}(\mathbb{C})$ such that $x^f := ([(t_f)^{-1},t_f]\otimes *)(u^f)$ is a unitary element of the tensor product of $M_{n_f}(\mathbb{C})$ and $(A,m,1,*)$, whose existence we have assumed per condition (ii) of Definition 2.27 (a).

Because Δ is a $*$-algebra morphism we can conclude for any $f \in N$,

$$\begin{aligned}(\mathrm{id} \otimes \Delta)(x^f) &= ([(t_f)^{-1}, t_f] \otimes (\Delta *))(u^f) \\ &= ([(t_f)^{-1}, t_f] \otimes ((* \otimes *)\Delta))(u^f) \\ &= ([(t_f)^{-1}, t_f] \otimes (* \otimes *))(u^f \ominus u^f) \\ &= x^f \ominus x^f,\end{aligned}$$

which is to say that x^f is a co-representation of (A, Δ), a unitary one of $(A, m, 1, *, \Delta)$ even.

By condition (v) of Definition 2.27 (a) the $*$-algebra $(A, m, 1, *)$ is generated by the set $\{u^f_{j,i} \mid f \in N \wedge \{i, j\} \subseteq [\![n_f]\!]\}$. In other words, A is the vector subspace of A generated by the entries of the elements of the monoid generated by $\{u^f, (\mathrm{id} \otimes *)(u^f)\}_{f \in N}$ with respect to the operation $\cdot$ of forming the product co-representation. Since $(\mathrm{id} \otimes *)(u^f) = ([t_f, (t_f)^{-1}] \otimes \mathrm{id})(x^f)$, the entries of $(\mathrm{id} \otimes *)(u^f)$ and x^f generate the same vector subspace of A for any $f \in N$. Thus, A is generated as a vector space by the entries of the elements of the monoid M generated by $\{u^f, x^f\}_{f \in N}$ with respect to the co-representation product.

Because the trivial co-representation 1 on the Hilbert space $\mathbb{C}$ is unitary and because products of unitary co-representations are unitary again, M actually consists exclusively of unitary co-representations of $(A, m, 1, *)$.

Now, let $R_{\mathrm{irr}} = (v^r)_{r \in R_{\mathrm{irr}}}$ be any maximal set of pairwise inequivalent unitary co-representations of $(A, m, 1, *)$ and for each $r \in R_{\mathrm{irr}}$ let $(v^r_{q,p})_{(q,p) \in [\![m_r]\!]^{\otimes 2}}$ be the matrix of v^r with respect to any orthonormal basis of the carrier Hilbert space of v^r. By Lemma 2.6 (a) the set $\bigcup_{r \in R_{\mathrm{irr}}} \{v^r_{q,p}\}_{p,q=1}^{m_r}$ is linearly independent in A. Moreover, by Lemma 2.6 (b) any element of M, being a unitary co-representation, is equivalent to a direct sum of pairwise inequivalent irreducible co-representations of $(A, m, 1, *)$. By the transitivity of the notion of equivalence of co-representations and the maximality of R_{irr} any element of M is thus equivalent to a direct sum of elements of R_{irr}. It follows that $\bigcup_{r \in R_{\mathrm{irr}}} \{v^r_{q,p}\}_{p,q=1}^{m_r}$ is a linear basis of A.

Hence, we can define a linear functional ϵ on A and a linear endomorphism S of A by requiring $\epsilon(v^r_{q,p}) \coloneqq \delta_{q,p}$ and $S(v^r_{q,p}) \coloneqq (v^r_{p,q})^*$ for any $\{p, q\} \subseteq [\![m_r]\!]$ and $r \in R_{\mathrm{irr}}$.

Step 2: The tuple $(A, m, 1, *, \Delta, \epsilon, S)$ is a CQG Hopf $*$-algebra. Then, obviously, $((\epsilon \otimes \mathrm{id}_A)\Delta)(v^r_{q,p}) = \sum_{s=1}^{m_r} (\epsilon \otimes \mathrm{id}_A)(v^r_{q,s} \otimes v^r_{s,p}) = \sum_{s=1}^{m_r} \delta_{q,s} v^r_{s,p} = v^r_{q,p}$ and by a similar argument, $((\mathrm{id}_A \otimes \epsilon)\Delta)(v^r_{q,p}) = v^r_{q,p}$ for any $\{p, q\} \subseteq [\![m_r]\!]$ and $r \in R_{\mathrm{irr}}$, which is to say $(\epsilon \otimes \mathrm{id}_A)\Delta = \mathrm{id}_A = (\mathrm{id}_A \otimes \epsilon)\Delta$. Furthermore, $(m(S \otimes \mathrm{id}_A)\Delta)(v^r_{q,p}) = \sum_{s=1}^{m_r} (m(S \otimes \mathrm{id}_A))(v^r_{q,s} \otimes v^r_{s,p}) = \sum_{s=1}^{m_r} (v^r_{s,q})^* v^r_{s,p} = \delta_{q,p} 1 = \epsilon(v^r_{q,p}) 1$ because v^r is unitary and, similarly, $(m(S \otimes \mathrm{id}_A)\Delta)(v^r_{q,p}) = \epsilon(v^r_{q,p}) 1$ for any $\{p, q\} \subseteq [\![m_r]\!]$ and $r \in R_{\mathrm{irr}}$, i.e., $m(S \otimes \mathrm{id}_A)\Delta = 1\epsilon = m(\mathrm{id}_A \otimes S)\Delta$. Hence, $(A, m, 1, *, \Delta, \epsilon, S)$ is a Hopf $*$-algebra.

Even more precisely, $(A, m, 1, *, \Delta, \epsilon, S)$ is a Hopf $*$-algebra generated by the matrix coefficients of the unitary co-representations $\{u^f\}_{f \in N}$ for which $\{(\mathrm{id}_f \otimes *)(u^f)\}_{f \in N}$ are unitarizable. That makes it a CQG Hopf $*$-algebra by [Bic17, Theorem 1.11].

Step 3: The CQG Hopf $*$-algebra $(A, m, 1, *, \Delta, \epsilon, S)$ has the asserted properties. Already by definition, condition (1) is satisfied. The remaining properties are also satisfied because the co-unit and antipode of a Hopf $*$-algebra are uniquely determined by the co-multiplication and multiplication and must have the asserted properties for any unitary co-representations.

(b) Follows quickly from the definitions. □

2.5. Tannaka-Krein duality. Section 2.5 gives an account of Woronowicz's Tannaka-Krein duality theorem from [Wor88]. Arguably, Woronowicz actually proved two Tannaka-Krein theorems in [Wor88], one which is ideally suited to studying easy quantum groups and one which is invaluable for theoretical purposes. Both will be presented in a version that addresses functorial aspects omitted in [Wor88]. The second theorem will be used to prove the claim about full subcategory inclusions mentioned earlier.

2.5.1. *Rigid concrete monoidal W^*-categories.* For the presentation of Tannaka-Krein duality the same axiomatization as the one used by Woronowicz in [Wor88] has been chosen since it is very to-the-point and accessible without any prior knowledge of category theory. Consequently, rather than tensor C^*-categories with fiber functors the following will be the central objects.

DEFINITION 2.29. (a) A *concrete W^*-category* is any tuple $R \equiv (R, (H_r)_{r\in R}, (\mathrm{Mor}(r,r'))_{(r,r')\in R\otimes R})$ such that

(i) R is a set, the collection of *objects* of R,

(ii) H_r is a finite-dimensional Hilbert space, the *fiber* space of r, for each object $r \in R$,

(iii) $\mathrm{Mor}(r,r')$, the *morphism space* $r \to r'$, is a vector subspace of $[V_r, V_{r'}]$, where V_r and $V_{r'}$ are the underlying vector space of H_r respectively $H_{r'}$, for any $\{r,r'\} \subseteq R$,

(iv) $\mathrm{id}_{H_r} \in \mathrm{Mor}(r,r)$ for any $r \in R$.

(v) $a'a \in \mathrm{Mor}(r,r'')$ for any $a \in \mathrm{Mor}(r,r')$, any $a' \in \mathrm{Mor}(r',r'')$ and any objects $\{r,r',r''\} \subseteq R$.

(vi) $a^* \in \mathrm{Mor}(r',r)$ for any $a \in \mathrm{Mor}(r,r')$ and $\{r,r'\} \subseteq R$,

(vii) for any $\{r,r'\} \subseteq R$, whenever $H_r = H_{r'}$ and $\mathrm{id}_{H_r} \in \mathrm{Mor}(r,r')$, then $r = r'$.

(b) Given any concrete W^*-categories $R \equiv (R, (H_{R,r})_{r\in R}, (\mathrm{Mor}_R(r,r'))_{(r,r')\in R\otimes R})$ and $S \equiv (S, (H_{S,s})_{s\in S}, (\mathrm{Mor}_S(s,s'))_{(s,s')\in S\otimes S})$, a *strict concrete W^*-functor* from R to S is any mapping F from the object set of R to that of S such that

(i) $H_{R,r} = H_{S,F(r)}$ for any $r \in R$,

(ii) $\mathrm{Mor}_R(r,r') \subseteq \mathrm{Mor}_S(F(r),F(r'))$ for any $\{r,r'\} \subseteq R$.

(c) A *concrete monoidal W^*-category* is by definition any tuple $R = (R, (H_r)_{r\in R}, (\mathrm{Mor}(r,r'))_{(r,r')\in R\otimes R}, \cdot)$ such that

(i) $(R, (H_r)_{r\in R}, (\mathrm{Mor}(r,r'))_{(r,r')\in R\otimes R})$ is a concrete W^*-category, the *underlying concrete W^*-category* of R,

(ii) $\cdot$ is a binary operation on the object class of R, the *monoidal product*,

(iii) $H_{r_1\cdot r_2} = H_{r_1} \otimes H_{r_2}$ for any $\{r,r'\} \subseteq R$,

(iv) $a_1 \otimes a_2 \in \mathrm{Mor}(r_1 \cdot r_2, r_1' \cdot r_2')$ for any $a_1 \in \mathrm{Mor}(r_1, r_1')$, any $a_2 \in \mathrm{Mor}(r_2, r_2')$ and any $\{r_1, r_2, r_1', r_2'\} \subseteq R$,

(v) $(r_1 \cdot r_2) \cdot r_3 = r_1 \cdot (r_2 \cdot r_3)$ for any $\{r_1, r_2, r_3\} \subseteq R$,

(vi) There exists a (then uniquely determined) object 1 of R such that $H_1 = \mathbb{C}$ and $1 \cdot r = r \cdot 1 = r$ for any $r \in R$, the *monoidal unit* of R.

(d) Whenever the tuples $R \equiv (R, (H_{R,r})_{r \in R}, (\mathrm{Mor}_R(r, r'))_{(r,r') \in R \otimes R}, \cdot_R)$ and $S \equiv (S, (H_{S,s})_{s \in S}, (\mathrm{Mor}_S(s, s'))_{(s,s') \in S \otimes S}, \cdot_S)$ are any two concrete monoidal W^*-categories with monoidal units 1_R and 1_S, respectively, a *strict concrete strict monoidal W^*-functor* from R to S is any F such that

(i) F is a strict concrete W^*-functor from the underlying concrete W^*-category of R to that of S,

(ii) $F(r_1 \cdot_R r_2) = F(r_1) \cdot_S F(r_2)$ for any $\{r_1, r_2\} \subseteq R$,

(iii) $F(1_R) = 1_S$, where 1_R and 1_S are the monoidal units of R and S, respectively.

(e) The *composition* of two strict concrete W^*-functors or strict concrete strict monoidal W^*-functors is simply the composition of the underlying mappings between the object sets. The *identity* on a concrete W^*-category or concrete monoidal W^*-category is the identity mapping on the set of objects.

(f) In any concrete monoidal W^*-category $(R, (H_r)_{r \in R}, (\mathrm{Mor}(r, r'))_{(r,r') \in R \otimes R}, \cdot)$ for any object $r \in R$ any object $\overline{r} \in R$ is called a *complex conjugate* of r if there exists an invertible linear map j from the underlying vector space of H_r to that of $\mathrm{cj}(H_{\overline{r}})$ such that $t_j \in \mathrm{Mor}(1, r \cdot \overline{r})$ and $\overline{t}_j \in \mathrm{Mor}(\overline{r} \cdot r, 1)$. A concrete monoidal W^*-category with a complex conjugate for each of its objects is said to be *rigid*.

2.5.2. *Representation theory of an algebraic compact quantum group.* Already in [Wor87], Woronowicz had presented a way of associating with any compact quantum group a rigid concrete monoidal W^*-algebra, a fact he acknowledges in [Wor88, Theorem 1.3]. The construction is as follows.

DEFINITION 2.30. (a) For any CQG Hopf $*$-algebra $(A, m, 1, *, \Delta, \epsilon, S)$ we call the tuple $(R, (H_r)_r, \mathrm{Mor}(r, r')_{(r,r') \in R \otimes R}, \cdot)$, where

(i) R is the set of all unitary co-representations of $(A, m, 1, *)$ on any finite-dimensional Hilbert space,

(ii) H_r is the carrier Hilbert space of r for each $r \in R$,

(iii) $\mathrm{Mor}(r, r')$ is the vector subspace of $[V_r, V_{r'}]$, where V_r and $V_{r'}$ are the vector spaces underlying H_r respectively $H_{r'}$, formed by the set of all intertwiners $r \to r'$ of (A, Δ) for any $(r, r') \in R \otimes R$,

(iv) $\cdot$ is the binary operation on R of forming the product co-representation $r_1 \cdot r_2$ of $(A, m, 1, \Delta)$ for any $(r_1, r_2) \in R \otimes R$,

the *co-representation theory* $\mathsf{C}(A, m, 1, *, \Delta, \epsilon, S)$ of $(A, m, 1, *, \Delta, \epsilon, S)$ and the *representation theory* of the formal dual of $(A, m, 1, *, \Delta, \epsilon, S)$.

(b) Given any morphism ψ from any CQG Hopf $*$-algebra $(A, m, 1, *, \Delta, \epsilon, S)$ to any CQG Hopf $*$-algebra $(A', m', 1', *', \Delta', \epsilon', S')$, if R and R' are the sets of all unitary co-representations of $(A, m, 1, *, \Delta)$ and $(A', m', 1', *', \Delta')$, respectively, on any finite-dimensional Hilbert spaces, then we call the mapping $R \to R'$ with $u^r \mapsto (\mathrm{id} \otimes \psi)(u^r)$ for any $u^r = r \in R$ the *co-representation theory* $\mathsf{C}(\psi)$ of ψ. We also say that $\mathsf{C}(\psi)$ is the *representation theory* of the formal dual of ψ.

C is a functor from the category of CQG Hopf $*$-algebras to the category of concrete monoidal W^*-categories and strict concrete monoidal W^*-functors.

PROPOSITION 2.31. *(a)* $\mathsf{C}(H)$ *is a rigid concrete monoidal* W^**-category for any CQG Hopf* $*$*-algebra.*

(b) $\mathsf{C}(\psi)$ *is a strict concrete monoidal* W^**-functor* $\mathsf{C}(H) \to \mathsf{C}(H')$ *for any morphism* $\psi\colon H \to H'$ *of CQG Hopf* $*$*-algebras and any CQG Hopf* $*$*-algebras* H *and* H'*.*

(c) $\mathsf{C}(\psi'\psi) = \mathsf{C}(\psi')\mathsf{C}(\psi)$ *for any morphisms* $\psi\colon H \to H'$ *and* $\psi'\colon H' \to H''$ *of CQG Hopf* $*$*-algebras and any CQG Hopf* $*$*-algebras* H*,* H' *and* H''*.*

(d) $\mathsf{C}(\mathrm{id}_H) = \mathrm{id}_{\mathsf{C}(H)}$ *for any CQG Hopf* $*$*-algebra* H*.*

The functor C is the identical right side to both Tannaka-Krein theorems contained in [Wor88]. It follows the two corresponding left sides.

2.5.3. *Tannaka-Krein representee.* The original article [Wor88] gives two equivalent ways of recovering a compact matrix quantum group from certain rigid monoidal W^*-category. They generalize to arbitrary compact quantum groups. Both have different advantages and disadvantages. The one that is particularly well-suited to easy quantum groups is the underlying CQG Hopf $*$-algebra of what Woronowicz might have called the "universal model".

In [Wor88], as in other discussions of Tannaka-Krein duality (see [NT13] and [MRT04]), the Tannaka dual of a rigid monoidal W^*-category is only essentially unique. However, for the purposes of this chapter, proper uniqueness would be more convenient. To achieve this, the following – admittedly strong – assumption is made hereafter.

ASSUMPTION 2.32. In the following, for any finite-dimensional Hilbert space H fix an orthonormal basis $(e_i^H)_{i=1}^{n_H}$.

The first left side to C, the one that is used for easy quantum groups, can then be introduced as explained below.

DEFINITION 2.33. (a) For any given rigid concrete monoidal W^*-category $R \equiv (R, (H_r)_{r\in R}, (\mathrm{Mor}(r, r'))_{(r,r')\in R\otimes R}, \cdot)$, if $n_r = \dim_{\mathbb{C}}(H_r)$ for each $r \in R$, then we call the tuple $(A, m, 1, *, \Delta, \epsilon, S)$, where

(i) $(A, m, 1, *)$ is the universal $*$-algebra over $\{u^r_{j,i} \,|\, r \in R \wedge \{i, j\} \subseteq [\![n_r]\!]\}$, where $u^r_{j,i}$ is short for (r, j, i) for any $\{i, j\} \subseteq [\![n_r]\!]$ and $r \in R$, subject to the relations

$$\{\textstyle\sum_{j=1}^{n_r} u^r_{k,j}(u^r_{i,j})^* - \delta_{k,i}1,\ \sum_{j=1}^{n_r}(u^r_{k,j})^* u^r_{i,j} - \delta_{k,i}1 \\ \qquad |\, r \in R \wedge \{i, k\} \subseteq [\![n_r]\!]\},$$

and the relations

$$\Big\{\textstyle\sum_{i=1}^{n_{r_1\cdot r_2}} w^{r_1,r_2}_{i,(g_1,g_2)}\, u^{r_1\cdot r_2}_{j,i} - \sum_{h_1=1}^{n_{r_1}} \sum_{h_2=1}^{n_{r_2}} w^{r_1,r_2}_{j,(h_1,h_2)}\, u^{r_1}_{h_1,g_1} u^{r_2}_{h_2,g_2} \\ \mid \{r_1,r_2\} \subseteq R \wedge g_1 \in [\![n_{r_1}]\!] \wedge g_2 \in [\![n_{r_2}]\!] \wedge j \in [\![n_{r_1\cdot r_2}]\!]\Big\},$$

where for any $\{r_1,r_2\} \subseteq R$, any $i_1 \in [\![n_{r_1}]\!]$, any $i_2 \in [\![n_{i_2}]\!]$ and $j \in [\![n_{r_1\cdot r_2}]\!]$,

$$w^{r_1,r_2}_{j,(i_1,i_2)} = \langle e^{H_{r_1\cdot r_2}}_j \mid e^{H_{r_1}}_{i_1} \otimes e^{H_{r_2}}_{i_2} \rangle_{H_{r_1\cdot r_2}},$$

as well as the relations

$$\Big\{\textstyle\sum_{i=1}^{n_{r'}} t_{i,g}\, u^{r'}_{j,i} - \sum_{h=1}^{n_r} t_{j,h}\, u^{r}_{h,g} \big| \{r,r'\} \subseteq R \wedge t \in \mathrm{Mor}(r,r') \\ \wedge\, g \in [\![n_r]\!] \wedge j \in [\![n_{r'}]\!]\Big\},$$

where for any $\{r,r'\} \subseteq R$, any $t \in \mathrm{Mor}(r,r')$, any $g \in [\![n_r]\!]$ and $j \in [\![n_{r'}]\!]$,

$$t_{j,g} = \langle e^{H_{r'}}_j \mid t(e^{H_r}_g) \rangle_{H_{r'}},$$

(ii) Δ is the unique $*$-algebra morphism from $(A,m,1,*)$ to the tensor product $*$-algebra of $(A,m,1,*)$ with itself such that for any $r \in R$ and any $\{i,k\} \subseteq [\![n_r]\!]$,

$$u^r_{k,i} \mapsto \textstyle\sum_{j=1}^{n_r} u^r_{k,j} \otimes u^r_{j,i},$$

(iii) ϵ is the unique $*$-algebra morphism from $(A,m,1,*)$ to $\mathbb{C}$ with for any $r \in R$ and any $\{i,j\} \subseteq [\![n_r]\!]$,

$$u^r_{j,i} \mapsto \delta_{j,i},$$

(iv) S is the unique algebra morphism from $(A,m,1)$ to its own opposite algebra such that for any $r \in R$, if $\overline{r}$ is any arbitrary complex conjugate of r in R via any j_r, then for any $\{i,j\} \subseteq [\![n_r]\!]$,

$$u^r_{j,i} \mapsto (u^r_{j,i})^* \quad \wedge \quad (u^r_{j,i})^* \mapsto \textstyle\sum_{k,\ell,r,s=1}^{n_r} a^r_{k,s} a^r_{k,i} b^r_{j,\ell} b^r_{r,\ell}\, u^r_{s,r},$$

where for any $\{i,j\} \subseteq [\![n_r]\!]$,

$$a^r_{j,i} = \langle e^{H_{\overline{r}}}_j \mid j_r(e^{H_r}_i) \rangle_{H_{\overline{r}}} \quad \wedge \quad b^r_{j,i} = \langle j_r^{-1}(e^{H_{\overline{r}}}_i) \mid e^{H_r}_j \rangle_{H_r},$$

the *Tannaka-Krein co-representee* $\mathsf{T}(R)$ of R. We also say that the formal dual of $\mathsf{T}(R)$ is the *Tannaka-Krein representee* of $\mathsf{T}(R)$.

(b) Given any rigid concrete monoidal W^*-categories R and S and any strict concrete strict monoidal W^*-functor F from R to S, we call the unique morphism of $*$-algebras from the underlying $*$-algebra of $\mathsf{T}(R)$ to that of $\mathsf{T}(S)$ with for any object r of R and any $\{i,j\} \subseteq [\![n_r]\!]$,

$$u^r_{j,i} \mapsto v^{F(r)}_{j,i}$$

where $u^r_{j,i}$ is short for (r,j,i) and $v^{F(r)}_{j,i}$ short for $(F(r),j,i)$, and where n_r is the dimension of the fiber space of r in R, the *Tannaka-Krein co-representee* $\mathsf{T}(F)$ of F. Analogously, we say that the formal dual of $\mathsf{T}(F)$ is the *Tannaka-Krein representee* of F.

That is well-defined because the underlying $*$-algebra of the Tannaka-Krein co-representee of a rigid concrete monoidal W^*-algebra together with the matrices of the $*$-algebra generators form a CMMQG $*$-algebra. Upon checking this we may thus apply Proposition 2.28 to infer that we have indeed constructed a CQG Hopf $*$-algebra.

T is a functor from the category of rigid monoidal W^*-categories and strict concrete strict monoidal W^*-functors to the categoy of CQG Hopf $*$-algebras.

PROPOSITION 2.34. *(a)* $\mathsf{T}(R)$ *is a CQG Hopf* $*$*-algebra for any rigid monoidal* W^**-category* R.

(b) $\mathsf{T}(F)$ *is a morphism of CQG Hopf* $*$*-algebras* $\mathsf{T}(R) \to \mathsf{T}(R')$ *for any strict concrete strict monoidal* W^**-functor* $F\colon R \to R'$ *and any rigid concrete monoidal* W^**-categories* R *and* R'.

(c) $\mathsf{T}(F' \circ F) = \mathsf{T}(F') \circ \mathsf{T}(F)$ *for any strict concrete strict monoidal* W^**-functors* $F\colon R \to R'$ *and* $F'\colon R' \to R''$ *and any rigid concrete monoidal* W^**-categories* R, R' *and* R''.

(d) $\mathsf{T}(\mathrm{id}_R) = \mathrm{id}_{\mathsf{T}(R)}$ *for any rigid concrete monoidal* W^**-category* R.

However, particularly for easy quantum groups, there is a way of obtaining the Tannaka-Krein co-representee (or rather an isomorphic CQG Hopf $*$-algebra) by a construction involving a universal $*$-algebra on far fewer generators. Again, this was shown by Woronowicz in [Wor88] for the case of compact matrix quantum groups. The next definition is, of course, independent of our fixing orthonormal bases above.

DEFINITION 2.35. (a) Let $R \equiv (R, (H_r)_{r\in R}, (\mathrm{Mor}(r,r'))_{(r,r')\in R\otimes R})$ be any concrete W^*-category.

(i) We call any two objects r and r' of R *equivalent* in R if $\mathrm{Mor}(r,r')$ contains a linear map that is a unitary operator $H_r \to H_{r'}$.

(ii) Any object r of R is called a *subobject* of any object r' in R if $\mathrm{Mor}(r,r')$ contains a linear map that is a partial isometry $p\colon H_r \to H_{r'}$ with $p^*p = \mathrm{id}_{H_r}$.

(iii) We say that any object r of R is a *direct sum* of $(r_1, r_2, \ldots, r_m)$ in R for any $m \in \mathbb{N}$ and any objects $\{r_i\}_{i=1}^m$ of R if there exist $\{p_i\}_{i=1}^m$ such that $p_i \in \mathrm{Mor}(r_i, r)$ and p_i is a partial isometry $H_{r_i} \to H_r$ and $p_i^* p_i = \mathrm{id}_{H_i}$ for each $i \in [\![m]\!]$ and such that $\sum_{i=1}^m p_i p_i^* = \mathrm{id}_{H_r}$.

(b) Any (not necessarily finite) set Q of objects of any given concrete monoidal W^*-category $R \equiv (R, (H_r)_{r\in R}, (\mathrm{Mor}(r,r'))_{(r,r')\in R\otimes R}, \cdot)$ is said to *generate* R if any object of R is equivalent to a subobject of a finite direct sum of objects contained in the submonoid of $(R, \cdot)$ generated by Q.

The following construction depends not only on the fixed bases but also the choice of generator set.

PROPOSITION 2.36. *(a) For any given rigid concrete monoidal* W^**-category* $R \equiv (R, (H_r)_{r\in R}, (\mathrm{Mor}(r,r'))_{(r,r')\in R\otimes R}, \cdot)$ *and any set* N *of objects of* R *for* *which there exists* $(\overline{f}, j_f)_{f\in N}$ *such that* $\overline{f}$ *is a complex conjugate of* f *in* R *via* j_f *for each* $f \in N$ *and such that* $\{f, \overline{f}\}_{f\in N}$ *generates* R*, if* $n_r = \dim_{\mathbb{C}}(H_r)$ *for each* $r \in \{f, \overline{f}\}_{f\in N}$*, then the tuple* $(A, m, 1, *, \Delta, \epsilon, S)$*, where*

(i) $(A, m, 1, *)$ *is the universal* $*$*-algebra over* $\{u^f_{j,i} \,|\, f \in N \wedge \{i, j\} \subseteq [\![n_f]\!]\}$*, where* $u^f_{j,i}$ *is short for* (f, j, i) *for any* $f \in N$ *and* $\{i, j\} \subseteq [\![n_f]\!]$*, subject to the relations*

$$\{\textstyle\sum_{j=1}^{n_r} u^r_{k,j}(u^r_{i,j})^* - \delta_{k,i}1,\ \sum_{j=1}^{n_r}(u^r_{k,j})^* u^r_{i,j} - \delta_{k,i}1 \\ |\, r \in \{f, \overline{f}\}_{f\in N} \wedge \{i, k\} \subseteq [\![n_r]\!]\},$$

and the relations

$$\Big\{\textstyle\sum_{i=1}^{n_{r_1\cdot r_2}} w^{r_1,r_2}_{i,(g_1,g_2)}\, u^{r_1\cdot r_2}_{j,i} - \sum_{h_1=1}^{n_{r_1}} \sum_{h_2=1}^{n_{r_2}} w^{r_1,r_2}_{j,(h_1,h_2)}\, u^{r_1}_{h_1,g_1} u^{r_2}_{h_2,g_2} \\ |\, \{r_1, r_2\} \subseteq \{f, \overline{f}\}_{f\in N} \wedge r_1 \cdot r_2 \in \{f, \overline{f}\}_{f\in N} \\ \wedge\, g_1 \in [\![n_{r_1}]\!] \wedge g_2 \in [\![n_{r_2}]\!] \wedge j \in [\![n_{r_1\cdot r_2}]\!]\Big\},$$

as well as the relations

$$\left\{\sum_{i_1=1}^{n_{q_1}} \cdots \sum_{i_\ell=1}^{n_{q_\ell}} t_{(i_1,\ldots,i_\ell),(g_1,\ldots,g_k)}\, u^{q_1^{\theta_1}}_{j_1,i_1} \cdots u^{q_\ell^{\theta_\ell}}_{j_\ell,i_\ell} \right. \\ - \sum_{h_1=1}^{n_{p_1}} \cdots \sum_{h_k=1}^{n_{p_k}} t_{(j_1,\ldots,j_\ell),(h_1,\ldots,h_k)}\, u^{p_1^{\eta_1}}_{h_1,g_1} \cdots u^{p_k^{\eta_k}}_{h_k,g_k} \Bigg| \\ \{k, \ell\} \subseteq \mathbb{N}_0 \wedge \{p_a\}_{a=1}^k \cup \{q_b\}_{b=1}^\ell \subseteq N \wedge \{\eta_a\}_{a=1}^k \cup \{\theta_b\}_{b=1}^\ell \subseteq \{\circ, \bullet\} \\ \wedge\, t \in \mathrm{Mor}(p_1^{\eta_1} \cdot \ldots \cdot p_k^{\eta_k}, q_1^{\theta_1} \cdot \ldots \cdot q_\ell^{\theta_\ell}) \\ \left. \wedge\, (\forall_{a=1}^k\colon g_a \in [\![n_{p_a}]\!]) \wedge (\forall_{b=1}^\ell\colon j_b \in [\![n_{q_b}]\!]) \right\},$$

where for any $f \in N$ *first* $f^\circ \equiv f$ *and* $f^\bullet \equiv \overline{f}$ *and for any* $\{i, j\} \subseteq [\![n_f]\!]$*, if*

$$a^f_{j,i} = \langle e^{H_{\overline{f}}}_j \mid j_f(e^{H_f}_i)\rangle_{H_{\overline{f}}} \quad \wedge \quad b^f_{j,i} = \langle j_f^{-1}(e^{H_{\overline{f}}}_i) \mid e^{H_f}_j\rangle_{H_f},$$

then

$$u^{\overline{f}}_{j,i} \equiv \textstyle\sum_{k,\ell=1}^{n_f} a^f_{j,\ell} b^f_{k,i}\,(u^f_{\ell,k})^*,$$

where for any $\{r_1, r_2\} \subseteq \{f, \overline{f}\}_{f\in N}$ *with* $r_1 \cdot r_2 \in \{f, \overline{f}\}_{f\in N}$*, any* $i_1 \in [\![n_{r_1}]\!]$*, any* $i_2 \in [\![n_{i_2}]\!]$ *and* $j \in [\![n_{r_1\cdot r_2}]\!]$*,*

$$w^{r_1,r_2}_{j,(i_1,i_2)} = \langle e^{H_{r_1\cdot r_2}}_j \mid e^{H_{r_1}}_{i_1} \otimes e^{H_{r_2}}_{i_2}\rangle_{H_{r_1\cdot r_2}},$$

where for any $\{k, \ell\} \subseteq \mathbb{N}_0$ *and* $\{p_a\}_{a=1}^k \subseteq N$ *and* $\{q_b\}_{b=1}^\ell \subseteq N$ *and* $\{\eta_a\}_{a=1}^k \subseteq \{\circ, \bullet\}$ *and* $\{\theta_b\}_{b=1}^\ell \subseteq \{\circ, \bullet\}$ *and* $t \in \mathrm{Mor}(p_1^{\eta_1} \cdot \ldots \cdot p_k^{\eta_k}, q_1^{\theta_1} \cdot \ldots \cdot q_\ell^{\theta_\ell})$

and any $\{i_a\}_{a=1}^{k}$ *and* $\{j_b\}_{b=1}^{\ell}$ *such that* $i_a \in [\![n_{p_a}]\!]$ *for any* $a \in [\![k]\!]$ *and* $j_b \in [\![n_{q_b}]\!]$ *for any* $b \in [\![\ell]\!]$,

$$t_{(j_1,\ldots,j_\ell),(i_1,\ldots,i_k)} = \langle t(e_{i_1}^{H_{p_1}} \otimes \ldots \otimes e_{i_k}^{H_{p_k}}) \mid e_{j_1}^{H_{q_1}} \otimes \ldots \otimes e_{j_\ell}^{H_{q_\ell}} \rangle_{H_{q_1^{\theta_1} \ldots q_\ell^{\theta_\ell}}},$$

(ii) Δ *is the unique* $*$*-algebra morphism from* $(A, m, 1, *)$ *to the tensor product* $*$*-algebra of* $(A, m, 1, *)$ *with itself such that for any* $\{i, k\} \subseteq [\![n_f]\!]$ *and any* $f \in N$,

$$u_{k,i}^f \mapsto \textstyle\sum_{j=1}^{n_f} u_{k,j}^f \otimes u_{j,i}^f,$$

(iii) ϵ *is the unique* $*$*-algebra morphism from* $(A, m, 1, *)$ *to* $\mathbb{C}$ *such that for any* $\{i, j\} \subseteq [\![n_f]\!]$ *and any* $f \in N$,

$$u_{j,i}^f \mapsto \delta_{j,i},$$

(iv) S *is the unique algebra morphism from* $(A, m, 1)$ *to its own opposite algebra such that for any* $\{i, j\} \subseteq [\![n_f]\!]$ *and any* $f \in N$,

$$u_{j,i}^f \mapsto (u_{j,i}^f)^* \quad \wedge \quad (u_{j,i}^f)^* \mapsto \textstyle\sum_{k,\ell,r,s=1}^{n_f} a_{k,s}^f a_{k,i}^f b_{j,\ell}^f b_{r,\ell}^f u_{s,r}^f$$

is CQG Hopf $*$*-algebra and the rule* $(f, j, i) \mapsto (f, j, i)$ *for any* $\{i, j\} \subseteq [\![n_f]\!]$ *and* $f \in N$ *defines an isomorphism of CQG Hopf* $*$*-algebras from* $(A, m, 1, *, \Delta, \epsilon, S)$ *to* $\mathsf{T}(R)$.

(b) Given any two rigid concrete monoidal W^**-categories* $R \equiv (R, (H_{R,r})_{r \in R}, (\mathrm{Mor}_R(r, r'))_{(r,r') \in R \otimes R}, \cdot_R)$ *and* $S \equiv (S, (H_{S,s})_{s \in S}, (\mathrm{Mor}_S(s, s'))_{(s,s') \in S \otimes S}, \cdot_S)$, *any strict concrete strict monoidal* W^**-functor* F *from* R *to* S, *and any sets* N *and* P *of objects of* R *respectively* S *such that there exist* $(\overline{f}, j^{R,f})_{f \in N}$ *and* $(\overline{g}, j^{S,g})_{g \in P}$ *such that* $\overline{f}$ *is a complex conjugate of* f *in* R *via* $j^{R,f}$ *for each* $f \in N$ *and* $\overline{g}$ *one of* g *in* S *via* $j^{S,g}$ *for each* $g \in P$, *and such that* $\{f, \overline{f}\}_{f \in N}$ *generates* R *and* $\{g, \overline{g}\}_{g \in P}$ *generates* S, *if* $n_f = \dim_{\mathbb{C}}(H_{R,f})$ *for each* $f \in N$, *if* $m_g = \dim_{\mathbb{C}}(H_{S,g})$ *for each* $g \in P$, *if* $u_{j,i}^f$ *is short for* (f, j, i) *for any* $f \in N$ *and* $\{i, j\} \subseteq [\![n_f]\!]$, *and if* $v_{y,x}^g$ *is short for* (g, y, x) *for any* $g \in P$ *and* $\{x, y\} \subseteq [\![m_g]\!]$, *then the morphism of CQG Hopf* $*$*-algebras* φ *from the CQG Hopf* $*$*-algebra of* R *constructed in (a) with respect to* N *to that of* S *constructed with respect to* P *such that for any* $f \in N$ *and any* $\{i, j\} \subseteq [\![n_f]\!]$, *if* $F(f) \neq 1$, *then*

$$u_{j,i}^f \mapsto \sum_{a=1}^{k} \sum_{x_1^a, y_1^a = 1}^{m_{g_{a,1}}} \cdots \sum_{x_{\ell_a}^a, y_{\ell_a}^a = 1}^{m_{g_{a,\ell_a}}} p_{j,(y_1^a, \ldots, y_{\ell_a}^a)}^a \overline{p_{i,(x_1^a, \ldots, x_{\ell_a}^a)}^a} \, v_{y_1^a, x_1^a}^{g_{a,1}^{\theta_{a,1}}} \cdots v_{y_{\ell_a}^a, x_{\ell_a}^a}^{g_{a,\ell_a}^{\theta_{a,\ell_a}}},$$

where, as before, $g^\circ = g$ *and* $g^\bullet = \overline{g}$ *for any* $g \in P$, *where* $(k, (s_a)_{a=1}^k, (p_a)_{a=1}^k)$ *and* $(\ell_a, (g_{a,b_a})_{b_a=1}^{\ell_a}, (\theta_{a,b_a})_{b_a=1}^{\ell_a})$ *for any* $a \in [\![k]\!]$ *are such that* $k \in \mathbb{N}$, *such that* $s_a \in S$ *and* $p_a \in \mathrm{Mor}_S(s_a, F(f))$ *for each* $a \in [\![k]\!]$, *such that* $\sum_{a=1}^k p_a p_a^* = \mathrm{id}_{H_{R,f}}$, *and such that* $\ell_a \in \mathbb{N}$ *and* $\{g_{a,b_a}\}_{b_a=1}^{\ell_a} \subseteq P$ *and* $\{\theta_{a,b_a}\}_{b_a=1}^{\ell_a} \subseteq \{\circ, \bullet\}$ *and* $s_a = g_{a,b_1}^{\theta_{a,b_1}} \cdot_S \ldots \cdot_S g_{a,b_{\ell_a}}^{\theta_{a,b_{\ell_a}}}$ *for each* $a \in [\![k]\!]$, *where for any* $a \in [\![k]\!]$ *and any*

$(z^a_{b_a})_{a=1}^{\ell_a}$ *such that* $z^a_{b_a} \in [\![m_{g_{a,b_a}}]\!]$ *for each* $b_a \in [\![\ell_a]\!]$,

$$p^a_{i,(z^a_1,\dots,z^a_{\ell_a})} = \langle e_i^{H_{R,f}} \mid p_a(e_{z^a_1}^{H_{S,g_{a,1}}^{\theta_{a,1}}} \otimes \dots \otimes e_{z^a_{\ell_a}}^{H_{S,g_{a,\ell_a}}^{\theta_{a,\ell_a}}}) \rangle_{H_{R,f}},$$

satisfies $\omega_S \varphi \omega_R^{-1} = \mathsf{T}(F)$, *where* ω_R *and* ω_S *are the isomorphisms from (a) for* R *respectively* S.

Again, that the construction in Proposition 2.36 (a) yields a CQG Hopf $*$-algebra can be verified using Proposition 2.28.

2.5.4. *Relationship between representation theory and Tannaka-Krein representee.* In a sense, forming the Tannaka-Krein representee of a rigid concrete monoidal W^*-category is a left inverse to the operation of taking the representation category of an algebraic compact quantum group. However, it is not a two-sided inverse. In detail, the relationship between the two assignments is as follows. Assumption 2.32 is still in effect.

DEFINITION 2.37. (a) For any CQG Hopf $*$-algebra $H = (A, m, 1, *, \Delta, \epsilon, S)$, if R is the object set of $\mathsf{C}(H)$, let $\mathrm{cu}_H^{\mathsf{T}\dashv\mathsf{C}}$ be the unique $*$-algebra morphism from the underlying $*$-algebra of $\mathsf{T}(\mathsf{C}(H))$ to $(A, m, 1, *)$ such that for any $r \in R$ and any $\{i, j\} \subseteq [\![n_r]\!]$,

$$(r, j, i) \mapsto v^r_{j,i},$$

where n_r is the dimension of the fiber of r in $\mathsf{C}(H)$ and where $(v^r_{j,i})_{(j,i)\in[\![n_r]\!]^{\otimes 2}}$ is the matrix of the co-representation r of (A, Δ) with respect to $(e_i^{H_r})_{i=1}^{n_r}$.

(b) Conversely, given any rigid concrete monoidal W^*-category $R \equiv (R, (H_r)_{r\in R}, (\mathrm{Mor}(r, r'))_{(r,r')\in R\otimes R}, \cdot)$, if $(A, m, 1, *, \Delta, \epsilon, S)$ is given by $\mathsf{T}(R)$, let $\mathrm{un}_R^{\mathsf{T}\dashv\mathsf{C}}$ be the the mapping from R to the set of all unitary co-representations of $(A, m, 1, *, \Delta)$ such that for any $r \in R$,

$$r \mapsto u^r$$

where, if $n_r = \dim_{\mathbb{C}}(H_r)$, if $(E^{H_r}_{j,i})_{(j,i)\in[\![n_r]\!]^{\otimes 2}}$ are the matrix units of $(e_i^{H_r})_{i\in[\![n_r]\!]}$ and if $u^r_{j,i}$ is short for the element (r, j, i) of A for any $\{i, j\} \subseteq [\![r]\!]$, then

$$u^r = \textstyle\sum_{i,j=1}^{n_f} E^{H_r}_{j,i} \otimes u^r_{j,i}.$$

The following is a variation on [Wor88, Theorem 1.3]. $\mathrm{cu}^{\mathsf{T}\dashv\mathsf{C}}$ and $\mathrm{un}^{\mathsf{T}\dashv\mathsf{C}}$ are the co-unit and unit, respectively, of an adjunction $\mathsf{T} \dashv \mathsf{C}$. Moreover, $\mathrm{cu}^{\mathsf{T}\dashv\mathsf{C}}$ is a natural isomorphism.

PROPOSITION 2.38. *(a)* $\mathrm{cu}_H^{\mathsf{T}\dashv\mathsf{C}}$ *is an isomorphism* $\mathsf{T}(\mathsf{C}(H)) \to H$ *of CQG Hopf* $*$*-algebras for any CQG Hopf* $*$*-algebra* H.

(b) $\mathrm{un}_R^{\mathsf{T}\dashv\mathsf{C}}$ *is a strict concrete strict monoidal* W^**-functor* $R \to \mathsf{C}(\mathsf{T}(R))$ *for any rigid concrete strict monoidal* W^**-category* R.

(c) $\mathrm{cu}_{H'}^{\mathsf{T}\dashv\mathsf{C}} \circ \mathsf{T}(\mathsf{C}(\psi)) = \psi \circ \mathrm{cu}_H^{\mathsf{T}\dashv\mathsf{C}}$ *for any morphism* $\psi\colon H \to H'$ *of CQG Hopf* $*$*-algebras and any CQG Hopf* $*$*-algebras* H *and* H'.

(d) $\mathrm{un}_{R'}^{\mathsf{T}\dashv\mathsf{C}} \circ F = \mathsf{C}(\mathsf{T}(F)) \circ \mathrm{un}_{R}^{\mathsf{T}\dashv\mathsf{C}}$ *for any strict concrete strict monoidal* W^**-functor* $F\colon R \to R'$ *and rigid concrete monoidal* W^**-categories* R *and* R'.

(e) $\mathrm{id}_{\mathsf{T}(R)} = \mathrm{cu}_{\mathsf{T}(R)}^{\mathsf{T}\dashv\mathsf{C}} \circ \mathsf{T}(\mathrm{un}_{R}^{\mathsf{T}\dashv\mathsf{C}})$ *for rigid concrete monoidal* W^**-category.*

(f) $\mathrm{id}_{\mathsf{C}(H)} = \mathsf{C}(\mathrm{cu}_{H}^{\mathsf{T}\dashv\mathsf{C}}) \circ \mathrm{un}_{\mathsf{C}(H)}^{\mathsf{T}\dashv\mathsf{C}}$ *for any CQG Hopf* $*$*-algebra* H.

While the co-unit of the adjunction $\mathsf{T} \dashv \mathsf{C}$ is a natural isomorphism, the unit is not. Answering the question for which rigid concrete monoidal W^*-categories the unit is invertible is tantamount to giving the second method of obtaining an algebraic compact quantum group from a rigid concrete monoidal W^*-category described in [Wor88].

2.5.5. *Completion of rigid concrete monoidal* W^**-categories.* This second construction factors through a third, namely the operation of forming the Cauchy completion, or completion, for short. As mentioned this crucial construction was already used by Woronowicz in [Wor88]. Our fixing bases has no import on the following definitions and results.

Definition 2.39. We say that any given concrete W^*-category $R \equiv (R, (H_r)_{r\in R}, (\mathrm{Mor}(r,r'))_{(r,r')\in R\otimes R})$ is *complete* if the following conditions are satisfied:

(a) For any $r \in R$, any finite-dimensional Hilbert space H and any unitary operator $v\colon H_r \to H$ there exists $r' \in R$ such that $H_{r'} = H$ and such that r and r' are equivalent in R.

(b) For any $r' \in R$ and any orthogonal projection $p \in B(H_{r'})$ with $p \in \mathrm{Mor}(r',r')$ there exists $r \in R$ such that H_r is the image Hilbert space of p and such that r is a subobject of r' in R.

(c) For any $m \in \mathbb{N}$ and any objects $\{r_i\}_{i=1}^m$ there exists $r \in R$ such that r is a direct sum of $(r_1, r_2, \ldots, r_m)$ in R.

If R is even a concrete monoidal W^*-category, we call R *complete* if the same is true of the underlying concrete W^*-category of R.

For LaTeX reasons a different symbol is used in the next definition in place of the double tilde accent employed in [Wor88, Proposition 2.7].

Definition 2.40. (a) The *completion* $\mathsf{D}(R)$ of any concrete monoidal W^*-category $R \equiv (R, (H_{R,r})_{r\in R}, (\mathrm{Mor}_R(r,r'))_{(r,r')\in R\otimes R}, \cdot_R)$ is the tuple

$$\widehat{R} = (\widehat{R}, (H_{\widehat{R},\widehat{r}})_{\widehat{r}\in\widehat{R}}, (\mathrm{Mor}_{\widehat{R}}(\widehat{r},\widehat{r}'))_{(\widehat{r},\widehat{r}')\in\widehat{R}\otimes\widehat{R}}, \cdot_{\widehat{R}})$$

such that, if

$$\widetilde{R} \equiv (\widetilde{R}, (H_{\widetilde{R},\widetilde{r}})_{\widetilde{r}\in\widetilde{R}}, (\mathrm{Mor}_{\widetilde{R}}(\widetilde{r},\widetilde{r}'))_{(\widetilde{r},\widetilde{r}')\in\widetilde{R}\otimes\widetilde{R}}, \cdot_{\widetilde{R}})$$

is the tuple such that

(i) $\widetilde{R}$ denotes the class

$$\begin{aligned}\{(H,(r_k,a_k)_{k\in\Delta})\,|\,H &\text{ is a fin.-dim. Hilbert space}\\ &\wedge\ \Delta \text{ is a finite set}\\ &\wedge\ (\forall_{k\in\Delta}\colon r_k\in R\wedge a_k\in[H_{R,r_k},H])\\ &\wedge\ (\forall_{(k,k')\in\Delta\otimes\Delta}\colon a_{k'}^*a_k\in \mathrm{Mor}_R(r_k,r_{k'}))\\ &\wedge\ \textstyle\sum_{k\in\Delta}a_ka_k^*=\mathrm{id}_H\},\end{aligned}$$

(ii) $H_{\widetilde{R},\widetilde{r}}=H$ for any element $\widetilde{r}=(H,(r_k,a_k)_{k\in\Delta})$ of $\widetilde{R}$,

(iii) for any elements $\widetilde{r}=(H,(r_k,a_k)_{k\in\Delta})$ and $\widetilde{r}'=(H',(r'_{k'},a'_{k'})_{k'\in\Delta'})$ of $\widetilde{R}$,

$$\begin{aligned}&\mathrm{Mor}_{\widetilde{R}}(\widetilde{r},\widetilde{r}')\\ &\quad=\{t\in[H,H']\wedge\forall_{(k,k')\in\Delta\otimes\Delta'}\colon (a'_{k'})^*ta_k\in\mathrm{Mor}_R(r_k,r'_{k'})\},\end{aligned}$$

(iv) $\cdot_{\widetilde{R}}$ is the binary operation on $\widetilde{R}$ with, for any elements $\widetilde{r}_1=(H_1,(r_{1,k_1},a_{1,k_1})_{k_1\in\Delta_1})$ and $\widetilde{r}_2=(H_2,(r_{2,k_2},a_{2,k_2})_{k_2\in\Delta_2})$ of $\widetilde{R}$,

$$\widetilde{r}_1\cdot_{\widetilde{R}}\widetilde{r}_2=(H_1\otimes H_2,(r_{1,k_1}\cdot_R r_{2,k_2},a_{1,k_1}\otimes a_{2,k_2})_{(k_1,k_2)\in\Delta_1\otimes\Delta_2}),$$

and if $\simeq_R$ is the equivalence relation

$$\{(\widetilde{r},\widetilde{r}')\in\widetilde{R}\otimes\widetilde{R}\wedge H_{\widetilde{R},\widetilde{r}}=H_{\widetilde{R},\widetilde{r}'}\wedge \mathrm{id}_{H_{\widetilde{R},\widetilde{r}}}\in\mathrm{Mor}_{\widetilde{R}}(\widetilde{r},\widetilde{r}')\}$$

on $\widetilde{R}$, then

(v) $\widehat{R}=\widetilde{R}/\simeq_R$,

(vi) $H_{\widehat{R},\widehat{r}}=H_{\widetilde{R},\widetilde{r}}$ for any $\widehat{r}\in\widehat{R}$, where $\widetilde{r}\in\widehat{r}$ can be arbitrary,

(vii) $\mathrm{Mor}_{\widehat{R}}(\widehat{r},\widehat{r}')=\mathrm{Mor}_{\widetilde{R}}(\widetilde{r},\widetilde{r}')$ for any $\{\widehat{r},\widehat{r}'\}\subseteq\widehat{R}$, where $\widetilde{r}\in\widehat{r}$ and $\widetilde{r}'\in\widehat{r}'$ can be arbitrary,

(viii) $\cdot_{\widehat{R}}$ is the binary operation on $\widehat{R}$ defined by

$$\widehat{r}_1\cdot_{\widehat{R}}\widehat{r}_2=\{\widetilde{r}'\in\widetilde{R}\wedge\widetilde{r}'\simeq_R\widetilde{r}_1\cdot_{\widetilde{R}}\widetilde{r}_2\}$$

for any $\{\widehat{r}_1,\widehat{r}_2\}\subseteq\widetilde{R}$, where $\widetilde{r}_1\in\widehat{r}_1$ and $\widetilde{r}_2\in\widehat{r}_2$ can be arbitrary.

(b) For any strict concrete strict monoidal W^*-functor F from any concrete monoidal W^*-cateogy $R\equiv(R,(H_{R,r})_{r\in R},(\mathrm{Mor}_R(r,r'))_{(r,r')\in R\otimes R},\cdot_R)$ to any concrete monoidal W^*-category $S\equiv(S,(H_{S,s})_{s\in S},(\mathrm{Mor}_S(s,s'))_{(s,s')\in S\otimes S},\cdot_S)$, if $\widetilde{F}$ is the mapping from $\widetilde{R}$ to $\widetilde{S}$ with, for any $\widetilde{r}=(H,(r_k,a_k)_{k\in\Delta})\in\widetilde{R}$,

$$\widetilde{F}(\widetilde{r})=(H,(F(r_k),a_k)_{k\in\Delta})$$

then the *completion* $\mathsf{D}(F)$ of F is the mapping $\widehat{F}$ from the object set of $\widehat{R}$ to that of $\widehat{S}$ with

$$\widehat{F}(\widehat{r})=\{\widetilde{s}'\in\widetilde{S}\wedge\widetilde{s}'\simeq_S\widetilde{F}(\widetilde{r})\},$$

for any $\widehat{r}\in\widehat{R}$, where $\widetilde{r}\in\widehat{r}$ can be arbitrary.

Note that the object set of a completion is a set while the intermediate class, the one before the identification, is not. The completion operation constitutes a functor D from the category of arbitrary, not necessarily complete concrete monoidal W^*-categories to the category of complete concrete monoidal W^*-categories. Moreover, it preserves rigidity. The following is implicit in [Wor88, Proposition 2.7].

PROPOSITION 2.41. *(a)* $\mathsf{D}(R)$ *is a complete rigid concrete monoidal* W^**-category* R *for any (not necessarily complete) rigid monoidal* W^**-category* R.

(b) $\mathsf{D}(F)$ *is a strict concrete strict monoidal* W^**-functor* $\mathsf{D}(R) \to \mathsf{D}(R')$ *for any strict concrete strict monoidal* W^**-functor* $F\colon R \to R'$ *and any (not necessarily complete) rigid concrete monoidal* W^**-categories* R *and* R'.

(c) $\mathsf{D}(F' \circ F) = \mathsf{D}(F') \circ \mathsf{D}(F)$ *for any strict concrete strict monoidal* W^**-functors* $F\colon R \to R''$ *and* $F'\colon R'' \to R'$ *and any (not necessarily complete) rigid concrete monoidal* W^**-categories* R, R'' *and* R'.

(d) $\mathsf{D}(\mathrm{id}_R) = \mathrm{id}_{\mathsf{D}(R)}$ *for any (not necessarily complete) rigid concrete monoidal* W^**-category.*

The following mappings are also left implicit in [Wor88, Proposition 2.7].

DEFINITION 2.42. (a) For any complete concrete monoidal W^*-category $R = (R, (H_{R,r})_{r\in R}, (\mathrm{Mor}_R(r,r'))_{(r,r')\in R\otimes R}, \cdot_R)$ let $\mathrm{cu}_R^{\mathsf{D}\dashv\subseteq}$ be the mapping from the object set of $\mathsf{D}(R)$ to that of R which assigns to any object $\widehat{r}$ of $\mathsf{D}(R)$ the unique object r of R with $H_{R,r} = H$ and $a_k \in \mathrm{Mor}_R(r_k, r)$ for any $k \in \Delta$, where $(H, (r_k, a_k)_{k\in\Delta}) \in \widehat{r}$ can be arbitrary.

(b) If $R = (R, (H_{R,r})_{r\in R}, (\mathrm{Mor}_R(r,r'))_{(r,r')\in R\otimes R}, \cdot_R)$ is any concrete monoidal W^*-category, let $\mathrm{un}_R^{\mathsf{D}\dashv\subseteq}$ be the mapping from the object class of R to that of $\mathsf{D}(R)$ with, for any object r of R,

$$r \mapsto \{\widetilde{r} \in \widetilde{R} \wedge \widetilde{r} \simeq_R (H_{R,r}, (r, \mathrm{id}_{H_{R,r}}))\}.$$

$\mathrm{cu}^{\mathsf{D}\dashv\subseteq}$ is the co-unit and $\mathrm{un}^{\mathsf{D}\dashv\subseteq}$ the unit of an adjunction between D as left adjoint and the inclusion functor which forgets that a concrete monoidal W^*-category is complete. In other words, the category of complete rigid monoidal W^*-categories is a reflective subcategory of the category of not necessarily complete rigid monoidal W^*-categories. In particular, $\mathrm{cu}^{\mathsf{D}\dashv\subseteq}$ is a natural isomorphism. Implict in [Wor88, Proposition 2.7] are the following statements.

PROPOSITION 2.43. *(a)* $\mathrm{cu}_S^{\mathsf{D}\dashv\subseteq}$ *is a strict concrete strict monoidal* W^**-functor* $\mathsf{D}(S) \to S$ *and a bijective mapping with inverse* $\mathrm{un}_S^{\mathsf{D}\dashv\subseteq}$ *for any complete rigid monoidal* W^**-category* S.

(b) $\mathrm{un}_R^{\mathsf{D}\dashv\subseteq}$ *is a strict concrete strict monoidal* W^**-functor* $R \to \mathsf{D}(R)$ *for any (not necessarily complete) rigid monoidal* W^**-category* R.

(c) $\mathrm{cu}_{S'}^{\mathsf{D}\dashv\subseteq} \circ \mathsf{D}(G) = G \circ \mathrm{cu}_S^{\mathsf{D}\dashv\subseteq}$ *for any strict concrete strict monoidal* W^**-functor* $G\colon S \to S'$ *and any complete rigid concrete monoidal* W^**-categories* S *and* S'.

(d) $\mathrm{un}_{R'}^{\mathsf{D}\dashv\subseteq}\circ F = \mathsf{D}(F)\circ\mathrm{un}_R^{\mathsf{D}\dashv\subseteq}$ *for any strict concrete strict monoidal W^*-functor $F\colon R\to R'$ and any (not necessarily complete) rigid concrete monoidal W^*-categories R and R'.*

(e) $\mathrm{id}_{\mathsf{D}(R)} = \mathrm{cu}_{\mathsf{D}(R)}^{\mathsf{D}\dashv\subseteq}\circ\mathsf{D}(\mathrm{un}_R^{\mathsf{D}\dashv\subseteq})$ *for any (not necessarily complete) rigid concrete monoidal W^*-category.*

(f) $\mathrm{id}_S = \mathrm{cu}_S^{\mathsf{D}\dashv\subseteq}\circ\mathrm{un}_S^{\mathsf{D}\dashv\subseteq}$ *(and also* $\mathrm{id}_{\mathsf{D}(S)} = \mathrm{un}_S^{\mathsf{D}\dashv\subseteq}\circ\mathrm{cu}_S^{\mathsf{D}\dashv\subseteq}$*) for any complete rigid concrete monoidal W^*-category.*

2.5.6. *Representees of complete rigid monoidal W^*-categories.* The definition of T yields a construction of the co-representee Hopf $*$-algebra in which the multiplication is convenient to describe and the vector space structure is not. For complete categories one can give a different construction which then has the opposite properties. However, this description is not constructive and purely relies on the axiom of choice. It will only be used for theoretical purposes in this chapter.

Definition 2.44. Any object r of any concrete W^*-category $R \equiv (R,(H_{r'})_{r'\in R},(\mathrm{Mor}(r',r''))_{(r',r'')\in R\otimes R})$ is called *irreducible* in R if $\mathrm{Mor}(r,r) = \mathbb{C}\mathrm{id}_{H_r}$. If R is even a concrete monoidal W^*-category, we call r *irreducible* in R if the same is true in the underlying concrete W^*-category of R. Moreover, any set R_{irr} of irreducible objects of R is called *complete* if any irreducible object of R is equivalent to an element of R_{irr}.

Equivalence preserves irreducibility. Conjugates of irreducible objects are irreducibles. In any complete category, any object is a direct sum of irreducible objects. (See [Wor88, Proposition 2.4].)

In order to make functorial statements, even stronger premises will be adopted than in Assumption 2.32.

Assumptions 2.45. As before, fix an orthonormal basis $(e_i^H)_{i=1}^{n_H}$ for any finite-dimensional Hilbert space H. In addition, for any complete rigid concrete monoidal W^*-category R single out a complete set R_{irr} of pairwise inequivalent irreducible objects of R.

Definition 2.46. (a) For any complete rigid monoidal W^*-category $R \equiv (R,(H_r)_{r\in R}),(\mathrm{Mor}(r,r'))_{(r,r')\in r\otimes r},\cdot)$, if $n_r = \dim_{\mathbb{C}}(H_r)$ for each $r\in R$, then we call the tuple $(A,m,1,*,\Delta,\epsilon,S)$, where

(i) A is the free vector space over the set $\{([r],j,i)\,|\,r\in R_{\mathrm{irr}}\wedge\{i,j\}\subseteq [\![n_r]\!]\}$, whose objects will from now on be addressed as $u_{j,i}^{[r]}\equiv([r],j,i)$ for any $r\in R_{\mathrm{irr}}$ and $\{i,j\}\subseteq[\![n_r]\!]$, where $[r]$ denotes the set of all objects of R equivalent to r in R for each $r\in R$,

(ii) m is the unique linear map $A\otimes A\to A$ such that for any $\{r,r'\}\subseteq R_{\mathrm{irr}}$, if $k\in\mathbb{N}$ and if $(r_s,p_s)_{s=1}^k$ is such $r_s\in R_{\mathrm{irr}}$ and $p_s\in\mathrm{Mor}(r_s,r\cdot r')$ for each $s\in[\![k]\!]$ and $\sum_{s=1}^k p_sp_s^* = \mathrm{id}_{H_{r\cdot r'}}$, then for any $\{i,j\}\subseteq[\![n_r]\!]$ and

$\{i', j'\} \subseteq [\![n_{r'}]\!]$,

$$u^{[r]}_{j,i} \otimes u^{[r']}_{j',i'} \mapsto \sum_{s=1}^{k} \sum_{x_s,y_s=1}^{n_{r_s}} p^s_{(j,j'),y_s} \overline{p^s_{(i,i'),x_s}}\, u^{[r_s]}_{y_s,x_s}$$

where for any $s \in [\![k]\!]$, any $x_s \in [\![n_{r_s}]\!]$, any $j \in [\![n_r]\!]$ and any $j' \in [\![n_{r'}]\!]$,

$$p^s_{(j,j'),x_s} = \langle e^{H_r}_j \otimes e^{H_{r'}}_{j'} \mid p_s e^{H_{r_s}}_{x_s} \rangle_{H_{r_s}},$$

(iii) 1 is given by $u^{[1]}_{1,1}$, where the 1 in brackets is the monoidal unit of R.

(iv) $*$ is the linear map $A \to A^{\mathrm{cj}}$ with for any $r \in R_{\mathrm{irr}}$ and $\{i, \ell\} \subseteq [\![n_r]\!]$,

$$u^{[r]}_{\ell,i} \mapsto \sum_{j,k}^{n_r} b^r_{\ell,k} a^r_{j,i}\, u^{[\overline{r}]}_{k,j},$$

where for any $\{i, j\} \subseteq [\![n_r]\!]$,

$$a^r_{j,i} = \langle e^{H_{\overline{r}}}_j \mid j_r(e^{H_r}_i) \rangle_{H_{\overline{r}}} \quad \wedge \quad b^r_{j,i} = \langle j_r^{-1}(e^{H_{\overline{r}}}_i) \mid e^{H_r}_j \rangle_{H_r}$$

(v) Δ is the linear map $A \to A \otimes A$ with for any $r \in R_{\mathrm{irr}}$ and $\{i, k\} \subseteq [\![n_r]\!]$,

$$u^{[r]}_{k,i} \mapsto \sum_{j=1}^{n_r} u^{[r]}_{k,j} \otimes u^{[r]}_{j,i}$$

(vi) ϵ is the linear functional on A with for any $r \in R_{\mathrm{irr}}$ and $\{i, j\} \subseteq [\![n_r]\!]$,

$$u^{[r]}_{j,i} \mapsto \delta_{j,i},$$

(vii) S is the linear endomorphism of A with for any $r \in R_{\mathrm{irr}}$ and $\{i, \ell\} \subseteq [\![n_r]\!]$,

$$u^{[r]}_{\ell,i} \mapsto \sum_{j,k=1}^{n_r} b^r_{i,j} a^r_{k,\ell}\, u^{[\overline{r}]}_{j,k},$$

the *Tannaka-Krein co-dual* $\mathsf{E}(R)$ of R. Moreover, we speak of the formal dual of $\mathsf{E}(R)$ as the *Tannaka-Krein dual* of R.

(b) For any strict concrete strict monoidal W^*-functor $F\colon R \to S$ and any pair of complete rigid concrete monoidal W^*-categories $R \equiv (R, (H_{R,r})_{r \in r}, (\mathrm{Mor}_R(r, r'))_{(r,r') \in R \otimes R}, \cdot_R)$ and $S \equiv (S, (H_{S,s})_{s \in s}, (\mathrm{Mor}_S(s, s'))_{(s,s') \in S \otimes S}, \cdot_S)$ we call the unique linear ψ map from the underlying vector space of $\mathsf{E}(R)$ to that of $\mathsf{E}(S)$ such that for any $r \in R_{\mathrm{irr}}$, if $k \in \mathbb{N}$ and if $(s_g, q_g)_{g=1}^k$ is such that $q_g \in \mathrm{Mor}_S(s_g, F(r))$ for each $g \in [\![k]\!]$ and $\sum_{g=1}^k q_g q_g^* = \mathrm{id}_{H_{S,F(r)}}$, then ψ satisfies for any $\{i, j\} \subseteq [\![n_r]\!]$,

$$u^{[r]}_{j,i} \mapsto \sum_{g=1}^{k} \sum_{a_g,b_g=1}^{m_{s_g}} q^g_{j,b_g} \overline{q^g_{i,a_g}}\, v^{[s_g]}_{b_g,a_g}$$

where $u^{[r]}_{j,i}$ is short for $([r], j, i)$ for any $\{i, j\} \subseteq [\![n_r]\!]$, where, similarly, $v^{[s_g]}_{b_g,a_g}$ is short for $([s_g], b_g, a_g)$ for any $\{b_g, a_g\} \subseteq [\![m_{s_g}]\!]$ and $g \in [\![k]\!]$, and where for any $g \in [\![k]\!]$, any $a_g \in [\![m_{s_g}]\!]$ and any $j \in [\![n_r]\!]$,

$$q^g_{j,a_g} = \langle e^{S,F(r)}_j \mid q_g e^{S,s_g}_{a_g} \rangle_{H_{S,s_g}},$$

the *Tannaka-Krein co-dual* $\mathsf{E}(F)$ of F. Likewise, the formal dual of $\mathsf{E}(F)$ is called the *Tannaka-Krein dual* of F.

E is a functor from the category of complete rigid monoidal W^*-categories and strict concrete strict monoidal W^*-functors to the category of CQG Hopf $*$-algebras.

PROPOSITION 2.47. *(a)* $\mathsf{E}(R)$ *is a CQG Hopf $*$-algebra for any complete rigid concrete monodial W^*-caetgory R.*

(b) $\mathsf{E}(F)$ *is a morphism* $\mathsf{E}(R) \to \mathsf{E}(R')$ *of CQG Hopf $*$-algebras for any strict concrete strict monoidal W^*-functor $F\colon R \to R'$.*

(c) $\mathsf{E}(F' \circ F) = \mathsf{E}(F') \circ \mathsf{E}(F)$ *for any strict concrete strict monoidal W^*-functors $F\colon R \to R'$ and $F'\colon R' \to R''$ and any complete rigid concrete monoidal W^*-categories R, R' and R''.*

(d) $\mathsf{E}(\mathrm{id}_R) = \mathrm{id}_{\mathsf{E}(R)}$ *for any complete rigid concrete monoidal W^*-category R.*

In fact, E is an equivalence, as explained below.

DEFINITION 2.48. (a) For any CQG Hopf $*$-algebra $H = (A, m, 1, *, \Delta, \epsilon, S)$, if $R \equiv (R, (H_r)_{r\in R}, (\mathrm{Mor}(r,r'))_{(r,r')\in r\otimes r}, \cdot)$ is given by $\mathsf{C}(H)$, let $\mathrm{cu}_H^{\mathsf{E}\dashv\mathsf{C}}$ be the unique linear map from the underlying vector space of $\mathsf{E}(\mathsf{C}(H))$ to A such that for $r \in R_{\mathrm{irr}}$ and any $\{i,j\} \subseteq [\![n_r]\!]$,

$$([r], j, i) \mapsto u^r_{j,i},$$

where $n_r = \dim_{\mathbb{C}}(H_r)$ and where $(u^r_{j,i})_{(j,i)\in[\![n_r]\!]^{\otimes 2}}$ is the matrix of the co-representation r of (A, Δ) with respect to $(e_i^{H_r})_{i=1}^{n_r}$.

(b) For any complete rigid concrete monoidal W^*-category $R \equiv (R, (H_r)_{r\in R}, (\mathrm{Mor}(r,r'))_{(r,r')\in R\otimes R}, \cdot)$, if $(A, m, 1, *, \Delta, \epsilon, S)$ is given by $\mathsf{E}(R)$, let $\mathrm{un}_R^{\mathsf{E}\dashv\mathsf{C}}$ be the the mapping from R to the set of all unitary co-representations of $(A, m, 1, *, \Delta)$ such that for any $r \in R$,

$$r \mapsto u^r,$$

where, if $n_r = \dim_{\mathbb{C}}(H_r)$, and if $(E^{H_r}_{j,i})_{(j,i)\in[\![n_r]\!]^{\otimes 2}}$ are the matrix units of $(e_i^{H_r})_{i\in[\![n_r]\!]}$, then

$$u^r = \textstyle\sum_{i,j=1}^{n_f} E^{H_r}_{j,i} \otimes u^r_{j,i},$$

where, if $k \in \mathbb{N}$ and if $(r_s, p_s)_{s=1}^k$ is such that $r_s \in R_{\mathrm{irr}}$ and $p_s \in \mathrm{Mor}(r_s, r)$ for each $s \in [\![k]\!]$ and $\sum_{s=1}^k p_s p_s^* = \mathrm{id}_{H_r}$, then for any $\{i,j\} \subseteq [\![n_r]\!]$,

$$u^r_{j,i} = \sum_{s=1}^{k} \sum_{x_s, y_s=1}^{n_{r_s}} p^s_{j,y_s} \overline{p^s_{i,x_s}}\, u^{[r_s]}_{y_s,x_s},$$

where $u^{[r_s]}_{y_s,x_s}$ is short for $([r_s], y_s, x_s)$ for any $\{x_s, y_s\} \subseteq [\![n_{r_s}]\!]$ and $s \in [\![k]\!]$, and where for any $s \in [\![k]\!]$, any $x_s \in [\![n_{r_s}]\!]$, and any $j \in [\![n_r]\!]$,

$$p^s_{j,x_s} = \langle e_j^{H_r} \mid p_s e_{x_s}^{H_{r_s}} \rangle_{H_r}.$$

(c) For any (not necessarily complete) rigid monoidal W^*-category $R \equiv (R, (H_{R,r})_{r\in R}, (\mathrm{Mor}_R(r,r'))_{(r,r')\in R\otimes R}, \cdot_R)$, if the complete rigid concrete monoidal W^*-category $S \equiv (S, (H_{S,s})_{s\in S}, (\mathrm{Mor}_S(s,s'))_{(s,s')\in S\otimes S}, \cdot_S)$ is given by $\mathsf{D}(R)$, then let w_R be the unique linear map from the vector space underlying $\mathsf{E}(S)$ to that underlying $\mathsf{T}(R)$ with the property that for any $s \in S_{\mathrm{irr}}$, if $(H_{S,s}, (r_k, a_k)_{k\in\Delta}) \in s$ is any representative of s, then for any $\{x,y\} \subseteq [\![m_s]\!]$,

$$v^{[s]}_{y,x} \mapsto \sum_{k\in\Delta} \sum_{i_k,j_k=1}^{n_{r_k}} a^k_{y,j_k} \overline{a^k_{x,i_k}}\, u^{r_k}_{j_k,i_k},$$

where $m_s = \dim_{\mathbb{C}}(H_{S,s})$, where $n_{r_k} = \dim_{\mathbb{C}}(H_{R,r_k})$ for any $k \in \Delta$, where $v^{[s]}_{y,x}$ is short for $([s], y, x)$, where $u^{r_k}_{j_k,i_k}$ is short for (r_k, j_k, i_k) for any $k \in \Delta$ and any $\{i_k, j_k\} \subseteq [\![n_{r_k}]\!]$, and where for any $k \in \Delta$, any $y \in [\![m_s]\!]$ and any $i_k \in [\![n_{r_k}]\!]$,

$$a^k_{y,i_k} = \langle e^{H_{S,s}}_y \mid a_k e^{H_{R,r_k}}_{i_k} \rangle_{H_{S,s}}.$$

$\mathrm{cu}^{\mathsf{E}\dashv|\mathsf{C}}$ is the co-unit and $\mathrm{un}^{\mathsf{E}\dashv|\mathsf{C}}$ the unit of an adjoint equivalence $\mathsf{E} \dashv |\mathsf{C}$, where $|\mathsf{C}$ is given by C, but interpreted as taking values in the category of complete rigid monoidal W^*-categories. Moreover, w is a natural isomorphism from $\mathsf{E}\circ\mathsf{D}$ to T. The following is a more explicit version of [Wor88, Theorem 1.3] and the combined implication of the statements in [Wor88, Section 3].

Proposition 2.49. *(a)* $\mathrm{cu}^{\mathsf{E}\dashv|\mathsf{C}}_H$ *is an isomorphism of CQG Hopf $*$-algebras from* $\mathsf{E}(\mathsf{C}(H))$ *to* H *for any CQG Hopf $*$-algebra* H.

(b) $\mathrm{un}^{\mathsf{E}\dashv|\mathsf{C}}_S$ *is a strict concrete strict monoidal W^*-functor from* $S \to \mathsf{C}(\mathsf{E}(S))$ *as well as a bijective mapping and its inverse mapping is a strict concrete strict monoidal W^*-functor* $\mathsf{C}(\mathsf{E}(S)) \to S$ *for any complete rigid concrete monoidal W^*-category* S.

(c) $\mathrm{cu}^{\mathsf{E}\dashv|\mathsf{C}}_{H'} \circ \mathsf{E}(\mathsf{C}(\psi)) = \psi \circ \mathrm{cu}^{\mathsf{E}\dashv|\mathsf{C}}_H$ *for any morphism* $\psi\colon H \to H'$ *of CQG Hopf $*$-algebras and any CQG Hopf $*$-algebras* H *and* H'.

(d) $\mathrm{un}^{\mathsf{E}\dashv|\mathsf{C}}_{S'} \circ G = \mathsf{C}(\mathsf{E}(G)) \circ \mathrm{un}^{\mathsf{E}\dashv|\mathsf{C}}_S$ *for any strict concrete strict monoidal W^*-functor* $G\colon S \to S'$ *and any complete rigid concrete monoidal W^*-categories* S *and* S'.

(e) $\mathrm{id}_{\mathsf{E}(S)} = \mathrm{cu}^{\mathsf{E}\dashv|\mathsf{C}}_{\mathsf{E}(S)} \circ \mathsf{E}(\mathrm{un}^{\mathsf{E}\dashv|\mathsf{C}}_S)$ *for any complete rigid concrete monoidal W^*-category* S.

(f) $\mathrm{id}_{\mathsf{C}(H)} = \mathsf{C}(\mathrm{cu}^{\mathsf{E}\dashv|\mathsf{C}}_H) \circ \mathrm{un}^{\mathsf{E}\dashv|\mathsf{C}}_{\mathsf{C}(H)}$ *for any CQG Hopf $*$-algebra.*

(g) w_R *is an isomorphism of CQG Hopf $*$-algebras from* $\mathsf{E}(\mathsf{D}(R))$ *to* $\mathsf{T}(R)$ *for any (not necessarily complete) rigid concrete monoidal W^*-category.*

(h) $\mathsf{w}_{R'} \circ \mathsf{E}(\mathsf{D}(F)) = \mathsf{T}(F) \circ \mathsf{w}_R$ *for any strict concrete strict monoidal W^*-functor* $F\colon R \to R'$ *and any (not necessarily complete) rigid concrete monoidal W^*-categories* R *and* R'.

2.6. Tannaka-Krein representees of full subcategory inclusions. From what has been said so far we only know that T is a left-adjoint functor and thus preserves co-limits. If we are interested in limits of CQG Hopf $*$-algebras this does not help us. However, we can still utilize T in order to find at least certain limits in the category CQG Hopf $*$-algebras. Namely, this section explains why Tannaka-Krein representees of "full subcategory inclusions" are injective CQG Hopf $*$-algebra morphisms.

DEFINITION 2.50. For any (not necessarily complete) concrete W^*-categories $R \equiv (R, (H_{R,r})_{r\in R}, (\mathrm{Mor}_R(r,r'))_{(r,r')\in R\otimes R})$ and $S \equiv (S, (H_{S,s})_{s\in S}, (\mathrm{Mor}_S(s,s'))_{(s,s')\in S\otimes S})$ any strict concrete W^*-functor $F\colon R \to S$ is called *full* if $\mathrm{Mor}_R(r,r') = \mathrm{Mor}_S(F(r), F(r'))$ for any $\{r,r'\} \subseteq R$.

If R and S are even concrete monoidal W^*-categories and F a strict concrete strict monoidal W^*-functor from R to S, we call F *full* if F is full considered as a concrete W^*-functor between the underlying concrete W^*-categories.

LEMMA 2.51. *For any full strict concrete strict monoidal W^*-functor $F\colon R \to S$ between any (not necessarily complete) rigid conrete monoidal W^*-cateogories the following are true:*

(a) *F is an injective mapping between the object sets.*
(b) *Any object r of R is irreducible in R if and only if $F(r)$ is irreducible in S.*
(c) *Any two objects r and r' of R are equivalent in R if and only if $F(r)$ and $F(r')$ are equivalent in S.*
(d) *The completion $\mathsf{D}(F)$ of F is full.*

PROOF. Let $R \equiv (R, (H_{R,r})_{r\in R}, (\mathrm{Mor}_R(r,r'))_{(r,r')\in R\otimes R})$ and $S \equiv (S, (H_{S,s})_{s\in S}, (\mathrm{Mor}_S(s,s'))_{(s,s')\in S\otimes S})$.

(a) Let $\{r,r'\} \subseteq R$ be such that $F(r) = F(r')$. Then, because F is a strict concrete W^*-functor, $H_{R,r} = H_{S,F(r)} = H_{S,F(r')} = H_{R,r'}$. Therefore and because S is a concrete W^*-category, $\mathrm{id}_{H_{R,r}} = \mathrm{id}_{H_{S,F(r)}} \in \mathrm{Mor}_S(F(r), F(r)) = \mathrm{Mor}_S(F(r), F(r'))$. And as F is full that implies $\mathrm{id}_{H_{R,r}} \in \mathrm{Mor}_S(F(r), F(r')) = \mathrm{Mor}_R(r,r')$. Thus we have shown $H_{R,r} = H_{R,r'}$ and $\mathrm{id}_{H_{R,r}} \in \mathrm{Mor}_R(r,r')$, which requires $r = r'$ by the definition of R being a concrete W^*-category. In conclusion, F is injective.

(b) By definition, any object r of R is irreducible in R if and only if any element of $\mathrm{Mor}_R(r,r)$ is a multiple of $\mathrm{id}_{H_{R,r}}$. Likewise, $F(r)$ is irreducible in S if and only if $\mathrm{Mor}_S(F(r), F(r))$ consists of multiples of $\mathrm{id}_{H_{S,F(r)}}$ exclusively. Since F is a full strict concrete W^*-functor, $H_{S,F(r)} = H_{R,r}$ and $\mathrm{Mor}_S(F(r), F(r)) = \mathrm{Mor}_R(r,r)$. Hence, the two conditions are actually equivalent.

(c) According to the definition, r and r' are equivalent in R if $\mathrm{Mor}_R(r,r')$ contains a unitary. Likewise, $F(r)$ and $F(r')$ are equivalent in S if the morphism space $\mathrm{Mor}_S(F(r), F(r'))$ contains a unitary. Since F is a full strict concrete W^*-functor, $\mathrm{Mor}_S(F(r), F(r')) = \mathrm{Mor}_R(r,r')$. Thus, the two conditions are equivalent.

(d) Let $\widehat{R} \equiv (\widehat{R}, (H_{\widehat{R},\widehat{r}})_{\widehat{r}\in\widehat{R}}, (\mathrm{Mor}_{\widehat{R}}(\widehat{r},\widehat{r}'))_{(\widehat{r},\widehat{r}')\in\widehat{R}\otimes\widehat{R}})$ be $\mathsf{D}(R)$, let $\widehat{S} \equiv (\widehat{S}, (H_{\widehat{S},\widehat{s}})_{\widehat{s}\in\widehat{S}}, (\mathrm{Mor}_{\widehat{S}}(\widehat{s},\widehat{s}'))_{(\widehat{s},\widehat{s}')\in\widehat{S}\otimes\widehat{S}})$ be $\mathsf{D}(S)$, let $\widehat{F}$ be $\mathsf{D}(F)$ and let $\{\widehat{r},\widehat{r}'\} \subseteq \widehat{R}$ be arbitrary. For any representatives $(H, (r_k, a_k)_{k\in\Delta}) \in \widehat{r}$ and $(H', (r'_{k'}, a'_{k'})_{k'\in\Delta'}) \in \widehat{r}'$ and any t the statement that $t \in \mathrm{Mor}_{\widehat{S}}(\widehat{F}(\widehat{r}), \widehat{F}(\widehat{r}'))$ by definition means that $t \in [H, H']$ and

$(a'_{k'})^* t a_k \in \mathrm{Mor}_S(F(r_k), F(r'_{k'}))$ for any $(k, k') \in \Delta \otimes \Delta'$. Since F is full and thus $\mathrm{Mor}_S(F(r_k), F(r'_{k'})) = \mathrm{Mor}_R(r_k, r'_{k'})$ for any $(k, k') \in \Delta \otimes \Delta'$ in this situation, we see that $t \in \mathrm{Mor}_{\widehat{S}}(F(\widehat{r}), F(\widehat{r}'))$ holds if and only $t \in [H, H']$ and $(a'_{k'})^* t a_k \in \mathrm{Mor}_R(r_k, r'_{k'})$ for any $(k, k') \in \Delta \otimes \Delta'$, which is to say $t \in \mathrm{Mor}_{\widehat{R}}(\widehat{r}, \widehat{r}')$. Thus, $\mathsf{D}(F)$ is full. □

2.6.1. *Injectivity on the Hopf $*$-algebraic level.* Using Woronowicz's two Tannaka-Krein theorems recalled in Section 2.5 and using Lemma 2.51 we prove in two steps that the Tannaka co-representees of full strict concrete strict monoidal W^*-functors are injective (see Proposition 2.53).

Lemma 2.52. $\mathsf{E}(F)$ *is injective for any full strict concrete strict monoidal W^*-functor between any complete rigid monoidal W^*-categories.*

Proof. Let $R \equiv (R, (H_{R,r})_{r \in R}, (\mathrm{Mor}_R(r, r'))_{(r,r') \in R \otimes R})$ and $S \equiv (S, (H_{S,s})_{s \in S}, (\mathrm{Mor}_S(s, s'))_{(s,s') \in S \otimes S})$ be be the complete rigid concrete monoidal W^*-categories with $F\colon R \to S$, let A_R and A_S be the vector spaces underlying the CQG Hopf $*$-algebras $\mathsf{E}(R)$ and $\mathsf{E}(S)$, respectively, and write ψ for $\mathsf{T}(F)$. We let $x \in A_R$ be arbitrary with $\psi(x) = 0$ and prove $x = 0$.

If $n_r = \dim_{\mathbb{C}}(H_{R,r})$ for any $r \in R$, then by definition the vector space A_R is free over $\{([r], j_r, i_r) \mid r \in R_{\mathrm{irr}} \wedge \{i_r, j_r\} \subseteq [\![n_r]\!]\}$. Hence, there exists a set $\{z^r_{j_r,i_r} \mid r \in R_{\mathrm{irr}} \wedge \{i_r, j_r\} \subseteq [\![n_r]\!]\} \subseteq \mathbb{C}$ with only finitely many non-zero elements such that

$$x = \sum_{r \in R_{\mathrm{irr}}} \sum_{i_r, j_r = 1}^{n_r} z^r_{j_r,i_r} \, u^{[r]}_{j_r,i_r},$$

where $u^{[r]}_{j_r,i_r}$ is short for $([r], j_r, i_r)$ for any $r \in R_{\mathrm{irr}}$ and $\{i_r, j_r\} \subseteq [\![n_r]\!]$. Since F is full for any $r \in R_{\mathrm{irr}}$ the object $F(r)$ is irreducible in S by Lemma 2.51 (b). Because S_{irr} is a complete system of irreducible objects of S there must then exist for each $r \in R_{\mathrm{irr}}$ an object $s_r \in S_{\mathrm{irr}}$ and a unitary operator $p_r\colon H_{S,s_r} \to H_{S,F(r)}$ with $p_r \in \mathrm{Mor}_S(s_r, F(r))$. In particular, then, $\dim_{\mathbb{C}}(H_{S,s_r}) = n_r$ for any $r \in R_{\mathrm{irr}}$ and the definition of ψ implies that then for any $r \in R_{\mathrm{irr}}$ and any $\{i_r, j_r\} \subseteq [\![n_r]\!]$,

$$\psi(u^{[r]}_{j_r,i_r}) = \sum_{x_r, y_r = 1}^{n_r} p^r_{j_r,y_r} z^r_{j_r,i_r} \overline{p^r_{i_r,x_r}} \, v^{[s_r]}_{y_r,x_r},$$

where $v^{[s_r]}_{y_r,x_r}$ is short for $([s_r], y_r, x_r)$ for any $\{x_r, y_r\} \subseteq [\![n_r]\!]$ and where for any $\{j_r, x_r\} \subseteq [\![n_r]\!]$,

$$p^r_{j_r,x_r} = \langle e^{S,F(r)}_{j_r} \mid p_r e^{S,s_r}_{x_r} \rangle_{H_{S,F(r)}}.$$

Thus, in total,

$$0 = \psi(x) = \sum_{r \in R_{\mathrm{irr}}} \sum_{i_r, j_r = 1}^{n_r} \sum_{x_r, y_r = 1}^{n_r} p^r_{j_r,y_r} z^r_{j_r,i_r} \overline{p^r_{i_r,x_r}} \, v^{[s_r]}_{y_r,x_r}.$$

Moreover, not only is F itself, seen as a mapping between the objects sets, injective by Lemma 2.51 (a), but by Lemma 2.51 (c) so is the induced mapping $R_{\mathrm{irr}} \to \{[s] \mid s \in S_{\mathrm{irr}}\}$ with $r \mapsto F(r)$ for any $r \in R_{\mathrm{irr}}$. Because A_S is free over

$\{v^{[s]}_{y_s,x_s} \,|\, s \in S_{\mathrm{irr}} \wedge \{x_s, y_s\} \subseteq [\![m_s]\!]\}$, where $m_s = \dim_{\mathbb{C}}(H_{S,s})$ for any $s \in S$, the above identity actually requires that for any $r \in R_{\mathrm{irr}}$ and any $\{x_r, y_r\} \subseteq [\![n_r]\!]$,

$$0 = \sum_{i_r, j_r = 1}^{n_r} p^r_{j_r,y_r} z^r_{j_r,i_r} \overline{p^r_{i_r,x_r}}.$$

In other words, for any $r \in R_{\mathrm{irr}}$, if $P_r = (p^r_{j_r,i_r})_{(j_r,i_r)\in[\![n_r]\!]^{\otimes 2}}$ and $Z_r = (z^r_{j_r,i_r})_{(j_r,i_r)\in[\![n_r]\!]^{\otimes 2}}$, then $(P_r)^*(Z_r)^{\mathrm{t}} P_r = 0$. Because P_r, being the coordinate matrix of the unitary operator p_r, is a unitary matrix, that demands $Z_r = 0$ for any $r \in R_{\mathrm{irr}}$. It follows $z^r_{j_r,i_r} = 0$ for any $r \in R_{\mathrm{irr}}$ and $\{i_r, j_r\} \subseteq [\![n_r]\!]$, which is to say $x = 0$. That is what we needed to see. □

Proposition 2.53. *$\mathsf{T}(F)$ is injective for any full strict concrete strict monoidal W^*-functor between any (not necessarily complete) rigid concrete monoidal W^*-categories.*

Proof. If R and S are such that $F\colon R \to S$, then $\mathsf{w}_S \circ \mathsf{E}(\mathsf{D}(F)) = \mathsf{T}(F) \circ \mathsf{w}_R$ by Proposition 2.49 (h), where both $\mathsf{w}_R\colon \mathsf{E}(\mathsf{D}(R)) \to \mathsf{T}(R)$ and $\mathsf{w}_S\colon \mathsf{E}(\mathsf{D}(S)) \to \mathsf{T}(S)$ are invertible CQG Hopf $*$-algebra morphisms by Proposition 2.49 (g) and thus in particular bijective. Consequently, $\mathsf{T}(F) = \mathsf{w}_S \circ \mathsf{E}(\mathsf{D}(F)) \circ \mathsf{w}_R^{-1}$ is injective if and only if $\mathsf{E}(\mathsf{D}(F))$ is injective. Because F being full implies that also $\mathsf{D}(F)$ is full by Lemma 2.51 (d) and because the domain $\mathsf{D}(R)$ and co-domain $\mathsf{D}(S)$ are complete rigid monoidal W^*-categories by Proposition 2.41 (a) the claim now follows from Lemma 2.52. □

2.6.2. *Injectivity on the reduced C^*-level.* It remains to prove that also the reduced CQG C^*-algebra morphisms induced by the Tannaka-Krein co-representees of full strict concrete strict monoidal W^*-functors are injective (see Proposition 2.57).

In terms of the matrix coefficients of irreducible co-representations the integral of any CQG Hopf $*$-algebra has very simple coordinates. The following characterization is implied by, for example, [Tim08, Corolary 3.2.7].

Proposition 2.54. *The integral of any CQG Hopf $*$-algebra $(A, m, 1, *, \Delta, \epsilon, S)$ is the unique linear functional h on A with the property that for any system I of pairwise inequivalent irreducible unitary co-representations of $(A, m, 1, *, \Delta)$ and any $u^r \in I$, if r is equivalent to the trivial co-representation 1, then $h(u^r) = 1$ and otherwise $h(u^r_{j,i}) = 0$ for any $\{i, j\} \subseteq [\![n_r]\!]$ and the matrix $(u^r_{j,i})_{(j,i)\in[\![n_r]\!]^{\otimes 2}}$ of u^r with respect to any basis of the carrier space.*

Lemma 2.55. *For any complete rigid concrete monoidal W^*-categories R and S and any full strict concrete strict monoidal W^*-functor $F\colon R \to S$, if $h_{\mathsf{E}(R)}$ and $h_{\mathsf{E}(S)}$ are the respective integrals of the CQG Hopf $*$-algebras $\mathsf{E}(R)$ and $\mathsf{E}(S)$, then*

$$h_{\mathsf{E}(S)} \circ \mathsf{E}(F) = h_{\mathsf{E}(R)}.$$

Proof. Proposition 2.49 (d) guarantees that $\mathrm{un}^{\mathsf{E}\dashv|\mathsf{C}}_S \circ F = \mathsf{C}(\mathsf{E}(F)) \circ \mathrm{un}^{\mathsf{E}\dashv|\mathsf{C}}_R$, where both $\mathrm{un}^{\mathsf{E}\dashv|\mathsf{C}}_R$ and $\mathrm{un}^{\mathsf{E}\dashv|\mathsf{C}}_S$ are invertible and have inverses that are also strict concrete monoidal W^*-functors by Proposition 2.49 (b). That makes $\mathrm{un}_R^{\mathsf{E}\dashv|\mathsf{C}}$ and $\mathrm{un}_S^{\mathsf{E}\dashv|\mathsf{C}}$ full, in particular. Because F is full by assumption and because compositions of full functors are full, $\mathsf{C}(\mathsf{E}(F)) = \mathrm{un}_S^{\mathsf{E}\dashv|\mathsf{C}} \circ F \circ (\mathrm{un}_R^{\mathsf{E}\dashv|\mathsf{C}})^{-1}$ is thus full as well. According to Lemma 2.51 (b) then, $\mathsf{C}(\mathsf{E}(R))$ is mapped by F to irreducible objects of $\mathsf{C}(\mathsf{E}(F))$. In fact, by Lemma 2.51 (c) the images of $\mathsf{C}(\mathsf{E}(R))$ under F form a set of pairwise inequivalent irreducible objects of S. By the axiom of choice we may thus complete this set to a complete system I of pairwise inequivalent irreducible objects of $\mathsf{C}(\mathsf{E}(F))$. For any $y \in I$ Proposition 2.54 implies $(\mathrm{id} \otimes h_{\mathsf{E}(S)})(y) = \delta_{1,y}$, where 1 is the monoidal unit of $\mathsf{C}(\mathsf{E}(S))$. In particular, $((\mathrm{id}\otimes h_{\mathsf{E}(S)})\circ\mathsf{C}(\mathsf{E}(F)))(x) = \delta_{1,x}$ for any $x \in \mathsf{C}(\mathsf{E}(R))$, where, now, 1 is the monoidal unit of $\mathsf{C}(\mathsf{E}(S))$. Since $((\mathrm{id} \otimes h_{\mathsf{E}(S)}) \circ \mathsf{C}(\mathsf{E}(F)))(x) = (\mathrm{id} \otimes (h_{\mathsf{E}(S)}\circ\mathsf{E}(F)))(x)$ by definition of C we have thus shown $(\mathrm{id}\otimes(h_{\mathsf{E}(S)}\circ\mathsf{E}(F)))(x) = \delta_{1,x}$ for any $x \in \mathsf{C}(\mathsf{E}(R))$. According to Proposition 2.54 that proves $(\mathrm{id} \otimes (h_{\mathsf{E}(S)} \circ \mathsf{E}(F)))(x) = (\mathrm{id} \otimes h_{\mathsf{E}(R)})(x)$ for any $x \in \mathsf{C}(\mathsf{E}(R))$ and thus the claim. □

LEMMA 2.56. *For any (not necessarily complete) rigid monoidal W^*-categories R and S and any full strict concrete strict monoidal W^*-functor $F\colon R \to S$, if $h_{\mathsf{T}(R)}$ and $h_{\mathsf{T}(S)}$ are the respective integrals of the CQG Hopf $*$-algebras $\mathsf{T}(R)$ and $\mathsf{T}(S)$, then*

$$h_{\mathsf{T}(S)} \circ \mathsf{T}(F) = h_{\mathsf{T}(R)}.$$

PROOF. $\mathsf{D}(R)$ and $\mathsf{D}(S)$ are complete rigid concrete monoidal W^*-categories and $\mathsf{D}(F)$ is a strict concrete strict monoidal W^*-functor $\mathsf{D}(R) \to \mathsf{D}(S)$ according to Proposition 2.41. Moreover, $\mathsf{D}(F)$ is full by Lemma 2.51 (d).Hence, if $h_{\mathsf{E}(\mathsf{D}(R))}$ and $h_{\mathsf{E}(\mathsf{D}(S))}$ denote the integrals of $\mathsf{E}(\mathsf{D}(R))$ and $\mathsf{E}(\mathsf{D}(S))$, respectively, then $h_{\mathsf{E}(\mathsf{D}(S))} \circ \mathsf{E}(\mathsf{T}(F)) = h_{\mathsf{E}(\mathsf{D}(R))}$ by Lemma 2.55. Since $\mathsf{w}_S \circ \mathsf{E}(\mathsf{D}(F)) = \mathsf{T}(F) \circ \mathsf{w}_R$ by Proposition 2.49 (h), where both $\mathsf{w}_R\colon \mathsf{E}(\mathsf{D}(R)) \to \mathsf{T}(R)$ and $\mathsf{w}_S\colon \mathsf{E}(\mathsf{D}(S)) \to \mathsf{T}(S)$ are invertible by Proposition 2.49 (g) it follows that $\mathsf{E}(\mathsf{D}(F)) = \mathsf{w}_S^{-1}\circ\mathsf{T}(F)\circ\mathsf{w}_R$. Thus we have shown the identity $h_{\mathsf{E}(\mathsf{D}(S))} \circ \mathsf{w}_S^{-1} \circ \mathsf{T}(F) \circ \mathsf{w}_R = h_{\mathsf{E}(\mathsf{D}(R))}$, which is equivalent to the statement that $h_{\mathsf{E}(\mathsf{D}(S))} \circ \mathsf{w}_S^{-1} \circ \mathsf{T}(F) = h_{\mathsf{E}(\mathsf{D}(R))} \circ \mathsf{w}_R^{-1}$. But, of course, since w_S^{-1} and w_R^{-1} are CQG Hopf $*$-algebra isomorphisms, they preserve the integrals, meaning that $h_{\mathsf{E}(\mathsf{D}(S))} \circ \mathsf{w}_S^{-1} = h_{\mathsf{T}(S)}$ and $h_{\mathsf{E}(\mathsf{D}(R))} \circ \mathsf{w}_R^{-1} = h_{\mathsf{T}(R)}$. And that proves the claim. □

PROPOSITION 2.57. $\mathsf{R}(\mathsf{T}(F))$ *is injective for any full strict concrete strict monoidal W^*-functor F.*

PROOF. Let $R \equiv (R, (H_{R,r})_{r\in R}, (\mathrm{Mor}_R(r, r'))_{(r,r')\in R\otimes R}, \cdot_R)$ and $S \equiv (S, (H_{S,r})_{r\in S}, (\mathrm{Mor}_S(r, r'))_{(r,r')\in S\otimes S}, \cdot_S)$ be the rigid concrete monoidal W^*-categories such that $F\colon R \to S$. Moreover, let $H_R \coloneqq (A_R, m_R, 1_R, *_R, \Delta_R, \epsilon_R, S_R)$ be given by $\mathsf{T}(R)$ and $H_S \coloneqq (A_S, m_S, 1_S, *_S, \Delta_S, \epsilon_S, S_S)$ by $\mathsf{T}(S)$ and abbreviate $\psi \coloneqq \mathsf{T}(F)$. Let h_R and h_S be the integrals of H_R respectively H_S and let $\langle\cdot\mid\cdot\rangle_R \coloneqq h_R m_R({*_R}^{\mathrm{cj}} \otimes \mathrm{id}_{A_R})$ and $\langle\cdot\mid\cdot\rangle_S \coloneqq h_S m_S({*_S}^{\mathrm{cj}} \otimes \mathrm{id}_{A_S})$ be the respective associated scalar products and $|\cdot|_R$ and $|\cdot|_S$ the norms on A_R induced by $\langle\cdot\mid\cdot\rangle_R$ respectively on A_S induced by $\langle\cdot\mid\cdot\rangle_S$. Let $\|\cdot\|_R$ be the left regular representation operator norm associated with

$|\cdot|_R$, i.e., the norm on A_R defined by $a \mapsto \sup\{|ab|_R \,|\, b \in A_R \wedge |b|_R = 1\}$ for any $a \in A_R$, and let, likewise, $\|\cdot\|_S$ be the left regular representation operator norm associated with $|\cdot|_S$. Let $C_R := (A'_R, m'_R, 1'_R, *'_R, \|\cdot\|'_R, \Delta'_R)$ be given by $\mathsf{R}(\mathsf{T}(R))$ and $C_S := (A'_S, m'_S, 1'_S, *'_S, \|\cdot\|'_S, \Delta'_S)$ by $\mathsf{R}(\mathsf{T}(S))$, let $\varphi := \mathsf{R}(\mathsf{T}(F))$ and let j_R and j_S be the respective inclusions of the normed spaces $(A_R, \|\cdot\|_R)$ and $(A_S, \|\cdot\|_S)$ into their respective Banach completions $(A'_R, \|\cdot\|'_R)$ and $(A'_S, \|\cdot\|'_S)$. We show in two steps that φ is injective.

Step 1: We prove that $\|a\|_R \leq \|\psi(a)\|_S$ for any $a \in A_R$. Indeed, because ψ is a $*$-algebra morphism, first,

$$\begin{aligned}(|\psi(a)|_S)^2 &= \langle \psi(a) \mid \psi(a)\rangle_S = h_S(\psi(a)^*\psi(a)) = h_S(\psi(a^*a)) = h_R(a^*a) = \langle a \mid a\rangle_R \\ &= (|a|_R)^2,\end{aligned}$$

where we have used use the fact that $h_S\psi = h_R$ by Lemma 2.56 in the fourth step.

In other words, ψ is an isometry from $(A_R, |\cdot|_R)$ to $(A_S, |\cdot|_S)$. If I is the image of ψ in A_S, it thus follows for any $a \in A_R$,

$$\begin{aligned}\|a\|_R &= \sup\{|ab|_R \,|\, b \in A_R \wedge |b|_R = 1\} \\ &= \sup\{|\psi(ab)|_S \,|\, b \in A_R \wedge |\psi(b)|_S = 1\} \\ &= \sup\{|\psi(a)y|_S \,|\, y \in I \wedge |y|_S = 1\} \\ &\leq \sup\{|\psi(a)y|_S \,|\, y \in A_S \wedge |y|_S = 1\} \\ &= \|\psi(a)\|_S,\end{aligned}$$

where we have used the fact that ψ is an algebra morphism in the third step.

Step 2: Using Step 1 we show that φ is injective. Given any $x \in A'_R$ with $\varphi(x) = 0$ and any $\delta \in \mathbb{R}$ with $0 < \delta$, by nature of $(A'_R, \|\cdot\|'_R)$ there exists $a_\delta \in A_R$ such that $\|x - j_R(a_\delta)\|'_R < \delta$. Because $\varphi = \mathsf{R}(\psi)$ is a morphism of C^*-algebras φ is a bounded linear map from $(A'_R, \|\cdot\|'_R)$ to $(A'_S, \|\cdot\|'_S)$. Hence, there exists $\lambda \in \mathbb{R}$ with $0 \leq \lambda$ such that $\|\varphi(b')\|'_S \leq \lambda\|b'\|'_R$ for any $b' \in A'_R$. Using what we have just shown and the facts that j_R is isometric from $(A_R, \|\cdot\|_R)$ to $(A'_R, \|\cdot\|'_R)$, that j_S is isometric from $(A_S, \|\cdot\|_S)$ to $(A'_S, \|\cdot\|'_S)$ and that $j_S\psi = \varphi j_R$, we conclude

$$\begin{aligned}\|j_R(a_\delta)\|'_R &= \|a_\delta\|_R \\ &\leq \|\psi(a_\delta)\|_S \\ &= \|j_S(\psi(a_\delta))\|'_S \\ &= \|0 - \varphi(j_R(a_\delta))\|'_S \\ &= \|\varphi(x) - \varphi(j_R(a_\delta))\|'_S \\ &= \|\varphi(x - j_R(a_\delta))\|'_S \\ &\leq \lambda\|x - j_R(a_\delta)\|'_R \\ &\leq \lambda\delta\end{aligned}$$

and thus by the reverse triangle inequality,

$$\|x\|'_R - \lambda\delta \leq \|x\|'_R - \|j_R(a_\delta)\|'_R \leq |\|x\|'_R - \|j_R(a_\delta)\|'_R| \leq \|x - j_R(a_\delta)\|'_R < \delta.$$

In consequence, $\|x\|'_R < (1+\lambda)\delta$ for any $\delta \in \mathbb{R}$ with $0 < \delta$. That requires $\|x\|'_R = 0$ and thus $x = 0$. Hence, φ is injective. □

3. General categories of partitions and easy quantum groups

Section 3 introduces categories of "labeled partitions" as studied by Freslon in [Fre17] and the so-called "easy" quantum groups they induce by Tannaka-Krein duality.

3.1. Fundamentals. Throughout the remainder of the chapter the following notation will be used for basic constructions around sets and mappings between them.

Definition 3.1. Let X and Y be any sets and $f : X \to Y$ any mapping.

(a) For any subset A of X write $f_{\to}(\mathsf{A}) \coloneqq \{f(\mathsf{a}) \,|\, \mathsf{a} \in \mathsf{A}\}$ for the *image* of A with respect to f. Also, let $\operatorname{ran}(f) \coloneqq f_{\to}(X)$.

(b) Dually, for any subset B of Y denote by $f^{\leftarrow}(\mathsf{B}) \coloneqq \{\mathsf{a} \in X \,|\, f(\mathsf{a}) \in \mathsf{B}\}$ the *pre-image* of B with respect to f.

(c) A *(set-theoretical) partition* of X is any non-empty subset p of the power set $\wp(X)$ of X such that $\mathsf{A} \cap \mathsf{B} = \varnothing$ for any $\{\mathsf{A}, \mathsf{B}\} \subseteq p$ with $\mathsf{A} \neq \mathsf{B}$ and such that $\bigcup p = X$. If so, the elements of p are referred to as its *blocks*.

(d) Given any partition p of X, the *quotient map associated to* p is defined as the mapping $\pi_p \colon X \to p$ with graph $\{(\mathsf{a}, \mathsf{B}) \,|\, \mathsf{B} \in p \wedge \mathsf{a} \in \mathsf{B}\}$.

(e) For any partition p of X the binary relation $\sim_p \coloneqq \{(\mathsf{a}, \mathsf{b}) \,|\, \mathsf{B} \in p \wedge \{\mathsf{a}, \mathsf{b}\} \subseteq \mathsf{B}\}$ on X is called the *equivalence relation associated with* p.

(f) For any partition p of X, if $f(\mathsf{a}) = f(\mathsf{b})$ for all $\{\mathsf{a}, \mathsf{b}\} \subseteq X$ with $\mathsf{a} \sim_p \mathsf{b}$, we write $f/p \coloneqq \{(\mathsf{B}, f(\mathsf{a})) \,|\, \mathsf{B} \in p \wedge \mathsf{a} \in \mathsf{B}\}$ for the *quotient* of f by p.

(g) For any set Y and any mapping $f : X \to Y$ the *kernel partition* of f is by definition $\ker(f) \coloneqq \{f^{\leftarrow}(\{\mathsf{a}\}) \,|\, \mathsf{a} \in Y\} \backslash \{\varnothing\}$.

(h) Given any two partitions p and q of X, we say that p is *finer* than q or, equivalently, that p *refines* q if for any $\mathsf{A} \in p$ there exists $\mathsf{B} \in q$ with $\mathsf{A} \subseteq \mathsf{B}$. The operation of forming the *join* of any two partitions, i.e., the finest partition refined by both, is denoted by $\vee$.

(i) For any subset W of X and any partition p of X we call $p|_{\mathsf{W}} \coloneqq \{\mathsf{B} \cap \mathsf{W} \,|\, \mathsf{B} \in p\} \backslash \{\varnothing\}$ the *restriction* of p to W.

(j) Given any partition q of Y, the partition $f^{\leftarrow\cdot}(q) \coloneqq \{f^{\leftarrow}(\mathsf{C}) \,|\, \mathsf{C} \in q\} \backslash \{\varnothing\}$ of X is called the *pull-back* of q by f.

The following elementary observations will be used countless times and, from now on, without further comment.

Remark 3.2. For any mapping $f \colon X \to Y$ between any sets X and Y with power sets $\wp(X)$ and $\wp(Y)$, respectively, the following are true.

(a) $\mathsf{A} \subseteq (f^{\leftarrow} \circ f_{\to})(\mathsf{A}) = \bigcup\{\mathsf{A}' \in \ker(f) \wedge \mathsf{A}' \cap \mathsf{A} \neq \varnothing\}$ for any $\mathsf{A} \in \wp(X)$.

(b) $(f_{\to} \circ f^{\leftarrow})(\mathsf{B}) = \mathsf{B} \cap \operatorname{ran}(f) \subseteq \mathsf{B}$ for any $\mathsf{B} \in \wp(Y)$.

(c) $f^{\leftarrow} \circ f_{\rightarrow} = \mathrm{id}_{\wp(X)}$ if and only if f is injective.
(d) $f_{\rightarrow} \circ f^{\leftarrow} = \mathrm{id}_{\wp(Y)}$ if and only if f is surjective.
(e) $f^{\leftarrow\text{-}}(q) \leq f^{\leftarrow\text{-}}(q')$ for any partitions q and q' of Y with $q \leq q'$.

Moreover, at times, we will encounter cyclic orders and related concepts.

DEFINITION 3.3. (a) A *partial cyclic order* on any set X is any ternary relation $\Gamma \equiv (\cdot \,|\, \cdot \,|\, \cdot)$ on X such that for any $\{\mathtt{a},\mathtt{b},\mathtt{c},\mathtt{d}\} \subseteq X$,

(i) if $(\mathtt{a} \,|\, \mathtt{b} \,|\, \mathtt{c})$, then $(\mathtt{b} \,|\, \mathtt{c} \,|\, \mathtt{a})$ (*cyclicity*)
(ii) if $(\mathtt{a} \,|\, \mathtt{b} \,|\, \mathtt{c})$, then not $(\mathtt{c} \,|\, \mathtt{b} \,|\, \mathtt{a})$ (*asymmetry*)
(iii) if $(\mathtt{a} \,|\, \mathtt{b} \,|\, \mathtt{c})$ and $(\mathtt{a} \,|\, \mathtt{c} \,|\, \mathtt{d})$, then $(\mathtt{a} \,|\, \mathtt{b} \,|\, \mathtt{d})$ (*transitivity*).

We then also call (X, Γ) a *(partially) cyclically ordered* set.

(b) Any partial cyclic order $\Gamma \equiv (\cdot \,|\, \cdot \,|\, \cdot)$ on any set X is called a *total cyclic order* on X if for any $\{\mathtt{a},\mathtt{b},\mathtt{c}\} \subseteq X$ with $\mathtt{c} \neq \mathtt{a} \neq \mathtt{b} \neq \mathtt{c}$ either $(\mathtt{a} \,|\, \mathtt{b} \,|\, \mathtt{c})$ or $(\mathtt{a} \,|\, \mathtt{c} \,|\, \mathtt{b})$. Then we also say that (X, Γ) is a *totally cyclically ordered* set.
(c) Given any two cyclically ordered sets $(X, (\cdot \,|\, \cdot \,|\, \cdot))$ and $(Y, [\cdot \,|\, \cdot \,|\, \cdot])$, any mapping f from X to Y is called *cyclically monotonic* with respect to $(\cdot \,|\, \cdot \,|\, \cdot)$ and $[\cdot \,|\, \cdot \,|\, \cdot]$ if for any $\{\mathtt{a},\mathtt{b},\mathtt{c}\} \subseteq X$, whenever $[f(\mathtt{a}) \,|\, f(\mathtt{b}) \,|\, f(\mathtt{c})]$, then also $(\mathtt{a} \,|\, \mathtt{b} \,|\, \mathtt{c})$.
(d) Any mapping between cyclically ordered sets is said to be *cyclically strictly monotonic* if it is both cyclically monotonic and injective.
(e) In any totally cyclically ordered set $(X, (\cdot \,|\, \cdot \,|\, \cdot))$ any subset $\mathsf{A} \subseteq X$ is called *convex* if for any $\{\mathtt{a},\mathtt{c}\} \subseteq \mathsf{A}$ with $\mathtt{a} \neq \mathtt{c}$ one of the sets $\{\mathtt{b} \in X \wedge (\mathtt{a} \,|\, \mathtt{b} \,|\, \mathtt{c})\}$ or $\{\mathtt{b} \in X \wedge (\mathtt{c} \,|\, \mathtt{b} \,|\, \mathtt{a})\}$ is contained in A.
(f) With respect to any total cyclic order $(\cdot \,|\, \cdot \,|\, \cdot)$ on any set X any two disjoint subsets $\mathsf{A} \subseteq X$ and $\mathsf{B} \subseteq X$ are said to *cross* each other if there exist $\{\mathtt{a}_1,\mathtt{a}_2\} \subseteq \mathsf{A}$ and $\{\mathtt{b}_1,\mathtt{b}_2\} \subseteq \mathsf{A}$ such that, simultaneously, $(\mathtt{a}_1 \,|\, \mathtt{b}_1 \,|\, \mathtt{a}_2)$ and $(\mathtt{b}_1 \,|\, \mathtt{a}_2 \,|\, \mathtt{b}_2)$ and $(\mathtt{a}_2 \,|\, \mathtt{b}_2 \,|\, \mathtt{a}_1)$. Otherwise, A and B are said to be *non-crossing* with respect to $(\cdot \,|\, \cdot \,|\, \cdot)$.
(g) For any totally cyclically ordered set $(X, (\cdot \,|\, \cdot \,|\, \cdot))$ and any $\mathtt{x} \in X$ the two total linear orders $\{(\mathtt{x},\mathtt{a}) \,|\, \mathtt{a} \in X\} \cup \{(\mathtt{a},\mathtt{b}) \,|\, \{\mathtt{a},\mathtt{b}\} \subseteq X \wedge ((\mathtt{x} \,|\, \mathtt{a} \,|\, \mathtt{b}) \vee \mathtt{a} = \mathtt{b})\}$ and $\{(\mathtt{a},\mathtt{x}) \,|\, \mathtt{a} \in X\} \cup \{(\mathtt{a},\mathtt{b}) \,|\, \{\mathtt{a},\mathtt{b}\} \subseteq X \wedge ((\mathtt{a} \,|\, \mathtt{b} \,|\, \mathtt{x}) \vee \mathtt{a} = \mathtt{b})\}$ on X are called the *left* and *right cut* of $(\cdot \,|\, \cdot \,|\, \cdot)$ at $\mathtt{x}$, respectively .
(h) Conversely, on any totally linearly ordered set $(X, \leq)$ (with associated strict oder $<$) we say that the total cyclic order $\{(\mathtt{a},\mathtt{b},\mathtt{c}) \,|\, \{\mathtt{a},\mathtt{b},\mathtt{c}\} \subseteq X \wedge (\mathtt{a} < \mathtt{b} < \mathtt{c} \vee \mathtt{b} < \mathtt{c} < \mathtt{a} \vee \mathtt{c} < \mathtt{a} < \mathtt{b})\}$ on X is *induced* by $\leq$.
(i) Finally, for any $n \in \mathbb{N}_0$ and any set X with $|X| = n$ and any permutation ν such that $X = \{\nu^{\circ k}(\mathtt{x})\}_{k=0}^{n-1}$ for some (or, equivalently, any) $\mathtt{x} \in X$ we say that the total cyclic order $\{(\mathtt{a}, \nu^{\circ k}(\mathtt{a}), \nu^{\circ \ell}(\mathtt{a})) \,|\, \mathtt{a} \in X \wedge \{k, \ell\} \subseteq [\![n-1]\!] \wedge k < \ell\}$ of X is *associated* with ν.

For example, the following will be important in the proof of Proposition 4.12.

REMARK 3.4. (a) For any totally cyclically ordered sets $(X, (\cdot \,|\, \cdot \,|\, \cdot))$ and $(Y, [\cdot \,|\, \cdot \,|\, \cdot])$, any mapping f from X to Y is cyclically monotonic with respect to $(\cdot\,|\,\cdot\,|\,\cdot)$ and $[\cdot\,|\,\cdot\,|\,\cdot]$ if and only if for any $\{\mathtt{a},\mathtt{b},\mathtt{c}\} \subseteq X$ with $f(\mathtt{c}) \neq f(\mathtt{a}) \neq f(\mathtt{b}) \neq f(\mathtt{c})$, whenever $(\mathtt{a}\,|\,\mathtt{b}\,|\,\mathtt{c})$, then also $[f(\mathtt{a})\,|\,f(\mathtt{b})\,|\,f(\mathtt{c})]$.

(b) In any totally cyclically ordered set (X,Γ) any subset $\mathsf{A} \subseteq X$ is convex if and only if the sets A and $X\backslash\mathsf{A}$ are non-crossing with respect to Γ.

3.2. Points. The "labeled partitions" to be defined will each have four ingredients. The first one will be their set of "points". The following definitions and symbols will be used to refer to those efficiently. (The other constituents of a labeled partition will be its "blocks", "tags" and "colors", covered in the next sections.)

Definition 3.5. Let $\{k,\ell\} \subseteq \mathbb{N}_0$ be arbitrary.

(a) Fix any two injections $i \mapsto {}^{\blacksquare}i$ and $j \mapsto {}_{\blacksquare}j$ defined on $\mathbb{N}$ with ${}^{\blacksquare}i \neq {}_{\blacksquare}j$ for any $\{i,j\} \subseteq \mathbb{N}$.

(b) We call $\Pi^k_\ell \coloneqq \{{}^{\blacksquare}i, {}_{\blacksquare}j \,|\, i \in [\![k]\!] \wedge j \in [\![\ell]\!]\}$ the *total set of* k upper and ℓ lower *points.*

(c) Given any set Y, any $\{k,\ell\} \subseteq \mathbb{N}_0$ and any mappings $f : [\![k]\!] \to Y$ and $g : [\![\ell]\!] \to Y$ we write $f\,{}^{\blacksquare}{}_{\blacksquare}\,g$ for the mapping $\Pi^k_\ell \to Y$ defined by ${}^{\blacksquare}i \mapsto f(i)$ and ${}_{\blacksquare}j \mapsto g(j)$ for any $i \in [\![k]\!]$ and $j \in [\![\ell]\!]$.

(d) The *successor function* for k upper and ℓ lower points is the permutation ν^k_ℓ of Π^k_ℓ defined by ${}^{\blacksquare}i \mapsto {}^{\blacksquare}(i-1)$ and ${}_{\blacksquare}j \mapsto {}_{\blacksquare}(j+1)$ for all $\{i,j\} \subseteq \mathbb{N}$ with $1 < i \leq k$ and $1 \leq j < \ell$, by ${}_{\blacksquare}\ell \mapsto {}^{\blacksquare}k$ if $k \neq 0 \neq \ell$ and ${}_{\blacksquare}\ell \mapsto {}_{\blacksquare}1$ if $k = 0 < \ell$ and by ${}^{\blacksquare}1 \mapsto {}_{\blacksquare}1$ if $k \neq 0 \neq \ell$ and ${}^{\blacksquare}1 \mapsto {}^{\blacksquare}k$ if $\ell = 0 < k$.

(e) By definition, the *cyclic order* $\Gamma^k_\ell \equiv (\cdot\,|\,\cdot\,|\,\cdot)^k_\ell$ for k upper and ℓ lower points is the total cyclic order associated with the successor function ν^k_ℓ.

(f) For any two sets $\mathsf{S} \subseteq \Pi^k_\ell$ and $\mathsf{T} \subseteq \Pi^k_\ell$ with $\mathsf{S} \cap \mathsf{T} = \varnothing$ we write $\mathsf{S} \mathbin{\rlap{+}\smallsetminus}^k_\ell \mathsf{T}$ if S and T cross with respesct to Γ^k_ℓ and $\mathsf{S} \mathbin{\smallsetminus\!\!\smallsetminus}^k_\ell \mathsf{T}$ otherwise.

(g) Given any $\{\mathtt{a},\mathtt{c}\} \subseteq \Pi^k_\ell$ with, importantly, $\mathtt{a} \neq \mathtt{c}$ the *(open) cyclic interval* with respect to Γ^k_ℓ from $\mathtt{a}$ to $\mathtt{c}$ is the set $]\mathtt{a},\mathtt{c}[^k_\ell \coloneqq \{\mathtt{b} \in \Pi^k_\ell \wedge (\mathtt{a}\,|\,\mathtt{b}\,|\,\mathtt{c})^k_\ell\}$. Moreover, we let $[\mathtt{a},\mathtt{c}[^k_\ell \coloneqq \{\mathtt{a}\} \cup]\mathtt{a},\mathtt{c}[^k_\ell$ and $]\mathtt{a},\mathtt{c}]^k_\ell \coloneqq]\mathtt{a},\mathtt{c}[^k_\ell \cup \{\mathtt{c}\}$ as well as $[\mathtt{a},\mathtt{c}]^k_\ell \coloneqq \{\mathtt{a}\} \cup]\mathtt{a},\mathtt{c}[^k_\ell \cup \{\mathtt{c}\}$, speaking of *right-open*, *left-open* and *closed intervals*, respectively.

(h) The *linear order* $\leq^k_\ell$ for k upper and ℓ lower points is defined as the left cut at ${}^{\blacksquare}k$ if $0 < k$ and the right cut at ${}_{\blacksquare}\ell$ if $0 < \ell$ (which is the same if both $0 < k$ and $0 < \ell$). We write $<^k_\ell$ for the associated strict linear order.

(i) The *horizontal reflection* onto k upper and ℓ lower points is the bijection $\kappa^k_\ell \colon \Pi^\ell_k \to \Pi^k_\ell$ with ${}^{\blacksquare}j \mapsto {}_{\blacksquare}j$ and ${}_{\blacksquare}i \mapsto {}^{\blacksquare}i$ for any $j \in [\![\ell]\!]$ and $i \in [\![k]\!]$.

(j) In contrast, the *vertical reflection* of k upper and ℓ lower points is the bijection $\rho^k_\ell \colon \Pi^k_\ell \to \Pi^k_\ell$ defined by ${}^{\blacksquare}i \mapsto {}^{\blacksquare}(k-i+1)$ and ${}_{\blacksquare}j \mapsto {}_{\blacksquare}(\ell-j+1)$ for any $i \in [\![k]\!]$ and $j \in [\![\ell]\!]$.

(k) For any $r \in \{⤹, ⤸, ⤺, ⤻\}$ the *r-rotation* $\omega^{r,k}_{\ \ \ell}$ onto k upper and ℓ lower points is defined,

   (i) if $0 < k$, as the bijection $\omega^{⤹,k}_{\ \ \ell} \colon \Pi^{k-1}_{\ell+1} \to \Pi^k_\ell$ which satisfies ${}^{\blacksquare}i \mapsto {}^{\blacksquare}(i+1)$ for any $i \in [\![k]\!]\backslash[\![1]\!]$ and ${}_{\blacksquare}1 \mapsto {}^{\blacksquare}1$ and ${}_{\blacksquare}j \mapsto {}_{\blacksquare}(j-1)$ for any $j \in [\![\ell]\!]$.

(ii) if $0 < \ell$, as the bijection $\omega^{\curvearrowleft,k}_{\ell}\colon \Pi^{k+1}_{\ell-1} \to \Pi^{k}_{\ell}$ which satisfies ${}^{\blacksquare}i \mapsto {}^{\blacksquare}i$ for any $i \in [\![k]\!]$ and ${}_{\blacksquare}j \mapsto {}_{\blacksquare}j$ for any $j \in [\![\ell - 1]\!]$ and ${}^{\blacksquare}(k+1) \mapsto {}_{\blacksquare}\ell$.

(iii) if $0 < \ell$, as the bijection $\omega^{\curvearrowright,k}_{\ell}\colon \Pi^{k+1}_{\ell-1} \to \Pi^{k}_{\ell}$ which satisfies ${}^{\blacksquare}i \mapsto {}^{\blacksquare}(i-1)$ for any $i \in [\![k]\!]$ and ${}^{\blacksquare}1 \mapsto {}_{\blacksquare}1$ and ${}_{\blacksquare}j \mapsto {}_{\blacksquare}(j+1)$ for any $j \in [\![\ell]\!]\backslash[\![1]\!]$.

(iv) if $0 < k$, as the bijection $\omega^{\looparrowleft,k}_{\ell}\colon \Pi^{k-1}_{\ell+1} \to \Pi^{k}_{\ell}$ which satisfies ${}_{\blacksquare}(\ell+1) \mapsto {}^{\blacksquare}k$ and ${}^{\blacksquare}i \mapsto {}^{\blacksquare}i$ for any $i \in [\![k-1]\!]$ and ${}_{\blacksquare}j \mapsto {}_{\blacksquare}j$ for any $j \in [\![\ell]\!]$.

(l) For any set $\mathsf{S} \subseteq \Pi^{k}_{\ell}$, firstly, let $\alpha(\mathsf{S}) = |\Pi^{k}_{0} \cap \mathsf{S}|$ be the *upper point count* of S and $\beta(\mathsf{S}) = |\Pi^{0}_{\ell} \cap \mathsf{S}|$ the *lower point count* of S, secondly, let the *upper enumeration* of S be the injection $\eta_{\mathsf{S},\ell}^{\;k} : [\![\alpha(\mathsf{S})]\!] \to \Pi^{k}_{\ell}$ with the graph

$$\{(|\Pi^{i}_{0} \cap \mathsf{S}|, {}^{\blacksquare}i) \mid i \in [\![k]\!] \wedge {}^{\blacksquare}i \in \mathsf{S}\}$$

and the *lower enumeration* of S the injection $\theta_{\mathsf{S},\ell}^{\;k} : [\![\beta(\mathsf{S})]\!] \to \Pi^{k}_{\ell}$ with the graph

$$\{(|\Pi^{0}_{j} \cap \mathsf{S}|, {}_{\blacksquare}j) \mid j \in [\![\ell]\!] \wedge {}_{\blacksquare}j \in \mathsf{S}\},$$

and, thirdly, let the *insertion* onto S for k upper and ℓ lower points be the injection $\Pi^{\alpha(\mathsf{S})}_{\beta(\mathsf{S})} \to \Pi^{k}_{\ell}$ defined by $\gamma_{\mathsf{S},\ell}^{\;k} \coloneqq \eta_{\mathsf{S},\ell}^{\;k} \,{}^{\blacksquare}{}_{\blacksquare}\, \theta_{\mathsf{S},\ell}^{\;k}$.

3.3. Tags, colors and labels. While [MW20] and [MW21a] only dealt with categories of two-colored partitions, now partitions with arbitrary labels, some two-colored, some uncolored, are required. The setting here is equivalent to the one first considered by Freslon in [Fre17].

Assumption 3.6. We fix any two sets $\circ$ and $\bullet$ with $\circ \neq \bullet$. Moreover, we let $\overline{\circ} \coloneqq \bullet$ and $\overline{\bullet} \coloneqq \circ$ and $\sigma(\circ) \coloneqq 1$ and $\sigma(\bullet) \coloneqq -1$.

Definition 3.7. A *choice of tags* is any pair $(\mathfrak{U}, \mathfrak{O})$ of countable sets $\mathfrak{U}$ and $\mathfrak{O}$ with $\mathfrak{U} \cap \mathfrak{O} = \varnothing$ and $(\mathfrak{U} \otimes \{\circ, \bullet\}) \cap \mathfrak{O} = \varnothing$.

Assumptions 3.8. In the following, let $(\mathfrak{U}, \mathfrak{O})$ be any choice of tags.

The role of the "color set" $\mathcal{A}$ in the sense of [Fre17, Definition 4] will be played by $(\mathfrak{U} \otimes \{\circ, \bullet\}) \uplus \mathfrak{O}$, where the involution is defined by $(x, c) \mapsto (x, \overline{c})$ for any $x \in \mathfrak{U}$ and $c \in \{\circ, \bullet\}$ and by $y \mapsto y$ for any $y \in \mathfrak{O}$.

Definition 3.9. Let $\{k, \ell\} \subseteq \mathbb{N}_0$ as well as $\mathfrak{c}\colon [\![k]\!] \to (\mathfrak{U} \otimes \{\circ, \bullet\}) \uplus \mathfrak{O}$ and $\mathfrak{d}\colon [\![\ell]\!] \to (\mathfrak{U} \otimes \{\circ, \bullet\}) \uplus \mathfrak{O}$ be arbitrary.

(a) The elements of $\mathfrak{U} \uplus \mathfrak{O}$ are called *tags*, those of $\mathfrak{U}$ *unitary tags* and those of $\mathfrak{O}$ *orthogonal tags*.

(b) The *tag function* for k upper $\mathfrak{c}$-labeled and ℓ lower $\mathfrak{d}$-labeld points is the mapping $\xi^{\mathfrak{c}}_{\mathfrak{d}}\colon \Pi^{k}_{\ell} \to \mathfrak{U} \uplus \mathfrak{O}$ which for any $x \in \mathfrak{U}$, any $y \in \mathfrak{O}$, any $i \in [\![k]\!]$ and any $j \in [\![\ell]\!]$ satisfies ${}^{\blacksquare}i \mapsto x$ if $\mathfrak{c}_i \in \{x\} \otimes \{\circ, \bullet\}$ and ${}^{\blacksquare}i \mapsto y$ if $\mathfrak{c}_i = y$ and ${}_{\blacksquare}j \mapsto x$ if $\mathfrak{d}_j \in \{x\} \otimes \{\circ, \bullet\}$ and ${}_{\blacksquare}j \mapsto y$ if $\mathfrak{d}_j = y$.

(c) For any $z \in \mathfrak{U} \uplus \mathfrak{O}$ we call $\xi^{\mathfrak{c}\leftarrow}_{\mathfrak{d}}(\{z\})$ the *z-area* of k upper $\mathfrak{c}$-labeled and ℓ lower $\mathfrak{d}$-labeled points. Moreover, we speak of $\xi^{\mathfrak{c}\leftarrow}_{\mathfrak{d}}(\mathfrak{U})$ and $\xi^{\mathfrak{c}\leftarrow}_{\mathfrak{d}}(\mathfrak{O})$ as the *unitary* and *orthogonal areas*, respectively.

(d) The *normalized color function* for k upper $\mathfrak{c}$-labeled and ℓ lower $\mathfrak{d}$-labeled points is the mapping $\zeta_{\mathfrak{d}}^{\mathfrak{c}}\colon \xi_{\ell}^{k\leftarrow}(\mathfrak{U}) \to \{\circ,\bullet\}$ which for any $c \in \{\circ,\bullet\}$, any $i \in [\![k]\!]$ and any $j \in [\![\ell]\!]$ satisfies $^{\blacksquare}i \mapsto \overline{c}$ if and only if $\mathfrak{c}_i \in \mathfrak{U} \otimes \{c\}$ and $_{\blacksquare}j \mapsto c$ if and only if $\mathfrak{d}_j \in \mathfrak{U} \otimes \{c\}$.

(e) By definition, for any $z \in \mathfrak{U}$, the *z-color sum* for k upper $\mathfrak{c}$-labeled and ℓ lower $\mathfrak{d}$-labeled points is the $\mathbb{Z}$-valued measure ${}_z\sigma_{\mathfrak{d}}^{\mathfrak{c}}$ on Π_{ℓ}^{k} which has density $\sigma \circ \zeta_{\mathfrak{d}}^{\mathfrak{c}}$ on $\xi_{\mathfrak{d}}^{\mathfrak{c}\leftarrow}(\{z\})$ and density 0 everywhere else.

(f) Moreover, for any $z \in \mathfrak{U}$ we write ${}_z\Sigma_{\mathfrak{d}}^{\mathfrak{c}} \coloneqq {}_z\sigma_{\mathfrak{d}}^{\mathfrak{c}}(\Pi_{\ell}^{k})$ for the *total z-color sum* of k upper $\mathfrak{c}$-labeled and ℓ lower $\mathfrak{d}$-labeled points.

(g) For any $z \in \mathfrak{U}$ we define the *z-color distance* for k upper $\mathfrak{c}$-labeled and ℓ lower $\mathfrak{d}$-labeled points as the mapping ${}_z\delta_{\mathfrak{d}}^{\mathfrak{c}}\colon \Pi_{\ell}^{k} \otimes \Pi_{\ell}^{k} \to \mathbb{Q}$ with

$$(\mathtt{a},\mathtt{b}) \mapsto \begin{cases} 0 & \text{if } \mathtt{a} = \mathtt{b}, \\ \frac{1}{2}{}_z\sigma_{\mathfrak{d}}^{\mathfrak{c}}(\{\mathtt{a}\}) + {}_z\sigma_{\mathfrak{d}}^{\mathfrak{c}}(]\mathtt{a},\mathtt{b}[_{\ell}^{k}) + \frac{1}{2}{}_z\sigma_{\mathfrak{d}}^{\mathfrak{c}}(\{\mathtt{b}\}) & \text{otherwise} \end{cases}$$

for any $\{\mathtt{a},\mathtt{b}\} \subseteq \Pi_{\ell}^{k}$.

The following is a generalization of [MW21a, Lemma 3.2].

LEMMA 3.10. *For any $\{k,\ell\} \subseteq \mathbb{N}_0$, any $\mathfrak{c}\colon [\![k]\!] \to (\mathfrak{U} \otimes \{\circ,\bullet\}) \cup \mathfrak{O}$ and $\mathfrak{d}\colon [\![\ell]\!] \to (\mathfrak{U} \otimes \{\circ,\bullet\}) \cup \mathfrak{O}$, any $x \in \mathfrak{U}$ and any $m \in \mathbb{Z}$, if ${}_x\Sigma_{\mathfrak{d}}^{\mathfrak{c}} \equiv_m 0$, then*

(a) ${}_x\delta_{\mathfrak{d}}^{\mathfrak{c}}(\mathtt{a},\mathtt{a}) \equiv_m 0$ for any $\mathtt{a} \in \Pi_{\ell}^{k}$.

(b) ${}_x\delta_{\mathfrak{d}}^{\mathfrak{c}}(\mathtt{b},\mathtt{a}) \equiv_m -{}_x\delta_{\mathfrak{d}}^{\mathfrak{c}}(\mathtt{a},\mathtt{b})$ for any $\{\mathtt{a},\mathtt{b}\} \subseteq \Pi_{\ell}^{k}$.

(c) ${}_x\delta_{\mathfrak{d}}^{\mathfrak{c}}(\mathtt{a},\mathtt{c}) \equiv_m {}_x\delta_{\mathfrak{d}}^{\mathfrak{c}}(\mathtt{a},\mathtt{b}) + {}_x\delta_{\mathfrak{d}}^{\mathfrak{c}}(\mathtt{b},\mathtt{c})$ for any $\{\mathtt{a},\mathtt{b},\mathtt{c}\} \subseteq \Pi_{\ell}^{k}$.

PROOF. (a) The even stronger ${}_x\delta_{\mathfrak{d}}^{\mathfrak{c}}(\mathtt{a},\mathtt{a}) = 0$ already holds by definition.

(b) The case $\mathtt{a} = \mathtt{b}$ is covered by (a). If $\mathtt{a} \neq \mathtt{b}$, then by the addivitiy of ${}_x\sigma_{\mathfrak{d}}^{\mathfrak{c}}$,

$$\begin{aligned} {}_x\sigma_{\mathfrak{d}}^{\mathfrak{c}}(]\mathtt{b},\mathtt{a}[_{\ell}^{k}) &= {}_x\sigma_{\mathfrak{d}}^{\mathfrak{c}}(\Pi_{\ell}^{k}\backslash[\mathtt{a},\mathtt{b}]_{\ell}^{k}) \\ &= {}_x\sigma_{\mathfrak{d}}^{\mathfrak{c}}(\Pi_{\ell}^{k}) - {}_x\sigma_{\mathfrak{d}}^{\mathfrak{c}}([\mathtt{a},\mathtt{b}]_{\ell}^{k}) \\ &= {}_x\Sigma_{\mathfrak{d}}^{\mathfrak{c}} - {}_x\sigma_{\mathfrak{d}}^{\mathfrak{c}}(\{\mathtt{a}\}) - {}_x\sigma_{\mathfrak{d}}^{\mathfrak{c}}(]\mathtt{a},\mathtt{b}[_{\ell}^{k}) - {}_x\sigma_{\mathfrak{d}}^{\mathfrak{c}}(\{\mathtt{b}\}) \end{aligned}$$

and thus

$$\begin{aligned} {}_x\delta_{\mathfrak{d}}^{\mathfrak{c}}(\mathtt{b},\mathtt{a}) &= \tfrac{1}{2}{}_x\sigma_{\mathfrak{d}}^{\mathfrak{c}}(\{\mathtt{b}\}) + {}_x\sigma_{\mathfrak{d}}^{\mathfrak{c}}(]\mathtt{b},\mathtt{a}[_{\ell}^{k}) + \tfrac{1}{2}{}_x\sigma_{\mathfrak{d}}^{\mathfrak{c}}(\{\mathtt{a}\}) \\ &= {}_x\Sigma_{\mathfrak{d}}^{\mathfrak{c}} - \tfrac{1}{2}{}_x\sigma_{\mathfrak{d}}^{\mathfrak{c}}(\{\mathtt{a}\}) - {}_x\sigma_{\mathfrak{d}}^{\mathfrak{c}}(]\mathtt{a},\mathtt{b}[_{\ell}^{k}) - \tfrac{1}{2}{}_x\sigma_{\mathfrak{d}}^{\mathfrak{c}}(\{\mathtt{b}\}) \\ &= {}_x\Sigma_{\mathfrak{d}}^{\mathfrak{c}} - {}_x\delta_{\mathfrak{d}}^{\mathfrak{c}}(\mathtt{a},\mathtt{b}), \end{aligned}$$

from which the assertion follows by ${}_x\Sigma_{\mathfrak{d}}^{\mathfrak{c}} \equiv_m 0$.

(c) Whenever at least two of $\mathtt{a}$, $\mathtt{b}$ and $\mathtt{c}$ coincide, the claim holds by (a) and (b). In the case where $\mathtt{a}$, $\mathtt{b}$ and $\mathtt{c}$ are pairwise distinct, either $(\mathtt{a}\mid\mathtt{b}\mid\mathtt{c})_{\ell}^{k}$ or $(\mathtt{a}\mid\mathtt{c}\mid\mathtt{b})_{\ell}^{k}$.

If $(\mathtt{a}\mid\mathtt{b}\mid\mathtt{c})_{\ell}^{k}$, then

$${}_x\sigma_{\mathfrak{d}}^{\mathfrak{c}}(]\mathtt{a},\mathtt{c}[_{\ell}^{k}) = {}_x\sigma_{\mathfrak{d}}^{\mathfrak{c}}(]\mathtt{a},\mathtt{b}[_{\ell}^{k} \cup \{\mathtt{b}\} \cup]\mathtt{b},\mathtt{c}[_{\ell}^{k}) = {}_x\sigma_{\mathfrak{d}}^{\mathfrak{c}}(]\mathtt{a},\mathtt{b}[_{\ell}^{k}) + {}_x\sigma_{\mathfrak{d}}^{\mathfrak{c}}(\{\mathtt{b}\}) + {}_x\sigma_{\mathfrak{d}}^{\mathfrak{c}}(]\mathtt{b},\mathtt{c}[_{\ell}^{k})$$

by the additivity of ${}_x\sigma^{\mathfrak{c}}_{\mathfrak{d}}$, and thus

$$\begin{aligned}
&{}_x\delta^{\mathfrak{c}}_{\mathfrak{d}}(\mathtt{a},\mathtt{c})\\
&= \tfrac{1}{2}{}_x\sigma^{\mathfrak{c}}_{\mathfrak{d}}(\{\mathtt{a}\}) + {}_x\sigma^{\mathfrak{c}}_{\mathfrak{d}}(]\mathtt{a},\mathtt{c}[^k_\ell) + \tfrac{1}{2}{}_x\sigma^{\mathfrak{c}}_{\mathfrak{d}}(\{\mathtt{c}\})\\
&= \big(\tfrac{1}{2}{}_x\sigma^{\mathfrak{c}}_{\mathfrak{d}}(\{\mathtt{a}\}) + {}_x\sigma^{\mathfrak{c}}_{\mathfrak{d}}(]\mathtt{a},\mathtt{b}[^k_\ell) + \tfrac{1}{2}{}_x\sigma^{\mathfrak{c}}_{\mathfrak{d}}(\{\mathtt{b}\})\big) + \big(\tfrac{1}{2}{}_x\sigma^{\mathfrak{c}}_{\mathfrak{d}}(\{\mathtt{b}\}) + {}_x\sigma^{\mathfrak{c}}_{\mathfrak{d}}(]\mathtt{b},\mathtt{c}[^k_\ell) + \tfrac{1}{2}{}_x\sigma^{\mathfrak{c}}_{\mathfrak{d}}(\{\mathtt{c}\})\big)\\
&= {}_x\delta^{\mathfrak{c}}_{\mathfrak{d}}(\mathtt{a},\mathtt{b}) + {}_x\delta^{\mathfrak{c}}_{\mathfrak{d}}(\mathtt{b},\mathtt{c}),
\end{aligned}$$

which proves the claim.

If, instead, $(\mathtt{a}\mid\mathtt{c}\mid\mathtt{b})^k_\ell$, then first,

$$\begin{aligned}
&{}_x\sigma^{\mathfrak{c}}_{\mathfrak{d}}([\mathtt{c},\mathtt{b}[^k_\ell)\\
&\quad= {}_x\sigma^{\mathfrak{c}}_{\mathfrak{d}}(\Pi^k_\ell\backslash[\mathtt{b},\mathtt{c}[^k_\ell) = {}_x\sigma^{\mathfrak{c}}_{\mathfrak{d}}(\Pi^k_\ell) - {}_x\sigma^{\mathfrak{c}}_{\mathfrak{d}}([\mathtt{b},\mathtt{c}[^k_\ell) = {}_x\Sigma^{\mathfrak{c}}_{\mathfrak{d}} - {}_x\sigma^{\mathfrak{c}}_{\mathfrak{d}}(\{\mathtt{b}\}) - {}_x\sigma^{\mathfrak{c}}_{\mathfrak{d}}(]\mathtt{b},\mathtt{c}[^k_\ell)
\end{aligned}$$

and thus

$$\begin{aligned}
{}_x\sigma^{\mathfrak{c}}_{\mathfrak{d}}(]\mathtt{a},\mathtt{c}[^k_\ell) &= {}_x\sigma^{\mathfrak{c}}_{\mathfrak{d}}(]\mathtt{a},\mathtt{b}[^k_\ell\backslash[\mathtt{c},\mathtt{b}[^k_\ell)\\
&= {}_x\sigma^{\mathfrak{c}}_{\mathfrak{d}}(]\mathtt{a},\mathtt{b}[^k_\ell) - {}_x\sigma^{\mathfrak{c}}_{\mathfrak{d}}([\mathtt{c},\mathtt{b}[^k_\ell)\\
&= {}_x\sigma^{\mathfrak{c}}_{\mathfrak{d}}(]\mathtt{a},\mathtt{b}[^k_\ell) - {}_x\Sigma^{\mathfrak{c}}_{\mathfrak{d}} + {}_x\sigma^{\mathfrak{c}}_{\mathfrak{d}}(\{\mathtt{b}\}) + {}_x\sigma^{\mathfrak{c}}_{\mathfrak{d}}(]\mathtt{b},\mathtt{c}[^k_\ell),
\end{aligned}$$

which then implies

$$\begin{aligned}
{}_x\delta^{\mathfrak{c}}_{\mathfrak{d}}(\mathtt{a},\mathtt{c}) &= \tfrac{1}{2}{}_x\sigma^{\mathfrak{c}}_{\mathfrak{d}}(\{\mathtt{a}\}) + {}_x\sigma^{\mathfrak{c}}_{\mathfrak{d}}(]\mathtt{a},\mathtt{c}[^k_\ell) + \tfrac{1}{2}{}_x\sigma^{\mathfrak{c}}_{\mathfrak{d}}(\{\mathtt{c}\})\\
&= -{}_x\Sigma^{\mathfrak{c}}_{\mathfrak{d}} + \big(\tfrac{1}{2}{}_x\sigma^{\mathfrak{c}}_{\mathfrak{d}}(\{\mathtt{a}\}) + {}_x\sigma^{\mathfrak{c}}_{\mathfrak{d}}(]\mathtt{a},\mathtt{b}[^k_\ell) + \tfrac{1}{2}{}_x\sigma^{\mathfrak{c}}_{\mathfrak{d}}(\{\mathtt{b}\})\big)\\
&\qquad + \big(\tfrac{1}{2}{}_x\sigma^{\mathfrak{c}}_{\mathfrak{d}}(\{\mathtt{b}\}) + {}_x\sigma^{\mathfrak{c}}_{\mathfrak{d}}(]\mathtt{b},\mathtt{c}[^k_\ell) + \tfrac{1}{2}{}_x\sigma^{\mathfrak{c}}_{\mathfrak{d}}(\{\mathtt{c}\})\big)\\
&= -{}_x\Sigma^{\mathfrak{c}}_{\mathfrak{d}} + {}_x\delta^{\mathfrak{c}}_{\mathfrak{d}}(\mathtt{a},\mathtt{b}) + {}_x\delta^{\mathfrak{c}}_{\mathfrak{d}}(\mathtt{b},\mathtt{c}),
\end{aligned}$$

whence the claim follows by ${}_x\Sigma^{\mathfrak{c}}_{\mathfrak{d}} \equiv_m 0$. □

3.4. Labeled partitions. With the definitions of points, blocks, tags and colors at hand, we are ready for the definition of labeled partitions. Again, these are called "colored partitions" in [Fre17].

DEFINITION 3.11. (a) For any $\{k,\ell\}\subseteq\mathbb{N}_0$ and any $\mathfrak{c}\colon [\![k]\!]\to(\mathfrak{U}\otimes\{\circ,\bullet\})\cup\mathfrak{O}$ and $\mathfrak{d}\colon [\![\ell]\!]\to(\mathfrak{U}\otimes\{\circ,\bullet\})\cup\mathfrak{O}$ a $(\mathfrak{U},\mathfrak{O})$-*tagged partition* of k upper $\mathfrak{c}$-labeled and ℓ lower $\mathfrak{d}$-labeled points is any triple $(\mathfrak{c},\mathfrak{d},p)$ where p is a set-theoretical partition of Π^k_ℓ. If the tags $(\mathfrak{U},\mathfrak{O})$ are clear from context, we also speak of a $(\mathfrak{c},\mathfrak{d})$-*labeled partition* for short.

(b) For any $\{k,\ell\}\subseteq\mathbb{N}_0$ and any $\mathfrak{c}\colon [\![k]\!]\to(\mathfrak{U}\otimes\{\circ,\bullet\})\cup\mathfrak{O}$ and $\mathfrak{d}\colon [\![\ell]\!]\to(\mathfrak{U}\otimes\{\circ,\bullet\})\cup\mathfrak{O}$ the set of all $(\mathfrak{c},\mathfrak{d})$-labeled partitions is denoted by ${}_{\mathfrak{U},\mathfrak{O}}\mathcal{S}(\mathfrak{c},\mathfrak{d})$.

(c) Moreover, let ${}_{\mathfrak{U},\mathfrak{O}}\mathcal{S} \coloneqq \bigcup\{{}_{\mathfrak{U},\mathfrak{O}}\mathcal{S}(\mathfrak{c},\mathfrak{d}) \mid \{k,\ell\}\subseteq\mathbb{N}_0 \wedge \mathfrak{c}\colon [\![k]\!]\to(\mathfrak{U}\otimes\{\circ,\bullet\})\cup\mathfrak{O} \wedge \mathfrak{d}\colon [\![\ell]\!]\to(\mathfrak{U}\otimes\{\circ,\bullet\})\cup\mathfrak{O}\}$.

REMARK 3.12. (a) If $\mathfrak{U}=\varnothing$ and $|\mathfrak{O}|=1$, a $(\mathfrak{U},\mathfrak{O})$-tagged labeled partition is the same as a *partition* in the sense of [BS09].

(b) In the case $|\mathfrak{U}|=1$ and $\mathfrak{O}=\varnothing$, the $(\mathfrak{U},\mathfrak{O})$-tagged labeled partitions are the same as *two-colored partitions* in the sense of [TW18].

3.5. Categories of labeled partitions. Categories of labeled partitions can be defined in several equivalent ways. For example, Proposition 3.26 shows that there at least two. The results of this section are straightforward generalizations of the corresponding statements for uncolored and two-colored partitions from [BS09], [TW18] and [Fre17].

Definition 3.13. For any $\{k,\ell,m\} \subseteq \mathbb{N}_0$, any $\mathfrak{a}\colon [\![k]\!] \to (\mathfrak{U}\otimes\{\circ,\bullet\})\cup\mathfrak{O}$ and $\mathfrak{b}\colon [\![\ell]\!] \to (\mathfrak{U}\otimes\{\circ,\bullet\})\cup\mathfrak{O}$ and $\mathfrak{c}\colon [\![m]\!] \to (\mathfrak{U}\otimes\{\circ,\bullet\})\cup\mathfrak{O}$ and any partitions p of Π^k_ℓ and q of Π^ℓ_m, the *composition* of $((\mathfrak{b},\mathfrak{c},q),(\mathfrak{a},\mathfrak{b},p))$ is defined as $(\mathfrak{a},\mathfrak{c},qp)$, where

$$
\begin{aligned}
&qp \coloneqq \\
&\{\mathsf{A}\in p \wedge \mathsf{A}\subseteq \Pi^k_0\}\\
&\cup\Big\{\bigcup\{\mathsf{A}\cap\Pi^k_0 \,|\, \mathsf{A}\in p\wedge \mathsf{A}\cap(\kappa^0_\ell)_{\rightarrow}(\mathsf{B})\neq\varnothing\}\cup\bigcup\{\mathsf{C}\cap\Pi^0_m\,|\,\mathsf{C}\in q\wedge \mathsf{C}\cap\mathsf{B}\neq\varnothing\}\Big\}_{\mathsf{B}\in s}\backslash\{\varnothing\}\\
&\cup\{\mathsf{C}\in q\wedge \mathsf{C}\subseteq\Pi^0_m\}
\end{aligned}
$$

of Π^k_m, where

$$s = ((\kappa^0_\ell)^{\leftarrow}(p|_{\Pi^0_\ell}))\vee(q|_{\Pi^\ell_0}).$$

Lastly, for the sake of uniformity, we can also write $(\mathfrak{b}\,{}^\blacksquare{}_\blacksquare\,\mathfrak{c})(\mathfrak{a}\,{}^\blacksquare{}_\blacksquare\,\mathfrak{b}) \coloneqq \mathfrak{a}\,{}^\blacksquare{}_\blacksquare\,\mathfrak{c}$.

Definition 3.14. For any $\{k,\ell\}\subseteq\mathbb{N}_0$, any $\mathfrak{c}\colon [\![k]\!]\to(\mathfrak{U}\otimes\{\circ,\bullet\})\cup\mathfrak{O}$ and $\mathfrak{d}\colon [\![\ell]\!]\to(\mathfrak{U}\otimes\{\circ,\bullet\})\cup\mathfrak{O}$ and any partition p of Π^k_ℓ the *adjoint* of $(\mathfrak{c},\mathfrak{d},p)$ is defined as $(\mathfrak{c},\mathfrak{d},p)^*\coloneqq(\mathfrak{d},\mathfrak{c},p^*)$, where $p^*\coloneqq(\kappa^k_\ell)^{\leftarrow}(p)$. Moreover, let $(\mathfrak{c}\,{}^\blacksquare{}_\blacksquare\,\mathfrak{d})^*\coloneqq\mathfrak{d}\,{}^\blacksquare{}_\blacksquare\,\mathfrak{c}$.

Definition 3.15. For any $k\in\mathbb{N}_0$ and $\mathfrak{c}\colon [\![k]\!]\to(\mathfrak{U}\otimes\{\circ,\bullet\})\cup\mathfrak{O}$ let $\mathrm{id}_\mathfrak{c}\coloneqq(\mathfrak{c},\mathfrak{c},\mathrm{id}_k)$ be the *identity* of $\mathfrak{c}$, where $\mathrm{id}_k\coloneqq |^{\otimes k}\equiv\{\{{}^\blacksquare i,{}_\blacksquare i\}\}^k_{i=1}$.

Notation 3.16. For any $\{m_1,m_2\}\subseteq\mathbb{N}_0$, any set X and any mappings $f_1\colon[\![m_1]\!]\to X$ and $f_2\colon[\![m_2]\!]\to X$ let $f_1 \vartriangle f_2$ denote the mapping $[\![m_1+m_2]\!]\to X$ with $i\mapsto f_1(i)$ if $i\le m_1$ and $i\mapsto f_2(i-m_1)$ if $m_1<i$ for any $i\in[\![m_1+m_2]\!]$.

Definition 3.17.
(a) For any $\{m_1,m_2\}\subseteq\mathbb{N}_0$ and $\mathfrak{a}_1\colon[\![m_1]\!]\to(\mathfrak{U}\otimes\{\circ,\bullet\})\cup\mathfrak{O}$ and $\mathfrak{a}_2\colon[\![m_2]\!]\to(\mathfrak{U}\otimes\{\circ,\bullet\})\cup\mathfrak{O}$ define the *tensor product* of $(\mathfrak{a}_1,\mathfrak{a}_2)$ as the mapping $\mathfrak{a}_1\otimes\mathfrak{a}_2\coloneqq\mathfrak{a}_1\vartriangle\mathfrak{a}_2$.
(b) For any $\{k_1,k_2,\ell_1,\ell_2\}\subseteq\mathbb{N}_0$, any $\mathfrak{c}_1\colon[\![k_1]\!]\to(\mathfrak{U}\otimes\{\circ,\bullet\})\cup\mathfrak{O}$ and $\mathfrak{c}_2\colon[\![k_2]\!]\to(\mathfrak{U}\otimes\{\circ,\bullet\})\cup\mathfrak{O}$ as well as any $\mathfrak{d}_1\colon[\![\ell_1]\!]\to(\mathfrak{U}\otimes\{\circ,\bullet\})\cup\mathfrak{O}$ and $\mathfrak{d}_2\colon[\![\ell_2]\!]\to(\mathfrak{U}\otimes\{\circ,\bullet\})\cup\mathfrak{O}$ and any partitions p_1 of $\Pi^{k_1}_{\ell_1}$ and p_2 of $\Pi^{k_2}_{\ell_2}$ let the *tensor product* of $((\mathfrak{c}_1,\mathfrak{d}_1,p_1),(\mathfrak{c}_2,\mathfrak{d}_2,p_2))$ be given by $(\mathfrak{c}_1,\mathfrak{d}_1,p_1)\otimes(\mathfrak{c}_2,\mathfrak{d}_2,p_2)\coloneqq(\mathfrak{c}_1\otimes\mathfrak{c}_2,\mathfrak{d}_1\otimes\mathfrak{d}_2,p_1\otimes p_2)$, where

$$p_1\otimes p_2\coloneqq\textstyle\bigcup_{t=1}^2\{\gamma_{\mathsf{H}_t,\ell_1+\ell_2}^{\;k_1+k_2}{}_{\rightarrow}(\mathsf{B})\,|\,\mathsf{B}\in p_t\},$$

where $\mathsf{H}_1=\Pi^{k_1}_{\ell_1}$ and $\mathsf{H}_2=\Pi^{k_1+k_2}_{\ell_1+\ell_2}\backslash\Pi^{k_1}_{\ell_1}$. Moreover, let $(\mathfrak{c}_1\,{}^\blacksquare{}_\blacksquare\,\mathfrak{d}_1)\otimes(\mathfrak{c}_2\,{}^\blacksquare{}_\blacksquare\,\mathfrak{d}_2)\coloneqq(\mathfrak{c}_1\otimes\mathfrak{c}_2)\,{}^\blacksquare{}_\blacksquare\,(\mathfrak{d}_1\otimes\mathfrak{d}_2)$.

Note the notational change in the following definition of the duals compared to [MW20] and [MW21a].

DEFINITION 3.18. (a) For any $m \in \mathbb{N}_0$ and $\mathfrak{a}\colon [\![m]\!] \to (\mathfrak{U} \otimes \{\circ, \bullet\}) \cup \mathfrak{O}$ define the *dual object* $\mathfrak{a}^\vee\colon [\![m]\!] \to (\mathfrak{U} \otimes \{\circ, \bullet\}) \cup \mathfrak{O}$ of $\mathfrak{a}$ to be the mapping which satisfies for any $i \in [\![m]\!]$ with $\mathfrak{a}(i) \in \mathfrak{U} \otimes \{\circ, \bullet\}$, if $\mathfrak{a}(m-i+1) = (x, c)$, then $i \mapsto (x, \overline{c})$ and which satisfies $i \mapsto \mathfrak{a}(m-i+1)$ for each $i \in [\![m]\!]$ with $\mathfrak{a}(i) \in \mathfrak{O}$.

(b) Moreover, for any $m \in \mathbb{N}_0$ and $\mathfrak{a}\colon [\![m]\!] \to (\mathfrak{U} \otimes \{\circ, \bullet\}) \cup \mathfrak{O}$ let the *(left) evaluation* of $\mathfrak{a}$ be defined as $\mathrm{ev}_\mathfrak{a} \coloneqq (\mathfrak{a}^\vee \otimes \mathfrak{a}, \varnothing, \mathrm{ev}_m)$, where

$$\mathrm{ev}_m \coloneqq \{\{{}^{\blacksquare}i, {}^{\blacksquare}(2m-i+1)\}\}_{i=1}^m.$$

(c) Likewise, for any $m \in \mathbb{N}_0$ and $\mathfrak{a}\colon [\![m]\!] \to (\mathfrak{U} \otimes \{\circ, \bullet\}) \cup \mathfrak{O}$ we call $\mathrm{coev}_\mathfrak{a} \coloneqq (\varnothing, \mathfrak{a} \otimes \mathfrak{a}^\vee, \mathrm{coev}_m)$, where

$$\mathrm{coev}_m \coloneqq \{\{{}_{\blacksquare}j, {}_{\blacksquare}(2m-j+1)\}\}_{j=1}^m,$$

the *(left) coevaluation* of $\mathfrak{a}$.

(d) For any $\{k, \ell\} \subseteq \mathbb{N}_0$, any $\mathfrak{c}\colon [\![k]\!] \to (\mathfrak{U} \otimes \{\circ, \bullet\}) \cup \mathfrak{O}$ and $\mathfrak{d}\colon [\![\ell]\!] \to (\mathfrak{U} \otimes \{\circ, \bullet\}) \cup \mathfrak{O}$ and any partition p of Π_ℓ^k the *(left) dual* of $(\mathfrak{c}, \mathfrak{d}, p)$ is defined as $(\mathfrak{c}, \mathfrak{d}, p)^\vee \coloneqq (\mathfrak{d}^\vee, \mathfrak{c}^\vee, p^\vee)$, where $p^\vee \coloneqq (\kappa_\ell^k \circ \rho_\ell^k)^{\leftarrow}(p)$. Moreover, let $(\mathfrak{c} \,{}^{\blacksquare}_{\blacksquare}\, \mathfrak{d})^\vee \coloneqq (\mathfrak{d}^\vee) \,{}^{\blacksquare}_{\blacksquare}\, (\mathfrak{c}^\vee)$.

DEFINITION 3.19. A category of $(\mathfrak{U}, \mathfrak{O})$-tagged labeled partitions is any subset of ${}_{\mathfrak{U},\mathfrak{O}}\mathcal{S}$ which includes for any $x \in \mathfrak{U}$ and $y \in \mathfrak{O}$ the labeled partitions

$$\mathrm{id}_{(x,\circ)},\ \mathrm{id}_{(x,\bullet)},\ \mathrm{id}_y,\ \mathrm{coev}_{(x,\circ)},\ \mathrm{coev}_{(x,\bullet)},\ \mathrm{coev}_y$$

and which is closed under composition, tensor products and forming adjoints.

LEMMA 3.20. *Any category of labeled partitions is closed under forming duals.*

PROOF. Analogous to the proof of [TW18, Lemma 1.1 (a)]. Note that the verticolor reflection is the adjoint of the dual. □

DEFINITION 3.21. Let $\{k, \ell\} \subseteq \mathbb{N}_0$ and $\mathfrak{c}\colon [\![k]\!] \to (\mathfrak{U} \otimes \{\circ, \bullet\}) \cup \mathfrak{O}$ and $\mathfrak{d}\colon [\![\ell]\!] \to (\mathfrak{U} \otimes \{\circ, \bullet\}) \cup \mathfrak{O}$ be arbitrary and let p be any partition of Π_ℓ^k.

(a) For any $r \in \{\Lsh, \Rsh, \looparrowleft, \looparrowright\}$ such that $\omega_\ell^{r,k}$ is defined, the *r-rotation* of $(\mathfrak{c}, \mathfrak{d}, p)$ is defined as $(\mathfrak{c}, \mathfrak{d}, p)^r \coloneqq (\mathfrak{a}, \mathfrak{b}, p^r)$, where $\mathfrak{a}$ and $\mathfrak{b}$ are given by
  (i) $\mathfrak{a} = (\mathfrak{c}_2, \mathfrak{c}_3 \ldots, \mathfrak{c}_k)$ and $\mathfrak{b} = (\mathfrak{c}_1{}^\vee, \mathfrak{d}_1, \mathfrak{d}_2, \ldots, \mathfrak{d}_\ell)$ if $r = \Lsh$,
  (ii) $\mathfrak{a} = (\mathfrak{c}_1, \mathfrak{c}_2 \ldots, \mathfrak{c}_k, \mathfrak{d}_\ell{}^\vee)$ and $\mathfrak{b} = (\mathfrak{d}_1, \mathfrak{d}_2, \ldots, \mathfrak{d}_{\ell-1})$ if $r = \Rsh$,
  (iii) $\mathfrak{a} = (\mathfrak{d}_1{}^\vee, \mathfrak{c}_1, \mathfrak{c}_2 \ldots, \mathfrak{c}_k)$ and $\mathfrak{b} = (\mathfrak{d}_2, \mathfrak{d}_3, \ldots, \mathfrak{d}_\ell)$ if $r = \looparrowleft$, and
  (iv) $\mathfrak{a} = (\mathfrak{c}_1, \mathfrak{c}_2 \ldots, \mathfrak{c}_{k-1})$ and $\mathfrak{b} = (\mathfrak{d}_1, \mathfrak{d}_2, \ldots, \mathfrak{d}_\ell, \mathfrak{c}_k{}^\vee)$ if $r = \looparrowright$,
  and where $p^r \coloneqq (\omega_\ell^{r,k})^{\leftarrow}(p)$. Moreover, let $(\mathfrak{c} \,{}^{\blacksquare}_{\blacksquare}\, \mathfrak{d})^r \coloneqq \mathfrak{a} \,{}^{\blacksquare}_{\blacksquare}\, \mathfrak{b}$.

(b) The *counter-clockwise cyclic rotation* $(\mathfrak{c}, \mathfrak{d}, p)^\circlearrowleft$ of p is given by $(\mathfrak{c}, \mathfrak{d}, p) = (\varnothing, \varnothing, \varnothing)$ if $k = \ell = 0$, by $((\mathfrak{c}, \mathfrak{d}, p)^\Lsh)^\Rsh$ if $0 < k$ and by $((\mathfrak{c}, \mathfrak{d}, p)^\Rsh)^\Lsh$ if $0 < \ell$.

(c) Analogously, the *clockwise cyclic rotation* $(\mathfrak{c}, \mathfrak{d}, p)^\circlearrowright$ of p is given by $(\mathfrak{c}, \mathfrak{d}, p) = (\varnothing, \varnothing, \varnothing)$ if $k = \ell = 0$, by $((\mathfrak{c}, \mathfrak{d}, p)^\looparrowright)^\looparrowleft$ if $0 < k$ and by $((\mathfrak{c}, \mathfrak{d}, p)^\looparrowleft)^\looparrowright$ if $0 < \ell$.

LEMMA 3.22. *Any category of labeled partitions is closed under any basic and cyclic rotations.*

PROOF. Follows in the same ways as in [TW18, Lemma 1.1 (a)]. □

Definition 3.23. Given any $\{k,\ell\} \subseteq \mathbb{N}_0$, any $\mathfrak{c}\colon [\![k]\!] \to (\mathfrak{U} \otimes \{\circ,\bullet\}) \uplus \mathfrak{O}$ and $\mathfrak{d}\colon [\![\ell]\!] \to (\mathfrak{U} \otimes \{\circ,\bullet\}) \uplus \mathfrak{O}$, any $\mathsf{S} \subseteq \Pi_\ell^k$ and any partition p of Π_ℓ^k, the *reindexed restriction* of $(\mathfrak{c},\mathfrak{d},p)$ to S is $R((\mathfrak{c},\mathfrak{d},p),\mathsf{S}) \coloneqq ((\mathfrak{c}\,{}^{\blacksquare}{}_{\blacksquare}\,\mathfrak{d}) \circ \eta_{\mathsf{S},\ell}{}^{k}, (\mathfrak{c}\,{}^{\blacksquare}{}_{\blacksquare}\,\mathfrak{d}) \circ \theta_{\mathsf{S},\ell}{}^{k}, R(p,\mathsf{S}))$, where $R(p,\mathsf{S}) \coloneqq (\gamma_{\mathsf{S},\ell}{}^{k})^{\leftarrow}(p)$. Moreover, let $R(\mathfrak{c}\,{}^{\blacksquare}{}_{\blacksquare}\,\mathfrak{d},\mathsf{S}) \coloneqq (\mathfrak{c}\,{}^{\blacksquare}{}_{\blacksquare}\,\mathfrak{d}) \circ \gamma_{\mathsf{S},\ell}{}^{k}$.

Generally, categories are *not* closed under reindexed restrictions.

Definition 3.24. For any $\{k,\ell\} \subseteq \mathbb{N}_0$, any $\mathfrak{c}\colon [\![k]\!] \to (\mathfrak{U} \otimes \{\circ,\bullet\}) \uplus \mathfrak{O}$ and $\mathfrak{d}\colon [\![\ell]\!] \to (\mathfrak{U} \otimes \{\circ,\bullet\}) \uplus \mathfrak{O}$, any $\mathsf{T} \subseteq \Pi_\ell^k$ and any partition p of Π_ℓ^k the *erasing* of T from $(\mathfrak{c},\mathfrak{d},p)$ is defined as $E((\mathfrak{c},\mathfrak{d},p),\mathsf{T}) \coloneqq R((\mathfrak{c},\mathfrak{d},q),\mathsf{M})$, where $\mathsf{M} = \Pi_\ell^k \backslash \mathsf{T}$ and where $q = \{\mathsf{B} \in p \wedge \mathsf{B} \cap \mathsf{T} = \varnothing\} \uplus \{\bigcup\{\mathsf{B} \in p \wedge \mathsf{B} \cap \mathsf{T} \neq \varnothing\}\}$. Also, let $E(\mathfrak{c}\,{}^{\blacksquare}{}_{\blacksquare}\,\mathfrak{d},\mathsf{T}) \coloneqq R(\mathfrak{c}\,{}^{\blacksquare}{}_{\blacksquare}\,\mathfrak{d},\mathsf{M})$.

Lemma 3.25. *Any category of $(\mathfrak{U},\mathfrak{O})$-tagged labeled partitions is closed under erasing any convex set T of size two for which there exists*

(a) $x \in \mathfrak{U}$ such that T is contained entirely in the x-area and has zero x-sum.
(b) $y \in \mathfrak{O}$ such that T is contained entirely in the y-area.

Proof. The proof of [TW18, Lemma 1.1 (b)] can be easily adapted to deduce the claim. □

The following equivalent definition of a category of labeled partitions is the one we will actually use to confirm categories because it is significantly easier to check.

Proposition 3.26. *Any subset of ${}_{\mathfrak{U},\mathfrak{O}}\mathcal{S}$ is a category of $(\mathfrak{U},\mathfrak{O})$-tagged labeled partitions if it includes $\{\mathrm{id}_{(x,\circ)}, \mathrm{id}_y \,|\, x \in \mathfrak{U} \wedge y \in \mathfrak{O}\}$ and is closed under the basic rotations $\curvearrowleft$ and $\curvearrowright$, forming adjoints, tensor products and erasing consecutive sets of two lower points which are either contained entirely in the x-tag area and have zero x-sum for some $x \in \mathfrak{U}$ or are contained entirely in the y-tag area for some $y \in \mathfrak{O}$.*

Proof. Analogous to the proof of [MW21b, Lemmka 4.3]. □

3.6. Linear categories of labeled partitions. As evident from [BS09, Proposition 1.9], when adding the usual fiber functors to categories of partitions the former preserve composition only up to a scalar factor. In other words, a scalar correction has been omitted in the definition of the composition operation for labeled partitions. This is a very convenient simplification when one is interested in finding categories of partitions (as was the case in [BS09] and [TW18]) because it completely avoids the need to work with linear categories, i.e., categories enriched in vector spaces. And in this chapter we will benefit from it as well in Section 4. In Section 9, however, it will be necessary to have the linear definitions and in particular the other composition at hand.

At the same time, an important generalization of [BS09] and [TW18] and even [Fre17] which enables to proof of the main result in the first place is the following assumption.

Definition 3.27. A *dimension profile* for $(\mathfrak{U},\mathfrak{O})$ is any mapping $\mathfrak{U} \uplus \mathfrak{O} \to \mathbb{N}$.

In other words, a dimension profile assigns a dimension to any tag. It will be the size of the fundamental representation of the factor quantum group associated with that tag.

ASSUMPTION 3.28. In the following, let N by any dimension profile for $(\mathfrak{U}, \mathfrak{O})$.

While [Fre17] only discusses the case where N is constant, we do *not* make that assumption. And, in fact, it will be crucial to have a non-constant dimension profile in Section 9.

DEFINITION 3.29. For any $\{k, \ell, m\} \subseteq \mathbb{N}_0$, any $\mathfrak{a}\colon [\![k]\!] \to (\mathfrak{U} \otimes \{\circ, \bullet\}) \uplus \mathfrak{O}$ and $\mathfrak{b}\colon [\![\ell]\!] \to (\mathfrak{U} \otimes \{\circ, \bullet\}) \uplus \mathfrak{O}$ and $\mathfrak{c}\colon [\![m]\!] \to (\mathfrak{U} \otimes \{\circ, \bullet\}) \uplus \mathfrak{O}$ and any partitions p of Π_ℓ^k and q of Π_m^ℓ such that $p \leq \ker(N \circ \xi_{\mathfrak{b}}^{\mathfrak{a}})$ and $q \leq \ker(N \circ \xi_{\mathfrak{c}}^{\mathfrak{b}})$ if,

$$s = (\kappa_\ell^{0\leftarrow}(p|_{\Pi_\ell^0})) \vee (q|_{\Pi_0^\ell})$$

and if for any $n \in \mathbb{N}_0$,

$$\begin{aligned} u_n = |\{\mathsf{B} \in s \wedge \mathsf{B} \subseteq (N \circ \xi_\varnothing^{\mathfrak{b}})^{\leftarrow}(\{n\}) \\ \wedge\ (\forall \mathsf{A}\colon \mathsf{A} \in p \wedge \mathsf{A} \cap \kappa_{\ell\to}^0(\mathsf{B}) \neq \varnothing \Rightarrow \mathsf{A} \cap \Pi_0^k = \varnothing) \\ \wedge\ (\forall \mathsf{C}\colon \mathsf{C} \in q \wedge \mathsf{C} \cap \mathsf{B} \neq \varnothing \Rightarrow \mathsf{C} \cap \Pi_m^0 = \varnothing)\}|, \end{aligned}$$

then we call the natural number

$$_N\mathrm{lf}((\mathfrak{b}, \mathfrak{c}, q), (\mathfrak{a}, \mathfrak{b}, p)) \coloneqq \prod_{n \in \mathbb{N}_0} n^{u_n}$$

the *linear (composition) factor* of $((\mathfrak{b}, \mathfrak{c}, q), (\mathfrak{a}, \mathfrak{b}, p))$ with respect to N.

Note that in Definition 3.29 the assumption cannot be dropped that the partitions which are to be composed be labeled in such a way that any two points belonging to the same block bear tags of identical dimension according to the dimension profile. Of course, that the definition make sense could also have been ensured by only allowing partitions where any block is contained in a single tag area. However, that would be too restrictive for our purposes.

A useful reformulation of the above definition connects the linear factor to the corresponding original notion from the theory of unlabeled partitions (see [BS09, p. 1469] and [FW16, p. 159]).

DEFINITION 3.30. For any $\{f, g, h\} \subseteq \mathbb{N}_0$ and any partitions v of Π_g^f and w of Π_h^g, if

$$t = (\kappa_g^{0\leftarrow}(v|_{\Pi_g^0})) \vee (w|_{\Pi_0^g}),$$

then we call the number

$$\begin{aligned} \mathrm{rl}(w, v) \coloneqq |\{\mathsf{G} \in t \wedge (\forall \mathsf{F}\colon \mathsf{F} \in v \wedge \mathsf{F} \cap \kappa_{g\to}^0(\mathsf{G}) \neq \varnothing \Rightarrow \mathsf{F} \cap \Pi_0^f = \varnothing) \\ \wedge\ (\forall \mathsf{H}\colon \mathsf{H} \in w \wedge \mathsf{H} \cap \mathsf{G} \neq \varnothing \Rightarrow \mathsf{H} \cap \Pi_h^0 = \varnothing)\}| \end{aligned}$$

the *(number of) removed loops* of (w, v).

Proposition 3.31. *For any* $(\mathfrak{a},\mathfrak{b},p)\in{}_{\mathfrak{U},\mathfrak{O}}\mathcal{S}$ *and* $(\mathfrak{b},\mathfrak{c},q)\in{}_{\mathfrak{U},\mathfrak{O}}\mathcal{S}$ *with* $p\leq\ker(N\circ\xi^{\mathfrak{a}}_{\mathfrak{b}})$ *and* $q\leq\ker(N\circ\xi^{\mathfrak{b}}_{\mathfrak{c}})$*, if for any* $n\in\mathbb{N}_0$,

$$ {}_n\mathsf{X}^{\mathfrak{a}}_{\mathfrak{b}}=(N\circ\xi^{\mathfrak{a}}_{\mathfrak{b}})^{\leftarrow}(\{n\}) \quad\wedge\quad {}_n\mathsf{X}^{\mathfrak{b}}_{\mathfrak{c}}=(N\circ\xi^{\mathfrak{b}}_{\mathfrak{c}})^{\leftarrow}(\{n\}) $$

and

$$ u_n=\mathrm{rl}(R(q,{}_n\mathsf{X}^{\mathfrak{b}}_{\mathfrak{c}}),R(p,{}_n\mathsf{X}^{\mathfrak{a}}_{\mathfrak{b}})), $$

then

$$ {}_N\mathrm{lf}((\mathfrak{b},\mathfrak{c},q),(\mathfrak{a},\mathfrak{b},p))=\prod_{n\in\mathbb{N}_0}n^{u_n}. $$

Proof. Follows quickly from the definitions. □

The linear version of the categories of partitions can then be defined as follows. Note that it now depends on the dimension profile.

Definition 3.32. For any category $\mathcal{C}$ of $(\mathfrak{U},\mathfrak{O})$-tagged labeled partitions such that any $(\mathfrak{c},\mathfrak{d},p)\in\mathcal{C}$ satisfies $p\leq\ker(N\circ\xi^{\mathfrak{c}}_{\mathfrak{d}})$ let

(a) $\mathrm{obj}_{{}^N\mathbb{C}\mathcal{C}}$ be given by $\bigcup_{k\in\mathbb{N}_0}\{\mathfrak{c}\colon [\![k]\!]\to(\mathfrak{U}\otimes\{\circ,\bullet\})\uplus\mathfrak{O}\}$,

(b) $\mathrm{mor}_{{}^N\mathbb{C}\mathcal{C}}(\mathfrak{c},\mathfrak{d})$ be the free vector space over the set of all $(\mathfrak{c},\mathfrak{d},p)\in\mathcal{C}$ for any $(\mathfrak{c},\mathfrak{d})\in{\mathrm{obj}_{{}^N\mathbb{C}\mathcal{C}}}^{\otimes 2}$,

(c) $\circ_{{}^N\mathbb{C}\mathcal{C},\mathfrak{a},\mathfrak{b},\mathfrak{c}}$ be the unique linear map $\mathrm{mor}_{{}^N\mathbb{C}\mathcal{C}}(\mathfrak{b},\mathfrak{c})\otimes\mathrm{mor}_{{}^N\mathbb{C}\mathcal{C}}(\mathfrak{a},\mathfrak{b})\to\mathrm{mor}_{{}^N\mathbb{C}\mathcal{C}}(\mathfrak{a},\mathfrak{c})$ which satisfies for any $(\mathfrak{a},\mathfrak{b},p)\in\mathcal{C}$ and any $(\mathfrak{b},\mathfrak{c},q)\in\mathcal{C}$,

$$ (\mathfrak{b},\mathfrak{c},q)\otimes(\mathfrak{a},\mathfrak{b},p)\mapsto{}_N\mathrm{lf}((\mathfrak{b},\mathfrak{c},q),(\mathfrak{a},\mathfrak{b},q))\,(\mathfrak{a},\mathfrak{c},qp), $$

for any $(\mathfrak{a},\mathfrak{b},\mathfrak{c})\in{\mathrm{obj}_{{}^N\mathbb{C}\mathcal{C}}}^{\otimes 3}$,

(d) $\mathrm{id}_{{}^N\mathbb{C}\mathcal{C},\mathfrak{c}}$ be the unique linear map $\mathbb{C}\to\mathrm{mor}_{{}^N\mathbb{C}\mathcal{C}}(\mathfrak{c},\mathfrak{c})$ with $1\mapsto\mathrm{id}_{\mathfrak{c}}$ for any $\mathfrak{c}\in\mathrm{obj}_{{}^N\mathbb{C}\mathcal{C}}$,

(e) $(*_{{}^N\mathbb{C}\mathcal{C}})_{1,\mathfrak{c},\mathfrak{d}}$ be the unique linear map $\mathrm{mor}_{{}^N\mathbb{C}\mathcal{C}}(\mathfrak{c},\mathfrak{d})\to\mathrm{mor}_{{}^N\mathbb{C}\mathcal{C}}(\mathfrak{d},\mathfrak{c})^{\mathrm{cj}}$ such that $(\mathfrak{c},\mathfrak{d},p)\mapsto(\mathfrak{d},\mathfrak{c},p^*)$ for any $(\mathfrak{c},\mathfrak{d},p)\in\mathcal{C}$ and any $(\mathfrak{c},\mathfrak{d})\in{\mathrm{obj}_{{}^N\mathbb{C}\mathcal{C}}}^{\otimes 2}$

(f) $(\otimes_{{}^N\mathbb{C}\mathcal{C}})_0$ be the mapping ${\mathrm{obj}_{{}^N\mathbb{C}\mathcal{C}}}^{\otimes 2}\to\mathrm{obj}_{{}^N\mathbb{C}\mathcal{C}}$ with $(\mathfrak{c}_1,\mathfrak{c}_2)\mapsto\mathfrak{c}_1\vartriangle\mathfrak{c}_2$ for any $(\mathfrak{c}_1,\mathfrak{c}_2)\in{\mathrm{obj}_{{}^N\mathbb{C}\mathcal{C}}}^{\otimes 2}$,

(g) $(\otimes_{{}^N\mathbb{C}\mathcal{C}})_{1,(\mathfrak{c}_1,\mathfrak{c}_2),(\mathfrak{d}_1,\mathfrak{d}_2)}$ be the linear map from $\mathrm{mor}_{{}^N\mathbb{C}\mathcal{C}}(\mathfrak{c}_1,\mathfrak{d}_1)\otimes\mathrm{mor}_{{}^N\mathbb{C}\mathcal{C}}(\mathfrak{c}_2,\mathfrak{d}_2)$ to $\mathrm{mor}_{{}^N\mathbb{C}\mathcal{C}}(\mathfrak{c}_1\vartriangle\mathfrak{c}_2,\mathfrak{d}_1\vartriangle\mathfrak{d}_2)$ with $(\mathfrak{c}_1,\mathfrak{d}_1,p_1)\otimes(\mathfrak{c}_2,\mathfrak{d}_2,p_2)\mapsto(\mathfrak{c}_1\vartriangle\mathfrak{c}_2,\mathfrak{d}_1\vartriangle\mathfrak{d}_2,p_1\otimes p_2)$ for any $(\mathfrak{c}_1,\mathfrak{d}_1,p_1)\in\mathcal{C}$ and $(\mathfrak{c}_2,\mathfrak{d}_2,p_2)\in\mathcal{C}$ and any $(\mathfrak{c}_1,\mathfrak{c}_2)\in{\mathrm{obj}_{{}^N\mathbb{C}\mathcal{C}}}^{\otimes 2}$ and $(\mathfrak{d}_1,\mathfrak{d}_2)\in{\mathrm{obj}_{{}^N\mathbb{C}\mathcal{C}}}^{\otimes 2}$,

(h) $I_{{}^N\mathbb{C}\mathcal{C}}$ be $\varnothing$.

Then ${}^N\mathbb{C}\mathcal{C}$ is a rigid strict monoidal $\mathbb{C}$-linear $*$-category for any category $\mathcal{C}$ of $(\mathfrak{U},\mathfrak{O})$-tagged labeled partition such that $(\mathfrak{c},\mathfrak{d},p)\in\mathcal{C}$ satisfies $p\leq\ker(N\circ\xi^{\mathfrak{c}}_{\mathfrak{d}})$.

Notation 3.33. Given any set X and any two set-theoretical partitions p and q of X, let

$$ \zeta(p,q)\coloneqq\begin{cases}1 & \text{if } p\leq q\\ 0 & \text{otherwise.}\end{cases} $$

Notation 3.34. For any $k \in \mathbb{N}_0$ and any $\mathfrak{c}\colon \llbracket k \rrbracket \to (\mathfrak{U} \otimes \{\circ, \bullet\}) \cup \mathfrak{O}$ let $J_{\mathfrak{c}}^N$ be the finite subset of $[\llbracket k \rrbracket, \mathbb{N}]$ made up by all $f\colon \llbracket k \rrbracket \to \mathbb{N}$ with the property that for any $i \in \llbracket k \rrbracket$, if $z_i \in \mathfrak{U} \cup \mathfrak{O}$ is such that $\mathfrak{c}(i) \in (\{z_i\} \otimes \{\circ, \bullet\}) \cup \{z_i\}$, then $f(i) \in \llbracket N(z_i) \rrbracket$.

Notation 3.35.
(a) For any finite set X let $\ell^2(X)$ be the Hilbert space given by the free vector space over X equipped with the scalar product defined by $x \otimes x' \mapsto \delta_{x,x'}$ for any $\{x, x'\} \subseteq X$.
(b) Given any finite sets X and Y and any mapping $f\colon X \to Y$, denote by $\ell^2(f)$ the unique operator $\ell^2(X) \to \ell^2(Y)$ with $x \mapsto f(x)$ for any $x \in X$.
(c) For any finite sets X_1 and X_2 let ${\ell^2}_{\otimes, X_1, X_2}$ be the unique operator $\ell^2(X_1) \otimes \ell^2(X_2) \to \ell^2(X_1 \otimes X_2)$ with $x_1 \otimes x_2 \mapsto (x_1, x_2)$.
(d) Finally, write ${\ell^2}_I$ for the unique operator $\mathbb{C} \to \ell^2(\{\varnothing\})$ with $1 \mapsto \varnothing$.

Definition 3.36.
(a) For any $k \in \mathbb{N}_0$ and any $\mathfrak{c}\colon \llbracket k \rrbracket \to (\mathfrak{U} \otimes \{\circ, \bullet\}) \cup \mathfrak{O}$ let

$$ {}_{\mathfrak{U},\mathfrak{O}}^{\;\;N}T(\mathfrak{c}) \equiv {}_{\mathfrak{U},\mathfrak{O}}^{\;\;N}T_0(\mathfrak{c}) \coloneqq \ell^2(J_{\mathfrak{c}}^N). $$

(b) Given any $\{k, \ell\} \subseteq \mathbb{N}_0$, any $\mathfrak{c}\colon \llbracket k \rrbracket \to (\mathfrak{U} \otimes \{\circ, \bullet\}) \cup \mathfrak{O}$ and $\mathfrak{d}\colon \llbracket \ell \rrbracket \to (\mathfrak{U} \otimes \{\circ, \bullet\}) \cup \mathfrak{O}$ and any $(\mathfrak{c}, \mathfrak{d}, p) \in {}_{\mathfrak{U},\mathfrak{O}}\mathcal{S}$ with $p \leq \ker(N \circ \xi_{\mathfrak{d}}^{\mathfrak{c}})$ let

$$ {}_{\mathfrak{U},\mathfrak{O}}^{\;\;N}T(\mathfrak{c}, \mathfrak{d}, p) \equiv {}_{\mathfrak{U},\mathfrak{O}}^{\;\;N}T_{1,\mathfrak{c},\mathfrak{d}}(\mathfrak{c}, \mathfrak{d}, p) $$

be the unique $\mathbb{C}$-linear operator ${}_{\mathfrak{U},\mathfrak{O}}^{\;\;N}T(\mathfrak{c}) \to {}_{\mathfrak{U},\mathfrak{O}}^{\;\;N}T(\mathfrak{d})$ such that for any $f \in J_{\mathfrak{c}}^N$,

$$ f \mapsto \sum_{g \in J_{\mathfrak{d}}^N} \zeta(p, \ker(f \mathbin{\blacksquare} g))\, g. $$

(c) For any $\{k_1, k_2\} \subseteq \mathbb{N}_0$ and any $\mathfrak{c}_1\colon \llbracket k_1 \rrbracket \to (\mathfrak{U} \otimes \{\circ, \bullet\}) \cup \mathfrak{O}$ and $\mathfrak{c}_2\colon \llbracket k_2 \rrbracket \to (\mathfrak{U} \otimes \{\circ, \bullet\}) \cup \mathfrak{O}$ let

$$ {T^N}_{\otimes, \mathfrak{c}_1, \mathfrak{c}_2} \coloneqq \ell^2(c) \circ {\ell^2}_{\otimes, J_{\mathfrak{c}_1}^N, J_{\mathfrak{c}_2}^N}, $$

where c is the mapping $J_{\mathfrak{c}_1}^N \otimes J_{\mathfrak{c}_2}^N \to J_{\mathfrak{c}_1 \otimes \mathfrak{c}_2}^N$ with

$$ (f_1, f_2) \mapsto f_1 \vartriangle f_2 $$

for any $f_1 \in J_{\mathfrak{c}_1}^N$ and $f_2 \in J_{\mathfrak{c}_2}^N$.
(d) Finally, let ${}_{\mathfrak{U},\mathfrak{O}}^{\;\;N}T_I \coloneqq {\ell^2}_I$.

For any category $\mathcal{C}$ of $(\mathfrak{U}, \mathfrak{O})$-tagged labeled partitions the rule ${}_{\mathfrak{U},\mathfrak{O}}^{\;\;N}T$ defines a strong monoidal $\mathbb{C}$-linear $*$-functor from ${}^N\mathbb{C}\mathcal{C}$ to the category of finite-dimensional Hilbert spaces.

3.7. General easy compact quantum groups. In the language of Section 2 one can now give the definition of Banica and Speicher's easy quantum groups from [BS09], or rather the generalization to labeled partitions, as follows.

Definition 3.37. For any category $\mathcal{C}$ of $(\mathfrak{U}, \mathfrak{O})$-tagged labeled partitions such that $p \leq \ker(N \circ \xi_{\mathfrak{d}}^{\mathfrak{c}})$ for any $(\mathfrak{c}, \mathfrak{d}, p) \in \mathcal{C}$ we call the tuple

$$ (R, (H_{\mathfrak{c}})_{\mathfrak{c} \in R}, (B_{\mathfrak{c},\mathfrak{d}})_{(\mathfrak{c},\mathfrak{d}) \in R^{\otimes 2}}, \otimes), $$

where the

(a) object set R is $\bigcup_{k\in\mathbb{N}_0}((\mathfrak{U}\otimes\{\circ,\bullet\})\uplus\mathfrak{D})^{\otimes k}$,
(b) fiber space $H_{\mathfrak{c}}$ of $\mathfrak{c}$ is given by $\ell^2(J_{\mathfrak{c}}^N)$ for any $\mathfrak{c}\in R$,
(c) space $B_{\mathfrak{c},\mathfrak{d}}$ of morphisms $\mathfrak{c}\to\mathfrak{d}$ is the vector subspace of $[\ell^2(J_{\mathfrak{c}}^N),\ell^2(J_{\mathfrak{d}}^N)]$ generated by the set $\{{}_{\mathfrak{U},\mathfrak{D}}^{\;N}T(\mathfrak{c},\mathfrak{d},p)\,|\,(\mathfrak{c},\mathfrak{d},p)\in\mathcal{C}\}$ for any $(\mathfrak{c},\mathfrak{d})\in R^{\otimes 2}$,
(d) monoidal product $\otimes$ is the mapping $R^{\otimes 2}\to R$ with $(\mathfrak{c}_1,\mathfrak{c}_2)\mapsto\mathfrak{c}_1\vartriangle\mathfrak{c}_2$ for any $(\mathfrak{c}_1,\mathfrak{c}_2)\in R^{\otimes 2}$,

the *rigid concrete monoidal* W^**-category* associated with $(\mathfrak{U},\mathfrak{D},\mathcal{C},N)$.

These concrete W^*-categories are generally *not* complete. For the next definition to make sense we need to fix an orthonormal basis of any finite-dimensional Hilbert space. We can make any such choice as long as we ensure that for any finite set X we pick the orthonormal basis X for $\ell^2(X)$.

Definition 3.38. For any category $\mathcal{C}$ of $(\mathfrak{U},\mathfrak{D})$-tagged labeled partitions such that $p\leq\ker(N\circ\xi_{\mathfrak{d}}^{\mathfrak{c}})$ for any $(\mathfrak{c},\mathfrak{d},p)\in\mathcal{C}$ we say that the Tannaka-Krein co-representee of the rigid concrete monoidal W^*-category associated with $(\mathfrak{U},\mathfrak{D},\mathcal{C},N)$ is the *easy CQG Hopf $*$-algebra* associated with $(\mathfrak{U},\mathfrak{D},\mathcal{C},N)$. Its formal dual is called the *easy algebraic compact quantum group* associated with $(\mathfrak{U},\mathfrak{D},\mathcal{C},N)$.

Notation 3.39. For any $\{k,\ell\}\subseteq\mathbb{N}_0$, any $\mathfrak{c}\colon[\![k]\!]\to(\mathfrak{U}\otimes\{\circ,\bullet\})\uplus\mathfrak{D}$, any $\mathfrak{d}\colon[\![\ell]\!]\to(\mathfrak{U}\otimes\{\circ,\bullet\})\uplus\mathfrak{D}$, any set-theoretical partition p of Π_ℓ^k such that $p\leq\ker(N\circ\xi_{\mathfrak{d}}^{\mathfrak{c}})$, any $g\in J_{\mathfrak{c}}^N$ and any $j\in J_{\mathfrak{d}}^N$ write $\mathrm{r}_{\mathfrak{d}}^{\mathfrak{c}}(p)_{j,g}$ for the element of the free $*$-algebra over $\{(z,j,i)\,|\,z\in\mathfrak{U}\uplus\mathfrak{D}\wedge\{i,j\}\subseteq[\![N(z)]\!]\}$ given by

$$\sum_{i\in J_{\mathfrak{d}}^N}\zeta(p,\ker(g\,{}^{\blacksquare}{}_{\blacksquare}\,i))\overrightarrow{\prod_{b=1}^{\ell}}\begin{Bmatrix}(u_{j_b,i_b}^{\xi_{\mathfrak{d}}^{\mathfrak{c}}({}_{\blacksquare}b)})^{\zeta_{\mathfrak{d}}^{\mathfrak{c}}({}_{\blacksquare}b)} & |\,\xi_{\mathfrak{d}}^{\mathfrak{c}}({}_{\blacksquare}b)\in\mathfrak{U}\\ u_{j_b,i_b}^{\xi_{\mathfrak{d}}^{\mathfrak{c}}({}_{\blacksquare}b)} & |\,\xi_{\mathfrak{d}}^{\mathfrak{c}}({}_{\blacksquare}b)\in\mathfrak{D}\end{Bmatrix}$$
$$-\sum_{h\in J_{\mathfrak{c}}^N}\zeta(p,\ker(h\,{}^{\blacksquare}{}_{\blacksquare}\,j))\overrightarrow{\prod_{a=1}^{k}}\begin{Bmatrix}(u_{h_a,g_a}^{\xi_{\mathfrak{d}}^{\mathfrak{c}}({}^{\blacksquare}a)})^{\overline{\zeta_{\mathfrak{d}}^{\mathfrak{c}}({}^{\blacksquare}a)}} & |\,\xi_{\mathfrak{d}}^{\mathfrak{c}}({}^{\blacksquare}a)\in\mathfrak{U}\\ u_{h_a,g_a}^{\xi_{\mathfrak{d}}^{\mathfrak{c}}({}^{\blacksquare}a)} & |\,\xi_{\mathfrak{d}}^{\mathfrak{c}}({}^{\blacksquare}a)\in\mathfrak{D}\end{Bmatrix},$$

where $u_{j,i}^z$ is short for (z,j,i) and where $(u_{j,i}^z)^\circ\coloneqq u_{j,i}^z$ and $(u_{j,i}^z)^\bullet\coloneqq(u_{j,i}^z)^*$ for any $z\in\mathfrak{U}\uplus\mathfrak{D}$ and $\{i,j\}\subseteq N(z)$.

Proposition 3.40. *For any category of $(\mathfrak{U},\mathfrak{D})$-tagged labeled partitions such that $p\leq\ker(N\circ\xi_{\mathfrak{d}}^{\mathfrak{c}})$ for any $(\mathfrak{c},\mathfrak{d},p)\in\mathcal{C}$ and any set $\mathcal{R}\subseteq\mathcal{C}$ with ${}_{\mathfrak{U},\mathfrak{D}}\langle\mathcal{R}\rangle=\mathcal{C}$ an isomorphism of CQG Hopf $*$-algebras from the easy CQG Hopf $*$-algebra associated with $(\mathfrak{U},\mathfrak{D},\mathcal{C},N)$ to the CQG Hopf $*$-algebra given by $(A,m,1,*,\Delta,\epsilon,S)$, where*

(a) $(A,m,1,*)$ is the universal $*$-algebra over $\{(z,j,i)\,|\,z\in\mathfrak{U}\uplus\mathfrak{D}\wedge\{i,j\}\subseteq N(z)\}$, whose elements are written as $u_{j,i}^z\equiv(z,j,i)$ below for any $z\in\mathfrak{U}\uplus\mathfrak{D}$ and $\{i,j\}\subseteq N(z)$, subject to the relations

$$\{(u_{j,i}^y)^*-u_{j,i}^y\,|\,y\in\mathfrak{D}\wedge\{i,j\}\subseteq N(y)\}$$

and the relations

$$\{\mathrm{r}_{\mathfrak{d}}^{\mathfrak{c}}(p)_{j,g}\,|\,\{k,\ell\}\subseteq\mathbb{N}_0\wedge\mathfrak{c}\colon[\![k]\!]\to(\mathfrak{U}\otimes\{\circ,\bullet\})\uplus\mathfrak{D}\wedge\mathfrak{d}\colon[\![\ell]\!]\to(\mathfrak{U}\otimes\{\circ,\bullet\})\uplus\mathfrak{D}$$
$$\wedge\,(\mathfrak{c},\mathfrak{d},p)\in\mathcal{R}\cup\{\mathrm{ev}_z,\mathrm{coev}_z\}_{z\in\mathfrak{U}\uplus\mathfrak{D}}\wedge g\in J_{\mathfrak{c}}^N\wedge j\in J_{\mathfrak{d}}^N\}$$

(b) *Δ is the unique morphism of $*$-algebras from $(A, m, 1, *)$ to the tensor product $*$-algebra of $(A, m, 1, *)$ with itself such that for any $z \in \mathfrak{U} \cup \mathfrak{O}$ and any $\{i, k\} \subseteq [\![N(z)]\!]$,*

$$u^z_{k,i} \mapsto \sum_{j=1}^{N(z)} u^z_{k,j} \otimes u^z_{j,i},$$

(c) *ϵ is the unique $*$-algebra morphism from $(A, m, 1, *)$ to $\mathbb{C}$ with for any $z \in \mathfrak{U} \cup \mathfrak{O}$ and any $\{i, j\} \subseteq [\![N(z)]\!]$,*

$$u^z_{j,i} \mapsto \delta_{j,i},$$

(d) *S is the unique algebra morphism from $(A, m, 1)$ to its opposite algebra such that for any $z \in \mathfrak{U} \cup \mathfrak{O}$ and any $\{i, j\} \subseteq [\![N(z)]\!]$,*

$$u^z_{j,i} \mapsto (u^z_{i,j})^* \quad \wedge \quad (u^z_{j,i})^* \mapsto u^z_{i,j},$$

is given by the unique $*$-algebra morphism such that $(z, j, i) \mapsto (z, j, i)$ for any $z \in \mathfrak{U} \cup \mathfrak{O}$ and any $\{i, j\} \subseteq [\![N(z)]\!]$.

Proof. If in the definition of the second set of relations $\mathcal{R} \cup \{\mathrm{ev}_z, \mathrm{coev}_z\}_{z \in \mathfrak{U} \cup \mathfrak{O}}$ is replaced by $\mathcal{C}$, then the statement follows immediately from Proposition 2.36 (a). In order to prove the stronger version above, one needs to check that the relations coming from a labeled partition which results from others by category operations can be obtained by $*$-ideal operations from the relations coming from the other partitions. $\square$

4. Co-products of categories of labeled partitions

Section 4 introduces direct, free, graph, crossed and wreath graph co-products of categories of labeled partitions. The ultimate goal is to show that these constructions correspond to the products of the same name for the associated easy compact quantum groups.

Assumption 4.1. In Section 4, let $(\mathfrak{U}, \mathfrak{O})$ be any choice of tags.

4.1. Direct Co-products. First, a category of labeled partitions is induced by the condition that each block be contained entirely in the area of a single tag. In fact, we show a bit more in Proposition 4.3. The below lemma will aid the proof as well as the proofs of several subsequent propositions.

Lemma 4.2. *Let $\{k, \ell\} \subseteq \mathbb{N}_0$, let $\mathfrak{c}\colon [\![k]\!] \to (\mathfrak{U} \otimes \{\circ, \bullet\}) \cup \mathfrak{O}$ and $\mathfrak{d}\colon [\![\ell]\!] \to (\mathfrak{U} \otimes \{\circ, \bullet\}) \cup \mathfrak{O}$ and for each $t \in [\![2]\!]$ let $\{k_t, \ell_t\} \subseteq \mathbb{N}_0$, and let $\mathfrak{c}_t\colon [\![k_t]\!] \to (\mathfrak{U} \otimes \{\circ, \bullet\}) \cup \mathfrak{O}$ and $\mathfrak{d}_t\colon [\![\ell_t]\!] \to (\mathfrak{U} \otimes \{\circ, \bullet\}) \cup \mathfrak{O}$.*

(a) *For any $r \in \{\curvearrowleft, \curvearrowright, \circlearrowleft, \circlearrowright\}$, if $\mathfrak{a} \blacksquare \mathfrak{b} = (\mathfrak{c} \blacksquare \mathfrak{d})^r$ is defined and if $\{x, y\} \subseteq \mathbb{N}_0$ are such that $\mathfrak{a}\colon [\![x]\!] \to (\mathfrak{U} \otimes \{\circ, \bullet\}) \cup \mathfrak{O}$ and $\mathfrak{b}\colon [\![y]\!] \to (\mathfrak{U} \otimes \{\circ, \bullet\}) \cup \mathfrak{O}$, then for any $z \in \mathfrak{U} \cup \mathfrak{O}$,*
   (i) *$\omega^{r,k}_{\ell}$ is strictly monotonic $\Gamma^x_y \to \Gamma^k_\ell$.*
   (ii) *$\xi^{\mathfrak{a}}_{\mathfrak{b}} = \xi^{\mathfrak{c}}_{\mathfrak{d}} \circ \omega^{r,k}_{\ell}$.*

(iii) $\zeta_{\mathfrak{b}}^{\mathfrak{a}} = \zeta_{\mathfrak{d}}^{\mathfrak{c}} \circ \omega^{r,k}_{\ell}$.
(iv) ${}_z\sigma_{\mathfrak{d}}^{\mathfrak{c}}$ *is the push-forward measure of* ${}_z\sigma_{\mathfrak{b}}^{\mathfrak{a}}$ *with respect to* $\omega^{r,k}_{\ell}$.

(b) *For any* $z \in \mathfrak{U} \cup \mathfrak{O}$,
(i) κ_{ℓ}^{k} *is strictly* anti-*monotonic* $\Gamma_k^{\ell} \to \Gamma_{\ell}^{k}$
(ii) $\xi_{\mathfrak{c}}^{\mathfrak{d}} = \xi_{\mathfrak{d}}^{\mathfrak{c}} \circ \kappa_{\ell}^{k}$.
(iii) $\overline{(\cdot)} \circ \zeta_{\mathfrak{c}}^{\mathfrak{d}} = \zeta_{\mathfrak{d}}^{\mathfrak{c}} \circ \kappa_{\ell}^{k}$.
(iv) $-{}_z\sigma_{\mathfrak{d}}^{\mathfrak{c}}$ *is the push-forward measure of* ${}_z\sigma_{\mathfrak{c}}^{\mathfrak{d}}$ *with respect to* κ_{ℓ}^{k}.

(c) *If* $\mathsf{H}_1 = \Pi_{\ell_1}^{k_1}$ *and* $\mathsf{H}_2 = \Pi_{\ell_1+\ell_2}^{k_1+k_2} \backslash \Pi_{\ell_1}^{k_1}$, *then for any* $z \in \mathfrak{U} \cup \mathfrak{O}$,
(i) $\gamma_{\mathsf{H}_t,\ell_1+\ell_2}^{\;\;k_1+k_2}$ *is monotonic* $\Gamma_{\ell_t}^{k_t} \to \Gamma_{\ell_1+\ell_2}^{k_1+k_2}$ *for any* $t \in [\![2]\!]$.
(ii) $\xi_{\mathfrak{d}_1 \otimes \mathfrak{d}_2}^{\mathfrak{c}_1 \otimes \mathfrak{c}_2} \circ \gamma_{\mathsf{H}_t,\ell_1+\ell_2}^{\;\;k_1+k_2} = \xi_{\mathfrak{d}_t}^{\mathfrak{c}_t}$ *for any* $t \in [\![2]\!]$.
(iii) $\zeta_{\mathfrak{d}_1 \otimes \mathfrak{d}_2}^{\mathfrak{c}_1 \otimes \mathfrak{c}_2} \circ \gamma_{\mathsf{H}_t,\ell_1+\ell_2}^{\;\;k_1+k_2} = \zeta_{\mathfrak{d}_t}^{\mathfrak{c}_t}$ *for any* $t \in [\![2]\!]$.
(iv) ${}_z\sigma_{\mathfrak{d}_1 \otimes \mathfrak{d}_2}^{\mathfrak{c}_1 \otimes \mathfrak{c}_2}$ *is the sum of the push-forward measures of* ${}_z\sigma_{\mathfrak{d}_1}^{\mathfrak{c}_1}$ *with respect to* $\gamma_{\mathsf{H}_1,\ell_1}^{\;\;k_1}$ *and* ${}_z\sigma_{\mathfrak{d}_2}^{\mathfrak{c}_2}$ *with respect to* $\gamma_{\mathsf{H}_2,\ell_2}^{\;\;k_2}$, *i.e.*,

$$ {}_z\sigma_{\mathfrak{d}_1 \otimes \mathfrak{d}_2}^{\mathfrak{c}_1 \otimes \mathfrak{c}_2}(\mathsf{Z}) = \textstyle\sum_{t=1}^{2} {}_z\sigma_{\mathfrak{d}_t}^{\mathfrak{c}_t}(\gamma_{\mathsf{H}_t,\ell_1+\ell_2}^{\;\;k_1+k_2 \leftarrow}(\mathsf{Z})) $$

for any $\mathsf{Z} \subseteq \Pi_{\ell_1+\ell_2}^{k_1+k_2}$.

(d) *For any* $\mathsf{S} \subseteq \Pi_{\ell}^{k}$, *if* $\mathfrak{a} \,\blacksquare\, \mathfrak{b} = R(\mathfrak{c} \,\blacksquare\, \mathfrak{d}, \mathsf{S})$ *and if* $\{x, y\}$ *are such that* $\mathfrak{a} \colon [\![x]\!] \to (\mathfrak{U} \otimes \{\circ, \bullet\}) \cup \mathfrak{O}$ *and* $\mathfrak{b} \colon [\![y]\!] \to (\mathfrak{U} \otimes \{\circ, \bullet\}) \cup \mathfrak{O}$, *then for any* $z \in \mathfrak{U} \cup \mathfrak{O}$,
(i) $\gamma_{\mathsf{S},\ell}^{\;\;k}$ *is monotonic* $\Gamma_y^x \to \Gamma_{\ell}^{k}$.
(ii) $\xi_{\mathfrak{b}}^{\mathfrak{a}} = \xi_{\mathfrak{d}}^{\mathfrak{c}} \circ \gamma_{\mathsf{S},\ell}^{\;\;k}$.
(iii) $\zeta_{\mathfrak{b}}^{\mathfrak{a}} = \zeta_{\mathfrak{d}}^{\mathfrak{c}} \circ \gamma_{\mathsf{S},\ell}^{\;\;k}$.
(iv) ${}_z\sigma_{\mathfrak{d}}^{\mathfrak{c}}(\mathsf{Z}) = {}_z\sigma_{\mathfrak{b}}^{\mathfrak{a}}(\gamma_{\mathsf{S},\ell}^{\;\;k \leftarrow}(\mathsf{Z})) + {}_z\sigma_{\mathfrak{d}}^{\mathfrak{c}}(\mathsf{Z} \backslash \mathsf{S})$ *for any* $\mathsf{Z} \subseteq \Pi_{\ell}^{k}$.

PROOF. The claims follow immediately from the definitions. □

PROPOSITION 4.3. *For any set-theoretical partition* h *of* $\mathfrak{U} \cup \mathfrak{O}$ *a category of* $(\mathfrak{U}, \mathfrak{O})$*-tagged labeled partitions is given by the set of all* $(\mathfrak{c}, \mathfrak{d}, p) \in {}_{\mathfrak{U},\mathfrak{O}}\mathcal{S}$ *such that* $p \leq \xi_{\mathfrak{d}}^{\mathfrak{c} \leftarrow}(h)$.

PROOF. Denote the alleged category by $\mathcal{C}$. We check the easier-to-verify conditions of Proposition 3.26.

Step 1: Identities. For any $x \in \mathfrak{U}$, any $y \in \mathfrak{O}$ and any $\mathfrak{c} \in \{(x, \circ), (x, \bullet), y\}$, since $\ker(\xi_{\mathfrak{c}}^{\mathfrak{c}}) = |$ and thus $\xi_{\mathfrak{c}}^{\mathfrak{c} \leftarrow}(h) = |$ by $\ker(\xi_{\mathfrak{c}}^{\mathfrak{c}}) \leq \xi_{\mathfrak{c}}^{\mathfrak{c} \leftarrow}(h)$ we infer $\mathrm{id}_{\mathfrak{c}} = (\mathfrak{c}, \mathfrak{c}, |) \in \mathcal{C}$.

Step 2: Rotations. Next, let $k \in \mathbb{N}$ and $\ell \in \mathbb{N}_0$ and $\mathfrak{c} \colon [\![k]\!] \to (\mathfrak{U} \otimes \{\circ, \bullet\}) \cup \mathfrak{O}$ and $\mathfrak{d} \colon [\![\ell]\!] \to (\mathfrak{U} \otimes \{\circ, \bullet\}) \cup \mathfrak{O}$ and $(\mathfrak{c}, \mathfrak{d}, p) \in \mathcal{C}$ arbitrary and let $r \in \{\Lsh, \Rsh\}$ be such that $(\mathfrak{a}, \mathfrak{b}, q) = (\mathfrak{c}, \mathfrak{d}, p)^r$ is defined. Then, Lemma 4.2 (a) implies $\xi_{\mathfrak{b}}^{\mathfrak{a} \leftarrow}(h) = (\xi_{\mathfrak{d}}^{\mathfrak{c}} \circ \omega^{r,k}_{\ell})^{\leftarrow}(h) = \omega^{r,k \leftarrow}_{\ell}(\xi_{\mathfrak{d}}^{\mathfrak{c} \leftarrow}(h))$. Since $q = \omega^{r,k \leftarrow}_{\ell}(p)$ by definition and since $\omega^{r,k \leftarrow}_{\ell}$ preserves $\leq$ this proves $(\mathfrak{a}, \mathfrak{b}, q) \in \mathcal{C}$.

Step 3: Adjoints. If $\{k, \ell\} \subseteq \mathbb{N}_0$ and $\mathfrak{c} \colon [\![k]\!] \to (\mathfrak{U} \otimes \{\circ, \bullet\}) \cup \mathfrak{O}$ and $\mathfrak{d} \colon [\![\ell]\!] \to (\mathfrak{U} \otimes \{\circ, \bullet\}) \cup \mathfrak{O}$ and $(\mathfrak{c}, \mathfrak{d}, p) \in \mathcal{C}$ and $(\mathfrak{a}, \mathfrak{b}, q) = (\mathfrak{c}, \mathfrak{d}, p)^*$, then, similarly, $\xi_{\mathfrak{b}}^{\mathfrak{a} \leftarrow}(h) = (\xi_{\mathfrak{d}}^{\mathfrak{c}} \circ \kappa_{\ell}^{k})^{\leftarrow}(h) = \kappa_{\ell}^{k \leftarrow}(\xi_{\mathfrak{d}}^{\mathfrak{c} \leftarrow}(h))$ by Lemma 4.2 (b). Hence, also $(\mathfrak{a}, \mathfrak{b}, q) \in \mathcal{C}$ because $q = \kappa_{\ell}^{k \leftarrow}(p)$ definitionally and because $\kappa_{\ell}^{k \leftarrow}$ preserves $\leq$.

Step 4: Tensor products. For each $t \in [\![2]\!]$ let $\{k_t, \ell_t\} \subseteq \mathbb{N}_0$ and $\mathfrak{c}_t\colon [\![k_t]\!] \to (\mathfrak{U} \otimes \{\circ, \bullet\}) \cup \mathfrak{D}$ and $\mathfrak{d}_t\colon [\![\ell_t]\!] \to (\mathfrak{U} \otimes \{\circ, \bullet\}) \cup \mathfrak{D}$ as well as $(\mathfrak{c}_t, \mathfrak{d}_t, p_t) \in \mathcal{C}$ be arbitrary. If $(\mathfrak{a}, \mathfrak{b}, q) = (\mathfrak{c}_1, \mathfrak{d}_1, p_1) \otimes (\mathfrak{c}_2, \mathfrak{d}_2, p_2)$ and if $\mathsf{H}_1 = \Pi^{k_1}_{\ell_1}$ and $\mathsf{H}_2 = \Pi^{k_1+k_2}_{\ell_1+\ell_2} \backslash \Pi^{k_1}_{\ell_1}$ and $\gamma_{\mathsf{H}_t} \equiv \gamma_{\mathsf{H}_t, \ell_1+\ell_2}^{\;\;k_1+k_2}$ for any $t \in [\![2]\!]$, then $\xi^{\mathfrak{a}}_{\mathfrak{b}} \circ \gamma_{\mathsf{H}_t} = \xi^{\mathfrak{c}_t}_{\mathfrak{d}_t}$ for any $t \in [\![2]\!]$ by Lemma 4.2 (c). Given any $\mathsf{B} \in q$, by definition of q, there exist $t \in [\![2]\!]$ and $\mathsf{B}_t \in p_t$ with $\mathsf{B} = \gamma_{\mathsf{H}_t \to}(\mathsf{B}_t)$. Due to $(\mathfrak{c}_t, \mathfrak{d}_t, p_t) \in \mathcal{C}$ we find $Z \in h$ such that $\mathsf{B}_t \subseteq \xi^{\mathfrak{c}_t \leftarrow}_{\mathfrak{d}_t}(Z)$. It follows $\mathsf{B} = \gamma_{\mathsf{H}_t \to}(\mathsf{B}_t) \subseteq \gamma_{\mathsf{H}_t \to}(\xi^{\mathfrak{c}_t \leftarrow}_{\mathfrak{d}_t}(Z)) \subseteq \gamma_{\mathsf{H}_t \to}((\xi^{\mathfrak{a}}_{\mathfrak{b}} \circ \gamma_{\mathsf{H}_t})^{\leftarrow}(Z)) = (\gamma_{\mathsf{H}_t \to} \circ \gamma_{\mathsf{H}_t}{}^{\leftarrow})(\xi^{\mathfrak{a} \leftarrow}_{\mathfrak{b}}(Z)) \subseteq \xi^{\mathfrak{a} \leftarrow}_{\mathfrak{b}}(Z)$. Thus, indeed, $q \leq \xi^{\mathfrak{a} \leftarrow}_{\mathfrak{b}}(h)$.

Step 5: Erasing. Finally, let $\{k, \ell\} \subseteq \mathbb{N}_0$ and $\mathfrak{c}\colon [\![k]\!] \to (\mathfrak{U} \otimes \{\circ, \bullet\}) \cup \mathfrak{D}$ and $\mathfrak{d}\colon [\![\ell]\!] \to (\mathfrak{U} \otimes \{\circ, \bullet\}) \cup \mathfrak{D}$ and $(\mathfrak{c}, \mathfrak{d}, p) \in \mathcal{C}$ be arbitrary and let $\mathsf{T} \subseteq \Pi^0_\ell$ and $z \in \mathfrak{U} \cup \mathfrak{D}$ be such that $|\mathsf{T}| = 2$ and $\mathsf{T} \subseteq \xi^{\mathfrak{c} \leftarrow}_{\mathfrak{d}}(\{z\})$ and such that, if $z \in \mathfrak{U}$, then ${}_z\sigma^{\mathfrak{c}}_{\mathfrak{d}}(\mathsf{T}) = 0$. If $\mathsf{M} = \Pi^k_\ell \backslash \mathsf{T}$ and $\gamma_{\mathsf{M}} \equiv \gamma_{\mathsf{M}, \ell}^{\;\;k}$ and $(\mathfrak{a}, \mathfrak{b}, E(p, \mathsf{T})) = E((\mathfrak{c}, \mathfrak{d}, p), \mathsf{T})$, then $\xi^{\mathfrak{a}}_{\mathfrak{b}} = \xi^{\mathfrak{c}}_{\mathfrak{d}} \circ \gamma_{\mathsf{M}}$ by Lemma 4.2 (d). For each $\mathsf{B} \in E(p, \mathsf{T})$, according to the definition of $E(p, \mathsf{T})$, there are now two cases to distinguish.

Case 5.1: If there is $\mathsf{A} \in p$ such that $\mathsf{A} \cap \mathsf{T} = \varnothing$ and $\mathsf{B} = \gamma_{\mathsf{M}}{}^{\leftarrow}(\mathsf{A})$, then by $p \leq \xi^{\mathfrak{c} \leftarrow}_{\mathfrak{d}}(h)$ we find $Z \in h$ with $\mathsf{A} \subseteq \xi^{\mathfrak{c} \leftarrow}_{\mathfrak{d}}(Z)$. Hence, $\mathsf{B} = \gamma_{\mathsf{M}}{}^{\leftarrow}(\mathsf{A}) \subseteq \gamma_{\mathsf{M}}{}^{\leftarrow}(\xi^{\mathfrak{c} \leftarrow}_{\mathfrak{d}}(Z)) = (\xi^{\mathfrak{c}}_{\mathfrak{d}} \circ \gamma_{\mathsf{M}})^{\leftarrow}(Z) = \xi^{\mathfrak{a} \leftarrow}_{\mathfrak{b}}(Z)$.

Case 5.2: The other possibility is that $\mathsf{B} = \gamma_{\mathsf{M}}{}^{\leftarrow}(\bigcup\{\mathsf{A} \in p \wedge \mathsf{A} \cap \mathsf{T} \neq \varnothing\})$. Since $\mathsf{T} \subseteq \xi^{\mathfrak{c} \leftarrow}_{\mathfrak{d}}(\{z\}) \in \ker(\xi^{\mathfrak{c}}_{\mathfrak{d}})$ and since $\ker(\xi^{\mathfrak{c}}_{\mathfrak{d}})$ is a set-theoretical partition of Π^k_ℓ, for any $\mathsf{A} \in p$, whenever $\mathsf{A} \cap \mathsf{T} \neq \varnothing$, then $\mathsf{A} \subseteq \xi^{\mathfrak{c} \leftarrow}_{\mathfrak{d}}(\{z\})$. Consequently, $\bigcup\{\mathsf{A} \in p \wedge \mathsf{A} \cap \mathsf{T} \neq \varnothing\} \subseteq \xi^{\mathfrak{c} \leftarrow}_{\mathfrak{d}}(\{z\})$. It follows $\mathsf{B} \subseteq \gamma_{\mathsf{M}}{}^{\leftarrow}(\xi^{\mathfrak{c} \leftarrow}_{\mathfrak{d}}(\{z\})) = (\xi^{\mathfrak{c}}_{\mathfrak{d}} \circ \gamma_{\mathsf{M}})^{\leftarrow}(\{z\}) = \xi^{\mathfrak{a} \leftarrow}_{\mathfrak{b}}(\{z\})$. Thus, if $Z \in h$ is such that $z \in Z$, then $\mathsf{B} \subseteq \xi^{\mathfrak{a} \leftarrow}_{\mathfrak{b}}(Z)$.

In conclusion, $\mathsf{B} \subseteq \xi^{\mathfrak{a} \leftarrow}_{\mathfrak{b}}(Z)$ for some $Z \in h$ in any case, which means that we have shown $E(p, \mathsf{T}) \leq \xi^{\mathfrak{a} \leftarrow}_{\mathfrak{b}}(h)$ as claimed. Overall then, $\mathcal{C}$ is indeed a category. □

Building on the preceding result, a category of labeled partition can be obtained by demanding that the restriction to any tag area belong to a category of singly tagged labeled partitions.

Definition 4.4. For any family $(\mathcal{X}_z)_{z \in \mathfrak{U} \cup \mathfrak{D}}$ such that $\mathcal{X}_z$ is a category of $(\{z\}, \varnothing)$-tagged labeled partitions if $z \in \mathfrak{U}$ and of of $(\varnothing, \{z\})$-tagged labeled partitions if $z \in \mathfrak{D}$ for any $z \in \mathfrak{U} \cup \mathfrak{D}$ the *direct co-product* $\times_{z \in \mathfrak{U} \cup \mathfrak{D}} \mathcal{X}_z$ of $(\mathcal{X}_z)_{z \in \mathfrak{U} \cup \mathfrak{D}}$ is the set of all $(\mathfrak{c}, \mathfrak{d}, p) \in {}_{\mathfrak{U}, \mathfrak{D}}\mathcal{S}$ such that $p \leq \ker(\xi^{\mathfrak{c}}_{\mathfrak{d}})$ and such that $R((\mathfrak{c}, \mathfrak{d}, p), \xi^{\mathfrak{c} \leftarrow}_{\mathfrak{d}}(\{z\})) \in \mathcal{X}_z$ for any $z \in \mathfrak{U} \cup \mathfrak{D}$.

The next lemma helps with the upcoming proof that direct products are actually categories and will be used later on as well.

Lemma 4.5. *Let $\{k, \ell\} \subseteq \mathbb{N}_0$, let $\mathfrak{c}\colon [\![k]\!] \to (\mathfrak{U} \otimes \{\circ, \bullet\}) \cup \mathfrak{D}$ and $\mathfrak{d}\colon [\![\ell]\!] \to (\mathfrak{U} \otimes \{\circ, \bullet\}) \cup \mathfrak{D}$, and let p be any partition of Π^k_ℓ. Moreover, for each $t \in [\![2]\!]$ let $\{k_t, \ell_t\} \subseteq \mathbb{N}_0$, let $\mathfrak{c}_t\colon [\![k_t]\!] \to (\mathfrak{U} \otimes \{\circ, \bullet\}) \cup \mathfrak{D}$ and $\mathfrak{d}_t\colon [\![\ell_t]\!] \to (\mathfrak{U} \otimes \{\circ, \bullet\}) \cup \mathfrak{D}$, and let p_t be any partition of $\Pi^{k_t}_{\ell_t}$.*

(a) *For any $r \in \{\text{⤹}, \text{⤸}, \text{⤺}, \text{⤻}\}$ and $\mathsf{Z} \subseteq \Pi^k_\ell$, if $(\mathfrak{a}, \mathfrak{b}, q) = (\mathfrak{c}, \mathfrak{d}, p)^r$ is defined, if $\mathsf{Y} = \omega^{r, k}_{\;\;\ell}{}^{\leftarrow}(\mathsf{Z})$, if $\{x, y\} \subseteq \mathbb{N}_0$ are such that $\mathfrak{a}\colon [\![x]\!] \to (\mathfrak{U} \otimes \{\circ, \bullet\}) \cup \mathfrak{D}$ and $\mathfrak{b}\colon [\![y]\!] \to$*

$(\mathfrak{U} \otimes \{\circ, \bullet\}) \cup \mathfrak{O}$, *if* $m = \alpha(\mathsf{Z})$ *and* $n = \beta(\mathsf{Z})$, *if* $(\mathfrak{u}, \mathfrak{v}, w) = R((\mathfrak{a}, \mathfrak{b}, q), \mathsf{Y})$, *if* $(\mathfrak{f}, \mathfrak{g}, s) = R((\mathfrak{c}, \mathfrak{d}, p), \mathsf{Z})$, *and if* e *is given by* ${}^{\blacksquare}1$ *if* $r = \curvearrowleft$, *by* ${}^{\blacksquare}k$ *if* $r = \curvearrowright$, *by* ${}_{\blacksquare}1$ *if* $r = \circlearrowleft$ *and by* ${}_{\blacksquare}\ell$ *if* $r = \circlearrowright$, *then*

(i) $\omega^{r,k}_{\ \ \ell} \circ \gamma_{\mathsf{Y},y}^{\ \ \ x} = \begin{cases} \gamma_{\mathsf{Z},\ell}^{\ \ \ k} & \textit{if } \mathsf{e} \notin \mathsf{Z} \\ \gamma_{\mathsf{Z},\ell}^{\ \ \ k} \circ \omega^{r,m}_{\ \ \ n} & \textit{otherwise.} \end{cases}$.

(ii) $\xi^{\mathfrak{u}}_{\mathfrak{v}} = \begin{cases} \xi^{\mathfrak{f}}_{\mathfrak{g}} & \textit{if } \mathsf{e} \notin \mathsf{Z} \\ \xi^{\mathfrak{f}}_{\mathfrak{g}} \circ \omega^{r,m}_{\ \ \ n} & \textit{otherwise.} \end{cases}$

(iii) $\zeta^{\mathfrak{u}}_{\mathfrak{v}} = \begin{cases} \zeta^{\mathfrak{f}}_{\mathfrak{g}} & \textit{if } \mathsf{e} \notin \mathsf{Z} \\ \zeta^{\mathfrak{f}}_{\mathfrak{g}} \circ \omega^{r,m}_{\ \ \ n} & \textit{otherwise.} \end{cases}$

(iv) $w = \begin{cases} s & \textit{if } \mathsf{e} \notin \mathsf{Z} \\ s^r & \textit{otherwise.} \end{cases}$

In other words,

$$R((\mathfrak{c}, \mathfrak{d}, p)^r, \mathsf{Y}) = \begin{cases} R((\mathfrak{c}, \mathfrak{d}, p), \mathsf{Z}) & \textit{if } \mathsf{e} \notin \mathsf{Z} \\ R((\mathfrak{c}, \mathfrak{d}, p), \mathsf{Z})^r & \textit{otherwise.} \end{cases}.$$

(b) For any $\mathsf{Z} \subseteq \Pi^k_\ell$, *if* $\mathsf{Y} = \kappa^{k\leftarrow}_\ell(\mathsf{Z})$, *and if* $m = \alpha(\mathsf{Z})$ *and* $n = \beta(\mathsf{Z})$, *then*

(i) $\kappa^k_\ell \circ \gamma_{\mathsf{Y},k}^{\ \ \ \ell} = \gamma_{\mathsf{Z},\ell}^{\ \ \ k} \circ \kappa^m_n$.

(ii) $\xi^{\mathfrak{u}}_{\mathfrak{v}} = \xi^{\mathfrak{g}}_{\mathfrak{f}}$.

(iii) $\zeta^{\mathfrak{u}}_{\mathfrak{v}} = \zeta^{\mathfrak{g}}_{\mathfrak{f}}$.

(iv) $w = s^*$.

In other words,

$$R((\mathfrak{c}, \mathfrak{d}, p)^*, \mathsf{Y}) = R((\mathfrak{c}, \mathfrak{d}, p), \mathsf{Z})^*.$$

(c) If $\mathsf{H}_1 = \Pi^{k_1}_{\ell_1}$ *and* $\mathsf{H}_2 = \Pi^{k_1+k_2}_{\ell_1+\ell_2} \backslash \Pi^{k_1}_{\ell_1}$, *then for any* $\mathsf{Z}_1 \subseteq \Pi^{k_1}_{\ell_1}$ *and* $\mathsf{Z}_2 \subseteq \Pi^{k_2}_{\ell_2}$, *if* $\mathsf{Y} \subseteq \Pi^{k_1+k_2}_{\ell_1+\ell_2}$ *the unique set with* $\mathsf{Z}_t = \gamma_{\mathsf{H}_t,\ell_1+\ell_2}^{\ \ \ \ \ \ \ k_1+k_2\leftarrow}(\mathsf{Y})$ *for each* $t \in [\![2]\!]$, *if* $(\mathfrak{u}, \mathfrak{v}, w) = R((\mathfrak{c}_1 \otimes \mathfrak{c}_2, \mathfrak{d}_1 \otimes \mathfrak{d}_2, q), \mathsf{Y})$, *if* $(\mathfrak{f}_t, \mathfrak{g}_t, s_t) = R((\mathfrak{c}_t, \mathfrak{d}_t, p_t), \mathsf{Z}_t)$ *and* $m_t = \alpha(\mathsf{Z}_t)$ *and* $n_t = \beta(\mathsf{Z}_t)$ *and for each* $t \in [\![2]\!]$, *and if* $\mathsf{R}_1 = \Pi^{m_1}_{n_1}$ *and* $\mathsf{R}_2 = \Pi^{m_1+m_2}_{n_1+n_2} \backslash \Pi^{m_1}_{n_1}$, *then for each* $t \in [\![2]\!]$,

(i) $\gamma_{\mathsf{Y},\ell_1+\ell_2}^{\ \ \ \ \ \ \ k_1+k_2} \circ \gamma_{\mathsf{R}_t,n_1+n_2}^{\ \ \ \ \ \ \ m_1+m_2} = \gamma_{\mathsf{H}_t,\ell_1+\ell_2}^{\ \ \ \ \ \ \ k_1+k_2} \circ \gamma_{\mathsf{Z}_t,\ell_t}^{\ \ \ \ k_t}$.

(ii) $\xi^{\mathfrak{u}}_{\mathfrak{v}} \circ \gamma_{\mathsf{R}_t,n_1+n_2}^{\ \ \ \ \ \ \ m_1+m_2} = \xi^{\mathfrak{f}_t}_{\mathfrak{g}_t}$.

(iii) $\zeta^{\mathfrak{u}}_{\mathfrak{v}} \circ \gamma_{\mathsf{R}_t,n_1+n_2}^{\ \ \ \ \ \ \ m_1+m_2} = \zeta^{\mathfrak{f}_t}_{\mathfrak{g}_t}$.

(iv) $w = s_1 \otimes s_2$.

In other words,

$$R((\mathfrak{c}_1, \mathfrak{d}_1, p_1) \otimes (\mathfrak{c}_2, \mathfrak{d}_2, p_2), \mathsf{Y}) \\ = R((\mathfrak{c}_1, \mathfrak{d}_1, p_1), \mathsf{Z}_1) \otimes R((\mathfrak{c}_2, \mathfrak{d}_2, p_2), \mathsf{Z}_2).$$

(d) For any $\mathsf{S} \subseteq \Pi^k_\ell$ *any* $\mathsf{Z} \subseteq \Pi^k_\ell$, *if* $\mathfrak{a}\,{}^{\blacksquare}{}_{\blacksquare}\,\mathfrak{b} = R(\mathfrak{c}\,{}^{\blacksquare}{}_{\blacksquare}\,\mathfrak{d}, \mathsf{S})$, *if* $x = \alpha(\mathsf{S})$ *and* $y = \beta(\mathsf{S})$, *if* $\mathsf{Y} = \gamma_{\mathsf{S},\ell}^{\ \ \ k\leftarrow}(\mathsf{Z})$, *if* $\mathfrak{u}\,{}^{\blacksquare}{}_{\blacksquare}\,\mathfrak{v} = R(\mathfrak{a}\,{}^{\blacksquare}{}_{\blacksquare}\,\mathfrak{b}, \mathsf{Y})$, *if* $\mathfrak{f}\,{}^{\blacksquare}{}_{\blacksquare}\,\mathfrak{g} = R(\mathfrak{c}\,{}^{\blacksquare}{}_{\blacksquare}\,\mathfrak{d}, \mathsf{Z})$, *and if* $m = \alpha(\mathsf{Z})$ *and* $n = \beta(\mathsf{Z})$ *and* $\mathsf{Q} = \gamma_{\mathsf{Z},\ell}^{\ \ \ k\leftarrow}(\mathsf{S})$, *then*

(i) ${\gamma_{\mathsf{S},\ell}}^{k} \circ {\gamma_{\mathsf{Y},y}}^{x} = {\gamma_{\mathsf{Z},\ell}}^{k} \circ {\gamma_{\mathsf{Q},n}}^{m}$.
(ii) $\xi_{\mathfrak{v}}^{\mathfrak{u}} = \xi_{\mathfrak{g}}^{\mathfrak{f}} \circ {\gamma_{\mathsf{Q},n}}^{m}$.
(iii) $\zeta_{\mathfrak{v}}^{\mathfrak{u}} = \zeta_{\mathfrak{g}}^{\mathfrak{f}} \circ {\gamma_{\mathsf{Q},n}}^{m}$.
(iv) $R(R(p,\mathsf{S}),\mathsf{Y}) = R(R(p,\mathsf{Z}),\mathsf{Q})$.
(v) $R(E(p,\mathsf{S}),\mathsf{T}) = E(R(p,\mathsf{Z}),\mathsf{V})$ *if* $\mathsf{T} = \Pi_\ell^k\backslash\mathsf{S}$ *and* $\mathsf{V} = \Pi_n^m\backslash\mathsf{Q}$.

In other words,

$$R(R((\mathfrak{c},\mathfrak{d},p),\mathsf{S}),\mathsf{Y}) = R(R((\mathfrak{c},\mathfrak{d},p),\mathsf{Z}),\mathsf{Q}).$$

Moreover, if $\mathsf{S} = \Pi_\ell^k\backslash\mathsf{T}$ *and* $\mathsf{V} = {\gamma_{\mathsf{Z},\ell}}^{k\leftarrow}(\mathsf{T})$ *for any* $\mathsf{T} \subseteq \Pi_\ell^k$*, then*

$$R(E((\mathfrak{c},\mathfrak{d},p),\mathsf{T}),\mathsf{Y}) = \begin{cases} R((\mathfrak{c},\mathfrak{d},p),\mathsf{Z}) & \text{if } \mathsf{Z}\cap\mathsf{T} = \varnothing \\ E(R((\mathfrak{c},\mathfrak{d},p),\mathsf{Z}),\mathsf{V}) & \text{otherwise.}\end{cases}$$

Proof. The claims follow immediately from the definitions. □

Proposition 4.6. *For any family* $(\mathcal{X}_z)_{z\in\mathfrak{U}\cup\mathfrak{O}}$ *such that* $\mathcal{X}_z$ *is a category of* $(\{z\},\varnothing)$*-tagged labeled partitions for each* $z\in\mathfrak{U}$ *and a category of* $(\varnothing,\{z\})$*-tagged labeled partitions for each* $z\in\mathfrak{O}$ *the direct co-product* $\times_{z\in\mathfrak{U}\cup\mathfrak{O}}\mathcal{X}_z$ *is a category of* $(\mathfrak{U},\mathfrak{O})$*-tagged labeled partitions.*

Proof. The set $\mathcal{D}$ of all $(\mathfrak{c},\mathfrak{d},p) \in {}_{\mathfrak{U},\mathfrak{O}}\mathcal{S}$ with $p \leq \ker(\xi_{\mathfrak{d}}^{\mathfrak{c}})$ is a category by Proposition 4.3. We only have to show the condition about the reindexed restrictions is stable under the operations of Proposition 3.26. Note that the claim would not have been altered if we had confined ourselves to asking $R((\mathfrak{c},\mathfrak{d},p),\xi_{\mathfrak{d}}^{\mathfrak{c}\leftarrow}(\{z\})) \in \mathcal{X}_z$ only for $z\in\mathfrak{U}\cup\mathfrak{O}$ with $\xi_{\mathfrak{d}}^{\mathfrak{c}\leftarrow}(\{z\}) \neq \varnothing$ because $R((\mathfrak{c},\mathfrak{d},p),\varnothing) = (\varnothing,\varnothing,\varnothing) \in \mathcal{X}_z$ for any $z\in\mathfrak{U}\cup\mathfrak{O}$.

Step 1: Identities. Given any $x\in\mathfrak{U}$, any $y\in\mathfrak{O}$ and any $\mathfrak{c}\in\{(x,\circ),(x,\bullet),y\}$, we prove $\mathrm{id}_{\mathfrak{c}} = (\mathfrak{c},\mathfrak{c},\sqcap) \in \times_{z\in\mathfrak{U}\cup\mathfrak{O}}\mathcal{X}_z$. Of course, $\ker(\xi_{\mathfrak{c}}^{\mathfrak{c}}) = \sqcap$. Hence, trivially, $\sqcap \leq \ker(\xi_{\mathfrak{c}}^{\mathfrak{c}})$. For the same reason, for any $z\in\mathfrak{U}\cup\mathfrak{O}$ with $\mathsf{Y} = \xi_{\mathfrak{c}}^{\mathfrak{c}\leftarrow}(\{z\}) \neq \varnothing$ we immediately know $\mathsf{Y} = \Pi_1^1$ and thus $R(\mathrm{id}_{\mathfrak{c}},\mathsf{Y}) = \mathrm{id}_{\mathfrak{c}} \in \mathcal{X}_z$ because $\mathcal{X}_z$ is a category. Thus, $\mathrm{id}_{\mathfrak{c}} \in \times_{z\in\mathfrak{U}\cup\mathfrak{O}}\mathcal{X}_z$

Step 2: Rotations. Let $k\in\mathbb{N}$ and $\ell\in\mathbb{N}_0$, let $\mathfrak{c}\colon [\![k]\!] \to (\mathfrak{U}\otimes\{\circ,\bullet\})\cup\mathfrak{O}$ and $\mathfrak{d}\colon [\![\ell]\!] \to (\mathfrak{U}\otimes\{\circ,\bullet\})\cup\mathfrak{O}$, let $(\mathfrak{c},\mathfrak{d},p) \in \times_{z\in\mathfrak{U}\cup\mathfrak{O}}\mathcal{X}_z$, let $r\in\{\circlearrowleft,\circlearrowright\}$, let $\mathsf{e} = {}^{\blacksquare}1$ if $r = \circlearrowleft$ and $\mathsf{e} = {}^{\blacksquare}k$ if $r = \circlearrowright$ and let $(\mathfrak{a},\mathfrak{b}) = (\mathfrak{c},\mathfrak{d})^r$. For any $z\in\mathfrak{U}\cup\mathfrak{O}$, if $\mathsf{Y} = \xi_{\mathfrak{b}}^{\mathfrak{a}\leftarrow}(\{z\})$ and $\mathsf{Z} = \xi_{\mathfrak{d}}^{\mathfrak{c}\leftarrow}(\{z\})$, then, because $\xi_{\mathfrak{b}}^{\mathfrak{a}} = \xi_{\mathfrak{d}}^{\mathfrak{c}}\circ{\omega^{r,k}_{\ell}}$ by Lemma 4.2 (a), necessarily, $\mathsf{Y} = {\omega^{r,k}_{\ell}}^{\leftarrow}(\mathsf{Z})$. Hence, by Lemma 4.5 (a) the partition $R((\mathfrak{a},\mathfrak{b},p^r),\mathsf{Y})$ is given by $R((\mathfrak{c},\mathfrak{d},p),\mathsf{Z})$ if $\mathsf{e}\notin\mathsf{Z}$ and by $R((\mathfrak{c},\mathfrak{d},p),\mathsf{Z})^r$ otherwise. And, of course, both $R((\mathfrak{c},\mathfrak{d},p),\mathsf{Z})$ and $R((\mathfrak{c},\mathfrak{d},p),\mathsf{Z})^r$ are elements of $\mathcal{X}_z$ since $(\mathfrak{c},\mathfrak{d},p) \in \times_{z\in\mathfrak{U}\cup\mathfrak{O}}\mathcal{X}_z$ by assumption and since categories are closed under rotations. Thus, indeed $R((\mathfrak{a},\mathfrak{b},p^r),\mathsf{Y}) \in \mathcal{X}_z$, which proves that, too, $\times_{z\in\mathfrak{U}\cup\mathfrak{O}}\mathcal{X}_z$ is stable under rotations.

Step 3: Adjoints. Given any $\{k,\ell\}\subseteq\mathbb{N}_0$, any $\mathfrak{c}\colon [\![k]\!] \to (\mathfrak{U}\otimes\{\circ,\bullet\})\cup\mathfrak{O}$ and $\mathfrak{d}\colon [\![\ell]\!] \to (\mathfrak{U}\otimes\{\circ,\bullet\})\cup\mathfrak{O}$, any $(\mathfrak{c},\mathfrak{d},p) \in \times_{z\in\mathfrak{U}\cup\mathfrak{O}}\mathcal{X}_z$ and any $z\in\mathfrak{U}\cup\mathfrak{O}$, if $\mathsf{Y} = \xi_{\mathfrak{c}}^{\mathfrak{d}\leftarrow}(\{z\})$ and $\mathsf{Z} = \xi_{\mathfrak{d}}^{\mathfrak{c}\leftarrow}(\{z\})$, then $\mathsf{Y} = \kappa_\ell^{k\leftarrow}(\mathsf{Z})$ because $\xi_{\mathfrak{c}}^{\mathfrak{d}} = \xi_{\mathfrak{d}}^{\mathfrak{c}}\circ\kappa_\ell^k$ by Lemma 4.2 (b). Thus $R((\mathfrak{d},\mathfrak{c},p^*),\mathsf{Y}) = R((\mathfrak{c},\mathfrak{d},p),\mathsf{Z})^*$according to Lemma 4.5 (b). Since $R((\mathfrak{c},\mathfrak{d},p),\mathsf{Z}) \in$

$\mathcal{X}_z$ by $(\mathfrak{c},\mathfrak{d},p) \in \times_{z\in\mathfrak{U}\cup\mathfrak{O}}\mathcal{X}_z$ and since $\mathcal{X}_z$ is closed under involution, this already proves $R((\mathfrak{d},\mathfrak{c},p^*),\mathsf{Y}) \in \mathcal{X}_z$. Hence, $\times_{z\in\mathfrak{U}\cup\mathfrak{O}}\mathcal{X}_z$ is also invariant under involution.

Step 4: Tensor products. For each $t \in [\![2]\!]$ let $\{k_t,\ell_t\} \subseteq \mathbb{N}_0$ an $\mathfrak{c}_t\colon [\![k_t]\!] \to (\mathfrak{U}\otimes\{\circ,\bullet\})\cup\mathfrak{O}$ and $\mathfrak{d}_t\colon [\![\ell_t]\!] \to (\mathfrak{U}\otimes\{\circ,\bullet\})\cup\mathfrak{O}$ as well as $(\mathfrak{c}_t,\mathfrak{d}_t,p_t) \in \times_{z\in\mathfrak{U}\cup\mathfrak{O}}\mathcal{X}_z$ and $z \in \mathfrak{U}\cup\mathfrak{O}$ be arbitrary. Moreover, let $\mathsf{H}_1 = \Pi^{k_1}_{\ell_1}$ and $\mathsf{H}_2 = \Pi^{k_1+k_2}_{\ell_1+\ell_2}\backslash\Pi^{k_1}_{\ell_1}$ and let $\mathsf{Y} = \xi^{\mathfrak{c}_1\otimes\mathfrak{c}_2}_{\mathfrak{d}_1\otimes\mathfrak{d}_2}{}^{\leftarrow}(\{z\})$. For any $t \in [\![2]\!]$, if $\mathsf{Z}_t = \xi^{\mathfrak{c}_t}_{\mathfrak{d}_t}{}^{\leftarrow}(\{z\})$, then $\mathsf{Z}_t = \gamma_{\mathsf{H}_t,\ell_1+\ell_2}^{\;k_1+k_2\leftarrow}(\mathsf{Y})$ since $\xi^{\mathfrak{c}_1\otimes\mathfrak{c}_2}_{\mathfrak{d}_1\otimes\mathfrak{d}_2} \circ \gamma_{\mathsf{H}_t,\ell_1+\ell_2}^{\;k_1+k_2} = \xi^{\mathfrak{c}_t}_{\mathfrak{d}_t}$ by Lemma 4.2 (c). Consequently, by Lemma 4.5 (b) the partitions $R((\mathfrak{c}_1,\mathfrak{d}_1,p_1)\otimes(\mathfrak{c}_2,\mathfrak{d}_2,p_2),\mathsf{Y})$ and $R((\mathfrak{c}_1,\mathfrak{d}_1,p_1),\mathsf{Z}_1)\otimes R((\mathfrak{c}_2,\mathfrak{d}_2,p_2),\mathsf{Z}_2)$ coincide. Because the assumption $(\mathfrak{c}_t,\mathfrak{d}_t,p_t) \in \times_{z\in\mathfrak{U}\cup\mathfrak{O}}\mathcal{X}_z$ ensures $R((\mathfrak{c}_t,\mathfrak{d}_t,p_t),\mathsf{Z}_t) \in \mathcal{X}_z$ for each $t \in [\![2]\!]$ and because $\mathcal{X}_z$ is closed under tensor product, we have thus shown $R((\mathfrak{c}_1,\mathfrak{d}_1,p_1)\otimes(\mathfrak{c}_2,\mathfrak{d}_2,p_2),\mathsf{Y}) \in \times_{z\in\mathfrak{U}\cup\mathfrak{O}}\mathcal{X}_z$. In conclusion, $\times_{z\in\mathfrak{U}\cup\mathfrak{O}}\mathcal{X}_z$ is stable under tensor products.

Step 5: Erasing. Last, let $\{k,\ell\} \subseteq \mathbb{N}_0$ and $\mathfrak{c}\colon [\![k]\!] \to (\mathfrak{U}\otimes\{\circ,\bullet\})\cup\mathfrak{O}$ and $\mathfrak{d}\colon [\![\ell]\!] \to (\mathfrak{U}\otimes\{\circ,\bullet\})\cup\mathfrak{O}$ and $(\mathfrak{c},\mathfrak{d},p) \in \times_{z\in\mathfrak{U}\cup\mathfrak{O}}\mathcal{X}_z$, let $\mathsf{T} \subseteq \Pi^0_\ell$ and $z_\mathsf{T} \in \mathfrak{U}\cup\mathfrak{O}$ be such that $|\mathsf{T}| = 2$ and $\mathsf{T} \subseteq \xi^{\mathfrak{c}}_{\mathfrak{d}}{}^{\leftarrow}(\{z_\mathsf{T}\})$ and such that, if $z_\mathsf{T} \in \mathfrak{U}$, then ${}_{z_\mathsf{T}}\sigma^{\mathfrak{c}}_{\mathfrak{d}}(\mathsf{T}) = 0$. Moreover, let $(\mathfrak{a},\mathfrak{b},E(p,\mathsf{T})) = E((\mathfrak{c},\mathfrak{d},p),\mathsf{T})$ and $\mathsf{S} = \Pi^k_\ell\backslash\mathsf{T}$. For any $z \in \mathfrak{U}\cup\mathfrak{O}$,if and $\mathsf{Y} = \xi^{\mathfrak{a}}_{\mathfrak{b}}{}^{\leftarrow}(\{z\})$ and $\mathsf{Z} = \xi^{\mathfrak{c}}_{\mathfrak{d}}{}^{\leftarrow}(\{z\})$, then $\mathsf{Y} = \gamma_{\mathsf{S},\ell}^{\;k\leftarrow}(\mathsf{Z})$ because $\xi^{\mathfrak{a}}_{\mathfrak{b}} = \xi \circ \gamma_{\mathsf{S},\ell}^{\;k}$ by Lemma 4.2 (d). With $\mathsf{V} = \gamma_{\mathsf{Z},\ell}^{\;k\leftarrow}(\mathsf{T})$, then by Lemma 4.5 (d) the partition $R((\mathfrak{a},\mathfrak{b},E(p,\mathsf{T})),\mathsf{Y})$ is given by $R((\mathfrak{c},\mathfrak{d},p),\mathsf{Z})$ if $\mathsf{Z}\cap\mathsf{T} = \varnothing$ and $E(R((\mathfrak{c},\mathfrak{d},p),\mathsf{Z}),\mathsf{V})$ otherwise. Of course, once more, $R((\mathfrak{c},\mathfrak{d},p),\mathsf{Z}) \in \mathcal{X}_z$ by $(\mathfrak{c},\mathfrak{d},p) \in \times_{z\in\mathfrak{U}\cup\mathfrak{O}}\mathcal{X}_z$. Hence, if $\mathsf{Z}\cap\mathsf{T} = \varnothing$, then $R((\mathfrak{a},\mathfrak{b},E(p,\mathsf{T})),\mathsf{Y}) \in \mathcal{X}_z$.

But this is also true if $\mathsf{Z}\cap\mathsf{T} \neq \varnothing$ because then $E(R((\mathfrak{c},\mathfrak{d},p),\mathsf{Z}),\mathsf{V}) \in \mathcal{X}_z$ for the following reasons. Because $\mathsf{T} \subseteq \xi^{\mathfrak{c}}_{\mathfrak{d}}{}^{\leftarrow}(\{z_\mathsf{T}\})$ and $\mathsf{Z} \in \ker(\xi^{\mathfrak{c}}_{\mathfrak{d}})$ and because $\ker(\xi^{\mathfrak{c}}_{\mathfrak{d}})$ is a set-theoretical partition of Π^k_ℓ, whenever $\mathsf{Z}\cap\mathsf{T} \neq \varnothing$, then already $\mathsf{T} \subseteq \mathsf{Z}$ and $z_\mathsf{T} = z$. Thus, in that case, $|\mathsf{V}| = |\mathsf{T}| = 2$ and, with $\mathfrak{f}\,\blacksquare\,\mathfrak{g} = R(\mathfrak{c}\,\blacksquare\,\mathfrak{d},\mathsf{Z})$, trivially, $\mathsf{V} \subseteq \xi^{\mathfrak{f}}_{\mathfrak{g}}{}^{\leftarrow}(\{z\})$ by $\xi^{\mathfrak{f}}_{\mathfrak{g}} = \xi^{\mathfrak{c}}_{\mathfrak{d}} \circ \gamma_{\mathsf{Z},\ell}^{\;k}$, and, likewise, if $z \in \mathfrak{U}$, then ${}_z\sigma^{\mathfrak{f}}_{\mathfrak{g}}(\mathsf{V}) = {}_z\sigma^{\mathfrak{c}}_{\mathfrak{d}}(\mathsf{T})$ Lemma 4.2 (d). Hence, erasing V is an operation under which $\mathcal{X}_z$ is closed according to Lemma 3.25. Altogether we have thus shown that $\times_{z\in\mathfrak{U}\cup\mathfrak{O}}\mathcal{X}_z$ is indeed a category. □

In a special case of the preceding proposition, custom notation is helpful.

Definition 4.7. For any countable set Z and any category $\mathcal{C}$ of two-colored or uncolored partitions, the *direct co-power category* $\mathcal{C}^{\times Z}$ is the category $\times_{z\in Z}\mathcal{X}_z$, where for each $z \in Z$ the category $\mathcal{X}_z$ is given by $\mathcal{C}$, seen as tagged with the single tag z.

4.2. Big graph co-products. As a refinement of the categories considered above, the demand that the blocks refine the tag areas and that the restriction to any tag area be a member of a particular singly-tagged category can be paired with the condition that two blocks belonging to specified combinations of tags may not cross.

Definition 4.8. A *partial commutation relation* on any given set X is any anti-reflexive symmetric binary relation on X.

DEFINITION 4.9. For any partial commutation relation r on $\mathfrak{U}\cup\mathfrak{D}$ and any family $(\mathcal{X}_z)_{z\in\mathfrak{U}\cup\mathfrak{D}}$ such that $\mathcal{X}_z$ is a category of $(\{z\},\varnothing)$-tagged labeled partitions if $z\in\mathfrak{U}$ and of $(\varnothing,\{z\})$-tagged labeled partitions if $z\in\mathfrak{D}$ for any $z\in\mathfrak{U}\cup\mathfrak{D}$ the *big graph co-product* $☆^r_{z\in\mathfrak{U}\cup\mathfrak{D}}\mathcal{X}_z$ of $(\mathcal{X}_z)_{z\in\mathfrak{U}\cup\mathfrak{D}}$ with respect to r is the set of all $(\mathfrak{c},\mathfrak{d},p)\in{}_{\mathfrak{U},\mathfrak{D}}\mathcal{S}$ for which there exist $\{k,\ell\}\subseteq\mathbb{N}_0$ such that $\mathfrak{c}\colon[\![k]\!]\to(\mathfrak{U}\otimes\{\circ,\bullet\})\cup\mathfrak{D}$ and $\mathfrak{d}\colon[\![\ell]\!]\to(\mathfrak{U}\otimes\{\circ,\bullet\})\cup\mathfrak{D}$, such that $p\leq\ker(\xi^{\mathfrak{c}}_{\mathfrak{d}})$, such that $R((\mathfrak{c},\mathfrak{d},p),\xi^{\mathfrak{c}\leftarrow}_{\mathfrak{d}}(\{z\}))\in\mathcal{X}_z$ for each $z\in\mathfrak{U}\cup\mathfrak{D}$ and such that for any $\{z,z'\}\subseteq\mathfrak{U}\cup\mathfrak{D}$ with $z\neq z'$, whenever $(z,z')\notin r$, then $\mathsf{B}\diagdown^k_\ell\mathsf{B}'$ for any $\{\mathsf{B},\mathsf{B}'\}\subseteq p$ with $\mathsf{B}\subseteq\xi^{\mathfrak{c}\leftarrow}_{\mathfrak{d}}(\{z\})$ and $\mathsf{B}'\subseteq\xi^{\mathfrak{c}\leftarrow}_{\mathfrak{d}}(\{z'\})$.

NOTATION 4.10. If $r=\varnothing$ in Definition 4.9, we also write $☆_{z\in\mathfrak{U}\cup\mathfrak{D}}\mathcal{X}_z$ instead of $☆^r_{z\in\mathfrak{U}\cup\mathfrak{D}}\mathcal{X}_z$ and speak of the *big free co-product* of $(\mathcal{X}_z)_{z\in\mathfrak{U}\cup\mathfrak{D}}$.

REMARK 4.11. In the setting of Definition 4.9, if r is the *trivial partial commutation relation* $\{(z,z')\,|\,\{z,z'\}\subseteq\mathfrak{U}\cup\mathfrak{D}\wedge z\neq z'\}$, then $☆^r_{z\in\mathfrak{U}\cup\mathfrak{D}}\mathcal{X}_z$ is actually the direct co-product $\times_{z\in\mathfrak{U}\cup\mathfrak{D}}\mathcal{X}_z$.

PROPOSITION 4.12. *For any partial commutation relation r on $\mathfrak{U}\cup\mathfrak{D}$ and any family $(\mathcal{X}_z)_{z\in\mathfrak{U}\cup\mathfrak{D}}$ such that $\mathcal{X}_z$ is a category of $(\{z\},\varnothing)$-tagged labeled partitions for each $z\in\mathfrak{U}$ and one of $(\varnothing,\{z\})$-tagged labeled partitions for each $z\in\mathfrak{D}$ the big graph co-product $☆^r_{z\in\mathfrak{U}\cup\mathfrak{D}}\mathcal{X}_z$ of $(\mathcal{X}_z)_{z\in\mathfrak{U}\cup\mathfrak{D}}$ with respect to r is a category of $(\mathfrak{U},\mathfrak{D})$-tagged labeled partitions.*

PROOF. By Proposition 4.6 we only need to check that the non-crossing condition distinguishing $☆^r_{z\in\mathfrak{U}\cup\mathfrak{D}}\mathcal{X}_z$ from $\times_{z\in\mathfrak{U}\cup\mathfrak{D}}\mathcal{X}_z$ has the required elements and is stable under the operations of Proposition 3.26.

Step 1: Identities. Given any $\mathfrak{c}\in(\mathfrak{U}\otimes\{\circ,\bullet\})\cup\mathfrak{D}$ we already know $\mathrm{id}_{\mathfrak{c}}\in\times_{z\in\mathfrak{U}\cup\mathfrak{D}}\mathcal{X}_z$. And because $\mathrm{id}_{\mathfrak{c}}$ only has a single block, trivially, $\mathrm{id}_{\mathfrak{c}}\in☆^r_{z\in\mathfrak{U}\cup\mathfrak{D}}\mathcal{X}_z$.

Step 2: Rotation. Let $k\in\mathbb{N}$ and $\ell\in\mathbb{N}_0$ and $\mathfrak{c}\colon[\![k]\!]\to(\mathfrak{U}\otimes\{\circ,\bullet\})\cup\mathfrak{D}$ and $\mathfrak{d}\colon[\![\ell]\!]\to(\mathfrak{U}\otimes\{\circ,\bullet\})\cup\mathfrak{D}$ and $(\mathfrak{c},\mathfrak{d},p)\in☆^r_{z\in\mathfrak{U}\cup\mathfrak{D}}\mathcal{X}_z$ as well as $r\in\{\Lsh,\Rsh\}$ be arbitrary, and let $\mathfrak{a}$ and $\mathfrak{b}$ be such that $\mathfrak{a}\,\blacksquare\,\mathfrak{b}=(\mathfrak{c}\,\blacksquare\,\mathfrak{d})^r$. Given any $\{z,z'\}\subseteq\mathfrak{U}\cup\mathfrak{D}$ with $z\neq z'$ and $(z,z')\notin r$ and any $\{\mathsf{B},\mathsf{B}'\}\subseteq p^r$ with $\mathsf{B}\subseteq\xi^{\mathfrak{a}\leftarrow}_{\mathfrak{b}}(\{z\})$ and $\mathsf{B}'\subseteq\xi^{\mathfrak{a}\leftarrow}_{\mathfrak{b}}(\{z'\})$, if $\mathsf{A}=\omega^{r,k}_{\ell\,\to}(\mathsf{B})$ and $\mathsf{A}'=\omega^{r,k}_{\ell\,\to}(\mathsf{B}')$, which is to say $\mathsf{B}=\omega^{r,k\leftarrow}_{\ell}(\mathsf{A})$ and $\mathsf{B}'=\omega^{r,k\leftarrow}_{\ell}(\mathsf{A}')$, then the definition $p^r=\omega^{r,k\leftarrow}_{\ell}(p)$ and the fact that $\omega^{r,k}_{\ell}$ is bijective imply $\{\mathsf{A},\mathsf{A}'\}\subseteq p$. Moreover, the identity $\xi^{\mathfrak{a}}_{\mathfrak{b}}=\xi^{\mathfrak{c}}_{\mathfrak{d}}\circ\omega^{r,k}_{\ell}$ from Lemma 4.2 (a) lets us infer that $\mathsf{A}\subseteq\xi^{\mathfrak{c}\leftarrow}_{\mathfrak{d}}(\{z\})$ and $\mathsf{A}'\subseteq\xi^{\mathfrak{c}\leftarrow}_{\mathfrak{d}}(\{z'\})$. Because $(\mathfrak{c},\mathfrak{d},p)\in☆^r_{z\in\mathfrak{U}\cup\mathfrak{D}}\mathcal{X}_z$ we conclude $\mathsf{A}\diagdown^k_\ell\mathsf{A}'$. And since $\omega^{r,k}_{\ell}$ is monotonic with respect to $\Gamma^{k-1}_{\ell+1}$ and Γ^k_ℓ by the same lemma, we may deduce $\mathsf{B}\diagdown^{k-1}_{\ell+1}\mathsf{B}'$ from that, proving $(\mathfrak{c},\mathfrak{d},p)^r\in☆^r_{z\in\mathfrak{U}\cup\mathfrak{D}}\mathcal{X}_z$.

Step 3: Adjoints. For any $\{k,\ell\}\subseteq\mathbb{N}_0$, any $\mathfrak{c}\colon[\![k]\!]\to(\mathfrak{U}\otimes\{\circ,\bullet\})\cup\mathfrak{D}$ and $\mathfrak{d}\colon[\![\ell]\!]\to(\mathfrak{U}\otimes\{\circ,\bullet\})\cup\mathfrak{D}$ and any $(\mathfrak{c},\mathfrak{d},p)\in☆^r_{z\in\mathfrak{U}\cup\mathfrak{D}}\mathcal{X}_z$, if $\{z,z'\}\subseteq\mathfrak{U}\cup\mathfrak{D}$ and $\{\mathsf{B},\mathsf{B}'\}\subseteq p^*$ are such that $z\neq z'$ and $(z,z')\notin r$ as well as $\mathsf{B}\subseteq\xi^{\mathfrak{d}\leftarrow}_{\mathfrak{c}}(\{z\})$ and $\mathsf{B}'\subseteq\xi^{\mathfrak{d}\leftarrow}_{\mathfrak{c}}(\{z'\})$, then putting $\mathsf{A}=\kappa^\ell_{k\to}(\mathsf{B})$ and $\mathsf{A}'=\kappa^\ell_{k\to}(\mathsf{B}')$ necessitates $\mathsf{B}=\kappa^{\ell\leftarrow}_k(\mathsf{A})\in p$ and $\mathsf{B}'=\kappa^{\ell\leftarrow}_k(\mathsf{A}')\in p$ because κ^ℓ_k is bijective and $p^*=\kappa^{\ell\leftarrow}_k(p)$. Since $\xi^{\mathfrak{d}}_{\mathfrak{c}}=\xi^{\mathfrak{c}}_{\mathfrak{d}}\circ\kappa^\ell_k$ by Lemma 4.2 (b) it follows that also $\mathsf{A}\subseteq\xi^{\mathfrak{c}\leftarrow}_{\mathfrak{d}}(\{z\})$ and $\mathsf{A}'\subseteq\xi^{\mathfrak{c}\leftarrow}_{\mathfrak{d}}(\{z'\})$ and thus $\mathsf{A}\diagdown^k_\ell\mathsf{A}'$ by $(\mathfrak{c},\mathfrak{d},p)\in☆^r_{z\in\mathfrak{U}\cup\mathfrak{D}}\mathcal{X}_z$. That implies $\mathsf{B}\diagdown^\ell_k\mathsf{B}'$ and thus $(\mathfrak{c},\mathfrak{d},p)^*\in☆^r_{z\in\mathfrak{U}\cup\mathfrak{D}}\mathcal{X}_z$ because κ^ℓ_k is anti-monotonic with respect to Γ^ℓ_k and Γ^k_ℓ by the same lemma.

Tensor products: Next, for each $t \in \llbracket 2\rrbracket$ let $\{k_t, \ell_t\} \subseteq \mathbb{N}_0$ and $\mathfrak{c}_t\colon \llbracket k_t\rrbracket \to (\mathfrak{U} \otimes \{\circ, \bullet\}) \uplus \mathfrak{D}$ and $\mathfrak{d}_t\colon \llbracket \ell_t\rrbracket \to (\mathfrak{U} \otimes \{\circ, \bullet\}) \uplus \mathfrak{D}$ as well as $(\mathfrak{c}_t, \mathfrak{d}_t, p_t) \in \star^r_{z\in\mathfrak{U}\uplus\mathfrak{D}} \mathcal{X}_z$ be arbitrary. Moreover, let $\{z, z'\} \subseteq \mathfrak{U} \uplus \mathfrak{D}$ and $\{\mathsf{B}, \mathsf{B}'\} \subseteq p_1 \otimes p_2$ be such that $z \neq z'$ and $(z, z') \notin r$ as well as $\mathsf{B} \subseteq \xi^{\mathfrak{c}_1\otimes\mathfrak{c}_2\leftarrow}_{\mathfrak{d}_1\otimes\mathfrak{d}_2}(\{z\})$ and $\mathsf{B}' \subseteq \xi^{\mathfrak{c}_1\otimes\mathfrak{c}_2\leftarrow}_{\mathfrak{d}_1\otimes\mathfrak{d}_2}(\{z'\})$. If we abbreviate $\mathsf{H}_1 = \Pi^{k_1}_{\ell_1}$ and $\mathsf{H}_2 = \Pi^{k_1+k_2}_{\ell_1+\ell_2} \backslash \Pi^{k_1}_{\ell_1}$, then by definition there exist $\{s, s'\} \subseteq \llbracket 2\rrbracket$ as well as $\mathsf{A} \in p_s$ and $\mathsf{A}' \in p_{s'}$ such that $\mathsf{B} = \gamma_{\mathsf{H}_s,\ell_1+\ell_2}^{\;k_1+k_2\rightarrow}(\mathsf{A})$ and $\mathsf{B}' = \gamma_{\mathsf{H}_{s'},\ell_1+\ell_2}^{\;k_1+k_2\rightarrow}(\mathsf{A}')$. Because $\gamma_{\mathsf{H}_1,\ell_1+\ell_2}^{\;k_1+k_2}$ and $\gamma_{\mathsf{H}_2,\ell_1+\ell_2}^{\;k_1+k_2}$ are injective that means $\mathsf{A} = \gamma_{\mathsf{H}_s,\ell_1+\ell_2}^{\;k_1+k_2\leftarrow}(\mathsf{B})$ and $\mathsf{A}' = \gamma_{\mathsf{H}_{s'},\ell_1+\ell_2}^{\;k_1+k_2\leftarrow}(\mathsf{B}')$. Moreover, because $\xi^{\mathfrak{c}_1\otimes\mathfrak{c}_2}_{\mathfrak{d}_1\otimes\mathfrak{d}_2} \circ \gamma_{\mathsf{H}_t,\ell_1+\ell_2}^{\;k_1+k_2} = \xi^{\mathfrak{c}_t}_{\mathfrak{d}_t}$ for each $t \in \llbracket 2\rrbracket$ Lemma 4.2 (c) we can infer $\mathsf{A} \subseteq \xi^{\mathfrak{c}_s\leftarrow}_{\mathfrak{d}_s}(\{z\})$ and $\mathsf{A}' \subseteq \xi^{\mathfrak{c}_{s'}\leftarrow}_{\mathfrak{d}_{s'}}(\{z'\})$. If $s \neq s'$, then $\mathsf{B} \mathrel{\diagdown\!\!\diagdown}^{k_1+k_2}_{\ell_1+\ell_2} \mathsf{B}'$ because H_1 and H_2 are convex and disjoint. And in the case that $s = s'$ we find $\mathsf{B} \mathrel{\diagdown\!\!\diagdown}^{k_1+k_2}_{\ell_1+\ell_2} \mathsf{B}'$ as well since $p_s \in \star^r_{z\in\mathfrak{U}\uplus\mathfrak{D}} \mathcal{X}_z$. Thus, $(\mathfrak{c}_1, \mathfrak{d}_1, p_1) \otimes (\mathfrak{c}_2, \mathfrak{d}_2, p_2) \in \star^r_{z\in\mathfrak{U}\uplus\mathfrak{D}} \mathcal{X}_z$.

Step 5: Erasing. Finally, let $\{k, \ell\} \subseteq \mathbb{N}_0$ and $\mathfrak{c}\colon \llbracket k\rrbracket \to (\mathfrak{U} \otimes \{\circ, \bullet\}) \uplus \mathfrak{D}$ and $\mathfrak{d}\colon \llbracket \ell\rrbracket \to (\mathfrak{U} \otimes \{\circ, \bullet\}) \uplus \mathfrak{D}$ and $(\mathfrak{c}, \mathfrak{d}, p) \in \star^r_{z\in\mathfrak{U}\uplus\mathfrak{D}} \mathcal{X}_z$, let $\mathsf{T} \subseteq \Pi^0_\ell$ and $z_\mathsf{T} \in \mathfrak{U} \uplus \mathfrak{D}$ be such that $|\mathsf{T}| = 2$ and $\mathsf{T} \subseteq \xi^{\mathfrak{c}\leftarrow}_{\mathfrak{d}}(\{z_\mathsf{T}\})$ and such that, if $z_\mathsf{T} \in \mathfrak{U}$, then ${}_{z_\mathsf{T}}\sigma^{\mathfrak{c}}_{\mathfrak{d}}(\mathsf{T}) = 0$ and let $\{z, z'\} \subseteq \mathfrak{U} \uplus \mathfrak{D}$, and $\{\mathsf{B}, \mathsf{B}'\} \subseteq E(p, \mathsf{T})$ be such that, if $\mathsf{M} = \Pi^k_\ell \backslash \mathsf{T}$ and if $\mathfrak{a}$ and $\mathfrak{b}$ are such that $\mathfrak{a} \blacksquare \mathfrak{b} = \mathfrak{c} \blacksquare \mathfrak{d} \circ \gamma^{\;k}_{\mathsf{M},\ell}$, then $\mathsf{B} \subseteq \xi^{\mathfrak{a}\leftarrow}_{\mathfrak{b}}(\{z\})$ and $\mathsf{B} \subseteq \xi^{\mathfrak{a}\leftarrow}_{\mathfrak{b}}(\{z'\})$. We have to show $\mathsf{B} \mathrel{\diagdown\!\!\diagdown}^k_{\ell-2} \mathsf{B}'$. According to the definition of $E(p, \mathsf{T})$ there are now two cases to distinguish.

Case 5.1: The first is that there exist $\{\mathsf{A}, \mathsf{A}'\} \subseteq p$ with $\mathsf{A} \cap \mathsf{T} = \varnothing = \mathsf{A}' \cap \mathsf{T}$ and $\mathsf{B} = \gamma^{\;k\leftarrow}_{\mathsf{M},\ell}(\mathsf{A})$ and $\mathsf{B}' = \gamma^{\;k\leftarrow}_{\mathsf{M},\ell}(\mathsf{A}')$. Because $\gamma^{\;k}_{\mathsf{M},\ell}$ is surjective onto M, because $\mathsf{A} \subseteq \mathsf{M}$ and $\mathsf{A}' \subseteq \mathsf{M}$ and because $\xi^{\mathfrak{a}}_{\mathfrak{b}} = \xi^{\mathfrak{c}}_{\mathfrak{d}} \circ \gamma^{\;k}_{\mathsf{M},\ell}$ by Lemma 4.2 (d), we can then conclude $\mathsf{A} = \gamma^{\;k\rightarrow}_{\mathsf{M},\ell}(\mathsf{B}) \subseteq \xi^{\mathfrak{c}\leftarrow}_{\mathfrak{d}}(\{z\})$ and $\mathsf{A}' = \gamma^{\;k\rightarrow}_{\mathsf{M},\ell}(\mathsf{B}') \subseteq \xi^{\mathfrak{c}\leftarrow}_{\mathfrak{d}}(\{z'\})$. Since $p \in \star^r_{z\in\mathfrak{U}\uplus\mathfrak{D}} \mathcal{X}_z$ that demands $\mathsf{A} \mathrel{\diagdown\!\!\diagdown}^k_\ell \mathsf{A}'$, from which $\mathsf{B} \mathrel{\diagdown\!\!\diagdown}^k_{\ell-2} \mathsf{B}'$ follows because $\gamma^{\;k}_{\mathsf{M},\ell}$ is monotonic with respect to $\Gamma^k_{\ell-2}$ and Γ^k_ℓ by the same lemma.

Case 5.2: The other possibility is that there exists $\mathsf{A} \in p$ with $\mathsf{A} \cap \mathsf{T} = \varnothing$ such that $\{\mathsf{B}, \mathsf{B}'\} = \{\gamma^{\;k\leftarrow}_{\mathsf{M},\ell}(\mathsf{A}), \gamma^{\;k\leftarrow}_{\mathsf{M},\ell}(\bigcup\{\mathsf{A}' \in p \wedge \mathsf{A}' \cap \mathsf{T} \neq \varnothing\})\}$. Since being non-crossing is a symmetric binary relation we can assume without loss of generality that $\mathsf{B} = \gamma^{\;k\leftarrow}_{\mathsf{M},\ell}(\mathsf{A})$ and $\mathsf{B}' = \gamma^{\;k\leftarrow}_{\mathsf{M},\ell}(\bigcup\{\mathsf{A}' \in p \wedge \mathsf{A}' \cap \mathsf{T} \neq \varnothing\})$. Then, as before, $\mathsf{A} = \gamma^{\;k\leftarrow}_{\mathsf{M},\ell}(\mathsf{B}) \subseteq \xi^{\mathfrak{c}\leftarrow}_{\mathfrak{d}}(\{z\})$ since $\gamma^{\;k}_{\mathsf{M},\ell}$ is surjective onto M and because $\xi^{\mathfrak{a}}_{\mathfrak{b}} = \xi^{\mathfrak{c}}_{\mathfrak{d}} \circ \gamma^{\;k}_{\mathsf{M},\ell}$ by Lemma 4.2 (d). Furthermore, because $|\mathsf{T}| = 2$ there can be either one or two $\mathsf{A}' \in p$ with $\mathsf{A}' \cap \mathsf{T} \neq \varnothing$.

Case 5.2.1: In the first situation, where there exists $\mathsf{A}' \in p$ with $\mathsf{T} \subseteq \mathsf{A}'$, and thus $\mathsf{B}' = \gamma^{\;k\leftarrow}_{\mathsf{M},\ell}(\mathsf{A}')$, we immediately conclude $\mathsf{A}' = \gamma^{\;k}_{\mathsf{M},\ell\rightarrow}(\mathsf{B}') \uplus \mathsf{T} \subseteq \xi^{\mathfrak{c}\leftarrow}_{\mathfrak{d}}(\{z'\})$ for the same reasons as before. And, once again, $p \in \star^r_{z\in\mathfrak{U}\uplus\mathfrak{D}} \mathcal{X}_z$ then implies $\mathsf{A} \mathrel{\diagdown\!\!\diagdown}^k_\ell \mathsf{A}'$ and thus $\mathsf{B} \mathrel{\diagdown\!\!\diagdown}^k_{\ell-2} \mathsf{B}'$ by the monotonicity of $\gamma^{\;k}_{\mathsf{M},\ell}$ guaranteed by Lemma 4.2 (d).

Case 5.2.2: The alternative is that there exist $\{\mathsf{A}'_1, \mathsf{A}'_2\} \subseteq p$ with $\mathsf{A}'_1 \neq \mathsf{A}'_2$ and $\mathsf{A}'_1 \cap \mathsf{T} \neq \varnothing \neq \mathsf{A}'_2 \cap \mathsf{T}$ and $\mathsf{T} \subseteq \mathsf{A}'_1 \uplus \mathsf{A}'_2$. In that case $\mathsf{A}'_1 \subseteq \xi^{\mathfrak{c}\leftarrow}_{\mathfrak{d}}(\{z_\mathsf{T}\})$ and $\mathsf{A}'_2 \subseteq \xi^{\mathfrak{c}\leftarrow}_{\mathfrak{d}}(\{z_\mathsf{T}\})$ because $\mathsf{T} \subseteq \xi^{\mathfrak{c}\leftarrow}_{\mathfrak{d}}(\{z_\mathsf{T}\})$ and because $p \leq \ker(\xi^{\mathfrak{c}}_{\mathfrak{d}})$ by $p \in \times_{z\in\mathfrak{U}\uplus\mathfrak{D}} \mathcal{X}_z$. On the other hand, the facts that $\gamma^{\;k}_{\mathsf{M},\ell}$ has image M and that $\mathsf{B}' = \gamma^{\;k\leftarrow}_{\mathsf{M},\ell}(\mathsf{A}'_1 \uplus \mathsf{A}'_2)$ imply $(\mathsf{A}'_1 \uplus \mathsf{A}'_2) \backslash \mathsf{T} \subseteq \xi^{\mathfrak{c}\leftarrow}_{\mathfrak{d}}(\{z'\})$ by $\xi^{\mathfrak{a}}_{\mathfrak{b}} = \xi^{\mathfrak{c}}_{\mathfrak{d}} \circ \gamma^{\;k}_{\mathsf{M},\ell}$. Hence, $z_\mathsf{T} = z' \neq z$ by $(\mathsf{A}'_1 \uplus \mathsf{A}'_2) \backslash \mathsf{T} = \gamma^{\;k\rightarrow}_{\mathsf{M},\ell}(\mathsf{B}') \neq \varnothing$. Thus we have shown altogether $\mathsf{A} \subseteq \xi_{\mathfrak{d}}^{\mathfrak{c}\leftarrow}(\{z\})$ and $\mathsf{A}'_1 \subseteq \xi_{\mathfrak{d}}^{\mathfrak{c}\leftarrow}(\{z'\})$ and $\mathsf{A}'_2 \subseteq \xi_{\mathfrak{d}}^{\mathfrak{c}\leftarrow}(\{z'\})$. By $p \in \text{☆}^r_{z\in\mathfrak{U}\cup\mathfrak{D}}\mathcal{X}_z$ that guarantees $\mathsf{A} \searrow^k_\ell \mathsf{A}'_1$ and $\mathsf{A} \searrow^k_\ell \mathsf{A}'_2$.

If we now let $\{\mathtt{b}_1, \mathtt{b}_2\} \subseteq \mathsf{B}$ and $\{\mathtt{b}'_1, \mathtt{b}'_2\} \subseteq \mathsf{B}'$ be arbitrary with $(\mathtt{b}_1 \mid \mathtt{b}'_1 \mid \mathtt{b}_2)^k_{\ell-2}$ and $(\mathtt{b}'_1 \mid \mathtt{b}_2 \mid \mathtt{b}'_2)^k_{\ell-2}$, then, by definition, in order to show $\mathsf{B} \searrow^k_{\ell-2} \mathsf{B}'$, what we must prove is $(\mathtt{b}_1 \mid \mathtt{b}'_2 \mid \mathtt{b}_2)^k_{\ell-2}$. In fact, if we define $\mathtt{a}_j = \gamma_{\mathsf{M},\ell}{}^k(\mathtt{b}_j)$ and $\mathtt{a}'_j = \gamma_{\mathsf{M},\ell}{}^k(\mathtt{b}'_j)$ for each $j \in [\![2]\!]$, then because $\gamma_{\mathsf{M},\ell}{}^k$ is monotonic and injective, we know $(\mathtt{a}_1 \mid \mathtt{a}'_1 \mid \mathtt{a}_2)^k_\ell$ and $(\mathtt{a}'_1 \mid \mathtt{a}_2 \mid \mathtt{a}'_2)^k_\ell$ and only have to prove $(\mathtt{a}_1 \mid \mathtt{a}'_2 \mid \mathtt{a}_2)^k_\ell$.

If there exists $i \in [\![2]\!]$ with $\{\mathtt{a}'_1, \mathtt{a}'_2\} \in \mathsf{A}'_i$, that is immediate by $\mathsf{A} \searrow^k_\ell \mathsf{A}'_i$. Hence, let us assume the opposite. By renaming $\mathsf{A}'_1 \leftrightarrow \mathsf{A}'_2$ if necessary, we can ensure then that $\mathtt{a}'_i \in \mathsf{A}'_i$ for each $i \in [\![2]\!]$ without affecting any of the assumptions. Furthermore, for each $i \in [\![2]\!]$ since $\mathsf{A}'_1 \cap \mathsf{T} \neq \varnothing \neq \mathsf{A}'_2 \cap \mathsf{T}$ we find an additional $\mathtt{t}_i \in \mathsf{A}'_i \cap \mathsf{T}$ and, necessarily, $\mathtt{t}_i \neq \mathtt{a}'_i$. Because T is convex with $|\mathsf{T}| = 2$ there exists $k \in [\![2]\!]$ such that $\mathtt{t}_{3-k} = \nu^k_\ell(\mathtt{t}_k)$. This choice has the consequence that $(\mathtt{t}_k \mid \mathtt{t}_{3-k} \mid \mathtt{x})^k_\ell$ for any $\mathtt{x} \in \mathsf{M}$.

Step 5.2.2.1: We show $(\mathtt{a}_1 \mid \mathtt{t}_1 \mid \mathtt{a}_2)^k_\ell$. Because Γ^k_ℓ is cyclic, $(\mathtt{a}_1 \mid \mathtt{a}'_1 \mid \mathtt{a}_2)^k_\ell$ implies $(\mathtt{a}_2 \mid \mathtt{a}_1 \mid \mathtt{a}'_1)^k_\ell$. Hence, if $(\mathtt{a}_2 \mid \mathtt{t}_1 \mid \mathtt{a}_1)^k_\ell$ held, then $(\mathtt{a}_2 \mid \mathtt{t}_1 \mid \mathtt{a}'_1)^k_\ell$ would as well by the transitivity of Γ^k_ℓ, and thus so would $(\mathtt{a}'_1 \mid \mathtt{a}_2 \mid \mathtt{t}_1)^k_\ell$ by cyclicity. Because $\{\mathtt{a}_1, \mathtt{a}_2\} \subseteq \mathsf{A}$ and $\{\mathtt{a}'_1, \mathtt{t}_1\} \subseteq \mathsf{A}'_1$ the relation $\mathsf{A} \searrow^k_\ell \mathsf{A}'_1$ allows us to infer that at least one of the three statements $(\mathtt{a}_1 \mid \mathtt{a}'_1 \mid \mathtt{a}_2)^k_\ell$ and $(\mathtt{a}'_1 \mid \mathtt{a}_2 \mid \mathtt{t}_1)^k_\ell$ and $(\mathtt{a}_2 \mid \mathtt{t}_1 \mid \mathtt{a}_1)^k_\ell$ must be false. Since $(\mathtt{a}_1 \mid \mathtt{a}'_1 \mid \mathtt{a}_2)^k_\ell$ is true and since, as just seen, $(\mathtt{a}_2 \mid \mathtt{t}_1 \mid \mathtt{a}_1)^k_\ell$ implies $(\mathtt{a}'_1 \mid \mathtt{a}_2 \mid \mathtt{t}_1)^k_\ell$ that means $(\mathtt{a}_2 \mid \mathtt{t}_1 \mid \mathtt{a}_1)^k_\ell$ is invalid. As Γ^k_ℓ is asymmetric we have thus indeed proved $(\mathtt{a}_1 \mid \mathtt{t}_1 \mid \mathtt{a}_2)^k_\ell$.

Step 5.2.2.2: The next step is to recognize that not only $(\mathtt{a}_1 \mid \mathtt{t}_1 \mid \mathtt{a}_2)^k_\ell$ but also $(\mathtt{a}_1 \mid \mathtt{t}_2 \mid \mathtt{a}_2)^k_\ell$. If $k = 1$, by the cyclicity of Γ^k_ℓ we can deduce from $(\mathtt{a}_1 \mid \mathtt{t}_1 \mid \mathtt{a}_2)^k_\ell$ and $(\mathtt{t}_1 \mid \mathtt{t}_2 \mid \mathtt{a}_2)^k_\ell$ that $(\mathtt{a}_2 \mid \mathtt{a}_1 \mid \mathtt{t}_1)^k_\ell$ and $(\mathtt{a}_2 \mid \mathtt{t}_1 \mid \mathtt{t}_2)^k_\ell$. From this it follows $(\mathtt{a}_2 \mid \mathtt{a}_1 \mid \mathtt{t}_2)^k_\ell$ by transitivity and thus $(\mathtt{a}_1 \mid \mathtt{t}_2 \mid \mathtt{a}_2)^k_\ell$ because Γ^k_ℓ is cyclic. Similarly, if $k = 2$ instead, then the relation $(\mathtt{a}_1 \mid \mathtt{t}_2 \mid \mathtt{t}_1)^k_\ell$, inferred from $(\mathtt{t}_2 \mid \mathtt{t}_1 \mid \mathtt{a}_1)^k_\ell$ by cyclicity, and the relation $(\mathtt{a}_1 \mid \mathtt{t}_1 \mid \mathtt{a}_2)^k_\ell$ together imply $(\mathtt{a}_1 \mid \mathtt{t}_2 \mid \mathtt{a}_2)^k_\ell$ immediately since Γ^k_ℓ is transitive.

Step 5.2.2.3: Now, we prove that $(\mathtt{t}_2 \mid \mathtt{a}_1 \mid \mathtt{a}'_2)^k_\ell$ or $(\mathtt{a}'_2 \mid \mathtt{a}_2 \mid \mathtt{t}_2)^k_\ell$. The facts that $\{\mathtt{a}_1, \mathtt{a}_2\} \subseteq \mathsf{A}$ and $\{\mathtt{a}'_2, \mathtt{t}_2\} \subseteq \mathsf{A}'_2$ and $\mathsf{A} \searrow^k_\ell \mathsf{A}'_2$ allow us to conclude that at least one of the three statements $(\mathtt{a}'_2 \mid \mathtt{a}_1 \mid \mathtt{t}_2)^k_\ell$ and $(\mathtt{a}_1 \mid \mathtt{t}_2 \mid \mathtt{a}_2)^k_\ell$ and $(\mathtt{t}_2 \mid \mathtt{a}_2 \mid \mathtt{a}'_2)^k_\ell$ is false. Of course, $(\mathtt{a}_1 \mid \mathtt{t}_2 \mid \mathtt{a}_2)^k_\ell$ is true by Step 5.2.2.2. Because Γ^k_ℓ is asymmetric we can thus infer that $(\mathtt{t}_2 \mid \mathtt{a}_1 \mid \mathtt{a}'_2)^k_\ell$ or $(\mathtt{a}'_2 \mid \mathtt{a}_2 \mid \mathtt{t}_2)^k_\ell$.

Step 5.2.2.4: Finally we distinguish the two cases from the previous step.

Case 5.2.2.4.1: If $(\mathtt{t}_2 \mid \mathtt{a}_1 \mid \mathtt{a}'_2)^k_\ell$, then also $(\mathtt{a}_1 \mid \mathtt{a}'_2 \mid \mathtt{t}_2)^k_\ell$ by cyclicity of Γ^k_ℓ, which is why the result $(\mathtt{a}_1 \mid \mathtt{t}_2 \mid \mathtt{a}_2)^k_\ell$ of Step 5.2.2.2 then lets us deduce $(\mathtt{a}_1 \mid \mathtt{a}'_2 \mid \mathtt{a}_2)^k_\ell$ by transitivity.

Case 5.2.2.4.2: And, if $(\mathtt{a}'_2 \mid \mathtt{a}_2 \mid \mathtt{t}_2)^k_\ell$, then because Γ^k_ℓ is cyclic and because of Step 5.2.2.2 both $(\mathtt{a}_2 \mid \mathtt{t}_2 \mid \mathtt{a}'_2)^k_\ell$ and $(\mathtt{a}_2 \mid \mathtt{a}_1 \mid \mathtt{t}_2)^k_\ell$. Transitivity lets us derive $(\mathtt{a}_2 \mid \mathtt{a}_1 \mid \mathtt{a}'_2)^k_\ell$ and thus $(\mathtt{a}_1 \mid \mathtt{a}'_2 \mid \mathtt{a}_2)^k_\ell$ by cyclicity.

In conclusion, $\mathsf{B} \searrow^k_{\ell-2} \mathsf{B}'$ holds in any case, proving $E((\mathfrak{c}, \mathfrak{d}, p), \mathsf{T}) \in \text{☆}^r_{z\in\mathfrak{U}\cup\mathfrak{D}}\mathcal{X}_z$ and thus making $\text{☆}^r_{z\in\mathfrak{U}\cup\mathfrak{D}}\mathcal{X}_z$ a category. ∎

For a particular special case of the preceding proposition it is convenient to have shorthand notation.

Definition 4.13. For any countable set Z, any category $\mathcal{C}$ of two-colored or uncolored partitions and any partial commutation relation r on Z, the *big graph co-power category* $\mathcal{C}^{\star(Z,r)}$ with respect to Z and r is the category $\star^r_{z\in Z}\mathcal{X}_z$, where for each $z \in Z$ the category $\mathcal{X}_z$ is given by $\mathcal{C}$, seen as tagged with the single tag z.

Notation 4.14. If $r = \varnothing$ in Definition 4.13, we also write $\mathcal{C}^{\star Z}$ instead of $\mathcal{C}^{\star(Z,r)}$ and speak of the Z-fold *big free co-power category* of $\mathcal{C}$.

Remark 4.15. In the setting of Definition 4.13, if r is the trivial commutation relation $\{(z,z')\,|\,\{z,z'\}\subseteq Z \wedge z\neq z'\}$, then $\mathcal{C}^{\star(Z,r)}$ coincides with $\mathcal{C}^{\times Z}$.

4.3. Little graph co-products. While the non-crossing conditions of the previous chapter constitute a notion of "graph co-product", in order to obtain the concept as it is defined in the algebraic setting, a smaller category than the big graph co-product needs to be considered.

Assumption 4.16. In Section 4.3, let r be any partial commutation relation on $\mathfrak{U}\cup\mathfrak{D}$ and let $(\mathcal{X}_z)_{z\in\mathfrak{U}\cup\mathfrak{D}}$ be any family such that $\mathcal{X}_z$ is a category of $(\{z\},\varnothing)$-tagged labeled partitions for any $z\in\mathfrak{U}$ and of $(\varnothing,\{z\})$-tagged labeled partitions for $z\in\mathfrak{D}$.

Definition 4.17. The *(little) graph co-product* $*^r_{z\in\mathfrak{U}\cup\mathfrak{D}}\mathcal{X}_z$ of $(\mathcal{X}_z)_{z\in\mathfrak{U}\cup\mathfrak{D}}$ with respect to r is the set of all $(\mathfrak{c},\mathfrak{d},p)\in{}_{\mathfrak{U},\mathfrak{D}}\mathcal{S}$ for which there exist $\{k,\ell\}\subseteq\mathbb{N}_0$ and a set-theoretical partition h of Π^k_ℓ, the *history* of $(\mathfrak{c},\mathfrak{d},p)$ with respect to $(\mathcal{X}_z)_{z\in\mathfrak{U}\cup\mathfrak{D}}$ and r, such that $\mathfrak{c}\colon [\![k]\!]\to(\mathfrak{U}\otimes\{\circ,\bullet\})\cup\mathfrak{D}$ and $\mathfrak{d}\colon [\![\ell]\!]\to(\mathfrak{U}\otimes\{\circ,\bullet\})\cup\mathfrak{D}$, such that $h\leq\ker(\xi^{\mathfrak{c}}_{\mathfrak{d}})$, such that for any $\{z,z'\}\subseteq\mathfrak{U}\cup\mathfrak{D}$, whenever $(z,z')\notin r$, then $\mathsf{H}\not\between^k_\ell\mathsf{H}'$ for any $\{\mathsf{H},\mathsf{H}'\}\subseteq h$ with $\mathsf{H}\subseteq\xi^{\mathfrak{c}\leftarrow}_{\mathfrak{d}}(\{z\})$ and $\mathsf{H}'\subseteq\xi^{\mathfrak{c}\leftarrow}_{\mathfrak{d}}(\{z'\})$, such that $p\leq h$, and such that $R((\mathfrak{c},\mathfrak{d},p),\mathsf{H})\in\mathcal{X}_z$ for any $z\in\mathfrak{U}\cup\mathfrak{D}$ and $\mathsf{H}\in h$ with $\mathsf{H}\subseteq\xi^{\mathfrak{c}\leftarrow}_{\mathfrak{d}}(\{z\})$.

Note in particular the difference from Definition 4.9 that in the non-crossing condition $z\neq z'$ is *not* required here.

Notation 4.18. If $r=\varnothing$ in Definition 4.17, we also write $*_{z\in\mathfrak{U}\cup\mathfrak{D}}\mathcal{X}_z$ instead of $*^r_{z\in\mathfrak{U}\cup\mathfrak{D}}\mathcal{X}_z$ and speak of the *(little) free co-product* of $(\mathcal{X}_z)_{z\in\mathfrak{U}\cup\mathfrak{D}}$.

Remark 4.19. In the setting of Definition 4.17, if r is the trivial partial commutation relation $\{(z,z')\,|\,\{z,z'\}\subseteq\mathfrak{U}\cup\mathfrak{D}\wedge z\neq z'\}$, then $*^r_{z\in\mathfrak{U}\cup\mathfrak{D}}\mathcal{X}_z$ is actually the direct co-product $\times_{z\in\mathfrak{U}\cup\mathfrak{D}}\mathcal{X}_z$. In particular, the categories $\star^r_{z\in\mathfrak{U}\cup\mathfrak{D}}\mathcal{X}_z$ and $*^r_{z\in\mathfrak{U}\cup\mathfrak{D}}\mathcal{X}_z$ coincide in that case. The difference between the big and the little graph co-product is a purely quantum phenomenon.

The nomenclature is consistent in the sense that the little graph co-product is actually a subset of the big one. The below lemma helps with proving that.

Lemma 4.20. *For any* $\{k,\ell\}\subseteq\mathbb{N}_0$*, any* $\mathfrak{c}\colon[\![k]\!]\to(\mathfrak{U}\otimes\{\circ,\bullet\})\cup\mathfrak{D}$ *and* $\mathfrak{d}\colon[\![\ell]\!]\to(\mathfrak{U}\otimes\{\circ,\bullet\})\cup\mathfrak{D}$*, any set-theoretical partition* p *of* Π^k_ℓ *and any set-theoretical partition* h *of* Π^k_ℓ *which is non-crossing with respect to* Γ^k_ℓ *and satisfies* $p\leq h$*,*

$$(\mathfrak{c},\mathfrak{d},p)\in{}_{\mathfrak{U},\mathfrak{D}}\langle R((\mathfrak{c},\mathfrak{d},p),\mathsf{S})\,|\,\mathsf{S}\in h\rangle.$$

Proof. The proof goes by induction over $|h|$. For $|h| = 1$ there is nothing to show. For general $|h|$ the assumption that h is non-crossing with respect to Γ^k_ℓ guarantees the existence of a block $\mathsf{S}_0 \in h$ which is convex with respect to Γ^k_ℓ. By Lemma 3.22 we can assume that $\blacksquare\ell \in \mathsf{S}_0$. Then, $(\mathfrak{c}, \mathfrak{d}, p)$ can be written as $R((\mathfrak{c}, \mathfrak{d}, p), \mathsf{M}) \otimes R((\mathfrak{c}, \mathfrak{d}, p), \mathsf{S}_0)$, where $\mathsf{M} \coloneqq \Pi^k_\ell \backslash \mathsf{S}_0$. Moreover, if $\{m, n\} \subseteq \mathbb{N}_0$ are such that, if $(\mathfrak{a}, \mathfrak{b}, q) \coloneqq R((\mathfrak{c}, \mathfrak{d}, p), \mathsf{M})$, then $\mathfrak{a}\colon [\![m]\!] \to (\mathfrak{U} \otimes \{\circ, \bullet\}) \cup \mathfrak{D}$ and $\mathfrak{b}\colon [\![n]\!] \to (\mathfrak{U} \otimes \{\circ, \bullet\}) \cup \mathfrak{D}$, then $g \coloneqq R(h, \mathsf{M})$ is a set-theoretical partition of Π^m_n which is non-crossing with respect to Γ^m_n and satisfies $q \leq g$ and $|g| < |h|$. Hence, the induction hypothesis lets us infer $(\mathfrak{a}, \mathfrak{b}, q) \in {}_{\mathfrak{U},\mathfrak{D}}\langle R((\mathfrak{a}, \mathfrak{b}, q), \mathsf{T}) \,|\, \mathsf{T} \in g\rangle$. Since for any $\mathsf{T} \in g$ there exists $\mathsf{S} \in h$ with $\mathsf{T} = \gamma_{\mathsf{M},\ell}^{\;k\leftarrow}(\mathsf{S})$ and thus $R((\mathfrak{a}, \mathfrak{b}, q), \mathsf{T}) = R((\mathfrak{c}, \mathfrak{d}, p), \mathsf{S})$ according to Lemma 4.5 (d) since $\mathsf{S} \subseteq \mathsf{M}$ we have shown $R((\mathfrak{c}, \mathfrak{d}, p), \mathsf{M}) \in {}_{\mathfrak{U},\mathfrak{D}}\langle R((\mathfrak{c}, \mathfrak{d}, p), \mathsf{S}) \,|\, \mathsf{S} \in h\rangle$. The decomposition $(\mathfrak{c}, \mathfrak{d}, p) = R((\mathfrak{c}, \mathfrak{d}, p), \mathsf{M}) \otimes R((\mathfrak{c}, \mathfrak{d}, p), \mathsf{S}_0)$ therefore proves the claim. □

Proposition 4.21. $\ast^r_{z \in \mathfrak{U} \cup \mathfrak{D}} \mathcal{X}_z \subseteq \star^r_{z \in \mathfrak{U} \cup \mathfrak{D}} \mathcal{X}_z$.

Proof. Let $(\mathfrak{c}, \mathfrak{d}, p) \in \ast^r_{z \in \mathfrak{U} \cup \mathfrak{D}} \mathcal{X}_z$ be arbitrary, let $\{k, \ell\} \subseteq \mathbb{N}_0$ be such that $\mathfrak{c}\colon [\![k]\!] \to (\mathfrak{U} \otimes \{\circ, \bullet\}) \cup \mathfrak{D}$ and $\mathfrak{d}\colon [\![\ell]\!] \to (\mathfrak{U} \otimes \{\circ, \bullet\}) \cup \mathfrak{D}$ and let h be a history of $(\mathfrak{c}, \mathfrak{d}, p)$ with respect to $(\mathcal{X}_z)_{z \in \mathfrak{U} \cup \mathfrak{D}}$ and r. We check that $(\mathfrak{c}, \mathfrak{d}, p)$ meets the conditions for membership in $\star^r_{z \in \mathfrak{U} \cup \mathfrak{D}} \mathcal{X}_z$ required by Definition 4.9.

Tags condition. First of all, the transitivity of $\leq$ guarantees $p \leq \ker(\xi^{\mathfrak{c}}_{\mathfrak{d}})$ because $p \leq h$ and $h \leq \ker(\xi^{\mathfrak{c}}_{\mathfrak{d}})$ by definition.

Restrictions condition. Let $z \in \mathfrak{U} \cup \mathfrak{D}$ be arbitrary and abbreviate $\mathsf{Y} \coloneqq \xi^{\mathfrak{c}\leftarrow}_{\mathfrak{d}}(\{z\})$ and $m \coloneqq \alpha(\mathsf{Y})$ and $n \coloneqq \beta(\mathsf{Y})$. We have to show that $(\mathfrak{a}, \mathfrak{b}, q) \coloneqq R((\mathfrak{c}, \mathfrak{d}, p), \mathsf{Y}) \in \mathcal{X}_z$.

First, we prove that the set-theoretical partition $g \coloneqq R(h, \mathsf{Y})$ of Π^m_n is non-crossing with respect to Γ^m_n. For any $\{\mathsf{G}, \mathsf{G}'\} \subseteq g$ with $\mathsf{G} \neq \mathsf{G}'$ there exist, by definition, $\{\mathsf{H}, \mathsf{H}'\} \subseteq p$ with $\mathsf{G} = \gamma_{\mathsf{Y},n}^{\;m\leftarrow}(\mathsf{H})$ and $\mathsf{G}' = \gamma_{\mathsf{Y},n}^{\;m\leftarrow}(\mathsf{H}')$. Because $\gamma_{\mathsf{Y},n}^{\;m}$ is injective it follows from $\varnothing = \mathsf{G} \cap \mathsf{G}'$ that also $\mathsf{H} \cap \mathsf{H}' = \gamma_{\mathsf{Y},n}^{\;m\leftarrow}(\mathsf{G}) \cap \gamma_{\mathsf{Y},n}^{\;m\leftarrow}(\mathsf{G}') = \gamma_{\mathsf{Y},n}^{\;m\leftarrow}(\mathsf{G} \cap \mathsf{G}') = \varnothing$, i.e., $\mathsf{H} \neq \mathsf{H}'$. Since $\mathsf{H} \subseteq \mathsf{Y}$ and $\mathsf{H}' \subseteq \mathsf{Y}$ and since $(z, z) \notin r$ the assumption that h is a history with respect to $(\mathcal{X}_z)_{z \in \mathfrak{U} \cup \mathfrak{D}}$ and r lets us conclude that $\mathsf{H} \mathbin{\lessdot\!\!\!\searrow}^k_\ell \mathsf{H}'$. And because $\gamma_{\mathsf{Y},\ell}^{\;k}$ is monotonic with respect to Γ^m_n and Γ^k_ℓ by Lemma 4.2 (d) that requires $\mathsf{G} \mathbin{\lessdot\!\!\!\searrow}^m_n \mathsf{G}'$. Hence, g is indeed non-crossing with respect to Γ^m_n.

Consequently, $(\mathfrak{a}, \mathfrak{b}, q) \in {}_{\mathfrak{U},\mathfrak{D}}\langle R((\mathfrak{a}, \mathfrak{b}, q), \mathsf{G}) \,|\, \mathsf{G} \in g\rangle$ by Lemma 4.20. For any $\mathsf{G} \in g$ there exists $\mathsf{H} \in h$ with $\mathsf{G} = \gamma_{\mathsf{Y},\ell}^{\;k\leftarrow}(\mathsf{H})$ and thus $R((\mathfrak{a}, \mathfrak{b}, q), \mathsf{G}) = R((\mathfrak{c}, \mathfrak{d}, p), \mathsf{H}) \in \mathcal{X}_z$ by Lemma 4.5 (d) and by h being a history for $(\mathfrak{c}, \mathfrak{d}, p)$ with respect to $(\mathcal{X}_z)_{z \in \mathfrak{U} \cup \mathfrak{D}}$ and r. It follows ${}_{\mathfrak{U},\mathfrak{D}}\langle R((\mathfrak{a}, \mathfrak{b}, q), \mathsf{G}) \,|\, \mathsf{G} \in g\rangle \subseteq \mathcal{X}_z$ and thus $(\mathfrak{a}, \mathfrak{b}, q) \in \mathcal{X}_z$, as claimed.

Non-crossing condition. Moreover, for any $\{z, z'\} \subseteq \mathfrak{U} \cup \mathfrak{D}$ with $(z, z') \notin r$ and $z \neq z'$ and any $\{\mathsf{B}, \mathsf{B}'\} \subseteq p$ with $\mathsf{B} \subseteq \xi^{\mathfrak{c}\leftarrow}_{\mathfrak{d}}(\{z\})$ and $\mathsf{B}' \subseteq \xi^{\mathfrak{c}\leftarrow}_{\mathfrak{d}}(\{z'\})$, by $p \leq h$, there exist $\{\mathsf{H}, \mathsf{H}'\} \subseteq h$ with $\mathsf{B} \subseteq \mathsf{H}$ and $\mathsf{B}' \subseteq \mathsf{H}'$. Since $h \leq \ker(\xi^{\mathfrak{c}}_{\mathfrak{d}})$ the facts that $\varnothing \neq \mathsf{B} \subseteq \mathsf{H} \cap \xi^{\mathfrak{c}\leftarrow}_{\mathfrak{d}}(\{z\})$ and $\varnothing \neq \mathsf{B}' \subseteq \mathsf{H}' \cap \xi^{\mathfrak{c}\leftarrow}_{\mathfrak{d}}(\{z'\})$ require that, actually, $\mathsf{H} \subseteq \xi^{\mathfrak{c}\leftarrow}_{\mathfrak{d}}(\{z\})$ and $\mathsf{H}' \subseteq \xi^{\mathfrak{c}\leftarrow}_{\mathfrak{d}}(\{z'\})$. Because h is a history with respect to $(\mathcal{X}_z)_{z \in \mathfrak{U} \cup \mathfrak{D}}$ and r therefore $\mathsf{H} \mathbin{\lessdot\!\!\!\searrow}^{\mathfrak{c}}_{\mathfrak{d}} \mathsf{H}'$. It follows $\mathsf{B} \mathbin{\lessdot\!\!\!\searrow}^{\mathfrak{c}}_{\mathfrak{d}} \mathsf{B}'$ by $\mathsf{B} \subseteq \mathsf{H}$ and $\mathsf{B}' \subseteq \mathsf{H}'$. Thus, $(\mathfrak{c}, \mathfrak{d}, p)$ also meets the non-crossing condition and is thus an element of $\star^r_{z \in \mathfrak{U} \cup \mathfrak{D}} \mathcal{X}_z$. □

The following observation will be helpful in proving that (little) graph co-products are categories of labeled partitions.

NOTATION 4.22. For any $z \in \mathfrak{U} \cup \mathfrak{O}$ let ${}_{\{z\},\varnothing}\mathcal{S}^+$ if $z \in \mathfrak{U}$ and ${}_{\varnothing,\{z\}}\mathcal{S}^+$ if $z \in \mathfrak{O}$ denote the set of all elements $(\mathfrak{c}, \mathfrak{d}, p)$ of ${}_{\{z\},\varnothing}\mathcal{S}$ respectively ${}_{\varnothing,\{z\}}\mathcal{S}$ such that, if $\{k,\ell\} \subseteq \mathbb{N}_0$ are such that $\mathfrak{c}\colon [\![k]\!] \to (\mathfrak{U} \otimes \{\circ, \bullet\}) \cup \mathfrak{O}$ and $\mathfrak{d}\colon [\![\ell]\!] \to (\mathfrak{U} \otimes \{\circ, \bullet\}) \cup \mathfrak{O}$, then p is non-crossing with respect to Γ_ℓ^k.

NOTATION 4.23. In Section 4.3, let ${}_{\mathfrak{U},\mathfrak{O}}^{\;r}\mathcal{S}^+$ denote the big graph co-product $\star^r_{z\in\mathfrak{U}\cup\mathfrak{O}} \mathcal{N}_z$, where for any $z \in \mathfrak{U} \cup \mathfrak{O}$ the category $\mathcal{N}_z$ is given by ${}_{\{z\},\varnothing}\mathcal{S}^+$ if $z \in \mathfrak{U}$ and by ${}_{\varnothing,\{z\}}\mathcal{S}^+$ if $z \in \mathfrak{O}$.

PROPOSITION 4.24. *The (little) graph co-product* $*^r_{z\in\mathfrak{U}\cup\mathfrak{O}} \mathcal{X}_z$ *of* $(\mathcal{X}_z)_{z\in\mathfrak{U}\cup\mathfrak{O}}$ *with respect to* r *can be expressed as the set of all* $(\mathfrak{c}, \mathfrak{d}, p) \in {}_{\mathfrak{U},\mathfrak{O}}\mathcal{S}$ *for which there exist* $\{k,\ell\} \subseteq \mathbb{N}_0$ *and a set-theoretical partition* h *of* Π_ℓ^k *such that* $\mathfrak{c}\colon [\![k]\!] \to (\mathfrak{U} \otimes \{\circ, \bullet\}) \cup \mathfrak{O}$ *and* $\mathfrak{d}\colon [\![\ell]\!] \to (\mathfrak{U} \otimes \{\circ, \bullet\}) \cup \mathfrak{O}$*, such that* $h \in {}_{\mathfrak{U},\mathfrak{O}}^{\;r}\mathcal{S}^+$ *such that* $p \leq h$*, and such that* $R((\mathfrak{c}, \mathfrak{d}, p), \mathsf{H}) \in \mathcal{X}_z$ *for any* $z \in \mathfrak{U} \cup \mathfrak{O}$ *and* $\mathsf{H} \in h$ *with* $\mathsf{H} \subseteq \xi_{\mathfrak{d}}^{\mathfrak{c}\leftarrow}(\{z\})$.

PROOF. An inspection of the definition immediately proves the claim. □

PROPOSITION 4.25. $*^r_{z\in\mathfrak{U}\cup\mathfrak{O}} \mathcal{X}_z$ *is a category of* $(\mathfrak{U}, \mathfrak{O})$*-tagged labeled partitions.*

PROOF. We show that $*^r_{z\in\mathfrak{U}\cup\mathfrak{O}} \mathcal{X}_z$ meets the conditions of Proposition 3.26. The proof makes extensive use of Propositions 4.24 and 4.12.

Step 1: Identities. For any $\mathfrak{c} \in (\mathfrak{U} \otimes \{\circ, \bullet\}) \cup \mathfrak{O}$ we know by Proposition 4.12 that $\mathrm{id}_\mathfrak{c} \in {}_{\mathfrak{U},\mathfrak{O}}^{\;r}\mathcal{S}^+$. Therefore the set-theoretical partition id_1 of Π_1^1 is a history for $\mathrm{id}_\mathfrak{c}$ with respect to $(\mathcal{X}_z)_{z\in\mathfrak{U}\cup\mathfrak{O}}$ and r.

Step 2: Rotation. Given any $k \in \mathbb{N}$ and $\ell \in \mathbb{N}_0$, any $\mathfrak{c}\colon [\![k]\!] \to (\mathfrak{U} \otimes \{\circ, \bullet\}) \cup \mathfrak{O}$ and $\mathfrak{d}\colon [\![\ell]\!] \to (\mathfrak{U}\otimes\{\circ, \bullet\})\cup\mathfrak{O}$, any set-theoretical partition p of Π_ℓ^k with $(\mathfrak{c}, \mathfrak{d}, p) \in *^r_{z\in\mathfrak{U}\cup\mathfrak{O}} \mathcal{X}_z$ and any $r \in \{\circlearrowleft, \circlearrowright\}$, if h is any history of $(\mathfrak{c}, \mathfrak{d}, p)$ with respect to $(\mathcal{X}_z)_{z\in\mathfrak{U}\cup\mathfrak{O}}$ and r, we show that $g \coloneqq h^r$ is a history for $(\mathfrak{a}, \mathfrak{b}, q) \coloneqq (\mathfrak{c}, \mathfrak{d}, p)^r$ with respect to $(\mathcal{X}_z)_{z\in\mathfrak{U}\cup\mathfrak{O}}$ and r. Since ${}_{\mathfrak{U},\mathfrak{O}}^{\;r}\mathcal{S}^+$ is closed under rotations by Proposition 4.12 and Lemma 3.22 the assumption that $(\mathfrak{c}, \mathfrak{d}, h) \in {}_{\mathfrak{U},\mathfrak{O}}^{\;r}\mathcal{S}^+$ ensures that also $(\mathfrak{a}, \mathfrak{b}, g) \in {}_{\mathfrak{U},\mathfrak{O}}^{\;r}\mathcal{S}^+$. Since $\omega^{r,k}_{\ell}{}^{\leftarrow}$ preserves $\leq$ we can conclude $q = \omega^{r,k}_{\ell}{}^{\leftarrow}(p) \leq \omega^{r,k}_{\ell}{}^{\leftarrow}(h) = g$ from $p \leq h$. For any $z \in \mathfrak{U}\cup\mathfrak{O}$ and any $\mathsf{G} \in g$ with $\mathsf{G} \subseteq \xi_{\mathfrak{b}}^{\mathfrak{a}\leftarrow}(\{z\})$, if $\mathsf{H} \in h$ is the block with $\mathsf{G} = \omega^{r,k}_{\ell}{}^{\leftarrow}(\mathsf{H})$, then the fact that $\xi_{\mathfrak{b}}^{\mathfrak{a}} = \xi_{\mathfrak{d}}^{\mathfrak{c}} \circ \omega^{r,k}_{\ell}$ by Lemma 4.2 (a) implies $\mathsf{H} = (\omega^{r,k}_{\ell\to} \circ \omega^{r,k}_{\ell}{}^{\leftarrow})(\mathsf{H}) = \omega^{r,k}_{\ell\to}(\mathsf{G}) \subseteq (\omega^{r,k}_{\ell\to}\circ\xi_{\mathfrak{b}}^{\mathfrak{a}\leftarrow})(\{z\}) = (\omega^{r,k}_{\ell\to}\circ\omega^{r,k}_{\ell}{}^{\leftarrow}\circ\xi_{\mathfrak{d}}^{\mathfrak{c}\leftarrow})(\{z\}) \subseteq \xi_{\mathfrak{d}}^{\mathfrak{c}\leftarrow}(\{z\})$, where we have also used the fact that $\omega^{r,k}_{\ell}$ is surjective. Because h is a history for $(\mathfrak{c}, \mathfrak{d}, p)$ we may thus conclude $R((\mathfrak{c}, \mathfrak{d}, p), \mathsf{H}) \in \mathcal{X}_z$. If $\mathsf{e} \coloneqq {}^\blacksquare 1$ in case $r = \circlearrowleft$ and $\mathsf{e} \coloneqq {}^\blacksquare k$ in case $r = \circlearrowright$, then $R((\mathfrak{a}, \mathfrak{b}, q), \mathsf{G})$ is given by $R((\mathfrak{c}, \mathfrak{d}, p), \mathsf{H})$ if $\mathsf{e} \notin \mathsf{H}$ and by $R((\mathfrak{c}, \mathfrak{d}, p), \mathsf{H})^r$ otherwise according to Lemma 4.5 (a). By Lemma 3.22 that proves $R((\mathfrak{a}, \mathfrak{b}, q), \mathsf{G}) \in \mathcal{X}_z$, making g a history for $(\mathfrak{a}, \mathfrak{b}, q)$ and thus verifying that $*^r_{z\in\mathfrak{U}\cup\mathfrak{O}} \mathcal{X}_z$ is closed under rotations.

Step 3: Adjoints. The proof that $*^r_{z\in\mathfrak{U}\cup\mathfrak{O}} \mathcal{X}_z$ is also invariant under forming adjoints is quite similar. For any $\{k,\ell\} \subseteq \mathbb{N}_0$, any $\mathfrak{c}\colon [\![k]\!] \to (\mathfrak{U} \otimes \{\circ, \bullet\}) \cup \mathfrak{O}$ and $\mathfrak{d}\colon [\![\ell]\!] \to (\mathfrak{U}\otimes\{\circ, \bullet\})\cup\mathfrak{O}$, any set-theoretical partition p of Π_ℓ^k with $(\mathfrak{c}, \mathfrak{d}, p) \in *^r_{z\in\mathfrak{U}\cup\mathfrak{O}} \mathcal{X}_z$

and any history h of $(\mathfrak{c},\mathfrak{d},p)$ with respect to $(\mathcal{X}_z)_{z\in\mathfrak{U}\cup\mathfrak{O}}$ and r, the set-theoretical partition $g \coloneqq h^*$ is a history for $(\mathfrak{d},\mathfrak{c},p^*)$ with respect to $(\mathcal{X}_z)_{z\in\mathfrak{U}\cup\mathfrak{O}}$ and r. Indeed, the invariance of ${}_{\mathfrak{U},\mathfrak{O}}^{\;r}\mathcal{S}^+$ under forming adjoints guaranteed by Proposition 4.12 lets us infer $(\mathfrak{d},\mathfrak{c},g)\in{}_{\mathfrak{U},\mathfrak{O}}^{\;r}\mathcal{S}^+$. And $p\leq h$ implies $p^*=\kappa_\ell^{k\leftarrow}(p)\leq\kappa_\ell^{k\leftarrow}(h)=g$ since $\kappa_\ell^{k\leftarrow}$ preserves $\leq$. Furthermore, given any $z\in\mathfrak{U}\cup\mathfrak{O}$ and any $\mathsf{G}\in g$ with $\mathsf{G}\subseteq\xi_{\mathfrak{c}}^{\mathfrak{d}\leftarrow}(\{z\})$ we find $\mathsf{H}\in h$ such that $\mathsf{G}=\kappa_\ell^{k\leftarrow}(\mathsf{H})$. Because $\xi_{\mathfrak{c}}^{\mathfrak{d}}=\xi_{\mathfrak{d}}^{\mathfrak{c}}\circ\kappa_\ell^k$ by Lemma 4.2 (b) and because κ_ℓ^k is surjective it follows $\mathsf{H}=(\kappa_{\ell\rightarrow}^k\circ\kappa_\ell^{k\leftarrow})(\mathsf{H})=\kappa_{\ell\rightarrow}^k(\mathsf{G})\subseteq(\kappa_{\ell\rightarrow}^k\circ\xi_{\mathfrak{c}}^{\mathfrak{d}\leftarrow})(\{z\})=(\kappa_{\ell\rightarrow}^k\circ\kappa_\ell^{k\leftarrow}\circ\xi_{\mathfrak{d}}^{\mathfrak{c}\leftarrow})(\{z\})\subseteq\xi_{\mathfrak{d}}^{\mathfrak{c}\leftarrow}(\{z\})$. The assumption that h is a history for $(\mathfrak{c},\mathfrak{d},p)$ allows us to infer $R((\mathfrak{c},\mathfrak{d},p),\mathsf{H})\in\mathcal{X}_z$. Because $R((\mathfrak{d},\mathfrak{c},p^*),\mathsf{G})=R((\mathfrak{c},\mathfrak{d},p),\mathsf{H})^*$ by Lemma 4.5 (b) that proves $R((\mathfrak{d},\mathfrak{c},p^*),\mathsf{G})\in\mathcal{X}_z$, which is what we needed to see.

Step 4: Monoidal product. Next, for each $t\in[\![2]\!]$ let $\{k_t,\ell_t\}\subseteq\mathbb{N}_0$, let $\mathfrak{c}_t\colon[\![k_t]\!]\to(\mathfrak{U}\otimes\{\circ,\bullet\})\cup\mathfrak{O}$ and $\mathfrak{d}_t\colon[\![\ell_t]\!]\to(\mathfrak{U}\otimes\{\circ,\bullet\})\cup\mathfrak{O}$, let $(\mathfrak{c}_t,\mathfrak{d}_t,p_t)\in *^r_{z\in\mathfrak{U}\cup\mathfrak{O}}\mathcal{X}_z$ and let h_t be any history for $(\mathfrak{c}_t,\mathfrak{d}_t,p_t)$ with respect to $(\mathcal{X}_z)_{z\in\mathfrak{U}\cup\mathfrak{O}}$ and r. Wr prove that $g\coloneqq h_1\otimes h_2$ is a history for $(\mathfrak{c}_1\otimes\mathfrak{c}_2,\mathfrak{d}_1\otimes\mathfrak{d}_2,p_1\otimes p_2)$ with respect to $(\mathcal{X}_z)_{z\in\mathfrak{U}\cup\mathfrak{O}}$ and r. Since $\{h_1,h_2\}\subseteq{}_{\mathfrak{U},\mathfrak{O}}^{\;r}\mathcal{S}^+$ by assumption, Proposition 4.12 ensures $g\in{}_{\mathfrak{U},\mathfrak{O}}^{\;r}\mathcal{S}^+$.

Since $\otimes$ preserves $\leq$ and since $p_t\leq h_t$ for each $t\in[\![2]\!]$, moreover, $p_1\otimes p_2\leq h_1\otimes h_2=g$. Given any $z\in\mathfrak{U}\cup\mathfrak{O}$ and any $\mathsf{G}\in g$ with $\mathsf{G}\subseteq\xi_{\mathfrak{d}_1\otimes\mathfrak{d}_2}^{\mathfrak{c}_1\otimes\mathfrak{c}_2\leftarrow}(\{z\})$ by definition there exist $t\in[\![2]\!]$ and $\mathsf{F}\in h_t$ such that $\mathsf{G}=\gamma_{\mathsf{H}_t,\ell_1+\ell_2\rightarrow}^{\;k_1+k_2}(\mathsf{F})$. Because $\xi_{\mathfrak{d}_1\otimes\mathfrak{d}_2}^{\mathfrak{c}_1\otimes\mathfrak{c}_2}\circ\gamma_{\mathsf{H}_t,\ell_1+\ell_2}^{\;k_1+k_2}=\xi_{\mathfrak{d}_t}^{\mathfrak{c}_t}$ by Lemma 4.2 (c) then $\mathsf{F}\subseteq(\gamma_{\mathsf{H}_t,\ell_1+\ell_2}^{\;k_1+k_2\leftarrow}\circ\gamma_{\mathsf{H}_t,\ell_1+\ell_2\rightarrow}^{\;k_1+k_2})(\mathsf{F})=\gamma_{\mathsf{H}_t,\ell_1+\ell_2}^{\;k_1+k_2\leftarrow}(\mathsf{G})\subseteq(\gamma_{\mathsf{H}_t,\ell_1+\ell_2}^{\;k_1+k_2\leftarrow}\circ\xi_{\mathfrak{d}_1\otimes\mathfrak{d}_2}^{\mathfrak{c}_1\otimes\mathfrak{c}_2\leftarrow})(\{z\})=\xi_{\mathfrak{d}_t}^{\mathfrak{c}_t\leftarrow}(\{z\})$. Moreover, since $\mathsf{H}_1\cap\mathsf{H}_2=\varnothing$ and since $\gamma_{\mathsf{H}_t,\ell_1+\ell_2}^{\;k_1+k_2}$ is injective, G is the unique subset of $\Pi_{\ell_1+\ell_2}^{k_1+k_2}$ with $\mathsf{F}=\gamma_{\mathsf{H}_t,\ell_1+\ell_2}^{\;k_1+k_2\leftarrow}(\mathsf{G})$ and $\varnothing=\gamma_{\mathsf{H}_{3-t},\ell_1+\ell_2}^{\;k_1+k_2\leftarrow}(\mathsf{G})$. Since $R((\mathfrak{c}_{3-t},\mathfrak{d}_{3-t},p_{3-t}),\varnothing)=(\varnothing,\varnothing,\varnothing)$ Lemma 4.5 (b) thus tells us that $R((\mathfrak{c}_1\otimes\mathfrak{c}_2,\mathfrak{d}_1\otimes\mathfrak{d}_2,p_1\otimes p_2),\mathsf{G})=R((\mathfrak{c}_t,\mathfrak{d}_t,p_t),\mathsf{F})$ and thus $R((\mathfrak{c}_1\otimes\mathfrak{c}_2,\mathfrak{d}_1\otimes\mathfrak{d}_2,p_1\otimes p_2),\mathsf{G})\in\mathcal{X}_z$ because h_t is a history for $(\mathfrak{c}_t,\mathfrak{d}_t,p_t)$. Hence, $*^r_{z\in\mathfrak{U}\cup\mathfrak{O}}\mathcal{X}_z$ is closed under monoidal products.

Step 5: Erasing. Finally, let $\{k,\ell\}\subseteq\mathbb{N}_0$, let $\mathfrak{c}\colon[\![k]\!]\to(\mathfrak{U}\otimes\{\circ,\bullet\})\cup\mathfrak{O}$ and $\mathfrak{d}\colon[\![\ell]\!]\to(\mathfrak{U}\otimes\{\circ,\bullet\})\cup\mathfrak{O}$, let $(\mathfrak{c},\mathfrak{d},p)\in *^r_{z\in\mathfrak{U}\cup\mathfrak{O}}\mathcal{X}_z$, let h be a history for $(\mathfrak{c},\mathfrak{d},p)$ with respect to $(\mathcal{X}_z)_{z\in\mathfrak{U}\cup\mathfrak{O}}$ and r, let $\mathsf{T}\subseteq\Pi_\ell^0$ be convex with respect to Γ_ℓ^k, let $|\mathsf{T}|=2$, let $z_{\mathsf{T}}\in\mathfrak{U}\cup\mathfrak{O}$, let $\mathsf{T}\subseteq\xi_{\mathfrak{d}}^{\mathfrak{c}\leftarrow}(\{z_{\mathsf{T}}\})$ and, if $z_{\mathsf{T}}\in\mathfrak{U}$, let ${}_{z_{\mathsf{T}}}\sigma_{\mathfrak{d}}^{\mathfrak{c}}(\mathsf{T})=0$. We prove that $g\coloneqq E(h,\mathsf{T})$, where $\mathsf{M}\coloneqq\Pi_\ell^k\backslash\mathsf{T}$, is a history for $(\mathfrak{a},\mathfrak{b},q)\coloneqq E((\mathfrak{c},\mathfrak{d},p),\mathsf{T})$ with respect to $(\mathcal{X}_z)_{z\in\mathfrak{U}\cup\mathfrak{O}}$ and r.

It is clear that $(\mathfrak{a},\mathfrak{b},g)=E((\mathfrak{c},\mathfrak{d},h),\mathsf{T})\in{}_{\mathfrak{U},\mathfrak{O}}^{\;r}\mathcal{S}^+$ since $(\mathfrak{c},\mathfrak{d},h)\in{}_{\mathfrak{U},\mathfrak{O}}^{\;r}\mathcal{S}^+$ by assumption and since ${}_{\mathfrak{U},\mathfrak{O}}^{\;r}\mathcal{S}^+$ is closed under erasing by Proposition 4.12 and Lemma 3.25. Moreover, the fact that erasing preserves $\leq$ ensures $q=E(p,\mathsf{T})\leq E(h,\mathsf{T})=g$.

Given any $z\in\mathfrak{U}\cup\mathfrak{O}$ and any $\mathsf{G}\in g$ with $\mathsf{G}\subseteq\xi_{\mathfrak{b}}^{\mathfrak{a}\leftarrow}(\{z\})$ we distinguish two cases in order to show $R((\mathfrak{a},\mathfrak{b},q),\mathsf{G})\in\mathcal{X}_z$.

Case 5.1: If there exists $\mathsf{H}\in h$ with $\mathsf{H}\cap\mathsf{T}=\varnothing$ and $\mathsf{G}=\gamma_{\mathsf{M},\ell}^{\;k\leftarrow}(\mathsf{H})$, then the fact that $\xi_{\mathfrak{b}}^{\mathfrak{a}}=\xi_{\mathfrak{d}}^{\mathfrak{c}}\circ\gamma_{\mathsf{M},\ell}^{\;k}$ by Lemma 4.2 (d) implies that $\mathsf{H}=(\gamma_{\mathsf{M},\ell\rightarrow}^{\;k}\circ\gamma_{\mathsf{M},\ell}^{\;k\leftarrow})(\mathsf{H})=\gamma_{\mathsf{M},\ell\rightarrow}^{\;k}(\mathsf{G})\subseteq(\gamma_{\mathsf{M},\ell\rightarrow}^{\;k}\circ\xi_{\mathfrak{b}}^{\mathfrak{a}\leftarrow})(\{z\})=(\gamma_{\mathsf{M},\ell\rightarrow}^{\;k}\circ\gamma_{\mathsf{M},\ell}^{\;k\leftarrow}\circ\xi_{\mathfrak{d}}^{\mathfrak{c}\leftarrow})(\{z\})\subseteq\xi_{\mathfrak{d}}^{\mathfrak{c}\leftarrow}(\{z\})$, where we have also used the fact that $\mathsf{H}\subseteq\mathrm{ran}(\gamma_{\mathsf{M},\ell}^{\;k})=\mathsf{M}$. The assumption that h is a history for $(\mathfrak{c},\mathfrak{d},p)$ therefore requies $R((\mathfrak{c},\mathfrak{d},p),\mathsf{H})\in\mathcal{X}_z$. Since, at the same time, $R((\mathfrak{a},\mathfrak{b},q),\mathsf{G})=R((\mathfrak{c},\mathfrak{d},p),\mathsf{H})$ by $\mathsf{H}\cap\mathsf{T}=\varnothing$ and by Lemma 4.5 (d) that proves $R((\mathfrak{a},\mathfrak{b},q),\mathsf{G})\in\mathcal{X}_z$ in that case.

Case 5.2: By definition of g, the alternative possibility is that $\mathsf{G} = \gamma_{\mathsf{M},\ell}^{k\leftarrow}(\mathsf{W})$ where $\mathsf{W} \coloneqq \bigcup\{\mathsf{H} \in h \wedge \mathsf{H}\cap\mathsf{T} \neq \varnothing\}$. In that case we let $\mathsf{V} \coloneqq \gamma_{\mathsf{W},\ell}^{k\leftarrow}(\mathsf{T})$ and prove that, if $(\mathfrak{f},\mathfrak{g},s) \coloneqq R((\mathfrak{c},\mathfrak{d},p),\mathsf{W})$, then $E((\mathfrak{f},\mathfrak{g},s),\mathsf{V}) \in \mathcal{X}_z$ and $R((\mathfrak{a},\mathfrak{b},q),\mathsf{G}) = E((\mathfrak{f},\mathfrak{g},s),\mathsf{V})$, thus proving $R((\mathfrak{a},\mathfrak{b},q),\mathsf{G}) \in \mathcal{X}_z$.

Step 5.2.1: First, as an auxiliary step, we show that $\mathsf{W} \subseteq \xi_{\mathfrak{d}}^{\mathfrak{c}\leftarrow}(\{z_\mathsf{T}\})$ and that $z = z_\mathsf{T}$. For any $\mathsf{H} \in h$ with $\mathsf{H} \cap \mathsf{T} \neq \varnothing$, because $h \subseteq \ker(\xi_{\mathfrak{d}}^{\mathfrak{c}})$ and $\mathsf{T} \subseteq \xi_{\mathfrak{d}}^{\mathfrak{c}\leftarrow}(\{z_\mathsf{T}\})$, the fact that $\varnothing \neq \mathsf{H}\cap\mathsf{T} \subseteq \mathsf{H}\cap\xi_{\mathfrak{d}}^{\mathfrak{c}\leftarrow}(\{z_\mathsf{T}\})$ requires $\mathsf{H} \subseteq \xi_{\mathfrak{d}}^{\mathfrak{c}\leftarrow}(\{z_\mathsf{T}\})$. Because h is a history for $(\mathfrak{c},\mathfrak{d},p)$ it follows $R((\mathfrak{c},\mathfrak{d},p),\mathsf{H}) \in \mathcal{X}_{z_\mathsf{T}}$ for any $\mathsf{H} \in h$ with $\mathsf{H}\cap\mathsf{T} \neq \varnothing$ and, consequently, $\mathsf{W} \subseteq \xi_{\mathfrak{d}}^{\mathfrak{c}\leftarrow}(\{z_\mathsf{T}\})$.

On the other hand, because $\xi_{\mathfrak{b}}^{\mathfrak{a}} = \xi_{\mathfrak{d}}^{\mathfrak{c}} \circ \gamma_{\mathsf{M},\ell}^{k}$ by Lemma 4.2 (d), also, $\mathsf{G} = \gamma_{\mathsf{M},\ell}^{k\leftarrow}(\mathsf{W}) \subseteq (\gamma_{\mathsf{M},\ell}^{k\leftarrow} \circ \xi_{\mathfrak{d}}^{\mathfrak{c}\leftarrow})(\{z_\mathsf{T}\}) = \xi_{\mathfrak{b}}^{\mathfrak{a}\leftarrow}(\{z_\mathsf{T}\})$ and thus $z = z_\mathsf{T}$ by $\varnothing \neq \mathsf{G} \subseteq \xi_{\mathfrak{b}}^{\mathfrak{a}\leftarrow}(\{z\}) \cap \xi_{\mathfrak{b}}^{\mathfrak{a}\leftarrow}(\{z_\mathsf{T}\})$.

Step 5.2.2: Next, as a second intermediate step we prove $(\mathfrak{f},\mathfrak{g},s) \in \mathcal{X}_z$. In order to show this, by Lemma 4.20 it suffices to let $t \coloneqq \gamma_{\mathsf{W},\ell}^{k\leftarrow}(h)$, to let $\{x,y\} \subseteq \mathbb{N}_0$ be such that $\mathfrak{f}\colon [\![x]\!] \to (\mathfrak{U} \otimes \{\circ,\bullet\}) \cup \mathfrak{O}$ and $\mathfrak{g}\colon [\![y]\!] \to (\mathfrak{U} \otimes \{\circ,\bullet\}) \cup \mathfrak{O}$ and to prove that t is non-crossing with respect to Γ_y^x and that $R((\mathfrak{f},\mathfrak{g},s),\mathsf{K}) \in \mathcal{X}_z$ for any $\mathsf{K} \in t$.

And, indeed, for any $\{\mathsf{K}_1,\mathsf{K}_2\} \subseteq t$ with $\mathsf{K}_1 \neq \mathsf{K}_2$, by definition, there exist $\{\mathsf{H}_1,\mathsf{H}_2\} \subseteq h$ with $\mathsf{K}_1 = \gamma_{\mathsf{W},\ell}^{k\leftarrow}(\mathsf{H}_1)$ and $\mathsf{K}_2 = \gamma_{\mathsf{W},\ell}^{k\leftarrow}(\mathsf{H}_2)$. It follows $\varnothing \neq \gamma_{\mathsf{W},\ell}^{k}{}_{\rightarrow}(\mathsf{K}_1) \subseteq (\gamma_{\mathsf{W},\ell}^{k}{}_{\rightarrow} \circ \gamma_{\mathsf{W},\ell}^{k\leftarrow})(\mathsf{H}_1) \subseteq \mathsf{H}_1$ and thus $\mathsf{K}_1 \cap \mathsf{W} \neq \varnothing$. Since W is a union of blocks of h and since $\mathsf{H}_1 \in h$ that requires $\mathsf{H}_1 \subseteq \mathsf{W}$. The same argument can be carried out for H_2 to prove H_2. Because $\mathsf{W} \subseteq \xi_{\mathfrak{d}}^{\mathfrak{c}\leftarrow}(\{z\})$ by Step 5.2.1 we have thus shown $\mathsf{H}_1 \subseteq \xi_{\mathfrak{d}}^{\mathfrak{c}\leftarrow}(\{z\})$ and $\xi_{\mathfrak{d}}^{\mathfrak{c}\leftarrow}(\{z\})$. Moreover, because $\gamma_{\mathsf{W},\ell}^{k}$ is injective, $\mathsf{H}_1\cap\mathsf{H}_2 = \gamma_{\mathsf{W},\ell}^{k\leftarrow}(\mathsf{K}_1)\cap\gamma_{\mathsf{W},\ell}^{k\leftarrow}(\mathsf{K}_2) = \gamma_{\mathsf{W},\ell}^{k\leftarrow}(\mathsf{K}_1 \cap \mathsf{K}_2) = \varnothing$ by $\mathsf{K}_1 \cap \mathsf{K}_2 = \varnothing$. Equivalently, $\mathsf{H}_1 \neq \mathsf{H}_2$. Therefore the assumption that h is a history for $(\mathfrak{c},\mathfrak{d},p)$ or, more precisely, that $(\mathfrak{c},\mathfrak{d},h) \in {}_{\mathfrak{U},\mathfrak{O}}^{\;r}\mathcal{S}^{+}$ allows us to conclude $\mathsf{H}_1 \mathbin{\diagdown\!\!\diagdown}_{\ell}^{k} \mathsf{H}_2$. Since $\gamma_{\mathsf{W},\ell}^{k}$ is monotonic with respect to Γ_y^x and Γ_ℓ^k that requires $\mathsf{K}_1 \mathbin{\diagdown\!\!\diagdown}_{y}^{x} \mathsf{K}_2$. Hence, t is non-crossing with respect to Γ_y^x.

Furthermore, given any $\mathsf{K} \in t$, if $\mathsf{H} \in h$ is the unique block with $\mathsf{K} = \gamma_{\mathsf{W},\ell}^{k\leftarrow}(\mathsf{H})$, then $\mathsf{H} \subseteq \mathsf{W} \subseteq \xi_{\mathfrak{d}}^{\mathfrak{c}\leftarrow}(\{z\})$. Because h is a history for $(\mathfrak{c},\mathfrak{d},p)$ it thus follows $R((\mathfrak{c},\mathfrak{d},p),\mathsf{H}) \in \mathcal{X}_z$. According to Lemma 4.5 (d), moreover, $R((\mathfrak{f},\mathfrak{g},s),\mathsf{K}) = R((\mathfrak{c},\mathfrak{d},p),\mathsf{H})$ because $\mathsf{H} \subseteq \mathsf{W}$. Hence, $(\mathfrak{f},\mathfrak{g},s) \in \mathcal{X}_z$, as claimed.

Step 5.2.3: $\gamma_{\mathsf{H},\ell}^{k}$ is strictly monotonic with respect to Γ_y^x and Γ_ℓ^k by Lemma 4.2 (d), which is why $\mathsf{V} = \gamma_{\mathsf{H},\ell}^{k\leftarrow}(\mathsf{T})$ is convex with respect to Γ_y^x and satisfies $|\mathsf{V}| = 2$. Moreover, of course, $\mathsf{V} = \gamma_{\mathsf{H},\ell}^{k\leftarrow}(\mathsf{T}) \subseteq (\gamma_{\mathsf{H},\ell}^{k}{}_{\rightarrow} \circ \xi_{\mathfrak{d}}^{\mathfrak{c}\leftarrow}) = \xi_{\mathfrak{g}}^{\mathfrak{f}\leftarrow}(\{z\})$ because $\xi_{\mathfrak{g}}^{\mathfrak{f}} = \xi_{\mathfrak{d}}^{\mathfrak{c}} \circ \gamma_{\mathsf{H},\ell}^{k}$. And, if $z \in \mathfrak{U}$, then ${}_z\sigma_{\mathfrak{g}}^{\mathfrak{f}}(\mathsf{V}) = {}_z\sigma_{\mathfrak{d}}^{\mathfrak{c}}(\mathsf{T}) = 0$ by Lemma 4.2 (d) and $\mathsf{T}\backslash\mathsf{H} = \varnothing$. It follows $E((\mathfrak{f},\mathfrak{g},s),\mathsf{V}) \in \mathcal{X}_z$ by Lemma 3.25 and Step 5.2.2.

Step 5.2.4: Because $R((\mathfrak{a},\mathfrak{b},q),\mathsf{G}) = E((\mathfrak{f},\mathfrak{g},s),\mathsf{V})$ by Lemma 4.5 (d) that proves $R((\mathfrak{a},\mathfrak{b},q),\mathsf{G}) \in \mathcal{X}_z$ in this case, which concludes the proof overall. ☐

Again, convenient shorthand notation is useful for a particular special case of Proposition 4.25.

DEFINITION 4.26. If there exists a category $\mathcal{C}$ of two-colored or uncolored partitions such that for each $z \in \mathfrak{U} \cup \mathfrak{O}$ the category $\mathcal{X}_z$ is given by $\mathcal{C}$, seen as tagged with the single tag z, then we call $*^r_{z\in\mathfrak{U}\cup\mathfrak{D}}\mathcal{X}_z$ the *(little) graph co-power category* of $\mathcal{C}$ with respect to $\mathfrak{U}\cup\mathfrak{D}$ and r and also write $\mathcal{C}^{*(\mathfrak{U}\cup\mathfrak{D},r)}$ instead of $*^r_{z\in\mathfrak{U}\cup\mathfrak{D}}\mathcal{X}_z$.

NOTATION 4.27. If $r=\varnothing$ in Definition 4.26, we also write $\mathcal{C}^{*\mathfrak{U}\cup\mathfrak{D}}$ instead of $\mathcal{C}^{*(\mathfrak{U}\cup\mathfrak{D},r)}$ and speak of the $\mathfrak{U}\cup\mathfrak{D}$-fold *(little) free co-power category* of $\mathcal{C}$.

REMARK 4.28. In the setting of Definition 4.26, if r is the trivial commutation relation $\{(z,z')\,|\,\{z,z'\}\subseteq\mathfrak{U}\cup\mathfrak{D}\wedge z\neq z'\}$, then $\mathcal{C}^{*(\mathfrak{U}\cup\mathfrak{D},r)}$ coincides with ($\mathcal{C}^{\star(\mathfrak{U}\cup\mathfrak{D},r)}$ and) $\mathcal{C}^{\times\mathfrak{U}\cup\mathfrak{D}}$.

4.4. Unique graph co-products. When it comes to proving that a certain partition belongs to the category, the little graph co-product is more complicated than the big one. Conversely, we will see in Section 5 that little graph co-products have very small generator sets. In order to best capitalize on the advantages of each co-product it is useful to know when the two coincide. In this section we show that this is the case if the factor categories have the following property.

DEFINITION 4.29. Any category $\mathcal{C}$ of $(\mathfrak{U},\mathfrak{D})$-tagged labeled partitions is called $\otimes$-*elbats* (with respect to $(\mathfrak{U},\mathfrak{D})$) if for any $(\mathfrak{c}_1,\mathfrak{d}_1,p_1)\in{}_{\mathfrak{U},\mathfrak{D}}\mathcal{S}$ and $(\mathfrak{c}_2,\mathfrak{d}_2,p_2)\in{}_{\mathfrak{U},\mathfrak{D}}\mathcal{S}$, whenever $(\mathfrak{c}_1,\mathfrak{d}_1,p_1)\otimes(\mathfrak{c}_2,\mathfrak{d}_2,p_2)\in\mathcal{C}$, then also $(\mathfrak{c}_1,\mathfrak{d}_1,p_1)\in\mathcal{C}$ and $(\mathfrak{c}_2,\mathfrak{d}_2,p_2)\in\mathcal{C}$.

Similarly to how Lemma 4.20 extends the statement that categories are closed under the monoidal product the next lemma expands on what it means to be $\otimes$-elbats.

LEMMA 4.30. *For any category $\mathcal{C}$ of $(\mathfrak{U},\mathfrak{D})$-tagged labeled partitions the following are equivalent:*

(i) $\mathcal{C}$ is $\otimes$-elbats with respect to $(\mathfrak{U},\mathfrak{D})$

(ii) For any $\{k,\ell\}\subseteq\mathbb{N}_0$, any $\mathfrak{c}\colon [\![k]\!]\to(\mathfrak{U}\otimes\{\circ,\bullet\})\cup\mathfrak{D}$ and $\mathfrak{d}\colon [\![\ell]\!]\to(\mathfrak{U}\otimes\{\circ,\bullet\})\cup\mathfrak{D}$, any set-theoretical partition p of Π^k_ℓ, any set-theoretical partition h of Π^k_ℓ which is non-crossing with respect to Γ^k_ℓ and satisfies $p\leq h$, if $(\mathfrak{c},\mathfrak{d},p)\in\mathcal{C}$, then for any $\mathsf{S}\in h$,

$$R((\mathfrak{c},\mathfrak{d},p),\mathsf{S})\in\mathcal{C}.$$

PROOF. It is clear that (ii) implies (i). The proof of the converse implication is very much similar to the proof of Lemma 4.20. Again, it goes by induction over $|h|$ and there is nothing to show for $|h|=1$. In the general case, there exists $\mathsf{S}_0\in h$ which is convex with respect to Γ^k_ℓ because h is non-crossing with respect to Γ^k_ℓ. Once more we can employ Lemma 3.22 to assume ${}_{\blacksquare}\ell\in\mathsf{S}_0$ and thus $(\mathfrak{c},\mathfrak{d},p)=R((\mathfrak{c},\mathfrak{d},p),\mathsf{M})\otimes R((\mathfrak{c},\mathfrak{d},p),\mathsf{S}_0)$, where $\mathsf{M}\coloneqq\Pi^k_\ell\backslash\mathsf{S}_0$. Since $\mathcal{C}$ is $\otimes$-elbats this decomposition allows us to infer both $R((\mathfrak{c},\mathfrak{d},p),\mathsf{M})\in\mathcal{C}$ and $R((\mathfrak{c},\mathfrak{d},p),\mathsf{S}_0)\in\mathcal{C}$. It remains to let $\mathsf{S}\in h$ be arbitrary with $\mathsf{S}\neq\mathsf{S}_0$ and show $R((\mathfrak{c},\mathfrak{d},p),\mathsf{S})\in\mathcal{C}$.

As before, if $(\mathfrak{a},\mathfrak{b},q)\coloneqq R((\mathfrak{c},\mathfrak{d},p),\mathsf{M})$, if $\{m,n\}\subseteq\mathbb{N}_0$ are such that $\mathfrak{a}\colon [\![m]\!]\to(\mathfrak{U}\otimes\{\circ,\bullet\})\cup\mathfrak{D}$ and $\mathfrak{b}\colon [\![n]\!]\to(\mathfrak{U}\otimes\{\circ,\bullet\})\cup\mathfrak{D}$ and if $g\coloneqq R(h,\mathsf{M})$, then g is a set-theoretical partition of Π^m_n which is non-crossing with respect to Γ^m_n and satisfies $q\leq g$ and $|g|<|h|$. By the definition of g and by the assumptions $p\leq h$ and

$\mathsf{S} \neq \mathsf{S}_0$ we know $\mathsf{T} := \gamma_{\mathsf{M},\ell \to}^{k}(\mathsf{S}) \in g$. Because $(\mathfrak{a}, \mathfrak{b}, q) \in \mathcal{C}$ and because $\mathcal{C}$ is $\otimes$-elbats the induction hypothesis therefore lets us conclude $R((\mathfrak{a}, \mathfrak{b}, q), \mathsf{T}) \in \mathcal{C}$. Because $R((\mathfrak{a}, \mathfrak{b}, q), \mathsf{T}) = R((\mathfrak{c}, \mathfrak{d}, p), \mathsf{S})$ by Lemma 4.5 (d) and by $\mathsf{S} \subseteq \mathsf{M}$ that means we have shown $R((\mathfrak{c}, \mathfrak{d}, p), \mathsf{S}) \in \mathcal{C}$. Hence, (i) really does imply (ii). □

In the proof that there is only one "graph co-product" for $\otimes$-elbats factors the following notion plays a crucial role.

Definition 4.31. Given any finite set X, any total cyclic order Γ on X, any set-theoretical partition p of X and any $\{\mathsf{a}, \mathsf{c}\} \subseteq X$, we say that c is *connected* to a with respect to Γ and p if there exist $m \in \mathbb{N}$ and pairwise distinct blocks $\mathsf{B}_1, \ldots, \mathsf{B}_m$ of p with $\mathsf{a} \in \mathsf{B}_1$ and $\mathsf{c} \in \mathsf{B}_m$ and $\mathsf{B}_i \not\perp \mathsf{B}_{i+1}$ with respect to Γ for any $i \in [\![m-1]\!]$.

This binary relation actually constitutes an equivalence.

Definition 4.32. For any finite set X, any total cyclic order Γ on X and any set-theoretical partition p of X the set-theoretical partition of X associated with the equivalence relation of being connected with respect to Γ and p is called the *connected components* of X with respect to Γ and p. We speak of any of its blocks as a *connected component* of X with respect to Γ and p.

The following is well-known. A proof can also be extracted from that of Case 3.2 in the proof of Proposition 4.35 below.

Proposition 4.33. *For any finite set X, any total cyclic order Γ on X and any set-theoretical partition p of X the connected components of Γ with respect to Γ and p are non-crossing with respect to Γ. More precisely, they form the finest non-crossing partition coarser than p.*

Assumption 4.34. In Section 4.4, let r be any partial commutation relation on $\mathfrak{U} \cup \mathfrak{D}$ and let $(\mathcal{X}_z)_{z \in \mathfrak{U} \cup \mathfrak{D}}$ be any family such that $\mathcal{X}_z$ is a category of $(\{z\}, \varnothing)$-tagged labeled partitions for any $z \in \mathfrak{U}$ and of $(\varnothing, \{z\})$-tagged labeled partitions for $z \in \mathfrak{D}$.

Proposition 4.35. *If $\mathcal{X}_z$ is $\otimes$-elbats with respect to $(\{z\}, \varnothing)$ for any $z \in \mathfrak{U}$ and with respect to $(\varnothing, \{z\})$ for any $z \in \mathfrak{D}$, then $*^{r}_{z \in \mathfrak{U} \cup \mathfrak{D}} \mathcal{X}_z = \star^{r}_{z \in \mathfrak{U} \cup \mathfrak{D}} \mathcal{X}_z$.*

Proof. The inclusion $*^{r}_{z \in \mathfrak{U} \cup \mathfrak{D}} \mathcal{X}_z \subseteq \star^{r}_{z \in \mathfrak{U} \cup \mathfrak{D}} \mathcal{X}_z$ was shown in Proposition 4.21. Let $(\mathfrak{c}, \mathfrak{d}, p) \in \star^{r}_{z \in \mathfrak{U} \cup \mathfrak{D}} \mathcal{X}_z$ be arbitrary and $\{k, \ell\} \subseteq \mathbb{N}_0$ such that $\mathfrak{c}\colon [\![k]\!] \to (\mathfrak{U} \otimes \{\circ, \bullet\}) \cup \mathfrak{D}$ and $\mathfrak{c}\colon [\![k]\!] \to (\mathfrak{U} \otimes \{\circ, \bullet\}) \cup \mathfrak{D}$. For any $z \in \mathfrak{U} \cup \mathfrak{D}$ abbreviate $\mathsf{Y}_z := \xi_{\mathfrak{d}}^{\mathfrak{c} \leftarrow}(\{z\})$ and $m_z := \alpha(\mathsf{Y}_z)$ and $n_z := \beta(\mathsf{Y}_z)$. We show that

$$h := \{\gamma_{\mathsf{Y}_z,\ell \to}^{k}(\mathsf{C}) \mid z \in \mathfrak{U} \cup \mathfrak{D} \wedge \mathsf{C} \text{ connected component of } \Pi_{n_z}^{m_z} \text{ w.r.t. } R(p, \mathsf{Y}_z) \text{ and } \Gamma_{n_z}^{m_z}\}$$

is a history for $(\mathfrak{c}, \mathfrak{d}, p)$ with respect to $(\mathcal{X}_z)_z$ and r.

Step 1: h is a set-theoretical partition of Π_ℓ^k. For any $\{z_1, z_2\} \subseteq \mathfrak{U} \cup \mathfrak{D}$, any connected components C_1 and C_2 of $\Pi_{n_{z_1}}^{m_{z_1}}$ with respect to $R(p, \mathsf{Y}_{z_1})$ and $\Gamma_{n_{z_1}}^{m_{z_1}}$ respectively of $\Pi_{n_{z_2}}^{m_{z_2}}$ with respect to $R(p, \mathsf{Y}_{z_2})$ and $\Gamma_{n_{z_2}}^{m_{z_2}}$, if $\mathsf{H}_1 := \gamma_{\mathsf{Y}_{z_1},\ell \to}^{k}(\mathsf{C}_1)$ and

$\mathsf{H}_2 \coloneqq \gamma_{\mathsf{Y}_{z_2},\ell}^{\;k}{}^{\rightarrow}(\mathsf{C}_2)$ satisfy $\mathsf{H}_1 \cap \mathsf{H}_2 \neq \varnothing$, then in particular $\mathsf{Y}_{z_1} \cap \mathsf{Y}_{z_2} \neq \varnothing$ because $\mathsf{H}_1 \subseteq \operatorname{ran}(\gamma_{\mathsf{Y}_{z_1},\ell}^{\;k}) = \mathsf{Y}_{z_1}$ and $\mathsf{H}_2 \subseteq \operatorname{ran}(\gamma_{\mathsf{Y}_{z_2},\ell}^{\;k}) = \mathsf{Y}_{z_2}$. Because $\{\mathsf{Y}_{z_1}, \mathsf{Y}_{z_2}\} \subseteq \ker(\xi_{\mathfrak{d}}^{\mathfrak{c}})$ that requires $\mathsf{Y}_{z_1} = \mathsf{Y}_{z_2}$, which is to say $z_1 = z_2$. In conclusion, because $\gamma_{\mathsf{Y}_{z_1},\ell}^{\;k}$ is injective, $\mathsf{C}_1 \cap \mathsf{C}_2 = \gamma_{\mathsf{Y}_{z_1},\ell}^{\;k}{}^{\leftarrow}(\mathsf{H}_1) \cap \gamma_{\mathsf{Y}_{z_2},\ell}^{\;k}{}^{\leftarrow}(\mathsf{H}_2) = \gamma_{\mathsf{Y}_{z_1},\ell}^{\;k}{}^{\leftarrow}(\mathsf{H}_1 \cap \mathsf{H}_2) \neq \varnothing$ because $\mathsf{H}_1 \cap \mathsf{H}_2 \subseteq \mathsf{Y}_{z_1}$. That requires $\mathsf{C}_1 = \mathsf{C}_2$ and thus $\mathsf{H}_1 = \mathsf{H}_2$.

For any $\mathtt{b} \in \Pi_\ell^k$, if $z \coloneqq \xi_{\mathfrak{d}}^{\mathfrak{c}}(\mathtt{b})$, then $\mathtt{b} \in \mathsf{Y}_z$. Because $\operatorname{ran}(\gamma_{\mathsf{Y}_z,\ell}^{\;k}) = \mathsf{Y}_z$ there is $\mathtt{a} \in \Pi_{n_z}^{m_z}$ with $\mathtt{b} = \gamma_{\mathsf{Y}_z,\ell}^{\;k}(\mathtt{a})$. If C is the connected component of $\mathtt{a}$ with respect to $R(p, \mathsf{Y}_z)$ and $\Gamma_{n_z}^{m_z}$, then $\mathtt{b} \in \gamma_{\mathsf{Y}_z,\ell}^{\;k}{}^{\rightarrow}(\mathsf{C})$. Hence, $\bigcup h = \Pi_\ell^k$, which makes h a set-theoretical partition of Π_ℓ^k.

Step 2: h respects the tags. For any $z \in \mathfrak{U} \cup \mathfrak{D}$ and any connected component of $\Pi_{n_z}^{m_z}$ with respect to $R(p, \mathsf{Y}_z)$ and $\Gamma_{n_z}^{m_z}$, obviously, $\gamma_{\mathsf{Y}_z,\ell}^{\;k}{}^{\rightarrow}(\mathsf{C}) \subseteq \mathsf{Y}_z \in \ker(\xi_{\mathfrak{d}}^{\mathfrak{c}})$. Hence, $h \leq \ker(\xi_{\mathfrak{d}}^{\mathfrak{c}})$.

Step 3: h meets the non-crossing conditions for being a history. Given any $\{z_1, z_2\} \subseteq \mathfrak{U} \cup \mathfrak{D}$ with $(z_1, z_2) \notin r$ and any connected components C_1 and C_2 of $\Pi_{n_{z_1}}^{m_{z_1}}$ with respect to $R(p, \mathsf{Y}_{z_1})$ and $\Gamma_{n_{z_1}}^{m_{z_1}}$ respectively of $\Pi_{n_{z_2}}^{m_{z_2}}$ with respect to $R(p, \mathsf{Y}_{z_2})$ and $\Gamma_{n_{z_2}}^{m_{z_2}}$, if $\mathsf{H}_1 \coloneqq \gamma_{\mathsf{Y}_{z_1},\ell}^{\;k}{}^{\rightarrow}(\mathsf{C}_1)$ and $\mathsf{H}_2 \coloneqq \gamma_{\mathsf{Y}_{z_2},\ell}^{\;k}{}^{\rightarrow}(\mathsf{C}_2)$ are such that $\mathsf{H}_1 \neq \mathsf{H}_2$ and, necessarily, $\mathsf{H}_1 \subseteq \mathsf{Y}_{z_1}$ and $\mathsf{H}_2 \subseteq \mathsf{Y}_{z_2}$, then we prove $\mathsf{H}_1 \mathrel{\diagdown\!\!\diagdown}_\ell^k \mathsf{H}_2$ by distinguishing two cases.

Case 3.1: Components belonging to identical tags. If $z \coloneqq z_1 = z_2$, then C_1 and C_2 are both connected components of $\Pi_{n_z}^{m_z}$ with respect to $R(p, \mathsf{Y}_z)$ and $\Gamma_{n_z}^{m_z}$. And the asssumption that $\mathsf{H}_1 \cap \mathsf{H}_2 = \varnothing$ then implies $\mathsf{C}_1 \cap \mathsf{C}_2 = \gamma_{\mathsf{Y}_z,\ell}^{\;k}{}^{\leftarrow}(\mathsf{H}_1 \cap \mathsf{H}_2) = \varnothing$, i.e., $\mathsf{C}_1 \neq \mathsf{C}_2$. Since the connected components of $\Pi_{n_z}^{m_z}$ with respect to $R(p, \mathsf{Y}_z)$ and $\Gamma_{n_z}^{m_z}$ are non-crossing with respect to $\Gamma_{n_z}^{m_z}$ by Proposition 4.33, we can thus conclude $\mathsf{C}_1 \mathrel{\diagdown\!\!\diagdown}_{n_z}^{m_z} \mathsf{C}_2$. Because $\gamma_{\mathsf{Y}_z,\ell}^{\;k}$ is strictly monotonic with respect to $\Gamma_{n_z}^{m_z}$ and Γ_ℓ^k it follows $\mathsf{H}_1 \mathrel{\diagdown\!\!\diagdown}_\ell^k \mathsf{H}_2$.

Case 3.2: Components belonging to distinct tags. In order to see that $\mathsf{H}_1 \mathrel{\diagdown\!\!\diagdown}_\ell^k \mathsf{H}_2$ in the case where $z_1 \neq z_2$, the proof of Proposition 4.33 can be adapted as follows. We suppose $\mathsf{H}_1 \not\!\!+_\ell^k \mathsf{H}_2$ and derive a contradiction. More precisely, we find blocks $\mathtt{F}_{i_0}^1$ and $\mathtt{F}_{j_0}^2$ of p with $\mathtt{F}_{i_0}^1 \subseteq \mathsf{Y}_{z_1}$ and $\mathtt{F}_{j_0}^2 \subseteq \mathsf{Y}_{z_2}$ and $\mathtt{F}_{i_0}^1 \not\!\!+_\ell^k \mathtt{F}_{j_0}^2$, which contradicts the assumption $p \in \text{☆}_{z \in \mathfrak{U} \cup \mathfrak{D}}^r \mathcal{X}_z$.

Step 3.2.1: Definition of the blocks $\mathtt{F}_1^1, \ldots, \mathtt{F}_{t_1}^1$ and $\mathtt{F}_1^2, \ldots, \mathtt{F}_{t_2}^2$. Under the assumption $\mathsf{H}_1 \not\!\!+_\ell^k \mathsf{H}_2$ there exist $\{\mathtt{b}_1^1, \mathtt{b}_2^1\} \subseteq \mathsf{H}_1$ and $\{\mathtt{b}_1^2, \mathtt{b}_2^2\} \subseteq \mathsf{H}_2$ with $(\mathtt{b}_1^1 \mid \mathtt{b}_1^2 \mid \mathtt{b}_2^1)_\ell^k$ and $(\mathtt{b}_1^2 \mid \mathtt{b}_2^1 \mid \mathtt{b}_2^2)_\ell^k$ and $(\mathtt{b}_2^1 \mid \mathtt{b}_2^2 \mid \mathtt{b}_1^1)_\ell^k$. By definition of H_1 and H_2 we find $\{\mathtt{a}_1^1, \mathtt{a}_2^1\} \subseteq \mathsf{C}_1$ and $\{\mathtt{a}_1^2, \mathtt{a}_2^2\} \subseteq \mathsf{C}_2$ with $\gamma_{\mathsf{Y}_{z_1},\ell}^{\;k}(\mathtt{a}_1^1) = \mathtt{b}_1^1$ and $\gamma_{\mathsf{Y}_{z_1},\ell}^{\;k}(\mathtt{a}_2^1) = \mathtt{b}_2^1$ and $\gamma_{\mathsf{Y}_{z_2},\ell}^{\;k}(\mathtt{a}_1^2) = \mathtt{b}_1^2$ and $\gamma_{\mathsf{Y}_{z_2},\ell}^{\;k}(\mathtt{a}_2^2) = \mathtt{b}_2^2$. By nature of C_1 and C_2 there must then exist $\{t_1, t_2\} \subseteq \mathbb{N}$ and pairwise distinct blocks $\mathtt{E}_1^1, \ldots, \mathtt{E}_{t_1}^1$ of $R(p, \mathsf{Y}_{z_1})$ and pairwise distinct blocks $\mathtt{E}_1^2, \ldots, \mathtt{E}_{t_2}^2$ of $R(p, \mathsf{Y}_{z_2})$ such that $\mathtt{a}_1^1 \in \mathtt{E}_1^1$ and $\mathtt{a}_2^1 \in \mathtt{E}_{t_1}^1$ and $\mathtt{E}_i^1 \not\!\!+_{n_{z_1}}^{m_{z_1}} \mathtt{E}_{i+1}^1$ for any $i \in [\![t_1 - 1]\!]$ and such that $\mathtt{a}_1^2 \in \mathtt{E}_1^2$ and $\mathtt{a}_2^2 \in \mathtt{E}_{t_2}^2$ and $\mathtt{E}_j^2 \not\!\!+_{n_{z_2}}^{m_{z_2}} \mathtt{E}_{j+1}^2$ for any $j \in [\![t_2 - 1]\!]$. We now let $\mathtt{F}_i^1 \coloneqq \gamma_{\mathsf{Y}_{z_1},\ell}^{\;k}{}^{\rightarrow}(\mathtt{E}_i^1)$ for any $i \in [\![t_1]\!]$ and $\mathtt{F}_j^2 \coloneqq \gamma_{\mathsf{Y}_{z_2},\ell}^{\;k}{}^{\rightarrow}(\mathtt{E}_j^2)$ for any $j \in [\![t_2]\!]$.

Step 3.2.2: Definition of j_0. Because $\mathtt{b}^2_1=\gamma_{\mathsf{Y}_{z_2},\ell}^{\;k}(\mathtt{a}^2_1)\in\gamma_{\mathsf{Y}_{z_2},\ell}^{\;k\to}(\mathtt{E}^2_1)=\mathtt{F}^2_1$ and because, by assumption, $(\mathtt{b}^1_1\mid\mathtt{b}^2_1\mid\mathtt{b}^1_2)^k_\ell$ the set $\{j\in[\![t_2]\!]\wedge\,]\mathtt{b}^1_1,\mathtt{b}^1_2[^k_\ell\cap\mathtt{F}^2_j\neq\varnothing\}$ is non-empty. Define j_0 to be its maximal element.

Step 3.2.3: Definition of auxiliary points $\mathtt{c}_1$ *and* $\mathtt{c}_2$. In order to define i_0, we first prove that $]\mathtt{b}^1_1,\mathtt{b}^1_2[^k_\ell\cap\mathtt{F}^2_{j_0}\neq\varnothing\neq\,]\mathtt{b}^1_2,\mathtt{b}^1_1[^k_\ell\cap\mathtt{F}^2_{j_0}$. That then allows us to find and fix some $\mathtt{c}_1\in\,]\mathtt{b}^1_1,\mathtt{b}^1_2[^k_\ell\cap\mathtt{F}^2_{j_0}$ and $\mathtt{c}_2\in\,]\mathtt{b}^1_2,\mathtt{b}^1_1[^k_\ell\cap\mathtt{F}^2_{j_0}$. Indeed, $]\mathtt{b}^1_1,\mathtt{b}^1_2[^k_\ell\cap\mathtt{F}^2_{j_0}\neq\varnothing$ already holds by the definition of j_0. And, if the set $]\mathtt{b}^1_2,\mathtt{b}^1_1[^k_\ell\cap\mathtt{F}^2_{j_0}$ was empty, that would require $j_0\neq t_2$ because $\mathtt{b}^2_2=\gamma_{\mathsf{Y}_{z_2},\ell}^{\;k}(\mathtt{a}^2_2)\in\gamma_{\mathsf{Y}_{z_2},\ell}^{\;k\to}(\mathtt{E}^2_{t_2})=\mathtt{F}^2_{t_2}$ by $\mathtt{a}^2_2\in\mathtt{E}^2_{t_2}$ and because, by assumption, $(\mathtt{b}^1_2\mid\mathtt{b}^2_2\mid\mathtt{b}^1_1)^k_\ell$. The maximality of j_0 would then demand $]\mathtt{b}^1_1,\mathtt{b}^1_2[^k_\ell\cap\mathtt{F}^2_{j_0+1}=\varnothing$. In other words, we would deduce that $\mathtt{F}^2_{j_0}\subseteq\,]\mathtt{b}^1_1,\mathtt{b}^1_2[^k_\ell$ and $\mathtt{F}^2_{j_0+1}\subseteq\,]\mathtt{b}^1_2,\mathtt{b}^1_1[^k_\ell$. Since the intervals $]\mathtt{b}^1_1,\mathtt{b}^1_2[^k_\ell$ and $]\mathtt{b}^1_2,\mathtt{b}^1_1[^k_\ell$ do not cross each other with respect to Γ^k_ℓ we would infer that also $\mathtt{F}^2_{j_0}\mathrel{\backslash\!\backslash}^k_\ell\mathtt{F}^2_{j_0+1}$. That would contradict the assumption $\mathtt{E}^2_{j_0}\pitchfork^{m_{z_2}}_{n_{z_2}}\mathtt{E}^2_{j_0+1}$ because $\gamma_{\mathsf{Y}_{z_2},\ell}^{\;k}$ is monotonic with respect to $\Gamma^{m_{z_2}}_{n_{z_2}}$ and Γ^k_ℓ by Lemma 4.2 (d). Hence, $]\mathtt{b}^1_2,\mathtt{b}^1_1[^k_\ell\cap\mathtt{F}^2_{j_0}$ is non-empty.

Step 3.2.4: Definition of i_0. We can define i_0 to be the maximal element of the set $\{i\in[\![t_1]\!]\wedge\,]\mathtt{c}_2,\mathtt{c}_1[^k_\ell\cap\mathtt{F}^1_i\neq\varnothing\}$ once we show that this set is non-empty. And, indeed, since $(\mathtt{b}^1_1\mid\mathtt{c}_1\mid\mathtt{b}^1_2)^k_\ell$ by the definition of $\mathtt{c}_1$ and since the cyclicity of Γ^k_ℓ lets us infer $(\mathtt{b}^1_1\mid\mathtt{b}^1_2\mid\mathtt{c}_2)^k_\ell$ from $(\mathtt{b}^1_2\mid\mathtt{c}_2\mid\mathtt{b}^1_1)^k_\ell$, which holds by the definition of $\mathtt{c}_2$, the transitivity of Γ^k_ℓ implies $(\mathtt{b}^1_1\mid\mathtt{c}_1\mid\mathtt{c}_2)^k_\ell$ or, equivalently, $(\mathtt{c}_2\mid\mathtt{b}^1_1\mid\mathtt{c}_1)^k_\ell$ by cyclicity. Because $\mathtt{b}^1_1=\gamma_{\mathsf{Y}_{z_1},\ell}^{\;k}(\mathtt{a}^1_1)\in\gamma_{\mathsf{Y}_{z_1},\ell}^{\;k\to}(\mathtt{E}^1_1)=\mathtt{F}^1_1$ by $\mathtt{a}^1_1\in\mathtt{E}^1_1$ that proves the asserted non-emptiness.

Step 3.2.5: Proof of $\mathtt{F}^1_{i_0}\pitchfork^k_\ell\mathtt{F}^2_{j_0}$. Already by definition it holds that $]\mathtt{c}_2,\mathtt{c}_1[^k_\ell\cap\mathtt{F}^1_{i_0}\neq\varnothing$. If $]\mathtt{c}_1,\mathtt{c}_2[^k_\ell\cap\mathtt{F}^1_{i_0}$ was empty, that would imply $i_0\neq t_1$ for the following two reasons. First, $\mathtt{b}^1_2=\gamma_{\mathsf{Y}_{z_1},\ell}^{\;k}(\mathtt{a}^1_2)\in\gamma_{\mathsf{Y}_{z_1},\ell}^{\;k\to}(\mathtt{E}^1_{t_1})=\mathtt{F}^1_{t_1}$ by $\mathtt{a}^1_2\in\mathtt{E}^1_{t_1}$. Second, since the definitions of $\mathtt{c}_2$ and $\mathtt{c}_1$ ensure $(\mathtt{b}^1_2\mid\mathtt{c}_2\mid\mathtt{b}^1_1)^k_\ell$ and $(\mathtt{b}^1_1\mid\mathtt{c}_1\mid\mathtt{b}^1_2)^k_\ell$, respectively, the latter of which is equivalent to $(\mathtt{b}^1_2\mid\mathtt{b}^1_1\mid\mathtt{c}_1)^k_\ell$ by cyclicity of Γ^k_ℓ, the transitivity of Γ^k_ℓ lets us conclude $(\mathtt{b}^1_2\mid\mathtt{c}_2\mid\mathtt{c}_1)^k_\ell$ or, equivalently, $(\mathtt{c}_1\mid\mathtt{b}^1_2\mid\mathtt{c}_2)^k_\ell$ by cyclicity. By $i_0<t_1$, the maximality of i_0 would then require $]\mathtt{c}_2,\mathtt{c}_1[^k_\ell\cap\mathtt{F}^1_{i_0+1}=\varnothing$, which would mean that $\mathtt{F}^1_{i_0}\subseteq\,]\mathtt{c}_2,\mathtt{c}_1[^k_\ell$ and $\mathtt{F}^2_{i_0}\subseteq\,]\mathtt{c}_1,\mathtt{c}_2[^k_\ell$. Because $]\mathtt{c}_2,\mathtt{c}_1[^k_\ell$ and $]\mathtt{c}_1,\mathtt{c}_2[^k_\ell$ are non-crossing with respect to Γ^k_ℓ it would follow that also $\mathtt{F}^1_{i_0}\mathrel{\backslash\!\backslash}^k_\ell\mathtt{F}^1_{i_0+1}$, contradicting the assumption $\mathtt{E}^1_{i_0}\pitchfork^{m_{z_1}}_{n_{z_1}}\mathtt{E}^1_{i_0+1}$. Therefore, $]\mathtt{c}_1,\mathtt{c}_2[^k_\ell\cap\mathtt{F}^1_{i_0}$ must be non-empty instead, proving that $\mathtt{F}^1_{i_0}\pitchfork^k_\ell\mathtt{F}^2_{j_0}$ since $\{\mathtt{c}_1,\mathtt{c}_2\}\subseteq\mathtt{F}^2_{j_0}$. That is the contradiction we sought. Thus, $\mathsf{H}_1\mathrel{\backslash\!\backslash}^k_\ell\mathsf{H}_2$ also in Case 3.2. Altogether, h meets the non-crossing conditions for being a history.

Step 4: h is coarser than p. Because $p\leq\ker(\xi^{\mathfrak{c}}_{\mathfrak{d}})$ by $p\in\star^r_{z\in\mathfrak{U}\cup\mathfrak{D}}\mathcal{X}_z$, for any $\mathsf{B}\in p$ there exists $z\in\mathfrak{U}\cup\mathfrak{D}$ such that $\mathsf{B}\subseteq\mathsf{Y}_z$. Because $\mathsf{B}\cap\mathsf{Y}_z=\mathsf{B}\neq\varnothing$ it follows $\mathsf{A}\coloneqq\gamma_{\mathsf{Y}_z,\ell}^{\;k\leftarrow}(\mathsf{B})\in R(p,\mathsf{Y}_z)$. Since $R(p,\mathsf{Y}_z)$ refines the connected components of $\Pi^{m_z}_{n_z}$ with respect to $\Gamma^{m_z}_{n_z}$ and $R(p,\mathsf{Y}_z)$ by Proposition 4.33, there must then exist such a connected component C with $\mathsf{A}\subseteq\mathsf{C}$. Because $\mathsf{B}\subseteq\mathsf{Y}_z$ that implies $\mathsf{B}=(\gamma_{\mathsf{Y}_z,\ell}^{\;k\to}\circ\gamma_{\mathsf{Y}_z,\ell}^{\;k\leftarrow})(\mathsf{B})=\gamma_{\mathsf{Y}_z,\ell}^{\;k\to}(\mathsf{A})\subseteq\gamma_{\mathsf{Y}_z,\ell}^{\;k\to}(\mathsf{C})\in h$. Hence, indeed, $p\leq h$.

Step 5: h meets the restriction conditions for being a history. Finally, let $z\in\mathfrak{U}\cup\mathfrak{D}$ and let C be any connected component of $\Pi^{m_z}_{n_z}$ with respect to $\Gamma^{m_z}_{n_z}$ and $R(p,\mathsf{Y}_z)$.

By assumption on $(\mathfrak{c},\mathfrak{d},p)$ then, $R((\mathfrak{c},\mathfrak{d},p),\mathsf{Y}_z)\in\mathcal{X}_z$. Since the connected components of $\Pi_{n_z}^{m_z}$ with respect to $\Gamma_{n_z}^{m_z}$ and $R(p,\mathsf{Y}_z)$ are non-crossing with respect to $\Gamma_{n_z}^{m_z}$ and coarser than $R(p,\mathsf{Y}_z)$ by Proposition 4.33 and since $\mathcal{X}_z$ is ⊗-elbats Lemma 4.30 guarantees that $R(R((\mathfrak{c},\mathfrak{d},p),\mathsf{Y}_z),\mathsf{C})\in\mathcal{X}_z$. Since $R((\mathfrak{c},\mathfrak{d},p),\gamma_{\mathsf{Y}_z,\ell\to}^{k}(\mathsf{C}))=R(R((\mathfrak{c},\mathfrak{d},p),\mathsf{Y}_z),\mathsf{C})$ by Lemma 4.5 (d) that proves $R((\mathfrak{c},\mathfrak{d},p),\gamma_{\mathsf{Y}_z,\ell\to}^{k}(\mathsf{C}))\in\mathcal{X}_z$. Thus h is a history for $(\mathfrak{c},\mathfrak{d},p)$, which concludes the proof. □

4.5. Crossed co-products with cyclic groups. The final general construction of categories of labeled partitions is that of a crossed co-product with a category representing a cyclic group. The partitions of the first factor are required to be tagged with the elements of the cyclic group. The second factor in the crossed co-product will be one of the following sets of two-colored partitions (in the sense of Remark 3.12 (b)) singly tagged with the unitary tag ℵ.

Definition 4.36. (a) For any $w\in\mathbb{N}$ let $\mathcal{Z}_w$ be the set of all $(\mathfrak{c},\mathfrak{d},p)\in{}_{\{\aleph\},\varnothing}\mathcal{S}$ such that ${}_{\aleph}\Sigma_{\mathfrak{d}}^{\mathfrak{c}}\equiv_w 0$ and such that for any $\mathsf{B}\in p$ always $|\mathsf{B}|\le 2$ and, if $|\mathsf{A}|=2$, then ${}_{\aleph}\sigma_{\mathfrak{d}}^{\mathfrak{c}}(\mathsf{A})=0$.

(b) Moreover, let $\mathcal{Z}_0$ be the set of all $(\mathfrak{c},\mathfrak{d},p)\in{}_{\{\aleph\},\varnothing}\mathcal{S}$ for any $\mathsf{B}\in p$ both $|\mathsf{A}|=2$ and ${}_{\aleph}\sigma_{\mathfrak{d}}^{\mathfrak{c}}(\mathsf{A})=0$.

Of course, $\mathcal{Z}_0$ is simply $\mathcal{U}$ from Definition 7.10 (a) below, seen as tagged with the single tag ℵ.

Remark 4.37. For $w\in\mathbb{N}$ the category $\mathcal{Z}_w$ was covered in terms of generators by the group case classification in [TW18, Theorem 8.3] under the name $\mathcal{B}_{\mathrm{grp,loc}}(w)$. It is also treated in [MW21b] as $\mathcal{R}_Q$ for $Q=(\{1,2\},\pm\{0,1\},w\mathbb{Z},\varnothing,w\mathbb{Z},\mathbb{Z})$.

Lemma 4.38. *(a) $\mathcal{Z}_w$ is a category of $(\{\aleph\},\varnothing)$-tagged labeled partitions for any $w\in\mathbb{N}$.*

(b) $\mathcal{Z}_0$ is a category of $(\{\aleph\},\varnothing)$-tagged labeled partitions.

Proof. A proof of (a) can be found in [MW21b, Theorem 6.20] (tenth row of the table, with $u=1$). And one of (b) is provided, e.g., in [MW20, Propostion 5.3], where $\mathcal{Z}_0$ is referred to as $\mathcal{S}_1$. □

The defining condition of the crossed co-product category will be that under a certain tag shift, the restriction to the area of the tags pertaining to the first factor category is an element of that category. The shift in question is achieved as follows.

Notation 4.39. If L is either $(\{\aleph\}\otimes\{\circ,\bullet\})\cup\mathbb{Z}$ or $(\mathbb{Z}\cup\{\aleph\})\otimes\{\circ,\bullet\}$, then for any $\{k,\ell\}\subseteq\mathbb{N}_0$ and any $\mathfrak{c}\colon [\![k]\!]\to L$ and $\mathfrak{d}\colon [\![\ell]\!]\to L$, in the following, we write $\varepsilon_{\mathfrak{d}}^{\mathfrak{c}}$ for the mapping $\xi_{\mathfrak{d}}^{\mathfrak{c}\leftarrow}(\mathbb{Z})\to\mathbb{Z}$ defined by

$$^{\blacksquare}i\mapsto\xi_{\mathfrak{d}}^{\mathfrak{c}}(^{\blacksquare}i)-\begin{cases}{}_{\aleph}\sigma_{\mathfrak{d}}^{\mathfrak{c}}(]^{\blacksquare}i,^{\blacksquare}1]_{\ell}^{k}) & |\, i\neq 1\\ 0 & |\, i=1\end{cases}\quad\text{and}\quad {}_{\blacksquare}j\mapsto\xi_{\mathfrak{d}}^{\mathfrak{c}}({}_{\blacksquare}j)+\begin{cases}{}_{\aleph}\sigma_{\mathfrak{d}}^{\mathfrak{c}}([{}_{\blacksquare}1,{}_{\blacksquare}j[_{\ell}^{k}) & |\, j\neq 1\\ 0 & |\, j=1\end{cases}$$

for any $i\in[\![k]\!]$ with $^{\blacksquare}i\in\xi_{\mathfrak{d}}^{\mathfrak{c}\leftarrow}(\mathbb{Z})$ and any $j\in[\![\ell]\!]$ with ${}_{\blacksquare}j\in\xi_{\mathfrak{d}}^{\mathfrak{c}\leftarrow}(\mathbb{Z})$.

In the interest of readability, the formulation of the definition of the crossed co-product category exploits a particular choice of presentations of the cyclic groups.

NOTATION 4.40. For any $w \in \mathbb{N}$ we work with the presentation $\mathbb{Z}_w$ of the cyclic group of order w given by the set $\{0, 1, \ldots, w-1\}$ equipped with the group law $+_w$ defined by $(s,t) \mapsto s+t$ if $s+t < w$ and $(s,t) \mapsto s+t-w$ otherwise. Also, let π_w be the group homomorphism $\mathbb{Z} \to \mathbb{Z}_w$ with $1 \mapsto 1$.

Moreover, we also write $\mathbb{Z}_0$ for $\mathbb{Z}$ (and $+_0$ for $+$) and π_0 for the identity on $\mathbb{Z}$.

In order for the defining condition of the crossed co-product category to be stable, the first factor category needs to be closed under shifting all tags of a given partition simultaneously, by any amount and in any direction.

DEFINITION 4.41. For any $w \in \mathbb{N}_0$, if $(\mathfrak{U}, \mathfrak{D}) \in \{(\mathbb{Z}_w, \varnothing), (\varnothing, \mathbb{Z}_w)\}$, if V is the left action of $\mathbb{Z}_w$ on $(\mathfrak{U} \otimes \{\circ, \bullet\}) \cup \mathfrak{D}$ with $(z,(x,c)) \mapsto V_z(x,c) \equiv (x +_w z, c)$ for any $x \in \mathfrak{U}$ and $c \in \{\circ, \bullet\}$ and with $(z,y) \mapsto V_z(y) \equiv y +_w z$ for any $y \in \mathfrak{D}$ and for any $z \in \mathbb{Z}_w$, then any category $\mathcal{X}$ of $(\mathfrak{U}, \mathfrak{D})$-tagged labeled partition will be called $\mathbb{Z}_w$*-invariant* if $(V_z \circ \mathfrak{f}, V_z \circ \mathfrak{g}, s) \in \mathcal{X}$ for any $(\mathfrak{f}, \mathfrak{g}, s) \in \mathcal{X}$ and any $z \in \mathbb{Z}_w$.

Because our particular choice of presentations for the cyclic groups enables us to say that $\mathbb{Z}_w$ is a subset of $\mathbb{Z}$ for any $w \in \mathbb{N}_0$, the following definition makes sense.

DEFINITION 4.42. For any $w \in \mathbb{N}_0$, if $(\mathfrak{U}, \mathfrak{D}) \in \{(\mathbb{Z}_w, \varnothing), (\varnothing, \mathbb{Z}_w)\}$, then for any $\mathbb{Z}_w$-invariant category $\mathcal{X}$ of $(\mathfrak{U}, \mathfrak{D})$-tagged labeled partitions the *crossed co-product category* $\mathcal{X} \rtimes \mathcal{Z}_w$ of $\mathcal{X}$ with $\mathcal{Z}_w$ is the set of all $(\mathfrak{c}, \mathfrak{d}, p) \in {}_{\mathfrak{U} \cup \{\aleph\}, \mathfrak{D}}\mathcal{S}$ for which there exist $\{k, \ell\} \subseteq \mathbb{N}_0$ such that $\mathfrak{c}\colon [\![k]\!] \to ((\mathfrak{U} \cup \{\aleph\}) \otimes \{\circ, \bullet\}) \cup \mathfrak{D}$ and $\mathfrak{d}\colon [\![\ell]\!] \to ((\mathfrak{U} \cup \{\aleph\}) \otimes \{\circ, \bullet\}) \cup \mathfrak{D}$, such that $p \leq \xi_{\mathfrak{d}}^{\mathfrak{c}\leftarrow}(\{\mathfrak{U} \cup \mathfrak{D}, \{\aleph\}\})$, such that $R((\mathfrak{c}, \mathfrak{d}, p), \xi_{\mathfrak{d}}^{\mathfrak{c}\leftarrow}(\{\aleph\})) \in \mathcal{Z}_w$ and such that, if $\mathsf{Y} = \xi_{\mathfrak{d}}^{\mathfrak{c}\leftarrow}(\mathfrak{U} \cup \mathfrak{D})$ and if the labelings $\mathfrak{u}$ and $\mathfrak{v}$ are such that $\xi_{\mathfrak{v}}^{\mathfrak{u}} = \pi_w \circ \varepsilon_{\mathfrak{d}}^{\mathfrak{c}} \circ \gamma_{\mathsf{Y},\ell}^{\;k}$ and $\zeta_{\mathfrak{v}}^{\mathfrak{u}} = \zeta_{\mathfrak{d}}^{\mathfrak{c}} \circ \gamma_{\mathsf{Y},\ell}^{\;k}$, then $(\mathfrak{u}, \mathfrak{v}, R(p, \mathsf{Y})) \in \mathcal{X}$.

The following auxiliary result will aid the proof that the above definition does indeed yield a category.

LEMMA 4.43. *If $\mathfrak{L}$ is either $(\{\aleph\} \otimes \{\circ, \bullet\}) \cup \mathbb{Z}$ or $(\mathbb{Z} \cup \{\aleph\}) \otimes \{\circ, \bullet\}$, then given any $\{k, \ell\} \subseteq \mathbb{N}_0$, any $\mathfrak{c}\colon [\![k]\!] \to \mathfrak{L}$ and $\mathfrak{d}\colon [\![\ell]\!] \to \mathfrak{L}$ as well as any $\{k_t, \ell_t\} \subseteq \mathbb{N}_0$ and any $\mathfrak{c}_t\colon [\![k_t]\!] \to \mathfrak{L}$ and $\mathfrak{d}_t\colon [\![\ell_t]\!] \to \mathfrak{L}$ for each $t \in [\![2]\!]$, the following hold.*

(a) *For any* $r \in \{\Lsh, \Rsh, \curvearrowleft, \curvearrowright\}$, *if* $\mathfrak{a}\,{}^{\bullet}_{\bullet}\,\mathfrak{b} = (\mathfrak{c}\,{}^{\bullet}_{\bullet}\,\mathfrak{d})^r$ *is defined and if* $\mathtt{e}$ *is given by* ${}^{\bullet}1$ *if* $r = \Lsh$, *by* ${}^{\bullet}k$ *if* $r = \curvearrowright$, *by* ${}_{\bullet}1$ *if* $r = \curvearrowleft$ *and by* ${}_{\bullet}\ell$ *if* $r = \Rsh$, *then*

$$(\varepsilon^{\mathfrak{a}}_{\mathfrak{b}} \circ (\omega^{r,k}_{\ell})^{-1})(\mathtt{t}) = \varepsilon^{\mathfrak{c}}_{\mathfrak{d}}(\mathtt{t}) + \begin{cases} {}_{\aleph}\sigma^{\mathfrak{c}}_{\mathfrak{d}}(\{\mathtt{e}\}) & \text{if } r = \Lsh \wedge \mathtt{t} \neq \mathtt{e} \\ 0 & \text{if } r = \Lsh \wedge \mathtt{t} = \mathtt{e} \\ 0 & \text{if } r = \curvearrowright \wedge \mathtt{t} \neq \mathtt{e} \\ -{}_{\aleph}\sigma^{\mathfrak{c}}_{\mathfrak{d}}(\{\mathtt{e}\}) + {}_{\aleph}\Sigma^{\mathfrak{c}}_{\mathfrak{d}} & \text{if } r = \curvearrowright \wedge \mathtt{t} = \mathtt{e} \\ -{}_{\aleph}\sigma^{\mathfrak{c}}_{\mathfrak{d}}(\{\mathtt{e}\}) & \text{if } r = \curvearrowleft \wedge \mathtt{t} \neq \mathtt{e} \\ 0 & \text{if } r = \curvearrowleft \wedge \mathtt{t} = \mathtt{e} \\ 0 & \text{if } r = \Rsh \wedge \mathtt{t} \neq \mathtt{e} \\ {}_{\aleph}\sigma^{\mathfrak{c}}_{\mathfrak{d}}(\{\mathtt{e}\}) - {}_{\aleph}\Sigma^{\mathfrak{c}}_{\mathfrak{d}} & \text{if } r = \Rsh \wedge \mathtt{t} = \mathtt{e} \end{cases}$$

for any $\mathtt{t} \in \xi^{\mathfrak{c}\leftarrow}_{\mathfrak{d}}(\mathbb{Z})$.

(b) *If* $\mathfrak{a}\,{}^{\bullet}_{\bullet}\,\mathfrak{b} = (\mathfrak{c}\,{}^{\bullet}_{\bullet}\,\mathfrak{d})^*$, *then*

$$\varepsilon^{\mathfrak{a}}_{\mathfrak{b}} = \varepsilon^{\mathfrak{c}}_{\mathfrak{d}} \circ \kappa^k_\ell.$$

(c) *If* $\mathsf{H}_1 = \Pi^{k_1}_{\ell_1}$ *and* $\mathsf{H}_2 = \Pi^{k_1+k_2}_{\ell_1+\ell_2} \backslash \Pi^{k_1}_{\ell_1}$ *and if* $\mathfrak{a}\,{}^{\bullet}_{\bullet}\,\mathfrak{b} = (\mathfrak{c}_1\,{}^{\bullet}_{\bullet}\,\mathfrak{d}_1) \otimes (\mathfrak{c}_2\,{}^{\bullet}_{\bullet}\,\mathfrak{d}_2)$, *then for each* $t \in [\![2]\!]$,

$$(\varepsilon^{\mathfrak{a}}_{\mathfrak{b}} \circ \gamma_{\mathsf{H}_t, \ell_1+\ell_2}^{\;k_1+k_2})(\mathtt{s}) = \varepsilon^{\mathfrak{c}_t}_{\mathfrak{d}_t}(\mathtt{s}) + \begin{cases} 0 & \text{if } t = 1 \\ -{}_{\aleph}\Sigma^{\mathfrak{c}_1}_{\varnothing} & \text{if } t = 2 \wedge \mathtt{s} \in \Pi^{k_2}_0 \\ -{}_{\aleph}\Sigma^{\mathfrak{c}_1}_{\varnothing} + {}_{\aleph}\Sigma^{\mathfrak{c}_1}_{\mathfrak{d}_1} & \text{if } t = 2 \wedge \mathtt{s} \in \Pi^0_{\ell_2} \end{cases}$$

for any $\mathtt{s} \in \xi^{\mathfrak{c}_t\leftarrow}_{\mathfrak{d}_t}(\mathbb{Z})$.

(d) *For any* $\mathsf{S} \subseteq \Pi^k_\ell$, *if* $\mathfrak{a}\,{}^{\bullet}_{\bullet}\,\mathfrak{b} = R(\mathfrak{c}\,{}^{\bullet}_{\bullet}\,\mathfrak{d}, \mathsf{S})$, *if* $x = \alpha(\mathsf{S})$ *and* $y = \beta(\mathsf{S})$, *then*

$$\begin{aligned}&(\varepsilon^{\mathfrak{c}}_{\mathfrak{d}} \circ \gamma_{\mathsf{S},\ell}^{\;k})(\mathtt{s}) \\ &= \varepsilon^{\mathfrak{a}}_{\mathfrak{b}}(\mathtt{s}) + \begin{cases} 0 & \text{if } ({}_{\bullet}1 \in \mathsf{S} \wedge \mathtt{s} = {}_{\bullet}1) \\ & \quad \vee ({}^{\bullet}1 \in \mathsf{S} \wedge \mathtt{s} = {}^{\bullet}1) \\ -{}_{\aleph}\sigma^{\mathfrak{c}}_{\mathfrak{d}}(]\gamma_{\mathsf{S},\ell}^{\;k}(\mathtt{s}), {}^{\bullet}1]^k_\ell \backslash \mathsf{S}) & \text{if } ({}^{\bullet}1 \notin \mathsf{S} \vee \mathtt{s} \neq {}^{\bullet}1) \wedge \mathtt{s} \in \Pi^x_0 \\ {}_{\aleph}\sigma^{\mathfrak{c}}_{\mathfrak{d}}([{}_{\bullet}1, \gamma_{\mathsf{S},\ell}^{\;k}(\mathtt{s})[^k_\ell \backslash \mathsf{S}) & \text{if } ({}_{\bullet}1 \notin \mathsf{S} \vee \mathtt{s} \neq {}_{\bullet}1) \wedge \mathtt{s} \in \Pi^0_y \end{cases}\end{aligned}$$

for any $\mathtt{s} \in \Pi^x_y$.

Proof. (a) Only the cases $r \in \{\Lsh, \curvearrowright\}$ need proving because the others then follow from (b) by exchaning the roles $k \leftrightarrow \ell$ and $\mathfrak{c} \leftrightarrow \mathfrak{d}$ and applying Lemma 4.2 (b).

Case $r = \Lsh$: Recall that $(\omega^{\Lsh,k}_\ell)^{-1}$ satisfies ${}^{\bullet}1 \mapsto {}_{\bullet}1$ and ${}^{\bullet}i \mapsto {}^{\bullet}(i-1)$ and ${}_{\bullet}j \mapsto {}_{\bullet}(j+1)$ for any $i \in [\![k]\!]$ with $i \neq 1$ and any $j \in [\![\ell]\!]$. It follows

$$(\varepsilon^{\mathfrak{a}}_{\mathfrak{b}} \circ (\omega^{\Lsh,k}_\ell)^{-1})({}^{\bullet}1) = \varepsilon^{\mathfrak{a}}_{\mathfrak{b}}({}_{\bullet}1) = \xi^{\mathfrak{a}}_{\mathfrak{b}}({}_{\bullet}1) = \xi^{\mathfrak{c}}_{\mathfrak{d}}(\omega^{\Lsh,k}_\ell({}_{\bullet}1)) = \xi^{\mathfrak{c}}_{\mathfrak{d}}({}^{\bullet}1) = \varepsilon^{\mathfrak{c}}_{\mathfrak{d}}({}^{\bullet}1),$$

where we have used the definitions of $\varepsilon^{\mathfrak{a}}_{\mathfrak{b}}$ and $\varepsilon^{\mathfrak{c}}_{\mathfrak{d}}$ in the second and last step and Lemma 4.2 (a) in the third step.

Because $\omega^{\Lsh,k}_\ell$ and its inverse are monotonic with respect to $\Gamma^{k-1}_{\ell+1}$ and Γ^k_ℓ and because ${}_{\aleph}\sigma^{\mathfrak{a}}_{\mathfrak{b}}$ is the pull-back measure of ${}_{\aleph}\sigma^{\mathfrak{c}}_{\mathfrak{d}}$ with respect to $\omega^{\Lsh,k}_\ell$, we find for any

$i \in [\![k]\!]$ with $i \neq 1$,

$$
\begin{aligned}
(\varepsilon^{\mathfrak{a}}_{\mathfrak{b}} \circ (\omega^{\circlearrowleft,k}_{\ell})^{-1})({}^{\blacksquare}i) &= \xi^{\mathfrak{a}}_{\mathfrak{b}}({}^{\blacksquare}(i-1)) - \begin{cases} {}_{\aleph}\sigma^{\mathfrak{a}}_{\mathfrak{b}}(]\,{}^{\blacksquare}(i-1), {}^{\blacksquare}1]^{k-1}_{\ell+1}) & \text{if } i-1 \neq 1 \\ 0 & \text{otherwise} \end{cases} \\
&= \xi^{\mathfrak{c}}_{\mathfrak{d}}({}^{\blacksquare}i) - \begin{cases} {}_{\aleph}\sigma^{\mathfrak{c}}_{\mathfrak{d}}(]\,{}^{\blacksquare}i, {}^{\blacksquare}2]^{k}_{\ell}) & \text{if } i \neq 2 \\ 0 & \text{otherwise} \end{cases} \\
&= \xi^{\mathfrak{c}}_{\mathfrak{d}}({}^{\blacksquare}i) - {}_{\aleph}\sigma^{\mathfrak{c}}_{\mathfrak{d}}(]\,{}^{\blacksquare}i, {}^{\blacksquare}1]^{k}_{\ell}) + {}_{\aleph}\sigma^{\mathfrak{c}}_{\mathfrak{d}}(\{{}^{\blacksquare}1\}) \\
&= \varepsilon^{\mathfrak{c}}_{\mathfrak{d}}({}^{\blacksquare}i) + {}_{\aleph}\sigma^{\mathfrak{c}}_{\mathfrak{d}}(\{{}^{\blacksquare}1\}),
\end{aligned}
$$

where we have used the decomposition $]\,{}^{\blacksquare}i, {}^{\blacksquare}1]^{k}_{\ell} =]\,{}^{\blacksquare}i, {}^{\blacksquare}2]^{k}_{\ell} \cup \{{}^{\blacksquare}1\}$ if $i \neq 2$ and the identity $]\,{}^{\blacksquare}2, {}^{\blacksquare}1]^{k}_{\ell} = \{{}^{\blacksquare}1\}$ as well as the additivity of ${}_{\aleph}\sigma^{\mathfrak{c}}_{\mathfrak{d}}$ in the next-to-last step.

For any $j \in [\![\ell]\!]$ we can compute similarly that

$$
\begin{aligned}
&(\varepsilon^{\mathfrak{a}}_{\mathfrak{b}} \circ (\omega^{\circlearrowleft,k}_{\ell})^{-1})({}_{\blacksquare}j) \\
&\quad= \xi^{\mathfrak{a}}_{\mathfrak{b}}({}_{\blacksquare}(j+1)) + {}_{\aleph}\sigma^{\mathfrak{a}}_{\mathfrak{b}}(]\,{}_{\blacksquare}1, {}_{\blacksquare}(j+1)]^{k-1}_{\ell+1}) \\
&\quad= \xi^{\mathfrak{a}}_{\mathfrak{b}}({}_{\blacksquare}(j+1)) + \left\{\begin{matrix} {}_{\aleph}\sigma^{\mathfrak{a}}_{\mathfrak{b}}(]\,{}_{\blacksquare}2, {}_{\blacksquare}(j+1)]^{k-1}_{\ell+1}) & \text{if } j+1 \neq 2 \\ 0 & \text{otherwise} \end{matrix}\right\} + {}_{\aleph}\sigma^{\mathfrak{a}}_{\mathfrak{b}}(\{{}_{\blacksquare}1\}) \\
&\quad= \xi^{\mathfrak{c}}_{\mathfrak{d}}({}_{\blacksquare}j) + \left\{\begin{matrix} {}_{\aleph}\sigma^{\mathfrak{c}}_{\mathfrak{d}}(]\,{}_{\blacksquare}1, {}_{\blacksquare}j]^{k}_{\ell}) & \text{if } j \neq 1 \\ 0 & \text{otherwise} \end{matrix}\right\} + {}_{\aleph}\sigma^{\mathfrak{c}}_{\mathfrak{d}}(\{{}^{\blacksquare}1\}) \\
&\quad= \varepsilon^{\mathfrak{c}}_{\mathfrak{d}}({}_{\blacksquare}j) + {}_{\aleph}\sigma^{\mathfrak{c}}_{\mathfrak{d}}(\{{}^{\blacksquare}1\}).
\end{aligned}
$$

Thus, the claim holds in this case.

Case $r = \circlearrowright$*:* The mapping $(\omega^{\circlearrowright,k}_{\ell})^{-1}$ is defined by the rule that ${}^{\blacksquare}i \mapsto {}^{\blacksquare}i$ and ${}^{\blacksquare}k \mapsto {}_{\blacksquare}(\ell+1)$ and ${}^{\blacksquare}j \mapsto {}^{\blacksquare}j$ for any $i \in [\![k]\!]$ with $i \neq k$ and any $j \in [\![\ell]\!]$. First of all, because ${}_{\aleph}\Sigma^{\mathfrak{c}}_{\mathfrak{d}} = {}_{\aleph}\sigma^{\mathfrak{c}}_{\mathfrak{d}}(\Pi^{k}_{0}) + {}_{\aleph}\sigma^{\mathfrak{c}}_{\mathfrak{d}}(\Pi^{0}_{\ell})$, similarly to the other computations,

$$
\begin{aligned}
(\varepsilon^{\mathfrak{a}}_{\mathfrak{b}} \circ (\omega^{\circlearrowleft,k}_{\ell})^{-1})({}^{\blacksquare}k) &= \xi^{\mathfrak{a}}_{\mathfrak{b}}({}_{\blacksquare}(\ell+1)) + \begin{cases} {}_{\aleph}\sigma^{\mathfrak{a}}_{\mathfrak{b}}([\,{}_{\blacksquare}1, {}_{\blacksquare}(\ell+1)[^{k-1}_{\ell+1}) & \text{if } \ell+1 \neq 1 \\ 0 & \text{otherwise} \end{cases} \\
&= \xi^{\mathfrak{a}}_{\mathfrak{b}}({}_{\blacksquare}(\ell+1)) + {}_{\aleph}\sigma^{\mathfrak{a}}_{\mathfrak{b}}(\Pi^{0}_{\ell}) \\
&= \xi^{\mathfrak{c}}_{\mathfrak{d}}({}^{\blacksquare}k) + {}_{\aleph}\sigma^{\mathfrak{c}}_{\mathfrak{d}}(\Pi^{0}_{\ell}) \\
&= \xi^{\mathfrak{c}}_{\mathfrak{d}}({}^{\blacksquare}k) - {}_{\aleph}\sigma^{\mathfrak{c}}_{\mathfrak{d}}(\Pi^{k}_{0}) + {}_{\aleph}\Sigma^{\mathfrak{c}}_{\mathfrak{d}} \\
&= \xi^{\mathfrak{c}}_{\mathfrak{d}}({}^{\blacksquare}k) - \left\{\begin{matrix} {}_{\aleph}\sigma^{\mathfrak{c}}_{\mathfrak{d}}([\,{}^{\blacksquare}k, {}^{\blacksquare}1]^{k}_{\ell}) & \text{if } k \neq 1 \\ {}_{\aleph}\sigma^{\mathfrak{c}}_{\mathfrak{d}}(\{{}^{\blacksquare}1\}) & \text{otherwise} \end{matrix}\right\} + {}_{\aleph}\Sigma^{\mathfrak{c}}_{\mathfrak{d}} \\
&= \xi^{\mathfrak{c}}_{\mathfrak{d}}({}^{\blacksquare}k) - \left\{\begin{matrix} {}_{\aleph}\sigma^{\mathfrak{c}}_{\mathfrak{d}}(]\,{}^{\blacksquare}k, {}^{\blacksquare}1]^{k}_{\ell}) & \text{if } k \neq 1 \\ 0 & \text{otherwise} \end{matrix}\right\} - {}_{\aleph}\sigma^{\mathfrak{c}}_{\mathfrak{d}}(\{{}^{\blacksquare}1\}) + {}_{\aleph}\Sigma^{\mathfrak{c}}_{\mathfrak{d}} \\
&= \varepsilon^{\mathfrak{c}}_{\mathfrak{d}}({}^{\blacksquare}k) - {}_{\aleph}\sigma^{\mathfrak{c}}_{\mathfrak{d}}(\{{}^{\blacksquare}k\}) + {}_{\aleph}\Sigma^{\mathfrak{c}}_{\mathfrak{d}}.
\end{aligned}
$$

On the other hand, for any $i \in \llbracket k \rrbracket$ with $i \neq k$,

$$
\begin{aligned}
(\varepsilon^{\mathfrak{a}}_{\mathfrak{b}} \circ (\omega^{\mathfrak{z},k}_{\ell})^{-1})({}^{\blacksquare}i) &= \xi^{\mathfrak{a}}_{\mathfrak{b}}({}^{\blacksquare}i) - \begin{cases} {}_{\aleph}\sigma^{\mathfrak{a}}_{\mathfrak{b}}(]{}^{\blacksquare}i, {}^{\blacksquare}1]^{k-1}_{\ell+1}) & \text{if } i \neq 1 \\ 0 & \text{otherwise} \end{cases} \\
&= \xi^{\mathfrak{c}}_{\mathfrak{d}}({}^{\blacksquare}i) - \begin{cases} {}_{\aleph}\sigma^{\mathfrak{c}}_{\mathfrak{d}}(]{}^{\blacksquare}i, {}^{\blacksquare}1]^{k}_{\ell}) & \text{if } i \neq 1 \\ 0 & \text{otherwise} \end{cases} \\
&= \varepsilon^{\mathfrak{c}}_{\mathfrak{d}}({}^{\blacksquare}i),
\end{aligned}
$$

and for any $j \in \llbracket \ell \rrbracket$, analogously,

$$
\begin{aligned}
(\varepsilon^{\mathfrak{a}}_{\mathfrak{b}} \circ (\omega^{\mathfrak{z},k}_{\ell})^{-1})({}_{\blacksquare}j) &= \xi^{\mathfrak{a}}_{\mathfrak{b}}({}_{\blacksquare}j) - \begin{cases} {}_{\aleph}\sigma^{\mathfrak{a}}_{\mathfrak{b}}([{}_{\blacksquare}1, {}_{\blacksquare}j[^{k-1}_{\ell+1}) & \text{if } j \neq 1 \\ 0 & \text{otherwise} \end{cases} \\
&= \xi^{\mathfrak{c}}_{\mathfrak{d}}({}_{\blacksquare}j) - \begin{cases} {}_{\aleph}\sigma^{\mathfrak{c}}_{\mathfrak{d}}([{}_{\blacksquare}1, {}_{\blacksquare}j[^{k}_{\ell}) & \text{if } j \neq 1 \\ 0 & \text{otherwise} \end{cases} \\
&= \varepsilon^{\mathfrak{c}}_{\mathfrak{d}}({}_{\blacksquare}j),
\end{aligned}
$$

which concludes the proof of (a).

(b) The mapping $\kappa^{\mathfrak{a}}_{\mathfrak{b}}$ is anti-monotonic with respect to Γ^{k}_{ℓ} and Γ^{ℓ}_{k}, and so is its inverse. Moreover, ${}_{\aleph}\sigma^{\mathfrak{d}}_{\mathfrak{c}}$ is the negative of the pull-back measure of ${}_{\aleph}\sigma^{\mathfrak{c}}_{\mathfrak{d}}$. In particular, ${}_{\aleph}\sigma^{\mathfrak{a}}_{\mathfrak{b}}([{}_{\blacksquare}1, {}_{\blacksquare}i[^{\ell}_{k}) = -{}_{\aleph}\sigma^{\mathfrak{c}}_{\mathfrak{d}}(]{}^{\blacksquare}i, {}^{\blacksquare}1]^{k}_{\ell})$ and thus

$$
\begin{aligned}
(\varepsilon^{\mathfrak{a}}_{\mathfrak{b}} \circ (\kappa^{k}_{\ell})^{-1})({}^{\blacksquare}i) &= \xi^{\mathfrak{a}}_{\mathfrak{b}}({}_{\blacksquare}i) + \begin{cases} {}_{\aleph}\sigma^{\mathfrak{a}}_{\mathfrak{b}}([{}_{\blacksquare}1, {}_{\blacksquare}i[^{\ell}_{k}) & \text{if } i \neq 1 \\ 0 & \text{otherwise} \end{cases} \\
&= \xi^{\mathfrak{c}}_{\mathfrak{d}}({}_{\blacksquare}i) - \begin{cases} {}_{\aleph}\sigma^{\mathfrak{c}}_{\mathfrak{d}}(]{}^{\blacksquare}i, {}^{\blacksquare}1]^{k}_{\ell}) & \text{if } i \neq 1 \\ 0 & \text{otherwise} \end{cases} \\
&= \varepsilon^{\mathfrak{c}}_{\mathfrak{d}}({}^{\blacksquare}i)
\end{aligned}
$$

for any $i \in \llbracket k \rrbracket$, where we have also used Lemma 4.2 (b) in the second step. S

Completely analogously, for any $j \in \llbracket \ell \rrbracket$, by ${}_{\aleph}\sigma^{\mathfrak{a}}_{\mathfrak{b}}(]{}^{\blacksquare}j, {}^{\blacksquare}1]^{\ell}_{k}) = -{}_{\aleph}\sigma^{\mathfrak{c}}_{\mathfrak{d}}([{}_{\blacksquare}1, {}_{\blacksquare}j[^{k}_{\ell})$,

$$
\begin{aligned}
(\varepsilon^{\mathfrak{a}}_{\mathfrak{b}} \circ (\kappa^{k}_{\ell})^{-1})({}_{\blacksquare}j) &= \xi^{\mathfrak{a}}_{\mathfrak{b}}({}^{\blacksquare}j) - \begin{cases} {}_{\aleph}\sigma^{\mathfrak{a}}_{\mathfrak{b}}(]{}^{\blacksquare}j, {}^{\blacksquare}1]^{\ell}_{k}) & \text{if } j \neq 1 \\ 0 & \text{otherwise} \end{cases} \\
&= \xi^{\mathfrak{c}}_{\mathfrak{d}}({}_{\blacksquare}j) + \begin{cases} {}_{\aleph}\sigma^{\mathfrak{c}}_{\mathfrak{d}}([{}_{\blacksquare}1, {}_{\blacksquare}j[^{k}_{\ell}) & \text{if } j \neq 1 \\ 0 & \text{otherwise} \end{cases} \\
&= \varepsilon^{\mathfrak{c}}_{\mathfrak{d}}({}_{\blacksquare}j),
\end{aligned}
$$

which then verifies (b).

(c) If we abbreviate $x \equiv k_1 + k_2$ and $y \equiv \ell_1 + \ell_2$ and $\gamma_{\mathsf{H}_t} \equiv \gamma_{\mathsf{H}_t,y}{}^{x}$, then γ_{H_t} is monotonic with respect to $\Gamma^{k_t}_{\ell_t}$ and Γ^{x}_{y} and ${}_{\aleph}\sigma^{\mathfrak{c}_t}_{\mathfrak{d}_t}$ is the pull-back measure of ${}_{\aleph}\sigma^{\mathfrak{a}}_{\mathfrak{b}}$ for each $t \in \llbracket 2 \rrbracket$. In fact, ${}_{\aleph}\sigma^{\mathfrak{a}}_{\mathfrak{b}}$ decomposes as ${}_{\aleph}\sigma^{\mathfrak{a}}_{\mathfrak{b}}(\mathsf{D}) = \sum_{t=1}^{2} {}_{\aleph}\sigma^{\mathfrak{c}_t}_{\mathfrak{d}_t}(\gamma^{\leftarrow}_{\mathsf{H}_t}(\mathsf{D}))$ for any $\mathsf{D} \subseteq \Pi^{x}_{y}$.

Hence and because γ_{H_1} is the identity on elements, for any $g \in [\![k_1]\!]$, by $]{}^{\blacksquare}g, {}^{\blacksquare}1]_y^x \cap \mathsf{H}_2 = \varnothing$,

$$\begin{aligned}(\varepsilon_{\mathfrak{b}}^{\mathfrak{a}} \circ \gamma_{\mathsf{H}_1})({}^{\blacksquare}g) &= \xi_{\mathfrak{b}}^{\mathfrak{a}}({}^{\blacksquare}g) - \begin{cases} {}_{\aleph}\sigma_{\mathfrak{b}}^{\mathfrak{a}}(]{}^{\blacksquare}g, {}^{\blacksquare}1]_y^x) & \text{if } g \neq 1 \\ 0 & \text{otherwise}\end{cases} \\ &= \xi_{\mathfrak{d}_1}^{\mathfrak{c}_1}({}^{\blacksquare}g) - \begin{cases} {}_{\aleph}\sigma_{\mathfrak{d}1}^{\mathfrak{c}_1}(]{}^{\blacksquare}g, {}^{\blacksquare}1]_{\ell_1}^{k_1}) & \text{if } g \neq 1 \\ 0 & \text{otherwise}\end{cases} \\ &= \varepsilon_{\mathfrak{d}_1}^{\mathfrak{c}_1}({}^{\blacksquare}g)\end{aligned}$$

where we have also employed Lemma 4.2 (c) at the end. Likewise, for any $h \in [\![\ell]\!]$, from $[{}_{\blacksquare}1, {}_{\blacksquare}h[_y^x \cap \mathsf{H}_2 = \varnothing$ it follows

$$\begin{aligned}(\varepsilon_{\mathfrak{b}}^{\mathfrak{a}} \circ \gamma_{\mathsf{H}_1})({}_{\blacksquare}h) &= \xi_{\mathfrak{b}}^{\mathfrak{a}}({}_{\blacksquare}h) - \begin{cases} {}_{\aleph}\sigma_{\mathfrak{b}}^{\mathfrak{a}}([{}_{\blacksquare}1, {}_{\blacksquare}h[_y^x) & \text{if } h \neq 1 \\ 0 & \text{otherwise}\end{cases} \\ &= \xi_{\mathfrak{d}_1}^{\mathfrak{c}_1}({}_{\blacksquare}h) - \begin{cases} {}_{\aleph}\sigma_{\mathfrak{d}1}^{\mathfrak{c}_1}([{}_{\blacksquare}1, {}_{\blacksquare}h[_{\ell_1}^{k_1}) & \text{if } h \neq 1 \\ 0 & \text{otherwise}\end{cases} \\ &= \varepsilon_{\mathfrak{d}_1}^{\mathfrak{c}_1}({}_{\blacksquare}h).\end{aligned}$$

In contrast, γ_{H_2} is generally not an identity. Namely, it satisfies ${}^{\blacksquare}g \mapsto {}^{\blacksquare}(k_1+g)$ and ${}_{\blacksquare}h \mapsto {}_{\blacksquare}(\ell_1+h)$ for any $g \in [\![k_2]\!]$ and $h \in [\![\ell_2]\!]$. Given any $g \in [\![k_2]\!]$ we have $k_1+g=1$ if and only if $k_1 = 0$ and $g = 1$. Second, given any such g, the set $]{}^{\blacksquare}(k_1+g), {}^{\blacksquare}1]_y^x$ can be decomposed as $]{}^{\blacksquare}(k_1+g), {}^{\blacksquare}(k_1+1)]_y^x \uplus \Pi_0^{k_1}$ if $g \neq 1$ (regardless of whether k_1 is zero or not) and is equal to $\Pi_0^{k_1}$ if $g = 1$ and $k_1 \neq 0$. For those reasons, for any $g \in [\![k_2]\!]$,

$$\begin{aligned}&(\varepsilon_{\mathfrak{b}}^{\mathfrak{a}} \circ \gamma_{\mathsf{H}_1})({}^{\blacksquare}g) \\ &\quad= \xi_{\mathfrak{b}}^{\mathfrak{a}}({}^{\blacksquare}(k_1+g)) - \begin{cases} {}_{\aleph}\sigma_{\mathfrak{b}}^{\mathfrak{a}}(]{}^{\blacksquare}(k_1+g), {}^{\blacksquare}1]_y^x) & \text{if } k_1+g \neq 1 \\ 0 & \text{otherwise}\end{cases} \\ &\quad= \xi_{\mathfrak{b}}^{\mathfrak{a}}({}^{\blacksquare}(k_1+g)) - \begin{cases} {}_{\aleph}\sigma_{\mathfrak{b}}^{\mathfrak{a}}(]{}^{\blacksquare}(k_1+g), {}^{\blacksquare}(k_1+1)]_y^x) + {}_{\aleph}\sigma_{\mathfrak{b}}^{\mathfrak{a}}(\Pi_0^{k_1}) & \text{if } g \neq 1 \\ {}_{\aleph}\sigma_{\mathfrak{b}}^{\mathfrak{a}}(\Pi_0^{k_1}) & \text{if } k_1 \neq 0 \wedge g = 1 \\ 0 & \text{if } k_1 = 0 \wedge g = 1\end{cases} \\ &\quad= \xi_{\mathfrak{d}_2}^{\mathfrak{c}_2}({}^{\blacksquare}g) - \begin{cases} {}_{\aleph}\sigma_{\mathfrak{d}_2}^{\mathfrak{c}_2}(]{}^{\blacksquare}g, {}^{\blacksquare}1]_{\ell_2}^{k_2}) + {}_{\aleph}\sigma_{\mathfrak{d}_1}^{\mathfrak{c}_1}(\Pi_0^{k_1}) & \text{if } g \neq 1 \\ {}_{\aleph}\sigma_{\mathfrak{d}_1}^{\mathfrak{c}_1}(\Pi_0^{k_1}) & \text{otherwise}\end{cases} \\ &\quad= \varepsilon_{\mathfrak{d}_2}^{\mathfrak{c}_2}({}^{\blacksquare}g) - {}_{\aleph}\Sigma_{\varnothing}^{\mathfrak{c}_1},\end{aligned}$$

where, in the third step, we have used the facts that ${}_{\aleph}\sigma_{\mathfrak{b}}^{\mathfrak{a}}(\Pi_0^{k_1}) = 0$ if $k_1 = 0$ that $]{}^{\blacksquare}(k_1+g), {}^{\blacksquare}(k_1+1)]_y^x \subseteq \mathsf{H}_2$ if $g \neq 1$ and that $\Pi_0^{k_1} \subseteq \mathsf{H}_1$.

For lower points the computation is largely analogous but differs in the last step. Namely, given any $h \in [\![\ell_2]\!]$,

$$
\begin{aligned}
&(\varepsilon^{\mathfrak{a}}_{\mathfrak{b}} \circ \gamma_{\mathsf{H}_1})({}_{\blacksquare}h)\\
&\quad = \xi^{\mathfrak{a}}_{\mathfrak{b}}({}_{\blacksquare}(\ell_1+h)) + \begin{cases} {}_{\aleph}\sigma^{\mathfrak{a}}_{\mathfrak{b}}([{}_{\blacksquare}1, {}_{\blacksquare}(\ell_1+h)[^x_y) & \text{if } \ell_1 + h \neq 1 \\ 0 & \text{otherwise}\end{cases}\\
&\quad = \xi^{\mathfrak{a}}_{\mathfrak{b}}({}_{\blacksquare}(\ell_1+h)) + \begin{cases} {}_{\aleph}\sigma^{\mathfrak{a}}_{\mathfrak{b}}(\Pi^0_{\ell_1}) + {}_{\aleph}\sigma^{\mathfrak{a}}_{\mathfrak{b}}([{}_{\blacksquare}(\ell_1+1), {}_{\blacksquare}(\ell_1+h)[^x_y) & \text{if } h \neq 1 \\ {}_{\aleph}\sigma^{\mathfrak{a}}_{\mathfrak{b}}(\Pi^0_{\ell_1}) & \text{if } \ell_1 \neq 0 \wedge h = 1 \\ 0 & \text{if } \ell_1 = 0 \wedge h = 1\end{cases}\\
&\quad = \xi^{\mathfrak{c}_2}_{\mathfrak{d}_2}({}_{\blacksquare}h) + \begin{cases} {}_{\aleph}\sigma^{\mathfrak{c}_1}_{\mathfrak{d}_1}(\Pi^0_{\ell_1}) + {}_{\aleph}\sigma^{\mathfrak{c}_2}_{\mathfrak{d}_2}(]{}^{\blacksquare}g, {}^{\blacksquare}1]^{k_2}_{\ell_2}) + & \text{if } h \neq 1 \\ {}_{\aleph}\sigma^{\mathfrak{c}_1}_{\mathfrak{d}_1}(\Pi^0_{\ell_1}) & \text{otherwise}\end{cases}\\
&\quad = \varepsilon^{\mathfrak{c}_2}_{\mathfrak{d}_2}({}_{\blacksquare}h) + {}_{\aleph}\Sigma^{\varnothing}_{\mathfrak{d}_1}\\
&\quad = \varepsilon^{\mathfrak{c}_2}_{\mathfrak{d}_2}({}_{\blacksquare}h) - {}_{\aleph}\Sigma^{\mathfrak{c}_1}_{\varnothing} + {}_{\aleph}\Sigma^{\mathfrak{c}_1}_{\mathfrak{d}_1},
\end{aligned}
$$

where we have used ${}_{\aleph}\Sigma^{\mathfrak{c}_1}_{\mathfrak{d}_1} = {}_{\aleph}\Sigma^{\mathfrak{c}_1}_{\varnothing} + {}_{\aleph}\Sigma^{\varnothing}_{\mathfrak{d}_1}$ in the last step. With that, the proof of (c) is complete.

(d) If we abbreviate $\gamma_{\mathsf{S}} \equiv \gamma_{\mathsf{S},\ell}^{\ k}$, then γ_{S} is monotonic with respect to Γ^x_y and Γ^k_ℓ and ${}_{\aleph}\sigma^{\mathfrak{a}}_{\mathfrak{b}}$ is the pull-back of ${}_{\aleph}\sigma^{\mathfrak{c}}_{\mathfrak{d}}$. Actually, ${}_{\aleph}\sigma^{\mathfrak{c}}_{\mathfrak{d}}(\mathsf{D}) = {}_{\aleph}\sigma^{\mathfrak{a}}_{\mathfrak{b}}(\gamma_{\mathsf{S}}{}^{\leftarrow}(\mathsf{D})) + {}_{\aleph}\sigma^{\mathfrak{c}}_{\mathfrak{d}}(\mathsf{D}\backslash\mathsf{S})$ for any $\mathsf{D} \subseteq \Pi^k_\ell$, which we will use momentarily. Moreover, given any $i \in [\![k]\!]$ and any $g \in [\![x]\!]$, by definition, $\gamma_{\mathsf{S}}({}^{\blacksquare}g) = {}^{\blacksquare}i$ if and only if ${}^{\blacksquare}i \in \mathsf{S}$ and $|\Pi^i_0 \cap \mathsf{S}| = g$. That has two consequences. First, if $\gamma_{\mathsf{S}}({}^{\blacksquare}g) = {}^{\blacksquare}i$, then $i = 1$ is equivalent to the conjunction of ${}^{\blacksquare}1 \in \mathsf{S}$ and $g = 1$. Second, if $\gamma_{\mathsf{S}}({}^{\blacksquare}g) = {}^{\blacksquare}i$ and if ${}^{\blacksquare}1 \notin \mathsf{S}$ and $g = 1$, then $]{}^{\blacksquare}i, {}^{\blacksquare}1]^k_\ell \cap \mathsf{S} = \varnothing$. Therefore, for any $g \in [\![x]\!]$, if ${}^{\blacksquare}i = \gamma_{\mathsf{S}}({}^{\blacksquare}g)$, then

$$
\begin{aligned}
(\varepsilon^{\mathfrak{c}}_{\mathfrak{d}} \circ \gamma_{\mathsf{S}})({}^{\blacksquare}g) &= \xi^{\mathfrak{c}}_{\mathfrak{d}}({}^{\blacksquare}i) - \begin{cases} {}_{\aleph}\sigma^{\mathfrak{c}}_{\mathfrak{d}}(]{}^{\blacksquare}i, {}^{\blacksquare}1]^k_\ell) & \text{if } i \neq 1 \\ 0 & \text{otherwise}\end{cases}\\
&= \xi^{\mathfrak{c}}_{\mathfrak{d}}({}^{\blacksquare}i) - \begin{cases} {}_{\aleph}\sigma^{\mathfrak{c}}_{\mathfrak{d}}(]{}^{\blacksquare}i, {}^{\blacksquare}1]^k_\ell \cap \mathsf{S}) + {}_{\aleph}\sigma^{\mathfrak{c}}_{\mathfrak{d}}(]{}^{\blacksquare}i, {}^{\blacksquare}1]^k_\ell\backslash\mathsf{S}) & \text{if } g \neq 1 \\ {}_{\aleph}\sigma^{\mathfrak{c}}_{\mathfrak{d}}(]{}^{\blacksquare}i, {}^{\blacksquare}1]^k_\ell\backslash\mathsf{S}) & \text{if } {}^{\blacksquare}1 \notin \mathsf{S} \wedge g = 1 \\ 0 & \text{if } {}^{\blacksquare}1 \in \mathsf{S} \wedge g = 1\end{cases}\\
&= \xi^{\mathfrak{a}}_{\mathfrak{b}}({}^{\blacksquare}g) - \begin{cases} {}_{\aleph}\sigma^{\mathfrak{a}}_{\mathfrak{b}}(]{}^{\blacksquare}g, {}^{\blacksquare}1]^k_\ell) + {}_{\aleph}\sigma^{\mathfrak{c}}_{\mathfrak{d}}(]{}^{\blacksquare}i, {}^{\blacksquare}1]^k_\ell\backslash\mathsf{S}) & \text{if } g \neq 1 \\ {}_{\aleph}\sigma^{\mathfrak{c}}_{\mathfrak{d}}(]{}^{\blacksquare}i, {}^{\blacksquare}1]^k_\ell\backslash\mathsf{S}) & \text{if } {}^{\blacksquare}1 \notin \mathsf{S} \wedge g = 1 \\ 0 & \text{if } {}^{\blacksquare}1 \in \mathsf{S} \wedge g = 1\end{cases}\\
&= \varepsilon^{\mathfrak{a}}_{\mathfrak{b}}({}^{\blacksquare}g) - \begin{cases} {}_{\aleph}\sigma^{\mathfrak{c}}_{\mathfrak{d}}(]{}^{\blacksquare}i, {}^{\blacksquare}1]^k_\ell\backslash\mathsf{S}) & \text{if } {}^{\blacksquare}1 \notin \mathsf{S} \vee g \neq 1 \\ 0 & \text{if } {}^{\blacksquare}1 \in \mathsf{S} \wedge g = 1\end{cases},
\end{aligned}
$$

where we havee used Lemma 4.2 (d) in the third step. And that is also what was claimed in this case.

The computation is very much analogous in the case of lower points. More precisely, for any $h \in [\![y]\!]$ and $j \in [\![\ell]\!]$, if ${}_{\blacksquare}j = \gamma_{\mathsf{S}}({}_{\blacksquare}h)$, then first, ${}_{\blacksquare}j = 1$ if and only if ${}_{\blacksquare}1 \in \mathsf{S}$ and $j = 1$, and, second $[{}_{\blacksquare}1, {}_{\blacksquare}j[^k_\ell \cap \mathsf{S} = \varnothing$ in case ${}_{\blacksquare}1 \notin \mathsf{S}$ and $g = 1$. Hence, for any

$h \in [\![y]\!]$, if ${}_{\blacksquare}j = \gamma_{\mathsf{S}}({}_{\blacksquare}h)$, then

$$\begin{aligned}(\varepsilon^{\mathfrak{c}}_{\mathfrak{d}} \circ \gamma_{\mathsf{S}})({}_{\blacksquare}h) &= \xi^{\mathfrak{c}}_{\mathfrak{d}}({}_{\blacksquare}j) + \begin{cases} {}_{\aleph}\sigma^{\mathfrak{c}}_{\mathfrak{d}}([{}^{\blacksquare}1, {}_{\blacksquare}j[^{k}_{\ell}) & \text{if } j \neq 1 \\ 0 & \text{otherwise}\end{cases} \\ &= \xi^{\mathfrak{c}}_{\mathfrak{d}}({}_{\blacksquare}j) + \begin{cases} {}_{\aleph}\sigma^{\mathfrak{c}}_{\mathfrak{d}}([{}_{\blacksquare}1, {}_{\blacksquare}j[^{k}_{\ell} \cap \mathsf{S}) + {}_{\aleph}\sigma^{\mathfrak{c}}_{\mathfrak{d}}([{}_{\blacksquare}1, {}_{\blacksquare}j[^{k}_{\ell} \backslash \mathsf{S}) & \text{if } h \neq 1 \\ {}_{\aleph}\sigma^{\mathfrak{c}}_{\mathfrak{d}}([{}_{\blacksquare}1, {}_{\blacksquare}j[^{k}_{\ell} \backslash \mathsf{S}) & \text{if } {}_{\blacksquare}1 \notin \mathsf{S} \wedge h = 1 \\ 0 & \text{if } {}_{\blacksquare}1 \in \mathsf{S} \wedge h = 1 \end{cases} \\ &= \xi^{\mathfrak{a}}_{\mathfrak{b}}({}_{\blacksquare}h) + \begin{cases} {}_{\aleph}\sigma^{\mathfrak{a}}_{\mathfrak{b}}([{}_{\blacksquare}1, {}_{\blacksquare}h[^{k}_{\ell}) + {}_{\aleph}\sigma^{\mathfrak{c}}_{\mathfrak{d}}([{}_{\blacksquare}1, {}_{\blacksquare}j[^{k}_{\ell} \backslash \mathsf{S}) & \text{if } h \neq 1 \\ {}_{\aleph}\sigma^{\mathfrak{c}}_{\mathfrak{d}}([{}_{\blacksquare}1, {}_{\blacksquare}j[^{k}_{\ell} \backslash \mathsf{S}) & \text{if } {}_{\blacksquare}1 \notin \mathsf{S} \wedge h = 1 \\ 0 & \text{if } {}_{\blacksquare}1 \in \mathsf{S} \wedge h = 1 \end{cases} \\ &= \varepsilon^{\mathfrak{a}}_{\mathfrak{b}}({}_{\blacksquare}h) + \begin{cases} {}_{\aleph}\sigma^{\mathfrak{c}}_{\mathfrak{d}}([{}_{\blacksquare}1, {}_{\blacksquare}j[^{k}_{\ell} \backslash \mathsf{S}) & \text{if } {}_{\blacksquare}1 \notin \mathsf{S} \vee h \neq 1 \\ 0 & \text{if } {}_{\blacksquare}1 \in \mathsf{S} \wedge h = 1 \end{cases},\end{aligned}$$

which we needed to prove. □

Proposition 4.44. *For any $w \in \mathbb{N}_0$, if $(\mathfrak{U}, \mathfrak{O}) \in \{(\mathbb{Z}_w, \varnothing), (\varnothing, \mathbb{Z}_w)\}$, then for any $\mathbb{Z}_w$-invariant category $\mathcal{X}$ of $(\mathfrak{U}, \mathfrak{O})$-tagged labeled partitions the crossed co-product $\mathcal{X} \rtimes \mathcal{Z}_w$ of $\mathcal{X}$ with $\mathcal{Z}_w$ is a category of $(\mathfrak{U} \cup \{\aleph\}, \mathfrak{O})$-tagged labeled partitions.*

Proof. If $\mathcal{D}$ is the set of all $(\mathfrak{c}, \mathfrak{d}, p) \in {}_{\mathfrak{U}\cup\{\aleph\},\mathfrak{O}}\mathcal{S}$ such that $p \leq \xi^{\mathfrak{c}\leftarrow}_{\mathfrak{d}}(\{\mathfrak{U} \cup \mathfrak{O}, \{\aleph\}\})$ and $R((\mathfrak{c}, \mathfrak{d}, p), \xi^{\mathfrak{c}\leftarrow}_{\mathfrak{d}}(\{\aleph\})) \in \mathcal{Z}_w$, then $\mathcal{D}$ is a category of $(\mathfrak{U} \cup \{\aleph\}, \mathfrak{O})$-tagged labeled partitions by Propositions 4.3 and 4.6. (Just choose there $\mathcal{X}_z = {}_{\{z\},\varnothing}\mathcal{S}$ for each $z \in \mathfrak{U}$ and $\mathcal{X}_z = {}_{\varnothing,\{z\}}\mathcal{S}$ for each $z \in \mathfrak{O}$.) Once again we show that the remaining property which turns a partition of $\mathcal{D}$ into one of $\mathcal{X} \rtimes \mathcal{Z}_w$ is satisfied by identitites and invariant under the operations of Proposition 3.26.

Step 1: Identities. For any $\mathfrak{c} \in ((\mathfrak{U} \cup \{\aleph\}) \otimes \{\circ, \bullet\}) \cup \mathfrak{O}$, by definition, $\varepsilon^{\mathfrak{c}}_{\mathfrak{c}}(\mathtt{t}) = \xi^{\mathfrak{c}}_{\mathfrak{c}}(\mathtt{t})$ for any $\mathtt{t} \in \xi^{\mathfrak{c}\leftarrow}_{\mathfrak{c}}(\mathfrak{U} \cup \mathfrak{O}) \subseteq \Pi^1_1$ and thus $\mathrm{id}_{\mathfrak{c}} \in \mathcal{X} \rtimes \mathcal{Z}_w$.

Step 2: Rotation. Let $k \in \mathbb{N}$ and $\ell \in \mathbb{N}_0$ and $\mathfrak{c}\colon [\![k]\!] \to ((\mathfrak{U} \cup \{\aleph\}) \otimes \{\circ, \bullet\}) \cup \mathfrak{O}$ and $\mathfrak{d}\colon [\![\ell]\!] \to \mathfrak{U} \cup \{\aleph\} \cup \mathfrak{O}$ and $(\mathfrak{c}, \mathfrak{d}, p) \in \mathcal{X} \rtimes \mathcal{Z}_w$ as well as $r \in \{\circlearrowleft, \circlearrowright\}$ be arbuirary. Moreover, let $\mathfrak{a}$ and $\mathfrak{b}$ be such that $\mathfrak{a}\,{}^{\blacksquare}{}_{\blacksquare}\,\mathfrak{b} = (\mathfrak{c}\,{}^{\blacksquare}{}_{\blacksquare}\,\mathfrak{d})^r$, let $\mathsf{Y} = \xi^{\mathfrak{a}\leftarrow}_{\mathfrak{b}}(\mathfrak{U} \cup \mathfrak{O})$ and let $\mathfrak{u}$ and $\mathfrak{v}$ be such that $\xi^{\mathfrak{u}}_{\mathfrak{v}} = \pi_w \circ \varepsilon^{\mathfrak{a}}_{\mathfrak{b}} \circ \gamma_{\mathsf{Y},k+1}^{\;k-1}$ and $\zeta^{\mathfrak{u}}_{\mathfrak{v}} = \zeta^{\mathfrak{a}}_{\mathfrak{b}} \circ \gamma_{\mathsf{Y},k+1}^{\;k-1}$. We have to prove $(\mathfrak{u}, \mathfrak{v}, R(p^r, \mathsf{Y})) \in \mathcal{X}$.

If $\mathsf{Z} = \xi^{\mathfrak{c}\leftarrow}_{\mathfrak{d}}(\mathfrak{U} \cup \mathfrak{O})$ and if $\mathfrak{f}$ and $\mathfrak{g}$ are such that $\xi^{\mathfrak{f}}_{\mathfrak{g}} = \pi_w \circ \varepsilon^{\mathfrak{c}}_{\mathfrak{d}} \circ \gamma_{\mathsf{Z},\ell}^{\;k}$ and $\zeta^{\mathfrak{f}}_{\mathfrak{g}} = \zeta^{\mathfrak{c}}_{\mathfrak{d}} \circ \gamma_{\mathsf{Z},\ell}^{\;k}$, then $(\mathfrak{f}, \mathfrak{g}, R(p, \mathsf{Z})) \in \mathcal{X}$ by $(\mathfrak{c}, \mathfrak{d}, p) \in \mathcal{X} \rtimes \mathcal{Z}_w$. If we write $\mathsf{e} = {}^{\blacksquare}1$ if $r = \circlearrowleft$ and $\mathsf{e} = {}^{\blacksquare}k$ if $r = \circlearrowright$, then $R(p^r, \mathsf{Y}) = R(p, \mathsf{Z})$ if $\mathsf{e} \notin \mathsf{Z}$ and $R(p^r, \mathsf{Y}) = R(p, \mathsf{Z})^r$ if $\mathsf{e} \in \mathsf{Z}$ by Lemma 4.5 (a). Hence, if we reprise the notation V for the action of $\mathbb{Z}_w$ on $(\mathfrak{U} \otimes \{\circ, \bullet\}) \cup \mathfrak{O}$, then by the $\mathbb{Z}_w$-invariance of $\mathcal{X}$ it suffices to find some $z \in \mathbb{Z}_w$ such that

$$\mathfrak{u}\,{}^{\blacksquare}{}_{\blacksquare}\,\mathfrak{v} = V_z \circ \begin{cases} \mathfrak{f}\,{}^{\blacksquare}{}_{\blacksquare}\,\mathfrak{g} & \text{if } \mathsf{e} \notin \mathsf{Z} \\ (\mathfrak{f}\,{}^{\blacksquare}{}_{\blacksquare}\,\mathfrak{g})^r & \text{otherwise}\end{cases}$$

in order to prove $(\mathfrak{u}, \mathfrak{v}, R(p^r, \mathsf{Y})) \in \mathcal{X}$. More precisely, z will be $\pi_w({}_{\aleph}\sigma^{\mathfrak{c}}_{\mathfrak{d}}(\{\mathsf{e}\}))$ if $r = \circlearrowleft$ and 0 if $r = \circlearrowright$. The proof is divided into two steps.

Step 2.1: First, we recognize that for any $\mathtt{t} \in \mathsf{Z}$,

$$(\pi_w \circ \varepsilon^{\mathtt{a}}_{\mathtt{b}} \circ (\omega^{r,k}_{\ell})^{-1})(\mathtt{t}) = (\pi_w \circ \varepsilon^{\mathfrak{c}}_{\mathfrak{d}})(\mathtt{t}) +_w z.$$

To see this, two observations are required. First, ${}_{\aleph}\Sigma^{\mathfrak{c}}_{\mathfrak{d}} \equiv_w 0$ for the following reasons. Because $(\mathfrak{c}, \mathfrak{d}, p) \in \mathcal{X} \rtimes \mathcal{Z}_w$, by definition of $\mathcal{Z}_w$ the total color sum of the labeled partition $R((\mathfrak{c}, \mathfrak{d}, p), \mathsf{Z})$ with respect to $\aleph$ must be a multiple of w. At the same time, by Lemma 4.2 (d) this total color sum is given by ${}_{\aleph}\sigma^{\mathfrak{c}}_{\mathfrak{d}}(\Pi^k_\ell) - {}_{\aleph}\sigma^{\mathfrak{c}}_{\mathfrak{d}}(\Pi^k_\ell \backslash \mathsf{S}) = {}_{\aleph}\Sigma^{\mathfrak{c}}_{\mathfrak{d}} - {}_{\aleph}\sigma^{\mathfrak{c}}_{\mathfrak{d}}(\mathsf{Z})$, where $\mathsf{S} = \Pi^k_\ell \backslash \mathsf{Z}$. Since $\mathsf{Z} \cap \xi^{\mathfrak{c}}_{\mathfrak{d}}{}^{\leftarrow}(\{\aleph\}) = \varnothing$ by definition, thus $\pi_w({}_{\aleph}\Sigma^{\mathfrak{c}}_{\mathfrak{d}}) = 0$.

The second observation is that for any $\mathtt{t} \in \mathsf{Z}$, if $\mathtt{t} = \mathtt{e}$, then ${}_{\aleph}\sigma^{\mathfrak{c}}_{\mathfrak{d}}(\{\mathtt{e}\}) = 0$ since $\mathsf{Z} \cap \xi^{\mathfrak{c}}_{\mathfrak{d}}{}^{\leftarrow}(\{\aleph\}) = \varnothing$.

With those at hand, we can infer the intermediate claim by using the implication of Lemma 4.43 (d) that for any $\mathtt{t} \in \mathsf{Z}$,

$$(\varepsilon^{\mathtt{a}}_{\mathtt{b}} \circ (\omega^{r,k}_{\ell})^{-1})(\mathtt{t}) = \varepsilon^{\mathtt{c}}_{\mathtt{d}}(\mathtt{t}) + \begin{cases} {}_{\aleph}\sigma^{\mathfrak{c}}_{\mathfrak{d}}(\{\mathtt{e}\}) & \text{if } r = \curvearrowleft \wedge \mathtt{t} \neq \mathtt{e} \\ 0 & \text{if } r = \curvearrowleft \wedge \mathtt{t} = \mathtt{e} \\ 0 & \text{if } r = \curvearrowright \wedge \mathtt{t} \neq \mathtt{e} \\ -{}_{\aleph}\sigma^{\mathfrak{c}}_{\mathfrak{d}}(\{\mathtt{e}\}) + {}_{\aleph}\Sigma^{\mathfrak{c}}_{\mathfrak{d}} & \text{if } r = \curvearrowright \wedge \mathtt{t} = \mathtt{e}. \end{cases}$$

Step 2.2: Because $\mathsf{Y} = \omega^{r,k}_{\ell}{}^{\leftarrow}(\mathsf{Z})$ by definition, for any $\mathtt{h} \in \Pi^{\alpha(\mathsf{Y})}_{\beta(\mathsf{Y})}$, if $\mathtt{t} = (\omega^{r,k}_{\ell} \circ \gamma_{\mathsf{Y},\ell+1}^{\;k-1})(\mathtt{h})$ and $m = \alpha(\mathsf{Z})$ and $n = \beta(\mathsf{Z})$, then $\mathtt{t} \in \mathsf{Z}$ and thus by Step 2.1:

$$\begin{aligned} \xi^{\mathfrak{u}}_{\mathfrak{v}}(\mathtt{h}) &= (\pi_w \circ \varepsilon^{\mathtt{a}}_{\mathtt{b}} \circ \gamma_{\mathsf{Y},\ell+1}^{\;k-1})(\mathtt{h}) \\ &= (\pi_w \circ \varepsilon^{\mathtt{a}}_{\mathtt{b}} \circ (\omega^{r,k}_{\ell})^{-1})(\mathtt{t}) \\ &= (\pi_w \circ \varepsilon^{\mathfrak{c}}_{\mathfrak{d}})(\mathtt{t}) +_w z \\ &= (\pi_w \circ \varepsilon^{\mathtt{c}}_{\mathtt{d}} \circ \omega^{r,k}_{\ell} \circ \gamma_{\mathsf{Y},\ell+1}^{\;k-1})(\mathtt{h}) +_w z \\ &= \begin{Bmatrix} (\pi_w \circ \varepsilon^{\mathtt{c}}_{\mathtt{d}} \circ \gamma_{\mathsf{Z},\ell}^{\;k})(\mathtt{h}) & \text{if } \mathtt{e} \notin \mathsf{Z} \\ (\pi_w \circ \varepsilon^{\mathtt{c}}_{\mathtt{d}} \circ \gamma_{\mathsf{Z},\ell}^{\;k} \circ \omega^{r,m}_{n})(\mathtt{h}) & \text{otherwise} \end{Bmatrix} +_w z \\ &= \begin{Bmatrix} \xi^{\mathfrak{f}}_{\mathfrak{g}}(\mathtt{h}) & \text{if } \mathtt{e} \notin \mathsf{Z} \\ \xi^{\mathfrak{f}}_{\mathfrak{g}}(\omega^{r,m}_{n}(\mathtt{h})) & \text{otherwise} \end{Bmatrix} +_w z, \end{aligned}$$

where we have used Lemma 4.5 (a) in the fifth step. Since by the same lemma

$$\begin{aligned} \zeta^{\mathfrak{u}}_{\mathfrak{v}} = \zeta^{\mathfrak{a}}_{\mathfrak{b}} \circ \gamma_{\mathsf{Y},\ell+1}^{\;k-1} = \zeta^{\mathfrak{c}}_{\mathfrak{d}} \circ \omega^{r,k}_{\ell} \circ \gamma_{\mathsf{Y},\ell+1}^{\;k-1} &= \begin{Bmatrix} \zeta^{\mathfrak{c}}_{\mathfrak{d}} \circ \gamma_{\mathsf{Z},\ell}^{\;k} & \text{if } \mathtt{e} \notin \mathsf{Z} \\ \zeta^{\mathfrak{c}}_{\mathfrak{d}} \circ \gamma_{\mathsf{Z},\ell}^{\;k} \circ \omega^{r,m}_{n} & \text{otherwise} \end{Bmatrix} \\ &= \begin{cases} \zeta^{\mathfrak{f}}_{\mathfrak{g}} & \text{if } \mathtt{e} \notin \mathsf{Z} \\ \zeta^{\mathfrak{f}}_{\mathfrak{g}} \circ \omega^{r,m}_{n} & \text{otherwise} \end{cases} \end{aligned}$$

that is all we needed to prove.

Step 3: Adjoints. Next, let $\{k, \ell\} \subseteq \mathbb{N}_0$ as well as $\mathfrak{c}\colon [\![k]\!] \to ((\mathfrak{U} \cup \{\aleph\}) \otimes \{\circ, \bullet\}) \cup \mathfrak{D}$ and $\mathfrak{d}\colon [\![\ell]\!] \to ((\mathfrak{U} \cup \{\aleph\}) \otimes \{\circ, \bullet\}) \cup \mathfrak{D}$ and finally $(\mathfrak{c}, \mathfrak{d}, p) \in \mathcal{X} \rtimes \mathcal{Z}_w$ be arbitrary. With $\mathsf{Y} = \xi^{\mathfrak{d}}_{\mathfrak{c}}{}^{\leftarrow}(\mathfrak{U} \cup \mathfrak{D})$ we let $\mathfrak{u}$ and $\mathfrak{v}$ be such that $\xi^{\mathfrak{u}}_{\mathfrak{v}} = \pi_w \circ \varepsilon^{\mathfrak{d}}_{\mathfrak{c}} \circ \gamma_{\mathsf{Y},k}^{\;\ell}$ and $\zeta^{\mathfrak{u}}_{\mathfrak{v}} = \zeta^{\mathfrak{d}}_{\mathfrak{c}} \circ \gamma_{\mathsf{Y},k}^{\;\ell}$ and then have to show $(\mathfrak{u}, \mathfrak{v}, R(p^*, \mathsf{Y})) \in \mathcal{X}$.

If $\mathsf{Z} = \xi_{\mathfrak{d}}^{\mathfrak{c}\leftarrow}(\mathfrak{U} \cup \mathfrak{O})$ and if $\mathfrak{f}$ and $\mathfrak{g}$ are such that $\xi_{\mathfrak{g}}^{\mathfrak{f}} = \pi_w \circ \varepsilon_{\mathfrak{d}}^{\mathfrak{c}} \circ \gamma_{\mathsf{Z},k}^{\;\;\ell}$ and $\zeta_{\mathfrak{g}}^{\mathfrak{f}} = \zeta_{\mathfrak{d}}^{\mathfrak{c}} \circ \gamma_{\mathsf{Z},\ell}^{\;\;k}$, then $(\mathfrak{f}, \mathfrak{g}, R(p, \mathsf{Z})) \in \mathcal{X}$ by $(\mathfrak{c}, \mathfrak{d}, p) \in \mathcal{X} \rtimes \mathcal{Z}_w$. Because $\mathcal{X}$ is closed under forming adjoints it suffices to show that $(\mathfrak{u}, \mathfrak{v}, R(p^*, \mathsf{Y})) = (\mathfrak{f}, \mathfrak{g}, R(p, \mathsf{Z}))^*$.

Moreover, $\mathsf{Y} = \kappa_\ell^{k\leftarrow}(\mathsf{Z})$ because $\xi_{\mathfrak{c}}^{\mathfrak{d}} = \xi_{\mathfrak{d}}^{\mathfrak{c}} \circ \kappa_\ell^k$ by Lemma 4.2 (b) and $R(p^*, \mathsf{Y}) = R(p, \mathsf{Z})^*$ by Lemma 4.5 (b). Hence, all that is left to show in order to verify $(\mathfrak{u}, \mathfrak{v}, R(p^*, \mathsf{Y})) = (\mathfrak{g}, \mathfrak{f}, R(p, \mathsf{Z})^*)$ is that

$$\mathfrak{u} \,\blacksquare\, \mathfrak{v} = \mathfrak{g} \,\blacksquare\, \mathfrak{f}.$$

And, indeed, if $m = \alpha(\mathsf{Z})$ and $n = \beta(\mathsf{Z})$, then because $\kappa_\ell^k \circ \gamma_{\mathsf{Y},k}^{\;\;\ell} = \gamma_{\mathsf{Z},\ell}^{\;\;k} \circ \kappa_n^m$ by Lemma 4.5 (b) and because $\varepsilon_{\mathfrak{c}}^{\mathfrak{d}} = \varepsilon_{\mathfrak{d}}^{\mathfrak{c}} \circ \kappa_\ell^k$ by Lemma 4.43 (b),

$$\begin{aligned}
\xi_{\mathfrak{v}}^{\mathfrak{u}} &= \pi_w \circ \varepsilon_{\mathfrak{c}}^{\mathfrak{d}} \circ \gamma_{\mathsf{Y},k}^{\;\;\ell} \\
&= \pi_w \circ \varepsilon_{\mathfrak{d}}^{\mathfrak{c}} \circ \kappa_\ell^k \circ \gamma_{\mathsf{Y},k}^{\;\;\ell} \\
&= \pi_w \circ \varepsilon_{\mathfrak{d}}^{\mathfrak{c}} \circ \gamma_{\mathsf{Z},\ell}^{\;\;k} \circ \kappa_n^m \\
&= \xi_{\mathfrak{g}}^{\mathfrak{f}} \circ \kappa_n^m \\
&= \xi_{\mathfrak{f}}^{\mathfrak{g}}
\end{aligned}$$

where the last step is also due to Lemma 4.5 (b). Similarly, and because by the same lemma, $\zeta_{\mathfrak{c}}^{\mathfrak{d}} = \overline{(\cdot)} \circ \xi_{\mathfrak{d}}^{\mathfrak{c}} \circ \kappa_\ell^k$ and $\zeta_{\mathfrak{f}}^{\mathfrak{g}} = \overline{(\cdot)} \circ \xi_{\mathfrak{g}}^{\mathfrak{f}} \circ \kappa_n^m$,

$$\begin{aligned}
\zeta_{\mathfrak{v}}^{\mathfrak{u}} &= \zeta_{\mathfrak{c}}^{\mathfrak{d}} \circ \gamma_{\mathsf{Y},k}^{\;\;\ell} \\
&= \overline{(\cdot)} \circ \zeta_{\mathfrak{d}}^{\mathfrak{c}} \circ \kappa_\ell^k \circ \gamma_{\mathsf{Y},k}^{\;\;\ell} \\
&= \overline{(\cdot)} \circ \zeta_{\mathfrak{d}}^{\mathfrak{c}} \circ \gamma_{\mathsf{Z},\ell}^{\;\;k} \circ \kappa_n^m \\
&= \overline{(\cdot)} \circ \zeta_{\mathfrak{g}}^{\mathfrak{f}} \circ \kappa_n^m \\
&= \zeta_{\mathfrak{f}}^{\mathfrak{g}}
\end{aligned}$$

which is all we had to show.

Step 4: Tensor products. For each $t \in [\![2]\!]$ let $\{k_t, \ell_t\} \subseteq \mathbb{N}_0$ as well as $\mathfrak{c}_t\colon [\![k_t]\!] \to ((\mathfrak{U} \cup \{\aleph\}) \otimes \{\circ, \bullet\}) \cup \mathfrak{O}$ and $\mathfrak{d}_t\colon [\![\ell_t]\!] \to ((\mathfrak{U} \cup \{\aleph\}) \otimes \{\circ, \bullet\}) \cup \mathfrak{O}$ and $(\mathfrak{c}_t, \mathfrak{d}_t, p_t) \in \mathcal{X} \rtimes \mathcal{Z}_w$ be arbitrary. If $\mathsf{Y} = \xi_{\mathfrak{d}_1 \otimes \mathfrak{d}_2}^{\mathfrak{c}_1 \otimes \mathfrak{c}_2 \leftarrow}(\mathfrak{U} \cup \mathfrak{O})$ and if $\mathfrak{u}$ and $\mathfrak{v}$ are such that $\xi_{\mathfrak{v}}^{\mathfrak{u}} = \pi_w \circ \varepsilon_{\mathfrak{d}_1 \otimes \mathfrak{d}_2}^{\mathfrak{c}_1 \otimes \mathfrak{c}_2} \circ \gamma_{\mathsf{Y},\ell_1+\ell_2}^{\;\;k_1+k_2}$, then we have to prove $(\mathfrak{u}, \mathfrak{v}, R(p_1 \otimes p_2, \mathsf{Y})) \in \mathcal{X}$.

For each $t \in [\![2]\!]$, if $\mathsf{Z}_t = \xi_{\mathfrak{d}_t}^{\mathfrak{c}_t\leftarrow}(\mathfrak{U} \cup \mathfrak{O})$ and if $\mathfrak{f}_t$ and $\mathfrak{g}_t$ are such that $\xi_{\mathfrak{g}_t}^{\mathfrak{f}_t} = \pi_t \circ \varepsilon_{\mathfrak{d}_t}^{\mathfrak{c}_t} \circ \gamma_{\mathsf{Z}_t,\ell_t}^{\;\;k_t}$ and $\zeta_{\mathfrak{g}_t}^{\mathfrak{f}_t} = \zeta_{\mathfrak{d}_t}^{\mathfrak{c}_t} \circ \gamma_{\mathsf{Z}_t,\ell_t}^{\;\;k_t}$, then $(\mathfrak{f}_t, \mathfrak{g}_t, R(p_t, \mathsf{Z}_t)) \in \mathcal{X}$ by $(\mathfrak{c}_t, \mathfrak{d}_t, p_t) \in \mathcal{X} \rtimes \mathcal{Z}_w$. Hence, because $\mathcal{X}$ is closed under tensor products and $\mathbb{Z}_w$-invariant it suffices to find $\{z_1, z_2\} \subseteq \mathbb{Z}_w$ such that

$$(\mathfrak{u}, \mathfrak{v}, R(p_1 \otimes p_2, \mathsf{Y})) = (V_{z_1} \circ \mathfrak{f}_1, V_{z_1} \circ \mathfrak{g}_1, R(p_1, \mathsf{Z}_1)) \otimes (V_{z_2} \circ \mathfrak{f}_2, V_{z_2} \circ \mathfrak{g}_2, R(p_2, \mathsf{Z}_2)).$$

Actually, since $R(p_1 \otimes p_2, \mathsf{Y}) = R(p_1, \mathsf{Z}_1) \otimes R(p_2, \mathsf{Z}_2)$ by Lemma 4.5 (c) it is enough to find z_1 and z_2 with

$$\mathfrak{u} \,\blacksquare\, \mathfrak{v} = ((V_{z_1} \circ \mathfrak{f}_1) \otimes (V_{z_2} \circ \mathfrak{f}_2)) \,\blacksquare\, ((V_{z_1} \circ \mathfrak{g}_1) \otimes (V_{z_2} \circ \mathfrak{g}_2)),$$

where, also, the right hand side is identical to $(V_{z_1} \circ (\mathfrak{f}_1 \,\blacksquare\, \mathfrak{g}_1)) \otimes (V_{z_2} \circ (\mathfrak{f}_2 \,\blacksquare\, \mathfrak{g}_2))$.

If $\mathsf{H}_1 = \Pi^{k_1}_{\ell_1}$ and $\mathsf{H}_2 = \Pi^{k_1+k_2}_{\ell_1+\ell_2}\backslash\Pi^{k_1}_{\ell_1}$, if $m_t = \alpha(\mathsf{Z}_t)$ and $n_t = \beta(\mathsf{Z}_t)$ for each $t \in [\![2]\!]$ and if $\mathsf{R}_1 = \Pi^{m_1}_{n_1}$ and $\mathsf{R}_2 = \Pi^{m_1+m_2}_{n_1+n_2}\backslash\Pi^{m_1}_{n_1}$, then $\mathsf{Z}_t = \gamma_{\mathsf{H}_t,n_1+n_2}^{\quad m_1+m_2}{}^{\leftarrow}(\mathsf{Y})$ since $\xi^{\mathfrak{c}_1\otimes\mathfrak{c}_2}_{\mathfrak{d}_1\otimes\mathfrak{d}_2}\circ\gamma_{\mathsf{H}_t,n_1+n_2}^{\quad m_1+m_2} = \xi^{\mathfrak{c}_t}_{\mathfrak{d}_t}$ by Lemma 4.5 (c) for each $t \in [\![2]\!]$. Moreover, then $\alpha(\mathsf{Y}) = m_1+m_2$ and $\beta(\mathsf{Y}) = n_1+n_2$ and the maps $(\gamma_{\mathsf{R}_t,n_1+n_2}^{\quad m_1+m_2})^2_{t=1}$ are jointly surjctive to the common domain $\Pi^{m_1+m_2}_{n_1+n_2}$ of $\mathfrak{u} \blacksquare \mathfrak{v}$ and $(V_{z_1}\circ(\mathfrak{f}_1 \blacksquare \mathfrak{g}_1))\otimes(V_{z_2}\circ(\mathfrak{f}_2 \blacksquare \mathfrak{g}_2))$. Therefore, all we have to prove is that there exist z_1 and z_2 with, for any $t \in [\![2]\!]$,

$$(\mathfrak{u} \blacksquare \mathfrak{v})\circ\gamma_{\mathsf{R}_t,n_1+n_2}^{\quad m_1+m_2} = V_{z_t}\circ(\mathfrak{f}_t \blacksquare \mathfrak{g}_t)$$

because the right hand side of this equation is identically $(V_{z_1}\circ(\mathfrak{f}_1 \blacksquare \mathfrak{g}_1))\otimes(V_{z_2}\circ(\mathfrak{f}_2 \blacksquare \mathfrak{g}_2))\circ\gamma_{\mathsf{R}_t,n_1+n_2}^{\quad m_1+m_2}$.

Because, in terms of unitary area and colors, $\mathfrak{u} \blacksquare \mathfrak{v}$ by definition coincides with $R((\mathfrak{c}_1\otimes\mathfrak{c}_2) \blacksquare (\mathfrak{d}_1\otimes\mathfrak{d}_2),\mathsf{Y})$ and since for each $t \in [\![2]\!]$, likewise, $V_{z_t}\circ(\mathfrak{f}_t \blacksquare \mathfrak{g}_t)$ coincides with $R(\mathfrak{f}_t \blacksquare \mathfrak{g}_t,\mathsf{Z}_t)$, no matter the value of z_t, we in fact only need to find z_1 and z_2 with

$$(\xi^{\mathtt{u}}_{\mathtt{v}}\circ\gamma_{\mathsf{R}_t,n_1+n_2}^{\quad m_1+m_2})(\mathtt{h}) = \xi^{\mathfrak{f}_t}_{\mathfrak{g}_t}(\mathtt{h}) +_w z_t$$

for any $\mathtt{h} \in \Pi^{m_t}_{n_t}$ and any $t \in [\![2]\!]$. More precisely, we will now prove this for $z_1 = 0$ and $z_2 = \pi_w(-_{\aleph}\Sigma^{\mathfrak{c}_1}_{\varnothing})$.

First, though, we once more need to recognize that ${}_{\aleph}\Sigma^{\mathfrak{c}_1}_{\mathfrak{d}_1} \equiv_w 0$. The argument is the same as before: Because $(\mathfrak{c}_1,\mathfrak{d}_1,p_1) \in \mathcal{X} \rtimes \mathcal{Z}_w$, by definition of $\mathcal{Z}_w$ the total $\aleph$-color-sum of $R((\mathfrak{c}_1,\mathfrak{d}_1,p_1),\mathsf{Z}_1)$ is divided by w. Simultaneously, by Lemma 4.2 (d) this sum is given by ${}_{\aleph}\sigma^{\mathfrak{c}_1}_{\mathfrak{d}_1}(\Pi^{k_1}_{\ell_1}) - {}_{\aleph}\sigma^{\mathfrak{c}_1}_{\mathfrak{d}_1}(\Pi^{k_1}_{\ell_1}\backslash\mathsf{S}) = {}_{\aleph}\Sigma^{\mathfrak{c}_1}_{\mathfrak{d}_1} - {}_{\aleph}\sigma^{\mathfrak{c}_1}_{\mathfrak{d}_1}(\mathsf{Z}_1)$, where $\mathsf{S} = \Pi^{k_1}_{\ell_1}\backslash\mathsf{Z}_1$, which proves $\pi_w({}_{\aleph}\Sigma^{\mathfrak{c}_1}_{\mathfrak{d}_1}) = 0$ by $\mathsf{Z}_1 \cap \xi^{\mathfrak{c}_1}_{\mathfrak{d}_1}{}^{\leftarrow}(\{\aleph\}) = \varnothing$.

In consequence, for any $t \in [\![2]\!]$ and $\mathtt{s} \in \mathsf{Z}_t$,

$$z_t \equiv_w \begin{cases} 0 & \text{if } t = 1 \\ -_{\aleph}\Sigma^{\mathfrak{c}_1}_{\varnothing} & \text{if } t = 2 \wedge \mathtt{s} \in \Pi^{k_2}_0 \\ -_{\aleph}\Sigma^{\mathfrak{c}_1}_{\varnothing} + {}_{\aleph}\Sigma^{\mathfrak{c}_1}_{\mathfrak{d}_1} & \text{if } t = 2 \wedge \mathtt{s} \in \Pi^0_{\ell_2} \end{cases}$$

and thus by Lemma 4.43 (c),

$$(\pi_t\circ\varepsilon^{\mathtt{c}_1\otimes\mathtt{c}_2}_{\mathtt{d}_1\otimes\mathtt{d}_2}\circ\gamma_{\mathsf{H}_t,\ell_1+\ell_2}^{\quad k_1+k_2})(\mathtt{s}) = (\pi_w\circ\varepsilon^{\mathfrak{c}_t}_{\mathfrak{d}_t})(\mathtt{s}) +_w z_t.$$

For any $t \in [\![2]\!]$ and $\mathtt{h} \in \Pi^{m_t}_{n_t}$, if we abbreviate $\mathtt{s} \equiv \gamma_{\mathsf{Z}_t,\ell_t}^{\quad k_t}(\mathtt{h})$, using that $\gamma_{\mathsf{Y},\ell_1+\ell_2}^{\quad k_1+k_2}\circ\gamma_{\mathsf{R}_t,n_1+n_2}^{\quad m_1+m_2} = \gamma_{\mathsf{H}_t,\ell_1+\ell_2}^{\quad k_1+k_2}\circ\gamma_{\mathsf{Z}_t,\ell_t}^{\quad k_t}$ by Lemma 4.5 (c),

$$\begin{aligned}(\xi^{\mathtt{u}}_{\mathtt{v}}\circ\gamma_{\mathsf{R}_t,n_1+n_2}^{\quad m_1+m_2})(\mathtt{h}) &= (\pi_w\circ\varepsilon^{\mathtt{c}_1\otimes\mathtt{c}_2}_{\mathtt{d}_1\otimes\mathtt{d}_2}\circ\gamma_{\mathsf{Y},\ell_1+\ell_2}^{\quad k_1+k_2}\circ\gamma_{\mathsf{R}_t,n_1+n_2}^{\quad m_1+m_2})(\mathtt{h}) \\ &= (\pi_w\circ\varepsilon^{\mathtt{c}_1\otimes\mathtt{c}_2}_{\mathtt{d}_1\otimes\mathtt{d}_2}\circ\gamma_{\mathsf{H}_t,\ell_1+\ell_2}^{\quad k_1+k_2}\circ\gamma_{\mathsf{Z}_t,\ell_t}^{\quad k_t})(\mathtt{h}) \\ &= (\pi_w\circ\varepsilon^{\mathtt{c}_1\otimes\mathtt{c}_2}_{\mathtt{d}_1\otimes\mathtt{d}_2}\circ\gamma_{\mathsf{H}_t,\ell_1+\ell_2}^{\quad k_1+k_2})(\mathtt{s}) \\ &= (\pi_w\circ\varepsilon^{\mathfrak{c}_t}_{\mathfrak{d}_t})(\mathtt{s}) +_w z_t \\ &= (\pi_w\circ\varepsilon^{\mathtt{c}_t}_{\mathtt{d}_t}\circ\gamma_{\mathsf{Z}_t,\ell_t}^{\quad k_t})(\mathtt{h}) +_w z_t \\ &= \xi^{\mathfrak{f}_t}_{\mathfrak{g}_t}(\mathtt{h}),\end{aligned}$$

which completes the proof that $\mathcal{X} \rtimes \mathcal{Z}_w$ is closed under tensor products.

Step 5: Erasing. Finally, let $k\in\mathbb{N}_0$ and $\ell\in\mathbb{N}$ as well as $\mathfrak{c}\colon [\![k]\!]\to((\mathfrak{U}\cup\{\aleph\})\otimes\{\circ,\bullet\})\cup\mathfrak{O}$ and $\mathfrak{d}\colon [\![\ell]\!]\to((\mathfrak{U}\cup\{\aleph\})\otimes\{\circ,\bullet\})\cup\mathfrak{O}$ and $(\mathfrak{c},\mathfrak{d},p)\in\mathcal{X}\rtimes\mathcal{Z}_w$ be arbitrary and let $z\in\mathfrak{U}\cup\{\aleph\}\cup\mathfrak{O}$ and $\mathsf{T}\subseteq\Pi_\ell^0$ be such that T is convex with respect to Γ_ℓ^k, such that $|\mathsf{T}|=2$, such that $\mathsf{T}\subseteq\xi_{\mathfrak{d}}^{\mathfrak{c}\leftarrow}(\{z\})$ and such that, if $\mathsf{T}\in\mathfrak{U}\cup\{\aleph\}$, then ${}_z\sigma_{\mathfrak{d}}^{\mathfrak{c}}(\mathsf{T})=0$. Moreover, let the labelings $\mathfrak{a}$ and $\mathfrak{b}$ be such that $\mathfrak{a}\,{}^\blacksquare_\blacksquare\,\mathfrak{b}=E(\mathfrak{c}\,{}^\blacksquare_\blacksquare\,\mathfrak{d},\mathsf{T})$, let $\mathsf{Y}=\xi_{\mathfrak{b}}^{\mathfrak{a}\leftarrow}(\mathfrak{U}\cup\mathfrak{O})$ and $\mathfrak{u}$ and $\mathfrak{v}$ such that $\xi_{\mathfrak{v}}^{\mathfrak{u}}=\pi_w\circ\varepsilon_{\mathfrak{b}}^{\mathfrak{a}}\circ\gamma_{\mathsf{Y},\ell-2}{}^{k}$ and $\zeta_{\mathfrak{v}}^{\mathfrak{u}}=\zeta_{\mathfrak{b}}^{\mathfrak{a}}\circ\gamma_{\mathsf{Y},\ell-2}{}^{k}$. We have to demonstrate $(\mathfrak{u},\mathfrak{v},R(E(p,\mathsf{T}),\mathsf{Y}))$.

If $\mathsf{Z}=\xi_{\mathfrak{d}}^{\mathfrak{c}\leftarrow}(\mathfrak{U}\cup\mathfrak{O})$ and if $\mathfrak{f}$ and $\mathfrak{g}$ are such that $\xi_{\mathfrak{g}}^{\mathfrak{f}}=\pi_w\circ\xi_{\mathfrak{d}}^{\mathfrak{c}}\circ\gamma_{\mathsf{Z},\ell}{}^{k}$ and $\zeta_{\mathfrak{g}}^{\mathfrak{f}}=\zeta_{\mathfrak{d}}^{\mathfrak{c}}\circ\gamma_{\mathsf{Z},\ell}{}^{k}$, then $(\mathfrak{f},\mathfrak{g},R(p,\mathsf{Z}))\in\mathcal{X}$ by $(\mathfrak{c},\mathfrak{d},p)\in\mathcal{X}\rtimes\mathcal{Z}_w$. Furthermore, either $\mathsf{T}\cap\mathsf{Z}=\varnothing$ or $\mathsf{T}\subseteq\mathsf{Z}$ because $\mathsf{T}\subseteq\xi_{\mathfrak{d}}^{\mathfrak{c}\leftarrow}(\{z\})$ and $\mathsf{Z}=\xi_{\mathfrak{d}}^{\mathfrak{c}\leftarrow}(\mathfrak{U}\cup\mathfrak{O})$. In addition, as $\xi_{\mathfrak{g}}^{\mathfrak{f}}=\xi_{\mathfrak{d}}^{\mathfrak{c}}\circ\gamma_{\mathsf{Z},\ell}{}^{k}$ by Lemma 4.2 (d), if we let $\mathsf{V}=\gamma_{\mathsf{Z},\ell}{}^{k\leftarrow}(\mathsf{T})$, then $\mathsf{V}\subseteq\xi_{\mathfrak{g}}^{\mathfrak{f}\leftarrow}(\{z\})$ and, if $\mathsf{T}\subseteq\mathsf{Z}$, then ${}_z\sigma_{\mathfrak{g}}^{\mathfrak{f}}(\mathsf{V})={}_z\sigma_{\mathfrak{d}}^{\mathfrak{c}}(\mathsf{T})$ by the same lemma. Because $\mathcal{X}$ is a category of labeled partitions it thus suffices to prove that $(\mathfrak{u},\mathfrak{v},R(E(p,\mathsf{T}),\mathsf{Y}))$ is identical to $(\mathfrak{f},\mathfrak{g},R(p,\mathsf{Z}))$ if $\mathsf{T}\cap\mathsf{Z}=\varnothing$ and to $E((\mathfrak{f},\mathfrak{g},R(p,\mathsf{Z})),\mathsf{V})$ if $\mathsf{T}\subseteq\mathsf{Z}$.

If we abbreviate $\mathsf{M}=\Pi_\ell^k\backslash\mathsf{T}$, then, as $\xi_{\mathfrak{d}}^{\mathfrak{c}}\circ\gamma_{\mathsf{M},\ell}{}^{k}=\xi_{\mathfrak{b}}^{\mathfrak{a}}$ by Lemma 4.2 (d), we can infer $\mathsf{Y}=\gamma_{\mathsf{M},\ell}{}^{k\leftarrow}(\mathsf{Z})$. Hence and since by Lemma 4.5 (d) the partition $R(E(p,\mathsf{T}),\mathsf{Y})$ is identical to $R(p,\mathsf{Z})$ if $\mathsf{T}\cap\mathsf{Z}=\varnothing$ and to $E(R(p,\mathsf{Z}),\mathsf{V})$ if $\mathsf{T}\subseteq\mathsf{Z}$, we actually only need to show

$$\mathfrak{u}\,{}^\blacksquare_\blacksquare\,\mathfrak{v}=\begin{cases}\mathfrak{f}\,{}^\blacksquare_\blacksquare\,\mathfrak{g} & \text{if } \mathsf{T}\cap\mathsf{Z}=\varnothing\\ E(\mathfrak{f}\,{}^\blacksquare_\blacksquare\,\mathfrak{g},\mathsf{V}) & \text{if } \mathsf{T}\subseteq\mathsf{Z}.\end{cases}$$

Moreover, because, in terms of colors, $\mathfrak{u}\,{}^\blacksquare_\blacksquare\,\mathfrak{v}$ coincides with $R(\mathfrak{a}\,{}^\blacksquare_\blacksquare\,\mathfrak{b},\mathsf{Y})$ and $\mathfrak{f}\,{}^\blacksquare_\blacksquare\,\mathfrak{g}$ with $R(\mathfrak{c}\,{}^\blacksquare_\blacksquare\,\mathfrak{d},\mathsf{Z})$, all we really need to prove is that

$$\xi_{\mathfrak{v}}^{\mathfrak{u}}=\xi_{\mathfrak{g}}^{\mathfrak{f}}\circ\gamma_{\mathsf{Q},n}{}^{m},$$

regardless of whether $\mathsf{T}\cap\mathsf{Z}=\varnothing$ or $\mathsf{T}\subseteq\mathsf{Z}$ (because $\gamma_{\mathsf{Q},n}{}^{m}$ is simply the identy on Π_n^m if $\mathsf{T}\cap\mathsf{Z}=\varnothing$). This we do in two steps.

Step 5.1: As an intermediate result we first show that for any $\mathsf{s}\in\mathsf{Y}$,

$$(\varepsilon_{\mathsf{d}}^{\mathsf{c}}\circ\gamma_{\mathsf{M},\ell}{}^{k})(\mathsf{s})=\varepsilon_{\mathfrak{b}}^{\mathfrak{a}}(\mathsf{s}).$$

By Lemma 4.43 (d), because $\mathsf{T}\subseteq\Pi_\ell^0$ and thus $]\gamma_{\mathsf{M},\ell}{}^{k}(\mathsf{s}),{}^\blacksquare 1]_\ell^k\cap\mathsf{T}=\varnothing$ for any $\mathsf{s}\in\Pi_\ell^k$ if ${}^\blacksquare 1\notin\mathsf{T}$ and $\mathsf{s}\neq{}^\blacksquare 1$, that is the same as proving for any $\mathsf{s}\in\Pi_\ell^0$ that, if ${}_\blacksquare 1\notin\mathsf{T}$ or $\mathsf{s}\neq{}_\blacksquare 1$, then

$${}_\aleph\sigma_{\mathsf{d}}^{\mathsf{c}}([{}_\blacksquare 1,\gamma_{\mathsf{M},\ell}{}^{k}(\mathsf{s})[_\ell^k\cap\mathsf{T})=0.$$

If $\mathsf{T}\subseteq\mathsf{Z}$, then $\mathsf{T}\subseteq\xi_{\mathfrak{d}}^{\mathfrak{c}\leftarrow}(\{z\})$ and $\mathsf{Z}=\xi_{\mathfrak{d}}^{\mathfrak{c}\leftarrow}(\mathfrak{U}\cup\mathfrak{O})$ demand $z\neq\aleph$ and thus $\mathsf{T}\cap\xi_{\mathfrak{d}}^{\mathfrak{c}\leftarrow}(\{\aleph\})=\varnothing$. Hence, indeed, ${}_\aleph\sigma_{\mathsf{d}}^{\mathsf{c}}([{}_\blacksquare 1,\gamma_{\mathsf{M},\ell}{}^{k}(\mathsf{s})[_\ell^k\cap\mathsf{T})=0$ in that case.

Should $\mathsf{T}\cap\mathsf{Z}=\varnothing$ instead, then by the same reasoning, $z=\aleph$. In that situation, if ${}_\blacksquare 1\in\mathsf{T}$, then, necessarily, $\mathsf{T}=\{{}_\blacksquare 1,{}_\blacksquare 2\}$ because $\mathsf{T}\subseteq\Pi_\ell^0$, because $|\mathsf{T}|=2$ and because T is convex with respect to Γ_ℓ^k. Hence, then $\mathsf{T}\subseteq[{}_\blacksquare 1,\gamma_{\mathsf{M},\ell}{}^{k}(\mathsf{s})[_\ell^k$ because $\gamma_{\mathsf{M},\ell}{}^{k}(\mathsf{s})\neq{}_\blacksquare 2\in\mathsf{T}$ by $\gamma_{\mathsf{M},\ell}{}^{k}(\mathsf{s})\in\mathsf{M}=\Pi_\ell^k\backslash\mathsf{T}$, which then implies ${}_\aleph\sigma_{\mathsf{d}}^{\mathsf{c}}([{}_\blacksquare 1,\gamma_{\mathsf{M},\ell}{}^{k}(\mathsf{s})[_\ell^k\cap\mathsf{T})={}_z\sigma_{\mathfrak{d}}^{\mathfrak{c}}(\mathsf{T})=0$ by assumption. If, alternatively, ${}_\blacksquare 1\notin\mathsf{T}$, then either $\mathsf{T}\subseteq[{}_\blacksquare 1,\gamma_{\mathsf{M},\ell}{}^{k}(\mathsf{s})[_\ell^k$ or $[{}_\blacksquare 1,\gamma_{\mathsf{M},\ell}{}^{k}(\mathsf{s})[_\ell^k\cap\mathsf{T}$

because $\blacksquare 1 \notin \mathsf{T}$ and $\gamma_{\mathsf{M},\ell}^{\;k}(\mathtt{s}) \notin \mathsf{T} = \varnothing$ and because T is convex. In consequence, ${}_{\aleph}\sigma^{\mathtt{c}}_{\mathtt{d}}([\blacksquare 1, \gamma_{\mathsf{M},\ell}^{\;k}(\mathtt{s})[^{k}_{\ell} \cap \mathsf{T})$ is given by ${}_{z}\sigma^{\mathfrak{c}}_{\mathfrak{d}}(\mathsf{T})$ or ${}_{z}\sigma^{\mathfrak{c}}_{\mathfrak{d}}(\varnothing)$, which is zero in both cases.

Step 5.2: By the first step, for any $\mathtt{h} \in \Pi^{m}_{n}$, if we abbreviate $\mathtt{s} \equiv \gamma_{\mathsf{Y},\ell-2}^{\;k}(\mathtt{h})$, then because $\gamma_{\mathsf{M},\ell}^{\;k} \circ \gamma_{\mathsf{Y},\ell-2}^{\;k} = \gamma_{\mathsf{Z},\ell}^{\;k} \circ \gamma_{\mathsf{Q},n}^{\;m}$ by Lemma 4.5 (d),

$$\begin{aligned}
\xi^{\mathfrak{u}}_{\mathfrak{v}}(\mathtt{h}) &= (\pi_w \circ \varepsilon^{\mathtt{a}}_{\mathtt{b}} \circ \gamma_{\mathsf{Y},\ell-2}^{\;k})(\mathtt{h}) \\
&= (\pi_w \circ \varepsilon^{\mathfrak{a}}_{\mathfrak{b}})(\mathtt{s}) \\
&= (\pi_w \circ \varepsilon^{\mathtt{c}}_{\mathtt{d}} \circ \gamma_{\mathsf{M},\ell}^{\;k})(\mathtt{s}) \\
&= (\pi_w \circ \varepsilon^{\mathtt{c}}_{\mathtt{d}} \circ \gamma_{\mathsf{M},\ell}^{\;k} \circ \gamma_{\mathsf{Y},\ell-2}^{\;k})(\mathtt{h}) \\
&= (\pi_w \circ \varepsilon^{\mathtt{c}}_{\mathtt{d}} \circ \gamma_{\mathsf{Z},\ell}^{\;k} \circ \gamma_{\mathsf{Q},n}^{\;m})(\mathtt{h}) \\
&= (\xi^{\mathtt{f}}_{\mathtt{g}} \circ \gamma_{\mathsf{Q},n}^{\;m})(\mathtt{h}),
\end{aligned}$$

as claimed. That concludes the proof that $\mathcal{X} \rtimes \mathcal{Z}_w$ is a category. □

Whereas the definition of the crossed co-product is convenient for the proof of Proposition 4.44, the following reformulation will be helpful in applications

Proposition 4.45. *For any $w \in \mathbb{N}_0$, if $(\mathfrak{U}, \mathfrak{O}) \in \{(\mathbb{Z}_w, \varnothing), (\varnothing, \mathbb{Z}_w)\}$, then for any $\mathbb{Z}_w$-invariant category $\mathcal{X}$ of $(\mathfrak{U}, \mathfrak{O})$-tagged labeled partitions the crossed co-product $\mathcal{X} \rtimes \mathcal{Z}_w$ of $\mathcal{X}$ with $\mathcal{Z}_w$ can be expressed as the set of all $(\mathfrak{c}, \mathfrak{d}, p) \in {}_{\mathfrak{U}\cup\{\aleph\},\mathfrak{O}}\mathcal{S}$ for which there exist $\{k, \ell\} \subseteq \mathbb{N}_0$ such that $\mathfrak{c}\colon [\![k]\!] \to ((\mathfrak{U} \cup \{\aleph\}) \otimes \{\circ, \bullet\}) \cup \mathfrak{O}$ and $\mathfrak{d}\colon [\![\ell]\!] \to ((\mathfrak{U} \cup \{\aleph\}) \otimes \{\circ, \bullet\}) \cup \mathfrak{O}$, such that $p \leq \xi^{\mathfrak{c}\,\leftarrow}_{\mathfrak{d}}(\{\mathfrak{U} \cup \mathfrak{O}, \{\aleph\}\})$, such that*

(i) if $w \in \mathbb{N}$, then ${}_{\aleph}\Sigma^{\mathfrak{c}}_{\mathfrak{d}} \equiv_w 0$ and for any $\mathsf{B} \in p$ with $\mathsf{B} \subseteq \xi^{\mathfrak{c}\,\leftarrow}_{\mathfrak{d}}(\{\aleph\})$ always $|\mathsf{B}| \leq 2$ and, if $|\mathsf{B}| = 2$, then ${}_{\aleph}\sigma^{\mathfrak{c}}_{\mathfrak{d}}(\mathsf{B}) = 0$,

(ii) if $w = 0$, then for any $\mathsf{B} \in p$ with $\mathsf{B} \subseteq \xi^{\mathfrak{c}\,\leftarrow}_{\mathfrak{d}}(\{\aleph\})$ both $|\mathsf{B}| = 2$ and ${}_{\aleph}\sigma^{\mathfrak{c}}_{\mathfrak{d}}(\mathsf{B}) = 0$,

and, in either case, such that, if $\mathsf{Y} = \xi^{\mathfrak{c}\,\leftarrow}_{\mathfrak{d}}(\mathfrak{U} \cup \mathfrak{O})$ and if the labelings $\mathfrak{f}$ and $\mathfrak{g}$ are such that $\xi^{\mathfrak{f}}_{\mathfrak{g}} = \pi_w \circ \varepsilon^{\mathfrak{c}}_{\mathfrak{d}} \circ \gamma_{\mathsf{Y},\ell}^{\;k}$ and $\zeta^{\mathfrak{f}}_{\mathfrak{g}} = \zeta^{\mathfrak{c}}_{\mathfrak{d}} \circ \gamma_{\mathsf{Y},\ell}^{\;k}$, then $(\mathfrak{f}, \mathfrak{g}, R(p, \mathsf{Y})) \in \mathcal{X}$,

Proof. Let $\{k, \ell\} \subseteq \mathbb{N}_0$, let $\mathfrak{c}\colon [\![k]\!] \to ((\mathfrak{U} \cup \{\aleph\}) \otimes \{\circ, \bullet\}) \cup \mathfrak{O}$ and $\mathfrak{d}\colon [\![\ell]\!] \to ((\mathfrak{U} \cup \{\aleph\}) \otimes \{\circ, \bullet\}) \cup \mathfrak{O}$, and let $(\mathfrak{c}, \mathfrak{d}, p) \in {}_{\mathfrak{U}\cup\{\aleph\},\mathfrak{O}}\mathcal{S}$ be such that $p \leq \xi^{\mathfrak{c}\,\leftarrow}_{\mathfrak{d}}(\{\mathfrak{U} \cup \mathfrak{O}, \{\aleph\}\})$, such that, if $\mathsf{Y} = \xi^{\mathfrak{c}\,\leftarrow}_{\mathfrak{d}}(\mathfrak{U} \cup \mathfrak{O})$ and if the labelings $\mathfrak{f}$ and $\mathfrak{g}$ satisfy $\xi^{\mathfrak{f}}_{\mathfrak{g}} = \pi_w \circ \varepsilon^{\mathfrak{c}}_{\mathfrak{d}} \circ \gamma_{\mathsf{Y},\ell}^{\;k}$ and $\zeta^{\mathfrak{f}}_{\mathfrak{g}} = \zeta^{\mathfrak{c}}_{\mathfrak{d}} \circ \gamma_{\mathsf{Y},\ell}^{\;k}$, then $(\mathfrak{f}, \mathfrak{g}, R(p, \mathsf{Y})) \in \mathcal{X}$. In order to prove the claim, we have to show that, if $\mathsf{W} = \xi^{\mathfrak{c}\,\leftarrow}_{\mathfrak{d}}(\{\aleph\})$ and $(\mathfrak{a}, \mathfrak{b}, q) = R((\mathfrak{c}, \mathfrak{d}, p), \mathsf{W})$, then $(\mathfrak{a}, \mathfrak{b}, q) \in \mathcal{Z}_w$ if and only if, depending on w, the conditions of (i) respectively (ii) are met.

First, we notice that Lemma 4.2 (d) implies ${}_{\aleph}\Sigma^{\mathfrak{a}}_{\mathfrak{b}} = {}_{\aleph}\sigma^{\mathfrak{a}}_{\mathfrak{b}}(\gamma_{\mathsf{W},\ell}^{\;k\leftarrow}(\Pi^{k}_{\ell})) = {}_{\aleph}\sigma^{\mathfrak{c}}_{\mathfrak{d}}(\Pi^{k}_{\ell}) = {}_{\aleph}\Sigma^{\mathfrak{c}}_{\mathfrak{d}}$ since ${}_{\aleph}\sigma^{\mathfrak{c}}_{\mathfrak{d}}(\Pi^{k}_{\ell} \backslash \mathsf{W}) = 0$ by definition of W.

Moreover, with $x \coloneqq \alpha(\mathsf{W})$ and $y \coloneqq \beta(\mathsf{W})$, consider any $\mathsf{C} \subseteq \Pi^{x}_{y}$ and $\mathsf{B} \subseteq \mathsf{W}$ with $\mathsf{C} = \gamma_{\mathsf{W},\ell}^{\;k\leftarrow}(\mathsf{B})$. Then, $|\mathsf{C}| = |\mathsf{B}|$ because $\gamma_{\mathsf{W},\ell}^{\;k}$ is injective. Furthermore, ${}_{\aleph}\sigma^{\mathfrak{c}}_{\mathfrak{d}}(\mathsf{B}) = {}_{\aleph}\sigma^{\mathfrak{a}}_{\mathfrak{b}}(\mathsf{C})$ by $\mathcal{Z}_w$ Lemma 4.2 (d) because $\mathsf{B} \backslash \mathsf{W} = \varnothing$ and thus ${}_{\aleph}\sigma^{\mathfrak{c}}_{\mathfrak{d}}(\mathsf{B} \backslash \mathsf{W}) = 0$.

Now, the claim follows immediately from the definition of $\mathcal{Z}_w$. □

4.6. Wreath graph co-products with cyclic groups. One special case of the crossed co-product construction deserves emphasis, namely the one where the other category is a (big or little) graph co-product.

Definition 4.46. For any $w \in \mathbb{N}_0$ any partial commutation relation r on $\mathbb{Z}_w$ is called $\mathbb{Z}_w$-*invariant* if for any $(z, z') \in r$ also $(z +_w s, z' +_w s) \in r$ for any $s \in \mathbb{Z}_w$.

Lemma 4.47. *For any category $\mathcal{C}$ of two-colored or uncolored partitions, any $w \in \mathbb{N}_0$ and any partial commutation relation r on $\mathbb{Z}_w$, if r is $\mathbb{Z}_w$-invariant, then*

(a) *the big graph power category $\mathcal{C}^{\star(\mathbb{Z}_w, r)}$ is $\mathbb{Z}_w$-invariant.*

(b) *the (little) graph power category $\mathcal{C}^{*(\mathbb{Z}_w, r)}$ is $\mathbb{Z}_w$-invariant.*

Proof. Recall that $\mathcal{C}^{\star(\mathbb{Z}_w, r)}$ and $\mathcal{C}^{*(\mathbb{Z}_w, r)}$ are given by $\star^r_{z\in\mathbb{Z}_w} \mathcal{X}_z$ respectively $*^r_{z\in\mathbb{Z}_w} \mathcal{X}_z$, where for each $z \in \mathbb{Z}_w$ the category $\mathcal{X}_z$ is given by $\mathcal{C}$, seen as tagged with the single tag z. If we reprise the notation V from Definition 4.41 for the actions of $\mathbb{Z}_w$ on the tag as well as the label set, then what the definition of $(\mathcal{X}_z)_{z\in\mathbb{Z}_w}$ means is that for any $s \in \mathbb{Z}_w$ and any $(\mathfrak{u}, \mathfrak{v}, w) \in \mathcal{X}_z$, also $(V_s \circ \mathfrak{u}, V_s \circ \mathfrak{v}, w) \in \mathcal{X}_{z+_w s}$. Write $(\mathfrak{U}, \mathfrak{D})$ for $(\mathbb{Z}_w, \varnothing)$ if $\mathcal{C}$ is two-colored and $(\varnothing, \mathbb{Z}_w)$ if $\mathcal{C}$ is uncolored.

(a) Let $\{k, \ell\} \subseteq \mathbb{N}_0$ and $\mathfrak{c}\colon [\![k]\!] \to (\mathfrak{U} \otimes \{\circ, \bullet\}) \cup \mathfrak{D}$ and $\mathfrak{d}\colon [\![\ell]\!] \to (\mathfrak{U} \otimes \{\circ, \bullet\}) \cup \mathfrak{D}$ and $(\mathfrak{c}, \mathfrak{d}, p) \in \star^r_{z\in\mathbb{Z}_w} \mathcal{X}_z$ and $s \in \mathbb{Z}_w$ be arbitrary. We have to prove $(V_s \circ \mathfrak{c}, V_s \circ \mathfrak{d}, p) \in \star^r_{z\in\mathbb{Z}_w} \mathcal{X}_z$.

First, we recognize that because $\xi^{V_s\circ\mathfrak{c}}_{V_s\circ\mathfrak{d}} = V_s \circ \xi^{\mathfrak{c}}_{\mathfrak{d}}$ and $\zeta^{V_s\circ\mathfrak{c}}_{V_s\circ\mathfrak{d}} = \zeta^{\mathfrak{c}}_{\mathfrak{d}}$. That implies in particular $\ker(\xi^{V_s\circ\mathfrak{c}}_{V_s\circ\mathfrak{d}}) = \xi^{\mathfrak{c}\leftarrow}_{\mathfrak{d}}(V_s^{\leftarrow}(\{z\}_{z\in\mathbb{Z}_w}))$ and thus $\ker(\xi^{V_s\circ\mathfrak{c}}_{V_s\circ\mathfrak{d}}) = \ker(\xi^{\mathfrak{c}}_{\mathfrak{d}})$ by $V_s^{\leftarrow}(\{z\}_{z\in\mathbb{Z}_w}) = \{z\}_{z\in\mathbb{Z}_w}$ by definition of V. It follows that $p \leq \ker(\xi^{V_s\circ\mathfrak{c}}_{V_s\circ\mathfrak{d}})$ since $p \leq \ker(\xi^{\mathfrak{c}}_{\mathfrak{d}})$ by $(\mathfrak{c}, \mathfrak{d}, p) \in \star^r_{z\in\mathbb{Z}_w} \mathcal{X}_z$.

Second, given any $z \in \mathbb{Z}_w$, if $\mathsf{Y} = \xi^{V_s\circ\mathfrak{c}\leftarrow}_{V_s\circ\mathfrak{d}}(\{z\})$, then we have to show $R((V_s \circ \mathfrak{c}, V_s \circ \mathfrak{d}, p), \mathsf{Y}) \in \mathcal{X}_z$. From $\xi^{V_s\circ\mathfrak{c}}_{V_s\circ\mathfrak{d}} = V_s \circ \xi^{\mathfrak{c}}_{\mathfrak{d}}$ it follows $\mathsf{Y} = \xi^{\mathfrak{c}\leftarrow}_{\mathfrak{d}}(V_s^{\leftarrow}(\{z\})) = \xi^{\mathfrak{c}\leftarrow}_{\mathfrak{d}}(\{z -_w s\})$. Hence, if $(\mathfrak{u}, \mathfrak{v}, w) = R((\mathfrak{c}, \mathfrak{d}, p), \mathsf{Y})$, we may infer that $(\mathfrak{u}, \mathfrak{v}, w) \in \mathcal{X}_{z -_w s}$ by our assumption that $(\mathfrak{c}, \mathfrak{d}, p) \in \star^r_{z\in\mathbb{Z}_w} \mathcal{X}_z$ and by the definition of $\star^r_{z\in\mathbb{Z}_w} \mathcal{X}_z$. According to our initial observation that implies $(V_s \circ \mathfrak{u}, V_s \circ \mathfrak{v}, w) \in \mathcal{X}_{(z-_w s)+_w s} = \mathcal{X}_z$. Moreover, since also $\xi^{V_s\circ\mathfrak{u}}_{V_s\circ\mathfrak{v}} = V_s \circ \xi^{\mathfrak{u}}_{\mathfrak{v}}$ and $\zeta^{V_s\circ\mathfrak{u}}_{V_s\circ\mathfrak{v}} = \zeta^{\mathfrak{u}}_{\mathfrak{v}}$ we find $\xi^{V_s\circ\mathfrak{u}}_{V_s\circ\mathfrak{v}} = V_s \circ \xi^{\mathfrak{u}}_{\mathfrak{v}} = V_s \circ \xi^{\mathfrak{c}}_{\mathfrak{d}} \circ \gamma_{\mathsf{Y},\ell}^{\,k} = \xi^{V_s\circ\mathfrak{c}}_{V_s\circ\mathfrak{d}} \circ \gamma_{\mathsf{Y},\ell}^{\,k}$ and, likewise, $\zeta^{V_s\circ\mathfrak{u}}_{V_s\circ\mathfrak{v}} = \zeta^{\mathfrak{u}}_{\mathfrak{v}} = \zeta^{\mathfrak{c}}_{\mathfrak{d}} \circ \gamma_{\mathsf{Y},\ell}^{\,k} = \zeta^{V_s\circ\mathfrak{c}}_{V_s\circ\mathfrak{d}} \circ \gamma_{\mathsf{Y},\ell}^{\,k}$, where in both cases we have used Lemma 4.2 (d). But that is to say $R((V_s \circ \mathfrak{c}, V_s \circ \mathfrak{d}, p), \mathsf{Y}) = (V_s \circ \mathfrak{u}, V_s \circ \mathfrak{v}, w)$. Thus, indeed, $R((V_s \circ \mathfrak{c}, V_s \circ \mathfrak{d}, p), \mathsf{Y}) \in \mathcal{X}_z$.

It remains to let $\{z, z'\} \subseteq \mathbb{Z}_w$ and $\{\mathsf{B}, \mathsf{B}'\} \subseteq p$ be arbitrary with $z \neq z'$ and $(z, z') \neq r$ and $\mathsf{B} \subseteq \xi^{V_s\circ\mathfrak{c}\leftarrow}_{V_s\circ\mathfrak{d}}(\{z\})$ and $\mathsf{B}' \subseteq \xi^{V_s\circ\mathfrak{c}\leftarrow}_{V_s\circ\mathfrak{d}}(\{z'\})$ and to prove $\mathsf{B} \not\!\!\diagdown^k_\ell \mathsf{B}'$. With $\xi^{V_s\circ\mathfrak{c}}_{V_s\circ\mathfrak{d}} = V_s \circ \xi^{\mathfrak{c}}_{\mathfrak{d}}$ we once more conclude $\mathsf{B} \subseteq \xi^{\mathfrak{c}\leftarrow}_{\mathfrak{d}}(\{z -_w s\})$ and $\mathsf{B}' \subseteq \xi^{\mathfrak{c}\leftarrow}_{\mathfrak{d}}(\{z' -_w s\})$. Of course, $z -_w s \neq z' -_w s$. If $(z -_w s, z' -_w s) \in r$ was true, the $\mathbb{Z}_w$-invariance of r would imply $(z, z') = ((z -_w s) +_s, (z' -_w s) +_s) \in r$, violating the assumption $(z, z') \notin r$. Hence also $(z -_w s, z' -_w s) \notin r$. Because $(\mathfrak{c}, \mathfrak{d}, p) \in \star^r_{z\in\mathbb{Z}_w} \mathcal{X}_z$ we can thus infer that $\mathsf{B} \not\!\!\diagdown^k_\ell \mathsf{B}'$ as claimed. Altogether, we have shown $\star^r_{z\in\mathbb{Z}_w} \mathcal{X}_z$ to be $\mathbb{Z}_w$-invariant.

(b) Now, let $\{k, \ell\} \subseteq \mathbb{N}_0$ and $\mathfrak{c}\colon [\![k]\!] \to (\mathfrak{U} \otimes \{\circ, \bullet\}) \cup \mathfrak{D}$ and $\mathfrak{d}\colon [\![\ell]\!] \to (\mathfrak{U} \otimes \{\circ, \bullet\}) \cup \mathfrak{D}$ and $(\mathfrak{c}, \mathfrak{d}, p) \in \star^r_{z\in\mathbb{Z}_w} \mathcal{X}_z$ and $s \in \mathbb{Z}_w$ and let h be a history for $(\mathfrak{c}, \mathfrak{d}, p)$ with respect to $(\mathcal{X}_z)_{z\in\mathfrak{U}\cup\mathfrak{D}}$ and r. We prove that h is also a history for $(V_s \circ \mathfrak{c}, V_s \circ \mathfrak{d}, p)$. If $\mathcal{N}_z \coloneqq {}_{\{z\},\varnothing}\mathcal{S}^+$ for any $z \in \mathfrak{U}$ and $\mathcal{N}_z \coloneqq {}_{\varnothing,\{z\}}\mathcal{S}^+$ for any $z \in \mathfrak{D}$, then $\star^r_{z\in\mathfrak{U}\cup\mathfrak{D}} \mathcal{N}_z$ is $\mathbb{Z}_w$-invariant by (a). The assumption that $(\mathfrak{c}, \mathfrak{d}, h) \in \star^r_{z\in\mathfrak{U}\cup\mathfrak{D}} \mathcal{N}_z$ therefore ensures that also $(V_s \circ \mathfrak{c}, V_s \circ \mathfrak{d}, h) \in \mathop{\Large\star}^r_{z\in\mathfrak{U}\cup\mathfrak{D}} \mathcal{N}_z$. Moreover, h being a history for $(\mathfrak{c},\mathfrak{d},p)$ tells us that for any $z \in \mathfrak{U} \cup \mathfrak{D}$ and any $\mathsf{H} \in h$ with $h \subseteq \xi^{V_s\circ\mathfrak{c}\leftarrow}_{V_s\circ\mathfrak{d}}(\{z\}) = \xi^{\mathfrak{c}\leftarrow}_{\mathfrak{d}}(\{z -_w s\})$ already $(\mathfrak{u},\mathfrak{v},w) \coloneqq R((\mathfrak{c},\mathfrak{d},p),\mathsf{H}) \in \mathcal{X}_{z-_w s}$. Since that implies $(V_s \circ \mathfrak{u}, V_s \circ \mathfrak{v}, w) \in \mathcal{X}_{(z-_w s)+_w s} = \mathcal{X}_z$ by our assumption from the very beginning of the proof and since $R((V_s \circ \mathfrak{c}, V_s \circ \mathfrak{d}, p),\mathsf{H}) = (V_s \circ \mathfrak{u}, V_s \circ \mathfrak{v}, w)$ it follows $R((V_s \circ \mathfrak{c}, V_s \circ \mathfrak{d}, p),\mathsf{H}) \in \mathcal{X}_z$. In other words, h is a history for $(V_s \circ \mathfrak{c}, V_s \circ \mathfrak{d}, p)$ and thus $(V_s \circ \mathfrak{c}, V_s \circ \mathfrak{d}, p)$ an element of $\ast^r_{z\in\mathbb{Z}_w} \mathcal{X}_z$ and thus the latter r-invariant. □

Hence, the following makes sense.

DEFINITION 4.48. Given any category $\mathcal{C}$ of two-colored or uncolored partitions, any $w \in \mathbb{N}_0$ and any $\mathbb{Z}_w$-invariant partial commutation relation r on $\mathbb{Z}_w$, the *(little) wreath graph co-product category* $\mathcal{C} \wr_r \mathcal{Z}_w$ of $\mathcal{C}$ and $\mathcal{Z}_w$ with respect to r is the crossed co-product category $\mathcal{C}^{*(\mathbb{Z}_w,r)} \rtimes \mathcal{Z}_w$.

Of course, one could also define a "big wreath graph co-product category". Since, in all our cases here, the two coincide, we can save on symbols, though, in this instance.

NOTATION 4.49. In Definition 4.48, if r is the trivial partial commutation relation $\{(z,z') \mid \{z,z'\} \subseteq \mathbb{Z}_w \wedge z \neq z'\}$, we also speak of simply the *wreath co-product* of $\mathcal{C}$ and $\mathcal{Z}_w$ and omit r from the notation, writing $\mathcal{C} \wr \mathcal{Z}_w$ for $\mathcal{C} \wr_r \mathcal{Z}_w$.

By the way, it is easy to classify all invariant partial commutation relations on cyclic groups.

PROPOSITION 4.50. *(a) $\varnothing$ is the only $\mathbb{Z}_1$-invariant partial commutation relation r on $\mathbb{Z}_1$.*

(b) For any $w \in \mathbb{N}$ with $2 \leq w$ and any $\mathbb{Z}_w$-invariant partial commutation relation r on $\mathbb{Z}_w$ there exists a unique $X \subseteq \{1,2,\ldots,\lfloor \frac{w}{2} \rfloor\}$ such that $r = \{(z,z') \mid \{z,z'\} \subseteq \mathbb{Z}_w \wedge (z' -_w z \in X \vee z -_w z' \in X)\}$.

(c) For any $\mathbb{Z}$-invariant partial commutation relation r on $\mathbb{Z}$ there exists a unique $X \subseteq \mathbb{N}$ such that $r = \{(z,z') \mid \{z,z'\} \subseteq \mathbb{Z} \wedge |z'-z| \in X\}$.

PROOF. Part (a) is clear because $\mathbb{Z}_1 = \{0\}$ is a singleton set and partial commutation relations must be anti-reflexive. We can prove (b) and (c) simultaneously. Hence, let $w \in \mathbb{N}_0 \backslash \{1\}$ and let r be any $\mathbb{Z}_w$-invariant partial commutation relation on $\mathbb{Z}_w$.

Existence: Let $Y_z \coloneqq \{z' -_w z \mid (z,z') \in r\}$ for any $z \in \mathbb{Z}_w$. Then, $Y_z = Y \coloneqq Y_0$ for any $z \in \mathbb{Z}_w$. Indeed, if $(z,z') \in r$ and $d = z' -_w z \in Y_z$, then also $(z \pm_w 1, z' \pm_w 1) \in r$ by the $\mathbb{Z}_w$-invariance of r, which implies $d = (z' \pm_w 1) -_w (z \pm_w 1) \in Y_{z\pm_w 1}$, i.e., $Y_z \subseteq Y_{z\pm_1}$. Hence, $Y_z = Y$ for any $z \in \mathbb{Z}_w$ by induction.

Moreover, $r = s \coloneqq \{(z,z') \mid \{z,z'\} \subseteq \mathbb{Z} \wedge z' -_w z \in Y\}$ for the following reasons. If $(z,z') \in r$, then $z' -_w z \in Y_z = Y$ by definition of Y and thus also $(z,z') \in r$ by definition of r. Conversely, if $(z,z') \in s$, then $z' -_w z \in Y = Y_z$ by definition of s, which requires $(z,z') \in r$ by definition of Y_z.

Furthermore, Y is closed under inversion in $\mathbb{Z}_w$. That is because for any $z = z -_w 0 \in Y$ first $(0,z) \in r$ by definition of $Y = Y_0$ and thus also $(z,0) \in r$ by the symmetry of r, from which it follows $-_w z = 0 -_w z \in Y_z = Y$ by definition of Y_z.

Finally, $0 \notin Y$ because by definition of $Y_0 = Y$ the opposite would require $(0,0) \in r$, contradicting the anti-reflexivity of r.

But that means $Y = X \uplus \{-_w z \,|\, z \in X\}$ for X defined as $Y \cap \{1,2,\ldots,\lfloor \frac{w}{2} \rfloor\}$ if $w \in \mathbb{N}$ and as $Y \cap \mathbb{N}$ if $w = 0$. In the case $w = 0$ this is clear. If $w \in \mathbb{N}$, then the inclusion $Y \supseteq X \uplus \{-_w z \,|\, z \in X\}$ is evident and the converse one is seen as follows. If $d \in Y \subseteq \{0,1,\ldots,w-1\}$, then, of course, $d \in X \uplus \{-_w z \,|\, z \in X\}$ if $d \leq \frac{w}{2}$. Should $\frac{w}{2} < d$, then also $d = -_w(-_w d) \in X \uplus \{-_w z \,|\, z \in X\}$ because then $-_w d \in X$ by $-_w d = w - d < w - \frac{w}{2} = \frac{w}{2}$.

As obviously, $s = \{(z,z') \,|\, \{z,z'\} \subseteq \mathbb{Z}_w \wedge (z' -_w z \in X \vee z -_w z' \in X)\}$ by definition of X, this proves the existence part of the claim, also in the case $w = 0$.

Uniqueness: For each $i \in [\![2]\!]$ let the set X_i be such that $r = \{(z,z') \,|\, \{z,z'\} \subseteq \mathbb{Z}_w \wedge (z' -_w z \in X_i \vee z -_w z' \in X_i)\}$. Given any $i \in [\![2]\!]$ and any $z \in X_i$ we can infer $(0,z) \in r$ because $z -_w 0 \in X_i$. By assumption that requires $z = z -_w 0 \in X_{3-i}$ or $-_w z = 0 -_w z \in X_{3-i}$. If $w = 0$, then $-_w z$ is not an element of X_{3-i} because $X_{3-i} \subseteq \mathbb{N}$ and $z \in X_i \subseteq \mathbb{N}$ per assumption. In the case $w \in \mathbb{N}$ we can also infer $z \in X_{3-i}$, however for more complicated reasons. If $z < \frac{w}{2}$, then $-_w z$ is not an element of X_{3-i} because $X_{3-i} \subseteq \{1,2,\ldots,\lfloor \frac{w}{2} \rfloor\}$ and $\frac{w}{2} = w - \frac{w}{2} < w - z = -_w z$. And, if $z = \frac{w}{2}$, then $-_w z = w - z = w - \frac{w}{2} = \frac{w}{2} = z$. Hence, $X_i \subseteq X_{3-i}$ in any case. That proves the uniqueness part of the claim. $\square$

In order to find a simpler description of the partitions of a wreath graph coproduct category the following lemma will be very helpful.

Lemma 4.51. *If $\mathfrak{L}$ is either $(\mathbb{Z} \uplus \{\aleph\}) \otimes \{\circ,\bullet\}$ or $(\{\aleph\} \otimes \{\circ,\bullet\}) \uplus \mathbb{Z}$, then for any $\{k,\ell\} \subseteq \mathbb{N}_0$, any $\mathfrak{c}\colon [\![k]\!] \to \mathfrak{L}$ and $\mathfrak{d}\colon [\![\ell]\!] \to \mathfrak{L}$, and any $\{\mathtt{a},\mathtt{b}\} \subseteq \Pi^k_\ell$,*

$$\varepsilon^{\mathfrak{c}}_{\mathfrak{d}}(\mathtt{b}) - \varepsilon^{\mathfrak{c}}_{\mathfrak{d}}(\mathtt{a}) \equiv \xi^{\mathfrak{c}}_{\mathfrak{d}}(\mathtt{b}) - \xi^{\mathfrak{c}}_{\mathfrak{d}}(\mathtt{a}) + {}_{\aleph}\delta^{\mathfrak{c}}_{\mathfrak{d}}(\mathtt{a},\mathtt{b})$$

with respect to the additive subgroup of $\mathbb{Q}$ generated by $\{{}_{\aleph}\Sigma^{\mathfrak{c}}_{\mathfrak{d}}, \frac{1}{2}{}_{\aleph}\sigma^{\mathfrak{c}}_{\mathfrak{d}}(\{\mathtt{a}\}), \frac{1}{2}{}_{\aleph}\sigma^{\mathfrak{c}}_{\mathfrak{d}}(\{\mathtt{b}\})\}$.

Proof. All congruences in the following will be modulo the subgroup from the claim. By definition, for any $\mathtt{t} \in \Pi^k_\ell$,

$$\varepsilon^{\mathfrak{c}}_{\mathfrak{d}}(\mathtt{t}) = \xi^{\mathtt{c}}_{\mathtt{d}}(\mathtt{t}) + \begin{cases} -{}_{\aleph}\sigma^{\mathfrak{c}}_{\mathfrak{d}}(]\mathtt{t}, {}^{\blacksquare}1]^k_\ell) & \text{if } \mathtt{t} \in \Pi^k_0 \backslash \{{}^{\blacksquare}1\} \\ {}_{\aleph}\sigma^{\mathtt{c}}_{\mathtt{d}}([{}_{\blacksquare}1, \mathtt{t}[^k_\ell) & \text{if } \mathtt{t} \in \Pi^0_\ell \backslash \{{}_{\blacksquare}1\} \\ 0 & \text{if } \mathtt{t} \in \{{}^{\blacksquare}1, {}_{\blacksquare}1\}, \end{cases}$$

where, if $\mathtt{t} \neq {}^{\blacksquare}1$, then

$$-{}_{\aleph}\sigma^{\mathfrak{c}}_{\mathfrak{d}}(]\mathtt{t}, {}^{\blacksquare}1]^k_\ell) = -{}_{\aleph}\delta^{\mathfrak{c}}_{\mathfrak{d}}(\mathtt{t}, {}^{\blacksquare}1) - \tfrac{1}{2}{}_{\aleph}\sigma^{\mathfrak{c}}_{\mathfrak{d}}(\{{}^{\blacksquare}1\}) + \tfrac{1}{2}{}_{\aleph}\sigma^{\mathfrak{c}}_{\mathfrak{d}}(\{\mathtt{t}\}),$$

and, likewise, if $\mathtt{t} \neq {}_{\blacksquare}1$, then

$${}_{\aleph}\sigma^{\mathfrak{c}}_{\mathfrak{d}}([{}_{\blacksquare}1, \mathtt{t}[^k_\ell) = {}_{\aleph}\delta^{\mathtt{c}}_{\mathtt{d}}({}_{\blacksquare}1, \mathtt{t}) + \tfrac{1}{2}{}_{\aleph}\sigma^{\mathtt{c}}_{\mathtt{d}}(\{{}_{\blacksquare}1\}) - \tfrac{1}{2}{}_{\aleph}\sigma^{\mathfrak{c}}_{\mathfrak{d}}(\{\mathtt{t}\}).$$

In particular, if $\mathtt{t}\in\{\mathtt{a},\mathtt{b}\}$, then

$$\varepsilon^{\mathfrak{c}}_{\mathfrak{d}}(\mathtt{t})\equiv\xi^{\mathtt{c}}_{\mathtt{d}}(\mathtt{t})+\begin{cases}-{}_\aleph\delta^{\mathfrak{c}}_{\mathfrak{d}}(\mathtt{t},{}^{\blacksquare}1)-\frac{1}{2}{}_\aleph\sigma^{\mathfrak{c}}_{\mathfrak{d}}(\{{}^{\blacksquare}1\}) & \text{if } \mathtt{t}\in\Pi^k_0\backslash\{{}^{\blacksquare}1\}\\ {}_\aleph\delta^{\mathtt{c}}_{\mathtt{d}}({}_{\blacksquare}1,\mathtt{t})+\frac{1}{2}{}_\aleph\sigma^{\mathfrak{c}}_{\mathfrak{d}}(\{{}_{\blacksquare}1\}) & \text{if } \mathtt{t}\in\Pi^0_\ell\backslash\{{}_{\blacksquare}1\}\\ 0 & \text{if } \mathtt{t}\in\{{}^{\blacksquare}1,{}_{\blacksquare}1\},\end{cases}$$

or, if we use that $\frac{1}{2}{}_\aleph\sigma^{\mathfrak{c}}_{\mathfrak{d}}(\{{}^{\blacksquare}1\})\equiv 0$ if $\mathtt{t}={}^{\blacksquare}1$ and that $\frac{1}{2}{}_\aleph\sigma^{\mathfrak{c}}_{\mathfrak{d}}(\{{}_{\blacksquare}1\})\equiv 0$ if $\mathtt{t}={}_{\blacksquare}1$, actually,

$$\varepsilon^{\mathfrak{c}}_{\mathfrak{d}}(\mathtt{t})\equiv\xi^{\mathtt{c}}_{\mathtt{d}}(\mathtt{t})+\begin{cases}-{}_\aleph\delta^{\mathfrak{c}}_{\mathfrak{d}}(\mathtt{t},{}^{\blacksquare}1)-\frac{1}{2}{}_\aleph\sigma^{\mathfrak{c}}_{\mathfrak{d}}(\{{}^{\blacksquare}1\}) & \text{if } \mathtt{t}\in\Pi^k_0\\ {}_\aleph\delta^{\mathtt{c}}_{\mathtt{d}}({}_{\blacksquare}1,\mathtt{t})+\frac{1}{2}{}_\aleph\sigma^{\mathfrak{c}}_{\mathfrak{d}}(\{{}_{\blacksquare}1\}) & \text{if } \mathtt{t}\in\Pi^0_\ell.\end{cases}$$

Thus, the number

$$(\varepsilon^{\mathfrak{c}}_{\mathfrak{d}}(\mathtt{b})-\varepsilon^{\mathfrak{c}}_{\mathfrak{d}}(\mathtt{a}))-(\xi^{\mathfrak{c}}_{\mathfrak{d}}(\mathtt{b})-\xi^{\mathfrak{c}}_{\mathfrak{d}}(\mathtt{a}))$$

is congruent with the expression

$$z:=\begin{Bmatrix}-{}_\aleph\delta^{\mathfrak{c}}_{\mathfrak{d}}(\mathtt{b},{}^{\blacksquare}1)-\frac{1}{2}{}_\aleph\sigma^{\mathfrak{c}}_{\mathfrak{d}}(\{{}^{\blacksquare}1\}) & \text{if } \mathtt{b}\in\Pi^k_0\\ {}_\aleph\delta^{\mathtt{c}}_{\mathtt{d}}({}_{\blacksquare}1,\mathtt{b})+\frac{1}{2}{}_\aleph\sigma^{\mathfrak{c}}_{\mathfrak{d}}(\{{}_{\blacksquare}1\}) & \text{if } \mathtt{b}\in\Pi^0_\ell\end{Bmatrix}-\begin{Bmatrix}-{}_\aleph\delta^{\mathfrak{c}}_{\mathfrak{d}}(\mathtt{a},{}^{\blacksquare}1)-\frac{1}{2}{}_\aleph\sigma^{\mathfrak{c}}_{\mathfrak{d}}(\{{}^{\blacksquare}1\}) & \text{if } \mathtt{a}\in\Pi^k_0\\ {}_\aleph\delta^{\mathtt{c}}_{\mathtt{d}}({}_{\blacksquare}1,\mathtt{a})+\frac{1}{2}{}_\aleph\sigma^{\mathfrak{c}}_{\mathfrak{d}}(\{{}_{\blacksquare}1\}) & \text{if } \mathtt{a}\in\Pi^0_\ell\end{Bmatrix}.$$

We prove the claim by showing $z\equiv{}_\aleph\delta^{\mathfrak{c}}_{\mathfrak{d}}(\mathtt{a},\mathtt{b})$. To do so, the four obvious cases need to be treated individually.

Case 1: First, let $\{\mathtt{a},\mathtt{b}\}\subseteq\Pi^k_0$. Then, z is given by $-{}_\aleph\delta^{\mathfrak{c}}_{\mathfrak{d}}(\mathtt{b},{}^{\blacksquare}1)+{}_\aleph\delta^{\mathfrak{c}}_{\mathfrak{d}}(\mathtt{a},{}^{\blacksquare}1)$. By Lemma 3.10 (b) that is congruent with ${}_\aleph\delta^{\mathtt{c}}_{\mathtt{d}}(\mathtt{a},{}^{\blacksquare}1)+{}_\aleph\delta^{\mathtt{c}}_{\mathtt{d}}({}^{\blacksquare}1,\mathtt{b})$, which in turn is congruent with ${}_\aleph\delta^{\mathfrak{c}}_{\mathfrak{d}}(\mathtt{a},\mathtt{b})$ by Lemma 3.10 (c).

Case 2: Similarly, in the case where $\{\mathtt{a},\mathtt{b}\}\subseteq\Pi^0_\ell$, the term z is ${}_\aleph\delta^{\mathtt{c}}_{\mathtt{d}}({}_{\blacksquare}1,\mathtt{b})-{}_\aleph\delta^{\mathfrak{c}}_{\mathfrak{d}}({}_{\blacksquare}1,\mathtt{a})$, which congrues with ${}_\aleph\delta^{\mathtt{c}}_{\mathtt{d}}(\mathtt{a},{}_{\blacksquare}1)+{}_\aleph\delta^{\mathtt{c}}_{\mathtt{d}}({}_{\blacksquare}1,\mathtt{b})$ by Lemma 3.10 (b). Thus, once more, Lemma 3.10 (c) reveals z to be congruent with ${}_\aleph\delta^{\mathfrak{c}}_{\mathfrak{d}}(\mathtt{a},\mathtt{b})$.

Case 3: However, if $\mathtt{a}\in\Pi^k_0$ and $\mathtt{b}\in\Pi^0_\ell$, at first nothing seems to cancel out. More precisely, then $z={}_\aleph\delta^{\mathfrak{c}}_{\mathfrak{d}}({}_{\blacksquare}1,\mathfrak{b})+\frac{1}{2}{}_\aleph\sigma^{\mathtt{c}}_{\mathtt{d}}(\{{}_{\blacksquare}1\})+{}_\aleph\delta^{\mathfrak{c}}_{\mathfrak{d}}(\mathtt{a},{}^{\blacksquare}1)+\frac{1}{2}{}_\aleph\sigma^{\mathfrak{c}}_{\mathfrak{d}}(\{{}^{\blacksquare}1\})$. But, since $]{}^{\blacksquare}1,{}_{\blacksquare}1[^k_\ell=\varnothing$, by definition, ${}_\aleph\delta^{\mathfrak{c}}_{\mathfrak{d}}({}^{\blacksquare}1,{}_{\blacksquare}1)=\frac{1}{2}{}_\aleph\sigma^{\mathfrak{c}}_{\mathfrak{d}}(\{{}^{\blacksquare}1\})+\frac{1}{2}{}_\aleph\sigma^{\mathfrak{c}}_{\mathfrak{d}}(\{{}_{\blacksquare}1\})$. Thus, actually $z={}_\aleph\delta^{\mathtt{c}}_{\mathtt{d}}(\mathtt{a},{}^{\blacksquare}1)+{}_\aleph\delta^{\mathtt{c}}_{\mathtt{d}}({}^{\blacksquare}1,{}_{\blacksquare}1)+{}_\aleph\delta^{\mathtt{c}}_{\mathtt{d}}({}_{\blacksquare}1,\mathtt{b})$, which is congruent with ${}_\aleph\delta^{\mathfrak{c}}_{\mathfrak{d}}(\mathtt{a},\mathtt{b})$ by two applications of Lemma 3.10 (c)

Case 4: Finally, if $\mathtt{a}\in\Pi^0_\ell$ and $\mathtt{b}\in\Pi^k_0$, then $z=-{}_\aleph\delta^{\mathtt{c}}_{\mathtt{d}}(\mathfrak{b},{}^{\blacksquare}1)-\frac{1}{2}{}_\aleph\sigma^{\mathtt{c}}_{\mathtt{d}}(\{{}^{\blacksquare}1\})-{}_\aleph\delta^{\mathtt{c}}_{\mathtt{d}}({}_{\blacksquare}1,\mathtt{a})-\frac{1}{2}{}_\aleph\sigma^{\mathfrak{c}}_{\mathfrak{d}}(\{{}_{\blacksquare}1\})$. For the same reason as in the previous case z can thus be rewritten as $-{}_\aleph\delta^{\mathtt{c}}_{\mathtt{d}}(\mathtt{b},{}^{\blacksquare}1)-{}_\aleph\delta^{\mathtt{c}}_{\mathtt{d}}({}^{\blacksquare}1,{}_{\blacksquare}1)-{}_\aleph\delta^{\mathtt{c}}_{\mathtt{d}}({}_{\blacksquare}1,\mathtt{a})$, which is congruent with $-{}_\aleph\delta^{\mathfrak{c}}_{\mathfrak{d}}(\mathtt{b},\mathtt{a})$ by Lemma 3.10 (c). Hence, a last application of Lemma 3.10 (b) concludes the proof. □

5. Generators of co-products of categories of labeled partitions

In order to be able to connect the co-products on the category level to the products on the quantum group level, Section 5 gives salient generators for (little) graph co-products and crossed co-products and thus in particular (little) wreath graph co-products.

5.1. Generators of graph co-products. The first major result of Section 5 is found in Section 5.1.6, namely Proposition 5.19 about the generators of graph co-product categories. The generating sets of partitions appearing there are introduced by Definition 5.12 in Section 5.1.4. The rest of Section 5.1 consists of extensive preparations for the proof of Proposition 5.19.

ASSUMPTIONS 5.1. In Section 5.1, let $(\mathfrak{U}, \mathfrak{O})$ be any choice of tags, r any partial commutation relation on $\mathfrak{U} \cup \mathfrak{O}$ and for each $z \in \mathfrak{U} \cup \mathfrak{O}$ let $\mathcal{X}_z$ be any category of $(\{z\}, \varnothing)$-tagged labeled partitions if $z \in \mathfrak{U}$ and of $(\varnothing, \{z\})$-tagged labeled partitions if $z \in \mathfrak{O}$.

The two main ideas behind the definitions and results of Section 5.1 can be summarized as follows:

(a) Any partition of $*^r_{z \in \mathfrak{U} \cup \mathfrak{O}} \mathcal{X}_z$ should result from ones of $\bigcup_{z \in \mathfrak{U} \cup \mathfrak{O}} \mathcal{X}_z$ by repeated tensor produts, rotations and "transpositions" of two lower points whose tags are allowed to commute according to r.

(b) "Transposition" operations of this kind can be implemented as precomposition with an appropriately labeled "crossing" partition $\times$.

Hence, $*^r_{z \in \mathfrak{U} \cup \mathfrak{O}} \mathcal{X}_z$ ought to be category generated by $\bigcup_{z \in \mathfrak{U} \cup \mathfrak{O}} \mathcal{X}_z$ and a set of the correct "crossing" partitions for r.

In Section 5.1 we check that this is indeed the case. However, it is convient to make some intermediate steps rather than to construct any partition of $*^r_{z \in \mathfrak{U} \cup \mathfrak{O}} \mathcal{X}_z$ in the way alluded to. Since many auxiliary results will not depend on $(\mathcal{X}_z)_{z \in \mathfrak{U} \cup \mathfrak{O}}$ in any way but only on r it will often simplify things to consider the following set of partitions.

NOTATION 5.2. In Section 5.1, let ${}^{\,r}_{\mathfrak{U},\mathfrak{O}}\mathcal{UO}^{++}$ denote the category $*^r_{z \in \mathfrak{U} \cup \mathfrak{O}} \mathcal{Y}_z$, where $\mathcal{Y}_z = \mathcal{U}^+$ for any $z \in \mathfrak{U}$ and $\mathcal{Y}_z = \mathcal{O}^+$ for any $z \in \mathfrak{O}$, and where in each case $\mathcal{U}^+$ respectively $\mathcal{O}^+$ is seen as tagged with the single tag z.

Note that ${}^{\,r}_{\mathfrak{U},\mathfrak{O}}\mathcal{UO}^{++}$ is contained in $*^r_{z \in \mathfrak{U} \cup \mathfrak{O}} \mathcal{X}_z$ without our having to make any assumptions on $(\mathcal{X}_z)_{z \in \mathfrak{U} \cup \mathfrak{O}}$. In this sense, ${}^{\,r}_{\mathfrak{U},\mathfrak{O}}\mathcal{UO}^{++}$ is the "minimal" graph co-product category with respect to r (whereas the word "minimal" here is *not* meant to refer to any distinction between different notions of "graph co-products" such as little vs. big graph co-products). The proof of Proposition 5.19 is now based on the following strategy:

Step 1 In Section 5.1.1 a certain class of operations is introduced (under which $*^r_{z \in \mathfrak{U} \cup \mathfrak{O}} \mathcal{X}_z$ is closed), the so-called ${}^{\,r}_{\mathfrak{U},\mathfrak{O}}\mathcal{UO}^{++}$*-allowed covariant rearrangements*.

Step 2 Section 5.1.2 classifies the isomorphisms of the category ${}^{\,r}_{\mathfrak{U},\mathfrak{O}}\mathcal{UO}^{++}$.

Step 3 In Section 5.1.3, using Step 2, it is shown that any category containing the isomorphisms of ${}^{\,r}_{\mathfrak{U},\mathfrak{O}}\mathcal{UO}^{++}$ is closed under ${}^{\,r}_{\mathfrak{U},\mathfrak{O}}\mathcal{UO}^{++}$-allowed covariant rearrangements.

Step 4 Section 5.1.4 defines supposed generators of $*^r_{z \in \mathfrak{U} \cup \mathfrak{O}} \mathcal{X}_z$, the *sets of generating crosses*, and shows how to find ones of minimal cardinality.

Step 5 In Section 5.1.5, with the help of Step 2, it is proved by induction that any category containing a set of generating crosses necessarily also includes all isomorphisms of ${}_{\mathfrak{U},\mathfrak{O}}^{\;r}\mathcal{UO}^{++}$.

Step 6 The final step in Section 5.1.6 is to prove via induction and using Step 3 that any category including the isomorphisms of ${}_{\mathfrak{U},\mathfrak{O}}^{\;r}\mathcal{UO}^{++}$ and $\bigcup_{z\in\mathfrak{U}\cup\mathfrak{O}}\mathcal{X}_z$ must be $*^{r}_{z\in\mathfrak{U}\cup\mathfrak{O}}\mathcal{X}_z$.

5.1.1. *Rearrangements in graph co-product categories.* Think of the following kind of operation as the permutations one can perform on a labeled partition by repeatedly "transposing" any two points whose tags commute according to r.

Definition 5.3. Let $\{k,\ell\}\subseteq\mathbb{N}_0$ as well as $\{m,n\}\subseteq\mathbb{N}_0$ and $\mathfrak{c}\colon[\![k]\!]\to(\mathfrak{U}\otimes\{\circ,\bullet\})\cup\mathfrak{O}$ and $\mathfrak{d}\colon[\![\ell]\!]\to(\mathfrak{U}\otimes\{\circ,\bullet\})\cup\mathfrak{O}$ as well as $\mathfrak{a}\colon[\![m]\!]\to(\mathfrak{U}\otimes\{\circ,\bullet\})\cup\mathfrak{O}$ and $\mathfrak{b}\colon[\![n]\!]\to(\mathfrak{U}\otimes\{\circ,\bullet\})\cup\mathfrak{O}$ and let p and q be any set-theoretical partitions of Π^k_ℓ and Π^m_n, respectively.

(a) A *covariant rearrangement* of $(\mathfrak{c},\mathfrak{d},p)$ into $(\mathfrak{a},\mathfrak{b},q)$ is any bijection $t\colon\Pi^m_n\to\Pi^k_\ell$ such that $\xi^{\mathfrak{a}}_{\mathfrak{b}}=\xi^{\mathfrak{c}}_{\mathfrak{d}}\circ t$ and $\zeta^{\mathfrak{a}}_{\mathfrak{b}}=\zeta^{\mathfrak{c}}_{\mathfrak{d}}\circ t$ and $q=t^{\leftarrow}(p)$.

(b) Any given covariant rearrangement t of $(\mathfrak{c},\mathfrak{d},p)$ into $(\mathfrak{a},\mathfrak{b},q)$ is said to be
   (i) a *rotation* if it is monotonic with respect to Γ^m_n and Γ^k_ℓ.
   (ii) ${}_{\mathfrak{U},\mathfrak{O}}^{\;r}\mathcal{UO}^{++}$-*allowed* if $k=\ell=0$ or there exist $\mathsf{b}\in\Pi^k_\ell$ and $\mathsf{a}\in\Pi^m_n$ such that for any $\{\mathsf{a}_1,\mathsf{a}_2\}\subseteq\Pi^m_n$ with $(\xi^{\mathfrak{c}}_{\mathfrak{d}}(t(\mathsf{a}_1)),\xi^{\mathfrak{c}}_{\mathfrak{d}}(t(\mathsf{a}_2)))\notin r$, if $t(\mathsf{a}_1)<t(\mathsf{a}_2)$ with respect to the cut of Γ^k_ℓ at b, then also $\mathsf{a}_1<\mathsf{a}_2$ with respect to the cut of Γ^m_n at a.

Note that in Definition 5.3 (b) (ii) the "if ..., then ..." could be replaced by "... if and only if ..." without altering the meanining.

At this point, the following well-known fact, which can be derived inductively from Lemma 3.22, deserves mentioning.

Proposition 5.4. *Any category of $(\mathfrak{U},\mathfrak{O})$-tagged labeled partitions is closed under rotations.*

We will now show that any graph co-product category (little or big, for that matter) with respect to r is closed under ${}_{\mathfrak{U},\mathfrak{O}}^{\;r}\mathcal{UO}^{++}$-allowed rearrangements, a fact that will simplify the induction argument in Proposition 5.19.

Lemma 5.5. *Let $\{k,\ell\}\subseteq\mathbb{N}_0$, let $\mathfrak{c}\colon[\![k]\!]\to(\mathfrak{U}\otimes\{\circ,\bullet\})\cup\mathfrak{O}$ and $\mathfrak{d}\colon[\![\ell]\!]\to(\mathfrak{U}\otimes\{\circ,\bullet\})\cup\mathfrak{O}$, let p be any set-theoretical partition of Π^k_ℓ, let $\{m,n\}\subseteq\mathbb{N}_0$, let $\mathfrak{a}\colon[\![m]\!]\to(\mathfrak{U}\otimes\{\circ,\bullet\})\cup\mathfrak{O}$ and $\mathfrak{b}\colon[\![n]\!]\to(\mathfrak{U}\otimes\{\circ,\bullet\})\cup\mathfrak{O}$, let q be any set-theoretical partition of Π^m_n, let t be any mapping $\Pi^m_n\to\Pi^k_\ell$, let $\mathsf{Z}\subseteq\Pi^k_\ell$ and $\mathsf{Z}=t^{\leftarrow}(\mathsf{Z})$.*

(a) There exists a unique mapping $s\colon\Pi^{\alpha(\mathsf{Y})}_{\beta(\mathsf{Y})}\to\Pi^{\alpha(\mathsf{Z})}_{\beta(\mathsf{Z})}$ with $t\circ\gamma_{\mathsf{Y},n}^{\;\;m}=\gamma_{\mathsf{Z},\ell}^{\;\;k}\circ s$.

(b) If $\mathsf{Z}\subseteq\operatorname{ran}(t)$, then s is surjective.

(c) If $t|_{\mathsf{Y}}$ is injective, then s is injective.

(d) If t is a covariant rearrangement of $(\mathfrak{c},\mathfrak{d},p)$ into $(\mathfrak{a},\mathfrak{b},q)$, then s is a covariant rearrangement of $R((\mathfrak{c},\mathfrak{d},p),\mathsf{Z})$ into $R((\mathfrak{a},\mathfrak{b},q),\mathsf{Y})$

(e) If t is a ${}^{r}_{\mathfrak{U},\mathfrak{O}}\mathcal{UO}^{++}$-allowed covariant rearrangement of $(\mathfrak{c},\mathfrak{d},p)$ into $(\mathfrak{a},\mathfrak{b},q)$ and if there exists $z\in\mathfrak{U}\cup\mathfrak{O}$ with $\mathsf{Z}\subseteq\xi^{\mathfrak{c}\leftarrow}_{\mathfrak{d}}(\{z\})$, then s is a rotation of $R((\mathfrak{c},\mathfrak{d},p),\mathsf{Z})$ into $R((\mathfrak{a},\mathfrak{b},q),\mathsf{Y})$.

Proof. (a) For any $\mathtt{u}\in\Pi^{\alpha(\mathsf{Y})}_{\beta(\mathsf{Y})}$ the definition of Y ensures that $t(\gamma_{\mathsf{Y},n}{}^{m}(\mathtt{u}))\in\mathsf{Z}$. Because $\mathrm{ran}(\gamma_{\mathsf{Z},\ell}{}^{k})=\mathsf{Z}$ we thus find $\mathtt{v}\in\Pi^{\alpha(\mathsf{Z})}_{\beta(\mathsf{Z})}$ with $\gamma_{\mathsf{Z},\ell}{}^{k}(\mathtt{v})=t(\gamma_{\mathsf{Y},n}{}^{m}(\mathtt{u}))$. If we now define $s(\mathtt{u})\coloneqq\mathtt{v}$, then the resulting mapping s has the desired property. If for each $i\in[\![2]\!]$ the mapping s_i satisfies $t\circ\gamma_{\mathsf{Y},n}{}^{m}=\gamma_{\mathsf{Z},\ell}{}^{k}\circ s_i$, then the identity $\gamma_{\mathsf{Z},\ell}{}^{k}\circ s_1=\gamma_{\mathsf{Z},\ell}{}^{k}\circ s_2$ requires $s_1=s_2$ because $\gamma_{\mathsf{Z},\ell}{}^{k}$ is injective. Hence, s is unique.

(b) Given any $\mathtt{u}\in\Pi^{\alpha(\mathsf{Z})}_{\beta(\mathsf{Z})}$, because $\mathsf{Z}\subseteq\mathrm{ran}(t)$ and because $\gamma_{\mathsf{Z},\ell}{}^{k}(\mathtt{u})\in\mathsf{Z}$, there exists $\mathtt{a}\in\Pi^{m}_{n}$ with $t(\mathtt{a})=\gamma_{\mathsf{Z},\ell}{}^{k}(\mathtt{u})$. Because $\mathtt{a}\in\mathsf{Y}=\mathrm{ran}(\gamma_{\mathsf{Y},n}{}^{m})$ we then find $\mathtt{v}\in\Pi^{\alpha(\mathsf{Y})}_{\beta(\mathsf{Y})}$ with $\gamma_{\mathsf{Y},n}{}^{m}(\mathtt{v})=\mathtt{a}$ and thus $s(\mathtt{v})=\mathtt{u}$. It follows $\gamma_{\mathsf{Z},\ell}{}^{k}(s(\mathtt{v}))=t(\gamma_{\mathsf{Y},n}{}^{m}(\mathtt{v}))=t(\mathtt{a})=\gamma_{\mathsf{Z},\ell}{}^{k}(\mathtt{u})$ by $t\circ\gamma_{\mathsf{Y},n}{}^{m}=\gamma_{\mathsf{Z},\ell}{}^{k}\circ s$ and thus $t(\mathtt{v})=\mathtt{u}$ since $\gamma_{\mathsf{Z},\ell}{}^{k}$ is injective.

(c) If $t|_{\mathsf{Y}}$ is injective, then so is $t\circ\gamma_{\mathsf{Y},n}{}^{m}$ because $\gamma_{\mathsf{Y},n}{}^{m}$ is and because $\mathsf{Y}=\mathrm{ran}(\gamma_{\mathsf{Y},n}{}^{m})$. Hence, $\gamma_{\mathsf{Z},\ell}{}^{k}\circ s$ is injective by $t\circ\gamma_{\mathsf{Y},n}{}^{m}=\gamma_{\mathsf{Z},\ell}{}^{k}\circ s$. Then, s must already be injective by itself.

(d) Parts (b) and (c) ensure that s is a bijection. If $(\mathfrak{u},\mathfrak{v},w)=R((\mathfrak{c},\mathfrak{d},p),\mathsf{Z})$ and $(\mathfrak{f},\mathfrak{g},h)=R((\mathfrak{a},\mathfrak{b},q),\mathsf{Y})$, then $\xi^{\mathfrak{f}}_{\mathfrak{g}}=\xi^{\mathfrak{a}}_{\mathfrak{b}}\circ\gamma_{\mathsf{Y},n}{}^{m}=\xi^{\mathfrak{c}}_{\mathfrak{d}}\circ t\circ\gamma_{\mathsf{Y},n}{}^{m}=\xi^{\mathfrak{c}}_{\mathfrak{d}}\circ\gamma_{\mathsf{Z},\ell}{}^{k}\circ s=\gamma_{\mathsf{Z},\mathfrak{v}}{}^{\mathfrak{u}}\circ s$, where the first identity holds by the definition of $(\mathfrak{f},\mathfrak{g},h)$, where the second is due to the assumption that t is a covariant rearrangement, where the third is implied by the defining property of s and where the last one holds by the definition of $(\mathfrak{u},\mathfrak{v},w)$. An analogous computation shows that $\zeta^{\mathfrak{f}}_{\mathfrak{g}}=\zeta^{\mathfrak{c}}_{\mathfrak{d}}\circ t$. And, similarly, $h=\gamma_{\mathsf{Y},n}{}^{m\leftarrow}(q)=(t\circ\gamma_{\mathsf{Y},n}{}^{m})^{\leftarrow}(p)=(\gamma_{\mathsf{Z},\ell}{}^{k}\circ s)^{\leftarrow}(p)=s^{\leftarrow}(w)$.

(e) By assumption there exist $\mathtt{b}\in\Pi^{k}_{\ell}$ and $\mathtt{a}\in\Pi^{m}_{n}$ such that for any $\{\mathtt{a}_1,\mathtt{a}_2\}\subseteq\Pi^{m}_{n}$, whenever $(\xi^{\mathfrak{c}}_{\mathfrak{d}}(t(\mathtt{a}_1)),\xi^{\mathfrak{c}}_{\mathfrak{d}}(t(\mathtt{a}_2)))\notin r$ and $t(\mathtt{a}_1)<t(\mathtt{a}_2)$ with respect to the cut of Γ^{k}_{ℓ} at $\mathtt{b}$, then $\mathtt{a}_1<\mathtt{a}_2$ with respect to the cut of Γ^{m}_{n} at $\mathtt{a}$. Given any $\{\mathtt{u}_1,\mathtt{u}_2,\mathtt{u}_3\}\subseteq\Pi^{\alpha(\mathsf{Y})}_{\beta(\mathsf{Y})}$ with $(s(\mathtt{u}_1)\mid s(\mathtt{u}_2)\mid s(\mathtt{u}_3))^{\alpha(\mathsf{Z})}_{\beta(\mathsf{Z})}$, we infer $(\gamma_{\mathsf{Z},\ell}{}^{k}(t(\mathtt{u}_1))\mid\gamma_{\mathsf{Z},\ell}{}^{k}(t(\mathtt{u}_2))\mid\gamma_{\mathsf{Z},\ell}{}^{k}(t(\mathtt{u}_3)))^{k}_{\ell}$ because $\gamma_{\mathsf{Z},\ell}{}^{k}$ is strictly monotonic with respect to $\Gamma^{\alpha(\mathsf{Z})}_{\beta(\mathsf{Z})}$ and Γ^{k}_{ℓ} by Lemma 4.2 (d). If $\mathtt{a}_i\coloneqq\gamma_{\mathsf{Y},n}{}^{m}(\mathtt{u}_i)$ for each $i\in[\![3]\!]$, then the defining property of s hence implies $(t(\mathtt{a}_1)\mid t(\mathtt{a}_2)\mid t(\mathtt{a}_3))^{k}_{\ell}$. With respect to the cut of Γ^{k}_{ℓ} at $\mathtt{b}$ that means that either $\mathtt{b}\leq t(\mathtt{a}_1)<t(\mathtt{a}_2)<t(\mathtt{a}_3)$ or $\mathtt{b}\leq t(\mathtt{a}_2)<t(\mathtt{a}_3)<t(\mathtt{a}_1)$ or $\mathtt{b}\leq t(\mathtt{a}_3)<t(\mathtt{a}_1)<t(\mathtt{a}_2)$. By assumption, moreover, $\xi^{\mathfrak{c}}_{\mathfrak{d}}(t(\mathtt{a}_i))=z$ for any $i\in[\![3]\!]$. Because r is anti-reflexive we thus conclude that $(\xi^{\mathfrak{c}}_{\mathfrak{d}}(t(\mathtt{a}_i)),\xi^{\mathfrak{c}}_{\mathfrak{d}}(t(\mathtt{a}_j)))\notin r$ for any $\{i,j\}\subseteq[\![3]\!]$. The assumptions on $\mathtt{b}$ and $\mathtt{a}$ therefore imply that with respect to the cut of Γ^{m}_{n} at $\mathtt{a}$ either $\mathtt{a}\leq\mathtt{a}_1<\mathtt{a}_2<\mathtt{a}_3$ or $\mathtt{a}\leq\mathtt{a}_2<\mathtt{a}_3<\mathtt{a}_1$ or $\mathtt{a}\leq\mathtt{a}_3<\mathtt{a}_1<\mathtt{a}_2$. In particular, thus, $(\mathtt{a}_1\mid\mathtt{a}_2\mid\mathtt{a}_3)^{m}_{n}$ in any case, i.e., $(\gamma_{\mathsf{Y},n}{}^{m}(\mathtt{u}_1)\mid\gamma_{\mathsf{Y},n}{}^{m}(\mathtt{u}_2)\mid\gamma_{\mathsf{Y},n}{}^{m}(\mathtt{u}_3))^{m}_{n}$. Because $\gamma_{\mathsf{Y},n}{}^{m}$ is monotonic with respect to $\Gamma^{\alpha(\mathsf{Y})}_{\beta(\mathsf{Y})}$ and Γ^{m}_{n} by Lemma 4.2 (d) it thus follows $(\mathtt{u}_1\mid\mathtt{u}_2\mid\mathtt{u}_3)^{\alpha(\mathsf{Y})}_{\beta(\mathsf{Y})}$, proving that s is a rotation. □

Proposition 5.6. *(a)* $\text{☆}^r_{z\in\mathfrak{U}\cup\mathfrak{D}}\mathcal{X}_z$ *is closed under* ${}^{\;r}_{\mathfrak{U},\mathfrak{D}}\mathcal{UO}^{++}$*-allowed covariant rearrangements.*

(b) $*^r_{z\in\mathfrak{U}\cup\mathfrak{D}}\mathcal{X}_z$ *is closed under* ${}^{\;r}_{\mathfrak{U},\mathfrak{D}}\mathcal{UO}^{++}$*-allowed covariant rearrangements. Moreover,* ${}^{\;r}_{\mathfrak{U},\mathfrak{D}}\mathcal{UO}^{++}$*-allowed covariant rearrangements preserve histories.*

Proof. Let $\{k,\ell\}\subseteq\mathbb{N}_0$, let $\mathfrak{c}\colon [\![k]\!]\to(\mathfrak{U}\otimes\{\circ,\bullet\})\cup\mathfrak{D}$ and $\mathfrak{d}\colon [\![\ell]\!]\to(\mathfrak{U}\otimes\{\circ,\bullet\})\cup\mathfrak{D}$, let p be any set-theoretical partition of Π^k_ℓ, let $\{m,n\}\subseteq\mathbb{N}_0$, let $\mathfrak{a}\colon [\![m]\!]\to(\mathfrak{U}\otimes\{\circ,\bullet\})\cup\mathfrak{D}$ and $\mathfrak{b}\colon [\![n]\!]\to(\mathfrak{U}\otimes\{\circ,\bullet\})\cup\mathfrak{D}$, let q be any set-theoretical partition of of Π^m_n and let t be any ${}^{\;r}_{\mathfrak{U},\mathfrak{D}}\mathcal{UO}^{++}$-allowed covariant rearrangement of $(\mathfrak{c},\mathfrak{d},p)$ into $(\mathfrak{a},\mathfrak{b},q)$.

(a) We suppose $(\mathfrak{c},\mathfrak{d},p)\in\mathcal{D}:=\text{☆}^r_{z\in\mathfrak{U}\cup\mathfrak{D}}\mathcal{X}_z$ and prove $(\mathfrak{a},\mathfrak{b},q)\in\mathcal{D}$. For any $z\in\mathfrak{U}\cup\mathfrak{D}$, if $\mathsf{Y}:=\xi^{\mathfrak{a}\leftarrow}_{\mathfrak{b}}(\{z\})$ and $\mathsf{Z}:=\xi^{\mathfrak{c}\leftarrow}_{\mathfrak{d}}(\{z\})$, then $\mathsf{Y}=t^{\leftarrow}(\mathsf{Z})$ by $\xi^{\mathfrak{a}}_{\mathfrak{b}}=\xi^{\mathfrak{c}}_{\mathfrak{d}}\circ t$. Hence, by all parts of Lemma 5.5 taken together there exists a rotation s of $R((\mathfrak{c},\mathfrak{d},p),\mathsf{Z})$ into $R((\mathfrak{a},\mathfrak{b},q),\mathsf{Y})$. Because $R((\mathfrak{c},\mathfrak{d},p),\mathsf{Z})\in\mathcal{X}_z$ by $(\mathfrak{c},\mathfrak{d},p)\in\mathcal{D}$ it thus follows $R((\mathfrak{a},\mathfrak{b},q),\mathsf{Y})\in\mathcal{X}_z$ by Propositon 5.4.

It remains to show that $(\mathfrak{a},\mathfrak{b},q)$ satisfies the non-crossing conditions of $\mathcal{D}$. Because t is a ${}^{\;r}_{\mathfrak{U},\mathfrak{D}}\mathcal{UO}^{++}$-allowed covariant rearrangement there exist $\mathsf{b}\in\Pi^k_\ell$ and $\mathsf{a}\in\Pi^m_n$ such that for any $\{\mathsf{a}_1,\mathsf{a}_2\}\subseteq\Pi^m_n$ with $(\xi^{\mathfrak{c}}_{\mathfrak{d}}(t(\mathsf{a}_1)),\xi^{\mathfrak{c}}_{\mathfrak{d}}(t(\mathsf{a}_2)))\notin r$ the statements that $t(\mathsf{a}_1)<t(\mathsf{a}_2)$ with respect to the cut of Γ^k_ℓ at b and that $\mathsf{a}_1<\mathsf{a}_2$ with respect to the cut of Γ^m_n at a are equivalent. Given any $\{z_1,z_2\}\subseteq\mathfrak{U}\cup\mathfrak{D}$ with $z_1\neq z_2$ and $(z_1,z_2)\notin r$ and any $\{\mathsf{A}_1,\mathsf{A}_2\}\subseteq q$ with $\mathsf{A}_1\neq\mathsf{A}_2$ and $\mathsf{A}_1\subseteq\xi^{\mathfrak{a}\leftarrow}_{\mathfrak{b}}(\{z_1\})$ and $\mathsf{A}_2\subseteq\xi^{\mathfrak{a}\leftarrow}_{\mathfrak{b}}(\{z_2\})$, there exist $\{\mathsf{B}_1,\mathsf{B}_2\}\subseteq p$ with $\mathsf{A}_1=t^{\leftarrow}(\mathsf{B}_1)$ and $\mathsf{A}_2=t^{\leftarrow}(\mathsf{B}_2)$ because $q=t^{\leftarrow}(p)$. Since t is bijective the fact that $\mathsf{A}_1\cap\mathsf{A}_2=\varnothing$ implies that also $\mathsf{B}_1\cap\mathsf{B}_2=\varnothing$, i.e., $\mathsf{B}_1\neq\mathsf{B}_2$. Moreover, $\mathsf{B}_1\subseteq\xi^{\mathfrak{c}\leftarrow}_{\mathfrak{d}}(\{z_1\})$ and $\mathsf{B}_2\subseteq\xi^{\mathfrak{c}\leftarrow}_{\mathfrak{d}}(\{z_1\})$ by $\xi^{\mathfrak{a}}_{\mathfrak{b}}=\xi^{\mathfrak{c}}_{\mathfrak{d}}\circ t$ because t is bijective. Since $(\mathfrak{c},\mathfrak{d},p)\in\mathcal{D}$ the assumption that $(z_1,z_2)\notin r$ requires $\mathsf{B}_1\mathbin{\not\times}^k_\ell\mathsf{B}_2$. We prove $\mathsf{A}_1\mathbin{\not\times}^m_n\mathsf{A}_2$ by contradiction. If $\mathsf{A}_1\times^m_n\mathsf{A}_2$, there exist $\{\mathsf{a}^1_1,\mathsf{a}^1_2\}\subseteq\mathsf{A}_1$ and $\{\mathsf{a}^2_1,\mathsf{a}^2_2\}\subseteq\mathsf{A}_2$ such that $(\mathsf{a}^1_1\mid\mathsf{a}^2_1\mid\mathsf{a}^1_2)^m_n$ and $(\mathsf{a}^2_1\mid\mathsf{a}^1_2\mid\mathsf{a}^2_2)^m_n$ and $(\mathsf{a}^1_2\mid\mathsf{a}^2_2\mid\mathsf{a}^1_1)^m_n$. Consequently, with respect to the cut of Γ^m_n at a, either $\mathsf{a}^1_1<\mathsf{a}^2_1<\mathsf{a}^1_2<\mathsf{a}^2_2$ or $\mathsf{a}^2_1<\mathsf{a}^1_2<\mathsf{a}^2_2<\mathsf{a}^1_1$ or $\mathsf{a}^1_2<\mathsf{a}^2_2<\mathsf{a}^1_1<\mathsf{a}^2_1$ or $\mathsf{a}^2_2<\mathsf{a}^1_1<\mathsf{a}^2_1<\mathsf{a}^1_2$. Hence, with respect to the cut of Γ^k_ℓ at b either $t(\mathsf{a}^1_1)<t(\mathsf{a}^2_1)<t(\mathsf{a}^1_2)<t(\mathsf{a}^2_2)$ or $t(\mathsf{a}^2_1)<t(\mathsf{a}^1_2)<t(\mathsf{a}^2_2)<t(\mathsf{a}^1_1)$ or $t(\mathsf{a}^1_2)<t(\mathsf{a}^2_2)<t(\mathsf{a}^1_1)<t(\mathsf{a}^2_1)$ or $t(\mathsf{a}^2_2)<t(\mathsf{a}^1_1)<t(\mathsf{a}^2_1)<t(\mathsf{a}^1_2)$ because $(\xi^{\mathfrak{c}}_{\mathfrak{d}}(t(\mathsf{a}^i_j)),\xi^{\mathfrak{c}}_{\mathfrak{d}}(t(\mathsf{a}^{3-i}_{j'})))=(z_i,z_{3-i})\notin r$ for any $\{i,j,j'\}\subseteq[\![2]\!]$. In conclusion, $(t(\mathsf{a}^1_1)\mid t(\mathsf{a}^2_1)\mid t(\mathsf{a}^1_2))^k_\ell$ and $(t(\mathsf{a}^2_1)\mid t(\mathsf{a}^1_2)\mid t(\mathsf{a}^2_2))^k_\ell$ and $(t(\mathsf{a}^1_2)\mid t(\mathsf{a}^2_2)\mid t(\mathsf{a}^1_1))^k_\ell$, which yields the contradiction $\mathsf{B}_1\times^k_\ell\mathsf{B}_2$ because $t_{\rightarrow}(\mathsf{A}_1)=\mathsf{B}_1$ and $t_{\rightarrow}(\mathsf{A}_2)=\mathsf{B}_2$ by the bijectivity of t. Hence, (a) is true.

(b) Now, let even $(\mathfrak{c},\mathfrak{d},p)\in\mathcal{C}:=*^r_{z\in\mathfrak{U}\cup\mathfrak{D}}\mathcal{X}_z$ and let h be any history for $(\mathfrak{c},\mathfrak{d},p)$ with respect to $(\mathcal{X}_z)_{z\in\mathfrak{U}\cup\mathfrak{D}}$ and r. We prove that $g:=t^{\leftarrow}(h)$ is a history for $(\mathfrak{a},\mathfrak{b},q)$ with respect to $(\mathcal{X}_z)_{z\in\mathfrak{U}\cup\mathfrak{D}}$ and r. By construction, t is a ${}^{\;r}_{\mathfrak{U},\mathfrak{D}}\mathcal{UO}^{++}$-allowed covariant rearrangement of $(\mathfrak{c},\mathfrak{d},h)$ into $(\mathfrak{a},\mathfrak{b},g)$. If $\mathcal{N}_z:={}_{\{z\},\varnothing}\mathcal{S}^+$ for any $z\in\mathfrak{U}$ and $\mathcal{N}_z:={}_{\varnothing,\{z\}}\mathcal{S}^+$ for any $z\in\mathfrak{D}$, then, because $(\mathfrak{c},\mathfrak{d},h)\in\text{☆}^r_{z\in\mathfrak{U}\cup\mathfrak{D}}\mathcal{N}_z$ by assumption that implies $(\mathfrak{a},\mathfrak{b},q)\in\text{☆}^r_{z\in\mathfrak{U}\cup\mathfrak{D}}\mathcal{N}_z$ by (a). Moreover, $q=t^{\leftarrow}(p)\leq t^{\leftarrow}(h)=g$ because $p\leq h$ by assumption and because $t^{\leftarrow}$ preserves $\leq$. Finally, for any $z\in\mathfrak{U}\cup\mathfrak{D}$ and $\mathsf{G}\in g$ with $\mathsf{G}\subseteq\xi^{\mathfrak{a}\leftarrow}_{\mathfrak{b}}(\{z\})$, if $\mathsf{H}\in h$ is the unique block with $\mathsf{G}=t^{\leftarrow}(\mathsf{H})$, then the assumption that $\xi^{\mathfrak{a}}_{\mathfrak{b}}=\xi^{\mathfrak{c}}_{\mathfrak{d}}\circ t$ ensures that also $\mathsf{H}\subseteq\xi^{\mathfrak{c}\leftarrow}_{\mathfrak{d}}(\{z\})$ because t is bijective. Since that guarantees

$R((\mathfrak{c},\mathfrak{d},p),\mathsf{H})\in\mathcal{X}_z$ the fact that $R((\mathfrak{a},\mathfrak{b},q),\mathsf{G})$ is a rotation of $R((\mathfrak{c},\mathfrak{d},p),\mathsf{H})$ by Lemma 5.5 proves that $R((\mathfrak{a},\mathfrak{b},q),\mathsf{G})\in\mathcal{X}_z$ by Proposition 5.4. And by Proposition 4.24 that is what we needed to see. □

While we will later also be able to infer (a) and the first part of (b) of Proposition 5.6 from Proposition 4.21 and Lemma 5.10, we really needed to verify Proposition 5.6 in the way we did for the second part of (b).

5.1.2. *Isomorphisms of the minimal graph co-product category.* Next, we classify all isomorphisms of ${}_{\mathfrak{U},\mathfrak{O}}^{\;r}\mathcal{UO}^{++}$. The following is well-known.

Proposition 5.7. *For any* $\{k,\ell\}\subseteq\mathbb{N}_0$, *any* $\mathfrak{c}\colon [\![k]\!]\to(\mathfrak{U}\otimes\{\circ,\bullet\})\cup\mathfrak{O}$ *and* $\mathfrak{d}\colon[\![\ell]\!]\to(\mathfrak{U}\otimes\{\circ,\bullet\})\cup\mathfrak{O}$ *and any set-theoretical partition* p *of* Π_ℓ^k *the labeled partition* $(\mathfrak{c},\mathfrak{d},p)$ *is an isomorphism of* ${}_{\mathfrak{U},\mathfrak{O}}\mathcal{S}$ *if and only if* $k=\ell$ *and* $|\mathsf{B}|=2$ *and* $\mathsf{B}\cap\Pi_0^k\neq\varnothing\neq\mathsf{B}\cap\Pi_\ell^0$ *for any* $\mathsf{B}\in p$.

Notation 5.8. Given any $k\in\mathbb{N}_0$, any $\mathfrak{c}\colon[\![k]\!]\to(\mathfrak{U}\otimes\{\circ,\bullet\})\cup\mathfrak{O}$ and any permutation s of $[\![k]\!]$ the *covariant isomorphism* induced by $\mathfrak{c}$ and s is the labeled partition $\mathrm{pm}_s^{\mathfrak{c}}\coloneqq(\mathfrak{c},\mathfrak{c}\circ s^{-1},\mathrm{pm}_s)$, where $\mathrm{pm}_s\coloneqq\{\{{}^{\blacksquare}i,{}_{\blacksquare}s(i)\}\mid i\in[\![k]\!]\}$.

Proposition 5.9. *For any* $k\in\mathbb{N}_0$, *any* $\mathfrak{c}\colon[\![k]\!]\to(\mathfrak{U}\otimes\{\circ,\bullet\})\cup\mathfrak{O}$ *and* $\mathfrak{d}\colon[\![k]\!]\to(\mathfrak{U}\otimes\{\circ,\bullet\})\cup\mathfrak{O}$ *and any set-theoretical partition* p *of* Π_k^k *such that* $(\mathfrak{c},\mathfrak{d},p)$ *is an isomorphism of* ${}_{\mathfrak{U},\mathfrak{O}}\mathcal{S}$ *the following are equivalent.*

(i) $(\mathfrak{c},\mathfrak{d},p)$ *is an isomorphism of* ${}_{\mathfrak{U},\mathfrak{O}}^{\;r}\mathcal{UO}^{++}$.

(ii) $p\leq\ker(\mathfrak{c}\,{}^{\blacksquare}{}_{\blacksquare}\,\mathfrak{d})$ *and for any* $\{z_1,z_2\}\subseteq\mathfrak{U}\cup\mathfrak{O}$ *and* $\{\mathsf{B}_1,\mathsf{B}_2\}\subseteq p$ *with* $\mathsf{B}_1\neq\mathsf{B}_2$ *and* $\mathsf{B}_1\subseteq\xi_{\mathfrak{d}}^{\mathfrak{c}\leftarrow}(\{z_1\})$ *and* $\mathsf{B}_2\subseteq\xi_{\mathfrak{d}}^{\mathfrak{c}\leftarrow}(\{z_2\})$, *whenever* $(z_1,z_2)\notin r$, *then* $\mathsf{B}_1\diagdown\!\!\diagdown_k^k\mathsf{B}_2$.

(iii) There exists a permutation s *of* $[\![k]\!]$ *such that* $(\mathfrak{c},\mathfrak{d},p)=\mathrm{pm}_s^{\mathfrak{c}}$ *and such that for any* $\{i_1,i_2\}\subseteq[\![k]\!]$ *with* $i_1<i_2$, *if* $\mathfrak{c}(i_1)\in(\{z_1\}\otimes\{\circ,\bullet\})\cup\{z_1\}$ *and* $\mathfrak{c}(i_2)\in(\{z_2\}\otimes\{\circ,\bullet\})\cup\{z_2\}$, *whenever* $(z_1,z_2)\notin r$, *then* $s(i_1)<s(i_2)$.

Proof. For any $z\in\mathfrak{U}\cup\mathfrak{O}$ abbreviate $\mathsf{Y}_z\coloneqq\xi_{\mathfrak{d}}^{\mathfrak{c}\leftarrow}(\{z\})$ and, as in Notation 5.2, let $\mathcal{Y}_z\coloneqq\mathcal{U}^+$ for any $z\in\mathfrak{U}$ and $\mathcal{Y}_z\coloneqq\mathcal{O}^+$ for any $z\in\mathfrak{O}$, where in each case $\mathcal{U}^+$ respectively $\mathcal{O}^+$ is seen as tagged with the single tag z.

Step 1: (i) implies (ii). Given any $\mathsf{B}\in p$, by Proposition 5.7 there exist $\{i,j\}\subseteq[\![k]\!]$ such that $\mathsf{B}=\{{}^{\blacksquare}i,{}_{\blacksquare}j\}$. If $(\mathfrak{c},\mathfrak{d},p)\in{}_{\mathfrak{U},\mathfrak{O}}^{\;r}\mathcal{UO}^{++}$, then by Definition 4.17 in particular, $p\leq\ker(\xi_{\mathfrak{d}}^{\mathfrak{c}})$. Hence, there exists $z\in\mathfrak{U}\cup\mathfrak{O}$ with $\mathsf{B}\subseteq\mathsf{Y}_z$, i.e., $\xi_{\mathfrak{d}}^{\mathfrak{c}}({}^{\blacksquare}i)=z=\xi_{\mathfrak{d}}^{\mathfrak{c}}({}_{\blacksquare}j)$. If $z\in\mathfrak{O}$, that is the same as saying $\mathfrak{c}(i)=z=\mathfrak{d}(j)$ or, in other words, $\mathsf{B}\subseteq(\mathfrak{c}\,{}^{\blacksquare}{}_{\blacksquare}\,\mathfrak{d})^{\leftarrow}(\{z\})\in\ker(\mathfrak{c}\,{}^{\blacksquare}{}_{\blacksquare}\,\mathfrak{d})$. In case $z\in\mathfrak{U}$, the assumption that $(\mathfrak{c},\mathfrak{d},p)\in{}_{\mathfrak{U},\mathfrak{O}}^{\;r}\mathcal{UO}^{++}$ and Definition 4.17 further imply $(\mathfrak{a},\mathfrak{b},q)\coloneqq R((\mathfrak{c},\mathfrak{d},p),\mathsf{Y}_z)\in\mathcal{Y}_z=\mathcal{U}^+$. According to Definition 7.10 (b), if $\mathsf{A}\coloneqq\gamma_{\mathsf{Y}_z,k}{}^{k\leftarrow}(\mathsf{B})$, then $\sigma_{\mathfrak{b}}^{\mathfrak{a}}(\mathsf{A})=0$ because $\mathsf{A}\in q$ by $\mathsf{B}\cap\mathsf{Y}_z=\mathsf{B}\neq\varnothing$. By Lemma 4.2 (d) that requires ${}_z\sigma_{\mathfrak{d}}^{\mathfrak{c}}(\mathsf{B})=0$ because $\mathsf{B}\backslash\mathsf{Y}_z=\varnothing$. We conclude $0={}_z\sigma_{\mathfrak{d}}^{\mathfrak{c}}(\{{}^{\blacksquare}i,{}_{\blacksquare}j\})=\sigma(\zeta_{\mathfrak{d}}^{\mathfrak{c}}({}^{\blacksquare}i))+\sigma(\zeta_{\mathfrak{d}}^{\mathfrak{c}}({}_{\blacksquare}j))$. That is only possible if $\zeta_{\mathfrak{d}}^{\mathfrak{c}}({}^{\blacksquare}i)\neq\zeta_{\mathfrak{d}}^{\mathfrak{c}}({}_{\blacksquare}j)$ or, equivalently, $c\coloneqq\mathfrak{c}(i)=\mathfrak{d}(j)$. In total, if $z\in\mathfrak{U}$, then $(\mathfrak{c}\,{}^{\blacksquare}{}_{\blacksquare}\,\mathfrak{d})({}^{\blacksquare}i)=(z,c)=(\mathfrak{c}\,{}^{\blacksquare}{}_{\blacksquare}\,\mathfrak{d})({}_{\blacksquare}j)$ and thus $\mathsf{B}\subseteq(\mathfrak{c}\,{}^{\blacksquare}{}_{\blacksquare}\,\mathfrak{d})^{\leftarrow}(\{(z,c)\})\in\ker(\mathfrak{c}\,{}^{\blacksquare}{}_{\blacksquare}\,\mathfrak{d})$. As $\mathsf{B}\in p$ was arbitrary, thus $p\leq\ker(\mathfrak{c}\,{}^{\blacksquare}{}_{\blacksquare}\,\mathfrak{d})$.

If now, $\{\mathsf{B}_1,\mathsf{B}_2\}\subseteq p$ with $\mathsf{B}_1\neq\mathsf{B}_2$ and $\mathsf{B}_1\subseteq\xi_{\mathfrak{d}}^{\mathfrak{c}\leftarrow}(\{z_1\})$ and $\mathsf{B}_2\subseteq\xi_{\mathfrak{d}}^{\mathfrak{c}\leftarrow}(\{z_2\})$ and $(z_1,z_2)\notin r$, then we distinguish two cases. If $z_1\neq z_2$, then already $\mathsf{B}_1\diagdown\!\!\diagdown_k^k\mathsf{B}_2$ by

$(\mathfrak{c},\mathfrak{d},p)\in {}_{\mathfrak{U},\mathfrak{O}}^{\;r}\mathcal{UO}^{++}$ and Definition 4.17. If $z\coloneqq z_1=z_2$ instead, then again $(\mathfrak{a},\mathfrak{b},q)\coloneqq R((\mathfrak{c},\mathfrak{d},p),\mathsf{Y}_z)\in\mathcal{Y}_z$. Since $\mathsf{B}_1\cap\mathsf{B}_2=\varnothing$, if $\mathsf{A}_1\coloneqq\gamma_{\mathsf{Y}_z,k}^{\;k\leftarrow}(\mathsf{B}_1)$ and $\mathsf{A}_2\coloneqq\gamma_{\mathsf{Y}_z,k}^{\;k\leftarrow}(\mathsf{B}_2)$, then $\mathsf{A}_1\cap\mathsf{A}_2=\gamma_{\mathsf{Y}_z,k}^{\;k\leftarrow}(\mathsf{B}_1\cap\mathsf{B}_2)=\varnothing$, i.e., $\mathsf{B}_1\neq\mathsf{B}_2$. Hence, if $\{m,n\}\subseteq\mathbb{N}_0$ are such that $\mathfrak{a}\colon[\![m]\!]\to(\mathfrak{U}\otimes\{\circ,\bullet\})\uplus\mathfrak{O}$ and $\mathfrak{b}\colon[\![n]\!]\to(\mathfrak{U}\otimes\{\circ,\bullet\})\uplus\mathfrak{O}$, then by Definitions 7.10 (b) and 7.3 (c), regardless of z, necessarily, $\mathsf{A}_1\mathbin{\backslash\!\backslash}^{m}_{n}\mathsf{A}_2$. Since $\gamma_{\mathsf{Y}_z,k}^{\;k}$ is strictly monotonic by Lemma 4.2 (d) then also $\mathsf{B}_1\mathbin{\backslash\!\backslash}^{k}_{k}\mathsf{B}_2$. Altogether, we have derived (ii).

Step 2: (ii) implies (iii). Since $(\mathfrak{c},\mathfrak{d},p)$ is assumed to be an isomorphism of ${}_{\mathfrak{U},\mathfrak{O}}\mathcal{S}$, by Proposition 5.7 for any $i\in[\![k]\!]$ there exists a unique $s(i)\in[\![k]\!]$ such that $\{{}^{\blacksquare}i,{}_{\blacksquare}s(i)\}\in p$. The rule $i\mapsto s(i)$ for any $i\in[\![k]\!]$ thus defines a permutation of $[\![k]\!]$ with $p=\mathrm{pm}_s$. Moreover, for any $i\in[\![k]\!]$ the assumption $p\leq\ker(\mathfrak{c}{}^{\blacksquare}{}_{\blacksquare}\mathfrak{d})$ requires $\mathfrak{d}(s(i))=(\mathfrak{c}{}^{\blacksquare}{}_{\blacksquare}\mathfrak{d})({}_{\blacksquare}s(i))=(\mathfrak{c}{}^{\blacksquare}{}_{\blacksquare}\mathfrak{d})({}^{\blacksquare}i)=\mathfrak{c}(i)$ because $\{{}^{\blacksquare}i,{}_{\blacksquare}s(i)\}\in p$. In other words, $\mathfrak{d}\circ s=\mathfrak{c}$ or, equivalently, $\mathfrak{d}=\mathfrak{c}\circ s^{-1}$. Hence, indeed, $(\mathfrak{c},\mathfrak{d},p)=\mathrm{pm}_s^{\mathfrak{c}}$.

Furthermore, for any $\{i_1,i_2\}\subseteq[\![k]\!]$ with $i_1<i_2$ and $(z_1,z_2)\notin r$, where $\mathfrak{c}(i_1)\in(\{z_1\}\otimes\{\circ,\bullet\})\uplus\{z_1\}$ and $\mathfrak{c}(i_2)\in(\{z_2\}\otimes\{\circ,\bullet\})\uplus\{z_2\}$, if $\mathsf{B}_1\coloneqq\{{}^{\blacksquare}i_1,{}_{\blacksquare}s(i_1)\}$ and $\mathsf{B}_2\coloneqq\{{}^{\blacksquare}i_2,{}_{\blacksquare}s(i_2)\}$, then $\{\mathsf{B}_1,\mathsf{B}_2\}\subseteq p$ and $\mathsf{B}_1\neq\mathsf{B}_2$ by $i_1\neq i_2$. Moreover, $\mathsf{B}_1\subseteq\xi_{\mathfrak{d}}^{\mathfrak{c}\leftarrow}(\{z_1\})$ and $\mathsf{B}_2\subseteq\xi_{\mathfrak{d}}^{\mathfrak{c}\leftarrow}(\{z_2\})$ because $p\leq\ker(\mathfrak{c}{}^{\blacksquare}{}_{\blacksquare}\mathfrak{d})$ and because $\xi_{\mathfrak{d}}^{\mathfrak{c}}({}^{\blacksquare}i_1)=z_1$ and $\xi_{\mathfrak{d}}^{\mathfrak{c}}({}^{\blacksquare}i_2)=z_2$. Therefore, by assumption, $\mathsf{B}_1\mathbin{\backslash\!\backslash}^{k}_{k}\mathsf{B}_2$. Hence and since both $({}_{\blacksquare}s(i_1)\mid{}^{\blacksquare}i_2\mid{}^{\blacksquare}i_1)^k_k$ and $({}^{\blacksquare}i_2\mid{}^{\blacksquare}i_1\mid{}_{\blacksquare}s(i_2))^k_k$ by $i_1<i_2$, necessarily, $({}^{\blacksquare}i_1\mid{}_{\blacksquare}s(i_1)\mid{}_{\blacksquare}s(i_2))^k_k$, which is to say $s(i_1)<s(i_2)$.

Step 3: (iii) implies (i). For any $i\in[\![k]\!]$, if $z\in\mathfrak{U}\uplus\mathfrak{O}$ is such that $\mathfrak{c}(i)\in(\{z\}\otimes\{\circ,\bullet\})\uplus\{z\}$, then the definition of $\mathrm{pm}_s^{\mathfrak{c}}$ implies $(\mathfrak{c}{}^{\blacksquare}{}_{\blacksquare}\mathfrak{d})({}^{\blacksquare}i)=\mathfrak{c}(i)=(\mathfrak{c}\circ s^{-1})(s(i))=\mathfrak{d}(s(i))=(\mathfrak{c}{}^{\blacksquare}{}_{\blacksquare}\mathfrak{d})({}_{\blacksquare}s(i))$ and thus, in particular, $\xi_{\mathfrak{d}}^{\mathfrak{c}}({}^{\blacksquare}i)=\xi_{\mathfrak{d}}^{\mathfrak{c}}({}_{\blacksquare}s(i))$ or, equivalently, $\{{}^{\blacksquare}i,{}_{\blacksquare}s(i)\}\subseteq\xi_{\mathfrak{d}}^{\mathfrak{c}\leftarrow}(\{z\})$. Since i was arbitrary and all blocks of p arise in this way it follows $p\leq\ker(\xi_{\mathfrak{d}}^{\mathfrak{c}})$.

Given any $z\in\mathfrak{U}\uplus\mathfrak{O}$, if $\mathsf{Y}_z\coloneqq\xi_{\mathfrak{d}}^{\mathfrak{c}\leftarrow}(\{z\})$ and $(\mathfrak{a},\mathfrak{b},q)\coloneqq R((\mathfrak{c},\mathfrak{d},p),\mathsf{Y}_z)$, we have to prove $(\mathfrak{a},\mathfrak{b},q)\in\mathcal{U}^+$ if $z\in\mathfrak{U}$ and $(\mathfrak{a},\mathfrak{b},q)\in\mathcal{O}^+$ if $z\in\mathfrak{O}$. For any $\mathsf{A}\in q$ there exists $\mathsf{B}\in p$ with $\mathsf{A}=\gamma_{\mathsf{Y}_z,k}^{\;k\leftarrow}(\mathsf{B})$ and $\mathsf{B}\cap\mathsf{Y}_z\neq\varnothing$ or, equivalently, $\mathsf{B}\subseteq\mathsf{Y}_z$ by $p\leq\ker(\xi_{\mathfrak{d}}^{\mathfrak{c}})$. According to the definition of $\mathrm{pm}_s^{\mathfrak{c}}$ we can find $i\in[\![k]\!]$ with $\mathsf{B}=\{{}^{\blacksquare}i,{}_{\blacksquare}s(i)\}$. Because $\gamma_{\mathsf{Y}_z,k}^{\;k}$ is injective, we conclude $|\mathsf{A}|=|\mathsf{B}|=2$.

Moreover, if $z\in\mathfrak{U}$, then the identity $(\mathfrak{c}{}^{\blacksquare}{}_{\blacksquare}\mathfrak{d})({}^{\blacksquare}i)=(\mathfrak{c}{}^{\blacksquare}{}_{\blacksquare}\mathfrak{d})({}_{\blacksquare}s(i))$ moreover implies $\zeta_{\mathfrak{d}}^{\mathfrak{c}}({}^{\blacksquare}i)\neq\zeta_{\mathfrak{d}}^{\mathfrak{c}}({}_{\blacksquare}s(i))$ and thus ${}_z\sigma_{\mathfrak{d}}^{\mathfrak{c}}(\mathsf{B})=\sigma(\zeta_{\mathfrak{d}}^{\mathfrak{c}}({}^{\blacksquare}i))+\sigma(\zeta_{\mathfrak{d}}^{\mathfrak{c}}({}_{\blacksquare}s(i)))=0$. It then follows $\sigma_{\mathfrak{b}}^{\mathfrak{a}}(\mathsf{A})={}_z\sigma_{\mathfrak{d}}^{\mathfrak{c}}(\mathsf{B})=0$ by Lemma 4.2 (d) since $\mathsf{B}\backslash\mathsf{Y}_z=\varnothing$.

Furthermore, regardless of whether $z\in\mathfrak{U}$ or $z\in\mathfrak{O}$, for any $\{\mathsf{A}_1,\mathsf{A}_2\}\subseteq q$ with $\mathsf{A}_1\neq\mathsf{A}_2$ we find $\{\mathsf{B}_1,\mathsf{B}_2\}\subseteq p$ with $\mathsf{A}_1=\gamma_{\mathsf{Y}_z,k}^{\;k\leftarrow}(\mathsf{B}_1)$ and $\mathsf{A}_2=\gamma_{\mathsf{Y}_z,k}^{\;k\leftarrow}(\mathsf{B}_2)$ and $\mathsf{B}_1\cap\mathsf{Y}_z\neq\varnothing\neq\mathsf{B}_2\cap\mathsf{Y}_z$, i.e., $\mathsf{B}_1\subseteq\mathsf{Y}_z$ and $\mathsf{B}_2\subseteq\mathsf{Y}_z$. By $p=\mathrm{pm}_s$ there must then exist $\{i_1,i_2\}\subseteq[\![k]\!]$ with $\mathsf{B}_1=\{{}^{\blacksquare}i_1,{}_{\blacksquare}s(i_1)\}$ and $\mathsf{B}_2=\{{}^{\blacksquare}i_2,{}_{\blacksquare}s(i_2)\}$. Because $\varnothing=\mathsf{A}_1\cap\mathsf{A}_2=\gamma_{\mathsf{Y}_z,k}^{\;k\leftarrow}(\mathsf{B}_1)\cap\gamma_{\mathsf{Y}_z,k}^{\;k\leftarrow}(\mathsf{B}_2)=\gamma_{\mathsf{Y}_z,k}^{\;k\leftarrow}(\mathsf{B}_1\cap\mathsf{B}_2)$, necessarily also, $\mathsf{B}_1\cap\mathsf{B}_2=\varnothing$ and thus, $i_1\neq i_2$. Because r, being a partial commutation relation, is anti-reflexive, $(z,z)\notin r$. In consequence, either both $i_1<i_2$ and $s(i_1)<s(i_2)$ or both $i_2<i_1$ and $s(i_2)<s(i_1)$. Either way, $\mathsf{B}_1\mathbin{\backslash\!\backslash}^{k}_{k}\mathsf{B}_2$ is the consequence. Hence, also $\mathsf{A}_1\mathbin{\backslash\!\backslash}^{m}_{n}\mathsf{A}_2$ because $\gamma_{\mathsf{Y}_z,k}^{\;k}$ is monotonic

with respect to Γ_n^m and Γ_k^k. Thus, we have shown $(\mathfrak{a},\mathfrak{b},q)$ to be an element of $\mathcal{U}^+$ if $z\in\mathfrak{U}$ and of $\mathcal{O}^+$ if $z\in\mathfrak{O}$.

Finally, let $\{z_1,z_2\}\subseteq\mathfrak{U}\cup\mathfrak{O}$ with $z_1\neq z_2$ and $(z_1,z_2)\notin r$ and $\{\mathsf{B}_1,\mathsf{B}_2\}\subseteq p$ with $\mathsf{B}_1\neq\mathsf{B}_2$ and $\mathsf{B}_1\subseteq\mathsf{Y}_{z_1}\coloneqq\xi_{\mathfrak{d}}^{\mathfrak{c}\leftarrow}(\{z_1\})$ and $\mathsf{B}_2\subseteq\mathsf{Y}_{z_2}\coloneqq\xi_{\mathfrak{d}}^{\mathfrak{c}\leftarrow}(\{z_2\})$ be arbitrary. Again, by $p=\mathrm{pm}_s$ there are $\{i_1,i_2\}\subseteq[\![k]\!]$ with $\mathsf{B}_1=\{{}^{\blacksquare}i_1,{}_{\blacksquare}s(i_1)\}$ and $\mathsf{B}_2=\{{}^{\blacksquare}i_2,{}_{\blacksquare}s(i_2)\}$ and, necessarily, $i_1\neq i_2$. By $(z_1,z_2)\notin r$, either both $i_1<i_2$ and $s(i_1)<s(i_2)$ or both $i_2<i_1$ and $s(i_2)<s(i_1)$. Thus, in any case, $\mathsf{B}_1\diagdown\!\!\diagdown_k^k\mathsf{B}_2$. That makes $(\mathfrak{c},\mathfrak{d},p)$ an element of ${}_{\mathfrak{U},\mathfrak{O}}^{\,r}\mathcal{UO}^{++}$. Hence, the claim is true. □

5.1.3. *Isomorphisms of the minimal graph category and rearrangements in graph co-product categories.* As the below lemma shows, the isomorphisms of ${}_{\mathfrak{U},\mathfrak{O}}^{\,r}\mathcal{UO}^{++}$, which were classified in Section 5.1.2, are the partitions which implement the operations defined in Section 5.1.1 when composed with.

Lemma 5.10. *Any category of $(\mathfrak{U},\mathfrak{O})$-tagged labeled partitions which contains the isomorphisms of ${}_{\mathfrak{U},\mathfrak{O}}^{\,r}\mathcal{UO}^{++}$ is closed under ${}_{\mathfrak{U},\mathfrak{O}}^{\,r}\mathcal{UO}^{++}$-allowed covariant rearrangements.*

Proof. Given any category $\mathcal{C}$ containing the isomorphisms of ${}_{\mathfrak{U},\mathfrak{O}}^{\,r}\mathcal{UO}^{++}$, any $\{k,\ell\}\subseteq\mathbb{N}_0$, any $\mathfrak{c}\colon[\![k]\!]\to(\mathfrak{U}\otimes\{\circ,\bullet\})\cup\mathfrak{O}$ and $\mathfrak{d}\colon[\![\ell]\!]\to(\mathfrak{U}\otimes\{\circ,\bullet\})\cup\mathfrak{O}$, any set-theoretical partition p with $(\mathfrak{c},\mathfrak{d},p)\in\mathcal{C}$ as well as any $\{m,n\}\subseteq\mathbb{N}_0$, any $\mathfrak{a}\colon[\![m]\!]\to(\mathfrak{U}\otimes\{\circ,\bullet\})\cup\mathfrak{O}$ and $\mathfrak{b}\colon[\![n]\!]\to(\mathfrak{U}\otimes\{\circ,\bullet\})\cup\mathfrak{O}$ and any set-theoretical partition q of Π_n^m and, finally, any ${}_{\mathfrak{U},\mathfrak{O}}^{\,r}\mathcal{UO}^{++}$-allowed covariant rearrangement t of $(\mathfrak{c},\mathfrak{d},p)$ into $(\mathfrak{a},\mathfrak{b},q)$, we have to show that $(\mathfrak{a},\mathfrak{b},q)\in\mathcal{C}$.

We can assume $0<k+\ell$ because, otherwise, $(\mathfrak{a},\mathfrak{b},q)=(\varnothing,\varnothing,\varnothing)=(\mathfrak{c},\mathfrak{d},p)\in\mathcal{C}$ by definition. Then, there exist $\mathsf{b}\in\Pi_\ell^k$ and $\mathsf{a}\in\Pi_n^m$ such that for any $\{\mathsf{a}_1,\mathsf{a}_2\}\subseteq\Pi_n^m$ with $(\xi_{\mathfrak{d}}^{\mathfrak{c}}(t(\mathsf{a}_1)),\xi_{\mathfrak{d}}^{\mathfrak{c}}(t(\mathsf{a}_2)))\notin r$, if $t(\mathsf{a}_1)<t(\mathsf{a}_2)$ with respect to the cut of Γ_ℓ^k at b, then also $\mathsf{a}_1<\mathsf{a}_2$ with respect to the cut of Γ_n^m at a. Let $x\colon\Pi_{k+\ell}^0\to\Pi_\ell^k$ be the unique rotation of $(\mathfrak{c},\mathfrak{d},p)$ with ${}_{\blacksquare}1\mapsto\mathsf{b}$ and let $y\colon\Pi_n^m\to\Pi_{m+n}^0$ be the unique rotation of $(\mathfrak{a},\mathfrak{b},q)$ with $\mathsf{a}\mapsto{}_{\blacksquare}1$. We can define a permutation s of $[\![k+\ell]\!]=[\![m+n]\!]$ by declaring for any $\{i,j\}\subseteq[\![k+\ell]\!]$ that $s(i)\coloneqq j$ if and only if $(x^{-1}\circ t\circ y^{-1})({}_{\blacksquare}i)={}_{\blacksquare}j$. Moreover, let $(\varnothing,\mathfrak{g},h)$ be the covariant rearrangement, i.e., rotation, $(\mathfrak{c},\mathfrak{d},p)$ by x. By Proposition 5.4 it is enough to show that $(\mathfrak{a},\mathfrak{b},q)$ is the covariant rearrangement, i.e., rotation, of of the composition $\mathrm{pm}_{s^{-1}}^{\mathfrak{g}}\,(\varnothing,\mathfrak{g},h)$ by y and that $\mathrm{pm}_{s^{-1}}^{\mathfrak{g}}\in\mathcal{C}$.

Step 1: $\mathrm{pm}_{s^{-1}}^{\mathfrak{g}}$ belongs to $\mathcal{C}$. In order to see $\mathrm{pm}_{s^{-1}}^{\mathfrak{g}}\in\mathcal{C}$, by Proposition 5.9 it is enough to prove that for any $\{j_1,j_2\}\subseteq[\![\ell]\!]$ with $j_1<j_2$, if $\{z_1,z_2\}\subseteq\mathfrak{U}\cup\mathfrak{O}$ are such that $\mathfrak{g}(j_1)\in(\{z_1\}\otimes\{\circ,\bullet\})\cup\{z_1\}$ and $\mathfrak{g}(j_2)\in(\{z_2\}\otimes\{\circ,\bullet\})\cup\{z_2\}$ and $(z_1,z_2)\notin r$, then $s^{-1}(j_1)<s^{-1}(j_2)$.

By construction, x is strictly monotonic with respect to the cut of $\Gamma_{k+\ell}^0$ at ${}_{\blacksquare}1$ and the cut of Γ_ℓ^k at b. Because $j_1<j_2$ means ${}_{\blacksquare}j_1<{}_{\blacksquare}j_2$ with respect to the cut of $\Gamma_{k+\ell}^0$ at ${}_{\blacksquare}1$ it thus follows $x({}_{\blacksquare}j_1)<x({}_{\blacksquare}j_2)$ with respect to the cut of Γ_ℓ^k at b. In other words, if $\mathsf{a}_1\coloneqq(t^{-1}\circ x)({}_{\blacksquare}j_1)$ and $\mathsf{a}_2\coloneqq(t^{-1}\circ x)({}_{\blacksquare}j_2)$, then $t(\mathsf{a}_1)=x({}_{\blacksquare}j_1)<x({}_{\blacksquare}j_2)=t(\mathsf{a}_2)$ with respect to the cut of Γ_ℓ^k at b. Moreover, the identity $\xi_{\mathfrak{g}}^{\varnothing}=\xi_{\mathfrak{d}}^{\mathfrak{c}}\circ x$ holding by definition of $(\varnothing,\mathfrak{g},h)$ implies that $\xi_{\mathfrak{d}}^{\mathfrak{c}}(t(\mathsf{a}_1))=\xi_{\mathfrak{d}}^{\mathfrak{c}}(x({}_{\blacksquare}j_1))=\xi_{\mathfrak{g}}^{\varnothing}({}_{\blacksquare}j_1)=z_1$ and, likewise,

$\xi^{\mathfrak{c}}_{\mathfrak{d}}(t(\mathsf{a}_2)) = z_2$. Hence, $(\xi^{\mathfrak{c}}_{\mathfrak{d}}(t(\mathsf{a}_1)), \xi^{\mathfrak{c}}_{\mathfrak{d}}(t(\mathsf{a}_2))) \notin r$ by $(z_1, z_2) \notin r$ and thus $\mathsf{a}_1 < \mathsf{a}_2$ with respect to the cut of Γ^m_n at a by the initial assumption on t and a and b. Because y is strictly monotonic with respect to the cut of Γ^m_n at a and the cut of Γ^0_{m+n} at $_\blacksquare 1$ we conclude that $y(\mathsf{a}_1) < y(\mathsf{a}_2)$ with respect to the cut of Γ^0_{m+n} at $_\blacksquare 1$. If $\{i_1, i_2\} \subseteq [\![m+n]\!]$ are such that $_\blacksquare i_1 = y(\mathsf{a}_1)$ and $_\blacksquare i_2 = y(\mathsf{a}_2)$, then we have hence shown $i_1 < i_2$. The definitions $_\blacksquare i_1 = (y \circ t^{-1} \circ x)(_\blacksquare j_1)$ and $_\blacksquare i_2 = (y \circ t^{-1} \circ x)(_\blacksquare j_2)$ can also be read as the identities $_\blacksquare j_1 = (x^{-1} \circ t \circ y^{-1})(_\blacksquare i_1)$ and $_\blacksquare j_2 = (x^{-1} \circ t \circ y^{-1})(_\blacksquare i_2)$, which is to say $s(i_1) = j_1$ and $s(i_2) = j_2$ or, equivalently, $i_1 = s^{-1}(j_1)$ and $i_2 = s^{-1}(j_2)$. Thus, $s^{-1}(j_1) < s^{-1}(j_2)$ as claimed, which proves $\mathrm{pm}^{\mathfrak{g}}_{s^{-1}} \in \mathcal{C}$ and hence ${}^{\,r}_{\mathfrak{U},\mathfrak{D}}\mathcal{UO}^{++} \in \mathcal{C}$.

Step 2: $(\mathfrak{a}, \mathfrak{b}, q)$ *is the rotation of* $\mathrm{pm}^{\mathfrak{g}}_{s^{-1}}(\varnothing, \mathfrak{g}, h)$ *by* y. If $(\varnothing, \mathfrak{v}, w) \coloneqq \mathrm{pm}^{\mathfrak{g}}_{s^{-1}}(\varnothing, \mathfrak{g}, h)$, then by definition for any $i \in [\![k+\ell]\!]$ the tag $\xi^{\varnothing}_{\mathfrak{v}}(_\blacksquare i)$ is given by $\xi^{\varnothing}_{\mathfrak{g}\circ(s^{-1})^{-1}}(_\blacksquare i)$ according to the definition of $\mathrm{pm}^{\mathfrak{g}}_{s^{-1}}$, which is, of course, the same as $\xi^{\varnothing}_{\mathfrak{g}\circ s}(_\blacksquare i)$ or, equivalently, $\xi^{\varnothing}_{\mathfrak{g}}(_\blacksquare s(i))$. Hence, the definition of s implies $\xi^{\varnothing}_{\mathfrak{v}} \circ y = \xi^{\varnothing}_{\mathfrak{g}} \circ x^{-1} \circ t = \xi^{\mathfrak{c}}_{\mathfrak{d}} \circ t = \xi^{\mathfrak{a}}_{\mathfrak{b}}$ because $\xi^{\varnothing}_{\mathfrak{g}} = \xi^{\mathfrak{c}}_{\mathfrak{d}} \circ x$. An analogous computation shows $\zeta^{\varnothing}_{\mathfrak{v}} \circ y = \zeta^{\mathfrak{a}}_{\mathfrak{b}}$. Thus, the claimed identity holds on the level of the labelings. We still need to prove it on the level of the blocks.

The supremum $e \coloneqq \kappa^{0}_{k+\ell}{}^{\leftarrow}(h|_{\Pi^0_{k+\ell}}) \vee (\mathrm{pm}_{s^{-1}}|_{\Pi^{k+\ell}_{\varnothing}})$ is actually just $\kappa^{k+\ell\leftarrow}_0(h)$. Indeed, since h has no upper points $h|_{\Pi^0_{k+\ell}} = h$. And, because $\mathrm{pm}_{s^{-1}} = \{{}^\blacksquare j, {}_\blacksquare s^{-1}(j)\}_{j=1}^{k+\ell}$ is an isomorphism, its restriction $\mathrm{pm}_{s^{-1}}|_{\Pi^{k+\ell}_0}$ to the upper row is given by $\{\{{}^\blacksquare j\}\}_{j=1}^{k+\ell}$. Since this is the unique minimal partition of $\Pi^{k+\ell}_0$ the supremum $e = \kappa^{k+\ell\leftarrow}_0(h) \vee \{\{{}^\blacksquare j\}\}_{j=1}^{k+\ell}$ is simply $\kappa^{k+\ell\leftarrow}_0(h)$ again. Taking into account that there are neither any $\mathsf{A} \in p$ with $\mathsf{A} \subseteq \Pi^0_{k+\ell}$ nor any $\mathsf{C} \in \mathrm{pm}_{s^{-1}}$ with $\mathsf{C} \subseteq \Pi^0_{k+\ell}$ and that $\mathsf{A} \cap \Pi^0_{k+\ell} = \varnothing$ for any $\mathsf{A} \in p$ with $\mathsf{A} \cap \kappa^0_{k+\ell}{}^{\leftarrow}(\mathsf{B}) \neq \varnothing$ for any $\mathsf{B} \in e$, the definition of w implies

$$w = \{\bigcup\{\mathsf{C} \cap \Pi^0_{k+\ell} \mid \mathsf{C} \in \mathrm{pm}_{s^{-1}} \wedge \mathsf{C} \cap \mathsf{B} \neq \varnothing\} \mid \mathsf{B} \in e\}.$$

Of course, for any $\mathsf{C} \in \mathrm{pm}_{s^{-1}}$ there exists $j \in [\![k+\ell]\!]$ with $\mathsf{C} = \{{}^\blacksquare j\, {}_\blacksquare s^{-1}(j)\}$, which is why $\mathsf{C} \cap \Pi^0_{k+\ell} = \{{}_\blacksquare s^{-1}(j)\}$ and why for any $\mathsf{B} \in e$ the set $\mathsf{C} \cap \mathsf{B}$ is non-empty if and only if ${}^\blacksquare j \in \mathsf{B}$. Hence, since $e = \kappa^0_{k+\ell}{}^{\leftarrow}(h)$ and since ${}_\blacksquare s^{-1}(j) = (y \circ t^{-1} \circ x)({}_\blacksquare j)$ for any $j \in [\![k+\ell]\!]$, as seen in Step 1, it follows

$$\begin{aligned} w &= \{\{{}_\blacksquare s^{-1}(j) \mid j \in [\![k+\ell]\!] \wedge {}^\blacksquare j \in \kappa^0_{k+\ell}{}^{\leftarrow}(\mathsf{A})\} \mid \mathsf{A} \in h\} \\ &= \{\{(y \circ t^{-1} \circ x)({}_\blacksquare j) \mid j \in [\![k+\ell]\!] \wedge {}_\blacksquare j \in \mathsf{A}\} \mid \mathsf{A} \in h\}. \end{aligned}$$

In other words, w is given by $\{(y \circ t^{-1} \circ x)_{\rightarrow}(\mathsf{A}) \mid \mathsf{A} \in h\}$ or, equivalently, $(x^{-1} \circ t \circ y^{-1})^{\leftarrow}(h)$. Because, on the one hand, $h = x^{\leftarrow}(p)$ and, on the other hand, $q = t^{\leftarrow}(p)$ we have thus proved $y^{\leftarrow}(w) = (x^{-1} \circ t)^{\leftarrow}(h) = t^{\leftarrow}(p) = q$. Hence, also on the block level, $(\mathfrak{a}, \mathfrak{b}, q)$ is the rotation of $(\varnothing, \mathfrak{v}, w)$ by y, which concludes the proof. □

Of course, by Lemma 5.10 in particular the categories ${}^{\,r}_{\mathfrak{U},\mathfrak{D}}\mathcal{UO}^{++}$ and $\ast^r_{z\in\mathfrak{U}\cup\mathfrak{D}}\mathcal{X}_z$ but also $\star^r_{z\in\mathfrak{U}\cup\mathfrak{D}}\mathcal{X}_z$ are closed under ${}^{\,r}_{\mathfrak{U},\mathfrak{D}}\mathcal{UO}^{++}$-allowed covariant rearrangements. That we had already seen in Proposition 5.6.

5.1.4. *Generating sets of crosses.* When it comes to the set of "crossing" partitions composition with which implements the "transposition" of two points whose tags commute according to r there is a certain flexibility, as we will see in this section.

DEFINITION 5.11. For any set X and any partial comutation relation r' on X any binary relation s on X is said to *generate* r' as a partial commutation relation if $r' = \{(x, x'), (x', x) \mid (x, x') \in s\}$.

DEFINITION 5.12. A *generating set of crosses for the r-graph co-product* with respect to $(\mathfrak{U}, \mathfrak{D})$ and r is any subset $\mathcal{R}$ of ${}_{\mathfrak{U},\mathfrak{D}}\mathcal{S}$ for which there exists a binary relation s on $\mathfrak{U} \cup \mathfrak{D}$ generating r as a partial commutation relation such that for any $(z, z') \in s$ there exist $\{c, c'\} \subseteq \{\circ, \bullet\}$ with, if

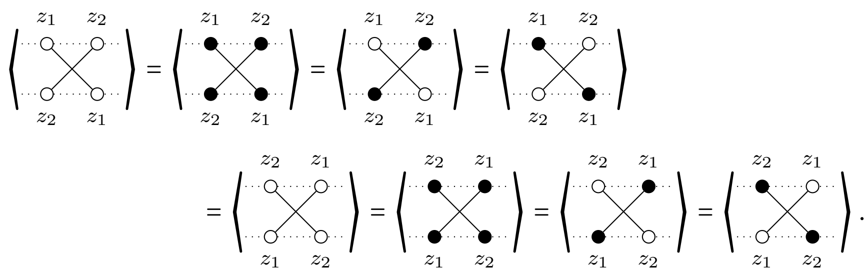
(a) $(z, z') \in \mathfrak{U} \otimes \mathfrak{U}$, then $((z, c) \otimes (z', c'), (z', c') \otimes (z, c), \times) \in \mathcal{R}$,
(b) $(z, z') \in \mathfrak{U} \otimes \mathfrak{D}$, then $((z, c) \otimes z', z' \otimes (z, c), \times) \in \mathcal{R}$,
(c) $(z, z') \in \mathfrak{D} \otimes \mathfrak{U}$, then $(z \otimes (z', c'), (z', c') \otimes z, \times) \in \mathcal{R}$,
(d) $(z, z') \in \mathfrak{D} \otimes \mathfrak{D}$, then $(z \otimes z', z' \otimes z, \times) \in \mathcal{R}$,

where $\times = \{\{{}^{\blacksquare}1, {}_{\blacksquare}2\}, \{{}^{\blacksquare}2, {}_{\blacksquare}1\}\}$. We say that s *induces* $\mathcal{R}$.

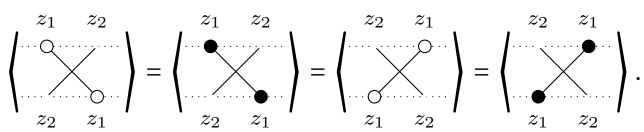
Clearly, $\ast^{r}_{z \in \mathfrak{U} \cup \mathfrak{D}} \mathcal{X}_z$ contains any set of generating crosses for the r-graph co-product with respect to $(\mathfrak{U}, \mathfrak{D})$.

LEMMA 5.13. *For arbitrary z_1 and z_2, if $\langle \cdot \rangle$ denotes the generated category of*

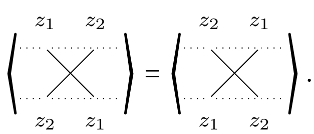
(a) $(\{z_1, z_2\}, \varnothing)$-tagged labeled partitions, then



(b) $(\{z_1\}, \{z_2\})$-tagged labeled partitions, then



(c) $(\varnothing, \{z_1, z_2\})$-tagged labeled partitions, then



PROOF. (a) Let the eight labeled partitions be enumerated left-to-right and top-to-bottom as $f_1, \ldots, f_8$. Then, $f_{i+1} = f_i^{\vee}$ (and thus $f_i = f_{i+1}^{\vee}$) for all $i \in \{1, 3, 5, 7\}$. Moreover, $f_{i+1} = (f_i^{\circlearrowleft})^*$ (and thus $f_i = (f_{i+1}^{*})^{\circlearrowright}$) for any $i \in \{2, 6\}$. Finally, $f_5 = f_4^{\circlearrowleft}$ (and thus $f_4 = f_5^{\circlearrowright}$). Hence, the claim follows by Lemmata 3.20 and 3.22.

(b) If we enumerate the four labeled partitions from left to right as $g_1, \dots, g_4$, then, again, $g_{i+1} = g_i{}^{\vee}$ (and thus $g_i = g_{i+1}{}^{\vee}$) for any $i \in \{1, 3\}$, and $g_3 = g_2{}^{\circlearrowleft}$ (and $g_2 = g_3{}^{\circlearrowright}$). In consequence, Lemmata 3.20 and 3.22 once more prove the claim.

(c) The two labeled partitions are adjoints of each other. □

LEMMA 5.14. *For any generating set of crosses $\mathcal{R}$ for the r-graph co-product with respect to $(\mathfrak{U}, \mathfrak{O})$, for any $(z, z') \in r$ and any $\{c, c'\} \subseteq \{\circ, \bullet\}$, if*

(a) $(z, z') \in \mathfrak{U} \otimes \mathfrak{U}$, then $((z, c) \otimes (z', c'), (z', c') \otimes (z, c), \times) \in {}_{\mathfrak{U},\mathfrak{O}}\langle\mathcal{R}\rangle$,

(b) $(z, z') \in \mathfrak{U} \otimes \mathfrak{O}$, then $((z, c) \otimes z', z' \otimes (z, c), \times) \in {}_{\mathfrak{U},\mathfrak{O}}\langle\mathcal{R}\rangle$,

(c) $(z, z') \in \mathfrak{O} \otimes \mathfrak{U}$, then $(z \otimes (z', c'), (z', c') \otimes z, \times) \in {}_{\mathfrak{U},\mathfrak{O}}\langle\mathcal{R}\rangle$,

(d) $(z, z') \in \mathfrak{O} \otimes \mathfrak{O}$, then $(z \otimes z', z' \otimes z, \times) \in {}_{\mathfrak{U},\mathfrak{O}}\langle\mathcal{R}\rangle$.

PROOF. Follows immediately from the definitions by Lemma 5.13. □

5.1.5. *From crosses to the isomorphisms of the minimal graph co-product category.* We now prove that it is possible to construct by tensor products and composition from any set of generating crosses for r (and all identities) all the isomorphisms of ${}^{r}_{\mathfrak{U},\mathfrak{O}}\mathcal{UO}^{++}$.

LEMMA 5.15. *For any $\{k, \ell\} \subseteq \mathbb{N}_0$, any partition p of Π_ℓ^k and any permutations s of $[\![k]\!]$ and t of $[\![\ell]\!]$,*

(a) $p\,\mathrm{pm}_s = \{(\mathsf{B} \cap \Pi_\ell^0) \cup \{{}^{\blacksquare}s^{-1}(i) \,|\, i \in [\![k]\!] \wedge {}^{\blacksquare}i \in \mathsf{B}\} \,|\, \mathsf{B} \in p\}$.

(b) $\mathrm{pm}_t\, p = \{\{{}_{\blacksquare}t(j) \,|\, j \in [\![\ell]\!] \wedge {}_{\blacksquare}j \in \mathsf{B}\} \cup (\mathsf{B} \cap \Pi_0^k) \,|\, \mathsf{B} \in p\}$.

PROOF. (a) Since pm_s is an isomorphism there are no $\mathsf{A} \in \mathrm{pm}_s$ with $\mathsf{A} \subseteq \Pi_0^k$ and the restriction $\mathrm{pm}_s|_{\Pi_k^0}$ is the minimal partition of Π_k^0 and thus the pull-back $\kappa_k^{0\leftarrow}(\mathrm{pm}_s|_{\Pi_k^0})$ the minimal partition of Π_0^k, which is why the supremum $\kappa_k^{0\leftarrow}(\mathrm{pm}_s|_{\Pi_k^0}) \vee p|_{\Pi_0^k}$ is simply $p|_{\Pi_0^k}$. Hence, by the definition of the composition operation,

$$p\,\mathrm{pm}_s = \{\mathsf{C} \in p \wedge \mathsf{C} \subseteq \Pi_\ell^0\} \cup \{\textstyle\dot\bigcup\{\mathsf{A} \cap \Pi_0^k \,|\, \mathsf{A} \in \mathrm{pm}_s \wedge \mathsf{A} \cap \kappa_0^{k\leftarrow}(\mathsf{B}) \neq \varnothing\} \cup \dot\bigcup\{\mathsf{C} \cap \Pi_\ell^0 \,|\, \mathsf{C} \in p \wedge \mathsf{C} \cap \mathsf{B} \neq \varnothing\} \,|\, \mathsf{B} \in p|_{\Pi_0^k}\}.$$

For any $\{\mathsf{C}, \mathsf{C}'\} \subseteq p$ with $\mathsf{C}' \cap \Pi_0^k \neq \varnothing$, if $\mathsf{B} = \mathsf{C}' \cap \Pi_0^k$, then $\mathsf{C} \cap \mathsf{B} \neq \varnothing$ if only if $\mathsf{C} \cap \mathsf{C}' \cap \Pi_0^k \neq \varnothing$, i.e., if and only if $\mathsf{C} = \mathsf{C}'$. Hence, actually,

$$p\,\mathrm{pm}_s = \{\mathsf{B} \in p \wedge \mathsf{B} \subseteq \Pi_\ell^0\} \cup \{\textstyle\dot\bigcup\{\mathsf{A} \cap \Pi_0^k \,|\, \mathsf{A} \in \mathrm{pm}_s \wedge \mathsf{A} \cap \kappa_0^{k\leftarrow}(\mathsf{B} \cap \Pi_0^k) \neq \varnothing\} \cup (\mathsf{B} \cap \Pi_\ell^0) \,|\, \mathsf{B} \in p \wedge \mathsf{B} \cap \Pi_0^k \neq \varnothing\}.$$

By definition, for any $\mathsf{A} \in \mathrm{pm}_s$ there exists $i \in [\![k]\!]$ with $\mathsf{A} = \{{}^{\blacksquare}i, {}_{\blacksquare}s(i)\}$ and thus $\mathsf{A} \cap \Pi_0^k = \{{}^{\blacksquare}i\}$. Moreover, then for any $\mathsf{B} \in p$ with $\mathsf{B} \cap \Pi_0^k \neq \varnothing$ because $\kappa_0^{k\leftarrow}(\mathsf{B} \cap \Pi_0^k) \subseteq \Pi_k^0$ saying $\mathsf{A} \cap \kappa_0^{k\leftarrow}(\mathsf{B} \cap \Pi_0^k) \neq \varnothing$ is the same as claiming ${}_{\blacksquare}s(i) \in \kappa_0^{k\leftarrow}(\mathsf{B} \cap \Pi_0^k)$, i.e., ${}^{\blacksquare}s(i) \in \mathsf{B}$. Thus, we have shown that

$$p\,\mathrm{pm}_s = \{\mathsf{B} \in p \wedge \mathsf{B} \cap \Pi_0^k = \varnothing\} \cup \{\{{}^{\blacksquare}i \,|\, i \in [\![k]\!] \wedge {}^{\blacksquare}s(i) \in \mathsf{B}\} \cup (\mathsf{B} \cap \Pi_\ell^0) \,|\, \mathsf{B} \in p \wedge \mathsf{B} \cap \Pi_0^k \neq \varnothing\}.$$

Of course, for any $\mathsf{B} \in p$ the sets $\{{}^{\blacksquare}i \mid i \in [\![k]\!] \wedge {}^{\blacksquare}s(i) \in \mathsf{B}\}$ and $\{{}^{\blacksquare}s^{-1}(i) \mid i \in [\![k]\!] \wedge {}^{\blacksquare}i \in \mathsf{B}\}$ coincide and, if $\mathsf{B} \cap \Pi_0^k = \varnothing$, then these two are empty. Hence, $p\,\mathrm{pm}_s = \{(\mathsf{B} \cap \Pi_\ell^0) \cup \{{}^{\blacksquare}s^{-1}(i) \mid i \in [\![k]\!] \wedge {}^{\blacksquare}i \in \mathsf{B}\} \mid \mathsf{B} \in p\}$, as claimed.

(b) Since $\mathrm{pm}_t^* = \mathrm{pm}_{t^{-1}}$, by (a) we can infer that $p^*\,\mathrm{pm}_t^* = \{(\mathsf{D} \cap \Pi_k^0) \cup \{{}^{\blacksquare}t(j) \mid j \in [\![\ell]\!] \wedge {}^{\blacksquare}j \in \mathsf{D}\} \mid \mathsf{D} \in p^*\}$. Because $\mathrm{pm}_t\, p = (p^*\,\mathrm{pm}_t^*)^*$ that proves $\mathrm{pm}_t\, p$ to be given by $\{\kappa_k^{\ell\leftarrow}((\mathsf{D} \cap \Pi_k^0) \cup \{{}^{\blacksquare}t(j) \mid j \in [\![\ell]\!] \wedge {}^{\blacksquare}j \in \mathsf{D}\}) \mid \mathsf{D} \in p^*\}$, which is precisely $\{\{{}_{\blacksquare}t(j) \mid j \in [\![\ell]\!] \wedge {}_{\blacksquare}j \in \mathsf{B}\} \cup (\mathsf{B} \cap \Pi_0^k) \mid \mathsf{B} \in p\}$. □

The next lemma was implicitly recognized in the proof of [FW16, Proposition 4.11].

LEMMA 5.16. *For any $k \in \mathbb{N}_0$, any $\mathfrak{a}\colon [\![k]\!] \to (\mathfrak{U} \otimes \{\circ, \bullet\}) \cup \mathfrak{O}$ and $\mathfrak{c}\colon [\![k]\!] \to (\mathfrak{U} \otimes \{\circ, \bullet\}) \cup \mathfrak{O}$ and any permutations s and t of $[\![k]\!]$, the pair $(\mathrm{pm}_t^{\mathfrak{c}}, \mathrm{pm}_s^{\mathfrak{a}})$ is composable if and only if $\mathfrak{c} = \mathfrak{a}^{-1} \circ s$ and, if so, then $\mathrm{pm}_t^{\mathfrak{c}}\,\mathrm{pm}_s^{\mathfrak{a}} = \mathrm{pm}_{t\circ s}^{\mathfrak{a}}$.*

PROOF. The claim about the composability is clear from the definitions. Lemma 5.15 (a) shows that the composition $\mathrm{pm}_t\,\mathrm{pm}_s$ is given by $\{(\mathsf{B} \cap \Pi_k^0) \cup \{{}^{\blacksquare}s^{-1}(i') \mid i' \in [\![k]\!] \wedge {}^{\blacksquare}i' \in \mathsf{B}\} \mid \mathsf{B} \in \mathrm{pm}_t\}$. By definition, $\mathrm{pm}_t = \{\{{}^{\blacksquare}j, {}_{\blacksquare}t(j)\}\}_{j=1}^k = \{\{{}^{\blacksquare}s(i), {}_{\blacksquare}t(s(i))\}\}_{i=1}^k$ and for any $\{i', i\} \subseteq [\![k]\!]$, if $\mathsf{B} = \{{}^{\blacksquare}s(i), {}_{\blacksquare}t(s(i))\}$, then ${}^{\blacksquare}i' \in \mathsf{B}$ if and only if $i' = s(i)$, in which case ${}^{\blacksquare}s^{-1}(i') = {}^{\blacksquare}i$ and $\mathsf{B} \cap \Pi_k^0 = \{{}_{\blacksquare}t(s(i))\}$. Thus we have shown $\mathrm{pm}_t\,\mathrm{pm}_s = \{\{{}^{\blacksquare}i, {}_{\blacksquare}(t \circ s)(i)\}\}_{i=1}^k = \mathrm{pm}_{t\circ s}$. □

LEMMA 5.17. *For any generating crosses $\mathcal{R}$ of the r-graph co-product with respect to $(\mathfrak{U}, \mathfrak{O})$ the category ${}_{\mathfrak{U},\mathfrak{O}}\langle\mathcal{R}\rangle$ contains all isomorphisms of ${}^{r}_{\mathfrak{U},\mathfrak{O}}\mathcal{UO}^{++}$.*

PROOF. We prove the claim by induction over the common length of the two objects linked by the isomorphism. For length 0, where $(\varnothing, \varnothing, \varnothing)$ is the only isomorphism, the claim is true by definition of ${}_{\mathfrak{U},\mathfrak{O}}\langle\mathcal{R}\rangle$. Let $k \in \mathbb{N}$ be arbitrary and suppose that ${}_{\mathfrak{U},\mathfrak{O}}\langle\mathcal{R}\rangle$ contains all isomorphisms of ${}^{r}_{\mathfrak{U},\mathfrak{O}}\mathcal{UO}^{++}$ between objects of lengths up to and including $k-1$. Moreover, let $\mathfrak{c}\colon [\![k]\!] \to (\mathfrak{U} \otimes \{\circ, \bullet\}) \cup \mathfrak{O}$ be arbitrary and let s be any permutation of $[\![k]\!]$ such that for any $\{i_1, i_2\} \subseteq [\![k]\!]$ with $i_1 < i_2$, if $\{z_1, z_2\} \subseteq \mathfrak{U} \cup \mathfrak{O}$ are such that $\mathfrak{c}(i_1) \in (\{z_1\} \otimes \{\circ, \bullet\}) \cup \{z_1\}$ and $\mathfrak{c}(i_2) \in (\{z_2\} \otimes \{\circ, \bullet\}) \cup \{z_2\}$ and if $(z_1, z_2) \notin r$, then $s(i_1) < s(i_2)$. Then, by Proposition 5.9 the claim is verified once we show $\mathrm{pm}_s^{\mathfrak{c}} \in {}_{\mathfrak{U},\mathfrak{O}}\langle\mathcal{R}\rangle$. To do so we distinguish three cases.

Case 1: Upper and lower last points are in the same block. First, let $s(k) = k$. In that case, we let $\mathfrak{a}\colon [\![k-1]\!] \to (\mathfrak{U} \otimes \{\circ, \bullet\}) \cup \mathfrak{O}$ and $t\colon [\![k-1]\!] \to \mathbb{N}$ and $\mathfrak{f} \in (\mathfrak{U} \otimes \{\circ, \bullet\}) \cup \mathfrak{O}$ be defined by $\mathfrak{a}(i) \coloneqq \mathfrak{c}(i)$ and $t(i) \coloneqq s(i)$ for any $i \in [\![k-1]\!]$ and by $\mathfrak{f} \coloneqq \mathfrak{c}(k)$. We now show that $\mathrm{pm}_t^{\mathfrak{a}}$ is well-defined isomorphism of ${}^{r}_{\mathfrak{U},\mathfrak{O}}\mathcal{UO}^{++}$ between objects of length $k-1$ and that $\mathrm{pm}_s^{\mathfrak{c}} = \mathrm{pm}_t^{\mathfrak{a}} \otimes \mathrm{id}_{\mathfrak{f}}$. Then, the induction hypothesis and the fact that $\mathrm{id}_{\mathfrak{f}} \in {}_{\mathfrak{U},\mathfrak{O}}\langle\mathcal{R}\rangle$ will imply that $\mathrm{pm}_s^{\mathfrak{c}} \in {}_{\mathfrak{U},\mathfrak{O}}\langle\mathcal{R}\rangle$, as claimed.

Step 1.1: The partition $\mathrm{pm}_t^{\mathfrak{a}}$ is well-defined isomorphism of ${}^{r}_{\mathfrak{U},\mathfrak{O}}\mathcal{UO}^{++}$ Because $s(k) = k$ and since s is a permutation of $[\![k]\!]$ the mapping t is actually a permutation of $[\![k-1]\!]$. Moreover, for any $\{i_1, i_2\} \subseteq [\![k-1]\!]$ with $i_1 < i_2$ and with $(z_1, z_2) \notin r$, where $\{z_1, z_2\} \subseteq \mathfrak{U} \cup \mathfrak{O}$ are such that $\mathfrak{a}(i_1) \in (\{z_1\} \otimes \{\circ, \bullet\}) \cup \{z_1\}$ and $\mathfrak{a}(i_2) \in (\{z_2\} \otimes \{\circ, \bullet\}) \cup \{z_2\}$, by definition, actually, $\mathfrak{c}(i_1) \in (\{z_1\} \otimes \{\circ, \bullet\}) \cup \{z_1\}$ and $\mathfrak{c}(i_2) \in$

$(\{z_2\}\otimes\{\circ,\bullet\})\cup\{z_2\}$ and thus $s(i_1)<s(i_2)$ by our initial assumption on s. But that is to say $t(i_1)<t(i_2)$ since $i_1<i_2<k$. Thus, $\mathrm{pm}_t^{\mathfrak{a}}\in{}_{\mathfrak{U},\mathfrak{O}}^{\;r}\mathcal{UO}^{++}$ by Proposition 5.9.

Step 1.2: The partitions $\mathrm{pm}_s^{\mathfrak{c}}$ *and* $\mathrm{pm}_t^{\mathfrak{a}}\otimes\mathrm{id}_{\mathfrak{f}}$ *are identical.* By definition, $\mathfrak{c}=\mathfrak{a}\otimes\mathfrak{f}$. Furthermore, according to the definitions, $\mathrm{pm}_s=\{\{{}^{\blacksquare}i,{}_{\blacksquare}s(i)\}\}_{i=1}^{k}=\{\{{}^{\blacksquare}i,{}_{\blacksquare}s(i)\}\}_{i=1}^{k-1}\cup\{\{{}^{\blacksquare}k,{}_{\blacksquare}s(k)\}\}=\{\{{}^{\blacksquare}i,{}_{\blacksquare}t(i)\}\}_{i=1}^{k-1}\cup\{\{{}^{\blacksquare}k,{}_{\blacksquare}k\}\}=\mathrm{pm}_t\otimes\mathrm{id}_1$. Lastly, for any $j\in[\![k-1]\!]$, because $t^{-1}(j)=s^{-1}(j)$, also $(\mathfrak{c}\circ s^{-1})(j)=(\mathfrak{c}\circ t^{-1})(j)=((\mathfrak{c}\circ t^{-1})\otimes\mathfrak{f})(j)$ and, of course, $(\mathfrak{c}\circ s^{-1})(k)=\mathfrak{c}(k)=\mathfrak{f}=((\mathfrak{c}\circ t^{-1})\otimes\mathfrak{f})(k)$. Hence, $\mathfrak{c}\circ s^{-1}=(\mathfrak{c}\circ t^{-1})\otimes\mathfrak{f}$. Altogether thus, indeed, $\mathrm{pm}_s^{\mathfrak{c}}=\mathrm{pm}_t^{\mathfrak{a}}\otimes\mathrm{id}_{\mathfrak{f}}$.

Case 2: Last upper point is in a block with neither first nor last lower points. Next, if $1<s(k)<k$, then let $\mathfrak{a}\colon[\![k-1]\!]\to(\mathfrak{U}\otimes\{\circ,\bullet\})\cup\mathfrak{O}$ be defined by $\mathfrak{a}(i)=\mathfrak{c}(i)$ for any $i\in[\![k-1]\!]$, let $t\colon[\![k-1]\!]\to\mathbb{N}$ satisfy for any $i\in[\![k-1]\!]$, if $s(i)<s(k)$, then $t(i)\coloneqq s(i)$ and, otherwise, $t(i)\coloneqq s(i)-1$, let $\mathfrak{f}\coloneqq\mathfrak{c}(k)$, let $\mathfrak{h}\colon[\![s(k)-1]\!]\to(\mathfrak{U}\otimes\{\circ,\bullet\})\cup\mathfrak{O}$ be given by the rule $\mathfrak{h}(i)\coloneqq(\mathfrak{c}\circ s^{-1})(i)$ for any $i\in[\![s(k)-1]\!]$, let $\mathfrak{u}\colon[\![k-s(k)+1]\!]\to(\mathfrak{U}\otimes\{\circ,\bullet\})\cup\mathfrak{O}$ be defined by $\mathfrak{u}(x)\coloneqq(\mathfrak{c}\circ s^{-1})(x+s(k))$ for any $x\in[\![k-s(k)]\!]$ and $\mathfrak{u}(k-s(k)+1)\coloneqq\mathfrak{c}(k)$ and let $m\colon[\![k-s(k)+1]\!]\to\mathbb{N}$ be defined by $m(x)=x+1$ for any $x\in[\![k-s(k)]\!]$ and by $m(k-s(k)+1)=1$. We prove that $\mathrm{pm}_t^{\mathfrak{a}}$ and $\mathrm{pm}_m^{\mathfrak{u}}$ are well-defined isomorphisms of ${}_{\mathfrak{U},\mathfrak{O}}^{\;r}\mathcal{UO}^{++}$ between objects of lengths $k-1$ respectively $k-s(k)+1$, that $(\mathrm{id}_{\mathfrak{h}}\otimes\mathrm{pm}_m^{\mathfrak{u}},\mathrm{pm}_t^{\mathfrak{a}}\otimes\mathrm{id}_{\mathfrak{f}})$ is composable and that $\mathrm{pm}_s^{\mathfrak{c}}=(\mathrm{id}_{\mathfrak{h}}\otimes\mathrm{pm}_m^{\mathfrak{u}})(\mathrm{pm}_t^{\mathfrak{a}}\otimes\mathrm{id}_{\mathfrak{f}})$. Then, once more, $\mathrm{pm}_s^{\mathfrak{c}}\in{}_{\mathfrak{U},\mathfrak{O}}\langle\mathcal{R}\rangle$ will follow immediately by the induction hypothesis and the fact that $\{\mathrm{id}_{\mathfrak{f}},\mathrm{id}_{\mathfrak{h}}\}\subseteq{}_{\mathfrak{U},\mathfrak{O}}\langle\mathcal{R}\rangle$.

Step 2.1: The partition $\mathrm{pm}_t^{\mathfrak{a}}$ *is a well-defined isomorphism of* ${}_{\mathfrak{U},\mathfrak{O}}^{\;r}\mathcal{UO}^{++}$. First of all, $\mathrm{ran}(t)\subseteq[\![k-1]\!]$ because for any $i\in[\![t]\!]$ by definition either $t(i)=s(i)<s(k)\leq k-1$ or $t(i)=s(i)-1\leq k-1$. Moreover, if $e\colon[\![k-1]\!]\to\mathbb{N}$ is defined by, for any $j\in[\![k-1]\!]$, if $j<s(k)$, then $e(j)\coloneqq s^{-1}(j)$ and, otherwise, $e(j)\coloneqq s^{-1}(j+1)$, then $e=t^{-1}$. Indeed, since $s(k)<k$ necessitates $s^{-1}(k)<k$ the definition ensures $\mathrm{ran}(e)\subseteq[\![k-1]\!]$ and for any $j\in[\![k-1]\!]$, if $j<s(k)$, then $(t\circ e)(j)=t(s^{-1}(j))=s(s^{-1}(j))=j$ because $s(s^{-1}(j))<s(k)$ and, otherwise, $(t\circ e)(j)=t(s^{-1}(j+1))=s(s^{-1}(j+1))-1=j$ because $s(k)\leq j<j+1=s(s^{-1}(j+1))$. And, evidently, for any $i\in[\![k-1]\!]$, if $s(i)<s(k)$, then $(e\circ t)(i)=e(s(i))=s^{-1}(s(i))=i$ and, otherwise, $(e\circ t)(i)=e(s(i)-1)=s^{-1}((s(i)-1)+1)=i$. Thus, $\mathrm{pm}_t^{\mathfrak{a}}$ is well-defined.

Given any $\{i_1,i_2\}\subseteq[\![k-1]\!]$ with $i_1<i_2$, if $\{z_1,z_2\}\subseteq\mathfrak{U}\cup\mathfrak{O}$ and $\mathfrak{a}(i_1)\in(\{z_1\}\otimes\{\circ,\bullet\})\cup\{z_1\}$ and $\mathfrak{a}(i_2)\in(\{z_2\}\otimes\{\circ,\bullet\})\cup\{z_2\}$ and $(z_1,z_2)\notin r$, then, once more, $\mathfrak{c}(i_1)\in(\{z_1\}\otimes\{\circ,\bullet\})\cup\{z_1\}$ and $\mathfrak{c}(i_2)\in(\{z_2\}\otimes\{\circ,\bullet\})\cup\{z_2\}$ by the definition of $\mathfrak{a}$. Per our assumption on s, that requires $s(i_1)<s(i_2)$. By $i_1\neq k\neq i_2$ there are now the three following possibilities.

Case 2.1.1: If $s(i_2)<s(k)$, then it follows $s(i_1)<s(k)$ as well as $t(i_1)=s(i_1)$ and $t(i_2)=s(i_2)$ by the definition of t. Hence, $t(i_1)<t(i_2)$ in this case.

Case 2.1.2: If $s(i_1)<s(k)<s(i_2)$, then in particular $s(i_1)<s(i_2)-1$. Because, this time, $t(i_1)=s(i_1)$ and $t(i_2)=s(i_2)-1$ according to the definition of t that proves $t(i_1)<t(i_2)$ here too.

Case 2.1.3: Lastly, if $s(k)<s(i_1)<s(i_2)$, then by the second inequality also, $s(i_1)-1<s(i_2)-1$ and thus by $t(i_1)=s(i_1)-1$ and $t(i_2)=s(i_2)-1$, indeed,

$t(i_1) < t(i_2)$, which hence holds in any case. In conclusion, $\mathrm{pm}^{\mathfrak{a}}_t \in {}^{\,r}_{\mathfrak{U},\mathfrak{D}}\mathcal{UO}^{++}$ by Proposition 5.9.

Step 2.2: The partition $\mathrm{pm}^{\mathfrak{u}}_m$ *is a well-defined isomorphism of* ${}^{\,r}_{\mathfrak{U},\mathfrak{D}}\mathcal{UO}^{++}$. It is clear that the cyclic shift m of $[\![k-s(k)+1]\!]$ by 1 to the right is a permutation. Hence, $\mathrm{pm}^{\mathfrak{u}}_m$ is well-defined.

Let $\{x_1,x_2\}\subseteq[\![k-s(k)+1]\!]$ satisfy $x_1<x_2$ and $(z_1,z_2)\notin r$, where $\{z_1,z_2\}\subseteq\mathfrak{U}\cup\mathfrak{D}$ are such that $\mathfrak{u}(x_1)\in(\{z_1\}\otimes\{\circ,\bullet\})\cup\{z_1\}$ and $\mathfrak{u}(x_2)\in(\{z_2\}\otimes\{\circ,\bullet\})\cup\{z_2\}$. We first recognize that $x_2<k-s(k)+1$. Indeed, if $i_1\coloneqq s^{-1}(x_1+s(k))$, then the fact that $x_1\neq 0$ implies that, first, $x_1+s(k)\neq s(k)$ and thus $i_1=s^{-1}(x_1+s(k))\neq s^{-1}(s(k))=k$ or, equivalently, $i_1<k$ and, second, $s(k)<x_1+s(k)=s(s^{-1}(x_1+s(k)))=s(i_1)$. Moreover, $\mathfrak{c}(i_1)=\mathfrak{c}(s^{-1}(x_1+s(k)))=\mathfrak{u}(x_1)\in(\{z_1\}\otimes\{\circ,\bullet\})\cup\{z_1\}$ by definition of $\mathfrak{u}$. If x_2 was $k-s(k)+1$, then the definition of $\mathfrak{u}$ would also imply $\mathfrak{c}(k)=\mathfrak{u}(x_2)\in(\{z_2\}\otimes\{\circ,\bullet\})\cup\{z_2\}$. However, that would contradict the initial assumption on s because $i_1<k$ and $s(k)<s(i_1)$ and $(z_1,z_2)\notin r$. Hence, $x_2<k-s(k)+1$ must hold instead. Consequently, not only $m(x_1)=x_1+1$ but also $m(x_2)=x_2+1$ and thus $m(x_1)<m(x_2)$ by $x_1<x_2$. In conclusion, $\mathrm{pm}^{\mathfrak{u}}_m\in{}^{\,r}_{\mathfrak{U},\mathfrak{D}}\mathcal{UO}^{++}$ by Proposition 5.9.

Step 2.3: The pairing $(\mathrm{pm}^{\mathfrak{a}}_t\otimes\mathrm{id}_{\mathfrak{f}},\mathrm{id}_{\mathfrak{h}}\otimes\mathrm{pm}^{\mathfrak{u}}_m)$ *is composable.* We have to prove $\mathfrak{h}\vartriangle\mathfrak{u}=(\mathfrak{a}\circ t^{-1})\vartriangle\mathfrak{f}$. By definition, for any $j\in[\![k]\!]$, if $j<s(k)$, then $(\mathfrak{h}\vartriangle\mathfrak{u})(j)=\mathfrak{h}(j)=\mathfrak{c}(s^{-1}(j))=\mathfrak{a}(t^{-1}(j))=((\mathfrak{a}\circ t^{-1})\vartriangle\mathfrak{f})(j)$. Similarly, if $s(k)\leq j<k-1$, then $(\mathfrak{h}\vartriangle\mathfrak{u})(j)=\mathfrak{u}(j-(s(k)-1))=\mathfrak{u}(j-s(k)+1)=\mathfrak{c}(s^{-1}((j-s(k)+1)+s(k)))=\mathfrak{c}(s^{-1}(j+1))=\mathfrak{a}(t^{-1}(j))=((\mathfrak{a}\circ t^{-1})\vartriangle\mathfrak{f})(j)$ because $t^{-1}(j)=s^{-1}(j+1)$, as seen in Step 2.1. Finally, $(\mathfrak{h}\vartriangle\mathfrak{u})(k)=\mathfrak{u}(k-(s(k)-1))=\mathfrak{u}(k-s(k)+1)=\mathfrak{c}(k)=\mathfrak{f}=((\mathfrak{a}\circ t^{-1})\vartriangle\mathfrak{f})(k)$.

Step 2.4: The partitions $\mathrm{pm}^{\mathfrak{c}}_s$ *and* $(\mathrm{id}_{\mathfrak{h}}\otimes\mathrm{pm}^{\mathfrak{u}}_m)(\mathrm{pm}^{\mathfrak{a}}_t\otimes\mathrm{id}_{\mathfrak{f}})$ *are identical.* Let $v\colon[\![k]\!]\to\mathbb{N}$ be defined for any $i\in[\![k-1]\!]$ by $v(i)\coloneqq s(i)$ if $s(i)<s(k)$, by $v(i)\coloneqq s(i)-1$ if $s(k)<s(i)$ and by $v(k)=k$. Moreover, let $w\colon[\![k]\!]\to[\![k]\!]$ satisfy for any $i\in[\![k]\!]$, if $i<s(k)$, then $w(i)=i$, and, if $s(k)\leq i<k$, then $w(i)=i+1$, and, $w(k)=s(k)$. In order to prove $(\mathrm{id}_{\mathfrak{h}}\otimes\mathrm{pm}^{\mathfrak{u}}_m)(\mathrm{pm}^{\mathfrak{a}}_t\otimes\mathrm{id}_{\mathfrak{f}})=\mathrm{pm}^{\mathfrak{c}}_s$, by Lemma 5.16 it suffices to show that $\mathrm{pm}^{\mathfrak{h}\otimes\mathfrak{u}}_w$ and $\mathrm{pm}^{\mathfrak{c}}_v$ are well-defined, that $\mathrm{id}_{\mathfrak{h}}\otimes\mathrm{pm}^{\mathfrak{u}}_m=\mathrm{pm}^{\mathfrak{h}\otimes\mathfrak{u}}_w$ and $\mathrm{pm}^{\mathfrak{a}}_t\otimes\mathrm{id}_{\mathfrak{f}}=\mathrm{pm}^{\mathfrak{c}}_v$, and that $w\circ v=s$.

Step 2.4.1: The partitions $\mathrm{pm}^{\mathfrak{c}}_v$ *is well-defined and coincides with* $\mathrm{pm}^{\mathfrak{a}}_t\otimes\mathrm{id}_{\mathfrak{f}}$. Since, on the one hand, $v(i)=t(i)$ for any $i\in[\![k-1]\!]$ and since t is a permutation of $[\![k-1]\!]$, as well as since, on the other hand, $v(k)=k$, it is clear that v is a permutation of $[\![k]\!]$. In particular, $v^{-1}(i)=t^{-1}(i)$ for any $i\in[\![k-1]\!]$ and $v^{-1}(k)=k$. Hence, $\mathrm{pm}^{\mathfrak{c}}_v$ is well-defined.

Already by defintion, $\mathfrak{a}\vartriangle\mathfrak{f}=\mathfrak{c}$. Moreover, using what we have already seen in Step 2.3 we find $\mathfrak{c}\circ v^{-1}=(\mathfrak{a}\vartriangle\mathfrak{f})\circ v^{-1}=(\mathfrak{a}\circ t^{-1})\vartriangle\mathfrak{f}$. And, finally, $\mathrm{pm}_t\otimes\mathrm{id}_1=\{\{{}^{\blacksquare}i,{}_{\blacksquare}t(i)\}\mid i\in[\![k-1]\!]\}\cup\{\{{}^{\blacksquare}k,{}_{\blacksquare}k\}\}=\mathrm{pm}_v$. Hence, indeed, $\mathrm{pm}^{\mathfrak{c}}_v=\mathrm{pm}^{\mathfrak{a}}_t\otimes\mathrm{id}_{\mathfrak{f}}$.

Step 2.4.2: The partition $\mathrm{pm}^{\mathfrak{h}\otimes\mathfrak{u}}_w$ *is well-defined and coincides with* $\mathrm{id}_{\mathfrak{h}}\otimes\mathrm{pm}^{\mathfrak{u}}_m$. By definition, for any $i\in[\![k]\!]$, if $i<s(k)$, then $w(i)=i$ and, if $s(k)\leq i<k$, then $w(i)=i+1=((i-s(k)+1)+1)+s(k)-1=m(i-s(k)+1)+s(k)-1$, and, likewise, $w(k)=s(k)=1+(s(k)-1)=m(k-s(k)+1)+s(k)-1$. Because the shift by $s(k)-1$ to the right is a bijection from $[\![k-s(k)+1]\!]$ to $[\![k]\!]\backslash[\![s(k)-1]\!]$ and because m is a

permutation of $[\![k-s(k)+1]\!]$ that proves w to be a permutation with $w(i)=i$ for any $i\in[\![s(k)-1]\!]$ and $w(i)=m(i-s^{-1}(k)+1)+s(k)-1$ for any $i\in[\![k]\!]\backslash[\![s(k)-1]\!]$. In particular, $\mathrm{pm}_w^{\mathfrak{h}\otimes\mathfrak{u}}$ is well-defined.

Moreover, we have shown incidentally that for any $i\in[\![k]\!]$, if $i<s(k)$, then $w^{-1}(i)=i$ and, if $s(k)\leq i$, then $w^{-1}(i)=m^{-1}(i-s(k)+1)+s(k)-1$. By what we have already seen in Step 2.3 it thus follows $(\mathfrak{h}\vartriangle\mathfrak{u})\circ w^{-1}=\mathfrak{h}\vartriangle(\mathfrak{u}\circ m^{-1})$.

Lastly, $\mathrm{id}_{s(k)-1}\otimes\mathrm{pm}_m=\{\{{}^{\blacksquare}i,{}_{\blacksquare}i\}\,|\,i\in[\![s(k)-1]\!]\}\cup\{{}^{\blacksquare}i,{}_{\blacksquare}(m(i-s(k)+1)+s(k)-1)\}\,|\,i\in\mathbb{N}\wedge s(k)\leq i\leq k\}=\{\{{}^{\blacksquare}i,{}_{\blacksquare}i\}\,|\,i\in[\![s(k)-1]\!]\}\cup\{{}^{\blacksquare}i,{}_{\blacksquare}(i+1)\}\,|\,i\in\mathbb{N}\wedge s(k)\leq i<k\}\cup\{\{{}^{\blacksquare}k,{}_{\blacksquare}s(k)\}\}=\mathrm{pm}_w$. Hence, indeed, $\mathrm{pm}_w^{\mathfrak{h}\otimes\mathfrak{u}}=\mathrm{id}_{\mathfrak{h}}\otimes\mathrm{pm}_m^{\mathfrak{u}}$.

Step 2.4.4: The permutations $w\circ v$ and s coincide. For any $i\in[\![k]\!]$, if $s(i)<s(k)$ (and thus, necessarily, $i<k$), then $v(i)=s(i)$ by the definition of v and thus $w(v(i))=s(i)$ by the definition of w. If both $s(k)\leq s(i)$ and $i<k$ (and thus, actually, $s(k)<s(i)$), then, $v(i)=s(i)-1$ according to the definition of v. Hence, in that case also, $w(v(i))=(s(i)-1)+1=s(i)$ by the definition of w because $s(k)\leq s(i)-1$ by $s(k)<s(i)$ and because, trivially, $s(i)-1\leq k-1<k$. Finally, too, $w(v(k))=w(k)=s(k)$ by definition. In total, hence, $w\circ v=s$. Overall, therefore, $\mathrm{pm}_s^{\mathfrak{c}}\in{}_{\mathfrak{U},\mathfrak{D}}\langle\mathcal{R}\rangle$ in this case as well.

Case 3: Last upper point and first lower point are in the same block. Finally, we consider the case that $s(k)=1$. Since, if $k=1$, then $\mathrm{pm}_s^{\mathfrak{c}}=\mathrm{id}_{\mathfrak{c}}$ and since $\mathrm{id}_{\mathfrak{c}}\in{}_{\mathfrak{U},\mathfrak{D}}\langle\mathcal{R}\rangle$ by definition of ${}_{\mathfrak{U},\mathfrak{D}}\langle\mathcal{R}\rangle$, we can assume $2\leq k$ in the following. Under this assumption, if $t\colon[\![k]\!]\to[\![k]\!]$ is the permutation defined by $1\mapsto 2$ and $2\mapsto 1$ and $i\mapsto i$ for any $i\in[\![k]\!]$ with $2<i$, then the pairing $(\mathrm{pm}_t^{\mathfrak{c}\circ s^{-1}\circ t^{-1}},\mathrm{pm}_{t\circ s}^{\mathfrak{c}})$ is composable and $\mathrm{pm}_t^{\mathfrak{c}\circ s^{-1}\circ t^{-1}}\mathrm{pm}_{t\circ s}^{\mathfrak{c}}=\mathrm{pm}_{t\circ t\circ s}^{\mathfrak{c}}=\mathrm{pm}_s^{\mathfrak{c}}$ by Lemma 5.16 because $t^{-1}=t$. Hence, in order to prove that $\mathrm{pm}_s^{\mathfrak{c}}\in{}_{\mathfrak{U},\mathfrak{D}}\langle\mathcal{R}\rangle$, it suffices to show that $\{\mathrm{pm}_t^{\mathfrak{c}\circ s^{-1}\circ t},\mathrm{pm}_{t\circ s}^{\mathfrak{c}}\}\subseteq{}_{\mathfrak{U},\mathfrak{D}}\langle\mathcal{R}\rangle$.

Step 3.1: The partition $\mathrm{pm}_{t\circ s}^{\mathfrak{c}}$ is an element of ${}_{\mathfrak{U},\mathfrak{D}}\langle\mathcal{R}\rangle$. Of course, $(t\circ s)(k)=t(s(k))=t(1)=2>1$ by assumption. Hence, cases 1 and 2 will imply that $\mathrm{pm}_{t\circ s}^{\mathfrak{c}}\in{}_{\mathfrak{U},\mathfrak{D}}\langle\mathcal{R}\rangle$ once we prove that $\mathrm{pm}_{t\circ s}^{\mathfrak{c}}$ is an isomorphism of ${}_{\mathfrak{U},\mathfrak{D}}^{\;r}\mathcal{UO}^{++}$.

Let $\{z_1,z_2\}\subseteq\mathfrak{U}\cup\mathfrak{D}$ with $(z_1,z_2)\notin r$ as well as $\{i_1,i_2\}\subseteq[\![k]\!]$ with $i_1<i_2$ and $\mathfrak{c}(i_1)\in(\{z_1\}\otimes\{\circ,\bullet\})\cup\{z_1\}$ and $\mathfrak{c}(i_2)\in(\{z_2\}\otimes\{\circ,\bullet\})\cup\{z_2\}$ be arbitrary. We show $(t\circ s)(i_1)<(t\circ s)(i_2)$. That then verifies $\mathrm{pm}_{t\circ s}^{\mathfrak{c}}\in{}_{\mathfrak{U},\mathfrak{D}}^{\;r}\mathcal{UO}^{++}$ by Proposition 5.9.

If $s(i_1)$ were 1, it would follow $i_1=k$, contradicting the assumption $i_1<i_2\leq k$. Hence, $s(i_1)\neq 1$, instead. Because the assumption on s guarantees $s(i_1)<s(i_2)$ it must futhermore hold that $s(i_2)\neq 1$. For the same reason it cannot be that simultaneously $2<s(i_1)$ and $s(i_2)=2$. We now distinguish two cases. If $s(i_1)\leq 2$, which is to say $s(i_1)=2$, then $t(s(i_1))=t(2)=1<t(s(i_2))$ by the definition of t. Alternatively, if $2<s(i_1)$, and thus, necessarily, $2<s(i_2)$, then also $t(s(i_1))=s(i_1)<s(i_2)=t(s(i_2))$ by the definition of t and the fact that $s(i_1)<s(i_2)$. In conclusion $(t\circ s)(i_1)<(t\circ s)(i_2)$ always holds, proving $\mathrm{pm}_{t\circ s}^{\mathfrak{c}}\in{}_{\mathfrak{U},\mathfrak{D}}\langle\mathcal{R}\rangle$.

Step 3.2: The partition $\mathrm{pm}_t^{\mathfrak{c}\circ s^{-1}\circ t}$ is an element of ${}_{\mathfrak{U},\mathfrak{D}}\langle\mathcal{R}\rangle$. Let $\mathfrak{h}\colon[\![2]\!]\to(\mathfrak{U}\otimes\{\circ,\bullet\})\cup\mathfrak{D}$ satisfy $\mathfrak{h}(1)\coloneqq\mathfrak{c}(s^{-1}(2))$ and $\mathfrak{h}(2)\coloneqq\mathfrak{c}(k)$, let $\mathfrak{u}\colon[\![k-2]\!]\to(\mathfrak{U}\otimes\{\circ,\bullet\})\cup\mathfrak{D}$ be defined by $\mathfrak{u}(x)\coloneqq\mathfrak{c}(s^{-1}(x+2))$ for any $x\in[\![k-2]\!]$ and let q be the permutation of $[\![2]\!]$

with $1 \mapsto 2$. Because $\mathrm{id}_{\mathfrak{u}} \in {}_{\mathfrak{U},\mathfrak{O}}\langle\mathcal{R}\rangle$ by definition, in order to show $\mathrm{pm}_t^{\mathfrak{c}\circ s^{-1}\circ t} \in {}_{\mathfrak{U},\mathfrak{O}}\langle\mathcal{R}\rangle$, it is enough to prove that $\mathrm{pm}_q^{\mathfrak{h}} \in {}_{\mathfrak{U},\mathfrak{O}}\langle\mathcal{R}\rangle$ and $\mathrm{pm}_q^{\mathfrak{h}} \otimes \mathrm{id}_{\mathfrak{u}} = \mathrm{pm}_t^{\mathfrak{c}\circ s^{-1}\circ t}$.

Step 3.2.1: The partition $\mathrm{pm}_q^{\mathfrak{h}}$ *is included in* ${}_{\mathfrak{U},\mathfrak{O}}\langle\mathcal{R}\rangle$. If $\{z_1, z_2\} \subseteq \mathfrak{U} \cup \mathfrak{O}$ are such that $\mathfrak{c}(s^{-1}(2)) \in (\{z_1\} \otimes \{\circ, \bullet\}) \cup \{z_1\}$ and $\mathfrak{c}(k) \in (\{z_2\} \otimes \{\circ, \bullet\}) \cup \{z_2\}$, then the fact that $s^{-1}(2) < k$ and, at the same time, $s(s^{-1}(2)) = 2 \geq 1 = s(k)$ demands $(z_1, z_2) \in r$ by assumption on s. Because $\mathrm{pm}_q^{\mathfrak{h}} = (\mathfrak{c}(s^{-1}(2)) \otimes \mathfrak{c}(k), \mathfrak{c}(k) \otimes \mathfrak{c}(s^{-1}(2)), \times)$ and because $z_1 \neq z_2$ by $s^{-1}(2) \neq k$ it hence follows $\mathrm{pm}_q^{\mathfrak{h}} \in {}_{\mathfrak{U},\mathfrak{O}}\langle\mathcal{R}\rangle$ by Lemma 5.14.

Step 3.2.2: The partitions $\mathrm{pm}_q^{\mathfrak{h}} \otimes \mathrm{id}_{\mathfrak{u}}$ *and* $\mathrm{pm}_t^{\mathfrak{c}\circ s^{-1}\circ t}$ *coincide.* By definition, $(\mathfrak{h} \vartriangle \mathfrak{u})(1) = \mathfrak{h}(1) = \mathfrak{c}(s^{-1}(2)) = (\mathfrak{c} \circ s^{-1} \circ t)(1)$ and $(\mathfrak{h} \vartriangle \mathfrak{u})(2) = \mathfrak{h}(2) = \mathfrak{c}(k) = \mathfrak{c}(s^{-1}(1)) = (\mathfrak{c} \circ s^{-1} \circ t)(2)$ and for any $i \in \mathbb{N}$ with $2 < i \leq k$, also, $(\mathfrak{h} \vartriangle \mathfrak{u})(i) = \mathfrak{u}(i-2) = \mathfrak{c}(s^{-1}((i-2)+2)) = \mathfrak{c}(s^{-1}(i)) = (\mathfrak{c} \circ s^{-1} \circ t)(i)$.

Similarly, because $q^{-1} = q$ the definitions imply $((\mathfrak{h} \circ q^{-1}) \vartriangle \mathfrak{u})(1) = \mathfrak{h}(q^{-1}(1)) = \mathfrak{h}(2) = \mathfrak{c}(k) = (\mathfrak{c} \circ s^{-1})(1)$ and $((\mathfrak{h} \circ q^{-1}) \vartriangle \mathfrak{u})(2) = \mathfrak{h}(q^{-1}(2)) = \mathfrak{h}(1) = (\mathfrak{c} \circ s^{-1})(2)$ and for any $i \in \mathbb{N}$ with $2 < i \leq k$, also, $((\mathfrak{h} \circ q^{-1}) \vartriangle \mathfrak{u})(i) = \mathfrak{u}(i-2) = \mathfrak{c}(s^{-1}((i-2)+2)) = (\mathfrak{c} \circ s^{-1})(i)$.

Since, of course, $\mathrm{pm}_q \otimes \mathrm{id}_{k-2} = \{\{{}^{\blacksquare}1, {}_{\blacksquare}2\}, \{{}^{\blacksquare}2, {}_{\blacksquare}1\}\} \cup \{\{{}^{\blacksquare}i, {}_{\blacksquare}i\} \mid i \in \mathbb{N} \wedge 2 < i \leq k\} = \mathrm{pm}_t$ we have thus shown $\mathrm{pm}_q^{\mathfrak{h}} \otimes \mathrm{id}_{\mathfrak{u}} = \mathrm{pm}_t^{\mathfrak{c}\circ s^{-1}\circ t}$. In conclusion, $\mathrm{pm}_s^{\mathfrak{c}} \in {}_{\mathfrak{U},\mathfrak{O}}\langle\mathcal{R}\rangle$ in case 3 too. That completes the induction step and thus the proof overall. □

5.1.6. *From the isomorphisms of the minimal graph co-product category to any graph co-product.* With the preparations of Sections 5.1.1– 5.1.5 we are almost ready to prove the main result of Section 5.1, giving natural generators for $*^r_{z\in\mathfrak{U}\cup\mathfrak{O}} \mathcal{X}_z$. The last ingredient we need is that $*^r_{z\in\mathfrak{U}\cup\mathfrak{O}} \mathcal{X}_z$ is not only closed under certain covariant rearrangements as seen in Proposition 5.6 but also under certain reindexed restrictions.

LEMMA 5.18. *For any* $\{k, \ell\} \subseteq \mathbb{N}_0$*, any* $\mathfrak{c}\colon [\![k]\!] \to (\mathfrak{U} \otimes \{\circ, \bullet\}) \cup \mathfrak{O}$ *and* $\mathfrak{d}\colon [\![\ell]\!] \to (\mathfrak{U} \otimes \{\circ, \bullet\}) \cup \mathfrak{O}$*, any set-theoretical partition* p *of* Π_ℓ^k *such that* $(\mathfrak{c}, \mathfrak{d}, p) \in *^r_{z\in\mathfrak{U}\cup\mathfrak{O}} \mathcal{X}_z$ *and any set-theoretical partition* o *of* Π_ℓ^k *with* $p \leq o$*, if there exists a history* h *for* $(\mathfrak{c}, \mathfrak{d}, p)$ *with respect to* $(\mathcal{X}_z)_{z\in\mathfrak{U}\cup\mathfrak{O}}$ *and* r *which satisfies* $h \leq o$*, then* $R((\mathfrak{c}, \mathfrak{d}, p), \mathsf{O}) \in *^r_{z\in\mathfrak{U}\cup\mathfrak{O}} \mathcal{X}_z$ *for any* $\mathsf{O} \in o$*.*

PROOF. Given any $\mathsf{O} \in o$, let $(\mathfrak{u}, \mathfrak{v}, w) \coloneqq R((\mathfrak{c}, \mathfrak{d}, p), \mathsf{O})$ and let $\{i, j\} \subseteq \mathbb{N}_0$ be such that $\mathfrak{u}\colon [\![i]\!] \to (\mathfrak{U} \otimes \{\circ, \bullet\}) \cup \mathfrak{O}$ and $\mathfrak{v}\colon [\![j]\!] \to (\mathfrak{U} \otimes \{\circ, \bullet\}) \cup \mathfrak{O}$. We show that $f \coloneqq R(h, \mathsf{O})$ is a history for $(\mathfrak{u}, \mathfrak{v}, w)$ with respect to $(\mathcal{X}_z)_{z\in\mathfrak{U}\cup\mathfrak{O}}$ and r.

Refinement conditions. Since $\xi_{\mathfrak{v}}^{\mathfrak{u}} = \xi_{\mathfrak{d}}^{\mathfrak{c}} \circ \gamma_{\mathsf{O},\ell}^{\,k}$ by Lemma 4.2 (d), since $h \leq \ker(\xi_{\mathfrak{d}}^{\mathfrak{c}})$ by assumption and since $\gamma_{\mathsf{O},\ell}^{\,k\leftarrow}$ preserves $\leq$ we can rest assured that $f = \gamma_{\mathsf{O},\ell}^{\,k\leftarrow}(h) \leq \gamma_{\mathsf{O},\ell}^{\,k\leftarrow}(\ker(\xi_{\mathfrak{d}}^{\mathfrak{c}})) = \ker(\xi_{\mathfrak{d}}^{\mathfrak{c}} \circ \gamma_{\mathsf{O},\ell}^{\,k}) = \ker(\xi_{\mathfrak{v}}^{\mathfrak{u}})$. Likewise, $p \leq h$ guarantees $w = \gamma_{\mathsf{O},\ell}^{\,k\leftarrow}(p) \leq \gamma_{\mathsf{O},\ell}^{\,k\leftarrow}(h) = f$. Hence, $w \leq f \leq \ker(\xi_{\mathfrak{v}}^{\mathfrak{u}})$.

Non-crossing conditions. Let $\{z_1, z_2\} \subseteq \mathfrak{U} \cup \mathfrak{O}$ with $(z_1, z_2) \notin r$ and $\{\mathsf{F}_1, \mathsf{F}_2\} \subseteq f$ with $\mathsf{F}_1 \subseteq \xi_{\mathfrak{v}}^{\mathfrak{u}\leftarrow}(\{z_1\})$ and $\mathsf{F}_2 \subseteq \xi_{\mathfrak{v}}^{\mathfrak{u}\leftarrow}(\{z_2\})$ be arbitrary. We need to show $\mathsf{F}_1 \diagdown\!\!\diagdown_j^i \mathsf{F}_2$.

By the definition of $f \coloneqq \gamma_{\mathsf{O},\ell}^{\,k\leftarrow}(h)$ there exist $\{\mathsf{H}_1, \mathsf{H}_2\} \subseteq h$ such that $\mathsf{F}_1 = \gamma_{\mathsf{O},\ell}^{\,k\leftarrow}(\mathsf{H}_1)$ and $\mathsf{F}_2 = \gamma_{\mathsf{O},\ell}^{\,k\leftarrow}(\mathsf{H}_2)$. Because $\mathrm{ran}(\gamma_{\mathsf{O},\ell}^{\,k}) = \mathsf{O}$ and thus $\varnothing \neq \gamma_{\mathsf{O},\ell}^{\,k\rightarrow}(\mathsf{F}_1) =$

$(\gamma_{\mathsf{O},\ell}^{k}{}_{\rightarrow} \circ \gamma_{\mathsf{O},\ell}^{k\leftarrow})(\mathsf{H}_1) = \mathsf{H}_1 \cap \mathsf{O}$ and, likewise, $\varnothing \neq \mathsf{H}_2 \cap \mathsf{O}$ the assumption that $h \leq o$ requires $\mathsf{H}_1 \subseteq \mathsf{O}$ and $\mathsf{H}_2 \subseteq \mathsf{O}$. In other words, $\mathsf{H}_1 = \gamma_{\mathsf{O},\ell}^{k}{}_{\rightarrow}(\mathsf{F}_1)$ and $\mathsf{H}_2 = \gamma_{\mathsf{O},\ell}^{k}{}_{\rightarrow}(\mathsf{F}_2)$. Since $\gamma_{\mathsf{O},\ell}^{k}$ is injective, the fact that $\varnothing = \mathsf{F}_1 \cap \mathsf{F}_2$ therefore ensures that also $\varnothing = \gamma_{\mathsf{O},\ell}^{k}{}_{\rightarrow}(\mathsf{F}_1 \cap \mathsf{F}_2) = \gamma_{\mathsf{O},\ell}^{k}{}_{\rightarrow}(\mathsf{F}_1) \cap \gamma_{\mathsf{O},\ell}^{k}{}_{\rightarrow}(\mathsf{F}_2) = \mathsf{H}_1 \cap \mathsf{H}_2$, i.e., $\mathsf{H}_1 \neq \mathsf{H}_2$. Moreover, by $\xi_{\mathfrak{v}}^{\mathfrak{u}} = \xi_{\mathfrak{d}}^{\mathfrak{c}} \circ \gamma_{\mathsf{O},\ell}^{k}$ we can infer from $\mathsf{F}_1 \subseteq \xi_{\mathfrak{v}}^{\mathfrak{u}\leftarrow}(\{z_1\})$ that $\mathsf{H}_1 = \gamma_{\mathsf{O},\ell}^{k}{}_{\rightarrow}(\mathsf{H}_1) \subseteq (\gamma_{\mathsf{O},\ell}^{k}{}_{\rightarrow} \circ \xi_{\mathfrak{v}}^{\mathfrak{u}\leftarrow})(\{z_1\}) = (\gamma_{\mathsf{O},\ell}^{k}{}_{\rightarrow} \circ \gamma_{\mathsf{O},\ell}^{k\leftarrow} \circ \xi_{\mathfrak{d}}^{\mathfrak{c}\leftarrow})(\{z_1\}) \subseteq \xi_{\mathfrak{d}}^{\mathfrak{c}\leftarrow}(\{z_1\})$ and, analogously, $\mathsf{H}_2 \subseteq \xi_{\mathfrak{d}}^{\mathfrak{c}\leftarrow}(\{z_2\})$. By the assumption on h that then demands $\mathsf{H}_1 \not\!\!\searrow_{\ell}^{k} \mathsf{H}_2$. Since $\gamma_{\mathsf{O},\ell}^{k}$ is monotonic with respect to Γ_j^i and Γ_ℓ^k it follows $\mathsf{F}_1 \not\!\!\searrow_{j}^{i} \mathsf{F}_2$.

Restriction conditions. Finally, for any $z \in \mathfrak{U} \cup \mathfrak{D}$ and any $\mathsf{F} \in f = \gamma_{\mathsf{O},\ell}^{k\leftarrow}(h)$ with $\mathsf{F} \subseteq \xi_{\mathfrak{d}}^{\mathfrak{c}\leftarrow}(\{z\})$ there exists $\mathsf{H} \in h$ with $\mathsf{F} = \gamma_{\mathsf{O},\ell}^{k\leftarrow}(\mathsf{H})$. The, $\mathsf{H} \cap \mathsf{O} = (\gamma_{\mathsf{O},\ell}^{k}{}_{\rightarrow} \cap \gamma_{\mathsf{O},\ell}^{k\leftarrow})(\mathsf{H}) = \gamma_{\mathsf{O},\ell}^{k}{}_{\rightarrow}(\mathsf{F}) \neq \varnothing$ by $\mathsf{F} \neq \varnothing$ and thus $\mathsf{H} \subseteq \mathsf{O}$, i.e., $\mathsf{H} = \gamma_{\mathsf{O},\ell}^{k}{}_{\rightarrow}(\mathsf{F})$, by $h \leq o$. Consequently, by $\mathsf{F} \subseteq \xi_{\mathfrak{d}}^{\mathfrak{c}\leftarrow}(\{z\})$ also $\mathsf{H} = \gamma_{\mathsf{O},\ell}^{k}{}_{\rightarrow}(\mathsf{F}) \subseteq (\gamma_{\mathsf{O},\ell}^{k}{}_{\rightarrow} \circ \xi_{\mathfrak{v}}^{\mathfrak{u}\leftarrow})(\{z\}) = (\gamma_{\mathsf{O},\ell}^{k}{}_{\rightarrow} \circ \gamma_{\mathsf{O},\ell}^{k\leftarrow} \circ \xi_{\mathfrak{d}}^{\mathfrak{c}\leftarrow})(\{z\}) \subseteq \xi_{\mathfrak{d}}^{\mathfrak{c}\leftarrow}(\{z\})$. It follows $R((\mathfrak{c},\mathfrak{d},p),\mathsf{H}) \in \mathcal{X}_z$ by the assumption on h. That proves $R((\mathfrak{u},\mathfrak{v},w),\mathsf{F}) \in \mathcal{X}_z$ since $R((\mathfrak{u},\mathfrak{v},w),\mathsf{F}) = R((\mathfrak{c},\mathfrak{d},p),\mathsf{H})$ according Lemma 4.5 (d) by $\mathsf{F} = \gamma_{\mathsf{O},\ell}^{k\leftarrow}(\mathsf{H})$ and $\mathsf{H} \subseteq \mathsf{O}$. In conclusion, f is a history for $(\mathfrak{u},\mathfrak{v},w)$. □

That leads us to the main result of Section 5.1.

Proposition 5.19. *If $\mathcal{G}_z \subseteq \mathcal{X}_z$ generates $\mathcal{X}_z$ as a category of $(\{z\},\varnothing)$-tagged labeled partitions for any $z \in \mathfrak{U}$ and as one of $(\varnothing,\{z\})$-tagged labeled partitions for any $z \in \mathfrak{D}$ and if $\mathcal{R}$ is a set of generating crosses for the r-graph co-product with respect to $(\mathfrak{U},\mathfrak{D})$, then $\mathcal{R} \cup \bigcup_{z\in\mathfrak{U}\cup\mathfrak{D}} \mathcal{G}_z$ generates $*^{r}_{z\in\mathfrak{U}\cup\mathfrak{D}} \mathcal{X}_z$ as a category of $(\mathfrak{U},\mathfrak{D})$-tagged labeled partitions.*

Proof. The proof goes by induction over the total number of points. Already by definition, in the base case of 0 points, the only element $(\varnothing,\varnothing,\varnothing)$ of $\mathcal{C} \coloneqq *^{r}_{z\in\mathfrak{U}\cup\mathfrak{D}} \mathcal{X}_z$ belongs to $\mathcal{B} \coloneqq {}_{\mathfrak{U},\mathfrak{D}}\langle \mathcal{R} \cup \bigcup_{z\in\mathfrak{U}\cup\mathfrak{D}} \mathcal{G}_z\rangle$. For general $\{k,\ell\} \subseteq \mathbb{N}_0$, any $\mathfrak{c}\colon [\![k]\!] \to (\mathfrak{U}\otimes\{\circ,\bullet\}) \cup \mathfrak{D}$ and $\mathfrak{d}\colon [\![\ell]\!] \to (\mathfrak{U}\otimes\{\circ,\bullet\}) \cup \mathfrak{D}$, any set-theoretical partition of Π_ℓ^k such that $(\mathfrak{c},\mathfrak{d},p) \in \mathcal{C}$ and any history h for $(\mathfrak{c},\mathfrak{d},p)$ with respect to $(\mathcal{X}_z)_{z\in\mathfrak{U}\cup\mathfrak{D}}$ and r we distiguish three cases.

Case 1: h consists of one (necessarily convex) block. If h is the maximal partition $\{\Pi_\ell^k\}$ of Π_ℓ^k, then the assumption $h \leq \ker(\xi_{\mathfrak{d}}^{\mathfrak{c}})$ requires the existence of $z \in \mathfrak{U} \cup \mathfrak{D}$ with $\Pi_\ell^k \subseteq \xi_{\mathfrak{d}}^{\mathfrak{c}\leftarrow}(\{z\})$. Hence, actually, $(\mathfrak{c},\mathfrak{d},p) \in \mathcal{X}_z$. Because $\mathcal{X}_z = {}_{\{z\},\varnothing}\langle\mathcal{G}_z\rangle$ if $z \in \mathfrak{U}$ and $\mathcal{X}_z = {}_{\varnothing,\{z\}}\langle\mathcal{G}_z\rangle$ if $z \in \mathfrak{D}$ and thus $\mathcal{X}_z \subseteq {}_{\mathfrak{U},\mathfrak{D}}\langle\mathcal{G}_z\rangle \subseteq \mathcal{B}$ that proves $(\mathfrak{c},\mathfrak{d},p) \in \mathcal{B}$ in this case.

Case 2: h has more than one block, at least one of them convex. Next, suppose that $2 \leq |h|$ and that there exists $\mathsf{H} \in h$ which is convex with respect to Γ_ℓ^k. Because then $o \coloneqq \{\mathsf{H}, \Pi_\ell^k \backslash \mathsf{H}\}$ is a set-theoretical partition of Π_ℓ^k with $h \leq o$ which, thanks to the convexity of H, is non-crossing with respect to Γ_ℓ^k, Lemma 4.20 proves that $(\mathfrak{c},\mathfrak{d},p) \in \mathcal{B}$ if we can show that $R((\mathfrak{c},\mathfrak{d},p),\mathsf{O}) \in \mathcal{B}$ for any $\mathsf{O} \in o$. But the latter is true by the induction hypothesis since for any $\mathsf{O} \in o$ the partition $R((\mathfrak{c},\mathfrak{d},p),\mathsf{O})$, which belongs to $\mathcal{C}$ by Lemma 5.18, has $|\mathsf{O}| < k + \ell$ many points, thanks to $2 \leq |h|$. Hence, $(\mathfrak{c},\mathfrak{d},p) \in \mathcal{B}$ also in this case.

Case 3: h has no convex blocks. The remaining possibility is that $2 \le |h|$ but that none of the blocks of h are convex with respect to Γ_ℓ^k. In order to show $(\mathfrak{c}, \mathfrak{d}, p) \in \mathcal{B}$ it suffices to prove that $\mathcal{B}$ is closed under ${}_{\mathfrak{U},\mathfrak{D}}^{\ r}\mathcal{UO}^{++}$-allowed covariant rearrangements and that $\mathcal{B}$ contains a partition $(\mathfrak{a}, \mathfrak{b}, q)$ that admits a ${}_{\mathfrak{U},\mathfrak{D}}^{\ r}\mathcal{UO}^{++}$-allowed covariant rearrangement into $(\mathfrak{c}, \mathfrak{d}, p)$.

And, indeed, by virtue of including a set of generating crosses for r with respect to $(\mathfrak{U}, \mathfrak{D})$, the category $\mathcal{B}$ contains all isomorphisms of ${}_{\mathfrak{U},\mathfrak{D}}^{\ r}\mathcal{UO}^{++}$ by Lemma 5.17 and is therefore closed under ${}_{\mathfrak{U},\mathfrak{D}}^{\ r}\mathcal{UO}^{++}$-allowed covariant rearrangements by Lemma 5.10.

Step 3.1: Definition of $(\mathfrak{a}, \mathfrak{b}, q)$*.* The assumption that h has at least two blocks, but none which are convex with respect to Γ_ℓ^k implies in particular that $2 \le |\mathsf{H}|$ for any $\mathsf{H} \in h$. Hence, the set

$$\{|[\mathtt{a},\mathtt{b}]_\ell^k \backslash \mathsf{H}| \mid \mathsf{H} \in h \wedge \{\mathtt{a},\mathtt{b}\} \subseteq \mathsf{H} \wedge \mathtt{a} \neq \mathtt{b} \wedge \mathsf{H} \subseteq [\mathtt{a},\mathtt{b}]_\ell^k\}$$

is non-empty. Moreover, the assumption ensures that this set only contains non-zero numbers. If its minimum is realized by $n := |[\mathtt{a},\mathtt{b}]_\ell^k \backslash \mathsf{H}|$ for $\mathsf{H} \in h$ and $\{\mathtt{a},\mathtt{b}\} \subseteq \mathsf{H}$ with $\mathtt{a} \neq \mathtt{b}$ and $\mathsf{H} \subseteq [\mathtt{a},\mathtt{b}]_\ell^k$, then let $m := |\mathsf{H}|$ and let $(\mathtt{e}_i)_{i=1}^m$, $(\mathtt{f}_j)_{j=1}^n$ and $(\mathtt{v}_s)_{s=1}^{n+m}$ be the enumerations of, respectively, H, $[\mathtt{a},\mathtt{b}]_\ell^k \backslash \mathsf{H}$ and $[\mathtt{a},\mathtt{b}]_\ell^k$ with respect to the cut of Γ_ℓ^k at $\mathtt{a}$. We can then define a permutation t of Π_ℓ^k by the rule that t is the identity on $]\mathtt{b},\mathtt{a}[_\ell^k$ and that for any $s \in [\![n+m]\!]$ the point $\mathtt{v}_s$ is mapped to $\mathtt{f}_s$ if $s \le n$ and to $\mathtt{e}_{s-n}$ if $n < s$. Indeed the inverse of t is the mapping which is the identity on $]\mathtt{b},\mathtt{a}[_\ell^k$ and satisfies $\mathtt{e}_i \mapsto \mathtt{v}_{n+i}$ for any $i \in [\![m]\!]$ and $\mathtt{f}_j \mapsto \mathtt{v}_j$ for any $j \in [\![n]\!]$.

Step 3.2: $(\mathfrak{c}, \mathfrak{d}, p)$ *and* $(\mathfrak{a}, \mathfrak{b}, q)$ *are* ${}_{\mathfrak{U},\mathfrak{D}}^{\ r}\mathcal{UO}^{++}$*-allowed covariant rearrangements of each other.* Because the inverse mapping of any ${}_{\mathfrak{U},\mathfrak{D}}^{\ r}\mathcal{UO}^{++}$-allowed covariant rearrangement is a ${}_{\mathfrak{U},\mathfrak{D}}^{\ r}\mathcal{UO}^{++}$-allowed covariant rearrangement too it does not matter whether we show $(\mathfrak{a}, \mathfrak{b}, q)$ to result from one of $(\mathfrak{c}, \mathfrak{d}, p)$ or the other way around. We prove that t is a ${}_{\mathfrak{U},\mathfrak{D}}^{\ r}\mathcal{UO}^{++}$-allowed covariant rearrangement of $(\mathfrak{c}, \mathfrak{d}, p)$ into $(\mathfrak{a}, \mathfrak{b}, q)$. More precisely we show that for any $\{\mathtt{a}_1,\mathtt{a}_2\} \subseteq \Pi_\ell^k$ with $(\xi_\mathfrak{d}^\mathfrak{c}(t(\mathtt{a}_1)), \xi_\mathfrak{d}^\mathfrak{c}(t(\mathtt{a}_2))) \notin r$, with respect to the cut of Γ_ℓ^k at $\mathtt{v}_1 = \mathtt{a}$, whenever $t(\mathtt{a}_1) < t(\mathtt{a}_2)$, then also $\mathtt{a}_1 < \mathtt{a}_2$.

Step 3.2.1: Auxiliary non-crossing statement. As an intermediate step we prove $(\xi_\mathfrak{d}^\mathfrak{c}(\mathtt{e}_i), \xi_\mathfrak{d}^\mathfrak{c}(\mathtt{f}_j)) \in r$ for any $i \in [\![m]\!]$ and $j \in [\![n]\!]$. Given such i and j, if $\mathsf{H}' \in h$ is the unique block with $\mathtt{f}_j \in \mathsf{H}'$, then $\mathsf{H} \neq \mathsf{H}'$ because $\mathtt{e}_i \in \mathsf{H}$ and $\mathtt{f}_j \in [\mathtt{a},\mathtt{b}]_\ell^k \backslash \mathsf{H}$ per definition. Hence, by the definition of $\mathcal{C}$ as a graph co-product, in order to verify that $(\xi_\mathfrak{d}^\mathfrak{c}(\mathtt{e}_i), \xi_\mathfrak{d}^\mathfrak{c}(\mathtt{f}_j)) \in r$, it suffices to show that $\mathsf{H} \not\multimap_\ell^k \mathsf{H}'$. Because $\{\mathtt{a},\mathtt{b}\} \subseteq \mathsf{H}$ and $\mathtt{f}_j \in]\mathtt{a},\mathtt{b}[_\ell^k \cap \mathsf{H}'$ that in turn is verified once we refute that H' is a subset of $]\mathtt{a},\mathtt{b}[_\ell^k$. If $\mathtt{a}'$ and $\mathtt{b}'$ are the minimal respectively maximal elements of the non-empty set $]\mathtt{a},\mathtt{b}[_\ell^k \cap \mathsf{H}'$ with respect to the cut of Γ_ℓ^k at $\mathtt{a}$, then disproving that $]\mathtt{a},\mathtt{b}[_\ell^k$ contains H' is the same as showing that $\mathsf{H}' \nsubseteq [\mathtt{a}',\mathtt{b}']_\ell^k$. And by the minimality of n we can prove the latter by demonstrating that $|[\mathtt{a}',\mathtt{b}']_\ell^k \backslash \mathsf{H}'| < n$.

And, indeed, if $\{j_\wedge, j_\vee\} \subseteq [\![n]\!]$ are the unique indices with $\mathtt{a}' = \mathtt{f}_{j_\wedge}$ and $\mathtt{b}' = \mathtt{f}_{j_\vee}$, then the assumption that $\{\mathtt{a}',\mathtt{b}'\} \subseteq \mathsf{H}'$ ensures that $[\mathtt{a}',\mathtt{b}']_\ell^k \backslash \mathsf{H}' =]\mathtt{a}',\mathtt{b}'[_\ell^k \backslash \mathsf{H}' =]\mathtt{f}_{j_\wedge},\mathtt{f}_{j_\vee}[_\ell^k \backslash \mathsf{H}'$ and thus that $|[\mathtt{a}',\mathtt{b}']_\ell^k \backslash \mathsf{H}'| = |]\mathtt{f}_{j_\wedge},\mathtt{f}_{j_\vee}[_\ell^k \backslash \mathsf{H}'| \le |]\mathtt{f}_{j_\wedge},\mathtt{f}_{j_\vee}[_\ell^k| < |[\mathtt{f}_{j_\wedge},\mathtt{f}_{j_\vee}]_\ell^k| = j_\vee - j_\wedge \le j_\vee \le n$. Hence, $(\xi_\mathfrak{d}^\mathfrak{c}(\mathtt{e}_i), \xi_\mathfrak{d}^\mathfrak{c}(\mathtt{f}_j)) \in r$, as claimed.

Step 3.2.2: t is ${}^{r}_{\mathfrak{U},\mathfrak{D}}\mathcal{UO}^{++}$-allowed. Now, let $\{\mathtt{a}_1,\mathtt{a}_2\} \subseteq \Pi^k_\ell$ be arbitrary with $(\xi^{\mathfrak{c}}_{\mathfrak{d}}(t(\mathtt{a}_1)),\xi^{\mathfrak{c}}_{\mathfrak{d}}(t(\mathtt{a}_2))) \notin r$ and $t(\mathtt{a}_1) < t(\mathtt{a}_2)$ with respect to the cut of Γ^k_ℓ at $\mathtt{v}_1 = \mathtt{a}$. We have to show that $\mathtt{a}_1 < \mathtt{a}_2$ with respect to the cut of Γ^k_ℓ at $\mathtt{v}_1 = \mathtt{a}$. By the reflexivity of r and the auxiliary statement from Step 3.2.1 we know that the assumption $(\xi^{\mathfrak{c}}_{\mathfrak{d}}(t(\mathtt{a}_1)),\xi^{\mathfrak{c}}_{\mathfrak{d}}(t(\mathtt{a}_2))) \notin r$ excludes the possiblity that there exist $i \in [\![m]\!]$ and $j \in [\![n]\!]$ such that $\{t(\mathtt{a}_1),t(\mathtt{a}_2)\}$ and $\{\mathtt{e}_i,\mathtt{f}_j\}$ are the same set. Hence, only the following four cases are possible.

Step 3.2.2.1: Both $t(\mathtt{a}_1)$ and $t(\mathtt{a}_2)$ lie outside of $[\mathtt{a},\mathtt{b}]^k_\ell$ If $\mathtt{b} < t(\mathtt{a}_1)$ (and thus also $\mathtt{b} < t(\mathtt{a}_2)$) with respect to the cut of Γ^k_ℓ at $\mathtt{a}$, then the definition that t restricts to the identity on $]\mathtt{b},\mathtt{a}[^k_\ell$ implies that, indeed, $\mathtt{a}_1 = t(\mathtt{a}_1) < t(\mathtt{a}_2) = \mathtt{a}_2$ with respect to the cut of Γ^k_ℓ at $\mathtt{a}$, as claimed.

Step 3.2.2.2: $t(\mathtt{a}_1)$ is inside and $t(\mathtt{a}_2)$ outside of $[\mathtt{a},\mathtt{b}]^k_\ell$. Similarly, if $t(\mathtt{a}_1) \leq \mathtt{b} < t(\mathtt{a}_2)$ with respect to the cut of Γ^k_ℓ at $\mathtt{a}$, then the fact that t restricts to permutations of $[\mathtt{a},\mathtt{b}]^k_\ell$ and of $]\mathtt{b},\mathtt{a}[^k_\ell$ lets us infer that also $\mathtt{a}_1 \leq \mathtt{b} < t(\mathtt{a}_2) = \mathtt{a}_2$ with respect to the cut of Γ^k_ℓ at $\mathtt{a}$.

Step 3.2.2.3: Both $t(\mathtt{a}_1)$ and $t(\mathtt{a}_2)$ belong to H. If there exist $\{i_1,i_2\} \subseteq [\![m]\!]$ with $t(\mathtt{a}_1) = \mathtt{e}_{i_1}$ and $t(\mathtt{a}_2) = \mathtt{e}_{i_2}$, then the assumption on $(\mathtt{e}_i)^m_{i=1}$ means that the inequality $\mathtt{e}_{i_1} < \mathtt{e}_{i_2}$ with respect to the cut of Γ^k_ℓ at $\mathtt{a}$ requires the inequality $i_1 < i_2$. Since then also $n+i_1 < n+i_2$ the assumption on $(\mathtt{v}_s)^{n+m}_{s=1}$ implies $\mathtt{a}_1 = t^{-1}(\mathtt{e}_{i_1}) = \mathtt{v}_{n+i_1} < \mathtt{v}_{n+i_2} = t^{-1}(\mathtt{e}_{i_2}) = \mathtt{a}_2$ with respect to the cut of Γ^k_ℓ at $\mathtt{a}$.

Step 3.2.2.4: Both $t(\mathtt{a}_1)$ and $t(\mathtt{a}_2)$ lie in $[\mathtt{a},\mathtt{b}]^k_\ell \backslash \mathsf{H}$. The only remaining possibility is that there are $\{j_1,j_2\} \subseteq [\![n]\!]$ with $t(\mathtt{a}_1) = \mathtt{f}_{j_1}$ and $t(\mathtt{a}_2) = \mathtt{f}_{j_2}$. By nature of $(\mathtt{f}_j)^n_{j=1}$, since $\mathtt{f}_{j_1} < \mathtt{f}_{j_2}$ with respect to the cut of Γ^k_ℓ at $\mathtt{a}$ then, necessarily, $j_1 < j_2$. By the assumption on $(\mathtt{v}_s)^{n+m}_{s=1}$ it thus follows $\mathtt{a}_1 = t^{-1}(\mathtt{f}_{j_1}) = \mathtt{v}_{j_1} < \mathtt{v}_{j_2} = t^{-1}(\mathtt{f}_{j_2}) = \mathtt{a}_2$ with respect to the cut of Γ^k_ℓ at $\mathtt{a}$. Hence, $\mathtt{a}_1 < \mathtt{a}_2$ with respect to the cut of Γ^k_ℓ at $\mathtt{a}$ holds in all cases. In conclusion, t is ${}^{r}_{\mathfrak{U},\mathfrak{D}}\mathcal{UO}^{++}$-allowed.

Step 3.3: $(\mathfrak{a},\mathfrak{b},q)$ belongs to $\mathcal{B}$. Because $(\mathfrak{a},\mathfrak{b},q)$ is a ${}^{r}_{\mathfrak{U},\mathfrak{D}}\mathcal{UO}^{++}$-allowed covariant rearrangement of $(\mathfrak{c},\mathfrak{d},p) \in \mathcal{C}$ by Step 3.2 and because $\mathcal{C}$ is closed under ${}^{r}_{\mathfrak{U},\mathfrak{D}}\mathcal{UO}^{++}$-allowed covariant rearrangements by Proposition 5.6 (b) the partition $(\mathfrak{a},\mathfrak{b},q)$ also belongs to $\mathcal{C}$. Furthermore, by the same proposition, $g \coloneqq t^{\leftarrow}(h)$ is a history for $(\mathfrak{a},\mathfrak{b},q)$ with respect to $(\mathcal{X}_z)_{z\in\mathfrak{U}\cup\mathfrak{D}}$ and r. Moreover, because t is bijective $2 \leq |h| = |t^{\leftarrow}(h)| = |g|$. Lastly, the definition of t implies $\mathsf{G} \coloneqq \{\mathtt{v}_{n+1},\ldots,\mathtt{v}_{n+m}\} = t^{\leftarrow}(\{\mathtt{e}_1,\ldots,\mathtt{e}_m\}) = t^{\leftarrow}(\mathsf{H}) \in g$. Because G is convex with respect to Γ^k_ℓ by definition of $(\mathtt{v}_s)^{n+m}_{s=1}$ that means that the labeled partition $(\mathfrak{a},\mathfrak{b},q)$ of $k+\ell$ many points falls under Case 2 and is thus an element of $\mathcal{B}$ by what we have already seen. As explained at the beginning of Step 3, that concludes the proof. □

5.2. Generators of crossed co-products. The second major result in Section 5 is Proposition 5.41 in Section 5.2.5 about the generators of crossed co-products with cyclic groups. The generating sets of partitions appearing there are introduced by Definition 5.38 in Section 5.2.4. Once again, the remaining definitions and results are necessary auxiliaries for the proof of Proposition 5.41.

ASSUMPTIONS 5.20. In Section 5.2, let $w \in \mathbb{N}_0$ be arbitrary, let $(\mathfrak{U}, \mathfrak{O})$ be either $(\mathbb{Z}_w, \varnothing)$ or $(\varnothing, \mathbb{Z}_w)$ and let $\mathcal{X}$ be any $\mathbb{Z}_w$-invariant category of $(\mathfrak{U}, \mathfrak{O})$-tagged labeled partitions.

Additional to the two main ideas of Section 5.1, which also guide Section 5.2, there are two more.

(a) The category $\mathcal{X} \rtimes \mathcal{Z}_w$ is $*$-equivalent to the $*$-categorical product of $\mathcal{X}$ and $\mathcal{Z}_w$ (whose morphisms are formal pairs of one morphism each from $\mathcal{X}$ and $\mathcal{Z}_w$, and likewise for objects).

(b) The natural isomorphisms inducing this equivalence can be built up via tensor products and compositions from appropriately labeled "crossing" partitions $\times$.

Hence, $\mathcal{X} \rtimes \mathcal{Z}_w$ should be generated by $\mathcal{X} \cup \mathcal{Z}_w$ and a set of "crossing" partitions. The strategy for the proof of this is as follows.

Step 1 Section 5.2.1 adds the more precise notation for mappings between sets of points required in the ensuing sections.

Step 2 Using the language of Step 1, in Section 5.2.2 (one side of) the aforementioned equivalence between $\mathcal{X} \rtimes \mathcal{Z}_w$ and the product of $\mathcal{X}$ and $\mathcal{Z}_w$ is introduced, the "normal form".

Step 3 That the "normal form" of Step 2 really is an equivalence is established in Section 5.2.3 with the help of Step 1.

Step 4 Section 5.2.4 defines the alleged generators of $\mathcal{X} \rtimes \mathcal{Z}_w$, again called the "sets of generating crosses", and gives insight into how many of those exist.

Step 5 Finally, Section 5.2.5 shows by induction and using the results of Step 3 how any category containing $\mathcal{X} \cup \mathcal{Z}_w$ and a generator from Step 4 must also contain $\mathcal{X} \rtimes \mathcal{Z}_w$.

5.2.1. *Additional notation for co-projections and co-products.* In many of the results and proofs in Section 5.2 (except Proposition 5.41 itsel) fthe categorical algebra of cocartesian categories shines through so much, that it makes no sense not to use the established notions from that field. More precisely, the following notation will be used for co-projections and co-products.

NOTATION 5.21. (a) For any $\{k, \ell\} \subseteq \mathbb{N}_0$ let $\blacktriangle_{k,\ell}$ be the mapping $[\![k]\!] \to \Pi_\ell^k$ with $i \mapsto {}^{\blacksquare}i$ for any $i \in [\![k]\!]$ and let $\blacktriangledown_{k,\ell}$ be the mapping $[\![\ell]\!] \to \Pi_\ell^k$ with $j \mapsto {}_{\blacksquare}j$ for any $j \in [\![\ell]\!]$.

(b) In the same vein, for any $\{k_t\}_{t=1}^2 \subseteq \mathbb{N}_0$ let $\triangleleft_{k_1,k_2}$ be the mapping $[\![k_1]\!] \to [\![k_1 + k_2]\!]$ with $i_1 \mapsto i_1$ for any $i_1 \in [\![k_1]\!]$ and let $\triangleright_{k_1,k_2}$ be the mapping $[\![k_2]\!] \to [\![k_1 + k_2]\!]$ with $i_2 \mapsto k_1 + i_2$ for any $i_2 \in [\![k_2]\!]$.

(c) Throughout, for any set X, given any $k_t \in \mathbb{N}_0$ and any $f_t\colon [\![k_t]\!] \to X$ for each $t \in [\![2]\!]$, let $f_1 \vartriangle f_2$ denote the map $[\![k_1 + k_2]\!] \to X$ with $(f_1 \vartriangle f_2) \circ \triangleleft_{k_1,k_2} = f_1$ and $(f_1 \vartriangle f_2) \circ \triangleright_{k_1,k_2} = f_2$.

(d) Moreover, given for each $t \in [\![2]\!]$ not only $k_t \in \mathbb{N}_0$ but also $x_t \in \mathbb{N}_0$ as well as $g_t\colon [\![k_t]\!] \to [\![x_t]\!]$, write $g_1 \circledcirc g_2 := (\triangleleft_{x_1,x_2} \circ g_1) \vartriangle (\triangleright_{x_1,x_2} \circ g_2)$.

(e) Similarly to (b), for any $\{k_t,\ell_t\}_{t=1}^2 \subseteq \mathbb{N}_0$ we can write $\lhd_{\ell_1,\ell_2}^{k_1,k_2} \coloneqq (\blacktriangle_{k_1+k_2,\ell_1+\ell_2} \circ \lhd_{k_1,k_2}) \,{}^{\blacksquare}_{\blacksquare}\, (\blacktriangledown_{k_1+k_2,\ell_1+\ell_2} \circ \lhd_{\ell_1,\ell_2})$ as well as, analogously, $\rhd_{\ell_1,\ell_2}^{k_1,k_2} \coloneqq (\blacktriangle_{k_1+k_2,\ell_1+\ell_2} \circ \rhd_{k_1,k_2}) \,{}^{\blacksquare}_{\blacksquare}\, (\blacktriangledown_{k_1+k_2,\ell_1+\ell_2} \circ \rhd_{\ell_1,\ell_2})$.

(f) In fact, for any set X and any $\{k_t,\ell_t\} \subseteq \mathbb{N}_0$ and $\mu_t\colon \Pi_{\ell_t}^{k_t} \to X$ for each $t \in [\![2]\!]$ we risk no misunderstandings if we extend the notation and also write $\mu_1 \vartriangle \mu_2$ for the unique mapping $\Pi_{\ell_1+\ell_2}^{k_1+k_2} \to X$ with $(\mu_1 \vartriangle \mu_2) \circ \lhd_{\ell_1,\ell_2}^{k_1,k_2} = \mu_1$ and $(\mu_1 \vartriangle \mu_2) \circ \rhd_{\ell_1,\ell_2}^{k_1,k_2} = \mu_2$.

(g) Also, for any $\{k_t\}_{t=1}^2 \subseteq \mathbb{N}_0$ let $\tau_{k_1,k_2} \coloneqq \rhd_{k_1,k_2} \vartriangle \lhd_{k_1,k_2}$.

(h) In the same manner, for any $\{k_t,\ell_t\}_{t=1}^2 \subseteq \mathbb{N}_0$ let $\tau_{\ell_1,\ell_2}^{k_1,k_2} \coloneqq \rhd_{\ell_1,\ell_2}^{k_1,k_2} \vartriangle \lhd_{\ell_1,\ell_2}^{k_1,k_2}$.

(i) Likewise, no confusion need be expected if, given any $\{k_t,\ell_t,x_t,y_t\} \subseteq \mathbb{N}_0$ and any $\omega_t\colon \Pi_{\ell_t}^{k_t} \to \Pi_{y_t}^{x_t}$ for each $t \in [\![2]\!]$, we let $\omega_1 \circledast \omega_2 \coloneqq (\lhd_{y_1,y_2}^{x_1,x_2} \circ \omega_1) \vartriangle (\rhd_{y_1,y_2}^{x_1,x_2} \circ \omega_2)$.

(j) Furthemore, for any $\{k_u\}_{u=1}^4 \subseteq \mathbb{N}_0$ let

$$\begin{aligned}\mu_{k_1,k_2,k_3,k_4} \coloneqq {}&((\lhd_{k_1+k_3,k_2+k_4} \circ \lhd_{k_1,k_3}) \vartriangle (\rhd_{k_1+k_3,k_2+k_4} \circ \lhd_{k_2,k_4}))\\ &\vartriangle ((\lhd_{k_1+k_3,k_2+k_4} \circ \rhd_{k_1,k_3}) \vartriangle (\rhd_{k_1+k_3,k_2+k_4} \circ \rhd_{k_2,k_4})).\end{aligned}$$

(k) Analogously, for any $\{k_u,\ell_u\}_{u=1}^4 \subseteq \mathbb{N}_0$, let

$$\begin{aligned}\mu_{\ell_1,\ell_2,\ell_3,\ell_4}^{k_1,k_2,k_3,k_4} \coloneqq {}&((\lhd_{\ell_1+\ell_3,\ell_2+\ell_4}^{k_1+k_3,k_2+k_4} \circ \lhd_{\ell_1,\ell_3}^{k_1,k_3}) \vartriangle (\rhd_{\ell_1+\ell_3,\ell_2+\ell_4}^{k_1+k_3,k_2+k_4} \circ \lhd_{\ell_2,\ell_4}^{k_2,k_4}))\\ &\vartriangle ((\lhd_{\ell_1+\ell_3,\ell_2+\ell_4}^{k_1+k_3,k_2+k_4} \circ \rhd_{\ell_1,\ell_3}^{k_1,k_3}) \vartriangle (\rhd_{\ell_1+\ell_3,\ell_2+\ell_4}^{k_1+k_3,k_2+k_4} \circ \rhd_{\ell_2,\ell_4}^{k_2,k_4})).\end{aligned}$$

Lemma 5.22. *For any $\{k_u,\ell_u\}_{u=1}^4 \subseteq \mathbb{N}_0$ the following hold. then*

(a) For any set X, when given any $f_u\colon \Pi_{\ell_u}^{k_u} \to X$ for each $u \in [\![4]\!]$, then

$$((f_1 \vartriangle f_2) \vartriangle (f_3 \vartriangle f_4)) \circ \mu_{\ell_1,\ell_2,\ell_3,\ell_4}^{k_1,k_2,k_3,k_4} = (f_1 \vartriangle f_3) \vartriangle (f_2 \vartriangle f_4).$$

(b) $\mu_{\ell_1,\ell_2,\ell_3,\ell_4}^{k_1,k_2,k_3,k_4} = f_1 \circledast \tau_{\ell_2,\ell_3}^{k_2,k_3} \circledast f_4$, where f_1 and f_4 are the identity mappings on $\Pi_{\ell_1}^{k_1}$ respectively $\Pi_{\ell_4}^{k_4}$.

Proof. Follows immediately from the definitions. □

5.2.2. *Definition of the normal form of labeled partitions in crossed co-products.* To any labeled partition of $\mathcal{X} \rtimes \mathcal{Z}_w$ we can associate one each of $\mathcal{X}$ and $\mathcal{Z}_w$ in a natural way which is already suggested by the very definition of $\mathcal{X} \rtimes \mathcal{Z}_w$.

Notation 5.23. In Section 5.2, for any $\{k,\ell\} \subseteq \mathbb{N}_0$ and any $\mathfrak{c}\colon [\![k]\!] \to ((\mathfrak{U} \cup \{\aleph\}) \otimes \{\circ,\bullet\}) \cup \mathfrak{O}$ and $\mathfrak{d}\colon [\![\ell]\!] \to ((\mathfrak{U} \cup \{\aleph\}) \otimes \{\circ,\bullet\}) \cup \mathfrak{O}$ let

$$\mathsf{U}(\mathfrak{c},\mathfrak{d}) \coloneqq \xi_{\mathfrak{d}}^{\mathfrak{c}\leftarrow}(\mathbb{Z}_w) \quad \text{and} \quad \mathsf{V}(\mathfrak{c},\mathfrak{d}) \coloneqq \xi_{\mathfrak{d}}^{\mathfrak{c}\leftarrow}(\{\aleph\}).$$

and

$$t_{\mathfrak{d}}^{\mathfrak{c}} \coloneqq \gamma_{\mathsf{U}(\mathfrak{c},\mathfrak{d}),\ell}^{\;k} \vartriangle \gamma_{\mathsf{V}(\mathfrak{c},\mathfrak{d}),\ell}^{\;k}.$$

Definition 5.24. For any $k \in \mathbb{N}_0$ and any $\mathfrak{c}\colon [\![k]\!] \to ((\mathfrak{U} \cup \{\aleph\}) \otimes \{\circ,\bullet\}) \cup \mathfrak{O}$, if $m_{\mathsf{U}} = \alpha(\mathsf{U}(\mathfrak{c},\varnothing))$ and $m_{\mathsf{V}} = \alpha(\mathsf{V}(\mathfrak{c},\varnothing))$, then let the labelings

(a) $L_{\mathsf{U}}(\mathfrak{c})\colon [\![m_{\mathsf{U}}]\!] \to (\mathfrak{U} \otimes \{\circ, \bullet\}) \cup \mathfrak{O}$ be such that $\xi_{\varnothing}^{L_{\mathsf{U}}(\mathfrak{c})} = \pi_w \circ \varepsilon_{\varnothing}^{\mathfrak{c}} \circ \gamma_{\mathsf{U}(\mathfrak{c},\varnothing),0}{}^{k}$ and $\zeta_{\varnothing}^{L_{\mathsf{U}}(\mathfrak{c})} = \zeta_{\varnothing}^{\mathfrak{c}} \circ \gamma_{\mathsf{U}(\mathfrak{c},\varnothing),0}{}^{k}$.

(b) $L_{\mathsf{V}}(\mathfrak{c})\colon [\![m_{\mathsf{V}}]\!] \to \{\aleph\} \otimes \{\circ, \bullet\}$ be such that $\zeta_{\varnothing}^{L_{\mathsf{V}}(\mathfrak{c})} = \zeta_{\varnothing}^{\mathfrak{c}} \circ \gamma_{\mathsf{V}(\mathfrak{c},\varnothing),0}{}^{k}$.

(c) $L(\mathfrak{c})\colon [\![k]\!] \to ((\mathfrak{U} \cup \{\aleph\}) \otimes \{\circ, \bullet\}) \cup \mathfrak{O}$ be given by $L_{\mathsf{U}}(\mathfrak{c}) \vartriangle L_{\mathsf{V}}(\mathfrak{c})$.

Definition 5.25. For any $\{k, \ell\} \subseteq \mathbb{N}_0$, any $\mathfrak{c}\colon [\![k]\!] \to ((\mathfrak{U} \cup \{\aleph\}) \otimes \{\circ, \bullet\}) \cup \mathfrak{O}$ and $\mathfrak{d}\colon [\![\ell]\!] \to ((\mathfrak{U} \cup \{\aleph\}) \otimes \{\circ, \bullet\}) \cup \mathfrak{O}$ and any set-theoretical partition p of Π_ℓ^k such that $(\mathfrak{c}, \mathfrak{d}, p) \in \mathcal{X} \rtimes \mathcal{Z}_w$ let

(a) $L_{\mathsf{U}}(\mathfrak{c}, \mathfrak{d}, p) \coloneqq (L_{\mathsf{U}}(\mathfrak{c}), L_{\mathsf{U}}(\mathfrak{d}), R(p, \mathsf{U}(\mathfrak{c}, \mathfrak{d})))$.

(b) $L_{\mathsf{V}}(\mathfrak{c}, \mathfrak{d}, p) \coloneqq (L_{\mathsf{V}}(\mathfrak{c}), L_{\mathsf{V}}(\mathfrak{d}), R(p, \mathsf{V}(\mathfrak{c}, \mathfrak{d})))$.

(c) $L(\mathfrak{c}, \mathfrak{d}, p) \coloneqq L_{\mathsf{U}}(\mathfrak{c}, \mathfrak{d}, p) \otimes L_{\mathsf{V}}(\mathfrak{c}, \mathfrak{d}, p)$.

Remark 5.26. In terms of Definitions 5.24 and 5.25 any $(\mathfrak{c}, \mathfrak{d}, p) \in {}_{\mathfrak{U}\cup\{\aleph\},\mathfrak{O}}\mathcal{S}$ is an element of $\mathcal{X} \rtimes \mathcal{Z}_w$ if and only if $p \leq \{\mathsf{U}(\mathfrak{c}, \mathfrak{d}), \mathsf{V}(\mathfrak{c}, \mathfrak{d})\}$ and $L_{\mathsf{U}}(\mathfrak{c}, \mathfrak{d}, p) \in \mathcal{X}$ and $L_{\mathsf{V}}(\mathfrak{c}, \mathfrak{d}, p) \in \mathcal{Z}_w$.

Definition 5.27. For any $k \in \mathbb{N}_0$ and any $\mathfrak{c}\colon [\![k]\!] \to ((\mathfrak{U} \cup \{\aleph\}) \otimes \{\circ, \bullet\}) \cup \mathfrak{O}$ let $\phi_{\mathfrak{c}} \coloneqq (L(\mathfrak{c}), \mathfrak{c}, \mathrm{pm}_s)$, where s is the permutation of $[\![k]\!]$ with $\blacktriangle_{k,0} \circ s = t_{\varnothing}^{\mathfrak{c}} \circ \blacktriangle_{k,0}$.

By Proposition 5.7 the morphism $\phi_{\mathfrak{c}}$ is clearly invertible for any object $\mathfrak{c}$ of $\mathcal{X} \rtimes \mathcal{Z}_w$.

Definition 5.28. Given for each $t \in [\![2]\!]$ any $k_t \in \mathbb{N}_0$ and any $\mathfrak{c}_t\colon [\![k_t]\!] \to ((\mathfrak{U} \cup \{\aleph\}) \otimes \{\circ, \bullet\}) \cup \mathfrak{O}$, let $L_{\otimes,\mathfrak{c}_1,\mathfrak{c}_2} \coloneqq (L(\mathfrak{c}_1) \vartriangle L(\mathfrak{c}_1), L(\mathfrak{c}_1 \vartriangle \mathfrak{c}_2), \mathrm{pm}_s)$, where, if $m_{\mathsf{Z}}^t = \alpha(\mathsf{Z}(\mathfrak{c}_t, \varnothing))$ for each $t \in [\![2]\!]$ and $\mathsf{Z} \in \{\mathsf{U}, \mathsf{V}\}$, then $s = \mu_{m_{\mathsf{U}}^1, m_{\mathsf{U}}^2, m_{\mathsf{V}}^1, m_{\mathsf{V}}^2}$.

Remark 5.29. It can be shown that L_{U}, L_{V} and L, are $*$-functors from $\mathcal{X} \rtimes \mathcal{Z}_w$ to, respectively, $\mathcal{X}$, $\mathcal{Z}_w$ and $\mathcal{X} \rtimes \mathcal{Z}_w$. Furthermore, one can prove that $L_{\otimes}$ turns L into a monoidal $*$-endofunctor of $\mathcal{X} \rtimes \mathcal{Z}_w$. It is possible to give natural transformations that make L_{U} and L_{V} into monoidal $*$-functors as well. Moreover, L can be seen to be a $*$-equivalence. More precisely, it can be checked that ϕ is a unitary monoidal natural isomorphism from L to the identity functor on $\mathcal{X} \rtimes \mathcal{Z}_w$.

However, the proof of Proposition 5.41 below will only use a small part of these properties, which is why no proofs are given for the remaining ones.

5.2.3. *Properties of the normal form in crossed co-products.* The next three lemmata aid in proving those of the results about L and ϕ mentioned in Remark 5.29 which we need in the following.

Lemma 5.30. *For any $\{k, \ell\} \subseteq \mathbb{N}_0$, any $\mathfrak{c}\colon [\![k]\!] \to ((\mathfrak{U} \cup \{\aleph\}) \otimes \{\circ, \bullet\}) \cup \mathfrak{O}$ and $\mathfrak{d}\colon [\![\ell]\!] \to ((\mathfrak{U} \cup \{\aleph\}) \otimes \{\circ, \bullet\}) \cup \mathfrak{O}$ and any set-theoretical partition p of Π_ℓ^k such that $(\mathfrak{c}, \mathfrak{d}, p) \in \mathcal{X} \rtimes \mathcal{Z}_w$,*

$$L(\mathfrak{c}, \mathfrak{d}, p) = (L(\mathfrak{c}), L(\mathfrak{d}), t_{\mathfrak{d}}^{\mathfrak{c}\,\leftarrow}(p)).$$

Proof. The claim is clear on the level of the labelings right from the definition. On the level of blocks, if $m_{\mathsf{Z}} = \alpha(\mathsf{Z}(\mathfrak{c}, \mathfrak{d}))$ and $n_{\mathsf{Z}} = \beta(\mathsf{Z}(\mathfrak{c}, \mathfrak{d}))$ for any $\mathsf{Z} \in \{\mathsf{U}, \mathsf{V}\}$ and if $\mathsf{H}_{\mathsf{U}} = \Pi_{n_{\mathsf{U}}}^{m_{\mathsf{U}}}$ and $\mathsf{H}_{\mathsf{V}} = \Pi_\ell^k \backslash \Pi_{n_{\mathsf{U}}}^{m_{\mathsf{U}}}$, then $\gamma_{\mathsf{H}_{\mathsf{U}},\ell}{}^{k} = \lhd_{n_{\mathsf{U}},n_{\mathsf{V}}}^{m_{\mathsf{U}},m_{\mathsf{V}}}$ and $\gamma_{\mathsf{H}_{\mathsf{V}},\ell}{}^{k} = \rhd_{n_{\mathsf{U}},n_{\mathsf{V}}}^{m_{\mathsf{U}},m_{\mathsf{V}}}$. Hence, the definition of $t^{\mathfrak{c}}_{\mathfrak{d}} = \gamma_{\mathsf{U}(\mathfrak{c},\mathfrak{d}),\ell}{}^{k} \vartriangle \gamma_{\mathsf{V}(\mathfrak{c},\mathfrak{d}),\ell}{}^{k}$ means precisely that $t^{\mathfrak{c}}_{\mathfrak{d}} \circ \gamma_{\mathsf{H_Z},\ell}{}^{k} = \gamma_{\mathsf{Z}(\mathfrak{c},\mathfrak{d}),\ell}{}^{k}$ for each $\mathsf{Z} \in \{\mathsf{U},\mathsf{V}\}$.

For any $\mathsf{Z} \in \{\mathsf{U},\mathsf{V}\}$ and any $\mathsf{B} \in p$ it follows that the set $(\gamma_{\mathsf{H_Z},\ell}{}^{k}{}_{\rightarrow} \circ \gamma_{\mathsf{Z}(\mathfrak{c},\mathfrak{d}),\ell}{}^{k\leftarrow})(\mathsf{B})$ is given by $(\gamma_{\mathsf{H_Z},\ell}{}^{k}{}_{\rightarrow} \circ (t^{\mathfrak{c}}_{\mathfrak{d}} \circ \gamma_{\mathsf{H_Z},\ell}{}^{k})^{\leftarrow})(\mathsf{B}) = (\gamma_{\mathsf{H_Z},\ell}{}^{k}{}_{\rightarrow} \circ \gamma_{\mathsf{H_Z},\ell}{}^{k\leftarrow})(t^{\mathfrak{c}\leftarrow}_{\mathfrak{d}}(\mathsf{B})) = t^{\mathfrak{c}\leftarrow}_{\mathfrak{d}}(\mathsf{B}) \cap \mathsf{H_Z}$, which amounts to $\varnothing$ if $\mathsf{B} \cap \mathsf{Z}(\mathfrak{c},\mathfrak{d}) = \varnothing$ and to $\gamma_{\mathsf{Z}(\mathfrak{c},\mathfrak{d}),\ell}{}^{k\leftarrow}(\mathsf{B})$ if $\mathsf{B} \subseteq \mathsf{Z}(\mathfrak{c},\mathfrak{d})$, where these two are the only possibilities since $p \leq \{\mathsf{U}(\mathfrak{c},\mathfrak{d}),\mathsf{V}(\mathfrak{c},\mathfrak{d})\}$ by $(\mathfrak{c},\mathfrak{d},p) \in \mathcal{X} \rtimes \mathcal{Z}_w$. Hence, by definition,

$$\begin{aligned}
&R(p,\mathsf{U}(\mathfrak{c},\mathfrak{d})) \otimes R(p,\mathsf{V}(\mathfrak{c},\mathfrak{d}))\\
&\qquad = \dot\bigcup_{\mathsf{Z}\in\{\mathsf{U},\mathsf{V}\}}\{\gamma_{\mathsf{H_Z},\ell}{}^{k}{}_{\rightarrow}(\mathsf{D}) \mid \mathsf{D} \in \gamma_{\mathsf{Z}(\mathfrak{c},\mathfrak{d}),\ell}{}^{k\leftarrow\cdot}(p)\}\\
&\qquad = \dot\bigcup_{\mathsf{Z}\in\{\mathsf{U},\mathsf{V}\}}\{(\gamma_{\mathsf{H_Z},\ell}{}^{k}{}_{\rightarrow} \circ \gamma_{\mathsf{Z}(\mathfrak{c},\mathfrak{d}),\ell}{}^{k\leftarrow})(\mathsf{B}) \mid \mathsf{B} \in p \wedge \mathsf{B} \subseteq \mathsf{Z}(\mathfrak{c},\mathfrak{d})\}\\
&\qquad = \dot\bigcup_{\mathsf{Z}\in\{\mathsf{U},\mathsf{V}\}}\{t^{\mathfrak{c}\leftarrow}_{\mathfrak{d}}(\mathsf{B}) \mid \mathsf{B} \in p \wedge \mathsf{B} \subseteq \mathsf{Z}(\mathfrak{c},\mathfrak{d})\}\\
&\qquad = \dot\bigcup_{\mathsf{Z}\in\{\mathsf{U},\mathsf{V}\}}\{\gamma_{\mathsf{Z}(\mathfrak{c},\mathfrak{d}),\ell}{}^{k\leftarrow}(\mathsf{B}) \mid \mathsf{B} \in p \wedge \mathsf{B} \subseteq \mathsf{Z}(\mathfrak{c},\mathfrak{d})\}\\
&\qquad = t^{\mathfrak{c}\leftarrow\cdot}_{\mathfrak{d}}(p),
\end{aligned}$$

which is what we needed to see. □

LEMMA 5.31. *(a) For any $\{k,\ell\} \subseteq \mathbb{N}_0$, any $\mathfrak{c}\colon [\![k]\!] \to ((\mathfrak{U} \cup \{\aleph\}) \otimes \{\circ,\bullet\}) \cup \mathfrak{O}$ and $\mathfrak{d}\colon [\![\ell]\!] \to ((\mathfrak{U} \cup \{\aleph\}) \otimes \{\circ,\bullet\}) \cup \mathfrak{O}$ and any $\mathsf{Z} \in \{\mathsf{U},\mathsf{V}\}$, if $m_{\mathsf{Z}} = \alpha(\mathsf{Z}(\mathfrak{c},\mathfrak{d}))$ and $n_{\mathsf{Z}} = \beta(\mathsf{Z}(\mathfrak{c},\mathfrak{d}))$ then the following hold.*

(i) $m_{\mathsf{Z}} = \alpha(\mathsf{Z}(\mathfrak{c},\varnothing))$ and $n_{\mathsf{Z}} = \beta(\mathsf{Z}(\varnothing,\mathfrak{d}))$ and for any $x \in [\![m_{\mathsf{Z}}]\!]$ and $y \in [\![n_{\mathsf{Z}}]\!]$,

$$\gamma_{\mathsf{Z}(\mathfrak{c},\mathfrak{d}),\ell}{}^{k}(^{\blacksquare}x) = \gamma_{\mathsf{Z}(\mathfrak{c},\varnothing),0}{}^{k}(^{\blacksquare}x) \quad \wedge \quad \gamma_{\mathsf{Z}(\mathfrak{c},\mathfrak{d}),\ell}{}^{k}(_{\blacksquare}y) = \gamma_{\mathsf{Z}(\varnothing,\mathfrak{d}),\ell}{}^{0}(_{\blacksquare}y).$$

(ii) $m_{\mathsf{Z}} = \beta(\mathsf{Z}(\mathfrak{d},\mathfrak{c}))$ and $n_{\mathsf{Z}} = \alpha(\mathsf{Z}(\mathfrak{d},\mathfrak{c}))$ and

$$\kappa^{k}_{\ell} \circ \gamma_{\mathsf{Z}(\mathfrak{d},\mathfrak{c}),k}{}^{\ell} = \gamma_{\mathsf{Z}(\mathfrak{c},\mathfrak{d}),\ell}{}^{k} \circ \kappa^{m_{\mathsf{Z}}}_{n_{\mathsf{Z}}}.$$

(b) Given for each $t \in [\![2]\!]$ any $\{k_t,\ell_t\} \subseteq \mathbb{N}_0$ and any $\mathfrak{c}_t\colon [\![k_t]\!] \to ((\mathfrak{U} \cup \{\aleph\}) \otimes \{\circ,\bullet\}) \cup \mathfrak{O}$ and $\mathfrak{d}_t\colon [\![\ell_t]\!] \to ((\mathfrak{U} \cup \{\aleph\}) \otimes \{\circ,\bullet\}) \cup \mathfrak{O}$, if $m^{t}_{\mathsf{Z}} = \alpha(\mathsf{Z}(\mathfrak{c}_t,\mathfrak{d}_t))$ and $n^{t}_{\mathsf{Z}} = \beta(\mathsf{Z}(\mathfrak{c}_t,\mathfrak{d}_t))$ for each $t \in [\![2]\!]$, then $m^1_{\mathsf{Z}} + m^2_{\mathsf{Z}} = \alpha(\mathsf{Z}(\mathfrak{c}_1 \vartriangle \mathfrak{c}_2,\mathfrak{d}_1 \vartriangle \mathfrak{d}_2))$ and $n^1_{\mathsf{Z}} + n^2_{\mathsf{Z}} = \beta(\mathsf{Z}(\mathfrak{c}_1 \vartriangle \mathfrak{c}_2,\mathfrak{d}_1 \vartriangle \mathfrak{d}_2))$ and

$$\begin{aligned}
\gamma_{\mathsf{Z}(\mathfrak{c}_1\vartriangle\mathfrak{c}_2,\mathfrak{d}_1\vartriangle\mathfrak{d}_2),\ell_1+\ell_2}{}^{k_1+k_2} \circ \lhd^{m^1_{\mathsf{Z}},m^2_{\mathsf{Z}}}_{n^1_{\mathsf{Z}},n^2_{\mathsf{Z}}} &= \lhd^{k_1,k_2}_{\ell_1,\ell_2} \circ \gamma_{\mathsf{Z}(\mathfrak{c}_1,\mathfrak{d}_1),\ell_1}{}^{k_1},\\
\gamma_{\mathsf{Z}(\mathfrak{c}_1\vartriangle\mathfrak{c}_2,\mathfrak{d}_1\vartriangle\mathfrak{d}_2),\ell_1+\ell_2}{}^{k_1+k_2} \circ \rhd^{m^1_{\mathsf{Z}},m^2_{\mathsf{Z}}}_{n^1_{\mathsf{Z}},n^2_{\mathsf{Z}}} &= \rhd^{k_1,k_2}_{\ell_1,\ell_2} \circ \gamma_{\mathsf{Z}(\mathfrak{c}_1,\mathfrak{d}_1),\ell_1}{}^{k_1}.
\end{aligned}$$

PROOF. (a) (i) Clear by definition.
(ii) Follows immediately from Lemma (a) (b).
(b) Implied by Lemma (a) (c). □

LEMMA 5.32. *(a) For any $\{k,\ell\} \subseteq \mathbb{N}_0$, any $\mathfrak{c}\colon [\![k]\!] \to ((\mathfrak{U} \cup \{\aleph\}) \otimes \{\circ,\bullet\}) \cup \mathfrak{O}$ and $\mathfrak{d}\colon [\![\ell]\!] \to ((\mathfrak{U} \cup \{\aleph\}) \otimes \{\circ,\bullet\}) \cup \mathfrak{O}$ and any $\mathsf{Z} \in \{\mathsf{U},\mathsf{V}\}$ the following hold.*

(i) $t^{\mathfrak{c}}_{\mathfrak{d}}(^{\blacksquare}i) = t^{\mathfrak{c}}_{\varnothing}(^{\blacksquare}i)$ and $t^{\mathfrak{c}}_{\mathfrak{d}}(_{\blacksquare}j) = t^{\varnothing}_{\mathfrak{d}}(_{\blacksquare}j)$ for any $i \in [\![k]\!]$ and $j \in [\![\ell]\!]$.

(ii) $t^{\mathfrak{c}}_{\mathfrak{d}} \circ \kappa^{k}_{\ell} = \kappa^{k}_{\ell} \circ t^{\mathfrak{d}}_{\mathfrak{c}}$.

(b) *Given for each* $t \in [\![2]\!]$ *any* $\{k_t, \ell_t\} \subseteq \mathbb{N}_0$, *any* $\mathfrak{c}_t\colon [\![k_t]\!] \to ((\mathfrak{U} \cup \{\aleph\}) \otimes \{\circ, \bullet\}) \cup \mathfrak{O}$ *and any* $\mathfrak{d}_t\colon [\![\ell_t]\!] \to ((\mathfrak{U} \cup \{\aleph\}) \otimes \{\circ, \bullet\}) \cup \mathfrak{O}$, *if* $m_{\mathsf{U}}^t = \alpha(\mathsf{U}(\mathfrak{c}_t, \mathfrak{d}_t))$ *and* $n_{\mathsf{U}}^t = \beta(\mathsf{U}(\mathfrak{c}_t, \mathfrak{d}_t))$ *and* $m_{\mathsf{V}}^t = \alpha(\mathsf{V}(\mathfrak{c}_t, \mathfrak{d}_t))$ *and* $n_{\mathsf{V}}^t = \beta(\mathsf{V}(\mathfrak{c}_t, \mathfrak{d}_t))$ *for each* $t \in [\![2]\!]$, *then*

$$t_{\mathfrak{d}_1\otimes\mathfrak{d}_2}^{\mathfrak{c}_1\otimes\mathfrak{c}_2} \circ \mu_{n_{\mathsf{U}}^1, n_{\mathsf{U}}^2, n_{\mathsf{V}}^1, n_{\mathsf{V}}^2}^{m_{\mathsf{U}}^1, m_{\mathsf{U}}^2, m_{\mathsf{V}}^1, m_{\mathsf{V}}^2} = t_{\mathfrak{d}_1}^{\mathfrak{c}_1} \circledcirc t_{\mathfrak{d}_2}^{\mathfrak{c}_2}.$$

Proof. (a) (i) Follows immediately from Lemma 5.32 (a) (i).

(ii) Likewise, but from Lemma 5.32 (a) (ii).

(b) By definition,

$$\begin{aligned}
t_{\mathfrak{d}_1\otimes\mathfrak{d}_2}^{\mathfrak{c}_1\otimes\mathfrak{c}_2} &= \gamma_{\mathsf{U}(\mathfrak{c}_1\otimes\mathfrak{c}_2,\mathfrak{d}_1\otimes\mathfrak{d}_2),\ell_1+\ell_2}^{k_1+k_2} \vartriangle \gamma_{\mathsf{V}(\mathfrak{c}_1\otimes\mathfrak{c}_2,\mathfrak{d}_1\otimes\mathfrak{d}_2),\ell_1+\ell_2}^{k_1+k_2} \\
&= ((\gamma_{\mathsf{U}(\mathfrak{c}_1\otimes\mathfrak{c}_2,\mathfrak{d}_1\otimes\mathfrak{d}_2),\ell_1+\ell_2}^{k_1+k_2} \circ \lhd_{n_{\mathsf{U}}^1,n_{\mathsf{U}}^2}^{m_{\mathsf{U}}^1,m_{\mathsf{U}}^2}) \vartriangle (\gamma_{\mathsf{U}(\mathfrak{c}_1\otimes\mathfrak{c}_2,\mathfrak{d}_1\otimes\mathfrak{d}_2),\ell_1+\ell_2}^{k_1+k_2} \circ \rhd_{n_{\mathsf{U}}^1,n_{\mathsf{U}}^2}^{m_{\mathsf{U}}^1,m_{\mathsf{U}}^2})) \\
&\qquad \vartriangle ((\gamma_{\mathsf{V}(\mathfrak{c}_1\otimes\mathfrak{c}_2,\mathfrak{d}_1\otimes\mathfrak{d}_2),\ell_1+\ell_2}^{k_1+k_2} \circ \lhd_{n_{\mathsf{V}}^1,n_{\mathsf{V}}^2}^{m_{\mathsf{V}}^1,m_{\mathsf{V}}^2}) \vartriangle (\gamma_{\mathsf{V}(\mathfrak{c}_1\otimes\mathfrak{c}_2,\mathfrak{d}_1\otimes\mathfrak{d}_2),\ell_1+\ell_2}^{k_1+k_2} \circ \rhd_{n_{\mathsf{V}}^1,n_{\mathsf{V}}^2}^{m_{\mathsf{V}}^1,m_{\mathsf{V}}^2})).
\end{aligned}$$

According to Lemma 5.32 (b) that proves

$$\begin{aligned}
t_{\mathfrak{d}_1\otimes\mathfrak{d}_2}^{\mathfrak{c}_1\otimes\mathfrak{c}_2} = ((\lhd_{\ell_1,\ell_2}^{k_1,k_2} \circ \gamma_{\mathsf{U}(\mathfrak{c}_1,\mathfrak{d}_1),\ell_1}^{k_1}) \vartriangle (\rhd_{\ell_1,\ell_2}^{k_1,k_2} \circ \gamma_{\mathsf{U}(\mathfrak{c}_2,\mathfrak{d}_2),\ell_2}^{k_2})) \\
\vartriangle ((\lhd_{\ell_1,\ell_2}^{k_1,k_2} \circ \gamma_{\mathsf{V}(\mathfrak{c}_1,\mathfrak{d}_1),\ell_1}^{k_1}) \vartriangle (\rhd_{\ell_1,\ell_2}^{k_1,k_2} \circ \gamma_{\mathsf{V}(\mathfrak{c}_2,\mathfrak{d}_2),\ell_2}^{k_2})).
\end{aligned}$$

Hence, by Lemma 5.22,

$$\begin{aligned}
&t_{\mathfrak{d}_1\otimes\mathfrak{d}_2}^{\mathfrak{c}_1\otimes\mathfrak{c}_2} \circ \mu_{n_{\mathsf{U}}^1,n_{\mathsf{U}}^2,n_{\mathsf{V}}^1,n_{\mathsf{V}}^2}^{m_{\mathsf{U}}^1,m_{\mathsf{U}}^2,m_{\mathsf{V}}^1,m_{\mathsf{V}}^2} \\
&= ((\lhd_{\ell_1,\ell_2}^{k_1,k_2} \circ \gamma_{\mathsf{U}(\mathfrak{c}_1,\mathfrak{d}_1),\ell_1}^{k_1}) \vartriangle (\lhd_{\ell_1,\ell_2}^{k_1,k_2} \circ \gamma_{\mathsf{V}(\mathfrak{c}_1,\mathfrak{d}_1),\ell_1}^{k_1})) \\
&\qquad \vartriangle ((\rhd_{\ell_1,\ell_2}^{k_1,k_2} \circ \gamma_{\mathsf{U}(\mathfrak{c}_2,\mathfrak{d}_2),\ell_2}^{k_2}) \vartriangle (\rhd_{\ell_1,\ell_2}^{k_1,k_2} \circ \gamma_{\mathsf{V}(\mathfrak{c}_2,\mathfrak{d}_2),\ell_2}^{k_2})) \\
&= (\lhd_{\ell_1,\ell_2}^{k_1,k_2} \circ (\gamma_{\mathsf{U}(\mathfrak{c}_1,\mathfrak{d}_1),\ell_1}^{k_1} \vartriangle \gamma_{\mathsf{V}(\mathfrak{c}_1,\mathfrak{d}_1),\ell_1}^{k_1})) \vartriangle (\rhd_{\ell_1,\ell_2}^{k_1,k_2} \circ (\gamma_{\mathsf{U}(\mathfrak{c}_2,\mathfrak{d}_2),\ell_2}^{k_2} \vartriangle \gamma_{\mathsf{V}(\mathfrak{c}_2,\mathfrak{d}_2),\ell_2}^{k_2})) \\
&= (\gamma_{\mathsf{U}(\mathfrak{c}_1,\mathfrak{d}_1),\ell_1}^{k_1} \vartriangle \gamma_{\mathsf{V}(\mathfrak{c}_1,\mathfrak{d}_1),\ell_1}^{k_1}) \circledcirc (\gamma_{\mathsf{U}(\mathfrak{c}_2,\mathfrak{d}_2),\ell_2}^{k_2} \vartriangle \gamma_{\mathsf{V}(\mathfrak{c}_2,\mathfrak{d}_2),\ell_2}^{k_2}) \\
&= t_{\mathfrak{d}_1}^{\mathfrak{c}_1} \circledcirc t_{\mathfrak{d}_2}^{\mathfrak{c}_2},
\end{aligned}$$

which is what was claimed. □

The first one of the claims made in Remark 5.29 we will actually need is that ϕ takes values in the morphisms of $\mathcal{X} \rtimes \mathcal{Z}_w$.

Lemma 5.33. $\phi_{\mathfrak{c}} \in \mathcal{X} \rtimes \mathcal{Z}_w$ *for any* $k \in \mathbb{N}_0$ *and any* $\mathfrak{c}\colon [\![k]\!] \to ((\mathfrak{U} \cup \{\aleph\}) \otimes \{\circ, \bullet\}) \cup \mathfrak{O}$.

Proof. If s is the permutation of $[\![k]\!]$ with ${}_{\blacksquare}s(i) = t_{\varnothing}^{\mathfrak{c}}({}^{\blacksquare}i)$ for any $i \in [\![k]\!]$, then by Remark 5.26 we have to show that $\mathrm{pm}_s \leq \{\mathsf{Z}(L(\mathfrak{c}), \mathfrak{c})\}_{\mathsf{Z}\in\{\mathsf{U},\mathsf{V}\}}$ and $L_{\mathsf{U}}(\phi_{\mathfrak{c}}) \in \mathcal{X}$ and $L_{\mathsf{V}}(\phi_{\mathfrak{c}}) \in \mathcal{X}$. Hence, it suffices to prove that $\mathrm{pm}_s \leq \{\mathsf{Z}(L(\mathfrak{c}), \mathfrak{c})\}_{\mathsf{Z}\in\{\mathsf{U},\mathsf{V}\}}$ and $L_{\mathsf{U}}(\phi_{\mathfrak{c}}) = \mathrm{id}_{L_{\mathsf{U}}(\mathfrak{c})}$ and $L_{\mathsf{V}}(\phi_{\mathfrak{c}}) = \mathrm{id}_{L_{\mathsf{V}}(\mathfrak{c})}$. Throughout, let $m_{\mathsf{Z}} \coloneqq \alpha(\mathsf{Z}(\mathfrak{c}, \varnothing))$ and then $\mathsf{H}_{\mathsf{Z}} \coloneqq \Pi_0^{m_{\mathsf{Z}}}$ for any $\mathsf{Z} \in \{\mathsf{U}, \mathsf{V}\}$.

Step 1: Characterizing $\mathsf{U}(L(\mathfrak{c}), \mathfrak{c})$ *and* $\mathsf{V}(L(\mathfrak{c}), \mathfrak{c})$. As a first intermediate step we show that $\mathsf{Z}(L(\mathfrak{c}), \mathfrak{c}) = \mathsf{H}_{\mathsf{Z}} \cup \kappa_0^{k\leftarrow}(\mathsf{Z}(\mathfrak{c}, \varnothing))$ for any $\mathsf{Z} \in \{\mathsf{U}, \mathsf{V}\}$. Let $X_{\mathsf{U}} \coloneqq \mathbb{Z}_w$ and $X_{\mathsf{V}} \coloneqq \{\aleph\}$ and let $\mathsf{Z} \in \{\mathsf{U}, \mathsf{V}\}$ be arbitrary.

On the one hand, we decompose $\xi_{\varnothing}^{L_{\mathsf{U}}(\mathfrak{c})\vartriangle L_{\mathsf{V}}(\mathfrak{c})\leftarrow}(X_{\mathsf{Z}})$ into $\dot\bigcup_{\mathsf{Z}'\in\{\mathsf{U},\mathsf{V}\}}\xi_{\varnothing}^{L_{\mathsf{U}}(\mathfrak{c})\vartriangle L_{\mathsf{V}}(\mathfrak{c})\leftarrow}(X_{\mathsf{Z}})\cap \mathsf{H}_{\mathsf{Z}'}$, i.e., $\dot\bigcup_{\mathsf{Z}'\in\{\mathsf{U},\mathsf{V}\}}(\gamma_{\mathsf{H}_{\mathsf{Z}'},0}{}^{k}_{\rightarrow}\circ\gamma_{\mathsf{H}_{\mathsf{Z}'},0}{}^{k\leftarrow}\circ\xi_{\varnothing}^{L_{\mathsf{U}}(\mathfrak{c})\vartriangle L_{\mathsf{V}}(\mathfrak{c})\leftarrow})(X_{\mathsf{Z}})$ or, identically, $\dot\bigcup_{\mathsf{Z}'\in\{\mathsf{U},\mathsf{V}\}}(\gamma_{\mathsf{H}_{\mathsf{Z}'},0}{}^{k}_{\rightarrow}\circ(\xi_{\varnothing}^{L_{\mathsf{U}}(\mathfrak{c})\vartriangle L_{\mathsf{V}}(\mathfrak{c})}\circ\gamma_{\mathsf{H}_{\mathsf{Z}'},0}{}^{k\leftarrow}))(X_{\mathsf{Z}})$. By Lemma 4.2 (c) that is the same as $\dot\bigcup_{\mathsf{Z}'\in\{\mathsf{U},\mathsf{V}\}}(\gamma_{\mathsf{H}_{\mathsf{Z}'},0}{}^{k}_{\rightarrow}\circ\xi_{\varnothing}^{L_{\mathsf{Z}'}(\mathfrak{c})\leftarrow})(X_{\mathsf{Z}})$. Since, by definition, $\xi_{\varnothing}^{L_{\mathsf{Z}'}(\mathfrak{c})\leftarrow}(X_{\mathsf{Z}})$ is $\Pi_0^{m_{\mathsf{Z}}}$ if $\mathsf{Z}'=\mathsf{Z}$ and $\varnothing$ if $\mathsf{Z}'\neq\mathsf{Z}$ we have thus shown $\xi_{\varnothing}^{L_{\mathsf{U}}(\mathfrak{c})\vartriangle L_{\mathsf{V}}(\mathfrak{c})\leftarrow}(X_{\mathsf{Z}})=\mathsf{H}_{\mathsf{Z}}$.

On the other hand, Lemma 5.31 (a) (ii) implies $\xi_{\mathfrak{c}}^{\varnothing\leftarrow}(X_{\mathsf{Z}})=\mathsf{Z}(\varnothing,\mathfrak{c})=\mathrm{ran}(\gamma_{\mathsf{Z}(\varnothing,\mathfrak{c}),k}{}^{0})=\mathrm{ran}(\gamma_{\mathsf{Z}(\varnothing,\mathfrak{c}),k}{}^{0}\circ\kappa_{m_{\mathsf{Z}}}^{0})=\mathrm{ran}(\gamma_{\mathsf{Z}(\varnothing,\mathfrak{c}),k}{}^{0}\circ\kappa_{m_{\mathsf{Z}}}^{0})=\mathrm{ran}(\kappa_k^0\circ\gamma_{\mathsf{Z}(\mathfrak{c},\varnothing),0}{}^{k})=\kappa_{k\rightarrow}^{0}(\mathrm{ran}(\gamma_{\mathsf{Z}(\mathfrak{c},\varnothing),0}{}^{k}))=\kappa_0^{k\leftarrow}(\mathsf{Z}(\mathfrak{c},\varnothing))$.

Finally, since $\mathsf{Z}(L(\mathfrak{c}),\mathfrak{c})=\xi_{\mathfrak{c}}^{L(\mathfrak{c})\leftarrow}(X_{\mathsf{Z}})=\xi_{\varnothing}^{L(\mathfrak{c})\leftarrow}(X_{\mathsf{Z}})\cup\xi_{\mathfrak{c}}^{\varnothing\leftarrow}(X_{\mathsf{Z}})$ that proves the initial auxiliary claim.

Step 2: Relating $\gamma_{\mathsf{U}(L(\mathfrak{c}),\mathfrak{c}),k}{}^{k}$ *and* $\gamma_{\mathsf{V}(L(\mathfrak{c}),\mathfrak{c}),k}{}^{k}$ *to* s. The second auxiliary statement we prove is that $\gamma_{\mathsf{U}(L(\mathfrak{c}),\mathfrak{c}),k}{}^{k}=(\blacktriangle_{k,k}\circ\lhd_{m_{\mathsf{U}},m_{\mathsf{V}}})^{\blacksquare}{}_{\blacksquare}(\blacktriangle_{k,k}\circ s\circ\lhd_{m_{\mathsf{U}},m_{\mathsf{V}}})$ and $\gamma_{\mathsf{V}(L(\mathfrak{c}),\mathfrak{c}),k}{}^{k}=(\blacktriangle_{k,k}\circ\rhd_{m_{\mathsf{U}},m_{\mathsf{V}}})^{\blacksquare}{}_{\blacksquare}(\blacktriangle_{k,k}\circ s\circ\rhd_{m_{\mathsf{U}},m_{\mathsf{V}}})$.

An equivalent way of expressing the outermost identity in the chain $t_{\varnothing}^{\mathfrak{c}}=\gamma_{\mathsf{U}(\mathfrak{c},\varnothing),0}{}^{k}\vartriangle\gamma_{\mathsf{V}(\mathfrak{c},\varnothing),0}{}^{k}=(\eta_{\mathsf{U}(\mathfrak{c},\varnothing),0}{}^{k}{}^{\blacksquare}{}_{\blacksquare}\varnothing)\vartriangle(\eta_{\mathsf{V}(\mathfrak{c},\varnothing),0}{}^{k}{}^{\blacksquare}{}_{\blacksquare}\varnothing)=(\eta_{\mathsf{U}(\mathfrak{c},\varnothing),0}{}^{k}\vartriangle\eta_{\mathsf{V}(\mathfrak{c},\varnothing),0}{}^{k})^{\blacksquare}{}_{\blacksquare}\varnothing$ is to say that both $t_{\varnothing}^{\mathfrak{c}}\circ\blacktriangle_{k,0}\circ\lhd_{m_{\mathsf{U}},m_{\mathsf{V}}}=\eta_{\mathsf{U}(\mathfrak{c},\varnothing),0}{}^{k}$ and $t_{\varnothing}^{\mathfrak{c}}\circ\blacktriangle_{k,0}\circ\rhd_{m_{\mathsf{U}},m_{\mathsf{V}}}=\eta_{\mathsf{V}(\mathfrak{c},\varnothing),0}{}^{k}$. The definition of s therefore implies $\blacktriangle_{k,k}\circ s\circ\lhd_{m_{\mathsf{U}},m_{\mathsf{V}}}=\eta_{\mathsf{U}(\mathfrak{c},\varnothing),k}{}^{k}$ and $\blacktriangle_{k,k}\circ s\circ\rhd_{m_{\mathsf{U}},m_{\mathsf{V}}}=\eta_{\mathsf{V}(\mathfrak{c},\varnothing),k}{}^{k}$, which by $\kappa_k^k\circ\blacktriangle_{k,k}=\blacktriangledown_{k,k}$ is equivalent to $\blacktriangledown_{k,k}\circ s\circ\lhd_{m_{\mathsf{U}},m_{\mathsf{V}}}=\kappa_k^k\circ\eta_{\mathsf{U}(\mathfrak{c},\varnothing),k}{}^{k}$ and $\blacktriangledown_{k,k}\circ s\circ\rhd_{m_{\mathsf{U}},m_{\mathsf{V}}}=\kappa_k^k\circ\eta_{\mathsf{V}(\mathfrak{c},\varnothing),k}{}^{k}$. By Step 1 we have thus shown $\gamma_{\mathsf{U}(L(\mathfrak{c}),\mathfrak{c}),k}{}^{k}\circ\blacktriangledown_{k,k}=\blacktriangledown_{k,k}\circ s\circ\lhd_{m_{\mathsf{U}},m_{\mathsf{V}}}$ and $\gamma_{\mathsf{V}(L(\mathfrak{c}),\mathfrak{c}),k}{}^{k}\circ\blacktriangledown_{k,k}=\blacktriangledown_{k,k}\circ s\circ\rhd_{m_{\mathsf{U}},m_{\mathsf{V}}}$.

Furthemore, Step 1 and the facts that $\eta_{\mathsf{H}_{\mathsf{U}},0}{}^{k}=\blacktriangle_{k,0}\circ\lhd_{m_{\mathsf{U}},m_{\mathsf{V}}}$ and $\eta_{\mathsf{H}_{\mathsf{V}},0}{}^{k}=\blacktriangle_{k,0}\circ\rhd_{m_{\mathsf{U}},m_{\mathsf{V}}}$ imply $\gamma_{\mathsf{U}(L(\mathfrak{c}),\mathfrak{c}),k}{}^{k}\circ\blacktriangle_{k,k}=\blacktriangledown_{k,k}\circ\lhd_{m_{\mathsf{U}},m_{\mathsf{V}}}$ and $\gamma_{\mathsf{V}(L(\mathfrak{c}),\mathfrak{c}),k}{}^{k}\circ\blacktriangle_{k,k}=\blacktriangledown_{k,k}\circ\rhd_{m_{\mathsf{U}},m_{\mathsf{V}}}$. Together the two four established identities prove the auxiliary claim.

Step 3: Formulating pm_s *in terms of* $\mathsf{U}(L(\mathfrak{c}),\mathfrak{c})$ *and* $\mathsf{V}(L(\mathfrak{c}),\mathfrak{c})$. On the one hand, the results of Step 2 say, equivalently, that $\gamma_{\mathsf{U}(L(\mathfrak{c}),\mathfrak{c}),k}{}^{k}({}^{\blacksquare}i_{\mathsf{U}})={}^{\blacksquare}i_{\mathsf{U}}$ and $\gamma_{\mathsf{U}(L(\mathfrak{c}),\mathfrak{c}),k}{}^{k}({}_{\blacksquare}i_{\mathsf{U}})={}_{\blacksquare}s(i_{\mathsf{U}})$ for any $i_{\mathsf{U}}\in[\![m_{\mathsf{U}}]\!]$ and that $\gamma_{\mathsf{V}(L(\mathfrak{c}),\mathfrak{c}),k}{}^{k}({}^{\blacksquare}i_{\mathsf{V}})={}^{\blacksquare}(m_{\mathsf{U}}+i_{\mathsf{V}})$ and $\gamma_{\mathsf{V}(L(\mathfrak{c}),\mathfrak{c}),k}{}^{k}({}_{\blacksquare}i_{\mathsf{V}})={}_{\blacksquare}s(m_{\mathsf{U}}+i_{\mathsf{V}})$ for any $i_{\mathsf{V}}\in[\![m_{\mathsf{V}}]\!]$. On the other, by definition, $\mathrm{pm}_s=\{\{{}^{\blacksquare}i,{}_{\blacksquare}s(i)\}\}_{i=1}^{k}=\{\{{}^{\blacksquare}i_{\mathsf{U}},{}_{\blacksquare}s(i_{\mathsf{U}})\}\}_{i_{\mathsf{U}}=1}^{m_{\mathsf{U}}}\cup\{\{{}^{\blacksquare}(m_{\mathsf{U}}+i_{\mathsf{V}}),{}_{\blacksquare}s(m_{\mathsf{U}}+i_{\mathsf{V}})\}\}_{i_{\mathsf{V}}=1}^{m_{\mathsf{V}}}$. Hence, Step 2 actually implies that $\mathrm{pm}_s=\dot\bigcup_{\mathsf{Z}\in\{\mathsf{U},\mathsf{V}\}}\{\gamma_{\mathsf{Z}(L(\mathfrak{c}),\mathfrak{c}),k}{}^{k}_{\rightarrow}(\mathsf{B})\,|\,\mathsf{B}\in\mathrm{id}_{m_{\mathsf{Z}}}\}$.

Step 4: Proving $\mathrm{pm}_s\leq\{\mathsf{Z}(L(\mathfrak{c}),\mathfrak{c})\}_{\mathsf{Z}\in\{\mathsf{U},\mathsf{V}\}}$. Since $\mathrm{ran}(\gamma_{\mathsf{Z}(L(\mathfrak{c}),\mathfrak{c}),k}{}^{k})=\mathsf{Z}(L(\mathfrak{c}),\mathfrak{c})$ for any $\mathsf{Z}\in\{\mathsf{U},\mathsf{V}\}$ Step 3 immediately lets us conclude that $\mathrm{pm}_s\leq\{\mathsf{Z}(L(\mathfrak{c}),\mathfrak{c})\}_{\mathsf{Z}\in\{\mathsf{U},\mathsf{V}\}}$.

Step 5: Proving $L_{\mathsf{U}}(L(\mathfrak{c}))=L_{\mathsf{U}}(\mathfrak{c})$ *and* $L_{\mathsf{V}}(L(\mathfrak{c}))=L_{\mathsf{V}}(\mathfrak{c})$. It suffices to prove that $\xi_{\varnothing}^{L_{\mathsf{Z}}(L(\mathfrak{c}))}=\xi_{\varnothing}^{L_{\mathsf{Z}}(\mathfrak{c})}$ and $\zeta_{\varnothing}^{L_{\mathsf{Z}}(L(\mathfrak{c}))}=\zeta_{\varnothing}^{L_{\mathsf{Z}}(\mathfrak{c})}$ for any $\mathsf{Z}\in\{\mathsf{U},\mathsf{V}\}$.

Step 5.1: Tags of $L_{\mathsf{U}}(L(\mathfrak{c}))$. By the definition of L_{U}, for any $i\in[\![k]\!]$ the tag $\xi_{\varnothing}^{L_{\mathsf{U}}(L(\mathfrak{c}))}({}^{\blacksquare}i)$ is given by $(\pi_w\circ\varepsilon_{\varnothing}^{L(\mathfrak{c})}\circ\gamma_{\mathsf{U}(L(\mathfrak{c}),\varnothing),0}{}^{k})({}^{\blacksquare}i)$. Since $\mathsf{U}(L(\mathfrak{c}),\varnothing)=\mathsf{H}_{\mathsf{U}}\cup\kappa_0^{k\leftarrow}(\mathsf{U}(\mathfrak{c},\varnothing))$ by Step 1 and since $\kappa_0^{k\leftarrow}(\mathsf{U}(\mathfrak{c},\varnothing))\subseteq\Pi_k^0$ the point $\gamma_{\mathsf{U}(L(\mathfrak{c}),\varnothing),0}{}^{k}({}^{\blacksquare}i)$ is the same as $\gamma_{\mathsf{H}_{\mathsf{U}},0}{}^{k}({}^{\blacksquare}i)$. Thereby and by the definition of L we have shown that

$\xi_{\varnothing}^{L_{\mathsf{U}}(L(\mathfrak{c}))}({}^{\blacksquare}i) = (\pi_w \circ \varepsilon_{\varnothing}^{L_{\mathsf{U}}(\mathfrak{c}) \vartriangle L_{\mathsf{V}}(\mathfrak{c})} \circ \gamma_{\mathsf{H}_{\mathsf{U}},0}{}^{k})({}^{\blacksquare}i)$. According to Lemma 4.43 (c) the tags $(\varepsilon_{\varnothing}^{L_{\mathsf{U}}(\mathfrak{c}) \vartriangle L_{\mathsf{V}}(\mathfrak{c})} \circ \gamma_{\mathsf{H}_{\mathsf{U}},0}{}^{k})({}^{\blacksquare}i)$ and $\varepsilon_{\varnothing}^{L_{\mathsf{U}}(\mathfrak{c})}({}^{\blacksquare}i)$ agree. And, of course, $\varepsilon_{\varnothing}^{L_{\mathsf{U}}(\mathfrak{c})}({}^{\blacksquare}i)$ is already an element of $\mathbb{Z}_w$, which proves . That proves that $\xi_{\varnothing}^{L_{\mathsf{U}}(L(\mathfrak{c}))}({}^{\blacksquare}i) = (\pi_w \circ \varepsilon_{\varnothing}^{L_{\mathsf{U}}(\mathfrak{c})})({}^{\blacksquare}i)$. Because $\operatorname{ran}(\xi_{\varnothing}^{L_{\mathsf{U}}(\mathfrak{c})}) \subseteq \mathbb{Z}_w$ the definition implies that $\varepsilon_{\varnothing}^{L_{\mathsf{U}}(\mathfrak{c})} = \xi_{\varnothing}^{L_{\mathsf{U}}(\mathfrak{c})} = \pi_w \circ \xi_{\varnothing}^{L_{\mathsf{U}}(\mathfrak{c})}$. In conclusion, $\xi_{\varnothing}^{L_{\mathsf{U}}(L(\mathfrak{c}))}({}^{\blacksquare}i) = \xi_{\varnothing}^{L_{\mathsf{U}}(\mathfrak{c})}({}^{\blacksquare}i)$, as claimed.

Step 5.2: Tags of $L_{\mathsf{V}}(L(\mathfrak{c}))$. The argument for $\xi_{\varnothing}^{L_{\mathsf{V}}(L(\mathfrak{c}))}$ is similar but easier. Given any $i \in [\![k]\!]$, the definition tells us that the tag $\xi_{\varnothing}^{L_{\mathsf{V}}(L(\mathfrak{c}))}({}^{\blacksquare}i)$ is given by $(\xi_{\varnothing}^{L_{\mathsf{U}}(\mathfrak{c}) \vartriangle L_{\mathsf{V}}(\mathfrak{c})} \circ \gamma_{\mathsf{V}(L(\mathfrak{c}),\varnothing),0}{}^{k})({}^{\blacksquare}i)$, which simplifies to $(\xi_{\varnothing}^{L_{\mathsf{U}}(\mathfrak{c}) \vartriangle L_{\mathsf{V}}(\mathfrak{c})} \circ \gamma_{\mathsf{H}_{\mathsf{V}},0}{}^{k})({}^{\blacksquare}i)$ because $\mathsf{V}(L(\mathfrak{c}),\varnothing) = \mathsf{H}_{\mathsf{V}} \cup \kappa_0^{k\leftarrow}(\mathsf{V}(\mathfrak{c},\varnothing))$ by Step 1 and because $\kappa_0^{k\leftarrow}(\mathsf{V}(\mathfrak{c},\varnothing)) \subseteq \Pi_k^0$. Now it is Lemma 4.2 (c) that lets us rewrite this as $\xi_{\varnothing}^{L_{\mathsf{V}}(\mathfrak{c})}({}^{\blacksquare}i)$, proving what we needed to see.

Step 5.3: Colors of $L_{\mathsf{U}}(L(\mathfrak{c}))$ *and* $L_{\mathsf{V}}(L(\mathfrak{c}))$. For any $\mathsf{Z} \in \{\mathsf{U},\mathsf{V}\}$ the proof that $\zeta_{\varnothing}^{L_{\mathsf{Z}}(L(\mathfrak{c}))} = \zeta_{\varnothing}^{L_{\mathsf{Z}}(\mathfrak{c})}$ is the same as the one that $\zeta_{\varnothing}^{L_{\mathsf{V}}(L(\mathfrak{c}))} = \zeta_{\varnothing}^{L_{\mathsf{V}}(\mathfrak{c})}$ once there ξ is replaced by ζ and V by Z. Thus, indeed, $L_{\mathsf{U}}(L(\mathfrak{c})) = L_{\mathsf{U}}(\mathfrak{c})$ and $L_{\mathsf{V}}(L(\mathfrak{c})) = L_{\mathsf{V}}(\mathfrak{c})$.

Step 6: Proving $R(\mathrm{pm}_s, \mathsf{U}(L(\mathfrak{c}),\mathfrak{c})) = \mathrm{id}_{m_{\mathsf{U}}}$ *and* $R(\mathrm{pm}_s, \mathsf{V}(L(\mathfrak{c}),\mathfrak{c})) = \mathrm{id}_{m_{\mathsf{V}}}$. Since $\mathrm{pm}_s \leq \{\mathsf{U}(L(\mathfrak{c}),\mathfrak{c}), \mathsf{V}(L(\mathfrak{c}),\mathfrak{c})\}$ by Step 4, since $\mathsf{U}(L(\mathfrak{c}),\mathfrak{c}) \cap \mathsf{V}(L(\mathfrak{c}),\mathfrak{c}) = \varnothing$ and since $\gamma_{\mathsf{Z}(L(\mathfrak{c}),\mathfrak{c}),k}{}^{k}$ is injective, Step 3 proves that $R(\mathrm{pm}_s, \mathsf{Z}(L(\mathfrak{c}),\mathfrak{c})) = \gamma_{\mathsf{Z}(L(\mathfrak{c}),\mathfrak{c}),k}{}^{k\leftarrow}(\mathrm{pm}_s) = \mathrm{id}_{m_{\mathsf{Z}}}$ for any $\mathsf{Z} \in \{\mathsf{U},\mathsf{V}\}$. □

The second result about ϕ we will require is that ϕ is a natural transformation from L to the identity functor (the former of which we have not even shown to be functorial at all, of course).

LEMMA 5.34. *For any* $\{k,\ell\} \subseteq \mathbb{N}_0$, *any* $\mathfrak{c}\colon [\![k]\!] \to ((\mathfrak{U} \cup \{\aleph\}) \otimes \{\circ,\bullet\}) \cup \mathfrak{O}$ *and* $\mathfrak{d}\colon [\![\ell]\!] \to ((\mathfrak{U} \cup \{\aleph\}) \otimes \{\circ,\bullet\}) \cup \mathfrak{O}$ *and any set-theoretical partition* p *of* Π_ℓ^k *such that* $(\mathfrak{c},\mathfrak{d},p) \in \mathcal{X} \rtimes \mathcal{Z}_w$,

$$\phi_{\mathfrak{d}}\, L(\mathfrak{c},\mathfrak{d},p) = (\mathfrak{c},\mathfrak{d},p)\, \phi_{\mathfrak{c}}.$$

PROOF. If f and g are the permutations of $[\![k]\!]$ respectively $[\![\ell]\!]$ with ${}^{\blacksquare}f(i) = t_{\varnothing}^{\mathfrak{c}}({}^{\blacksquare}i)$ for any $i \in [\![k]\!]$and ${}^{\blacksquare}g(j) = t_{\varnothing}^{\mathfrak{d}}({}_{\blacksquare}j)$ for any $j \in [\![\ell]\!]$ and if $v = t_{\mathfrak{d}}^{\mathfrak{c}\leftarrow}(p)$, then it suffices to show that

$$p\,\mathrm{pm}_f = \{t_{\mathfrak{d}}^{\mathfrak{c}\leftarrow}(\mathsf{B} \cap \Pi_0^k) \cup (\mathsf{B} \cap \Pi_\ell^0) \mid \mathsf{B} \in p\} = \mathrm{pm}_g\, v.$$

where each identity will be proved separately.

Step 1: Left identity. By Lemma 5.15 (b) and the assumption on f,

$$\begin{aligned} p\,\mathrm{pm}_f &= \{\{{}^{\blacksquare}f^{-1}(i) \mid i \in [\![k]\!] \wedge {}^{\blacksquare}i \in \mathsf{B}\} \cup (\mathsf{B} \cap \Pi_\ell^0) \mid \mathsf{B} \in p\} \\ &= \{\{{}^{\blacksquare}i \mid i \in [\![k]\!] \wedge {}^{\blacksquare}f(i) \in \mathsf{B}\} \cup (\mathsf{B} \cap \Pi_\ell^0) \mid \mathsf{B} \in p\} \\ &= \{\{\mathsf{a} \in \Pi_0^k \wedge t_{\varnothing}^{\mathfrak{c}}(\mathsf{a}) \in \mathsf{B}\} \cup (\mathsf{B} \cap \Pi_\ell^0) \mid \mathsf{B} \in p\} \\ &= \{t_{\varnothing}^{\mathfrak{c}\leftarrow}(\mathsf{B} \cap \Pi_0^k) \cup (\mathsf{B} \cap \Pi_\ell^0) \mid \mathsf{B} \in p\}, \end{aligned}$$

which proves the first identity by Lemma 5.32 (a) (i).

Step 2: Right identity. With the help of both parts of Lemma 5.32 (a) we can conclude from the assumption on g that ${}_{\blacksquare}g(j) = \kappa^0_\ell({}^{\blacksquare}g(j)) = (\kappa^0_\ell \circ t^{\mathfrak{d}}_{\varnothing})({}^{\blacksquare}j) = (t^{\varnothing}_{\mathfrak{d}} \circ \kappa^0_\ell)({}^{\blacksquare}j) = t^{\varnothing}_{\mathfrak{d}}({}_{\blacksquare}j) = t^{\mathfrak{c}}_{\mathfrak{d}}({}_{\blacksquare}j)$ for any $j \in [\![\ell]\!]$. Hence, Lemma 5.15 (a) implies that

$$\begin{aligned}\mathrm{pm}_g\, v &= \{(\mathsf{C} \cap \Pi^k_0) \cup \{{}_{\blacksquare}g(j) \mid j \in [\![\ell]\!] \wedge {}_{\blacksquare}j \in \mathsf{C}\} \mid \mathsf{C} \in v\}\\ &= \{(t^{\mathfrak{c}\leftarrow}_{\mathfrak{d}}(\mathsf{B}) \cap \Pi^k_0) \cup \{t^{\mathfrak{c}}_{\mathfrak{d}}({}_{\blacksquare}j) \mid j \in [\![\ell]\!] \wedge {}_{\blacksquare}j \in t^{\mathfrak{c}\leftarrow}_{\mathfrak{d}}(\mathsf{B})\} \mid \mathsf{B} \in p\}\\ &= \{t^{\mathfrak{c}\leftarrow}_{\mathfrak{d}}(\mathsf{B} \cap \Pi^k_0) \cup \{t^{\mathfrak{c}}_{\mathfrak{d}}(\mathsf{b}) \mid \mathsf{b} \in \Pi^0_\ell \wedge t^{\mathfrak{c}}_{\mathfrak{d}}(\mathsf{b}) \in \mathsf{B}\} \mid \mathsf{B} \in p\}\\ &= \{t^{\mathfrak{c}\leftarrow}_{\varnothing}(\mathsf{B} \cap \Pi^k_0) \cup (\mathsf{B} \cap \Pi^0_\ell) \mid \mathsf{B} \in p\},\end{aligned}$$

where we have also used the facts that $t^{\mathfrak{c}}_{\mathfrak{d}}$ is bijective and preserves upper and lower points. □

The final one of the statements about L_{U}, L_{V}, L and ϕ alluded to in Remark 5.29 which we will need is that the natural transformation ϕ from L to the identity functor is monoidal. The proof of this will use the following fact.

LEMMA 5.35. *Given for each $t \in [\![2]\!]$ any $k_t \in \mathbb{N}_0$, any $\mathfrak{c}_t\colon [\![k_t]\!] \to (\mathfrak{U} \otimes \{\circ, \bullet\}) \cup \mathfrak{O}$ and any permutation s_t of $[\![k_t]\!]$,*

$$\mathrm{pm}^{\mathfrak{c}_1}_{s_1} \otimes \mathrm{pm}^{\mathfrak{c}_2}_{s_2} = \mathrm{pm}^{\mathfrak{c}_1 \vartriangle \mathfrak{c}_2}_{s_1 \circledcirc s_2}.$$

PROOF. The claim is clear on the level of labels. Of course, $\lhd^{k_1,k_2}_{k_1,k_2\to}(\{{}^{\blacksquare}i_1, {}_{\blacksquare}i_1\}) = \{{}^{\blacksquare}i_1, {}_{\blacksquare}s_1(i_1)\}$ for any $i_1 \in [\![k_1]\!]$ and $\rhd^{k_1,k_2}_{k_1,k_2\to}(\{{}^{\blacksquare}i_1, {}_{\blacksquare}i_1\}) = \{{}^{\blacksquare}(k_1+i_2), {}_{\blacksquare}s_2(k_1+i_2)\}$ for any $i_2 \in [\![k_2]\!]$. Therefore, $\mathrm{pm}_{s_1} \otimes \mathrm{pm}_{s_2} = \{\lhd^{k_1,k_2}_{k_1,k_2\to}(\mathsf{B}_1) \mid \mathsf{B}_1 \in \mathrm{pm}_{s_1}\} \cup \{\rhd^{k_1,k_2}_{k_1,k_2\to}(\mathsf{B}_2) \mid \mathsf{B}_2 \in \mathrm{pm}_{s_2}\}$ is given by $\{\{{}^{\blacksquare}i_1, {}_{\blacksquare}s_1(i_1)\}, \{{}^{\blacksquare}(k_1+i_2), {}_{\blacksquare}s_2(k_1+i_2)\} \mid i_1 \in [\![k_1]\!], i_2 \in [\![k_2]\!]\}$, i.e., by $\{\{{}^{\blacksquare}i, {}_{\blacksquare}(s_1 \circledcirc s_2)(i)\} \mid i \in [\![k_1+k_2]\!]\} = \mathrm{pm}_{s_1 \circledcirc s_2}$. □

LEMMA 5.36. *For any $\{k_1, k_2\} \subseteq \mathbb{N}_0$ and any $\mathfrak{c}_1\colon [\![k_1]\!] \to ((\mathfrak{U} \cup \{\aleph\}) \otimes \{\circ, \bullet\}) \cup \mathfrak{O}$ and $\mathfrak{c}_2\colon [\![k_2]\!] \to ((\mathfrak{U} \cup \{\aleph\}) \otimes \{\circ, \bullet\}) \cup \mathfrak{O}$,*

$$\phi_{\mathfrak{c}_1 \otimes \mathfrak{c}_2}\, L_{\otimes, \mathfrak{c}_1, \mathfrak{c}_2} = \phi_{\mathfrak{c}_1} \otimes \phi_{\mathfrak{c}_2}.$$

PROOF. For any $\mathsf{Z} \in \{\mathsf{U}, \mathsf{V}\}$ and $t \in [\![2]\!]$ let $m^t_{\mathsf{Z}} \coloneqq \alpha(\mathsf{Z}(\mathfrak{c}_t, \varnothing))$. Moreover, let h be the permutation of $[\![k_1+k_2]\!]$ with $\blacktriangle_{k_1+k_2,0} \circ h = t^{\mathfrak{c}_1 \vartriangle \mathfrak{c}_2}_{\varnothing} \circ \blacktriangle_{k_1+k_2,0}$, let $g \coloneqq \mu_{m^1_{\mathsf{U}}, m^2_{\mathsf{U}}, m^1_{\mathsf{V}}, m^2_{\mathsf{V}}}$ and for each $t \in [\![2]\!]$ let f_t be the permutation of $[\![k_t]\!]$ with $\blacktriangle_{k_t,0} \circ f_t = t^{\mathfrak{c}_t}_{\varnothing} \circ \blacktriangle_{k_t,0}$. Then, by definition, $\phi_{\mathfrak{c}_1 \vartriangle \mathfrak{c}_2} = (L(\mathfrak{c}_1 \vartriangle \mathfrak{c}_2), \mathfrak{c}_1 \vartriangle \mathfrak{c}_2, \mathrm{pm}_h)$ and $L_{\otimes, \mathfrak{c}_1, \mathfrak{c}_2} = (L(\mathfrak{c}_1) \vartriangle L(\mathfrak{c}_2), L(\mathfrak{c}_1 \vartriangle \mathfrak{c}_2), \mathrm{pm}_g)$ and $\phi_{\mathfrak{c}_t} = (L(\mathfrak{c}_t), \mathfrak{c}_t, \mathrm{pm}_{f_t})$ for each $t \in [\![2]\!]$. That implies $\phi_{\mathfrak{c}_1 \vartriangle \mathfrak{c}_2}\, L_{\otimes, \mathfrak{c}_1, \mathfrak{c}_2} = (L(\mathfrak{c}_1) \vartriangle L(\mathfrak{c}_2), \mathfrak{c}_1 \vartriangle \mathfrak{c}_2, \mathrm{pm}_h\, \mathrm{pm}_g)$ and $\phi_{\mathfrak{c}_1} \otimes \phi_{\mathfrak{c}_2} = (L(\mathfrak{c}_1) \vartriangle L(\mathfrak{c}_2), \mathfrak{c}_1 \vartriangle \mathfrak{c}_2, \mathrm{pm}_{f_1} \otimes \mathrm{pm}_{f_2})$. Hence, by Lemmata 5.16 and 5.35 all we have to prove is that $h \circ g = f_1 \circledcirc f_2$.

Because $\blacktriangle_{k_1+k_2,0}\circ g = \mu_{0,0,0,0}^{m_{\mathsf{U}}^1,m_{\mathsf{U}}^2,m_{\mathsf{V}}^1,m_{\mathsf{V}}^2}\circ\blacktriangle_{k_1+k_2,0}$ and because $\blacktriangle_{k_1+k_2,0}\circ\lhd_{k_1,k_2} = \lhd_{0,0}^{k_1,k_2}\circ\blacktriangle_{k_1,0}$ and $\blacktriangle_{k_1+k_2,0}\circ\rhd_{k_1,k_2} = \rhd_{0,0}^{k_1,k_2}\circ\blacktriangle_{k_2,0}$ by definition, Lemma 5.32 (b) lets us infer

$$\begin{aligned}
\blacktriangle_{k_1+k_2,0}\circ h\circ g &= t_{\varnothing}^{\mathfrak{c}_1\vartriangle\mathfrak{c}_2}\circ\blacktriangle_{k_1+k_2,0}\circ g\\
&= t_{\varnothing}^{\mathfrak{c}_1\vartriangle\mathfrak{c}_2}\circ\mu_{0,0,0,0}^{m_{\mathsf{U}}^1,m_{\mathsf{U}}^2,m_{\mathsf{V}}^1,m_{\mathsf{V}}^2}\circ\blacktriangle_{k_1+k_2,0}\\
&= (t_{\varnothing}^{\mathfrak{c}_1}\circledcirc t_{\varnothing}^{\mathfrak{c}_2})\circ\blacktriangle_{k_1+k_2,0}\\
&= (t_{\varnothing}^{\mathfrak{c}_1}\circledcirc t_{\varnothing}^{\mathfrak{c}_2})\circ((\blacktriangle_{k_1+k_2,0}\circ\lhd_{k_1,k_2})\vartriangle(\blacktriangle_{k_1+k_2,0}\circ\rhd_{k_1,k_2}))\\
&= (t_{\varnothing}^{\mathfrak{c}_1}\circledcirc t_{\varnothing}^{\mathfrak{c}_2})\circ((\lhd_{0,0}^{k_1,k_2}\circ\blacktriangle_{k_1,0})\vartriangle(\rhd_{0,0}^{k_1,k_2}\circ\blacktriangle_{k_2,0}))\\
&= (((t_{\varnothing}^{\mathfrak{c}_1}\circledcirc t_{\varnothing}^{\mathfrak{c}_2})\circ\lhd_{0,0}^{k_1,k_2}\circ\blacktriangle_{k_1,0})\vartriangle((t_{\varnothing}^{\mathfrak{c}_1}\circledcirc t_{\varnothing}^{\mathfrak{c}_2})\circ\rhd_{0,0}^{k_1,k_2}\circ\blacktriangle_{k_2,0}))\\
&= (\lhd_{0,0}^{k_1,k_2}\circ t_{\varnothing}^{\mathfrak{c}_1}\circ\blacktriangle_{k_1,0})\vartriangle(\rhd_{0,0}^{k_1,k_2}\circ t_{\varnothing}^{\mathfrak{c}_2}\circ\blacktriangle_{k_2,0})\\
&= (\lhd_{0,0}^{k_1,k_2}\circ\blacktriangle_{k_1,0}\circ f_1)\vartriangle(\rhd_{0,0}^{k_1,k_2}\circ\blacktriangle_{k_2,0}\circ f_2)\\
&= (\blacktriangle_{k_1+k_2,0}\circ\lhd_{k_1,k_2}\circ f_1)\vartriangle(\blacktriangle_{k_1+k_2,0}\circ\rhd_{k_1,k_2}\circ f_2)\\
&= \blacktriangle_{k_1+k_2,0}\circ((\lhd_{k_1,k_2}\circ f_1)\vartriangle(\rhd_{k_1,k_2}\circ f_2))\\
&= \blacktriangle_{k_1+k_2,0}\circ(f_1\circledcirc f_2),
\end{aligned}$$

which is all we needed to see because $\blacktriangle_{k_1+k_2,0}$ is injective. □

The last preparation we need to make for the proof of the main result of Section 5.2 about the generators of $\mathcal{X}\rtimes\mathcal{Z}_w$ goes beyond what was claimed in Remark 5.29. It is a result expressing $L_\otimes$ in terms of ϕ.

LEMMA 5.37. *When given for each $t\in[\![2]\!]$ any $k_t\in\mathbb{N}_0$ and any $\mathfrak{c}_t\colon[\![k_t]\!]\to((\mathfrak{U}\cup\{\aleph\})\otimes\{\circ,\bullet\})\cup\mathfrak{O}$, then*

$$L_{\otimes,\mathfrak{c}_1,\mathfrak{c}_2} = \mathrm{id}_{L_{\mathsf{U}}(\mathfrak{c}_1)}\otimes\phi_{L_{\mathsf{V}}(\mathfrak{c}_1)\otimes L_{\mathsf{U}}(\mathfrak{c}_2)}{}^*\otimes\mathrm{id}_{L_{\mathsf{V}}(\mathfrak{c}_2)}.$$

PROOF. For any $\mathsf{Z}\in\{\mathsf{U},\mathsf{V}\}$ let $m_{\mathsf{Z}}^t\coloneqq\alpha(\mathsf{Z}(\mathfrak{c}_t,\varnothing))$ for each $t\in[\![2]\!]$ and let $m_{\mathsf{Z}}\coloneqq\alpha(\mathsf{Z}(\mathfrak{c}_1\vartriangle\mathfrak{c}_2,\varnothing))$. If s is the permutation of $[\![m_{\mathsf{V}}^1+m_{\mathsf{U}}^2]\!]$ with $\blacktriangle_{m_{\mathsf{V}}^1+m_{\mathsf{U}}^2,0}\circ s = t_{\varnothing}^{L_{\mathsf{V}}(\mathfrak{c}_1)\vartriangle L_{\mathsf{U}}(\mathfrak{c}_2)}\circ\blacktriangle_{m_{\mathsf{V}}^1+m_{\mathsf{U}}^2,0}$, then $\phi_{L_{\mathsf{V}}(\mathfrak{c}_1)\vartriangle L_{\mathsf{U}}(\mathfrak{c}_2)} = (L(L_{\mathsf{V}}(\mathfrak{c}_1)\vartriangle L_{\mathsf{U}}(\mathfrak{c}_2)), L_{\mathsf{V}}(\mathfrak{c}_1)\vartriangle L_{\mathsf{U}}(\mathfrak{c}_2),\mathrm{pm}_s)$ and, consequently, $\phi_{L_{\mathsf{V}}(\mathfrak{c}_1)\vartriangle L_{\mathsf{U}}(\mathfrak{c}_2)}{}^* = (L_{\mathsf{V}}(\mathfrak{c}_1)\vartriangle L_{\mathsf{U}}(\mathfrak{c}_2), L(L_{\mathsf{V}}(\mathfrak{c}_1)\vartriangle L_{\mathsf{U}}(\mathfrak{c}_2)),\mathrm{pm}_{s^{-1}})$ by Lemma 5.16. If, moreover, f_1 and f_2 are the identity mappings of $[\![m_{\mathsf{U}}^1]\!]$ respectively $[\![m_{\mathsf{V}}^2]\!]$, then the partition $\mathrm{id}_{L_{\mathsf{U}}(\mathfrak{c}_1)}\otimes\phi_{L_{\mathsf{V}}(\mathfrak{c}_1)\vartriangle L_{\mathsf{U}}(\mathfrak{c}_2)}{}^*\otimes\mathrm{id}_{L_{\mathsf{V}}(\mathfrak{c}_2)}$ can thus be written as $(L_{\mathsf{U}}(\mathfrak{c}_1)\vartriangle(L_{\mathsf{V}}(\mathfrak{c}_1)\vartriangle L_{\mathsf{U}}(\mathfrak{c}_2))\vartriangle L_{\mathsf{V}}(\mathfrak{c}_2), L_{\mathsf{U}}(\mathfrak{c}_1)\vartriangle L(L_{\mathsf{V}}(\mathfrak{c}_1)\vartriangle L_{\mathsf{U}}(\mathfrak{c}_2))\vartriangle L_{\mathsf{V}}(\mathfrak{c}_2),\mathrm{pm}_{f_1\circledcirc s^{-1}\circledcirc f_2})$ by Lemma 5.35. Similarly, if $e\coloneqq\tau_{m_{\mathsf{U}}^2,m_{\mathsf{V}}^1}$, then $L_{\otimes,\mathfrak{c}_1,\mathfrak{c}_2} = (L(\mathfrak{c}_1)\vartriangle L(\mathfrak{c}_2), L(\mathfrak{c}_1\vartriangle\mathfrak{c}_2),\mathrm{pm}_{f_1\circledcirc e\circledcirc f_2})$ by Lemma 5.35 because $\mu_{m_{\mathsf{U}}^1,m_{\mathsf{U}}^2,m_{\mathsf{V}}^1,m_{\mathsf{V}}^2} = f_1\circledcirc e\circledcirc f_2$ by Lemma 5.22 (b). Thus, since $L(\mathfrak{c}_1)\vartriangle L(\mathfrak{c}_2) = (L_{\mathsf{U}}(\mathfrak{c}_1)\vartriangle L_{\mathsf{V}}(\mathfrak{c}_2))\vartriangle(L_{\mathsf{U}}(\mathfrak{c}_1)\vartriangle L_{\mathsf{V}}(\mathfrak{c}_2))$ by definition, the claimed identity holds if and only if both $L(\mathfrak{c}_1\vartriangle\mathfrak{c}_2) = L_{\mathsf{U}}(\mathfrak{c}_1)\vartriangle L(L_{\mathsf{V}}(\mathfrak{c}_1)\vartriangle L_{\mathsf{U}}(\mathfrak{c}_2))\vartriangle L_{\mathsf{V}}(\mathfrak{c}_2)$ and $e = s^{-1}$.

Step 1: Characterizing $\mathsf{U}(L_{\mathsf{V}}(\mathfrak{c}_1)\vartriangle L_{\mathsf{U}}(\mathfrak{c}_2),\varnothing)$ *and* $\mathsf{V}(L_{\mathsf{V}}(\mathfrak{c}_1)\vartriangle L_{\mathsf{U}}(\mathfrak{c}_2),\varnothing)$. As a first auxiliary statement, we show that, if $\mathsf{S}_1\coloneqq\Pi_0^{m_{\mathsf{V}}^1}$ and $\mathsf{S}_2\coloneqq\Pi_0^{m_{\mathsf{V}}^1+m_{\mathsf{U}}^2}\backslash\Pi_0^{m_{\mathsf{V}}^1}$, then $\mathsf{U}(L_{\mathsf{V}}(\mathfrak{c}_1)\vartriangle L_{\mathsf{U}}(\mathfrak{c}_2),\varnothing) = \mathsf{S}_2$ and $\mathsf{V}(L_{\mathsf{V}}(\mathfrak{c}_1)\vartriangle L_{\mathsf{U}}(\mathfrak{c}_2),\varnothing) = \mathsf{S}_1$.

Because $\mathsf{S}_1 \uplus \mathsf{S}_2$ is the whole area of definition $\Pi_0^{m_{\mathsf{V}}^1+m_{\mathsf{U}}^2}$ of $L_{\mathsf{V}}(\mathfrak{c}_1) \vartriangle L_{\mathsf{U}}(\mathfrak{c}_2)$ we can decompose $\mathsf{U}(L_{\mathsf{V}}(\mathfrak{c}_1) \vartriangle L_{\mathsf{U}}(\mathfrak{c}_2),\varnothing)$ into $\bigcup_{t=1}^2(\mathsf{U}(L_{\mathsf{V}}(\mathfrak{c}_1) \vartriangle L_{\mathsf{U}}(\mathfrak{c}_2),\varnothing) \cap \mathsf{S}_t)$. As $\mathsf{S}_t = \operatorname{ran}(\gamma_{\mathsf{S}_t,0}{}^{m_{\mathsf{V}}^1+m_{\mathsf{U}}^2})$ for each $t \in [\![2]\!]$ and as $\mathsf{U}(L_{\mathsf{V}}(\mathfrak{c}_1) \vartriangle L_{\mathsf{U}}(\mathfrak{c}_2),\varnothing) = \xi_\varnothing^{L_{\mathsf{V}}(\mathfrak{c}_1)\vartriangle L_{\mathsf{U}}(\mathfrak{c}_2)\leftarrow}(\mathbb{Z}_w)$ by definition that is identical to $\bigcup_{t=1}^2(\gamma_{\mathsf{S}_t,0}{}^{m_{\mathsf{V}}^1+m_{\mathsf{U}}^2}{}_{\rightarrow}\circ\gamma_{\mathsf{S}_t,0}{}^{m_{\mathsf{V}}^1+m_{\mathsf{U}}^2\leftarrow}\circ\xi_\varnothing^{L_{\mathsf{V}}(\mathfrak{c}_1)\vartriangle L_{\mathsf{U}}(\mathfrak{c}_2)\leftarrow})(\mathbb{Z}_w)$ or, equivalently, $\bigcup_{t=1}^2(\gamma_{\mathsf{S}_t,0}{}^{m_{\mathsf{V}}^1+m_{\mathsf{U}}^2}{}_{\rightarrow}\circ(\xi_\varnothing^{L_{\mathsf{V}}(\mathfrak{c}_1)\vartriangle L_{\mathsf{U}}(\mathfrak{c}_2)}\circ\gamma_{\mathsf{S}_t,0}{}^{m_{\mathsf{V}}^1+m_{\mathsf{U}}^2})^{\leftarrow})(\mathbb{Z}_w)$. Since $\xi_\varnothing^{L_{\mathsf{V}}(\mathfrak{c}_1)\vartriangle L_{\mathsf{U}}(\mathfrak{c}_2)}\circ \gamma_{\mathsf{S}_1,0}{}^{m_{\mathsf{V}}^1+m_{\mathsf{U}}^2} = \xi_\varnothing^{L_{\mathsf{V}}(\mathfrak{c}_1)}$ and $\xi_\varnothing^{L_{\mathsf{V}}(\mathfrak{c}_1)\vartriangle L_{\mathsf{U}}(\mathfrak{c}_2)} \circ \gamma_{\mathsf{S}_2,0}{}^{m_{\mathsf{V}}^1+m_{\mathsf{U}}^2} = \xi_\varnothing^{L_{\mathsf{U}}(\mathfrak{c}_2)}$ by Lemma 4.2 (c) we have thus shown that $\mathsf{U}(L_{\mathsf{V}}(\mathfrak{c}_1) \vartriangle L_{\mathsf{U}}(\mathfrak{c}_2),\varnothing)$ is given by $(\gamma_{\mathsf{S}_1,0}{}^{m_{\mathsf{V}}^1+m_{\mathsf{U}}^2}{}_{\rightarrow} \circ \xi_\varnothing^{L_{\mathsf{V}}(\mathfrak{c}_1)\leftarrow})(\mathbb{Z}_w) \uplus (\gamma_{\mathsf{S}_2,0}{}^{m_{\mathsf{V}}^1+m_{\mathsf{U}}^2}{}_{\rightarrow} \circ \xi_\varnothing^{L_{\mathsf{U}}(\mathfrak{c}_2)\leftarrow})(\mathbb{Z}_w)$. Of course, $\xi_\varnothing^{L_{\mathsf{V}}(\mathfrak{c}_1)\leftarrow}(\mathbb{Z}_w) = \varnothing$ and $\xi_\varnothing^{L_{\mathsf{U}}(\mathfrak{c}_2)\leftarrow}(\mathbb{Z}_w) = \Pi_0^{m_{\mathsf{U}}^2}$ by definition. Thus we conclude that $\mathsf{U}(L_{\mathsf{V}}(\mathfrak{c}_1) \vartriangle L_{\mathsf{U}}(\mathfrak{c}_2),\varnothing) = \gamma_{\mathsf{S}_2,0}{}^{m_{\mathsf{V}}^1+m_{\mathsf{U}}^2}{}_{\rightarrow}(\Pi_0^{m_{\mathsf{U}}^2}) = \operatorname{ran}(\gamma_{\mathsf{S}_2,0}{}^{m_{\mathsf{V}}^1+m_{\mathsf{U}}^2}) = \mathsf{S}_2$, which proves the first one of the two claimed identities.

In an analogous manner we can decompose the set $\mathsf{V}(L_{\mathsf{V}}(\mathfrak{c}_1) \vartriangle L_{\mathsf{U}}(\mathfrak{c}_2),\varnothing) = \xi_\varnothing^{L_{\mathsf{V}}(\mathfrak{c}_1)\vartriangle L_{\mathsf{U}}(\mathfrak{c}_2)\leftarrow}(\{\aleph\})$ into $\bigcup_{t=1}^2(\gamma_{\mathsf{S}_t,0}{}^{m_{\mathsf{V}}^1+m_{\mathsf{U}}^2}{}_{\rightarrow}\circ(\xi_\varnothing^{L_{\mathsf{V}}(\mathfrak{c}_1)\vartriangle L_{\mathsf{U}}(\mathfrak{c}_2)}\circ\gamma_{\mathsf{S}_t,0}{}^{m_{\mathsf{V}}^1+m_{\mathsf{U}}^2})^{\leftarrow})(\{\aleph\})$ or, equivalently, $(\gamma_{\mathsf{S}_1,0}{}^{m_{\mathsf{V}}^1+m_{\mathsf{U}}^2}{}_{\rightarrow}\circ\xi_\varnothing^{L_{\mathsf{V}}(\mathfrak{c}_1)\leftarrow})(\{\aleph\})\uplus(\gamma_{\mathsf{S}_2,0}{}^{m_{\mathsf{V}}^1+m_{\mathsf{U}}^2}{}_{\rightarrow}\circ\xi_\varnothing^{L_{\mathsf{U}}(\mathfrak{c}_2)\leftarrow})(\{\aleph\})$. Hence, the facts that $\xi_\varnothing^{L_{\mathsf{V}}(\mathfrak{c}_1)\leftarrow}(\{\aleph\}) = \Pi_0^{m_{\mathsf{V}}^1}$ and $\xi_\varnothing^{L_{\mathsf{U}}(\mathfrak{c}_2)\leftarrow}(\{\aleph\}) = \varnothing$ imply that $\mathsf{V}(L_{\mathsf{V}}(\mathfrak{c}_1) \vartriangle L_{\mathsf{U}}(\mathfrak{c}_2),\varnothing)$ is given by S_1. That is what we needed to see.

Step 2: Proving $e = s^{-1}$. Because $\mathsf{S}_1 = \operatorname{ran}(\lhd_{0,0}^{m_{\mathsf{V}}^1,m_{\mathsf{U}}^2})$ and $\mathsf{S}_2 = \operatorname{ran}(\rhd_{0,0}^{m_{\mathsf{V}}^1,m_{\mathsf{U}}^2})$ a different way of expressing the result of Step 1 is to say that $\gamma_{\mathsf{U}(L_{\mathsf{V}}(\mathfrak{c}_1)\vartriangle L_{\mathsf{U}}(\mathfrak{c}_2),\varnothing),0}{}^{m_{\mathsf{V}}^1+m_{\mathsf{U}}^2} = \rhd_{0,0}^{m_{\mathsf{V}}^1,m_{\mathsf{U}}^2}$ and $\gamma_{\mathsf{V}(L_{\mathsf{V}}(\mathfrak{c}_1)\vartriangle L_{\mathsf{U}}(\mathfrak{c}_2),\varnothing),0}{}^{m_{\mathsf{V}}^1+m_{\mathsf{U}}^2} = \lhd_{0,0}^{m_{\mathsf{V}}^1,m_{\mathsf{U}}^2}$. Hence, the definitions imply that $t_\varnothing^{L_{\mathsf{V}}(\mathfrak{c}_1)\vartriangle L_{\mathsf{U}}(\mathfrak{c}_2)} = \gamma_{\mathsf{U}(L_{\mathsf{V}}(\mathfrak{c}_1)\vartriangle L_{\mathsf{U}}(\mathfrak{c}_2),\varnothing),0}{}^{m_{\mathsf{V}}^1+m_{\mathsf{U}}^2} \vartriangle \gamma_{\mathsf{V}(L_{\mathsf{V}}(\mathfrak{c}_1)\vartriangle L_{\mathsf{U}}(\mathfrak{c}_2),\varnothing),0}{}^{m_{\mathsf{V}}^1+m_{\mathsf{U}}^2} = \rhd_{0,0}^{m_{\mathsf{V}}^1,m_{\mathsf{U}}^2} \vartriangle \lhd_{0,0}^{m_{\mathsf{V}}^1,m_{\mathsf{U}}^2} = \tau_{0,0}^{m_{\mathsf{V}}^1,m_{\mathsf{U}}^2}$. Since $\tau_{0,0}^{m_{\mathsf{V}}^1,m_{\mathsf{U}}^2} \circ \blacktriangle_{m_{\mathsf{V}}^1+m_{\mathsf{U}}^2,0} = \blacktriangle_{m_{\mathsf{V}}^1+m_{\mathsf{U}}^2,0} \circ \tau_{m_{\mathsf{V}}^1,m_{\mathsf{U}}^2} = \blacktriangle_{m_{\mathsf{V}}^1+m_{\mathsf{U}}^2,0} \circ e^{-1}$ we can thus infer from the definition of s the identity $\blacktriangle_{m_{\mathsf{V}}^1+m_{\mathsf{U}}^2,0} \circ s = t_\varnothing^{L_{\mathsf{V}}(\mathfrak{c}_1)\vartriangle L_{\mathsf{U}}(\mathfrak{c}_2)} \circ \blacktriangle_{m_{\mathsf{V}}^1+m_{\mathsf{U}}^2,0} = \blacktriangle_{m_{\mathsf{V}}^1+m_{\mathsf{U}}^2,0} \circ e^{-1}$. That proves $s = e^{-1}$ because $\blacktriangle_{m_{\mathsf{V}}^1+m_{\mathsf{U}}^2,0}$ is injective. Hence, $e = s^{-1}$, as claimed.

Step 3: Proving $L(\mathfrak{c}_1 \vartriangle \mathfrak{c}_2) = L_{\mathsf{U}}(\mathfrak{c}_1) \vartriangle L(L_{\mathsf{V}}(\mathfrak{c}_1) \vartriangle L_{\mathsf{U}}(\mathfrak{c}_2)) \vartriangle L_{\mathsf{V}}(\mathfrak{c}_2)$. If we abbreviate

$$\mathsf{T}_1 \coloneqq \Pi_0^{m_{\mathsf{U}}^1}, \qquad \mathsf{T}_2 \coloneqq \Pi_0^{m_{\mathsf{U}}^1+m_{\mathsf{U}}^2}\backslash\Pi_0^{m_{\mathsf{U}}^1},$$
$$\mathsf{T}_3 \coloneqq \Pi_0^{m_{\mathsf{U}}^1+m_{\mathsf{U}}^2+m_{\mathsf{V}}^1}\backslash\Pi_0^{m_{\mathsf{U}}^1+m_{\mathsf{U}}^2}, \quad \mathsf{T}_4 \coloneqq \Pi_0^{m_{\mathsf{U}}^1+m_{\mathsf{U}}^2+m_{\mathsf{V}}^1+m_{\mathsf{V}}^2}\backslash\Pi_0^{m_{\mathsf{U}}^1+m_{\mathsf{U}}^2+m_{\mathsf{V}}^1},$$

then $\bigcup_{u=1}^4 \mathsf{T}_u = \Pi_0^{k_1+k_2}$. Therefore we can prove that $L(\mathfrak{c}_1 \vartriangle \mathfrak{c}_2)$ and $\mathfrak{e} \coloneqq L_{\mathsf{U}}(\mathfrak{c}_1) \vartriangle L(L_{\mathsf{V}}(\mathfrak{c}_1) \vartriangle L_{\mathsf{U}}(\mathfrak{c}_2)) \vartriangle L_{\mathsf{V}}(\mathfrak{c}_2)$ agree by showing that for any $u \in [\![4]\!]$,

$$\xi_\varnothing^{L(\mathfrak{c}_1\vartriangle\mathfrak{c}_2)} \circ \gamma_{\mathsf{T}_u,0}{}^{k_1+k_2} = \xi_\varnothing^{\mathfrak{e}} \circ \gamma_{\mathsf{T}_u,0}{}^{k_1+k_2} \quad \wedge \quad \zeta_\varnothing^{L(\mathfrak{c}_1\vartriangle\mathfrak{c}_2)} \circ \gamma_{\mathsf{T}_u,0}{}^{k_1+k_2} = \zeta_\varnothing^{\mathfrak{e}} \circ \gamma_{\mathsf{T}_u,0}{}^{k_1+k_2}.$$

Those are the eight identities we will check in the following.

Step 3.1: Key auxiliary statements. We will use the facts that

$${}_{\aleph}\Sigma_\varnothing^{L_{\mathsf{V}}(\mathfrak{c}_1)} = {}_{\aleph}\Sigma_\varnothing^{\mathfrak{c}_1} \quad \wedge \quad \xi_\varnothing^{L_{\mathsf{U}}(\mathfrak{c}_2)} = \varepsilon_\varnothing^{L_{\mathsf{U}}(\mathfrak{c}_2)}$$

Indeed, because, by definition, $\xi_{\varnothing}^{L_{\mathsf{V}}(\mathfrak{c}_1)} = \xi_{\varnothing}^{\mathfrak{c}_1} \circ \gamma_{\mathsf{V}(\mathfrak{c}_1,\varnothing),0}{}^{k_1}$ and $\zeta_{\varnothing}^{L_{\mathsf{V}}(\mathfrak{c}_1)} = \zeta_{\varnothing}^{\mathfrak{c}_1} \circ \gamma_{\mathsf{V}(\mathfrak{c}_1,\varnothing),0}{}^{k_1}$ Lemma (a) (a) implies ${}_{\aleph}\Sigma_{\varnothing}^{L_{\mathsf{V}}(\mathfrak{c}_1)} = {}_{\aleph}\sigma_{\varnothing}^{L_{\mathsf{V}}(\mathfrak{c}_1)}(\Pi_0^{m_{\mathsf{V}}^1}) = {}_{\aleph}\sigma_{\varnothing}^{\mathfrak{c}_1}(\Pi_0^{k_1}) = {}_{\aleph}\Sigma_{\varnothing}^{\mathfrak{c}_1}$ by $\Pi_0^{m_{\mathsf{V}}^1} = \gamma_{\mathsf{V}(\mathfrak{c}_1,\varnothing),0}{}^{k_1\leftarrow}(\Pi_0^{k_1})$ and ${}_{\aleph}\sigma_{\varnothing}^{\mathfrak{c}_1}(\Pi_0^{k_1}\backslash\mathsf{V}(\mathfrak{c}_1,\varnothing)) = {}_{\aleph}\sigma_{\varnothing}^{\mathfrak{c}_1}(\mathsf{U}(\mathfrak{c}_1,\varnothing)) = 0$.

Similarly, because $\mathsf{V}(L_{\mathsf{U}}(\mathfrak{c}_2),\varnothing) = \varnothing$, for any $i \in [\![m_{\mathsf{U}}^2]\!]$ with $i \neq 1$, necessarily, ${}_{\aleph}\sigma_{\varnothing}^{L_{\mathsf{U}}(\mathfrak{c}_2)}(]\,{}^{\blacksquare}i, {}^{\blacksquare}1]_0^{m_{\mathsf{U}}^2}) = 0$ and thus $\varepsilon_{\varnothing}^{L_{\mathsf{U}}(\mathfrak{c}_2)}({}^{\blacksquare}i) = \xi_{\varnothing}^{L_{\mathsf{U}}(\mathfrak{c}_2)}({}^{\blacksquare}i)$ by definition.

Step 3.2: Auxiliary definitions. In addition to $\{\mathsf{T}_u\}_{u=1}^4$ we will need to consider other set-theoretical partitions of $\Pi_0^{k_1+k_2}$ as well, and to use certain relations between them and $\{\mathsf{T}_u\}_{u=1}^4$.

Throughout, let also $\mathsf{H}_1 \coloneqq \Pi_0^{k_1}$ and $\mathsf{H}_2 \coloneqq \Pi_0^{k_1+k_2}\backslash\Pi_0^{k_1}$, and $\mathsf{G}_{\mathsf{U}} \coloneqq \mathsf{T}_1 \uplus \mathsf{T}_2$ and $\mathsf{G}_{\mathsf{V}} \coloneqq \mathsf{T}_3 \uplus \mathsf{T}_4$, and $\mathsf{W}_1 = \mathsf{T}_1$ and $\mathsf{W}_2 = \mathsf{T}_2 \uplus \mathsf{T}_3$ and $\mathsf{W}_3 = \mathsf{T}_4$, and $\mathsf{I}_{\mathsf{U}}^1 \coloneqq \Pi_0^{m_{\mathsf{U}}^1}$ and $\mathsf{I}_{\mathsf{U}}^2 \coloneqq \Pi_0^{m_{\mathsf{U}}}\backslash\Pi_0^{m_{\mathsf{U}}^1}$ and $\mathsf{I}_{\mathsf{V}}^1 \coloneqq \Pi_0^{m_{\mathsf{V}}^1}$ and $\mathsf{I}_{\mathsf{V}}^2 \coloneqq \Pi_0^{m_{\mathsf{V}}}\Pi_0^{m_{\mathsf{V}}^1}$, and $\mathsf{K}_{\mathsf{U}} \coloneqq \Pi_0^{m_{\mathsf{U}}^2}$ and $\mathsf{K}_{\mathsf{V}} \coloneqq \Pi_0^{m_{\mathsf{V}}^1+m_{\mathsf{U}}^2}\backslash\Pi_0^{m_{\mathsf{U}}^2}$. Note that then,

$$
\begin{aligned}
\gamma_{\mathsf{T}_1,0}{}^{k_1+k_2} &= \gamma_{\mathsf{G}_{\mathsf{U}},0}{}^{k_1+k_2} \circ \gamma_{\mathsf{I}_{\mathsf{U}}^1,0}{}^{m_{\mathsf{U}}} = \gamma_{\mathsf{W}_1,0}{}^{k_1+k_2}\\
\gamma_{\mathsf{T}_2,0}{}^{k_1+k_2} &= \gamma_{\mathsf{G}_{\mathsf{U}},0}{}^{k_1+k_2} \circ \gamma_{\mathsf{I}_{\mathsf{U}}^2,0}{}^{m_{\mathsf{U}}} = \gamma_{\mathsf{W}_2,0}{}^{k_1+k_2} \circ \gamma_{\mathsf{K}_{\mathsf{U}},0}{}^{m_{\mathsf{V}}^1+m_{\mathsf{U}}^2}\\
\gamma_{\mathsf{T}_3,0}{}^{k_1+k_2} &= \gamma_{\mathsf{G}_{\mathsf{V}},0}{}^{k_1+k_2} \circ \gamma_{\mathsf{I}_{\mathsf{V}}^1,0}{}^{m_{\mathsf{V}}} = \gamma_{\mathsf{W}_2,0}{}^{k_1+k_2} \circ \gamma_{\mathsf{K}_{\mathsf{V}},0}{}^{m_{\mathsf{V}}^1+m_{\mathsf{U}}^2}\\
\gamma_{\mathsf{T}_4,0}{}^{k_1+k_2} &= \gamma_{\mathsf{G}_{\mathsf{V}},0}{}^{k_1+k_2} \circ \gamma_{\mathsf{I}_{\mathsf{V}}^2,0}{}^{m_{\mathsf{V}}} = \gamma_{\mathsf{W}_3,0}{}^{k_1+k_2}.
\end{aligned}
$$

Step 3.3: First unwinding the definitions. Furthemore, recognize that by Lemma 4.2 (c) the definition $L(\mathfrak{c}_1 \vartriangle \mathfrak{c}_2) = L_{\mathsf{U}}(\mathfrak{c}_1 \vartriangle \mathfrak{c}_2) \vartriangle L_{\mathsf{V}}(\mathfrak{c}_1 \vartriangle \mathfrak{c}_2)$ ensures that

$$
\begin{aligned}
&\xi_{\varnothing}^{L(\mathfrak{c}_1\vartriangle\mathfrak{c}_2)} \circ \gamma_{\mathsf{G}_{\mathsf{U}},0}{}^{k_1+k_2} = \xi_{\varnothing}^{L_{\mathsf{U}}(\mathfrak{c}_1\vartriangle\mathfrak{c}_2)} \quad &&\wedge \quad \zeta_{\varnothing}^{L(\mathfrak{c}_1\vartriangle\mathfrak{c}_2)} \circ \gamma_{\mathsf{G}_{\mathsf{U}},0}{}^{k_1+k_2} = \zeta_{\varnothing}^{L_{\mathsf{U}}(\mathfrak{c}_1\vartriangle\mathfrak{c}_2)}\\
&\xi_{\varnothing}^{L(\mathfrak{c}_1\vartriangle\mathfrak{c}_2)} \circ \gamma_{\mathsf{G}_{\mathsf{V}},0}{}^{k_1+k_2} = \xi_{\varnothing}^{L_{\mathsf{V}}(\mathfrak{c}_1\vartriangle\mathfrak{c}_2)} \quad &&\wedge \quad \zeta_{\varnothing}^{L(\mathfrak{c}_1\vartriangle\mathfrak{c}_2)} \circ \gamma_{\mathsf{G}_{\mathsf{V}},0}{}^{k_1+k_2} = \zeta_{\varnothing}^{L_{\mathsf{V}}(\mathfrak{c}_1\vartriangle\mathfrak{c}_2)}.
\end{aligned}
$$

By the same reasoning, moreover,

$$
\begin{aligned}
\xi_{\varnothing}^{L(L_{\mathsf{V}}(\mathfrak{c}_1)\vartriangle L_{\mathsf{U}}(\mathfrak{c}_2))} \circ \gamma_{\mathsf{K}_{\mathsf{U}},0}{}^{m_{\mathsf{V}}^1+m_{\mathsf{U}}^2} &= \xi_{\varnothing}^{L_{\mathsf{U}}(L_{\mathsf{V}}(\mathfrak{c}_1)\vartriangle L_{\mathsf{U}}(\mathfrak{c}_2))}\\
\zeta_{\varnothing}^{L(L_{\mathsf{V}}(\mathfrak{c}_1)\vartriangle L_{\mathsf{U}}(\mathfrak{c}_2))} \circ \gamma_{\mathsf{K}_{\mathsf{U}},0}{}^{m_{\mathsf{V}}^1+m_{\mathsf{U}}^2} &= \zeta_{\varnothing}^{L_{\mathsf{U}}(L_{\mathsf{V}}(\mathfrak{c}_1)\vartriangle L_{\mathsf{U}}(\mathfrak{c}_2))}\\
\xi_{\varnothing}^{L(L_{\mathsf{V}}(\mathfrak{c}_1)\vartriangle L_{\mathsf{U}}(\mathfrak{c}_2))} \circ \gamma_{\mathsf{K}_{\mathsf{V}},0}{}^{m_{\mathsf{V}}^1+m_{\mathsf{U}}^2} &= \xi_{\varnothing}^{L_{\mathsf{V}}(L_{\mathsf{V}}(\mathfrak{c}_1)\vartriangle L_{\mathsf{U}}(\mathfrak{c}_2))}\\
\zeta_{\varnothing}^{L(L_{\mathsf{V}}(\mathfrak{c}_1)\vartriangle L_{\mathsf{U}}(\mathfrak{c}_2))} \circ \gamma_{\mathsf{K}_{\mathsf{V}},0}{}^{m_{\mathsf{V}}^1+m_{\mathsf{U}}^2} &= \zeta_{\varnothing}^{L_{\mathsf{V}}(L_{\mathsf{V}}(\mathfrak{c}_1)\vartriangle L_{\mathsf{U}}(\mathfrak{c}_2))}.
\end{aligned}
$$

And, finally, an iterated application of Lemma 4.2 (c) to $\mathfrak{e} = L_{\mathsf{U}}(\mathfrak{c}_1) \vartriangle (L_{\mathsf{V}}(\mathfrak{c}_1) \vartriangle L_{\mathsf{U}}(\mathfrak{c}_2)) \vartriangle L_{\mathsf{V}}(\mathfrak{c}_2)$ shows that

$$
\begin{aligned}
&\xi_{\varnothing}^{\mathfrak{e}} \circ \gamma_{\mathsf{W}_1,0}{}^{k_1+k_2} = \xi_{\varnothing}^{L_{\mathsf{U}}(\mathfrak{c}_1)} \quad &&\wedge \quad \zeta_{\varnothing}^{\mathfrak{e}} \circ \gamma_{\mathsf{W}_1,0}{}^{k_1+k_2} = \zeta_{\varnothing}^{L_{\mathsf{U}}(\mathfrak{c}_1)}\\
&\xi_{\varnothing}^{\mathfrak{e}} \circ \gamma_{\mathsf{W}_2,0}{}^{k_1+k_2} = \xi_{\varnothing}^{L(L_{\mathsf{V}}(\mathfrak{c}_1)\vartriangle L_{\mathsf{U}}(\mathfrak{c}_2))} \quad &&\wedge \quad \zeta_{\varnothing}^{\mathfrak{e}} \circ \gamma_{\mathsf{W}_2,0}{}^{k_1+k_2} = \zeta_{\varnothing}^{L(L_{\mathsf{V}}(\mathfrak{c}_1)\vartriangle L_{\mathsf{U}}(\mathfrak{c}_2))}\\
&\xi_{\varnothing}^{\mathfrak{e}} \circ \gamma_{\mathsf{W}_3,0}{}^{k_1+k_2} = \xi_{\varnothing}^{L_{\mathsf{V}}(\mathfrak{c}_2)} \quad &&\wedge \quad \zeta_{\varnothing}^{\mathfrak{e}} \circ \gamma_{\mathsf{W}_3,0}{}^{k_1+k_2} = \zeta_{\varnothing}^{L_{\mathsf{V}}(\mathfrak{c}_2)}.
\end{aligned}
$$

Step 3.4: Proving the eight identities. It remains to use the definitions of Step 3.2 and the results of Steps 3.1 and 3.3 to prove the eight identities. Let V denote the action of $\mathbb{Z}_w$ on itself.

Step 3.4.1: Tags in area T_1. By Step 3.2 we can rewrite the mapping $\xi_\varnothing^{L(\mathfrak{c}_1 \vartriangle \mathfrak{c}_2)} \circ \gamma_{\mathsf{T}_1,0}{}^{k_1+k_2}$ as $\xi_\varnothing^{L(\mathfrak{c}_1 \vartriangle \mathfrak{c}_2)} \circ \gamma_{\mathsf{G}_\mathsf{U},0}{}^{k_1+k_2} \circ \gamma_{\mathsf{I}^1_\mathsf{U},0}{}^{m_\mathsf{U}}$. According to Step 3.3 that is the same as $\xi_\varnothing^{L_\mathsf{U}(\mathfrak{c}_1 \vartriangle \mathfrak{c}_2)} \circ \gamma_{\mathsf{I}^1_\mathsf{U},0}{}^{m_\mathsf{U}}$. By the definition of L_U that mapping in turn can be expressed as $\pi_w \circ \varepsilon_\varnothing^{\mathfrak{c}_1 \vartriangle \mathfrak{c}_2} \circ \gamma_{\mathsf{U}(\mathfrak{c}_1 \vartriangle \mathfrak{c}_2,\varnothing),0}{}^{k_1+k_2} \circ \gamma_{\mathsf{I}^1_\mathsf{U},0}{}^{m_\mathsf{U}}$. By Lemma 5.31 (b) this coincides with $\pi_w \circ \varepsilon_\varnothing^{\mathfrak{c}_1 \vartriangle \mathfrak{c}_2} \circ \gamma_{\mathsf{H}_1,0}{}^{k_1+k_2} \circ \gamma_{\mathsf{U}(\mathfrak{c}_1,\varnothing),0}{}^{k_1}$. Since the restrictions of $\varepsilon_\varnothing^{\mathfrak{c}_1 \vartriangle \mathfrak{c}_2} \circ \gamma_{\mathsf{H}_1,0}{}^{k_1+k_2}$ and $\varepsilon_\varnothing^{\mathfrak{c}_1}$ to I^1_U coincide by Lemma 4.43 (d), we can rewrite the previous expression as $\pi_w \circ \varepsilon_\varnothing^{\mathfrak{c}_1} \circ \gamma_{\mathsf{U}(\mathfrak{c}_1,\varnothing),0}{}^{k_1}$. Of course, by the definition of L_U that mapping is precisely $\xi_\varnothing^{L_\mathsf{U}(\mathfrak{c}_1)}$. By Step 3.3 this in turn is another way of writing the mapping $\xi_\varnothing^{\mathfrak{c}} \circ \gamma_{\mathsf{W}_1,0}{}^{k_1+k_2}$. Hence, by Step 3.2 the mappings $\xi_\varnothing^{L(\mathfrak{c}_1 \vartriangle \mathfrak{c}_2)} \circ \gamma_{\mathsf{T}_1,0}{}^{k_1+k_2}$ and $\xi_\varnothing^{\mathfrak{c}} \circ \gamma_{\mathsf{T}_1,0}{}^{k_1+k_2}$ are one and the same.

Step 3.4.2: Colors in area T_1. The argument for the colors is similar but easier. Namely, $\zeta_\varnothing^{L(\mathfrak{c}_1 \vartriangle \mathfrak{c}_2)} \circ \gamma_{\mathsf{T}_1,0}{}^{k_1+k_2}$ is the same as $\zeta_\varnothing^{L(\mathfrak{c}_1 \vartriangle \mathfrak{c}_2)} \circ \gamma_{\mathsf{G}_\mathsf{U},0}{}^{k_1+k_2} \circ \gamma_{\mathsf{I}^1_\mathsf{U},0}{}^{m_\mathsf{U}}$ by Step 3.2 and thus as $\zeta_\varnothing^{L_\mathsf{U}(\mathfrak{c}_1 \vartriangle \mathfrak{c}_2)} \circ \gamma_{\mathsf{I}^1_\mathsf{U},0}{}^{m_\mathsf{U}}$ by Step 3.3. By the definition of L_U that is $\zeta_\varnothing^{\mathfrak{c}_1 \vartriangle \mathfrak{c}_2} \circ \gamma_{\mathsf{U}(\mathfrak{c}_1 \vartriangle \mathfrak{c}_2,\varnothing),0}{}^{k_1+k_2} \circ \gamma_{\mathsf{I}^1_\mathsf{U},0}{}^{m_\mathsf{U}}$ or, equivalently, $\zeta_\varnothing^{\mathfrak{c}_1 \vartriangle \mathfrak{c}_2} \circ \gamma_{\mathsf{H}_1,0}{}^{k_1+k_2} \circ \gamma_{\mathsf{U}(\mathfrak{c}_1,\varnothing),0}{}^{k_1}$ by Lemma 5.31 (b). According to Lemma 4.2 (c) that agrees with $\zeta_\varnothing^{\mathfrak{c}_1} \circ \gamma_{\mathsf{U}(\mathfrak{c}_1,\varnothing),0}{}^{k_1}$, i.e., with $\zeta_\varnothing^{L_\mathsf{U}(\mathfrak{c}_1)}$ by the definition of L_U. And this is the same as $\zeta_\varnothing^{\mathfrak{c}} \circ \gamma_{\mathsf{W}_1,0}{}^{k_1+k_2}$ by Step 3.3 and thus as $\xi_\varnothing^{\mathfrak{c}} \circ \gamma_{\mathsf{T}_1,0}{}^{k_1+k_2}$ by Step 3.2.

Step 3.4.3: Tags in area T_2. For the tags in T_2 we can adapt the argument from T_1 with the help of Step 3.1. The mapping $\xi_\varnothing^{L(\mathfrak{c}_1 \vartriangle \mathfrak{c}_2)} \circ \gamma_{\mathsf{T}_2,0}{}^{k_1+k_2}$ is identical to $\xi_\varnothing^{L(\mathfrak{c}_1 \vartriangle \mathfrak{c}_2)} \circ \gamma_{\mathsf{G}_\mathsf{U},0}{}^{k_1+k_2} \circ \gamma_{\mathsf{I}^2_\mathsf{U},0}{}^{m_\mathsf{U}}$ by Step 3.2 and thus to $\xi_\varnothing^{L_\mathsf{U}(\mathfrak{c}_1 \vartriangle \mathfrak{c}_2)} \circ \gamma_{\mathsf{I}^2_\mathsf{U},0}{}^{m_\mathsf{U}}$ by Step 3.3. This is the same as $\pi_w \circ \varepsilon_\varnothing^{\mathfrak{c}_1 \vartriangle \mathfrak{c}_2} \circ \gamma_{\mathsf{U}(\mathfrak{c}_1 \vartriangle \mathfrak{c}_2,\varnothing),0}{}^{k_1+k_2} \circ \gamma_{\mathsf{I}^2_\mathsf{U},0}{}^{m_\mathsf{U}}$ by the definition of L_U, which can be rewritten in the form $\pi_w \circ \varepsilon_\varnothing^{\mathfrak{c}_1 \vartriangle \mathfrak{c}_2} \circ \gamma_{\mathsf{H}_2,0}{}^{k_1+k_2} \circ \gamma_{\mathsf{U}(\mathfrak{c}_2,\varnothing),0}{}^{k_2}$ by Lemma 5.31 (b). Differently from the case of T_1 the implication of Lemma 4.43 (d) can now be expressed by saying that $\pi_w \circ \varepsilon_\varnothing^{\mathfrak{c}_1 \vartriangle \mathfrak{c}_2} \circ \gamma_{\mathsf{H}_2,0}{}^{k_1+k_2}$ and $V_{-\aleph\Sigma_\varnothing^{\mathfrak{c}_1}} \circ \pi_w \circ \varepsilon_\varnothing^{\mathfrak{c}_2}$ assume the same values on the set I^2_U. Hence, the previous mapping is identical to $V_{-\aleph\Sigma_\varnothing^{\mathfrak{c}_1}} \circ \pi_w \circ \varepsilon_\varnothing^{\mathfrak{c}_2} \circ \gamma_{\mathsf{U}(\mathfrak{c}_2,\varnothing),0}{}^{k_2}$ or, indeed, $V_{-\aleph\Sigma_\varnothing^{\mathfrak{c}_1}} \circ \xi_\varnothing^{L_\mathsf{U}(\mathfrak{c}_2)}$ by the definition of L_U. By both the results of Step 3.1 this is another way of writing $V_{-\aleph\Sigma_\varnothing^{L_\mathsf{V}(\mathfrak{c}_1)}} \circ \varepsilon_\varnothing^{L_\mathsf{U}(\mathfrak{c}_2)}$, which is the same as $V_{-\aleph\Sigma_\varnothing^{L_\mathsf{V}(\mathfrak{c}_1)}} \circ \pi_w \circ \varepsilon_\varnothing^{L_\mathsf{U}(\mathfrak{c}_2)}$ because $\varepsilon_\varnothing^{L_\mathsf{U}(\mathfrak{c}_2)}$ already only takes values in $\mathbb{Z}_w$. Because, by applying Lemma 4.43 (d) in reverse, we know that the mappings $V_{-\aleph\Sigma_\varnothing^{L_\mathsf{V}(\mathfrak{c}_1)}} \circ \pi_w \circ \varepsilon_\varnothing^{L_\mathsf{U}(\mathfrak{c}_2)}$ and $\pi_w \circ \varepsilon_\varnothing^{L_\mathsf{V}(\mathfrak{c}_1) \vartriangle L_\mathsf{U}(\mathfrak{c}_2)} \circ \gamma_{\mathsf{S}_2,0}{}^{m^1_\mathsf{V}+m^2_\mathsf{U}}$ agree on the set $\mathsf{U}(L_\mathsf{U}(\mathfrak{c}_2),\varnothing)$, i.e., everywhere, that is identical to $\pi_w \circ \varepsilon_\varnothing^{L_\mathsf{V}(\mathfrak{c}_1) \vartriangle L_\mathsf{U}(\mathfrak{c}_2)} \circ \gamma_{\mathsf{S}_2,0}{}^{m^1_\mathsf{V}+m^2_\mathsf{U}}$. According to Step 1 that is the same as $\pi_w \circ \varepsilon_\varnothing^{L_{\mathsf{V}}(\mathfrak{c}_1)\vartriangle L_{\mathsf{U}}(\mathfrak{c}_2)} \circ \gamma_{\mathsf{U}(L_{\mathsf{V}}(\mathfrak{c}_1)\vartriangle L_{\mathsf{U}}(\mathfrak{c}_2),\varnothing),\varnothing}{}^{m^1_{\mathsf{V}}+m^2_{\mathsf{U}}}$. Since this is precisely $\xi_\varnothing^{L_{\mathsf{U}}(L_{\mathsf{V}}(\mathfrak{c}_1)\vartriangle L_{\mathsf{U}}(\mathfrak{c}_2))}$ by the definition of L_{U} it can also be expressed as $\xi_\varnothing^{L(L_{\mathsf{V}}(\mathfrak{c}_1)\vartriangle L_{\mathsf{U}}(\mathfrak{c}_2))} \circ \gamma_{\mathsf{K}_{\mathsf{U}},0}{}^{m^1_{\mathsf{V}}+m^2_{\mathsf{U}}}$ by Step 3.3. Because, also by Step 3.3, this is the same as $\xi_\varnothing^{\mathfrak{c}} \circ \gamma_{\mathsf{W}_2,0}{}^{k_1+k_2} \circ \gamma_{\mathsf{K}_{\mathsf{U}},0}{}^{m^1_{\mathsf{V}}+m^2_{\mathsf{U}}}$ Step 3.2 shows that we have thus proved $\xi_\varnothing^{L(\mathfrak{c}_1\vartriangle\mathfrak{c}_2)} \circ \gamma_{\mathsf{T}_2,0}{}^{k_1+k_2}$ and $\xi_\varnothing^{\mathfrak{c}} \circ \gamma_{\mathsf{T}_2,0}{}^{k_1+k_2}$ to coincide.

Step 3.4.4: Colors in area T_2. Once more, the reasoning for the colors is very similar to the one for the tags but easier in key respects. Steps 3.2, 3.3 and the definition of L_{U}, respectively, let us rewrite $\zeta_\varnothing^{L(\mathfrak{c}_1\vartriangle\mathfrak{c}_2)} \circ \gamma_{\mathsf{T}_2,0}{}^{k_1+k_2}$ first as $\zeta_\varnothing^{L(\mathfrak{c}_1\vartriangle\mathfrak{c}_2)} \circ \gamma_{\mathsf{G}_{\mathsf{U}},0}{}^{k_1+k_2} \circ \gamma_{\mathsf{I}^2_{\mathsf{U}},0}{}^{m_{\mathsf{U}}}$, then as $\zeta_\varnothing^{L_{\mathsf{U}}(\mathfrak{c}_1\vartriangle\mathfrak{c}_2)} \circ \gamma_{\mathsf{I}^2_{\mathsf{U}},0}{}^{m_{\mathsf{U}}}$ and eventually as $\zeta_\varnothing^{\mathfrak{c}_1\vartriangle\mathfrak{c}_2} \circ \gamma_{\mathsf{U}(\mathfrak{c}_1\vartriangle\mathfrak{c}_2,\varnothing),0}{}^{k_1+k_2} \circ \gamma_{\mathsf{I}^2_{\mathsf{U}},0}{}^{m_{\mathsf{U}}}$. By Lemma 5.31 (b) that is the same as $\zeta_\varnothing^{\mathfrak{c}_1\vartriangle\mathfrak{c}_2} \circ \gamma_{\mathsf{H}_2,0}{}^{k_1+k_2} \circ \gamma_{\mathsf{U}(\mathfrak{c}_2,\varnothing),0}{}^{k_2}$, which by Lemma 4.2 (c) is identical to $\zeta_\varnothing^{\mathfrak{c}_2} \circ \gamma_{\mathsf{U}(\mathfrak{c}_2,\varnothing),0}{}^{k_2}$, i.e., to $\zeta_\varnothing^{L_{\mathsf{U}}(\mathfrak{c}_2)}$ according to the definition of L_{U}. Using, in succession, Step 3.3, Step 1 and the definition of L_{U} again, we can transform this into first $\zeta_\varnothing^{L_{\mathsf{V}}(\mathfrak{c}_1)\vartriangle L_{\mathsf{U}}(\mathfrak{c}_2)} \circ \gamma_{\mathsf{S}_2,0}{}^{m^1_{\mathsf{V}}+m^2_{\mathsf{U}}}$, then $\zeta_\varnothing^{L_{\mathsf{V}}(\mathfrak{c}_1)\vartriangle L_{\mathsf{U}}(\mathfrak{c}_2)} \circ \gamma_{\mathsf{U}(L_{\mathsf{V}}(\mathfrak{c}_1)\vartriangle L_{\mathsf{U}}(\mathfrak{c}_2),\varnothing),\varnothing}{}^{m^1_{\mathsf{V}}+m^2_{\mathsf{U}}}$ and then $\zeta_\varnothing^{L_{\mathsf{U}}(L_{\mathsf{V}}(\mathfrak{c}_1)\vartriangle L_{\mathsf{U}}(\mathfrak{c}_2))}$. By two identities from Step 3.3 this is the same as $\zeta_\varnothing^{L(L_{\mathsf{V}}(\mathfrak{c}_1)\vartriangle L_{\mathsf{U}}(\mathfrak{c}_2))} \circ \gamma_{\mathsf{K}_{\mathsf{U}},0}{}^{m^1_{\mathsf{V}}+m^2_{\mathsf{U}}}$ and thus $\zeta_\varnothing^{\mathfrak{c}} \circ \gamma_{\mathsf{W}_2,0}{}^{k_1+k_2} \circ \gamma_{\mathsf{K}_{\mathsf{U}},0}{}^{m^1_{\mathsf{V}}+m^2_{\mathsf{U}}}$ or, indeed, $\zeta_\varnothing^{\mathfrak{c}} \circ \gamma_{\mathsf{T}_2,0}{}^{k_1+k_2}$ by Step 3.2.

Step 3.4.5: Tags in area T_3. The cases T_3 and T_4 are thankfully easier. We rewrite $\xi_\varnothing^{L(\mathfrak{c}_1\vartriangle\mathfrak{c}_2)} \circ \gamma_{\mathsf{T}_3,0}{}^{k_1+k_2}$ as $\xi_\varnothing^{L(\mathfrak{c}_1\vartriangle\mathfrak{c}_2)} \circ \gamma_{\mathsf{G}_{\mathsf{V}},0}{}^{k_1+k_2} \circ \gamma_{\mathsf{I}^1_{\mathsf{V}},0}{}^{m_{\mathsf{V}}}$ with the help Step 3.2 and thus as $\xi_\varnothing^{L_{\mathsf{V}}(\mathfrak{c}_1\vartriangle\mathfrak{c}_2)} \circ \gamma_{\mathsf{I}^1_{\mathsf{V}},0}{}^{m_{\mathsf{V}}}$, according to Step 3.3. By the definition of L_{V} that is the same as $\xi_\varnothing^{\mathfrak{c}_1\vartriangle\mathfrak{c}_2} \circ \gamma_{\mathsf{V}(\mathfrak{c}_1\vartriangle\mathfrak{c}_2,\varnothing),0}{}^{k_1+k_2} \circ \gamma_{\mathsf{I}^1_{\mathsf{V}},0}{}^{m_{\mathsf{V}}}$ or, equivalently, $\xi_\varnothing^{\mathfrak{c}_1\vartriangle\mathfrak{c}_2} \circ \gamma_{\mathsf{H}_1,0}{}^{k_1+k_2} \circ \gamma_{\mathsf{V}(\mathfrak{c}_1,\varnothing),0}{}^{k_1}$ by Lemma 5.31 (b). Lemma 4.2 (c) lets us transform this into $\xi_\varnothing^{\mathfrak{c}_1} \circ \gamma_{\mathsf{V}(\mathfrak{c}_1,\varnothing),0}{}^{k_1}$, which is to say $\xi_\varnothing^{L_{\mathsf{V}}(\mathfrak{c}_1)}$ by the definition of L_{V}. A second application Lemma 4.2 (c) rewrites this as $\xi_\varnothing^{L_{\mathsf{V}}(\mathfrak{c}_1)\vartriangle L_{\mathsf{U}}(\mathfrak{c}_2)} \circ \gamma_{\mathsf{S}_1,0}{}^{m^1_{\mathsf{V}}+m^2_{\mathsf{U}}}$, which is $\xi_\varnothing^{L_{\mathsf{V}}(\mathfrak{c}_1)\vartriangle L_{\mathsf{U}}(\mathfrak{c}_2)} \circ \gamma_{\mathsf{V}(L_{\mathsf{V}}(\mathfrak{c}_1)\vartriangle L_{\mathsf{U}}(\mathfrak{c}_2),\varnothing),0}{}^{m^1_{\mathsf{V}}+m^2_{\mathsf{U}}}$ by Step 1. The definition of L_{V} is such that this mapping is precisely $\xi_\varnothing^{L_{\mathsf{V}}(L_{\mathsf{V}}(\mathfrak{c}_1)\vartriangle L_{\mathsf{U}}(\mathfrak{c}_2))}$, i.e., $\xi_\varnothing^{L(L_{\mathsf{V}}(\mathfrak{c}_1)\vartriangle L_{\mathsf{U}}(\mathfrak{c}_2))} \circ \gamma_{\mathsf{K}_{\mathsf{V}},0}{}^{m^1_{\mathsf{V}}+m^2_{\mathsf{U}}}$ by Step 3.3. Since this is identical to $\xi_\varnothing^{\mathfrak{c}} \circ \gamma_{\mathsf{W}_2,0}{}^{k_1+k_2} \circ \gamma_{\mathsf{K}_{\mathsf{V}},0}{}^{m^1_{\mathsf{V}}+m^2_{\mathsf{U}}}$ by Step 3.3 and thus $\xi_\varnothing^{\mathfrak{c}} \circ \gamma_{\mathsf{T}_3,0}{}^{k_1+k_2}$ by Step 3.2 the tags in T_3 do agree.

Step 3.4.6: Colors in area T_3. The proof for the colors is exactly the same as for the tags. Merely replace ξ with ζ everywhere.

Step 3.4.7: Tags in area T_4. By Step 3.2 the mapping $\xi_\varnothing^{L(\mathfrak{c}_1\vartriangle\mathfrak{c}_2)} \circ \gamma_{\mathsf{T}_4,0}{}^{k_1+k_2}$ is the same as $\xi_\varnothing^{L(\mathfrak{c}_1\vartriangle\mathfrak{c}_2)} \circ \gamma_{\mathsf{G}_{\mathsf{V}},0}{}^{k_1+k_2} \circ \gamma_{\mathsf{I}^2_{\mathsf{V}},0}{}^{m_{\mathsf{V}}}$, which by Step 3.3 is identical to $\xi_\varnothing^{L_{\mathsf{V}}(\mathfrak{c}_1\vartriangle\mathfrak{c}_2)} \circ \gamma_{\mathsf{I}^2_{\mathsf{V}},0}{}^{m_{\mathsf{V}}}$. This is another way of writing $\xi_\varnothing^{\mathfrak{c}_1\vartriangle\mathfrak{c}_2} \circ \gamma_{\mathsf{V}(\mathfrak{c}_1\vartriangle\mathfrak{c}_2,\varnothing),0}{}^{k_1+k_2} \circ \gamma_{\mathsf{I}^2_{\mathsf{V}},0}{}^{m_{\mathsf{V}}}$ according to the definition of L_{V}. By Lemma 5.31 (b) this can also be expressed as $\xi_\varnothing^{\mathfrak{c}_1\vartriangle\mathfrak{c}_2} \circ \gamma_{\mathsf{H}_2,0}{}^{k_1+k_2} \circ \gamma_{\mathsf{V}(\mathfrak{c}_2,\varnothing),0}{}^{k_2}$, which in turn is the same as $\xi_\varnothing^{\mathfrak{c}_2} \circ \gamma_{\mathsf{V}(\mathfrak{c}_2,\varnothing),0}{}^{k_2}$ by Lemma 4.2 (c), i.e., as $\xi_\varnothing^{L_{\mathsf{V}}(\mathfrak{c}_2)}$ by the definition of L_{V}. And that coincides with $\xi^{\mathfrak{c}}_{\varnothing} \circ \gamma_{\mathsf{W}_3,0}{}^{k_1+k_2}$ by Step 3.3 and thus with $\xi^{\mathfrak{c}}_{\varnothing} \circ \gamma_{\mathsf{T}_4,0}{}^{k_1+k_2}$ by Step 3.2.

Step 3.4.8: Colors in area T_4. As in the case of T_3 the proof for the colors in T_4 is identical to the one for the tags upon replacing ξ with ζ. And that concludes the proof. ☐

5.2.4. *Generating crosses for crossed co-products.* Similar to the generating crosses for graph co-products, "crossing" partitions implement the label-altering transpositions allowed in a crossed co-product. However, closer attention is required to the possible labelings of these crosses than in the graph co-product case. Nonetheless, there is still a lot of flexibility when choosing generators.

DEFINITION 5.38. A *generating set of crosses for the* $\mathbb{Z}_w$*-crossed co-product* with respect to $(\mathfrak{U}, \mathfrak{O})$ is any subset $\mathcal{R}$ of ${}_{\mathfrak{U}\cup\{\aleph\},\mathfrak{O}}\mathcal{S}$ such that for any $z \in \mathfrak{U} \cup \mathfrak{O}$ there exist $\{c', c\} \subseteq \{\circ, \bullet\}$ such that,

(a) if $z \in \mathfrak{U}$, then $((\aleph, c) \otimes (\pi_w(z + \delta_{c,\bullet}), c'), (\pi_w(z + \delta_{c,\circ}), c') \otimes (\aleph, c), \times) \in \mathcal{R}$ or $((\pi_w(z + \delta_{c,\circ}), c') \otimes (\aleph, c), (\aleph, c) \otimes (\pi_w(z + \delta_{c,\bullet}), c'), \times) \in \mathcal{R}$,

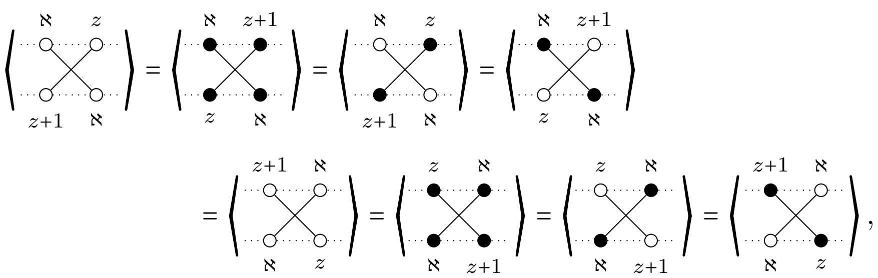
(b) if $z \in \mathfrak{O}$, then $((\aleph, c) \otimes \pi_w(z + \delta_{c,\bullet}), \pi_w(z + \delta_{c,\circ}) \otimes (\aleph, c), \times) \in \mathcal{R}$ or $(\pi_w(z + \delta_{c,\circ}) \otimes (\aleph, c), (\aleph, c) \otimes \pi_w(z + \delta_{c,\bullet}), \times) \in \mathcal{R}$,

where $\times = \{\{{}^{\blacksquare}1, {}_{\blacksquare}2\}, \{{}^{\blacksquare}2, {}_{\blacksquare}1\}\}$.

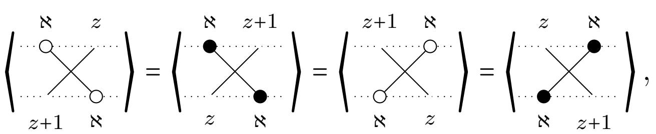
Again, it is clear from the definition that any generating set of crosses for the $\mathbb{Z}_w$-crossed co-product with respect to $(\mathfrak{U}, \mathfrak{O})$ is contained in $\mathcal{X} \rtimes \mathcal{Z}_w$.

LEMMA 5.39. *For any* $z \in \mathbb{Z}_w$*, if* $\langle\,\cdot\,\rangle$ *denotes the generated category of*

(a) $(\{\aleph, z\}, \varnothing)$*-tagged labeled partitions, then*



(b) $(\{\aleph\}, \{z\})$*-tagged labeled partitions, then*



where everywhere, by an abuse of notation, $z + 1$ *stands for* $\pi_w(z + 1)$ *(and thus, in particular, actually for just* z *if* $w = 1$*).*

PROOF. The proofs of the parts (a) and (b) of Lemma 5.13 apply respectively. ☐

LEMMA 5.40. *For any generating set of crosses $\mathcal{R}$ for the $\mathbb{Z}_w$-crossed co-product with respect to $(\mathfrak{U},\mathfrak{O})$, any $z\in\mathfrak{U}\cup\mathfrak{O}$ and any $\{c',c\}\subseteq\{\circ,\bullet\}$,*

(a) if $z\in\mathfrak{U}$, then $((\aleph,c)\otimes(\pi_w(z+\delta_{c,\bullet}),c'),(\pi_w(z+\delta_{c,\circ}),c')\otimes(\aleph,c),\times)\in{}_{\mathfrak{U}\cup\{\aleph\},\mathfrak{O}}\langle\mathcal{R}\rangle$ and $((\pi_w(z+\delta_{c,\circ}),c')\otimes(\aleph,c),(\aleph,c)\otimes(\pi_w(z+\delta_{c,\bullet}),c'),\times)\in{}_{\mathfrak{U}\cup\{\aleph\},\mathfrak{O}}\langle\mathcal{R}\rangle$,

(b) if $z\in\mathfrak{U}$, then $((\aleph,c)\otimes\pi_w(z+\delta_{c,\bullet}),\pi_w(z+\delta_{c,\circ})\otimes(\aleph,c),\times)\in{}_{\mathfrak{U}\cup\{\aleph\},\mathfrak{O}}\langle\mathcal{R}\rangle$ and $(\pi_w(z+\delta_{c,\circ})\otimes(\aleph,c),(\aleph,c)\otimes\pi_w(z+\delta_{c,\bullet}),\times)\in{}_{\mathfrak{U}\cup\{\aleph\},\mathfrak{O}}\langle\mathcal{R}\rangle$,

PROOF. Follows immediately from the definitions and Lemma 5.39. □

5.2.5. *From generating crosses to all partitions in crossed co-products.* Capitalizing on the results of Sections 5.2.3 and 5.2.4 we can now prove the main result of Section 5.2.

PROPOSITION 5.41. *For any $\mathcal{G}_1\subseteq\mathcal{X}$ generating $\mathcal{X}$ as a category of $(\mathfrak{U},\mathfrak{O})$-tagged labeled partitions, any $\mathcal{G}_2\subseteq\mathcal{Z}_w$ generating $\mathcal{Z}_w$ as a category of $(\{\aleph\},\varnothing)$-tagged labeled partitions and any set $\mathcal{R}$ of generating crosses for the $\mathbb{Z}_w$-crossed co-product with respect to $(\mathfrak{U},\mathfrak{O})$, the set $\mathcal{G}_1\cup\mathcal{G}_2\cup\mathcal{R}$ generates $\mathcal{X}\rtimes\mathcal{Z}_w$ as a category of $(\mathfrak{U}\cup\{\aleph\},\mathfrak{O})$-tagged labeled partitions.*

PROOF. Abbreviate $\mathcal{C}:={}_{\mathfrak{U}\cup\{\aleph\},\mathfrak{O}}\langle\mathcal{G}_1\cup\mathcal{G}_2\cup\mathcal{R}\rangle$. The proof is divided into two steps.

Step 1: Simplification. We justify that it is enough to prove $\phi_{\mathfrak{c}}\in\mathcal{C}$ for any object $\mathfrak{c}$ of $\mathcal{X}\rtimes\mathcal{Z}_w$. Indeed, if so, then for any $(\mathfrak{c},\mathfrak{d},p)\in\mathcal{X}\rtimes\mathcal{Z}_w$ since $(\mathfrak{c},\mathfrak{d},p)=\phi_{\mathfrak{d}}\,L(\mathfrak{c},\mathfrak{d},p)\,\phi_{\mathfrak{c}}^{-1}$ by Lemma 5.34 the conclusion $(\mathfrak{c},\mathfrak{d},p)\in\mathcal{C}$ is clear once we prove $L(\mathfrak{c},\mathfrak{d},p)\in\mathcal{C}$. But that is obvious since, by definition, $L(\mathfrak{c},\mathfrak{d},p)=L_{\mathsf{U}}(\mathfrak{c},\mathfrak{d},p)\otimes L_{\mathsf{V}}(\mathfrak{c},\mathfrak{d},p)$, where $L_{\mathsf{U}}(\mathfrak{c},\mathfrak{d},p)\in\mathcal{X}={}_{\mathfrak{U},\mathfrak{O}}\langle\mathcal{G}_1\rangle\subseteq{}_{\mathfrak{U}\cup\{\aleph\},\mathfrak{O}}\langle\mathcal{G}_1\rangle\subseteq\mathcal{C}$ and, likewise, $L_{\mathsf{V}}(\mathfrak{c},\mathfrak{d},p)\in\mathcal{Z}_w={}_{\{\aleph\},\varnothing}\langle\mathcal{G}_2\rangle\subseteq{}_{\mathfrak{U}\cup\{\aleph\},\mathfrak{O}}\langle\mathcal{G}_2\rangle\subseteq\mathcal{C}$.

Step 2: Proof of the simplified claim. We show $\phi_{\mathfrak{c}}\in\mathcal{C}$ for any object $\mathfrak{c}$ of $\mathcal{X}\rtimes\mathcal{Z}_w$ by induction over the length of $\mathfrak{c}$.

Step 2.1: Base case: As the base for our induction step we will need to know that the claim holds for at least the lengths zero, one and two. The only object of length 0 is $\varnothing$ and, obviously, the definitions ensure $\phi_{\varnothing}=(\varnothing,\varnothing,\varnothing)\in\mathcal{C}$. For any object $\mathfrak{c}$ of $\mathcal{X}\rtimes\mathcal{Z}_w$ of length 1, i.e., any $\mathfrak{c}\in((\mathfrak{U}\cup\{\aleph\})\otimes\{\circ,\bullet\})\cup\mathfrak{O}$, the pair $(L_{\mathsf{U}}(\mathfrak{c}),L_{\mathsf{V}}(\mathfrak{c}))$ is either $(\mathfrak{c},\varnothing)$ or $(\varnothing,\mathfrak{c})$. Then, necessarily, $\phi_{\varnothing}=\mathrm{id}_{\mathfrak{c}}\in\mathcal{C}$ by definition. The first interesting case is that of length 2. For any $\mathfrak{c}\colon[\![2]\!]\to((\mathfrak{U}\cup\{\aleph\})\otimes\{\circ,\bullet\})\cup\mathfrak{O}$, if $\{z_1,z_2\}\subseteq\mathfrak{U}\cup\{\aleph\}\cup\mathfrak{O}$ and $\{c_1,c_2\}\subseteq\{\circ,\bullet\}$ are such that $\xi^{\mathfrak{c}}_{\varnothing}(^{\blacksquare}1)=z_1$ and $\xi^{\mathfrak{c}}_{\varnothing}(^{\blacksquare}2)=z_2$ and such that, if $z_1\in\mathfrak{U}\cup\{\aleph\}$, then $\zeta^{\mathfrak{c}}_{\varnothing}(^{\blacksquare}1)=\overline{c_1}$ and, if $z_2\in\mathfrak{U}\cup\{\aleph\}$, then $\zeta^{\mathfrak{c}}_{\varnothing}(^{\blacksquare}2)=\overline{c_2}$, then there are eight cases to distinguish. Abbreviate $\mathsf{Y}:=\xi^{\mathfrak{c}\leftarrow}_{\varnothing}(\mathfrak{U}\cup\mathfrak{O})$ and $\mathsf{Z}:=\xi^{\mathfrak{c}\leftarrow}_{\varnothing}(\{\aleph\})$.

Case 2.1.1: If $\{z_1,z_2\}\subseteq\mathfrak{U}\cup\mathfrak{O}$, then $\mathsf{Y}=\Pi^2_0$ and $\mathsf{Z}=\varnothing$. Hence, the definitions imply $\varepsilon^{\mathfrak{c}}_{\varnothing}=\xi^{\mathfrak{c}}_{\varnothing}$ and thus $L_{\mathsf{U}}(\mathfrak{c})=\mathfrak{c}$ and $L_{\mathsf{V}}(\mathfrak{c})=\varnothing$ and thus $L(\mathfrak{c})=\mathfrak{c}$. Hence and since $\gamma_{\mathsf{Y},0}{}^k$ is then the identity on Π^2_0 and $\gamma_{\mathsf{Z},0}{}^k$ the empty map, ultimately, $\phi_{\mathfrak{c}}=(\mathfrak{c},\mathfrak{c},\mathrm{id}_2)=\mathrm{id}_{\mathfrak{c}}$. Hence, $\phi_{\mathfrak{c}}\in\mathcal{X}\subseteq\mathcal{C}$ in that case.

Case 2.1.2: Similarly, if $z_1=z_2=\aleph$, then $\mathsf{Y}=\varnothing$ and $\mathsf{Z}=\Pi^2_0$. It follows tthat $L_{\mathsf{U}}(\mathfrak{c})=\varnothing$ and $L_{\mathsf{V}}(\mathfrak{c})=\mathfrak{c}$ and that $\gamma_{\mathsf{Y},0}{}^k$ is the empty map and $\gamma_{\mathsf{Z},0}{}^k$ the identity on Π^2_0. Therefore, $L(\mathfrak{c})=\mathfrak{c}$ and $\phi_{\mathfrak{c}}=(\mathfrak{c},\mathfrak{c},\mathrm{id}_2)=\mathrm{id}_{\mathfrak{c}}\in\mathcal{Z}_w\subseteq\mathcal{C}$.

Case 2.1.3: In the situation that $z_1 \in \mathfrak{U} \uplus \mathfrak{O}$ and $z_2 = \aleph$, of course, $\mathsf{Y} = \{{}^{\blacksquare}1\}$ and $\mathsf{Z} = \{{}^{\blacksquare}2\}$. Moreover, then $\varepsilon^{\mathfrak{c}}_{\varnothing}({}^{\blacksquare}1) = z_1$ by definition and thus $L_{\mathsf{U}}(\mathfrak{c}) = \mathfrak{c}(1)$ and $L_{\mathsf{V}}(\mathfrak{c}) = \mathfrak{c}(2)$ and thus $L = \mathfrak{c}$. Furthermore, $\gamma_{\mathsf{Y},0}{}^{k}$ is then the identity on $\{{}^{\blacksquare}1\}$ and $\gamma_{\mathsf{Z},0}{}^{k}$ is defined by ${}^{\blacksquare}1 \mapsto {}^{\blacksquare}2$. It follows that $\phi_{\mathfrak{c}} = (\mathfrak{c}, \mathfrak{c}, \mathrm{id}_1 \otimes \mathrm{id}_1) = \mathrm{id}_{\mathfrak{c}(1)} \otimes \mathrm{id}_{\mathfrak{c}(2)} = \mathrm{id}_{\mathfrak{c}}$. In conclusion, $\phi_{\mathfrak{c}} \in \mathcal{C}$ also in this case.

Case 2.1.4: The final possibility is that $z_1 = \aleph$ and $z_2 \in \mathfrak{U} \uplus \mathfrak{O}$, in which case, $\mathsf{Y} = \{{}^{\blacksquare}2\}$ and $\mathsf{Z} = \{{}^{\blacksquare}1\}$. Now, $\varepsilon^{\mathfrak{c}}_{\varnothing}({}^{\blacksquare}2) = \xi^{\mathfrak{c}}_{\varnothing}({}^{\blacksquare}2) - {}_{\aleph}\sigma^{\mathfrak{c}}_{\varnothing}(]{}^{\blacksquare}2, {}^{\blacksquare}1]^{2}_{0}) = z_2 - {}_{\aleph}\sigma^{\mathfrak{c}}_{\varnothing}(\{{}^{\blacksquare}1\}) = z_2 - \sigma(\zeta^{\mathfrak{c}}_{\varnothing}({}^{\blacksquare}1)) = z_2 + \sigma(c_1)$ and thus $L_{\mathsf{U}}(\mathfrak{c}) = (\pi_w(z_2 + \sigma(c_1)), c_2)$ if $z_2 \in \mathfrak{U}$ and $L_{\mathsf{U}}(\mathfrak{c}) = \pi_w(z_2 + \sigma(c_1))$ if $z_2 \in \mathfrak{O}$ as well as $L_{\mathsf{V}}(\mathfrak{c}) = (\aleph, c_1)$. Furthermore, $\gamma_{\mathsf{Y},0}{}^{k}$ is then defined by ${}^{\blacksquare}1 \mapsto {}^{\blacksquare}2$ and $\gamma_{\mathsf{Z},0}{}^{k}$ is the identity on $\{{}^{\blacksquare}1\}$, whence $\{\{\gamma_{\mathsf{Y},0}{}^{2}({}^{\blacksquare}1), {}_{\blacksquare}1\}, \{\gamma_{\mathsf{Z},0}{}^{2}({}^{\blacksquare}1), {}_{\blacksquare}(1+1)\}\} = \{\{{}^{\blacksquare}2, {}_{\blacksquare}1\}, \{{}^{\blacksquare}1, {}_{\blacksquare}2\}\} = \times$. Thus, $\phi_{\mathfrak{c}}$ is $(((\aleph, c_1), (z_2, c_2)), ((\pi_w(z_2 + \sigma(c_1)), c_2), (\aleph, c_1)), \times)$ if $z_2 \in \mathfrak{U}$ and $(((\aleph, c_1), z_2), (\pi_w(z_2 + \sigma(c_1)), (\aleph, c_1)), \times)$ if $z_2 \in \mathfrak{O}$.

Note that, if $c \coloneqq c_1$ and $z \coloneqq \pi_w(z_2 - \delta_{c_1,\bullet})$, then $z \in \mathbb{Z}_w$ and $z_2 = \pi_w(z + \delta_{c,\bullet})$ and $\pi_w(z_2 + \sigma(c_1)) = \pi_w(z + \delta_{c,\circ})$ by $\sigma(c_1) = -\delta_{c,\bullet} + \delta_{c,\circ}$. Therefore, if $c' \coloneqq c_2$, then $\phi_{\mathfrak{c}}$ can also be written as $((\aleph, c) \otimes (\pi_w(z + \delta_{c,\bullet}), c'), (\pi_w(z + \delta_{c,\circ}), c') \otimes (\aleph, c), \times) \in \mathcal{R}$ if $z \in \mathfrak{U}$ and as $((\aleph, c) \otimes \pi_w(z + \delta_{c,\bullet}), \pi_w(z + \delta_{c,\circ}) \otimes (\aleph, c), \times) \in \mathcal{R}$ if $z \in \mathfrak{U}$. Because $\mathcal{G}_1 \cup \mathcal{G}_2 \cup \mathcal{R}$ includes the set of generating crosses $\mathcal{R}$ for the $\mathbb{Z}_w$-crossed product with respect to $(\mathfrak{U}, \mathfrak{O})$ the category $\mathcal{C}$ thus contains $\phi_{\mathfrak{c}}$ by Lemma 5.40. Thus the claim holds for all lengths less than or equal to 2.

Step 2.2: Induction step: Let now $k \in \mathbb{N}_0$ with $2 < k$ and $\mathfrak{c}\colon [\![k]\!] \to ((\mathfrak{U} \uplus \{\aleph\}) \otimes \{\circ, \bullet\}) \uplus \mathfrak{O}$ be arbitrary. In two steps we show that $\phi_{\mathfrak{c}} \in \mathcal{C}$. The assumption $2 < k$ will be crucial to the first one.

Step 2.2.1: First, we prove that there exists $k_1 \in \mathbb{N}_0$ such that, first, $k_1 < k$, second, $k_2 \coloneqq k - k_1 < k$, and, third, if $\mathfrak{c}_1\colon [\![k_1]\!] \to ((\mathfrak{U} \uplus \{\aleph\}) \otimes \{\circ, \bullet\}) \uplus \mathfrak{O}$ and $\mathfrak{c}_2\colon [\![k_2]\!] \to ((\mathfrak{U} \uplus \{\aleph\}) \otimes \{\circ, \bullet\}) \uplus \mathfrak{O}$ are such that $\mathfrak{c} = \mathfrak{c}_1 \vartriangle \mathfrak{c}_2$ and if $m_{1,2} \coloneqq \alpha(\xi^{\mathfrak{c}_1}_{\varnothing}{}^{\leftarrow}(\{\aleph\}))$ and $m_{2,1} \coloneqq \alpha(\xi^{\mathfrak{c}_2}_{\varnothing}{}^{\leftarrow}(\mathfrak{U} \uplus \mathfrak{O}))$, then $m_{1,2} + m_{2,1} < k$.

Indeed, if $k'_1 \coloneqq 1$ and $k'_2 \coloneqq k - k'_1$, then both $k'_1 < k$ (for which assuming $2 \leq k$ would have sufficed) and $k'_2 < k$. Moreover, let $\mathfrak{c}'_1\colon [\![k'_1]\!] \to ((\mathfrak{U} \uplus \{\aleph\}) \otimes \{\circ, \bullet\}) \uplus \mathfrak{O}$ and $\mathfrak{c}'_2\colon [\![k'_2]\!] \to ((\mathfrak{U} \uplus \{\aleph\}) \otimes \{\circ, \bullet\}) \uplus \mathfrak{O}$ be the unique labelings with $\mathfrak{c} = \mathfrak{c}'_1 \vartriangle \mathfrak{c}'_2$ and let $m'_{t,1} \coloneqq \alpha(\xi^{\mathfrak{c}'_t}_{\varnothing}{}^{\leftarrow}(\mathfrak{U} \uplus \mathfrak{O}))$ and $m'_{t,2} \coloneqq \alpha(\xi^{\mathfrak{c}'_t}_{\varnothing}{}^{\leftarrow}(\{\aleph\}))$ for each $t \in [\![2]\!]$. If $m'_{1,2} + m'_{2,1} < k$, then $k_1 \coloneqq k'_1$ and $k_2 \coloneqq k'_2$ obviously satisfy all three requirements set out in the beginning. Otherwise, we put $k_1 \coloneqq k'_1 + 1 = 2$ and $k_2 \coloneqq k'_2 - 1 = k - 2$. The assumption that $2 < k$ then allows us to conclude that, still, $k_1 < k$. (That is why we needed lengths zero, one and two as the base for the induction.) Of course, $k_2 < k$ holds true as well. What remains to be shown is that, if now $\mathfrak{c}_1\colon [\![k_1]\!] \to ((\mathfrak{U} \uplus \{\aleph\}) \otimes \{\circ, \bullet\}) \uplus \mathfrak{O}$ and $\mathfrak{c}_2\colon [\![k_2]\!] \to ((\mathfrak{U} \uplus \{\aleph\}) \otimes \{\circ, \bullet\}) \uplus \mathfrak{O}$ are such that $\mathfrak{c} = \mathfrak{c}_1 \vartriangle \mathfrak{c}_2$ and if $m_{1,2} \coloneqq \alpha(\xi^{\mathfrak{c}_1}_{\varnothing}{}^{\leftarrow}(\{\aleph\}))$ and $m_{2,1} \coloneqq \alpha(\xi^{\mathfrak{c}_2}_{\varnothing}{}^{\leftarrow}(\mathfrak{U} \uplus \mathfrak{O}))$, then $m_{1,2} + m_{2,1} < k$.

Because $m'_{1,2} \leq k'_1$ and $m'_{2,1} \leq k'_2$ and because $k'_1 + k'_2 = k$, of course, $m'_{1,2} + m'_{2,1} \leq k'_1 + k'_2 = k$. Hence, the assumption that $k \leq m'_{1,2} + m'_{2,1}$ actually means that $m'_{1,2} + m'_{2,1} = k$. We conclude that $m'_{2,1} = k - m'_{1,2} \geq k - k'_1 = k'_2$ and, likewise, $m'_{1,2} = k - m'_{2,1} \geq k - k'_2 = k'_1$. Because $m'_{t,1} + m'_{t,2} = k'_t$ for each $t \in [\![2]\!]$, that only leaves the possibility that $m'_{1,2} = k'_1$

and $m'_{2,1} = k'_2$ and $m'_{1,1} = m'_{2,2} = 0$. In other words, $\xi_{\varnothing}^{\mathfrak{c}'_1}$ is constant with value $\aleph$ and $\xi_{\varnothing}^{\mathfrak{c}'_2}$ only takes values in $\mathfrak{U} \cup \mathfrak{D}$. Thus, on the one hand, $\xi_{\varnothing}^{\mathfrak{c}_1}(^{\blacksquare}i) = \xi_{\varnothing}^{\mathfrak{c}'_1}(^{\blacksquare}i) = \aleph$ for each $i \in [\![k'_1]\!]$ and $\xi_{\varnothing}^{\mathfrak{c}_1}(^{\blacksquare}k_1) = \xi_{\varnothing}^{\mathfrak{c}'_2}(^{\blacksquare}1) \in \mathfrak{U} \cup \mathfrak{D}$ and, hence, $m_{1,2} = |\xi_{\varnothing}^{\mathfrak{c}_1 \leftarrow}(\{\aleph\})| = |\xi_{\varnothing}^{\mathfrak{c}'_1 \leftarrow}(\{\aleph\})| = m'_{1,2}$. On the other hand, $\xi_{\varnothing}^{\mathfrak{c}_2}(^{\blacksquare}i) = \xi_{\varnothing}^{\mathfrak{c}'_2}(^{\blacksquare}i+1) \in \mathfrak{U} \cup \mathfrak{D}$ for any $i \in [\![k_2]\!]$ and $\xi_{\varnothing}^{\mathfrak{c}'_1}(^{\blacksquare}1) \in \mathfrak{U} \cup \mathfrak{D}$ and, hence, $m_{2,1} = |\xi_{\varnothing}^{\mathfrak{c}_1 \leftarrow}(\mathfrak{U} \cup \mathfrak{D})| = |\xi_{\varnothing}^{\mathfrak{c}'_1 \leftarrow}(\mathfrak{U} \cup \mathfrak{D}) \backslash \{^{\blacksquare}1\}| = |\xi_{\varnothing}^{\mathfrak{c}'_1 \leftarrow}(\mathfrak{U} \cup \mathfrak{D})| - |\{^{\blacksquare}1\}| = m'_{1,2} - 1$. In conclusion, $m_{1,2} + m_{2,1} = m'_{1,2} + (m'_{2,1} - 1) = k - 1 < k$, which is what we needed to prove.

Step 2.2.2: Because $k_1 < k$ and $k_2 < k$ the induction hypothesis guarantees $\{\phi_{\mathfrak{c}_1}, \phi_{\mathfrak{c}_2}\} \subseteq \mathcal{C}$. And the third property of k_1 and k_2, namely that the number $m_{1,2} + m_{2,1}$, the sum of the lengths of $L_{\vee}(\mathfrak{c}_1)$ and $L_{\cup}(\mathfrak{c}_2)$, is also less than k, lets us apply the induction hypothesis one more time to infer that also $\phi_{L_{\vee}(\mathfrak{c}_1) \otimes L_{\vee}(\mathfrak{c}_2)} \in \mathcal{C}$. Consequently, $L_{\otimes, \mathfrak{c}_1, \mathfrak{c}_2} = \mathrm{id}_{L_{\cup}(\mathfrak{c}_1)} \otimes \phi_{L_{\vee}(\mathfrak{c}_1) \otimes L_{\vee}(\mathfrak{c}_2)}{}^{*} \otimes \mathrm{id}_{L_{\vee}(\mathfrak{c}_2)}$ is an element of $\mathcal{C}$ as well. Because $\phi_{\mathfrak{c}_1 \otimes \mathfrak{c}_2} = (\phi_{\mathfrak{c}_1} \otimes \phi_{\mathfrak{c}_2}) L_{\otimes, \mathfrak{c}_1, \mathfrak{c}_2}{}^{*}$ by Lemma 5.36 that proves $\phi_{\mathfrak{c}} = \phi_{\mathfrak{c}_1 \otimes \mathfrak{c}_2} \in \mathcal{C}$, concluding the induction step and thus the proof. □

5.3. Generators of wreath graph co-products. Naturally, the definitions and results of Sections 5.1 and 5.2 in combination give us a statement about the generators of wreath graph co-products.

ASSUMPTIONS 5.42. In Section 5.3, let $(w \in \mathbb{N}_0)$ be arbitrary, let r be any $\mathbb{Z}_w$-invariant partial commutation relation on $\mathbb{Z}_w$, let $\mathcal{C}$ be any category of uncolored or two-colored partitions, for each $z \in \mathbb{Z}_w$ let $\mathcal{X}_z$ be given by $\mathcal{C}$ interpreted as tagged with z and let $(\mathfrak{U}, \mathfrak{D})$ be $(\mathbb{Z}_w, \varnothing)$ if $\mathcal{C}$ is uncolored and $(\varnothing, \mathbb{Z}_w)$ if $\mathcal{C}$ is two-colored.

PROPOSITION 5.43. *For any $\mathcal{G}_{\mathcal{C}} \subseteq \mathcal{C}$ generating $\mathcal{C}$ as a category of uncolored partition if $\mathcal{C}$ is uncolored and as a category of two-colored partitions if $\mathcal{C}$ is two-colored, any $\mathcal{G}_{\mathcal{Z}_w} \subseteq \mathcal{Z}_w$ generating $\mathcal{Z}_w$ as a category of $(\{\aleph\}, \varnothing)$-tagged labeled partitions, any set $\mathcal{R}_{*_r}$ of generating crosses for the r-graph co-product with respect to $(\mathfrak{U}, \mathfrak{D})$ and any set $\mathcal{R}_{\rtimes}$ of generating crosses for the $\mathbb{Z}_w$-crossed co-product with respect to $(\mathfrak{U}, \mathfrak{D})$, if for each $z \in \mathbb{Z}_w$ the set $\mathcal{G}_{\mathcal{X}_z}$ is given by $\mathcal{G}_{\mathcal{C}}$ interpreted as tagged with z, then $\bigcup_{z \in \mathbb{Z}_w} \mathcal{G}_{\mathcal{X}_z} \cup \mathcal{G}_{\mathcal{Z}_w} \cup \mathcal{R}_{*_r} \cup \mathcal{R}_{\rtimes}$ generates $\mathcal{C} \wr_r \mathcal{Z}_w$ as a category of $(\mathfrak{U} \cup \{\aleph\}, \mathfrak{D})$-tagged labeled partitions.*

PROOF. Because $\mathcal{G}_{\mathcal{C}}$ generates $\mathcal{C}$, for each $z \in \mathbb{Z}_z$ the set $\mathcal{G}_{\mathcal{X}_z}$ generates $\mathcal{X}_z$ as a category of $(\{z\}, \varnothing)$-tagged labeled partitions if $z \in \mathfrak{U}$ and as a category of $(\varnothing, \{z\})$-tagged labeled partitions if $z \in \mathfrak{D}$. Therefore, by Proposition 5.19 the set $\mathcal{G}_{\mathcal{X}} \coloneqq \bigcup_{z \in \mathbb{Z}_w} \mathcal{G}_{\mathcal{X}_z} \cup \mathcal{R}_{*_r}$ generates $\mathcal{C}^{*(\mathbb{Z}_w, r)} = *^{r}_{z \in \mathbb{Z}_w} \mathcal{X}_z$ as a category of $(\mathfrak{U}, \mathfrak{D})$-tagged labeled partitions. By Proposition 5.41 it thus follows that a generator of $\mathcal{C} \wr_r \mathcal{Z}_w = \mathcal{C}^{*(\mathbb{Z}_w, r)} \rtimes \mathcal{Z}_w$ as a category of $(\mathfrak{U} \cup \{\aleph\}, \mathfrak{D})$-tagged labeled partitions is given by $\mathcal{G}_{\mathcal{X}} \cup \mathcal{G}_{\mathcal{Z}_w} \cup \mathcal{R}_{\rtimes} = \bigcup_{z \in \mathbb{Z}_w} \mathcal{G}_{\mathcal{X}_z} \cup \mathcal{G}_{\mathcal{Z}_w} \cup \mathcal{R}_{*_r} \cup \mathcal{R}_{\rtimes}$. □

6. Relating the products on the category and the quantum group level

The results of 5 now allow us to show that the easy CQG Hopf $*$-algebra associated with the (graph, crossed, wreath graph) co-product category of a family of input categories is isomorphic to the respective CQG Hopf $*$-algebra co-product of the easy CQG Hopf $*$-algebras associated with the input categories.

6.1. Relating the graph co-products.

For the graph co-product it is Proposition 5.19 that enables us to make the connection.

LEMMA 6.1. *For any* $\{z,z'\}\subseteq\mathfrak{U}\cup\mathfrak{D}$, *any* $\{c,c'\}\subseteq\{\circ,\bullet\}$, *and any* $\{N(z),N(z')\}\subseteq\mathbb{N}$, *if* $p=\times$,

(a) *if* $(z,z')\in\mathfrak{U}\otimes\mathfrak{U}$ *and* $\mathfrak{c}=(z,c)\otimes(z',c')$ *and* $\mathfrak{d}=(z',c')\otimes(z,c)$, *then* $\mathrm{r}_{\mathfrak{d}}^{\mathfrak{c}}(p)_{j,g}=(u^{z'}_{j_1,g_2})^{c'}(u^{z}_{j_2,g_1})^{c}-(u^{z}_{j_2,g_1})^{c}(u^{z'}_{j_1,g_2})^{c'}$ *for any* $g\in J^N_{\mathfrak{c}}$ *and* $j\in J^N_{\mathfrak{d}}$.

(b) *if* $(z,z')\in\mathfrak{U}\otimes\mathfrak{D}$ *and* $\mathfrak{c}=(z,c)\otimes z'$ *and* $\mathfrak{d}=z'\otimes(z,c)$, *then* $\mathrm{r}_{\mathfrak{d}}^{\mathfrak{c}}(p)_{j,g}=u^{z'}_{j_1,g_2}(u^{z}_{j_2,g_1})^{c}-(u^{z}_{j_2,g_1})^{c}u^{z'}_{j_1,g_2}$ *for any* $g\in J^N_{\mathfrak{c}}$ *and* $j\in J^N_{\mathfrak{d}}$.

(c) *if* $(z,z')\in\mathfrak{D}\otimes\mathfrak{U}$ *and* $\mathfrak{c}=z\otimes(z',c')$ *and* $\mathfrak{d}=(z',c')\otimes z$, *then* $\mathrm{r}_{\mathfrak{d}}^{\mathfrak{c}}(p)_{j,g}=(u^{z'}_{j_1,g_2})^{c'}u^{z}_{j_2,g_1}-u^{z}_{j_2,g_1}(u^{z'}_{j_1,g_2})^{c'}$ *for any* $g\in J^N_{\mathfrak{c}}$ *and* $j\in J^N_{\mathfrak{d}}$.

(d) *if* $(z,z')\in\mathfrak{D}\otimes\mathfrak{D}$ *and* $\mathfrak{c}=z\otimes z'$ *and* $\mathfrak{d}=z'\otimes z$, *then* $\mathrm{r}_{\mathfrak{d}}^{\mathfrak{c}}(p)_{j,g}=u^{z'}_{j_1,g_2}u^{z}_{j_2,g_1}-u^{z}_{j_2,g_1}u^{z'}_{j_1,g_2}$ *for any* $g\in J^N_{\mathfrak{c}}$ *and* $j\in J^N_{\mathfrak{d}}$.

PROOF. Immediate from the definitions. ☐

PROPOSITION 6.2. *Given any choice of tags* $(\mathfrak{U},\mathfrak{D})$, *any partial commutation relation* r *on* $\mathfrak{U}\cup\mathfrak{D}$, *any category* $\mathcal{X}_z$ *of* $(\{z\},\varnothing)$*-tagged labeled partitions for each* $z\in\mathfrak{U}$, *any category* $\mathcal{X}_z$ *of* $(\varnothing,\{z\})$*-tagged labeled partitions for each* $z\in\mathfrak{D}$, *as well as any dimension* $N_z\in\mathbb{N}$ *for each* $z\in\mathfrak{U}\cup\mathfrak{D}$, *if* H_z *is the CQG Hopf* $*$*-algebra associated with* $(\{z\},\varnothing,\mathcal{X}_z,N_z)$ *for each* $z\in\mathfrak{U}$ *and the one associated with* $(\varnothing,\{z\},\mathcal{X}_z,N_z)$ *for each* $z\in\mathfrak{D}$ *and if* H' *is the CQG Hopf* $*$*-algebra associated with* $(\mathfrak{U},\mathfrak{D},*^r_{z\in\mathfrak{U}\cup\mathfrak{D}}\mathcal{X}_z,N)$, *where* $N=(N_z)_{z\in\mathfrak{U}\cup\mathfrak{D}}$, *if* $(\iota_z)_{z\in\mathfrak{U}\cup\mathfrak{D}}$ *are the associated co-projections of the graph co-product CQG Hopf* $*$*-algebra* $*^r_{z\in\mathfrak{U}\cup\mathfrak{D}}H_z$ *of* $(H_z)_{z\in\mathfrak{U}\cup\mathfrak{D}}$ *with respect to* r, *then there exists an isomorphism of CQG Hopf* $*$*-algebras from* $*^r_{z\in\mathfrak{U}\cup\mathfrak{D}}H_z$ *to* H' *such that for any* $x\in\mathfrak{U}$ *and any* $\{i_x,j_x\}\subseteq[\![N_x]\!]$ *and any* $y\in\mathfrak{D}$ *and any* $\{i_y,j_y\}\subseteq[\![N_y]\!]$,

$$\iota_x(u_{j_x,i_x})\mapsto v^{(x,\circ)}_{j_x,i_x}\quad\wedge\quad\iota_y(u_{j_y,i_y})\mapsto v^{y}_{j_y,i_y},$$

where u_{j_x,i_x} *and* $v^{(x,\circ)}_{j_x,i_x}$ *are both short for* $((x,\circ),j_x,i_x)$, *and where* u_{j_y,i_y} *and* $v^{y}_{j_y,i_y}$ *are both short for* (y,j_y,i_y). *Moreover, this isomorphism is the unique* $*$*-algebra morphism betwen the underlying* $*$*-algebras with the given property.*

PROOF. For any $z\in\mathfrak{U}\cup\mathfrak{D}$, by Proposition 3.40 we can identify H_z with the underlying CQG Hopf $*$-algebra of the CMMQG $*$-algebra of profile N_z given by (A_z,u^z) where A_z is the universal $*$-algebra over $\{u_{j,i}\}_{i,j=1}^{N_z}$, where $u_{j,i}$ means (z,j,i)

for any $\{i,j\} \subseteq [\![N_z]\!]$, subject to the relations

$$\{\mathrm{r}_{\mathfrak{d}}^{\mathfrak{c}}(p)_{j,g}(u^z) \,|\, \{k,\ell\} \subseteq \mathbb{N}_0 \wedge \mathfrak{c}\colon [\![k]\!] \to ((\{z\} \cap \mathfrak{U}) \otimes \{\circ,\bullet\}) \cup (\{z\} \cap \mathfrak{O}) \\ \wedge\, \mathfrak{d}\colon [\![\ell]\!] \to ((\{z\} \cap \mathfrak{U}) \otimes \{\circ,\bullet\}) \cup (\{z\} \cap \mathfrak{O}) \\ \wedge\, (\mathfrak{c},\mathfrak{d},p) \in \mathcal{X}_z \wedge g \in [\![N_z]\!]^{\otimes k} \wedge j \in [\![N_z]\!]^{\otimes \ell}\}$$

and, if $z \in \mathfrak{O}$, the relations $\{u_{j,i}^* - u_{j,i} \,|\, \{i,j\} \subseteq [\![N_z]\!]\}$, and where $u^z = (u_{j,i})_{(j,i)\in[\![N_z]\!]^{\otimes 2}}$.

If $\mathcal{R}_r$ is the maximal set of generating crosses for the r-graph co-product with respect to $(\mathfrak{U},\mathfrak{O})$, then Proposition 5.19 guarantees that $\mathcal{R}_r \cup \bigcup_{z\in\mathfrak{U}\cup\mathfrak{O}} \mathcal{X}_z$ generates $*^r_{z\in\mathfrak{U}\cup\mathfrak{O}} \mathcal{X}_z$ as a category of $(\mathfrak{U},\mathfrak{O})$-tagged labeled partitions. Hence, Proposition 3.40 lets us identity H' with the underlying CQG Hopf $*$-algebra of the CMMQG $*$-algebra of profile N given by $(A',(v^z)_{z\in\mathfrak{U}\cup\mathfrak{O}})$, where A' is the universal $*$-algebra over $\{(z,j_z,i_z) \,|\, z \in \mathfrak{U}\cup\mathfrak{O} \wedge \{i_z,j_z\} \subseteq [\![N_z]\!]\}$, whose elements we write as $v^z_{j_z,i_z} = (z,j_z,i_z)$ for any $z \in \mathfrak{U}\cup\mathfrak{O}$ and any $\{i_z,j_z\} \subseteq [\![N_z]\!]$, subject to the relations

$$\{\mathrm{r}_{\mathfrak{d}}^{\mathfrak{c}}(p)_{j,g}((v^z)_{z\in\mathfrak{U}\cup\mathfrak{O}}) \,|\, \{k,\ell\} \subseteq \mathbb{N}_0 \wedge \mathfrak{c}\colon [\![k]\!] \to (\mathfrak{U} \otimes \{\circ,\bullet\}) \cup \mathfrak{O} \\ \wedge\, \mathfrak{d}\colon [\![\ell]\!] \to (\mathfrak{U} \otimes \{\circ,\bullet\}) \cup \mathfrak{O} \wedge (\mathfrak{c},\mathfrak{d},p) \in \mathcal{R}_r \cup \textstyle\bigcup_{z\in\mathfrak{U}\cup\mathfrak{O}} \mathcal{X}_z \\ \wedge\, g \in J^N_{\mathfrak{c}} \wedge j \in J^N_{\mathfrak{d}}\}$$

and, additionally, the relations $\{(v^y_{j_y,i_y})^* - v^y_{j_y,i_y} \,|\, \{i_y,j_y\} \subseteq [\![N_y]\!] \wedge y \in \mathfrak{O}\}$, and where $v^z = (v^z_{j_z,i_z})_{(j_z,i_z)\in[\![N_z]\!]^{\otimes 2}}$ for any $z \in \mathfrak{U}\cup\mathfrak{O}$.

By Lemma 6.1 the set

$$\{\mathrm{r}_{\mathfrak{d}}^{\mathfrak{c}}(p)_{j,g}((v^z)_{z\in\mathfrak{U}\cup\mathfrak{O}}) \,|\, \{k,\ell\} \subseteq \mathbb{N}_0 \wedge \mathfrak{c}\colon [\![k]\!] \to (\mathfrak{U} \otimes \{\circ,\bullet\}) \cup \mathfrak{O} \\ \wedge\, \mathfrak{d}\colon [\![\ell]\!] \to (\mathfrak{U} \otimes \{\circ,\bullet\}) \cup \mathfrak{O} \wedge (\mathfrak{c},\mathfrak{d},p) \in \mathcal{R}_r \\ \wedge\, g \in J^N_{\mathfrak{c}} \wedge j \in J^N_{\mathfrak{d}}\}$$

is given by

$$\begin{aligned} \bigcup\{ &\{(v^{z_1}_{j_1,i_1})^{c_1}(v^{z_2}_{j_2,i_2})^{c_2} - (v^{z_2}_{j_2,i_2})^{c_2}(v^{z_1}_{j_1,i_1})^{c_1} \,|\, (z_1,z_2) \in r \cap (\mathfrak{U}\otimes\mathfrak{U})\}, \\ &\{(v^{z_1}_{j_1,i_1})^{c_1} v^{z_2}_{j_2,i_2} - v^{z_2}_{j_2,i_2}(v^{z_1}_{j_1,i_1})^{c_1} \,|\, (z_1,z_2) \in r \cap (\mathfrak{U}\otimes\mathfrak{O})\}, \\ &\{v^{z_1}_{j_1,i_1}(v^{z_2}_{j_2,i_2})^{c_2} - (v^{z_2}_{j_2,i_2})^{c_2} v^{z_1}_{j_1,i_1} \,|\, (z_1,z_2) \in r \cap (\mathfrak{O}\otimes\mathfrak{U})\}, \\ &\{v^{z_1}_{j_1,i_1} v^{z_2}_{j_2,i_2} - v^{z_2}_{j_2,i_2} v^{z_1}_{j_1,i_1} \,|\, (z_1,z_2) \in r \cap (\mathfrak{O}\otimes\mathfrak{O})\} \\ &|\, \{i_1,j_1\} \subseteq [\![N_{z_1}]\!] \wedge \{i_2,j_2\} \subseteq [\![N_{z_2}]\!] \wedge \{c_1,c_2\} \subseteq \{\circ,\bullet\}\}. \end{aligned}$$

If in the free algebra over $\{v^z_{j_z,i_z} \,|\, z \in \mathfrak{U}\cup\mathfrak{O} \wedge \{i_z,j_z\} \subseteq [\![N_z]\!]\}$ for any $z \in \mathfrak{U}\cup\mathfrak{O}$ the $*$-subalgebra generated by $\{v^z_{j,i}\}_{i,j=1}^{N_z}$ is denoted by A'_z, then the above set, the relations induced by $\mathcal{R}_r$, generates the same $*$-ideal as the set

$$\{a_1a_2 - a_2a_1 \,|\, a_1 \in A'_{z_1} \wedge a_2 \in A'_{z_2} \wedge (z_1,z_2) \in r\}.$$

In the light of Definition 2.22 that proves the claim. $\square$

6.2. Relating the crossed co-products. In the case of the crossed co-product the link between the category and the CQG Hopf $*$-algebra level is established by Proposition 5.41. However, first, it is important to note that the easy CQG Hopf $*$-algebras associated with the category $\mathcal{Z}_w$ is actually the group CQG Hopf $*$-algebra of $\mathbb{Z}_w$ for eadch $w \in \mathbb{N}_0$.

LEMMA 6.3. *For any $w \in \mathbb{N}_0$ the category $\mathcal{Z}_w$ is generated as a category of two-colored partitions by the set $\{⌘\}$ if $w = 0$ and by $\{⌘, \updownarrow^{\otimes w}\}$ if $1 \le w$.*

PROOF. Because $\langle ⁂ \rangle = \langle ⌘ \rangle$ the case $w = 0$ of the claim is shown in [MW20, Proposition 7.3 (a)] where $\mathcal{Z}_0$ is referred to as $\mathcal{S}_1$. For $1 \le w$ a proof can be found in [TW18, Theorem 8.3] since $\mathcal{Z}_w$ is the same category as $\mathcal{B}_{\mathrm{grp,loc}}(w)$ there and since $\updownarrow \otimes \updownarrow \in \langle \updownarrow^{\otimes w} \rangle$. □

For the next lemma to make sense, recall that $\mathbb{C}$ is a zero element in the category of CQG Hopf $*$-algebras.

LEMMA 6.4. *For any $w \in \mathbb{N}_0$, if Z_w is the easy CQG Hopf $*$-algebra associated with $(\{\aleph\}, \varnothing, \mathcal{Z}_w, 1)$, then there exists an isomorphism of CQG Hopf $*$-algebras from Z_w to $\mathbb{C}[\mathbb{Z}_w]$ such that, if $w \neq 1$, then*

$$\aleph \mapsto \aleph,$$

where the left $\aleph$ is short for $((\aleph, \circ), 1, 1)$ and where the right $\aleph$ is the basis vector of the free vector space over $\mathbb{Z}_w$ belonging to the group element 1. Moreover, this isomorphism is the unique $*$-algebra morphism between the underlying $*$-algebras with the given property.

PROOF. Let $\updownarrow^{\otimes 0} \coloneqq \varnothing$ denote the empty partition. By Proposition 3.40 we can identify Z_w with the underlying CQG Hopf $*$-algebra of the CMMQG $*$-algebra of profile 1 given by $(A, \aleph)$, where A is the universal $*$-algebra over $\{\aleph\}$ subject to the relations

$$\{\mathrm{r}_{\mathfrak{d}}^{\mathfrak{c}}(p)_{j,g}(\aleph) \mid \{k, \ell\} \subseteq \mathbb{N}_0 \wedge \mathfrak{c}\colon [\![k]\!] \to \{\aleph\} \otimes \{\circ, \bullet\} \wedge \mathfrak{d}\colon [\![\ell]\!] \to \{\aleph\} \otimes \{\circ, \bullet\}$$
$$\wedge\ (\mathfrak{c}, \mathfrak{d}, p) \in \{\sqcap_{\circ\bullet}, \sqcap_{\bullet\circ}, \sqcup^{\circ\bullet}, \sqcup^{\bullet\circ}, \updownarrow^{\otimes w}\} \wedge g \in [\![1]\!]^{\otimes k} \wedge j \in [\![1]\!]^{\otimes \ell}\}.$$

By definition, this set is the same as

$$\{\aleph\aleph^* - 1, \aleph^*\aleph - 1, 1 - \aleph\aleph^*, 1 - \aleph^*\aleph, \aleph^w - 1\},$$

which now proves the claim. □

LEMMA 6.5. *For any $w \in \mathbb{N}_0$ and any $n \in \mathbb{N}$, if the dimension profile N is such that $N(z) = n$ for any $z \in \mathbb{Z}_w$ and $N(\aleph) = 1$, if $p = \times$, if $\aleph$ is short for $(\aleph, 1, 1)$, if $u_{j,i}^z$ is short for (z, j, i) for any $\{i, j\} \subseteq [\![n]\!]$ and any $z \in \mathbb{Z}_w$, then for any $z \in \mathbb{Z}_w$ and any $\{c, c'\} \subseteq \{\circ, \bullet\}$,*

(a) if $z \in \mathfrak{U}$ and $\mathfrak{c} = (\aleph, c) \otimes (\pi_w(z + \delta_{c,\bullet}), c')$ and $\mathfrak{d} = (\pi_w(z + \delta_{c,\circ}), c') \otimes (\aleph, c)$, then $\mathrm{r}_{\mathfrak{d}}^{\mathfrak{c}}(p)_{j,g} = (u_{j_1,g_2}^{\pi_w(z+\delta_{c,\circ})})^{c'} \aleph^c - \aleph^c (u_{j_1,g_2}^{\pi_w(z+\delta_{c,\bullet})})^{c'}$ for any $g \in J_{\mathfrak{c}}^N$ and $j \in J_{\mathfrak{d}}^N$.

(b) *if* $z \in \mathfrak{U}$ *and* $\mathfrak{c} = (\pi_w(z + \delta_{c,\circ}), c') \otimes (\aleph, c)$ *and* $\mathfrak{d} = (\aleph, c) \otimes (\pi_w(z + \delta_{c,\bullet}), c')$, *then* $\mathrm{r}^{\mathfrak{c}}_{\mathfrak{d}}(p)_{j,g} = \aleph^c (u^{\pi_w(z+\delta_{c,\bullet})}_{j_2,g_1})^{c'} - (u^{\pi_w(z+\delta_{c,\circ})}_{j_2,g_1})^{c'} \aleph^c$ *for any* $g \in J^N_{\mathfrak{c}}$ *and* $j \in J^N_{\mathfrak{d}}$.

(c) *if* $z \in \mathfrak{D}$ *and* $\mathfrak{c} = (\aleph, c) \otimes \pi_w(z + \delta_{c,\bullet})$ *and* $\mathfrak{d} = \pi_w(z + \delta_{c,\circ}) \otimes (\aleph, c)$, *then* $\mathrm{r}^{\mathfrak{c}}_{\mathfrak{d}}(p)_{j,g} = u^{\pi_w(z+\delta_{c,\circ})}_{j_1,g_2} \aleph^c - \aleph^c u^{\pi_w(z+\delta_{c,\bullet})}_{j_1,g_2}$ *for any* $g \in J^N_{\mathfrak{c}}$ *and* $j \in J^N_{\mathfrak{d}}$.

(d) *if* $z \in \mathfrak{D}$ *and* $\mathfrak{c} = \pi_w(z + \delta_{c,\circ}) \otimes (\aleph, c)$ *and* $\mathfrak{d} = (\aleph, c) \otimes \pi_w(z + \delta_{c,\bullet})$, *then* $\mathrm{r}^{\mathfrak{c}}_{\mathfrak{d}}(p)_{j,g} = \aleph^c u^{\pi_w(z+\delta_{c,\bullet})}_{j_2,g_1} - u^{\pi_w(z+\delta_{c,\circ})}_{j_2,g_1} \aleph^c$ *for any* $g \in J^N_{\mathfrak{c}}$ *and* $j \in J^N_{\mathfrak{d}}$.

Proof. Immediate from the definitions. □

Again, in the next proposition, remember that $\mathbb{C}$ is the zero object of the category of CQG Hopf $*$-algebras.

Proposition 6.6. *For any* $w \in \mathbb{N}_0$ *and any* $n \in \mathbb{N}$, *if* $(\mathfrak{U}, \mathfrak{D})$ *is either* $(\mathbb{Z}_w, \varnothing)$ *or* $(\varnothing, \mathbb{Z}_w)$, *if* V *is the action of* $\mathbb{Z}_w$ *on* $(\mathfrak{U} \otimes \{\circ, \bullet\}) \cup \mathfrak{D}$, *then for any* $\mathbb{Z}_w$*-invariant category* $\mathcal{X}$ *of* $(\mathfrak{U}, \mathfrak{D})$*-tagged labeled partitions, if* H *is the CQG Hopf* $*$*-algebra associated with* $(\mathfrak{U}, \mathfrak{D}, \mathcal{X}, (n)_{z \in \mathbb{Z}_w})$, *if* α *is the group homomorphism from* $\mathbb{Z}_w$ *to the group of CQG Hopf* $*$*-algebra automorphisms of* H *such that* α_z *is given by the unique* $*$*-algebra endomorphism of the underlying* $*$*-algebra of* H *with* $u^{\mathfrak{c}}_{j,i} \mapsto u^{V_z \circ \mathfrak{c}}_{j,i}$ *for any* $k \in \mathbb{N}_0$, *any* $\mathfrak{c}\colon [\![k]\!] \to (\mathfrak{U} \otimes \{\circ, \bullet\}) \cup \mathfrak{D}$, *any* $\{i, j\} \subseteq [\![n]\!]$ *and any* $z \in \mathbb{Z}_w$, *and if* H' *is the CQG Hopf* $*$*-algebra associated with* $(\mathfrak{U} \cup \{\aleph\}, \mathfrak{D}, \mathcal{X} \rtimes \mathcal{Z}_w, N)$, *where the dimension profile* N *is such that* $N_z = n$ *for any* $z \in \mathbb{Z}_w$ *and* $N_\aleph = 1$, *if* $(\iota_H, \iota_{\mathbb{Z}_w})$ *are the associated co-projections of the crossed co-product CQG Hopf* $*$*-algebra* $H \rtimes_\alpha \mathbb{C}[\mathbb{Z}_w]$ *of* H *and* $\mathbb{C}[\mathbb{Z}_w]$ *with respect to* α, *then there exists an isomorphism of CQG Hopf* $*$*-algebras from* $H \rtimes_\alpha \mathbb{C}[\mathbb{Z}_w]$ *to* H' *such that for any* $x \in \mathfrak{U}$, *any* $y \in \mathfrak{D}$ *and any* $\{i, j\} \subseteq [\![n]\!]$,

$$\iota_H(u^{(x,\circ)}_{j,i}) \mapsto v^{(x,\circ)}_{j,i} \quad \wedge \quad \iota_H(u^y_{j,i}) \mapsto v^y_{j,i} \quad \wedge \quad (w \neq 1 \Rightarrow \iota_{\mathbb{Z}_w}(1) \mapsto v^{\aleph}_{1,1}),$$

where $u^{(x,\circ)}_{j,i}$ *and* $v^{(x,\circ)}_{j,i}$ *are both short for* $((x, \circ), j, i)$, *where* $u^y_{j,i}$ *and* $v^y_{j,i}$ *are both short for* (y, j, i), *where* 1 *is the basis vector of the free vector space over* $\mathbb{Z}_w$ *belonging to the group element* 1, *and where* $v^{\aleph}_{1,1}$ *is short for* $(\aleph, 1, 1)$. *Moreover, this isomorphism is the unique* $*$*-algebra morphism betwen the underlying* $*$*-algebras with the given property.*

Proof. By Proposition 3.40 we can identify H with the underlying CQG Hopf $*$-algebra of the CMMQG $*$-algebra of profile $(n)_{z \in \mathbb{Z}_w}$ given by $(A, (u^z)_{z \in \mathbb{Z}_w})$ where A is the universal $*$-algebra over $\{u^z_{j,i} \mid z \in \mathbb{Z}_w \wedge \{i, j\} \subseteq [\![n]\!]\}$, where $u^z_{j,i}$ means (z, j, i) for any $\{i, j\} \subseteq [\![n]\!]$ and $z \in \mathbb{Z}_w$, subject to the relations

$$\begin{aligned}\{\mathrm{r}^{\mathfrak{c}}_{\mathfrak{d}}(p)_{j,g}((u^z)_{z \in \mathbb{Z}_w}) \mid \{k, \ell\} \subseteq \mathbb{N}_0 \wedge \mathfrak{c}\colon [\![k]\!] \to (\mathfrak{U} \otimes \{\circ, \bullet\}) \cup \mathfrak{D} \\ \wedge\ \mathfrak{d}\colon [\![\ell]\!] \to (\mathfrak{U} \otimes \{\circ, \bullet\}) \cup \mathfrak{D} \wedge (\mathfrak{c}, \mathfrak{d}, p) \in \mathcal{X} \\ \wedge\ g \in [\![n]\!]^{\otimes k} \wedge j \in [\![n]\!]^{\otimes \ell}\}\end{aligned}$$

and, additionally, the relations $\{(u^y_{j,i})^* - u^y_{j,i} \mid \{i, j\} \subseteq [\![n]\!] \wedge y \in \mathfrak{D}\}$, and where $u^z = (u^z_{j,i})_{(j,i) \in [\![n]\!]^{\otimes 2}}$ for any $z \in \mathbb{Z}_w$.

If $\mathcal{R}_w$ is the union of all sets of generating crosses for the $\mathbb{Z}_w$-crossed co-product with respect to $(\mathfrak{U}, \mathfrak{O})$, then $\mathcal{X} \cup \mathcal{Z}_w \cup \mathcal{R}_w$ generates $\mathcal{X} \rtimes \mathcal{Z}_w$ as a category of $(\mathfrak{U} \uplus \{\aleph\}, \mathfrak{O})$-tagged labeled partitions by Proposition 5.41. By applying Proposition 3.40 a second time we can thus identify H' with the underlying CQG Hopf $*$-algebra of the CMMQG $*$-algebra of profile N given by $(A', (v^z)_{z \in \mathbb{Z}_w \uplus \{\aleph\}})$, where A' is the universal $*$-algebra over $\{(z, j, i) \,|\, z \in \mathbb{Z}_w \wedge \{i, j\} \subseteq [\![n]\!]\} \cup \{(\aleph, 1, 1)\}$, whose elements we write as $v^z_{j,i} = (z, j, i)$ for any $z \in \mathbb{Z}_w$ and any $\{i, j\} \subseteq [\![n]\!]$ and as $v^{\aleph}_{1,1} = (\aleph, 1, 1)$, subject to the relations

$$\begin{aligned}\{\mathrm{r}^{\mathfrak{c}}_{\mathfrak{d}}(p)_{j,g}((v^z)_{z \in \mathbb{Z}_w \uplus \{\aleph\}}) \,|\, \{k, \ell\} \subseteq \mathbb{N}_0 \wedge \mathfrak{c} &\colon [\![k]\!] \to ((\mathfrak{U} \uplus \{\aleph\}) \otimes \{\circ, \bullet\}) \uplus \mathfrak{O} \\ \wedge\, \mathfrak{d} &\colon [\![\ell]\!] \to ((\mathfrak{U} \uplus \{\aleph\}) \otimes \{\circ, \bullet\}) \uplus \mathfrak{O} \\ &\wedge (\mathfrak{c}, \mathfrak{d}, p) \in \mathcal{X} \cup \mathcal{Z}_w \cup \mathcal{R}_w \wedge g \in J^N_{\mathfrak{c}} \wedge j \in J^N_{\mathfrak{d}}\}\end{aligned}$$

and, additionally, the relations $\{(v^y_{j,i})^* - v^y_{j,i} \,|\, \{i, j\} \subseteq [\![n]\!] \wedge y \in \mathfrak{O}\}$, and where $v^z = (v^z_{j,i})_{(j,i) \in [\![n]\!]^{\otimes 2}}$ for any $z \in \mathbb{Z}_w$ and $v^{\aleph} = v^{\aleph}_{1,1}$.

According to Lemma 6.5 another way of writing the set

$$\begin{aligned}\{\mathrm{r}^{\mathfrak{c}}_{\mathfrak{d}}(p)_{j,g}((v^z)_{z \in \mathbb{Z}_w \uplus \{\aleph\}}) \,|\, \{k, \ell\} \subseteq \mathbb{N}_0 \wedge \mathfrak{c} &\colon [\![k]\!] \to ((\mathfrak{U} \uplus \{\aleph\}) \otimes \{\circ, \bullet\}) \uplus \mathfrak{O} \\ \wedge\, \mathfrak{d} \colon [\![\ell]\!] \to ((\mathfrak{U} \uplus \{\aleph\}) \otimes \{\circ, \bullet\}) &\uplus \mathfrak{O} \wedge (\mathfrak{c}, \mathfrak{d}, p) \in \mathcal{R}_w \\ &\wedge g \in J^N_{\mathfrak{c}} \wedge j \in J^N_{\mathfrak{d}}\}\end{aligned}$$

is the following:

$$\begin{aligned}\bigcup\{\{&(v^{\pi_w(z+\delta_{c,\circ})}_{j,i})^{c'} (v^{\aleph}_{1,1})^c - (v^{\aleph}_{1,1})^c (v^{\pi_w(z+\delta_{c,\bullet})}_{j,i})^{c'}, \\ &(v^{\aleph}_{1,1})^c (v^{\pi_w(z+\delta_{c,\bullet})}_{j,i})^{c'} - (v^{\pi_w(z+\delta_{c,\circ})}_{j,i})^{c'} (v^{\aleph}_{1,1})^c \;|\, z \in \mathfrak{U}\}, \\ \{\quad &v^{\pi_w(z+\delta_{c,\circ})}_{j,i} (v^{\aleph}_{1,1})^c - (v^{\aleph}_{1,1})^c v^{\pi_w(z+\delta_{c,\bullet})}_{j,i}, \\ &(v^{\aleph}_{1,1})^c v^{\pi_w(z+\delta_{c,\bullet})}_{j,i} - v^{\pi_w(z+\delta_{c,\circ})}_{j,i} (v^{\aleph}_{1,1})^c \qquad |\, z \in \mathfrak{O}\} \\ |\, z \in \mathbb{Z}_w &\wedge \{c, c'\} \subseteq \{\circ, \bullet\} \wedge \{i, j\} \subseteq [\![n]\!]\}.\end{aligned}$$

Simplified, if $(\mathfrak{U}, \mathfrak{O}) = (\mathbb{Z}_w, \varnothing)$, then this set is the same as the relations

$$\begin{aligned}\{\pm(v^{\aleph}_{1,1}(v^z_{j,i})^{c'} - (v^{\pi_w(z+1)}_{j,i})^{c'} v^{\aleph}_{1,1}), \pm((v^{\aleph}_{1,1})^* (v^z_{j,i})^{c'} - (v^{\pi_w(z-1)}_{j,i})^{c'} (v^{\aleph}_{1,1})^*) \\ |\, z \in \mathbb{Z}_w \wedge \{i, j\} \subseteq [\![n]\!] \wedge c' \in \{\circ, \bullet\}\}\end{aligned}$$

and, if $(\mathfrak{U}, \mathfrak{O}) = (\varnothing, \mathbb{Z}_w)$, then this set is the same as

$$\begin{aligned}\{\pm(v^{\aleph}_{1,1} v^z_{j,i} - v^{\pi_w(z+1)}_{j,i} v^{\aleph}_{1,1}), \pm((v^{\aleph}_{1,1})^* v^z_{j,i} - v^{\pi_w(z-1)}_{j,i} (v^{\aleph}_{1,1})^*), \\ |\, z \in \mathbb{Z}_w \wedge \{i, j\} \subseteq [\![n]\!]\}\end{aligned}$$

If in the free algebra over $\{v^z_{j,i} \,|\, z \in \mathbb{Z}_w \wedge \{i, j\} \subseteq [\![n]\!]\} \cup \{v^{\aleph}_{1,1}\}$ the $*$-subalgebra generated by $\{v^z_{j,i} \,|\, z \in \mathbb{Z}_w \wedge \{i, j\} \subseteq [\![n]\!]\}$ is denoted by A', then the above set, the relations induced by $\mathcal{R}_w$, generates the same $*$-ideal as the set

$$\{v^{\aleph}_{1,1} a - \alpha_1(a) v^{\aleph}_{1,1}, (v^{\aleph}_{1,1})^* a - \alpha_{w-1}(a)(v^{\aleph}_{1,1})^* \,|\, a \in A'\}.$$

By Definition 2.24 and Lemma 6.4 that is what we needed to see. ☐

6.3. Relating the wreath graph co-products. Combining the results of Sections 6.1 and 6.2 then yields the following characterization of the easy quantum group associated with the a wreath graph co-product category.

PROPOSITION 6.7. *For any* $w \in \mathbb{N}_0$*, any* $n \in \mathbb{N}$*, any category* $\mathcal{C}$ *of either two-colored or uncolored partitions, seen as singly tagged with* $\beth$*, for any* $\mathbb{Z}_w$*-invariant partial commutation relation* r *on* $\mathbb{Z}_w$*, if* H *is the easy CQG Hopf* $*$*-algebra associated with* $(\{\beth\},\varnothing,\mathcal{C},n)$ *if* $\mathcal{C}$ *is two-colored and with* $(\varnothing,\{\beth\},\mathcal{C},n)$ *if* $\mathcal{C}$ *is uncolored, if* $(\iota_z)_{z\in\mathbb{Z}_w\cup\{\aleph\}}$ *are the co-projections associated with the wreath graph co-product CQG Hopf* $*$*-algebra of* H *and* $\mathbb{C}[\mathbb{Z}_w]$ *with respect to* r*, if* $(\mathfrak{U},\mathfrak{O})$ *is given by* $(\mathbb{Z}_w,\varnothing)$ *if* $\mathcal{C}$ *is two-colored and by* $(\varnothing,\mathbb{Z}_w)$ *if* $\mathcal{C}$ *is uncolored, if the dimension profile* N *is such that* $N_z = n$ *for any* $z \in \mathbb{Z}_w$ *and* $N_\aleph = 1$*, if* H' *is the easy CQG Hopf* $*$*-algebra associated with* $(\mathfrak{U}\cup\{\aleph\},\mathfrak{O},\mathcal{C}\wr_r\mathcal{Z}_w,N)$*, then there exists an isomorphism of CQG Hopf* $*$*-algebras from* $H\wr_r\mathbb{C}[\mathbb{Z}_w]$ *to* H' *such that for any* $x\in\mathfrak{U}$*, any* $y\in\mathfrak{O}$ *and any* $\{i,j\}\subseteq[\![n]\!]$*,*

$$\iota_x(u^{(\beth,\circ)}_{j,i}) \mapsto v^{(x,\circ)}_{j,i} \quad\wedge\quad \iota_y(u^{\beth}_{j,i}) \mapsto v^{y}_{j,i} \quad\wedge\quad \iota_\aleph(1)\mapsto v^{\aleph}_{1,1},$$

where $u^{(\beth,\circ)}_{j,i}$ *is short for* $((\beth,\circ),j,i)$ *and* $u^{\beth}_{j,i}$ *short for* $(\beth,j,i)$*, where* $v^{(x,\circ)}_{j,i}$ *is short for* $((x,\circ),j,i)$ *and* $v^{y}_{j,i}$ *short for* (y,j,i)*, where* 1 *is the basis vector of the free vector space over the set* $\mathbb{Z}_w$ *belonging to the group element* 1*, and where* $v^{\aleph}_{1,1}$ *is short for* $((\aleph,\circ),1,1)$*. Moreover, this isomorphism is the unique* $*$*-algebra morphism betwen the underlying* $*$*-algebras with the given property.*

PROOF. By Proposition 3.40 we can identify H with the underlying CQG Hopf $*$-algebra of the CMMQG $*$-algebra of profile n given by $(A,u^{\beth})$ where A is the universal $*$-algebra over $\{u^{\beth}_{j,i}\,|\,\{i,j\}\subseteq[\![n]\!]\}$, where $u^{\beth}_{j,i}$ means $(\beth,j,i)$ for any $\{i,j\}\subseteq[\![n]\!]$, subject to the relations

$$\begin{aligned}\{\mathrm{r}^{\mathfrak{c}}_{\mathfrak{d}}(p)_{j,g}(u^{\beth})\,|\,\{k,\ell\}\subseteq\mathbb{N}_0 \wedge \mathfrak{c}\colon [\![k]\!]\to((\{\beth\}\cap\mathfrak{U})\otimes\{\circ,\bullet\})\cup(\{\beth\}\cap\mathfrak{O})\\ \wedge\ \mathfrak{d}\colon[\![\ell]\!]\to((\{\beth\}\cap\mathfrak{U})\otimes\{\circ,\bullet\})\cup(\{\beth\}\cap\mathfrak{O})\wedge(\mathfrak{c},\mathfrak{d},p)\in\mathcal{C}\\ \wedge\ g\in[\![n]\!]^{\otimes k}\wedge j\in[\![n]\!]^{\otimes\ell}\}\end{aligned}$$

and, if $\beth\in\mathfrak{O}$, additionally, the relations $\{(u^{\beth}_{j,i})^*-u^{\beth}_{j,i}\,|\,\{i,j\}\subseteq[\![n]\!]\}$, and where $u^{\beth}=(u^{\beth}_{j,i})_{(j,i)\in[\![n]\!]^{\otimes 2}}$.

If $\mathcal{R}_{*_r}$ is the largest generating set of crosses for the r-graph product with respect to $(\mathfrak{U},\mathfrak{O})$ and if $\mathcal{R}_{\rtimes}$ is the largest generating set of crosses for the $\mathbb{Z}_w$-crossed product with respect to $(\mathfrak{U},\mathfrak{O})$, if $\mathcal{X}_z$ is given by $\mathcal{C}$, but singly tagged with z, for any $z\in\mathbb{Z}_w$. then $(\bigcup_{z\in\mathbb{Z}_w}\mathcal{X}_z)\cup\mathcal{Z}_w\cup\mathcal{R}_{*_r}\cup\mathcal{R}_{\rtimes}$ generates $\mathcal{C}\wr_r\mathcal{Z}_w$ as a category of $(\mathfrak{U}\cup\{\aleph\},\mathfrak{O})$-tagged labeled partitions by Proposition 5.43. Hence, Proposition 3.40 lets us identitfy H' with the underlying CQG Hopf $*$-algebra of the CMMQG $*$-algebra of profile N given by $(A',(v^z)_{z\in\mathbb{Z}_w\cup\{\aleph\}})$, where A' is the universal $*$-algebra over $\{(z,j,i)\,|\,z\in\mathbb{Z}_w\wedge\{i,j\}\subseteq[\![n]\!]\}\cup\{(\aleph,1,1)\}$, whose elements we write as $v^z_{j,i}=(z,j,i)$ for any

$z \in \mathbb{Z}_w$ and any $\{i,j\} \subseteq [\![n]\!]$ and as $v^{\aleph}_{1,1} = (\aleph, 1, 1)$, subject to the relations

$$\{\mathrm{r}^{\mathfrak{c}}_{\mathfrak{d}}(p)_{j,g}((v^z)_{z\in\mathbb{Z}_w\cup\{\aleph\}}) \,|\, \{k,\ell\} \subseteq \mathbb{N}_0 \wedge \mathfrak{c}\colon [\![k]\!] \to ((\mathfrak{U} \cup \{\aleph\}) \otimes \{\circ,\bullet\}) \cup \mathfrak{O} \wedge \mathfrak{d}\colon [\![\ell]\!] \to ((\mathfrak{U} \cup \{\aleph\}) \otimes \{\circ,\bullet\}) \cup \mathfrak{O} \wedge (\mathfrak{c},\mathfrak{d},p) \in (\textstyle\bigcup_{z\in\mathbb{Z}_w} \mathcal{X}_z) \cup \mathcal{Z}_w \cup \mathcal{R}_{*_r} \cup \mathcal{R}_{\rtimes} \wedge g \in J^N_{\mathfrak{c}} \wedge j \in J^N_{\mathfrak{d}}\}$$

and, if $\mathcal{C}$ is uncolored, additionally, the relations $\{(v^z_{j,i})^* - v^z_{j,i} \,|\, \{i,j\} \subseteq [\![n]\!] \wedge z \in \mathbb{Z}_w\}$, and where $v^z = (v^z_{j,i})_{(j,i)\in[\![n]\!]^{\otimes 2}}$ for any $z \in \mathbb{Z}_w$ and $v^{\aleph} = v^{\aleph}_{1,1}$.

By Lemma 6.1 the set

$$\{\mathrm{r}^{\mathfrak{c}}_{\mathfrak{d}}(p)_{j,g}((v^z)_{z\in\mathbb{Z}_w\cup\{\aleph\}}) \,|\, \{k,\ell\} \subseteq \mathbb{N}_0 \wedge \mathfrak{c}\colon [\![k]\!] \to ((\mathfrak{U} \cup \{\aleph\}) \otimes \{\circ,\bullet\}) \cup \mathfrak{O} \wedge \mathfrak{d}\colon [\![\ell]\!] \to ((\mathfrak{U} \cup \{\aleph\}) \otimes \{\circ,\bullet\}) \cup \mathfrak{O} \wedge (\mathfrak{c},\mathfrak{d},p) \in \mathcal{R}_{*_r} \wedge g \in J^N_{\mathfrak{c}} \wedge j \in J^N_{\mathfrak{d}}\},$$

if $\mathcal{C}$ is two-colored, is given by

$$\{(v^{z_1}_{j_1,i_1})^{c_1}(v^{z_2}_{j_2,i_2})^{c_2} - (v^{z_2}_{j_2,i_2})^{c_2}(v^{z_1}_{j_1,i_1})^{c_1} \,|\, (z_1,z_2) \in r \wedge \{i_1,i_2,j_1,j_2\} \subseteq [\![n]\!] \wedge \{c_1,c_2\} \subseteq \{\circ,\bullet\}\},$$

and, if $\mathcal{C}$ is uncolored, by

$$\{v^{z_1}_{j_1,i_1}v^{z_2}_{j_2,i_2} - v^{z_2}_{j_2,i_2}v^{z_1}_{j_1,i_1} \,|\, (z_1,z_2) \in r \wedge \{i_1,i_2,j_1,j_2\} \subseteq [\![n]\!]\}.$$

Furthermore, according to Lemma 6.5 the relations

$$\{\mathrm{r}^{\mathfrak{c}}_{\mathfrak{d}}(p)_{j,g}((v^z)_{z\in\mathbb{Z}_w\cup\{\aleph\}}) \,|\, \{k,\ell\} \subseteq \mathbb{N}_0 \wedge \mathfrak{c}\colon [\![k]\!] \to ((\mathfrak{U} \cup \{\aleph\}) \otimes \{\circ,\bullet\}) \cup \mathfrak{O} \wedge \mathfrak{d}\colon [\![\ell]\!] \to ((\mathfrak{U} \cup \{\aleph\}) \otimes \{\circ,\bullet\}) \cup \mathfrak{O} \wedge (\mathfrak{c},\mathfrak{d},p) \in \mathcal{R}_{\rtimes} \wedge g \in J^N_{\mathfrak{c}} \wedge j \in J^N_{\mathfrak{d}}\},$$

if $\mathcal{C}$ is two-colored, are the same as

$$\{\pm(v^{\aleph}_{1,1}(v^z_{j,i})^{c'} - (v^{\pi_w(z+1)}_{j,i})^{c'}v^{\aleph}_{1,1}), \pm((v^{\aleph}_{1,1})^*(v^z_{j,i})^{c'} - (v^{\pi_w(z-1)}_{j,i})^{c'}(v^{\aleph}_{1,1})^*) \,|\, z \in \mathbb{Z}_w \wedge \{i,j\} \subseteq [\![n]\!] \wedge c' \in \{\circ,\bullet\}\},$$

and, if $\mathcal{C}$ is uncolored, the same as

$$\{\pm(v^{\aleph}_{1,1}v^z_{j,i} - v^{\pi_w(z+1)}_{j,i}v^{\aleph}_{1,1}), \pm((v^{\aleph}_{1,1})^*v^z_{j,i} - v^{\pi_w(z-1)}_{j,i}(v^{\aleph}_{1,1})^*), \,|\, z \in \mathbb{Z}_w \wedge \{i,j\} \subseteq [\![n]\!]\}.$$

Thus, a comparison with Definition 2.26 and Lemma 6.4 proves the claim. □

7. The partitions of the orthogonal and unitary half-liberations

Section 7 recalls the definitions of the categories of un- and two-colored partitions inducing the representation theories of the orthogonal respectively unitary half-liberations. In the latter case, a new formulation of those definitions is given.

7.1. The partitions of the orthogonal half-liberations. The original case of labeled partitions considered already by Brauer in the group case [Bra37] and famously by Banica and Speicher for quantum groups [BS09] is the "orthogonal" one, where there is only one tag and no colors.

ASSUMPTION 7.1. In Section 7.1, our choice of tags will be $(\mathfrak{U}, \mathfrak{O}) = (\varnothing, \{z\})$, where z can be arbitrary (but fixed in the following).

NOTATION 7.2. Because $\mathfrak{U} \uplus \mathfrak{O}$ consists of only a single orthogonal tag, actually, the only information retained by labels is their lengths. However, those can already be recovererd from the partition itself. Hence, in Section 7.1 labels, both tags and colors, will be omitted altogether. A labeled partition is then only an ordinary, set-theoretical partition. Instead of $(\mathfrak{U}, \mathfrak{O})$-tagged labeled partitions we simply speak of *partitions.*

In the orthogonal case, there are only three "half-liberations", as shown by Banica and Speicher [BS09].

DEFINITION 7.3. (a) Let $\mathcal{O}$ be the set of all partitions p such that $|\mathsf{B}| = 2$ for any $\mathsf{B} \in p$.
(b) Let $\mathcal{O}^*$ be the set of all partitions p for which there exist $\{k, \ell\} \subseteq \mathbb{N}_0$ such that p is a set-theoretical partition of Π^k_ℓ and such that $|\mathsf{B}| = 2$ and $|]\mathsf{b}, \mathsf{b}'[^k_\ell| \equiv_2 0$ for any $\{\mathsf{b}, \mathsf{b}'\} \subseteq \mathsf{B}$ with $\mathsf{b} \neq \mathsf{b}'$ and any $\mathsf{B} \in p$.
(c) Let $\mathcal{O}^+$ be the set of all partitions p for which there exist $\{k, \ell\} \subseteq \mathbb{N}_0$ such that p is a set-theoretical partition of Π^k_ℓ, such that $|\mathsf{B}| = 2$ for any $\mathsf{B} \in p$ and such that $\mathsf{B}_1 \not\diagdown^k_\ell \mathsf{B}_2$ for any $\{\mathsf{B}_1, \mathsf{B}_2\} \subseteq p$ with $\mathsf{B}_1 \neq \mathsf{B}_2$.

PROPOSITION 7.4. *(a) $\mathcal{O}$, $\mathcal{O}^*$ and $\mathcal{O}^+$ are categories of partitions.*
(b) Those three are the only categories $\mathcal{C}$ of partitions with $\mathcal{O}^+ \subseteq \mathcal{C} \subseteq \mathcal{O}$.
(c) $\mathcal{O}^+ \subsetneq \mathcal{O}^* \subsetneq \mathcal{O}$.
(d) $\mathcal{O}^+ = \langle \varnothing \rangle$ and $\mathcal{O}^* = \langle ⧻ \rangle$ and $\mathcal{O} = \langle ⧺ \rangle$.

DEFINITION 7.5. Any partition p *has no odd blocks* if $|\mathsf{B}| \equiv_2 0$ for any $\mathsf{B} \in p$.

The following is well-known (compare also [TW18, Lemma 1.1 (c)]).

PROPOSITION 7.6. *Any category of partitions consisting only of partitions which have no odd blocks is $\otimes$-elbats. In particular, so are $\mathcal{O}$, $\mathcal{O}^*$ and $\mathcal{O}^+$.*

PROOF. For each $t \in [\![2]\!]$, if $\{k_t, \ell_t\} \subseteq \mathbb{N}_0$ and if p_t is any set-theoretical partition of $\Pi^{k_t}_{\ell_t}$ with even blocks, then $k_t + \ell_t = \sum_{\mathsf{B} \in p_t} |\mathsf{B}| \equiv_2 0$. For that reason, if $q_1 \coloneqq p_1{}^{\curvearrowleft k_1}$ and $q_2 \coloneqq p_2{}^{\wr k_2}$, then $(q_1 \otimes q_2, \mathrm{id}_{k_1+\ell_1} \otimes \mathrm{ev}_{(k_2+\ell_2)/2})$ and $(q_1 \otimes q_2, \mathrm{ev}_{(k_1+\ell_1)/2} \otimes \mathrm{id}_{k_2+\ell_2})$ are well-defined composable pairings. Because the compositions of those pairings are precisely q_1 and q_2, respectively, that proves $\{q_1, q_2\} \subseteq \langle q_1 \otimes q_2 \rangle$. Since $p_1 = q_1{}^{\curvearrowright k_1}$ and $p_2 = q_2{}^{\rotatebox{180}{$\wr$} k_2}$ and since $q_1 \otimes q_2 = ((p_1 \otimes p_2)^{\curvearrowleft k_1})^{\wr k_2}$ we have thus shown $\{p_1, p_2\} \subseteq \langle p_1 \otimes p_2 \rangle$. $\square$

By Proposition 4.35 it follows that for such categories there is no difference between big and little graph power or wreath graph power categories.

Definition 7.7. For any $n \in \mathbb{N}$ we call the easy algebraic compact quantum group asssociated with

(a) $(\mathcal{O}, n)$ the orthogonal group O_n
(b) $(\mathcal{O}^*, n)$ the half-liberated orthogonal quantum group O_n^*
(c) $(\mathcal{O}^+, n)$ the fre orthogonal quantum group O_n^+

in dimension n.

7.2. The partitions of the unitary half-liberations. We recall the definition of the categories inducing the representations of the unitary half-liberations in the sense of [MW20] and [MW21a]. Moreover, it is convenient to give a reformulation of their definitions in terms of reindexed restrictions.

Assumption 7.8. In Section 7.2, our choice of tags will be $(\mathfrak{U}, \mathfrak{D}) = (\{z\}, \varnothing)$, where z can be arbitrary (but fixed in the following).

Notation 7.9. Since the tag set $\mathfrak{U} \cup \mathfrak{D}$ is a singleton, tags will be omitted from the notation altogether in Section 7.2. Instead of $(\mathfrak{U}, \mathfrak{D})$-tagged labeled partitions we simply speak of *two-colored partitions.*

The two extreme unitary half-liberations are well-known.

Definition 7.10. (a) Let $\mathcal{U}$ be the set of all $(\mathfrak{c}, \mathfrak{d}, p) \in \mathcal{S}$ with $|\mathsf{B}| = 2$ and $\sigma_{\mathfrak{d}}^{\mathfrak{c}}(\mathsf{B}) = 0$ for any $\mathsf{B} \in p$.

(b) Let $\mathcal{U}^+$ be the set of all $(\mathfrak{c}, \mathfrak{d}, p) \in \mathcal{S}$ for which there exist $\{k, \ell\} \subseteq \mathbb{N}_0$ with $\mathfrak{c}\colon [\![k]\!] \to \{\circ, \bullet\}$ and $\mathfrak{d}\colon [\![\ell]\!] \to \{\circ, \bullet\}$ such that $|\mathsf{B}| = 2$ and $\sigma_{\mathfrak{d}}^{\mathfrak{c}}(\mathsf{B}) = 0$ for any $\mathsf{B} \in p$ and such that $\mathsf{B}_1 \mathbin{\diagdown\!\diagdown}_{\ell}^{k} \mathsf{B}_2$ for any $\{\mathsf{B}_1, \mathsf{B}_2\} \subseteq p$ with $\mathsf{B}_1 \neq \mathsf{B}_2$.

Of course, they correspond respectively to the classical unitary groups $(U_n)_{n \in \mathbb{N}}$ and Wang's [Wan95a] free unitary quantum groups $(U_n^+)_{n \in \mathbb{N}}$.

Definition 7.11. (a) For any $w \in \mathbb{N}$ let $\mathcal{U}_w^*$ be the set of all $(\mathfrak{c}, \mathfrak{d}, p) \in \mathcal{S}$ such that $|\mathsf{B}| = 2$ and $\sigma_{\mathfrak{d}}^{\mathfrak{c}}(\mathsf{B}) = 0$ and $\delta_{\mathfrak{d}}^{\mathfrak{c}}(\mathsf{b}, \mathsf{b}') \equiv_w 0$ for any $\mathsf{B} \in p$ and any $\{\mathsf{b}, \mathsf{b}'\} \subseteq \mathsf{B}$.

(b) For any additive subsemigroup D of $\mathbb{N}$ let $\mathcal{U}_D^{\times}$ be the set of all $(\mathfrak{c}, \mathfrak{d}, p) \in \mathcal{S}$ for which there exist $\{k, \ell\} \subseteq \mathbb{N}_0$ with $\mathfrak{c}\colon [\![k]\!] \to \{\circ, \bullet\}$ and $\mathfrak{d}\colon [\![\ell]\!] \to \{\circ, \bullet\}$ such that $|\mathsf{B}| = 2$ and $\sigma_{\mathfrak{d}}^{\mathfrak{c}}(\mathsf{B}) = 0$ and $\delta_{\mathfrak{d}}^{\mathfrak{c}}(\mathsf{b}, \mathsf{b}') = 0$ for any $\mathsf{B} \in p$ and any $\{\mathsf{b}, \mathsf{b}'\} \subseteq \mathsf{B}$, and such that for any $\{\mathsf{B}_1, \mathsf{B}_2\} \subseteq p$ with $\mathsf{B}_1 \neq \mathsf{B}_2$, whenever there exist $\mathsf{b}_1 \in \mathsf{B}_1$ and $\mathsf{b}_2 \in \mathsf{B}_2$ with $|\delta_{\mathfrak{d}}^{\mathfrak{c}}(\mathsf{b}_1, \mathsf{b}_2)| \in D$, then $\mathsf{B}_1 \mathbin{\diagdown\!\diagdown}_{\ell}^{k} \mathsf{B}_2$.

(c) For any additive subsemigroup D of $\mathbb{N}$ let $\mathcal{U}_D^{\times +}$ be the set of all $(\mathfrak{c}, \mathfrak{d}, p) \in \mathcal{S}$ for which there exist $\{k, \ell\} \subseteq \mathbb{N}_0$ with $\mathfrak{c}\colon [\![k]\!] \to \{\circ, \bullet\}$ and $\mathfrak{d}\colon [\![\ell]\!] \to \{\circ, \bullet\}$ such that $|\mathsf{B}| = 2$ and $\sigma_{\mathfrak{d}}^{\mathfrak{c}}(\mathsf{B}) = 0$ and $\delta_{\mathfrak{d}}^{\mathfrak{c}}(\mathsf{b}, \mathsf{b}') = 0$ for any $\mathsf{B} \in p$ and any $\{\mathsf{b}, \mathsf{b}'\} \subseteq \mathsf{B}$, and such that for any $\{\mathsf{B}_1, \mathsf{B}_2\} \subseteq p$ with $\mathsf{B}_1 \neq \mathsf{B}_2$, whenever there exist $\mathsf{b}_1 \in \mathsf{B}_1$ and $\mathsf{b}_2 \in \mathsf{B}_2$ with $|\delta_{\mathfrak{d}}^{\mathfrak{c}}(\mathsf{b}_1, \mathsf{b}_2)| \in D \cup \{0\}$, then $\mathsf{B}_1 \mathbin{\diagdown\!\diagdown}_{\ell}^{k} \mathsf{B}_2$.

The sets $\mathcal{U}_w^*$ for $w \in \mathbb{N}$ were denoted by $\mathcal{S}_w$ in [MW20] and [MW21a]. And for any subsemigroup D of $(\mathbb{N}, +)$ the sets $\mathcal{U}_D^{\times}$ and $\mathcal{U}_D^{\times +}$ were called $\mathcal{I}_D$ and $\mathcal{I}_{D \cup \{0\}}$, respectively. In contrast to [MW20] and [MW21a], the category $\mathcal{I}_0$ there is *not* addressed as "$\mathcal{U}_0^*$" here, only as $\mathcal{U}_\varnothing^\times$.

The following was shown in [MW20] and [MW21a].

PROPOSITION 7.12. *(a)* $\mathcal{U}_w^*$, $\mathcal{U}_D^\times$ *and* $\mathcal{U}_D^{\times +}$ *are categories of two-colored partitions for any* $w \in \mathbb{N}$ *and any additive subsemigroup* D *of* $\mathbb{N}$.

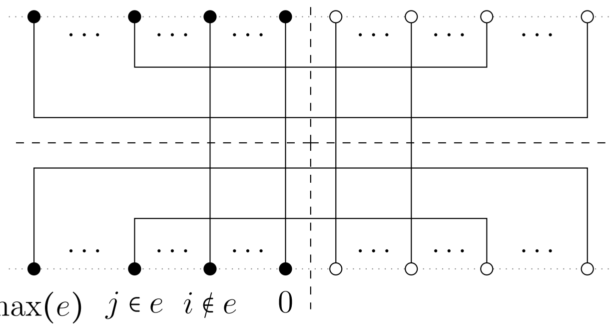
(b) For any category $\mathcal{C}$ *of two-colored partitions with* $\mathcal{U}^+ \subseteq \mathcal{C} \subseteq \mathcal{U}$ *there exists* $w \in \mathbb{N}$ *such that* $\mathcal{C} = \mathcal{U}_w^*$ *or an additive subsemigroup* D *of* $\mathbb{N}$ *such that* $\mathcal{C} = \mathcal{U}_D^\times$ *or* $\mathcal{C} = \mathcal{U}_D^{\times +}$.

(c) For any $w \in \mathbb{N}$ *and any additive subsemigroup* D *of* $\mathbb{N}$*:*

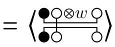
$$\begin{array}{ccccccccccc} & & \mathcal{U}_{\mathbb{N}}^\times & \subset & \mathcal{U}_D^\times & \subset & \mathcal{U}_\varnothing^\times & \subset & \mathcal{U}_w^* & \subset & \mathcal{U}_1^* = \mathcal{U} \\ & & \cup & & \cup & & \cup & & & & \\ \mathcal{U}^+ & = & \mathcal{U}_{\mathbb{N}}^{\times +} & \subset & \mathcal{U}_D^{\times +} & \subset & \mathcal{U}_\varnothing^{\times +} & & & & \end{array}$$

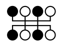
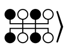
Moreover, all these inclusions are strict if $w \neq 1$ *and* $D \notin \{\varnothing, \mathbb{N}\}$.

(d) If $\mathrm{Br}_\bullet(\varnothing) \coloneqq (\varnothing, \varnothing, \varnothing)$ *and if for any finite* $e \subseteq \mathbb{N}$ *with* $e \neq \varnothing$ *the partition*



is denoted by $\mathrm{Br}_\bullet(e)$*, then for any* $w \in \mathbb{N}$ *and any additive subsemigroup* D *of* $\mathbb{N}$,

$\mathcal{U}_w^* = \langle$  $\rangle$,

$\mathcal{U}_D^\times = \begin{cases} \langle \mathrm{Br}_\bullet([\![k]\!] \backslash D), \end{cases}$  $\mid k \in \mathbb{N}\backslash D\rangle$ *if* $|\mathbb{N}\backslash D| = \infty$; $\langle \mathrm{Br}_\bullet(\mathbb{N}\backslash D),$  $\rangle$ *otherwise*,

$$\mathcal{U}_D^{\times +} = \begin{cases} \langle \mathrm{Br}_\bullet([\![k]\!] \backslash D) \mid k \in \mathbb{N}\backslash D\rangle & \text{if } |\mathbb{N}\backslash D| = \infty \\ \langle \mathrm{Br}_\bullet(\mathbb{N}\backslash D)\rangle & \text{otherwise} \end{cases}.$$

The reformulation of the definitions of these categories developed below (see Proposition 7.19) motivates all the results in this chapter.

By Lemma 3.10 the following definitions make sense.

DEFINITION 7.13. Let $w \in \mathbb{N}_0$ and let $\{k, \ell\} \subseteq \mathbb{N}_0$ and $\mathfrak{c}\colon [\![k]\!] \to \{\circ, \bullet\}$ and $\mathfrak{d}\colon [\![\ell]\!] \to \{\circ, \bullet\}$ be such that $\Sigma_{\mathfrak{d}}^{\mathfrak{c}} \equiv_w 0$.

(a) We call the set-theoretical partition ${}^w\Delta_{\mathfrak{d}}^{\mathfrak{c}}$ of Π_ℓ^k associated with the equivalence relation which for any $\{\mathtt{b}, \mathtt{b}'\} \subseteq \Pi_\ell^k$ calls $\mathtt{b}$ and $\mathtt{b}'$ equivalent if and only if $\delta^{\mathfrak{c}}_{\mathfrak{d}}(\mathtt{b},\mathtt{b}') \equiv_w 0$ the *w-parts* for k upper $\mathfrak{c}$-colored and ℓ lower $\mathfrak{d}$-colored points.

(b) For any $\{\mathsf{S}_1,\mathsf{S}_2\} \subseteq {}^w\Delta^{\mathfrak{c}}_{\mathfrak{d}}$ we write ${}^w\delta^{\mathfrak{c}}_{\mathfrak{d}}(\mathsf{S}_1,\mathsf{S}_2) \coloneqq \delta^{\mathfrak{c}}_{\mathfrak{d}}(\mathtt{b}_1,\mathtt{b}_2)$, where $\mathtt{b}_1 \in \mathsf{S}_1$ and $\mathtt{b}_2 \in \mathsf{S}_2$ can be arbitrary.

As the next proposition shows, with respect to a pair of labelings with vanishing total color sum, any part also has color sum zero.

PROPOSITION 7.14. *For any* $w \in \mathbb{N}_0$*, any* $\{k,\ell\} \subseteq \mathbb{N}_0$ *and any* $\mathfrak{c}\colon [\![k]\!] \to \{\circ,\bullet\}$ *and* $\mathfrak{d}\colon [\![\ell]\!] \to \{\circ,\bullet\}$*, if* $\Sigma^{\mathfrak{c}}_{\mathfrak{d}} = 0$*, then* $\sigma^{\mathfrak{c}}_{\mathfrak{d}}(\mathsf{S}) = 0$ *for any* $\mathsf{S} \in {}^w\Delta^{\mathfrak{c}}_{\mathfrak{d}}$*.*

PROOF. The proof goes by induction over $k+\ell$. It is vacuously true for $k+\ell = 0$ because then ${}^w\Delta^{\mathfrak{c}}_{\mathfrak{d}} = \varnothing$. For general $0 < k+\ell$ we are going to construct $\{m,n\} \subseteq \mathbb{N}_0$ and $\mathfrak{a}\colon [\![m]\!] \to \{\circ,\bullet\}$ and $\mathfrak{b}\colon [\![n]\!] \to \{\circ,\bullet\}$ with $m+n < k+\ell$ and $\Sigma^{\mathfrak{a}}_{\mathfrak{b}} = 0$ as well as $\mathsf{R} \in {}^w\Delta^{\mathfrak{a}}_{\mathfrak{b}} \cup \{\varnothing\}$ with $\sigma^{\mathfrak{a}}_{\mathfrak{b}}(\mathsf{R}) = \sigma^{\mathfrak{c}}_{\mathfrak{d}}(\mathsf{S})$. In the case that $\mathsf{R} = \varnothing$ the claim will then be trivially true. And, in the alternative case it will then follow immediately by applying the induction hypothesis to $\mathfrak{a}$, $\mathfrak{b}$ and R.

Step 1: Definition of $\mathfrak{a}$*,* $\mathfrak{b}$ *and* R*.* More precisely, we are going to define a proper subset M of Π^k_ℓ and then let $m \coloneqq \alpha(\mathsf{M})$ and $n \coloneqq \beta(\mathsf{M})$ and $\mathfrak{a} \coloneqq (\mathfrak{c} \,\blacksquare\, \mathfrak{d}) \circ \eta_{\mathsf{M},\ell}{}^{k}$ and $\mathfrak{b} \coloneqq (\mathfrak{c} \,\blacksquare\, \mathfrak{d}) \circ \theta_{\mathsf{M},\ell}{}^{k}$ and $\mathsf{R} = \gamma_{\mathsf{M},\ell}{}^{k\leftarrow}(\mathsf{S})$. Actually, M is going to be the complement of a set $\mathsf{T} \subseteq \Pi^k_\ell$ with $|\mathsf{T}| = 2$ and $\sigma^{\mathfrak{c}}_{\mathfrak{d}}(\mathsf{T}) = 0$ which is convex with respect to Γ^k_ℓ.

Such a set T exists because $\Sigma^{\mathfrak{c}}_{\mathfrak{d}} = 0$. Indeed, if $\{\mathtt{t} \in \Pi^k_\ell \wedge \zeta^{\mathfrak{c}}_{\mathfrak{d}}(\nu^k_\ell(\mathtt{t})) \neq \zeta^{\mathfrak{c}}_{\mathfrak{d}}(\mathtt{t})\}$ was empty, it would inductively follow $\zeta^{\mathfrak{c}}_{\mathfrak{d}}(\mathtt{a}) = \zeta^{\mathfrak{c}}_{\mathfrak{d}}(\mathtt{b})$ for any $\{\mathtt{a},\mathtt{b}\} \subseteq \Pi^k_\ell$ and thus $\Sigma^{\mathfrak{c}}_{\mathfrak{d}} = \sum_{\mathtt{a}\in\Pi^k_\ell} \sigma(\zeta^{\mathfrak{c}}_{\mathfrak{d}}(\mathtt{a})) = \varepsilon \cdot (k+\ell) \neq 0$ for some $\varepsilon \in \{-1,1\}$. If $\mathtt{t} \in \Pi^k_\ell$ is such that $\zeta^{\mathfrak{c}}_{\mathfrak{d}}(\nu^k_\ell(\mathtt{t})) \neq \zeta^{\mathfrak{c}}_{\mathfrak{d}}(\mathtt{t})$, then the two-elemental set $\mathsf{T} \coloneqq \{\mathtt{t},\nu^k_\ell(\mathtt{t})\}$ is convex with respect to Γ^k_ℓ and satisfies $\sigma^{\mathfrak{c}}_{\mathfrak{d}}(\mathsf{T}) = \sigma(\zeta^{\mathfrak{c}}_{\mathfrak{d}}(\mathtt{t})) + \sigma(\zeta^{\mathfrak{c}}_{\mathfrak{d}}(\nu^k_\ell(\mathtt{t}))) = 0$.

Step 2: Verifying the asserted properties of $\mathfrak{a}$*,* $\mathfrak{b}$ *and* R*.* It remains to prove that $\Sigma^{\mathfrak{a}}_{\mathfrak{b}} = 0$ and $\mathsf{R} \in {}^w\Delta^{\mathfrak{a}}_{\mathfrak{b}} \cup \{\varnothing\}$ and $\sigma^{\mathfrak{a}}_{\mathfrak{b}}(\mathsf{R}) = \sigma^{\mathfrak{c}}_{\mathfrak{d}}(\mathsf{S})$.

Step 2.1: Proving $\Sigma^{\mathfrak{a}}_{\mathfrak{b}} = 0$*.* The definition $\zeta^{\mathfrak{a}}_{\mathfrak{b}} = \zeta^{\mathfrak{c}}_{\mathfrak{d}} \circ \gamma_{\mathsf{M},\ell}{}^{k}$ allows us to compute

$$\begin{aligned}
\Sigma^{\mathfrak{a}}_{\mathfrak{b}} &= \sigma^{\mathfrak{a}}_{\mathfrak{b}}(\Pi^m_n) \\
&= \textstyle\sum_{\mathtt{z}\in\Pi^m_n} \sigma(\zeta^{\mathfrak{a}}_{\mathfrak{b}}(\mathtt{z})) \\
&= \textstyle\sum_{\mathtt{z}\in\Pi^m_n} \sigma(\zeta^{\mathfrak{c}}_{\mathfrak{d}}(\gamma_{\mathsf{M},\ell}{}^{k}(\mathtt{z}))) \\
&= \textstyle\sum_{\mathtt{e}\in\mathsf{M}} \sigma(\zeta^{\mathfrak{c}}_{\mathfrak{d}}(\mathtt{e})) \\
&= \sigma^{\mathfrak{c}}_{\mathfrak{d}}(\mathsf{M}) \\
&= \sigma^{\mathfrak{c}}_{\mathfrak{d}}(\Pi^k_\ell) - \sigma^{\mathfrak{c}}_{\mathfrak{d}}(\mathsf{T})
\end{aligned}$$

and because $\sigma^{\mathfrak{c}}_{\mathfrak{d}}(\Pi^k_\ell) = \Sigma^{\mathfrak{c}}_{\mathfrak{d}} = 0$ by assumption and $\sigma^{\mathfrak{c}}_{\mathfrak{d}}(\mathsf{T}) = 0$ by construction.

Step 2.2: Proving $\sigma^{\mathfrak{a}}_{\mathfrak{b}}(\mathsf{R}) = \sigma^{\mathfrak{c}}_{\mathfrak{d}}(\mathsf{S})$*.* Since $]\mathtt{t},\nu^k_\ell(\mathtt{t})[^k_\ell = \varnothing$, by definition, $\delta^{\mathfrak{c}}_{\mathfrak{d}}(\mathtt{t},\nu^k_\ell(\mathtt{t})) = \frac{1}{2}\sigma^{\mathfrak{c}}_{\mathfrak{d}}(\{\mathtt{t},\nu^k_\ell(\mathtt{t})\}) + \sigma^{\mathfrak{c}}_{\mathfrak{d}}(]\mathtt{t},\nu^k_\ell(\mathtt{t})[^k_\ell) = \frac{1}{2}\sigma^{\mathfrak{c}}_{\mathfrak{d}}(\mathsf{T}) = 0$. In other words, $\mathtt{t}$ and $\nu^k_\ell(\mathtt{t})$ belong to the same w-part. Consequently, either $\mathsf{T} \subseteq \mathsf{S}$ or $\mathsf{T} \cap \mathsf{S} = \varnothing$ because ${}^w\Delta^{\mathfrak{c}}_{\mathfrak{d}}$ is a set-theoretical partition of Π_ℓ^k. Furthermore, we conclude from $\zeta_\mathfrak{b}^\mathfrak{a} = \zeta_\mathfrak{d}^\mathfrak{c} \circ \gamma_{\mathsf{M},\ell}^{\;k}$ that

$$\begin{aligned}\sigma_\mathfrak{b}^\mathfrak{a}(\mathsf{R}) &= \textstyle\sum_{\mathsf{z}\in\mathsf{R}} \sigma(\zeta_\mathfrak{b}^\mathfrak{a}(\mathsf{z}))\\ &= \textstyle\sum_{\mathsf{z}\in\mathsf{R}} \sigma(\zeta_\mathfrak{d}^\mathfrak{c}(\gamma_{\mathsf{M},\ell}^{\;k}(\mathsf{z})))\\ &= \textstyle\sum_{\mathsf{e}\in\mathsf{S}\cap\mathsf{M}} \sigma(\zeta_\mathfrak{d}^\mathfrak{c}(\mathsf{e}))\\ &= \sigma_\mathfrak{d}^\mathfrak{c}(\mathsf{S}\cap\mathsf{M})\\ &= \sigma_\mathfrak{d}^\mathfrak{c}(\mathsf{S}\backslash\mathsf{T}),\end{aligned}$$

which is $\sigma_\mathfrak{d}^\mathfrak{c}(\mathsf{S})$ if $\mathsf{T}\cap\mathsf{S} = \varnothing$ and $\sigma_\mathfrak{d}^\mathfrak{c}(\mathsf{S}) - \sigma_\mathfrak{d}^\mathfrak{c}(\mathsf{T})$ if $\mathsf{T} \subseteq \mathsf{S}$ and thus by $\sigma_\mathfrak{d}^\mathfrak{c}(\mathsf{T}) = 0$, ultimately, $\sigma_\mathfrak{d}^\mathfrak{c}(\mathsf{S})$ in any case.

Step 2.3: Proving $\mathsf{R} \in {}^w\Delta_\mathfrak{b}^\mathfrak{a} \cup \{\varnothing\}$. If $\mathsf{S} \subseteq \mathsf{T}$, then $\mathsf{R} = \gamma_{\mathsf{M},\ell}^{\;k\leftarrow}(\mathsf{S}) \neq \varnothing$ and the proof is complete. Hence, we can assume $\mathsf{S} \notin \mathsf{T}$, i.e., $\mathsf{S}\cap\mathsf{M} \neq \varnothing$. Because then $\mathsf{R} \in R({}^w\Delta_\mathfrak{d}^\mathfrak{c}, \mathsf{M})$ it suffices to show that, actually, ${}^w\Delta_\mathfrak{b}^\mathfrak{a} = R({}^w\Delta_\mathfrak{d}^\mathfrak{c}, \mathsf{M})$. That is the same as proving for any $\{\mathtt{x},\mathtt{y}\} \subseteq \Pi_n^m$ that $\delta_\mathfrak{b}^\mathfrak{a}(\mathtt{x},\mathtt{y}) \equiv_w 0$ if and only if $\delta_\mathfrak{d}^\mathfrak{c}(\gamma_{\mathsf{M},\ell}^{\;k}(\mathtt{x}), \gamma_{\mathsf{M},\ell}^{\;k}(\mathtt{y})) \equiv_w 0$. This is certainly true if for any $\{\mathtt{x},\mathtt{y}\} \subseteq \Pi_n^m$ already $\delta_\mathfrak{b}^\mathfrak{a}(\mathtt{x},\mathtt{y}) = \delta_\mathfrak{d}^\mathfrak{c}(\gamma_{\mathsf{M},\ell}^{\;k}(\mathtt{x}), \gamma_{\mathsf{M},\ell}^{\;k}(\mathtt{y}))$, which we now show.

For $\mathtt{x} = \mathtt{y}$ this is clear. If $\mathtt{x} \neq \mathtt{y}$ and $\mathtt{a} \coloneqq \gamma_{\mathsf{M},\ell}^{\;k}(\mathtt{x})$ and $\mathtt{b} \coloneqq \gamma_{\mathsf{M},\ell}^{\;k}(\mathtt{y})$, then also $\mathtt{a} \neq \mathtt{b}$ because $\gamma_{\mathsf{M},\ell}^{\;k}$ is injective. Since $\gamma_{\mathsf{M},\ell}^{\;k}$ is also cyclically monotonic with respect to Γ_n^m and Γ_ℓ^k, moreover, $\gamma_{\mathsf{M},\ell}^{\;k}{}_{\rightarrow}(]\mathtt{x},\mathtt{y}[_n^m) =]\mathtt{a},\mathtt{b}[_\ell^k \cap \mathsf{M}$. Because $\{\mathtt{a},\mathtt{b}\}\cap\mathsf{T} = \varnothing$ and because T is convex, either $\mathsf{T}\cap\,]\mathtt{a},\mathtt{b}[_\ell^k = \varnothing$ or $\mathsf{T} \subseteq\,]\mathtt{a},\mathtt{b}[_\ell^k$. In conclusion, $\sigma_\mathfrak{d}^\mathfrak{c}(]\mathtt{a},\mathtt{b}[_\ell^k \cap \mathsf{M})$ is given by $\sigma_\mathfrak{d}^\mathfrak{c}(]\mathtt{a},\mathtt{b}[_\ell^k)$ in the former case, by $\sigma_\mathfrak{d}^\mathfrak{c}(]\mathtt{a},\mathtt{b}[_\ell^k) - \sigma_\mathfrak{d}^\mathfrak{c}(\mathsf{T})$ in the latter case and by virtue of $\sigma_\mathfrak{d}^\mathfrak{c}(\mathsf{T}) = 0$ thus by $\sigma_\mathfrak{d}^\mathfrak{c}(]\mathtt{a},\mathtt{b}[_\ell^k)$ in any case.

Hence, the definition $\zeta_\mathfrak{b}^\mathfrak{a} = \zeta_\mathfrak{d}^\mathfrak{c} \circ \gamma_{\mathsf{M},\ell}^{\;k}$ implies first

$$\begin{aligned}\sigma_\mathfrak{b}^\mathfrak{a}(]\mathtt{x},\mathtt{y}[_n^m) &= \textstyle\sum_{\mathtt{z}\in]\mathtt{x},\mathtt{y}[_n^m} \sigma(\zeta_\mathfrak{b}^\mathfrak{a}(\mathtt{z}))\\ &= \textstyle\sum_{\mathtt{z}\in]\mathtt{x},\mathtt{y}[_n^m} \sigma(\zeta_\mathfrak{d}^\mathfrak{c}(\gamma_{\mathsf{M},\ell}^{\;k}(\mathtt{z})))\\ &= \textstyle\sum_{\mathtt{e}\in]\mathtt{a},\mathtt{b}[_\ell^k\cap\mathsf{M}} \sigma(\zeta_\mathfrak{d}^\mathfrak{c}(\mathtt{e}))\\ &= \sigma_\mathfrak{d}^\mathfrak{c}(]\mathtt{a},\mathtt{b}[_\ell^k \cap \mathsf{M})\\ &= \sigma_\mathfrak{d}^\mathfrak{c}(]\mathtt{a},\mathtt{b}[_\ell^k)\end{aligned}$$

and $\sigma_\mathfrak{b}^\mathfrak{a}(\{\mathtt{x}\}) = \sigma_\mathfrak{d}^\mathfrak{c}(\{\mathtt{a}\})$ and $\sigma_\mathfrak{b}^\mathfrak{a}(\{\mathtt{y}\}) = \sigma_\mathfrak{d}^\mathfrak{c}(\{\mathtt{b}\})$, and thus

$$\begin{aligned}\delta_\mathfrak{b}^\mathfrak{a}(\mathtt{x},\mathtt{y}) &= \tfrac{1}{2}\sigma_\mathfrak{b}^\mathfrak{a}(\{\mathtt{x}\}) + \sigma_\mathfrak{b}^\mathfrak{a}(]\mathtt{x},\mathtt{y}[_n^m) + \tfrac{1}{2}\sigma_\mathfrak{b}^\mathfrak{a}(\{\mathtt{y}\})\\ &= \tfrac{1}{2}\sigma_\mathfrak{d}^\mathfrak{c}(\{\mathtt{a}\}) + \sigma_\mathfrak{d}^\mathfrak{c}(]\mathtt{a},\mathtt{b}[_\ell^k) + \tfrac{1}{2}\sigma_\mathfrak{d}^\mathfrak{c}(\{\mathtt{b}\})\\ &= \delta_\mathfrak{d}^\mathfrak{c}(\mathtt{a},\mathtt{b}).\end{aligned}$$

And that is all we needed to see. □

Next, we prove that the points of any 0-part alternate in normalized color and that any two subsequent points of distinct 0-parts have identical normalized colors. The proof of this will require the following discrete intermediate value theorem.

LEMMA 7.15. *For any* $\{a,b\} \subseteq \mathbb{Z}$ *with* $a < b$ *and any* $f\colon \{i \in \mathbb{Z} \,|\, a \le i \le b\} \to \mathbb{Z}$, *if* $\partial f\colon \{i \in \mathbb{Z} \,|\, a < i \le b\} \to \mathbb{Z}$, $j \mapsto f(j) - f(j-1)$ *satisfies* $\operatorname{ran}(\partial f) \subseteq \{-1,0,1\}$ *and if* $f(a)\cdot f(b) < 0$, *then there exists* $x \in \mathbb{Z}$ *with* $a < x < b$ *such that* $f(x) = 0$.

PROOF. First, note that for any $x \in \mathbb{Z}$ with $a < x \le b$, necessarily, $f(x-1)\cdot f(x) \ge 0$ because, if $f(x)$ and $f(x-1)$ were both non-zero and had opposite signs, it would follow $|f(x) - f(x-1)| = |f(x)| + |f(x-1)| \ge 1 + 1 = 2$, contradicting the assumption $|\partial f(x)| \le 1$. In particular, the premise $f(a)\cdot f(b) < 0$ guarantees $1 < b - a$.

The proof goes by induction over $b-a$. In the base case $2 = b - a$, by the initial remark, both $f(a)\cdot f(a+1) \ge 0$ and $f(a+1)\cdot f(b) \ge 0$. Hence, if $f(a+1)$ was non-zero, $f(a)$ and $f(b)$ would have the same sign, in contradiction to our assumption.

For general $2 \le b-a$ we can assume $f(b-1) \ne 0$. If so, then by the observation at the beginning of the proof $f(b-1)$ has the same sign as $f(b)$. Hence, $f(a)\cdot f(b) < 0$ and the induction hypothesis yields an $x \in \mathbb{Z}$ with $a < x < b-1$ and $f(x) = 0$, which concludes the proof. □

LEMMA 7.16. *For any* $\{k,\ell\} \subseteq \mathbb{N}_0$, *any* $\mathfrak{c} : [\![k]\!] \to \{\circ,\bullet\}$ *and* $\mathfrak{d} : [\![\ell]\!] \to \{\circ,\bullet\}$ *with* $\Sigma_{\mathfrak{d}}^{\mathfrak{c}} = 0$, *any* $\{\mathsf{S},\mathsf{S}'\} \subseteq {}^{0}\Delta_{\mathfrak{d}}^{\mathfrak{c}}$ *and any* $\mathsf{a} \in \mathsf{S}$ *and* $\mathsf{a}' \in \mathsf{S}'$ *with* $\mathsf{a} \ne \mathsf{a}'$ *and* $]\mathsf{a},\mathsf{a}'[_{\ell}^{k} \cap (\mathsf{S} \cup \mathsf{S}') = \varnothing$,

$$\mathsf{S} = \mathsf{S}' \Leftrightarrow \zeta_{\mathfrak{d}}^{\mathfrak{c}}(\mathsf{a}) \ne \zeta_{\mathfrak{d}}^{\mathfrak{c}}(\mathsf{a}').$$

PROOF. We prove both implications simultaneously and distinguish two cases.

Case 1: a *is the predecessor of* a'. If $]\mathsf{a},\mathsf{a}'[_{\ell}^{k} = \varnothing$, then, by definition, $\delta_{\mathfrak{d}}^{\mathfrak{c}}(\mathsf{a},\mathsf{a}') = \frac{1}{2}\sigma_{\mathfrak{d}}^{\mathfrak{c}}(\{\mathsf{a},\mathsf{a}'\})$, which is 0 if and only if $\sigma_{\mathfrak{d}}^{\mathfrak{c}}(\{\mathsf{a},\mathsf{a}'\}) = 0$, i.e., if and only if $\zeta_{\mathfrak{d}}^{\mathfrak{c}}(\mathsf{a}) \ne \zeta_{\mathfrak{d}}^{\mathfrak{c}}(\mathsf{a}')$. Because the identity $\mathsf{S} = \mathsf{S}'$ holds if and only if $\delta_{\mathfrak{d}}^{\mathfrak{c}}(\mathsf{a},\mathsf{a}') = 0$ that proves the assertion in this case.

Case 2: a *is not the predecessor of* a'. Now, let $]\mathsf{a},\mathsf{a}'[_{\ell}^{k} \ne \varnothing$ instead. Then, there exist $m \in \mathbb{N}$ with $1 < m$ and pairwise distinct points $\mathsf{b}_0, \mathsf{b}_1, \ldots, \mathsf{b}_m$ such that $(\mathsf{b}_n \mid \mathsf{b}_{n+1} \mid \mathsf{b}_{n+2})_{\ell}^{k}$ for all $n \in \mathbb{N}_0$ with $n < m-1$ and $(\mathsf{b}_{m-1} \mid \mathsf{b}_m \mid \mathsf{b}_0)_{\ell}^{k}$ and such that $\{\mathsf{b}_0, \mathsf{b}_1, \ldots, \mathsf{b}_m\} = [\mathsf{a},\mathsf{a}']_{\ell}^{k}$, in particular, $\mathsf{b}_0 = \mathsf{a}$ and $\mathsf{b}_m = \mathsf{a}'$. For every $n \in \mathbb{N}_0$ with $n \le m$ let $x_n \coloneqq \sigma_{\mathfrak{d}}^{\mathfrak{c}}(\{\mathsf{b}_n\})$. Then, we must show $x_0 \ne x_m$ if $\mathsf{S} = \mathsf{S}'$ and $x_0 = x_m$ if $\mathsf{S} \ne \mathsf{S}'$. Define the function $f\colon \{0\} \cup [\![\ell]\!] \to \mathbb{Z}$, $n \mapsto \delta_{\mathfrak{d}}^{\mathfrak{c}}(\mathsf{b}_n, \mathsf{b}_m)$.

Step 2.1: $f^{\leftarrow}(\{0\})$ *is* $\{0,m\}$ *if* $\mathsf{S} = \mathsf{S}'$ *and* $\{m\}$ *if* $\mathsf{S} \ne \mathsf{S}'$. By definition, for any $n \in \mathbb{N}_0$ with $n \le m$, the statement $f(n) = 0$ is equivalent to proposing $\delta_{\mathfrak{d}}^{\mathfrak{c}}(\mathsf{b}_n, \mathsf{b}_m) = 0$, which in turn is the same as saying $\mathsf{b}_n \in \mathsf{S}'$ because $\mathsf{b}_m = \mathsf{a}' \in \mathsf{S}'$. Because $]\mathsf{b}_0, \mathsf{b}_m[_{\ell}^{k} \cap \mathsf{S}' =]\mathsf{a},\mathsf{a}'[_{\ell}^{k} \cap \mathsf{S}' = \varnothing$ per assumption, the inclusion $f^{\leftarrow}(\{0\}) \subseteq \{0,m\}$ has thus been shown. Moreover, we can conclude that $m \in f^{\leftarrow}(\{0\})$ is always true and that $0 \in f^{\leftarrow}(\{0\})$ holds if and only if $\mathsf{S} = \mathsf{S}'$ (because $\mathsf{b}_0 = \mathsf{a} \in \mathsf{S}$).

Step 2.2: $f^{\leftarrow}(\{f(0)\})$ *is* $\{0,m\}$ *if* $\mathsf{S} = \mathsf{S}'$ *and* $\{0\}$ *if* $\mathsf{S} \ne \mathsf{S}'$. For all $n \in \mathbb{N}_0$ with $n \le m$ the statement $f(n) = f(0)$ is equivalent to claiming $\delta_{\mathfrak{d}}^{\mathfrak{c}}(\mathsf{b}_n, \mathsf{b}_m) = \delta_{\mathfrak{d}}^{\mathfrak{c}}(\mathsf{b}_0, \mathsf{b}_m)$; and this in turn is true if and only if $0 = \delta_{\mathfrak{d}}^{\mathfrak{c}}(\mathsf{b}_0, \mathsf{b}_m) - \delta_{\mathfrak{d}}^{\mathfrak{c}}(\mathsf{b}_n, \mathsf{b}_m) = \delta_{\mathfrak{d}}^{\mathfrak{c}}(\mathsf{b}_0, \mathsf{b}_m) + \delta_{\mathfrak{d}}^{\mathfrak{c}}(\mathsf{b}_m, \mathsf{b}_n) = \delta_{\mathfrak{d}}^{\mathfrak{c}}(\mathsf{b}_0, \mathsf{b}_n)$ by Lemma 3.10 and $\Sigma_{\mathfrak{d}}^{\mathfrak{c}} = 0$. As $\mathsf{b}_0 = \mathsf{a} \in \mathsf{S}$ we have thus checked for any $n \in [\![m]\!]$ that $f(n) = f(0)$ if and only if $\mathsf{b}_n \in \mathsf{S}$. It follows, on the one hand, that $f^{\leftarrow}(\{0\}) \subseteq \{0,m\}$ because $]\mathsf{b}_0, \mathsf{b}_m[_{\ell}^{k} \cap \mathsf{S} =]\mathsf{a},\mathsf{a}'[_{\ell}^{k} \cap \mathsf{S} = \varnothing$, and, on the other hand, that $m \in f^{\leftarrow}(\{0\})$ if and only if $\mathsf{S} = \mathsf{S}'$ (because $\mathsf{b}_m = \mathsf{a}' \in \mathsf{S}'$).

Step 2.3: Proving $\operatorname{ran}(\partial f) \subseteq \{-1,0,1\}$ *and a formula for* ∂f. We show that $\partial f\colon [\![\ell]\!] \to \mathbb{Z}$, $n \mapsto f(n) - f(n-1)$ satisfies $\operatorname{ran}(\partial f) \subseteq \{-1,0,1\}$ and find a formula for ∂f. Per definition and by Lemma 3.10 and $\Sigma_{\mathfrak{d}}^{\mathfrak{c}} = 0$, for any $n \in [\![m]\!]$,

$$\begin{aligned} f(n) - f(n-1) &= \delta_{\mathfrak{d}}^{\mathfrak{c}}(\mathtt{b}_n, \mathtt{b}_m) - \delta_{\mathfrak{d}}^{\mathfrak{c}}(\mathtt{b}_{n-1}, \mathtt{b}_m) = \delta_{\mathfrak{d}}^{\mathfrak{c}}(\mathtt{b}_n, \mathtt{b}_m) + \delta_{\mathfrak{d}}^{\mathfrak{c}}(\mathtt{b}_m, \mathtt{b}_{n-1}) \\ &= \delta_{\mathfrak{d}}^{\mathfrak{c}}(\mathtt{b}_n, \mathtt{b}_{n-1}) = -\delta_{\mathfrak{d}}^{\mathfrak{c}}(\mathtt{b}_{n-1}, \mathtt{b}_n) = -\tfrac{1}{2}\sigma_{\mathfrak{d}}^{\mathfrak{c}}(\{\mathtt{b}_{n-1}, \mathtt{b}_n\}) \\ &= -\tfrac{1}{2}(x_{n-1} + x_n) = \begin{cases} -1 & \text{if } x_{n-1} = x_n = 1, \\ 1 & \text{if } x_{n-1} = x_n = -1, \\ 0 & \text{if } x_{n-1} \neq x_n, \end{cases} \end{aligned}$$

which in particular proves $\operatorname{ran}(\partial f) \subseteq \{-1,0,1\}$.

Step 2.4: Proving $\partial f(1) \neq 0 \neq \partial f(m)$. From $\{m\} \subseteq f^{\leftarrow}(\{0\}) \subseteq \{0, m\}$ by Step 2.1 it follows $\partial f(m) = f(m) - f(m-1) = -f(m-1) \neq 0$ because $1 < m$. Likewise, $\{0\} \subseteq f^{\leftarrow}(\{f(0)\}) \subseteq \{0, m\}$ by Step 2.2 proves $\partial f(1) = f(1) - f(0) \neq 0$, also since $1 < m$.

Step 2.5: Sign of f *and definition of* ε. We show by contradiction that there exists $\varepsilon \in \{-1,1\}$ such that $\varepsilon f(n) > 0$ for all $n \in [\![m{-}1]\!]$.

Suppose that there exist $\{n, n'\} \subseteq [\![m{-}1]\!]$ with $n < n'$ such that $f(n)f(n') < 0$. Because $\operatorname{ran}(\partial f) \subseteq \{-1,0,1\}$, Lemma 7.15 then ensures the existence of $i \in [\![m]\!]$ with $0 < n < i < n' < m$ and $f(i) = 0$. That contradicts the result $f^{\leftarrow}(\{0\}) \subseteq \{0, m\}$ of Step 2.1. Hence, an $\varepsilon \in \{-1,1\}$ such that $\varepsilon f(n) > 0$ for all $n \in [\![m{-}1]\!]$ exists.

Step 2.6: Sign of $f(0) - f$. Now, we prove that, if $f(0) \neq 0$, then $\varepsilon(f(0) - f(n)) > 0$ for all $n \in [\![m{-}1]\!]$.

Again, assume that $f(0) \neq 0$ and that there exist $\{n, n'\} \subseteq [\![m{-}1]\!]$ with $n < n'$ such that $(f(0) - f(n))(f(0) - f(n')) < 0$. Because $\operatorname{ran}(\partial(f(0) - f)) = -\operatorname{ran}(\partial f) = -\{-1,0,1\} = \{-1,0,1\}$, by Lemma 7.15 there is $i \in [\![m]\!]$ such that $0 < n < i < n' < m$ and $f(0) - f(i) = 0$. This is the contradiction we sought because $f^{\leftarrow}(\{f(0)\}) \subseteq \{0, m\}$ by Step 2.2.

Step 2.7: $\partial f(1) \neq \partial f(m)$ *if and only if* $\mathsf{S} = \mathsf{S}'$. According to Step 2.1 the assumption $\mathsf{S} = \mathsf{S}'$ is equivalent to $f(0) = 0$. Because $\partial f(1) \neq 0 \neq \partial f(m)$ by Step 2.4 and because $\operatorname{ran}(\partial f) \subseteq \{-1,0,1\}$ by Step 2.3, it suffices to prove that $\partial f(1)$ and $\partial f(m)$ have different signs if and only if $f(0) = 0$ in order to see that $\partial f(1) \neq \partial f(m)$ if and only if $\mathsf{S} = \mathsf{S}'$.

First, suppose $f(0) = 0$. Then, $\partial f(1) = f(1) - f(0) = f(1)$ and $\partial f(m) = f(m) - f(m{-}1) = -f(m{-}1)$ by Step 2.1 and thus $\varepsilon\partial f(1) = \varepsilon f(1)$ and $\varepsilon\partial f(m) = -\varepsilon f(m{-}1)$. Because $\varepsilon f(1) > 0$ and $\varepsilon f(m{-}1) > 0$ by Step 2.5, it follows $\varepsilon\partial f(1) > 0$ and $\varepsilon\partial f(m) < 0$. In conclusion, $\partial f(1) \neq \partial f(m)$.

Alternatively, let $f(0) \neq 0$. Then, $\varepsilon\partial f(1) = -\varepsilon(f(0) - f(1))$ and, still, $\varepsilon\partial f(m) = -\varepsilon f(m-1)$ because $f(m) = 0$ by Step 2.1. Now, $\varepsilon f(m-1) > 0$ by Step 2.5 and $\varepsilon(f(0) - f(1)) > 0$ by Step 2.6 imply $\varepsilon\partial f(1) < 0$ and $\varepsilon\partial f(m) < 0$. Thus, in this case, $\partial f(1) = \partial f(m)$.

Step 2.8: $x_0 \neq x_m$ *if and only if* $\mathsf{S} = \mathsf{S}'$. From the statement $0 \neq \partial f(1) \neq \partial f(m) \neq 0$ which holds in case $\mathsf{S} = \mathsf{S}'$ by Steps 2.4 and 2.7 and from our formula for ∂f found in Step 2.3 it follows $0 \neq -\frac{1}{2}(x_0 + x_1) \neq -\frac{1}{2}(x_{m-1} + x_m) \neq 0$ and thus $0 \neq x_0 + x_1 \neq x_{m-1} + x_m \neq 0$. That is only possible if $x_0 = x_1 \neq x_{m-1} = x_m$.

Likewise, if $\mathsf{S} \neq \mathsf{S}'$, then $\partial f(1) = \partial f(m) \neq 0$, as shown in Steps 2.4 and 2.7, then $x_0 + x_1 = x_{m-1} + x_m \neq 0$ by Step 2.3 requires $x_0 = x_1 = x_{m-1} = x_m$. That is what we needed to see. □

In particular, we can now prove the following.

PROPOSITION 7.17. *For any* $\{k, \ell\} \subseteq \mathbb{N}_0$, *any* $\mathfrak{c}\colon [\![k]\!] \to \{\circ, \bullet\}$ *and* $\mathfrak{d}\colon [\![\ell]\!] \to \{\circ, \bullet\}$ *with* $\Sigma_{\mathfrak{d}}^{\mathfrak{c}} = 0$, *any* $\mathsf{S} \in {}^0\Delta_{\mathfrak{d}}^{\mathfrak{c}}$ *and any* $\{\mathsf{a}, \mathsf{b}\} \subseteq \mathsf{S}$ *with* $\mathsf{a} \neq \mathsf{b}$ *the statements* $\sigma_{\mathfrak{d}}^{\mathfrak{c}}(\{\mathsf{a}, \mathsf{b}\}) = 0$ *and* $|]\mathsf{a}, \mathsf{b}[_\ell^k \cap \mathsf{S}| \equiv_2 0$ *are equivalent.*

PROOF. If $m \in \mathbb{N}$ and $\{\mathsf{c}_i\}_{i=0}^m \subseteq \Pi_\ell^k$ are such that $[\mathsf{a}, \mathsf{b}]_\ell^k \cap \mathsf{S} = \{\mathsf{c}_0, \mathsf{c}_1, \ldots, \mathsf{c}_m\}$, i.e., $\mathsf{c}_0 = \mathsf{a}$ and $\mathsf{c}_m = \mathsf{b}$ in particular, such that $(\mathsf{c}_i \mid \mathsf{c}_{i+1} \mid \mathsf{c}_{i+2})_\ell^k$ for any $i \in \mathbb{N}_0$ with $i < m-1$ and such that $(\mathsf{c}_{m-1} \mid \mathsf{c}_1 \mid \mathsf{c}_0)_\ell^k$, then $]\mathsf{c}_i, \mathsf{c}_{i+1}[_\ell^k \cap \mathsf{S} = \varnothing$ for each $i \in \mathbb{N}_0$ with $i < m$. Thus, $\zeta_{\mathfrak{d}}^{\mathfrak{c}}(\mathsf{c}_i) \neq \zeta_{\mathfrak{d}}^{\mathfrak{c}}(\mathsf{c}_{i+1})$ for each $i \in \mathbb{N}_0$ with $i < m$ by Lemma 7.16. Inductively, it follows that $\zeta_{\mathfrak{d}}^{\mathfrak{c}}(\mathsf{a}) \neq \zeta_{\mathfrak{d}}^{\mathfrak{c}}(\mathsf{b})$ if $m \in 2\mathbb{N}_0$ and $\zeta_{\mathfrak{d}}^{\mathfrak{c}}(\mathsf{a}) = \zeta_{\mathfrak{d}}^{\mathfrak{c}}(\mathsf{b})$ if $m \in 2\mathbb{N}_0 + 1$, which is what was claimed. □

With that result at hand, we can now give a reformulation of the definition of the categories of the unitary half-liberations. The following lemma unclutters the proof of Proposition 7.19.

LEMMA 7.18. *For any* $w \in \mathbb{N}_0$, *any* $D \subseteq \mathbb{N}$, *any* $\{k, \ell\} \subseteq \mathbb{N}_0$, *any* $\mathfrak{c}\colon [\![k]\!] \to \{\circ, \bullet\}$ *and* $\mathfrak{d}\colon [\![\ell]\!] \to \{\circ, \bullet\}$ *and any* $(\mathfrak{c}, \mathfrak{d}, p) \in \mathcal{S}$, *if* $\Sigma_{\mathfrak{d}}^{\mathfrak{c}} \equiv_w 0$, *then the following hold.*

- *(a) The following are equivalent:*
  - *(i)* $\delta_{\mathfrak{d}}^{\mathfrak{c}}(\mathsf{b}, \mathsf{b}') \equiv_w 0$ *for any* $\{\mathsf{b}, \mathsf{b}'\} \subseteq \mathsf{B}$ *and any* $\mathsf{B} \in p$.
  - *(ii)* $p \leq {}^w\Delta_{\mathfrak{d}}^{\mathfrak{c}}$.
- *(b) If* $p \leq {}^w\Delta_{\mathfrak{d}}^{\mathfrak{c}}$, *then the following are equivalent.*
  - *(i)* $|\mathsf{B}| = 2$ *for any* $\mathsf{B} \in p$.
  - *(ii) For any* $\mathsf{S} \in {}^w\Delta_{\mathfrak{d}}^{\mathfrak{c}}$, *if* $s = R(p, \mathsf{S})$, *then* $|\mathsf{A}| = 2$ *for any* $\mathsf{A} \in s$.
- *(c) If* $p \leq {}^w\Delta_{\mathfrak{d}}^{\mathfrak{c}}$, *then following hold:*
  - *(i)* $\sigma_{\mathfrak{d}}^{\mathfrak{c}}(\mathsf{B}) = 0$ *for any* $\mathsf{B} \in p$.
  - *(ii) For any* $\mathsf{S} \in {}^w\Delta_{\mathfrak{d}}^{\mathfrak{c}}$, *if* $(\mathfrak{f}, \mathfrak{g}, s) = R((\mathfrak{c}, \mathfrak{d}, p), \mathsf{S})$, *then* $\sigma_{\mathfrak{g}}^{\mathfrak{f}}(\mathsf{A}) = 0$ *for any* $\mathsf{A} \in s$.
- *(d) If* $w = 0$, *if* $p \leq {}^w\Delta_{\mathfrak{d}}^{\mathfrak{c}}$ *and if* $|\mathsf{B}| = 2$ *and* $\sigma_{\mathfrak{d}}^{\mathfrak{c}}(\mathsf{B}) = 0$ *for any* $\mathsf{B} \in p$, *then for any* $\mathsf{S} \in {}^w\Delta_{\mathfrak{d}}^{\mathfrak{c}}$, *if* $s = R(p, \mathsf{S})$ *and if* $\{m, n\} \subseteq \mathbb{N}_0$ *are such that* s *is a partition of* Π_n^m, *then* $|]\mathsf{a}, \mathsf{a}'[_n^m| \equiv_2 0$ *for any* $\{\mathsf{a}, \mathsf{a}'\} \subseteq \mathsf{A}$ *with* $\mathsf{a} \neq \mathsf{a}'$ *and any* $\mathsf{A} \in s$.
- *(e) If* $w = 0$ *and if* $p \leq {}^w\Delta_{\mathfrak{d}}^{\mathfrak{c}}$, *then the following are equivalent:*
  - *(i) For any* $\{\mathsf{B}_1, \mathsf{B}_2\} \subseteq p$, *whenever there exist* $\mathsf{b}_1 \in \mathsf{B}_1$ *and* $\mathsf{b}_2 \in \mathsf{B}_2$ *with* $|\delta_{\mathfrak{d}}^{\mathfrak{c}}(\mathsf{B}_1, \mathsf{B}_2)| \notin D$, *then* $\mathsf{B}_1 \mathbin{\diagdown\!\!\diagdown}_\ell^k \mathsf{B}_2$.
  - *(ii) For any* $\{\mathsf{S}_1, \mathsf{S}_2\} \subseteq {}^0\Delta_{\mathfrak{d}}^{\mathfrak{c}}$, *if* $|{}^0\delta_{\mathfrak{d}}^{\mathfrak{c}}(\mathsf{S}_1, \mathsf{S}_2)| \notin D$, *then* $\mathsf{B}_1 \mathbin{\diagdown\!\!\diagdown}_\ell^k \mathsf{B}_2$ *for any* $\{\mathsf{B}_1, \mathsf{B}_2\} \subseteq p$ *with* $\mathsf{B}_1 \subseteq \mathsf{S}_1$ *and* $\mathsf{B}_2 \subseteq \mathsf{S}_2$.
- *(f) If* $w = 0$ *and if* $p \leq {}^w\Delta_{\mathfrak{d}}^{\mathfrak{c}}$, *then the following are equivalent:*
  - *(i) For any* $\{\mathsf{B}_1, \mathsf{B}_2\} \subseteq p$ *with* $\mathsf{B}_1 \neq \mathsf{B}_2$, *whenever there exist* $\mathsf{b}_1 \in \mathsf{B}_1$ *and* $\mathsf{b}_2 \in \mathsf{B}_2$ *with* $|\delta_{\mathfrak{d}}^{\mathfrak{c}}(\mathsf{b}_1, \mathsf{b}_2)| = 0$, *then* $\mathsf{B}_1 \mathbin{\diagdown\!\!\diagdown}_\ell^k \mathsf{B}_2$.

(ii) For any $\mathsf{S} \in {}^{0}\Delta^{\mathfrak{c}}_{\mathfrak{d}}$*, if* $s = R(p,\mathsf{S})$ *and if* $\{m,n\} \subseteq \mathbb{N}_0$ *are such that* s *is a partition of* Π^{m}_{n}*, then* $\mathsf{A}_1 \mathrel{\diagdown\!\!\diagdown}^{m}_{n} \mathsf{A}_2$ *for any* $\{\mathsf{A}_1,\mathsf{A}_2\} \subseteq s$ *with* $\mathsf{A}_1 \neq \mathsf{A}_2$.

Proof. (a) That follows immediately from the definition of ${}^{w}\Delta^{\mathfrak{c}}_{\mathfrak{d}}$ and is only stated for emphasis.

(b)+(c) We can show (b) and (c) simultaneously. For any $\mathsf{B} \in p$, for any $\mathsf{S} \in p$ and, if $(\mathfrak{f},\mathfrak{g},s) = R((\mathfrak{c},\mathfrak{d},p),\mathsf{S})$, for any $\mathsf{A} \in s$, if $\mathsf{B} \subseteq \mathsf{S}$ and $\mathsf{A} = \gamma_{\mathsf{S},\ell}^{\;k\leftarrow}(\mathsf{B})$, then, on the one hand, $|\mathsf{A}| = |\gamma_{\mathsf{S},\ell}^{\;k\leftarrow}(\mathsf{B})| = |\mathsf{B}|$ because $\gamma_{\mathsf{S},\ell}^{\;k}$ is injective. On the other hand, also $\sigma^{\mathfrak{c}}_{\mathfrak{d}}(\mathsf{B}) = \sigma^{\mathfrak{f}}_{\mathfrak{g}}(\gamma_{\mathsf{S},\ell}^{\;k\leftarrow}(\mathsf{B})) = \sigma^{\mathfrak{f}}_{\mathfrak{g}}(\mathsf{A})$ by Lemma 4.2 (d) because $\mathsf{B}\backslash\mathsf{S} = \varnothing$.

The assumption $p \leq {}^{w}\Delta^{\mathfrak{c}}_{\mathfrak{d}}$ ensures that for any $\mathsf{B} \in p$ there exist exactly such $\mathsf{S} \in {}^{w}\Delta^{\mathfrak{c}}_{\mathfrak{d}}$ and $\mathsf{A} \in R(p,\mathsf{S})$ with $\mathsf{B} \subseteq \mathsf{S}$ and $\mathsf{A} = \gamma_{\mathsf{S},\ell}^{\;k\leftarrow}(\mathsf{B})$ and that, conversely, for any $\mathsf{S} \in {}^{w}\Delta^{\mathfrak{c}}_{\mathfrak{d}}$ and any $\mathsf{A} \in R(p,\mathsf{S})$ we can always find such a $\mathsf{B} \in p$ that $\mathsf{B} \subseteq \mathsf{S}$ and $\mathsf{A} = \gamma_{\mathsf{S},\ell}^{\;k\leftarrow}(\mathsf{B})$. Hence, the claims are true.

(d) Given any $\mathsf{S} \in {}^{0}\Delta^{\mathfrak{c}}_{\mathfrak{d}}$, any $\mathsf{A} \in s \coloneqq R(p,\mathsf{S})$ and any $\{\mathsf{a},\mathsf{a}'\} \subseteq \mathsf{A}$ with $\mathsf{a} \neq \mathsf{a}'$, there exists by definition $\mathsf{B} \in p$ such that $\mathsf{A} = \gamma_{\mathsf{S},\ell}^{\;k\leftarrow}(\mathsf{B})$ and, consequently, $\mathsf{b} \coloneqq \gamma_{\mathsf{S},\ell}^{\;k}(\mathsf{a}) \in \mathsf{B}$ and $\mathsf{b}' \coloneqq \gamma_{\mathsf{S},\ell}^{\;k}(\mathsf{a}') \in \mathsf{B}$. The fact that thus, in particular, $\mathsf{B} \cap \mathsf{S} \neq \varnothing$ demands that already $\mathsf{B} \subseteq \mathsf{S}$ because $p \leq {}^{w}\Delta^{\mathfrak{c}}_{\mathfrak{d}}$. Moreover, by assumption, then $\sigma^{\mathfrak{c}}_{\mathfrak{d}}(\mathsf{B}) = 0$ and $|\mathsf{B}| = 2$. Since $\gamma_{\mathsf{S},\ell}^{\;k}$ is injective, furthermore, $\mathsf{b} \neq \mathsf{b}'$. From $|\mathsf{B}| = 2$ it thus follows $\mathsf{B} = \{\mathsf{b},\mathsf{b}'\}$ and hence $\sigma^{\mathfrak{c}}_{\mathfrak{d}}(\{\mathsf{b},\mathsf{b}'\}) = 0$ by $\sigma^{\mathfrak{c}}_{\mathfrak{d}}(\mathsf{B}) = 0$. Proposition 7.17 therefore implies $|]\mathsf{b},\mathsf{b}'[^{k}_{\ell} \cap \mathsf{S}| \equiv_2 0$. Because $\gamma_{\mathsf{S},\ell}^{\;k}$ is monotonic with respect to Γ^{m}_{n} and Γ^{k}_{ℓ} by Lemma 4.2 (d), moreover, $]\mathsf{a},\mathsf{a}'[^{m}_{n} = \gamma_{\mathsf{S},\ell}^{\;k\leftarrow}(]\mathsf{b},\mathsf{b}'[^{k}_{\ell} \cap \mathsf{S})$. By the injectivity of $\gamma_{\mathsf{S},\ell}^{\;k}$, thus, $|]\mathsf{a},\mathsf{a}'[^{m}_{n}| = |]\mathsf{b},\mathsf{b}'[^{k}_{\ell} \cap \mathsf{S}| \equiv_2 0$, as claimed.

(e) Again, this is clear from the definition of ${}^{0}\Delta^{\mathfrak{c}}_{\mathfrak{d}}$ and ${}^{0}\delta^{\mathfrak{c}}_{\mathfrak{d}}$ and only stated in the interest of emphasis.

(f) By definition of ${}^{0}\Delta^{\mathfrak{c}}_{\mathfrak{d}}$ the statement (i) is equivalent to the requirement that $\mathsf{B}_1 \mathrel{\diagdown\!\!\diagdown}^{k}_{\ell} \mathsf{B}_2$ for any $\mathsf{S} \in {}^{0}\Delta^{\mathfrak{c}}_{\mathfrak{d}}$ and any $\{\mathsf{B}_1,\mathsf{B}_2\} \subseteq p$ with $\mathsf{B}_1 \subseteq \mathsf{S}$ and $\mathsf{B}_2 \subseteq \mathsf{S}$ and $\mathsf{B}_1 \neq \mathsf{B}_2$.

For any $\mathsf{S} \in {}^{0}\Delta^{\mathfrak{c}}_{\mathfrak{d}}$, if $s = R(p,\mathsf{S})$ and if $\{m,n\} \subseteq \mathbb{N}_0$ are such that s is a partition of Π^{k}_{ℓ}, then for any $\{\mathsf{B}_1,\mathsf{B}_2\} \subseteq p$ with $\mathsf{B}_1 \subseteq \mathsf{S}$ and $\mathsf{B}_2 \subseteq \mathsf{S}$ and $\mathsf{B}_1 \neq \mathsf{B}_2$, and any $\{\mathsf{A}_1,\mathsf{A}_2\} \subseteq s$ with $\mathsf{A}_1 \neq \mathsf{A}_2$, whenever $\mathsf{A}_1 = \gamma_{\mathsf{S},\ell}^{\;k\leftarrow}(\mathsf{B}_1)$ and $\mathsf{A}_2 = \gamma_{\mathsf{S},\ell}^{\;k\leftarrow}(\mathsf{B}_2)$, then $\mathsf{B}_1 \mathrel{\not\diagdown}^{k}_{\ell} \mathsf{B}_2$ if and only if $\mathsf{A}_1 \mathrel{\not\diagdown}^{m}_{n} \mathsf{A}_2$ because $\gamma_{\mathsf{S},\ell}^{\;k}$ is strictly monotonic by Lemma 4.2 (d).

And, once more, the assumption $p \leq {}^{0}\Delta^{\mathfrak{c}}_{\mathfrak{d}}$ guarantees that for any $\{\mathsf{B}_1,\mathsf{B}_2\}$ as above there exist $\{\mathsf{A}_1,\mathsf{A}_2\}$ with the above properties, and vice versa. Hence, (f) is valid. □

Proposition 7.19. *Let* $w \in \mathbb{N}$ *and let* D *be any additive subsemigroup of* $\mathbb{N}$.

(a) For any $w \in \mathbb{N}$ *the category* $\mathcal{U}^{*}_{w}$ *can be expressed as the set of all* $(\mathfrak{c},\mathfrak{d},p) \in \mathcal{S}$ *such that* $\Sigma^{\mathfrak{c}}_{\mathfrak{d}} \equiv_w 0$ *and* $p \leq {}^{w}\Delta^{\mathfrak{c}}_{\mathfrak{d}}$ *and* $R((\mathfrak{c},\mathfrak{d},p),\mathsf{S}) \in \mathcal{U}$ *for any* $\mathsf{S} \in {}^{w}\Delta^{\mathfrak{c}}_{\mathfrak{d}}$.

(b) The category $\mathcal{U}^{\times}_{D}$ *can be expressed as the set of all* $(\mathfrak{c},\mathfrak{d},p) \in \mathcal{S}$ *for which there exist* $\{k,\ell\} \subseteq \mathbb{N}_0$ *such that* $\mathfrak{c}\colon [\![k]\!] \to \{\circ,\bullet\}$ *and* $\mathfrak{d}\colon [\![\ell]\!] \to \{\circ,\bullet\}$*, such that* $\Sigma^{\mathfrak{c}}_{\mathfrak{d}} = 0$*, such that* $p \leq {}^{0}\Delta^{\mathfrak{c}}_{\mathfrak{d}}$*, such that* $R((\mathfrak{c},\mathfrak{d},p),\mathsf{S}) \in \mathcal{U}$ *for any* $\mathsf{S} \in {}^{0}\Delta^{\mathfrak{c}}_{\mathfrak{d}}$ *and such that for any* $\{\mathsf{S}_1,\mathsf{S}_2\} \subseteq {}^{0}\Delta^{\mathfrak{c}}_{\mathfrak{d}}$*, if* $|{}^{0}\delta^{\mathfrak{c}}_{\mathfrak{d}}(\mathsf{S}_1,\mathsf{S}_2)| \notin D$*, then* $\mathsf{B}_1 \mathrel{\diagdown\!\!\diagdown}^{k}_{\ell} \mathsf{B}_2$ *for any* $\{\mathsf{B}_1,\mathsf{B}_2\} \subseteq p$ *with* $\mathsf{B}_1 \subseteq \mathsf{S}_1$ *and* $\mathsf{B}_2 \subseteq \mathsf{S}_2$.

(c) *The category* $\mathcal{U}_D^{\times}$ *can be expressed as the set of all* $(\mathfrak{c},\mathfrak{d},p)\in\mathcal{S}$ *for which there exist* $\{k,\ell\}\subseteq\mathbb{N}_0$ *such that* $\mathfrak{c}\colon [\![k]\!]\to\{\circ,\bullet\}$ *and* $\mathfrak{d}\colon [\![\ell]\!]\to\{\circ,\bullet\}$*, such that* $\Sigma_{\mathfrak{d}}^{\mathfrak{c}}=0$*, such that* $p\leq {}^0\Delta_{\mathfrak{d}}^{\mathfrak{c}}$*, such that* $R(p,\mathsf{S})\in\mathcal{O}^*$ *for any* $\mathsf{S}\in{}^0\Delta_{\mathfrak{d}}^{\mathfrak{c}}$ *and such that for any* $\{\mathsf{S}_1,\mathsf{S}_2\}\subseteq{}^0\Delta_{\mathfrak{d}}^{\mathfrak{c}}$*, if* $|{}^0\delta_{\mathfrak{d}}^{\mathfrak{c}}(\mathsf{S}_1,\mathsf{S}_2)|\notin D$*, then* $\mathsf{B}_1 \diagdown\!\!\diagdown_{\ell}^{k} \mathsf{B}_2$ *for any* $\{\mathsf{B}_1,\mathsf{B}_2\}\subseteq p$ *with* $\mathsf{B}_1\subseteq\mathsf{S}_1$ *and* $\mathsf{B}_2\subseteq\mathsf{S}_2$*.*

(d) *The category* $\mathcal{U}_D^{\times+}$ *can be expressed as the set of all* $(\mathfrak{c},\mathfrak{d},p)\in\mathcal{S}$ *for which there exist* $\{k,\ell\}\subseteq\mathbb{N}_0$ *such that* $\mathfrak{c}\colon [\![k]\!]\to\{\circ,\bullet\}$ *and* $\mathfrak{d}\colon [\![\ell]\!]\to\{\circ,\bullet\}$*, such that* $\Sigma_{\mathfrak{d}}^{\mathfrak{c}}=0$*, such that* $p\leq {}^0\Delta_{\mathfrak{d}}^{\mathfrak{c}}$*, such that* $R((\mathfrak{c},\mathfrak{d},p),\mathsf{S})\in\mathcal{U}^+$ *for any* $\mathsf{S}\in{}^0\Delta_{\mathfrak{d}}^{\mathfrak{c}}$ *and such that for any* $\{\mathsf{S}_1,\mathsf{S}_2\}\subseteq{}^0\Delta_{\mathfrak{d}}^{\mathfrak{c}}$*, if* $|{}^0\delta_{\mathfrak{d}}^{\mathfrak{c}}(\mathsf{S}_1,\mathsf{S}_2)|\notin D$*, then* $\mathsf{B}_1 \diagdown\!\!\diagdown_{\ell}^{k} \mathsf{B}_2$ *for any* $\{\mathsf{B}_1,\mathsf{B}_2\}\subseteq p$ *with* $\mathsf{B}_1\subseteq\mathsf{S}_1$ *and* $\mathsf{B}_2\subseteq\mathsf{S}_2$*.*

(e) *The category* $\mathcal{U}_D^{\times+}$ *can be expressed as the set of all* $(\mathfrak{c},\mathfrak{d},p)\in\mathcal{S}$ *for which there exist* $\{k,\ell\}\subseteq\mathbb{N}_0$ *such that* $\mathfrak{c}\colon [\![k]\!]\to\{\circ,\bullet\}$ *and* $\mathfrak{d}\colon [\![\ell]\!]\to\{\circ,\bullet\}$*, such that* $\Sigma_{\mathfrak{d}}^{\mathfrak{c}}=0$*, such that* $p\leq {}^0\Delta_{\mathfrak{d}}^{\mathfrak{c}}$*, such that* $R(p,\mathsf{S})\in\mathcal{O}^+$ *for any* $\mathsf{S}\in{}^0\Delta_{\mathfrak{d}}^{\mathfrak{c}}$ *and such that for any* $\{\mathsf{S}_1,\mathsf{S}_2\}\subseteq{}^0\Delta_{\mathfrak{d}}^{\mathfrak{c}}$*, if* $|{}^0\delta_{\mathfrak{d}}^{\mathfrak{c}}(\mathsf{S}_1,\mathsf{S}_2)|\notin D$*, then* $\mathsf{B}_1 \diagdown\!\!\diagdown_{\ell}^{k} \mathsf{B}_2$ *for any* $\{\mathsf{B}_1,\mathsf{B}_2\}\subseteq p$ *with* $\mathsf{B}_1\subseteq\mathsf{S}_1$ *and* $\mathsf{B}_2\subseteq\mathsf{S}_2$*.*

PROOF. With the help of Lemma 7.18 all the claims follow immediately from Definitions 7.3, 7.10 and 7.11. More, precisely:

(a) Direct implication of Definition 7.11 (a) on the one hand and Definition 7.10 (a) and Lemma 7.18 (a) on the other.

(b) Consequence of Definition 7.11 (b) as well as Definition 7.10 (a) and Lemma 7.18 (a)–(d).

(c) Follows from Definition 7.11 (b) combined with Definition 7.3 (b) and all parts except (b) and (f) of Lemma 7.18.

(d) Implied by Definition 7.11 (c) and Definition 7.10 (b) and all parts besides (e) of Lemma 7.18.

(e) Can be inferred immediately from Definition 7.11 (c) in combination with Definition 7.3 (c) and all parts other than (b) of Lemma 7.18. □

DEFINITION 7.20. Any two-colored partition $(\mathfrak{c},\mathfrak{d},p)$ is said to *have no non-neutral blocks* if $\sigma_{\mathfrak{d}}^{\mathfrak{c}}(\mathsf{B})=0$ for any $\mathsf{B}\in p$.

The below result parallels Proposition 7.6. Again, this was recognized before (see [TW18, Lemma 1.1 (c)]).

PROPOSITION 7.21. *Any category of two-colored partitions consisting only of partitions which have no non-neutral blocks is* $\otimes$*-elbats. In particular, so are* $\mathcal{U}_w^{*}$*,* $\mathcal{U}_D^{\times}$ *and* $\mathcal{U}_D^{\times+}$ *for any* $w\in\mathbb{N}$ *and any additive subsemigroup* D *of* $\mathbb{N}$*.*

PROOF. The proof of [MW20, Lemma 6.8] applies. The assumption there that any block have only two legs is immaterial. □

Thus, also for those categories big and little graph power or wreath graph coproduct categories coincide by Proposition 4.35.

DEFINITION 7.22. For any $n\in\mathbb{N}$, any $w\in\mathbb{N}$ and any additive subsemigroup of $\mathbb{N}$ the easy algebraic compact quantum group asssociated with

(a) $(\mathcal{U}_w^*, n)$ is denoted by $U_{w,n}^*$.
(b) $(\mathcal{U}_D^\times, n)$ is denoted by $U_{D,n}^\times$.
(c) $(\mathcal{U}_D^{\times+}, n)$ is denoted by $U_{D,n}^{\times+}$.

8. The related categories of labeled partitions

While it would have been possible to give an ad-hoc definition of the supercategories of the unitary half-liberations, verifying that these are indeed categories would have been complicated and repetitive. With the general theorems about categories of labeled partitions from Section 4 at hand, however, all that is left to do is to reformulate a little the descriptions obtained from those results. This is achieved in Proposition 8.5. In order to make its proof as efficient as possible, an intermediate step is taken.

8.1. Reformulating the occurring graph co-products. This intermediate step consists in reformulating the definitions of certain graph co-product categories, see Proposition 8.3 below. The next lemma simplifies its proof.

LEMMA 8.1. *For any* $w \in \mathbb{N}_0$*, if* $(\mathfrak{U}, \mathfrak{O}) \in \{(\mathbb{Z}_w, \varnothing), (\varnothing, \mathbb{Z}_w)\}$*, then for any partial commutation relation* r *on* $\mathbb{Z}_w$*, any* $\{m, n\} \subseteq \mathbb{N}_0$*, any* $\mathfrak{f}\colon [\![m]\!] \to (\mathfrak{U} \otimes \{\circ, \bullet\}) \cup \mathfrak{O}$ *and* $\mathfrak{g}\colon [\![n]\!] \to (\mathfrak{U} \otimes \{\circ, \bullet\}) \cup \mathfrak{O}$ *and any* $(\mathfrak{f}, \mathfrak{g}, s) \in {}_{\mathfrak{U},\mathfrak{O}}\mathcal{S}$ *with* $s \leq \ker(\xi_{\mathfrak{g}}^{\mathfrak{f}})$ *the following hold.*

(a) The following are equivalent:
  - *(i)* $|\mathsf{A}| = 2$ *for any* $\mathsf{A} \in s$.
  - *(ii) For any* $z \in \mathbb{Z}_w$*, if* $(\mathfrak{u}, \mathfrak{v}, w) = R((\mathfrak{f}, \mathfrak{g}, s), \xi_{\mathfrak{g}}^{\mathfrak{f}\leftarrow}(\{z\}))$*, then* $|\mathsf{D}| = 2$ *for any* $\mathsf{D} \in w$.

(b) If $(\mathfrak{U}, \mathfrak{O}) = (\mathbb{Z}_w, \varnothing)$*, then the following are equivalent:*
  - *(i)* $\sum_{z \in \mathbb{Z}_w} {}_z\sigma_{\mathfrak{g}}^{\mathfrak{f}}(\mathsf{A}) = 0$ *for any* $\mathsf{A} \in s$.
  - *(ii) For any* $z \in \mathbb{Z}_w$*, if* $(\mathfrak{u}, \mathfrak{v}, w) = R((\mathfrak{f}, \mathfrak{g}, s), \xi_{\mathfrak{g}}^{\mathfrak{f}\leftarrow}(\{z\}))$*, then* ${}_z\sigma_{\mathfrak{g}}^{\mathfrak{f}}(\mathsf{D}) = 0$ *for any* $\mathsf{D} \in w$.

(c) The following are equivalent:
  - *(i) For any* $\mathsf{A} \in s$ *and any* $\{\mathsf{a}, \mathsf{a}''\} \subseteq \mathsf{A}$*, if* $\mathsf{a} \neq \mathsf{a}''$*, then* $|]\mathsf{a}, \mathsf{a}''[_n^m \cap \{\mathsf{a}' \in \Pi_n^m \wedge \xi_{\mathfrak{g}}^{\mathfrak{f}}(\mathsf{a}') - \xi_{\mathfrak{g}}^{\mathfrak{f}}(\mathsf{a}) = 0\}| \equiv_2 0$.
  - *(ii) For any* $z \in \mathbb{Z}_w$*, if* $\{i, j\} \subseteq \mathbb{N}_0$ *and* $\mathfrak{u}\colon [\![i]\!] \to (\mathfrak{U} \otimes \{\circ, \bullet\}) \cup \mathfrak{O}$ *and* $\mathfrak{v}\colon [\![j]\!] \to (\mathfrak{U} \otimes \{\circ, \bullet\}) \cup \mathfrak{O}$ *and* $(\mathfrak{u}, \mathfrak{v}, w) = R((\mathfrak{f}, \mathfrak{g}, s), \xi_{\mathfrak{g}}^{\mathfrak{f}\leftarrow}(\{z\}))$*, then for any* $\mathsf{D} \in w$ *and any* $\{\mathsf{x}, \mathsf{x}'\} \subseteq \mathsf{D}$*, if* $\mathsf{x} \neq \mathsf{x}'$*, then* $|]\mathsf{x}, \mathsf{x}'[_j^i| \equiv_2 0$.

(d) The following are equivalent:
  - *(i) For any* $\{\mathsf{A}_1, \mathsf{A}_2\} \subseteq s$ *with* $\mathsf{A}_1 \neq \mathsf{A}_2$*, whenever there exist* $\mathsf{a}_1 \in \mathsf{A}_1$ *and* $\mathsf{a}_2 \in \mathsf{A}_2$*, such that* $\xi_{\mathfrak{g}}^{\mathfrak{f}}(\mathsf{a}_2) - \xi_{\mathfrak{g}}^{\mathfrak{f}}(\mathsf{a}_1) = 0$*, then* $\mathsf{A}_1 \not\!\!\searrow_n^m \mathsf{A}_2$.
  - *(ii) For any* $z \in \mathbb{Z}_w$*, if* $\{i, j\} \subseteq \mathbb{N}_0$ *and* $\mathfrak{u}\colon [\![i]\!] \to (\mathfrak{U} \otimes \{\circ, \bullet\}) \cup \mathfrak{O}$ *and* $\mathfrak{v}\colon [\![j]\!] \to (\mathfrak{U} \otimes \{\circ, \bullet\}) \cup \mathfrak{O}$ *and* $(\mathfrak{u}, \mathfrak{v}, w) = R((\mathfrak{f}, \mathfrak{g}, s), \xi_{\mathfrak{g}}^{\mathfrak{f}\leftarrow}(\{z\}))$*, then* $\mathsf{D}_1 \not\!\!\searrow_j^i \mathsf{D}_2$ *for any* $\{\mathsf{D}_1, \mathsf{D}_2\} \subseteq w$ *with* $\mathsf{D}_1 \neq \mathsf{D}_2$.

(e) The following are equivalent:

(i) *For any* $\{\mathsf{A}_1,\mathsf{A}_2\}\subseteq s$ *with* $\mathsf{A}_1\neq\mathsf{A}_2$*, whenever there exist* $\mathsf{a}_1\in\mathsf{A}_1$ *and* $\mathsf{a}_2\in\mathsf{A}_2$ *such that* $\xi^{\mathfrak{f}}_{\mathfrak{g}}(\mathsf{a}_1)\neq\xi^{\mathfrak{f}}_{\mathfrak{g}}(\mathsf{a}_2)$ *and* $(\xi^{\mathfrak{f}}_{\mathfrak{g}}(\mathsf{a}_1),\xi^{\mathfrak{f}}_{\mathfrak{g}}(\mathsf{a}_2))\notin r$*, then* $\mathsf{A}_1\not\searrow^m_n\mathsf{A}_2$.

(ii) *For any* $\{z_1,z_2\}\subseteq\mathbb{Z}_w$ *with* $z_1\neq z_2$*, whenever* $(z_1,z_2)\notin r$*, then* $\mathsf{A}_1\not\searrow^m_n\mathsf{A}_2$ *for any* $\{\mathsf{A}_1,\mathsf{A}_2\}\subseteq s$ *with* $\mathsf{A}_1\subseteq\xi^{\mathfrak{f}\leftarrow}_{\mathfrak{g}}(\{z_1\})$ *and* $\mathsf{A}_2\subseteq\xi^{\mathfrak{f}\leftarrow}_{\mathfrak{g}}(\{z_2\})$.

Proof. (a),(b) We can prove (a) and (b) simultaneously.

If $z\in\mathbb{Z}_w$, if $\mathsf{Y}=\xi^{\mathfrak{f}\leftarrow}_{\mathfrak{g}}(\{z\})$, if $(\mathfrak{u},\mathfrak{v},w)=R((\mathfrak{f},\mathfrak{g},s),\mathsf{Y})$ and if $\mathsf{D}\in w$, then by definition there exists $\mathsf{A}\in s$ with $\mathsf{A}\cap\mathsf{Y}\neq\varnothing$ and $\mathsf{D}=\gamma_{\mathsf{Y},n}^{\;m\leftarrow}(\mathsf{A})$. As $s\leq\ker(\xi^{\mathfrak{f}}_{\mathfrak{g}})$, that demands $\mathsf{A}\subseteq\mathsf{Y}$ and thus $|\mathsf{D}|=|\mathsf{A}|$ because $\gamma_{\mathsf{Y},n}^{\;m}$ is injective. Hence, if $|\mathsf{A}|=2$, then also $|\mathsf{D}|=2$. Moreover, if $(\mathfrak{U},\mathfrak{D})=(\mathbb{Z}_w,\varnothing)$, then Lemma 4.2 (d) implies that $\sum_{x\in\mathbb{Z}_w}{}_x\sigma^{\mathfrak{f}}_{\mathfrak{g}}(\mathsf{A})=\sum_{x\in\mathbb{Z}_w}{}_x\sigma^{\mathfrak{f}}_{\mathfrak{g}}(\mathsf{D})$ because $\mathsf{Y}\backslash\mathsf{A}=\varnothing$. As $\xi^{\mathfrak{u}}_{\mathfrak{v}}$ is constant with value z by definition of Y, of course, $\sum_{x\in\mathbb{Z}_w}{}_x\sigma^{\mathfrak{f}}_{\mathfrak{g}}(\mathsf{D})={}_z\sigma^{\mathfrak{u}}_{\mathfrak{v}}(\mathsf{D})$ and thus ${}_z\sigma^{\mathfrak{u}}_{\mathfrak{v}}(\mathsf{D})=0$ if $\sum_{z\in\mathbb{Z}_w}{}_x\sigma^{\mathfrak{f}}_{\mathfrak{g}}(\mathsf{A})=0$.

Conversely, if $\mathsf{A}\in s$, then by $s\leq\ker(\xi^{\mathfrak{f}}_{\mathfrak{g}})$ there is $z\in\mathbb{Z}_w$ with $\mathsf{A}\subseteq\mathsf{Y}\coloneqq\xi^{\mathfrak{f}\leftarrow}_{\mathfrak{g}}(\{z\})$. If $(\mathfrak{u},\mathfrak{v},w)=R((\mathfrak{f},\mathfrak{g},s),\mathsf{Y})$, then $\mathsf{D}\coloneqq\gamma_{\mathsf{Y},n}^{\;m\leftarrow}(\mathsf{A})\in w$ and thus $|\mathsf{A}|=|\mathsf{D}|$ since $\gamma_{\mathsf{Y},n}^{\;m}$ is injective. In conclusion, if $|\mathsf{D}|=2$, then $|\mathsf{A}|=2$ as well. Furthermore, because the definition of Y implies that $\xi^{\mathfrak{u}}_{\mathfrak{v}}$ is constant with value z we can infer ${}_z\sigma^{\mathfrak{u}}_{\mathfrak{v}}(\mathsf{D})=\sum_{x\in\mathbb{Z}_w}{}_x\sigma^{\mathfrak{f}}_{\mathfrak{g}}(\mathsf{D})$ and because $\sum_{x\in\mathbb{Z}_w}{}_x\sigma^{\mathfrak{f}}_{\mathfrak{g}}(\mathsf{D})=\sum_{x\in\mathbb{Z}_w}{}_x\sigma^{\mathfrak{f}}_{\mathfrak{g}}(\mathsf{A})$ by Lemma 4.2 (d), if ${}_z\sigma^{\mathfrak{u}}_{\mathfrak{v}}(\mathsf{D})=0$, then also $\sum_{z\in\mathbb{Z}_w}{}_x\sigma^{\mathfrak{f}}_{\mathfrak{g}}(\mathsf{A})=0$.

(c) If $z\in\mathbb{Z}_w$, if $\{i,j\}\subseteq\mathbb{N}_0$, if $\mathfrak{u}\colon[\![i]\!]\to(\mathfrak{U}\otimes\{\circ,\bullet\})\cup\mathfrak{D}$ and $\mathfrak{v}\colon[\![j]\!]\to(\mathfrak{U}\otimes\{\circ,\bullet\})\cup\mathfrak{D}$, if $\mathsf{Y}=\xi^{\mathfrak{f}\leftarrow}_{\mathfrak{g}}(\{z\})$, if $(\mathfrak{u},\mathfrak{v},w)=R((\mathfrak{f},\mathfrak{g},s),\mathsf{Y})$, if $\mathsf{D}\in w$, if $\{\mathsf{x},\mathsf{x}'\}\subseteq\mathsf{D}$ and if $\mathsf{x}\neq\mathsf{x}'$, then by definition of w there exists $\mathsf{A}\in s$ with $\mathsf{D}=\gamma_{\mathsf{Y},n}^{\;m\leftarrow}(\mathsf{A})$. If $\mathsf{a}\coloneqq\gamma_{\mathsf{Y},n}^{\;m}(\mathsf{x})$ and $\mathsf{a}''\coloneqq\gamma_{\mathsf{Y},n}^{\;m}(\mathsf{x}')$, then $\{\mathsf{a},\mathsf{a}''\}\subseteq\mathsf{A}$ and $\mathsf{a}\neq\mathsf{a}''$ since $\gamma_{\mathsf{Y},n}^{\;m}$ is injective. Moreover, $]\mathsf{x},\mathsf{x}'[^i_j=\gamma_{\mathsf{Y},n}^{\;m\leftarrow}(]\mathsf{a},\mathsf{a}''[^m_n)=\gamma_{\mathsf{Y},n}^{\;m\leftarrow}(]\mathsf{a},\mathsf{a}''[^m_n\cap\mathsf{Y})$ because $\gamma_{\mathsf{Y},n}^{\;m}$ is also monotonic with respect to Γ^i_j and Γ^m_n by Lemma 4.2 (d). In particular, $|]\mathsf{x},\mathsf{x}'[^i_j|=|]\mathsf{a},\mathsf{a}''[^m_n\cap\{\mathsf{a}'\in\Pi^m_n\wedge\xi^{\mathfrak{f}}_{\mathfrak{g}}(\mathsf{a}')-\xi^{\mathfrak{f}}_{\mathfrak{g}}(\mathsf{a})=0\}|$ and because $\mathsf{Y}=\{\mathsf{a}'\in\Pi^m_n\wedge\xi^{\mathfrak{f}}_{\mathfrak{g}}(\mathsf{a}')-\xi^{\mathfrak{f}}_{\mathfrak{g}}(\mathsf{a})=0\}$ and because $\gamma_{\mathsf{Y},n}^{\;m}$ is injective. That proves one implication.

Conversely, if $\mathsf{A}\in s$, if $\{\mathsf{a},\mathsf{a}''\}\subseteq\mathsf{A}$, if $\mathsf{a}\neq\mathsf{a}''$, then by $s\leq\ker(\xi^{\mathfrak{f}}_{\mathfrak{g}})$ there exists $z\in\mathbb{Z}_w$ with $\mathsf{A}\subseteq\mathsf{Y}\coloneqq\xi^{\mathfrak{f}\leftarrow}_{\mathfrak{g}}(\{z\})$. Consequently, if $\{i,j\}\subseteq\mathbb{N}_0$, if $\mathfrak{u}\colon[\![i]\!]\to(\mathfrak{U}\otimes\{\circ,\bullet\})\cup\mathfrak{D}$ and $\mathfrak{v}\colon[\![j]\!]\to(\mathfrak{U}\otimes\{\circ,\bullet\})\cup\mathfrak{D}$, if $\mathsf{Y}=\xi^{\mathfrak{f}\leftarrow}_{\mathfrak{g}}(\{z\})$ and if $(\mathfrak{u},\mathfrak{v},w)=R((\mathfrak{f},\mathfrak{g},s),\mathsf{Y})$, then $\mathsf{D}\coloneqq\gamma_{\mathsf{Y},n}^{\;m\leftarrow}(\mathsf{A})\in w$. Because $\mathsf{A}\subseteq\mathsf{Y}$ we find $\{\mathsf{x},\mathsf{x}'\}\subseteq\mathsf{A}$ with $\gamma_{\mathsf{Y},n}^{\;m}(\mathsf{x})=\mathsf{a}$ and $\gamma_{\mathsf{Y},n}^{\;m}(\mathsf{x}')=\mathsf{a}''$. Because $\mathsf{a}\neq\mathsf{a}''$ also $\mathsf{x}\neq\mathsf{x}'$. And since $\gamma_{\mathsf{Y},n}^{\;m}$ is monotonic with respect to Γ^i_j and Γ^m_n by Lemma 4.2 (d), again, $]\mathsf{x},\mathsf{x}'[^i_j=\gamma_{\mathsf{Y},n}^{\;m\leftarrow}(]\mathsf{a},\mathsf{a}''[^m_n\cap\mathsf{Y})$ and thus $|]\mathsf{x},\mathsf{x}'[^i_j|=|]\mathsf{a},\mathsf{a}''[^m_n\cap\mathsf{Y}|$, which proves the other implication.

(d) If (i) holds and if $z\in\mathbb{Z}_w$, if $\{i,j\}\subseteq\mathbb{N}_0$, if $\mathsf{Y}=\xi^{\mathfrak{f}\leftarrow}_{\mathfrak{g}}(\{z\})$, if $\mathfrak{u}\colon[\![i]\!]\to(\mathfrak{U}\otimes\{\circ,\bullet\})\cup\mathfrak{D}$ and $\mathfrak{v}\colon[\![j]\!]\to(\mathfrak{U}\otimes\{\circ,\bullet\})\cup\mathfrak{D}$, if $(\mathfrak{u},\mathfrak{v},w)=R((\mathfrak{f},\mathfrak{g},s),\mathsf{Y})$ and if $\{\mathsf{D}_1,\mathsf{D}_2\}\subseteq w$ and $\mathsf{D}_1\neq\mathsf{D}_2$, then by definition of w there exist $\{\mathsf{A}_1,\mathsf{A}_2\}\subseteq s$ with $\mathsf{D}_1=\gamma_{\mathsf{Y},n}^{\;m\leftarrow}(\mathsf{A}_1)$ and $\mathsf{D}_2=\gamma_{\mathsf{Y},n}^{\;m\leftarrow}(\mathsf{A}_2)$. Since $\gamma_{\mathsf{Y},n}^{\;m}$ is injective, $\mathsf{A}_1\cap\mathsf{A}_2=\gamma_{\mathsf{Y},n}^{\;m}{}_{\to}(\mathsf{D}_1)\cap\gamma_{\mathsf{Y},n}^{\;m}{}_{\to}(\mathsf{D}_2)=\gamma_{\mathsf{Y},n}^{\;m}{}_{\to}(\mathsf{D}_1\cap\mathsf{D}_2)$ and thus $\mathsf{A}_1\cap\mathsf{A}_2=\varnothing$ by $\mathsf{D}_1\cap\mathsf{D}_2=\varnothing$, or, equivalently, $\mathsf{A}_1\neq\mathsf{A}_2$. Moreover, because $\mathsf{D}_1\neq\varnothing\neq\mathsf{D}_2$ there exist $\mathsf{x}_1\in\mathsf{D}_1$ and $\mathsf{x}_2\in\mathsf{D}_2$. For $\mathsf{a}_1\coloneqq\gamma_{\mathsf{Y},n}^{\;m}(\mathsf{x}_1)$

and $\mathsf{a}_2 \coloneqq \gamma_{\mathsf{Y},n}^{\ m}(\mathsf{x}_2)$ then, $\mathsf{a}_1 \in \mathsf{A}_1$ and $\mathsf{a}_2 \in \mathsf{A}_2$ and $\xi_{\mathfrak{g}}^{\mathfrak{f}}(\mathsf{a}_1) = \xi_{\mathfrak{g}}^{\mathfrak{f}}(\mathsf{a}_2) = z$ by definition of Y, which is to say $\xi_{\mathfrak{g}}^{\mathfrak{f}}(\mathsf{a}_2) - \xi_{\mathfrak{g}}^{\mathfrak{f}}(\mathsf{a}_1) = 0$. Hence, assumption (i) guarantees $\mathsf{A}_1 \diagdown\!\!\diagdown_n^m \mathsf{A}_2$. Because $\gamma_{\mathsf{Y},n}^{\ m}$ is strictly monotonic with respect to Γ_j^i and Γ_n^m by Lemma 4.2 (d) this requires that, too, $\mathsf{D}_1 \diagdown\!\!\diagdown_j^i \mathsf{D}_2$.

Conversely, if (ii) is true, if $\{\mathsf{A}_1, \mathsf{A}_2\} \subseteq s$ and $\mathsf{a}_1 \in \mathsf{A}_1$ and $\mathsf{a}_2 \in \mathsf{A}_2$ are such that $\mathsf{A}_1 \neq \mathsf{A}_2$ and $\xi_{\mathfrak{g}}^{\mathfrak{f}}(\mathsf{a}_2) - \xi_{\mathfrak{g}}^{\mathfrak{f}}(\mathsf{a}_1) = 0$, then defining $z \coloneqq \xi_{\mathfrak{g}}^{\mathfrak{f}}(\mathsf{a}_1)$ and $\mathsf{Y} \coloneqq \xi_{\mathfrak{g}}^{\mathfrak{f}\leftarrow}(\{z\})$ implies $\mathsf{a}_1 \in \mathsf{A}_1 \cap \mathsf{Y} \neq \varnothing$ and $\mathsf{a}_2 \in \mathsf{A}_2 \cap \mathsf{Y} \neq \varnothing$. Consequently, if $\{i, j\} \subseteq \mathbb{N}_0$ and $\mathfrak{u}\colon [\![i]\!] \to (\mathfrak{U} \otimes \{\circ, \bullet\}) \cup \mathfrak{O}$ and $\mathfrak{v}\colon [\![j]\!] \to (\mathfrak{U} \otimes \{\circ, \bullet\}) \cup \mathfrak{O}$ and $(\mathfrak{u}, \mathfrak{v}, w) = R((\mathfrak{f}, \mathfrak{g}, s), \mathsf{Y})$, then $\mathsf{D}_1 \coloneqq \gamma_{\mathsf{Y},n}^{\ m\leftarrow}(\mathsf{A}_1) \in w$ and $\mathsf{D}_2 \coloneqq \gamma_{\mathsf{Y},n}^{\ m\leftarrow}(\mathsf{A}_2) \in w$. Moreover, since $\mathsf{A}_1 \cap \mathsf{A}_2 = \varnothing$, also $\mathsf{D}_1 \cap \mathsf{D}_2 = \gamma_{\mathsf{Y},j}^{\ i\leftarrow}(\mathsf{A}_1) \cap \gamma_{\mathsf{Y},j}^{\ i\leftarrow}(\mathsf{A}_2) = \gamma_{\mathsf{Y},j}^{\ i\leftarrow}(\mathsf{A}_1 \cap \mathsf{A}_2) = \varnothing$, i.e., $\mathsf{D}_1 \neq \mathsf{D}_2$. Hence, by assumption (ii), necessarily, $\mathsf{D}_1 \diagdown\!\!\diagdown_j^i \mathsf{D}_2$. Since $\gamma_{\mathsf{Y},n}^{\ m}$ is injective we may conclude $\mathsf{A}_1 \diagdown\!\!\diagdown_n^m \mathsf{A}_2$, as claimed.

(e) That is clear because $s \leq \ker(\xi_{\mathfrak{g}}^{\mathfrak{f}})$ and only stated for emphasis. □

NOTATION 8.2. For any additive subsemigroup D of $\mathbb{N}$ let

$$r_D \coloneqq \{(z, z') \mid \{z, z'\} \subseteq \mathbb{Z} \wedge |z' - z| \notin D \cup \{0\}\}.$$

PROPOSITION 8.3. *Let $w \in \mathbb{N}$ and let D be any additive subsemigroup of $\mathbb{N}$.*

(a) For any $w \in \mathbb{N}$ the category $\mathcal{U}^{\times \mathbb{Z}_w}$ can be expressed as the set of all $(\mathfrak{f}, \mathfrak{g}, s) \in {}_{\mathbb{Z}_w,\varnothing}\mathcal{S}$ such that $s \leq \ker(\xi_{\mathfrak{g}}^{\mathfrak{f}})$ and such that $|\mathsf{A}| = 2$ and $\sum_{z \in \mathbb{Z}_w} z\sigma_{\mathfrak{g}}^{\mathfrak{f}}(\mathsf{A}) = 0$ for any $\mathsf{A} \in s$.

(b) The category $\mathcal{U}^{*(\mathbb{Z}, r_D)}$ can be expressed as the set of all $(\mathfrak{f}, \mathfrak{g}, s) \in {}_{\mathbb{Z},\varnothing}\mathcal{S}$ for which there exist $\{m, n\} \subseteq \mathbb{N}_0$ such that $\mathfrak{f}\colon [\![m]\!] \to \mathbb{Z} \otimes \{\circ, \bullet\}$ and $\mathfrak{g}\colon [\![n]\!] \to \mathbb{Z} \otimes \{\circ, \bullet\}$, such that $s \leq \ker(\xi_{\mathfrak{g}}^{\mathfrak{f}})$, such that $|\mathsf{A}| = 2$ and $\sum_{z \in \mathbb{Z}} z\sigma_{\mathfrak{g}}^{\mathfrak{f}}(\mathsf{A}) = 0$ for any $\mathsf{A} \in s$, and such that for any $\{\mathsf{A}_1, \mathsf{A}_2\} \subseteq s$ with $\mathsf{A}_1 \neq \mathsf{A}_2$, whenever there exist $\mathsf{a}_1 \in \mathsf{A}_1$ and $\mathsf{a}_2 \in \mathsf{A}_2$ such that $|\xi_{\mathfrak{g}}^{\mathfrak{f}}(\mathsf{a}_2) - \xi_{\mathfrak{g}}^{\mathfrak{f}}(\mathsf{a}_1)| \in D$, then $\mathsf{A}_1 \diagdown\!\!\diagdown_n^m \mathsf{A}_2$.

(c) The category $(\mathcal{U}^+)^{*(\mathbb{Z}, r_D)}$ can be expressed as the set of all $(\mathfrak{f}, \mathfrak{g}, s) \in {}_{\mathbb{Z},\varnothing}\mathcal{S}$ for which there exist $\{m, n\} \subseteq \mathbb{N}_0$ such that $\mathfrak{f}\colon [\![m]\!] \to \mathbb{Z} \otimes \{\circ, \bullet\}$ and $\mathfrak{g}\colon [\![n]\!] \to \mathbb{Z} \otimes \{\circ, \bullet\}$, such that $s \leq \ker(\xi_{\mathfrak{g}}^{\mathfrak{f}})$, such that $|\mathsf{A}| = 2$ and $\sum_{z \in \mathbb{Z}} z\sigma_{\mathfrak{g}}^{\mathfrak{f}}(\mathsf{A}) = 0$ for any $\mathsf{A} \in s$, and such that for any $\{\mathsf{A}_1, \mathsf{A}_2\} \subseteq s$ with $\mathsf{A}_1 \neq \mathsf{A}_2$, whenever there exist $\mathsf{a}_1 \in \mathsf{A}_1$ and $\mathsf{a}_2 \in \mathsf{A}_2$ such that $|\xi_{\mathfrak{g}}^{\mathfrak{f}}(\mathsf{a}_2) - \xi_{\mathfrak{g}}^{\mathfrak{f}}(\mathsf{a}_1)| \in D \cup \{0\}$, then $\mathsf{A}_1 \diagdown\!\!\diagdown_n^m \mathsf{A}_2$.

(d) The category $(\mathcal{O}^*)^{*(\mathbb{Z}, r_D)}$ can be expressed as the set of all $(\mathfrak{f}, \mathfrak{g}, s) \in {}_{\varnothing,\mathbb{Z}}\mathcal{S}$ for which there exist $\{m, n\} \subseteq \mathbb{N}_0$ such that $\mathfrak{f}\colon [\![m]\!] \to \mathbb{Z}$ and $\mathfrak{g}\colon [\![n]\!] \to \mathbb{Z}$, such that $s \leq \ker(\xi_{\mathfrak{g}}^{\mathfrak{f}})$, such that $|\mathsf{A}| = 2$ for any $\mathsf{A} \in s$, such that $|]\mathsf{a}, \mathsf{c}[_n^m \cap \{\mathsf{b} \in \Pi_n^m \wedge \xi_{\mathfrak{g}}^{\mathfrak{f}}(\mathsf{b}) - \xi_{\mathfrak{g}}^{\mathfrak{f}}(\mathsf{a}) = 0\}| \equiv_2 0$ for any $\{\mathsf{a}, \mathsf{c}\} \subseteq \mathsf{A}$ with $\mathsf{a} \neq \mathsf{c}$ and any $\mathsf{A} \in s$, and such that for any $\{\mathsf{A}_1, \mathsf{A}_2\} \subseteq s$ with $\mathsf{A}_1 \neq \mathsf{A}_2$, whenever there exist $\mathsf{a}_1 \in \mathsf{A}_1$ and $\mathsf{a}_2 \in \mathsf{A}_2$ such that $|\xi_{\mathfrak{g}}^{\mathfrak{f}}(\mathsf{a}_2) - \xi_{\mathfrak{g}}^{\mathfrak{f}}(\mathsf{a}_1)| \in D$, then $\mathsf{A}_1 \diagdown\!\!\diagdown_n^m \mathsf{A}_2$.

(e) The category $(\mathcal{O}^+)^{*(\mathbb{Z}, r_D)}$ can be expressed as the set of all $(\mathfrak{f}, \mathfrak{g}, s) \in {}_{\varnothing,\mathbb{Z}}\mathcal{S}$ for which there exist $\{m, n\} \subseteq \mathbb{N}_0$ such that $\mathfrak{f}\colon [\![m]\!] \to \mathbb{Z}$ and $\mathfrak{g}\colon [\![n]\!] \to \mathbb{Z}$, such that $s \leq \ker(\xi_{\mathfrak{g}}^{\mathfrak{f}})$, such that $|\mathsf{A}| = 2$ for any $\mathsf{A} \in s$, and such that for any $\{\mathsf{A}_1, \mathsf{A}_2\} \subseteq s$ with $\mathsf{A}_1 \neq \mathsf{A}_2$, whenever there exist $\mathsf{a}_1 \in \mathsf{A}_1$ and $\mathsf{a}_2 \in \mathsf{A}_2$ such that $|\xi_{\mathfrak{g}}^{\mathfrak{f}}(\mathsf{a}_2) - \xi_{\mathfrak{g}}^{\mathfrak{f}}(\mathsf{a}_1)| \in D \cup \{0\}$, then $\mathsf{A}_1 \diagdown\!\!\diagdown_n^m \mathsf{A}_2$.

PROOF. The claims all follow from the definitions and Lemma 8.1.

(a) By Definitions 7.10 (a) and 4.7 the category $\mathcal{U}^{\times \mathbb{Z}_w}$ is given by the set of all $(\mathfrak{f}, \mathfrak{g}, s) \in {}_{\mathbb{Z}_w, \varnothing}\mathcal{S}$ such that $s \leq \ker(\xi^{\mathfrak{f}}_{\mathfrak{g}})$ and such that for any $z \in \mathbb{Z}_w$, if $(\mathfrak{u}, \mathfrak{v}, w) = R((\mathfrak{f}, \mathfrak{g}, s), \xi^{\mathfrak{f}\leftarrow}_{\mathfrak{g}}(\{z\}))$, then $|\mathsf{D}| = 2$ and ${}_z\sigma^{\mathfrak{u}}_{\mathfrak{v}}(\mathsf{D}) = 0$ for any $\mathsf{D} \in w$. Hence, if we remember Remark 4.13, we can apply Lemma 8.1 (a) and (b), which then yields the assertion.

(b) Similarly, according to Definitions 7.10 (a) and 4.13 the category $\mathcal{U}^{*(\mathbb{Z}, r_D)}$ is given by the set of all $(\mathfrak{f}, \mathfrak{g}, s) \in {}_{\mathbb{Z}, \varnothing}\mathcal{S}$ for which there exist $\{m, n\} \subseteq \mathbb{N}_0$ such that $\mathfrak{f}\colon [\![m]\!] \to \mathbb{Z} \otimes \{\circ, \bullet\}$ and $\mathfrak{g}\colon [\![n]\!] \to \mathbb{Z} \otimes \{\circ, \bullet\}$, such that $s \leq \ker(\xi^{\mathfrak{f}}_{\mathfrak{g}})$, such that for any $z \in \mathbb{Z}$, if $(\mathfrak{u}, \mathfrak{v}, w) = R((\mathfrak{f}, \mathfrak{g}, s), \xi^{\mathfrak{f}\leftarrow}_{\mathfrak{g}}(\{z\}))$, then $|\mathsf{D}| = 2$ and ${}_z\sigma^{\mathfrak{u}}_{\mathfrak{v}}(\mathsf{D}) = 0$ for any $\mathsf{D} \in w$, and such that for any $\{z_1, z_2\} \subseteq \mathbb{Z}$ with $z_1 \neq z_2$, whenever $(z_1, z_2) \notin r_D$, then $\mathsf{A}_1 \between^m_n \mathsf{A}_2$ for any $\{\mathsf{A}_1, \mathsf{A}_2\} \subseteq s$ with $\mathsf{A}_1 \subseteq \xi^{\mathfrak{f}\leftarrow}_{\mathfrak{g}}(\{z_1\})$ and $\mathsf{A}_2 \subseteq \xi^{\mathfrak{f}\leftarrow}_{\mathfrak{g}}(\{z_2\})$. Thus, the claim follows by Lemma 8.1 (a), (b) and (e).

(c) Definitions 7.10 (b) and 4.13 tell us that $(\mathcal{U}^{+})^{*(\mathbb{Z}, r_D)}$ consists precisely of all $(\mathfrak{f}, \mathfrak{g}, s) \in {}_{\mathbb{Z}, \varnothing}\mathcal{S}$ for which there exist $\{m, n\} \subseteq \mathbb{N}_0$ such that $\mathfrak{f}\colon [\![m]\!] \to \mathbb{Z} \otimes \{\circ, \bullet\}$ and $\mathfrak{g}\colon [\![n]\!] \to \mathbb{Z} \otimes \{\circ, \bullet\}$, such that $s \leq \ker(\xi^{\mathfrak{f}}_{\mathfrak{g}})$, such that for any $z \in \mathbb{Z}$, if $\{i, j\} \subseteq \mathbb{N}_0$ and $\mathfrak{u}\colon [\![i]\!] \to \mathbb{Z} \otimes \{\circ, \bullet\}$ and $\mathfrak{v}\colon [\![j]\!] \to \mathbb{Z} \otimes \{\circ, \bullet\}$ and $(\mathfrak{u}, \mathfrak{v}, w) = R((\mathfrak{f}, \mathfrak{g}, s), \xi^{\mathfrak{f}\leftarrow}_{\mathfrak{g}}(\{z\}))$, then $|\mathsf{D}| = 2$ and ${}_z\sigma^{\mathfrak{u}}_{\mathfrak{v}}(\mathsf{D}) = 0$ for any $\mathsf{D} \in w$ and $\mathsf{D}_1 \between^i_j \mathsf{D}_2$ for any $\{\mathsf{D}_1, \mathsf{D}_2\} \subseteq w$ with $\mathsf{D}_1 \neq \mathsf{D}_2$, and such that for any $\{z_1, z_2\} \subseteq \mathbb{Z}$ with $z_1 \neq z_2$, whenever $(z_1, z_2) \notin r_D$, then $\mathsf{A}_1 \between^m_n \mathsf{A}_2$ for any $\{\mathsf{A}_1, \mathsf{A}_2\} \subseteq s$ with $\mathsf{A}_1 \subseteq \xi^{\mathfrak{f}\leftarrow}_{\mathfrak{g}}(\{z_1\})$ and $\mathsf{A}_2 \subseteq \xi^{\mathfrak{f}\leftarrow}_{\mathfrak{g}}(\{z_2\})$. Hence, the claim follows by Lemma 8.1 (a), (b), (d) and (e). Note that the conditions coming from Parts (d) and (e) have been combined in the claim.

(d) By Definitions 7.3 (b) and 4.13 the category $(\mathcal{O}^{*})^{*(\mathbb{Z}, r_D)}$ comprises exactly all $(\mathfrak{f}, \mathfrak{g}, s) \in {}_{\varnothing, \mathbb{Z}}\mathcal{S}$ for which there exist $\{m, n\} \subseteq \mathbb{N}_0$ such that $\mathfrak{f}\colon [\![m]\!] \to \mathbb{Z}$ and $\mathfrak{g}\colon [\![n]\!] \to \mathbb{Z}$, such that $s \leq \ker(\xi^{\mathfrak{f}}_{\mathfrak{g}})$, such that for any $z \in \mathbb{Z}$, if $\{i, j\} \subseteq \mathbb{N}_0$ and $\mathfrak{u}\colon [\![i]\!] \to \mathbb{Z}$ and $\mathfrak{v}\colon [\![j]\!] \to \mathbb{Z}$ and $(\mathfrak{u}, \mathfrak{v}, w) = R((\mathfrak{f}, \mathfrak{g}, s), \xi^{\mathfrak{f}\leftarrow}_{\mathfrak{g}}(\{z\}))$, then $|\mathsf{D}| = 2$ and $|]\mathsf{x}, \mathsf{x}'[^i_j| \equiv_2 0$ for any $\{\mathsf{x}, \mathsf{x}'\} \subseteq \mathsf{D}$ with $\mathsf{x} \neq \mathsf{x}'$ and $\mathsf{D} \in w$, and such that for any $\{z_1, z_2\} \subseteq \mathbb{Z}$ with $z_1 \neq z_2$, whenever $(z_1, z_2) \notin r_D$, then $\mathsf{A}_1 \between^m_n \mathsf{A}_2$ for any $\{\mathsf{A}_1, \mathsf{A}_2\} \subseteq s$ with $\mathsf{A}_1 \subseteq \xi^{\mathfrak{f}\leftarrow}_{\mathfrak{g}}(\{z_1\})$ and $\mathsf{A}_2 \subseteq \xi^{\mathfrak{f}\leftarrow}_{\mathfrak{g}}(\{z_2\})$. Therefore, we can infer the claimed identity with the help of Lemma 8.1 (a), (c) and (e).

(e) Finally, it is by Definitions 7.3 (c) and 4.13 that we know $(\mathcal{O}^{+})^{*(\mathbb{Z}, r_D)}$ to be made up of all $(\mathfrak{f}, \mathfrak{g}, s) \in {}_{\varnothing, \mathbb{Z}}\mathcal{S}$ for which there exist $\{m, n\} \subseteq \mathbb{N}_0$ such that $\mathfrak{f}\colon [\![m]\!] \to \mathbb{Z}$ and $\mathfrak{g}\colon [\![n]\!] \to \mathbb{Z}$, such that $s \leq \ker(\xi^{\mathfrak{f}}_{\mathfrak{g}})$, such that for any $z \in \mathbb{Z}$, if $\{i, j\} \subseteq \mathbb{N}_0$ and $\mathfrak{u}\colon [\![i]\!] \to \mathbb{Z}$ and $\mathfrak{v}\colon [\![j]\!] \to \mathbb{Z}$ and $(\mathfrak{u}, \mathfrak{v}, w) = R((\mathfrak{f}, \mathfrak{g}, s), \xi^{\mathfrak{f}\leftarrow}_{\mathfrak{g}}(\{z\}))$, then $|\mathsf{D}| = 2$ and $|]\mathsf{x}, \mathsf{x}'[^i_j| \equiv_2 0$ for any $\{\mathsf{x}, \mathsf{x}'\} \subseteq \mathsf{D}$ with $\mathsf{x} \neq \mathsf{x}'$ and $\mathsf{D} \in w$ and $\mathsf{D}_1 \between^i_j \mathsf{D}_2$ for any $\{\mathsf{D}_1, \mathsf{D}_2\} \subseteq w$ with $\mathsf{D}_1 \neq \mathsf{D}_2$, and such that for any $\{z_1, z_2\} \subseteq \mathbb{Z}$ with $z_1 \neq z_2$, whenever $(z_1, z_2) \notin r_D$, then $\mathsf{A}_1 \between^m_n \mathsf{A}_2$ for any $\{\mathsf{A}_1, \mathsf{A}_2\} \subseteq s$ with $\mathsf{A}_1 \subseteq \xi^{\mathfrak{f}\leftarrow}_{\mathfrak{g}}(\{z_1\})$ and $\mathsf{A}_2 \subseteq \xi^{\mathfrak{f}\leftarrow}_{\mathfrak{g}}(\{z_2\})$. In conclusion, Lemma 8.1 (a), (c), (d) and (e) proves the assertion if we once more combine the conditions coming from Parts (d) and (e) into one. $\square$

8.2. Reformulating the wreath graph co-products. With the help of Proposition 8.3 we are now able to reformulate the definitions of the wreath graph co-product supercategories of the unitary half-liberations. The ensuing lemma systematizes the required proofs.

LEMMA 8.4. *For any* $w \in \mathbb{N}_0$, *if* $(\mathfrak{U},\mathfrak{D}) \in \{(\mathbb{Z}_w,\varnothing),(\varnothing,\mathbb{Z}_w)\}$, *then for any* $\mathbb{Z}_w$-*invariant partial commutation relation* r *on* $\mathbb{Z}_w$, *any* $\{k,\ell\}\subseteq\mathbb{N}_0$, *any* $\mathfrak{c}\colon [\![k]\!] \to ((\mathfrak{U}\cup\{\aleph\})\otimes\{\circ,\bullet\})\cup\mathfrak{D}$ *and* $\mathfrak{d}\colon [\![\ell]\!]\to((\mathfrak{U}\cup\{\aleph\})\otimes\{\circ,\bullet\})\cup\mathfrak{D}$ *and any* $(\mathfrak{c},\mathfrak{d},p)\in {}_{\mathfrak{U}\cup\{\aleph\},\mathfrak{D}}\mathcal{S}$ *with* ${}_{\aleph}\Sigma^{\mathfrak{c}}_{\mathfrak{d}} \equiv_w 0$ *and* $p \le \xi^{\mathfrak{c}\leftarrow}_{\mathfrak{d}}(\{\mathbb{Z}_w,\{\aleph\}\})$ *the following hold if* $\mathsf{Y} = \xi^{\mathfrak{c}\leftarrow}_{\mathfrak{d}}(\mathbb{Z}_w)$, *if* $\{m,n\}\subseteq\mathbb{N}_0$ *and if* $\mathfrak{f}\colon[\![m]\!]\to(\mathfrak{U}\otimes\{\circ,\bullet\})\cup\mathfrak{D}$ *and* $\mathfrak{g}\colon[\![n]\!]\to(\mathfrak{U}\otimes\{\circ,\bullet\})\cup\mathfrak{D}$ *are such that* $\xi^{\mathfrak{f}}_{\mathfrak{g}} = \pi_w\circ\varepsilon^{\mathfrak{c}}_{\mathfrak{d}}\circ\gamma_{\mathsf{Y},\ell}^{\ k}$ *and* $\zeta^{\mathfrak{f}}_{\mathfrak{g}} = \zeta^{\mathfrak{c}}_{\mathfrak{d}}\circ\gamma_{\mathsf{Y},\ell}^{\ k}$ *and if* $s = R(p,\mathsf{Y})$.

(a) The following are equivalent:
- *(i)* $\xi^{\mathfrak{c}}_{\mathfrak{d}}(\mathsf{b}') - \xi^{\mathfrak{c}}_{\mathfrak{d}}(\mathsf{b}) + {}_{\aleph}\delta^{\mathfrak{c}}_{\mathfrak{d}}(\mathsf{b},\mathsf{b}') \equiv_w 0$ *for any* $\{\mathsf{b},\mathsf{b}'\}\subseteq\mathsf{B}$ *and any* $\mathsf{B}\in p$ *with* $\mathsf{B}\subseteq\xi^{\mathfrak{c}\leftarrow}_{\mathfrak{d}}(\mathbb{Z}_w)$.
- *(ii)* $s\le\ker(\xi^{\mathfrak{f}}_{\mathfrak{g}})$.

(b) The following are equivalent:
- *(i)* $|\mathsf{B}| = 2$ *for any* $\mathsf{B}\in p$ *with* $\mathsf{B}\subseteq\xi^{\mathfrak{c}\leftarrow}_{\mathfrak{d}}(\mathbb{Z}_w)$.
- *(ii)* $|\mathsf{A}| = 2$ *for any* $\mathsf{A}\in s$.

(c) If $(\mathfrak{U},\mathfrak{D}) = (\mathbb{Z}_w,\varnothing)$, *then the following are equivalent:*
- *(i)* $\sum_{z\in\mathbb{Z}_w} {}_z\sigma^{\mathfrak{c}}_{\mathfrak{d}}(\mathsf{B}) = 0$ *for any* $\mathsf{B}\in p$ *with* $\mathsf{B}\subseteq\xi^{\mathfrak{c}\leftarrow}_{\mathfrak{d}}(\mathbb{Z}_w)$.
- *(ii)* $\sum_{z\in\mathbb{Z}_w} {}_z\sigma^{\mathfrak{f}}_{\mathfrak{g}}(\mathsf{A}) = 0$ *for any* $\mathsf{A}\in s$.

(d) If $s\le\ker(\xi^{\mathfrak{f}}_{\mathfrak{g}})$, *then the following are equivalent:*
- *(i) For any* $\mathsf{B}\in p$ *with* $\mathsf{B}\subseteq\xi^{\mathfrak{c}\leftarrow}_{\mathfrak{d}}(\mathbb{Z}_w)$ *and any* $\{\mathsf{b},\mathsf{b}''\}\subseteq\mathsf{B}$, *if* $\mathsf{b}\neq\mathsf{b}''$, *then* $|]\mathsf{b},\mathsf{b}''[^{k}_{\ell}\cap\{\mathsf{b}'\in\Pi^{k}_{\ell}\wedge\xi^{\mathfrak{c}}_{\mathfrak{d}}(\mathsf{b}')\in\mathbb{Z}_w\wedge\xi^{\mathfrak{c}}_{\mathfrak{d}}(\mathsf{b}')-\xi^{\mathfrak{c}}_{\mathfrak{d}}(\mathsf{b})+{}_{\aleph}\delta^{\mathfrak{c}}_{\mathfrak{d}}(\mathsf{b},\mathsf{b}')\equiv_w 0\}|\equiv_2 0$.
- *(ii) For any* $\mathsf{A}\in s$ *and any* $\{\mathsf{a},\mathsf{a}''\}\subseteq\mathsf{A}$, *if* $\mathsf{a}\neq\mathsf{a}''$, *then* $|]\mathsf{a},\mathsf{a}''[^{m}_{n}\cap\{\mathsf{a}'\in\Pi^{m}_{n}\wedge\xi^{\mathfrak{f}}_{\mathfrak{g}}(\mathsf{a}')-\xi^{\mathfrak{f}}_{\mathfrak{g}}(\mathsf{a})=0\}|\equiv_2 0$.

(e) If $s\le\ker(\xi^{\mathfrak{f}}_{\mathfrak{g}})$, *then the following are equivalent :*
- *(i) For any* $\{\mathsf{B}_1,\mathsf{B}_2\}\subseteq p$ *with* $\mathsf{B}_1\subseteq\xi^{\mathfrak{c}\leftarrow}_{\mathfrak{d}}(\mathbb{Z}_w)$ *and* $\mathsf{B}_2\subseteq\xi^{\mathfrak{c}\leftarrow}_{\mathfrak{d}}(\mathbb{Z}_w)$ *and* $\mathsf{B}_1\neq\mathsf{B}_2$, *whenever there exist* $\mathsf{b}_1\in\mathsf{B}_1$ *and* $\mathsf{b}_2\in\mathsf{B}_2$ *such that* $\xi^{\mathfrak{c}}_{\mathfrak{d}}(\mathsf{b}_2)-\xi^{\mathfrak{c}}_{\mathfrak{d}}(\mathsf{b}_1)+{}_{\aleph}\delta^{\mathfrak{c}}_{\mathfrak{d}}(\mathsf{b}_1,\mathsf{b}_2)\equiv_w 0$, *then* $\mathsf{B}_1 \between^{k}_{\ell} \mathsf{B}_2$.
- *(ii) For any* $\{\mathsf{A}_1,\mathsf{A}_2\}\subseteq s$ *with* $\mathsf{A}_1\neq\mathsf{A}_2$, *whenever there exist* $\mathsf{a}_1\in\mathsf{A}_1$ *and* $\mathsf{a}_2\in\mathsf{A}_2$ *such that* $\xi^{\mathfrak{f}}_{\mathfrak{g}}(\mathsf{a}_2)-\xi^{\mathfrak{f}}_{\mathfrak{g}}(\mathsf{a}_1)=0$, *then* $\mathsf{A}_1\between^{m}_{n}\mathsf{A}_2$.

(f) If $s\le\ker(\xi^{\mathfrak{f}}_{\mathfrak{g}})$, *if* $w=0$, *if* D *is any additive subsemigroup of* $\mathbb{N}$ *and if* $r=r_D$, *then the following are equivalent:*
- *(i) For any* $\{\mathsf{B}_1,\mathsf{B}_2\}\subseteq p$ *with* $\mathsf{B}_1\subseteq\xi^{\mathfrak{c}\leftarrow}_{\mathfrak{d}}(\mathbb{Z}_w)$ *and* $\mathsf{B}_2\subseteq\xi^{\mathfrak{c}\leftarrow}_{\mathfrak{d}}(\mathbb{Z}_w)$ *and* $\mathsf{B}_1\neq\mathsf{B}_2$, *whenever there exist* $\mathsf{b}_1\in\mathsf{B}_1$ *and* $\mathsf{b}_2\in\mathsf{B}_2$ *such that* $|\xi^{\mathfrak{c}}_{\mathfrak{d}}(\mathsf{b}_2)-\xi^{\mathfrak{c}}_{\mathfrak{d}}(\mathsf{b}_1)+{}_{\aleph}\delta^{\mathfrak{c}}_{\mathfrak{d}}(\mathsf{b}_1,\mathsf{b}_2)|\notin D$, *then* $\mathsf{B}_1\between^{k}_{\ell}\mathsf{B}_2$.
- *(ii) For any* $\{\mathsf{A}_1,\mathsf{A}_2\}\subseteq s$ *with* $\mathsf{A}_1\neq\mathsf{A}_2$, *whenever there exist* $\mathsf{a}_1\in\mathsf{A}_1$ *and* $\mathsf{a}_2\in\mathsf{A}_2$ *such that* $\xi^{\mathfrak{f}}_{\mathfrak{g}}(\mathsf{a}_1)\neq\xi^{\mathfrak{f}}_{\mathfrak{g}}(\mathsf{a}_2)$ *and* $(\xi^{\mathfrak{f}}_{\mathfrak{g}}(\mathsf{a}_1),\xi^{\mathfrak{f}}_{\mathfrak{g}}(\mathsf{a}_2))\notin r_D$, *then* $\mathsf{A}_1\between^{m}_{n}\mathsf{A}_2$.

PROOF. The following observation will be used in the proofs of all claims except (b) and (c). For any $\{\mathsf{a},\mathsf{a}'\}\subseteq\Pi^{m}_{n}$ and any $\{\mathsf{b},\mathsf{b}'\}\subseteq\mathsf{Y}$, if $\mathsf{b}=\gamma_{\mathsf{Y},\ell}^{\ k}(\mathsf{a})$ and $\mathsf{b}'=\gamma_{\mathsf{Y},\ell}^{\ k}(\mathsf{a}')$, then

$$\xi_{\mathfrak{g}}^{\mathfrak{f}}(\mathsf{a}) = \xi_{\mathfrak{g}}^{\mathfrak{f}}(\mathsf{a}') \Leftrightarrow \xi_{\mathfrak{d}}^{\mathfrak{c}}(\mathsf{b}') - \xi_{\mathfrak{d}}^{\mathfrak{c}}(\mathsf{b}) + {}_{\aleph}\delta_{\mathfrak{d}}^{\mathfrak{c}}(\mathsf{b},\mathsf{b}') \equiv_w 0.$$

Indeed, since ${}_{\aleph}\sigma_{\mathfrak{d}}^{\mathfrak{c}}(\{\mathsf{b}\}) = {}_{\aleph}\sigma_{\mathfrak{d}}^{\mathfrak{c}}(\{\mathsf{b}'\}) = 0$ by $\{\mathsf{b},\mathsf{b}'\} \cap \xi_{\mathfrak{d}}^{\mathfrak{c}\leftarrow}(\{\aleph\}) = \varnothing$ and since ${}_{\aleph}\Sigma_{\mathfrak{d}}^{\mathfrak{c}} \equiv_w 0$ by assumption, Lemma 4.51 implies $\varepsilon_{\mathfrak{d}}^{\mathfrak{c}}(\mathsf{b}') - \varepsilon_{\mathfrak{d}}^{\mathfrak{c}}(\mathsf{b}) \equiv_w \xi_{\mathfrak{d}}^{\mathfrak{c}}(\mathsf{b}') - \xi_{\mathfrak{d}}^{\mathfrak{c}}(\mathsf{b}) + {}_{\aleph}\delta_{\mathfrak{d}}^{\mathfrak{c}}(\mathsf{b},\mathsf{b}')$. And, by definition, $\xi_{\mathfrak{g}}^{\mathfrak{f}}(\mathsf{a}) = \xi_{\mathfrak{g}}^{\mathfrak{f}}(\mathsf{a}')$ if and only if $\varepsilon_{\mathfrak{d}}^{\mathfrak{c}}(\mathsf{b}') - \varepsilon_{\mathfrak{d}}^{\mathfrak{c}}(\mathsf{b}) \equiv_w 0$.

(a) First suppose that (i) holds. Given any $\mathsf{A} \in s$ the definition of s guarantees the existence of some $\mathsf{B} \in p$ with $\mathsf{B} \subseteq \mathsf{Y}$ and $\mathsf{A} = \gamma_{\mathsf{Y},\ell}^{k\leftarrow}(\mathsf{B})$. Hence, for any $\{\mathsf{a},\mathsf{a}'\} \subseteq \mathsf{A}$, if $\mathsf{b} \coloneqq \gamma_{\mathsf{Y},\ell}^{k}(\mathsf{a})$ and $\mathsf{b}' \coloneqq \gamma_{\mathsf{Y},\ell}^{k}(\mathsf{a}')$, then $\{\mathsf{a},\mathsf{a}'\} \subseteq \mathsf{B}$ and thus $\xi_{\mathfrak{d}}^{\mathfrak{c}}(\mathsf{b}') - \xi_{\mathfrak{d}}^{\mathfrak{c}}(\mathsf{b}) + {}_{\aleph}\delta_{\mathfrak{d}}^{\mathfrak{c}}(\mathsf{b},\mathsf{b}') \equiv_w 0$ by (i). Since that requries $\xi_{\mathfrak{g}}^{\mathfrak{f}}(\mathsf{a}) = \xi_{\mathfrak{g}}^{\mathfrak{f}}(\mathsf{a}')$ by the initial remark we have thus shown $s \leq \ker(\xi_{\mathfrak{g}}^{\mathfrak{f}})$.

Conversely, if (ii) holds and if $\mathsf{B} \in p$ with $\mathsf{B} \subseteq \mathsf{Y}$ and $\{\mathsf{b},\mathsf{b}'\} \subseteq \mathsf{B}$ are arbitrary, then $\mathsf{A} \coloneqq \gamma_{\mathsf{Y},\ell}^{k\leftarrow}(\mathsf{B}) \in s$. Moreover, then there exist $\{\mathsf{a},\mathsf{a}'\} \subseteq \mathsf{A}$ with $\mathsf{b} = \gamma_{\mathsf{Y},\ell}^{k}(\mathsf{a})$ and $\mathsf{b}' = \gamma_{\mathsf{Y},\ell}^{k}(\mathsf{a}')$ and with $\xi_{\mathfrak{g}}^{\mathfrak{f}}(\mathsf{a}) = \xi_{\mathfrak{g}}^{\mathfrak{f}}(\mathsf{a}')$ by (ii). The observation at the beginning of the proof therefore implies $\xi_{\mathfrak{d}}^{\mathfrak{c}}(\mathsf{b}') - \xi_{\mathfrak{d}}^{\mathfrak{c}}(\mathsf{b}) + {}_{\aleph}\delta_{\mathfrak{d}}^{\mathfrak{c}}(\mathsf{b},\mathsf{b}') \equiv_w 0$, as claimed.

(b)+(c) We can prove (b) and (c) at the same time. If $(\mathfrak{U},\mathfrak{O}) = (\mathbb{Z}_w,\varnothing)$, then, given any $\mathsf{B} \in p$ and $\mathsf{A} \in s$ with $\mathsf{B} \subseteq \mathsf{Y}$ and $\mathsf{A} = \gamma_{\mathsf{Y},\ell}^{k\leftarrow}(\mathsf{B})$, since $\gamma_{\mathsf{Y},\ell}^{k}$ is injective, $|\mathsf{B}| = |\gamma_{\mathsf{Y},\ell}^{k\leftarrow}(\mathsf{B})| = |\mathsf{A}|$. simultaneously, since $\mathsf{B} \cap \xi_{\mathfrak{d}}^{\mathfrak{c}\leftarrow}(\{\aleph\}) = \varnothing$ and $\zeta_{\mathfrak{g}}^{\mathfrak{f}} = \zeta_{\mathfrak{d}}^{\mathfrak{c}} \circ \gamma_{\mathsf{Y},\ell}^{k}$,

$$\sum_{z\in\mathbb{Z}_w} {}_{\aleph}\sigma_{\mathfrak{d}}^{\mathfrak{c}}(\mathsf{B}) = \sum_{z\in\mathfrak{U}\cup\{\aleph\}\cup\mathfrak{O}} {}_{\aleph}\sigma_{\mathfrak{d}}^{\mathfrak{c}}(\mathsf{B}) = \sum_{\mathfrak{b}\in\mathsf{B}} \sigma(\zeta_{\mathfrak{d}}^{\mathfrak{c}}(\mathfrak{b})) = \sum_{\mathfrak{a}\in\mathsf{A}} \sigma(\zeta_{\mathfrak{d}}^{\mathfrak{c}}(\gamma_{\mathsf{Y},\ell}^{k}(\mathfrak{b}))) = \sum_{z\in\mathbb{Z}_w} {}_{z}\sigma_{\mathfrak{g}}^{\mathfrak{f}}(\mathsf{A}).$$

By definition of s for any B as above there exists A of the above description, and vice versa. Hence, (i) and (ii) are equivalent.

(d) The remark right at the beginning of the proof is equivalent to saying that for any $\mathsf{a} \in \Pi_n^m$ and $\mathsf{b} \in \mathsf{Y}$ with $\mathsf{b} = \gamma_{\mathsf{Y},\ell}^{k}(\mathsf{a})$,

$$\{\mathsf{a}' \in \Pi_n^m \wedge \xi_{\mathfrak{g}}^{\mathfrak{f}}(\mathsf{a}') - \xi_{\mathfrak{g}}^{\mathfrak{f}}(\mathsf{a}) = 0\} = \gamma_{\mathsf{Y},\ell}^{k\leftarrow}(\{\mathsf{b}' \in \mathsf{Y} \wedge \xi_{\mathfrak{d}}^{\mathfrak{c}}(\mathsf{b}') - \xi_{\mathfrak{d}}^{\mathfrak{c}}(\mathsf{b}') + {}_{\aleph}\delta_{\mathfrak{d}}^{\mathfrak{c}}(\mathsf{b},\mathsf{b}) \equiv_w 0\}).$$

Given any $\mathsf{a}'' \in \Pi_n^m$ and $\mathsf{b}'' \in \mathsf{Y}$ with $\mathsf{a} \neq \mathsf{a}''$ and $\mathsf{b} \neq \mathsf{b}''$ and $\mathsf{b}'' = \gamma_{\mathsf{Y},\ell}^{k}(\mathsf{a}'')$, moreover, $]\mathsf{a},\mathsf{a}''[_n^m = \gamma_{\mathsf{Y},\ell}^{k\leftarrow}(]\mathsf{b},\mathsf{b}''[_\ell^k)$ since $\gamma_{\mathsf{Y},\ell}^{k}$ is strictly monotonic with respect to Γ_n^m and Γ_ℓ^k. Hence,

$$\begin{aligned}]\mathsf{a},\mathsf{a}''[_n^m \cap \{\mathsf{a}' \in \Pi_n^m \wedge \xi_{\mathfrak{g}}^{\mathfrak{f}}(\mathsf{a}') - \xi_{\mathfrak{g}}^{\mathfrak{f}}(\mathsf{a}) = 0\} \\ = \gamma_{\mathsf{Y},\ell}^{k\leftarrow}(]\mathsf{b},\mathsf{b}''[_\ell^k \cap \{\mathsf{b}' \in \mathsf{Y} \wedge \xi_{\mathfrak{d}}^{\mathfrak{c}}(\mathsf{b}') - \xi_{\mathfrak{d}}^{\mathfrak{c}}(\mathsf{b}') + {}_{\aleph}\delta_{\mathfrak{d}}^{\mathfrak{c}}(\mathsf{b},\mathsf{b}) \equiv_w 0\}). \end{aligned}$$

Because the set $]\mathsf{b},\mathsf{b}''[_\ell^k \cap \{\mathsf{b}' \in \mathsf{Y} \wedge \xi_{\mathfrak{d}}^{\mathfrak{c}}(\mathsf{b}') - \xi_{\mathfrak{d}}^{\mathfrak{c}}(\mathsf{b}') + {}_{\aleph}\delta_{\mathfrak{d}}^{\mathfrak{c}}(\mathsf{b},\mathsf{b}) \equiv_w 0\}$ is contained in Y and because $\gamma_{\mathsf{Y},\ell}^{k}$ is injective the above identity proves that $|]\mathsf{a},\mathsf{a}''[_n^m \cap \{\mathsf{a}' \in \Pi_n^m \wedge \xi_{\mathfrak{g}}^{\mathfrak{f}}(\mathsf{a}') - \xi_{\mathfrak{g}}^{\mathfrak{f}}(\mathsf{a}) = 0\}|$ and $|]\mathsf{b},\mathsf{b}''[_\ell^k \cap \{\mathsf{b}' \in \mathsf{Y} \wedge \xi_{\mathfrak{d}}^{\mathfrak{c}}(\mathsf{b}') - \xi_{\mathfrak{d}}^{\mathfrak{c}}(\mathsf{b}') + {}_{\aleph}\delta_{\mathfrak{d}}^{\mathfrak{c}}(\mathsf{b},\mathsf{b}) \equiv_w 0\}|$ coincide.

Since by definition of s for any $\mathsf{B} \in p$ with $\mathsf{B} \subseteq \mathsf{Y}$ and any $\{\mathsf{b},\mathsf{b}''\} \subseteq \mathsf{B}$ with $\mathsf{b} \neq \mathsf{b}''$ there exist A, a and a'' as above, (i) implies (ii). The converse implication holds as well, since also for any $\mathsf{A} \in s$ and any $\{\mathsf{a},\mathsf{a}''\} \subseteq \mathsf{A}$ with $\mathsf{a} \neq \mathsf{a}''$ we can always find B, b and b'' as above.

(e) If (i) holds and if $\{\mathsf{A}_1, \mathsf{A}_2\} \subseteq s$ with $\mathsf{A}_1 \neq \mathsf{A}_2$ and if there exist $\mathsf{a}_1 \in \mathsf{A}_1$ and $\mathsf{a}_2 \in \mathsf{A}_2$ such that $\xi^{\mathfrak{f}}_{\mathfrak{g}}(\mathsf{a}_2) - \xi^{\mathfrak{f}}_{\mathfrak{g}}(\mathsf{a}_1) = 0$, then by definition of s there exist $\{\mathsf{B}_1, \mathsf{B}_2\} \subseteq p$ with $\mathsf{B}_1 \subseteq \mathsf{Y}$ and $\mathsf{B}_2 \subseteq \mathsf{Y}$ and $\mathsf{A}_1 = \gamma_{\mathsf{Y},\ell}^{k\leftarrow}(\mathsf{B}_1)$ and $\mathsf{A}_2 = \gamma_{\mathsf{Y},\ell}^{k\leftarrow}(\mathsf{B}_2)$ as well as, consequently, $\mathsf{b}_1 \coloneqq \gamma_{\mathsf{Y},\ell}^{k}(\mathsf{a}_1) \in \mathsf{B}_1$ and $\mathsf{b}_2 \coloneqq \gamma_{\mathsf{Y},\ell}^{k}(\mathsf{a}_2) \in \mathsf{B}_2$. Moreover, because $\mathsf{A}_1 \cap \mathsf{A}_2 = \varnothing$ and because $\gamma_{\mathsf{Y},\ell}^{k}$ is injective, $\mathsf{B}_1 \cap \mathsf{B}_2 = \gamma_{\mathsf{Y},\ell}^{k\rightarrow}(\mathsf{A}_1) \cap \gamma_{\mathsf{Y},\ell}^{k\rightarrow}(\mathsf{A}_2) = \gamma_{\mathsf{Y},\ell}^{k\rightarrow}(\mathsf{A}_1 \cap \mathsf{A}_2) = \varnothing$, i.e., $\mathsf{B}_1 \neq \mathsf{B}_2$. By the initial remark, the assumption that $\xi^{\mathfrak{f}}_{\mathfrak{g}}(\mathsf{a}_2) - \xi^{\mathfrak{f}}_{\mathfrak{g}}(\mathsf{a}_1) = 0$ ensures that $\xi^{\mathfrak{c}}_{\mathfrak{d}}(\mathsf{b}_2) - \xi^{\mathfrak{c}}_{\mathfrak{d}}(\mathsf{b}_1) + {}_{\aleph}\delta^{\mathfrak{c}}_{\mathfrak{d}}(\mathsf{b}_1, \mathsf{b}_2) \equiv_w 0$. Hence, $\mathsf{B}_1 \mathbin{\diagdown\!\!\diagdown}^{k}_{\ell} \mathsf{B}_2$. Since $\gamma_{\mathsf{Y},\ell}^{k}$ is monotonic with respect to Γ^m_n and Γ^k_ℓ that requires $\mathsf{A}_1 \mathbin{\diagdown\!\!\diagdown}^{m}_{n} \mathsf{A}_2$. Since $\gamma_{\mathsf{Y},\ell}^{k}$. Hence, (i) implies (ii).

Conversely, if (ii) is true and if $\{\mathsf{B}_1, \mathsf{B}_2\} \subseteq p$ and $\mathsf{B}_1 \neq \mathsf{B}_2$ and $\mathsf{B}_1 \subseteq \mathsf{Y}$ and $\mathsf{B}_2 \subseteq \mathsf{Y}$ and if $\mathsf{b}_1 \in \mathsf{B}_1$ and $\mathsf{b}_2 \in \mathsf{B}_2$ are such that $\xi^{\mathfrak{c}}_{\mathfrak{d}}(\mathsf{b}_2) - \xi^{\mathfrak{c}}_{\mathfrak{d}}(\mathsf{b}_1) + {}_{\aleph}\delta^{\mathfrak{c}}_{\mathfrak{d}}(\mathsf{b}_1, \mathsf{b}_2) \equiv_w 0$, then $\mathsf{A}_1 \coloneqq \gamma_{\mathsf{Y},\ell}^{k\leftarrow}(\mathsf{B}_1) \in s$ and $\mathsf{A}_2 \coloneqq \gamma_{\mathsf{Y},\ell}^{k\leftarrow}(\mathsf{B}_2) \in s$. Moreover, there are then $\mathsf{a}_1 \in \mathsf{A}_1$ and $\mathsf{a}_2 \in \mathsf{A}_2$ with $\mathsf{b}_1 = \gamma_{\mathsf{Y},\ell}^{k}(\mathsf{a}_1)$ and $\mathsf{b}_2 = \gamma_{\mathsf{Y},\ell}^{k}(\mathsf{a}_2)$. By the observation at the beginning of the proof the assumption $\xi^{\mathfrak{c}}_{\mathfrak{d}}(\mathsf{b}_2) - \xi^{\mathfrak{c}}_{\mathfrak{d}}(\mathsf{b}_1) + {}_{\aleph}\delta^{\mathfrak{c}}_{\mathfrak{d}}(\mathsf{b}_1, \mathsf{b}_2) \equiv_w 0$ implies $\xi^{\mathfrak{f}}_{\mathfrak{g}}(\mathsf{a}_2) - \xi^{\mathfrak{f}}_{\mathfrak{g}}(\mathsf{a}_1) = 0$. By (ii) we conclude $\mathsf{A}_1 \mathbin{\diagdown\!\!\diagdown}^{m}_{n} \mathsf{A}_2$. Because $\gamma_{\mathsf{Y},\ell}^{k}$ is strictly monotonic with respect to Γ^m_n and Γ^k_ℓ by Lemma 4.2 (d) it follows $\mathsf{B}_1 \mathbin{\diagdown\!\!\diagdown}^{k}_{\ell} \mathsf{B}_2$. In other words, (ii) requires (i).

(f) Let (i) be true and let $\{\mathsf{A}_1, \mathsf{A}_2\} \subseteq s$, let $\mathsf{A}_1 \neq \mathsf{A}_2$, let $\mathsf{a}_1 \in \mathsf{A}_1$ and $\mathsf{a}_2 \in \mathsf{A}_2$, let $\xi^{\mathfrak{f}}_{\mathfrak{g}}(\mathsf{a}_1) \neq \xi^{\mathfrak{f}}_{\mathfrak{g}}(\mathsf{a}_2)$ and let $(\xi^{\mathfrak{f}}_{\mathfrak{g}}(\mathsf{a}_1), \xi^{\mathfrak{f}}_{\mathfrak{g}}(\mathsf{a}_2)) \notin r_D$. By definition of s there exist $\{\mathsf{B}_1, \mathsf{B}_2\} \subseteq p$ with $\mathsf{A}_1 = \gamma_{\mathsf{Y},\ell}^{k\leftarrow}(\mathsf{B}_1)$ and $\mathsf{A}_2 = \gamma_{\mathsf{Y},\ell}^{k\leftarrow}(\mathsf{B}_2)$ and, consequently, $\mathsf{b}_1 \coloneqq \gamma_{\mathsf{Y},\ell}^{k}(\mathsf{a}_1) \in \mathsf{B}_1$ and $\mathsf{b}_2 \coloneqq \gamma_{\mathsf{Y},\ell}^{k}(\mathsf{a}_2) \in \mathsf{B}_2$. Moreover, because $p \leq \xi^{\mathfrak{c}\leftarrow}_{\mathfrak{d}}(\{\mathbb{Z}, \{\aleph\}\})$ and $\mathsf{B}_1 \cap \mathsf{Y} \neq \varnothing$ and $\mathsf{B}_2 \cap \mathsf{Y} \neq \varnothing$, necessarily, $\mathsf{B}_1 \subseteq \mathsf{Y}$ and $\mathsf{B}_2 \subseteq \mathsf{Y}$. By definition, the assumption $(\xi^{\mathfrak{f}}_{\mathfrak{g}}(\mathsf{a}_1), \xi^{\mathfrak{f}}_{\mathfrak{g}}(\mathsf{a}_2)) \notin r_D$ implies $|\xi^{\mathfrak{f}}_{\mathfrak{g}}(\mathsf{a}_2) - \xi^{\mathfrak{f}}_{\mathfrak{g}}(\mathsf{a}_1)| \in D \uplus \{0\}$. Hence, the assumption $\xi^{\mathfrak{f}}_{\mathfrak{g}}(\mathsf{a}_1) \neq \xi^{\mathfrak{f}}_{\mathfrak{g}}(\mathsf{a}_2)$ lets us infer $|\xi^{\mathfrak{f}}_{\mathfrak{g}}(\mathsf{a}_2) - \xi^{\mathfrak{f}}_{\mathfrak{g}}(\mathsf{a}_1)| \in D$. By the initial remark that means $|\xi^{\mathfrak{c}}_{\mathfrak{d}}(\mathsf{a}_2) - \xi^{\mathfrak{c}}_{\mathfrak{d}}(\mathsf{a}_1) + {}_{\aleph}\delta^{\mathfrak{c}}_{\mathfrak{d}}(\mathsf{a}_1, \mathsf{a}_2)| \in D$. Therefore, (i) guarantees $\mathsf{B}_1 \mathbin{\diagdown\!\!\diagdown}^{k}_{\ell} \mathsf{B}_2$. Since $\gamma_{\mathsf{Y},\ell}^{k}$ is monotonic with respect to Γ^m_n and Γ^k_ℓ by Lemma 4.2 (d) that warrants that $\mathsf{A}_1 \mathbin{\diagdown\!\!\diagdown}^{m}_{n} \mathsf{A}_2$ too. Thus, we have shown that (i) implies (ii).

If, conversely, (ii) holds and $\{\mathsf{B}_1, \mathsf{B}_2\} \subseteq p$ and $\mathsf{b}_1 \in \mathsf{B}_1$ and $\mathsf{b}_2 \in \mathsf{B}_2$ are such that $\mathsf{B}_1 \neq \mathsf{B}_2$ and $\mathsf{B}_1 \subseteq \mathsf{Y}$ and $\mathsf{B}_2 \subseteq \mathsf{Y}$ and $|\xi^{\mathfrak{c}}_{\mathfrak{d}}(\mathsf{b}_2) - \xi^{\mathfrak{c}}_{\mathfrak{d}}(\mathsf{b}_1) + {}_{\aleph}\delta^{\mathfrak{c}}_{\mathfrak{d}}(\mathsf{b}_1, \mathsf{b}_2)| \in D$, then $\mathsf{A}_1 \coloneqq \gamma_{\mathsf{Y},\ell}^{k\leftarrow}(\mathsf{B}_1) \in s$ and $\mathsf{A}_2 \coloneqq \gamma_{\mathsf{Y},\ell}^{k\leftarrow}(\mathsf{B}_2) \in s$ and there are $\mathsf{a}_1 \in \mathsf{A}_1$ and $\mathsf{a}_2 \in \mathsf{A}_2$ with $\mathsf{b}_1 = \gamma_{\mathsf{Y},\ell}^{k}(\mathsf{a}_1)$ and $\mathsf{b}_2 = \gamma_{\mathsf{Y},\ell}^{k}(\mathsf{a}_2)$. By the auxiliary statement at the very beginning of the proof the assumption $|\xi^{\mathfrak{c}}_{\mathfrak{d}}(\mathsf{b}_2) - \xi^{\mathfrak{c}}_{\mathfrak{d}}(\mathsf{b}_1) + {}_{\aleph}\delta^{\mathfrak{c}}_{\mathfrak{d}}(\mathsf{b}_1, \mathsf{b}_2)| \in D$ implies $|\xi^{\mathfrak{f}}_{\mathfrak{g}}(\mathsf{a}_2) - \xi^{\mathfrak{f}}_{\mathfrak{g}}(\mathsf{a}_1)| \in D$. In particular, $0 \notin D$ requires $\xi^{\mathfrak{f}}_{\mathfrak{g}}(\mathsf{a}_2) \neq \xi^{\mathfrak{f}}_{\mathfrak{g}}(\mathsf{a}_1)$. Moreover, $|\xi^{\mathfrak{f}}_{\mathfrak{g}}(\mathsf{a}_2) - \xi^{\mathfrak{f}}_{\mathfrak{g}}(\mathsf{a}_1)| \in D$ necessitates $(\xi^{\mathfrak{f}}_{\mathfrak{g}}(\mathsf{a}_2), \xi^{\mathfrak{f}}_{\mathfrak{g}}(\mathsf{a}_1)) \notin r_D$ by definition of r_D. Hence, (ii) ensures $\mathsf{A}_1 \mathbin{\diagdown\!\!\diagdown}^{m}_{n} \mathsf{A}_2$. Because $\gamma_{\mathsf{Y},\ell}^{k}$ is strictly monotonic with respect to Γ^m_n and Γ^k_ℓ by Lemma 4.2 (d) we can infer that also $\mathsf{B}_1 \mathbin{\diagdown\!\!\diagdown}^{k}_{\ell} \mathsf{B}_2$, i.e., have derived (i). □

PROPOSITION 8.5. *Let $w \in \mathbb{N}$ and let D be any additive subsemigroup of $\mathbb{N}$.*

(a) The category $\mathcal{U} \wr \mathcal{Z}_w$ can be expressed as the set of all $(\mathfrak{c}, \mathfrak{d}, p) \in {}_{\mathbb{Z}_w \cup \{\aleph\}, \varnothing}\mathcal{S}$ such that ${}_{\aleph}\Sigma^{\mathfrak{c}}_{\mathfrak{d}} \equiv_w 0$, such that $p \leq \xi^{\mathfrak{c}\leftarrow}_{\mathfrak{d}}(\{\mathbb{Z}_w, \{\aleph\}\})$, such that for any $\mathsf{B} \in p$ with $\mathsf{B} \subseteq \xi^{\mathfrak{c}\leftarrow}_{\mathfrak{d}}(\{\aleph\})$ always $|\mathsf{B}| \leq 2$ and, if $|\mathsf{B}| = 2$, then ${}_{\aleph}\sigma^{\mathfrak{c}}_{\mathfrak{d}}(\mathsf{B}) = 0$, and such that for any $\mathsf{B} \in p$ with $\mathsf{B} \subseteq \xi^{\mathfrak{c}\leftarrow}_{\mathfrak{d}}(\mathbb{Z}_w)$ not only $|\mathsf{B}| = 2$ and $\sum_{z \in \mathbb{Z}_w} {}_{z}\sigma^{\mathfrak{c}}_{\mathfrak{d}}(\mathsf{B}) = 0$ but also $\xi^{\mathfrak{c}}_{\mathfrak{d}}(\mathsf{b}') - \xi^{\mathfrak{c}}_{\mathfrak{d}}(\mathsf{b}) + {}_{\aleph}\delta^{\mathfrak{c}}_{\mathfrak{d}}(\mathsf{b}, \mathsf{b}') \equiv_w 0$ for any $\{\mathsf{b}, \mathsf{b}'\} \subseteq \mathsf{B}$.

(b) *The category* $\mathcal{U} \wr_{r_D} \mathcal{Z}_0$ *can be expressed as the set of all* $(\mathfrak{c}, \mathfrak{d}, p) \in {}_{\mathbb{Z}\uplus\{\aleph\},\varnothing}\mathcal{S}$ *for which there exist* $\{k,\ell\} \subseteq \mathbb{N}_0$ *such that* $\mathfrak{c}\colon [\![k]\!] \to (\mathbb{Z}\uplus\{\aleph\}) \otimes \{\circ,\bullet\}$ *and* $\mathfrak{d}\colon [\![\ell]\!] \to (\mathbb{Z}\uplus\{\aleph\}) \otimes \{\circ,\bullet\}$, *such that* $p \leq \xi^{\mathfrak{c}\leftarrow}_{\mathfrak{d}}(\{\mathbb{Z},\{\aleph\}\})$, *such that* $|\mathsf{B}| = 2$ *for any* $\mathsf{B} \in p$, *such that* ${}_{\aleph}\sigma^{\mathfrak{c}}_{\mathfrak{d}}(\mathsf{B}) = 0$ *for any* $\mathsf{B} \in p$ *with* $\mathsf{B} \subseteq \xi^{\mathfrak{c}\leftarrow}_{\mathfrak{d}}(\{\aleph\})$, *such that* $\sum_{z\in\mathbb{Z}} {}_{z}\sigma^{\mathfrak{c}}_{\mathfrak{d}}(\mathsf{B}) = 0$ *and* $\xi^{\mathfrak{c}}_{\mathfrak{d}}(\mathsf{b}') - \xi^{\mathfrak{c}}_{\mathfrak{d}}(\mathsf{b}) + {}_{\aleph}\delta^{\mathfrak{c}}_{\mathfrak{d}}(\mathsf{b},\mathsf{b}') = 0$ *for any* $\{\mathsf{b},\mathsf{b}'\} \subseteq \mathsf{B}$ *and any* $\mathsf{B} \in p$ *with* $\mathsf{B} \subseteq \xi^{\mathfrak{c}\leftarrow}_{\mathfrak{d}}(\mathbb{Z})$, *and such that for any* $\{\mathsf{B}_1,\mathsf{B}_2\} \subseteq p$ *with* $\mathsf{B}_1 \subseteq \xi^{\mathfrak{c}\leftarrow}_{\mathfrak{d}}(\mathbb{Z})$ *and* $\mathsf{B}_2 \subseteq \xi^{\mathfrak{c}\leftarrow}_{\mathfrak{d}}(\mathbb{Z})$ *and* $\mathsf{B}_1 \neq \mathsf{B}_2$, *whenever there exist* $\mathsf{b}_1 \in \mathsf{B}_1$ *and* $\mathsf{b}_2 \in \mathsf{B}_2$ *such that* $|\xi^{\mathfrak{c}}_{\mathfrak{d}}(\mathsf{b}_2) - \xi^{\mathfrak{c}}_{\mathfrak{d}}(\mathsf{b}_1) + {}_{\aleph}\delta^{\mathfrak{c}}_{\mathfrak{d}}(\mathsf{b}_1,\mathsf{b}_2)| \in D$, *then* $\mathsf{B}_1 \diagdown\!\diagdown^{k}_{\ell} \mathsf{B}_2$.

(c) *The category* $\mathcal{U}^{+} \wr_{r_D} \mathcal{Z}_0$ *can be expressed as the set of all* $(\mathfrak{c}, \mathfrak{d}, p) \in {}_{\mathbb{Z}\uplus\{\aleph\},\varnothing}\mathcal{S}$ *for which there exist* $\{k,\ell\} \subseteq \mathbb{N}_0$ *such that* $\mathfrak{c}\colon [\![k]\!] \to (\mathbb{Z}\uplus\{\aleph\}) \otimes \{\circ,\bullet\}$ *and* $\mathfrak{d}\colon [\![\ell]\!] \to (\mathbb{Z}\uplus\{\aleph\}) \otimes \{\circ,\bullet\}$, *such that* $p \leq \xi^{\mathfrak{c}\leftarrow}_{\mathfrak{d}}(\{\mathbb{Z},\{\aleph\}\})$, *such that* $|\mathsf{B}| = 2$ *for any* $\mathsf{B} \in p$, *such that* ${}_{\aleph}\sigma^{\mathfrak{c}}_{\mathfrak{d}}(\mathsf{B}) = 0$ *for any* $\mathsf{B} \in p$ *with* $\mathsf{B} \subseteq \xi^{\mathfrak{c}\leftarrow}_{\mathfrak{d}}(\{\aleph\})$, *such that* $\sum_{z\in\mathbb{Z}} {}_{z}\sigma^{\mathfrak{c}}_{\mathfrak{d}}(\mathsf{B}) = 0$ *and* $\xi^{\mathfrak{c}}_{\mathfrak{d}}(\mathsf{b}') - \xi^{\mathfrak{c}}_{\mathfrak{d}}(\mathsf{b}) + {}_{\aleph}\delta^{\mathfrak{c}}_{\mathfrak{d}}(\mathsf{b},\mathsf{b}') = 0$ *for any* $\{\mathsf{b},\mathsf{b}'\} \subseteq \mathsf{B}$ *and any* $\mathsf{B} \in p$ *with* $\mathsf{B} \subseteq \xi^{\mathfrak{c}\leftarrow}_{\mathfrak{d}}(\mathbb{Z})$, *and such that for any* $\{\mathsf{B}_1,\mathsf{B}_2\} \subseteq p$ *with* $\mathsf{B}_1 \subseteq \xi^{\mathfrak{c}\leftarrow}_{\mathfrak{d}}(\mathbb{Z})$ *and* $\mathsf{B}_2 \subseteq \xi^{\mathfrak{c}\leftarrow}_{\mathfrak{d}}(\mathbb{Z})$ *with* $\mathsf{B}_1 \neq \mathsf{B}_2$, *whenever there exist* $\mathsf{b}_1 \in \mathsf{B}_1$ *and* $\mathsf{b}_2 \in \mathsf{B}_2$ *such that* $|\xi^{\mathfrak{c}}_{\mathfrak{d}}(\mathsf{b}_2) - \xi^{\mathfrak{c}}_{\mathfrak{d}}(\mathsf{b}_1) + {}_{\aleph}\delta^{\mathfrak{c}}_{\mathfrak{d}}(\mathsf{b}_1,\mathsf{b}_2)| \in D \uplus \{0\}$, *then* $\mathsf{B}_1 \diagdown\!\diagdown^{k}_{\ell} \mathsf{B}_2$.

(d) *The category* $\mathcal{O}^{*} \wr_{r_D} \mathcal{Z}_0$ *can be expresssed as the set of all* $(\mathfrak{c}, \mathfrak{d}, p) \in {}_{\{\aleph\},\mathbb{Z}}\mathcal{S}$ *for which there exist* $\{k,\ell\} \subseteq \mathbb{N}_0$ *such that* $\mathfrak{c}\colon [\![k]\!] \to (\{\aleph\} \otimes \{\circ,\bullet\}) \uplus \mathbb{Z}$ *and* $\mathfrak{d}\colon [\![\ell]\!] \to (\{\aleph\} \otimes \{\circ,\bullet\}) \uplus \mathbb{Z}$, *such that* $p \leq \xi^{\mathfrak{c}\leftarrow}_{\mathfrak{d}}(\{\mathbb{Z},\{\aleph\}\})$, *such that* $|\mathsf{B}| = 2$ *for any* $\mathsf{B} \in p$, *such that* ${}_{\aleph}\sigma^{\mathfrak{c}}_{\mathfrak{d}}(\mathsf{B}) = 0$ *for any* $\mathsf{B} \in p$ *with* $\mathsf{B} \subseteq \xi^{\mathfrak{c}\leftarrow}_{\mathfrak{d}}(\{\aleph\})$, *such that* $\xi^{\mathfrak{c}}_{\mathfrak{d}}(\mathsf{b}') - \xi^{\mathfrak{c}}_{\mathfrak{d}}(\mathsf{b}) + {}_{\aleph}\delta^{\mathfrak{c}}_{\mathfrak{d}}(\mathsf{b},\mathsf{b}') = 0$ *for any* $\{\mathsf{b},\mathsf{b}'\} \in \mathsf{B}$ *and* $\mathsf{B} \in p$ *with* $\mathsf{B} \subseteq \xi^{\mathfrak{c}\leftarrow}_{\mathfrak{d}}(\mathbb{Z})$, *such that* $|]\mathsf{b},\mathsf{b}''[^{k}_{\ell} \cap \{\mathsf{b}' \in \Pi^{k}_{\ell} \wedge \xi^{\mathfrak{c}}_{\mathfrak{d}}(\mathsf{b}') \in \mathbb{Z} \wedge \xi^{\mathfrak{c}}_{\mathfrak{d}}(\mathsf{b}') - \xi^{\mathfrak{c}}_{\mathfrak{d}}(\mathsf{b}) + {}_{\aleph}\delta^{\mathfrak{c}}_{\mathfrak{d}}(\mathsf{b},\mathsf{b}') = 0\}| \equiv_2 0$ *for any* $\{\mathsf{b},\mathsf{b}''\} \in \mathsf{B}$ *with* $\mathsf{b} \neq \mathsf{b}''$ *and* $\mathsf{B} \in p$ *with* $\mathsf{B} \subseteq \xi^{\mathfrak{c}\leftarrow}_{\mathfrak{d}}(\mathbb{Z})$, *and such that for any* $\{\mathsf{B}_1,\mathsf{B}_2\} \subseteq p$ *with* $\mathsf{B}_1 \subseteq \xi^{\mathfrak{c}\leftarrow}_{\mathfrak{d}}(\mathbb{Z})$ *and* $\mathsf{B}_2 \subseteq \xi^{\mathfrak{c}\leftarrow}_{\mathfrak{d}}(\mathbb{Z})$ *with* $\mathsf{B}_1 \neq \mathsf{B}_2$, *whenever there exist* $\mathsf{b}_1 \in \mathsf{B}_1$ *and* $\mathsf{b}_2 \in \mathsf{B}_2$ *such that* $|\xi^{\mathfrak{c}}_{\mathfrak{d}}(\mathsf{b}_2) - \xi^{\mathfrak{c}}_{\mathfrak{d}}(\mathsf{b}_1) + {}_{\aleph}\delta^{\mathfrak{c}}_{\mathfrak{d}}(\mathsf{b}_1,\mathsf{b}_2)| \in D$, *then* $\mathsf{B}_1 \diagdown\!\diagdown^{k}_{\ell} \mathsf{B}_2$.

(e) *The category* $\mathcal{O}^{+} \wr_{r_D} \mathcal{Z}_0$ *can be expresssed as the set of all* $(\mathfrak{c}, \mathfrak{d}, p) \in {}_{\{\aleph\},\mathbb{Z}}\mathcal{S}$ *for which there exist* $\{k,\ell\} \subseteq \mathbb{N}_0$ *such that* $\mathfrak{c}\colon [\![k]\!] \to (\{\aleph\} \otimes \{\circ,\bullet\}) \uplus \mathbb{Z}$ *and* $\mathfrak{d}\colon [\![\ell]\!] \to (\{\aleph\} \otimes \{\circ,\bullet\}) \uplus \mathbb{Z}$, *such that* $p \leq \xi^{\mathfrak{c}\leftarrow}_{\mathfrak{d}}(\{\mathbb{Z},\{\aleph\}\})$, *such that* $|\mathsf{B}| = 2$ *for any* $\mathsf{B} \in p$, *such that* ${}_{\aleph}\sigma^{\mathfrak{c}}_{\mathfrak{d}}(\mathsf{B}) = 0$ *for any* $\mathsf{B} \in p$ *with* $\mathsf{B} \subseteq \xi^{\mathfrak{c}\leftarrow}_{\mathfrak{d}}(\{\aleph\})$, *such that* $\xi^{\mathfrak{c}}_{\mathfrak{d}}(\mathsf{b}') - \xi^{\mathfrak{c}}_{\mathfrak{d}}(\mathsf{b}) + {}_{\aleph}\delta^{\mathfrak{c}}_{\mathfrak{d}}(\mathsf{b},\mathsf{b}') = 0$ *for any* $\{\mathsf{b},\mathsf{b}'\} \in \mathsf{B}$ *and* $\mathsf{B} \in p$ *with* $\mathsf{B} \subseteq \xi^{\mathfrak{c}\leftarrow}_{\mathfrak{d}}(\mathbb{Z})$, *such that* $|]\mathsf{b},\mathsf{b}''[^{k}_{\ell} \cap \{\mathsf{b}' \in \Pi^{k}_{\ell} \wedge \xi^{\mathfrak{c}}_{\mathfrak{d}}(\mathsf{b}') \in \mathbb{Z} \wedge \xi^{\mathfrak{c}}_{\mathfrak{d}}(\mathsf{b}') - \xi^{\mathfrak{c}}_{\mathfrak{d}}(\mathsf{b}) + {}_{\aleph}\delta^{\mathfrak{c}}_{\mathfrak{d}}(\mathsf{b},\mathsf{b}') = 0\}| \equiv_2 0$ *for any* $\{\mathsf{b},\mathsf{b}''\} \in \mathsf{B}$ *with* $\mathsf{b} \neq \mathsf{b}''$ *and* $\mathsf{B} \in p$ *with* $\mathsf{B} \subseteq \xi^{\mathfrak{c}\leftarrow}_{\mathfrak{d}}(\mathbb{Z})$, *and such that for any* $\{\mathsf{B}_1,\mathsf{B}_2\} \subseteq p$ *with* $\mathsf{B}_1 \subseteq \xi^{\mathfrak{c}\leftarrow}_{\mathfrak{d}}(\mathbb{Z})$ *and* $\mathsf{B}_2 \subseteq \xi^{\mathfrak{c}\leftarrow}_{\mathfrak{d}}(\mathbb{Z})$, *whenever there exist* $\mathsf{b}_1 \in \mathsf{B}_1$ *and* $\mathsf{b}_2 \in \mathsf{B}_2$ *such that* $|\xi^{\mathfrak{c}}_{\mathfrak{d}}(\mathsf{b}_2) - \xi^{\mathfrak{c}}_{\mathfrak{d}}(\mathsf{b}_1) + {}_{\aleph}\delta^{\mathfrak{c}}_{\mathfrak{d}}(\mathsf{b}_1,\mathsf{b}_2)| \in D \uplus \{0\}$, *then* $\mathsf{B}_1 \diagdown\!\diagdown^{k}_{\ell} \mathsf{B}_2$.

PROOF. All claims follow from Propositions 4.45 and 8.3 in combination with Lemma 8.4. More precisely:

(a) If we take Propositions 4.45 and 8.3 (a) into account, then by definition the category $\mathcal{U} \wr \mathcal{Z}_w = (\mathcal{U}^{\times \mathbb{Z}_w}) \rtimes \mathcal{Z}_w$ is given by the set of all $(\mathfrak{c}, \mathfrak{d}, p) \in {}_{\mathbb{Z}_w\uplus\{\aleph\},\varnothing}\mathcal{S}$ such that ${}_{\aleph}\Sigma^{\mathfrak{c}}_{\mathfrak{d}} \equiv_w 0$, such that $p \leq \xi^{\mathfrak{c}\leftarrow}_{\mathfrak{d}}(\{\mathbb{Z}_w,\{\aleph\}\})$, such that for any $\mathsf{B} \in p$ with $\mathsf{B} \subseteq \xi^{\mathfrak{c}\leftarrow}_{\mathfrak{d}}(\{\aleph\})$ always $|\mathsf{B}| \leq 2$ and, if $|\mathsf{B}| = 2$, then ${}_{\aleph}\sigma^{\mathfrak{c}}_{\mathfrak{d}}(\mathsf{B}) = 0$, and such that, if $\{k,\ell\} \subseteq \mathbb{N}_0$ are such that $\mathfrak{c}\colon [\![k]\!] \to (\mathbb{Z}_w \cup \{\aleph\}) \otimes \{\circ,\bullet\}$ and $\mathfrak{d}\colon [\![\ell]\!] \to (\mathbb{Z}_w \cup \{\aleph\}) \otimes \{\circ,\bullet\}$, if $\mathsf{Y} = \xi_{\mathfrak{d}}^{\mathfrak{c}\leftarrow}(\mathbb{Z}_w)$, if the labelings $\mathfrak{f}$ and $\mathfrak{g}$ are such that $\xi_{\mathfrak{g}}^{\mathfrak{f}} = \pi_w \circ \varepsilon_{\mathfrak{d}}^{\mathfrak{c}} \circ \gamma_{\mathsf{Y},\ell}{}^{k}$ and $\zeta_{\mathfrak{g}}^{\mathfrak{f}} = \zeta_{\mathfrak{d}}^{\mathfrak{c}} \circ \gamma_{\mathsf{Y},\ell}{}^{k}$ and if $s = R(p,\mathsf{Y})$, then $s \leq \ker(\xi_{\mathfrak{g}}^{\mathfrak{f}})$ and $|\mathsf{A}| = 2$ and $\sum_{z\in\mathbb{Z}_w} {}_z\sigma_{\mathfrak{g}}^{\mathfrak{f}}(\mathsf{A}) = 0$ for any $\mathsf{A} \in s$. Thus, the claim follows from Lemma 8.4 (a)–(c).

(b) By Propositions 4.45 and 8.3 (b) the category $\mathcal{U} \wr_{r_D} \mathcal{Z}_0 = (\mathcal{U}^{*(\mathbb{Z},r_D)}) \rtimes \mathcal{Z}_0$ is the set of all $(\mathfrak{c},\mathfrak{d},p) \in {}_{\mathbb{Z}\cup\{\aleph\},\varnothing}\mathcal{S}$ for which there exist $\{k,\ell\} \subseteq \mathbb{N}_0$ such that $\mathfrak{c}\colon [\![k]\!] \to (\mathbb{Z} \cup \{\aleph\}) \otimes \{\circ,\bullet\}$ and $\mathfrak{d}\colon [\![\ell]\!] \to (\mathbb{Z} \cup \{\aleph\}) \otimes \{\circ,\bullet\}$, such that $p \leq \xi_{\mathfrak{d}}^{\mathfrak{c}\leftarrow}(\{\mathbb{Z},\{\aleph\}\})$, such that for any $\mathsf{B} \in p$ with $\mathsf{B} \subseteq \xi_{\mathfrak{d}}^{\mathfrak{c}\leftarrow}(\{\aleph\})$ both $|\mathsf{B}| = 2$ and ${}_\aleph\sigma_{\mathfrak{d}}^{\mathfrak{c}}(\mathsf{B}) = 0$, and such that, if $\mathsf{Y} = \xi_{\mathfrak{d}}^{\mathfrak{c}\leftarrow}(\mathbb{Z})$, if $\{m,n\} \subseteq \mathbb{N}_0$ and $\mathfrak{f}\colon [\![m]\!] \to \mathbb{Z} \otimes \{\circ,\bullet\}$ and $\mathfrak{g}\colon [\![n]\!] \to \mathbb{Z} \otimes \{\circ,\bullet\}$ are such that $\xi_{\mathfrak{g}}^{\mathfrak{f}} = \pi_w \circ \varepsilon_{\mathfrak{d}}^{\mathfrak{c}} \circ \gamma_{\mathsf{Y},\ell}{}^{k}$ and $\zeta_{\mathfrak{g}}^{\mathfrak{f}} = \zeta_{\mathfrak{d}}^{\mathfrak{c}} \circ \gamma_{\mathsf{Y},\ell}{}^{k}$, and if $s = R(p,\mathsf{Y})$, then $s \leq \ker(\xi_{\mathfrak{g}}^{\mathfrak{f}})$, and $|\mathsf{A}| = 2$ and $\sum_{z\in\mathbb{Z}} {}_z\sigma_{\mathfrak{g}}^{\mathfrak{f}}(\mathsf{A}) = 0$ for any $\mathsf{A} \in p$, and for any $\{\mathsf{A}_1,\mathsf{A}_2\} \subseteq s$ with $\mathsf{A}_1 \neq \mathsf{A}_2$, whenever there exist $\mathsf{a}_1 \in \mathsf{A}_1$ and $\mathsf{a}_2 \in \mathsf{A}_2$ such that $|\xi_{\mathfrak{g}}^{\mathfrak{f}}(\mathsf{a}_2) - \xi_{\mathfrak{g}}^{\mathfrak{f}}(\mathsf{a}_1)| \in D$, then $\mathsf{A}_1 \not\!\!\!\diagdown_n^m \mathsf{A}_2$. Hence, Lemma 8.4 (a)–(c) and (e) prove the claim.

(c) According to Propositions 4.45 and 8.3 (c) the category $\mathcal{U}^+ \wr_{r_D} \mathcal{Z}_0$, which is to say $((\mathcal{U}^+)^{*(\mathbb{Z},r_D)}) \rtimes \mathcal{Z}_0$, is the set of all $(\mathfrak{c},\mathfrak{d},p) \in {}_{\mathbb{Z}\cup\{\aleph\},\varnothing}\mathcal{S}$ for which there exist $\{k,\ell\} \subseteq \mathbb{N}_0$ such that $\mathfrak{c}\colon [\![k]\!] \to (\mathbb{Z} \cup \{\aleph\}) \otimes \{\circ,\bullet\}$ and $\mathfrak{d}\colon [\![\ell]\!] \to (\mathbb{Z} \cup \{\aleph\}) \otimes \{\circ,\bullet\}$, such that $p \leq \xi_{\mathfrak{d}}^{\mathfrak{c}\leftarrow}(\{\mathbb{Z},\{\aleph\}\})$, such that for any $\mathsf{B} \in p$ with $\mathsf{B} \subseteq \xi_{\mathfrak{d}}^{\mathfrak{c}\leftarrow}(\{\aleph\})$ both $|\mathsf{B}| = 2$ and ${}_\aleph\sigma_{\mathfrak{d}}^{\mathfrak{c}}(\mathsf{B}) = 0$, and such that, if $\mathsf{Y} = \xi_{\mathfrak{d}}^{\mathfrak{c}\leftarrow}(\mathbb{Z})$, if $\{m,n\} \subseteq \mathbb{N}_0$ and $\mathfrak{f}\colon [\![m]\!] \to \mathbb{Z} \otimes \{\circ,\bullet\}$ and $\mathfrak{g}\colon [\![n]\!] \to \mathbb{Z} \otimes \{\circ,\bullet\}$ are such that $\xi_{\mathfrak{g}}^{\mathfrak{f}} = \pi_w \circ \varepsilon_{\mathfrak{d}}^{\mathfrak{c}} \circ \gamma_{\mathsf{Y},\ell}{}^{k}$ and $\zeta_{\mathfrak{g}}^{\mathfrak{f}} = \zeta_{\mathfrak{d}}^{\mathfrak{c}} \circ \gamma_{\mathsf{Y},\ell}{}^{k}$, and if $s = R(p,\mathsf{Y})$, then $s \leq \ker(\xi_{\mathfrak{g}}^{\mathfrak{f}})$, and $|\mathsf{A}| = 2$ and $\sum_{z\in\mathbb{Z}} {}_z\sigma_{\mathfrak{g}}^{\mathfrak{f}}(\mathsf{A}) = 0$ for any $\mathsf{A} \in p$, and for any $\{\mathsf{A}_1,\mathsf{A}_2\} \subseteq s$ with $\mathsf{A}_1 \neq \mathsf{A}_2$, whenever there exist $\mathsf{a}_1 \in \mathsf{A}_1$ and $\mathsf{a}_2 \in \mathsf{A}_2$ such that $|\xi_{\mathfrak{g}}^{\mathfrak{f}}(\mathsf{a}_2) - \xi_{\mathfrak{g}}^{\mathfrak{f}}(\mathsf{a}_1)| \in D \cup \{0\}$, then $\mathsf{A}_1 \not\!\!\!\diagdown_n^m \mathsf{A}_2$. Therefore, Lemma 8.4 (a)–(c), (d) and (e) verify the assertion.

(d) Per Propositions 4.45 and 8.3 (d) the category $\mathcal{O}^* \wr_{r_D} \mathcal{Z}_0 = ((\mathcal{O}^*)^{*(\mathbb{Z},r_D)}) \rtimes \mathcal{Z}_0$ is the set of all $(\mathfrak{c},\mathfrak{d},p) \in {}_{\{\aleph\},\mathbb{Z}}\mathcal{S}$ for which there exist $\{k,\ell\} \subseteq \mathbb{N}_0$ such that $\mathfrak{c}\colon [\![k]\!] \to (\{\aleph\} \otimes \{\circ,\bullet\}) \cup \mathbb{Z}$ and $\mathfrak{d}\colon [\![\ell]\!] \to (\{\aleph\} \otimes \{\circ,\bullet\}) \cup \mathbb{Z}$, such that $p \leq \xi_{\mathfrak{d}}^{\mathfrak{c}\leftarrow}(\{\mathbb{Z},\{\aleph\}\})$, such that for any $\mathsf{B} \in p$ with $\mathsf{B} \subseteq \xi_{\mathfrak{d}}^{\mathfrak{c}\leftarrow}(\{\aleph\})$ both $|\mathsf{B}| = 2$ and ${}_\aleph\sigma_{\mathfrak{d}}^{\mathfrak{c}}(\mathsf{B}) = 0$, and such that, if $\mathsf{Y} = \xi_{\mathfrak{d}}^{\mathfrak{c}\leftarrow}(\mathbb{Z})$, if $\{m,n\} \subseteq \mathbb{N}_0$ and $\mathfrak{f}\colon [\![m]\!] \to \mathbb{Z}$ and $\mathfrak{g}\colon [\![n]\!] \to \mathbb{Z}$ are such that $\xi_{\mathfrak{g}}^{\mathfrak{f}} = \pi_w \circ \varepsilon_{\mathfrak{d}}^{\mathfrak{c}} \circ \gamma_{\mathsf{Y},\ell}{}^{k}$ and if $s = R(p,\mathsf{Y})$, then $s \leq \ker(\xi_{\mathfrak{g}}^{\mathfrak{f}})$ and $|\mathsf{A}| = 2$ for any $\mathsf{A} \in s$ and $|]\mathsf{a},\mathsf{a}''[_n^m \cap \{\mathsf{a}' \in \Pi_n^m \wedge \xi_{\mathfrak{g}}^{\mathfrak{f}}(\mathsf{a}') - \xi_{\mathfrak{g}}^{\mathfrak{f}}(\mathsf{a}) = 0\}| \equiv_2 0$ for any $\{\mathsf{a},\mathsf{a}''\} \subseteq \mathsf{A}$ with $\mathsf{a} \neq \mathsf{a}''$ and any $\mathsf{A} \in s$, and for any $\{\mathsf{A}_1,\mathsf{A}_2\} \subseteq s$ with $\mathsf{A}_1 \neq \mathsf{A}_2$, whenever there exist $\mathsf{a}_1 \in \mathsf{A}_1$ and $\mathsf{a}_2 \in \mathsf{A}_2$ such that $|\xi_{\mathfrak{g}}^{\mathfrak{f}}(\mathsf{a}_2) - \xi_{\mathfrak{g}}^{\mathfrak{f}}(\mathsf{a}_1)| \in D$, then $\mathsf{A}_1 \not\!\!\!\diagdown_n^m \mathsf{A}_2$. Thus, the claim follows by Lemma 8.4 (a), (b), (d) and (e).

(e) Finally, Propositions 4.45 and 8.3 (e) tell us that the category $\mathcal{O}^+ \wr_{r_D} \mathcal{Z}_0 = ((\mathcal{O}^+)^{*(\mathbb{Z},r_D)}) \rtimes \mathcal{Z}_0$ is the set of all $(\mathfrak{c},\mathfrak{d},p) \in {}_{\{\aleph\},\mathbb{Z}}\mathcal{S}$ for which there exist $\{k,\ell\} \subseteq \mathbb{N}_0$ such that $\mathfrak{c}\colon [\![k]\!] \to (\{\aleph\} \otimes \{\circ,\bullet\}) \cup \mathbb{Z}$ and $\mathfrak{d}\colon [\![\ell]\!] \to (\{\aleph\} \otimes \{\circ,\bullet\}) \cup \mathbb{Z}$, such that $p \leq \xi_{\mathfrak{d}}^{\mathfrak{c}\leftarrow}(\{\mathbb{Z},\{\aleph\}\})$, such that for any $\mathsf{B} \in p$ with $\mathsf{B} \subseteq \xi_{\mathfrak{d}}^{\mathfrak{c}\leftarrow}(\{\aleph\})$ both $|\mathsf{B}| = 2$ and ${}_\aleph\sigma_{\mathfrak{d}}^{\mathfrak{c}}(\mathsf{B}) = 0$, and such that, if $\mathsf{Y} = \xi_{\mathfrak{d}}^{\mathfrak{c}\leftarrow}(\mathbb{Z})$, if $\{m,n\} \subseteq \mathbb{N}_0$ and $\mathfrak{f}\colon [\![m]\!] \to \mathbb{Z}$ and $\mathfrak{g}\colon [\![n]\!] \to \mathbb{Z}$ are such that $\xi_{\mathfrak{g}}^{\mathfrak{f}} = \pi_w \circ \varepsilon_{\mathfrak{d}}^{\mathfrak{c}} \circ \gamma_{\mathsf{Y},\ell}{}^{k}$ and if $s = R(p,\mathsf{Y})$, then $s \leq \ker(\xi_{\mathfrak{g}}^{\mathfrak{f}})$ and $|\mathsf{A}| = 2$ for any $\mathsf{A} \in s$ and $|]\mathsf{a},\mathsf{a}''[_n^m \cap \{\mathsf{a}' \in \Pi_n^m \wedge \xi_{\mathfrak{g}}^{\mathfrak{f}}(\mathsf{a}') - \xi_{\mathfrak{g}}^{\mathfrak{f}}(\mathsf{a}) = 0\}| \equiv_2 0$ for any $\{\mathsf{a},\mathsf{a}''\} \subseteq \mathsf{A}$ with $\mathsf{a} \neq \mathsf{a}''$ and any $\mathsf{A} \in s$, and for any $\{\mathsf{A}_1,\mathsf{A}_2\} \subseteq s$ with $\mathsf{A}_1 \neq \mathsf{A}_2$, whenever there exist $\mathsf{a}_1 \in \mathsf{A}_1$ and $\mathsf{a}_2 \in \mathsf{A}_2$ such that $|\xi^{\mathfrak{f}}_{\mathfrak{g}}(\mathsf{a}_2) - \xi^{\mathfrak{f}}_{\mathfrak{g}}(\mathsf{a}_1)| \in D \cup \{0\}$, then $\mathsf{A}_1 \diagdown^m_n \mathsf{A}_2$. Hence, Lemma 8.4 (a), (b) and (d)–(f) prove the claim. □

9. The functor between the two-colored and the labeled categories

In this section, a faithful strict monoidal functor is constructed from each category of two-colored partitions from Section 7.2, i.e., one representing a unitary half-liberation, into its associated supercategory from Section 8. It is convenient to define it on the set of all two-colored partitions. Concretely, on objects it will be completely determined by the condition that $\circ$ is sent to $(0,\circ)\otimes(\aleph,\circ)$ respectively $0\otimes(\aleph,\circ)$, depending on whether the tags other than $\aleph$ are unitary or orthogonal in the supercategory. On partitions, it will simply "double up" the original partition, one copy on the 0-area, the other on the $\aleph$-area.

9.1. Definition of the general functor. On objects the alleged functor is defined as follows.

DEFINITION 9.1. For any $w \in \mathbb{N}_0$, if $(\mathfrak{U},\mathfrak{O}) \in \{(\mathbb{Z}_w,\varnothing),(\varnothing,\mathbb{Z}_w)\}$, then let $F \equiv F_0 \equiv {}_{\mathfrak{U},\mathfrak{O}}F_0$ be the mapping with the property that for any $k \in \mathbb{N}_0$ and any $\mathfrak{c}\colon [\![k]\!] \to \{\circ,\bullet\}$ the labeling $F(\mathfrak{c})\colon [\![2k]\!] \to ((\mathfrak{U}\cup\{\aleph\})\otimes\{\circ,\bullet\})\cup\mathfrak{O}$ satisfies

$$x \mapsto \begin{cases} (\aleph,\bullet) & \text{if } x \text{ odd} \wedge \mathfrak{c}(\frac{x+1}{2}) = \bullet \\ (0,\circ) & \text{if } x \text{ odd} \wedge \mathfrak{c}(\frac{x+1}{2}) = \circ \wedge (\mathfrak{U},\mathfrak{O}) = (\mathbb{Z}_w,\varnothing) \\ 0 & \text{if } x \text{ odd} \wedge \mathfrak{c}(\frac{x+1}{2}) = \circ \wedge (\mathfrak{U},\mathfrak{O}) = (\varnothing,\mathbb{Z}_w) \\ (0,\bullet) & \text{if } x \text{ even} \wedge \mathfrak{c}(\frac{x}{2}) = \bullet \wedge (\mathfrak{U},\mathfrak{O}) = (\mathbb{Z}_w,\varnothing) \\ 0 & \text{if } x \text{ even} \wedge \mathfrak{c}(\frac{x}{2}) = \bullet \wedge (\mathfrak{U},\mathfrak{O}) = (\varnothing,\mathbb{Z}_w) \\ (\aleph,\circ) & \text{if } x \text{ even} \wedge \mathfrak{c}(\frac{x}{2}) = \circ \end{cases}$$

for any $x \in [\![2k]\!]$.

Defining the functor on morphisms only requires part (c) of the following lemma. However, it is convenient to prove the other statements at this point as well.

LEMMA 9.2. *For any $w \in \mathbb{N}_0$, if $(\mathfrak{U},\mathfrak{O}) \in \{(\mathbb{Z}_w,\varnothing),(\varnothing,\mathbb{Z}_w)\}$, then for any $\{k,\ell\} \subseteq \mathbb{N}_0$ and any $\mathfrak{c}\colon [\![k]\!] \to \{\circ,\bullet\}$ and $\mathfrak{d}\colon [\![\ell]\!] \to \{\circ,\bullet\}$, if $\mathfrak{a} = F(\mathfrak{c})$ and $\mathfrak{b} = F(\mathfrak{d})$ and $\mathsf{Y} = \xi^{\mathfrak{a}\leftarrow}_{\mathfrak{b}}(\mathfrak{U}\cup\mathfrak{O})$ and $\mathsf{Z} = \xi^{\mathfrak{a}\leftarrow}_{\mathfrak{b}}(\{\aleph\})$, the following hold.*

(a) For any $x \in [\![2k]\!]$ and $y \in [\![2\ell]\!]$,

$$\xi^{\mathfrak{a}}_{\mathfrak{b}}(^{\blacksquare}x) = \begin{cases} 0 & |\,(x \text{ odd} \wedge \mathfrak{c}(\frac{x+1}{2}) = \circ) \vee (x \text{ even} \wedge \mathfrak{c}(\frac{x}{2}) = \bullet) \\ \aleph & |\,(x \text{ odd} \wedge \mathfrak{c}(\frac{x+1}{2}) = \bullet) \vee (x \text{ even} \wedge \mathfrak{c}(\frac{x}{2}) = \circ) \end{cases},$$

$$\xi^{\mathfrak{a}}_{\mathfrak{b}}(_{\blacksquare}y) = \begin{cases} 0 & |\,(y \text{ odd} \wedge \mathfrak{d}(\frac{y+1}{2}) = \circ) \vee (y \text{ even} \wedge \mathfrak{d}(\frac{y}{2}) = \bullet) \\ \aleph & |\,(y \text{ odd} \wedge \mathfrak{d}(\frac{y+1}{2}) = \bullet) \vee (y \text{ even} \wedge \mathfrak{d}(\frac{y}{2}) = \circ) \end{cases}.$$

In particular,

$$\begin{aligned} \mathsf{Y} = \{^{\blacksquare}x \,|\, x \in [\![2k]\!] \wedge ((x \text{ odd} \wedge \mathfrak{c}(\tfrac{x+1}{2}) = \circ) \vee (x \text{ even} \wedge \mathfrak{c}(\tfrac{x}{2}) = \bullet))\} \\ \cup \{_{\blacksquare}y \,|\, y \in [\![2\ell]\!] \wedge ((y \text{ odd} \wedge \mathfrak{d}(\tfrac{y+1}{2}) = \circ) \vee (y \text{ even} \wedge \mathfrak{d}(\tfrac{y}{2}) = \bullet))\} \end{aligned}$$

and

$$\begin{aligned}\mathsf{Z} = \{{}^{\blacksquare}x \,|\, x \in [\![2k]\!] \wedge ((x \textit{ odd} \wedge \mathfrak{c}(\tfrac{x+1}{2}) = \bullet) \vee (x \textit{ even} \wedge \mathfrak{c}(\tfrac{x}{2}) = \circ))\}\\ \uplus \{{}_{\blacksquare}y \,|\, y \in [\![2\ell]\!] \wedge ((y \textit{ odd} \wedge \mathfrak{d}(\tfrac{y+1}{2}) = \bullet) \vee (y \textit{ even} \wedge \mathfrak{d}(\tfrac{y}{2}) = \circ))\}.\end{aligned}$$

(b) *For any* $x \in [\![2k]\!]$ *and* $y \in [\![2\ell]\!]$, *if* $(\mathfrak{U}, \mathfrak{O}) = (\mathbb{Z}_w, \varnothing)$ *and if* ${}^{\blacksquare}x \in \mathsf{Y}$ *and* ${}_{\blacksquare}y \in \mathsf{Y}$, *then*

$$\zeta^{\mathfrak{a}}_{\mathfrak{b}}({}^{\blacksquare}x) = \begin{cases} \overline{\circ} & |\, x \textit{ odd} \wedge \mathfrak{c}(\frac{x+1}{2}) = \circ \\ \overline{\bullet} & |\, x \textit{ even} \wedge \mathfrak{c}(\frac{x}{2}) = \bullet \end{cases}, \quad \zeta^{\mathfrak{a}}_{\mathfrak{b}}({}_{\blacksquare}y) = \begin{cases} \circ & |\, y \textit{ odd} \wedge \mathfrak{d}(\frac{y+1}{2}) = \circ \\ \bullet & |\, y \textit{ even} \wedge \mathfrak{d}(\frac{y}{2}) = \bullet \end{cases},$$

and, if ${}^{\blacksquare}x \in \mathsf{Z}$ *and* ${}_{\blacksquare}y \in \mathsf{Z}$, *then*

$$\zeta^{\mathfrak{a}}_{\mathfrak{b}}({}^{\blacksquare}x) = \begin{cases} \overline{\bullet} & |\, x \textit{ odd} \wedge \mathfrak{c}(\frac{x+1}{2}) = \bullet \\ \overline{\circ} & |\, x \textit{ even} \wedge \mathfrak{c}(\frac{x}{2}) = \circ \end{cases}, \quad \zeta^{\mathfrak{a}}_{\mathfrak{b}}({}_{\blacksquare}y) = \begin{cases} \bullet & |\, y \textit{ odd} \wedge \mathfrak{d}(\frac{y+1}{2}) = \bullet \\ \circ & |\, y \textit{ even} \wedge \mathfrak{d}(\frac{y}{2}) = \circ \end{cases}.$$

(c) $\alpha(\mathsf{Y}) = \alpha(\mathsf{Z}) = k$ *and* $\beta(\mathsf{Y}) = \beta(\mathsf{Z}) = \ell$.

(d) *For any* $i \in [\![k]\!]$ *and any* $j \in [\![\ell]\!]$,

$$\gamma_{\mathsf{Y},2\ell}^{\,2k}({}^{\blacksquare}i) = \begin{cases} {}^{\blacksquare}(2i) & |\, \mathfrak{c}(i) = \bullet \\ {}^{\blacksquare}(2i-1) & |\, \mathfrak{c}(i) = \circ \end{cases}, \quad \gamma_{\mathsf{Y},2\ell}^{\,2k}({}_{\blacksquare}j) = \begin{cases} {}_{\blacksquare}(2j) & |\, \mathfrak{d}(j) = \bullet \\ {}_{\blacksquare}(2j-1) & |\, \mathfrak{d}(j) = \circ \end{cases}$$

and

$$\gamma_{\mathsf{Z},2\ell}^{\,2k}({}^{\blacksquare}i) = \begin{cases} {}^{\blacksquare}(2i-1) & |\, \mathfrak{c}(i) = \bullet \\ {}^{\blacksquare}(2i) & |\, \mathfrak{c}(i) = \circ \end{cases}, \quad \gamma_{\mathsf{Z},2\ell}^{\,2k}({}_{\blacksquare}j) = \begin{cases} {}_{\blacksquare}(2j-1) & |\, \mathfrak{d}(j) = \bullet \\ {}_{\blacksquare}(2j) & |\, \mathfrak{d}(j) = \circ \end{cases}.$$

(e) *For any* $\mathsf{b} \in \Pi^k_\ell$, *if* $\mathsf{a} = \gamma_{\mathsf{Y},2\ell}^{\,2k}(\mathsf{b})$ *and* $\mathsf{z} = \gamma_{\mathsf{Z},2\ell}^{\,2k}(\mathsf{b})$, *then*

$$(\mathsf{z} = \nu^{2k}_{2\ell}(\mathsf{a}) \Leftrightarrow \zeta^{\mathfrak{c}}_{\mathfrak{d}}(\mathsf{b}) = \circ) \wedge (\mathsf{a} = \nu^{2k}_{2\ell}(\mathsf{z}) \Leftrightarrow \zeta^{\mathfrak{c}}_{\mathfrak{d}}(\mathsf{b}) = \bullet).$$

Proof. (a) and (b) are evident from the definition. In order to facilitate the proofs of (c) and (d) we first show the auxiliary statement that for any $x \in [\![2k]\!]$ and $y \in [\![2\ell]\!]$,

$$|\mathsf{Y} \cap \Pi^x_0| = \begin{cases} \frac{x-1}{2} & |\, x \text{ odd} \wedge \mathfrak{c}(\frac{x+1}{2}) = \bullet \\ \frac{x+1}{2} & |\, x \text{ odd} \wedge \mathfrak{c}(\frac{x+1}{2}) = \circ \\ \frac{x}{2} & |\, x \text{ even} \end{cases}, \quad |\mathsf{Y} \cap \Pi^0_y| = \begin{cases} \frac{y-1}{2} & |\, y \text{ odd} \wedge \mathfrak{d}(\frac{y+1}{2}) = \bullet \\ \frac{y+1}{2} & |\, y \text{ odd} \wedge \mathfrak{d}(\frac{y+1}{2}) = \circ \\ \frac{y}{2} & |\, y \text{ even} \end{cases}$$

and

$$|\mathsf{Z} \cap \Pi^x_0| = \begin{cases} \frac{x+1}{2} & |\, x \text{ odd} \wedge \mathfrak{c}(\frac{x+1}{2}) = \bullet \\ \frac{x-1}{2} & |\, x \text{ odd} \wedge \mathfrak{c}(\frac{x+1}{2}) = \circ \\ \frac{x}{2} & |\, x \text{ even} \end{cases}, \quad |\mathsf{Z} \cap \Pi^0_y| = \begin{cases} \frac{y+1}{2} & |\, y \text{ odd} \wedge \mathfrak{d}(\frac{y+1}{2}) = \bullet \\ \frac{y-1}{2} & |\, y \text{ odd} \wedge \mathfrak{d}(\frac{y+1}{2}) = \circ \\ \frac{y}{2} & |\, y \text{ even} \end{cases}.$$

Indeed, e.g., for any $x \in [\![2k]\!]$ part (a) shows that the number $|\mathsf{Y} \cap \Pi^x_0|$ is given by

$$|\{t \in [\![x]\!] \wedge ((t \text{ odd} \wedge \mathfrak{c}(\tfrac{t+1}{2}) = \circ) \vee (t \text{ even} \wedge \mathfrak{c}(\tfrac{t}{2}) = \bullet))\}|,$$

which, of course, is the same

$$|\{t \in [\![x]\!] \wedge t \text{ odd} \wedge \mathfrak{c}(\tfrac{t+1}{2}) = \circ\}| + |\{t \in [\![x]\!] \wedge t \text{ even} \wedge \mathfrak{c}(\tfrac{t}{2}) = \bullet\}|.$$

By the change of variable $t = 2i - 1$ in the first and $t = 2i$ in the second term this number can also be expressed as

$$|\{i \in [\![k]\!] \wedge i \leq \tfrac{x+1}{2} \wedge \mathfrak{c}(i) = \circ\}| + |\{i \in [\![k]\!] \wedge i \leq \tfrac{x}{2} \wedge \mathfrak{c}(i) = \bullet\}|.$$

Distinguishing the two cases $i \leq \frac{x}{2}$ and $\frac{x}{2} < i \leq \frac{x+1}{2}$ in the first case and then combining two of three resulting overall cases let us rewrite $|\mathsf{Y} \cap \Pi_0^x|$ as

$$|\{i \in [\![k]\!] \wedge i \leq \tfrac{x}{2} \wedge (\mathfrak{c}(i) = \circ \vee \mathfrak{c}(i) = \bullet)\}| + |\{i \in [\![k]\!] \wedge \tfrac{x}{2} < i \leq \tfrac{x+1}{2} \wedge \mathfrak{c}(i) = \circ\}|.$$

The condition in the second expression can only be satisfied if x is odd. Otherwise, this case yields no contribution. Thus, we have shown,

$$|\mathsf{Y} \cap \Pi_0^x| = \lfloor \tfrac{x}{2} \rfloor + \begin{cases} 1 & |\, x \text{ odd} \wedge \mathfrak{c}(\frac{x+1}{2}) = \circ \\ 0 & |\, \text{otherwise,} \end{cases}$$

which is what was claimed. The proof for $|\mathsf{Y} \cap \Pi_y^0|$ is analogous. And for $|\mathsf{Z} \cap \Pi_0^x|$ and $|\mathsf{Z} \cap \Pi_y^0|$ the roles of $\circ$ and $\bullet$ in the above proof are exchanged.

(c) The preceding observation implies in particular that $\alpha(\mathsf{Y}) = |\mathsf{Y} \cap \Pi_0^{2k}| = \frac{2k}{2} = k$ and, likewise, $\beta(\mathsf{Y}) = |\mathsf{Y} \cap \Pi_{2\ell}^0| = \ell$. It follows $\alpha(\mathsf{Z}) = k$ and $\beta(\mathsf{Z}) = \ell$ by $\mathsf{Z} = \Pi_\ell^k \backslash \mathsf{Y}$.

(d) Given any $i \in [\![k]\!]$ and $x \in [\![2k]\!]$ according to the definition $\gamma_{\mathsf{Y},2\ell}^{\,2k}({}^{\blacksquare}i) = {}^{\blacksquare}x$ if and only if both $|\mathsf{Y} \cap \Pi_0^x| = i$ and ${}^{\blacksquare}x \in \mathsf{Y}$. By (a) that is equivalent to the condition that

$$i = \left.\begin{cases} \frac{x-1}{2} & |\, x \text{ odd} \wedge \mathfrak{c}(\frac{x+1}{2}) = \bullet \\ \frac{x+1}{2} & |\, x \text{ odd} \wedge \mathfrak{c}(\frac{x+1}{2}) = \circ \\ \frac{x}{2} & |\, x \text{ even} \end{cases}\right\} \wedge \left((x \text{ odd} \wedge \mathfrak{c}(\tfrac{x+1}{2}) = \circ) \vee (x \text{ even} \wedge \mathfrak{c}(\tfrac{x}{2}) = \bullet)\right),$$

which immediately simplifies to the demand that

$$i = \begin{cases} \frac{x+1}{2} & |\, x \text{ odd} \wedge \mathfrak{c}(\frac{x+1}{2}) = \circ \\ \frac{x}{2} & |\, x \text{ even} \wedge \mathfrak{c}(\frac{x}{2}) = \bullet \end{cases}.$$

And that proves the claim about $\gamma_{\mathsf{Y},2\ell}^{\,2k}({}^{\blacksquare}i)$. The proof of the claim about $\gamma_{\mathsf{Y},2\ell}^{\,2k}({}_{\blacksquare}j)$ for $j \in [\![2\ell]\!]$ is completely analogous. Just replace i with j and $\mathfrak{c}$ with $\mathfrak{d}$.

Moreover, for any $i \in [\![k]\!]$ and $x \in [\![2k]\!]$ it holds $\gamma_{\mathsf{Z},2\ell}^{\,2k}({}^{\blacksquare}i) = {}^{\blacksquare}x$ if and only if both $|\mathsf{Z} \cap \Pi_0^x| = i$ and ${}^{\blacksquare}x \in \mathsf{Z}$, which by (a) is satisfied in turn if and only if

$$i = \left.\begin{cases} \frac{x+1}{2} & |\, x \text{ odd} \wedge \mathfrak{c}(\frac{x+1}{2}) = \bullet \\ \frac{x-1}{2} & |\, x \text{ odd} \wedge \mathfrak{c}(\frac{x+1}{2}) = \circ \\ \frac{x}{2} & |\, x \text{ even} \end{cases}\right\} \wedge \left((x \text{ odd} \wedge \mathfrak{c}(\tfrac{x+1}{2}) = \bullet) \vee (x \text{ even} \wedge \mathfrak{c}(\tfrac{x}{2}) = \circ)\right),$$

Since that is equivalent to the condition that

$$i = \begin{cases} \frac{x+1}{2} & |\, x \text{ odd} \wedge \mathfrak{c}(\frac{x+1}{2}) = \bullet \\ \frac{x}{2} & |\, x \text{ even} \wedge \mathfrak{c}(\frac{x}{2}) = \circ \end{cases}.$$

the claim about $\gamma_{\mathsf{Z},2\ell}^{\,2k}({}^{\blacksquare}i)$ is true as well. Once more, the claim about $\gamma_{\mathsf{Z},2\ell}^{\,2k}({}_{\blacksquare}j)$ for $j \in [\![2\ell]\!]$ is analogous.

(e) If $\mathtt{b} = {}_{\blacksquare}j$ for some $j \in [\![\ell]\!]$, then by (d), if $\mathfrak{d}(j) = \bullet$, then $\mathtt{a} = {}_{\blacksquare}(2j)$ and $\mathtt{z} = {}_{\blacksquare}(2j-1)$ and, if $\mathfrak{d}(j) = \circ$, then $\mathtt{a} = {}_{\blacksquare}(2j-1)$ and $\mathtt{z} = {}_{\blacksquare}(2j)$. As $\nu_{2\ell}^{2k}({}_{\blacksquare}(2j-1)) = {}_{\blacksquare}(2j)$ and $\zeta_{\mathfrak{d}}^{\mathfrak{c}}(\mathtt{b}) = \mathfrak{d}(j)$ then, this is the proof in that case. In the opposite situation, where $\mathtt{b} = {}^{\blacksquare}i$ for some $i \in [\![k]\!]$, part (d) shows that, likewise, if $\mathfrak{c}(i) = \bullet$, then $\mathtt{a} = {}^{\blacksquare}(2j)$ and $\mathtt{z} = {}^{\blacksquare}(2j-1)$ and, if $\mathfrak{c}(j) = \circ$, then $\mathtt{a} = {}^{\blacksquare}(2j-1)$ and $\mathtt{z} = {}^{\blacksquare}(2j)$. Contrary to the previous case, though, $\nu_{2\ell}^{2k}({}^{\blacksquare}(2j)) = {}^{\blacksquare}(2j-1)$, here. However, now also $\zeta_{\mathfrak{d}}^{\mathfrak{c}}(\mathtt{b}) = \overline{\mathfrak{c}(j)}$. Thus, the two "inversions" cancel out and we see that the claim is true in both cases. □

By Lemma 9.2 (c) the following makes sense.

Definition 9.3. For any $w \in \mathbb{N}_0$, if $(\mathfrak{U}, \mathfrak{D}) \in \{(\mathbb{Z}_w, \varnothing), (\varnothing, \mathbb{Z}_w)\}$, let $F \equiv {}_{\mathfrak{U},\mathfrak{D}}F$ be defined as the pair (F_0, F_1), where F_0 is the mapping from Definition 9.1 and where F_1 is the mapping from $\mathcal{S}$ to ${}_{\mathfrak{U}\cup\{\aleph\},\mathfrak{D}}\mathcal{S}$ which for any $\{k, \ell\} \subseteq \mathbb{N}_0$, any $\mathfrak{c}\colon [\![k]\!] \to \{\circ, \bullet\}$ and $\mathfrak{d}\colon [\![\ell]\!] \to \{\circ, \bullet\}$ and any $(\mathfrak{c}, \mathfrak{d}, p) \in \mathcal{S}$ satisfies $(\mathfrak{c}, \mathfrak{d}, p) \mapsto (\mathfrak{a}, \mathfrak{b}, q)$, where $\mathfrak{a} = F_0(\mathfrak{c})$ and $\mathfrak{b} = F_0(\mathfrak{d})$ and where, if $\mathsf{Y} = \xi_{\mathfrak{b}}^{\mathfrak{a}\leftarrow}(\mathfrak{U} \cup \mathfrak{D})$ and $\mathsf{Z} = \xi_{\mathfrak{b}}^{\mathfrak{a}\leftarrow}(\{\aleph\})$, then

$$q = \{\gamma_{\mathsf{Y},2\ell\rightarrow}^{\;2k}(\mathsf{B}), \gamma_{\mathsf{Z},2\ell\rightarrow}^{\;2k}(\mathsf{B}) \mid \mathsf{B} \in p\}.$$

9.2. Functoriality of the construction. Next, we show that F is indeed a faithful strict monoidal $*$-functor. Once more, only part (f) of the next lemma will be required to prove this. However, the other properties of F (which will be needed later) are convenient to prove at the same time.

Lemma 9.4. *For any $w \in \mathbb{N}_0$, if $(\mathfrak{U}, \mathfrak{D}) \in \{(\mathbb{Z}_w, \varnothing), (\varnothing, \mathbb{Z}_w)\}$, then for any $\{k, \ell\} \subseteq \mathbb{N}_0$, any $\mathfrak{c}\colon [\![k]\!] \to \{\circ, \bullet\}$ and $\mathfrak{d}\colon [\![\ell]\!] \to \{\circ, \bullet\}$ and any $(\mathfrak{c}, \mathfrak{d}, p) \in \mathcal{S}$, if $(\mathfrak{a}, \mathfrak{b}, q) = F(\mathfrak{c}, \mathfrak{d}, p)$ and $\mathsf{Y} = \xi_{\mathfrak{b}}^{\mathfrak{a}\leftarrow}(\mathfrak{U} \cup \mathfrak{D})$ and $\mathsf{Z} = \xi_{\mathfrak{b}}^{\mathfrak{a}\leftarrow}(\{\aleph\})$, then*

(a) $q \leq \xi_{\mathfrak{b}}^{\mathfrak{a}\dashleftarrow}(\{\mathfrak{U} \cup \mathfrak{D}, \{\aleph\}\})$.

(b) If $(\mathfrak{U}, \mathfrak{D}) = (\mathbb{Z}_w, \varnothing)$, *then* $\zeta_{\mathfrak{b}}^{\mathfrak{a}} \circ \gamma_{\mathsf{Y},2\ell}^{\;2k} = \zeta_{\mathfrak{d}}^{\mathfrak{c}}$. *Moreover,* $\zeta_{\mathfrak{b}}^{\mathfrak{a}} \circ \gamma_{\mathsf{Z},2\ell}^{\;2k} = \zeta_{\mathfrak{d}}^{\mathfrak{c}}$.

(c) $\{|\mathsf{A}| \mid \mathsf{A} \in q \wedge \mathsf{A} \subseteq \mathsf{Y}\} = \{|\mathsf{C}| \mid \mathsf{C} \in q \wedge \mathsf{C} \subseteq \mathsf{Z}\} = \{|\mathsf{B}| \mid \mathsf{B} \in p\}$.

(d) If $(\mathfrak{U}, \mathfrak{D}) = (\mathbb{Z}_w, \varnothing)$, *then* $\{\sum_{z\in\mathbb{Z}_w} {}_z\sigma_{\mathfrak{b}}^{\mathfrak{a}}(\mathsf{A}) \mid \mathsf{A} \in q \wedge \mathsf{A} \subseteq \mathsf{Y}\} = \{\sigma_{\mathfrak{d}}^{\mathfrak{c}}(\mathsf{B}) \mid \mathsf{B} \in p\}$. *Moreover,* $\{{}_{\aleph}\sigma_{\mathfrak{b}}^{\mathfrak{a}}(\mathsf{C}) \mid \mathsf{C} \in q \wedge \mathsf{C} \subseteq \mathsf{Z}\} = \{\sigma_{\mathfrak{d}}^{\mathfrak{c}}(\mathsf{B}) \mid \mathsf{B} \in p\}$.

(e) If $(\mathfrak{U}, \mathfrak{D}) = (\mathbb{Z}_w, \varnothing)$, *then* $\sum_{z\in\mathbb{Z}_w} {}_z\Sigma_{\mathfrak{b}}^{\mathfrak{a}} = \Sigma_{\mathfrak{d}}^{\mathfrak{c}}$. *Moreover,* ${}_{\aleph}\Sigma_{\mathfrak{b}}^{\mathfrak{a}} = \Sigma_{\mathfrak{d}}^{\mathfrak{c}}$.

(f) $R(q, \mathsf{Y}) = R(q, \mathsf{Z}) = p$.

(g) If $\Sigma_{\mathfrak{d}}^{\mathfrak{c}} \equiv_w 0$, *then for any* $\{\mathtt{b}, \mathtt{b}'\} \subseteq \Pi_{\ell}^{k}$, *if* $\mathtt{a} = \gamma_{\mathsf{Y},2\ell}^{\;2k}(\mathtt{b})$ *and* $\mathtt{a}' = \gamma_{\mathsf{Y},2\ell}^{\;2k}(\mathtt{b}')$, *then*

$$\xi_{\mathfrak{b}}^{\mathfrak{a}}(\mathtt{a}') - \xi_{\mathfrak{b}}^{\mathfrak{a}}(\mathtt{a}) + {}_{\aleph}\delta_{\mathfrak{b}}^{\mathfrak{a}}(\mathtt{a}, \mathtt{a}') = \delta_{\mathfrak{d}}^{\mathfrak{c}}(\mathtt{b}, \mathtt{b}').$$

(h) If $\Sigma_{\mathfrak{d}}^{\mathfrak{c}} \equiv_w 0$, *then for any* $\{\mathtt{b}, \mathtt{b}''\} \subseteq \Pi_{\ell}^{k}$ *with* $\mathtt{b} \neq \mathtt{b}''$, *if* $\mathtt{a} = \gamma_{\mathsf{Y},2\ell}^{\;2k}(\mathtt{b})$ *and* $\mathtt{a}'' = \gamma_{\mathsf{Y},2\ell}^{\;2k}(\mathtt{b}'')$, *then*

$$\begin{aligned}|\,]\mathtt{a}, \mathtt{a}''[_{2\ell}^{2k} \cap \{\mathtt{a}' \in \Pi_{2\ell}^{2k} \wedge \xi_{\mathfrak{b}}^{\mathfrak{a}}(\mathtt{a}') \in \mathbb{Z}_w \\ \wedge\, \xi_{\mathfrak{b}}^{\mathfrak{a}}(\mathtt{a}') - \xi_{\mathfrak{b}}^{\mathfrak{a}}(\mathtt{a}) + {}_{\aleph}\delta_{\mathfrak{b}}^{\mathfrak{a}}(\mathtt{a}, \mathtt{a}') \equiv_w 0\}| \\ = |\,]\mathtt{b}, \mathtt{b}''[_{\ell}^{k} \cap \{\mathtt{b}' \in \Pi_{\ell}^{k} \wedge \delta_{\mathfrak{d}}^{\mathfrak{c}}(\mathtt{b}, \mathtt{b}') \equiv_w 0\}|.\end{aligned}$$

Proof. (a) If $k = \ell = 0$ and thus $q = \varnothing = \xi_{\mathfrak{b}}^{\mathfrak{a}\dashleftarrow}(\{\mathfrak{U} \cup \mathfrak{D}, \{\aleph\}\})$, there is nothing to show. Otherwise, $\xi_{\mathfrak{b}}^{\mathfrak{a}\dashleftarrow}(\{\mathfrak{U} \cup \mathfrak{D}, \{\aleph\}\}) = \{\mathsf{Y}, \mathsf{Z}\}$. But that is clear since for any $\mathsf{A} \in q$

by definition there exists $\mathsf{B}\in p$ such that either $\mathsf{A}=\gamma_{\mathsf{Y},2\ell\to}^{2k}(\mathsf{B})$ or $\mathsf{A}=\gamma_{\mathsf{Z},2\ell\to}^{2k}(\mathsf{B})$ and since $\gamma_{\mathsf{Y},2\ell\to}^{2k}(\mathsf{B})\subseteq\mathsf{Y}$ and $\gamma_{\mathsf{Z},2\ell\to}^{2k}(\mathsf{B})\subseteq\mathsf{Z}$.

(b) If $(\mathfrak{U},\mathfrak{D})=(\mathbb{Z}_w,\varnothing)$, then for any $i\in[\![k]\!]$ by Lemma 9.2 (d) the point $\gamma_{\mathsf{Y},2\ell}^{2k}({}^\blacksquare i)$ is given by ${}^\blacksquare(2i)$ if $\mathfrak{c}(i)=\bullet$ and by ${}^\blacksquare(2i-1)$ if $\mathfrak{c}(i)=\circ$. That is why by Lemma 9.2 (b), the color $\zeta_{\mathfrak{b}}^{\mathfrak{a}}(\gamma_{\mathsf{Y},2\ell}^{2k}({}^\blacksquare i))$ is equal to $\circ$ if $\mathfrak{c}(i)=\bullet$ and to $\bullet$ if $\mathfrak{c}(i)=\circ$, i.e., to $\zeta_{\mathfrak{d}}^{\mathfrak{c}}({}^\blacksquare i)$. Likewise, Lemma 9.2 (d) shows that for any $j\in[\![\ell]\!]$ the point $\gamma_{\mathsf{Y},2\ell}^{2k}({}_\blacksquare j)$ is ${}_\blacksquare(2j)$ if $\mathfrak{d}(j)=\bullet$ and ${}_\blacksquare(2j-1)$ if $\mathfrak{d}(j)=\circ$, whence we conclude with the help of Lemma 9.2 (b) that $\zeta_{\mathfrak{b}}^{\mathfrak{a}}(\gamma_{\mathsf{Y},2\ell}^{2k}({}_\blacksquare j))$ is $\bullet$ if $\mathfrak{d}(j)=\bullet$ and $\circ$ if $\mathfrak{d}(j)=\circ$, which, of course, is $\zeta_{\mathfrak{d}}^{\mathfrak{c}}({}_\blacksquare j)$. The proves the first claim.

Analogously, for any $i\in[\![k]\!]$ by Lemma 9.2 (d) the point $\gamma_{\mathsf{Z},2\ell}^{2k}({}^\blacksquare i)$ is given by ${}^\blacksquare(2i-1)$ if $\mathfrak{c}(i)=\bullet$ and by ${}^\blacksquare(2i)$ if $\mathfrak{c}(i)=\circ$ and thus by Lemma 9.2 (b) the color $\zeta_{\mathfrak{b}}^{\mathfrak{a}}(\gamma_{\mathsf{Z},2\ell}^{2k}({}^\blacksquare i))$ by $\circ$ if $\mathfrak{c}(i)=\bullet$ and by $\bullet$ if $\mathfrak{c}(i)=\circ$. And, for any $j\in[\![\ell]\!]$ Lemma 9.2 (d) tells us that $\gamma_{\mathsf{Z},2\ell}^{2k}({}_\blacksquare j)$ is ${}_\blacksquare(2j-1)$ if $\mathfrak{d}(j)=\bullet$ and ${}_\blacksquare(2j)$ if $\mathfrak{d}(j)=\circ$, which is why by the Lemma 9.2 (b) the color $\zeta_{\mathfrak{b}}^{\mathfrak{a}}(\gamma_{\mathsf{Z},2\ell}^{2k}({}_\blacksquare j))$ is $\bullet$ if $\mathfrak{d}(j)=\bullet$ and $\circ$ if $\mathfrak{d}(j)=\circ$. Hence, the second claim is true as well.

(c) Because $\mathsf{Y}\cap\mathsf{Z}=\varnothing$ and because Y is the image of $\gamma_{\mathsf{Y},2\ell}^{2k}$ and Z is the image of $\gamma_{\mathsf{Z},2\ell}^{2k}$ the definition of q implies $\{\mathsf{A}\in q\wedge\mathsf{A}\subseteq\mathsf{Y}\}=\{\gamma_{\mathsf{Y},2\ell\to}^{2k}(\mathsf{B})\,|\,\mathsf{B}\in p\}$ and $\{\mathsf{A}\in q\wedge\mathsf{A}\subseteq\mathsf{Z}\}=\{\gamma_{\mathsf{Z},2\ell\to}^{2k}(\mathsf{B})\,|\,\mathsf{B}\in p\}$. Hence, the fact that $\gamma_{\mathsf{Y},2\ell}^{2k}$ and $\gamma_{\mathsf{Z},2\ell}^{2k}$ are injective proves the claim.

(d) If $(\mathfrak{U},\mathfrak{D})=(\mathbb{Z}_w,\varnothing)$, then for any $\mathsf{B}\in p$ by definition of $\mathfrak{a}$ and $\mathfrak{b}$ the block $\mathsf{A}=\gamma_{\mathsf{Y},2\ell\to}^{2k}(\mathsf{B})$ of q is contained in $\mathsf{Y}=\xi_{\mathfrak{b}}^{\mathfrak{a}\leftarrow}(\mathbb{Z}_w)$ and thus satisfies by the first part of (b),

$$\textstyle\sum_{z\in\mathbb{Z}_w}{}_z\sigma_{\mathfrak{b}}^{\mathfrak{a}}(\mathsf{A})=\sum_{\mathsf{e}\in\mathsf{A}}\sigma(\zeta_{\mathfrak{b}}^{\mathfrak{a}}(\mathsf{e}))=\sum_{\mathsf{d}\in\mathsf{B}}\sigma(\zeta_{\mathfrak{b}}^{\mathfrak{a}}(\gamma_{\mathsf{Y},2\ell}^{2k}(\mathsf{d})))=\sum_{\mathsf{d}\in\mathsf{B}}\sigma(\zeta_{\mathfrak{b}}^{\mathfrak{c}}(\mathsf{d}))=\sigma_{\mathfrak{d}}^{\mathfrak{c}}(\mathsf{B}).$$

Since any $\mathsf{A}\in q$ with $\mathsf{A}\subseteq\mathsf{Y}$ arises in this way, that proves the first assertion.

Similarly, if, instead, $\mathsf{A}=\gamma_{\mathsf{Z},2\ell\to}^{2k}(\mathsf{B})$, then $\mathsf{A}\subseteq\mathsf{Z}=\xi_{\mathfrak{b}}^{\mathfrak{a}\leftarrow}(\{\aleph\})$ and by the second part of (b),

$$\textstyle{}_\aleph\sigma_{\mathfrak{b}}^{\mathfrak{c}}(\mathsf{A})=\sum_{\mathsf{e}\in\mathsf{A}}\sigma(\zeta_{\mathfrak{b}}^{\mathfrak{a}}(\mathsf{e}))=\sum_{\mathsf{d}\in\mathsf{B}}\sigma(\zeta_{\mathfrak{b}}^{\mathfrak{a}}(\gamma_{\mathsf{Y},2\ell}^{2k}(\mathsf{d})))=\sum_{\mathsf{d}\in\mathsf{B}}\sigma(\zeta_{\mathfrak{b}}^{\mathfrak{c}}(\mathsf{d}))=\sigma_{\mathfrak{d}}^{\mathfrak{c}}(\mathsf{B}).$$

And because all $\mathsf{A}\in q$ with $\mathsf{A}\subseteq\mathsf{Z}$ are of this kind, we have thus confirmed the second claim too.

(e) If $(\mathfrak{U},\mathfrak{D})=(\mathbb{Z}_w,\varnothing)$, then the first part of (d) and $\Pi_{2\ell}^{2k}=\bigcup_{\mathsf{A}\in q}\mathsf{A}$ imply

$$\textstyle\sum_{z\in\mathbb{Z}_w}{}_z\Sigma_{\mathfrak{b}}^{\mathfrak{a}}=\sum_{z\in\mathbb{Z}_w}\sum_{\mathsf{A}\in q}{}_z\sigma_{\mathfrak{b}}^{\mathfrak{a}}(\mathsf{B})=\sum_{\mathsf{A}\in q\wedge\mathsf{A}\subseteq\mathsf{Y}}\sum_{z\in\mathbb{Z}_w}{}_z\sigma_{\mathfrak{b}}^{\mathfrak{a}}(\mathsf{B})=\sum_{\mathsf{B}\in p}\sigma_{\mathfrak{d}}^{\mathfrak{c}}(p)=\Sigma_{\mathfrak{d}}^{\mathfrak{c}}.$$

Likewise, the second part of (d) lets us conclude

$$\textstyle{}_\aleph\Sigma_{\mathfrak{b}}^{\mathfrak{a}}=\sum_{\mathsf{A}\in q}{}_\aleph\sigma_{\mathfrak{b}}^{\mathfrak{a}}(\mathsf{B})=\sum_{\mathsf{A}\in q\wedge\mathsf{A}\subseteq\mathsf{Z}}{}_\aleph\sigma_{\mathfrak{b}}^{\mathfrak{a}}(\mathsf{B})=\sum_{\mathsf{B}\in p}\sigma_{\mathfrak{d}}^{\mathfrak{c}}(p)=\Sigma_{\mathfrak{d}}^{\mathfrak{c}}.$$

(f) For any $\mathsf{B} \in p$ both $\gamma_{\mathsf{Y},2\ell}^{2k\leftarrow}(\gamma_{\mathsf{Y},2\ell}^{2k}{}_{\rightarrow}(\mathsf{B})) = \mathsf{B} \neq \varnothing$ because $\gamma_{\mathsf{Y},2\ell}^{2k}$ is injective and $\gamma_{\mathsf{Y},2\ell}^{2k\leftarrow}(\gamma_{\mathsf{Z},2\ell}^{2k}{}_{\rightarrow}(\mathsf{B})) = \varnothing$ because $\mathsf{Y} \cap \mathsf{Z} = \varnothing$. Hence, indeed

$$\begin{aligned} R(q,\mathsf{Y}) &= \gamma_{\mathsf{Y},2\ell}^{2k\leftarrow}(\{\gamma_{\mathsf{Y},2\ell}^{2k}{}_{\rightarrow}(\mathsf{B}), \gamma_{\mathsf{Z},2\ell}^{2k}{}_{\rightarrow}(\mathsf{B})\}_{\mathsf{B}\in p}) \\ &= \{\gamma_{\mathsf{Y},2\ell}^{2k\leftarrow}(\gamma_{\mathsf{Y},2\ell}^{2k}{}_{\rightarrow}(\mathsf{B})), \gamma_{\mathsf{Y},2\ell}^{2k\leftarrow}(\gamma_{\mathsf{Z},2\ell}^{2k}{}_{\rightarrow}(\mathsf{B}))\}_{\mathsf{B}\in p}\backslash\{\varnothing\} \\ &= p. \end{aligned}$$

Exchanging the roles of Y and Z proves $R(q,\mathsf{Z}) = p$.

(g) For any $\{\mathsf{b},\mathsf{b}'\} \subseteq \Pi_\ell^k$, if $\mathsf{a} = \gamma_{\mathsf{Y},2\ell}^{2k}(\mathsf{b})$ and $\mathsf{a}' = \gamma_{\mathsf{Y},2\ell}^{2k}(\mathsf{b}')$, then $\xi_{\mathfrak{b}}^{\mathfrak{a}}(\mathsf{a}) = \xi_{\mathfrak{b}}^{\mathfrak{a}}(\mathsf{a}') = 0$ because $\mathsf{Y} = \xi_{\mathfrak{b}}^{\mathfrak{a}\leftarrow}(\{0\})$ by Lemma 9.2 (a). Hence, if $\Sigma_{\mathfrak{d}}^{\mathfrak{c}} \equiv_w 0$ and thus also ${}_{\aleph}\Sigma_{\mathfrak{b}}^{\mathfrak{a}} \equiv_w 0$ by the second part of (e), it is sufficient to show is that ${}_{\aleph}\delta_{\mathfrak{b}}^{\mathfrak{a}}(\mathsf{a},\mathsf{a}')$ and coincide $\delta_{\mathfrak{d}}^{\mathfrak{c}}(\mathsf{b},\mathsf{b}')$ coincide. If $\mathsf{b} = \mathsf{b}'$, that is clear. Hence, let $\mathsf{b} \neq \mathsf{b}'$ in the following. Because ${}_{\aleph}\sigma_{\mathfrak{b}}^{\mathfrak{a}}(\{\mathsf{a}\}) = {}_{\aleph}\sigma_{\mathfrak{b}}^{\mathfrak{a}}(\{\mathsf{a}'\}) = 0$ by $\{\mathsf{a},\mathsf{a}'\} \subseteq \mathsf{Y}$ it is enough to to prove that ${}_{\aleph}\sigma_{\mathfrak{b}}^{\mathfrak{a}}(]\mathsf{a},\mathsf{a}'[_{2\ell}^{2k})$ is equal to $\delta_{\mathfrak{d}}^{\mathfrak{c}}(\mathsf{b},\mathsf{b}')$.

If $\mathsf{z} \coloneqq \gamma_{\mathsf{Z},2\ell}^{2k}(\mathsf{b})$ and $\mathsf{z}' \coloneqq \gamma_{\mathsf{Z},2\ell}^{2k}(\mathsf{b}')$, then ${}_{\aleph}\delta_{\mathfrak{b}}^{\mathfrak{a}}(\mathsf{z},\mathsf{z}') = \delta_{\mathfrak{d}}^{\mathfrak{c}}(\mathsf{b},\mathsf{b}')$ for the the following reasons. Since $\zeta_{\mathfrak{b}}^{\mathfrak{a}} = \zeta_{\mathfrak{d}}^{\mathfrak{c}} \circ \gamma_{\mathsf{Z},2\ell}^{2k}$ by the second part of (b) the measure ${}_{\aleph}\sigma_{\mathfrak{b}}^{\mathfrak{a}}$ is the push-forward of $\sigma_{\mathfrak{d}}^{\mathfrak{c}}$, i.e., ${}_{\aleph}\sigma_{\mathfrak{b}}^{\mathfrak{a}}(\mathsf{X}) = \sigma_{\mathfrak{d}}^{\mathfrak{c}}(\gamma_{\mathsf{Z},2\ell}^{2k\leftarrow}(\mathsf{X}))$ for any $\mathsf{X} \subseteq \Pi_{2\ell}^{2k}$. Moreover, $\gamma_{\mathsf{Z},2\ell}^{2k\leftarrow}(]\mathsf{z},\mathsf{z}'[_{2\ell}^{2k}) =]\mathsf{b},\mathsf{b}'[_\ell^k$ since $\gamma_{\mathsf{Z},2\ell}^{2k}$ is strictly monotonic with respect to Γ_ℓ^k and $\Gamma_{2\ell}^{2k}$ by Lemma 4.2 (a). Thus,

$$\begin{aligned} {}_{\aleph}\delta_{\mathfrak{b}}^{\mathfrak{a}}(\mathsf{z},\mathsf{z}') &= \tfrac{1}{2}{}_{\aleph}\sigma_{\mathfrak{b}}^{\mathfrak{a}}(\{\mathsf{z}\}) + {}_{\aleph}\sigma_{\mathfrak{b}}^{\mathfrak{a}}(]\mathsf{z},\mathsf{z}'[_{2\ell}^{2k}) + \tfrac{1}{2}{}_{\aleph}\sigma_{\mathfrak{b}}^{\mathfrak{a}}(\{\mathsf{z}'\}) \\ &= \tfrac{1}{2}\sigma_{\mathfrak{d}}^{\mathfrak{c}}(\{\mathsf{b}\}) + \sigma_{\mathfrak{d}}^{\mathfrak{c}}(]\mathsf{b},\mathsf{b}'[_\ell^k) + \tfrac{1}{2}\sigma_{\mathfrak{d}}^{\mathfrak{c}}(\{\mathsf{b}'\}) \\ &= \delta_{\mathfrak{d}}^{\mathfrak{c}}(\mathsf{b},\mathsf{b}'). \end{aligned}$$

Hence, it suffices to prove that ${}_{\aleph}\sigma_{\mathfrak{b}}^{\mathfrak{a}}(]\mathsf{a},\mathsf{a}'[_{2\ell}^{2k})$ and ${}_{\aleph}\delta_{\mathfrak{b}}^{\mathfrak{a}}(\mathsf{z},\mathsf{z}')$ are the same.

Furthermore, by two applications of Lemma 9.2 (e),

$$\begin{cases} \mathsf{a} = \nu_{2\ell}^{2k}(\mathsf{z}) \wedge \mathsf{z}' = \nu_{2\ell}^{2k}(\mathsf{a}') & \text{if } \zeta_{\mathfrak{d}}^{\mathfrak{c}}(\mathsf{b}) = \bullet \wedge \zeta_{\mathfrak{d}}^{\mathfrak{c}}(\mathsf{b}') = \circ \\ \mathsf{a} = \nu_{2\ell}^{2k}(\mathsf{z}) \wedge \mathsf{a}' = \nu_{2\ell}^{2k}(\mathsf{z}') & \text{if } \zeta_{\mathfrak{d}}^{\mathfrak{c}}(\mathsf{b}) = \bullet \wedge \zeta_{\mathfrak{d}}^{\mathfrak{c}}(\mathsf{b}') = \bullet \\ \mathsf{z} = \nu_{2\ell}^{2k}(\mathsf{a}) \wedge \mathsf{z}' = \nu_{2\ell}^{2k}(\mathsf{a}') & \text{if } \zeta_{\mathfrak{d}}^{\mathfrak{c}}(\mathsf{b}) = \circ \wedge \zeta_{\mathfrak{d}}^{\mathfrak{c}}(\mathsf{b}') = \circ \\ \mathsf{z} = \nu_{2\ell}^{2k}(\mathsf{a}) \wedge \mathsf{a}' = \nu_{2\ell}^{2k}(\mathsf{z}') & \text{if } \zeta_{\mathfrak{d}}^{\mathfrak{c}}(\mathsf{b}) = \circ \wedge \zeta_{\mathfrak{d}}^{\mathfrak{c}}(\mathsf{b}') = \bullet. \end{cases}$$

On the other hand, naturally,

$$\begin{cases} \{\mathsf{a}\} \cup]\mathsf{a},\mathsf{a}'[_{2\ell}^{2k} \cup \{\mathsf{a}'\} = \;]\mathsf{z},\mathsf{z}'[_{2\ell}^{2k} & \text{if } \mathsf{a} = \nu_{2\ell}^{2k}(\mathsf{z}) \wedge \mathsf{z}' = \nu_{2\ell}^{2k}(\mathsf{a}') \\ \{\mathsf{a}\} \cup]\mathsf{a},\mathsf{a}'[_{2\ell}^{2k} = \;]\mathsf{z},\mathsf{z}'[_{2\ell}^{2k} \cup \{\mathsf{z}'\} & \text{if } \mathsf{a} = \nu_{2\ell}^{2k}(\mathsf{z}) \wedge \mathsf{a}' = \nu_{2\ell}^{2k}(\mathsf{z}') \\ ]\mathsf{a},\mathsf{a}'[_{2\ell}^{2k} \cup \{\mathsf{a}'\} = \{\mathsf{z}\} \cup]\mathsf{z},\mathsf{z}'[_{2\ell}^{2k} & \text{if } \mathsf{z} = \nu_{2\ell}^{2k}(\mathsf{a}) \wedge \mathsf{z}' = \nu_{2\ell}^{2k}(\mathsf{a}') \\ ]\mathsf{a},\mathsf{a}'[_{2\ell}^{2k} = \{\mathsf{z}\} \cup]\mathsf{z},\mathsf{z}'[_{2\ell}^{2k} \cup \{\mathsf{z}'\} & \text{if } \mathsf{z} = \nu_{2\ell}^{2k}(\mathsf{a}) \wedge \mathsf{a}' = \nu_{2\ell}^{2k}(\mathsf{z}'). \end{cases}$$

Combining the two statements and using the additivity of ${}_{\aleph}\sigma^{\mathfrak{a}}_{\mathfrak{b}}$ and the fact that ${}_{\aleph}\sigma^{\mathfrak{a}}_{\mathfrak{b}}(\{\mathsf{a}\}) = {}_{\aleph}\sigma^{\mathfrak{a}}_{\mathfrak{b}}(\{\mathsf{a}'\}) = 0$ by $\{\mathsf{a},\mathsf{a}'\} \subseteq \mathsf{Y}$ thus yields

$$ {}_{\aleph}\sigma^{\mathfrak{a}}_{\mathfrak{b}}(]\mathsf{a},\mathsf{a}'[^{2k}_{2\ell}) = \begin{cases} {}_{\aleph}\sigma^{\mathfrak{a}}_{\mathfrak{b}}(]\mathsf{z},\mathsf{z}'[^{2k}_{2\ell}) & \text{if } \zeta^{\mathfrak{c}}_{\mathfrak{d}}(\mathsf{b}) = \bullet \wedge \zeta^{\mathfrak{c}}_{\mathfrak{d}}(\mathsf{b}') = \circ \\ {}_{\aleph}\sigma^{\mathfrak{a}}_{\mathfrak{b}}(]\mathsf{z},\mathsf{z}'[^{2k}_{2\ell}) + {}_{\aleph}\sigma^{\mathfrak{a}}_{\mathfrak{b}}(\{\mathsf{z}'\}) & \text{if } \zeta^{\mathfrak{c}}_{\mathfrak{d}}(\mathsf{b}) = \bullet \wedge \zeta^{\mathfrak{c}}_{\mathfrak{d}}(\mathsf{b}') = \bullet \\ {}_{\aleph}\sigma^{\mathfrak{a}}_{\mathfrak{b}}(\{\mathsf{z}\}) + {}_{\aleph}\sigma^{\mathfrak{a}}_{\mathfrak{b}}(]\mathsf{z},\mathsf{z}'[^{2k}_{2\ell}) & \text{if } \zeta^{\mathfrak{c}}_{\mathfrak{d}}(\mathsf{b}) = \circ \wedge \zeta^{\mathfrak{c}}_{\mathfrak{d}}(\mathsf{b}') = \circ \\ {}_{\aleph}\sigma^{\mathfrak{a}}_{\mathfrak{b}}(\{\mathsf{z}\}) + {}_{\aleph}\sigma^{\mathfrak{a}}_{\mathfrak{b}}(]\mathsf{z},\mathsf{z}'[^{2k}_{2\ell}) + {}_{\aleph}\sigma^{\mathfrak{a}}_{\mathfrak{b}}(\{\mathsf{z}'\}) & \text{if } \zeta^{\mathfrak{c}}_{\mathfrak{d}}(\mathsf{b}) = \circ \wedge \zeta^{\mathfrak{c}}_{\mathfrak{d}}(\mathsf{b}') = \bullet. \end{cases} $$

Actually, the right hand side is identical to $\frac{1}{2}{}_{\aleph}\sigma^{\mathfrak{a}}_{\mathfrak{b}}(\{\mathsf{z}\}) + {}_{\aleph}\sigma^{\mathfrak{a}}_{\mathfrak{b}}(]\mathsf{z},\mathsf{z}'[^{2k}_{2\ell}) + \frac{1}{2}{}_{\aleph}\sigma^{\mathfrak{a}}_{\mathfrak{b}}(\{\mathsf{z}'\})$ because ${}_{\aleph}\sigma^{\mathfrak{a}}_{\mathfrak{b}}(\{\mathsf{z}\}) = \sigma(\zeta^{\mathfrak{c}}_{\mathfrak{d}}(\mathsf{b}))$ and ${}_{\aleph}\sigma^{\mathfrak{a}}_{\mathfrak{b}}(\{\mathsf{z}'\}) = \sigma(\zeta^{\mathfrak{c}}_{\mathfrak{d}}(\mathsf{b}'))$. Thus, the claim is true.

(h) By (g) the mapping $\gamma^{2k}_{\mathsf{Y},2\ell}$ restricts to a bijection from the set $\{\mathsf{b}' \in \Pi^k_\ell \wedge \delta^{\mathfrak{c}}_{\mathfrak{d}}(\mathsf{b},\mathsf{b}') \equiv_w 0\}$ to $\{\mathsf{a}' \in \mathsf{Y} \wedge \xi^{\mathfrak{a}}_{\mathfrak{b}}(\mathsf{a}') - \xi^{\mathfrak{a}}_{\mathfrak{b}}(\mathsf{a}) + {}_{\aleph}\delta^{\mathfrak{a}}_{\mathfrak{b}}(\mathsf{a},\mathsf{a}') \equiv_w 0\}$. By Lemma 4.2 (a) it is also strictly monotonic with respect to Γ^k_ℓ and $\Gamma^{2k}_{2\ell}$ and thus restricts to a bijection from $]\mathsf{b},\mathsf{b}''[^k_\ell$ to $]\mathsf{a},\mathsf{a}''[^{2k}_{2\ell} \cap \mathsf{Y}$. And that concludes the proof. □

We are now ready to show that the construction really does yield a functor. More precisely, below we show that for any $w \in \mathbb{N}_0$ and any $n \in \mathbb{N}$, if $(\mathfrak{U},\mathfrak{D}) \in \{(\mathbb{Z}_w,\varnothing),(\varnothing,\mathbb{Z}_w)\}$ and if $N\colon \mathbb{Z}_w \uplus \{\aleph\} \to \mathbb{N}$ is such that $N(z) = n$ for any $z \in \mathbb{Z}_w$ and $N(\aleph) = 1$, then

- □ F is a faithful strict monoidal $*$-functor from $\mathcal{S}$ to ${}_{\mathfrak{U}\uplus\{\aleph\},\mathfrak{D}}\mathcal{S}$.
- □ ${}^{n,N}\mathbb{C}F$ is a faithful $\mathbb{C}$-linear strict monoidal $*$-functor from ${}^{n}\mathbb{C}\mathcal{S}$ to ${}^{N}\mathbb{C}_{\mathfrak{U}\uplus\{\aleph\},\mathfrak{D}}\mathcal{S}$. faithful strict monoidal linear $*$-functor between the linear versions of the categories.

Proposition 9.5. *For any $w \in \mathbb{N}_0$ and any $n \in \mathbb{N}$, if $(\mathfrak{U},\mathfrak{D}) \in \{(\mathbb{Z}_w,\varnothing),(\varnothing,\mathbb{Z}_w)\}$ and if $N\colon \mathbb{Z}_w \uplus \{\aleph\} \to \mathbb{N}$ is such that $N(z) = n$ for any $z \in \mathbb{Z}_w$ and $N(\aleph) = 1$, then the following are true.*

(a) For any $\{(\mathfrak{c},\mathfrak{d},p),(\mathfrak{c}',\mathfrak{d}',p')\} \subseteq \mathcal{S}$, if $F(\mathfrak{c},\mathfrak{d},p) = F(\mathfrak{c}',\mathfrak{d}',p')$, then already $(\mathfrak{c},\mathfrak{d},p) = (\mathfrak{c}',\mathfrak{d}',p')$.

(b) $F(\mathrm{id}_{\mathfrak{c}}) = \mathrm{id}_{F(\mathfrak{c})}$ for any $\mathfrak{c}\colon [\![k]\!] \to \{\circ,\bullet\}$ and any $k \in \mathbb{N}_0$

(c) $F((\mathfrak{c},\mathfrak{d},p)^*) = (F(\mathfrak{c},\mathfrak{d},p))^*$ for any $(\mathfrak{c},\mathfrak{d},p) \in \mathcal{S}$.

(d) $F(\mathfrak{c}_1 \otimes \mathfrak{c}_2) = F(\mathfrak{c}_1) \otimes F(\mathfrak{c}_2)$ for any $\mathfrak{c}_1\colon [\![k_1]\!] \to \{\circ,\bullet\}$ and $\mathfrak{c}_2\colon [\![k_2]\!] \to \{\circ,\bullet\}$ and any $\{k_1,k_2\} \subseteq \mathbb{N}_0$.

(e) $F((\mathfrak{c}_1,\mathfrak{d}_1,p_1) \otimes (\mathfrak{c}_2,\mathfrak{d}_2,p_2)) = F(\mathfrak{c}_1,\mathfrak{d}_1,p_1) \otimes F(\mathfrak{c}_2,\mathfrak{d}_2,p_2)$ for any $\{(\mathfrak{c}_1,\mathfrak{d}_1,p_1), (\mathfrak{c}_2,\mathfrak{d}_2,p_2)\} \subseteq \mathcal{S}$.

(f) $F((\mathfrak{d},\mathfrak{e},q)(\mathfrak{c},\mathfrak{d},p)) = F(\mathfrak{d},\mathfrak{e},q)F(\mathfrak{c},\mathfrak{d},p)$ as well as ${}_{n}\mathrm{lf}((\mathfrak{d},\mathfrak{e},q),(\mathfrak{c},\mathfrak{d},p)) = {}_{N}\mathrm{lf}(F(\mathfrak{d},\mathfrak{e},q),F(\mathfrak{c},\mathfrak{d},p))$ for any $\{(\mathfrak{c},\mathfrak{d},p),(\mathfrak{d},\mathfrak{e},q)\} \subseteq \mathcal{S}$.

Proof. Throughout, the following shorthand will be used.

Notation in this proof. For any $\{k,\ell\} \subseteq \mathbb{N}_0$ and any two-colorings $\mathfrak{c}\colon [\![k]\!] \to \{\circ,\bullet\}$ and $\mathfrak{d}\colon [\![\ell]\!] \to \{\circ,\bullet\}$ let

$$ \mathsf{Y}(\mathfrak{c},\mathfrak{d}) := \xi^{F(\mathfrak{c})\leftarrow}_{F(\mathfrak{d})}(\mathbb{Z}_w) \quad \wedge \quad \mathsf{Z}(\mathfrak{c},\mathfrak{d}) := \xi^{F(\mathfrak{c})\leftarrow}_{F(\mathfrak{d})}(\{\aleph\}). $$

Auxiliary statements. It will be used on numerous occasions that then for any $\mathsf{X} \in \{\mathsf{Y}, \mathsf{Z}\}$ and $i \in [\![k]\!]$ and $j \in [\![\ell]\!]$,

$$\gamma_{\mathsf{X}(\mathfrak{c},\mathfrak{d}),2\ell}^{\;2k}({}^{\blacksquare}i) = \gamma_{\mathsf{X}(\mathfrak{c},\varnothing),0}^{\;2k}({}^{\blacksquare}i) \quad \wedge \quad \gamma_{\mathsf{X}(\mathfrak{c},\mathfrak{d}),2\ell}^{\;2k}({}_{\blacksquare}j) = \gamma_{\mathsf{X}(\varnothing,\mathfrak{d}),2\ell}^{\;0}({}_{\blacksquare}j)$$

and

$$\kappa_{2k}^{2\ell} \circ \gamma_{\mathsf{X}(\mathfrak{c},\mathfrak{d}),2\ell}^{\;2k} = \gamma_{\mathsf{X}(\mathfrak{d},\mathfrak{c}),2k}^{\;2\ell} \circ \kappa_k^\ell.$$

Indeed, for $\mathsf{X} = \mathsf{Y}$ and any $x \in [\![2k]\!]$, by definition, $\gamma_{\mathsf{Y}(\mathfrak{c},\mathfrak{d}),2\ell}^{\;2k}({}^{\blacksquare}i) = {}^{\blacksquare}x$ if and only if ${}^{\blacksquare}x \in \mathsf{Y}(\mathfrak{c},\mathfrak{d})$ and $|\mathsf{Y}(\mathfrak{c},\mathfrak{d}) \cap \Pi_0^x| = i$, which is to say $\xi_{F(\mathfrak{d})}^{F(\mathfrak{c})}({}^{\blacksquare}i) \in \mathbb{Z}_w$ and $|\{t \in [\![x]\!] \wedge \xi_{F(\mathfrak{d})}^{F(\mathfrak{c})}({}^{\blacksquare}t) \in \mathbb{Z}_w\}| = i$ or, equivalently, $F(\mathfrak{c})(x) \in \mathbb{Z}_w$ and $|\{t \in [\![x]\!] \wedge F(\mathfrak{c})(t) \in \mathbb{Z}_w\}| = i$. But exactly that is also what it means for $\gamma_{\mathsf{Y}(\mathfrak{c},\varnothing),0}^{\;2k}({}^{\blacksquare}i) = {}^{\blacksquare}x$ to hold. The proofs for ${}_{\blacksquare}j$ and for the case $\mathsf{X} = \mathsf{Z}$ are analogous. The second identity follows similarly. And, indeed, the fact that $\xi_{F(\mathfrak{c})}^{F(\mathfrak{d})} = \xi_{F(\mathfrak{d})}^{F(\mathfrak{c})} \circ \kappa_{2\ell}^{2k}$ by Lemmy 4.2 (b) implies $\mathsf{X}(\mathfrak{d},\mathfrak{c}) = \kappa_{2\ell}^{2k\leftarrow}(\mathsf{X}(\mathfrak{c},\mathfrak{d}))$, which then proves $\kappa_{2k}^{2\ell} \circ \gamma_{\mathsf{X}(\mathfrak{c},\mathfrak{d}),2\ell}^{\;2k} = \gamma_{\mathsf{X}(\mathfrak{d},\mathfrak{c}),2k}^{\;2\ell} \circ \kappa_k^\ell$ by Lemmy 4.5 (b).

(a) Given any $\{k,\ell\} \subseteq \mathbb{N}_0$ and $\mathfrak{c}\colon [\![k]\!] \to \{\circ,\bullet\}$ and $\mathfrak{d}\colon [\![\ell]\!] \to \{\circ,\bullet\}$ and $(\mathfrak{c},\mathfrak{d},p) \in \mathcal{S}$, if $(\mathfrak{a},\mathfrak{b},q) = F(\mathfrak{c},\mathfrak{d},p)$, then p can be recovered from q because $R(q,\mathsf{Z}(\mathfrak{c},\mathfrak{d})) = p$ by Lemma 9.4 (f). That proves the first claim.

(b) For any $k \in \mathbb{N}_0$ and $\mathfrak{c}\colon [\![k]\!] \to \{\circ,\bullet\}$, if $(F(\mathfrak{c}),F(\mathfrak{c}),q) = F(\mathrm{id}_{\mathfrak{c}})$, then by definition $q = \{\gamma_{\mathsf{X}(\mathfrak{c},\mathfrak{c}),2k\to}^{\;2k}(\{{}^{\blacksquare}i,{}_{\blacksquare}i\}) \,|\, \mathsf{X} \in \{\mathsf{Y},\mathsf{Z}\} \wedge i \in [\![k]\!]\}$. According to Lemma 9.2 (d) the set $\gamma_{\mathsf{X}(\mathfrak{c},\mathfrak{c}),2k\to}^{\;2k}(\{{}^{\blacksquare}i,{}_{\blacksquare}i\})$ is given by $\{{}^{\blacksquare}(2i),{}_{\blacksquare}(2i)\}$ if either $\mathfrak{c}(i) = \bullet$ and $\mathsf{X} = \mathsf{Y}$ or $\mathfrak{c}(i) = \circ$ and $\mathsf{X} = \mathsf{Z}$ and by $\{{}^{\blacksquare}(2i-1),{}_{\blacksquare}(2i-1)\}$ if either $\mathfrak{c}(i) = \circ$ and $\mathsf{X} = \mathsf{Y}$ or $\mathfrak{c}(i) = \bullet$ and $\mathsf{X} = \mathsf{Z}$. In consequence, $q = \{\{{}^{\blacksquare}(2i),{}_{\blacksquare}(2i)\},\{{}^{\blacksquare}(2i-1),{}_{\blacksquare}(2i-1)\} \,|\, i \in [\![k]\!]\} = \mathrm{id}_{2k}$, which proves $F(\mathrm{id}_{\mathfrak{c}}) = \mathrm{id}_{F(\mathfrak{c})}$.

(c) Given any $\{k,\ell\} \subseteq \mathbb{N}_0$ and $\mathfrak{c}\colon [\![k]\!] \to \{\circ,\bullet\}$ and $\mathfrak{d}\colon [\![\ell]\!] \to \{\circ,\bullet\}$ and $(\mathfrak{c},\mathfrak{d},p) \in \mathcal{S}$, if $(\mathfrak{a},\mathfrak{b},q) = F(\mathfrak{c},\mathfrak{d},p)$ and $(\mathfrak{b},\mathfrak{a},r) = F(\mathfrak{d},\mathfrak{c},p^*)$, then $r = \{\gamma_{\mathsf{X}(\mathfrak{d},\mathfrak{c}),2k\to}^{\;2\ell}(\mathsf{E}) \,|\, \mathsf{X} \in \{\mathsf{Y},\mathsf{Z}\} \wedge \mathsf{E} \in p^*\} = \{\gamma_{\mathsf{X}(\mathfrak{d},\mathfrak{c}),2k\to}^{\;2\ell}(\kappa_\ell^{k\leftarrow}(\mathsf{B})) \,|\, \mathsf{X} \in \{\mathsf{Y},\mathsf{Z}\} \wedge \mathsf{B} \in p\}$ by definition. Since $(\kappa_\ell^k)^{-1} = \kappa_k^\ell$ the mapping $\gamma_{\mathsf{X}(\mathfrak{d},\mathfrak{c}),2k\to}^{\;2\ell} \circ \kappa_\ell^{k\leftarrow}$ is the same as $\gamma_{\mathsf{X}(\mathfrak{d},\mathfrak{c}),2k\to}^{\;2\ell} \circ \kappa_{k\to}^\ell = (\gamma_{\mathsf{X}(\mathfrak{d},\mathfrak{c}),2k}^{\;2\ell} \circ \kappa_k^\ell)_\to$, which, because $\kappa_{2k}^{2\ell} \circ \gamma_{\mathsf{X}(\mathfrak{c},\mathfrak{d}),2\ell}^{\;2k} = \gamma_{\mathsf{X}(\mathfrak{d},\mathfrak{c}),2k}^{\;2\ell} \circ \kappa_k^\ell$ by the initial remark, is in turn identical to $(\kappa_{2k}^{2\ell} \circ \gamma_{\mathsf{X}(\mathfrak{c},\mathfrak{d}),2\ell}^{\;2k})_\to = \kappa_{2k\to}^{2\ell} \circ \gamma_{\mathsf{X}(\mathfrak{c},\mathfrak{d}),2\ell\to}^{\;2k}$. Hence, considering that $(\kappa_{2k}^{2\ell})^{-1} = \kappa_{2\ell}^{2k}$, we have shown $r = \{\kappa_{2\ell}^{2k\leftarrow}(\gamma_{\mathsf{X}(\mathfrak{c},\mathfrak{d}),2\ell\to}^{\;2k}(\mathsf{B})) \,|\, \mathsf{X} \in \{\mathsf{Y},\mathsf{Z}\} \wedge \mathsf{B} \in p\} = \{\gamma_{\mathsf{X}(\mathfrak{c},\mathfrak{d}),2\ell\to}^{\;2k}(\mathsf{B}) \,|\, \mathsf{X} \in \{\mathsf{Y},\mathsf{Z}\} \wedge \mathsf{B} \in p\}^* = q^*$. In other words, $F((\mathfrak{c},\mathfrak{d},p)^*) = F(\mathfrak{c},\mathfrak{d},p)^*$.

(d) If $\mathfrak{a}_t = F(\mathfrak{c}_t)$ for each $t \in [\![2]\!]$ and if $\mathfrak{e} = F(\mathfrak{c}_1 \otimes \mathfrak{c}_2)$, then we have to prove $\mathfrak{e} = \mathfrak{a}_1 \otimes \mathfrak{a}_2$. If the statements $(\mathfrak{U},\mathfrak{O}) = (\mathbb{Z}_w,\varnothing)$ and $(\mathfrak{U},\mathfrak{O}) = (\varnothing,\mathbb{Z}_w)$ are abbreviated by U and O, respectively, then for any $x \in [\![2]\!]$, by definition, on the one hand,

$$(\mathfrak{a}_1 \otimes \mathfrak{a}_2)(x) = \begin{cases} (\aleph, \bullet) & | \quad x \leq 2k_1 \wedge \quad x \text{ odd} \quad \wedge \quad \mathfrak{c}_1(\frac{x+1}{2}) = \bullet \\ (0, \circ) & | \quad x \leq 2k_1 \wedge \quad x \text{ odd} \quad \wedge \quad \mathfrak{c}_1(\frac{x+1}{2}) = \circ \wedge U \\ 0 & | \quad x \leq 2k_1 \wedge \quad x \text{ odd} \quad \wedge \quad \mathfrak{c}_1(\frac{x+1}{2}) = \circ \wedge O \\ (0, \bullet) & | \quad x \leq 2k_1 \wedge \quad x \text{ even} \quad \wedge \quad \mathfrak{c}_1(\frac{x}{2}) = \bullet \wedge U \\ 0 & | \quad x \leq 2k_1 \wedge \quad x \text{ even} \quad \wedge \quad \mathfrak{c}_1(\frac{x}{2}) = \bullet \wedge O \\ (\aleph, \circ) & | \quad x \leq 2k_1 \wedge \quad x \text{ even} \quad \wedge \quad \mathfrak{c}_1(\frac{x}{2}) = \circ \\ (\aleph, \bullet) & | \; 2k_1 < x \quad \wedge \; x - 2k_1 \text{ odd} \wedge \mathfrak{c}_2(\frac{x-2k_1+1}{2}) = \bullet \\ (0, \circ) & | \; 2k_1 < x \quad \wedge \; x - 2k_1 \text{ odd} \wedge \mathfrak{c}_2(\frac{x-2k_1+1}{2}) = \circ \wedge U \\ 0 & | \; 2k_1 < x \quad \wedge \; x - 2k_1 \text{ odd} \wedge \mathfrak{c}_2(\frac{x-2k_1+1}{2}) = \circ \wedge O \\ (0, \bullet) & | \; 2k_1 < x \quad \wedge \; x - 2k_1 \text{ even} \wedge \mathfrak{c}_2(\frac{x-2k_1}{2}) = \bullet \wedge U \\ 0 & | \; 2k_1 < x \quad \wedge \; x - 2k_1 \text{ even} \wedge \mathfrak{c}_2(\frac{x-2k_1}{2}) = \bullet \wedge O \\ (\aleph, \circ) & | \; 2k_1 < x \quad \wedge \; x - 2k_1 \text{ even} \wedge \mathfrak{c}_2(\frac{x-2k_1}{2}) = \circ \end{cases}$$

and, on the other hand,

$$\mathfrak{e}(x) = \begin{cases} (\aleph, \bullet) & | \; x \text{ odd} \wedge \quad \mathfrak{c}_1(\frac{x+1}{2}) = \bullet \quad \wedge \; \frac{x+1}{2} \leq k_1 \\ (\aleph, \bullet) & | \; x \text{ odd} \wedge \mathfrak{c}_2(\frac{x+1}{2} - k_1) = \bullet \quad \wedge \; k_1 < \frac{x+1}{2} \\ (0, \circ) & | \; x \text{ odd} \wedge \quad \mathfrak{c}_1(\frac{x+1}{2}) = \circ \wedge U \wedge \frac{x+1}{2} \leq k_1 \\ (0, \circ) & | \; x \text{ odd} \wedge \mathfrak{c}_2(\frac{x+1}{2} - k_1) = \circ \wedge U \wedge \; k_1 < \frac{x+1}{2} \\ 0 & | \; x \text{ odd} \wedge \quad \mathfrak{c}_1(\frac{x+1}{2}) = \circ \wedge O \wedge \frac{x+1}{2} \leq k_1 \\ 0 & | \; x \text{ odd} \wedge \mathfrak{c}_2(\frac{x+1}{2} - k_1) = \circ \wedge O \wedge \; k_1 < \frac{x+1}{2} \\ (0, \bullet) & | \; x \text{ even} \wedge \quad \mathfrak{c}_1(\frac{x}{2}) = \bullet \wedge U \wedge \; \frac{x}{2} \leq k_1 \\ (0, \bullet) & | \; x \text{ even} \wedge \mathfrak{c}_2(\frac{x}{2} - k_1) = \bullet \wedge U \wedge \; k_1 < \frac{x}{2} \\ 0 & | \; x \text{ even} \wedge \quad \mathfrak{c}_1(\frac{x}{2}) = \bullet \wedge O \wedge \; \frac{x}{2} \leq k_1 \\ 0 & | \; x \text{ even} \wedge \mathfrak{c}_2(\frac{x}{2} - k_1) = \bullet \wedge O \wedge \; k_1 < \frac{x}{2} \\ (\aleph, \circ) & | \; x \text{ even} \wedge \quad \mathfrak{c}_1(\frac{x}{2}) = \circ \quad \wedge \; \frac{x}{2} \leq k_1 \\ (\aleph, \circ) & | \; x \text{ even} \wedge \mathfrak{c}_1(\frac{x}{2} - k_1) = \circ \quad \wedge \; k_1 < \frac{x}{2} \end{cases}.$$

Of course, $x - 2k_1 \equiv_2 x$ and $\frac{x-2k_1+1}{2} = \frac{x+1}{2} - k_1$ and $\frac{x-2k_1}{2} = \frac{x}{2} - k_1$. Moreover, if x is odd, then the two statements $\frac{x+1}{2} \leq k_1$ and $x \leq 2k_1$ are equivalent (and thus also the statements $k_1 < \frac{x+1}{2}$ and $2k_1 < x$). Hence, $\mathfrak{e} = \mathfrak{a}_1 \otimes \mathfrak{a}_2$.

(e) If $\{k_t, \ell_t\} \subseteq \mathbb{N}_0$ are such that $\mathfrak{c}_t\colon [\![k_t]\!] \to \{\circ, \bullet\}$ and $\mathfrak{d}_t\colon [\![\ell_t]\!] \to \{\circ, \bullet\}$ for each $t \in [\![2]\!]$, if $(\mathfrak{a}_t, \mathfrak{b}_t, q_t) = F(\mathfrak{c}_t, \mathfrak{d}_t, p_t)$ for each $t \in [\![2]\!]$ and if $(\mathfrak{a}_1 \otimes \mathfrak{a}_2, \mathfrak{b}_1 \otimes \mathfrak{b}_2, r) = F(\mathfrak{c}_1 \otimes \mathfrak{c}_2, \mathfrak{d}_1 \otimes \mathfrak{d}_2, p_1 \otimes p_2)$, we are to show $r = q_1 \otimes q_2$.

If $\mathsf{H}_1 = \Pi^{k_1}_{\ell_1}$ and $\mathsf{H}_2 = \Pi^{k_1+k_2}_{\ell_1+\ell_2} \backslash \Pi^{k_1}_{\ell_1}$ and if $\mathsf{S}_1 = \Pi^{2k_1}_{2\ell_1}$ and $\mathsf{S}_2 = \Pi^{2(k_1+k_2)}_{2(\ell_1+\ell_2)} \backslash \Pi^{2k_1}_{2\ell_1}$, then for any $\mathsf{X} \in \{\mathsf{Y}, \mathsf{Z}\}$ and any $t \in [\![2]\!]$,

$$\gamma^{2(k_1+k_2)}_{\mathsf{X}(\mathfrak{c}_1 \otimes \mathfrak{c}_2, \mathfrak{d}_1 \otimes \mathfrak{d}_2), 2(\ell_1+\ell_2)} \circ \gamma^{k_1+k_2}_{\mathsf{H}_t, \ell_1+\ell_2} = \gamma^{2k_1+2k_1}_{\mathsf{S}_t, 2\ell_1+2\ell_1} \circ \gamma^{2k_t}_{\mathsf{X}(\mathfrak{c}_t, \mathfrak{d}_t), 2\ell_t}.$$

Indeed, if X stands for $\mathbb{Z}_w$ if $\mathsf{X} = \mathsf{Y}$ and for $\{\aleph\}$ if $\mathsf{X} = \mathsf{Z}$, then $\mathsf{X}(\mathfrak{c}_1 \otimes \mathfrak{c}_2, \mathfrak{d}_1 \otimes \mathfrak{d}_2) = \xi^{F(\mathfrak{c}_1\otimes\mathfrak{c}_2)}_{F(\mathfrak{d}_1\otimes\mathfrak{d}_2)}{}^{\leftarrow}(X) = \xi^{F(\mathfrak{c}_1)\otimes F(\mathfrak{c}_2)}_{F(\mathfrak{d}_1)\otimes\mathfrak{d}_2}{}^{\leftarrow}(X) = \xi^{\mathfrak{a}_1\otimes\mathfrak{a}_2}_{\mathfrak{b}_1\otimes\mathfrak{b}_2}{}^{\leftarrow}(X)$ since F respects the monoidal product on the level of objects by Step 3.1. For that reason and because $\xi^{\mathfrak{a}_1\otimes\mathfrak{a}_2}_{\mathfrak{b}_1\otimes\mathfrak{b}_2} \circ \gamma_{\mathsf{S}_t,2\ell_1+2\ell_1}^{\;2k_1+2k_1} = \xi^{\mathfrak{a}_t}_{\mathfrak{b}_t}$ for each $t \in [\![2]\!]$ by Lemma 4.2 (c) we can infer $\mathsf{X}(\mathfrak{c}_t, \mathfrak{d}_t) = \xi^{\mathfrak{a}_t}_{\mathfrak{b}_t}{}^{\leftarrow}(X) = (\xi^{\mathfrak{a}_1\otimes\mathfrak{a}_2}_{\mathfrak{b}_1\otimes\mathfrak{b}_2} \circ \gamma_{\mathsf{S}_t,2\ell_1+2\ell_1}^{\;2k_1+2k_1})^{\leftarrow}(X) = \gamma_{\mathsf{S}_t,2\ell_1+2\ell_1}^{\;2k_1+2k_1}{}^{\leftarrow}(\xi^{\mathfrak{a}_1\otimes\mathfrak{a}_2}_{\mathfrak{b}_1\otimes\mathfrak{b}_2}{}^{\leftarrow}(X)) = \gamma_{\mathsf{S}_t,2\ell_1+2\ell_1}^{\;2k_1+2k_1}{}^{\leftarrow}(\mathsf{X}(\mathfrak{c}_1 \otimes \mathfrak{c}_2, \mathfrak{d}_1 \otimes \mathfrak{d}_2))$ for each $t \in [\![2]\!]$. Hence, the above identity follows by Lemma 4.5 (c).

Consequently and since by definition, $p_1 \otimes p_2 = \{\gamma_{\mathsf{H}_t,\ell_1+\ell_2}^{\;k_1+k_2}{}_{\rightarrow}(\mathsf{B}_t) \mid t \in [\![2]\!] \wedge \mathsf{B}_t \in p_t\}$,

$$\begin{aligned}
r &= \{\gamma_{\mathsf{X}(\mathfrak{c}_1\otimes\mathfrak{c}_2,\mathfrak{d}_1\otimes\mathfrak{d}_2),2(\ell_1+\ell_2)}^{\;2(k_1+k_2)}{}_{\rightarrow}(\mathsf{A}) \mid \mathsf{X} \in \{\mathsf{Y},\mathsf{Z}\} \wedge \mathsf{A} \in p_1 \otimes p_2\} \\
&= \{(\gamma_{\mathsf{X}(\mathfrak{c}_1\otimes\mathfrak{c}_2,\mathfrak{d}_1\otimes\mathfrak{d}_2),2(\ell_1+\ell_2)}^{\;2(k_1+k_2)} \circ \gamma_{\mathsf{H}_t,\ell_1+\ell_2}^{\;k_1+k_2})_{\rightarrow}(\mathsf{B}_t) \mid \mathsf{X} \in \{\mathsf{Y},\mathsf{Z}\} \wedge t \in [\![2]\!] \wedge \mathsf{B}_t \in p_t\} \\
&= \{(\gamma_{\mathsf{S}_t,2\ell_1+2\ell_1}^{\;2k_1+2k_1} \circ \gamma_{\mathsf{X}(\mathfrak{c}_t,\mathfrak{d}_t),2\ell_t}^{\;2k_t})_{\rightarrow}(\mathsf{B}_t) \mid \mathsf{X} \in \{\mathsf{Y},\mathsf{Z}\} \wedge t \in [\![2]\!] \wedge \mathsf{B}_t \in p_t\} \\
&= \{\gamma_{\mathsf{S}_t,2\ell_1+2\ell_1}^{\;2k_1+2k_1}{}_{\rightarrow}(\mathsf{A}_t) \mid t \in [\![2]\!] \wedge \mathsf{A}_t \in q_t\} \\
&= q_1 \otimes q_2,
\end{aligned}$$

where the last equality is the definition.

(f) Let $\{k, \ell, m\} \subseteq \mathbb{N}_0$ be such that $\mathfrak{c}\colon [\![k]\!] \to \{\circ, \bullet\}$ and $\mathfrak{d}\colon [\![\ell]\!] \to \{\circ, \bullet\}$ and $\mathfrak{e}\colon [\![m]\!] \to \{\circ, \bullet\}$. If $(\mathfrak{f}, \mathfrak{g}, s) = F(\mathfrak{c}, \mathfrak{d}, p)$ and $(\mathfrak{g}, \mathfrak{h}, t) = F(\mathfrak{d}, \mathfrak{e}, q)$ and $(\mathfrak{f}, \mathfrak{h}, r) = F(\mathfrak{c}, \mathfrak{e}, qp)$, then we have to prove that $r = ts$ as well as

$$_n\mathrm{lf}((\mathfrak{d}, \mathfrak{e}, q), (\mathfrak{c}, \mathfrak{d}, p)) = {}_N\mathrm{lf}((\mathfrak{g}, \mathfrak{h}, t), (\mathfrak{f}, \mathfrak{g}, s)).$$

On the one hand, by definition,

$$r = \{\gamma_{\mathsf{Y}(\mathfrak{c},\mathfrak{e}),2m}^{\;2k}{}_{\rightarrow}(\mathsf{D}), \gamma_{\mathsf{Z}(\mathfrak{c},\mathfrak{e}),2m}^{\;2k}{}_{\rightarrow}(\mathsf{D}) \mid \mathsf{D} \in qp\},$$

where, if

$$u = (\kappa_\ell^{0}{}^{\leftarrow}(p|_{\Pi_\ell^0})) \vee (q|_{\Pi_0^\ell}),$$

then

$$\begin{aligned}
qp = \{&\mathsf{A} \in p \wedge \mathsf{A} \subseteq \Pi_0^k\} \\
&\uplus \Big\{\bigcup\{\mathsf{A} \cap \Pi_0^k \mid \mathsf{A} \in p \wedge \mathsf{A} \cap \kappa_{\ell\rightarrow}^0(\mathsf{B}) \neq \emptyset\} \\
&\qquad\qquad \uplus \bigcup\{\mathsf{C} \cap \Pi_m^0 \mid \mathsf{C} \in q \wedge \mathsf{C} \cap \mathsf{B} \neq \emptyset\} \mid \mathsf{B} \in u\Big\} \backslash \{\emptyset\} \\
&\uplus \{\mathsf{C} \in q \wedge \mathsf{C} \subseteq \Pi_m^0\}
\end{aligned}$$

And, on the other hand, if

$$v = (\kappa_{2\ell}^{0}{}^{\leftarrow}(s|_{\Pi_{2\ell}^0})) \vee (t|_{\Pi_0^{2\ell}}),$$

then

$$\begin{aligned}
ts = \{&\mathsf{E} \in s \wedge \mathsf{E} \subseteq \Pi_0^{2k}\} \\
&\uplus \Big\{\bigcup\{\mathsf{E} \cap \Pi_0^{2k} \mid \mathsf{E} \in s \wedge \mathsf{E} \cap \kappa_{2\ell\rightarrow}^0(\mathsf{F}) \neq \emptyset\} \\
&\qquad\qquad \uplus \bigcup\{\mathsf{G} \cap \Pi_{2m}^0 \mid \mathsf{G} \in t \wedge \mathsf{G} \cap \mathsf{F} \neq \emptyset\} \mid \mathsf{F} \in v\Big\} \backslash \{\emptyset\} \\
&\uplus \{\mathsf{G} \in t \wedge \mathsf{G} \subseteq \Pi_m^0\}.
\end{aligned}$$

where

$$s=\{\gamma_{\mathsf{Y}(\mathfrak{c},\mathfrak{d}),2\ell}^{2k\rightarrow}(\mathsf{A}),\gamma_{\mathsf{Z}(\mathfrak{c},\mathfrak{d}),2\ell}^{2k\rightarrow}(\mathsf{A})\,|\,\mathsf{A}\in p\}$$

and

$$t=\{\gamma_{\mathsf{Y}(\mathfrak{d},\mathfrak{e}),2m}^{2\ell\rightarrow}(\mathsf{C}),\gamma_{\mathsf{Z}(\mathfrak{d},\mathfrak{e}),2m}^{2\ell\rightarrow}(\mathsf{C})\,|\,\mathsf{C}\in q\}.$$

We prove the two claims in four steps, beginning with the assertion $r=ts$.

Step 1: Relating v and u. First, we show that

$$v=\{\gamma_{\mathsf{Y}(\mathfrak{d},\varnothing),0}^{2\ell\rightarrow}(\mathsf{B}),\gamma_{\mathsf{Z}(\mathfrak{d},\varnothing),0}^{2\ell\rightarrow}(\mathsf{B})\,|\,\mathsf{B}\in u\}$$

by verifying that the right-hand side has the universal property of v.

Step 1.1: We begin by proving $t|_{\Pi_0^{2\ell}}\leq\{\gamma_{\mathsf{Y}(\mathfrak{d},\varnothing),0}^{2\ell\rightarrow}(\mathsf{B}),\gamma_{\mathsf{Z}(\mathfrak{d},\varnothing),0}^{2\ell\rightarrow}(\mathsf{B})\,|\,\mathsf{B}\in u\}$. Given any $\mathsf{F}\in t|_{\Pi_0^{2\ell}}$, by definition, there exists $\mathsf{G}\in t$ with $\mathsf{F}=\mathsf{G}\cap\Pi_0^{2\ell}$. By the definition of t we then find $\mathsf{C}\in q$ and $\mathsf{X}\in\{\mathsf{Y},\mathsf{Z}\}$ such that $\mathsf{G}=\gamma_{\mathsf{X}(\mathfrak{d},\mathfrak{e}),2m}^{2\ell\rightarrow}(\mathsf{C})$. Since $\gamma_{\mathsf{X}(\mathfrak{d},\mathfrak{e}),2m}^{2\ell}$ is injective and since, trivially, $\gamma_{\mathsf{X}(\mathfrak{d},\mathfrak{e}),2m}^{2\ell\leftarrow}(\Pi_0^{2\ell})=\Pi_0^{\ell}$ it follows $\mathsf{F}=\gamma_{\mathsf{X}(\mathfrak{d},\mathfrak{e}),2m}^{2\ell\rightarrow}(\mathsf{C})\cap\Pi_0^{2\ell}=\gamma_{\mathsf{X}(\mathfrak{d},\mathfrak{e}),2m}^{2\ell\rightarrow}(\mathsf{C}\cap\Pi_0^{\ell})$. Because $\mathsf{F}\neq\varnothing$ we can thus infer $\mathsf{C}\cap\Pi_0^{\ell}\neq\varnothing$, which is to say $\mathsf{C}\cap\Pi_0^{\ell}\in q|_{\Pi_0^{\ell}}$. Hence, by the universal property of u there exists $\mathsf{B}\in u$ such that $\mathsf{C}\cap\Pi_0^{\ell}\subseteq\mathsf{B}$. It follows $\mathsf{F}=\gamma_{\mathsf{X}(\mathfrak{d},\mathfrak{e}),2m}^{2\ell\rightarrow}(\mathsf{C}\cap\Pi_0^{\ell})\subseteq\gamma_{\mathsf{X}(\mathfrak{d},\mathfrak{e}),2m}^{2\ell\rightarrow}(\mathsf{B})$. Because $\mathsf{B}\subseteq\Pi_0^{\ell}$, moreover, $\gamma_{\mathsf{X}(\mathfrak{d},\mathfrak{e}),2m}^{2\ell\rightarrow}(\mathsf{B})=\gamma_{\mathsf{X}(\mathfrak{d},\varnothing),0}^{2\ell\rightarrow}(\mathsf{B})$, as seen initially. Hence, $\mathsf{F}\subseteq\gamma_{\mathsf{X}(\mathfrak{d},\varnothing),0}^{2\ell\rightarrow}(\mathsf{B})$, which is what we needed to see.

Step 1.2: Next, we show $\kappa_{2\ell}^{0\dashleftarrow}(s|_{\Pi_{2\ell}^0})\leq\{\gamma_{\mathsf{Y}(\mathfrak{d},\varnothing),0}^{2\ell\rightarrow}(\mathsf{B}),\gamma_{\mathsf{Z}(\mathfrak{d},\varnothing),0}^{2\ell\rightarrow}(\mathsf{B})\,|\,\mathsf{B}\in u\}$, the proof of which is very similar to the one in Step 1.1. For any $\mathsf{F}\in\kappa_{2\ell}^{0\dashleftarrow}(s|_{\Pi_{2\ell}^0})$ we find, by definition, $\mathsf{E}\in s$ such that $\mathsf{F}=\kappa_{2\ell}^{0\leftarrow}(\mathsf{E}\cap\Pi_{2\ell}^0)$. Hence, by nature of s there exist $\mathsf{A}\in p$ and $\mathsf{X}\in\{\mathsf{Y},\mathsf{Z}\}$ with $\mathsf{E}=\gamma_{\mathsf{X}(\mathfrak{c},\mathfrak{d}),2\ell}^{2k\rightarrow}(\mathsf{A})$. Because $\gamma_{\mathsf{X}(\mathfrak{c},\mathfrak{d}),2\ell}^{2k}$ is injective and because $\gamma_{\mathsf{X}(\mathfrak{c},\mathfrak{d}),2\ell}^{2k\leftarrow}(\Pi_{2\ell}^0)=\Pi_{\ell}^0$ we thus infer $\mathsf{F}=\kappa_{2\ell}^{0\leftarrow}(\gamma_{\mathsf{X}(\mathfrak{c},\mathfrak{d}),2\ell}^{2k\rightarrow}(\mathsf{A})\cap\Pi_{2\ell}^0)=\kappa_{2\ell}^{2k\leftarrow}(\gamma_{\mathsf{X}(\mathfrak{c},\mathfrak{d}),2\ell}^{2k\rightarrow}(\mathsf{A}\cap\Pi_{\ell}^0))=(\kappa_{2k}^{2\ell}\circ\gamma_{\mathsf{X}(\mathfrak{c},\mathfrak{d}),2\ell}^{2k})_{\rightarrow}(\mathsf{A}\cap\Pi_{\ell}^0)$, where we have also used that $\gamma_{\mathsf{X}(\mathfrak{c},\mathfrak{d}),2\ell}^{2k\rightarrow}(\mathsf{A}\cap\Pi_{\ell}^0)\subseteq\Pi_{2\ell}^0$ and that $(\kappa_{2\ell}^{2k})^{-1}=\kappa_{2k}^{2\ell}$. Furthermore, since $\kappa_{2k}^{2\ell}\circ\gamma_{\mathsf{X}(\mathfrak{c},\mathfrak{d}),2\ell}^{2k}=\gamma_{\mathsf{X}(\mathfrak{d},\mathfrak{c}),2k}^{2\ell}\circ\kappa_{k}^{\ell}$, as recognized at the very beginning, we have hence shown $\mathsf{F}=(\gamma_{\mathsf{X}(\mathfrak{d},\mathfrak{c}),2k}^{2\ell}\circ\kappa_{k}^{\ell})_{\rightarrow}(\mathsf{A}\cap\Pi_{\ell}^0)$ or, equivalently, $\mathsf{F}=\gamma_{\mathsf{X}(\mathfrak{d},\varnothing),0}^{2\ell\rightarrow}(\kappa_{\ell}^{0\leftarrow}(\mathsf{A}\cap\Pi_{\ell}^0))$ once we take into account that $(\kappa_{k}^{\ell})^{-1}=\kappa_{\ell}^{k}$, that $\mathsf{A}\cap\Pi_{\ell}^0\subseteq\Pi_{\ell}^0$ and that $\kappa_{\ell}^{0\leftarrow}(\mathsf{A}\cap\Pi_{\ell}^0)\subseteq\Pi_0^{\ell}$ and thus $\gamma_{\mathsf{X}(\mathfrak{d},\mathfrak{c}),2k}^{2\ell\rightarrow}(\kappa_{\ell}^{0\leftarrow}(\mathsf{A}\cap\Pi_{\ell}^0))=\gamma_{\mathsf{X}(\mathfrak{d},\varnothing),0}^{2\ell\rightarrow}(\kappa_{\ell}^{0\leftarrow}(\mathsf{A}\cap\Pi_{\ell}^0))$ by what we saw initially. Since $\varnothing\neq\mathsf{F}=\gamma_{\mathsf{X}(\mathfrak{d},\varnothing),0}^{2\ell\rightarrow}(\kappa_{\ell}^{0\leftarrow}(\mathsf{A}\cap\Pi_{\ell}^0))$, in particular, $\mathsf{A}\cap\Pi_{\ell}^0\neq\varnothing$. Consequently, $\mathsf{A}\cap\Pi_{\ell}^0\in p|_{\Pi_{\ell}^0}$ and thus $\kappa_{\ell}^{0\leftarrow}(\mathsf{A}\cap\Pi_{\ell}^0)\in\kappa_{\ell}^{0\dashleftarrow}(p|_{\Pi_{\ell}^0})$. The definition of u therefore implies the existence of $\mathsf{B}\in u$ with $\kappa_{\ell}^{0\leftarrow}(\mathsf{A}\cap\Pi_{\ell}^0)\subseteq\mathsf{B}$. We conclude $\gamma_{\mathsf{X}(\mathfrak{d},\varnothing),0}^{2\ell\rightarrow}(\kappa_{\ell}^{0\leftarrow}(\mathsf{A}\cap\Pi_{\ell}^0))\subseteq\gamma_{\mathsf{X}(\mathfrak{d},\varnothing),0}^{2\ell\rightarrow}(\mathsf{B})$. And that is what we had to show.

Step 1.3: Finally, we let w be any set-theoretical partition of $\Pi_0^{2\ell}$ satisfying $\kappa_{2\ell}^{0\dashleftarrow}(s|_{\Pi_{2\ell}^0})\leq w$ and $t|_{\Pi_0^{2\ell}}\leq w$ and prove $\{\gamma_{\mathsf{Y}(\mathfrak{d},\varnothing),0}^{2\ell\rightarrow}(\mathsf{B}),\gamma_{\mathsf{Z}(\mathfrak{d},\varnothing),0}^{2\ell\rightarrow}(\mathsf{B})\,|\,\mathsf{B}\in u\}\leq w$. Hence, let $\mathsf{X}\in\{\mathsf{Y},\mathsf{Z}\}$ and $\mathsf{B}\in u$ be arbitrary. We have to find $\mathsf{F}\in w$ with $\gamma_{\mathsf{X}(\mathfrak{d},\varnothing),0}^{2\ell\rightarrow}(\mathsf{B})\subseteq\mathsf{F}$. That requires three steps in itself.

Step 1.3.1: First, we prove the auxiliary result that ${\gamma_{\mathsf{X}(\mathfrak{d},\varnothing),0}}^{2\ell\dashleftarrow}(t|_{\Pi_0^{2\ell}}) = q|_{\Pi_0^\ell}$. Indeed, by definition, any $\mathsf{B}' \subseteq \Pi_0^\ell$ is an element of ${\gamma_{\mathsf{X}(\mathfrak{d},\varnothing),0}}^{2\ell\dashleftarrow}(t|_{\Pi_0^{2\ell}})$ if and only if there exists $\mathsf{C} \in q$ with both $\mathsf{C}\cap\Pi_0^\ell \neq \varnothing$ and with either both ${\gamma_{\mathsf{Y}(\mathfrak{d},\mathfrak{e}),2m}}^{2\ell}{}_{\rightarrow}(\mathsf{C}\cap\Pi_0^\ell)\cap\mathsf{X}(\mathfrak{d},\varnothing) \neq \varnothing$ and $\mathsf{B}' = {\gamma_{\mathsf{X}(\mathfrak{d},\varnothing),0}}^{2\ell\leftarrow}({\gamma_{\mathsf{Y}(\mathfrak{d},\mathfrak{e}),2m}}^{2\ell}{}_{\rightarrow}(\mathsf{C}\cap\Pi_0^\ell))$ or both ${\gamma_{\mathsf{Z}(\mathfrak{d},\mathfrak{e}),2m}}^{2\ell}{}_{\rightarrow}(\mathsf{C}\cap\Pi_0^\ell)\cap\mathsf{X}(\mathfrak{d},\varnothing) \neq \varnothing$ and $\mathsf{B}' = {\gamma_{\mathsf{X}(\mathfrak{d},\varnothing),0}}^{2\ell\leftarrow}({\gamma_{\mathsf{Z}(\mathfrak{d},\mathfrak{e}),2m}}^{2\ell}{}_{\rightarrow}(\mathsf{C}\cap\Pi_0^\ell))$. Because $\mathsf{X}(\mathfrak{d},\varnothing) \subseteq \mathsf{X}(\mathfrak{d},\mathfrak{e})$ and $\mathsf{Y}(\mathfrak{d},\mathfrak{e}) \cap \mathsf{Z}(\mathfrak{d},\mathfrak{e}) = \varnothing$ that is equivalent to there existing $\mathsf{C} \in q$ with both $\mathsf{C}\cap\Pi_0^\ell \neq \varnothing$ and $\mathsf{B}' = {\gamma_{\mathsf{X}(\mathfrak{d},\varnothing),0}}^{2\ell\leftarrow}({\gamma_{\mathsf{X}(\mathfrak{d},\mathfrak{e}),2m}}^{2\ell}{}_{\rightarrow}(\mathsf{C}\cap\Pi_0^\ell))$. And because the injectivity of ${\gamma_{\mathsf{X}(\mathfrak{d},\varnothing),0}}^{2\ell}$ of course implies ${\gamma_{\mathsf{X}(\mathfrak{d},\varnothing),0}}^{2\ell\leftarrow}({\gamma_{\mathsf{X}(\mathfrak{d},\mathfrak{e}),2m}}^{2\ell}{}_{\rightarrow}(\mathsf{C}\cap\Pi_0^\ell))$ for any $\mathsf{C} \subseteq \Pi_m^\ell$ that proves the assertion.

Step 1.3.2: We will also need to know that ${\gamma_{\mathsf{X}(\mathfrak{d},\varnothing),0}}^{2\ell\dashleftarrow}({\kappa_{2\ell}^0}^{\dashleftarrow}(s|_{\Pi_{2\ell}^0})) = {\kappa_\ell^0}^{\dashleftarrow}(p|_{\Pi_\ell^0})$. According to the observation at the very beginning of the proof, $\kappa_{2\ell}^0 \circ {\gamma_{\mathsf{X}(\mathfrak{d},\varnothing),0}}^{2\ell} = {\gamma_{\mathsf{X}(\varnothing,\mathfrak{d}),2\ell}}^{0} \circ \kappa_\ell^0$ and thus ${\gamma_{\mathsf{X}(\mathfrak{d},\varnothing),0}}^{2\ell\dashleftarrow} \circ {\kappa_{2\ell}^0}^{\dashleftarrow} = (\kappa_{2\ell}^0 \circ {\gamma_{\mathsf{X}(\mathfrak{d},\varnothing),0}}^{2\ell})^{\dashleftarrow} = ({\gamma_{\mathsf{X}(\varnothing,\mathfrak{d}),2\ell}}^{0} \circ \kappa_\ell^0)^{\dashleftarrow} = {\kappa_\ell^0}^{\dashleftarrow} \circ {\gamma_{\mathsf{X}(\varnothing,\mathfrak{d}),2\ell}}^{0\dashleftarrow}$. Hence, it suffices to show ${\gamma_{\mathsf{X}(\varnothing,\mathfrak{d}),2\ell}}^{0\dashleftarrow}(s|_{\Pi_{2\ell}^0}) = p|_{\Pi_\ell^0}$. And the proof of this is similar to the one in Step 1.3.1: Any $\mathsf{B}' \subseteq \Pi_\ell^0$ belongs to ${\gamma_{\mathsf{X}(\varnothing,\mathfrak{d}),2\ell}}^{0\dashleftarrow}(s|_{\Pi_{2\ell}^0})$ if and only if there is $\mathsf{A} \in p$ with $\mathsf{A}\cap\Pi_\ell^0 \neq \varnothing$ and with either both ${\gamma_{\mathsf{Y}(\mathfrak{c},\mathfrak{d}),2\ell}}^{2k}{}_{\rightarrow}(\mathsf{A}\cap\Pi_\ell^0)\cap\mathsf{X}(\varnothing,\mathfrak{d}) \neq \varnothing$ and $\mathsf{B}' = {\gamma_{\mathsf{X}(\varnothing,\mathfrak{d}),2\ell}}^{0\leftarrow}({\gamma_{\mathsf{Y}(\mathfrak{c},\mathfrak{d}),2\ell}}^{2k}{}_{\rightarrow}(\mathsf{A}\cap\Pi_\ell^0))$ or both ${\gamma_{\mathsf{Z}(\mathfrak{c},\mathfrak{d}),2\ell}}^{2k}{}_{\rightarrow}(\mathsf{A}\cap\Pi_\ell^0)\cap\mathsf{X}(\varnothing,\mathfrak{d}) \neq \varnothing$ and $\mathsf{B}' = {\gamma_{\mathsf{X}(\varnothing,\mathfrak{d}),2\ell}}^{0\leftarrow}({\gamma_{\mathsf{Z}(\mathfrak{c},\mathfrak{d}),2\ell}}^{2k}{}_{\rightarrow}(\mathsf{A}\cap\Pi_\ell^0))$. Since $\mathsf{X}(\varnothing,\mathfrak{d}) \subseteq \mathsf{X}(\mathfrak{c},\mathfrak{d})$ and $\mathsf{Y}(\mathfrak{c},\mathfrak{d}) \cap \mathsf{Z}(\mathfrak{c},\mathfrak{d}) = \varnothing$ that is true if and only if there exists $\mathsf{A} \in p$ with both $\mathsf{A}\cap\Pi_\ell^0 \neq \varnothing$ and $\mathsf{B}' = {\gamma_{\mathsf{X}(\varnothing,\mathfrak{d}),2\ell}}^{0\leftarrow}({\gamma_{\mathsf{X}(\mathfrak{c},\mathfrak{d}),2\ell}}^{2k}{}_{\rightarrow}(\mathsf{C}\cap\Pi_\ell^0))$, which is to say $\mathsf{B}' \in p|_{\Pi_\ell^0}$.

Step 1.3.3: With the results of Steps 1.3.1 and 4.1.3.2 at hand, we can now construct F. Namely, they allow us to conlude from the assumptions ${\kappa_{2\ell}^0}^{\dashleftarrow}(s|_{\Pi_{2\ell}^0}) \leq w$ and $t|_{\Pi_0^{2\ell}} \leq w$ that ${\kappa_\ell^0}^{\dashleftarrow}(p|_{\Pi_\ell^0}) \leq {\gamma_{\mathsf{X}(\mathfrak{d},\varnothing),0}}^{2\ell\dashleftarrow}(w)$ and $q|_{\Pi_0^\ell} \leq {\gamma_{\mathsf{X}(\mathfrak{d},\varnothing),0}}^{2\ell\dashleftarrow}(w)$. Hence, $u \leq {\gamma_{\mathsf{X}(\mathfrak{d},\varnothing),0}}^{2\ell\dashleftarrow}(w)$ by the universal property of u, which implies that there is $\mathsf{F} \in w$ with $\mathsf{B} \subseteq {\gamma_{\mathsf{X}(\mathfrak{d},\varnothing),0}}^{2\ell\leftarrow}(\mathsf{F})$. Because, trivially, ${\gamma_{\mathsf{X}(\mathfrak{d},\varnothing),0}}^{2\ell}{}_{\rightarrow}({\gamma_{\mathsf{X}(\mathfrak{d},\varnothing),0}}^{2\ell\leftarrow}(\mathsf{F})) \subseteq \mathsf{F}$ it follows that F has the desired property ${\gamma_{\mathsf{X}(\mathfrak{d},\varnothing),0}}^{2\ell}{}_{\rightarrow}(\mathsf{B}) \subseteq \mathsf{F}$.

Step 2: Conclusions of the relationship between u and v. As the second preparatory step to proving that r and ts coincide we show that for any $\mathsf{B} \in u$ and $\mathsf{X} \in \{\mathsf{Y},\mathsf{Z}\}$ and $\mathsf{F} \in v$, if $\mathsf{F} = {\gamma_{\mathsf{X}(\mathfrak{d},\varnothing),0}}^{2\ell}{}_{\rightarrow}(\mathsf{B})$, then

$$\begin{aligned}&\bigcup\{\mathsf{E}\cap\Pi_0^{2k} \,|\, \mathsf{E}\in s \wedge \mathsf{E}\cap\kappa^0_{2\ell\rightarrow}(\mathsf{F}) \neq \varnothing\}\\ &\qquad\qquad \uplus \bigcup\{\mathsf{G}\cap\Pi_{2m}^0 \,|\, \mathsf{G}\in t \wedge \mathsf{G}\cap\mathsf{F} \neq \varnothing\}\\ &\qquad\qquad\qquad = {\gamma_{\mathsf{X}(\mathfrak{c},\mathfrak{e}),2m}}^{2k}{}_{\rightarrow}\Big(\bigcup\{\mathsf{A}\cap\Pi_0^k \,|\, \mathsf{A}\in p \wedge \mathsf{A}\cap\kappa^0_{\ell\rightarrow}(\mathsf{B}) \neq \varnothing\}\\ &\qquad\qquad\qquad\qquad\qquad \uplus \bigcup\{\mathsf{C}\cap\Pi_m^0 \,|\, \mathsf{C}\in q \wedge \mathsf{C}\cap\mathsf{B} \neq \varnothing\}\Big).\end{aligned}$$

To that end it suffices to show that already,

$$\begin{aligned}&\{\mathsf{E}\cap\Pi_0^{2k} \,|\, \mathsf{E}\in s \wedge \mathsf{E}\cap\kappa^0_{2\ell\rightarrow}(\mathsf{F}) \neq \varnothing\}\\ &\qquad\qquad = \{{\gamma_{\mathsf{X}(\mathfrak{c},\mathfrak{e}),2m}}^{2k}{}_{\rightarrow}(\mathsf{A}\cap\Pi_0^k) \,|\, \mathsf{A}\in p \wedge \mathsf{A}\cap\kappa^0_{\ell\rightarrow}(\mathsf{B}) \neq \varnothing\}\end{aligned}$$

and

$$\{\mathsf{G}\cap\Pi^0_{2m}\,|\,\mathsf{G}\in t\wedge\mathsf{G}\cap\mathsf{F}\neq\varnothing\}=\{\gamma_{\mathsf{X}(\mathfrak{c},\mathfrak{e}),2m}{}^{2k}{}_{\to}(\mathsf{C}\cap\Pi^0_m)\,|\,\mathsf{C}\in q\wedge\mathsf{C}\cap\mathsf{B}\neq\varnothing\}.$$

Step 2.1: As an intermediate step we first note $\kappa^0_{2\ell\to}(\mathsf{F})=\gamma_{\mathsf{X}(\mathfrak{c},\mathfrak{d}),2\ell}{}^{2k}{}_{\to}(\kappa^0_{\ell\to}(\mathsf{B}))$ and $\mathsf{F}=\gamma_{\mathsf{X}(\mathfrak{d},\mathfrak{e}),2m}{}^{2\ell}{}_{\to}(\mathsf{B})$. Indeed, because $\mathsf{B}\subseteq\Pi^\ell_0$, by the remark at the very beginning of the proof, $\mathsf{F}=\gamma_{\mathsf{X}(\mathfrak{d},\varnothing),0}{}^{2\ell}{}_{\to}(\mathsf{B})=\gamma_{\mathsf{X}(\mathfrak{d},\mathfrak{e}),2m}{}^{2\ell}{}_{\to}(\mathsf{B})$. And, per assumption the set $\kappa^0_{2\ell\to}(\mathsf{F})$ is given by $(\kappa^0_{2\ell}\circ\gamma_{\mathsf{X}(\mathfrak{d},\varnothing),0}{}^{2\ell})_{\to}(\mathsf{B})$, which can also be written as $(\gamma_{\mathsf{X}(\varnothing,\mathfrak{d}),2\ell}{}^{0}\circ\kappa^0_\ell)_{\to}(\mathsf{B})$ since $\kappa^0_{2\ell}\circ\gamma_{\mathsf{X}(\mathfrak{d},\varnothing),0}{}^{2\ell}=\gamma_{\mathsf{X}(\varnothing,\mathfrak{d}),2\ell}{}^{0}\circ\kappa^0_\ell$ as seen at the beginning. In fact, since $\mathsf{B}\subseteq\Pi^\ell_0$ and thus $\kappa^0_{\ell\to}(\mathsf{B})\subseteq\Pi^0_\ell$ the other part of the initial observation now tells us that $\kappa^0_{2\ell\to}(\mathsf{F})$ is identical to $\gamma_{\mathsf{X}(\mathfrak{c},\mathfrak{d}),2\ell}{}^{2k}{}_{\to}(\kappa^0_{\ell\to}(\mathsf{B}))$.

Step 2.2: The second intermediate step consists in recognizing that the first one implies

$$\begin{aligned}&\{\mathsf{E}\cap\Pi^{2k}_0\,|\,\mathsf{E}\in s\wedge\mathsf{E}\cap\kappa^0_{2\ell\to}(\mathsf{F})\neq\varnothing\}\\&\quad=\{\gamma_{\mathsf{X}(\mathfrak{c},\mathfrak{d}),2\ell}{}^{2k}{}_{\to}(\mathsf{A})\cap\Pi^{2k}_0\,|\,\mathsf{A}\in p\wedge\gamma_{\mathsf{X}(\mathfrak{c},\mathfrak{d}),2\ell}{}^{2k}{}_{\to}(\mathsf{A})\cap\gamma_{\mathsf{X}(\mathfrak{c},\mathfrak{d}),2\ell}{}^{2k}{}_{\to}(\kappa^0_{\ell\to}(\mathsf{B}))\neq\varnothing\}\end{aligned}$$

and

$$\begin{aligned}&\{\mathsf{G}\cap\Pi^0_{2m}\,|\,\mathsf{G}\in t\wedge\mathsf{G}\cap\mathsf{F}\neq\varnothing\}\\&\quad=\{\gamma_{\mathsf{X}(\mathfrak{d},\mathfrak{e}),2m}{}^{2\ell}{}_{\to}(\mathsf{C})\cap\Pi^0_{2m}\,|\,\mathsf{C}\in q\wedge\gamma_{\mathsf{X}(\mathfrak{d},\mathfrak{e}),2m}{}^{2\ell}{}_{\to}(\mathsf{C})\cap\gamma_{\mathsf{X}(\mathfrak{d},\mathfrak{e}),2m}{}^{2\ell}{}_{\to}(\mathsf{B})\neq\varnothing\}.\end{aligned}$$

In the case of the second identity that is for the following reasons. By definition any G is an element of t if and only if there exists $\mathsf{C}\in q$ such that $\mathsf{G}=\gamma_{\mathsf{Y}(\mathfrak{d},\mathfrak{e}),2m}{}^{2\ell}{}_{\to}(\mathsf{C})$ or $\mathsf{G}=\gamma_{\mathsf{Z}(\mathfrak{d},\mathfrak{e}),2m}{}^{2\ell}{}_{\to}(\mathsf{C})$. However, since $\mathsf{F}=\gamma_{\mathsf{X}(\mathfrak{d},\mathfrak{e}),2m}{}^{2\ell}{}_{\to}(\mathsf{B})\subseteq\mathsf{X}(\mathfrak{d},\mathfrak{e})$ by Step 2.1 and since $\mathsf{Y}(\mathfrak{d},\mathfrak{e})\cap\mathsf{Z}(\mathfrak{d},\mathfrak{e})=\varnothing$, whenever such a G satisfies $\mathsf{G}\cap\mathsf{F}\neq\varnothing$, it must already be of the form $\gamma_{\mathsf{X}(\mathfrak{d},\mathfrak{e}),2m}{}^{2\ell}{}_{\to}(\mathsf{C})$.

Similarly, any E belongs to s if and only if there is $\mathsf{A}\in p$ with $\mathsf{E}=\gamma_{\mathsf{Y}(\mathfrak{c},\mathfrak{d}),2\ell}{}^{2k}{}_{\to}(\mathsf{A})$ or $\mathsf{E}=\gamma_{\mathsf{Z}(\mathfrak{c},\mathfrak{d}),2\ell}{}^{2k}{}_{\to}(\mathsf{A})$, which narrows to just $\mathsf{E}=\gamma_{\mathsf{X}(\mathfrak{c},\mathfrak{d}),2\ell}{}^{2k}{}_{\to}(\mathsf{A})$ as soon as $\mathsf{E}\cap\kappa^0_{2\ell\to}(\mathsf{F})\neq\varnothing$ because $\kappa^0_{2\ell\to}(\mathsf{F})=\gamma_{\mathsf{X}(\mathfrak{c},\mathfrak{d}),2\ell}{}^{2k}{}_{\to}(\kappa^0_{\ell\to}(\mathsf{B}))\subseteq\mathsf{X}(\mathfrak{c},\mathfrak{d})$ by Step 2.1 and because $\mathsf{Y}(\mathfrak{c},\mathfrak{d})\cap\mathsf{Z}(\mathfrak{c},\mathfrak{d})=\varnothing$.

Step 2.3: Now, we can conclude from the preceding step that

$$\begin{aligned}&\{\mathsf{E}\cap\Pi^{2k}_0\,|\,\mathsf{E}\in s\wedge\mathsf{E}\cap\kappa^0_{2\ell\to}(\mathsf{F})\neq\varnothing\}\\&\quad=\{\gamma_{\mathsf{X}(\mathfrak{c},\mathfrak{d}),2\ell}{}^{2k}{}_{\to}(\mathsf{A})\cap\Pi^{2k}_0\,|\,\mathsf{A}\in p\wedge\mathsf{A}\cap\kappa^0_{\ell\to}(\mathsf{B})\neq\varnothing\}\end{aligned}$$

and

$$\{\mathsf{G}\cap\Pi^0_{2m}\,|\,\mathsf{G}\in t\wedge\mathsf{G}\cap\mathsf{F}\neq\varnothing\}=\{\gamma_{\mathsf{X}(\mathfrak{d},\mathfrak{e}),2m}{}^{2\ell}{}_{\to}(\mathsf{C})\cap\Pi^0_{2m}\,|\,\mathsf{C}\in q\wedge\mathsf{C}\cap\mathsf{B}\neq\varnothing\}.$$

Indeed, the injectivity of $\gamma_{\mathsf{X}(\mathfrak{c},\mathfrak{e}),2\ell}{}^{2k}$ ensures that $\gamma_{\mathsf{X}(\mathfrak{d},\mathfrak{e}),2m}{}^{2\ell}{}_{\to}(\mathsf{C})\cap\gamma_{\mathsf{X}(\mathfrak{d},\mathfrak{e}),2m}{}^{2\ell}{}_{\to}(\mathsf{B})$ and $\gamma_{\mathsf{X}(\mathfrak{d},\mathfrak{e}),2m}{}^{2\ell}{}_{\to}(\mathsf{C}\cap\mathsf{B})$ coincide for any $\mathsf{C}\in q$. And, of course, $\gamma_{\mathsf{X}(\mathfrak{d},\mathfrak{e}),2m}{}^{2\ell}{}_{\to}(\mathsf{C}\cap\mathsf{B})\neq\varnothing$ if and only if $\mathsf{C}\cap\mathsf{B}\neq\varnothing$. Likewise, the sets $\gamma_{\mathsf{X}(\mathfrak{c},\mathfrak{d}),2\ell}{}^{2k}{}_{\to}(\mathsf{A})\cap\gamma_{\mathsf{X}(\mathfrak{c},\mathfrak{d}),2\ell}{}^{2k}{}_{\to}(\kappa^0_{\ell\to}(\mathsf{B}))$ and $\gamma_{\mathsf{X}(\mathfrak{c},\mathfrak{d}),2\ell}{}^{2k}{}_{\to}(\mathsf{A}\cap\kappa^0_{\ell\to}(\mathsf{B}))$ are the same for any $\mathsf{A}\in p$ because $\gamma_{\mathsf{X}(\mathfrak{d},\mathfrak{e}),2m}{}^{2\ell}$ is injective, whence $\gamma_{\mathsf{X}(\mathfrak{c},\mathfrak{d}),2\ell}{}^{2k}{}_{\to}(\mathsf{A})\cap\gamma_{\mathsf{X}(\mathfrak{c},\mathfrak{d}),2\ell}{}^{2k}{}_{\to}(\kappa^0_{\ell\to}(\mathsf{B}))\neq\varnothing$ if and only if $\mathsf{A}\cap\kappa^0_{\ell\to}(\mathsf{B})\neq\varnothing$.

Step 2.4: Given the results of Step 2.3, the assertion is certainly true if $\gamma_{\mathsf{X}(\mathfrak{c},\mathfrak{d}),2\ell}^{2k\rightarrow}(\mathsf{A})\cap\Pi_0^{2k}=\gamma_{\mathsf{X}(\mathfrak{c},\mathfrak{e}),2m}^{2k\rightarrow}(\mathsf{A}\cap\Pi_0^k)$ for any $\mathsf{A}\in p$ and $\gamma_{\mathsf{X}(\mathfrak{d},\mathfrak{e}),2m}^{2\ell\rightarrow}(\mathsf{C})\cap\Pi_{2m}^0=\gamma_{\mathsf{X}(\mathfrak{c},\mathfrak{e}),2m}^{2k\rightarrow}(\mathsf{C}\cap\Pi_m^0)$ for any $\mathsf{C}\in q$.

Because, trivially, $\gamma_{\mathsf{X}(\mathfrak{d},\mathfrak{e}),2m}^{2\ell\rightarrow}(\Pi_m^0)=\mathsf{X}(\mathfrak{d},\mathfrak{e})\cap\Pi_{2m}^0$ the facts that $\gamma_{\mathsf{X}(\mathfrak{d},\mathfrak{e}),2m}^{2\ell}$ is injective and that $\gamma_{\mathsf{X}(\mathfrak{d},\mathfrak{e}),2m}^{2\ell\rightarrow}(\mathsf{C})\subseteq\mathsf{X}(\mathfrak{d},\mathfrak{e})$ allow us to conclude that $\gamma_{\mathsf{X}(\mathfrak{d},\mathfrak{e}),2m}^{2\ell\rightarrow}(\mathsf{C})\cap\Pi_{2m}^0=\gamma_{\mathsf{X}(\mathfrak{d},\mathfrak{e}),2m}^{2\ell\rightarrow}(\mathsf{C}\cap\Pi_m^0)$. And because $\mathsf{C}\cap\Pi_m^0\subseteq\Pi_m^0$ the remark made at the very beginning of the proof implies $\gamma_{\mathsf{X}(\mathfrak{d},\mathfrak{e}),2m}^{2\ell\rightarrow}(\mathsf{C}\cap\Pi_m^0)=\gamma_{\mathsf{X}(\varnothing,\mathfrak{e}),2m}^{0\rightarrow}(\mathsf{C}\cap\Pi_m^0)=\gamma_{\mathsf{X}(\mathfrak{c},\mathfrak{e}),2m}^{2k\rightarrow}(\mathsf{C}\cap\Pi_m^0)$. Analogously, $\gamma_{\mathsf{X}(\mathfrak{c},\mathfrak{d}),2\ell}^{2k\rightarrow}(\mathsf{A})\cap\Pi_0^{2k}=\gamma_{\mathsf{X}(\mathfrak{c},\mathfrak{d}),2\ell}^{2k\rightarrow}(\mathsf{A}\cap\Pi_0^k)=\gamma_{\mathsf{X}(\mathfrak{c},\varnothing),0}^{2k\rightarrow}(\mathsf{A}\cap\Pi_0^k)=\gamma_{\mathsf{X}(\mathfrak{c},\mathfrak{e}),2m}^{2k\rightarrow}(\mathsf{A}\cap\Pi_0^k)$.

Step 3: Showing $r=ts$. With the help of the results of Steps 1 and 2 we can now verify $r\subseteq ts$ and $r\supseteq ts$.

Step 3.1: Given any $\mathsf{H}\in r$, there exist $\mathsf{X}\in\{\mathsf{Y},\mathsf{Z}\}$ and $\mathsf{D}\in qp$ such that $\mathsf{H}=\gamma_{\mathsf{X}(\mathfrak{c},\mathfrak{e}),2m}^{2k\rightarrow}(\mathsf{D})$. According to the definition of qp there are now three cases to consider.

Case 3.1.1: If there exists $\mathsf{A}\in p$ with $\mathsf{A}\subseteq\Pi_0^k$ and $\mathsf{D}=\mathsf{A}$, then $\mathsf{H}=\gamma_{\mathsf{X}(\mathfrak{c},\mathfrak{e}),2m}^{2k\rightarrow}(\mathsf{A})=\gamma_{\mathsf{X}(\mathfrak{c},\varnothing),0}^{2k\rightarrow}(\mathsf{A})=\gamma_{\mathsf{X}(\mathfrak{c},\mathfrak{d}),2\ell}^{2k\rightarrow}(\mathsf{A})$ by two subsequent application of the auxiliary statement at the beginning of the proof. Hence, actually, $\mathsf{H}\in s$ by definition of s. Because $\mathsf{A}\subseteq\Pi_0^k$ also ensures $\mathsf{H}=\gamma_{\mathsf{X}(\mathfrak{c},\mathfrak{d}),2\ell}^{2k\rightarrow}(\mathsf{A})\subseteq\Pi_0^{2k}$ we conclude $\mathsf{H}\in ts$ by definition of ts.

Case 3.1.2: Similarly, if there is $\mathsf{C}\in q$ with $\mathsf{C}\subseteq\Pi_m^0$ and $\mathsf{D}=\mathsf{C}$, then $\mathsf{H}=\gamma_{\mathsf{X}(\mathfrak{c},\mathfrak{e}),2m}^{2k\rightarrow}(\mathsf{C})=\gamma_{\mathsf{X}(\varnothing,\mathfrak{e}),2m}^{0\rightarrow}(\mathsf{C})=\gamma_{\mathsf{X}(\mathfrak{d},\mathfrak{e}),2m}^{2\ell\rightarrow}(\mathsf{C})\in t$ and thus $\mathsf{H}\in qp$ because $\mathsf{H}\subseteq\Pi_{2m}^0$.

Case 3.1.3: The only remaining possibility is that there exists $\mathsf{B}\in u$ such that $\mathsf{D}=\bigcup\{\mathsf{A}\cap\Pi_0^k\,|\,\mathsf{A}\in p\wedge\mathsf{A}\cap\kappa_{\ell\rightarrow}^0(\mathsf{B})\neq\varnothing\}\uplus\bigcup\{\mathsf{C}\cap\Pi_m^0\,|\,\mathsf{C}\in q\wedge\mathsf{C}\cap\mathsf{B}\neq\varnothing\}$. If so, then $\gamma_{\mathsf{X}(\mathfrak{d},\varnothing),0}^{2\ell\rightarrow}(\mathsf{B})\in v$ by Step 1 and thus $\mathsf{H}=\gamma_{\mathsf{X}(\mathfrak{c},\mathfrak{e}),2m}^{2k\rightarrow}(\mathsf{D})\in ts$ by Step 2. Hence, indeed, $r\subseteq ts$.

Step 3.2: For any $\mathsf{H}\in ts$ the definition of ts requires us to distinguish three cases as well if we want to prove $\mathsf{H}\in ts$.

Case 3.2.1: If there exists $\mathsf{E}\in s$ with $\mathsf{E}\subseteq\Pi_0^{2k}$ and $\mathsf{H}=\mathsf{E}$, then by definition of s there are $\mathsf{X}\in\{\mathsf{Y},\mathsf{Z}\}$ and $\mathsf{A}\in p$ such that $\mathsf{H}=\gamma_{\mathsf{X}(\mathfrak{c},\mathfrak{d}),2\ell}^{2k\rightarrow}(\mathsf{A})$. From $\mathsf{E}\subseteq\Pi_0^{2k}$ it follows that $\mathsf{A}\subseteq\gamma_{\mathsf{X}(\mathfrak{c},\mathfrak{d}),2\ell}^{2k\leftarrow}(\mathsf{H})\subseteq\gamma_{\mathsf{X}(\mathfrak{c},\mathfrak{d}),2\ell}^{2k\rightarrow}(\Pi_0^{2k})=\Pi_0^k$ and thus $\mathsf{A}\in qp$ by definition of qp. Another consequence of $\mathsf{A}\subseteq\Pi_0^k$ is that $\mathsf{H}=\gamma_{\mathsf{X}(\mathfrak{c},\mathfrak{d}),2\ell}^{2k\rightarrow}(\mathsf{A})=\gamma_{\mathsf{X}(\mathfrak{c},\varnothing),0}^{2k\rightarrow}(\mathsf{A})=\gamma_{\mathsf{X}(\mathfrak{c},\mathfrak{e}),2m}^{2k\rightarrow}(\mathsf{A})$ by the auxiliary statement at the beginning of the proof. Hence, by definition, $\mathsf{H}\in r$.

Case 3.2.2: Analogously, in the case of there being $\mathsf{G}\in t$ with $\mathsf{H}=\mathsf{G}\subseteq\Pi_{2m}^0$, we find $\mathsf{X}\in\{\mathsf{Y},\mathsf{Z}\}$ and $\mathsf{C}\in q$ such that $\mathsf{G}=\gamma_{\mathsf{X}(\mathfrak{d},\mathfrak{e}),2m}^{2\ell\rightarrow}(\mathsf{C})$. Then, $\mathsf{G}\subseteq\Pi_{2m}^0$ demands, on the one hand, $\mathsf{C}\subseteq\gamma_{\mathsf{X}(\mathfrak{d},\mathfrak{e}),2m}^{2\ell\leftarrow}(\mathsf{H})\subseteq\gamma_{\mathsf{X}(\mathfrak{d},\mathfrak{e}),2m}^{2\ell\rightarrow}(\Pi_{2m}^0)=\Pi_m^0$, ensuring $\mathsf{C}\in qp$, and, on the other hand, $\mathsf{G}=\gamma_{\mathsf{X}(\mathfrak{d},\mathfrak{e}),2m}^{2\ell\rightarrow}(\mathsf{C})=\gamma_{\mathsf{X}(\varnothing,\mathfrak{e}),2m}^{0\rightarrow}(\mathsf{C})=\gamma_{\mathsf{X}(\mathfrak{c},\mathfrak{e}),2m}^{2k\rightarrow}(\mathsf{C})$ and thus $\mathsf{G}\in r$.

Case 3.2.3: Lastly, there can exist $\mathsf{F}\in v$ such that $\mathsf{H}=\bigcup\{\mathsf{E}\cap\Pi_0^{2k}\,|\,\mathsf{E}\in s\wedge\mathsf{E}\cap\kappa_{2\ell\rightarrow}^0(\mathsf{F})\neq\varnothing\}\uplus\bigcup\{\mathsf{G}\cap\Pi_{2m}^0\,|\,\mathsf{G}\in t\wedge\mathsf{G}\cap\mathsf{F}\neq\varnothing\}$. According to Step 1 we then find $\mathsf{X}\in\{\mathsf{Y},\mathsf{Z}\}$ and $\mathsf{B}\in u$ such that $\mathsf{F}=\gamma_{\mathsf{X}(\mathfrak{d},\varnothing),0}^{2\ell\rightarrow}(\mathsf{B})$. In consequence, if $\mathsf{D}=\bigcup\{\mathsf{A}\cap\Pi_0^k\,|\,\mathsf{A}\in p\wedge\mathsf{A}\cap\kappa_{\ell\rightarrow}^0(\mathsf{B})\neq\varnothing\}\uplus\bigcup\{\mathsf{C}\cap\Pi_m^0\,|\,\mathsf{C}\in q\wedge\mathsf{C}\cap\mathsf{B}\neq\varnothing\}$, then $\mathsf{D}\in qp$ by definition of qp

and $\mathsf{H} = \gamma_{\mathsf{X}(\mathfrak{c},\mathfrak{e}),2m\rightarrow}^{2k}(\mathsf{D})$ by Step 2. It follows $\mathsf{H} \in r$ by definition of r, which concludes the proof of $r = ts$.

Step 4: It remains to prove

$$_{n}\mathrm{lf}((\mathfrak{d},\mathfrak{e},q),(\mathfrak{c},\mathfrak{d},p)) = {}_{N}\mathrm{lf}((\mathfrak{g},\mathfrak{h},t),(\mathfrak{f},\mathfrak{g},s)).$$

According to Proposition 3.31, on the one hand, of course,

$$_{n}\mathrm{lf}((\mathfrak{d},\mathfrak{e},q),(\mathfrak{c},\mathfrak{d},p)) = n^{\mathrm{rl}(q,p)}.$$

On the other hand, $\ker(N) = \{\mathbb{Z}_w, \{\aleph\}\}$ implies $\ker(N \circ \xi_{\mathfrak{g}}^{\mathfrak{f}}) = \{\mathsf{Y}(\mathfrak{c},\mathfrak{d}), \mathsf{Z}(\mathfrak{c},\mathfrak{d})\}\backslash\{\varnothing\}$ and, likewise, $\ker(N \circ \xi_{\mathfrak{h}}^{\mathfrak{g}}) = \{\mathsf{Y}(\mathfrak{d},\mathfrak{e}), \mathsf{Z}(\mathfrak{d},\mathfrak{e})\}\backslash\{\varnothing\}$. Moreover, $R(q,\mathsf{Y}(\mathfrak{d},\mathfrak{e})) = q$ and $R(p,\mathsf{Y}(\mathfrak{c},\mathfrak{d})) = p$ by Lemma 9.4 (f). Hence, Proposition 3.31 here yields

$$\begin{aligned} {}_{N}\mathrm{lf}((\mathfrak{g},\mathfrak{h},t),(\mathfrak{f},\mathfrak{g},s)) &= n^{\mathrm{rl}(R(q,\mathsf{Y}(\mathfrak{d},\mathfrak{e})),R(p,\mathsf{Y}(\mathfrak{c},\mathfrak{d})))} \cdot 1^{\mathrm{rl}(R(q,\mathsf{Z}(\mathfrak{d},\mathfrak{e})),R(p,\mathsf{Z}(\mathfrak{c},\mathfrak{d})))} \\ &= n^{\mathrm{rl}(q,p)}. \end{aligned}$$

Thus, all the claims are true. □

9.3. Restriction to unitary half-liberations. Finally, with the help of Lemma 9.4 we show that the functor restricts to a functor between each category inducing a unitary half-liberation and the associated supercategory of labeled partitions.

More precisely, we prove that for any $w \in \mathbb{N}$ and any additive subsemigroup D of $\mathbb{N}$ the following are well-defined faithful strict monoidal $*$-functors by ways of restriction:

- □ ${}_{\mathbb{Z}_w,\varnothing}F$ from $\mathcal{U}_w^*$ to $\mathcal{U} \wr \mathcal{Z}_w$.
- □ ${}_{\mathbb{Z},\varnothing}F$ from $\mathcal{U}_D^\times$ to $\mathcal{U} \wr_{r_D} \mathcal{Z}_0$.
- □ ${}_{\mathbb{Z},\varnothing}F$ from $\mathcal{U}_D^{\times+}$ to $\mathcal{U}^+ \wr_{r_D} \mathcal{Z}_0$.
- □ ${}_{\varnothing,\mathbb{Z}}F$ from $\mathcal{U}_D^\times$ to $\mathcal{O}^* \wr_{r_D} \mathcal{Z}_0$.
- □ ${}_{\varnothing,\mathbb{Z}}F$ from $\mathcal{U}_D^{\times+}$ to $\mathcal{O}^+ \wr_{r_D} \mathcal{Z}_0$.

It then follows immediately that for any $n \in \mathbb{N}$, if the profile N is such that $N(\aleph) = 1$ and $N(z) = n$ for any $z \neq \aleph$, then also following restrictions are well-defined faithful strict monoidal $\mathbb{C}$-linear $*$-functors:

- □ ${}^{n,N}\mathbb{C}_{\mathbb{Z}_w,\varnothing}F$ from ${}^{n}\mathbb{C}\mathcal{U}_w^*$ to ${}^{N}\mathbb{C}(\mathcal{U} \wr \mathcal{Z}_w)$.
- □ ${}^{n,N}\mathbb{C}_{\mathbb{Z},\varnothing}F$ from ${}^{n}\mathbb{C}\mathcal{U}_D^\times$ to ${}^{N}\mathbb{C}(\mathcal{U} \wr_{r_D} \mathcal{Z}_0)$.
- □ ${}^{n,N}\mathbb{C}_{\mathbb{Z},\varnothing}F$ from ${}^{n}\mathbb{C}\mathcal{U}_D^{\times+}$ to ${}^{N}\mathbb{C}(\mathcal{U}^+ \wr_{r_D} \mathcal{Z}_0)$.
- □ ${}^{n,N}\mathbb{C}_{\varnothing,\mathbb{Z}}F$ from ${}^{n}\mathbb{C}\mathcal{U}_D^\times$ to ${}^{N}\mathbb{C}(\mathcal{O}^* \wr_{r_D} \mathcal{Z}_0)$.
- □ ${}^{n,N}\mathbb{C}_{\varnothing,\mathbb{Z}}F$ from ${}^{n}\mathbb{C}\mathcal{U}_D^{\times+}$ to ${}^{N}\mathbb{C}(\mathcal{O}^+ \wr_{r_D} \mathcal{Z}_0)$.

PROPOSITION 9.6. *For any $w \in \mathbb{N}$ and any additive subsemigroup D of $\mathbb{N}$,*

(a) ${}_{\mathbb{Z}_w,\varnothing}F$ *maps* $\mathcal{U}_w^*$ *into* $\mathcal{U} \wr \mathcal{Z}_w$.
(b) ${}_{\mathbb{Z},\varnothing}F$ *maps* $\mathcal{U}_D^\times$ *into* $\mathcal{U} \wr_{r_D} \mathcal{Z}_0$.
(c) ${}_{\mathbb{Z},\varnothing}F$ *maps* $\mathcal{U}_D^{\times+}$ *into* $\mathcal{U}^+ \wr_{r_D} \mathcal{Z}_0$.
(d) ${}_{\varnothing,\mathbb{Z}}F$ *maps* $\mathcal{U}_D^\times$ *into* $\mathcal{O}^* \wr_{r_D} \mathcal{Z}_0$.
(e) ${}_{\varnothing,\mathbb{Z}}F$ *maps* $\mathcal{U}_D^{\times+}$ *into* $\mathcal{O}^+ \wr_{r_D} \mathcal{Z}_0$.

PROOF. We can verify all five claims simultaneously. In each case (a)–(e), let $\mathcal{C}$ be the respective domain category, $\mathcal{D}$ the co-domain category and F the functor. Furthemore, let $w \coloneqq 0$ in cases (b)–(e). Finally, let E stand for $\varnothing$ in case (a), for D in cases (b) and (d) and for $D \cup \{0\}$ in cases (c) and (e).

Given any $\{k, \ell\} \subseteq \mathbb{N}_0$, any $\mathfrak{c}\colon [\![k]\!] \to \{\circ, \bullet\}$ and $\mathfrak{d}\colon [\![\ell]\!] \to \{\circ, \bullet\}$ and any $(\mathfrak{c}, \mathfrak{d}, p) \in \mathcal{C}$, by Proposition 9.5 we only have to prove $(\mathfrak{a}, \mathfrak{b}, q) = F(\mathfrak{c}, \mathfrak{d}, p) \in \mathcal{D}$, i.e., we need to check that $(\mathfrak{a}, \mathfrak{b}, q)$ has the properties listed in the respective claim of Proposition 8.5.

If $\mathsf{Z} = \xi^{\mathfrak{a}\leftarrow}_{\mathfrak{b}}(\{\aleph\})$ and $\mathsf{Y} = \xi^{\mathfrak{a}\leftarrow}_{\mathfrak{b}}(\mathbb{Z}_w)$, then it suffices to show that ${}_{\aleph}\Sigma^{\mathfrak{a}}_{\mathfrak{b}} \equiv_w 0$, that $q \leq \{\mathsf{Y}, \mathsf{Z}\}$, that $|\mathsf{A}| = 2$ for each $\mathsf{A} \in q$, that ${}_{\aleph}\sigma^{\mathfrak{a}}_{\mathfrak{b}}(\mathsf{C}) = 0$ for any $\mathsf{C} \in q$ with $\mathsf{C} \subseteq \mathsf{Z}$, that $\xi^{\mathfrak{a}}_{\mathfrak{b}}(\mathsf{a}') - \xi^{\mathfrak{a}}_{\mathfrak{b}}(\mathsf{a}) + {}_{\aleph}\delta^{\mathfrak{a}}_{\mathfrak{b}}(\mathsf{a}, \mathsf{a}') \equiv_w 0$ for any $\{\mathsf{a}, \mathsf{a}'\} \subseteq \mathsf{A}$ and $\mathsf{A} \in q$, that $\sum_{z \in \mathbb{Z}_w} {}_z\sigma^{\mathfrak{a}}_{\mathfrak{b}}(\mathsf{A}) = 0$ for any $\mathsf{A} \in q$ with $\mathsf{A} \subseteq \mathsf{Y}$ in cases (a)–(c), that $|]\mathsf{a}, \mathsf{a}''[^{2k}_{2\ell} \cap \{\mathsf{a}' \in \mathsf{Y} \wedge \xi^{\mathfrak{a}}_{\mathfrak{b}}(\mathsf{a}') - \xi^{\mathfrak{a}}_{\mathfrak{b}}(\mathsf{a}) + {}_{\aleph}\delta^{\mathfrak{a}}_{\mathfrak{b}}(\mathsf{a}, \mathsf{a}') \equiv_w 0\}| \equiv_2 0$ for any $\{\mathsf{a}, \mathsf{a}''\} \subseteq \mathsf{A}$ with $\mathsf{a} \neq \mathsf{a}''$ and $\mathsf{A} \in q$ with $\mathsf{A} \subseteq \mathsf{Y}$ in cases (d) and (e), that $\mathsf{A}_1 \not\!\!\diagdown^{2k}_{2\ell} \mathsf{A}_2$ for any $\{\mathsf{A}_1, \mathsf{A}_2\} \subseteq q$ such that $\mathsf{A}_1 \subseteq \mathsf{Y}$ and $\mathsf{A}_2 \subseteq \mathsf{Y}$, such that $\mathsf{A}_1 \neq \mathsf{A}_2$ and such that there exist $\mathsf{a}_1 \in \mathsf{A}_1$ and $\mathsf{a}_2 \in \mathsf{A}_2$ with $|\xi^{\mathfrak{a}}_{\mathfrak{b}}(\mathsf{a}_2) - \xi^{\mathfrak{a}}_{\mathfrak{b}}(\mathsf{a}_1) + {}_{\aleph}\delta^{\mathfrak{a}}_{\mathfrak{b}}(\mathsf{a}_1, \mathsf{a}_2)| \in E$.

Total color sum. First of all, ${}_{\aleph}\Sigma^{\mathfrak{a}}_{\mathfrak{b}} = \Sigma^{\mathfrak{c}}_{\mathfrak{d}}$ by the second part of Lemma 9.4 (e) and thus ${}_{\aleph}\Sigma^{\mathfrak{a}}_{\mathfrak{b}} \equiv_w 0$ because $\Sigma^{\mathfrak{c}}_{\mathfrak{d}} = \sum_{\mathsf{B} \in p} \sigma^{\mathfrak{c}}_{\mathfrak{d}}(\mathsf{B}) = 0$ by Definition 7.11.

Tag areas. Lemma 9.4 (a) guarantees $q \leq \{\mathsf{Y}, \mathsf{Z}\}$, regardless of $\mathcal{C}$.

Block sizes. Since $|\mathsf{B}| = 2$ for any $\mathsf{B} \in p$ by Definition 7.11, Lemma 9.4 (c) implies $|\mathsf{A}| = 2$ for any $\mathsf{A} \in q$.

Block color sums in Z. For any $\mathsf{C} \in q$ with $\mathsf{C} \subseteq \mathsf{Y}$ the second part of Lemma 9.4 (d) guarantees ${}_{\aleph}\sigma^{\mathfrak{a}}_{\mathfrak{b}}(\mathsf{C}) = 0$ since, as we have already used, $\sigma^{\mathfrak{c}}_{\mathfrak{d}}(\mathsf{B}) = 0$ for any $\mathsf{B} \in p$ by Definition 7.11.

Block leg tag distances. For any $\mathsf{A} \in q$ with $\mathsf{A} \subseteq \mathsf{Y}$ there exists $\mathsf{B} \in p$ with $\mathsf{B} = \gamma^{2k\leftarrow}_{\mathsf{Y},2\ell}(\mathsf{A})$ and because $p = R(q, \mathsf{Y})$ by Lemma 9.4 (e). Consequently, for any $\{\mathsf{a}, \mathsf{a}'\} \subseteq \mathsf{A}$ there are $\{\mathsf{b}, \mathsf{b}'\} \subseteq \mathsf{B}$ with $\mathsf{a} = \gamma^{2k}_{\mathsf{Y},2\ell}(\mathsf{b})$ and $\mathsf{a}' = \gamma^{2k}_{\mathsf{Y},2\ell}(\mathsf{b}')$. By Definition 7.11 then $\delta^{\mathfrak{c}}_{\mathfrak{d}}(\mathsf{b}, \mathsf{b}') \equiv_w 0$. According to Lemma 9.4 (g) that proves $\xi^{\mathfrak{a}}_{\mathfrak{b}}(\mathsf{a}') - \xi^{\mathfrak{a}}_{\mathfrak{b}}(\mathsf{a}) + {}_{\aleph}\delta^{\mathfrak{a}}_{\mathfrak{b}}(\mathsf{a}, \mathsf{a}') \equiv_w 0$.

Block color sums in Y. In cases (a)–(c), for any $\mathsf{A} \in q$ with $\mathsf{A} \subseteq \mathsf{Y}$ the first part of Lemma 9.4 (d) is applicable and allows us to conclude $\sum_{z \in \mathbb{Z}_w} {}_z\sigma^{\mathfrak{a}}_{\mathfrak{b}}(\mathsf{A}) = 0$ since, as we already know, $\sigma^{\mathfrak{c}}_{\mathfrak{d}}(\mathsf{B}) = 0$ for any $\mathsf{B} \in p$ by Definition 7.11.

Block spread parities. Now consider the cases (d) and (e) instead. For any $\mathsf{A} \in q$ with $\mathsf{A} \subseteq \mathsf{Y}$ and any $\{\mathsf{a}, \mathsf{a}''\} \subseteq \mathsf{A}$ with $\mathsf{a} \neq \mathsf{a}''$, because $p = R(q, \mathsf{Y})$ by Lemma 9.4 (e), there exist $\mathsf{B} \in p$ with $\mathsf{B} = \gamma^{2k\leftarrow}_{\mathsf{Y},2\ell}(\mathsf{A})$ and thus $\{\mathsf{b}, \mathsf{b}''\} \subseteq \mathsf{B}$ with $\mathsf{a} = \gamma^{2k}_{\mathsf{Y},2\ell}(\mathsf{b})$ and $\mathsf{a}'' = \gamma^{2k}_{\mathsf{Y},2\ell}(\mathsf{b}'')$. Since $\mathsf{a} \neq \mathsf{a}''$ and since $\gamma^{2k}_{\mathsf{Y},2\ell}$ is injective, $\mathsf{b} \neq \mathsf{b}''$. If $\mathsf{S} = \{\mathsf{b}' \in \Pi^k_\ell \wedge \delta^{\mathfrak{c}}_{\mathfrak{d}}(\mathsf{b}, \mathsf{b}') \equiv_0 0\}$, then $\mathsf{S} \in {}^0\Delta^{\mathfrak{c}}_{\mathfrak{d}}$ and $\mathsf{B} \subseteq \mathsf{S}$ because $p \leq {}^0\Delta^{\mathfrak{c}}_{\mathfrak{d}}$ by Definition 7.11. Definition 7.11 also implies $\sigma^{\mathfrak{c}}_{\mathfrak{d}}(\mathsf{B}) = 0$. Finally, $|\mathsf{B}| = 2$ by Definition 7.11 and thus, actually, $\mathsf{B} = \{\mathsf{b}, \mathsf{b}''\}$ and $\sigma^{\mathfrak{c}}_{\mathfrak{d}}(\{\mathsf{b}, \mathsf{b}''\}) = 0$. Hence, $|]\mathsf{b}, \mathsf{b}''[^k_\ell \cap \mathsf{S}| \equiv_2 0$ by Proposition 7.17. By Lemma 9.4 (h) that means $|]\mathsf{a}, \mathsf{a}''[^{2k}_{2\ell} \cap \{\mathsf{a}' \in \mathsf{Y} \wedge \xi^{\mathfrak{a}}_{\mathfrak{b}}(\mathsf{a}') - \xi^{\mathfrak{a}}_{\mathfrak{b}}(\mathsf{a}) + {}_{\aleph}\delta^{\mathfrak{a}}_{\mathfrak{b}}(\mathsf{a}, \mathsf{a}') \equiv_0 0\}| \equiv_2 0$.

Non-crossing conditions. Given any $\{\mathsf{A}_1, \mathsf{A}_2\} \subseteq q$ with $\mathsf{A}_1 \subseteq \mathsf{Y}$ and $\mathsf{A}_2 \subseteq \mathsf{Y}$ and $\mathsf{A}_1 \neq \mathsf{A}_2$ and any $\mathsf{a}_1 \in \mathsf{A}_1$ and $\mathsf{a}_2 \in \mathsf{A}_2$ with $|\xi^{\mathfrak{a}}_{\mathfrak{b}}(\mathsf{a}_2) - \xi^{\mathfrak{a}}_{\mathfrak{b}}(\mathsf{a}_1) + {}_{\aleph}\delta^{\mathfrak{a}}_{\mathfrak{b}}(\mathsf{a}_1, \mathsf{a}_2)| \in E$, by Lemma 9.4 (e) there exist $\{\mathsf{B}_1, \mathsf{B}_2\} \subseteq p$ with $\mathsf{A}_1 = \gamma^{2k\leftarrow}_{\mathsf{Y},2\ell}(\mathsf{B}_1)$ and $\mathsf{A}_2 = \gamma^{2k\leftarrow}_{\mathsf{Y},2\ell}(\mathsf{B}_2)$ and, consequently, $\mathsf{b}_1 \in \mathsf{B}_1$ and $\mathsf{b}_2 \in \mathsf{B}_2$ with $\mathsf{a}_1 = \gamma^{2k}_{\mathsf{Y},2\ell}(\mathsf{b})$ and $\mathsf{a}_2 = \gamma^{2k}_{\mathsf{Y},2\ell}(\mathsf{b}'')$. Since

$\mathsf{A}_1 \cap \mathsf{A}_2 = \varnothing$, also $\mathsf{B}_1 \cap \mathsf{B}_2 = \gamma_{\mathsf{Y},2\ell}^{\;2k\leftarrow}(\mathsf{A}_1) \cap \gamma_{\mathsf{Y},2\ell}^{\;2k\leftarrow}(\mathsf{A}_2) = \gamma_{\mathsf{Y},2\ell}^{\;2k\leftarrow}(\mathsf{A}_1 \cap \mathsf{A}_2) = \varnothing$ and thus $\mathsf{B}_1 \neq \mathsf{B}_2$. Moreover, $\delta_{\mathfrak{d}}^{\mathfrak{c}}(\mathsf{b}_1, \mathsf{b}_2) = \xi_{\mathfrak{b}}^{\mathfrak{a}}(\mathsf{a}_2) - \xi_{\mathfrak{b}}^{\mathfrak{a}}(\mathsf{a}_1) + {}_{\aleph}\delta_{\mathfrak{b}}^{\mathfrak{a}}(\mathsf{a}_1, \mathsf{a}_2)$ by Lemma 9.4 (g) and thus $|\delta_{\mathfrak{d}}^{\mathfrak{c}}(\mathsf{b}_1, \mathsf{b}_2)| \in E$. Hence, $\mathsf{B}_1 \mathbin{\diagdown\!\!\diagdown}_{\ell}^{k} \mathsf{B}_2$ by Definition 7.11. Since $\gamma_{\mathsf{Y},2\ell}^{\;2k}$ is strictly monotonic by Lemma 4.2 (a) that warrants $\mathsf{A}_1 \mathbin{\diagdown\!\!\diagdown}_{2\ell}^{2k} \mathsf{A}_2$ and thus allows us to conclude $(\mathfrak{a}, \mathfrak{b}, q) \in \mathcal{D}$, as claimed. ☐

9.4. Preservation of the fiber functor. The rule F does not only map the categories of the categories of the unitary half-liberations into their respetive super-categories if labeled partitions, it also preserves the fiber functors. More precisely, the functor is not strictly preserved. We show that a unitary natural transformation is involved.

ASSUMPTION 9.7. In Section 9.4, let $w \in \mathbb{N}_0$, let $n \in \mathbb{N}$, let $(\mathfrak{U}, \mathfrak{O})$ be either $(\mathbb{Z}_w, \varnothing)$ or $(\varnothing, \mathbb{Z}_w)$ and let the profile N be such that $N(\aleph) = 1$ and $N(z) = n$ for any $z \in \mathbb{Z}_w$.

NOTATION 9.8. For any $k \in \mathbb{N}_0$ and $\mathfrak{c}\colon [\![k]\!] \to \{\circ, \bullet\}$ let $\omega_{\mathfrak{c}} \equiv {}_{\mathfrak{U},\mathfrak{O}}^{\;\;n}\omega_{\mathfrak{c}}$ be the mapping $J_{\mathfrak{c}}^{n} \to [[\![2k]\!], \mathbb{N}]$ such that for any $f \in J_{\mathfrak{c}}^{n}$ the mapping $\omega_{\mathfrak{c}}(f)\colon [\![2k]\!] \to \mathbb{N}$ satisfies

$$x \mapsto \begin{cases} f(\frac{x+1}{2}) & \text{if } x \text{ odd} \wedge \mathfrak{c}(\frac{x+1}{2}) = \circ \\ f(\frac{x}{2}) & \text{if } x \text{ even} \wedge \mathfrak{c}(\frac{x}{2}) = \bullet \\ 1 & \text{otherwise} \end{cases}$$

for any $x \in [\![2k]\!]$.

LEMMA 9.9. *Let $k \in \mathbb{N}_0$ and $\mathfrak{c}\colon [\![k]\!] \to \{\circ, \bullet\}$ be arbitrary.*

(a) For any $\{k, \ell\} \subseteq \mathbb{N}_0$, any $\mathfrak{c}\colon [\![k]\!] \to \{\circ, \bullet\}$ and $\mathfrak{d}\colon [\![\ell]\!] \to \{\circ, \bullet\}$ and any $f \in J_{\mathfrak{c}}^{n}$ and $g \in J_{\mathfrak{d}}^{n}$, if $\mathsf{Y} = \xi_{\mathfrak{d}}^{\mathfrak{c}\leftarrow}(\mathbb{Z}_w)$, then

$$(\omega_{\mathfrak{c}}(f) \,{}^{\blacksquare}\!{}_{\blacksquare}\, \omega_{\mathfrak{d}}(g)) \circ \gamma_{\mathsf{Y},2\ell}^{\;2k} = f \,{}^{\blacksquare}\!{}_{\blacksquare}\, g,$$

and, if $\mathsf{Z} = \xi_{\mathfrak{d}}^{\mathfrak{c}\leftarrow}(\{\aleph\})$, then the mapping $(\omega_{\mathfrak{c}}(f) \,{}^{\blacksquare}\!{}_{\blacksquare}\, \omega_{\mathfrak{d}}(g)) \circ \gamma_{\mathsf{Z},2\ell}^{\;2k}$ is constant with value 1.

(b) $\omega_{\mathfrak{c}}$ is a bijection from $J_{\mathfrak{c}}^{n}$ to $J_{F(\mathfrak{c})}^{N}$ for any $k \in \mathbb{N}_0$ and $\mathfrak{c}\colon [\![k]\!] \to \{\circ, \bullet\}$.

PROOF. (a) If $u = \omega_{\mathfrak{c}}(f)$ and $v = \omega_{\mathfrak{c}}(g)$, then for any $i \in [\![k]\!]$, if $\mathfrak{c}(i) = \bullet$, then $\gamma_{\mathsf{Y},2\ell}^{\;2k}({}^{\blacksquare}i) = {}^{\blacksquare}(2i)$ by Lemma 9.2 (d). If so and if $x = 2i$, then x is even and, by $\frac{x}{2} = i$, first $\mathfrak{c}(\frac{x}{2}) = \mathfrak{c}(i) = \bullet$ and thus $u(x) = f(\frac{x}{2}) = f(i)$ by definition of u. It follows $((u \,{}^{\blacksquare}\!{}_{\blacksquare}\, v) \circ \gamma_{\mathsf{Y},2\ell}^{\;2k})({}^{\blacksquare}i) = (u \,{}^{\blacksquare}\!{}_{\blacksquare}\, v)({}^{\blacksquare}x) = u(x) = f(i) = (f \,{}^{\blacksquare}\!{}_{\blacksquare}\, g)({}^{\blacksquare}i)$ in this case. Under the same assumptions on i, on the other hand, $\gamma_{\mathsf{Z},2\ell}^{\;2k}({}^{\blacksquare}i) = {}^{\blacksquare}(2i-1)$ Lemma 9.2 (d), which implies that, if, instead, $x = 2i - 1$, then x is odd and, by $i = \frac{x+1}{2}$, first $\mathfrak{c}(\frac{x+1}{2}) = \mathfrak{c}(i) = \bullet$ and thus $u(x) = 1$ by definition of u. Hence, $((u \,{}^{\blacksquare}\!{}_{\blacksquare}\, v) \circ \gamma_{\mathsf{Z},2\ell}^{\;2k})({}^{\blacksquare}i) = (u \,{}^{\blacksquare}\!{}_{\blacksquare}\, v)({}^{\blacksquare}x) = u(x) = 1$ then.

Alternatively, if $\mathfrak{c}(i) = \circ$, then Lemma 9.2 (d) implies $\gamma_{\mathsf{Y},2\ell}^{\;2k}({}^{\blacksquare}i) = {}^{\blacksquare}(2i-1)$ and thus for the odd number $x = 2i - 1$ because of $i = \frac{x+1}{2}$ first $\mathfrak{c}(\frac{x+1}{2}) = \circ$ and thus $u(x) = f(\frac{x+1}{2}) = f(i)$, ensuring $((u \,{}^{\blacksquare}\!{}_{\blacksquare}\, v) \circ \gamma_{\mathsf{Y},2\ell}^{\;2k})({}^{\blacksquare}i) = (f \,{}^{\blacksquare}\!{}_{\blacksquare}\, g)({}^{\blacksquare}i)$ in the same way as before. Similarly, if, still, $\mathfrak{c}({}^{\blacksquare}i) = \circ$ but if now $x = 2i$, then x is even with $\mathfrak{c}(\frac{x}{2}) = \circ$ and thus $u(x) = 1$. Because then $\gamma_{\mathsf{Z},2\ell}^{2k}({}^{\blacksquare}i) = {}^{\blacksquare}x$ by Lemma 9.2 (d) that proves $((u\,{}^{\blacksquare}_{\blacksquare}\,v) \circ \gamma_{\mathsf{Z},2\ell}^{2k})({}^{\blacksquare}i) = 1$ in this case as well. The remaining cases are analogous.

(b) Abbreviate $\mathfrak{a} \coloneqq F(\mathfrak{c})$ and for each $x \in [\![2k]\!]$ let the tag $z_x \in \mathbb{Z}_w \cup \{\aleph\}$ be such that $\mathfrak{a}(x) \in (\{z_x\} \otimes \{\circ, \bullet\}) \cup \{z_x\}$.

Well-defined. We begin by showing that the mapping $\omega_{\mathfrak{c}}$ maps $J_{\mathfrak{c}}^n$ into $J_{\mathfrak{a}}^N$, where $\mathfrak{a} = F(\mathfrak{c})$. Given any $f \in J_{\mathfrak{c}}^n$, if $u = \omega_{\mathfrak{c}}(f)$, we have to show $u(x) \in [\![N(z_x)]\!]$. Since $\mathrm{ran}(N) \subseteq \{1, n\}$, since $N(\aleph) = 1$ and $\mathrm{ran}(u) \subseteq \mathrm{ran}(f) \cup \{1\} \subseteq [\![n]\!]$ by definition of $\omega_{\mathfrak{c}}$, it is enough to show that $u(x) = 1$ for any $x \in [\![2k]\!]$ with $z_x = \aleph$. By definition of F, whenever $z_x = \aleph$, then either x is odd and $\mathfrak{c}(\frac{x+1}{2}) = \bullet$ or x is even and $\mathfrak{c}(\frac{x}{2}) = \circ$. Of course, the definition of $\omega_{\mathfrak{c}}$ was chosen in such a way that in these cases, indeed, $u(x) = 1$. Hence, $\omega_{\mathfrak{c}}$ is well-defined.

Injective. That $\omega_{\mathfrak{c}}$ is injective follows from (a). Indeed, given any $\{f, f'\} \subseteq J_{\mathfrak{c}}^n$ such that $u = \omega_{\mathfrak{c}}(f)$ and $u' = \omega_{\mathfrak{c}}(f')$ coincide, we can apply (a) with $\ell = 0$ and $g = \varnothing$ to prove $f\,{}^{\blacksquare}_{\blacksquare}\,\varnothing = (u\,{}^{\blacksquare}_{\blacksquare}\,\varnothing) \circ \gamma_{\mathsf{Y},0}^{2k} = (u'\,{}^{\blacksquare}_{\blacksquare}\,\varnothing) \circ \gamma_{\mathsf{Y},0}^{2k} = f'\,{}^{\blacksquare}_{\blacksquare}\,\varnothing$ and thus $f = f'$.

Surjective. It remains to prove that $\omega_{\mathfrak{c}}$ is surjective. If presented with any $u \in J_{\mathfrak{c}}^N$, we define $f(i)$ as $u(2i)$ if $\mathfrak{c}(i) = \bullet$ and as $u(2i-1)$ if $\mathfrak{c}(\circ)$. From the definition of $J_{\mathfrak{c}}^N$ we know that $u(x) = 1$ for any $x \in [\![2k]\!]$ with $z_x = \aleph$ as well as $u(x) \leq n$ for any $x \in [\![2k]\!]$ with $z_x \in \mathbb{Z}_w$. In particular, $f(i) \leq n$ for any $i \in [\![k]\!]$ and thus $f \in J_{\mathfrak{c}}^n = [[\![k]\!], [\![n]\!]]$. Moreover, $u'(x) \coloneqq \omega_{\mathfrak{c}}(f)(x) = u(x)$ for any $x \in [\![2k]\!]$. Indeed, if either x is odd and $\mathfrak{c}(\frac{x+1}{2}) = \bullet$ or x is even and $\mathfrak{c}(\frac{x}{2}) = \circ$, then, on the one hand, $z_x = \aleph$ by definition of F and thus, as seen, $u(x) = 1$ and, on the other hand, $u'(x) = 1$ by definition of $\omega_{\mathfrak{c}}$. Hence, $u(x) = u'(x)$ in these cases. If x is odd and $\mathfrak{c}(\frac{x+1}{2}) = \bullet$, then the definitions of u' and f imply $u'(x) = f(\frac{x+1}{2}) = u(2\frac{x+1}{2} - 1) = u(x)$. Likewise, if x is even and $\mathfrak{c}(\frac{x}{2}) = \bullet$, then $u'(x) = f(\frac{x}{2}) = u(2\frac{x}{2}) = u(x)$ for the same reasons. □

LEMMA 9.10. *For any $\{k, \ell\} \subseteq \mathbb{N}_0$, any $[\![\mathfrak{c}]\!][\![k]\!]\{\circ, \bullet\}$ and $[\![\mathfrak{d}]\!][\![\ell]\!]\{\circ, \bullet\}$ and any $(\mathfrak{c}, \mathfrak{d}, p) \in \mathcal{S}$, if $\mathfrak{a} = F(\mathfrak{c})$ and $\mathfrak{b} = F(\mathfrak{d})$ and $\mathsf{Y} = \xi_{\mathfrak{b}}^{\mathfrak{a}\leftarrow}(\mathbb{Z}_w)$ and $\mathsf{Z} = \xi_{\mathfrak{b}}^{\mathfrak{a}\leftarrow}(\{\aleph\})$, then for any set-theoretical partition q of $\Pi_{2\ell}^{2k}$ with $q \leq \{\mathsf{Y}, \mathsf{Z}\}$ and $p = R(q, \mathsf{Y})$ and any $f \in J_{\mathfrak{c}}^n$ and $g \in J_{\mathfrak{d}}^n$,*

$$\zeta(p, \ker(f\,{}^{\blacksquare}_{\blacksquare}\,g)) = \zeta(q, \ker(\omega_{\mathfrak{c}}(f)\,{}^{\blacksquare}_{\blacksquare}\,\omega_{\mathfrak{d}}(g))).$$

PROOF. If we let $u \coloneqq \omega_{\mathfrak{c}}(f)$ and $v \coloneqq \omega_{\mathfrak{d}}(g)$, then we have to show that $p \leq \ker(f\,{}^{\blacksquare}_{\blacksquare}\,g)$ if and only $q \leq \ker(u\,{}^{\blacksquare}_{\blacksquare}\,v)$.

First, suppose $p \leq \ker(f\,{}^{\blacksquare}_{\blacksquare}\,g)$ and let $\mathsf{A} \in q$ be arbitrary. Then, either $\mathsf{A} \subseteq \mathsf{Y}$ or $\mathsf{A} \subseteq \mathsf{Z}$ because $q \leq \{\mathsf{Y}, \mathsf{Z}\}$. If $\mathsf{A} \subseteq \mathsf{Z}$, then $\varnothing \neq \mathsf{A} \subseteq (u\,{}^{\blacksquare}_{\blacksquare}\,v)^{\leftarrow}(\{1\}) \in \ker(u\,{}^{\blacksquare}_{\blacksquare}\,v)$ because $(u\,{}^{\blacksquare}_{\blacksquare}\,v) \circ \gamma_{\mathsf{Z},2\ell}^{2k}$ and thus $(u\,{}^{\blacksquare}_{\blacksquare}\,v)|_{\mathsf{Z}}$ is constant with value 1 by Lemma 9.9 (a). Alternatively, if $\mathsf{A} \subseteq \mathsf{Y}$, then $\mathsf{A} \cap \mathsf{Y} = \mathsf{A} \neq \varnothing$ and thus $\mathsf{B} \coloneqq \gamma_{\mathsf{Y},2\ell}^{2k\leftarrow}(\mathsf{A}) \in p$ by $p = R(q, \mathsf{Y})$. Hence, by $p \leq \ker(f\,{}^{\blacksquare}_{\blacksquare}\,g)$ there exists $m \in \mathbb{N}$ with $\mathsf{B} \subseteq (f\,{}^{\blacksquare}_{\blacksquare}\,g)^{\leftarrow}(\{m\})$. Because $\gamma_{\mathsf{Y},2\ell}^{2k}$ is surjective onto Y and because $(u\,{}^{\blacksquare}_{\blacksquare}\,v) \circ \gamma_{\mathsf{Y},2\ell}^{2k} = f\,{}^{\blacksquare}_{\blacksquare}\,g$ by Lemma 9.9 (a) it thus follows $\mathsf{A} = \gamma_{\mathsf{Y},2\ell}^{2k\rightarrow}(\gamma_{\mathsf{Y},2\ell}^{2k\leftarrow}(\mathsf{A})) = \gamma_{\mathsf{Y},2\ell}^{2k\rightarrow}(\mathsf{B}) \subseteq \gamma_{\mathsf{Y},2\ell}^{2k\rightarrow}((f\,{}^{\blacksquare}_{\blacksquare}\,g)^{\leftarrow}(\{m\})) = (\gamma_{\mathsf{Y},2\ell}^{2k\rightarrow} \circ \gamma_{\mathsf{Y},2\ell}^{2k\leftarrow})((u\,{}^{\blacksquare}_{\blacksquare}\,v)^{\leftarrow}(\{m\})) \subseteq (u\,{}^{\blacksquare}_{\blacksquare}\,v)^{\leftarrow}(\{m\}) \in \ker(u\,{}^{\blacksquare}_{\blacksquare}\,v)$. That proves one implication.

Conversely, if $q \leq \ker(u \mathbin{\blacksquare} v)$ and if $\mathsf{B} \in p$, then by $p = R(q, \mathsf{Y})$ there exists $\mathsf{A} \in q$ with $\mathsf{A} \subseteq \mathsf{Y}$ and $\mathsf{B} = \gamma_{\mathsf{Y},2\ell}^{2k \leftarrow}(\mathsf{A})$. Hence, by $q \leq \ker(u \mathbin{\blacksquare} v)$ there exists $m \in \mathbb{N}$ with $\mathsf{A} \subseteq (u \mathbin{\blacksquare} v)^{\leftarrow}(\{m\})$. Since $(u \mathbin{\blacksquare} v) \circ \gamma_{\mathsf{Y},2\ell}^{2k} = f \mathbin{\blacksquare} g$ by Lemma 9.9 (a) we conclude $\mathsf{B} = \gamma_{\mathsf{Y},2\ell}^{2k \leftarrow}(\mathsf{A}) \subseteq \gamma_{\mathsf{Y},2\ell}^{2k \leftarrow}((u \mathbin{\blacksquare} v)^{\leftarrow}(\{m\})) = (f \mathbin{\blacksquare} g)^{\leftarrow}(\{m\}) \in \ker(f \mathbin{\blacksquare} g)$. Therefore, the other implication holds too. ☐

Definition 9.11. For each $k \in \mathbb{N}_0$ and each $\mathfrak{c}\colon [\![k]\!] \to \{\circ, \bullet\}$ let

$$\chi_{\mathfrak{c}} \equiv {}_{\mathfrak{U},\mathfrak{D}}^{\;n}\chi_{\mathfrak{c}} \coloneqq \ell^2(\omega_{\mathfrak{c}})\colon \ell^2(J_{\mathfrak{c}}^n) \to \ell^2(J_{F(\mathfrak{c})}^N).$$

Now we prove that ${}_{\mathfrak{U},\mathfrak{D}}^{\;n}\chi$ is a unitary monoidal $\mathbb{C}$-linear natural transformation from ${}^nT$ to ${}_{\mathfrak{U},\mathfrak{D}}^{\;N}T \circ {}^{n,N}\mathbb{C}({}_{\mathfrak{U},\mathfrak{D}}F)$.

Proposition 9.12. *(a)* ${}_{\mathfrak{U},\mathfrak{D}}^{\;n}\chi_{\mathfrak{d}} \circ {}^nT(\mathfrak{c}, \mathfrak{d}, p) = {}_{\mathfrak{U},\mathfrak{D}}^{\;N}T({}_{\mathfrak{U},\mathfrak{D}}F(\mathfrak{c}, \mathfrak{d}, p)) \circ {}_{\mathfrak{U},\mathfrak{D}}^{\;n}\chi_{\mathfrak{c}}$ *for any* $(\mathfrak{c}, \mathfrak{d}, p) \in \mathcal{S}$.

(b) ${}_{\mathfrak{U},\mathfrak{D}}^{\;n}\chi_{\mathfrak{c}}$ *is a unitary* $\ell^2(J_{\mathfrak{c}}^n) \to \ell^2(J_{F(\mathfrak{c})}^N)$ *for any* $k \in \mathbb{N}_0$ *and* $\mathfrak{c}\colon [\![k]\!] \to \{\circ, \bullet\}$.

(c) ${}_{\mathfrak{U},\mathfrak{D}}^{\;N}T_{\otimes, {}_{\mathfrak{U},\mathfrak{D}}F(\mathfrak{c}_1), {}_{\mathfrak{U},\mathfrak{D}}F(\mathfrak{c}_2)} \circ ({}_{\mathfrak{U},\mathfrak{D}}^{\;n}\chi_{\mathfrak{c}_1} \otimes {}_{\mathfrak{U},\mathfrak{D}}^{\;n}\chi_{\mathfrak{c}_2}) = {}_{\mathfrak{U},\mathfrak{D}}^{\;n}\chi_{\mathfrak{c}_1 \otimes \mathfrak{c}_2} \circ {}^nT_{\otimes, \mathfrak{c}_1, \mathfrak{c}_2}$ *for any* $\{k_1, k_2\} \subseteq \mathbb{N}_0$ *and any* $\mathfrak{c}_1\colon [\![k_1]\!] \to \{\circ, \bullet\}$ *and* $\mathfrak{c}_2\colon [\![k_2]\!] \to \{\circ, \bullet\}$.

Proof. (a) Let $\{k, \ell\} \subseteq \mathbb{N}_0$ be such that $\mathfrak{c}\colon [\![k]\!] \to \{\circ, \bullet\}$ and $\mathfrak{d}\colon [\![\ell]\!] \to \{\circ, \bullet\}$. If $(\mathfrak{a}, \mathfrak{b}, q) = F(\mathfrak{c}, \mathfrak{d}, p)$, then we have to prove

$$\chi_{\mathfrak{d}} \circ {}^nT(\mathfrak{c}, \mathfrak{d}, p) = {}_{\mathfrak{U},\mathfrak{D}}^{\;N}T(\mathfrak{a}, \mathfrak{b}, q) \circ \chi_{\mathfrak{c}}.$$

Indeed, for any $f \in J_{\mathfrak{c}}^n$, according to the definitons of ${}^nT$ and $\chi_{\mathfrak{d}} = \ell^2(\omega_{\mathfrak{d}})$, the vector $(\chi_{\mathfrak{d}} \circ {}^nT(\mathfrak{c}, \mathfrak{d}, p))(f)$ is given by $\sum_{g \in J_{\mathfrak{d}}^n} \zeta(p, \ker(f \mathbin{\blacksquare} g))\, \omega_{\mathfrak{d}}(g)$, where we have also used the linearity of $\chi_{\mathfrak{d}}$. Because $q \leq \xi_{\mathfrak{b}}^{\mathfrak{a}\leftarrow}(\{\mathbb{Z}_w, \{\aleph\}\})$ and $p = R(q, \xi_{\mathfrak{b}}^{\mathfrak{a}\leftarrow}(\mathbb{Z}_w))$ by Lemma 9.4 (a) and (f), Lemma 9.10 lets this equal $\sum_{g \in J_{\mathfrak{d}}^n} \zeta(q, \ker(\omega_{\mathfrak{c}}(f) \mathbin{\blacksquare} \omega_{\mathfrak{d}}(g)))\, \omega_{\mathfrak{d}}(g)$. Since $\omega_{\mathfrak{d}}$ is a bijection from $J_{\mathfrak{d}}^n$ to $J_{\mathfrak{b}}^N$ by Lemma 9.9 (b) we can reindex the sum and see that this vector is the same as $\sum_{v \in J_{\mathfrak{b}}^N} \zeta(q, \ker(\omega_{\mathfrak{c}}(f) \mathbin{\blacksquare} v))\, v$. And, by definition of ${}_{\mathfrak{U},\mathfrak{D}}^{\;N}T$ that is precisely ${}_{\mathfrak{U},\mathfrak{D}}^{\;N}T(\mathfrak{a}, \mathfrak{b}, q)(\omega_{\mathfrak{c}}(f))$ or, equivalently, $({}_{\mathfrak{U},\mathfrak{D}}^{\;N}T(\mathfrak{a}, \mathfrak{b}, q) \circ \chi_{\mathfrak{c}})(f)$ by the definition of $\chi_{\mathfrak{c}} = \ell^2(\omega_{\mathfrak{c}})$.

(b) That $\chi_{\mathfrak{c}}$ is in fact an isomorphism is clear because $\omega_{\mathfrak{c}}$ is bijective by Lemma 9.9 (b) and $\chi_{\mathfrak{c}} = \ell^2(\omega_{\mathfrak{c}})$ by definition. Moreover, if $\mathfrak{a} = F(\mathfrak{c})$, then for any $f \in J_{\mathfrak{c}}^n$ and any $u \in J_{\mathfrak{a}}^N$, by definition of ℓ^2,

$$\langle u \mid \chi_{\mathfrak{c}}(f) \rangle_{\ell^2(J_{\mathfrak{a}}^N)} = \delta_{u, \omega_{\mathfrak{c}}(f)} = \delta_{\omega_{\mathfrak{c}}^{-1}(u), f} = \langle \omega_{\mathfrak{c}}^{-1}(u) \mid f \rangle_{\ell^2(J_{\mathfrak{c}}^n)}$$

and thus $\chi_{\mathfrak{c}}^* = \ell^2(\omega_{\mathfrak{c}}^{-1}) = \chi_{\mathfrak{c}}^{-1}$ because $J_{\mathfrak{a}}^N$ is an orthonormal basis of $\ell^2(J_{\mathfrak{a}}^N)$ and $J_{\mathfrak{c}}^n$ one of $\ell^2(J_{\mathfrak{c}}^n)$. Hence, $\chi_{\mathfrak{c}}$ is indeed unitary.

(c) We have to prove that, if $\mathfrak{a}_t = F(\mathfrak{c}_t)$ for each $t \in [\![2]\!]$, then

$${}_{\mathfrak{U},\mathfrak{D}}^{\;N}T_{\otimes, \mathfrak{a}_1, \mathfrak{a}_2} \circ (\chi_{\mathfrak{c}_1} \otimes \chi_{\mathfrak{c}_2}) = \chi_{\mathfrak{c}_1 \otimes \mathfrak{c}_2} \circ {}^nT_{\otimes, \mathfrak{c}_1, \mathfrak{c}_2}.$$

Step 1: First, we justify that it suffices to show for any $f_1 \in J_{\mathfrak{c}_1}^n$ and $f_2 \in J_{\mathfrak{c}_2}^n$,

$$\omega_{\mathfrak{c}_1}(f_1) \vartriangle \omega_{\mathfrak{c}_2}(f_2) = \omega_{\mathfrak{c}_1 \otimes \mathfrak{c}_2}(f_1 \vartriangle f_2).$$

If r is the mapping $J^n_{\mathfrak{c}_1} \otimes J^n_{\mathfrak{c}_2} \to J^n_{\mathfrak{c}_1\otimes\mathfrak{c}_2}$ with $(f_1, f_2) \mapsto f_1 \vartriangle f_2$ for any $f_1 \in J^n_{\mathfrak{c}_1}$ and $f_2 \in J^n_{\mathfrak{c}_2}$ and if s is the mapping $J^N_{\mathfrak{a}_1} \otimes J^N_{\mathfrak{a}_2} \to J^N_{\mathfrak{a}_1\otimes\mathfrak{a}_2}$ with $(h_1, h_2) \mapsto h_1 \vartriangle h_2$ for any $h_1 \in J^n_{\mathfrak{a}_1}$ and $h_2 \in J^n_{\mathfrak{a}_2}$, then ${}^nT_{\otimes,\mathfrak{c}_1,\mathfrak{c}_2} = \ell^2(r)\circ\ell^2_{\otimes,J^n_{\mathfrak{c}_1},J^n_{\mathfrak{c}_2}}$ and ${}_{\mathfrak{U},\mathfrak{O}}^{\;N}T_{\otimes,\mathfrak{a}_1,\mathfrak{a}_2} = \ell^2(s)\circ\ell^2_{\otimes,J^N_{\mathfrak{a}_1},J^N_{\mathfrak{a}_2}}$ by definition. Also using the functoriality of ℓ^2, hence, ${}_{\mathfrak{U},\mathfrak{O}}^{\;N}T_{\otimes,\mathfrak{a}_1,\mathfrak{a}_2} \circ (\chi_{\mathfrak{c}_1} \otimes \chi_{\mathfrak{c}_2}) = \ell^2(\omega_{\mathfrak{c}_1\otimes\mathfrak{c}_2} \circ r) \circ \ell^2_{\otimes,J^n_{\mathfrak{c}_1},J^n_{\mathfrak{c}_2}}$ and $\chi_{\mathfrak{c}_1\otimes\mathfrak{c}_2} \circ {}^nT_{\otimes,\mathfrak{c}_1,\mathfrak{c}_2} = \ell^2(s) \circ \ell^2_{\otimes,J^N_{\mathfrak{a}_1},J^N_{\mathfrak{a}_2}} \circ (\ell^2(\omega_{\mathfrak{c}_1}) \otimes \ell^2(\omega_{\mathfrak{c}_2}))$. Because ℓ^2 is a monoidal functor, $\ell^2_{\otimes,J^N_{\mathfrak{a}_1},J^N_{\mathfrak{a}_2}} \circ (\ell^2(\omega_{\mathfrak{c}_1}) \otimes \ell^2(\omega_{\mathfrak{c}_2})) = \ell^2(\omega_{\mathfrak{c}_1} \otimes \omega_{\mathfrak{c}_2}) \circ \ell^2_{\otimes,J^N_{\mathfrak{c}_1},J^N_{\mathfrak{c}_2}}$. Therefore, $\chi_{\mathfrak{c}_1\otimes\mathfrak{c}_2}\circ{}^nT_{\otimes,\mathfrak{c}_1,\mathfrak{c}_2} = \ell^2(s\circ(\omega_{\mathfrak{c}_1}\otimes\omega_{\mathfrak{c}_2}))\circ\ell^2_{\otimes,J^N_{\mathfrak{c}_1},J^N_{\mathfrak{c}_2}}$. Thus, the identity ${}_{\mathfrak{U},\mathfrak{O}}^{\;N}T_{\otimes,\mathfrak{a}_1,\mathfrak{a}_2} \circ (\chi_{\mathfrak{c}_1} \otimes \chi_{\mathfrak{c}_2}) = \chi_{\mathfrak{c}_1\otimes\mathfrak{c}_2} \circ {}^nT_{\otimes,\mathfrak{c}_1,\mathfrak{c}_2}$ holds if $\ell^2(\omega_{\mathfrak{c}_1\otimes\mathfrak{c}_2} \circ r) = \ell^2(s \circ (\omega_{\mathfrak{c}_1} \otimes \omega_{\mathfrak{c}_2}))$. And that is true if $\omega_{\mathfrak{c}_1\otimes\mathfrak{c}_2} \circ r = s \circ (\omega_{\mathfrak{c}_1} \otimes \omega_{\mathfrak{c}_2})$. But, in terms of elements, that is exactly what the above identity says.

Step 2: By Step 1 we only need to prove $\omega_{\mathfrak{c}_1}(f_1) \vartriangle \omega_{\mathfrak{c}_2}(f_2) = \omega_{\mathfrak{c}_1\otimes\mathfrak{c}_2}(f_1 \vartriangle f_2)$ for any $f_1 \in J^n_{\mathfrak{c}_1}$ and $f_2 \in J^n_{\mathfrak{c}_2}$. Given any such f_1 and f_2, for any $x \in [\![2k_1 + 2k_2]\!]$, by definition, on the one hand,

$$\omega_{\mathfrak{c}_1\otimes\mathfrak{c}_2}(f_1 \vartriangle f_2) = \begin{cases} f_1(\frac{x+1}{2}) & | \, x \text{ odd } \wedge \quad \mathfrak{c}_1(\frac{x+1}{2}) = \circ \wedge \frac{x+1}{2} \le k_1 \\ f_2(\frac{x+1}{2} - k_1) & | \, x \text{ odd } \wedge \mathfrak{c}_2(\frac{x+1}{2} - k_1) = \circ \wedge \; k_1 < \frac{x+1}{2} \\ f_1(\frac{x}{2}) & | \, x \text{ even } \wedge \quad \mathfrak{c}_1(\frac{x}{2}) = \bullet \wedge \; \frac{x}{2} \le k_1 \\ f_2(\frac{x}{2} - k_1) & | \, x \text{ even } \wedge \; \mathfrak{c}_2(\frac{x}{2} - k_1) = \bullet \wedge \; k_1 < \frac{x}{2} \\ 1 & | \, \text{otherwise} \end{cases}$$

and, on the other hand,

$$\omega_{\mathfrak{c}_1}(f_1) \vartriangle \omega_{\mathfrak{c}_2}(f_2) = \begin{cases} f_1(\frac{x+1}{2}) & | \; x \le 2k_1 \wedge \quad x \text{ odd } \quad \wedge \quad \mathfrak{c}_1(\frac{x+1}{2}) = \circ \\ f_1(\frac{x}{2}) & | \; x \le 2k_1 \wedge \quad x \text{ even } \quad \wedge \quad \mathfrak{c}_1(\frac{x}{2}) = \bullet \\ f_2(\frac{x-2k_1+1}{2}) & | \, 2k_1 < x \quad \wedge \; x - 2k_1 \text{ odd } \wedge \; \mathfrak{c}_1(\frac{x-2k_1+1}{2}) = \circ \\ f_2(\frac{x-2k_1}{2}) & | \, 2k_1 < x \quad \wedge \; x - 2k_1 \text{ even } \wedge \; \mathfrak{c}_1(\frac{x-2k_1}{2}) = \bullet \\ 1 & | \, \text{otherwise} \end{cases}.$$

Because $x \equiv_2 x - 2k_1$, because $\frac{x-2k_1+1}{2} = \frac{x+1}{2} - k_1$ and $\frac{x-2k_1}{2} = \frac{x}{2} - k_1$ and because, if x is odd, then $\frac{x+1}{2} \le k_1$ if and only if $x \le 2k_1$, the two agree. Hence, χ is indeed a unitary monoidal $\mathbb{C}$-linear natural isomorphism. ☐

9.5. Fullness of image functor. While, as is not difficult to see, the functors of Proposition 9.6 are never full, together with the appropriate restrictions of the natural isomorphisms of Proposition 9.12 they induce full functors between the full images of the fiber functors.

More precisely, we show that for any $w \in \mathbb{N}$, any additive subsemigroup D of $\mathbb{N}$ and any $n \in \mathbb{N}$, if the profile N is such that $N(\aleph) = 1$ and $N(z) = n$ for any $z \neq \aleph$, then the following restrictions are full on the operator level.

- ☐ $({}^{n,N}\mathbb{C}_{\mathbb{Z}_w,\varnothing}F, {}_{\mathbb{Z}_w,\varnothing}^{\;n}\chi)$ from the restriction of ${}^nT$ to ${}^n\mathbb{C}\mathcal{U}^*_w$ to the restriction of ${}_{\mathbb{Z}_w\cup\{\aleph\},\varnothing}^{\;N}T$ to ${}^N\mathbb{C}(\mathcal{U} \wr \mathcal{Z}_w)$.

- ◻ $({}^{n,N}\mathbb{C}_{\mathbb{Z},\varnothing}F, {}_{\mathbb{Z},\varnothing}^{\;n}\chi)$ from the restriction of ${}^nT$ to ${}^n\mathbb{C}\mathcal{U}_D^{\times}$ to the restriction of ${}_{\mathbb{Z}\uplus\{\aleph\},\varnothing}^{\;N}T$ to ${}^N\mathbb{C}(\mathcal{U} \wr_{r_D} \mathcal{Z}_0)$.
- ◻ $({}^{n,N}\mathbb{C}_{\mathbb{Z},\varnothing}F, {}_{\mathbb{Z},\varnothing}^{\;n}\chi)$ from the restriction of ${}^nT$ to ${}^n\mathbb{C}\mathcal{U}_D^{\times+}$ to the restriction of ${}_{\mathbb{Z}\uplus\{\aleph\},\varnothing}^{\;N}T$ to ${}^N\mathbb{C}(\mathcal{U}^+ \wr_{r_D} \mathcal{Z}_0)$.
- ◻ $({}^{n,N}\mathbb{C}_{\varnothing,\mathbb{Z}}F, {}_{\varnothing,\mathbb{Z}}^{\;n}\chi)$ from the restriction of ${}^nT$ to ${}^n\mathbb{C}\mathcal{U}_D^{\times}$ to the restriction of ${}_{\{\aleph\},\mathbb{Z}}^{\;N}T$ to ${}^N\mathbb{C}(\mathcal{O}^* \wr_{r_D} \mathcal{Z}_0)$.
- ◻ $({}^{n,N}\mathbb{C}_{\varnothing,\mathbb{Z}}F, {}_{\varnothing,\mathbb{Z}}^{\;n}\chi)$ from the restriction of ${}^nT$ to ${}^n\mathbb{C}\mathcal{U}_D^{\times+}$ to the restriction of ${}_{\{\aleph\},\mathbb{Z}}^{\;N}T$ to ${}^N\mathbb{C}(\mathcal{O}^+ \wr_{r_D} \mathcal{Z}_0)$.

That is equivalent to the following proposition.

PROPOSITION 9.13. *For any $w \in \mathbb{N}$, any additive subsemigroup D of $\mathbb{N}$ and any $n \in \mathbb{N}$, if the profile N is such that $N(\aleph) = 1$ and $N(z) = n$ for any $z \neq \aleph$, and if the tuple $(\mathcal{C}, R \mid \mathcal{D}, S \mid F, \chi)$ is one of the following*

(a) $(\mathcal{U}_w^*, {}^nT \mid \mathcal{U} \wr \mathcal{Z}_w, {}_{\mathbb{Z}_w\uplus\{\aleph\},\varnothing}^{\;N}T \mid {}_{\mathbb{Z}_w,\varnothing}F, {}_{\mathbb{Z}_w,\varnothing}^{\;n}\chi)$

(b) $(\mathcal{U}_D^{\times}, {}^nT \mid \mathcal{U} \wr_{r_D} \mathcal{Z}_0, {}_{\mathbb{Z}\uplus\{\aleph\},\varnothing}^{\;N}T \mid {}_{\mathbb{Z},\varnothing}F, {}_{\mathbb{Z},\varnothing}^{\;n}\chi)$

(c) $(\mathcal{U}_D^{\times+}, {}^nT \mid \mathcal{U}^+ \wr_{r_D} \mathcal{Z}_0, {}_{\mathbb{Z}\uplus\{\aleph\},\varnothing}^{\;N}T \mid {}_{\mathbb{Z},\varnothing}F, {}_{\mathbb{Z},\varnothing}^{\;n}\chi)$

(d) $(\mathcal{U}_D^{\times}, {}^nT \mid \mathcal{O}^* \wr_{r_D} \mathcal{Z}_0, {}_{\{\aleph\},\mathbb{Z}}^{\;N}T \mid {}_{\varnothing,\mathbb{Z}}F, {}_{\varnothing,\mathbb{Z}}^{\;n}\chi)$

(e) $(\mathcal{U}_D^{\times+}, {}^nT \mid \mathcal{O}^+ \wr_{r_D} \mathcal{Z}_0, {}_{\{\aleph\},\mathbb{Z}}^{\;N}T \mid {}_{\varnothing,\mathbb{Z}}F, {}_{\varnothing,\mathbb{Z}}^{\;n}\chi)$,

then for any $\{k,\ell\} \subseteq \mathbb{N}_0$ and any $\mathfrak{c}\colon [\![k]\!] \to \{\circ,\bullet\}$ and $\mathfrak{d}\colon [\![\ell]\!] \to \{\circ,\bullet\}$, if $\mathfrak{a} = F(\mathfrak{c})$ and $\mathfrak{b} = F(\mathfrak{d})$, then for any set-theoretical partition q of $\Pi_{2\ell}^{2k}$ with $(\mathfrak{a},\mathfrak{b},q) \in \mathcal{D}$ there exists a set-theoretical partition p of Π_ℓ^k such that $(\mathfrak{c},\mathfrak{d},p) \in \mathcal{C}$ and

$$R(\mathfrak{c},\mathfrak{d},p) = \chi_{\mathfrak{d}}^{-1} \circ S(\mathfrak{a},\mathfrak{b},q) \circ \chi_{\mathfrak{c}}.$$

PROOF. We can prove all claims simultaneously. Let $w \coloneqq 0$ in cases (b)–(e) and let E be given by $\varnothing$ in case (a), by D in cases (b) and (d) and by $D \uplus \{0\}$ in cases (c) and (e).

We show that for any $\{k,\ell\} \subseteq \mathbb{N}_0$ and any $\mathfrak{c}\colon [\![k]\!] \to \{\circ,\bullet\}$ and $\mathfrak{d}\colon [\![\ell]\!] \to \{\circ,\bullet\}$, if $\mathfrak{a} = F(\mathfrak{c})$ and $\mathfrak{b} = F(\mathfrak{d})$, then for any set-theoretical partition q of $\Pi_{2\ell}^{2k}$ with $(\mathfrak{a},\mathfrak{b},q) \in \mathcal{D}$ there exists a set-theoretical partition p of Π_ℓ^k such that $(\mathfrak{c},\mathfrak{d},p) \in \mathcal{C}$ and

$$\chi_{\mathfrak{d}} \circ R(\mathfrak{c},\mathfrak{d},p) = S(\mathfrak{a},\mathfrak{b},q) \circ \chi_{\mathfrak{c}}.$$

More precisely, if $\mathsf{Y} = \xi_{\mathfrak{b}}^{\mathfrak{a}\leftarrow}(\mathbb{Z}_w)$ and $\mathsf{Z} = \xi_{\mathfrak{b}}^{\mathfrak{a}\leftarrow}(\{\aleph\})$, we prove that $p \coloneqq R(q,\mathsf{Y})$ has the desired properties.

Step 1: We first check that $(\mathfrak{c},\mathfrak{d},p) \in p$, i.e., that $|\mathsf{B}| = 2$ and $\sigma_{\mathfrak{d}}^{\mathfrak{c}}(\mathsf{B}) = 0$ and $\delta_{\mathfrak{d}}^{\mathfrak{c}}(\mathsf{b},\mathsf{b}') \equiv_w 0$ for any $\{\mathsf{b},\mathsf{b}'\} \subseteq \mathsf{B}$ and $\mathsf{B} \in p$, and that for any $\{\mathsf{B}_1,\mathsf{B}_2\} \subseteq p$ with $\mathsf{B}_1 \neq \mathsf{B}_2$, whenever there exist $\mathsf{b}_1 \in \mathsf{B}_1$ and $\mathsf{b}_2 \in \mathsf{B}_2$ with $|\delta_{\mathfrak{d}}^{\mathfrak{c}}(\mathsf{b}_1,\mathsf{b}_2)| \in E$, then $\mathsf{B}_1 \between_\ell^k \mathsf{B}_2$.

By definition of $\mathcal{D}$ the assumption that $(\mathfrak{a},\mathfrak{b},q) \in \mathcal{D}$ ensures that ${}_\aleph\Sigma_{\mathfrak{b}}^{\mathfrak{a}} \equiv_w 0$, that $q \leq \{\mathsf{Y},\mathsf{Z}\}$, that for any $\mathsf{C} \in q$ with $\mathsf{C} \subseteq \mathsf{Z}$ always $|\mathsf{C}| \leq 2$ and, if $|\mathsf{C}| = 2$, then ${}_\aleph\sigma_{\mathfrak{b}}^{\mathfrak{a}}(\mathsf{C}) = 0$, as well as $|\mathsf{C}| \geq 2$ in cases (b)–(e), that any $\mathsf{A} \in q$ with $\mathsf{A} \subseteq \mathsf{Y}$ satisfies $|\mathsf{A}| = 2$ and $\xi_{\mathfrak{b}}^{\mathfrak{a}}(\mathsf{a}') - \xi_{\mathfrak{b}}^{\mathfrak{a}}(\mathsf{a}) + {}_\aleph\delta_{\mathfrak{b}}^{\mathfrak{a}}(\mathsf{a},\mathsf{a}') \equiv_w 0$ for any $\{\mathsf{a},\mathsf{a}'\} \subseteq \mathsf{A}$, and $\sum_{z\in\mathbb{Z}_w} {}_\aleph\sigma_{\mathfrak{b}}^{\mathfrak{a}}(\mathsf{A}) = 0$ in cases (a)–(c), and $|]\mathsf{a},\mathsf{a}''[_{2\ell}^{2k} \cap \{\mathsf{a}' \in \mathsf{Y} \wedge \xi_{\mathfrak{b}}^{\mathfrak{a}}(\mathsf{a}') - \xi_{\mathfrak{b}}^{\mathfrak{a}}(\mathsf{a}) + {}_\aleph\delta_{\mathfrak{b}}^{\mathfrak{a}}(\mathsf{a},\mathsf{a}') = 0\}| \equiv_2 0$ for any $\{\mathsf{a},\mathsf{a}''\} \subseteq \mathsf{A}$ with $\mathsf{a} \neq \mathsf{a}''$ in cases (d) and (e), and that for any $\{\mathsf{A}_1,\mathsf{A}_2\} \subseteq q$ with

$\mathsf{A}_1 \subseteq \mathsf{Y}$ and $\mathsf{A}_2 \subseteq \mathsf{Y}$ and $\mathsf{A}_1 \neq \mathsf{A}_2$, whenever there exist $\mathsf{a}_1 \in \mathsf{A}_1$ and $\mathsf{a}_2 \in \mathsf{A}_2$ with $|\xi^{\mathfrak{a}}_{\mathfrak{b}}(\mathsf{a}_2) - \xi^{\mathfrak{a}}_{\mathfrak{b}}(\mathsf{a}_1) + {}_{\aleph}\delta^{\mathfrak{a}}_{\mathfrak{b}}(\mathsf{a}_1, \mathsf{a}_2)| \in E$, then $\mathsf{A}_1 \not\between^{2k}_{2\ell} \mathsf{A}_2$.

Step 1.1: Block size. Given any $\mathsf{B} \in p$, by definition of p there exists $\mathsf{A} \subseteq q$ with $\mathsf{B} = \gamma^{2k\leftarrow}_{\mathsf{Y},2\ell}(\mathsf{A})$. Because $q \leq \{\mathsf{Y}, \mathsf{Z}\}$ by $(\mathfrak{a}, \mathfrak{b}, q) \in \mathcal{D}$ the fact that $\mathsf{B} \neq \varnothing$ and thus $\mathsf{A} \cap \mathsf{Y} \supseteq \gamma^{2k}_{\mathsf{Y},2\ell\rightarrow}(\mathsf{B}) \neq \varnothing$ ensures $\mathsf{A} \subseteq \mathsf{Y}$. Hence, $|\mathsf{B}| = 2$ because $\gamma^{2k}_{\mathsf{Y},2\ell}$ is injective and $|\mathsf{A}| = 2$ by $(\mathfrak{a}, \mathfrak{b}, q) \in \mathcal{D}$.

Step 1.2: Block spread. Moreover, if $\mathsf{B} \in p$ and A are as in the preceding step, and if $\{\mathsf{b}, \mathsf{b}'\} \subseteq \mathsf{B}$ are arbitrary, then $\mathsf{a} \coloneqq \gamma^{2k}_{\mathsf{Y},2\ell}(\mathsf{b}) \in \mathsf{A}$ and $\mathsf{a}' \coloneqq \gamma^{2k}_{\mathsf{Y},2\ell}(\mathsf{b}') \in \mathsf{A}$. Hence, $\xi^{\mathfrak{a}}_{\mathfrak{b}}(\mathsf{a}') - \xi^{\mathfrak{a}}_{\mathfrak{b}}(\mathsf{a}) + {}_{\aleph}\delta^{\mathfrak{a}}_{\mathfrak{b}}(\mathsf{a}, \mathsf{a}') \equiv_w 0$ by $(\mathfrak{a}, \mathfrak{b}, q) \in \mathcal{D}$. And, according to Lemma 9.4 (g) that implies $\delta^{\mathfrak{c}}_{\mathfrak{d}}(\mathsf{b}, \mathsf{b}') \equiv_w 0$.

Step 1.3: Block color sum. If, still, B and A are as before, then the assumption $(\mathfrak{a}, \mathfrak{b}, q) \in \mathcal{D}$ guarantees in cases (a)–(c) that $\sum_{z \in \mathbb{Z}_w} {}_z\sigma^{\mathfrak{a}}_{\mathfrak{b}}(\mathsf{A}) = 0$. Since $\zeta^{\mathfrak{a}}_{\mathfrak{b}} \circ \gamma^{2k}_{\mathsf{Y},2\ell} = \xi^{\mathfrak{c}}_{\mathfrak{d}}$ by Lemma 9.4 (b) we can thus conclude $\sigma^{\mathfrak{c}}_{\mathfrak{d}}(\mathsf{B}) = \sum_{\mathsf{b} \in \mathsf{B}} \sigma(\zeta^{\mathfrak{c}}_{\mathfrak{d}}(\mathsf{b})) = \sum_{\mathsf{a} \in \mathsf{A}} \sigma(\zeta^{\mathfrak{a}}_{\mathfrak{b}}(\mathsf{a})) = \sum_{z \in \mathbb{Z}_w} {}_z\sigma^{\mathfrak{a}}_{\mathfrak{b}}(\mathsf{A}) = 0$ in those cases. In cases (d) and (e) by $|\mathsf{B}| = 2$ there exist $\{\mathsf{b}, \mathsf{b}''\} \subseteq \Pi^k_\ell$ with $\mathsf{b} \neq \mathsf{b}''$ and $\{\mathsf{b}, \mathsf{b}''\} = \mathsf{B}$. Since $\gamma^{2k}_{\mathsf{Y},2\ell}$ is injective the points $\mathsf{a} \coloneqq \gamma^{2k}_{\mathsf{Y},2\ell}(\mathsf{b})$ and $\mathsf{a} \coloneqq \gamma^{2k}_{\mathsf{Y},2\ell}(\mathsf{b}'')$ satisfy $\mathsf{a} \neq \mathsf{a}''$. Hence, the facts that $w = 0$ and $\{\mathsf{a}, \mathsf{a}''\} \subseteq \mathsf{A} \in q$ imply $|]\mathsf{a}, \mathsf{a}''[^{2k}_{2\ell} \cap \{\mathsf{a}' \in \mathsf{Y} \wedge \xi^{\mathfrak{a}}_{\mathfrak{b}}(\mathsf{a}') - \xi^{\mathfrak{a}}_{\mathfrak{b}}(\mathsf{a}) + {}_{\aleph}\delta^{\mathfrak{a}}_{\mathfrak{b}}(\mathsf{a}, \mathsf{a}') = 0\}| \equiv_2 0$ by $(\mathfrak{a}, \mathfrak{b}, q) \in \mathcal{D}$. Lemma 9.4 (h) tells us that this requires $|]\mathsf{b}, \mathsf{b}''[^k_\ell \cap \{\mathsf{b}' \in \Pi^k_\ell \wedge \delta^{\mathfrak{c}}_{\mathfrak{d}}(\mathsf{b}, \mathsf{b}') = 0\}| \equiv_2 0$. Since $\mathsf{S} \coloneqq \{\mathsf{b}' \in \Pi^k_\ell \wedge \delta^{\mathfrak{c}}_{\mathfrak{d}}(\mathsf{b}, \mathsf{b}') = 0\} \in {}^0\Delta^{\mathfrak{c}}_{\mathfrak{d}}$ and since, as we have seen, $\mathsf{B} \subseteq \mathsf{S}$, that implies $\sigma^{\mathfrak{c}}_{\mathfrak{d}}(\{\mathsf{b}, \mathsf{b}''\}) = 0$ by Proposition 7.17. Hence, $\sigma^{\mathfrak{c}}_{\mathfrak{d}}(\mathsf{B}) = 0$ in any case.

Step 1.4: Non-crossing conditions. For any $\{\mathsf{B}_1, \mathsf{B}_2\} \subseteq p$ with $\mathsf{B}_1 \neq \mathsf{B}_2$ and any $\mathsf{b}_1 \in \mathsf{B}_1$ and $\mathsf{b}_2 \in \mathsf{B}_2$ with $|\delta^{\mathfrak{c}}_{\mathfrak{d}}(\mathsf{b}_1, \mathsf{b}_2)| \in E$, by definition of p, there exist $\mathsf{A}_1 \in q$ and $\mathsf{A}_2 \in q$ such that $\mathsf{B}_1 = \gamma^{2k\leftarrow}_{\mathsf{Y},2\ell}(\mathsf{A}_1)$ and $\mathsf{B}_1 = \gamma^{2k\leftarrow}_{\mathsf{Y},2\ell}(\mathsf{A}_2)$. Again, that demands $\mathsf{B}_1 \subseteq \mathsf{Y}$ and $\mathsf{B}_2 \subseteq \mathsf{Y}$ because $q \leq \{\mathsf{Y}, \mathsf{Z}\}$ by $(\mathfrak{a}, \mathfrak{b}, q) \in \mathcal{D}$. Moreover, since $\mathsf{A}_1 \cap \mathsf{A}_2 \subseteq \mathsf{Y}$ the fact that $\varnothing = \mathsf{B}_1 \cap \mathsf{B}_2 = \gamma^{2k\leftarrow}_{\mathsf{Y},2\ell}(\mathsf{A}_1) \cap \gamma^{2k\leftarrow}_{\mathsf{Y},2\ell}(\mathsf{A}_2) = \gamma^{2k\leftarrow}_{\mathsf{Y},2\ell}(\mathsf{A}_1 \cap \mathsf{A}_2)$ requires $\mathsf{A}_1 \cap \mathsf{A}_2 = \varnothing$. Finally, the points $\mathsf{a}_1 \coloneqq \gamma^{2k}_{\mathsf{Y},2\ell}(\mathsf{b}_1) \in \mathsf{A}_1$ and $\mathsf{a}_2 \coloneqq \gamma^{2k}_{\mathsf{Y},2\ell}(\mathsf{b}_2) \in \mathsf{A}_2$ satisfy $|\xi^{\mathfrak{a}}_{\mathfrak{b}}(\mathsf{a}_2) - \xi^{\mathfrak{a}}_{\mathfrak{b}}(\mathsf{a}_1) + {}_{\aleph}\delta^{\mathfrak{a}}_{\mathfrak{b}}(\mathsf{a}_1, \mathsf{a}_2)| \in E$ by Lemma 9.4 (g). The assumption that $(\mathfrak{a}, \mathfrak{b}, q) \in \mathcal{D}$ therefore necessitates $\mathsf{A}_1 \not\between^{2k}_{2\ell} \mathsf{A}_2$. Because $\gamma^{2k}_{\mathsf{Y},2\ell}$ is monotonic with respect to Γ^k_ℓ and $\Gamma^{2k}_{2\ell}$ by Lemma 4.2 4.2, we can thus infer $\mathsf{B}_1 \not\between^k_\ell \mathsf{B}_2$. In conclusion, $(\mathfrak{c}, \mathfrak{d}, p) \in \mathcal{C}$.

Step 2: It remains to prove that $(\chi_{\mathfrak{d}} \circ R(\mathfrak{c}, \mathfrak{d}, p))(f) = (S(\mathfrak{a}, \mathfrak{b}, q) \circ \chi_{\mathfrak{c}})(f)$ for any $f \in J^n_{\mathfrak{c}}$, which, by definition, is to say

$$\sum_{g \in J^n_{\mathfrak{d}}} \zeta(p, \ker(f \,\text{▘▖}\, g))\ \omega_{\mathfrak{d}}(g) = \sum_{v \in J^N_{\mathfrak{b}}} \zeta(q, \ker(\omega_{\mathfrak{c}}(f) \,\text{▘▖}\, v))\ v.$$

Since $\omega_{\mathfrak{d}}$ is a bijection from $J^n_{\mathfrak{d}}$ to $J^N_{\mathfrak{b}}$ by Lemma 9.9 (b), we can reindex the sum on the right-hand side and see that it is identical to $\sum_{g \in J^n_{\mathfrak{d}}} \zeta(q, \ker(\omega_{\mathfrak{c}}(f) \,\text{▘▖}\, \omega_{\mathfrak{d}}(g)))\, \omega_{\mathfrak{d}}(g)$. Moreover, $\zeta(p, \ker(f \,\text{▘▖}\, g)) = \zeta(q, \ker(\omega_{\mathfrak{c}}(f) \,\text{▘▖}\, \omega_{\mathfrak{d}}(g)))$ for any $g \in J^n_{\mathfrak{d}}$ according to Lemma 9.10 because $p = R(q, \mathsf{Y})$ by construction and $q \leq \{\mathsf{Y}, \mathsf{Z}\}$ by $(\mathfrak{a}, \mathfrak{b}, q) \in \mathcal{D}$. Thus, the above identity is indeed satisfied for any $f \in J^n_{\mathfrak{c}}$, which is what we needed to prove. $\square$

10. Characterizing the unitary half-liberations

ASSUMPTION 10.1. In Section 10, for any $w \in \mathbb{N}_0$, any $n \in \mathbb{N}$, if $(\mathfrak{U}, \mathfrak{O})$ is either $(\mathbb{Z}_w, \varnothing)$ or $(\varnothing, \mathbb{Z}_w)$, if the dimension profile N for $(\mathfrak{U}, \mathfrak{O})$ is such that $N(z) = n$ for any $z \in \mathbb{Z}_w$ and $N(\aleph) = 1$, then for any $k \in \mathbb{N}_0$ and any $\mathfrak{c}\colon [\![k]\!] \to \{\circ, \bullet\}$ we identify the Hilbert spaces $\ell^2(J^n_{\mathfrak{c}})$ and $\ell^2(J^N_{\mathfrak{U},\mathfrak{O}F(\mathfrak{c})})$ via the unitary operator ${}^{\;n}_{\mathfrak{U},\mathfrak{O}}\chi_{\mathfrak{c}}$, as we can do by Proposition 9.12.

At last, all requirements have been compiled to prove the Main Result.

PROOF OF THE MAIN RESULT. It suffices to prove (a) because then (b) follows by Proposition 2.53 and (c) by Proposition 2.57 and (d) by by Proposition 2.11. Depending on (G, K_G) define $(\mathcal{C}, \mathcal{D}_{\mathcal{C}}, \mathfrak{U}, \mathfrak{O}, F)$ as follows

(G, K_G)	$\mathcal{C}$	$\mathcal{D}_{\mathcal{C}}$	$\mathfrak{U}$	$\mathfrak{O}$	F
$(U^*_{w,n}, U_n)$	$\mathcal{U}^*_w$	$\mathcal{U} \wr \mathcal{Z}_w$	$\mathbb{Z}_w$	$\varnothing$	${}_{\mathbb{Z}_w,\varnothing}F$
$(U^\times_{D,n}, U_n)$	$\mathcal{U}^\times_D$	$\mathcal{U} \wr_{r_D} \mathcal{Z}_0$	$\mathbb{Z}$	$\varnothing$	${}_{\mathbb{Z},\varnothing}F$
$(U^\times_{D,n}, O^*_n)$	$\mathcal{U}^\times_D$	$\mathcal{O}^* \wr_{r_D} \mathcal{Z}_0$	$\varnothing$	$\mathbb{Z}$	${}_{\varnothing,\mathbb{Z}}F$
$(U^{\times+}_{D,n}, U^+_n)$	$\mathcal{U}^{\times+}_D$	$\mathcal{U}^+ \wr_{r_D} \mathcal{Z}_0$	$\mathbb{Z}$	$\varnothing$	${}_{\mathbb{Z},\varnothing}F$
$(U^{\times+}_{D,n}, O^+_n)$	$\mathcal{U}^{\times+}_D$	$\mathcal{O}^+ \wr_{r_D} \mathcal{Z}_0$	$\varnothing$	$\mathbb{Z}$	${}_{\varnothing,\mathbb{Z}}F$

Then, by Proposition 6.7 we can identify $K_G \hat{\wr}_{r_G} \widehat{\mathbb{Z}_{m_G}}$ with the easy algebraic compact quantum group associated with $(\mathfrak{U} \cup \{\aleph\}, \mathfrak{O}, \mathcal{D}_{\mathcal{C}}, N)$, where N is such that $N(z) = n$ for any $z \in \mathfrak{U} \cup \mathfrak{O}$ and $N(\aleph) = 1$. Let R and S be the easy rigid concrete monoidal W^*-categories associated with $(\{\sqsupset\}, \varnothing, \mathcal{C}, n)$ adnd $(\mathfrak{U} \cup \{\aleph\}, \mathfrak{O}, \mathcal{D}_{\mathcal{C}}, N)$, respectively. By Propositions 9.5, 9.6 and 9.13 then F is a full strict concrete monoidal W^*-functor $R \to S$ which satisfies $(\sqsupset, \circ) \mapsto (0, \circ) \vartriangle (\aleph, \circ)$. And that is what (a) claims. □

11. Concluding remarks

The chapter concludes with three remarks on the main result and its implications.

11.1. Co-amenability. Banica and Bichon note in [BB17] that their characterization of the compact quantum group $U^*_{w,n}$ for $w \in \mathbb{N}$ and $n \in \mathbb{N}$ as a quotient compact quantum group of $U_n \hat{\wr} \mathbb{Z}_w$ implies that $U^*_{w,n}$ is co-amenable. The same logic applies to at least one other half-liberated unitary easy quantum group.

THEOREM 11.1. *$U^\times_{\varnothing,n}$ is co-amenable for any $n \in \mathbb{N}$.*

PROOF. *Step 1: Auxiliary statement.* We first recognize that any CQG Hopf $*$-algebra has a co-amenable dual if it admits a CQG Hopf $*$-algebra morphism into the formal dual of a co-amenable algebraic compact quantum group. This follows from [KR17, Lemma 1.11], which says that any CQG Hopf $*$-algebra has a co-amenable formal dual if and only if no element of the kernel of its co-unit is invertible in the reduced CQG C^*-algebra induced. Namely, for the following reasons.

Let H and K be any CQG Hopf $*$-algebras, let ϵ_H and ϵ_K be their respective co-units, let j_H and j_K be the inclusions of H into $\mathsf{R}(H)$ respectively of K into

$\mathsf{R}(K)$, let ψ be any CQG Hopf $*$-algebra morphism $H \to K$, let φ denote $\mathsf{R}(\psi)$, let the dual of K be co-amenable and let a be any vector of H with $\epsilon_H(a) = 0$. Then $\epsilon_K(\psi(a)) = 0$ by $\epsilon_K\psi = \epsilon_H$. Hence, $j_K(\psi(a))$ is not invertible in $\mathsf{R}(K)$ by [KR17, Lemma 1.11] since K has a co-amenable dual. Because $\varphi j_H = j_K\psi$, that means that $\varphi(j_H(a))$ has no inverse in $\mathsf{R}(K)$. Since φ is an algebra morphism $\mathsf{R}(H) \to \mathsf{R}(K)$ it must map invertible elements to invertible elements. Thus, by contraposition, $j_H(a)$ cannot have an inverse in $\mathsf{R}(H)$. Since a was arbitrary the formal dual of H thus has to be co-amenable by [KR17, Lemma 1.11].

Step 2: Proof of the claim. By the main result the CQG Hopf $*$-algebra of $U^{\times}_{\varnothing,n}$ admits a CQG Hopf $*$-algebra into the CQG Hopf $*$-algebra of $(U_n)^{\hat{\times}\mathbb{Z}} \hat{\rtimes} \widehat{\mathbb{Z}}$. Since $(U_n)^{\hat{\times}\mathbb{Z}}$ is a compact group and since $\mathbb{Z}$ is an amenable discrete group, $(U_n)^{\hat{\times}\mathbb{Z}} \hat{\rtimes} \widehat{\mathbb{Z}}$ is co-amenable by [Kye08a, Proposition 7.4]. Hence, $U^{\times}_{\varnothing,n}$ is co-amenable by Step 1. □

11.2. Degenerate cases. Graph products can degenerate into free products of direct products or into direct products or free products. One may ask whether this can happen for any of the graph products occurring in the main result.

PROPOSITION 11.2. *For any countable set I, any partial commutation relation r on I and any family $(G_i)_{i\in I}$ of algebraic compact quantum groups, the cases where $\hat{*}^r_{i\in I} G_i$ can be written as a combination of free and tensor products of the factors $(G_i)_{i\in I}$ are precisely those where $\overline{r} \equiv r \cup \{(i,i) \,|\, i \in I\}$ or $\neg r \equiv (I\otimes I)\backslash r$ is transitive.*

(a) If $\overline{r}$ is an equivalence on I, then

$$\hat{\underset{i\in I}{\overset{r}{*}}} G_i \cong \hat{\underset{B\in I/\overline{r}}{*}} (\hat{\underset{i\in B}{\times}} G_i).$$

(b) If $\neg r$ is an equivalence on I, then

$$\hat{\underset{i\in I}{\overset{r}{*}}} G_i \cong \hat{\underset{B\in I/\neg r}{\times}} (\hat{\underset{i\in B}{*}} G_i).$$

PROOF. Follows immediately from the universal properties. □

PROPOSITION 11.3. *For any subsemigroup D of $(\mathbb{N},+)$, if*

$$r_D = \{(s,t) \,|\, \{s,t\} \subseteq \mathbb{Z} \wedge |t-s| \notin \{0\} \cup D\},$$

then

(a) $\overline{r_D} \equiv r_D \cup \{(s,s) \,|\, s \in \mathbb{Z}\}$ is an equivalence on $\mathbb{Z}$ if and only if $D \in \{\varnothing, \mathbb{N}\}$,

(b) $\neg r_D \equiv (\mathbb{Z}\otimes\mathbb{Z})\backslash r_D$ is an equivalence on $\mathbb{Z}$ if and only if $D \in \{\varnothing, d\mathbb{N} \,|\, d \in \mathbb{N}\}$.

PROOF. (a) For any $\{s,t\} \subseteq \mathbb{Z}$, by definition, $(s,t) \in \overline{r_D}$ if and only if $|t-s| \notin D$ or $s = t$. Thus, $\overline{r_\varnothing} = \mathbb{Z}\otimes\mathbb{Z}$ and $\overline{r_\mathbb{N}} = \{(s,s) \,|\, s \in \mathbb{Z}\}$, which are clearly equivalences. If $D \notin \{\varnothing, \mathbb{N}\}$, then $1 < g \coloneqq \min D$ because D is a subsemigroup of $(\mathbb{N},+)$. Then, $(-1,0) \in \overline{r_D}$ because $|0-(-1)| = 1 \notin D$ and $(0,g-1) \in \overline{r_D}$ because $|(g-1)-0| = g-1 \notin D$ but $(-1,g-1) \notin \overline{r_D}$ as $|(g-1)-(-1)| = g \in D$ and $-1 \neq g-1$, which proves r_D intranstive.

(b) Now, for any $\{s,t\} \subseteq \mathbb{Z}$, by definition, $(s,t) \in \neg r_D$ if and only if $|t-s| \in \{0\} \cup D$. Hence, $\neg r_\varnothing = \{(s,s) \,|\, s \in \mathbb{Z}\}$, which is obviously an equivalence. For any $d \in \mathbb{N}$

the relation $\neg r_{d\mathbb{N}} = \{(s,t) \mid t - s \in d\mathbb{Z}\}$ is simply congruence modulo d and thus an equivalence relation as well.

If $D \notin \{\varnothing, d\mathbb{N} \mid d \in \mathbb{N}\}$ and $g \coloneqq \min D$, then $g\mathbb{N} \subseteq D$ because D is a subsemigroup of $(\mathbb{N}, +)$. Because $D \neq g\mathbb{N}$ by assumption, $h \coloneqq \min(D \backslash g\mathbb{N})$ is a well-defined element of D with $g < h$. If $|h-g| = h-g$ was in D, then the inequality $h-g<h$ would imply $h-g \in g\mathbb{N}$ by the minimality of h and thus $h = (h-g)+g \in g\mathbb{N}$ – a contradiction. Now, $(g,0) \in \neg r_D$ because $|0-g| = g \in D$ and $(0,h) \in \neg r_D$ because $|h-0| = h \in D$. However, $(g,h) \notin \neg r_D$ because $|h-g| \notin \{0\} \cup D$. Hence, $\neg r_D$ fails to be transitive. □

11.3. Generalization to other non-hyperoctahedral categories. The functor ${}_{\mathfrak{U},\mathfrak{O}}F$ constructed in Section 9 was defined on the set $\mathcal{S}$ of *all* two-colored partitions and shown to restrict to functors mapping each category of the unitary half-liberation into a particular subcategory of ${}_{\mathfrak{U}\uplus\{\aleph\},\mathfrak{O}}\mathcal{S}$. That only the three families of categories of two-colored partitions associated with the unitary half-liberations were considered in this chapter was mainly an issue of practical feasibility, not of mathematical necessity. Of course, we can also restrict ${}_{\mathfrak{U},\mathfrak{O}}F$ to other, say, non-hyperoctahedral categories $\mathcal{C}$ of two-colored partitions and see what the resulting images ${}_{\mathfrak{U},\mathfrak{O}}F_{\rightarrow}(\mathcal{C})$ are. Where ${}_{\mathfrak{U},\mathfrak{O}}F_{\rightarrow}(\mathcal{C})$ turns out to be a subcategory of ${}_{\mathfrak{U}\uplus\{\aleph\},\mathfrak{O}}\mathcal{S}$ we have found yet another new relationship between two easy compact quantum groups.

The appropriate choice of tags $(\mathfrak{U}, \mathfrak{O})$ probably depends on the parameters denoted by $L(\mathcal{C})$ and $K(\mathcal{C})$ in [MW21b; MW21c]: Depending on whether $(L,K)(\mathcal{C})$ is given by $(\varnothing, w\mathbb{Z})$, $(w\mathbb{Z}, w\mathbb{Z})$ or $(w+2w\mathbb{Z}, 2w\mathbb{Z})$ for some $w \in \mathbb{N}_0$, one might want to consider $(\mathbb{Z}_w, \varnothing)$, $(\varnothing, \mathbb{Z}_w)$ and $(\varnothing, \mathbb{Z}_w)$, respectively. If ${}_{\mathfrak{U},\mathfrak{O}}F_{\rightarrow}(\mathcal{C})$ can be written as a wreath graph co-product $\mathcal{E} \wr_r \mathcal{Z}_w$ for some category $\mathcal{E}$ of two-colored or uncolored partitions, then the parameter $X(\mathcal{C})$ from [MW21b; MW21c] will likely indicate the partial commutation relation r as $\{(s,t) \mid \{s,t\} \subseteq \mathbb{Z}_w \wedge s \neq t \wedge (t -_w s \in X(\mathcal{C}) \vee s -_w t \in X(\mathcal{C}))\}$. The category $\mathcal{E}$ will be non-crossing if and only if $0 \notin X(\mathcal{C})$. Moreover, one should be able to find $\mathcal{E}$ as $\{R((\mathfrak{c}, \mathfrak{d}, p), \mathsf{S}) \mid (\mathfrak{c}, \mathfrak{d}, p) \in \mathcal{C} \wedge \mathsf{S} \in {}^{w}\Delta_{\mathfrak{d}}^{\mathfrak{c}}\}$ if $(\mathfrak{U}, \mathfrak{O}) = (\mathbb{Z}_w, \varnothing)$ and as $\{R(p, \mathsf{S}) \mid (\mathfrak{c}, \mathfrak{d}, p) \in \mathcal{C} \wedge \mathsf{S} \in {}^{w}\Delta_{\mathfrak{d}}^{\mathfrak{c}}\}$ if $(\mathfrak{U}, \mathfrak{O}) = (\varnothing, \mathbb{Z}_w)$.

In fact, this set of rules for determining $(\mathfrak{U}, \mathfrak{O}, w, \mathcal{E}, r)$ seems likely to exhibit $\mathcal{C}$ as a full subcategory of $\mathcal{E} \wr_r \mathcal{Z}_w$ if and only if the parameter $\Sigma(\mathcal{C})$ from [MW21b; MW21c] is $\{0\}$. For other values of $\Sigma(\mathcal{C})$ it is less clear what to expect. Judging by [TW16], nested wreath graph co-products may make an appearance. Likely, a different definition of ${}_{\mathfrak{U},\mathfrak{O}}F$ is in order there.

Finally, for hyperoctahedral $\mathcal{C}$, it is quite unclear what a good functor ${}_{\mathfrak{U},\mathfrak{O}}F$ might look like or if there even is a helpful one to be found.

Part 2

Results about homological invariants

CHAPTER 4

First cohomology of unitary easy quantum group duals

1. Introduction

The first quantum group cohomology with trivial coefficients of the discrete dual of any unitary easy quantum group is computed. Even those potential quantum groups whose associated categories of two-colored partitions have not yet been found. Some remarks on computing the second cohomology are offered.

1.1. Background and context. In [BS09] Banica and Speicher provided a way of constructing compact quantum groups (in the sense of [Wor87; Wor91; Wor98]) by solving infinite combinatorics puzzles: They introduced three operations on the collection of all equivalence relations of disjoint unions of finite sets and showed that each subset which is closed under these operations gives rise to a compact quantum group. An uncountable number of such sets and of the resulting so-called "easy" quantum groups and, in fact, all there can be, have since been found in [BS09; BCS10; Web13; RW14; RW16a; RW16b]. In [TW18], Tarrago and Weber extended Banica and Speicher's operations to the collection of all "two-colored" partitions, thus providing even more quantum groups to find. The classification program they initiated to determine all sets closed under the operations is still ongoing (see [TW18; Gro18; MW20; MW21a; MW21b; MW21c; Maa21]). The construction has since been further extended to partitions with arbitrarily many "colors" by Freslon in [Fre17], to "three-dimensional" sets in by Cébron and Weber in [CW16] and to equivalence relations on graphs instead of sets by Mančinska and Roberson in [MR20].

An issue that all these constructions share is that it is difficult to tell which of the resulting compact quantum groups are new and which are isomorphic to already known ones. In particular, each solution to the combinatorics puzzle does not only provide one quantum group but an entire countably infinite series, one for each dimension of its fundamental representation. And already Banica and Speicher themselves observed in [BS09, Proposition 2.4 (4)] that, at least in some cases, the quantum groups of one solution are isomorphic to those of another, just shifted by one dimension. That underlines the importance of studying quantum group invariants with the potential of telling easy quantum groups apart. Of course, these are often very difficult to compute like, e.g., the L^2-cohomology of [Kye08c] of discrete quantum groups. But perhaps at least the cohomology with trivial coefficients is a reasonable goal to strive for. The present chapter computes its first order for all of Tarrago and Weber's so-called unitary easy quantum groups, even the potential ones whose combinatorics puzzles have not been solved yet.

Moreover, it provides a few remarks on the computation of the second order begun by Bichon, Das, Franz, Kula and Skalski in [BFG17; Das+18] as well as Wendel in [Wen20]. The former five investigated the cohomology of certain easy quantum groups out of a different motivation. In particular, they were interested in the Calabi-Yau property of [Gin06], a generalization of Poincaré duality, and the classification of Schürmann triples. Namely, a quantum group whose second cohomology vanishes has the AC property, defined in [FGT15], which is important in the study of quantum Lévy processes because it guarantees the existence of an associated Schürmann triple.

In [BFG17; Das+18], Bichon, Das, Franz, Kula and Skalski had already laid out a potential strategy for computing the second cohomology of any easy quantum group. This strategy is based on two key insights and goes as follows. They interpreted quantum group cohomology as Hochschild cohomology and chose the Hochschild complex as their resolution. Thus, they were faced with having to compute the quotient space of the 2-cocycles by the 2-coboundaries. By a very clever use of the universal property of the quantum groups in question they managed to solve the linear system of equations determining the space of 2-coboundaries. This use of the universal property is the first key tool (see [BFG17, Lemma 5.4] and [Das+18, Lemma 4.1]).

Understanding the 2-cocycles then allowed them to define a "defect map", an injective linear map from 2-cohomology to a certain finite-dimensional vector space of matrices. Thus, at this point they only needed to determine the image of this defect map in order to compute the second cohomology. This is where their second key insight comes into play. Namely, although being interested only in the second order cohomology, they incidentally also computed the first. That is because they wanted to make use of the multiplicative structure of the cohomology ring. They showed that, at least for the specific quantum groups they investigated, each 2-cocycle was cohomologous to a cup product of two 1-cocycles. Thus, rather than having to probe the infinite-dimensional vector space of all 2-cocycles as the domain of the defect map they could confine themselves to determining the image of the restriction to the cup-products, a finite-dimensional space.

In short, when trying to compute the second cohomology of any easy quantum group it might be helpful, perhaps even necessary, to know the first cohomology. Hence, the main result of the present chapter might also constitute an intermediate step in computing the higher cohomologies of all easy quantum groups.

1.2. Main results. The next result extends and generalizes those of [BFG17] and [Das+18] an [Wen20].

Main results. *For any $n \in \mathbb{N}$, any unitary easy compact $n \times n$-matrix quantum group G with fundamental representation u and any category $\mathcal{C}$ of two-colored* *partitions associated with G an isomorphism of vector spaces over $\mathbb{C}$ is given by*

$$H^1(\widehat{G}) \hookrightarrow\!\!\!\!\twoheadrightarrow \{v \in M_n(\mathbb{C}) \wedge A(\mathcal{C}, v)\} \cong \mathbb{C}^{\oplus\beta_1(\widehat{G})}$$

$$\eta + B^1(\widehat{G}) \longmapsto (\eta(u_{j,i}))_{(j,i)\in\{1,\ldots,n\}\otimes\{1,\ldots,n\}},$$

where $M_n(\mathbb{C})$ is the $\mathbb{C}$-vector space of $n\times n$-matrices with complex entries and where, if we say that $\mathcal{C}$ has property

1 *if and only if every block of every partition of $\mathcal{C}$ has at most two points,*
2 *if and only if every block of every partition of $\mathcal{C}$ has at least two points,*
3 *if and only if each block of each partition of $\mathcal{C}$ with at least two points contains as many white lower and black upper points as it does black lower and white upper points,*
4 *if and only if each partition of $\mathcal{C}$ has as many white lower and black upper points as it has black lower and white upper points,*

then $\beta_1(\widehat{G}) \in \mathbb{N}_0$ and for any $v \in M_n(\mathbb{C})$ the predicate $A(\mathcal{C}, v)$ are as follows:

If $\mathcal{C}$ is ...,	*then $A(\mathcal{C}, v)$ is "..."*	*and $\beta_1(\widehat{G})$ is ...*
$1 \wedge 2 \wedge 3$	$\top$	n^2
$1 \wedge \neg 2 \wedge 3$	$\exists\lambda \in \mathbb{C}\colon v - \lambda I$ *is small*	$(n-1)^2 + 1$
$1 \wedge \neg 2 \wedge 3 \wedge \neg 4$	v *is small*	$(n-1)^2$
$1 \wedge 2 \wedge \neg 3 \wedge 4$	$\exists\lambda \in \mathbb{C}\colon v - \lambda I$ *is skew-symmetric*	$\frac{1}{2}n(n-1) + 1$
$1 \wedge 2 \wedge \neg 3 \wedge \neg 4$	v *is skew-symmetric*	$\frac{1}{2}n(n-1)$
$1 \wedge \neg 2 \wedge \neg 3 \wedge 4$	$\exists\lambda \in \mathbb{C}\colon v - \lambda I$ *is skew-symmetric and small*	$\frac{1}{2}(n-1)(n-2) + 1$
$1 \wedge \neg 2 \wedge \neg 3 \wedge \neg 4$	v *is skew-symmetric and small*	$\frac{1}{2}(n-1)(n-2)$
$\neg 1 \wedge 2 \wedge 3 \wedge 4$	v *is diagonal*	n
$\neg 1 \wedge \neg 3 \wedge 4$	$\exists\lambda \in \mathbb{C}\colon v = \lambda I$	1
$\neg 1 \wedge \neg 3 \wedge \neg 4$	$v = 0$	0

And these are all the cases that can occur. Here, a matrix is called "small" if each of its rows and each of its columns sums to 0. And I denotes the identity $n\times n$-matrix.

Note that if G is an *orthogonal* easy compact $n \times n$-matrix quantum group, i.e., if $\sqcap_{\circ\circ} \in \mathcal{C}$, then $\mathcal{C}$ obviously has neither property 3 nor property 4. Hence, $\beta_1(\widehat{G})$ can only take the three values $\frac{1}{2}n(n-1)$, $\frac{1}{2}(n-1)(n-2)$ and 0.

1.3. Structure of the chapter. Section 2 recalls the definition of compact quantum groups and the quantum group cohomology with trivial coefficients of their discrete duals.

Following that, Section 3 provides particular examples of compact quantum groups by presenting the definitions of categories of two-colored partitions and unitary easy quantum groups.

For the convenience of the reader, the definitions of the first and second order Hochschild cohomology and important results about them are stated and proved in Section 4. The section also contains a characterization of those cohomologies for universal algebras.

Section 5 defines the vector spaces of matrices appearing in the main result and computes their dimensions.

The proof of the main theorem is contained in Section 6. Starting from the characterization of the first cohomology of a universal algebra from Section 4 and using the auxiliary results from 5 the first quantum group cohomology as defined in Section 2 is computed of the discrete duals of the quantum groups defined in Section 3.

The chapter concludes with Section 7 offering a few remarks on computing the second cohomology of all unitary easy compact quantum groups.

1.4. Notation. In the following, all algebras are meant to be associative and unital. Throughout, the symbols $\triangleright$ and $\triangleleft$ are used to denote the left respectively right actions of any algebra on any bimodule. Moreover, given any vector spaces V and W over any field the symbol $[V,W]$ will stand for the vector space of linear maps from V to W. For any vector space X the name $\mathrm{End}(X)$ will be used for $[X,X]$ considered as an algebra via composition. Furthermore, for any vector space X and any (possibly infinite) set E the notation $X^{\times E}$ will be used for the E-fold direct product vector space of X (not to be confused with the direct sum $X^{\oplus E}$). Finally, for any field $\mathbb{K}$ and any set E the free $\mathbb{K}$-algebra over E will be denoted by $\mathbb{K}\langle E\rangle$. For any $R \subseteq \mathbb{K}\langle E\rangle$ we will write $\mathbb{K}\langle E\,|\,R\rangle$ for the universal $\mathbb{K}$-algebra with generators E and relations R. For any complex vector space V the conjugate vector space is denoted by V^{cj} and any element f of $[V,W]$ between any complex vector spaces V and W if interpreted as a linear map from V^{cj} to W^{cj} is referred to as f^{cj}.

2. Quantum groups and their cohomology

The most general kind of "quantum group" in the sense considered here are the locally compact quantum groups introduced by Kustermans and Vaes in [KV00; Kus01; KV03; Kus05]. Two subcategories of these are Woronowicz's compact quantum groups defined in [Wor87; Wor91; Wor98] and Van Daele's discrete quantum groups studied in [Van96; Van98].

While of those two each is equivalent to the dual category of the other via Pontryagin duality, it is customary to ascribe the cohomology discussed in the present chapter to the discrete quantum group rather than its compact dual in order to preserve the analogy with the group case. At the same time, the particular quantum groups treated in this chapter are usually considered to be compact rather than discrete.

To keep the presentation as short as possible only the definition of compact quantum groups will be given and the fact that the quantum group cohomology is actually that of discrete quantum groups will be glossed over by only giving the definition of the composition of the cohomology functor with the Pontryagin transformation. However, the custom will be respected when it comes to notation.

2.1. Compact quantum groups. Quantum groups can be defined both on an analytic, namely von-Neumann- or C^*-algebraic level, and on a purely algebraic level. The two definitions give rise to intimately related but ultimately non-equivalent categories. However, for the purposes of quantum group cohomology this is unimportant. Instead, it fully suffices and is easiest to consider the purely algebraic definition, which reads as follows.

DEFINITION 2.1. (a) An *(algebraic) compact quantum group* G is the formal dual of any tuple $(A, m, 1, \Delta, *, \epsilon, S)$, then denoted by $\mathbb{C}[\widehat{G}]$, consisting of

- □ a $\mathbb{C}$-algebra $(A, m, 1)$ (with underlying vector space A, multiplication $m\colon A \otimes_{\mathbb{C}} A \to A$ and unit $1\colon \mathbb{C} \to A$),
- □ a linear map $*\colon A \to A^{\mathrm{cj}}$, the $*$-*operation*,
- □ a $\mathbb{C}$-linear map $\Delta\colon A \to A \otimes_{\mathbb{C}} A$ and morphism of $\mathbb{C}$-algebras from $(A, m, 1)$ to the tensor product $\mathbb{C}$-algebra of $(A, m, 1)$ with itself, the *comultiplication*,
- □ a $\mathbb{C}$-linear functional ϵ on A and morphism of $\mathbb{C}$-algebras from $(A, m, 1)$ to $\mathbb{C}$, the *counit*,
- □ a $\mathbb{C}$-linear map $S\colon A \to A$, the *antipode* or *coinverse*,

such that the following conditions are met

(i) $*^{\mathrm{cj}} \circ * = \mathrm{id}_A$,
(ii) $* \circ m \circ \gamma_{A,A} = m^{\mathrm{cj}} \circ (* \otimes *)$, where $\gamma_{A,A}$ is the unique $\mathbb{C}$-linear endomorphism of $A \otimes A$ with $a \otimes b \mapsto b \otimes a$ for any $\{a, b\} \subseteq A$,
(iii) $(* \otimes *) \circ \Delta = \Delta^{\mathrm{cj}} \circ *$,
(iv) $(\mathrm{id}_A \otimes \Delta) \circ \Delta = \alpha_{A,A,A} \circ (\Delta \otimes \mathrm{id}_A) \circ \Delta$, where $\alpha_{A,A,A}$ is the unique $\mathbb{C}$-linear map $(A \otimes A) \otimes A \to A \otimes (A \otimes A)$ with $(a \otimes b) \otimes c \mapsto a \otimes (b \otimes c)$ for any $\{a, b, c\} \subseteq A$.
(v) $\mathrm{id}_A = \lambda_A \circ (\epsilon \otimes \mathrm{id}_A) \circ \Delta$, where λ_A is the $\mathbb{C}$-linear map $\mathbb{C} \otimes_{\mathbb{C}} A \to A$ with $z \otimes a \mapsto z\, a$ for any $z \in \mathbb{C}$ and $a \in A$,
(vi) $\mathrm{id}_A = \rho_A \circ (\mathrm{id}_A \otimes \epsilon) \circ \Delta$, where ρ_A is the $\mathbb{C}$-linear map $A \otimes_{\mathbb{C}} \mathbb{C} \to A$ with $a \otimes z \mapsto z\, a$ for any $z \in \mathbb{C}$ and $a \in A$,
(vii) $m \circ (S \otimes \mathrm{id}_A) \circ \Delta = 1 \circ \epsilon$,
(viii) $m \circ (\mathrm{id}_A \otimes S) \circ \Delta = 1 \circ \epsilon$.
(ix) there exists a (then uniquely determined) $\mathbb{C}$-linear functional h, the *integral*, such that
  (1) $h \circ 1 = \mathrm{id}_{\mathbb{C}}$,
  (2) $h(a^* a) \in \mathbb{R}$ and $0 \leq h(a^* a)$ for any $a \in A$,
  (3) $a = 0$ for any $a \in A$ with $h(a^* a) = 0$,
  (4) $1 \circ h = \lambda_A \circ (h \otimes \mathrm{id}_A) \circ \Delta$,
  (5) $1 \circ h = \rho_A \circ (\mathrm{id}_A \otimes h) \circ \Delta$.

(b) A *morphism* $\varphi\colon G' \to G$ *of (algebraic) compact quantum groups* from any compact quantum group G' with formal dual $(A', m', 1', *', \Delta', \epsilon', S')$ to any compact quantum group G with formal dual $(A, m, 1, *, \Delta, \epsilon, S)$ is the formal dual of any f, denoted by $\mathbb{C}[\widehat{\varphi}]$, such that

(i) f is a morphism of $\mathbb{C}$-algebras from $(A, m, 1)$ to $(A', m', 1')$,
(ii) $*' \circ f = f^{\mathrm{cj}} \circ *$,
(iii) $(f \otimes f) \circ \Delta = \Delta' \circ f$,
(iv) $\epsilon' \circ f = \epsilon$,
(v) $f \circ S = S' \circ f$.

2.2. Quantum group cohomology. For any discrete group Γ the group cohomology of Γ can be defined in many equivalent ways, e.g., as the right derived functor of the functor $(\,\cdot\,)^\Gamma$ from the category of Γ-modules to the category of abelian groups which sends any Γ-module M to the abelian subgroup M^Γ of M given by the invariant elements $\{m \in M \wedge g \triangleright m = m\}$ and which sends any Γ-module morphism $h\colon M \to N$ to the morphism of abelian groups $f^\Gamma\colon M^\Gamma \to N^\Gamma$ with $m \mapsto h(m)$ for any $m \in M^\Gamma$.

A naturally isomorphic definition is to say that the p-th group cohomology of Γ with coefficients in M is given by $\mathrm{Ext}^p_{\mathbb{C}[\Gamma]}(\mathbb{C}, M)$, the image of M under the right derivative of the functor $\mathrm{Hom}_{\mathbb{C}[\Gamma]}(\mathbb{C}, \cdot)$ which maps a left $\mathbb{C}[\Gamma]$-module to the abelian group of left $\mathbb{C}[\Gamma]$-module morphisms from the trivial left $\mathbb{C}[\Gamma]$-module $\mathbb{C}$ to M and which maps any morphism $h\colon M \to N$ of left $\mathbb{C}[\Gamma]$-modules to the group homomorphism which assigns to any group homomorphism $f\colon \mathbb{C} \to M$ the group homomorphism $h \circ f\colon \mathbb{C} \to N$.

Neither of those is the position taken in the present chapter. Rather, a third naturally isomorphic definition of group cohomology justifies the language adopted below. Namely, using Shapiro's Lemma (see [Ben98, Lemma 2.8.4]) it is possible to consider instead the p-th Hochschild cohomology $H^p_{\mathrm{HS}}(\mathbb{C}[\Gamma], M)$ of $\mathbb{C}[\Gamma]$ with coefficients in the bi-module M with the trivial right action of $\mathbb{C}[\Gamma]$. Of course, in this chapter, we are only interested in quantum group cohomology with trivial coefficients. An ad-hoc definition at least the first two orders of Hochschild cohomology is provided in Section 4.

Definition 2.2. For any compact quantum group G and any $p \in \mathbb{N}_0$, if A is the underlying algebra and ϵ the co-unit of $\mathbb{C}[\widehat{G}]$ and if X denotes the A-bimodule given by the complex vector space $\mathbb{C}$ equipped with the left and right A-actions defined by $a \otimes \lambda \mapsto \epsilon(a)\lambda$ respectively $\lambda \otimes a \mapsto \lambda\epsilon(a)$ for any $a \in A$ and $\lambda \in \mathbb{C}$, then the *p-th quantum group cohomology with trivial coefficients* of the discrete dual $\widehat{G}$ of G is defined as

$$H^p(\widehat{G}) = H^p_{\mathrm{HS}}(A, X),$$

the p-th Hochschild cohomology of A with coefficients in X.

3. Categories of two-colored partitions and unitary easy quantum groups

The quantum groups whose quantum group cohomology is investigated in the present chapters are the discrete duals of so-called easy quantum groups. They can be defined via Tannaka-Krein duality (see [Wor88]) using the combinatorics of partitions. An ad-hoc definition is given below. Note that the word "partition" here refers to something (similar but) different from its meanings in other contexts. To avoid confusion the following language is adopted throughout the chapter.

NOTATION 3.1. (a) For any mapping $f\colon X \to Y$ between any sets X and Y and any subsets $A \subseteq X$ and $B \subseteq Y$ let $f_{\rightarrow}(A) := \{f(x) \,|\, x \in A\}$ and $f^{\leftarrow}(B) := \{x \in X \wedge f(x) \in B\}$ denote the *image of* A respectively the *pre-image of* B *under* f. Moreover, let $\operatorname{ran}(f) := f_{\rightarrow}(X)$ and $\ker(f) := \{f^{\leftarrow}(\{y\}) \,|\, y \in \operatorname{ran}(f)\}$ be the *image* and *kernel* of f.

(b) For any *set-theoretical partition* p of, i.e., quotient set of an equivalence relation on, X write π_p for the associated *projection*, the mapping $X \to p$ which maps any $\mathsf{x} \in X$ to the unique $\mathsf{B} \in p$ with $\mathsf{x} \in \mathsf{B}$. And for any second set Y and any mapping $f\colon X \to Y$ with $p \leq \ker(f)$ let f/p denote the *quotient mapping*, the unique mapping $p \to Y$ with $(f/p) \circ \pi_p = f$.

(c) Given any two set-theoretical partitions p and q of any common set X, write $p \leq q$ if p is *finer* than q, i.e., if for any $\mathsf{B} \in p$ there exists $\mathsf{C} \in q$ with $\mathsf{B} \subseteq \mathsf{C}$. In that case, let $\zeta(p,q) := 1$ and let $\zeta(p,q) := 0$ otherwise. Furthermore, for any set-theoretical partitions p_1 and p_2 of X let $p_1 \vee p_2$ denote the *join* of p_1 and p_2, the unique set-theoretical partition s of X which satisfies $p_1 \leq s$ and $p_2 \leq s$ and which is minimal with that property with respect to the partial order $\leq$.

Any "partition" will be a set-theoretical partition but not vice versa.

3.1. Two-colored partitions and their categories. Rather, (two-colored) partitions can be defined as follows. For further details see [TW18], where two-colored partitions were first introduced, generalizing the "uncolored" partitions considered in [BS09].

ASSUMPTIONS 3.2. (a) Let $^{\blacksquare}(\cdot)$ and $_{\blacksquare}(\cdot)$ be any two injections with common domain $\mathbb{N}$ and with disjoint ranges.

(b) Let $\circ$ and $\bullet$ be any two sets with $\circ \neq \bullet$.

DEFINITION 3.3. (a) For any $\{k,\ell\} \subseteq \mathbb{N}_0$ we call $\Pi^k_\ell := \{^{\blacksquare}i, {_{\blacksquare}j} \,|\, i \in [\![k]\!] \wedge j \in [\![\ell]\!]\}$ the set of k upper and ℓ lower *points*.

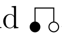
(b) Given any $\{k,\ell\} \subseteq \mathbb{N}_0$, any set X and any mappings $g\colon [\![k]\!] \to X$ and $j\colon [\![\ell]\!] \to X$ denote by $g\,^{\blacksquare}{_{\blacksquare}}\,j$ the mapping $\Pi^k_\ell \to X$ with $^{\blacksquare}i \mapsto g(i)$ for any $i \in [\![k]\!]$ and ${_{\blacksquare}j} \mapsto j(j)$ for any $j \in [\![\ell]\!]$.

(c) $\circ$ and $\bullet$ are called the two *colors* and are said to be *dual* to each other, in symbols, $\overline{\circ} := \bullet$ and $\overline{\bullet} := \circ$. They moreover have the *color values* $\sigma(\circ) := 1$ and $\sigma(\bullet) := -1$.

(d) For any $\{k,\ell\} \subseteq \mathbb{N}_0$, any $\mathfrak{c}\colon [\![k]\!] \to \{\circ,\bullet\}$ and any $\mathfrak{d}\colon [\![\ell]\!] \to \{\circ,\bullet\}$ the *color sum* of $(\mathfrak{c},\mathfrak{d})$ is the $\mathbb{Z}$-valued measure $\sigma^{\mathfrak{c}}_{\mathfrak{d}}$ on Π^k_ℓ with density $-\sigma(\mathfrak{c}_i)$ on $^{\blacksquare}i$ for any $i \in [\![k]\!]$ and density $\sigma(\mathfrak{d}_j)$ on ${_{\blacksquare}j}$ for any $j \in [\![\ell]\!]$. Moreover, $\Sigma^{\mathfrak{c}}_{\mathfrak{d}} := \sigma^{\mathfrak{c}}_{\mathfrak{d}}(\Pi^k_\ell)$ is called the *total color sum* of $(\mathfrak{c},\mathfrak{d})$.

Definition 3.4. (a) A *two-colored partition* is any triple $(\mathfrak{c}, \mathfrak{d}, p)$ for which there exist $\{k, \ell\} \subseteq \mathbb{N}_0$ such that $\mathfrak{c}$ and $\mathfrak{d}$ are mappings from $[\![k]\!]$ respectively $[\![\ell]\!]$ to $\{\circ, \bullet\}$, the upper and lower *colorings*, and such that p, the collection of *blocks*, is a set-theoretical partition of of points.

(b) Any set $\mathcal{C}$ of two-colored partitions meeting the following conditions is called a *category of two-colored partitions*:

(i) $\mathcal{C}$ contains , ,  and .

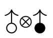
(ii) $(\mathfrak{d}, \mathfrak{c}, p^*) \in \mathcal{C}$ for any $(\mathfrak{c}, \mathfrak{d}, p) \in \mathcal{C}$, where, if $\{k, \ell\} \subseteq \mathbb{N}_0$ are such that p is a set-theoretical partition of Π_ℓ^k, then $p^* := \{\{{}^{\blacksquare}j \,|\, j \in [\![\ell]\!] \wedge {}_{\blacksquare}j \in \mathsf{B}\} \cup \{{}_{\blacksquare}i \,|\, i \in [\![k]\!] \wedge {}^{\blacksquare}i \in \mathsf{B}\}\}_{\mathsf{B} \in p}$ is the *adjoint* of p.

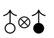
(iii) $(\mathfrak{c}_1 \otimes \mathfrak{c}_2, \mathfrak{d}_1 \otimes \mathfrak{d}_2, p_1 \otimes p_2) \in \mathcal{C}$ for any $(\mathfrak{c}_1, \mathfrak{d}_1, p_1) \in \mathcal{C}$ and $(\mathfrak{c}_2, \mathfrak{d}_2, p_2) \in \mathcal{C}$, where, if k_t and ℓ_t are such that p_t is a set-theoretical partition of $\Pi_{\ell_t}^{k_t}$ for each $t \in [\![2]\!]$, then $\mathfrak{c}_1 \otimes \mathfrak{c}_2 \in \{\circ, \bullet\}^{\otimes(k_1+k_2)}$ is defined by $i \mapsto \mathfrak{c}_1(i)$ if $i \leq k_1$ and $i \mapsto \mathfrak{c}_2(i - k_1)$ if $k_1 < i$ and, analogously, $\mathfrak{d}_1 \otimes \mathfrak{d}_2 \in \{\circ, \bullet\}^{\otimes(\ell_1+\ell_2)}$ is defined by $j \mapsto \mathfrak{d}_1(j)$ if $j \leq \ell_1$ and $j \mapsto \mathfrak{d}_2(j - \ell_1)$ if $\ell_1 < j$, and where $p_1 \otimes p_2 := p_1 \cup \{\{{}^{\blacksquare}(k_1 + i) \,|\, i \in [\![k_2]\!] \wedge {}^{\blacksquare}i \in \mathsf{B}\} \cup \{{}_{\blacksquare}(\ell_1 + j) \,|\, j \in [\![\ell_2]\!] \wedge {}_{\blacksquare}j \in \mathsf{B}\}\}_{\mathsf{B} \in p_2}$ is the *tensor product* of (p_1, p_2).

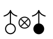
(iv) $(\mathfrak{c}, \mathfrak{e}, qp) \in \mathcal{C}$ for any $(\mathfrak{c}, \mathfrak{d}, p) \in \mathcal{C}$ and $(\mathfrak{d}, \mathfrak{e}, q) \in \mathcal{C}$, where if $\{k, \ell, m\} \subseteq \mathbb{N}_0$ are such that p is a set-theoretical partition of Π_ℓ^k and q one of Π_m^ℓ, and if s is the join of the two set-theoretical partitions $\{\{j \in [\![\ell]\!] \wedge {}_{\blacksquare}j \in \mathsf{A}\}\}_{\mathsf{A} \in p}$ and $\{\{i \in [\![\ell]\!] \wedge {}^{\blacksquare}i \in \mathsf{C}\}\}_{\mathsf{C} \in q}$ of $[\![\ell]\!]$, then $qp := \{\mathsf{A} \in p \wedge \mathsf{A} \subseteq \Pi_0^k\} \cup \{\mathsf{C} \in q \wedge \mathsf{C} \subseteq \Pi_0^m\} \cup \{\bigcup\{\mathsf{A} \cap \Pi_0^k \,|\, \mathsf{A} \in p \wedge \exists j \in B\colon {}_{\blacksquare}j \in \mathsf{A}\} \cup \bigcup\{\mathsf{C} \cap \Pi_m^0 \,|\, \mathsf{C} \in q \wedge \exists i \in B\colon {}^{\blacksquare}i \in \mathsf{C}\}\}_{B \in s} \backslash \{\varnothing\}$ is the *composition* of (q, p).

(c) For any set $\mathcal{G}$ of two-colored partition we write $\langle \mathcal{G} \rangle$ for the intersection of all categories of partitions containing $\mathcal{G}$ and we say that $\mathcal{G}$ *generates* $\langle \mathcal{G} \rangle$.

Definition 3.5. We say that any category $\mathcal{C}$ of two-colored partitions

(a) is *case* $\mathcal{O}$ if  $\notin \mathcal{C}$ and  $\notin \mathcal{C}$.

(b) is *case* $\mathcal{B}$ if  $\in \mathcal{C}$ and  $\notin \mathcal{C}$.

(c) is *case* $\mathcal{H}$ if  $\notin \mathcal{C}$ and  $\in \mathcal{C}$.

(d) is *case* $\mathcal{S}$ if  $\in \mathcal{C}$ and  $\in \mathcal{C}$.

(e) *has only neutral non-singleton blocks* if $\sigma_{\mathfrak{d}}^{\mathfrak{c}}(\mathsf{B}) = 0$ for any $(\mathfrak{c}, \mathfrak{d}, p) \in \mathcal{C}$ and any $\mathsf{B} \in p$ with $2 \leq |\mathsf{B}|$ and that $\mathcal{C}$ *has some non-neutral non-singleton blocks* otherwise.

(f) *has only neutral partitions* if $\Sigma_{\mathfrak{d}}^{\mathfrak{c}} = 0$ for any $(\mathfrak{c}, \mathfrak{d}, p) \in \mathcal{C}$ and that $\mathcal{C}$ *has some non-neutral partitions* otherwise.

We will need to know the following elementary facts about categories of two-colored partitions.

Definition 3.6. Given any $\{k, \ell\} \subseteq \mathbb{N}_0$, any $\mathfrak{c} \in \{\circ, \bullet\}^{\otimes k}$, any $\mathfrak{d} \in \{\circ, \bullet\}^{\otimes \ell}$ and any set-theoretical partition p of Π_ℓ^k the *dual* of $(\mathfrak{c}, \mathfrak{d}, p)$ is the triple $(\overline{\mathfrak{d}}, \overline{\mathfrak{c}}, \overline{p})$, where $\overline{\mathfrak{d}} \in \{\circ, \bullet\}^{\otimes \ell}$ is defined by $j \mapsto \overline{\mathfrak{d}_{\ell-j+1}}$, where $\overline{\mathfrak{c}} \in \{\circ, \bullet\}^{\otimes k}$ is defined by $i \mapsto \overline{\mathfrak{c}_{k-i+1}}$, and where $\overline{p} \coloneqq \{\{{}^{\blacksquare}(\ell - j + 1) \,|\, j \in [\![\ell]\!] \wedge {}_{\blacksquare}j \in \mathsf{B}\} \cup \{{}_{\blacksquare}(k - i + 1) \,|\, i \in [\![k]\!] \wedge {}^{\blacksquare}i \in \mathsf{B}\}\}_{\mathsf{B}\in p}$ is the *dual* of p.

LEMMA 3.7. *Let $\mathcal{C}$ be any category of two-colored partitions.*
(a) $(\overline{\mathfrak{d}}, \overline{\mathfrak{c}}, \overline{p}) \in \mathcal{C}$ *for any* $(\mathfrak{c}, \mathfrak{d}, p) \in \mathcal{C}$.
(b) $\uparrow_{\circ}\otimes\uparrow_{\bullet} \in \mathcal{C}$ *if and only if there exist* $(\mathfrak{c}, \mathfrak{d}, p) \in \mathcal{C}$ *and* $\mathsf{B} \in p$ *such that* $|\mathsf{B}| < 2$.
(c) $\sqcap\!\sqcap\!\sqcap_{\circ\bullet\circ\bullet} \in \mathcal{C}$ *if and only if there exist* $(\mathfrak{c}, \mathfrak{d}, p) \in \mathcal{C}$ *and* $\mathsf{B} \in p$ *such that* $|\mathsf{B}| > 2$.
(d) If $\uparrow_{\circ}\otimes\uparrow_{\bullet} \in \mathcal{C}$ *and* $\sqcap\!\sqcap\!\sqcap_{\circ\bullet\circ\bullet} \in \mathcal{C}$*, then* $\sqcap\!\!\uparrow\!\!\sqcap\!\uparrow_{\circ\bullet\circ\bullet} \in \mathcal{C}$.
(e) If $\uparrow_{\circ}\otimes\uparrow_{\bullet} \in \mathcal{C}$*, then* $\uparrow_{\circ}^{\otimes|\Sigma_{\mathfrak{d}}^{\mathfrak{c}}|} \in \mathcal{C}$ *for any* $(\mathfrak{c}, \mathfrak{d}, p) \in \mathcal{C}$.

PROOF. Part (a) is implied by [TW18, Lemmata 1.1 (a)]. Parts (b) and (c) are [TW18, Lemmata 1.3 (b), 2.1 (a)]. and [TW18, Lemmata 1.3 (d), 2.1 (b)], respectively. Parts (d) and (e) follow immediately from [TW18, Lemmata 1.3 (b), 2.1 (a)] and [TW18, Lemmata 1.1 (a), (b)]. □

3.2. Unitary easy quantum groups. "Easy" quantum groups are now defined by transforming the elements of a given category of partitions into relations for the generators of a universal algebra that can be given the structure of a compact quantum group. To be more precise, an entire series of compact quantum groups indexed by $\mathbb{N}$ arises in this way.

ASSUMPTIONS 3.8. In the following, fix any $n \in \mathbb{N}$ and any $2n^2$-elemental set $E = \{u^{\circ}_{j,i}, u^{\bullet}_{j,i}\}_{i,j=1}^{n}$ and define the two families $u^{\circ} \coloneqq (u^{\circ}_{j,i})_{(j,i)\in[\![n]\!]^{\otimes 2}}$ and $u^{\bullet} \coloneqq (u^{\bullet}_{j,i})_{(j,i)\in[\![n]\!]^{\otimes 2}}$.

The transformation of partitions into relations is accomplished by the following formula.

NOTATION 3.9. For any $\{k, \ell\} \subseteq \mathbb{N}_0$, any $\mathfrak{c} \in \{\circ, \bullet\}^{\otimes k}$ and $\mathfrak{d} \in \{\circ, \bullet\}^{\otimes \ell}$, any partition p of Π_{ℓ}^{k} and any $g \in [\![n]\!]^{\otimes k}$ and $j \in [\![n]\!]^{\otimes \ell}$, let, in $\mathbb{C}\langle E\rangle$,

$$\mathrm{r}_{\mathfrak{d}}^{\mathfrak{c}}(p)_{j,g} \coloneqq \sum_{i\in[\![n]\!]^{\otimes \ell}} \zeta(p, \ker(g\,{}^{\blacksquare}{}_{\blacksquare}\, i)) \overrightarrow{\prod_{b=1}^{\ell}} u^{\mathfrak{d}_b}_{j_b, i_b} - \sum_{h\in[\![n]\!]^{\otimes k}} \zeta(p, \ker(h\,{}^{\blacksquare}{}_{\blacksquare}\, j)) \overrightarrow{\prod_{a=1}^{k}} u^{\mathfrak{c}_a}_{h_a, g_a}.$$

For example, the relations induced by $\sqcap_{\circ\bullet}$ and $\sqcap_{\bullet\circ}$ will be of the utmost importance.

LEMMA 3.10. *For any $g \in [\![n]\!]^{\otimes 2}$ and $j \in [\![n]\!]^{\otimes 2}$ the following hold.*

$\mathrm{r}^{\varnothing}_{\bullet\circ}(\sqcap)_{j,\varnothing} = \sum_{i=1}^{n} u^{\bullet}_{j_1,i} u^{\circ}_{j_2,i} - \delta_{j_1,j_2} 1$	$\mathrm{r}^{\bullet\circ}_{\varnothing}(\sqcup)_{\varnothing,g} = \delta_{g_1,g_2} 1 - \sum_{h=1}^{n} u^{\bullet}_{h,g_1} u^{\circ}_{h,g_2}$
$\mathrm{r}^{\varnothing}_{\circ\bullet}(\sqcap)_{j,\varnothing} = \sum_{i=1}^{n} u^{\circ}_{j_1,i} u^{\bullet}_{j_2,i} - \delta_{j_1,j_2} 1$	$\mathrm{r}^{\circ\bullet}_{\varnothing}(\sqcup)_{\varnothing,g} = \delta_{g_1,g_2} 1 - \sum_{h=1}^{n} u^{\circ}_{h,g_1} u^{\bullet}_{h,g_2}$

PROOF. Immediate from the definition. □

The definition of "easy" quantum groups now goes as follows.

DEFINITION 3.11. For any category $\mathcal{C}$ of two-colored partitions the *unitary easy (algebraic) compact quantum group* of $(\mathcal{C}, n)$ is $(A, m, 1, *, \Delta, \epsilon, S)$, where

(a) $(A, m, 1)$ is given by $\mathbb{C}\langle E \,|\, R\rangle$, where

$$R = \big\{ \mathrm{r}^{\mathfrak{c}}_{\mathfrak{d}}(p)_{g,j} \,|\, \{k,\ell\} \subseteq \mathbb{N}_0 \wedge \mathfrak{c} \in \{\circ,\bullet\}^{\otimes k} \wedge \mathfrak{d} \in \{\circ,\bullet\}^{\otimes \ell} \\ \wedge (\mathfrak{c},\mathfrak{d},p) \in \mathcal{R} \wedge g \in [\![n]\!]^{\otimes k} \wedge j \in [\![n]\!]^{\otimes \ell} \big\},$$

where

$$\mathcal{R} = \mathcal{G} \cup \big\{ (\overline{\mathfrak{c}}, \overline{\mathfrak{d}}, (\overline{p})^*) \,|\, (\mathfrak{c},\mathfrak{d},p) \in \mathcal{G} \big\} \cup \{\sqcap_{\circ\bullet}, \sqcap_{\bullet\circ}\},$$

where $\mathcal{G}$ can be any set of two-colored partitions generating $\mathcal{C}$,

(b) $*\colon A \to A^{\mathrm{cj}}$ is the unique $\mathbb{C}$-algebra morphism from $(A,m,1)$ to the opposite algebra of $(A,m,1)$ with $u^{\mathfrak{c}}_{j,i} \mapsto u^{\overline{\mathfrak{c}}}_{j,i}$ for any $\{i,j\} \subseteq [\![n]\!]$ and $\mathfrak{c} \in \{\circ,\bullet\}$,

(c) $\Delta\colon A \to A \otimes \mathbb{C} A$ is the unique morphism of $\mathbb{C}$-algebras from $(A,m,1)$ to the tensor product of $(A,m,1)$ with itself such that $u^{\mathfrak{c}}_{j,i} \mapsto \sum_{s=1}^{n} u^{\mathfrak{c}}_{j,s} \otimes u^{\mathfrak{c}}_{s,i}$ for any $\{i,j\} \subseteq [\![n]\!]$ and $\mathfrak{c} \in \{\circ,\bullet\}$,

(d) $\epsilon\colon A \to \mathbb{C}$ is the unique morphism of $\mathbb{C}$-algebras from $(A,m,1)$ to $\mathbb{C}$ with $u^{\mathfrak{c}}_{j,i} \mapsto \delta_{j,i}$ for any $\{i,j\} \subseteq [\![n]\!]$ and $\mathfrak{c} \in \{\circ,\bullet\}$,

(e) $S\colon A \to A$ is the unique morphism of $\mathbb{C}$-algebra from $(A,m,1)$ to the opposite algebra of $(A,m,1)$ with $u^{\mathfrak{c}}_{j,i} \mapsto u^{\overline{\mathfrak{c}}}_{i,j}$ for any $\{i,j\} \subseteq [\![n]\!]$ and $\mathfrak{c} \in \{\circ,\bullet\}$.

Remark 3.12. The definition of unitary easy quantum groups is usually given in terms of universal $*$-algebras. In order to see that the variant given above is equivalent observe that for any $\{k,\ell\} \subseteq \mathbb{N}_0$, any $\mathfrak{c} \in \{\circ,\bullet\}^{\otimes k}$, any $\mathfrak{d} \in \{\circ,\bullet\}^{\otimes \ell}$, any $p \in \mathcal{C}(\mathfrak{c},\mathfrak{d})$, any $g \in [\![n]\!]^{\otimes k}$ and any $j \in [\![n]\!]^{\otimes \ell}$, if for any $m \in \mathbb{N}_0$ and any $e \in [\![n]\!]^{\otimes m}$ the tuple $\overline{e} \in [\![n]\!]^{\otimes m}$ is defined by $i \mapsto e_{m-i+1}$ for any $i \in [\![m]\!]$, then, with respect to the $*$-map in Definition 3.11,

$$\begin{aligned}(\mathrm{r}^{\mathfrak{c}}_{\mathfrak{d}}(p)_{j,g})^* &= \sum_{i \in [\![n]\!]^{\otimes \ell}} \zeta(p, \ker(g \centerdot i)) \overleftarrow{\prod_{b=1}^{\ell}} (u^{\mathfrak{d}_b}_{j_b,i_b})^* - \sum_{h \in [\![n]\!]^{\otimes k}} \zeta(p, \ker(h \centerdot j)) \overleftarrow{\prod_{a=1}^{k}} (u^{\mathfrak{c}_a}_{h_a,g_a})^* \\ &= \sum_{i \in [\![n]\!]^{\otimes \ell}} \zeta((\overline{p})^*, \ker(\overline{g} \centerdot \overline{i})) \overrightarrow{\prod_{b=1}^{\ell}} u^{\overline{\mathfrak{d}}_b}_{\overline{j}_b,\overline{i}_b} - \sum_{h \in [\![n]\!]^{\otimes k}} \zeta((\overline{p})^*, \ker(\overline{h} \centerdot \overline{j})) \overrightarrow{\prod_{a=1}^{k}} u^{\overline{\mathfrak{c}}_a}_{\overline{h}_a,\overline{g}_a} \\ &= \mathrm{r}^{\overline{\mathfrak{c}}}_{\overline{\mathfrak{d}}}((\overline{p})^*)_{\overline{j},\overline{g}}.\end{aligned}$$

That shows that adding the relations $\big\{ (\overline{\mathfrak{c}}, \overline{\mathfrak{d}}, (\overline{p})^*) \,|\, (\mathfrak{c},\mathfrak{d},p) \in \mathcal{G} \big\}$ to $\mathcal{G}$ in the definition of $\mathcal{R}$ compensates for the switch from the universal $*$-algebra to the universal algebra.

4. First and second Hochschild cohomology of universal algebras

For the convenience of the reader Section 4 recalls the definitions of and some elementary results about the first and second Hochschild cohomology of arbitrary algebras (Sections 4.1 and 4.3), including the normalization of 2-coycles (Section 4.3.1) and the reduction of second cohomology to first cohomology with different coefficients (Section 4.3.2). Moreover, it provides an algebraic characterization of 2-coboundaries (Section 4.3.3) as well as a way of producing 2-coycles from 1-cocycles (Section 4.3.4) via so-called "cup-products".

However, the main goal of the section is to address very generally the first and second Hochschild cohomologies of universal algebras, i.e., algebras defined in terms of generators and relations, in Sections 4.2 and 4.4.

No claim to originality is made with respect to the material of this section.

4.1. First Hochschild cohomology of arbitrary algebras. The following definitions were first given by Hochschild in [Hoc56].

ASSUMPTIONS 4.1. In Section 4.1, let $\mathbb{K}$ be any field, A any $\mathbb{K}$-algebra and X any A-bimodule.

DEFINITION 4.2. (a) The *X-valued Hochschild 1-cocycles of A* are the $\mathbb{K}$-vector subspace $Z^1_{\text{HS}}(A,X)$ of $[A,X]$ formed by all elements η such that

$$(\partial\eta)(a_1\otimes a_2) \coloneqq a_1 \rhd \eta(a_2) - \eta(a_1a_2) + \eta(a_1) \lhd a_2 = 0$$

for any $\{a_1,a_2\}\subseteq A$.

(b) The *X-valued Hochschild 1-coboundaries of A* are the $\mathbb{K}$-vector subspace $B^1_{\text{HS}}(A,X)$ of $[A,X]$ formed by all elements η such that there exists $x\in X$ with

$$\eta(a) = (\partial x)(a) \coloneqq a \rhd x - x \lhd a$$

for any $a\in A$.

LEMMA 4.3. *$B^1_{\text{HS}}(A,X)$ is a $\mathbb{K}$-vector subspace of $Z^1_{\text{HS}}(A,X)$.*

PROOF. For any $x\in X$, if $\eta=\partial x$, then

$$\begin{aligned}
&(\partial\eta)(a_1\otimes a_2)\\
&\quad= a_1 \rhd (\partial x)(a_2) - (\partial x)(a_1a_2) + (\partial x)(a_1) \lhd a_2\\
&\quad= a_1 \rhd (a_2 \rhd x - x \lhd a_2) - (a_1a_2 \rhd x - x \lhd a_1a_2) + (a_1 \rhd x - x \lhd a_1) \lhd a_2\\
&\quad= 0
\end{aligned}$$

by $a_1 \rhd (a_2 \rhd x) = a_1a_2 \rhd x$ and $a_1 \rhd (x \lhd a_2) = (a_1 \rhd x) \lhd a_2$ and $x \lhd a_1a_2 = (x \lhd a_1) \lhd a_2$. □

DEFINITION 4.4. We call the quotient $\mathbb{K}$-vector space $H^1_{\text{HS}}(A,X)$ of $Z^1_{\text{HS}}(A,X)$ with respect to $B^1_{\text{HS}}(A,X)$ the *first Hochschild cohomology of A with X-coefficients.*

REMARK 4.5. It is important to the proof of the next result to observe that for any $\eta\in Z^1_{\text{HS}}(A,X)$ necessarily, $\eta(1)=0$ because $\eta(1)=\eta(1\cdot 1)=\eta(1)\lhd 1+1\rhd\eta(1)=2\eta(1)$.

The following is a reformulation and explication of [KR17, Lemma 1.9].

LEMMA 4.6. *(a) The $\mathbb{K}$-vector space $A \oplus X$ becomes a $\mathbb{K}$-algebra $A \oplus^1 X$ when equipped with the unit $(1,0)$ and the multiplication defined by*

$$(a_1, x_1) \otimes (a_2, x_2) \mapsto (a_1 a_2, a_1 \rhd x_2 + x_1 \lhd a_2)$$

for any $\{a_1, a_2\} \subseteq A$ and $\{x_1, x_2\} \subseteq X$.

(b) For any $m \in \mathbb{N}$, any $\{a_i\}_{i=1}^m \subseteq A$ and any $\{x_i\}_{i=1}^m \subseteq X$, in $A \oplus^1 X$,

$$\overrightarrow{\prod}_{i=1}^m (a_i, x_i) = \left(\overrightarrow{\prod}_{i=1}^m a_i, \sum_{i=1}^m \left(\overrightarrow{\prod}_{j=1}^{i-1} a_j\right) \rhd x_i \lhd \left(\overrightarrow{\prod}_{j=i+1}^m a_j\right)\right).$$

(c) If $\pi^1 \colon A \oplus X \to A$, $(a,x) \mapsto a$ and $\pi^2 \colon A \oplus X \to X$, $(a,x) \mapsto x$ are the projections, then there exists a bijection

$$Z^1_{\mathrm{HS}}(A,X) \hookrightarrow\!\!\!\rightarrow \{f \ \mathbb{K}\text{-algebra homomorphism } A \to A \oplus^1 X \wedge \pi^1 \circ f = \mathrm{id}_A\}$$

which assigns to any $\eta \in Z^1_{\mathrm{HS}}(A,X)$ the mapping $A \to A \oplus X$ with $a \mapsto (a, \eta(a))$ for any $a \in A$. Its inverse obeys the rule $f \mapsto \pi^2 \circ f$ for any $\mathbb{K}$-algebra homomorphism f from A to $A \oplus^1 X$ with $\pi^1 \circ f = \mathrm{id}_A$.

PROOF. (a) For any $a \in A$ and any $x \in X$,

$$(a,x)(1,0) = (a1, a \rhd 0 + x \lhd 1) = (a,x) = (1a, 1 \rhd x + 0 \lhd a) = (1,0)(a,x),$$

which proves that $(1,0)$ is a unit. The multiplication is associative because for any $\{a_1, a_2, a_3\} \subseteq A$ and any $\{x_1, x_2, x_3\} \subseteq X$, on the one hand,

$$\begin{aligned}((a_1,x_1)\cdot(a_2,x_2))\cdot(a_3,x_3) &= (a_1a_2, a_1 \rhd x_2 + x_1 \lhd a_2)\cdot(a_3,x_3)\\ &= (a_1a_2a_3, (a_1a_2) \rhd x_3 + (a_1 \rhd x_2 + x_1 \lhd a_2) \lhd a_3),\end{aligned}$$

and, on the other hand,

$$\begin{aligned}(a_1,x_1)\cdot((a_2,x_2)\cdot(a_3,x_3)) &= (a_1,x_1)\cdot(a_2a_3, a_2 \rhd x_3 + x_2 \lhd a_3)\\ &= (a_1a_2a_3, a_1 \rhd (a_2 \rhd x_3 + x_2 \lhd a_3) + x_1 \lhd (a_2a_3)),\end{aligned}$$

which is identically $(a_1a_2a_3, a_1a_2 \rhd x_3 + a_1 \rhd x_2 \lhd a_3 + x_1 \lhd a_2a_3)$ in both cases.

(b) Respectively, the cases $m = 1,2,3$ are trivial, the definition of the multiplication and a result in the proof of (a). We assume the claim holds for $m-1$ and prove it for m. Indeed, the product

$$\overrightarrow{\prod}_{i=1}^m (a_i, x_i) = \left(\overrightarrow{\prod}_{i=1}^{m-1} a_i, \sum_{i=1}^{m-1} \left(\overrightarrow{\prod}_{j=1}^{i-1} a_j\right) \rhd x_i \lhd \left(\overrightarrow{\prod}_{j=i+1}^{m-1} a_j\right)\right)\cdot(a_m, x_m),$$

by definition, is given by

$$\left(\left(\overrightarrow{\prod}_{i=1}^{m-1} a_i\right) a_m, \left(\overrightarrow{\prod}_{i=1}^{m-1} a_i\right) \rhd x_m + \left(\sum_{i=1}^{m-1} \left(\overrightarrow{\prod}_{j=1}^{i-1} a_j\right) \rhd x_i \lhd \left(\overrightarrow{\prod}_{j=i+1}^{m-1} a_j\right)\right) \lhd a_m\right)$$

which can be rewritten as

$$\left(\overrightarrow{\prod}_{i=1}^m a_i, \sum_{i=1}^m \left(\overrightarrow{\prod}_{j=1}^{i-1} a_j\right) \rhd x_i \lhd \left(\overrightarrow{\prod}_{j=i+1}^m a_j\right)\right).$$

(c) For any $\mathbb{K}$-linear map f from A to $A \oplus X$ with $f^1 = \pi^1 \circ f$ and $f^2 = \pi^2 \circ f$, being a homomorphism of $\mathbb{K}$-algebras from A to $A \oplus^1 X$ is equivalent to the conjunction of $f^1(1) = 1$ and $f^2(1) = 0$ and $f^1(a_1 a_2) = f^1(a_1) f^1(a_2)$ and

$$f^2(a_1 a_2) = f^1(a_1) \triangleright f^2(a_2) + f^2(a_1) \triangleleft f^1(a_2)$$

holding for any $\{a_1, a_2\} \subseteq A$. Hence, f is a homomorphism of $\mathbb{K}$-algebras from A to $A \oplus^1 X$ with $f^1 = \mathrm{id}_A$ if and only if $f^2(1) = 0$ and $\partial f^2 = 0$. By definition and by Remark 4.5 that is the case if and only if $f^2 \in Z^1_{\mathrm{HS}}(A, X)$. □

REMARK 4.7. If $A \oplus^1 X$ is the $\mathbb{K}$-algebra structure on $A \oplus X$ from Lemma 4.6 and if $M = \{f \text{ } \mathbb{K}\text{-algebra hom. } A \to A \oplus^1 X \wedge \pi^1 \circ f = \mathrm{id}_A\} \subseteq [A, A \oplus X]$, then, even though the rule $f \mapsto \pi^2 \circ f$ defines a $\mathbb{K}$-linear map from $[A, A \oplus X]$ to $[A, X]$ and even though this mapping restricts to a bijection from M to the $\mathbb{K}$-vector subspace $Z^1_{\mathrm{HS}}(A, X)$ of $[A, X]$, the set M is generally *not* a $\mathbb{K}$-vector subspace of $[A, A \oplus X]$.

4.2. First Hochschild cohomology of universal algebras. Using Lemma 4.6, it is possible to give an equational characterization of the 1-cocycles on any universal algebra.

ASSUMPTIONS 4.8. In Section 4.2, let $\mathbb{K}$ be any field, E any (not necessarily finite) set, $R \subseteq \mathbb{K}\langle E \rangle$ any subset, J the two-sided $\mathbb{K}$-ideal of $\mathbb{K}\langle E \rangle$ generated by R, and X any $\mathbb{K}\langle E \,|\, R \rangle$-bimodule.

DEFINITION 4.9. Let

$$F^1_{E,R,X} \colon \mathbb{K}\langle E \rangle \to [X^{\times E}, X],\ p \mapsto F^{1,p}_{E,R,X}$$

be the unique $\mathbb{K}$-linear map with for any $m \in \mathbb{N}$ and any $\{e_i\}_{i=1}^m \subseteq E$, if $p = \overrightarrow{\prod}_{i=1}^m e_i$, then for any $x \in X^{\times E}$,

$$F^{1,p}_{E,R,X}(x) = \textstyle\sum_{i=1}^m \left(\overrightarrow{\prod}_{j=1}^{i-1} e_j + J\right) \triangleright x_{e_i} \triangleleft \left(\overrightarrow{\prod}_{j=i+1}^{m} e_j + J\right),$$

and with $F^{1,1}_{E,R,X} = 0$.

LEMMA 4.10. *A commutative diagram of $\mathbb{K}$-linear maps is given by*

$$\begin{array}{ccc} Z^1_{\mathrm{HS}}(\mathbb{K}\langle E \,|\, R \rangle, X) & \hookrightarrow\!\!\!\!\twoheadrightarrow & \{x \in X^{\times E} \wedge \forall r \in R \colon F^{1,r}_{E,R,X}(x) = 0\} \\ \subseteq \uparrow & & \uparrow \subseteq \\ B^1_{\mathrm{HS}}(\mathbb{K}\langle E \,|\, R \rangle, X) & \hookrightarrow\!\!\!\!\twoheadrightarrow & \{((e+J) \triangleright z - z \triangleleft (e+J))_{e \in E} \,|\, z \in X\} \end{array},$$

where the sets on the right are $\mathbb{K}$-vector subspaces of $X^{\times E}$ and where the horizontal arrows both assign to any element η of their respective domains the tuple

$$(\eta(e+J))_{e \in E}.$$

In particular, the lower right space is a subspace of the one above. Moreover, the horizontal arrows are both $\mathbb{K}$-linear isomorphisms. Their respective inverses both *assign to any element x of their respective domains the mapping $\mathbb{K}\langle E\,|\,R\rangle \to X$ with*

$$p + J \mapsto F^{1,p}_{E,R,X}(x)$$

for any $p \in \mathbb{K}\langle E\rangle$. For the lower row of the diagram that is the same as assigning $((e+J) \triangleright z - z \triangleleft (e+J))_{e\in E} \mapsto \partial z$ for any $z \in X$.

Proof. We prove the many statements constituting the claim in five steps. Throughout, abbreviate $A \equiv \mathbb{K}\langle E\,|\,R\rangle$, let $B \equiv A \oplus^1 X$ be the algebra structure on $A \oplus X$ from Lemma 4.6 and let π^1 and π^2 be the projections $A \oplus X \to A$ and $A \oplus X \to X$, respectively.

Step 1: First auxiliary statement: We prove for any $z \in X$ that, if $x = ((e+J) \triangleright z - z \triangleleft (e+J))_{e\in E}$, then $\partial z(p+J) = F^{1,p}_{E,R,X}(x)$ for any $p \in \mathbb{K}\langle E\rangle$. Indeed, for any $m \in \mathbb{N}_0$ and any $\{e_i\}_{i=1}^m \subseteq E$, if $p = e_1 \ldots e_m$, then by definition,

$$\begin{aligned}
F^{1,p}_{E,R,X}(x) &= \textstyle\sum_{i=1}^m \left(\overrightarrow{\prod}_{j=1}^{i-1} e_j + J\right) \triangleright x_{e_i} \triangleleft \left(\overrightarrow{\prod}_{j=i+1}^{m} e_j + J\right)\\
&= \textstyle\sum_{i=1}^m \left(\overrightarrow{\prod}_{j=1}^{i-1} e_j + J\right) \triangleright (e_i + J \triangleright z - z \triangleleft e_i + J) \triangleleft \left(\overrightarrow{\prod}_{j=i+1}^{m} e_j + J\right)\\
&= \left(\textstyle\sum_{i=2}^{m+1} \left(\overrightarrow{\prod}_{j=1}^{i-1} e_j + J\right) \triangleright z \triangleleft \left(\overrightarrow{\prod}_{j=i}^{m} e_j + J\right)\right)\\
&\qquad\qquad - \left(\textstyle\sum_{i=1}^{m} \left(\overrightarrow{\prod}_{j=1}^{i-1} e_j + J\right) \triangleright z \triangleleft \left(\overrightarrow{\prod}_{j=i}^{m} e_j + J\right)\right)\\
&= \left(\overrightarrow{\prod}_{i=1}^{m} e_i + J\right) \triangleright z - z \triangleleft \left(\overrightarrow{\prod}_{i=1}^{m} e_i + J\right)
\end{aligned}$$

and thus by $\mathbb{K}$-linearity, actually, $F^{1,p}_{E,R,X}(x) = (p+J) \triangleright z - z \triangleleft (p+J) = \partial z(p+J)$ for any $p \in \mathbb{K}\langle E\rangle$.

Step 2: Vertical arrows. By Lemma 4.3 the left vertical arrow is well-defined and injective. The same is true for the right one: Given any $z \in X$, if $x = ((e+J) \triangleright z - z \triangleleft (e+J))_{e\in E}$, then $F^{1,r}_{E,R,X}(x) = 0$ for any $r \in R$ because $F^{1,r}_{E,R,X}(x) = \partial z(r+J)$ by Step 1 and because $r \in J$.

Step 3: Upper horizontal arrow. Next, we prove that the upper horizontal arrow is a well-defined bijection and that it has the alleged inverse. Namely, we prove that it can be written as $W \circ V \circ U$ for well-defined bijections W, V and U and that $U^{-1} \circ V^{-1} \circ W^{-1}$ is the claimed inverse. Actually, we will construct V out of another bijection V_0.

Step 3.1: Second auxiliary statement. Before the introduction of the various bijections it is convenient to observe the following. For any $b \in B^{\times E}$, if $(a_e, x_e) = b_e$ for each $e \in E$, then for any $m \in \mathbb{N}_0$ and any $\{e_i\}_{i=1}^m \subseteq E$ by Lemma 4.6 (b),

$$\left(\overrightarrow{\prod}_{i=1}^{m} e_i\right)(b) = \left(\overrightarrow{\prod}_{i=1}^{m} a_{e_i}, \textstyle\sum_{i=1}^m \left(\overrightarrow{\prod}_{j=1}^{i-1} e_j + J\right) \triangleright x_{e_i} \triangleleft \left(\overrightarrow{\prod}_{j=i+1}^{m} e_j + J\right)\right).$$

Thus, by $\mathbb{K}$-linearity, for any $b \in B^{\times E}$, if $a \in A^{\times E}$ and $x \in X^{\times E}$ are such that $b_e = (a_e, x_e)$ for each $e \in E$, then for any $p \in \mathbb{K}\langle E\rangle$,

$$p(b) = (p(a), F^{1,p}_{E,R,X}(x)).$$

Step 3.2: Definition of V_0*.* The universal property of $A = \mathbb{K}\langle E\,|\,R\rangle$ applied to our B says precisely that a certain bijection

$$V_0\colon \{f\ \mathbb{K}\text{-algebra hom. } A \to B\} \hookrightarrow\!\!\!\!\!\rightarrow \{b \in B^{\times E} \wedge \forall r \in R\colon r(b) = 0\},$$

is defined by $f \mapsto (f(e+J))_{e\in E}$ for any $\mathbb{K}$-algebra homomorphism f from A to B and that the inverse assigns to any $b \in B^{\times E}$ wth $r(b) = 0$ for any $r \in R$ the mapping $A \to B$ with $p + J \mapsto p(b)$ for any $p \in \mathbb{K}\langle E\rangle$.

Step 3.3: Definition of V*.* The bijection V_0 from Step 3.2 restricts to a bijection

$$V\colon \{f\ \mathbb{K}\text{-algebra hom. } A \to B \wedge \pi^1 \circ f = \mathrm{id}_A\}$$
$$\hookrightarrow\!\!\!\!\!\rightarrow \{b \in B^{\times E} \wedge \forall r \in R\colon r(b) = 0 \wedge \forall p \in \mathbb{K}\langle E\rangle\colon \pi^1(p(b)) = p + J\}$$

for the following reasons.

On the one hand, for any $\mathbb{K}$-algebra homomorphism f from A to B with $\pi^1 \circ f = \mathrm{id}_A$ and any $p \in \mathbb{K}\langle E\rangle$, if $b = (f(e+J))_{e\in E}$ and $a = (\pi^1(b_e))_{e\in E}$, then $a = (e+J)_{e\in E}$ by $\pi^1 \circ f = \mathrm{id}_A$, which means $p(a) = p + J$, and thus $\pi^1(p(b)) = p + J$ since Step 3.1 implies $\pi^1(p(b)) = p(a)$.

On the other hand, given any $b \in B^{\times E}$ with $r(b) = 0$ for each $r \in R$ and $\pi^1(p(b)) = p + J$ for any $p \in \mathbb{K}\langle E\rangle$, if the $\mathbb{K}$-algebra homomorphism f from A to B is such that $p + J \mapsto p(b)$ for each $p \in \mathbb{K}\langle E\rangle$, then, of course, the assumption that $\pi^1(p(b)) = p + J$ for any $p \in \mathbb{K}\langle E\rangle$ is exactly what $\pi^1 \circ f = \mathrm{id}_A$ means.

Step 3.4: Definition of W*.* Next, we justify that a particular bijection

$$W\colon \{b \in B^{\times E} \wedge \forall r \in R\colon r(b) = 0 \wedge \forall p \in \mathbb{K}\langle E\rangle\colon \pi^1(p(b)) = p + J\}$$
$$\hookrightarrow\!\!\!\!\!\rightarrow \{x \in X^{\times E} \wedge \forall r \in R\colon F^{1,r}_{E,R,X}(x) = 0\}$$

is given by $b \mapsto (\pi^2(b_e))_{e\in E}$ for any $b \in B^{\times E}$ with $r(b) = 0$ for each $r \in R$ and $\pi^1(p(b)) = p + J$ for any $p \in \mathbb{K}\langle E\rangle$ and that its inverse maps $x \mapsto ((e+J, x_e))_{e\in E}$ for any $x \in X^{\times E}$ with $F^{1,r}_{E,R,X}(x) = 0$ for all $r \in R$.

Indeed, given any $b \in B^{\times E}$ such that $r(b) = 0$ for any $r \in R$ and $\pi^1(p(b)) = p + J$ for any $p \in \mathbb{K}\langle E\rangle$, if $x = (\pi^2(b_e))_{e\in E}$, then Step 3.1 lets us infer $F^{1,p}_{E,R,X}(x) = \pi^2(p(b))$ for any $p \in \mathbb{K}\langle E\rangle$ and thus, in particular, $F^{1,r}_{E,R,X}(x) = 0$ by $r(b) = 0$ for any $r \in R$.

Conversely, given any $x \in X^{\times E}$ with $F^{1,r}_{E,R,X}(x) = 0$ for any $r \in R$, if $b = ((e + J, x_e))_{e\in E}$, then $p(b) = (p + J, F^{1,p}_{E,R,X}(x))$ for any $p \in \mathbb{K}\langle E\rangle$ by Step 3.1. In particular, for any $r \in R$, since $r \in J$ and $F^{1,r}_{E,R,X}(x) = 0$, this implies $r(b) = 0$. Besides that, it also ensures $\pi^1(p(b)) = p + J$ and $\pi^2(p(b)) = F^{1,p}_{E,R,X}(x)$ for any $p \in \mathbb{K}\langle E\rangle$.

Step 3.5: Definition of U*.* According to Lemma 4.6 (c) a certain bijection

$$U\colon Z^1_{\mathrm{HS}}(A, X) \hookrightarrow\!\!\!\!\!\rightarrow \{f\ \mathbb{K}\text{-algebra hom. } A \to B \wedge \pi^1 \circ f = \mathrm{id}_A\}.$$

sends any $\eta \in Z^1_{\mathrm{HS}}(A, X)$ to the mapping $A \to B$ with $a \mapsto (a, \eta(a))$ for any $a \in A$ and has its inverse satisfy $f \mapsto \pi^2 \circ f$ for any $\mathbb{K}$-algebra homomorphism f from A to B with $f = \mathrm{id}_A$.

Step 3.6: Synthesis for upper arrow. In conclusion, the mapping $W \circ V \circ U$ yields a well-defined bijection

$$Z^1_{\mathrm{HS}}(A, X) \longleftrightarrow \{x \in X^{\times E} \wedge \forall r \in R\colon F^{1,r}_{E,R,X}(x) = 0\}$$

with $\eta \mapsto (\eta(e+J))_{e\in E}$ for any $\eta \in Z^1_{\mathrm{HS}}(A, X)$ and its inverse assigns to any $x \in X^{\times E}$ with $F^{1,r}_{E,R,X}(x) = 0$ for all $r \in R$ the mapping $A \to X$ with $p + J \mapsto F^{1,p}_{E,R,X}(x)$ for any $p \in \mathbb{K}\langle E\rangle$.

Step 4: Lower horizontal arrow. We prove that the lower horizontal arrow is a well-defined bijection and has the inverse stated in the claim.

For any $\eta \in B^1_{\mathrm{HS}}(A, X)$ and any $z \in X$ with $\eta = \partial z$ the upper horizontal arrow by definition maps η to the tuple $x \in X^{\times E}$ with $x_e = \eta(e+J) = (\partial z)(e+J) = (e+J) \triangleright z - z \triangleleft (e+J)$. Hence, the lower horizontal arrow is well-defined.

Conversely, given any $z \in X$, if $x = ((e+J) \triangleright z - z \triangleleft (e+J))_{e\in E}$ and if η is the image of x under the inverse of the upper horizontal arorw, i.e., the mapping $A \to X$ with $p + J \mapsto F^{1,p}_{E,R,X}(x)$ for any $p \in \mathbb{K}\langle E\rangle$, then $\eta = \partial z$ by Step 1.

Step 5: Commutativity. Since the lower horizontal arrow is defined by the same rule as the upper one and since the vertical arrows are set inclusions, it is clear that the diagram commutes. □

PROPOSITION 4.11. *There exists an isomorphism of $\mathbb{K}$-vector spaces*

$$H^1_{\mathrm{HS}}(\mathbb{K}\langle E \,|\, R\rangle, X) \longleftrightarrow \frac{\{x \in X^{\times E} \wedge \forall r \in R\colon F^{1,r}_{E,R,X}(x) = 0\}}{\{((e+J) \triangleright z - z \triangleleft (e+J))_{e\in E} \,|\, z \in X\}}$$

where the sets on the right-hand side are $\mathbb{K}$-vector subspaces of $X^{\times E}$, such that for any $\eta \in Z^1_{\mathrm{HS}}(\mathbb{K}\langle E \,|\, R\rangle, X)$ the class of η is sent to the class of x with $x_e = \eta(e+J)$ for any $e \in E$. For any $x \in X^{\times E}$ with $F^{1,r}_{E,R,X}(x) = 0$ for each $r \in R$ the inverse isomorphism sends the class of x to the class of η with $\eta(p+J) = F^{1,p}_{E,R,X}(x)$ for any $p \in \mathbb{K}\langle E\rangle$.

PROOF. Follows immediately from Lemma 4.10. □

4.3. Second Hochschild cohomology of arbitrary algebras. The second Hochschild cohomology is defined in Section 4.3.1 for arbitrary algebras. That is also where the normalization of 2-cocycles is addressed and where certain basic identities are proved for later use. Following that, Section 4.3.2 discusses how to express the second Hochschild cohomology in terms of the first. Section 4.3.3 characterizes 2-coboundaries in terms of algebra homomorphisms. And Section 4.3.4 gives a way of producing explicit 2-cocycles.

ASSUMPTIONS 4.12. In Section 4.3, let $\mathbb{K}$ be any field, A any algebra and X any A-bimodule.

4.3.1. *Definition, normalization and basic identities.* Again, the following definitions are taken from [Hoc56].

DEFINITION 4.13. (a) The *X-valued Hochschild 2-cocycles of A* are the $\mathbb{K}$-vector subspace $Z^2_{\mathrm{HS}}(A,X)$ of $[A\otimes A,X]$ formed by all elements c such that

$$(\partial c)(a_1\otimes a_2\otimes a_3) \coloneqq$$
$$a_1 \triangleright c(a_2\otimes a_3) - c(a_1a_2\otimes a_3) + c(a_1\otimes a_2a_3) - c(a_1\otimes a_2)\triangleleft a_3 = 0$$

for any $\{a_1,a_2,a_3\}\subseteq A$.

(b) The *X-valued Hochschild 2-coboundaries of A* are the $\mathbb{K}$-vector subspace $B^2_{\mathrm{HS}}(A,X)$ of $[A\otimes A,X]$ formed by all elements c such that there exists $\psi\in[A,X]$ with

$$c(a_1\otimes a_2) = (\partial\psi)(a_1\otimes a_2) \coloneqq a_1\triangleright\psi(a_2) - \psi(a_1a_2) + \psi(a_1)\triangleleft a_2$$

for any $\{a_1,a_2\}\subseteq A$.

LEMMA 4.14. *$B^2_{\mathrm{HS}}(A,X)$ is a $\mathbb{K}$-vector subspace of $Z^2_{\mathrm{HS}}(A,X)$.*

PROOF. For any $\psi\in[A,X]$, if $c=\partial\psi$, then

$$\begin{aligned}
&(\partial c)(a_1\otimes a_2\otimes a_3)\\
&\quad = a_1\triangleright(\partial\psi)(a_2\otimes a_3) - (\partial\psi)(a_1a_2\otimes a_3) + (\partial\psi)(a_1\otimes a_2a_3) - (\partial\psi)(a_1\otimes a_2)\triangleleft a_3\\
&\quad = a_1\triangleright(a_2\triangleright\psi(a_3) - \psi(a_2a_3) + \psi(a_2)\triangleleft a_3)\\
&\qquad\qquad - (a_1a_2\triangleright\psi(a_3) - \psi(a_1a_2a_3) + \psi(a_1a_2)\triangleleft a_3)\\
&\qquad\qquad\qquad + (a_1\triangleright\psi(a_2a_3) - \psi(a_1a_2a_3) + \psi(a_1)\triangleleft a_2a_3)\\
&\qquad\qquad\qquad\qquad - (a_1\triangleright\psi(a_2) - \psi(a_1a_2) + \psi(a_1)\triangleleft a_2)\triangleleft a_3,
\end{aligned}$$

which is zero by $a_1\triangleright(a_2\triangleright\psi(a_3)) = a_1a_2\triangleright\psi(a_3)$ and $a_1\triangleright(\psi(a_2)\triangleleft a_3) = (a_1\triangleright\psi(a_2))\triangleleft a_3$ and $\psi(a_1)\triangleleft a_2a_3 = (\psi(a_1)\triangleleft a_2)\triangleleft a_3$. □

DEFINITION 4.15. We call the quotient $\mathbb{K}$-vector space $H^2_{\mathrm{HS}}(A,X)$ of $Z^2_{\mathrm{HS}}(A,X)$ with respect to $B^2_{\mathrm{HS}}(A,X)$ the *second Hochschild cohomology of A with X-coefficients.*

Differently from 1-cocycles, a 2-cocycle need not vanish on the identity. However, they can be normalized to do so without affecting their cohomology class. The following is the first step in seeing how.

LEMMA 4.16. *Let $c\in Z^2_{\mathrm{HS}}(A,X)$ be arbitrary.*

(a) The following are equivalent:
  - *(i) $c(1\otimes 1)=0$.*
  - *(ii) $c(a\otimes 1)=0$ for any $a\in A$.*
  - *(iii) $c(1\otimes a)=0$ for any $a\in A$.*

(b) If there exists $\psi\in[A,X]$ such that $c=\partial\psi$, then $c(1\otimes 1)=\psi(1)$. In particular, $c(1\otimes 1)=0$ if and only if $\psi(1)=0$.

PROOF. (a) It is clear that (ii) and (iii) each imply (i). Because c is a 2-cocycle we infer for any $a \in A$,

$$0 = (\partial c)(a \otimes 1 \otimes 1) = a \triangleright c(1 \otimes 1) - c(a \cdot 1 \otimes 1) + c(a \otimes 1 \cdot 1) - c(a \otimes 1) \triangleleft 1$$

which is to say $c(a \otimes 1) = a \triangleright c(1 \otimes 1)$. From this we can conclude that (i) requires (ii). Similarly, for any $a \in A$,

$$0 = (\partial c)(1 \otimes 1 \otimes a) = 1 \triangleright c(1 \otimes a) - c(1 \cdot 1 \otimes a) + c(1 \otimes 1 \cdot a) - c(1 \otimes 1) \triangleleft a$$

and thus $c(1 \otimes a) = c(1 \otimes 1) \triangleleft a$, proving that (i) necessitates (ii). That is all we needed to see.

(b) By definition, $c(1 \otimes 1) = (\partial \psi)(1 \otimes 1) = 1 \triangleright \psi(1) - \psi(1 \cdot 1) + \psi(1) \triangleleft 1 = \psi(1)$. □

DEFINITION 4.17. (a) Any $c \in Z^2_{\mathrm{HS}}(A, X)$ is called a *normalized X-valued Hochschild 2-cocycle* if any (and thus all) of the three conditions of Lemma 4.16 (a) are satisfied. Likewise, if any $c \in B^2_{\mathrm{HS}}(A, X)$ meets the condition of Lemma 4.16 (b), we speak of a *normalized X-valued Hochschild 2-coboundary.*

(b) We write $Z^{2,0}_{\mathrm{HS}}(A, X)$ and $B^{2,0}_{\mathrm{HS}}(A, X)$ for the $\mathbb{K}$-vector subspaces of $Z^2_{\mathrm{HS}}(A, X)$ respectively $B^2_{\mathrm{HS}}(A, X)$ formed by all normalized elements.

(c) Finally, we let $H^2_{\mathrm{HS}}(A, X)$ stand for the quotient $\mathbb{K}$-vector space of $Z^{2,0}_{\mathrm{HS}}(A, X)$ with respect to $B^{2,0}_{\mathrm{HS}}(A, X)$.

The next lemma shows how for any normalized 2-cocycle products in the first argument can be shifted to the second one, an auxiliary result we will need in the proof of Lemma 4.30.

LEMMA 4.18. *For any $c \in Z^{2,0}_{\mathrm{HS}}(A, X)$, any $m \in \mathbb{N}_0$, any $\{a_i\}_{i=1}^m \subseteq A$ and any $b \in A$,*

$$c\left(\left(\overrightarrow{\prod}_{i=1}^m a_i\right) \otimes b\right) = \sum_{i=1}^m \left(\overrightarrow{\prod}_{j=1}^{i-1} a_j\right) \triangleright \left[c\left(a_i \otimes \left(\overrightarrow{\prod}_{j=i+1}^m a_j\right) b\right) - c\left(a_i \otimes \left(\overrightarrow{\prod}_{j=i+1}^m a_j\right)\right) \triangleleft b\right].$$

PROOF. In the case $m = 0$ the claim is equivalent to the statement $c(1 \otimes b) = 0$ which holds by $c(1 \otimes 1) = 0$ according to Lemma 4.16 (a). Hence, we can assume $1 \leq m$ in the following. For each $i \in [\![m]\!]$, because c is a 2-cocycle,

$$\partial c\left(a_i \otimes \left(\overrightarrow{\prod}_{j=i+1}^m a_j\right) \otimes b\right) = 0,$$

which, when written out, is equivalent to

$$c\left(a_i \otimes \left(\overrightarrow{\prod}_{j=i+1}^m a_j\right) b\right) - c\left(a_i \otimes \left(\overrightarrow{\prod}_{j=i+1}^m a_j\right)\right) \triangleleft b = c\left(\left(\overrightarrow{\prod}_{j=i}^m a_j\right) \otimes b\right) - a_i \triangleright c\left(\left(\overrightarrow{\prod}_{j=i+1}^m a_j\right) \otimes b\right).$$

Multiplying the equation for $i \in [\![m]\!]$ with $\overrightarrow{\prod}_{j=1}^{i-1} a_j$ from the left and then summing up the m many equations shows that the right-hand side of the claimed identity is equal to

$$\sum_{i=1}^m \left(\overrightarrow{\prod}_{j=1}^{i-1} a_j\right) \triangleright c\left(\left(\overrightarrow{\prod}_{j=i}^m a_j\right) \otimes b\right) - \sum_{i=1}^m \left(\overrightarrow{\prod}_{j=1}^{i} a_j\right) \triangleright c\left(\left(\overrightarrow{\prod}_{j=i+1}^m a_j\right) \otimes b\right).$$

Upon shifting the summation index in the first term by 1 to the left, this expression takes the form

$$\sum_{i=0}^{m-1}\Big(\overrightarrow{\prod}_{j=1}^{i}a_j\Big)\triangleright c\Big(\Big(\overrightarrow{\prod}_{j=i+1}^{m}a_j\Big)\otimes b\Big)-\sum_{i=1}^{m}\Big(\overrightarrow{\prod}_{j=1}^{i}a_j\Big)\triangleright c\Big(\Big(\overrightarrow{\prod}_{j=i+1}^{m}a_j\Big)\otimes b\Big).$$

Of this telescoping sum only

$$c\Big(\Big(\overrightarrow{\prod}_{j=1}^{m}a_j\Big)\otimes b\Big)-\Big(\overrightarrow{\prod}_{j=1}^{m}a_j\Big)\triangleright c\,(1\otimes b)$$

remains, which is identical to the left-hand side of the asserted equality because $c(1\otimes b)=0$ by $c(1\otimes 1)=0$ and Lemma 4.16 (a). Thus, the claim holds. □

LEMMA 4.19. *(a) A commutative diagram of $\mathbb{K}$-linear maps is given by*

$$\begin{array}{ccc} Z^2_{\mathrm{HS}}(A,X) & \hookrightarrow\!\!\!\!\twoheadrightarrow & Z^{2,0}_{\mathrm{HS}}(A,X)\oplus X \\ {\scriptstyle\subseteq}\uparrow & & \uparrow{\scriptstyle\subseteq} \\ B^2_{\mathrm{HS}}(A,X) & \hookrightarrow\!\!\!\!\twoheadrightarrow & B^{2,0}_{\mathrm{HS}}(A,X)\oplus X \end{array},$$

where each one of the horizontal arrows assigns to any element c of its domain the pair whose first component is given by the unqiue $\mathbb{K}$-linear mapping $A\otimes A\to X$ with

$$a_1\otimes a_2\mapsto c(a_1\otimes a_2)-a_1\triangleright c(1\otimes 1)\triangleleft a_2$$

for any $\{a_1,a_2\}\subseteq A$ and whose second component is given by $c(1\otimes 1)$. Moreover, the horizontal arrows are $\mathbb{K}$-linear isomorphisms whose inverses map any element $(\bar{c},x)$ of their respective domains to the unique $\mathbb{K}$-linear mapping $A\otimes A\to X$ with

$$a_1\otimes a_2\mapsto \bar{c}(a_1\otimes a_2)+a_1\triangleright x\triangleleft a_2$$

for any $\{a_1,a_2\}\subseteq A$.

(b) There exists an isomorphism of $\mathbb{K}$-vector spaces

$$H^2_{\mathrm{HS}}(A,X)\hookrightarrow\!\!\!\!\twoheadrightarrow H^{2,0}_{\mathrm{HS}}(A,X)$$

which for any $c\in Z^2_{\mathrm{HS}}(A,X)$ maps the class of c with respect to $B^2_{\mathrm{HS}}(A,X)$ to the class with respect to $B^{2,0}_{\mathrm{HS}}(A,X)$ of the uniquely determined $\mathbb{K}$-linear mapping $A\otimes A\to X$ with

$$a_1\otimes a_2\mapsto c(a_1\otimes a_2)-a_1\triangleright c(1\otimes 1)\triangleleft a_2$$

for any $\{a_1,a_2\}\subseteq A$. For any $\bar{c}\in Z^{2,0}_{\mathrm{HS}}(A,X)$ the inverse assigns to the class of $\bar{c}$ with respect to $B^{2,0}_{\mathrm{HS}}(A,X)$ its class with respect to $B^2_{\mathrm{HS}}(A,X)$.

PROOF. The proofs of all claims will use the following fact: For any $x\in X$ the map $w_x\in[A\otimes A,X]$ defined by $w_x(a_1\otimes a_2):=a_1\triangleright x\triangleleft a_2$ for any $\{a_1,a_2\}\subseteq A$ is an element of $B^2_{\mathrm{HS}}(A,X)$ because, if we consider $\nu_x \in [A,X]$ with $a \mapsto a \triangleright x$ for any $a \in A$, then for any $\{a_1, a_2\} \subseteq A$,

$$(\partial \nu_x)(a_1 \otimes a_2) = a_1 \triangleright a_2 \triangleright x - a_1 a_2 \triangleright x + a_1 \triangleright x \triangleleft a_2 = w_x(a_1 \otimes a_2).$$

Moreover, $w_x(1 \otimes 1) = 1 \triangleright x \triangleleft 1 = x$ by definition.

(a) *Step 1:* We begin with the supposed map from $Z^2_{\mathrm{HS}}(A,X)$ to $Z^{2,0}_{\mathrm{HS}}(A,X) \oplus X$. It assigns to any $c \in Z^2_{\mathrm{HS}}(A,X)$ the tuple

$$(c - w_{c(1\otimes 1)}, c(1 \otimes 1)).$$

For any $c \in Z^2_{\mathrm{HS}}(A,X)$, clearly, $c - w_{c(1\otimes 1)} \in Z^2_{\mathrm{HS}}(A,X)$ because $w_{c(1\otimes 1)} \in B^2_{\mathrm{HS}}(A,X) \subseteq Z^2_{\mathrm{HS}}(A,X)$, and $(c - w_{c(1\otimes 1)})(1\otimes 1) = 0$ by $w_{c(1\otimes 1)}(1\otimes 1) = c(1\otimes 1)$. Thus, the mapping is well-defined.

Conversely, a well-defined linear map from $Z^{2,0}_{\mathrm{HS}}(A,X) \oplus X$ to $Z^2_{\mathrm{HS}}(A,X)$ is obtained via the rule $(\tilde{c}, x) \mapsto \tilde{c} + w_x$ for any $\tilde{c} \in Z^{2,0}_{\mathrm{HS}}(A,X)$ and $x \in X$ because $w_x \in B^2_{\mathrm{HS}}(A,X)$.

Now, on the one hand, for any $c \in Z^2_{\mathrm{HS}}(A,X)$, if $x = \tilde{c}(1 \otimes 1)$ and $\tilde{c} = c - w_x$, then $\tilde{c} + w_x = (c - w_x) + w_x = c$. And, on the other hand, for any $\tilde{c} \in Z^2_{\mathrm{HS}}(A,X)$ with $\tilde{c}(1 \otimes 1) = 0$ and any $x \in X$, if $c = \tilde{c} + w_x$, then $c(1 \otimes 1) = (\tilde{c} + w_x)(1 \otimes 1) = x$ by $\tilde{c}(1 \otimes 1) = 0$ and $w_x(1 \otimes 1) = x$, and thus $c - w_{c(1\otimes 1)} = (\tilde{c} + w_x) - w_x = \tilde{c}$. In conclusion, the two maps are inverse to each other.

Step 2: Since $w_{c(1\otimes 1)} \in B^2_{\mathrm{HS}}(A,X)$ for any $c \in Z^2_{\mathrm{HS}}(A,X)$, let alone $c \in B^2_{\mathrm{HS}}(A,X)$, the rule $c \mapsto (c - w_{c(1\otimes 1)}, c(1 \otimes 1))$ for any $c \in Z^2_{\mathrm{HS}}(A,X)$ by restriction also yields a map from $B^2_{\mathrm{HS}}(A,X)$ to $B^{2,0}_{\mathrm{HS}}(A,X) \oplus X$. It is an isomorphism because also the inverse of the map from Step 1, defined by the rule $(\tilde{c}, x) \mapsto \tilde{c} + w_x$ for any $\tilde{c} \in Z^{2,0}_{\mathrm{HS}}(A,X)$ and $x \in X$, restricts to a map from $B^{2,0}_{\mathrm{HS}}(A,X) \oplus X$ to $B^2_{\mathrm{HS}}(A,X)$ since $w_x \in B^2_{\mathrm{HS}}(A,X)$ for any $x \in X$.

(b) By (a) the rule $c + B^{2,0}_{\mathrm{HS}}(A,X) \mapsto (c - w_{c(1\otimes 1)}, c(1 \otimes 1)) + B^{2,0}_{\mathrm{HS}}(A,X)$ for any $c \in Z^{2,0}_{\mathrm{HS}}(A,X)$ defines an isomorphism from $H^2_{\mathrm{HS}}(A,X)$ to the quotient space Q of $Z^{2,0}_{\mathrm{HS}}(A,X) \oplus X$ with respect to $B^{2,0}_{\mathrm{HS}}(A,X) \oplus X$. Hence, it suffices to prove that an isomorphism from Q to $H^{2,0}_{\mathrm{HS}}(A,X)$ is obtained via the assignment $(\tilde{c}, x) + B^{2,0}_{\mathrm{HS}}(A,X) \oplus X \mapsto \tilde{c} + B^{2,0}_{\mathrm{HS}}(A,X)$ for any $\tilde{c} \in Z^{2,0}_{\mathrm{HS}}(A,X)$ and $x \in X$. And that is indeed the case because assigning $\bar{c} + B^{2,0}_{\mathrm{HS}}(A,X) \mapsto (\bar{c}, 0) + B^{2,0}_{\mathrm{HS}}(A,X) \oplus X$ for any $\bar{c} \in Z^{2,0}_{\mathrm{HS}}(A,X)$ defines a map from $H^{2,0}_{\mathrm{HS}}(A,X)$ to Q and because the two are inverse to each other by $(\tilde{c}, x) + B^{2,0}_{\mathrm{HS}}(A,X) \oplus X = (\tilde{c}, 0) + B^{2,0}_{\mathrm{HS}}(A,X) \oplus X$ holding for any $\tilde{c} \in Z^{2,0}_{\mathrm{HS}}(A,X)$ and any $x \in X$. □

4.3.2. *Reduction to first Hochschild cohomology with other coefficients.* It is possible to express normalized 2-cocycles and 2-coboundaries as 1-cocycles respectively 1-coboundaries valued in an appropriate module of normalized linear maps.

NOTATION 4.20. For any $\mathbb{K}$-vector spaces A, B and C write $\theta_{B,A,C}$ for the *currying* $\mathbb{K}$-linear isomorphism $[A \otimes B, C] \to [A, [B, C]]$ which assigns to any $f\colon A \otimes B \to C$ the mapping $A \to [B, C]$ which sends any $a \in A$ to the mapping $B \to C$ with $b \mapsto f(a \otimes b)$

for any $b \in B$ (and whose inverse assigns to any $g\colon A \to [B, C]$ the unique $\mathbb{K}$-linear mapping $A \otimes B \to C$ with $a \otimes b \mapsto (g(a))(b)$ for any $a \in A$ and $b \in B$.

LEMMA 4.21. *(a) The $\mathbb{K}$-vector subspace $\{\gamma \in [A, X] \wedge \gamma(1) = 0\}$ of $[A, X]$ becomes an A-bimodule $A|X$ when equipped with the left action given by the unique $\mathbb{K}$-linear map which for any $a \in A$ and $\gamma \in [A, X]$ with $\gamma(1) = 0$ assigns to $a \otimes \gamma$ the mapping $A \to X$ with*

$$b \mapsto a \rhd \gamma(b)$$

for any $b \in A$ and with the right action given by the unique $\mathbb{K}$-linear map which for any $a \in A$ and $\gamma \in [A, X]$ with $\gamma(1) = 0$ assigns to $\gamma \otimes a$ the mapping $A \to X$ with

$$b \mapsto \gamma(ab) - \gamma(a) \lhd b$$

for any $b \in A$.

(b) A commutative diagram of $\mathbb{K}$-linear maps is given by

$$\begin{array}{ccc} Z^{2,0}_{\mathrm{HS}}(A, X) & \hookrightarrow\!\!\!\!\to & Z^1_{\mathrm{HS}}(A, A|X) \\ \subseteq\uparrow & & \uparrow\subseteq \\ B^{2,0}_{\mathrm{HS}}(A, X) & \hookrightarrow\!\!\!\!\to & B^1_{\mathrm{HS}}(A, A|X) \end{array},$$

where the horizontal arrows are restrictions of the currying isomorphism $\theta_{A,A,X}$. Moreover, the horizontal arrows are $\mathbb{K}$-linear isomorphisms whose inverses are given by the respective restrictions of ${\theta_{A,A,X}}^{-1}$.

(c) There exists an isomorphism of $\mathbb{K}$-vector spaces

$$H^{2,0}_{\mathrm{HS}}(A, X) \hookrightarrow\!\!\!\!\to H^1_{\mathrm{HS}}(A, A|X)$$

with

$$\overline{c} + B^{2,0}_{\mathrm{HS}}(A, X) \mapsto \theta_{A,A,X}(\overline{c}) + B^1_{\mathrm{HS}}(A, A|X)$$

for any $c \in Z^{2,0}_{\mathrm{HS}}(A, X)$. Its inverse satisfies

$$\tilde{c} + B^1_{\mathrm{HS}}(A, A|X) \mapsto {\theta_{A,A,X}}^{-1}(\tilde{c}) + B^{2,0}_{\mathrm{HS}}(A, X)$$

for any $\tilde{c} \in Z^1_{\mathrm{HS}}(A, A|X)$.

PROOF. (a) The alleged left action $\blacktriangleright$ and right action $\blacktriangleleft$ of A on $\{\gamma \in [A, X] \wedge \gamma(1) = 0\}$ are well-defined. Indeed, for any $a \in A$ and $\gamma \in [A, X]$ with $\gamma(1) = 0$ both $(a \blacktriangleright \gamma)(1) = a \rhd \gamma(1) = 0$ and $(\gamma \blacktriangleleft a)(1) = \gamma(a1) - \gamma(a) \lhd 1 = 0$.

That $\blacktriangleright$ and $\blacktriangleleft$ do define commuting left respectively right actions is evidenced by the facts that for any $\{a_1, a_2\} \subseteq A$, any $\gamma \in [A, X]$ with $\gamma(1) = 0$ and any $b \in A$,

$$\begin{aligned}&(a_1 \blacktriangleright (a_2 \blacktriangleright \gamma))(b)\\ &\qquad = a_1 \rhd (a_2 \blacktriangleright \gamma)(b) = a_1 \rhd (a_2 \rhd \gamma(b)) = a_1a_2 \rhd \gamma(b) = (a_1a_2 \blacktriangleright \gamma)(b)\end{aligned}$$

and, thanks to $\gamma(a_1) \triangleleft a_2 b = (\gamma(a_1) \triangleleft a_2) \triangleleft b$,

$$\begin{aligned} ((\gamma \blacktriangleleft a_1) \blacktriangleleft a_2)(b) &= (\gamma \blacktriangleleft a_1)(a_2 b) - (\gamma \blacktriangleleft a_1)(a_2) \triangleleft b \\ &= \gamma(a_1(a_2 b)) - \gamma(a_1) \triangleleft a_2 b - (\gamma(a_1 a_2) - \gamma(a_1) \triangleleft a_2) \triangleleft b \\ &= \gamma((a_1 a_2) b) - \gamma(a_1 a_2) \triangleleft b \\ &= (\gamma \blacktriangleleft a_1 a_2)(b) \end{aligned}$$

as well as $(1 \blacktriangleright \gamma)(b) = 1 \triangleright \gamma(b) = \gamma(b)$ and $(\gamma \blacktriangleleft 1)(b) = \gamma(1b) - \gamma(1) \triangleleft b = \gamma(b)$ by virtue of $\gamma(1) = 0$ and, lastly, thanks to $(a_1 \triangleright \gamma(a_2)) \triangleleft b = a_1 \triangleright (\gamma(a_2) \triangleleft b)$,

$$\begin{aligned} ((a_1 \blacktriangleright \gamma) \blacktriangleleft a_2)(b) &= (a_1 \blacktriangleright \gamma)(a_2 b) - (a_1 \blacktriangleright \gamma)(a_2) \triangleleft b \\ &= a_1 \triangleright \gamma(a_2 b) - (a_1 \triangleright \gamma(a_2)) \triangleleft b \\ &= a_1 \triangleright (\gamma(a_2 b) - \gamma(a_2) \triangleleft b) \\ &= a_1 \triangleright (\gamma \blacktriangleleft a_2)(b) \\ &= (a_1 \blacktriangleright (\gamma \blacktriangleleft a_2))(b). \end{aligned}$$

(b) *Step 1:* First, we show that $\theta \equiv \theta_{A,A,X}$ restricts to a map from $Z^{2,0}_{\mathrm{HS}}(A, X)$ to $Z^1_{\mathrm{HS}}(A, A|X)$.

Indeed, given any $c \in Z^{2,0}_{\mathrm{HS}}(A, X)$, if $\tilde{c} = \theta(c)$ and if $\tilde{c}^a \equiv \tilde{c}(a)$ for any $a \in A$, then Lemma 4.16 (a) implies that $\tilde{c}^a(1) = c(a \otimes 1) = 0$ for any $a \in A$, which is to say $\tilde{c} \in [A, A|X]$. Moreover, then for any $\{a_1, a_2, a_3\} \subseteq A$,

$$\begin{aligned} &((\partial^1 \tilde{c})(a_1 \otimes a_2))(a_3) \\ &\quad = (a_1 \blacktriangleright \tilde{c}^{a_2})(a_3) - (\tilde{c}^{a_1 a_2})(a_3) + (\tilde{c}^{a_1} \blacktriangleleft a_2)(a_3) \\ &\quad = a_1 \triangleright \tilde{c}^{a_2}(a_3) - \tilde{c}^{a_1 a_2}(a_3) + (\tilde{c}^{a_1}(a_2 a_3) - \tilde{c}^{a_1}(a_2) \triangleleft a_3) \\ &\quad = a_1 \triangleright c(a_2 \otimes a_3) - c(a_1 a_2 \otimes a_3) + c(a_1 \otimes a_2 a_3) - c(a_1 \otimes a_2) \triangleleft a_3 \\ &\quad = (\partial^2 c)(a_1 \otimes a_2 \otimes a_3) \\ &\quad = 0, \end{aligned}$$

i.e., $\tilde{c} \in Z^1_{\mathrm{HS}}(A, A|X)$.

Moreover, θ^{-1} restricts to a map from $Z^1_{\mathrm{HS}}(A, A|X)$ to $Z^{2,0}_{\mathrm{HS}}(A, X)$, thus confirming that θ is an isomorphism. Indeed, given any $\tilde{c} \in Z^1_{\mathrm{HS}}(A, A|X)$, if $c = \theta^{-1}(\tilde{c})$, then we can read the above computation backwards to find $(\partial^2 c)(a_1 \otimes a_2 \otimes a_3) = ((\partial^1 \tilde{c})(a_1 \otimes a_2))(a_3) = 0$ for any $\{a_1, a_2, a_3\} \subseteq A$ and so infer $c \in Z^2_{\mathrm{HS}}(A, A|X)$. And, of course, $c(1 \otimes 1) = \tilde{c}(1)(1) = 0$ by $\tilde{c}(1) \in A|X$.

Step 2: It remains to prove that θ also gives an isomorphism from $B^{2,0}_{\mathrm{HS}}(A, X)$ to $B^1_{\mathrm{HS}}(A, A|X)$. For any $\psi \in [A, X]$, if $c = \partial^1 \psi \in Z^2_{\mathrm{HS}}(A, X)$ satisfies $c(1 \otimes 1) = 0$, then also $\psi(1) = 0$ by Lemma 4.16 (b), i.e., $\psi \in A|X$. Furthermore, if $\tilde{c} = \theta(c)$, then for any $\{a_1, a_2\} \subseteq A$,

$$\begin{aligned}\tilde{c}(a_1)(a_2) &= c(a_1 \otimes a_2)\\ &= (\partial^1\psi)(a_1 \otimes a_2)\\ &= a_1 \triangleright \psi(a_2) - \psi(a_1a_2) + \psi(a_1) \triangleleft a_2\\ &= (a_1 \blacktriangleright \psi)(a_2) - (\psi \blacktriangleleft a_1)(a_2)\\ &= ((\partial^0\psi)(a_1))(a_2),\end{aligned}$$

which means $\tilde{c} = \partial\psi \in B^1_{\mathrm{HS}}(A, A|X)$. So, θ becomes a map from $B^{2,0}_{\mathrm{HS}}(A, X)$ to $B^1_{\mathrm{HS}}(A, A|X)$.

Conversely, also θ^{-1} restricts to a map from $B^1_{\mathrm{HS}}(A, A|X)$ to $B^{2,0}_{\mathrm{HS}}(A, X)$. That is because for any $\psi \in A|X$, if $\tilde{c} = \partial\psi \in B^1_{\mathrm{HS}}(A, A|X)$ and $c = \theta^{-1}(\tilde{c})$, then the last three identities of the above computation still hold and show $c = \partial\psi$, i.e., $c \in Z^2_{\mathrm{HS}}(A, X)$. Naturally, also $c(1 \otimes 1) = \tilde{c}(1)(1) = ((\partial\psi)(1))(1) = (1 \blacktriangleright \psi)(1) - (1 \blacktriangleleft \psi)(1) = 1 \triangleright \psi(1) - \psi(1) + \psi(1) \triangleleft 1 = \psi(1) = 0$ by $\psi \in A|X$. Thus, all claims of (b) are true.

(c) Follows immediately from (b). ☐

4.3.3. *Algebra homomorphism characterization of* 2*-coboundaries.* The following lemma is adapted from [BFG17, Lemma 5.4].

LEMMA 4.22. *For any $c \in Z^{2,0}_{\mathrm{HS}}(A, X)$ and any $\psi \in [A, X]$ with $\psi(1) = 0$, the $\mathbb{K}$-linear map $T_{c,\psi}\colon A \to [A \oplus X, A \oplus X]$ which assigns to any $a \in A$ the mapping $A \oplus X \to A \oplus X$ with*

$$(b, y) \mapsto (ab, c(a \otimes b) - \psi(a) \triangleleft b + a \triangleright y)$$

for any $b \in A$ and $y \in X$ is a $\mathbb{K}$-algebra homomorphism $A \to \mathrm{End}(A \oplus X)$ if and only if $c = \partial\psi$.

PROOF. Because $c(1 \otimes b) = c(1 \otimes 1) = \psi(1) = \psi(1) \triangleleft b = 0$ by Lemma 4.16 (b), always,

$$T_{c,\psi}(1)(b, y) = (1b, c(1 \otimes b) - \psi(1) \triangleleft b + 1 \triangleright y) = (b, y)$$

for any $b \in A$ and $y \in X$, which is to say that $T_{c,\psi}$ is unital in any case. Thus, it only matters whether $T_{c,\psi}$ is multiplicative. For any $\{a_1, a_2\} \subseteq A$ and any $b \in A$ and $y \in X$, on the one hand, $T_{c,\psi}(a_1a_2)$ sends (b, y) to

$$(a_1a_2b, c(a_1a_2 \otimes b) - \psi(a_1a_2) \triangleleft b + a_1a_2 \triangleright y)$$

and, on the other hand, $T_{c,\psi}(a_1)T_{c,\psi}(a_2)$ maps (b, y) to

$$\begin{aligned}&T_{c,\psi}(a_1)(a_2b, c(a_2 \otimes b) - \psi(a_2) \triangleleft b + a_2 \triangleright y)\\ &= (a_1(a_2b), c(a_1 \otimes a_2b) - \psi(a_1) \triangleleft a_2b + a_1 \triangleright (c(a_2 \otimes b) - \psi(a_2) \triangleleft b + a_2 \triangleright y))\\ &= (a_1a_2b, a_1 \triangleright c(a_2 \otimes b) + c(a_1 \otimes a_2b) - a_1 \triangleright \psi(a_2) \triangleleft b - \psi(a_1) \triangleleft a_2b + a_1a_2 \triangleright y).\end{aligned}$$

Hence, $T_{c,\psi}(a_1a_2) = T_{c,\psi}(a_1)T_{c,\psi}(a_2)$ holds if and only if for any $b \in A$,

$$\begin{aligned}&a_1 \triangleright c(a_2 \otimes b) - c(a_1a_2 \otimes b) + c(a_1 \otimes a_2b)\\ &\qquad = a_1 \triangleright \psi(a_2) \triangleleft b - \psi(a_1a_2) \triangleleft b + \psi(a_1) \triangleleft a_2b.\end{aligned}$$

Because $\partial c(a_1 \otimes a_2 \otimes b) = 0$, which is to say,

$$a_1 \triangleright c(a_2 \otimes b) - c(a_1 a_2 \otimes b) + c(a_1 \otimes a_2 b) - c(a_1 \otimes a_2) \triangleleft b = 0,$$

the previous statement is equivalent to

$$c(a_1 \otimes a_2) \triangleleft b = (a_1 \triangleright \psi(a_2) - \psi(a_1 a_2) + \psi(a_1) \triangleleft a_2) \triangleleft b.$$

Thus, we have shown that $T_{c,\psi}$ is a $\mathbb{K}$-algebra homomorphism from A to $\mathrm{End}(A \oplus X)$ if and only if $c(a_1 \otimes a_2) \triangleleft b = \partial\psi(a_1 \otimes a_2) \triangleleft b$ for any $\{a_1, a_2, b\} \subseteq A$. Because A has a unit, this last condition is equivalent to $c = \partial\psi$. Thus, the proof is complete. □

The following auxiliary result will be used to prove a refinement of Lemma 4.22 in Lemma 4.30.

LEMMA 4.23. *For any $c \in [A \otimes A, X]$, for any $m \in \mathbb{N}$, any $\{a_i\}_{i=1}^m \subseteq A$ and any $\{x_{a_i}\}_{i=1}^m \subseteq X$, if t_i is the $\mathbb{K}$-linear endomorphism of $A \oplus X$ with*

$$(b, y) \mapsto (a_i b, c(a_i \otimes b) - x_{a_i} \triangleleft b + a_i \triangleright b)$$

for any $b \in A$, any $y \in X$ and any $i \in [\![m]\!]$, then the composition $t_1 \circ t_2 \circ \ldots \circ t_m$ is given by the $\mathbb{K}$-linear endomorphism of $A \oplus X$ which for any $b \in A$ and any $y \in X$ maps (b, y) to the pair with first component $a_1 a_2 \ldots a_m b$ and second component

$$\sum_{i=1}^{m} \left(\overrightarrow{\prod}_{j=1}^{i-1} a_j\right) \triangleright \left[c\left(a_i \otimes \left(\overrightarrow{\prod}_{j=i+1}^{m} a_j\right) b\right) - x_{a_i} \triangleleft \left(\left(\overrightarrow{\prod}_{j=i+1}^{m} a_j\right) b\right)\right] + \left(\overrightarrow{\prod}_{i=1}^{m} a_i\right) \triangleright y.$$

PROOF. The claim holds for $m = 1$ by definition. In order to prove it for $m+1$ we compose t_{m+1} with the expression claimed to be $t_1 \circ t_2 \circ \ldots \circ t_m$. For any $b \in A$ and $y \in X$ the resulting map sends (b, y) to the pair with first component $(a_1 a_2 \ldots a_m) a_{m+1} b$ and second component

$$\begin{aligned}
&\sum_{i=1}^{m} \left(\overrightarrow{\prod}_{j=1}^{i-1} a_j\right) \triangleright \left[c\left(a_i \otimes \left(\overrightarrow{\prod}_{j=i+1}^{m} a_j\right) a_{m+1} b\right) - x_{a_i} \triangleleft \left(\left(\overrightarrow{\prod}_{j=i+1}^{m} a_j\right) a_{m+1} b\right)\right] \\
&\qquad + \left(\overrightarrow{\prod}_{i=1}^{m} a_i\right) \triangleright (c(a_{m+1} \otimes b) - x_{a_{m+1}} \triangleleft b) + \left(\overrightarrow{\prod}_{i=1}^{m} a_i\right) \triangleright a_{m+1} \triangleright y \\
&= \sum_{i=1}^{m+1} \left(\overrightarrow{\prod}_{j=1}^{i-1} a_j\right) \triangleright \left[c\left(a_i \otimes \left(\overrightarrow{\prod}_{j=i+1}^{m+1} a_j\right) b\right) - x_{a_i} \triangleleft \left(\left(\overrightarrow{\prod}_{j=i+1}^{m+1} a_j\right) b\right)\right] + \left(\overrightarrow{\prod}_{i=1}^{m+1} a_i\right) \triangleright y,
\end{aligned}$$

which concludes the proof. □

4.3.4. *Constructing particular 2-cocycles from 1-cocycles.* If the coefficient bimodule is equipped with an algebra structure compatible with the bimodule actions, there is a simple way of obtaining 2-cocycles, normalized ones, from 1-cocycles.

LEMMA 4.24. *If X is also a $\mathbb{K}$-algebra with multiplication $\cdot$ such that for any $a \in A$ and any $\{x_1, x_2\} \subseteq X$,*

$$\begin{aligned}
&(a \triangleright x_1) \cdot x_2 = a \triangleright (x_1 \cdot x_2) \\
\textit{and}\quad &(x_1 \triangleleft a) \cdot x_2 = x_1 \cdot (a \triangleright x_2) \\
\textit{and}\quad &x_1 \cdot (x_2 \triangleleft a) = (x_1 \cdot x_2) \triangleleft a,
\end{aligned}$$

then there exists a unique $\mathbb{K}$*-linear map*

$$Z^1_{\mathrm{HS}}(A,X)\otimes Z^1_{\mathrm{HS}}(A,X)\to Z^{2,0}_{\mathrm{HS}}(A,X).$$

which for any $\{\eta,\eta'\}\subseteq Z^1_{\mathrm{HS}}(A,X)$ *assigns to* $\eta\otimes\eta'$ *the unique* $\mathbb{K}$*-linear mapping* $A\otimes A\to X$ *with*

$$a\otimes b\mapsto \eta(a)\cdot\eta'(b)$$

for any $\{a,b\}\subseteq A$*.*

Proof. All we have to show is that the mapping really does yield normalized 2-cocycles. For any $\{\eta,\eta'\}\subseteq Z^1_{\mathrm{HS}}(A,X)$, if $c\in[A\otimes A,X]$ is defined by $a\otimes b\mapsto\eta(a)\cdot\eta'(b)$ for any $\{a,b\}\subseteq A$, then for any $\{a_1,a_2,a_3\}\subseteq A$ the vector $(\partial c)(a_1\otimes a_2\otimes a_3)$ is given by

$$a_1\rhd(\eta(a_2)\cdot\eta'(a_3))-\eta(a_1a_2)\cdot\eta'(a_3)+\eta(a_1)\cdot\eta'(a_2a_3)-(\eta(a_1)\cdot\eta(a_2))\lhd a_3.$$

Because the 1-cocycle properties of η and η' imply $\eta(a_1a_2)=\eta(a_1)\lhd a_2+a_1\rhd\eta(a_2)$ and $\eta'(a_2a_3)=\eta'(a_2)\lhd a_3+a_2\rhd\eta'(a_3)$, after switching the two middle ones of the resulting six terms, this can be rewritten as

$$\begin{aligned}&a_1\rhd(\eta(a_2)\cdot\eta'(a_3))-(\eta(a_1)\lhd a_2)\cdot\eta'(a_3)+\eta(a_1)\cdot(\eta'(a_2)\lhd a_3)\\-&(a_1\rhd\eta(a_2))\cdot\eta'(a_3)+\eta(a_1)\cdot(a_2\rhd\eta'(a_3))-(\eta(a_1)\cdot\eta(a_2))\lhd a_3.\end{aligned}$$

The assumptions on the algebra structure of X ensure that each of the three terms in the first row cancels the one below it, proving $(\partial c)(a_1\otimes a_2\otimes a_3)=0$ as asserted. Moreover, $c(1\otimes 1)=\eta(1)\cdot\eta'(1)=0$ by $\eta(1)=\eta'(1)=0$ by Remark 4.5. □

Remark 4.25. If in Lemma 4.24 the bimodule X lacks an appropriate algebra structure, one can also search for A-subbimodules M of $A|X$ from Lemma 4.21 (a). The inclusion of M into $A|X$ then yields a linear map $Z^1_{\mathrm{HS}}(A,M)\to Z^{2,0}_{\mathrm{HS}}(A,X)$ via the isomorphism from Lemma 4.21 (c).

In particular, if $ab\rhd x=ba\rhd x$ for any $\{a,b\}\subseteq A$ and $x\in X$, then $Z^1_{\mathrm{HS}}(A,X)$ is a viable choice for M.

4.4. Second Hochschild cohomology of universal algebras. As with the first Hochschild cohomology, the results obtained for arbitrary algebras can be improved in the case of universal algebras.

Assumptions 4.26. In Section 4.4, let $\mathbb{K}$ be any field, E any (not necessarily finite) set, $R\subseteq\mathbb{K}\langle E\rangle$ any subset, J the two-sided $\mathbb{K}$-ideal of $\mathbb{K}\langle E\rangle$ generated by R and X any $\mathbb{K}\langle E\,|\,R\rangle$-bimodule.

Definition 4.27. (a) Let

$$F^2_{E,R,X}\colon\mathbb{K}\langle E\rangle\to[[\mathbb{K}\langle E\,|\,R\rangle,X]^{\times E},[\mathbb{K}\langle E\,|\,R\rangle,X]],\ r\mapsto F^{2,r}_{E,R,X}$$

be the $\mathbb{K}$-linear map with for any $m \in \mathbb{N}$ and any $\{e_i\}_{i=1}^m \subseteq E$, if $r = \overrightarrow{\prod}_{i=1}^m e_i$, then for any $\gamma \in [\mathbb{K}\langle E \,|\, R\rangle, X]^{\times E}$ and any $a \in \mathbb{K}\langle E \,|\, R\rangle$,

$$\begin{aligned} &F_{E,R,X}^{2,r}(\gamma)(a) \\ &= \sum_{i=1}^{m} \left(\overrightarrow{\prod}_{j=1}^{i-1} e_j + J\right) \triangleright \left[\gamma_{e_i}\left(\left(\overrightarrow{\prod}_{j=i+1}^{m} e_j + J\right) a\right) - \gamma_{e_i}\left(\overrightarrow{\prod}_{j=i+1}^{m} e_j + J\right) \triangleleft a\right], \end{aligned}$$

and with $F_{E,R,X}^{2,1} = 0$.

(b) Similarly, let

$$G_{E,R,X}^{2} \colon \mathbb{K}\langle E\rangle \to [[\mathbb{K}\langle E \,|\, R\rangle, X]^{\times E}, X],\ r \mapsto G_{E,R,X}^{2,r}$$

be the $\mathbb{K}$-linear map with for any $m \in \mathbb{N}_0$ and any $\{e_i\}_{i=1}^m \subseteq E$, if $r = \overrightarrow{\prod}_{i=1}^m e_i$, then for any $\gamma \in [\mathbb{K}\langle E \,|\, R\rangle, X]^{\times E}$,

$$G_{E,R,X}^{2,r}(\gamma) = \sum_{i=1}^{m} \left(\overrightarrow{\prod}_{j=1}^{i-1} e_j + J\right) \triangleright \gamma_{e_i}\left(\overrightarrow{\prod}_{j=i+1}^{m} e_j + J\right),$$

and with $G_{E,R,X}^{2,1} = 0$.

LEMMA 4.28. *A commutative diagram of $\mathbb{K}$-linear maps is given by*

$$\begin{array}{ccc} Z_{\mathrm{HS}}^{2,0}(\mathbb{K}\langle E \,|\, R\rangle, X) & \hookrightarrow\!\!\!\!\rightarrow & \begin{array}{l}\{\gamma \in [\mathbb{K}\langle E \,|\, R\rangle, X]^{\times E} \wedge \forall e \in E\colon \gamma_e(1+J) = 0 \\ \qquad \wedge\ \forall r \in R\colon F_{E,R,X}^{2,r}(\gamma) = 0\}\end{array} \\ \uparrow \subseteq & & \uparrow \subseteq \\ B_{\mathrm{HS}}^{2,0}(\mathbb{K}\langle E \,|\, R\rangle, X) & \hookrightarrow\!\!\!\!\rightarrow & \begin{array}{l}\{\gamma \in [\mathbb{K}\langle E \,|\, R\rangle, X]^{\times E} \wedge \exists \psi \in [\mathbb{K}\langle E \,|\, R\rangle, X]\colon \\ \ \psi(1+J) = 0 \wedge \forall b \in \mathbb{K}\langle E \,|\, R\rangle\colon \forall e \in E\colon \\ \ \gamma_e(b) = (e+J) \triangleright \psi(b) - \psi((e+J)b) + \psi(e+J) \triangleleft b\}\end{array} \end{array},$$

where the sets on the right-hand side are $\mathbb{K}$-vector subspaces of $[\mathbb{K}\langle E \,|\, R\rangle, X]^{\times E}$ and where the horizontal arrows both assign to any element c of their respective domains the tuple γ with γ_e for each $e \in E$ being the mapping $\mathbb{K}\langle E \,|\, R\rangle \to X$ with

$$b \mapsto c((e+J) \otimes b)$$

for any $b \in \mathbb{K}\langle E \,|\, R\rangle$.
In particular, the lower right space is contained in the above one. Moreover, the horizontal arrows are both $\mathbb{K}$-linear isomorphisms. Their respective inverses both assign to any element γ of their respective domains the unique $\mathbb{K}$-linear map $\mathbb{K}\langle E \,|\, R\rangle \otimes \mathbb{K}\langle E \,|\, R\rangle \to X$ with

$$(p+J) \otimes b \mapsto F_{E,R,X}^{2,p}(\gamma)(b)$$

for any $p \in \mathbb{K}\langle E\rangle$ and $b \in \mathbb{K}\langle E \,|\, R\rangle$. For the lower row of the diagram that is the same as assigning $\gamma \mapsto \partial\psi$ for any $\psi \in [\mathbb{K}\langle E \,|\, R\rangle, X]$ with $\psi(1+J) = 0$ such that $\gamma_e = (e+J) \triangleright \psi(b) - \psi((e+J)b) + \psi(e+J) \triangleleft b$ for any $b \in \mathbb{K}\langle E \,|\, R\rangle$ and $e \in E$.

PROOF. If we abbreviate $A \equiv \mathbb{K}\langle E \,|\, R\rangle$ and if $A|X$ is then the A-bimodule from Lemma 4.21 (a) with left respectively right A-actions $\blacktriangleright$ and $\blacktriangleleft$, then by the combination of Lemma 4.21 (b) and Lemma 4.10 the diagram

$$\begin{array}{ccccc} Z^{2,0}_{\mathrm{HS}}(A,X) & \hookrightarrow\!\!\!\twoheadrightarrow & Z^{1}_{\mathrm{HS}}(A,A|X) & \hookrightarrow\!\!\!\twoheadrightarrow & \{\gamma \in (A|X)^{\times E} \wedge \forall r \in R\colon F^{1,r}_{E,R,A|X}(\gamma) = 0\} \\ \uparrow\subseteq & & \uparrow\subseteq & & \uparrow\subseteq \\ B^{2,0}_{\mathrm{HS}}(A,X) & \hookrightarrow\!\!\!\twoheadrightarrow & B^{1}_{\mathrm{HS}}(A,A|X) & \hookrightarrow\!\!\!\twoheadrightarrow & \{(e+J \blacktriangleright \psi - \psi \blacktriangleleft e+J)_{e\in E} \,|\, \psi \in A|X\} \end{array}$$

of $\mathbb{K}$-linear maps commutes, where the left horizontal arrows are the restrictions of the currying isomorphism $\theta_{A,A,X}$, where the inverses of the left horizontal arrows are the restrictions of $\theta_{A,A,X}{}^{-1}$, where the right horizontal are both $\mathbb{K}$-linear isomorphisms assigning to any element η of their respective domains the tuple $(\eta(e+J))_{e\in E}$ and whose respective inverses map any element γ of their respective domains to the mapping $A \to A|X$ with $p+J \mapsto F^{1,p}_{E,R,A|X}(\gamma)$ for any $p \in \mathbb{K}\langle E\rangle$, which in the case of the lower right horizontal arrow is the same as assigning $((e+J) \blacktriangleright \psi - \psi \blacktriangleleft (e+J))_{e\in E} \mapsto \partial\psi$ for any $\psi \in A|X$.

We first show that the right upper corner of the above diagram is the same as the one from the diagram in the claim. To see this we first remember that, as a $\mathbb{K}$-vector space, $A|X$ is by definition $\{\gamma \in [A,X] \wedge \gamma(1+J) = 0\}$. Hence, demanding $\gamma \in (A|X)^{\times E}$ is the same as asking both $\gamma \in [A,X]^{\times E}$ and $\gamma_e(1+J) = 0$ for any $e \in E$. Moreover, $F^{1}_{E,R,A|X} = F^{2}_{E,R,X}$. Indeed, $F^{1,1}_{E,R,A|X} = F^{2,1}_{E,R,X} = 0$, of course, and for any $m \in \mathbb{N}$ and $\{e_i\}_{i=1}^m \subseteq E$, by definition,

$$\begin{aligned} &F^{1,r}_{E,R,A|X}(\gamma)(b) \\ &\quad= \textstyle\sum_{i=1}^m \left(\left(\overrightarrow{\prod}_{j=1}^{i-1} e_j + J \right) \blacktriangleright \gamma_{e_i} \blacktriangleleft \left(\overrightarrow{\prod}_{j=i+1}^{m} e_j + J \right) \right)(b) \\ &\quad= \textstyle\sum_{i=1}^m \left(\overrightarrow{\prod}_{j=1}^{i-1} e_j + J \right) \triangleright \left[\left(\gamma_{e_i} \blacktriangleleft \left(\overrightarrow{\prod}_{j=i+1}^{m} e_j + J \right) \right)(b) \right] \\ &\quad= \textstyle\sum_{i=1}^m \left(\overrightarrow{\prod}_{j=1}^{i-1} e_j + J \right) \triangleright \left[\gamma_{e_i} \left(\left(\overrightarrow{\prod}_{j=i+1}^{m} e_j + J \right) b \right) - \gamma_{e_i} \left(\overrightarrow{\prod}_{j=i+1}^{m} e_j + J \right) \triangleleft b \right] \\ &\quad= F^{2,r}_{E,R,X}(\gamma)(b). \end{aligned}$$

for any $\gamma \in (A|X)^{\times E}$ and $b \in A$, from which $F^{1}_{E,R,A|X} = F^{2}_{E,R,X}$ follows by $\mathbb{K}$-linearity. Hence, the right upper corner of the diagrams agree.

The same is true for the lower right corners because for any $\psi \in A|X$, any $e \in E$ and any $b \in A$, by definition,

$$(e+J \blacktriangleright \psi - \psi \blacktriangleleft e+J)(b) = e+J \triangleright \psi(b) - \psi((e+J)b) + \psi(e+J) \triangleleft b.$$

Furthermore, the composition of the horizontal arrows in the composite diagram above yields exactly the horizontal arrows of the diagram in the assertion. And the same holds for the inverses. □

Lemma 4.28 can be refined by characterizing the 2-coboundaries with the help of Lemma 4.22. Doing so will require the following auxiliary result.

Lemma 4.29. *For any* $c \in Z^{2,0}_{\mathrm{HS}}(\mathbb{K}\langle E\,|\,R\rangle, X)$ *and any* $x \in X^{\times E}$ *there exists a* $\mathbb{K}$-*algebra homomorphism* $\mathbb{K}\langle E\,|\,R\rangle \to \mathrm{End}(\mathbb{K}\langle E\,|\,R\rangle \oplus X)$ *with the property that for any* $e \in E$ *the element* $e + J$ *is mapped to the endomorphism with for any* $b \in \mathbb{K}\langle E\,|\,R\rangle$ *and* $y \in X$,

$$(b, y) \mapsto ((e + J)b, c((e + J) \otimes b) - x_e \lhd b + (e + J) \rhd y)$$

if and only if for any $r \in R$, *if* $r = \sum_{m\in\mathbb{N}_0} \sum_{e\in E^{\otimes m}} \lambda_{m,e} \overrightarrow{\prod}_{i=1}^{m} e_i$, *then*

$$0 = \sum_{m\in\mathbb{N}_0} \sum_{e\in E^{\otimes m}} \lambda_{m,e} \sum_{i=1}^{m} \Big(\overrightarrow{\prod}_{j=1}^{i-1} e_j + J\Big) \rhd \Big[c\Big((e_i + J) \otimes \Big(\overrightarrow{\prod}_{j=i+1}^{m} e_j + J\Big)\Big) - x_{e_i} \lhd \Big(\overrightarrow{\prod}_{j=i+1}^{m} e_j + J\Big)\Big].$$

Proof. By the universal property of $\mathbb{K}\langle E\,|\,R\rangle$ the existence of a $\mathbb{K}$-algebra homomorphism $t\colon \mathbb{K}\langle E\,|\,R\rangle \to \mathrm{End}(\mathbb{K}\langle E\,|\,R\rangle \oplus X)$ with the stated property is equivalent to $r((t_e)_{e\in E}) = 0$ holding for any $r \in R$, where for any $e \in E$ the endomorphism t_e of $\mathbb{K}\langle E\,|\,R\rangle \oplus X$ satisfies

$$(b, y) \mapsto ((e + J)b, c((e + J) \otimes b) - x_e \lhd b + (e + J) \rhd y)$$

for any $b \in \mathbb{K}\langle E\,|\,R\rangle$ and $y \in X$. If $r = \sum_{m\in\mathbb{N}_0} \sum_{e\in E^{\otimes m}} \lambda_{m,e} \overrightarrow{\prod}_{i=1}^{m} e_i \in R$ is any relation, then by the definition of $\mathrm{End}(\mathbb{K}\langle E\,|\,R\rangle \oplus X)$, by Lemma 4.23 and by $\mathbb{K}$-linearity the element $r((t_e)_{e\in E}) = \sum_{m\in\mathbb{N}_0} \sum_{e\in E^{\otimes m}} \lambda_{m,e}\, t_1 \circ t_2 \circ \ldots \circ t_m$ is given by the mapping which for any $b \in \mathbb{K}\langle E\,|\,R\rangle$ and $y \in X$ sends (b, y) to the pair with first component

$$\textstyle\sum_{m\in\mathbb{N}_0} \sum_{e\in E^{\otimes m}} \lambda_{m,e} \Big(\overrightarrow{\prod}_{i=1}^{m} e_i + J\Big) b = r((e + J)_{e\in E}) b = 0.$$

and second component

$$\sum_{m\in\mathbb{N}_0} \sum_{e\in E^{\otimes m}} \lambda_{m,e} \sum_{i=1}^{m} \Big(\overrightarrow{\prod}_{j=1}^{i-1} e_j + J\Big) \rhd \Big[c\Big((e_i + J) \otimes \Big(\overrightarrow{\prod}_{j=i+1}^{m} e_j + J\Big) b\Big) - x_{e_i} \lhd \Big(\Big(\overrightarrow{\prod}_{j=i+1}^{m} e_j + J\Big) b\Big)\Big] + \sum_{m\in\mathbb{N}_0} \sum_{e\in E^{\otimes m}} \lambda_{m,e} \Big(\overrightarrow{\prod}_{i=1}^{m} e_i + J\Big) \rhd y,$$

where, actually, the last summand can be ignored because

$$\sum_{m\in\mathbb{N}_0} \sum_{e\in E^{\otimes m}} \lambda_{m,e} \Big(\overrightarrow{\prod}_{i=1}^{m} e_i + J\Big) \rhd y = r((e + J)_{e\in E}) \rhd y = 0.$$

Hence, $r((t_e)_{e\in E}) = 0$ if and only if for any $b \in \mathbb{K}\langle E\,|\,R\rangle$,

$$0 = \sum_{m\in\mathbb{N}_0} \sum_{e\in E^{\otimes m}} \lambda_{m,e} \sum_{i=1}^{m} \Big(\overrightarrow{\prod}_{j=1}^{i-1} e_j + J\Big) \rhd \Big[c\Big((e_i + J) \otimes \Big(\overrightarrow{\prod}_{j=i+1}^{m} e_j + J\Big) b\Big) - x_{e_i} \lhd \Big(\Big(\overrightarrow{\prod}_{j=i+1}^{m} e_j + J\Big) b\Big)\Big].$$

On the other hand, because $c \in Z^{2,0}_{\mathrm{HS}}(\mathbb{K}\langle E\,|\,R\rangle, X)$ Lemma 4.28 assures us that for any $b \in \mathbb{K}\langle E\,|\,R\rangle$, if $\gamma \in [\mathbb{K}\langle E\,|\,R\rangle, X]^{\times E}$ is such that for any $e \in E$ the component

γ_e is given by the mapping with $b \mapsto c((e+J)\otimes b)$ for any $b \in \mathbb{K}\langle E\,|\,R\rangle$, then $F^{2,r}_{E,R,X}(\gamma)(b) = 0$, which is to say

$$0 = \sum_{m\in\mathbb{N}_0}\sum_{e\in E^{\otimes m}} \lambda_{m,e} \sum_{i=1}^{m} \Big(\overrightarrow{\prod}{}_{j=1}^{i-1} e_j + J\Big) \triangleright \Big[c\Big((e_i+J)\otimes\Big(\overrightarrow{\prod}{}_{j=i+1}^{m} e_j + J\Big) b\Big) - c\Big((e_i+J)\otimes\Big(\overrightarrow{\prod}{}_{j=i+1}^{m} e_j + J\Big)\Big) \triangleleft b\Big].$$

By subtracting this equation from the one preceding it we see that $r((t_e)_{e\in E}) = 0$ if and only if for any $b \in \mathbb{K}\langle E\,|\,R\rangle$,

$$0 = \sum_{m\in\mathbb{N}_0}\sum_{e\in E^{\otimes m}} \lambda_{m,e} \sum_{i=1}^{m} \Big(\overrightarrow{\prod}{}_{j=1}^{i-1} e_j + J\Big) \triangleright \Big[c\Big((e_i+J)\otimes\Big(\overrightarrow{\prod}{}_{j=i+1}^{m} e_j + J\Big)\Big) - x_{e_i} \triangleleft \Big(\overrightarrow{\prod}{}_{j=i+1}^{m} e_j + J\Big)\Big] \triangleleft b.$$

And that proves the claim because $\mathbb{K}\langle E\,|\,R\rangle$ has a unit. □

It follows a first formulation of the anounced characterization of 2-coboundaries to be integrated in Lemma 4.28.

LEMMA 4.30. *Any* $c \in Z^{2,0}_{\mathrm{HS}}(\mathbb{K}\langle E\,|\,R\rangle, X)$ *satisfies* $c \in B^{2,0}_{\mathrm{HS}}(\mathbb{K}\langle E\,|\,R\rangle, X)$ *if and only if there exists* $x \in X^{\times E}$ *such that*

$$0 = \sum_{m\in\mathbb{N}_0}\sum_{e\in E^{\otimes m}} \lambda_{m,e} \sum_{i=1}^{m} \Big(\overrightarrow{\prod}{}_{j=1}^{i-1} e_j + J\Big) \triangleright \Big[c\Big((e_i+J)\otimes\Big(\overrightarrow{\prod}{}_{j=i+1}^{m} e_j + J\Big)\Big) - x_{e_i} \triangleleft \Big(\overrightarrow{\prod}{}_{j=i+1}^{m} e_j + J\Big)\Big]$$

for any $r = \sum_{m\in\mathbb{N}_0}\sum_{e\in E^{\otimes m}} \lambda_{m,e} \overrightarrow{\prod}{}_{i=1}^{m} e_i \in R$.

PROOF. We show each implication separately.

Coboundary implies solution. First, suppose that there exists $\psi \in [\mathbb{K}\langle E\,|\,R\rangle, X]$ with $\psi(1+J) = 0$ such that $c = \partial\psi$ and let $x \in X^{\times E}$ be such that $x_e := \psi(e+J)$ for any $e \in E$. Then, given any $r \in R$ as in the claim, the expression on the right hand side of the equation in the claim, by

$$\partial\psi\Big((e_i+J)\otimes\Big(\overrightarrow{\prod}{}_{j=i+1}^{m} e_j + J\Big)\Big) = (e_i+J) \triangleright \psi\Big(\overrightarrow{\prod}{}_{j=i+1}^{m} e_j + J\Big) - \psi\Big(\overrightarrow{\prod}{}_{j=i}^{m} e_j + J\Big) + \psi(e_i+J) \triangleleft \Big(\overrightarrow{\prod}{}_{j=i+1}^{m} e_j + J\Big)$$

holding for any $m \in \mathbb{N}_0$, any $e \in E^{\otimes m}$ and any $i \in [\![m]\!]$, is identical to

$$\sum_{m\in\mathbb{N}_0}\sum_{e\in E^{\otimes m}} \lambda_{m,e} \sum_{i=1}^{m} \Big(\overrightarrow{\prod}{}_{j=1}^{i-1} e_j + J\Big) \triangleright \Big[(e_i+J) \triangleright \psi\Big(\overrightarrow{\prod}{}_{j=i+1}^{m} e_j + J\Big) - \psi\Big(\overrightarrow{\prod}{}_{j=i}^{m} e_j + J\Big)\Big].$$

Distributing the left action shows that this is the same as

$$\sum_{m\in\mathbb{N}_0}\sum_{e\in E^{\otimes m}} \lambda_{m,e} \sum_{i=1}^{m} \Big[\Big(\overrightarrow{\prod}{}_{j=1}^{i} e_j + J\Big) \triangleright \psi\Big(\overrightarrow{\prod}{}_{j=i+1}^{m} e_j + J\Big) - \Big(\overrightarrow{\prod}{}_{j=1}^{i-1} e_j + J\Big) \triangleright \psi\Big(\overrightarrow{\prod}{}_{j=i}^{m} e_j + J\Big)\Big],$$

or, after reassociating the outer sum and shifting the summation index in the second sum by 1 to the left, identically,

$$\sum_{m\in\mathbb{N}_0}\sum_{e\in E^{\otimes m}}\lambda_{m,e}\Bigg[\sum_{i=1}^{m}\Big(\overrightarrow{\prod}_{j=1}^{i}e_j+J\Big)\triangleright\psi\Big(\overrightarrow{\prod}_{j=i+1}^{m}e_j+J\Big)\\ -\sum_{i=0}^{m-1}\Big(\overrightarrow{\prod}_{j=1}^{i}e_j+J\Big)\triangleright\psi\Big(\overrightarrow{\prod}_{j=i+1}^{m}e_j+J\Big)\Bigg].$$

By $\psi(1+J)=0$ this elements coincides with

$$-\sum_{m\in\mathbb{N}_0}\sum_{e\in E^{\otimes m}}\lambda_{m,e}\,\psi\Big(\overrightarrow{\prod}_{j=1}^{m}e_j+J\Big)$$

or, in other words, $-\psi(r((e+J)_{e\in E}))$, which is zero by $r((e+J)_{e\in E})=0$. Hence, this part of the claim is true.

Solution implies coboundary. Conversely, let there now exist $x\in X^{\times E}$ such that the identities in the claim are satisfied. Then, by Lemma 4.29 there exists an algebra homomorphism $t\colon\mathbb{K}\langle E\,|\,R\rangle\to\mathrm{End}(\mathbb{K}\langle E\,|\,R\rangle\oplus X)$ such that for any $e\in E$ the endomorphism $t(e+J)$ satisfies

$$(b,y)\mapsto((e+J)b,c((e+J)\otimes b)-x_e\triangleleft b+(e+J)\triangleright y)$$

for any $b\in\mathbb{K}\langle E\,|\,R\rangle$ and $y\in X$. Thus, if π^2 is the projection $\mathbb{K}\langle E\,|\,R\rangle\oplus X\to X$, $(b,y)\mapsto y$, we obtain a $\mathbb{K}$-linear map $\psi\colon\mathbb{K}\langle E\,|\,R\rangle\to X$ by defining

$$\psi(a):=-\pi^2(t(a)(1+J,0))$$

for any $a\in\mathbb{K}\langle E\,|\,R\rangle$. By Lemma 4.22, in order to show $c=\partial\psi$ it suffices to prove that the $\mathbb{K}$-linear map $T_{c,\psi}\colon\mathbb{K}\langle E\,|\,R\rangle\to[\mathbb{K}\langle E\,|\,R\rangle\oplus X,\mathbb{K}\langle E\,|\,R\rangle\oplus X]$ which assigns to any $a\in\mathbb{K}\langle E\,|\,R\rangle$ the endomorphism with

$$(b,y)\mapsto(ab,c(a\otimes b)-\psi(a)\triangleleft b+a\triangleright y)$$

for any $b\in\mathbb{K}\langle E\,|\,R\rangle$ and $y\in X$ is a $\mathbb{K}$-algebra homomorphism from $\mathbb{K}\langle E\,|\,R\rangle$ to $\mathrm{End}(\mathbb{K}\langle E\,|\,R\rangle\oplus X)$. Thus, we can prove our claim by verifying that $T_{c,\psi}$ coincides with the known algebra homomorphism t.

By $\mathbb{K}$-linearity of $T_{c,\psi}$ and t it suffices to show $T_{c,\psi}(\overrightarrow{\prod}_{i=1}^{m}e_i+J)=t(\overrightarrow{\prod}_{i=1}^{m}e_i+J)$ for any $m\in\mathbb{N}_0$ and $e\in E^{\otimes m}$. Since t is a $\mathbb{K}$-algebra homomorphism from $\mathbb{K}\langle E\,|\,R\rangle$ to $\mathrm{End}(\mathbb{K}\langle E\,|\,R\rangle\oplus X)$ and since E generates $\mathbb{K}\langle E\,|\,R\rangle$ we find by the defining property of t and by Lemma 4.23 that the element

$$t(\overrightarrow{\prod}_{i=1}^{m}e_i+J)=\overrightarrow{\prod}_{i=1}^{m}t(e_i+J)$$

of $\mathrm{End}(\mathbb{K}\langle E\,|\,R\rangle\oplus X)$ is given by the mapping which for any $b\in\mathbb{K}\langle E\,|\,R\rangle$ and $y\in X$ sends (b,y) to the pair with first component $(e_1e_2\ldots e_m+J)b$ and with second component

$$\sum_{i=1}^{m}\Big(\overrightarrow{\prod}_{j=1}^{i-1}e_j+J\Big)\triangleright\Big[c\Big((e_i+J)\otimes\Big(\overrightarrow{\prod}_{j=i+1}^{m}e_j+J\Big)b\Big)\\ -x_{e_i}\triangleleft\Big(\Big(\overrightarrow{\prod}_{j=i+1}^{m}e_j+J\Big)b\Big)\Big]+\Big(\overrightarrow{\prod}_{j=1}^{m}e_j+J\Big)\triangleright y.$$

In particular, we can infer

$$\begin{aligned}&-\psi\left(\overrightarrow{\prod}_{i=1}^{m}e_i+J\right)\\&\quad=\pi^2\left(t\left(\overrightarrow{\prod}_{i=1}^{m}e_i+J\right)(1+J,0)\right)\\&\quad=\sum_{i=1}^{m}\left(\overrightarrow{\prod}_{j=1}^{i-1}e_j+J\right)\rhd\left[c\left((e_i+J)\otimes\left(\overrightarrow{\prod}_{j=i+1}^{m}e_j+J\right)\right)-x_{e_i}\lhd\left(\overrightarrow{\prod}_{j=i+1}^{m}e_j+J\right)\right].\end{aligned}$$

On the other hand, by the definition of $T_{c,\psi}$, the endomorphism $T_{c,\psi}(\overrightarrow{\prod}_{i=1}^{m}e_i+J)$ for any $b\in\mathbb{K}\langle E\,|\,R\rangle$ and $y\in X$ maps (b,y) to the pair with first entry $(e_1e_2\dots e_m+J)b$ and with second entry

$$c\left(\left(\overrightarrow{\prod}_{i=1}^{m}e_i+J\right)\otimes b\right)-\psi\left(\overrightarrow{\prod}_{i=1}^{m}e_i+J\right)\lhd b+\left(\overrightarrow{\prod}_{i=1}^{m}e_i+J\right)\rhd y.$$

It follows that all we have to prove is that for any $b\in\mathbb{K}\langle E\,|\,R\rangle$,

$$\begin{aligned}\sum_{i=1}^{m}\left(\overrightarrow{\prod}_{j=1}^{i-1}e_j+J\right)\rhd\left[c\left((e_i+J)\otimes\left(\overrightarrow{\prod}_{j=i+1}^{m}e_j+J\right)b\right)-x_{e_i}\lhd\left(\left(\overrightarrow{\prod}_{j=i+1}^{m}e_j+J\right)b\right)\right]\\=c\left(\left(\overrightarrow{\prod}_{i=1}^{m}e_i+J\right)\otimes b\right)-\psi\left(\overrightarrow{\prod}_{i=1}^{m}e_i+J\right)\lhd b.\end{aligned}$$

Inserting the expression for $-\psi(e_1e_2\dots e_m+J)$ obtained above shows that the right-hand side of this equation is identical to

$$\begin{aligned}&c\left(\left(\overrightarrow{\prod}_{i=1}^{m}e_i+J\right)\otimes b\right)\\&\quad+\sum_{i=1}^{m}\left(\overrightarrow{\prod}_{j=1}^{i-1}e_j+J\right)\rhd\left[c\left((e_i+J)\otimes\left(\overrightarrow{\prod}_{j=i+1}^{m}e_j+J\right)\right)-x_{e_i}\lhd\left(\overrightarrow{\prod}_{j=i+1}^{m}e_j+J\right)\right]\lhd b.\end{aligned}$$

Hence, $T_{c,\psi}(\overrightarrow{\prod}_{i=1}^{m}e_i+J)=t(\overrightarrow{\prod}_{i=1}^{m}e_i+J)$ holds if and only if

$$\begin{aligned}c\left(\left(\overrightarrow{\prod}_{i=1}^{m}e_i+J\right)\otimes b\right)=\sum_{i=1}^{m}\left(\overrightarrow{\prod}_{j=1}^{i-1}e_j+J\right)\rhd\Big[c\left((e_i+J)\otimes\left(\overrightarrow{\prod}_{j=i+1}^{m}e_j+J\right)b\right)\\-c\left((e_i+J)\otimes\left(\overrightarrow{\prod}_{j=i+1}^{m}e_j+J\right)\right)\lhd b\Big],\end{aligned}$$

which is true by Lemma 4.18. Thus, the proof is complete. □

For integrating Lemma 4.30 into Lemma 4.28 it is convenient to reformulate it in the following way

LEMMA 4.31. *For any $\gamma\in[\mathbb{K}\langle E\,|\,R\rangle,X]^{\times E}$ such that $\gamma_e(1+J)=0$ for any $e\in E$ and such that $F^{2,r}_{E,R,X}(\gamma)=0$ for any $r\in R$ there exists $\psi\in[\mathbb{K}\langle E\,|\,R\rangle,X]$ such that $\psi(1+J)=0$ and*

$$\gamma_e(b)=(e+J)\rhd\psi(b)-\psi((e+J)b)+\psi(e+J)\lhd b$$

for any $e\in E$ if and only if there exists $x\in X^{\times E}$ such that for any $r\in R$,

$$G^{2,r}_{E,R,X}(\gamma)=F^{1,r}_{E,R,X}(x).$$

Proof. By Lemma 4.28 there exists $c \in Z^{2,0}_{\mathrm{HS}}(\mathbb{K}\langle E \,|\, R\rangle, X)$ with $c((e+J)\otimes b) = \gamma_e(b)$ for any $e \in E$ and $b \in \mathbb{K}\langle E \,|\, R\rangle$. Consequently, Lemma 4.30 tells us that $c \in B^2_{\mathrm{HS}}(\mathbb{K}\langle E \,|\, R\rangle, X)$ if and only if there exists $x \in X^{\times E}$ such that

$$\begin{aligned} 0 &= \sum_{m\in\mathbb{N}_0}\sum_{e\in E^{\otimes m}} \lambda_{m,e} \sum_{i=1}^{m} \Big(\overrightarrow{\prod}_{j=1}^{i-1} e_j + J\Big) \triangleright \Big[c\Big((e_i + J)\otimes \Big(\overrightarrow{\prod}_{j=i+1}^{m} e_j + J\Big)\Big) \\ &\qquad\qquad - x_{e_i} \triangleleft \Big(\overrightarrow{\prod}_{j=i+1}^{m} e_j + J\Big)\Big] \\ &= \sum_{m\in\mathbb{N}_0}\sum_{e\in E^{\otimes m}} \lambda_{m,e} \sum_{i=1}^{m} \Big(\overrightarrow{\prod}_{j=1}^{i-1} e_j + J\Big) \triangleright \Big[\gamma_{e_i}\Big(\overrightarrow{\prod}_{j=i+1}^{m} e_j + J\Big) - x_{e_i} \triangleleft \Big(\overrightarrow{\prod}_{j=i+1}^{m} e_j + J\Big)\Big] \\ &= G^{2,r}_{E,R,X}(\gamma) - F^{1,r}_{E,R,X}(x) \end{aligned}$$

for any $r = \sum_{m\in\mathbb{N}_0}\sum_{e\in E^{\otimes m}} \lambda_{m,e} \overrightarrow{\prod}_{i=1}^{m} e_i \in R$, i.e., if and only if the condition in the assertion is satisfied. And, by Lemma 4.28 the map c being an element of $B^2_{\mathrm{HS}}(\mathbb{K}\langle E \,|\, R\rangle, X)$ is equivalent to there existing $\psi \in [\mathbb{K}\langle E \,|\, R\rangle, X]$ such that $\psi(1+J) = 0$ and $\gamma_e(b) = ((e+J) \triangleright \psi(b) - \psi((e+J)b) + \psi(e+J) \triangleleft b)$ for any $e \in E$ and $b \in \mathbb{K}\langle E \,|\, R\rangle$. Hence, the claim is true. □

The following proposition now gives the combined results of Lemma 4.28 and Lemma 4.30 and is the characterization of the second Hochschild cohomology of universal algebras we sought.

Proposition 4.32. *There exists an isomorphism of* $\mathbb{K}$*-vector spaces*

$$H^2_{\mathrm{HS}}(\mathbb{K}\langle E \,|\, R\rangle, X) \hookrightarrow\!\!\!\!\!\twoheadrightarrow \frac{\{\gamma \in [\mathbb{K}\langle E \,|\, R\rangle, X]^{\times E} \ \wedge\ \forall e \in E\colon \gamma_e(1+J) = 0 \ \wedge\ \forall r \in R\colon F^{2,r}_{E,R,X}(\gamma) = 0\}}{\{\gamma \text{ as above } \wedge\ \exists x \in X^{\times E}\colon \forall r \in R\colon G^{2,r}_{E,R,X}(\gamma) = F^{1,r}_{E,R,X}(x)\}},$$

where the sets on the right-hand side are $\mathbb{K}$*-vector subspaces of* $[\mathbb{K}\langle E \,|\, R\rangle, X]^{\times E}$*, which for any* $c \in Z^2_{\mathrm{HS}}(\mathbb{K}\langle E \,|\, R\rangle, X)$ *assigns to the class of* c *the class of the tuple* γ *with* γ_e *for each* $e \in E$ *being the mapping with*

$$b \mapsto c((e+J)\otimes b) - (e+J) \triangleright c((1+J)\otimes(1+J)) \triangleleft b$$

for any $b \in \mathbb{K}\langle E \,|\, R\rangle$*. For any* $\gamma \in [\mathbb{K}\langle E \,|\, R\rangle, X]^{\times E}$ *with* $\gamma_e(1+J) = 0$ *for any* $e \in E$ *and with* $F^{2,r}_{E,R,X}(\gamma) = 0$ *for each* $r \in R$ *the inverse isomorphism maps the class of* γ *to the class of the unique* $\mathbb{K}$*-linear map* $\mathbb{K}\langle E \,|\, R\rangle \otimes \mathbb{K}\langle E \,|\, R\rangle \to X$ *with*

$$(p+J)\otimes b \mapsto F^{2,p}_{E,R,X}(\gamma)(b)$$

for any $b \in \mathbb{K}\langle E \,|\, R\rangle$ *and* $p \in \mathbb{K}\langle E\rangle$*.*

Proof. Composing the isomorphism implied by Lemma 4.28 with the one from Lemma 4.21 (c) and using Lemma 4.30 yields the claim. □

Lastly, we can refine Lemma 4.24 to give a construction of particular 2-cocycles on universal algebras.

PROPOSITION 4.33. *If X is simultaneously a $\mathbb{K}$-algebra with multiplication $\cdot$ such that for any $a \in \mathbb{K}\langle E \,|\, R\rangle$ and any $\{x, x'\} \subseteq X$,*

$$\begin{aligned} & (a \triangleright x) \cdot x' = a \triangleright (x \cdot x') \\ \textit{and} \quad & (x \triangleleft a) \cdot x' = x \cdot (a \triangleright x') \\ \textit{and} \quad & x \cdot (x' \triangleleft a) = (x \cdot x') \triangleleft a, \end{aligned}$$

then there exists a unique $\mathbb{K}$-linear map

$$\begin{aligned} & \{x \in X^{\times E} \wedge \forall r \in R\colon F^{1,r}_{E,R,X}(x) = 0\}^{\otimes 2} \\ & \quad \longrightarrow \{\gamma \in [\mathbb{K}\langle E \,|\, R\rangle, X]^{\times E} \wedge \forall e \in E\colon \gamma_e(1 + J) = 0 \wedge \forall r \in R\colon F^{2,r}_{E,R,X}(\gamma) = 0\} \end{aligned}$$

which for any $\{x, x'\} \subseteq X^{\times E}$ with $F^{1,r}_{E,R,X}(x) = F^{1,r}_{E,R,X}(x') = 0$ for each $r \in R$ maps $x \otimes x'$ the tuple γ with for any $e \in E$ the component γ_e given by the mapping with

$$p + J \mapsto x_e \cdot F^{1,p}_{E,R,X}(x')$$

for any $p \in \mathbb{K}\langle E\rangle$.

PROOF. By Lemmata 4.10, 4.24 and 4.28 there is a diagram of three composable $\mathbb{K}$-linear maps as below

$$\begin{array}{ccc} \{x \in X^{\times E} \wedge \forall r \in R\colon F^{1,r}_{E,R,X}(x) = 0\}^{\otimes 2} & \dashrightarrow & \begin{array}{l}\{\gamma \in [\mathbb{K}\langle E \,|\, R\rangle, X]^{\times E} \wedge \forall e \in E\colon \\ \gamma_e(1 + J) = 0 \wedge \forall r \in R\colon F^{2,r}_{E,R,X}(\gamma) = 0\}\end{array} \\ \downarrow & & \uparrow \\ Z^1_{\mathrm{HS}}(\mathbb{K}\langle E \,|\, R\rangle, X)^{\otimes 2} & \longrightarrow & Z^{2,0}_{\mathrm{HS}}(\mathbb{K}\langle E \,|\, R\rangle, X), \end{array}$$

where the left vertical arrow is the unique $\mathbb{K}$-linear mapping which for any $\{x, x'\} \subseteq X^{\times E}$ with $F^{1,r}_{E,R,X}(x) = F^{1,r}_{E,R,X}(x') = 0$ for each $r \in R$ maps $x \otimes x'$ to $\eta \otimes \eta'$ with $\eta(p+J) = F^{1,p}_{E,R,X}(x)$ and $\eta'(p + J) = F^{1,p}_{E,R,X}(x')$ for any $p \in \mathbb{K}\langle E\rangle$, where the lower horizontal arrow is the unique $\mathbb{K}$-linear mapping which for any $\{\eta, \eta'\} \subseteq Z^1_{\mathrm{HS}}(\mathbb{K}\langle E \,|\, R\rangle, X)$ maps $\eta \otimes \eta'$ to the unique $\mathbb{K}$-linear mapping c with $c(a \otimes b) = \eta(a) \cdot \eta'(b)$ for any $\{a, b\} \subseteq \mathbb{K}\langle E \,|\, R\rangle$, and where the right horizontal arrow maps any $c \in Z^{2,0}_{\mathrm{HS}}(\mathbb{K}\langle E \,|\, R\rangle, X)$ to the tuple γ with for any $e \in E$ the component γ_e being the mapping with $b \mapsto c((e+J)\otimes b)$. We prove that the composition of the three arrows is exactly the rule in the claim.

Indeed, for any $\{x, x'\} \subseteq X^{\times E}$ with $F^{1,r}_{E,R,X}(x) = F^{1,r}_{E,R,X}(x') = 0$ for each $r \in R$, if $\eta \otimes \eta'$ is the image of $x \otimes x'$ under the first arrow, if c is the image of $\eta \otimes \eta'$ under the second arrow and if γ is the image of c under third arrow, then for any $e \in E$,

$$\begin{aligned} \gamma_e(e + J) = c((e + J) \otimes (p + J)) = \eta(e + J) \cdot \eta'(p + J) = F^{1,e}_{E,R,X}(x) \cdot F^{1,p}_{E,R,X}(x') \\ = x_e \cdot F^{1,p}_{E,R,X}(x') \end{aligned}$$

for any $p \in \mathbb{K}\langle E\rangle$. That is what we needed to prove. □

To do.example of a non-trivial submodule $\mathbb{K}\langle E \,|\, R\rangle|X$ as a way of producing elements of $Z^2_{\mathrm{HS}}(A, X)$

5. Certain spaces of scalar matrices and their dimensions

Recall that for any $n \in \mathbb{N}$ any $v \in M_n(\mathbb{C})$ is called *skew-symmetric* if $v = -v^{\mathrm{t}}$.

DEFINITION 5.1. For the sake of brevity we call any $v \in M_n(\mathbb{C})$ *small* if $\sum_{i=1}^n v_{j,i} = 0$ for any $j \in [\![n]\!]$ and $\sum_{j=1}^n v_{j,i} = 0$ for any $i \in [\![n]\!]$, i.e., if each row and each column sums to zero.

LEMMA 5.2. *For any $n \in \mathbb{N}$ and $v \in M_n(\mathbb{C})$ the following equivalences hold.*

(a) There is $\lambda \in \mathbb{C}$ such that $v - \lambda I$ is small if and only if $\sum_{s=1}^n v_{j,s} - \sum_{s=1}^n v_{s,i} = 0$ for any $\{i,j\} \subseteq [\![n]\!]$. Moreover, then $\lambda = \sum_{s=1}^n v_{j,s} = \sum_{s=1}^n v_{s,i}$ for any $\{i,j\} \subseteq [\![n]\!]$.

(b) There is $\lambda \in \mathbb{C}$ such that $v - \lambda I$ is skew-symmetric if and only if for any $\{i,j\} \subseteq [\![n]\!]$ with $i \neq j$ both $v_{j,i} + v_{i,j} = 0$ and $v_{j,j} - v_{i,i} = 0$. Moreover, then $\lambda = v_{i,i}$ for any $i \in [\![n]\!]$.

(c) There are $\{\lambda_1, \lambda_2\} \subseteq \mathbb{C}$ such that $v - \lambda_1 I$ is skew-symmetric and $v - \lambda_2 I$ small if and only if there is $\lambda \in \mathbb{C}$ such that $v - \lambda I$ is both skew-symmetric and small. Moreover, then $\lambda = \lambda_1 = \lambda_2$.

PROOF. (a) If $\lambda \in \mathbb{C}$ is such that $w \equiv v - \lambda I$ is small, then for any $\{i,j\} \subseteq [\![n]\!]$ it follows $0 = \sum_{s=1}^n w_{j,s} = \sum_{s=1}^n (v_{j,s} - \lambda\delta_{j,s}) = \sum_{s=1}^n v_{j,s} - \lambda$ and $0 = \sum_{s=1}^n w_{s,i} = \sum_{s=1}^n (v_{s,i} - \lambda\delta_{s,i}) = \sum_{s=1}^n v_{s,i} - \lambda$, which proves $\sum_{s=1}^n v_{j,s} = \lambda = \sum_{s=1}^n v_{s,i}$. Of course, then $\sum_{s=1}^n v_{j,s} - \sum_{s=1}^n v_{s,i} = \lambda - \lambda = 0$ for any $\{i,j\} \subseteq [\![n]\!]$.

Conversely, if $\sum_{s=1}^n v_{j,s} - \sum_{s=1}^n v_{s,i} = 0$ for any $\{i,j\} \subseteq [\![n]\!]$ and if we let $\lambda \equiv \sum_{s=1}^n v_{1,s}$ and $w \equiv v - \lambda I$, then for any $\{i,j\} \subseteq [\![n]\!]$, first, $\lambda = \sum_{s=1}^n v_{j,s} = \sum_{s=1}^n v_{s,i}$ and thus, second, $\sum_{s=1}^n w_{j,s} = \sum_{s=1}^n (v_{j,s} - \lambda\delta_{j,s}) = \sum_{s=1}^n v_{j,s} - \lambda = 0$ and, likewise, $\sum_{s=1}^n w_{s,i} = \sum_{s=1}^n (v_{s,i} - \lambda\delta_{s,i}) = \sum_{s=1}^n v_{s,i} - \lambda = 0$. Hence, w is small then.

(b) If for $\lambda \in \mathbb{C}$ the matrix $w \equiv v - \lambda I$ is skew-symmetric, then $0 = w_{j,i} + w_{i,j} = (v_{j,i} - \lambda\delta_{j,i}) + (v_{i,j} - \lambda\delta_{i,j}) = v_{j,i} + v_{i,j} - 2\lambda\delta_{j,i}$ for any $\{i,j\} \subseteq [\![n]\!]$. Consequently, if $i \neq j$, this means $0 = v_{j,i} + v_{i,j}$ and, if $i = j$, we find $0 = 2v_{i,i} - 2\lambda$, i.e., $\lambda = v_{i,i}$. And that implies in particular $v_{j,j} - v_{i,i} = \lambda - \lambda = 0$.

If, conversely, $v_{j,i} + v_{i,j} = 0$ and $v_{j,j} - v_{i,i} = 0$ for any $\{i,j\} \subseteq [\![n]\!]$ with $i \neq j$ and if we let $\lambda \equiv v_{1,1}$ and $w \equiv v - \lambda I$, then, on the one hand, $\lambda = v_{i,i}$ for any $i \in [\![n]\!]$ and, on the other hand, for any $\{i,j\} \subseteq [\![n]\!]$, generally, $w_{j,i} + w_{i,j} = (v_{j,i} - \lambda\delta_{j,i}) + (v_{i,j} - \lambda\delta_{i,j}) = v_{j,i} + v_{i,j} - 2\lambda\delta_{j,i}$, which in case $i \neq j$ simply means $w_{j,i} + w_{i,j} = v_{j,i} + v_{i,j} = 0$ and which for $i = j$ amounts to $w_{j,i} + w_{i,j} = 2v_{i,i} - 2\lambda = 2\lambda - 2\lambda = 0$. In conclusion, w is skew-symmetric then.

(c) One implication is clear. If, conversely, $\{\lambda_1, \lambda_2\} \subseteq \mathbb{C}$ are such that $v - \lambda_1 I$ is skew-symmetric and $v - \lambda_2 I$ is small, then $\lambda_1 = v_{1,1}$ by (b) and $\lambda_2 = \sum_{j=1}^n v_{j,1} = \sum_{i=1}^n v_{1,j}$ by (a). Subtracting the two identities $\sum_{j=1}^n v_{j,1} = \lambda_1 + \sum_{j=2}^n v_{j,1}$ and $\sum_{i=1}^n v_{1,i} = \lambda_1 + \sum_{i=2}^n v_{1,i}$ from each other therefore yields $0 = \sum_{j=2}^n v_{j,1} - \sum_{i=2}^n v_{1,i}$. Since also $v_{i,1} = -v_{1,i}$ for each $i \in [\![n]\!]$ with $1 < i$ by (b), that is the same as saying $0 = 2\sum_{j=2}^n v_{j,1}$. And $\sum_{j=2}^n v_{j,1} = 0$ then implies $\lambda_2 = \lambda_1 + \sum_{j=2}^n v_{j,1} = \lambda_1$, which is all we needed to see. ☐

LEMMA 5.3. *For any $n \in \mathbb{N}$ and each statement A below the set $\{v \in M_n(\mathbb{C}) \wedge A(v)\}$ is a complex vector subspace of $M_n(\mathbb{C})$ and has the listed dimension.*

$A(v)$	$\dim_{\mathbb{C}}\{v \in M_n(\mathbb{C}) \wedge A(v)\}$
$\top$	n^2
$\exists\lambda \in \mathbb{C}$: $v - \lambda I$ *is small*	$(n-1)^2 + 1$
v *is small*	$(n-1)^2$
$\exists\lambda \in \mathbb{C}$: $v - \lambda I$ *is skew-symmetric*	$\frac{1}{2}n(n-1) + 1$
v *is skew-symmetric*	$\frac{1}{2}n(n-1)$
$\exists\lambda \in \mathbb{C}$: $v - \lambda I$ *is skew-symmetric and small*	$\frac{1}{2}(n-1)(n-2) + 1$
v *is skew-symmetric and small*	$\frac{1}{2}(n-1)(n-2)$
v *is diagonal*	n
$\exists\lambda \in \mathbb{C}$: $v = \lambda I$	1
$v = 0$	0

Proof. (a) It is well known that, if for any $\{k, \ell\} \subseteq [\![n]\!]$ the matrix $E^n_{\ell,k} \in M_n(\mathbb{C})$ has $\delta_{\ell,j}\delta_{k,i}$ as its (j,i)-entry for any $\{i,j\} \subseteq [\![n]\!]$, then the family $(E^n_{\ell,k})_{(\ell,k)\in[\![n]\!]^{\otimes 2}}$ is a $\mathbb{C}$-linear basis of $\{v \in M_n(\mathbb{C}) \wedge A(v)\} = M_n(\mathbb{C})$.

(b) Since A can be expressed by a homogenous system of linear equations by Lemma 5.2 (a) the set $\{v \in M_n(\mathbb{C}) \wedge A(v)\}$ is indeed a vector space. Hence, it suffices to show that a $\mathbb{C}$-linear isomorphism $\varphi_n\colon M_{n-1}(\mathbb{C}) \oplus \mathbb{C} \to \{v \in M_n(\mathbb{C}) \wedge A(v)\}$ is defined by the rule that $(u, \lambda) \mapsto v$, where for any $\{i, j\} \subseteq [\![n]\!]$,

$$v_{j,i} = \begin{cases} u_{j,i} + \lambda\delta_{j,i} & |\, j < n \wedge i < n \\ -\sum_{\ell=1}^{n-1} u_{\ell,i} & |\, j = n \wedge i < n \\ -\sum_{k=1}^{n-1} u_{j,k} & |\, j < n \wedge i = n \\ \sum_{k,\ell=1}^{n-1} u_{\ell,k} + \lambda & |\, j = n \wedge i = n \end{cases},$$

for any $u \in M_{n-1}(\mathbb{C})$ and $\lambda \in \mathbb{C}$. We begin by proving that φ_n is well-defined. For any $u \in M_{n-1}(\mathbb{C})$ and $\lambda \in \mathbb{C}$, if $\varphi_n(u, \lambda) = v$ and if $w = v - \lambda I$, then for any $i \in [\![n]\!]$ with $i < n$, by definition,

$$\textstyle\sum_{j=1}^{n} w_{j,i} = \sum_{j=1}^{n-1}(v_{j,i} - \lambda\delta_{j,i}) + v_{n,i} = \sum_{j=1}^{n-1} u_{j,i} + (-\sum_{\ell=1}^{n-1} u_{\ell,i}) = 0$$

and also,

$$\textstyle\sum_{j=1}^{n} w_{j,n} = \sum_{j=1}^{n-1}(v_{j,n} - \lambda\delta_{j,n}) + (v_{n,n} - \lambda) = \sum_{j=1}^{n-1}(-\sum_{k=1}^{n-1} u_{j,k}) + \sum_{k,\ell=1}^{n-1} u_{\ell,k} = 0,$$

as well as for any $j \in [\![n]\!]$ with $j < n$,

$$\textstyle\sum_{i=1}^{n} w_{j,i} = \sum_{i=1}^{n-1}(v_{j,i} - \lambda\delta_{j,i}) + v_{j,n} = \sum_{i=1}^{n-1} u_{j,i} + (-\sum_{k=1}^{n-1} u_{j,k}) = 0$$

and also,

$$\textstyle\sum_{i=1}^{n} w_{n,i} = \sum_{i=1}^{n-1}(v_{n,i} - \lambda\delta_{n,i}) + (v_{n,n} - \lambda) = \sum_{i=1}^{n-1}(-\sum_{\ell=1}^{n-1} u_{\ell,i}) + \sum_{k,\ell=1}^{n-1} u_{\ell,k} = 0,$$

proving that w is small, i.e., that $A(v)$ holds.

On the other hand, by Lemma 5.2 (a) a well-defined $\mathbb{C}$-linear map $\psi_n\colon \{v \in M_n(\mathbb{C}) \wedge A(v)\} \to M_{n-1}(\mathbb{C}) \oplus \mathbb{C}$ is given by the rule that for any $v \in M_n(\mathbb{C})$ satisfying $A(v)$, if $\lambda \in \mathbb{C}$ is such that $v - \lambda I$ is small, then $v \mapsto (u, \lambda)$, where for any

$\{k,\ell\} \subseteq [\![n-1]\!]$,

$$u_{\ell,k} = v_{\ell,k} - \lambda\delta_{\ell,k}.$$

It remains to show $\psi_n \circ \varphi_n = \mathrm{id}$ and $\varphi_n \circ \psi_n = \mathrm{id}$. And, indeed, for any $u \in M_{n-1}(\mathbb{C})$ and $\lambda \in \mathbb{C}$, if $v = \varphi_n(u,\lambda)$, then we have already seen that $w = v - \lambda I$ is small. For any $\{k,\ell\} \subseteq [\![n]\!]$, by definition, $w_{\ell,k} = v_{\ell,k} - \lambda\delta_{\ell,k} = (u_{\ell,k} + \lambda\delta_{\ell,k}) - \lambda\delta_{\ell,k} = u_{\ell,k}$, which proves $\varphi_n(v) = (u,\lambda)$ and thus $\psi_n \circ \varphi_n = \mathrm{id}$.

Conversely, for any $v \in M_n(\mathbb{C})$ such that $A(v)$ is satisfied, if $(u,\lambda) = \psi_n(v)$, then we already know $\lambda = \sum_{\ell=1}^{n} v_{\ell,i} = \sum_{k=1}^{n} v_{j,k}$ for any $\{k,\ell\} \subseteq [\![n]\!]$ by Lemma 5.2 (a). If $v' = \psi_n(u,\lambda)$, then for any $\{i,j\} \subseteq [\![n]\!]$ with $i < n$ and $j < n$ it hence follows by definition $v'_{j,i} = u_{j,i} + \lambda\delta_{j,i} = (v_{j,i} - \lambda\delta_{j,i}) + \lambda\delta_{j,i} = v_{j,i}$ as well as by $\lambda = \sum_{\ell=1}^{n} v_{\ell,i}$,

$$v'_{n,i} = -\textstyle\sum_{\ell=1}^{n-1} u_{\ell,i} = -\sum_{\ell=1}^{n-1}(v_{\ell,i} - \lambda\delta_{\ell,i}) = \lambda - \sum_{\ell=1}^{n-1} v_{\ell,i} = v_{n,i}$$

and by $\lambda = \sum_{k=1}^{n} v_{k,j}$,

$$v'_{j,n} = -\textstyle\sum_{k=1}^{n-1} u_{j,k} = -\sum_{k=1}^{n-1}(v_{j,k} - \lambda\delta_{j,k}) = \lambda - \sum_{k=1}^{n-1} v_{j,k} = v_{j,n}$$

and, lastly,

$$\begin{aligned} v'_{n,n} &= \textstyle\sum_{k,\ell=1}^{n-1} u_{\ell,k} + \lambda = \sum_{k,\ell=1}^{n-1}(v_{\ell,k} - \lambda\delta_{\ell,k}) + \lambda = \sum_{\ell=1}^{n-1}\left(\sum_{k=1}^{n-1} v_{\ell,k} - \lambda\right) + \lambda \\ &= \textstyle\sum_{\ell=1}^{n-1}(-v_{\ell,n}) + \lambda = v_{n,n}, \end{aligned}$$

where we have used $\lambda = \sum_{k=1}^{n} v_{\ell,k}$ for any $\ell \in [\![n]\!]$ in the next-to-last step and $\lambda = \sum_{\ell=1}^{n} v_{\ell,n}$ in the last. Thus, we have shown $v' = u$ and thus $\varphi_n \circ \psi_n = \mathrm{id}$, which concludes the proof in this case.

(c) By by Lemma 5.2 (a) the space $\{v \in M_n(\mathbb{C}) \wedge A(v)\}$ is exactly the image of $M_{n-1}(\mathbb{C}) \oplus \{0\}$ under φ_n.

(d) Lemma 5.2 (b) showed that A can be equivalently expressed as a system of homogenous linear equations, thus proving $\{v \in M_n(\mathbb{C}) \wedge A(v)\}$ to be a vector space. Let $\Gamma_n = \{(j,i) \,|\, \{i,j\} \subseteq [\![n]\!] \wedge j < i\} \cup \{\varnothing\}$ as well as $B^n_{(j,i)} = T^n_{j,i} = E^n_{j,i} - E^n_{i,j}$ for any $\{i,j\} \subseteq [\![n]\!]$ with $j < i$ and $B^n_\varnothing = \mathrm{id}$. Then, the claim will be verified once we show that $(B^n_\gamma)_{\gamma\in\Gamma_n}$ is a $\mathbb{C}$-linear basis of $\{v \in M_n(\mathbb{C}) \wedge A(v)\}$.

The family $(B^n_\gamma)_{\gamma\in\Gamma_n}$ is $\mathbb{C}$-linearly independent. Indeed, if $(a_\gamma)_{\gamma\in\Gamma_n} \in \mathbb{C}^{\times\Gamma_n}$ is such that $\sum_{\gamma\in\Gamma_n} a_\gamma B^n_\gamma = 0$, then by $I = \sum_{i=1}^{n} E^n_{i,i}$,

$$0 = \sum_{\substack{(j,i)\in[\![n]\!]^{\otimes 2} \\ \wedge j<i}} a_{(j,i)} \left(E^n_{j,i} - E^n_{i,j}\right) + a_\varnothing \sum_{i=1}^{n} E^n_{i,i} = \sum_{(j,i)\in[\![n]\!]^{\otimes 2}} \begin{Bmatrix} a_{(j,i)} & | \, j < i \\ -a_{(i,j)} & | \, i < j \\ a_\varnothing & | \, j = i \end{Bmatrix} E^n_{j,i},$$

which demands $(a_\gamma)_{\gamma\in\Gamma_n} = 0$ since $(E^n_{\ell,k})_{(\ell,k)\in[\![n]\!]^{\otimes 2}}$ is $\mathbb{C}$-linearly independent.

It remains to prove that $\{B^n_\gamma \,|\, \gamma \in \Gamma_n\}$ spans $\{v \in M_n(\mathbb{C}) \wedge A(v)\}$. If $v \in M_n(\mathbb{C})$ and $\lambda \in \mathbb{C}$ are such that $w = v - \lambda I$ is skew-symmetric, then $v_{j,i} = -v_{i,j}$ and $\lambda = v_{j,j} = v_{i,i}$ for any $\{i,j\} \subseteq [\![n]\!]$ with $j \neq i$ by Lemma 5.2 (b). Hence, if we let $a_\varnothing = \lambda$ and

$a_{(j,i)} = w_{j,i} = v_{j,i}$ for any $\{i,j\} \subseteq [\![n]\!]$ with $j < i$, then

$$\sum_{\gamma\in\Gamma_n} a_\gamma\, B^n_\gamma = \sum_{(j,i)\in[\![n]\!]^{\otimes 2}} \begin{Bmatrix} a_{(j,i)} & | \, j<i \\ -a_{(i,j)} & | \, i<j \\ a_\varnothing & | \, j=i \end{Bmatrix} E^n_{j,i} = \sum_{(j,i)\in[\![n]\!]^{\otimes 2}} \begin{Bmatrix} v_{j,i} & | \, j<i \\ -v_{i,j} & | \, i<j \\ \lambda & | \, j=i \end{Bmatrix} E^n_{j,i} = v.$$

Thus, $(B^n_\gamma)_{\gamma\in\Gamma_n}$ is a $\mathbb{C}$-linear basis.

(e) The proof of the previous claim shows that any $v \in M_n(\mathbb{C})$ is skew-symmetric if and only if it is in the span of $\{B^n_\gamma \,|\, \gamma \in \Gamma_n\}$ and has coefficient 0 with respect to $B^n_\varnothing$. Hence, $\{T^n_{j,i} \,|\, \{i,j\} \subseteq [\![n]\!] \wedge j < i\}$ is a $\mathbb{C}$-linear basis of $\{v \in M_n(\mathbb{C}) \wedge A(v)\}$.

(f) All three parts (a)–(c) of Lemma 5.2 combined imply that $\{v \in M_n(\mathbb{C}) \wedge A(v)\}$ is the solution set to a homogenous system of linear equations and thus a vector space. Hence, it suffices to prove that φ_n restricts to a mapping $\{u \in M_{n-1}(\mathbb{C}) \wedge u = -u^{\mathrm{t}}\} \to \{v \in M_n(\mathbb{C}) \wedge A(v)\}$ and ψ_n to one in the reverse direction.

For any skew-symmetric $u \in M_{n-1}(\mathbb{C})$ and any $\lambda \in \mathbb{C}$, if $v = \varphi_n(u,\lambda)$ and $w = v - \lambda I$, then for any $\{i,j\} \subseteq [\![n]\!]$ with $i < n$ and $j < n$ we have already seen that $w_{j,i} = u_{j,i}$, implying $w_{j,i} + w_{i,j} = u_{j,i} + u_{i,j} = 0$ by $u = -u^{\mathrm{t}}$. Moreover, for the same reason,

$$\begin{aligned} w_{n,i} + w_{i,n} &= (v_{n,i} - \lambda\delta_{n,i}) + (v_{i,n} - \lambda\delta_{i,n}) = v_{n,i} + v_{i,n} \\ &= (-\textstyle\sum_{\ell=1}^{n-1} u_{\ell,i}) + (-\sum_{k=1}^{n-1} u_{i,k}) = -\sum_{\ell=1}^{n-1}(u_{\ell,i} + u_{i,\ell}) = 0 \end{aligned}$$

and

$$\begin{aligned} w_{j,n} + w_{n,j} &= (v_{j,n} - \lambda\delta_{j,n}) + (v_{n,j} - \lambda\delta_{n,j}) = v_{j,n} + v_{n,j} \\ &= (-\textstyle\sum_{k=1}^{n-1} u_{j,k}) + (-\sum_{\ell=1}^{n-1} u_{\ell,j}) = -\sum_{k=1}^{n-1}(u_{j,k} + u_{k,j}) = 0 \end{aligned}$$

as well as

$$w_{n,n} + w_{n,n} = 2(v_{n,n} - \lambda) = 2\textstyle\sum_{k,\ell=1}^{n-1} u_{\ell,k} = \sum_{k,\ell=1}^{n-1}(u_{\ell,k} + u_{k,\ell}) = 0,$$

which completes the proof that φ_n restricts to a map into $\{v \in M_n(\mathbb{C}) \wedge A(v)\}$.

Conversely, if $v \in M_n(\mathbb{C})$ and $\lambda \in \mathbb{C}$ are such that $w = v - \lambda I$ is skew-symmetric and small, then $\lambda = v_{i,i}$ for any $i \in [\![n]\!]$ by Lemma 5.2 (b). For $(u,\lambda) = \psi_n(v)$ and any $\{k,\ell\} \subseteq [\![n-1]\!]$, by definition, $u_{\ell,k} = w_{\ell,k}$ and thus $u_{\ell,k} + u_{k,\ell} = w_{\ell,k} + w_{k,\ell} = 0$ by $w = -w^{\mathrm{t}}$. Hence, ψ_n maps $\{v \in M_n(\mathbb{C}) \wedge A(v)\}$ into $\{u \in M_{n-1}(\mathbb{C}) \wedge u = -u^{\mathrm{t}}\} \oplus \mathbb{C}$.

(g) As we have just shown, any $v \in M_{\mathbb{C}}(n)$ is skew-symmetric and small if and only if it lies in the image of $\{u \in M_{n-1}(\mathbb{C}) \wedge u = -u^{\mathrm{t}}\} \oplus \{0\}$ under φ_n. And φ_n is a $\mathbb{C}$-linear isomorphism from this space to $\{v \in M_n(\mathbb{C}) \wedge A(v)\}$.

(h) It is well-known that $(E^n_{i,i})_{i\in[\![n]\!]}$ is a $\mathbb{C}$-linear basis of $\{v \in M_n(\mathbb{C}) \wedge A(v)\}$.

(i) In this case, $\{v \in M_n(\mathbb{C}) \wedge A(v)\}$ is the $\mathbb{C}$-linear span of I in $M_n(\mathbb{C})$.

(j) Here, $\{v \in M_n(\mathbb{C}) \wedge A(v)\}$ is the zero $\mathbb{C}$-linear space. □

6. First cohomology of unitary easy quantum group duals

This section computes the first quantum group cohomology with trivial coefficients as defined in Section 2 of the discrete duals of all unitary easy compact quantum groups, which were defined in Section 3. That is achieved by applying the method of Section 4 while using the results of Section 5 as auxiliaries.

More precisely, because unitary easy quantum groups can be expressed in terms of universal algebras Proposition 4.11 gives a way of expressing the first cohomology as the set of solutions to a linear equation for scalar vectors with as many components as there are generators, modulo the set of solutions to a second set of such equations. However, the latter will turn out to be trivial. Hence, only the first set of solutions will have to be solved. Doing so, first by successively simplifying the equations and then by distinuishing cases, is the main contents of this section.

6.1. Equational characterization of first cohomology. In addition to resuming the Assumptions 3.8 and the abbreviations from Notation 3.9 the following shorthand will be used.

NOTATION 6.1. (a) Let Y be the $\mathbb{C}\langle E\rangle$-bimodule $\mathbb{C}$ with left and right actions given by $u^{\mathfrak{c}}_{j,i}\otimes x\mapsto \delta_{j,i}x$ respectively $x\otimes u^{\mathfrak{c}}_{j,i}\mapsto \delta_{j,i}x$ for any $\{i,j\}\subseteq [\![n]\!]$, any $\mathfrak{c}\in\{\circ,\bullet\}$ and any $x\in\mathbb{C}$.

(b) For any $v\in M_n(\mathbb{C})$ let the vector $x^v\in\mathbb{C}^{\times E}$ be such that for any $\{i,j\}\subseteq[\![n]\!]$,

$$x^v_{u^\circ_{j,i}} \coloneqq v_{j,i} \quad\text{and}\quad x^v_{u^\bullet_{j,i}} \coloneqq -v_{i,j}.$$

Not only will the first step in determining the first cohomology of the duals of all unitary easy quantum groups apply Proposition 4.11 to Definition 3.11, it will also make an immediate first simplification. More precisely, the implications of the relations coming from $\sqcap_{\circ\bullet}$ and $\sqcap_{\bullet\circ}$ will be acounted for.

LEMMA 6.2. *For any $g\in[\![n]\!]^{\otimes 2}$ and $j\in[\![n]\!]^{\otimes 2}$ and any $x\in\mathbb{C}^{\times E}$, if r is given by*

(a) $\mathrm{r}^{\varnothing}_{\bullet\circ}(\sqcap)_{j,\varnothing}$, then $F^{1,r}_{E,\varnothing,Y}(x) = x_{u^\bullet_{j_1,j_2}} + x_{u^\circ_{j_2,j_1}}$

(b) $\mathrm{r}^{\varnothing}_{\circ\bullet}(\sqcap)_{j,\varnothing}$, then $F^{1,r}_{E,\varnothing,Y}(x) = x_{u^\circ_{j_1,j_2}} + x_{u^\bullet_{j_2,j_1}}$

(c) $\mathrm{r}^{\bullet\circ}_{\varnothing}(\sqcup)_{\varnothing,g}$, then $F^{1,r}_{E,\varnothing,Y}(x) = -x_{u^\bullet_{g_2,g_1}} - x_{u^\circ_{g_1,g_2}}$

(d) $\mathrm{r}^{\circ\bullet}_{\varnothing}(\sqcup)_{\varnothing,g}$, then $F^{1,r}_{E,\varnothing,Y}(x) = -x_{u^\circ_{g_2,g_1}} - x_{u^\bullet_{g_1,g_2}}$

PROOF. Starting from Lemma 3.10 the claims all follow immediately from Definition 4.9. ☐

PROPOSITION 6.3. *For any category $\mathcal{C}$ of two-colored partitions, any generator set $\mathcal{G}$ of $\mathcal{C}$, if G is the unitary easy compact quantum group associated with $(\mathcal{C},n)$, if*

$$\mathcal{R}=\mathcal{G}\cup\left\{(\overline{\mathfrak{c}},\overline{\mathfrak{d}},(\overline{p})^*)\,|\,(\mathfrak{c},\mathfrak{d},p)\in\mathcal{G}\right\},$$

and if

$$R=\left\{\mathrm{r}^{\mathfrak{c}}_{\mathfrak{d}}(p)_{j,g}|\ \{k,\ell\}\subseteq\mathbb{N}_0\wedge\mathfrak{c}\in\{\circ,\bullet\}^{\otimes k}\wedge\mathfrak{d}\in\{\circ,\bullet\}^{\otimes\ell}\wedge(\mathfrak{c},\mathfrak{d},p)\in\mathcal{R}\right.$$
$$\left.\wedge\ g\in[\![n]\!]^{\otimes k}\wedge j\in[\![n]\!]^{\otimes\ell}\right\},$$

then there exists an isomorphism of $\mathbb{C}$-vector spaces

$$H^1(\widehat{G}) \hookrightarrow\!\!\!\!\to \{v\in M_n(\mathbb{C})\,|\,\forall r\in R\colon F^{1,r}_{E,\varnothing,Y}(x^v)=0\},$$

where the set on the right is regarded as a $\mathbb{C}$*-vector subspace of* $M_n(\mathbb{C})$*, with*

$$\eta + B^1(\widehat{G}) \longmapsto (\eta(u^{\circ}_{j,i}))_{(j,i)\in[\![n]\!]^{\otimes 2}}.$$

for any $\eta \in Z^1(\widehat{G})$*.*

Proof. Let R' be the auxiliary relation set $R \cup \{\mathrm{r}^{\varnothing}_{\circ\bullet}(\sqcap)_{j,\varnothing}, \mathrm{r}^{\varnothing}_{\bullet\circ}(\sqcap)_{j,\varnothing} \,|\, j \in [\![n]\!]^{\otimes 2}\}$. The proof is organized into two parts.

Step 1: First, we show that there exists an isomorphism of $\mathbb{C}$-vector spaces from $H^1(\widehat{G})$ to $\{x \in \mathbb{C}^{\times E} \,|\, \forall r \in R'\colon F^{1,r}_{E,\varnothing,Y}(x) = 0\}$, the latter seen as a $\mathbb{C}$-vector subspace of $\mathbb{C}^{\times E}$, which for any $\eta \in Z^1(\widehat{G})$ maps $\eta + B^1(\widehat{G})$ to the E-indexed family of complex number with $u^{\circ}_{j,i} \mapsto \eta(u^{\circ}_{j,i})$ and $u^{\bullet}_{j,i} \mapsto \eta(u^{\bullet}_{j,i})$ for any $\{i,j\} \subseteq [\![n]\!]$.

By definition, $H^1(\widehat{G}) = H^1_{\mathrm{HS}}(\mathbb{C}[\widehat{G}], X)$, where X is the $\mathbb{C}$-vector space $\mathbb{C}$ equipped with the $\mathbb{C}[\widehat{G}]$-bimodule structure defined by $a \otimes \lambda \mapsto \varepsilon(a)\cdot\lambda$ and $\lambda \otimes a \mapsto \lambda\cdot\varepsilon(a)$ for any $\lambda \in \mathbb{C}$ and $a \in \mathbb{C}[\widehat{G}]$, where ε is the co-unit of $\mathbb{C}[\widehat{G}]$.

By Proposition 4.11, there exists a $\mathbb{C}$-linear isomorphism from $Z^1_{\mathrm{HS}}(\mathbb{C}\langle E \,|\, R'\rangle, X)$ to the quotient $\mathbb{C}$-vector space of $\{x \in \mathbb{C}^{\times E} \,|\, \forall r \in R'\colon F^{1,r}_{E,R',X}(x) = 0\}$ with respect to $\{((e+J) \triangleright \lambda - \lambda \triangleleft (e+J))_{e\in E} \,|\, \lambda \in \mathbb{C}\}$, where both are seen as subspace of $\mathbb{C}^{\times E}$, such that for any $\eta \in Z^1(\widehat{G})$ the class of η is sent to the class of $(\eta(e+J))_{e\in E}$.

But, of course, by definition of X, for any $\lambda \in \mathbb{C}$, any $\mathfrak{c} \in \{\circ,\bullet\}$ and any $\{j,i\} \subseteq [\![n]\!]$, the number $(u^{\mathfrak{c}}_{j,i} + J) \triangleright \lambda - \lambda \triangleleft (u^{\mathfrak{c}}_{j,i} + J) = \varepsilon(u^{\mathfrak{c}}_{j,i})\lambda - \lambda\varepsilon(u^{\mathfrak{c}}_{j,i}) = 0$ because $\mathbb{C}$ is commutative. Hence, the subspace $\{((e+J) \triangleright \lambda - \lambda \triangleleft (e+J))_{e\in E} \,|\, \lambda \in \mathbb{C}\}$ of $\mathbb{C}^{\times E}$ is actually the trivial one, $\{0\}$.

Lastly, because Y is precisely the restriction of scalars of X along the projection $\mathbb{C}\langle E\rangle \to \mathbb{C}\langle E \,|\, R'\rangle$, $p \mapsto p + J$, actually, $F^{1,p}_{E,R',X}(x) = F^{1,p}_{E,\varnothing,Y}(x)$ for any $p \in \mathbb{C}\langle E\rangle$ and $x \in \mathbb{C}^E$. Hence, we have verified what we wanted to show in this first part of the proof.

Step 2: It remains to prove that the rule $x \mapsto (x_{u^{\circ}_{j,i}})_{(j,i)\in[\![n]\!]^{\otimes 2}}$ gives a $\mathbb{C}$-linear isomorphism

$$\{x \in \mathbb{C}^{\times E} \,|\, \forall r \in R'\colon F^{1,r}_{E,\varnothing,Y}(x) = 0\} \lhook\joinrel\twoheadrightarrow \{v \in M_n(\mathbb{C}) \,|\, \forall r \in R\colon F^{1,r}_{E,\varnothing,Y}(x^v) = 0\}.$$

The claimed isomorphism is well-defined: Let any $x \in \mathbb{C}^{\times E}$ be such that $F^{1,r}_{E,\varnothing,Y}(x) = 0$ for any $r \in R'$. Then, for any $j \in [\![n]\!]^{\otimes 2}$ because $\mathrm{r}^{\varnothing}_{\circ\bullet}(\sqcap)_{j,\varnothing} \in R'$ in particular

$$x_{u^{\circ}_{j_1,j_2}} + x_{u^{\bullet}_{j_2,j_1}} = 0$$

by Lemma 6.2, i.e., $x_{u^{\bullet}_{j_2,j_1}} = -x_{u^{\circ}_{j_1,j_2}}$. Hence, if we let $v \equiv (x_{u^{\circ}_{j,i}})_{(j,i)\in[\![n]\!]^{\otimes 2}}$, then for any $\{i,j\} \subseteq [\![n]\!]$ by definition not only $x^v_{u^{\circ}_{j,i}} = v_{j,i} = x_{u^{\circ}_{j,i}}$ but also

$$x^v_{u^{\bullet}_{j,i}} = -v_{i,j} = -x_{u^{\circ}_{i,j}} = x_{u^{\bullet}_{j,i}},$$

which is to say $x^v = x$. Thus, per assumption, in particular $F^{1,r}_{E,\varnothing,Y}(x^v) = F^{1,r}_{E,\varnothing,Y}(x) = 0$ for any $r \in R$ since $R \subseteq R'$. That proves that the map is well-defined.

It is clear that the mapping is $\mathbb{C}$-linear. Moreover, it is injective because, if again $x \in \mathbb{C}^{\times E}$ is such that $F^{1,r}_{E,\varnothing,Y}(x) = 0$ for any $r \in R'$ and if again $v \equiv (x_{u^{\circ}_{j,i}})_{(j,i)\in [\![n]\!]^{\otimes 2}}$, then $v = 0$ necessitates $x^v = 0$ by definition of x^v and thus $x = 0$ by the identity $x^v = x$ established in the preceding paragraph.

To show surjectivity we let $v \in M_n(\mathbb{C})$ be arbitrary with $F^{1,r}_{E,\varnothing,Y}(x^v) = 0$ for any $r \in R$ and abbreviate $x \equiv x^v$. Then, of course, $F^{1,r}_{E,\varnothing,Y}(x) = 0$ for any $r \in R$ since $R \subseteq R'$. But this is also true for any $r \in R' \backslash R$. Namely, by Lemma 6.2, if $j \in [\![n]\!]^{\otimes 2}$ and $r = \mathrm{r}^{\varnothing}_{\circ\bullet}(\sqcap)_{j,\varnothing}$, then

$$F^{1,r}_{E,\varnothing,Y}(x) = x_{u^{\circ}_{j_1,j_2}} + x_{u^{\bullet}_{j_2,j_1}} = x^v_{u^{\circ}_{j_1,j_2}} + x^v_{u^{\bullet}_{j_2,j_1}} = v_{j_1,j_2} - v_{j_1,j_2} = 0$$

and, likewise, if $r = \mathrm{r}^{\varnothing}_{\bullet\circ}(\sqcap)_{j,\varnothing}$, then $F^{1,r}_{E,\varnothing,Y}(x) = 0$. Hence, x is a preimage of v. That is all we needed to prove. □

The task laid out by Proposition 6.3 is clear. We need to solve the set of linear equations in $M_n(\mathbb{C})$ on the right hand side of the isomorphism there – for each category of two-colored partitions.

6.2. Simplifying the equations. Eventually, in Section 6.3 namely, solving the equations of Proposition 6.3 will require case distinctions for different kinds of categories of two-colored partitions. However, there are a great number of simplification we can make to the equation system before it needs to come to that. Moreover, this reduces the number of cases we eventually have to consider immensely. That is what Section 6.2 is about.

6.2.1. *First round of simplifications.* The first simplification is achieved in Lemma 6.8 below. We will see that for $r = \mathrm{r}^{\mathfrak{c}}_{\mathfrak{d}}(p)_{j,g}$ for any $\{k,\ell\} \subseteq \mathbb{N}_0$, any $\mathfrak{c} \in \{\circ,\bullet\}^{\otimes k}$, any $\mathfrak{d} \in \{\circ,\bullet\}^{\otimes \ell}$, any two-colored partition $(\mathfrak{c},\mathfrak{d},p)$, any $g \in [\![n]\!]^{\otimes k}$ and any $j \in [\![n]\!]^{\otimes \ell}$ at most those summands in $F^{1,r}_{E,\varnothing,Y}$ possibly survive where the linear coefficient involving ζ inherited from r compares p to the kernel of a mapping $\Pi^k_\ell \to [\![n]\!]$ which differs from $g \,{}^{\bullet}\!{}_{\bullet}\, j$ on at most a single element of Π^k_ℓ. The next three lemmata explicate when such a comparison can possibly yield 1 and what that then implies about p, g and j.

Notation 6.4. For any set S, any $\{k,\ell\} \subseteq \mathbb{N}_0$, any mapping $f\colon \Pi^k_\ell \to S$, any $\mathtt{x} \in \Pi^k_\ell$ and any $s \in S$ write $f \downarrow_{\mathtt{x}} s$ for the mapping $\Pi^k_\ell \to S$ with $\mathtt{y} \mapsto f(\mathtt{y})$ for any $\mathtt{y} \in \Pi^k_\ell \backslash \{\mathtt{x}\}$ and with $\mathtt{x} \mapsto s$.

Lemma 6.5. *For any $n \in \mathbb{N}$, any $\{k,\ell\} \subseteq \mathbb{N}_0$, any mapping $f\colon \Pi^k_\ell \to [\![n]\!]$, any partition p of Π^k_ℓ, any $\mathtt{x} \in \Pi^k_\ell$ and any $s \in [\![n]\!]$, the statements $p \leq \ker(f \downarrow_{\mathtt{x}} s)$ and*

$$p \backslash \{\pi_p(\mathtt{x})\} \cup \{\pi_p(\mathtt{x}) \backslash \{\mathtt{x}\}, \{\mathtt{x}\}\} \backslash \{\varnothing\} \leq \ker(f) \wedge \pi_p(\mathtt{x}) \backslash \{\mathtt{x}\} \subseteq f^{\leftarrow}(\{s\}).$$

are equivalent.

PROOF. We show each implication separately. We will use many times the fact that for any $r \in [\![n]\!]$,

$$\begin{aligned}(f \downarrow_{\mathtt{x}} s)^{\leftarrow}(\{r\})\backslash\{\mathtt{x}\} &= \{\mathtt{a} \in \Pi_\ell^k \wedge (f \downarrow_{\mathtt{x}} s)(\mathtt{a}) = r \wedge \mathtt{a} \neq \mathtt{x}\}\\ &= \{\mathtt{a} \in \Pi_\ell^k \wedge f(\mathtt{a}) = r \wedge \mathtt{a} \neq \mathtt{x}\}\\ &= f^{\leftarrow}(\{r\})\backslash\{\mathtt{x}\}.\end{aligned}$$

Step 1: First, suppose $p \leq \ker(f \downarrow_{\mathtt{x}} s)$. Then, there exists $r \in \operatorname{ran}(f \downarrow_{\mathtt{x}} s)$ such that $\pi_p(\mathtt{x}) \subseteq (f \downarrow_{\mathtt{x}} s)^{\leftarrow}(\{r\})$. Because $\mathtt{x} \in \pi_p(\mathtt{x})$ this requires $\mathtt{x} \in (f \downarrow_{\mathtt{x}} s)^{\leftarrow}(\{r\})$ and thus $r = s$ by $(f \downarrow_{\mathtt{x}} s)(\mathtt{x}) = s$. It follows $\pi_p(\mathtt{x}) \subseteq (f \downarrow_{\mathtt{x}} s)^{\leftarrow}(\{s\})$ and thus in particular $\pi_p(\mathtt{x})\backslash\{\mathtt{x}\} \subseteq f^{\leftarrow}(\{s\})\backslash\{\mathtt{x}\} = f^{\leftarrow}(\{s\})\backslash\{\mathtt{x}\} \subseteq f^{\leftarrow}(\{s\})$, which is one half of what we had to show.

It is trivially true that $\{\mathtt{x}\} \subseteq f^{\leftarrow}(\{f(\mathtt{x})\}) \in \ker(f)$. We have already seen $\pi_p(\mathtt{x})\backslash\{\mathtt{x}\} \subseteq f^{\leftarrow}(\{s\}) \in \ker(f)$. For any $\mathsf{B} \in p$ with $\mathsf{B} \neq \pi_p(\mathtt{x})$, i.e., $\mathtt{x} \notin \mathsf{B}$, there exists by assumption $r' \in \operatorname{ran}(f \downarrow_{\mathtt{x}} s)$ with $\mathsf{B} \subseteq (f \downarrow_{\mathtt{x}} s)^{\leftarrow}(\{r'\})$. We conclude $\mathsf{B} = \mathsf{B}\backslash\{\mathtt{x}\} \subseteq (f \downarrow_{\mathtt{x}} s)^{\leftarrow}(\{r'\})\backslash\{\mathtt{x}\} = f^{\leftarrow}(\{r'\})\backslash\{\mathtt{x}\} \subseteq f^{\leftarrow}(\{r'\}) \in \ker(f)$. Thus, the other half of the claim, $p\backslash\{\pi_p(\mathtt{x})\} \cup \{\pi_p(\mathtt{x})\backslash\{\mathtt{x}\}, \{\mathtt{x}\}\}\backslash\{\varnothing\} \leq \ker(f)$ holds as well. That proves one implication.

Step 2: In order to show the converse implication we assume that both $p\backslash\{\pi_p(\mathtt{x})\} \cup \{\pi_p(\mathtt{x})\backslash\{\mathtt{x}\}, \{\mathtt{x}\}\}\backslash\{\varnothing\} \leq \ker(f)$ and $\pi_p(\mathtt{x})\backslash\{\mathtt{x}\} \subseteq f^{\leftarrow}(\{s\})$ and then we distinguish two cases.

Case 2.1: If $\{\mathtt{x}\} \in p$ and thus $\pi_p(\mathtt{x}) = \{\mathtt{x}\}$ and $\pi_p(\mathtt{x})\backslash\{\mathtt{x}\} = \varnothing$, then the assumption is simply equivalent to the statement $p \leq \ker(f)$. Naturally, $\{\mathtt{x}\} \subseteq (f \downarrow_{\mathtt{x}} s)^{\leftarrow}(\{s\}) \in \ker(f)$ by $(f \downarrow_{\mathtt{x}} s)(\mathtt{x}) = s$. For any $\mathsf{B} \in p$ with $\mathsf{B} \neq \{\mathtt{x}\}$ there exists by our premise a value $r \in \operatorname{ran}(f)$ with $\mathsf{B} \subseteq f^{\leftarrow}(\{r\})$. Thus, also $\mathsf{B} = \mathsf{B}\backslash\{\mathtt{x}\} \subseteq f^{\leftarrow}(\{r\})\backslash\{\mathtt{x}\} = (f \downarrow_{\mathtt{x}} s)^{\leftarrow}(\{r\})\backslash\{\mathtt{x}\} \in \ker(f \downarrow_{\mathtt{x}} s)$. In conclusion, $p \leq \ker(f \downarrow_{\mathtt{x}} s)$.

Case 2.2: In the instance that $\{\mathtt{x}\} \notin p$ the initial assumption simplifies to the statement $p\backslash\{\pi_p(\mathtt{x})\} \cup \{\pi_p(\mathtt{x})\backslash\{\mathtt{x}\}, \{\mathtt{x}\}\} \leq \ker(f)$ and $\pi_p(\mathtt{x})\backslash\{\mathtt{x}\} \subseteq f^{\leftarrow}(\{s\})$. The latter condition implies $\pi_p(\mathtt{x})\backslash\{\mathtt{x}\} \subseteq f^{\leftarrow}(\{s\})\backslash\{\mathtt{x}\} = (f \downarrow_{\mathtt{x}} s)^{\leftarrow}(\{s\})\backslash\{\mathtt{x}\} \subseteq (f \downarrow_{\mathtt{x}} s)^{\leftarrow}(\{s\})$ and thus by $(f \downarrow_{\mathtt{x}} s)(\mathtt{x}) = s$ also $\pi_p(\mathtt{x}) = \pi_p(\mathtt{x})\backslash\{\mathtt{x}\} \cup \{\mathtt{x}\} \subseteq (f \downarrow_{\mathtt{x}} s)^{\leftarrow}(\{s\}) \cup \{\mathtt{x}\} \subseteq (f \downarrow_{\mathtt{x}} s)^{\leftarrow}(\{s\}) \in \ker(f \downarrow_{\mathtt{x}} s)$. On the other hand, for any $\mathsf{B} \in p$ with $\mathsf{B} \neq \pi_p(\mathtt{x})$, which is to say $\mathtt{x} \notin \mathsf{B}$, there exists by assumption $r \in \operatorname{ran}(f)$ with $\mathsf{B} \subseteq f^{\leftarrow}(\{r\})$. It follows $\mathsf{B} = \mathsf{B}\backslash\{\mathtt{x}\} \subseteq f^{\leftarrow}(\{r\})\backslash\{\mathtt{x}\} = (f \downarrow_{\mathtt{x}} s)^{\leftarrow}(\{r\})\backslash\{\mathtt{x}\} \subseteq (f \downarrow_{\mathtt{x}} s)^{\leftarrow}(\{r\}) \in \ker(f \downarrow_{\mathtt{x}} s)$. Hence, altogether, $p \leq \ker(f \downarrow_{\mathtt{x}} s)$, which concludes the proof. □

LEMMA 6.6. *For any $n \in \mathbb{N}$, any $\{k, \ell\} \subseteq \mathbb{N}_0$, any partition p of Π_ℓ^k and any mapping $f\colon \Pi_\ell^k \to [\![n]\!]$ there exist $\mathtt{x} \in \Pi_\ell^k$ and $s \in [\![n]\!]$ such that $p \leq \ker(f \downarrow_{\mathtt{x}} s)$ if and only if one of the following mutually exclusive statements is true.*

(i) $p \neq \varnothing$ and $p \leq \ker(f)$.

(ii) There exists a (necessarily unique) $\{\mathtt{x}_1, \mathtt{x}_2\} \in p$ such that for any $\mathsf{A} \in p$ with $\mathsf{A} \neq \{\mathtt{x}_1, \mathtt{x}_2\}$ there exists $\mathsf{B} \in \ker(f)$ with $\mathsf{A} \subseteq \mathsf{B}$, and such that $f(\mathtt{x}_1) \neq f(\mathtt{x}_2)$.

(iii) There exist (necessarily unique) $\mathsf{X} \in p$ and $\mathtt{x} \in \mathsf{X}$ and $s \in [\![n]\!]$ such that $3 \leq |\mathsf{X}|$, such that for any $\mathsf{A} \in p$ with $\mathsf{A} \neq \mathsf{X}$ there exists $\mathsf{B} \in \ker(f)$ with $\mathsf{A} \subseteq \mathsf{B}$, such that $s \neq f(\mathtt{x})$ and such that $f(\mathtt{y}) = s$ for any $\mathtt{y} \in \mathsf{X}$ with $\mathtt{y} \neq \mathtt{x}$.

Proof. *Step 1: Equivalence.* Ignoring, for now, the claim that (i)– (iii) are mutually exclusive and the uniqness assertions in (ii) and (iii), we prove that there exist $\mathtt{x} \in \Pi_\ell^k$ and $s \in [\![n]\!]$ such that $p \leq \ker(f \downarrow_{\mathtt{x}} s)$ if and only if one of (i)– (iii) holds. Each implication is shown individually.

Step 1.1: First, we suppose that at least one of the three statements (i)– (iii) is true and deduce the existence of $\mathtt{x} \in \Pi_\ell^k$ and $s \in [\![n]\!]$ with $p \leq \ker(f \downarrow_{\mathtt{x}} s)$. By Lemma 6.5 that is the same as finding $\mathtt{x} \in \Pi_\ell^k$ and $s \in [\![n]\!]$ such that $p\backslash\{\pi_p(\mathtt{x})\} \cup \{\pi_p(\mathtt{x})\backslash\{\mathtt{x}\}, \{\mathtt{x}\}\}\backslash\{\varnothing\} \leq \ker(f)$ and $\pi_p(\mathtt{x})\backslash\{\mathtt{x}\} \subseteq f^{\leftarrow}(\{s\})$. The three cases need to be treated individually.

Case 1.1.1: Suppose $p \neq \varnothing$ and $p \leq \ker(f)$. Then, by $p \neq \varnothing$, we can find and fix some $\mathtt{x} \in \Pi_\ell^k$ and put $s := f(\mathtt{x})$. From $p \leq \ker(f)$ it then follows $\pi_p(\mathtt{x}) \subseteq f^{\leftarrow}(\{s\})$ and thus in particular $\pi_p(\mathtt{x})\backslash\{\mathtt{x}\} \subseteq f^{\leftarrow}(\{s\})$, which is one part of what we have to show. The other part, $p\backslash\{\pi_p(\mathtt{x})\} \cup \{\pi_p(\mathtt{x})\backslash\{\mathtt{x}\}, \{\mathtt{x}\}\}\backslash\{\varnothing\} \leq \ker(f)$ is a consequence of the fact $p\backslash\{\pi_p(\mathtt{x})\} \cup \{\pi_p(\mathtt{x})\backslash\{\mathtt{x}\}, \{\mathtt{x}\}\}\backslash\{\varnothing\} \leq p$ and the assumption $p \leq \ker(f)$.

Case 1.1.2: Next, let there exist $\{\mathtt{x}_1, \mathtt{x}_2\}$ such that for any $\mathsf{A} \in p$ with $\mathsf{A} \neq \{\mathtt{x}_1, \mathtt{x}_2\}$ there exists $\mathsf{B} \in \ker(f)$ with $\mathsf{A} \subseteq \mathsf{B}$, and such that $f(\mathtt{x}_1) \neq f(\mathtt{x}_2)$. If we define $\mathtt{x} := \mathtt{x}_1$ and $s := f(\mathtt{x}_2)$, then $\pi_p(\mathtt{x}) = \{\mathtt{x}_1, \mathtt{x}_2\}$ and thus $\pi_p(\mathtt{x})\backslash\{\mathtt{x}\} = \{\mathtt{x}_2\} \subseteq f^{\leftarrow}(\{s\})$. On the other hand, $p\backslash\{\pi_p(\mathtt{x})\} \cup \{\pi_p(\mathtt{x})\backslash\{\mathtt{x}\}, \{\mathtt{x}\}\}\backslash\{\varnothing\} = p\backslash\{\{\mathtt{x}_1, \mathtt{x}_2\}\} \cup \{\{\mathtt{x}_1\}, \{\mathtt{x}_2\}\} \leq \ker(f)$ because, by assumption, for each $\mathsf{A} \in p\backslash\{\{\mathtt{x}_1, \mathtt{x}_2\}\}$ there exists $\mathsf{B} \in \ker(f)$ with $\mathsf{A} \subseteq \mathsf{B} \in \ker(f)$ and, of course, $\{\mathtt{x}_1\} \subseteq f^{\leftarrow}(\{f(\mathtt{x}_1)\}) \in \ker(f)$ and $\{\mathtt{x}_2\} \subseteq f^{\leftarrow}(\{s\})$.

Case 1.1.3: Finally, let $\mathsf{X} \in p$ and $\mathtt{x} \in \mathsf{X}$ and $s \in [\![n]\!]$ be such that $3 \leq |\mathsf{X}|$, such that for any $\mathsf{A} \in p$ with $\mathsf{A} \neq \mathsf{X}$ there exists $\mathsf{B} \in \ker(f)$ with $\mathsf{A} \subseteq \mathsf{B}$, such that $s \neq f(\mathtt{x})$ and such that $f(\mathtt{y}) = s$ for any $\mathtt{y} \in \mathsf{X}$ with $\mathtt{y} \neq \mathtt{x}$. Then, obviously, $\pi_p(\mathtt{x})\backslash\{\mathtt{x}\} = \mathsf{X}\backslash\{\mathtt{x}\} \subseteq f^{\leftarrow}(\{s\})$ by assumption. And, $p\backslash\{\pi_p(\mathtt{x})\} \cup \{\pi_p(\mathtt{x})\backslash\{\mathtt{x}\}, \{\mathtt{x}\}\}\backslash\{\varnothing\} = p\backslash\{\mathsf{X}\} \cup \{\mathsf{X}\backslash\{\mathtt{x}\}, \{\mathtt{x}\}\} \leq \ker(f)$ because, by assumption, for any $\mathsf{A} \in p\backslash\{\mathsf{X}\}$ there exists $\mathsf{B} \in \ker(f)$ with $\mathsf{A} \subseteq \mathsf{B}$ and because $\mathsf{X}\backslash\{\mathtt{x}\} \subseteq f^{\leftarrow}(\{s\}) \in \ker(f)$ and $\{\mathtt{x}\} \subseteq f^{\leftarrow}(\{f(\mathtt{x})\}) \in \ker(f)$.

Altogether, if (i), (ii) or (iii) hold, there are $\mathtt{x} \in \Pi_\ell^k$ and $s \in [\![n]\!]$ with $p \leq \ker(f \downarrow_{\mathtt{x}} s)$.

Step 1.2: In order to show the converse we assume that there exist $\mathtt{x} \in \Pi_\ell^k$ and $s \in [\![n]\!]$ such that $p\backslash\{\pi_p(\mathtt{x})\} \cup \{\pi_p(\mathtt{x})\backslash\{\mathtt{x}\}, \{\mathtt{x}\}\}\backslash\{\varnothing\} \leq \ker(f)$ and $\pi_p(\mathtt{x})\backslash\{\mathtt{x}\} \subseteq f^{\leftarrow}(\{s\})$ (which we can by Lemma 6.5) and derive one of (i)– (iii). Again, a case distinction is in order. Note that the existence of $\mathtt{x}$ requires $p \neq \varnothing$.

Case 1.2.1: First, let $f(\mathtt{x}) = s$. Then $\pi_p(\mathtt{x}) = \pi_p(\mathtt{x})\backslash\{\mathtt{x}\} \cup \{\mathtt{x}\} \subseteq f^{\leftarrow}(\{s\}) \in \ker(f)$ by $\pi_p(\mathtt{x})\backslash\{\mathtt{x}\} \subseteq f^{\leftarrow}(\{s\})$. Thus $p \leq \ker(f)$ by $p\backslash\{\pi_p(\mathtt{x})\} \cup \{\pi_p(\mathtt{x})\backslash\{\mathtt{x}\}, \{\mathtt{x}\}\}\backslash\{\varnothing\} \leq \ker(f)$. In other words, we have shown (i) in this case.

Case 1.2.2: Similarly, if $\pi_p(\mathtt{x}) = \{\mathtt{x}\}$, then $p = p\backslash\{\pi_p(\mathtt{x})\} \cup \{\pi_p(\mathtt{x})\backslash\{\mathtt{x}\}, \{\mathtt{x}\}\}\backslash\{\varnothing\} \leq \ker(f)$ and thus (i) holds.

Case 1.2.3: If $f(\mathtt{x}) \neq s$ and $|\pi_p(\mathtt{x})| = 2$, then we put $\mathtt{x}_1 := \mathtt{x}$ and we let $\mathtt{x}_2$ be the unique element of $\pi_p(\mathtt{x})\backslash\{\mathtt{x}\}$. It follows $\{\mathtt{x}_2\} = \pi_p(\mathtt{x})\backslash\{\mathtt{x}\} \subseteq f^{\leftarrow}(\{s\})$ and thus $f(\mathtt{x}_2) = s \neq f(\mathtt{x}) = f(\mathtt{x}_1)$ by our assumptions. And, the premise $p\backslash\{\{\mathtt{x}_1, \mathtt{x}_2\}\} \cup \{\{\mathtt{x}_1\}, \{\mathtt{x}_2\}\} = p\backslash\{\pi_p(\mathtt{x})\} \cup \{\pi_p(\mathtt{x})\backslash\{\mathtt{x}\}, \{\mathtt{x}\}\}\backslash\{\varnothing\} \leq \ker(f)$ means that for any $\mathsf{A} \in p$ with $\mathsf{A} \neq \{\mathtt{x}_1, \mathtt{x}_2\}$ there exists $\mathsf{B} \in \ker(f)$ with $\mathsf{A} \subseteq \mathsf{B}$. Hence, this case implies (ii).

Case 1.2.4: The last remaining possibility is that $f(\mathtt{x}) \neq s$ and $3 \leq |\pi_p(\mathtt{x})|$. Putting $\mathsf{X} \coloneqq \pi_p(\mathtt{x})$ implies $\mathsf{X}\backslash\{\mathtt{x}\} = \pi_p(\mathtt{x})\backslash\{\mathtt{x}\} \subseteq f^{\leftarrow}(\{s\})$ by assumption, which is to say $f(\mathtt{y}) = s \neq f(\mathtt{x})$ for any $\mathtt{y} \in \mathsf{X}$ with $\mathtt{y} \neq \mathtt{x}$. On the other hand, since $p\backslash\{\mathsf{X}\} \cup \{\mathsf{X}\backslash\{\mathtt{x}\}, \{\mathtt{x}\}\} = p\backslash\{\pi_p(\mathtt{x})\} \cup \{\pi_p(\mathtt{x})\backslash\{\mathtt{x}\}, \{\mathtt{x}\}\}\backslash\{\varnothing\} \leq \ker(f)$, for any $\mathsf{A} \in p$ with $\mathsf{A} \neq \mathsf{X}$ there exists $\mathsf{B} \in \ker(f)$ with $\mathsf{A} \subseteq \mathsf{B}$. In other words, (iii) holds.

That concludes the verification of the equivalence of one of (i)–(iii) being true and there existing $\mathtt{x} \in \Pi_\ell^k$ and $s \in [\![n]\!]$ with $p \leq \ker(f \downarrow_{\mathtt{x}} s)$.

Step 2: Uniqueness. Next, we need to prove the uniqueness assertions contained in (ii) and (iii).

Step 2.1: In case (ii), suppose $\{\mathtt{x}_1', \mathtt{x}_2'\} \in p$ also has the property that for any $\mathsf{A} \in p$ with $\mathsf{B} \neq \{\mathtt{x}_1', \mathtt{x}_2'\}$ there exists $\mathsf{B} \in \ker(f)$ with $\mathsf{A} \subseteq \mathsf{B}$ and that $f(\mathtt{x}_1') = f(\mathtt{x}_2')$. Then, if $\{\mathtt{x}_1, \mathtt{x}_2\} \neq \{\mathtt{x}_1', \mathtt{x}_2'\}$ were true, then by assumption on $\{\mathtt{x}_1, \mathtt{x}_2\}$ there would exist $\mathsf{B} \in \ker(f)$ with $\{\mathtt{x}_1', \mathtt{x}_2'\} \subseteq \mathsf{B}$, meaning $f(\mathtt{x}_1') = f(\mathtt{x}_2')$ in contradiction to our assumption. Hence, $\{\mathtt{x}_1, \mathtt{x}_2\} = \{\mathtt{x}_1', \mathtt{x}_2'\}$ must be true instead. Hence, $\{\mathtt{x}_1, \mathtt{x}_2\}$ is unique.

Step 2.2: Now, let (iii) hold and let $\mathsf{X}' \in p$ and $\mathtt{x}' \in \mathsf{X}'$ and $s' \in [\![n]\!]$ too be such that $3 \leq |\mathsf{X}'|$, such that for any $\mathsf{A} \in p$ with $\mathsf{A} \neq \mathsf{X}'$ there exists $\mathsf{B} \in \ker(f)$ with $\mathsf{A} \subseteq \mathsf{B}$, such that $s' \neq f(\mathtt{x}')$ and such that $f(\mathtt{y}') = s'$ for any $\mathtt{y}' \in \mathsf{X}'$ with $\mathtt{y}' \neq \mathtt{x}'$.

If $\mathsf{X} \neq \mathsf{X}'$ held, the assumption on X would imply the existence of $\mathsf{B} \in p$ with $\mathsf{X}' \subseteq \mathsf{B}$. In particular, it would follow $f(\mathtt{y}') = f(\mathtt{x}')$ for any $\mathtt{y}' \in \mathsf{X}'$ with $\mathtt{y}' \neq \mathtt{x}'$, of which there exists at least one by $3 \leq |\mathsf{X}'|$. Because that would contradict the assumption, we must have $\mathsf{X} = \mathsf{X}'$ instead.

Furthermore, supposing $\mathtt{x} \neq \mathtt{x}'$ demands of any $\mathtt{y} \in \mathsf{X}\backslash\{\mathtt{x}, \mathtt{x}'\}$ both $f(\mathtt{y}) = s$ by the assumption on $\mathtt{x}$ and s and $f(\mathtt{y}) = s'$ by the one on $\mathtt{x}'$ and s'. Hence, as $\mathsf{X}\backslash\{\mathtt{x}, \mathtt{x}'\} \neq \varnothing$ by $3 \leq |\mathsf{X}'|$, if $\mathtt{x}' \neq \mathtt{x}$, then $s = s'$. That would be a contradiction because the property of $\mathtt{x}'$ also requires $s \neq f(\mathtt{x}) = s'$ in that case. Hence, only $\mathtt{x} = \mathtt{x}'$ can be true.

Lastly, because the assumptions on s and s' imply $f(\mathtt{y}) = s$ respectively $f(\mathtt{y}) = s'$ for any $\mathtt{y} \in \mathsf{X}$ with $\mathtt{y} \neq \mathtt{x} = \mathtt{x}'$ and because $\mathsf{X}\backslash\{\mathtt{x}\} \neq \varnothing$, we must have $s = s'$ as well. Thus, the uniqueness claim of (iii) has been demonstrated.

Step 3: Mutual exclusivity. Finally, we show that the three statements (i)– (iii) are mutually exclusive. If (ii) holds, then (i) cannot be true because $f(\mathtt{x}_1) \neq f(\mathtt{x}_2)$ excludes the existence of $\mathsf{B} \in \ker(f)$ with $\{\mathtt{x}_1, \mathtt{x}_2\} \subseteq \mathsf{B}$, which would be necessary for $p \leq \ker(f)$ to hold. Similarly, (iii) forbids (i) because the existence of $\mathtt{y} \in \mathsf{X}\backslash\{\mathtt{x}\} \neq \varnothing$ with $f(\mathtt{x}) \neq s = f(\mathtt{y})$ does not allow for any $\mathsf{B} \in \ker(f)$ with $\mathsf{X} \subseteq \mathsf{B}$, which $p \leq \ker(f)$ would require. Lastly, if (i) and (iii) were both true, then $\{\mathtt{x}_1, \mathtt{x}_2\} \neq \mathsf{X}$ would follow from $3 \leq |\mathsf{X}|$, thus demanding by the property of X the existence of $\mathsf{B} \in \ker(f)$ with $\{\mathtt{x}_1, \mathtt{x}_2\} \subseteq \mathsf{B}$, in contradiction to $f(\mathtt{x}_1) \neq f(\mathtt{x}_2)$. In summary, we have shown that (i)– (iii) are mutually exclusive, which concludes the proof overall. □

LEMMA 6.7. *Let $n \in \mathbb{N}$, let $\{k, \ell\} \subseteq \mathbb{N}_0$, let p be any partition of Π_ℓ^k, let $f\colon \Pi_\ell^k \to [\![n]\!]$ be any mapping and let $\mathtt{z} \in \Pi_\ell^k$ and $t \in [\![n]\!]$ be arbitrary.*

(a) If $p \neq \varnothing$ and $p \leq \ker(f)$, then $p \leq \ker(f \downarrow_{\mathtt{z}} t)$ if and only if either $|\pi_p(\mathtt{z})| = 1$ or both $2 \leq |\pi_p(\mathtt{z})|$ and $t = f(\mathtt{z})$.

(b) *If there exists (a necessarily unique)* $\{\mathtt{x}_1, \mathtt{x}_2\} \in p$ *such that for any* $\mathsf{A} \in p$ *with* $\mathsf{A} \neq \{\mathtt{x}_1, \mathtt{x}_2\}$ *there exists* $\mathsf{B} \in \ker(f)$ *with* $\mathsf{A} \subseteq \mathsf{B}$*, and such that* $f(\mathtt{x}_1) \neq f(\mathtt{x}_2)$*, then* $p \leq \ker(f \downarrow_{\mathtt{z}} t)$ *if and only if either both* $\mathtt{z} = \mathtt{x}_1$ *and* $t = f(\mathtt{x}_2)$ *or both* $\mathtt{z} = \mathtt{x}_2$ *and* $t = f(\mathtt{x}_1)$*.*

(c) *If there exist (necessarily unique)* $\mathsf{X} \in p$ *and* $\mathtt{x} \in \mathsf{X}$ *and* $s \in [\![n]\!]$ *such that* $3 \leq |\mathsf{X}|$*, such that for any* $\mathsf{A} \in p$ *with* $\mathsf{A} \neq \mathsf{X}$ *there exists* $\mathsf{B} \in \ker(f)$ *with* $\mathsf{A} \subseteq \mathsf{B}$*, such that* $s \neq f(\mathtt{x})$ *and such that* $f(\mathtt{y}) = s$ *for any* $\mathtt{y} \in \mathsf{X}$ *with* $\mathtt{y} \neq \mathtt{x}$*, then* $p \leq \ker(f \downarrow_{\mathtt{z}} t)$ *if and only if* $\mathtt{z} = \mathtt{x}$ *and* $t = s$*.*

Proof. By Lemma 6.5 the statement $p \leq \ker(f \downarrow_{\mathtt{z}} t)$ is equivalent to the conjunction of $p\backslash\{\pi_p(\mathtt{z})\} \cup \{\pi_p(\mathtt{z})\backslash\{\mathtt{z}\}, \{\mathtt{z}\}\}\backslash\{\varnothing\} \leq \ker(f)$ and $\pi_p(\mathtt{z})\backslash\{\mathtt{z}\} \subseteq f^{\leftarrow}(\{t\})$.

(a) Because $p\backslash\{\pi_p(\mathtt{z})\} \cup \{\pi_p(\mathtt{z})\backslash\{\mathtt{z}\}, \{\mathtt{z}\}\}\backslash\{\varnothing\} \leq p$, in the situation of (a), where $p \leq \ker(f)$, we only need to determine when $\pi_p(\mathtt{z})\backslash\{\mathtt{z}\} \subseteq f^{\leftarrow}(\{t\})$. If $|\pi_p(\mathtt{z})| = 1$, i.e., $\pi_p(\mathtt{z})\backslash\{\mathtt{z}\} = \varnothing$, this condition is trivially satisfied. And if $2 \leq |\pi_p(\mathtt{z})|$, then $\pi_p(\mathtt{z})\backslash\{\mathtt{z}\} \subseteq f^{\leftarrow}(\{t\})$ holds if and only if $t = f(\mathtt{z})$ because $\pi_p(\mathtt{z})\backslash\{\mathtt{z}\} \subseteq \pi_p(\mathtt{z}) \subseteq f^{\leftarrow}(\{f(\mathtt{z})\})$ by assumption. That proves (a).

(b) In case (b), if $\mathtt{z} \notin \{\mathtt{x}_1, \mathtt{x}_2\}$, then $\{\mathtt{x}_1, \mathtt{x}_2\} \in p\backslash\{\pi_p(\mathtt{z})\} \cup \{\pi_p(\mathtt{z})\backslash\{\mathtt{z}\}, \{\mathtt{z}\}\}\backslash\{\varnothing\}$. However, because $f(\mathtt{x}_1) \neq f(\mathtt{x}_2)$ there cannot exist any $\mathsf{B} \in \ker(f)$ with $\{\mathtt{x}_1, \mathtt{x}_2\} \subseteq \mathsf{B}$. Hence, $\mathtt{z} \notin \{\mathtt{x}_1, \mathtt{x}_2\}$ excludes $\{\mathtt{x}_1, \mathtt{x}_2\} \in p\backslash\{\pi_p(\mathtt{z})\} \cup \{\pi_p(\mathtt{z})\backslash\{\mathtt{z}\}, \{\mathtt{z}\}\}\backslash\{\varnothing\} \leq \ker(f)$ and thus $p \leq \ker(f \downarrow_{\mathtt{z}} t)$.

Hence, $p \leq \ker(f \downarrow_{\mathtt{z}} t)$ requires the existence of $i \in [\![2]\!]$ with $\mathtt{z} = \mathtt{x}_i$. If so, then $p\backslash\{\pi_p(\mathtt{z})\} \cup \{\pi_p(\mathtt{z})\backslash\{\mathtt{z}\}, \{\mathtt{z}\}\}\backslash\{\varnothing\} = p\backslash\{\{\mathtt{x}_1, \mathtt{x}_2\}\} \cup \{\{\mathtt{x}_1\}, \{\mathtt{x}_2\}\} \leq \ker(f)$ since by assumption for any $\mathsf{A} \in p$ with $\mathsf{A} \notin \{\mathtt{x}_1, \mathtt{x}_2\}$ there exists $\mathsf{B} \in \ker(f)$ with $\mathsf{A} \subseteq \mathsf{B}$. Thus, in this case, $p \leq \ker(f \downarrow_{\mathtt{z}} t)$ is equivalent to $\{\mathtt{x}_{3-i}\} = \{\mathtt{x}_1, \mathtt{x}_2\}\backslash\{\mathtt{x}_i\} = \pi_p(\mathtt{z})\backslash\{\mathtt{z}\} \subseteq f^{\leftarrow}(\{t\})$, i.e., to $f(\mathtt{x}_{3-i}) = t$, just as (b) claimed.

(c) Finally, under the assumptions of (c), whenever $\mathtt{z} \notin \mathsf{X}$, then $\mathsf{X} \in p\backslash\{\pi_p(\mathtt{z})\} \cup \{\pi_p(\mathtt{z})\backslash\{\mathtt{z}\}, \{\mathtt{z}\}\}\backslash\{\varnothing\} \nleq \ker(f)$ by the existence of $\mathtt{y} \in \mathsf{X}\backslash\{\mathtt{x}\} \neq \varnothing$ with $f(\mathtt{x}) \neq s = f(\mathtt{y})$. Consequently, $p \nleq \ker(f \downarrow_{\mathtt{z}} t)$ if $\mathtt{z} \notin \mathsf{X}$.

For $\mathtt{z} \in \mathsf{X}$, because by assumption there is for any $\mathsf{A} \in p$ with $\mathsf{A} \neq \mathsf{X} = \pi_p(\mathtt{z})$ a $\mathsf{B} \in \ker(f)$ with $\mathsf{A} \subseteq \mathsf{B}$ the condition $p\backslash\{\pi_p(\mathtt{z})\} \cup \{\pi_p(\mathtt{z})\backslash\{\mathtt{z}\}, \{\mathtt{z}\}\}\backslash\{\varnothing\} \leq \ker(f)$ simplifies to the existence of $\mathsf{B} \in \ker(f)$ with $\pi_p(\mathtt{z})\backslash\{\mathtt{z}\} \subseteq \mathsf{B}$, which is subsumed by the second condition. In other words, if $\mathtt{z} \in \mathsf{X}$, then $p \leq \ker(f \downarrow_{\mathtt{z}} t)$ if and only if $\pi_p(\mathtt{z})\backslash\{\mathtt{z}\} \subseteq f^{\leftarrow}(\{t\})$.

If $\mathtt{z} \neq \mathtt{x}$, then $\pi_p(\mathtt{z})\backslash\{\mathtt{z}\} \nsubseteq f^{\leftarrow}(\{t\})$ because, by $3 \leq |\mathsf{X}|$, there exist $\mathtt{y} \in \mathsf{X}\backslash\{\mathtt{x}, \mathtt{z}\}$ with $f(\mathtt{y}) = s \neq f(\mathtt{x})$ by assumption. Hence, $p \leq \ker(f \downarrow_{\mathtt{z}} t)$ requires $\mathtt{z} = \mathtt{x}$. And in that case is is equivalent to $\mathsf{X}\backslash\{\mathtt{x}\} = \pi_p(\mathtt{z})\backslash\{\mathtt{z}\} \subseteq f^{\leftarrow}(\{t\})$, which is satisfied if and only if $t = s$ because $f(\mathtt{y}) = s$ for any $\mathtt{y} \in \mathsf{X}\backslash\{\mathtt{x}\} \neq \varnothing$. Thus, the assertion of (c) is true as well and, thus, so is the claim overall. □

In regard of the preceding three lemmata we can make the first round of simplifications to the equation system we need to solve.

LEMMA 6.8. *For any* $\{k,\ell\}\subseteq\mathbb{N}_0$, *any* $\mathfrak{c}\in\{\circ,\bullet\}^{\otimes k}$ *and* $\mathfrak{d}\in\{\circ,\bullet\}^{\otimes\ell}$, *any partition* p *of* Π^k_ℓ, *any* $g\in[\![n]\!]^{\otimes k}$ *and any* $j\in[\![n]\!]^{\otimes\ell}$, *if* $r\equiv \mathrm{r}^{\mathfrak{c}}_{\mathfrak{d}}(p)_{j,g}$ *and if we abbreviate* $f\equiv g\,\blacktriangledown\, j$ *and* $\mathfrak{w}\equiv\mathfrak{c}\,\blacktriangledown\,\mathfrak{d}$, *then for any* $x\in\mathbb{C}^{\times E}$ *the number* $F^{r,1}_{E,\varnothing,Y}(x)$ *is given by,*

(i) *if* $p\neq\varnothing$ *and* $p\leq\ker(f)$,

$$\sum_{\substack{\mathtt{x}\in\Pi^k_\ell\\ \wedge|\pi_p(\mathtt{x})|=1}}\sum_{s=1}^n\begin{Bmatrix}-x_{u^{\mathfrak{w}(\mathtt{x})}_{s,f(\mathtt{x})}} & \text{if } \mathtt{x}\in\Pi^k_0\\ x_{u^{\mathfrak{w}(\mathtt{x})}_{f(\mathtt{x}),s}} & \text{if } \mathtt{x}\in\Pi^0_\ell\end{Bmatrix}+\sum_{\substack{\mathtt{x}\in\Pi^k_\ell\\ \wedge 2\leq|\pi_p(\mathtt{x})|}}\begin{Bmatrix}-1 & \text{if } \mathtt{x}\in\Pi^k_0\\ 1 & \text{if } \mathtt{x}\in\Pi^0_\ell\end{Bmatrix}x_{u^{\mathfrak{w}(\mathtt{x})}_{f(\mathtt{x}),f(\mathtt{x})}},$$

(ii) *if (necessarily* $p\neq\varnothing$ *and* $p\nleq\ker(f)$ *and) there exists a (necessarily unique)* $\{\mathtt{x}_1,\mathtt{x}_2\}\in p$ *such that for any* $\mathsf{A}\in p$ *with* $\mathsf{A}\neq\{\mathtt{x}_1,\mathtt{x}_2\}$ *there exists* $\mathsf{B}\in\ker(f)$ *with* $\mathsf{A}\subseteq\mathsf{B}$, *and such that* $f(\mathtt{x}_1)\neq f(\mathtt{x}_2)$,

$$\begin{Bmatrix}-x_{u^{\mathfrak{w}(\mathtt{x}_1)}_{f(\mathtt{x}_2),f(\mathtt{x}_1)}} & \text{if } \mathtt{x}_1\in\Pi^k_0\\ x_{u^{\mathfrak{w}(\mathtt{x}_1)}_{f(\mathtt{x}_1),f(\mathtt{x}_2)}} & \text{if } \mathtt{x}_1\in\Pi^0_\ell\end{Bmatrix}+\begin{Bmatrix}-x_{u^{\mathfrak{w}(\mathtt{x}_2)}_{f(\mathtt{x}_1),f(\mathtt{x}_2)}} & \text{if } \mathtt{x}_2\in\Pi^k_0\\ x_{u^{\mathfrak{w}(\mathtt{x}_2)}_{f(\mathtt{x}_2),f(\mathtt{x}_1)}} & \text{if } \mathtt{x}_2\in\Pi^0_\ell\end{Bmatrix},$$

(iii) *if (necessarily* $p\neq\varnothing$ *and* $p\nleq\ker(f)$ *and) there exist (necessarily unique)* $\mathsf{X}\in p$ *and* $\mathtt{x}\in\mathsf{X}$ *and* $s\in[\![n]\!]$ *such that* $3\leq|\mathsf{X}|$, *such that for any* $\mathsf{A}\in p$ *with* $\mathsf{A}\neq\mathsf{X}$ *there exists* $\mathsf{B}\in\ker(f)$ *with* $\mathsf{A}\subseteq\mathsf{B}$, *such that* $f(\mathtt{x})\neq s$ *and such that* $f(\mathtt{y})=s$ *for any* $\mathtt{y}\in\mathsf{X}$ *with* $\mathtt{y}\neq\mathtt{x}$,

$$\begin{cases}-x_{u^{\mathfrak{w}(\mathtt{x})}_{s,f(\mathtt{x})}} & \text{if } \mathtt{x}\in\Pi^k_0\\ x_{u^{\mathfrak{w}(\mathtt{x})}_{f(\mathtt{x}),s}} & \text{if } \mathtt{x}\in\Pi^0_\ell,\end{cases}$$

(iv) *and* 0 *otherwise.*

PROOF. By definition of $F^{r,1}_{E,\varnothing,Y}$ and Y, for any $x\in\mathbb{C}^{\times E}$ (with the same abbreviations as in the claim),

$$F^{r,1}_{E,\varnothing,Y}(x)=\sum_{i\in[\![n]\!]^{\otimes\ell}}\zeta(p,\ker(g\,\blacktriangledown\, i))\sum_{b=1}^{\ell}\left(\prod_{\substack{q\in[\![\ell]\!]\\ \wedge q\neq b}}\delta_{j_b,i_b}\right)x_{u^{\mathfrak{d}_b}_{j_b,i_b}}$$
$$-\sum_{h\in[\![n]\!]^{\otimes k}}\zeta(p,\ker(h\,\blacktriangledown\, j))\sum_{a=1}^{k}\left(\prod_{\substack{q\in[\![k]\!]\\ \wedge q\neq a}}\delta_{h_a,g_a}\right)x_{u^{\mathfrak{c}_a}_{h_a,g_a}},$$

which after commuting the sums and evaluating the sums over i respectively h (as far as possible) is identical to

$$\sum_{b=1}^{\ell}\sum_{i_b=1}^{n}\zeta(p,\ker(g\,\blacktriangledown\,(j_1,\ldots,j_{b-1},i_b,j_{b+1},\ldots,j_\ell)))\,x_{u^{\mathfrak{d}_b}_{j_b,i_b}}$$
$$-\sum_{a=1}^{k}\sum_{h_a=1}^{n}\zeta(p,\ker((g_1,\ldots,g_{a-1},h_a,g_{a+1},\ldots,g_k)\,\blacktriangledown\, j))\,x_{u^{\mathfrak{c}_a}_{h_a,g_a}}.$$

In other words, for any $x \in \mathbb{C}^{\times E}$,

$$F^{r,1}_{E,\varnothing,Y}(x) = \sum_{\mathtt{z}\in\Pi^k_\ell} \sum_{t=1}^{n} \zeta(p, \ker(f \downarrow_{\mathtt{z}} t)) \begin{cases} -x_{u^{\mathfrak{w}(\mathtt{z})}_{t,f(\mathtt{z})}} & \text{if } \mathtt{z} \in \Pi^k_0 \\ x_{u^{\mathfrak{w}(\mathtt{z})}_{f(\mathtt{z}),t}} & \text{if } \mathtt{z} \in \Pi^0_\ell. \end{cases}$$

From this identity we see immediately that $F^{r,1}_{E,\varnothing,Y} \neq 0$ requires the existence of $\mathtt{x} \in \Pi^k_\ell$ and $s \in [\![n]\!]$ with $p \leq \ker(f \downarrow_{\mathtt{x}} s)$. Thus, Lemma 6.6 verifies (iv). It remains to treat the cases (i)–(iii).

(i) In the situation of (i), for any $\mathtt{z} \in \Pi^k_\ell$ and $t \in [\![n]\!]$ we know from Lemma 6.7 (a) that $p \leq \ker(f \downarrow_{\mathtt{z}} t)$ if and only if either $|\pi_p(\mathtt{z})| = 1$ or both $2 \leq |\pi_p(\mathtt{z})|$ and $t = f(\mathtt{z})$. Consequently, for any $x \in \mathbb{C}^{\times E}$,

$$F^{r,1}_{E,\varnothing,Y}(x) = \sum_{\substack{\mathtt{x}\in\Pi^k_\ell \\ \wedge |\pi_p(\mathtt{x})|=1}} \sum_{s=1}^{n} \begin{Bmatrix} -x_{u^{\mathfrak{w}(\mathtt{x})}_{s,f(\mathtt{x})}} & \text{if } \mathtt{x} \in \Pi^k_0 \\ x_{u^{\mathfrak{w}(\mathtt{x})}_{f(\mathtt{x}),s}} & \text{if } \mathtt{x} \in \Pi^0_\ell \end{Bmatrix} + \sum_{\substack{\mathtt{x}\in\Pi^k_\ell \\ \wedge 2\leq|\pi_p(\mathtt{x})|}} \begin{Bmatrix} -1 & \text{if } \mathtt{x} \in \Pi^k_0 \\ 1 & \text{if } \mathtt{x} \in \Pi^0_\ell \end{Bmatrix} x_{u^{\mathfrak{w}(\mathtt{x})}_{f(\mathtt{x}),f(\mathtt{x})}},$$

as claimed in (i).

(ii) Under the assumptions of (ii), Lemma 6.7 (b) tells us for any $\mathtt{z} \in \Pi^k_\ell$ and $t \in [\![n]\!]$ that $p \leq \ker(f \downarrow_{\mathtt{z}} t)$ if and only if either both $\mathtt{z} = \mathtt{x}_1$ and $t = f(\mathtt{x}_2)$ or both $\mathtt{z} = \mathtt{x}_2$ and $t = f(\mathtt{x}_1)$. It follows for any $x \in \mathbb{C}^{\times E}$,

$$F^{r,1}_{E,\varnothing,Y}(x) = \begin{Bmatrix} -x_{u^{\mathfrak{w}(\mathtt{x}_1)}_{f(\mathtt{x}_2),f(\mathtt{x}_1)}} & \text{if } \mathtt{x}_1 \in \Pi^k_0 \\ x_{u^{\mathfrak{w}(\mathtt{x}_1)}_{f(\mathtt{x}_1),f(\mathtt{x}_2)}} & \text{if } \mathtt{x}_1 \in \Pi^0_\ell \end{Bmatrix} + \begin{Bmatrix} -x_{u^{\mathfrak{w}(\mathtt{x}_2)}_{f(\mathtt{x}_1),f(\mathtt{x}_2)}} & \text{if } \mathtt{x}_2 \in \Pi^k_0 \\ x_{u^{\mathfrak{w}(\mathtt{x}_2)}_{f(\mathtt{x}_2),f(\mathtt{x}_1)}} & \text{if } \mathtt{x}_2 \in \Pi^0_\ell \end{Bmatrix},$$

which is what we had to prove in this case.

(iii) Finally, if the premises of (iii) are satisfied, then for any $\mathtt{z} \in \Pi^k_\ell$ and $t \in [\![n]\!]$ Lemma 6.7 (c) lets us infer that $p \leq \ker(f \downarrow_{\mathtt{z}} t)$ if and only if $\mathtt{z} = \mathtt{x}$ and $t = s$. Hence, in this case, for any $x \in \mathbb{C}^{\times E}$,

$$F^{r,1}_{E,\varnothing,Y}(x) = \begin{cases} -x_{u^{\mathfrak{w}(\mathtt{x})}_{s,f(\mathtt{x})}} & \text{if } \mathtt{x} \in \Pi^k_0 \\ x_{u^{\mathfrak{w}(\mathtt{x})}_{f(\mathtt{x}),s}} & \text{if } \mathtt{x} \in \Pi^0_\ell \end{cases},$$

which concludes the proof. □

6.2.2. *Second round of simplifications.* Whereas Lemma 6.8 still considered the numbers $F^{1,r}_{E,\varnothing,Y}(x)$ for arbitrary $x \in \mathbb{C}^{\times E}$, Lemma 6.9 now takes into account that, by Proposition 4.11, we only need to solve the equations $F^{1,r}_{E,\varnothing,Y}(x) = 0$ for those x for which there exists $v \in M_n(\mathbb{C})$ with $x = x^v$. That further simplifies the system of equations.

LEMMA 6.9. *For any $\{k, \ell\} \subseteq \mathbb{N}_0$, any $\mathfrak{c} \in \{\circ, \bullet\}^{\otimes k}$ and $\mathfrak{d} \in \{\circ, \bullet\}^{\otimes \ell}$, any partition p of Π^k_ℓ, any $g \in [\![n]\!]^{\otimes k}$, any $j \in [\![n]\!]^{\otimes \ell}$ and any $v \in M_n(\mathbb{C})$, if $r \equiv \mathrm{r}^{\mathfrak{c}}_{\mathfrak{d}}(p)_{j,g}$ and if $f \equiv g \,^\centerdot\!\!{}_\centerdot\, j$, then $F^{1,r}_{E,\varnothing,Y}(x^v)$ is given by,* *(i) if $p \neq \emptyset$ and $p \leq \ker(f)$,*

$$\sum_{\substack{\mathsf{A}\in p\\ \wedge |\mathsf{A}|=1}} \sigma_{\mathfrak{d}}^{\mathfrak{c}}(\mathsf{A}) \sum_{s=1}^{n} \begin{Bmatrix} v_{(f/p)(\mathsf{A}),s} & \text{if } \sigma_{\mathfrak{d}}^{\mathfrak{c}}(\mathsf{A})=1 \\ v_{s,(f/p)(\mathsf{A})} & \text{otherwise} \end{Bmatrix} + \sum_{\substack{\mathsf{A}\in p\\ \wedge 2\leq |\mathsf{A}|}} \sigma_{\mathfrak{d}}^{\mathfrak{c}}(\mathsf{A})\, v_{(f/p)(\mathsf{A}),(f/p)(\mathsf{A})},$$

(ii) if (necessarily $p \neq \emptyset$ and $p \nleq \ker(f)$ and) there exists a (necessarily unique) $\{\mathtt{x}_1,\mathtt{x}_2\} \in p$ such that for any $\mathsf{A}\in p$ with $\mathsf{A} \neq \{\mathtt{x}_1,\mathtt{x}_2\}$ there exists $\mathsf{B} \in \ker(f)$ with $\mathsf{A} \subseteq \mathsf{B}$, and such that $f(\mathtt{x}_1) \neq f(\mathtt{x}_2)$,

$$\tfrac{1}{2}\sigma_{\mathfrak{d}}^{\mathfrak{c}}(\{\mathtt{x}_1,\mathtt{x}_2\})\,(v_{f(\mathtt{x}_1),f(\mathtt{x}_2)} + v_{f(\mathtt{x}_2),f(\mathtt{x}_1)}),$$

(iii) if (necessarily $p \neq \emptyset$ and $p \nleq \ker(f)$ and) there exist (necessarily unique) $\mathsf{X}\in p$ and $\mathtt{x}\in\mathsf{X}$ and $s \in [\![n]\!]$ such that $3 \leq |\mathsf{X}|$, such that for any $\mathsf{A}\in p$ with $\mathsf{A}\neq\mathsf{X}$ there exists $\mathsf{B}\in\ker(f)$ with $\mathsf{A}\subseteq\mathsf{B}$, such that $f(\mathtt{x}) \neq s$ and such that $f(\mathtt{y}) = s$ for any $\mathtt{y}\in\mathsf{X}$ with $\mathtt{y}\neq\mathtt{x}$,

$$\sigma_{\mathsf{d}}^{\mathsf{c}}(\{\mathtt{x}\}) \begin{cases} v_{(f/p)(\mathsf{X}),s} & \text{if } \sigma_{\mathfrak{d}}^{\mathfrak{c}}(\{\mathtt{x}\}) = 1 \\ v_{s,(f/p)(\mathsf{X})} & \text{otherwise,} \end{cases}$$

(iv) and 0 otherwise.

Proof. The claim follows from Lemma 6.8. We only have to show that in each of the first three cases the numbers coincide. For the purposes of this proof, let $v^{\mathrm{b}(1)} = v$ and $v^{\mathrm{b}(-1)} = v^{\mathrm{t}}$ and recall $\sigma(\circ) = 1$ and $\sigma(\bullet) = -1$. Then, for any $c \in \{\circ,\bullet\}$ and $\{i,j\} \subseteq [\![n]\!]$ the definitions imply

$$x^{v}_{u^{c}_{j,i}} = \begin{Bmatrix} v_{j,i} & \text{if } c=\circ \\ -v_{i,j} & \text{if } c=\bullet \end{Bmatrix} = (-1)^{\sigma(c)}(v^{\mathrm{b}(\sigma(c))})_{j,i}.$$

If $\mathfrak{w} = \mathfrak{c}^{\bullet}\!\!{}_{\bullet}\mathfrak{d}$, then it follows for any $\mathtt{x}\in\Pi_\ell^k$ and any $z\in[\![n]\!]$ that

$$\begin{aligned}
\begin{Bmatrix} -x^{v}_{u^{\mathfrak{w}(\mathtt{x})}_{z,f(\mathtt{x})}} & \text{if } \mathtt{x}\in\Pi_0^k \\ x^{v}_{u^{\mathfrak{w}(\mathtt{x})}_{f(\mathtt{x}),z}} & \text{if } \mathtt{x}\in\Pi_\ell^0 \end{Bmatrix}
&= \begin{Bmatrix} -(-1)^{\sigma(\mathfrak{w}(\mathtt{x}))}(v^{\mathrm{b}(\sigma(\mathfrak{w}(\mathtt{x})))})_{z,f(\mathtt{x})} & \text{if } \mathtt{x}\in\Pi_0^k \\ (-1)^{\sigma(\mathfrak{w}(\mathtt{x}))}(v^{\mathrm{b}(\sigma(\mathfrak{w}(\mathtt{x})))})_{f(\mathtt{x}),z} & \text{if } \mathtt{x}\in\Pi_\ell^0 \end{Bmatrix} \\
&= \begin{Bmatrix} (-1)^{\sigma(\overline{\mathfrak{w}(\mathtt{x})})}(v^{\mathrm{b}(\sigma(\overline{\mathfrak{w}(\mathtt{x})}))})_{f(\mathtt{x}),z} & \text{if } \mathtt{x}\in\Pi_0^k \\ (-1)^{\sigma(\mathfrak{w}(\mathtt{x}))}(v^{\mathrm{b}(\sigma(\mathfrak{w}(\mathtt{x})))})_{f(\mathtt{x}),z} & \text{if } \mathtt{x}\in\Pi_\ell^0 \end{Bmatrix} \\
&= \sigma_{\mathfrak{d}}^{\mathfrak{c}}(\{\mathtt{x}\})\,(v^{\mathrm{b}(\sigma_{\mathfrak{d}}^{\mathfrak{c}}(\{\mathtt{x}\}))})_{f(\mathtt{x}),z}
\end{aligned}$$

by the definition of the color sum and, analogously,

$$\begin{Bmatrix} -x^{v}_{u^{\mathfrak{w}(\mathtt{x})}_{f(\mathtt{x}),z}} & \text{if } \mathtt{x}\in\Pi_0^k \\ x^{v}_{u^{\mathfrak{w}(\mathtt{x})}_{z,f(\mathtt{x})}} & \text{if } \mathtt{x}\in\Pi_\ell^0 \end{Bmatrix} = \sigma_{\mathfrak{d}}^{\mathfrak{c}}(\{\mathtt{x}\})\,(v^{\mathrm{b}(\sigma_{\mathfrak{d}}^{\mathfrak{c}}(\{\mathtt{x}\}))})_{z,f(\mathtt{x})}.$$

We now distinguish the three relevant cases.

(i) In the situation of (i), by Lemma 6.8 the number $F^{1,r}_{E,\varnothing,Y}(x^v)$ is given by,

$$\sum_{\substack{\mathtt{x}\in\Pi^k_\ell\\ \wedge|\pi_p(\mathtt{x})|=1}}\sum_{s=1}^n\begin{Bmatrix}-x^v_{u^{\mathfrak{w}(\mathtt{x})}_{s,f(\mathtt{x})}} & \text{if } \mathtt{x}\in\Pi^k_0\\ x^v_{u^{\mathfrak{w}(\mathtt{x})}_{f(\mathtt{x}),s}} & \text{if } \mathtt{x}\in\Pi^0_\ell\end{Bmatrix}+\sum_{\substack{\mathtt{x}\in\Pi^k_\ell\\ \wedge 2\le|\pi_p(\mathtt{x})|}}\begin{Bmatrix}-x^v_{u^{\mathfrak{w}(\mathtt{x})}_{f(\mathtt{x}),f(\mathtt{x})}} & \text{if } \mathtt{x}\in\Pi^k_0\\ x^v_{u^{\mathfrak{w}(\mathtt{x})}_{f(\mathtt{x}),f(\mathtt{x})}} & \text{if } \mathtt{x}\in\Pi^0_\ell.\end{Bmatrix}$$

By what was shown initially, this can be rewritten identically as

$$\sum_{\substack{\mathtt{x}\in\Pi^k_\ell\\ \wedge|\pi_p(\mathtt{x})|=1}}\sum_{s=1}^n\sigma^{\mathfrak{c}}_{\mathfrak{d}}(\{\mathtt{x}\})\,(v^{\mathrm{b}(\sigma^{\mathfrak{c}}_{\mathfrak{d}}(\{\mathtt{x}\}))})_{f(\mathtt{x}),s}+\sum_{\substack{\mathtt{x}\in\Pi^k_\ell\\ \wedge 2\le|\pi_p(\mathtt{x})|}}\sigma^{\mathfrak{c}}_{\mathfrak{d}}(\{\mathtt{x}\})\,(v^{\mathrm{b}(\sigma^{\mathfrak{c}}_{\mathfrak{d}}(\{\mathtt{x}\}))})_{f(\mathtt{x}),f(\mathtt{x})}.$$

And that is excatly what was claimed because $\ker(f)\le p$ and $\sum_{\mathtt{x}\in\mathsf{A}}\sigma^{\mathfrak{c}}_{\mathfrak{d}}(\{\mathtt{x}\})=\sigma^{\mathfrak{c}}_{\mathfrak{d}}(\mathsf{A})$ for any $\mathsf{A}\in p$.

(ii) Under the assumptions of (ii), Lemma 6.8 tells us that $F^{1,r}_{E,\varnothing,Y}(x^v)$ can be computed as

$$\begin{Bmatrix}-x^v_{u^{\mathfrak{w}(\mathtt{x}_1)}_{f(\mathtt{x}_2),f(\mathtt{x}_1)}} & \text{if } \mathtt{x}_1\in\Pi^k_0\\ x^v_{u^{\mathfrak{w}(\mathtt{x}_1)}_{f(\mathtt{x}_1),f(\mathtt{x}_2)}} & \text{if } \mathtt{x}_1\in\Pi^0_\ell\end{Bmatrix}+\begin{Bmatrix}-x^v_{u^{\mathfrak{w}(\mathtt{x}_2)}_{f(\mathtt{x}_1),f(\mathtt{x}_2)}} & \text{if } \mathtt{x}_2\in\Pi^k_0\\ x^v_{u^{\mathfrak{w}(\mathtt{x}_2)}_{f(\mathtt{x}_2),f(\mathtt{x}_1)}} & \text{if } \mathtt{x}_2\in\Pi^0_\ell\end{Bmatrix},$$

which, by our initial observerations, is identical to

$$\sigma^{\mathfrak{c}}_{\mathfrak{d}}(\{\mathtt{x}_1\})\,(v^{\mathrm{b}(\sigma^{\mathfrak{c}}_{\mathfrak{d}}(\{\mathtt{x}_1\}))})_{f(\mathtt{x}_2),f(\mathtt{x}_1)}+\sigma^{\mathfrak{c}}_{\mathfrak{d}}(\{\mathtt{x}_2\})\,(v^{\mathrm{b}(\sigma^{\mathfrak{c}}_{\mathfrak{d}}(\{\mathtt{x}_2\}))})_{f(\mathtt{x}_1),f(\mathtt{x}_2)}.$$

Since $\sigma^{\mathfrak{c}}_{\mathfrak{d}}(\{\mathtt{x}_i\})\in\{-1,1\}$ for each $i\in[\![2]\!]$, either $\sigma^{\mathfrak{c}}_{\mathfrak{d}}(\{\mathtt{x}_1\})=\sigma^{\mathfrak{c}}_{\mathfrak{d}}(\{\mathtt{x}_2\})$, in which case we infer

$$\begin{aligned}F^{1,r}_{E,\varnothing,Y}(x^v)&=2\sigma^{\mathfrak{c}}_{\mathfrak{d}}(\{\mathtt{x}_1\})\,((v^{\mathrm{b}(\sigma^{\mathfrak{c}}_{\mathfrak{d}}(\{\mathtt{x}_1\}))})_{f(\mathtt{x}_2),f(\mathtt{x}_1)}+(v^{\mathrm{b}(\sigma^{\mathfrak{c}}_{\mathfrak{d}}(\{\mathtt{x}_1\}))})_{f(\mathtt{x}_1),f(\mathtt{x}_2)}),\\&=\tfrac{1}{2}\sigma^{\mathfrak{c}}_{\mathfrak{d}}(\{\mathtt{x}_1,\mathtt{x}_2\})\,(v_{f(\mathtt{x}_1),f(\mathtt{x}_2)}+v_{f(\mathtt{x}_2),f(\mathtt{x}_1)}),\end{aligned}$$

or $\sigma^{\mathfrak{c}}_{\mathfrak{d}}(\{\mathtt{x}_1\})=-\sigma^{\mathfrak{c}}_{\mathfrak{d}}(\{\mathtt{x}_2\})$, implying

$$F^{1,r}_{E,\varnothing,Y}(x^v)=\sigma^{\mathfrak{c}}_{\mathfrak{d}}(\{\mathtt{x}_1\})\,(v^{\mathrm{b}(\sigma^{\mathfrak{c}}_{\mathfrak{d}}(\{\mathtt{x}_1\}))})_{f(\mathtt{x}_2),f(\mathtt{x}_1)}-\sigma^{\mathfrak{c}}_{\mathfrak{d}}(\{\mathtt{x}_1\})\,(v^{\mathrm{b}(-\sigma^{\mathfrak{c}}_{\mathfrak{d}}(\{\mathtt{x}_1\}))})_{f(\mathtt{x}_1),f(\mathtt{x}_2)}=0.$$

And that is precisely what we needed to show in this case.

(iii) Finally, if the premises of (iii) are satisfied, according to Lemma 6.8 and by our initial findings,

$$F^{1,r}_{E,\varnothing,Y}(x^v)=\begin{Bmatrix}-x^v_{u^{\mathfrak{w}(\mathtt{x})}_{s,f(\mathtt{x})}} & \text{if } \mathtt{x}\in\Pi^k_0\\ x^v_{u^{\mathfrak{w}(\mathtt{x})}_{f(\mathtt{x}),s}} & \text{if } \mathtt{x}\in\Pi^0_\ell,\end{Bmatrix}=\sigma^{\mathfrak{c}}_{\mathfrak{d}}(\{\mathtt{x}\})\,(v^{\mathrm{b}(\sigma^{\mathfrak{c}}_{\mathfrak{d}}(\{\mathtt{x}\}))})_{f(\mathtt{x}),s}.$$

Since this is just what we claimed, that concludes the proof. ☐

6.2.3. *Third round of simplifications.* Up to now we have only analyzed the numbers $F^{1,r}_{E,\varnothing,Y}(x^v)$ for one single $r = \mathrm{r}^{\mathfrak{c}}_{\mathfrak{d}}(p)_{j,g}$ at a time, i.e., for a fixed parameter tuple $(\mathfrak{c},\mathfrak{d},p,g,j)$. However, the parameter set indexing our equations has many symmetry properties. The first we now take into acount with Lemma 6.10 below is that, with $(\mathfrak{c},\mathfrak{d},p,g,j)$, it also includes $(\mathfrak{c},\mathfrak{d},p,j',g')$ for arbitrary $g' \in [\![n]\!]^{\otimes k}$ and $j' \in [\![n]\!]^{\otimes \ell}$, where $\{k,\ell\} \subseteq \mathbb{N}_0$ are such that $\mathfrak{c} \in \{\circ,\bullet\}^{\otimes k}$ and $\mathfrak{d} \in \{\circ,\bullet\}^{\otimes \ell}$. We ask what it means for $F^{1,r}_{E,\varnothing,Y}(x^v)$ to vanish for all such $r = \mathrm{r}^{\mathfrak{c}}_{\mathfrak{d}}(p)_{j',g'}$ simultaneously.

LEMMA 6.10. *For any $\{k,\ell\} \subseteq \mathbb{N}_0$, any $\mathfrak{c} \in \{\circ,\bullet\}^{\otimes k}$ and $\mathfrak{d} \in \{\circ,\bullet\}^{\otimes \ell}$, any partition p of Π^k_ℓ and any $v \in M_n(\mathbb{C})$, if*

$$R = \left\{\mathrm{r}^{\mathfrak{c}}_{\mathfrak{d}}(p)_{j,g} \,\middle|\, g \in [\![n]\!]^{\otimes k} \wedge j \in [\![n]\!]^{\otimes \ell}\right\},$$

then $F^{1,r}_{E,\varnothing,Y}(x^v) = 0$ holds for any $r \in R$ if and only if all of the following are satisfied.

(i) For any $h\colon p \to [\![n]\!]$,

$$\sum_{\substack{\mathsf{B}\in p\\ \wedge |\mathsf{B}|=1}} \sigma^{\mathfrak{c}}_{\mathfrak{d}}(\mathsf{B}) \sum_{s=1}^{n} \begin{Bmatrix} v_{h(\mathsf{B}),s} & \text{if } \sigma^{\mathfrak{c}}_{\mathfrak{d}}(\mathsf{B}) = 1 \\ v_{s,h(\mathsf{B})} & \text{otherwise} \end{Bmatrix} + \sum_{\substack{\mathsf{B}\in p\\ \wedge 2\le|\mathsf{B}|}} \sigma^{\mathfrak{c}}_{\mathfrak{d}}(\mathsf{B})\, v_{h(\mathsf{B}),h(\mathsf{B})} = 0.$$

(ii) If there exists $\mathsf{B} \in p$ with $|\mathsf{B}| = 2$ and $\sigma^{\mathfrak{c}}_{\mathfrak{d}}(\mathsf{B}) \neq 0$, then $v_{b,a} + v_{a,b} = 0$ for any $\{a,b\} \subseteq [\![n]\!]$ with $a \neq b$.

(iii) If there exists $\mathsf{B} \in p$ with $2 < |\mathsf{B}|$, then $v_{b,a} = 0$ for any $\{a,b\} \subseteq [\![n]\!]$ with $a \neq b$.

PROOF. First, suppose that conditions (i)–(iii) are satisfied, let $g \in [\![n]\!]^{\otimes k}$ and $j \in [\![n]\!]^{\otimes \ell}$ be arbitrary and abbreviate $r \equiv \mathrm{r}^{\mathfrak{c}}_{\mathfrak{d}}(p)_{j,g}$ and $f \equiv g \,\overset{\bullet}{\scriptscriptstyle\bullet}\, j$. We show that $F^{1,r}_{E,\varnothing,Y}(x^v) = 0$. If f is as in case (i) of Lemma 6.9 and if we let $h = f/p$, then condition (i) says precisely that $F^{1,r}_{E,\varnothing,Y}(x^v) = 0$. If we find $\mathtt{x}_1$ and $\mathtt{x}_2$ as in case (ii) of Lemma 6.9, then $F^{1,r}_{E,\varnothing,Y}(x^v) = \frac{1}{2}\sigma^{\mathfrak{c}}_{\mathfrak{d}}(\{\mathtt{x}_1,\mathtt{x}_2\})\,(v_{f(\mathtt{x}_1),f(\mathtt{x}_2)} + v_{f(\mathtt{x}_2),f(\mathtt{x}_1)})$. Hence, if $\sigma^{\mathfrak{c}}_{\mathfrak{d}}(\{\mathtt{x}_1,\mathtt{x}_2\}) = 0$ we have nothing to prove. Otherwise, condition (ii) guarantees that $v_{b,a} + v_{a,b} = 0$ for any $\{a,b\} \subseteq [\![n]\!]$, thus showing $F^{1,r}_{E,\varnothing,Y}(x^v) = 0$ since $f(\mathtt{x}_2) \neq f(\mathtt{x}_1)$. If there exist s, X and $\mathtt{x}$ as in case (iii) of Lemma 6.9, then condition (iii) implies that v is diagonal. Since $F^{1,r}_{E,\varnothing,Y}(x^v)$ is given by $\sigma^{\mathfrak{c}}_{\mathfrak{d}}(\{\mathtt{x}\})\,v_{(f/p)(\mathsf{X}),s}$ or $\sigma^{\mathfrak{c}}_{\mathfrak{d}}(\{\mathtt{x}\})\,v_{s,(f/p)(\mathsf{X})}$ that proves $F^{1,r}_{E,\varnothing,Y}(x^v) = 0$ in this case since $s \neq (f/p)(\mathsf{X})$. Finally, if f is as in case (iv) of Lemma 6.9 there is nothing to show. Hence, $F^{1,r}_{E,\varnothing,Y}(x^v)$ as asserted.

To show the converse we assume $F^{1,r}_{E,\varnothing,Y}(x^v) = 0$ for any $r \in R$. For any $h\colon p \to [\![n]\!]$, if we let $f = h \circ \pi_{\sim_p}$ and $r = \mathrm{r}^{\mathfrak{c}}_{\mathfrak{d}}(p)_{j,g}$, then $p \le \ker(f)$ and $F^{1,r}_{E,\varnothing,Y}(x^v)$ is exactly the left-hand side of the equation in condition (i) by case (i) of Lemma 6.9, which proves condition (i) to be satisfied.

Now, suppose that there exists $\mathsf{B} \in p$ with $|\mathsf{B}| = 2$, i.e., $\mathsf{B} = \{\mathtt{x}_1,\mathtt{x}_2\}$ for some $\{\mathtt{x}_1,\mathtt{x}_2\} \subseteq \Pi^k_\ell$ with $\mathtt{x}_1 \neq \mathtt{x}_2$, and with $\sigma^{\mathfrak{c}}_{\mathfrak{d}}(\mathsf{B}) \neq 0$ and let $\{a,b\} \subseteq [\![n]\!]$ with $a \neq b$ be arbitrary. If f is such that $\mathtt{x}_1 \mapsto a$ and $\mathtt{y} \mapsto b$ for any $\mathtt{y} \in \Pi^k_\ell \backslash \{\mathtt{x}_1\}$, case (ii) of Lemma 6.9 tells us that $F^{1,r}_{E,\varnothing,Y}(x^v) = \frac{1}{2}\sigma^{\mathfrak{c}}_{\mathfrak{d}}(\{\mathtt{x}_1,\mathtt{x}_2\})\,(v_{b,a} + v_{a,b})$. Since $\sigma^{\mathfrak{c}}_{\mathfrak{d}}(\{\mathtt{x}_1,\mathtt{x}_2\}) \neq 0$ by assumption, (ii) is thus true as well.

Lastly, let there exist $\mathsf{B} \in p$ with $2 < |\mathsf{B}|$ and let $\{a,b\} \subseteq [\![n]\!]$ be arbitrary with $a \neq b$. We can choose any $\mathtt{x} \in \mathsf{X} := \mathsf{B}$ and define f by demanding $\mathtt{x} \mapsto a$ and $\mathtt{y} \mapsto s := b$ for any $\mathtt{y} \in \Pi_\ell^k \backslash \{\mathtt{x}\}$ to ensure that f, s, X and $\mathtt{x}$ are as in case (iii) of Lemma 6.9 and thus that $F^{1,r}_{E,\varnothing,Y}(x^v)$ is given by $\sigma^{\mathfrak{c}}_{\mathfrak{d}}(\{\mathtt{x}\})\, v_{a,b}$ or $\sigma^{\mathfrak{c}}_{\mathfrak{d}}(\{\mathtt{x}\})\, v_{b,a}$. Thus, also the last condition (iii) is satisfied and the proof is complete. $\square$

6.2.4. *Fourth round of simplifications.* In Lemma 6.11 below, continuing what we began with Lemma 6.10, we capitalize on further symmetries of the parameter set indexing the equations. Namely, with any $(\mathfrak{c},\mathfrak{d},p,g,j)$ it includes also $(\bar{\mathfrak{c}},\bar{\mathfrak{d}},\overline{p}^*,g',j')$ for any g' and j'. Moreover, we only need to consider special $(\mathfrak{c},\mathfrak{d},p)$.

Lemma 6.11. *For any $\{k,\ell\} \subseteq \mathbb{N}_0$, any $\mathfrak{c} \in \{\circ,\bullet\}^{\otimes k}$, any $\mathfrak{d} \in \{\circ,\bullet\}^{\otimes \ell}$, any partition p of Π_ℓ^k and any $v \in M_n(\mathbb{C})$, if*

$$R = \left\{ \mathrm{r}^{\mathfrak{c}}_{\mathfrak{d}}(p)_{j,g}, \mathrm{r}^{\bar{\mathfrak{c}}}_{\bar{\mathfrak{d}}}((\overline{p})^*)_{j,g} \,|\, g \in [\![n]\!]^{\otimes k} \wedge j \in [\![n]\!]^{\otimes \ell} \right\},$$

then the statement that $F^{r,1}_{E,\varnothing,Y}(x^v) = 0$ holds for any $r \in R$ is equivalent in the case that $(\mathfrak{c},\mathfrak{d},p)$ is given by

(a) $\sqcap\!\sqcap\!\sqcap_{\circ\bullet\circ\bullet}$ to v being diagonal.

(b) $\uparrow_\circ\otimes\uparrow_\bullet$ to there existing $\lambda \in \mathbb{C}$ such that $v - \lambda I$ is small.

(c) $\uparrow_\circ^{\otimes t}$ for some $t \in \mathbb{N}$ to v being small.

Proof. (a) Because $(\bar{\mathfrak{c}},\bar{\mathfrak{d}},(\overline{p})^*) = \sqcap\!\sqcap\!\sqcap_{\circ\bullet\circ\bullet}$ we only need to consider the conditions coming from $\sqcap\!\sqcap\!\sqcap_{\circ\bullet\circ\bullet}$, i.e., $R = \{\mathrm{r}^{\varnothing}_{\circ\bullet\circ\bullet}(\sqcap\!\sqcap\!\sqcap)_{j,\varnothing} \,|\, j \in [\![n]\!]^{\otimes 4}\}$.

Because $|\mathsf{B}| \neq 1$ and $\sigma^{\varnothing}_{\circ\bullet\circ\bullet}(\mathsf{B}) = 0$ for the only $\mathsf{B} \in \sqcap\!\sqcap\!\sqcap$ condition (i) of Lemma 6.10 is satisfied for any $h\colon \sqcap\!\sqcap\!\sqcap \to [\![n]\!]$, regardless of whether v is diagonal or not. Similarly, since also $|\mathsf{B}| \neq 2$ for the only $\mathsf{B} \in \sqcap\!\sqcap\!\sqcap$ the same is true about condition (ii) of Lemma 6.10. It is condition (iii) of Lemma 6.10 alone which is relevant. Namely, since there is $\mathsf{B} \in \sqcap\!\sqcap\!\sqcap$ with $3 < |\mathsf{B}|$ it implies that $F^{r,1}_{E,\varnothing,Y}(x^v) = 0$ for any $r \in R$ if and only if v is diagonal.

(b) Once more, the fact that $(\bar{\mathfrak{c}},\bar{\mathfrak{d}},(\overline{p})^*)$ is identical to $\uparrow_\circ\otimes\uparrow_\bullet$ allows us to draw the simplifying conclusion $R = \{\mathrm{r}^{\varnothing}_{\circ\bullet}(\uparrow\uparrow)_{j,\varnothing} \,|\, j \in [\![n]\!]^{\otimes 2}\}$. Furthermore, because $|\mathsf{B}| = 1$ for any $\mathsf{B} \in \uparrow\uparrow$ and because $\sigma^{\varnothing}_{\circ\bullet}(\{_{\blacksquare}1\}) = 1$ and $\sigma^{\varnothing}_{\circ\bullet}(\{_{\blacksquare}2\}) = -1$, for any $h\colon \uparrow\uparrow \to [\![n]\!]$ what condition (i) of Lemma 6.10 demands is that $\sum_{s=1}^n v_{h(\{_{\blacksquare}1\}),s} - \sum_{s=1}^n v_{s,h(\{_{\blacksquare}2\})}$ be zero. At the same time, condition (ii) and condition (iii) of Lemma 6.10 are always trivially satisfied since there are no $\mathsf{B} \in \uparrow\uparrow$ with $|\mathsf{B}| = 2$ or even $2 < |\mathsf{B}|$. Hence, $F^{r,1}_{E,\varnothing,Y}(x^v) = 0$ holds for any $r \in R$ if and only if $\sum_{s=1}^n v_{j,s} - \sum_{s=1}^n v_{s,i}$ for any $\{i,j\} \subseteq [\![n]\!]$, i.e., by Lemma 5.2 (a), if and only if there exists $\lambda \in \mathbb{C}$ such that $v - \lambda I$ is small.

(c) In this case, the partitions $(\bar{\mathfrak{c}},\bar{\mathfrak{d}},(\overline{p})^*) = \uparrow_\bullet^{\otimes t}$ and $(\mathfrak{c},\mathfrak{d},p)$ are distinct and must thus both be considered.

Since $|\mathsf{B}| = 1$ and $\sigma^{\varnothing}_{c^{\otimes t}}(\mathsf{B}) = \sigma(c)$ for any $\mathsf{B} \in \uparrow^{\otimes t}$ and any $c \in \{\circ,\bullet\}$ condition (i) of Lemma 6.10 is satisfied for $\uparrow_\circ^{\otimes t}$ and $\uparrow_\bullet^{\otimes t}$ if and only if $\sum_{d=1}^t \sum_{s=1}^n v_{h(\{_{\blacksquare}d\}),s} = 0$ respectively $-\sum_{d=1}^t \sum_{s=1}^n v_{s,h(\{_{\blacksquare}d\})} = 0$ for any $h\colon \uparrow^{\otimes t} \to [\![n]\!]$. Moreover, conditions (ii) and (iii) of Lemma 6.10 are trivially satisfied by the absence of any $\mathsf{B} \in \uparrow^{\otimes t}$ with $2 \leq |\mathsf{B}|$.

Consequently, if v is small and thus $\sum_{s=1}^{n} v_{h(\{\centerdot d\}),s} = \sum_{s=1}^{n} v_{s,h(\{\centerdot d\})} = 0$ for any $d \in [\![t]\!]$ all three conditions of Lemma 6.10 are met for both $\uparrow_{\circ}^{\otimes t}$ and $\uparrow_{\bullet}^{\otimes t}$.

If, conversely, $F^{r,1}_{E,\varnothing,Y}(x^v) = 0$ for any $r \in R$, then for any $i \in [\![n]\!]$, if $h\colon \uparrow^{\otimes t} \to [\![n]\!]$ is constant with value i, condition (i) of Lemma 6.10 for $\uparrow_{\circ}^{\otimes t}$ and $\uparrow_{\bullet}^{\otimes t}$ lets us infer $0 = t \sum_{s=1}^{n} v_{i,s}$ respectively $0 = t \sum_{s=1}^{n} v_{s,i}$, proving v to be small. □

6.3. Case distinctions. Having studied first single equations and then larger subsystems of the linear equations of Proposition 4.11 we are now at the point where we can solve the system for all categories of two-colored partitions simultaneously by distinguishing just twelve cases.

In a sense this is yet another extension of the process begun in Lemma 6.11 and continued in Lemma 6.10 of considering ever more equations simultaneously in a way inspired by the symmetries of the parameter set. The symmetries we now take into account are much deeper than the ones used before. They rely on combinatorial analyses and are listed in Lemma 3.7.

PROPOSITION 6.12. *For any category $\mathcal{C}$ of two-colored partitions and any $v \in M_n(\mathbb{C})$ if*

$$R = \big\{ \mathrm{r}^{\mathfrak{c}}_{\mathfrak{d}}(p)_{j,g} \,\big|\, \{k,\ell\} \subseteq \mathbb{N}_0 \wedge \mathfrak{c} \in \{\circ,\bullet\}^{\otimes k} \wedge \mathfrak{d} \in \{\circ,\bullet\}^{\otimes \ell} \wedge (\mathfrak{c},\mathfrak{d},p) \in \mathcal{C} \wedge g \in [\![n]\!]^{\otimes k} \wedge j \in [\![n]\!]^{\otimes \ell} \big\},$$

then $F^{1,r}_{E,\varnothing,Y}(x^v) = 0$ holding for any $r \in R$ is equivalent,

(i) if $\mathcal{C}$ is case $\mathcal{O}$ and
   (1) has only neutral non-singleton blocks, to the absolutely true statement.
   (2) has some non-neutral non-singleton blocks but only neutral partitions, to there existing $\lambda \in \mathbb{C}$ such that $v - \lambda I$ is skew-symmetric.
   (3) has some non-neutral partitions, to v being skew-symmetric.

(ii) if $\mathcal{C}$ is case $\mathcal{B}$ and
   (1) has only neutral non-singleton blocks and only neutral partitions, to there existing $\lambda \in \mathbb{C}$ such that $v - \lambda I$ is small.
   (2) has only neutral non-singleton blocks and some non-neutral partitions, to v being small
   (3) has some non-neutral non-singleton blocks and only neutral partitions, to there existing $\lambda \in \mathbb{C}$ such that $v - \lambda I$ is skew-symmetric and small.
   (4) has some non-neutral non-singleton blocks and some non-neutral partitions, to v being skew-symmetric and small.

(iii) if $\mathcal{C}$ is case $\mathcal{H}$ and
   (1) has only neutral non-singleton blocks, to v being diagonal.
   (2) has some non-neutral non-singleton blocks but only neutral partitions, to there existing $\lambda \in \mathbb{C}$ such that $v = \lambda I$.
   (3) has some non-neutral partitions, to $v = 0$.

(iv) if $\mathcal{C}$ is case $\mathcal{S}$ and
   (1) has only neutral partitions, to there existing $\lambda \in \mathbb{C}$ such that $v = \lambda I$.

(2) has some non-neutral partitions, to $v = 0$.
Other cases cannot occur.

Proof. We need to treat each case individually. By Lemma 6.10 what we have to show is that the asserted statement about v is true if and only if for each $(\mathfrak{c}, \mathfrak{d}, p) \in \mathcal{C}$ the conditions (i)–(iii) of Lemma 6.10 are satisfied.

(i) First, suppose that $\mathcal{C}$ is case $\mathcal{O}$. By Lemma 3.7 (b) and (c) that means $|\mathsf{B}| = 2$ for any $\mathsf{B} \in p$ and any $(\mathfrak{c}, \mathfrak{d}, p) \in \mathcal{C}$.

(1) Let $\mathcal{C}$ have only neutral non-singleton blocks. All we have to show is that for any $(\mathfrak{c}, \mathfrak{d}, p) \in \mathcal{C}$ all three conditions (i)–(iii) of Lemma 6.10 are satisfied. Indeed, if $(\mathfrak{c}, \mathfrak{d}, p) \in \mathcal{C}$ and $h\colon p \to [\![n]\!]$, then, since $|\mathsf{B}| = 2$ for any $\mathsf{B} \in p$, condition (i) of Lemma 6.10 is met if $\sum_{\mathsf{B} \in p} \sigma_{\mathfrak{d}}^{\mathfrak{c}}(\mathsf{B})\, v_{h(\mathsf{B}),h(\mathsf{B})} = 0$. And because also $\sigma_{\mathfrak{d}}^{\mathfrak{c}}(\mathsf{B}) = 0$ for any $\mathsf{B} \in p$ by $\mathcal{C}$ having only neutral singleton blocks this is very much the case. For the same reason condition (i) of Lemma 6.10 is trivially satisfied. The same is true about condition (i) of Lemma 6.10 since there are no $\mathsf{B} \in p$ with $2 < |\mathsf{B}|$.

(2) The next case is that $\mathcal{C}$ has some non-neutral non-singleton blocks but still only neutral partitions.

Suppose that there exists $\lambda \in \mathbb{C}$ such that $v - \lambda I$ is skew-symmetric and let $(\mathfrak{c}, \mathfrak{d}, p) \in \mathcal{C}$ and $h\colon p \to [\![n]\!]$ be arbitrary. Because $|\mathsf{B}| = 2$ for any $\mathsf{B} \in p$ condition (i) of Lemma 6.10 is satisfied if and only if $\sum_{\mathsf{B} \in p} \sigma_{\mathfrak{d}}^{\mathfrak{c}}(\mathsf{B})\, v_{h(\mathsf{B}),h(\mathsf{B})} = 0$. Since $v - \lambda I$ is skew-symmetric $v_{j,j} = v_{i,i}$ for any $\{i, j\} \subseteq [\![n]\!]$ with $j \neq i$ by Lemma 5.2 (b). Thus, what condition (i) of Lemma 6.10 actually demands is that the term $\sum_{\mathsf{B} \in p} \sigma_{\mathfrak{d}}^{\mathfrak{c}}(\mathsf{B})\, v_{1,1} = \Sigma_{\mathfrak{d}}^{\mathfrak{c}}\, v_{1,1}$ be zero, which it is since $\mathcal{C}$ having only neutral partitions ensures $\Sigma_{\mathfrak{d}}^{\mathfrak{c}} = 0$. Lemma 5.2 (b) furthermore guarantees that $v_{j,i} + v_{i,j} = 0$ for any $\{i, j\} \subseteq [\![n]\!]$ with $j \neq i$, which is why condition (ii) of Lemma 6.10 is met, regardless of whether there is $\mathsf{B} \in p$ with $\sigma_{\mathfrak{d}}^{\mathfrak{c}}(\mathsf{B}) \neq 0$ or not. And since there are no $\mathsf{B} \in p$ with $2 < |\mathsf{B}|$ condition (iii) of Lemma 6.10 is trivially satisfied.

Conversely, let now $F_{E,\varnothing,Y}^{1,r}(x^v) = 0$ hold for any $r \in R$. By assumption we find $(\mathfrak{c}, \mathfrak{d}, p) \in \mathcal{C}$ and $\mathsf{A} \in p$ with $\sigma_{\mathfrak{d}}^{\mathfrak{c}}(\mathsf{A}) \neq 0$ but still $\Sigma_{\mathfrak{d}}^{\mathfrak{c}} = 0$ and, of course, with $|\mathsf{A}| = 2$ since $\mathcal{C}$ is case $\mathcal{O}$. Hence, $v_{j,i} + v_{i,j} = 0$ for any $\{i, j\} \subseteq [\![n]\!]$ with $i \neq j$ by condition (ii) of Lemma 6.10. But also, given any $\{i, j\} \subseteq [\![n]\!]$ with $i \neq j$, if $h\colon p \to [\![n]\!]$ is such that $\mathsf{A} \mapsto j$ and $\mathsf{B} \mapsto i$ for any $\mathsf{B} \in p \backslash \{\mathsf{A}\}$, condition (i) of Lemma 6.10 implies then $\sigma_{\mathfrak{d}}^{\mathfrak{c}}(\mathsf{A})\, v_{j,j} + \sum_{\mathsf{B} \in p \wedge \mathsf{B} \neq \mathsf{A}} \sigma_{\mathfrak{d}}^{\mathfrak{c}}(\mathsf{B})\, v_{i,i} = 0$. Since $0 = \Sigma_{\mathfrak{d}}^{\mathfrak{c}} = \sum_{\mathsf{B} \in p} \sigma_{\mathfrak{d}}^{\mathfrak{c}}(\mathsf{B}) = \sigma_{\mathfrak{d}}^{\mathfrak{c}}(\mathsf{A}) + \sum_{\mathsf{B} \in p \wedge \mathsf{B} \neq \mathsf{A}} \sigma_{\mathfrak{d}}^{\mathfrak{c}}(\mathsf{B})$, i.e., $\sum_{\mathsf{B} \in p \wedge \mathsf{B} \neq \mathsf{A}} \sigma_{\mathfrak{d}}^{\mathfrak{c}}(\mathsf{B}) = -\sigma_{\mathfrak{d}}^{\mathfrak{c}}(\mathsf{A})$, this means $\sigma_{\mathfrak{d}}^{\mathfrak{c}}(\mathsf{A})\,(v_{j,j} - v_{i,i}) = 0$, which implies $v_{j,j} - v_{i,i} = 0$ by $\sigma_{\mathfrak{d}}^{\mathfrak{c}}(\mathsf{A}) \neq 0$. Hence, there exists $\lambda \in \mathbb{C}$ such that $v - \lambda I$ is skew-symmetric by Lemma 5.2 (b).

(3) The last possiblity in case $\mathcal{O}$ is for $\mathcal{C}$ to have some non-neutral partitions.

Assume that v is skew-symmetric and let $(\mathfrak{c}, \mathfrak{d}, p) \in \mathcal{C}$ and $h\colon p \to [\![n]\!]$ be arbitrary. Since there are no $\mathsf{B} \in p$ with $|\mathsf{B}| \neq 1$ condition (i) of Lemma 6.10 is met if $\sum_{\mathsf{B} \in p} \sigma_{\mathfrak{d}}^{\mathfrak{c}}(\mathsf{B})\, v_{h(\mathsf{B}),h(\mathsf{B})} = 0$ Since v being skew-symmetric implies $v_{h(\mathsf{B}),h(\mathsf{B})} = 0$ for any $\mathsf{B} \in p$ this is indeed true. But v being skew-symmetric also implies $v_{j,i} + v_{i,j} = 0$ for any $\{i, j\} \subseteq [\![n]\!]$ with $j \neq i$, which is why condition (ii) of Lemma 6.10 is satisfied no matter whether there is $\mathsf{B} \in p$ with $\sigma_{\mathfrak{d}}^{\mathfrak{c}}(\mathsf{B}) \neq 0$ or not. Finally, condition (i) of Lemma 6.10 is of course fulfilled by $\mathcal{C}$ being case $\mathcal{O}$.

To see the converse, we assume that $F^{1,r}_{E,\varnothing,Y}(x^v) = 0$ holds for any $r \in R$. Because $\mathcal{C}$ has some non-neutral partitions there exists $(\mathfrak{c},\mathfrak{d},p) \in \mathcal{C}$ with $\Sigma_{\mathfrak{d}}^{\mathfrak{c}} \neq 0$. For any $i \in [\![n]\!]$, if $h\colon p \to [\![n]\!]$ is constant with value i, then condition (ii) of Lemma 6.10 implies $0 = \sum_{\mathsf{B}\in p} \sigma_{\mathfrak{d}}^{\mathfrak{c}}(\mathsf{B})\, v_{i,i} = \Sigma_{\mathfrak{d}}^{\mathfrak{c}}\, v_{i,i}$ and thus $v_{i,i} = 0$ by $\Sigma_{\mathfrak{d}}^{\mathfrak{c}} \neq 0$. Futhermore, Since $\Sigma_{\mathfrak{d}}^{\mathfrak{c}} = \sum_{\mathsf{B}\in p} \sigma_{\mathfrak{d}}^{\mathfrak{c}}(\mathsf{B})$ the assumption $\Sigma_{\mathfrak{d}}^{\mathfrak{c}} \neq 0$ also requires the existence of at least one $\mathsf{B} \in p$ with $\sigma_{\mathfrak{d}}^{\mathfrak{c}}(\mathsf{B}) \neq 0$. Because $\mathcal{C}$ is case $\mathcal{O}$ condition (ii) of Lemma 6.10 therefore yields $v_{j,i}+v_{i,j} = 0$ for any $\{i,j\} \subseteq [\![n]\!]$ with $j \neq i$. In other words, v is skew-symmetric.

(ii) If $\mathcal{C}$ is case $\mathcal{B}$, then $|\mathsf{B}| \leq 2$ for any $\mathsf{B} \in p$ and any $(\mathfrak{c},\mathfrak{d},p) \in p$ by Lemma 3.7 (c) and, of course, $\uparrow^{\circ}\otimes\uparrow^{\bullet} \in \mathcal{C}$ by definition.

(1) As the first subcase, let $\mathcal{C}$ have only neutral non-singleton blocks and only neutral partitions.

Suppose that $\lambda \in \mathbb{C}$ is such that $v - \lambda I$ is small and let $(\mathfrak{c},\mathfrak{d},p) \in \mathcal{C}$ and $h\colon p \to [\![n]\!]$ be arbitrary. By Lemma 5.2 (a) then $\lambda = \sum_{s=1}^{n} v_{h(\mathsf{B}),s} = \sum_{s=1}^{n} v_{s,h(\mathsf{B})}$ for any $\mathsf{B} \in p$. Hence, and because $\sigma_{\mathfrak{d}}^{\mathfrak{c}}(\mathsf{B}) = 0$ for any $\mathsf{B} \in p$ with $2 \leq |\mathsf{B}|$ by $\mathcal{C}$ having only neutral non-singleton blocks, in order to satisfy condition (i) of Lemma 6.10 the term $\sum_{\mathsf{B}\in p \wedge |\mathsf{B}|=1} \sigma_{\mathfrak{d}}^{\mathfrak{c}}(\mathsf{B})\,\lambda$ has to vanish. And, of course, it does since $\mathcal{C}$ having only neutral partitions ensures $0 = \Sigma_{\mathfrak{d}}^{\mathfrak{c}} = \sum_{\mathsf{B}\in p \wedge |\mathsf{B}|=1} \sigma_{\mathfrak{d}}^{\mathfrak{c}}(\mathsf{B}) + \sum_{\mathsf{B}\in p \wedge 2\leq|\mathsf{B}|} \sigma_{\mathfrak{d}}^{\mathfrak{c}}(\mathsf{B}) = \sum_{\mathsf{B}\in p \wedge |\mathsf{B}|=1} \sigma_{\mathfrak{d}}^{\mathfrak{c}}(\mathsf{B})$, where the last step is due to $\mathcal{C}$ having only neutral non-singleton blocks again.

If on the other hand $F^{1,r}_{E,\varnothing,Y}(x^v) = 0$ for any $r \in R$, then Lemma 6.11 (b) the fact that $\uparrow^{\circ}\otimes\uparrow^{\bullet} \in \mathcal{C}$ already implies by that there exists $\lambda \in \mathbb{C}$ such that $v - \lambda I$ is small.

(2) Next, suppose that $\mathcal{C}$ has only neutral non-singleton blocks and some non-neutral partitions.

Let v be small and let $(\mathfrak{c},\mathfrak{d},p) \in \mathcal{C}$ and $h\colon p \to [\![n]\!]$ be arbitrary. Then $\sum_{s=1}^{n} v_{h(\mathsf{B}),s} = \sum_{s=1}^{n} v_{s,h(\mathsf{B})} = 0$ for any $\mathsf{B} \in p$. For that reason the first sum on the left hand side of the equation in condition (i) of Lemma 6.10 vanishes. And since $\mathcal{C}$ having only neutral non-singleton blocks means $\sigma_{\mathfrak{d}}^{\mathfrak{c}}(\mathsf{B}) = 0$ for any $\mathsf{B} \in p$ with $2 \leq |\mathsf{B}|$ the second term does as well. Hence, condition (i) of Lemma 6.10 is satisfied. The fact that $\sigma_{\mathfrak{d}}^{\mathfrak{c}}(\mathsf{B}) = 0$ for any $\mathsf{B} \in p$ with $2 \leq |\mathsf{B}|$ also implies that condition (ii) of Lemma 6.10 is trivially fulfilled. Finally, condition (iii) of Lemma 6.10 is met as well since $\mathcal{C}$ is case $\mathcal{B}$.

Because $\mathcal{C}$ has some non-neutral partitions we find a $(\mathfrak{c},\mathfrak{d},p) \in \mathcal{C}$ with $t = |\Sigma_{\mathfrak{d}}^{\mathfrak{c}}| \neq 0$. By Lemma 3.7 (e) that necessitates $\uparrow^{\circ\,\otimes t} \in \mathcal{C}$. Hence, if $F^{1,r}_{E,\varnothing,Y}(x^v) = 0$ for any $r \in R$, then v is small by Lemma 6.11 (c).

(3) Now, let $\mathcal{C}$ have some non-neutral non-singleton blocks but only neutral partitions.

If $\lambda \in \mathbb{C}$ is such that $v - \lambda I$ is both skew-symmetric and small, then given any $(\mathfrak{c},\mathfrak{d},p) \in \mathcal{C}$ and $h\colon p \to [\![n]\!]$, we infer for any $\mathsf{B} \in p$, first, $\lambda = \sum_{s=1}^{n} v_{h(\mathsf{B}),s} = \sum_{s=1}^{n} v_{s,h(\mathsf{B})}$ by Lemma 5.2 (a) and, second, $\lambda = v_{h(\mathsf{B}),h(\mathsf{B})}$ by Lemma 5.2 (b). Consequently, condition (i) of Lemma 6.10 is satisfied if and only if the term $\sum_{\mathsf{B}\in p \wedge |\mathsf{B}|=1} \sigma_{\mathfrak{d}}^{\mathfrak{c}}(\mathsf{B})\,\lambda + \sum_{\mathsf{B}\in p \wedge 2\leq|\mathsf{B}|} \sigma_{\mathfrak{d}}^{\mathfrak{c}}(\mathsf{B})\,\lambda = \sum_{\mathsf{B}\in p} \sigma_{\mathfrak{d}}^{\mathfrak{c}}(\mathsf{B})\,\lambda = \Sigma_{\mathfrak{d}}^{\mathfrak{c}}\,\lambda$ is zero, which, of course, it is since $\mathcal{C}$ having only neutral partitions guarantees $\Sigma^{\mathfrak{c}}_{\mathfrak{d}} = 0$. Because Lemma 5.2 (b) also tells us that $v_{j,i} + v_{i,j} = 0$ for any $\{i,j\} \subseteq [\![n]\!]$ with $i \neq j$ condition (ii) of Lemma 6.10 is met, irrespective of whether there actually is some $\mathsf{B} \in p$ with $|\mathsf{B}| = 2$ and $\sigma^{\mathfrak{c}}_{\mathfrak{d}}(\mathsf{B}) \neq 0$. Lastly, condition (iii) of Lemma 6.10 is trivially fulfilled since $\mathcal{C}$ is case $\mathcal{B}$.

Conversely, if $F^{1,r}_{E,\varnothing,Y}(x^v) = 0$ for any $r \in R$, then the fact that $\uparrow\otimes\downarrow \in \mathcal{C}$ ensures the existence of $\lambda_1 \in \mathbb{C}$ such that $v - \lambda_1 I$ is small by Lemma 6.11 (b). Additionally, since $\mathcal{C}$ has some non-neutral non-singleton blocks we find a $(\mathfrak{c},\mathfrak{d},p) \in \mathcal{C}$ with the property that there is $\mathsf{A} \in p$ with $2 \leq |\mathsf{A}|$ and $\sigma^{\mathfrak{c}}_{\mathfrak{d}}(\mathsf{A}) \neq 0$. Because $\mathcal{C}$ being case $\mathcal{B}$ then also demands $|\mathsf{A}| \leq 2$ that means $v_{j,i}+v_{i,j} = 0$ for any $\{i,j\} \subseteq [\![n]\!]$ with $i \neq j$ by condition (ii) of Lemma 6.10. Moreover, given any $\{i,j\} \subseteq [\![n]\!]$ with $i \neq j$, if $h\colon p \to [\![n]\!]$ is such that $\mathsf{A} \mapsto j$ and $\mathsf{B} \mapsto i$ for any $\mathsf{B} \in p \backslash \{\mathsf{A}\}$ and if $h'\colon p \to [\![n]\!]$ is constant with value i, then, considering that $\lambda_1 = \sum_{s=1}^n v_{i,s} = \sum_{s=1}^n v_{s,i}$ by Lemma 5.2 (a), condition (i) of Lemma 6.10 yields the identities $\sum_{\mathsf{B}\in p \wedge |\mathsf{B}|=1} \sigma^{\mathfrak{c}}_{\mathfrak{d}}(\mathsf{B})\, \lambda_1 + \sigma^{\mathfrak{c}}_{\mathfrak{d}}(\mathsf{A})\, v_{j,j} + \sum_{\mathsf{B}\in p \wedge 2\leq|\mathsf{B}| \wedge \mathsf{B}\neq\mathsf{A}} \sigma^{\mathfrak{c}}_{\mathfrak{d}}(\mathsf{B})\, v_{i,i} = 0$ and $\sum_{\mathsf{B}\in p \wedge |\mathsf{B}|=1} \sigma^{\mathfrak{c}}_{\mathfrak{d}}(\mathsf{B})\, \lambda_1 + \sum_{\mathsf{B}\in p \wedge 2\leq|\mathsf{B}|} \sigma^{\mathfrak{c}}_{\mathfrak{d}}(\mathsf{B})\, v_{i,i} = 0$. Subtracting the second from the first thus yields the identity $\sigma^{\mathfrak{c}}_{\mathfrak{d}}(\mathsf{A})\,(v_{j,j} - v_{i,i}) = 0$. Since $\sigma^{\mathfrak{c}}_{\mathfrak{d}}(\mathsf{A}) \neq 0$ we can infer $v_{j,j} = v_{i,i}$ for any $\{i,j\} \subseteq [\![n]\!]$ with $i \neq j$. Since also $v_{j,i} + v_{j,i} = 0$ for any $\{i,j\} \subseteq [\![n]\!]$ with $i \neq j$ by condition (i) of Lemma 6.10, by Lemma 5.2 (b) we have thus shown that there exists $\lambda_2 \in \mathbb{C}$ such that $v - \lambda_2 I$ is skew-symmetric. According to Lemma 5.2 (c) that is all we needed to see.

(4) As the final subcase for case $\mathcal{B}$ let $\mathcal{C}$ have some non-neutral non-singleton blocks as well as some non-neutral partitions.

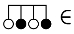
If v is skew-symmetric and small and if $(\mathfrak{c},\mathfrak{d},p) \in \mathcal{C}$ and $h\colon p \to [\![n]\!]$ are arbitrary, then by definition, $\sum_{s=1}^n v_{h(\mathsf{B}),s} = \sum_{s=1}^n v_{s,h(\mathsf{B})} = 0$ and $v_{h(\mathsf{B}),h(\mathsf{B})} = 0$ for any $\mathsf{B} \in p$. For that reason condition (i) of Lemma 6.10 is trivially satisfied. Because also $v_{j,i}+v_{i,j} = 0$ for any $\{i,j\} \subseteq [\![n]\!]$ with $i \neq j$ condition (ii) of Lemma 6.10 is met as well, no matter whether there exists $\mathsf{B} \in p$ with $|\mathsf{B}| = 2$ and $\sigma^{\mathfrak{c}}_{\mathfrak{d}}(\mathsf{B}) \neq 0$. And, of course, condition (iii) of Lemma 6.10 is vacuous for $(\mathfrak{c},\mathfrak{d},p)$ since $\mathcal{C}$ is case $\mathcal{B}$.

In order to prove the converse, let $F^{1,r}_{E,\varnothing,Y}(x^v) = 0$ for any $r \in R$. Since $\mathcal{C}$ has some non-neutral partitions we find a $(\mathfrak{c},\mathfrak{d},p) \in \mathcal{C}$ with $t \equiv |\Sigma^{\mathfrak{c}}_{\mathfrak{d}}| \neq 0$. As $\uparrow\otimes\downarrow \in \mathcal{C}$ we conclude $\uparrow^{\otimes t} \in \mathcal{C}$ by Lemma 3.7 (e). It follows that v is small by Lemma 6.11 (c). Furthermore, the assumption of $\mathcal{C}$ having some non-neutral non-singleton blocks implies the existence of $(\mathfrak{a},\mathfrak{b},q) \in \mathcal{C}$ and $\mathsf{A} \in q$ with $2 \leq |\mathsf{A}|$ and $\sigma^{\mathfrak{a}}_{\mathfrak{b}}(\mathsf{A}) \neq 0$. If now for any $\{i,j\} \subseteq [\![n]\!]$ with $i \neq j$ the mapping $h\colon q \to [\![n]\!]$ is such that $\mathsf{A} \mapsto j$ and $\mathsf{B} \mapsto i$ for any $\mathsf{B} \in p\backslash\{\mathsf{A}\}$ and if $h'\colon q \to [\![n]\!]$ is constant with value i, then condition (i) of Lemma 6.10 implies the identities $\sigma^{\mathfrak{a}}_{\mathfrak{b}}(\mathsf{A})\, v_{j,j} + \sum_{\mathsf{B}\in q \wedge 2\leq|\mathsf{B}| \wedge \mathsf{B}\neq\mathsf{A}} \sigma^{\mathfrak{a}}_{\mathfrak{b}}(\mathsf{B})\, v_{i,i} = 0$ and $\sum_{\mathsf{B}\in q \wedge 2\leq|\mathsf{B}|} \sigma^{\mathfrak{a}}_{\mathfrak{b}}(\mathsf{B})\, v_{i,i} = 0$ because $\sum_{s=1}^n v_{i,s} = \sum_{s=1}^n v_{s,i} = 0$ by v being small. Subtracting the second from the first yieds $\sigma^{\mathfrak{a}}_{\mathfrak{b}}(\mathsf{A})\,(v_{j,j} - v_{i,i}) = 0$ and thus $v_{j,j} = v_{i,i}$ by $\sigma^{\mathfrak{a}}_{\mathfrak{b}}(\mathsf{A}) \neq 0$. Because the presence of A in q also ensures $v_{j,i} + v_{i,j} = 0$ for any $\{i,j\} \subseteq [\![n]\!]$ with $i \neq j$ by condition (ii) of Lemma 6.10 we have thus shown that there exists $\lambda_2 \in \mathbb{C}$ such that $v - \lambda_2 I$ is skew-symmetric by Lemma 5.2 (b). Because v is also small, applying Lemma 5.2 (c) (with $\lambda_1 = 0$) we see that $\lambda_2 = 0$, i.e., that v is skew-symmetric and small.

(iii) If $\mathcal{C}$ is case $\mathcal{H}$, then  $\in \mathcal{C}$ and $2 \leq |\mathsf{B}|$ for any $\mathsf{B} \in p$ and any $(\mathfrak{c}, \mathfrak{d}, p) \in \mathcal{C}$ by Lemma 3.7 (b) and  $\notin \mathcal{C}$.

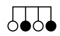
(1) Suppose first that $\mathcal{C}$ has only neutral non-singleton blocks.

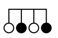
If v is diagonal, if $(\mathfrak{c}, \mathfrak{d}, p) \in \mathcal{C}$ and if $h\colon p \to [\![n]\!]$, then because both $2 \leq |\mathsf{B}|$ and $\sigma_{\mathfrak{d}}^{\mathfrak{c}}(\mathsf{B}) = 0$ for any $\mathsf{B} \in p$ by assumption, condition (i) of Lemma 6.10 is satisfied trivially. For the same reason, condition (ii) of Lemma 6.10 is vacuous. And condition (iii) of Lemma 6.10 is met as well, regardless of whether there is $\mathsf{B} \in p$ with $2 < |\mathsf{B}|$, because v is diagonal per assumption.

If, conversely, $F_{E,\varnothing,Y}^{1,r}(x^v) = 0$ for any $r \in R$, then  $\in \mathcal{C}$ by Lemma 6.11 (a) forces v to be diagonal.

(2) Next, suppose that v has some non-neutral non-singleton blocks but only neutral partitions.

Let $\lambda \in \mathbb{C}$ be such that $v = \lambda I$ and let $(\mathfrak{c}, \mathfrak{d}, p) \in \mathcal{C}$ and $h\colon p \to [\![n]\!]$ be arbitrary. As there are no $\mathsf{B} \in p$ with $|\mathsf{B}| = 1$ condition (i) of Lemma 6.10 demands precisely that the term $\sum_{\mathsf{B}\in p} \sigma_{\mathfrak{d}}^{\mathfrak{c}}(\mathsf{B})\,\lambda = \Sigma_{\mathfrak{d}}^{\mathfrak{c}}\,\lambda$ vanishes, which, of course, it does by $\mathcal{C}$ having only neutral partitions. Moreover, since $v_{j,i} = 0$ for any $\{i, j\} \subseteq [\![n]\!]$ with $i \neq j$ condition (ii) of Lemma 6.10 is certainly satisfied, even if there is $\mathsf{B} \in p$ with $|\mathsf{B}| = 2$ and $\sigma_{\mathfrak{d}}^{\mathfrak{c}}(\mathsf{B}) \neq 0$. The same reason also ensures that condition (iii) of Lemma 6.10 is met, irrespective of whether there exists $\mathsf{B} \in p$ with $2 < |\mathsf{B}|$ or not.

Conversely, if $F_{E,\varnothing,Y}^{1,r}(x^v) = 0$ for any $r \in R$, then v is diagonal by  $\in \mathcal{C}$ according to Lemma 6.11 (a). Moreover, $\mathcal{C}$ having some non-neutral non-singleton blocks lets us find a $(\mathfrak{c}, \mathfrak{d}, p) \in \mathcal{C}$ and an $\mathsf{A} \in p$ with $2 \leq |p|$ and $\sigma_{\mathfrak{d}}^{\mathfrak{c}}(\mathsf{A}) \neq 0$. If, given any $\{i, j\} \subseteq [\![n]\!]$ with $i \neq j$ we let $h\colon p \to [\![n]\!]$ be such that $\mathsf{A} \mapsto j$ and $\mathsf{B} \mapsto i$ for any $\mathsf{B} \in p \backslash \{\mathsf{A}\}$, then condition (i) of Lemma 6.10 lets us know that $\sigma_{\mathfrak{d}}^{\mathfrak{c}}(\mathsf{A})\, v_{j,j} + \sum_{\mathsf{B}\in p \wedge \mathsf{B}\neq\mathsf{A}} \sigma_{\mathfrak{d}}^{\mathfrak{c}}(\mathsf{B})\, v_{i,i} = 0$. Since $\mathcal{C}$ having only neutral partitions implies $0 = \Sigma_{\mathfrak{d}}^{\mathfrak{c}} = \sigma_{\mathfrak{d}}^{\mathfrak{c}}(\mathsf{A}) + \sum_{\mathsf{B}\in p \wedge \mathsf{B}\neq\mathsf{A}} \sigma_{\mathfrak{d}}^{\mathfrak{c}}(\mathsf{B})$ that is the same as saying $\sigma_{\mathfrak{d}}^{\mathfrak{c}}(\mathsf{A})\,(v_{j,j} - v_{i,i}) = 0$, which means $v_{j,j} = v_{i,i}$ by $\sigma_{\mathfrak{d}}^{\mathfrak{c}}(\mathsf{A}) \neq 0$. Hence, if $\lambda = v_{1,1}$, then $v = \lambda I$ as claimed.

(3) Lastly for case $\mathcal{H}$, let $\mathcal{C}$ have non-neutral partitions. If $v = 0$, then conditions (i)–(iii) of Lemma 6.10 are trivially satisfied. We only need to prove the converse. If $F_{E,\varnothing,Y}^{1,r}(x^v) = 0$ for any $r \in R$, then, as before, v is diagonal by  $\in \mathcal{C}$ and Lemma 6.11 (a). By assumption there exists $(\mathfrak{c}, \mathfrak{d}, p) \in \mathcal{C}$ with $\Sigma_{\mathfrak{d}}^{\mathfrak{c}} \neq 0$. Hence, for any $i \in [\![n]\!]$, if $h\colon p \to [\![n]\!]$ is constant with value i, then by $2 \leq |\mathsf{B}|$ for any $\mathsf{B} \in p$ condition (i) of Lemma 6.10 shows that $0 = \sum_{\mathsf{B}\in p} \sigma_{\mathfrak{d}}^{\mathfrak{c}}(\mathsf{B})\, v_{i,i} = \Sigma_{\mathfrak{d}}^{\mathfrak{c}}\, v_{i,i}$, i.e., that $v_{i,i} = 0$. In other words, $v = 0$.

(iv) It only remains to consider the eventuality that $\mathcal{C}$ is case $\mathcal{S}$. There are only two possibilities here to treat.

(1) First, let $\mathcal{C}$ have only neutral partitions.

If there is $\lambda \in \mathbb{C}$ such that $v = \lambda I$ and if $(\mathfrak{c}, \mathfrak{d}, p) \in \mathcal{C}$ and $h\colon p \to [\![n]\!]$ are arbitrary, then what condition (i) of Lemma 6.10 demands is that the sum $\sum_{\mathsf{B}\in p \wedge |\mathsf{B}|=1} \sigma_{\mathfrak{d}}^{\mathfrak{c}}(\mathsf{B})\,\lambda + \sum_{\mathsf{B}\in p \wedge 2\leq|\mathsf{B}|} \sigma_{\mathfrak{d}}^{\mathfrak{c}}(\mathsf{B})\,\lambda = \sum_{\mathsf{B}\in p} \sigma_{\mathfrak{d}}^{\mathfrak{c}}(\mathsf{B})\,\lambda = \Sigma_{\mathfrak{d}}^{\mathfrak{c}}\,\lambda$ vanish. And because $\mathcal{C}$ having neutral partitions implies $\Sigma_{\mathfrak{d}}^{\mathfrak{c}} = 0$ this is indeed the case. Moreover, v being diagonal of course guarantees that conditions (ii) and (iii) of Lemma 6.10 are satisfied, no matter what the blocks of p.

If, conversely, $F^{1,r}_{E,\varnothing,Y}(x^v) = 0$ for any $r \in R$, then v is diagonal by $\sqcap\sqcap\sqcap_{\circ\bullet\circ\bullet} \in \mathcal{C}$ and Lemma 6.11 (a).

(2) The alternative is that $\mathcal{C}$ has some non-neutral partitions.

Of course, if $v = 0$, then conditions (i)–(iii) of Lemma 6.10 are satisfied trivially.

Conversely, assuming $F^{1,r}_{E,\varnothing,Y}(x^v) = 0$ for any $r \in R$, from $\sqcap\sqcap\sqcap_{\circ\bullet\circ\bullet} \in \mathcal{C}$ it again follows that v is diagonal by Lemma 6.11 (a). In addition, for any $i \in [\![n]\!]$, if $(\mathfrak{c}, \mathfrak{d}, p) \in \mathcal{C}$ is such that $\Sigma^{\mathfrak{c}}_{\mathfrak{d}} \neq 0$, as there must exist by assumption, and if $h\colon p \to [\![n]\!]$ is constant with value i, then condition (i) of Lemma 6.10 lets us know that $0 = \sum_{\mathsf{B} \in p \wedge |\mathsf{B}|=1} \sigma^{\mathfrak{c}}_{\mathfrak{d}}(\mathsf{B})\, v_{i,i} + \sum_{\mathsf{B} \in p \wedge 2 \leq |\mathsf{B}|} \sigma^{\mathfrak{c}}_{\mathfrak{d}}(\mathsf{B})\, v_{i,i} = \sum_{\mathsf{B} \in p} \sigma^{\mathfrak{c}}_{\mathfrak{d}}(\mathsf{B})\, v_{i,i} = \Sigma^{\mathfrak{c}}_{\mathfrak{d}}\, v_{i,i}$. Because $\Sigma^{\mathfrak{c}}_{\mathfrak{d}} \neq 0$ that requires $v_{i,i} = 0$ and thus $v = 0$, which concludes the proof. □

Now, we have all the ingredients required to prove the main theorem.

Proof of the Main Result. The claims that the sets of matrices are vector spaces of the given dimensions $\beta_1(\widehat{G})$ were shown in Lemma 5.3. The remainder of the assertions are the combined result of Propositions 6.3 and 6.12. □

7. Remarks on the second cohomology

In the following, let us make the Assumptions 3.8 and use the abbreviations from both Notation 3.9 and Notation 6.1. While, for the first cohomology it was convenient to pull every thing back to $\mathbb{C}\langle E\rangle$ and work with the $\mathbb{C}\langle E\rangle$-bimodule Y, this is much less often so for the second cohomology. Instead, we shall make the following assumptions throughout. Section 7.

Assumptions 7.1. Let $n \in \mathbb{N}$, let $\mathcal{C}$ be any category of two-colored partitions and let $\mathcal{G}$ be any generator set of $\mathcal{C}$.

Notation 7.2. (a) Write G for the unitary easy compact quantum group associated with $(\mathcal{C}, n)$.

(b) Denote

$$\mathcal{R} := \mathcal{G} \cup \{(\bar{\mathfrak{c}}, \bar{\mathfrak{d}}, \bar{p}^*) \,|\, (\mathfrak{c}, \mathfrak{d}, p) \in \mathcal{G}\} \cup \{\sqcap_{\circ\bullet}, \sqcap_{\bullet\circ}\}.$$

(c) And then let

$$R := \{\mathrm{r}^{\mathfrak{c}}_{\mathfrak{d}}(p)_{j,g} \,|\, \{k,\ell\} \subseteq \mathbb{N}_0 \wedge \mathfrak{c} \in \{\circ,\bullet\}^{\otimes k} \wedge \mathfrak{d} \in \{\circ,\bullet\}^{\otimes \ell} \wedge (\mathfrak{c}, \mathfrak{d}, p) \in \mathcal{R} \wedge g \in [\![n]\!]^{\otimes k} \wedge j \in [\![n]\!]^{\otimes \ell}\},$$

(d) Finally, write X for the $\mathbb{C}\langle E\,|\,R\rangle$-module $\mathbb{C}$ equipped with the left and right actions respectively defined by $u^{\mathfrak{c}}_{j,i} \otimes \lambda \mapsto \delta_{j,i}\lambda$ and $\lambda \otimes u^{\mathfrak{c}}_{j,i} \mapsto \delta_{j,i}\lambda$ for any $\lambda \in \mathbb{C}$, any $\mathfrak{c} \in \{\circ,\bullet\}$ and any $\{i,j\} \subseteq [\![n]\!]$.

Lemma 7.3. *For any $g \in [\![n]\!]^{\otimes 2}$ and $j \in [\![n]\!]^{\otimes 2}$, any $\gamma \in [\mathbb{C}\langle E\rangle, \mathbb{C}]^{\times E}$ with $\gamma(1) = 0$ and any $a \in \mathbb{C}\langle E\rangle$, if $\mathring{a} \equiv a - \varepsilon(a)1$, then the following identities hold:*

r	$F^{2,r}_{E,\varnothing,Y}(\gamma)(a)$	$G^{2,r}_{E,\varnothing,Y}(\gamma)$
$\mathrm{r}^{\varnothing}_{\bullet\circ}(\sqcap)_{j,\varnothing}$	$\sum_{i=1}^{n}\gamma_{u^{\bullet}_{j_1,i}}(u^{\circ}_{j_2,i}\mathring{a})+\gamma_{u^{\circ}_{j_2,j_1}}(\mathring{a})$	$\sum_{i=1}^{n}\gamma_{u^{\bullet}_{j_1,i}}(u^{\circ}_{j_2,i})$
$\mathrm{r}^{\varnothing}_{\circ\bullet}(\sqcap)_{j,\varnothing}$	$\sum_{i=1}^{n}\gamma_{u^{\circ}_{j_1,i}}(u^{\bullet}_{j_2,i}\mathring{a})+\gamma_{u^{\bullet}_{j_2,j_1}}(\mathring{a})$	$\sum_{i=1}^{n}\gamma_{u^{\circ}_{j_1,i}}(u^{\bullet}_{j_2,i})$
$\mathrm{r}^{\bullet\circ}_{\varnothing}(\sqcup)_{\varnothing,g}$	$-\sum_{h=1}^{n}\gamma_{u^{\bullet}_{h,g_1}}(u^{\circ}_{h,g_2}\mathring{a})-\gamma_{u^{\circ}_{g_1,g_2}}(\mathring{a})$	$-\sum_{h=1}^{n}\gamma_{u^{\bullet}_{h,g_1}}(u^{\circ}_{h,g_2})$
$\mathrm{r}^{\circ\bullet}_{\varnothing}(\sqcup)_{\varnothing,g}$	$-\sum_{h=1}^{n}\gamma_{u^{\circ}_{h,g_1}}(u^{\bullet}_{h,g_2}\mathring{a})-\gamma_{u^{\bullet}_{g_1,g_2}}(\mathring{a})$	$-\sum_{h=1}^{n}\gamma_{u^{\circ}_{h,g_1}}(u^{\bullet}_{h,g_2})$

PROOF. Starting from Lemma 3.10 all the claims follow immediately from Definition 4.27. □

PROPOSITION 7.4. *There exists an isomorphism of* $\mathbb{C}$*-vector spaces*

$$H^2(\widehat{G}) \hookrightarrow\!\!\!\!\rightarrow \frac{\begin{array}{r}\{\gamma\in[\mathbb{C}\langle E\,|\,R\rangle,\mathbb{C}]^{\times E}\,\wedge\,\forall e\in E\colon\ \gamma_e(1+J)=0\\ \wedge\ \forall r\in R\colon\ F^{2,r}_{E,R,X}(\gamma)=0\}\end{array}}{\begin{array}{l}\{\gamma\ \text{as above}\ \wedge\ \exists x\in\mathbb{C}^{\times E}\colon\\ \qquad\forall r\in R\colon\ G^{2,r}_{E,R,X}(\gamma)=F^{1,r}_{E,R,X}(x)\}\end{array}},$$

where the sets on the right-hand side are $\mathbb{C}$*-vector subspaces of* $[\mathbb{C}\langle E\,|\,R\rangle,\mathbb{C}]^{\times E}$*, which for any* $c\in Z^2(\widehat{G})$ *assigns to the class of* c *the class of the tuple* γ *with* γ_e *for each* $e\in E$ *being the mapping with*

$$a\mapsto c((e+J)\otimes a)-(e+J)\rhd c((1+J)\otimes(1+J))\lhd a$$

for any $a\in\mathbb{C}\langle E\,|\,R\rangle$*. For any* $\gamma\in[\mathbb{C}\langle E\,|\,R\rangle,\mathbb{C}]^{\times E}$ *with* $\gamma_e(1+J)=0$ *for any* $e\in E$ *and with* $F^{2,r}_{E,R,X}(\gamma)=0$ *for each* $r\in R$ *the inverse isomorphism maps the class of* γ *to the class of the unique* $\mathbb{C}$*-linear map* $\mathbb{C}\langle E\,|\,R\rangle\otimes\mathbb{C}\langle E\,|\,R\rangle\to\mathbb{C}$ *with*

$$(p+J)\otimes a\mapsto F^{2,p}_{E,R,X}(\gamma)(a)$$

for any $a\in\mathbb{C}\langle E\,|\,R\rangle$ *and* $p\in\mathbb{C}\langle E\rangle$*.*

PROOF. Follows immediately from Definition 3.11 Proposition 4.32. □

Now, the results of Lemma 6.8 tell us a lot about the right-hand sides of the equations determining the denominator in the quotient from Proposition 7.4.

CHAPTER 5

L^2-Betti numbers of certain unitary easy quantum group duals

As shown by Bichon, Kyed and Raum it is possible to determine the L^2-Betti numbers of the discrete duals of certain unitary easy compact quantum groups by presenting those compact quantum groups as graded twists by the cyclic group of order two of the free product of two copies of appropriately chosen orthogonal easy compact quantum groups. In particular, that is how the higher L^2-Betti numbers of the free unitary quantum group were first computed by the three authors. The present chapter is a very minor extension of their work. By applying exactly the same method as Bichon, Kyed and Raum the L^2-Betti numbers of six near-free unitary easy quantum groups are computed.

1. Introduction

1.1. Background and Context. Easy quantum groups, introduced in the orthogonal case in [BS09] and in the unitary case in [TW18], are particular compact quantum groups in Woronowicz's sense, as defined in [Wor87; Wor91; Wor98].

They result by applying the Tannaka-Krein duality theorem from [Wor88] to so-called categories of uncolored respectively two-colored partitions. These are combinatorial in nature: With horizontal concatenation as composition, vertical concatenation as tensor product and horizontal reflection as star operation, the set of all set-theoretical partitions of any two distinguished disjoint totally ordered sets, addtionally equipped with a bivalent map in the two-colored case, becomes a rigid monoidal $*$-category. Finding rigid monoidal $*$-subcategories then yields the easy quantum groups, compact quantum subgroups of Wang's free orthogonal respectively unitary quantum groups, which were defined in [Wan95a].

While this construction produces vast numbers of compact quantum groups of which there was a distinct dirth of examples beforehand, there is a significant disadvantage to it. It is difficult to say which of the resulting quantum groups are isomorphic and to relate them to compact quantum groups defined by other means than Tannaka-Krein reconstruction. One way of addressing this issue is to study the quantum group invariants of easy quantum groups such as the L^2-Betti numbers of their discrete duals.

These invariants were first introduced for quantum groups by Kyed in [Kye08b] but have been studied for Riemannian manifolds since the work of Atiyah in [Ati76] and for discrete groups since Cheeger and Gromov's work in [CG86]. Thus, highly analytical in their origins L^2-Betti numbers at first glance may seem difficult to link to the combinatorics of easy quantum groups. But in [Lüc97; Lüc98a; Lüc98b] Lück managed to give a far more algebraic definition of L^2-Betti numbers, opening the door to the use of classical results from homological algebra.

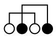
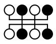
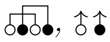
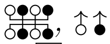
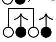
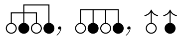
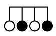
It is via this route, also incorporating extensions by Thom from [Tho08] that Kyed and Raum computed the first L^2-Betti number of the discrete dual of the free unitary quantum group in [KR17]. Further refining this approach Bichon, Kyed and Raum were able to compute also the higher order numbers of that quantum group in [BKR18]. The prior corresponding computation for the case of the free orthogonal quantum group by Vergnioux, Collins, Härtel and Thom in [Ver12] and [CHT09] had made use of very intricate custom-made tools, quantum cayley trees, and even computer algebra. But using these results, Bichon, Kyed and Raum managed to do their computations by more or less purely algebraic considerations.

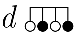
Besides the free unitary quantum group there were two other unitary easy quantum groups of whose discrete duals Bichon, Kyed and Raum determined all L^2-Betti numbers. The method of proof was the same in all three cases. The present work is merely the observation of the fact, of which Bichon, Kyed and Raum were surely aware themselves, that their proof strategy applies to even more unitary easy quantum groups than these three. More precisely, the L^2-Betti numbers of six other unitary quantum groups are computed in this chapter. For the discrete dual of a seventh unitary easy quantum group Bichon, Kyed and Raum's method can at least be used to express the L^2-Betti numbers through those of a certain orthogonal easy quantum group which might be easier to compute.

It should be noted that after their work in [BKR18] Kyed and Raum together with Vaes and Valvekens found a way to generalize their method even further in [Kye+17] by using the L^2-theory of quasi-regular inclusions of von Neumann algebras developed by Popa, Shlyakhtenko and Vaes in [PSV18]. Likely, it is possible to give an alternative proof of the results in this article be using this method. See Section 8 for more details.

1.2. Main Result. The results represented by the first three rows of the below table were proved by Bichon, Kyed and Raum in [BKR18]. To my knowledge, the remaining ones are new.

MAIN RESULT. *For any $p \in \mathbb{N}_0$, any $n \in \mathbb{N}$ and any of the sets $\mathcal{G}_G$ of two-colored partitions listed below the discrete dual of the unitary easy compact quantum group G associated with the category $\langle \mathcal{G}_G \rangle$ of two-colored partitions generated by $\mathcal{G}_G$ and with n has the following p-th L^2-Betti number:*

G	$\mathcal{G}_G$	$\beta_p^{(2)}(\widehat{G})$
		$(1-\delta_{n,1})\delta_{p,1}$
$U_{\mathbb{N},n}^{\times}$		$(1-\delta_{n,1})\delta_{p,1}$
$U_n^+ = U_{\mathbb{N},n}^{\times +}$	$\varnothing$	$(1-\delta_{n,1})\delta_{p,1}$
		$(1-\delta_{n,1}-\frac{1}{2}\delta_{n,2})\delta_{p,1}$
		$2b_p+(1-2b_0)\delta_{p,1}-2b_0\delta_{p,0}$
		$(2-2\delta_{n,1}-\delta_{n,2})\delta_{p,1}$
		$(1-\delta_{n,1}-\frac{1}{2}\delta_{n,2})\delta_{p,1}$
		$(1-\frac{1}{n!})\delta_{p,1}$
		$(1-\delta_{n,1}-\frac{1}{2}\delta_{n,2}-\frac{1}{3}\delta_{n,6})\delta_{p,1}$
		$(2-\delta_{n,1}-\delta_{n,2}-\delta_{n,3})\delta_{p,1}$

Here, $(b_p)_{p\in\mathbb{N}_0}$ are the L^2-Betti numbers of the discrete dual of $B_n^{\#*}$, the half-liberated bistochastic quantum group introduced in [Web13, Definition 3.2 (3)], and  *is the only partition with a block of size greater than two.*

1.3. Structure of the Chapter. Section 2 recalls the necessary definitions of Hopf $*$-algebras and (algebraic) compact quantum groups, including direct and free products of the latter.

The key notion used in the proof given by Bichon, Kyed and Raum is that of a graded twist of a Hopf $*$-algebra by an invariant co-central action of a discrete group. The required definitions and results about this construction are provided in Section 3.

Following that, the concepts and theorems about both orthogonal and unitary easy compact quantum groups which are needed in order to express and prove the main result are given in Section 4.

Section 5 treats L^2-Betti numbers of the discrete duals of compact quantum groups.

The strategy employed by Bichon, Kyed and Raum is summarized in Section 6. Proposition 6.1 lists all the statements about the input quantum group one needs to prove in order to infer the L^2-Betti numbers of its discrete dual.

That the new inputs to Bichon, Kyed and Raum's method actually meet its requirements is then verified in Section 7, proving the main result.

Section 8 closes with a few remarks, including one about a conjectured alternative proof of the main theorem.

2. Compact quantum groups

In the following, all vector spaces are complex and all (*-)algebras unital.

DEFINITION 2.1. (a) A *Hopf $*$-algebra* is any $*$-algebra A additionally equipped with three mappings Δ, ϵ and S such that, if m is the multiplication of A and 1 the unit, then

(i) Δ, the *comultiplication*, is a linear map $A \to A \otimes A$ and a $*$-algebra morphism from A to the tensor product of A with itself,

(ii) ϵ, the *counit*, is a linear functional on A and a $*$-algebra morphism from A to $\mathbb{C}$,
(iii) S is a linear map $A \to A$, the *antipode* or *coinverse*,
(iv) $(\mathrm{id}_A \otimes \Delta) \circ \Delta = (\Delta \otimes \mathrm{id}_A) \circ \Delta$,
(v) $\mathrm{id}_A = (\epsilon \otimes \mathrm{id}_A) \circ \Delta$,
(vi) $\mathrm{id}_A = (\mathrm{id}_A \otimes \epsilon) \circ \Delta$,
(vii) $m \circ (S \otimes \mathrm{id}_A) \circ \Delta = 1 \circ \epsilon$,
(viii) $m \circ (\mathrm{id}_A \otimes S) \circ \Delta = 1 \circ \epsilon$.

(b) A *morphism of Hopf-$*$-algebras* from any Hopf $*$-algebra A with co-multiplication Δ, co-unit ϵ and antipode S to any Hopf $*$-algebra A' with co-multplication Δ', co-unit ϵ' and antipode S' is any morphism f of $*$-algebras from A to A' such that
(i) $(f \otimes f) \circ \Delta = \Delta' \circ f$,
(ii) $\epsilon' \circ f = \epsilon$,
(iii) $f \circ S = S' \circ f$,

(c) For any Hopf-$*$-algebra A with co-multiplication Δ and unit 1 an *integral* of A is any faithful state h such that
(i) $1 \circ h = (h \otimes \mathrm{id}_A) \circ \Delta$,
(ii) $1 \circ h = (\mathrm{id}_A \otimes h) \circ \Delta$.

Antipodes are anti-multiplicative. On any Hopf $*$-algebra there can exist at most one integral.

DEFINITION 2.2. (a) An (algebraic) *compact quantum group* G is the formal dual of any Hopf $*$-algebra, denoted by $\mathrm{Pol}(G)$, which admits an integral. Equivalently, we say that $\mathrm{Pol}(G)$ is a *CQG algebra*.
(b) For any compact quantum groups G' and G a *morphism* u *of* (algebraic) *compact quantum groups* from G' to G is the formal dual of any morphism of Hopf $*$-algebras, denoted by $\mathrm{Pol}(u)$, from G to G'.

DEFINITION 2.3. (a) Any compact quantum group G is said to be of *Kac type* if the antipode of $\mathrm{Pol}(G)$ is involutive.
(b) For each $g \in [\![2]\!]$ let G_g be any compact quantum group, A_g its formal dual and Δ_g, ϵ_g and S_g, respectively, the co-multplication, co-unit and antipode of A_g.
(i) The *direct product* $G_1 \hat{\times} G_2$ of (G_1, G_2) is the formal dual of the CQG algebra given by the tensor product $*$-algebra $A_1 \otimes A_2$ equipped with the co-multiplication $(\mathrm{id} \otimes \gamma_{A_1,A_2} \otimes \mathrm{id}) \circ (\Delta_1 \otimes \Delta_2)$, where γ_{A_1,A_2} is the linear isomorphism which swaps the two tensor factors in $A_1 \otimes A_2$, with the co-unit $\epsilon_1 \otimes \epsilon_2$ and with the antipode $S_1 \otimes S_2$.
(ii) The *free product* $G_1 \hat{*} G_2$ of (G_1, G_2) is the formal dual of the CQG algebra given by the free product (i.e., categorically speaking, co-product) $*$-algebra $A_1 * A_2$ equipped with the co-multiplication given by the unique $*$-algebra morphism Δ from $A_1 * A_2$ to $(A_1 * A_2) \otimes (A_1 * A_2)$

with $\Delta \circ \iota_g = (\iota_g \otimes \iota_g) \circ \Delta_g$, where ι_g is the co-projection of A_g into $A_1 * A_2$, for each $g \in [\![2]\!]$, with the co-unit given by the unique $*$-algebra morphism ϵ from $A_1 * A_2$ to $\mathbb{C}$ with $\epsilon \circ \iota_g = \epsilon_g$ for each $g \in [\![2]\!]$ and with the unique anti-multiplicative linear endomorphism S of $A_1 * A_2$ with $S \circ \iota_g = \iota_g \circ S_g$ for each $g \in [\![2]\!]$.

The direct or free product of two Kac type compact quantum groups is evidently again of Kac type.

3. Graded Twists

The key notion used in Bichon, Kyed and Raum's strategy in [BKR18] is that of a graded twist, to be explained in this section. As a background the following general Hopf algebra terms are needed (see [AD95]).

Definition 3.1. (a) Any Hopf $*$-algebra morphism $p\colon A \to B$ for any Hopf $*$-algebras A and B is called *co-central* if

$$(\mathrm{id}_A \otimes p) \circ \Delta_A = (\mathrm{id}_A \otimes p) \circ \gamma_{A,A} \circ \Delta_A,$$

where $\gamma_{A,A}$ is the symmetry of (A, A), i.e., the linear endomorphism of $A \otimes A$ with $a_1 \otimes a_2 \mapsto a_2 \otimes a_1$ for any $\{a_1, a_2\} \subseteq A$, and where Δ_A is the co-multiplication of A.

(b) For any Hopf $*$-algebras A and B any Hopf $*$-algebra morphism $p\colon A \to B$ the *Hopf kernel* of p is the Hopf $*$-subalgebra $\mathrm{hker}(p)$ of A formed by the kernel of the linear map

$$(\mathrm{id}_A \otimes (p - 1_B \epsilon_A) \otimes \mathrm{id}_A) \circ (\Delta_A \otimes \mathrm{id}_A) \circ \Delta_A$$

where Δ_A and ϵ_A are the co-multplication and co-unit of A, respectively, and where 1_B is the unit of B.

(c) Any short sequence of Hopf $*$-algebras

$$\mathbb{C} \longrightarrow A \xrightarrow{\;i\;} B \xrightarrow{\;p\;} C \longrightarrow \mathbb{C}$$

is called *exact* if, where
- □ $\ker(p)$ is the kernel of p as a linear map,
- □ $\mathrm{im}(i)$ is the image of p as a linear map,
- □ $i(A)^+$ is the subspace of B formed by the set $\{i(a) \,|\, a \in A \wedge \epsilon_A(a) = 0\}$, where ϵ_A is the co-unit of A,
- □ $Bi(A)^+$ and $i(A)^+B$ are the subspaces of B generated by the sets $\{bb_0 \,|\, b \in B \wedge b_0 \in i(A)^+\}$ respectively $\{b_0 b \,|\, b \in B \wedge b_0 \in i(A)^+\}$,
- □ $B^{\mathrm{co}C}$ and ${}^{\mathrm{co}C}B$ are the subspaces of B formed by $\{b \in B \wedge ((\mathrm{id} \otimes p) \circ \Delta_B)(b) = b \otimes 1\}$ respectively $\{b \in B \wedge ((\mathrm{id} \otimes p) \circ \gamma_{B,B} \circ \Delta_B)(b) = b \otimes 1\}$, where Δ_B is the co-multiplication of B and where $\gamma_{B,B}$ is the symmetry of (B, B),

the following conditions are satisfied cumulatively:

(i) i is injective,

(ii) p is surjective,
(iii) $\ker(p) = Bi(A)^+ = i(A)^+B$,
(iv) $\mathrm{im}(i) = B^{\mathrm{co}C} = {}^{\mathrm{co}C}B$.

The following definitions and the results ensuring they make sense are taken from [BNY16].

DEFINITION 3.2. Let A be any Hopf $*$-algebra, Δ its co-multiplication, ϵ its co-unit, S its antipode, Γ any discrete group and e its neutral element.

(a) A *Hopf $*$-algebra Γ-grading* of A is any family $(A_g)_{g\in\Gamma}$ of vector subspaces of A such that $A \cong \bigoplus_{g\in\Gamma} A_g$ as vector spaces, such that $\Delta(a) \in A_g \otimes A_g$ for any $a \in A_g$ and $g \in \Gamma$, such that $ab \in A_{gh}$ for any $a \in A_g$, any $h \in A_h$ and any $\{g, h\} \subseteq \Gamma$, and such that $a^* \in A_{g^{-1}}$ for any $a \in A_g$ and $g \in \Gamma$.

(b) For any co-central Hopf $*$-algebra morphism $p\colon A \to \mathbb{C}[\Gamma]$, if

$$A_g \coloneqq \{a \in A \wedge ((\mathrm{id}_A \otimes p) \circ \Delta)(a) = a \otimes g\}$$

for any $g \in \Gamma$, then the Hopf $*$-algebra Γ-grading of A given by $(A_g)_{g\in\Gamma}$ is called the *grading associated with p*.

(c) For any action α of Γ on A by Hopf $*$-algebra automorphisms the *crossed product $A \rtimes_\alpha \mathbb{C}[\Gamma]$* of A and $\mathbb{C}[\Gamma]$ with respect to α is the Hopf $*$-algebra given by the vector space $A\otimes\mathbb{C}[\Gamma]$ equipped with the multiplication defined by

$$(a \otimes g) \otimes (b \otimes h) \mapsto a\alpha_g(b) \otimes gh$$

for any $\{a, b\} \subseteq A$ and $\{g, h\} \subseteq \Gamma$, with the unit $1 \otimes e$, with the co-multiplication defined by

$$a \otimes g \mapsto (\mathrm{id} \otimes \gamma_{A,\mathbb{C}[\Gamma]} \otimes \mathrm{id})(\Delta(a) \otimes g \otimes g)$$

for any $a \in A$ and $g \in \Gamma$, where $\gamma_{A,\mathbb{C}[\Gamma]}$ is the mapping which swaps the two tensor factors in $A \otimes \mathbb{C}[\Gamma]$, with the co-unit defined by

$$a \otimes g \mapsto \epsilon(a)$$

for any $a \in A$ and $g \in \Gamma$, with the antipode defined by

$$(a \otimes g) \mapsto S(\alpha_{g^{-1}}(a)) \otimes g^{-1}$$

for any $a \in A$ and $g \in \Gamma$ and with the $*$-operation defined by

$$(a \otimes g) \mapsto \alpha_{g^{-1}}(a^*) \otimes g^{-1}$$

for any $a \in A$ and $g \in \Gamma$.

(d) For any action α of Γ on A by Hopf $*$-algebra automorphisms any Hopf $*$-algebra Γ-grading $(A_g)_{g\in\Gamma}$ of A is called *α-invariant* if $\alpha_h(a) \in A_g$ for any $a \in A_g$ and $\{g, h\} \subseteq \Gamma$.

(e) Any pair (p, α) of a co-central Hopf $*$-algebra morphism $p\colon A \to \mathbb{C}[\Gamma]$ and an action α of Γ on A by Hopf $*$-algebra automorphisms is called an *invariant co-central action* of Γ on A if the grading associated with p is α-invariant.

The next proposition is a combination of [BNY16, Lemma 2.13] and [BNY18, Proposition 1.2].

PROPOSITION 3.3. *Let Γ be any discrete group, e its neutral element, A any Hopf $*$-algebra, Δ its co-multiplication, ϵ its co-unit, p any surjective co-central Hopf $*$-algebra morphism $A \to \mathbb{C}[\Gamma]$, and $(A_a)_{a\in\Gamma}$ the associated grading.*

(a) Then the following are true.

(i) The sequence of Hopf $*$-algebras

$$\mathbb{C} \hookrightarrow \mathrm{hker}(p) \overset{\subseteq}{\hookrightarrow} A \overset{p}{\twoheadrightarrow} \mathbb{C}[\Gamma] \twoheadrightarrow \mathbb{C}.$$

is exact.

(ii) $\mathrm{hker}(p)$ is the Hopf $*$-subalgebra of A formed by the space A_e.

(iii) If A is a CQG algebra, then so is $\mathrm{hker}(p)$.

(b) For any action α of Γ on A by Hopf $*$-automorphisms which turns (p,α) into a co-central invariant action the following are true.

(i) There also exists an exact sequence of Hopf $*$-algebras

$$\mathbb{C} \hookrightarrow \mathrm{hker}(p) \overset{j}{\hookrightarrow} A^{t,(p,\alpha)} \overset{\tilde{p}}{\twoheadrightarrow} \mathbb{C}[\Gamma] \twoheadrightarrow \mathbb{C},$$

where j is defined by $a \mapsto a\otimes e$ for any $a \in \mathrm{hker}(p)$ and where $\tilde{p}$ is given by $a \otimes g \mapsto \epsilon(a)g$ for any $a \in A_g$ and $g \in \Gamma$.

(ii) If A is a CQG algebra, then so is $A^{t,(p,\alpha)}$.

NOTATION 3.4. In the context of any discrete group Γ denote by $-\iota$ the inversion of Γ, i.e., the mapping $\Gamma \to \Gamma$ with $g \mapsto g^{-1}$ for any $g \in \Gamma$.

Unwinding the definitions used in the statement of [BNY16, Lemma 2.14] and transfer to Hopf $*$-algebras produces the following lemma.

PROPOSITION 3.5. *For any invariant co-central actions (p,α) and (q,β) of any discrete abelian group Γ on any Hopf $*$-algebras A respectively B by Hopf $*$-algebra automorphisms and any Hopf $*$-algebra morphism $f\colon B \to A^{t,(p,\alpha)}$, if Δ_A and ϵ_A are the co-multiplication respectively co-unit of A, if Δ_B and ϵ_B are those of B, if Δ is the co-multiplication of $\mathbb{C}[\Gamma]$, i.e., the linear map with $g \mapsto g\otimes g$ for any $g \in \Gamma$, if ι and ι' are the inclusions of $A^{t,(p,\alpha)}$ in $A \rtimes_\alpha \mathbb{C}[\Gamma]$ respectively $B^{t,(q,\beta(-\iota))}$ in $B \rtimes_{\beta(-\iota)} \mathbb{C}[\Gamma]$, and if there exists a Hopf $*$-algebra morphism $f'\colon A \to B^{t,(q,\beta(-\iota))}$ such that*

(a) $(\epsilon_A \otimes \mathrm{id}) \circ \iota \circ f = q$,

(b) $(\epsilon_B \otimes \mathrm{id}) \circ \iota' \circ f' = p$,

(c) $(\alpha_h \otimes \mathrm{id}) \circ \iota \circ f = \iota \circ f \circ \beta_h$ for any $h \in \Gamma$,

(d) $(\beta_{g^{-1}} \otimes \mathrm{id}) \circ \iota' \circ f' = \iota' \circ f' \circ \alpha_g$ for any $g \in \Gamma$,

(e) $((\iota' \circ f') \otimes \mathrm{id}) \circ \iota \circ f = (\mathrm{id} \otimes (\Delta \circ q)) \circ \Delta_B$,

(f) $((\iota \circ f) \otimes \mathrm{id}) \circ \iota' \circ f' = (\mathrm{id} \otimes (\Delta \circ p)) \circ \Delta_A$,

then f is an isomorphism of Hopf $*$-algebras $B \to A^{t,(p,\alpha)}$.

4. Easy quantum groups

Let ${}^{\blacksquare}(\cdot)$ and ${}_{\blacksquare}(\cdot)$ be two fixed injections with disjoint ranges and common domain $\mathbb{N}$. Moreover, let $\{\circ, \bullet\}$ be any two-elemental set.

DEFINITION 4.1. (a) A (uncolored) *partition* is any triple (k, ℓ, p) such that $\{k, \ell\} \subseteq \mathbb{N}_0$ and such that p, the collection of *blocks*, is a set-theoretical partition of (i.e., quotient set of an equivalence relation on) the set $\Pi_\ell^k \coloneqq \{{}^{\blacksquare}i\}_{i=1}^k \uplus \{{}_{\blacksquare}j\}_{j=1}^\ell$, the collection of *points*, consisting of the *upper points* $\{{}^{\blacksquare}i\}_{i=1}^k$ and the *lower points* $\{{}_{\blacksquare}j\}_{j=1}^\ell$.

(b) Similarly, a *two-colored partition* is any triple $(\mathfrak{c}, \mathfrak{d}, p)$ for which there exist $\{k, \ell\} \subseteq \mathbb{N}_0$ such that (k, ℓ, p) is an uncolored partition, and such that $\mathfrak{c}$ and $\mathfrak{d}$ are k- respectively ℓ-tuples of elements of $\{\circ, \bullet\}$, the *colorings* of the upper and lower points, respectively.

Uncolored or two-colored partitions can be depicted graphically. The points are arranged in two rows, the upper ones above the lower ones. Curves are drawn between points belonging to the same block. It will be clear from the context whether intersecting curves indicate that more than two points are included in a common block or that two or more distinct blocks "cross" each other.

DEFINITION 4.2. (a) A *category of partitions* is any set $\mathcal{C}$ of partitions such that the following conditions are met:

(i) $\mathcal{C}$ contains $|$ and $\sqcap$ and $\sqcup$.

(ii) $(\ell, k, p^*) \in \mathcal{C}$ for any $(k, \ell, p) \in \mathcal{C}$, where $p^* \coloneqq \{\{{}^{\blacksquare}j \,|\, j \in [\![\ell]\!] \wedge {}_{\blacksquare}j \in \mathsf{B}\} \uplus \{{}_{\blacksquare}i \,|\, i \in [\![k]\!] \wedge {}^{\blacksquare}i \in \mathsf{B}\}\}_{\mathsf{B} \in p}$ is the *adjoint* of p.

(iii) $(k_1 + k_2, \ell_1 + \ell_2, p_1 \otimes p_2) \in \mathcal{C}$ for any $(k_1, \ell_1, p_1) \in \mathcal{C}$ and $(k_2, \ell_2, p_2) \in \mathcal{C}$, where $p_1 \otimes p_2 \coloneqq p_1 \uplus \{\{{}^{\blacksquare}(k_1 + i) \,|\, i \in [\![k_2]\!] \wedge {}^{\blacksquare}i \in \mathsf{B}\} \uplus \{{}_{\blacksquare}(\ell_1 + j) \,|\, j \in [\![\ell_2]\!] \wedge {}_{\blacksquare}j \in \mathsf{B}\}\}_{\mathsf{B} \in p_2}$ is the *tensor product* of (p_1, p_2).

(iv) $(k, m, qp) \in \mathcal{C}$ for any $(k, \ell, p) \in \mathcal{C}$ and $(\ell, m, q) \in \mathcal{C}$, where, if s is the join of the two set-theoretical partitions $\{\{j \in [\![\ell]\!] \wedge {}_{\blacksquare}j \in \mathsf{A}\}\}_{\mathsf{A} \in p}$ and $\{\{i \in [\![\ell]\!] \wedge {}^{\blacksquare}i \in \mathsf{C}\}\}_{\mathsf{C} \in q}$ of $[\![\ell]\!]$, i.e., the quotient set of the finest equivalence relation on $[\![\ell]\!]$ containing the equivalence relations associated with these two, then $qp \coloneqq \{\mathsf{A} \in p \wedge \mathsf{A} \subseteq \Pi_0^k\} \uplus \{\mathsf{C} \in q \wedge \mathsf{C} \subseteq \Pi_0^m\} \uplus \{\biguplus\{\mathsf{A} \cap \Pi_0^k \,|\, \mathsf{A} \in p \wedge \exists j \in B\colon {}_{\blacksquare}j \in \mathsf{A}\} \uplus \biguplus\{\mathsf{C} \cap \Pi_m^0 \,|\, \mathsf{C} \in q \wedge \exists i \in B\colon {}^{\blacksquare}i \in \mathsf{C}\}\}_{B \in s} \backslash \{\varnothing\}$ is the *composition* of (q, p).

(b) Similarly, a *category of two-colored partitions* is any set $\mathcal{C}$ of two-colored partitions with the ensuing properties.

(i) $\mathcal{C}$ contains $\overset{\circ}{\underset{\circ}{|}}$, $\overset{\bullet}{\underset{\bullet}{|}}$, $\sqcap_{\circ\bullet}$, $\sqcap_{\bullet\circ}$, $\sqcup^{\circ\bullet}$ and $\sqcup^{\bullet\circ}$.

(ii) $(\mathfrak{d}, \mathfrak{c}, p^*)$ for any $(\mathfrak{c}, \mathfrak{d}, p) \in \mathcal{C}$.

(iii) $(\mathfrak{c}_1 \otimes \mathfrak{c}_2, \mathfrak{d}_1 \otimes \mathfrak{d}_2, p_1 \otimes p_2) \in \mathcal{C}$ for any $(\mathfrak{c}_1, \mathfrak{d}_1, p_1) \in \mathcal{C}$ and $(\mathfrak{c}_2, \mathfrak{d}_2, p_2) \in \mathcal{C}$, where, if k_t and ℓ_t are the lengths of the tuples $\mathfrak{c}_t$ respectively $\mathfrak{d}_t$ for each $t \in [\![2]\!]$, then $\mathfrak{c}_1 \otimes \mathfrak{c}_2$ is a $(k_1 + k_2)$-tuple in $\{\circ, \bullet\}$ with $i \mapsto \mathfrak{c}_1(i)$ if $i \le k_1$ and $i \mapsto \mathfrak{c}_2(i - k_1)$ if $k_1 < i$ and, analogously, $\mathfrak{d}_1 \otimes \mathfrak{d}_2$ is a $(\ell_1 + \ell_2)$-tuple with $j \mapsto \mathfrak{d}_1(j)$ if $j \le \ell_1$ and $j \mapsto \mathfrak{d}_2(j - \ell_1)$ if $\ell_1 < j$.

(iv) $(\mathfrak{c},\mathfrak{e},qp)\in\mathcal{C}$ for any $(\mathfrak{c},\mathfrak{d},p)\in\mathcal{C}$ and $(\mathfrak{d},\mathfrak{e},q)\in\mathcal{C}$.

(c) Finally, for any set $\mathcal{G}$ of only uncolored or only two-colored partitions we write $\langle\mathcal{G}\rangle$ for the intersection of all categories of uncolored respectively two-colored partitions containing $\mathcal{G}$ and we say that $\mathcal{G}$ *generates* $\langle\mathcal{G}\rangle$.

Categories of partitions can be used to construct compact quantum groups in the following way.

NOTATION 4.3. Let $n\in\mathbb{N}$ be arbitrary, let $\{v_{j,i}\}_{i,j=1}^n$ and $\{u^\circ_{j,i},u^\bullet_{j,i}\}_{i,j=1}^n$ be any n^2-respectively $2n^2$-elemental sets, and let $v \coloneqq (v_{j,i})_{(j,i)\in[\![n]\!]^{\otimes n}}$ and $u^\circ \coloneqq (u^\circ_{j,i})_{(j,i)\in[\![n]\!]^{\otimes 2}}$ and $u^\bullet \coloneqq (u^\bullet_{j,i})_{(j,i)\in[\![n]\!]^{\otimes 2}}$. Moreover, let $\{k,\ell\}\subseteq\mathbb{N}_0$ and let p be any set-theoretical partition of Π^k_ℓ.

(a) For any $e\in[\![n]\!]^{\otimes k}$ and any $f\in[\![n]\!]^{\otimes\ell}$ let the number $\zeta(p,\ker(e\,{}^{\blacksquare}{}_{\blacksquare}\,f))$ be either 0 or 1 and let it be 1 if and only if for any $\{i,i'\}\subseteq[\![k]\!]$, whenever there is $\mathsf{B}\in p$ with $\{{}^{\blacksquare}i,{}^{\blacksquare}i'\}\subseteq\mathsf{B}$, then $e_i=e_{i'}$, and for any $\{j,j\}\subseteq[\![\ell]\!]$, whenever there is $\mathsf{B}\in p$ with $\{{}_{\blacksquare}j,{}_{\blacksquare}j'\}\subseteq\mathsf{B}$, then $f_j=f_{j'}$, and for any $i\in[\![k]\!]$ and any $j\in[\![\ell]\!]$, whenever there exists $\mathsf{B}\in p$ with $\{{}^{\blacksquare}i,{}_{\blacksquare}j\}\subseteq\mathsf{B}$, then $e_i=f_j$.

(b) For any $g\in[\![n]\!]^{\otimes k}$ and any $j\in[\![n]\!]^{\otimes\ell}$ let $\mathrm{r}^k_\ell(p)_{j,g}(v)$ be the polynomial in the indeterminates $\{v_{j,i}\}_{i,j=1}^n$ given by

$$\textstyle\sum_{i\in[\![n]\!]^{\otimes\ell}}\zeta(p,\ker(g\,{}^{\blacksquare}{}_{\blacksquare}\,i))\overrightarrow{\prod}_{b=1}^{\ell}v_{j_b,i_b} \\ \qquad\qquad -\textstyle\sum_{h\in[\![n]\!]^{\otimes k}}\zeta(p,\ker(h\,{}^{\blacksquare}{}_{\blacksquare}\,j))\overrightarrow{\prod}_{a=1}^{k}v_{h_a,g_a}.$$

(c) For any $\mathfrak{c}\in\{\circ,\bullet\}^{\otimes k}$, any $\mathfrak{d}\in\{\circ,\bullet\}^{\otimes\ell}$, any $g\in[\![n]\!]^{\otimes k}$ and any $j\in[\![n]\!]^{\otimes\ell}$, let $\mathrm{r}^{\mathfrak{c}}_{\mathfrak{d}}(p)_{j,g}(u^\circ,u^\bullet)$ be the polynomial in the indeterminates $\{u^\circ_{j,i},u^\bullet_{j,i}\}_{i,j=1}^n$ given by

$$\textstyle\sum_{i\in[\![n]\!]^{\otimes\ell}}\zeta(p,\ker(g\,{}^{\blacksquare}{}_{\blacksquare}\,i))\overrightarrow{\prod}_{b=1}^{\ell}u^{\mathfrak{d}_b}_{j_b,i_b} \\ \qquad\qquad -\textstyle\sum_{h\in[\![n]\!]^{\otimes k}}\zeta(p,\ker(h\,{}^{\blacksquare}{}_{\blacksquare}\,j))\overrightarrow{\prod}_{a=1}^{k}u^{\mathfrak{c}_a}_{h_a,g_a}.$$

DEFINITION 4.4. Let n be arbitrary.

(a) For any category $\mathcal{C}$ of uncolored partitions the *orthogonal easy* compact quantum group associated with $\mathcal{C}$ and n is the formal dual of the CQG algebra given by the universal $*$-algebra A with n^2 many generators $\{v_{j,i}\}_{i,j=1}^n$ subject to the relations

$$\{v^*_{j,i}-v_{j,i}\}_{i,j=1}^n \\ \qquad \cup\{\mathrm{r}^k_\ell(p)_{j,g}(v)\,|\,(k,\ell,p)\in\mathcal{G}\cup\{\sqcap,\sqcup\}\wedge g\in[\![n]\!]^{\otimes k}\wedge j\in[\![n]\!]^{\otimes\ell}\},$$

where $\mathcal{G}$ can be any generating set of $\mathcal{C}$ and where $v=(v_{j,i})_{(j,i)\in[\![n]\!]^{\otimes 2}}$, equipped with the unique morphism of $*$-algebras from A to $A\otimes A$ with $v_{j,i}\mapsto\sum_{s=1}^n v_{j,s}\otimes v_{s,i}$ for any $\{i,j\}\subseteq[\![n]\!]$ as co-multiplication, the unique morphism of $*$-algebras from A to $\mathbb{C}$ with $v_{j,i}\mapsto\delta_{j,i}$ for any $\{i,j\}\subseteq[\![n]\!]$ as co-unit and the unique antimultiplicative linear map from A to A with $v_{j,i}\mapsto v_{i,j}$ and $v^*_{j,i}\mapsto v^*_{i,j}$ for any $\{i,j\}\subseteq[\![n]\!]$ as antipode.

(b) For any category $\mathcal{C}$ of two-colored partitions the *unitary easy* compact quantum group associated with $\mathcal{C}$ and n is the formal dual of the CQG algebra given by the universal $*$-algebra with n^2 many generators $\{u_{j,i}\}_{i,j=1}^n$ subject to the relations

$$\begin{aligned}\{\mathrm{r}_{\mathfrak{d}}^{\mathfrak{c}}(p)_{j,g}(u,\overline{u}) \,|\, \{k,\ell\} \subseteq \mathbb{N}_0 \wedge \mathfrak{c} \in \{\circ,\bullet\}^{\otimes k} \wedge \mathfrak{d} \in \{\circ,\bullet\}^{\otimes \ell} \\ \wedge\ (\mathfrak{c},\mathfrak{d},p) \in \mathcal{G} \cup \{\sqcap_{\circ\bullet}, \sqcap_{\bullet\circ}, \sqcup^{\circ\bullet}, \sqcup^{\bullet\circ}\} \\ \wedge\ g \in [\![n]\!]^{\otimes k} \wedge j \in [\![n]\!]^{\otimes \ell}\},\end{aligned}$$

where $\mathcal{G}$ can be any generating set of $\mathcal{C}$ and where $u = (u_{j,i})_{(j,i)\in[\![n]\!]^{\otimes n}}$ and $\overline{u} = (u_{j,i}^*)_{(j,i)\in[\![n]\!]^{\otimes n}}$, equipped with the unique morphism of $*$-algebras from A to $A \otimes A$ with $u_{j,i} \mapsto \sum_{s=1}^n u_{j,s} \otimes u_{s,i}$ for any $\{i,j\} \subseteq [\![n]\!]$ as co-multplication, the unique morphism of $*$-algebras from A to $\mathbb{C}$ with $u_{j,i} \mapsto \delta_{j,i}$ for any $\{i,j\} \subseteq [\![n]\!]$ as co-unit and with the unique anti-multiplicative linear map from A to A with $u_{j,i} \mapsto u_{i,j}^*$ and $u_{j,i}^* \mapsto u_{i,j}$ for any $\{i,j\} \subseteq [\![n]\!]$ as antipode.

Any easy orthogonal or easy unitary compact quantum group is evidently of Kac type.

5. L^2-Betti numbers

When it comes to the definition, given in [Kye08b], of the p-th L^2-Betti number

$$\beta_p^{(2)}(\widehat{G}) \coloneqq \dim_{\mathrm{L}^\infty(G)} \mathrm{Tor}_p^{\mathrm{Pol}(G)}(\mathbb{C}, \mathrm{L}^2(G))$$

of the discrete dual of any compact quantum group G of Kac type, we can afford to be agnostic. For the purposes of this chapter it suffices entirely to know that the number only depends on the isomorphism class of G and that one can prove certain results about it which we take as axioms.

In the below proposition, (a), (b) and (c) were shown by Kyed in [Kye08b, Proposition 2.9] in combination with [Kye11, Theorem 2.1], in [Kye08a, Corollary 6.2] and in [Kye12, Corollary 3.2], respectively. Parts (d) and (e) were proved by Bichon, Kyed and Raum in Theorems C respectively D of [BKR18]. Keep in mind that $0 \notin \mathbb{N}$. Just as with the definition of the L^2-Betti numbers themselves we can be agnostic about the notion of co-amenability appearing below (which was defined in [BMT01]). We only need to know its consequence.

PROPOSITION 5.1. *(a) For any compact quantum group G of Kac type,*

$$\beta_0^{(2)}(\widehat{G}) = \begin{cases} \frac{1}{\dim_{\mathbb{C}}(\mathrm{Pol}(G))} & \textit{if } \dim_{\mathbb{C}}(\mathrm{Pol}(G)) < \infty \\ 0 & \textit{otherwise.} \end{cases}$$

(b) For any compact quantum group G of Kac type, if G is co-amenable, then for any $p \in \mathbb{N}$,

$$\beta_p^{(2)}(\widehat{G}) = 0.$$

However, $\beta_0^{(2)}(\widehat{G})$ may well be non-zero still.

(c) *For any finite set* Γ, *any family* $(G_g)_{g\in\Gamma}$ *of compact quantum groups of Kac type and any* $p \in \mathbb{N}_0$,

$$\beta_p^{(2)}(\times_{g\in\Gamma}\widehat{G_g}) = \sum_{\substack{q:\Gamma\to\mathbb{N}_0\\ \sum_{g\in\Gamma} q_g = p}} \prod_{g\in\Gamma} \beta_{q_g}^{(2)}(\widehat{G_g}).$$

(d) *For any finite set* Γ, *any family* $(G_g)_{g\in\Gamma}$ *of non-trivial compact quantum groups of Kac type and any* $p \in \mathbb{N}_0$,

$$\beta_p^{(2)}(\underset{g\in\Gamma}{*}\widehat{G_g}) = \begin{cases} 0 & \text{if } p = 0\\ |\Gamma| - 1 + \sum_{g\in\Gamma}(\beta_1^{(2)}(\widehat{G_g}) - \beta_0^{(2)}(\widehat{G_g})) & \text{if } p = 1\\ \sum_{g\in\Gamma}\beta_p^{(2)}(\widehat{G_g}) & \text{if } 2 \leq p.\end{cases}$$

(e) *For any compact quantum groups* G *and* H *of Kac type and any finite abelian group* Γ, *if there exists an exact sequence of Hopf* $*$*-algebras*

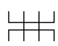
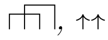
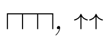
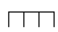
$$\mathbb{C} \hookrightarrow \mathrm{Pol}(H) \hookrightarrow \mathrm{Pol}(G) \twoheadrightarrow \mathbb{C}[\Gamma] \twoheadrightarrow \mathbb{C}$$

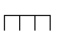
then for any $p \in \mathbb{N}_0$,

$$\beta_p^{(2)}(\widehat{H}) = |\Gamma|\,\beta_p^{(2)}(\widehat{G}).$$

Besides on these results we also take as axiomatic inputs the following values of L^2-Betti numbers of certain quantum groups. They were computated by or easily result from other computations by various other authors, as explained below. Note that the quantum groups $(B_n^{\#*})_{n\in\mathbb{N}}$ are *not* listed in the table.

PROPOSITION 5.2. *For any* $p \in \mathbb{N}_0$, *any* $n \in \mathbb{N}$ *and the orthogonal easy compact quantum group* K *associated with the category* $\mathcal{C} = \langle \mathcal{G}_K \rangle$ *of partitions listed in the below table and n the discrete dual of* K *has the p-th* L^2*-Betti number given in the corresponding row.*

K	$\mathcal{G}_K$	$\beta_p^{(2)}(\widehat{K})$
O_n		$\frac{1}{2}\delta_{n,1}\delta_{p,0}$
O_n^*		$\frac{1}{2}\delta_{n,1}\delta_{p,0}$
O_n^+		$\frac{1}{2}\delta_{n,1}\delta_{p,0}$
B_n'		$\frac{1}{2}(\delta_{n,1} + \frac{1}{2}\delta_{n,2})\delta_{p,0}$
$B_n'^+$		$\frac{1}{2}(\delta_{n,1} + \frac{1}{2}\delta_{n,2})\delta_{p,0}$
$B_n^{\#+}$	↑↑	$\frac{1}{2}\delta_{n,1}\delta_{p,0} + \frac{1}{2}(1 - \delta_{n,1} - \delta_{n,2})\delta_{p,1}$
S_n'		$\frac{1}{2}\frac{1}{n!}\delta_{p,0}$
$S_n'^+$		$\frac{1}{2}(\delta_{n,1} + \frac{1}{2}\delta_{n,2} + \frac{1}{6}\delta_{n,3})\delta_{p,0}$
H_n^+		$\frac{1}{2}\delta_{n,1}\delta_{p,0} + \frac{1}{2}(1 - \delta_{n,2} - \frac{1}{3}\delta_{n,3})\delta_{p,1}$

In particular, K *is non-trivial. Here,*  *is the only partition with a block of size greater than two.*

PROOF. The claim about the L^2-Betti numbers is all we have to prove since it implies that K is non-trivial because $\beta_0^{(2)}(\widehat{K}) \neq 1$ and because triviality of K would require the opposite by Proposition 5.1 (a).

Case O_n. If $n = 1$, then O_n is the finite multiplicative group $\{-1, 1\}$. If $2 \leq n$ the group O_n is infinite. Hence, the zeroth L^2-Betti number of its discrete dual is $\frac{1}{2}$ if $n = 1$ and 0 otherwise by Proposition 5.1 (a). And because O_n is a compact group its dual is amenable and thus has vanishing L^2-Betti numbers of all positive orders by Proposition 5.1 (b).

Case O_n^*. For $n = 1$ the compact quantum groups O_n and O_n^* coincide. If $2 \leq n$, then O_n^* is infinite since even O_n is. Thus the zeroth L^2-Betti number of the dual of O_n^* is the same as that of the dual of O_n. Since O_n^* is co-amenable by [BV10, Corollary 9.3] the remaining numbers are zero too by Proposition 5.1 (b).

Case O_n^+. In the case $n = 1$, in fact, all three compact quantum groups O_n, O_n^* and O_n^+ are identical. And, again, if $2 \leq n$, then, like O_n and O_n^*, the compact quantum group O_n^+ is infinite. That proves the claim about the zeroth L^2-Betti number of its discrete dual. That the numbers also vanish in positive order was computed in [CHT09, Section 4] and [Ver12, Corollary 5.3].

Case B_n'. The compact group B_n' is isomorphic to $B_n \hat{\times} \widehat{\mathbb{Z}_2}$ by [BS09, Proposition 2.4 (4)], where, if we let O_0 be the trivial group, then B_n is isomorphic to O_{n-1} by [BS09, Proposition 2.4 (6)]. Hence, B_n' is isomorphic to $O_{n-1} \hat{\times} \widehat{\mathbb{Z}_2}$. Of course, $\beta_q^{(2)}(\widehat{O_0}) = \delta_{q,0}$ by Proposition 5.1 (a) and thus $\beta_q^{(2)}(\widehat{O_{n-1}}) = (\delta_{n-1,0} + \frac{1}{2}\delta_{n-1,1})\delta_{q,0} = (\delta_{n,1} + \frac{1}{2}\delta_{n,2})\delta_{q,0}$ for any $q \in \mathbb{N}_0$ by what we have already seen. Furthermore, $\beta_q^{(2)}(\mathbb{Z}_2) = \frac{1}{2}\delta_{q,0}$ for any $q \in \mathbb{N}_0$ by Proposition 5.1 (a). Thus Proposition 5.1 (c) implies that

$$\begin{aligned}
\beta_p^{(2)}(\widehat{B_n'}) &= \textstyle\sum_{\substack{(p_1,p_2)\in\mathbb{N}_0^{\otimes 2}\\ p_1+p_2=p}} \beta_{p_1}^{(2)}(\widehat{O_{n-1}})\beta_{p_2}^{(2)}(\mathbb{Z}_2) \\
&= \textstyle\sum_{\substack{(p_1,p_2)\in\mathbb{N}_0^{\otimes 2}\\ p_1+p_2=p}} (\delta_{n,1} + \frac{1}{2}\delta_{n,2})\delta_{p_1,0} \cdot \frac{1}{2}\delta_{p_2,0} \\
&= \tfrac{1}{2}(\delta_{n,1} + \tfrac{1}{2}\delta_{n,2}) \textstyle\sum_{\substack{(p_1,p_2)\in\mathbb{N}_0^{\otimes 2}\\ p_1+p_2=p}} \delta_{p_1,0}\delta_{p_2,0} \\
&= \tfrac{1}{2}(\delta_{n,1} + \tfrac{1}{2}\delta_{n,2})\delta_{p,0}.
\end{aligned}$$

Case $B_n'^+$. By [Web13, Proposition 5.1 (a)] or [Web13, Proposition 5.2 (b)] the compact quantum group $B_n'^+$ is isomorphic to the direct product $B_n^+ \hat{\times} \widehat{\mathbb{Z}_2}$, where, if B_0^+ is the trivial group, then B_n^+ is isomorphic to O_{n-1}^+ by [Rau12, Theorem 4.1 (1)]. In conclusion, $B_n'^+$ is isomorphic to $O_{n-1}^+ \hat{\times} \widehat{\mathbb{Z}_2}$. Since, as already seen, the duals of O_{n-1}^+ and O_{n-1} have identical L^2-Betti numbers, the same proof as in the case of B_n' applies and proves the assertion.

Case $B_n^{\#+}$. According to [Web13, Proposition 5.2 (b)] or [Rau12, Theorem 4.1 (3)] (where $B_n^{\#+}$ is called $B_n'^+$) the compact quantum groups $B_n^{\#+}$ and $B_n^+ \hat{*} \widehat{\mathbb{Z}_2}$ are isomorphic, where, as already noted, $B_n^+ \cong O_{n-1}^+$, with O_0^+ being the trivial group. Thus, $B_n^{\#+}$ is isomorphic to $O_{n-1}^+ \hat{*} \widehat{\mathbb{Z}_2}$.

This implies $B_1^{\#+} \cong \widehat{\mathbb{Z}_2}$ from which we conclude that $\beta_p^{(2)}(\widehat{B_n^{\#+}}) = \frac{1}{2}\delta_{p,0}$. If $2 \leq n$, then O_{n-1}^+ is non-trivial because even O_{n-1} is, and so is $\widehat{\mathbb{Z}_2}$. Hence, Proposition 5.1 (d) is applicable in that case. By Proposition 5.1 (a) and by what we proved previously, if $2 \le n$, then $\beta^{(2)}_q(\widehat{O^+_{n-1}}) = \frac{1}{2}\delta_{n,2}\delta_{q,0}$ and $\beta^{(2)}_q(\mathbb{Z}_2) = \frac{1}{2}\delta_{q,0}$ for any $q \in \mathbb{N}_0$. Therefore, $\beta^{(2)}_0(\widehat{B^{\#+}_n}) = 0$ and

$$\begin{aligned}\beta^{(2)}_1(\widehat{B^{\#+}_n}) &= 1 + \beta^{(2)}_1(\widehat{O^+_{n-1}}) - \beta^{(2)}_0(\widehat{O^+_{n-1}}) + \beta^{(2)}_1(\mathbb{Z}_2) - \beta^{(2)}_0(\mathbb{Z}_2)\\ &= 1 + 0 - \tfrac{1}{2}\delta_{n,2} + 0 - \tfrac{1}{2}\\ &= \tfrac{1}{2}(1 - \delta_{n,2})\end{aligned}$$

as well as, if $2 \le p$, then $\beta^{(2)}_p(\widehat{B^{\#+}_n}) = \beta^{(2)}_p(\widehat{O^+_{n-1}} * \mathbb{Z}_2) = \beta^{(2)}_p(\widehat{O^+_{n-1}}) + \beta^{(2)}_p(\mathbb{Z}_2) = 0$. Combining all the cases into one statement now yields the claim.

Case S'_n. By [BS09, Proposition 2.4 (5)] there is an isomorphism between S'_n and $S_n \hat{\times} \widehat{\mathbb{Z}_2}$. Proposition 5.1 (a) tells us that $\beta^{(2)}_q(\widehat{S_n}) = \frac{1}{n!}\delta_{q,0}$ and $\beta^{(2)}_q(\mathbb{Z}_2) = \frac{1}{2}\delta_{q,0}$. Hence, Proposition 5.1 (c) implies

$$\begin{aligned}\beta^{(2)}_p(\widehat{S'_n}) &= \textstyle\sum_{\substack{(p_1,p_2)\in\mathbb{N}_0{}^{\otimes 2}\\ p_1+p_2=p}} \beta^{(2)}_{p_1}(\widehat{S_n})\beta^{(2)}_{p_2}(\mathbb{Z}_2)\\ &= \textstyle\sum_{\substack{(p_1,p_2)\in\mathbb{N}_0{}^{\otimes 2}\\ p_1+p_2=p}} \frac{1}{n!}\delta_{p_1,0}\cdot\frac{1}{2}\delta_{p_2,0}\\ &= \tfrac{1}{2}\tfrac{1}{n!}\textstyle\sum_{\substack{(p_1,p_2)\in\mathbb{N}_0{}^{\otimes 2}\\ p_1+p_2=p}} \delta_{p_1,0}\delta_{p_2,0}\\ &= \tfrac{1}{2}\tfrac{1}{n!}\delta_{p,0}.\end{aligned}$$

Case S'^+_n. By [Web13, Proposition 5.1 (a)] or [Rau12, Theorem 4.1 (2)] the compact quantum group S'^+_n is isomorphic to $S^+_n \hat{\times} \widehat{\mathbb{Z}_2}$, where S^+_n is the quantum permutation group of [Wan98]. If $n \le 3$, then the compact quantum groups S^+_n and S_n coincide by [Wan98] and [Ban05], which is why $\beta^{(2)}_q(\widehat{S^+_n}) = \frac{1}{n!}\delta_{q,0}$ for any $q \in \mathbb{N}_0$ in that case. It was shown in [Kye+17, Theorem 5.2 (ii)] that, if $4 \le n$, then $\beta^{(2)}_q(\widehat{S^+_n}) = 0$ for any $q \in \mathbb{N}_0$. In total, $\beta^{(2)}_q(\widehat{S^+_n}) = (\delta_{n,1} + \frac{1}{2}\delta_{n,2} + \frac{1}{6}\delta_{n,3})\delta_{q,0}$ for any $q \in \mathbb{N}_0$. Of course, $\beta^{(2)}_q(\mathbb{Z}_2) = \frac{1}{2}\delta_{q,0}$ for any $q \in \mathbb{N}_0$ by Proposition 5.1 (a). Hence, by Proposition 5.1 (c),

$$\begin{aligned}\beta^{(2)}_p(\widehat{S'^+_n}) &= \textstyle\sum_{\substack{(p_1,p_2)\in\mathbb{N}_0{}^{\otimes 2}\\ p_1+p_2=p}} \beta^{(2)}_{p_1}(\widehat{S^+_n})\beta^{(2)}_{p_2}(\mathbb{Z}_2)\\ &= \textstyle\sum_{\substack{(p_1,p_2)\in\mathbb{N}_0{}^{\otimes 2}\\ p_1+p_2=p}} (\delta_{n,1} + \frac{1}{2}\delta_{n,2} + \frac{1}{6}\delta_{n,3})\delta_{p_1,0}\cdot\frac{1}{2}\delta_{p_2,0}\\ &= \tfrac{1}{2}(\delta_{n,1} + \tfrac{1}{2}\delta_{n,2} + \tfrac{1}{6}\delta_{n,3})\textstyle\sum_{\substack{(p_1,p_2)\in\mathbb{N}_0{}^{\otimes 2}\\ p_1+p_2=p}} \delta_{p_1,0}\delta_{p_2,0}\\ &= \tfrac{1}{2}(\delta_{n,1} + \tfrac{1}{2}\delta_{n,2} + \tfrac{1}{6}\delta_{n,3})\delta_{p,0}.\end{aligned}$$

Case H^+_n. The compact quantum group H^+_1 is isomorphic to $\widehat{\mathbb{Z}_2}$ by [BBC07, p. 15]. Thus, $\beta^{(2)}_p(\widehat{H^+_1}) = \frac{1}{2}\delta_{p,0}$. It was proved in [Kye+17, Theorem 5.2 (v)] that, if $4 \le n$, then $\beta^{(2)}_q(\widehat{H^+_n}) = \frac{1}{2}\delta_{q,1}$ for any $q \in \mathbb{N}_0$. But, in fact, the reasoning behind the proof is valid also in the cases $n = 2$ and $n = 3$. If $2 \le n$, then by [BBC07, Theorem 5.5] the compact quantum group H^+_n is isomorphic to the free wreath product $\widehat{\mathbb{Z}_2} \wr_* S^+_n$, defined in [Bic04], of $\widehat{\mathbb{Z}_2}$ and the quantum group S^+_n we encountered in the case S'^+_n. The action of S^+_n on the space of n many points is ergodic, no matter the value of $2 \le n$. Moreover, S_n^+ is non-trivial because already S_n is, for $2 \le n$. For that reason [Kye+17, Theorem 5.2 (iv)] applies and tells us that the L^2-Betti numbers of the duals of $\widehat{\mathbb{Z}_2} \wr_* S_n^+$ and $\widehat{\mathbb{Z}_2} * S_n^+$ coincide. As already seen, if $2 \le n$, then $\beta_q^{(2)}(\widehat{S_n^+}) = (\frac{1}{2}\delta_{n,2} + \frac{1}{6}\delta_{n,3})\delta_{q,0}$ and $\beta_q^{(2)}(\mathbb{Z}_2) = \frac{1}{2}\delta_{q,0}$ for any $q \in \mathbb{N}_0$. Hence Proposition 5.1 (d) implies that $\beta_0^{(2)}(\widehat{H_n^+}) = 0$, that

$$\begin{aligned}\beta_1^{(2)}(\widehat{H_n^+}) &= 1 + \beta_1^{(2)}(\mathbb{Z}_2) - \beta_0^{(2)}(\mathbb{Z}_2) + \beta_1^{(2)}(\widehat{S_n^+}) - \beta_0^{(2)}(\widehat{S_n^+}) \\ &= 1 + 0 - \tfrac{1}{2} + 0 - (\tfrac{1}{2}\delta_{n,2} + \tfrac{1}{6}\delta_{n,3}) \\ &= \tfrac{1}{2} - \tfrac{1}{2}\delta_{n,2} - \tfrac{1}{6}\delta_{n,3}\end{aligned}$$

and that, if $2 \le p$, then $\beta_p^{(2)}(\widehat{H_n^+}) = \beta_p^{(2)}(\mathbb{Z}_2) + \beta_p^{(2)}(\widehat{S_n^+}) = 0$. That concludes the proof. □

6. The proof strategy

Bichon, Kyed and Raum's strategy for computing the L^2-Betti numbers from [BKR18] and [BNY16, Example 2.18] can be summarized as follows.

PROPOSITION 6.1. *For any two compact quantum groups G and K of Kac type such that K is non-trivial, if $B = \mathrm{Pol}(G)$ and $A = \mathrm{Pol}(K) * \mathrm{Pol}(K)$, if there exist $(p, \alpha, q, \beta, f, f')$ such that*

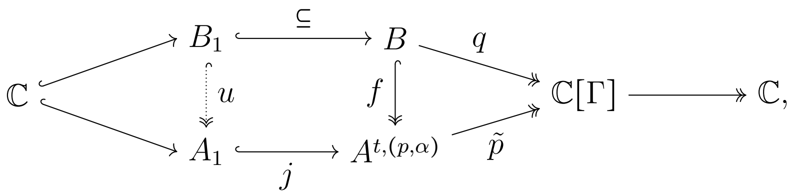
(a) (p, α) is an invariant co-central action of $\mathbb{Z}_2$ on A,
(b) (q, β) is an invariant co-central action of $\mathbb{Z}_2$ on B,
(c) p is surjective,
(d) q is surjective,
(e) f is a Hopf $*$-algebras morphism from B to $A^{t,(p,\alpha)}$,
(f) f' is a Hopf $*$-algebras morphism from A to $B^{t,(q,\beta(-\iota))}$,
(g) $(\epsilon_A \otimes \mathrm{id}) \circ \iota \circ f = q$,
(h) $(\epsilon_B \otimes \mathrm{id}) \circ \iota' \circ f' = p$,
(i) $(\alpha_{g^{-1}} \otimes \mathrm{id}) \circ \iota \circ f = \iota \circ f \circ \beta_g$ for any $g \in \mathbb{Z}_2$,
(j) $(\beta_g \otimes \mathrm{id}) \circ \iota' \circ f' = \iota' \circ f' \circ \alpha_g$ for any $g \in \mathbb{Z}_2$,
(k) $((\iota' \circ f') \otimes \mathrm{id}) \circ \iota \circ f = (\mathrm{id} \otimes (\Delta \circ q)) \circ \Delta_B$,
(l) $((\iota \circ f) \otimes \mathrm{id}) \circ \iota' \circ f' = (\mathrm{id} \otimes (\Delta \circ p)) \circ \Delta_A$,

where Δ_A and ϵ_A are the co-multiplication respectively co-unit of A, where Δ_B and ϵ_B are those of B, where Δ is the co-multiplication of $\mathbb{Z}_2$, and where ι and ι' are the inclusions of $A^{t,(p,\alpha)}$ into $A \rtimes_\alpha \mathbb{C}[\Gamma]$ respectively of $B^{t,(q,\beta(-\iota))}$ in $B \rtimes_{\beta(-\iota)} \mathbb{C}[\Gamma]$, then

$$\beta_p^{(2)}(\widehat{G}) = \begin{cases} 0 & \text{if } p = 0 \\ 1 + 2\beta_1^{(2)}(\widehat{K}) - 2\beta_0^{(2)}(\widehat{K}) & \text{if } p = 1 \\ 2\beta_p^{(2)}(\widehat{K}) & \text{if } 2 \le p \end{cases}$$

PROOF. Let 1 be the neutral and z the other element of $\mathbb{Z}_2$ and let $(A_g)_{g \in \mathbb{Z}_2}$ and $(B_g)_{g \in \mathbb{Z}_2}$ be the $\mathbb{Z}_2$-gradings of A associated with p respectively of B associated with q. By Proposition 3.3, due to assumptions (a)– (d) the Hopf $*$-algebras $\mathrm{hker}(p) = A_1$

and $\operatorname{hker}(q) = B_1$ as well as $A^{t,(p,\alpha)}$ are CQG algebras and the two maximal horizontal strings of arrows form exact sequences of Hopf $*$-algebras in the diagram



where $\tilde{p} = (\epsilon_A \otimes \mathrm{id}) \circ \iota$ and where j is the unique linear map with $a \mapsto a \otimes 1$ for any $a \in A_1$.

Moreover, because $\mathbb{Z}_2$ is abelian, by Proposition 3.5 assumptions (e)– (l) imply that f is actually an isomorphism of Hopf $*$-algebras from B to $A^{t,(p,\alpha)}$.

That ensures that there exists an isomorphism u of Hopf $*$-algebras from B_1 to A_1 with $j(u(b)) = f(b)$ for any $b \in B_1$. Indeed, if Δ_t denotes the co-multiplication of $A^{t,(p,\alpha)}$, then assumptions (e) and (g) allow us to infer that

$$\begin{aligned}(\mathrm{id} \otimes \tilde{p}) \circ \Delta_t \circ f &= (f \otimes ((\epsilon_A \otimes \mathrm{id}) \circ \iota \circ f)) \circ \Delta_B \\ &= (f \otimes q) \circ \Delta_B,\end{aligned}$$

and thus for any $b \in B_1$ that $((\mathrm{id} \otimes \tilde{p}) \circ \Delta_t)(f(b)) = ((f \otimes q) \circ \Delta_B)(b) = (f \otimes q)(b \otimes 1_B) = f(b) \otimes 1$, where 1_B is the unit of B. Hence, u is well-defined and injective. It is also surjective because, if Δ_B is the co-multplication of B, if Δ_A, and ϵ_A are the co-multiplication respectively co-unit of A, if Δ_{A_1} is the co-mulitplication of A_1, if i is the inclusion of A_1 in A, and if v is the linear map $A \to A^{t,(p,\alpha)}$ with $a \mapsto a \otimes g$ for any $a \in A_g$ and any $g \in \mathbb{Z}_2$, then $(\epsilon_A \otimes \mathrm{id}) \circ \iota \circ j = 1 \circ \epsilon_A \circ i$ and $j = v \circ i$ and thus by a second application of (g),

$$\begin{aligned}(\mathrm{id} \otimes q) \circ \Delta_B \circ f^{-1} \circ j &= (\mathrm{id} \otimes q) \circ (f^{-1} \otimes f^{-1}) \circ \Delta_t \circ j \\ &= (f^{-1} \otimes ((\epsilon_A \otimes \mathrm{id}) \circ \iota)) \circ (j \otimes j) \circ \Delta_{A_1} \\ &= ((f^{-1} \circ v \circ i) \otimes (1 \circ \epsilon_A \circ i)) \circ \Delta_{A_1} \\ &= ((f^{-1} \circ v) \otimes 1) \circ (\mathrm{id} \otimes \epsilon_A) \circ \Delta_A \circ i \\ &= ((f^{-1} \circ v) \otimes 1) \circ i \\ &= (f^{-1} \circ j) \otimes 1.\end{aligned}$$

Hence, for any $a \in A_1$, if $b \coloneqq f^{-1}(j(a))$, then clearly $u(b) = a$ by the injectivity of j, and $b \in B_1$ by $((\mathrm{id} \otimes q) \circ \Delta_B)(b) = ((\mathrm{id} \otimes q) \circ \Delta_B \circ f^{-1} \circ j)(a) = ((f^{-1} \circ j) \otimes 1)(a) = b \otimes 1$, as just seen.

Consequently, if H is the compact quantum group with $\operatorname{Pol}(H) \cong B_1 \cong A_1$, then Proposition 5.1 (e) informs us that $2\beta_p^{(2)}(\widehat{K} * \widehat{K}) = \beta_p^{(2)}(\widehat{H}) = 2\beta_p^{(2)}(\widehat{G})$ for any $p \in \mathbb{N}_0$. Since K is non-trivial the claim now follows by Proposition 5.1 (d). $\square$

7. The computation

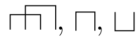
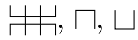
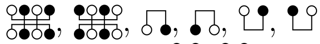
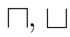
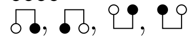
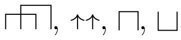
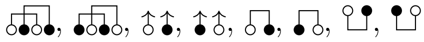
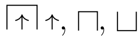
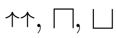
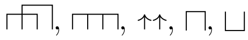
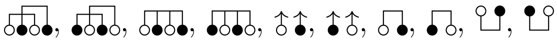
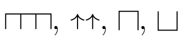
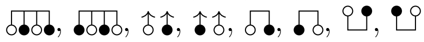
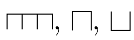
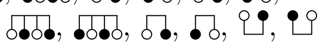
To prove the main result it suffices to check the conditions of Propositon 6.1 for the following inputs.

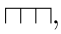
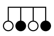
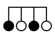
Assumptions 7.1. In Section 7, let

(a) 1 be the neutral and z the other element of $\mathbb{Z}_2$, whose law we write multiplicatively,

(b) $n \in \mathbb{N}$ be arbitrary,

(c) $(\mathcal{G}_K, \mathcal{G}_G)$ be one of the following pairs of sets of uncolored and two-colored partitions:

$\langle\mathcal{G}_K\rangle$	$\mathcal{G}_K$	$\mathcal{G}_G$
O		
O*		
O+		
B′		
B#*		
B′+		
B#+		
S′		
S′+		
H+		

where ,  and  are the only partitions with blocks of sizes greater than two,

(d) K be the orthogonal easy compact $n \times n$-matrix quantum group associated with the category $\langle\mathcal{G}_K\rangle$ of uncolored partitions,

(e) G be the unitary easy compact $n\times n$-matrix quantum group associated with the category $\langle\mathcal{G}_G\rangle$ of two-colored partitions,

(f) $\{u_{j,i}\}_{i,j=1}^n$ be the generators of the universal $*$-algebra

$$\begin{aligned} B \coloneqq \mathbb{C}^*\langle \{u_{j,i}\}_{i,j=1}^n \mid \forall\{k,\ell\} \subseteq \mathbb{N}_0\colon\ &\forall \mathfrak{c} \in \{\circ,\bullet\}^{\otimes k}\colon\ \forall \mathfrak{d} \in \{\circ,\bullet\}^{\otimes \ell}\colon \\ &\forall(\mathfrak{c},\mathfrak{d},p) \in \mathcal{G}_G\colon\ \forall g \in [\![n]\!]^{\otimes k}\colon\ \forall j \in [\![n]\!]^{\otimes \ell}\colon \\ &\mathrm{r}_{\mathfrak{d}}^{\mathfrak{c}}(p)_{j,g}(u,\overline{u}) = 0\rangle, \end{aligned}$$

i.e., the Hopf $*$-algebra $\mathrm{Pol}(G)$ given by the regular functions on G, where $u \coloneqq (u_{j,i})_{(j,i)\in[\![n]\!]^{\otimes 2}}$ and $\overline{u} \coloneqq (u_{j,i}^*)_{(j,i)\in[\![n]\!]^{\otimes 2}}$.

(g) $\{v_{j,i}^{(1)}, v_{j,i}^{(2)}\}_{i,j=1}^n$ be the generators of the universal $*$-algebra

$$\begin{aligned} A \coloneqq \mathbb{C}^*\langle \{v_{j,i}^{(1)}, v_{j,i}^{(2)}\}_{i,j=1}^n \mid \forall_{q=1}^2\colon\ \forall_{i,j=1}^n\colon\ &(v_{j,i}^{(q)})^* = v_{j,i}^{(q)} \\ &\wedge\ \forall(k,\ell,p) \in \mathcal{G}_K\colon \\ &\quad\forall g \in [\![n]\!]^{\otimes k}\colon\ \forall j \in [\![n]\!]^{\otimes \ell}\colon \\ &\quad\mathrm{r}_\ell^k(p)_{j,g}(v^{(q)}) = 0\rangle, \end{aligned}$$

i.e., the Hopf $*$-algebra $\mathrm{Pol}(K \hat{*} K)$ given by the regular functions on the free product of G with itself, where $v^{(q)} \coloneqq (v^{(q)}_{j,i})_{(j,i)\in[\![n]\!]\otimes[\![n]\!]}$ for each $q \in [\![2]\!]$.

Remark 7.2. (a) The sets of partitions given as generators in Assumptions 7.1 are non-standard. To recognize the more familiar generators note that

(i) $\langle ⊓⊓ \rangle = \langle \times \rangle$ because, on the one hand, $(|\otimes|\otimes ⊔⊔)(⊓⊓\otimes|\otimes|) = \times$ and, on the other hand, $(\times\otimes|\otimes|)⊓⊓ = ⊓⊓$ and because $⊓⊓ = (|\otimes ⊔⊓ \otimes|)(⊓\otimes⊓)$.

(ii) $\langle ⊓⊓⊓ \rangle = \langle \times\!| \rangle$ because $(|\otimes|\otimes⊔\otimes|)(⊓⊓⊓\otimes|)(|\otimes|\otimes|\otimes⊓) = \times\!|$ and because $(\times\!|\otimes|)(|\otimes|\otimes ⊔⊓)(\times\!|\otimes|) = ⊓⊓⊓$.

(iii) $\langle ⊓⊓⊓_{●○●○} \rangle = \langle \times_{●○} \rangle = \langle \times_{○●} \rangle = \langle ⊓⊓⊓_{○●○●} \rangle$ since $(○\otimes●\otimes⊔\otimes○)(⊓⊓⊓\otimes○)(○\otimes●\otimes○\otimes⊓) = \times$ and $(●\otimes○\otimes●\otimes⊔)(●\otimes\times\otimes●)(⊓\otimes●\otimes○\otimes●) = \times$ and $(\times\otimes○)(●\otimes○\otimes ⊔⊓)(\times\otimes○) = ⊓⊓⊓$ and since the same computation can be carried out with the roles of $\bullet$ and $\circ$ reversed.

(iv) $\langle ⊓⊓_{●○●○} \rangle = \langle ⊓⊓_{○●●○} \rangle$ because $(⊔\otimes○\otimes●\otimes○\otimes●)(○\otimes⊓⊓\otimes●)⊓ = ⊓⊓$ and because of the same identity with $\circ$ and $\bullet$ exchanged.

(v) $\langle ⊓↑↑ \rangle = \langle ⊓↑↑ \rangle$ because $(○\cdot●\cdot○\cdot●\cdot⊔)(○\cdot⊓↑\cdot●\cdot○\cdot●\cdot○\cdot●)(○\cdot●\cdot○\cdot●\cdot⊓\cdot○\cdot●\cdot○\cdot●)(○\cdot●\cdot○\cdot⊓\cdot●\cdot○\cdot●)(○\cdot●\cdot⊓\cdot○\cdot●)(○\cdot⊓\cdot●)⊓$, where $\cdot$ is short for $\otimes$, and because the same can be done with $\circ$ in place of $\bullet$ and vice versa.

(vi) $\langle ↕↕ \rangle = \langle ↕↕ \rangle$ because $(⊔\otimes○\otimes●)(○\otimes↕↕\otimes●)⊓ = ↕↕$ and because of the analogous computation with $\circ \leftrightarrow \bullet$.

(vii) $\langle ⊓⊓⊓ \rangle = \langle ⊓⊓⊓ \rangle$ since $(⊔\otimes○\otimes●\otimes○\otimes●)(○\otimes⊓⊓⊓\otimes●)⊓ = ⊓⊓⊓$ and the same with $\circ \leftrightarrow \bullet$.

(b) Except in the case of the last row of the table in Assumption 7.1 (c) the category $\langle \mathcal{G}_G \rangle$ is non-hyperoctahedral an thus included in the scope of [MW21b]. In the language of that article, $\langle \mathcal{G}_G \rangle$ is the category $\mathcal{R}_Q$, where $Q = (f, v, s, l, k, x)$ is as follows.

$\langle \mathcal{G}_K \rangle$	f	v	s	l	k	x
O	$\{2\}$	$\pm\{0,2\}$	$\{0\}$	$\{0\}$	$\{0\}$	$\{0\}$
O*	$\{2\}$	$\{0\}$	$\{0\}$	$\varnothing$	$\{0\}$	$\{0\}$
O+	$\{2\}$	$\{0\}$	$\{0\}$	$\varnothing$	$\{0\}$	$\varnothing$
B′	$\{1,2\}$	$\pm\{0,1,2\}$	$\{0\}$	$\{0\}$	$\{0\}$	$\{0\}$
B#*	$\{1,2\}$	$\pm\{0,1\}$	$\{0\}$	$\varnothing$	$\{0\}$	$\{0\}$
B′+	$\{1,2\}$	$\pm\{0,1,2\}$	$\{0\}$	$\{0\}$	$\{0\}$	$\varnothing$
B#+	$\{1,2\}$	$\pm\{0,1\}$	$\{0\}$	$\varnothing$	$\{0\}$	$\varnothing$
S′	$\mathbb{N}$	$\mathbb{Z}$	$\{0\}$	$\{0\}$	$\{0\}$	$\{0\}$
S′+	$\mathbb{N}$	$\mathbb{Z}$	$\{0\}$	$\{0\}$	$\{0\}$	$\varnothing$

(c) Moreover, in the case of the last row of the table in Assumption 7.1 (c) the category $\langle \mathcal{G}_G \rangle$ is $\mathcal{H}'_{\mathrm{loc}} = \langle ⊓⊓⊓ \rangle$ in the language of [TW18, Proposition 4.3] and $\mathcal{W}_{\{\circ,\bullet\}}$ in that of Chapter 5.

(d) By definition, the Hopf $*$-algebra A is also the co-product Hopf $*$-algebra $\mathrm{Pol}(K) * \mathrm{Pol}(K)$ of the regular functions

$$\mathrm{Pol}(K) = \mathbb{C}^*\langle \{v_{j,i}\}_{i,j=1}^n \,|\, \forall_{i,j=1}^n\colon v_{j,i}^* = v_{j,i} \wedge \forall (k,\ell,p) \in \mathcal{G}_K\colon \\ \forall g \in [\![n]\!]^{\otimes k}\colon \forall j \in [\![n]\!]^{\otimes \ell}\colon \mathrm{r}_\ell^k(p)_{j,g}(v) = 0\rangle,$$

on K with itself.

Throughout we will use the fact that the sets $\{\circ,\bullet\} \cong \{1,*\}$ and $[\![2]\!]$ are equinumerous and admit (left) actions of $\mathbb{Z}_2$ equivariant under the bijection.

NOTATION 7.3. (a) $\phi_\circ \coloneqq 1$ and $\phi_\bullet \coloneqq 2$ as well as $\psi_1 \coloneqq \circ$ and $\psi_2 \coloneqq \bullet$.

(b) $1.\circ = \circ$ and $1.\bullet = \bullet$ and $z.\circ = \bullet$ and $z.\bullet = \circ$.

(c) $1.1 = 1$ and $1.2 = 2$ and $z.1 = 2$ and $z.2 = 1$.

(d) For any $k \in \mathbb{N}_0$, any $\mathfrak{c} \in \{\circ,\bullet\}^{\otimes k}$ and any $g \in \mathbb{Z}_2$ let $g.\mathfrak{c} \in \{\circ,\bullet\}^{\otimes k}$ be such that $(g.\mathfrak{c})_i = g.(\mathfrak{c}_i)$ for any $i \in [\![k]\!]$.

(e) For any $k \in \mathbb{N}_0$, any $q \in [\![2]\!]^{\otimes k}$ and any $g \in \mathbb{Z}_2$ let $g.q \in [\![2]\!]^{\otimes k}$ be such that $(g.q)_i = g.(q_i)$ for any $i \in [\![k]\!]$.

(f) From now on, in any $*$-algebra, for any element a let $a^\circ \coloneqq a$ and $a^\bullet \coloneqq a^*$.

The key to the proofs in this section are the following properties of $(\mathcal{G}_K, \mathcal{G}_G)$ (and of $(\langle\mathcal{G}_K\rangle, \langle\mathcal{G}_G\rangle)$ actually):

- □ All partitions occurring in either set have evenly many points.
- □ The points of any two-colored partition in $\mathcal{G}_G$ alternate in normalized color.
- □ Forgetting all colors in $\mathcal{G}_G$ produces exactly the partitions of $\mathcal{G}_K$.
- □ Conversely, coloring the partitions of $\mathcal{G}_K$ in any way such that normalized colors alternate yields precisely the partitions of $\mathcal{G}_G$.

In more detail, we can record the following facts.

LEMMA 7.4. *Let $\{k,\ell\} \subseteq \mathbb{N}_0$ be arbitrary and let p be any partition of Π_ℓ^k.*

(a) For any $\mathfrak{c} \in \{\circ,\bullet\}^{\otimes k}$ and $\mathfrak{d} \in \{\circ,\bullet\}^{\otimes \ell}$, if $(\mathfrak{c},\mathfrak{d},p) \in \mathcal{G}_G$, then

(i) $k+\ell$ is even.

(ii) There exists $x \in \{\circ,\bullet\}$ such that $\mathfrak{c}_a = z^{a-1}.x$ for any $a \in [\![k]\!]$ and $\mathfrak{d}_b = z^{b-1}.x$ for any $b \in [\![\ell]\!]$.

(iii) $(z.\mathfrak{c}, z.\mathfrak{d}, p) \in \mathcal{G}_G$.

(iv) $(k,\ell,p) \in \mathcal{G}_K$.

(b) If $(k,\ell,p) \in \mathcal{G}_K$, then

(i) $k+\ell$ is even.

(ii) For any $x \in \{\circ,\bullet\}$, if $\mathfrak{c} \in \{\circ,\bullet\}^{\otimes k}$ and $\mathfrak{d} \in \{\circ,\bullet\}^{\otimes \ell}$ are such that $\mathfrak{c}_a = z^{a-1}.x$ for any $a \in [\![k]\!]$ and $\mathfrak{d}_b = z^{b-1}.x$ for any $b \in [\![\ell]\!]$, then $(\mathfrak{c},\mathfrak{d},p) \in \mathcal{G}_G$.

PROOF. Obvious from the definitions. □

Let us now start carrying out the proof strategy of Proposition 6.1.

LEMMA 7.5. *There exists a unique surjective co-central morphism $q\colon B \to \mathbb{C}[\Gamma]$ of Hopf $*$-algebras such that for any $\{i,j\} \subseteq [\![n]\!]$,*

$$u_{i,j} \longmapsto \delta_{i,j} z.$$

Proof. First, we show that there exists a unique morphism q of $*$-algebras with the prescribed property. Then, we prove this q to also respect the Hopf $*$-algebra operations and be co-central. It will then automatically be surjective because its image will be a $*$-subalgebra of $\mathbb{C}[\mathbb{Z}_2]$ containing the $*$-generator $q(u_{1,1}) = z$ of the latter.

Step 1: q exists and is unique as a $*$-algebra morphism. The desired property of q can be rephrased by saying that $\mathrm{id} \otimes q$ is supposed to map u to $1_{n\times n} \otimes z$, where $1_{n\times n}$ is the identity scalar $n \times n$-matrix. The matrix $(1_{n\times n} \otimes z)^{\bullet}$ in $\mathbb{C}[\mathbb{Z}_2]$ is given by $1_{n\times n} \otimes z$ because $z^* = z$. Hence, by the universal property of B it suffices to check that $\mathrm{r}^{\mathfrak{c}}_{\mathfrak{d}}(p)_{j,g}(1_{n\times n} \otimes z, 1_{n\times n} \otimes z) = 0$ for any $\{k, \ell\} \subseteq \mathbb{N}_0$, any $\mathfrak{c} : [\![k]\!] \to \{\circ, \bullet\}$, $\mathfrak{d} : [\![\ell]\!] \to \{\circ, \bullet\}$, any partition p of Π^k_ℓ such that $(\mathfrak{c}, \mathfrak{d}, p) \in \mathcal{G}_G$, any $g \in [\![n]\!]^{\otimes k}$ and any $j \in [\![n]\!]^{\otimes \ell}$.

By definition, the element $\mathrm{r}^{\mathfrak{c}}_{\mathfrak{d}}(p)_{j,g}(1_{n\times n} \otimes z, 1_{n\times n} \otimes z)$ of $\mathbb{C}[\mathbb{Z}_2]$ is given by

$$\textstyle\sum_{i\in[\![n]\!]^{\otimes \ell}} \zeta(p, \ker(g \mathbin{{}^{\blacksquare}{}_{\blacksquare}} i)) \overrightarrow{\prod}_{b=1}^{\ell} (\delta_{j_b, i_b} z)^{\mathfrak{d}_b} \\ \qquad\qquad - \textstyle\sum_{h\in[\![n]\!]^{\otimes k}} \zeta(p, \ker(h \mathbin{{}^{\blacksquare}{}_{\blacksquare}} j)) \overrightarrow{\prod}_{a=1}^{k} (\delta_{h_a, g_a} z)^{\mathfrak{c}_a}$$

which by $z = z^*$ simplifies to

$$\sum_{i\in[\![n]\!]^{\otimes \ell}} \zeta(p, \ker(g \mathbin{{}^{\blacksquare}{}_{\blacksquare}} i))\, \delta_{j,i}\, z^{\ell} - \sum_{h\in[\![n]\!]^{\otimes k}} \zeta(p, \ker(h \mathbin{{}^{\blacksquare}{}_{\blacksquare}} j))\, \delta_{h,g}\, z^{k}$$

or, even more simply,

$$\zeta(p, \ker(g \mathbin{{}^{\blacksquare}{}_{\blacksquare}} j))\, z^{\ell} - \zeta(p, g \mathbin{{}^{\blacksquare}{}_{\blacksquare}} j)\, z^{k}.$$

Because $k + \ell$ is even by Lemma 7.4 (a) (i) and thus $z^{\ell} = 1z^{-\ell} = z^{k+\ell} z^{-\ell} = z^k$ we have thus shown $\mathrm{r}^{\mathfrak{c}}_{\mathfrak{d}}(p)_{j,g}(1_{n\times n} \otimes z, 1_{n\times n} \otimes z) = 0$, which proves that q exists and is unique as a $*$-algebra morphism.

Step 2: q is a Hopf $*$-morphism. Because q is known to be a morphism of $*$-algebras by Step 1 it is enough that we check the conditions for being a morphism of Hopf $*$-algebra on the $*$-generators of B. If Δ_B, ϵ_B and S_B are the co-multiplication, co-unit and antipode of B, respectively, and if Δ, ϵ and S are those of $\mathbb{C}[\mathbb{Z}_2]$, then for any $\{i, j\} \subseteq [\![n]\!]$,

$$\begin{aligned}((q \otimes q) \circ \Delta_B)(u_{j,i}) &= (q \otimes q)(\textstyle\sum_{s=1}^{n} u_{j,s} \otimes u_{s,i}) \\ &= \textstyle\sum_{s=1}^{n} \delta_{j,s}\, z \otimes \delta_{s,i} z \\ &= \delta_{j,i}\, z \otimes z \\ &= \Delta(\delta_{j,i} z) \\ &= (\Delta \circ q)(u_{j,i})\end{aligned}$$

and

$$\epsilon_B(u_{j,i}) = \delta_{j,i} = \epsilon(\delta_{j,i} z) = (\epsilon \circ q)(u_{j,i})$$

and

$$\begin{aligned}(q\circ S_B)(u_{j,i}) &= q(u_{i,j}^*)\\ &= \delta_{i,j}\, z^*\\ &= \delta_{j,i}\, z^{-1}\\ &= S(\delta_{j,i}\, z)\\ &= (S\circ q)(u_{j,i}).\end{aligned}$$

Thus, q is also a morphism of Hopf $*$-algebras from B to $\mathbb{C}[\Gamma]$.

Step 3: q is co-central. Once more, Step 1 allows to confine ourselves to $*$-generators of B when verifying the co-centrality condition. If $\gamma_{B,B}$ is the flip of the tensor factors in $B\otimes B$, then for any $\{i,j\}\subseteq [\![n]\!]$,

$$\begin{aligned}((\mathrm{id}\otimes q)\circ\Delta_B)(u_{j,i}) &= (\mathrm{id}\otimes q)(\textstyle\sum_{s=1}^n u_{j,s}\otimes u_{s,i})\\ &= \textstyle\sum_{s=1}^n u_{j,s}\otimes \delta_{s,i}\, z\\ &= u_{j,i}\otimes z\\ &= \textstyle\sum_{s=1}^n u_{s,i}\otimes \delta_{j,s}\, z\\ &= (\mathrm{id}\otimes q)(\textstyle\sum_{s=1}^n u_{s,i}\otimes u_{j,s})\\ &= ((\mathrm{id}\otimes q)\circ\gamma_{B,B})(\textstyle\sum_{s=1}^n u_{j,s}\otimes u_{s,i})\\ &= ((\mathrm{id}\otimes q)\circ\gamma_{B,B}\circ\Delta_B)(u_{j,i}).\end{aligned}$$

Thus, the morphism q is also co-central. ☐

LEMMA 7.6. *There exists a unique action β of $\mathbb{Z}_2$ on B by Hopf $*$-automorphisms such that β_z is the unique Hopf $*$-algebra endomorphism of B such that for any $\{i,j\}\subseteq [\![n]\!]$,*

$$u_{j,i}\mapsto u_{j,i}^*.$$

PROOF. If β_z exists, it is obviously involutive. Hence, it suffices to show that β_z exists and is unique as a Hopf $*$-algebra morphism. Once more, we first show that β_z exists and is unique as a $*$-algebra morphism and then verify that it is also a morphism of Hopf $*$-algebras.

Step 1: β_z exists and is unique as a $*$-algebra morphism. Given any $\{k,\ell\}\subseteq\mathbb{N}_0$, any $\mathfrak{c}\in\{\circ,\bullet\}^{\otimes k}$, any $\mathfrak{d}\in\{\circ,\bullet\}^{\otimes \ell}$, any partition p of Π_ℓ^k such that $(\mathfrak{c},\mathfrak{d},p)\in\mathcal{G}_G$, any $g\in[\![n]\!]^{\otimes k}$ and any $j\in[\![n]\!]^{\otimes\ell}$, we have to prove that $\mathrm{r}_{\mathfrak{d}}^{\mathfrak{c}}(p)_{j,g}(\overline{u},u)=0$ (note the inverted order of u and $\overline{u}$ compared to the definition of B) in order to establish that β_z exists as a morphism of $*$-algebras. By Lemma 7.4 (a) (iii) the assumption that $(\mathfrak{c},\mathfrak{d},p)\in\mathcal{G}_G$ guarantees that also $(z.\mathfrak{c},z.\mathfrak{d},p)\in\mathcal{G}_G$ and thus $\mathrm{r}_{z.\mathfrak{d}}^{z.\mathfrak{c}}(p)_{j,g}(u,\overline{u})=0$ by the definition of B. In consequence, it is enough to prove that $\mathrm{r}_{\mathfrak{d}}^{\mathfrak{c}}(p)_{j,g}(\overline{u},u)=\mathrm{r}_{z.\mathfrak{d}}^{z.\mathfrak{c}}(p)_{j,g}(u,\overline{u})$.

And, indeed, by definition, the element $\mathrm{r}_{\mathfrak{d}}^{\mathfrak{c}}(p)_{j,g}(\overline{u},u)$ of B is given by

$$\textstyle\sum_{i\in[\![n]\!]^{\otimes\ell}}\zeta(p,\ker(g\,\blacksquare\, i))\,\overrightarrow{\prod}_{b=1}^{\ell}(u_{j_b,i_b}^*)^{\mathfrak{d}_b}-\sum_{h\in[\![n]\!]^{\otimes k}}\zeta(p,\ker(h\,\blacksquare\, j))\,\overrightarrow{\prod}_{a=1}^{k}(u_{h_a,g_a}^*)^{\mathfrak{c}_a},$$

which we can obviously just as well write as

$$\textstyle\sum_{i\in[\![n]\!]^{\otimes\ell}}\zeta(p,\ker(g\,\blacksquare\,i))\overrightarrow{\prod}_{b=1}^{\ell}u_{j_b,i_b}^{z.\mathfrak{d}_b}-\sum_{h\in[\![n]\!]^{\otimes k}}\zeta(p,\ker(h\,\blacksquare\,j))\overrightarrow{\prod}_{a=1}^{k}u_{h_a,g_a}^{z.\mathfrak{c}_a},$$

i.e., as $\mathrm{r}_{z.\mathfrak{d}}^{z.\mathfrak{c}}(p)_{j,g}(u,\overline{u})$.

Step 2: β_z is a Hopf $*$-algebra morphism. By Step 1 it suffices to check the conditions on the $*$-generators of B. If Δ_B, ϵ_B and S_B are the co-multiplication, co-unit and antipode of B, respectively, then for any $\{i,j\}\subseteq[\![n]\!]$,

$$\begin{aligned}((\beta_z\otimes\beta_z)\circ\Delta_B)(u_{j,i})&=\textstyle\sum_{s=1}^{n}(\beta_z\otimes\beta_z)(u_{j,s}\otimes u_{s,i})\\&=\textstyle\sum_{s=1}^{n}(u_{j,s}^{*}\otimes u_{s,i}^{*})\\&=\Delta_B(u_{j,i}^{*})\\&=(\Delta_B\circ\beta_z)(u_{j,i}).\end{aligned}$$

Likewise, for any $\{i,j\}\subseteq[\![n]\!]$,

$$(\epsilon_B\circ\beta_z)(u_{j,i})=\epsilon_B(u_{j,i}^{*})=\delta_{j,i}=\epsilon_B(u_{j,i})$$

and

$$(S_B\circ\beta_z)(u_{j,i})=S_B(u_{j,i}^{*})=u_{i,j}=\beta_z(u_{i,j}^{*})=(\beta_z\circ S_B)(u_{j,i})$$

since u and $\overline{u}$ are both unitary. Thus, β_z is a Hopf $*$-morphism, which is all we needed to prove. ☐

LEMMA 7.7. *(q,β) is a co-central invariant action of $\mathbb{Z}_2$ by Hopf $*$-algebra automorphisms on B.*

PROOF. If $(B_g)_{g\in\mathbb{Z}_2}$ denotes the $\mathbb{Z}_2$-grading of B associated with q, then by $\beta_1=\mathrm{id}_B$ it suffices to prove for any $b\in B$ that $\beta_z(b)\in B_1$ if $b\in B_1$ and that $\beta_z(b)\in B_z$ if $b\in B_z$. Moreover, it is enough to check this on linear generators.

If Δ_B is the co-multiplication of B, then for any $g\in\mathbb{Z}_2$, any $m\in\mathbb{N}_0$, any $\mathfrak{c}\in\{\circ,\bullet\}^{\otimes m}$ and any $\{i,j\}\subseteq[\![n]\!]^{\otimes m}$ the element $b\coloneqq\overrightarrow{\prod}_{t=1}^{m}u_{j_t,i_t}^{\mathfrak{c}_t}$ belongs to B_g if and only if $((\mathrm{id}\otimes q)\circ\Delta_B)(b)=b\otimes g$, where

$$\begin{aligned}((\mathrm{id}\otimes q)\circ\Delta_B)(b)&=(\mathrm{id}\otimes q)(\Delta_B(\overrightarrow{\prod}_{t=1}^{m}u_{j_t,i_t}^{\mathfrak{c}_t}))\\&=\overrightarrow{\prod}_{t=1}^{m}((\mathrm{id}\otimes q)\circ\Delta_B)(u_{j_t,i_t}^{\mathfrak{c}_t})\\&=\overrightarrow{\prod}_{t=1}^{m}\left(\textstyle\sum_{s_t=1}^{m}(\mathrm{id}\otimes q)(u_{j_t,s_t}^{\mathfrak{c}_t}\otimes u_{s_t,i_t}^{\mathfrak{c}_t})\right)\\&=\overrightarrow{\prod}_{t=1}^{m}\left(\textstyle\sum_{s_t=1}^{m}\delta_{s_t,i_t}\,u_{j_t,s_t}^{\mathfrak{c}_t}\otimes z^{\mathfrak{c}_t}\right)\\&=\overrightarrow{\prod}_{t=1}^{m}\left(u_{j_t,i_t}^{\mathfrak{c}_t}\otimes z^{\mathfrak{c}_t}\right)\\&=\left(\overrightarrow{\prod}_{t=1}^{m}u_{j_t,i_t}^{\mathfrak{c}_t}\right)\otimes z^{\sum_{t=1}^{m}\sigma(\mathfrak{c}_t)}\\&=b\otimes z^{\Sigma_{\mathfrak{c}}^{\varnothing}}.\end{aligned}$$

Furthermore, by definition,

$$\beta_z(b)=\overrightarrow{\prod}_{t=1}^{n}\beta_z(u_{j_t,i_t}^{\mathfrak{c}_t})=\overrightarrow{\prod}_{t=1}^{n}u_{j_t,i_t}^{z.\mathfrak{c}_t}$$

and therefore

$$((\mathrm{id}\otimes q)\circ\Delta_B)(\beta_z(b)) = \beta_z(b)\otimes z^{\sum_{t=1}^m \sigma(z.\mathfrak{c}_t)} = \beta_z(b)\otimes z^{-\Sigma_{\mathfrak{c}}^{\varnothing}}.$$

Because $z^{-1} = z$ that proves the claim. ☐

LEMMA 7.8. *There exists a unique surjective co-central morphism $p\colon A\to\mathbb{C}[\Gamma]$ of Hopf $*$-algebras such that for any $q\in[\![2]\!]$ and any $\{i,j\}\subseteq[\![n]\!]$,*

$$v_{j,i}^{(q)}\mapsto\delta_{j,i}z.$$

PROOF. In the same way as before we show that a unique morphism of $*$-algebras with the prescribed values exists and then prove that to be a co-central Hopf $*$-algebra morphism. It will then obviously also be surjective.

Step 1: p exists and is unique as a $*$-algebra morphism. Because $z=z^*$, obviously, $v_{j,i}^{(q)}$ is mapped to a self-adjoint element of $\mathbb{C}[\mathbb{Z}_2]$ for any $\{i,j\}\subseteq[\![n]\!]$ and $q\in[\![2]\!]$. Thus, in order to prove the existence of p as a $*$-algebra morphism it is enough to check the relations coming from the partitions.

For any $\{k,\ell\}$, any set-theoretical partition p of Π_ℓ^k with $(k,\ell,p)\in\mathcal{G}_K$, any $g\in[\![n]\!]^{\otimes k}$, any $j\in[\![n]\!]^{\otimes\ell}$, and any $q\in[\![2]\!]$, if $1_{n\times n}$ is the identity $n\times n$-matrix, then $v^{(q)}$ would be mapped to $1_{n\times n}\otimes z$ by $\mathrm{id}\otimes p$. Thus we need to prove that the element

$$\begin{aligned}
\mathrm{r}_\ell^k(p)_{j,g}(1_{n\times n}\otimes z) &= \textstyle\sum_{i\in[\![n]\!]^{\otimes\ell}}\zeta(p,\ker(g\,^{\blacksquare}_{\blacksquare}\,i))\overrightarrow{\prod}_{b=1}^{\ell}(\delta_{j_b,i_b}\,z)\\
&\qquad -\textstyle\sum_{h\in[\![n]\!]^{\otimes k}}\zeta(p,\ker(h\,^{\blacksquare}_{\blacksquare}\,j))\overrightarrow{\prod}_{a=1}^{k}(\delta_{h_a,g_a}\,z)\\
&= \textstyle\sum_{i\in[\![n]\!]^{\otimes\ell}}\delta_{j,i}\,\zeta(p,\ker(g\,^{\blacksquare}_{\blacksquare}\,i))\,z^k\\
&\qquad -\textstyle\sum_{h\in[\![n]\!]^{\otimes k}}\delta_{h,g}\,\zeta(p,\ker(h\,^{\blacksquare}_{\blacksquare}\,j))\,z^\ell\\
&= \zeta(p,\ker(g\,^{\blacksquare}_{\blacksquare}\,j))\,(z^\ell-z^k)
\end{aligned}$$

of $\mathbb{C}[\mathbb{Z}_2]$ is zero. And, indeed, because the assumption that $(k,\ell,p)\in\mathcal{G}_K$ requires $k+\ell$ to be even by Lemma 7.4 (a) (i) we can infer that $z^\ell = 1z^{-\ell} = z^{k+\ell}z^{-\ell} = z^k$ and thus $\mathrm{r}_\ell^k(p)_{j,g}(1_{n\times n}\otimes z)=0$. Hence, such a morphism p of $*$-algebras does exist and is unique.

Step 2: p is a Hopf $*$-algebra morphism. By Step 1 it is enough to verify the conditions on the $*$-generators. If Δ_A, ϵ_A and S_A are the co-multplication, co-unit and antipode of A, respectively, and if Δ, ϵ and S are those of $\mathbb{C}[\mathbb{Z}_2]$, then for any $q\in[\![2]\!]$ and any $\{i,j\}\subseteq[\![n]\!]$,

$$\begin{aligned}
((p\otimes p)\circ\Delta_A)(v_{j,i}^{(q)}) &= \textstyle\sum_{s=1}^n(p\otimes p)(v_{j,s}^{(q)}\otimes v_{s,i}^{(q)})\\
&= \textstyle\sum_{s=1}^n(\delta_{j,s}\,z)\otimes(\delta_{s,i}\,z)\\
&= \delta_{j,i}\,z\otimes z\\
&= \Delta(\delta_{j,i}\,z)\\
&= (\Delta\circ p)(v_{j,i}^{(q)})
\end{aligned}$$

and

$$(\epsilon\circ p)(v_{j,i}^{(q)}) = \epsilon(\delta_{j,i}\,z) = \delta_{j,i} = (\epsilon_A)(v_{j,i}^{(q)})$$

and

$$(S \circ p)(v_{j,i}^{(q)}) = S(\delta_{j,i}\, z) = \delta_{j,i}\, z^{-1} = \delta_{i,j}\, z = p(v_{i,j}^{(q)}) = (p \circ S_A)(v_{j,i}^{(q)}).$$

Thus, p is a morphism of Hopf $*$-algebras.

Step 3: p is co-central. It is enough to prove the co-centrality condition on $*$-generators since the mappings on both sides are $*$-algebra morphisms. If $\gamma_{A,A}$ is the symmetry isomorphism of $A \otimes A$, then for any $q \in [\![2]\!]$ and any $\{i,j\} \subseteq [\![n]\!]$,

$$\begin{aligned}
((\mathrm{id} \otimes p) \circ \Delta_A)(v_{j,i}^{(q)}) &= \textstyle\sum_{s=1}^{n} (\mathrm{id} \otimes p)(v_{j,s}^{(q)} \otimes v_{s,i}^{(q)}) \\
&= \textstyle\sum_{s=1}^{n} \delta_{s,i}\, v_{j,s}^{(q)} \otimes z \\
&= v_{j,i}^{(q)} \otimes z \\
&= \textstyle\sum_{s=1}^{n} \delta_{j,s}\, v_{s,i}^{(q)} \otimes z \\
&= \textstyle\sum_{s=1}^{n} (\mathrm{id} \otimes p)(v_{s,i}^{(q)} \otimes v_{j,s}^{(q)}) \\
&= \textstyle\sum_{s=1}^{n} ((\mathrm{id} \otimes p) \circ \gamma_{A,A})(v_{j,s}^{(q)} \otimes v_{s,i}^{(q)}) \\
&= ((\mathrm{id} \otimes p) \circ \gamma_{A,A} \circ \Delta_A)(v_{j,i}^{(q)}).
\end{aligned}$$

That is what we needed to see. □

LEMMA 7.9. *There exists a unique action α of $\mathbb{Z}_2$ on A by Hopf $*$-automorphisms such that α_z is the unique Hopf $*$-algebra endomorphism of A such that for any $q \in [\![2]\!]$ and any $\{i,j\} \subseteq [\![n]\!]$,*

$$v_{j,i}^{(q)} \mapsto v_{j,i}^{(3-q)}.$$

PROOF. The fact that α_z exists and is a Hopf $*$-algebra endomorphism of $A = \mathrm{Pol}(K) * \mathrm{Pol}(K)$ is guaranteed by the universal property of the co-product of Hopf $*$-algebras. It is obviously involutive by definition, which is why α is a well-defined action of $\mathbb{Z}_2$ on A by Hopf $*$-algebra automorphisms. □

LEMMA 7.10. *(p, α) is a co-central invariant action of $\mathbb{Z}_2$ by Hopf $*$-algebra automorphisms on A.*

PROOF. Let $(A_g)_{g \in \mathbb{Z}_2}$ be the $\mathbb{Z}_2$-grading of A associated with p. Then, all we have to prove is that $\alpha_z(a) \in A_1$ for any $a \in A_1$ and that $\alpha_z(a) \in A_z$ for any $a \in A_z$. In fact, we can confine ourselves to checking this on elements a of a linear generator set of A.

Given any $m \in \mathbb{N}_0$, any $q \in [\![2]\!]^{\otimes m}$ and any $\{i,j\} \subseteq [\![n]\!]^{\otimes m}$, if $a := \overrightarrow{\prod}_{t=1}^{m} v_{j_t,i_t}^{(q_t)}$ and if Δ_A is the co-multplication of A, then

$$\begin{aligned}((\mathrm{id}\otimes p)\circ\Delta_A)(a) &= \overrightarrow{\prod}_{t=1}^{m}(\mathrm{id}\otimes p)(\Delta_A(v_{j_t,i_t}^{(q_t)}))\\ &= \overrightarrow{\prod}_{t=1}^{m}\left(\textstyle\sum_{s_t=1}^{n}(\mathrm{id}\otimes p)(v_{j_t,s_t}^{(q_t)}\otimes v_{s_t,i_t}^{(q_t)})\right)\\ &= \overrightarrow{\prod}_{t=1}^{m}\left(\textstyle\sum_{s_t=1}^{n}\delta_{s_t,i_t}\, v_{j_t,s_t}^{(q_t)}\otimes z\right)\\ &= \overrightarrow{\prod}_{t=1}^{m}\left(v_{j_t,i_t}^{(q_t)}\otimes z\right)\\ &= a\otimes z^m.\end{aligned}$$

This shows that $a \in A_1$ if and only if m is even and that $a \in A_z$ if and only if m is odd. Since

$$\alpha_z(a) = \overrightarrow{\prod}_{t=1}^{m}\alpha_z(v_{j_t,i_t}^{(q_t)}) = \overrightarrow{\prod}_{t=1}^{m} v_{j_t,i_t}^{(z.q_t)},$$

i.e., since α does not affect the length of monomials in the $*$-generators, it is clear that, necessarily, $\{a, \alpha_z(a)\} \subseteq A_1$ or $\{a, \alpha_z(a)\} \subseteq A_z$. In conclusion, $(A_g)_{g\in\mathbb{Z}_2}$ is α-invariant, which is what we had to prove. ☐

LEMMA 7.11. *There exists a unique morphism of Hopf $*$-algebras $f\colon B \to A^{t,(p,\alpha(-\iota))}$ such that for any $\{i,j\} \subseteq [\![n]\!]$,*

$$u_{j,i} \mapsto v_{j,i}^{(1)} \otimes z.$$

PROOF. By the definition of $A^{t,(p,\alpha)}$ as a Hopf $*$-subalgebra of $A \rtimes_\alpha \mathbb{C}[\Gamma]$ it is enough to prove that there exists a unique morphism $g\colon B \to A \rtimes_\alpha \mathbb{C}[\Gamma]$ of Hopf $*$-algebras which satisfies $u_{j,i} \mapsto v_{j,i}^{(1)} \otimes z$ for any $\{i,j\} \subseteq [\![n]\!]$ and whose image lies within $A^{t,(p,\alpha)}$. As many times before we first show that such a map g exists and is unique as a morphism of $*$-algebras and then check that it respects Hopf $*$-algebra operations.

Step 1: g exists and is unique as a $*$-algebra morphism. By the universal property of B the alleged g exists and is unique at least as a morphism of $*$-algebras if $v^{(1)} \otimes z$ satisfies the defining $*$-relations for u.

For any $\{i,j\} \subseteq [\![n]\!]$ the definition of $A \rtimes_\alpha \mathbb{C}[\Gamma]$ implies that the adjoint of the element $v_{j,i}^{(1)} \otimes z$ is given by

$$(v_{j,i}^{(1)} \otimes z)^* = \alpha_{z^{-1}}((v_{j,i}^{(1)})^*) \otimes z^{-1} = \alpha_z(v_{j,i}^{(1)}) \otimes z = v_{j,i}^{(z.1)} \otimes z = v_{j,i}^{(2)} \otimes z.$$

Hence, in order to establish the existence of g we have to check that for any $\{k,\ell\} \subseteq \mathbb{N}_0$, any $\mathfrak{c} \in \{\circ,\bullet\}^{\otimes k}$, any $\mathfrak{d} \in \{\circ,\bullet\}^{\otimes \ell}$, any partition p of Π_ℓ^k such that $(\mathfrak{c},\mathfrak{d},p) \in \mathcal{G}_G$, any $g \in [\![n]\!]^{\otimes k}$ and any $j \in [\![n]\!]^{\otimes \ell}$ the element $\mathrm{r}_{\mathfrak{d}}^{\mathfrak{c}}(p)_{j,g}(v^{(1)} \otimes z, v^{(2)} \otimes z)$ of $A \rtimes_\alpha \mathbb{C}[\Gamma]$ is zero.

Because, by Lemma 7.4 (a) (iv), our assumption that $(\mathfrak{c},\mathfrak{d},p) \in \mathcal{G}_G$ implies that $(k,\ell,p) \in \mathcal{G}_K$ and thus $\mathrm{r}_\ell^k(p)_{j,g}(v^{(q)}) = 0$ for any $q \in [\![2]\!]$ we can prove that $\mathrm{r}_{\mathfrak{d}}^{\mathfrak{c}}(p)_{j,g}(v^{(1)} \otimes z, v^{(2)} \otimes z) = 0$ by giving some $x \in \{\circ,\bullet\}$ and $m \in \mathbb{N}_0$ such that $\mathrm{r}_{\mathfrak{d}}^{\mathfrak{c}}(p)_{j,g}(v^{(1)} \otimes z, v^{(2)} \otimes z)$ is equal to $\mathrm{r}_\ell^k(p)_{j,g}(v^{(\phi_x)} \otimes z) \otimes z^m$. More precisely, we will choose $m \coloneqq k$ and let x be that element of $\{\circ, \bullet\}$ with the property that $\mathfrak{c}_a = z^{a-1}.x$ for any $a \in [\![k]\!]$ and $\mathfrak{d}_b = z^{b-1}.x$ for any $b \in [\![\ell]\!]$ guaranteed to exist by Lemma 7.4 (a) (ii).

And, indeed, by definition, $\mathrm{r}^{\mathfrak{c}}_{\mathfrak{d}}(p)_{j,g}(v^{(1)} \otimes z, v^{(2)} \otimes z)$ is given by

$$\textstyle\sum_{i \in [\![n]\!]^{\otimes \ell}} \zeta(p, \ker(g \blacksquare i)) \overrightarrow{\prod}_{b=1}^{\ell} (v^{(1)}_{j_b, i_b} \otimes z)^{\mathfrak{d}_b} \\ - \textstyle\sum_{h \in [\![n]\!]^{\otimes k}} \zeta(p, \ker(h \blacksquare j)) \overrightarrow{\prod}_{a=1}^{k} (v^{(1)}_{h_a, g_a} \otimes z)^{\mathfrak{c}_a}.$$

By what we saw initially about adjoints in $A \rtimes_\alpha \mathbb{C}[\Gamma]$ this can be simplified to

$$\textstyle\sum_{i \in [\![n]\!]^{\otimes \ell}} \zeta(p, \ker(g \blacksquare i)) \overrightarrow{\prod}_{b=1}^{\ell} (v^{(\phi_{\mathfrak{d}_b})}_{j_b, i_b} \otimes z) \\ - \textstyle\sum_{h \in [\![n]\!]^{\otimes k}} \zeta(p, \ker(h \blacksquare j)) \overrightarrow{\prod}_{a=1}^{k} (v^{(\phi_{\mathfrak{c}_a})}_{h_a, g_a} \otimes z).$$

The definition of the multiplication of $A \rtimes_\alpha \mathbb{C}[\Gamma]$ implies that this element is the same as

$$\textstyle\sum_{i \in [\![n]\!]^{\otimes \ell}} \zeta(p, \ker(g \blacksquare i)) \left(\overrightarrow{\prod}_{b=1}^{\ell} \alpha_{z^{b-1}}(v^{(\phi_{\mathfrak{d}_b})}_{j_b, i_b})\right) \otimes z^{\ell} \\ - \textstyle\sum_{h \in [\![n]\!]^{\otimes k}} \zeta(p, \ker(h \blacksquare j)) \left(\overrightarrow{\prod}_{a=1}^{k} \alpha_{z^{a-1}}(v^{(\phi_{\mathfrak{c}_a})}_{h_a, g_a})\right) \otimes z^{k},$$

i.e., by the definition of α the same as

$$\textstyle\sum_{i \in [\![n]\!]^{\otimes \ell}} \zeta(p, \ker(g \blacksquare i)) \left(\overrightarrow{\prod}_{b=1}^{\ell} v^{(z^{b-1}.\phi_{\mathfrak{d}_b})}_{j_b, i_b}\right) \otimes z^{\ell} \\ - \textstyle\sum_{h \in [\![n]\!]^{\otimes k}} \zeta(p, \ker(h \blacksquare j)) \left(\overrightarrow{\prod}_{a=1}^{k} v^{(z^{a-1}.\phi_{\mathfrak{c}_a})}_{h_a, g_a}\right) \otimes z^{k}.$$

From the definition of x it now follows that $z^{a-1}.\phi_{\mathfrak{c}_a} = z^{a-1}.\phi_{z^{a-1}.x} = (z^{a-1}z^{a-1}).\phi_x = z^{2a-2}.\phi_x = \phi_x$ and likewise $z^{b-1}.\phi_{\mathfrak{d}_b} = \phi_x$ for any $a \in [\![k]\!]$ and $b \in [\![\ell]\!]$. Moreover, by Lemma 7.4 (a) (i) the assumption that $(\mathfrak{c}, \mathfrak{d}, p) \in \mathcal{G}_G$ also requires that $k + \ell$ is even and thus that $z^m = z^k = z^\ell$. Hence, we have shown that $\mathrm{r}^{\mathfrak{c}}_{\mathfrak{d}}(p)_{j,g}(v^{(1)} \otimes z, v^{(2)} \otimes z)$ is given by

$$\textstyle\sum_{i \in [\![n]\!]^{\otimes \ell}} \zeta(p, \ker(g \blacksquare i)) \left(\overrightarrow{\prod}_{b=1}^{\ell} v^{(\phi_x)}_{j_b, i_b}\right) \otimes z^{m} \\ - \textstyle\sum_{h \in [\![n]\!]^{\otimes k}} \zeta(p, \ker(h \blacksquare j)) \left(\overrightarrow{\prod}_{a=1}^{k} v^{(\phi_x)}_{h_a, g_a}\right) \otimes z^{m},$$

which is precisely $\mathrm{r}^{k}_{\ell}(p)_{j,g}(v^{(\phi_x)} \otimes z) \otimes z^m$. That is what we needed to see.

Step 2: g is a Hopf $*$-algebra morphism. According to Step 1 it suffices to check the conditions on the $*$-generators of B. Let Δ_B, ϵ_B and S_B be, respectively, the co-multplication, the co-unit and the antipode of B, let $\Delta_{\rtimes}$, $\epsilon_{\rtimes}$ and $S_{\rtimes}$ be those of $A \rtimes_\alpha \mathbb{C}[\Gamma]$, moreover, Δ_A, ϵ_A and S_A those of A and, finally, Δ, ϵ and S those of $\mathbb{C}[\mathbb{Z}_2]$. If, furthermore, $\gamma_{A, \mathbb{C}[\mathbb{Z}_2]}$ is the linear map which flips the two tensor factors in $A \otimes \mathbb{C}[\mathbb{Z}_2]$ then for any $\{i,j\} \subseteq [\![n]\!]$,

$$\begin{aligned}((g \otimes g) \circ \Delta_B)(u_{j,i}) &= \textstyle\sum_{s=1}^n (g \otimes g)(u_{j,s} \otimes u_{s,i}) \\ &= \textstyle\sum_{s=1}^n v_{j,s}^{(1)} \otimes z \otimes v_{s,i}^{(1)} \otimes z \\ &= (\mathrm{id} \otimes \gamma_{A,\mathbb{C}[\mathbb{Z}_2]} \otimes \mathrm{id})(\textstyle\sum_{s=1}^n v_{j,s}^{(1)} \otimes v_{s,i}^{(1)} \otimes z \otimes z) \\ &= (\mathrm{id} \otimes \gamma_{A,\mathbb{C}[\mathbb{Z}_2]} \otimes \mathrm{id})(\Delta_A \otimes \Delta)(v_{j,i}^{(1)} \otimes z) \\ &= \Delta_{\rtimes}(v_{j,i}^{(1)} \otimes z) \\ &= (\Delta_{\rtimes} \circ g)(u_{j,i})\end{aligned}$$

and

$$(\epsilon_{\rtimes} \circ g)(u_{j,i}) = \epsilon_{\rtimes}(v_{j,i}^{(1)} \otimes z) = (\epsilon_A \otimes \epsilon)(v_{j,i}^{(1)} \otimes z) = \delta_{j,i} = \epsilon_B(u_{j,i})$$

as well as, using what we showed initially about the adjoints in $A \rtimes_\alpha \mathbb{C}[\Gamma]$,

$$\begin{aligned}(S_{\rtimes} \circ g)(u_{j,i}) &= S_{\rtimes}(v_{j,i}^{(1)} \otimes z) \\ &= S_A(\alpha_{z^{-1}}(v_{j,i}^{(1)})) \otimes S(z) \\ &= S_A(v_{j,i}^{(z^{-1}.1)}) \otimes z^{-1} \\ &= S_A(v_{j,i}^{(2)}) \otimes z \\ &= v_{i,j}^{(2)} \otimes z \\ &= (v_{i,j}^{(1)} \otimes z)^* \\ &= g(u_{i,j})^* \\ &= g(u_{i,j}^*) \\ &= (g \circ S_B)(u_{j,i}).\end{aligned}$$

Hence, g is indeed a morphism of Hopf $*$-algebras.

Step 2: The image of g lies inside $A^{t,(p,\alpha)}$. For any $\{i,j\} \subseteq [\![n]\!]$ the element $v_{j,i}^{(1)} \otimes z$ of $A \rtimes_\alpha \mathbb{C}[\Gamma]$ belongs to the Hopf $*$-subalgebra $A^{t,(p,\alpha)}$ because $((\mathrm{id} \otimes p) \circ \Delta_A)(v_{j,i}^{(1)}) = \sum_{s=1}^n (\mathrm{id} \otimes p)(v_{j,s}^{(1)} \otimes v_{s,i}^{(1)}) = \sum_{s=1}^n \delta_{s,i}\, v_{j,s}^{(1)} \otimes z = v_{j,i}^{(1)} \otimes z$. Since g is a morphism of $*$-algebras and since $\{v_{j,i}^{(1)} \otimes z\}_{i,j=1}^n$ is a $*$-generator set of its image this proves the image in its entirety to lie in $A \rtimes_\alpha \mathbb{C}[\Gamma]$. By the initial remark this proves the existence and uniqueness of f with all the asserted properties. □

LEMMA 7.12. *There exists a unique morphism of Hopf $*$-algebras $f'\colon A \to B^{t,(q,\beta(-\iota))}$ such that for any $\{i,j\} \subseteq [\![n]\!]$,*

$$v_{j,i}^{(1)} \mapsto u_{j,i} \otimes z \quad \wedge \quad v_{j,i}^{(2)} \mapsto u_{j,i}^* \otimes z.$$

PROOF. Again, because $B^{t,(q,\beta(-\iota))}$ is a Hopf $*$-subalgebra of $B \rtimes_{\beta(-\iota)} \mathbb{C}[\Gamma]$ it suffices to give a morphism $g'\colon A \to B \rtimes_{\beta(-\iota)} \mathbb{C}[\Gamma]$ of Hopf $*$-algebras which satisfies $v_{j,i}^{(q)} \mapsto u_{j,i}^{\psi_q} \otimes z$ for any $q \in [\![2]\!]$ and $\{i,j\} \subseteq [\![n]\!]$ and which maps to $B^{t,(q,\beta(-\iota))}$.

Step 1: g' exists as a $*$-algebra morphism. To prove that there is at least a unique morphism g' of $*$-algebras with the prescribed values we need to check that each of $u \otimes z$ and $\overline{u} \otimes z$ has only self-adjoint entries and that $u \otimes z$ satisfies the relations of $v^{(1)}$ and $\overline{u} \otimes z$ those of $v^{(2)}$.

Indeed, for any $x \in \{\circ, \bullet\}$ and $\{i, j\} \subseteq [\![n]\!]$ by the definition of $B \rtimes_{\beta(-\iota)} \mathbb{C}[\Gamma]$,

$$(u_{j,i}^{x} \otimes z)^{*} = (\beta(-\iota))_{z^{-1}}((u_{j,i}^{x})^{*}) \otimes z^{-1} = \beta_{z}(u_{j,i}^{z.x}) \otimes z = u_{j,i}^{z^{2}.x} \otimes z = u_{j,i}^{x} \otimes z.$$

It remains to check the relations induced by the partitions. Equivalently, given any $\{k, \ell\} \subseteq \mathbb{N}_0$, any partition p of Π_{ℓ}^{k} such that $(k, \ell, p) \in \mathcal{G}_K$, any $g \in [\![n]\!]^{\otimes k}$, any $j \in [\![n]\!]^{\otimes \ell}$ and any $q \in [\![2]\!]$ we have to prove that $\mathrm{r}_{\ell}^{k}(p)_{j,g}(u^{\psi_q} \otimes z) = 0$.

Let $m \coloneqq k$, let $c \coloneqq \psi_q$ and let $\mathfrak{c} \in \{\circ, \bullet\}^{\otimes k}$ and $\mathfrak{d} \in \{\circ, \bullet\}^{\otimes \ell}$ be such that $\mathfrak{c}_a = z^{a-1}.c$ for any $a \in [\![k]\!]$ and $\mathfrak{d}_b = z^{b-1}.c$ for any $b \in [\![\ell]\!]$. Then by Lemma 7.4 (a) (ii) the assumption that $(k, \ell, p) \in \mathcal{G}_K$ guarantees that $(\mathfrak{c}, \mathfrak{d}, p) \in \mathcal{G}_G$. Because thus $\mathrm{r}_{\mathfrak{d}}^{\mathfrak{c}}(p)_{j,g}(u, \overline{u}) = 0$ by the definition of B it suffices to prove that $\mathrm{r}_{\ell}^{k}(p)_{j,g}(u^{\psi_q} \otimes z)$ is the same as $\mathrm{r}_{\mathfrak{d}}^{\mathfrak{c}}(p)_{j,g}(u, \overline{u}) \otimes z^{m}$.

And, indeed, by definition the element $\mathrm{r}_{\ell}^{k}(p)_{j,g}(u^{\psi_q} \otimes z)$ of $B \rtimes_{\beta(-\iota)} \mathbb{C}[\Gamma]$ is given by

$$\textstyle\sum_{i \in [\![n]\!]^{\otimes \ell}} \zeta(p, \ker(g \,\blacksquare\, i)) \overrightarrow{\prod}_{b=1}^{\ell} (u_{j_b, i_b}^{\psi_q} \otimes z) \\ \qquad - \textstyle\sum_{h \in [\![n]\!]^{\otimes k}} \zeta(p, \ker(h \,\blacksquare\, j)) \overrightarrow{\prod}_{a=1}^{k} (u_{h_a, g_a}^{\psi_q} \otimes z).$$

The definition of the multiplication in $B \rtimes_{\beta(-\iota)} \mathbb{C}[\Gamma]$ implies that this is the same as

$$\textstyle\sum_{i \in [\![n]\!]^{\otimes \ell}} \zeta(p, \ker(g \,\blacksquare\, i)) \left(\overrightarrow{\prod}_{b=1}^{\ell} (\beta(-\iota))_{z^{b}}(u_{j_b, i_b}^{\psi_q})\right) \otimes z^{\ell} \\ \qquad - \textstyle\sum_{h \in [\![n]\!]^{\otimes k}} \zeta(p, \ker(h \,\blacksquare\, j)) \left(\overrightarrow{\prod}_{a=1}^{k} (\beta(-\iota))_{z^{a}}(u_{h_a, g_a}^{\psi_q})\right) \otimes z^{k},$$

which agrees with

$$\textstyle\sum_{i \in [\![n]\!]^{\otimes \ell}} \zeta(p, \ker(g \,\blacksquare\, i)) \left(\overrightarrow{\prod}_{b=1}^{\ell} u_{j_b, i_b}^{z^{-b}.\psi_q}\right) \otimes z^{\ell} \\ \qquad - \textstyle\sum_{h \in [\![n]\!]^{\otimes k}} \zeta(p, \ker(h \,\blacksquare\, j)) \left(\overrightarrow{\prod}_{a=1}^{k} u_{h_a, g_a}^{z^{-a}.\psi_q}\right) \otimes z^{k}.$$

Since the assumption that $(k, \ell, p) \in \mathcal{G}_K$ demands that $k+\ell$ is even by Lemma 7.4 (a) (i) and thus that $z^{m} = z^{\ell} = z^{k}$, we can rewrite the above as

$$\textstyle\sum_{i \in [\![n]\!]^{\otimes \ell}} \zeta(p, \ker(g \,\blacksquare\, i)) \left(\overrightarrow{\prod}_{b=1}^{\ell} u_{j_b, i_b}^{z^{b}.c}\right) \otimes z^{m} \\ \qquad - \textstyle\sum_{h \in [\![n]\!]^{\otimes k}} \zeta(p, \ker(h \,\blacksquare\, j)) \left(\overrightarrow{\prod}_{a=1}^{k} u_{h_a, g_a}^{z^{a}.c}\right) \otimes z^{m},$$

which is the same as $\mathrm{r}_{\mathfrak{d}}^{\mathfrak{c}}(p)_{j,g}(u, \overline{u}) \otimes z^{m}$, as we needed to see.

Step 2: g' is a Hopf $*$-algebra morphism. By Step 1 we only need to check the conditions on the $*$-generators of B. Let Δ_A, ϵ_A and S_A be, respectively, the co-multplication, the co-unit and the antipode of A, let $\Delta_{\rtimes}$, $\epsilon_{\rtimes}$ and $S_{\rtimes}$ be those of $B \rtimes_{\beta(-\iota)} \mathbb{C}[\Gamma]$, moreover, Δ_B, ϵ_B and S_B those of B and, finally, Δ, ϵ and S those of

$\mathbb{C}[\mathbb{Z}_2]$. If, furthermore, $\gamma_{B,\mathbb{C}[\mathbb{Z}_2]}$ is the linear map which flips the two tensor factors in $B \otimes \mathbb{C}[\mathbb{Z}_2]$ then for any $q \in [\![2]\!]$ and $\{i,j\} \subseteq [\![n]\!]$,

$$\begin{aligned}((g' \otimes g') \circ \Delta_A)(v_{j,i}^{(q)}) &= \textstyle\sum_{s=1}^n (g' \otimes g')(v_{j,s}^{(q)} \otimes v_{s,i}^{(q)}) \\ &= \textstyle\sum_{s=1}^n u_{j,s}^{\psi_q} \otimes z \otimes u_{s,i}^{\psi_q} \otimes z \\ &= (\mathrm{id} \otimes \gamma_{B,\mathbb{C}[\mathbb{Z}_2]} \otimes \mathrm{id})(\textstyle\sum_{s=1}^n u_{j,s}^{\psi_q} \otimes u_{s,i}^{\psi_q} \otimes z \otimes z) \\ &= (\mathrm{id} \otimes \gamma_{B,\mathbb{C}[\mathbb{Z}_2]} \otimes \mathrm{id})(\Delta_B \otimes \Delta)(u_{j,i}^{\psi_q} \otimes z) \\ &= \Delta_\rtimes(u_{j,i}^{\psi_q} \otimes z) \\ &= (\Delta_\rtimes \circ g')(v_{j,i}^{(q)})\end{aligned}$$

and

$$(\epsilon_\rtimes \circ g')(v_{j,i}^{(q)}) = \epsilon_\rtimes(u_{j,i}^{\psi_q} \otimes z) = (\epsilon_B \otimes \epsilon)(u_{j,i}^{\psi_q} \otimes z) = \delta_{j,i} = \epsilon_A(v_{j,i}^{(q)})$$

as well as

$$\begin{aligned}(S_\rtimes \circ g')(v_{j,i}^{(q)}) &= S_\rtimes(u_{j,i}^{\psi_q} \otimes z) \\ &= S_B((\beta(-\iota))_{z^{-1}}(u_{j,i}^{\psi_q})) \otimes S(z) \\ &= S_B(u_{j,i}^{z.\psi_q}) \otimes z^{-1} \\ &= u_{i,j}^{z^2.\psi_q} \otimes z \\ &= u_{i,j}^{\psi_q} \otimes z \\ &= g'(v_{i,j}^{(q)}) \\ &= (g' \circ S_A)(v_{j,i}^{(q)}).\end{aligned}$$

Hence, g' is indeed a morphism of Hopf $\star$-algebras.

Step 3: The image of g' lies inside $B^{t,(q,\beta(-\iota))}$. For any $q \in [\![2]\!]$ and any $\{i,j\} \subseteq [\![n]\!]$ the element $u_{j,i}^{\psi_q}$ satisfies $((\mathrm{id} \otimes q) \circ \Delta_B)(u_{j,i}^{\psi_q}) = \sum_{s=1}^n (\mathrm{id} \otimes q)(u_{j,s}^{\psi_q} \otimes u_{s,i}^{\psi_q}) = \sum_{s=1}^n \delta_{s,i}\, u_{j,s}^{\psi_q} \otimes z = u_{j,i}^{\psi_q} \otimes z$, rendering $u_{j,i}^{\psi_q} \otimes z$ an element of $B^{t,(q,\beta(-\iota))}$. Because the image of the $\star$-algebra morphism g' is generated as $\star$-algebra by $\{u_{j,i}^{\psi_q} \otimes z \,|\, q \in [\![2]\!] \wedge \{i,j\} \subseteq [\![n]\!]\} = \{u_{j,i} \otimes z, u_{j,i}^* \otimes z\}_{i,j=1}^n$ this proves this image to be contained entirely in $B^{t,(q,\beta(-\iota))}$. By what was said in the beginning that is all we needed to see. ☐

LEMMA 7.13. *If Δ_A and ϵ_A are the co-multiplication respectively co-unit of A, if Δ_B and ϵ_B are those of B, if Δ is the co-multiplication of $\mathbb{C}[\mathbb{Z}_2]$, and if ι and ι' are the inclusions of $A^{t,(p,\alpha)}$ into $A \rtimes_\alpha \mathbb{C}[\Gamma]$ respectively of $B^{t,(q,\beta(-\iota))}$ in $B \rtimes_{\beta(-\iota)} \mathbb{C}[\Gamma]$, then*

(a) $(\epsilon_A \otimes \mathrm{id}) \circ \iota \circ f = q$.
(b) $(\epsilon_B \otimes \mathrm{id}) \circ \iota' \circ f' = p$.
(c) $(\alpha_{g^{-1}} \otimes \mathrm{id}) \circ \iota \circ f = \iota \circ f \circ \beta_g$ *for any* $g \in \mathbb{Z}_2$.
(d) $(\beta_g \otimes \mathrm{id}) \circ \iota' \circ f' = \iota' \circ f' \circ \alpha_g$ *for any* $g \in \mathbb{Z}_2$.
(e) $((\iota' \circ f') \otimes \mathrm{id}) \circ \iota \circ f = (\mathrm{id} \otimes (\Delta \circ q)) \circ \Delta_B$.

(f) $((\iota \circ f) \otimes \mathrm{id}) \circ \iota' \circ f' = (\mathrm{id} \otimes (\Delta \circ p)) \circ \Delta_A$.

Proof. Since the mappings on both sides of the asserted identities are morphisms of $*$-algebras it suffices to prove the claims on $*$-generators. Moreover, since $\alpha_1 = \mathrm{id}_A$ and $\beta_1 = \mathrm{id}_B$ it is enough to check claims (c) and (d) only for $g = z$. Let $\{i, j\} \subseteq [\![n]\!]$ and $x \in \{\circ, \bullet\}$ as well as $q \in [\![2]\!]$ be arbitrary.

(a) $((\epsilon_A \otimes \mathrm{id}) \circ \iota \circ f)(u_{j,i}^x) = (\epsilon_A \otimes \mathrm{id})(v_{j,i}^{(\phi_x)} \otimes z) = \delta_{j,i}\, z = q(u_{j,i}^x)$.

(b) $((\epsilon_B \otimes \mathrm{id}) \circ \iota' \circ f')(v_{j,i}^{(q)}) = (\epsilon_B \otimes \mathrm{id})(u_{j,i}^{\psi_q} \otimes z) = \delta_{j,i}\, z = p(v_{j,i}^{(q)})$.

(c) We can compute, using only the definitions,

$$\begin{aligned}
((\alpha_{z^{-1}} \otimes \mathrm{id}) \circ \iota \circ f)(u_{j,i}^x) &= (\alpha_z \otimes \mathrm{id})(v_{j,i}^{(\phi_x)} \otimes z) \\
&= v_{j,i}^{(z.\phi_x)} \otimes z \\
&= v_{j,i}^{(\phi_{z.x})} \otimes z \\
&= (\iota \circ f)(u_{j,i}^{z.x}) \\
&= (\iota \circ f \circ \beta_z)(u_{j,i}^x).
\end{aligned}$$

(d) Similarly,

$$\begin{aligned}
((\beta_z \otimes \mathrm{id}) \circ \iota' \circ f')(v_{j,i}^{(q)}) &= (\beta_z \otimes \mathrm{id})(u_{j,i}^{\psi_q} \otimes z) \\
&= u_{j,i}^{z.\psi_q} \otimes z \\
&= u_{j,i}^{\psi_{z.q}} \otimes z \\
&= (\iota' \circ f')(v_{j,i}^{(z.q)}) \\
&= (\iota' \circ f' \circ \alpha_z)(v_{j,i}^{(q)}).
\end{aligned}$$

(e) The definitions imply that

$$\begin{aligned}
(((\iota' \circ f') \otimes \mathrm{id}) \circ \iota \circ f)(u_{j,i}^x) &= ((\iota' \circ f') \otimes \mathrm{id})(v_{j,i}^{(\phi_x)} \otimes z) \\
&= u_{j,i}^x \otimes z \otimes z \\
&= (\mathrm{id} \otimes \Delta)(u_{j,i}^x \otimes z) \\
&= \textstyle\sum_{s=1}^n \delta_{s,i}\, (\mathrm{id} \otimes \Delta)(u_{j,s}^x \otimes z) \\
&= \textstyle\sum_{s=1}^n (\mathrm{id} \otimes (\Delta \circ q))(u_{j,s}^x \otimes u_{s,i}^x) \\
&= ((\mathrm{id} \otimes (\Delta \circ q)) \circ \Delta_B)(u_{j,i}^x).
\end{aligned}$$

(f) In the same way,

$$\begin{aligned}(((\iota\circ f)\otimes\mathrm{id})\circ\iota'\circ f')(v_{j,i}^{(q)}) &= ((\iota\circ f)\otimes\mathrm{id})(u_{j,i}^{\psi_q}\otimes z)\\ &= v_{j,i}^{(q)}\otimes z\otimes z\\ &= (\mathrm{id}\otimes\Delta)(v_{j,i}^{(q)}\otimes z)\\ &= \textstyle\sum_{s=1}^{n}\delta_{s,i}\,(\mathrm{id}\otimes\Delta)(v_{j,s}^{(q)}\otimes z)\\ &= \textstyle\sum_{s=1}^{n}(\mathrm{id}\otimes(\Delta\circ p))(v_{j,s}^{(q)}\otimes v_{s,i}^{(q)})\\ &= ((\mathrm{id}\otimes(\Delta\circ p))\circ\Delta_A)(v_{j,i}^{(q)}).\end{aligned}$$

Thus, all the claims are true. ☐

Proof of the main result. The claim from Section 1.2 follows from Lemmata 7.5–7.13 in combination with Proposition 6.1 and Proposition 5.2. ☐

8. Concluding remarks

8.1. Unitary easy quantum groups as graded twists. It should be emphasized that in the process of proving the main result any unitary easy quantum group G listed in the main result has been shown to be isomorphic as a compact quantum group to the graded twist by $\mathbb{Z}_2$ of the free product of two copies of the orthogonal quantum group K obtained by forgetting the colors of the associated category of partitions.

8.2. Alternative proof. Another way of proving the main result might go as follows. As developed by Freslon in [Fre17], easy quantum groups can also be constructed via Tannaka-Krein duality from categories of partitions labeled not only with two but any number of colors. In particular, one can define direct and free products of categories of partitions by tagging partitions with the factor category they come from, concatenating them horizontally and then permuting points in any respectively such a way that, roughly said, there are no crossings between blocks with distinct tags. (In truth, it is more complicated than that.) As seen in Chapter 3, it is even possible to introduce a crossed product of the category of any cyclic group with any category of partitions equipped with an invariant action of that group.

By employing the same functor F used there, $\langle\mathcal{G}_K\rangle$ can be embedded in the crossed product of $\langle\mathcal{G}_G\rangle * \langle\mathcal{G}_G\rangle$ with the category of $\mathbb{Z}_2$ as a subcategory, which becomes full on the operator level. Via Tannaka-Krein duality F gives rise exactly to the morphism $\iota\circ f$ of Hopf $*$-algebras from B to $A\rtimes_\alpha\mathbb{C}[\Gamma]$. Similarly, once can exhibit $\langle\mathcal{G}_G\rangle * \langle\mathcal{G}_G\rangle$ as a full subcategory of the crossed product of $\langle\mathcal{G}_K\rangle$ with the category $\mathbb{Z}_2$ on the operator level, which then induces the map f'. And also the maps q and p can be implemented as functors of partitions. Actually, in the end, the diagram from the proof of Proposition 6.1 is fully reproduced purely in terms of categories of multi-colored partitions and functors between them.

An analog to Proposition 5.1 (e) is then provided by [Kye+17, Theorem 3.9]. The assumption there that the tensor C^*-categories be Cauchy complete is immaterial because the functors between the categories of partitions extend to fully faithful and injective-on-objects functors between the Cauchy completions. If there is also an analog of Proposition 5.1 (d) for rigid tensor C^*-categories, one only needs to check that the categories of partitions corresponding to B_1 and A_1 are equivalent in order to reproduce the argument from Proposition 6.1. In this way, the algebraic proof could then even be devolved to the level of pure combinatorics.

8.3. Generalizations. While there are vastly more categories of two-colored partitions than the ones considered as inputs $\langle\mathcal{G}_K\rangle$ in the main result of this chapter the strategy of proof employed here seems to rely critically on the properties of $(\langle\mathcal{G}_K\rangle,\langle\mathcal{G}_G\rangle)$ listed in Lemma 7.4. Some of them generalize to other pairs of uncolored and two-colored categories, others do not.

More precisely, there are no known other pairs $(\langle\mathcal{G}_K\rangle,\langle\mathcal{G}_G\rangle)$ with all the properties of Lemma 7.4. One property which is satisfied more broadly, though, is the one from Lemma 7.4 (a) (iii). It remains true for many more general $\langle\mathcal{G}_K\rangle$. For example, all hyperoctahedral categories of two-colored partitions are closed under color inversion. Likewise, Lemma 7.4 (a) (i) is valid for many more categories.

Of course, Lemma 7.4 (a) (i) and Lemma 7.4 (a) (iii) are implied by all the remaining parts of Lemma 7.4. Unfortunately, it is these remaining parts which seem present nowhere else. At best, it seems, something similar but less favorable can be found: The relationship between $\langle\mathcal{G}_K\rangle$ and $\langle\mathcal{G}_G\rangle$ expressed in those parts of Lemma 7.4, for example between the categories $\mathcal{U}^+=\mathcal{U}_{\mathbb{N}}^{\times+}$ and O^+ associated with $U_n^+=U_{\mathbb{N},n}^{\times+}$ respectively O_n^+, is not so different from the one between the categories $\mathcal{U}_D^{\times+}$ and O^+ demonstrated in Chapter 3 (or between $\mathcal{U}_D^{\times}$ and O^*), for arbitrary additive subsemigroups D of $\mathbb{N}$: In any element of $\langle\mathcal{G}_K\rangle$ the restriction to each block of the set-theoretical partition of the set of points which is induced by the condition of having zero color distance is an element of $\langle\mathcal{G}_G\rangle$. What makes the pairs $(\langle\mathcal{G}_K\rangle,\langle\mathcal{G}_G\rangle)$ covered by the main result special is that there all points have color distance zero.

As seen in Chapter 3, the categories $\mathcal{U}_D^{\times+}$ can be understood (at least on the operator level) as full subcategories of a wreath graph product of O^+ and the category of $\mathbb{Z}$. Unfortunately, the fact that $\mathbb{Z}$ is infinite means that there is no scaling formula analogous to Proposition 5.1 (e). Besides that, there is no Künneth formula for finite, let alone infinite graph products generalizing Proposition 5.1 (c) and Proposition 5.1 (d). Hence, the analogous strategy to Proposition 6.1 leads nowhere.

Part 3

Compact quantum groups of combinatorial type

CHAPTER 6

Compact quantum groups of combinatorial type

1. Introduction

This final chapter studies a generalization of the tensor envelope of a regular category (and the resulting group) introduced by Friedrich Knop. The generalization developed here provides a uniform framework which encompasses the categories of partitions and associated so-called easy compact quantum groups of Banica and speicher, their generalizations by Tarrago and Weber and by Freslon as well as the categories of bi-labeled graphs and induced graph-theoretic compact quantum groups of Mančinska and Roberson. In particular the latter are not covered by Knop's construction, which was the main motivation of this joint project with Johannes Flake and Moritz Weber. Unfortunately, due to time constraints not all conjectures could be verified. Any claims for which there was no time left to work out a proof are clearly marked.

1.1. Background and context. Quantum group objects in the sense of formal duals of not necessarily commutative Hopf objects are linked by a bi-adjunction, variously named Tannaka or Tannaka-Krein duality, to certain functors from rigid monoidal enriched categories to the self-enriched cosmos (see, e.g., [Del07a] in an algebraic and [Wor88] in an analytic context). Many known examples of quantum group objects are defined in this way by first providing a functor rather than the Hopf object itself. In these cases the functors in question often have a combinatorial nature. For example, the classical compact groups are related to functors which are determined by the combinatorics of certain "partitions" of finite sets (see [Bra37] for the special case of the orthogonal groups). As another example, Banica and Speicher's easy quantum groups from [BS09] are also intimately linked to partitions. Actually, until Mančinska and Roberson's graph-theoretical quantum groups, which are defined in [MR19] through the combinatorics of finite "bi-labeled graphs", all the known examples of such "quantum groups of combinatorial type" could be subsumed under a single common construction procedure: forming Knop's tensor envelope of a regular category, as defined in [Kno07]. The aim of this chapter was to provide a true generalization of Knop's work that accounts for Mančinska and Roberson's quantum groups and hopefully leads to a large number of new functors/quantum groups which have not been studied yet.

Starting from a regular category, Knop modifies the associated category of *relations*. The latter were first introduced, not for arbitrary regular categories, but only for abelian categories by Mac Lane in [Mac61] and by Puppe in [Pup62]. It was Barr, Grillet and Osdol who extended the definition to regular categories in [Bar70; BGO71]. Roughly, a relation in this sense is any monomorphism into a product of two objects. Two such monomorphisms can be composed if they share a common object. In that case, the two components with respect to the common object are pulled back. Composing the pull-backs with the other two components and forming the product morphism yields another relation. Or, rather, it almost does. The resulting morphism need not be monic. Instead it must first be factorized into a regular epimorphism followed by a monorphism. And it is this mono which is then named the composed relation. Generalizing Deligne's idea from [Del07b], Knop considered the base change of the set-enriched category of relations along the free functor to modules over a given ring and then rescaled the composition operation. Namely rather than simply forgetting the regular epimorphism part of the factorization, this morphism is transformed into a scalar and kept as a linear factor of the ordinary composition result. Simplifying broadly, a functor defined on this modified relation category is then obtained by a construction involving the hom functor of the regular input category.

As noted by Knop in [Kno07] applying this construction to the regular category given by a skeleton of the dual category of finite sets produces exactly the "partitions" used by Deligne in [Del07b]. Moreover, Banica and Speicher's "categories of partitions" from [BS09] are revealed to be special subcategories of Deligne's. It is very tempting to conjecture that Mančinska and Roberson's "graph categories" from [MR19] can be understood as certain subcategories of Knop's tensor envelope of the dual category of finite graphs. However this is *not* the case. The reasons for this are explained in Section 1.2 below. Suffice it to say that the hom functor of this regular category does not have the properties required by Knop in order for the construction of the fiber functor to succeed.

That Mančinska and Roberson's graph categories and graph-theoretical quantum groups from [MR19] cannot be explained through Knop's tensor envelopes of regular categories is not just a cosmetic issue. In [Kno07], beyond the construction itself, Knop manages to draw far-reaching conclusions about the groups resulting via Tannaka-Krein from the regular category. For particularly favorable input categories he can show that the tensor envelope is a semi-simple tensor category in the sense of [Eti+16] and can even classify the simple objects. These results were even extended to the case of, not the whole, but certain subcategories of Deligne's category by Flake and Maaßen in [FM21]. Having results of this kind available for Mančinska and Roberson's categories would be massively helpful. And the hope for the work presented in this chapter was that the generalization of Knop's construction would allow for such theorems to be proved.

This chapter puts forth a construction of a "tensor envelope" that applies to more general inputs than regular categories with suitable hom functors. In particular, it can be used to produce Mančinska and Roberson's graph categories and associated graph-theoretical quantum groups. The idea behind the construction presented is to loosen the requirements for what counts as a "relation", which is not a new one. It was noted early on in category theory that many of the properties of regular epimorphisms and monomorphisms which are used in the construction of the relations category are shared by more general structures called "factorization systems". Those and the resulting generalized "relations" were intensively studied, especially by Klein in [Kle70], Freyd and Kelly in [FK72], Meisen in [Mei74a; Mei74b], Pavlovic in [Pav95; Pav96] and Jaywardene and Wyler in [JW00]. Though, this list is far from complete.

Generalizing Knop's construction, the concept of a regular category will be replaced by that of a cartesian monoidal category with pull-backs and a pull-back-stable orthogonal factorization system. In the theory of relations, it has been known for some time that this step does not necessarily provide much of a generalization at all. Kelly showed in [Kel91] that the category of relations with respect to a pull-back-stable proper factorization system is always isomorphic to a category of relations in the classical sense, i.e., one constructed from a regular category. However, *proper* is the crucial term here, meaning that every morphism is decomposed into an epimorphism and a monomorphism. Our construction manages to provide a true generalization of [Kno07] exactly because we allow "improper" factorization systems, the Mančinska-Roberson case being a prime example.

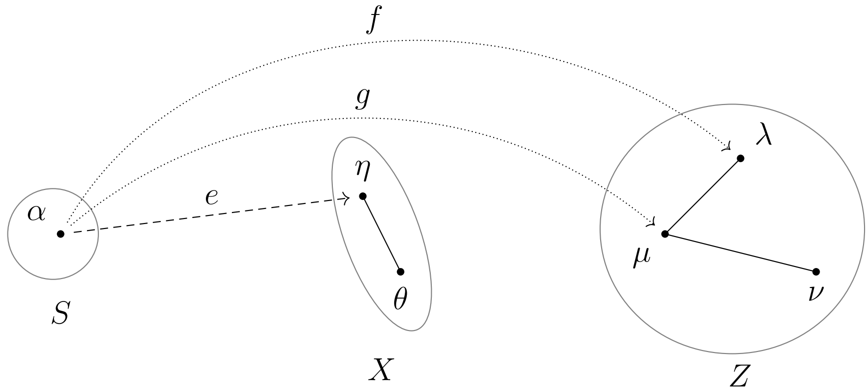
However, there are three caveats.

1. The construction does generally not yield a tensor category. Rather, the best that can be guaranteed in general is a strong concrete monoidal †-functor defined on a rigid monoidal †-category, both enriched in modules over a commutative $\star$-ring. In particular, the morphism spaces of the category are generally *in*finite-dimensional. (Summary in Section 2.)
2. That the resulting categories and functors have all these properties just laid out has *not* yet been fully verified due to the high number of rather lengthy and at times quite difficult computations involved in proving those claims and a correspondingly large lack of time before the due date of this thesis. (Partial proof in Section 5.)
3. While the properties of the particularly favorable input categories which enable Knop to prove semisimplicity and classify the simple objects admit generalizations, the latter have less powerful implications. In particular, *no* criterion for semisimplicity is presented. However, it is shown that a crucial ingredient of Knop's proof, the so-called "through block" or "core" factorization, generalizes under fairly benign assumptions. (See Sections 6 and 7.)

1.2. Why an extension of Knop's construction is needed. For the reader familiar with [Kno07] it is explained in the following why Knop's construction does not explain Mančinska and Roberson's.

Let fGr be the category of all finite undirected graphs with or without loops as objects and with adjacency-preserving maps between vertex sets as morphisms. The category $\mathsf{fGr}^{\mathrm{op}}$ is essentially small, locally finite and regular. Regular epimorphisms of $\mathsf{fGr}^{\mathrm{op}}$ are equivalently graph embeddings. For any graph Z the functor $\hom_{\mathsf{fGr}}(\cdot, Z)$: $\mathsf{fGr}^{\mathrm{op}} \to \mathsf{fSet}$ is left-exact. However, there do exist graphs Z for which $\hom_{\mathsf{fGr}}(\cdot, Z)$ *fails to be uniform*, as the following counterexample illustrates.

Let $Z = (\{\lambda, \mu, \nu\}, \{\{\lambda, \mu\}, \{\mu, \nu\}\})$, where λ, μ, and ν are pairwise distinct, be the line graph on three vertices, $X = (\{\eta, \theta\}, \{\{\eta, \theta\}\})$, where $\eta \neq \theta$, the line graph on two vertices and $S = (\{\alpha\}, \varnothing)$ the one on a single vertex. The mapping $\alpha \mapsto \eta$ defines a graph embedding of S into X, i.e., a regular epimorphism e: $X \to S$ of $\mathsf{fGr}^{\mathrm{op}}$. We show that $\hom_{\mathsf{fGr}}(e, Z)$ is not a uniform mapping.



Of course, both $f : \alpha \mapsto \lambda$ and $g : \alpha \mapsto \mu$ are graph homomorphisms from S to Z, i.e., morphisms $Z \to S$ of $\mathsf{fGr}^{\mathrm{op}}$. The fibers of f and g with respect to $\hom_{\mathsf{fGr}}(e, Z)$ have different cardinalities: There is only a single graph homomorphism from X to Z which, when composed with e, yields f, namely the mapping with $\eta \mapsto \lambda$ and $\theta \mapsto \mu$. However, two graph homomorphisms compose with e to produce g: Naturally we map $\eta \mapsto \mu$. But now we have two choices, $\theta \mapsto \lambda$ or $\theta \mapsto \nu$. Thus, $\hom_{\mathsf{fGr}}(\cdot, Z)$ is not uniform.

Hence, if one is interested in constructing fiber functors for $\mathcal{T}$ from $\hom_{\mathsf{fGr}}(\cdot, Z)$ for such Z, then not only does one need to choose a degree function adapted to that functor but also to select a factorization system for $\mathsf{fGr}^{\mathrm{op}}$ which turns $\hom_{\mathsf{fGr}}(\cdot, Z)$ into a uniform functor in the first place.

1.3. Technical caveats. One technicality should be pointed out rightaway: In [Kno07], Knop takes care to carry out his construction not directly on his input category $\mathcal{A}$ but on $\mathcal{A}^{\varnothing}$, the input with an *absolutely initial object* $\varnothing$ freely adjoined to it. For example, he does not require $\mathcal{A}$ to be a regular category – as one would usually do when considering relations – but $\mathcal{A}^{\varnothing}$ (see [Kno07, Definition 2.2]). Likewise, when producing fiber functors for his category $\mathcal{A}$, it is not the input functor P: $\mathcal{A} \to \mathsf{Set}$ which he demands be left-exact but $P^{\varnothing}$, the extension of P to $\mathcal{A}^{\varnothing}$ via $\varnothing \mapsto \varnothing$ (see [Kno07, Definition 9.2]). In the present chapter it was chosen to abstain from adjoining $\varnothing$ to $\mathcal{A}$, not because the same could not be done in this setting but only because it makes for an easier read. Moreover, for the same reason, only the analogs of what Knop calls *non-degenerate* uniform functors (see [Kno07, Remark, p. 599]) will be considered here. Thus, technically, the results of this chapter generalize Knop's only insofar as $\mathcal{A}$ is *complete and regular* in his sense (and not just *regular*) and P is left-exact and non-degenerate.

1.4. Structure of the chapter. Section 2 gives a quick overview of the generalization of Knop's construction from [Kno07] proposed here. It includes definitions and some proofs for the most important standard concepts and the new ones introduced here. However, it does not spell out the proofs that the definitions, which often may seem to depend on particular choices of representatives of equivalence classes, in actuality do not. Nor is it shown in Section 2 that the constructions have all the properties asserted by Conjectures 2.14 and 2.19. It must be emphasized that, while most of these properties were checked, *not* all of them could be, due to time constraints.

The ensuing Section 3 explains how the known constructions by Banica and Speicher, by Tarrago and Weber, by Freslon, by Deligne, by Knop himself, and, crucially, by Mančinska and Roberson are special cases of the construction from Section 2. Moreover, it hints at the vast wealth of examples not yet considered.

Section 5 gives a *partial* proof that the construction is well-defined and has all the properties asserted. This partial proof is prepared by Section 4, which offers additional background knowledge on crucial concepts used in Section 5

A major feature of Knop's construction is that the quotient of the Cauchy completion of his modified relation category with respect to the tensor radical forms a semi-simple tensor category. This result does *not* seem to pass to the more general setting considered here. However, Sections 6 and 7 investigate how two important ingredients of Knop's semi-simplicity proof generalize or fail to do so, the theory of subquotients and the property of being both (Barr-)exact and Mal'cev.

Finally, Section 8 points out some research questions raised by the generalized construction.

2. Overview of the construction

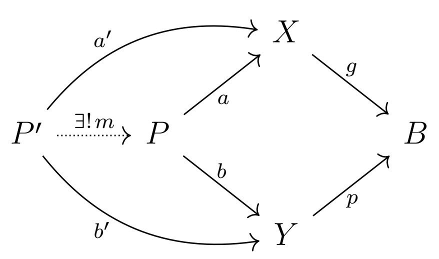
Philosophically, Deligne and Knop's construction can be understood as *partial horizontal decategorification*, the collapsing of a multiple-object category into a single-object one – but applied not to an entire category but only a subcategory. In somewhat greater detail: In any regular category $\mathcal{A}$, as mentioned, any morphism decomposes into a product of a regular epimorphism and a monomorphism. Thus, with every hom set $X \to A \otimes B$ one can associate collections $\mathcal{E}_{A,B}$ of morphisms $X \to \cdot$ of the subcategory $\mathcal{E}$ of regular epimorphisms and $\mathcal{M}_{A,B}$ of morphisms $\cdot \to A \otimes B$ of the subcategory $\mathcal{M}$ of monomorphisms by considering all the $\mathcal{E}$- and $\mathcal{M}$-parts of decompositions of $\mathcal{A}$-morphisms $X \to A \otimes B$. In a loose, non-technical sense $\mathcal{E}_{A,B}$ "acts" by composition on $\mathcal{M}_{A,B}$ to produce the hom set. This describes a situation similar in spirit to a category enriched in a monoidal category of module objects over a monoid object. However, the collections $\mathcal{E}_{A,B}$, firstly, depend on A and B and, secondly, are not monoids but something more complicated, involving potentially all objects of $\mathcal{A}$. The Deligne-Knop procedure constructs from $\mathcal{E}$ via the means of a *degree function* a monoid object R and produces an honest R-modules-enriched category with the hom object $A \to B$ given by a free action of R on $\mathcal{M}_{A,B}$.

NOTATION 2.1. Always, $0 \notin \mathbb{N}$. Rather, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. We will write $\mathrm{epi}_\mathcal{C}$, $\mathrm{rep}_\mathcal{C}$, $\mathrm{mon}_\mathcal{C}$, $\mathrm{rmo}_\mathcal{C}$ and $\mathrm{iso}_\mathcal{C}$ for the subcategories of epimorphisms, regular epimorphisms, monomorphisms, regular epimorphisms and isomorphisms, respectively, of any category $\mathcal{C}$. Given any category $\mathcal{C}$ equipped with monoidal structure the monoidal product of $\mathcal{C}$ will be denoted by $\otimes_\mathcal{C}$, the monoidal unit by $I_\mathcal{C}$, the associator by $\alpha_\mathcal{C}$ and the left and right unitors by $\lambda_\mathcal{C}$ and $\rho_\mathcal{C}$, respectively. If the category in question is clear from the context the index may be omitted. If $\mathcal{C}$ is endowed with a braiding, the latter will be addressed by $\gamma_\mathcal{C}$. For any monoidal functor F the symbols $F_\otimes$ and F_I will be used, respectively, for the product and the unit coherence transformations. For any given $*$-monoid M we will write O_M for the underlying object, $\otimes_M$ for the operation, I_M for the unit and $*_M$ for the star of M. Finally, the push-forward of a monoid M along a monoidal functor F is denoted by $F_\triangleright(M)$.

2.1. Ingredients for the construction. The following definitions are required to formulate the generalized construction.

2.1.1. *Cartesian monoidal categories.* Any symmetric monoidal category is said to be *cartesian monoidal* if I is a terminal object and if for any objects A and B, if ω_A and ω_B denote the terminal morphisms of A and B, respectively, the left projection $\pi^1_{A,B} \coloneqq \rho_A \circ (\mathrm{id}_A \otimes \omega_B)$ and the right projection $\pi^2_{A,B} \coloneqq \lambda_B \circ (\omega_A \otimes \mathrm{id}_B)$ form a product of A and B. If so, given any $f\colon X \to A$ and $g\colon X \to B$ their product morphism $X \to A \otimes B$ with respect to this product will be denoted by $f \times g$.

2.1.2. *Pull-backs.* Recall that a *span* is any pair of morphisms with common domain and, dually, a *co-span* any pair with common codomain. Given a co-span (g,p) with $g\colon X \to B$ and $p\colon Y \to B$ a *pull-back* of (g,p) is any span (a,b) such that there exists P with $a\colon P \to X$ and $b\colon P \to Y$ and $g \circ a = p \circ b$ and such that (a,b) is universal with this property.



In this situation we say that a is a *pulled-back version* or, simply, a *pull-back* of p *along* g (and, likewise, b a pull-back of g along p).

We call any wide subcategory $\mathcal{E}$ of $\mathcal{A}$ *pull-back-stable* if e' belongs to $\mathcal{E}$ for any $e \in \mathrm{mor}_\mathcal{E}(A,B)$ and any pull-back $e'\colon A' \to B'$ of e in $\mathcal{A}$ along any morphism $B' \to B$ of $\mathcal{A}$.

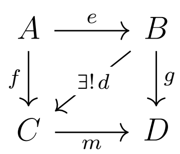
LEMMA 2.2. *Let $\mathcal{E}$ be any pull-back-stable wide subcategory of any category $\mathcal{A}$.*

(a) Every isomorphism of $\mathcal{A}$ is a morphism of $\mathcal{E}$.
(b) If $\mathcal{A}$ is equipped with a cartesian monoidal structure, then $\mathcal{E}$ is even a symmetric monoidal subcategory.

PROOF. Claim (a) holds because for any isomorphism $u: A \to X$ of $\mathcal{A}$ a pull-back in $\mathcal{A}$ of the co-span $(\mathrm{id}_X, \mathrm{id}_X)$ of $\mathcal{E}$ is given by (u, u). If $\mathcal{A}$ is cartesian monoidal and if $e_i \in \mathrm{mor}_{\mathcal{E}}(A_i, X_i)$ for each $i \in \{1,2\}$, then, first, $(e_1 \otimes \mathrm{id}_{A_2}, \pi^1_{A_1,A_2})$ is a pull-back of $(\pi^1_{X_1,A_2}, e_1)$ in $\mathcal{A}$, which makes $e_1 \otimes \mathrm{id}_{A_2}$ a morphism of $\mathcal{E}$, and, second, $(\mathrm{id}_{X_1} \otimes e_2, \pi^2_{X_1,A_2})$ is a pull-back of $(\pi^2_{X_1,X_2}, e_2)$, ensuring that $\mathrm{id}_{X_1} \otimes e_2$ belongs to $\mathcal{E}$. Now, the identity $(\mathrm{id}_{X_1} \otimes e_2) \circ (e_1 \otimes \mathrm{id}_{A_2}) = e_1 \otimes e_2$ proves (b). □

A functor $H: \mathcal{A} \to \mathcal{A}'$ between categories with pull-backs is said to *preserve pull-backs* if for any pull-back (a,b) of any co-span (g,p) in $\mathcal{A}$ the span $(H(a), H(b))$ is a pull-back of $(H(g), H(p))$ in $\mathcal{A}'$.

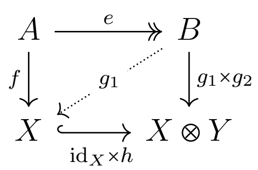
Given pull-back-preserving functors $H, K: \mathcal{A} \to \mathcal{A}'$ any natural transformation $\eta: H \Rightarrow K$ is called *equifibered* if for any morphism $f: X \to Y$ of $\mathcal{A}$ the span $(H(f), \eta_X)$ is a pull-back of $(\eta_Y, K(f))$ in $\mathcal{A}'$.

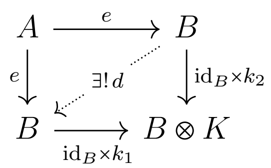


2.1.3. *Factorization systems.* In any category $\mathcal{A}$ we say that any morphism $e: A \to B$ is *left-orthogonal* to any morphism $m: C \to D$ or, equivalently, that m is *right-orthogonal* to e, in symbols: $e \perp m$, if for any $f: A \to C$ and $g: B \to D$ such that $g \circ e = m \circ f$ there exists a unique $d: B \to C$, a *diagonal*, such that $f = d \circ e$ and $g = m \circ d$. For any systems $\mathcal{E}$ and $\mathcal{M}$ of morphisms of $\mathcal{A}$ let $\mathcal{E}^{\perp} := \{m \mid \forall e \in \mathcal{E}: e \perp m\}$ and ${}^{\perp}\mathcal{M} := \{e \mid \forall m \in \mathcal{M}: e \perp m\}$. Then, $\mathcal{E}^{\perp}$ and ${}^{\perp}\mathcal{M}$ are wide subcategories of $\mathcal{A}$ containing the core $\mathrm{iso}_{\mathcal{A}}$ of $\mathcal{A}$ by [FK72, Propositions 2.1.1 (a) and 2.1.2]. Any pair $(\mathcal{E}, \mathcal{M})$ of systems of morphisms is called a *pre-factorization-system* if $\mathcal{E}^{\perp} = \mathcal{M}$ and $\mathcal{E} = {}^{\perp}\mathcal{M}$. If so, then $\mathcal{M}$ is pull-back-stable by [FK72, Proposition 2.1.1 (b)]. Finally, we call any pre-factorization-system $(\mathcal{E}, \mathcal{M})$ a *factorization system* of $\mathcal{A}$ if for any morphism $f: X \to Y$ of $\mathcal{A}$ there exists an object S, an *image object*, and (e, m), a *factorization*, such that $e: X \to S$ is a morphism of $\mathcal{E}$ and $m: S \to Y$ one of $\mathcal{M}$ and such that $f = m \circ e$. Completely analogous is the definition of a *weak factorization system*, with the sole difference that we do not demand that the diagonal be unique. Rather than speaking of orthogonality, we say that the morphisms have the *lifting property* with respect to each other and write $\pitchfork$ instead of $\perp$.

LEMMA 2.3. *[JW00, Proposition 1.1.1] For any cartesian monoidal category $\mathcal{A}$ and any factorization system $(\mathcal{E}, \mathcal{M})$ of $\mathcal{A}$ all morphisms of $\mathcal{E}$ are epimorphisms of $\mathcal{A}$ if and only if $\mathrm{id}_{\mathcal{A},X} \times_{\mathcal{A}} h$ is a morphism of $\mathcal{M}$ for any $h \in \mathrm{mor}_{\mathcal{A}}(X, Y)$.*

Proof. If $\mathcal{E} \hookrightarrow \mathrm{epi}_{\mathcal{A}}$, then, given any $e \in \mathrm{mor}_{\mathcal{E}}(A,B)$ and any f and $g_1 \times g_2$ such that $(g_1 \times g_2) \circ e = (\mathrm{id}_X \times h) \circ f$, the diagonal $g_1\colon B \to X$ already satisfies $g_1 \circ e = f$ trivially. And, the other identity $(\mathrm{id}_X \times h) \circ g_1 = g_1 \times g_2$, i.e., effectively, $g_2 = h \circ g_1$, follows from $g_2 \circ e = h \circ f = h \circ g_1 \circ e$ because e is an epimorphism. For the same reason the diagonal is unique.





To show the converse, if $e \in \mathrm{mor}_{\mathcal{E}}(A,B)$ and $k_i\colon B \to K$ for each $i \in \{1,2\}$ are such that $k_1 \circ e = k_2 \circ e$, then, because $(\mathrm{id}_B \times k_1) \circ e = (\mathrm{id}_B \times k_2) \circ e$ and because $\mathrm{id}_B \times k_1 \in \mathrm{mor}_{\mathcal{M}}(B, B \otimes K)$ by assumption, there exists $d\colon B \to B$ such that, in particular, $(\mathrm{id}_B \times k_1) \circ d = \mathrm{id}_B \times k_2$. But that already implies $d = \mathrm{id}_B$ and, thus, $k_1 = k_2$. □

Finally, given any wide subcategory $\mathcal{M}$ of $\mathcal{A}$ we say that $\mathcal{A}$ is $\mathcal{M}$*-subobject-small* respectively $\mathcal{M}$*-subobject-finite* if the slice category $\mathcal{M}/X$ of $\mathcal{M}$ over X is essentially small respectively finite, i.e., equivalent to a small respectively finite category, for any object X of $\mathcal{A}$.

2.1.4. *Generalized degree functions.* The next definition encompasses [Kno07, Definition 3.1] as the special case where S is given by the abelian groups Ab, where $\mathcal{A}$ is regular and where $\mathcal{E}$ is the subcategory $\mathrm{rep}_{\mathcal{A}}$ of regular epimorphisms of $\mathcal{A}$.

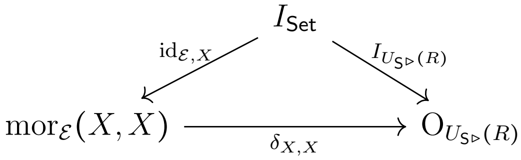
Definition 2.4. Let S with $U_{\mathsf{S}}\colon \mathsf{S} \to \mathsf{Set}$ be any concrete symmetric monoidal category and let $\mathbf{Md}(\mathsf{S})$ be the symmetric monoidal category of commutative monoid objects of S and $U_{\mathbf{Md}(\mathsf{S})}$ its forgetful symmetric monoidal functor $\mathbf{Md}(\mathsf{S}) \to \mathsf{Set}$.

(a) For any object R of $\mathbf{Md}(\mathsf{S})$, any cartesian monoidal category $\mathcal{A}$ with pull-backs and any pull-back-stable wide subcategory $\mathcal{E}$ of $\mathcal{A}$, an $(\mathsf{S}, U_{\mathsf{S}})$*-type* R*-valued* $\mathcal{E}$*-degree function* is any pull-back-invariant functor from $\mathcal{E}$ to the delooping of the monoid object $U_{\mathsf{S}^{\triangleright}}(R)$ of Set underlying R, i.e., any δ such that

(i) δ is a family of morphisms $\delta_{A,B}$ of Set indexed by all pairs (A,B) of objects of $\mathcal{A}$ such that

$$\delta_{A,B} \in \mathrm{mor}_{\mathsf{Set}}(\mathrm{mor}_{\mathcal{E}}(A,B), \mathrm{O}_{U_{\mathsf{S}^{\triangleright}}(R)}),$$

(ii) $\delta_{X,X} \circ_{\mathsf{Set}} \mathrm{id}_{\mathcal{E},X} = I_{U_{\mathsf{S}^{\triangleright}}(R)}$ for any object X of $\mathcal{A}$, where we have considered $\mathrm{id}_{\mathcal{E},X}$ to be a morphism $I_{\mathsf{Set}} \to \mathrm{mor}_{\mathcal{E}}(X,X)$ of Set,



(iii) $\otimes_{U_{\mathsf{S}_\triangleright}(R)} \circ_{\mathsf{Set}} (\delta_{B,C} \otimes_{\mathsf{Set}} \delta_{A,B}) = \delta_{A,C} \circ_{\mathsf{Set}} \circ_{\mathcal{E},A,B,C}$ for any objects A, B and C of $\mathcal{A}$, and

$$\begin{array}{ccc} \operatorname{mor}_{\mathcal{E}}(B,C) \otimes_{\mathsf{Set}} \operatorname{mor}_{\mathcal{E}}(A,B) & \xrightarrow{\circ_{\mathcal{E},A,B,C}} & \operatorname{mor}_{\mathcal{E}}(A,C) \\ {\scriptstyle \delta_{B,C}\otimes_{\mathsf{Set}}\delta_{A,B}}\downarrow & & \downarrow{\scriptstyle \delta_{A,C}} \\ \mathrm{O}_{U_{\mathsf{S}_\triangleright}(R)} \otimes_{\mathsf{Set}} \mathrm{O}_{U_{\mathsf{S}_\triangleright}(R)} & \xrightarrow[\otimes_{U_{\mathsf{S}_\triangleright}(R)}]{} & \mathrm{O}_{U_{\mathsf{S}_\triangleright}(R)} \end{array}$$

(iv) $\delta_{A',B'}(e') = \delta_{A,B}(e)$ for any $e \in \operatorname{mor}_{\mathcal{E}}(A,B)$ and any pull-back $e' \in \operatorname{mor}_{\mathcal{E}}(A',B')$ along any morphism $B' \to B$ of $\mathcal{A}$.

(b) We say that δ is a *co-universal* $(\mathsf{S},U_{\mathsf{S}})$-type $\mathcal{E}$-degree function for $\mathcal{A}$ if for any object R' of $\mathbf{Md}(\mathsf{S})$ and any $(\mathsf{S},U_{\mathsf{S}})$-type R'-valued $\mathcal{E}$-degree function δ' for $\mathcal{A}$ there exists a unique $f \in \operatorname{mor}_{\mathbf{Md}(\mathsf{S})}(R,R')$ with $\delta'_{A,B} = U_{\mathbf{Md}(\mathsf{S})}(f) \circ_{\mathsf{Set}} \delta_{A,B}$ for any objects A and B of $\mathcal{A}$.

(c) For any further concrete symmetric monoidal category $(\mathsf{S}',U_{\mathsf{S}'})$, any functor $F\colon \mathsf{S}' \to \mathsf{S}$ with $U_{\mathsf{S}} \circ F = U_{\mathsf{S}'}$, any commutative monoid object R' of S' and any $j \in \operatorname{mor}_{\mathbf{Md}(\mathsf{S})}(R, F_\triangleright(R'))$ the *transformation* $(F,j)^{\circledast}(\delta)$ of δ with respect to F and j is the $(\mathsf{S}',U_{\mathsf{S}'})$-type R'-valued $\mathcal{E}$-degree function for $\mathcal{A}$ with

$$((F,j)^{\circledast}(\delta))_{A,B} \coloneqq U_{\mathbf{Md}(\mathsf{S})}(j) \circ_{\mathsf{Set}} \delta_{A,B},$$

for any objects A and B of $\mathcal{A}$.

(d) The *trivial* $(\mathsf{S},U_{\mathsf{S}})$-type $\mathcal{E}$-degree function for $\mathcal{A}$ is the unique $I_{\mathbf{Md}(\mathsf{S})}$-valued one.

By the same reasoning as in [Kno07, Section 8, p. 593], if a cartesian monoidal category with pull-backs is essentially small, then any pull-back-stable wide subcategory has (at least) a co-universal Set-type as well as Ab-type degree function.

LEMMA 2.5. *Let $(\mathsf{S},U_{\mathsf{S}})$ be any concrete symmetric monoidal category, R any commutative S-monoid, $\mathcal{A}$ any cartesian monoidal category, $\mathcal{E}$ any pull-back-stable wide subcategory of $\mathcal{A}$ and δ any $(\mathsf{S},U_{\mathsf{S}})$-type R-valued $\mathcal{E}$-degree function for $\mathcal{A}$.*

(a) $\delta_{A,X}(u) = I_{U_{\mathsf{S}_\triangleright}(R)}$ for any isomorphism $u\colon A \to X$ of $\mathcal{A}$.

(b) $\otimes_{U_{\mathsf{S}_\triangleright}(R)} \circ_{\mathsf{Set}} (\delta_{A_1,X_1} \otimes_{\mathsf{Set}} \delta_{A_2,X_2}) = \delta_{A_1\otimes_{\mathcal{A}}A_2, X_1\otimes_{\mathcal{A}}X_2} \circ_{\mathsf{Set}} (\otimes_{\mathcal{E}})_{1,(A_1,A_2),(X_1,X_2)}$, where $\otimes_{\mathcal{E}}$ is the restriction of $\otimes_{\mathcal{A}}$.

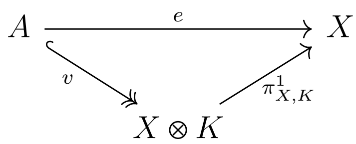
$$\begin{array}{ccc} \operatorname{mor}_{\mathcal{E}}(A_1,X_1) \otimes_{\mathsf{Set}} \operatorname{mor}_{\mathcal{E}}(A_2,X_2) & \xrightarrow{(\otimes_{\mathcal{E}})_{1,(A_1,A_2),(X_1,X_2)}} & \operatorname{mor}_{\mathcal{E}}(A_1 \otimes_{\mathcal{A}} A_2, X_1 \otimes_{\mathcal{A}} X_2) \\ {\scriptstyle \delta_{A_1,X_1}\otimes_{\mathsf{Set}}\delta_{A_2,X_2}}\downarrow & & \downarrow{\scriptstyle \delta_{A_1\otimes_{\mathcal{A}}A_2,X_1\otimes_{\mathcal{A}}X_2}} \\ \mathrm{O}_{U_{\mathsf{S}_\triangleright}(R)} \otimes_{\mathsf{Set}} \mathrm{O}_{U_{\mathsf{S}_\triangleright}(R)} & \xrightarrow[\otimes_{U_{\mathsf{S}_\triangleright}(R)}]{} & \mathrm{O}_{U_{\mathsf{S}_\triangleright}(R)} \end{array}$$

PROOF. The proof of Lemma 2.2 (a) showed that u can be written as a pull-back of id_X, which proves (a). If $e_i \in \operatorname{mor}_{\mathcal{E}}(A_i,X_i)$ for each $i \in \{1,2\}$, we saw in the proof of Lemma 2.2 (b) that $e_1 \otimes \mathrm{id}_{A_2}$ is a pull-back of e_1 and $\mathrm{id}_{X_1} \otimes e_2$ one of e_2. It follows $\delta(e_1 \otimes e_2) = \delta((\mathrm{id}_{X_1} \otimes e_2) \circ (e_1 \otimes \mathrm{id}_{A_2})) = \delta(\mathrm{id}_{X_1} \otimes e_2) \otimes_{U_{\mathsf{S}^\triangleright}(R)} \delta(e_1 \otimes \mathrm{id}_{A_2}) = \delta(e_2) \otimes_{U_{\mathsf{S}^\triangleright}(R)} \delta(e_1) = \delta(e_1) \otimes_{U_{\mathsf{S}^\triangleright}(R)} \delta(e_2)$, verifying (b). □

Lemma 2.5 (a) shows in particular that for any cartesian monoidal category $\mathcal{A}$ and any concrete symmetric monoidal category $(\mathsf{S}, U_{\mathsf{S}})$ there is exactly one $\mathrm{iso}_{\mathcal{A}}$-degree function, the trivial one.

2.1.5. *Quasi-projections.* We exhibit a natural pull-back-stable wide subcategory on any given monoidal category and determine its universal Set-type degree function in good cases.

Definition 2.6. In any cartesian monoidal category a *quasi-projection* is any morphism of the form $e = \pi^1_{X,K} \circ v$ for any objects A, X and K and any isomorphism $v\colon A \to X \otimes K$.



We write $\mathrm{qpr}_{\mathcal{A}}$ for the collection of all quasi-projections of any given cartesian monoidal category $\mathcal{A}$.

For any essentially small cartesian monoidal category $\mathcal{A}$ the set $\mathcal{A}/\cong$ of all isomorphism classes $[X]$ of objects X of $\mathcal{A}$ becomes a commutative monoid object $\mathrm{M}_{\mathcal{A}}$ of Set, the *object monoid* of $\mathcal{A}$, when equipped with the operation $[X_1] \otimes_{\mathrm{M}_{\mathcal{A}}} [X_2] := [X_1 \otimes_{\mathcal{A}} X_2]$ and the unit $[I_{\mathcal{A}}]$.

Definition 2.7. Any essentially small cartesian monoidal category $\mathcal{A}$ is said to be *monoidally cancellative* if the commutative monoid $\mathrm{M}_{\mathcal{A}}$ is cancellative.

In such an $\mathcal{A}$, for any object X and any isomorphisms $v\colon A \to X \otimes K$ and $v'\colon A \to X \otimes K'$, whenever $\pi^1_{X,K} \circ v = \pi^1_{X,K'} \circ v'$, then $[K] = [K']$ because $v' \circ v^{-1}$ is an isomorphism $X \otimes K \to X \otimes K'$. Hence, the following definition makes sense.

Definition 2.8. In any monoidally cancellative essentially small cartesian monoidal category $\mathcal{A}$ the *degree* of any quasi-projection $e\colon A \to X$ is the unique element $\mathrm{qker}_{\mathcal{A},A,X}(e) := [K]$ of $\mathrm{M}_{\mathcal{A}}$ for which there exists an invertible $v\colon A \to X \otimes K$ such that $e = \pi^1_{X,K} \circ v$.

Proposition 2.9. *Let $\mathcal{A}$ be any cartesian monoidal category.*

(a) $\mathrm{qpr}_{\mathcal{A}}$ *is a pull-back-stable wide subcategory of $\mathcal{A}$.*

(b) If $\mathcal{A}$ is essentially small and monoidally cancellative, then $\mathrm{qker}_{\mathcal{A}}$ *is a co-universal* $(\mathsf{Set}, \mathrm{id}_{\mathsf{Set}})$*-type* $\mathrm{qpr}_{\mathcal{A}}$*-degree function for $\mathcal{A}$.*

Proof. *Identities.* For any object X the decomposition $\mathrm{id}_X = \rho_X \circ \rho_X^{-1} = \pi^1_{X,I} \circ \rho_X^{-1}$ shows that $\mathrm{qpr}_{\mathcal{A}}$ contains all identities and, under the premises of (b), that $\mathrm{qker}_{\mathcal{A}}(\mathrm{id}_X) = I_{\mathrm{M}_{\mathcal{A}}}$.

Composition. Given any isomorphisms $v\colon A \to B \otimes K$ and $u\colon B \to X \otimes L$ the morphism $w := \alpha_{X,L,K} \circ (u \otimes \mathrm{id}_K) \circ v$ is invertible and satisfies $\pi^1_{X,L\otimes K} \circ w = (\pi^1_{X,L} \circ u) \circ (\pi^1_{B,K} \circ v)$, proving that $\mathrm{qpr}_{\mathcal{A}}$ is closed under composition. Under the additional assumptions of (b), we can moreover infer $\mathrm{qker}_{\mathcal{A}}((\pi^1_{X,L} \circ u) \circ (\pi^1_{B,K} \circ v)) = [L \otimes K] = [L] \otimes_{\mathrm{M}_{\mathcal{A}}} [K] = \mathrm{qker}_{\mathcal{A}}(\pi^1_{X,L} \circ u) \otimes_{\mathrm{M}_{\mathcal{A}}} \mathrm{qker}_{\mathcal{A}}(\pi^1_{B,K} \circ v)$.

Pullbacks. If $v: A \to X \otimes K$ is any isomorphism and $f: C \to X$ any morphism of $\mathcal{A}$, then $(v^{-1} \circ (f \otimes \mathrm{id}_K), \pi^1_{C,K})$ is a pull-back of $(\pi^1_{X,K} \circ v, f)$. Hence, if (f', e') is an arbitrary pull-back of $(\pi^1_{X,K} \circ v, f)$ we find an isomorphism u such that $f' = v^{-1} \circ (f \otimes \mathrm{id}_K) \circ u$ and $e' = \pi^1_{C,K} \circ u$, which exhibits e' as a quasi-projection. Thus $\mathrm{qpr}_{\mathcal{A}}$ is pull-back-stable. Moreover, the identity $\mathrm{qker}_{\mathcal{A}}(e') = [K] = \mathrm{qker}_{\mathcal{A}}(\pi^1_{X,K} \circ v)$ in the situation of (b) completes the proof that $\mathrm{qker}_{\mathcal{A}}$ is a degree function.

Co-universality It only remains to show that $\mathrm{qker}_{\mathcal{A}}$ is co-universal. If R is any commutative monoid of Set and δ any Set-type R-valued $\mathrm{qpr}_{\mathcal{A}}$-degree function, then at most one monoid morphism $u: \mathrm{M}_{\mathcal{A}} \to R$ with $\delta_{A,X} = U_{\mathbf{Md}(\mathsf{Set})}(u) \circ \mathrm{qker}_{\mathcal{A},A,X}$ for all objects A and X can exist because the Set morphisms $\mathrm{qker}_{\mathcal{A},A,X}$ are by construction jointly epimorphic. We exhibit such a u.

The rule $[K] \mapsto \delta(\pi^1_{I,K})$ defines a morphism $\tilde{u}: \mathcal{A}/_{\cong} \to \mathrm{O}_{U_{\mathsf{Set}^{\triangleright}}(R)}$ of Set because $\pi^1_{I,K}$ is a quasi-projection and because for any isomorphism $w: K \to K'$, as $\mathrm{id}_I \otimes w$ is invertible, $\delta(\pi^1_{I,K'} \circ (\mathrm{id}_I \otimes w)) = \delta(\pi^1_{I,K})$ by Lemmata 2.2 (a) and 2.5 (a).

Moreover, for any isomorphism $v: A \to X \otimes K$ Lemma 2.2 (b) and Part (a) ensure that $\mathrm{id}_X \otimes \pi^1_{I,K}$ belongs to $\mathrm{qpr}_{\mathcal{A}}$ and Lemma 2.2 (b) guarantees $\delta(\mathrm{id}_X \otimes \pi^1_{I,K}) = \delta(\pi^1_{I,K})$. As ρ_X, $(\mathrm{id}_X \otimes \lambda_K^{-1})$ and v are all invertible and thus by Lemma 2.2 (a) and Part (a) morphisms of $\mathrm{qpr}_{\mathcal{A}}$ as well, δ is defined on every factor in the decomposition $\rho_X \circ (\mathrm{id}_X \otimes \pi^1_{I,K}) \circ (\mathrm{id}_X \otimes \lambda_K^{-1}) \circ v$ of $e := \pi^1_{X,K} \circ v$. Thus, $\delta(e) = \delta(\pi^1_{I,K}) = \tilde{u}([K]) = \tilde{u}(\mathrm{qker}_{\mathcal{A}}(e))$ by both parts of Lemma 2.5.

As $\pi^1_{I,I} = \lambda_I$ is an isomorphism, $\tilde{u}(I_{\mathrm{M}_{\mathcal{A}}}) = \tilde{u}([I]) = \delta(\pi^1_{I,I}) = I_R$ by Part (a) and Lemmata 2.2 (a) and 2.5 (a). Given any objects K_1 and K_2, the three morphisms $e_1 := \pi^1_{I\otimes K_2,K_1} \circ \alpha^{-1}_{I,K_2,K_1}$, $e_2 := \pi^1_{I,K_2}$ and $e := \pi^1_{I,K_2\otimes K_1}$ are all quasi-projections, which is why, by what we saw in the preceding paragraph, $\delta(e_1) = \tilde{u}(\mathrm{qker}_{\mathcal{A}}(e_1)) = \tilde{u}([K_1])$ and $\delta(e_2) = \tilde{u}([K_2])$ as well as $\delta(e) = \tilde{u}([K_1 \otimes K_2])$. From $e = e_2 \circ e_1$ it now follows $\tilde{u}([K_1] \otimes_{\mathrm{M}_{\mathcal{A}}} [K_2]) = \tilde{u}([K_1 \otimes K_2]) = \delta(e) = \delta(e_2 \circ e_1) = \delta(e_2) \otimes_R \delta(e_1) = \tilde{u}([K_2]) \otimes_R \tilde{u}([K_1])$. Thus, $\tilde{u}$ indeed lifts to a morphism $u: \mathrm{M}_{\mathcal{A}} \to R$ with the desired properties. □

In [Kno07, Section 9, p. 599] Knop defines a *uniform map* as a mapping between finite sets all of whose fibers have the same cardinality. This cardinality he calls the *degree* of the uniform map (but only if the co-domain of the latter is non-empty). In our language: The full subcategory nefSet of Set generated by all non-empty finite sets is a cartesian monoidal category; its class $\mathrm{qpr}_{\mathsf{nefSet}}$ of quasi-projections, given by all uniform maps between non-empty sets, is a pull-back-stable subcategory of nefSet by Proposition 2.9 (a); its universal degree function is $\mathrm{qker}_{\mathsf{nefSet}}$ by Proposition 2.9 (b), which via the isomorphism of $\mathrm{M}_{\mathsf{nefSet}} \to (\mathbb{N}, \cdot), [K] \mapsto |K|$ corresponds to the degree of a uniform map in Knop's sense.

2.2. The generalized Deligne-Knop functor. With the terms and results of Section 2.1 at hand, we can formulate the definition of the generalization of Knop's construction from [Kno07], which in turn generalized Deligne's from [Del07b]. More precisely, there will be two versions, one without and one with fiber functors to be used for Tannaka-Krein duality.

2.2.1. *Non-Tannakian version.* Throughout, this section let S with $U_{\mathsf{S}}\colon \mathsf{S} \to \mathsf{Set}$ be any concrete symmetric monoidal category with co-equalizers such that $(\,\cdot\,) \otimes_{\mathsf{S}} X$ preserves co-equalizers for each object X of S and such that U_{S} admits a symmetric (necessarily strong) monoidal left adjoint $G_{\mathsf{S}}\colon \mathsf{Set} \to \mathsf{S}$, let R be any commutative $*$-monoid object of S, let $\mathsf{M} \equiv \mathbf{Mod}_R(\mathsf{S})$ be the symmetric monoidal category of module objects over R in S and let $G_{\mathsf{M}}\colon \mathsf{Set} \to \mathsf{M}$ and $U_{\mathsf{M}}\colon \mathsf{M} \to \mathsf{Set}$ be, respectively, the free and the forgetful symmetric monoidal functors induced by G_{S} and U_{S}, let $\mathrm{cu}_{\mathsf{M}}\colon G_{\mathsf{M}} \circ U_{\mathsf{M}} \Rightarrow \mathrm{id}_{\mathsf{M}}$ be the co-unit and $\mathrm{un}_{\mathsf{M}}\colon \mathrm{id}_{\mathsf{Set}} \Rightarrow U_{\mathsf{M}} \circ G_{\mathsf{M}}$ the unit of the adjunction $G_{\mathsf{M}} \dashv U_{\mathsf{M}}$, and let cj_{M} be the endofunctor of M which replaces any module by the module with the conjugate action of R (and which is the identity on morphisms). Notable examples of S are Set or Ab.

The following definition describes the inputs as well as the outputs of the generalized Deligne-Knop construction.

Definition 2.10. (a) Write $\mathsf{d}_{\mathsf{S},R}\mathsf{esmCAT}^{\mathrm{cart,fc}}$ for the strict 2-category whose 0-cells are triples $(\mathcal{A}, \mathcal{E}, \delta)$ such that
- (i) $\mathcal{A}$ is any cartesian monoidal category with pull-backs,
- (ii) $\mathcal{E}$ is any wide pull-back-stable subcategory of $\mathcal{A}$ such that $\mathcal{E}$ consists of epimorphisms of $\mathcal{A}$
- (iii) δ is any $(\mathsf{S}, U_{\mathsf{S}})$-type R-valued $\mathcal{E}$-degree function for $\mathcal{A}$,

whose 1-cells from $(\mathcal{A}, \mathcal{E}, \delta)$ and $(\mathcal{A}', \mathcal{E}', \delta')$ are all H such that
- (i) H is a symmetric strong monoidal pull-back-preserving functor $\mathcal{A} \to \mathcal{A}'$,
- (ii) H restricts to a functor $\mathcal{E} \to \mathcal{E}'$, and
- (iii) $\delta'_{H(A),H(B)} \circ_{\mathsf{Set}} H_{1,A,B} = \delta_{A,B}$ for any objects A and B of $\mathcal{A}$,

$$\begin{array}{ccc} \mathrm{mor}_{\mathcal{E}}(A,B) & \xrightarrow{\;H_{1,A,B}\;} & \mathrm{mor}_{\mathcal{E}'}(H(A),H(B)) \\ & \underset{\delta_{A,B}}{\searrow} \quad \quad \underset{\delta'_{H(A),H(B)}}{\swarrow} & \\ & U_{\mathbf{Md}(\mathsf{S})}(R) & \end{array}$$

whose 2-cells of 1-cells from $(\mathcal{A}, \mathcal{E}, \delta)$ to $(\mathcal{A}', \mathcal{E}', \delta')$ from H to G are given by all equifibered monoidal natural transformations $H \Rightarrow G$, and whose identities and operations are inherited from the 2-category smCAT of symmetric monoidal categories.

(b) Let $\mathsf{d}_{\mathsf{S},R}\mathsf{esmCAT}^{\mathrm{cart,fc}}_{\mathrm{fs}}$ be the sub-2-category of $\mathsf{d}_{\mathsf{S},R}\mathsf{esmCAT}^{\mathrm{cart,fc}}$ with all
- (i) 0-cells $(\mathcal{A}, \mathcal{E}, \delta)$ such that $(\mathcal{E}, \mathcal{E}^{\perp})$ is a factorization system for $\mathcal{A}$ and such that $\mathcal{A}$ is $\mathcal{E}^{\perp}$-subobject-small,
- (ii) 1-cells H between such 0-cells, say from $(\mathcal{A}, \mathcal{E}, \delta)$ to $(\mathcal{A}', \mathcal{E}', \delta')$, such that H restricts to a functor $\mathcal{E}^{\perp} \to (\mathcal{E}')^{\perp}$, and
- (iii) 2-cells between such 1-cells.

(c) Write $\mathsf{sm}\mathbf{Mod}_R(\mathsf{S})\dagger\mathsf{CAT}^{\mathrm{r}}$ for the (strict) 2-category of rigid symmetric monoidal $\mathbf{Mod}_R(\mathsf{S})$-enriched †-categories (with symmetric monoidal $\mathbf{Mod}_R(\mathsf{S})$-enriched †-functors as 1-cells and monoidal $\mathbf{Mod}_R(\mathsf{S})$-enriched natural transformations as 2-cells).

Because any commutative monoid object is trivially a commutative $*$-monoid object when equipped with the identity, no effective restriction is imposed by requiring $*$-monoid objects instead of just monoid objects in Definition 2.10 (a).

Lemma 2.3 ensures that the following definitions make sense. It can be checked that they are independent of the choices of representatives of equivalence classes.

DEFINITION 2.11. For any 0-cell $(\mathcal{A},\mathcal{E},\delta)$ of $\mathsf{d}_{\mathsf{S},R}\mathsf{esmCAT}^{\mathrm{cart,fc}}_{\mathrm{fs}}$ make the following definitions, where $\mathcal{M} := \mathcal{E}^{\perp}$ and where $\mathcal{T}^0$ is short for $\mathcal{T}^0(\mathcal{A},\mathcal{E},\delta)$:

(a) **Objects.** Let $\mathrm{obj}_{\mathcal{T}^0}$ be the same as $\mathrm{obj}_{\mathcal{A}}$.

(b) **Morphisms.** For any objects A and B of $\mathcal{A}$ let, temporarily,

$$r(A,B) \equiv r_{\mathcal{A}}(A,B) := \left(\mathcal{M} \diagup (A \otimes_{\mathcal{A}} B)\right) \diagup_{\cong} \in \mathrm{obj}_{\mathsf{Set}}$$

denote the set of all isomorphism classes of objects of the slice category of $\mathcal{M}$ over the object $A \otimes_{\mathcal{A}} B$ and then let

$$\mathrm{mor}_{\mathcal{T}^0}(A,B) := (G_{\mathsf{M}})_0(r(A,B)) \in \mathrm{obj}_{\mathsf{M}},$$

be the free R-module over that.

(c) **Composition.** For any objects A, B and C of $\mathcal{A}$ let, temporarily,

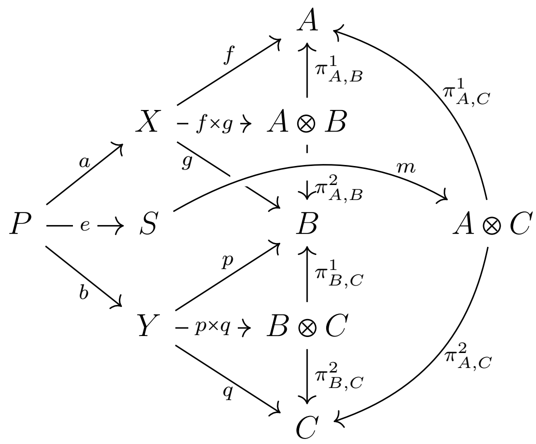
$$c_{A,B,C} \in \mathrm{mor}_{\mathsf{Set}}(r(B,C) \otimes_{\mathsf{Set}} r(A,B), U_{\mathsf{M}}(I_{\mathsf{M}}) \otimes_{\mathsf{Set}} r(A,C))$$

be given by

$$(y,x) \mapsto (\delta(e),[m])$$

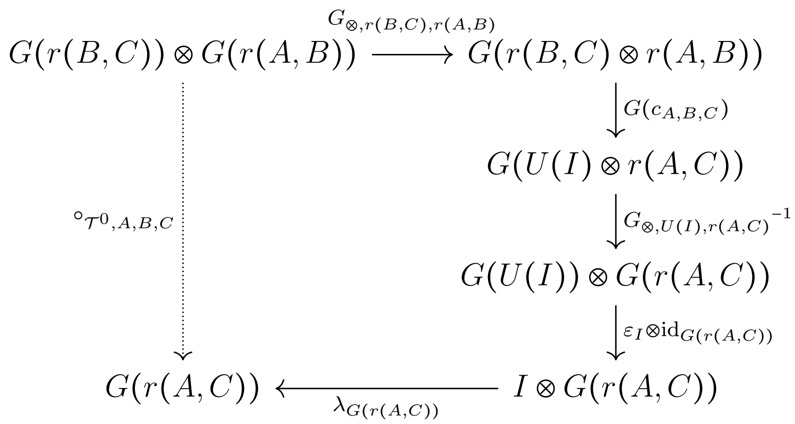
for any $f \in \mathrm{mor}_{\mathcal{A}}(X,A)$ and $g \in \mathrm{mor}_{\mathcal{A}}(X,B)$ such that $f \times_{\mathcal{A}} g \in x$, any $p \in \mathrm{mor}_{\mathcal{A}}(Y,B)$ and $q \in \mathrm{mor}_{\mathcal{A}}(Y,C)$ such that $p \times_{\mathcal{A}} q \in y$, any $a \in \mathrm{mor}_{\mathcal{A}}(P,X)$ and $b \in \mathrm{mor}_{\mathcal{A}}(P,Y)$ such that (a,b) is a pull-back of (g,p) and any $e \in$

$\mathrm{mor}_{\mathcal{E}}(P,S)$ and $m \in \mathrm{mor}_{\mathcal{M}}(S, A \otimes_{\mathcal{A}} C)$ such that (e,m) is an $(\mathcal{E},\mathcal{M})$-factorization of $(f \circ_{\mathcal{A}} a) \times_{\mathcal{A}} (q \circ_{\mathcal{A}} b)$,



and then, suppressing everywhere the index M, define

$$\circ_{\mathcal{T}^0,A,B,C} := \lambda_{G(r(A,C))} \circ (\varepsilon_I \otimes \mathrm{id}_{G(r(A,C))}) \circ {G_{\otimes,U(I),r(A,C)}}^{-1} \circ G(c_{A,B,C}) \circ G_{\otimes,r(B,C),r(A,B)}.$$



(d) **Identities.** For any object A of $\mathcal{A}$, if we interpret the isomorphism class $[\mathrm{id}_{\mathcal{A},A} \times_{\mathcal{A}} \mathrm{id}_{\mathcal{A},A}]$ of objects of $\mathcal{M}/(A \otimes_{\mathcal{A}} A)$ as a morphism $I_{\mathsf{Set}} \to \mathrm{mor}_{\mathcal{T}^0}(A,A)$, define

$$\mathrm{id}_{\mathcal{T}^0,A} := G_{\mathsf{M}}([\mathrm{id}_{\mathcal{A},A} \times_{\mathcal{A}} \mathrm{id}_{\mathcal{A},A}]) \circ_{\mathsf{M}} (G_{\mathsf{M}})_I.$$

(e) **Monoidal product.** For any objects A_1 and A_2 of $\mathcal{A}$ let

$$A_1 \otimes_{\mathcal{T}^0} A_2 := A_1 \otimes_{\mathcal{A}} A_2.$$

Moreover, for any objects A_1, A_2, B_1 and B_2 of $\mathcal{A}$ if, temporarily,

$$m_{A_1,A_2,B_1,B_2} \in \mathrm{mor}_{\mathsf{Set}}(r(A_1,B_1) \otimes_{\mathsf{Set}} r(A_2,B_2), r(A_1 \otimes_{\mathcal{A}} A_2, B_1 \otimes_{\mathcal{A}} B_2))$$

denotes the mapping with

$$(x_1,x_2) \mapsto [(f_1 \otimes_{\mathcal{A}} f_2) \times_{\mathcal{A}} (g_1 \otimes_{\mathcal{A}} g_2)]$$

for any $f_i \in \mathrm{mor}_{\mathcal{A}}(X_i,A_i)$ and $g_i \in \mathrm{mor}_{\mathcal{A}}(X_i,B_i)$ such that $f_i \times_{\mathcal{A}} g_i \in x_i$ for each $i \in \{1,2\}$, let

$$(\otimes_{\mathcal{T}^0})_{1,(A_1,A_2),(B_1,B_2)} := G_{\mathsf{M}}(m_{A_1,A_2,B_1,B_2}) \circ_{\mathsf{M}} (G_{\mathsf{M}})_{\otimes,r(A_1,B_1),r(A_2,B_2)}.$$

(f) **Monoidal Unit.** Let $I_{\mathcal{T}^0}$ be the same as $I_{\mathcal{A}}$.

(g) **Associator.** For any objects A_1, A_2 and A_3 of $\mathcal{A}$ if we consider the isomorphism class

$$[\mathrm{id}_{\mathcal{A},(A_1\otimes_{\mathcal{A}}A_2)\otimes_{\mathcal{A}}A_3} \times_{\mathcal{A}} \alpha_{\mathcal{A},A_1,A_2,A_3}]$$

of objects of the category

$$\mathcal{M}\big/(((A_1 \otimes_{\mathcal{A}} A_2) \otimes_{\mathcal{A}} A_3) \otimes_{\mathcal{A}} (A_1 \otimes_{\mathcal{A}} (A_2 \otimes_{\mathcal{A}} A_3)))$$

to be a morphism $I_{\mathsf{Set}} \to r((A_1 \otimes_{\mathcal{A}} A_2) \otimes_{\mathcal{A}} A_3, A_1 \otimes_{\mathcal{A}} (A_2 \otimes_{\mathcal{A}} A_3))$ of Set, we can put

$$\alpha_{\mathcal{T}^0,A_1,A_2,A_3} := G([\mathrm{id}_{\mathcal{A},(A_1\otimes_{\mathcal{A}}A_2)\otimes_{\mathcal{A}}A_3} \times_{\mathcal{A}} \alpha_{\mathcal{A},A_1,A_2,A_3}]) \circ_{\mathsf{M}} G_I.$$

(h) **Left unitor.** For any object A of $\mathcal{A}$, if the class $[\mathrm{id}_{\mathcal{A},I_{\mathcal{A}}\otimes_{\mathcal{A}}A} \times_{\mathcal{A}} \lambda_{\mathcal{A},A}]$ of objects of $\mathcal{M}/((I_{\mathcal{A}} \otimes_{\mathcal{A}} A) \otimes_{\mathcal{A}} A)$ is thought of as a morphism $I_{\mathsf{Set}} \to r(I_{\mathcal{A}}\otimes_{\mathcal{A}}A, A)$ of Set, one can define

$$\lambda_{\mathcal{A},A} := G_{\mathsf{M}}([\mathrm{id}_{\mathcal{A},I_{\mathcal{A}}\otimes_{\mathcal{A}}A} \times_{\mathcal{A}} \lambda_{\mathcal{A},A}]) \circ_{\mathsf{M}} (G_{\mathsf{M}})_I.$$

(i) **Right unitor.** Likewise, for any object A of $\mathcal{A}$ it makes sense to define

$$\rho_{\mathcal{A},A} := G_{\mathsf{M}}([\mathrm{id}_{\mathcal{A},A\otimes_{\mathcal{A}}I_{\mathcal{A}}} \times_{\mathcal{A}} \rho_{\mathcal{A},A}]) \circ_{\mathsf{M}} (G_{\mathsf{M}})_I.$$

(j) **Braiding.** For any objects A_1 and A_2 of $\mathcal{A}$, if we interpret the isomorphism class $[\mathrm{id}_{\mathcal{A},A_1\otimes_{\mathcal{A}}A_2} \times_{\mathcal{A}} \gamma_{\mathcal{A},A_1,A_2}]$ of objects of $\mathcal{M}/(A_1 \otimes_{\mathcal{A}} A_2) \otimes_{\mathcal{A}} (A_2 \otimes_{\mathcal{A}} A_1)$ as a morphism $I_{\mathsf{Set}} \to r(A_1 \otimes_{\mathcal{A}} A_2, A_2 \otimes_{\mathcal{A}} A_1)$, we can let

$$\gamma_{\mathcal{T}^0,A_1,A_2} := G_{\mathsf{M}}([\mathrm{id}_{\mathcal{A},A_1\otimes_{\mathcal{A}}A_2} \times_{\mathcal{A}} \gamma_{\mathcal{A},A_1,A_2}]) \circ_{\mathsf{M}} (G_{\mathsf{M}})_I.$$

(k) **Dagger.** Finally, for any objects A and B of $\mathcal{A}$, if, temporarily,

$$d_{A,B} \in \mathrm{mor}_{\mathsf{Set}}(r(A,B), r(B,A))$$

is the mapping with

$$x \mapsto [g \times_{\mathcal{A}} f]$$

for any $f \in \mathrm{mor}_{\mathcal{A}}(X,A)$ and $g \in \mathrm{mor}_{\mathcal{A}}(X,B)$ such that $f \times_{\mathcal{A}} g \in x$, then let $(\dagger_{\mathcal{T}^0})_0$ be the identity on $\mathrm{obj}_{\mathcal{A}}$ and let

$$(\dagger_{\mathcal{T}^0})_{1,A,B} := \varepsilon_{\mathsf{M},\mathrm{cj}_{\mathsf{M}}(G_{\mathsf{M}}(r(B,A)))} \circ_{\mathsf{M}} G_{\mathsf{M}}(\eta_{\mathsf{M},r(B,A)} \circ_{\mathsf{Set}} d_{A,B}).$$

DEFINITION 2.12. For any 1-cell H from any 0-cell $(\mathcal{A}, \mathcal{E}, \delta)$ to any 0-cell $(\mathcal{A}', \mathcal{E}', \delta')$ of $\mathsf{d}_{\mathsf{S},R}\mathsf{esmCAT}_{\mathrm{fs}}^{\mathrm{cart,fc}}$, define $\mathcal{T}^0(H)$ as follows, where $\mathcal{M}' \coloneqq (\mathcal{E}')^{\perp}$.

(a) **On objects:** $(\mathcal{T}^0(H))_0$ is the same as H_0.

(b) **On morphisms:** For any objects A and B of $\mathcal{A}$, if, temporarily,

$$v_{A,B} \in \mathrm{mor}_{\mathsf{Set}}(r_{\mathcal{A}}(A,B), r_{\mathcal{A}'}(H_0(A), H_0(B)))$$

denotes the mapping

$$x \mapsto [H_{1,X,A}(f) \times_{\mathcal{A}'} H_{1,X,B}(g)]$$

for any $f \in \mathrm{mor}_{\mathcal{A}}(X,A)$ and $g \in \mathrm{mor}_{\mathcal{A}}(X,B)$ such that $f \times_{\mathcal{A}} g \in x$, then

$$(\mathcal{T}^0(H))_{1,A,B} \coloneqq G_{\mathsf{M}}(v_{A,B}).$$

(c) **Monoidal product coherence:** For any objects A_1 and A_2 of $\mathcal{A}$, if we interpret the isomorphism class

$$[\mathrm{id}_{\mathcal{A}',H(A_1)\otimes_{\mathcal{A}'}H(A_2)} \otimes_{\mathcal{A}'} H_{\otimes,A_1,A_2}]$$

of objects of

$$\mathcal{M}' \diagup ((H(A_1) \otimes_{\mathcal{A}'} H(A_2)) \otimes_{\mathcal{A}'} (H(A_1 \otimes_{\mathcal{A}} A_2)))$$

as a morphism $I_{\mathsf{Set}} \to r_{\mathcal{A}'}(H(A_1) \otimes_{\mathcal{A}'} H(A_2), H(A_1 \otimes_{\mathcal{A}} A_2))$, then

$$(\mathcal{T}^0(H))_{\otimes,A_1,A_2} \coloneqq G_{\mathsf{M}}([\mathrm{id}_{\mathcal{A}',H(A_1)\otimes_{\mathcal{A}'}H(A_2)} \otimes_{\mathcal{A}'} H_{\otimes,A_1,A_2}]) \circ_{\mathsf{M}} (G_{\mathsf{M}})_I.$$

(d) **Monoidal unit coherence:** Finally, if we consider the isomorphism class

$$[\mathrm{id}_{\mathcal{A}',I_{\mathcal{A}'}} \times_{\mathcal{A}'} H_I]$$

of objects of

$$\mathcal{M}' \diagup (I_{\mathcal{A}'} \otimes_{\mathcal{A}'} H(I_{\mathcal{A}}))$$

a morphism $I_{\mathsf{Set}} \to r_{\mathcal{A}'}(I_{\mathcal{A}'}, H(I_{\mathcal{A}}))$, then

$$(\mathcal{T}^0(H))_I \coloneqq \mathsf{M}([\mathrm{id}_{\mathcal{A}',I_{\mathcal{A}'}} \times_{\mathcal{A}'} H_I]) \circ_{\mathsf{M}} (G_{\mathsf{M}})_I.$$

DEFINITION 2.13. For any 1-cells H and K from any 0-cell $(\mathcal{A}, \mathcal{E}, \delta)$ to any 0-cell $(\mathcal{A}', \mathcal{E}', \delta')$ and for any 2-cell η from H to K of $\mathsf{d}_{\mathsf{S},R}\mathsf{esmCAT}_{\mathrm{fs}}^{\mathrm{cart,fc}}$, if for any object A of $\mathcal{A}$ the isomorphism class

$$[\mathrm{id}_{\mathcal{A}',H(A)} \times_{\mathcal{A}'} \eta_A]$$

of objects of the category

$$\mathcal{M}' \diagup (H(A) \otimes_{\mathcal{A}'} K(A))$$

is interepreted as a morphism $I_{\mathsf{Set}} \to r_{\mathcal{A}'}(H(A), K(A))$, then define

$$(\mathcal{T}^0(\eta))_A \coloneqq G_{\mathsf{M}}([\mathrm{id}_{\mathcal{A}',H(A)} \times_{\mathcal{A}'} \eta_A]) \circ_{\mathsf{M}} G_{\mathsf{M}I}.$$

Unfortunately, I was not able to check all the constituent claims of the following assertion because I ran out of time.

CONJECTURE 2.14. *$\mathcal{T}^0$ is a 2-functor* $\mathsf{d}_{\mathsf{S},R}\mathsf{esmCAT}^{\mathrm{cart,fc}}_{\mathrm{fs}} \to \mathsf{sm}\mathbf{Mod}_R(\mathsf{S})\dagger\mathsf{CAT}^{\mathrm{r}}$.

Of course, we are then free to consider a composition of $\mathcal{T}^0$ with any or all of (at least) the following 2-functors, in various orders:

(1) going from $\mathsf{sm}\mathbf{Mod}_R(\mathsf{S})\dagger\mathsf{CAT}^{\mathrm{r}}$ to $\mathsf{sm}\mathbf{Mod}_K(\mathbf{V})\dagger\mathsf{CAT}^{\mathrm{r}}$ a change $W_{\rightharpoondown}$ of enriching category along any symmetric monoidal functor $W\colon \mathbf{Mod}_R(\mathsf{S}) \to \mathbf{Mod}_K(\mathbf{V})$, in particular, if $\mathsf{S} = \mathbf{V}$, restriction, extension or co-extension of scalars $R \leftrightarrow K$, or, a W induced by any symmetric monoidal functor $\mathsf{S} \to \mathbf{V}$, notably, if $\mathsf{S} = \mathsf{Set}$ and $\mathbf{V} = \mathsf{Ab}$, via the free functor $G_{\mathsf{Ab}}\colon \mathsf{Set} \to \mathsf{Ab}$,

(2) the 2-functor that forgets the †-structure,

(3) provided the current enriching category is complete and co-complete, the *Cauchy completion* 2-functor (for rigid symmetric monoidal (†-)categories), e.g., if enriched over Ab-modules the 2-functor also known as the *Karoubian* or *pseudo-abelian* closure.

2.2.2. *Tannakian version.* As with the non-Tannakian version of the construction, there are numerous variants to consider with respect to different enriching categories and points in time when a change of enriching category may occur. For the sake of brevity only one scenario is treated. Namely, let R be a commutative unital $*$-ring, i.e., a commutative $*$-monoid object in Ab and let j be the unique morphism of $*$-monoid objects $\mathrm{M}_{\mathsf{nefSet}} \cong (\mathbb{N},\cdot) \to U_{\mathsf{Ab}\triangleright}(R)$.

DEFINITION 2.15. (a) Of $\mathsf{d}_{\mathsf{Ab},R}\mathsf{esmCAT}^{\mathrm{cart,fc}}_{\mathrm{fs}}$ let $\mathsf{d}_{\mathsf{Ab},R}\mathsf{esmCat}^{\mathrm{cart,fc}}_{\mathrm{fs}}$ be the full sub-2-category generated by all 0-cells $(\mathcal{A},\mathcal{E},\delta)$ such that $\mathcal{A}$ is small.

(b) Let $\mathsf{sm}\mathbf{Mod}_R(\mathsf{Ab})\dagger\mathsf{Cat}^{\mathrm{r}}$ be the full sub-2-category of $\mathsf{sm}\mathbf{Mod}_R(\mathsf{Ab})\dagger\mathsf{CAT}^{\mathrm{r}}$ on all small 0-cells.

We describe a 2-functor, an extension of $\mathcal{T}^0$, denoted by $\mathcal{T}^0_T$,

$$\begin{array}{c}
\mathsf{d}_{\mathsf{Ab},R}\mathsf{esmCat}^{\mathrm{cart,fc}}_{\mathrm{fs}} \big/ (\mathsf{nefSet}, \mathrm{qpr}_{\mathsf{nefSet}}, (U_{\mathsf{Ab}}, j)^{\circledast}(\mathrm{qker}_{\mathsf{nefSet}})) \\
\Big\downarrow \mathcal{T}^0_T \\
\mathsf{sm}\mathbf{Mod}_R(\mathsf{Ab})\dagger\mathsf{Cat}^{\mathrm{r}} \big/ {}^{\uparrow}\mathbf{Mod}_R(\mathsf{Ab})^{\mathrm{fgp}} \\
\big\downarrow \;\dashv\; \big\uparrow \mathrm{Comod}_{\mathrm{uni}} \\
*\mathsf{Hopf}(\mathbf{Mod}_R(\mathsf{Ab})),
\end{array}$$

where the top is the sub-2-category of the slice-2-category of $\mathsf{d}_{\mathsf{Ab},R}\mathsf{esmCAT}^{\mathrm{cart,fc}}$ over the 0-cell $(\mathsf{nefSet}, \mathrm{qpr}_{\mathsf{nefSet}}, (U_{\mathsf{Ab}}, j)^{\circledast}(\mathrm{qker}_{\mathsf{nefSet}}))$ induced by $\mathsf{d}_{\mathsf{Ab},R}\mathsf{esmCat}^{\mathrm{cart,fc}}_{\mathrm{fs}}$, and where the middle is the slice 2-category of $\mathsf{sm}\mathbf{Mod}_R(\mathsf{S})\dagger\mathsf{CAT}^{\mathrm{r}}$ over the 0-cell ${}^{\uparrow}\mathbf{Mod}_R(\mathsf{Ab})^{\mathrm{fgp}}$, the category $\mathbf{Mod}_R(\mathsf{Ab})^{\mathrm{fgp}}$ of finitely generated projective R-modules considered (via self-enrichment) as a $\mathbf{Mod}_R(\mathsf{Ab})$-enriched category, induced by $\mathsf{sm}\mathbf{Mod}_R(\mathsf{Ab})\dagger\mathsf{Cat}^{\mathrm{r}}$. At least if $R = \mathbb{C}$, then by Tannaka-Krein duality there is a 2-adjunction between this 2-category and $*\mathsf{Hopf}(\mathbf{Mod}_R(\mathsf{Ab}))$, the 2-category of $*$-Hopf-monoids (and only trivial 2-cells).

Resume the abbreviations from the beginning of Section 2.2.1 with $\mathsf{S} = \mathsf{Ab}$. Moreover, let $\nu_{\mathsf{M}}\colon U_{\mathsf{M}} \circ [\,\cdot\,,\,\cdot\,]_{\mathsf{M}} \Rightarrow \hom_{\mathsf{M}}$ be the natural isomorphism connecting the internal and external hom functors and the forgetful functor and let $(\,\cdot\,)^{\vee_{\mathsf{M}}}$ denote the dualization with respect to the canonical dual objects $X^{\vee_{\mathsf{M}}} = X$ for any dualizable object X of M. Finally, for any object X of M let $\mathrm{ev}_{\mathsf{M},X}\colon [X,\,\cdot\,]_{\mathsf{M}} \otimes_{\mathsf{M}} X \Rightarrow \mathrm{id}_{\mathsf{M}}$ and $\mathrm{ce}_{\mathsf{M},X}\colon \mathrm{id}_{\mathsf{M}} \Rightarrow [X, (\,\cdot\,) \otimes X]_{\mathsf{M}}$ be the evaluations respectively co-evaluations with respect to $\otimes_{\mathsf{M}}$ and $[\,\cdot\,,\,\cdot\,]_{\mathsf{M}}$.

Definition 2.16. For any 0-cell $(\mathcal{A}, \mathcal{E}, \delta)$ of $\mathsf{d}_{\mathsf{Ab},R}\mathsf{esmCat}^{\mathrm{cart,fc}}_{\mathrm{fs}}$ and any 1-cell P of $\mathsf{d}_{\mathsf{Ab},R}\mathsf{esmCAT}^{\mathrm{cart,fc}}$ from $(\mathcal{A}, \mathcal{E}, \delta)$ to $(\mathsf{nefSet}, \mathrm{qpr}_{\mathsf{nefSet}}, (U_{\mathsf{Ab}}, j)^{\circledast}(\mathrm{qker}_{\mathsf{nefSet}}))$ make the following definitions for $\mathcal{T}^0_T(P)$.

(a) **On objects.** For any object A of $\mathcal{A}$ let $(\mathcal{T}^0_T(P))_0(A) \coloneqq G_{\mathsf{M}}(P(A))$.

(b) **On morphisms.** If for any objects A and B of $\mathcal{A}$, temporarily,

$$t_{A,B} \in \mathrm{mor}_{\mathsf{Set}}(r(A,B), \mathrm{mor}_{\mathsf{M}}(G_{\mathsf{M}}(P(A)), G_{\mathsf{M}}(P(B))))$$

denotes the mapping given by

$$x \mapsto G_{\mathsf{M}}(P(g)) \circ_{\mathsf{M}} G_{\mathsf{M}}(P(f))^{\vee_{\mathsf{M}}}$$

for any $f \in \mathrm{mor}_{\mathcal{A}}(X,A)$ and $g \in \mathrm{mor}_{\mathcal{A}}(X,B)$ such that $f \times_{\mathcal{A}} g \in x$, then define

$$(\mathcal{T}^0_T(P))_1 \coloneqq \mathrm{cu}_{\mathsf{M},[G_{\mathsf{M}}(P(A)),G_{\mathsf{M}}(P(B))]_{\mathsf{M}}} \circ_{\mathsf{M}} G_{\mathsf{M}}(\nu^{-1}_{\mathsf{M},G_{\mathsf{M}}(P(A)),G_{\mathsf{M}}(P(B))} \circ_{\mathsf{Set}} t_{A,B}).$$

(c) **Monoidal product coherence.** For any objects A_1 and A_2 of $\mathcal{A}$ let

$$\begin{aligned}(\mathcal{T}^0_T(P))_{\otimes,A_1,A_2} &\coloneqq [\mathrm{id}_{\mathsf{M},G_{\mathsf{M}}(P(A_1))\otimes_{\mathsf{M}}G_{\mathsf{M}}(P(A_2))}, (G_{\mathsf{M}} \circ P)_{\otimes,A_1,A_2}]_{\mathsf{M}} \\ &\quad \circ_{\mathsf{M}} [\mathrm{id}_{\mathsf{M},G_{\mathsf{M}}(P(A_1))\otimes_{\mathsf{M}}G_{\mathsf{M}}(P(A_2))}, \lambda_{\mathsf{M},G_{\mathsf{M}}(P(A_1))\otimes_{\mathsf{M}}G_{\mathsf{M}}(P(A_2))}]_{\mathsf{M}} \\ &\quad\quad \circ_{\mathsf{M}} \mathrm{ce}_{\mathsf{M},G_{\mathsf{M}}(P(A_1))\otimes_{\mathsf{M}}G_{\mathsf{M}}(P(A_2)),I_{\mathsf{M}}}.\end{aligned}$$

(d) **Monoidal unit coherence.** Finally, define

$$(\mathcal{T}^0_T(P))_I \coloneqq [\mathrm{id}_{\mathsf{M},I_{\mathsf{M}}}, (G_{\mathsf{M}} \circ P)_I \circ_{\mathsf{M}} \lambda_{\mathsf{M},I_{\mathsf{M}}}]_{\mathsf{M}} \circ_{\mathsf{M}} \mathrm{ce}_{\mathsf{M},I_{\mathsf{M}},I_{\mathsf{M}}}.$$

Definition 2.17. Given, in $\mathsf{d}_{\mathsf{Ab},R}\mathsf{esmCat}^{\mathrm{cart,fc}}_{\mathrm{fs}}$, any 0-cells $(\mathcal{A}, \mathcal{E}, \delta)$ and $(\mathcal{A}', \mathcal{E}', \delta')$, any 1-cell H from $(\mathcal{A}', \mathcal{E}', \delta')$ to $(\mathcal{A}, \mathcal{E}, \delta)$, and, in $\mathsf{d}_{\mathsf{Ab},R}\mathsf{esmCAT}^{\mathrm{cart,fc}}$, any 1-cells P from $(\mathcal{A}, \mathcal{E}, \delta)$ and P' from $(\mathcal{A}', \mathcal{E}', \delta')$ to $(\mathsf{nefSet}, \mathrm{qpr}_{\mathsf{nefSet}}, (U_{\mathsf{Ab}}, j)^{\circledast}(\mathrm{qker}_{\mathsf{nefSet}}))$, and any invertible 2-cell η from P to $P' \circ H$, define $\mathcal{T}^0_T(H, \eta)$ as the pair with

(a) first component given by $\mathcal{T}^0(H)$ and

(b) second component given by the family of M-morphisms indexed over $\mathrm{obj}_{\mathcal{A}}$ such that, if for any object A of $\mathcal{A}$, if we interpret $G_{\mathsf{M}}(\eta_A)$ as a morphism

$I_{\mathsf{Set}} \to \mathrm{mor}_{\mathsf{M}}(G_{\mathsf{M}}(P(A)), G_{\mathsf{M}}(P'(H(A))))$ of Set, the A-component is given by

$$\mathrm{cu}_{\mathsf{M},[G_{\mathsf{M}}(P(A)),G_{\mathsf{M}}(P'(H(A)))]_{\mathsf{M}}} \circ_{\mathsf{M}} G_{\mathsf{M}}(\nu^{-1}_{\mathsf{M},G_{\mathsf{M}}(P(A)),G_{\mathsf{M}}(P'(H(A)))} \circ_{\mathsf{Set}} \eta_A) \circ_{\mathsf{M}} (G_{\mathsf{M}})_I.$$

We use the symbol $\diamond$ for the horizontal composition of 2-cells and $\triangleright$ for the right whiskering of 1-cell and 2-cells.

Definition 2.18. Given, in $\mathsf{d}_{\mathsf{Ab},R}\mathsf{esmCat}^{\mathrm{cart,fc}}_{\mathrm{fs}}$, any 0-cells $(\mathcal{A}, \mathcal{E}, \delta)$ and $(\mathcal{A}', \mathcal{E}', \delta')$ and any 1-cells H and K from $(\mathcal{A}', \mathcal{E}', \delta')$ to $(\mathcal{A}, \mathcal{E}, \delta)$, as well as, in $\mathsf{d}_{\mathsf{Ab},R}\mathsf{esmCAT}^{\mathrm{cart,fc}}$, any 1-cells P from $(\mathcal{A}, \mathcal{E}, \delta)$ and P' from $(\mathcal{A}', \mathcal{E}', \delta')$ to the 0-cell $(\mathsf{nefSet}, \mathrm{qpr}_{\mathsf{nefSet}}, (U_{\mathsf{Ab}}, j)^{\circledast}(\mathrm{qker}_{\mathsf{nefSet}}))$, any invertible 2-cells η and θ from P to $P' \circ H$ respectively $P' \circ K$, and any 2-cell ξ from H to K such that $(P' \triangleright \xi) \diamond \eta = \theta$, let $\mathcal{T}^0_T(\xi) := \mathcal{T}^0(\xi)$.

Again, since I ran out of time, I was not able to fully confirm that the following is true.

Conjecture 2.19. *$\mathcal{T}^0_T$ defines a 2-functor as indicated.*

Beware that, differently from [Kno07, Theorem 9.4 (a)] we have only considered "non-degenerate uniform functors" in Knop's sense by working in the slice category over nefSet rather than just fSet.

3. Examples

Section 3 illustrates the reach of the construction presented by listing how it subsumes (posibly via subcategories) the various relation categories studied in the tensor category and compact quantum group literature (up to (unitary) monoidal linear equivalence). Beware $0 \notin \mathbb{N}$.

Examples 3.1.
(a) Of course, **Deligne's category** $\mathrm{Rep}(S_t)$ from [Del07b] had already been shown to be a special case of Knop's construction in [Kno07, pp. 579, 590, 596, 599]. It is obtained as the Cauchy completion of $\mathcal{T}^0(\mathcal{A}, \mathcal{E}, \delta)$ for $\mathcal{A}$ being the skeleton of $\mathsf{fSet}^{\mathrm{op}}$ with objects $\{\varnothing, \{1, \ldots, n\} \,|\, n \in \mathbb{N}\}$ and with the cartesian monoidal structure induced by $\{1, \ldots, n_1\} \otimes \{1, \ldots, n_2\} := \{1, \ldots, n_1{+}n_2\}$ and $\pi^1_{\{1,\ldots,n_1\},\{1,\ldots,n_2\}} : j \mapsto j$ and $\pi^2_{\{1,\ldots,n_1\},\{1,\ldots,n_2\}} : j \mapsto n_1 + j$ for all $\{n_1, n_2\} \subseteq \mathbb{N}$ and for $\mathcal{E} = \mathrm{qpr}_{\mathcal{A}}$ and $\mathsf{S} = \mathsf{Ab}$ and $R = \mathbb{C}[t]$ and δ being the transformation of $\mathrm{qker}_{\mathcal{A}}$ with respect to $U_{\mathsf{Ab}} \colon \mathsf{Ab} \to \mathsf{Set}$ and $\mathrm{M}_{\mathcal{A}} \cong (\mathbb{N}_0, +) \to \mathbb{C}[t], 1 \mapsto t$. To construct its Tannakian version (meaning $t \in \mathbb{N}$) one considers $R = \mathbb{C}$ and $P = \hom_{\mathcal{A}}(\{1, \ldots, t\}, \cdot)$.

(b) But, naturally, most (see Section 1.3) of **Knop's categories** $\mathcal{T}^0$ of [Kno07, Theorem 3.4] can be obtained as the categories $\mathcal{T}^0(\mathcal{A}, \mathcal{E}, \delta)$ for $\mathcal{A}$ being any regular (in the usual sense, "complete and regular" in Knop's sense) category (with any cartesian monoidal structure chosen, as Knop only defines up to symmetric monoidal equivalence) and $\mathcal{E} = \mathrm{rep}_{\mathcal{A}}$ and $\mathsf{S} = \mathsf{Ab}$ and R

being any commutative unital ring and δ being any Ab-type R-valued $\mathrm{rep}_{\mathcal{A}}$-degree function. For the Tannakian version of [Kno07, Theorem 9.4 (a)], we are limited to considering P to be any left-exact non-degenerate uniform functor in Knop's sense.

(c) The categories of **partitions** defined by Banica and Speicher in [BS09] and subsequently classified in [Web13],[RW16a], [RW14], [RW16b] by Raum and the third author can be found as subcategories of $\mathcal{T}^0(\mathcal{A},\mathcal{E},\delta)$ for $\mathcal{A}$ being as in (a), for $\mathcal{E} = \mathrm{qpr}_{\mathcal{A}}$, for $\mathsf{S} = \mathsf{Set}$, for δ being the Set-type trivial $\mathrm{qpr}_{\mathcal{A}}$-degree function. To obtain the Tannakian version for $n \in \mathbb{N}$ one considers instead $\mathsf{S} = \mathsf{Ab}$ and $R = \mathbb{C}$ and $P = \hom_{\mathcal{A}}(\{1,\ldots,n\},\cdot)$.

(d) In order to find the categories of **two-colored partitions** introduced by Tarrago and the second author in [TW18] and further studied in [Gro18], [MW20], [MW21a], [MW21b] and [MW20] one considers subcategories of $\mathcal{T}^0(\mathcal{A},\mathcal{E},\delta)$, where $\mathcal{A}$ is the category (equivalent but, crucially, not isomorphic to $\mathsf{fSet}^{\mathrm{op}}$) with object set $\{\varnothing,(c_1,\ldots,c_n)\,|\,n \in \mathbb{N},\{c_1,\ldots,c_n\} \subseteq C\}$, where $C = \{\circ,\bullet\}$ and where $(c_1,\ldots,c_n) \coloneqq \{\{1,c_1\},\ldots,\{n,c_n\}\}$ and $C \cap \mathbb{N} = \varnothing$ and $\bullet \neq \circ$, and with the cartesian monoidal structure induced by $(c_1,\ldots,c_m)\otimes(c'_1,\ldots,c'_n) \coloneqq (c_1,\ldots,c_m,c'_1,\ldots,c'_n)$ and the projections $\pi^1_{(c_1,\ldots,c_m),(c'_1,\ldots,c'_n)} : \{j,c_j\} \mapsto \{j,c_j\}$ and $\pi^2_{(c_1,\ldots,c_m),(c'_1,\ldots,c'_n)} : \{j,c'_j\} \mapsto \{m+j,c'_j\}$, where $\mathcal{E} = \mathrm{qpr}_{\mathcal{A}}$, where $\mathsf{S} = \mathsf{Set}$ and where δ is the trivial Set-type $\mathrm{qpr}_{\mathcal{A}}$-degree function. "Subcategories" here are only those which, if $\overline{\circ} \coloneqq \bullet$ and $\overline{\bullet} \coloneqq \circ$ contain for each $c \in C$ the morphism $\varnothing \to (c,\overline{c})$ given by the classes of the unique maps $(c) \mapsto (c,\overline{c})$ of $\mathcal{A}$. As in the one-colored case, to get the Tannakian version for $n \in \mathbb{N}$, one chooses $\mathsf{S} = \mathsf{Ab}$ and $R = \mathbb{C}$ and $P = \hom_{\mathcal{A}}((\circ,\ldots,\circ),\cdot)$, where $(\circ,\ldots,\circ)$ has cardinality n.

(e) For Freslon's categories of **colored partitions** from [Fre17, Definition 5] one does the same as in (d) with the abstract color set C (in Freslon's notation $\mathcal{A}$) and its involution $c \mapsto \overline{c}$ playing the roles of the set and map of the same names there.

(f) Mančinska and Roberson's **graph** categories from [MR20, Definition 8.1] are subcategories of $\mathcal{T}^0(\mathcal{A},\mathcal{E},R)$ for the following choices of $\mathcal{A}$, $\mathcal{E}$, δ: If fGr is the cartesian monoidal category of simple undirected finite graphs (with or without loops) and adjacency-preserving maps between their vertex sets and if $U_{\mathsf{fGr}}\colon \mathsf{fGr} \to \mathsf{fSet}$ is the functor which sends each graph to its set of vertices, then $\mathcal{A}$ is the full subcategory of $\mathsf{fGr}^{\mathrm{op}}$ on all graphs which are mapped by $(U_{\mathsf{fGr}})^{\mathrm{op}}$ to the category $\mathcal{A}$ from Example 3.1 (a). It is equipped with a cartesian monoidal structure with respect to which $(U_{\mathsf{fGr}})^{\mathrm{op}}$ becomes a strict monoidal functor. We choose $\mathcal{E} = \mathrm{iso}_{\mathcal{A}}$ and $\mathsf{S} = \mathsf{Set}$ and let δ be the unique $\mathrm{iso}_{\mathcal{A}}$-degree function. In particular, Mančinska and Roberson's graph category $\mathcal{G}$ of all bi-labeled graphs is then given by the full subcategory of $\mathcal{T}^0(\mathcal{A},\mathcal{E},R)$ generated by all co-free objects (with respect to the right adjoint $\mathsf{fSet}^{\mathrm{op}} \to \mathsf{fGr}^{\mathrm{op}}$ of $(U_{\mathsf{fGr}})^{\mathrm{op}}$). For the Tannakian version, one considers $\mathsf{S} = \mathsf{Ab}$ and $R = \mathbb{C}$ and $P = \hom_{\mathcal{A}}(G, \cdot)$ for any graph G.

If Conjectures 2.14 and 2.19 are correct, though, we can immediately recognize a sheer unlimited number of further categories.

Examples 3.2. (a) Because the right adjoint of $(U_{\mathsf{fGr}})^{\mathrm{op}}$ embeds $\mathsf{fSet}^{\mathrm{op}}$ into $\mathsf{fGr}^{\mathrm{op}}$ Mančinska and Roberson's graph categories (Example 3.1 (f)) generalize Banica and Speicher's partition categories (Example 3.1 (c)). However, this embedding does not send the $\mathcal{E}$ of the former, quasi-projections, to those of the latter, isomorphisms. This can be remedied in two ways. First, we can consider "categories of bi-labeled sets" in Mančinska and Roberson's language, i.e., in Example 3.1 (c), replace $\mathrm{qpr}_{\mathcal{A}}$ by $\mathrm{iso}_{\mathcal{A}}$ (and δ with the trivial degree function).

Alternatively, one may consider subcategories of $\mathcal{T}^0(\mathcal{A}, \mathcal{E}, \delta)$ for $\mathcal{A}$ as in Example 3.1 (f) and $\mathcal{E} = \mathrm{qpr}_{\mathcal{A}}$ and $\delta = \mathrm{qker}_{\mathcal{A}}$. Like $\mathrm{M}_{\mathsf{fSet}^{\mathrm{op}}}$, the object monoid $\mathrm{M}_{\mathsf{fGr}^{\mathrm{op}}}$ is free, however not singly generated but free over the set of all isomorphism classes of connected graphs instead.

(b) We can give a choice of $\mathcal{A}$, $\mathcal{E}$ and δ that generalizes Mančinska and Roberson's graph categories (Example 3.1 (f)) in the same way that Freslon's categories (Example 3.1 (e)) generalize those of Banica and Speicher (Example 3.1 (c)). Given a "color set" C we let $\mathcal{A}$ be the full subcategory of $\mathsf{fGr}^{\mathrm{op}}$ (equivalent but not isomorphic to $\mathsf{fGr}^{\mathrm{op}}$) generated by all objects which are mapped by $(U_{\mathsf{fGr}})^{\mathrm{op}}$ to objects of the category $\mathcal{A}$ from Example 3.1 (e). We can then choose $\mathcal{E} = \mathrm{qpr}_{\mathcal{A}}$ and $\delta = \mathrm{qker}_{\mathcal{A}}$ and $\mathsf{S} = \mathsf{Set}$ to obtain a $\mathcal{T}^0(\mathcal{A}, \mathcal{E}, \delta)$ whose subcategories correspond to categories of "colored graphs". However, this is *not* to be confused with colored graphs in the sense (of graph theory) that vertices or edges are labeled and homomorphisms are required to preserve these labels (as sketched in [MR20, Section 8.2]). The colors are only relevant for controlling the number of isomorphic objects for the purpose of then carrying out constructions which violate the principle of equivalence, such as taking subcategories.

4. Auxiliaries for the proof

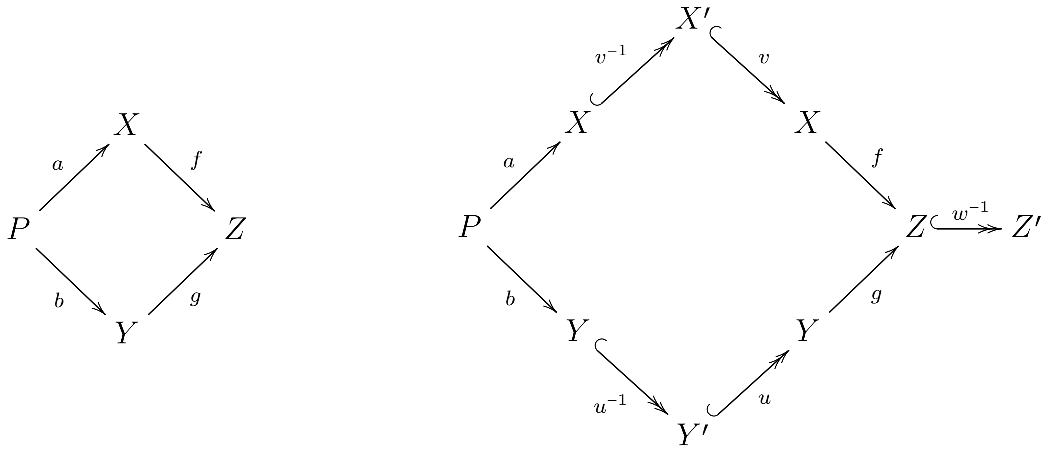
The partial proof offered in Section 5 that the construction from Section 2 has all the properties asserted requires more theory about pull-backs, cartesian monoidal categories and orthogonal factorization than was provided in Section 2. The material in Section 4 is *not* new.

4.1. Pull-backs. On many occasions proving the main results will require us to compute pull-backs, particularly in cartesian monoidal categories. The next few results facilitate such computations in many situations.

LEMMA 4.1. *In any category, for any* $\{X, X', Y, Y', Z, Z'\} \subseteq \mathrm{obj}$*, any morphisms* $f \in \mathrm{mor}(X, Z)$ *and* $g \in \mathrm{mor}(Y, Z)$ *and* $a \in \mathrm{mor}(P, X)$ *and* $b \in \mathrm{mor}(P, Y)$*, and any* $v \in \mathrm{iso}(X', X)$ *and* $u \in \mathrm{iso}(Y', Y)$ *and* $w \in \mathrm{iso}(Z', Z)$ *the following are equivalent:*

(a) (a, b) *is a pull-back of* (f, g).

(b) $(v^{-1} \circ a, u^{-1} \circ b)$ *is a pull-back of* $(w^{-1} \circ f \circ v, w^{-1} \circ g \circ u)$.



PROOF. Because w is an isomorphism it is clear that $f \circ a = g \circ b$ is true if and only if $(w^{-1} \circ f \circ v) \circ (v^{-1} \circ a) = (w^{-1} \circ g \circ u) \circ (u^{-1} \circ b)$ holds. Hence, the left diagram commutes if and only if the right ones is commutative.

Now, suppose that (a, b) is a pull-back of (f, g) and let Q and $r\colon Q \to X'$ and $s\colon Q \to Y'$ be such that $(w^{-1} \circ f \circ v) \circ r = (w^{-1} \circ g \circ u) \circ s$. It follows $f \circ (v \circ r) = g \circ (u \circ s)$. Thus, by the assumption there exists a morphism $t\colon Q \to P$ with $r = a \circ t$ and $s = b \circ t$ and, moreover, there is only such morphism. Since the pair of identities $v \circ r = a \circ t$ and $u \circ s = b \circ t$ is equivalent to the pair $r = v^{-1} \circ t$ and $s = u^{-1} \circ t$, we have thus shown that $(v^{-1} \circ a, u^{-1} \circ b)$ is a pull-back of $(w^{-1} \circ f \circ v, w^{-1} \circ g \circ u)$.

Conversely, let now $(v^{-1} \circ a, u^{-1} \circ b)$ be a pull-back of $(w^{-1} \circ f \circ v, w^{-1} \circ g \circ u)$ and let C and $d\colon C \to X$ and $e\colon C \to Y$ be such that $f \circ d = g \circ e$. Then, because w is an isomorphism, $(w^{-1} \circ f \circ v) \circ (v^{-1} \circ d) = (w^{-1} \circ g \circ u) \circ (u^{-1} \circ e)$. Hence, the assumption implies the existence of $h\colon C \to P$ such that $v^{-1} \circ d = (v^{-1} \circ a) \circ h$ and $u^{-1} \circ e = (u^{-1} \circ b) \circ h$ and its uniqueness with these properties. However, this pair of identities is clearly equivalent to the pair $d = a \circ h$ and $e = b \circ h$, which proves that (a, b) is a pull-back of (f, g). □

LEMMA 4.2. *In any category, for any* $\{X, Y, Z, P, P'\} \subseteq \mathrm{obj}$ *and any* $f \in \mathrm{mor}(X, Z)$ *and* $g \in \mathrm{mor}(Y, Z)$ *and* $a \in \mathrm{mor}(P, X)$ *and* $b \in \mathrm{mor}(P, Y)$*, and let* $w \in \mathrm{iso}(P', P)$*, the span* (a, b) *is a pull-back of* (f, g) *if and only if* $(a \circ w, b \circ w)$ *is one.*

PROOF. Follows from the definitions. □

The following important associativity law for pull-backs was proved in [Kel69, Lemma 5.1].

LEMMA 4.3. *The following statements hold in any category.*

(a) If a commutative diagram is given by

$$\begin{array}{ccccc} A & \xrightarrow{b} & B & \xrightarrow{q} & C \\ {\scriptstyle a}\downarrow & & \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle h} \\ D & \xrightarrow[f]{} & E & \xrightarrow[g]{} & F \end{array} \quad \text{and thus also by} \quad \begin{array}{ccc} A & \xrightarrow{q\circ b} & C \\ {\scriptstyle a}\downarrow & & \downarrow{\scriptstyle h} \\ D & \xrightarrow[g\circ f]{} & F \end{array}$$

and if (p,q) is a pull-back of (g,h), then the following are equivalent:
(i) (a,b) is a pull-back of (f,p).
(ii) $(a,q\circ b)$ is a pull-back of $(g\circ f,h)$.

(b) If commutative diagrams are given by

$$\begin{array}{ccc} B & \xrightarrow{q} & C \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle h} \\ E & \xrightarrow[g]{} & F \end{array} \quad \text{and} \quad \begin{array}{ccc} A & \xrightarrow{r} & C \\ {\scriptstyle a}\downarrow & & \downarrow{\scriptstyle h} \\ D & \xrightarrow[g\circ f]{} & F \end{array}$$

and if (p,q) is a pull-back of (g,h) and (a,r) is a pull-back of $(g\circ f,h)$, then there exists a unique morphism b

$$\begin{array}{ccccc} & & \overset{r}{\frown} & & \\ A & \xrightarrow[b]{} & B & \xrightarrow[q]{} & C \\ {\scriptstyle a}\downarrow & & \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle h} \\ D & \xrightarrow[f]{} & E & \xrightarrow[g]{} & F \end{array}$$

such that $r = q\circ b$ and $f\circ a = p\circ b$.

LEMMA 4.4. *In any category, for any $n\in\mathbb{N}$, for any objects $B_1,\ldots,B_{n+1}$ and $C_1,\ldots,C_{n+1}$ and for any morphisms $a_1,\ldots,a_{n+1}$ and $b_1,\ldots,b_n$ and $c_1,\ldots,c_n$ such that $b_i\colon B_i\to B_{i+1}$ and $c_i\colon C_i\to C_{i+1}$ for each $i\in\{1,\ldots,n\}$, and such that $a_i\colon B_i\to C_i$ for each $i\in\{1,\ldots,n+1\}$, whenever (a_i,b_i) is a pull-back of (c_i,a_{i+1}) for each $i\in\{1,\ldots,n\}$, then $(a_1,b_n\circ\ldots\circ b_1)$ is a pull-back of $(c_n\circ\ldots\circ c_1,a_{n+1})$.*

$$\begin{array}{ccccccccccccc} C_1 & \xrightarrow{c_1} & C_2 & \xrightarrow{c_2} & C_3 & \xrightarrow{c_3} & C_4 & \dashrightarrow & C_{n-1} & \xrightarrow{c_{n-1}} & C_n & \xrightarrow{c_n} & C_{n+1} \\ {\scriptstyle a_1}\uparrow & & {\scriptstyle a_2}\uparrow & & {\scriptstyle a_3}\uparrow & & {\scriptstyle a_4}\uparrow & \cdots & {\scriptstyle a_{n-1}}\uparrow & & {\scriptstyle a_n}\uparrow & & {\scriptstyle a_{n+1}}\uparrow \\ B_1 & \xrightarrow[b_1]{} & B_2 & \xrightarrow[b_2]{} & B_3 & \xrightarrow[b_3]{} & B_4 & \dashrightarrow & B_{n-1} & \xrightarrow[b_{n-1}]{} & B_n & \xrightarrow[b_n]{} & B_{n+1} \end{array}$$

PROOF. For $n=1$ the claim is entailed by Lemma 4.3 (a). Suppose it is true for all such diagrams of length n. Then, in particular in the current diagram of length

$(n+1)$ we can conclude that $(a_1, b_{n-1} \circ \ldots \circ b_1)$ is a pull-back of $(c_{n-1} \circ \ldots \circ b_1, a_n)$.

$$\begin{array}{ccccc} C_1 & \xrightarrow{c_{n-1}\circ\ldots\circ c_1} & C_n & \xrightarrow{c_n} & C_{n+1} \\ {\scriptstyle a_1}\uparrow & & {\scriptstyle a_n}\uparrow & & {\scriptstyle a_{n+1}}\uparrow \\ B_1 & \xrightarrow[b_{n-1}\circ\ldots\circ b_1]{} & B_n & \xrightarrow[b_n]{} & B_{n+1} \end{array}$$

Applying Lemma 4.3 (a) to the remaining length 2 diagram now yields that the pair $(a_1, b_n \circ \ldots \circ b_1)$ is a pull-back of $(c_n \circ \ldots \circ c_1, a_{n+1})$. □

We need the following extension of Lemma 4.3.

LEMMA 4.5. *If a commutative diagram is given by*

$$\begin{array}{ccccc} A & \xrightarrow{b} & B & \xrightarrow{q} & C \\ {\scriptstyle a}\downarrow & & \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle k} \\ D & \xrightarrow[d]{} & E & \xrightarrow[t]{} & F \\ {\scriptstyle c}\downarrow & & \downarrow{\scriptstyle s} & & \downarrow{\scriptstyle h} \\ G & \xrightarrow[f]{} & H & \xrightarrow[g]{} & K \end{array} \qquad \textit{and thus also by} \qquad \begin{array}{ccc} A & \xrightarrow{q\circ b} & C \\ {\scriptstyle a\circ c}\downarrow & & \downarrow{\scriptstyle k\circ h} \\ G & \xrightarrow[g\circ f]{} & K \end{array}$$

and if (s,t) is a pull-back of (g,h), if (c,d) is one of (f,s), if (p,q) is one of (t,k), and if (a,b) is one of (d,p), then $(c \circ a, q \circ b)$ is a pull-back of $(g \circ f, h \circ k)$.

PROOF. By Lemma 4.3 (a), applied to each of the diagrams

$$\begin{array}{ccccc} A & \xrightarrow{b} & B & \xrightarrow{q} & C \\ {\scriptstyle a}\downarrow & & \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle k} \\ D & \xrightarrow[d]{} & E & \xrightarrow[t]{} & F \end{array} \qquad \text{and} \qquad \begin{array}{ccccc} D & \xrightarrow{d} & E & \xrightarrow{t} & F \\ {\scriptstyle c}\downarrow & & \downarrow{\scriptstyle s} & & \downarrow{\scriptstyle h} \\ G & \xrightarrow[f]{} & H & \xrightarrow[g]{} & K \end{array},$$

the pair $(a, q \circ b)$ is a pull-back of $(t \circ d, k)$ and $(c, t \circ d)$ one of $(g \circ f, h)$. Now, applying Lemma 4.3 (a) a third time to the diagram

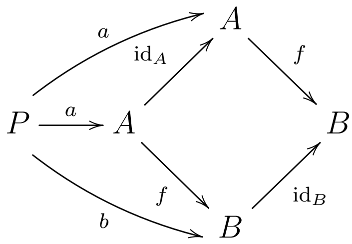
$$\begin{array}{ccccc} A & \xrightarrow{a} & D & \xrightarrow{c} & G \\ {\scriptstyle q\circ b}\downarrow & & \downarrow{\scriptstyle t\circ d} & & \downarrow{\scriptstyle g\circ f} \\ C & \xrightarrow[k]{} & F & \xrightarrow[h]{} & K \end{array}$$

yields the claim. □

The following shows that identities also act as neutral elements for pull-backs and will be used countless times.

LEMMA 4.6. *In any category,* (id_A, f) *is a pullback of* (f, id_B) *for any* $f \in \mathrm{mor}(A, B)$ *and any* $\{A, B\} \subseteq \mathrm{obj}$.

Proof. Of course, $\mathrm{id}_B \circ f = f \circ \mathrm{id}_A$. If $a\colon P \to A$ and $b\colon P \to B$ are such that $f \circ a = \mathrm{id}_B \circ b$, then $u \coloneqq a$ is a morphism $P \to A$ with $a = \mathrm{id}_A \circ u$ and $b = f \circ u$.

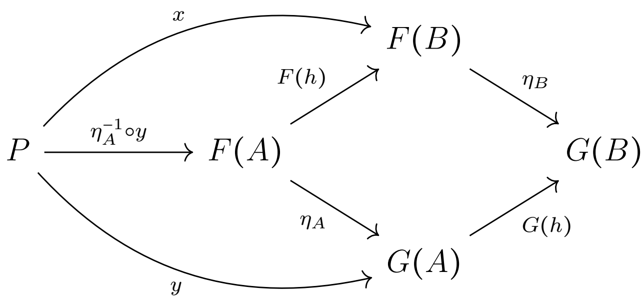


Moreover, if $u'\colon P \to A$ satisfies $a = \mathrm{id}_A \circ u'$ and $b = f \circ u'$, then $u' = a = u$ by the first equality. □

Finally, for the 2-functorial part of the main results it will be important to know that natural isomorphisms are equifibered.

Lemma 4.7. *Any natural isomorphism η from any functor F to any functor G from any category $\mathcal{C}$ to any category $\mathcal{D}$ is equifibered.*

Proof. Let $\{A, B\} \subseteq \mathrm{obj}_{\mathcal{C}}$ and any $h \in \mathrm{mor}_{\mathcal{C}}(A, B)$ be arbitrary and let $P \in \mathrm{obj}_{\mathcal{D}}$ and $x \in \mathrm{mor}_{\mathcal{D}}(P, F(B))$ and $y \in \mathrm{mor}_{\mathcal{D}}(P, G(A))$ be such that $\eta_B \circ_{\mathcal{D}} x = G(h) \circ_{\mathcal{D}} y$.



Since η is a natural isomorphism the naturality condition $\eta_B \circ_{\mathcal{D}} F(h) = G(h) \circ_{\mathcal{D}} \eta_A$ implies $\eta_B^{-1} \circ_{\mathcal{D}} G(h) = F(h) \circ_{\mathcal{D}} \eta_A^{-1}$. Hence, if we define $m \coloneqq \eta_A^{-1} \circ_{\mathcal{D}} y \in \mathrm{mor}_{\mathcal{D}}(P, F(A))$, then, on the one hand, of course, $\eta_A \circ_{\mathcal{D}} m = y$, but on the other hand also $F(h) \circ_{\mathcal{D}} m = F(h) \circ_{\mathcal{D}} (\eta_A^{-1} \circ_{\mathcal{D}} y) = (F(h) \circ_{\mathcal{D}} \eta_A^{-1}) \circ_{\mathcal{D}} y = (\eta_B^{-1} \circ_{\mathcal{D}} G(h)) \circ_{\mathcal{D}} y = \eta_B^{-1} \circ_{\mathcal{D}} (G(h) \circ_{\mathcal{D}} y) = \eta_B^{-1} \circ_{\mathcal{D}} \eta_B \circ_{\mathcal{D}} x = x$. Moreover, m is unique with these properties because η_A is a monomorphism of $\mathcal{D}$. □

4.2. Cartesian Monoidal Categories. For the proofs in Section 5 it is especially important to know how to compute pull-backs involving the structure morphisms of a cartesian monoidal category. The next lemma gives an overview of how those structure morphisms arise from the projections.

Notation 4.8. In any symmetric monoidal category $\mathcal{C}$ denote by

$$\mu_{\mathcal{C},A,B,C,D} = \alpha^{-1}_{\mathcal{C},A,C,B\otimes_{\mathcal{C}} D} \circ_{\mathcal{C}} (\mathrm{id}_{\mathcal{C},A} \otimes_{\mathcal{C}} \alpha_{\mathcal{C},C,B,D}) \circ_{\mathcal{C}} (\mathrm{id}_{\mathcal{C},A} \otimes_{\mathcal{C}} (\gamma_{\mathcal{C},B,C} \otimes_{\mathcal{C}} \mathrm{id}_{\mathcal{C},D})) \circ (\mathrm{id}_A \otimes \alpha^{-1}_{B,C,D}) \circ \alpha_{A,B,C\otimes D}$$

the *four middle interchnage* of (A, B, C, D). If $\mathcal{C}$ is clear from the context, the index $\mathcal{C}$ may be omitted.

LEMMA 4.9. *In any cartesian monoidal category*

(a) $(\pi^1_{A,B}, \pi^2_{A,B})$ *is a product of* (A, B) *for any* $\{A, B\} \subseteq \mathrm{obj}$.

(b) ω_A *is the unique morphism* $A \to I$ *for each* $A \in \mathrm{obj}$.

(c) $f_1 \otimes f_2 = (f_1 \circ \pi^1_{A_1,A_2}) \times (f_2 \circ \pi^2_{A_1,A_2})$ *for any* $f_1 \colon A_1 \to B_1$ *and* $f_2 \colon A_2 \to B_2$.

(d) $\alpha_{A,B,C} = (\pi^1_{A,B} \circ \pi^1_{A\otimes B,C}) \times ((\pi^2_{A,B} \circ \pi^1_{A\otimes B,C}) \times \pi^2_{A\otimes B,C})$ *as well as* $\alpha^{-1}_{A,B,C} = \pi^1_{A,B\otimes C} \times ((\pi^1_{B,C} \circ \pi^2_{A,B\otimes C}) \times (\pi^2_{B,C} \circ \pi^2_{A,B\otimes C}))$ *for any* $\{A, B, C\} \subseteq \mathrm{obj}$.

(e) $\lambda_A = \pi^2_{I,A}$ *and* $\lambda^{-1}_A = \omega_A \times \mathrm{id}_A$ *for any* $A \in \mathrm{obj}$.

(f) $\rho_A = \pi^1_{A,I}$ *and* $\rho^{-1}_A = \mathrm{id}_A \times \omega_A$*for any* $A \in \mathrm{obj}$.

(g) $\gamma_{A,B} = \pi^2_{A,B} \times \pi^1_{A,B}$ *for any objects* $\{A, B\} \subseteq \mathrm{obj}$.

(h) *for any* $\{A, B, C, D\} \subseteq \mathrm{obj}$,

$$\begin{aligned}\mu_{A,B,C,D} = ((\pi^1_{A,B} \circ \pi^1_{A\otimes B,C\otimes D}) \times (\pi^1_{C,D} \circ \pi^2_{A\otimes B,C\otimes D}))\\ \times ((\pi^2_{A,B} \circ \pi^1_{A\otimes B,C\otimes D}) \times (\pi^2_{C,D} \circ \pi^2_{A\otimes B,C\otimes D})).\end{aligned}$$

PROOF. The proof is omitted. □

LEMMA 4.10. *If commutative diagrams in any cartesian monoidal category are given by*

$$\begin{array}{ccc} A_1 & \xrightarrow{a_1} & B_1 \\ {\scriptstyle b_1}\downarrow & & \downarrow{\scriptstyle g_1} \\ C_1 & \xrightarrow[h_1]{} & D_1 \end{array} \quad \textit{and} \quad \begin{array}{ccc} A_2 & \xrightarrow{a_2} & B_2 \\ {\scriptstyle b_2}\downarrow & & \downarrow{\scriptstyle g_2} \\ C_2 & \xrightarrow[h_2]{} & D_2 \end{array} \quad \textit{and thus also by} \quad \begin{array}{ccc} A_1 \otimes A_2 & \xrightarrow{a_1\otimes a_2} & B_1 \otimes B_2 \\ {\scriptstyle b_1\otimes b_2}\downarrow & & \downarrow{\scriptstyle g_1\otimes g_2} \\ C_1 \otimes C_2 & \xrightarrow[h_1\otimes h_2]{} & D_1 \otimes D_2 \end{array},$$

and if (a_1, b_1) *is a pull-back of* (g_1, h_1) *and* (a_2, b_2) *one of* (g_2, h_2), *then the pair* $(a_1 \otimes a_2, b_1 \otimes b_2)$ *is a pull-back of* $(g_1 \otimes g_2, h_1 \otimes h_2)$.

PROOF. By assumption, $(g_1 \otimes g_2) \circ (a_1 \otimes a_2) = (g_1 \circ a_1) \otimes (g_2 \circ a_2) = (h_1 \circ b_1) \otimes (h_2 \circ b_2) = (h_1 \otimes h_2) \circ (b_1 \otimes b_2)$ because $\otimes$ is a functor. We prove that $(a_1 \otimes a_2, b_1 \otimes b_2)$ is universal with that property.

For each $i \in \{1, 2\}$ let the objects X_i, Y_i, and B_i be such that $g_i \colon X_i \to B_i$ and $h_i \colon Y_i \to B_i$, let P_i be the common domain of (a_i, b_i) and let (a', b') be a pair of morphisms with common domain P' and with $(g_1 \otimes g_2) \circ a' = (h_1 \otimes h_2) \circ b'$. By the universal property of (a_i, b_i) for each $i \in \{1, 2\}$ there exists a unique morphism $u_i \colon P' \to P_i$ with $\pi^i_{X_1,X_2} \circ a' = a_i \circ u_i$ and $\pi^i_{Y_1,Y_2} \circ b' = b_i \circ u_i$: Indeed,

$$\begin{aligned} g_i \circ (\pi^i_{X_1,X_2} \circ a') &= (g_i \circ \pi^i_{X_1,X_2}) \circ a' = (\pi^i_{Y_1,Y_2} \circ ((g_1 \circ \pi^1_{X_1,X_2}) \times (g_2 \circ \pi^2_{X_1,X_2}))) \circ a' \\ &= (\pi^i_{Y_1,Y_2} \circ (g_1 \otimes g_2)) \circ a' = \pi^i_{Y_1,Y_2} \circ ((g_1 \otimes g_2) \circ a') \\ &= \pi^i_{Y_1,Y_2} \circ ((h_1 \otimes h_2) \circ a') \\ &= h_i \circ (\pi^i_{Y_1,Y_2} \circ a'), \end{aligned}$$

where the last step is analogous to the ones leading up to it, only in reverse and with the roles of g_i and h_i exchanged. For $u_1 \times u_2 \colon P' \to P_1 \otimes P_2$ we find

$$\begin{aligned} a' &= (\pi^1_{X_1,X_2} \circ a') \times (\pi^2_{X_1,X_2} \circ a') = (a_1 \circ u_1) \times (a_2 \circ u_2) \\ &= (a_1 \circ (\pi^1_{P_1,P_2} \circ (u_1 \times u_2))) \times (a_2 \circ (\pi^2_{P_1,P_2} \circ (u_1 \times u_2))) \\ &= ((a_1 \circ \pi^1_{P_1,P_2}) \times (a_2 \circ \pi^2_{P_1,P_2})) \circ (u_1 \times u_2) \\ &= (a_1 \otimes a_2) \circ (u_1 \times u_2) \end{aligned}$$

and by an analogous computation $b' = (b_1 \otimes b_2) \circ (u_1 \times u_2)$. It remains to prove that $u_1 \times u_2$ is unique with that property. Let $u' \colon P' \to P_1 \otimes P_2$ be such that $a' = (a_1 \otimes a_2) \circ u'$ and $b' = (b_1 \otimes b_2) \circ u'$. For each $i \in \{1,2\}$ the morphism $\pi^i_{P_1,P_2} \circ u' \colon P' \to P_i$ satisfies

$$\begin{aligned} \pi^i_{X_1,X_2} \circ a' &= \pi^i_{X_1,X_2} \circ ((a_1 \otimes a_2) \circ u') = \pi^i_{X_1,X_2} \circ (((a_1 \circ \pi^1_{P_1,P_2}) \times (a_2 \circ \pi^2_{P_1,P_2})) \circ u') \\ &= (\pi^i_{X_1,X_2} \circ ((a_1 \circ \pi^1_{P_1,P_2}) \times (a_2 \circ \pi^2_{P_1,P_2}))) \circ u' = (a_i \circ \pi^i_{P_1,P_2}) \circ u' \\ &= a_i \circ (\pi^i_{P_1,P_2} \circ u') \end{aligned}$$

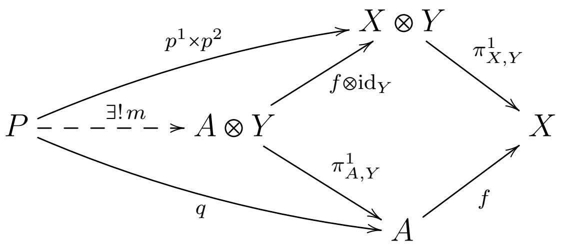
and, analogously, $\pi^i_{X_1,X_2} \circ b' = b_i \circ (\pi^i_{P_1,P_2} \circ u')$, the properties which characterized u_i uniquely. From $u_i = \pi^i_{P_1,P_2} \circ u'$ for each $i \in \{1,2\}$ it now follows $u' = u_1 \times u_2$ by the uniqueness of product morphisms. And that is what we needed to see. $\square$

Lemma 4.11. *In any cartesian monoidal category, the following statements hold for any* $\{X,Y\} \subseteq \mathrm{obj}$.

(a) *For any* $f \in \mathrm{mor}(A,X)$ *a pull-back of* $(\pi^1_{X,Y}, f)$ *is given by* $(f \otimes \mathrm{id}_Y, \pi^1_{A,Y})$.

(b) *For any* $g \in \mathrm{mor}(B,Y)$ *a pull-back of* $(\pi^2_{X,Y}, g)$ *is given by* $(\mathrm{id}_X \otimes g, \pi^2_{X,B})$.

Proof. (a) Because $\mathcal{C}$ is cartesian monoidal, $f \otimes \mathrm{id}_Y = (f \circ \pi^1_{A,Y}) \times (\mathrm{id}_Y \circ \pi^2_{A,Y})$, obviously, $\pi^1_{X,Y} \circ (f \otimes \mathrm{id}_Y) = f \circ \pi^1_{A,Y}$. Now, let the object P and the morphisms $p^1 \times p^2 \colon P \to X \otimes Y$ and $q \colon P \to A$ be such that $\pi^1_{X,Y} \circ (p^1 \times p^2) = p^1 = f \circ q$.



If we define $m \coloneqq q \times p^2$, then, evidently, $\pi^1_{A,Y} \circ m = q$. But also, $(f \otimes \mathrm{id}_Y) \circ m = (f \circ q) \times (\mathrm{id}_Y \circ p^2) = p^1 \times p^2$. Moreover, m is unique with this property: If $m' \colon P \to A \otimes Y$ satisfies $\pi^1_{A,Y} \circ m' = q$ and $(f \otimes \mathrm{id}_Y) \circ m' = p^1 \times p^2$, then the first identity implies $\pi^1_{A,Y} \circ m = q = \pi^1_{A,Y} \circ m'$ and the second $\pi^2_{A,Y} \circ m = p^2 = \pi^2_{A,Y} \circ m'$ and thus $m = m'$.

(b) If we apply Part (a) with B in the role of A and g in that of f (and, accordingly, the roles of X and Y reversed), we infer that $(g \otimes \mathrm{id}_X, \pi^1_{B,X})$ is a pull-back of $(\pi^1_{Y,X}, g)$. It follows by Lemma 4.1 that $(\gamma^{-1}_{X,Y} \circ (g \otimes \mathrm{id}_X), \pi^1_{B,X})$ is a pull-back of $(\pi^1_{Y,X} \circ \gamma_{X,Y}, g)$. By nature of the braiding, that is equivalent to saying that

$((\mathrm{id}_X \otimes g) \circ \gamma_{B,X}, \pi^2_{X,B} \circ \gamma_{B,X})$ is a pull-back of $(\pi^2_{X,Y}, g)$. now, Lemma 4.2 yields the claim. □

LEMMA 4.12. *In any cartesian monoidal category, for any* $\{X, A, B, C\} \subseteq \mathrm{obj}$ *and any* $f \in \mathrm{mor}(X, A)$*, any* $g \in \mathrm{mor}(X, B)$ *and any* $h \in \mathrm{mor}(X, C)$*,*

$$\alpha_{A,B,C} \circ ((f \times g) \times h) = (f \times (g \times h)).$$

PROOF. Since, per assumption, $\alpha_{A,B,C} = (\pi^1_{A,B} \circ \pi^1_{A\otimes B,C}) \times ((\pi^2_{A,B} \circ \pi^1_{A\otimes B,C}) \times \pi^2_{A\otimes B,C})$, we can compute directly

$$\begin{aligned}
&\pi^1_{A,B\otimes C} \circ \alpha_{A,B,C} \circ ((f \times g) \times h) \\
&\qquad = \pi^1_{A,B\otimes C} \circ ((\pi^1_{A,B} \circ \pi^1_{A\otimes B,C}) \times ((\pi^2_{A,B} \circ \pi^1_{A\otimes B,C}) \times \pi^2_{A\otimes B,C})) \circ ((f \times g) \times h) \\
&\qquad = \pi^1_{A,B} \circ \pi^1_{A\otimes B,C} \circ ((f \times g) \times h) \\
&\qquad = \pi^1_{A,B} \circ (f \times g) \\
&\qquad = f.
\end{aligned}$$

and

$$\begin{aligned}
&\pi^1_{B,C} \circ \pi^2_{A,B\otimes C} \circ \alpha_{A,B,C} \circ ((f \times g) \times h) \\
&\quad = \pi^1_{B,C} \circ \pi^2_{A,B\otimes C} \circ ((\pi^1_{A,B} \circ \pi^1_{A\otimes B,C}) \times ((\pi^2_{A,B} \circ \pi^1_{A\otimes B,C}) \times \pi^2_{A\otimes B,C})) \circ ((f \times g) \times h) \\
&\quad = \pi^1_{B,C} \circ ((\pi^2_{A,B} \circ \pi^1_{A\otimes B,C}) \times \pi^2_{A\otimes B,C}) \circ ((f \times g) \times h) \\
&\quad = \pi^2_{A,B} \circ \pi^1_{A\otimes B,C} \circ ((f \times g) \times h) \\
&\quad = \pi^2_{A,B} \circ (f \times g) \\
&\quad = g
\end{aligned}$$

as well as

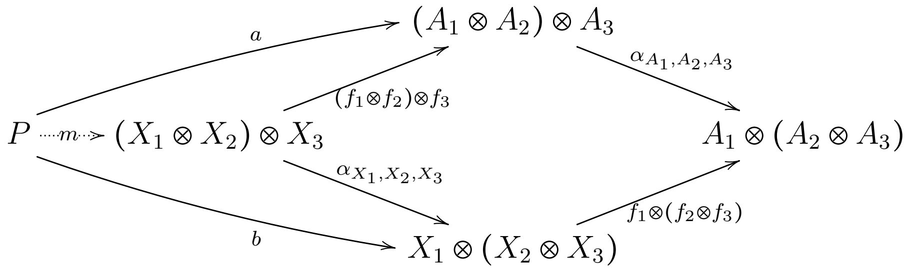
$$\begin{aligned}
&\pi^2_{B,C} \circ \pi^2_{A,B\otimes C} \circ \alpha_{A,B,C} \circ ((f \times g) \times h) \\
&\quad = \pi^2_{B,C} \circ \pi^2_{A,B\otimes C} \circ ((\pi^1_{A,B} \circ \pi^1_{A\otimes B,C}) \times ((\pi^2_{A,B} \circ \pi^1_{A\otimes B,C}) \times \pi^2_{A\otimes B,C})) \circ ((f \times g) \times h) \\
&\quad = \pi^2_{B,C} \circ ((\pi^2_{A,B} \circ \pi^1_{A\otimes B,C}) \times \pi^2_{A\otimes B,C}) \circ ((f \times g) \times h) \\
&\quad = \pi^2_{A\otimes B,C} \circ ((f \times g) \times h) \\
&\quad = h.
\end{aligned}$$

The last two identities prove $\pi^2_{A,B\otimes C} \circ \alpha_{A,B,C} \circ ((f \times g) \times h) = g \times h$ by the universal property of $(\pi^1_{B,C}, \pi^2_{B,C})$. Together with the first identity this proves the claim by the universal property of $(\pi^1_{A,B\otimes C}, \pi^2_{A,B\otimes C})$. □

LEMMA 4.13. *In any cartesian monoidal category, let* $\{X_i, A_i\} \subseteq \mathrm{obj}$ *and* $f_i \in \mathrm{mor}(X_i, A_i)$ *for each* $i \in \{1, 2, 3\}$*, and let* $\{X, A\} \subseteq \mathrm{obj}$ *and* $f \in \mathrm{mor}(X, A)$*.*

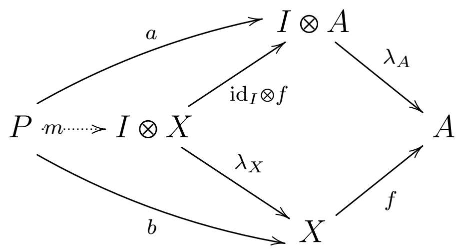
(a) $((f_1 \otimes f_2) \otimes f_3, \alpha_{X_1,X_2,X_3})$ *is a pull-back of* $(\alpha_{A_1,A_2,A_3}, f_1 \otimes (f_2 \otimes f_3))$*.*
(b) $(\mathrm{id}_I \otimes f, \lambda_X)$ *is a pull-back of* (λ_A, f)*.*
(c) $(f \otimes \mathrm{id}_I, \rho_X)$ *is a pull-back of* (ρ_A, f)*.*

Proof. (a) The identity $\alpha_{A_1,A_2,A_3}\circ((f_1\otimes f_2)\otimes f_3) = (f_1\otimes(f_2\otimes f_3))\circ\alpha_{X_1,X_2,X_3}$ holds as $\alpha_{\mathcal{A}}$ is a natural transformation from $((\cdot_1)\otimes_{\mathcal{A}}(\cdot_2))\otimes_{\mathcal{A}}(\cdot_3)$ to $(\cdot_1)\otimes_{\mathcal{A}}((\cdot_2)\otimes_{\mathcal{A}}(\cdot_3))$. We just have to show that $((f_1\otimes f_2)\otimes f_3, \alpha_{X_1,X_2,X_3})$ is universal with that property. Let $P\in\text{obj}$ and let $a\colon P\to(A_1\otimes A_2)\otimes A_3$ and $b\colon P\to X_1\otimes(X_2\otimes X_3)$ satisfy $\alpha_{A_1,A_2,A_3}\circ a = (f_1\otimes(f_2\otimes f_3))\circ b$.



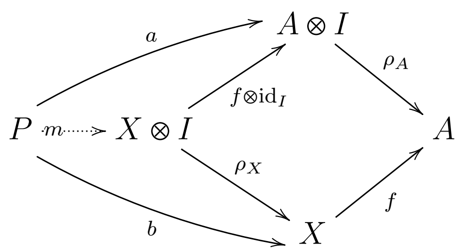
Define $m := \alpha^{-1}_{X_1,X_2,X_3}\circ b$. Then, all that is left to show is that $a \overset{!}{=} ((f_1\otimes f_2)\otimes f_3)\circ m$. We have already seen that $\alpha_{A_1,A_2,A_3}\circ((f_1\otimes f_2)\otimes f_3) = (f_1\otimes(f_2\otimes f_3))\circ\alpha_{X_1,X_2,X_3}$. This can be rewritten as $((f_1\otimes f_2)\otimes f_3)\circ\alpha^{-1}_{X_1,X_2,X_3} = \alpha^{-1}_{A_1,A_2,A_3}\circ(f_1\otimes(f_2\otimes f_3))$. It follows $((f_1\otimes f_2)\otimes f_3)\circ m = \alpha^{-1}_{A_1,A_2,A_3}\circ(f_1\otimes(f_2\otimes f_3))\circ b$. And the right hand side of that is just a per assumption.

(b) Again, the identity $\lambda_A\circ(\text{id}_I\otimes f) = f\circ\lambda_X$ follows from the axiom that λ is a natural transformation $I\otimes(\,\cdot\,)\to(\,\cdot\,)$. Only universality needs proving. Hence, let P and $a\colon P\to I\otimes A$ and $b\colon P\to X$ be such that $\lambda_A\circ a = f\circ b$.



If we put $m := \lambda^{-1}_X\circ b$, then $a \overset{!}{=} (\text{id}_I\otimes f)\circ m$ is all that is left to prove. We can rewrite $\lambda_A\circ(\text{id}_I\otimes f) = f\circ\lambda_X$ as $(\text{id}_I\otimes f)\circ\lambda^{-1}_X = \lambda^{-1}_A\circ f$. Hence, $(\text{id}_I\otimes f)\circ m = \lambda^{-1}_A\circ f\circ b$. And this equals a per assumption.

(c) $\rho_A\circ(f\otimes\text{id}_I) = f\circ\rho_X$ because λ is a natural transformation $I\otimes(\,\cdot\,)\to(\,\cdot\,)$. We show universality. Let P and $a\colon P\to I\otimes A$ and $b\colon P\to X$ satisfy $\rho_A\circ a = f\circ b$.



We let $m := \rho_X^{-1} \circ b$ and prove $a \overset{!}{=} (f \otimes \mathrm{id}_I) \circ m$. From $(f \otimes \mathrm{id}_I) \circ \rho_X^{-1} = \rho_A^{-1} \circ f$ it follows $(f \otimes \mathrm{id}_I) \circ m = \rho_A^{-1} \circ f \circ b = a$, which concludes the proof. $\square$

LEMMA 4.14. *In any cartesian monoidal the following are true for any $A \in \mathrm{obj}$.*

(a) A pull-back of $(\alpha_{A,A,A} \circ ((\mathrm{id}_A \times \mathrm{id}_A) \otimes \mathrm{id}_A), \mathrm{id}_A \otimes (\mathrm{id}_A \times \mathrm{id}_A))$ is given by $(\mathrm{id}_A \times \mathrm{id}_A, \mathrm{id}_A \times \mathrm{id}_A)$.

(b) A pull-back of $(\alpha^{-1}_{A,A,A} \circ (\mathrm{id}_A \otimes (\mathrm{id}_A \times \mathrm{id}_A)), (\mathrm{id}_A \times \mathrm{id}_A) \otimes \mathrm{id}_A)$ is given by $(\mathrm{id}_A \times \mathrm{id}_A, \mathrm{id}_A \times \mathrm{id}_A)$.

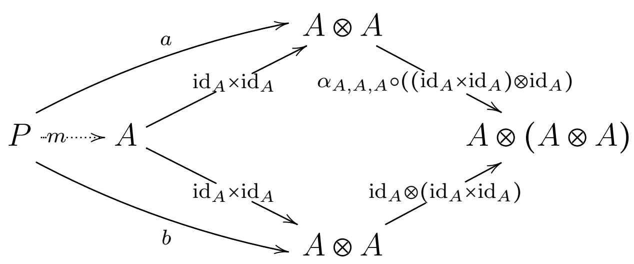
(c) $\omega_A \times \mathrm{id}_A$ is an isomorphism $A \to I \otimes A$.

(d) $\mathrm{id}_A \times \omega_A$ is an isomorphism $A \to A \otimes I$.

PROOF. (a) Because $(\mathrm{id}_A \otimes (\mathrm{id}_A \times \mathrm{id}_A)) \circ (\mathrm{id}_A \times \mathrm{id}_A) = \mathrm{id}_A \times (\mathrm{id}_A \times \mathrm{id}_A)$ and, likewise, $((\mathrm{id}_A \times \mathrm{id}_A) \otimes \mathrm{id}_A) \circ (\mathrm{id}_A \times \mathrm{id}_A) = (\mathrm{id}_A \times \mathrm{id}_A) \times \mathrm{id}_A$, the alleged pull-back $(\mathrm{id}_A \times \mathrm{id}_A, \mathrm{id}_A \times \mathrm{id}_A)$ satisfies

$$\begin{aligned}
&(\alpha_{A,A,A} \circ ((\mathrm{id}_A \times \mathrm{id}_A) \otimes \mathrm{id}_A)) \circ (\mathrm{id}_A \times \mathrm{id}_A) \\
&\qquad = \alpha_{A,A,A} \circ (((\mathrm{id}_A \times \mathrm{id}_A) \otimes \mathrm{id}_A) \circ (\mathrm{id}_A \times \mathrm{id}_A)) \\
&\qquad = \alpha_{A,A,A} \circ ((\mathrm{id}_A \times \mathrm{id}_A) \times \mathrm{id}_A) \\
&\qquad = \mathrm{id}_A \times (\mathrm{id}_A \times \mathrm{id}_A) \\
&\qquad = (\mathrm{id}_A \otimes (\mathrm{id}_A \times \mathrm{id}_A)) \circ (\mathrm{id}_A \times \mathrm{id}_A),
\end{aligned}$$

where Lemma 4.12 justifies the third step. We prove that $(\mathrm{id}_A \times \mathrm{id}_A, \mathrm{id}_A \times \mathrm{id}_A)$ is unique with that property. Let $P \in \mathrm{obj}$ and $a\colon P \to A \otimes A$ and $b\colon P \to A \otimes A$ satisfy $(\alpha_{A,A,A} \circ ((\mathrm{id}_A \times \mathrm{id}_A) \otimes \mathrm{id}_A)) \circ a = (\mathrm{id}_A \otimes (\mathrm{id}_A \times \mathrm{id}_A)) \circ b$.

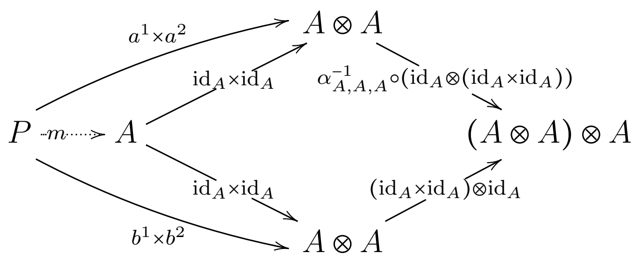


We prove for $m := \pi^1_{A,A} \circ b$ that $b \overset{!}{=} (\mathrm{id}_A \times \mathrm{id}_A) \circ m = m \times m \overset{!}{=} a$ and that m is the only morphism $P \to A$ which satisfies these equations. Abbreviate $a^1 := \pi^1_{A,A} \circ a$ and $a^2 := \pi^2_{A,A} \circ a$ as well as $b^1 := \pi^1_{A,A} \circ b$ and $b^2 := \pi^2_{A,A} \circ b$, i.e., $a = a^1 \times a^2$ and $b = b^1 \times b^2$. Then, using Lemma 4.12, the assumptions on a and b can be expressed as $a^1 \times (a^1 \times a^2) = \alpha_{A,A,A} \circ ((a^1 \times a^1) \times a^2) = (\alpha_{A,A,A} \circ ((\mathrm{id}_A \times \mathrm{id}_A) \otimes \mathrm{id}_A)) \circ (a^1 \times a^2) = (\mathrm{id}_A \otimes (\mathrm{id}_A \times \mathrm{id}_A)) \circ (b^1 \times b^2) = b^1 \times (b^2 \times b^2)$. That demands $a^1 = b^1$ and $(a^1 \times a^2) = (b^2 \times b^2)$, which in turn requires $a^1 = a^2 = b^2$. Thus, in combination, $m = b^1 = b^2 = a^1 = a^2$ and thus $b = a = m \times m$, as claimed. Of course, m is the only possible morphism with this property.

(b) The proof is almost the same as for Part (a). Here, too, Lemma 4.12 yields

$$\begin{aligned}(\alpha^{-1}_{A,A,A} \circ (\mathrm{id}_A \otimes (\mathrm{id}_A \times \mathrm{id}_A))) \circ (\mathrm{id}_A \times \mathrm{id}_A)\\ &= \alpha^{-1}_{A,A,A} \circ (\mathrm{id}_A \times (\mathrm{id}_A \times \mathrm{id}_A)) = (\mathrm{id}_A \times \mathrm{id}_A) \times \mathrm{id}_A\\ &= ((\mathrm{id}_A \times \mathrm{id}_A) \otimes \mathrm{id}_A) \circ (\mathrm{id}_A \times \mathrm{id}_A),\end{aligned}$$

If $a^1 \times a^2\colon P \to A \otimes A$ and $b^1 \times b^2\colon P \to A \otimes A$ also satisfy $(a^1 \times a^2) \times a^2 = (\alpha^{-1}_{A,A,A} \circ (\mathrm{id}_A \otimes (\mathrm{id}_A \times \mathrm{id}_A))) \circ (a^1 \times a^2) = ((\mathrm{id}_A \times \mathrm{id}_A) \otimes \mathrm{id}_A) \circ (b^1 \times b^2) = (b^1 \times b^1) \times b^2$,



which is to say $a^1 = a^2 = b^1 = b^2$, then $m := b^1$ gives a unique mediating morphism.

(c) We show that $\pi^2_{I,A}$ is the inverse of $\omega_A \times \mathrm{id}_A$. Of course, $\pi^2_{I,A} \circ (\omega_A \times \mathrm{id}_A) = \mathrm{id}_A$. As there is only one morphism $I \otimes A \to I$ the two morphisms $\pi^1_{I,A} \circ ((\omega_A \times \mathrm{id}_A) \circ \pi^2_{I,A})$ and $\pi^1_{I,A}$ must be the same. On the other hand, $\pi^2_{I,A} \circ ((\omega_A \times \mathrm{id}_A) \circ \pi^2_{I,A}) = (\pi^2_{I,A} \circ (\omega_A \times \mathrm{id}_A)) \circ \pi^2_{I,A} = \mathrm{id}_A \circ \pi^2_{I,A} = \pi^2_{I,A}$. That proves $(\omega_A \times \mathrm{id}_A) \circ \pi^2_{I,A} = \mathrm{id}_{I \otimes A}$.

(d) The proof is analogous to that of Part (c); the inverse of $\mathrm{id}_A \times \omega_A$ is $\pi^1_{A,I}$. □

LEMMA 4.15. *In any cartesian monoidal category, for any* $\{X, A, B\} \subseteq \mathrm{obj}$ *and* $f \in \mathrm{mor}(X, A)$ *and* $g \in \mathrm{mor}(X, B)$,

$$g \times f = \gamma_{A,B} \circ (f \times g).$$

PROOF. Because $\gamma_{A,B} = \pi^2_{A,B} \times \pi^1_{A,B}$ we can compute immediately

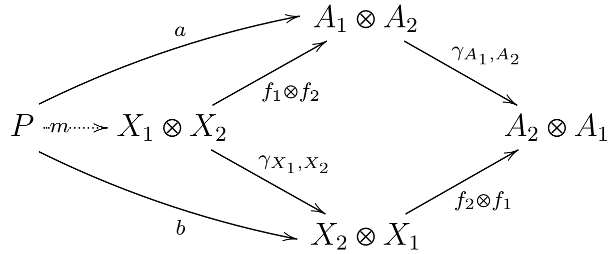
$$\begin{aligned}\gamma_{A,B} \circ (f \times g) &= (\pi^2_{A,B} \times \pi^1_{A,B}) \circ (f \times g)\\ &= (\pi^2_{A,B} \circ (f \times g)) \times (\pi^1_{A,B} \circ (f \times g))\\ &= g \times f,\end{aligned}$$

thus proving the claim. □

LEMMA 4.16. *In any cartesian monoidal category, if* $\{X_i, A_i\} \subseteq \mathrm{obj}$ *and* $f_i \in \mathrm{mor}(X_i, A_i)$ *for each* $i \in \{1, 2\}$, *then the span* $(f_1 \otimes f_2, \gamma_{X_1,X_2})$ *is a pull-back of the co-span* $(\gamma_{A_1,A_2}, f_2 \otimes f_1)$.

PROOF. Because γ is a natural transformation $(\cdot_1) \otimes (\cdot_2) \to (\cdot_2) \otimes (\cdot_1)$ the equation $\gamma_{A_1,A_2} \circ (f_1 \otimes f_2) = (f_2 \otimes f_1) \circ \gamma_{X_1,X_2}$ is automcatically satisfied. We just need to verify that $(f_1 \otimes f_2, \gamma_{X_1,X_2})$ is universal with that property. Let P and $a\colon P \to X_1 \otimes X_2$

and b: $P \to X_2 \otimes X_1$ with $a = a^1 \times a^2$ and $b = b^1 \times b^2$ be such that $\gamma_{A_1,A_2} \circ a = (f_2 \otimes f_1) \circ b$.



This means $a^2 \times a^1 = (\pi^2_{A_1,A_2} \circ a) \times (\pi^1_{A_1,A_2} \circ a) = (\pi^2_{A_1,A_2} \times \pi^1_{A_1,A_2}) \circ a = \gamma_{A_1,A_2} \circ a = (f_2 \otimes f_1) \circ (b^1 \times b^2) = (f_2 \circ b^1) \times (f_1 \circ b^2)$, which is to say $a^2 = f_2 \circ b^1$ and $a^1 = f_1 \circ b^2$. If we define $m := b^2 \times b^1$, then m: $P \to X_1 \otimes X_2$ satisfies $(f_1 \otimes f_2) \circ m = (f_1 \circ b^2) \times (f_2 \circ b^1) = a^1 \times a^2 = a$ and $\gamma_{X_1,X_2} \circ m = (\pi^2_{X_1,X_2} \times \pi^1_{X_1,X_2}) \circ (b^2 \times b^1) = (\pi^2_{X_1,X_2} \circ (b^2 \times b^1)) \times (\pi^1_{X_1,X_2} \circ (b^2 \times b^1)) = b^1 \times b^2 = b$. Moreover, m is unique with that property: If m': $P \to X_1 \otimes X_2$ is any morphism with $a = (f_1 \otimes f_2) \circ m'$ and $b = \gamma_{X_1,X_2} \circ m'$ and if $m' = m'_1 \times m'_2$, then $b^1 \times b^2 = b = \gamma_{X_1,X_2} \circ m' = (\pi^2_{X_1,X_2} \times \pi^1_{X_1,X_2}) \circ (m'_1 \times m'_2) = m'_2 \times m'_1$ implies $m'_1 = b^2$ and $m'_2 = b^1$, i.e., $m' = b^2 \times b^1 = m$. □

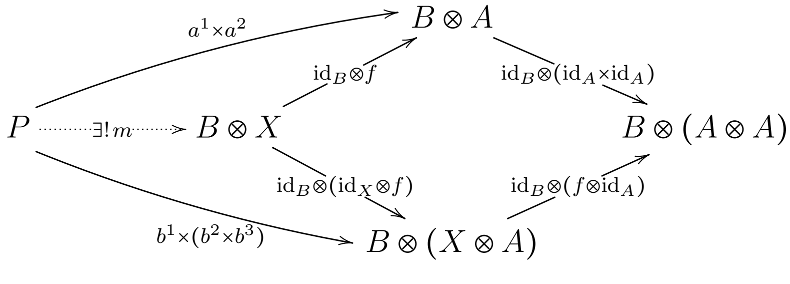
LEMMA 4.17. *In any cartesian monoidal category, for any* $\{X, A, B\} \subseteq \mathrm{obj}$*, any* $f \in \mathrm{mor}(X, A)$ *and any* $g \in \mathrm{mor}(X, B)$ *a pull-back of*

(a) $(\mathrm{id}_B \otimes (\mathrm{id}_A \times \mathrm{id}_A), \mathrm{id}_B \otimes (f \otimes \mathrm{id}_A))$ *is given by* $(\mathrm{id}_B \otimes f, \mathrm{id}_B \otimes (\mathrm{id}_X \times f))$*,*
(b) $((\mathrm{id}_B \otimes g) \otimes \mathrm{id}_A, (\mathrm{id}_B \times \mathrm{id}_B) \otimes \mathrm{id}_A)$ *is given by* $((g \times \mathrm{id}_X) \otimes \mathrm{id}_A, g \otimes \mathrm{id}_A)$*,*
(c) $(\mathrm{id}_B \otimes (\mathrm{id}_X \times f), \alpha_{B,X,A} \circ ((g \times \mathrm{id}_X) \otimes \mathrm{id}_A))$ *is given by* $(g \times \mathrm{id}_X, \mathrm{id}_X \times f)$*.*

PROOF. (a) The alleged pull-back satisfies

$$\begin{aligned}(\mathrm{id}_B \otimes (\mathrm{id}_A \times \mathrm{id}_A)) \circ (\mathrm{id}_B \otimes f) &= (\mathrm{id}_B \circ \mathrm{id}_B) \otimes ((\mathrm{id}_A \times \mathrm{id}_A) \circ f) \\ &= \mathrm{id}_B \otimes (f \times f) \\ &= \mathrm{id}_B \otimes ((f \circ \mathrm{id}_X) \times (\mathrm{id}_A \circ f)) \\ &= (\mathrm{id}_B \circ \mathrm{id}_B) \otimes ((f \otimes \mathrm{id}_A) \circ (\mathrm{id}_X \times f)) \\ &= (\mathrm{id}_B \otimes (f \otimes \mathrm{id}_A)) \circ (\mathrm{id}_B \otimes (\mathrm{id}_X \times f)).\end{aligned}$$

Hence, we only need to show that it is universal with that property. Let $P \in \mathrm{obj}$ and $a = a^1 \times a^2 \in \mathrm{mor}(P, B \otimes A)$ and $b = b^1 \times (b^2 \times b^3) \in \mathrm{mor}(P, B \otimes (X \otimes A))$ be such that $(\mathrm{id}_B \otimes (\mathrm{id}_A \times \mathrm{id}_A)) \circ a = (\mathrm{id}_B \otimes (f \otimes \mathrm{id}_A)) \circ b$.



The assumptions on a and b imply

$$\begin{aligned} a^1 \times (a^2 \times a^2) &= (\mathrm{id}_B \circ a^1) \times ((\mathrm{id}_A \times \mathrm{id}_A) \circ a^2) \\ &= (\mathrm{id}_B \otimes (\mathrm{id}_A \times \mathrm{id}_A)) \circ (a^1 \times a^2) \\ &= (\mathrm{id}_B \otimes (f \otimes \mathrm{id}_A)) \circ (b^1 \times (b^2 \times b^3)) \\ &= (\mathrm{id}_B \circ b^1) \times ((f \otimes \mathrm{id}_A) \circ (b^2 \times b^3)) \\ &= b^1 \times ((f \circ b^2) \times (\mathrm{id}_A \circ b^3)) \\ &= b^1 \times ((f \circ b^2) \times b^3) \end{aligned}$$

and thus $a^1 = b^1$ and $a^2 = f \circ b^2 = b^3$.

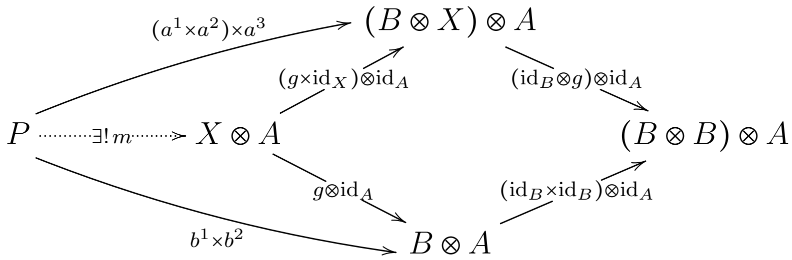
Hence, if we define $m := a^1 \times b^2 \in \mathrm{mor}(P, B \otimes X)$, then $(\mathrm{id}_B \otimes f) \circ m = (\mathrm{id}_B \circ a^1) \times (f \circ b^2) = a^1 \times a^2 = a$ and $(\mathrm{id}_B \otimes (\mathrm{id}_X \times f)) \circ m = (\mathrm{id}_B \circ a^1) \times ((\mathrm{id}_X \times f) \circ b^2) = a^1 \times ((\mathrm{id}_X \circ b^2) \times (f \circ b^2)) = b^1 \times (b^2 \times b^3) = b$.

Moreover, if any $m' = m'_1 \times m'_2 \in \mathrm{mor}(P, B \otimes X)$ also has the properties $a = (\mathrm{id}_B \otimes f) \circ m'$ and $b^1 \times (b^2 \times b^3) = b = (\mathrm{id}_B \otimes (\mathrm{id}_X \times f)) \circ m' = m'_1 \times (m'_2 \times (f \circ m'_2))$, then second equation demands $a^1 = b^1 = m'_1$ and $b^2 = m'_2$, which is to say $m' = m$. That proves the claim.

(b) The proof is similar to that of Part (a). First, we can verify by direct computation that

$$\begin{aligned} ((\mathrm{id}_B \otimes g) \otimes \mathrm{id}_A) \circ ((g \times \mathrm{id}_X) \otimes \mathrm{id}_A) &= ((\mathrm{id}_B \otimes g) \circ (g \times \mathrm{id}_X)) \otimes (\mathrm{id}_A \circ \mathrm{id}_A) \\ &= (g \times g) \otimes \mathrm{id}_A \\ &= ((\mathrm{id}_B \times \mathrm{id}_B) \otimes \mathrm{id}_A) \circ (g \otimes \mathrm{id}_A). \end{aligned}$$

And if $P \in \mathrm{obj}$ and $a = (a^1 \times a^2) \times a^3 \in \mathrm{mor}(P, (B \otimes X) \otimes A)$ and $b = b^1 \times b^2 \in \mathrm{mor}(P, B \otimes A)$ satisfy $((\mathrm{id}_B \otimes g) \otimes \mathrm{id}_A) \circ a = ((\mathrm{id}_B \times \mathrm{id}_B) \otimes \mathrm{id}_A) \circ b$,



then this means

$$\begin{aligned} (a^1 \times (g \circ a^2)) \times a^3 &= ((\mathrm{id}_B \circ g) \circ (a^1 \times a^2)) \times (\mathrm{id}_A \circ a^3) \\ &= (\mathrm{id}_B \otimes g) \otimes \mathrm{id}_A) \circ ((a^1 \times a^2) \times a^3) \\ &= ((\mathrm{id}_B \times \mathrm{id}_B) \otimes \mathrm{id}_A) \circ (b^1 \times b^2) \\ &= (b^1 \times b^1) \times b^2, \end{aligned}$$

or, equivalently, $a^1 = b^1 = g \circ a^2$ and $a^3 = b^2$.

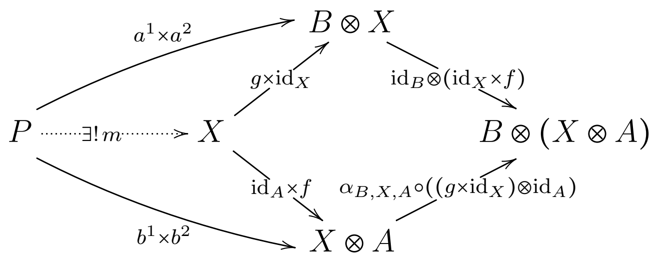
Therefore, we can conclude for $m := a^2 \times b^2 \in \mathrm{mor}(P, X \otimes A)$ that $((g \times \mathrm{id}_X) \otimes \mathrm{id}_A) \circ m = ((g \circ a^2) \times a^2) \times b^2 = (a^1 \times a^2) \times a^3 = a$ and $(g \otimes \mathrm{id}_X) \circ m = (g \circ a^2) \times b^2 = b^1 \times b^2 = b$.

And m is unique with that property as for any $m' = m_1' \times m_2' \in \mathrm{mor}(P, X \otimes A)$ with $(a^1 \times a^2) \times a^3 = a = ((g \times \mathrm{id}_X) \otimes \mathrm{id}_A) \circ m' = ((g \circ m_1') \times m_1') \times m_2'$ and $(g \otimes \mathrm{id}_A) \circ m' = b$, the first equation already implies $a^2 = m_1'$ and $b^2 = a^3 = m_2'$. That is what we needed to see.

(c) Also the third proof is similar to the two previous ones. With the help of Lemma 4.12 we compute

$$\begin{aligned}(\mathrm{id}_B \otimes (\mathrm{id}_X \times f)) \circ (g \times \mathrm{id}_X) &= (\mathrm{id}_B \circ g) \times ((\mathrm{id}_X \times f) \circ \mathrm{id}_X) \\ &= g \times (\mathrm{id}_X \times f) \\ &= \alpha^{-1}_{B,X,A} \circ ((g \times \mathrm{id}_X) \times f) \\ &= \alpha^{-1}_{B,X,A} \circ (((g \times \mathrm{id}_X) \circ \mathrm{id}_X) \times (\mathrm{id}_A \times f)) \\ &= (\alpha^{-1}_{B,X,A} \circ ((g \times \mathrm{id}_X) \otimes \mathrm{id}_A)) \circ (\mathrm{id}_X \times f).\end{aligned}$$

If $P \in \mathrm{obj}$ and $a = a^1 \times a^2 \in \mathrm{mor}(P, B \otimes X)$ and $b = b^1 \times b^2 \in \mathrm{mor}(P, X \otimes A)$ satisfy $(\mathrm{id}_B \otimes (\mathrm{id}_X \times f)) \circ a = (\alpha_{B,X,A} \circ ((g \times \mathrm{id}_X) \otimes \mathrm{id}_A)) \circ b$,



then we find, again using Lemma 4.12,

$$\begin{aligned}a^1 \times (a^2 \times (f \circ a^2)) &= (\mathrm{id}_B \circ a^1) \times ((\mathrm{id}_X \times f) \circ a^2) \\ &= (\mathrm{id}_B \otimes (\mathrm{id}_X \times f)) \circ (a^1 \times a^2) \\ &= (\alpha^{-1}_{B,X,A} \circ ((g \times \mathrm{id}_X) \otimes \mathrm{id}_A)) \circ (b^1 \times b^2) \\ &= \alpha^{-1}_{B,X,A} \circ (((g \circ b^1) \times b^1) \times b^2) \\ &= (g \circ b^1) \times (b^1 \times b^2),\end{aligned}$$

and thus $a^1 = g \circ b^1$ and $a^2 = b^1$ and $f \circ a^2 = b^2$.

Consequently, $m := a^2 \in \mathrm{mor}(P, X)$ satisfies $(g \times \mathrm{id}_X) \circ m = (g \circ a^2) \times a^2 = (g \circ b^1) \times a^2 = a$ and $(\mathrm{id}_X \times f) \circ m = a^2 \times (f \circ a^2) = b^1 \times b^2 = b$.

Finally, if $m' \in \mathrm{mor}(P, X)$ is any morphism with $a^1 \times a^2 = a = (g \times \mathrm{id}_X) \circ m' = (g \circ m') \times m'$ and $b = (\mathrm{id}_X \times f) \circ m'$, it follows $m = a^2 = m'$ from the first identity. That concludes the proof. ☐

LEMMA 4.18. *In any cartesian monoidal category, for any* $\{X, A_1, A_2, B_1, B_1\} \subseteq \mathrm{obj}$, *if* $p_i \in \mathrm{mor}(X, A_i)$ *and* $q_i \in \mathrm{mor}(X, B_i)$ *for any* $i \in \{1, 2\}$, *then*

$$\mu_{A_1,B_1,A_2,B_2} \circ ((p_1 \times q_1) \times (p_2 \times q_2)) = (p_1 \times p_2) \times (q_1 \times q_2).$$

PROOF. Recall that

$$\mu_{A_1,B_1,A_2,B_2} = \alpha^{-1}_{A_1,A_2,B_1\otimes B_2} \circ (\mathrm{id}_{A_1} \otimes \alpha_{A_2,B_1,B_2}) \circ (\mathrm{id}_{A_1} \otimes (\gamma_{B_1,A_2} \otimes \mathrm{id}_{B_2})) \circ (\mathrm{id}_{A_1} \otimes \alpha^{-1}_{B_1,A_2,B_2}) \circ \alpha_{A_1,B_1,A_2\otimes B_2}.$$

Using Lemma 4.12, we compute

$$\alpha_{A_1,B_1,A_2\otimes B_2} \circ ((p_1 \times q_1) \times (p_2 \times q_2)) = p_1 \times (q_1 \times (p_2 \times q_2))$$

and

$$(\mathrm{id}_{A_1} \otimes \alpha^{-1}_{B_1,A_2,B_2}) \circ (p_1 \times (q_1 \times (p_2 \times q_2))) = p_1 \times ((q_1 \times p_2) \times q_2)$$

and, because $\gamma_{B_1,A_2} \circ (q_1 \times p_2) = (\pi^2_{B_1,A_2} \times \pi^1_{B_1,A_2}) \circ (q_1 \times p_2) = p_2 \times q_1$,

$$(\mathrm{id}_{A_1} \otimes (\gamma_{B_1,A_2} \otimes \mathrm{id}_{B_2})) \circ (p_1 \times ((q_1 \times p_2) \times q_2)) = (p_1 \times ((p_2 \times q_1) \times q_2))$$

and

$$(\mathrm{id}_{A_1} \otimes \alpha_{A_2,B_1,B_2}) \circ (p_1 \times ((p_2 \times q_1) \times q_2)) = (p_1 \times (p_2 \times (q_1 \times q_2)))$$

and, finally,

$$\alpha^{-1}_{A_1,A_2,B_1\otimes B_2} \circ (p_1 \times (p_2 \times (q_1 \times q_2))) = ((p_1 \times p_2) \times (q_1 \times q_2)),$$

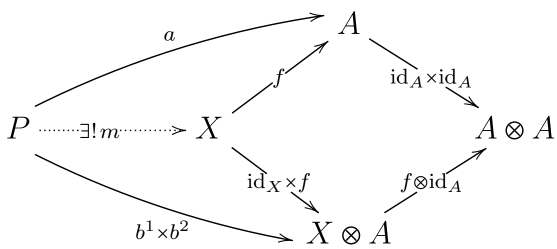
which concludes the proof. □

LEMMA 4.19. *In any cartesian monoidal category, for any objects A and X and for any morphism $f\colon X \to A$ a pull-back of $(\mathrm{id}_A \times \mathrm{id}_A, f \otimes \mathrm{id}_A)$ is given by $(f, \mathrm{id}_X \times f)$.*

PROOF. That the supposed pull-back completes a commutative square is seen by

$$(\mathrm{id}_A \times \mathrm{id}_A) \circ f = f \times f = (f \circ \mathrm{id}_X) \times (\mathrm{id}_A \circ f) = (f \otimes \mathrm{id}_A) \circ (\mathrm{id}_X \times f).$$

Let the object P and the morphisms $a\colon P \to A$ and $b^1 \times b^2\colon P \to X \times A$ be such that $(\mathrm{id}_A \times \mathrm{id}_A) \circ a = (f \otimes \mathrm{id}_A) \circ (b^1 \times b^2)$.



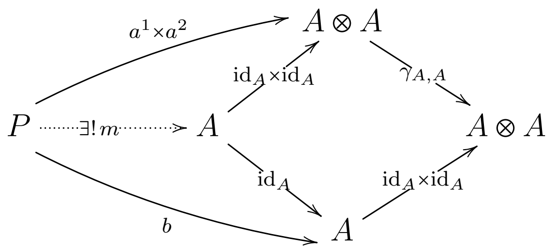
That is to say $a \times a = (f \circ b^1) \times b^2$ or, equivalently, $a = f \circ b^1 = b^2$. Hence, if we define $m := b^1$, then $a = f \circ m$ and $b^1 \times b^2 = (\mathrm{id}_X \circ b^1) \times (f \circ b^1) = (\mathrm{id}_X \times f) \circ m$. Moreover, any $m'\colon P \to X$ with $a = f \circ m'$ and $b^1 \times b^2 = (\mathrm{id}_X \times f) \circ m'$ necessarily satisfies $m' = b^1$ due to the second identity. That is what we needed to see. □

LEMMA 4.20. *In any cartesian monoidal category, for any object A a pull-back of $(\gamma_{A,A}, \mathrm{id}_A \times \mathrm{id}_A)$ is given by $(\mathrm{id}_A \times \mathrm{id}_A, \mathrm{id}_A)$.*

Proof. From Lemma 4.15 it follows

$$\gamma_{A,A} \circ (\mathrm{id}_A \times \mathrm{id}_A) = \mathrm{id}_A \times \mathrm{id}_A = (\mathrm{id}_A \times \mathrm{id}_A) \circ \mathrm{id}_A.$$

If we let P and $a^1 \times a^2 \colon P \to A \otimes A$ and $b\colon P \to A$ be such that $\gamma_{A,A} \circ (a^1 \times a^2) = (\mathrm{id}_A \times \mathrm{id}_A) \circ b$, then



another application of Lemma 4.15 yields $a^2 \times a^1 = \gamma_{A,A} \circ (a^1 \times a^2) = (\mathrm{id}_A \times \mathrm{id}_A) \circ b = b \times b$ and thus $a^1 = a^2 = b$. Consequently, for $m := a^1$ we obtain $a^1 \times a^2 = a^1 \times a^1 = (\mathrm{id}_A \times \mathrm{id}_A) \circ m$ and $b = a^1 = \mathrm{id}_A \circ m$. And, finally, any $m' \colon P \to A$ with $a^2 \times a^2 = (\mathrm{id}_A \times \mathrm{id}_A) \circ m'$ and $b = \mathrm{id}_A \circ m'$ of course satisfies $m' = b = a^1 = m$. Hence, the claim is true. □

Lemma 4.21. *For any cartesian monoidal category $\mathcal{C}$ and any pull-back-stable wide subcategory $\mathcal{M}$ of $\mathcal{C}$ and any object X of $\mathcal{C}$ the following are equivalent:*

(a) $m_1 \times m_2 \in \mathrm{mor}_{\mathcal{M}}$ *for any* $\{m_1, m_2\} \subseteq \mathrm{mor}_{\mathcal{M}}$ *with common domain X.*

(b) $\mathrm{id}_X \times m \in \mathrm{mor}_{\mathcal{M}}$ *and* $m \times \mathrm{id}_X \in \mathrm{mor}_{\mathcal{M}}$ *for any* $m \in \mathrm{mor}_{\mathcal{M}}$ *with domain X.*

(c) $\mathrm{id}_X \times \mathrm{id}_X \in \mathrm{mor}_{\mathcal{M}}$.

Proof. Because $\mathcal{M}$ is a wide subcategory, $\mathrm{id}_X \in \mathrm{mor}_{\mathcal{M}}$. Hence, the implications (a) $\Rightarrow$ (b) $\Rightarrow$ (c) are clear. We suppose (c) is true and prove (a). Let Y_1 and Y_2 be the co-domains of m_1 and m_2, respectively. By Lemma 4.11 (a) the span $(m_1 \otimes \mathrm{id}_X, \pi^1_{X,X})$ is a pull-back of $(\pi^1_{Y_1,X}, m_1)$. Because $\mathcal{M}$ is pull-back-stable and because $m_1 \in \mathrm{mor}_{\mathcal{M}}$ we thus conclude $m_1 \otimes \mathrm{id}_X \in \mathrm{mor}_{\mathcal{M}}$. Similarly, $(\mathrm{id}_{Y_2} \otimes m_2, \pi^2_{Y_1,X})$ is a pull-back of $(\pi^2_{Y_1,Y_2}, m_2)$ by Lemma 4.11 (b), implying $\mathrm{id}_{Y_2} \otimes m_2 \in \mathrm{mor}_{\mathcal{M}}$. Since $\mathcal{M}$ is a subcategory and since $\mathrm{id}_X \times \mathrm{id}_X$ by assumption, these two conclusions allow us to infer $m_1 \times m_2 = (\mathrm{id}_{Y_1} \otimes m_2) \circ (m_1 \otimes \mathrm{id}_X) \circ (\mathrm{id}_X \times \mathrm{id}_X) \in \mathrm{mor}_{\mathcal{M}}$. That is all we needed to see. □

4.3. Orthogonality and (Pre-)Factorization Systems. Section 5 but also Section 7 will require us to know more properties of orthogonality, pre-factorization and factorization systems, in particular in cartesian monoidal categories, than were given in Section 2. Moreover, the present section provides some proofs for claims in Section 2 whose proofs were omitted there.

Lemma 4.22. *For any morphisms e and m in any category $\mathcal{A}$ the two statements $e \perp_{\mathcal{A}} m$ and $m \perp_{\mathcal{A}^{\mathrm{op}}} e$ are equivalent.*

PROOF. It suffices to show one implication. Let $e \perp_{\mathcal{A}} m$ and let h and k be such that $k \circ_{\mathcal{A}^{\mathrm{op}}} m = e \circ_{\mathcal{A}^{\mathrm{op}}} h$.

$$\text{in } \mathcal{A}^{\mathrm{op}}: \begin{array}{ccc} A & \xrightarrow{m} & B \\ {\scriptstyle h}\downarrow & \swarrow {\scriptstyle \exists! v} & \downarrow{\scriptstyle k} \\ X & \xrightarrow[e]{} & Y \end{array} \iff \text{in } \mathcal{A}: \begin{array}{ccc} B & \xleftarrow{m} & A \\ {\scriptstyle h}\uparrow & \nearrow {\scriptstyle \exists! v} & \uparrow{\scriptstyle k} \\ Y & \xleftarrow[e]{} & X \end{array} \iff \text{in } \mathcal{A}: \begin{array}{ccc} X & \xrightarrow{e} & Y \\ {\scriptstyle k}\downarrow & \swarrow {\scriptstyle \exists! v} & \downarrow{\scriptstyle h} \\ A & \xrightarrow[m]{} & B \end{array}$$

That means $h \circ_{\mathcal{A}} e = m \circ_{\mathcal{A}} k$. Hence, by $e \perp_{\mathcal{A}} m$ there exists a unique morphism v of $\mathcal{A}$ such that $k = v \circ_{\mathcal{A}} e$ and $h = m \circ_{\mathcal{A}} v$. However, this pair of equations is equivalent to the identities $h = v \circ_{\mathcal{A}^{\mathrm{op}}} m$ and $k = e \circ_{\mathcal{A}^{\mathrm{op}}} v$. In other words, $m \perp_{\mathcal{A}^{\mathrm{op}}} e$. □

LEMMA 4.23. *For any category $\mathcal{A}$ and any system $\mathcal{E}$ and $\mathcal{M}$ of morphisms the following are equivalent:*

(a) $(\mathcal{E}, \mathcal{M})$ is a pre-factorization-system of $\mathcal{A}$.

(b) $(\mathcal{M}^{\mathrm{op}}, \mathcal{E}^{\mathrm{op}})$ is a pre-factorization-system of $\mathcal{A}^{\mathrm{op}}$.

PROOF. Follows from Lemma 4.22. □

One of the morphism classes of a pre-factorization system already fully determines the other. This is also implied by [FK72, p. 173].

LEMMA 4.24. *Let $(\mathcal{E}, \mathcal{M})$ and $(\mathcal{E}', \mathcal{M}')$ be any two pre-factorization systems of the same category.*

(a) If $\mathcal{E} = \mathcal{E}'$, then $(\mathcal{E}, \mathcal{M}) = (\mathcal{E}', \mathcal{M}')$.

(b) If $\mathcal{M} = \mathcal{M}'$, then $(\mathcal{E}, \mathcal{M}) = (\mathcal{E}', \mathcal{M}')$.

PROOF. (a) From $\mathcal{E} = \mathcal{E}'$ it follows $\mathcal{M} = \mathcal{E}^{\perp} = \mathcal{E}'^{\perp} = \mathcal{M}'$.
(b) Follows by Lemma 4.23 and Part (a). □

LEMMA 4.25. *For any isomorphism u in any category $\mathcal{A}$, both $u \perp_{\mathcal{A}} m$ and $e \perp_{\mathcal{A}} u$ for any morphisms e and m of $\mathcal{A}$.*

PROOF. By Lemma 4.22 we only need to prove $u \perp m$ for arbitrary m. Let f and g be such that $g \circ u = m \circ f$. Then, $w := f \circ u^{-1}$ is the unique solution of the equation $f = w \circ u$. Since it satisfies $m \circ w = m \circ (f \circ u^{-1}) = (m \circ f) \circ u^{-1} = (g \circ u) \circ u^{-1} = g$, we have shown $u \perp m$. □

The following very remarkable property of pre-factorization systems was shown in [FK72, Proposition 2.1.1 (e)].

LEMMA 4.26. *Let $(\mathcal{E}, \mathcal{M})$ be any pre-factorization-system of any category $\mathcal{A}$.*

(a) For any $\{X, Y, Z\} \subseteq \mathrm{obj}_{\mathcal{A}}$ and any $e \in \mathrm{mor}_{\mathcal{A}}(X, Y)$ and $e' \in \mathrm{mor}_{\mathcal{A}}(Y, Z)$, whenever $e' \circ e \in \mathrm{mor}_{\mathcal{E}}(X, Z)$ and $e \in \mathrm{mor}_{\mathcal{E}}(X, Y) \cup \mathrm{epi}_{\mathcal{A}}(X, Y)$, then $e' \in \mathrm{mor}_{\mathcal{E}}(Y, Z)$.

(b) For any $\{X, Y, Z\} \subseteq \mathrm{obj}_{\mathcal{A}}$ and any $m \in \mathrm{mor}_{\mathcal{A}}(X, Y)$ and $m' \in \mathrm{mor}_{\mathcal{A}}(Y, Z)$, whenever $m' \circ m \in \mathrm{mor}_{\mathcal{M}}(X, Z)$ and $m' \in \mathrm{mor}_{\mathcal{M}}(Y, Z) \cup \mathrm{mon}_{\mathcal{A}}(Y, Z)$, then $m \in \mathrm{mor}_{\mathcal{M}}(X, Y)$.

Proof. (a) Let $\{A, B\} \subseteq \text{obj}_{\mathcal{A}}$ and $m \in \text{mor}_{\mathcal{M}}(A, B)$ and $f \in \text{mor}_{\mathcal{A}}(Y, A)$ and $g \in \text{mor}_{\mathcal{A}}(Z, B)$ be such that $g \circ e' = m \circ f$. We need to prove that there exists a unique $w \in \text{mor}_{\mathcal{A}}(Z, A)$ such that $w \circ e' = f$ and $m \circ w = g$.

Step 1: Existence of w. The assumption $g \circ e' = m \circ f$ implies $g \circ (e' \circ e) = (g \circ e') \circ e = (m \circ f) \circ e = m \circ (f \circ e)$. Because $e' \circ e \in \text{mor}_{\mathcal{E}}(X, Z)$ and $m \in \text{mor}_{\mathcal{M}}(A, B)$ and thus $e' \circ e \perp m$ we find $w \in \text{mor}_{\mathcal{A}}(Z, A)$ with $w \circ (e' \circ e) = f \circ e$ and $m \circ w = g$. Moreover, w is unique with these properties.

$$\begin{array}{ccc} X & \xrightarrow{e' \circ e} & Z \\ {\scriptstyle f \circ e}\downarrow & \exists!\, w \swarrow & \downarrow{\scriptstyle g} \\ A & \xrightarrow[m]{} & B \end{array}$$

Note that the second identity is already one of the properties we seek w to have. Now, we prove $w \circ e' = f$. To do so, we need to distinguish the two cases of whether e is a morphism of $\mathcal{E}$ or an epimorphism of $\mathcal{A}$.

Case 1.1: If $e \in \text{epi}_{\mathcal{A}}(X, Y)$, then $(w \circ e') \circ e = w \circ (e' \circ e) = f \circ e$ ensures this rightaway.

Case 1.2: Hence, we suppose $e \in \text{mor}_{\mathcal{E}}(X, Y)$ and prove the same for this case. Another implication of our premise $g \circ e' = m \circ f$ is that $(g \circ e') \circ e = (m \circ f) \circ e = m \circ (f \circ e)$. Therefore, $e \in \text{mor}_{\mathcal{E}}(X, Y)$ and $m \in \text{mor}_{\mathcal{M}}(A, B)$ and thus $e \perp m$ allow us to infer the existence of $d \in \text{mor}_{\mathcal{A}}(Y, A)$ with $d \circ e = f \circ e$ and $m \circ d = g \circ e'$, which is also uniquely determined by these two properties.

$$\begin{array}{ccc} X & \xrightarrow{e} & Y \\ {\scriptstyle f \circ e}\downarrow & \exists!\, d \swarrow & \downarrow{\scriptstyle g \circ e'} \\ A & \xrightarrow[m]{} & B \end{array}$$

In fact, we can recognize that $d = f$: It is clear that f satisfies the first characteristic equation of d; and the second one, $m \circ f = g \circ e'$, holds per our initial assumptions on f, g and m.

Thus, in order to show $w \circ e' = f$ it suffices to prove $d = w \circ e'$, which we do by verifying that the latter satisfies the defining equations of the former. Firstly, $(w \circ e') \circ e = w \circ (e' \circ e) = f \circ e$ we have only just observed. Secondly, $m \circ (w \circ e') = (m \circ w) \circ e' = g \circ e'$ by the other unique property of w. Hence, indeed, $f = d = w \circ e'$, as claimed.

Step 2: Uniqueness of w. It remains to show that w is unique with $w \circ e' = f$ and $m \circ w = g$. Once more, the justification depends on whether e is a morphism of $\mathcal{E}$ or an epimorphism of $\mathcal{A}$.

Case 2.1: If $e \in \text{epi}_{\mathcal{A}}(X, Y)$, then the characteristic properties $(w \circ e') \circ e = f \circ e$ and $m \circ w = g$ which w has by our assumptions are obviously equivalent to the ones we claim it to have.

Case 2.2: Thus, we only need to consider the case $e \in \text{mor}_{\mathcal{E}}(X, Y)$. If so, let $\tilde{w} \in \text{mor}_{\mathcal{A}}(Z, A)$ be arbitrary with $\tilde{w} \circ e' = f$ and $m \circ \tilde{w} = g$. The second identity means that $\tilde{w}$ has one of the two universal properties determining w. Furthermore, the first identity implies that $\tilde{w}$ also has the other, $\tilde{w} \circ (e' \circ e) = (\tilde{w} \circ e') \circ e = f \circ e$. Hence, $\tilde{w} = w$, which concludes the proof of (a).

(b) Follows from (a) and Lemma 4.23. □

LEMMA 4.27. *For any category $\mathcal{A}$ and any pre-factorization-system $(\mathcal{E}, \mathcal{M})$ of $\mathcal{A}$ the following are equivalent:*

(a) $\mathrm{mor}_{\mathcal{E}}(X,Y) \subseteq \mathrm{epi}_{\mathcal{A}}(X,Y)$ *for any* $\{X,Y\} \subseteq \mathrm{obj}_{\mathcal{A}}$.

(b) For any $\{X,Y,Z\} \subseteq \mathrm{obj}_{\mathcal{A}}$ *and any* $m \in \mathrm{mor}_{\mathcal{M}}(X,Y)$ *and* $m' \in \mathrm{mor}_{\mathcal{M}}(Y,Z)$, *whenever* $m' \circ m \in \mathrm{mor}_{\mathcal{M}}(X,Z)$, *then* $m \in \mathrm{mor}_{\mathcal{M}}(Y,Z)$.

(c) $m \in \mathrm{mor}_{\mathcal{M}}(X,Y)$ *for any* $\{Y,C\} \subseteq \mathrm{obj}_{\mathcal{A}}$ *and* $\{h_1,h_2\} \subseteq \mathrm{mor}_{\mathcal{A}}(Y,C)$ *and any equalizer m of (h_1,h_2) in $\mathcal{A}$ with equalizer object X.*

(d) $m \in \mathrm{mor}_{\mathcal{M}}(X,Y)$ *for any* $\{X,Y\} \subseteq \mathrm{obj}_{\mathcal{A}}$ *and any* $m \in \mathrm{mor}_{\mathcal{A}}(X,Y)$ *which is a section in $\mathcal{A}$.*

PROOF. Follows by Lemma 4.23 from [FK72, Proposition 2.1.4]. □

DEFINITION 4.28. For any two pre-factzorization systems $(\mathcal{E}, \mathcal{M})$ and $(\mathcal{E}', \mathcal{M}')$ of any category $\mathcal{C}$ we write $(\mathcal{E}, \mathcal{M}) \leq (\mathcal{E}', \mathcal{M}')$ if $\mathcal{E} \subseteq \mathcal{E}'$ and $\mathcal{M}' \subseteq \mathcal{M}$.

Also the following is noted in [FK72, p. 173].

LEMMA 4.29. *(a) For any pre-factzorization systems $(\mathcal{E}, \mathcal{M})$ and $(\mathcal{E}', \mathcal{M}')$ of the same category the statements $\mathcal{E} \subseteq \mathcal{E}'$ and $\mathcal{M}' \subseteq \mathcal{M}$ are actually equivalent.*

(b) The binary relation $\leq$ is a partial order on the class of all pre-factorization systems of the same category.

(c) The partially ordered class of all pre-factorization systems is a (possibly large) complete lattice.

(d) If $(\mathcal{E}_i, \mathcal{M}_i)_{i\in I}$ is any set-indexed family of pre-factorization systems of the same category, then its meet is given by

$$\left(\bigcap\nolimits_{i\in I}\mathcal{E}_i, \left(\bigcap\nolimits_{i\in I}\mathcal{E}_i\right)^{\perp}\right)$$

and its join by

$$\left({}^{\perp}\left(\bigcap\nolimits_{i\in I}\mathcal{M}_i\right), \bigcap\nolimits_{i\in I}\mathcal{M}_i\right).$$

The next important properties of pre-factorization system was proved in [FK72, Proposition 2.1.1 (b)].

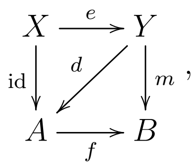
LEMMA 4.30. *Let $\mathcal{A}$ be any category and $(\mathcal{E}, \mathcal{M})$ a pre-factorization system in $\mathcal{A}$.*

(a) For any span (e,a) and any push-out (a',e') of (e,a) in $\mathcal{A}$, whenever $e \in \mathcal{E}$, then $e' \in \mathcal{E}$.

(b) For any co-span (m,f) and any pull-back (f',m') of (m,f) in $\mathcal{A}$, whenever $m \in \mathcal{M}$, then $m' \in \mathcal{M}$.

The next lemma can be found in [AHS04, Lemma 14.5].

LEMMA 4.31. *For any pre-factorization system* $(\mathcal{E}, \mathcal{M})$ *of any category* $\mathcal{A}$*, for any* $e \in \mathcal{E}$*, any* $m \in \mathcal{M}$ *and any commutative diagram in* $\mathcal{A}$

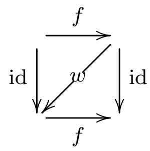


e is an isomorphism and $f \in \mathcal{M}$.

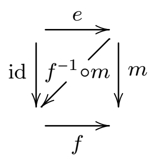
One inclusion of the following identity was proved in [FK72, Proposition 2.1.1 (1)]. The other is shown in [AHS04, Proposition 14.6 (1)].

LEMMA 4.32. $\mathcal{E} \cap \mathcal{M} = \mathrm{iso}_{\mathcal{A}}$ *for any pre-factorization system* $(\mathcal{E}, \mathcal{M})$ *of any category* $\mathcal{A}$.

PROOF. For any $f \in \mathcal{E} \cap \mathcal{M}$ the diagonal property of $(\mathcal{E}, \mathcal{M})$ ensures the existence of a morphism w such that $\mathrm{id} = w \circ f$ and $\mathrm{id} = f \circ w$.



Conversely, let f be an isomorphism of $\mathcal{A}$ and let (e, m) be an $(\mathcal{E}, \mathcal{M})$-factorization of f. Then, in $\mathcal{A}$ the diagram



commutes, implying $f \in \mathcal{M}$ by Lemma 4.31. Considering the fact that $(\mathcal{M}, \mathcal{E})$ is a factorization system of $\mathcal{A}^{\mathrm{op}}$, this argument also proves $f \in \mathcal{E}$. □

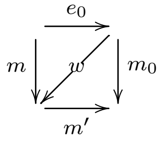
The following was shown in [FK72, Proposition 2.1.1 (a)]. See also [AHS04, Proposition 14.6 (2)].

LEMMA 4.33. *For any pre-factorization system* $(\mathcal{E}, \mathcal{M})$ *of any category* $\mathcal{A}$ *both* $\mathcal{E}$ *and* $\mathcal{M}$ *are each closed under composition, i.e.,*

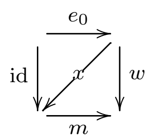
(a) *where defined,* $e' \circ_{\mathcal{A}} e \in \mathcal{E}$ *for* $e' \in \mathcal{E}$ *and* $e \in \mathcal{E}$*, and,*

(b) *where defined,* $m' \circ_{\mathcal{A}} m \in \mathcal{M}$ *for* $m' \in \mathcal{M}$ *and* $m \in \mathcal{M}$.

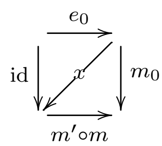
PROOF. Again, by duality it suffices to prove Part (a). Let (e_0, m_0) be an $(\mathcal{E}, \mathcal{M})$-factorization of $m' \circ m$. By the diagonal property of $(\mathcal{E}, \mathcal{M})$ there exists a morphism w such that $m = w \circ e_0$ and $m_0 = m' \circ w$.



In particular, $w \circ e_0 = m \circ \mathrm{id}$ allows a second application of the diagonal property axiom, yielding a morphism x such that $\mathrm{id} = x \circ e_0$ and $w = m \circ x$.



In consequence, the identity $m_0 = m' \circ w = m' \circ (m \circ x) = (m' \circ m) \circ x$ renders the diagram

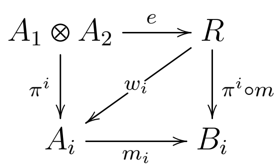


commutative. It follows $m' \circ m \in \mathcal{M}$ by Lemma 4.31. □

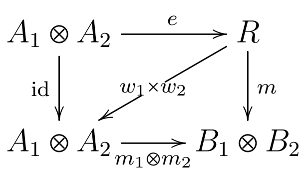
The following is proved in [AHS04, Proposition 14.15].

LEMMA 4.34. *For any pre-factorization system $(\mathcal{E}, \mathcal{M})$ of any cartesian monoidal category $\mathcal{A}$ the category $\mathcal{M}$ is closed under monoidal products, i.e., $m_1 \otimes_{\mathcal{A}} m_2 \in \mathcal{M}$ for any $m_1 \in \mathcal{M}$ and $m_2 \in \mathcal{M}$.*

PROOF. For each $i \in \{1,2\}$ let A_i and B_i be such that $m_i\colon A_i \to B_i$, let (e, m) be an $(\mathcal{E}, \mathcal{M})$-factorization of $m_1 \otimes m_2$ and let R be the domain of m. Then, for each $i \in \{1,2\}$ because $m_i \circ \pi^i_{A_1,A_2} = (\pi^i_{B_1,B_2} \circ m) \circ e$ by the diagonal property of $(\mathcal{E}, \mathcal{M})$ there exists a morphism $w_i\colon R \to A_i$ such that $\pi^i_{A_1,A_2} = w_i \circ e$ and $\pi^i_{B_1,B_2} \circ m = m_i \circ w_i$.



It follows $(w_1 \times w_2) \circ e = (w_1 \circ e) \times (w_2 \circ e) = \pi^1_{A_1,A_2} \times \pi^2_{B_1,B_2} = \mathrm{id}_{A_1 \otimes A_2}$ as well as $(m_1 \otimes m_2) \circ (w_1 \times w_2) = (m_1 \circ w_1) \times (m_2 \circ w_2) = (\pi^1_{B_1,B_2} \circ m) \times (\pi^2_{B_1,B_2} \circ m) = m$. In other words, the diagram



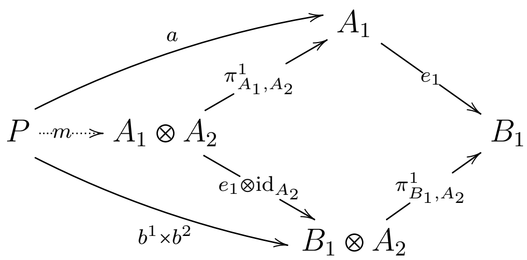
is commutative. Because $e \in \mathcal{E}$ and $m \in \mathcal{M}$ Lemma 4.31 hence implies $m_1 \otimes m_2 \in \mathcal{M}$, as claimed. □

For the next lemma, the assumed stability of $(\mathcal{E}, \mathcal{M})$ is crucial.

LEMMA 4.35. *For any pre-factorization system $(\mathcal{E},\mathcal{M})$ of any cartesian monoidal category $\mathcal{A}$, if the subcategory $\mathcal{E}$ is pull-back stable, then $\mathcal{E}$ is closed under monoidal products, i.e., $e_1 \otimes_{\mathcal{A}} e_2 \in \mathcal{E}$ for any $e_1 \in \mathcal{E}$ and $e_2 \in \mathcal{E}$.*

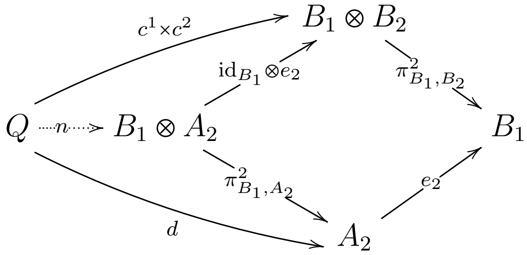
PROOF. For each $i \in \{1,2\}$ let the objects A_i and B_i be such that $e_i\colon A_i \to B_i$. Once we show that $e_1 \otimes \mathrm{id}_{A_2}$ and $\mathrm{id}_{B_1} \otimes e_2$ are elements of $\mathcal{E}$, Lemma 4.33 (a) will imply the claim because $(\mathrm{id}_{B_1} \otimes e_2) \circ (e_1 \otimes \mathrm{id}_{A_2}) = (\mathrm{id}_{B_1} \circ e_1) \otimes (e_2 \circ \mathrm{id}_{A_2}) = e_1 \otimes e_2$.

Step 1: Since $(\mathcal{E},\mathcal{M})$ is stable and $e_1 \in \mathcal{E}$ we can prove that $e_1 \otimes \mathrm{id}_{A_2}$ belongs to $\mathcal{E}$ by showing that $(\pi^1_{A_1,A_2}, e_1 \otimes \mathrm{id}_{A_2})$ is a pull-back of $(e_1, \pi^1_{B_1,A_2})$. The diamond in the diagram



commutes as $\pi^1_{B_1,A_2} \circ (e_1 \otimes \mathrm{id}_{A_2}) = \pi^1_{B_1,A_2} \circ ((e_1 \circ \pi^1_{A_1,A_2}) \times (\mathrm{id}_{A_2} \circ \pi^2_{A_1,A_2})) = e_1 \circ \pi^1_{A_1,A_2}$. Let P be an object and let $a\colon P \to A_1$ and $b^1 \times b^2\colon P \to B_1 \otimes A_2$ be morphisms such that $e_1 \circ a = \pi^1_{B_1,A_2} \circ (b^1 \times b^2)$. If we define $m \coloneqq a \times b^2$, then, immediately, $\pi^1_{A_1,A_2} \circ m = a$, and $e_1 \circ a = b^1$ implies $(e_1 \otimes \mathrm{id}_{A_2}) \circ m = (e_1 \circ a) \times (\mathrm{id}_{A_2} \circ b^2) = b^1 \times b^2$. Moreover, if $m'_1 \times m'_2\colon P \to A_1 \otimes A_2$ is any morphism with $a = \pi^1_{A_1,A_2} \circ (m'_1 \times m'_2)$ and $b^1 \times b^2 = (e_2 \otimes \mathrm{id}_{A_2}) \circ (m'_1 \times m'_2)$, then the first of these identities requires $m'_1 = a$ and the second $m'_2 = b^2$, which is to say $m'_1 \times m'_2 = m$.

Step 2: Again, we use the assumptions that $(\mathcal{E},\mathcal{M})$ is stable and that $e_2 \in \mathcal{E}$ and prove that $\mathrm{id}_{B_1} \otimes e_2$ is an element of $\mathcal{E}$ by showing that $(\mathrm{id}_{B_1} \otimes e_2, \pi^2_{B_1,A_2})$ is a pull-back of $(\pi^2_{B_1,B_2}, e_2)$. Because $\pi^2_{B_1,B_2} \circ (\mathrm{id}_{B_1} \otimes e_2) = \pi^2_{B_1,B_2} \circ ((\mathrm{id}_{B_1} \circ \pi^1_{B_1,A_2}) \times (e_2 \circ \pi^2_{B_1,A_2})) = e_2 \circ \pi^2_{B_1,A_2}$ the diamond in



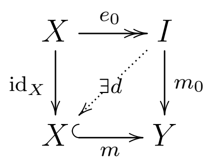
commutes. If $c^1 \times c^2\colon Q \to B_1 \otimes B_2$ and $d\colon Q \to A_2$ also satisfy $\pi^2_{B_1,B_2} \circ (c^1 \times c^2) = e_2 \circ d$, then defining $n \coloneqq c^1 \times d$ leads to $d = \pi^2_{B_1,A_2} \circ n$ and $(\mathrm{id}_{B_1} \otimes e_2) \circ n = c^1 \times (e_2 \circ d) = c^1 \times c^2$. And n is unique with this property because any $n'_1 \times n'_2\colon Q \to B_1 \otimes A_2$ with $c^1 \times c^2 = (\mathrm{id}_{B_1} \otimes e_2) \circ (n'_1 \times n'_2) = n'_1 \times (e_2 \circ n'_2)$ and $d = \pi^2_{B_1,A_2} \circ (n'_1 \times n'_2) = n'_2$ has to satisfy $n'_1 \times n'_2 = c^1 \times d = n$. That concludes the proof. □

The following extended version of Lemma 2.2 is given in [JW00, Proposition 1.1.1].

LEMMA 4.36. *For any cartesian monoidal category $\mathcal{A}$ with pull-backs and any factorization system $(\mathcal{E}, \mathcal{M})$ of $\mathcal{A}$ the following are equivalent:*

(i) *All elements of $\mathcal{E}$ are epimorphisms of $\mathcal{A}$.*
(ii) *Any strong monomorphism of $\mathcal{A}$ is an element of $\mathcal{M}$.*
(iii) *Any equalizer of $\mathcal{A}$ is an element of $\mathcal{M}$.*
(iv) *For any morphisms f and g of $\mathcal{A}$ with common co-domain and any pull-back (u, v) of (f, g) in $\mathcal{A}$ the morphism $u \times_{\mathcal{A}} v$ is an element of $\mathcal{M}$.*
(v) $\mathrm{id}_{\mathcal{A},A} \times_{\mathcal{A}} f \in \mathcal{M}$ *for any morphism f of $\mathcal{A}$ with domain A.*
(vi) $\mathrm{id}_{\mathcal{A},X} \times_{\mathcal{A}} \mathrm{id}_{\mathcal{A},X} \in \mathcal{M}$ *for any $X \in \mathrm{obj}_{\mathcal{A}}$.*

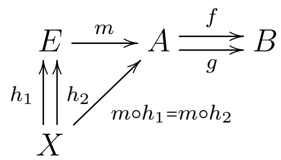
PROOF. *(i) ⇒ (ii):* Let $\mathcal{E}$ consist of epimorphisms and let $m: X \to Y$ be a strong monomorphism. If (e_0, m_0) is an $(\mathcal{E}, \mathcal{M})$-factorization of m with image object I, then $m_0 \circ e_0 = m \circ \mathrm{id}_X$.



Since m is a strong monomorphism and since e_0 is an epimorphism there must then exist a morphism $d: I \to X$ with $d \circ e_0 = \mathrm{id}_X$ and $m \circ d = m_0$. It follows $m \in \mathcal{M}$ by Lemma 4.31.

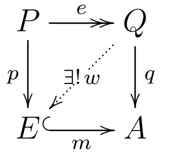
(ii) ⇒ (iii): Suppose that $\mathcal{M}$ contains all strong monomorphisms and let m with domain E be an equalizer of $f: A \to B$ and $g: A \to B$. In order to show $m \in \mathcal{M}$ it suffices to show that m is a strong monomorphism.

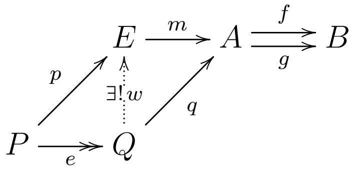
We prove that m is a monomorphism: If $h_1: X \to E$ and $h_2: X \to E$ are such that $m \circ h_1 = m \circ h_2$,



then $m \circ h_1$ equalizes (f, g) because $f \circ (m \circ h_1) = (f \circ m) \circ h_1 = (g \circ m) \circ h_1 = g \circ (m \circ h_1)$. Hence, by the universal property of equalizers any morphism $u: X \to E$ with $m \circ h_1 = m \circ u$ is unique with that property. It follows $h_1 = h_2$, making m a monomorphism.

The monomorphism m is in fact strong: Let $e: P \to Q$ be an epimorphism and let $p: P \to E$ and $q: Q \to A$ be such that $q \circ e = m \circ p$.

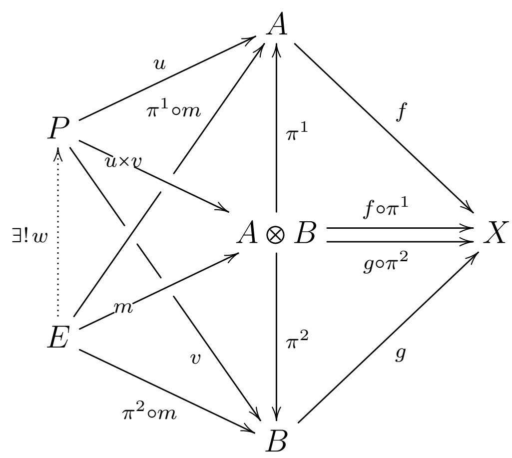




Because e is an epimorphism the identity $(f\circ q)\circ e = f\circ(q\circ e) = f\circ(m\circ p) = (f\circ m)\circ p = (g\circ m)\circ p = g\circ(m\circ p) = g\circ(q\circ e) = (g\circ q)\circ e$ allows us to conclude that $f\circ q = g\circ q$. Hence, by the universality of m as a morphism equalizing (f,g) there exists a unique $w\colon Q\to E$ such that $q = m\circ w$. Since m is a monomorphism it follows from $m\circ(w\circ e) = (m\circ w)\circ e = q\circ e = m\circ p$ that $w\circ e = p$. Thus, m is a strong monomorphism.

(iii) ⇒ (iv): Let $\mathcal{M}$ contain all equalizers and let (u,v) be a pull-back of (f,g) with pull-back object P, where $f\colon A\to X$ and $g\colon B\to X$. Once we show that $u\times v$ is an equalizer of $(f\circ\pi^1_{A,B}, g\circ\pi^2_{A,B})$ the claim $u\times v\in\mathcal{M}$ will have been proven.

Clearly, $(f\circ\pi^1_{A,B})\circ(u\times v) = f\circ(\pi^1_{A,B}\circ(u\times v)) = f\circ u = g\circ v = g\circ(\pi^2_{A,B}\circ(u\times v)) = (g\circ\pi^2_{A,B})\circ(u\times v)$. Let $m\colon E\to A\otimes B$ be such that $(f\circ\pi^1_{A,B})\circ m = (g\circ\pi^2_{A,B})\circ m$.



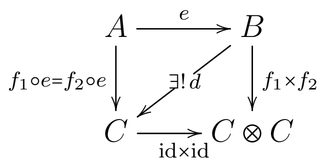
Since this is equivalent to $f\circ(\pi^1_{A,B}\circ m) = g\circ(\pi^2_{A,B}\circ m)$ there exists by the universal property of (u,v) as a pull-back of (f,g) exactly one morphism $w\colon E\to P$ such that $\pi^1_{A,B}\circ m = u\circ w$ and $\pi^2_{A,B}\circ m = v\circ w$. Because the latter two identities are equivalent to $m = (u\circ w)\times(v\circ w) = (u\times v)\circ w$ we have thus shown $u\times v$ is an equalizer of $(f\circ\pi^1_{A,B}, g\circ\pi^2_{A,B})$.

(iv) ⇒ (v): Suppose that $\mathcal{M}$ contains all products of pull-back pairs and let $f\colon A\to B$ be arbitrary. By Lemma 4.6 the pair (id_A, f) is a pull-back of (f,id_B). Hence, the assumption implies $\mathrm{id}_A\times f\in\mathcal{M}$.

(v) ⇒ (vi): If $\mathrm{id}_A\times f\in\mathcal{M}$ for any morphisms f with domain A, then the choice $A = X$ and $f = \mathrm{id}_X$ implies $\mathrm{id}_X\times\mathrm{id}_X\in\mathcal{M}$ for any object X.

(vi) ⇒ (i): Let $\mathrm{id}_X\times\mathrm{id}_X\in\mathcal{M}$ for any object X and let $e\colon A\to B$ with $e\in\mathcal{E}$ be arbitrary. We show that e is an epimorphism.

Hence, let $f_1\colon B \to C$ and $f_2\colon B \to C$ be such that $f_1 \circ e = f_2 \circ e$. Because $(f_1 \times f_2) \circ e = (f_1 \circ e) \times (f_2 \circ e) = (f_1 \circ e) \times (f_1 \circ e) = (\mathrm{id}_C \times \mathrm{id}_C) \circ (f_1 \circ e)$



and because $e \in \mathcal{E}$ and $\mathrm{id}_C \times \mathrm{id}_C \in \mathcal{M}$ per our assumptions, the diagonal property of $(\mathcal{E}, \mathcal{M})$ implies the existence of a morphism $d\colon B \to C$ such that $d \circ e = f_1 \circ e$ and $(\mathrm{id}_C \times \mathrm{id}_C) \circ d = f_1 \times f_2$, which is also unique with that property. From the identity $f_1 \times f_2 = (\mathrm{id}_C \times \mathrm{id}_C) \circ d = d \times d$ we can conclude $f_1 = d = f_2$. Thus, e is indeed an an epimorphism. □

5. Partial proof of the construction

Section 5 offers partial proofs are of Conjectures 2.14 and 2.19. Unfortunately, I was unable to check everything in the time allotted.

The construction of the generalized relation envelope proceeds in three stages, which themselves are divided into several steps each. We first construct the so-called span category, then the generalized relation category and only then the actual category from Section 2.2.

The results presented here about spans and relations are *not* new. N claims to originality are made for any of the first two stages of the construction. Some of the proofs can also be found in the literature (see [Kle70; Mei74a; Mei74b; Pav95; Pav96; Jay95; JW00]), however, far from all of them. In particular, since spans and relations were mainly studied there out of an interest in bicategories, the rigid monoidal structure is seldom considered in the literature.

5.1. Stage 1a: Spans. The starting point for the construction is a category of (classes of) "spans". We obtain it from Bénabou's bicategory of spans, which he introduced in [Bén67, Section 2.6], generalizing an idea of Yoneda's from [Yon60, § 3, Section 3.0].

5.1.1. *Category.* The inputs to the span category stage of the category can be more general than the ones allowed for the final stage of the construction presented in Section 2.

Definition 5.1. (a) Let smCAT denote the strict 2-category of all symmetric monoidal categories, symmetric monoidal functors and monoidal natural transformations.

(b) Write $\mathsf{smCAT}^{\mathrm{cart,fc}}$ for the sub-2-category of smCAT whose 0-cells are all cartesian monoidal categories with pull-backs, whose 1-cells are all symmetric strong monoidal pull-back-preserving functors between these, and whose 2-cells are all equifibered monoidal natural transformations between those.

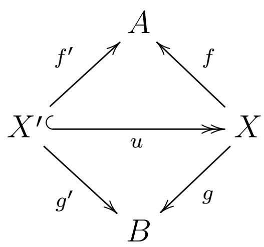
(c) Denote by $\mathsf{smCAT}^{\mathrm{cart,fc}}_{\mathrm{sos}}$ the full sub-2-category of $\mathsf{smCAT}^{\mathrm{cart,fc}}$ generated by all 0-cells $\mathcal{A}$ which are $\mathcal{A}$-subobject-small.

(d) Finally, let $\mathsf{rsm\dagger CAT}$ be the strict 2-category of rigid symmetric monoidal †-categories, symmetric monoidal †-functors and monoidal natural transformations.

Definition 5.2. For any two objects A and B of any category a *span from A to B* is any pair (f,g) such that there exists an object X, the *base*, with $f\colon X\to A$ and $g\colon X\to B$.

If one reframes isomorphy of 1-cells under 2-cells in Bénabou's bicategory of spans, one is lead to make the following definition.

Definition 5.3. For any objects A and B in any category any two spans (f,g) and (f',g') from A to B with bases X and X', respectively, are said to be *equivalent* if there exists an isomorphism $u\colon X'\to X$ such that $f'=f\circ_{\mathcal{A}} u$ and $g'=g\circ_{\mathcal{A}} u$.



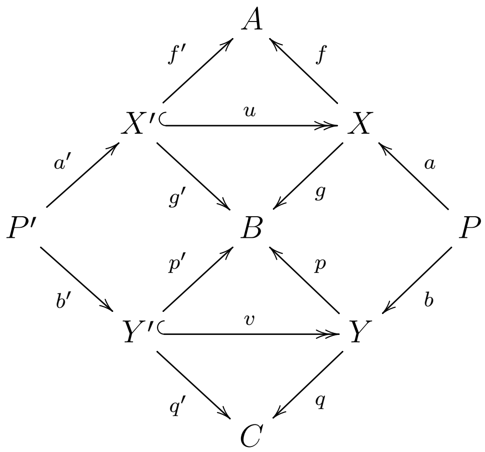
This does indeed define an equivalence relation on spans.

Notation 5.4. For any span (f,g) in $\mathcal{A}$ we write $[f,g]$ for the class of all spans in $\mathcal{A}$ equivalent to (f,g).

Classes of spans will be composed via pull-backs. We need to check that this makes sense.

Lemma 5.5. *In any category, let $\{A,B,C\}\subseteq \mathrm{obj}$ let (f',g') be spans from A to B, let (p,q) and (p',q') be spans from B to C, let (a,b) be a pull-back of (g,p) and let (a',b') be a pull-back of (g',p'). If (f,g) and (f',g') are equivalent and (p,q) and (p',q') are equivalent, then $(f\circ a, q\circ b)$ and $(f'\circ a', q'\circ b')$ are equivalent spans from A to C.*

Proof. Let the objects X, X', Y, Y', P and P' be as in the below diagram. Moreover, let u be an isomorphism $X'\to X$ with $f\circ u=f'$ and $g\circ u=g'$ and, likewise, v an isomorphism $Y'\to Y$ with $p\circ v=p'$ and $q\circ v=q'$.



Step 1: We justify that it suffices to prove that $(u \circ a', v \circ b')$ is a pull-back of (g, p). Indeed, if so, then by the essential uniqueness of (a, b) we find a morphism $w: P' \to P$ with $u \circ a' = a \circ w$ and $v \circ b' = b \circ w$ and w is unique with that property and an isomorphism. Moreover,

$$(f \circ a) \circ w = f \circ (a \circ w) = f \circ (u \circ a') = (f \circ u) \circ a' = f' \circ a'$$

and, analogously, $(q \circ b) \circ w = q' \circ b'$. And that is what we needed to see.

Step 2: Now, we actually show that $(u \circ a', v \circ b')$ is a pull-back of (g, p). Firstly,

$$g \circ (u \circ a') = (g \circ u) \circ a' = g' \circ a' = p' \circ b' = (p \circ v) \circ b' = p \circ (v \circ b'),$$

which is necessary. Secondly, let (c, d) be a pair of morphisms with common domain Q and with $g \circ c = p \circ d$. We need to prove that there exists a unique morphism $s: Q \to P'$ such that $c = (u \circ a') \circ s$ and $d = (v \circ b') \circ s$. Because

$$g' \circ (u^{-1} \circ c) = (g' \circ u^{-1}) \circ c = g \circ c = p \circ d = (p' \circ v^{-1}) \circ d = p' \circ (v^{-1} \circ d)$$

and because (a', b') is a pull-back of (g', p') there exists a unique morphism $s: Q \to P'$ with $u^{-1} \circ c = a' \circ s$ and $v^{-1} \circ d = b' \circ s$, i.e., with $c = (u \circ a') \circ s$ and $d = (v \circ b') \circ s$, as desired. And, in fact, s is unique with the latter property: Any $s': Q \to P'$ with $c = (u \circ a') \circ s'$ and $d = (v \circ b') \circ s'$ also satisfies $u^{-1} \circ c = a' \circ s'$ and $v^{-1} \circ d = b' \circ s'$, from which $s' = s$ follows by the uniqueness of s. That concludes the proof. □

Now we can give the definition of the span category, Part (d) of course enabled by Lemma 5.5.

DEFINITION 5.6. For any 0-cell $\mathcal{A}$ of $\mathsf{smCAT}^{\text{cart,fc}}_{\text{sos}}$ make the following definitions:

$$\mathrm{S}(\mathcal{A}) := (\mathrm{obj}_{\mathrm{S}(\mathcal{A})}, \mathrm{mor}_{\mathrm{S}(\mathcal{A})}, \circ_{\mathrm{S}(\mathcal{A})}, \mathrm{id}_{\mathrm{S}(\mathcal{A})})$$

(a) Let $\mathrm{obj}_{\mathrm{S}(\mathcal{A})} := \mathrm{obj}_{\mathcal{A}}$,
(b) For any $\{A, B\} \subseteq \mathrm{obj}_{\mathrm{S}(\mathcal{A})}$ let

$$\mathrm{mor}_{\mathrm{S}(\mathcal{A})}(A, B) := \left(\mathcal{A} \diagup (A \otimes_{\mathcal{A}} B) \right) \diagup \cong$$

be the set of all equivalence classes of spans in $\mathcal{A}$ from A to B.

(c) For each $A \in \mathrm{obj}_{\mathrm{S}(\mathcal{A})}$ let $\mathrm{id}_{\mathrm{S}(\mathcal{A}),A} := [\mathrm{id}_{\mathcal{A},A}, \mathrm{id}_{\mathcal{A},A}]$.

(d) For any $\{A, B, C\} \subseteq \mathrm{obj}_{\mathrm{S}(\mathcal{A})}$, any $[f, g] \in \mathrm{mor}_{\mathrm{S}(\mathcal{A})}(A, B)$ and any $[p, q] \in \mathrm{mor}_{\mathrm{S}(\mathcal{A})}(B, C)$ let

$$[p, q] \circ_{\mathrm{S}(\mathcal{A})} [f, g] := [f \circ_{\mathcal{A}} a, q \circ_{\mathcal{A}} b]$$

for any pull-back (a, b) of (g, p) in $\mathcal{A}$.

The following is claimed in [Bén67, Section 2.6] without proof. A proof of a similar result can be found at [Kle70, Theorem 2.5].

LEMMA 5.7. *For any* 0*-cell* $\mathcal{A}$ *of* $\mathsf{smCAT}^{\mathrm{cart,fc}}_{\mathrm{sos}}$ *the following are true:*

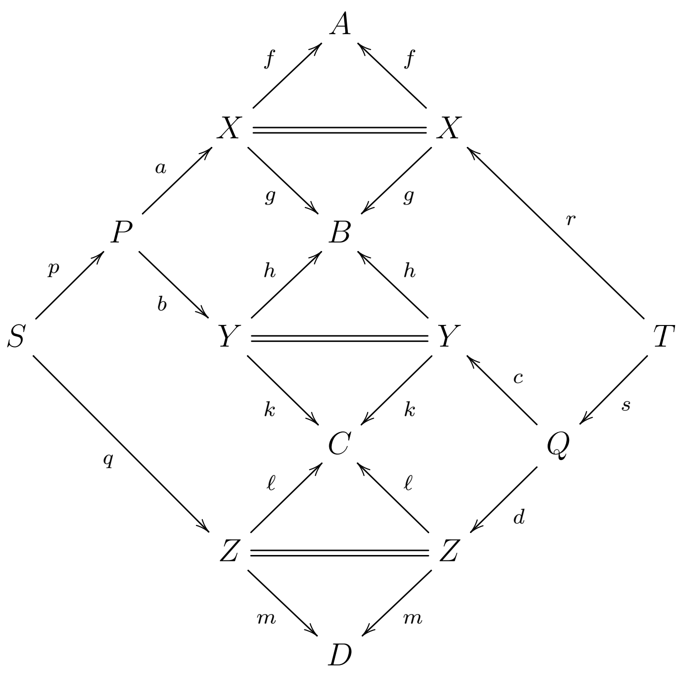
(a) For any $\{A, B, C\} \subseteq \mathrm{obj}_{\mathrm{S}(\mathcal{A})}$, *any* $x \in \mathrm{mor}_{\mathrm{S}(\mathcal{A})}(A, B)$, *any* $y \in \mathrm{mor}_{\mathrm{S}(\mathcal{A})}(B, C)$ *and any* $z \in \mathrm{mor}_{\mathrm{S}(\mathcal{A})}(C, D)$,

$$(z \circ_{\mathrm{S}(\mathcal{A})} y) \circ_{\mathrm{S}(\mathcal{A})} x = z \circ_{\mathrm{S}(\mathcal{A})} (y \circ_{\mathrm{S}(\mathcal{A})} x).$$

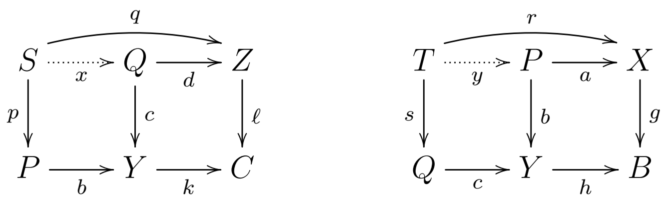
(b) For any $\{A, B\} \subseteq \mathrm{obj}_{\mathrm{S}(\mathcal{A})}$ *and any* $x \in \mathrm{mor}_{\mathrm{S}(\mathcal{A})}(A, B)$,

$$\mathrm{id}_{\mathrm{S}(\mathcal{A}),B} \circ_{\mathrm{S}(\mathcal{A})} x = x \quad \text{and} \quad x \circ_{\mathrm{S}(\mathcal{A})} \mathrm{id}_{\mathrm{S}(\mathcal{A}),A} = x.$$

PROOF. (a) Let A, B, C and D be objects of $\mathcal{A}$ and let $[f, g]$ be a class of spans from A to B, let $[h, k]$ be one from B to C and let $[\ell, m]$ be one from C to D. By Lemma 5.5 it suffices to show that $(f \circ a \circ p, m \circ q)$ and $(f \circ r, m \circ d \circ s)$ are equivalent spans from A to D for a pull-back (a, b) of (g, h), a pull-back (c, d) of (k, ℓ), a pull-back (p, q) of $(k \circ b, \ell)$, and a pull-back (r, s) of $(g, h \circ c)$. Let the objects X, Y, Z, P, Q, S and T be as in the diagram.



By applying Part (b) of Lemma 4.3 to each of the diagrams, where in each case both the large square and the right small square are pull-backs by assumption,



we find unique morphisms $x\colon S \to Q$ with $q = d \circ x$ and $b \circ p = c \circ x$ and $y\colon T \to P$ with $r = a \circ y$ and $b \circ y = c \circ s$. Moreover, both (p, x) and (y, s) are each pull-backs of (b, c) by Part (a) of Lemma 4.3.

By the essential uniqueness of pull-backs there must then exist an isomorphism $w\colon S \to T$ with $p = y \circ w$ and $x = s \circ w$. It follows $r \circ w = (a \circ y) \circ w = a \circ (y \circ w) = a \circ p$ and $q = d \circ x = d \circ (s \circ w) = (d \circ s) \circ w$. Because that implies in particular $f \circ a \circ p = (f \circ r) \circ w$ and $m \circ q = (m \circ d \circ s) \circ w$, the spans $(f \circ a \circ p, m \circ q)$ and $(f \circ r, m \circ d \circ s)$ have thus been shown to be equivalent.

(b) Let A and B be objects of $\mathcal{A}$ and let $[f, g]$ be any class of spans from A to B. If X is the base of (f, g), then by Lemma 4.6 a pull-back of (id_A, f) is given by (f, id_X). Lemma 5.5 hence assures us that $[f, g] \circ [\mathrm{id}_A, \mathrm{id}_A] = [\mathrm{id}_A \circ f, g \circ \mathrm{id}_X] = [f, g]$. The proof of the other identity is analogous. □

PROPOSITION 5.8. $(\mathrm{obj}_{\mathrm{S}(\mathcal{A})}, \mathrm{mor}_{\mathrm{S}(\mathcal{A})}, \circ_{\mathrm{S}(\mathcal{A})}, \mathrm{id}_{\mathrm{S}(\mathcal{A})})$ *is a category for any* 0*-cell* $\mathcal{A}$ *of* $\mathsf{smCAT}^{\mathrm{cart,fc}}_{\mathrm{sos}}$.

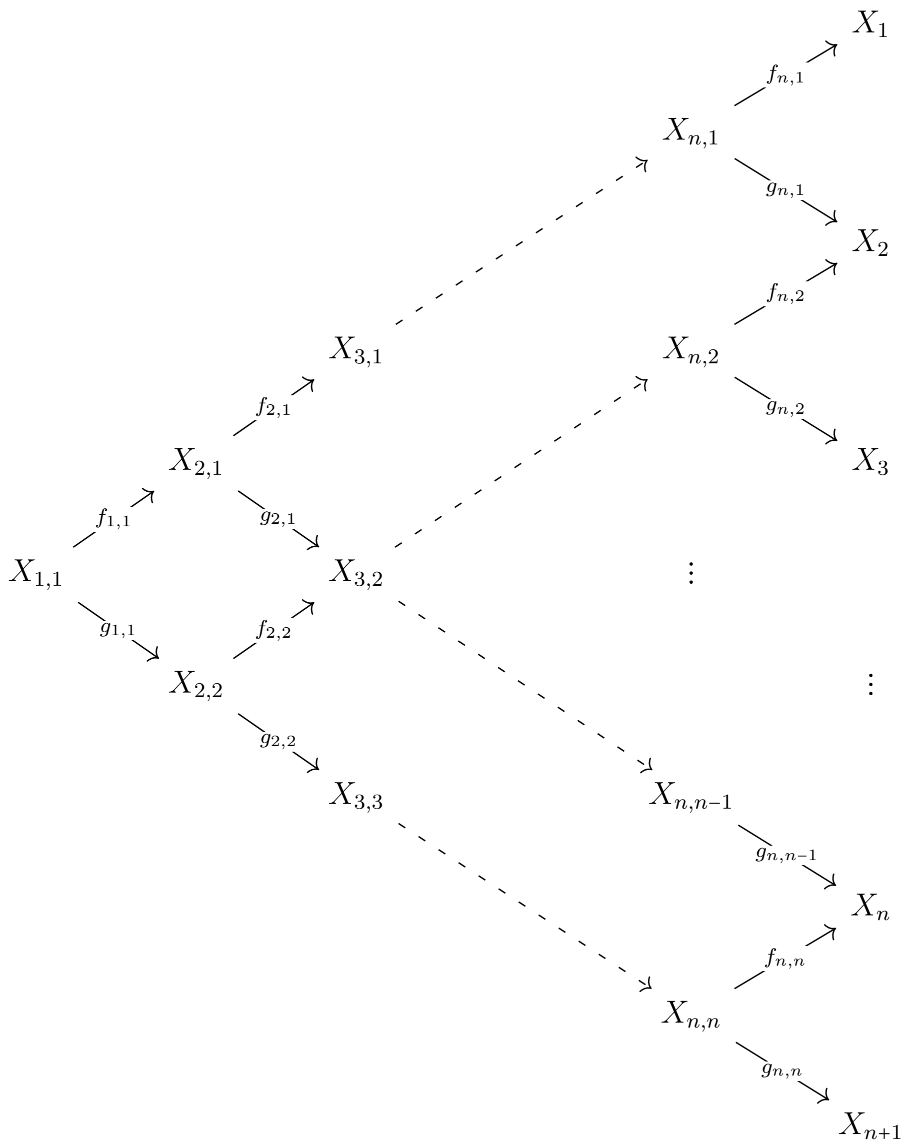
PROOF. That is the implication of Lemma 5.7. □

On countless occasions we will have to compute products in span categories with more than two factors. The following is the key result for doing so efficiently. It will be used in later computations without explicit reference.

LEMMA 5.9. *For any* 0*-cell* $\mathcal{A}$ *of* $\mathsf{smCAT}^{\mathrm{cart,fc}}_{\mathrm{sos}}$ *and any* $n \in \mathbb{N}$*, if* $\{X_1, \ldots, X_{n+1}\} \subseteq \mathrm{obj}_{\mathrm{S}(\mathcal{A})}$ *and if* $x_i \in \mathrm{mor}_{\mathrm{S}(\mathcal{A})}(X_i, X_{i+1})$ *for each* $i \in \{1, \ldots, n\}$*, then*

$$x_n \circ_{\mathrm{S}(\mathcal{A})} \cdots \circ_{\mathrm{S}(\mathcal{A})} x_1 = [f_{n,1} \circ_{\mathcal{A}} f_{n-1,1} \circ_{\mathcal{A}} \cdots \circ_{\mathcal{A}} f_{1,1}, g_{n,n} \circ_{\mathcal{A}} g_{n-1,n-1} \circ_{\mathcal{A}} \cdots \circ_{\mathcal{A}} g_{1,1}],$$

where $(f_{n,j}, g_{n,j}) \in x_j$ *for each* $j \in \{1, \ldots, n\}$ *are arbitrary representatives and where for each* $i \in \{1, \ldots, n-1\}$ *and each* $j \in \{1, \ldots, i\}$ *the span* $(f_{i,j}, g_{i,j})$ *is any pull-back* *of* $(g_{i+1,j}, f_{i+1,j+1})$ *in* $\mathcal{A}$.



PROOF. For $n = 1$ the claim is simply the definition of $\circ_{\mathrm{S}(\mathcal{A})}$. Suppose it is true for $n-1$, where $n \geq 2$. We prove it for n. By the induction hypothesis,

$$x_{n-1} \circ \ldots \circ x_1 = [f_{n,1} \circ f_{n-1,1} \circ \ldots \circ f_{2,1}, g_{n,n-1} \circ g_{n-1,n-2} \circ \ldots \circ g_{2,1}].$$

And it remains to prove that

$$[f_{n,n}, g_{n,n}] \circ [f_{n,1} \circ f_{n-1,1} \circ \ldots \circ f_{2,1}, g_{n,n-1} \circ g_{n-1,n-2} \circ \ldots \circ g_{2,1}] \stackrel{!}{=} [f_{n,1} \circ f_{n-1,1} \circ \ldots \circ f_{1,1}, g_{n,n} \circ g_{n-1,n-1} \circ \ldots \circ g_{1,1}].$$

By the definition of $\circ_{\mathrm{S}(\mathcal{A})}$ and Lemma 5.5 that in turn is true if a pull-back of $(g_{n,n-1} \circ g_{n-1,n-2} \circ \ldots \circ g_{2,1}, f_{n,n})$ in $\mathcal{A}$ is given by $(f_{1,1}, g_{n-1,n-1} \circ g_{n-2,n-2} \circ \ldots \circ g_{1,1})$. And that this is the case is guaranteed by Lemma 4.4. ☐

Definition 5.10. For any $X \in \mathrm{obj}_{\mathcal{A}}$ let $\Theta_{\mathcal{A}}(X) := X$ and for any $\{A, B\} \subseteq \mathrm{obj}_{\mathcal{A}}$ and any $f \in \mathrm{mor}_{\mathcal{A}}(A, B)$ define $\Theta_{\mathcal{A}}(f) := [\mathrm{id}_{\mathcal{A},A}, f]$.

Lemma 5.11. *For any 0-cell $\mathcal{A}$ of $\mathsf{smCAT}^{\mathrm{cart,fc}}_{\mathrm{sos}}$ the following are true:*

(a) For any $\{A, B\} \subseteq \mathrm{obj}_{\mathcal{A}}$, any $f \in \mathrm{mor}_{\mathcal{A}}(A, B)$ and any $g \in \mathrm{mor}_{\mathcal{A}}(B, C)$,

$$\Theta_{\mathcal{A}}(g) \circ_{\mathrm{S}(\mathcal{A})} \Theta_{\mathcal{A}}(f) = \Theta_{\mathcal{A}}(g \circ_{\mathcal{A}} f).$$

(b) $\Theta_{\mathcal{A}}(\mathrm{id}_{\mathcal{A},X}) = \mathrm{id}_{\mathrm{S}(\mathcal{A}),X}$ for any $X \in \mathrm{obj}_{\mathcal{A}}$.

(c) $(\Theta_{\mathcal{A}})_0$ is injective.

(d) $(\Theta_{\mathcal{A}})_{1,A,B}$ is injective for any $\{A, B\} \subseteq \mathrm{obj}_{\mathcal{A}}$.

Proof. (a) Given any objects A, B and C and morphisms $f\colon A \to B$ and $g\colon B \to C$, a pull-back of (f, id_B) is given by (id_A, f) according to Lemma 4.6. Hence, by Lemma 5.5, it follows $[\mathrm{id}_B, g] \circ [\mathrm{id}_A, f] = [\mathrm{id}_A \circ \mathrm{id}_A, g \circ f]$. And that is what we needed to see.

(b) Clear from the definition.

(c) Trivially true.

(d) For any objects A and B and morphisms f and f' from A to B in $\mathcal{A}$, the assumption $[\mathrm{id}_A, f] = [\mathrm{id}_A, f']$ implies the existence of an automorphism u of A with $\mathrm{id}_A \circ u = \mathrm{id}_A$ and $f' \circ u = f$. Because the first identity requires $u = \mathrm{id}_A$, the second shows $f' = f$. ☐

Proposition 5.12. *$\Theta_{\mathcal{A}}$ is a categorial embedding $\mathcal{A} \to \mathrm{S}(\mathcal{A})$ for any 0-cell $\mathcal{A}$ of $\mathsf{smCAT}^{\mathrm{cart,fc}}_{\mathrm{sos}}$.*

Proof. This is what Lemma 5.11 spells out. ☐

Lemma 5.13. *For any categories $\mathcal{A}$ and $\mathcal{B}$ and any functor $H\colon \mathcal{A} \to \mathcal{B}$ and $\{A, B\} \subseteq \mathrm{obj}_{\mathcal{A}}$, if any (f, g) and (f', g') of $\mathcal{A}$ are equivalent spans of $\mathcal{A}$ from A to B, then $(H(f), H(g))$ and $(H(f'), H(g'))$ are equivalent spans of $\mathcal{B}$ from $H(A)$ to $H(B)$.*

Proof. If X and X' are the bases of (f, g) respectively (f', g'), then equivalence of (f, g) and (f', g') implies the existence of an isomorphism u of $\mathcal{A}$ from X to X' with $f' \circ u = f$ and $g' \circ u = g$. Because H is a functor, $H(u)$ is invertible in $\mathcal{B}$. Hence, $H(f') \circ H(u) = H(f' \circ u) = H(f)$ and $H(g') \circ H(u) = H(g' \circ u) = H(g)$ prove that $(H(f), H(g))$ and $(H(f'), H(g'))$ are equivalent as well. ☐

The following definition makes sense by Lemma 5.13

Definition 5.14. For any 0-cells $\mathcal{A}$ and $\mathcal{B}$ and any 1-cell H from $\mathcal{A}$ to $\mathcal{B}$ of $\mathsf{smCAT}^{\mathrm{cart,fc}}_{\mathrm{sos}}$ define

(a) $\mathrm{S}(H)_0 := H_0$ and

(b) for any $\{A, B\} \subseteq \mathrm{obj}_{\mathrm{S}(\mathcal{A})}$ let $\mathrm{S}(H)_{1,A,B}$ be the morphism

$$\mathrm{mor}_{\mathrm{S}(\mathcal{A})}(A, B) \to \mathrm{mor}_{\mathrm{S}(\mathcal{B})}(H_0(A), H_0(B))$$

of Set defined by

$$[f, g] \mapsto [H_1(f), H_1(g)].$$

LEMMA 5.15. *For any* 0*-cells* $\mathcal{A}$ *and* $\mathcal{B}$ *and any* 1*-cell* H *from* $\mathcal{A}$ *to* $\mathcal{B}$ *of* $\mathsf{smCAT}^{\mathrm{cart,fc}}_{\mathrm{sos}}$ *the following are true:*

(a) For any $\{A, B, C\} \subseteq \mathrm{obj}_{\mathrm{S}(\mathcal{A})}$*, any* $x \in \mathrm{mor}_{\mathrm{S}(\mathcal{A})}(A, B)$ *and* $y \in \mathrm{mor}_{\mathrm{S}(\mathcal{A})}(B, C)$*,*

$$\mathrm{S}(H)(y) \circ_{\mathrm{S}(\mathcal{B})} \mathrm{S}(H)(x) = \mathrm{S}(H)(y \circ_{\mathrm{S}(\mathcal{A})} x).$$

(b) $\mathrm{S}(H)(\mathrm{id}_{\mathrm{S}(\mathcal{A}),X}) = \mathrm{id}_{\mathrm{S}(\mathcal{B}),\mathrm{S}(H)(X)}$ *for any* $X \in \mathrm{obj}_{\mathrm{S}(\mathcal{A})}$.

PROOF. (a) Given any objects A, B and C of $\mathcal{A}$ and any spans $[f, g]$ from A to B with base X and $[p, q]$ from B to C with base Y in $\mathcal{A}$ as well as any pull-back (a, b) of (g, p) in $\mathcal{A}$ with pull-back object P the span $[f \circ_{\mathcal{A}} a, q \circ_{\mathcal{B}} \circ b]$ represents $[p, q] \circ_{\mathrm{S}(\mathcal{A})} [f, g]$ by Lemma 5.5. Because H preserves pull-backs $(H(a), H(b))$ is a pull-back in $\mathcal{B}$ of $(H(g), H(p))$. Thus, by a second application of Lemma 5.5, the span $(H(f \circ_{\mathcal{A}} a), H(f \circ_{\mathcal{A}} a)) = (H(f) \circ_{\mathcal{B}} H(a), H(f) \circ_{\mathcal{B}} H(a))$ represents the morphism $\mathrm{S}(H)([p, q]) \circ_{\mathrm{S}(\mathcal{B})} \mathrm{S}(H)([f, g]) = [H(p), H(q)] \circ_{\mathrm{S}(\mathcal{B})} [H(f), H(g)]$.

(b) Clearly, $\mathrm{S}(H)(\mathrm{id}_{\mathrm{S}(\mathcal{A}),X}) = \mathrm{S}(H)([\mathrm{id}_{\mathcal{A},X}, \mathrm{id}_{\mathcal{A},X}]) = [H(\mathrm{id}_{\mathcal{A},X}), H(\mathrm{id}_{\mathcal{A},X})] = [\mathrm{id}_{\mathcal{B},H(X)}, \mathrm{id}_{\mathcal{B},H(X)}] = \mathrm{id}_{\mathrm{S}(\mathcal{B}),\mathrm{S}(H)(X)}$. □

PROPOSITION 5.16. *For any* 0*-cells* $\mathcal{A}$ *and* $\mathcal{B}$ *and any* 1*-cell* H *from* $\mathcal{A}$ *to* $\mathcal{B}$ *of* $\mathsf{smCAT}^{\mathrm{cart,fc}}_{\mathrm{sos}}$ *the pair* $(\mathrm{S}(H)_0, \mathrm{S}(H)_1)$ *is a functor from* $(\mathrm{obj}_{\mathrm{S}(\mathcal{A})}, \mathrm{mor}_{\mathrm{S}(\mathcal{A})}, \circ_{\mathrm{S}(\mathcal{A})}, \mathrm{id}_{\mathrm{S}(\mathcal{A})})$ *to* $(\mathrm{obj}_{\mathrm{S}(\mathcal{B})}, \mathrm{mor}_{\mathrm{S}(\mathcal{B})}, \circ_{\mathrm{S}(\mathcal{B})}, \mathrm{id}_{\mathrm{S}(\mathcal{B})})$

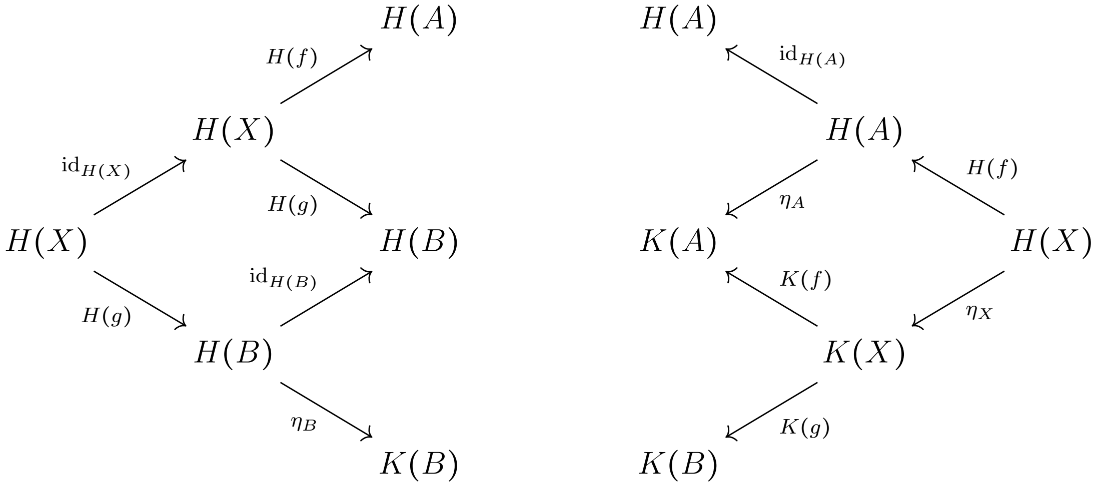
PROOF. This is a summary of the results of Lemma 5.15. □

DEFINITION 5.17. For any 0-cells $\mathcal{A}$ and $\mathcal{B}$, any 1-cells H and K from $\mathcal{A}$ to $\mathcal{B}$ and for any 2-cell from H to K of $\mathsf{smCAT}^{\mathrm{cart,fc}}_{\mathrm{sos}}$ let $\mathrm{S}(\eta)$ be the $\mathrm{obj}_{\mathcal{A}}$-indexed family with components $(\mathrm{S}(\eta))_A := \Theta_{\mathcal{B}}(\eta_A)$ for any object A of $\mathcal{A}$.

LEMMA 5.18. *For any* 0*-cells* $\mathcal{A}$ *and* $\mathcal{B}$*, any* 1*-cells* H *and* K *from* $\mathcal{A}$ *to* $\mathcal{B}$*, any* 2*-cell from* H *to* K *of* $\mathsf{smCAT}^{\mathrm{cart,fc}}_{\mathrm{sos}}$*, any* $\{A, B\} \subseteq \mathrm{obj}_{\mathrm{S}(\mathcal{A})}$ *and any* $x \in \mathrm{mor}_{\mathrm{S}(\mathcal{A})}(A, B)$*,*

$$\mathrm{S}(\eta)_B \circ_{\mathrm{S}(\mathcal{B})} \mathrm{S}(H)(x) = \mathrm{S}(K)(x) \circ_{\mathrm{S}(\mathcal{B})} \mathrm{S}(\eta)_A.$$

PROOF. Let A and B be any objects and (f, g) any span of $\mathcal{A}$ from A to B and let X be its base.



Then, because $(\mathrm{id}_{\mathcal{B},H(X)}, H(g))$ is a pull-back of $(H(g), \mathrm{id}_{\mathcal{B},\mathrm{id}_{\mathcal{B},H(B)}})$ in $\mathcal{B}$ by Lemma 4.6,

$$\begin{aligned} \mathrm{S}(\eta)_B \circ_{\mathrm{S}(\mathcal{B})} \mathrm{S}(H)([f,g]) &= [\mathrm{id}_{\mathcal{B},H(B)}, \eta_B] \circ_{\mathrm{S}(\mathcal{B})} [H(f), H(g)] \\ &= [H(f) \circ_{\mathcal{B}} \mathrm{id}_{\mathcal{B},H(X)}, \eta_B \circ_{\mathcal{B}} H(g)] \\ &= [\mathrm{id}_{\mathcal{B},H(A)} \circ_{\mathcal{B}} H(f), K(g) \circ_{\mathcal{B}} \eta_X], \end{aligned}$$

where we have used the assumption that η is a natural transformation from H to K in the last step.

Because η is even equifibered $(H(f), \eta_X)$ is a pull-back of $(\eta_A, K(f))$ in $\mathcal{B}$. For that reason,

$$\begin{aligned} \mathrm{S}(K)([f,g]) \circ_{\mathrm{S}(\mathcal{B})} \mathrm{S}(\eta)_A &= [K(f), K(g)] \circ_{\mathrm{S}(\mathcal{B})} [\mathrm{id}_{\mathcal{B},H(A)}, \eta_A] \\ &= [\mathrm{id}_{\mathcal{B},H(A)} \circ_{\mathcal{B}} H(f), K(g) \circ_{\mathcal{B}} \eta_X], \end{aligned}$$

which proves the claim. □

PROPOSITION 5.19. *For any* 0*-cells* $\mathcal{A}$ *and* $\mathcal{B}$*, any* 1*-cells* H *and* K *from* $\mathcal{A}$ *to* $\mathcal{B}$ *and for any* 2*-cell from* H *to* K *of* $\mathsf{smCAT}^{\mathrm{cart,fc}}_{\mathrm{sos}}$ *the family* $\mathrm{S}(\eta)$ *is a natural transformation from* $\mathrm{S}(H)$ *to* $\mathrm{S}(K)$.

PROOF. Follows from Lemma 5.18. □

The next lemma will be a useful tool for later computations.

LEMMA 5.20. *For any* 0*-cells* $\mathcal{A}$ *and* $\mathcal{B}$ *and any* 1*-cell* H *from* $\mathcal{A}$ *to* $\mathcal{B}$ *of* $\mathsf{smCAT}^{\mathrm{cart,fc}}_{\mathrm{sos}}$,

$$\mathrm{S}(H) \circ_{\mathsf{CAT}} \Theta_{\mathcal{A}} = \Theta_{\mathcal{B}} \circ_{\mathsf{CAT}} H.$$

PROOF. For any $X \in \mathrm{obj}_{\mathcal{A}}$ the definitions imply

$$\begin{aligned}(\mathrm{S}(H) \circ \Theta_{\mathcal{A}})(X) &= \mathrm{S}(H)(\Theta_{\mathcal{A}}(X)) = \mathrm{S}(H)(X) = H(X) = \Theta_{\mathcal{B}}(H(X)) \\ &= (\Theta_{\mathcal{B}} \circ H)(X).\end{aligned}$$

Likewise, for any $\{A, B\} \subseteq \mathrm{obj}_{\mathcal{A}}$ and any $f \in \mathrm{mor}_{\mathcal{A}}(A, B)$,

$$\begin{aligned}(\mathrm{S}(H) \circ \Theta_{\mathcal{A}})(f) &= \mathrm{S}(H)(\Theta_{\mathcal{A}}(f)) = \mathrm{S}(H)([\mathrm{id}_{\mathcal{A},A}, f]) = [H(\mathrm{id}_{\mathcal{B},H(A)}), H(f)] \\ &= \Theta_{\mathcal{B}}(H(f)),\end{aligned}$$

which completes the proof. □

5.1.2. *Monoidal Category.* The next step is to prove that the span category can be given a rigid monoidal structure if the input has products and a terminal object.

LEMMA 5.21. *In any cartesian monoidal category, for each $i \in \{1, 2\}$ let $\{A_i, B_i\} \subseteq$ obj and let (f_i, g_i) and (f_i', g_i') be two spans from A_i to B_i. If for each $i \in \{1, 2\}$ the spans (f_i, g_i) and (f_i', g_i') are equivalent, then $(f_1 \otimes f_2, g_1 \otimes g_2)$ and $(f_1' \otimes f_2', g_1' \otimes g_2')$ are equivalent spans from $A_1 \otimes A_2$ to $B_1 \otimes B_2$.*

PROOF. For each $i \in \{1, 2\}$, if X_i denotes the base of (f_i, g_i) and X_i' the base of (f_i', g_i'), then by assumption there exists an isomorphism u_i: $X_i' \to X_i$ with $f_i' = f_i \circ u_i$ and $g_i' = g_i \circ u_i$. Since $\otimes$ is a functor, the morphism $u_1 \otimes u_2$ is an isomorphism $X_1' \otimes X_2' \to X_i \otimes X_2$. For the same reason, $u_1 \otimes u_2$ satisfies $f_1' \otimes f_2' = (f_1 \circ u_1) \otimes (f_2 \circ u_2) = (f_1 \otimes f_2) \circ (u_1 \otimes u_2)$ and, likewise, $g_1' \otimes g_2' = (g_1 \otimes g_2) \circ (u_1 \otimes u_2)$, which proves the asserted equivalence. □

By the preceding lemma, Part (c) in the following Definition makes sense. For Parts (d)–(f) recall the Definition 5.10 of the embedding $\Theta_{\mathcal{A}}$ for any 0-cell $\mathcal{A}$ of $\mathsf{smCAT}_{\mathrm{sos}}^{\mathrm{cart,fc}}$.

DEFINITION 5.22. For any 0-cell $\mathcal{A}$ of $\mathsf{smCAT}_{\mathrm{sos}}^{\mathrm{cart,fc}}$, any objects A, A_1, A_2, B, B_1, B_2 and C of $\mathcal{A}$ and any $[f_1, g_1] \in \mathrm{mor}_{\mathrm{S}(\mathcal{A})}(A_1, B_1)$ and $[f_2, g_2] \in \mathrm{mor}_{\mathrm{S}(\mathcal{A})}(A_2, B_2)$ define

(a) $A_1 \otimes_{\mathrm{S}(\mathcal{A})} A_2 \coloneqq A_1 \otimes_{\mathcal{A}} A_2.$,
(b) $I_{\mathrm{S}(\mathcal{A})} \coloneqq I_{\mathcal{A}}$,
(c) $[f_1, g_1] \otimes_{\mathrm{S}(\mathcal{A})} [f_2, g_2] \coloneqq [f_1 \otimes_{\mathcal{A}} f_2, g_1 \otimes_{\mathcal{A}} g_2]$,
(d) $\alpha_{\mathrm{S}(\mathcal{A}),A,B,C} \coloneqq \Theta_{\mathcal{A}}(\alpha_{\mathcal{A},A,B,C})$,
(e) $\lambda_{\mathrm{S}(\mathcal{A}),A} \coloneqq \Theta_{\mathcal{A}}(\lambda_{\mathcal{A},A})$,
(f) $\rho_{\mathrm{S}(\mathcal{A}),A} \coloneqq \Theta_{\mathcal{A}}(\rho_{\mathcal{A}} A)$.

The following lemma is not part of the main claim but an important tool for later proofs.

LEMMA 5.23. *For any 0-cell $\mathcal{A}$ of $\mathsf{smCAT}_{\mathrm{sos}}^{\mathrm{cart,fc}}$ and any two morphisms f_1 and f_2 of $\mathcal{A}$,*

$$\Theta_{\mathcal{A}}(f_1 \otimes_{\mathcal{A}} f_2) = \Theta_{\mathcal{A}}(f_1) \otimes_{\mathrm{S}} \Theta_{\mathcal{A}}(f_2).$$

Proof. For each $i \in \{1,2\}$ let X_i and A_i be such that $f_i\colon X_i \to A_i$. Per definition and by Lemma 5.21,

$$\begin{aligned}\Theta_{\mathcal{A}}(f_1 \otimes_{\mathcal{A}} f_2) &= [\mathrm{id}_{\mathcal{A},X_1 \otimes_{\mathcal{A}} X_2}, f_1 \otimes_{\mathcal{A}} f_2] = [\mathrm{id}_{\mathcal{A},X_1} \otimes_{\mathcal{A}} \mathrm{id}_{\mathcal{A},X_2}, f_1 \otimes_{\mathcal{A}} f_2] \\ &= [\mathrm{id}_{\mathcal{A},X_1}, f_1] \otimes_{\mathrm{S}} [\mathrm{id}_{\mathcal{A},X_2}, f_2] = \Theta_{\mathcal{A}}(f_1) \otimes_{\mathrm{S}} \Theta_{\mathcal{A}}(f_2),\end{aligned}$$

where we have used that $\otimes_{\mathcal{A}}$ is a functor $\mathcal{A} \otimes_{\mathsf{CAT}} \mathcal{A} \to \mathcal{A}$ in the second step. □

Lemma 5.24. *Let $\mathcal{A}$ be any 0-cell of $\mathsf{smCAT}^{\mathrm{cart,fc}}_{\mathrm{sos}}$ and abbreviate $\mathrm{S}(\mathcal{A})$ by S.*

(a) $\otimes_{\mathrm{S}}$ is a functor $\mathrm{S} \otimes_{\mathsf{CAT}} \mathrm{S} \to \mathrm{S}$.

(b) α_{S} is a natural isomorphism of functors $\mathrm{S} \otimes_{\mathsf{CAT}} \mathrm{S} \otimes_{\mathsf{CAT}} \mathrm{S} \to \mathrm{S}$ from $((\cdot_1) \otimes_{\mathrm{S}} (\cdot_2)) \otimes_{\mathrm{S}} (\cdot_3)$ to $(\cdot_1) \otimes_{\mathrm{S}} ((\cdot_2) \otimes_{\mathrm{S}} (\cdot_3))$.

(c) λ_{S} is a natural isomorphism of S-endofunctors from $I_{\mathrm{S}} \otimes_{\mathrm{S}} (\cdot)$ to $(\cdot)$.

(d) ρ_{S} is a natural isomorphism of S-endofunctors from $(\cdot) \otimes_{\mathrm{S}} I_{\mathrm{S}}$ to $(\cdot)$.

(e) For any objects A, B, C and D of S a commutative diagram in S is given by

$$\begin{array}{ccc} & (A \otimes B) \otimes (C \otimes D) & \\ {}^{\alpha_{A\otimes B,C,D}}\nearrow & & \searrow^{\alpha_{A,B,C\otimes D}} \\ ((A \otimes B) \otimes C) \otimes D & & A \otimes (B \otimes (C \otimes D)) \\ \downarrow{\scriptstyle \alpha_{A,B,C} \otimes \mathrm{id}_D} & & \uparrow{\scriptstyle \mathrm{id}_A \otimes \alpha_{B,C,D}} \\ (A \otimes (B \otimes C)) \otimes D & \xrightarrow[\alpha_{A,B\otimes C,D}]{} & A \otimes ((B \otimes C) \otimes D) \end{array}$$

(f) For any objects A and B of S a commutative diagram in S is given by

$$\begin{array}{ccc} (A \otimes I) \otimes B & \xrightarrow{\alpha_{A,I,B}} & A \otimes (I \otimes B) \\ {}_{\rho_A \otimes \mathrm{id}_B}\searrow & & \swarrow_{\mathrm{id}_A \otimes \lambda_B} \\ & A \otimes B & \end{array}$$

Proof. (a) We have to show that $\otimes_{\mathrm{S}}$ respects composition and identities.

Composition. For each $i \in \{1,2\}$ let A_i, B_i and C_i be objects of $\mathcal{A}$, let (f_i, g_i) be a span in $\mathcal{A}$ from A to B and let (p_i, q_i) be one from B to C and let (a_i, b_i) be a pull-back of (g_i, p_i). Hence, by definition and Lemma 5.5,

$$\begin{aligned}&([p_1, q_1] \circ_{\mathrm{S}} [f_1, g_1]) \otimes_{\mathrm{S}} ([p_2, q_2] \circ_{\mathrm{S}} [f_2, g_2]) \\ &\qquad = [f_1 \circ_{\mathcal{A}} a_1, q_1 \circ_{\mathcal{A}} b_1] \otimes_{\mathrm{S}} [f_2 \circ_{\mathcal{A}} a_2, q_2 \circ_{\mathcal{A}} b_2] \\ &\qquad = [(f_1 \circ_{\mathcal{A}} a_1) \otimes_{\mathcal{A}} (f_2 \circ_{\mathcal{A}} a_2), (q_1 \circ_{\mathcal{A}} b_1) \otimes_{\mathcal{A}} (q_2 \circ_{\mathcal{A}} b_2)] \\ &\qquad = [(f_1 \otimes_{\mathcal{A}} f_2) \circ_{\mathcal{A}} (a_1 \otimes_{\mathcal{A}} a_2), (q_1 \otimes_{\mathcal{A}} q_2) \circ_{\mathcal{A}} (b_1 \otimes_{\mathcal{A}} b_2)],\end{aligned}$$

where we have used that $\otimes_{\mathcal{A}}$ is a functor $\mathcal{A}\times\mathcal{A}\to\mathcal{A}$ in the last step. On the other hand, because $(a_1\otimes_{\mathcal{A}} a_2, b_1\otimes_{\mathcal{A}} b_2)$ is a pull-back of $(g_1\otimes_{\mathcal{A}} g_2, p_1\otimes_{\mathcal{A}} p_2)$ by Lemma 4.10 the definition of $\circ_{\mathrm{S}}$ allows us to conclude

$$\begin{aligned}&= [p_1\otimes_{\mathcal{A}} p_2, q_1\otimes_{\mathcal{A}} q_2]\circ_{\mathrm{S}} [f_1\otimes_{\mathcal{A}} f_2, g_1\otimes_{\mathcal{A}} g_2]\\ &= ([p_1,q_1]\otimes_{\mathrm{S}}[p_2,q_2])\circ_{\mathrm{S}}([f_1,g_1]\otimes_{\mathrm{S}}[f_2,g_2]).\end{aligned}$$

Thus, $\otimes_{\mathrm{S}}$ does indeed respect composition.

Identities. For any objects X and Y of $\mathcal{A}$, by definition,

$$\begin{aligned}\mathrm{id}_{\mathrm{S},X}\otimes_{\mathrm{S}}\mathrm{id}_{\mathrm{S},Y} &= [\mathrm{id}_{\mathcal{A},X},\mathrm{id}_{\mathcal{A},X}]\otimes_{\mathrm{S}}[\mathrm{id}_{\mathcal{A},Y},\mathrm{id}_{\mathcal{A},Y}]\\ &= [\mathrm{id}_{\mathcal{A},X}\otimes_{\mathcal{A}}\mathrm{id}_{\mathcal{A},Y},\mathrm{id}_{\mathcal{A},X}\otimes_{\mathcal{A}}\mathrm{id}_{\mathcal{A},Y}] = [\mathrm{id}_{\mathcal{A},X\otimes_{\mathcal{A}}Y},\mathrm{id}_{\mathcal{A},X\otimes_{\mathcal{A}}Y}],\end{aligned}$$

where again we have used the assumption that $\otimes_{\mathcal{A}}$ is a functor $\mathcal{A}\times\mathcal{A}\to\mathcal{A}$.

(b) By Proposition 5.12 the morphisms making up α_{S} are isomorphisms because $\alpha_{\mathcal{A}}$ consists of isomorphisms. We only need to show that α_{S} is a natural transformation. For each $i\in\{1,2,3\}$ let (f_i,g_i) be a span in $\mathcal{A}$ from object A_i to B_i with base X_i. By Lemma 4.6 a pull-back of $((g_1\otimes_{\mathcal{A}} g_2)\otimes_{\mathcal{A}} g_3, \mathrm{id}_{\mathcal{A},(B_1\otimes_{\mathcal{A}}B_2)\otimes_{\mathcal{A}}B_3})$ is given by $(\mathrm{id}_{\mathcal{A},(A_1\otimes_{\mathcal{A}}A_2)\otimes_{\mathcal{A}}A_3}, (g_1\otimes_{\mathcal{A}} g_2)\otimes_{\mathcal{A}} g_3)$. Hence, by definition and Lemma 5.5,

$$\begin{aligned}&\alpha_{\mathrm{S},B_1,B_2,B_3}\circ_{\mathrm{S}}(([f_1,g_1]\otimes_{\mathrm{S}}[f_2,g_2])\otimes_{\mathrm{S}}[f_3,g_3])\\ &\quad= [\mathrm{id}_{\mathcal{A},(B_1\otimes_{\mathcal{A}}B_2)\otimes_{\mathcal{A}}B_3},\alpha_{\mathcal{A},B_1,B_2,B_3}]\circ_{\mathrm{S}}[(f_1\otimes_{\mathcal{A}} f_2)\otimes_{\mathcal{A}} f_3,(g_1\otimes_{\mathcal{A}} g_2)\otimes_{\mathcal{A}} g_3]\\ &\quad= [((f_1\otimes_{\mathcal{A}} f_2)\otimes_{\mathcal{A}} f_3)\circ_{\mathcal{A}}\mathrm{id}_{\mathcal{A},(A_1\otimes_{\mathcal{A}}A_2)\otimes_{\mathcal{A}}A_3},\alpha_{\mathcal{A},B_1,B_2,B_3}\circ_{\mathcal{A}}((g_1\otimes_{\mathcal{A}} g_2)\otimes_{\mathcal{A}} g_3)]\\ &\quad= [(f_1\otimes_{\mathcal{A}} f_2)\otimes_{\mathcal{A}} f_3,(g_1\otimes_{\mathcal{A}}(g_2\otimes_{\mathcal{A}} g_3))\circ_{\mathcal{A}}\alpha_{\mathcal{A},X_1,X_2,X_3}],\end{aligned}$$

where we have used that $\alpha_{\mathcal{A}}$ is a natural transformation from $((\cdot_1)\otimes_{\mathcal{A}}(\cdot_2))\otimes_{\mathcal{A}}(\cdot_3)$ to $(\cdot_1)\otimes_{\mathcal{A}}((\cdot_2)\otimes_{\mathcal{A}}(\cdot_3))$. As by Lemma 4.13 (a) the pair $((f_1\otimes_{\mathcal{A}} f_2)\otimes_{\mathcal{A}} f_3,\alpha_{\mathcal{A},X_1,X_2,X_3})$ is a pull-back of $(\alpha_{\mathcal{A},A_1,A_2,A_3}, f_1\otimes_{\mathcal{A}}(f_2\otimes_{\mathcal{A}} f_3))$,

$$\begin{aligned}&= [f_1\otimes_{\mathcal{A}}(f_2\otimes_{\mathcal{A}} f_3), g_1\otimes_{\mathcal{A}}(g_2\otimes_{\mathcal{A}} g_3)]\circ_{\mathrm{S}}[\mathrm{id}_{\mathcal{A},(A_1\otimes_{\mathcal{A}}A_2)\otimes_{\mathcal{A}}A_3},\alpha_{\mathcal{A},A_1,A_2,A_3}]\\ &= ([f_1,g_1]\otimes_{\mathrm{S}}([f_2,g_2]\otimes_{\mathrm{S}}[f_3,g_3]))\circ_{\mathrm{S}}\alpha_{\mathrm{S},A_1,A_2,A_3},\end{aligned}$$

which is what we needed to see.

(c) Again, Proposition 5.12 provides us with inverses for the morphisms λ_{S}. Hence, all we have to show is that λ_{S} is natural. Let (f,g) be a span in $\mathcal{A}$ from object A to object B with base X. A pull-back of $(\mathrm{id}_{\mathcal{A},I_{\mathcal{A}}}\otimes_{\mathcal{A}} g, \mathrm{id}_{\mathcal{A},I_{\mathcal{A}}\otimes_{\mathcal{A}}B})$ is given by $(\mathrm{id}_{\mathcal{A},I_{\mathcal{A}}\otimes_{\mathcal{A}}X},\mathrm{id}_{\mathcal{A},I_{\mathcal{A}}}\otimes_{\mathcal{A}} g)$ according to Lemma 4.6. Thus, by Lemma 5.5,

$$\begin{aligned}\lambda_{\mathrm{S},B}\circ_{\mathrm{S}}(\mathrm{id}_{\mathrm{S},I_{\mathrm{S}}}\otimes_{\mathrm{S}}[f,g]) &= [\mathrm{id}_{\mathcal{A},I_{\mathcal{A}}\otimes_{\mathcal{A}}B},\lambda_{\mathcal{A},B}]\circ_{\mathrm{S}}[\mathrm{id}_{\mathcal{A},I_{\mathcal{A}}}\otimes_{\mathcal{A}} f,\mathrm{id}_{\mathcal{A},I_{\mathcal{A}}}\otimes_{\mathcal{A}} g]\\ &= [(\mathrm{id}_{\mathcal{A},I_{\mathcal{A}}}\otimes_{\mathcal{A}} f)\circ_{\mathcal{A}}\mathrm{id}_{\mathcal{A},I_{\mathcal{A}}\otimes_{\mathcal{A}}X},\lambda_{\mathcal{A},B}\circ_{\mathcal{A}}(\mathrm{id}_{\mathcal{A},I_{\mathcal{A}}}\otimes_{\mathcal{A}} g)]\\ &= [\mathrm{id}_{\mathcal{A},I_{\mathcal{A}}}\otimes_{\mathcal{A}} f, g\circ_{\mathcal{A}}\lambda_{\mathcal{A},X}],\end{aligned}$$

where the last step is due to $\lambda_{\mathcal{A}}$ being a natural transformation $I_{\mathcal{A}}\otimes_{\mathcal{A}}(\cdot)\to(\cdot)$. Lemma 4.13 (b) tells us that $(\mathrm{id}_{\mathcal{A},I_{\mathcal{A}}}\otimes_{\mathcal{A}} f,\lambda_{\mathcal{A},X})$ is a pull-back of $(\lambda_{\mathcal{A},A}, f)$. Hence, the above is identical to $[f,g]\circ_{\mathrm{S}}[\mathrm{id}_{\mathcal{A},I_{\mathcal{A}}\otimes_{\mathcal{A}}A},\lambda_{\mathcal{A},A}] = [f,g]\circ_{\mathrm{S}}\lambda_{\mathrm{S},A}$, which is precisely what we needed to prove.

(d) Proposition 5.12 relieves of us of proving that ρ_{S} is comprised of isomorphisms. To see naturality we let (f, g) be any span in $\mathcal{A}$ from A to B with base X. Lemma 4.6 lets us find a pull-back of $(g \otimes_{\mathcal{A}} \mathrm{id}_{\mathcal{A},I_{\mathcal{A}}}, \mathrm{id}_{\mathcal{A},B\otimes_{\mathcal{A}} I_{\mathcal{A}}})$ in $(\mathrm{id}_{\mathcal{A},X\otimes_{\mathcal{A}} I_{\mathcal{A}}}, g \otimes_{\mathcal{A}} \mathrm{id}_{\mathcal{A},I_{\mathcal{A}}})$. With the help of Lemma 5.5 we then compute

$$\begin{aligned}\rho_{\mathrm{S},B} \circ_{\mathrm{S}} ([f,g] \otimes_{\mathrm{S}} \mathrm{id}_{\mathrm{S},I_{\mathrm{S}}}) &= [\mathrm{id}_{\mathcal{A},B\otimes_{\mathcal{A}} I_{\mathcal{A}}}, \rho_{\mathcal{A},B}] \circ_{\mathrm{S}} [f \otimes_{\mathcal{A}} \mathrm{id}_{\mathcal{A},I_{\mathcal{A}}}, g \otimes_{\mathcal{A}} \mathrm{id}_{\mathcal{A},I_{\mathcal{A}}}] \\ &= [(f \otimes_{\mathcal{A}} \mathrm{id}_{\mathcal{A},I_{\mathcal{A}}}) \circ_{\mathcal{A}} \mathrm{id}_{\mathcal{A},X\otimes_{\mathcal{A}} I_{\mathcal{A}}}, \rho_{\mathcal{A},B} \circ_{\mathcal{A}} (g \otimes_{\mathcal{A}} \mathrm{id}_{\mathcal{A},I_{\mathcal{A}}})] \\ &= [f \otimes_{\mathcal{A}} \mathrm{id}_{\mathcal{A},I_{\mathcal{A}}}, g \circ_{\mathcal{A}} \rho_{\mathcal{A},X}],\end{aligned}$$

using the naturality of $\rho_{\mathcal{A}}$. Since $(f \otimes_{\mathcal{A}} \mathrm{id}_{\mathcal{A},I_{\mathcal{A}}}, \rho_{\mathcal{A},X})$ is a pull-back of $(\rho_{\mathcal{A},A}, f)$ by Lemma 4.13 (c), the above is the same as $[f, g] \circ_{\mathrm{S}} [\mathrm{id}_{\mathcal{A},A\otimes_{\mathcal{A}} I_{\mathcal{A}}}, \rho_{\mathcal{A},A}] = [f, g] \circ_{\mathrm{S}} \rho_{\mathrm{S},A}$, proving the claim.

(e) Associators in S are defined as images of the corresponding associators of $\mathcal{A}$ under $\Theta_{\mathcal{A}}$. With the help of Proposition 5.12 and, crucially, Lemma 5.23 we can recognize

$$\begin{aligned} \alpha_{\mathrm{S},A,B,C} \otimes_{\mathrm{S}} \mathrm{id}_{\mathrm{S},D} &= \Theta_{\mathcal{A}}(\alpha_{\mathcal{A},A,B,C} \otimes_{\mathcal{A}} \mathrm{id}_{\mathcal{A},D}) \\ \text{and} \quad \mathrm{id}_{\mathrm{S},A} \otimes_{\mathrm{S}} \alpha_{\mathrm{S},B,C,D} &= \Theta_{\mathcal{A}}(\mathrm{id}_{\mathcal{A},A} \otimes_{\mathcal{A}} \alpha_{\mathcal{A},B,C,D}).\end{aligned}$$

Hence, what we actually claim is that the image of the corresponding diagram in $\mathcal{A}$ under the functor $\Theta_{\mathcal{A}}$ commutes. Because $\mathcal{A}$ is a monoidal category, of course, the diagram in question is commutative in $\mathcal{A}$. By Proposition 5.12 the same is then true for the image of that diagram in S.

(f) Again, by Proposition 5.12 and Lemma 5.23 the diagram in S is simply the image under $\Theta_{\mathcal{A}}$ of the corresponding diagram in $\mathcal{A}$. Since it commutes in $\mathcal{A}$ and since $\Theta_{\mathcal{A}}$ is a functor it also commutes in S. □

PROPOSITION 5.25. $\mathrm{S}(\mathcal{A})$, *when equipped with* $(\otimes_{\mathrm{S}(\mathcal{A})}, I_{\mathrm{S}(\mathcal{A})}, \alpha_{\mathrm{S}(\mathcal{A})}, \lambda_{\mathrm{S}(\mathcal{A})}, \rho_{\mathrm{S}(\mathcal{A})})$, *is a monoidal category for any* 0*-cell* $\mathcal{A}$ *of* $\mathsf{smCAT}^{\mathrm{cart,fc}}_{\mathrm{sos}}$.

PROOF. That is the combined implication of Lemma 5.24 □

PROPOSITION 5.26. $\Theta_{\mathcal{A}}$ *is a strict monoidal functor* $\mathcal{A} \to \mathrm{S}(\mathcal{A})$ *for any* 0*-cell* $\mathcal{A}$ *of* $\mathsf{smCAT}^{\mathrm{cart,fc}}_{\mathrm{sos}}$.

PROOF. Follows immediately from Proposition 5.25 and Lemma 5.23. □

DEFINITION 5.27. For any 0-cells $\mathcal{A}$ and $\mathcal{B}$ and any 1-cell H from $\mathcal{A}$ to $\mathcal{B}$ of $\mathsf{smCAT}^{\mathrm{cart,fc}}_{\mathrm{sos}}$ define

(a) $\mathrm{S}(H)_{\otimes,A_1,A_2} \coloneqq \Theta_{\mathcal{B}}(H_{\otimes,A_1,A_2})$ for any $\{A_1, A_2\} \subseteq \mathrm{obj}_{\mathcal{A}}$, and

(b) $\mathrm{S}(H)_I \coloneqq \Theta_{\mathcal{A}}(H_I)$.

LEMMA 5.28. *For any* 0*-cells* $\mathcal{A}$ *and* $\mathcal{B}$ *and any* 1*-cell* H *from* $\mathcal{A}$ *to* $\mathcal{B}$ *of* $\mathsf{smCAT}^{\mathrm{cart,fc}}_{\mathrm{sos}}$ *the following hold:*

(a) For any $\{A_1, A_2\} \subseteq \mathrm{obj}_{\mathcal{A}}$ *the morphism* $\mathrm{S}(H)_{\otimes,A_1,A_2}$ *is invertible in* $\mathrm{S}(\mathcal{B})$.

(b) *For any* $\{A_1, A_2, B_1, B_2\} \subseteq \mathrm{obj}_{\mathrm{S}(\mathcal{A})}$ *and any* $x_1 \in \mathrm{mor}_{\mathrm{S}(\mathcal{A})}(A_1, B_1)$ *and any* $x_2 \in \mathrm{mor}_{\mathrm{S}(\mathcal{A})}(A_2, B_2)$,

$$\begin{aligned}\mathrm{S}(H)_{\otimes,B_1,B_2} &\circ_{\mathrm{S}(\mathcal{B})} (\mathrm{S}(H)(x_1) \otimes_{\mathrm{S}(\mathcal{B})} \mathrm{S}(H)(x_2)) \\ &= \mathrm{S}(H)(x_1 \otimes_{\mathrm{S}(\mathcal{A})} x_2) \circ_{\mathrm{S}(\mathcal{B})} \mathrm{S}(H)_{\otimes,A_1,A_2}.\end{aligned}$$

$$\begin{array}{ccc} \mathrm{S}(H)(A_1) \otimes_{\mathrm{S}(\mathcal{B})} \mathrm{S}(H)(A_2) & \xrightarrow{\mathrm{S}(H)(x_1)\otimes_{\mathrm{S}(\mathcal{B})}\mathrm{S}(H)(x_2)} & \mathrm{S}(H)(B_1) \otimes_{\mathrm{S}(\mathcal{B})} \mathrm{S}(H)(B_2) \\ {\scriptstyle \mathrm{S}(H)_{\otimes,A_1,A_2}}\downarrow & & \downarrow{\scriptstyle \mathrm{S}(H)_{\otimes,B_1,B_2}} \\ \mathrm{S}(H)(A_1 \otimes_{\mathrm{S}(\mathcal{A})} A_2) & \xrightarrow[\mathrm{S}(H)(x_1\otimes_{\mathrm{S}(\mathcal{A})}x_2)]{} & \mathrm{S}(H)(B_1 \otimes_{\mathrm{S}(\mathcal{A})} B_2) \end{array}$$

(c) *The morphism* $\mathrm{S}(H)_I$ *is invertible in* $\mathrm{S}(\mathcal{B})$.

(d) *For any* $\{A_1, A_2, A_3\} \subseteq \mathrm{obj}_{\mathrm{S}(\mathcal{A})}$,

$$\begin{aligned}\mathrm{S}(H)(\alpha_{\mathrm{S}(\mathcal{A}),A_1,A_2,A_3}) &\circ_{\mathrm{S}(\mathcal{B})} \mathrm{S}(H)_{\otimes,A_1\otimes_{\mathrm{S}(\mathcal{A})}A_2,A_3} \\ &\circ_{\mathrm{S}(\mathcal{B})} (\mathrm{S}(H)_{\otimes,A_1,A_2} \otimes_{\mathrm{S}(\mathcal{B})} \mathrm{id}_{\mathrm{S}(\mathcal{B}),\mathrm{S}(H)(A_3)}) \\ = \mathrm{S}(H)_{\otimes,A_1,A_2\otimes_{\mathrm{S}(\mathcal{A})}A_3} &\circ_{\mathrm{S}(\mathcal{B})} (\mathrm{id}_{\mathrm{S}(\mathcal{B}),\mathrm{S}(H)(A_1)} \otimes_{\mathrm{S}(\mathcal{B})} \mathrm{S}(H)_{\otimes,A_2,A_3}) \\ &\circ_{\mathrm{S}(\mathcal{B})} \alpha_{\mathrm{S}(\mathcal{B}),\mathrm{S}(H)(A_1),\mathrm{S}(H)(A_2),\mathrm{S}(H)(A_3)}.\end{aligned}$$

$$\begin{array}{ccc} (\mathrm{S}(H)(A_1) \otimes_{\mathrm{S}(\mathcal{B})} \mathrm{S}(H)(A_2)) \otimes_{\mathrm{S}(\mathcal{B})} \mathrm{S}(H)(A_3) & \xrightarrow{\alpha_{\mathrm{S}(\mathcal{B}),\mathrm{S}(H)(A_1),\mathrm{S}(H)(A_2),\mathrm{S}(H)(A_3)}} & \mathrm{S}(H)(A_1) \otimes_{\mathrm{S}(\mathcal{B})} (\mathrm{S}(H)(A_2) \otimes_{\mathrm{S}(\mathcal{B})} \mathrm{S}(H)(A_3)) \\ \downarrow{\scriptstyle \mathrm{S}(H)_{\otimes,A_1,A_2}\otimes_{\mathrm{S}(\mathcal{B})}\mathrm{id}_{\mathrm{S}(\mathcal{B}),\mathrm{S}(H)(A_3)}} & & \downarrow{\scriptstyle \mathrm{id}_{\mathrm{S}(\mathcal{B}),\mathrm{S}(H)(A_1)}\otimes_{\mathrm{S}(\mathcal{B})}\mathrm{S}(H)_{\otimes,A_2,A_3}} \\ \mathrm{S}(H)(A_1 \otimes_{\mathrm{S}(\mathcal{A})} A_2) \otimes_{\mathrm{S}(\mathcal{B})} \mathrm{S}(H)(A_3) & & \mathrm{S}(H)(A_1) \otimes_{\mathrm{S}(\mathcal{B})} \mathrm{S}(H)(A_2 \otimes_{\mathrm{S}(\mathcal{A})} A_3) \\ \downarrow{\scriptstyle \mathrm{S}(H)_{\otimes,A_1\otimes_{\mathrm{S}(\mathcal{A})}A_2,A_3}} & & \downarrow{\scriptstyle \mathrm{S}(H)_{\otimes,A_1,A_2\otimes_{\mathrm{S}(\mathcal{A})}A_3}} \\ \mathrm{S}(H)((A_1 \otimes_{\mathrm{S}(\mathcal{A})} A_2) \otimes_{\mathrm{S}(\mathcal{A})} A_3) & \xrightarrow[\mathrm{S}(H)(\alpha_{\mathrm{S}(\mathcal{A}),A_1,A_2,A_3})]{} & \mathrm{S}(H)(A_1 \otimes_{\mathrm{S}(\mathcal{A})} (A_2 \otimes_{\mathrm{S}(\mathcal{A})} A_3)) \end{array}$$

(e) *For any* $X \in \mathrm{obj}_{\mathrm{S}(\mathcal{A})}$,

$$\begin{aligned}\lambda_{\mathrm{S}(\mathcal{B}),\mathrm{S}(H)(X)} = \mathrm{S}(H)(\lambda_{\mathrm{S}(\mathcal{A}),X}) &\circ_{\mathrm{S}(\mathcal{B})} \mathrm{S}(H)_{\otimes,I_{\mathrm{S}(\mathcal{A})},X} \\ &\circ_{\mathrm{S}(\mathcal{B})} (\mathrm{S}(H)_I \otimes_{\mathrm{S}(\mathcal{B})} \mathrm{id}_{\mathrm{S}(\mathcal{B}),\mathrm{S}(H)(X)}).\end{aligned}$$

$$\begin{array}{ccc} I_{\mathrm{S}(\mathcal{B})} \otimes_{\mathrm{S}(\mathcal{B})} \mathrm{S}(H)(X) & \xrightarrow{\mathrm{S}(H)_I \otimes_{\mathrm{S}(\mathcal{B})} \mathrm{id}_{\mathrm{S}(\mathcal{B}),\mathrm{S}(H)(X)}} & \mathrm{S}(H)(I_{\mathrm{S}(\mathcal{A})}) \otimes_{\mathrm{S}(\mathcal{B})} \mathrm{S}(H)(X) \\ {\scriptstyle \lambda_{\mathrm{S}(\mathcal{B}),\mathrm{S}(H)(X)}} \downarrow & & \downarrow {\scriptstyle \mathrm{S}(H)_{\otimes, I_{\mathrm{S}(\mathcal{A})}, X}} \\ \mathrm{S}(H)(X) & \xleftarrow[\mathrm{S}(H)(\lambda_{\mathrm{S}(\mathcal{A}),X})]{} & \mathrm{S}(H)(I_{\mathrm{S}(\mathcal{A})} \otimes_{\mathrm{S}(\mathcal{B})} X) \end{array}$$

(f) For any $X \in \mathrm{obj}_{\mathrm{S}(\mathcal{A})}$,

$$\rho_{\mathrm{S}(\mathcal{B}),\mathrm{S}(H)(X)} = \mathrm{S}(H)(\rho_{\mathrm{S}(\mathcal{A}),X}) \circ_{\mathrm{S}(\mathcal{B})} \mathrm{S}(H)_{\otimes,X,I_{\mathrm{S}(\mathcal{A})}} \circ_{\mathrm{S}(\mathcal{B})} (\mathrm{id}_{\mathrm{S}(\mathcal{B}),\mathrm{S}(H)(X)} \otimes_{\mathrm{S}(\mathcal{B})} \mathrm{S}(H)_I).$$

$$\begin{array}{ccc} \mathrm{S}(H)(X) \otimes_{\mathrm{S}(\mathcal{B})} I_{\mathrm{S}(\mathcal{B})} & \xrightarrow{\mathrm{id}_{\mathrm{S}(\mathcal{B}),\mathrm{S}(H)(X)} \otimes_{\mathrm{S}(\mathcal{B})} \mathrm{S}(H)_I} & \mathrm{S}(H)(X) \otimes_{\mathrm{S}(\mathcal{B})} \mathrm{S}(H)(I_{\mathrm{S}(\mathcal{A})}) \\ {\scriptstyle \rho_{\mathrm{S}(\mathcal{B}),\mathrm{S}(H)(X)}} \downarrow & & \downarrow {\scriptstyle \mathrm{S}(H)_{\otimes, X, I_{\mathrm{S}(\mathcal{A})}}} \\ \mathrm{S}(H)(X) & \xleftarrow[\mathrm{S}(H)(\rho_{\mathrm{S}(\mathcal{A}),X})]{} & \mathrm{S}(H)(X \otimes_{\mathrm{S}(\mathcal{B})} I_{\mathrm{S}(\mathcal{A})}) \end{array}$$

PROOF. (a) Since H is by assumption a strong monoidal functor the morphism $H_{\otimes,A_1,A_2}$ is invertible in $\mathcal{B}$. Hence, the definition $\mathrm{S}(H)_{\otimes,A_1,A_2} = \Theta_{\mathcal{B}}(H_{\otimes,A_1,A_2})$ and Proposition 5.12 prove the assertion.

(b) For each $i \in \{1,2\}$ let $(f_i, g_i) \in x_i$ be arbitrary and let X_i its base. Unwinding the definitions yields

$$\begin{aligned} &\mathrm{S}(H)_{\otimes,B_1,B_2} \circ_{\mathrm{S}(\mathcal{B})} (\mathrm{S}(H)(x_1) \otimes_{\mathrm{S}(\mathcal{B})} \mathrm{S}(H)(x_2)) \\ &\quad = [\mathrm{id}_{\mathcal{B},H(B_1)\otimes_{\mathcal{B}} H(B_2)}, H_{\otimes,B_1,B_2}] \circ_{\mathrm{S}(\mathcal{B})} [H(f_1) \otimes_{\mathcal{B}} H(f_2), H(g_1) \otimes_{\mathcal{B}} H(g_2)]. \end{aligned}$$

Because the span $(\mathrm{id}_{\mathcal{B},H(X_1)\otimes_{\mathcal{B}} H(X_2)}, H(g_1) \otimes_{\mathcal{B}} H(g_2))$ is a pull-back of the co-span $(H(g_1) \otimes_{\mathcal{B}} H(g_2), \mathrm{id}_{\mathcal{B},H(B_1)\otimes_{\mathcal{B}} H(B_2)})$ in $\mathcal{B}$ by Lemma 4.6 the above is identical to

$$\begin{aligned} &[(H(f_1) \otimes_{\mathcal{B}} H(f_2)) \circ_{\mathcal{B}} \mathrm{id}_{\mathcal{B},H(X_1)\otimes_{\mathcal{B}} H(X_2)}, H_{\otimes,B_1,B_2} \circ_{\mathcal{B}} (H(g_1) \otimes_{\mathcal{B}} H(g_2))] \\ &\quad = [\mathrm{id}_{\mathcal{B},H(A_1)\otimes_{\mathcal{B}} H(A_2)} \circ_{\mathcal{B}} (H(f_1) \otimes_{\mathcal{B}} H(f_2)), H(g_1 \otimes_{\mathcal{A}} g_2) \circ_{\mathcal{B}} H_{\otimes,X_1,X_2}]. \end{aligned}$$

Because H is a strong monoidal functor $H_{\otimes}$ is a natural isomorphism and thus equifibered by Lemma 4.7. Consequently, $(H(f_1) \otimes_{\mathcal{B}} H(f_2), H_{\otimes,X_1,X_2})$ is a pull-back of $(H_{\otimes,A_1,A_2}, H(f_1 \otimes_{\mathcal{A}} f_2))$ in $\mathcal{B}$, which lends the following form to the previous expression:

$$\begin{aligned} &[H(f_1 \otimes_{\mathcal{A}} f_2), H(g_1 \otimes_{\mathcal{A}} g_2)] \circ_{\mathrm{S}(\mathcal{B})} [\mathrm{id}_{\mathcal{B},H(A_1)\otimes_{\mathcal{B}} H(A_2)}, H_{\otimes,A_1,A_2}] \\ &\quad = \mathrm{S}(H)(x_1 \otimes_{\mathrm{S}(H)\mathcal{A}} x_2) \circ_{\mathrm{S}(\mathcal{B})} \mathrm{S}(H)_{\otimes,A_1,A_2}. \end{aligned}$$

That is what we needed to see.

(c) Again, the assumption that H is strong monoidal makes H_I invertible in $\mathcal{B}$ and thus $\mathrm{S}(H)_I = \Theta_{\mathcal{B}}(H_I)$ invertible in $\mathrm{S}(\mathcal{B})$ by Proposition 5.12.

(d) Proposition 5.26, and Lemma 5.20 allow a quick computation:

$$\begin{aligned}
&\mathrm{S}(H)(\alpha_{\mathrm{S}(\mathcal{A}),A_1,A_2,A_3}) \circ_{\mathrm{S}(\mathcal{B})} \mathrm{S}(H)_{\otimes,A_1\otimes_{\mathrm{S}(\mathcal{A})}A_2,A_3} \\
&\qquad\qquad \circ_{\mathrm{S}(\mathcal{B})} (\mathrm{S}(H)_{\otimes,A_1,A_2} \otimes_{\mathrm{S}(\mathcal{B})} \mathrm{id}_{\mathrm{S}(\mathcal{B}),\mathrm{S}(H)(A_3)}) \\
&= \mathrm{S}(H)(\Theta_{\mathcal{A}}(\alpha_{\mathcal{A},A_1,A_2,A_3})) \circ_{\mathrm{S}(\mathcal{B})} \Theta_{\mathcal{B}}(H_{\otimes,A_1\otimes_{\mathcal{A}}A_2,A_3}) \\
&\qquad\qquad \circ_{\mathrm{S}(\mathcal{B})} (\Theta_{\mathcal{B}}(H_{\otimes,A_1,A_2}) \otimes_{\mathrm{S}(\mathcal{B})} \Theta_{\mathcal{B}}(\mathrm{id}_{\mathcal{B},H(A_3)})) \\
&\underset{5.20}{\overset{5.26}{=}} \Theta_{\mathcal{B}}(H(\alpha_{\mathcal{A},A_1,A_2,A_3})) \circ_{\mathrm{S}(\mathcal{B})} \Theta_{\mathcal{B}}(H_{\otimes,A_1\otimes_{\mathcal{A}}A_2,A_3}) \circ_{\mathrm{S}(\mathcal{B})} \Theta_{\mathcal{B}}(H_{\otimes,A_1,A_2} \otimes_{\mathcal{B}} \mathrm{id}_{\mathcal{B},H(A_3)}) \\
&\overset{5.12}{=} \Theta_{\mathcal{B}}(H(\alpha_{\mathcal{A},A_1,A_2,A_3}) \circ_{\mathcal{B}} H_{\otimes,A_1\otimes_{\mathcal{A}}A_2,A_3} \circ_{\mathcal{B}} (H_{\otimes,A_1,A_2} \otimes_{\mathcal{B}} \mathrm{id}_{\mathcal{B},H(A_3)})) \\
&= \Theta_{\mathcal{B}}(H_{\otimes,A_1,A_2\otimes_{\mathcal{A}}A_3} \circ_{\mathcal{B}} (\mathrm{id}_{\mathcal{B},H(A_1)} \otimes_{\mathcal{B}} H_{\otimes,A_2,A_3}) \circ_{\mathcal{B}} \alpha_{\mathcal{B},H(A_1),H(A_2),H(A_3)}) \\
&\overset{5.12}{=} \Theta_{\mathcal{B}}(H_{\otimes,A_1,A_2\otimes_{\mathcal{A}}A_3}) \circ_{\mathrm{S}(\mathcal{B})} \Theta_{\mathcal{B}}(\mathrm{id}_{\mathcal{B},H(A_1)} \otimes_{\mathcal{B}} H_{\otimes,A_2,A_3}) \circ_{\mathrm{S}(\mathcal{B})} \Theta_{\mathcal{B}}(\alpha_{\mathcal{B},H(A_1),H(A_2),H(A_3)}) \\
&\overset{5.26}{=} \Theta_{\mathcal{B}}(H_{\otimes,A_1,A_2\otimes_{\mathcal{A}}A_3}) \circ_{\mathrm{S}(\mathcal{B})} (\Theta_{\mathcal{B}}(\mathrm{id}_{\mathcal{B},H(A_1)}) \otimes_{\mathrm{S}(\mathcal{B})} \Theta_{\mathcal{B}}(H_{\otimes,A_2,A_3})) \\
&\qquad\qquad \circ_{\mathrm{S}(\mathcal{B})} \Theta_{\mathcal{B}}(\alpha_{\mathcal{B},H(A_1),H(A_2),H(A_3)}) \\
&= \mathrm{S}(H)_{\otimes,A_1,A_2\otimes_{\mathrm{S}(\mathcal{A})}A_3} \circ_{\mathrm{S}(\mathcal{B})} (\mathrm{id}_{\mathrm{S}(\mathcal{B}),\mathrm{S}(H)(A_1)} \otimes_{\mathrm{S}(\mathcal{B})} \mathrm{S}(H)_{\otimes,A_2,A_3}) \\
&\qquad\qquad \circ_{\mathrm{S}(\mathcal{B})} \alpha_{\mathrm{S}(\mathcal{B}),\mathrm{S}(H)(A_1),\mathrm{S}(H)(A_2),\mathrm{S}(H)(A_3)},
\end{aligned}$$

where the fourth step is enabled by H being a monoidal functor from $\mathcal{A}$ to $\mathcal{B}$.

(e) Similarly to Part (d), we compute with the help of Proposition 5.26, and Lemma 5.20:

$$\begin{aligned}
&\mathrm{S}(H)(\lambda_{\mathrm{S}(\mathcal{A}),X}) \circ_{\mathrm{S}(\mathcal{B})} \mathrm{S}(H)_{\otimes,I_{\mathrm{S}(\mathcal{A})},X} \circ_{\mathrm{S}(\mathcal{B})} (\mathrm{S}(H)_I \otimes_{\mathrm{S}(\mathcal{B})} \mathrm{id}_{\mathrm{S}(\mathcal{B}),\mathrm{S}(H)(X)}) \\
&= \mathrm{S}(H)(\Theta_{\mathcal{A}}(\lambda_{\mathcal{A},X})) \circ_{\mathrm{S}(\mathcal{B})} \Theta_{\mathcal{B}}(H_{\otimes,I_{\mathcal{A}},X}) \circ_{\mathrm{S}(\mathcal{B})} (\Theta_{\mathcal{B}}(H_I) \otimes_{\mathrm{S}(\mathcal{B})} \Theta_{\mathcal{B}}(\mathrm{id}_{\mathcal{B},H(X)})) \\
&\underset{5.20}{\overset{5.26}{=}} \Theta_{\mathcal{B}}(H(\lambda_{\mathcal{A},X})) \circ_{\mathrm{S}(\mathcal{B})} \Theta_{\mathcal{B}}(H_{\otimes,I_{\mathcal{A}},X}) \circ_{\mathrm{S}(\mathcal{B})} \Theta_{\mathcal{B}}(H_I \otimes_{\mathcal{B}} \mathrm{id}_{\mathcal{B},H(X)}) \\
&\overset{5.12}{=} \Theta_{\mathcal{B}}(H(\lambda_{\mathcal{A},X}) \circ_{\mathcal{B}} H_{\otimes,I_{\mathcal{A}},X} \circ_{\mathcal{B}} (H_I \otimes_{\mathcal{B}} \mathrm{id}_{\mathcal{B},H(X)})) \\
&= \Theta_{\mathcal{B}}(\lambda_{\mathcal{B},H(X)}) \\
&= \lambda_{\mathrm{S}(\mathcal{B}),\mathrm{S}(H)(X)},
\end{aligned}$$

where the fourth step is due to H being a monoidal functor from $\mathcal{A}$ to $\mathcal{B}$.

(f) The proof is analogous to that of Part (e):

$$\begin{aligned}
&\mathrm{S}(H)(\rho_{\mathrm{S}(\mathcal{A}),X}) \circ_{\mathrm{S}(\mathcal{B})} \mathrm{S}(H)_{\otimes,X,I_{\mathrm{S}(\mathcal{A})}} \circ_{\mathrm{S}(\mathcal{B})} (\mathrm{id}_{\mathrm{S}(\mathcal{B}),\mathrm{S}(H)(X)} \otimes_{\mathrm{S}(\mathcal{B})} \mathrm{S}(H)_I) \\
&= \mathrm{S}(H)(\Theta_{\mathcal{A}}(\rho_{\mathcal{A},X})) \circ_{\mathrm{S}(\mathcal{B})} \Theta_{\mathcal{B}}(H_{\otimes,X,I_{\mathcal{A}}}) \circ_{\mathrm{S}(\mathcal{B})} (\Theta_{\mathcal{B}}(\mathrm{id}_{\mathcal{B},H(X)}) \otimes_{\mathrm{S}(\mathcal{B})} \Theta_{\mathcal{B}}(H_I)) \\
&\underset{5.20}{\overset{5.26}{=}} \Theta_{\mathcal{B}}(H(\rho_{\mathcal{A},X})) \circ_{\mathrm{S}(\mathcal{B})} \Theta_{\mathcal{B}}(H_{\otimes,X,I_{\mathcal{A}}}) \circ_{\mathrm{S}(\mathcal{B})} \Theta_{\mathcal{B}}(\mathrm{id}_{\mathcal{B},H(X)} \otimes_{\mathcal{B}} H_I) \\
&\overset{5.12}{=} \Theta_{\mathcal{B}}(H(\rho_{\mathcal{A},X}) \circ_{\mathcal{B}} H_{\otimes,X,I_{\mathcal{A}}} \circ_{\mathcal{B}} (\mathrm{id}_{\mathcal{B},H(X)} \otimes_{\mathcal{B}} H_I)) \\
&= \Theta_{\mathcal{B}}(\rho_{\mathcal{B},H(X)}) \\
&= \rho_{\mathrm{S}(\mathcal{B}),\mathrm{S}(H)(X)},
\end{aligned}$$

where, again, we can make the fourth step becausee H is a monoidal functor from $\mathcal{A}$ to $\mathcal{B}$. □

PROPOSITION 5.29. *$\mathrm{S}(H)$, when equipped with $(\mathrm{S}(H)_{\otimes}, \mathrm{S}(H)_{I})$, is a strong monoidal functor from $\mathrm{S}(\mathcal{A})$ to $\mathrm{S}(\mathcal{B})$ for any 0-cells $\mathcal{A}$ and $\mathcal{B}$ and any 1-cell H from $\mathcal{A}$ to $\mathcal{B}$ of $\mathsf{smCAT}^{\mathrm{cart,fc}}_{\mathrm{sos}}$*

PROOF. Implied by the entirety of Lemma 5.28. □

LEMMA 5.30. *For any 0-cells $\mathcal{A}$ and $\mathcal{B}$, any 1-cells H and K from $\mathcal{A}$ to $\mathcal{B}$ and any 2-cell η from H to K of $\mathsf{smCAT}^{\mathrm{cart,fc}}_{\mathrm{sos}}$ the following hold:*

(a) For any $\{A_1, A_2\} \subseteq \mathrm{obj}_{\mathrm{S}(\mathcal{A})}$,

$$\mathrm{S}(K)_{\otimes,A_1,A_2} \circ_{\mathrm{S}(\mathcal{B})} (\mathrm{S}(\eta)_{A_1} \otimes_{\mathrm{S}(\mathcal{B})} \mathrm{S}(\eta)_{A_2}) = \mathrm{S}(\eta)_{A_1 \otimes_{\mathrm{S}(\mathcal{A})} A_2} \circ_{\mathrm{S}(\mathcal{B})} \mathrm{S}(H)_{\otimes,A_1,A_2}.$$

$$\begin{array}{ccc}
\mathrm{S}(H)(A_1) \otimes_{\mathrm{S}(\mathcal{B})} \mathrm{S}(H)(A_2) & \xrightarrow{\mathrm{S}(\eta)_{A_1} \otimes_{\mathrm{S}(\mathcal{B})} \mathrm{S}(\eta)_{A_2}} & \mathrm{S}(K)(A_1) \otimes_{\mathrm{S}(\mathcal{B})} \mathrm{S}(K)(A_2) \\
\Big\downarrow {\scriptstyle \mathrm{S}(H)_{\otimes,A_1,A_2}} & & \Big\downarrow {\scriptstyle \mathrm{S}(K)_{\otimes,A_1,A_2}} \\
\mathrm{S}(H)(A_1 \otimes_{\mathrm{S}(\mathcal{A})} A_2) & \xrightarrow[\mathrm{S}(\eta)_{A_1 \otimes_{\mathrm{S}(\mathcal{A})} A_2}]{} & \mathrm{S}(K)(A_1 \otimes_{\mathrm{S}(\mathcal{A})} A_2)
\end{array}$$

(b) $\mathrm{S}(K)_{I} = \mathrm{S}(\eta)_{I_{\mathrm{S}(\mathcal{A})}} \circ_{\mathrm{S}(\mathcal{B})} \mathrm{S}(H)_{I}$.

$$\begin{array}{ccc}
 & I_{\mathrm{S}(\mathcal{B})} & \\
{\scriptstyle \mathrm{S}(H)_I} \swarrow & & \searrow {\scriptstyle \mathrm{S}(K)_I} \\
\mathrm{S}(H)(I_{\mathrm{S}(\mathcal{A})}) & \xrightarrow[\mathrm{S}(\eta)_{I_{\mathrm{S}(\mathcal{A})}}]{} & \mathrm{S}(K)(I_{\mathrm{S}(\mathcal{A})})
\end{array}$$

PROOF. (a) Direct computation, relying on Proposition 5.26, shows:

$$\begin{aligned}
&\mathrm{S}(K)_{\otimes,A_1,A_2} \circ_{\mathrm{S}(\mathcal{B})} (\mathrm{S}(\eta)_{A_1} \otimes_{\mathrm{S}(\mathcal{B})} \mathrm{S}(\eta)_{A_2}) \\
&\quad = \Theta_{\mathcal{B}}(K_{\otimes,A_1,A_2}) \circ_{\mathrm{S}(\mathcal{B})} (\Theta_{\mathcal{B}}(\eta_{A_1}) \otimes_{\mathrm{S}(\mathcal{B})} \Theta_{\mathcal{B}}(\eta_{A_2})) \\
&\quad \overset{5.26}{=} \Theta_{\mathcal{B}}(K_{\otimes,A_1,A_2} \circ_{\mathcal{B}} (\eta_{A_1} \otimes_{\mathcal{B}} \eta_{A_2})) \\
&\quad = \Theta_{\mathcal{B}}(\eta_{A_1 \otimes_{\mathcal{A}} A_2} \circ_{\mathcal{B}} H_{\otimes,A_1,A_2}) \\
&\quad \overset{5.12}{=} \Theta_{\mathcal{B}}(\eta_{A_1 \otimes_{\mathcal{A}} A_2}) \circ_{\mathrm{S}(\mathcal{A})} \Theta_{\mathcal{B}}(H_{\otimes,A_1,A_2}) \\
&\quad = \mathrm{S}(\eta)_{A_1 \otimes_{\mathrm{S}(\mathcal{A})} A_2} \circ_{\mathrm{S}(\mathcal{B})} \mathrm{S}(H)_{\otimes,A_1,A_2},
\end{aligned}$$

where the third step is enabled by η being a monoidal natural transformation from H to K.

(b) Again, we can infer immediately, with the help of Proposition 5.26,

$$\begin{aligned}
\mathrm{S}(\eta)_{I_{\mathrm{S}(\mathcal{A})}} \circ_{\mathrm{S}(\mathcal{B})} \mathrm{S}(H)_{I} &= \Theta_{\mathcal{B}}(\eta_{I_{\mathcal{A}}}) \circ_{\mathrm{S}(\mathcal{B})} \Theta_{\mathcal{B}}(H_I) \\
&\overset{5.26}{=} \Theta_{\mathcal{B}}(\eta_{I_{\mathcal{A}}} \circ_{\mathcal{B}} H_I) \\
&= \Theta_{\mathcal{B}}(K_I) \\
&= \mathrm{S}(K)_{I},
\end{aligned}$$

where we have used the fact that η is a monoidal natural transformation from H to K in the third step. □

PROPOSITION 5.31. $\mathrm{S}(\eta)$ *is a monoidal natural transformation of monoida functors from* $\mathrm{S}(\mathcal{A})$ *to* $\mathrm{S}(\mathcal{B})$ *from* $\mathrm{S}(H)$ *to* $\mathrm{S}(K)$ *for any* 0*-cells* $\mathcal{A}$ *and* $\mathcal{B}$*, any* 1*-cells* H *and* K *from* $\mathcal{A}$ *to* $\mathcal{B}$ *and any* 2*-cell* η *from* H *to* K *of* $\mathsf{smCAT}^{\mathrm{cart,fc}}_{\mathrm{sos}}$.

PROOF. The claim summarizes Lemma 5.30. □

5.1.3. *Symmetric Monoidal Category.* The monoidal structure defined on the span category inherits its symmetric nature from the cartesian monoidal input category.

DEFINITION 5.32. For any $\{A, B\} \subseteq \mathrm{obj}_{\mathrm{S}(\mathcal{A})}$ define $\gamma_{\mathrm{S}(\mathcal{A}),A,B} \coloneqq \Theta_{\mathcal{A}}(\gamma_{\mathcal{A},A,B})$ for any 0-cell $\mathcal{A}$ of $\mathsf{smCAT}^{\mathrm{cart,fc}}_{\mathrm{sos}}$.

LEMMA 5.33. *Let* $\mathcal{A}$ *be any* 0*-cell of* $\mathsf{smCAT}^{\mathrm{cart,fc}}_{\mathrm{sos}}$ *and abbreviate* $\mathrm{S}(\mathcal{A})$ *by* S.

(a) γ_{S} *is a natural isomorphism of functors* $\mathrm{S} \otimes_{\mathsf{CAT}} \mathrm{S} \to \mathrm{S}$ *from* $(\cdot_1) \otimes_{\mathrm{S}} (\cdot_2)$ *to* $(\cdot_2) \otimes_{\mathrm{S}} (\cdot_1)$.

(b) For any objects A*,* B *and* C *of* S *commutative diagrams in* S *are given by*

$$\begin{array}{ccccc} (A \otimes B) \otimes C & \xrightarrow{\alpha_{A,B,C}} & A \otimes (B \otimes C) & \xrightarrow{\gamma_{A,B\otimes C}} & (B \otimes C) \otimes A \\ {\scriptstyle \gamma_{A,B}\otimes \mathrm{id}_C}\downarrow & & & & \downarrow{\scriptstyle \alpha_{B,C,A}} \\ (B \otimes A) \otimes C & \xrightarrow[\alpha_{B,A,C}]{} & B \otimes (A \otimes C) & \xrightarrow[\mathrm{id}_B \otimes \gamma_{A,C}]{} & B \otimes (C \otimes A) \end{array}$$

and

$$\begin{array}{ccccc} A \otimes (B \otimes C) & \xrightarrow{\alpha^{-1}_{A,B,C}} & (A \otimes B) \otimes C & \xrightarrow{\gamma_{A\otimes B,C}} & C \otimes (A \otimes B) \\ {\scriptstyle \mathrm{id}_A\otimes \gamma_{B,C}}\downarrow & & & & \downarrow{\scriptstyle \alpha^{-1}_{C,A,B}} \\ A \otimes (C \otimes B) & \xrightarrow[\alpha^{-1}_{A,C,B}]{} & (A \otimes C) \otimes B & \xrightarrow[\gamma_{A,C} \otimes \mathrm{id}_B]{} & (C \otimes A) \otimes B \end{array}.$$

(c) $\gamma_{\mathrm{S},B,A} \circ_{\mathrm{S}} \gamma_{\mathrm{S},A,B} = \mathrm{id}_{\mathrm{S},A\otimes_{\mathrm{S}} B}$ *for any objects* A *and* B *of* S.

PROOF. (a) Proposition 5.12 ensures that γ_{S} is made up of isomorphisms because $\gamma_{\mathcal{A}}$ is. What we have to prove is that γ_{S} is a natural transformation. For each $i \in \{1, 2\}$ let (f_i, g_i) be a span in $\mathcal{A}$ from A_i to B_i with base object X_i. Lemma 4.6 implies that a pull-back of $(g_1 \otimes_{\mathcal{A}} g_2, \mathrm{id}_{\mathcal{A},B_1\otimes_{\mathcal{A}} B_2})$ is given by $(\mathrm{id}_{\mathcal{A},X_1\otimes_{\mathcal{A}}X_2}, g_1 \otimes_{\mathcal{A}} g_2)$. Hence, by Lemmata 5.5 and 5.21,

$$\begin{aligned} &\gamma_{\mathrm{S},B_1,B_2} \circ_{\mathrm{S}} ([f_1, g_1] \otimes_{\mathrm{S}} [f_2, g_2]) \\ &\qquad = [\mathrm{id}_{\mathcal{A},B_1\otimes_{\mathcal{A}}B_2}, \gamma_{\mathcal{A},B_1,B_2}] \circ_{\mathrm{S}} [f_1 \otimes_{\mathcal{A}} f_2, g_1 \otimes_{\mathcal{A}} g_2] \\ &\qquad = [(f_1 \otimes_{\mathcal{A}} f_2) \circ_{\mathcal{A}} \mathrm{id}_{\mathcal{A},X_1\otimes_{\mathcal{A}}X_2}, \gamma_{\mathcal{A},B_1,B_2} \circ_{\mathcal{A}} (g_1 \otimes_{\mathcal{A}} g_2)] \\ &\qquad = [\mathrm{id}_{\mathcal{A},A_1\otimes_{\mathcal{A}}A_2} \circ (f_1 \otimes_{\mathcal{A}} f_2), (g_2 \otimes_{\mathcal{A}} g_1) \circ \gamma_{\mathcal{A},X_1,X_2}], \end{aligned}$$

where we have used that $\gamma_{\mathcal{A}}$ is a natural transformation $(\cdot_1) \otimes_{\mathcal{A}} (\cdot_2) \to (\cdot_2) \otimes_{\mathcal{A}} (\cdot_1)$.

Because $(f_1 \otimes_{\mathcal{A}} f_2, \gamma_{\mathcal{A},X_1,X_2})$ is a pull-back of $(\gamma_{\mathcal{A},A_1,A_2}, f_2 \otimes_{\mathcal{A}} f_1)$ by Lemma 4.16 it follows

$$\begin{aligned} &= [f_2 \otimes_{\mathcal{A}} f_1, g_2 \otimes_{\mathcal{A}} g_1] \circ_{\mathrm{S}} [\mathrm{id}_{\mathcal{A},A_1 \otimes_{\mathcal{A}} A_2}, \gamma_{\mathcal{A},A_1,A_2}] \\ &= ([f_2, g_2] \otimes_{\mathrm{S}} [f_1, g_1]) \circ_{\mathrm{S}} \gamma_{\mathrm{S},A_1,A_2} \end{aligned}$$

which is what we needed to prove.

(b) For any objects X, Y and Z of $\mathcal{A}$ Proposition 5.12 implies that $\alpha^{-1}_{\mathrm{S},X,Y,Z} = \Theta_{\mathcal{A}}(\alpha_{\mathcal{A},X,Y,Z})^{-1} = \Theta_{\mathcal{A}}(\alpha^{-1}_{\mathcal{A},X,Y,Z})$ and, together with Lemma 5.23, that $\gamma_{\mathrm{S},X,Y} \otimes_{\mathrm{S}} \mathrm{id}_{\mathrm{S},Z} = \Theta_{\mathcal{A}}(\gamma_{\mathcal{A},X,Y}) \otimes_{\mathrm{S}} \Theta_{\mathcal{A}}(\mathrm{id}_{\mathcal{A},Z}) = \Theta_{\mathcal{A}}(\gamma_{\mathcal{A},X,Y} \otimes_{\mathcal{A}} \mathrm{id}_{\mathcal{A},Z})$ and, likewise, $\mathrm{id}_{\mathrm{S},Z} \otimes_{\mathrm{S}} \gamma_{\mathrm{S},X,Y} = \Theta_{\mathcal{A}}(\mathrm{id}_{\mathcal{A},Z} \otimes_{\mathcal{A}} \gamma_{\mathcal{A},X,Y})$. Hence, what we claim is that the images of the corresponding diagrams of $\mathcal{A}$ under $\Theta_{\mathcal{A}}$ commute. And because $\Theta_{\mathcal{A}}$ is a functor by Proposition 5.12 this is indeed true.

(c) Proposition 5.12 lets us compute $\gamma_{\mathrm{S},B,A} \circ_{\mathrm{S}} \gamma_{\mathrm{S},A,B} = \Theta_{\mathcal{A}}(\gamma_{\mathcal{A},B,A}) \circ_{\mathrm{S}} \Theta_{\mathcal{A}}(\gamma_{\mathcal{A},A,B}) = \Theta_{\mathcal{A}}(\gamma_{\mathcal{A},B,A} \circ_{\mathcal{A}} \gamma_{\mathcal{A},A,B}) = \Theta_{\mathcal{A}}(\mathrm{id}_{\mathcal{A},A \otimes_{\mathcal{A}} B}) = \mathrm{id}_{\mathrm{S},A \otimes_{\mathrm{S}} B}$, where we have used that $\mathcal{A}$ is symmetric. □

Proposition 5.34. *$\gamma_{\mathrm{S}(\mathcal{A})}$ is a symmetric braiding for the monoidal category $\mathrm{S}(\mathcal{A})$ for any 0-cell $\mathcal{A}$ of $\mathsf{smCAT}^{\mathrm{cart,fc}}_{\mathrm{sos}}$.*

Proof. The claim is the conclusion from all parts of Lemma 5.33 taken together. □

Lemma 5.35. *For any 0-cells $\mathcal{A}$ and $\mathcal{B}$ and any 1-cell H from $\mathcal{A}$ to $\mathcal{B}$ in $\mathsf{smCAT}^{\mathrm{cart,fc}}_{\mathrm{sos}}$ it holds for any $\{A_1, A_2\} \subseteq \mathrm{obj}_{\mathrm{S}(\mathcal{A})}$ that*

$$\mathrm{S}(H)_{\otimes,A_2,A_1} \circ_{\mathrm{S}(\mathcal{B})} \gamma_{\mathrm{S}(\mathcal{B}),\mathrm{S}(H)(A_1),\mathrm{S}(H)(A_2)} = \mathrm{S}(H)(\gamma_{\mathrm{S}(\mathcal{A}),A_1,A_2}) \circ_{\mathrm{S}(\mathcal{B})} \mathrm{S}(H)_{\otimes,A_1,A_2}.$$

$$\begin{array}{ccc} \mathrm{S}(H)(A_1) \otimes_{\mathrm{S}(\mathcal{B})} \mathrm{S}(H)(A_2) & \xrightarrow{\gamma_{\mathrm{S}(\mathcal{B}),\mathrm{S}(H)(A_1),\mathrm{S}(H)(A_2)}} & \mathrm{S}(H)(A_2) \otimes_{\mathrm{S}(\mathcal{B})} \mathrm{S}(H)(A_1) \\ \downarrow{\scriptstyle \mathrm{S}(H)_{\otimes,A_1,A_2}} & & \downarrow{\scriptstyle \mathrm{S}(H)_{\otimes,A_2,A_1}} \\ \mathrm{S}(H)(A_1 \otimes_{\mathrm{S}(\mathcal{A})} A_2) & \xrightarrow[\mathrm{S}(H)(\gamma_{\mathrm{S}(\mathcal{A}),A_1,A_2})]{} & \mathrm{S}(H)(A_2 \otimes_{\mathrm{S}(\mathcal{A})} A_1) \end{array}$$

Proof. The proof is a direct computation employing Lemma 5.20:

$$\begin{aligned} &\mathrm{S}(H)_{\otimes,A_2,A_1} \circ_{\mathrm{S}(\mathcal{B})} \gamma_{\mathrm{S}(\mathcal{B}),\mathrm{S}(H)(A_1),\mathrm{S}(H)(A_2)} \\ &\quad = \Theta_{\mathcal{B}}(H_{\otimes,A_2,A_1}) \circ_{\mathrm{S}(\mathcal{B})} \Theta_{\mathcal{B}}(\gamma_{\mathcal{B},H(A_1),H(A_2)}) \\ &\quad \overset{5.12}{=} \Theta_{\mathcal{B}}(H_{\otimes,A_2,A_1} \circ_{\mathcal{B}} \gamma_{\mathcal{B},H(A_1),H(A_2)}) \\ &\quad = \Theta_{\mathcal{B}}(H(\gamma_{\mathcal{A},A_1,A_2}) \circ_{\mathcal{B}} H_{\otimes,A_1,A_2}) \\ &\quad \overset{5.12}{=} \Theta_{\mathcal{B}}(H(\gamma_{\mathcal{A},A_1,A_2})) \circ_{\mathrm{S}(\mathcal{B})} \Theta_{\mathcal{B}}(H_{\otimes,A_1,A_2}) \\ &\quad \overset{5.20}{=} \mathrm{S}(H)(\Theta_{\mathcal{A}}(\gamma_{\mathcal{A},A_1,A_2})) \circ_{\mathrm{S}(\mathcal{B})} \Theta_{\mathcal{B}}(H_{\otimes,A_1,A_2}) \\ &\quad = \mathrm{S}(H)(\gamma_{\mathrm{S}(\mathcal{A}),A_1,A_2}) \circ_{\mathrm{S}(\mathcal{B})} \mathrm{S}(H)_{\otimes,A_1,A_2}, \end{aligned}$$

where we have used that H is a symmetric monoidal functor from $\mathcal{A}$ to $\mathcal{B}$ in the third step. □

PROPOSITION 5.36. *$\mathrm{S}(H)$ is a symmetric monoidal functor from $\mathrm{S}(\mathcal{A})$ to $\mathrm{S}(\mathcal{B})$ for any 0-cells $\mathcal{A}$ and $\mathcal{B}$ and any 1-cell H from $\mathcal{A}$ to $\mathcal{B}$ of $\mathsf{smCAT}^{\mathrm{cart,fc}}_{\mathrm{sos}}$.*

PROOF. That is what Lemma 5.35 says in summary. □

The next result will again be a powerful tool in proving upcoming claims.

PROPOSITION 5.37. *$\Theta_{\mathcal{A}}$ is a symmetric monoidal functor $\mathcal{A} \to \mathrm{S}(\mathcal{A})$ for any 0-cell $\mathcal{A}$ of $\mathsf{smCAT}^{\mathrm{cart,fc}}_{\mathrm{sos}}$.*

PROOF. By Proposition 5.26, the monoidal functor $\Theta_{\mathcal{A}}\colon \mathcal{A} \to \mathrm{S}(\mathcal{A})$ is strict. Hence, we only have to show $\Theta_{\mathcal{A}}(\gamma_{\mathcal{A},A_1,A_2}) = \gamma_{\mathrm{S}(\mathcal{A}),\Theta_{\mathcal{A}}(A_1),\Theta_{\mathcal{A}}(A_2)}$ for any $\{A_1, A_2\} \subseteq \mathrm{obj}_{\mathcal{A}}$.

$$\begin{array}{ccc} \Theta(A_1)\otimes\Theta(A_2) & \xrightarrow{\gamma_{\Theta(A_1),\Theta(A_2)}} & \Theta(A_2)\otimes\Theta(A_1) \\ \| & & \| \\ \Theta(A_1\otimes A_2) & \xrightarrow[\Theta(\gamma_{A_1,A_2})]{} & \Theta(A_2\otimes A_1)\end{array}$$

However, because $\Theta_{\mathcal{A}}$ is the identity on objects, this is just the definition of $\gamma_{\mathrm{S}(\mathcal{A})}$. □

5.1.4. *Rigid Symmetric Monoidal Category.* Crucially, via terminal and diagonal morphisms in the cartesian monoidal input category the span category can be turned into a rigid symmetric monoidal category.

DEFINITION 5.38. For any 0-cell $\mathcal{A}$ of $\mathsf{smCAT}^{\mathrm{cart,fc}}_{\mathrm{sos}}$ and any $A \in \mathrm{obj}_{\mathrm{S}(\mathcal{A})}$ define

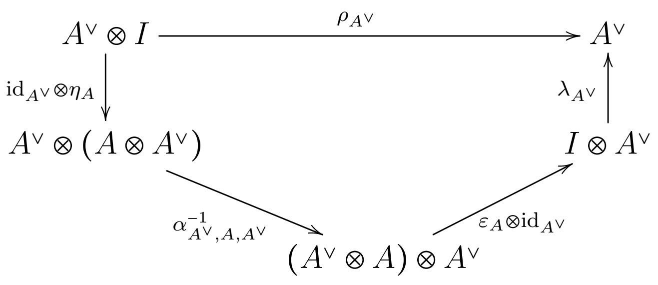
(a) $A^{\vee_{\mathrm{S}(\mathcal{A})}} := A$.
(b) $\varepsilon_{\mathrm{S}(\mathcal{A}),A} := [\mathrm{id}_{\mathcal{A},A} \times_{\mathcal{A}} \mathrm{id}_{\mathcal{A},A}, \omega_{\mathcal{A},A}]$, and
(c) $\eta_{\mathrm{S}(\mathcal{A}),A} := [\omega_{\mathcal{A},A}, \mathrm{id}_{\mathcal{A},A} \times_{\mathcal{A}} \mathrm{id}_{\mathcal{A},A}]$,

where $\omega_{\mathcal{A},A}$ is the unique morphism $A \to I_{\mathcal{A}}$ of $\mathcal{A}$.

LEMMA 5.39. *For any 0-cell $\mathcal{A}$ of $\mathsf{smCAT}^{\mathrm{cart,fc}}_{\mathrm{sos}}$ and for any object A of $\mathrm{S}(\mathcal{A})$ a commutative diagrams in $\mathrm{S}(\mathcal{A})$ are given by*

$$\begin{array}{ccccc} I\otimes A & & \xrightarrow{\lambda_A} & & A \\ \downarrow{\scriptstyle \eta_A\otimes \mathrm{id}_A} & & & & \uparrow{\scriptstyle\rho_A} \\ (A\otimes A^\vee)\otimes A & & & & A\otimes I \\ & \searrow{\scriptstyle \alpha_{A,A^\vee,A}} & & \nearrow{\scriptstyle \mathrm{id}_A\otimes\varepsilon_A} & \\ & & A\otimes(A^\vee\otimes A) & & \end{array}$$

and

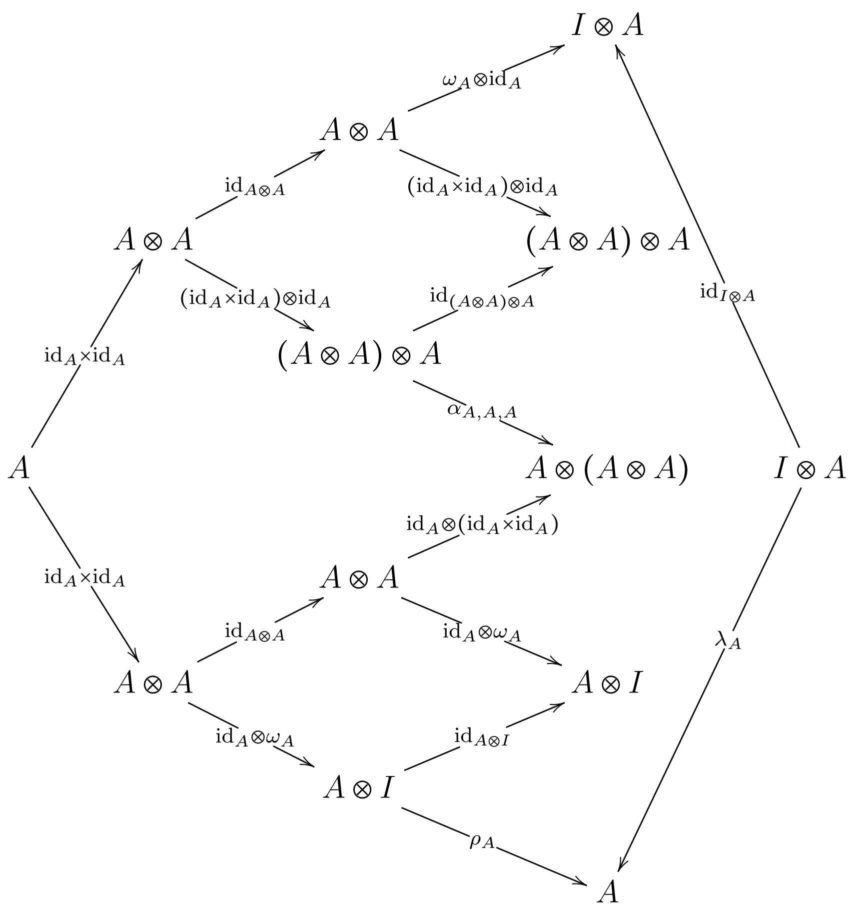


Proof. We treat the two diagrams separately. Abbreviate $\mathrm{S}(\mathcal{A})$ by S in the following.

First diagram. In order to verify our claim

$$\lambda_{\mathrm{S},A} \stackrel{!}{=} (\rho_{\mathrm{S},A} \circ_{\mathrm{S}} (\mathrm{id}_{\mathrm{S},A} \otimes_{\mathrm{S}} \varepsilon_{\mathrm{S},A})) \circ_{\mathrm{S}} (\alpha_{\mathrm{S},A,A^{\vee_{\mathrm{S}}},A} \circ_{\mathrm{S}} (\eta_{\mathrm{S},A} \otimes_{\mathrm{S}} \mathrm{id}_{\mathrm{S},A}))$$

we compute directly the composition on the right hand side. This we do by evaluating first the composition of the first two morphisms and that of the last two and then composing the results.



By definition, if $\omega_{\mathcal{A},A}$ is the unique morphism $A \to I_{\mathcal{A}}$,

$$\begin{aligned}
&\rho_{S,A} \circ_S (\mathrm{id}_{S,A} \otimes_S \varepsilon_{S,A}) \\
&\qquad = [\mathrm{id}_{\mathcal{A},A\otimes_{\mathcal{A}} I_{\mathcal{A}}}, \rho_{\mathcal{A},A}] \circ_S ([\mathrm{id}_{\mathcal{A},A}, \mathrm{id}_{\mathcal{A},A}] \otimes_S [\mathrm{id}_{\mathcal{A},A} \times_{\mathcal{A}} \mathrm{id}_{\mathcal{A},A}, \omega_{\mathcal{A},A}]) \\
&\qquad = [\mathrm{id}_{\mathcal{A},A\otimes_{\mathcal{A}} I_{\mathcal{A}}}, \rho_{\mathcal{A},A}] \circ_S [\mathrm{id}_{\mathcal{A},A} \otimes_{\mathcal{A}} (\mathrm{id}_{\mathcal{A},A} \times_{\mathcal{A}} \mathrm{id}_{\mathcal{A},A}), \mathrm{id}_{\mathcal{A},A} \otimes_{\mathcal{A}} \omega_{\mathcal{A},A}] \\
&\qquad = [\mathrm{id}_{\mathcal{A},A} \otimes_{\mathcal{A}} (\mathrm{id}_{\mathcal{A},A} \times_{\mathcal{A}} \mathrm{id}_{\mathcal{A},A}), \rho_{\mathcal{A},A} \circ_{\mathcal{A}} (\mathrm{id}_{\mathcal{A},A} \otimes_{\mathcal{A}} \omega_{\mathcal{A},A})],
\end{aligned}$$

where the last step is due to Lemma 4.6, which informs us that a pull-back of $(\mathrm{id}_{\mathcal{A},A} \otimes_{\mathcal{A}} \omega_{\mathcal{A},A}, \mathrm{id}_{\mathcal{A},A\otimes_{\mathcal{A}} I_{\mathcal{A}}})$ is given by $(\mathrm{id}_{\mathcal{A},A\otimes_{\mathcal{A}} A}, \mathrm{id}_{\mathcal{A},A} \otimes_{\mathcal{A}} \omega_{\mathcal{A},A})$.

Similarly,

$$\begin{aligned}
&\alpha_{S,A,A^{\vee_S},A} \circ_S (\eta_{S,A} \otimes_S \mathrm{id}_{S,A}) \\
&\qquad = [\mathrm{id}_{\mathcal{A},(A\otimes_{\mathcal{A}} A)\otimes_{\mathcal{A}} A}, \alpha_{\mathcal{A},A,A,A}] \circ_S ([\omega_{\mathcal{A},A}, \mathrm{id}_{\mathcal{A},A} \times_{\mathcal{A}} \mathrm{id}_{\mathcal{A},A}] \otimes_S [\mathrm{id}_{\mathcal{A},A}, \mathrm{id}_{\mathcal{A},A}]) \\
&\qquad = [\mathrm{id}_{\mathcal{A},(A\otimes_{\mathcal{A}} A)\otimes_{\mathcal{A}} A}, \alpha_{\mathcal{A},A,A,A}] \circ_S [\omega_{\mathcal{A},A} \otimes_{\mathcal{A}} \mathrm{id}_{\mathcal{A},A}, (\mathrm{id}_{\mathcal{A},A} \times_{\mathcal{A}} \mathrm{id}_{\mathcal{A},A}) \otimes_{\mathcal{A}} \mathrm{id}_{\mathcal{A},A}] \\
&\qquad = [\omega_{\mathcal{A},A} \otimes_{\mathcal{A}} \mathrm{id}_{\mathcal{A},A}, \alpha_{\mathcal{A},A,A,A} \circ_{\mathcal{A}} ((\mathrm{id}_{\mathcal{A},A} \times_{\mathcal{A}} \mathrm{id}_{\mathcal{A},A}) \otimes_{\mathcal{A}} \mathrm{id}_{\mathcal{A},A})],
\end{aligned}$$

where, again, the last step is justified by Lemma 4.6, according to which a pullback of $((\mathrm{id}_{\mathcal{A},A} \times_{\mathcal{A}} \mathrm{id}_{\mathcal{A},A}) \otimes_{\mathcal{A}} \mathrm{id}_{\mathcal{A},A}, \mathrm{id}_{\mathcal{A},(A\otimes_{\mathcal{A}} A)\otimes_{\mathcal{A}} A})$ is $(\mathrm{id}_{\mathcal{A},A\otimes_{\mathcal{A}} A}, (\mathrm{id}_{\mathcal{A},A} \times_{\mathcal{A}} \mathrm{id}_{\mathcal{A},A}) \otimes_{\mathcal{A}} \mathrm{id}_{\mathcal{A},A})$.

Now, we can compose these two results. By Lemma 4.14 (a) a pull-back of the pair $(\alpha_{\mathcal{A},A,A,A} \circ_{\mathcal{A}} ((\mathrm{id}_{\mathcal{A},A} \times_{\mathcal{A}} \mathrm{id}_{\mathcal{A},A}) \otimes_{\mathcal{A}} \mathrm{id}_{\mathcal{A},A}), \mathrm{id}_{\mathcal{A},A} \otimes_{\mathcal{A}} (\mathrm{id}_{\mathcal{A},A} \times_{\mathcal{A}} \mathrm{id}_{\mathcal{A},A}))$ is given by $(\mathrm{id}_{\mathcal{A},A} \times_{\mathcal{A}} \mathrm{id}_{\mathcal{A},A}, \mathrm{id}_{\mathcal{A},A} \times_{\mathcal{A}} \mathrm{id}_{\mathcal{A},A})$. Thus, we obtain for the final composition,

$$\begin{aligned}
&(\rho_{S,A} \circ_S (\mathrm{id}_{S,A} \otimes_S \varepsilon_{S,A})) \circ_S (\alpha_{S,A,A^{\vee_S},A} \circ_S (\eta_{S,A} \otimes_S \mathrm{id}_{S,A})) \\
&= [(\omega_{\mathcal{A},A} \otimes_{\mathcal{A}} \mathrm{id}_{\mathcal{A},A}) \circ_{\mathcal{A}} (\mathrm{id}_{\mathcal{A},A} \times_{\mathcal{A}} \mathrm{id}_{\mathcal{A},A}), (\rho_{\mathcal{A},A} \circ_{\mathcal{A}} (\mathrm{id}_{\mathcal{A},A} \otimes_{\mathcal{A}} \omega_{\mathcal{A},A})) \circ_{\mathcal{A}} (\mathrm{id}_{\mathcal{A},A} \times_{\mathcal{A}} \mathrm{id}_{\mathcal{A},A})] \\
&= [\omega_{\mathcal{A},A} \times_{\mathcal{A}} \mathrm{id}_{\mathcal{A},A}, \rho_{\mathcal{A},A} \circ_{\mathcal{A}} (\mathrm{id}_{\mathcal{A},A} \times_{\mathcal{A}} \omega_{\mathcal{A},A})].
\end{aligned}$$

Exploiting the special form of the left and right unitors of $\mathcal{A}$, we see

$$\begin{aligned}
&\rho_{\mathcal{A},A} \circ_{\mathcal{A}} (\mathrm{id}_{\mathcal{A},A} \times_{\mathcal{A}} \omega_{\mathcal{A},A}) \\
&\qquad = \pi^1_{\mathcal{A},A,I_{\mathcal{A}}} \circ_{\mathcal{A}} (\mathrm{id}_{\mathcal{A},A} \times_{\mathcal{A}} \omega_{\mathcal{A},A}) = \mathrm{id}_{\mathcal{A},A} = \pi^2_{\mathcal{A},I_{\mathcal{A}},A} \circ_{\mathcal{A}} (\omega_{\mathcal{A},A} \times_{\mathcal{A}} \mathrm{id}_{\mathcal{A},A}) \\
&\qquad = \lambda_{\mathcal{A},A} \circ_{\mathcal{A}} (\omega_{\mathcal{A},A} \times_{\mathcal{A}} \mathrm{id}_{\mathcal{A},A})
\end{aligned}$$

Hence, the right hand side of the claimed identity evaluates to

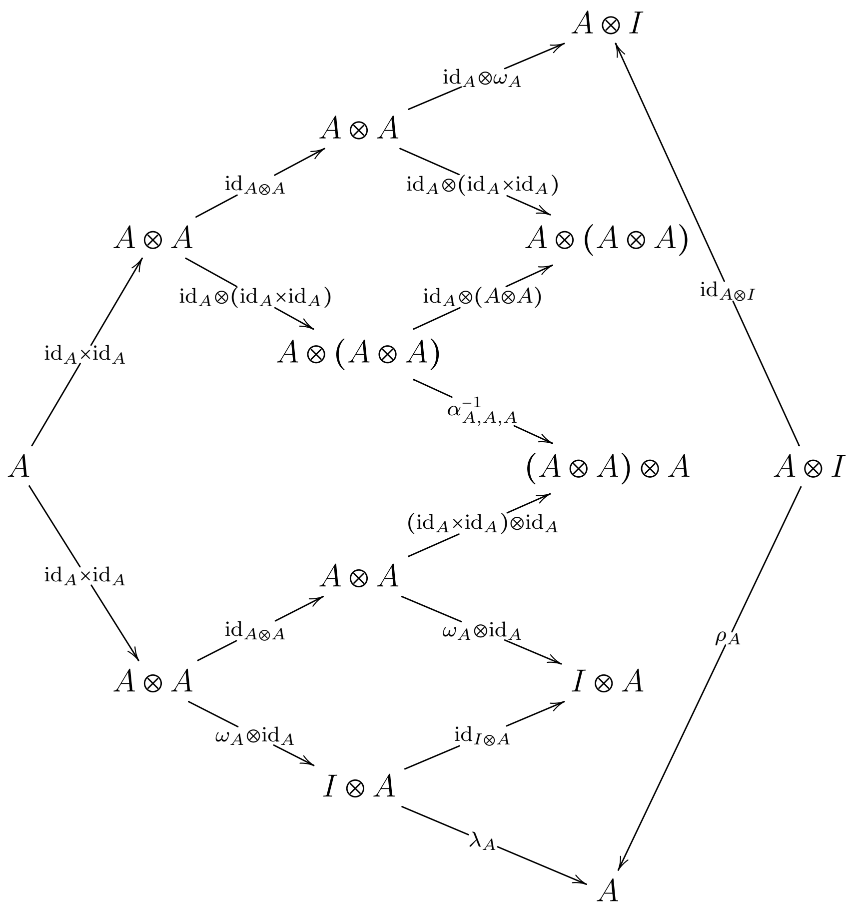
$$[\mathrm{id}_{\mathcal{A},I_{\mathcal{A}}\otimes_{\mathcal{A}} A} \circ_{\mathcal{A}} (\omega_{\mathcal{A},A} \times_{\mathcal{A}} \mathrm{id}_{\mathcal{A},A}), \lambda_{\mathcal{A},A} \circ_{\mathcal{A}} (\omega_{\mathcal{A},A} \times_{\mathcal{A}} \mathrm{id}_{\mathcal{A},A})].$$

Because $\omega_{\mathcal{A},A} \times_{\mathcal{A}} \mathrm{id}_{\mathcal{A},A}$ is an isomorphism $A \to I_{\mathcal{A}} \otimes_{\mathcal{A}} A$ by Lemma 4.14 (c), this is nothing but $[\mathrm{id}_{\mathcal{A},I_{\mathcal{A}}\otimes_{\mathcal{A}} A}, \lambda_{\mathcal{A},A}] = \lambda_{S,A}$. And thus the first diagram is commutative.

Second diagram. The proof is very much analogous to the one for the first diagram. This time we have to show

$$\begin{aligned}
\rho_{S,A^{\vee_S}} \overset{!}{=} (&\lambda_{S,A^{\vee_S}} \circ_S (\varepsilon_{S,A} \otimes_S \mathrm{id}_{S,A^{\vee_S}})) \\
&\circ_S (\alpha^{-1}_{S,A^{\vee_S},A,A^{\vee_S}} \circ_S (\mathrm{id}_{S,A^{\vee_S}} \otimes_S \eta_{S,A}))
\end{aligned}$$

and we apply the same strategy.



The composition of the first two morphisms on the right hand side evaluates to

$$
\begin{aligned}
&(\lambda_{\mathrm{S},A^{\vee_{\mathrm{S}}}} \circ_{\mathrm{S}} (\varepsilon_{\mathrm{S},A} \otimes_{\mathrm{S}} \mathrm{id}_{\mathrm{S},A^{\vee_{\mathrm{S}}}}))\\
&\qquad = [\mathrm{id}_{\mathcal{A},I_{\mathcal{A}}\otimes_{\mathcal{A}}A}, \lambda_{\mathcal{A},A}] \circ_{\mathrm{S}} [(\mathrm{id}_{\mathcal{A},A} \times_{\mathcal{A}} \mathrm{id}_{\mathcal{A},A}) \otimes_{\mathcal{A}} \mathrm{id}_{\mathcal{A},A}, \omega_{\mathcal{A},A} \otimes_{\mathcal{A}} \mathrm{id}_{\mathcal{A},A}]\\
&\qquad = [(\mathrm{id}_{\mathcal{A},A} \times_{\mathcal{A}} \mathrm{id}_{\mathcal{A},A}) \otimes_{\mathcal{A}} \mathrm{id}_{\mathcal{A},A}, \lambda_{\mathcal{A},A} \circ_{\mathcal{A}} (\omega_{\mathcal{A},A} \otimes_{\mathcal{A}} \mathrm{id}_{\mathcal{A},A})]
\end{aligned}
$$

because $(\mathrm{id}_{\mathcal{A},A\otimes_{\mathcal{A}}A}, \omega_{\mathcal{A},A} \otimes_{\mathcal{A}} \mathrm{id}_{\mathcal{A},A})$ is a pull-back of $(\omega_{\mathcal{A},A} \otimes_{\mathcal{A}} \mathrm{id}_{\mathcal{A},A}, \mathrm{id}_{\mathcal{A},I_{\mathcal{A}}\otimes_{\mathcal{A}}A})$ by Lemma 4.6.

And, because $\Theta_{\mathcal{A}}(\alpha_{\mathcal{A},A,A,A})^{-1} = \Theta_{\mathcal{A}}(\alpha^{-1}_{\mathcal{A},A,A,A})$ by Proposition 5.12, composing the second pair of morphisms on the right-hand side of the assertion gives

$$
\begin{aligned}
&\alpha^{-1}_{\mathrm{S},A^{\vee_{\mathrm{S}}},A,A^{\vee_{\mathrm{S}}}} \circ_{\mathrm{S}} (\mathrm{id}_{\mathrm{S},A^{\vee_{\mathrm{S}}}} \otimes_{\mathrm{S}} \eta_{\mathrm{S},A})\\
&\qquad = [\mathrm{id}_{\mathcal{A},A\otimes_{\mathcal{A}}(A\otimes_{\mathcal{A}}A)}, \alpha^{-1}_{\mathcal{A},A,A,A}] \circ_{\mathrm{S}} [\mathrm{id}_{\mathcal{A},A} \otimes_{\mathcal{A}} \omega_{\mathcal{A},A}, \mathrm{id}_{\mathcal{A},A} \otimes_{\mathcal{A}} (\mathrm{id}_{\mathcal{A},A} \times_{\mathcal{A}} \mathrm{id}_{\mathcal{A},A})]\\
&\qquad = [\mathrm{id}_{\mathcal{A},A} \otimes_{\mathcal{A}} \omega_{\mathcal{A},A}, \alpha^{-1}_{\mathcal{A},A,A,A} \circ_{\mathcal{A}} (\mathrm{id}_{\mathcal{A},A} \otimes_{\mathcal{A}} (\mathrm{id}_{\mathcal{A},A} \times_{\mathcal{A}} \mathrm{id}_{\mathcal{A},A}))],
\end{aligned}
$$

since a pull-back of $(\mathrm{id}_{\mathcal{A},A} \otimes_{\mathcal{A}} (\mathrm{id}_{\mathcal{A},A} \times_{\mathcal{A}} \mathrm{id}_{\mathcal{A},A}), \mathrm{id}_{\mathcal{A},A\otimes_{\mathcal{A}}(A\otimes_{\mathcal{A}}A)})$, according to Lemma 4.6, is given by $(\mathrm{id}_{\mathcal{A},A\otimes_{\mathcal{A}}A}, \mathrm{id}_{\mathcal{A},A} \otimes_{\mathcal{A}} (\mathrm{id}_{\mathcal{A},A} \times_{\mathcal{A}} \mathrm{id}_{\mathcal{A},A}))$.

With the knowledge from Lemma 4.14 (b) that $(\mathrm{id}_{\mathcal{A},A} \times_{\mathcal{A}} \mathrm{id}_{\mathcal{A},A}, \mathrm{id}_{\mathcal{A},A} \times_{\mathcal{A}} \mathrm{id}_{\mathcal{A},A})$ is a pull-back of $(\alpha^{-1}_{\mathcal{A},A,A,A} \circ_{\mathcal{A}} (\mathrm{id}_{\mathcal{A},A} \otimes_{\mathcal{A}} (\mathrm{id}_{\mathcal{A},A} \times_{\mathcal{A}} \mathrm{id}_{\mathcal{A},A})), (\mathrm{id}_{\mathcal{A},A} \times_{\mathcal{A}} \mathrm{id}_{\mathcal{A},A}) \otimes_{\mathcal{A}} \mathrm{id}_{\mathcal{A},A})$ the right-hand of the claim as a whole is then given by

$$\begin{aligned}
&[(\mathrm{id}_{\mathcal{A},A} \otimes_{\mathcal{A}} \omega_{\mathcal{A},A}) \circ_{\mathcal{A}} (\mathrm{id}_{\mathcal{A},A} \times_{\mathcal{A}} \mathrm{id}_{\mathcal{A},A}), (\lambda_{\mathcal{A},A} \circ_{\mathcal{A}} (\omega_{\mathcal{A},A} \otimes_{\mathcal{A}} \mathrm{id}_{\mathcal{A},A})) \circ_{\mathcal{A}} (\mathrm{id}_{\mathcal{A},A} \times_{\mathcal{A}} \mathrm{id}_{\mathcal{A},A})] \\
&= [\mathrm{id}_{\mathcal{A},A} \times_{\mathcal{A}} \omega_{\mathcal{A},A}, \lambda_{\mathcal{A},A} \circ_{\mathcal{A}} (\omega_{\mathcal{A},A} \times_{\mathcal{A}} \mathrm{id}_{\mathcal{A},A})] \\
&= [\mathrm{id}_{\mathcal{A},A} \times_{\mathcal{A}} \omega_{\mathcal{A},A}, \rho_{\mathcal{A},A} \circ_{\mathcal{A}} (\mathrm{id}_{\mathcal{A},A} \times_{\mathcal{A}} \omega_{\mathcal{A},A})] \\
&= \rho_{\mathrm{S},A^{\vee_{\mathrm{S}}}},
\end{aligned}$$

where the last step is due to the fact that $\mathrm{id}_{\mathcal{A},A} \times_{\mathcal{A}} \omega_{\mathcal{A},A}$ is an isomorphism $A \to A \otimes_{\mathcal{A}} I_{\mathcal{A}}$ by Lemma 4.14 (d). That concludes the proof. □

PROPOSITION 5.40. *The symmetric monoidal category* $\mathrm{S}(\mathcal{A})$ *is rigid for any* 0-*cell* $\mathcal{A}$ *of* $\mathsf{smCAT}^{\mathrm{cart,fc}}_{\mathrm{sos}}$. *Left duals with associated evaluations and co-evaluations are given by* $(\,\cdot\,)^{\vee_{\mathrm{S}(\mathcal{A})}}$, $\varepsilon_{\mathrm{S}(\mathcal{A})}$ *and* $\eta_{\mathrm{S}(\mathcal{A})}$, *respectively.*

PROOF. That is the combined result of Lemma 5.39. □

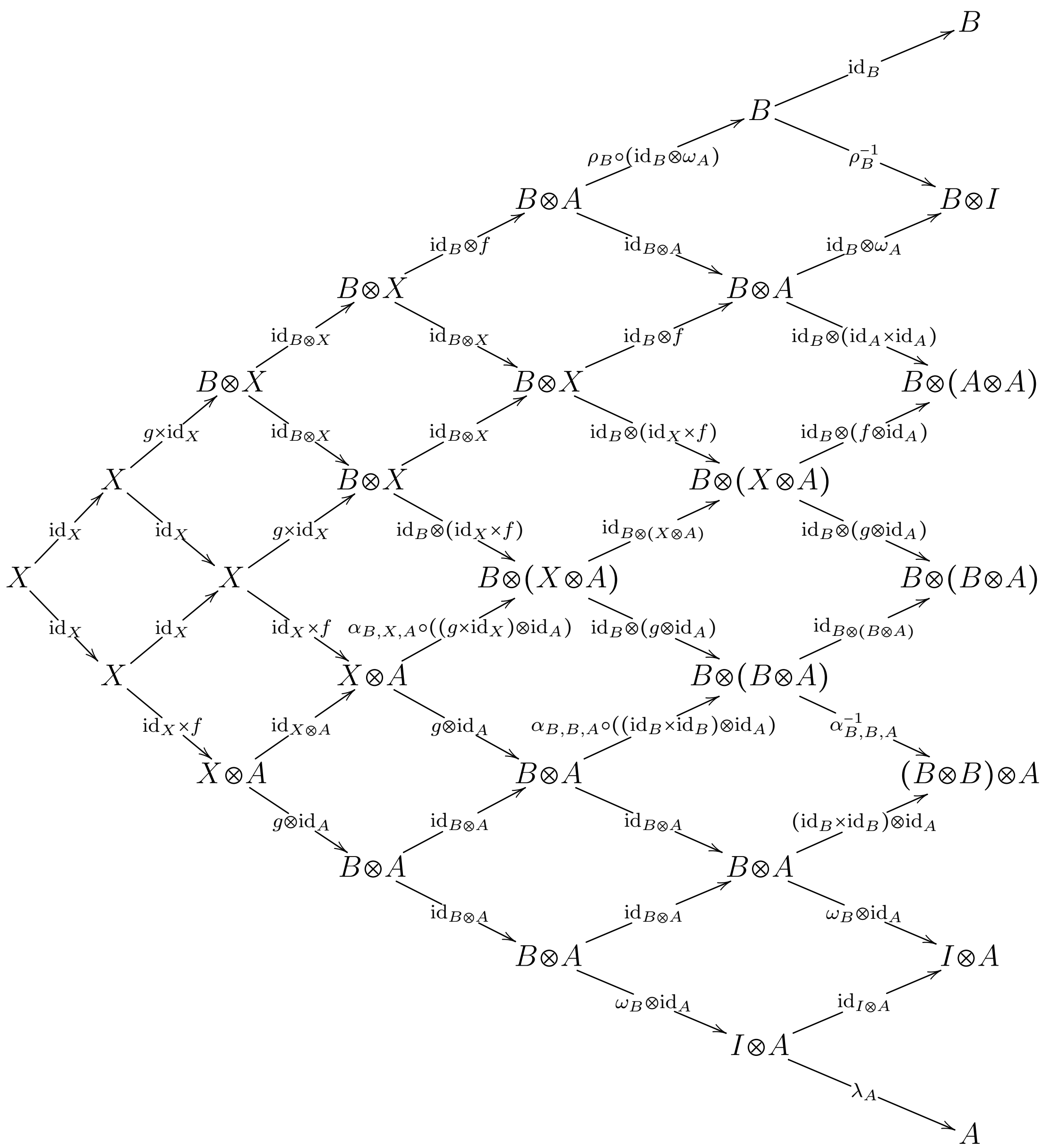
For later steps in the construction process it is convenient to now determine the dual morphisms with respect to the self-dual structure of S.

PROPOSITION 5.41. *For any* 0-*cell* $\mathcal{A}$ *of* S, *any objects* A *and* B *and any morphism* $[f,g]\colon A \to B$ *of* $\mathrm{S}(\mathcal{A})$ *the dual morphism* $[f,g]^{\vee_{\mathrm{S}(\mathcal{A})}}$ *of* $[f,g]$ *with respect to the dualization* $((\,\cdot\,)^{\vee_{\mathrm{S}(\mathcal{A})}}, \varepsilon_{\mathrm{S}(\mathcal{A})}, \eta_{\mathrm{S}(\mathcal{A})})$ *is given by* $[g,f]$.

PROOF. Abbreviate $\mathrm{S}(\mathcal{A})$ by S. We have to verify that the following identity of morphisms of S (the index S is omitted here)

$$[g,f] \overset{!}{=} \lambda_{A^\vee} \circ (\varepsilon_B \otimes \mathrm{id}_{A^\vee}) \circ \alpha^{-1}_{B^\vee,B,A^\vee} \circ (\mathrm{id}_{B^\vee} \otimes ([f,g] \otimes \mathrm{id}_{A^\vee})) \circ (\mathrm{id}_{B^\vee} \otimes \eta_A) \circ \rho^{-1}_{B^\vee}.$$

If X is the base object of the span (f,g), then the composition of the morphisms on the right-hand side of the claim is the class of the span formed by the two outermost arms of the following diagram (from now on the index $\mathcal{A}$ is omitted instead):



That requires justifying. We start at the top and move downward, then passing to the next column to left, again starting from the top, and so on. A pull-back of

- ◻ $(\rho_B^{-1}, \mathrm{id}_B \otimes \omega_A)$ is given by $(\rho_B \circ (\mathrm{id}_B \otimes \omega_A), \mathrm{id}_{B\otimes A})$ by Lemma 4.1 because this is a pull-back of $(\mathrm{id}_B, \rho_B \circ (\mathrm{id}_B \otimes \omega_A))$ by Lemma 4.6,
- ◻ $(\mathrm{id}_B \otimes (\mathrm{id}_A \times \mathrm{id}_A), \mathrm{id}_B \otimes (f \otimes \mathrm{id}_A))$ is $(\mathrm{id}_B \otimes f, \mathrm{id}_B \otimes (\mathrm{id}_X \times f))$ according to Lemma 4.17 (a),
- ◻ $(\mathrm{id}_B \otimes (g \otimes \mathrm{id}_A), \mathrm{id}_{B\otimes(B\otimes A)})$ is $(\mathrm{id}_{B\otimes(X\otimes A)}, \mathrm{id}_B \otimes (g \otimes \mathrm{id}_A))$ by Lemma 4.6,

- □ $(\alpha^{-1}_{B,B,A}, (\mathrm{id}_B \times \mathrm{id}_B) \otimes \mathrm{id}_A)$ is $(\alpha_{B,B,A} \circ ((\mathrm{id}_B \times \mathrm{id}_B) \otimes \mathrm{id}_A), \mathrm{id}_{B\otimes A})$ by Lemma 4.1 because, according to Lemma 4.6, in $((\mathrm{id}_B \times \mathrm{id}_B) \otimes \mathrm{id}_A, \mathrm{id}_{B\otimes A})$ we have a pull-back of $(\mathrm{id}_{(B\otimes B)\otimes A}, (\mathrm{id}_B \times \mathrm{id}_B) \otimes \mathrm{id}_A)$,
- □ $(\omega_B \otimes \mathrm{id}_A, \mathrm{id}_{I\otimes A})$ is given by $(\mathrm{id}_{B\otimes A}, \omega_B \otimes \mathrm{id}_A)$ by Lemma 4.6,
- □ $(\mathrm{id}_{B\otimes A}, \mathrm{id}_B \otimes f)$ is given by $(\mathrm{id}_B \otimes f, \mathrm{id}_{B\otimes X})$ by Lemma 4.6,
- □ $(\mathrm{id}_B \otimes (\mathrm{id}_X \times f), \mathrm{id}_{B\otimes(X\otimes A)})$ is $(\mathrm{id}_{B\otimes X}, \mathrm{id}_B \otimes (\mathrm{id}_X \times f))$ by Lemma 4.6,
- □ $(\mathrm{id}_B \otimes (g \otimes \mathrm{id}_A), \alpha_{B,B,A} \circ ((\mathrm{id}_B \times \mathrm{id}_B) \otimes \mathrm{id}_A))$ is given, according to Lemma 4.1, by $(\alpha_{B,X,A} \circ ((g \times \mathrm{id}_X) \otimes \mathrm{id}_A), g \otimes \mathrm{id}_A)$ because, by Lemma 4.17 (b), one of $((\mathrm{id}_B \otimes g) \otimes \mathrm{id}_A, (\mathrm{id}_B \times \mathrm{id}_B) \otimes \mathrm{id}_A)$ is $((g \times \mathrm{id}_X) \otimes \mathrm{id}_A, g \otimes \mathrm{id}_A)$ and because $\alpha_{B,B,A} \circ ((\mathrm{id}_B \otimes g) \otimes \mathrm{id}_A) = (\mathrm{id}_B \otimes (g \otimes \mathrm{id}_A)) \circ \alpha_{B,X,A}$,
- □ $(\mathrm{id}_{B\otimes A}, \mathrm{id}_{B\otimes A})$ is given by $(\mathrm{id}_{B\otimes A}, \mathrm{id}_{B\otimes A})$ by Lemma 4.6,
- □ $(\mathrm{id}_{B\otimes X}, \mathrm{id}_{B\otimes X})$ is given by $(\mathrm{id}_{B\otimes X}, \mathrm{id}_{B\otimes X})$ by Lemma 4.6,
- □ $(\mathrm{id}_B \otimes (\mathrm{id}_X \times f), \alpha_{B,X,A} \circ ((g \times \mathrm{id}_X) \otimes \mathrm{id}_A))$ is $(g \times \mathrm{id}_X, \mathrm{id}_X \times f)$ according to Lemma 4.17 (c),
- □ $(g \otimes \mathrm{id}_A, \mathrm{id}_{B\otimes A})$ is given by $(\mathrm{id}_{X\otimes A}, g \otimes \mathrm{id}_A)$ by Lemma 4.6,
- □ $(\mathrm{id}_{B\otimes X}, g \times \mathrm{id}_X)$ is given by $(g \times \mathrm{id}_X, \mathrm{id}_X)$ by Lemma 4.6,
- □ $(\mathrm{id}_X \times f, \mathrm{id}_{X\otimes A})$ is given by $(\mathrm{id}_X, \mathrm{id}_X \times f)$ by Lemma 4.6,
- □ $(\mathrm{id}_X, \mathrm{id}_X)$ is given by $(\mathrm{id}_X, \mathrm{id}_X)$ by Lemma 4.6.

Because $\rho_B \circ (\mathrm{id}_B \otimes \omega_A) = \pi^1_{B,I} \circ (\pi^1_{B,A} \times (\omega_A \circ \pi^2_{B,A})) = \pi^1_{B,A}$ we obtain for the uppermost arm of the diagram

$$\begin{aligned}
&\mathrm{id}_B \circ \rho_B \circ (\mathrm{id}_B \otimes \omega_A) \circ (\mathrm{id}_B \otimes f) \circ \mathrm{id}_{B\otimes X} \circ (g \times \mathrm{id}_X) \circ \mathrm{id}_X \\
&\quad = \pi^1_{B,A} \circ (\mathrm{id}_B \otimes f) \circ (g \times \mathrm{id}_X) \\
&\quad = \pi^1_{B,A} \circ (g \times f) \\
&\quad = g.
\end{aligned}$$

Similarly, since $\lambda_A \circ (\omega_B \otimes \mathrm{id}_A) = \pi^2_{B,A}$, the lowermost arm evaluates to

$$\begin{aligned}
&\lambda_A \circ (\omega_B \otimes \mathrm{id}_A) \circ \mathrm{id}_{B\otimes A} \circ (g \otimes \mathrm{id}_A) \circ (\mathrm{id}_X \times f) \circ \mathrm{id}_X \\
&\quad = \pi^2_{B,A} \circ (g \otimes \mathrm{id}_A) \circ (\mathrm{id}_X \times f) \\
&\quad = \pi^2_{B,A} \circ (g \times f) \\
&\quad = f.
\end{aligned}$$

That is what we needed to see. □

Moreover, we characterize the traces with respect to the dualization defined above, also for later use.

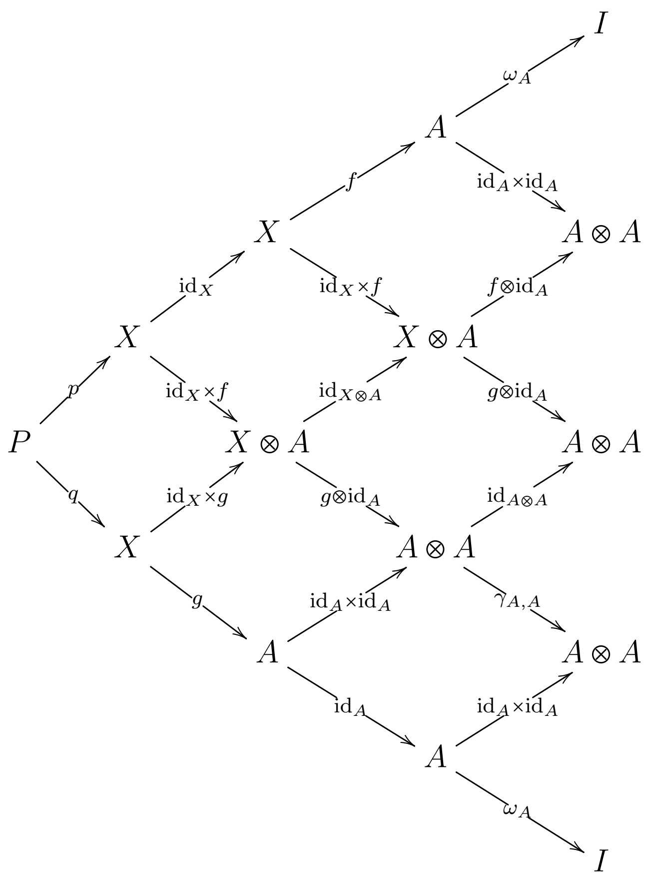
Proposition 5.42. *For any 0-cell $\mathcal{A}$ of $\mathsf{smCAT}^{\mathrm{cart,fc}}_{\mathrm{sos}}$, any For any $A \in \mathrm{obj}_{\mathrm{S}(\mathcal{A})}$ and any endomorphism $[f,g]$ of A in $\mathrm{S}(\mathcal{A})$ the trace $\mathrm{tr}_{\mathrm{S}(\mathcal{A})}([f,g])$ of $[f,g]$ with respect to the dualization $((\cdot)^{\vee_{\mathrm{S}(\mathcal{A})}}, \varepsilon_{\mathrm{S}(\mathcal{A})}, \eta_{\mathrm{S}(\mathcal{A})})$ is given by*

$$[\omega_{\mathcal{A},A} \circ_{\mathcal{A}} f \circ_{\mathcal{A}} p, \omega_{\mathcal{A},A} \circ_{\mathcal{A}} g \circ_{\mathcal{A}} q]$$

for any pull-back (p,q) of $(\mathrm{id}_{\mathcal{A},X} \times_{\mathcal{A}} f, \mathrm{id}_{\mathcal{A},X} \times_{\mathcal{A}} g)$, where X is the base object of the span (f,g) in $\mathcal{A}$.

Proof. Abbreviate $\mathrm{S}(\mathcal{A})$ by S. We need to confirm that the term given for $\mathrm{tr}_{\mathrm{S}}([f,g])$ in the claim is identical to the morphism

$$\varepsilon_{\mathrm{S},A} \circ_{\mathrm{S}} \gamma_{\mathrm{S},A,A^{\vee_{\mathrm{S}}}} \circ_{\mathrm{S}} ([f,g] \otimes_{\mathrm{S}} \mathrm{id}_{\mathrm{S},A^{\vee_{\mathrm{S}}}}) \circ_{\mathrm{S}} \eta_{\mathrm{S},A}.$$



By unwinding the definitions, this assertion is seen to be equivalent to the identity

$$\begin{aligned}[t] &[\omega_{\mathcal{A},A} \circ_{\mathcal{A}} f \circ_{\mathcal{A}} p, \omega_{\mathcal{A},A} \circ_{\mathcal{A}} g \circ_{\mathcal{A}} q] \\ &\qquad \overset{!}{=} [\mathrm{id}_{\mathcal{A},A} \times_{\mathcal{A}} \mathrm{id}_{\mathcal{A},A}, \omega_{\mathcal{A},A}] \circ_{\mathrm{S}} [\mathrm{id}_{\mathcal{A},A\otimes_{\mathcal{A}}A}, \gamma_{\mathcal{A},A,A}] \\ &\qquad\qquad \circ_{\mathrm{S}} [f \otimes_{\mathcal{A}} \mathrm{id}_{\mathcal{A},A}, g \otimes_{\mathcal{A}} \mathrm{id}_{\mathcal{A},A}] \circ_{\mathrm{S}} [\omega_{\mathcal{A},A}, \mathrm{id}_{\mathcal{A},A} \times_{\mathcal{A}} \mathrm{id}_{\mathcal{A},A}]. \end{aligned}$$

In the above $\mathcal{A}$-diagram, all small squares are pull-backs for the following reasons:

- □ $(f, \mathrm{id}_{\mathcal{A},X} \times_{\mathcal{A}} f)$ is a pull-back of $(\mathrm{id}_{\mathcal{A},A} \times_{\mathcal{A}} \mathrm{id}_{\mathcal{A},A}, f \otimes_{\mathcal{A}} \mathrm{id}_{\mathcal{A},A})$ by Lemma 4.19.
- □ $(\mathrm{id}_{\mathcal{A},X\otimes_{\mathcal{A}}A}, g \otimes_{\mathcal{A}} \mathrm{id}_{\mathcal{A},X})$ is one of $(g \otimes_{\mathcal{A}} \mathrm{id}_{\mathcal{A},A}, \mathrm{id}_{\mathcal{A},A\otimes_{\mathcal{A}}A})$ by Lemma 4.6.
- □ $(\mathrm{id}_{\mathcal{A},A} \times_{\mathcal{A}} \mathrm{id}_{\mathcal{A},A}, \mathrm{id}_{\mathcal{A},A})$ is one of $(\gamma_{\mathcal{A},A,A}, \mathrm{id}_{\mathcal{A},A} \times_{\mathcal{A}} \mathrm{id}_{\mathcal{A},A})$ by Lemma 4.20.
- □ $(\mathrm{id}_{\mathcal{A},X}, \mathrm{id}_{\mathcal{A},X} \times_{\mathcal{A}} f)$ is one of $(\mathrm{id}_{\mathcal{A},A} \times_{\mathcal{A}} f, \mathrm{id}_{\mathcal{A},X\otimes_{\mathcal{A}}A})$ by Lemma 4.6.
- □ $(\mathrm{id}_{\mathcal{A},X} \times_{\mathcal{A}} g, g)$ is one of $(g \otimes_{\mathcal{A}} \mathrm{id}_{\mathcal{A},A}, \mathrm{id}_{\mathcal{A},A} \times_{\mathcal{A}} \mathrm{id}_{\mathcal{A},A})$ by a second application of Lemma 4.19.

Now, the claim easily follows by Lemmata 4.3 and 4.5. □

Lastly, it is convenient to observe the following property of S for which the fact that all objects are self-dual is crucial.

PROPOSITION 5.43. *For any* $\{A,B\}\in \mathrm{obj}_{\mathcal{A}}$ *and any span* (f,g) *of* $\mathcal{A}$ *from* A *to* B*,*

$$[f,g]=\Theta_{\mathcal{A}}(g)\circ_{\mathrm{S}}\Theta_{\mathcal{A}}(f)^{\vee_{\mathrm{S}}}.$$

PROOF. If X denotes the base object of the span (f,g) in $\mathcal{A}$, then by Proposition 5.41,

$$\begin{aligned}\Theta_{\mathcal{A}}(g)\circ_{\mathrm{S}}\Theta_{\mathcal{A}}(f)^{\vee_{\mathrm{S}}}&=[\mathrm{id}_{\mathcal{A},X},g]\circ_{\mathrm{S}}[\mathrm{id}_{\mathcal{A},X},f]^{\vee_{\mathrm{S}}}\\&=[\mathrm{id}_{\mathcal{A},X},g]\circ_{\mathrm{S}}[f,\mathrm{id}_{\mathcal{A},X}]\\&=[f\circ_{\mathcal{A}}\mathrm{id}_{\mathcal{A},X},g\circ_{\mathcal{A}}\mathrm{id}_{\mathcal{A},X}]\\&=[f,g],\end{aligned}$$

where we have used in the next-to-last step that $(\mathrm{id}_{\mathcal{A},X},\mathrm{id}_{\mathcal{A},X})$ is a pull-back of itself in $\mathcal{A}$ by Lemma 4.6. □

5.2. Stage 1b: Spans (with †). In the previous section, for any 0-cell $\mathcal{A}$ of $\mathsf{smCAT}^{\mathrm{cart,fc}}_{\mathrm{sos}}$ we saw that exchanging the roles of input and output morphism in a span corresponds to forming the dual morphism with respect to the self-dualization $((\cdot)^{\vee_{\mathrm{S}(\mathcal{A})}},\varepsilon_{\mathrm{S}(\mathcal{A})},\eta_{\mathrm{S}(\mathcal{A})})$. From this point of view the following definitions and results might seem superfluous because the functors $(\cdot)^{\vee_{\mathrm{S}(\mathcal{A})}}$ and $(\cdot)^{\dagger_{\mathrm{S}(\mathcal{A})}}$ will be the same. However, we will be interested in rigid monoidal †-subcategories of $\mathrm{S}(\mathcal{A})$ which will not necessarily contain the evaluation and co-evaluation morphisms needed in order for all objects to be self-dual. Therefore, $(\cdot)^{\dagger_{\mathrm{S}(\mathcal{A})}}$ is crucial.

5.2.1. *†-category.* We first have to check that the supposed † is well-defined.

LEMMA 5.44. *For any category* $\mathcal{A}$*, any* $\{A,B\}\subseteq\mathrm{obj}_{\mathcal{A}}$*, any spans* (f,g) *and* (f',g') *of* $\mathcal{A}$ *from* A *to* B*, whenever* (f,g) *and* (f',g') *are equivalent,* (g,f) *and* (g',f') *are equivalent as spans of* $\mathcal{A}$ *from* B *to* A*.*

PROOF. Let X and X' be the bases of (f,g) and (f',g'), respectively. Because (f,g) and (f',g') are equivalent we find an isomorphism u from X to X' such that $f'\circ u=f$ and $g'\circ u=g$. We can read these equations equally as proof of (g,f) and (g',f') being equivalent. □

DEFINITION 5.45. For any 0-cell of $\mathsf{smCAT}^{\mathrm{cart,fc}}_{\mathrm{sos}}$ let

(a) $X^{\dagger_{\mathrm{S}(\mathcal{A})}}:=X$ for any $X\in\mathrm{obj}_{\mathrm{S}(\mathcal{A})}$,

(b) $[f,g]^{\dagger_{\mathrm{S}(\mathcal{A})}}:=[g,f]$ for any $\{A,B\}\subseteq\mathrm{obj}_{\mathrm{S}(\mathcal{A})}$ and any $[f,g]\in\mathrm{mor}_{\mathrm{S}(\mathcal{A})}(A,B)$.

LEMMA 5.46. *Let* $\mathcal{A}$ *be any* 0*-cell of* $\mathsf{smCAT}^{\mathrm{cart,fc}}_{\mathrm{sos}}$*. Abbreviate* $\mathrm{S}(\mathcal{A})$ *by* S*.*

(a) $(\cdot)^{\dagger_{\mathrm{S}}}$ *is a contravariant endofunctor of* S*.*

(b) $(\cdot)^{\dagger_{\mathrm{S}}}$ *is the identity on objects.*

(c) $((\cdot)^{\dagger_{\mathrm{S}}})^{\mathrm{op}_{\mathsf{CAT}}}\circ_{\mathsf{CAT}}(\cdot)^{\dagger_{\mathrm{S}}}=\mathrm{id}_{\mathsf{CAT},\mathrm{S}}$.

PROOF. (a) Clearly, ${\mathrm{id}_{\mathrm{S},X}}^{\dagger_\mathrm{S}} = [\mathrm{id}_{\mathcal{A},X}, \mathrm{id}_{\mathcal{A},X}]^{\dagger_\mathrm{S}} = [\mathrm{id}_{\mathcal{A},X}, \mathrm{id}_{\mathcal{A},X}] = \mathrm{id}_{\mathrm{S},X}$ for any $X \in \mathrm{obj}_\mathcal{A}$. For any $\{A, B, C\} \subseteq \mathrm{mor}_\mathrm{S}$ and $[f, g] \in \mathrm{mor}_\mathrm{S}(A, B)$ and $[p, q] \in \mathrm{mor}_\mathrm{S}(B, C)$, if (a, b) is any pull-back of (g, p), then (b, a) is a pull-back of (p, g). Hence, by definition,

$$\begin{aligned}([p, q] \circ_\mathrm{S} [f, g])^{\dagger_\mathrm{S}} &= [f \circ_\mathcal{A} a, q \circ_\mathcal{A} b]^{\dagger_\mathrm{S}} = [q \circ_\mathcal{A} b, f \circ_\mathcal{A} a] = [g, f] \circ_\mathrm{S} [q, p] \\ &= [f, g]^{\dagger_\mathrm{S}} \circ_\mathrm{S} [p, q]^{\dagger_\mathrm{S}},\end{aligned}$$

which is what we needed to show.

(b) Holds by definition.

(c) For objects the identity is clear by (b). If $\{A, B\} \subseteq \mathrm{obj}_\mathrm{S}$ and $[f, g] \in \mathrm{mor}_\mathrm{S}(A, B)$, then, by definition, $([f, g]^{\dagger_\mathrm{S}})^{\dagger_\mathrm{S}} = [g, f]^{\dagger_\mathrm{S}} = [f, g]$. □

PROPOSITION 5.47. $\mathrm{S}(\mathcal{A})$*, when equipped with* $(\cdot)^{\dagger_{\mathrm{S}(\mathcal{A})}}$*, is a †-category for any* 0*-cell* $\mathcal{A}$ *of* $\mathsf{smCAT}^{\mathrm{cart,fc}}_{\mathrm{sos}}$.

PROOF. The claim is the combined implication of Lemma 5.46. □

LEMMA 5.48. *For any* 0*-cells* $\mathcal{A}$ *and* $\mathcal{B}$ *and any* 1*-cell* H *from* $\mathcal{A}$ *to* $\mathcal{B}$*,*

$$\mathrm{S}(H)(x^{\dagger_{\mathrm{S}(\mathcal{A})}}) = \mathrm{S}(H)(x)^{\dagger_{\mathrm{S}(\mathcal{B})}}.$$

PROOF. For any $(f, g) \in x$, by definition

$$\begin{aligned}\mathrm{S}(H)(x^{\dagger_{\mathrm{S}(\mathcal{A})}}) &= \mathrm{S}(H)([f, g]^{\dagger_{\mathrm{S}(\mathcal{A})}}) \\ &= \mathrm{S}(H)([g, f]) \\ &= [H(g), H(f)] \\ &= [H(f), H(g)]^{\dagger_{\mathrm{S}(\mathcal{B})}} \\ &= \mathrm{S}(H)([f, g])^{\dagger_{\mathrm{S}(\mathcal{B})}} \\ &= \mathrm{S}(H)(x)^{\dagger_{\mathrm{S}(\mathcal{B})}},\end{aligned}$$

as claimed. □

PROPOSITION 5.49. $\mathrm{S}(H)$ *is a †-functor from* $\mathrm{S}(\mathcal{A})$ *to* $\mathrm{S}(\mathcal{B})$ *for any* 0*-cells* $\mathcal{A}$ *and* $\mathcal{B}$ *and any* 1*-cell* H *from* $\mathcal{A}$ *to* $\mathcal{B}$ *of* $\mathsf{smCAT}^{\mathrm{cart,fc}}_{\mathrm{sos}}$.

PROOF. The claim is a reformulation of Lemma 5.48. □

5.2.2. *Monoidal †-category.* Next, we prove that the span category with its already defined monoidal structure also becomes a monoidal †-category. Denote by $*\mathsf{CAT}$ the (large) monoidal category of †-categories.

LEMMA 5.50. *Let* $\mathcal{A}$ *be any* 0*-cell of* $\mathsf{smCAT}^{\mathrm{cart,fc}}_{\mathrm{sos}}$ *and abbreviate* $\mathrm{S}(\mathcal{A})$ *by* S.

(a) $\otimes_\mathrm{S}$ *is a †-functor* $\mathrm{S} \otimes_{*\mathsf{CAT}} \mathrm{S} \to \mathrm{S}$.

(b) ${\alpha_{\mathrm{S},A_1,A_2,A_3}}^{\dagger_\mathrm{S}} = {\alpha_{\mathrm{S},A_1,A_2,A_3}}^{-1_\mathrm{S}}$ *for any* $\{A_1, A_2, A_3\} \subseteq \mathrm{obj}_\mathrm{S}$.

(c) ${\lambda_{\mathrm{S},A}}^{\dagger_\mathrm{S}} = {\lambda_{\mathrm{S},A}}^{-1_\mathrm{S}}$ *for any* $A \in \mathrm{obj}_\mathrm{S}$.

(d) ${\rho_{\mathrm{S},A}}^{\dagger_\mathrm{S}} = {\rho_{\mathrm{S},A}}^{-1_\mathrm{S}}$ *for any* $A \in \mathrm{obj}_\mathrm{S}$.

PROOF. (a) We already know that $\otimes_{\mathrm{S}}$ is a functor $\mathrm{S} \otimes_{\mathsf{CAT}} \mathrm{S} \to \mathrm{S}$. Hence, the only part of the claim which needs proving is that $\otimes_{\mathrm{S}}$ respects $(\cdot)^{\dagger_{\mathrm{S}}}$.

And this is immediately clear on the level of objects because there $(\cdot)^{\dagger_{\mathrm{S}}}$ is simply the identity. Hence, let $\{A_1, A_2, B_1, B_2\} \subseteq \mathrm{obj}_{\mathrm{S}}$ and $[f_1, g_1] \in \mathrm{mor}_{\mathrm{S}}(A_1, B_1)$ and $[f_2, g_2] \in \mathrm{mor}_{\mathrm{S}}(A_2, B_2)$ be arbitrary. By definition, then,

$$\begin{aligned}([f_1, g_1] \otimes_{\mathrm{S}} [f_2, g_2])^{\dagger_{\mathrm{S}}} &= [f_1 \otimes_{\mathcal{A}} f_2, g_1 \otimes_{\mathcal{A}} g_2]^{\dagger_{\mathrm{S}}} = [g_1 \otimes_{\mathcal{A}} g_2, f_1 \otimes_{\mathcal{A}} f_2] \\ &= [g_1, f_1] \otimes_{\mathrm{S}} [g_2, f_2] = [f_1, g_1]^{\dagger_{\mathrm{S}}} \otimes_{\mathrm{S}} [f_2, g_2]^{\dagger_{\mathrm{S}}},\end{aligned}$$

which is what we needed to see.

(b) We compute immediately,

$$\begin{aligned}{\alpha_{\mathrm{S},A_1,A_2,A_3}}^{-1_{\mathrm{S}}} &= \Theta_{\mathcal{A}}(\alpha_{\mathcal{A},A_1,A_2,A_3})^{-1_{\mathrm{S}}} \\ &= \Theta_{\mathcal{A}}({\alpha_{\mathcal{A},A_1,A_2,A_3}}^{-1_{\mathcal{A}}}) \\ &= [\mathrm{id}_{\mathcal{A},A_1 \otimes_{\mathcal{A}} (A_2 \otimes_{\mathcal{A}} A_3)}, {\alpha_{\mathcal{A},A_1,A_2,A_3}}^{-1_{\mathcal{A}}}] \\ &= [\mathrm{id}_{\mathcal{A},A_1 \otimes_{\mathcal{A}} (A_2 \otimes_{\mathcal{A}} A_3)} \circ_{\mathcal{A}} \alpha_{\mathcal{A},A_1,A_2,A_3}, {\alpha_{\mathcal{A},A_1,A_2,A_3}}^{-1_{\mathcal{A}}} \circ_{\mathcal{A}} \alpha_{\mathcal{A},A_1,A_2,A_3}] \\ &= [\alpha_{\mathcal{A},A_1,A_2,A_3}, \mathrm{id}_{\mathcal{A},(A_1 \otimes_{\mathcal{A}} A_2) \otimes_{\mathcal{A}} A_3}] \\ &= [\mathrm{id}_{\mathcal{A},A_1 \otimes_{\mathcal{A}} (A_2 \otimes_{\mathcal{A}} A_3)}, \alpha_{\mathcal{A},A_1,A_2,A_3}]^{\dagger_{\mathrm{S}}} \\ &= \Theta_{\mathcal{A}}(\alpha_{\mathcal{A},A_1,A_2,A_3})^{\dagger_{\mathrm{S}}} \\ &= {\alpha_{\mathrm{S},A_1,A_2,A_3}}^{\dagger_{\mathrm{S}}},\end{aligned}$$

where the second step is due to Proposition 5.12 and the fourth to the definition of span equivalence.

(c) The proof is completely analogous to that of (b): We find,

$$\begin{aligned}{\lambda_{\mathrm{S},A}}^{-1_{\mathrm{S}}} &= \Theta_{\mathcal{A}}(\lambda_{\mathcal{A},A})^{-1_{\mathrm{S}}} \\ &= \Theta_{\mathcal{A}}({\lambda_{\mathcal{A},A}}^{-1_{\mathcal{A}}}) \\ &= [\mathrm{id}_{\mathcal{A},A}, {\lambda_{\mathcal{A},A}}^{-1_{\mathcal{A}}}] \\ &= [\mathrm{id}_{\mathcal{A},A} \circ_{\mathcal{A}} \lambda_{\mathcal{A},A}, {\lambda_{\mathcal{A},A}}^{-1_{\mathcal{A}}} \circ_{\mathcal{A}} \lambda_{\mathcal{A},A}] \\ &= [\lambda_{\mathcal{A},A}, \mathrm{id}_{\mathcal{A},I_{\mathcal{A}} \otimes_{\mathcal{A}} A}] \\ &= [\mathrm{id}_{\mathcal{A},I_{\mathcal{A}} \otimes_{\mathcal{A}} A}, \lambda_{\mathcal{A},A}]^{\dagger_{\mathrm{S}}} \\ &= \Theta_{\mathcal{A}}(\lambda_{\mathcal{A},A})^{\dagger_{\mathrm{S}}} \\ &= {\lambda_{\mathrm{S},A}}^{\dagger_{\mathrm{S}}},\end{aligned}$$

where again we have employed Proposition 5.12.

(d) As before,

$$\begin{aligned}
\rho_{\mathrm{S},A}{}^{-1_{\mathrm{S}}} &= \Theta_{\mathcal{A}}(\rho_{\mathcal{A},A})^{-1_{\mathrm{S}}} \\
&= \Theta_{\mathcal{A}}(\rho_{\mathcal{A},A}{}^{-1_{\mathcal{A}}}) \\
&= [\mathrm{id}_{\mathcal{A},A}, \rho_{\mathcal{A},A}{}^{-1_{\mathcal{A}}}] \\
&= [\mathrm{id}_{\mathcal{A},A} \circ_{\mathcal{A}} \rho_{\mathcal{A},A}, \rho_{\mathcal{A},A}{}^{-1_{\mathcal{A}}} \circ_{\mathcal{A}} \rho_{\mathcal{A},A}] \\
&= [\rho_{\mathcal{A},A}, \mathrm{id}_{\mathcal{A},A\otimes_{\mathcal{A}} I_{\mathcal{A}}}] \\
&= [\mathrm{id}_{\mathcal{A},A\otimes_{\mathcal{A}} I_{\mathcal{A}}}, \rho_{\mathcal{A},A}]^{\dagger_{\mathrm{S}}} \\
&= \Theta_{\mathcal{A}}(\rho_{\mathcal{A},A})^{\dagger_{\mathrm{S}}} \\
&= \rho_{\mathrm{S},A}{}^{\dagger_{\mathrm{S}}}
\end{aligned}$$

by Proposition 5.12. That concludes the proof. □

PROPOSITION 5.51. $\mathrm{S}(\mathcal{A})$ *is a monoidal †-category for any* 0*-cell* $\mathcal{A}$ *of* $\mathsf{smCAT}^{\mathrm{cart,fc}}_{\mathrm{sos}}$.

PROOF. That is the combined implication of Lemma 5.50. □

It is worth noting for the future that the embedding of any cartesian monoidal input category turns isomorphisms into unitary morphisms of the span category.

LEMMA 5.52. *For any* 0*-cell* $\mathcal{A}$ *of* $\mathsf{smCAT}^{\mathrm{cart,fc}}_{\mathrm{sos}}$*, any* $\{X, Y\} \subseteq \mathrm{obj}_{\mathcal{A}}$ *and any* $h \in \mathrm{mor}_{\mathcal{A}}(X, Y)$*, if* h *is invertible in* $\mathcal{A}$*, then*

$$\Theta_{\mathcal{A}}(h)^{\dagger_{\mathrm{S}(\mathcal{A})}} = \Theta_{\mathcal{A}}(h^{-1_{\mathcal{A}}}).$$

PROOF. By definition,

$$\begin{aligned}
\Theta_{\mathcal{A}}(h)^{\dagger_{\mathrm{S}(\mathcal{A})}} &= [\mathrm{id}_{\mathcal{A},X}, h]^{\dagger_{\mathrm{S}(\mathcal{A})}} = [h, \mathrm{id}_{\mathcal{A},X}] = [h \circ_{\mathcal{A}} h^{-1_{\mathcal{A}}}, \mathrm{id}_{\mathcal{A},X} \circ_{\mathcal{A}} h^{-1_{\mathcal{A}}}] = [\mathrm{id}_{\mathcal{A},Y}, h^{-1_{\mathcal{A}}}] \\
&= \Theta_{\mathcal{A}}(h^{-1_{\mathcal{A}}}),
\end{aligned}$$

as asserted. □

LEMMA 5.53. *For any* 0*-cells* $\mathcal{A}$ *and* $\mathcal{B}$ *and any* 1*-cell* H *from* $\mathcal{A}$ *to* $\mathcal{B}$ *of* $\mathsf{smCAT}^{\mathrm{cart,fc}}_{\mathrm{sos}}$,

(a) for any $\{X_1, X_2\} \subseteq \mathrm{obj}_{\mathrm{S}(\mathcal{A})}$,

$$(\mathrm{S}(H)_{\otimes,X_1,X_2})^{\dagger_{\mathrm{S}(\mathcal{B})}} \circ_{\mathrm{S}(\mathcal{B})} \mathrm{S}(H)_{\otimes,X_1,X_2} = \mathrm{id}_{\mathrm{S}(\mathcal{B}),\mathrm{S}(H)(X_1)\otimes_{\mathrm{S}(\mathcal{B})}\mathrm{S}(H)(X_2)}$$

and

$$\mathrm{S}(H)_{\otimes,X_1,X_2} \circ_{\mathrm{S}(\mathcal{B})} (\mathrm{S}(H)_{\otimes,X_1,X_2})^{\dagger_{\mathrm{S}(\mathcal{B})}} = \mathrm{id}_{\mathrm{S}(\mathcal{B}),\mathrm{S}(H)(X_1\otimes_{\mathrm{S}(\mathcal{A})}X_2)}.$$

(b) as well as

$$(\mathrm{S}(H)_I)^{\dagger_{\mathrm{S}(\mathcal{B})}} \circ_{\mathrm{S}(\mathcal{B})} \mathrm{S}(H)_I = \mathrm{id}_{\mathrm{S}(\mathcal{B}),I_{\mathrm{S}(\mathcal{B})}}$$

and

$$\mathrm{S}(H)_I \circ_{\mathrm{S}(\mathcal{B})} (\mathrm{S}(H)_I)^{\dagger_{\mathrm{S}(\mathcal{B})}} = \mathrm{id}_{\mathrm{S}(\mathcal{B}),\mathrm{S}(H)(I_{\mathrm{S}(\mathcal{A})})}.$$

PROOF. Follows from Lemma 5.52. □

PROPOSITION 5.54. $\mathrm{S}(H)$ *is a monoidal †-functor from* $\mathrm{S}(\mathcal{A})$ *to* $\mathrm{S}(\mathcal{B})$ *for any* 0*-cells* $\mathcal{A}$ *and* $\mathcal{B}$ *and any* 1*-cell* H *from* $\mathcal{A}$ *to* $\mathcal{B}$ *of* $\mathsf{smCAT}^{\mathrm{cart,fc}}_{\mathrm{sos}}$*.*

PROOF. That claim is another way of expressing Lemma 5.53. □

5.2.3. *Symmetric Monoidal †-category.* Also the braiding of the span category is a unitary morphism.

LEMMA 5.55. $\gamma_{\mathrm{S}(\mathcal{A}),A_1,A_2}{}^{\dagger_{\mathrm{S}(\mathcal{A})}} = \gamma_{\mathrm{S}(\mathcal{A}),A_1,A_2}{}^{-1_{\mathrm{S}(\mathcal{A})}}$ *for any* $\{A_1, A_2\} \subseteq \mathrm{obj}_{\mathrm{S}(\mathcal{A})}$ *and any* 0*-cell* $\mathcal{A}$ *of* $\mathsf{smCAT}^{\mathrm{cart,fc}}_{\mathrm{sos}}$*.*

PROOF. As in the proof of Lemma 5.50, we infer

$$\begin{aligned}
\gamma_{\mathrm{S}(\mathcal{A}),A_1,A_2}{}^{-1_{\mathrm{S}(\mathcal{A})}} &= \Theta_{\mathcal{A}}(\gamma_{\mathcal{A},A_2,A_1})^{-1_{\mathrm{S}(\mathcal{A})}} \\
&= \Theta_{\mathcal{A}}(\gamma_{\mathcal{A},A_1,A_2}{}^{-1_{\mathcal{A}}}) \\
&= [\mathrm{id}_{\mathcal{A},A_2\otimes_{\mathcal{A}}A_1}, \gamma_{\mathcal{A},A_1,A_2}{}^{-1_{\mathcal{A}}}] \\
&= [\mathrm{id}_{\mathcal{A},A_2\otimes_{\mathcal{A}}A_1} \circ_{\mathcal{A}} \gamma_{\mathcal{A},A_1,A_2}, \gamma_{\mathcal{A},A_1,A_2}{}^{-1_{\mathcal{A}}} \circ_{\mathcal{A}} \gamma_{\mathcal{A},A_1,A_2}] \\
&= [\gamma_{\mathcal{A},A_1,A_2}, \mathrm{id}_{\mathcal{A},A_1\otimes_{\mathcal{A}}A_2}] \\
&= [\mathrm{id}_{\mathcal{A},A_1\otimes_{\mathcal{A}}A_2}, \gamma_{\mathcal{A},A_1,A_2}]^{\dagger_{\mathrm{S}}} \\
&= \Theta_{\mathcal{A}}(\gamma_{\mathcal{A},A_1,A_2})^{\dagger_{\mathrm{S}}} \\
&= \gamma_{\mathrm{S}(\mathcal{A}),A_1,A_2}{}^{\dagger_{\mathrm{S}}},
\end{aligned}$$

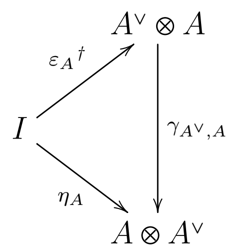
employing Proposition 5.12. □

PROPOSITION 5.56. $\mathrm{S}(\mathcal{A})$ *is a symmetric monoidal †-category for any* 0*-cell* $\mathcal{A}$ *of* $\mathsf{smCAT}^{\mathrm{cart,fc}}_{\mathrm{sos}}$*.*

PROOF. Follows from Lemma 5.55. □

5.2.4. *Rigid Symmetric Monoidal †-category.* Finally, the rigid monoidal structure of the span category is compatible with its †-structure.

LEMMA 5.57. $\gamma_{\mathrm{S}(\mathcal{A}),A^{\vee_{\mathrm{S}(\mathcal{A})}},A} \circ_{\mathrm{S}(\mathcal{A})} \varepsilon_{\mathrm{S}(\mathcal{A}),A}{}^{\dagger_{\mathrm{S}(\mathcal{A})}} = \eta_{\mathrm{S}(\mathcal{A}),A}$ *for any* $A \in \mathrm{obj}_{\mathrm{S}(\mathcal{A})}$ *and any* 0*-cell* $\mathcal{A}$ *of* $\mathsf{smCAT}^{\mathrm{cart,fc}}_{\mathrm{sos}}$*. In other words, in* $\mathrm{S}(\mathcal{A})$ *the diagram*



commutes.

PROOF. By Lemma 4.6 a pull-back of $(\mathrm{id}_{\mathcal{A},A} \times_{\mathcal{A}} \mathrm{id}_{\mathcal{A},A}, \mathrm{id}_{\mathcal{A},A\otimes_{\mathcal{A}}A})$ in $\mathcal{A}$ is given by $(\mathrm{id}_{\mathcal{A},A}, \mathrm{id}_{\mathcal{A},A} \times_{\mathcal{A}} \mathrm{id}_{\mathcal{A},A})$. Moreover, $\gamma_{\mathcal{A},A,A} \circ_{\mathcal{A}} (\mathrm{id}_{\mathcal{A},A} \times_{\mathcal{A}} \mathrm{id}_{\mathcal{A},A}) = \mathrm{id}_{\mathcal{A},A} \times_{\mathcal{A}} \mathrm{id}_{\mathcal{A},A}$ by

Lemma 4.16. Abbreviate $\mathrm{S}(\mathcal{A})$ by S. Hence, using $A^{\vee_\mathrm{S}} = A$, we compute directly

$$\begin{aligned}\gamma_{\mathrm{S},A^{\vee_\mathrm{S}},A} \circ_\mathrm{S} {\varepsilon_{\mathrm{S},A}}^{\dagger_\mathrm{S}} &= [\mathrm{id}_{\mathcal{A},A\otimes_\mathcal{A}A}, \gamma_{\mathcal{A},A,A}] \circ_\mathrm{S} [\mathrm{id}_{\mathcal{A},A} \times_\mathcal{A} \mathrm{id}_{\mathcal{A},A}, \omega_{\mathcal{A},A}]^{\dagger_\mathrm{S}}\\ &= [\mathrm{id}_{\mathcal{A},A\otimes_\mathcal{A}A}, \gamma_{\mathcal{A},A,A}] \circ_\mathrm{S} [\omega_{\mathcal{A},A}, \mathrm{id}_{\mathcal{A},A} \times_\mathcal{A} \mathrm{id}_{\mathcal{A},A}]\\ &= [\omega_{\mathcal{A},A} \circ_\mathcal{A} \mathrm{id}_{\mathcal{A},A}, \gamma_{\mathcal{A},A,A} \circ_\mathcal{A} (\mathrm{id}_{\mathcal{A},A} \times_\mathcal{A} \mathrm{id}_{\mathcal{A},A})]\\ &= [\omega_{\mathcal{A},A}, \mathrm{id}_{\mathcal{A},A} \times_\mathcal{A} \mathrm{id}_{\mathcal{A},A}]\\ &= \eta_{\mathrm{S},A},\end{aligned}$$

and obtain the claimed identity. □

PROPOSITION 5.58. *$\mathrm{S}(\mathcal{A})$ is a rigid symmetric monoidal †-category for any 0-cell $\mathcal{A}$ of $\mathsf{smCAT}^{\mathrm{cart,fc}}_{\mathrm{sos}}$.*

PROOF. Was shown in Lemma 5.57. □

That concludes the first stage of the construction. All the claims for this state that will be required in order to show the main result on later stages have been proved. Again, it must be emphasized, that *none* of this was new.

5.3. Stage 2a: Relations. In the second step of the construction, we pass from the span category to the generalized relations category with respect to a pull-back stable factorization system. While all proofs for the span stage of the construction could still be given, this is unfortunately no longer the case for the this stage. However, times was still enough to prove all the claim for the 0-cells of the 2-functor. Since the construction on 1-cells and 2-cells is not covered, there is no problem making the below assumption in the following.

ASSUMPTIONS 5.59. Throughout this section, let $\mathcal{A}$ be a cartesian monoidal category with pull-backs and let $(\mathcal{E},\mathcal{M})$ be any pull-back-stable factorization system of $\mathcal{A}$. Moreover, $\mathrm{S}(\mathcal{A})$ will be abbreviated by S if there is no risk of confusion.

As explained at the beginning of the section, the construction of the relation category is well-known. However, only few of the involved proofs can be found in the literature. That makes it quite difficult to convince oneself that the construction works. In order to remedy this situation, the proofs are provided in this chapter.

When referring to the literature it is important to keep track of the different assumptions made on $(\mathcal{E},\mathcal{M})$. For example, [Kle70] only treats the case where $(\mathcal{E},\mathcal{M})$ is proper, i.e., where $\mathcal{E} \hookrightarrow \mathrm{epi}_\mathcal{A}$ and $\mathcal{M} \hookrightarrow \mathrm{mon}_\mathcal{A}$. In contrast, [Jay95; JW00] mostly assumes $\mathcal{M} \hookrightarrow \mathrm{mon}_\mathcal{A}$. And [Mei74a; Mei74b] and [Pav95; Pav96] assume neither. For the third stage we will later assume $\mathcal{E} \hookrightarrow \mathrm{epi}_\mathcal{A}$.

5.3.1. *Category.* First, we define the basic category structure for $(\mathcal{E},\mathcal{M})$-relations.

LEMMA 5.60. *For any objects A and B of $\mathcal{A}$ and any two spans (f,g) and (f',g') in $\mathcal{A}$ from A to B, whenever (f,g) and (f',g') are equivalent, then $f \times_\mathcal{A} g \in \mathcal{M}$ if and only if $f' \times_\mathcal{A} g' \in \mathcal{M}$.*

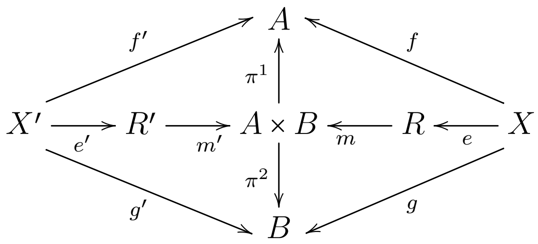
Proof. By symmetry it suffices to assume $f \times g \in \mathcal{M}$ and prove $f' \times g' \in \mathcal{M}$. Let X be the base of (f, g) and X' that of (f', g'). By assumption there exists an isomorphism $u: X' \to X$ with $f' = f \circ u$ and $g' = g \circ u$. It follows $f' \times g' = (f \circ u) \times (g \circ u) = (f \times g) \circ u$. Because $\mathcal{M}$ is closed under appending its elements to isomorphisms of $\mathcal{A}$ we have thus shown $f' \times g' \in \mathcal{M}$ as claimed. □

Definition 5.61. We call a class $[f, g]$ of spans in $\mathcal{A}$ from object A to object B of $\mathcal{A}$ an $(\mathcal{E}, \mathcal{M})$-*relation* in $\mathcal{A}$ if $f \times_{\mathcal{A}} g \in \mathcal{M}$.

Obviously, every relation is a span but generally not vice versa. There is a kind of "projection" operation which allows us to turn any span into a relation.

Lemma 5.62. *Let A and B be objects of $\mathcal{A}$, let (f, g) and (f', g') be spans in $\mathcal{A}$ from A to B and let (e, m) be an $(\mathcal{E}, \mathcal{M})$-factorization of $f \times_{\mathcal{A}} g$ and (e', m') one of $f' \times_{\mathcal{A}} g'$. If (f, g) and (f', g') form a pair of equivalent spans in $\mathcal{A}$ from A to B, then so do $(\pi^1_{\mathcal{A},A,B} \circ_{\mathcal{A}} m, \pi^2_{\mathcal{A},A,B} \circ_{\mathcal{A}} m)$ and $(\pi^1_{\mathcal{A},A,B} \circ_{\mathcal{A}} m', \pi^2_{\mathcal{A},A,B} \circ_{\mathcal{A}} m')$.*

Proof. Let X and X' be the bases of (f, g) and (f', g'), respectively. By assumption there exists an isomorphsim $u: X' \to X$ such that $f' = f \circ u$ and $g' = g \circ u$.



It follows, $f' \times g' = (f \circ u) \times (g \circ u) = (f \times g) \circ u = (m \circ e) \circ u = m \circ (e \circ u)$. Moreover, $u \in \mathcal{E}$ by Lemma 4.32 since u is an isomorphism. Hence, also $e \circ u \in \mathcal{E}$ by Lemma 4.33 (b). Thus, $(e \circ u, m) \in \mathcal{E} \times \mathcal{M}$ is an $(\mathcal{E}, \mathcal{M})$-factorization of $f' \times g'$. By the essential uniqueness of factorizations we can infer the existence of an isomorphism $w: R' \to R$ with $e \circ u = w \circ e'$ and $m' = m \circ w$. The latter identity of course implies in particular $\pi^i_{A,B} \circ m' = \pi^i_{A,B} \circ m \circ w$ for each $i \in \{1, 2\}$ and thus shows $(\pi^1_{A,B} \circ m, \pi^2_{A,B} \circ m)$ and $(\pi^1_{A,B} \circ m', \pi^2_{A,B} \circ m')$ to be equivalent. □

By the preceding lemma the following is a well-defined mapping.

Definition 5.63. For any object $X \in \mathcal{A}$ define $\Phi_{\mathcal{A},\mathcal{E}}(X) := X$ and for any objects A and B and any class $[f, g]$ of spans in $\mathcal{A}$ from A to B let $\Phi_{\mathcal{A},\mathcal{E}}([f, g]) := [\pi^1_{\mathcal{A},A,B} \circ_{\mathcal{A}} m, \pi^2_{\mathcal{A},A,B} \circ_{\mathcal{A}} m]$, where (e, m) is any $(\mathcal{E}, \mathcal{M})$-factorization of $f \times_{\mathcal{A}} g$.

Relations are precisely the "fixed points" of this "projection" operation.

Lemma 5.64. *Any class $[f, g]$ of spans in $\mathcal{A}$ from any object A to any object B is an $(\mathcal{E}, \mathcal{M})$-relation in $\mathcal{A}$ if and only if $\Phi_{\mathcal{A},\mathcal{E}}([f, g]) = [f, g]$.*

PROOF. If (e, m) is any $(\mathcal{E}, \mathcal{M})$-factorization of $f \times g$, by definition, $\Phi([f, g]) = [\pi^1_{A,B} \circ m, \pi^2_{A,B} \circ m]$. Because $(\pi^1_{A,B} \circ m) \times (\pi^2_{A,B} \circ m) = (\pi^1_{A,B} \times \pi^2_{A,B}) \circ m = m$, Lemma 5.60 assures us that $\Phi([f, g])$ is an $(\mathcal{E}, \mathcal{M})$-relation. Hence, if $[f, g] = \Phi([f, g])$, then $[f, g]$ is an $(\mathcal{E}, \mathcal{M})$-relation. We only need to show the converse.

Let $[f, g]$ be an $(\mathcal{E}, \mathcal{M})$-relation. We prove that (f, g) and $(\pi^1_{A,B} \circ m, \pi^2_{A,B} \circ m)$ are equivalent. Let X be the base of (f, g) and R the domain of e. By Lemma 4.32 the class $\mathcal{E}$ contains the isomorphism id_X. Since $f \times g \in \mathcal{M}$ according to Lemma 5.60, the pair $(\mathrm{id}_X, f \times g)$ is an $(\mathcal{E}, \mathcal{M})$-factorization of $f \times g$. By the essential uniqueness of factorizations there must then exist an isomorphism $u\colon R \to X$ with $\mathrm{id}_X = u \circ e$ and $m = (f \times g) \circ u$. The latter identity implies $\pi^1_{A,B} \circ m = \pi^1_{A,B} \circ (f \times g) \circ u = f \circ u$ and, likewise, $\pi^2_{A,B} \circ m = g \circ u$. And that is what we needed to see. □

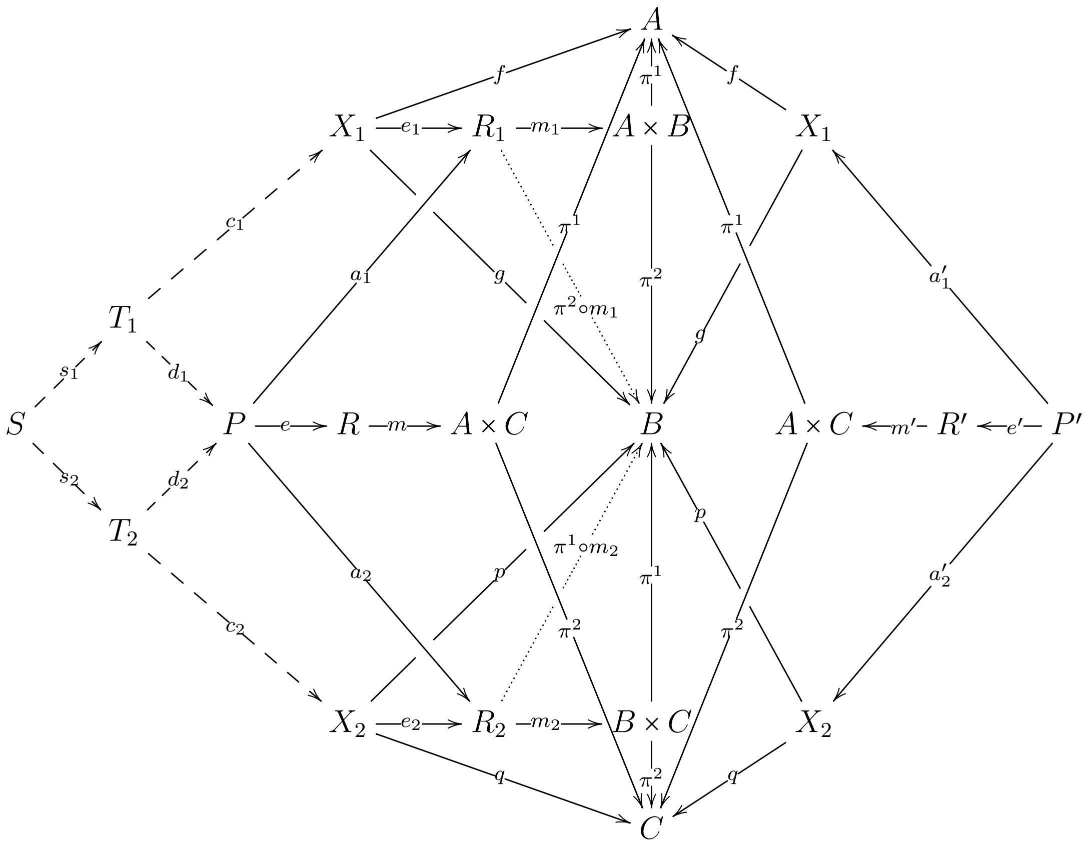
The following crucial result about the "projection" being idempotent is shown in [JW00, Section 2.4] under even weaker assumptions.

LEMMA 5.65. *For any objects A, B and C of $\mathcal{A}$ and any classes $[f, g]$ of spans in $\mathcal{A}$ from A to B and $[p, q]$ from B to C,*

$$\Phi_{\mathcal{A},\mathcal{E}}(\Phi_{\mathcal{A},\mathcal{E}}([p, q]) \circ_{\mathrm{S}} \Phi_{\mathcal{A},\mathcal{E}}([f, g])) = \Phi_{\mathcal{A},\mathcal{E}}([p, q] \circ_{\mathrm{S}} [f, g]).$$

PROOF. Let X_1 be the base of (f, g) and X_2 that of (p, q), let (e_1, m_1) be an $(\mathcal{E}, \mathcal{M})$-factorization of $f \times g$, let (e_2, m_2) be one of $p \times q$, let R_1 be the domain of m_1 and R_2 that of m_2, let (a_1, a_2) be a pull-back of $(\pi^1_{A,B} \circ m_1, \pi^2_{B,C} \circ m_2)$ with pull-back object P and (a'_1, a'_2) a pull-back of (g, p) with pull-back object P', let (e, m) be an $(\mathcal{E}, \mathcal{M})$-factorization of $(\pi^1_{A,B} \circ m_1 \circ a_1) \times (\pi^2_{B,C} \circ m_2 \circ a_2)$ and (e', m') one of $(f \circ a'_1) \times (q \circ a'_2)$ and let R be the domain of m and R' that of m'. By Lemma 5.5 it suffices to prove that the two spans $(\pi^1_{A,C} \circ m, \pi^2_{A,C} \circ m)$ and $(\pi^1_{A,C} \circ m', \pi^2_{A,C} \circ m')$ from A to C are equivalent. More precisely, we construct an isomorphism $u\colon R' \to R$

with $\pi^1_{A,C} \circ m' = (\pi^1_{A,C} \circ m) \circ u$ and $\pi^2_{A,C} \circ m' = (\pi^2_{A,C} \circ m) \circ u$, i.e., with $m' = m \circ u$.



Step 1: Construction of v. As an intermediate step we find a second pull-back of (g,p). Since $\mathcal{A}$ has all pull-backs there exist in particular pull-backs (c_1,d_1) of (e_1,a_1) with pull-back object T_1 and (c_2,d_2) of (e_2,a_2) with pull-back object T_2. Then, there also exists a pull-back (s_1,s_2) of (d_1,d_2) with pull-back object S. Since all of the four small squares in the resulting diagram

$$\begin{array}{ccccc} S & \xrightarrow{s_1} & T_1 & \xrightarrow{c_1} & X_1 \\ \downarrow{\scriptstyle s_2} & & \downarrow{\scriptstyle d_1} & & \downarrow{\scriptstyle e_1} \\ T_2 & \xrightarrow[d_2]{} & P & \xrightarrow[a_1]{} & R_1 \\ \downarrow{\scriptstyle c_2} & & \downarrow{\scriptstyle a_2} & & \downarrow{\scriptstyle \pi^2\circ m_1} \\ X_2 & \xrightarrow[e_2]{} & R_2 & \xrightarrow[\pi^1\circ m_2]{} & B \end{array}$$

are pull-backs by assumption, Lemma 4.5 guarantees that $(c_1 \circ s_1, c_2 \circ s_2)$ is a pull-back of $((\pi^2_{A,B} \circ m_1) \circ e_1, (\pi^1_{B,C} \circ m_2) \circ e_2)$. Because $(\pi^2_{A,B} \circ m_1) \circ e_1 = \pi^2_{A,B} \circ (m_1 \circ e_1) = \pi^2_{A,B} \circ (f \times g) = g$ and, likewise, $(\pi^1_{B,C} \circ m_2) \circ e_2 = p$, the pair $(c_1 \circ s_1, c_2 \circ s_2)$ is indeed a pull-back of (g, p). By the essential uniqueness of pull-backs there then exists an isomorphism $v\colon P' \to S$ with $a_1' = (c_1 \circ s_1) \circ v$ and $a_2' = (c_2 \circ s_2) \circ v$.

Step 2: Construction of u. In order to find u we construct a second $(\mathcal{E}, \mathcal{M})$-factorization of $(f \circ a_1') \times (q \circ a_2')$. Because the factorization system $(\mathcal{E}, \mathcal{M})$ is stable the morphisms d_1 and d_2, being pull-backs of $e_1 \in \mathcal{E}$ and $e_2 \in \mathcal{E}$, respectively, are elements of $\mathcal{E}$ themselves. By the same reasoning, s_1 as a pull-back of $d_2 \in \mathcal{E}$ also belongs to $\mathcal{E}$. In conclusion, $d_1 \circ s_1 \in \mathcal{E}$ since $\mathcal{E}$ is closed under composition by Lemma 4.33. Applying that same lemma once more then yields $e \circ (d_1 \circ s_1) \in \mathcal{E}$ because $e \in \mathcal{E}$. Since $\mathcal{E}$ contains all isomorphisms by Lemma 4.32 and thus also v in particular we can infer $(e \circ d_1 \circ s_1) \circ v \in \mathcal{E}$ by Lemma 4.33. Moreover, by construction,

$$\begin{aligned}
&(\pi^1_{A,C} \circ m) \circ (e \circ d_1 \circ s_1 \circ v)\\
&\quad = (\pi^1_{A,C} \circ m \circ e) \circ d_1 \circ s_1 \circ v = (\pi^1_{A,C} \circ m_1 \circ a_1) \circ d_1 \circ s_1 \circ v\\
&\quad = \pi^1_{A,C} \circ m_1 \circ (a_1 \circ d_1) \circ s_1 \circ v = \pi^1_{A,C} \circ m_1 \circ (e_1 \circ c_1) \circ s_1 \circ v\\
&\quad = \pi^1_{A,C} \circ (m_1 \circ e_1) \circ (c_1 \circ s_1 \circ v) = \pi^1_{A,B} \circ f \circ a_1',
\end{aligned}$$

and, by an analogous computation, $(\pi^2_{A,C} \circ m) \circ (e \circ d_1 \circ s_1 \circ v) = \pi^1_{A,C} \circ q \circ a_2'$. In other words, $m \circ (e \circ d_1 \circ s_1 \circ v) = (f \circ a_1') \times (q \circ a_2')$. Thus, $(e \circ d_1 \circ s_1 \circ v, m)$ is an $(\mathcal{E}, \mathcal{M})$-factorization of $(f \circ a_1') \times (q \circ a_2')$ as intended. Since those are essentially unique there must then exist an isomorphism $u\colon R' \to R$ with $e \circ d_1 \circ s_1 \circ v = u \circ e'$ and $m' = m \circ u$. And the second identity is just what we needed to prove. ☐

We can now give the definition of the basic category structure of the relations categories.

DEFINITION 5.66. (a) Define $\mathrm{obj}_{\mathrm{Rel}(\mathcal{A},\mathcal{E})} := \mathrm{obj}_{\mathcal{A}}$.

(b) For any objects A and B of $\mathcal{A}$ let $\mathrm{mor}_{\mathrm{Rel}(\mathcal{A},\mathcal{E})}(A, B)$ be the class of all $(\mathcal{E}, \mathcal{M})$-relations from A to B in $\mathcal{A}$.
(c) For any object X of $\mathcal{A}$ define $\mathrm{id}_{\mathrm{Rel}(\mathcal{A},\mathcal{E}),X} := \Phi_{\mathcal{A},\mathcal{E}}(\mathrm{id}_{\mathrm{S}(\mathcal{A})}X)$.
(d) Given any objects A, B and B of $\mathcal{A}$ and any $(\mathcal{E}, \mathcal{M})$-relations $[f, g]$ from A to B and $[p, q]$ from B to C in $\mathcal{A}$ define

$$[p, q] \circ_{\mathrm{Rel}(\mathcal{A},\mathcal{E})} [f, g] := \Phi_{\mathcal{A},\mathcal{E}}([p, q] \circ_{\mathrm{S}(\mathcal{A})} [f, g]).$$

PROPOSITION 5.67. $(\mathrm{obj}_{\mathrm{Rel}(\mathcal{A},\mathcal{E})}, \mathrm{mor}_{\mathrm{Rel}(\mathcal{A},\mathcal{E})}, \circ_{\mathrm{Rel}(\mathcal{A},\mathcal{E})}, \mathrm{id}_{\mathrm{Rel}(\mathcal{A},\mathcal{E})})$ *is a category.*

PROOF. Let Rel be short for $\mathrm{Rel}(\mathcal{A}, \mathcal{E})$. We need to confirm that $\circ_{\mathrm{Rel}}$ is an associative operation and that $\mathrm{id}_{\mathrm{Rel}}$ provide neutral elements for it.

Composition is associative. Let A, B, C and D be objects of $\mathcal{A}$ and let x be an $(\mathcal{E}, \mathcal{M})$-relation in $\mathcal{A}$ from A to B, let y be one from B to C and let z be one from C to D. Then, since $z = \Phi(z)$ and $x = \Phi(x)$ by Lemma 5.64, applying Lemma 5.65

yields

$$\begin{aligned} z \circ_{\mathrm{Rel}} (y \circ_{\mathrm{Rel}} x) &= \Phi(z \circ_{\mathrm{S}} \Phi(y \circ_{\mathrm{S}} x)) = \Phi(\Phi(z) \circ_{\mathrm{S}} \Phi(y \circ_{\mathrm{S}} x)) \\ &= \Phi(z \circ_{\mathrm{S}} (y \circ_{\mathrm{S}} x)) = \Phi((z \circ_{\mathrm{S}} y) \circ_{\mathrm{S}} x) \\ &= \Phi(\Phi(z \circ_{\mathrm{S}} y) \circ_{\mathrm{S}} \Phi(x)) = \Phi(\Phi(z \circ_{\mathrm{S}} y) \circ_{\mathrm{S}} x) \\ &= (z \circ_{\mathrm{Rel}} y) \circ_{\mathrm{Rel}} x, \end{aligned}$$

where we have used that $\circ_{\mathrm{S}}$ is associative in the fourth step.

Identities are neutral elements. If A and B are any objects of $\mathcal{A}$ and x any relation in $\mathcal{A}$ from A to B, then, since $x = \Phi(x)$ by Lemma 5.64, Lemma 5.65 implies $x \circ_{\mathrm{Rel}} \mathrm{id}_{\mathrm{Rel},A} = \Phi(x \circ_{\mathrm{S}} \Phi(\mathrm{id}_{\mathrm{S},A})) = \Phi(\Phi(x) \circ_{\mathrm{S}} \Phi(\mathrm{id}_{\mathrm{S},A})) = \Phi(x \circ_{\mathrm{S}} \mathrm{id}_{\mathrm{S},A}) = \Phi(x) = x$ because id_{S} gives neutral elements of $\circ_{\mathrm{S}}$. Analogously, one computes $\mathrm{id}_{\mathrm{Rel},B} \circ_{\mathrm{Rel}} x = x$. □

PROPOSITION 5.68. *$\Phi_{\mathcal{A},\mathcal{E}}$ is a full functor* $\mathrm{S}(\mathcal{A}) \to \mathrm{Rel}(\mathcal{A},\mathcal{E})$ *which is surjective on objects.*

PROOF. This is literally the definition of $\mathrm{Rel}(\mathcal{A},\mathcal{E})$. □

The equivalence relation $x \sim x'$ between morphisms x and x' defined by $\Phi_{\mathcal{A},\mathcal{E}}(x) = \Phi_{\mathcal{A},\mathcal{E}}(x')$ of $\mathrm{S}(\mathcal{A})$ is a congruence on $\mathrm{S}(\mathcal{A})$ by Lemma 5.65. Hence, $\Phi_{\mathcal{A},\mathcal{E}}$ actually exhibits $\mathrm{Rel}(\mathcal{A},\mathcal{E})$ as (one representative of) a *quotient category* of $\mathrm{S}(\mathcal{A})$ (a co-subobject of $\mathrm{S}(\mathcal{A})$ in the the category of categories).

PROPOSITION 5.69. *If $\mathcal{A}$ is $\mathcal{M}$-subobject-small,* $\mathrm{Rel}(\mathcal{A},\mathcal{E})$ *is locally small.*

PROOF. Let A and B be any objects of S, i.e., of $\mathcal{A}$. Let $\mathcal{M}_{A\otimes_{\mathcal{A}}B}$ denote the class of all elements of $\mathcal{M}$ with co-domain $A \otimes_{\mathcal{A}} B$ and let $\sim$ be the equivalence relation on $\mathcal{M}_{A\otimes_{\mathcal{A}}B}$ which calls two such morphisms m and m' equivalent if there exists an isomorphism u of $\mathcal{A}$ with $m' = m \circ_{\mathcal{A}} u$. Then the assumption that $\mathcal{A}$ be $\mathcal{M}$-subobject-small ensures that $\mathcal{M}_{A\otimes_{\mathcal{A}}B}/\sim$ is a set. By the universal property of products and Lemma 5.60 the rule $[f,g] \mapsto [f \times_{\mathcal{A}} g]_{\sim}$ defines a bijection of classes $\mathrm{mor}_{\mathrm{Rel}}(A,B) \to \mathcal{M}_{A\otimes_{\mathcal{A}}B}/\sim$. □

5.3.2. *Monoidal Category.* The relation category inherits a monoidal structure from the span category.

In regard of Lemma 5.64 the next result shows in particular that the monoidal product of two relations, viewed as span classes, is again a relation.

LEMMA 5.70. *For any objects A_1, A_2, B_1 and B_2 of $\mathcal{A}$ and any $(\mathcal{E},\mathcal{M})$-relations $[f_1,g_1]$ from A_1 to B_1 and $[f_2,g_2]$ from A_2 to B_2 in $\mathcal{A}$,*

$$\Phi_{\mathcal{A},\mathcal{E}}([f_1,g_1]) \otimes_{\mathcal{A}} \Phi_{\mathcal{A},\mathcal{E}}([f_2,g_2]) = \Phi_{\mathcal{A},\mathcal{E}}([f_1,g_1] \otimes_{\mathrm{S}} [f_2,g_2]).$$

PROOF. For each $i \in \{1,2\}$ let (e_i,m_i) be an $(\mathcal{E},\mathcal{M})$-factorization of $f_i \times g_i$ with image object R_i, and let (e,m) be one of $(f_1 \otimes f_2) \times (g_1 \otimes g_2)$ with image object

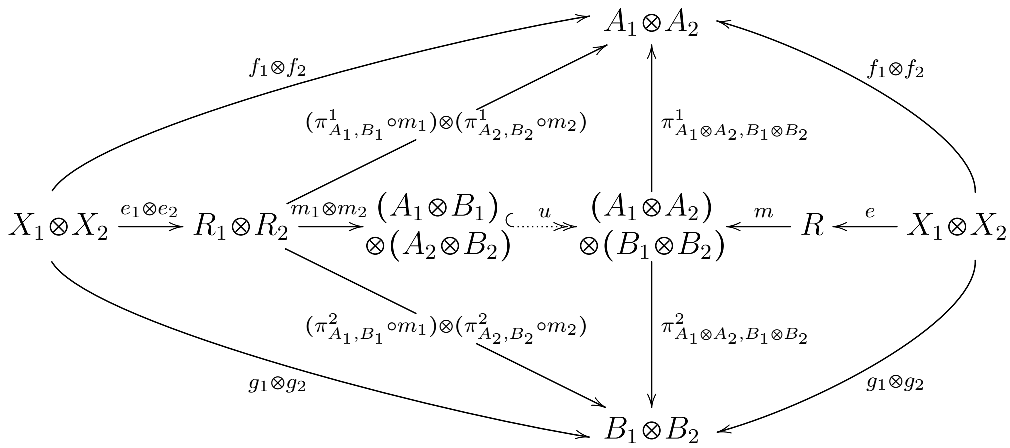
R. Then, we need to find an isomorphism $w\colon R_1 \otimes R_2 \to R$ of $\mathcal{A}$ such that for each $i \in \{1,2\}$

$$(\pi^i_{A_1,B_1} \circ m_1) \otimes (\pi^i_{A_2,B_2} \circ m_2) \stackrel{!}{=} (\pi^i_{A_1 \otimes A_2, B_1 \otimes B_2} \circ m) \circ w.$$

Step 1: Reformulation. Abbreviate $m_1 = m_1^1 \times m_1^2$ and $m_2 = m_2^1 \times m_2^2$. Using the universal property of products, this assertion can then be rephrased as

$$(m_1^1 \otimes m_2^1) \times (m_1^2 \otimes m_2^2) \stackrel{!}{=} m \circ w.$$

For brevity, let $u := \mu_{A_1,B_1,A_2,B_2}$ be the middle four interchange isomorphism $(A_1 \otimes B_1) \otimes (A_2 \otimes B_2) \to (A_1 \otimes A_2) \otimes (B_1 \otimes B_2)$.



By using Lemma 4.18 we can recognize

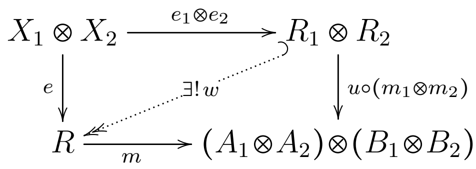
$$\begin{aligned}
&(m_1^1 \otimes m_2^1) \times (m_1^2 \otimes m_2^2)\\
&\quad = ((m_1^1 \circ \pi^1_{R_1,R_2}) \times (m_2^1 \circ \pi^2_{R_1,R_2})) \times ((m_1^2 \circ \pi^1_{R_1,R_2}) \times (m_2^2 \circ \pi^2_{R_1,R_2}))\\
&\quad = u \circ (((m_1^1 \circ \pi^1_{R_1,R_2}) \times (m_1^2 \circ \pi^1_{R_1,R_2})) \times ((m_2^1 \circ \pi^2_{R_1,R_2}) \times (m_2^2 \circ \pi^2_{R_1,R_2})))\\
&\quad = u \circ (((m_1^1 \times m_1^2) \circ \pi^1_{R_1,R_2}) \times ((m_2^1 \times m_2^2) \circ \pi^2_{R_1,R_2}))\\
&\quad = u \circ ((m_1 \circ \pi^1_{R_1,R_2}) \times (m_2 \circ \pi^2_{R_1,R_2}))\\
&\quad = u \circ (m_1 \otimes m_2).
\end{aligned}$$

Consequently, we can reformulate the claim as

$$u \circ (m_1 \otimes m_2) \stackrel{!}{=} m \circ w.$$

Step 2: Construction of w. If we show that $(e_1 \otimes e_2, u \circ (m_1 \otimes m_2))$ is an $(\mathcal{E},\mathcal{M})$-factorization of $(f_1 \otimes f_2) \times (g_1 \otimes g_2)$, then by the essential uniqueness of factorizations we will have shown the existence of an isomorphism $w\colon R_1 \otimes R_2 \to R$ with $e =$

$w \circ (e_1 \otimes e_2)$ and $u \circ (m_1 \otimes m_2) = m \circ w$ – and thus, by Step 1, the claim.



Lemma 4.35 assures us that $e_1 \otimes e_2 \in \mathcal{E}$ because $e_1 \in \mathcal{E}$ and $e_2 \in \mathcal{E}$. Likewise, $m_1 \otimes m_2 \in \mathcal{M}$ follows from $m_1 \in \mathcal{M}$ and $m_2 \in \mathcal{M}$ by Lemma 4.34. Since, u is an isomorphism Lemmata 4.32 and 4.33 (b) therefore allow us to conclude $u \circ (m_1 \otimes m_2) \in \mathcal{M}$. Finally, by Lemma 4.18,

$$\begin{aligned}
&(u \circ (m_1 \otimes m_2)) \circ (m_1 \otimes m_2) \\
&\qquad = u \circ ((m_1 \otimes m_2) \circ (m_1 \otimes m_2)) \\
&\qquad = u \circ ((m_1 \circ e_1) \otimes (m_2 \circ e_2)) \\
&\qquad = u \circ ((f_1 \times g_1) \otimes (f_2 \times g_2)) \\
&\qquad = u \circ (((f_1 \times g_1) \circ \pi^1_{X_1,X_2}) \times ((f_2 \times g_2) \circ \pi^2_{X_1,X_2})) \\
&\qquad = u \circ (((f_1 \circ \pi^1_{X_1,X_2}) \times (g_1 \circ \pi^1_{X_1,X_2})) \times ((f_2 \circ \pi^2_{X_1,X_2}) \times (g_2 \circ \pi^2_{X_1,X_2}))) \\
&\qquad = (((f_1 \circ \pi^1_{X_1,X_2}) \times (f_2 \circ \pi^2_{X_1,X_2})) \times ((g_1 \circ \pi^1_{X_1,X_2}) \times (g_2 \circ \pi^2_{X_1,X_2}))) \\
&\qquad = (f_1 \otimes f_2) \times (g_1 \otimes g_2).
\end{aligned}$$

And that is all we needed to see. □

As announced, $\mathrm{Rel}(\mathcal{A}, \mathcal{E})$ inherits its monoidal structure from $\mathrm{S}(\mathcal{A})$.

DEFINITION 5.71. For any objects A, A_1, A_2, B, B_1, B_2 and C of $\mathcal{A}$ and any $(\mathcal{E}, \mathcal{M})$-relations $[f_1, g_1] \in \mathrm{mor}_{\mathrm{Rel}(\mathcal{A},\mathcal{E})}(A_1, B_1)$ and $[f_2, g_2] \in \mathrm{mor}_{\mathrm{Rel}(\mathcal{A},\mathcal{E})}(A_2, B_2)$ in $\mathcal{A}$ define

(a) $A_1 \otimes_{\mathrm{Rel}(\mathcal{A},\mathcal{E})} A_2 := \Phi_{\mathcal{A},\mathcal{E}}(A_1 \otimes_{\mathrm{S}(\mathcal{A})} A_2) = A_1 \otimes_{\mathcal{A}} A_2$,
(b) $I_{\mathrm{Rel}(\mathcal{A},\mathcal{E})} := \Phi_{\mathcal{A},\mathcal{E}}(I_{\mathrm{S}}) = I_{\mathcal{A}}$,
(c) $[f_1, g_1] \otimes_{\mathrm{Rel}(\mathcal{A},\mathcal{E})} [f_2, g_2] := \Phi_{\mathcal{A},\mathcal{E}}([f_1, g_1] \otimes_{\mathrm{S}\mathcal{A}} [f_2, g_2]) = [f_1, g_1] \otimes_{\mathrm{S}(\mathcal{A})} [f_2, g_2]$,
(d) $\alpha_{\mathrm{Rel}(\mathcal{A},\mathcal{E}),A,B,C} := \Phi_{\mathcal{A},\mathcal{E}}(\alpha_{\mathrm{S}(\mathcal{A}),A,B,C})$,
(e) $\lambda_{\mathrm{Rel}(\mathcal{A},\mathcal{E}),A} := \Phi_{\mathcal{A},\mathcal{E}}(\lambda_{\mathrm{S}(\mathcal{A}),A})$,
(f) $\rho_{\mathrm{Rel}(\mathcal{A},\mathcal{E}),A} := \Phi_{\mathcal{A},\mathcal{E}}(\rho_{\mathrm{S}(\mathcal{A}),A})$,

LEMMA 5.72. *Let* Rel *be short for* $\mathrm{Rel}(\mathcal{A}, \mathcal{E})$.
(a) $\otimes_{\mathrm{Rel}}$ *is a functor* $\mathrm{Rel} \otimes_{\mathsf{CAT}} \mathrm{Rel} \to \mathrm{Rel}$.
(b) α_{Rel} *is a natural isomorphism of functors* $\mathrm{Rel} \otimes_{\mathsf{CAT}} \mathrm{Rel} \otimes_{\mathsf{CAT}} \mathrm{Rel} \to \mathrm{Rel}$ *from* $((\cdot_1) \otimes_{\mathrm{Rel}} (\cdot_2)) \otimes_{\mathrm{Rel}} (\cdot_3)$ *to* $(\cdot_1) \otimes_{\mathrm{Rel}} ((\cdot_2) \otimes_{\mathrm{Rel}} (\cdot_3))$.
(c) λ_{Rel} *is a natural isomorphism of* Rel-*endofunctors* $I_{\mathrm{Rel}} \otimes_{\mathrm{Rel}} (\,\cdot\,) \to (\,\cdot\,)$.
(d) ρ_{Rel} *is a natural isomorphism of* Rel-*endofunctors* $(\,\cdot\,) \otimes_{\mathrm{Rel}} I_{\mathrm{Rel}} \to (\,\cdot\,)$.

(e) *For any objects A, B, C and D of* Rel *a commutative diagram in* Rel *is given by*

$$\begin{array}{ccccc} & & (A\otimes B)\otimes(C\otimes D) & & \\ & \nearrow^{\alpha_{A\otimes B,C,D}} & & \searrow^{\alpha_{A,B,C\otimes D}} & \\ ((A\otimes B)\otimes C)\otimes D & & & & A\otimes(B\otimes(C\otimes D)) \\ \downarrow{\scriptstyle \alpha_{A,B,C}\otimes \mathrm{id}_D} & & & & \uparrow{\scriptstyle \mathrm{id}_A\otimes\alpha_{B,C,D}} \\ (A\otimes(B\otimes C))\otimes D & & \xrightarrow[\alpha_{A,B\otimes C,D}]{} & & A\otimes((B\otimes C)\otimes D)\end{array}$$

(f) *For any objects A and B of* Rel *a commutative diagram in* Rel *is given by*

$$\begin{array}{ccc} (A\otimes I)\otimes B & \xrightarrow{\alpha_{A,I,B}} & A\otimes(I\otimes B) \\ & {\scriptstyle \rho_A\otimes\mathrm{id}_B}\searrow \quad \swarrow{\scriptstyle \mathrm{id}_A\otimes\lambda_B} & \\ & A\otimes B & \end{array}$$

PROOF. (a) We have to prove that $\otimes_{\mathrm{Rel}}$ respects identities and composition.

Step 1: Identities respected. For any $\{A_1, A_2\} \subseteq \mathrm{obj}_{\mathrm{Rel}}$ we conclude, using that S is a monoidal category,

$$\begin{aligned} \mathrm{id}_{\mathrm{Rel},A_1} \otimes_{\mathrm{Rel}} \mathrm{id}_{\mathrm{Rel},A_2} &= \Phi(\mathrm{id}_{\mathrm{S},A_1}) \otimes_{\mathrm{S}} \Phi(\mathrm{id}_{\mathrm{S},A_2}) \\ &\overset{5.70}{=} \Phi(\mathrm{id}_{\mathrm{S},A_1} \otimes_{\mathrm{S}} \mathrm{id}_{\mathrm{S},A_2}) \\ &\overset{5.25}{=} \Phi(\mathrm{id}_{\mathrm{S},A_1 \otimes_{\mathrm{S}} A_2}) \\ &= \mathrm{id}_{\mathrm{Rel},A_1 \otimes_{\mathrm{Rel}} A_2}, \end{aligned}$$

where are all other identities hold by definition.

Step 2: Composition respected. For any $i \in \{1,2\}$ let $\{A_i, B_i, C_i\} \subseteq \mathrm{obj}_{\mathrm{Rel}}$ as well as $x_i \in \mathrm{mor}_{\mathrm{Rel}}(A_i, B_i)$ and $y_i \in \mathrm{mor}_{\mathrm{Rel}}(B_i, C_i)$. Again, S being a monoidal category lets us infer,

$$\begin{aligned} (y_1 \otimes_{\mathrm{Rel}} y_2) \circ_{\mathrm{Rel}} (x_1 \otimes_{\mathrm{Rel}} x_2) &= \Phi(y_1 \otimes_{\mathrm{S}} y_2) \circ_{\mathrm{Rel}} \Phi(x_1 \otimes_{\mathrm{S}} x_2) \\ &\overset{5.68}{=} \Phi((y_1 \otimes_{\mathrm{S}} y_2) \circ_{\mathrm{S}} (x_1 \otimes_{\mathrm{S}} x_2)) \\ &\overset{5.25}{=} \Phi((y_1 \circ_{\mathrm{S}} x_1) \otimes_{\mathrm{S}} (y_2 \circ_{\mathrm{S}} x_2)) \\ &\overset{5.70}{=} \Phi(y_1 \circ_{\mathrm{S}} x_1) \otimes_{\mathrm{S}} \Phi(y_2 \circ_{\mathrm{S}} x_2) \\ &= (y_1 \circ_{\mathrm{Rel}} x_1) \otimes_{\mathrm{S}} (y_2 \circ_{\mathrm{Rel}} x_2) \\ &= (y_1 \circ_{\mathrm{Rel}} x_1) \otimes_{\mathrm{Rel}} (y_2 \circ_{\mathrm{Rel}} x_2), \end{aligned}$$

where we have used that Φ is a functor $\mathrm{S} \to \mathrm{Rel}$ by Proposition 5.68. That proves (a).

(b) First, we show that α_{Rel} is a natural transformation. For each $i \in \{1,2,3\}$ let $\{A_i, B_i\} \subseteq \text{obj}_{\text{Rel}}$ and $x_i \in \text{mor}_{\text{Rel}}(A_i, B_i)$. Then, we find

$$\begin{aligned}
&\alpha_{\text{Rel},B_1,B_2,B_3} \circ_{\text{Rel}} ((x_1 \otimes_{\text{Rel}} x_2) \otimes_{\text{Rel}} x_3) \\
&\overset{5.64,\,5.70}{=} \Phi(\alpha_{\text{S},B_1,B_2,B_3}) \circ_{\text{Rel}} \Phi((x_1 \otimes_{\text{S}} x_2) \otimes_{\text{S}} x_3) \\
&\overset{5.68}{=} \Phi(\alpha_{\text{S},B_1,B_2,B_3} \circ_{\text{S}} ((x_1 \otimes_{\text{S}} x_2) \otimes_{\text{S}} x_3)) \\
&\overset{5.25}{=} \Phi((x_1 \otimes_{\text{S}} (x_2 \otimes_{\text{S}} x_3)) \circ_{\text{S}} \alpha_{\text{S},A_1,A_2,A_3}) \\
&\overset{5.68}{=} \Phi(x_1 \otimes_{\text{S}} (x_2 \otimes_{\text{S}} x_3)) \circ_{\text{Rel}} \Phi(\alpha_{\text{S},A_1,A_2,A_3}) \\
&\overset{5.64,\,5.70}{=} (x_1 \otimes_{\text{Rel}} (x_2 \otimes_{\text{Rel}} x_3)) \circ_{\text{Rel}} \alpha_{\text{Rel},A_1,A_2,A_3},
\end{aligned}$$

as required. Moreover, because α_{S} is a natural isomorphism by Proposition 5.25, the definitions and Proposition 5.68 imply that α_{Rel} is a natural isomorphism as well.

(c) By Proposition 5.68 and the definition of λ_{Rel} it suffices to prove that λ_{Rel} is a natural transformation. And, indeed, for any $\{A, B\} \subseteq \text{obj}_{\text{Rel}}$ and $x \in \text{mor}_{\text{Rel}}(A, B)$,

$$\begin{aligned}
\lambda_{\text{Rel},B} \circ_{\text{Rel}} (\text{id}_{\text{Rel},I_{\text{Rel}}} \otimes_{\text{Rel}} x) &\overset{5.64}{=} \Phi(\lambda_{\text{S},B}) \circ_{\text{Rel}} (\Phi(\text{id}_{\text{S},I_{\text{S}}}) \otimes_{\text{S}} \Phi(x)) \\
&\overset{5.70}{=} \Phi(\lambda_{\text{S},B}) \circ_{\text{Rel}} \Phi(\text{id}_{\text{S},I_{\text{S}}} \otimes_{\text{S}} x) \\
&\overset{5.68}{=} \Phi(\lambda_{\text{S},B} \circ_{\text{S}} (\text{id}_{\text{S},I_{\text{S}}} \otimes_{\text{S}} x)) \\
&\overset{5.25}{=} \Phi(x \circ_{\text{S}} \lambda_{\text{S},I_{\text{S}}}) \\
&\overset{5.68}{=} \Phi(x) \circ_{\text{Rel}} \Phi(\lambda_{\text{S},I_{\text{S}}}) \\
&\overset{5.64}{=} x \circ_{\text{Rel}} \lambda_{\text{Rel},A},
\end{aligned}$$

which is what we needed to see.

(d) The proof is analogous to that of (c): For any $\{A, B\} \subseteq \text{obj}_{\text{Rel}}$ and $x \in \text{mor}_{\text{Rel}}(A, B)$,

$$\begin{aligned}
\rho_{\text{Rel},B} \circ_{\text{Rel}} (x \otimes_{\text{Rel}} \text{id}_{\text{Rel},I_{\text{Rel}}}) &\overset{5.64}{=} \Phi(\rho_{\text{S},B}) \circ_{\text{Rel}} (\Phi(x) \otimes_{\text{S}} \Phi(\text{id}_{\text{S},I_{\text{S}}})) \\
&\overset{5.70}{=} \Phi(\rho_{\text{S},B}) \circ_{\text{Rel}} \Phi(x \otimes_{\text{S}} \text{id}_{\text{S},I_{\text{S}}}) \\
&\overset{5.68}{=} \Phi(\rho_{\text{S},B} \circ_{\text{S}} (x \otimes_{\text{S}} \text{id}_{\text{S},I_{\text{S}}})) \\
&\overset{5.25}{=} \Phi(x \circ_{\text{S}} \rho_{\text{S},I_{\text{S}}}) \\
&\overset{5.68}{=} \Phi(x) \circ_{\text{Rel}} \Phi(\rho_{\text{S},I_{\text{S}}}) \\
&\overset{5.64}{=} x \circ_{\text{Rel}} \rho_{\text{Rel},A},
\end{aligned}$$

from which the assertion follows by Proposition 5.68.

(e) For any $\{A, B, C, D\} \subseteq \text{obj}_{\text{Rel}}$ the definitions and Lemma 5.70 imply

$$\begin{aligned}
\alpha_{\text{Rel},A,B,C} \otimes_{\text{Rel}} \text{id}_{\text{Rel},D} &= \Phi(\alpha_{\text{S},A,B,C}) \otimes_{\text{S}} \Phi(\text{id}_{\text{S},D}) \\
&= \Phi(\alpha_{\text{S},A,B,C} \otimes_{\text{S}} \text{id}_{\text{S},D})
\end{aligned}$$

and, likewise,

$$\begin{aligned}\mathrm{id}_{\mathrm{Rel},A} \otimes_{\mathrm{Rel}} \alpha_{\mathrm{Rel},B,C,D} &= \Phi(\mathrm{id}_{\mathrm{S},A}) \otimes_{\mathrm{S}} \Phi(\alpha_{\mathrm{S},B,C,D}) \\ &= \Phi(\alpha_{\mathrm{S},A,B,C} \otimes_{\mathrm{S}} \mathrm{id}_{\mathrm{S},D}).\end{aligned}$$

Moreover, by definition, $\alpha_{\mathrm{Rel},A\otimes_{\mathrm{Rel}}B,C,D} = \Phi(\alpha_{\mathrm{S},A\otimes_{\mathrm{S}}B,C,D})$ as well as $\alpha_{\mathrm{Rel},A,B\otimes_{\mathrm{Rel}}C,D} = \Phi(\alpha_{\mathrm{S},A,B\otimes_{\mathrm{Rel}}C,D})$ and $\alpha_{\mathrm{Rel},A,B,C\otimes_{\mathrm{Rel}}D} = \Phi(\alpha_{\mathrm{S},A,B,C\otimes_{\mathrm{S}}D})$. Hence, the claimed identity is precisely the assertion that the image of the pentagon diagram in S under Φ commutes in Rel. Because Φ is a functor $\mathrm{S} \to \mathrm{Rel}$ by Proposition 5.68 and S a monoidal category by Proposition 5.25, this is indeed true.

(f) Because Φ is the identity on objects, because $\otimes_{\mathrm{Rel}}$ and $\otimes_{\mathrm{S}}$ agree on objects, because

$$\mathrm{id}_{\mathrm{Rel},A} \otimes_{\mathrm{Rel}} \lambda_{\mathrm{Rel},B} = \Phi(\mathrm{id}_{\mathrm{S},A}) \otimes_{\mathrm{S}} \Phi(\lambda_{\mathrm{S},B}) = \Phi(\mathrm{id}_{\mathrm{S},A} \otimes_{\mathrm{S}} \lambda_{\mathrm{S},B})$$

and, likewise,

$$\rho_{\mathrm{Rel},A} \otimes_{\mathrm{Rel}} \mathrm{id}_{\mathrm{Rel},B} = \Phi(\rho_{\mathrm{S},A}) \otimes_{\mathrm{S}} \Phi(\mathrm{id}_{\mathrm{S},B}) = \Phi(\rho_{\mathrm{S},B} \otimes_{\mathrm{S}} \mathrm{id}_{\mathrm{S},A}),$$

and because $\alpha_{\mathrm{Rel},A,I_{\mathrm{Rel}},B} = \Phi(\alpha_{\mathrm{S},A,I_{\mathrm{S}},B})$ by definition, the assertion that the triangle identity holds in Rel is equivalent to the claim that image of triangle diagram in S under Φ commutes. Because S is a monoidal category by Proposition 5.25 and because Φ is a functor $\mathrm{S} \to \mathrm{Rel}$ by Proposition 5.68 that already proves the claim. ☐

Proposition 5.73. *With* $(\otimes_{\mathrm{Rel}}, I_{\mathrm{Rel}}, \alpha_{\mathrm{Rel}}, \lambda_{\mathrm{Rel}}, \rho_{\mathrm{Rel}})$ *the relations* Rel *become a monoidal category, where* Rel *is short for* $\mathrm{Rel}(\mathcal{A}, \mathcal{E})$.

Proof. All parts of Lemma 5.72 taken together warrant this conclusion. ☐

Proposition 5.74. $\Phi_{\mathcal{A},\mathcal{E}}$ *is a strict monoidal functor* $\mathrm{S}(\mathcal{A}) \to \mathrm{Rel}(\mathcal{A}, \mathcal{E})$.

Proof. That $\Phi_{\mathcal{A},\mathcal{E}}$ is functorial was shown in Proposition 5.68. That it strictly preserves monoidal units is clear because it is the identity on objects. And Lemmata 5.70 and 5.64 together prove that ☐

5.3.3. *Symmetric Monoidal Category.* The monoidal category $\mathrm{Rel}(\mathcal{A}, \mathcal{E})$ inherits a braiding from $\mathrm{S}(\mathcal{A})$ via $\Phi_{\mathcal{A},\mathcal{E}}$.

Definition 5.75. For any $\{A, B\} \subseteq \mathrm{obj}_{\mathrm{Rel}(\mathcal{A},\mathcal{E})}$ let $\gamma_{\mathrm{Rel}(\mathcal{A},\mathcal{E}),A,B} \coloneqq \Phi_{\mathcal{A},\mathcal{E}}(\gamma_{\mathrm{S}(\mathcal{A}),A,B})$.

Lemma 5.76. *Let* Rel *be short for* $\mathrm{Rel}(\mathcal{A}, \mathcal{E})$.

(a) γ_{Rel} *is a natural isomorphism of functors* $\mathrm{Rel} \otimes_{\mathsf{CAT}} \mathrm{Rel} \to \mathrm{Rel}$ *from* $(\cdot_1) \otimes_{\mathrm{Rel}} (\cdot_2)$ *to* $(\cdot_2) \otimes_{\mathrm{Rel}} (\cdot_1)$.

(b) For any objects A, B *and* C *of* Rel *commutative diagrams in* Rel *are given by*

$$\begin{array}{ccccc} (A \otimes B) \otimes C & \xrightarrow{\alpha_{A,B,C}} & A \otimes (B \otimes C) & \xrightarrow{\gamma_{A,B\otimes C}} & (B \otimes C) \otimes A \\ {\scriptstyle \gamma_{A,B}\otimes \mathrm{id}_C}\downarrow & & & & \downarrow{\scriptstyle \alpha_{B,C,A}} \\ (B \otimes A) \otimes C & \xrightarrow[\alpha_{B,A,C}]{} & B \otimes (A \otimes C) & \xrightarrow[\mathrm{id}_B\otimes\gamma_{A,C}]{} & B \otimes (C \otimes A) \end{array}$$

and

$$\begin{array}{ccccc} A\otimes(B\otimes C) & \xrightarrow{\alpha^{-1}_{A,B,C}} & (A\otimes B)\otimes C & \xrightarrow{\gamma_{A\otimes B,C}} & C\otimes(A\otimes B) \\ {\scriptstyle \mathrm{id}_A\otimes\gamma_{B,C}}\big\downarrow & & & & \big\downarrow{\scriptstyle \alpha^{-1}_{C,A,B}} \\ A\otimes(C\otimes B) & \xrightarrow[\alpha^{-1}_{A,C,B}]{} & (A\otimes C)\otimes B & \xrightarrow[\gamma_{A,C}\otimes\mathrm{id}_B]{} & (C\otimes A)\otimes B \end{array}.$$

(c) $\gamma_{\mathrm{Rel},B,A}\circ_{\mathrm{Rel}}\gamma_{\mathrm{Rel},A,B}=\mathrm{id}_{\mathrm{Rel},A\otimes_{\mathrm{Rel}}B}$ *for any objects* A *and* B *of* Rel.

PROOF. (a) First, we show that γ_{Rel} is a natural transformation. For each $i\in\{1,2\}$ let $\{A_i,B_i\}\subseteq\mathrm{obj}_{\mathrm{Rel}}$ and $x_i\in\mathrm{mor}_{\mathrm{Rel}}(A_i,B_i)$. Then,

$$x_1\otimes_{\mathrm{Rel}}x_2\overset{5.64}{=}\Phi(x_1)\otimes_{\mathrm{Rel}}\Phi(x_2)=\Phi(x_1)\otimes_{\mathrm{S}}\Phi(x_2)\overset{5.70}{=}\Phi(x_1\otimes_{\mathrm{S}}x_2)$$

and, likewise, $x_2\otimes_{\mathrm{Rel}}x_1=\Phi(x_2\otimes_{\mathrm{S}}x_1)$. Thus, we conclude

$$\begin{aligned}\gamma_{\mathrm{Rel},B_1,B_2}\circ_{\mathrm{Rel}}(x_1\otimes_{\mathrm{Rel}}x_2) &= \Phi(\gamma_{\mathrm{S},B_1,B_2})\circ_{\mathrm{Rel}}\Phi(x_1\otimes_{\mathrm{S}}x_2)\\ &\overset{5.68}{=}\Phi(\gamma_{\mathrm{S},B_1,B_2}\circ_{\mathrm{S}}(x_1\otimes_{\mathrm{S}}x_2))\\ &\overset{5.34}{=}\Phi((x_2\otimes_{\mathrm{S}}x_1)\circ_{\mathrm{S}}\gamma_{\mathrm{S},A_1,A_2})\\ &\overset{5.68}{=}\Phi(x_2\otimes_{\mathrm{S}}x_1)\circ_{\mathrm{Rel}}\Phi(\gamma_{\mathrm{S},A_1,A_2})\\ &= (x_2\otimes_{\mathrm{Rel}}x_1)\circ_{\mathrm{Rel}}\gamma_{\mathrm{Rel},A_1,A_2}.\end{aligned}$$

Moreover, because γ_{S} is a natural isomorphism and Φ a functor $\mathrm{S}\to\mathrm{Rel}$ by Proposition 5.68, the image γ_{Rel} of γ_{S} under Φ is a natural isomorphism as well.

(b) According to Proposition 5.34 the corresponding diagrams in S commute. Because Φ is a functor $\mathrm{S}\to\mathrm{Rel}$ by Proposition 5.68 it suffices to prove that the Rel-diagrams in the claim are the images of the corresponding S-diagrams under Φ.

By definition, $\alpha_{\mathrm{Rel},A_1,A_2,A_3}=\Phi(\alpha_{\mathrm{S},A_1,A_2,A_3})$ for any $\{A_1,A_2,A_3\}\subseteq\mathrm{obj}_{\mathrm{Rel}}$. And because Φ is a functor, also ${\alpha_{\mathrm{Rel},A_1,A_2,A_3}}^{-1_{\mathrm{Rel}}}=\Phi({\alpha_{\mathrm{S},A_1,A_2,A_3}}^{-1_{\mathrm{S}}})$ for any $\{A_1,A_2,A_3\}\subseteq\mathrm{obj}_{\mathrm{Rel}}$. Of course, $\gamma_{\mathrm{Rel},A_1,A_2}=\Phi(\gamma_{\mathrm{S},A_1,A_2})$ for any $\{A_1,A_2\}\subseteq\mathrm{obj}_{\mathrm{Rel}}$ is the very definition of γ_{Rel}. Finally,

$$\begin{aligned}\mathrm{id}_{\mathrm{Rel},A_1}\otimes_{\mathrm{Rel}}\gamma_{\mathrm{Rel},A_2,A_3} &= \Phi(\mathrm{id}_{\mathrm{S},A_1})\otimes_{\mathrm{Rel}}\Phi(\gamma_{\mathrm{S},A_2,A_3})\\ &\overset{5.74}{=}\Phi(\mathrm{id}_{\mathrm{S},A_1}\otimes_{\mathrm{S}}\gamma_{\mathrm{S},A_2,A_3})\end{aligned}$$

and, likewise, $\gamma_{\mathrm{Rel},A_1,A_2}\otimes_{\mathrm{Rel}}\mathrm{id}_{\mathrm{Rel},A_3}=\Phi(\gamma_{\mathrm{S},A_1,A_2}\otimes_{\mathrm{S}}\mathrm{id}_{\mathrm{S},A_3})$ for any $\{A_1,A_2,A_3\}\subseteq\mathrm{obj}_{\mathrm{Rel}}$. Hence, the claim is indeed true.

(c) Because S is symmetric and Φ a functor $\mathrm{S}\to\mathrm{Rel}$ the definitions imply

$$\begin{aligned}\gamma_{\mathrm{Rel},B,A}\circ_{\mathrm{Rel}}\gamma_{\mathrm{Rel},A,B} &= \Phi(\gamma_{\mathrm{S},B,A})\circ_{\mathrm{Rel}}\Phi(\gamma_{\mathrm{S},A,B})\\ &\overset{5.68}{=}\Phi(\gamma_{\mathrm{S},B,A}\circ_{\mathrm{S}}\gamma_{\mathrm{S},A,B})\\ &\overset{5.34}{=}\Phi(\mathrm{id}_{\mathrm{S},A\otimes_{\mathrm{S}}B})\\ &= \mathrm{id}_{\mathrm{Rel},A\otimes_{\mathrm{Rel}}B},\end{aligned}$$

thus proving the claim. □

PROPOSITION 5.77. *$\gamma_{\mathrm{Rel}(\mathcal{A},\mathcal{E})}$ is a symmetric braiding for the monoidal category $\mathrm{Rel}(\mathcal{A},\mathcal{E})$.*

PROOF. Follows from all parts of Lemma 5.76 read together. □

PROPOSITION 5.78. *$\Phi_{\mathcal{A},\mathcal{E}}$ is a symmetric functor $\mathrm{S}(\mathcal{A}) \to \mathrm{Rel}\mathcal{A},\mathcal{E}$.*

PROOF. We already know from Proposition 5.74 that the functor $\Phi\colon \mathrm{S} \to \mathrm{Rel}$ is strict monoidal. We still have to prove that $\Phi(\gamma_{\mathrm{S},A_1,A_2}) = \gamma_{\mathrm{Rel},\Phi(A_1),\Phi(A_2)}$ for any $\{A_1, A_2\} \subseteq \mathrm{obj}_{\mathrm{S}}$.

$$\begin{array}{ccc}
\Phi(A_1) \otimes \Phi(A_2) & \xrightarrow{\gamma_{\Phi(A_1),\Phi(A_2)}} & \Phi(A_2) \otimes \Phi(A_1) \\
\| & & \| \\
\Phi(A_1 \otimes A_2) & \xrightarrow[\Phi(\gamma_{A_1,A_2})]{} & \Phi(A_2 \otimes A_1)
\end{array}$$

But, of course, this is just the definition of γ_{Rel}. □

5.3.4. *Rigid Symmetric Monoidal Category.* Also the rigid structure of the span category passes to the relation category.

DEFINITION 5.79. For any $A \in \mathrm{obj}_{\mathrm{Rel}(\mathcal{A},\mathcal{E})}$ define
(a) $A^{\vee_{\mathrm{Rel}(\mathcal{A},\mathcal{E})}} := \Phi_{\mathcal{A},\mathcal{E}}(A^{\vee_{\mathrm{S}(\mathcal{A})}}) = A$.
(b) $\varepsilon_{\mathrm{Rel}(\mathcal{A},\mathcal{E}),A} := \Phi_{\mathcal{A},\mathcal{E}}(\varepsilon_{\mathrm{S}(\mathcal{A}),A})$, and
(c) $\eta_{\mathrm{Rel}(\mathcal{A},\mathcal{E}),A} := \Phi_{\mathcal{A},\mathcal{E}}(\eta_{\mathrm{S}(\mathcal{A}),A})$.

LEMMA 5.80. *For any object A of $\mathrm{Rel}(\mathcal{A},\mathcal{E})$ commutative diagrams in $\mathrm{Rel}(\mathcal{A},\mathcal{E})$ are given by*

$$\begin{array}{ccccc}
I \otimes A & & \xrightarrow{\lambda_A} & & A \\
\downarrow{\scriptstyle \eta_A \otimes \mathrm{id}_A} & & & & \uparrow{\scriptstyle \rho_A} \\
(A \otimes A^\vee) \otimes A & & & & A \otimes I \\
& \searrow^{\alpha_{A,A^\vee,A}} & & \nearrow^{\mathrm{id}_A \otimes \varepsilon_A} & \\
& & A \otimes (A^\vee \otimes A) & &
\end{array}$$

and

$$\begin{array}{ccccc}
A^\vee \otimes I & & \xrightarrow{\rho_{A^\vee}} & & A^\vee \\
\downarrow{\scriptstyle \mathrm{id}_{A^\vee} \otimes \eta_A} & & & & \uparrow{\scriptstyle \lambda_{A^\vee}} \\
A^\vee \otimes (A \otimes A^\vee) & & & & I \otimes A^\vee \\
& \searrow^{\alpha^{-1}_{A^\vee,A,A^\vee}} & & \nearrow^{\varepsilon_A \otimes \mathrm{id}_{A^\vee}} & \\
& & (A^\vee \otimes A) \otimes A^\vee & &
\end{array}$$

PROOF. According to Proposition 5.40 the corresponding diagrams in S commute. Moreover, Φ is a functor $\mathrm{S} \to \mathrm{Rel}$ which is the identity on objects by Proposition 5.68. Hence, it suffices to show that the Rel-diagrams in the assertion are the images of their respective corresponding S-diagram under Φ.

By definition, $\lambda_{\mathrm{Rel},X} = \Phi(\lambda_{\mathrm{S},X})$ and $\rho_{\mathrm{Rel},X} = \Phi(\rho_{\mathrm{S},X})$ for any $X \in \mathrm{obj}_{\mathrm{Rel}}$. Likewise, $\alpha_{\mathrm{S},X_1,X_2,X_3} = \Phi(\alpha_{\mathrm{S},X_1,X_2,X_3})$ for any $\{X_1, X_2, X_3\} \subseteq \mathrm{obj}_{\mathrm{Rel}}$. And, because Φ is a functor, also $\alpha_{\mathrm{Rel},X_1,X_2,X_3}{}^{-1_{\mathrm{Rel}}} = \Phi(\alpha_{\mathrm{S},X_1,X_2,X_3}{}^{-1_{\mathrm{S}}})$ for any $\{X_1, X_2, X_3\} \subseteq \mathrm{obj}_{\mathrm{Rel}}$. Of course, $X^{\vee_{\mathrm{Rel}}} = \Phi(X^{\vee_{\mathrm{S}}})$ and $\varepsilon_{\mathrm{Rel},X} = \Phi(\varepsilon_{\mathrm{S},X})$ and $\eta_{\mathrm{Rel},X} = \Phi(\eta_{\mathrm{S},X})$ are the very definitions for any $X \in \mathrm{obj}_{\mathrm{Rel}}$. Finally,

$$\eta_{\mathrm{Rel},A} \otimes_{\mathrm{Rel}} \mathrm{id}_{\mathrm{Rel},A} = \Phi(\eta_{\mathrm{S},A}) \otimes_{\mathrm{Rel}} \mathrm{id}_{\mathrm{Rel},A} \overset{5.74}{=} \Phi(\eta_{\mathrm{S},A} \otimes_{\mathrm{S}} \mathrm{id}_{\mathrm{S},A})$$

and, likewise, $\mathrm{id}_{\mathrm{Rel},A} \otimes_{\mathrm{Rel}} \varepsilon_{\mathrm{Rel},A} = \Phi(\mathrm{id}_{\mathrm{S},A} \otimes_{\mathrm{S}} \varepsilon_{\mathrm{S},A})$ and $\mathrm{id}_{\mathrm{Rel},A^{\vee_{\mathrm{Rel}}}} \otimes_{\mathrm{Rel}} \eta_{\mathrm{Rel},A} = \Phi(\mathrm{id}_{\mathrm{S},A^{\vee_{\mathrm{S}}}} \otimes_{\mathrm{S}} \eta_{\mathrm{S},A})$ and $\varepsilon_{\mathrm{Rel},A} \otimes_{\mathrm{Rel}} \mathrm{id}_{\mathrm{Rel},A^{\vee_{\mathrm{Rel}}}} = \Phi(\varepsilon_{\mathrm{S},A} \otimes_{\mathrm{S}} \mathrm{id}_{\mathrm{S},A^{\vee_{\mathrm{S}}}})$. Thus, the claim is true. ☐

PROPOSITION 5.81. *The symmetric monoidal category* $\mathrm{Rel}(\mathcal{A},\mathcal{E})$ *is rigid. Left duals with associated evaluations and co-evaluations are given by* $(\,\cdot\,)^{\vee_{\mathrm{Rel}(\mathcal{A},\mathcal{E})}}$, $\varepsilon_{\mathrm{Rel}(\mathcal{A},\mathcal{E})}$ *and* $\eta_{\mathrm{Rel}(\mathcal{A},\mathcal{E})}$, *respectively.*

PROOF. That is the combined results of 5.80. ☐

Again, we compute the dual morphisms with respect to the chosen dualization. The following lemma shows that the ensuing proposition makes sense.

LEMMA 5.82. *For any* $\{A, B\} \subseteq \mathrm{obj}_{\mathcal{A}}$ *and any span* (f, g) *in* $\mathcal{A}$ *from* A *to* B *the two statements* $f \times_{\mathcal{A}} g \in \mathcal{M}$ *and* $g \times_{\mathcal{A}} f \in \mathcal{M}$ *are equivalent.*

PROOF. By symmetry it suffices to show one implication. Hence, suppose that $f \times g \in \mathcal{M}$. Because $\mathcal{M}$ is closed under composition with isomorphisms by Lemmata 4.34 and 4.32 and because $\gamma_{A,B}$ is an isomorphism, $\gamma_{A,B} \circ (f \times g) \in \mathcal{M}$. By Lemma 4.15 that proves $g \times f \in \mathcal{M}$, which is what we needed to see. ☐

PROPOSITION 5.83. *For any* $\{A, B\} \subseteq \mathrm{obj}_{\mathrm{Rel}(\mathcal{A},\mathcal{E})}$ *and any* $[f, g] \in \mathrm{mor}_{\mathrm{Rel}(\mathcal{A},\mathcal{E})}(A, B)$ *the dual morphism* $[f, g]^{\vee_{\mathrm{Rel}(\mathcal{A},\mathcal{E})}}$ *of* $[f, g]$ *with respect to* $((\,\cdot\,)^{\vee_{\mathrm{Rel}(\mathcal{A},\mathcal{E})}}, \varepsilon_{\mathrm{Rel}(\mathcal{A},\mathcal{E})}, \eta_{\mathrm{Rel}(\mathcal{A},\mathcal{E})})$ *is given by* $[g, f]$.

PROOF. If we read the equation

$$[g, f] = \lambda_{A^\vee} \circ (\varepsilon_B \otimes \mathrm{id}_{A^\vee}) \circ \alpha^{-1}_{B^\vee,B,A^\vee} \circ (\mathrm{id}_{B^\vee} \otimes ([f, g] \otimes \mathrm{id}_{A^\vee})) \circ (\mathrm{id}_{B^\vee} \otimes \eta_A) \circ \rho^{-1}_{B^\vee}$$

as an identity in S, then this is the result of Proposition 5.41; and if we read it as an identity in Rel, then it is precisely our claim.

By definition,

$$\lambda_{\mathrm{Rel},A^{\vee_{\mathrm{Rel}}}} = \Phi(\lambda_{\mathrm{S},A^{\vee_{\mathrm{S}}}}).$$

Proposition 5.74 implies

$$\varepsilon_{\mathrm{Rel},B} \otimes_{\mathrm{Rel}} \mathrm{id}_{\mathrm{Rel},A^{\vee_{\mathrm{Rel}}}} = \Phi(\varepsilon_{\mathrm{S},B} \otimes_{\mathrm{S}} \mathrm{id}_{\mathrm{S},A^{\vee_{\mathrm{S}}}})$$

Also by definition and Proposition 5.68,

$$\alpha^{-1}_{\mathrm{Rel},B^{\vee_{\mathrm{Rel}}},B,A^{\vee_{\mathrm{Rel}}}} = \Phi(\alpha^{-1}_{\mathrm{S},B^{\vee_{\mathrm{S}}},B,A^{\vee_{\mathrm{S}}}}).$$

Lemma 5.64 shows $[f,g] = \Phi([f,g])$, which is why Proposition 5.74 lets us infer

$$\mathrm{id}_{\mathrm{Rel},B^{\vee_{\mathrm{Rel}}}} \otimes_{\mathrm{Rel}} ([f,g] \otimes_{\mathrm{Rel}} \mathrm{id}_{\mathrm{Rel},A^{\vee_{\mathrm{Rel}}}}) = \Phi(\mathrm{id}_{\mathrm{S},B^{\vee_{\mathrm{S}}}} \otimes_{\mathrm{S}} ([f,g] \otimes_{\mathrm{S}} \mathrm{id}_{\mathrm{S},A^{\vee_{\mathrm{S}}}})).$$

Another application of that proposition yields

$$\mathrm{id}_{\mathrm{Rel},B^{\vee_{\mathrm{Rel}}}} \otimes_{\mathrm{Rel}} \eta_{\mathrm{Rel},A} = \Phi(\mathrm{id}_{\mathrm{S},B^{\vee_{\mathrm{S}}}} \otimes_{\mathrm{S}} \eta_{\mathrm{S},A})$$

And

$$\rho_{\mathrm{Rel},B^{\vee_{\mathrm{S}}}}{}^{-1_{\mathrm{Rel}}} = \Phi(\rho_{\mathrm{S},B^{\vee_{\mathrm{S}}}}{}^{-1_{\mathrm{S}}})$$

is true by Proposition 5.68 and by definition of ρ_{Rel}.

Hence, the right hand side of the claimed identity is a composition in Rel of images of morphisms of S under Φ. Because Φ is a functor $\mathrm{S} \to \mathrm{Rel}$ by Proposition 5.68, the right hand side of the assertion is thus identical to

$$\Phi(\lambda_{A^\vee} \circ (\varepsilon_B \otimes \mathrm{id}_{A^\vee}) \circ \alpha^{-1}_{B^\vee,B,A^\vee} \circ (\mathrm{id}_{B^\vee} \otimes ([f,g] \otimes \mathrm{id}_{A^\vee})) \circ (\mathrm{id}_{B^\vee} \otimes \eta_A) \circ \rho^{-1}_{B^\vee}),$$

where the index S was suppresed. Since the argument of Φ in this term evaluates to $[g,f]$ by the S-version of the initial identity, our claim is equivalent to the identity $[g,f] = \Phi([g,f])$. However, this is clear by Lemmata 5.64 and 5.82. Hence, the assertion is true. □

Moreover, we can link the traces in the span category to those in the relation category.

Proposition 5.84. *For any* $A \in \mathrm{obj}_{\mathrm{Rel}(\mathcal{A},\mathcal{E})}$ *and any endomorphism* x *of* A *in* $\mathrm{Rel}(\mathcal{A},\mathcal{E})$ *the trace* $\mathrm{tr}_{\mathrm{Rel}(\mathcal{A},\mathcal{E})}(x)$ *of* x *with respect to* $((\cdot)^{\vee_{\mathrm{Rel}(\mathcal{A},\mathcal{E})}}, \varepsilon_{\mathrm{Rel}(\mathcal{A},\mathcal{E})}, \eta_{\mathrm{Rel}(\mathcal{A},\mathcal{E})})$ *is given by*

$$\mathrm{tr}_{\mathrm{Rel}(\mathcal{A},\mathcal{E})}(x) = \Phi_{\mathcal{A},\mathcal{E}}(\mathrm{tr}_{\mathrm{S}(\mathcal{A})}(x)).$$

Proof. The definitions, Lemma 5.64 and Proposition 5.78 imply

$$\begin{aligned}
&\mathrm{tr}_{\mathrm{Rel}}(x)\\
&\quad = \varepsilon_{\mathrm{Rel},A} \circ_{\mathrm{Rel}} \gamma_{\mathrm{Rel},A,A^{\vee_{\mathrm{Rel}}}} \circ_{\mathrm{Rel}} ([f,g] \otimes_{\mathrm{Rel}} \mathrm{id}_{\mathrm{Rel},A^{\vee_{\mathrm{Rel}}}}) \circ_{\mathrm{Rel}} \eta_{\mathrm{Rel},A}\\
&\quad = \Phi(\varepsilon_{\mathrm{S},A}) \circ_{\mathrm{Rel}} \Phi(\gamma_{\mathrm{S},A,A^{\vee_{\mathrm{S}}}}) \circ_{\mathrm{Rel}} (\Phi([f,g]) \otimes_{\mathrm{Rel}} \Phi(\mathrm{id}_{\mathrm{S},A^{\vee_{\mathrm{S}}}}))\\
&\qquad \circ_{\mathrm{Rel}} \Phi(\eta_{\mathrm{S},A})\\
&\quad = \Phi(\varepsilon_{\mathrm{S},A}) \circ_{\mathrm{Rel}} \Phi(\gamma_{\mathrm{S},A,A^{\vee_{\mathrm{S}}}}) \circ_{\mathrm{Rel}} \Phi([f,g] \otimes_{\mathrm{S}} \mathrm{id}_{\mathrm{S},A^{\vee_{\mathrm{S}}}}) \circ_{\mathrm{Rel}} \Phi(\eta_{\mathrm{S},A})\\
&\quad = \Phi(\varepsilon_{\mathrm{S},A} \circ_{\mathrm{S}} \gamma_{\mathrm{S},A,A^{\vee_{\mathrm{S}}}} \circ_{\mathrm{S}} ([f,g] \otimes_{\mathrm{S}} \mathrm{id}_{\mathrm{S},A^{\vee_{\mathrm{S}}}}) \circ_{\mathrm{S}} \eta_{\mathrm{S},A})\\
&\quad = \Phi(\mathrm{tr}_{\mathrm{S}}(x)),
\end{aligned}$$

as claimed. □

5.4. Stage 2b: Relations (with †). As for spans, we can consider the category of relations as equipped with the extra structure of a †-operation.

5.4.1. *†-category.* Next, we show that $\mathrm{Rel}(\mathcal{A},\mathcal{E})$ also inherits via $\Phi_{\mathcal{A},\mathcal{E}}$ the †-structure of $\mathrm{S}(\mathcal{A})$ and that $\Phi_{\mathcal{A},\mathcal{E}}$ thus becomes a †-functor. The following definition makes sense by Lemmata 5.64 and 5.82.

Definition 5.85. Let

(a) $X^{\dagger_{\mathrm{Rel}(\mathcal{A},\mathcal{E})}} := \Phi_{\mathcal{A},\mathcal{E}}(X^{\dagger_{\mathrm{S}(\mathcal{A})}}) = X$ for any $X \in \mathrm{obj}_{\mathrm{Rel}(\mathcal{A},\mathcal{E})}$,

(b) $[f,g]^{\dagger_{\mathrm{Rel}(\mathcal{A},\mathcal{E})}} := \Phi_{\mathcal{A},\mathcal{E}}([f,g]^{\dagger_{\mathrm{S}(\mathcal{A})}}) = [g,f]$ for any $\{A,B\} \subseteq \mathrm{obj}_{\mathrm{Rel}(\mathcal{A},\mathcal{E})}$ and $[f,g] \in \mathrm{mor}_{\mathrm{Rel}(\mathcal{A},\mathcal{E})}(A,B)$.

Note that for the relation category it is significantly less immediate than for the span category that the †-operation is functorial.

Lemma 5.86. *$\Phi_{\mathcal{A},\mathcal{E}}(x^{\dagger_{\mathrm{S}(\mathcal{A})}}) = \Phi_{\mathcal{A},\mathcal{E}}(x)^{\dagger_{\mathrm{S}(\mathcal{A})}}$ for any morphism x of $\mathrm{S}(\mathcal{A})$.*

Proof. Let $\{X,A,B\} \subseteq \mathrm{obj}_{\mathcal{A}}$ and $f \in \mathrm{mor}_{\mathcal{A}}(X,A)$ and $g \in \mathrm{mor}_{\mathcal{A}}(X,B)$ be such that $x = [f,g]$, and let (e,m) be an $(\mathcal{E},\mathcal{M})$-factorization of $f \times_{\mathcal{A}} g$.

Because $\gamma_{\mathcal{A},A,B} \in \mathrm{iso}_{\mathcal{A}} \subseteq \mathcal{M}$ by Lemma 4.32 and thus also $\gamma_{\mathcal{A},A,B} \circ_{\mathcal{A}} m \in \mathcal{M}$ by Lemma 4.33 (b) and because

$$(\gamma_{\mathcal{A},A,B} \circ_{\mathcal{A}} m) \circ_{\mathcal{A}} e = \gamma_{\mathcal{A},A,B} \circ_{\mathcal{A}} (m \circ_{\mathcal{A}} e) = \gamma_{\mathcal{A},A,B} \circ_{\mathcal{A}} (f \times_{\mathcal{A}} g) = g \times_{\mathcal{A}} f$$

by Lemma 4.15, the pair $(e, \gamma_{\mathcal{A},A,B} \circ_{\mathcal{A}} m)$ is an $(\mathcal{E},\mathcal{M})$-factorization of $g \times_{\mathcal{A}} f$. Hence, by definition (and Lemma 5.62),

$$\Phi([g,f]) = [\pi^1_{\mathcal{A},B,A} \circ_{\mathcal{A}} (\gamma_{\mathcal{A},A,B} \circ_{\mathcal{A}} m), \pi^1_{\mathcal{A},B,A} \circ_{\mathcal{A}} (\gamma_{\mathcal{A},A,B} \circ_{\mathcal{A}} m)].$$

From $\pi^1_{\mathcal{A},B,A} \circ_{\mathcal{A}} \gamma_{\mathcal{A},A,B} = \pi^1_{\mathcal{A},B,A} \circ_{\mathcal{A}} (\pi^2_{\mathcal{A},A,B} \times_{\mathcal{A}} \pi^1_{\mathcal{A},A,B}) = \pi^2_{\mathcal{A},A,B}$ and, likewise, $\pi^2_{\mathcal{A},B,A} \circ_{\mathcal{A}} \gamma_{\mathcal{A},A,B} = \pi^1_{\mathcal{A},A,B}$ it then follows

$$\begin{aligned}\Phi([f,g]^{\dagger_{\mathrm{S}}}) &= \Phi([g,f]) \\ &= [\pi^2_{\mathcal{A},A,B} \circ_{\mathcal{A}} m, \pi^1_{\mathcal{A},A,B} \circ_{\mathcal{A}} m] \\ &= [\pi^1_{\mathcal{A},A,B} \circ_{\mathcal{A}} m, \pi^2_{\mathcal{A},A,B} \circ_{\mathcal{A}} m]^{\dagger_{\mathrm{S}}} \\ &= \Phi[f,g]^{\dagger_{\mathrm{S}}},\end{aligned}$$

which concludes the proof □

Lemma 5.87. *(a) $(\cdot)^{\dagger_{\mathrm{Rel}(\mathcal{A},\mathcal{E})}}$ is a contravariant endofunctor of* Rel.

(b) $(\cdot)^{\dagger_{\mathrm{Rel}(\mathcal{A},\mathcal{E})}}$ is the identity on objects.

(c) $((\cdot)^{\dagger_{\mathrm{Rel}(\mathcal{A},\mathcal{E})}})^{\mathrm{op}_{\mathsf{CAT}}} \circ_{\mathsf{CAT}} (\cdot)^{\dagger_{\mathrm{Rel}(\mathcal{A},\mathcal{E})}} = \mathrm{id}_{\mathsf{CAT},\mathrm{Rel}(\mathcal{A},\mathcal{E})}$.

Proof. (a) For any $X \in \mathrm{obj}_{\mathrm{Rel}}$, by definition and Lemmata 5.64 and 5.82,

$$\mathrm{id}_{\mathrm{Rel},X}{}^{\dagger_{\mathrm{Rel}}} = \Phi(\mathrm{id}_{\mathrm{S},X})^{\dagger_{\mathrm{S}}} \overset{5.86}{=} \Phi(\mathrm{id}_{\mathrm{S},X}{}^{\dagger_{\mathrm{S}}}) \overset{5.47}{=} \Phi(\mathrm{id}_{\mathrm{S},X}) = \mathrm{id}_{\mathrm{Rel},X}.$$

For any $\{A,B,C\} \subseteq \mathrm{mor}_{\mathrm{S}}$ and any $x \in \mathrm{mor}_{\mathrm{Rel}}(A,B)$ and $y \in \mathrm{mor}_{\mathrm{Rel}}(B,C)$ the definitions and Lemmata 5.64 and 5.82 once more allow us to conclude

$$\begin{aligned}(y \circ_{\mathrm{Rel}} x)^{\dagger_{\mathrm{Rel}}} &= (\Phi(y \circ_{\mathrm{S}} x))^{\dagger_{\mathrm{S}}} \overset{5.86}{=} \Phi((y \circ_{\mathrm{S}} x)^{\dagger_{\mathrm{S}}}) \overset{5.47}{=} \Phi(x^{\dagger_{\mathrm{S}}} \circ_{\mathrm{S}} y^{\dagger_{\mathrm{S}}}) \\ &= x^{\dagger_{\mathrm{Rel}}} \circ_{\mathrm{Rel}} y^{\dagger_{\mathrm{Rel}}}.\end{aligned}$$

Both results together prove (a).

(b) Holds by definition.

(c) On the level of objects the claim is clear. Because on morphisms, $(\,\cdot\,)^{\dagger_{\mathrm{Rel}}}$ is merely a restriction of $(\,\cdot\,)^{\dagger_{\mathrm{S}}}$, Proposition 5.47 implies the assertion. □

PROPOSITION 5.88. $\mathrm{Rel}(\mathcal{A},\mathcal{E})$*, when equipped with* $(\,\cdot\,)^{\dagger_{\mathrm{Rel}(\mathcal{A},\mathcal{E})}}$*, is a †-category.*

PROOF. The claim is the combined implication of Lemma 5.87. □

Moreover, the "projection" even respects the †-operation.

PROPOSITION 5.89. $\Phi_{\mathcal{A},\mathcal{E}}$ *is a †-functor* $\mathrm{S}(\mathcal{A}) \to \mathrm{Rel}(\mathcal{A},\mathcal{E})$*.*

PROOF. Because Φ is the identity on objects, we only need to prove the identity $(\,\cdot\,)^{\dagger_{\mathrm{Rel}}} \circ_{\mathsf{CAT}} \Phi = \Phi^{\mathrm{op}_{\mathsf{CAT}}} \circ_{\mathsf{CAT}} (\,\cdot\,)^{\dagger_{\mathrm{S}}}$ on morphisms. But, there, the claim is merely Lemma 5.86. □

5.4.2. *Monoidal †-category.* In fact, just like the span category the relation category is even a monoidal †-category. Recall that $*\mathsf{CAT}$ denotes the (large) monoidal category of †-categories.

LEMMA 5.90. *Let* Rel *be short for* $\mathrm{Rel}(\mathcal{A},\mathcal{E})$*.*
(a) $\otimes_{\mathrm{Rel}}$ *is a †-functor* $\mathrm{Rel} \otimes_{*\mathsf{CAT}} \mathrm{Rel} \to \mathrm{Rel}$.
(b) ${\alpha_{\mathrm{Rel},A_1,A_2,A_3}}^{\dagger_{\mathrm{Rel}}} = {\alpha_{\mathrm{Rel},A_1,A_2,A_3}}^{-1_{\mathrm{Rel}}}$ *for any* $\{A_1,A_2,A_3\} \subseteq \mathrm{obj}_{\mathrm{Rel}}$.
(c) ${\lambda_{\mathrm{Rel},A}}^{\dagger_{\mathrm{Rel}}} = {\lambda_{\mathrm{Rel},A}}^{-1_{\mathrm{Rel}}}$ *for any* $A \in \mathrm{obj}_{\mathrm{Rel}}$.
(d) ${\rho_{\mathrm{Rel},A}}^{\dagger_{\mathrm{Rel}}} = {\rho_{\mathrm{Rel},A}}^{-1_{\mathrm{Rel}}}$ *for any* $A \in \mathrm{obj}_{\mathrm{Rel}}$.

PROOF. (a) For any $\{X_1,X_2\} \subseteq \mathrm{obj}_{\mathrm{Rel}}$ the definitions imply

$$(X_1 \otimes_{\mathrm{Rel}} X_2)^{\dagger_{\mathrm{Rel}}} = (X_1 \otimes_{\mathrm{S}} X_2)^{\dagger_{\mathrm{S}}} \overset{5.51}{=} {X_1}^{\dagger_{\mathrm{S}}} \otimes_{\mathrm{S}} {X_2}^{\dagger_{\mathrm{S}}} = {X_1}^{\dagger_{\mathrm{Rel}}} \otimes_{\mathrm{Rel}} {X_2}^{\dagger_{\mathrm{Rel}}}.$$

Similarly, if $\{A_i,B_i\} \subseteq \mathrm{obj}_{\mathrm{Rel}}$ and $x_i \in \mathrm{mor}_{\mathrm{Rel}}(A_i,B_i)$ for each $i \in \{1,2\}$, then, by definition,

$$(x_1 \otimes_{\mathrm{Rel}} x_2)^{\dagger_{\mathrm{Rel}}} = (x_1 \otimes_{\mathrm{S}} x_2)^{\dagger_{\mathrm{S}}} \overset{5.51}{=} {x_1}^{\dagger_{\mathrm{S}}} \otimes_{\mathrm{S}} {x_2}^{\dagger_{\mathrm{S}}} = {x_1}^{\dagger_{\mathrm{Rel}}} \otimes_{\mathrm{Rel}} {x_2}^{\dagger_{\mathrm{Rel}}}.$$

Hence, (a) is true.

(b) For any $\{A_1,A_2,A_3\} \subseteq \mathrm{obj}_{\mathrm{Rel}}$,

$$\begin{aligned} {\alpha_{\mathrm{Rel},A_1,A_2,A_3}}^{\dagger_{\mathrm{Rel}}} &= \Phi(\alpha_{\mathrm{S},A_1,A_2,A_3})^{\dagger_{\mathrm{Rel}}} \\ &\overset{5.89}{=} \Phi({\alpha_{\mathrm{S},A_1,A_2,A_3}}^{\dagger_{\mathrm{S}}}) \\ &\overset{5.51}{=} \Phi({\alpha_{\mathrm{S},A_1,A_2,A_3}}^{-1_{\mathrm{S}}}) \\ &\overset{5.68}{=} \Phi(\alpha_{\mathrm{S},A_1,A_2,A_3})^{-1_{\mathrm{Rel}}} \\ &= {\alpha_{\mathrm{Rel},A_1,A_2,A_3}}^{-1_{\mathrm{Rel}}}. \end{aligned}$$

(c) In full analogy to the proof of (b), for any $A \in \mathrm{obj}_{\mathrm{Rel}}$,

$$\begin{aligned}
{\lambda_{\mathrm{Rel},A}}^{\dagger_{\mathrm{Rel}}} &= \Phi(\lambda_{\mathrm{S},A})^{\dagger_{\mathrm{Rel}}} \\
&\overset{5.89}{=} \Phi({\lambda_{\mathrm{S},A}}^{\dagger_{\mathrm{S}}}) \\
&\overset{5.51}{=} \Phi({\lambda_{\mathrm{S},A}}^{-1_{\mathrm{S}}}) \\
&\overset{5.68}{=} \Phi(\lambda_{\mathrm{S},A})^{-1_{\mathrm{Rel}}} \\
&= {\lambda_{\mathrm{Rel},A}}^{-1_{\mathrm{Rel}}}.
\end{aligned}$$

(d) Once more, an analogous computation yields for any $A \in \mathrm{obj}_{\mathrm{Rel}}$,

$$\begin{aligned}
{\rho_{\mathrm{Rel},A}}^{\dagger_{\mathrm{Rel}}} &= \Phi(\rho_{\mathrm{S},A})^{\dagger_{\mathrm{Rel}}} \\
&\overset{5.89}{=} \Phi({\rho_{\mathrm{S},A}}^{\dagger_{\mathrm{S}}}) \\
&\overset{5.51}{=} \Phi({\rho_{\mathrm{S},A}}^{-1_{\mathrm{S}}}) \\
&\overset{5.68}{=} \Phi(\rho_{\mathrm{S},A})^{-1_{\mathrm{Rel}}} \\
&= {\rho_{\mathrm{Rel},A}}^{-1_{\mathrm{Rel}}}.
\end{aligned}$$

That concludes the proof. □

PROPOSITION 5.91. $\mathrm{Rel}(\mathcal{A},\mathcal{E})$ *is a monoidal †-category.*

PROOF. That is the combined implication of Lemma 5.90. □

5.4.3. *Symmetric Monoidal †-category.* As is to be expected, the relation category is actually a symmetric monoidal †-category like the span category.

LEMMA 5.92. ${\gamma_{\mathrm{Rel}(\mathcal{A},\mathcal{E}),A_1,A_2}}^{\dagger_{\mathrm{Rel}(\mathcal{A},\mathcal{E})}} = {\gamma_{\mathrm{Rel}(\mathcal{A},\mathcal{E}),A_1,A_2}}^{-1_{\mathrm{Rel}(\mathcal{A},\mathcal{E})}}$ *for any* $\{A_1, A_2\} \subseteq \mathrm{obj}_{\mathrm{Rel}(\mathcal{A},\mathcal{E})}$.

PROOF. The proof is completely analogous those of Parts (b)–(d) of Lemma 5.90. For any $\{A_1, A_2\} \subseteq \mathrm{obj}_{\mathrm{Rel}}$,

$$\begin{aligned}
{\gamma_{\mathrm{Rel},A_1,A_2}}^{\dagger_{\mathrm{Rel}}} &= \Phi(\gamma_{\mathrm{S},A_1,A_2})^{\dagger_{\mathrm{Rel}}} \\
&\overset{5.89}{=} \Phi({\gamma_{\mathrm{S},A_1,A_2}}^{\dagger_{\mathrm{S}}}) \\
&\overset{5.51}{=} \Phi({\gamma_{\mathrm{S},A_1,A_2}}^{-1_{\mathrm{S}}}) \\
&\overset{5.68}{=} \Phi(\gamma_{\mathrm{S},A_1,A_2})^{-1_{\mathrm{Rel}}} \\
&= {\gamma_{\mathrm{Rel},A_1,A_2}}^{-1_{\mathrm{Rel}}},
\end{aligned}$$

which is what we needed to see. □

PROPOSITION 5.93. $\mathrm{Rel}(\mathcal{A},\mathcal{E})$ *is a symmetric monoidal †-category.*

PROOF. Follows from Lemma 5.92. □

5.4.4. *Rigid Symmetric Monoidal †-category.* Finally, the rigid structure of the relation category is compatible with its †-structure, just as it was the case for the span category.

LEMMA 5.94. *If* Rel *is short for* $\mathrm{Rel}(\mathcal{A},\mathcal{E})$, *then* $\gamma_{\mathrm{Rel},A^{\vee_{\mathrm{Rel}}},A} \circ_{\mathrm{Rel}} {\varepsilon_{\mathrm{Rel},A}}^{\dagger_{\mathrm{Rel}}} = \eta_{\mathrm{Rel},A}$ *for any* $A \in \mathrm{obj}_{\mathrm{Rel}}$. *In other words, in* Rel *the diagram*

$$\begin{array}{ccc} & & A^\vee \otimes A \\ & \overset{{\varepsilon_A}^\dagger}{\nearrow} & \\ I & & \Big\downarrow \gamma_{A^\vee,A} \\ & \underset{\eta_A}{\searrow} & \\ & & A \otimes A^\vee \end{array}$$

commutes.

PROOF. We compute immediately

$$\begin{aligned} \gamma_{\mathrm{Rel},A^{\vee_{\mathrm{Rel}}},A} \circ_{\mathrm{Rel}} {\varepsilon_{\mathrm{Rel},A}}^{\dagger_{\mathrm{Rel}}} &= \Phi(\gamma_{\mathrm{S},A^{\vee_\mathrm{S}},A}) \circ_{\mathrm{Rel}} \Phi(\varepsilon_{\mathrm{S},A})^{\dagger_{\mathrm{Rel}}} \\ &\overset{5.89}{=} \Phi(\gamma_{\mathrm{S},A^{\vee_\mathrm{S}},A}) \circ_{\mathrm{Rel}} \Phi({\varepsilon_{\mathrm{S},A}}^{\dagger_\mathrm{S}}) \\ &\overset{5.68}{=} \Phi(\gamma_{\mathrm{S},A^{\vee_\mathrm{S}},A} \circ_\mathrm{S} {\varepsilon_{\mathrm{S},A}}^{\dagger_\mathrm{S}}) \\ &\overset{5.58}{=} \Phi(\eta_{\mathrm{S},A}) \\ &= \eta_{\mathrm{Rel},A}, \end{aligned}$$

which is what we needed to see. □

PROPOSITION 5.95. $\mathrm{Rel}(\mathcal{A},\mathcal{E})$ *is a rigid symmetric monoidal †-category.*

PROOF. Was shown in Lemma 5.94. □

That completes the second stage of the construction – at least on 0-cells. As explained, none of this was new. However, this changes in the next section.

5.5. Stage 3a: Linearized Relations. With the results of the preceding two stages at hand, one should be able to prove at least Conjecture 2.14. However, I was not able to confirm this before the due date of the thesis. The computational effort greatly increases from the second to the third stage because of the change in enriching category. For that reason, most claims on this stage have to remain conjectures for now. Not even for 0-cells the construction is complete.

The construction carried out below generalizes [Kno07]. The corresponding proofs are however not given in [Kno07]. Hence, it is not possible to simply refer to them and point out the necessary modifications.

ASSUMPTIONS 5.96. Make the same assumptions about $(\mathsf{S}, U_\mathsf{S})$ and R and M as well as $G_\mathsf{M} \dashv U_\mathsf{M}$ via cu_M and un_M as in Section 2, let B be the symmetric monoidal bicategory of small M-enriched categories, let $(\mathcal{A},\mathcal{E},\delta)$ be any 0-cell of

$\mathsf{d}_{\mathsf{S},R}\mathsf{esmCAT}^{\mathrm{cart,fc}}_{\mathrm{fs}}$ and let $\mathcal{M} := \mathcal{E}^{\perp}$. Moreover, as before, S will be short for $\mathrm{S}(\mathcal{A})$ on occasion. Likewise, Rel will abbreviate $\mathrm{Rel}(\mathcal{A},\mathcal{E})$ sometimes.

Mind that we have thus assumed $\mathcal{E} \hookrightarrow \mathrm{epi}_{\mathcal{A}}$ in particular.

5.5.1. *Linear Category.* The first step is to define the generalized tensor envelope just as an enriched category, i.e., to introduce the composition and identities. In that context, it is important to note that our assumption about $\mathcal{E}$ consisting of epimorphisms of $\mathcal{A}$ allows us to draw the following conclusions about the identities. Note that this was never claimed anywhere on the second stage (and also not required).

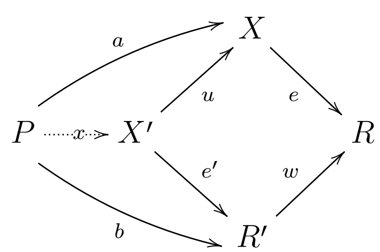
LEMMA 5.97. $\mathrm{id}_{\mathrm{Rel}(\mathcal{A},\mathcal{E}),X} = \mathrm{id}_{\mathrm{S}(\mathcal{A}),X}$ *for any object X of $\mathcal{A}$.*

PROOF. Since $\mathcal{E}$ consists of epimorphisms of $\mathcal{A}$ Lemma 4.36 guarantees that $\mathrm{id}_{\mathcal{A},X} \times_{\mathcal{A}} \mathrm{id}_{\mathcal{A},X} \in \mathcal{M}$. Hence, $\mathrm{id}_{\mathrm{Rel},X} = \Phi(\mathrm{id}_{\mathrm{S},X}) = \Phi([\mathrm{id}_{\mathcal{A},X}, \mathrm{id}_{\mathcal{A},X}]) = [\mathrm{id}_{\mathcal{A},X}, \mathrm{id}_{\mathcal{A},X}]$ by Lemma 5.64. □

It is no surprise that we will indeed inherit the identities from the relations and thus, by the above lemma, from the spans. In order to define the composition, we first introduce an analog of the "projection" $\Phi_{\mathcal{A},\mathcal{E}}$. It will send any span, not to a relation, but to a scalar. The next lemma ensures that this definition makes sense. It is analogous to Lemma 5.62.

LEMMA 5.98. *Let A and B be objects of $\mathcal{A}$, let (f,g) and (f',g') be spans in $\mathcal{A}$ from A to B and let (e,m) be an $(\mathcal{E},\mathcal{M})$-factorization of $f \times_{\mathcal{A}} g$ and (e',m') one of $f' \times_{\mathcal{A}} g'$. If (f,g) and (f',g') form a pair of equivalent spans in $\mathcal{A}$ from A to B, then $\delta(e) = \delta(e')$.*

PROOF. The assumptions are the same as in Lemma 5.62. In the proof of that lemma it was shown that, if $u\colon X' \to X$ is an isomorphism such that $f' = f \circ u$ and $g' = g \circ u$, then (since $(e \circ u, m) \in \mathcal{E} \times \mathcal{M}$ is an $(\mathcal{E},\mathcal{M})$-factorization of $f' \times g'$) there exists an isomorphism $w\colon R' \to R$ with $e \circ u = w \circ e'$ and $m' = m \circ w$. Once we prove that (u, e') is a pull-back of (e, w) the claim will be clear because δ, as a degree-function, is invariant under pull-backs.



We only need to show that (u, e') is universal with the property $e \circ u = w \circ e'$. Let $a\colon P \to X$ and $b\colon P \to R'$ be such that $e \circ a = w \circ b$. If we define $x := u^{-1} \circ a$, then obviously $u \circ x = a$. But $w^{-1} \circ e = e' \circ u^{-1}$ also implies $e' \circ x = e' \circ (u^{-1} \circ a) = (e' \circ u^{-1}) \circ a =$

$(w^{-1} \circ e) \circ a = w^{-1} \circ (e \circ a) = w^{-1} \circ (w \circ b) = b$. And it is clear that the condition $a = u \circ x$ determines x uniquely. Hence, the claim is true. □

Hence, we can introduce the "functional" as follows by using the degree function.

DEFINITION 5.99. For any objects A and B and any class $[f, g]$ of spans in $\mathcal{A}$ from A to B let $\Delta_{\mathcal{A},\mathcal{E},\delta}([f, g]) := \delta(e)$, where (e, m) is any $(\mathcal{E}, \mathcal{M})$-factorization of $f \times_{\mathcal{A}} g$.

Conversely, the degree function δ can be recovered from $\Delta_{\mathcal{A},\mathcal{E},\delta}$ via the identity $\delta(e) = \Delta_{\mathcal{A},\mathcal{E},\delta}([e, e])$, which holds for any $e \in \mathcal{E}$ because, if X is the co-domain of e, then $(e, \mathrm{id}_{\mathcal{A},X} \times_{\mathcal{A}} \mathrm{id}_{\mathcal{A},X})$ is an $(\mathcal{E}, \mathcal{M})$-factorization of $e \times_{\mathcal{A}} e$. In analogy to Lemma 5.64, which showed that $(\mathcal{E}, \mathcal{M})$-relations are fixed points of the "projection" $\Phi_{\mathcal{A},\mathcal{E}}$, below we show that the "funcional" $\Delta_{\mathcal{A},\mathcal{E},\delta}$ maps $(\mathcal{E}, \mathcal{M})$-relations to the unit.

The following is a side remark unrelated to the proof of the construction. It shows that the "projection" and the "functional" together are able to discriminate whether a morphism belongs to $\mathcal{E}$.

REMARK 5.100. For any $\{X, Y\} \subseteq \mathrm{obj}_{\mathcal{A}}$, if we resume the terms from Definition 5.103, then the inclusion

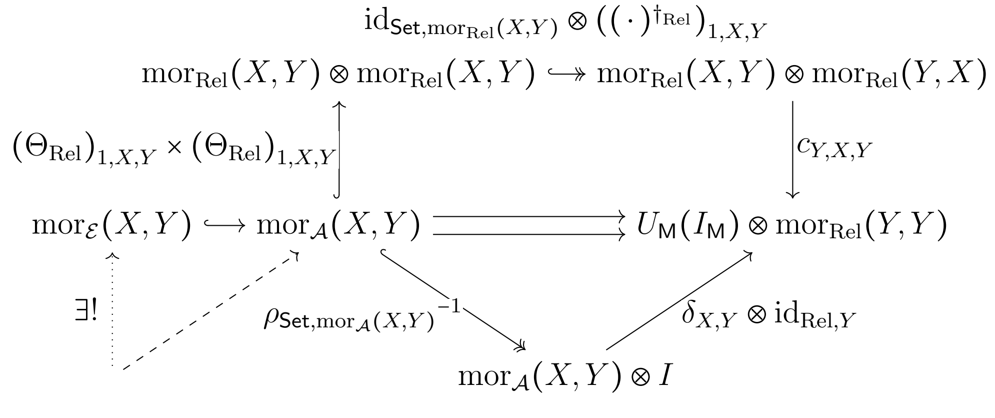
$$\mathrm{mor}_{\mathcal{E}}(X, Y) \to \mathrm{mor}_{\mathcal{A}}(X, Y)$$

is the equalizer in Set of the two morphisms

$$(\delta_{X,Y} \otimes_{\mathsf{Set}} \mathrm{id}_{\mathrm{Rel},Y}) \circ_{\mathsf{Set}} \rho_{\mathsf{Set},\mathrm{mor}_{\mathcal{A}}(X,Y)}{}^{-1_{\mathsf{Set}}},$$

where we interpret $\mathrm{id}_{\mathrm{Rel},Y}$ as a morphsim $I_{\mathsf{Set}} \to \mathrm{mor}_{\mathrm{Rel}}(Y, Y)$, and

$$c_{Y,X,Y} \circ_{\mathsf{Set}} (\mathrm{id}_{\mathsf{Set},\mathrm{mor}_{\mathrm{Rel}}(X,Y)} \otimes_{\mathsf{Set}} ((\,\cdot\,)^{\dagger_{\mathrm{Rel}}})_{1,X,Y}) \circ_{\mathsf{Set}} ((\Theta_{\mathrm{Rel}})_{1,X,Y} \times_{\mathsf{Set}} (\Theta_{\mathrm{Rel}})_{1,X,Y}).$$



Or, in terms of elements, for any $e \in \mathrm{mor}_{\mathcal{A}}(X, Y)$,

$$\left.\begin{array}{rr} & \Delta_{\mathcal{A},\mathcal{E},\delta}(\Theta_{\mathcal{A}}(e) \circ_{\mathrm{S}} \Theta_{\mathcal{A}}(e)^{\dagger_{\mathrm{S}}}) = \delta(e) \\ \text{and} & \Phi_{\mathcal{A},\mathcal{E}}(\Theta_{\mathcal{A}}(e) \circ_{\mathrm{S}} \Theta_{\mathcal{A}}(e)^{\dagger_{\mathrm{S}}}) = \mathrm{id}_{\mathrm{S},Y} \end{array}\right. \iff e \in \mathrm{mor}_{\mathcal{E}}(X, Y).$$

PROOF. By Lemma 4.36 we know $\Theta_{\mathcal{A}}(e) \circ_{\mathrm{S}} \Theta_{\mathcal{A}}(e)^{\dagger_{\mathrm{S}}} = [e, e]$. Hence, if (e_0, m_0) is any $(\mathcal{E}, \mathcal{M})$-factorization of $e \times_{\mathcal{A}} e$ with image object S, then

$$\Delta_{\mathcal{A},\mathcal{E},\delta}(\Theta_{\mathcal{A}}(e) \circ_{\mathrm{S}} \Theta_{\mathcal{A}}(e)^{\dagger_{\mathrm{S}}}) = \delta(e_0)$$
$$\text{and} \quad \Phi_{\mathcal{A},\mathcal{E}}(\Theta_{\mathcal{A}}(e) \circ_{\mathrm{S}} \Theta_{\mathcal{A}}(e)^{\dagger_{\mathrm{S}}}) = [\pi^1_{\mathcal{A},Y,Y} \circ_{\mathcal{A}} m_0, \pi^2_{\mathcal{A},Y,Y} \circ_{\mathcal{A}} m_0].$$

In consequence, the claim is equivalent to the assertion that $e \in \mathrm{mor}_{\mathcal{M}}(X, Y)$ holds if and only if $\delta(e) = \delta(e_0)$ is true and simultaneously the spans $(\mathrm{id}_{\mathcal{A},Y}, \mathrm{id}_{\mathcal{A},Y})$ and $(\pi^1_{\mathcal{A},Y,Y} \circ_{\mathcal{A}} m_0, \pi^2_{\mathcal{A},Y,Y} \circ_{\mathcal{A}} m_0)$ are equivalent. The latter is true if and only if there exists $u \in \mathrm{iso}_{\mathcal{A}}(Y, S)$ such that $m_0 = (\mathrm{id}_{\mathcal{A},Y} \times_{\mathcal{A}} \mathrm{id}_{\mathcal{A},Y}) \circ_{\mathcal{A}} u$.

Now, if $e \in \mathrm{mor}_{\mathcal{E}}(X, Y)$, we can choose $(e_0, m_0) = (e, \mathrm{id}_{\mathcal{A},Y} \times_{\mathcal{A}} \mathrm{id}_{\mathcal{A},Y})$ by Lemma 4.36 and $S = Y$ and $u = \mathrm{id}_{\mathcal{A},Y}$ and thus find the conditions trivially satisfied.

Conversely, if $\delta(e) = \delta(e_0)$ and $m_0 = (\mathrm{id}_{\mathcal{A},Y} \times_{\mathcal{A}} \mathrm{id}_{\mathcal{A},Y}) \circ_{\mathcal{A}} u$ for some $u \in \mathrm{iso}_{\mathcal{A}}(Y, S)$, then it follows $e \times_{\mathcal{A}} e = m_0 \circ_{\mathcal{A}} e_0 = ((\mathrm{id}_{\mathcal{A},Y} \times_{\mathcal{A}} \mathrm{id}_{\mathcal{A},Y}) \circ_{\mathcal{A}} u) \circ_{\mathcal{A}} e_0 = (u \circ_{\mathcal{A}} e_0) \times_{\mathcal{A}} (u \circ_{\mathcal{A}} e_0)$. The consequence $e = u \circ_{\mathcal{A}} e_0$ proves $e \in \mathrm{mor}_{\mathcal{E}}(X, Y)$ by Lemma 4.33 (a) because $e_0 \in \mathrm{mor}_{\mathcal{E}}(X, S)$ and $u \in \mathrm{mor}_{\mathcal{E}}(Y, S)$ by Lemma 4.32. □

LEMMA 5.101. *For any objects A and B of $\mathcal{A}$ and any $(\mathcal{E}, \mathcal{M})$-relation $[f, g]$, if we interpret $\Delta_{\mathcal{A},\mathcal{E},\delta}([f, g])$ as a mapping $I_{\mathsf{Set}} \to U_{\mathsf{S}}(\mathrm{O}_R)$, then*

$$\Delta_{\mathcal{A},\mathcal{E},\delta}([f, g]) = I_{(U_{\mathsf{S}})_{\triangleright}(R)}.$$

PROOF. Let X be the base of (f, g). Then, $\mathrm{id}_{\mathcal{A},X} \in \mathcal{E}$ by Lemma 4.32 and $f \times_{\mathcal{A}} g \in \mathcal{M}$ because $[f, g]$ is an $(\mathcal{E}, \mathcal{M})$-relation. Hence, $(\mathrm{id}_{\mathcal{A},X}, f \times_{\mathcal{A}} g)$ is an $(\mathcal{E}, \mathcal{M})$-factorization of $f \times_{\mathcal{A}} g$. It follows $\Delta_{\mathcal{A},\mathcal{E},\delta}([f, g]) = \delta(\mathrm{id}_{\mathcal{A},X}) = U_{\mathsf{S}}(I_R) \circ_{\mathsf{Set}} (U_{\mathsf{S}})_I$ by the axioms of δ. □

And, once more, an analogy to the second stage of the construction arises. The following lemma can be understood as the counterpart to 5.65. It is the crucial ingredient to the proof that composition is associative.

LEMMA 5.102. *For any $\{A, B, C\} \subseteq \mathrm{obj}_{\mathcal{A}}$,*

$$\begin{aligned}&(\Delta_{\mathcal{A},\mathcal{E},\delta})_{1,A,C} \circ_{\mathsf{Set}} \circ_{\mathrm{S},A,B,C} \\ &\quad = \otimes_{(U_{\mathsf{S}})_{\triangleright}(R)} \circ_{\mathsf{Set}} \Big(\big(\otimes_{(U_{\mathsf{S}})_{\triangleright}(R)} \circ_{\mathsf{Set}} ((\Delta_{\mathcal{A},\mathcal{E},\delta})_{1,B,C} \otimes_{\mathsf{Set}} (\Delta_{\mathcal{A},\mathcal{E},\delta})_{1,A,B}) \big) \\ &\qquad\qquad \times_{\mathsf{Set}} \big((\Delta_{\mathcal{A},\mathcal{E},\delta})_{1,A,C} \circ_{\mathsf{Set}} \circ_{\mathrm{S},A,B,C} \circ_{\mathsf{Set}} ((\Phi_{\mathcal{A},\mathcal{E}})_{1,B,C} \otimes_{\mathsf{Set}} (\Phi_{\mathcal{A},\mathcal{E}})_{1,A,B}) \big) \Big)\end{aligned}$$

where we interpret $(\Phi_{\mathcal{A},\mathcal{E}})_{1,A,B}$ *and* $(\Phi_{\mathcal{A},\mathcal{E}})_{1,B,C}$ *as self-mappings of* $\mathrm{mor}_{\mathrm{S}}(A,B)$ *respectively* $\mathrm{mor}_{\mathrm{S}}(B,C)$*.*

$$
\begin{array}{ccc}
\mathrm{mor}(B,C)\otimes\mathrm{mor}(A,B) & \xrightarrow{\mathrm{id}_{\mathrm{mor}(B,C)\otimes\mathrm{mor}(A,B)}\times\mathrm{id}_{\mathrm{mor}(B,C)\otimes\mathrm{mor}(A,B)}} & (\mathrm{mor}(B,C)\otimes\mathrm{mor}(A,B))\otimes(\mathrm{mor}(B,C)\otimes\mathrm{mor}(A,B)) \\
\downarrow{\scriptstyle \circ_{\mathrm{S},A,B,C}} & & \downarrow{\scriptstyle \mathrm{id}_{\mathrm{mor}(B,C)\otimes\mathrm{mor}(A,B)}\otimes((\Phi_{\mathcal{A},\mathcal{E}})_{1,B,C}\otimes_{\mathsf{Set}}(\Phi_{\mathcal{A},\mathcal{E}})_{1,A,B})} \\
\mathrm{mor}(A,C) & & (\mathrm{mor}(B,C)\otimes\mathrm{mor}(A,B))\otimes(\mathrm{mor}(B,C)\otimes\mathrm{mor}(A,B)) \\
 & & \downarrow{\scriptstyle ((\Delta_{\mathcal{A},\mathcal{E},\delta})_{1,B,C}\otimes_{\mathsf{Set}}(\Delta_{\mathcal{A},\mathcal{E},\delta})_{1,A,B})\otimes\circ_{\mathrm{S},A,B,C}} \\
\downarrow{\scriptstyle (\Delta_{\mathcal{A},\mathcal{E},\delta})_{1,A,C}} & & (U(I)\otimes U(I))\otimes\mathrm{mor}(A,C) \\
 & & \downarrow{\scriptstyle \otimes_{(U_{\mathsf{S}})_{\triangleright}(R)}\otimes(\Delta_{\mathcal{A},\mathcal{E},\delta})_{1,A,C}} \\
U(I) & \xleftarrow[\otimes_{(U_{\mathsf{S}})_{\triangleright}(R)}]{} & U(I)\otimes U(I)
\end{array}
$$

Or, in terms of elements, for any classes $x\in\mathrm{mor}_{\mathrm{S}}(A,B)$ *and* $y\in\mathrm{mor}_{\mathrm{S}}(B,C)$*,*

$$\Delta_{\mathcal{A},\mathcal{E},\delta}(y\circ_{\mathrm{S}}x)=(\Delta_{\mathcal{A},\mathcal{E},\delta}(y)\otimes_{(U_{\mathsf{S}})_{\triangleright}(R)}\Delta_{\mathcal{A},\mathcal{E},\delta}(x))\otimes_{(U_{\mathsf{S}})_{\triangleright}(R)}\Delta_{\mathcal{A},\mathcal{E},\delta}(\Phi_{\mathcal{A},\mathcal{E}}(y)\circ_{\mathrm{S}}\Phi_{\mathcal{A},\mathcal{E}}(x)).$$

PROOF. If we let $[f,g]$ be a class of spans in $\mathcal{A}$ from A to B and $[p,q]$ one of spans from B to C, then the premises are identical with those of Lemma 5.65. Resume the definitions from the proof there. Then, $\Delta([p,q]\circ_{\mathrm{S}}[f,g])=\delta(e')$ and $\Delta([p,q])=\delta(e_2)$ and $\Delta([f,g])=\delta(e_1)$ as well as $\Delta(\Phi([p,q])\circ_{\mathrm{S}}\Phi([f,g]))=\delta(e)$. Hence what we have to show is that

$$\delta(e')\overset{!}{=}(\delta(e_2)\otimes_{(U_{\mathsf{S}})_{\triangleright}(R)}\delta(e_1))\otimes_{(U_{\mathsf{S}})_{\triangleright}(R)}\delta(e).$$

The assumption that δ is invariant under pull-backs has the following implications: $\delta(d_1)=\delta(e_1)$ because (c_1,d_1) is a pull-back of (e_1,a_1) and, likewise, $\delta(d_2)=\delta(e_2)$ because (c_2,d_2) is one of (e_2,a_2); moreover, $\delta(s_1)=\delta(d_2)$ since (s_1,s_2) is a pull-back of (d_1,d_2). In conclusion, $\delta(e_1)=\delta(d_1)$ and $\delta(e_2)=\delta(s_1)$.

Because u and v are isomorphisms, $\delta(u)=\delta(v)=I_{(U_{\mathsf{S}})_{\triangleright}(R)}$ by Lemma 2.5 (a). We have already seen $e\circ d_1\circ s_1\circ v\in\mathcal{E}$ and $e\circ d_1\circ s_1\circ v=u\circ e'$. Hence, δ being multiplicative lets us infer

$$
\begin{aligned}
\delta(e')&=\delta(u)\otimes_{(U_{\mathsf{S}})_{\triangleright}(R)}\delta(e')\\
&=\delta(u\circ e')\\
&=\delta(e\circ d_1\circ s_1\circ v)\\
&=(\delta(e)\otimes_{(U_{\mathsf{S}})_{\triangleright}(R)}(\delta(d_1)\otimes_{(U_{\mathsf{S}})_{\triangleright}(R)}\delta(s_1)))\otimes_{(U_{\mathsf{S}})_{\triangleright}(R)}\delta(v)\\
&=(\delta(e_2)\otimes_{(U_{\mathsf{S}})_{\triangleright}(R)}\delta(e_1))\otimes_{(U_{\mathsf{S}})_{\triangleright}(R)}\delta(e),
\end{aligned}
$$

where we have used that $(U_{\mathsf{S}})_{\triangleright}(R)$ is commutative. That concludes the proof. □

Since $\mathrm{Rel}(\mathcal{A},\mathcal{E})$ is locally small by Proposition 5.69 the following makes sense. It is the same definition as given in Section 2.2.

DEFINITION 5.103. Let $\mathcal{T}^0$ be short for $\mathcal{T}^0(\mathcal{A},\mathcal{E},\delta)$.

(a) Define $\mathrm{obj}_{\mathcal{T}^0} \coloneqq \mathrm{obj}_{\mathrm{Rel}}$.

(b) For any $\{A,B\} \subseteq \mathrm{obj}_{\mathrm{Rel}}$ let $\mathrm{mor}_{\mathcal{T}^0}(A,B) \coloneqq G_{\mathsf{M}}(\mathrm{mor}_{\mathrm{Rel}}(A,B))$.

(c) Given any $\{A,B,C\} \subseteq \mathrm{obj}_{\mathcal{T}^0}$, if $c_{A,B,C}$ temporarily stands for the mapping

$$\mathrm{mor}_{\mathrm{Rel}}(B,C) \otimes_{\mathsf{Set}} \mathrm{mor}_{\mathrm{Rel}}(A,B) \to U_{\mathsf{M}}(I_{\mathsf{M}}) \otimes_{\mathsf{Set}} \mathrm{mor}_{\mathrm{Rel}}(A,C),$$
$$(y,x) \mapsto (\Delta_{\mathcal{A},\mathcal{E},\delta}(y \circ_{\mathrm{S}} x), y \circ_{\mathrm{Rel}} x),$$

then define

$$\begin{aligned}\circ_{\mathcal{T}^0,A,B,C} \coloneqq \lambda_{\mathsf{M},\mathrm{mor}_{\mathcal{T}^0}(A,C)} \circ_{\mathsf{M}} (\mathrm{cu}_{\mathsf{M},I_{\mathsf{M}}} \otimes_{\mathsf{M}} \mathrm{id}_{\mathsf{M},\mathrm{mor}_{\mathcal{T}^0}(A,C)}) \\ \circ_{\mathsf{M}} (G_{\mathsf{M}})_{\otimes,U_{\mathsf{M}}(I_{\mathsf{M}}),\mathrm{mor}_{\mathrm{Rel}}(A,C)}^{-1_{\mathsf{M}}} \circ_{\mathsf{M}} G_{\mathsf{M}}(c_{A,B,C}) \\ \circ_{\mathsf{M}} (G_{\mathsf{M}})_{\otimes,\mathrm{mor}_{\mathrm{Rel}}(B,C),\mathrm{mor}_{\mathrm{Rel}}(A,B)}\end{aligned}$$

(d) For any $A \in \mathrm{obj}_{\mathcal{T}^0}$, considering $\mathrm{id}_{\mathrm{Rel},A}$ a mapping $I_{\mathsf{Set}} \to \mathrm{mor}_{\mathrm{Rel}}(A,A)$, let

$$\mathrm{id}_{\mathcal{T}^0,A} \coloneqq G_{\mathsf{M}}(\mathrm{id}_{\mathrm{Rel},A}) \circ_{\mathsf{M}} (G_{\mathsf{M}})_I.$$

The following is the centerpiece of the proof that the composition of $\mathcal{T}^0(\mathcal{A},\mathcal{E},\delta)$ is associative. It is enabled by Lemmata 5.101 and 5.102.

LEMMA 5.104. *For any $\{A,B,C,D\} \subseteq \mathrm{obj}_{\mathrm{S}}$, if we resume the names from Definition 5.103, then*

$$\begin{aligned}&(\otimes_{(U_{\mathsf{S}})_{\triangleright}(R)} \otimes_{\mathsf{Set}} \mathrm{id}_{\mathsf{Set},\mathrm{mor}_{\mathrm{Rel}}(A,B)}) \circ_{\mathsf{Set}} \alpha_{\mathsf{Set},U_{\mathsf{M}}(I_{\mathsf{M}}),U_{\mathsf{M}}(I_{\mathsf{M}}),\mathrm{mor}_{\mathrm{Rel}}(A,D)}^{-1_{\mathsf{Set}}} \\ &\circ_{\mathsf{Set}}(\mathrm{id}_{\mathsf{Set},U_{\mathsf{M}}(I_{\mathsf{M}})} \otimes_{\mathsf{Set}} c_{A,B,D}) \circ \alpha_{\mathsf{Set},U_{\mathsf{M}}(I_{\mathsf{M}}),\mathrm{mor}_{\mathrm{Rel}}(B,D),\mathrm{mor}_{\mathrm{Rel}}(A,B)} \\ &\circ_{\mathsf{Set}}(c_{B,C,D} \otimes_{\mathsf{Set}} \mathrm{id}_{\mathsf{Set},\mathrm{mor}_{\mathrm{Rel}}(A,B)}) \\ &= (\otimes_{(U_{\mathsf{S}})_{\triangleright}(R)} \otimes_{\mathsf{Set}} \mathrm{id}_{\mathsf{Set},\mathrm{mor}_{\mathrm{Rel}}(A,B)}) \circ_{\mathsf{Set}} \alpha_{\mathsf{Set},U_{\mathsf{M}}(I_{\mathsf{M}}),U_{\mathsf{M}}(I_{\mathsf{M}}),\mathrm{mor}_{\mathrm{Rel}}(A,D)}^{-1_{\mathsf{Set}}} \\ &\quad \circ_{\mathsf{Set}} (\mathrm{id}_{\mathsf{Set},U_{\mathsf{M}}(I_{\mathsf{M}})} \otimes_{\mathsf{Set}} c_{A,C,D}) \circ \alpha_{\mathsf{Set},U_{\mathsf{M}}(I_{\mathsf{M}}),\mathrm{mor}_{\mathrm{Rel}}(C,D),\mathrm{mor}_{\mathrm{Rel}}(A,C)} \\ &\quad \circ_{\mathsf{Set}} (\gamma_{\mathsf{Set},\mathrm{mor}_{\mathrm{Rel}}(C,D),U_{\mathsf{M}}(I_{\mathsf{M}})} \otimes_{\mathsf{Set}} \mathrm{id}_{\mathsf{Set},\mathrm{mor}_{\mathrm{Rel}}(A,C)}) \circ_{\mathsf{Set}} \alpha_{\mathsf{Set},\mathrm{mor}_{\mathrm{Rel}}(C,D),U_{\mathsf{M}}(I_{\mathsf{M}}),\mathrm{mor}_{\mathrm{Rel}}(A,C)}^{-1_{\mathsf{Set}}} \\ &\quad \circ_{\mathsf{Set}} (\mathrm{id}_{\mathsf{Set},\mathrm{mor}_{\mathrm{Rel}}(C,D)} \otimes_{\mathsf{Set}} c_{A,B,C}) \circ_{\mathsf{Set}} \alpha_{\mathsf{Set},\mathrm{mor}_{\mathrm{Rel}}(C,D),\mathrm{mor}_{\mathrm{Rel}}(B,C),\mathrm{mor}_{\mathrm{Rel}}(A,B)}.\end{aligned}$$

$$
\begin{array}{ccc}
(\mathrm{mor}(C,D) \otimes \mathrm{mor}(B,C)) \otimes \mathrm{mor}(A,B) & \xrightarrow{\alpha_{\mathrm{mor}(C,D),\mathrm{mor}(B,C),\mathrm{mor}(A,B)}} & \mathrm{mor}(C,D) \otimes (\mathrm{mor}(B,C) \otimes \mathrm{mor}(A,B)) \\
\Big\downarrow {\scriptstyle c_{B,C,D}\otimes \mathrm{id}_{\mathrm{mor}(A,B)}} & & \Big\downarrow {\scriptstyle \mathrm{id}_{\mathrm{mor}(C,D)}\otimes c_{A,B,C}} \\
(U(I) \otimes \mathrm{mor}(B,D)) \otimes \mathrm{mor}(A,B) & & \mathrm{mor}(C,D) \otimes (U(I) \otimes \mathrm{mor}(A,C)) \\
 & & \Big\downarrow {\scriptstyle \alpha^{-1}_{\mathrm{mor}(C,D),U(I),\mathrm{mor}(A,C)}} \\
 & & (\mathrm{mor}(C,D) \otimes U(I)) \otimes \mathrm{mor}(A,C) \\
\Big\downarrow {\scriptstyle \alpha_{U(I),\mathrm{mor}(B,D),\mathrm{mor}(A,B)}} & & \Big\downarrow {\scriptstyle \gamma_{U(I),\mathrm{mor}(C,D)}\otimes \mathrm{id}_{\mathrm{mor}(A,C)}} \\
 & & (U(I) \otimes \mathrm{mor}(C,D)) \otimes \mathrm{mor}(A,C) \\
 & & \Big\downarrow {\scriptstyle \alpha_{U(I),\mathrm{mor}(C,D),\mathrm{mor}(A,C)}} \\
U(I) \otimes (\mathrm{mor}(B,D) \otimes \mathrm{mor}(A,B)) & & U(I) \otimes (\mathrm{mor}(C,D) \otimes \mathrm{mor}(A,C)) \\
\Big\downarrow {\scriptstyle \mathrm{id}_{U(I)}\otimes c_{A,B,D}} & & \Big\downarrow {\scriptstyle \mathrm{id}_{U(I)}\otimes c_{A,C,D}} \\
U(I) \otimes (U(I) \otimes \mathrm{mor}(A,D)) & & U(I) \otimes (U(I) \otimes \mathrm{mor}(A,D)) \\
\Big\downarrow {\scriptstyle \alpha^{-1}_{U(I),U(I),\mathrm{mor}(A,D)}} & & \Big\downarrow {\scriptstyle \alpha^{-1}_{U(I),U(I),\mathrm{mor}(A,D)}} \\
(U(I) \otimes U(I)) \otimes \mathrm{mor}(A,D) & & (U(I) \otimes U(I)) \otimes \mathrm{mor}(A,D) \\
\Big\downarrow {\scriptstyle \otimes_{U_{\triangleright}(R)}\otimes \mathrm{id}_{\mathrm{mor}(A,D)}} & & \Big\downarrow {\scriptstyle \otimes_{U_{\triangleright}(R)}\otimes \mathrm{id}_{\mathrm{mor}(A,D)}} \\
U(I) \otimes \mathrm{mor}(A,D) & = & U(I) \otimes \mathrm{mor}(A,D)
\end{array}
$$

Or, in terms of elements, for any $x \in \mathrm{mor}_{\mathrm{Rel}}(A,B)$ *and* $y \in \mathrm{mor}_{\mathrm{Rel}}(B,C)$ *and* $z \in \mathrm{mor}_{\mathrm{Rel}}(C,D)$*,*

$$
\begin{aligned}
&\Delta_{\mathcal{A},\mathcal{E},\delta}(z \circ_{\mathrm{S}} y) \otimes_{(U_{\mathrm{S}})_{\triangleright}(R)} \Delta_{\mathcal{A},\mathcal{E},\delta}(\Phi_{\mathcal{A},\mathcal{E}}(z \circ_{\mathrm{S}} y) \circ_{\mathrm{S}} x) \\
&\qquad = \Delta_{\mathcal{A},\mathcal{E},\delta}(y \circ_{\mathrm{S}} x) \otimes_{(U_{\mathrm{S}})_{\triangleright}(R)} \Delta_{\mathcal{A},\mathcal{E},\delta}(z \circ_{\mathrm{S}} \Phi_{\mathcal{A},\mathcal{E}}(y \circ_{\mathrm{S}} x))
\end{aligned}
$$

and

$$\Phi_{\mathcal{A},\mathcal{E}}(\Phi_{\mathcal{A},\mathcal{E}}(z \circ_{\mathrm{S}} y) \circ_{\mathrm{S}} x) = \Phi_{\mathcal{A},\mathcal{E}}(z \circ_{\mathrm{S}} \Phi_{\mathcal{A},\mathcal{E}}(y \circ_{\mathrm{S}} x)).$$

Proof. We show both identities separately. Firstly, since z and x are $(\mathcal{E},\mathcal{M})$-relations Lemma 5.64 implies $\Phi(z) = z$ and $\Phi(x) = x$. For the same reason, $\Delta(z) = \Delta(x) = I_{(U_{\mathrm{S}})_{\triangleright}(R)}$ by Lemma 5.101. Hence we infer with the help of Lemma 5.102

$$
\begin{aligned}
&\Delta(y \circ_{\mathrm{S}} x) \otimes_{(U_{\mathrm{S}})_{\triangleright}(R)} \Delta(z \circ_{\mathrm{S}} \Phi(y \circ_{\mathrm{S}} x)) \\
&\qquad = (\Delta(z) \otimes_{(U_{\mathrm{S}})_{\triangleright}(R)} \Delta(y \circ_{\mathrm{S}} x)) \otimes_{(U_{\mathrm{S}})_{\triangleright}(R)} \Delta(\Phi(z) \circ_{\mathrm{S}} \Phi(y \circ_{\mathrm{S}} x)) \\
&\qquad = \Delta(z \circ_{\mathrm{S}} (y \circ_{\mathrm{S}} x)) \\
&\qquad = \Delta((z \circ_{\mathrm{S}} y) \circ_{\mathrm{S}} x) \\
&\qquad = (\Delta(z \circ_{\mathrm{S}} y) \otimes_{(U_{\mathrm{S}})_{\triangleright}(R)} \Delta(x)) \otimes_{(U_{\mathrm{S}})_{\triangleright}(R)} \Delta(\Phi(z \circ_{\mathrm{S}} y) \circ_{\mathrm{S}} \Phi(x)) \\
&\qquad = \Delta(z \circ_{\mathrm{S}} y) \otimes_{(U_{\mathrm{S}})_{\triangleright}(R)} \Delta(\Phi(z \circ_{\mathrm{S}} y) \circ_{\mathrm{S}} x),
\end{aligned}
$$

where we have used Proposition 5.67 in the third step.

Similarly, $\Phi(z) = z$ and $\Phi(x) = x$ also prove, by Lemma 5.65,

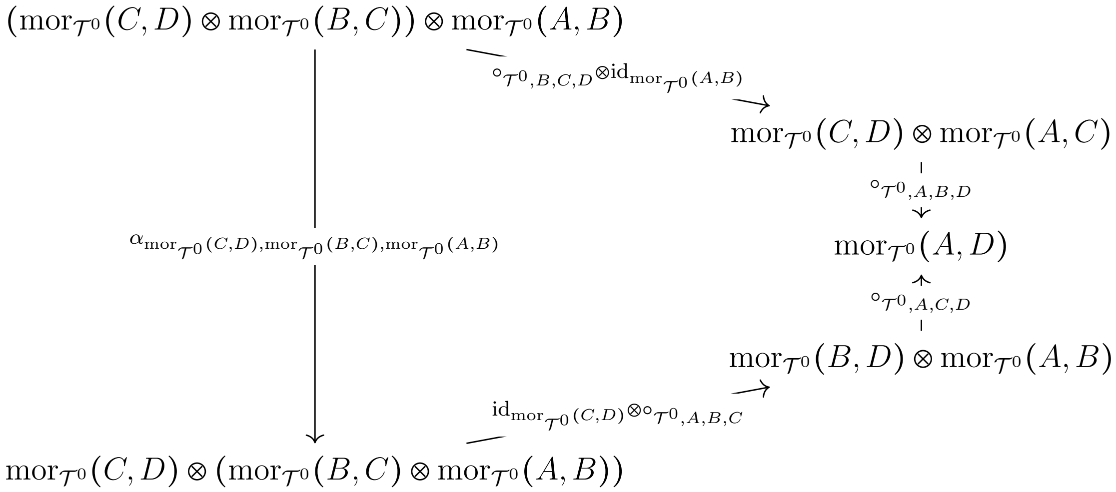
$$\begin{aligned}\Phi(\Phi(z \circ_{\mathrm{S}} y) \circ_{\mathrm{S}} x) &= \Phi(\Phi(z \circ_{\mathrm{S}} y) \circ_{\mathrm{S}} \Phi(x)) \\ &= \Phi((z \circ_{\mathrm{S}} y) \circ_{\mathrm{S}} x) \\ &= \Phi(z \circ_{\mathrm{S}} (y \circ_{\mathrm{S}} x)) \\ &= \Phi(\Phi(z) \circ_{\mathrm{S}} \Phi((y \circ_{\mathrm{S}} x))) \\ &= \Phi(z \circ_{\mathrm{S}} \Phi(y \circ_{\mathrm{S}} x)),\end{aligned}$$

where again the third step used Proposition 5.67. ☐

The proof that the composition of $\mathcal{T}^0(\mathcal{A},\mathcal{E},\delta)$ is associative a very long computation. Due to time constraints, only half of the proof is given here.

LEMMA 5.105. *For any* $\{A,B,C,D\} \subseteq \mathrm{obj}_{\mathcal{T}^0}$,

$$\begin{aligned}&\circ_{\mathcal{T}^0,A,C,D} \circ_{\mathsf{M}} (\mathrm{id}_{\mathsf{M},\mathrm{mor}_{\mathcal{T}^0}(C,D)} \otimes_{\mathsf{M}} \circ_{\mathcal{T}^0,A,B,C}) \circ_{\mathsf{M}} \alpha_{\mathsf{M},\mathrm{mor}_{\mathcal{T}^0}(C,D),\mathrm{mor}_{\mathcal{T}^0}(B,C),\mathrm{mor}_{\mathcal{T}^0}(A,B)} \\ &= \circ_{\mathcal{T}^0,A,B,D} \circ_{\mathsf{M}} (\circ_{\mathcal{T}^0,B,C,D} \otimes_{\mathsf{M}} \mathrm{id}_{\mathsf{M},\mathrm{mor}_{\mathcal{T}^0}(A,B)})\end{aligned}$$



PROOF. The proof strategy is to transform each side of the claimed identity into the same expression. Unfortunately, there only was time to write this down for the left-hand side. The rest of the proof is very similar. Transforming the left-hand side into the intended midway point between both sides takes 25 steps. In each step, the part of the expression that will be substituted in the next step is printed in red. Insofar as the current form of the left-hand side already agrees with the target expression it is colored green.

The key to the proof is Lemma 5.104. The notation from Definition 5.103 is continued. Due to the long formulas occurring, the following abbreviations are used throughout the proof: Firstly, all category indices $\mathcal{T}^0$, M, S and Set are suppressed. Secondly, for any $\{X,Y\} \subseteq \mathrm{obj}_{\mathcal{T}^0}$ we write XY instead of $\mathrm{mor}_{\mathrm{Rel}}(X,Y)$.

Step 1: By definition, the left-hand side of the claimed identity is given by

$$\lambda_{G(AD)} \circ (\mathrm{cu}_I \otimes \mathrm{id}_{G(AD)}) \circ G^{-1}_{\otimes,U(I),AD} \circ G(c_{A,C,D}) \circ G_{\otimes,CD,AC} \\ \circ (\mathrm{id}_{G(CD)} \otimes (\lambda_{G(AC)} \circ (\mathrm{cu}_I \otimes \mathrm{id}_{G(AC)}) \circ G^{-1}_{\otimes,U(I),AC} \circ G(c_{A,B,C}) \circ G_{\otimes,BC,AB})) \\ \circ \alpha_{G(CD),G(BC),G(AB)}.$$

Step 2: Since, by the general theory of monoidal adjunctions, $G^{-1}_{\otimes,U(I),AC} = \mathrm{cu}_{G(U(I))\otimes G(AC)} \circ G(U_{\otimes,G(U(I)),G(AC)}) \circ G(\mathrm{un}_{U(I)} \otimes \mathrm{un}_{BD})$, this is the same as

$$\lambda_{G(AD)} \circ (\mathrm{cu}_I \otimes \mathrm{id}_{G(AD)}) \circ G^{-1}_{\otimes,U(I),AD} \circ G(c_{A,C,D}) \circ G_{\otimes,CD,AC} \\ \circ (\mathrm{id}_{G(CD)} \otimes (\lambda_{G(AC)} \circ (\mathrm{cu}_I \otimes \mathrm{id}_{G(AC)}) \circ \mathrm{cu}_{G(U(I))\otimes G(AC)} \circ G(U_{\otimes,G(U(I)),G(AC)}) \\ \circ (\mathrm{un}_{U(I)} \otimes \mathrm{un}_{BD}) \circ c_{A,B,C}) \circ G_{\otimes,BC,AB})) \circ \alpha_{G(CD),G(BC),G(AB)}.$$

Step 3: Because cu is a natural transformation from $G \circ U$ to id we can use the identity $\lambda_{G(AC)} \circ (\mathrm{cu}_I \otimes \mathrm{id}_{G(AC)}) \circ \mathrm{cu}_{G(U(I))\otimes G(AC)} = \mathrm{cu}_{G(AC)} \circ G(U(\lambda_{G(AC)} \circ (\mathrm{cu}_I \otimes \mathrm{id}_{G(AC)})))$ and reassociate the terms to transform this into

$$\lambda_{G(AD)} \circ (\mathrm{cu}_I \otimes \mathrm{id}_{G(AD)}) \circ G^{-1}_{\otimes,U(I),AD} \circ G(c_{A,C,D}) \circ G_{\otimes,CD,AC} \\ \circ (\mathrm{id}_{G(CD)} \otimes (\mathrm{cu}_{G(AC)} \circ G(U(\lambda_{G(AC)} \circ (\mathrm{cu}_I \otimes \mathrm{id}_{G(AC)})) \circ U_{\otimes,G(U(I)),G(AC)} \\ \circ (\mathrm{un}_{U(I)} \otimes \mathrm{un}_{BD}) \circ c_{A,B,C}) \circ G_{\otimes,BC,AB})) \circ \alpha_{G(CD),G(BC),G(AB)}.$$

Step 4: By the counit-unit equation, $\mathrm{id}_{G(CD)} = \mathrm{cu}_{G(CD)} \circ G(\mathrm{un}_{CD})$, which is why the above can be rewritten as

$$\lambda_{G(AD)} \circ (\mathrm{cu}_I \otimes \mathrm{id}_{G(AD)}) \circ G^{-1}_{\otimes,U(I),AD} \circ G(c_{A,C,D}) \circ G_{\otimes,CD,AC} \\ \circ (\mathrm{cu}_{G(CD)} \otimes \mathrm{cu}_{G(AC)}) \circ (G(\mathrm{un}_{CD}) \otimes G(U(\lambda_{G(AC)} \circ (\mathrm{cu}_I \otimes \mathrm{id}_{G(AC)})) \\ \circ U_{\otimes,G(U(I)),G(AC)} \circ (\mathrm{un}_{U(I)} \otimes \mathrm{un}_{BD}) \circ c_{A,B,C}) \circ G_{\otimes,BC,AB})) \circ \alpha_{G(CD),G(BC),G(AB)}.$$

Step 5: The identity

$$\mathrm{cu}_{G(CD)} \otimes \mathrm{cu}_{G(AC)} = \mathrm{cu}_{G(CD)\otimes G(AC)} \circ G(U_{\otimes,G(CD),G(AC)}) \circ G_{\otimes,U(G(CD)),U(G(AC))},$$

which is due to the fact that cu is a monoidal natural transformation from $G \circ U$ to id, lets us express this term as

$$\lambda_{G(AD)} \circ (\mathrm{cu}_I \otimes \mathrm{id}_{G(AD)}) \circ G^{-1}_{\otimes,U(I),AD} \circ G(c_{A,C,D}) \circ G_{\otimes,CD,AC} \circ \mathrm{cu}_{G(CD)\otimes G(AC)} \\ \circ G(U_{\otimes,G(CD),G(AC)}) \circ G_{\otimes,U(G(CD)),U(G(AC))} \circ (G(\mathrm{un}_{CD}) \otimes G(U(\lambda_{G(AC)} \\ \circ (\mathrm{cu}_I \otimes \mathrm{id}_{G(AC)})) \circ U_{\otimes,G(U(I)),G(AC)} \circ (\mathrm{un}_{U(I)} \otimes \mathrm{un}_{BD}) \circ c_{A,B,C})) \\ \circ (\mathrm{id}_{G(CD)} \otimes G_{\otimes,BC,AB}) \circ \alpha_{G(CD),G(BC),G(AB)}.$$

Step 6: If we now make use of the fact that G is a monoidal functor and thus,

$$G_{\otimes,U(G(CD)),U(G(AC))} \circ (G(\mathrm{un}_{CD}) \otimes G(U(\lambda_{G(AC)} \circ (\mathrm{cu}_I \otimes \mathrm{id}_{G(AC)})) \\ \circ U_{\otimes,G(U(I)),G(AC)} \circ (\mathrm{un}_{U(I)} \otimes \mathrm{un}_{BD}) \circ c_{A,B,C})) \\ = G(\mathrm{un}_{CD} \otimes (U(\lambda_{G(AC)} \circ (\mathrm{cu}_I \otimes \mathrm{id}_{G(AC)})) \circ U_{\otimes,G(U(I)),G(AC)} \circ (\mathrm{un}_{U(I)} \otimes \mathrm{un}_{BD}) \\ \circ c_{A,B,C})) \circ G_{\otimes,CD,BC\otimes AB}$$

the above expression takes on the form

$$\lambda_{G(AD)} \circ (\mathrm{cu}_I \otimes \mathrm{id}_{G(AD)}) \circ G^{-1}_{\otimes,U(I),AD} \circ G(c_{A,C,D}) \circ G_{\otimes,CD,AC} \circ \mathrm{cu}_{G(CD)\otimes G(AC)}$$
$$\circ\, G(U_{\otimes,G(CD),G(AC)} \circ (\mathrm{un}_{CD} \otimes (U(\lambda_{G(AC)} \circ (\mathrm{cu}_I \otimes \mathrm{id}_{G(AC)})) \circ U_{\otimes,G(U(I)),G(AC)}$$
$$\circ\, (\mathrm{un}_{U(I)} \otimes \mathrm{un}_{BD}) \circ c_{A,B,C}))) \circ G_{\otimes,CD,BC\otimes AB} \circ (\mathrm{id}_{G(CD)} \otimes G_{\otimes,BC,AB})$$
$$\circ\, \alpha_{G(CD),G(BC),G(AB)}.$$

Step 7: By the associativity axiom for monoidal functors, $G_{\otimes,CD,BC\otimes AB} \circ (\mathrm{id}_{G(CD)} \otimes G_{\otimes,BC,AB}) \circ \alpha_{G(CD),G(BC),G(AB)} = G(\alpha_{CD,BC,AB}) \circ G_{\otimes,CD\otimes BC,AB} \circ (G_{\otimes,CD,BC} \otimes \mathrm{id}_{G(AB)})$. Hence, we can reformulate our morphism as

$$\lambda_{G(AD)} \circ (\mathrm{cu}_I \otimes \mathrm{id}_{G(AD)}) \circ G^{-1}_{\otimes,U(I),AD} \circ G(c_{A,C,D}) \circ G_{\otimes,CD,AC} \circ \mathrm{cu}_{G(CD)\otimes G(AC)}$$
$$\circ\, G(U_{\otimes,G(CD),G(AC)} \circ (\mathrm{id}_{U(G(CD))} \otimes (U(\lambda_{G(AC)}) \circ U(\mathrm{cu}_I \otimes \mathrm{id}_{G(AC)})$$
$$\circ\, U_{\otimes,G(U(I)),G(AC)})) \circ (\mathrm{un}_{CD} \otimes (\mathrm{un}_{U(I)} \otimes \mathrm{un}_{AC})) \circ (\mathrm{id}_{CD} \otimes c_{A,B,C}) \circ \alpha_{CD,BC,AB})$$
$$\circ\, G_{\otimes,CD\otimes BC,AB} \circ (G_{\otimes,CD,BC} \otimes \mathrm{id}_{G(AB)}),$$

where also several terms have been reassociated.

Step 8: Because U is a monoidal functor, $U(\mathrm{cu}_I \otimes \mathrm{id}_{G(AC)}) \circ U_{\otimes,G(U(I)),G(AC)} = U_{\otimes,I,G(AC)} \circ (U(\mathrm{cu}_I) \otimes \mathrm{id}_{U(G(AC))})$, which, in combination with some reassociation, makes the above equal to

$$\lambda_{G(AD)} \circ (\mathrm{cu}_I \otimes \mathrm{id}_{G(AD)}) \circ G^{-1}_{\otimes,U(I),AD} \circ G(c_{A,C,D}) \circ G_{\otimes,CD,AC} \circ \mathrm{cu}_{G(CD)\otimes G(AC)}$$
$$\circ\, G(U_{\otimes,G(CD),G(AC)} \circ (U(\mathrm{id}_{G(CD)}) \otimes U(\lambda_{G(AC)})) \circ (\mathrm{id}_{U(G(CD))} \otimes U_{\otimes,I,G(AC)})$$
$$\circ\, (\mathrm{un}_{CD} \otimes ((U(\mathrm{cu}_I) \circ \mathrm{un}_{U(I)}) \otimes \mathrm{un}_{AC})) \circ (\mathrm{id}_{CD} \otimes c_{A,B,C}) \circ \alpha_{CD,BC,AB})$$
$$\circ\, G_{\otimes,CD\otimes BC,AB} \circ (G_{\otimes,CD,BC} \otimes \mathrm{id}_{G(AB)}).$$

Step 9: Once more, a counit-unit equation yields $U(\mathrm{cu}_I) \circ \mathrm{un}_{U(I)} = \mathrm{id}_{U(I)}$. At the same time, because U is a monoidal functor, $U_{\otimes,G(CD),G(AC)} \circ (U(\mathrm{id}_{G(CD)}) \otimes U(\lambda_{G(AC)})) = U(\mathrm{id}_{G(CD)} \otimes \lambda_{G(AC)}) \circ U_{\otimes,G(CD),I\otimes G(AC)}$. Therefore, the previous term is identical to

$$\lambda_{G(AD)} \circ (\mathrm{cu}_I \otimes \mathrm{id}_{G(AD)}) \circ G^{-1}_{\otimes,U(I),AD} \circ G(c_{A,C,D}) \circ G_{\otimes,CD,AC} \circ \mathrm{cu}_{G(CD)\otimes G(AC)}$$
$$\circ\, G(U(\mathrm{id}_{G(CD)} \otimes \lambda_{G(AC)}) \circ U_{\otimes,G(CD),I\otimes G(AC)} \circ (\mathrm{id}_{U(G(CD))} \otimes U_{\otimes,I,G(AC)})$$
$$\circ\, (\mathrm{un}_{CD} \otimes (\mathrm{id}_{U(I)} \otimes \mathrm{un}_{AC})) \circ (\mathrm{id}_{CD} \otimes c_{A,B,C}) \circ \alpha_{CD,BC,AB})$$
$$\circ\, G_{\otimes,CD\otimes BC,AB} \circ (G_{\otimes,CD,BC} \otimes \mathrm{id}_{G(AB)}).$$

Step 10: Since M is a symmetric monoidal category,

$$(\mathrm{id}_{G(CD)} \otimes \lambda_{G(AC)}) \circ \alpha_{G(CD),I,G(AC)} = \rho_{G(CD)} \otimes \mathrm{id}_{AC}.$$

That is why our morphism can also be written as

$$\lambda_{G(AD)} \circ (\mathrm{cu}_I \otimes \mathrm{id}_{G(AD)}) \circ G^{-1}_{\otimes,U(I),AD} \circ G(c_{A,C,D}) \circ G_{\otimes,CD,AC} \circ \mathrm{cu}_{G(CD)\otimes G(AC)}$$
$$\circ\, G(U(\rho_{G(CD)} \otimes \mathrm{id}_{AC}) \circ U(\alpha^{-1}_{G(CD),I,G(AC)}) \circ U_{\otimes,G(CD),I\otimes G(AC)}$$
$$\circ\, (\mathrm{id}_{U(G(CD))} \otimes U_{\otimes,I,G(AC)}) \circ (\mathrm{un}_{CD} \otimes (\mathrm{id}_{U(I)} \otimes \mathrm{un}_{AC})) \circ (\mathrm{id}_{CD} \otimes c_{A,B,C})$$
$$\circ\, \alpha_{CD,BC,AB}) \circ G_{\otimes,CD\otimes BC,AB} \circ (G_{\otimes,CD,BC} \otimes \mathrm{id}_{G(AB)}).$$

Step 11: The associativity axiom for monoidal functors tells us that

$$\begin{aligned}&U_{\otimes,G(CD),I\otimes G(AC)}\circ(\mathrm{id}_{U(G(CD))}\otimes U_{\otimes,I,G(AC)})\circ\alpha_{U(G(CD)),U(I),U(G(AC))}\\&=U(\alpha_{G(CD),I,G(AC)})\circ U_{\otimes,G(CD)\otimes I,G(AC)}\circ(U_{\otimes,G(CD),I}\otimes\mathrm{id}_{U(G(AC))})\end{aligned}$$

Thus, the above term agrees with

$$\begin{aligned}&\lambda_{G(AD)}\circ(\mathrm{cu}_I\otimes\mathrm{id}_{G(AD)})\circ G^{-1}_{\otimes,U(I),AD}\circ G(c_{A,C,D})\circ G_{\otimes,CD,AC}\circ\mathrm{cu}_{G(CD)\otimes G(AC)}\\&\circ G(U(\rho_{G(CD)}\otimes\mathrm{id}_{AC})\circ U_{\otimes,G(CD)\otimes I,G(AC)}\circ(U_{\otimes,G(CD),I}\otimes\mathrm{id}_{U(G(AC))})\\&\circ\alpha^{-1}_{U(G(CD)),U(I),U(G(AC))}\circ(\mathrm{un}_{CD}\otimes(\mathrm{id}_{U(I)}\otimes\mathrm{un}_{AC}))\circ(\mathrm{id}_{CD}\otimes c_{A,B,C})\\&\circ\alpha_{CD,BC,AB})\circ G_{\otimes,CD\otimes BC,AB}\circ(G_{\otimes,CD,BC}\otimes\mathrm{id}_{G(AB)}).\end{aligned}$$

Step 12: Using that $\rho_{G(CD)}=\lambda_{G(CD)}\circ\gamma_{G(CD),I}$ in the braided monoidal category M and the fact that $\alpha_{U(G(CD)),U(I),U(G(AC))}\circ((\mathrm{un}_{CD}\otimes\mathrm{un}_{U(I)})\otimes\mathrm{un}_{AC})=(\mathrm{un}_{CD}\otimes(\mathrm{id}_{U(I)}\otimes\mathrm{un}_{AC}))\circ\alpha_{CD,U(I),AC}$, we rewrite the morphism as

$$\begin{aligned}&\lambda_{G(AD)}\circ(\mathrm{cu}_I\otimes\mathrm{id}_{G(AD)})\circ G^{-1}_{\otimes,U(I),AD}\circ G(c_{A,C,D})\circ G_{\otimes,CD,AC}\circ\mathrm{cu}_{G(CD)\otimes G(AC)}\\&\circ G(U(\lambda_{G(CD)}\otimes\mathrm{id}_{AC})\circ U(\gamma_{G(CD),I}\otimes\mathrm{id}_{AC})\circ U_{\otimes,G(CD)\otimes I,G(AC)}\\&\circ(U_{\otimes,G(CD),I}\otimes\mathrm{id}_{U(G(AC))})\circ((\mathrm{un}_{CD}\otimes\mathrm{id}_{U(I)})\otimes\mathrm{un}_{AC})\circ\alpha^{-1}_{CD,U(I),AC}\\&\circ(\mathrm{id}_{CD}\otimes c_{A,B,C})\circ\alpha_{CD,BC,AB})\circ G_{\otimes,CD\otimes BC,AB}\circ(G_{\otimes,CD,BC}\otimes\mathrm{id}_{G(AB)}).\end{aligned}$$

Step 13: Now $\lambda_{G(CD)}\otimes\mathrm{id}_{AC}=\lambda_{G(CD)\otimes G(AC)}\circ\alpha_{I,G(CD),G(AC)}$. At the same time, because U is a monoidal functor,

$$\begin{aligned}&U(\gamma_{G(CD),I}\otimes\mathrm{id}_{AC})\circ U_{\otimes,G(CD)\otimes I,G(AC)}\\&\qquad=U_{\otimes,I\otimes G(CD),G(AC)}\circ(U(\gamma_{G(CD),I})\otimes\mathrm{id}_{U(G(AC))}).\end{aligned}$$

Hence, the previous term is the same as

$$\begin{aligned}&\lambda_{G(AD)}\circ(\mathrm{cu}_I\otimes\mathrm{id}_{G(AD)})\circ G^{-1}_{\otimes,U(I),AD}\circ G(c_{A,C,D})\circ G_{\otimes,CD,AC}\circ\mathrm{cu}_{G(CD)\otimes G(AC)}\\&\circ G(U(\lambda_{G(CD)\otimes G(AC)}\circ\alpha_{I,G(CD),G(AC)})\circ U_{\otimes,I\otimes G(CD),G(AC)}\\&\circ((U(\gamma_{G(CD),I})\circ U_{\otimes,G(CD),I})\otimes\mathrm{id}_{U(G(AC))})\circ((\mathrm{un}_{CD}\otimes\mathrm{id}_{U(I)})\otimes\mathrm{un}_{AC})\\&\circ\alpha^{-1}_{CD,U(I),AC}\circ(\mathrm{id}_{CD}\otimes c_{A,B,C})\circ\alpha_{CD,BC,AB})\circ G_{\otimes,CD\otimes BC,AB}\\&\circ(G_{\otimes,CD,BC}\otimes\mathrm{id}_{G(AB)}).\end{aligned}$$

Step 14: Since U is a braided monoidal functor,

$$U(\gamma_{G(CD),I})\circ U_{\otimes,G(CD),I}=U_{\otimes,I,G(CD)}\circ\gamma_{U(G(CD)),U(I)},$$

which, together with some reassociation, lets us rewrite the above as

$$\begin{aligned}&\lambda_{G(AD)}\circ(\mathrm{cu}_I\otimes\mathrm{id}_{G(AD)})\circ G^{-1}_{\otimes,U(I),AD}\circ G(c_{A,C,D})\circ G_{\otimes,CD,AC}\circ\mathrm{cu}_{G(CD)\otimes G(AC)}\\&\circ G(U(\lambda_{G(CD)\otimes G(AC)})\circ U(\alpha_{I,G(CD),G(AC)})\circ U_{\otimes,I\otimes G(CD),G(AC)}\\&\circ(U_{\otimes,I,G(CD)}\otimes\mathrm{id}_{U(G(AC))})\circ((\gamma_{U(G(CD)),U(I)}\circ(\mathrm{un}_{CD}\otimes\mathrm{id}_{U(I)}))\otimes\mathrm{un}_{AC})\\&\circ\alpha^{-1}_{CD,U(I),AC}\circ(\mathrm{id}_{CD}\otimes c_{A,B,C})\circ\alpha_{CD,BC,AB})\circ G_{\otimes,CD\otimes BC,AB}\\&\circ(G_{\otimes,CD,BC}\otimes\mathrm{id}_{G(AB)}).\end{aligned}$$

Step 15: For the next transformation of the term we use $\gamma_{U(G(CD)),U(I)} \circ (\mathrm{un}_{CD} \otimes \mathrm{id}_{U(I)}) = (\mathrm{id}_{U(I)} \otimes \mathrm{un}_{CD}) \circ \gamma_{CD,U(I)}$. Moreover, the associativity axiom for the monoidal functor U yields

$$\begin{aligned}&U(\alpha_{I,G(CD),G(AC)}) \circ U_{\otimes,I\otimes G(CD),G(AC)} \circ (U_{\otimes,I,G(CD)} \otimes \mathrm{id}_{U(G(AC))})\\&= U_{\otimes,I,G(CD)\otimes G(AC)} \circ (\mathrm{id}_{U(I)} \otimes U_{\otimes,G(CD),G(AC)}) \circ \alpha_{U(I),U(G(CD)),U(G(AC))},\end{aligned}$$

allowing us to rewrite the previous expression as

$$\begin{aligned}&\lambda_{G(AD)} \circ (\mathrm{cu}_I \otimes \mathrm{id}_{G(AD)}) \circ G^{-1}_{\otimes,U(I),AD} \circ G(c_{A,C,D}) \circ G_{\otimes,CD,AC} \circ \mathrm{cu}_{G(CD)\otimes G(AC)}\\&\circ G(U(\lambda_{G(CD)\otimes G(AC)}) \circ U_{\otimes,I,G(CD)\otimes G(AC)} \circ (\mathrm{id}_{U(I)} \otimes U_{\otimes,G(CD),G(AC)})\\&\circ \alpha_{U(I),U(G(CD)),U(G(AC))} \circ ((\mathrm{id}_{U(I)} \otimes \mathrm{un}_{CD}) \otimes \mathrm{un}_{AC}) \circ (\gamma_{CD,U(I)} \otimes \mathrm{id}_{AC})\\&\circ \alpha^{-1}_{CD,U(I),AC} \circ (\mathrm{id}_{CD} \otimes c_{A,B,C}) \circ \alpha_{CD,BC,AB}) \circ G_{\otimes,CD\otimes BC,AB}\\&\circ (G_{\otimes,CD,BC} \otimes \mathrm{id}_{G(AB)}),\end{aligned}$$

where also several terms have been reassociated.

Step 16: At this point, cu being a transformation from $G \circ U$ to id lets us make the substitution

$$\begin{aligned}&((\mathrm{cu}_I \otimes \mathrm{id}_{G(AD)}) \circ G^{-1}_{\otimes,U(I),AD} \circ G(c_{A,C,D}) \circ G_{\otimes,CD,AC} \circ)\mathrm{cu}_{G(CD)\otimes G(AC)}\\&= \mathrm{cu}_{I\otimes G(AD)} \circ G(U((\mathrm{cu}_I \otimes \mathrm{id}_{G(AD)}) \circ G^{-1}_{\otimes,U(I),AD} \circ G(c_{A,C,D}) \circ G_{\otimes,CD,AC})).\end{aligned}$$

Using, additionally,

$$\begin{aligned}&\alpha_{U(I),U(G(CD)),U(G(AC))} \circ ((\mathrm{id}_{U(I)} \otimes \mathrm{un}_{CD}) \otimes \mathrm{un}_{AC})\\&\qquad = (\mathrm{id}_{U(I)} \otimes (\mathrm{un}_{CD} \otimes \mathrm{un}_{AC})) \circ \alpha_{U(I),CD,AC}\end{aligned}$$

thus gives, after some reassociation,

$$\begin{aligned}&\lambda_{G(AD)} \circ \mathrm{cu}_{I\otimes G(AD)} \circ G(U((\mathrm{cu}_I \otimes \mathrm{id}_{G(AD)}) \circ G^{-1}_{\otimes,U(I),AD} \circ G(c_{A,C,D})\\&\circ G_{\otimes,CD,AC} \circ \lambda_{G(CD)\otimes G(AC)}) \circ U_{\otimes,I,G(CD)\otimes G(AC)} \circ (\mathrm{id}_{U(I)} \otimes U_{\otimes,G(CD),G(AC)})\\&\circ (\mathrm{id}_{U(I)} \otimes (\mathrm{un}_{CD} \otimes \mathrm{un}_{AC})) \circ \alpha_{U(I),CD,AC} \circ (\gamma_{CD,U(I)} \otimes \mathrm{id}_{AC}) \circ \alpha^{-1}_{CD,U(I),AC}\\&\circ (\mathrm{id}_{CD} \otimes c_{A,B,C}) \circ \alpha_{CD,BC,AB}) \circ G_{\otimes,CD\otimes BC,AB} \circ (G_{\otimes,CD,BC} \otimes \mathrm{id}_{G(AB)}).\end{aligned}$$

Step 17: After the switch

$$\begin{aligned}&((\mathrm{cu}_I \otimes \mathrm{id}_{G(AD)}) \circ G^{-1}_{\otimes,U(I),AD} \circ G(c_{A,C,D}) \circ G_{\otimes,CD,AC}) \circ \lambda_{G(CD)\otimes G(AC)}\\&= \lambda_{I\otimes G(AD)} \circ (\mathrm{id}_I \otimes ((\mathrm{cu}_I \otimes \mathrm{id}_{G(AD)}) \circ G^{-1}_{\otimes,U(I),AD} \circ G(c_{A,C,D}) \circ G_{\otimes,CD,AC}))\end{aligned}$$

this is identical to

$$\begin{aligned}&\lambda_{G(AD)} \circ \mathrm{cu}_{I\otimes G(AD)} \circ G(U(\lambda_{I\otimes G(AD)}) \circ U(\mathrm{id}_I \otimes ((\mathrm{cu}_I \otimes \mathrm{id}_{G(AD)}) \circ G^{-1}_{\otimes,U(I),AD}\\&\circ G(c_{A,C,D}) \circ G_{\otimes,CD,AC})) \circ U_{\otimes,I,G(CD)\otimes G(AC)} \circ (\mathrm{id}_{U(I)} \otimes U_{\otimes,G(CD),G(AC)})\\&\circ (\mathrm{id}_{U(I)} \otimes (\mathrm{un}_{CD} \otimes \mathrm{un}_{AC})) \circ \alpha_{U(I),CD,AC} \circ (\gamma_{CD,U(I)} \otimes \mathrm{id}_{AC}) \circ \alpha^{-1}_{CD,U(I),AC}\\&\circ (\mathrm{id}_{CD} \otimes c_{A,B,C}) \circ \alpha_{CD,BC,AB}) \circ G_{\otimes,CD\otimes BC,AB} \circ (G_{\otimes,CD,BC} \otimes \mathrm{id}_{G(AB)}).\end{aligned}$$

Step 18: From U being a monoidal functor it follows

$$U(\mathrm{id}_I \otimes ((\mathrm{cu}_I \otimes \mathrm{id}_{G(AD)}) \circ G^{-1}_{\otimes,U(I),AD} \circ G(c_{A,C,D}) \circ G_{\otimes,CD,AC})) \circ U_{\otimes,I,G(CD)\otimes G(AC)}$$
$$= U_{\otimes,I,I\otimes G(AD)} \circ (\mathrm{id}_{U(I)} \otimes U((\mathrm{cu}_I \otimes \mathrm{id}_{G(AD)}) \circ G^{-1}_{\otimes,U(I),AD} \circ G(c_{A,C,D}) \circ G_{\otimes,CD,AC})).$$

This identity and reassociation of terms show that the above morphism is the same as

$$\lambda_{G(AD)} \circ \mathrm{cu}_{I\otimes G(AD)} \circ G(U(\lambda_{I\otimes G(AD)}) \circ U_{\otimes,I,I\otimes G(AD)} \circ (\mathrm{id}_{U(I)} \otimes U((\mathrm{cu}_I \otimes \mathrm{id}_{G(AD)})$$
$$\circ G^{-1}_{\otimes,U(I),AD} \circ G(c_{A,C,D}))) \circ (\mathrm{id}_{U(I)} \otimes (U(G_{\otimes,CD,AC}) \circ U_{\otimes,G(CD),G(AC)}$$
$$\circ (\mathrm{un}_{CD} \otimes \mathrm{un}_{AC}))) \circ \alpha_{U(I),CD,AC} \circ (\gamma_{CD,U(I)} \otimes \mathrm{id}_{AC}) \circ \alpha^{-1}_{CD,U(I),AC}$$
$$\circ (\mathrm{id}_{CD} \otimes c_{A,B,C}) \circ \alpha_{CD,BC,AB}) \circ G_{\otimes,CD\otimes BC,AB} \circ (G_{\otimes,CD,BC} \otimes \mathrm{id}_{G(AB)}).$$

Step 19: Because $G \dashv U$ is a monoidal adjunction,

$$\mathrm{un}_{CD\otimes AC} = U(G_{\otimes,CD,AC}) \circ U_{\otimes,G(CD),G(AC)} \circ (\mathrm{un}_{CD} \otimes \mathrm{un}_{AC}),$$

which is why the previous expression can be rewritten as

$$\lambda_{G(AD)} \circ \mathrm{cu}_{I\otimes G(AD)} \circ G(U(\lambda_{I\otimes G(AD)}) \circ U_{\otimes,I,I\otimes G(AD)} \circ (\mathrm{id}_{U(I)} \otimes U((\mathrm{cu}_I \otimes \mathrm{id}_{G(AD)})$$
$$\circ G^{-1}_{\otimes,U(I),AD})) \circ (\mathrm{id}_{U(I)} \otimes (U(G(c_{A,C,D})) \circ \mathrm{un}_{CD\otimes AC})) \circ \alpha_{U(I),CD,AC}$$
$$\circ (\gamma_{CD,U(I)} \otimes \mathrm{id}_{AC}) \circ \alpha^{-1}_{CD,U(I),AC} \circ (\mathrm{id}_{CD} \otimes c_{A,B,C}) \circ \alpha_{CD,BC,AB})$$
$$\circ G_{\otimes,CD\otimes BC,AB} \circ (G_{\otimes,CD,BC} \otimes \mathrm{id}_{G(AB)}).$$

Step 20: The preceding step finally allows us to move the second composition map past the units because $U(G(c_{A,C,D})) \circ \mathrm{un}_{CD\otimes AC} = \mathrm{un}_{U(I)\otimes AC} \circ c_{A,C,D}$, yielding

$$\lambda_{G(AD)} \circ \mathrm{cu}_{I\otimes G(AD)} \circ G(U(\lambda_{I\otimes G(AD)}) \circ U_{\otimes,I,I\otimes G(AD)} \circ (\mathrm{id}_{U(I)} \otimes U(\mathrm{cu}_I \otimes \mathrm{id}_{G(AD)}))$$
$$\circ (\mathrm{id}_{U(I)} \otimes (U(G^{-1}_{\otimes,U(I),AD}) \circ \mathrm{un}_{U(I)\otimes AD})) \circ (\mathrm{id}_{U(I)} \otimes c_{A,C,D}) \circ \alpha_{U(I),CD,AC}$$
$$\circ (\gamma_{CD,U(I)} \otimes \mathrm{id}_{AC}) \circ \alpha^{-1}_{CD,U(I),AC} \circ (\mathrm{id}_{CD} \otimes c_{A,B,C}) \circ \alpha_{CD,BC,AB})$$
$$\circ G_{\otimes,CD\otimes BC,AB} \circ (G_{\otimes,CD,BC} \otimes \mathrm{id}_{G(AB)}).$$

Step 21: Having commuted unit and composition map, we immediately reverse the operation which allowed us to do this in the first place. Once more, $G \dashv U$ being a monoidal adjunction tells us that

$$\mathrm{un}_{U(I)\otimes AD} = U(G_{\otimes,U(I),AD}) \circ U_{\otimes,G(U(I)),G(AD)} \circ (\mathrm{un}_{U(I)} \otimes \mathrm{un}_{AD}).$$

Hence our morphism can also be expressed as

$$\lambda_{G(AD)} \circ \mathrm{cu}_{I\otimes G(AD)} \circ G(U(\lambda_{I\otimes G(AD)}) \circ U_{\otimes,I,I\otimes G(AD)} \circ (\mathrm{id}_{U(I)} \otimes (U(\mathrm{cu}_I \otimes \mathrm{id}_{G(AD)})$$
$$\circ U_{\otimes,G(U(I)),G(AD)})) \circ (\mathrm{id}_{U(I)} \otimes (\mathrm{un}_{U(I)} \otimes \mathrm{un}_{AD})) \circ (\mathrm{id}_{U(I)} \otimes c_{A,C,D}) \circ \alpha_{U(I),CD,AC}$$
$$\circ (\gamma_{CD,U(I)} \otimes \mathrm{id}_{AC}) \circ \alpha^{-1}_{CD,U(I),AC} \circ (\mathrm{id}_{CD} \otimes c_{A,B,C}) \circ \alpha_{CD,BC,AB})$$
$$\circ G_{\otimes,CD\otimes BC,AB} \circ (G_{\otimes,CD,BC} \otimes \mathrm{id}_{G(AB)}).$$

Step 22: Since U is a monoidal functor we can infer

$$U(\mathrm{cu}_I \otimes \mathrm{id}_{G(AD)}) \circ U_{\otimes,G(U(I)),G(AD)} = U_{\otimes,I,G(AD)} \circ (U(\mathrm{cu}_I) \otimes \mathrm{id}_{U(G(AD))}).$$

At the same time, $\lambda_{I\otimes G(AD)}\circ\alpha_{I,I,G(AD)} = \lambda_I\otimes \mathrm{id}_{G(AD)}$. Hence, our morphism concurs with

$$\begin{aligned}&\lambda_{G(AD)}\circ \mathrm{cu}_{I\otimes G(AD)}\circ G\big(U(\lambda_I\otimes \mathrm{id}_{G(AD)})\circ U(\alpha^{-1}_{I,I,G(AD)})\circ U_{\otimes,I,I\otimes G(AD)}\\ &\quad\circ(\mathrm{id}_{U(I)}\otimes U_{\otimes,I,G(AD)})\circ(\mathrm{id}_{U(I)}\otimes((U(\mathrm{cu}_I)\circ \mathrm{un}_{U(I)})\otimes \mathrm{un}_{AD}))\circ(\mathrm{id}_{U(I)}\otimes c_{A,C,D})\\ &\qquad\circ\alpha_{U(I),CD,AC}\circ(\gamma_{CD,U(I)}\otimes \mathrm{id}_{AC})\circ\alpha^{-1}_{CD,U(I),AC}\circ(\mathrm{id}_{CD}\otimes c_{A,B,C})\circ\alpha_{CD,BC,AB}\big)\\ &\qquad\quad\circ G_{\otimes,CD\otimes BC,AB}\circ(G_{\otimes,CD,BC}\otimes \mathrm{id}_{G(AB)}).\end{aligned}$$

Step 23: The associativity axiom satisfied by the monoidal functor U implies

$$\begin{aligned}&U(\alpha_{I,I,G(AD)})\circ U_{\otimes,I\otimes I,G(AD)}\circ(U_{\otimes,I,I}\otimes \mathrm{id}_{U(G(AD))})\\ &\quad= U_{\otimes,I,I\otimes G(AD)}\circ(\mathrm{id}_{U(I)}\otimes U_{\otimes,I,G(AD)})\circ\alpha_{U(I),U(I),U(G(AD))}.\end{aligned}$$

Moreover, by one of the two co-unit-unit-equations, $U(\mathrm{cu}_I)\circ \mathrm{un}_{U(I)} = \mathrm{id}_{U(I)}$. Hence, the above expression is identical to

$$\begin{aligned}&\lambda_{G(AD)}\circ \mathrm{cu}_{I\otimes G(AD)}\circ G\big(U(\lambda_I\otimes \mathrm{id}_{G(AD)})\circ U_{\otimes,I\otimes I,G(AD)}\circ(U_{\otimes,I,I}\otimes \mathrm{id}_{U(G(AD))})\\ &\quad\circ\alpha^{-1}_{U(I),U(I),U(G(AD))}\circ(\mathrm{id}_{U(I)}\otimes(\mathrm{id}_{U(I)}\otimes \mathrm{un}_{AD}))\circ(\mathrm{id}_{U(I)}\otimes c_{A,C,D})\\ &\qquad\circ\alpha_{U(I),CD,AC}\circ(\gamma_{CD,U(I)}\otimes \mathrm{id}_{AC})\circ\alpha^{-1}_{CD,U(I),AC}\circ(\mathrm{id}_{CD}\otimes c_{A,B,C})\circ\alpha_{CD,BC,AB}\big)\\ &\qquad\quad\circ G_{\otimes,CD\otimes BC,AB}\circ(G_{\otimes,CD,BC}\otimes \mathrm{id}_{G(AB)}).\end{aligned}$$

Step 24: Because, on the one hand,

$$\alpha_{U(I),U(I),U(G(AD))}\circ(\mathrm{id}_{I\otimes I}\otimes \mathrm{un}_{AD}) = (\mathrm{id}_{U(I)}\otimes(\mathrm{id}_{U(I)}\otimes \mathrm{un}_{AD}))\circ\alpha_{U(I),U(I),AD}$$

and, on the other hand,

$$U(\lambda_I\otimes \mathrm{id}_{G(AD)})\circ U_{\otimes,I\otimes I,G(AD)} = U_{\otimes,I,G(AD)}\circ(U(\lambda_I)\otimes \mathrm{id}_{U(G(AD))}),$$

we can rewrite the previous term as

$$\begin{aligned}&\lambda_{G(AD)}\circ \mathrm{cu}_{I\otimes G(AD)}\circ G\big(U_{\otimes,I,G(AD)}\circ((U(\lambda_I)\circ U_{\otimes,I,I})\otimes \mathrm{un}_{AD})\circ\alpha^{-1}_{U(I),U(I),AD}\\ &\quad\circ(\mathrm{id}_{U(I)}\otimes c_{A,C,D})\circ\alpha_{U(I),CD,AC}\circ(\gamma_{CD,U(I)}\otimes \mathrm{id}_{AC})\circ\alpha^{-1}_{CD,U(I),AC}\\ &\qquad\circ(\mathrm{id}_{CD}\otimes c_{A,B,C})\circ\alpha_{CD,BC,AB}\big)\circ G_{\otimes,CD\otimes BC,AB}\circ(G_{\otimes,CD,BC}\otimes \mathrm{id}_{G(AB)}).\end{aligned}$$

Step 25: Halfway point. Since $U(\lambda_I)\circ U_{\otimes,I,I} = \otimes_{U(R)}$ we have transformed our morphism into the form

$$\begin{aligned}&\lambda_{G(AD)}\circ \mathrm{cu}_{I\otimes G(AD)}\circ G\big(U_{\otimes,I,G(AD)}\circ(\mathrm{id}_{U(I)}\otimes \mathrm{un}_{AD})\circ(\otimes_{U(R)}\otimes \mathrm{id}_{AD})\\ &\quad\circ\alpha^{-1}_{U(I),U(I),AD}\circ(\mathrm{id}_{U(I)}\otimes c_{A,C,D})\circ\alpha_{U(I),CD,AC}\circ(\gamma_{CD,U(I)}\otimes \mathrm{id}_{AC})\circ\alpha^{-1}_{CD,U(I),AC}\\ &\qquad\circ(\mathrm{id}_{CD}\otimes c_{A,B,C})\circ\alpha_{CD,BC,AB}\big)\circ G_{\otimes,CD\otimes BC,AB}\circ(G_{\otimes,CD,BC}\otimes \mathrm{id}_{G(AB)}),\end{aligned}$$

which allows us to apply Lemma 5.104 in order to see that this equals

$$\begin{aligned}&\lambda_{G(AD)}\circ \mathrm{cu}_{I\otimes G(AD)}\circ G\big(U_{\otimes,I,G(AD)}\circ(\mathrm{id}_{U(I)}\otimes \mathrm{un}_{AD})\circ(\otimes_{U(R)}\otimes \mathrm{id}_{AD})\\ &\quad\circ\alpha^{-1}_{U(I),U(I),AD}\circ(\mathrm{id}_{U(I)}\otimes c_{A,B,D})\circ\alpha_{U(I),BD,AB}\circ(c_{B,C,D}\otimes \mathrm{id}_{AB})\big)\\ &\qquad\circ G_{\otimes,CD\otimes BC,AB}\circ(G_{\otimes,CD,BC}\otimes \mathrm{id}_{G(AB)}).\end{aligned}$$

That was the crucial step. It is possible to transform the right-hand side of the asserted identity into the same form using similar arguments. As mentioned, that part of the proof had to be omitted. □

There is some anecdotal evidence that of the many axioms to check for $\mathcal{T}^0(\mathcal{A},\mathcal{E},\delta)$ the associativity of the composition is the most complicated. For example, confirming the identity axioms for the composition is comparatively simple. The following lemma is the key to proving that.

LEMMA 5.106. *For any* $\{A,B\} \subseteq \mathrm{obj}_{\mathrm{Rel}}$*, in the terms of Definition 5.103, and if we interpret* $\mathrm{id}_{\mathrm{Rel},A}$ *and* $\mathrm{id}_{\mathrm{Rel},B}$ *as mappings* $I_{\mathsf{Set}} \to \mathrm{mor}_{\mathrm{Rel}}(A,A)$ *respectively* $I_{\mathsf{Set}} \to \mathrm{mor}_{\mathrm{Rel}}(B,B)$*, then*

$$c_{A,B,B} \circ_{\mathsf{Set}} (\mathrm{id}_{\mathrm{Rel},B} \otimes_{\mathsf{Set}} \mathrm{id}_{\mathsf{Set},\mathrm{mor}_{\mathrm{Rel}}(A,B)}) = I_{(U_{\mathsf{S}})_\triangleright(R)} \otimes_{\mathsf{Set}} \mathrm{id}_{\mathsf{Set},\mathrm{mor}_{\mathrm{Rel}}(A,B)}$$

$$\begin{array}{ccc}
I \otimes I & \xrightarrow{I_{U_\triangleright(R)} \otimes \mathrm{id}_{\mathrm{mor}(A,B)}} & U(I) \otimes \mathrm{mor}(A,B) \\
{\scriptstyle \mathrm{id}_{\mathrm{Rel},B} \otimes \mathrm{id}_{\mathrm{mor}(A,B)}}\big\downarrow & \nearrow {\scriptstyle c_{A,B,B}} & \\
\mathrm{mor}(B,B) \otimes \mathrm{mor}(A,B) & &
\end{array}$$

and

$$\begin{aligned} c_{A,A,B} \circ_{\mathsf{Set}} (\mathrm{id}_{\mathsf{Set},\mathrm{mor}_{\mathrm{Rel}}(A,B)} \otimes_{\mathsf{Set}} \mathrm{id}_{\mathrm{Rel},A}) \\ = \gamma_{\mathsf{Set},\mathrm{mor}_{\mathrm{Rel}}(A,B),U_{\mathsf{M}}(I_{\mathsf{M}})} \circ_{\mathsf{Set}} (\mathrm{id}_{\mathsf{Set},\mathrm{mor}_{\mathrm{Rel}}(A,B)} \otimes_{\mathsf{Set}} I_{(U_{\mathsf{S}})_\triangleright(R)}). \end{aligned}$$

$$\begin{array}{ccc}
I \otimes I & \xrightarrow{\mathrm{id}_{\mathrm{mor}(A,B)} \otimes I_{U_\triangleright(R)}} & \mathrm{mor}(A,B) \otimes U(I) \\
{\scriptstyle \mathrm{id}_{\mathrm{mor}(A,B)} \otimes \mathrm{id}_{\mathrm{Rel},A}}\big\downarrow & & \big\downarrow{\scriptstyle \gamma_{\mathrm{mor}(A,B),U(I)}} \\
\mathrm{mor}(A,B) \otimes \mathrm{mor}(A,A) & \xrightarrow[c_{A,A,B}]{} & U(I) \otimes \mathrm{mor}(A,B)
\end{array}$$

Or, in terms of elements, for any $x \in \mathrm{mor}_{\mathrm{Rel}}(A,B)$*,*

$$\Delta_{\mathcal{A},\mathcal{E},\delta}(\mathrm{id}_{\mathrm{Rel},B} \circ_{\mathrm{S}} x) = \Delta_{\mathcal{A},\mathcal{E},\delta}(x \circ_{\mathrm{S}} \mathrm{id}_{\mathrm{Rel},A}) = I_{(U_{\mathsf{S}})_\triangleright(R)}$$

and

$$\Phi_{\mathcal{A},\mathcal{E}}(\mathrm{id}_{\mathrm{Rel},B} \circ_{\mathrm{S}} x) = \Phi_{\mathcal{A},\mathcal{E}}(x \circ_{\mathrm{S}} \mathrm{id}_{\mathrm{Rel},A}) = x.$$

PROOF. Because Lemma 5.97 ensures $\mathrm{id}_{\mathrm{Rel},A} = \mathrm{id}_{\mathrm{S},A}$ and $\mathrm{id}_{\mathrm{Rel},B} = \mathrm{id}_{\mathrm{S},B}$ we can indeed infer $\Delta(\mathrm{id}_{\mathrm{Rel},B} \circ_{\mathrm{S}} x) = \Delta(\mathrm{id}_{\mathrm{S},B} \circ_{\mathrm{S}} x) = \Delta(x) = I_{(U_{\mathsf{S}})_\triangleright(R)}$ and, likewise, $\Delta(x \circ_{\mathrm{S}} \mathrm{id}_{\mathrm{Rel},A}) = I_{(U_{\mathsf{S}})_\triangleright(R)}$ by Lemma 5.101.

Similarly, Lemma 5.97 implies $\Phi(\mathrm{id}_{\mathrm{Rel},B} \circ_{\mathrm{S}} x) = \Phi(\mathrm{id}_{\mathrm{S},B} \circ_{\mathrm{S}} x) = \Phi(x) = x$ and, likewise, $\Phi(x \circ_{\mathrm{S}} \mathrm{id}_{\mathrm{Rel},A}) = x$ by Lemma 5.64. □

The following lemma shows that the identities of the span category are left and right neutral elements even for the linearly modified composition of relations.

LEMMA 5.107. *For any* $\{A, B\} \in \mathrm{obj}_{\mathcal{T}^0}$,

$$\circ_{\mathcal{T}^0,A,B,B} \circ_{\mathsf{M}} \left(\mathrm{id}_{\mathcal{T}^0,B} \otimes_{\mathsf{M}} \mathrm{id}_{\mathsf{M},\mathrm{mor}_{\mathcal{T}^0}(A,B)}\right) = \lambda_{\mathsf{M},\mathrm{mor}_{\mathcal{T}^0}(A,B)}$$

$$\begin{array}{ccc} I \otimes \mathrm{mor}_{\mathcal{T}^0}(A,B) & \xrightarrow{\mathrm{id}_{\mathcal{T}^0,B} \otimes \mathrm{id}_{\mathrm{mor}_{\mathcal{T}^0}(A,B)}} & \mathrm{mor}_{\mathcal{T}^0}(B,B) \otimes \mathrm{mor}_{\mathcal{T}^0}(A,B) \\ & {\scriptstyle \lambda_{\mathrm{mor}_{\mathcal{T}^0}(A,B)}} \searrow \qquad \swarrow {\scriptstyle \circ_{\mathcal{T}^0,A,B,B}} & \\ & \mathrm{mor}_{\mathcal{T}^0}(A,B) & \end{array}$$

and

$$\circ_{\mathcal{T}^0,A,A,B} \circ_{\mathsf{M}} \left(\mathrm{id}_{\mathsf{M},\mathrm{mor}_{\mathcal{T}^0}(A,B)} \otimes_{\mathsf{M}} \mathrm{id}_{\mathcal{T}^0,A}\right) = \rho_{\mathsf{M},\mathrm{mor}_{\mathcal{T}^0}(A,B)}.$$

$$\begin{array}{ccc} \mathrm{mor}_{\mathcal{T}^0}(A,B) \otimes I & \xrightarrow{\mathrm{id}_{\mathrm{mor}_{\mathcal{T}^0}(A,B)} \otimes \mathrm{id}_{\mathcal{T}^0,A}} & \mathrm{mor}_{\mathcal{T}^0}(A,B) \otimes \mathrm{mor}_{\mathcal{T}^0}(A,A) \\ & {\scriptstyle \rho_{\mathrm{mor}_{\mathcal{T}^0}(A,B)}} \searrow \qquad \swarrow {\scriptstyle \circ_{\mathcal{T}^0,A,A,B}} & \\ & \mathrm{mor}_{\mathcal{T}^0}(A,B) & \end{array}$$

PROOF. As in the proof of Lemma 5.105 the category indices $\mathcal{T}^0$, Rel, M, S and Set will be suppressed and $\mathrm{mor}_{\mathrm{Rel}}(X,Y) \equiv XY$ for any $\{X,Y\} \subseteq \mathrm{obj}_{\mathcal{A}}$. For emphasis, the index Rel is kept for $\mathrm{id}_{\mathrm{Rel},A}$ and $\mathrm{id}_{\mathrm{Rel},B}$, though.

First identity. By definition the left hand side of the first identity is equal to

$$\begin{aligned} &\lambda_{G(AB)} \circ (\mathrm{cu}_I \otimes \mathrm{id}_{G(AB)}) \circ G^{-1}_{\otimes,U(I),AB} \circ G(c_{A,B,B}) \\ &\qquad \circ G_{\otimes,BB,AB} \circ (G(\mathrm{id}_{\mathrm{Rel},B}) \otimes G(\mathrm{id}_{AB})) \circ (G_I \otimes \mathrm{id}_{G(AB)}). \end{aligned}$$

Because G is a monoidal functor, $G_{\otimes,BB,AB} \circ (G(\mathrm{id}_{\mathrm{Rel},A}) \otimes G(\mathrm{id}_{AB})) = G(\mathrm{id}_{\mathrm{Rel},B} \otimes \mathrm{id}_{AB}) \circ G_{\otimes,I,AB}$, where I is the monoidal unit of Set. Hence, the above expression is the same as

$$\begin{aligned} &\lambda_{G(AB)} \circ (\mathrm{cu}_I \otimes \mathrm{id}_{G(AB)}) \circ G^{-1}_{\otimes,U(I),AB} \circ G(c_{A,B,B} \circ (\mathrm{id}_{\mathrm{Rel},B} \otimes \mathrm{id}_{AB})) \\ &\qquad \circ G_{\otimes,I,AB} \circ (G_I \otimes \mathrm{id}_{G(AB)}). \end{aligned}$$

Since $c_{A,B,B} \circ (\mathrm{id}_{\mathrm{Rel},B} \otimes \mathrm{id}_{AB}) = U_I \otimes \mathrm{id}_{AB}$ by Lemma 5.106, we have thus transformed the left hand side of the assertion into

$$\lambda_{G(AB)} \circ (\mathrm{cu}_I \otimes \mathrm{id}_{G(AB)}) \circ G^{-1}_{\otimes,U(I),AB} \circ G(U_I \otimes \mathrm{id}_{AB}) \circ G_{\otimes,I,AB} \circ (G_I \otimes \mathrm{id}_{G(AB)}).$$

With the identity $G^{-1}_{\otimes,U(I),AB} \circ G(U_I \otimes \mathrm{id}_{AB}) = (G(U_I) \otimes \mathrm{id}_{G(AB)}) \circ G^{-1}_{\otimes,I,AB}$, which is due to G being strong monoidal, we can reverse the previous switch and rewrite the above as

$$\lambda_{G(AB)} \circ ((\mathrm{cu}_I \circ G(U_I) \circ G_I) \otimes \mathrm{id}_{G(AB)}).$$

Now, the assumption that $G \dashv U$ is a monoidal adjunction implies $\mathrm{cu}_I \circ (G(U_I) \circ G_I) = \mathrm{id}_I$. Thus, the first claim is true.

Second identity. The computation is largely analogous: The left hand side of the second claimed identity is given by

$$\lambda_{G(AB)} \circ (\mathrm{cu}_I \otimes \mathrm{id}_{G(AB)}) \circ G^{-1}_{\otimes,U(I),AB} \circ G(c_{A,A,B}) \\ \circ G_{\otimes,AB,AA} \circ (G(\mathrm{id}_{AB}) \otimes G(\mathrm{id}_{\mathrm{Rel},A})) \circ (\mathrm{id}_{G(AB)} \otimes G_I),$$

which is the same as

$$\lambda_{G(AB)} \circ (\mathrm{cu}_I \otimes \mathrm{id}_{G(AB)}) \circ G^{-1}_{\otimes,U(I),AB} \circ G(c_{A,A,B} \circ (\mathrm{id}_{AB} \otimes \mathrm{id}_{\mathrm{Rel},A})) \\ \circ G_{\otimes,AB,I} \circ (\mathrm{id}_{G(AB)} \otimes G_I).$$

Thus, Lemma 5.106 allows us to rewrite it as

$$\lambda_{G(AB)} \circ (\mathrm{cu}_I \otimes \mathrm{id}_{G(AB)}) \circ G^{-1}_{\otimes,U(I),AB} \circ G(\gamma_{AB,U(I)}) \circ G(\mathrm{id}_{AB} \otimes U_I) \\ \circ G_{\otimes,AB,I} \circ (\mathrm{id}_{G(AB)} \otimes G_I).$$

Again we reverse the previous switch via $G(\mathrm{id}_{AB} \otimes U_I) \circ G_{\otimes,AB,I} = G_{\otimes,AB,U(I)} \circ (\mathrm{id}_{G(AB)} \otimes G(U_I))$. Now, though, we use in addition the assumption that G is a strong symmetric monoidal functor and conclude $G(\gamma_{AB,U(I)}) \circ G_{\otimes,AB,U(I)} = G_{\otimes,U(I),AB} \circ \gamma_{G(AB),G(U(I))}$. Hence, the left hand side of the claim is identical to

$$\lambda_{G(AB)} \circ (\mathrm{cu}_I \otimes \mathrm{id}_{G(AB)}) \circ \gamma_{G(AB),G(U(I))} \circ (\mathrm{id}_{G(AB)} \otimes (G(U_I) \circ G_I)),$$

Since M is symmetric monoidal, $(\mathrm{cu}_I \otimes \mathrm{id}_{G(AB)}) \circ \gamma_{G(AB),G(U(I))} = \gamma_{G(AB),I} \circ (\mathrm{id}_{G(AB)} \otimes \mathrm{cu}_I)$, which is why the above is equal to

$$\lambda_{G(AB)} \circ \gamma_{G(AB),I} \circ (\mathrm{id}_{G(AB)} \otimes (\mathrm{cu}_I \circ (G(U_I) \circ G_I))).$$

The identity $\mathrm{cu}_I \circ (G(U_I) \circ G_I) = \mathrm{id}_I$ and now, additionally, the identity $\lambda_{G(AB)} \circ \gamma_{G(AB),I} = \rho_{G(AB)}$, which holds because M is symmetric monoidal, prove the second claim. □

The following is the last part of the constuction which I was able to verify (even though a part of the proof is not given here). For the remaining axioms there was no time left.

PROPOSITION 5.108. $(\mathrm{obj}_{\mathcal{T}^0}, \mathrm{mor}_{\mathcal{T}^0}, \circ_{\mathcal{T}^0}, \mathrm{id}_{\mathcal{T}^0})$ *is an* M*-enriched category.*

PROOF. That is the combined implication of Lemmata 5.105 and 5.107. □

Useful in confirming the remaining properties might be the following functor from $G_{\mathsf{M}}(\mathrm{S}(\mathcal{A}))$ (the base change of $\mathrm{S}(\mathcal{A})$ along the monoidal functor G_{M}, producing an M-enriched category) to $\mathcal{T}^0(\mathcal{A},\mathcal{E},\delta)$.

CONJECTURE 5.109. *A full* M*-enriched functor* $G_{\mathsf{M}}(\mathrm{S}) \to \mathcal{T}^0$ *which is bijective on objects is defined by* $X \mapsto X$ *for objects* $X \in \mathrm{obj}_{G_{\mathsf{M}}(\mathrm{S})}$ *and by*

$$(A,B) \mapsto \lambda_{\mathsf{M},G_{\mathsf{M}}(\mathrm{mor}_{\mathrm{Rel}}(A,B))} \circ_{\mathsf{M}} (\mathrm{cu}_{\mathsf{M},I_{\mathsf{M}}} \otimes_{\mathsf{M}} \mathrm{id}_{\mathsf{M},G_{\mathsf{M}}(\mathrm{mor}_{\mathrm{Rel}}(A,B))}) \\ \circ_{\mathsf{M}} (G_{\mathsf{M}})_{\otimes,U_{\mathsf{M}}(I_{\mathsf{M}}),\mathrm{mor}_{\mathrm{Rel}}(A,B)}^{-1_{\mathsf{M}}} \circ_{\mathsf{M}} G_{\mathsf{M}}((\Delta_{\mathcal{A},\mathcal{E},\delta})_{1,A,B} \times_{\mathsf{Set}} (\Phi_{\mathcal{A},\mathcal{E}})_{1,A,B})$$

for morphisms.

5.5.2. *Monoidal Linear Category.* The monoidal structure of $\mathrm{Rel}(\mathcal{A},\mathcal{E})$ should be passed on to $\mathcal{T}^0(\mathcal{A},\mathcal{E},\delta)$. The following is probably the key lemma for proving that.

LEMMA 5.110. *For any* $\{A_1, A_2, B_1, B_2\} \subseteq \mathrm{obj}_{\mathrm{S}}$,

$$\begin{aligned}&\otimes_{(U_{\mathsf{S}})_{\triangleright}(R)} \circ_{\mathsf{Set}} \big((\Delta_{\mathcal{A},\mathcal{E},\delta})_{1,A_1,B_1} \otimes_{\mathsf{Set}} (\Delta_{\mathcal{A},\mathcal{E},\delta})_{1,A_2,B_2}\big)\\ &\qquad = (\Delta_{\mathcal{A},\mathcal{E},\delta})_{1,A_1\otimes_{\mathrm{S}}A_2,B_1\otimes_{\mathrm{S}}B_2} \circ_{\mathsf{Set}} (\otimes_{\mathrm{S}})_{1,(A_1,A_2),(B_1,B_2)}\end{aligned}$$

$$\begin{array}{ccc}\mathrm{mor}_{\mathrm{S}}(A_1,B_1)\otimes\mathrm{mor}_{\mathrm{S}}(A_2,B_2) & \xrightarrow{(\Delta_{\mathcal{A},\mathcal{E},\delta})_{1,A_1,B_1}\otimes(\Delta_{\mathcal{A},\mathcal{E},\delta})_{1,A_2,B_2}} & U(I)\otimes U(I)\\ {\scriptstyle (\otimes_{\mathrm{S}})_{1,(A_1,A_2),(B_1,B_2)}}\big\downarrow & & \big\downarrow{\scriptstyle \otimes_{U_{\triangleright}(R)}}\\ \mathrm{mor}_{\mathrm{S}}(A_1\otimes_{\mathrm{S}}A_2, B_1\otimes_{\mathrm{S}}B_2) & \xrightarrow[(\Delta_{\mathcal{A},\mathcal{E},\delta})_{1,A_1\otimes_{\mathrm{S}}A_2,B_1\otimes_{\mathrm{S}}B_2}]{} & U(I)\end{array}$$

Or, in terms of elements, for any $x_1 \in \mathrm{mor}_{\mathrm{S}}(A_1,B_1)$ *and* $x_2 \in \mathrm{mor}_{\mathrm{S}}(A_2,B_2)$,

$$\Delta_{\mathcal{A},\mathcal{E},\delta}(x_1 \otimes_{\mathrm{S}} x_2) = \Delta_{\mathcal{A},\mathcal{E},\delta}(x_1) \otimes_{(U_{\mathsf{S}})_{\triangleright}(R)} \Delta_{\mathcal{A},\mathcal{E},\delta}(x_2).$$

PROOF. Resume the definitions of the proof of Lemma 5.70 with $x_1 := [f_1, g_1]$ and $x_2 = [f_2, g_2]$. Then, $\Delta([f_i, g_i]) = \delta(e_i)$ for each $i \in \{1,2\}$ per assumption. In the proof of Lemma 5.70 we showed that $(e_1 \otimes_{\mathcal{A}} e_2, u \circ_{\mathcal{A}} (m_1 \otimes_{\mathcal{A}} m_2))$ is an $(\mathcal{E},\mathcal{M})$-factorization of $(f_1 \otimes_{\mathcal{A}} f_2) \times_{\mathcal{A}} (g_1 \otimes_{\mathcal{A}} g_2)$. Hence, by definition $\Delta([f_1,g_1] \otimes_{\mathrm{S}} [f_2,g_2]) = \delta(e_1 \otimes_{\mathcal{A}} e_2)$. Because δ is a $\mathcal{E}$-degree function, Lemma 2.5 (b) guarantees $\delta(e_1 \otimes_{\mathcal{A}} e_2) = \delta(e_1) \otimes_{(U_{\mathsf{S}})_{\triangleright}(R)} \delta(e_2)$. Thus, $\Delta(x_1 \otimes_{\mathrm{S}} x_2) = \delta(e_1 \otimes_{\mathcal{A}} e_2) = \delta(e_1) \otimes_{(U_{\mathsf{S}})_{\triangleright}(R)} \delta(e_2) = \Delta(x_1) \otimes_{(U_{\mathsf{S}})_{\triangleright}(R)} \Delta(x_2)$ as claimed. □

Using this lemma it should be straightforward to show the following result, which is likely the centerpiece to showing that the monoidal product is functorial.

CONJECTURE 5.111. *If* $\{A_i, B_i, C_i\} \subseteq \mathrm{obj}_{\mathrm{S}}$ *and for each* $i \in \{1,2\}$ *and if we resume the definition from Definition 5.103, then*

$$\begin{aligned}&c_{A_1\otimes_{\mathrm{S}}A_2,B_1\otimes_{\mathrm{S}}B_2,C_1\otimes_{\mathrm{S}}C_2}\\ &\quad\circ_{\mathsf{Set}} \big((\otimes_{\mathrm{S}})_{1,(B_1,B_2),(C_1,C_2)} \otimes_{\mathsf{Set}} (\otimes_{\mathrm{S}})_{1,(A_1,A_2),(B_1,B_2)}\big)\\ &\qquad\circ_{\mathsf{Set}} \mu_{\mathsf{Set},\mathrm{mor}_{\mathrm{S}}(B_1,C_1),\mathrm{mor}_{\mathrm{S}}(A_1,B_1),\mathrm{mor}_{\mathrm{S}}(B_2,C_2),\mathrm{mor}_{\mathrm{S}}(A_2,B_2)}\\ &\qquad= \big(\otimes_{(U_{\mathsf{S}})_{\triangleright}(R)} \otimes_{\mathsf{Set}} (\otimes_{\mathrm{S}})_{1,(A_1,A_2),(C_1,C_2)}\big)\\ &\qquad\qquad\circ_{\mathsf{Set}} \mu_{\mathsf{Set},U_{\mathsf{M}}(I_{\mathsf{M}}),U_{\mathsf{M}}(I_{\mathsf{M}}),\mathrm{mor}_{\mathrm{S}}(A_1,C_1),\mathrm{mor}_{\mathrm{S}}(A_2,C_2)}\\ &\qquad\qquad\quad\circ_{\mathsf{Set}} \big(c_{A_1,B_1,C_1} \otimes_{\mathsf{Set}} c_{A_2,B_2,C_2}\big).\end{aligned}$$

$$\begin{array}{ccc}
\begin{matrix}(\mathrm{mor}(B_1,C_1)\otimes \mathrm{mor}(A_1,B_1))\\ \otimes(\mathrm{mor}(B_2,C_2)\otimes \mathrm{mor}(A_2,B_2))\end{matrix} & \xrightarrow{\mu_{\mathrm{mor}(B_1,C_1),\mathrm{mor}(A_1,B_1),\mathrm{mor}(B_2,C_2),\mathrm{mor}(A_2,B_2)}} & \begin{matrix}(\mathrm{mor}(B_1,C_1)\otimes \mathrm{mor}(B_2,C_2))\\ \otimes(\mathrm{mor}(A_1,B_1)\otimes \mathrm{mor}(A_2,B_2))\end{matrix}\\
\downarrow{\scriptstyle c_{A_1,B_1,C_1}\otimes c_{A_2,B_2,C_2}} & & \downarrow{\scriptstyle (\otimes_{\mathrm{S}})_{1,(B_1,B_2),(C_1,C_2)}\otimes(\otimes_{\mathrm{S}})_{1,(A_1,A_2),(B_1,B_2)}}\\
\begin{matrix}(U(I)\otimes \mathrm{mor}(A_1,C_1))\\ \otimes(U(I)\otimes \mathrm{mor}(A_2,C_2))\end{matrix} & & \begin{matrix}\mathrm{mor}(B_1\otimes B_2,C_1\otimes C_2)\\ \otimes\,\mathrm{mor}(A_1\otimes A_2,B_1\otimes B_2)\end{matrix}\\
\downarrow{\scriptstyle \mu_{U(I),\mathrm{mor}(A_1,C_1),U(I),\mathrm{mor}(A_2,C_2)}} & & \downarrow{\scriptstyle c_{A_1\otimes A_2,B_1\otimes B_2,C_1\otimes C_2}}\\
\begin{matrix}(U(I)\otimes U(I))\\ \otimes(\mathrm{mor}(A_1,C_1)\otimes \mathrm{mor}(A_2,C_2))\end{matrix} & \xrightarrow[\otimes_{U_\triangleright(R)}\otimes(\otimes_{\mathrm{S}})_{1,(A_1,A_2),(C_1,C_2)}]{} & U(I)\otimes \mathrm{mor}(A_1\otimes A_2,C_1\otimes C_2)
\end{array}$$

Or, in terms of elements, if $x_i \in \mathrm{mor}_{\mathrm{S}}(A_i,B_i)$ *and* $y_i \in \mathrm{mor}_{\mathrm{S}}(B_i,C_i)$ *for each* $i \in \{1,2\}$*, then for any*

$$\Delta_{\mathcal{A},\mathcal{E},\delta}((y_1 \otimes_{\mathrm{S}} y_2)\circ_{\mathrm{S}}(x_1\otimes_{\mathrm{S}} x_2)) = \Delta_{\mathcal{A},\mathcal{E},\delta}(y_1\circ_{\mathrm{S}} x_1)\otimes_{(U_{\mathrm{S}})_\triangleright(R)}\Delta_{\mathcal{A},\mathcal{E},\delta}(y_2\circ_{\mathrm{S}} x_2)$$

and

$$\Phi_{\mathcal{A},\mathcal{E}}((y_1 \otimes_{\mathrm{S}} y_2)\circ_{\mathrm{S}}(x_1\otimes_{\mathrm{S}} x_2)) = \Phi_{\mathcal{A},\mathcal{E}}(y_1\circ_{\mathrm{S}} x_1)\otimes_{\mathrm{S}}\Phi_{\mathcal{A},\mathcal{E}}(y_2\circ_{\mathrm{S}} x_2).$$

The following definition of the monoidal structure of $\mathcal{T}^0(\mathcal{A},\mathcal{E},\delta)$ is the same as the one given in Section 2.

DEFINITION 5.112. (a) For any $\{A_1,A_2\}\subseteq \mathrm{obj}_{\mathcal{T}^0}$ let $A_1\otimes_{\mathcal{T}^0} A_2 := A_1\otimes_{\mathrm{Rel}} A_2$.

(b) Given any $\{A_i,B_i\}\subseteq \mathrm{obj}_{\mathcal{T}^0}$ for each $i\in\{1,2\}$, let

$$(\otimes_{\mathcal{T}^0})_{1,(A_1,A_2),(B_1,B_2)} := G_{\mathsf{M}}((\otimes_{\mathrm{Rel}})_{1,(A_1,A_2),(B_1,B_2)})\circ_{\mathsf{M}}(G_{\mathsf{M}})_{\otimes,\mathrm{mor}_{\mathrm{Rel}}(A_1,B_1),\mathrm{mor}_{\mathrm{Rel}}(A_2,B_2)}.$$

(c) Let $I_{\mathcal{T}^0} := I_{\mathrm{Rel}}$.

(d) For any $\{A_1,A_2,A_3\}\subseteq \mathrm{obj}_{\mathcal{T}^0}$, considering $\alpha_{\mathrm{Rel},A_1,A_2,A_3}$ a mapping $I_{\mathsf{Set}}\to \mathrm{mor}_{\mathrm{Rel}}((A_1\otimes_{\mathrm{Rel}} A_2)\otimes_{\mathrm{Rel}} A_3, A_1\otimes_{\mathrm{Rel}}(A_2\otimes_{\mathrm{Rel}} A_3))$, define

$$\alpha_{\mathcal{T}^0,A_1,A_2,A_3} := G_{\mathsf{M}}(\alpha_{\mathrm{Rel},A_1,A_2,A_3})\circ_{\mathsf{M}}(G_{\mathsf{M}})_I.$$

(e) For any $A\in \mathrm{obj}_{\mathcal{T}^0}$, viewing $\lambda_{\mathrm{Rel},A}$ a map $I_{\mathsf{Set}}\to \mathrm{mor}_{\mathrm{Rel}}(I_{\mathrm{Rel}}\otimes_{\mathrm{Rel}} A, A)$, let

$$\lambda_{\mathcal{T}^0,A} := G_{\mathsf{M}}(\lambda_{\mathrm{Rel},A})\circ_{\mathsf{M}}(G_{\mathsf{M}})_I.$$

(f) For any $A\in \mathrm{obj}_{\mathcal{T}^0}$, viewing $\rho_{\mathrm{Rel},A}$ a map $I_{\mathsf{Set}}\to \mathrm{mor}_{\mathrm{Rel}}(A\otimes_{\mathrm{Rel}} I_{\mathrm{Rel}}, A)$, let

$$\rho_{\mathcal{T}^0,A} := G_{\mathsf{M}}(\rho_{\mathrm{Rel},A})\circ_{\mathsf{M}}(G_{\mathsf{M}})_I.$$

One then has to check the following in order to prove that $\mathcal{T}^0(\mathcal{A},\mathcal{E},\delta)$ is a monoidal M-enriched category. Recall that B denotes the symmetric monoidal 2-category of small M-enriched categories. The composition of 1-cells in B is written as $\circ_{\mathsf{B}}$ and the vertical composition of 2-cells as $\cdot_{\mathsf{B}}$. The symbols $\lhd_{\mathsf{B}}$ and $\rhd_{\mathsf{B}}$ are used for the left respectively right whiskering in B. Moreover, we write $\oslash_{\mathsf{B}}$ and $\ominus_{\mathsf{B}}$ for the left and right tensor whiskering in B, respectively, (forming the monoidal product of a given 2-cell with the identity 2-cell of a given 1-cell).

Conjecture 5.113. *(a) $\otimes_{\mathcal{T}^0}$ is an M-enriched functor $\mathcal{T}^0 \otimes_{\mathsf{B}} \mathcal{T}^0$ to $\mathcal{T}^0$.*

(b) $I_{\mathcal{T}^0}$ is an M-enriched functor from I_{B} to $\mathcal{T}^0$.

(c) $\alpha_{\mathcal{T}^0}$ is an M-enriched natural isomorphism of M-enriched functors from $(\mathcal{T}^0 \otimes_{\mathsf{B}} \mathcal{T}^0) \otimes_{\mathsf{B}} \mathcal{T}^0$ to $\mathcal{T}^0$ from $\otimes_{\mathcal{T}^0} \circ_{\mathsf{B}} (\otimes_{\mathcal{T}^0} \otimes_{\mathsf{B}} \mathrm{id}_{\mathsf{B},\mathcal{T}^0})$ to $\otimes_{\mathcal{T}^0} \circ_{\mathsf{B}} (\mathrm{id}_{\mathsf{B},\mathcal{T}^0} \otimes_{\mathsf{B}} \otimes_{\mathcal{T}^0}) \circ_{\mathsf{B}} \alpha_{\mathsf{B},\mathcal{T}^0,\mathcal{T}^0,\mathcal{T}^0}$.

(d) $\lambda_{\mathcal{T}^0}$ is an M-enriched natural isomorphism of M-enriched functors from $I_{\mathsf{B}} \otimes_{\mathsf{B}} \mathcal{T}^0$ to $\mathcal{T}^0$ from $\otimes_{\mathcal{T}^0} \circ_{\mathsf{B}} (I_{\mathcal{T}^0} \otimes_{\mathsf{B}} \mathrm{id}_{\mathsf{B},\mathcal{T}^0})$ to $\lambda_{\mathsf{B},\mathcal{T}^0}$.

(e) $\rho_{\mathcal{T}^0}$ is an M-enriched natural isomorphism of M-enriched functors from $\mathcal{T}^0 \otimes_{\mathsf{B}} I_{\mathsf{B}}$ to $\mathcal{T}^0$ from $\otimes_{\mathcal{T}^0} \circ_{\mathsf{B}} (\mathrm{id}_{\mathsf{B},\mathcal{T}^0} \otimes_{\mathsf{B}} I_{\mathcal{T}^0})$ to $\rho_{\mathsf{B},\mathcal{T}^0}$.

(f) The pentagon identity holds:

$$\begin{aligned}
&(\alpha_{\mathcal{T}^0} \lhd_{\mathsf{B}} (((\mathrm{id}_{\mathsf{B},\mathcal{T}^0} \otimes_{\mathsf{B}} \mathrm{id}_{\mathsf{B},\mathcal{T}^0}) \otimes_{\mathsf{B}} \otimes_{\mathcal{T}^0}) \circ_{\mathsf{B}} \alpha_{\mathsf{B},\mathcal{T}^0\otimes_{\mathsf{B}}\mathcal{T}^0,\mathcal{T}^0,\mathcal{T}^0}))\\
&\cdot_{\mathsf{B}} (\alpha_{\mathcal{T}^0} \lhd_{\mathsf{B}} ((\otimes_{\mathcal{T}^0} \otimes_{\mathsf{B}} \mathrm{id}_{\mathsf{B},\mathcal{T}^0}) \otimes_{\mathsf{B}} \mathrm{id}_{\mathsf{B},\mathcal{T}^0}))\\
&\quad = (\otimes_{\mathcal{T}^0} \rhd_{\mathsf{B}} (\mathrm{id}_{\mathsf{B},\mathcal{T}^0} \ominus_{\mathsf{B}} \alpha_{\mathcal{T}^0}) \lhd_{\mathsf{B}} (\alpha_{\mathsf{B},\mathcal{T}^0,\mathcal{T}^0\otimes_{\mathsf{B}}\mathcal{T}^0,\mathcal{T}^0} \circ_{\mathsf{B}} (\alpha_{\mathsf{B},\mathcal{T}^0,\mathcal{T}^0,\mathcal{T}^0} \otimes_{\mathsf{B}} \mathrm{id}_{\mathsf{B},\mathcal{T}^0})))\\
&\quad\quad \cdot_{\mathsf{B}} (\alpha_{\mathcal{T}^0} \lhd_{\mathsf{B}} (((\mathrm{id}_{\mathsf{B},\mathcal{T}^0} \otimes_{\mathsf{B}} \otimes_{\mathcal{T}^0}) \otimes_{\mathsf{B}} \mathrm{id}_{\mathsf{B},\mathcal{T}^0}) \circ_{\mathsf{B}} (\alpha_{\mathsf{B},\mathcal{T}^0,\mathcal{T}^0,\mathcal{T}^0} \otimes_{\mathsf{B}} \mathrm{id}_{\mathsf{B},\mathcal{T}^0})))\\
&\quad\quad \cdot_{\mathsf{B}} (\otimes_{\mathcal{T}^0} \rhd_{\mathsf{B}} (\alpha_{\mathcal{T}^0} \oslash_{\mathsf{B}} \mathrm{id}_{\mathsf{B},\mathcal{T}^0})).
\end{aligned}$$

(g) The triangle identity holds:

$$\begin{aligned}
&\otimes_{\mathcal{T}^0} \rhd_{\mathsf{B}} (\rho_{\mathcal{T}^0} \oslash_{\mathsf{B}} \mathrm{id}_{\mathsf{B},\mathcal{T}^0})\\
&\quad = (\otimes_{\mathcal{T}^0} \rhd_{\mathsf{B}} (\mathrm{id}_{\mathsf{B},\mathcal{T}^0} \ominus_{\mathsf{B}} \lambda_{\mathcal{T}^0}) \lhd_{\mathsf{B}} \alpha_{\mathsf{B},\mathcal{T}^0,I_{\mathsf{B}},\mathcal{T}^0})\\
&\quad\quad \cdot_{\mathsf{B}} (\alpha_{\mathcal{T}^0} \lhd_{\mathsf{B}} ((\mathrm{id}_{\mathsf{B},\mathcal{T}^0} \otimes_{\mathsf{B}} I_{\mathcal{T}^0}) \otimes_{\mathsf{B}} \mathrm{id}_{\mathsf{B},\mathcal{T}^0})).
\end{aligned}$$

Once that conjecture is confirmed, we will have shown the following.

Conjecture 5.114. *$\mathcal{T}^0(\mathcal{A},\mathcal{E},\delta)$ is a monoidal M-enriched category.*

5.5.3. *Symmetric Monoidal Linear Category.* The symmetric braiding of $\mathrm{S}(\mathcal{A})$ should also induce one for $\mathcal{T}^0(\mathcal{A},\mathcal{E},\delta)$. The next definition is the same as the one given in Section 2.

Definition 5.115. For any $\{A_1, A_2\} \subseteq \mathrm{obj}_{\mathcal{T}^0}$, considering $\gamma_{\mathrm{Rel},A_1,A_2}$ a mapping $I_{\mathsf{Set}} \to \mathrm{mor}_{\mathrm{Rel}}(A_1 \otimes_{\mathrm{Rel}} A_2, A_2 \otimes_{\mathrm{Rel}} A_1)$ define

$$\gamma_{\mathcal{T}^0,A_1,A_2} \coloneqq G_{\mathsf{M}}(\gamma_{\mathrm{Rel},A_1,A_2}) \circ_{\mathsf{M}} (G_{\mathsf{M}})_I.$$

What needs to be checked are the following statements. There, the symbol i_{B} is used for the identity 2-cells of B.

Conjecture 5.116. *(a) $\gamma_{\mathcal{T}^0}$ is an M-enriched natural transformation of M-enriched functors from $\mathcal{T}^0 \otimes_{\mathsf{B}} \mathcal{T}^0$ to $\mathcal{T}^0$ from $\otimes_{\mathcal{T}^0}$ to $\otimes_{\mathcal{T}^0} \circ_{\mathsf{B}} \gamma_{\mathsf{B},\mathcal{T}^0,\mathcal{T}^0}$.*

(b) *The* first hexagon identity *holds:*

$$(\alpha_{\mathcal{T}^0} \lhd_{\mathsf{B}} (\gamma_{\mathsf{B},\mathcal{T}^0,\mathcal{T}^0 \otimes_{\mathsf{B}} \mathcal{T}^0} \circ_{\mathsf{B}} \alpha_{\mathsf{B},\mathcal{T}^0,\mathcal{T}^0,\mathcal{T}^0}))$$
$$\cdot_{\mathsf{B}} (\gamma_{\mathcal{T}^0} \lhd_{\mathsf{B}} ((\mathrm{id}_{\mathsf{B},\mathcal{T}^0} \otimes_{\mathsf{B}} \otimes_{\mathcal{T}^0}) \circ_{\mathsf{B}} \alpha_{\mathsf{B},\mathcal{T}^0,\mathcal{T}^0,\mathcal{T}^0}))$$
$$\cdot_{\mathsf{B}} \alpha_{\mathcal{T}^0}$$
$$= (\otimes_{\mathcal{T}^0} \rhd_{\mathsf{B}} (\mathrm{id}_{\mathsf{B},\mathcal{T}^0} \oslash_{\mathsf{B}} \gamma_{\mathcal{T}^0}) \lhd_{\mathsf{B}} (\alpha_{\mathsf{B},\mathcal{T}^0,\mathcal{T}^0,\mathcal{T}^0} \circ_{\mathsf{B}} (\gamma_{\mathsf{B},\mathcal{T}^0,\mathcal{T}^0} \otimes_{\mathsf{B}} \mathrm{id}_{\mathsf{B},\mathcal{T}^0})))$$
$$\cdot_{\mathsf{B}} (\alpha_{\mathcal{T}^0} \lhd_{\mathsf{B}} (\gamma_{\mathsf{B},\mathcal{T}^0,\mathcal{T}^0} \otimes_{\mathsf{B}} \mathrm{id}_{\mathsf{B},\mathcal{T}^0}))$$
$$\cdot_{\mathsf{B}} (\otimes_{\mathcal{T}^0} \rhd_{\mathsf{B}} (\gamma_{\mathcal{T}^0} \oslash_{\mathsf{B}} \mathrm{id}_{\mathsf{B},\mathcal{T}^0})).$$

(c) *The* second hexagon identity *holds:*

$$(\alpha_{\mathcal{T}^0}{}^{-1_{\mathsf{B}}} \lhd_{\mathsf{B}} (\alpha_{\mathsf{B},\mathcal{T}^0,\mathcal{T}^0,\mathcal{T}^0}{}^{-1_{\mathsf{B}}} \circ_{\mathsf{B}} \gamma_{\mathsf{B},\mathcal{T}^0 \otimes_{\mathsf{B}} \mathcal{T}^0,\mathcal{T}^0} \circ_{\mathsf{B}} \alpha_{\mathsf{B},\mathcal{T}^0,\mathcal{T}^0,\mathcal{T}^0}{}^{-1_{\mathsf{B}}}))$$
$$\cdot_{\mathsf{B}} (\gamma_{\mathcal{T}^0} \lhd_{\mathsf{B}} ((\otimes_{\mathcal{T}^0} \otimes_{\mathsf{B}} \mathrm{id}_{\mathsf{B},\mathcal{T}^0}) \circ_{\mathsf{B}} \alpha_{\mathsf{B},\mathcal{T}^0,\mathcal{T}^0,\mathcal{T}^0}{}^{-1_{\mathsf{B}}}))$$
$$\cdot_{\mathsf{B}} (\alpha_{\mathcal{T}^0}{}^{-1_{\mathsf{B}}} \lhd_{\mathsf{B}} \alpha_{\mathsf{B},\mathcal{T}^0,\mathcal{T}^0,\mathcal{T}^0}{}^{-1_{\mathsf{B}}})$$
$$= (\otimes_{\mathcal{T}^0} \rhd_{\mathsf{B}} (\gamma_{\mathcal{T}^0} \oslash_{\mathsf{B}} \mathrm{id}_{\mathsf{B},\mathcal{T}^0}) \lhd_{\mathsf{B}} (\alpha_{\mathsf{B},\mathcal{T}^0,\mathcal{T}^0,\mathcal{T}^0}{}^{-1_{\mathsf{B}}} \circ_{\mathsf{B}} (\mathrm{id}_{\mathsf{B},\mathcal{T}^0} \otimes_{\mathsf{B}} \gamma_{\mathsf{B},\mathcal{T}^0,\mathcal{T}^0})))$$
$$\cdot_{\mathsf{B}} (\alpha_{\mathcal{T}^0}{}^{-1_{\mathsf{B}}} \lhd_{\mathsf{B}} (\alpha_{\mathsf{B},\mathcal{T}^0,\mathcal{T}^0,\mathcal{T}^0}{}^{-1_{\mathsf{B}}} \circ_{\mathsf{B}} (\mathrm{id}_{\mathsf{B},\mathcal{T}^0} \otimes_{\mathsf{B}} \gamma_{\mathsf{B},\mathcal{T}^0,\mathcal{T}^0})))$$
$$\cdot_{\mathsf{B}} (\otimes_{\mathcal{T}^0} \rhd_{\mathsf{B}} (\mathrm{id}_{\mathsf{B},\mathcal{T}^0} \oslash_{\mathsf{B}} \gamma_{\mathcal{T}^0})).$$

(d) $(\gamma_{\mathcal{T}^0} \lhd_{\mathsf{B}} \gamma_{\mathsf{B},\mathcal{T}^0,\mathcal{T}^0}) \cdot_{\mathsf{B}} \gamma_{\mathcal{T}^0} = \mathrm{i}_{\mathsf{B},\otimes_{\mathcal{T}^0}}$.

Then the following will have been confirmed.

PROPOSITION 5.117. *$\mathcal{T}^0(\mathcal{A},\mathcal{E},\delta)$ is a symmetric monoidal M-enriched category.*

5.5.4. *Rigid Symmetric Monoidal Linear Category.* Of course, the most important part of the construction from the quantum group perspective is the rigidity. This property should be inherited by $\mathcal{T}^0(\mathcal{A},\mathcal{E},\delta)$ from $\mathrm{S}(\mathcal{A})$. Explicitly, it should be true that one always choose the following dualizations.

DEFINITION 5.118. For any $X \in \mathrm{obj}_{\mathcal{T}^0}$ let

$$X^{\vee_{\mathcal{T}^0}} := X^{\vee_{\mathrm{Rel}}}$$

and, considering $\varepsilon_{\mathrm{Rel},X}$ a mapping $I_{\mathsf{Set}} \to \mathrm{mor}_{\mathrm{Rel}}(X^{\vee_{\mathrm{Rel}}} \otimes_{\mathrm{Rel}} X, I_{\mathrm{Rel}})$ and $\eta_{\mathrm{Rel},X}$ a mapping $I_{\mathsf{Set}} \to \mathrm{mor}_{\mathrm{Rel}}(I_{\mathcal{T}^0}, X \otimes_{\mathcal{T}^0} X^{\vee_{\mathcal{T}^0}})$, define

$$\varepsilon_{\mathcal{T}^0,X} := G_{\mathsf{M}}(\varepsilon_{\mathrm{Rel},X}) \circ_{\mathsf{M}} (G_{\mathsf{M}})_I$$

and

$$\eta_{\mathcal{T}^0,X} := G_{\mathsf{M}}(\eta_{\mathrm{Rel},X}) \circ_{\mathsf{M}} (G_{\mathsf{M}})_I.$$

To confirm that this definition really does provide duals for all objects of $\mathcal{T}^0(\mathcal{A},\mathcal{E},\delta)$ we need to prove the below conjectures.

CONJECTURE 5.119. *(a) $\varepsilon_{\mathcal{T}^0,X}$ is an M-enriched natural transformation of M-enriched functors $I_{\mathsf{B}} \to \mathcal{T}^0$ from $\otimes_{\mathcal{T}^0} \circ_{\mathsf{B}} (X^{\vee_{\mathcal{T}^0}} \otimes_{\mathsf{B}} X) \circ_{\mathsf{B}} \lambda_{\mathsf{B},I_{\mathsf{B}}}{}^{-1_{\mathsf{B}}}$ to $I_{\mathcal{T}^0}$,*

(b) $\eta_{\mathcal{T}^0,X}$ is an M-enriched natural transformation of M-enriched functors $I_{\mathsf{B}} \to \mathcal{T}^0$ from $I_{\mathcal{T}^0}$ to $\otimes_{\mathcal{T}^0} \circ_{\mathsf{B}} (X \otimes_{\mathsf{B}} X^{\vee_{\mathcal{T}^0}}) \circ_{\mathsf{B}} \rho_{\mathsf{B},I_{\mathsf{B}}}{}^{-1_{\mathsf{B}}}$,

(c) *The first triangle identity holds,*

$$\begin{aligned}&\lambda_{\mathcal{T}^0} \lhd_{\mathsf{B}} ((\mathrm{id}_{\mathsf{B},I_{\mathsf{B}}} \otimes_{\mathsf{B}} X) \circ_{\mathsf{B}} \lambda_{\mathsf{B},I_{\mathsf{B}}}{}^{-1_{\mathsf{B}}})\\ &= \cdot_{\mathsf{B}}(\rho_{\mathcal{T}^0} \lhd_{\mathsf{B}} ((X \otimes_{\mathsf{B}} \mathrm{id}_{\mathsf{B},I_{\mathsf{B}}}) \circ_{\mathsf{B}} \rho_{\mathsf{B},I_{\mathsf{B}}}{}^{-1_{\mathsf{B}}}))\\ &\quad \cdot_{\mathsf{B}} (\otimes_{\mathcal{T}^0} \rhd_{\mathsf{B}} (X \oslash_{\mathsf{B}} \varepsilon_{\mathcal{T}^0,X}) \lhd_{\mathsf{B}} \lambda_{\mathsf{B},I_{\mathsf{B}}}{}^{-1_{\mathsf{B}}})\\ &\quad \cdot_{\mathsf{B}} (\alpha_{\mathcal{T}^0} \lhd_{\mathsf{B}} ((((X \otimes_{\mathsf{B}} X^{\vee_{\mathcal{T}^0}}) \circ_{\mathsf{B}} \rho_{\mathsf{B},I_{\mathsf{B}}}{}^{-1_{\mathsf{B}}}) \otimes_{\mathsf{B}} X) \circ_{\mathsf{B}} \lambda_{\mathsf{B},I_{\mathsf{B}}}{}^{-1_{\mathsf{B}}}))\\ &\quad \cdot_{\mathsf{B}} (\otimes_{\mathcal{T}^0} \rhd_{\mathsf{B}} (\eta_{\mathcal{T}^0,X} \oslash_{\mathsf{B}} X) \lhd_{\mathsf{B}} \lambda_{\mathsf{B},I_{\mathsf{B}}}{}^{-1_{\mathsf{B}}}).\end{aligned}$$

(d) *The second triangle identity holds,*

$$\begin{aligned}&\rho_{\mathcal{T}^0} \lhd_{\mathsf{B}} ((X^{\vee_{\mathcal{T}^0}} \otimes_{\mathsf{B}} \mathrm{id}_{\mathsf{B},I_{\mathsf{B}}}) \circ_{\mathsf{B}} \rho_{\mathsf{B},I_{\mathsf{B}}}{}^{-1_{\mathsf{B}}})\\ &= \cdot_{\mathsf{B}}(\lambda_{\mathcal{T}^0} \lhd_{\mathsf{B}} ((\mathrm{id}_{\mathsf{B},I_{\mathsf{B}}} \otimes_{\mathsf{B}} X^{\vee_{\mathcal{T}^0}}) \circ_{\mathsf{B}} \lambda_{\mathsf{B},I_{\mathsf{B}}}{}^{-1_{\mathsf{B}}}))\\ &\quad \cdot_{\mathsf{B}} (\otimes_{\mathcal{T}^0} \rhd_{\mathsf{B}} (\varepsilon_{\mathcal{T}^0,X} \oslash_{\mathsf{B}} X^{\vee_{\mathcal{T}^0}}) \lhd_{\mathsf{B}} \rho_{\mathsf{B},I_{\mathsf{B}}}{}^{-1_{\mathsf{B}}})\\ &\quad \cdot_{\mathsf{B}} (\alpha_{\mathcal{T}^0}{}^{-1_{\mathsf{B}}} \lhd_{\mathsf{B}} ((((X^{\vee_{\mathcal{T}^0}} \otimes_{\mathsf{B}} X) \circ_{\mathsf{B}} \lambda_{\mathsf{B},I_{\mathsf{B}}}{}^{-1_{\mathsf{B}}}) \otimes_{\mathsf{B}} X^{\vee_{\mathcal{T}^0}}) \circ_{\mathsf{B}} \rho_{\mathsf{B},I_{\mathsf{B}}}{}^{-1_{\mathsf{B}}}))\\ &\quad \cdot_{\mathsf{B}} (\otimes_{\mathcal{T}^0} \rhd_{\mathsf{B}} (X^{\vee_{\mathcal{T}^0}} \oslash_{\mathsf{B}} \eta_{\mathcal{T}^0,X}) \lhd_{\mathsf{B}} \lambda_{\mathsf{B},I_{\mathsf{B}}}{}^{-1_{\mathsf{B}}}).\end{aligned}$$

Then, the below conclusion will be immediate.

Conjecture 5.120. *$\mathcal{T}^0(\mathcal{A},\mathcal{E},\delta)$ is a rigid symmetric monoidal M-enriched category.*

5.6. Stage 3b: $*$-Linear Relations (with †). An important particularity of the span category is, obviously, the natural †-structure. This should also pass to the generalized tensor envelope. This is the point where the $*$-structure of R and the induced endofunctor cj_{M} of M come into play. Recall that cj_{M} maps any module object over R to the module object with the same underlying S-object but with the conjugate action of R.

5.6.1. *$*$-Linear †-category.* Note that the †-functor of $\mathcal{T}^0(\mathcal{A},\mathcal{E},\delta)$ is supposed to be anti-linear. Hence, cj_{M} appears in the definition. Remember that $U_{\mathsf{M}} \circ_{\mathsf{CAT}} \mathrm{cj}_{\mathsf{M}} = U_{\mathsf{M}}$.

Definition 5.121. Define $X^{\dagger_{\mathcal{T}^0}} := X$ for any X and let

$$((\,\cdot\,)^{\dagger_{\mathcal{T}^0}})_{1,A,B} := \mathrm{cu}_{\mathsf{M},\mathrm{cj}_{\mathsf{M}}(\mathrm{mor}_{\mathcal{T}^0}(B,A))} \circ_{\mathsf{M}} G_{\mathsf{M}}(\mathrm{un}_{\mathsf{M},\mathrm{mor}_{\mathrm{Rel}}(B,A)} \circ_{\mathsf{Set}} ((\,\cdot\,)^{\dagger_{\mathrm{Rel}}})_{1,A,B})$$

for any $\{A,B\} \subseteq \mathrm{obj}_{\mathcal{T}^0}$.

In the following conjecture $*_{\mathsf{B}}$ is the opconjugation 2-functor. It maps any M-enriched category to the base change of the opposite M-enriched category along the monoidal functor cj_{B}.

Conjecture 5.122. (a) *$(\,\cdot\,)^{\dagger_{\mathcal{T}^0}}$ is an M-enriched functor $\mathcal{T}^0 \to (\mathcal{T}^0)^{*_{\mathsf{B}}}$.*
(b) *$X^{\dagger_{\mathcal{T}^0}} = X$ for any $X \in \mathrm{obj}_{\mathcal{T}^0}$.*
(c) *$((\,\cdot\,)^{\dagger_{\mathcal{T}^0}})^{*_{\mathsf{B}}} \circ_{\mathsf{B}} (\,\cdot\,)^{\dagger_{\mathcal{T}^0}} = \mathrm{id}_{\mathsf{B},\mathcal{T}^0}$.*

Proving these conjecture is by definition equvivalent to showing the following.

Conjecture 5.123. *$\mathcal{T}^0$ is an M-enriched †-category.*

5.6.2. *Monoidal $*$-Linear †-category.* As with $\mathrm{S}(\mathcal{A})$ the †-structure should be compatible with the monoidal one. We have to check that the monoidal structure maps are unitary. Here, $*\mathsf{B}$ is the symmetric monoidal 2-category of M-enriched †-categories.

CONJECTURE 5.124. *(a)* $\otimes_{\mathcal{T}^0}$ *is an* M*-enriched †-functor* $\mathcal{T}^0 \otimes_{*\mathsf{B}} \mathcal{T}^0 \to \mathcal{T}^0$.
(b) $\alpha_{\mathcal{T}^0}$ *is a natural unitary* M*-enriched isomorphism.*
(c) $\lambda_{\mathcal{T}^0}$ *is a natural unitary* M*-enriched isomorphism.*
(d) $\rho_{\mathcal{T}^0}$ *is a natural unitary* M*-enriched isomorphism.*

Then the following will be true.

CONJECTURE 5.125. $\mathcal{T}^0(\mathcal{A},\mathcal{E},\delta)$ *is a monoidal* M*-enriched †-category.*

5.6.3. *Symmetric Monoidal $*$-Linear †-category.* There is no reason why the braiding of $\mathcal{T}^0(\mathcal{A},\mathcal{E},\delta)$ should not be compatible with the †-structure since this is the case for $\mathrm{S}(\mathcal{A})$ and $\mathrm{Rel}(\mathcal{A},\mathcal{E})$. Concretely, one would have to check the following.

CONJECTURE 5.126. $\gamma_{\mathcal{T}^0}$ *is a unitary* M*-enriched isomorphism.*

Then, the below conjecture would be confirmed.

CONJECTURE 5.127. $\mathcal{T}^0(\mathcal{A},\mathcal{E},\delta)$ *is a symmetric monoidal* M*-enriched †-category.*

5.6.4. *Rigid Symmetric Monoidal $*$-Linear †-category.* Finally, the particular relationship between the duals and the † that is present in $\mathrm{S}(\mathcal{A})$ and $\mathrm{Rel}(\mathcal{A},\mathcal{E})$ should be inherited by $\mathcal{T}^0(\mathcal{A},\mathcal{E},\delta)$.

CONJECTURE 5.128. *For any* $X \in \mathrm{obj}_{\mathcal{T}^0}$ *and any dual* $(X^{\vee_{\mathcal{T}^0}}, \varepsilon_{\mathcal{T}^0,X}, \eta_{\mathcal{T}^0,X})$ *of* X *in* $\mathcal{T}^0$*,*

$$(\gamma_{\mathcal{T}^0} \triangleleft_{\mathsf{B}} (X^{\vee_{\mathcal{T}^0}} \otimes_{\mathsf{B}} X) \circ_{\mathsf{B}} \lambda_{\mathsf{B},I_{\mathsf{B}}}{}^{-1_{\mathsf{B}}}) \cdot_{\mathsf{B}} (\varepsilon_{\mathcal{T}^0,X})^{\dagger_{\mathsf{B}}} = \eta_{\mathcal{T}^0,X}.$$

In other words, the following should hold.

CONJECTURE 5.129. $\mathcal{T}^0$ *is a rigid symmetric monoidal* M*-enriched †-category.*

After that Conjecture 2.14 will be confirmed – at the level of 0-cells. Of course, much more is asserted, which still needs to be checked as well. Unfortunately, this was not possible in the allotted. The same is true about Conjecture 2.19.

6. Subquotients

A crucial ingredient in the proof of Knop's semisimplicity result in [Kno07, Theorem 6.1] is the theory of subquotients and Lemmata 2.5 and 2.6 especially. Unfortunately, as mentioned in the introduction, the generalization from uniform functors on regular categories to the inputs considered here comes at a price. The present section investigates which results about subquotients carry over from Knop's setting and which do not.

The following is the natural generalization of the notion of subquotient object considered by Knop (see [Kno07, p. 577] after Definition 2.4).

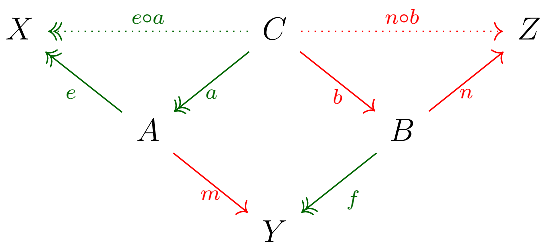
Definition 6.1. Let $\mathcal{A}$ be any category, let $\mathcal{E}$ and $\mathcal{M}$ be any subcategories of $\mathcal{A}$ and let $\{X,Y\} \subseteq \mathrm{obj}_{\mathcal{A}}$ be arbitrary. X is said to be an $(\mathcal{E},\mathcal{M})$-*subquotient* of Y, in symbols: $X \lesssim_{\mathcal{A},\mathcal{E},\mathcal{M}} Y$, if there exists $A \in \mathrm{obj}_{\mathcal{A}}$ such that $\mathrm{mor}_{\mathcal{M}}(A,Y) \neq \emptyset$ and $\mathrm{mor}_{\mathcal{E}}(A,X) \neq \emptyset$.

$$X \xleftarrow{\;e\;} A \xrightarrow{\;m\;} Y$$

And, in fact, the first half of [Kno07, Lemma 2.5] remains valid under far more general assumptions, as the below reasoning shows.

Lemma 6.2. *For any category $\mathcal{A}$ with pull-backs and any subcategories $\mathcal{E}$ and $\mathcal{M}$ of $\mathcal{A}$ such that $\mathcal{E}$ is $\mathcal{M}$-pull-back-stable and $\mathcal{M}$ is $\mathcal{E}$-pull-back-stable $\lesssim_{\mathcal{A},\mathcal{E},\mathcal{M}}$ is a pre-order on* $\mathrm{obj}_{\mathcal{A}}$.

Proof. Clearly, $\lesssim$ is reflexive. We need to show that $\lesssim$ is transitive. If $\{X,Y,Z\} \subseteq \mathrm{obj}_{\mathcal{A}}$ are such that $X \lesssim Y$ and $Y \lesssim Z$, then there exist $\{A,B\} \subseteq \mathrm{obj}_{\mathcal{A}}$ and $m \in \mathrm{mor}_{\mathcal{M}}(A,Y)$ and $e \in \mathrm{mor}_{\mathcal{E}}(A,X)$ as well as $n \in \mathrm{mor}_{\mathcal{M}}(B,Z)$ and $f \in \mathrm{mor}_{\mathcal{E}}(B,Y)$. Because $\mathcal{A}$ has pull-backs we find a pull-back (a,b) of (m,f) with pull-back object C.



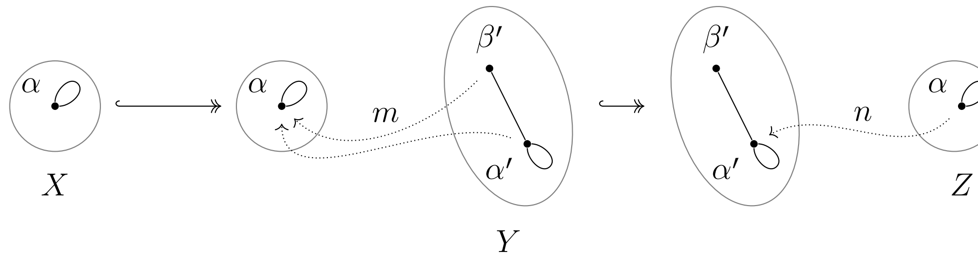
Since $\mathcal{M}$ is stable under pull-backs along $\mathcal{E}$ and since $m \in \mathrm{mor}_{\mathcal{M}}(A,Y)$ and $f \in \mathrm{mor}_{\mathcal{E}}(B,Y)$ we can infer $a \in \mathrm{mor}_{\mathcal{M}}(C,A)$. Likewise, because $\mathcal{E}$ is $\mathcal{M}$-pull-back-stable, $b \in \mathrm{mor}_{\mathcal{E}}(C,B)$. As $\mathcal{E}$ and $\mathcal{M}$ are subcategories it follows $n \circ b \in \mathrm{mor}_{\mathcal{M}}(C,Z)$ and $e \circ a \in \mathrm{mor}_{\mathcal{E}}(C,X)$. Thus, $X \lesssim Z$. □

However, the second half of [Kno07, Lemma 2.5] is generally false in the more general setting, as the next remark shows.

Remark 6.3. Given any category $\mathcal{A}$ and any subcategories $\mathcal{E}$ and $\mathcal{M}$ of $\mathcal{A}$ saying, for any $\{X,Y\} \subseteq \mathrm{obj}_{\mathcal{A}}$, that X is a *proper* $(\mathcal{E},\mathcal{M})$-*subquotient* of Y, in symbols: $X \prec_{\mathcal{A},\mathcal{E},\mathcal{M}} Y$, if there exists $A \in \mathrm{obj}_{\mathcal{A}}$ such that $\mathrm{mor}_{\mathcal{M}}(A,Y) \neq \emptyset$ and $\mathrm{mor}_{\mathcal{E}}(A,X) \neq \emptyset$ and, additionally, $\mathrm{mor}_{\mathcal{E}}(A,X)\backslash\mathrm{iso}_{\mathcal{A}}(A,X) \neq \emptyset$ or $\mathrm{mor}_{\mathcal{M}}(A,Y)\backslash\mathrm{iso}_{\mathcal{A}}(A,Y) \neq \emptyset$, does not yield a transitive relation, even if $\mathcal{E}$ is $\mathcal{M}$-pull-back-stable and $\mathcal{M}$ is $\mathcal{E}$-pull-back-stable, unless both $\mathcal{E} \hookrightarrow \mathrm{epi}_{\mathcal{A}}$ and $\mathcal{M} \hookrightarrow \mathrm{mon}_{\mathcal{A}}$.

As a counterexample consider the second case of Example 3.2 (a). In other words, we choose the opposite category $\mathsf{fGr}^{\mathrm{op}}$ of all finite undirected graphs with or without loops as $\mathcal{A}$, the class of all graph embeddings whose images are unions of connected components as $\mathcal{E}$ and the the class of all graph homomorphisms whose image intersects each connected component of their target as $\mathcal{M}$. Then, consider the graphs $X = Z = (\{\alpha\},\{\{\alpha\}\})$ and $Y = (\{\alpha',\beta'\},\{\{\alpha'\},\{\alpha',\beta'\}\})$, where $\alpha' \neq \beta'$.

The unique graph morphism $m\colon Y \to X$ and the graph morphism $n\colon Z \to Y$ with $\alpha \mapsto \alpha'$ each hit all components of their respective co-domains. Since neither of them is a graph isomorphism, that makes X a proper $(\mathcal{E},\mathcal{M})$-subquotient of Y and Y a proper $(\mathcal{E},\mathcal{M})$-subquotient of Z in $\mathsf{fGr}^{\mathrm{op}}$.



However, X is *not* a proper subquotient of Z. Indeed, suppose that C is a graph which admits a graph embedding g of X into C whose image is a union of connected components as well as a graph homomorphism p from Z to C whose image intersects each component of Z. Since $\{g(\alpha)\}$ is a connected component of C, which p has to hit, we must have $p(\alpha) = g(\alpha)$. That forces $\{g(\alpha)\}$ to be the only component of C because we have run out of vertices of Z which we could map to further components under p. Hence, $C = (\{g(\alpha)\},\{\{g(\alpha)\}\})$ and $g = p$ are both isomorphisms.

Thus, for our more general inputs a more detailed analysis is required. Perhaps the following notion with more favorable properties plays a role there. Of course, the inherent asymmetry complicates things.

Definition 6.4. Let $\mathcal{A}$ be any category, let $\mathcal{E}$ and $\mathcal{M}$ be any subcategories of $\mathcal{A}$ and let $\{X,Y\} \subseteq \mathrm{obj}_{\mathcal{A}}$ be arbitrary. W call X a *half-proper* $(\mathcal{E},\mathcal{M})$*-subquotient* of Y, in symbols: $X \lesssim_{\mathcal{A},\mathcal{E},\mathcal{M}} Y$, if there exists $A \in \mathrm{obj}_{\mathcal{A}}$ such that $\mathrm{mor}_{\mathcal{M}}(A,Y) \neq \varnothing$ and $\mathrm{mor}_{\mathcal{E}}(A,X)\backslash\mathrm{iso}_{\mathcal{A}}(A,X) \neq \varnothing$.

Lemma 6.5. *For any category $\mathcal{A}$ with pull-backs and any subcategories $\mathcal{E}$ and $\mathcal{M}$ of $\mathcal{A}$ such that $\mathcal{E}$ is $\mathcal{M}$-pull-back-stable and $\mathcal{M}$ is $\mathcal{E}$-pull-back-stable and such that $\mathcal{E} \hookrightarrow \mathrm{epi}_{\mathcal{A}}$ the binary relation $\lesssim_{\mathcal{A},\mathcal{E},\mathcal{M}}$ on $\mathrm{obj}_{\mathcal{A}}$ is transitive.*

Proof. Make the same assumptions as in the proof of Lemma 6.2. If we suppose in addition that $X \lesssim Y \lesssim Z$, we can choose e and f non-invertible. In order to prove $X \lesssim Z$ it suffices to show that $e \circ a$ is not an isomorphism. This we show by contradiction. If $u \in \mathrm{iso}_{\mathcal{A}}(X,C)$ is an inverse of $e \circ a$, then the identity $\mathrm{id}_C = u \circ (e \circ a) = (u \circ e) \circ a$ exhibits $u \circ e$ as a left-inverse of a. Since the premise $\mathcal{E} \hookrightarrow \mathrm{epi}_{\mathcal{A}}$ implies that a is an epimorphism, a is thus invertible with $a^{-1} = u \circ e$. It follows $e \circ (a \circ u) = (e \circ a) \circ u = \mathrm{id}_A$ and $(a \circ u) \circ e = a \circ (u \circ e) = \mathrm{id}_X$, which means that a is invertible, contradictory to our assumption. Thus, $\lesssim$ is transitive. ∎

7. Core factorization

Next to [Kno07, Lemmata 2.5, 2.6], another prerequisite to Knop's semisimplicity result in [Kno07, Theorem 6.1] is the theory of core factorizations and [Kno07, Lemma 5.2] in particular. Although it is never explicit expressed in this way, Knop recognizes an orthogonal factorization system for the category of relations that is inherited by his tensor envelope (see (i) of [Kno07, Lemma 5.2]). Moreover, this factorization system has a certain stability property with respect to the subquotient relation (see (ii)). Section 7 examines which parts of [Kno07, Lemma 5.2] hold true in the more general setting considered in this chapter.

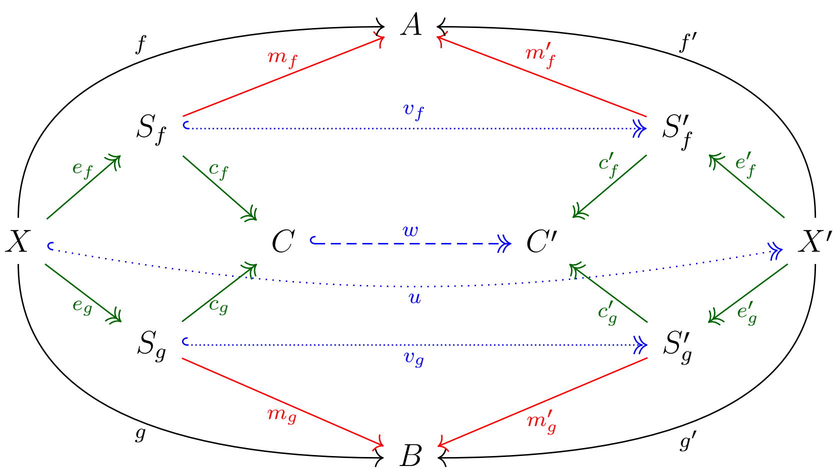
7.1. Cores. We begin by showing that it at least makes sense to speak of the core (see [Kno07, Definition 5.1]) of a relation in our context. (However, we will see that the existence and properties of such a core are a different matter.)

Assumptions 7.1. For the remainder of Section 7 let $(\mathcal{A},\mathcal{E},\delta)$ be any 0-cell of $\mathsf{d}_{\mathsf{S},R}\mathsf{esmCAT}^{\mathrm{cart,fc}}_{\mathrm{fs}}$ and $\mathcal{M} \coloneqq \mathcal{E}^{\perp}$ and let S be short for $\mathrm{S}(\mathcal{A})$ and Rel for $\mathrm{Rel}(\mathcal{A},\mathcal{E})$ and $\mathcal{T}^0$ for $\mathcal{T}^0(\mathcal{A},\mathcal{E},\delta)$.

The next result justifies that the definition following it works.

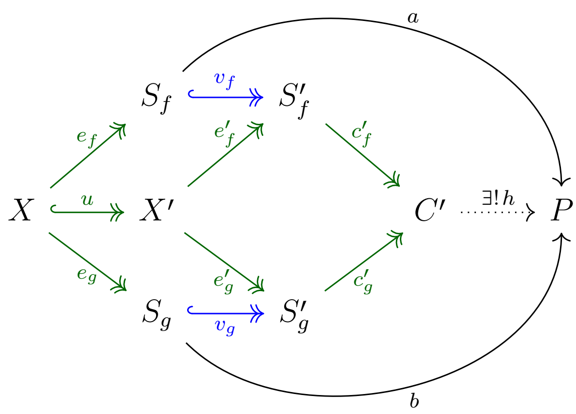
Lemma 7.2. *If $\{A,B\} \subseteq \mathrm{obj}_{\mathcal{A}}$, if (f,g) and (f',g') are spans in $\mathcal{A}$ from A to B with base objects X and X', respectively, if (e_f,m_f) is an $(\mathcal{E},\mathcal{M})$-factorization of f with image object S_f and (e_g,m_g) one of g with object S_g and (e'_f,m'_f) one of f' with object S'_f and (e'_g,m'_g) one of g' with object S'_g and if (c_f,c_g) if a push-out of (e_f,e_g) in $\mathcal{A}$ with push-out object C and (c'_f,c'_g) one of (e'_f,e'_g) with object C', then, if (f,g) and (f',g') are equivalent as spans in $\mathcal{A}$ from A to B, then there exists $w \in \mathrm{iso}_{\mathcal{A}}(C,C')$ such that $(m_f, w\circ c_f)$ and (m'_f,c'_f) are equivalent spans in $\mathcal{A}$ from A to C' and such that $(w\circ c_g, m_g)$ and (c'_g,m'_g) are equivalent spans in $\mathcal{A}$ from C' to B.*

Proof. The assumption that (f,g) and (f',g') are equivalent spans allows us to find $u \in \mathrm{iso}(X,X')$ such that $f = f'\circ u$ and $g = g'\circ u$.



Since u is an isomorphism, $e'_f \circ u \in \mathrm{mor}_{\mathcal{E}}(X, S'_f)$ and $e'_g \circ u \in \mathrm{mor}_{\mathcal{E}}(X, S'_g)$ by Lemma 4.33 (a). Hence, by $f = f' \circ u = (m'_f \circ e'_f) \circ u = m'_f \circ (e'_f \circ u)$ the pair $(m'_f, e'_f \circ u)$ is an $(\mathcal{E}, \mathcal{M})$-factorization of f. Likewise, the identity $g = g' \circ u = (m'_g \circ e'_g) \circ u = m'_g \circ (e'_g \circ u)$ proves $(m'_g, e'_g \circ u)$ to be an $(\mathcal{E}, \mathcal{M})$-factorization of g. Because $(\mathcal{E}, \mathcal{M})$-factorizations are essentially unique we hence find unique isomorphisms $v_f \in \mathrm{iso}_{\mathcal{A}}(S_f, S'_f)$ and $v_g \in \mathrm{iso}_{\mathcal{A}}(S_g, S'_g)$ such that $v_f \circ e_f = e'_f \circ u$ and $m_f = m'_f \circ v_f$ and, likewise, $v_g \circ e_g = e'_g \circ u$ and $m_g = m'_g \circ v_g$.

We show that $(c'_f \circ v_f, c'_g \circ v_g)$ is a push-out of (e_f, e_g). Indeed, $(c'_f \circ v_f) \circ e_f = c'_f \circ (v_f \circ e_f) = c'_f \circ (e'_f \circ u) = (c'_f \circ e'_f) \circ u = (c'_g \circ e'_g) \circ u = c'_g \circ (e'_g \circ u) = c'_g \circ (v_g \circ e_g) = (c'_g \circ v_g) \circ e_g$. Let $P \in \mathrm{obj}_{\mathcal{A}}$ and $a \in \mathrm{mor}_{\mathcal{A}}(S_f, P)$ and $b \in \mathrm{mor}_{\mathcal{A}}(S_g, P)$ be such that $a \circ e_f = b \circ e_g$.

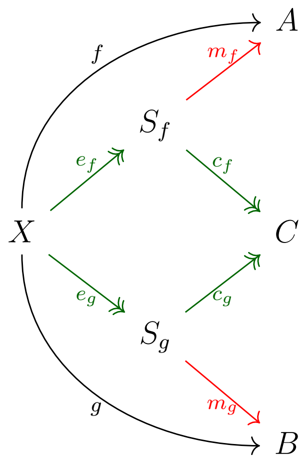


The identities $v_f \circ e_f = e'_f \circ u$ and $v_g \circ e_g = e'_g \circ u$ prove ${v_f}^{-1} \circ e'_f = e_f \circ u^{-1}$ and, likewise, ${v_g}^{-1} \circ e'_g = e_g \circ u^{-1}$. Hence, we find $(a \circ {v_f}^{-1}) \circ e'_f = a \circ ({v_f}^{-1} \circ e'_f) = a \circ (e_f \circ u^{-1}) = (a \circ e_f) \circ u^{-1} = (b \circ e_g) \circ u^{-1} = b \circ (e_g \circ u^{-1}) = b \circ ({v_g}^{-1} \circ e'_g) = (b \circ {v_g}^{-1}) \circ e'_g$. Since (c'_f, c'_g) is a push-out of (e'_f, e'_g) there then exists a unique $h \in \mathrm{mor}_{\mathcal{A}}(C', P)$ such that $h \circ c'_f = a \circ {v_f}^{-1}$ and $h \circ c'_g = b \circ {v_g}^{-1}$. But the latter is equivalent to $h \circ (c'_f \circ v_f) = a$ and $h \circ (c'_g \circ v_g) = b$ holding. Thus we have shown $(c'_f \circ v_f, c'_g \circ v_g)$ to be a push-out of (e_f, e_g).

By the essential uniqueness of push-outs we therefore find a unique $w \in \mathrm{iso}_{\mathcal{A}}(C, C')$ such that $w \circ c_f = c'_f \circ v_f$ and $w \circ c_g = c'_g \circ v_g$. Now the identities $m'_f \circ v_f = m_f$ and $c'_f \circ v_f = w \circ c_f$ and the fact that v_f is invertible together imply that the spans $(m_f, w \circ c_f)$ and (m'_f, c'_f) are equivalent. Likewise, $(m_g, w \circ c_g)$ and (m'_g, c'_g) are equivalent because $m'_g \circ v_g = m_g$ and $c'_g \circ v_g = w \circ c_g$ and because v_g is an isomorphism. In other words, w has the desired property. □

Note that the isomorphism in Lemma 7.2 will generally *not* be unqiue. Nonetheless, at least the following makes sense.

Definition 7.3. If $\mathcal{E}$ has push-outs, then for any $\{A, B\} \subseteq \mathrm{obj}_{\mathcal{A}}$ and any $x \in \mathrm{mor}_{\mathrm{Rel}}(A, B)$ the *core (object class) of* x is $[C]_{\mathcal{A}}$, where C is the push-out object of any push-out (c_f, c_g) of (e_f, e_g) for any $(\mathcal{E}, \mathcal{M})$-factorizations (e_f, m_f) of f and (e_g, m_g) of g for any $(f, g) \in x$.



The core of a relation will obviously be the image object of the factorization. We proceed to define the two wide subcategories making up the factorization system.

Lemma 7.4. *For any* $\{A, B\} \subseteq \mathrm{obj}_{\mathcal{A}}$ *and for any spans* (f, g) *and* (f', g') *of* $\mathcal{A}$ *from* A *to* B *with base objects* X *and* X'*, respectively, the following hold if* (f, g) *and* (f', g') *are equivalent:*

(a) $(f, g) \in \mathrm{mor}_{\mathcal{M}}(X, A) \otimes_{\mathsf{Set}} \mathrm{mor}_{\mathcal{E}}(X, B)$ *if and only if* $(f', g') \in \mathrm{mor}_{\mathcal{M}}(X', A) \otimes_{\mathsf{Set}} \mathrm{mor}_{\mathcal{E}}(X', B)$

(b) $(f, g) \in \mathrm{mor}_{\mathcal{E}}(X, A) \otimes_{\mathsf{Set}} \mathrm{mor}_{\mathcal{M}}(X, B)$ *if and only if* $(f', g') \in \mathrm{mor}_{\mathcal{E}}(X', A) \otimes_{\mathsf{Set}} \mathrm{mor}_{\mathcal{M}}(X', B)$

Proof. (a) By symmetry it suffices to show one implication. Hence, suppose $f \in \mathrm{mor}_{\mathcal{M}}(X, A)$ and $g \in \mathrm{mor}_{\mathcal{E}}(X, B)$. Because (f, g) and (f', g') are equivalent there exists $u \in \mathrm{iso}_{\mathcal{A}}(X', X)$ such that $f \circ u = f'$ and $g \circ u = g'$. By Lemma 4.33 that means both $f' \in \mathrm{mor}_{\mathcal{M}}(X', A)$ and $g' \in \mathrm{mor}_{\mathcal{E}}(X', B)$.

(b) The proof is analogous to the one for Part (a). □

Hence, the following definition makes sense.

Definition 7.5. For any $\{A, B\} \subseteq \mathrm{obj}_{\mathcal{A}}$ any $x \in \mathrm{mor}_{\mathrm{S}}(A, B)$ is called

(a) *pre-core* if there exist $X \in \mathrm{obj}_{\mathcal{A}}$ and $(f, g) \in \mathrm{mor}_{\mathcal{M}}(X, A) \otimes_{\mathsf{Set}} \mathrm{mor}_{\mathcal{E}}(X, B)$ such that $(f, g) \in x$, and

(b) *post-core* if there exist $X \in \mathrm{obj}_{\mathcal{A}}$ and $(f, g) \in \mathrm{mor}_{\mathcal{E}}(X, A) \otimes_{\mathsf{Set}} \mathrm{mor}_{\mathcal{M}}(X, B)$ such that $(f, g) \in x$.

The pre-cores and post-cores are to be the two families of morphisms into which any relation factorizes, ideally. The next result confirms that those kinds of spans are indeed relations.

LEMMA 7.6. *For any* $\{A, B\} \subseteq \mathrm{obj}_{\mathcal{A}}$ *and any* $x \in \mathrm{mor}_{\mathrm{S}}(A, B)$ *if* x *is pre-core or post-core, then* $x \in \mathrm{mor}_{\mathrm{Rel}}(A, B)$.

PROOF. If x is pre-core, there exist $X \in \mathrm{obj}_{\mathcal{A}}$ as well as $f \in \mathrm{mor}_{\mathcal{M}}(X, A)$ and $g \in \mathrm{mor}_{\mathcal{E}}(X, B)$ with $(f, g) \in x$. As $\mathcal{E} \hookrightarrow \mathrm{epi}_{\mathcal{A}}$, Lemma 4.36 guarantees $\mathrm{id}_{\mathcal{A},X} \times_{\mathcal{A}} g \in \mathrm{mor}_{\mathcal{M}}(X, X \otimes_{\mathcal{A}} B)$. And, on the other hand, Lemma 4.34 lets us deduce $f \otimes_{\mathcal{A}} \mathrm{id}_{\mathcal{A},B} \in \mathrm{mor}_{\mathcal{M}}(X \otimes_{\mathcal{A}} B, A \otimes_{\mathcal{A}} B)$. Hence, $f \times_{\mathcal{A}} g = (f \otimes_{\mathcal{A}} \mathrm{id}_{\mathcal{A},B}) \circ_{\mathcal{A}} (\mathrm{id}_{\mathcal{A},X} \times_{\mathcal{A}} g) \in \mathrm{mor}_{\mathcal{M}}(X, A \otimes_{\mathcal{A}} B)$ by Lemma 4.33. Thus, $x = [f, g] \in \mathrm{mor}_{\mathrm{Rel}}(A, B)$. The proof for the case that x is post-core is analogous. □

Moreover, as required for a factorization system the isomorphisms belong to both halfs.

LEMMA 7.7. *For any* $\{A, B\} \subseteq \mathrm{obj}_{\mathrm{Rel}}$ *any* $x \in \mathrm{iso}_{\mathrm{S}}(A, B)$ *is both pre-core and post-core.*

PROOF. The assumption that x is invertible in S implies the existence of an $f \in \mathrm{iso}_{\mathcal{A}}(A, B)$ such that $(\mathrm{id}_{\mathcal{A},A}, f) \in x$. Since $\mathrm{iso}_{\mathcal{A}} \hookrightarrow \mathcal{E}$ and $\mathrm{iso}_{\mathcal{A}} \hookrightarrow \mathcal{M}$ by Lemma 4.32 that already proves the claim. □

Finally, we can verify that the classes of pre-cores and post-cores do form subcategories of the relations. Moreover, very conveniently, relation composition is the same as span composition for any pairs of pre-cores or post-cores.

LEMMA 7.8. *For any* $\{A, B, C\} \subseteq \mathrm{obj}_{\mathcal{A}}$, *any* $x \in \mathrm{mor}_{\mathrm{Rel}}(A, B)$ *and any* $y \in \mathrm{mor}_{\mathrm{Rel}}(B, C)$,

(a) *if* x *and* y *are pre-core, then* $y \circ_{\mathrm{Rel}} x$ *is pre-core, and*

(b) *if* x *and* y *are post-core, then* $y \circ_{\mathrm{Rel}} x$ *is post-core.*

Moreover, in both cases, $x \circ_{\mathrm{Rel}} y = x \circ_{\mathrm{S}} y$.

PROOF. Let $(f, g) \in x$ and $(p, q) \in y$ be arbitrary, let X and Y be the base objects of (f, g) and (p, q), respectively, let (p', g') be any pull-back of (g, p) and let U be the pull-back object of (p', g'). We only show (a) since the proof of (b) is analogous.

If x and y are pre-core, then $f \in \mathrm{mor}_{\mathcal{M}}(X, A)$ and $p \in \mathrm{mor}_{\mathcal{M}}(Y, B)$ while $g \in \mathrm{mor}_{\mathcal{E}}(X, B)$ and $q \in \mathrm{mor}_{\mathcal{E}}(Y, C)$. Because $\mathcal{M}$ is naturally closed under pull-backs as the right part of a factorization system, the fact $p \in \mathrm{mor}_{\mathcal{M}}(Y, B)$ implies $p' \in \mathrm{mor}_{\mathcal{M}}(U, X)$. Likewise, since also $\mathcal{E}$ is pull-back-stable per our assumption, $g' \in \mathrm{mor}_{\mathcal{E}}(U, Y)$ is ensured by $g \in \mathrm{mor}_{\mathcal{E}}(X, B)$. It follows $f \circ_{\mathcal{A}} p' \in \mathrm{mor}_{\mathcal{M}}(U, A)$ and $q \circ_{\mathcal{A}} g' \in \mathrm{mor}_{\mathcal{E}}(U, C)$ by Lemma 4.33. Thus we have shown $y \circ_{\mathrm{S}} x = [f \circ_{\mathcal{A}} p', q \circ_{\mathcal{A}} g']$ to be pre-core. Because $y \circ_{\mathrm{Rel}} x = \Phi_{\mathcal{A},\mathcal{E}}(y \circ_{\mathrm{S}} x)$ the remainder of the claim follows from Lemmata 7.6 and 5.64. □

For later, let us fix symbols for the two parts of the to-be factorization system.

DEFINITION 7.9. Let $\mathfrak{E}$ denote the wide subcategory of Rel given by all pre-core relations and let $\mathfrak{M}$ be the one of all post-core relations.

7.2. Pseudo-exact-Mal'cev properties. Actually, to be precise, Knop's semisimplicity result, [Kno07, Theorem 6.1], is formulated not for arbitrary regular categories but only for such which are exact and Mal'cev. Recalling the individual definitions of the exactness and Mal'cev properties would lead us far into the theory of equivalences on regular categories, which is actually not required in the following. The interested reader may refer to [CKP93] for those details. On its own, each of those two definitions does not seem to make too much sense for the general factorization systems $(\mathcal{E}, \mathcal{M})$ considered in this chapter. However, technically, Knop himself does not use the definitions individually, either. Rather, he only works with a reformulation of the combination of *both* properties which he gives in [Kno07, Proposition 1.2]. And for this reformulation a natural generalization to our present context comes to mind readily, as we will see (in Definition 7.14).

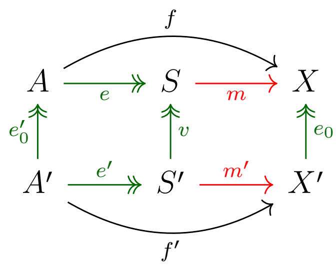
The truth is, though, that this adapted property has much less of an impact than its analog has for regular categories. Just supposing that $\mathcal{A}$ is regular and that the abstact version of [Kno07, Proposition 1.2] is satisfied does not produce all the benefical effects Knop requires to prove [Kno07, Lemma 5.2]. In fact, even adding the assumption that $(\mathcal{E}, \mathcal{M})$ is proper does not achieve this. For that reason, in the following, not just one "pseudo-exact-Mal'cev property" but three of them will be defined and examined for their implications (see Definitions 7.10, 7.14 and 7.21).

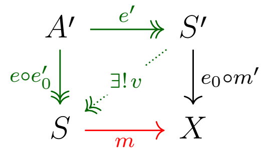
The second pseudo-exact-Mal'cev property will guarantee the existence of $(\mathfrak{E}, \mathfrak{M})$-(factorizations (see Lemma 7.16) but does not ensure $\mathfrak{E} \pitchfork \mathfrak{M}$, let alone $\mathfrak{E} \perp \mathfrak{M}$. However, adding the first pseudo-exact-Mal'cev property will allow us to infer that at least $\mathfrak{E} \pitchfork \mathfrak{M}$ (see Lemma 7.18). Even then, though, $\mathfrak{E}$ and $\mathfrak{M}$ need not be orthogonal to each other (see Remark 7.20 for a counterexample). In other words, the combination of the first two pseudo-exact-Mal'cev properties yields only a weakened version of [Kno07, Lemma 5.2 (i)] (in Lemma 7.17). It does not seem to imply a version of [Kno07, Lemma 5.2 (ii)], though. That is where the third pseudo-exact-Mal'cev property comes it. Making this third assumption will let us conclude a weak analog of [Kno07, Lemma 5.2 (ii)] (in Lemma 7.24).

7.2.1. *First pseudo-exact-Mal'cev property.* The first property an input $(\mathcal{A}, \mathcal{E})$ may have that give it properties similar to the ones of exact Mal'cev regular categories is the following.

Definition 7.10. We say that $\mathcal{A}$ has the *first* $(\mathcal{E}, \mathcal{M})$*-pseudo-exact-Mal'cev property* if for any $\{A, X, X'\} \subseteq \mathrm{obj}_{\mathcal{A}}$, any $e_0 \in \mathrm{mor}_{\mathcal{E}}(X', X)$, any $f \in \mathrm{mor}_{\mathcal{A}}(A, X)$, any pull-back (f', e_0') of (e_0, f) with object A', any $(\mathcal{E}, \mathcal{M})$-factorization of (e, m) of f with object S and any $(\mathcal{E}, \mathcal{M})$-factorization (e', m') of f' with object S', if $v \in \mathrm{mor}_{\mathcal{A}}(S', S)$ is the unique diagonal with $e_0 \circ m' = m \circ v$ and $v \circ e' = e \circ e_0'$, then

(m', v) is a pull-back of (e_0, m).





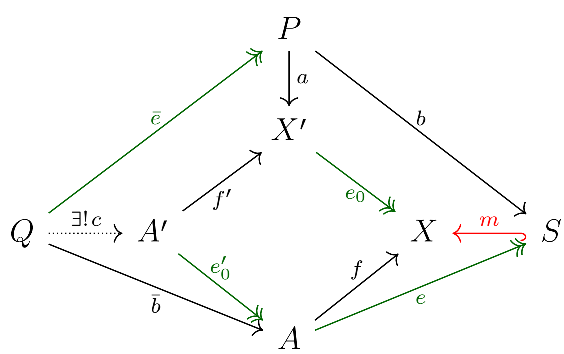
REMARK 7.11. Note that in Definition 7.10 the diagonal v is necessarily a morphism of $\mathcal{E}$ by Lemma 4.26 and that, if (m', v) is a pull-back of (e_0, m), then (e'_0, e') is also a pull-back of (e, v) by Lemma 4.3.

The next lemma shows that the first pseudo-exact-Mal'cev property captures an aspect of what it means for $(\mathcal{E}, \mathcal{M})$ to be proper.

LEMMA 7.12. *The category $\mathcal{A}$ has the first $(\mathcal{E}, \mathcal{M})$-pseudo-exact-Mal'cev property if $\mathcal{M} \hookrightarrow \mathrm{mon}_{\mathcal{A}}$.*

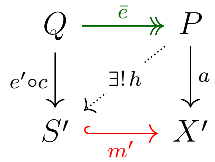
PROOF. Let A, A', X, X', S and S' as well as f, f', e_0, e'_0, e, e', m, m' and v all be as in Definition 7.10. In order to show that (m', v) is a pull-back of (e_0, m) we let $P \in \mathrm{obj}_{\mathcal{A}}$ and $a \in \mathrm{mor}_{\mathcal{A}}(P, X')$ and $b \in \mathrm{mor}_{\mathcal{A}}(P, S)$ be arbitrary with $e_0 \circ a = m \circ b$ and we prove the existence of a unique $h \in \mathrm{mor}_{\mathcal{A}}(P, S')$ with $a = m' \circ h$ and $b = v \circ h$.

Step 1: Construction of auxiliary morphism c. Let $(\bar{e}, \bar{b})$ be any pull-back of (b, e) with object Q. Because $e_0 \circ (a \circ \bar{e}) = (e_0 \circ a) \circ \bar{e} = (m \circ b) \circ \bar{e} = m \circ (b \circ \bar{e}) = m \circ (e \circ \bar{b}) = (m \circ e) \circ \bar{b} = f \circ \bar{b}$ the assumption that (e'_0, f') is a pull-back of (f, e_0) implies that there is some $c \in \mathrm{mor}_{\mathcal{A}}(Q, A')$ with $f' \circ c = a \circ \bar{e}$ and $e'_0 \circ c = \bar{b}$ and that c is unique with these properties.



Step 2: Construction of h from c. Since $\bar{e}$ is the pull-back of e along b, since $e \in \mathrm{mor}_{\mathcal{E}}(A, S)$ and since $\mathcal{E}$ is pull-back-stable we can infer $\bar{e} \in \mathrm{mor}_{\mathcal{E}}(Q, P)$ and thus, in particular, $\bar{e} \perp m'$ as $m' \in \mathrm{mor}_{\mathcal{M}}(S', X')$. Because $m' \circ (e' \circ c) = (m' \circ e') \circ c = f' \circ c = a \circ \bar{e}$ we hence find a $h \in \mathrm{mor}_{\mathcal{A}}(Q, A')$ with $e' \circ c = h \circ \bar{e}$ and $a = m' \circ h$ and

which is in fact the only morphism satisfying these equations.

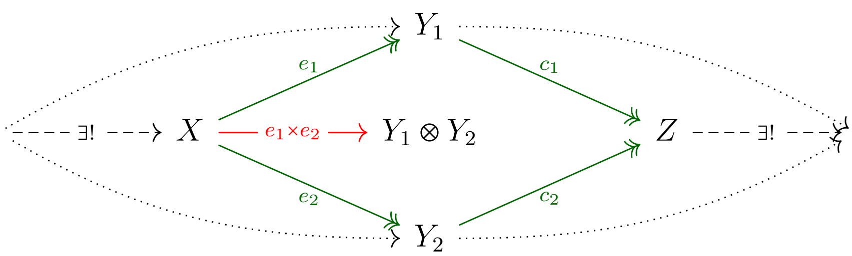


Step 3: Verifying that h is as desired. While we have thus already seen that h has the first asserted property, $a = m' \circ h$, we still need to show that it also has the second one, $b = v \circ h$. This is where the assumption $\mathcal{M} \hookrightarrow \mathrm{mon}_{\mathcal{A}}$ comes into play: Because $m \in \mathrm{mor}_{\mathcal{M}}(S, X)$ is a monomorphism, proving $b = v \circ h$ is the same as showing $m \circ b = m \circ (v \circ h)$. And, indeed, $m \circ b = e_0 \circ a = e_0 \circ (m' \circ h) = (e_0 \circ m') \circ h = (m \circ v) \circ h = m \circ (v \circ h)$. Moreover, h is unique: Let $k \in \mathrm{mor}_{\mathcal{A}}(P, S')$ be any morphism with $a = m' \circ k$ and $b = v \circ k$. Because the square from Step 2 commutes, $m' \circ (k \circ \bar{e}) = (m' \circ k) \circ \bar{e} = a \circ \bar{e} = m' \circ (e' \circ c)$. Because $m' \in \mathrm{mor}_{\mathcal{M}}(S', X')$ is a monomorphism by $\mathcal{M} \hookrightarrow \mathrm{mon}_{\mathcal{A}}$, that allows us to infer $k \circ \bar{e} = e' \circ c$. But $m' \circ k = a$ and $k \circ \bar{e} = e' \circ c$ demand $k = h$ by the universality of h. That concludes the proof. □

REMARK 7.13. In particular, Lemma 7.12 holds if $\mathcal{A}$ is regular and $\mathcal{E}$ and $\mathcal{M}$ are given by the regular epimorphisms and monomorphisms of $\mathcal{A}$, respectively. Crucially, though, there exist $\mathcal{A}$ which have the first $(\mathcal{E}, \mathcal{M})$-pseudo-exact-Mal'cev property such that $\mathcal{M} \not\hookrightarrow \mathrm{mon}_{\mathcal{A}}$, for example, $\mathcal{A} = \mathsf{fGr}^{\mathrm{op}}$ with $\mathcal{E} = \mathrm{qpr}_{\mathsf{fGr}^{\mathrm{op}}}$.

7.2.2. *Second pseudo-exact-Mal'cev property.* The next property is the announced analog of [Kno07, Proposition A1.2].

DEFINITION 7.14. $\mathcal{A}$ is said to have the *second* $(\mathcal{E}, \mathcal{M})$*-pseudo-exact-Mal'cev property* if $\mathcal{E}$ has push-outs and if for any $\{X, Y_1, Y_2, Z\} \subseteq \mathrm{obj}_{\mathcal{A}}$ and any $e_i \in \mathrm{mor}_{\mathcal{E}}(X, Y_i)$ and $c_i \in \mathrm{mor}_{\mathcal{E}}(Y_i, Z)$ for each $i \in \{1, 2\}$ such that $c_1 \circ e_1 = c_2 \circ e_2$ it holds that (e_1, e_2) is a pull-back of (c_1, c_2) in $\mathcal{A}$ if and only if both (c_1, c_2) is a push-out of (e_1, e_2) and $e_1 \times e_2 \in \mathrm{mor}_{\mathcal{M}}(X, Y_1 \otimes Y_2)$.



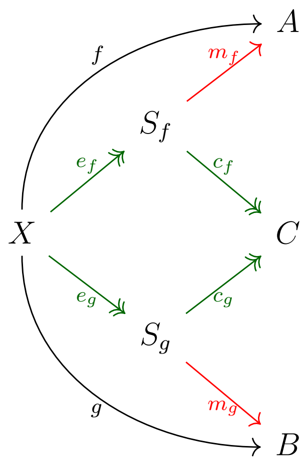
PROPOSITION 7.15. *$\mathcal{A}$ has the second $(\mathcal{E}, \mathcal{M})$-pseudo-exact-Mal'cev property if $\mathcal{A}$ is regular exact Mal'cev and if $\mathcal{E}$ and $\mathcal{M}$ are given by the regular epimorphisms and the monomorphisms of $\mathcal{A}$, respectively.*

PROOF. That is precisely [Kno07, Proposition A.1.2]. □

The second pseudo-exact-Mal'cev property on its own ensures the existence of core factorizations.

Lemma 7.16. *If $\mathcal{A}$ has the second $(\mathcal{E},\mathcal{M})$-pseudo-exact-Mal'cev property, then for any $\{A,B\} \subseteq \mathrm{obj}_{\mathrm{Rel}}$ and any $x \in \mathrm{mor}_{\mathrm{Rel}}(A,B)$ there exist $C \in \mathrm{obj}_{\mathrm{Rel}}$ and (y,z) such that $y \in \mathrm{mor}_{\mathfrak{E}}(A,C)$ and $z \in \mathrm{mor}_{\mathfrak{M}}(C,B)$ and $x = z \circ_{\mathrm{Rel}} y = z \circ_{\mathrm{S}} y$.*

Proof. Let $(f,g) \in x$ be arbitary, let X be the base object of (f,g), let (e_f,m_f) be any $(\mathcal{E},\mathcal{M})$-factorization of f, let (e_g,m_g) be any $(\mathcal{E},\mathcal{M})$-factorization of g and let S_f and S_g be the image objects of (e_f,m_f) and (e_g,m_g), respectively. Since $\mathcal{A}$ has the second $(\mathcal{E},\mathcal{M})$-pseudo-exact-Mal'cev property there exists a push-out (c_f,c_g) of (e_f,e_g). Call its push-out object C. Because $\mathcal{E}$ is closed under push-outs we can infer $c_f \in \mathrm{mor}_{\mathcal{E}}(S_f,C)$ and $c_g \in \mathrm{mor}_{\mathcal{E}}(S_g,C)$ from $e_g \in \mathrm{mor}_{\mathcal{E}}(X,S_g)$ and $e_f \in \mathrm{mor}_{\mathcal{E}}(X,S_f)$. Hence, $y := [m_f,c_f] \in \mathrm{mor}_{\mathrm{S}}(A,C)$ is pre-core and $z := [c_g,m_g] \in \mathrm{mor}_{\mathrm{S}}(C,B)$ is post-core. Lemma 7.6 guarantees that y and z are indeed morphisms of Rel. It remains to prove $x = z \circ_{\mathrm{Rel}} y = z \circ_{\mathrm{S}} y$.



Because $\mathcal{A}$ has the second $(\mathcal{E},\mathcal{M})$-pseudo-exact-Mal'cev property the assumption that (c_f,c_g) is a push-out of (e_f,e_g) implies that, in turn, (e_f,e_g) is a pull-back of (c_f,c_g). In consequence, $(f,g) = (m_f \circ e_f, m_g \circ e_g) \in z \circ_{\mathrm{S}} y$ by the definition of span class composition. Because $x = [f,g] \in \mathrm{mor}_{\mathrm{Rel}}(A,B)$ by assumption and because $z \circ_{\mathrm{Rel}} y = \Phi_{\mathcal{A},\mathcal{E}}(z \circ_{\mathrm{S}} y)$ Lemma 5.64 thus proves the claim. □

The ensuing result generalizes [Kno07, Lemma 5.2 (i)], saying that the factorization from the preceding lemma is not just one in Rel but also one in $\mathcal{T}^0$.

Lemma 7.17. *If $\mathcal{A}$ has the second $(\mathcal{E},\mathcal{M})$-pseudo-exact-Mal'cev property, then for any $\{A,B\} \subseteq \mathrm{obj}_{\mathcal{T}^0}$ and $x \in \mathrm{mor}_{\mathrm{Rel}}(A,B)$ there exist $C \in \mathrm{obj}_{\mathcal{T}^0}$ and (y,z) such that $y \in \mathrm{mor}_{\mathfrak{E}}(A,C)$ and $z \in \mathrm{mor}_{\mathfrak{M}}(C,B)$ and such that, if we interpret x, y and z as morphisms from I_{Set} to $\mathrm{mor}_{\mathrm{Rel}}(A,B)$, $\mathrm{mor}_{\mathrm{Rel}}(A,C)$ and $\mathrm{mor}_{\mathrm{Rel}}(C,B)$, respectively, then*

$$G_{\mathsf{M}}(x) \circ_{\mathsf{M}} (G_{\mathsf{M}})_I \\ = \circ_{\mathcal{T}^0,A,C,B} \circ_{\mathsf{M}} ((G_{\mathsf{M}}(z) \circ_{\mathsf{M}} (G_{\mathsf{M}})_I) \otimes_{\mathsf{M}} (G_{\mathsf{M}}(y) \circ_{\mathsf{M}} (G_{\mathsf{M}})_I)) \circ_{\mathsf{M}} {\lambda_{\mathsf{M},I_{\mathsf{M}}}}^{-1_{\mathsf{M}}}.$$

PROOF. By Lemma 7.16 there exist y and z with all the asserted properties, except that the identity $x = z \circ_{\mathrm{Rel}} y = z \circ_{\mathrm{S}} y$ is satisfied instead of the one we claim. Thus, we only need to prove the missing identity.

Step 1: Resuming the definitions from 5.103, note that because of

$$c_{A,C,B}(z,y) = (\Delta_{\mathcal{A},\mathcal{E},\delta}(z \circ_{\mathrm{S}} y), \Phi_{\mathcal{A},\mathcal{E}}(z \circ_{\mathrm{S}} y)) = (\Delta_{\mathcal{A},\mathcal{E},\delta}(x), \Phi_{\mathcal{A},\mathcal{E}}(x))$$

and because of $x \in \mathrm{mor}_{\mathrm{Rel}}(A,B)$ Lemmata 5.64 and 5.101 imply

$$c_{A,C,B} \circ_{\mathsf{Set}} (z \otimes_{\mathsf{Set}} y) = I_{(U_{\mathsf{S}})_{\triangleright}(R)} \otimes_{\mathsf{Set}} x.$$

Step 2: By definition the right hand side of the claimed identity is equal to

$$\begin{aligned}\lambda_{\mathsf{M},G_{\mathsf{M}}(\mathrm{mor}_{\mathrm{Rel}}(A,B))} &\circ_{\mathsf{M}} (\mathrm{cu}_{\mathsf{M},I_{\mathsf{M}}} \otimes_{\mathsf{M}} \mathrm{id}_{\mathsf{M},G_{\mathsf{M}}(\mathrm{mor}_{\mathrm{Rel}}(A,B))}) \circ_{\mathsf{M}} {(G_{\mathsf{M}})_{\otimes,U_{\mathsf{M}}(I_{\mathsf{M}}),\mathrm{mor}_{\mathrm{Rel}}(A,B)}}^{-1_{\mathsf{M}}}\\ &\circ_{\mathsf{M}} G_{\mathsf{M}}(c_{A,C,B}) \circ_{\mathsf{M}} (G_{\mathsf{M}})_{\otimes,\mathrm{mor}_{\mathrm{Rel}}(C,B),\mathrm{mor}_{\mathrm{Rel}}(A,C)} \circ_{\mathsf{M}} (G_{\mathsf{M}}(z) \otimes G_{\mathsf{M}}(y))\\ &\circ_{\mathsf{M}} ((G_{\mathsf{M}})_I \otimes_{\mathsf{M}} (G_{\mathsf{M}})_I) \circ_{\mathsf{M}} {\lambda_{\mathsf{M},I_{\mathsf{M}}}}^{-1_{\mathsf{M}}}.\end{aligned}$$

Step 3: Because

$$(G_{\mathsf{M}})_{\otimes,\mathrm{mor}_{\mathrm{Rel}}(C,B),\mathrm{mor}_{\mathrm{Rel}}(A,C)} \circ_{\mathsf{M}} (G_{\mathsf{M}}(z) \otimes G_{\mathsf{M}}(y)) = G_{\mathsf{M}}(z \otimes_{\mathsf{Set}} y) \circ_{\mathsf{M}} (G_{\mathsf{M}})_{\otimes,I_{\mathsf{Set}},I_{\mathsf{Set}}}$$

and because $G_{\mathsf{M}}(c_{A,C,B} \circ_{\mathsf{Set}} (z \otimes_{\mathsf{Set}} y)) = G_{\mathsf{M}}(I_{(U_{\mathsf{S}})_{\triangleright}(R)} \otimes_{\mathsf{Set}} x)$ this is the same as

$$\begin{aligned}\lambda_{\mathsf{M},G_{\mathsf{M}}(\mathrm{mor}_{\mathrm{Rel}}(A,B))} &\circ_{\mathsf{M}} (\mathrm{cu}_{\mathsf{M},I_{\mathsf{M}}} \otimes_{\mathsf{M}} \mathrm{id}_{\mathsf{M},G_{\mathsf{M}}(\mathrm{mor}_{\mathrm{Rel}}(A,B))}) \circ_{\mathsf{M}} {(G_{\mathsf{M}})_{\otimes,U_{\mathsf{M}}(I_{\mathsf{M}}),\mathrm{mor}_{\mathrm{Rel}}(A,B)}}^{-1_{\mathsf{M}}}\\ &\circ_{\mathsf{M}} G_{\mathsf{M}}(I_{(U_{\mathsf{S}})_{\triangleright}(R)} \otimes_{\mathsf{Set}} x) \circ_{\mathsf{M}} (G_{\mathsf{M}})_{\otimes,I_{\mathsf{Set}},I_{\mathsf{Set}}} \circ_{\mathsf{M}} ((G_{\mathsf{M}})_I \otimes_{\mathsf{M}} (G_{\mathsf{M}})_I) \circ_{\mathsf{M}} {\lambda_{\mathsf{M},I_{\mathsf{M}}}}^{-1_{\mathsf{M}}}.\end{aligned}$$

Step 4: Now, the fact that

$$\begin{aligned}(G_{\mathsf{M}})_{\otimes,U_{\mathsf{M}}(I_{\mathsf{M}}),\mathrm{mor}_{\mathrm{Rel}}(A,B)} &\circ_{\mathsf{M}} (G_{\mathsf{M}}(I_{(U_{\mathsf{S}})_{\triangleright}(R)}) \otimes_{\mathsf{M}} G_{\mathsf{M}}(x))\\ &= G_{\mathsf{M}}(I_{(U_{\mathsf{S}})_{\triangleright}(R)} \otimes_{\mathsf{Set}} x) \circ_{\mathsf{M}} (G_{\mathsf{M}})_{\otimes,I_{\mathsf{Set}},I_{\mathsf{Set}}}\end{aligned}$$

transforms this into the morphism

$$\begin{aligned}&\lambda_{\mathsf{M},G_{\mathsf{M}}(\mathrm{mor}_{\mathrm{Rel}}(A,B))}\\ &\quad\circ_{\mathsf{M}} ((\mathrm{cu}_{\mathsf{M},I_{\mathsf{M}}} \circ_{\mathsf{M}} G_{\mathsf{M}}(I_{(U_{\mathsf{S}})_{\triangleright}(R)}) \circ_{\mathsf{M}} (G_{\mathsf{M}})_I) \otimes_{\mathsf{M}} (G_{\mathsf{M}}(x) \circ_{\mathsf{M}} (G_{\mathsf{M}})_I)) \circ_{\mathsf{M}} {\lambda_{\mathsf{M},I_{\mathsf{M}}}}^{-1_{\mathsf{M}}}.\end{aligned}$$

Step 5: Since cu_{M} is a monoidal transformation from $(G_{\mathsf{M}} \circ_{\mathsf{Cat}} U_{\mathsf{M}})$ to $\mathrm{id}_{\mathsf{Cat},\mathsf{M}}$,

$$\mathrm{cu}_{\mathsf{M},I_{\mathsf{M}}} \circ_{\mathsf{M}} G_{\mathsf{M}}((U_{\mathsf{M}})_I) \circ_{\mathsf{M}} (G_{\mathsf{M}})_I = \mathrm{id}_{\mathsf{M},I_{\mathsf{M}}}$$

and since

$$I_{(U_{\mathsf{S}})_{\triangleright}(R)} = U_{\mathsf{S}}(I_R) \circ_{\mathsf{Set}} (U_{\mathsf{S}})_I = U_{\mathsf{S}}((A_{\mathsf{M}})_I) \circ_{\mathsf{Set}} (U_{\mathsf{S}})_I = (U_{\mathsf{S}} \circ_{\mathsf{Cat}} A_{\mathsf{M}})_I = (U_{\mathsf{M}})_I$$

we have thus shown the right hand side of the claim to be equal to

$$\lambda_{\mathsf{M},G_{\mathsf{M}}(\mathrm{mor}_{\mathrm{Rel}}(A,B))} \circ_{\mathsf{M}} (\mathrm{id}_{\mathsf{M},I_{\mathsf{M}}} \otimes_{\mathsf{M}} (G_{\mathsf{M}}(x) \circ_{\mathsf{M}} (G_{\mathsf{M}})_I)) \circ_{\mathsf{M}} {\lambda_{\mathsf{M},I_{\mathsf{M}}}}^{-1_{\mathsf{M}}}.$$

From this the claim now follows immediately. □

Combining the first and second pseudo-exact-Mal'cev properties gives pre-cores and post-cores the lifting property against each other.

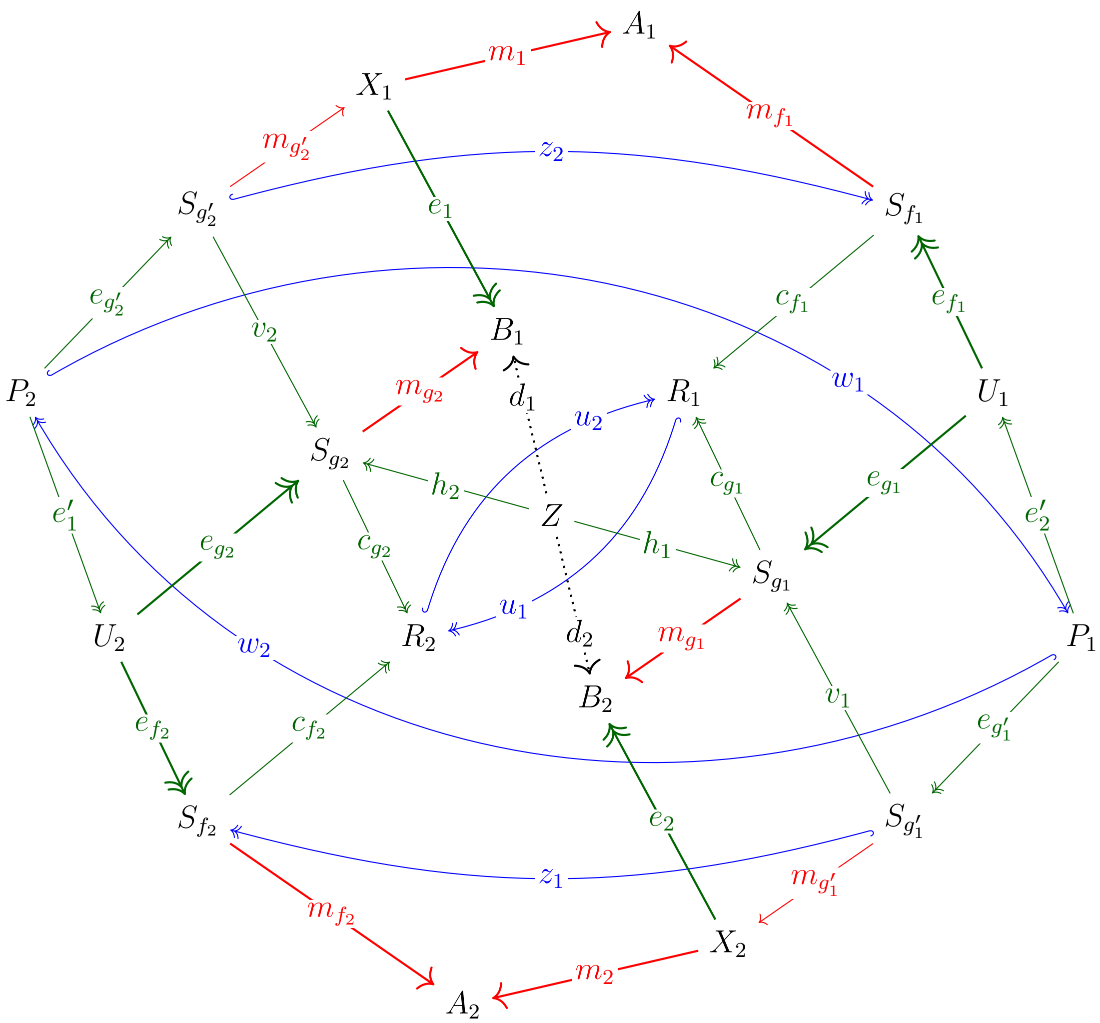
LEMMA 7.18. *If $\mathcal{A}$ has the first and second $(\mathcal{E},\mathcal{M})$-pseudo-exact-Mal'cev properties, then $\mathfrak{E} \pitchfork \mathfrak{M}$ in* Rel.

Proof. We let $\{A_1, A_2, B_1, B_2\} \subseteq \mathrm{obj}_{\mathrm{Rel}}$ and $[m_1, e_1] \in \mathrm{mor}_{\mathfrak{E}}(A_1, B_1)$ and $[e_2, m_2] \in \mathrm{mor}_{\mathfrak{M}}(B_2, A_2)$ be arbitrary and prove $[m_1, e_1] \pitchfork_{\mathrm{Rel}} [e_2, m_2]$. Hence, let $[f_1, g_1] \in \mathrm{mor}_{\mathrm{Rel}}(A_1, B_2)$ and $[g_2, f_2] \in \mathrm{mor}_{\mathrm{Rel}}(B_1, A_2)$ be arbitrary with $[g_2, f_2] \circ_{\mathrm{Rel}} [m_1, e_1] = [e_2, m_2] \circ_{\mathrm{Rel}} [f_1, g_1]$. We show that there exists $[d_1, d_2] \in \mathrm{mor}_{\mathrm{Rel}}(B_1, B_2)$ such that $[f_1, g_1] = [d_1, d_2] \circ_{\mathrm{Rel}} [m_1, e_1]$ and $[e_2, m_2] \circ_{\mathrm{Rel}} [d_1, d_2] = [g_2, f_2]$.

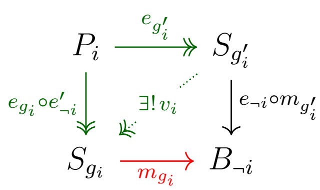
$$\begin{array}{ccc} A_1 & \xrightarrow{[m_1,e_1]} & B_1 \\ {\scriptstyle [f_1,g_1]}\big\downarrow & \exists!\,[d_1,d_2] \swarrow & \big\downarrow{\scriptstyle [g_2,f_2]} \\ B_2 & \xrightarrow[{[e_2,m_2]}]{} & A_2 \end{array}$$

The proof is divided into two steps. In the first, we construct $[d_1, d_2]$. The second is dedicated to showing that $[d_1, d_2]$ has the desired properties. For brevity, $\neg i := 3 - i$ for each $i \in \{1, 2\}$.

Step 1: Constructing the relation $[d_1, d_2]$. The construction of $[d_1, d_2]$ will itself be divided into several steps. First, some definitions used throughout the entire proof. Let X_1 denote the base of (m_1, e_1) and X_2 that of (e_2, m_2). Likewise, let U_1 and U_2 be the base objects of (f_1, g_1) and (g_2, f_2), respectively. Moreover, for each $i \in \{1, 2\}$ let $(g'_i, e'_{\neg i})$ be any pull-back of $(e_{\neg i}, g_i)$ and let P_i be its pull-back object. In addition, for each $i \in \{1, 2\}$ let (e_{f_i}, m_{f_i}) be any $(\mathcal{E}, \mathcal{M})$-factorization of f_i, let (e_{g_i}, m_{g_i}) be any of g_i and $(e_{g'_i}, m_{g'_i})$ any of g'_i. Furthermore, for each $i \in \{1, 2\}$ let S_{f_i}, S_{g_i} and $S_{g'_i}$ be the image objects of (e_{f_i}, m_{f_i}), (e_{g_i}, m_{g_i}) and $(e_{g'_i}, m_{g'_i})$, respectively. Because $\mathcal{A}$ has the second $(\mathcal{E}, \mathcal{M})$-pseudo-exact-Mal'cev property, for each $i \in \{1, 2\}$ we find a push-out (c_{g_i}, c_{f_i}) of the $\mathcal{E}$-span (e_{g_i}, e_{f_i}), whose push-out object we then denote by R_i.



Step 1.1: Construction of v_1 and v_2. For each $i \in \{1,2\}$ since $(e_{\neg i} \circ m_{g'_i}) \circ e_{g'_i} = e_{\neg i} \circ (m_{g'_i} \circ e_{g'_i}) = e_{\neg i} \circ g'_i = g_i \circ e'_{\neg i} = (m_{g_i} \circ e_{g_i}) \circ e'_{\neg i} = m_{g_i} \circ (e_{g_i} \circ e'_{\neg i})$ and

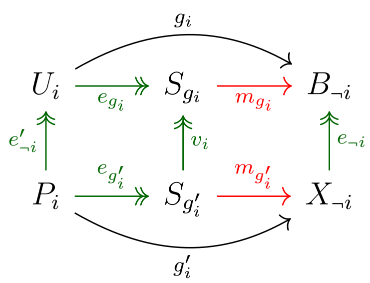


since $e_{g'_i} \in \mathrm{mor}_{\mathcal{E}}(P_i, S_{g'_i})$ and $m_{g_i} \in \mathrm{mor}_{\mathcal{M}}(S_{g_i}, B_{\neg i})$ there exists a unique $v_i \in \mathrm{mor}_{\mathcal{A}}(S_{g'_i}, S_{g_i})$ such that

$$v_i \circ e_{g'_i} = e_{g_i} \circ e'_{\neg i} \quad \text{and} \quad m_{g_i} \circ v_i = e_{\neg i} \circ m_{g'_i}.$$

Moreover, because, on the one hand, $e_{g_i} \in \mathrm{mor}_{\mathcal{E}}(U_i, S_{g_i})$ and $e'_{\neg i} \in \mathrm{mor}_{\mathcal{E}}(P_i, U_i)$, and thus $e_{g_i} \circ e'_{\neg i} \in \mathrm{mor}_{\mathcal{E}}(P_i, S_{g_i})$ by Lemma 4.33, and, on the other hand, $e_{g'_i} \in \mathrm{mor}_{\mathcal{E}}(P_i, S_{g'_i})$, the first equation implies $v_i \in \mathrm{mor}_{\mathcal{E}}(S_{g'_i}, S_{g_i})$ by Lemma 4.26.

Step 1.2: Construction of w_1 and w_2. Because $\mathcal{A}$ has the first $(\mathcal{E},\mathcal{M})$-pseudo-exact-Mal'cev property the assumptions that $(e'_{\neg i}, g'_i)$ is a pull-back of $(g_i, e_{\neg i})$, that $e_{\neg i} \in \mathrm{mor}_{\mathcal{E}}(X_{\neg i}, B_{\neg i})$, that (e_{g_i}, m_{g_i}) is a factorization of g_i, that $(e_{g'_i}, m_{g'_i})$ is one of g'_i and that v_i is the unique diagonal with $v_i \circ e_{g'_i} = e_{g_i} \circ e'_{\neg i}$ and $m_{g_i} \circ v_i = e_{\neg i} \circ m_{g'_i}$ allows us to conclude that $(v_i, m_{g'_i})$ is a pull-back of $(m_{g_i}, e_{\neg i})$ for each $i \in \{1,2\}$.

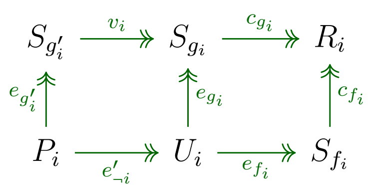


Now that we know that both $(e'_{\neg i}, g'_i)$ is a pull-back of $(g_i, e_{\neg i})$ and $(v_i, m_{g'_i})$ is a pull-back of $(m_{g_i}, e_{\neg i})$ we can infer by Lemma 4.3 that also $(e'_{\neg i}, e_{g'_i})$ is a pull-back of (e_{g_i}, v_i) for each $i \in \{1,2\}$.

As $[f_1, g_1] \in \mathrm{mor}_{\mathrm{Rel}}(A_1, B_2)$ and $[g_2, f_2] \in \mathrm{mor}_{\mathrm{Rel}}(B_1, A_2)$, which by Lemma 5.88 is to say $[g_i, f_i] \in \mathrm{mor}_{\mathrm{Rel}}(B_i, A_i)$ for each $i \in \{1,2\}$, we are assured that $g_i \times f_i \in \mathrm{mor}_{\mathcal{M}}(U_i, B_i \otimes A_i)$ for each $i \in \{1,2\}$. For that reason, for each $i \in \{1,2\}$, once we write $g_i \times f_i = (m_{g_i} \circ e_{g_i}) \times (m_{f_i} \circ e_{f_i}) = (m_{g_i} \otimes m_{f_i}) \circ (e_{g_i} \times e_{f_i})$ we can recognize $e_{g_i} \times e_{f_i} \in \mathrm{mor}_{\mathcal{M}}(U_i, S_{g_i} \otimes S_{f_i})$ by Lemma 4.26 because also $m_{g_i} \otimes m_{f_i} \in \mathrm{mor}_{\mathcal{M}}(S_{g_i} \otimes S_{f_i}, B_i \otimes A_i)$ by Lemma 4.34.

Since $\mathcal{A}$ has the second $(\mathcal{E},\mathcal{M})$-pseudo-exact-Mal'cev property the assumption that (c_{g_i}, c_{f_i}) is a push-out of (e_{g_i}, e_{f_i}) and since all the morphisms e_{g_i}, e_{f_i}, c_{g_i} and c_{f_i} belong to $\mathcal{E}$ our conclusion $e_{g_i} \times e_{f_i} \in \mathrm{mor}_{\mathcal{M}}(U_i, S_{g_i} \otimes S_{f_i})$ implies that (e_{g_i}, e_{f_i}) is a pull-back of (c_{g_i}, c_{f_i}) for each $i \in \{1,2\}$.

For each $i \in \{1,2\}$ because $(c_{g_i} \circ v_i) \circ e_{g'_i} = c_{g_i} \circ (v_i \circ e_{g'_i}) = c_{g_i} \circ (e_{g_i} \circ e'_{\neg i}) = (c_{g_i} \circ e_{g_i}) \circ e'_{\neg i} = (c_{f_i} \circ e_{f_i}) \circ e'_{\neg i} = c_{f_i} \circ (e_{f_i} \circ e'_{\neg i})$, because $(e_{g'_i}, e'_{\neg i})$ is a pull-back of (v_i, e_{g_i}) and (e_{g_i}, e_{f_i}) is one of (c_{g_i}, c_{f_i}) another application of Lemma 4.3 now shows that $(e_{g'_i}, e_{f_i} \circ e'_{\neg i})$ is a pull-back of $(c_{g_i} \circ v_i, e_{g'_i})$.



Since $\mathcal{A}$ has the second $(\mathcal{E},\mathcal{M})$-pseudo-exact-Mal'cev property, since the four morphisms $e_{g_i'}$, c_{f_i}, $c_{g_i}\circ v_i$ and $e_{f_i}\circ e'_{\neg i}$ are in $\mathcal{E}$, in the case of the latter two by Lemma 4.33, and since $(e_{g_i'}, e_{f_i}\circ e'_{\neg i})$ is a pull-back of $(c_{g_i}\circ v_i, c_{f_i})$ we know in particular that $e_{g_i'}\times(e_{f_i}\circ e'_{\neg i})\in \mathrm{mor}_{\mathcal{M}}(P_i, S_{g_i'}\otimes S_{f_i})$ for each $i\in\{1,2\}$.

Because, naturally $m_{\neg i}\otimes m_{f_i}\in \mathrm{mor}_{\mathcal{M}}(X_{\neg i}\otimes S_{f_i}, A_{\neg i}\otimes A_i)$ by Lemma 4.34, this consequence proves $(m_{\neg i}\circ g_i')\times(f_i\circ e'_{\neg i}) = (m_{\neg i}\circ(m_{g_i'}\circ e_{g_i'}))\times((m_{f_i}\circ e_{f_i})\circ e'_{\neg i}) = ((m_{\neg i}\circ m_{g_i'})\otimes m_{f_i})\circ(e_{g_i'}\times(e_{f_i}\circ e'_{\neg i}))\in \mathrm{mor}_{\mathcal{M}}(P_i, A_{\neg i}\otimes A_i)$ by Lemma 4.33 for each $i\in\{1,2\}$. Thus, we have shown $[m_{\neg i}\circ g_i', f_i\circ e'_{\neg i}]\in \mathrm{mor}_{\mathrm{Rel}}(A_{\neg i}, A_i)$ for both $i\in\{1,2\}$.

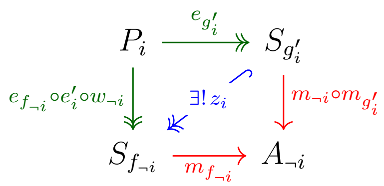
By definition of the composition of span classes this means $[e_2,m_2]\circ_{\mathrm{S}}[f_1,g_1]\in \mathrm{mor}_{\mathrm{Rel}}(A_1,A_2)$ and $[g_2,f_2]\circ_{\mathrm{S}}[m_1,e_1]\in \mathrm{mor}_{\mathrm{Rel}}(A_1,A_2)$, in the latter case via an application of 5.87. It follows $[e_2,m_2]\circ_{\mathrm{Rel}}[f_1,g_1] = \Phi_{\mathcal{A},\mathcal{E}}([e_2,m_2]\circ_{\mathrm{S}}[f_1,g_1]) = [e_2,m_2]\circ_{\mathrm{S}}[f_1,g_1]$ and, likewise, $[g_2,f_2]\circ_{\mathrm{Rel}}[m_1,e_1] = [g_2,f_2]\circ_{\mathrm{S}}[m_1,e_1]$ by Lemma 5.64.

Consequently, our assumption that $[e_2,m_2]\circ_{\mathrm{Rel}}[f_1,g_1] = [g_2,f_2]\circ_{\mathrm{Rel}}[m_1,e_1]$ implies $[e_2,m_2]\circ_{\mathrm{S}}[f_1,g_1] = [g_2,f_2]\circ_{\mathrm{S}}[m_1,e_1]$, which then yields the existence of $w_2\in \mathrm{iso}_{\mathcal{A}}(P_1,P_2)$ with $f_1\circ e_2' = (m_1\circ g_2')\circ w_2$ and $m_2\circ g_1' = (f_2\circ e_1')\circ w_2$. If we define $w_1 := w_2{}^{-1}$, then we can express these two identities equivalently by saying that

$$m_i\circ g'_{\neg i} = (f_i\circ e'_{\neg i})\circ w_i$$

for each $i\in\{1,2\}$, where $w_i\in \mathrm{iso}_{\mathcal{A}}(P_{\neg i}, P_i)$.

Step 1.3: Construction of z_1 and z_2. For each $i\in\{1,2\}$, because $(m_{\neg i}\circ m_{g_i'})\circ e_{g_i'} = m_{\neg i}\circ(m_{g_i'}\circ e_{g_i'}) = m_{\neg i}\circ g_i' = (f_{\neg i}\circ e_i')\circ w_{\neg i} = (m_{g'_{\neg i}}\circ e_{g'_{\neg i}})\circ e_i'\circ w_{\neg i} = m_{g'_{\neg i}}\circ(e_{g'_{\neg i}}\circ e_i'\circ w_{\neg i})$,

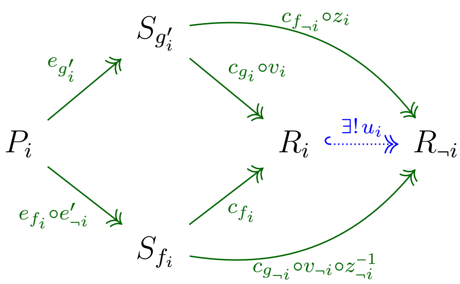


because $e_{g_i'}\in \mathrm{mor}_{\mathcal{E}}(P_i, S_{g_i'})$ and $m_{\neg i}\circ m_{g_i'}\in \mathrm{mor}_{\mathcal{M}}(S_{g_i'}, A_{\neg i})$by Lemma 4.33, and because, likewise, $e_{f_{\neg i}}\circ e_i'\circ w_{\neg i}\in \mathrm{mor}_{\mathcal{E}}(P_i, S_{f_{\neg i}})$, again by Lemma 4.33 and now also by Lemma 4.32, and $m_{f_{\neg i}}\in \mathrm{mor}_{\mathcal{M}}(S_{f_{\neg i}}, A_{\neg i})$ the pairs $(e_{g_i'}, m_{\neg i}\circ m_{g_i'})$ and $(e_{f_{\neg i}}\circ e_i'\circ w_{\neg i}, m_{f_{\neg i}})$ are two $(\mathcal{E},\mathcal{M})$-factorizations of the same morphism. Hence, for each $i\in\{1,2\}$ there exists $z_i\in \mathrm{iso}_{\mathcal{A}}(S_{g_i'}, S_{f_{\neg i}})$ such that

$$z_i\circ e_{g_i'} = e_{f_{\neg i}}\circ e_i'\circ w_{\neg i} \quad\text{ and }\quad m_{f_{\neg i}}\circ z_i = m_{\neg i}\circ m_{g_i'}.$$

Step 1.4: Construction of u_1 and u_2. Since the pair $(e_{g_i'}, e'_{\neg i}\circ e_{f_i})$ is a pull-back of $(c_{g_i}\circ v_i, c_{f_i})$, as seen in the second step, and since $\mathcal{A}$ has the second $(\mathcal{E},\mathcal{M})$-pseudo-exact-Mal'cev property, conversely $(c_{g_i}\circ v_i, c_{f_i})$ is a push-out of $(e_{g_i'}, e'_{\neg i}\circ e_{f_i})$ for each $i\in\{1,2\}$. Hence, from the identity $(c_{f_{\neg i}}\circ z_i)\circ e_{g_i'} = c_{f_{\neg i}}\circ(z_i\circ e_{g_i'}) = c_{f_{\neg i}}\circ(e_{f_{\neg i}}\circ e_i'\circ w_{\neg i}) = (c_{f_{\neg i}}\circ e_{f_{\neg i}})\circ e_i'\circ w_{\neg i} = (c_{g_{\neg i}}\circ e_{g_{\neg i}})\circ e_i'\circ w_{\neg i} = c_{g_{\neg i}}\circ(e_{g_{\neg i}}\circ e_i')\circ w_{\neg i} = c_{g_{\neg i}}\circ(v_{\neg i}\circ e_{g'_{\neg i}})\circ w_{\neg i} = c_{g_{\neg i}}\circ v_{\neg i}\circ z_{\neg i}^{-1}\circ(z_{\neg i}\circ e_{g'_{\neg i}})\circ w_{\neg i} = c_{g_{\neg i}}\circ v_{\neg i}\circ z_{\neg i}^{-1}\circ(e_{f_i}\circ e'_{\neg i}\circ w_i)\circ w_{\neg i} =$

$(c_{g_{\neg i}} \circ v_{\neg i} \circ z_{\neg i}^{-1}) \circ (e_{f_i} \circ e'_{\neg i})$



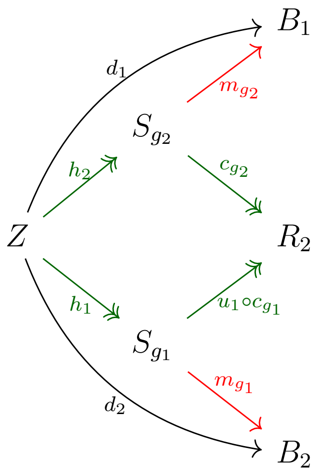
it follows that there exists $u_i \in \mathrm{iso}_{\mathcal{A}}(R_i, R_{\neg i})$ such that

$$c_{f_{\neg i}} \circ z_i = u_i \circ (c_{g_i} \circ v_i) \quad \text{and} \quad c_{g_{\neg i}} \circ v_{\neg i} \circ z_{\neg i}^{-1} = u_i \circ c_{f_i}$$

for each $i \in \{1,2\}$.

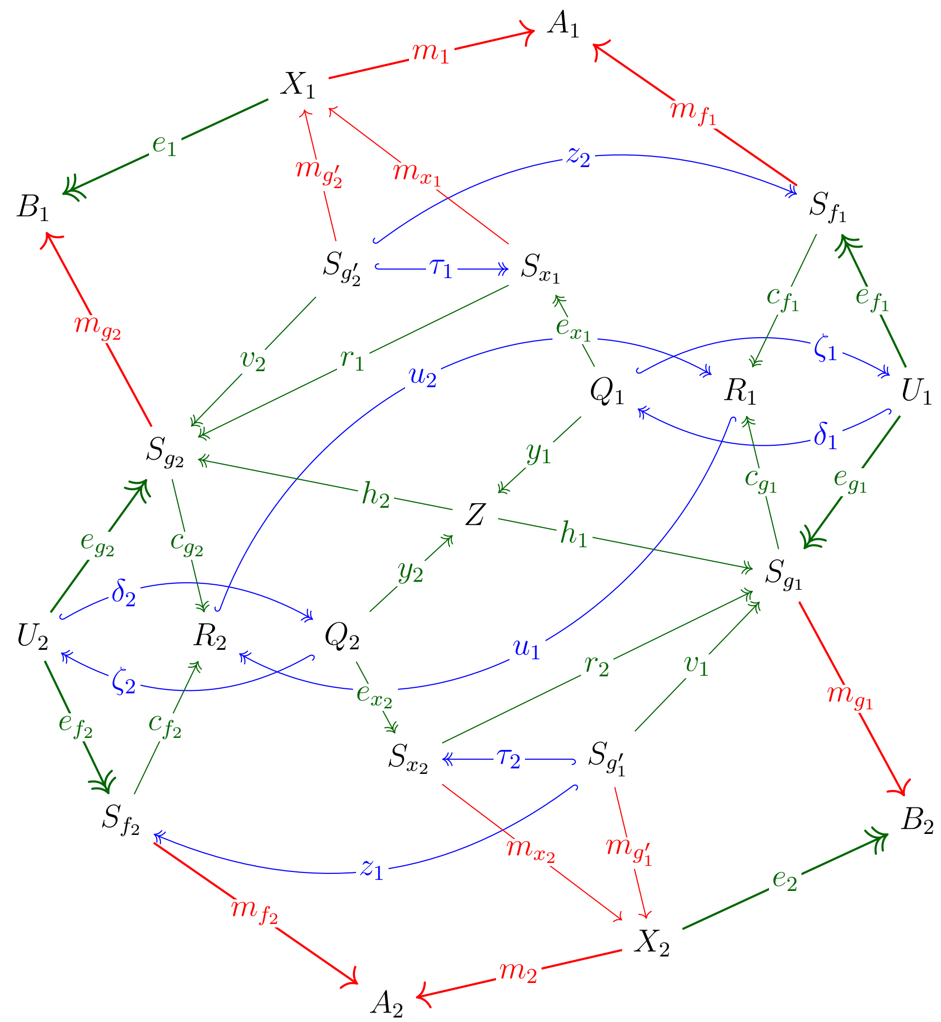
These identities imply for each $i \in \{1,2\}$ that $(u_{\neg i} \circ u_i) \circ (c_{g_i} \circ v_i) = u_{\neg i} \circ (u_i \circ c_{g_i} \circ v_i) = u_{\neg i} \circ (c_{f_{\neg i}} \circ z_i) = (u_{\neg i} \circ c_{f_{\neg i}}) \circ z_i = (c_{g_i} \circ v_i \circ z_i^{-1}) \circ z_i = c_{g_i} \circ v_i$ and $(u_{\neg i} \circ u_i) \circ c_{f_i} = u_{\neg i} \circ (u_i \circ c_{f_i}) = u_{\neg i} \circ (c_{g_{\neg i}} \circ v_{\neg i} \circ z_{\neg i}^{-1}) = (u_{\neg i} \circ c_{g_{\neg i}} \circ v_{\neg i}) \circ z_{\neg i}^{-1} = (c_{f_i} \circ z_{\neg i}) \circ z_{\neg i}^{-1} = c_{f_i}$. But id_{R_i} is the only morphism $\omega \in \mathrm{mor}_{\mathcal{A}}(R_i, R_i)$ with $\omega \circ (c_{g_i} \circ v_i) = c_{g_i} \circ v_i$ and $\omega \circ c_{f_i} = c_{f_i}$ because $(c_{g_i} \circ v_i, c_{f_i})$ is a push-out of $(e_{g'_i}, e_{f_i} \circ e'_{\neg i})$. Thus, we have shown $u_i \in \mathrm{iso}_{\mathcal{A}}(R_i, R_{\neg i})$ and $u_{\neg i}^{-1} = u_i$ for each $i \in \{1,2\}$.

Step 1.5: Definition of $[d_1, d_2]$. Because $\mathcal{A}$ is finitely complete we can find a pull-back (h_1, h_2) of the co-span $(u_1 \circ c_{g_1}, c_{g_2})$. Denote its pull-back object by Z and define $d_1 := m_{g_2} \circ h_2$ and $d_2 := m_{g_1} \circ h_1$.

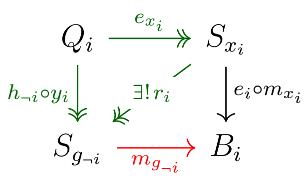


Since $c_{g_2} \in \mathrm{mor}_{\mathcal{E}}(S_{g_2}, R_2)$, since $u_1 \circ c_{g_1} \in \mathrm{mor}_{\mathcal{E}}(S_{g_1}, R_2)$ by Lemmata 4.33 and 4.32 and since $\mathcal{E}$ is pull-back-stable the definition of (h_1, h_2) implies $h_1 \in \mathrm{mor}_{\mathcal{E}}(Z, S_{g_1})$ and $h_2 \in \mathrm{mor}_{\mathcal{E}}(Z, S_{g_2})$. In particular, the square formed by the morphisms h_2, h_1, c_{g_2} and $u_1 \circ c_{g_1}$ lies entirely in $\mathcal{E}$. Hence, as $\mathcal{A}$ has the second $(\mathcal{E}, \mathcal{M})$-pseudo-exact-Mal'cev property the assumption that (h_1, h_2) is a pull-back of $(u_1 \circ c_{g_1}, c_{g_2})$ implies that, in turn, $(u_1 \circ c_{g_1}, c_{g_2})$ is a push-out of (h_1, h_2) and that $h_2 \times h_1 \in \mathrm{mor}_{\mathcal{M}}(Z, S_{g_2} \otimes S_{g_1})$. From this, from $m_{g_2} \otimes m_{g_1} \in \mathrm{mor}_{\mathcal{M}}(S_{g_2} \otimes S_{g_1}, B_1 \otimes B_2)$ and from the decomposition $d_1 \times d_2 = (m_{g_2} \circ h_2) \times (m_{g_1} \circ h_1) = (m_{g_2} \otimes m_{g_1}) \circ (h_2 \times h_1)$ it follows $d_1 \times d_2 \in \mathrm{mor}_{\mathcal{M}}(Z, B_1 \otimes B_2)$. Thus, indeed, $[d_1, d_2] \in \mathrm{mor}_{\mathrm{Rel}}(B_1, B_2)$ as needed.

Step 2: Proving that $[d_1, d_2]$ *is a diagonal.* Just like the construction of $[d_1, d_2]$, the proof that $[d_1, d_2]$ is the diagonal we seek requires several steps. Again, further definitions common to all the steps. For each $i \in \{1, 2\}$ let (y_i, x_i) be any pull-back of (d_i, e_i) and let Q_i be its pull-back object. Morover, let (e_{x_i}, m_{x_i}) be any $(\mathcal{E}, \mathcal{M})$-factorization of x_i for each $i \in \{1, 2\}$ and call its image object S_{x_i}.



Step 2.1: Construction of r_1 *and* r_2. For each $i \in \{1,2\}$ the identity $(e_i \circ m_{x_i}) \circ e_{x_i} = e_i \circ (m_{x_i} \circ e_{x_i}) = e_i \circ x_i = d_i \circ y_i = (m_{g_{\neg i}} \circ h_{\neg i}) \circ y_i = m_{g_{\neg i}} \circ (h_{\neg i} \circ y_i)$ and



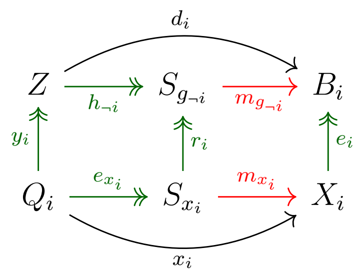
the assumptions that $e_{x_i} \in \mathrm{mor}_{\mathcal{E}}(Q_i, S_{x_i})$ and $m_{g_{\neg i}} \in \mathrm{mor}_{\mathcal{M}}(S_{g_{\neg i}}, B_i)$ imply that there exists some $r_i \in \mathrm{mor}_{\mathcal{A}}(S_{x_i}, S_{g_{\neg i}})$ with

$$r_i \circ e_{x_i} = h_{\neg i} \circ y_i \quad \text{and} \quad m_{g_{\neg i}} \circ r_i = e_i \circ m_{x_i}$$

and that r_i is unique with these properties.

Moreover, because $\mathcal{E}$ is pull-back-stable and because $e_i \in \mathrm{mor}_{\mathcal{E}}(X, B)$ also $y_i \in \mathrm{mor}_{\mathcal{E}}(Q_i, Z)$ and thus $h_{\neg i} \circ y_i \in \mathrm{mor}_{\mathcal{E}}(Q_i, S_{g_{\neg i}})$ by Lemma 4.33 for each $i \in \{1,2\}$. Consequently, from $r_i \circ e_{x_i} = h_{\neg i} \circ y_i$ it follows $r_i \in \mathrm{mor}_{\mathcal{E}}(S_{x_i}, S_{g_{\neg i}})$ for each $i \in \{1,2\}$ by Lemma 4.26.

Step 2.2: Construction of τ_1 *and* τ_2. Because $\mathcal{A}$ has the first $(\mathcal{E},\mathcal{M})$-pseudo-exact-Mal'cev property, because (y_i, x_i) is a pull-back of (d_i, e_i), because $e_i \in \mathrm{mor}_{\mathcal{E}}(X_i, B_i)$, because $(h_{\neg i}, m_{g_{\neg i}})$ is an $(\mathcal{E},\mathcal{M})$-factorization of d_i and (e_{x_i}, m_{x_i}) one of x_i and because r_i is the unique diagonal with $r_i \circ e_{x_i} = h_{\neg i} \circ y_i$ and $m_{g_{\neg i}} \circ r_i = e_i \circ m_{x_i}$ we can infer that (r_i, m_{x_i}) is a pull-back of $(m_{g_{\neg i}}, e_i)$ for each $i \in \{1,2\}$.



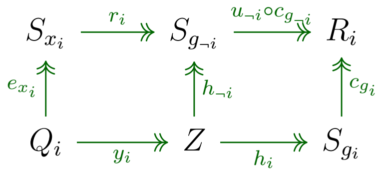
Thus, for each $i \in \{1,2\}$, both $(v_{\neg i}, m_{g'_{\neg i}})$ and (r_i, m_{x_i}) are pull-backs of $(m_{g_{\neg i}}, e_i)$, whence, by essential uniqueness, there exists an isomorphism $\tau_i \in \mathrm{iso}_{\mathcal{A}}(S_{g'_{\neg i}}, S_{x_i})$ with

$$v_{\neg i} = r_i \circ \tau_i \quad \text{and} \quad m_{g'_{\neg i}} = m_{x_i} \circ \tau_i.$$

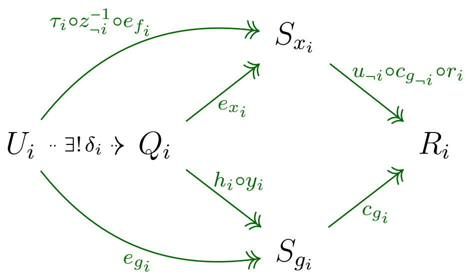
Step 2.3: Construction of δ_1 *and* δ_2. By Lemma 4.3, since $(y_i, m_{x_i} \circ e_{x_i})$ is a pull-back of $(m_{g_{\neg i}} \circ h_{\neg i}, e_i)$ by assumption, since (r_i, m_{x_i}) is a pull-back of $(m_{g_{\neg i}}, e_i)$ by the previous step and because $r_i \circ e_{x_i} = h_{\neg i} \circ y_i$ and $m_{g_{\neg i}} \circ r_i = e_i \circ m_{x_i}$ the pair (y_i, e_{x_i}) is a pull-back of $(h_{\neg i}, r_i)$ for each $i \in \{1,2\}$.

On the other hand, thanks to Lemma 4.1 the assumption that (h_2, h_1) is a pull-back of $(c_{g_2}, u_1 \circ c_{g_1})$ implies $(h_{\neg i}, h_i)$ is a pull-back of $(u_{\neg i} \circ c_{g_{\neg i}}, c_{g_i})$ for each $i \in \{1,2\}$.

As $((u_{\neg i} \circ c_{g_{\neg i}}) \circ r_i) \circ e_{x_i} = u_{\neg i} \circ c_{g_{\neg i}} \circ (r_i \circ e_{x_i}) = u_{\neg i} \circ c_{g_{\neg i}} \circ (h_{\neg i} \circ y_i) = ((u_{\neg i} \circ c_{g_{\neg i}}) \circ h_{\neg i}) \circ y_i = (c_{g_i} \circ h_i) \circ y_i = c_{g_i} \circ (h_i \circ y_i)$, as $(h_{\neg i}, h_i)$ is a pull-back of $(u_{\neg i} \circ c_{g_{\neg i}}, c_{g_i})$ and as (e_{x_i}, y_i) is a pull-back of $(r_i, h_{\neg i})$ Lemma 4.3 ensures that also $(e_{x_i}, h_i \circ y_i)$ is a pull-back of $((u_{\neg i} \circ c_{g_{\neg i}}) \circ r_i, c_{g_i})$ for each $i \in \{1, 2\}$.



For that reason, for each $i \in \{1, 2\}$, we can infer from $((u_{\neg i} \circ c_{g_{\neg i}}) \circ r_i) \circ (\tau_i \circ z_{\neg i}^{-1} \circ e_{f_i}) = u_{\neg i} \circ c_{g \neg i} \circ (r_i \circ \tau_i) \circ z_{\neg i}^{-1} \circ e_{f_i} = u_{\neg i} \circ c_{g \neg i} \circ v_{\neg i} \circ z_{\neg i}^{-1} \circ e_{f_i} = u_{\neg i} \circ (c_{g \neg i} \circ v_{\neg i} \circ z_{\neg i}^{-1}) \circ e_{f_i} = u_{\neg i} \circ (u_i \circ c_{f_i}) \circ e_{f_i} = c_{f_i} \circ e_{f_i} = c_{g_i} \circ e_{g_i}$

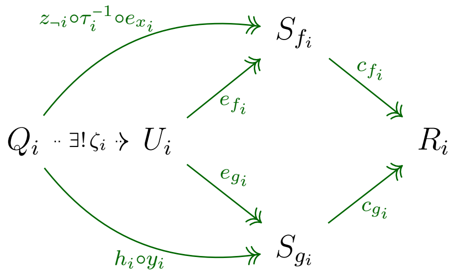


that there exists $\delta_i \in \mathrm{mor}_{\mathcal{A}}(U_i, Q_i)$ with

$$e_{x_i} \circ \delta_i = \tau_i \circ z_{\neg i}^{-1} \circ e_{f_i} \quad \text{and} \quad (h_i \circ y_i) \circ \delta_i = e_{g_i}$$

and that δ_i is the only morphism satisfying these equations.

Step 2.4: Construction of ζ_1 and ζ_2. Conversely, since $c_{f_i} \circ (z_{\neg i} \circ \tau_i^{-1} \circ e_{x_i}) = (c_{f_i} \circ z_{\neg i}) \circ \tau_i^{-1} \circ e_{x_i} = (u_{\neg i} \circ c_{g_{\neg i}} \circ v_{\neg i}) \circ \tau_i^{-1} \circ e_{x_i} = u_{\neg i} \circ c_{g_{\neg i}} \circ (v_{\neg i} \circ \tau_i^{-1}) \circ e_{x_i} = u_{\neg i} \circ c_{g_{\neg i}} \circ r_i \circ e_{x_i} = u_{\neg i} \circ c_{g_{\neg i}} \circ (r_i \circ e_{x_i}) = u_{\neg i} \circ c_{g_{\neg i}} \circ (h_{\neg i} \circ y_i) = (u_{\neg i} \circ c_{g_{\neg i}} \circ h_{\neg i}) \circ y_i = (c_{g_i} \circ h_i) \circ y_i = c_{g_i} \circ (h_i \circ y_i)$, where we have used that $(h_{\neg i}, h_i)$ is a pull-back of $(u_{\neg i} \circ c_{g_{\neg i}}, c_{g_i})$,



and because (e_{f_i}, e_{g_i}) is a pull-back of (c_{f_i}, c_{g_i}) there exists $\zeta_i \in \mathrm{mor}_{\mathcal{A}}(Q_i, U_i)$ such that

$$e_{f_i} \circ \zeta_i = z_{\neg i} \circ \tau_i^{-1} \circ e_{x_i} \quad \text{and} \quad e_{g_i} \circ \zeta_i = h_i \circ y_i$$

for each $i \in \{1, 2\}$ and such that no other morphism has these properties.

Step 2.5: δ_i and ζ_i are inverses of each other. Since $(e_{x_i}, h_i \circ x_i)$ is a pull-back id_{Q_i} is the only endomorphism ω of Q_i with $e_{x_i} \circ \omega = e_{x_i}$ and $(h_i \circ y_i) \circ \omega = h_i \circ y_i$. Hence, the identities $e_{x_i} \circ (\delta_i \circ \zeta_i) = (e_{x_i} \circ \delta_i) \circ \zeta_i = (\tau_i \circ z_{\neg i}^{-1} \circ e_{f_i}) \circ \zeta_i = \tau_i \circ z_{\neg i}^{-1} \circ (e_{f_i} \circ \zeta_i) =$

$\tau_i \circ z_{\neg i}^{-1} \circ (z_{\neg i} \circ \tau_i^{-1} \circ e_{x_i}) = e_{x_i}$ and $(h_i \circ y_i) \circ (\delta_i \circ \zeta_i) = (h_i \circ y_i \circ \delta_i) \circ \zeta_i = e_{g_i} \circ \zeta_i = h_i \circ y_i$ prove $\delta_i \circ \zeta_i = \mathrm{id}_{Q_i}$.

Likewise, because (e_{f_i}, e_{g_i}) is a pull-back and thus id_{U_i} the only $\omega \in \mathrm{mor}_{\mathcal{A}}(U_i, U_i)$ with $e_{f_i} \circ \mathrm{id}_{U_i} = e_{f_i}$ and $e_{g_i} \circ \mathrm{id}_{U_i} = e_{g_i}$ we can infer from $e_{f_i} \circ (\zeta_i \circ \delta_i) = (e_{f_i} \circ \zeta_i) \circ \delta_i = (z_{\neg i} \circ \tau_i^{-1} \circ e_{x_i}) \circ \delta_i = z_{\neg i} \circ \tau_i^{-1} \circ (e_{x_i} \circ \delta_i) = z_{\neg i} \circ \tau_i^{-1} \circ (\tau_i \circ z_{\neg i}^{-1} \circ e_{f_i}) = e_{f_i}$ and $e_{g_i} \circ (\zeta_i \circ \delta_i) = (e_{g_i} \circ \zeta_i) \circ \delta_i = (h_i \circ y_i) \circ \delta_i = e_{g_i}$ that $\zeta_i \circ \delta_i = \mathrm{id}_{U_i}$ for each $i \in \{1, 2\}$. Thus, we have shown $\zeta_1 = \delta_1^{-1}$ and $\zeta_2 = \delta_2^{-1}$.

Step 2.6: δ_i induces an equivalence of spans. Since $\delta_i \in \mathrm{iso}_{\mathcal{A}}(U_i, Q_i)$ and since $(m_i \circ x_i) \circ \delta_i = m_i \circ (m_{x_i} \circ e_{x_i}) \circ \delta_i = m_i \circ m_{x_i} \circ (e_{x_i} \circ \delta_i) = m_i \circ m_{x_i} \circ (\tau_i \circ z_{\neg i}^{-1} \circ e_{f_i}) = m_i \circ (m_{x_i} \circ \tau_i) \circ z_{\neg i}^{-1} \circ e_{f_i} = m_i \circ m_{g'_{\neg i}} \circ z_{\neg i}^{-1} \circ e_{f_i} = (m_i \circ m_{g'_{\neg i}} \circ z_{\neg i}^{-1}) \circ e_{f_i} = m_{f_i} \circ e_{f_i} = f_i$ and $(d_{\neg i} \circ y_i) \circ \delta_i = (m_{g_i} \circ h_i \circ y_i) \circ \delta_i = m_{g_i} \circ (h_i \circ y_i \circ \delta_i) = m_{g_i} \circ e_{g_i} = g_i$ the spans (f_i, g_i) and $(m_i \circ x_i, d_{\neg i} \circ y_i)$ are equivalent for each $i \in \{1, 2\}$.

Since (x_i, y_i) is a pull-back of (e_i, d_i) we have thus verified for each $i \in \{1, 2\}$ that $[f_i, g_i] = [d_i, d_{\neg i}] \circ_{\mathrm{S}} [m_i, e_i]$. Since $[f_i, g_i] \in \mathrm{mor}_{\mathrm{Rel}}(A_i, B_{\neg i})$, by assumption in the case of $[f_1, g_1]$ and by Proposition 5.88 in the case of $[f_2, g_2]$, that actually means $[d_i, d_{\neg i}] \circ_{\mathrm{Rel}} [m_i, e_i] = \Phi_{\mathcal{A},\mathcal{E}}([d_i, d_{\neg i}] \circ_{\mathrm{S}} [m_i, e_i]) = \Phi_{\mathcal{A},\mathcal{E}}([f_i, g_i]) = [f_i, g_i]$ by Lemma 5.64. By Lemma 5.47 that is the same as saying $[f_1, g_1] = [d_1, d_2] \circ_{\mathrm{Rel}} [m_1, e_1]$ and $[g_2, f_2] = [e_2, m_2] \circ_{\mathrm{Rel}} [d_1, d_2]$. Thus, $[d_1, d_2]$ is indeed the diagonal we sought. ☐

By combining everything we have seen so far, we can prove a weakened version of [Kno07, Lemma 5.2 (i)].

PROPOSITION 7.19. *If $\mathcal{A}$ has the first and second $(\mathcal{E}, \mathcal{M})$-pseudo-exact-Mal'cev properties, then $(\mathfrak{E}, \mathfrak{M})$, the categories of pre-core and post-core relations, form a weak factorization system of* Rel.

PROOF. Follows from Lemmata 7.8, 7.7, 7.18 and 7.16. ☐

Unfortunately, even in good cases, this factorization is only a weak one.

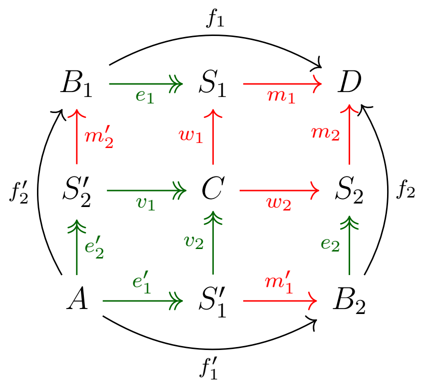
REMARK 7.20. If $\mathcal{M} \hookrightarrow \mathrm{mon}_{\mathcal{A}}$ it can be seen that $(\mathfrak{E}, \mathfrak{M})$ is actually an orthogonal factorization system of Rel. However, that is generally false otherwise. As a counterexample consider the case $\mathcal{A} = \mathsf{fGr}^{\mathrm{op}}$ and $\mathcal{E} = \mathrm{qpr}_{\mathsf{fGr}^{\mathrm{op}}}$:

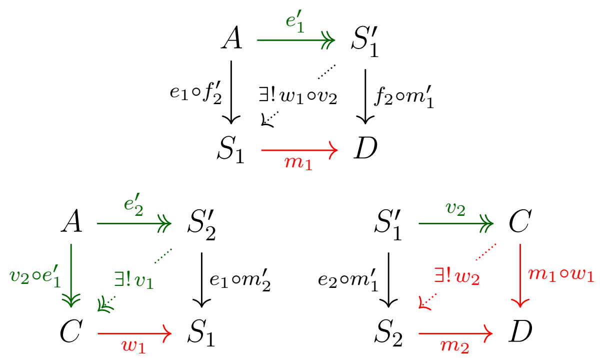
In the notation of the proof of Lemma 7.18 let $A_1 = A_2 = (\{1\}, \varnothing)$, let $B_1 = B_2 = X_1 = X_2 = (\{1, 2\}, \{\{1, 2\}\})$, let $U_1 = U_2 = (\{1, 2, 3\}, \{\{1, 2\}, \{2, 3\}\})$, let $e_1 = e_2 = \{(1, 1), (2, 2)\}$, let $m_1 = \{(1, 1)\}$ and $m_2 = \{(1, 2)\}$, let $f_1 = f_2 = \{(1, 2)\}$, let $g_1 = \{(1, 1), (2, 2)\}$ and $g_2 = \{(1, 2), (2, 3)\}$. Then there exist two diagonals $[d_1, d_2]$ and $[k_1, k_2]$ with $[d_1, d_2] \neq [k_1, k_2]$. Namely, let $d_1 = \{(1, 2), (2, 3)\}$ and $d_2 = \{(1, 1), (2, 2)\}$ as well as $k_1 = \{(1, 2), (2, 1)\}$ and $k_2 = \{(1, 1), (2, 2)\}$ and consider both (d_1, d_2) and (k_1, k_2) as spans $B_1 \to B_2$ over the base $U_1 = U_2$.

7.2.3. *Third pseudo-exact-Mal'cev property.* The third and final property studied in this chapter is the following. The distinction between push-out in $\mathcal{E}$ and $\mathcal{A}$ is crucial.

DEFINITION 7.21. If $\mathcal{A}$ is any category and $(\mathcal{E}, \mathcal{M})$ any factorization system of $\mathcal{A}$, then $\mathcal{A}$ is said to have the *third $(\mathcal{E}, \mathcal{M})$-pseudo-exact-Mal'cev property* if for any

$\{B_1, B_2, D\} \subseteq \mathrm{obj}_{\mathcal{A}}$, any $f_1 \in \mathrm{mor}_{\mathcal{A}}(S_1, D)$ and $f_2 \in \mathrm{mor}_{\mathcal{A}}(S_2, D)$, and any pull-back (f_2', f_1') of (f_1, f_2) in $\mathcal{A}$, if A is the pull-back object of (f_2', f_1'), if for any $i \in \{1,2\}$ the pair (e_i, m_i) is any $(\mathcal{E},\mathcal{M})$-factorization of f_i with image object S_i and (e_i', m_i') one of f_i' with image object S_i' and if the morphisms $v_1 \in \mathrm{mor}_{\mathcal{E}}(S_2', C)$ and $v_2 \in \mathrm{mor}_{\mathcal{E}}(S_1', C)$ and $w_1 \in \mathrm{mor}_{\mathcal{M}}(C, S_1)$ and $w_2 \in \mathrm{mor}_{\mathcal{M}}(C, S_2)$ satisfy $m_1 \circ w_1 = m_2 \circ w_2$ and $v_1 \circ e_2' = v_2 \circ e_1'$ and $e_1 \circ m_2' = w_1 \circ v_1$ and $e_2 \circ m_1' = w_2 \circ v_2$, then (v_1, v_2) is a push-out of (e_2', e_1') in $\mathcal{E}$ (although, not necessarily in $\mathcal{A}$).



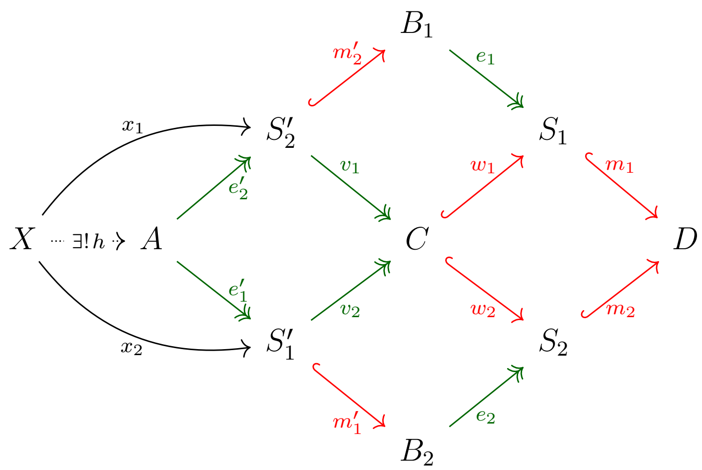


Proposition 7.22. *$\mathcal{A}$ has the third $(\mathcal{E},\mathcal{M})$-pseudo-exact-Mal'cev property if $\mathcal{A}$ is regular exact Mal'cev and if $\mathcal{E}$ and $\mathcal{M}$ are given by the regular epimorphisms and the monomorphisms of $\mathcal{A}$, respectively.*

Proof. Suppose we find ourselves in the situation of Definition 7.21. We prove that (v_1, v_2) is a push-out of (e_2', e_1') in $\mathcal{E}$.

Step 1: In fact, our first step is to show that in this case, where $\mathcal{A}$ is regular exact Mal'cev, (v_1, v_2) is a push-out of (e_2', e_1') in $\mathcal{A}$. Since $\mathcal{A}$ has the second $(\mathcal{E},\mathcal{M})$-pseudo-exact-Mal'cev property by Proposition 7.15 it suffices to verify that (e_2', e_1') is a pull-back of (v_1, v_2) in $\mathcal{A}$ in order to show this.

Hence, let $X \in \mathrm{obj}_{\mathcal{A}}$ and $x_1 \in \mathrm{mor}_{\mathcal{A}}(X, S_2')$ and $x_2 \in \mathrm{mor}_{\mathcal{A}}(X, S_1')$ be arbitrary with $v_1 \circ x_1 = v_2 \circ x_2$.



Because $f_1 \circ (m_2' \circ x_1) = (m_1 \circ e_1) \circ (m_2' \circ x_1) = m_1 \circ (e_1 \circ m_2') \circ x_1 = m_1 \circ (w_1 \circ v_1) \circ x_1 = (m_1 \circ w_1) \circ (v_1 \circ x_1) = (m_2 \circ w_2) \circ (v_2 \circ x_2) = m_2 \circ (w_2 \circ v_2) \circ x_2 = m_2 \circ (e_2 \circ m_1') \circ x_2 = (m_2 \circ e_2) \circ (m_1' \circ x_2) = f_2 \circ (m_1' \circ x_2)$ and because $(m_2' \circ e_2', m_1' \circ e_1')$ is a pull-back of (f_1, f_2) in $\mathcal{A}$ there exists a unique $h \in \mathrm{mor}_{\mathcal{A}}(X, A)$ with $(m_2' \circ e_2') \circ h = m_2' \circ x_1$ and $(m_1' \circ e_1') \circ h = m_1' \circ x_2$. Since m_2' and m_1' are monomorphisms these last two conditions are satisfied if and only if $e_2' \circ h = x_1$ and $e_1' \circ h = x_2$. Thus, we have shown that (e_2', e_1') is a pull-back of (v_1, v_2) in $\mathcal{A}$.

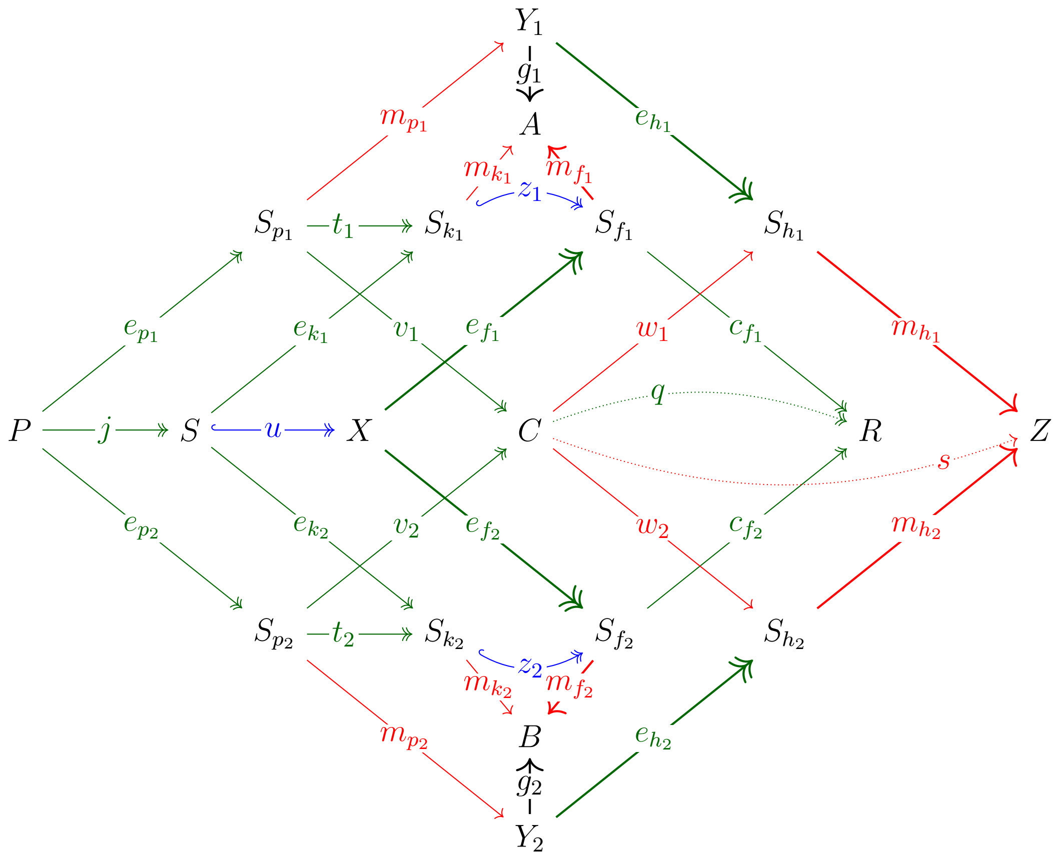
Step 2: Having seen that (v_1, v_2) is a push-out of (e_2', e_1') in $\mathcal{A}$ it is now immediate that (v_1, v_2) is also a push-out of (e_2', e_1') in $\mathcal{E}$: Let $Y \in \mathrm{obj}_{\mathcal{A}}$ and $y_2 \in \mathrm{mor}_{\mathcal{E}}(S_2', Y)$ and $y_1 \in \mathrm{mor}_{\mathcal{E}}(S_1', Y)$ be arbitrary with $y_2 \circ e_2' = y_1 \circ e_1'$. By Step 1 there exists a unique $k \in \mathrm{mor}_{\mathcal{A}}(C, Y)$ with $k \circ v_1 = y_2$ and $k \circ v_2 = y_1$. Since $\mathcal{E}$ consists of epimorphisms and since v_1 and y_2 are both morphisms of $\mathcal{E}$, Lemma 4.36 tells us that the identity $k \circ v_1 = y_2$ ensures $k \in \mathrm{mor}_{\mathcal{E}}(C, Y)$. Since k is already unique in all of $\mathcal{A}$ it is so in $\mathcal{E}$ in particular. □

REMARK 7.23. Importantly, $\mathcal{A}$ *can* have the third $(\mathcal{E}, \mathcal{M})$-pseudo-exact-Mal'cev property even if $\mathcal{A}$ is not regular, e.g., if $\mathcal{A} = \mathsf{fGr}^{\mathrm{op}}$ and $\mathcal{E} = \mathrm{qpr}_{\mathsf{fGr}^{\mathrm{op}}}$.

If both the second and third pseudo-exact-Mal'cev properties are satisfied a weaker form of [Kno07, Lemma 5.2 (ii)] holds.

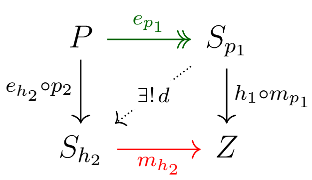
LEMMA 7.24. *If $\mathcal{A}$ has the second and third $(\mathcal{E}, \mathcal{M})$-pseudo-exact-Mal'cev properties, then for any $\{A, B, Z\} \subseteq \mathrm{obj}_{\mathcal{A}}$ and any $x \in \mathrm{mor}_{\mathrm{Rel}}(A, B)$ and $y \in \mathrm{mor}_{\mathrm{Rel}}(A, Z)$ and $z \in \mathrm{mor}_{\mathrm{Rel}}(Z, B)$, if $x = z \circ_{\mathrm{Rel}} y$, then $R \lesssim_{\mathcal{A}, \mathcal{E}, \mathcal{M}} Z$ for any core object R of x.*

PROOF. It suffices to prove the claim for *some* R, rather than *any* R. If $(f_1, f_2) \in x$ and if for each $i \in \{1, 2\}$ the pair (e_{f_i}, m_{f_i}) is any $(\mathcal{E}, \mathcal{M})$-factorization of f_i and S_{f_i} its image object, then the co-span (e_{f_1}, e_{f_2}) in $\mathcal{E}$ has a push-out (c_{f_1}, c_{f_2}) in $\mathcal{E}$ because $\mathcal{A}$ has the second $(\mathcal{E}, \mathcal{M})$-pseudo-exact-Mal'cev property. By definition the push-out object of (c_{f_1}, c_{f_2}), call it R, is a core object of x. We construct $C \in \mathrm{obj}_{\mathcal{A}}$ such that there exist $s \in \mathrm{mor}_{\mathcal{M}}(C, Z)$ and $q \in \mathrm{mor}_{\mathcal{E}}(C, R)$, which then proves the claim.



Step 1: Construction of R. Let Y_1 and Y_2 be the bases of $(g_1, h_1) \in y$ and $(h_2, g_2) \in z$, respectively, let (p_1, p_2) be any pull-back of (h_1, h_2) with pull-back object P. Moreover, for each $i \in \{1, 2\}$ let (e_{h_i}, m_{h_i}) be any $(\mathcal{E}, \mathcal{M})$-factorization of h_i with object S_{h_i}, let (e_{p_i}, m_{p_i}) be any of p_i with object S_{p_i} and let (e_{k_i}, m_{k_i}) be any of k_i with object S_{k_i}.

Since $(h_1 \circ m_{p_1}) \circ e_{p_1} = h_1 \circ (m_{p_1} \circ e_{p_1}) = h_1 \circ p_1 = h_2 \circ p_2 = (m_{h_2} \circ e_{h_2}) \circ p_2 = m_{h_2} \circ (e_{h_2} \circ p_2)$ and because $e_{p_1} \in \mathrm{mor}_{\mathcal{E}}(P, S_{p_1})$ and $m_{h_2} \in \mathrm{mor}_{\mathcal{M}}(S_{h_2}, Z)$, and thus $e_{p_1} \perp m_{h_2}$,

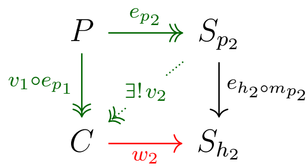


there exists a unique $d \in \mathrm{mor}_{\mathcal{A}}(S_{p_1}, S_{h_2})$ with $e_{h_2} \circ p_2 = d \circ e_{p_1}$ and $m_{h_2} \circ d = h_1 \circ m_{p_1}$. Let (v_1, w_2) be any $(\mathcal{E}, \mathcal{M})$-factorization of d and let C be its image object.

Step 2: Construction of s. Given that $m_{h_2} \in \mathrm{mor}_{\mathcal{M}}(S_{h_2}, Z)$ and $w_2 \in \mathrm{mor}_{\mathcal{M}}(C, S_{h_2})$, Lemma 4.33 guarantees $s := m_{h_2} \circ w_2 \in \mathrm{mor}_{\mathcal{M}}(C, Z)$.

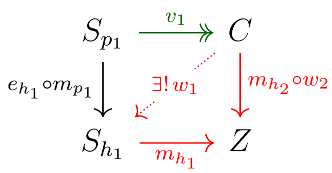
Step 3: Construction of q. Constructing $q \in \mathrm{mor}_{\mathcal{E}}(C,R)$ requires multiple steps. The third $(\mathcal{E},\mathcal{M})$-pseudo-exact-Mal'cev property is used only in the last one.

Step 3.1: Construction of v_2. Because $(e_{h_2} \circ m_{p_2}) \circ e_{p_2} = e_{h_2} \circ (m_{p_2}) \circ e_{p_2}) = e_{h_2} \circ p_2 = d \circ e_{p_1} = (w_2 \circ v_1) \circ e_{p_1} = w_2 \circ (v_1 \circ e_{p_1})$ and because $e_{p_2} \in \mathrm{mor}_{\mathcal{E}}(P, S_{p_2})$ and $w_2 \in \mathrm{mor}_{\mathcal{M}}(C, S_{h_2})$, implying $e_{p_2} \perp w_2$,



we find a unique $v_2 \in \mathrm{mor}_{\mathcal{A}}(S_{p_2}, C)$ with $v_1 \circ e_{p_1} = v_2 \circ e_{p_2}$ and $w_2 \circ v_2 = e_{h_2} \circ m_{p_2}$. And by Lemma 4.26 the former of these identities ensures $v_2 \in \mathrm{mor}_{\mathcal{E}}(S_{p_2}, C)$ because $v_1 \circ e_{p_1} \in \mathrm{mor}_{\mathcal{E}}(P, C)$ and $e_{p_2} \in \mathrm{mor}_{\mathcal{E}}(P, S_{p_2})$.

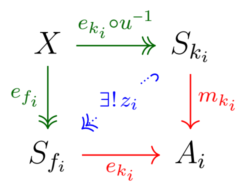
Step 3.2: Construction of w_1. The identity $(m_{h_2} \circ w_2) \circ v_1 = m_{h_2} \circ (w_2 \circ v_1) = m_{h_2} \circ d = h_1 \circ m_{p_1} = (m_{h_1} \circ e_{h_1}) \circ m_{p_1} = m_{h_1} \circ (e_{h_1} \circ m_{p_1})$ and the assumptions $v_1 \in \mathrm{mor}_{\mathcal{E}}(S_{p_1}, C)$ and $m_{h_1} \in \mathrm{mor}_{\mathcal{M}}(S_{h_1}, Z)$, which let us infer $v_1 \perp m_{h_1}$,



prove the existence of a unique $w_1 \in \mathrm{mor}_{\mathcal{A}}(C, S_{h_1})$ with $e_{h_1} \circ m_{p_1} = w_1 \circ v_1$ and $m_{h_1} \circ w_1 = m_{h_2} \circ w_2$. We may further conclude $w_1 \in \mathrm{mor}_{\mathcal{M}}(C, S_{h_1})$ from $m_{h_2} \circ w_2 \in \mathrm{mor}_{\mathcal{M}}(C, Z)$ and $m_{h_1} \in \mathrm{mor}_{\mathcal{M}}(S_{h_1}, Z)$ by Lemma 4.26.

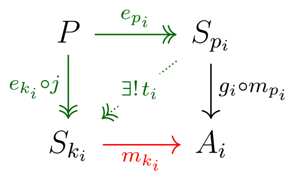
Step 3.3: Construction of u. If $(j, k_1 \times k_2)$ is any $(\mathcal{E},\mathcal{M})$-factorization of $(g_1 \circ p_1) \times (g_2 \circ p_2)$ with image object S, then, by definition, $(k_1, k_2) \in z \circ_{\mathrm{Rel}} y$. Hence, if X is the base of (f_1, f_2), the assumption $x = z \circ_{\mathrm{Rel}} y$ allows us to find $u \in \mathrm{iso}_{\mathcal{A}}(S, X)$ with $k_i = f_i \circ u$ for each $i \in \{1,2\}$.

Step 3.4: Construction of z_1 and z_2. If for each $i \in \{1,2\}$ any $(\mathcal{E},\mathcal{M})$-factorization of k_i is given by (e_{k_i}, m_{k_i}) and if S_{k_i} is the corresponding image object, then $m_{k_i} \circ (e_{k_i} \circ u^{-1}) = (m_{k_i} \circ e_{k_i}) \circ u^{-1} = k_i \circ u^{-1} = f_i$ for each i. As, for each $i \in \{1,2\}$, both $e_{k_i} \circ u^{-1} \in \mathrm{mor}_{\mathcal{E}}(X, S_{k_i})$ and $m_{k_i} \in \mathrm{mor}_{\mathcal{M}}(S_{k_i}, A_i)$, where $A_1 = A$ and $A_2 = B$, the pair $(e_{k_i} \circ u^{-1}, m_i)$ is an $(\mathcal{E},\mathcal{M})$-factoriation of f_i. Since those are essentially unique,



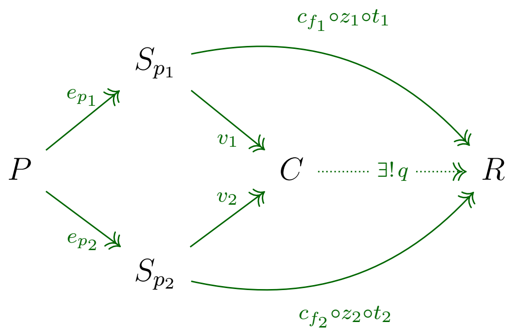
for each $i \in \{1,2\}$ there exists a unique $z_i \in \mathrm{mor}_{\mathcal{A}}(S_{k_i}, S_{f_i})$ with $e_{f_i} = z_i \circ e_{k_i} \circ u^{-1}$ and $m_{k_i} \circ z_i = m_{f_i}$.

Step 3.5: Construction of t_1 and t_2. For each $i \in \{1,2\}$ since $(g_i \circ m_{p_i}) \circ e_{p_i} = g_i \circ (m_{p_i} \circ e_{p_i}) = g_i \circ p_i = k_i \circ j = (m_{k_i} \circ e_{k_i}) \circ j = m_{k_i} \circ (e_{k_i} \circ j)$ and since $e_{p_i} \in \mathrm{mor}_{\mathcal{E}}(P, S_{p_i})$ and $m_{k_i} \in \mathrm{mor}_{\mathcal{M}}(S_{p_i}, A_i)$, where again $A_1 = A$ and $A_2 = B$, from which it follows $e_{p_i} \perp m_{k_i}$,



we can find $t_i \in \mathrm{mor}_{\mathcal{A}}(S_{p_i}, S_{k_i})$ with $e_{k_i} \circ j = t_i \circ e_{p_i}$ and $m_{k_i} \circ t_i = g_i \circ m_{p_i}$. In addition, Lemma 4.26 allows us to infer $t_i \in \mathrm{mor}_{\mathcal{E}}(S_{p_i}, S_{k_i})$ from $e_{k_i} \circ j \in \mathrm{mor}_{\mathcal{E}}(P, S_{k_i})$ and $e_{p_i} \in \mathrm{mor}_{\mathcal{E}}(P, S_{p_i})$.

Step 3.6: Construction of q. Since $\mathcal{A}$ has the third $(\mathcal{E}, \mathcal{M})$-pseudo-exact-Mal'cev property and since (p_1, p_2) is a pull-back of (h_1, h_2) in $\mathcal{A}$ the co-span (v_1, v_2) is a push-out of (e_{p_1}, e_{p_2}) in $\mathcal{A}$. Moreover, $(c_{f_1} \circ z_1 \circ t_1, c_{f_2} \circ z_2 \circ t_2)$ is a co-span in $\mathcal{E}$ by Lemma 4.33. Therefore, the identity



implies the existence of a unique $q \in \mathrm{mor}_{\mathcal{E}}(C, R)$ with $c_i \circ z_i \circ t_i = v_i$ for each $i \in \{1,2\}$. That concludes the proof. ☐

For the reader familiar with the semisimplicity proof of [Kno07] the following remark sketches where the analogous argument fails for the more general inputs considered in this chapter.

REMARK 7.25. As explained in Example 3.1 (f), for Mančinska and Roberson's categories input $(\mathcal{A}, \mathcal{E}, \delta)$ the right complement $\mathcal{M}$ of $\mathcal{E}$ does *not* consist of monomorphisms anymore. Then, the proof given by Knop for his critical Lemma 5.2 (ii) no longer works: It is no longer possible to conclude in (5.3) from $\tilde{r} \to \overline{z}$ being in E that also $\overline{s} \to \overline{z}$ and $\overline{t} \to \overline{z}$ are elements of E. This step rests on M consisting of monomorphisms. Namely it uses [FK72, Proposition 2.1.4 (c)] to infer this. But this statement that for composable morphisms p and q whenever $pq \in E$, then already $p \in E$, is only true in case M is a class of monomorphisms.

8. Future Research Directions

QUESTIONS 8.1. (a) Most importantly, if S is closed symmetric monoidal and has equalizers, what are the epic 1-cells of $\mathsf{sm}\mathbf{Mod}_R(\mathsf{S})\dagger\mathsf{CAT}^{\mathrm{r}}$ from $\mathcal{T}^0(\mathcal{A},\mathcal{E},\delta)$ to ${}^{\uparrow}\mathbf{Mod}_R(\mathsf{S})^{\mathrm{fgp}}$, the dualizable R-module objects of S considered as enriched over themselves?

(b) Is there a right 2-adjoint to the Tannakian version $\mathcal{T}^0_T$ of the Deligne-Knop 2-functor, thus completing the diagram following Definition 2.15 and yielding a 2-adjunction between the compositions?

(c) The Tannakian biadjunction in the said diagram is even monoidal. For which monoidal structures on its domain and co-domain can $\mathcal{T}^0_T$ be extended to a monoidal 2-functor?

(d) What is a or, ideally, the most general setting for $\mathcal{A}$ which makes it possible to perform "liberation" in the sense of [BS09] in $\mathcal{T}^0(\mathcal{A},\mathcal{E},\delta)$? In other words, what are the abstract analogs of non-crossing partitions or planar bi-labeled graphs?

(e) Is there a wide subcategory of $\mathcal{T}^0(\mathsf{fGr}^{\mathrm{op}},\mathrm{qpr}_{\mathsf{fGr}^{\mathrm{op}}},\mathrm{qker}_{\mathsf{fGr}^{\mathrm{op}}})$ which restricts to the graph category of all planar bi-labeled graphs in the sense of [MR20]?

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