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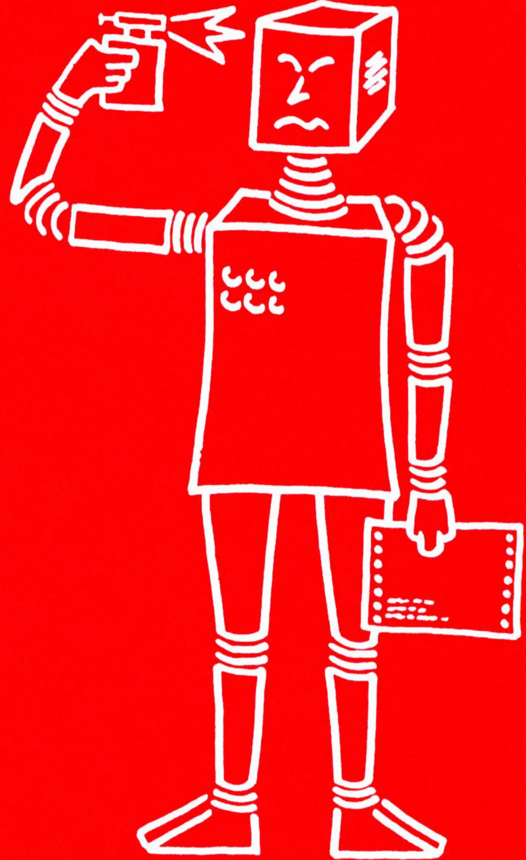

A Many-Sorted Calculus Based on Resolution and Paramodulation

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Abstract

The first-order calculus whose well formed formulas are clauses and whose sole inference rules are factorization, resolution and paramodulation is extended to a many-sorted calculus. As a basis for Automated Theorem Proving, this many-sorted calculus leads to a remarkable reduction of the search space and also to simpler proofs. Soundness and completeness of the new calculus and the Sort-Theorem, which relates the many-sorted calculus to its one-sorted counterpart, are shown. In addition results about term rewriting and unification in a many-sorted calculus are obtained. The practical consequences for an implementation of an automated theorem prover based on the many-sorted calculus are described.

Contents

"As a rule," said Holmes,"the more bizarre a thing is the less mysterious it proves to be. It is your commonplace, featureless cases which are really puzzling."

A.C. Doyle, The Red-Headed League

1. Introduction

Sorts are frequently used in practical applications of the first-order predicate calculus. For example we write formulas like

(i) $\forall x{:}S.\ \Phi(x)$ and $\exists x{:}S.\ \Phi(x)$

and treat them *formally* as *abbreviations* for

(ii) $\forall x.\ S(x) \supset \Phi(x)$ and $\exists x.\ S(x) \wedge \Phi(x)$.

We use *well sorted* formulas, because they provide a convenient shorthand notation for ordinary first-order formulas. But sorts also *influence the deductions* from a given set of well sorted formulas. For instance, if P is a predicate only defined on the sort $\mathbb{Z}$ of integers, we will never perform a deduction like $\forall x{:}\mathbb{Z}.\ P(x) \vdash P(\sqrt{2})$. Proofs are simplified, because a *many-sorted calculus* is more adapted to a *many-sorted theory* and hence not surprisingly *deductions which respect sorts* as well as the *usage of well sorted formulas* reflect the everyday usage of predicate logic. But *which advantages* do we really have by using sorts and *what kind* of calculus are we actually working in?

Let us sketch how a *many-sorted* (mehrsortig) calculus is developed from a given (sound and complete) first-order one-sorted (einsortig) calculus: Assume we have a set of *sort symbols*, ordered by a given *subsort order*. Variable and function symbols (of our given calculus) are associated with a certain sort symbol. The *sort of a term*, which is different from a variable, is determined by the sort of its outermost function symbol. In the construction of *well sorted* (sortenrecht) formulas, we allow for each argument position of a function or predicate symbol only well sorted terms of a certain *domainsort* or of a *subsort* of this domainsort.

The *inference rules* of the many-sorted calculus are the inference rules of the given calculus, but with the restriction that *only well sorted formulas* can be deduced by an application of the restricted inference rules. Starting with well sorted formulas this guarantees that only well sorted formulas are derived in a deduction. Now let $\vdash_{\Sigma} \Phi$ denote that Φ is a theorem of the many-sorted calculus. We write AX $\vdash_{\Sigma} \Phi$ to indicate that there is a deduction of Φ from the hypotheses AX. Further let us assume that we have a *notion of truth* for well sorted formulas. We write $\Vdash_{\Sigma} \Phi$ to indicate the validity of the well sorted formula Φ and let AX $\Vdash_{\Sigma} \Phi$ denote the semantic implication. Obviously we are only interested in a many-sorted calculus which is sound and complete, i.e. we allow only definitions of $\vdash_{\Sigma}$ and $\Vdash_{\Sigma}$ which guarantee

(1) $\Vdash_{\Sigma} \Phi$ iff $\vdash_{\Sigma} \Phi$, for each well sorted formula Φ.

Let us assume our definitions satisfy (1). Then we may ask, which formulas do we expect as theorems of the many-sorted calculus *compared* to its one-sorted counterpart? To facilitate a comparison between the calculi, we represent the relations between the function symbols and the sort symbols as well as the subsort order by the set A^{Σ} of *sort axioms* (Sortenaxiome), i.e. a set of first-order formulas. For a well sorted formula Φ, e.g. (i), the *relativization* $\hat{\Phi}$ (Sortenbeschränkung, Relativierung) of Φ is the *unabbreviated* version of Φ, e.g. (ii), where sort symbols are used as unary predicate symbols to express the sort of a variable. Now we can state what kind of theorems we expect in a many-sorted calculus: Our definitions of $\vdash_{\Sigma}$ and $\Vdash_{\Sigma}$ should ensure

(2.1) $\Vdash_{\Sigma} \Phi$ iff $A^{\Sigma} \Vdash \hat{\Phi}$

(2) and

(2.2) $\vdash_{\Sigma} \Phi$ iff $A^{\Sigma} \vdash \hat{\Phi}$, for each well sorted formula Φ.

Condition (2) is called the *Sort-Theorem* (Sortensatz), (2.1) is its *modeltheoretic part* and (2.2) its *prooftheoretic part*.

The Sort-Theorem also shows the advantages we have using a many-sorted calculus: We obtain a *shorter deduction* with *smaller formulas* from a *smaller set of hypotheses*, when proving $\vdash_{\Sigma} \Phi$ instead of $A^{\Sigma} \vdash \hat{\Phi}$.

The reason is that deductions about sortrelationships, which are performed explicitly in the one-sorted calculus, are *built into the inference mechanism* in the many-sorted calculus.

The connection between a first-order one-sorted calculus and its many-sorted counterpart can be summarized as follows:

$$\begin{array}{ccc} \models_{\Sigma} \Phi & \overset{(1)}{\longleftrightarrow} & \vdash_{\Sigma} \Phi \\ (2.1)\ \updownarrow & & \updownarrow\ (2.2) \\ A^{\Sigma} \models \hat{\Phi} & \underset{(3)}{\longleftrightarrow} & A^{\Sigma} \vdash \hat{\Phi} \end{array}$$

Suppose soundness and completeness of the given calculus (3) are known. Then in order to show the commutativity of the above diagram we either need a proof of both parts of the Sort-Theorem (2.1 and 2.2) or a proof of one of its parts (2.1 or 2.2) together with a proof of the soundness and completeness of the many-sorted calculus (1).

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In his thesis, *J. Herbrand* presented a many-sorted version of his calculus and proved the prooftheoretic part of the Sort-Theorem [Her30]. However *Herbrand's* proof is inadequate, because he did not consider that certain deductions in his one-sorted calculus cannot be translated to deductions in the many-sorted calculus. This was pointed out by *A. Schmidt* [Sch38], who proposed a many-sorted version of a Hilbert-Calculus without subsorts and proved the prooftheoretic part of the Sort-Theorem for this calulus [Sch38, Sch51].

H. Wang defined a many-sorted version of a Hilbert-Calculus without function symbols and subsorts [Wan52]. He proved the soundness and completeness of his calculus and the modeltheoretic part of the Sort-Theorem. *Wang* also gave an alternative proof of the prooftheoretic part of the Sort-Theorem by an application of the Herbrand-Theorem.

P.C. Gilmore pointed out that this proof is inadequate. He extended the many-sorted calculus of *Wang* by the introduction of subsorts and presented an improved version of the proof-theoretic part of the Sort-Theorem for this extended calculus [Gil58].

T. Hailperin presented a calculus which can be viewed as a generalization of *Wang's* many-sorted calculus [Hai57]. In this calculus sortrelationships can be expressed by arbitrary first-order formulas instead of atomar formulas, i.e. unary predicates. *Hailperin* proved a theorem which corresponds to the prooftheoretic part of the Sort-Theorem.

A. Oberschelp [Obe62] proposed several many-sorted versions of a calculus of *Montague* and *Henkin* [MH56]. In these calculi function symbols and subsorts are admitted. *Oberschelp* proved the soundness and completeness of his calculi and also gave the proofs for the modeltheoretic parts of the Sort-Theorems.

A.V. Idel'son discussed forms of many-sorted calculi of constructive mathematical logic [Ide64], which are based on the calculus of natural deduction [Gen34].

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With the emerging field of *Automated Theorem Proving* a first-order calculus becomes *a practical tool* to find mathematical proofs. The advantages of a many-sorted calculus are well recognized within this field, e.g. [Hay71, Hen72]. Also several theorem proving programs have been based on some kind of a many-sorted calculus, e.g. [Wey77,Cha78,BM79] (unfortunately without a sound theoretical foundation). Thereby the works cited above become of *practical* significance. Most theorem proving programs are based on a first-order calculus whose inference rules are *factorization*, *resolution* and *paramodulation* [Rob65, WR73] and whose formulas (called *clauses*) are in skolemized conjunctive normal form [Lov78]. We call such a calculus an *RP-calculus*.

In this paper, we define the *ΣRP-calculus*, i.e. a many-sorted version of the RP-calculus, and introduce a notion of unsatisfiability of sets of well sorted clauses.

We prove soundness and completeness of the ΣRP-calculus, as well as the modeltheoretic part of the Sort-Theorem, i.e. we show that the following diagram is commutative:

$$\begin{array}{ccc} S \text{ is } \Sigma E\text{-unsatisfiable} & \overset{(1)}{\longleftrightarrow} & S^E \vdash_{\Sigma RP} \square \\ (2.1)\ \updownarrow & & \updownarrow\ (2.2) \\ \hat{S} \cup A^{\Sigma} \text{ is } E\text{-unsatisfiable} & \underset{(3)}{\longleftrightarrow} & \hat{S}^E \cup A^{\Sigma} \vdash_{RP} \square \end{array}$$

Here S^E denotes the extension of the set S of well sorted clauses by all functionally-reflexive axioms [WR73] and □ denotes the empty clause.

We consider *term rewriting under sorts* because important aspects of paramodulation are related to term rewriting.

We exhibit that the ΣRP-calculus is only complete provided the subsort order imposes a certain structure on the set of sort symbols. Moreover in the case of paramodulation the set of well sorted clauses to be refuted has to be in a certain format to ensure completeness. These restrictions are *specific* to the ΣRP-calculus, because they are imposed by the principle of *most generality*, which is essential for the RP-calculus.

We show that these restrictions can be abandoned without loosing completeness, if the ΣRP-calculus is extended by an additional inference rule, the so called *weakening rule*. This rule is specific to a *many*-sorted calculus, because it cannot be applied if only *one* sort is given, and hence in our formulation is the RP-calculus but a special case of the ΣRP-calculus. We present special results about *unification under sorts*, which are necessitated by the weakening rule.

The *practical application* of the ΣRP-calculus in Automated Theorem Proving, leads to a *drastic reduction of the search space* and to *shorter refutations* of *smaller sets* of *shorter clauses* as compared to the RP-calculus. We describe all necessary modifications to extend an automated theorem prover based on the RP-calculus, yielding an automated theorem prover for the ΣRP-calculus and it can be seen that the advantages of the ΣRP-calculus hardly cause any additional costs by the new inference rules.

The practical usefulness of the ΣRP-calculus has been demonstrated by an implementation in an existing proof procedure [BES81,Ohl82].

Throughout the paper we use the following standard mathematical notation:

id	identity function
$f_{\mid M}$	function f restricted to a subset M of its domain
$f(t)\downarrow$	t is the domain of the function f
$f(t)\uparrow$	*not* $f(t)\downarrow$
$\circ$	composition of functions
|	negation, e.g. $x \not\leq y$ means *not* $x \leq y$
|M|	cardinality of set M
$M \backslash N$	set theoretic difference of M and N
M-L	abbreviates $M \backslash \{L\}$
⍁	end of case in a proof by cases
⊠	end of an example, definition or proof
∇	contradiction

2. *Basis Notions of the RP-Calculus*

Syntactic Notions Given pairwise disjoint alphabets, the infinite set of *variable symbols* $\mathfrak{V}$, the non-empty set of *function symbols* $\mathfrak{F}$ and the non-empty set of *predicate symbols* $\mathfrak{P}$, together with an *arity-function* for function and predicate symbols, we let T denote the set of all well formed *terms* over $\mathfrak{V}$ and $\mathfrak{F}$ and let AT denote the set of all well formed *atoms* over $\mathfrak{V}$, $\mathfrak{F}$ and $\mathfrak{P}$. $\mathfrak{C}$ stands for the set of all *constant symbols*, i.e. function symbols with arity 0.

A *literal* is an atom (also called a *positive* literal) or an expression of the form *not* A, where A is an atom (also called a *negative* literal). A pair of literals is called *complementary*, if one of the literals is positive and the other is negative. Given a literal L, $|L|$ denotes the *atom* of L and L^c denotes L's *complement*. The *predicate letter* of L is P iff $|L| = P(t_1 \ldots t_n)$ for some $t_i \in T$. LIT denotes the set of all literals. As usual a *clause* is a finite set of literals and $\square$ denotes the *empty clause*. The *clause language* $\mathfrak{L}$ is the set of all clauses over $\mathfrak{V}$, $\mathfrak{F}$ and $\mathfrak{P}$.

For a set D of terms, literals or clauses, *vars*(D) is defined as the set of all variable symbols in D. D is *variable disjoint* iff for all $q,r \in D$, $vars(\{q\}) \cap vars(\{r\}) = \emptyset$, provided that $q \neq r$.

The subscript *gr* abbreviates *ground*, which stands for *variable free*, e.g. a *ground term* is a variable free term and T_{gr} is the set of all ground terms. AT_{gr}, LIT_{gr} and $\mathfrak{L}_{gr}$ are defined in a similar way.

When concerned with equality reasoning, we use E as the syntactic *equality sign* and assert $E \in \mathfrak{P}$. S^E denotes the *extension* of the clause set S by all functionally-reflexive axioms [WR73]. The set of all *equality atoms* AT^E is defined as $AT^E = \{E(q\ r) \mid q,r \in T\}$.

Substitutions and Unifiers A *substitution* σ is a function which maps terms to terms and satisfies

(1) $\sigma \circ \sigma = \sigma$,

(2) $\sigma_{|\mathfrak{C}} = id$,

(3) $\sigma f(t_1 \ldots t_n) = f(\sigma t_1 \ldots \sigma t_n)$, and

(4) $\{x \in \mathfrak{V} \mid \sigma x \neq x\}$ is finite .

By conditions (1), (2) and (3) each substitution σ is completely determined by its restriction $\sigma_{|\mathfrak{V}}$. We make frequently use of this property, for instance we write $\{x_1 \leftarrow t_1, \ldots, x_n \leftarrow t_n\}$ to *represent* a substitution σ with $\sigma_{|\mathfrak{V}}\, x_i = t_i$. ε is the *identity substitution* and SUB denotes the set of all substitutions. Applications of substitutions to literals, to sets of terms and to sets of literals are defined in the obvious way.

The *domain* of a substitution σ, denoted DOM(σ), is the set of all variable symbols x with $\sigma x \neq x$. The *codomain* of σ, denoted COD(σ), is defined as σ(DOM(σ)). We say two substitutions θ and λ *agree* on a subset V of $\mathfrak{V}$, denoted $\theta = \lambda[V]$ iff $\theta x = \lambda x$ for each $x \in V$. The following lemma is frequently used throughout this paper:

Lemma 2.1 Let $\theta, \lambda \in$ SUB, $t \in T$ and $V, W \subset \mathfrak{V}$. Then

(1) $= [V]$ is an equivalence relation ,

(2) $\theta = \lambda[V \cup W]$ iff $\theta = \lambda[V]$ and $\theta = \lambda[W]$, and

(3) if vars($\{t\}$) $\subset V$ and $\theta = \lambda[V]$, then $\theta t = \lambda t$.

For a given subset V of $\mathfrak{V}$, a *renaming substitution* for V is a substitution ν satisfying

(1) DOM(ν) = V ,

(2) COD(ν) $\subset \mathfrak{V}$, and

(3) $\nu_{|V}$ is injective .

A renaming substitution for a clause C or for a set of clauses S is a renaming substitution for the set of variable symbols in C or in S.

We say a substitution σ is a *ground substitution* iff COD(σ) $\subset T_{gr}$. SUB_{gr} denotes the set of all ground substitutions. For a clause C and a substitution σ, σC is called an *instance* of C. If $\sigma C \in \mathfrak{L}_{gr}$, then σC is a *ground instance* of C.

Given a non-empty set D of terms or atoms, we call a substitution σ a *unifier* of D and say that D is *unifiable* iff $|\sigma D| = 1$. σ is called a *most general unifier* (or *mgu*) of D iff σ unifies D and satisfies $\theta = \theta \circ \sigma$ for each unifier θ of D.

Subterm selectors A partial function α which maps terms to terms is called an *argument selector* iff there exists a natural number k for α such that for *each* term $f(t_1,\dots,t_n), \alpha(f(t_1\dots t_n)) = t_k$, provided $k \le n$. SEL denotes the set of all argument selectors. The identity function on terms, an argument selector or a finite composition of argument selectors is called a subterm selector, or *selector* for short. We let SEL* denote the set of all selectors and define SEL^+ as $SEL^* \setminus \{id_{|T}\}$. Selectors are sometimes called *occurrences* or *positions* in the literature and are often represented by finite strings of natural numbers (cf. [Ros73]).

Each selector α induces a symmetric and transitive relation $\underset{\alpha}{\sim}$ on T by $q \underset{\alpha}{\sim} r$ iff $\alpha(q){\downarrow}$, $\alpha(r){\downarrow}$ and q differs from r at most on the subterms of q and r selected by α. $\lessdot$ is a *partial order* on SEL* defined as $\alpha \lessdot \beta$ iff $\alpha = \delta \circ \beta$ for some $\delta \in SEL^+$, i.e. α selects a subterm of the subterm selected by β.

A pair of selectors α and β are called *weakly independent*, denoted $\alpha \mathrel{\underline{\perp}} \beta$, iff $\alpha \not\lessdot \beta$ and $\alpha \not\gtrdot \beta$. α and β are *strongly independent*, written $\alpha \perp \beta$, iff $\alpha \not\mathrel{\underline{\lessdot}} \beta$ and $\alpha \not\mathrel{\underline{\gtrdot}} \beta$, where $\alpha \mathrel{\underline{\lessdot}} \beta$ abbreviates $\alpha \lessdot \beta$ or $\alpha = \beta$.

For a given set of literals I, a pair of terms q,r and a selector α, the expression $q \underset{\alpha}{\rightarrow}_I r$ is an abbreviation for $q \underset{\alpha}{\sim} r$ and $E(\alpha(q)\alpha(r)) \in I$. The following lemma is frequently used in the subsequent sections (cf. [Ros73]):

Lemma 2.2 Let $q,r \in T$, $\alpha,\beta \in SEL^*$, $I \subset LIT$ and $\sigma \in SUB$. Then

(1) if $q \underset{\alpha}{\sim} r$ and $\alpha(q) = \alpha(r)$, then $q = r$,

(2) $q \underset{\beta\circ\alpha}{\sim} r$ iff $\alpha(q) \underset{\beta}{\sim} \alpha(r)$ and $q \underset{\alpha}{\sim} r$,

(3) $q \underset{\beta\circ\alpha}{\longrightarrow}_I r$ iff $\alpha(q) \underset{\beta}{\rightarrow}_I \alpha(r)$ and $q \underset{\alpha}{\sim} r$,

(4) if $q \underset{\alpha}{\sim} r$, then $\sigma q \underset{\alpha}{\sim} \sigma r$,

(5) if $\alpha(q){\downarrow}$, then $\sigma\alpha(q) = \alpha(\sigma q)$,

(6) if $\alpha \perp \beta$ and $q \underset{\beta}{\sim} r$, then $\alpha(q) = \alpha(r)$, and

(7) if $\alpha \mathrel{\underline{\perp}} \beta$ and $q \underset{\alpha}{\rightarrow}_I q' \underset{\beta}{\rightarrow}_I r$ for some $q' \in T$, then $q \underset{\beta}{\rightarrow}_I r' \underset{\alpha}{\rightarrow}_I r$ for some $r' \in T$.

Selectors different from $id_{|T}$ are applied to atoms as to terms. For a literal L we assert $\alpha(L) = \alpha(|L|)$. For a pair of literals L and K, we define $L \underset{\alpha}{\sim} K$ as for terms but with the proviso that either both literals have to be positive or both have to be negative. Additionally we assert that for a literal L, $id_{|T}(L)\downarrow$ is always false. Then for a pair of literals L and K, $L \underset{\alpha}{\rightarrow}_I K$ is defined as for terms and Lemma 2.2 holds for literals as well.

Term Rewriting A *(ground) term rewriting system* is a set of directed equations $\mathcal{R} = \{E(q_i r_i) \in AT^E_{gr} | i \in J\}$ where $J \subset \mathbb{N}$. We define $\Rightarrow_{\mathcal{R}}$ by $q \Rightarrow_{\mathcal{R}} r$ iff $E(q\ r) \in \mathcal{R}$ and we use $\longrightarrow_{\mathcal{R}}$ to denote the *reduction relation* associated with $\mathcal{R}$, that is $q \longrightarrow_{\mathcal{R}} r$ iff $q \underset{\alpha}{\rightarrow}_{\mathcal{R}} r$ for some $\alpha \in SEL^*$. We use the standard notation $\overset{+}{\longrightarrow}_{\mathcal{R}}$ for the *transitive closure* of $\longrightarrow_{\mathcal{R}}$ and $\overset{*}{\longrightarrow}_{\mathcal{R}}$ for the *reflexive closure* of $\overset{+}{\longrightarrow}_{\mathcal{R}}$. If for two ground terms q and r there exists a sequence $q_1,\ldots,q_{n+1} \in T_{gr}$ and a sequence $\alpha_1,\ldots,\alpha_n \in SEL^*$ such that
$q = q_1 \underset{\alpha_1}{\longrightarrow}_{\mathcal{R}} q_2 \ldots q_n \underset{\alpha_n}{\longrightarrow}_{\mathcal{R}} q_{n+1} = r$, we call $q_1 \underset{\alpha_1}{\longrightarrow} \ldots \underset{\alpha_n}{\longrightarrow} q_{n+1}$ an $\mathcal{R}$-*rewrite of r from q with length n.*

$\mathcal{R}$ is called *symmetric* iff $\Rightarrow_{\mathcal{R}}$ is a symmetric relation. $\overset{*}{\longrightarrow}_{\mathcal{R}}$ is an *equivalence relation* if $\mathcal{R}$ is symmetric.

We like to manipulate *ground literals* by a term rewriting system and extend $\longrightarrow_{\mathcal{R}}$ in the obvious way, i.e. $L \longrightarrow_{\mathcal{R}} K$ for a pair of ground literals L and K iff $L \underset{\alpha}{\rightarrow}_{\mathcal{R}} K$ for some $\alpha \in SEL^+$.

For a set of literals I, the *term rewriting system* $\mathcal{R}(I)$ *contained in I* is defined as $\mathcal{R}(I) = I \cap AT^E_{gr}$.

Inference Rules and Deductions $Res(C,L,D,K,\sigma) = \sigma(C-L) \cup \sigma(D-K)$ is the *resolvent* of clauses C and D upon literals L and K, where σ is an mgu of $\{|L|,|K|\}$. A substitution σ *factors* a clause C and σC is a *factor* of C iff σ is an mgu of some subset of C or $\sigma = \gamma \circ \tau$, τ factors C and γ is an mgu of some subset of τC.
$Par(C,L,D,E(q\ r),\alpha,\sigma) = \sigma(C-L) \cup \sigma(D-E(q\ r)) \cup \{\sigma K\}$ is a *paramodulant* of clauses C and D upon L and E(q r) iff σ is an mgu of $\{\alpha(L),q\}$, $\sigma L \underset{\alpha}{\sim} \sigma K$ and $\alpha(\sigma K) = \sigma r$ (or σ is an mgu of $\{\alpha(L),r\}$, $\sigma L \underset{\alpha}{\sim} \sigma K$ and $\alpha(\sigma K) = \sigma q$), where σK is the *modulant literal* [Lov78].

Given a variable disjoint set S of clauses and a clause C, $S \vdash C$ denotes the existence of a *deduction* of C from S, i.e. there exists a list of clauses $\langle B_1,\ldots,B_n\rangle$ such that $B_n = C$ and $B_i \in S$

or $B_i = \nu_i R$, where $1 \le i \le n$, R is a resolvent, paramodulant or factor of clauses preceeding B_i in the list and ν_i is a renaming substitution for $S \cup \{B_1, \dots, B_{i-1}\}$. As usual, a *refutation* of S is a deduction of the empty clause from S. $S \vdash_R C$ is a deduction without paramodulation and $S \vdash_P$ is a deduction without resolution.

Semantic Notions Given a set of clauses S, S_{gr} denotes the set of all *ground instances* of the clauses in S. Computing S_{gr} for a given clause set S, we will agree that $\mathfrak{F}$ and $\mathfrak{P}$ are minimal, i.e. each symbol from $\mathfrak{F}$ and $\mathfrak{P}$ is used in at least one of the clauses of S. This guarantees that T_{gr} is the *Herbrand Universe* of S and AT_{gr} is the *Herbrand Base* of S [Lov78], if we assume in addition that $\mathfrak{C} = \{c\}$ for the case that S contains no constant symbol at all.

A (possibly infinite) subset I of LIT_{gr} is called an *interpretation* iff for each $L \in I$, $L^c \notin I$. I is called *reflexive* iff for each $t \in T_{gr}$, $E(t\,t) \in I$. We say that I is *E-closed* iff for each $L \in I$, $K \in LIT_{gr}$ and some $\alpha \in SEL^+$, $K \in I$ whenever $L \xrightarrow[\alpha]{}_I K$. A reflexive and E-closed interpretation is an *E-interpretation*. The following lemma is used constantly:

Lemma 2.3 Let $I \subset LIT_{gr}$ be E-closed. Then

(1) if I is reflexive, then $E(q\,r) \in I$ iff $E(r\,q) \in I$, and

(2) if $L \in I$, $K \in LIT_{gr}$ and $L \xrightarrow{*}_{\mathfrak{R}(I)} K$, then $K \in I$.

An interpretation I *satisfies a ground clause* C iff $I \cap C \neq \emptyset$. I *satisfies a clause* C iff I satisfies each ground instance σC of C. I *satisfies a set of clauses* S iff I satisfies *each* clause in S. In this case I is a *model* of S and S is *satisfiable*. If I is an E-interpretation, I *E-satisfies* S, I is an *E-model* of S and S is *E-satisfiable*.

3. *Basic Notions of the ΣRP-Calculus*

<u>*Sorts and Signatures*</u> Let $\mathfrak{S}$ be a finite and non-empty set of *sort symbols*. The *subsort order* $\leq_{\mathfrak{S}}$ imposed on $\mathfrak{S}$ is a partial order on $\mathfrak{S}$. We say s_1 is a *subsort* of s_2 and s_2 is a *super-sort* of s_1 iff $s_1 \leq_{\mathfrak{S}} s_2$. We use $s_1 <_{\mathfrak{S}} s_2$ as an abbreviation for $s_1 \leq_{\mathfrak{S}} s_2$ and $s_1 \neq s_2$. s_1 is a *direct subsort* of s_2, $s_1 \ll_{\mathfrak{S}} s_2$, iff $s_1 <_{\mathfrak{S}} s_2$ and there is no s with $s_1 <_{\mathfrak{S}} s <_{\mathfrak{S}} s_2$. If $\mathfrak{S}$ is known from the context we shall sometimes omit the indices, e.g. we write $\leq$ for $\leq_{\mathfrak{S}}$.

$\langle \mathfrak{S}, < \rangle$ is a *well founded set*, because $\mathfrak{S}$ is a finite and $<$ is irreflexive and assymmetric. This fact will be used in later proofs. Subsequently we only consider ordered sets of sorts symbols $\langle \mathfrak{S}, \leq \rangle$, which posess a *maximal element* s_o, i.e. $s \leq s_o$ for each $s \in \mathfrak{S}$. We say that $\langle \mathfrak{S}, \leq \rangle$ is a *tree structure*, whenever $s_1 \geq s \leq s_2$ implies $s_1 \leq s_2$ or $s_1 \geq s_2$ for all sort symbols $s, s_1, s_2 \in \mathfrak{S}$.

For a given $\langle \mathfrak{S}, \leq \rangle$ let $\mathfrak{S}^*$ be the set of all finite strings from $\mathfrak{S}$, including the empty string e. Now for each $s \in \mathfrak{S}$ and for each $w \in \mathfrak{S}^*$ let $\mathfrak{V}_s$ be a set of variable symbols, $\mathcal{F}_{w,s}$ a set of function symbols and $\mathfrak{P}_w$ a set of predicate symbols such that all these sets are pairwise disjoint. Additionally we shall assume, that for each s which is *minimal* in $\langle \mathfrak{S}, \leq \rangle$, $\mathcal{F}_{e,s} \neq \emptyset$ and we assert that $E \in \mathfrak{P}_{s_o s_o}$. Then an *$\mathfrak{S}$-sorted signature* Σ is a family $\Sigma_{w,s}$ of sets such that $\Sigma_{w,s} = \mathfrak{V}_s \cup \mathcal{F}_{w,s} \cup \mathfrak{P}_w$. Setting $\mathfrak{V} = \bigcup \mathfrak{V}_s$, $\mathcal{F} = \bigcup \mathcal{F}_{w,s}$, $\mathfrak{C} = \mathcal{F}_{e,s}$ and $\mathfrak{P} = \bigcup \mathfrak{P}_w$ for each $s \in \mathfrak{S}$ and each $w \in \mathfrak{S}^*$, we define terms, atoms, literals e.t.c. as in section 2.

<u>*Syntactic Notions*</u> For a variable or function symbol a, the *rangesort* [a] of a is s iff $a \in \mathfrak{V}_s \cup \mathcal{F}_{w,s}$ (for some $w \in \mathfrak{S}^*$). For a function or predicate symbol m, the *ith domainsort* $[m]_i$ of m is s_i iff $m \in \mathcal{F}_{s_1 \ldots s_k, s} \cup \mathfrak{P}_{s_1 \ldots s_k}$, provided $1 \leq i \leq k$. The *sort* [t] of a term t is s iff $t \in \mathfrak{V}_s \cup \mathcal{F}_{e,s}$ or $t = f(t_1 \ldots t_k)$ and $[f] = s$.

For a term t and a selector α with $\alpha(t)\downarrow$ we define the *α-maximal sort* of t, denoted $[t]_\alpha$, by

(1) $[t]_\alpha = [t]$, if $\alpha = id_{|T}$,

(2) $[t]_\alpha = [f]_i$, if $t = f(t_1 \ldots t_n)$, $\alpha(t) = t_i$ and $\alpha \in SEL$, and

(3) $[t]_\alpha = [\delta(t)]_\beta$, if $\alpha = \beta \circ \delta$ with $\beta \in SEL$ and $\delta \in SEL^+$.

For $\alpha \in SEL^+$ we apply this definition to atoms as well and assert for each literal L that $[L]_\alpha = [|L|]_\alpha$. The following lemma is easy to prove:

<u>*Lemma 3.1*</u> Let $q, r \in T$, $\alpha, \beta \in SEL^*$ and $\sigma \in SUB$. Then

(1) if $\beta \in SEL^+$, then $[q]_{\beta \circ \alpha} = [\alpha(q)]_\beta$,

(2) if $q \notin \mathfrak{V}$ and $\alpha(q)\downarrow$, then $[q]_\alpha = [\sigma q]_\alpha$,

(3) if $q \underset{\beta}{\sim} r$ and $\alpha \not\leq\!\cdot\, \beta$, then $[\alpha(q)] = [\alpha(r)]$, and

(4) if $q \underset{\beta}{\sim} r$ and $\alpha \not<\!\cdot\, \beta$, then $[q]_\alpha = [r]_\alpha$.

Given an $\mathcal{S}$-sorted signature Σ, a term t is called a *well sorted* term or a Σ-*term* iff $[\alpha(t)] \leq [t]_\alpha$ for *each* selector α with $\alpha(t)\downarrow$. We say an atom A is *well sorted* or A is a Σ-*atom* iff $A \in \mathcal{P}_e$ or $[\alpha(A)] \leq [A]_\alpha$ for *each* selector α with $\alpha(A)\downarrow$. T_Σ denotes the set of all Σ-terms, AT_Σ denotes the set of all Σ-atoms and LIT_Σ is set of all Σ-*literals*. Later we shall frequently use the following lemmata:

<u>*Lemma 3.2*</u> Let $q \in T_\Sigma$, $r \in T$ and $\alpha \in SEL^*$.

If $q \underset{\alpha}{\sim} r$, $\alpha(r) \in T_\Sigma$ and $[\alpha(r)] \leq [r]_\alpha$, then $r \in T_\Sigma$.

<u>*Lemma 3.3*</u> Let $s \in \mathcal{S}$, $f \in \mathcal{F}_{s_1 \ldots s_n, s}$ and $f(q_1 \ldots q_n) \in T$. Then

(1) $\mathfrak{V}_s \cup \mathcal{F}_{e,s} \subset T_\Sigma$,and

(2) $f(q_1 \ldots q_n) \in T_\Sigma$ iff $q_i \in T_\Sigma$ and $[q_i] \leq s_i$ for each i with $1 \leq i \leq n$.

Obviously these lemmata hold for literals as well. A *well sorted* clause or a Σ-*clause* is a finite set of Σ-literals. The *many-sorted language* $\mathcal{L}_\Sigma$ is the set of all Σ-clauses. $T_{\Sigma gr}$ denotes the set of all variable free Σ-terms. $AT_{\Sigma gr}$, $LIT_{\Sigma gr}$ and $\mathcal{L}_{\Sigma gr}$ are defined in a similar way.

Sometimes we use sort symbols also as unary predicate symbols. We assume that $s \in \mathcal{P}_s$ for each $s \in \mathcal{S}$ and define $LIT^{\mathcal{S}}$ ($LIT^{\mathcal{S}}_\Sigma$) as the set of all (Σ-) literals in the *extended language* $\mathcal{L}^{\mathcal{S}}$ ($\mathcal{L}^{\mathcal{S}}_\Sigma$). An atom or a literal whose predicate letter is a sort symbol is called a *sort atom* or a *sort literal* respectively.

Sort Axioms and Relativizations Given an $\mathcal{S}$-sorted signature Σ, we define the set A^Σ of all *sort axioms* of Σ as the smallest subset of $\mathcal{L}^{\mathcal{S}}_\Sigma$ which satisfies

(1) $\{s(a)\} \in A^\Sigma$, if $a \in \mathcal{F}_{e,s}$,

(2) $\{not\ s_1(x_1),\dots,not\ s_k(x_k),\ s(f(x_1\dots x_k))\} \in A^\Sigma$, if $x_i \in \mathcal{V}_{s_i}$, $f \in \mathcal{F}_{s_1\dots s_k,s}$ and $x_i \neq x_j$,

(3) $\{not\ s_1(y),\ s_2(y)\} \in A^\Sigma$, if $y \in \mathcal{V}_{s_1}$ and $s_1 \ll s_2$, and

(4) no clause in A^Σ is a variant (i.e. can be obtained by a renaming substitution) of another clause in A^Σ and A^Σ is variable disjoint.

If $\mathcal{F}$ is finite, by condition (4) A^Σ is finite.

For a Σ-clause C, the *relativization* $\hat{C}^\Sigma$ *of clause* C is a clause in $\mathcal{L}^{\mathcal{S}}_\Sigma$ and defined as

$$\hat{C}^\Sigma = \{not\ s_1(x_1),\dots,not\ s_n\ (x_n)\} \cup C$$

where $x_i \in \mathcal{V}_{s_i}$ and $\{x_1,\dots,x_n\} = vars(C)$. The *relativization of a* Σ-*clause set* S, denoted $\hat{S}^\Sigma$, is the subset of $\mathcal{L}^{\mathcal{S}}_\Sigma$ defined as

$$\hat{S}^\Sigma = \{\hat{C}^\Sigma \in \mathcal{L}^{\mathcal{S}}_\Sigma \mid C \in S\} \quad .$$

If Σ is known from the context we write $\hat{C}$ instead of $\hat{C}^\Sigma$ and $\hat{S}$ for $\hat{S}^\Sigma$.

Substitutions and Unifiers A Σ-*substitution* σ is a substitution satisfying $\sigma(T_\Sigma) \subset T_\Sigma$. SUB_Σ denotes the set of all Σ-substitutions. A Σ-*renaming substitution* ν for a set D of variables, literals or clauses is a renaming substitution for D such that $[\nu x] = [x]$ for each $x \in \mathfrak{V}$. A Σ-*ground substitution* σ is a Σ-substitution with $COD(\sigma) \subset T_{\Sigma gr}$. $SUB_{\Sigma gr}$ denotes the set of all Σ-ground substitutions.

For a Σ-clause C and a Σ-substitution σ, σC is called a Σ-*instance* of C. If $\sigma C \in \mathfrak{L}_{\Sigma gr}$, then σC is a Σ-*ground instance* of C. The following lemma is easily shown:

Lemma 3.4 Let $\theta, \sigma \in SUB_\Sigma$ and $\lambda \in SUB$. Then

(1) if $\theta \circ \sigma \in SUB$, then $\theta \circ \sigma \in SUB_\Sigma$,

(2) if $\theta = \lambda \circ \sigma$, then $\theta = \delta \circ \sigma$ for some $\delta \in SUB_\Sigma$,

(3) $\sigma(LIT_\Sigma) \subset LIT_\Sigma$, and

(4) $\sigma(\mathfrak{L}_\Sigma) \subset \mathfrak{L}_\Sigma$.

A set D of Σ-terms or Σ-atoms is Σ-*unifiable* iff D is unifiable with a Σ-substitution σ. Then σ is a Σ-*unifier* of D. σ is a Σ-*mgu* of D iff σ is mgu of D and $\sigma \in SUB_\Sigma$.

A Σ-substitution μ is a *weakening substitution* for a set $V \subset \mathfrak{V}$ iff μ satisfies

(1) $COD(\mu) \subset \mathfrak{V}$,

(2) $COD(\mu) \cap V = \emptyset$,

(3) $\mu_{|V}$ is injective , and

(4) $[\mu x] < [x]$, if $x \in DOM(\mu)$.

For each $V \subset \mathfrak{V}$, $WSUB(V)$ denotes the set of all weakening substitutions for V. Obviously $\varepsilon \in WSUB(V)$ and $WSUB(V) \subset SUB_\Sigma$.

Term Rewriting For a (ground) term rewriting system $\mathcal{R}$ we define the *Σ-reduction relation* $\to_{\Sigma\mathcal{R}}$ associated with $\mathcal{R}$ and a signature Σ by $\to_{\Sigma\mathcal{R}} = \underset{*}{\to}_{\mathcal{R}} \cap (T_{\Sigma gr} \times T_{\Sigma gr})$. $\xrightarrow{+}_{\Sigma\mathcal{R}}$ is the transitive closure of $\to_{\Sigma\mathcal{R}}$ and $\longrightarrow_{\Sigma\mathcal{R}}$ is the reflexive closure of $\xrightarrow{+}_{\Sigma\mathcal{R}}$. $\mathcal{R}$ is a *Σ-maximal* term rewriting system iff $\Rightarrow_{\mathcal{R}} = \xrightarrow{+}_{\Sigma\mathcal{R}}$. Note that in general a Σ-maximal term rewriting system is *infinite*. The following lemma is frequently used throughout this paper:

Lemma 3.5 Let $\mathcal{R}$ be a Σ-maximal term rewriting system, $q,r \in T_{\Sigma gr}$, $t \in \{q,r\}$ and $\alpha,\beta \in SEL^*$. Then

(1) if $q \Rightarrow_{\mathcal{R}} r$, then $t \in T_{\Sigma gr}$, and

(2) if $q \xrightarrow[\beta]{}_{\mathcal{R}} r$, $\alpha \lessdot \beta$ and $\alpha(t)\downarrow$, then $[\alpha(t)] \le [t]_\alpha$.

Inference Rules and Deductions A resolvent R of two Σ-clauses is a *Σ-resolvent* iff the substitution used to form R is a Σ-substitution. If a Σ-substitution factors a Σ-clause, then this factor is a *Σ-factor*. If $P = Par(C,L,D,E(q\ r),\alpha,\sigma)$ is a parmodulant of the Σ-clauses C and D, $\sigma \in SUB_\Sigma$ and $[\sigma r] \le [\sigma L]_\alpha$ (or $[\sigma q] \le [\sigma L]_\alpha$ if we replace σr by σq), then P is called a *Σ-paramodulant*. If C is a Σ-clause and μ is a weakening substitution for some $V \supset vars(C)$, then μC is a *weakened variant* of C. Obviously, each Σ-resolvent, Σ-factor, Σ-paramodulant and each weakened variant is a Σ-clause.

Given a variable disjoint set of Σ-clauses S, $S \vdash_{\Sigma} C$ denotes the existence of a Σ-deduction of C from S, i.e. there exists a list of Σ-clauses $\langle B_1,\ldots,B_n \rangle$ such that $B_n = C$ and $B_i \in S$ or $B_i = \nu_i R$, where $1 \le i \le n$, R is a Σ-resolvent, Σ-factor, Σ-paramodulant or a weakened variant of clauses preceeding B_i in the list and ν_i is a Σ-renaming substitution for $S \cup \{B_1,\ldots,B_{i-1}\}$. A *$\Sigma$-refutation* is a Σ-deduction of the empty clause. $S \vdash_{\Sigma R} C$ denotes a Σ-deduction without Σ-paramodulants and $S \vdash_{\Sigma P} C$ is a Σ-deduction without Σ-resolution.

Semantic Notions Given a set of Σ-clauses S, $S_{\Sigma gr}$ denotes the set of all Σ-ground instances of the Σ-clauses in S. An interpretation I *Σ-satisfies a Σ-clause* C iff I satisfies each Σ-ground instance σC of C. I *Σ-satisfies a set of Σ-clauses* S iff I Σ-satisfies each clause in S. In this case, I is a *Σ-model* of S and S is *Σ-satisfiable*. If in addition I is an E-interpretation, then I *ΣE-satisfies* S, I is a *ΣE-model* of S and S is *ΣE-satisfiable*. It is easy to prove that:

Lemma 3.6 Let $S \subset \mathcal{L}_{\Sigma}$ and $I \subset LIT_{gr}$ be an interpretation. Then

(1) S is Σ-unsatisfiable iff $S_{\Sigma gr}$ is unsatisfiable ,

(2) S is ΣE-unsatisfiable iff $S_{\Sigma gr}$ is E-unsatisfiable ,

(3) $S_{\Sigma gr} \subset S_{gr}$, and

(4) if I Σ-satisfies S, then $I \cap LIT_{\Sigma gr}$ Σ-satisfies S .

Note that 3.6 (4) in general does *not* hold for ΣE-satisfiability, i.e. there exist E-interpretations I such that $I \cap LIT_{\Sigma gr}$ neither is reflexive nor is E-closed and hence is no E-interpretation.

Throughout the paper $\langle \mathcal{S}, \leq \rangle$ is a partially ordered set of sort-symbols, Σ is some $\mathcal{S}$-sorted signature and S stands for any variable disjoint set of Σ-clauses.

4. Two Examples

The following examples should provide some motivation for our work illustrating also the notions introduced so far. Also the examples demonstrate that the ΣRP-calculus is incomplete without the weakening rule. Essentially there are two reasons for this incompleteness:

- the Unification Theorem does not hold in the ΣRP-calculus,
- paramodulation is incomplete in the ΣRP-calculus (without the weakening rule).

Example 4.1 Let $\mathcal{S} = \{A,B,C,D\}$ with $D \ll B \ll A$ and $D \ll C \ll A$. Let $P \in \mathcal{P}_A$, $d \in \mathcal{F}_{e,D}$, $u \in \mathfrak{V}_B$, $v \in \mathfrak{V}_C$ and $w \in \mathfrak{V}_D$. Now consider the set of Σ-clauses $S = \{\{P(u)\}, \{\textit{not}\ P(v)\}\}$. S is Σ-unsatisfiable, because $S_{\Sigma gr} = \{\{P(d)\}, \{\textit{not}\ P(d)\}\}$ is unsatisfiable. But neither σ with $\sigma|_{\mathfrak{V}} = \{u \leftarrow v\}$ nor τ with $\tau|_{\mathfrak{V}} = \{v \leftarrow u\}$ are Σ-substitutions, i.e. no Σ-resolvent can be derived from the two clauses in S. But with the weakening rule, we find a Σ-refutation from S:

(C1)	$\forall u.$	$\{P(u)\}$	, given
(C2)	$\forall v.$	$\{\textit{not}\ P(v)\}$	, given
(R1)	$\forall w.$	$\{P(w)\}$	, weakened variant $\mu C1$ of C1, where $\mu|_{\mathfrak{V}} = \{u \leftarrow w\}$
(R2)		$\square$	, Σ-resolvent of C2 and R1, because Θ with $\Theta|_{\mathfrak{V}} = \{v \leftarrow w\}$ is a Σ-substitution.

Now let us consider the RP-calculus. Firstly we replace S by its relativization $\hat{S}^{\Sigma}$:

(C1')	$\forall u.$	$\{\textit{not}\ B(u), P(u)\}$	, $\hat{C1}^{\Sigma}$
(C2')	$\forall v.$	$\{\textit{not}\ C(v), \textit{not}\ P(v)\}$	, $\hat{C2}^{\Sigma}$

The set of sort axioms A^{Σ} is obtained as

(C3')	$\forall x.$	$\{\textit{not}\ D(x), B(x)\}$	, since $D \ll B$
(C4')	$\forall y.$	$\{\textit{not}\ D(y), C(y)\}$	, since $D \ll C$
(C5')		$\{D(d)\}$	, since $d \in \mathcal{F}_{e,D}$
(C6')	$\forall i.$	$\{\textit{not}\ B(i), A(i)\}$	, since $B \ll A$
(C7')	$\forall j.$	$\{\textit{not}\ C(j), A(j)\}$	, since $C \ll A$

Here is a refutation of $(\hat{S}^{\Sigma} \cup A^{\Sigma})$:

(R1') $\forall w.$ {*not* D(w), P(w)} , Res(C1',C3')

(R2') $\forall v.$ {*not* C(v), *not* D(v)} , Res(R1',C2')

(R3') $\forall y.$ {*not* D(y)} , Res(R2',C4')

(R4') $\square$, Res(R3',C5')

If we remove the literals whose predicate letters are sort symbols from the clauses C1', C2', R1' and R2' we obtain the previous Σ-refutation of S. The advantage of the ΣRP-calculus is obvious now: We get a *shorter* refutation (R1 and R2 instead of R1',...,R4') of *shorter* clauses (C_i,R_i instead of C_i',R_i') of a *smaller* set of clauses (S instead of ($\hat{S}^{\Sigma}$ $\cup$ A^{Σ})).

Note that we propose the weakening rule as an additional inference rule only in order to isolate the crucial point and to obtain completeness results. In a *proof procedure* this rule is realized by a modification of the *unification algorithm* (see section 11),i.e.in our system [BES81, Ohl82] the empty clause is derived from C1 and C2 by a single resolution step using the substitution $\Theta \circ \mu|_{\mathfrak{V}} = \{u \leftarrow w, v \leftarrow w\}$.

In order to compare the search spaces involved with the many-sorted calculus and its one-sorted counterpart we find *one* initial resolvent in S in contrast to *seven* initial resolvents in ($\hat{S}^{\Sigma}$ $\cup$ A^{Σ}). This demonstrates particularily well the drastic reduction of the search space, when working in the ΣRP-calculus instead of the RP-calculus. ⊠

However the modification of the unification algorithm only covers applications of the weakening rule as in the above example. Unfortunately there are cases which cannot be solved by the modified unification (cf. section 11):

Example 4.2 Let $\mathcal{S} = \{A,B\}$ with $B \ll A$ and let $P \in \mathcal{P}_B$, $\{b_1,b_2\} \subset \mathcal{F}_{e,B}$, $\{x,y\} \subset \mathfrak{V}_A$ and $z \in \mathfrak{V}_B$. S = {{P(b_1)}, {E(x y)}, {*not* P(b_2)}} is a ΣE-unsatisfiable set of Σ-clauses because $S_{\Sigma gr}$ = {{P(b_1)},{E(b_1 b_2)},..., {*not* P(b_2)}} is E-unsatisfiable. We can derive four paramodulants from S, namely {P(x)}, {P(y)}, {*not* P(x)} and {*not* P(y)} neither of which is a Σ-clause, i.e. not a Σ-paramodulant. But with the weakening rule we find a Σ-refutation of S:

(C1) {P(b_1)} , given

(C2) $\forall x,y.$ {E(x y)} , given

(C3) {*not* P(b_2)} , given

(C4) $\forall x,z.$ {E(x z)} , weakened variant μC2 of C2,where $\mu|_{\mathfrak{V}} = \{y \leftarrow z\}$

(C5) $\forall z.$ {P(z)} , Σ-paramodulant of C1 and C4

(C6) $\square$, Σ-resolvent of C5 and C3.

⊠

5. *Term Rewriting under Sorts*

Since certain aspects of paramodulation can best be described using *term rewriting systems*, we present some results for term rewriting under sorts. For our purposes we can restrict ourselves to the ground case. In this section we prove the

Σ-Rewrite Theorem If $\mathcal{R}$ is a Σ-maximal term rewriting system, then $\xrightarrow{+}_{\mathcal{R}} \cap (T_{\Sigma gr} \times T_{\Sigma gr}) = \xrightarrow{+}_{\Sigma\mathcal{R}}$

For each pair $q_1, q_{n+1} \in T_{\Sigma gr}$ with $q_1 \xrightarrow{+}_{\mathcal{R}} q_{n+1}$ by the Σ-Rewrite Theorem we can find an $\mathcal{R}$-rewrite of q_{n+1} from q_1 such that each term in this $\mathcal{R}$-rewrite is a Σ-ground term, provided $\mathcal{R}$ is Σ-maximal. This is illustrated in the following diagram:

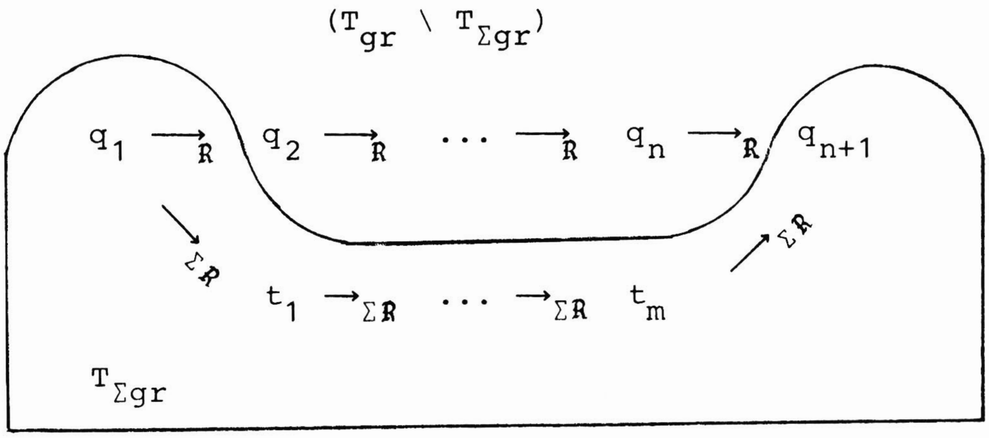

Figure 5.1 The Σ-Rewrite Theorem

Let us sketch the proof of the Σ-Rewrite Theorem before we go into details. The main problem is to prove that $q_1 \xrightarrow{+}_{\Sigma\mathcal{R}} q_{n+1}$, if $\{q_2,\ldots,q_n\} \not\subseteq T_{\Sigma gr}$. We give a constructive proof to find an $\mathcal{R}$-rewrite $q_1 \to r_2 \to \ldots \to r_{n-1} \to q_{n+1}$, which is *shorter* than the initially given $\mathcal{R}$-rewrite

(1) $$q_1 \xrightarrow[\alpha_1]{} q_2 \ldots q_n \xrightarrow[\alpha_n]{} q_{n+1}.$$

So we can successively *remove* those terms in the $\mathcal{R}$-rewrite, which are not in $T_{\Sigma gr}$.

The construction works as follows: By the *Σ-Rewrite Lemma* we can single out from (1) a certain $\mathcal{R}$-rewrite

(2) $$q_{i-1} \xrightarrow[\alpha_{i-1}]{} q_i \ldots q_{j-1} \xrightarrow[\alpha_{j-1}]{} q_j, \quad (2 \le i < j \le n+1)$$

and it is shown in the *Shift-Up Lemma* that (2) is still an $\mathcal{R}$-rewrite, if we replace each selector α_h $(i \le h \le j-1)$ by the selector α_{i-1}, provided that $\alpha_h \lessdot \alpha_{i-1}$.

By the *Shift-Left Lemma* the selector α_{j-1} can be moved to the left yielding a new $\mathcal{R}$-rewrite

(3) $$q_{i-1} \xrightarrow[\alpha_{j-1}]{} r_i \xrightarrow[\alpha_{i-1}]{} r_{i+1} \ldots r_{j-1} \xrightarrow[\alpha_{j-2}]{} q_j \quad .$$

Additionally we can show that $\alpha_{j-1} = \alpha_{i-1}$, i.e.

(4) $$q_{i-1} \xrightarrow[\alpha_{i-1}]{} r_i \xrightarrow[\alpha_{i-1}]{} r_{i+1} \ldots r_{j-1} \xrightarrow[\alpha_{j-2}]{} q_j$$

is an $\mathcal{R}$-rewrite. Finally we use the *Reduction Lemma* to reduce (4) to

(5) $$q_{i-1} \xrightarrow[\alpha_{i-1}]{}_{\mathcal{R}} r_{i+1} \ldots r_{j-1} \xrightarrow[\alpha_{j-2}]{}_{\mathcal{R}} q_j \quad .$$

Thus we have found an $\mathcal{R}$-rewrite

(6) $$q_1 \xrightarrow[\alpha_1]{} q_2 \ldots q_{i-1} \xrightarrow[\alpha_{i-1}]{} r_{i+1} \ldots r_{j-1} \xrightarrow[\alpha_{j-2}]{} q_j \ldots q_{n-1} \xrightarrow[\alpha_n]{} q_n$$

of length n-1.

<u>*Lemma 5.1*</u> (Σ-Rewrite Lemma) Let $\mathcal{R}$ be a Σ-maximal term rewriting system and $q_1 \xrightarrow[\alpha_1]{} q_2 \ldots q_n \xrightarrow[\alpha_n]{} q_{n+1}$, $n \ge 1$, be an $\mathcal{R}$-rewrite such that $q_1, q_{n+1} \in T_{\Sigma gr}$. If $\{q_2,\ldots,q_n\} \not\subseteq T_{\Sigma gr}$ then there exist indices i and j with $2 \le i < j \le n+1$ such that for each h with $i \le h \le j-1$

(1) $\alpha_{j-1} = \alpha_{i-1}$,
(2) $\alpha_{i-1}(q_h) \in T_{\Sigma gr}$, and
(3) $[\alpha_{i-1}(q_h)] \nleq [q_h]_{\alpha_{i-1}}$.

Proof Since $\{q_2,\ldots,q_n\} \not\subset T_{\Sigma gr}$, we know for some $q \in \{q_2,\ldots,q_n\}$ and some $\beta \in SEL^*$ that $\beta(q)\downarrow$ but $[\beta(q)] \not\leq [q]_\beta$. Among those selectors β we can choose a selector α which is minimal w.r.t. $\lessdot$. Let q_m be a term of the $\mathcal{R}$-rewrite with

(4) $\quad \alpha(q_m)\downarrow$ and $[\alpha(q_m)] \not\leq [q_m]_\alpha$, (where $2 \leq m \leq n$) .

Now starting on q_m we *move left* in the $\mathcal{R}$-rewrite until we find the *first* term q_{i-1} with

(5) $\quad [\alpha(q_{i-1})] \leq [q_{i-1}]_\alpha$, if $\alpha(q_{i-1})\downarrow$ (where $i-1<m$)

and starting again on q_m we *move right* in the $\mathcal{R}$-rewrite until we find the *first* term q_j with

(6) $\quad [\alpha(q_j)] \leq [q_j]_\alpha$, if $\alpha(q_j)\downarrow$ (where $m<j$).

The existence of q_{i-1} is guaranteed, because the *leftmost* term q_1 in the $\mathcal{R}$-rewrite is a Σ-term and hence satisfies $[\alpha(q_1)] \leq [q_1]_\alpha$, if $\alpha(q_1)\downarrow$. By the same argument the *rightmost* term q_{n+1} in the $\mathcal{R}$-rewrite guarantees the existence of q_j.

Suppose that $\alpha(q_{i-1})\uparrow$. Since q_{i-1} is the *first* term to the *left* of q_m which satisfies (5), we know that $\alpha(q_i)\downarrow$ and $[\alpha(q_i)] \not\leq [q_i]_\alpha$.

<u>*Case (i)*</u> $\quad \alpha \lessdot \alpha_{i-1}$: Using $q_{i-1} \xrightarrow[\alpha_{i-1}]{}_{\mathcal{R}} q_i$ and $\alpha(q_i)\downarrow$ we infer by Lemma 3.5 (2) that $[\alpha(q_i)] \leq [q_i]_\alpha$. ▽ ☐

<u>*Case (ii)*</u> $\quad \alpha \cdot\!\geq \alpha_{i-1}$: From $q_{i-1} \xrightarrow[\alpha_{i-1}]{}_{\mathcal{R}} q_i$ we infer that $\alpha_{i-1}(q_{i-1})\downarrow$, hence $\alpha(q_{i-1})\downarrow$. ▽ ☐

<u>*Case (iii)*</u> $\quad \alpha \perp \alpha_{i-1}$: From $q_{i-1} \xrightarrow[\alpha_{i-1}]{}_{\mathcal{R}} q_i$ we infer by Lemma 2.2 (6) that $\alpha(q_{i-1}) = \alpha(q_i)$ and with $\alpha(q_i)\downarrow$ we obtain $\alpha(q_{i-1})\downarrow$. ▽ ☐

Hence we have proved that $\alpha(q_{i-1})\downarrow$ and using (5) we can write

(7) $\quad \alpha(q_{i-1})\downarrow$ and $[\alpha(q_{i-1})] \leq [q_{i-1}]_\alpha$ (where $i-1<m$) .

By a similar argument we prove that $\alpha(q_j)\downarrow$ and using (6) we can write

(8) $\quad \alpha(q_j)\downarrow$ and $[\alpha(q_j)] \leq [q_j]_\alpha$ (where $m<j$) .

Now suppose that for some h' with $i \leq h' \leq j-1$ $[\alpha(q_{h'})] \leq [q_{h'}]_\alpha$, provided $\alpha(q_{h'})\downarrow$:

Case (i) $i \leq h' < m$: Then $q_{h'}$ instead of q_{i-1} is the *first* term to the *left* of q_m with $[\alpha(q_{h'})] \leq [q_{h'}]_\alpha$, if $\alpha(q_{h'})\downarrow$. ∇ ☒

Case (ii) $h' = m$: This case is impossible by (4). ∇ ☒

Case (iii) $m < h' \leq j-1$: Then $q_{h'}$ instead of q_j is the *first* term to the *right* of q_m with $[\alpha(q_{h'})] \leq [q_{h'}]_\alpha$, if $\alpha(q_{h'})\downarrow$. ∇ ☒

Thus we have proved

(9) $\alpha(q_h)\downarrow$ and $[\alpha(q_h)] \leq [q_h]_\alpha$, for each h with $i \leq h \leq j-1$.

Let us assume by way of contradiction that $\alpha(q_{h'}) \notin T_{\Sigma gr}$ for some h' with $i \leq h' \leq j-1$. Since $\alpha(q_{h'})\downarrow$ by (9), there exists some $\delta \in SEL^+$ such that $\delta\alpha(q_{h'})\downarrow$ and $[\delta\alpha(q_{h'})] \nleq [\alpha(q_{h'})]_\delta = [q_{h'}]_{\delta\circ\alpha}$. But $\delta\circ\alpha \lessdot \alpha$ contradicts the minimality of α. ∇ Thus we have established that

(10) $\alpha(q_h) \in T_{\Sigma gr}$, for each h with $i \leq h \leq j-1$.

Now suppose that $\alpha \neq \alpha_{i-1}$. Then $\alpha \lessdot \alpha_{i-1}$, $\alpha \perp \alpha_{i-1}$ or $\alpha \gtrdot \alpha_{i-1}$, i.e. $\alpha \lessdot \alpha_1$ or $\alpha \not\lessdot \alpha_{i-1}$:

Case (i) $\alpha \lessdot \alpha_{i-1}$: Since $q_{i-1} \xrightarrow[\alpha_{i-1}]{} _R q_i$ and $\alpha(q_i)\downarrow$ by (9), we can use Lemma 3.5 (2) to infer that $[\alpha(q_i)] \leq [q_i]_\alpha$ which contradicts (9). ∇ ☒

Case (ii) $\alpha \not\lessdot \alpha_{i-1}$: Using $q_{i-1} \xrightarrow[\alpha_{i-1}]{} _R q_i$ we obtain $[\alpha(q_{i-1})] = [\alpha(q_i)]$ by Lemma 3.1 (3) and $[q_{i-1}]_\alpha = [q_i]_\alpha$ by Lemma 3.1 (4). Hence by (7) $[\alpha(q_i)] \leq [q_i]_\alpha$ which contradicts (9). ∇ ☒

Hence

(11) $\alpha = \alpha_{i-1}$

and by a similar argument we can prove that

(12) $\alpha = \alpha_{j-1}$.

From (11) and (12) we infer (1), (11) and (10) gives us (2) and finally we obtain (3) by (11) and (9). ☒

Lemma 5.2 (Shift-Up Lemma) Let $\mathfrak{R}$ be a Σ-maximal term rewriting system and $q_1 \xrightarrow[\alpha_1]{} q_2 \ldots q_n \xrightarrow[\alpha_n]{} q_{n+1}$, $n \geq 1$, be an $\mathfrak{R}$-rewrite. If $\{\alpha_1(q_1),\ldots,\alpha_1(q_{n+1})\} \subset T_{\Sigma gr}$, then there exist $\beta_1,\ldots,\beta_n \in SEL^*$ such that for each h with $1 \leq h \leq n$:

(1) $q_1 \xrightarrow[\beta_1]{}_{\mathfrak{R}} q_2 \xrightarrow[\beta_2]{}_{\mathfrak{R}} q_3 \ldots\ldots q_n \xrightarrow[\beta_n]{}_{\mathfrak{R}} q_{n+1}$,

(2) $\beta_h = \alpha_1$, if $\alpha_h \lessdot \alpha_1$, and

(3) $\beta_h = \alpha_h$, if $\alpha_h \not\lessdot \alpha_1$.

Proof The proof is by induction on n.

Base Case n = 1: The lemma holds trivially. ∅

Induction Step: Let $q_1 \xrightarrow[\alpha_1]{} q_2 \ldots q_n \xrightarrow[\alpha_n]{} q_{n+1} \xrightarrow[\alpha_{n+1}]{} q_{n+2}$ be an $\mathfrak{R}$-rewrite with $\{\alpha_1(q_1),\ldots,\alpha_1(q_{n+1}),\alpha_1(q_{n+2})\} \subset T_{\Sigma gr}$. Our induction hypotheses is to assume, that there exist some $\beta_1,\ldots,\beta_n \in SEL^*$ such that conditions (1), (2) and (3) are satisfied for each h with $1 \leq h \leq n$.

For $\alpha_{n+1} \not\lessdot \alpha_1$, we define $\beta_{n+1} = \alpha_{n+1}$ and $\beta_1,\ldots,\beta_{n+1}$ is a sequence of selectors with the desired properties.

If $\alpha_{n+1} \lessdot \alpha_1$, then $\alpha_{n+1} = \beta \circ \alpha_1$ for some $\beta \in SEL^+$, i.e. $q_{n+1} \xrightarrow[\beta\circ\alpha_1]{}_{\mathfrak{R}} q_{n+2}$. Now by Lemma 2.2(3) $\alpha_1(q_{n+1}) \xrightarrow[\beta]{}_{\mathfrak{R}} \alpha_1(q_{n+2})$ and $q_{n+1} \underset{\alpha_1}{\sim} q_{n+2}$. Since $\alpha_1(q_{n+1}),\ \alpha_1(q_{n+2}) \in T_{\Sigma gr}$ by assumption, we can write $\alpha_1(q_{n+1}) \longrightarrow_{\Sigma\mathfrak{R}} \alpha_1(q_{n+1})$ and because $\mathfrak{R}$ is Σ-maximal we have $\alpha_1(q_{n+1}) \Rightarrow_{\mathfrak{R}} \alpha_1(q_{n+2})$.

With $q_{n+1} \underset{\alpha_1}{\sim} q_{n+2}$, we obtain $q_{n+1} \xrightarrow[\alpha_1]{}_{\mathfrak{R}} q_{n+2}$ and setting $\beta_{n+1} = \alpha_1$, $\beta_1,\ldots,\beta_{n+1}$ is a sequence of selectors such that for each h with $1 \leq h \leq n+1$ conditions (1), (2) and (3) are satisfied. ∅ ⊠

Lemma 5.3 (Shift-Left Lemma) Let $\mathcal{R}$ be a term rewriting system and $q_1 \xrightarrow[\alpha_1]{} q_2 \ldots q_n \xrightarrow[\alpha_n]{} q_{n+1}$, $n \geq 1$, be an $\mathcal{R}$-rewrite. If for each i with $1 \leq i \leq n$ $\alpha_i \perp \alpha_n$, then there exist $r_2, \ldots, r_n \in T_{gr}$ such that $q_1 \xrightarrow[\alpha_n]{}_{\mathcal{R}} r_2 \xrightarrow[\alpha_1]{}_{\mathcal{R}} r_3 \ldots r_n \xrightarrow[\alpha_{n-1}]{}_{\mathcal{R}} q_{n+1}$.

Proof The proof is by induction on n.

Base Case n = 1: The lemma holds trivially. ▨

Induction Step: Let $q_1 \xrightarrow[\alpha_1]{} q_2 \ldots q_{n+1} \xrightarrow[\alpha_{n+1}]{} q_{n+2}$ be an $\mathcal{R}$-rewrite such that $\alpha_i \perp \alpha_{n+1}$ for each i with $1 \leq i \leq n+1$.

Our induction hypotheses is to assume that the lemma holds for all $\mathcal{R}$-rewrites with length at most n. Hence we are allowed to assume that $q_2 \xrightarrow[\alpha_{n+1}]{}_{\mathcal{R}} r_3 \xrightarrow[\alpha_2]{}_{\mathcal{R}} r_4 \ldots r_{n+1} \xrightarrow[\alpha_n]{}_{\mathcal{R}} q_{n+2}$ for some $r_3, \ldots, r_{n+1} \in T_{gr}$. From $q_1 \xrightarrow[\alpha_1]{}_{\mathcal{R}} q_2 \xrightarrow[\alpha_{n+1}]{}_{\mathcal{R}} r_3$ and $\alpha_1 \perp \alpha_{n+1}$ we infer by Lemma 2.2 (7) the existence of some $r_2 \in T_{gr}$ such that $q_1 \xrightarrow[\alpha_{n+1}]{}_{\mathcal{R}} r_2 \xrightarrow[\alpha_1]{}_{\mathcal{R}} r_3$. Hence we have found some $r_2, r_3, \ldots, r_{n+1} \in T_{gr}$ such that $q_1 \xrightarrow[\alpha_{n+1}]{}_{\mathcal{R}} r_2 \xrightarrow[\alpha_1]{}_{\mathcal{R}} r_3 \ldots r_{n+1} \xrightarrow[\alpha_n]{}_{\mathcal{R}} q_{n+2}$. ▨ ⊠

Lemma 5.4 (Reduction Lemma) Let $\mathcal{R}$ be a Σ-maximal term rewriting system and $q_1 \xrightarrow[\alpha]{} q_2 \ldots q_n \xrightarrow[\alpha]{} q_{n+1}$, $n \geq 1$, be an $\mathcal{R}$-rewrite. Then $q_1 \xrightarrow[\alpha]{}_{\mathcal{R}} q_{n+1}$.

Proof For each i with $1 \leq i \leq n$ we know that $q_i \underset{\alpha}{\sim} q_{i+1}$ and $\alpha(q_i) \Rightarrow_{\mathcal{R}} \alpha(q_{i+1})$, hence $\alpha(q_i) \xrightarrow{+}_{\Sigma\mathcal{R}} \alpha(q_{i+1})$ because $\mathcal{R}$ is Σ-maximal. But then $\alpha(q_1) \xrightarrow{+}_{\Sigma\mathcal{R}} \alpha(q_{n+1})$ since $\xrightarrow{+}_{\Sigma\mathcal{R}}$ is transitive and finally $\alpha(q_1) \Rightarrow_{\mathcal{R}} \alpha(q_{n+1})$ by the Σ-maximality of $\mathcal{R}$. Because of $q_i \underset{\alpha}{\sim} q_{i+1}$ and since $\underset{\alpha}{\sim}$ is transitive we have $q_1 \underset{\alpha}{\sim} q_{n+1}$, hence with $\alpha(q_1) \Rightarrow_{\mathcal{R}} \alpha(q_{n+1})$ we infer $q_1 \xrightarrow[\alpha]{}_{\mathcal{R}} q_{n+1}$. ⊠

Theorem 5.5 (Σ-Rewrite Theorem) If $\mathcal{R}$ is a Σ-maximal term rewriting system, then $\xrightarrow{+}_{\mathcal{R}} \cap (T_{\Sigma gr} \times T_{\Sigma gr}) = \xrightarrow{+}_{\Sigma\mathcal{R}}$

Proof "$\supset$" Obvious, because $\xrightarrow{+}_{\Sigma\mathcal{R}} \subset \xrightarrow{+}_{\mathcal{R}}$ and $\xrightarrow{+}_{\Sigma\mathcal{R}} \subset (T_{\Sigma gr} \times T_{\Sigma gr})$.

"$\subset$" Let us assume by way of contradiction that there exists a pair of Σ-ground terms q_1 and q_{n+1} such that $q_1 \xrightarrow{+}_{\mathcal{R}} q_{n+1}$ but $q_1 \not\xrightarrow{+}_{\Sigma\mathcal{R}} q_{n+1}$. Let

$$q_1 \xrightarrow[\alpha_1]{} q_2 \cdots q_n \xrightarrow[\alpha_n]{} q_{n+1}$$

be an $\mathcal{R}$-rewrite of q_{n+1} from q_1 with *minimal* length. We know that $n \geq 2$, since $q_1, q_{n+1} \in T_{\Sigma gr}$ and $q_1 \not\xrightarrow{+}_{\Sigma\mathcal{R}} q_{n+1}$. But then $\{q_2,\dots,q_n\} \not\subset T_{\Sigma gr}$ and by the Σ-Rewrite Lemma (5.1) there exist indices i and j with $2 \leq i < j \leq n+1$ such that for each h with $i \leq h \leq j-1$

(1) $\alpha_{j-1} = \alpha_{i-1}$,

(2) $\alpha_{i-1}(q_h) \in T_{\Sigma gr}$, and

(3) $[\alpha_{i-1}(q_h)] \nleq [q_h]_{\alpha_{i-1}}$.

Consider the $\mathcal{R}$-rewrite

(4) $q_{i-1} \xrightarrow[\alpha_{i-1}]{} q_i \cdots q_{j-1} \xrightarrow[\alpha_{j-1}]{} q_j$.

Since $\alpha_{i-1}(q_{i-1}) \Rightarrow_{\mathcal{R}} \alpha_{i-1}(q_i)$ and $\alpha_{j-1}(q_{j-1}) \Rightarrow_{\mathcal{R}} \alpha_{j-1}(q_j)$, we know by Lemma 3.5 (1) that $\alpha_{i-1}(q_{i-1}) \in T_{\Sigma gr}$ and that $\alpha_{j-1}(q_j) \in T_{\Sigma gr}$, because $\mathcal{R}$ is Σ-maximal. Now by (1) and (2) $\{\alpha_{i-1}(q_{i-1}),\dots,\alpha_{i-1}(q_j)\} \subset T_{\Sigma gr}$ and hence by the Shift-Up Lemma (5.2) there exist $\beta_{i-1},\dots,\beta_{j-1} \in SEL^*$ such that for each h with $i-1 \leq h \leq j-1$

(5) $q_{i-1} \xrightarrow[\beta_{i-1}]{}_{\mathcal{R}} q_i \xrightarrow[\beta_i]{}_{\mathcal{R}} q_{i+1} \cdots q_{j-1} \xrightarrow[\beta_{j-1}]{}_{\mathcal{R}} q_j$,

(6) $\beta_h = \alpha_{i-1}$, if $\alpha_h \lessdot \alpha_{i-1}$, and

(7) $\beta_h = \alpha_h$, if $\alpha_h \not\lessdot \alpha_{i-1}$.

From (1) and (7) we infer

(8) $\beta_{j-1} = \alpha_{i-1}$.

Now we prove that $\beta_h \perp \alpha_{i-1}$ for each h with $i \leq h \leq j-1$:

Case (i) $\alpha_{i-1} \triangleright \alpha_h$: Then by (6) $\beta_h = \alpha_{i-1}$, i.e. $\beta_h \perp \alpha_{i-1}$ by definition of $\perp$. ∅

Case (ii) $\alpha_{i-1} \perp \alpha_h$: Then $\alpha_h \ntriangleleft \alpha_{i-1}$, hence by (7) $\beta_h = \alpha_h$, i.e. $\beta_h \perp \alpha_{i-1}$. ∅

Case (iii) $\alpha_{i-1} \triangleleft \alpha_h$: Using $q_h \xrightarrow[\alpha_h]{}_R q_{h+1}$ and $\alpha_{i-1}(q_h)\downarrow$ by (2), we obtain by Lemma 3.5 (2) that $[\alpha_{i-1}(q_h)] \leq [q_h]_{\alpha_{i-1}}$, i.e. a contradiction to (3). ∇ ∅

Hence, using (7) and (8), we can write

(9) $\beta_h \perp \beta_{j-1}$, for each h with $i-1 \leq h \leq j-1$.

Now with (5) and (9) we can use the Shift-Left Lemma (5.3) to infer the existence of some $r_i,\ldots,r_{j-1} \in T_{gr}$ such that

(10) $q_{i-1} \xrightarrow[\beta_{j-1}]{}_R r_i \xrightarrow[\beta_{i-1}]{}_R r_{i+1} \cdots r_{j-1} \xrightarrow[\beta_{j-2}]{}_R q_j$ and in particular

(11) $q_{i-1} \xrightarrow[\alpha_{i-1}]{}_R r_i \xrightarrow[\alpha_{i-1}]{}_R r_{i+1}$

because $\beta_{j-1} = \alpha_{i-1}$ by (8) and $\beta_{i-1} = \alpha_{i-1}$ by (7). But with (11) we can use the Reduction Lemma (5.4) to obtain

(12) $q_{i-1} \xrightarrow[\alpha_{i-1}]{}_R r_{i+1}$.

Summarizing we have found an R-rewrite

$$q_1 \xrightarrow[\alpha_1]{} q_2 \cdots q_{i-1} \xrightarrow[\alpha_{i-1}]{} r_{i+1} \xrightarrow[\beta_i]{} r_{i+2} \cdots r_{j-1} \xrightarrow[\beta_{j-2}]{} q_j \xrightarrow[\alpha_j]{} q_{j+1} \cdots q_n \xrightarrow[\alpha_n]{} q_{n+1}$$

of q_{n+1} from q_1 with length n-1, i.e. the R-rewrite $q_1 \xrightarrow[\alpha_1]{} q_2 \ldots q_n \xrightarrow[\alpha_n]{} q_{n+1}$ initially given was not with minimal length. ∇ ⊠

Note that the Σ-Rewrite Theorem obviously also holds for R-rewrites of ground literals, i.e.

$$\xrightarrow{+}_R \cap (LIT_{\Sigma gr} \times LIT_{\Sigma gr}) = \xrightarrow{+}_{\Sigma R} .$$

6. *Completeness of the ΣRP-Calculus - The Ground Case*

The following main theorem is shown in this section:

Ground Completeness Theorem for ΣRP If $S_{\Sigma gr}$ is E-unsatisfiable, then $S^E_{\Sigma gr} \vdash_{\Sigma} \square$.

At the Σ-ground level there is no difference between resolution and Σ-resolution. Hence the main effort is in showing the result for paramodulation, i.e. to prove a result which links Σ-deductions $\vdash_{\Sigma P}$ to deductions $\vdash_{P}$ from a set of Σ-ground clauses. To this effect we define:

Definition 6.1

$Par(S) = \{C \in \mathcal{L}_{gr} \mid S^E_{\Sigma gr} \vdash_P C\}$,

$Par_\Sigma(S) = \{C \in \mathcal{L}_{\Sigma gr} \mid S^E_{\Sigma gr} \vdash_{\Sigma P} C\}$, and

$RPar(S) = \{C \in \mathcal{L}_{gr} \mid S^E_{\Sigma gr} \vdash_P C$, such that no clause in $\vdash_P$ is obtained by paramodulating into a positive equality literal$\}$. ⊠

As a prerequisite for the proof of the Ground Completeness Theorem we show that if $Par_\Sigma(S)$ is satisfiable, then RPar(S) is satisfiable. This is achieved in the following way:

First we introduce the notion of a ΣE-*restricted interpretation* and we show that $Par_\Sigma(S)$ possesses a model, which is a ΣE-restricted interpretation, whenever $Par_\Sigma(S)$ is satisfiable. Next we prove that each ΣE-restricted interpretation contains a Σ-maximal and symmetric term rewriting system. This fact is used to introduce the *rewrite-closure* I* of a ΣE-restricted interpretation I and to prove that I* is again an interpretation. Moreover we show that the rewrite-closure I* of I is a model of RPar(S), provided I satisfies $Par_\Sigma(S)$.

For each E-interpretation I, I ∩ $LIT_{\Sigma gr}$ is a Σ-*restricted* interpretation. But it is more useful to define:

Definition 6.2 An interpretation I is called *Σ-reflexive* iff $E(q\ q) \in I$ for each $q \in T_{\Sigma gr}$. We say that I is *ΣE-closed* iff for each $L \in I$ and for each $K \in LIT_{\Sigma gr}$, $K \in I$ whenever $L \xrightarrow[\alpha]{}_I K$ for some $\alpha \in SEL^+$. I is called a *ΣE-restricted* interpretation iff I is Σ-reflexive, I is ΣE-closed and $I \subset LIT_{\Sigma gr}$. ⊠

Lemma 6.1 If $Par_\Sigma(S)$ is satisfiable, then it possesses a model, which is a ΣE-restricted interpretation.

Proof Let M be a minimal model of $Par_\Sigma(S)$. We show that M is a ΣE-restricted interpretation:

M is Σ-reflexive: Obvious, because $\{E(q\ q)\} \in S^E_{\Sigma gr} \subset Par_\Sigma(S)$ for each $q \in T_{\Sigma gr}$. ◻

M is ΣE-closed: Let $L \in M$ and $K \in LIT_{\Sigma gr}$ such that $L \xrightarrow[\alpha]{}_M K$ for some $\alpha \in SEL^+$. Now assume by way of contradiction, that $M \cap C \neq \{L\}$ for each $C \in Par_\Sigma(S)$. With $M \cap C \neq \emptyset$ for each $C \in Par_\Sigma(S)$ we obtain that $(M-L) \cap C \neq \emptyset$, i.e. $M-L$ satisfies $Par_\Sigma(S)$ and therefore M is not minimal. ∇ Hence

(1) $M \cap C_L = \{L\}$, for some $C_L \in Par_\Sigma(S)$,

and by an analogue argument

(2) $M \cap C_E = \{E(\alpha(L)\alpha(K))\}$, for some $C_E \in Par_\Sigma(S)$.

Let C be the paramodulant of C_L and C_E upon L and $E(\alpha(L)\alpha(K))$, i.e.

(3) $C = (C_L-L) \cup (C_E-E(\alpha(L)\alpha(K)) \cup \{K\}$.

We know that $[\alpha(K)] \leq [K]_\alpha$ because $K \in LIT_{\Sigma gr}$ and that $[K]_\alpha = [L]_\alpha$ by Lemma 3.1 (4), i.e. $[\alpha(K)] \leq [L]_\alpha$. Hence C is a Σ-paramodulant, i.e. $C \in Par_\Sigma(S)$ and therefore

(4) $M \cap C \neq \emptyset$.

Using (1), (2) and (3) we infer that $M \cap C = M \cap \{K\}$, hence by (4) $K \in M$, thus M is ΣE-closed. ◻

$M \subset LIT_{\Sigma gr}$: Suppose that $L \in M$ for some $L \notin LIT_{\Sigma gr}$. Then $L \notin M \cap C$ for each $C \in Par_\Sigma(S)$, because each clause in $Par_\Sigma(S)$ contains only Σ-literals. Hence $M-L$ is also a model of $Par_\Sigma(S)$, i.e. M is not minimal. ∇ ◻ ⊠

Lemma 6.2 If I is a ΣE-restricted interpretation, then $\mathfrak{R}(I)$ is a Σ-maximal and symmetric term rewriting system.

Proof First we prove that $\mathfrak{R}(I)$ is Σ-maximal, i.e.
$\Rightarrow_{\mathfrak{R}(I)} = \xrightarrow{+}_{\Sigma\mathfrak{R}(I)}$:

"⊂" Obviously $\Rightarrow_{\mathfrak{R}(I)} \subset \rightarrow_{\mathfrak{R}(I)}$ and $\Rightarrow_{\mathfrak{R}(I)} \subset (T_{\Sigma gr} \times T_{\Sigma gr})$ because $I \subset LIT_{\Sigma gr}$. Hence $\Rightarrow_{\mathfrak{R}(I)} \subset \longrightarrow_{\Sigma\mathfrak{R}(I)} \subset \xrightarrow{+}_{\Sigma\mathfrak{R}(I)}$. ∎

"⊃" Let $q_1, q_{n+1} \in T_{\Sigma gr}$ such that $q_1 \xrightarrow{+}_{\Sigma\mathfrak{R}(I)} q_{n+1}$, i.e. there exists an $\mathfrak{R}(I)$-rewrite

$$q_1 \xrightarrow[\alpha_1]{} q_2 \cdots q_n \xrightarrow[\alpha_n]{} q_{n+1} \quad .$$

We prove by induction on the length n of the $\mathfrak{R}(I)$-rewrite, that

$$q_1 \Rightarrow_{\mathfrak{R}(I)} q_{n+1} \quad .$$

Base Case n=1: Let $\alpha \in SEL^+$ such that $\alpha(E(t_1\, t_2)) = t_2$. Then $\alpha(E(q_1\, q_1)) = q_1 \xrightarrow[\alpha_1]{}_{\mathfrak{R}(I)} q_2 = \alpha(E(q_1\, q_2))$ and with $E(q_1\, q_1) \underset{\alpha}{\sim} E(q_1\, q_2)$ we infer by Lemma 2.2 (3) that

$$E(q_1\, q_1) \xrightarrow[\alpha_1 \circ \alpha]{}_{\mathfrak{R}(I)} E(q_1\, q_2) \quad .$$

Since $q_1, q_2 \in T_{\Sigma gr}$ we know that $E(q_1\, q_2) \in LIT_{\Sigma gr}$ and by the Σ-reflexivity of I we obtain $E(q_1\, q_1) \in I$.

Hence $E(q_1\, q_2) \in I$, because I is ΣE-closed, i.e. $q_1 \Rightarrow_{\mathfrak{R}(I)} q_2$. ∎

Induction Step: Our induction hypotheses is to assume that $q_1 \Rightarrow_{\mathfrak{R}(I)} q_{n+1}$ for each $\mathfrak{R}(I)$-rewrite of q_{n+1} from q_1 with length n, provided $q_1 \xrightarrow{+}_{\Sigma\mathfrak{R}(I)} q_{n+1}$. Let

$$q_1 \xrightarrow[\alpha_1]{} q_2 \cdots q_{n+1} \xrightarrow[\alpha_{n+1}]{} q_{n+2}$$

be an $\mathfrak{R}(I)$-rewrite such that

$$q_1 \xrightarrow{+}_{\Sigma\mathfrak{R}(I)} q_{n+2} \quad .$$

Then with the same argument as in the base case we infer that $E(q_1\, q_2) \in I$ and by the induction hypotheses we obtain that

$q_2 \Rightarrow_{\mathfrak{R}(I)} q_{n+2}$, i.e. $E(q_2\, q_{n+2}) \in I$.

Since $q_1, q_{n+2} \in T_{\Sigma gr}$, we know that $E(q_1\, q_{n+2}) \in LIT_{\Sigma gr}$ and choosing $\alpha \in SEL^+$ as in the base case we obtain $E(q_1\, q_2) \underset{\alpha}{\sim} E(q_1\, q_{n+2})$ and $E(\alpha(E(q_1\, q_2))\, \alpha(E(q_1\, q_{n+2}))) = E(q_2\, q_{n+2}) \in I$, i.e.

$$E(q_1\, q_2) \xrightarrow[\alpha I]{} E(q_1\, q_{n+2}) \ .$$

But then by the ΣE-closure of I, $E(q_1\, q_{n+2}) \in I$, i.e. $q_1 \Rightarrow_{\mathfrak{R}(I)} q_{n+2}$. ☐

To prove that <u>$\mathfrak{R}(I)$ is symmetric</u>, suppose that $q \Rightarrow_{\mathfrak{R}(I)} r$ for any $q,r \in T_{\Sigma gr}$. Let $\alpha \in SEL^+$ such that $\alpha(E(t_1\, t_2)) = t_1$. Then $E(q\, q) \underset{\alpha}{\sim} E(r\, q)$ and $E(\alpha(E(q\, q))\, \alpha(E(r\, q))) = E(q\, r) \in I$, i.e.

$$E(q\, q) \xrightarrow[\alpha I]{} E(r\, q) \ .$$

Obviously $E(r\, q) \in LIT_{\Sigma gr}$ and by the Σ-reflexivity of I, $E(q\, q) \in I$. Hence $E(r\, q) \in I$, because I is ΣE-closed, i.e. $r \Rightarrow_{\mathfrak{R}(I)} q$. ☐ ☒

Definition 6.3 For a ΣE-restricted interpretation I we define the *rewrite-closure* I^* of I as

$$I^* \;=\; I \cup \{K \in LIT_{gr} \backslash AT^E_{gr} \mid L \xrightarrow{*}_{\mathfrak{R}(I)} K \quad \text{for some } L \in I\}. \boxtimes$$

An important property of the rewrite-closure is that we do not introduce any *new* Σ-ground literals:

Lemma 6.3 If I is a ΣE-restricted interpretation, then $I = I^* \cap LIT_{\Sigma gr}$.

Proof "$\subset$" Obvious, because $I \subset I^*$ and $I \subset LIT_{\Sigma gr}$.

"$\supset$" Let $K \in I^* \cap LIT_{\Sigma gr}$. If $K \in I$ we are finished. For $K \in I^* \backslash I$, there exists some $L \in I$ such that $L \xrightarrow{+}_{\mathfrak{R}(I)} K$. By assumption $K \in LIT_{\Sigma gr}$ and with $I \subset LIT_{\Sigma gr}$ we know that $L \in LIT_{\Sigma gr}$. By Lemma 6.2 we find that $\mathfrak{R}(I)$ is Σ-maximal, hence by the Σ-Rewrite Theorem (5.5), $L \xrightarrow{+}_{\Sigma\mathfrak{R}(I)} K$, i.e. there exists an $\mathfrak{R}(I)$-rewrite $L = L_1 \xrightarrow[\alpha_1]{} L_2 \ldots L_n \xrightarrow[\alpha_n]{} L_{n+1} = K$, such that $\{L_1,\ldots,L_{n+1}\} \subset LIT_{\Sigma gr}$.

It is easily verified by induction on n using the ΣE-closure of I, that $\{L_1,\ldots,L_{n+1}\} \subset I$, hence $K \in I$. ⊠

Now we prove that the rewrite closure of a ΣE-restricted interpretation is always an interpretation:

Lemma 6.4 If I is a ΣE-restricted interpretation, then I* is an interpretation.

Proof Assume by contradiction that I* is not an interpretation, i.e. $\{Q,Q^c\} \subset I^*$ for some ground literal Q. Then by Definition 6.3 there exist literals L and K^c in I such that

$$L \xrightarrow{*}_{\mathfrak{R}(I)} Q \text{ and } K^c \xrightarrow{*}_{\mathfrak{R}(I)} Q^c.$$

With $K^c \xrightarrow{*}_{\mathfrak{R}(I)} Q^c$ we obtain $K \xrightarrow{*}_{\mathfrak{R}(I)} Q$ and since $\xrightarrow{*}_{\mathfrak{R}(I)}$ is symmetric by Lemma 6.2 we have $Q \xrightarrow{*}_{\mathfrak{R}(I)} K$. Finally with $L \xrightarrow{*}_{\mathfrak{R}(I)} Q$ we find by the transitivity of $\xrightarrow{*}_{\mathfrak{R}(I)}$ that $L \xrightarrow{*}_{\mathfrak{R}(I)} K$.

Now suppose that $K \in AT^E_{gr}$. Then $Q \in AT^E_{gr}$ and by Definition 6.3 $Q \in I$. But then $Q \in LIT_{\Sigma gr}$, i.e. $Q^c \in LIT_{\Sigma gr}$, hence by Lemma 6.3 $Q^c \in I$ contradicting that I is an interpretation. ∇

So let us assume that $K \notin AT^E_{gr}$. Then by Definition 6.3 $K \in I^*$ and since $K^c \in I$, $K^c \in LIT_{\Sigma gr}$, i.e. $K \in LIT_{\Sigma gr}$. Hence by Lemma 6.3 $K \in I$ and again I would not be an interpretation. ∇ ⊠

Using Lemma 6.4 we can construct a model of RPAR(S) from a model of $Par_\Sigma(S)$:

Lemma 6.5 If $Par_\Sigma(S)$ is satisfiable, then RPar(S) is satisfiable.

Proof By Lemma 6.1 there exists a model M of $Par_\Sigma(S)$, which is a ΣE-restricted interpretation. Hence by Lemma 6.4, M* is an interpretation. We prove by induction on the length n of a deduction $S^E_{\Sigma gr} \vdash_{\overline{P}} C$, that M* satisfies each clause $C \in RPar(S)$.

Base Case n=0: Then $C \in S^E_{\Sigma gr} \subset Par_\Sigma(S)$ and $M^* \cap C \neq \emptyset$, because M satisfies $Par_\Sigma(S)$ and $M \subset M^*$. ⍁

Induction Step: Let $C = (C_L - L) \cup (C_E - E(\alpha(L)\alpha(K))) \cup \{K\}$ be a paramodulant of the clauses C_L, $C_E \in RPar(S)$ upon L and $E(\alpha(L)\alpha(K))$. Then

$$L \underset{\alpha}{\sim} K \text{ and } L \notin AT^E_{gr}$$

by Definition 6.1. Our induction hypotheses is to assume that $M^* \cap C_L \neq \emptyset$ and $M^* \cap C_E \neq \emptyset$.

If $M^* \cap (C_L - L) \neq \emptyset$ or $M^* \cap (C_E - E(\alpha(L)\alpha(K))) \neq \emptyset$ then $M^* \cap C \neq \emptyset$ and we are finished.

So let us assume that $M^* \cap (C_L - L) = \emptyset$ and $M^* \cap (C_E - E(\alpha(L)\alpha(K))) = \emptyset$. Then $L \in M^*$ and $E(\alpha(L)\alpha(K)) \in M^*$ and by Definition 6.3 $E(\alpha(L)\alpha(K)) \in M$. Using $L \underset{\alpha}{\sim} K$ we obtain $L \xrightarrow[\alpha]{}_M K$, i.e. $L \xrightarrow{*}_{\mathfrak{R}(M)} K$.

With $L \in M^*$ we find some $Q \in M$ such that $Q \xrightarrow{*}_{\mathfrak{R}(M)} L$, hence $Q \xrightarrow{*}_{\mathfrak{R}(M)} K$. From $L \notin AT^E_{gr}$ we obtain $K \notin AT^E_{gr}$. Hence by Definition 6.3 $K \in M^*$, i.e. $M^* \cap C = \{K\} \neq \emptyset$. ⍁ ⊠

Using Lemma 6.5 we can prove

> *Theorem 6.6* (Ground Completeness Theorem for ΣRP)
> If $S_{\Sigma gr}$ is E-unsatisfiable, then $S^E_{\Sigma gr} \vdash_{\Sigma} \square$.

Proof If $S_{\Sigma gr}$ is E-unsatisfiable, then Par(S) is E-unsatisfiable because $S_{\Sigma gr} \subset Par(S)$. By Theorem 1 from [WR73] we infer that Par(S) is unsatisfiable, hence RPar(S) is unsatisfiable [Lov78]. But then by contraposition of Lemma 6.5 $Par_\Sigma(S)$ is unsatisfiable and there exists a finite and unsatisfiable subset P of $Par_\Sigma(S)$ by the Compactness Theorem [Lov78]. Hence $P \vdash_{R} \square$ by the completeness of the RP-calculus [Rob65] and since there is no difference between a Σ-deduction $\vdash_{\Sigma R}$ and a deduction $\vdash_{R}$ from a Σ-ground clause set, we can write $P \vdash_{\Sigma R} \square$. From $P \subset Par_\Sigma(S)$ we infer that $S^E_{\Sigma gr} \vdash_{\Sigma P} C$ for each $C \in P$ and we obtain finally $S^E_{\Sigma gr} \vdash_{\Sigma} \square$. ⊠

7. Unification under Sorts

An important result of first-order unification theory is the *Unification Theorem* [Rob65] which states the existence of a most general unifier for a set of unifiable terms.

Unfortunately for unification under sorts, the Unification Theorem only holds for signatures, where $<\mathcal{S},\leq>$ is a tree structure. In general we must content ourselves with a weaker result.

We start with a lemma, which allows to distinguish Σ-substitutions from ordinary substitutions by inspecting their restriction on $\mathfrak{V}$:

> *Lemma 7.1* If $\sigma \in SUB$, then $\sigma \in SUB_\Sigma$ iff $\sigma x \in T_\Sigma$ and $[\sigma x] \leq [x]$ for each $x \in \mathfrak{V}$.

Proof "⇒" Since $\sigma \in SUB_\Sigma$ iff $\sigma(T_\Sigma) \subset T_\Sigma$, we know that $\sigma(\mathfrak{V}) \subset T_\Sigma$, i.e. $\sigma x \in T_\Sigma$ for each $x \in \mathfrak{V}$. Now assume by way of contradiction that $[\sigma x_o] \nleq [x_o]$ for some $x_o \in \mathfrak{V}$. If $f \in \mathcal{F}_{[x_o],s}$ for some $s \in \mathcal{S}$, then $f(x_o) \in T_\Sigma$ but $\sigma f(x_o) = f(\sigma x_o) \notin T_\Sigma$ by Lemma 3.3 because $[\sigma x_o] \nleq [x_o] = [f]_1$. Hence $\sigma(T_\Sigma) \not\subset T_\Sigma$, i.e. $\sigma \notin SUB_\Sigma$. ∇ ☑

"⇐" We prove by structural induction on t that $\sigma t \in T_\Sigma$ for each $t \in T_\Sigma$:

Base Case $t \in \mathfrak{C}$: Then $\sigma t = t \in T_\Sigma$ by Lemma 3.3 (1). ☑

Base Case $t \in \mathfrak{V}$: Then $\sigma t \in T_\Sigma$ by assumption. ☑

Induction Step: Suppose that $t = f(t_1 ... t_n) \in T_\Sigma$ and $\sigma t_i \in T_\Sigma$ for each i with $1 \leq i \leq n$. If $t_i \in \mathfrak{V}$, then $[\sigma t_i] \leq [t_i]$ by assumption, and if $t_i \notin \mathfrak{V}$, then $[\sigma t_i] = [t_i]$ by Lemma 3.1 (2). Hence $[\sigma t_i] \leq [t_i] \leq [f]_i$ for each i, i.e. $\sigma t \in T_\Sigma$ by Lemma 3.3 (2). ☑ ☒

> *Corollary 7.2* If $\sigma \in SUB_\Sigma$ and $t \in T$, then $[\sigma t] \leq [t]$.

Proof If $t \in \mathfrak{V}$, then $[\sigma t] \leq [t]$ by Lemma 7.1 and if $t \notin \mathfrak{V}$, then $[\sigma t] = [t]$ by Lemma 3.1 (2). ☒

For unification under sorts, the notion of *Σ-compatibility* plays a central role:

Definition 7.1 Let $D \subset T_\Sigma$ be unifiable. D is *Σ-compatible* iff $[x] \leq [y]$ or $[x] \geq [y]$ for all $x, y \in vars(D)$ with $\tau x = \tau y$, where τ is an mgu of D. ⊠

Note that this definition is independent of the choice of the mgu τ of D.

We show that each Σ-unifiable set of Σ-terms which is Σ-compatible possesses a Σ-mgu:

Lemma 7.3 Let $D \subset T_\Sigma$ be Σ-unifiable. If D is Σ-compatible, then there exists a Σ-mgu of D.

Proof Let $\theta \in SUB_\Sigma$ be a unifier of D and $\tau \in SUB$ an mgu of D. Then there exist $\tau_1, \ldots, \tau_n \in SUB$, $n \geq 1$, such that

(1) $\tau = \tau_1 \circ \ldots \circ \tau_n$,

(2) $COD(\tau_i) = \{y_i\}$, for each i with $1 \leq i < n$ and some $y_i \in \mathfrak{V}$

(3) $COD(\tau_n) \cap \mathfrak{V} = \emptyset$,

(4) $COD(\tau_i) \cap COD(\tau_j) = \emptyset$, for each i,j with $1 \leq i,j < n$ and $i \neq j$, and

(5) $DOM(\tau_k) \cap DOM(\tau_l) = \emptyset$, for each k, l with $1 \leq k, l \leq n$ and $k \neq l$.

For each i with $1 \leq i < n$ we define an *order relation* $\leq_i$ on $DOM(\tau_i) \cup \{y_i\}$ by

(6) $u \leq_i v$ iff $[u] \leq [v]$, for each $u, v \in DOM(\tau_i) \cup \{y_i\}$.

$\leq_i$ is *connex*, i.e. $u \leq_i v$ or $u \geq_i v$ for each $u, v \in DOM(\tau_i) \cup \{y_i\}$, because $\tau u = \tau_i u = y_i = \tau_i v = \tau v$ and hence by the Σ-compatibility of D, $[u] \leq [v]$ or $[u] \geq [v]$.

Since $\leq_i$ is connex, there exists at least one minimal element x_i w.r.t. $\leq_i$ in $DOM(\tau_i) \cup \{y_i\}$.

We define $\sigma_1,\ldots,\sigma_{n-1},\sigma \in SUB$ by

(7) $\sigma_{i|\mathfrak{V}} = \{y_i \leftarrow x_i\} \circ \tau_{i|\mathfrak{V}}$, for each i with $1 \le i < n$, and

(8) $\sigma = \sigma_1 \circ \ldots \circ \sigma_{n-1}$.

Since we have obtained σ_i from τ_i by a variable renaming, we know from the unification theory that $\sigma_1,\ldots,\sigma_{n-1},\sigma$ are in fact substitutions and that

(9) $\sigma \circ \tau_n$ is an mgu of D.

We show that

(10) $[\sigma_i x] \le [x]$, for each $x \in \mathfrak{V}$ and for each i with $1 \le i < n$.

Case (i) $x \notin DOM(\tau_i) \cup \{y_i\}$: Then $\tau_i x = x \neq y_i$, i.e. $\sigma_i x = x$ and in particular $[\sigma_i x] = [x]$. ∅

Case (ii) $x \in DOM(\tau_i) \cup \{y_i\}$: Then by (2), $\tau_i x = y_i$, and by (7), $\sigma_i x = x_i$. We know that $x_i \le_i x$, because x_i is minimal w.r.t. $\le_i$ in $DOM(\tau_i) \cup \{y_i\}$, hence $[\sigma_i x] = [x_i] \le [x]$. ∅

From (2) and (7) we can infer that $COD(\sigma_i) = \{x_i\}$, i.e. $\sigma_i x \in \mathfrak{V} \subset T_\Sigma$ for each $x \in \mathfrak{V}$. Now using (10) we obtain by Lemma 7.1 that

(11) $\sigma_i \in SUB_\Sigma$, for each i with $1 \le i < n$,

and using (8) we have by Lemma 3.4 (1)

(12) $\sigma \in SUB_\Sigma$.

Suppose that $\tau_n x \in \mathfrak{V}$ for some $x \in \mathfrak{V}$. Then by (3) $\tau_n x = x$, i.e.

$$[\sigma\tau_n x] = [\sigma x] \le [x]$$

by (12) and Lemma 7.1.

If $\tau_n x \notin \mathfrak{V}$ for some $x \in \mathfrak{V}$, then $\sigma\tau_n x \notin \mathfrak{V}$ and therefore

$$[\sigma\tau_n x] = [\theta\sigma\tau_n x] = [\theta x] \le [x]$$

by Lemma 3.1 (2) and Lemma 7.1 using (9).

Thus we have established that

(13) $[\sigma\tau_n x] \le [x]$ for each $x \in \mathfrak{V}$.

We prove that

(14) $\sigma\tau_n x \in T_\Sigma$ for each $x \in \mathfrak{V}$,

i.e. $[\alpha(\sigma\tau_n x)] \le [\sigma\tau_n x]_\alpha$ for each $\alpha \in SEL^+$ with $\alpha(\sigma\tau_n x)\downarrow$.

Case (i) $\alpha(\sigma\tau_n x) \notin \mathfrak{V}$: Then $\sigma\tau_n x \notin \mathfrak{V}$ and by Lemma 2.2 (5) $\alpha(\theta\sigma\tau_n x)\downarrow$. From Lemma 3.1 (2) we know that $[\alpha(\sigma\tau_n x)] = [\theta\alpha(\sigma\tau_n x)] = [\alpha(\theta\sigma\tau_n x)]$ and that $[\sigma\tau_n x]_\alpha = [\theta\sigma\tau_n x]_\alpha$. But $\theta = \theta\circ\sigma\circ\tau_n$ by (9), hence $[\alpha(\sigma\tau_n x)] = [\alpha(\theta x)] \le [\theta x]_\alpha = [\sigma\tau_n x]_\alpha$, because $\theta x \in T_\Sigma$ and $\alpha(\theta x)\downarrow$. ▨

Case (ii) $\alpha(\sigma\tau_n x) \in \mathfrak{V}$: Then $\alpha(\tau_n x) \in \mathfrak{V}$ and since τ is an mgu of D, there exists a subterm q of some term in D such that

(15) $\tau q = \tau_n x = \tau x$, and

(16) $\alpha(q)\downarrow$.

From $D \subset T_\Sigma$ and $\alpha \in SEL^+$ we obtain that $q \in T_\Sigma\setminus\mathfrak{V}$ and in particular $[\alpha(q)] \le [q]_\alpha = [\sigma\tau q]_\alpha = [\sigma\tau_n x]_\alpha$, i.e.

(17) $[\alpha(q)] \le [\sigma\tau_n x]_\alpha$.

If $\alpha(q) = \alpha(\tau_n x)$, then $[\alpha(\sigma\tau_n x)] = [\sigma\alpha(\tau_n x)] = [\sigma\alpha(q)] \le [\alpha(q)]$ by Corollary 7.2, i.e. using (17) $[\alpha(\sigma\tau_n x)] \le [\sigma\tau_n x]_\alpha$ and we are finished.

So let us assume that $\alpha(q) \neq \alpha(\tau_n x)$. Since $\alpha(q)\downarrow$ and $\tau\alpha(q) = \alpha(\tau q) = \alpha(\tau_n x) \in \mathfrak{V}$, we know that $\alpha(q) \in DOM(\tau_i)$, $\alpha(\tau_n x) = y_i$ and $[x_i] \le [\alpha(q)]$ for some i with $1 \le i < n$.
Hence $[\alpha(\sigma\tau_n x)] = [\sigma\alpha(\tau_n x)] = [\sigma y_i] = [\sigma_i y_i] = [x_i] \le [\alpha(q)]$ and with (17) we obtain $[\alpha(\sigma\tau_n x)] \le [\sigma\tau_n x]_\alpha$. ▨

From (13) and (14) we obtain with Lemma 7.1 that $\sigma\circ\tau_n \in SUB_\Sigma$, hence by (9) $\sigma\circ\tau_n$ is a Σ-mgu of D. ⊠

Now we can prove that the Unification Theorem holds for signatures, where $<\mathfrak{S},\le>$ is a tree structure:

Theorem 7.4 Let $D \subset T_\Sigma$ be Σ-unifiable. If $<\mathfrak{S},\le>$ is a tree structure, then there exists a Σ-mgu of D.

Proof We show that D is Σ-compatible: Let $\theta \in SUB_\Sigma$ be a unifier of D and let $x,y \in vars(D)$ such that $\tau x = \tau y$ for an mgu τ of D. Then $\theta x = \theta y$ and with $\theta \in SUB_\Sigma$ we have by Lemma 7.1 $[x] \ge [\theta x] = [\theta y] \le [y]$. But then $[x] \le [y]$ or $[x] \ge [y]$ since $<\mathfrak{S},\le>$ is a tree structure. Hence D is Σ-compatible and by Lemma 7.3 there exists a Σ-mgu of D. ⊠

For signatures, where $<\mathfrak{S},\le>$ is not a tree structure we enforce the existence of a Σ-mgu for a set of Σ-unifiable terms using a weakening substitution:

Theorem 7.5 (Σ-Unification Theorem) Given $D \subset T_\Sigma$, $V \subset \mathfrak{V}$ and $\theta \in SUB_\Sigma$ with $vars(D) \subset V$ and θ unifies D, there exist $\mu,\sigma,\lambda \in SUB_\Sigma$ such that

(1) $\mu \in WSUB(V)$,

(2) σ is an mgu of μD , and

(3) $\theta = \lambda \circ \sigma \circ \mu [V]$.

Proof Let $\{x_1,\ldots,x_n\}$ be the set of all variables of vars(D) such that $[\theta x_i] < [x_i]$ and let $\{z_1,\ldots,z_n\}$ be a subset of $\mathfrak{V} \backslash V$ satisfying $z_i \in \mathfrak{V}_{[\theta x_i]}$, where $1 \le i \le n$. We define two Σ-substitutions μ and $\bar{\mu}$ by

(4) $\mu_{|\mathfrak{V}} = \{x_1 \leftarrow z_1,\ldots,x_n \leftarrow z_n\}$, and

(5) $\bar{\mu}_{|\mathfrak{V}} = \{z_1 \leftarrow \theta x_1,\ldots,z_n \leftarrow \theta x_n\}$.

Obviously $\mu \in WSUB(V)$, i.e. condition (1) is satisfied. We prove that

(6) $\theta \circ \bar{\mu} \circ \mu = \theta [V]$,

i.e. $\theta \bar{\mu} \mu x = \theta x$ for each $x \in V$:

Case (i) $x \in DOM(\mu)$: Then $x = x_i$ for some i with $1 \le i \le n$ and using (4) and (5) we obtain $\theta\bar{\mu}\mu x_i = \theta\bar{\mu}z_i = \theta\theta x_i = \theta x_i$. ◻

Case (ii) $x \notin DOM(\mu)$: Since $x \in V$ and $\mu \in WSUB(V)$ we know that $x \notin COD(\mu) = DOM(\bar{\mu})$, hence $\theta\bar{\mu}\mu x = \theta\bar{\mu}x = \theta x$. ◻

Since θ unifies D, by (6) $\theta\circ\bar{\mu}\circ\mu$ unifies D, hence $\theta\circ\bar{\mu}$ is a unifier of μD.

We show that

(7) $[\theta\bar{\mu}x] = [x]$ for each $x \in \mathfrak{V}\setminus DOM(\mu)$.

Case (i) $x \in DOM(\bar{\mu})$: Then $x = z_i$ for some i with $1 \le i \le n$ and using (5) we obtain $[\theta\bar{\mu}z_i] = [\theta\theta x_i] = [\theta x_i] = [z_i]$. ◻

Case (ii) $x \notin DOM(\bar{\mu})$: Then $[\theta\bar{\mu}x] = [\theta x] = [x]$ because $x \notin DOM(\mu)$. ◻

Now we can prove that μD is Σ-compatible: Let $x,y \in vars(\mu D)$ such that $\tau x = \tau y$ for an mgu τ of μD. Then $\theta\bar{\mu}x = \theta\bar{\mu}y$ because $\theta\circ\bar{\mu}$ unifies μD and with $vars(\mu D) \cap DOM(\mu) = \emptyset$ we obtain by (7) that $[x] = [\theta\bar{\mu}x] = [\theta\bar{\mu}y] = [y]$, i.e. μD is Σ-compatible.

Obviously $\theta\circ\bar{\mu} \in SUB_\Sigma$, i.e. μD is Σ-unifiable, hence by Lemma 7.3 there exists a Σ-mgu σ of μD and condition (2) is satisfied. Since $\theta\circ\bar{\mu}$ unifies μD and σ is an mgu of μD, we know that $\theta\circ\bar{\mu}\circ\sigma = \theta\circ\bar{\mu}$, hence $\theta\circ\bar{\mu}\circ\sigma\circ\mu = \theta\circ\bar{\mu}\circ\mu$ and using (6) we obtain $\theta\circ\bar{\mu}\circ\sigma\circ\mu = \theta[V]$. Setting $\lambda = \theta\circ\bar{\mu} \in SUB_\Sigma$ condition (3) is satisfied. ⊠

Note that the Σ-Unification Theorem obviously also holds for Σ-unifiable sets of Σ-atoms.

8. Completeness of the ΣRP-Calculus - The General Case

In this section the completeness theorem for the general case is shown, i.e. we prove

> *Completeness Theorem for ΣRP*
>
> If S is ΣE-unsatisfiable, then $S^E \vdash_{\Sigma} \square$.

This completeness theorem is shown as usual for the one-sorted calculus: We prove the Lifting Lemmata for Σ-Resolution and for Σ-Paramodulation in order to justify the Lifting Theorem for Σ-Deductions. To ease notation we shall omit in Σ-deductions the explicit mentioning of Σ-renaming substitutions and assume instead that the set of clauses in a Σ-deduction is always variable disjoint.

> *Lemma 8.1* (Lifting Lemma for Σ-Resolution) Given $A,B \in \mathcal{L}_{\Sigma}$, $L_A \in A$, $L_B \in B$ and $\theta \in SUB_{\Sigma}$ such that A and B share no variable symbols, L_A and L_B are complementary and θ unifies $\{|L_A|,|L_B|\}$, there exist a Σ-factor A* of a weakened variant of A, a Σ-factor B* of a weakened variant of B, a pair of complementary literals $L_A^* \in A^*$ and $L_B^* \in B^*$, weakened variants ρA* and ρB* and some $\lambda, \sigma \in SUB_{\Sigma}$ such that
>
> $$Res(\theta A,\theta L_A,\theta B,\theta L_B,\varepsilon) = \lambda \; Res(\rho A^*,\rho L_A^*,\rho B^*,\rho L_B^*,\sigma).$$

The statement of this lemma can be summarized by a diagram. The expressions in the diagram are defined as in the lemma:

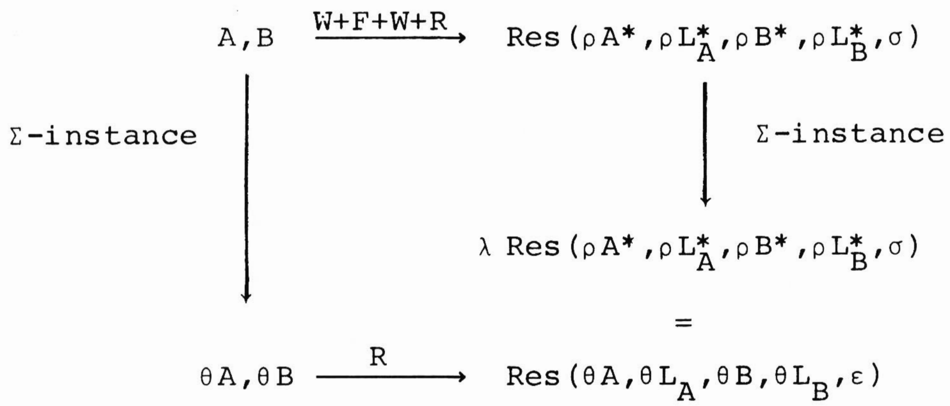

Figure 8.1 Lifting Lemma for Σ-Resolution

Proof Setting $V = vars(A \cup B)$, by the Σ-Unification Theorem (7.5) there exist some $\mu, \gamma, \lambda_A \in SUB_\Sigma$ such that

(1) $\mu \in WSUB(V)$,

(2) γ factors μA ,

(3) $\theta = \lambda_A \circ \gamma \circ \mu [V]$, and

(4) $\gamma\mu L = \gamma\mu K$ for all $L, K \in A$ with $\theta L = \theta K$.

By (1) μA is a weakened variant of A and by (2) $\gamma\mu A$ is a Σ-factor of μA. By the same argument there exist some $\nu, \delta, \lambda_B \in SUB_\Sigma$, such that νB is a weakened variant of B, $\delta\nu B$ is a Σ-factor of νB, and

(5) $\theta = \lambda_B \circ \delta \circ \nu \; [V]$.

Since we are allowed to assume that $\gamma\mu A$ and $\delta\nu B$ have no variables in common by (3) and (5) there exists some $\lambda^* \in SUB_\Sigma$ such that

(6) $\lambda^* = \lambda_A \; [vars(\gamma\mu A)]$, and

(7) $\lambda^* = \lambda_B \; [vars(\delta\nu B)]$.

Let $A^* = \gamma\mu A$, $B^* = \delta\nu B$, $V^* = vars(A^* \cup B^*)$ and L^* an abbreviation for $\gamma\mu L$ if $L \in A$ or for $\delta\nu L$ if $L \in B$. Then for each $L \in A$, $\lambda^* L^* = \lambda^* \gamma\mu L = \lambda_A \gamma\mu L = \theta L$ by (6) and (3), and by an analogue argumentation $\lambda^* L^* = \theta L$ for each $L \in B$, i.e.

(8) $\lambda^* L^* = \theta L$, for each $L \in (A \cup B)$.

Hence $\lambda^* |L_A^*| = \theta |L_A| = \theta |L_B| = \lambda^* |L_B^*|$, i.e. λ^* is a Σ-unifier of $\{|L_A^*|, |L_B^*|\}$, and by the Σ-Unification Theorem (7.5) there exist $\rho, \sigma, \lambda \in SUB_\Sigma$ such that

(9) $\rho \in WSUB(V^*)$,

(10) σ is an mgu of $\{\rho |L_A^*|, \rho |L_B^*|\}$, and

(11) $\lambda^* = \lambda \circ \sigma \circ \rho \; [V^*]$.

By (9) we know that ρA^* and ρB^* are weakened variants and by (10) we can form a Σ-resolvent of ρA^* and ρB^* upon ρL_A^* and ρL_B^*.

Now suppose that $\lambda^*(A^* - L^*_A) \neq \lambda^*A^* - \lambda^*L^*_A$: Then for some literal $L^* \in A^*$ we find $L^* \neq L^*_A$ but $\lambda^*L^* = \lambda^*L^*_A$. From (8) we obtain $\theta L = \theta L_A$, hence by (4) $L^* = \gamma\mu L = \gamma\mu L_A = L^*_A$. ∇
Thus we have proved that

(12) $\lambda^*(A^* - L^*_A) = \lambda^*A^* - \lambda^*L^*_A$

and we obtain by a similar argument that

(13) $\lambda^*(B^* - L^*_B) = \lambda^*B^* - \lambda^*L^*_B$.

But then

$\lambda\ \mathrm{Res}(\rho A^*, \rho L^*_A, \rho B^*, \rho L^*_B, \sigma)$

$= \lambda\sigma(\rho A^* - \rho L^*_A) \cup \lambda\sigma(\rho B^* - \rho L^*_B)$,

$= \lambda\sigma\rho(A^* - L^*_A) \cup \lambda\sigma\rho(B^* - L^*_B)$, because $\rho_{|V^*}$ is injective by (9),

$= \lambda^*(A^* - L^*_A) \cup \lambda^*(B^* - L^*_B)$, by (11)

$= (\lambda^*A^* - \lambda^*L^*_A) \cup (\lambda^*B^* - \lambda^*L^*_B)$, by (12) and (13)

$= (\theta A - \theta L_A) \cup (\theta B - \theta L_B)$, by (8)

$= \mathrm{Res}(\theta A, \theta L_A, \theta B, \theta L_B, \varepsilon)$. ⊠

Lemma 8.2 (Lifting Lemma for Σ-Paramodulation) Given $A, B \in \mathcal{L}_\Sigma$, $L_A \in A$, $L_B = E(q\ r) \in B$, $\alpha \in SEL^+$ and $\theta \in SUB_\Sigma$ such that A and B share no variable symbols, θ unifies $\{\alpha(L_A), q\}$ and $[\theta r] \leq [\theta L_A]_\alpha$, there exist a Σ-factor A^* of a weakened variant of A, a Σ-factor B^* of a weakened variant of B, a pair of literals $L^*_A \in A^*$ and $L^*_B = E(q^*r^*) \in B^*$, weakened variants ρA^* and ρB^* and some $\lambda, \sigma \in SUB_\Sigma$ such that

$\mathrm{Par}(\theta A, \theta L_A, \theta B, \theta L_B, \alpha, \varepsilon) = \lambda\ \mathrm{Par}(\rho A^*, \rho L^*_A, \rho B^*, \rho L^*_B, \alpha, \sigma)$ and $[\sigma\rho r^*] \leq [\sigma\rho L^*_A]_\alpha$.

We illustrate the statement of this lemma by a diagram:

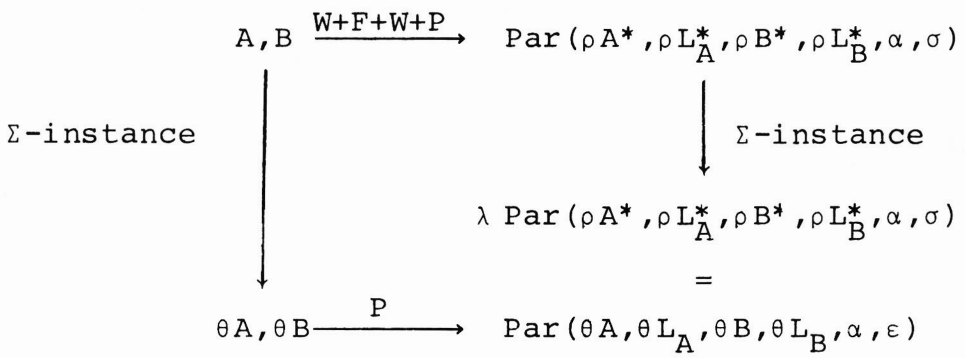

Figure 8.2 Lifting Lemma for Σ-paramodulation

Proof Suppose that $r \in \mathfrak{V}$ and $[r] \nleq [\theta L_A]_\alpha$. Let $\tilde{V} = vars(A \cup B)$, $r' \in \mathfrak{V}_{\widetilde{[\theta r]}} \backslash \tilde{V}$, and $\tau, \bar{\tau}, \tilde{\theta} \in SUB_\Sigma$ such that $\tau_{|\mathfrak{V}} = \{r \leftarrow r'\}$, $\bar{\tau}_{|\mathfrak{V}} = \{r' \leftarrow \theta r\}$ and $\tilde{\theta} = \theta \circ \bar{\tau}$. Since $\theta \in SUB_\Sigma$ we know that $[\theta r] \leq [r]$. If $[\theta r] = [r]$, then $[r] \leq [\theta L_A]_\alpha$, because $[\theta r] \leq [\theta L_A]_\alpha$. ∇
Hence $[\tau r] = [r'] = [\theta r] < [r]$ and therefore

(1) $\tau \in WSUB(\tilde{V})$

It is easily verified that

(2) $\tilde{\theta} = \theta \ [\tilde{V}]$

and that

(3) $\tilde{\theta} \circ \tau = \theta \ [\tilde{V}]$.

For $r \notin \mathfrak{V}$ or $[r] \leq [\theta L_A]_\alpha$ we let $\tau = \bar{\tau} = \varepsilon$ and $\theta = \tilde{\theta}$. Then obviously (1), (2) and (3) still hold and in either case τB is a weakened variant of B.

From (2) and (3) we infer that $\tilde{\theta}$ is a Σ-unifier of $\{\alpha(L_A), \tau q\}$. By the same argument as in the proof of the Lifting Lemma for Σ-Resolution (8.1), there exist $\mu, \gamma, \nu, \delta, \rho, \lambda$ and $\lambda^* \in SUB_\Sigma$ such that μA and $\nu \tau B$ are weakened variants of A and τB respectively, $A^* = \gamma\mu A$ is a Σ-factor of μA, $B^* = \delta\nu\tau B$ is a Σ-factor of $\nu\tau B$,

(5) $\rho \in WSUB(V^*)$,

(6) σ is an mgu of $\{\rho\alpha(L_A^*), \rho q^*\}$,

(7) $\lambda^* = \lambda \circ \sigma \circ \rho \ [V^*]$,

(8) $\lambda^* L^* = \tilde{\theta} L$, for $L \in (A \cup \tau B)$,

(9) $\lambda^*(A^* - L_A^*) = \lambda^* A^* - \lambda^* L_A^*$, and

(10) $\lambda^*(B^* - L_B^*) = \lambda^* B^* - \lambda^* L_B^*$,

where $V^* = vars(A^* \cup B^*)$, $q^* = \gamma\mu q$ and L^* abbreviates $\gamma\mu L$ or $\delta\nu\tau L$ for $L \in A$ or $L \in B$ respectively.

Let $K, K' \in LIT$ such that $L_A \underset{\alpha}{\sim} K$, $\alpha(K) = r$, $L_A^* \underset{\alpha}{\sim} K'$ and $\alpha(K') = r^*$, where $r^* = \delta\nu\tau r$.

Then by Lemma 2.2 (4, 5)

(11) $\theta L_A \underset{\alpha}{\sim} \theta K$ and $\alpha(\theta K) = \theta r$,

(12) $\sigma\rho L_A^* \underset{\alpha}{\sim} \sigma\rho K'$ and $\alpha(\sigma\rho K') = \sigma\rho r^*$, and

(13) $\lambda^* L_A^* \underset{\alpha}{\sim} \lambda^* K'$ and $\alpha(\lambda^* K') = \lambda^* r^*$.

Using (8) we obtain from (13) $\tilde{\theta} L_A \underset{\alpha}{\sim} \lambda^* K'$ and $\alpha(\lambda^* K') = \tilde{\theta}\tau r$, hence by (2) and (3) $\Theta L_A \underset{\alpha}{\sim} \lambda^* K'$ and $\alpha(\lambda^* K') = \theta r$, and using (11) we infer $\lambda^* K' \underset{\alpha}{\sim} \theta K$ and $\alpha(\lambda^* K') = \alpha(\theta K)$. Hence by Lemma 2.2 (1)

(14) $\lambda^* K' = \Theta K$.

But then

$\lambda\ \mathrm{Par}(\rho A^*, \rho L_A^*, \rho B^*, \rho L_B^*, \alpha, \sigma)$

$= \lambda\sigma(\rho A^* - \rho L_A^*) \cup \lambda\sigma(\rho B^* - \rho L_B^*) \cup \lambda\{\sigma\rho K'\}$, by (12)

$= \lambda\sigma\rho(A^* - L_A^*) \cup \lambda\sigma\rho(B^* - L_B^*) \cup \{\lambda\sigma\rho K'\}$, because $\rho_{|V^*}$ is injective by (5)

$= \lambda^*(A^* - L_A^*) \cup \lambda^*(B^* - L_B^*) \cup \{\lambda^* K'\}$, by (7)

$= (\lambda^* A^* - \lambda^* L_A^*) \cup (\lambda^* B^* - \lambda^* L_B^*) \cup \{\lambda^* K'\}$, by (9) and (10)

$= (\theta A - \theta L_A) \cup (\theta B - \theta L_B) \cup \{\theta K\}$, by (8), (2), (3) and (14)

$= \mathrm{Par}(\theta A, \theta L_A, \theta B, \theta L_B, \alpha, \varepsilon)$, by (11).

In order to prove that $[\sigma\rho r^*] \le [\sigma\rho L_A^*]_\alpha$ we infer

$[\sigma\rho r^*] = [\sigma\rho\delta\nu\tau r] \le [\tau r]$, by Corollary 7.2,

$[\tau r] \le [\theta L_A]_\alpha$, by definition of τ and by assumption, and

$[\theta L_A]_\alpha = [\sigma\rho\gamma\mu L_A]_\alpha = [\sigma\rho L_A^*]_\alpha$, by Lemma 3.1 (2). ⊠

Theorem 8.3 (Lifting Theorem for Σ-Deductions) For each Σ-deduction $<B_1,\ldots,B_n>$ from $S^E_{\Sigma gr}$ there exists a Σ-deduction $<C_1,\ldots,C_m>$ from S^E such that for each clause B_i, $1 \le i \le n$, there exists a clause C_k, $1 \le k \le m$, and some $\lambda_k \in SUB_{\Sigma gr}$ with $B_i = \lambda_k C_k$.

Proof The proof is by induction upon the length n of the Σ-deduction $<B_1,\ldots,B_n>$ from $S^E_{\Sigma gr}$:

Base Case n=1: Then $B_1 \in S^E_{\Sigma gr}$, i.e. $B_1 = \lambda C$ for some $C \in S^E$ and some $\lambda \in SUB_{\Sigma gr}$. $<C>$ is a Σ-deduction from S^E with the desired properties. ∅

Induction Step n>1: Our induction hypotheses is to assume that the theorem holds for all Σ-deductions from $S^E_{\Sigma gr}$ with length less or equal to n:

Case (i) $B_{n+1} \in S^E_{\Sigma gr}$: As for the base case we find some $C \in S^E$ and some $\lambda \in SUB_{\Sigma gr}$ such that $B_{n+1} = \lambda C$. By the induction hypotheses $<C_1,\ldots,C_m,C>$ is a Σ-deduction from S^E with the desired properties. ∅

Case (ii) $B_{n+1} = Res(B_i,L_i,B_j,L_j,\varepsilon)$ with $i, j \le n$: By the induction hypotheses there exist clauses $C_k, C_l, 1 \le k, l \le m$, and some $\lambda_k, \lambda_l \in SUB_{\Sigma gr}$ such that $B_i = \lambda_k C_k$ and $B_j = \lambda_l C_l$. Since C_k and C_l share no variable symbols, there exists some $\theta \in SUB_{\Sigma gr}$ such that $\theta C_k = \lambda_k C_k = B_i$ and $\theta C_l = \lambda_l C_l = B_j$.

Now by the Lifting Lemma for Σ-Resolution (8.1), there exist weakened variants C_{m+1} and C_{m+2} of C_k and C_l respectively, Σ-factors C_{m+3} and C_{m+4} of C_{m+1} and C_{m+2} respectively, weakened variants C_{m+5} and C_{m+6} of C_{m+3} and C_{m+4} respectively, a Σ-resolvent C_{m+7} of C_{m+5} and C_{m+6}, and some $\lambda \in SUB_{\Sigma gr}$ such that $B_{n+1} = \lambda C_{m+7}$. By the induction hypotheses $<C_1,\ldots,C_m,C_{m+1},\ldots,C_{m+7}>$ is a Σ-deduction from S^E with the desired properties. ∅

Case (iii) $B_{n+1} = Par(B_i, L_i, B_j, L_j, \alpha, \varepsilon)$ with $i, j \leq n$, $L_j = E(q\ r)$ and $[r] \leq [L_i]_\alpha$: By the induction hypotheses there exist clauses C_k, C_l, $1 \leq k$, $l \leq m$ and some $\lambda_k, \lambda_l \in SUB_{\Sigma gr}$ such that $B_i = \lambda_k C_k$ and $B_j = \lambda_l C_l$. Since C_k and C_l share no variable symbols, there exists some $\theta \in SUB_{\Sigma gr}$ such that $\theta C_k = \lambda_k C_k = B_i$ and $\theta C_l = \lambda_l C_l = B_j$.

Moreover, since S^E contains a functionally-reflexive axiom for each function symbol in B_i we are allowed to assume that $\alpha(L_k)\downarrow$ for some $L_k \in C_k$ with $\theta L_k = L_i$. Let $L_l = E(q'r') \in C_l$ such that $\theta L_l = L_j$.

Then $[\theta r'] = [r] \leq [L_i]_\alpha = [\theta L_k]_\alpha$ and by the Lifting Lemma for Σ-paramodulation (8.2), there exist weakened variants C_{m+1} and C_{m+2} of C_k and C_l respectively, Σ-factors C_{m+3} and C_{m+4} of C_{m+1} and C_{m+2} respectively, weakened variants C_{m+5} and C_{m+6} of C_{m+3} and C_{m+4} respectively, a Σ-paramodulant C_{m+7} of C_{m+5} and C_{m+6}, and some $\lambda \in SUB_{\Sigma gr}$ such that $B_{n+1} = \lambda C_{m+7}$.

By the induction hypotheses $\langle C_1, \ldots, C_m, C_{m+1}, \ldots, C_{m+7}\rangle$ is a Σ-deduction from S^E with the desired properties. ☒ ☒

Now we are prepared to prove the completeness of the ΣRP-calculus:

> *Theorem 8.4* (Completeness Theorem for ΣRP) If S is ΣE-unsatisfiable, then $S^E \vdash_\Sigma \square$.

Proof If S is ΣE-unsatisfiable, then $S_{\Sigma gr}$ is E-unsatisfiable and $S^E_{\Sigma gr} \vdash_\Sigma \square$ by the Ground Completeness Theorem for ΣRP (6.6), i.e. there exists a Σ-deduction $\langle B_1, \ldots, B_n\rangle$ from $S^E_{\Sigma gr}$ such that $B_n = \square$.

Now by the Lifting Theorem for Σ-deductions (8.3), there exists a Σ-deduction $\langle C_1, \ldots, C_k, \ldots, C_m\rangle$ from S^E and some $\lambda_k \in SUB_{\Sigma gr}$ such that $B_n = \lambda_k C_k$. With $B_n = \square$, C_k must be the empty clause also. Hence $\langle C_1, \ldots, C_k\rangle$ is a Σ-refutation of S^E, i.e. $S^E \vdash_\Sigma \square$. ☒

Remarks

Functionally-Reflexive Axioms The completeness of the ΣRP-calculus is proved here only for sets of Σ-clauses, which contain all functionally-reflexive axioms. For the RP-calculus it is known that these axioms are not a necessary prerequisite for the completeness results to hold [Bra75]. We conjecture that this result holds for the ΣRP-calculus also.

The Weakening Rule In section 4 it is shown that the ΣRP-calculus is incomplete without the weakening rule. We believe that the introduction of the weakening rule is the *weakest extension* of the RP-calculus which guarantees completeness:

In particular the RP-calculus is nothing but a special case of the ΣRP-calculus, where the set of sort symbols $\mathcal{S}$ is a singleton $\{s_o\}$. For each weakening substitution μ, $[\mu x] < [x]$ has to be satisfied for every $x \in DOM(\mu)$. But $[\mu x] < [x]$ is always false if only one sort symbol is present, hence μ must be the empty substitution ε. But then each weakened variant of a clause is the clause itself and we are back to the RP-calculus.

If the weakening rule is abandoned, completeness can be maintaine by some restrictions:

A many-sorted resolution calculus (i.e. without paramodulation) is complete, if $<\mathcal{S},\leq>$ is a *tree structure*. This is an immediate consequence of Theorem 7.4.

The full ΣRP-calculus is complete without the weakening rule, if $<\mathcal{S},\leq>$ is a tree structure and the set S of Σ-clauses to be refuted contains all functionally-reflexive axioms and all *constant-reflexive axioms* (i.e. clauses of the form $\{E(c\ c)\}$ for each $c \in \mathfrak{C}$), whenever S contains at least one literal of the form $E(x\ t)$ or $E(t\ x)$, where x is a variable symbol:

In the proof of the Lifting Lemma for Σ-Paramodulation (8.2) we have seen that (in contrast to Σ-resolution) in general an additional application of the weakening rule is necessary, if by a paramodulation step some term is replaced by a *variable*.

It is easy to see that such a paramodulation can always be avoided, if all reflexive axioms are present, i.e. the additional application of the weakening rule is not necessary then.

Refinements Σ-Paramodulation is so restrictive that we loose an important refinement: The RP-calculus is still complete, if we never paramodulate into positive equality literals [Lov78]. The consequence is, that the transitive closure of the predicate E need not be computed, i.e. in the RP-calculus we do not need deductions of the form: E(q r), E(r s) $\vdash$ E(q s). But unfortunately these deductions are necessary in the ΣRP-calculus:

Example 8.1 Let S $= \{\{P(b_1)\},\{\textit{not}\ P(b_2)\},\{E(b_1 a)\},\{E(a\ b_2)\}\}$ be a set of Σ-clauses such that $\mathcal{S} = \{A,B\}$, $B \ll A$, $P \in \mathcal{P}_B$, $\{b_1,b_2\} \subset \mathcal{F}_{e,B}$ and $a \in \mathcal{F}_{e,A}$. Obviously neither $\{P(a)\}$ nor $\{\textit{not}\ P(a)\}$ is a Σ-clause, i.e. they can not be obtained from S by Σ-paramodulation. But with the Σ-paramodulant $\{E(b_1 b_2)\}$ of $\{E(b_1 a)\}$ and $\{E(a\ b_2)\}$ we obtain the Σ-paramodulants $\{P(b_2)\}$ and $\{\textit{not}\ P(b_1)\}$ each of which leads to a Σ-refutation of S. ⊠

The problem is the transitive closure of E, hence we conjecture that Σ-paramodulation is still complete if we never paramodulate into subterms of positive equality literals unless these subterms are in an *argument position*.

9. *Soundness of the* ΣRP-*Calculus*

In the following it is shown that the ΣRP-calculus is sound, i.e. we prove the

<u>*Soundness Theorem for* ΣRP</u> If $S \vdash_{\Sigma} \square$, then S is ΣE-unsatisfiable.

To ease notation we shall omit in Σ-deductions the explicit mentioning of Σ-renaming substitutions and assume instead that the clauses in a Σ-deduction share no variable symbols.

The following lemma is frequently used in this section:

<u>*Lemma 9.1*</u> Let $A,B \in \mathfrak{L}_{\Sigma}$ and $\theta \in SUB_{\Sigma}$ such that $A \subset B$ and $\theta A \in \mathfrak{L}_{\Sigma gr}$. Then there exists some $\lambda \in SUB_{\Sigma}$ such that

(1) $\theta = \lambda[vars(A)]$, and

(2) $\lambda B \in \mathfrak{L}_{\Sigma gr}$.

Proof Let $\{x_1,...,x_n\} = vars(B) \setminus vars(A)$ and let $\lambda, \delta \in SUB_{\Sigma}$ such that $\lambda = \theta \circ \delta$, where $\delta_{|\mathfrak{V}} = \{x_1 \leftarrow t_1,...,x_n \leftarrow t_n\}$ for some $t_i \in T_{\Sigma gr}$ with $[t_i] \leq [x_i], 1 \leq i \leq n$. It is obvious that λ satisfies conditions (1) and (2). ⊠

Although this lemma is trivial, it exhibits a crucial point regarding the soundness of ΣRP:

The existence of the above terms t_i is guaranteed, since for each sort symbol s which is minimal in $\mathfrak{S}$, there exists a constant symbol $c \in \mathcal{F}_{e,s}$ (see section 3). If this requirement is *not* fulfilled, i.e. there do exist "empty" sorts, then this lemma and in turn the results of this section do not hold. The ΣRP-calculus is then *not sound*, because for a non-empty Σ-clause set S, $S_{\Sigma gr}$ may be empty.

Note that for the RP-calculus a similar requirement, i.e. $\mathfrak{C} = \{c\}$ if S contains no constant symbol (see section 2), prevents that S_{gr} is empty for some non-empty clause set S, thus guaranteeing the soundness of the RP-calculus.

To prove the soundness of the ΣRP-calculus we use

> *Lemma 9.2* (Soundness Lemma for ΣRP) Let M be an E-interpretation and $C \in \mathcal{L}_\Sigma$. If M Σ-satisfies S and $S \vdash_\Sigma C$, then M Σ-satisfies C.

Proof The proof is by induction on the length n of the Σ-deduction of C from S.

Base Case n=1: Then $C \in S$ and M Σ-satisfies C by assumption. ☐

Induction Step n>1: Let $C_{n+1} \in \mathcal{L}_\Sigma$ such that the length of the Σ-deduction $S \vdash_\Sigma C_{n+1}$ is n+1. If $C_{n+1} \in S$, then M Σ-satisfies C_{n+1} by the same argument as in the base case, hence we suppose that $C_{n+1} \notin S$.

Our induction hypotheses is to assume that M Σ-satisfies each Σ-clause C_i with $S \vdash_\Sigma C_i$, where $i \leq n$. Let $\theta \in SUB_\Sigma$ such that $\theta C_{n+1} \in \mathcal{L}_{\Sigma gr}$. We prove that $M \cap \theta C \neq \emptyset$:

Case (i) Weakening and Factoring: Let $C_{n+1} = \sigma C_i$, $i \leq n$, such that $S \vdash_\Sigma C_i$ and C_{n+1} is a weakened variant or a Σ-factor of C_i.

Then $\theta\sigma C_i \in \mathcal{L}_{\Sigma gr}$ and $M \cap \theta C_{n+1} = M \cap \theta\sigma C_i \neq \emptyset$ by the induction hypotheses. ☐

Case (ii) Resolution: Let $C_{n+1} = Res(C_i, L_i, C_j, L_j, \sigma)$, $i,j \leq n$, such that $S \vdash_\Sigma C_i$, $S \vdash_\Sigma C_j$ and $\sigma \in SUB_\Sigma$.

Since $C_{n+1} \subset \sigma(C_i \cup C_j)$, by Lemma 9.1 there exists some $\lambda \in SUB_\Sigma$ such that $\theta C_{n+1} = \lambda C_{n+1}$ and $\lambda\sigma(C_i \cup C_j) \in \mathcal{L}_{\Sigma gr}$.

From our induction hypotheses we obtain that $M \cap \lambda\sigma C_i \neq \emptyset$ and that $M \cap \lambda\sigma C_j \neq \emptyset$.

Case (ii.1) $\lambda\sigma L_i \notin M$: Then $M \cap \lambda\sigma(C_i - L_i) \neq \emptyset$ and with $\lambda\sigma(C_i - L_i) \subset \theta C_{n+1}$ we obtain $M \cap \theta C_{n+1} \neq \emptyset$. ☐

Case (ii.2) $\lambda\sigma L_i \in M$: Then $\lambda\sigma L_j \notin M$ because $\lambda\sigma L_i^c = \lambda\sigma L_j$, hence $M \cap \lambda\sigma(C_j - L_j) \neq \emptyset$ and with $\lambda\sigma(C_j - L_j) \subset \theta C_{n+1}$ we obtain $M \cap \theta C_{n+1} \neq \emptyset$. ☐ ☒

Case (iii) Paramodulation: Let $C_{n+1} = Par(C_i, L, C_j, E(q\ r), \alpha, \sigma)$, $i, j \le n$, and $K \in LIT_\Sigma$ such that $S \vdash_\Sigma C_i$, $S \vdash_\Sigma C_j$, $\sigma \in SUB_\Sigma$, $\sigma L \underset{\alpha}{\sim} \sigma K$ and $\alpha(\sigma K) = \sigma r$.

Since $C_{n+1} \subset \sigma(C_i \cup C_j \cup \{K\})$, by Lemma 9.1 there exists some $\lambda \in SUB_\Sigma$ such that $\theta C_{n+1} = \lambda C_{n+1}$ and $\lambda\sigma(C_i \cup C_j \cup \{K\}) \in \mathcal{L}_{\Sigma gr}$.

From our induction hypotheses we obtain that $M \cap \lambda\sigma C_i \neq \emptyset$ and that $M \cap \lambda\sigma C_j \neq \emptyset$.

Case (iii.1) $\lambda\sigma L \notin M$ or $\lambda\sigma E(q\ r) \notin M$. Then by the same argument as in case (ii.1), $M \cap \lambda\sigma(C_i - L) \neq \emptyset$ or $M \cap \lambda\sigma(C_j - E(q\ r)) \neq \emptyset$, hence $M \cap \theta C_{n+1} \neq \emptyset$. ◻

Case (iii.2) $\lambda\sigma L \in M$ and $\lambda\sigma E(q\ r) \in M$. Then by Lemma 2.2 (4) $\lambda\sigma L \underset{\alpha}{\sim} \lambda\sigma K$ and by Lemma 2.2 (5) $\alpha(\lambda\sigma L) = \lambda\sigma q$ and $\alpha(\lambda\sigma K) = \lambda\sigma r$, i.e. $\lambda\sigma L \xrightarrow[\alpha]{}_M \lambda\sigma K$, hence $\lambda\sigma K \in M$ because M is E-closed.

But then $M \cap \{\lambda\sigma K\} \neq \emptyset$ and with $\{\lambda\sigma K\} \subset \theta C_{n+1}$ we obtain that $M \cap \theta C_{n+1} \neq \emptyset$. ◻ ◻ ⊠

The soundness of the ΣRP-calculus is now an immediate consequence:

Corollary 9.3 (Soundness Theorem for ΣRP) If $S \vdash_\Sigma \square$, then S is ΣE-unsatisfiable.

Proof From $S \vdash_\Sigma \square$ we infer by the Soundness Lemma (9.2) that no E-interpretation Σ-satisfies S, i.e. S is ΣE-unsatisfiable. ◻

10. The Sort-Theorem

In this section the connection between the RP- and the ΣRP-calculus is established, i.e. we prove the

> *Sort-Theorem for the* ΣRP-*Calculus*
> S is ΣE-unsatisfiable iff $(\hat{S} \cup A^{\Sigma})$ is E-unsatisfiable.

We prove both directions of this equivalence independently using

> *Lemma 10.4* If M is an E-model of $(\hat{S} \cup A^{\Sigma})_{gr}$, then M E-satisfies $S_{\Sigma gr}$.

and

> *Lemma 10.7* If $S_{\Sigma gr}$ is E-satisfiable, then $(\hat{S} \cup A^{\Sigma})_{gr}$ is E-satisfiable.

In order to justify Lemma 10.4, we start with some facts about models of the set A^{Σ} of all sort axioms.

> *Lemma 10.1* If M is a model of A^{Σ}, $c \in \mathfrak{C}$ and $s \in \mathfrak{S}$ such that $[c] \leq s$, then $s(c) \in M$.

Proof We prove by structural induction on s that $s(c) \in M$.

Base Case s is minimal in $<\mathfrak{S}, \leq>$: Then $[c] = s$, hence $\{s(c)\} \in A^{\Sigma}$ by definition of A^{Σ} and $s(c) \in M$ because M satisfies A^{Σ}. ☒

Induction Step s is not minimal in $<\mathfrak{S}, \leq>$: Our induction hypotheses is to assume that the lemma holds for each $s_i \in \mathfrak{S}$ such that $s_i \ll s$. If $[c] = s$, then $s(c) \in M$ by the same argument as in the base case and we are finished.

So let us assume that $[c] < s$. Then $[c] \leq s_i$ for some $s_i \in \$$ with $s_i \ll s$ and $\{not\ s_i(x), s(x)\} \in A^\Sigma$ by definition of A^Σ, i.e.

$$\{not\ s_i(c), s(c)\} \in A^\Sigma_{gr} .$$

By the induction hypotheses we obtain $s_i(c) \in M$, i.e. $not\ s_i(c) \notin M$ because M is an interpretation. But then $s(c) \in M$, because M satisfies A^Σ. ◻ ⊠

Lemma 10.2 If M is a model of A^Σ, $t \in T_{gr}$ and $s_1, s \in \$$ such that $s_1 \leq s$ and $s_1(t) \in M$, then $s(t) \in M$.

Proof We prove by structural induction on s that $s(t) \in M$.

Base Case s is minimal in $<\$, \leq>$: Then $s_1 = s$ and $s(t) \in M$ by assumption. ◻

Induction Step s is not minimal in $<\$, \leq>$: Our induction hypotheses is to assume that the lemma holds for each $s_i \in \$$ such that $s_i \ll s$. If $s_1 = s$, then $s(t) \in M$ by the same argument as in the base case and we are finished.

So let us assume that $s_1 < s$. Then $s_1 \leq s_i$ for some $s_i \in \$$ with $s_i \ll s$ and $\{not\ s_i(x), s(x)\} \in A^\Sigma$ by definition of A^Σ, i.e.

$$\{not\ s_i(t),\ s(t)\} \in A^\Sigma_{gr} .$$

By the induction hypotheses we obtain $s_i(t) \in M$, i.e. $not\ s_i(t) \notin M$ because M is an interpretation. But then $s(t) \in M$, because M satisfies A^Σ. ◻ ⊠

Definition 10.1 The *kernel* M^Σ of A^Σ is defined as

$M^\Sigma = \{L \in LIT^\$_{\Sigma gr} \mid L = s(q)$ and $[q] \leq s$ for some $s \in \$$ and $q \in T_{\Sigma gr}\}$. ⊠

The kernel M^Σ is an interpretation, because it contains only positive literals. Moreover M^Σ is contained in *every* model of A^Σ:

Lemma 10.3 If M is a model of A^{Σ}, then $M^{\Sigma} \subset M$.

Proof Let M be a model of A^{Σ} and $q \in T_{\Sigma gr}$. We show by structural induction on q that $s(q) \in M$ for each $s \in \mathcal{S}$ satisfying $[q] \leq s$.

Base Case $q \in \mathfrak{C}$: Then $s(q) \in M$ by Lemma 10.1. ☐

Induction Step: Let $q = f(q_1 \ldots q_k)$, $f \in \mathcal{F}_{s_1 \ldots s_k, s_{k+1}}$, $s_{k+1} \leq s$, $q_i \in T_{\Sigma gr}$ and $[q_i] \leq s_i$ for each i with $1 \leq i \leq k$. Our induction hypotheses is to assume that $s_i(q_i) \in M$ for each i with $1 \leq i \leq k$. By definition of A^{Σ} we know that $\{not\ s_1(x_1), \ldots, not\ s_k(x_k), s_{k+1}(f(x_1 \ldots x_k))\} \in A^{\Sigma}$, i.e.

$$\{not\ s_1(q_1), \ldots, not\ s_k(q_k), s_{k+1}(f(q_1 \ldots q_k))\} \in A^{\Sigma}_{gr} .$$

By the induction hypotheses we obtain that $not\ s_i(q_i) \notin M$, hence $s_{k+1}(f(q_1 \ldots q_k)) = s_{k+1}(q) \in M$ because M satisfies A^{Σ}. With $s_{k+1} \leq s$ we finally infer by Lemma 10.2 that $s(q) \in M$. ☐ ☒

With the following lemma we can prove one direction of the equivalence stated in the Sort-Theorem:

Lemma 10.4 If M is an E-model of $(\hat{S} \cup A^{\Sigma})_{gr}$, then M E-satisfies $S_{\Sigma gr}$.

Proof Let $C \in S$ and $\theta \in SUB_{\Sigma}$ such that $\theta C \in S_{\Sigma gr}$. Then

$$\theta\hat{C} = (\{not\ s_1(q_1), \ldots, not\ s_n(q_n)\} \cup \theta C) \in \hat{S}_{gr} ,$$

where $q_i \in T_{\Sigma gr}$, $s_i \in \mathcal{S}$ and $[q_i] \leq s_i$ for each i with $1 \leq i \leq n$.

Let M be an E-model of $(\hat{S} \cup A^{\Sigma})_{gr}$. Then M is a model of A^{Σ} and by Lemma 10.3 $M^{\Sigma} \subset M$. Hence by Definition 10.1 $s_i(q_i) \in M$, i.e. $not\ s_i(q_i) \notin M$ because M is an interpretation, and therefore $M \cap \theta\hat{C} = M \cap \theta C$. Since $\theta\hat{C} \in \hat{S}_{gr}$ we know that M satisfies $\theta\hat{C}$, hence $M \cap \theta C \neq \emptyset$. ☒

To prove the other direction of the equivalence stated in the Sort-Theorem, we construct an E-model M* of $(\hat{S} \cup A^{\Sigma})_{gr}$ from an E-model M of $S_{\Sigma gr}$. This construction is carried out in two steps:

First we extend M to an E-interpretation M' which satisfies $(\hat{S} \cup A^{\Sigma})_{\Sigma gr}$. In the second step M' is extended to an E-interpretation M* in which each sort atom L is false, i.e. $L^c \in M^*$, if L is not true in M', i.e. $L \notin M'$, and we prove in Lemma 10.7 that M* satisfies $(\hat{S} \cup A^{\Sigma})_{gr}$.

Definition 10.2 Let I be an E-interpretation. The I-*kernel* $\lceil I \rceil$ of A^{Σ} is defired as

$$\lceil I \rceil = \{L \in LIT^{\$}_{gr} \mid L = s(t) \text{ such that } s(q) \in M^{\Sigma} \text{ and } E(q\ t) \in I \text{ for some } s \in \$,\ t \in T_{gr} \text{ and } q \in T_{\Sigma gr}\}.$$ ☒

Definition 10.3 Let I be an E-interpretation. The *$-complement* of I, denoted I^c, is defined as

$$I^c = \{L \in LIT^{\$}_{gr} \mid L = \textit{not}\ s(t) \text{ and } s(t) \notin I \text{ for some } s \in \$ \text{ and } t \in T_{gr}\}.$$ ☒

If we extend an E-interpretation I by the I-kernel or by the $-complement of I, we obtain an E-interpretation. This is proved by the following two lemmata:

Lemma 10.5 If $I \subset LIT_{gr}$ is an E-interpretation, then $(I \cup \lceil I \rceil)$ is an E-interpretation.

Proof $(I \cup \lceil I \rceil)$ is an interpretation: We prove that $L^c \notin (I \cup \lceil I \rceil)$ for each $L \in (I \cup \lceil I \rceil)$.

Case (i) $L \in I$: Then $L^c \notin I$ because I is an interpretation. $L^c \notin \lceil I \rceil$ since $\lceil I \rceil$ contains only sort atoms, but L^c cannot be a sort atom because $L \in LIT_{gr}$ by assumption. Hence $L^c \notin (I \cup \lceil I \rceil)$. ☒

Case (ii) $L \in \lceil I \rceil$: Then $L^c \notin \lceil I \rceil$ because $\lceil I \rceil$ contains only positive literals. $L^c \notin I$ because L^c is a sort literal and $I \subset LIT_{gr}$ by assumption. Hence $L^c \notin (I \cup \lceil I \rceil)$. ☐ ☐

I ∪ ⌈I⌉ is reflexive: Obvious, because I is. ☐

I ∪ ⌈I⌉ is E-closed: Let $L \in (I \cup \lceil I \rceil)$, $K \in LIT^{\$}_{gr}$ and $\alpha \in SEL^+$ such that $L \xrightarrow[\alpha]{} _{(I \cup \lceil I \rceil)} K$. Then $E(\alpha(L)\alpha(K)) \in (I \cup \lceil I \rceil)$, and since $\lceil I \rceil$ contains only sort atoms we obtain $E(\alpha(L)\alpha(K)) \in I$, i.e.

$$L \xrightarrow[\alpha]{}_I K \quad .$$

If $L \in I$, then $K \in I$ by the E-closure of I and we are finished.

So let us assume that $L \in \lceil I \rceil$, i.e. by Definition 10.2 there exist $s \in \$$, $t \in T_{gr}$ and $q \in T_{\Sigma gr}$ such that

$$L = s(t),\ s(q) \in M_\Sigma \text{ and } E(q\ t) \in I.$$

From $L \xrightarrow[\alpha]{}_I K$ we obtain $L \underset{\alpha}{\sim} K$, hence

$$K = s(r)$$

for some $r \in T_{gr}$. Let $\alpha_1, \alpha_2 \in SEL$ such that $\alpha_1(s(t)) = t$ and $\alpha_2(E(q\ t)) = t$. Then $\alpha = \beta \circ \alpha_1$ for some $\beta \in SEL^*$ and with $L \xrightarrow[\alpha]{}_I K$ we obtain by Lemma 2.2 (3) $t \xrightarrow[\beta]{}_I r$. Hence $\alpha_2(E(q\ t)) \xrightarrow[\beta]{}_I \alpha_2(E(q\ r))$ and with $E(q\ t) \underset{\alpha_2}{\sim} E(q\ r)$ we infer by Lemma 2.2 (3)

$$E(q\ t) \xrightarrow[\beta \circ \alpha_2]{}_I E(q\ r).$$

Hence $E(q\ r) \in I$, because $E(q\ t) \in I$ and I is E-closed. Using $s(q) \in M_\Sigma$, by Definition 10.2 we finally obtain $s(r) = K \in \lceil I \rceil$. ☐ ☒

Lemma 10.6 If $I \subset LIT^{\$}_{gr}$ is an E-interpretation, then $(I \cup I^c)$ is an E-interpretation.

Proof $(I \cup I^c)$ is an interpretation: Let $L \in (I \cup I^c)$. If $L \in I$, then $L^c \notin I$ because I is an interpretation, and $L^c \notin I^c$ by Definition 10.3. If $L \in I^c$, then $L^c \notin I$ by definition, and $L^c \notin I^c$ because I^c contains only negative literals. Hence in either case $L^c \notin (I \cup I^c)$. ☐

<u>$(I \cup I^c)$is reflexive:</u> Obvious, because I is. ⍁

<u>$(I \cup I^c)$ is E-closed:</u> Let $L \in (I \cup I^c)$, $K \in LIT^{\$}_{gr}$ and $\alpha \in SEL^+$ such that $L \xrightarrow[\alpha]{}_{(I \cup I^c)} K$. Then $E(\alpha(L)\alpha(K)) \in (I \cup I^c)$, and since I^c contains only negative literals we obtain $E(\alpha(L)\alpha(K)) \in I$, i.e.

$$L \xrightarrow[\alpha]{}_I K .$$

If $L \in I$, then $K \in I$ by the E-closure of I and we are finished.

So let us assume that $L \in I^c$. Suppose that $K^c \in I$. From $L \xrightarrow[\alpha]{}_I K$ we obtain $L^c \xrightarrow[\alpha]{}_I K^c$, hence by Lemma 2.3 (1) $K^c \xrightarrow[\alpha]{}_I L^c$, i.e. $L^c \in I$ because I is E-closed. But then $L \notin I^c$. ∇ Hence $K^c \notin I$, i.e. $K \in I^c$. ⍁ ⊠

Lemma 10.7 If $S_{\Sigma gr}$ is E-satisfiable, then $(\hat{S} \cup A^{\Sigma})_{gr}$ is E-satisfiable.

Proof Let M be an E-model of $S_{\Sigma gr}$. Since $S_{\Sigma gr} \subset \mathcal{L}_{\Sigma gr}$, we are allowed to assume that $M \subset LIT_{gr}$. Hence by Lemma 10.5 $(M \cup \lceil M \rceil)$ is an E-interpretation and using Lemma 10.6 we obtain that

$$M^* = (M \cup \lceil M \rceil) \cup (M \cup \lceil M \rceil)^c$$

is an E-interpretation. Let $C \in (\hat{S} \cup A^{\Sigma})$, i.e.

$$C = \{not\ s_1(x_1), \ldots, not\ s_n(x_n)\} \cup D$$

where $D \in \mathcal{L}^{\$}_{\Sigma}$ and no literal in D is a negative sort literal, $vars(D) = \{x_1, \ldots, x_n\}$ and $[x_i] = s_i$ for each i with $1 \le i \le n$, and let $\theta \in SUB$ such that $\theta_{|\mathfrak{V}} = \{x_1 \leftarrow t_1, \ldots, x_n \leftarrow t_n\}$ and $COD(\theta) \in T_{gr}$, i.e.

$$\theta C = (\{not\ s_1(t_1), \ldots, not\ s_n(t_n)\} \cup \theta D) \in (\hat{S} \cup A^{\Sigma})_{gr} .$$

We prove that M^* satisfies θC.

Case (i) $s_i(t_i) \notin \lceil M \rceil$ for some i with $1 \le i \le n$: Then $s_i(t_i) \notin (M \cup \lceil M \rceil)$ because M contains no sort literals, hence *not* $s_i(t_i) \in (M \cup \lceil M \rceil)^c$ by Definition 10.3. But then *not* $s_i(t_i) \in M^* \cap \theta C$, i.e. M^* satisfies θC. ⍁

Case (ii) $s_i(t_i) \in \lceil M \rceil$ for each i with $1 \le i \le n$: Then by Definition 10.2 there exist $q_1,\dots,q_n \in T_{\Sigma gr}$ such that

$$s_i(q_i) \in M^{\Sigma} \text{ and } E(q_i\ t_i) \in M$$

for each i with $1 \le i \le n$. Let $\lambda \in SUB_{\Sigma}$ such that $\lambda_{|\mathfrak{V}} = \{x_1 \leftarrow q_1,\dots,x_n \leftarrow q_n\}$.

We prove that

$$(M \cup \lceil M \rceil) \cap \lambda D \neq \emptyset \ .$$

Case (ii.1) $s(r) \in D$ for some $s \in \mathcal{S}$ and some $r \in T$. Then $r \in T_{\Sigma}$ and $[r] \le s$ since $D \in \mathcal{L}^{\mathcal{S}}_{\Sigma}$. Hence $s(\lambda r) \in \lambda D$, $\lambda r \in T_{\Sigma gr}$ and $[\lambda r] \le s$. By Definition 10.1 we obtain that $s(\lambda r) \in M^{\Sigma}$, hence by Definition 10.2 $s(\lambda r) \in \lceil M \rceil$, i.e. $s(\lambda r) \in (M \cup \lceil M \rceil) \cap \lambda D$. ☐

Case (ii.2) $s(r) \notin D$ for each $s \in \mathcal{S}$ and each $r \in T$: Then $C = \hat{D}$, i.e. $D \in S$ and $\lambda D \in S_{\Sigma gr}$. Since M satisfies $S_{\Sigma gr}$ we know that $M \cap \lambda D \neq \emptyset$, i.e. $(M \cup \lceil M \rceil) \cap \lambda D \neq \emptyset$. ☐

Let $L \in D$ such that $\lambda L \in (M \cup \lceil M \rceil)$, let $\{\alpha_1,\dots,\alpha_h\}$ = $\{\alpha \in SEL^+ \mid \alpha(L) = x_i$ for some $x_i \in vars(D)\}$ and let $K_1,\dots,K_{h+1} \in LIT^{\mathcal{S}}_{gr}$ such that $K_1 = \lambda L$, $K_j \underset{\alpha_j}{\sim} K_{j+1}$, $\alpha_j(K_j) = \lambda\alpha_j(L)$ and $\alpha_j(K_{j+1}) = \theta\alpha_j(L)$ for each j with $1 \le j \le h$.

Then $K_{h+1} = \theta L$ and $K_1 \xrightarrow[\alpha_1]{} (M \cup \lceil M \rceil)\ K_2 \dots K_h \xrightarrow[\alpha_h]{} (M \cup \lceil M \rceil)\ K_{h+1}$, i.e.

$$K_1 \xrightarrow{*}_{\mathcal{R}(M \cup \lceil M \rceil)} K_{h+1}$$

because $\alpha_j(K_j) = \lambda x_i = q_i$, $\alpha_j(K_{j+1}) = \theta x_i = t_i$ and $E(q_i\ t_i) \in M$.

With $K_1 = \lambda L \in (M \cup \lceil M \rceil)$ we obtain by Lemma 2.3 (2) that $K_{h+1} = \theta L \in (M \cup \lceil M \rceil)$. Hence $\theta L \in (M \cup \lceil M \rceil) \cap \theta D$, i.e. M* satisfies θC. ☐ ☒

The Sort-Theorem is now an immediate consequence:

Theorem 10.8 (Sort-Theorem for the ΣRP-Calculus) S is ΣE-unsatisfiable iff $(\hat{S} \cup A^{\Sigma})$ is E-unsatisfiable.

Proof "⇒" If S is ΣE-unsatisfiable, then $S_{\Sigma gr}$ is E-unsatisfiable, hence by contraposition of Lemma 10.4 $(\hat{S} \cup A^{\Sigma})_{gr}$ is E-unsatisfiable, i.e. $(\hat{S} \cup A^{\Sigma})$ is E-unsatisfiable. ◪

"⇐" If $(\hat{S} \cup A^{\Sigma})$ is E-unsatisfiable, then $(\hat{S} \cup A^{\Sigma})_{gr}$ is E-unsatisfiable, hence by contraposition of Lemma 10.7 $S_{\Sigma gr}$ is E-unsatisfiable, i.e. S is ΣE-unsatisfiable. ◪ ⊠

11. An Automated Theorem Prover for the ΣRP-*Calculus*

In this section a brief overview is presented of how an automated theorem prover ATP based on the RP-calculus can be modified to obtain an automated theorem prover ATP_Σ for the ΣRP-calculus. The necessary modifications of the ATP concern

- the input-language compiler ,
- the skolemization routine ,
- the unification algorithm , and
- the computation of factors, resolvents and paramodulants.

A protocol of an example run of an existing ATP_Σ is exhibited at the end of this section.

The Compiler The compiler tests whether a given input string satisfies the rules of syntax and those of the 'static semantics' (i.e. that function symbols are used with a proper arity e.t.c.) of a (somewhere defined) first-order language $\mathcal{K}$ with the usual junctors, universal and existential quantifiers (and produces as 'code' a first-order formula in a certain representation, but this is of course irrelevant here).

The rules of the static semantics have to be extended such that only formulas from the set of all *well sorted* first-order formulas $\mathcal{K}_\Sigma \subset \mathcal{K}$ will be accepted: For each atomar formula A in a formula given as input, the compiler has to determine whether A is a *well sorted* atomar formula, i.e.

$$[\alpha(A)] \leq [A]_\alpha \text{ for each } \alpha \in SEL^+ \text{ satisfying } \alpha(A)\downarrow, \text{ or } A \in \mathcal{P}_e .$$

This problem is the same as for programming languages with sorts (often called *types*), e.g. PASCAL or ADA, and hence can be solved using the well known techniques of compiler construction.

In addition a device is required to define a set of sort symbols $\mathcal{S}$, a subsort order $\leq_\mathcal{S}$ and some $\mathcal{S}$-sorted signature Σ. This is achieved extending the language $\mathcal{K}_\Sigma$ by certain constructs which allow the definition of $\mathcal{S}$, $\leq_\mathcal{S}$ and Σ [Wal82]. For this extension the compiler has to perform additional 'semantic' tests, e.g. to check whether $\leq_\mathcal{S}$ is in fact an *order* relation.

The Skolemization Routine On skolemization of a first-order formula (given in a certain format) each occurence of an existentially quantified variable symbol y in an atomar formula is replaced by a *skolem term* t, and all existential quantifiers are removed.

The *skolem term* t consists of a *new* function symbol f followed by a (possibly empty) sequence $x_1,\ldots,x_n$ of variable symbols as arguments, where each x_i is a universally quantified variable symbol and the variable symbol y, which was replaced by t, is in the scope of exactly the universal quantifiers for the variable symbols x_i.

For the *Σ-skolemization*, i.e. skolemization under sorts, this process is the same for each formula in K_Σ, but in addition the signature Σ has to be extended, yielding a signature Σ* for the new function symbols introduced by the skolemization: We assert

$$f \in \mathcal{F}_{s_1\ldots s_n,s} \quad \text{iff} \quad [x_i] = s_i \quad \text{and} \quad [y] = s \quad ,$$

where f, x_i and y are defined as above, and it is obvious that Σ-skolemization transforms well sorted formulas (of K_Σ) into well sorted formulas (of $K_{\Sigma*}$).

To be correct, we have to show that Σ-skolemization *maintains* Σ-(un)satisfiability: From [Obe62] we obtain the semantic notions for K_Σ, and in particular a notion of Σ-unsatisfiability for formulas $\phi \in K_\Sigma$. Let $K^\$$ be the extended language of K, where sort symbols may be used as unary predicate symbols. Then by the Sort-Theorem of [Obe62], each formula

$$\phi \in K_\Sigma \text{ is } \Sigma\text{-unsatisfiable iff } (\{\hat{\phi}\} \cup A^\Sigma) \subset K^\$ \text{ is unsatisfiable.}$$

Consider the following diagram:

$$\begin{array}{ccc}
\langle \phi \in K_\Sigma, \Sigma \rangle & \xrightarrow{(1)} & (\{\hat{\phi}\} \cup A^\Sigma) \subset K^\$ \\
\Big\downarrow (5) & & \Big\downarrow (2) \\
\langle \tilde{\phi} \in K_{\Sigma*}, \Sigma* \rangle & \xrightarrow[(4)]{} & (\{\hat{\tilde{\phi}}\} \cup A^{\Sigma*}) \underset{(3)}{\equiv} (\{\tilde{\hat{\phi}}\} \cup A^\Sigma) \subset K^\$
\end{array}$$

Figure 11.1 Skolemization and Σ-Skolemization

Here $\tilde{\phi}$ denotes the formula which is obtained from the formula ϕ by (Σ-)skolemization on (2) and (5). On (1) and (4) a formula is replaced by its relativization and the set of sort axioms.

With a proof of the equivalence (3) (which is technical and omitted here), we obtain from figure 11.1 that each Σ-skolemized formula $\tilde{\phi} \in K_{\Sigma*}$ is $\Sigma*$-unsatisfiable iff $\phi \in K_{\Sigma}$ is Σ-unsatisfiable, because (1) and (4) maintain (un)satisfiability by the Sort-Theorem of [Obe62], and (2) is the skolemization in the one-sorted calculus and hence leaves (un)satisfiability unchanged [Lov78].

The Unification Algorithm At the very heart of each (Robinson) unification algorithm, variable symbols x have to be unified with terms t. The resulting substitution, represented by $\{x \leftarrow t\}$, is composed with other substitutions of this kind, yielding finally an mgu for the set of terms (or atoms) initially given to the unification algorithm (provided the set is unifiable). Hence each unification algorithm contains a sequence of statements like

(1) *if* $x = t$ *then return* ({})
(2) *if* $x \in vars(\{t\})$ *then* stop/failure
(3) *return* ($\{x \leftarrow t\}$)

Figure 11.2 Unification of variables and terms

On unification under sorts, by the Σ-Unification Theorem a Σ-mgu may exist for a set of Σ-unifiable terms only after an application of a weakening substitution. We modify the unification algorithm to obtain a *Σ-unification algorithm* by replacing statement (3) in figure 11.2 by the sequence of statements

(3.1) *if* $[t] \leq_{\mathcal{S}} [x]$ *then return* ($\{\{x \leftarrow t\}\}$)
(3.2) *if* $t \notin \mathcal{V}$ *or* $[t] \sqcap_{\mathcal{S}} [x] = \emptyset$ *then* stop/failure
(3.3) *if* $[x] <_{\mathcal{S}} [t]$ *then return* ($\{\{t \leftarrow x\}\}$)
(3.4) *let* $\{s_1, \ldots, s_k\} = \max([t] \sqcap_{\mathcal{S}} [x])$
(3.5) *let* $\{z_1, \ldots, z_k\} \subset \mathcal{V}$ such that no z_i is used before and $[z_i] = s$
(3.6) *return* ($\{\{x \leftarrow z_1, t \leftarrow z_1\}, \ldots, \{x \leftarrow z_k, t \leftarrow z_k\}\}$)

where $s_1 \sqcap_{\mathcal{S}} s_2 = \{s \in \mathcal{S} \mid s \leq_{\mathcal{S}} s_1 \text{ and } s \leq_{\mathcal{S}} s_2\}$ and
$\max(S) = \{s \in S \mid s \not\leq_{\mathcal{S}} s' \text{ for each } s' \in S\}$

Figure 11.3 Σ-unification of variables and terms

For each Σ-unifiable set of Σ-terms the Σ-unification algorithm returns a *set* of Σ-unifiers and not a single unifier as usual, because a unification problem may have *several* most general solutions under sorts.

It can be verified that for each set D of Σ-terms (or Σ-atoms) given as input, the Σ-unification algorithm terminates with a failure indication, if D is not Σ-unifiable, and else terminates with a finite set of Σ-substitutions $U(D) = \{\tau_1,\dots,\tau_n\}$ as output, such that for each i with $1 \le i \le n$:

(1) $\mu_i \in WSUB(vars(D))$,

(2) σ_i is a Σ-mgu of $\mu_i D$,

(3) $\tau_i = \sigma_i \circ \mu_i\ [vars(D)]$, and

(4) for each Σ-unifier θ of D, there exist some $\lambda \in SUB_\Sigma$ and some $\tau_j \in U(D)$ such that
$\theta = \lambda \circ \tau_j [vars(D)]$.

If D is Σ-unifiable and in addition is Σ-*compatible*, then $U(D) = \{\tau\}$, where τ is a Σ-mgu of D, because by the Σ-compatibility exactly one of the conditions in the statements (3.1) and (3.3) of figure 11.3 is always satisfied.

Computation of Factors, Resolvents and Paramodulants We outline an implementation of an ATP_Σ, which avoids the explicit computation of weakened variants:

Let A be a clause in a Σ-deduction and let $B \subset A$ such that $|B| \ge 2$ and

$$U(B) = \{\tau_1,\dots,\tau_n\} \ .$$

Then in order to be complete, the ATP_Σ has to compute *each* Σ-clause

$$\tau_i A \ ,$$

each of which is a Σ-factor of weakened variant of A. This is an immediate consequence of the Lifting Lemmata for Σ-Resolution and Σ-Paramodulation.

Let A, B be clauses in a Σ-deduction, $L_A \in A$ and $L_B \in B$ such that L_A and L_B are complementary and

$$U(\{|L_A|,|L_B|\}) = \{\tau_1,\ldots,\tau_n\} \quad .$$

To be complete, the ATP_Σ has to compute *each* Σ-clause

$$\tau_i(A - L_A) \cup \tau_i(B - L_B) \qquad ,$$

each of which is a Σ-resolvent of some weakened variants of A and B. This is justified by the Lifting Lemma for Σ-Resolution.

Let A, B be clauses in a Σ-deduction, $L \in A$, $E(q\ r) \in B$ and $\alpha \in SEL^+$ such that $\{\alpha(L),q\}$ is Σ-unifiable. If $r \in \mathfrak{V}$ and $[r] \sqcap_S [L]_\alpha = \emptyset$ or $r \notin \mathfrak{V}$ and $[r] \not\leq_S [L]_\alpha$, then there exists no Σ-paramodulant of A and (possibly some weakened variant of) B, because $[\theta r] \not\leq_S [\theta L]_\alpha$ for each $\theta \in SUB_\Sigma$.

Hence a Σ-paramodulant can only be obtained, if $r \in \mathfrak{V}$ implies $[r] \sqcap_S [L]_\alpha \neq \emptyset$ and $r \notin \mathfrak{V}$ implies $[r] \leq_S [L]_\alpha$. Let $\{s_1,\ldots,s_k\} = \max([r] \sqcap_S [L]_\alpha)$ and $\{z_1,\ldots,z_k\} \subset \mathfrak{V}$ such that each z_i is never used before and $[z_i] = s_i$. If $[r] \not\leq_S [L]_\alpha$ we define $\mu_1,\ldots,\mu_k \in SUB_\Sigma$ by $\mu_{j|\mathfrak{V}} = \{r \leftarrow z_j\}$. For $[r] \leq_S [L]_\alpha$, we set k=1 and $\mu_1 = \varepsilon$. It is obvious that $\mu_j B$ is a weakened variant of B and that $[\mu_j r] \leq [L]_\alpha$ for each j with $1 \leq j \leq k$.

We know from the proof of the Lifting Lemma for Σ-Paramodulation that $\{\alpha(L),\mu_j q\}$ is Σ-unifiable, hence

$$U(\{\alpha(L),\mu_j q\}) = \{\tau_1^j,\ldots,\tau_{n_j}^j\}$$

is not empty. To be complete, the ATP_Σ has to compute for each j and i with $1 \leq j \leq k$ and $1 \leq i \leq n_j$ the Σ-clauses

$$\tau_i^j(A - L) \cup \tau_i^j(B - E(q\ r)) \cup \{\tau_i^j K\}$$

(where $\tau_i^j K$ is a modulant literal), each of which is a Σ-paramodulant of some weakened variants of A and B by the Lifting Lemma for Σ-Paramodulation.

After the computation of a Σ-factor, Σ-resolvent or Σ-paramodulant the variable symbols of these Σ-clauses have to be renamed using an appropriate Σ-*renaming substitution*.

The Markgraf Karl Refutation Procedure [BES81, Ohl82], a theorem proving system developed at the University of Karlsruhe, was adapted to the ΣRP-calculus according to the modifications stated above. We exhibit a proof protocol of the new system, proving a many-sorted version of the well known *monkey-banana-problem* [Lov78]:

```
****************************************************************************************************
*                                                                                                  *
*    ATP SYSTEM:   MARKGRAF KARL REFUTATION PROCEDURE, UNI KARLSRUHE, VERSION  12-OCT-82           *
*                                                                                                  *
*         DATE:   2-NOV-82 16:46:27                                                                *
*                                                                                                  *
****************************************************************************************************


FORMULAE GIVEN TO THE THEOREM PROVER:

          AXIOMS:         SORT ANIMAL,TALL:IN.ROOM
                          TYPE BANANA,FLOOR:IN.ROOM
                          TYPE CHAIR:TALL
                          TYPE MONKEY:ANIMAL
                          TYPE CAN.REACH(ANIMAL IN.ROOM)
                          TYPE CLOSE.TO(IN.ROOM IN.ROOM)
                          TYPE ON(IN.ROOM IN.ROOM)
                          TYPE UNDER(IN.ROOM IN.ROOM)
                          TYPE CAN.MOVE.NEAR(ANIMAL IN.ROOM IN.ROOM)
                          TYPE CAN.CLIMB(ANIMAL TALL)
                          ALL X:ANIMAL ALL Y:IN.ROOM CLOSE.TO (X Y) IMPL CAN.REACH (X Y)
                          ALL X:ANIMAL ALL Y:TALL ON (X Y) AND UNDER (Y BANANA) IMPL CLOSE.TO (X BANANA)
                          ALL X:ANIMAL ALL Y,Z:IN.ROOM CAN.MOVE.NEAR (X Y Z)
                                                       IMPL (CLOSE.TO (Z FLOOR) OR UNDER (Y Z))
                          ALL X:ANIMAL ALL Y:TALL CAN.CLIMB (X Y) IMPL ON (X Y)
                          CAN.MOVE.NEAR (MONKEY CHAIR BANANA)
                          NOT CLOSE.TO (BANANA FLOOR)
                          CAN.CLIMB (MONKEY CHAIR)

          THEOREM:        CAN.REACH (MONKEY BANANA)


*************************
*   INITIAL GRAPH       *
*************************


CLAUSES:

     AXM1 :  ALL X:ANIMAL Y:IN.ROOM NOT CLOSE.TO(X Y)  OR  CAN.REACH(X Y)
     AXM2 :  ALL X:ANIMAL Y:TALL NOT ON(X Y)  OR  NOT UNDER(Y BANANA)  OR  CLOSE.TO(X BANANA)
     AXM3 :  ALL X:ANIMAL Y:IN.ROOM Z:IN.ROOM NOT CAN.MOVE.NEAR(X Y Z)  OR  CLOSE.TO(Z FLOOR)
                                                OR UNDER(Y Z)
     AXM4 :  ALL X:ANIMAL Y:TALL NOT CAN.CLIMB(X Y)  OR  ON(X Y)
     AXM5 :  CAN.MOVE.NEAR(MONKEY CHAIR BANANA)
     AXM6 :  NOT CLOSE.TO(BANANA FLOOR)
     AXM7 :  CAN.CLIMB(MONKEY CHAIR)
     THM8 :  NOT CAN.REACH(MONKEY BANANA)


AXM2    AND AXM3    IMPLIES     RES1 :  ALL X:ANIMAL Y:TALL Z:ANIMAL CLOSE.TO(X BANANA)  OR  NOT ON(X Y)
                                                                     OR  CLOSE.TO(BANANA FLOOR)
                                                                     OR  NOT CAN.MOVE.NEAR(Z Y BANANA)
RES1    AND AXM5    IMPLIES     RES2 :  ALL X:ANIMAL CLOSE.TO(BANANA FLOOR)  OR  NOT ON(X CHAIR)
                                                     OR  CLOSE.TO(X BANANA)
AXM1    AND RES2    IMPLIES     RES3 :  ALL X:ANIMAL CAN.REACH(X BANANA)  OR  NOT ON(X CHAIR)
                                                     OR  CLOSE.TO(BANANA FLOOR)
AXM6    AND RES3    IMPLIES     RES4 :  ALL X:ANIMAL NOT ON(X CHAIR)  OR  CAN.REACH(X BANANA)
THM8    AND RES4    IMPLIES     RES5 :  NOT ON(MONKEY CHAIR)
RES5    AND AXM4    IMPLIES     RES6 :  NOT CAN.CLIMB(MONKEY CHAIR)
RES6    AND AXM7    IMPLIES     RES7 :  EMPTY
```

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```
GRAPH SUCCESSFULLY REFUTED .

CPU-TIME USED:                              3.320000 SECONDS
NUMBER OF STEPS EXECUTED:                   7
NUMBER OF LINKS GENERATED:                  22
     RLINKS:                                22
     PLINKS:                                0
     FLINKS:                                0
     NUMBER OF LINKS IN INITIAL GRAPH:      8
     RLINKS:                                8
     PLINKS:                                0
     FLINKS:                                0
NUMBER OF CLAUSES GENERATED:                15
     INITIAL CLAUSES:                       8
     DEDUCED CLAUSES:                       7
          RESOLVENTS:                       7
          PARAMODULANTS:                    0
          FACTORS:                          0
NUMBER OF CLAUSES DELETED:                  10
LEVEL OF PROOF:                             7
NUMBER OF CLAUSES IN PROOF:                 15
     INITIAL:                               8
     DEDUCED:                               7
G-PENETRANCE:                               1.000000   (# OF CLAUSES IN PROOF  /  # OF CLAUSES GENERATED)
D-PENETRANCE:                               1.000000   (# OF DEDUCED CLAUSES IN PROOF  /  # OF CLAUSES DEDUCED)
R-VALUE:                                    0.666667   (# OF CLAUSES DELETED  /  # OF CLAUSES GENERATED)

THE FOLLOWING CLAUSES WERE USED IN THE PROOF:

AXM7 AXM4 AXM5 AXM3 AXM2 RES1 RES2 AXM1 RES3 AXM6 RES4 THM8 RES5 RES6 RES7 .


THE THEOREM IS PROVED.
END  OF  PROOF:     2-NOV-82 16:47:22
```

Figure 11.4 A proof of the monkey-banana problem, using a many-sorted axiomatization

In our system, we use the expressions (cf. [Wal82])

SORT $s_1,\ldots,s_n$:s to denote $s_1 \ll_{\mathfrak{s}} s \ldots s_n \ll_{\mathfrak{s}} s$,

TYPE $c_1,\ldots,c_n$:s to denote $c_1 \in \mathcal{F}_{e,s} \ldots c_n \in \mathcal{F}_{e,s}$,

TYPE $P(s_1 \ldots s_n)$ to denote $P \in \mathfrak{P}_{s_1 \ldots s_n}$, and

ALL x:s to denote the universal quantification of a variable symbol $x \in \mathfrak{V}_s$.

In the proof statistics, the value for 'number of links generated' corresponds to the size of the search space, the value for 'number of steps executed' is a measure for the expense of the actual search and 'level of proof' represents the search depth.

Let us compare the above protocol with a proof protocol of the same problem, using the one-sorted axiomatization obtained from [Lov78]:

```
***************************************************************************************************
*                                                                                                 *
*   ATP SYSTEM:   MARKGRAF KARL REFUTATION PROCEDURE, UNI KARLSRUHE, VERSION  12-OCT-82            *
*                                                                                                 *
*        DATE:    2-NOV-82 16:40:31                                                               *
*                                                                                                 *
***************************************************************************************************


FORMULAE GIVEN TO THE THEOREM PROVER:

        AXIOMS:       ALL X,Y ANIMAL (X) AND CLOSE.TO (X Y) IMPL CAN.REACH (X Y)
                      ALL X,Y ON (X Y) AND UNDER (Y BANANA) AND TALL (Y) IMPL CLOSE.TO (X BANANA)
                      ALL X,Y,Z IN.ROOM (X) AND IN.ROOM (Y) AND IN.ROOM (Z) AND CAN.MOVE.NEAR (X Y Z)
                            IMPL (CLOSE.TO (Z FLOOR) OR UNDER (Y Z))
                      ALL X,Y CAN.CLIMB (X Y) IMPL ON (X Y)
                      ANIMAL (MONKEY)
                      TALL (CHAIR)
                      IN.ROOM (MONKEY)
                      IN.ROOM (BANANA)
                      IN.ROOM (CHAIR)
                      CAN.MOVE.NEAR (MONKEY CHAIR BANANA)
                      NOT CLOSE.TO (BANANA FLOOR)
                      CAN.CLIMB (MONKEY CHAIR)

        THEOREM:      CAN.REACH (MONKEY BANANA)


***************************
*   INITIAL GRAPH         *
***************************


CLAUSES:

     AXM1  :  ALL X:ANY Y:ANY NOT ANIMAL(X)  OR  NOT CLOSE.TO(X Y)  OR  CAN.REACH(X Y)
     AXM2  :  ALL X:ANY Y:ANY NOT ON(X Y)  OR  NOT UNDER(Y BANANA)  OR  NOT TALL(Y)  OR  CLOSE.TO(X BANANA)
     AXM3  :  ALL X:ANY Y:ANY Z:ANY NOT IN.ROOM(X)  OR  NOT IN.ROOM(Y)  OR  NOT IN.ROOM(Z)
                                 OR  NOT CAN.MOVE.NEAR(X Y Z)  OR CLOSE.TO(Z FLOOR)  OR  UNDER(Y Z)
     AXM4  :  ALL X:ANY Y:ANY NOT CAN.CLIMB(X Y)  OR  ON(X Y)
     AXM5  :  ANIMAL(MONKEY)
     AXM6  :  TALL(CHAIR)
     AXM7  :  IN.ROOM(MONKEY)
     AXM8  :  IN.ROOM(BANANA)
     AXM9  :  IN.ROOM(CHAIR)
     AXM10  :  CAN.MOVE.NEAR(MONKEY CHAIR BANANA)
     AXM11  :  NOT CLOSE.TO(BANANA FLOOR)
     AXM12  :  CAN.CLIMB(MONKEY CHAIR)
     THM13  :  NOT CAN.REACH(MONKEY BANANA)


 AXM3             IMPLIES     AXM3.FAC1 :  ALL X:ANY UNDER(X X)  OR  CLOSE.TO(X FLOOR)
                                                  OR  NOT CAN.MOVE.NEAR(X X X)  OR  NOT IN.ROOM(X)
 AXM3             IMPLIES     AXM3.FAC2 :  ALL X:ANY Y:ANY UNDER(X Y)  OR  CLOSE.TO(Y FLOOR)
                                                  OR  NOT CAN.MOVE.NEAR(Y X Y)  OR  NOT IN.ROOM(Y)
                                                  OR  NOT IN.ROOM(X)
 AXM3             IMPLIES     AXM3.FAC3 :  ALL X:ANY Y:ANY UNDER(X X)  OR  CLOSE.TO(X FLOOR)
                                                  OR  NOT CAN.MOVE.NEAR(Y X X)  OR  NOT IN.ROOM(X)
                                                  OR  NOT IN.ROOM(Y)
 AXM3             IMPLIES     AXM3.FAC4 :  ALL X:ANY Y:ANY UNDER(X Y)  OR  CLOSE.TO(Y FLOOR)
                                                  OR  NOT CAN.MOVE.NEAR(X X Y)  OR  NOT IN.ROOM(Y)
                                                  OR  NOT IN.ROOM(X)
 AXM1   AND AXM5   IMPLIES    RES1  :  ALL X:ANY CAN.REACH(MONKEY X)  OR  NOT CLOSE.TO(MONKEY X)
 THM13  AND RES1   IMPLIES    RES2  :  NOT CLOSE.TO(MONKEY BANANA)
 RES2   AND AXM2   IMPLIES    RES3  :  ALL X:ANY NOT TALL(X)  OR  NOT UNDER(X BANANA)  OR  NOT ON(MONKEY X)
ES3   AND AXM4   IMPLIES     RES4 :  ALL X:ANY NOT UNDER(X BANANA)  OR  NOT TALL(X)
                                          OR  NOT CAN.CLIMB(MONKEY X)
ES4   AND AXM12  IMPLIES     RES5 :  NOT TALL(CHAIR)  OR  NOT UNDER(CHAIR BANANA)
ES5   AND AXM3   IMPLIES     RES6 :  ALL X:ANY NOT TALL(CHAIR)  OR  CLOSE.TO(BANANA FLOOR)
                                          OR  NOT CAN.MOVE.NEAR(X CHAIR BANANA)  OR  NOT IN.ROOM(BANANA)
                                          OR  NOT IN.ROOM(CHAIR)  OR  NOT IN.ROOM(X)
ES6   AND AXM9   IMPLIES     RES7 :  ALL X:ANY NOT IN.ROOM(X)  OR  NOT IN.ROOM(BANANA)
                                          OR  NOT CAN.MOVE.NEAR(X CHAIR BANANA)
                                          OR  CLOSE.TO(BANANA FLOOR)  OR  NOT TALL(CHAIR)
ES7   AND AXM6   IMPLIES     RES8 :  ALL X:ANY CLOSE.TO(BANANA FLOOR)  OR  NOT CAN.MOVE.NEAR(X CHAIR BANANA)
                                          OR  NOT IN.ROOM(BANANA)  OR  NOT IN.ROOM(X)
ES8   AND AXM8   IMPLIES     RES9 :  ALL X:ANY NOT IN.ROOM(X)  OR  NOT CAN.MOVE.NEAR(X CHAIR BANANA)
                                          OR  CLOSE.TO(BANANA FLOOR)
XM11  AND RES9   IMPLIES     RES10 :  ALL X:ANY NOT CAN.MOVE.NEAR(X CHAIR BANANA)  OR  NOT IN.ROOM(X)
ES10  AND AXM10  IMPLIES     RES11 :  NOT IN.ROOM(MONKEY)
ES11  AND AXM7   IMPLIES     RES12 :  EMPTY
```

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```
GRAPH SUCCESSFULLY REFUTED .

CPU-TIME USED:                              11.375999 SECONDS
NUMBER OF STEPS EXECUTED:                   16
NUMBER OF LINKS GENERATED:                  99
     RLINKS:                                95
     PLINKS:                                0
     FLINKS:                                4
     NUMBER OF LINKS IN INITIAL GRAPH:      23
     RLINKS:                                19
     PLINKS:                                0
     FLINKS:                                4
NUMBER OF CLAUSES GENERATED:                29
     INITIAL CLAUSES:                       13
     DEDUCED CLAUSES:                       16
          RESOLVENTS:                       12
          PARAMODULANTS:                    0
          FACTORS:                          4
NUMBER OF CLAUSES DELETED:                  22
LEVEL OF PROOF:                             12
NUMBER OF CLAUSES IN PROOF:                 25
     INITIAL:                               13
     DEDUCED:                               12
G-PENETRANCE:                               0.862069   (# OF CLAUSES IN PROOF  /  # OF CLAUSES GENERATED)
D-PENETRANCE:                               0.750000   (# OF DEDUCED CLAUSES IN PROOF  /  # OF CLAUSES DEDUCED)
R-VALUE:                                    0.758621   (# OF CLAUSES DELETED  /  # OF CLAUSES GENERATED)

THE FOLLOWING CLAUSES WERE USED IN THE PROOF:

AXM7 AXM10 AXM8 AXM6 AXM9 AXM3 AXM12 AXM4 AXM2 AXM5 AXM1 RES1 THM13 RES2 RES3 RES4 RES5 RES6 RES7 RES8 RES9
AXM11 RES10 RES11 RES12 .


THE THEOREM IS PROVED.
END  OF  PROOF:     2-NOV-82 16:42:18
```

Figure 11.5 A proof of the monkey-banana problem, using a one-sorted axiomatization

A comparison between the statistical values of both protocols immediately reveals the advantages using an automated theorem prover based on the ΣRP-calculus.

12. Conclusion

For the RP-calculus many *refinements* like set-of-support, linear resolution, hyperresolution e.t.c. have been proposed and their behaviour regarding completeness was investigated [Lov78]. Instead of reinvestigating these results for the ΣRP-calculus, it seems advantageous to have a *direct* proof of the *prooftheoretic part* of the Sort-Theorem, i.e.

$$\hat{S}^E \cup A^\Sigma \vdash \square \text{ iff } S^E \vdash_\Sigma \square \quad ,$$

for each signature Σ and each Σ-clause set S.

This direct proof would provide a *constructive way* to translate a refutation in the RP-calculus to a refutation in the ΣRP-calculus (whenever this is possible). We conjecture that such a construction would show (in most cases), whether completeness results for refinements in the RP-calculus also hold for the ΣRP-calculus.

At present the *$-Logik*, the *Σ-Logik* and the *S-Logik* of *Oberschelp* appear to be the most expressive many-sorted calculi, for which soundness, completeness and the Sort-Theorem have been proved [Obe62]:

Sets of function symbols do not need to be disjoint in the *$-Logik*. For instance scalar-addition and vector-addition may share the same function symbol, e.g.

$$\text{plus} \in \mathcal{F}_{ss,s} \cap \mathcal{F}_{vv,v} \;, \quad \text{where } s \sqcap_{\$} v = \emptyset \;.$$

It is obvious that the respective 'intended' function symbol can be determined by inspecting the sorts of the arguments it is applied to. (Note that in this paper we have made frequent use of this feature. For instance |D| were used to denote the cardinality of the set D while |L| stood for the atom of the literal L.)

We believe that this useful practical device should *not* be incorporated in a calculus and its inference machinery. It would be reasonably implemented in an automated theorem prover, if shared function symbols are renamed by a compiler on input and are re-renamed by the protocol facility on output.

The *Σ-Logik* is not a many-sorted calculus in the strong sense, because *non-well sorted* formulas are admitted as axioms and hence appear as theorems. *Oberschelp* gave an example for a non-well sorted formula which nevertheless is a meaningfull expressio in the given context.

However we believe that this proposal is not advantageous for Automated Theorem Proving, because it destroys (especially for paramodulation) the advantages of a strongly restricted search space by the restriction of the inference rules to well sorted formulas.

The *S-Logik* corresponds directly to the *ΣRP-calculus*, however with one exception: The sets of function symbols may have non-empty intersections. For instance one may assert

$$\text{plus} \in \mathcal{F}_{zz,z} \cap \mathcal{F}_{nn,n}, \quad \text{where } n <_s z \quad .$$

If *plus* is applied to a pair of terms such that at least one of this terms has sort z, then the whole term has sort z. But its sort is n, if *both* arguments have sort n.

By such an extension the sorts of *all subterms* of a term have to be computed to determine the sort of the given term. (Note that in the ΣRP-calculus the sort of a term is independent of its arguments, because it is determined by the outermost symbol of the term.)

Obviously this feature enriches the expressive power of a many sorted calculus. In the ΣRP-calculus we need *additional axioms* (and also additional function symbols) to state similar facts.

For instance

$$\text{plus} \in \mathcal{F}_{zz,z} \quad ,$$

$$\text{add} \in \mathcal{F}_{nn,n} \quad , \text{ and}$$

$$\forall x,y{:}n.\ \text{plus}(x\ y) \equiv \text{add}(x\ y) \quad , \text{ where } n <_s z \quad ,$$

is an equivalent formulation in the ΣRP-calculus for the fact stated in the example above. All inferences (within the ΣRP-calculus) using the above equality are *shifted to the inference mechanism* in the *S*-Logik.

To ensure completeness, a similar extension of the ΣRP-calculus presupposes a corresponding reformulation of the *Σ-Rewrite Theorem* and of the *Σ-Unification Theorem*, which does not appear to be straightforward.

There are other proposals for various kinds of many-sorted calculi which are superior in their expressive power (at least on certain aspects). For instance *Hailperin's 'Theory of Restricted Quantification'* [Hai57] allows to express sortrelationships using arbitrary first-order formulas, whereas usually the only relationship between sort symbols is given by the subsort order. Moreover it is possible to write formulas like (in our notation)

$$\forall x{:}\bar{s}\ .\ \Phi(x)$$

with the intended meaning that x may have every sort, *except* sort s. One should take great care in adapting these (fundamental) extensions for Automated Theorem Proving, because they generally involve that deductions about sortrelationships can no longer be built into the inference mechanism of the system.

In fact, *Hailperin's* calculus *contains* the one-sorted calculus and hence a translation of a many-sorted version of a theorem to the one-sorted version of the theorem (and vice versa) is effected *within* his calculus rather then by a translation from *one* calculus to *another* using sort axioms and relativizations.

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