



On the global regularity for minimizers of variational integrals: splitting-type problems in 2D and extensions to the general anisotropic setting

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Abstract

We mainly discuss superquadratic minimization problems for splitting-type variational integrals on a bounded Lipschitz domain $\Omega \subset \mathbb{R}^2$ and prove higher integrability of the gradient up to the boundary by incorporating an appropriate weight-function measuring the distance of the solution to the boundary data. As a corollary, the local Hölder coefficient with respect to some improved Hölder continuity is quantified in terms of the function $\text{dist}(\cdot, \partial\Omega)$. The results are extended to anisotropic problems without splitting structure under natural growth and ellipticity conditions. In both cases we argue with variants of Caccioppoli's inequality involving small weights.

Keywords Global higher integrability · Splitting-type energies · Anisotropic growth

Mathematics Subject Classification 49N60 · 49Q20 · 49J45

1 Introduction

In the main part of our note we consider splitting-type energies as a particular class of variational integrals

$$J[u] := \int_{\Omega} f(\nabla u) \, dx \quad (1)$$

defined for functions $u: \mathbb{R}^n \supset \Omega \rightarrow \mathbb{R}$. In the splitting case, the energy density $f: \mathbb{R}^n \rightarrow \mathbb{R}$ admits an additive decomposition into different parts depending on the various first partial derivatives of the admissible functions u , for example, we can consider the case

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$$f(\nabla u) = \sum_{i=1}^n f_i(\partial_i u) \quad (2)$$

with functions $f_i: \mathbb{R} \rightarrow \mathbb{R}$, $i = 1, \dots, n$, of possibly different growth rates.

This also leads us to the more general category of variational problems with non-standard growth, where in the simplest case one studies an energy functional J defined in equation (1) with convex density f being bounded from above and from below by different powers of $|\nabla u|$.

For an overview of the various aspects of variational problems with non-standard growth including aspects as existence and regularity we refer the interested reader, e.g., to [1–8] and to the recent paper [9] together with the references quoted therein.

Regarding the question of interior regularity of J -minimizers in the non-standard setting, the above references provide rather complete answers, while the problem of boundary regularity for the solution of

$$J[u] \rightarrow \min, \quad u = u_0 \quad \text{on } \partial\Omega, \quad (3)$$

with sufficiently smooth boundary datum u_0 seems to be not so well investigated.

We mention the contribution [10], where the higher integrability of the minimizer u up to the boundary is established for densities f being in some sense close to the splitting-class defined in (2). A precise formulation of the assumptions concerning the density f is given in inequality (2.1) of the paper [10], and we also like to stress that a survey of the known global regularity results is provided in the introduction of this reference.

Even more recently the contribution of Koch [11] addresses the global higher integrability of the gradient of solutions of variational problems with (p, q) -growth, which to our knowledge is the first result to improve global integrability of the gradient allowing full anisotropy with different growth rates with respect to different partial derivatives. The boundary data are supposed to belong to some fractional order spaces (see, e.g., [12] or [13]) and roughly speaking are handled like an additional x -dependence following ideas as outlined in, e.g., [14] for the interior situation. Hence, the admissible energy densities correspond to the examples presented in [14]. On the other hand the case of vectorial functions $u: \mathbb{R}^n \supset \Omega \rightarrow \mathbb{R}^m$, $n \geq 2$, $m \geq 1$, as well as the subquadratic situation are included.

In our note we first discuss the variational problem (3) on a bounded Lipschitz domain $\Omega \subset \mathbb{R}^2$ with energy functional J defined in equation (1) under the following assumptions on the data: for $i = 1, 2$ let $f_i \in C^2(\mathbb{R})$ satisfy

$$a_i(1+t^2)^{\frac{q_i-2}{2}} \leq f_i''(t) \leq A_i(1+t^2)^{\frac{q_i-2}{2}}, \quad t \in \mathbb{R}, \quad (4)$$

with constants $a_i, A_i > 0$ and exponents

$$q_i > 2. \quad (5)$$

We define the energy density $f: \mathbb{R}^2 \rightarrow \mathbb{R}$ according to (recall the “splitting condition” (2))

$$f(Z) := f_1(z_1) + f_2(z_2), \quad Z = (z_1, z_2) \in \mathbb{R}^2. \tag{6}$$

Note that (4) yields the anisotropic ellipticity condition

$$c_1(1 + |Z|^2)^{\frac{p-2}{2}} |Y|^2 \leq D^2f(Z)(Y, Y) \leq c_2(1 + |Z|^2)^{\frac{q-2}{2}} |Y|^2 \tag{7}$$

valid for $Z, Y \in \mathbb{R}^2$ with positive constants c_1, c_2 and for the choice of exponents $p = 2$ and $q := \max\{q_1, q_2\}$. In fact, (7) is a direct consequence of (5) and the formula

$$D^2f(Z)(Y, Y) = \sum_{i=1}^2 f_i''(z_i) |y_i|^2, \quad Z, Y \in \mathbb{R}^2. \tag{8}$$

Finally, we fix a function

$$u_0 \in W^{1,\infty}(\Omega) \tag{9}$$

remarking that actually $u_0 \in W^{1,t}(\Omega)$ with exponent t being sufficiently large (depending on q_1 and q_2) is needed in our calculations. For a definition of the Sobolev spaces $W^{k,r}(\Omega)$ and their local variants we refer to [12].

The function u_0 acts as a prescribed boundary datum in the minimization problem

$$J[u] := \int_{\Omega} f(\nabla u) \, dx \rightarrow \min \quad \text{in } u_0 + W_0^{1,1}(\Omega). \tag{10}$$

Now we state our result in the splitting case by the way establishing global higher integrability without relating q_1 and q_2 and under quite weak assumptions on u_0 which means that we do not have to impose additional hypotheses concerning the regularity of the trace of u_0 in the sense that $u_0|_{\partial\Omega}$ belongs to some fractional Sobolev space. The price we have to pay is a weight function which has to be incorporated.

Theorem 1.1 *Let the assumptions (4)–(6) and (9) hold. Moreover, let $u \in u_0 + W_0^{1,1}(\Omega)$ denote the unique solution of problem (10). Then it holds*

$$|u - u_0|^t \left(|\partial_1 u|^{\frac{3}{2}q_1} + |\partial_2 u|^{\frac{3}{2}q_2} \right) \in L^1(\Omega) \tag{11}$$

for any choice of the exponent t such that

$$t > T(q_1, q_2) \tag{12}$$

with a finite lower bound $T(q_1, q_2) \geq 1$ depending on the growth rates q_1, q_2 .

Remark 1.1 An admissible choice for the quantity $T(q_1, q_2)$ from (12) is

$$T(q_1, q_2) = 6 \max \left\{ 1, \frac{1}{2} \frac{q_{\max} q_{\min} - 1}{q_{\min} q_{\min} - 2}, \frac{q_{\min}}{q_{\min} - 2} - \frac{1}{6} \right\} \frac{q_{\min}}{q_{\min} - 2}. \tag{13}$$

Here and in the following we let

$$q_{\min} := \min\{q_1, q_2\}, \quad q_{\max} := \max\{q_1, q_2\}.$$

In the case $q_i > 5$, $i = 1, 2$, we can replace (13) by the expression

$$T(q_1, q_2) = 6q_{\max} \max \left\{ \frac{1}{2q_{\min} - 4}, \frac{1}{q_{\max} + 2} \right\} \quad (14)$$

which means that in the case of higher initial integrability we can decrease the exponent t in (11) leading to a stronger result. We also note that for the choice $q_i > 5$, $i = 1, 2$, the condition (9) can be replaced by the requirement $u_0 \in W^{1, q_{\min}}(\Omega)$ together with $\partial_i u_0 \in L^{q_i}(\Omega)$, $i = 1, 2$.

Remark 1.2 The existence and the uniqueness of a solution u to problem (10) easily follow from the growth properties of f and its strict convexity combined with our assumption (9) on the boundary datum u_0 . Obviously the minimizer satisfies $\partial_i u \in L^{q_i}(\Omega)$, $i = 1, 2$, and from Theorem 1.1 ii) in [8] we deduce interior Hölder continuity of the first partial derivatives. This in turn implies that u is in the space $W_{\text{loc}}^{2,2}(\Omega)$.

Remark 1.3 Of course we expect that our theorem can be extended to the case $n > 2$, i.e. to densities

$$f(Z) = \sum_{i=1}^n f_i(z_i) \quad Z \in \mathbb{R}^n,$$

with functions f_i satisfying (4) and (5) being replaced by $q_i > n$, $i = 1, \dots, n$. We restrict our considerations in the splitting case to the 2D-situation for two reasons: first, a generalization to the case $n \geq 2$ would be accompanied by a tremendous technical effort without promising a deeper insight. Secondly, the interior regularity results of [8] as the starting point for our arguments would have to be generalized to the full splitting case in n dimensions which goes beyond the aim of this note.

Remark 1.4 Inequality (11) can be seen as weighted global higher integrability result for ∇u . In fact, on account of (5), u is Hölder continuous up to the boundary, which means $|u(x) - u_0(x)| \approx \text{dist}^\alpha(x, \partial\Omega)$ for points x near $\partial\Omega$. Thus, in this vague sense, Theorem 1.1 yields $|\partial_i u|^{t_i} \text{dist}^\beta(\cdot, \partial\Omega) \in L^1(\Omega)$ for a certain range of exponents $\beta > 0$ and $t_i > q_i$.

Another interpretation will be given in Corollary 1.1 on the behaviour of a suitable Hölder coefficient.

In the second part of our paper we drop the splitting condition (6) and look at densities f with anisotropic (p, q) -growth in the sense of (7) and for a moment we still restrict our considerations to the two-dimensional case. Then the arguments of the first part can be carried over provided that p and q are not too far apart.

Theorem 1.2 Assume that $f: \mathbb{R}^2 \rightarrow \mathbb{R}$ is of class $C^2(\mathbb{R}^2)$ satisfying (7) with exponents $2 < p \leq q$ such that

$$q < \min \left\{ p + \frac{1}{2}, 2p - 2 \right\}. \tag{15}$$

Let (9) hold for the boundary datum u_0 and let $u \in u_0 + W_0^{1,1}(\Omega)$ denote the unique solution of problem (10). Then we have $(\Gamma := 1 + |\nabla u|^2)$

$$|u - u_0|^{2\kappa} \Gamma^s \in L^1(\Omega)$$

for any exponents κ, s such that

$$s \in \left(\frac{q}{2} + \frac{3}{4}, p - \frac{1}{4} \right) \tag{16}$$

and

$$\kappa > \max \left\{ \frac{p}{p-2}, \frac{s}{s - \left(\frac{q}{2} + \frac{3}{4}\right)}, \frac{2s(p-1)}{(p-2)^2} \right\}. \tag{17}$$

Remark 1.5 Since f is strictly convex and since we have (9), existence and uniqueness of a solution u to problem (10) are immediate. Note that (15) implies (4.2) in Marcellini’s paper [15], hence $u \in W_{loc}^{1,q}(\Omega)$ on account of Theorem 4.1 in [15]. Quoting Corollary 2.2 from this reference we deduce $u \in C^{1,\alpha}(\Omega)$ and $u \in W_{loc}^{2,2}(\Omega)$.

In our final theorem we incorporate two new features: we modify the idea of the proofs of the previous results by starting with a more subtle inequality being applied to terms on the left-hand sides of our foregoing calculations by the way extending the range of admissible exponents p and q . Moreover, we include the general case of n -dimensions, $n \geq 2$. This can be done without technical efforts comparable to the ones which are needed to generalize the 2D splitting case.

Now the precise formulation of our last result is as follows:

Theorem 1.3 Consider the variational problem (10) on a bounded Lipschitz domain $\Omega \subset \mathbb{R}^n$ now just assuming (p,q) -growth of the C^2 -density $f: \mathbb{R}^n \rightarrow \mathbb{R}$ in the sense of (7) without splitting structure. Let the boundary datum u_0 satisfy (9). Suppose that

$$2 \leq n < p \leq q \quad \text{and} \quad q < p + \frac{2(p-n)}{n}. \tag{18}$$

By assumption (18) we may choose a real number $\kappa > 1$ sufficiently large such that

$$q < p + \frac{2(p-n)}{n} - \frac{2p-1}{n\kappa}. \tag{19}$$

Moreover, let

$$\bar{s} < p \frac{\kappa(n+2) - 2}{2n\kappa} - \frac{\kappa - 1}{2n\kappa}. \quad (20)$$

Increasing κ , if necessary, we suppose in addition that

$$\bar{s} < (\kappa - 1) \frac{p - n}{n} + \frac{p}{2} \quad (21)$$

and that we have the inequalities (103) below. Then it holds

$$\int_{\Omega} \Gamma^{\bar{s}} |u - u_0|^{2(\kappa-1)} dx \leq c, \quad \Gamma := 1 + |\nabla u|^2,$$

for a finite constant c depending on n, p, q, κ and \bar{s} .

Remark 1.6 Passing to the limit $\kappa \rightarrow \infty$, (20) becomes

$$\bar{s} \leq \frac{p}{2} + \frac{2p - 1}{2n}.$$

Remark 1.7 We note that Remark 1.5 extends to the setting studied in Theorem 1.3, since now condition (4.2) from [15] is a consequence of the more restrictive inequality (18).

Our paper is organized as follows: in Section 2 we present the proof of Theorem 1.1 for a given Lipschitz boundary datum u_0 (recall (9)) and just supposing (5). In the case $q_1, q_2 > 5$ the lower bound $T(q_1, q_2)$ occurring in (12) looks much nicer and (9) can be considerably weakened. We sketch some arguments in Section 3. The two-dimensional non-splitting case, i.e. the setting described in Theorem 1.2, is discussed in Section 4 and in Section 5 we discuss the higher dimensional situation of Theorem 1.3.

During the proofs of our results we make essential use of Caccioppoli-type inequalities. As a main new feature these inequalities involve “small weights”, for instance weight functions like $(1 + |\partial_i u|)^{\alpha}$, $i = 1, 2$, with a certain range of negative exponents α . A rather general analysis of these tools is presented in Section 6.

Before we get into all these technical details we finish the introduction by adding some extra comments concerning the interpretation of our results. For simplicity we restrict ourselves to the setting described in Theorem 1.1 and denote by u the unique solution of problem (10) under the hypothesis of Theorem 1.1.

We start with a local observation: by adding a distance function if necessary, we find an appropriate representative of the boundary data respecting a lower bound for $|u - u_0|$ in a neighbourhood of a fixed $x_0 \in \Omega$.

Proposition 1.1 Fix $x_0 \in \Omega$. Then there exists a neighbourhood $U = U(x_0) \Subset \Omega$ of x_0 such that we may assume that for all $x \in U$

$$|\hat{u}|(x) = |u - u_0|(x) \geq \underline{d} := \frac{1}{2}d(x_0), \quad \hat{u} := u - u_0,$$

where

$$d(x) = \text{dist}(x, \partial\Omega),$$

denotes the distance function to the boundary $\partial\Omega$.

Proof of Proposition 1.1 The distance function d is discussed, e.g., in [16], Appendix 14.6. In particular, if Ω is a domain in \mathbb{R}^n with non-empty boundary, then by (14.91) of [16] d is uniformly Lipschitz continuous.

Let us suppose w.l.o.g. that $\hat{u}(x_0) \geq 0$. If $\hat{u}(x_0) < d(x_0)$, then we choose $\tilde{u}_0 := u_0 - d$ as admissible Lipschitz boundary datum with $u_0 = \tilde{u}_0$ on $\partial\Omega$. Hence both u_0 and \tilde{u}_0 produce the same minimizer u of the Dirichlet problem (10) and by definition we have

$$(u - \tilde{u}_0)(x_0) = (u - u_0 + d)(x_0) \geq d(x_0).$$

The considerations of our note completely remain unchanged if we replace u_0 by \tilde{u}_0 . Hence we have w.l.o.g. that $\hat{u}(x_0) \geq d(x_0)$. Proposition 1.1 then follows from the continuity of u and u_0 . □

Let us fix $x_0 \in \Omega$, and define U, \hat{u} as in Proposition 1.1. Moreover we let $h: \Omega \rightarrow \mathbb{R}, h := |\hat{u}|^{\frac{4}{3} \frac{\kappa}{q_{\min}} + 1}$, hence

$$|\nabla h| \leq \left[\frac{4}{3} \frac{\kappa}{q_{\min}} + 1 \right] |\hat{u}|^{\frac{4}{3} \frac{\kappa}{q_{\min}}} |\nabla \hat{u}|.$$

Theorem 1.1 gives

$$\int_{\Omega} |\nabla h|^{\frac{3}{2} q_{\min}} dx \leq c, \quad \text{i.e.} \quad h \in W^{1, \frac{3}{2} q_{\min}}(\Omega).$$

Letting $\zeta = \frac{4}{3} \frac{\kappa}{q_{\min}} + 1$, the imbedding into Hölder spaces yields $|\hat{u}|^{\zeta} \in C^{0, \mu}(\bar{\Omega})$ with $\mu = 1 - \frac{4}{3q_{\min}}$.

Now we consider the Taylor expansion of the function $g: \mathbb{R}^+ \rightarrow \mathbb{R}, w \mapsto w^{\zeta}$ around a fixed $\tilde{w} > 0$: for w sufficiently close to \tilde{w} we have

$$w^{\zeta} = \tilde{w}^{\zeta} + \zeta \tilde{w}^{\zeta-1} (w - \tilde{w}) + O(|w - \tilde{w}|^2).$$

Suppose w.l.o.g. that $\hat{u} > 0$ in U . Inserting $\tilde{w} = \hat{u}(x_0)$ and $w = \hat{u}(x)$, x sufficiently close to x_0 , in the Taylor expansion we obtain using Proposition 1.1

$$\begin{aligned} \frac{|\hat{u}^{\zeta}(x) - \hat{u}^{\zeta}(x_0)|}{\zeta \underline{d}^{\zeta-1}} &\geq \frac{|\hat{u}^{\zeta}(x) - \hat{u}^{\zeta}(x_0)|}{\zeta \hat{u}^{\zeta-1}(x_0)} \\ &\geq u(x) - u(x_0) + u_0(x_0) - u_0(x) + O(|\hat{u}(x) - \hat{u}(x_0)|^2). \end{aligned} \tag{22}$$

Since $|\hat{u}|^{\zeta}$ is of class $C^{0, \mu}(\bar{\Omega})$ and since according to (9) u_0 is Lipschitz, by (22) we find a constant $c > 0$ not depending on x_0 such that

$$u(x) - u(x_0) \leq c \frac{1}{\zeta d^{\zeta-1}} |x - x_0|^\mu + O(|x - x_0|^{2\mu}). \tag{23}$$

Changing the roles of x and x_0 and following the arguments leading to (23), we find a constant $c > 0$ not depending on x_0 such that for all x sufficiently close to x_0 we have

$$|u(x) - u(x_0)| \leq c \frac{1}{\zeta d^{\zeta-1}} |x - x_0|^\mu. \tag{24}$$

Summarizing the results we obtain the following corollary to Theorem 1.1.

Corollary 1.1 *Under the assumptions of Theorem 1.1 we let*

$$\zeta := \frac{4}{3} \frac{\kappa}{q_{\min}} + 1, \quad \mu := 1 - \frac{4}{3q_{\min}}.$$

If $x_0 \in \Omega$ is fixed, then for every sufficiently small neighbourhood $U = U(x_0) \Subset \Omega$ of x_0 the Hölder coefficient of u (see [16] for notation) satisfies

$$[u]_{\mu; x_0} := \sup_U \frac{|u(x) - u(x_0)|}{|x - x_0|^\mu} \leq c \frac{1}{\zeta d^{\zeta-1}(x_0)},$$

where the constant c is not depending on x_0 .

2 Proof of Theorem 1.1

We proceed in several steps assuming from now on the validity of the hypotheses (4), (5), (6) and (9). Let u denote the unique solution of problem (10) and recall the (interior) regularity properties of u stated in Remark 1.2. We start with the following elementary observation:

Proposition 2.1 *Consider numbers $\kappa, s_1, s_2 \geq 1$. Then there is a finite constant c such that for $i = 1, 2$ and any $\eta \in C_0^1(\Omega)$*

$$\begin{aligned} \int_{\Omega} \Gamma_i^{s_i} |\eta|^{2\kappa} \, dx &\leq c \left\{ \left[\int_{\Omega} |\nabla \Gamma_i^{\frac{s_i}{2}}| |\eta|^\kappa \, dx \right]^2 + \left[\int_{\Omega} \Gamma_i^{\frac{s_i}{2}} |\nabla \eta| |\eta|^{\kappa-1} \, dx \right]^2 \right\} \\ &=: c \{ (I)_i + (II)_i \}, \quad \Gamma_i := 1 + |\partial_i u|^2. \end{aligned} \tag{25}$$

Proof of Proposition 2.1 Let $v := \Gamma_i^{s_i/2} |\eta|^\kappa$ and observe that due to $u \in W_{\text{loc}}^{2,2} \cap W_{\text{loc}}^{1,\infty}(\Omega)$ the function v is in the space $W_0^{1,1}(\Omega)$, hence the Sobolev-Poincaré inequality implies

$$\int_{\Omega} v^2 \, dx \leq c \left[\int_{\Omega} |\nabla v| \, dx \right]^2,$$

and inequality (25) directly follows by observing that $|\nabla|\eta|| \leq |\nabla\eta|$. □

Next we replace η by a sequence $\eta_m, m \in \mathbb{N}$, being defined through

$$\eta_m := \varphi_m(u - u_0) \tag{26}$$

with $\varphi_m \in C_0^1(\Omega), 0 \leq \varphi_m \leq 1$,

$$\begin{aligned} \varphi_m = 1 \text{ on } \left\{ x \in \Omega : \text{dist}(x, \partial\Omega) \geq \frac{1}{m} \right\}, \quad \varphi_m = 0 \text{ on } \left\{ x \in \Omega : \text{dist}(x, \partial\Omega) \leq \frac{1}{2m} \right\}, \\ |\nabla\varphi_m| \leq cm, \quad c = c(\partial\Omega). \end{aligned} \tag{27}$$

Note that due to (5) and (9) inequality (25) extends to η_m , moreover, according to [16], Theorem 7.17, we have

$$|u(x) - u_0(x)| \leq c\rho^{1-\frac{2}{q_{\min}}}, \quad q_{\min} := \min\{q_1, q_2\}, \tag{28}$$

for points $x \in \Omega$ at distance $\leq \rho$ to $\partial\Omega$, provided ρ is sufficiently small.

The quantity $(II)_i$ defined in (25) with respect to the choice $\eta = \eta_m$ behaves as follows:

Proposition 2.2 *Let $\delta := q_{\min} - 2$ and consider numbers $s_i, \kappa \geq 1, i = 1, 2$, such that*

$$s_i > \frac{\delta}{2}, \quad \kappa \geq \hat{\kappa} := \frac{q_{\min} - 1}{q_{\min} - 2}. \tag{29}$$

Then it holds for $(II)_i = (II_m)_i, i = 1, 2$, defined in (25) with respect to the function η_m from (26)

$$(II)_i \leq c \int_{\Omega} \Gamma_i^{s_i - \frac{\delta}{2}} |\eta_m|^{2\kappa - 2\hat{\kappa}} \, dx =: c(III)_i, \quad i = 1, 2, \tag{30}$$

c denoting a finite constant being uniform in m.

Proof of Proposition 2.2 By Hölder’s inequality we have

$$\begin{aligned} (II)_i &= \left[\int_{\Omega} \Gamma_i^{\frac{s_i}{2} - \frac{\delta}{4}} |\eta_m|^{\kappa - \hat{\kappa}} |\eta_m|^{\hat{\kappa} - 1} |\nabla\eta_m| \Gamma_i^{\frac{\delta}{4}} \, dx \right]^2 \\ &\leq \left[\int_{\Omega} \Gamma_i^{s_i - \frac{\delta}{2}} |\eta_m|^{2\kappa - 2\hat{\kappa}} \, dx \right] \left[\int_{\Omega} \Gamma_i^{\frac{\delta}{2}} |\eta_m|^{2\hat{\kappa} - 2} |\nabla\eta_m|^2 \, dx \right] \end{aligned}$$

and for the second integral on the r.h.s. we observe (recall (26) and (27), (28))

$$\begin{aligned}
 |\eta_m|^{2\hat{\kappa}-2} |\nabla \eta_m|^2 &\leq c \left[|\nabla(u - u_0)|^2 + |\nabla \varphi_m|^2 |u - u_0|^{2\hat{\kappa}} \right] \\
 &\leq c \left[|\nabla u|^2 + |\nabla u_0|^2 + m^2 \left(\frac{1}{m} \right)^{2\hat{\kappa}(1-2/q_{\min})} \right].
 \end{aligned}
 \tag{31}$$

Since Ω is Lipschitz we observe that $|\text{spt} \nabla \varphi_m| \leq c/m$ and obtain

$$\int_{\text{spt} \nabla \varphi_m} \Gamma_i^{\frac{\delta}{2}} m^{2-2\hat{\kappa}(1-\frac{2}{q_{\min}})} \, dx \leq \int_{\Omega} \Gamma_i^{\frac{q_{\min}}{2}} \, dx + m^{[2-2\hat{\kappa}(1-\frac{2}{q_{\min}})]\frac{q_{\min}}{2}} m^{-1}
 \tag{32}$$

The choice (29) of $\hat{\kappa}$ shows that the right-hand side of (32) is bounded. Letting $\Gamma := 1 + |\nabla u|^2$ we arrive at

$$\begin{aligned}
 (II)_i &\leq c (III)_i \left[\int_{\Omega} \left[\Gamma^{\frac{\delta}{2}} \Gamma + \Gamma^{\frac{\delta}{2}} |\nabla u_0|^2 \right] \, dx + 1 \right] \\
 &\leq c (III)_i \left[\int_{\Omega} \Gamma^{\frac{q_{\min}}{2}} \, dx + \int_{\Omega} |\nabla u_0|^{q_{\min}} \, dx + 1 \right],
 \end{aligned}$$

where in the final step we have used the definition of δ . Now (30) clearly is a consequence of (9) (in fact, $u_0 \in W^{1,q_{\min}}(\Omega)$ would be sufficient). □

In the following we will discuss the quantity $(III)_i$ defined in (30) under the assumptions of Proposition 2.2.

We have by Young’s inequality for any $\varepsilon > 0$

$$\begin{aligned}
 (III)_i &= \int_{\Omega} \Gamma_i^{s_i \frac{s_i - \delta/2}{s_i}} |\eta_m|^{2\kappa - 2\hat{\kappa}} \, dx = \int_{\Omega} \left[\Gamma_i^{s_i} |\eta_m|^{2\kappa} \right]^{\frac{s_i - \delta/2}{s_i}} |\eta_m|^{2\kappa - 2\hat{\kappa} - 2\kappa \frac{s_i - \delta/2}{s_i}} \, dx \\
 &\leq \varepsilon \int_{\Omega} \Gamma_i^{s_i} |\eta_m|^{2\kappa} \, dx + c(\varepsilon) \int_{\Omega} |\eta_m|^{\vartheta} \, dx
 \end{aligned}$$

with exponent

$$\vartheta := \left[\frac{s_i}{s_i - \delta/2} \right]^* \left(2\kappa - 2\hat{\kappa} - 2\kappa \frac{s_i - \delta/2}{s_i} \right),$$

[...]^{*} denoting the exponent conjugate to $\frac{s_i}{s_i - \delta/2}$. Note that $\vartheta \geq 0$ provided we additionally assume that $\kappa \geq 1$ satisfies the inequality

$$\kappa \geq \frac{2s_i}{\delta} \hat{\kappa} = 2s_i \frac{q_{\min} - 1}{(q_{\min} - 2)^2}.
 \tag{33}$$

Inserting the above estimate for $(III)_i$ into (30), we find

$$(II)_i \leq \varepsilon \int_{\Omega} \Gamma_i^{s_i} |\eta_m|^{2\kappa} \, dx + c(\varepsilon),$$

and if we choose ε sufficiently small, we see that (25) yields the following result:

Proposition 2.3 *Let $s_i, \kappa \geq 1$ satisfy (29) and (33). Then there exists a constant c being independent of m such that ($i = 1, 2$)*

$$\int_{\Omega} \Gamma_i^{s_i} |\eta_m|^{2\kappa} dx \leq c \left[(I)_i + 1 \right], \quad (I)_i := \left[\int_{\Omega} |\nabla \Gamma_i^{\frac{s_i}{2}}| |\eta_m|^{\kappa} dx \right]^2. \tag{34}$$

In order to prove Theorem 1.1 it remains to discuss the quantity $(I) = (I)_i$ for $i = 1, 2$, i.e. now second derivatives of u have to be handled in an appropriate way. It holds

$$\begin{aligned} (I)_1 &\leq c \left[\int_{\Omega} \Gamma_1^{\frac{s_1}{2}-1} |\nabla \Gamma_1| |\eta_m|^{\kappa} dx \right]^2 \\ &\leq c \left[\int_{\Omega} \Gamma_1^{\frac{s_1-1}{2}} |\partial_1 \partial_1 u| |\eta_m|^{\kappa} dx \right]^2 + c \left[\int_{\Omega} \Gamma_1^{\frac{s_1-1}{2}} |\partial_1 \partial_2 u| |\eta_m|^{\kappa} dx \right]^2 =: T_1 + \tilde{T}_1, \end{aligned}$$

and in the same manner

$$(I)_2 \leq c \left[\int_{\Omega} \Gamma_2^{\frac{s_2-1}{2}} |\partial_2 \partial_2 u| |\eta_m|^{\kappa} dx \right]^2 + c \left[\int_{\Omega} \Gamma_2^{\frac{s_2-1}{2}} |\partial_1 \partial_2 u| |\eta_m|^{\kappa} dx \right]^2 =: T_2 + \tilde{T}_2.$$

We have

$$\begin{aligned} T_i &= c \left[\int_{\Omega} \Gamma_i^{\frac{q_i-2}{4}} |\partial_i \partial_i u| |\eta_m|^{\kappa} \Gamma_i^{\frac{\alpha_i}{2}} \Gamma_i^{\frac{s_i-1}{2} - \frac{q_i-2}{4} - \frac{\alpha_i}{2}} dx \right]^2 \\ &\leq c \left[\int_{\Omega} \Gamma_i^{\frac{q_i-2}{2}} |\partial_i \partial_i u|^2 |\eta_m|^{2\kappa} \Gamma_i^{\alpha_i} dx \right] \left[\int_{\Omega} \Gamma_i^{s_i-1 - \frac{q_i-2}{2} - \alpha_i} dx \right], \end{aligned}$$

where the last estimate follows from Hölder’s inequality and $\alpha_i, i = 1, 2$, denote real numbers such that for the moment

$$s_i \leq q_i + \alpha_i, \quad i = 1, 2. \tag{35}$$

Note that the condition (35) guarantees the validity of

$$\int_{\Omega} \Gamma_i^{s_i-1 - \frac{q_i-2}{2} - \alpha_i} dx \leq c, \quad i = 1, 2, \tag{36}$$

for a finite constant c . Recalling (4) we see that (36) yields the bound

$$T_i \leq c \int_{\Omega} f_i''(\partial_i u) |\partial_i \partial_i u|^2 |\eta_m|^{2\kappa} \Gamma_i^{\alpha_i} dx \tag{37}$$

again for $i = 1, 2$.

Let us look at the quantities \tilde{T}_i : we have by Hölder’s inequality

$$\begin{aligned}
 \tilde{T}_1 &= c \left[\int_{\Omega} \Gamma_1^{\frac{q_1-2}{4}} |\partial_1 \partial_2 u| |\eta_m|^\kappa \Gamma_1^{\frac{s_1-1}{2}} \Gamma_2^{\frac{\alpha}{2}} \Gamma_1^{-\frac{q_1-2}{4}} \Gamma_2^{-\frac{\alpha_2}{2}} dx \right]^2 \\
 &\leq c \left[\int_{\Omega} \Gamma_1^{\frac{q_1-2}{2}} |\partial_1 \partial_2 u|^2 |\eta_m|^{2\kappa} \Gamma_2^{\alpha_2} dx \right] \left[\int_{\Omega} \Gamma_1^{s_1-1-\frac{q_1-2}{2}} \Gamma_2^{-\alpha_2} dx \right] \\
 &\leq c \left[\int_{\Omega} f_1''(\partial_1 u) |\partial_1 \partial_2 u|^2 |\eta_m|^{2\kappa} \Gamma_2^{\alpha_2} dx \right] \left[\int_{\Omega} \Gamma_1^{s_1-1-\frac{q_1-2}{2}} \Gamma_2^{-\alpha_2} dx \right] \\
 &=: cS_1' \cdot S_1'' .
 \end{aligned}
 \tag{38}$$

In order to benefit from the inequality (38), we replace (35) by the stronger bound

$$s_i \leq \frac{3}{4}q_i, \quad i = 1, 2,
 \tag{39}$$

together with the requirement

$$\alpha_i \in (-1/2, 0), \quad i = 1, 2.
 \tag{40}$$

Here we note that (39) together with (40) yields (35). We then obtain on account of Young’s inequality

$$S_1'' \leq c \left[\int_{\Omega} \Gamma_1^{\frac{q_1}{2}} dx + \int_{\Omega} \Gamma_2^\beta dx \right]$$

with exponent

$$\beta := -\alpha_2 \left[\frac{q_1/2}{s_1 - q_1/2} \right]^* = -\alpha_2 \frac{q_1/2}{q_1 - s_1} \leq -2\alpha_2 \leq \frac{q_2}{2},$$

where we used (39), (40) and (40), (5), respectively, for the last inequalities. Thus (38) reduces to

$$\tilde{T}_1 \leq c \int_{\Omega} f_1''(\partial_1 u) |\partial_1 \partial_2 u|^2 |\eta_m|^{2\kappa} \Gamma_2^{\alpha_2} dx .
 \tag{41}$$

In the same spirit it follows

$$\tilde{T}_2 \leq c \int_{\Omega} f_2''(\partial_2 u) |\partial_1 \partial_2 u|^2 |\eta_m|^{2\kappa} \Gamma_1^{\alpha_1} dx .
 \tag{42}$$

Let us return to (34): we have by (37), (41) and (42) and by (8)

$$\begin{aligned}
 & \int_{\Omega} \left[\Gamma_1^{s_1} + \Gamma_2^{s_2} \right] |\eta_m|^{2\kappa} dx \leq c [1 + T_1 + \tilde{T}_2 + \tilde{T}_1 + T_2] \\
 & = c \left[1 + \int_{\Omega} \left(f_1''(\partial_1 u) |\partial_1 \partial_1 u|^2 + f_2''(\partial_2 u) |\partial_1 \partial_2 u|^2 \right) |\eta_m|^{2\kappa} \Gamma_1^{\alpha_1} dx \right. \\
 & \quad \left. + \int_{\Omega} \left(f_1''(\partial_1 u) |\partial_1 \partial_2 u|^2 + f_2''(\partial_2 u) |\partial_2 \partial_2 u|^2 \right) |\eta_m|^{2\kappa} \Gamma_2^{\alpha_2} dx \right] \\
 & = c \left[1 + \int_{\Omega} D^2 f(\nabla u) (\partial_1 \nabla u, \partial_1 \nabla u) |\eta_m|^{2\kappa} \Gamma_1^{\alpha_1} dx \right. \\
 & \quad \left. + \int_{\Omega} D^2 f(\nabla u) (\partial_2 \nabla u, \partial_2 \nabla u) |\eta_m|^{2\kappa} \Gamma_2^{\alpha_2} dx \right].
 \end{aligned}$$

The remaining integrals are handled with the help of Proposition 6.1 (compare also inequality (4.6) from [17]) replacing l by κ and η by $|\eta_m|$, respectively. We note that the proof of Proposition 6.1 obviously remains valid with these replacements. We emphasize that (40) is an essential assumption to apply Proposition 6.1. We get

$$\begin{aligned}
 \int_{\Omega} \left[\Gamma_1^{s_1} + \Gamma_2^{s_2} \right] |\eta_m|^{2\kappa} dx & \leq c \left[1 + \int_{\Omega} D^2 f(\nabla u) (\nabla |\eta_m|, \nabla |\eta_m|) \Gamma_1^{1+\alpha_1} |\eta_m|^{2\kappa-2} dx \right. \\
 & \quad \left. + \int_{\Omega} D^2 f(\nabla u) (\nabla |\eta_m|, \nabla |\eta_m|) \Gamma_2^{1+\alpha_2} |\eta_m|^{2\kappa-2} dx \right].
 \end{aligned} \tag{43}$$

Finally we let $s_i = \frac{3}{4}q_i, i = 1, 2$, which is the optimal choice with respect to (39). We note that with this choice (33) follows from

$$\kappa \geq \frac{3}{2} \frac{q_{\max}(q_{\min} - 1)}{(q_{\min} - 2)^2}, \tag{44}$$

and (44) is valid if κ is chosen according to assumption (12) from Theorem 1.1 with $T(q_1, q_2)$ defined in (13).

From (4) and (8) we deduce

$$\begin{aligned}
 \text{r.h.s. of (2.19)} & \leq c \left[1 + \sum_{i=1}^2 \int_{\Omega} \Gamma_i^{\frac{q_i}{2} + \alpha_i} |\partial_i \eta_m|^2 |\eta_m|^{2\kappa-2} dx \right. \\
 & \quad + \int_{\Omega} \Gamma_2^{\frac{q_2-2}{2}} |\partial_2 \eta_m|^2 |\eta_m|^{2\kappa-2} \Gamma_1^{1+\alpha_1} dx \\
 & \quad \left. + \int_{\Omega} \Gamma_1^{\frac{q_1-2}{2}} |\partial_1 \eta_m|^2 |\eta_m|^{2\kappa-2} \Gamma_2^{1+\alpha_2} dx \right].
 \end{aligned} \tag{45}$$

We recall the definition (26) of η_m and the gradient bound for φ_m stated in (27). This yields for integrand of the first integral on the right-hand side of (45):

$$\Gamma_i^{\frac{q_i}{2} + \alpha_i} |\partial_i \eta_m|^2 |\eta_m|^{2\kappa-2} \leq c \left[\Gamma_i^{\frac{q_i}{2} + \alpha_i} |\partial_i (u - u_0)|^2 |\eta_m|^{2\kappa-2} + \Gamma_i^{\frac{q_i}{2} + \alpha_i} |\partial_i \varphi_m|^2 |u - u_0|^{2\kappa} \right].$$

We quote inequality (28) and recall that $\nabla\varphi_m$ has support in the set $\{x \in \Omega : \text{dist}(x, \partial\Omega) \leq 1/m\}$ satisfying $|\text{spt}\nabla\varphi_m| \leq c/m$, hence

$$\begin{aligned} \int_{\Omega} \Gamma_i^{\frac{q_i}{2} + \alpha_i} |\partial_i \varphi_m|^2 |u - u_0|^{2\kappa} \, dx &\leq \int_{\text{spt}\nabla\varphi_m} \Gamma_i^{\frac{q_i}{2} + \alpha_i} m^{2-2\kappa \frac{q_{\min}-2}{q_{\min}}} \, dx \\ &\leq c \int_{\Omega} \Gamma_i^{\frac{q_i}{2}} \, dx + c \int_{\text{spt}\nabla\varphi_m} m^{[2-2\kappa \frac{q_{\min}-2}{q_{\min}}] \gamma_i^*} \, dx, \end{aligned}$$

where (recall (40))

$$\gamma_i = \frac{q_i}{q_i + 2\alpha_i}, \quad \gamma_i^* = \left[\frac{q_i}{q_i + 2\alpha_i} \right]^* = -\frac{q_i}{2\alpha_i}.$$

Thus the second integral on the right-hand side is bounded if, with α_i sufficiently close to $-1/2$, we have

$$\kappa > \frac{2q_{\max} - 1}{2q_{\max}} \frac{q_{\min}}{q_{\min} - 2}. \tag{46}$$

Assuming (46) we arrive at

$$\int_{\Omega} \Gamma_i^{\frac{q_i}{2} + \alpha_i} |\partial_i \eta_m|^2 |\eta_m|^{2\kappa-2} \, dx \leq c \left[1 + \int_{\Omega} \Gamma_i^{\frac{q_i}{2} + \alpha_i} |\partial_i u - \partial_i u_0|^2 |\eta_m|^{2\kappa-2} \, dx \right] \tag{47}$$

for any α_i sufficiently close to $-1/2$. Assuming this we next let

$$\beta_i = \frac{3q_i}{2q_i + 4(1 + \alpha_i)} \quad \text{with conjugate exponent} \quad \beta_i^* = \frac{3q_i}{q_i - 4(1 + \alpha_i)}$$

and apply Young’s inequality in an obvious way to get

$$\int_{\Omega} \Gamma_i^{\frac{q_i}{2} + \alpha_i + 1} |\eta_m|^{2\kappa-2} \, dx \leq \varepsilon \int_{\Omega} \Gamma_i^{s_i} |\eta_m|^{2\kappa} \, dx + c(\varepsilon) \int_{\Omega} |\eta_m|^{2\kappa-2\beta_i^*} \, dx,$$

where the first term can be absorbed in the left-hand side of (43), while the second one bounded under the assumption

$$\kappa > \frac{3q_{\min}}{q_{\min} - 2}. \tag{48}$$

Here we used the fact that for $q > 2$ the function $q/(q - 2)$ is a decreasing function. Altogether we have shown that for exponents α_i close to $-1/2$ the first integral on the right-hand side of (45) splits into two parts, where the first one can be absorbed in the left-hand side of (45) and the second one stays bounded. During our calculations we evidently used (9), however (9) can be replaced by weaker integrability assumptions concerning $\partial_i u_0$. We leave the details to the reader.

Let us finally consider the “mixed terms” on the right-hand side of (45). We first observe the inequality

$$\begin{aligned}
 & \int_{\Omega} \Gamma_2^{\frac{q_2-2}{2}} |\partial_2 \eta_m|^2 |\eta_m|^{2\kappa-2} \Gamma_1^{1+\alpha_1} \, dx \\
 & \leq c \int_{\Omega} \Gamma_2^{\frac{q_2-2}{2}} (\partial_2(u - u_0))^2 |\eta_m|^{2\kappa-2} \Gamma_1^{1+\alpha_1} \, dx \\
 & \quad + \int_{\Omega} \Gamma_2^{\frac{q_2-2}{2}} |\partial_2 \varphi_m|^2 |u - u_0|^2 |\eta_m|^{2\kappa-2} \Gamma_1^{1+\alpha_1} \, dx.
 \end{aligned} \tag{49}$$

Considering the limit case $\alpha_1 = -1/2$, the first integral on the right-hand side of (49) basically is of the form

$$\int_{\Omega} \Gamma_2^{\frac{q_2}{2}} \Gamma_1^{\frac{1}{2}} |\eta_m|^{2\kappa-2} \, dx$$

and this integral directly results from an application of Caccioppoli’s inequality. If we like to show an integrability result for $\Gamma_2^{t_2}$ with some power $t_2 > q_2/2$, then the idea is to apply Young’s inequality choosing

$$\beta = \frac{2t_2}{q_2} \quad \text{with conjugate exponent} \quad \beta^* = \frac{2t_2}{2t_2 - q_2}$$

leading to the quantities $\Gamma_2^{t_2}$ and $\Gamma_1^{\frac{t_2}{2t_2 - q_2}}$. If $t_1 = t_1(q_1) > q_1/2$ denotes the desired integrability exponent for Γ_1 , then this requires the bound

$$\frac{t_2}{2t_2 - q_2} \leq t_1(q_1) \quad \text{for all } q_1 > 2$$

and of course we need the same condition changing the roles of t_1 and t_2 . With the symmetric Ansatz $t_i = \theta q_i$ for some $\theta > 1/2$ we are immediately led to $t_i = \frac{3}{4} q_i$, $i = 1, 2$, which again motivates our choice of s_i .

More precisely: discussing the first integral on the right-hand side of (49) we choose $\beta_1 = 3/2$ with conjugate exponent $\beta_1^* = 3$ and obtain

$$\begin{aligned}
 \int_{\Omega} \Gamma_2^{\frac{q_2}{2}} |\eta_m|^{2\kappa-2} \Gamma_1^{1+\alpha_1} \, dx &= \int_{\Omega} \Gamma_2^{\frac{q_2}{2}} \Gamma_1^{1+\alpha_1} |\eta_m|^{\frac{2\kappa}{\beta_1}} |\eta_m|^{\frac{2\kappa}{\beta_1^*} - 2} \, dx \\
 &\leq \varepsilon \int_{\Omega} \Gamma_2^{s_2} |\eta_m|^{2\kappa} \, dx + c(\varepsilon) \int_{\Omega} \Gamma_1^{3(1+\alpha_1)} |\eta_m|^{2\kappa-6} \, dx.
 \end{aligned}$$

Here the first integral is absorbed in the left-hand side of (43) and since α_1 is chosen sufficiently close to $-1/2$, the second integral is bounded provided that we suppose in addition

$$q_1 > 3 \quad \text{and} \quad \kappa \geq 3. \tag{50}$$

If $q_1 < 3$, using

$$\beta_2 = \frac{q_1}{4(1 + \alpha_1)}, \quad \beta_2^* = \frac{q_1}{q_1 - 4(1 + \alpha_1)},$$

we are led to ($\tilde{\varepsilon} \ll \varepsilon$)

$$c(\varepsilon) \int_{\Omega} \Gamma_1^{3(1+\alpha_1)} \eta_m^{2\kappa-6} dx \leq \tilde{\varepsilon} \int_{\Omega} \Gamma_1^{s_1} \eta^{2\kappa} dx + c(\tilde{\varepsilon}) \int_{\Omega} \eta^{2\kappa-6\beta_2^*} dx,$$

where the first integral is absorbed in the right-hand side of (43) and if α_1 sufficiently close to $-1/2$ we now suppose

$$\kappa > \frac{3q_1}{q_1 - 2}. \quad (51)$$

Note that the condition (51) is a consequence of (48).

It remains to discuss the second integral on the right-hand side of (49). Using Young's inequality with

$$\beta_3 = \frac{3}{2} \frac{q_2}{q_2 - 2}, \quad \beta_3^* = \frac{3q_2}{q_2 + 4},$$

we obtain

$$\begin{aligned} & \int_{\Omega} \Gamma_2^{\frac{q_2-2}{2}} |\partial_2 \varphi_m|^2 |u - u_0|^2 |\eta_m|^{2\kappa-2} \Gamma_1^{1+\alpha_1} dx \\ & \leq \varepsilon \int_{\Omega} \Gamma_2^{s_2} |\eta_m|^{2\kappa} dx \\ & \quad + c(\varepsilon) \int_{\Omega} \Gamma_1^{(1+\alpha_1)\frac{3q_2}{q_2+4}} |\partial_2 \varphi_m|^{2\beta_3^*} |u - u_0|^{2\beta_3^*} |\eta_m|^{2\kappa-2\beta_3^*} dx, \end{aligned} \quad (52)$$

where again the first integral is absorbed in the left-hand side of (43). Considering the second integral we choose

$$\beta_4 = \frac{q_1(q_2 + 4)}{4q_2(1 + \alpha_1)} \quad \text{with conjugate exponent} \quad \beta_4^* = \frac{q_1(q_2 + 4)}{q_2[q_1 - 4(1 + \alpha_1)] + 4q_1}.$$

This gives ($\tilde{\varepsilon} \ll \varepsilon$)

$$\begin{aligned} & c(\varepsilon) \int_{\Omega} \Gamma_1^{(1+\alpha_1)\frac{3q_2}{q_2+4}} |\partial_2 \varphi_m|^{2\beta_3^*} |u - u_0|^{2\beta_3^*} |\eta_m|^{2\kappa-2\beta_3^*} dx \\ & \leq \tilde{\varepsilon} \int_{\Omega} \Gamma_1^{s_1} |\eta_m|^{2\kappa} dx \\ & \quad + c(\tilde{\varepsilon}) \int_{\text{spt} \nabla \varphi_m} |\partial_2 \varphi_m|^{2\beta_3^* \beta_4^*} |u - u_0|^{2\beta_3^* \beta_4^*} |\eta_m|^{2\kappa-2\beta_3^* \beta_4^*} dx. \end{aligned} \quad (53)$$

As usual the first integral on the right-hand side of (53) is absorbed in the left-hand side of (43) and we calculate

$$\beta_3^* \beta_4^* = \frac{3q_1 q_2}{q_2[q_1 - 4(1 + \alpha_1)] + 4q_1}. \quad (54)$$

We obtain recalling (27) and (28)

$$\begin{aligned}
 & \int_{\text{spt} \nabla \varphi_m} |\partial_2 \varphi_m|^{2\beta_3^* \beta_4^*} |u - u_0|^{2\beta_3^* \beta_4^*} |\eta_m|^{2\kappa - 2\beta_3^* \beta_4^*} dx \\
 & \leq \int_{\text{spt} \nabla \varphi_m} |\partial_2 \varphi_m|^{2\beta_3^* \beta_4^*} |u - u_0|^{2\beta_3^* \beta_4^*} |u - u_0|^{2\kappa - 2\beta_3^* \beta_4^*} dx \\
 & \leq cm^{-1} m^{\frac{6q_1 q_2}{q_2(q_1 - 4(1 + \alpha_1)) + 4q_1}} m^{-\frac{q_{\min} - 2}{q_{\min}} 2\kappa}.
 \end{aligned} \tag{55}$$

For α_1 sufficiently close to $-1/2$ this leads to the requirement

$$\kappa > \frac{q_{\min}}{q_{\min} - 2} \frac{1}{2} \left[\frac{6q_1 q_2}{q_2(q_1 - 2) + 4q_1} - 1 \right]. \tag{56}$$

With $q_1 > 2$ fixed and for $q_2 > 2$ we consider the function $g(q_2) = \frac{6q_1 q_2}{q_2(q_1 - 2) + 4q_1}$. We have

$$\begin{aligned}
 g'(q_2) &= \frac{6q_1}{q_2(q_1 - 2) + 4q_1} - \frac{6q_1 q_2}{(q_2(q_1 - 2) + 4q_1)^2} (q_1 - 2) \\
 &= \frac{24q_1^2}{(q_2(q_1 - 2) + 4q_1)^2} > 0,
 \end{aligned}$$

hence g is an increasing function and

$$g(q_2) \leq \lim_{t \rightarrow \infty} \frac{6q_1 t}{t(q_1 - 2) + 4q_1} = \frac{6q_1}{q_1 - 2}.$$

Thus,

$$\kappa \geq \frac{q_{\min}}{q_{\min} - 2} \frac{1}{2} \left[6 \frac{q_{\min}}{q_{\min} - 2} - 1 \right] \tag{57}$$

implies the validity of (56).

Summarizing the conditions imposed on κ during our calculations, i.e. recalling the bounds (29), (44), (46), (48), (57) we are led to the lower bound

$$\kappa > 3 \max \left\{ 1, \frac{1}{2} \frac{q_{\max}}{q_{\min}} \frac{q_{\min} - 1}{q_{\min} - 2}, \frac{q_{\min}}{q_{\min} - 2} - \frac{1}{6} \right\} \frac{q_{\min}}{q_{\min} - 2} \tag{58}$$

for the exponent κ . Assuming the validity of (58) and returning to (43) we now have shown that for α_i sufficiently close to $-1/2$ the right-hand side of (43) can be splitted into terms which either can be absorbed in the left-hand side of (43) or stay uniformly bounded, hence

$$\int_{\Omega} \left[\Gamma_1^{\frac{3}{4} q_1} + \Gamma_2^{\frac{3}{4} q_2} \right] |\eta_m|^{2\kappa} dx \leq c \tag{59}$$

for a finite constant c independent of m . Passing to the limit $m \rightarrow \infty$ in (59) our claim (11) follows. Obviously (12) is a consequence of (58) and the definition of

$T(q_1, q_2)$ stated in (13), which completes the proof of Theorem 1.1 for arbitrary exponents $q_1, q_2 > 2$. \square

Remark 2.1 Let us add some comments on the behaviour of κ , which means that we look at the lower bound for the exponent κ given by the right-hand side of inequality (58).

- i) Since $u - u_0 \in W^{1, q_{\min}}(\Omega)$ and thereby $u - u_0 \in C^{0, \nu}(\overline{\Omega})$, $\nu := 1 - 2/q_{\min}$, the Hölder exponent enters (58), which also corresponds to the natural effect that
- ii) $\kappa \rightarrow \infty$ as $q_{\min} \rightarrow 2$.
- iii) The ratio q_{\max}/q_{\min} determines the growth of κ .
- iv) In the limit $q_{\max} = q_{\min} \rightarrow \infty$ condition (58) reduces to $\kappa > 3$.

3 Comments on the case $q_{\min} > 5$

We now choose the sequence $\eta_m \in C_0^1(\Omega)$ according to

$$\begin{aligned} \partial_i \eta_m &\rightarrow \partial_i u, \text{ in } L^{q_i}(\Omega), \quad i = 1, 2, \\ \eta_m &\rightarrow u - u_0 \text{ uniformly as } m \rightarrow \infty. \end{aligned} \quad (60)$$

For some elementary properties of anisotropic Sobolev spaces including an appropriate version of this density result we refer, e.g., to [18, 19]). We emphasize that during the following calculations condition (9) can be replaced by the weaker requirement $\partial_i u_0 \in L^{q_i}(\Omega)$, $i = 1, 2$. Proposition 2.1 obviously holds for η_m , and with δ as in Proposition 2.2 we now obtain (30) with the choice $\hat{\kappa} = 1$ observing that going through the proof of (30) the quantity $(II)_i$ can be handled as follows: first we note

$$(II)_i \leq \left[\int_{\Omega} \Gamma^{s_i - \frac{\delta}{2}} |\eta_m|^{2\kappa - 2} dx \right] \cdot \left[\int_{\Omega} |\nabla \eta_m|^2 \Gamma_i^{\frac{\delta}{2}} dx \right],$$

and then we use

$$\int_{\Omega} |\nabla \eta_m|^2 \Gamma_i^{\frac{\delta}{2}} dx \leq c \left[1 + \int_{\Omega} |\nabla u|^{q_{\min}} dx \right],$$

which is a consequence of (60).

We continue with the discussion of $(III)_i$ for the choice $\hat{\kappa} = 1$, and observe that (33) has to be replaced by

$$\kappa \geq \frac{2}{\delta} s_i. \quad (61)$$

Exactly as in Section 2 we obtain the inequalities (43), (45) now being valid for any exponents s_i, α_i, κ such that ($i = 1, 2$)

$$\alpha_i \in (-1/2, 0), \quad s_i \leq \frac{3}{4}q_i, \quad \kappa \geq \frac{2}{\delta}s_i. \tag{62}$$

Still following the lines of Section 2 we let $s_i = \frac{3}{4}q_i, i = 1, 2$, and replace (61) by

$$\kappa \geq \frac{3}{2\delta}q_{\max}. \tag{63}$$

We let

$$\rho_i := 2\kappa - 2 - 2\kappa \left[\frac{q_i}{2} + \alpha_i \right] \frac{1}{s_i}$$

and consider the terms on the right-hand side of (45). Young’s inequality yields ($0 < \varepsilon < 1$)

$$\begin{aligned} & \sum_{i=1}^2 \int_{\Omega} \Gamma_i^{\frac{q_i}{2} + \alpha_i} |\partial_i \eta_m|^2 |\eta_m|^{2\kappa - 2} \, dx \\ &= \sum_{i=1}^2 \int_{\Omega} \left[\Gamma_i^{s_i} |\eta_m|^{2\kappa} \right]^{\frac{1}{s_i} \left[\frac{q_i}{2} + \alpha_i \right]} |\partial_i \eta_m|^2 |\eta_m|^{\rho_i} \, dx \\ &\leq \sum_{i=1}^2 \left[\varepsilon \int_{\Omega} \Gamma_i^{s_i} |\eta_m|^{2\kappa} \, dx + c(\varepsilon) \int_{\Omega} |\eta_m|^{\rho_i \gamma_i^*} |\partial_i \eta_m|^{2\gamma_i^*} \, dx \right] \end{aligned} \tag{64}$$

with

$$\gamma_i := \frac{s_i}{\frac{q_i}{2} + \alpha_i} = \frac{3q_i}{2q_i + 4\alpha_i}, \quad \gamma_i^* = \frac{3q_i}{q_i - 4\alpha_i}.$$

While the ε -part of (64) can be absorbed, the remaining integrals are bounded if for $i = 1, 2$

$$\rho_i \geq 0 \quad \text{and} \quad 2\gamma_i^* \leq q_i.$$

Noting that the function $t \mapsto t/(t + 2), t \geq 0$, is increasing and by choosing $\alpha_i, i = 1, 2$, sufficiently close to $-1/2$, we see that these requirements are consequences of the strict inequalities

$$q_{\max} > 4, \quad \kappa > \frac{3q_{\max}}{q_{\max} + 2}. \tag{65}$$

We thus have assuming (63) and (65)

$$\begin{aligned} \int_{\Omega} \left[\Gamma_1^{s_1} + \Gamma_2^{s_2} \right] |\eta_m|^{2\kappa} \, dx &\leq c \left[1 + \int_{\Omega} \Gamma_2^{\frac{q_2 - 2}{2}} |\partial_2 \eta_m|^2 |\eta_m|^{2\kappa - 2} \Gamma_1^{1 + \alpha_1} \, dx \right. \\ &\quad \left. + \int_{\Omega} \Gamma_1^{\frac{q_1 - 2}{2}} |\partial_1 \eta_m|^2 |\eta_m|^{2\kappa - 2} \Gamma_2^{1 + \alpha_2} \, dx \right]. \end{aligned} \tag{66}$$

Let us have closer look at the first integral on the right-hand side of (66). Youngs’s inequality gives

$$\begin{aligned}
 & \int_{\Omega} \Gamma_2^{\frac{q_2-2}{2}} |\partial_2 \eta_m|^2 |\eta_m|^{2\kappa-2} \Gamma_1^{1+\alpha_1} \, dx \\
 & \leq \int_{\Omega} |\partial_2 \eta_m|^{q_2} \, dx + \int_{\Omega} \Gamma_2^{\frac{q_2}{2}} |\eta_m|^{(2\kappa-2)\frac{q_2}{q_2-2} \Gamma_1^{(1+\alpha_1)\frac{q_2}{q_2-2}} \, dx \\
 & \leq c \left[1 + \int_{\Omega} \left[\Gamma_2^{s_2} |\eta_m|^{2\kappa} \right]^{\frac{2}{3}} |\eta_m|^{(2\kappa-2)\frac{q_2}{q_2-2} - \frac{4}{3}\kappa} \Gamma_1^{(1+\alpha_1)\frac{q_2}{q_2-2}} \, dx \right] \tag{67} \\
 & \leq c + \varepsilon \int_{\Omega} \Gamma_2^{s_2} |\eta_m|^{2\kappa} \, dx \\
 & \quad + c(\varepsilon) \int_{\Omega} |\eta_m|^{3\left[(2\kappa-2)\frac{q_2}{q_2-2} - \frac{4}{3}\kappa\right]} \Gamma_1^{3(1+\alpha_1)\frac{q_2}{q_2-2}} \, dx.
 \end{aligned}$$

Here the first integral is absorbed and the second is bounded if we have (again α_2 being sufficiently close to $-1/2$)

$$(2\kappa - 2)\frac{q_2}{q_2 - 2} - \frac{4}{3}\kappa \geq 0 \quad \text{and} \quad \frac{3}{2}\frac{q_2}{q_2 - 2} < \frac{q_1}{2}.$$

The first condition follows from the requirement (65), the second one holds if we assume in addition that $q_{\min} > 5$.

In the same way the last term on the right-hand side of (66) is handled and by combining (63), (65) we have shown Theorem 1.1 together with the formula (14). □

4 Proof of Theorem 1.2

We proceed along the lines of Section 2 assuming that all the hypotheses of Theorem 1.2 are satisfied. In place of (25) we have $(\Gamma := 1 + |\nabla u|^2)$

$$\begin{aligned}
 \int_{\Omega} \Gamma^s |\eta|^{2\kappa} \, dx & \leq c \left\{ \left[\int_{\Omega} |\nabla \Gamma^{\frac{s}{2}}| |\eta|^{\kappa} \, dx \right]^2 + \left[\int_{\Omega} \Gamma^{\frac{s}{2}} |\nabla \eta| |\eta|^{\kappa-1} \, dx \right]^2 \right\} \tag{68} \\
 & =: c\{(I) + (II)\},
 \end{aligned}$$

and (68) holds for any $s, \kappa \geq 1$ and $\eta \in C_0^1(\Omega)$. We define the sequence η_m as in (26) observing that (28) holds with q_{\min} replaced by p . Letting $\delta := p - 2$, defining $\hat{\kappa} := \frac{p-1}{p-2}$ we obtain for

$$s > \frac{\delta}{2}, \quad \kappa \geq \hat{\kappa} \tag{69}$$

applying obvious modifications in the proof of (30)

$$(II) \leq c \int_{\Omega} \Gamma^{s-\frac{\delta}{2}} |\eta_m|^{2\kappa-2\hat{\kappa}} dx =: c(III). \tag{70}$$

Here (II) just denotes the term occurring on the right-hand side of (68) with η being replaced by η_m . The quantity (III) is discussed analogously to the term (III)_i; in accordance with the calculations presented after the proof of Proposition 2.2 we get for any $\varepsilon > 0$

$$(III) \leq \varepsilon \int_{\Omega} \Gamma^s |\eta_m|^{2\kappa} dx + c(\varepsilon) \int_{\Omega} |\eta_m|^{\vartheta} dx, \tag{71}$$

$$\vartheta := \left[\frac{s}{s-\frac{\delta}{2}} \right]^* \left(2\kappa - 2\hat{\kappa} - 2\kappa \frac{s-\frac{\delta}{2}}{s} \right),$$

with exponent $\vartheta \geq 0$, if we assume that

$$\kappa \geq 2 \frac{s}{\delta} \hat{\kappa} \tag{72}$$

is satisfied. Choosing ε sufficiently small, inserting (71) in (70) and returning to (68) it is shown (compare (35))

$$\int_{\Omega} \Gamma^s |\eta_m|^{2\kappa} dx \leq c \{ (I) + 1 \} \quad (I) := \left[\int_{\Omega} |\nabla \Gamma^{\frac{s}{2}}| |\eta_m|^{\kappa} dx \right]^2. \tag{73}$$

Adjusting the calculations presented after inequality (35) to the situation at hand we consider numbers s, α satisfying

$$s \leq p + \alpha \tag{74}$$

and find recalling (7) and using (74)

$$(I) \leq c \left[\int_{\Omega} \Gamma^{\frac{s-1}{2}} |\nabla^2 u| |\eta_m|^{\kappa} dx \right]^2$$

$$\leq c \left[\int_{\Omega} \Gamma^{\frac{p-2}{2}} |\nabla^2 u|^2 |\eta_m|^{2\kappa} \Gamma^{\alpha} dx \right] \cdot \left[\int_{\Omega} \Gamma^{s-1-\frac{p-2}{2}-\alpha} dx \right]$$

$$\leq c \int_{\Omega} D^2 f(\nabla u) (\partial_i \nabla u, \partial_i \nabla u) |\eta_m|^{2\kappa} \Gamma^{\alpha} dx,$$

where from now on the sum is taken with respect to the index i . Let us assume in addition to (74) that we have

$$\alpha > -\frac{1}{4} \tag{75}$$

as lower bound for the parameter α . Quoting (107) from Proposition 6.2 (with η being replaced by $|\eta_m|^{\kappa}$) and returning to (73) we find:

$$\int_{\Omega} \Gamma^s |\eta_m|^{2\kappa} \, dx \leq c \int_{\Omega} D^2 f(\nabla u) (\nabla |\eta_m|, \nabla |\eta_m|) \Gamma^{1+\alpha} |\eta_m|^{2\kappa-2} \, dx. \tag{76}$$

On the right-hand side of (76) we use the second inequality from the ellipticity condition (7) to obtain (in analogy to the inequalities (43), (45))

$$\int_{\Omega} \Gamma^s |\eta_m|^{2\kappa} \, dx \leq c \int_{\Omega} \Gamma^{\frac{q}{2}+\alpha} |\nabla \eta_m|^2 |\eta_m|^{2\kappa-2} \, dx. \tag{77}$$

The right-hand side of (77) is discussed following the arguments presented after (45): we first have (recall (28))

$$\begin{aligned} \int_{\Omega} \Gamma^{\frac{q}{2}+\alpha} |\nabla \varphi_m|^2 |u - u_0|^{2\kappa} \, dx &\leq \int_{\text{spt} \nabla \varphi_m} \Gamma^{\frac{q}{2}+\alpha} m^{2-2\kappa \frac{p-2}{p}} \, dx \\ &\leq c \int_{\Omega} \Gamma^{\frac{p}{2}} \, dx + \int_{\text{spt} \nabla \varphi_m} m^{\gamma^* [2-2\kappa \frac{p-2}{p}]} \, dx, \end{aligned}$$

where we have defined

$$\gamma := \frac{p/2}{\frac{q}{2} + \alpha}, \quad \gamma^* := \frac{\gamma}{\gamma - 1}.$$

Obviously this requires the bound $\frac{p}{2} \geq \frac{q}{2} + \alpha$, i.e. the condition

$$q < p + \frac{1}{2} \tag{78}$$

in combination with (75) yields the maximal range of anisotropy, since then we can choose α sufficiently close to $-1/4$ to guarantee $p > q + 2\alpha$. Moreover, we assume the validity of $-1 + \gamma^* [2 - 2\kappa(p - 2)/p] \leq 0$. This is true again for α chosen sufficiently close to $-1/4$, provided that

$$\kappa > \frac{p + q - \frac{1}{2}}{2(p - 2)}. \tag{79}$$

Recalling (78) we see that (79) is a consequence of the stronger bound

$$\kappa > \frac{p}{p - 2}. \tag{80}$$

In accordance with (47) we therefore arrive at

$$\text{r.h.s. of (4.10)} \leq c \left[1 + \int_{\Omega} \Gamma^{\frac{q}{2}+\alpha} |\nabla u - \nabla u_0|^2 |\eta_m|^{2\kappa-2} \, dx \right]. \tag{81}$$

Neglecting the contribution resulting from ∇u_0 and under the additional hypothesis

$$s > \frac{q}{2} + \frac{3}{4}, \tag{82}$$

which guarantees the validity of $s > \frac{q}{2} + 1 + \alpha$ for $\alpha > -1/4$, we have

$$\int_{\Omega} \Gamma^{\frac{q}{2}+1+\alpha} |\eta_m|^{2\kappa-2} dx \leq \varepsilon \int_{\Omega} \Gamma^s |\eta_m|^{2\kappa} dx + c(\varepsilon) \int_{\Omega} |\eta_m|^{2\kappa-2\beta^*} dx, \tag{83}$$

$$\beta^* := \frac{\beta}{\beta - 1}, \quad \beta := \frac{s}{1 + \frac{q}{2} + \alpha}. \tag{84}$$

Let us add a comment concerning the conditions imposed on s : as remarked after (78) and during the subsequent calculations the parameter α has to be adjusted in an appropriate way and might become very close to the critical value $-1/4$. For this reason inequality (74) is replaced by the stronger one (recall (16))

$$s < p - \frac{1}{4}. \tag{85}$$

In order to find numbers s satisfying (82) and (85) we need the bound $q < 2p - 2$ being more restrictive for $p \in (2, 5/2)$ than inequality (78).

Finally we note that the exponent $2\kappa - 2\beta^*$ occurring in the second integral on the right-hand side of (83) is non-negative provided $\kappa \geq s / (s - [1 + \frac{q}{2} + \alpha])$, and the latter inequality holds for α sufficiently close to $-1/4$, if (compare (17))

$$\kappa > \frac{s}{s - [\frac{q}{2} + \frac{3}{4}]}. \tag{86}$$

Assuming (86) our claim follows by inserting (83) into (81) and choosing ε sufficiently small. □

5 Proof of Theorem 1.3

Let all the assumptions of Theorem 1.3 hold. We define the number

$$\gamma := \frac{2n}{n+2}, \quad 1 \leq \gamma < 2, \quad \text{with Sobolev conjugate} \quad \frac{n\gamma}{n-\gamma} = 2,$$

and note that

$$\frac{n-\gamma}{n} = \frac{n}{n+2} = \frac{\gamma}{2}, \quad \frac{\gamma}{2-\gamma} = \frac{n}{2}.$$

We also remark that for any fixed α and for any $\kappa > 1$ the inequality

$$\frac{p}{2} < p \frac{\kappa(n+2)-2}{2n\kappa} + \alpha \frac{\kappa-1}{\kappa}. \tag{87}$$

is equivalent to the requirement $\alpha > -p/n$.

Lemma 5.1 Fix $\alpha > -1/(2n)$, $\kappa > 1$ with (87) and choose a real number \bar{s}_α satisfying

$$\frac{p}{2} < \bar{s}_\alpha < p \frac{\kappa(n+2) - 2}{2n\kappa} + \alpha \frac{\kappa - 1}{\kappa}. \quad (88)$$

Then we have for any $\varepsilon > 0$ and for $\eta \in C_0^1(\Omega)$

$$\begin{aligned} \int_{\Omega} \Gamma^{\bar{s}_\alpha} |\eta|^{2(\kappa-1)} dx &\leq \varepsilon \int_{\Omega} \Gamma^{\frac{q}{2} + \alpha} |\nabla \eta|^2 |\eta|^{2(\kappa-1)} dx \\ &+ \varepsilon \int_{\Omega} \Gamma^{-\frac{p}{n} + s} |\nabla \eta|^2 |\eta|^{2(\kappa-1)} dx + c \end{aligned} \quad (89)$$

for a constant being independent on η , where s is given by (90).

Proof of Lemma 5.1 We first observe that for

$$s = \bar{s}_\alpha \frac{\kappa}{\kappa - 1} - \frac{p}{2(\kappa - 1)} \quad (90)$$

we have after an application of Young's inequality

$$\int_{\Omega} \Gamma^{\bar{s}_\alpha} |\eta|^{2(\kappa-1)} dx = \int_{\Omega} \Gamma^{\bar{s}_\alpha - \frac{p}{2\kappa}} |\eta|^{2(\kappa-1)} \Gamma^{\frac{p}{2\kappa}} dx \leq \tilde{\varepsilon} \int_{\Omega} \Gamma^s |\eta|^{2\kappa} dx + c(\tilde{\varepsilon}) \int_{\Omega} \Gamma^{\frac{p}{2}} dx \quad (91)$$

for any $\tilde{\varepsilon} > 0$. We then estimate using Sobolev's inequality

$$\begin{aligned} \int_{\Omega} \Gamma^s |\eta|^{2\kappa} dx &= \int_{\Omega} [\Gamma^{\frac{s}{2}} |\eta|^\kappa]^2 dx \leq c \left[\int_{\Omega} |\nabla [\Gamma^{\frac{s}{2}} |\eta|^\kappa]|^\gamma dx \right]^{\frac{2}{\gamma}} \\ &\leq c \left[\int_{\Omega} |\nabla \Gamma^{\frac{s}{2}}|^\gamma |\eta|^{k\gamma} dx \right]^{\frac{2}{\gamma}} + c(\kappa) \left[\int_{\Omega} \Gamma^{\frac{s\gamma}{2}} |\eta|^{(\kappa-1)\gamma} |\nabla \eta|^\gamma dx \right]^{\frac{2}{\gamma}} \\ &= cT_1^{\frac{2}{\gamma}} + cT_2^{\frac{2}{\gamma}}. \end{aligned} \quad (92)$$

Let us first consider T_1 . Hölder's inequality gives

$$\begin{aligned} T_1^{\frac{2}{\gamma}} &\leq c \left[\int_{\Omega} |\nabla^2 u|^\gamma \Gamma^{\gamma \frac{s-1}{2}} |\eta|^{k\gamma} dx \right]^{\frac{2}{\gamma}} \\ &= c \left[\int_{\Omega} |\nabla^2 u|^\gamma \Gamma^{\gamma \frac{p-2}{4}} \Gamma^{\gamma \frac{\alpha}{2}} \Gamma^{\gamma \frac{2-p}{4}} \Gamma^{-\gamma \frac{\alpha}{2}} \Gamma^{\gamma \frac{s-1}{2}} |\eta|^{k\gamma} dx \right]^{\frac{2}{\gamma}} \\ &\leq c \left[\int_{\Omega} |\nabla^2 u|^2 \Gamma^{\frac{p-2}{2}} \Gamma^\alpha |\eta|^{2k} dx \right] \cdot \left[\int_{\Omega} \Gamma^{\frac{\gamma}{2-\gamma} (\frac{2-p}{2} - \alpha + s - 1)} dx \right]^{\frac{2-\gamma}{\gamma}} \\ &= cT_{1,1} \cdot T_{1,2}^{\frac{2-\gamma}{\gamma}}. \end{aligned} \quad (93)$$

For $T_{1,2}$ we observe recalling (90)

$$\frac{n}{2} \left[\frac{2-p}{2} - \alpha + s - 1 \right] < \frac{p}{2} \Leftrightarrow \bar{s}_\alpha < p \frac{\kappa(n+2) - 2}{2n\kappa} + \alpha \frac{\kappa - 1}{\kappa},$$

which is true by the choice (88) of \bar{s}_α , hence $T_{1,2}$ is uniformly bounded.

We handle $T_{1,1}$ with the help of Proposition 6.2 (replacing η by $|\eta|^\kappa$):

$$\begin{aligned} T_{1,1} &\leq c \int_{\Omega} D^2 f(\nabla u) (\nabla \partial_\gamma u, \nabla \partial_\gamma u) \Gamma^\alpha |\eta|^{2\kappa} \, dx \\ &\leq c \int_{\Omega} D^2 f(\nabla u) (\nabla |\eta|^\kappa, \nabla |\eta|^\kappa) |\nabla u|^2 \Gamma^\alpha \, dx \\ &\leq c \int_{\Omega} \Gamma^{\frac{q-2}{2}} |\eta|^{2\kappa-2} |\nabla \eta|^2 \Gamma^{1+\alpha} \, dx = c \int_{\Omega} \Gamma^{\frac{q}{2}+\alpha} |\nabla \eta|^2 |\eta|^{2\kappa-2} \, dx. \end{aligned} \tag{94}$$

From (92) - (94) we conclude

$$\int_{\Omega} \Gamma^s |\eta|^{2\kappa} \, dx \leq c \int_{\Omega} \Gamma^{\frac{q}{2}+\alpha} |\nabla \eta|^2 |\eta|^{2(\kappa-1)} \, dx + c T_2^{\frac{2}{\gamma}} \tag{95}$$

with constants c being independent of η and it remains to discuss T_2 in (95): we have for $\hat{\mu} = 2/\gamma$, $\mu = 2/(2 - \gamma)$,

$$\begin{aligned} T_2^{\frac{2}{\gamma}} &= \left[\int_{\Omega} \Gamma^{\frac{s\gamma}{2}} |\nabla \eta|^\gamma |\eta|^{(\kappa-1)\gamma} \, dx \right]^{\frac{2}{\gamma}} = \left[\int_{\Omega} \Gamma^{\frac{p}{2\mu}} \Gamma^{-\frac{p}{2\mu}} \Gamma^{\frac{s\gamma}{2}} |\nabla \eta|^\gamma |\eta|^{(\kappa-1)\gamma} \, dx \right]^{\frac{2}{\gamma}} \\ &\leq c \left[\int_{\Omega} \Gamma^{\frac{p}{2}} \, dx \right]^{\frac{2}{n}} \cdot \int_{\Omega} \Gamma^{-\frac{p}{n}+s} |\nabla \eta|^2 |\eta|^{2(\kappa-1)} \, dx. \end{aligned} \tag{96}$$

With (91) for $\tilde{\varepsilon} < \varepsilon/c$ sufficiently small, (95) and (96) the proof of Lemma 5.1 is completed. □

Now we come to the proof of the theorem: we choose $\eta_m = (u - u_0)\varphi_m$ with φ_m defined after (26). Then Lemma 5.1 yields

$$\begin{aligned} \int_{\Omega} \Gamma^{\bar{s}_\alpha} |\eta_m|^{2(\kappa-1)} \, dx &\leq \varepsilon \int_{\Omega} \Gamma^{\frac{q}{2}+\alpha} [|\nabla u| + |\nabla u_0|]^2 |\eta_m|^{2(\kappa-1)} \, dx \\ &\quad + \varepsilon \int_{\Omega} \Gamma^{-\frac{p}{n}+s} [|\nabla u| + |\nabla u_0|]^2 |\eta_m|^{2(\kappa-1)} \, dx \\ &\quad + \varepsilon \int_{\Omega} \Gamma^{(\frac{q}{2}+\alpha)} |\nabla \varphi_m|^2 |u - u_0|^2 |\eta_m|^{2(\kappa-1)} \, dx \\ &\quad + \varepsilon \int_{\Omega} \Gamma^{-\frac{p}{n}+s} |\nabla \varphi_m|^2 |u - u_0|^2 |\eta_m|^{2(\kappa-1)} \, dx + c. \end{aligned} \tag{97}$$

Here the first two integrals on the right-hand side can be absorbed in the left-hand side provided that we have (recall that u_0 is Lipschitz)

$$\max \left\{ \frac{q}{2} + \alpha, -\frac{p}{n} + s \right\} < \bar{s}_\alpha - 1. \tag{98}$$

Also on account of (98) we can handle the remaining integrals on the right-hand side of (97) with the help of Young’s inequality. We obtain

$$\begin{aligned} & \int_{\Omega} \Gamma^{\frac{q}{2} + \alpha} |\nabla \varphi_m|^2 |u - u_0|^2 |\eta_m|^{2(\kappa-1)} \, dx \\ & \leq \int_{\Omega} \Gamma^{\bar{s}_\alpha} |\eta_m|^{2(\kappa-1)} \, dx + c \int_{\text{spt} \nabla \varphi_m} |\nabla \varphi_m|^{2\beta_1^*} |u - u_0|^{2\beta_1^*} |\eta_m|^{2(\kappa-1)} \, dx. \end{aligned} \tag{99}$$

as well as

$$\begin{aligned} & \int_{\Omega} \Gamma^{-\frac{p}{n} + s} |\nabla \varphi_m|^2 |u - u_0|^2 \eta_m^{2(\kappa-1)} \, dx \\ & \leq \int_{\Omega} \Gamma^{\bar{s}_\alpha} \eta_m^{2(\kappa-1)} \, dx + c \int_{\text{spt} \nabla \varphi_m} |\nabla \varphi_m|^{2\beta_2^*} |u - u_0|^{2\beta_2^*} \eta_m^{2(\kappa-1)} \, dx. \end{aligned} \tag{100}$$

In (99) we choose β_1 such that

$$1 < \beta_1 < \left[p \frac{\kappa(n+2) - 2}{\kappa} - \frac{\kappa - 1}{\kappa} \right] \frac{1}{qn - 1}$$

with conjugate exponent (recall $q - p < 2p/n$ on account of (18))

$$\beta_1^* > \frac{\kappa [p(n+2) - 1] - 2p + 1}{\kappa [2p - (q-p)n] - 2p + 1}. \tag{101}$$

Note that for α sufficiently close to $-1/(2n)$ it holds (recall 88)

$$1 < \beta_1 < \bar{s}_\alpha \left[\frac{q}{2} - \frac{1}{2n} \right]^{-1}.$$

In (100) we define β_2 according to

$$1 < \beta_2 < \left[p \frac{\kappa(n+2) - 2}{2\kappa} - \frac{\kappa - 1}{2\kappa} \right] \frac{1}{sn - p}$$

with conjugate exponent

$$\beta_2^* > \frac{\kappa [pn + 4p - 1 - 2sn] - 2p + 1}{\kappa [2p - (q-p)n] - 2p + 1}. \tag{102}$$

and choosing α close to $-1/(2n)$ we can arrange that (recall 88)

$$1 < \beta_2 < \bar{s}_\alpha \left[-\frac{p}{n} + s \right]^{-1}.$$

Returning to (99) and (100), respectively, the first integral again is absorbed in the left-hand side of (97). The remaining integrals stay bounded if we have for $i = 1, 2$

$$m^{-1} m_i^{\beta_i^*} [2^{-2(1-n/p)}]^{-2(\kappa-1)(1-n/p)} \leq c,$$

Thus we require the condition

$$\beta_i^* < (\kappa - 1) \frac{p - n}{n} + \frac{p}{2n}, \quad i = 1, 2, \tag{103}$$

which on account of $p > n$ is satisfied for κ sufficiently large.

It remains to arrange (98) together with the inequality on the right-hand side of (88). Here we first observe

$$\begin{aligned} \frac{q}{2} + \alpha < p \frac{\kappa(n+2) - 2}{2n\kappa} + \alpha \frac{\kappa - 1}{\kappa} - 1 &\Leftrightarrow q < p \frac{\kappa(n+2) - 2}{n\kappa} - 2 - 2 \frac{\alpha}{\kappa} \\ &\Leftrightarrow q - p < \frac{2}{n}(p - n) - \frac{2p + 2n\alpha}{n\kappa} \end{aligned} \tag{104}$$

For α sufficiently close to $-1/(2n)$, (104) is a consequence of (19). We finally have to discuss

$$\begin{aligned} -\frac{p}{n} + s \leq \bar{s}_\alpha - 1 &\Leftrightarrow -\frac{p}{n} + \bar{s}_\alpha \frac{\kappa}{\kappa - 1} - \frac{p}{2(\kappa - 1)} < \bar{s}_\alpha - 1 \\ &\Leftrightarrow \bar{s}_\alpha < (\kappa - 1) \frac{p - n}{n} + \frac{p}{2}. \end{aligned} \tag{105}$$

Assumption (21) implies (105). Hence, (97), (99) and (100) prove Theorem 1.3 passing to the limit $m \rightarrow \infty$. □

6 Appendix. Caccioppoli-type inequalities

We prove two Caccioppoli-type inequalities with small weights (i.e. involving powers of $\Gamma = 1 + |\nabla u|^2$ or of $\Gamma_i = 1 + |\partial_i u|^2$, $i = 1, \dots, n$, with a certain range of negative exponents), where the first one is the appropriate version in the splitting context.

There is no need to restrict the following considerations to the case $n = 2$. Thus, throughout this appendix, we suppose that $\Omega \subset \mathbb{R}^n$ is a bounded Lipschitz domain and that $f: \mathbb{R}^n \rightarrow \mathbb{R}$ is of class C^2 satisfying $D^2f(Z)(Y, Y) > 0$ for all $Z, Y \in \mathbb{R}^n$.

Moreover we suppose that $u \in W_{loc}^{2,2}(\Omega) \cap C^1(\Omega)$ solves the differentiated Euler equation

$$0 = \int_{\Omega} D^2f(\nabla u)(\nabla \partial_i u, \nabla \psi) \, dx \quad \text{for all } \psi \in C_0^\infty(\Omega) \tag{106}$$

and for any $1 \leq i \leq n$ fixed.

Proposition 6.1 Fix $l \in \mathbb{N}$ and suppose that $\eta \in C_0^\infty(\Omega)$, $0 \leq \eta \leq 1$. Then the inequality

$$\begin{aligned} & \int_{\Omega} D^2 f(\nabla u)(\nabla \partial_i u, \nabla \partial_i u) \eta^{2l} \Gamma_i^\alpha \, dx \\ & \leq c \int_{\Omega} D^2 f(\nabla u)(\nabla \eta, \nabla \eta) \eta^{2l-2} \Gamma_i^{\alpha+1} \, dx, \quad \Gamma_i := 1 + |\partial_i u|^2, \end{aligned} \quad (107)$$

holds for any $\alpha > -1/2$ and for any fixed $1 \leq i \leq n$.

Proof of Proposition 6.1 Suppose that $-1/2 < \alpha$ and fix $1 \leq i \leq n$ (no summation with respect to i). Using approximation arguments we may insert

$$\psi := \eta^{2l} \partial_i u \Gamma_i^\alpha$$

in the equation (106) with the result

$$\begin{aligned} \int_{\Omega} D^2 f(\nabla u)(\nabla \partial_i u, \nabla \partial_i u) \eta^{2l} \Gamma_i^\alpha \, dx &= - \int_{\Omega} D^2 f(\nabla u)(\nabla \partial_i u, \nabla \Gamma_i^\alpha) \partial_i u \eta^{2l} \, dx \\ &\quad - \int_{\Omega} D^2 f(\nabla u)(\nabla \partial_i u, \nabla(\eta^{2l})) \partial_i u \Gamma_i^\alpha \, dx \\ &=: S_1 + S_2. \end{aligned} \quad (108)$$

In (108) we have

$$S_1 = -2\alpha \int_{\Omega} D^2 f(\nabla u)(\nabla \partial_i u, \nabla \partial_i u) |\partial_i u|^2 \Gamma_i^{\alpha-1} \eta^{2l} \, dx$$

which gives $S_1 \leq 0$ if $\alpha \geq 0$. In this case we will just neglect S_1 in the following. In the case $-1/2 < \alpha < 0$ we estimate

$$\begin{aligned} |S_1| &= 2|\alpha| \int_{\Omega} D^2 f(\nabla u)(\nabla \partial_i u, \nabla \partial_i u) |\partial_i u|^2 \Gamma_i^{\alpha-1} \eta^{2l} \, dx \\ &\leq 2|\alpha| \int_{\Omega} D^2 f(\nabla u)(\nabla \partial_i u, \nabla \partial_i u) \Gamma_i^\alpha \eta^{2l} \, dx. \end{aligned}$$

Since we have $2|\alpha| < 1$ we may absorb $|S_1|$ in the left-hand side of (108), hence

$$\int_{\Omega} D^2 f(\nabla u)(\nabla \partial_i u, \nabla \partial_i u) \eta^{2l} \Gamma_i^\alpha \, dx \leq c |S_2|.$$

For $0 < \varepsilon$ sufficiently small we apply the Cauchy-Schwarz inequality to discuss S_2 :

$$\begin{aligned} & \int_{\Omega} D^2f(\nabla u)(\nabla\partial_i u, \nabla\eta)\eta^{2l-1}\Gamma_i^\alpha\partial_i u \, dx \\ & \leq \varepsilon \int_{\Omega} D^2f(\nabla u)(\nabla\partial_i u, \nabla\partial_i u)\eta^{2l}\Gamma_i^\alpha \, dx \\ & \quad + c(\varepsilon) \int_{\Omega} D^2f(\nabla u)(\nabla\eta, \nabla\eta)\eta^{2l-2}\Gamma_i^\alpha|\partial_i u|^2 \, dx. \end{aligned}$$

After absorbing the first term in the right-hand side of (108) we have established our claim (107). □

Instead of the quantities Γ_i our second inequality (compare [20] for the discussion in two dimensions) involves the full derivative, i.e. we incorporate certain negative powers of $\Gamma = 1 + |\nabla u|^2$. As a consequence we do not obtain the range $-1/2 < \alpha$ and have to replace this condition by the requirement $-1/(2n) < \alpha$.

Proposition 6.2 *Suppose that $\eta \in C_0^\infty(\Omega)$ and fix some real number α such that $-1/(2n) < \alpha$. Then we have (summation with respect to $i = 1, \dots, n$)*

$$\begin{aligned} & [1 + 2\alpha n] \int_{\Omega} D^2f(\nabla u)(\nabla\partial_i u, \nabla\partial_i u)\Gamma^\alpha\eta^2 \, dx \\ & \leq c \left[\int_{\text{spt}\nabla\eta} D^2f(\nabla u)(\nabla\partial_i u, \nabla\partial_i u)\Gamma^\alpha\eta^2 \, dx \right]^{\frac{1}{2}} \\ & \quad \cdot \left[\int_{\text{spt}\nabla\eta} D^2f(\nabla u)(\nabla\eta, \nabla\eta)|\nabla u|^2\Gamma^\alpha \, dx \right]^{\frac{1}{2}}, \end{aligned} \tag{109}$$

where the constant c is not depending on η . In particular it holds

$$\int_{\Omega} D^2f(\nabla u)(\nabla\partial_i u, \nabla\partial_i u)\Gamma^\alpha\eta^2 \, dx \leq c \int_{\text{spt}\nabla\eta} D^2f(\nabla u)(\nabla\eta, \nabla\eta)|\nabla u|^2\Gamma^\alpha \, dx. \tag{110}$$

Proof For $i = 1, \dots, n$ and any η as above we deduce from (106)

$$\begin{aligned} & \int_{\Omega} D^2f(\nabla u)(\nabla\partial_i u, \nabla\partial_i u)\Gamma^\alpha\eta^2 \, dx \\ & = - \int_{\Omega} D^2f(\nabla u)(\nabla\partial_i u, \partial_i u\nabla\Gamma^\alpha)\eta^2 \, dx \\ & \quad - 2 \int_{\Omega} D^2f(\nabla u)(\nabla\partial_i u, \nabla\eta)\partial_i u\Gamma^\alpha\eta \, dx =: I + II, \end{aligned} \tag{111}$$

where at this stage no summation with respect to the index i is performed. Following [20] we denote the bilinear form $D^2f(\cdot, \cdot)$ by $\langle \cdot, \cdot \rangle$. The first observation is the inequality

$$\begin{aligned}
\sum_{i=1}^n \langle \partial_i \nabla u, \partial_i \nabla u \rangle \Gamma^\alpha &\geq \sum_{i=1}^n \langle \partial_i \nabla u, \partial_i \nabla u \rangle \sum_{j=1}^n (\partial_j u)^2 \Gamma^{\alpha-1} \\
&\geq \sum_{i=1}^n \langle \partial_i \nabla u, \partial_i \nabla u \rangle (\partial_i u)^2 \Gamma^{\alpha-1} \\
&= \sum_{i=1}^n \langle \partial_i u \partial_i \nabla u, \partial_i u \partial_i \nabla u \rangle \Gamma^{\alpha-1}.
\end{aligned} \tag{112}$$

For handling the integrand of the term I from (111) we use the identity

$$\begin{aligned}
-\sum_{i=1}^n \langle \partial_i \nabla u, \partial_i u \nabla \Gamma^\alpha \rangle &= -\alpha \sum_{i=1}^n \langle \partial_i u \partial_i \nabla u, \nabla \sum_{j=1}^n (\partial_j u)^2 \rangle \Gamma^{\alpha-1} \\
&= -2\alpha \sum_{i=1}^n \langle \partial_i u \partial_i \nabla u, \sum_{j=1}^n \partial_j u \partial_j \nabla u \rangle \Gamma^{\alpha-1} \\
&= -2\alpha \sum_{i=1}^n \langle \partial_i u \partial_i \nabla u, \partial_i u \partial_i \nabla u \rangle \Gamma^{\alpha-1} \\
&\quad - 2\alpha \sum_{i=1}^n \langle \partial_i u \partial_i \nabla u, \sum_{j \neq i} \partial_j u \partial_j \nabla u \rangle \Gamma^{\alpha-1}.
\end{aligned} \tag{113}$$

The last term on the right-hand side of (113) is estimated as follows

$$\begin{aligned}
&\sum_{i=1}^n \langle \partial_i u \partial_i \nabla u, \sum_{j \neq i} \partial_j u \partial_j \nabla u \rangle \\
&\leq \sum_{i=1}^n \left[\sum_{j \neq i} \frac{1}{2} \left[\langle \partial_i u \partial_i \nabla u, \partial_i u \partial_i \nabla u \rangle + \langle \partial_j u \partial_j \nabla u, \partial_j u \partial_j \nabla u \rangle \right] \right] \\
&= \frac{1}{2} \sum_{i=1}^n \left[(n-1) \langle \partial_i u \partial_i \nabla u, \partial_i u \partial_i \nabla u \rangle + \sum_{j \neq i} \langle \partial_j u \partial_j \nabla u, \partial_j u \partial_j \nabla u \rangle \right] \\
&= \frac{1}{2} \sum_{i=1}^n \left[(n-2) \langle \partial_i u \partial_i \nabla u, \partial_i u \partial_i \nabla u \rangle + \sum_{j=1}^n \langle \partial_j u \partial_j \nabla u, \partial_j u \partial_j \nabla u \rangle \right] \\
&= \frac{1}{2} \left[(n-2) \sum_{i=1}^n \langle \partial_i u \partial_i \nabla u, \partial_i u \partial_i \nabla u \rangle + n \sum_{j=1}^n \langle \partial_j u \partial_j \nabla u, \partial_j u \partial_j \nabla u \rangle \right] \\
&= (n-1) \sum_{i=1}^n \langle \partial_i u \partial_i \nabla u, \partial_i u \partial_i \nabla u \rangle.
\end{aligned}$$

This, together with (113) gives (recall $-2\alpha > 0$)

$$\begin{aligned}
 - \sum_{i=1}^n \left\langle \partial_i \nabla u, \partial_i u \nabla \Gamma^\alpha \right\rangle &= - 2\alpha \sum_{i=1}^n \left\langle \partial_i u \partial_i \nabla u, \partial_i u \partial_i \nabla u \right\rangle \Gamma^{\alpha-1} \\
 &\quad - 2\alpha \sum_{i=1}^n \left\langle \partial_i u \partial_i \nabla u, \sum_{j \neq i} \partial_j u \partial_j \nabla u \right\rangle \Gamma^{\alpha-1} \\
 &\leq - 2\alpha \sum_{i=1}^n \left\langle \partial_i u \partial_i \nabla u, \partial_i u \partial_i \nabla u \right\rangle \Gamma^{\alpha-1} \tag{114} \\
 &\quad - 2\alpha(n-1) \sum_{i=1}^n \left\langle \partial_i u \partial_i \nabla u, \partial_i u \partial_i \nabla u \right\rangle \Gamma^{\alpha-1} \\
 &= - 2\alpha n \sum_{i=1}^n \left\langle \partial_i u \partial_i \nabla u, \partial_i u \partial_i \nabla u \right\rangle \Gamma^{\alpha-1}.
 \end{aligned}$$

Combining (114) and (112) we get

$$\begin{aligned}
 - \sum_{i=1}^n \int_{\Omega} D^2 f(\nabla u) (\partial_i \nabla u, \partial_i u \nabla \Gamma^\alpha) \eta^2 \, dx \\
 \leq - 2\alpha n \sum_{i=1}^n \int_{\Omega} D^2 f(\nabla u) (\partial_i \nabla u, \partial_i \nabla u) \Gamma^\alpha \eta^2 \, dx.
 \end{aligned} \tag{115}$$

Returning to (111) and using (115) we get (from now on summation with respect to i)

$$\begin{aligned}
 [1 + 2\alpha n] \int_{\Omega} D^2 f(\nabla u) (\nabla \partial_i u, \nabla \partial_i u) \Gamma^\alpha \eta^2 \, dx \\
 \leq - 2 \int_{\text{spt} \nabla \eta} D^2 f(\nabla u) (\eta \nabla \partial_i u, \partial_i u \nabla \eta) \Gamma^\alpha \, dx.
 \end{aligned} \tag{116}$$

This finishes the proof by applying the Cauchy-Schwarz inequality. □

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References

1. Giaquinta, M.: Growth conditions and regularity, a counterexample. *Manuscripta Math.* **59**(2), 245–248 (1987)
2. Marcellini, P.: Regularity of minimizers of integrals of the calculus of variations with nonstandard growth conditions. *Arch. Ration. Mech. Anal.* **3**, 267–284 (1989)
3. Choe, H.J.: Interior behaviour of minimizers for certain functionals with nonstandard growth. *Non-linear Anal.* **19**(10), 933–945 (1992)
4. Esposito, L., Leonetti, F., Mingione, G.: Higher integrability for minimizers of integral functionals with (p, q) growth. *J. Differ. Equ.* **157**(2), 414–438 (1999)
5. Fusco, N., Sbordone, C.: Some remarks on the regularity of minima of anisotropic integrals. *Commun. Partial Differ. Equ.* **18**(1–2), 153–167 (1993)
6. Bildhauer, M.: *Convex, Variational Problems. Linear, Nearly Linear and Anisotropic Growth Conditions*, Volume of Lecture Notes in Mathematics, p. 2003. Springer, Berlin (1818)
7. Bildhauer, M.: Fuchs, higher integrability of the gradient for vectorial minimizers of decomposable variational integrals. *Manuscripta Math.* **123**(3), 269–283 (2007)
8. Bildhauer, M., Fuchs, M., Zhong, X.: A regularity theory for scalar local minimizers of splitting-type variational integrals. *Ann. Sc. Norm. Super. Pisa Cl. Sci. (5)* **6**(3), 385–404 (2007)
9. Beck, L., Mingione, G.: Lipschitz bounds and nonuniform ellipticity. *Commun. Pure Appl. Math.* LXXIII:944–1034 (2020)
10. Leonetti, F., Siepe, F.: Global integrability for minimizers of anisotropic functionals. *Manuscripta Math.* **144**(1–2), 91–98 (2014)
11. Koch, L.: Global higher integrability for minimisers of convex functionals with (p, q) -growth. *Calc. Var. Partial Differ. Equ.* 60(2):Paper No. 63 (2021)
12. Adams, R.A., Fournier, J.: *Sobolev Spaces*, Volume 140 of Pure and Applied Mathematics, , second edition. Elsevier, Amsterdam (2003)
13. Demengel, F., Demengel G.: *Functional Spaces for the Theory of Elliptic Partial Differential Equations* (Translation from French Original). Universitext. Springer (EDP Sciences), London (Les Ulis), 2012 (2007)
14. Esposito, L., Leonetti, F., Mingione, G.: Sharp regularity for functionals with (p, q) growth. *J. Differ. Equ.* **204**(1), 5–55 (2004)
15. Marcellini, P.: Regularity and existence of solutions of elliptic equations with p, q -growth conditions. *J. Differ. Equ.* **90**(1), 1–30 (1991)
16. Gilbarg, D., Trudinger, N.: *Elliptic Partial Differential Equations of Second Order*, Volume 224 of Grundlehren der math. Wiss., Revised Third Printing, second edition. Springer, Berlin (1998)
17. Bildhauer, M., Fuchs, M.: Splitting type variational problems with linear growth conditions. *J. Math. Sci. (N.Y.) Probl. Math. Anal. No. 105* **250**(2), 45–58 (2020)
18. Rákosnik, J.: Some remarks to anisotropic Sobolev spaces. I. *Beiträge Anal.* **13**, 55–68 (1979)
19. Rákosnik, J.: Some remarks to anisotropic Sobolev spaces. II. *Beiträge Anal.* **15**, 127–140 (1981)
20. Bildhauer, M., Fuchs, M.: Small weights in Caccioppoli’s inequality and applications to Liouville-type theorems for non-standard problems. *Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI)* **508**, 73–88 (2021)

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