



An extension of a theorem of Bers and Finn on the removability of isolated singularities to the Euler–Lagrange equations related to general linear growth problems

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Abstract

A famous theorem of Bers and Finn states that isolated singularities of solutions to the non-parametric minimal surface equation are removable. We show that this result remains valid, if the area functional is replaced by a general functional of linear growth depending on the modulus of the gradient.

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1 Introduction

We discuss solutions $u \in C^2(D)$ defined on an open set $D \subset \mathbb{R}^n$ of the equation

$$\operatorname{div} \left[\frac{g'(|\nabla u|)}{|\nabla u|} \nabla u \right] = 0 \quad (1.1)$$

arising as the Euler–Lagrange equation of the variational problem

$$\int_{\Omega} g(|\nabla v|) \, dx \rightarrow \min \quad (1.2)$$

among functions $v: \Omega \rightarrow \mathbb{R}$ with prescribed boundary data. The assumptions concerning the density g are as follows:

we consider functions $g: [0, \infty) \rightarrow \mathbb{R}$ of class $C^{2,\alpha}([0, \infty))$ for some exponent $0 < \alpha < 1$ being of linear growth in the sense that with suitable constants $a, A > 0, b, B \geq 0$ the inequality

$$at - b \leq g(t) \leq At + B \quad (1.3)$$

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holds for any $t \geq 0$. Moreover, we require strict convexity of g by imposing the condition

$$g''(t) > 0 \quad \text{for all } t \geq 0. \tag{1.4}$$

Finally, we assume

$$g'(0) = 0. \tag{1.5}$$

We then will prove that the famous theorem of Bers and Finn (see [1,2]) on the removability of isolated singularities for solutions of the non-parametric minimal surface equation extends to any solution of (1.1) provided that g satisfies these hypotheses.

In more detail we have the following result:

Theorem 1 *Consider an open set $\Omega \subset \mathbb{R}^n$, fix some point $x_0 \in \Omega$ and assume that $u \in C^2(\Omega \setminus \{x_0\})$ is a solution of Eq. (1.1) on the set $D := \Omega \setminus \{x_0\}$ with g satisfying (1.3)–(1.5). Then u admits an extension $\bar{u} \in C^2(\Omega)$ and \bar{u} solves Eq. (1.1) on the set Ω .*

In the case of minimal surfaces, i.e. for the choice $g(t) = \sqrt{1+t^2}$ in Eq. (1.1) and for $n = 2$, the result of the theorem was proved independently by Bers [1] and Finn [2]. Concerning solutions of the non-parametric minimal surface equation in dimensions $n > 2$ the removability of singular sets K being closed subsets of Ω such that $\mathcal{H}^{n-1}(K) = 0$ was established by DeGiorgi and Stampacchia [3], by Simon [4], Anzellotti [5] and Miranda [6]. As a matter of fact the removability of (isolated) singularities essentially depends on the growth rate of the density g , which means that in the case of superlinear growth non-removable (isolated) singularities exist. At the same time our arguments essentially use the observation that convexity together with linear growth implies the boundedness of g' , in particular g' is a one-to-one mapping $g': [0, \infty) \rightarrow [0, g'_\infty)$, where $g'_\infty := \lim_{t \rightarrow \infty} g'(t)$.

During the proof of Theorem 1 we will have to distinguish two essentially different cases, where the first one is closely related to the minimal surface setting in the sense that we suppose

$$\int_0^\infty t g''(t) dt < \infty \tag{1.6}$$

restricting the growth of g'' at infinity. Note that (1.6) is a consequence of the pointwise inequality

$$g''(t) \leq c(1+t)^{-\mu}, \quad t \geq 0, \tag{1.7}$$

provided we choose $\mu > 2$. In the minimal surface case, i.e. for the choice $g(t) = \sqrt{1+t^2}$, we can choose $\mu = 3$ in estimate (1.7), and by a “ μ -surface in \mathbb{R}^{n+1} ” we denote the graph $\{(x, u(x)) \in \mathbb{R}^{n+1} : x \in D\}$ of a solution $u: D \rightarrow \mathbb{R}$ of Eq. (1.1), provided that g satisfies the conditions (1.3), (1.4) and (1.7) for some exponent $\mu > 2$. We refer to the recent manuscript [7] on some geometric properties of μ -surfaces in the case $n = 2$. Adopting this notation we deduce from Theorem 1 that μ -surfaces do not admit isolated singular points.

However, this removability property does not depend on any geometric features. As it is formulated in Theorem 1, the non-existence of isolated singularities is just a consequence of the linear growth of g which is also exploited in the second case

$$\int_0^\infty t g''(t) dt = \infty. \tag{1.8}$$

This condition already occurs, e.g., in [8] (compare also [9]) in a quite different setting: in Theorem 1.1 of [8], equation (1.8) together with some kind of balancing condition serves as

a criterion for the solvability of a classical Dirichlet-problem, where the authors argue with the help of suitable barrier functions. Both in [8] and in [9] generalized catenoids are used as basic tools, which is also the case in our considerations. Depending on the conditions (1.6) and (1.8), respectively, these catenoids are of infinite height or uniformly bounded.

We close the introduction by remarking that Theorem 1 easily implies a zero-order Liouville type-result for entire solutions of certain non-autonomous equations in the plane.

Corollary 1 *Let the function g satisfy (1.3)–(1.5) and define $a(t) := g'(t)/t$. Let $u \in C^2(\mathbb{R}^2)$ denote a solution of*

$$\operatorname{div} \left[a(|z|^2 |\nabla u(z)|) \nabla u(z) \right] = 0 \tag{1.9}$$

for all $z = (x, y) = x + iy \in \mathbb{R}^2 = \mathbb{C}$. Then u must be a constant function.

Proof For $z \in \mathbb{C} \setminus \{0\}$ let $v(z) := u(1/z)$. From (1.9) it follows that v is a solution of (1.1) on $\mathbb{C} \setminus \{0\}$. Theorem 1 shows that v extends to a smooth solution of (1.1) on the whole plane. Since u is smooth in the origin, we obtain the boundedness of v , and the constancy of v follows from Theorem 1.1 in [10]. □

2 Proof of Theorem 1 under condition (1.6)

In the following we consider energy densities $g: [0, \infty) \rightarrow \mathbb{R}$ of class $C^{2,\alpha}$ such that (1.3)–(1.5) hold. In particular g' is a bounded function and strictly increasing, thus

$$0 = g'(0) < g'(t) \rightarrow g'_\infty \quad \text{as } t \rightarrow \infty. \tag{2.1}$$

W.l.o.g. it is assumed that

$$g'_\infty = 1. \tag{2.2}$$

Moreover, $g \in C^2$ together with $g'(0) = 0$ yields that the function $G: \mathbb{R}^n \rightarrow \mathbb{R}$, $G(p) := g(|p|)$ is of class $C^2(\mathbb{R}^n)$ satisfying

$$\sum_{i,j=1}^n \frac{\partial^2 G}{\partial p_i \partial p_j}(p) q_i q_j > 0 \quad \text{for all } p, q \in \mathbb{R}^n, \quad q \neq 0. \tag{2.3}$$

Step 1. Maximum principle.

We observe that in the subsequent considerations we may not assume Lipschitz continuity of solutions up to the boundary, hence Theorem 1.2 of [11] does not apply. We will make use of the following variant:

Lemma 1 *Suppose that D is a bounded Lipschitz domain in \mathbb{R}^n and that we have (1.3)–(1.5). Moreover suppose that $u, v \in C^2(D) \cap C^0(\overline{D})$ satisfy Eq. (1.1). Then we have:*

$$u \leq v + M \quad \text{on } \partial D \text{ for some real number } M \quad \Rightarrow \quad u \leq v + M \quad \text{in } D.$$

With Lemma 1 the following corollary is immediate:

Corollary 2 *The Dirichlet-problem associated to (1.1) within the class $C^2(D) \cap C^0(\overline{D})$ admits at most one solution.*

Proof of Lemma 1. From (1.1) one obtains

$$0 = \sum_{i,j=1}^n \frac{\partial^2 G}{\partial p_i \partial p_j} (\nabla u)_{\partial x_i} \partial x_j u \text{ on } D. \tag{2.4}$$

Now we refer to Theorem 10.1, p. 263, of [12] with coefficients $(x \in D, y \in \mathbb{R}, p \in \mathbb{R}^n)$

$$a_{ij}(x, y, p) := \frac{\partial^2 G}{\partial p_i \partial p_j}(p)$$

which, by (2.3), are seen to be elliptic. Since we consider the admissible function space $C^2(D) \cap C^0(\overline{D})$ the proof is complete with the above mentioned reference. \square

Step 2. Generalized catenoids as comparison surfaces.

Let g satisfy (1.3)–(1.5) and (1.6) and recall (2.1), which implies that g' maps $[0, \infty)$ in a one-to-one way onto the interval $[0, 1)$.

For numbers $\alpha > 0$ and constants $a \in \mathbb{R}$ we define for $x \in \mathbb{R}^n, |x| > \alpha^{1/(n-1)}$,

$$\begin{aligned} k_{\alpha,a}^{\pm}(x) &:= l_{\alpha,a}^{\pm}(|x|) \\ &:= \pm \int_{\alpha^{1/(n-1)}}^{|x|} (g')^{-1}\left(\frac{\alpha}{r^{n-1}}\right) dr + a. \end{aligned} \tag{2.5}$$

We have:

Lemma 2 *The functions $k_{\alpha,a}^{\pm}$ are solutions of problem (1.1) on $|x| > \alpha^{1/(n-1)}$ with continuous extension (through the value a) to the boundary $|x| = \alpha^{1/(n-1)}$.*

Proof of Lemma 2. W.l.o.g. we let $\alpha = 1, a = 0$ in the definition of $k_{\alpha,a}^{\pm}$ and $l_{\alpha,a}^{\pm}$, respectively. Dropping the indices α and a we have for $t > 1$ (letting $r^{n-1} = 1/g'(s)$)

$$\begin{aligned} l^+(t) &= \int_1^t (g')^{-1}\left(\frac{1}{r^{n-1}}\right) dr \\ &= \int_{\infty}^{s^*(t)} s g''(s) \left[-\frac{1}{n-1} (g'(s))^{-\frac{n}{n-1}} \right] ds, \\ s^* &= s^*(t) := (g')^{-1}\left(\frac{1}{t^{n-1}}\right), \end{aligned}$$

hence

$$l^+(t) = \int_{s^*(t)}^{\infty} s g''(s) \left[\frac{1}{n-1} (g'(s))^{-\frac{n}{n-1}} \right] ds \tag{2.6}$$

is well defined at least for $t > 1$ on account of assumption (1.6) and due to the behaviour of g' as stated in (2.1).

Moreover, from (2.6) it immediately follows that

$$\lim_{t \downarrow 1} l^+(t) = 0,$$

thus l^+ has a continuous extension to $t = 1$ by letting $l^+(1) = 0$.

Let us look at Eq. (1.1) in the case

$$D = B_R(0) \setminus \overline{B_r(0)}$$

for balls centered at 0 with radii $0 < r < R \leq \infty$. Suppose further that we have a solution $u(x)$ of the form $u(x) = \varphi(\rho)$, $\rho = |x|$. Then (1.1) is equivalent to the ODE

$$\frac{d}{d\rho} \left[\rho^{n-1} \frac{g'(|\varphi'(\rho)|)}{|\varphi'(\rho)|} \varphi'(\rho) \right] = 0, \quad \rho \in (r, R), \tag{2.7}$$

and obviously l^+ solves (2.7) for the choices $r = 1, R = \infty$. This proves Lemma 2, since with obvious modifications the above calculations can be adjusted to the functions $k_{\alpha,a}^-$. \square

Step 3. Comparison principle.

Lemma 3 *Let g satisfy the assumptions (1.3)–(1.6). For $0 < r < R < \infty$ let $D = B_R(0) - \overline{B_r(0)}$ in Eq. (1.1) and consider a solution $u \in C^2(D) \cap C^1(\overline{D})$ such that for some $a \in \mathbb{R}$ it holds with $\alpha := r^{n-1}$*

$$u \leq k_{\alpha,a}^- \quad \text{on} \quad \partial B_R(0). \tag{2.8}$$

Then we have

$$u \leq k_{\alpha,a}^- \quad \text{throughout} \quad \overline{D}. \tag{2.9}$$

Proof of Lemma 3. By Lemmas 1 and 2 it is enough to show that

$$u \leq k_{\alpha,a}^- \quad \text{on} \quad \partial B_r(0). \tag{2.10}$$

W.l.o.g. let $r = 1$ and $a = 0$ and write k^- in place of $k_{1,a}^-$. Following a standard reasoning known from the minimal surface case (compare [13]) we assume that (2.10) is wrong. Then we can choose $x_0 \in \partial B_1(0)$ satisfying (on account of $k^- \equiv 0$ on $\partial B_1(0)$)

$$0 < u(x_0) = \max_{|x|=1} u(x) =: M. \tag{2.11}$$

For $t > 1$ we let

$$\phi(t) := u(tx_0) - k^-(tx_0)$$

and get

$$\phi'(t) = x_0 \cdot \nabla u(tx_0) + (g')^{-1} \left(\frac{1}{t^{n-1}} \right).$$

Since we assume $u \in C^1(\overline{D})$ and since we have

$$(g')^{-1} \left(\frac{1}{t} \right) \rightarrow \infty \quad \text{as} \quad t \downarrow 1,$$

there exists $\varepsilon > 0$ such that $\phi'(t) > 0$ for all $t \in (1, 1 + \varepsilon)$. This implies

$$u(tx_0) - k^-(tx_0) > u(x_0) \quad \text{on} \quad (1, 1 + \varepsilon). \tag{2.12}$$

Recalling the definition of M and our assumption (2.8), Lemma 1 yields

$$u - k^- \leq M \quad \text{on} \quad \overline{D}. \tag{2.13}$$

Obviously (2.13) contradicts (2.12), thus we have (2.10) and the proof is complete. \square

Step 4. Removability of isolated singularities.

Now we are going to prove the first part of the theorem. Let g satisfy (1.3)–(1.6) and consider a solution $u \in C^2(B_R(0) \setminus \{0\})$ of Eq. (1.1) on the punctured ball $B_R(0) \setminus \{0\}$. W.l.o.g. we assume that $u \in C^1(\overline{B_R(0)} \setminus B_r(0))$ for any radius $0 < r < R$. Following standard arguments (compare [13]) we claim

$$\min_{|x|=R} u \leq u(y) \leq \max_{|x|=R} u(x) \quad \text{for all } y \neq 0, \quad |y| < R. \tag{2.14}$$

In fact, we let

$$M(r) := \max_{|x|=r} u(x), \quad 0 < r \leq R,$$

and define for $0 < r < R$

$$a := M(R) + \int_r^R (g')^{-1}\left(\frac{\alpha}{t^{n-1}}\right) dt,$$

i.e. we have (again with $\alpha = r^{n-1}$)

$$k_{\alpha,a}^- \equiv M(R) \quad \text{on } \partial B_R(0), \quad \text{hence } u \leq k_{\alpha,a}^- \quad \text{on } \partial B_R(0). \tag{2.15}$$

Quoting Lemma 3 and observing that (2.15) corresponds to hypothesis (2.8), we obtain (compare (2.9))

$$u \leq k_{\alpha,a}^- \quad \text{on } \overline{B_R(0)} \setminus B_r(0). \tag{2.16}$$

Fix a point x such that $0 < |x| < R$. Then (2.16) implies for any $0 < r = \alpha^{1/(n-1)} < |x|$ (recall (2.5) and (2.15))

$$u(x) \leq k_{\alpha,a}^-(x) = M(R) + \int_{|x|}^R (g')^{-1}\left(\frac{\alpha}{t^{n-1}}\right) dt. \tag{2.17}$$

Recall that x is fixed and that we have (1.5). Hence, passing to the limit $r \rightarrow 0$ in (2.17), we obtain

$$u(x) \leq M(R) \quad \text{for any } x \in B_R(0) \setminus \{0\},$$

thus the second inequality stated in (2.14) is established. The first inequality in (2.14) follows with obvious modifications.

Finally, let $\tilde{u} \in C^1(\overline{B_R(0)})$ be a smooth extension of $u|_{\partial B_R(0)}$. From the Hilbert-Haar theory (w.r.t. the convex domain $B_R(0)$) we find a unique Lipschitz-minimizer $v: \overline{B_R(0)} \rightarrow \mathbb{R}$ of the energy

$$\int_{B_R(0)} G(\nabla w) \, dx = \int_{B_R(0)} g(|\nabla w|) \, dx$$

subject to the boundary data $\tilde{u}|_{\partial B_R(0)} = u|_{\partial B_R(0)}$, which due to our hypotheses (recall that $g \in C^{2,\alpha}([0, \infty))$) turns out to be of class $C^{2,\beta}(B_R(0))$ for some $\beta \in (0, 1)$. In fact, the Hilbert-Haar minimizer v has Hölder continuous first derivatives (see, e.g. [11], Theorem 1.7) and standard arguments from regularity theory applied to Eq. (2.4) imply $v \in C^{2,\beta}$.

We claim $v = u$ on $B_R(0) \setminus \{0\}$, which means that v is the desired C^2 -extension of u .

In fact, for $0 < \varepsilon \ll 1$ it holds (using (2.4) for the functions u and v on $D = B_R(0) - \{0\}$ and with ν denoting the exterior normal on $\partial(B_R(0) - B_\varepsilon(0))$)

$$\int_{B_R(0) \setminus B_\varepsilon(0)} (\nabla u - \nabla v)(DG(\nabla u) - DG(\nabla v)) \, dx$$

$$\begin{aligned}
 &= \int_{\partial(B_R(0) \setminus B_\varepsilon(0))} (u - v)(DG(\nabla u) - DG(\nabla v)) \cdot \nu \, d\mathcal{H}^1 \\
 &\rightarrow 0 \quad \text{as } \varepsilon \rightarrow 0
 \end{aligned}$$

on account of $u, v \in L^\infty(B_R(0))$. By ellipticity this implies $\nabla u = \nabla v$ on $B_R(0) \setminus \{0\}$ and our claim follows. □

3 Proof of Theorem 1 under condition (1.8)

Let the density g satisfy the same assumptions as stated in the beginning of the previous section, in particular we have (2.2), but now we replace (1.6) by condition (1.8). W.l.o.g. we may assume that $\Omega = B_2(0)$, $x_0 = 0$ in Theorem 1, in particular $B_1(x_0) \Subset \Omega$. Replacing (2.5) we now fix $0 < r < 1$ and let

$$\begin{aligned}
 k_{r,a}^\pm(x) &:= l_{r,a}^\pm(|x|) \\
 &:= a \pm \int_1^{|x|} (g')^{-1}\left(\frac{r}{t^{n-1}}\right) dt, \quad |x| > r^{\frac{1}{n-1}}.
 \end{aligned} \tag{3.1}$$

Then we have for $|x| > r^{1/(n-1)}$ (letting $t^{n-1} = r/g'(s)$ for the fixed number $0 < r < 1$)

$$\begin{aligned}
 k_{r,a}^\pm(x) &= a \mp r^{\frac{1}{n-1}} \int_{(g')^{-1}(r)}^{s^*(|x|)} s g''(s) \left[\frac{1}{n-1} (g'(s))^{-\frac{n}{n-1}} \right] ds, \\
 s^* = s^*(|x|) &:= (g')^{-1}\left(\frac{r}{|x|^{n-1}}\right).
 \end{aligned}$$

We recall (1.8) and note that $k_{r,a}^\pm(x)$ is defined for $|x| > r^{1/(n-1)}$ with limit

$$k_{r,a}^\pm(x) \rightarrow \mp \infty \quad \text{as } |x| \rightarrow r^{\frac{1}{n-1}}. \tag{3.2}$$

Here $a \in \mathbb{R}$ is chosen according to (note $k_{r,a}^\pm(x) = a$ for $|x| = 1$)

$$u \leq k_{r,a}^- = a := \max_{\partial B_1(0)} u \quad \text{on } \partial B_1(0). \tag{3.3}$$

Since u is bounded on $\partial B_{r^{1/(n-1)}}(0)$ and since we have (3.2), we may choose $\varepsilon > 0$ sufficiently small such that

$$u \leq k_{r,a}^- \quad \text{on } \partial B_{(r+\varepsilon)^{1/(n-1)}}(0). \tag{3.4}$$

With (3.3) and (3.4) we now directly apply Lemma 1 to obtain

$$u \leq k_{r,a}^- \quad \text{on } \overline{B_1(0)} \setminus B_{(r+\varepsilon)^{1/(n-1)}}(0). \tag{3.5}$$

W.l.o.g. we may suppose $\varepsilon < r/2$, hence (3.5) gives

$$u \leq k_{r,a}^- \quad \text{on } \overline{B_1(0)} \setminus B_{(3r/2)^{1/(n-1)}}(0). \tag{3.6}$$

Now we fix $x \in B_1(0)$, $x \neq 0$, and recall that the real number a chosen in (3.3) is not depending on the radius r considered above. We let

$$r := \frac{1}{2}|x|^{n-1}, \quad \text{i.e. } |x| = (2r)^{1/(n-1)}. \tag{3.7}$$

Then (3.6) together with the choice (3.7) finally yields

$$u(x) \leq k_{r,a}^-(x). \tag{3.8}$$

On the other hand definition (3.1) gives together with the monotonicity of g'

$$\begin{aligned} k_{r,a}^-(x) &= l_{r,a}^-(|x|) = a - \int_1^{|x|} (g')^{-1} \left(\frac{1}{2} \frac{|x|^{n-1}}{t^{n-1}} \right) dt \\ &= a + \int_{|x|}^1 (g')^{-1} \left(\frac{1}{2} \frac{|x|^{n-1}}{t^{n-1}} \right) dt \\ &\leq a + (1 - |x|)(g')^{-1}(1/2) \leq c, \end{aligned} \quad (3.9)$$

where the constant c is not depending on x , hence we have established a uniform upper bound for u and a uniform lower bound follows along similar lines. Proceeding exactly as done in the first case at the end of Step 4 the theorem is proved. \square

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