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On Minimal Models and the Termination of Flips for Generalized Pairs

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To my family and friends,
who always believed in me.

Zusammenfassung

Das Ziel dieser Arbeit ist die Untersuchung zweier offener Probleme in der höherdimensionalen birationalen Geometrie, nämlich der Vermutung zur Existenz von minimalen Modellen und der Vermutung zur Terminierung von Flips. Wir arbeiten hauptsächlich mit verallgemeinerten Paaren und untersuchen demzufolge die entsprechenden Versionen der oben genannten Vermutungen des Minimal-Modell-Programms in diesem breiteren Rahmen. Der erste Teil der Dissertation widmet sich daher der Entwicklung der grundlegenden Aspekte der Theorie der verallgemeinerten Paare.

Um die Vermutung zur Existenz von minimalen Modellen anzugehen, betrachten wir zunächst bestimmte Zariski-Zerlegungen in höheren Dimensionen, die sogennanten schwachen Zariski-Zerlegungen und NQC Nakayama-Zariski-Zerlegungen. Anschließend beweisen wir, dass die Existenz von minimalen Modellen für log-kanonische verallgemeinerte Paare aus der Existenz von minimalen Modellen für glatte Varietäten folgt, und ferner, dass die Existenz von minimalen Modellen im Wesentlichen zur Existenz dieser Zariski-Zerlegungen äquivalent ist.

Der letzte Teil dieser Arbeit befasst sich mit der Vermutung zur Terminierung von Flips. Wir zeigen zuerst die Spezielle Terminierung für log-kanonische verallgemeinerte Paare. Danach beweisen wir die Terminierung von Flips für log-kanonische verallgemeinerte Paare der Dimension 3 sowie für pseudo-effektive log-kanonische verallgemeinerte Paare der Dimension 4.

Abstract

The aim of this thesis is the investigation of two open problems in higher-dimensional birational geometry, namely the existence of minimal models conjecture and the termination of flips conjecture. We mainly work with generalized pairs and we therefore study the corresponding versions of the aforementioned conjectures of the Minimal Model Program in this wider context. Consequently, the first part of the thesis is devoted to the development of the basic aspects of the theory of generalized pairs.

In order to deal with the existence of minimal models conjecture, we first study particular Zariski decompositions in higher dimensions, the so-called weak Zariski decompositions and NQC Nakayama-Zariski decompositions. Subsequently, we prove that the existence of minimal models for log canonical generalized pairs follows from the existence of minimal models for smooth varieties, and we also demonstrate that the existence of minimal models is essentially equivalent to the existence of those Zariski decompositions.

The last part of this thesis focuses on the termination of flips conjecture. First, we show the special termination for log canonical generalized pairs. Afterwards, we establish the termination of flips for log canonical generalized pairs of dimension 3 as well as for pseudo-effective log canonical generalized pairs of dimension 4.

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Introduction

Algebraic geometry studies comprehensively geometric objects that arise as solutions of systems of polynomial equations in several variables, i.e., *algebraic varieties*. Its ultimate goal is arguably the complete classification of these objects. More precisely, the basic aim is to classify varieties up to birational equivalence, since many quantities associated with a variety that capture its geometry are preserved when two varieties are *birationally equivalent*, i.e., if they have isomorphic dense open subsets or, equivalently, isomorphic function fields. Thus, the central idea towards this goal is to first search for a "good" representative in each birational equivalence class, that is, for a variety which has better geometric features in comparison to other members of the class, and then to study thoroughly the good representatives of the class.

The Minimal Model Program, also known as Mori's program and abbreviated as MMP, lies at the core of the birational classification theory of (complex) algebraic varieties. Indeed, one of its main objectives is to construct good representatives in any fixed birational equivalence class, known as *minimal models*, by performing certain well-understood birational operations, which are called *divisorial contractions* and *flips*. The former contract subvarieties of codimension one, whereas the latter can be thought of as surgery operations in higher codimension and first occur in dimension 3. In short, the process is the following: given a projective variety X, one obtains via a sequence of divisorial contractions and flips another projective variety Y, which is birationally equivalent to X and which has better (global) geometric properties than X; the resulting variety Y is regarded as a good representative of the birational equivalence class of X. Hence, two fundamental problems in the MMP are, first, whether we can always find *some* finite sequence of divisorial contractions and flips that ends with a minimal model, as desired, and, second, whether *every* such sequence of maps is finite, that is, whether the procedure in question always terminates.

The MMP in dimension 2 is nowadays considered classical, as it was established in the early 20th century by the Italian school of algebraic geometry, and actually leads to the birational classification of smooth projective surfaces. On the other hand, in order to pursue the birational classification of higher-dimensional algebraic varieties, it turns out that it does not suffice to deal only with varieties, but one actually needs to work in a wider context instead, namely with *pairs* (X, B) consisting of a normal projective variety X and a Weil $\mathbb{Q}$-divisor B on X such that $m(K_X+B)$ is Cartier for some $m \geq 1$, where K_X is the canonical divisor of X. Moreover, one is also forced to allow certain "mild" singularities, e.g., *Kawamata log terminal* (klt) or *log canonical* singularities. In this setting, the MMP in dimension 3 was successfully completed in the '80s and '90s with the contribution of many mathematicians, including Kawamata, Kollár, Miyaoka, Mori, Reid and Shokurov. Additionally, the recent papers [BCHM10, CL12, CL13] proved that the MMP works in arbitrary dimension for pairs *of general type*, that is, for pairs (X, B) such that the global sections of $\mathcal{O}_X\big(m(K_X + B)\big)$ grow maximally for sufficiently large and divisible $m \geq 1$. On the other hand, despite the fact that the MMP has already experienced considerable progress in dimension 4, it remains widely open in higher dimensions, at least in the non-general-type case. Finally, it is worth mentioning that the MMP predicts that there are three "building blocks" in birational geometry, namely, *Fano*, *Calabi-Yau* and *canonically polarized* varieties, whose canonical class is, loosely speaking, negative, trivial and positive, respectively.

The purpose of this thesis is to address two central conjectures of the MMP, namely *the existence of minimal models conjecture* and *the termination of flips conjecture*. However, instead of working in the category of pairs that was mentioned above, we work primarily in an even more general context and we consider *generalized pairs*, namely couples of the form $(X, B+M)$, where the divisor M has additional positivity properties; for example, M is often a nef $\mathbb{Q}$-divisor on X, but in general it is determined by some nef $\mathbb{Q}$-divisor that sits on some birational model of X. Accordingly, we investigate the corresponding versions of the aforementioned conjectures in the context of generalized pairs. The precise statements of those conjectures as well as the basic reasons why we work with generalized pairs will be discussed below. Before we proceed, we highlight that this thesis mainly comprizes our joint papers [LT19, LMT20, CT20] with Vladimir Lazić, Joaquín Moraga and Guodu Chen, but it also contains some new material, which will be indicated in the text.

Generalized pairs were recently introduced by Birkar and Zhang in [BZ16], where they studied the effectivity of Iitaka fibrations, although some special cases of this concept had appeared earlier in works of Birkar [Bir12b] and Birkar and Hu [BH14]. They generalize the usual notion of pairs and the motivation for their definition stems mainly from the so-called *canonical bundle formula*; roughly, generalized pairs model the structure of the base of an *lc-trivial fibration*. Nevertheless, since their introduction, generalized pairs have been implemented successfully in a wide variety of contexts, ranging from *the BAB conjecture* [Bir16, Bir19] and *Fujita's spectrum conjecture* [HL20a] to *the termination of flips conjecture* [HM18, Mor18, LMT20, CT20] and *the generalized non-vanishing conjecture* [Has20, HL20b, LP20a, LP20b]. It is therefore fair to say that generalized pairs underlie many of the latest developments in birational geometry. Besides, several recent papers, including [HL18, HM18, Mor18, LT19, CT20], indicate that it is actually essential to understand their birational geometry, even if one is only interested in problems involving varieties or pairs. For instance, the only proof that exists to this day of the termination of flips for pseudo-effective log canonical pairs of dimension 4 exploits crucially the machinery of generalized pairs.

Consequently, a significant part of this thesis is devoted to the development of the basic aspects of the theory of generalized pairs, building on the previous works [BZ16, HL18]. In the remainder of the thesis we deal with the following two open problems of the MMP in the setting of generalized pairs.

Existence of Minimal Models Conjecture. *Let $(X, B+M)$ be a projective log canonical generalized pair with data $f\colon X' \to X$ and M', where f' is a projective birational morphism, M' is an $\mathbb{R}_{\geq 0}$-linear combination of nef $\mathbb{Q}$-Cartier divisors on X' and $M := f_*M'$. If $K_X + B + M$ is pseudo-effective, then $(X, B+M)$ has a minimal model.*

Termination of Flips Conjecture. *Let $(X, B + M)$ be a projective log canonical generalized pair with data $f: X' \to X$ and M' as above. Then any sequence of flips starting from $(X, B + M)$ terminates.*

Note that by taking $M' = 0$ (hence $M = 0$) and $f = \mathrm{id}_X$ we recover the standard versions of the above conjectures. Furthermore, generalized pairs as above, that is, such that the nef $\mathbb{R}$-divisor M' has this special form, are called *NQC*; this acronym stands for *nef $\mathbb{Q}$-Cartier combinations*. The main reason why M' is required to be NQC is that NQC generalized pairs behave better in proofs; this was first realized in [BZ16, HL18].

Next, we present the main results of this thesis. However, for simplicity we do not state them in their most general form (the same is also true for the formulation of the above conjectures). We also illustrate in parallel the basic reasons why we work primarily in the category of generalized pairs. Finally, to avoid confusion, we stress that all varieties below are assumed to be projective.

First of all, as far as the existence of minimal models conjecture is concerned, our main result is the following surprising reduction theorem.

Theorem. *The existence of minimal models for smooth varieties of dimension n implies the existence of minimal models for*

(a) *log canonical pairs of dimension n, and*

(b) *NQC log canonical generalized pairs of dimension n whose underlying variety is $\mathbb{Q}$-factorial klt.*

Furthermore, assuming the existence of minimal models for smooth varieties of dimension $n - 1$, we deduce the existence of minimal models for log canonical pairs (X, B) such that $K_X + B$ is pseudo-effective, X is *uniruled*, i.e., it is covered by rational curves, and $\dim X = n$. Additionally, we obtain an analogous result for uniruled NQC generalized pairs.

The theory of *weak Zariski decompositions*, initiated by Birkar in [Bir12b] and further developed in [HL18, LT19], plays a fundamental role in the proof of the above results. Roughly speaking, a generalized pair $(X, B + M)$ admits a weak Zariski decomposition if the pullback of (the $\mathbb{R}$-Cartier divisor) $K_X + B + M$ to some resolution of X can be expressed numerically as the sum of a nef and an effective divisor. In this thesis we study the fundamental properties of *NQC* weak Zariski decompositions, namely weak Zariski decompositions whose nef part is actually NQC. In particular, we obtain the following result, which improves considerably on [HM18, Theorem 2].

Theorem. *The existence of NQC weak Zariski decompositions for smooth varieties of dimension n implies the existence of NQC weak Zariski decompositions for NQC log canonical generalized pairs of dimension n.*

For the proof of the above result we use crucially the theory of generalized pairs and in particular the canonical bundle formula, which allows us to proceed by induction on the dimension. Actually, this proof is an instance that demonstrates how powerful the machinery of generalized pairs is. Besides, the methods developed for this proof have been utilized by Lazić [Laz19] and Lazić and Meng [LM19], leading to considerable progress towards two other central conjectures of the MMP, namely *the non-vanishing conjecture* and *the abundance conjecture*.

It is also worthwhile to mention that NQC weak Zariski decompositions appear naturally and that their existence is closely related to the existence of minimal models. Indeed, if a (generalized) pair has a minimal model, then it is quite easy to see that it admits an NQC weak Zariski decomposition, hence conjecturally NQC weak Zariski decompositions always exist. In addition, Birkar [Bir12b] proved that a log canonical pair has a minimal model (albeit in a weaker sense than the usual one) if it admits an NQC weak Zariski decomposition, assuming the termination of flips in lower dimensions. Furthermore, Han and Li [HL18] showed that such an equivalence holds also in the context of NQC generalized pairs under weaker assumptions in lower dimensions. In this thesis, building on the papers [Bir12b, HL18], we refine the aforementioned results, establishing the equivalence between the existence of minimal models and the existence of NQC weak Zariski decompositions under mild assumptions in lower dimensions, namely the existence of minimal models for smooth varieties. Besides, we also discuss NQC Nakayama-Zariski decompositions and we obtain a similar and actually unconditional equivalence, which generalizes a previous result of Birkar and Hu [BH14].

Now, we turn to the termination of flips conjecture. Our first step towards the resolution of this problem is to deal with the so-called *special termination*, which, roughly speaking, claims that in any sequence of flips with respect to a log canonical pair the locus of curves contracted at a step avoids the locus of log canonical singularities eventually. In this thesis we demonstrate that the termination of flips for klt pairs of dimension at most $n-1$ implies the special termination for log canonical pairs of dimension n, and we also obtain an analogous statement concerning NQC generalized pairs. Thus, we vastly generalize Fujino's theorem on the special termination for dlt pairs [Fuj07, Theorem 4.2.1]. Finally, we emphasize that our proof is the first complete and rigorous proof of the special termination for *log canonical* pairs in the literature.

Next, by applying the special termination for NQC log canonical generalized pairs and the ascending chain condition for log canonical thresholds [BZ16, Theorem 1.5], we prove that the termination of flips for pseudo-effective NQC log canonical generalized pairs which admit NQC weak Zariski decompositions follows from the termination of flips in lower dimensions. We remark that this is an analog of Birkar's termination result [Bir07, Theorem 1.3] in the context of generalized pairs, and that it also extends [HM18, Theorem 1] to the setting of $\mathbb{R}$-divisors with a different approach.

Furthermore, we establish the following two special cases of the termination of flips conjecture.

Theorem. *The termination of flips conjecture holds for*

(a) *NQC log canonical generalized pairs of dimension* 3*, and*

(b) *pseudo-effective NQC log canonical generalized pairs of dimension* 4*.*

For the proof of (a) we utilize several ideas from the earlier works [K+92, Kaw92, Sho96] on the subject; in particular, the notion of *difficulty* plays a key role in our arguments. To deduce (b), we combine (a) with our aforementioned inductive termination result, using the fact that pseudo-effective NQC log canonical generalized pairs of dimension 4 admit NQC weak Zariski decompositions; note that this follows from our previous results, since minimal models exist for terminal 4-folds by [KMM87, Theorem 5-1-15].

Last but not least, observe that the termination of flips for pseudo-effective log canonical pairs of dimension 4 is a special case of (b) of the above theorem, cf. [Mor18]. As already mentioned above, currently there exists no proof of this statement that does not utilize the machinery of generalized pairs. This indicates that the category of generalized pairs is the right setting in which one could tackle the termination of flips conjecture. Besides, we anticipate that generalized pairs will play a central role in future developments in birational geometry and the Minimal Model Program.

Overview of the Contents

The thesis is organized as follows.

In Chapter 1 we recall several basic definitions and we gather some general and well-known results which do not fit elsewhere in the thesis, but which will be used in the sequel. The primary purpose of this chapter is to render the thesis somewhat more self-contained, but secondarily also to provide specific references for standard notions in birational geometry.

In Chapter 2 we discuss exhaustively the basics of generalized pairs. We made a serious effort to deliver an as complete picture of the theory of generalized pairs as possible. On the one hand, as explained in the introduction, generalized pairs play a fundamental role in the thesis, so this chapter should be regarded as its backbone. On the other hand, the various definitions and results concerning generalized pairs are currently scattered in several papers, including [BZ16, HL18, Bir19, Fil19, LT19, CT20, Fil20, HL20b, HL20c, LMT20], and we thus regarded it as worthy to collect and organize everything in a single chapter. There are, however, a few topics, such as the canonical bundle formula or adjunction for generalized pairs, that are not treated thoroughly in the thesis, yet we have provided appropriate references for such topics in the text. Overall, Chapter 2 contains a plethora of information about generalized pairs and could therefore serve as a general reference.

In Chapter 3 we study the basic properties of NQC weak Zariski decompositions (in the relative setting) and NQC Nakayama-Zariski decompositions (in the absolute setting). In particular, we prove that the existence of NQC weak Zariski decompositions for smooth varieties implies the existence of NQC weak Zariski decompositions for NQC log canonical generalized pairs.

In Chapter 4 we address the existence of minimal models conjecture using crucially the theory of Zariski decompositions that was developed in Chapter 3. Among others, we show that the existence of minimal models for smooth varieties implies the existence of minimal models for NQC log canonical generalized pairs.

In Chapter 5 we discuss in detail the special termination for (NQC) log canonical (generalized) pairs and we also present some immediate applications towards the termination of flips conjecture. In particular, we reduce the special termination for (NQC) log canonical (generalized) pairs of dimension n to the termination of flips for (NQC) klt (generalized) pairs of dimension at most $n-1$.

In Chapter 6 we investigate the termination of flips conjecture using as one of our basic tools the special termination for NQC log canonical generalized pairs that we establish in Chapter 5. In particular, we verify this conjecture both for NQC log canonical generalized pairs of dimension 3 and for pseudo-effective NQC log canonical generalized pairs of dimension 4.

A more detailed description of the contents of each chapter is given at its beginning. In particular, our sources for the material to be presented are always stated clearly there. Moreover, in many cases even further details are provided at the beginning of individual sections and subsections in order to guide the reader through the text.

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Notation and Conventions

We follow generally accepted notation and terminology, as in [Har77, KM98, Laz04]. Moreover, we denote by $\mathbb{Z}_{\geq 0}$ (resp. $\mathbb{Q}_{\geq 0}$, $\mathbb{R}_{\geq 0}$) the set of non-negative integers (resp. rationals, reals) and we define analogously the sets $\mathbb{Z}_{\geq 1}$, $\mathbb{R}_{>0}$. Finally, we adopt the following conventions:

- We work over the field $\mathbb{C}$ of complex numbers.
- A *variety* is an integral separated scheme of finite type over $\mathbb{C}$.
- Given a variety X, by a *point* of X we mean a closed point $x \in X$. A point $x \in X$ is called *general* if it belongs to the complement of a proper algebraic subset of X, and *very general* if it belongs to the complement of a countable union of proper algebraic subsets of X. Furthermore, given a morphism $f\colon X \to Y$ of varieties and a point $y \in Y$, the *fiber* $f^{-1}(y)$ of f over y is called *general* if y is a general point of Y, and *very general* if y is a very general point of Y.
- By a *divisor* we understand a Weil divisor. Unless otherwise stated, we work with $\mathbb{R}$-*divisors*, i.e., with finite formal $\mathbb{R}$-linear combinations $D = \sum d_i D_i$ of distinct prime divisors. Usually we also work with $\mathbb{R}$-*Cartier* divisors, i.e., with finite formal $\mathbb{R}$-linear combinations of Cartier divisors.
- If $\mathscr{F}$ is a coherent sheaf on a variety X, then we denote by $h^i(X, \mathscr{F})$ the dimension of the i-th cohomology group $H^i(X, \mathscr{F})$ of $\mathscr{F}$, that is,

$$h^i(X, \mathscr{F}) := \dim_{\mathbb{C}} H^i(X, \mathscr{F}).$$

- A *higher model* of a normal variety X is a normal variety Y together with a projective birational morphism $f\colon Y \to X$.

1

Preliminaries

In this chapter we collect the definitions of several basic notions that occur frequently in the thesis and we establish some general results that will be used at certain points in the next chapters of the thesis. In particular, we recall the various notions of positivity of $\mathbb{R}$-divisors in the relative setting, e.g., relative nefness, and the various types of maps that one usually encounters in higher-dimensional birational geometry, e.g., small contractions. The material covered below is taken from [KM98, Nak04, Cor07, HK10, Kol13, Fuj17] and we refer to these works for further information.

1.1 Divisors

Given an $\mathbb{R}$-divisor $D = \sum d_i D_i$, where the D_i are distinct prime divisors, we set

$$D^{<1} := \sum_{i\,:\,d_i<1} d_i D_i, \quad D^{=1} := \sum_{i\,:\,d_i=1} D_i, \quad \text{and} \quad \lfloor D \rfloor := \sum \lfloor d_i \rfloor D_i,$$

where as usual for $x \in \mathbb{R}$ we denote by $\lfloor x \rfloor$ the *round-down* of x, i.e., the greatest integer $\leq x$. In addition, given a set $\Gamma \subseteq \mathbb{R}$, the notation $D \in \Gamma$ (used in Section 6.2) means that the coefficients d_i of D belong to the set Γ. In particular, we say that D is a *boundary* $\mathbb{R}$*-divisor* or simply a *boundary* if $D \in [0,1]$. In this case we use the notation $D^{=1}$ and $\lfloor D \rfloor$ interchangeably.

Definition 1.1. Let $\pi\colon X \to Z$ be a projective morphism of normal varieties. Two $\mathbb{R}$-divisors D_1 and D_2 on X are said to be $\mathbb{R}$*-linearly equivalent over* Z, denoted by $D_1 \sim_{\mathbb{R},Z} D_2$, if there is an $\mathbb{R}$-Cartier $\mathbb{R}$-divisor B on Z such that $D_1 \sim_{\mathbb{R}} D_2 + \pi^* B$.

Definition 1.2. Let $\pi\colon X \to Z$ be a projective morphism of normal varieties and let D be an $\mathbb{R}$-divisor on X.

(a) *The* $\mathbb{R}$*-linear system associated with* D *over* Z is defined as

$$|D/Z|_{\mathbb{R}} := \{G \geq 0 \mid G \sim_{\mathbb{R},Z} D\}.$$

(b) *The stable base locus of* D *over* Z is defined as

$$\mathbf{B}(D/Z) := \bigcap_{E \in |D/Z|_{\mathbb{R}}} \operatorname{Supp} E.$$

Definition 1.3. Let $\pi\colon X \to Z$ be a projective morphism of normal varieties. Two $\mathbb{R}$-Cartier $\mathbb{R}$-divisors D_1 and D_2 on X are said to be *numerically equivalent over* Z, denoted by $D_1 \equiv_Z D_2$, if $D_1 \cdot C = D_2 \cdot C$ for any curve C contained in a fiber of π.

Let $\pi\colon X \to Z$ be a projective morphism of normal varieties. We denote by $N^1(X/Z)$ the $\mathbb{R}$-vector space of numerical equivalence classes over Z of $\mathbb{R}$-Cartier $\mathbb{R}$-divisors on X, that is, $N^1(X/Z) := (\mathrm{Pic}(X)/\equiv_Z) \otimes_{\mathbb{Z}} \mathbb{R}$, where $\mathrm{Pic}(X)$ is *the Picard group of* X. Note that $N^1(X/Z)$ is a finite-dimensional $\mathbb{R}$-vector space, its dimension is denoted by $\rho(X/Z)$ and is called *the relative Picard number of* X *over* Z. Moreover, $N^1(X/Z)$ is dual to the $\mathbb{R}$-vector space $N_1(X/Z)$ generated by numerical equivalence classes of integral curves on X which are mapped to points on Z by π, see [KM98, Example 2.16] and [Fuj17, Section 2.2]. In particular, we have

$$\rho(X/Z) = \dim_{\mathbb{R}} N^1(X/Z) = \dim_{\mathbb{R}} N_1(X/Z).$$

Finally, *the Kleiman-Mori cone of* π is denoted by $\overline{\mathrm{NE}}(X/Z)$ and is defined as the closed convex cone in $N_1(X/Z)$ generated by integral curves on X which are mapped to points on Z by π.

Notions of Positivity in the Relative Setting

Definition 1.4. Let $\pi\colon X \to Z$ be a projective morphism of normal varieties. An $\mathbb{R}$-Cartier $\mathbb{R}$-divisor D on X is called:

(a) *ample over* Z or π*-ample* if it is ample on every fiber of π,

(b) *semi-ample over* Z or π*-semi-ample* if there exists a morphism $f\colon X \to Y$ over Z such that $D \sim_{\mathbb{R},Z} f^*A$, where A is a π-ample $\mathbb{R}$-Cartier divisor on Y,

(c) *nef over* Z or π*-nef* if it is nef on every fiber of π, i.e., $D \cdot C \geq 0$ for any curve $C \subseteq X$ such that $\pi(C)$ is a point,

(d) *big over* Z or π*-big* if it is big on a very general fiber of π,

(e) *pseudo-effective over* Z or π*-pseudo-effective* if it is pseudo-effective on a very general fiber of π.

Remark 1.5. Let $\pi\colon X \to Z$ be a projective morphism of normal varieties.

(1) An $\mathbb{R}$-Cartier $\mathbb{R}$-divisor A on X is ample over Z if and only if $A = \sum \alpha_i A_i$, where $\alpha_i \in \mathbb{R}_{>0}$ and A_i is a π-ample Cartier divisor on X for every i.

(2) An $\mathbb{R}$-Cartier $\mathbb{R}$-divisor S on X is semi-ample over Z if and only if $S = \sum \alpha_i S_i$, where $\alpha_i \in \mathbb{R}_{>0}$ and S_i is a π-semi-ample Cartier divisor on X for every i.

(3) An $\mathbb{R}$-Cartier $\mathbb{R}$-divisor B on X is big over Z if and only if $B \sim_{\mathbb{R},Z} A + E$, where A is a π-ample divisor on X and E is an effective $\mathbb{R}$-divisor on X.

(4) An $\mathbb{R}$-Cartier $\mathbb{R}$-divisor P on X is pseudo-effective over Z if and only if $P + A$ is big over Z for any π-ample divisor A on X.

Definition 1.6. Let $\pi\colon X \to Z$ be a projective morphism of normal varieties. An $\mathbb{R}$-Cartier $\mathbb{R}$-divisor P on X is called *NQC (over Z)* if it is an $\mathbb{R}_{\geq 0}$-linear combination of $\mathbb{Q}$-Cartier divisors on X which are nef over Z. The acronym NQC stands for *nef* $\mathbb{Q}$*-Cartier combinations*.

Definition 1.7. Let $\pi\colon X \to Z$ be a projective morphism of normal varieties.

(a) *The movable cone of* π is denoted by $\overline{\mathrm{Mov}}(X/Z)$ and is defined as the closed convex cone in $N^1(X/Z)$ generated by Cartier divisors M on X such that $\pi_*\mathcal{O}_X(M) \neq 0$ and the cokernel of the natural homomorphism $\pi^*\pi_*\mathcal{O}_X(M) \to \mathcal{O}_X(M)$ has support of codimension ≥ 2.

(b) An $\mathbb{R}$-Cartier $\mathbb{R}$-divisor M on X is called *movable over* Z or π*-movable* if the numerical equivalence class over Z of M belongs to $\overline{\mathrm{Mov}}(X/Z)$.

Example 1.8.

(1) Let X be a normal projective variety. Let $M \geq 0$ be a Cartier divisor on X such that $\operatorname{codim}_X \mathbf{B}(M) \geq 2$, i.e., there exists an integer $m \geq 1$ such that the linear system $|mM|$ is fixed-part-free. Then M is a movable divisor on X.

(2) If D is a pseudo-effective $\mathbb{R}$-divisor on a smooth projective variety X and if $D = P_\sigma(D)+N_\sigma(D)$ is *the Nakayama-Zariski decomposition of* D (see Section 3.2), then $P_\sigma(D)$ is movable.

The Canonical Class

Definition 1.9. Let X be a normal variety of dimension n. A *canonical divisor* on X is a Weil divisor K_X on X such that $\mathcal{O}_{X_{\mathrm{reg}}}\big(K_X|_{X_{\mathrm{reg}}}\big) \cong \omega_{X_{\mathrm{reg}}} := \bigwedge^n \Omega^1_{X_{\mathrm{reg}}}$, where X_{reg} is the smooth locus of X.

Note that K_X is well-defined up to linear equivalence, since $\operatorname{codim}_X (X \setminus X_{\mathrm{reg}}) \geq 2$. Nevertheless, it is often called *the* canonical divisor of X.

Iitaka Dimension

Let X be a normal projective variety and let D be an $\mathbb{R}$-Cartier $\mathbb{R}$-divisor on X. We denote by $\kappa(X, D)$ *the Iitaka dimension of* D. For the definition and the various properties of the Iitaka dimension we refer to [Nak04, Chapter II, Section 3.b] and [Fuj17, Section 2.5].

Numerical Dimension

Definition 1.10. Let X be a normal projective variety and let D be an $\mathbb{R}$-Cartier $\mathbb{R}$-divisor on X. We define *the numerical dimension* $\nu(X, D)$ *of* D as follows:

(a) Assume that D is pseudo-effective.

- If A is a Cartier divisor on X and we have $H^0\big(X, \mathcal{O}_X\big(\lfloor mD\rfloor + A\big)\big) \neq 0$ for infinitely many $m \in \mathbb{N}$, then we set

$$\sigma(D; A) := \max\left\{ k \in \mathbb{N} \;\middle|\; \limsup_{m\to\infty} \frac{h^0\big(X, \mathcal{O}_X\big(\lfloor mD\rfloor + A\big)\big)}{m^k} > 0 \right\}.$$

- If A is a Cartier divisor on X and we have $H^0\big(X, \mathcal{O}_X\big(\lfloor mD\rfloor + A\big)\big) \neq 0$ only for finitely many $m \in \mathbb{N}$, then we set $\sigma(D; A) := -\infty$.

Now, we define

$$\nu(X, D) := \max \big\{ \sigma(D; A) \mid A \text{ is a Cartier divisor on } X \big\}.$$

(b) Assume that D is not pseudo-effective. Then we set $\nu(X, D) := -\infty$.

Note that $\nu(X, D) \leq \dim X$. In addition, if D is pseudo-effective, then $\nu(X, D) \geq 0$, see [Nak04, Remark V.2.6(5)]. For the various properties of the numerical dimension we refer to [Kaw85, Nak04, Leh13, Fuj17, LP20a]. Finally, we remark that the numerical dimension of D is often denoted by $\kappa_\sigma(X, D)$ instead.

Exceptional Divisors

Definition 1.11. Let $\pi \colon X \to Z$ be a projective morphism of normal varieties. An $\mathbb{R}$-divisor E on X is called *exceptional over Z* or *π-exceptional* if

$$\operatorname{codim}_Z \pi\big(\operatorname{Supp}(E)\big) \geq 2.$$

Note that if $\pi \colon X \to Z$ is a projective birational morphism of normal varieties, then every π-exceptional prime divisor is contained in the exceptional locus $\operatorname{Exc}(\pi)$ of π.

Divisorial Valuations

Definition 1.12. Let X be a normal variety. A discrete valuation v of the function field $\mathbb{C}(X)$ of X is called *divisorial* if there exist a higher model $f \colon Y \to X$ of X and a prime divisor E on Y such that $v = \operatorname{mult}_E$, i.e., $v(\varphi) = \operatorname{mult}_E(\varphi)$ is the order (of zeros or poles) of a rational function $\varphi \in \mathbb{C}(X)^*$ along E. *The center of v on X* is defined as $c_X(v) := f(E) \subseteq X$. If, moreover, $\operatorname{codim}_X \big(c_X(v)\big) \geq 2$, then the divisorial valuation v is called *exceptional*.

It is customary to abuse notation and identify a divisorial valuation $v = \operatorname{mult}_E$ of $\mathbb{C}(X)$ with (a particular choice of a prime divisor) E and say instead that E is a *divisorial valuation over X with center $c_X(E)$ on X*, see [KM98, Remark 2.23]. We will use exclusively this terminology in the thesis and we will often encounter *exceptional divisorial valuations E over X*, i.e., divisorial valuations E over X such that $\operatorname{codim}_X \big(c_X(E)\big) \geq 2$.

Lastly, note that divisorial valuations can be reached by a finite sequence of blow-ups (by repeatedly blowing-up their centers), see [KM98, Lemma 2.45].

1.2 Maps

Definition 1.13. Let X be a normal variety.

(a) A *resolution* of X is a higher model $f \colon Y \to X$ of X such that Y is smooth.

(b) A *log resolution* of (X, D), where D is a Weil divisor on X, is a resolution $f \colon Y \to X$ of X such that the exceptional locus $\operatorname{Exc}(f)$ of f has pure codimension one and $\operatorname{Supp}(f_*^{-1}D) \cup \operatorname{Exc}(f)$ is an *SNC divisor*, i.e., its irreducible components are smooth and intersect transversally. The acronym SNC stands for *simple normal crossings*.

Recall that the existence of log resolutions was established by Hironaka, see [Kol13, Theorem 10.45(1)]. We will sometimes use implicitly a strengthening of Hironaka's theorem on the resolution of singularities due to Szabó [Sza94], namely, the existence of a log resolution of (X, D) which is an isomorphism over the locus where X is smooth and D is SNC, see [Kol13, Theorem 10.45(2)]. Furthermore, in Section 6.2 we will also use the existence of *the minimal resolution* of a surface, see [Mat02, Theorem 4-6-2] and [Kol13, Theorem 2.25].

Definition 1.14. Let X and Y be normal varieties.

(a) A projective surjective morphism $f\colon X \to Y$ with connected fibers is called a *fibration*.

(b) A birational map $\varphi\colon X \dashrightarrow Y$ whose inverse $\varphi^{-1}\colon Y \dashrightarrow X$ does not contract any divisors is called a *birational contraction*.

(c) Let $\varphi\colon X \dashrightarrow Y$ be a birational map. If both φ and φ^{-1} are birational contractions, then φ is called a *small map* or an *isomorphism in codimension one*.

(d) Let D be an $\mathbb{R}$-Cartier $\mathbb{R}$-divisor on X, let $\varphi\colon X \dashrightarrow Y$ be a birational contraction, and assume that $\varphi_* D$ is $\mathbb{R}$-Cartier. The map φ is called *D-non-positive* (resp. *D-negative*) if there exists a resolution of indeterminacies of φ

$$\begin{array}{ccc} & W & \\ {}^{p}\swarrow & & \searrow^{q} \\ X & \overset{\varphi}{\dashrightarrow} & Y \end{array}$$

such that W is smooth and

$$p^* D \sim_{\mathbb{R}} q^* (\varphi_* D) + E,$$

where E is an effective q-exceptional $\mathbb{R}$-divisor (resp. E is an effective q-exceptional $\mathbb{R}$-divisor whose support contains the strict transform of every φ-exceptional prime divisor).

Example 1.15. Let $f\colon X \to Y$ be a projective birational morphism of varieties, and assume that Y is normal. It follows from [Har77, Corollary III.11.4] that f is a fibration. Moreover, it holds that $\operatorname{codim}_Y \operatorname{Exc}(f^{-1}) \geq 2$, hence f is also a birational contraction, see [Deb01, Paragraph 1.40].

1.3 Auxiliary Results

The next result is a version of the so-called *Zariski's main theorem*. It will be used several times (implicitly or explicitly) in the thesis.

Lemma 1.16. *Let $f\colon X \to Y$ be a finite birational morphism of varieties, and assume that Y is normal. Then f is an isomorphism.*

Proof. By the proof of [Har77, Corollary III.11.4] we see that $f_* \mathcal{O}_X = \mathcal{O}_Y$, that is, the morphism of sheaves $f^\sharp\colon \mathcal{O}_Y \to f_* \mathcal{O}_X$ is an isomorphism. It follows now from [Har77, Corollary III.11.3] that f has connected fibers, and since f is quasi-finite by assumption, we infer that f is bijective. Moreover, f is closed by assumption. But a continuous, bijective and closed map is a homeomorphism. Hence, the morphism $(f, f^\sharp)$ of varieties is an isomorphism. $\square$

Using the above lemma, we deduce readily the following well-known result in the case of surfaces, which shows in particular that there are no flipping contractions in dimension 2. We refer to Section 2.3 for more information.

Corollary 1.17. *Let $f\colon X \to Y$ be a projective birational morphism between varieties of dimension 2, and assume that Y is normal. If f is not an isomorphism, then*

$$\operatorname{codim}_X \operatorname{Exc}(f) = 1.$$

Proof. If $\operatorname{codim}_X \operatorname{Exc}(f) \geq 2$, then $\operatorname{Exc}(f)$ is a non-empty finite subset of X. Hence, f is quasi-finite, since it contracts no curves, and as f is projective, it is actually finite. But it follows then from Lemma 1.16 that f is an isomorphism, a contradiction. □

The following two results are [LT19, Lemmas 2.2 and 2.3], respectively. They will be used in the proof of Lemma 2.50.

Lemma 1.18. *Let $f\colon X \to Y$ be a projective surjective morphism between normal quasi-projective varieties. Then there exists an open subset $U \subseteq Y$ with the following property: if D is an $\mathbb{R}$-Cartier $\mathbb{R}$-divisor on X which is numerically trivial on a fiber over some point of U, then $D|_{f^{-1}(U)} \equiv_U 0$.*

Proof. Let $\pi\colon W \to X$ be a resolution of X and let U be an open subset of Y such that $f \circ \pi$ is smooth over U, see [Har77, Corollary III.10.7]. Let D be an $\mathbb{R}$-Cartier $\mathbb{R}$-divisor on X such that $D|_{f^{-1}(s)} \equiv 0$ for some $s \in U$. Then $(\pi^* D)|_{(f\circ\pi)^{-1}(s)} \equiv 0$. By [Nak04, Lemma II.5.15(3)] we obtain $\pi^* D|_{(f\circ\pi)^{-1}(U)} \equiv_U 0$ and consequently $D|_{f^{-1}(U)} \equiv_U 0$. □

Lemma 1.19. *Let $f\colon X \to Y$ be a fibration between normal quasi-projective varieties. Let D be a $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor on X.*

(i) *If $\kappa(F, D|_F) \geq 0$ for a very general fiber F of f, then there exists an effective $\mathbb{Q}$-divisor E on X such that $D \sim_{\mathbb{Q},Y} E$.*

(ii) *If $D|_F \sim_{\mathbb{Q}} 0$ for a very general fiber F of f, then there exists a non-empty open subset $U \subseteq Y$ such that $D|_{f^{-1}(U)} \sim_{\mathbb{Q},U} 0$.*

Proof.
(i) We may assume that D is a Cartier divisor. By assumption there exists a subset $V \subseteq Y$, which is the intersection of countably many dense open subsets of Y, such that $\kappa(X_y, D|_{X_y}) \geq 0$ for the fiber X_y of f over every closed point $y \in V$.

Let X_η be the generic fiber of f and assume that $h^0\big(X_\eta, \mathcal{O}_{X_\eta}(mD)\big) = 0$ for all $m \in \mathbb{Z}_{\geq 1}$. By [Har77, Theorem III.12.8] for each $m \in \mathbb{Z}_{\geq 1}$ there exists a non-empty open subset $U_m \subseteq Y$ such that $h^0\big(X_y, \mathcal{O}_{X_y}(mD)\big) = 0$ for every point $y \in U_m$. But then for all $m \in \mathbb{Z}_{\geq 1}$ and all $y \in V \cap \bigcap_{m=1}^{\infty} U_m$ we have

$$h^0\big(X_y, \mathcal{O}_{X_y}(mD)\big) = 0,$$

a contradiction.

Therefore, there exists an effective divisor G on X_η such that $D|_{X_\eta} \sim_{\mathbb{Q}} G$, and (i) follows from [BCHM10, Lemma 3.2.1].

(ii) By (i) there exists an effective $\mathbb{Q}$-divisor E on X such that $D \sim_{\mathbb{Q},Y} E$. Then $E|_F \sim_{\mathbb{Q}} 0$ for a very general fiber F of f, hence $E|_F = 0$, since F is projective. Therefore, E cannot be dominant over Y, and we note that the set $U := Y \setminus f(E)$ has the desired properties. □

The last result of this section is [LMT20, Lemma 2.3]. It will be used in the proof of Lemma 5.2.

Lemma 1.20. *Let $f\colon Y \to X$ and $g\colon X \to Z$ be projective morphisms between normal varieties, and assume that f is birational. Let D be an $\mathbb{R}$-Cartier $\mathbb{R}$-divisor on X. Then*

$$\mathbf{B}(f^*D/Z) = f^{-1}\big(\mathbf{B}(D/Z)\big).$$

Proof. By the definition of the relative stable base locus, to prove the lemma, it suffices to show that $f^*|D/Z|_\mathbb{R} = |f^*D/Z|_\mathbb{R}$. It is clear that $f^*|D/Z|_\mathbb{R} \subseteq |f^*D/Z|_\mathbb{R}$. For the converse inclusion, let $G \in |f^*D/Z|_\mathbb{R}$. We may write $f^*f_*G = G + E$, where E is f-exceptional. There exists an $\mathbb{R}$-Cartier $\mathbb{R}$-divisor L on Z such that

$$G \sim_\mathbb{R} f^*D + (g \circ f)^*L = f^*(D + g^*L),$$

and thus

$$E \sim_\mathbb{R} f^*(f_*G - D - g^*L).$$

Therefore $E = 0$ by the Negativity lemma [KM98, Lemma 3.39(1)], and consequently $f^*f_*G = G$. Hence $f^*|D/Z|_\mathbb{R} \supseteq |f^*D/Z|_\mathbb{R}$. This completes the proof. ☐

2

Generalized Pairs and their Singularities

In this chapter we present the fundamentals of the theory of generalized pairs. In order to give a glimpse of the topics that will be discussed below, we outline now the contents of each section of this chapter.

In Section 2.1 we define generalized pairs and the various classes of their singularities and we establish their basic properties. In particular, we investigate monotonicity properties of discrepancies, we study dlt generalized pairs and we establish the existence of certain useful modifications of generalized pairs, e.g., dlt blow-ups.

In Section 2.2 we define various types of models of generalized pairs, e.g., minimal models and Mori fiber spaces, both in the usual sense and in the sense of Birkar-Shokurov. In addition, we investigate how they are related to each other. Moreover, we deal with the problem of the existence of Mori fiber spaces for non-pseudo-effective generalized pairs.

In Section 2.3 we discuss thoroughly the Minimal Model Program (MMP) in the setting of generalized pairs. In particular, we recall the definitions of divisorial contractions and flips for generalized pairs, we discuss briefly the problem of their existence, we show how to run an MMP in this setting and, finally, we prove several results that will be utilized in the next chapters of the thesis.

We assume that the reader is familiar with the definitions and basic results concerning usual pairs and their singularities, and we refer to [KMM87, KM98, Mat02, HK10, Kol13, Fuj17] for further details in the usual setting. In particular, our main reference for the MMP in this setting is [KM98]. Additionally, the various results from [KM98] that are extended below to the context of generalized pairs are always mentioned explicitly for the convenience of the reader.

Furthermore, our main references for the material presented in this chapter are [BZ16, Section 4], where several aspects of the theory of generalized pairs were originally developed, [HL18, Sections 2 and 3], where dlt singularieties of generalized pairs were treated in detail for the first time, and various parts of our papers [LT19, LMT20, CT20], where several basic properties of generalized pairs were first supplemented, building mainly on the previous two works.

Last but not least, although we work throughout the thesis with $\mathbb{R}$-divisors, we have cited several results from [KM98], which are stated and proved only for $\mathbb{Q}$-divisors. However, all of them hold also for $\mathbb{R}$-divisors and the corresponding statements with essentially identical proofs can be found in [Fuj17]. A typical example is the Negativity lemma [KM98, Lemma 3.39]; its version for $\mathbb{R}$-divisors is [Fuj17, Lemma 2.3.26].

2.1 Definitions and Fundamental Properties

Definition 2.1. A *generalized pair*, abbreviated as *g-pair*, consists of

- a normal variety X, equipped with projective morphisms
$$X' \xrightarrow{f} X \longrightarrow Z,$$
where f is birational and X' is a normal variety,
- an effective $\mathbb{R}$-divisor B on X, and
- an $\mathbb{R}$-Cartier $\mathbb{R}$-divisor M' on X' which is nef over Z,

such that $K_X + B + M$ is $\mathbb{R}$-Cartier, where $M := f_* M'$.

We usually refer to a g-pair as above by saying that $(X, B+M)$ is a g-pair with data $X' \xrightarrow{f} X \to Z$ and M'. However, we often denote such a g-pair simply by $(X/Z, B+M)$, but remember the whole g-pair structure. Moreover, the divisor B (resp. M') is called *the boundary part* (resp. *the nef part*) of the g-pair.

Furthermore, we emphasize that the definition is flexible with respect to X' and M', namely, if $g\colon X'' \to X'$ is a projective birational morphism from a normal variety X'', then we may replace X' with X'' and M' with $g^* M'$, see [Bir19, §2.13(1)]. Therefore, the variety X' may always be chosen as a sufficiently high birational model of X; in practice, $f\colon X' \to X$ will usually be a log resolution of (X, B).

Finally, we recall that a g-pair $(X, B + M)$ with data $X \xrightarrow{\mathrm{id}_X} X \to Z$ and $M' = M$ is called a *polarized pair*, see [BH14]. If, additionally, $M' = M = 0$, then $(X/Z, B)$ is a usual *pair*. In other words, a pair $(X/Z, B)$ consists of a normal variety X, equipped with a projective morphism $X \to Z$, and an effective $\mathbb{R}$-divisor B on X such that K_X+B is $\mathbb{R}$-Cartier; and a polarized pair $(X/Z, B + P)$ consists of a usual pair $(X/Z, B)$ and a nef $\mathbb{R}$-Cartier $\mathbb{R}$-divisor P on X.

Definition 2.2. A g-pair $(X/Z, B + M)$ is called:

(a) *effective over* Z if there exists an effective $\mathbb{R}$-Cartier $\mathbb{R}$-divisor G on X such that $K_X + B + M \equiv_Z G$,

(b) *pseudo-effective over* Z if the divisor $K_X + B + M$ is pseudo-effective over Z, and *non-pseudo-effective over* Z otherwise,

(c) *NQC* if its nef part M' is an NQC divisor on X', i.e., $M' = \sum_{j=1}^{\ell} \mu_j M'_j$, where $\mu_j \in \mathbb{R}_{\geq 0}$ and the M'_j are $\mathbb{Q}$-Cartier divisors on X' which are nef over Z,

(d) *log smooth* if X is smooth, with data $X \xrightarrow{\mathrm{id}_X} X \to Z$ and $M' = M$, and if $B + M$ has SNC support,

(e) *uniruled* if X is uniruled, i.e., X is covered by rational curves, see [Deb01, Section 4.1].

Definition 2.3. Let $(X, B + M)$ be a g-pair with data $X' \xrightarrow{f} X \to Z$ and M'. Let E be a divisorial valuation over X. We may assume that E is a prime divisor on X'. We may write
$$K_{X'} + B' + M' \sim_{\mathbb{R}} f^*(K_X + B + M)$$

for some $\mathbb{R}$-divisor B' on X'. *The discrepancy of E with respect to* $(X, B+M)$[1] is defined as

$$a(E, X, B+M) := -\operatorname{mult}_E B'.$$

The next result generalizes [KM98, Lemma 2.27] to the context of g-pairs and shows that discrepancies possess a monotonicity property. Its proof is straightforward and therefore omitted.

Lemma 2.4. *With the same notation as in Definition 2.3, if Δ is an effective $\mathbb{R}$-Cartier $\mathbb{R}$-divisor on X, then*

$$a(E, X, B+M) = a(E, X, B+\Delta+M) + \operatorname{mult}_E f^*\Delta.$$

In particular,

$$a(E, X, B+M) \geq a(E, X, B+\Delta+M)$$

and the strict inequality holds if and only if $c_X(E) \subseteq \operatorname{Supp}\Delta$.

We define now the six classes of singularities that are most important for the MMP.

Definition 2.5. A g-pair $(X/Z, B+M)$ is called:

(a) *terminal* if $a(E, X, B+M) > 0$ for any exceptional divisorial valuation E over X,

(b) *canonical* if $a(E, X, B+M) \geq 0$ for any exceptional divisorial valuation E over X,

(c) *klt* if $a(E, X, B+M) > -1$ for any divisorial valuation E over X,

(d) *lc* if $a(E, X, B+M) \geq -1$ for any divisorial valuation E over X,

(e) *dlt*[2] if it is lc and if there exists a closed subset $V \subseteq X$ such that

 (1) $(X \setminus V, B|_{X\setminus V})$ is a log smooth pair, and
 (2) if $a(E, X, B+M) = -1$ for some divisorial valuation E over X, then $c_X(E) \nsubseteq V$ and $c_X(E) \setminus V$ is an lc center of $(X \setminus V, B|_{X\setminus V})$,

(f) *plt* if it is dlt and if $\lfloor B \rfloor$ is the disjoint union of its irreducible components.

[1] In the literature one often encounters the notion of *the log discrepancy of a divisorial valuation E with respect to the g-pair* $(X, B+M)$, which is defined as $a_\ell(E, X, B+M) := 1 + a(E, X, B+M)$. Throughout this thesis we use exclusively discrepancies.

[2] As insinuated in the introduction to this chapter, we have adopted the definition of dlt singularities from [HL18]. Another definition of dlt singularities is given in [Bir19]. The difference between [HL18, Definition 2.2] and the definition from [Bir19, §2.13(2)] is quite delicate. Specifically, according to the second one, M is nef in a neighborhood of the generic point of every lc center of $(X, B+M)$, whereas according to the first one this need not be the case, that is, $M|_{X\setminus V}$ is not necessarily nef. Therefore, if a g-pair is dlt according to Birkar's definition, then it is also dlt according to Han-Li's definition; indeed, the condition from Definition 2.5(e)(2) that "$c_X(E) \setminus V$ is an lc center of $(X \setminus V, B|_{X\setminus V})$" follows automatically from the nefness of M on $X \setminus V$, since it guarantees that

$$-1 = a(E, X, B+M) = a(E, X\setminus V, B|_{X\setminus V} + M|_{X\setminus V}) = a(E, X\setminus V, B|_{X\setminus V}).$$

However, a dlt g-pair according to Han-Li's definition need not be dlt according to Birkar's definition.

Recall that klt (resp. dlt, plt, lc) is the abbreviation of "Kawamata log terminal" (resp. "divisorial log terminal", "purely log terminal", "log canonical").

Note that the variety Z does not play any role in the definition of singularities of g-pairs. Indeed, singularities are local in nature over X, see [KM98, Section 2.3] or [Kol13, Section 2.1], so we may assume that $X \to Z$ is the identity map. This is why Z is suppressed in the notation.

Furthermore, if (X, B) is a usual pair, then by taking $M = 0$ in Definition 2.5 we recover the various definitions of singularities of (X, B); in particular, as far as dlt and plt singularities are concerned, see [KM98, Definition 2.37 and Proposition 5.51], respectively. If, additionally, $(X, B + P)$ is a polarized pair, then it follows readily from Definition 2.3 that P does not contribute to the singularities of the pair (see also Remark 2.7 below), that is, for any divisorial valuation E over X we have $a(E, X, B + P) = a(E, X, B)$, and thus $(X, B + P)$ is klt (lc and so forth) if and only if (X, B) is klt (lc and so forth). This fact will be used without explicit mention in the thesis.

Finally, it is worth mentioning that there is no difference between klt (resp. lc) polarized pairs and klt (resp. lc) generalized pairs in dimension two, but this is no longer true in higher dimensions, see [HL20b, Lemma 2.4 and Remark 2.5].

Comment. If in Definition 2.1 we drop the assumption that the boundary part is effective, then we obtain the notion of a *generalized sub-pair* (or *g-sub-pair* for short), and accordingly the notions of usual *sub-pairs* and *polarized sub-pairs*. Moreover, we may define the *discrepancy* of a divisorial valuation with respect to a g-sub-pair exactly as in Definition 2.3. In particular, we obtain the notions of *klt* and *lc g-sub-pairs* as in Definition 2.5. We note that all these notions occur frequently in proofs – many definitions and results in this chapter could actually have been stated for g-sub-pairs instead, as is done, for instance, in [KM98] – and a typical example, where all the concepts under discussion occur and combine, is the following:

Let $(X, B+M)$ be a klt (resp. lc) g-pair with data $X' \xrightarrow{f} X \to Z$ and M'. We may assume that f is a log resolution of (X, B) and we may write

$$K_{X'} + B' + M' \sim_{\mathbb{R}} f^*(K_X + B + M)$$

for some $\mathbb{R}$-divisor B' on X' with coefficients < 1 (resp. ≤ 1). Since (X', B') is a log smooth klt (resp. lc) sub-pair by [KM98, Corollary 2.31(3)], we conclude that $(X'/Z, B' + M')$ is a klt (resp. lc) polarized sub-pair.

Remark 2.6. Let $(X, B+M)$ be a g-pair with data $X' \xrightarrow{f} X \to Z$ and M'. If f is a log resolution of (X, B), then the g-pair $(X, B+M)$ is klt (resp. lc) if and only if the coefficients of B' are < 1 (resp. ≤ 1).

Remark 2.7. Let $(X, B+M)$ be a g-pair with data $X' \xrightarrow{f} X \to Z$ and M'. Note that the divisor $K_X + B$ is $\mathbb{R}$-Cartier if and only if the divisor M is $\mathbb{R}$-Cartier. In this case, (X, B) is a pair in the usual sense. Moreover, we may write $K_{X'} + \widetilde{B} \sim_{\mathbb{R}} f^*(K_X + B)$ and $f^*M = M' + F$, where F is an f-exceptional $\mathbb{R}$-divisor. By the Negativity lemma [KM98, Lemma 3.39(1)] we infer that $F \geq 0$. Hence, $K_{X'}+B'+M' \sim_{\mathbb{R}} f^*(K_X+B+M)$, where $B' := \widetilde{B} + F$ and $B' \geq \widetilde{B}$. Consequently, if E is a divisorial valuation over X, then $a(E, X, B + M) = a(E, X, B) - \operatorname{mult}_E F$, and thus

$$a(E, X, B + M) \leq a(E, X, B).$$

Hence, if the g-pair $(X, B+M)$ is terminal (resp. canonical, klt, dlt, plt, lc) and X is $\mathbb{Q}$-factorial, then the underlying pair (X, B) is also terminal (resp. canonical, klt, dlt, plt, lc).

In particular, if $(X, B+M)$ is a $\mathbb{Q}$-factorial terminal g-pair, then $\lfloor B \rfloor = 0$ and X is smooth in codimension two by [KM98, Theorem 4.5 and Corollary 5.18].

Finally, recall that a normal variety V has *rational singularities* if there exists a resolution $\varphi\colon W \to V$ of V such that $R^i\varphi_*\mathcal{O}_W = 0$ for all $i > 0$, see [KM98, Definition 5.8 and Theorem 5.10]. It follows from the above and from [KM98, Theorem 5.22] that the underlying variety X of a $\mathbb{Q}$-factorial dlt g-pair $(X, B+M)$ has rational singularities. For further information about rational singularities we refer to [KM98, Section 5.1], [Kol13, Section 2.5] and [Fuj16].

Definition 2.8. Let $(X, B+M)$ be an lc g-pair with data $X' \xrightarrow{f} X \to Z$ and M'. Let P' be an $\mathbb{R}$-divisor on X' which is nef over Z and let N be an effective $\mathbb{R}$-divisor on X such that $P+N$ is $\mathbb{R}$-Cartier, where $P := f_*P'$. *The lc threshold of $P+N$ with respect to* $(X, B+M)$ is defined as

$$\sup\left\{t \in \mathbb{R}_{\geq 0} \mid \big(X, (B+tN) + (M+tP)\big) \text{ is lc}\right\},$$

where the g-pair in the definition has boundary part $B+tN$ and nef part $M'+tP'$.

2.1.1 The Non-KLT Locus of a Generalized Pair

Definition 2.9. Let $(X/Z, B+M)$ be an lc g-pair.

(a) A subvariety S of X is called an *lc center of* X if there exists a divisorial valuation E over X such that $a(E, X, B+M) = -1$ and $c_X(E) = S$.

(b) The union of all lc centers of X is denoted by $\mathrm{Nklt}(X, B+M)$ and is called *the non-klt locus of* $(X, B+M)$.

We refer to [Kol13, Chapter 4] for a thorough discussion about the non-klt locus of a usual pair. We remark, though, that the lc centers of a log smooth pair (X, B) are the *strata* of $\lfloor B \rfloor = \sum D_i$, i.e., the irreducible components of the various intersections $\bigcap D_{i_\ell}$ of its irreducible components D_i, see [Kol13, Last Paragraph of Definition 4.15]. In addition, we reproduce below [Fuj07, Proposition 3.9.2], which describes the lc centers and adjunction for (higher-codimensional) lc centers of a dlt pair. All these results will be used without explicit mention in the thesis.

Proposition 2.10. *Let (X, B) be a dlt pair.*

(i) *Let $\lfloor B \rfloor = \sum_{i \in I} D_i$ be the decomposition of $\lfloor B \rfloor$ into irreducible components. Then S is an lc center of (X, B) with $\operatorname{codim}_X S = k$ if and only if S is an irreducible component of $D_{i_1} \cap \ldots \cap D_{i_k}$ for some $\{i_1, \ldots, i_k\} \subseteq I$.*

(ii) *Let S be an lc center of (X, B). Then S is normal and we obtain a dlt pair (S, B_S) by* adjunction*, i.e., by the formula $K_S + B_S = (K_X + B)|_S$.*

We gather below the basic properties of the non-klt locus of a g-pair, which will play a significant role in Chapters 5 and 6. Moreover, we refer to [Sva19, Bir20, FS20] for recent developments regarding the properties of the non-klt locus of a g-pair with focus on the so-called *connectedness principle*, and to [KM98, Section 5.4] and [Kol13, Section 4.4] for a relevant discussion in the context of usual pairs.

Remark 2.11.

(1) If $(X, B+M)$ is a $\mathbb{Q}$-factorial dlt g-pair, then by definition and by Remark 2.7 the underlying pair (X, B) is also $\mathbb{Q}$-factorial dlt and the lc centers of $(X, B+M)$ coincide with those of (X, B). In particular, by Proposition 2.10(i) we obtain

$$\operatorname{Nklt}(X, B+M) = \operatorname{Nklt}(X, B) = \operatorname{Supp}\lfloor B \rfloor.$$

(2) If $(X, B+M)$ is an lc g-pair and if $h\colon (Y, \Delta+\Xi) \to (X, B+M)$ is a dlt blow-up of $(X, B+M)$ (see Lemma 2.24(iii)), then by Lemma 2.13 and by (1) we deduce that

$$\operatorname{Nklt}(X, B+M) = h\big(\operatorname{Nklt}(Y, \Delta+\Xi)\big) = h\big(\operatorname{Supp}\lfloor \Delta \rfloor\big).$$

In particular, the number of lc centers of an lc g-pair is finite.

2.1.2 Computing and Comparing Discrepancies

We prove below analogs of [KM98, Lemma 2.29, Lemma 2.30, Corollary 2.35(1) and Proposition 2.36(2)], respectively, in the context of g-pairs. These results are taken from [CT20, Section 2.1], except for Lemma 2.13, which has been included here for the sake of completeness.

Lemma 2.12. *Let $(X, B+M)$ be an NQC g-pair with data $X' \to X \to Z$ and $M' = \sum_{j=1}^{l} \mu_j M_j'$, where $B = \sum_{i=1}^{s} b_i B_i$ and the B_i are distinct prime divisors, $\mu_j \geq 0$ and the M_j' are $\mathbb{Q}$-Cartier divisors which are nef over Z. Assume that X is $\mathbb{Q}$-factorial and smooth near a codimension $k \geq 2$ closed subvariety V of X. Consider the blow-up of X along V and let E be the irreducible component of the exceptional divisor that dominates V. Then*

$$a(E, X, B+M) = k - 1 - \sum_{i=1}^{s} n_i b_i - \sum_{j=1}^{l} m_j \mu_j$$

for some non-negative integers $n_1, \dots, n_s, m_1, \dots, m_l$.

Proof. We may assume that E is a prime divisor on X'. By assumption and by [KM98, Lemma 2.29] we obtain $a(E, X, B) = k - 1 - \sum_{i=1}^{s} n_i b_i$ for some non-negative integers $n_1, \dots, n_s$; note that $n_i = \operatorname{mult}_V B_i$. Furthermore, by assumption and by Remark 2.7 we have $\operatorname{mult}_E(f^*M - M') = \sum_{j=1}^{l} m_j \mu_j$ for some non-negative integers $m_1, \dots, m_l$. Since

$$a(E, X, B+M) = a(E, X, B) - \operatorname{mult}_E(f^*M - M')$$

by Remark 2.7, we obtain the statement. □

Lemma 2.13. *Let $(X, B+M)$ be a g-sub-pair with data $X' \xrightarrow{f} X \to Z$ and M'. Let $h\colon Y \to X$ be a proper birational morphism from a normal variety Y and let $(Y, \Delta+\Xi)$ be a g-sub-pair with data $X' \xrightarrow{g} Y \to Z$ and M' such that*

$$K_Y + \Delta + \Xi \sim_{\mathbb{R}} h^*(K_X + B + M) \quad \text{and} \quad h_*\Delta = B.$$

Then, for any divisorial valuation E over X, we have

$$a(E, Y, \Delta+\Xi) = a(E, X, B+M).$$

Proof. Let E be a divisorial valuation over X. By replacing X' with a higher model, we may assume that E is a prime divisor on X' and that $f = h \circ g$. We may write

$$K_{X'} + B' + M' \sim_{\mathbb{R}} f^*(K_X + B + M) \tag{2.1}$$

for some $\mathbb{R}$-divisor B' on X' such that $f_*B' = B$, and

$$K_{X'} + \Delta' + M' \sim_{\mathbb{R}} g^*(K_Y + \Delta + \Xi) \tag{2.2}$$

for some $\mathbb{R}$-divisor Δ' on X' such that $g_*\Delta' = \Delta$. Set $F := B' - \Delta'$ and observe that

$$\operatorname{mult}_E F = a(E, Y, \Delta + \Xi) - a(E, X, B + M).$$

Moreover, $f_*F = f_*B' - f_*\Delta' = B - h_*\Delta = 0$, so F is an f-exceptional divisor. We claim that $F = 0$, which clearly yields the statement. Indeed, by the assumptions we obtain

$$f^*(K_X + B + M) \sim_{\mathbb{R}} g^*(K_Y + \Delta + \Xi)$$

and it follows now from (2.1) and (2.2) that $B' \sim_{\mathbb{R}} \Delta'$, hence $F \sim_{\mathbb{R}} 0$. By the Negativity lemma [KM98, Lemma 3.39(1)] we infer that $F = 0$. This completes the proof. □

Lemma 2.14. *Let $\big(X, (B+N) + (M+P)\big)$ be a $\mathbb{Q}$-factorial klt (resp. lc) g-pair with data $X' \xrightarrow{f} X \to Z$ and $M' + P'$. Then the g-pair $(X, B + M)$ is also klt (resp. lc).*

Proof. We may assume that f is a log resolution of $(X, B + N)$ and we may write

$$K_{X'} + B' + M' \sim_{\mathbb{R}} f^*(K_X + B + M)$$

for some $\mathbb{R}$-divisor B' on X'. We may also write

$$f^*P = P' + E_1 \quad \text{and} \quad f^*N = f_*^{-1}N + E_2,$$

where E_1 is an effective f-exceptional $\mathbb{R}$-divisor by the Negativity lemma [KM98, Lemma 3.39(1)], as P' is f-nef, and E_2 is an effective f-exceptional $\mathbb{R}$-divisor, as $N \geq 0$. Therefore,

$$K_{X'} + (B' + f_*^{-1}N + E_1 + E_2) + (M' + P') \sim_{\mathbb{R}} f^*\big(K_X + (B+N) + (M+P)\big).$$

Since the g-pair $\big(X, (B+N) + (M+P)\big)$ is klt (resp. lc), the coefficients of the divisor $B' + f_*^{-1}N + E_1 + E_2$ are < 1 (resp. ≤ 1). The statement follows from Remark 2.6. □

Proposition 2.15. *Let $(X, B + M)$ be a klt g-pair with data $X' \xrightarrow{f} X \to Z$ and M'. Then there exists a positive real number ε such that there are only finitely many exceptional divisorial valuations E over X with discrepancy $a(E, X, B + M) < \varepsilon$.*

Proof. We may assume that f is a log resolution of (X, B) such that if we write

$$K_{X'} + B' + M' \sim_{\mathbb{R}} f^*(K_X + B + M)$$

for some $\mathbb{R}$-divisor $B' = (B')^+ - (B')^-$ on X' with coefficients < 1, where the divisors $(B')^+$ and $(B')^-$ are effective and have no common components, then $\operatorname{Supp}(B')^+$ is smooth, cf. [KM98, Proposition 2.36(1)]. We write $(B')^+ = \sum_{i=1}^r \gamma_i G_i$, where the G_i are distinct prime divisors on X' and $\gamma_i \in (0,1)$ for every $1 \leq i \leq r$, and we set $\varepsilon := 1 - \max\{\gamma_i \mid 1 \leq i \leq r\} > 0$. If F is an exceptional divisorial valuation over X', then by Lemma 2.13 and by [KM98, Lemma 2.27 and Corollary 2.31(3)] we obtain

$$a(F, X, B + M) = a(F, X', B' + M') = a(F, X', B') \geq a\big(F, X', (B')^+\big) \geq \varepsilon.$$

This yields the statement. □

Remark 2.16. If $(X, B+M)$ is a terminal g-pair and if we write $B = \sum_{i=1}^{s} b_i B_i$, where $b_i \in [0,1)$ and the B_i are distinct prime divisors on X, then we may take $\varepsilon := 1 - \max\{b_i \mid 1 \leq i \leq s\}$ in Proposition 2.15, since the divisor $(B')^+$ in the previous proof is precisely the strict transform of B on X'. This observation will play a key role in the proof of Lemma 6.2.

Monotonicity of Discrepancies

The following result is [LMT20, Lemma 2.9]. It allows us to compare discrepancies in several different settings. Parts (i) and (iii) are analogs of [KM98, Proposition 3.51 and Lemma 3.38], respectively, in the context of g-pairs. Note that Lemmas 2.36, 2.44 and 2.45 constitute their main applications. Parts (ii) and (iv) will only be needed in Chapter 5, when we will deal with a version of the so-called *difficulty* of g-pairs. The following notation will be used in these two parts (and thus in Chapter 5 as well).

Notation 2.17. Let $(X, B+M)$ be a dlt g-pair and let S be an lc center of $(X, B+M)$. We define a dlt g-pair $(S, B_S + M_S)$ by *adjunction*, i.e., by the formula

$$K_S + B_S + M_S = (K_X + B + M)|_S$$

as in [HL18, Proposition 2.8]. We note that S is normal by Proposition 2.23.

Lemma 2.18. *Let $(X, B+M)$ and $(X', B'+M')$ be g-pairs such that there exists a diagram*

$$\begin{array}{ccccc} & & W & & \\ & \swarrow^{g} & & \searrow^{g'} & \\ X & & \overset{\varphi}{\dashrightarrow} & & X' \\ & \searrow_{f} & & \swarrow_{f'} & \\ & & Y, & & \end{array}$$

where Y and W are normal varieties, all morphisms are proper birational, $K_{X'}+B'+M'$ is f'-nef, and there exists a nef $\mathbb{R}$-Cartier $\mathbb{R}$-divisor M_W on W with $g_* M_W = M$ and $g'_* M_W = M'$.

(i) *Assume that $B' = \varphi_* B + E$, where E is the sum of all the φ^{-1}-exceptional prime divisors on X', and that*

$$a(F, X, B+M) \leq a(F, X', B'+M')$$

for every φ-exceptional prime divisor F on X. Then for any divisorial valuation F over X we have

$$a(F, X, B+M) \leq a(F, X', B'+M').$$

(ii) *Under the assumptions of (i), assume additionally that $(X, B+M)$ is dlt and let S be an lc center of $(X, B+M)$. Assume that φ is an isomorphism at the generic point of S and define S' as the strict transform of S on X'. Then for any divisorial valuation F over S we have*

$$a(F, S, B_S + M_S) \leq a(F, S', B'_{S'} + M'_{S'}).$$

(iii) *Assume that* $-(K_X + B + M)$ *is* f*-nef and that* $f_*B \geq f'_*B'$. *Then for any divisorial valuation* F *over* Y *we have*

$$a(F, X, B + M) \leq a(F, X', B' + M'),$$

and the strict inequality holds if either

(a) $-(K_X + B + M)$ *is* f*-ample and* f *is not an isomorphism above the generic point of* $c_Y(F)$*, or*

(b) $K_{X'} + B' + M'$ *is* f'*-ample and* f' *is not an isomorphism above the generic point of* $c_Y(F)$.

In particular, if $(X, B + M)$ *is lc and if either (a) or (b) holds, then* $c_{X'}(F)$ *is not an lc center of the g-pair* $(X', B' + M')$.

(iv) *Assume that* $-(K_X + B + M)$ *is* f*-nef, that* $f_*B = f'_*B'$ *and that* $(X, B + M)$ *is dlt. Let* S *be an lc center of* $(X, B + M)$. *Assume that* φ *is an isomorphism at the generic point of* S *and define* S' *as the strict transform of* S *on* X'. *Let* T *be the normalization of* $f(S)$*, so that we have the following diagram:*

$$\begin{array}{ccc} (S, B_S + M_S) & \overset{\varphi|_S}{\dashrightarrow} & (S', B'_{S'} + M'_{S'}) \\ & {\scriptstyle f|_S} \searrow \quad \swarrow {\scriptstyle f'|_{S'}} & \\ & T & \end{array}$$

Then for any divisorial valuation F *over* T *we have*

$$a(F, S, B_S + M_S) \leq a(F, S', B'_{S'} + M'_{S'}),$$

and the strict inequality holds if either

(a) $-(K_X + B + M)$ *is* f*-ample and* $f|_S$ *is not an isomorphism above the generic point of* $c_T(F)$*, or*

(b) $K_{X'} + B' + M'$ *is* f'*-ample and* $f'|_{S'}$ *is not an isomorphism above the generic point of* $c_T(F)$.

In particular, if either (a) or (b) holds, then $c_{S'}(F)$ *is not an lc center of the g-pair* $(S', B'_{S'} + M'_{S'})$.

Proof. The proofs of (i) and (iii) are similar to the proofs of [KM98, Proposition 3.51 and Lemma 3.38], respectively. Nevertheless, we provide the details for the benefit of the reader.

By possibly replacing W with a higher model, we may assume that $c_W(F)$ is a divisor on W. Set $h := f \circ g = f' \circ g'$. Then

$$\begin{aligned} K_W + M_W &\sim_{\mathbb{R}} g^*(K_X + B + M) + \sum a(F_i, X, B + M) F_i \\ &\sim_{\mathbb{R}} (g')^*(K_{X'} + B' + M') + \sum a(F_i, X', B' + M') F_i. \end{aligned}$$

Consider the $\mathbb{R}$-Cartier $\mathbb{R}$-divisor

$$\begin{aligned} H &:= \sum \big(a(F_i, X, B + M) - a(F_i, X', B' + M')\big) F_i \\ &\sim_{\mathbb{R}} (g')^*(K_{X'} + B' + M') - g^*(K_X + B + M). \end{aligned} \tag{2.3}$$

Under the assumptions of (i) the divisor H is g-nef and $g_*H \leq 0$, hence $H \leq 0$ by the Negativity lemma [KM98, Lemma 3.39(1)]. Under the assumptions of (iii) the divisor H is h-nef and $h_*H = f'_*B' - f_*B \leq 0$, hence $H \leq 0$ by the Negativity lemma [KM98, Lemma 3.39(1)]. This yields (i) and the first part of (iii).

Now, assume that the case (a) of (iii) holds; we argue similarly if (b) holds. Since f is not an isomorphism above the generic point η of $c_Y(F)$, we may find a curve $C_X \subseteq f^{-1}(\eta)$, and therefore a curve $C_Z \subseteq h^{-1}(\eta)$ such that $g(C_Z) = C_X$. By the assumptions and by (2.3) we obtain $H \cdot C_Z > 0$, and since $-H \geq 0$, we deduce that $C_Z \subseteq \operatorname{Supp} H$. Then [KM98, Lemma 3.39(2)] implies that $h^{-1}(\eta) \subseteq \operatorname{Supp} H$, and thus $F \subseteq \operatorname{Supp} H$. This yields the second statement of (iii).

For (ii) and (iv), by [KM98, Lemma 2.45] there is a sequence of blow-ups of S along the centers of F such that the center of F becomes a divisor. By considering these blow-ups as blow-ups of X and possibly blowing up further, we may assume that, on W, the center $c_{S_W}(F)$ is a divisor, where S_W is the strict transform of S on W, and that there exist finitely many prime divisors $\widehat{F}_i$ on W such that $\widehat{F}_i|_{S_W} = c_{S_W}(F)$ and $c_T(F) = c_Y\big(\widehat{F}_i\big)$ for each such $\widehat{F}_i$. Then (ii) follows by restricting (2.3) to S_W; indeed, we know that $H \leq 0$ and, taking [KM98, Remark 2.23] into account, we note that $S_W \nsubseteq \operatorname{Supp} H$, since φ is an isomorphism at the generic point of S.

Now, to prove (iv)(a), we observe that f is not an isomorphism above the generic point of each $c_Y\big(\widehat{F}_i\big)$, so (iv)(a) follows from (iii) (see the proof of (ii) above). We obtain (iv)(b) similarly, by first blowing up along the centers of F on S' instead. □

2.1.3 Basic Properties of DLT Generalized Pairs

We now turn our attention to dlt g-pairs. In particular, we generalize below [KM98, Proposition 2.41, Proposition 5.51, Proposition 2.40, Proposition 2.43 and Corollary 5.52], respectively, to the setting of g-pairs.

Proposition 2.19. *A dlt g-pair $(X, B+M)$ is klt if and only if $\lfloor B \rfloor = 0$.*

Proof. If $(X, B+M)$ is klt, then it is clearly dlt with $V = X$ in Definition 2.5(e). Conversely, assume that $\lfloor B \rfloor = 0$. Let E be a divisorial valuation over X and let $V \subseteq X$ be the closed subset from Definition 2.5(e). If $c_X(E) \subseteq V$, then we have $a(E, X, B+M) > -1$ by definition. If $c_X(E) \nsubseteq V$, then again $a(E, X, B+M) > -1$, since we would otherwise have $a(E, X, B+M) = -1$, so $c_X(E) \setminus V$ would be an lc center of $(X \setminus V, B|_{X \setminus V})$ by the definition of dlt, and it would therefore hold that $c_X(E) \setminus V \subseteq \operatorname{Supp}\lfloor B|_{X\setminus V} \rfloor$, which contradicts our assumption that $\lfloor B \rfloor = 0$. In conclusion, $(X, B+M)$ is klt. □

Proposition 2.20. *Let $(X, B+M)$ be a dlt g-pair. The following are equivalent:*

(i) *$(X, B+M)$ is plt,*

(ii) *for every exceptional divisorial valuation E over X we have $a(E, X, B+M) > -1$,*

(iii) *$\lfloor B \rfloor$ is normal.*

Proof.
(i) $\Longrightarrow$ (ii): Let E be an exceptional divisorial valuation over X and let $V \subseteq X$ be the closed subset from Definition 2.5(e). If $c_X(E) \subseteq V$, then $a(E, X, B+M) > -1$ by the definition of dlt. If $c_X(E) \nsubseteq V$, then again $a(E, X, B+M) > -1$; indeed, we

would otherwise have $a(E, X, B + M) = -1$, so $c_X(E) \setminus V$ would be an lc center of $(X \setminus V, B|_{X\setminus V})$ by the definition of dlt, but then, since the irreducible components of $\lfloor B \rfloor$ do not meet by assumption, there would exist an irreducible component D of $\lfloor B \rfloor$ such that ($D \nsubseteq V$ by the definition of dlt and) $c_X(F) \setminus V = D \setminus V$, which is absurd, as $\operatorname{codim}_X E \geq 2$. This proves (ii).

(ii) $\implies$ (iii): We may repeat verbatim the arguments from [KM98, Proposition 5.51, Proof of (1) $\implies$ (2)] to deduce that $\lfloor B \rfloor$ is normal.

(iii) $\implies$ (i): Since $\lfloor B \rfloor$ is normal, its irreducible components do not intersect, and thus $\lfloor B \rfloor$ is the disjoint union of its irreducible components. This shows (i). □

Lemma 2.21. *Let $(X, B + M)$ be a g-pair with data $X' \xrightarrow{f} X \to Z$ and M'. The following are equivalent:*

(i) *$(X, B + M)$ is dlt,*

(ii) *$(X, B + M)$ is lc and there exists a closed subset $V \subseteq X$ such that*

   (1) *$(X \setminus V, B|_{X\setminus V})$ is a log smooth pair, and*

   (2) *after possibly replacing X' with a higher model, f is a log resolution of (X, B) such that $f^{-1}(V)$ has pure codimension 1 and if $a(F, X, B + M) = -1$ for some prime divisor F on X', then $F \nsubseteq f^{-1}(V)$ (or equivalently, $c_X(F) \nsubseteq V$) and $c_X(F) \setminus V$ is an lc center of $(X \setminus V, B|_{X\setminus V})$.*

Proof. First of all, for any divisorial valuation E over X, it is easy to see that

$$c_{X'}(E) \nsubseteq f^{-1}(V) \iff c_X(E) \nsubseteq V,$$

since $c_X(E) = f\big(c_{X'}(E)\big)$ and since f is surjective. Due to the above equivalence it is obvious that (i) implies (ii), so we prove below the converse implication. Clearly, by assumption it remains to show that if $a(E, X, B + M) = -1$ for some divisorial valuation E, then $c_X(E) \nsubseteq V$ and $c_X(E) \setminus V$ is an lc center of $(X \setminus V, B|_{X\setminus V})$.

Let E be a divisorial valuation over X such that $a(E, X, B + M) = -1$. We may write

$$K_{X'} + B' + M' \sim_{\mathbb{R}} f^*(K_X + B + M)$$

for some $\mathbb{R}$-divisor B' on X' with coefficients ≤ 1. If Δ' is an effective $\mathbb{R}$-divisor on X' such that $\operatorname{Supp}(\Delta') = f^{-1}(V)$, then every irreducible component of Δ' has coefficient < 1 in B' by (ii)(2). Thus, we may find a sufficiently small $\varepsilon > 0$ such that every irreducible component of $B' + \varepsilon\Delta'$ has coefficient ≤ 1, and hence $(X', B' + \varepsilon\Delta')$ is a log smooth lc sub-pair by [KM98, Corollary 2.31(3)].

Furthermore, we observe that $c_X(E) \nsubseteq V$, since otherwise it would hold that $c_{X'}(E) \subseteq f^{-1}(V) = \operatorname{Supp}(\Delta')$, and thus by Lemma 2.13 and by [KM98, Lemma 2.27] we would obtain

$$-1 = a(E, X, B + M) = a(E, X', B' + M') = a(E, X', B') > a(E, X', B' + \varepsilon\Delta') \geq -1,$$

a contradiction.

Now, if $\lfloor B \rfloor = \sum_{i=1}^r \Delta_i$, then $\lfloor B' \rfloor = \sum_{i=1}^r f_*^{-1}\Delta_i + \sum_{j=1}^s F_j = \sum_{k=1}^{r+s} G_k$, where the F_j are f-exceptional prime divisors. For every $k \in \{1, \dots, r+s\}$, by (ii)(2) we infer that $G_k \nsubseteq f^{-1}(V)$ and that $f(G_k) \setminus V$ is an lc center of the log smooth pair $(X \setminus V, B|_{X\setminus V})$.

Moreover, the above (valid by Lemma 2.13) equalities of discrepancies show that $c_{X'}(E)$ is an lc center of the log smooth lc sub-pair (X', B'). It follows that there is a subset $K \subseteq \{1, \dots, r+s\}$ such that $c_{X'}(E)$ is an irreducible component of $\bigcap_{k\in K} G_k$, and that for every $k \in K$ there is a subset $J_k \subseteq \{1, \dots, r\}$ such that $f(G_k) \setminus V$ is an irreducible component of $\bigcap_{i\in J_k} \big(\Delta_i \setminus V\big)$. Therefore, there is a subset $I \subseteq \{1, \dots, r\}$ such that $c_X(E) \setminus V$ is an irreducible component of $\bigcap_{i\in I} \big(\Delta_i \setminus V\big)$, and thus $c_X(E) \setminus V$ is an lc center of $(X \setminus V, B|_{X\setminus V})$. This concludes the proof. □

Proposition 2.22. *Let $(X, B+M)$ be a dlt g-pair with data $X' \xrightarrow{f} X \to Z$ and M'. Assume that X is quasi-projective and let H be an ample divisor on X.*[3] *Let Δ be an effective $\mathbb{Q}$-divisor (not necessarily $\mathbb{Q}$-Cartier) such that $B - \Delta \geq 0$. Then there exist a positive rational number c and an effective $\mathbb{Q}$-divisor $D \equiv \Delta + cH$ such that $(X, B - \varepsilon\Delta + \varepsilon D + M)$ is dlt for every real number $0 < \varepsilon \ll 1$.*

Proof. Choose integers $m, n \geq 1$ such that $m\Delta$ is a $\mathbb{Z}$-divisor and the sheaf $\mathcal{O}(m\Delta+nH)$ is globally generated. Let $D' \in |m\Delta+nH|$ be a general member (that does not contain any lc center of $(X, B+M)$) and set $D := \frac{1}{m}D'$. Since $-m\Delta + mD \sim nH$ is Cartier, the divisor $-\Delta + D$ is $\mathbb{Q}$-Cartier, and therefore the divisor $K_X + B - \varepsilon\Delta + \varepsilon D + M$ is $\mathbb{R}$-Cartier for any $\varepsilon \geq 0$.

Let $V \subseteq X$ be the closed subset from Definition 2.5(e). Then $m\Delta$ is Cartier on $X\setminus V$, thus $|m\Delta+nH|$ is base-point-free on $X\setminus Z$. Since D' is a general member of this linear system, the divisor $B+D$ has SNC support on $X\setminus V$ by [Laz04, Lemma 9.1.9], and since $\operatorname{Supp}\Delta \subseteq \operatorname{Supp} B$ by assumption, we conclude that $\big(X \setminus V, (B - \varepsilon\Delta + \varepsilon D)|_{X\setminus V}\big)$ is log smooth for any $\varepsilon \geq 0$.

Now, we will show that $(X, B - \varepsilon\Delta + \varepsilon D + M)$ is lc for sufficiently small $\varepsilon > 0$. Possibly replacing X' with a higher model, we may assume that f is a log resolution of $(X, B+D)$, and thus of $(X, B - \varepsilon\Delta + \varepsilon D)$. Since $(X, B+M)$ is lc, we may write

$$K_{X'} + B' + M' \sim_{\mathbb{R}} f^*(K_X + B + M)$$

for some $\mathbb{R}$-divisor B' on X' with coefficients ≤ 1. Moreover, since $-\Delta+D$ is $\mathbb{Q}$-Cartier, given $\varepsilon > 0$ we may add $\varepsilon f^*(-\Delta + D)$ to both sides of the above relation to obtain

$$K_{X'} + \Gamma' + M' \sim_{\mathbb{R}} f^*(K_X + B - \varepsilon\Delta + \varepsilon D + M),$$

where $\Gamma' := B' + f^*(-\varepsilon\Delta + \varepsilon D)$. We write $B = \sum b_i B_i$ and $\Delta = \sum \delta_i B_i$, where $0 \leq \delta_i \leq b_i \leq 1$ by assumption. Let E_j be the f-exceptional prime divisors. Then

$$B' = \sum b_i f_*^{-1} B_i - \sum a(E_j, X, B+M) E_j,$$

where $a(E_j, X, B+M) \geq -1$, and

$$\varepsilon f^*(-\Delta + D) = \sum (-\varepsilon\delta_i) f_*^{-1} B_i + \varepsilon f_*^{-1} D + \sum \varepsilon e_j E_j, \tag{2.4}$$

where $e_j \in \mathbb{R}$. Therefore,

$$\Gamma' = \sum \big(b_i - \varepsilon\delta_i\big) f_*^{-1} B_i + \varepsilon f_*^{-1} D + \sum \big(\varepsilon e_j - a(E_j, X, B+M)\big) E_j. \tag{2.5}$$

[3] We refer to [Laz04, Proof of Theorem 4.1.10, Footnote 5] for the precise meaning of ampleness in this setting.

To prove the assertion, by Remark 2.6 it suffices to show that for sufficiently small $\varepsilon > 0$ all the coefficients of Γ' are ≤ 1. Suppose that there exists a prime divisor G on X' such that $\operatorname{mult}_G \Gamma' = -a(G, X, B - \varepsilon\Delta + \varepsilon D + M) > 1$. By (2.5) and for $0 < \varepsilon \ll 1$, we deduce that G is an f-exceptional prime divisor $G = E_{j_0}$ with $a(E_{j_0}, X, B + M) = -1$ and $e_{j_0} > 0$. On the one hand, $c_X(E_{j_0}) \nsubseteq V$ by the definition of dlt. On the other hand, by (2.4) we infer that $E_{j_0} \subseteq \operatorname{Supp} f^*(-\Delta + D) = f^{-1}\big(\operatorname{Supp}(\Delta + D)\big)$, and hence $c_X(E_{j_0}) = f(E_{j_0}) \subseteq \operatorname{Supp}\Delta$, since $c_X(E_{j_0}) \nsubseteq \operatorname{Supp} D$ by the choice of D. Hence, setting $U := X \setminus V$ and $U' := f^{-1}(U)$ to simplify the following formulas, we obtain

$$\begin{aligned}
-1 &= a(E_{j_0}, X, B + M) = a(E_{j_0}, U, (B + M)|_U) \\
&= a\big(E_{j_0}, U, (B - \varepsilon\Delta + \varepsilon D + M)|_U\big) - \varepsilon \operatorname{mult}_{E_{j_0}} \big((f^*\Delta)|_{U'}\big) + \varepsilon \operatorname{mult}_{E_{j_0}} \big((f^*D)|_{U'}\big) \\
&= a(E_{j_0}, X, B - \varepsilon\Delta + \varepsilon D + M) - \varepsilon \operatorname{mult}_{E_{j_0}} \big(f^*\Delta|_{U'}\big) \\
&< -1,
\end{aligned}$$

which is a contradiction, and thus such a divisor $G = E_{j_0}$ does not exist. By (2.5) we conclude that $(X, B - \varepsilon\Delta + \varepsilon D + M)$ is lc for any $0 < \varepsilon \ll 1$.

According to Lemma 2.21, to prove the statement, it remains to show that, for any prime divisor F on X' with $a(F, X, B - \varepsilon\Delta + \varepsilon D + M) = -1$, we have that $F \nsubseteq f^{-1}(V)$ and that $c_X(F) \setminus V$ is an lc center of $\big(X \setminus V, (B - \varepsilon\Delta + \varepsilon D)|_{X\setminus V}\big)$. Fix such an F. By (2.5) we infer that there are only two possibilities for F: either $F = f_*^{-1} B_i$ for some i such that $b_i = -a(F, X, B + M) = 1$ and $\delta_i = 0$ (that is, F is a component of $f_*^{-1}\lfloor B \rfloor$ which is not a component of $f_*^{-1}\Delta$) or $F = E_j$ for some j such that $a(E_j, X, B + M) = -1$ and $e_j = 0$ (and thus F is not a component of $f^*(-\Delta + D)$ by (2.4)). Since in any case it holds that $a(F, X, B + M) = -1$, it follows that $F \nsubseteq f^{-1}(V)$, that $c_X(F) \setminus V$ is an lc center of $(X \setminus V, B|_{X\setminus V})$, and that $c_X(F) \nsubseteq \operatorname{Supp} D$ by the choice of D. Additionally, we have that $c_X(F) \nsubseteq \operatorname{Supp}\Delta$; indeed, this is clear if F is not f-exceptional, whereas if F is f-exceptional, then $F \nsubseteq \operatorname{Supp}\big(f^*(-\Delta + D)\big) = f^{-1}\big(\operatorname{Supp}(\Delta + D)\big)$, and hence $c_X(F) = f(F) \nsubseteq \operatorname{Supp}(\Delta + D) = \operatorname{Supp}\Delta \cup \operatorname{Supp} D$. By all the above, we deduce that

$$\begin{aligned}
-1 &= a(F, U, B|_U) \\
&= a\big(F, U, (B - \varepsilon\Delta + \varepsilon D)|_U\big) - \varepsilon \operatorname{mult}_F \big((f^*\Delta)|_{U'}\big) + \varepsilon \operatorname{mult}_F \big((f^*D)|_{U'}\big) \\
&= a\big(F, U, (B - \varepsilon\Delta + \varepsilon D)|_U\big),
\end{aligned}$$

where $U = X \setminus V$ and $U' = f^{-1}(U)$, which implies that $c_X(F) \setminus V$ is an lc center of $\big(X \setminus V, (B - \varepsilon\Delta + \varepsilon D)|_{X\setminus V}\big)$, as desired. This finishes the proof. □

Proposition 2.23. *Let $(X, B + M)$ be a dlt g-pair such that X is quasi-projective. Then every irreducible component of $\lfloor B \rfloor$ is normal.*

Proof. Let S be an irreducible component of $\lfloor B \rfloor$. Write $B = \sum_{i=1}^r b_i B_i + \sum_{j=1}^s D_j$, where $b_i \in (0, 1)$ for every $i \in \{1, \dots, r\}$ and, say, $D_1 := S$. Pick $b_i' \in \mathbb{Q}$ such that $0 < b_i' \leq b_i$ for every $i \in \{1, \dots, r\}$ and $\Delta := \sum_{i=1}^r b_i' B_i + \sum_{j=2}^s D_j$ is a $\mathbb{Q}$-divisor satisfying $0 \leq \Delta \leq B$. Observe also that $\lfloor \Delta \rfloor = \lfloor B \rfloor - S$. By Proposition 2.22 there exist an effective $\mathbb{Q}$-divisor Γ and a real number $0 < \varepsilon \ll 1$ such that $(X, B - \varepsilon\Delta + \varepsilon\Gamma + M)$ is a dlt g-pair. Note also that $\lfloor B - \varepsilon\Delta + \varepsilon\Gamma \rfloor = S$. Therefore, $(X, B - \varepsilon\Delta + \varepsilon\Gamma + M)$ is actually plt, and now it follows from Proposition 2.20 that S is normal. This concludes the proof. □

Finally, the author would like to thank Jingjun Han for showing him a proof of the above result and for informing him that this result will be incorporated in a new version of his joint paper [HL18] with Zhan Li.

2.1.4 Modifications of a Generalized Pair

The following lemma establishes the existence of several useful modifications of a given g-pair, namely, the existence of a *$\mathbb{Q}$-factorial terminal modification* of a klt g-pair, of a *small $\mathbb{Q}$-factorial modification* of a klt g-pair, and, finally, of a *$\mathbb{Q}$-factorial dlt modification* of an lc g-pair; the latter will be used frequently in the thesis. We refer to [Kol13, Section 1.4] for the corresponding results in the setting of usual pairs.

Part (i) of the lemma is [CT20, Lemma 2.10(i)] and we reproduce its proof below, see also [Mor18, Proposition 1.47]. Part (ii) is [Mor18, Proposition 1.48]; however, we give here a rather different proof, along the lines of [Bir09, Remark 2.3]. Part (iii) is [HL18, Proposition 3.9], but we incorporate its proof for both the sake of completeness and since it is similar to the previous ones.

Finally, we note that in order to prove the lemma we need to run certain MMPs with scaling of an ample divisor in the context of g-pairs; we refer to Section 2.3 for the details, but we also include all the relevant references below. We are forced to assume that all varieties involved are quasi-projective, since the termination of these MMPs relies on [BCHM10] and [Bir12a]. We also remark that the key result for the following proofs is [HL18, Proposition 3.8], which generalizes [Bir12a, Theorems 3.4 and 3.5] to the context of g-pairs.

Lemma 2.24. *Let $(X, B+M)$ be a g-pair with data $X' \xrightarrow{f} X \to Z$ and M'.*

(i) *If $(X, B+M)$ is klt, then, after possibly replacing X' with a higher model, there exists a $\mathbb{Q}$-factorial terminal g-pair $(Y, \Delta+\Xi)$ with data $X' \xrightarrow{g} Y \to Z$ and M', and a projective birational morphism $h\colon Y \to X$ such that*

$$K_Y + \Delta + \Xi \sim_{\mathbb{R}} h^*(K_X + B + M).$$

Moreover, each h-exceptional prime divisor E satisfies $a(E, X, B+M) \in (-1, 0]$. The g-pair $(Y, \Delta+\Xi)$ is called a $\mathbb{Q}$-factorial terminalization *of $(X, B+M)$.*

(ii) *If $(X, B+M)$ is klt, then, after possibly replacing X' with a higher model, there exists a $\mathbb{Q}$-factorial klt g-pair $(Y, \Delta+\Xi)$ with data $X' \xrightarrow{g} Y \to Z$ and M', and a small projective birational morphism $h\colon Y \to X$ such that*

$$K_Y + \Delta + \Xi \sim_{\mathbb{R}} h^*(K_X + B + M),$$

where $\Delta = h_*^{-1}B$. The g-pair $(Y, \Delta+\Xi)$ is called a small $\mathbb{Q}$-factorialization *of $(X, B+M)$.*

(iii) *If $(X, B+M)$ is lc, then, after possibly replacing X' with a higher model, there exist a $\mathbb{Q}$-factorial dlt g-pair $(Y, \Delta+\Xi)$ with data $X' \xrightarrow{g} Y \to Z$ and M', and a projective birational morphism $h\colon Y \to X$ such that*

$$K_Y + \Delta + \Xi \sim_{\mathbb{R}} h^*(K_X + B + M)$$

and each h-exceptional prime divisor is an irreducible component of $\lfloor \Delta \rfloor$. The g-pair $(Y, \Delta+\Xi)$ is called a dlt blow-up *of $(X, B+M)$.*

Proof.
(i) By Proposition 2.15 we know that there are only finitely many exceptional divisorial valuations $E_1, \dots, E_k$ over X with $a(E_i, X, B+M) \in (-1, 0]$ for any $1 \le i \le k$. We

may assume that $f\colon X' \to X$ is a log resolution of (X, B) such that each E_i is a prime divisor on X'. Thus, we may write

$$K_{X'} + B' + M' \sim_{\mathbb{R}} f^*(K_X + B + M) + F',$$

where B' is an effective $\mathbb{R}$-divisor with coefficients < 1 supported on $f_*^{-1}B$ and the f-exceptional prime divisors $E_1, \dots, E_k$, and F' is an effective f-exceptional $\mathbb{R}$-divisor that has no common components with B'. In particular, $(X', B' + M')$ is a $\mathbb{Q}$-factorial klt g-pair such that $K_{X'} + B' + M' \equiv_X F'$. Hence, by [HL18, Lemma 3.5 and Proposition 3.8] we may run a $(K_{X'} + B' + M')$-MMP with scaling of an ample divisor over X which contracts only F' and terminates with a model $h\colon Y \to X$ such that $K_Y + \Delta + \Xi \sim_{\mathbb{R}} h^*(K_X + B + M)$, where Δ is the strict transform of B' on Y and Ξ is a pushforward of M'. Furthermore, by construction the h-exceptional prime divisors are the strict transforms of the E_i on Y. It follows now from Lemma 2.13 that the $\mathbb{Q}$-factorial g-pair $(Y, \Delta + \Xi)$ is actually terminal. This completes the proof of (i).

(ii) We may assume that $f\colon X' \to X$ is a log resolution of (X, B). Let $E_1, \dots, E_k$ be the f-exceptional prime divisors on X'. Since $(X, B + M)$ is klt, we may pick $\varepsilon \in \mathbb{R}$ such that

$$0 < \varepsilon < 1 + \min\big\{a(E_i, X, B + M) \mid 1 \leq i \leq k\big\}.$$

Now, set

$$\Gamma' := f_*^{-1}B + (1 - \varepsilon)\sum_{i=1}^{k} E_i \quad \text{ and } \quad F' := \sum_{i=1}^{k} \big(a(E_i, X, B + M) + 1 - \varepsilon\big)E_i,$$

and observe that F' is an effective f-exceptional $\mathbb{R}$-divisor whose support contains each E_i due to the choice of ε. Then $(X', \Gamma' + M')$ is a $\mathbb{Q}$-factorial klt g-pair such that

$$K_{X'} + \Gamma' + M' \sim_{\mathbb{R}} f^*(K_X + B + M) + F',$$

and in particular, $K_{X'} + \Gamma' + M' \equiv_X F'$. Hence, by [HL18, Lemma 3.5 and Proposition 3.8] we may run a $(K_{X'} + \Gamma' + M')$-MMP with scaling of an ample divisor over X which contracts only F' and terminates with a model $h\colon Y \to X$ such that $K_Y + \Delta + \Xi \sim_{\mathbb{R}} h^*(K_X + B + M)$, where Δ is the strict transform of Γ' on Y and Ξ is a pushforward of M'. By construction the g-pair $(Y, \Delta + \Xi)$ is $\mathbb{Q}$-factorial klt and the map h is small. This concludes the proof of (ii).

(iii) We may assume that $f\colon X' \to X$ is a log resolution of (X, B). Let $E_1, \dots, E_k$ be the f-exceptional prime divisors and set $B' := f_*^{-1}B + \sum_{i=1}^{k} E_i$. Then we may write

$$K_{X'} + B' + M' \sim_{\mathbb{R}} f^*(K_X + B + M) + F',$$

where F' is an effective f-exceptional $\mathbb{R}$-divisor supported only on those E_i with discrepancy $a(E_i, X, B + M) > -1$. Note that $(X', B' + M')$ is a $\mathbb{Q}$-factorial dlt g-pair such that $K_{X'} + B' + M' \equiv_X F'$. Hence, by [HL18, Remark 2.3, Lemma 3.5 and Proposition 3.8] we may run a $(K_{X'} + B' + M')$-MMP with scaling of an ample divisor over X which contracts only F' and terminates with a model $h\colon Y \to X$ such that $K_Y + \Delta + \Xi \sim_{\mathbb{R}} h^*(K_X + B + M)$, where Δ is the strict transform of B' on Y and Ξ is a pushforward of M'. Moreover, $(Y, \Delta + \Xi)$ is a $\mathbb{Q}$-factorial dlt g-pair by [HL18, Lemma 3.7], and by construction the h-exceptional prime divisors are the strict transforms on Y of those E_i with discrepancy $a(E_i, X, B + M) = -1$; clearly, such E_i are components of $\lfloor \Delta \rfloor$. This finishes the proof of (iii). □

It is worthwhile to mention that Lemma 2.24(ii) also holds for dlt g-pairs. This was first observed in [HM18, Lemma 1.25]. We give a detailed proof of this fact below for the benefit of the reader.

Corollary 2.25. *Let $(X, B+M)$ be a dlt g-pair with data $X' \xrightarrow{f} X \to Z$ and M'. Assume that X and Z are quasi-projective. Then, after possibly replacing X' with a higher model, there exists a $\mathbb{Q}$-factorial dlt g-pair $(Y, \Delta+\Xi)$ with data $X' \xrightarrow{g} Y \to Z$ and M', and a small projective birational morphism $h\colon Y \to X$ such that*

$$K_Y + \Delta + \Xi \sim_{\mathbb{R}} h^*(K_X + B + M),$$

where $\Delta = h_*^{-1}B$. The g-pair $(Y, \Delta+\Xi)$ is called a small $\mathbb{Q}$-factorialization *of the dlt g-pair $(X, B+M)$.*

Proof. By Propositions 2.19 and 2.22 we deduce that there exist an effective $\mathbb{Q}$-divisor D and a sufficiently small positive real number ε such that $-\lfloor B \rfloor + D$ is $\mathbb{Q}$-Cartier and $(X, B - \varepsilon\lfloor B \rfloor + \varepsilon D + M)$ is a klt g-pair. By Lemma 2.24(ii) we may consider a small $\mathbb{Q}$-factorialization $h\colon (Y, \Gamma+\Xi) \to (X, B - \varepsilon\lfloor B \rfloor + \varepsilon D + M)$ of the aforementioned g-pair, where $\Gamma := h_*^{-1}(B - \varepsilon\lfloor B \rfloor + \varepsilon D)$. Set $\Delta := h_*^{-1}B$ and observe that

$$K_Y + \Delta + \Xi \sim_{\mathbb{R}} h^*(K_X + B + M).$$

We will show now that $(Y, \Delta+\Xi)$ is a dlt g-pair. First, it follows from Lemma 2.13 that $(Y, \Delta+\Xi)$ is lc. Now, let $V \subseteq X$ be the closed subset from Definition 2.5(e). Set $U := X \setminus V$ and $W := Y \setminus h^{-1}(U)$. Since U is $\mathbb{Q}$-factorial and since the restriction $h|_{h^{-1}(U)}\colon h^{-1}(U) \to U$ is also small, by [Fuj17, Lemma 2.1.4] we conclude that the map $h|_{h^{-1}(U)}$ is in fact an isomorphism. Therefore, $\big(Y \setminus W, \Delta|_{Y\setminus W}\big)$ is a log smooth pair. In addition, for every divisorial valuation F over Y with $a(F, Y, \Delta+\Xi) = -1$ we have that $c_Y(F) \nsubseteq W$ and $c_Y(F) \setminus W$ is an lc center of $\big(Y \setminus W, \Delta|_{Y\setminus W}\big)$. Indeed, as far as the first assertion of the two is concerned, if $c_Y(F) \subseteq W$, then $c_X(F) \subseteq V$, so by Lemma 2.13 and by the definition of dlt we would obtain

$$-1 = a(F, Y, \Delta+\Xi) = a(F, X, B+M) > -1,$$

a contradiction. The second assertion above is derived similarly, using the facts that $c_X(F) \nsubseteq V$ and $a(F, X, B+M) = -1$. This concludes the proof. □

2.2 Types of Models of a Generalized Pair

In this section we recall the definitions of various types of models of a g-pair, e.g., minimal models and Mori fiber spaces, both *in the usual sense* and *in the sense of Birkar-Shokurov*.[4] We also compare these notions and discuss their differences in great detail. In addition, we investigate how unique minimal and canonical models of g-pairs are and we actually obtain results analogous to the ones concerning minimal and canonical models of usual pairs. Finally, we demonstrate in Subsection 2.2.5 that the problem of the existence of Mori fiber spaces for non-pseudo-effective lc g-pairs has essentially already been resolved. On the other hand, we remark that the dual

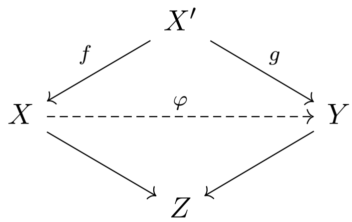
[4]Minimal models in the usual sense (resp. in the sense of Birkar-Shokurov) are often called *log terminal models* (resp. *log minimal models*) in the literature.

problem of the existence of minimal models for pseudo-effective lc g-pairs remains open in general, and we will be primarily concerned with it in Chapter 4.

Below we follow rather closely the presentation in [Bir12a, Section 2] and [Bir12b, Section 2]. However, as in [KM98, Kol13], we adopt the following convention: we drop the frequently used prefix “log” when referring to minimal and canonical models or even to the MMP; in other words, instead of “log canonical models” or “log Minimal Model Program (LMMP)”, we will simply talk about “canonical models” and the “Minimal Model Program (MMP)”, since it is highly unlikely that this will cause any confusion.

2.2.1 Models in the Sense of Birkar-Shokurov

Definition 2.26. Let $(X, B + M)$ be a g-pair with data $X' \xrightarrow{f} X \to Z$ and M'. Let $(Y/Z, B_Y + M_Y)$ be a g-pair together with a birational map $\varphi\colon X \dashrightarrow Y$ over Z and assume that X' is a sufficiently high birational model of X so that the induced map $g\colon X' \dashrightarrow Y$ is a morphism.



The g-pair $(Y, B_Y + M_Y)$ is called a *birational model in the sense of Birkar-Shokurov of* $(X, B + M)$ *over* Z if

- $B_Y = \varphi_* B + E$, where E is the sum of the φ^{-1}-exceptional prime divisors, and
- $M_Y = g_* M'$.

If, moreover,

- $K_Y + B_Y + M_Y$ is nef over Z, and
- $a(F, X, B + M) \leq a(F, Y, B_Y + M_Y)$ for any φ-exceptional prime divisor F,

then $(Y, B_Y + M_Y)$ is called a *weak canonical model in the sense of Birkar-Shokurov of* $(X, B + M)$ *over* Z.

Assume now that $(X, B + M)$ is lc. A weak canonical model in the sense of Birkar-Shokurov $\varphi\colon (X, B + M) \dashrightarrow (Y, B_Y + M_Y)$ of $(X, B + M)$ over Z is called a *minimal model in the sense of Birkar-Shokurov of* $(X, B + M)$ *over* Z if

- $(Y, B_Y + M_Y)$ is $\mathbb{Q}$-factorial dlt, and
- the above inequality on discrepancies is strict, i.e., for any φ-exceptional prime divisor F on X we have

$$a(F, X, B + M) < a(F, Y, B_Y + M_Y).$$

If, additionally, $K_Y + B_Y + M_Y$ is semi-ample over Z, then $(Y, B_Y + M_Y)$ is called a *good minimal model in the sense of Birkar-Shokurov of* $(X, B + M)$ *over* Z.

Definition 2.27. Let $(X/Z, B+M)$ be an lc g-pair. A birational model in the sense of Birkar-Shokurov $\varphi\colon (X, B+M) \dashrightarrow (Y, B_Y+M_Y)$ of $(X, B+M)$ over Z is called a *Mori fiber space in the sense of Birkar-Shokurov of* $(X, B+M)$ *over* Z if

- (Y, B_Y+M_Y) is $\mathbb{Q}$-factorial dlt,
- there exists a $(K_Y+B_Y+M_Y)$-negative extremal contraction $Y \to T$ over Z with $\dim Y > \dim T$, and
- for any divisorial valuation F over X we have

$$a(F, X, B+M) \leq a(F, Y, B_Y+M_Y),$$

and the strict inequality holds if $c_X(F)$ is a φ-exceptional prime divisor.

Comment. If in Definition 2.26 (resp. Definition 2.27) we drop the assumption that (Y, B_Y+M_Y) is dlt, then we obtain the notion of a *(good) minimal model in the sense of Birkar-Hashizume* (resp. *Mori fiber space in the sense of Birkar-Hashizume*), see [Has18a, Definition 2.2]. We stress, however, that the difference between these definitions is intrinsically not important; the justification in the setting of g-pairs will be provided in the next two paragraphs, see also [Has18a, Remark 2.4]. For this reason, we will not distinguish between (good) minimal models (resp. Mori fiber spaces) in the sense of Birkar-Shokurov and (good) minimal models (resp. Mori fiber spaces) in the sense of Birkar-Hashizume.

As far as minimal models are concerned, a (good) minimal model in the sense of Birkar-Shokurov is obviously a (good) minimal model in the sense of Birkar-Hashizume; and, conversely, any dlt blow-up of a (good) minimal model in the sense of Birkar-Hashizume is a (good) minimal model in the sense of Birkar-Shokurov.

As far as Mori fiber spaces are concerned, a Mori fiber space in the sense of Birkar-Shokurov is obviously a Mori fiber space in the sense of Birkar-Hashizume. Conversely, if an lc g-pair has a Mori fiber space in the sense of Birkar-Hashizume, then it has a Mori fiber space in the sense of Birkar-Shokurov by Theorem 2.40(iii).

Log Smooth Models

Log smooth models of lc pairs have been exploited effectively in [Bir10a, Bir12b, BH14] in order to construct minimal models in the sense of Birkar-Shokurov of lc pairs. Here, an analogous concept in the context of g-pairs is introduced, which will be used in Section 4.2 for similar purposes. Note that the following definition has already appeared implicitly in the proof of [HL18, Theorem 5.4].

Definition 2.28. Let $(X, B+M)$ be an lc g-pair with data $X' \xrightarrow{f} X \to Z$ and M'. We may assume that f is a sufficiently high log resolution of (X, B) so that $\operatorname{Exc}(f) \cup \operatorname{Supp}(f_*^{-1}B) \cup \operatorname{Supp} M'$ is an SNC divisor. Set $B^{ls} := f_*^{-1}B + E$, where E is the sum of the f-exceptional prime divisors on X'. Then the log smooth polarized pair $(X'/Z, B^{ls}+M')$ is called a *log smooth model of* $(X/Z, B+M)$.

Note that if $M' = 0$ in the above definition, then we recover the notion of a log smooth model of an lc pair given, for instance, in [Bir10a, Construction 2.3], see also [Bir12a, Definition 2.3] and [Bir12b, Definition 2.2].

Notation 2.29. Let $(X/Z, B+M)$ be an lc g-pair. From now on we will denote a log smooth model of $(X/Z, B+M)$ by $(W/Z, B_W+M_W)$. It should be understood then that the g-pair $(X, B+M)$ comes with data $W \xrightarrow{f} X \to Z$ and M_W and that $B_W := f_*^{-1}B + E$, where E is the sum of the f-exceptional prime divisors.

Remark 2.30. If $(W/Z, B_W+M_W)$ is a log smooth model of an lc g-pair $(X/Z, B+M)$, then we may write

$$K_W + B_W + M_W \sim_{\mathbb{R}} f^*(K_X + B + M) + F,$$

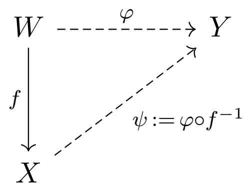
where F is an effective f-exceptional $\mathbb{R}$-divisor supported on the f-exceptional prime divisors D with $a(D, X, B+M) > -1$.

Constructing Minimal Models in the Sense of Birkar-Shokurov

The following lemma indicates how log smooth models are usually used in practice and plays a key role in the proofs of Proposition 2.33, Theorem 3.11(i) and Theorem 4.18.

Lemma 2.31. *Let $(X/Z, B+M)$ be an lc g-pair. Let (W, B_W+M_W) be a log smooth model of $(X, B+M)$ and let (Y, B_Y+M_Y) be a minimal model in the sense of Birkar-Shokurov of (W, B_W+M_W) over Z. Then (Y, B_Y+M_Y) is also a minimal model in the sense of Birkar-Shokurov of $(X, B+M)$ over Z.*

Proof. Consider the following diagram:



It follows by the assumptions that $B_Y = \psi_*^{-1}B + E_Y$, where E_Y is the sum of the ψ^{-1}-exceptional prime divisors, and that there is a higher model $f' \colon W' \to W$ of W such that the induced map $g \colon W' \to W$ is a morphism and $g_*\big((f')^*M_W\big) = M_Y$. Therefore, $\psi \colon (X, B+M) \dashrightarrow (Y, B_Y+M_Y)$ is a birational model in the sense of Birkar-Shokurov of $(X, B+M)$ over Z such that (Y, B_Y+M_Y) is $\mathbb{Q}$-factorial dlt and $K_Y+B_Y+M_Y$ is nef over Z by the assumptions. Moreover, if F is a ψ-exceptional prime divisor, then $f_*^{-1}F$ is a φ-exceptional prime divisor, and thus we have

$$a(F, X, B+M) = a(F, W, B_W+M_W) < a(F, Y, B_Y+M_Y).$$

This finishes the proof. □

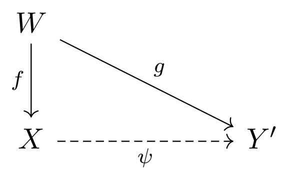
We remark, however, that the corresponding statement for Mori fiber spaces is not true in general, since the required condition on discrepancies might not be satisfied (by f-exceptional prime divisors). On the other hand, the following observation, which is easily derived by invoking Lemma 2.13, is very useful when one tries to construct minimal models (or Mori fiber spaces) in the sense of Birkar-Shokurov of lc g-pairs. For such an application, see Theorem 2.40(iii). Incidentally, further results in this direction are [HX13, Lemma 2.10] and [Has19, Lemma 2.15], just to name a few.

Remark 2.32. Let $(X/Z, B+M)$ be an lc g-pair and let $h\colon (T, B_T+M_T) \to (X, B+M)$ be a dlt blow-up of $(X, B+M)$. If (Y, B_Y+M_Y) is a minimal model (resp. Mori fiber space) in the sense of Birkar-Shokurov of (T, B_T+M_T) over Z, then (Y, B_Y+M_Y) is also a minimal model (resp. Mori fiber space) in the sense of Birkar-Shokurov of $(X, B+M)$ over Z.

To the best of our knowledge, the following result has not appeared elsewhere in the literature. It generalizes [Bir12a, Corollary 3.7] to the context of g-pairs and complements Lemma 2.39(ii). It tells us that in order to construct minimal models in the sense of Birkar-Shokurov it suffices to construct weak canonical models in the sense of Birkar-Shokurov.

Proposition 2.33. *Let $(X, B+M)$ be an lc g-pair with data $X' \xrightarrow{f} X \to Z$ and M'. Assume that X and Z are quasi-projective. If $(X, B+M)$ has a weak canonical model in the sense of Birkar-Shokurov over Z, then $(X, B+M)$ has a minimal model in the sense of Birkar-Shokurov over Z.*

Proof. Let $\psi\colon (X, B+M) \dashrightarrow (Y', B_{Y'}+M_{Y'})$ be a weak canonical model in the sense of Birkar-Shokurov of $(X, B+M)$ over Z. Let $f\colon (W, B_W+M_W) \to (X, B+M)$ be a sufficiently high log smooth model of $(X, B+M)$ so that the induced map $g\colon W \dashrightarrow Y'$ is actually a morphism.



We may write

$$K_W + B_W + M_W \sim_{\mathbb{R}} f^*(K_X + B + M) + F, \tag{2.6}$$

where F is an effective f-exceptional $\mathbb{R}$-divisor, see Remark 2.30, and

$$f^*(K_X + B + M) \sim_{\mathbb{R}} g^*(K_{Y'} + B_{Y'} + M_{Y'}) + G, \tag{2.7}$$

where $G := \sum \big(a(D, Y', B_{Y'}+M_{Y'}) - a(D, X, B+M)\big)D$ is an effective $\mathbb{R}$-divisor by Lemma 2.18(i). We claim that G is g-exceptional. Indeed, if there were an irreducible component D of G that were not g-exceptional, then D should be f-exceptional, since otherwise we would have $a(D, X, B+M) = a(D, Y', B_{Y'}+M_{Y'})$, and thus D could not be an irreducible component of G. But then $g(D)$ would be a ψ^{-1}-exceptional prime divisor, so by the definition of weak canonical models in the sense of Birkar-Shokurov we would have $a(D, Y', B_{Y'}+M_{Y'}) = -1$. Since G is effective, we should also have $a(D, X, B+M) = -1$, which shows again that D could not be an irreducible component of G and contradicts our assumption. This proves our assertion.

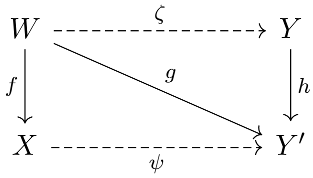
By (2.6) and (2.7) we obtain

$$K_W + B_W + M_W \equiv_{Y'} F + G.$$

We will show that $F + G \geq 0$ is g-exceptional. Since G is g-exceptional, it suffices to prove that F is g-exceptional. By Remark 2.30 we know that the irreducible components of F are f-exceptional prime divisors D on W with $a(D, X, B+M) > -1$. If there were an irreducible component D of F that were not g-exceptional, then (as above) $g(D)$

would be a ψ^{-1}-exceptional prime divisor and we would have $a(D, Y', B_{Y'}+M_{Y'}) = -1$, but this contradicts the fact that G is effective.

Now, by [HL18, Remark 2.3, Lemma 3.5 and Proposition 3.8] we may run some $(K_W + B_W + M_W)$-MMP with scaling of an ample divisor over Y' which contracts only $F + G$ and terminates with a model $h\colon Y \to Y'$ such that $K_Y + B_Y + M_Y \sim_{\mathbb{R}} h^*(K_{Y'} + B_{Y'} + M_{Y'})$, where B_Y is the strict transform of B_W on Y and M_Y is a pushforward of M_W.



In particular, $(Y, B_Y + M_Y)$ is a $\mathbb{Q}$-factorial dlt g-pair by [HL18, Lemma 3.7], and $K_Y + B_Y + M_Y$ is nef over Z. Hence, $(Y, B_Y + M_Y)$ is a minimal model of $(W, B_W + M_W)$ over Z, and it follows now from Lemma 2.31 that $(Y, B_Y + M_Y)$ is a minimal model in the sense of Birkar-Shokurov of $(X, B + M)$ over Z. □

2.2.2 Models in the Usual Sense

Definition 2.34. Let $(X/Z, B + M)$ be a g-pair.

(a) A birational model in the sense of Birkar-Shokurov $\varphi\colon (X, B+M) \dashrightarrow (Y, B_Y+M_Y)$ of $(X, B + M)$ over Z such that the map $\varphi\colon X \dashrightarrow Y$ is a birational contraction is called *birational model of* $(X, B + M)$ *over* Z.

(b) A weak canonical model in the sense of Birkar-Shokurov $\varphi\colon (X, B+M) \dashrightarrow (Y, B_Y+M_Y)$ of $(X, B+M)$ over Z such that the map $\varphi\colon X \dashrightarrow Y$ is a birational contraction is called a *weak canonical model of* $(X, B + M)$ *over* Z.

(c) A weak canonical model $\varphi\colon (X, B + M) \dashrightarrow (Y, B_Y + M_Y)$ of $(X, B + M)$ over Z is called a *minimal model of* $(X, B + M)$ *over* Z if

- Y is not necessarily $\mathbb{Q}$-factorial if X is not $\mathbb{Q}$-factorial, but Y is (required to be) $\mathbb{Q}$-factorial if X is $\mathbb{Q}$-factorial, and
- $a(F, X, B + M) < a(F, Y, B_Y + M_Y)$ for any φ-exceptional prime divisor F.

If, additionally, $K_Y + B_Y + M_Y$ is semi-ample over Z, then $(Y, B_Y + M_Y)$ is called a *good minimal model of* $(X, B + M)$ *over* Z.

(d) A weak canonical model $\varphi\colon (X, B + M) \dashrightarrow (Y, B_Y + M_Y)$ of $(X, B + M)$ over Z is called a *canonical model of* $(X, B + M)$ *over* Z if, additionally, $K_Y + B_Y + M_Y$ is ample over Z.

Note that the definitions of (good) minimal and canonical models given above differ slightly from the ones appearing in our papers [LT19, Section 2], [LMT20, Section 2] and [CT20, Section 2] in the sense that no assumptions are made on the singularities of the g-pairs involved. The precise relation between the singularities of a g-pair and those of a weak canonical model of that g-pair will be investigated below in Proposition 2.36. In addition, the relations between minimal and canonical models as well as the question of their uniqueness will also be studied in the next subsection.

Definition 2.35. Let $(X/Z, B + M)$ be a g-pair. A birational model

$$\varphi\colon (X, B + M) \dashrightarrow (Y, B_Y + M_Y)$$

of $(X, B + M)$ over Z is called a *Mori fiber space of* $(X, B + M)$ *over* Z if

- Y is not necessarily $\mathbb{Q}$-factorial if X is not $\mathbb{Q}$-factorial, but Y is (required to be) $\mathbb{Q}$-factorial if X is $\mathbb{Q}$-factorial,
- there exists a $(K_Y + B_Y + M_Y)$-negative extremal contraction $Y \to T$ over Z with $\dim Y > \dim T$, and
- for any divisorial valuation F over X we have

  $$a(F, X, B + M) \leq a(F, Y, B_Y + M_Y),$$

  and the strict inequality holds if $c_X(F)$ is a φ-exceptional prime divisor.

2.2.3 Basic Properties of Minimal and Canonical Models

We generalize [KM98, Proposition 3.51 and Theorem 3.52] and [Fuj17, Lemma 4.8.4], respectively, to the setting of g-pairs.

Proposition 2.36. *Let $(X/Z, B + M)$ be a g-pair and let $(X^w, B^w + M^w)$ be a weak canonical model (in the sense of Birkar-Shokurov) of $(X, B+M)$ over Z. Then for any divisorial valuation F over X we have*

$$a(F, X, B + M) \leq a(F, X^w, B^w + M^w).$$

In particular, if $(X, B + M)$ is klt (resp. lc), then $(X^w, B^w + M^w)$ is also klt (resp. lc).

Proof. The first part of the statement is essentially a reformulation of Lemma 2.18(i), while the second part of the statement is an immediate consequence of the first one. □

The next two results correspond to [CT20, Lemma 2.12] and [LMT20, Lemma 2.13], respectively, and concern the uniqueness of minimal and canonical models of a given g-pair as well as the relation between them.

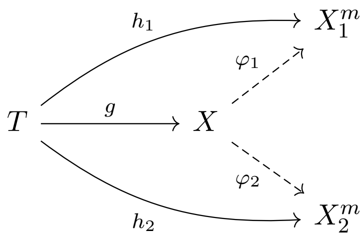
Proposition 2.37. *Let $(X/Z, B + M)$ be an lc g-pair and let*

$$\varphi_i\colon (X, B + M) \dashrightarrow (X_i^m, B_i^m + M_i^m),\ i \in \{1, 2\},$$

be two minimal models of $(X, B+M)$ over Z. Then the map $\varphi_2 \circ \varphi_1^{-1}\colon X_1^m \dashrightarrow X_2^m$ is an isomorphism in codimension one.

Proof. The proof is analogous to the proof of [KM98, Theorem 3.52(2)]. Nevertheless, we provide the details for the benefit of the reader.

Let $g\colon T \to X$ be a sufficiently high model so that the induced maps $h_i\colon T \to X_i^m$ are actually morphisms and there exists an $\mathbb{R}$-Cartier $\mathbb{R}$-divisor M_T on T which is nef over Z and satisfies $(h_i)_* M_T = M_i^m$, where $i \in \{1, 2\}$. We obtain the following diagram:



Set $E_i := g^*(K_X + B + M) - h_i^*(K_{X_i^m} + B_i^m + M_i^m)$, $i \in \{1,2\}$. By Lemma 2.18(i) and by the definition of a minimal model, E_i is an effective h_i-exceptional $\mathbb{R}$-divisor and $\operatorname{Supp} E_i$ contains the strict transforms of all the φ_i-exceptional prime divisors. Subtracting the two formulas, we obtain

$$h_1^*(K_{X_1^m} + B_1^m + M_1^m) - h_2^*(K_{X_2^m} + B_2^m + M_2^m) = E_2 - E_1.$$

Since $(h_1)_*(E_2 - E_1) \geq 0$ and $-(E_2 - E_1)$ is h_1-nef, by the Negativity lemma [KM98, Lemma 3.39(1)] we obtain $E_2 - E_1 \geq 0$. Similarly, $E_1 - E_2 \geq 0$. Thus, $E_1 = E_2$, so φ_1 and φ_2 have the same exceptional divisors, whence the statement. □

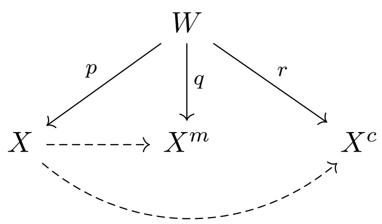
Proposition 2.38. *Let $(X/Z, B+M)$ be an lc g-pair. Let $(X^m, B^m + M^m)$ be a minimal model of $(X, B+M)$ over Z and let $(X^c, B^c + M^c)$ be a canonical model of $(X, B+M)$ over Z. Then there exists a birational morphism $\alpha\colon X^m \to X^c$ such that*

$$K_{X^m} + B^m + M^m \sim_{\mathbb{R}} \alpha^*(K_{X^c} + B^c + M^c).$$

In particular, $K_{X^m} + B^m + M^m$ is semi-ample over Z and there exists a unique, up to isomorphism, canonical model of $(X, B+M)$ over Z.

Proof. The proof is analogous to the proof of [Fuj17, Lemma 4.8.4]. Nevertheless, we provide the details for the benefit of the reader.

Let W be a common resolution of X, X^m and X^c, together with morphisms $p\colon W \to X$, $q\colon W \to X^m$ and $r\colon W \to X^c$.



We may write

$$p^*(K_X + B + M) \sim_{\mathbb{R}} q^*(K_{X^m} + B^m + M^m) + F$$

and

$$p^*(K_X + B + M) \sim_{\mathbb{R}} r^*(K_{X^c} + B^c + M^c) + G,$$

where F is effective and q-exceptional and G is effective and r-exceptional, see Lemma 2.18(i). Therefore,

$$q^*(K_{X^m} + B^m + M^m) + F \sim_{\mathbb{R}} r^*(K_{X^c} + B^c + M^c) + G.$$

Note that $q_*(G - F) \geq 0$ and $-(G - F)$ is q-nef, and that $r_*(F - G) \geq 0$ and $-(F - G)$ is r-nef. It follows from the Negativity lemma [KM98, Lemma 3.39(1)] that $F = G$, and thus

$$q^*(K_{X^m} + B^m + M^m) \sim_{\mathbb{R}} r^*(K_{X^c} + B^c + M^c). \tag{2.8}$$

Let C be a curve on W which is contracted by q. Then

$$\begin{aligned} 0 &= q^*(K_{X^m} + B^m + M^m) \cdot C = r^*(K_{X^c} + B^c + M^c) \cdot C \\ &= (K_{X^c} + B^c + M^c) \cdot r_*C, \end{aligned}$$

hence C is also contracted by r, since $K_{X^c} + B^c + M^c$ is ample over Z. Thus, by the Rigidity lemma [Deb01, Lemma 1.15] there exists a birational morphism $\alpha\colon X^m \to X^c$ such that $r = \alpha \circ q$, and the first statement follows from (2.8).

Assume now that there exists another canonical model $(Y, B_Y + M_Y)$ of $(X, B+M)$. Then, analogously as above, there exists a birational morphism $\beta\colon X^c \to Y$ such that

$$K_{X^c} + B^c + M^c \sim_{\mathbb{R}} \beta^*(K_Y + B_Y + M_Y).$$

Since $K_{X^c} + B^c + M^c$ is ample over Z, the map β must be finite, hence an isomorphism by Lemma 1.16. □

2.2.4 What are the Differences?

Consider an lc g-pair $(X, B+M)$. Observe that the differences between minimal models (resp. Mori fiber spaces) in the usual sense and in the sense of Birkar-Shokurov of $(X, B+M)$ lie in the type of the map φ to the model $(Y, B_Y + M_Y)$ – whether it is a birational contraction or not – and the kind of the singularities of the model $(Y, B_Y + M_Y)$ – whether they are dlt and/or $\mathbb{Q}$-factorial or not; note that they are at least lc by Proposition 2.36 (resp. by definition). Actually, we can say more about the map φ, and hence about the differences between the notions in question.

To begin with, let $\varphi\colon (X, B+M) \dashrightarrow (Y, B_Y + M_Y)$ be a minimal model (resp. Mori fiber space) in the sense of Birkar-Shokurov of $(X, B+M)$ over Z. Since $\varphi\colon X \dashrightarrow Y$ may not be a birational contraction, its inverse $\varphi^{-1}\colon Y \dashrightarrow X$ is in general allowed to contract some divisors. However, these divisors are actually very special, namely they determine lc centers of $(X, B+M)$. Indeed, if D is a φ^{-1}-exceptional prime divisor, then D is an irreducible component of B_Y with coefficient one by definition, and it follows from Proposition 2.36 (resp. by definition) that

$$-1 \leq a(D, X, B+M) \leq a(D, Y, B_Y + M_Y) = -1,$$

and hence $a(D, X, B+M) = -1$; in other words, $c_X(D)$ is an lc center of $(X, B+M)$. In conclusion, φ is only allowed to *extract* lc centers of $(X, B+M)$.

Furthermore, if $(X, B+M)$ is plt, then φ is necessarily a birational contraction, since otherwise there would exist a φ^{-1}-exceptional prime divisor D, and then by the previous paragraph and by Proposition 2.20 we would obtain

$$-1 < a(D, X, B+M) = a(D, Y, B_Y + M_Y) = -1,$$

a contradiction. Additionally, we claim that for any exceptional divisorial valuation E over Y we have $a(E, Y, B_Y + M_Y) > -1$. Indeed, if E is exceptional over X, then

$$-1 < a(E, X, B+M) \leq a(E, Y, B_Y + M_Y)$$

by Propositions 2.20 and 2.36, while if $c_X(E)$ is a φ-exceptional prime divisor, then

$$-1 \leq a(E, X, B+M) < a(E, Y, B_Y + M_Y)$$

by definition. Hence, $(Y, B_Y + M_Y)$ is also plt by Proposition 2.20. (This is clearly also true if $(Y, B_Y + M_Y)$ were instead a minimal model (resp. Mori fiber space) of $(X, B + M)$ over Z.) To summarize, minimal models (resp. Mori fiber spaces) in the usual sense and in the sense of Birkar-Shokurov coincide (modulo $\mathbb{Q}$-factoriality) in the plt case.

Having already discussed the differences between minimal models in the usual sense and minimal models in the sense of Birkar-Shokurov and having also discovered that they essentially coincide in the plt case, it is reasonable to wonder whether they actually coincide in more general contexts. The purpose of the next result is to shed light on this question.

Lemma 2.39. *Assuming that all varieties below are quasi-projective, the following statements hold.*

(i) *If $(X/Z, B)$ is an lc pair, then it has a minimal model over Z if and only if it has a minimal model in the sense of Birkar-Shokurov over Z.*

(ii) *If $(X/Z, B + M)$ is an NQC lc g-pair such that $(X, 0)$ is $\mathbb{Q}$-factorial klt, then it has a minimal model over Z if and only if it has a minimal model in the sense of Birkar-Shokurov over Z.*

Proof. This is [LT19, Lemma 2.9] and we reproduce its proof for the convenience of the reader.

(i) If (Y, B_Y) is a minimal model of (X, B) over Z, then a dlt blow-up of (Y, B_Y) is a minimal model in the sense of Birkar-Shokurov of (X, B) over Z. The converse follows immediately from [HH19, Theorem 1.7].

(ii) If $(Y, B_Y + M_Y)$ is a minimal model of $(X, B + M)$ over Z, then a dlt blow-up of $(Y, B_Y + M_Y)$ is a minimal model in the sense of Birkar-Shokurov of $(X, B + M)$ over Z. We emphasize that the assumption "$(X, 0)$ is $\mathbb{Q}$-factorial klt" does not play any role in this implication. The converse follows by repeating verbatim the proof of [HL18, Theorem 1.7], except that our assumption replaces [HL18, Theorem 5.4] in that proof. □

Comment. If in Lemma 2.39(ii) we drop the assumption that $(X, 0)$ is $\mathbb{Q}$-factorial klt and ask for the exact analog of Lemma 2.39(i) in the context of g-pairs, then the answer turns out to be much more complicated. Of course, one direction is easy, namely, if $(Y, B_Y + M_Y)$ is a minimal model of $(X, B + M)$ over Z, then any dlt blow-up of $(Y, B_Y + M_Y)$ is a minimal model in the sense of Birkar-Shokurov of $(X, B + M)$ over Z, as already mentioned above. However, regarding the converse, which is indisputably the most important implication of the two, currently we can at least give the following conditional answer: assuming the termination of flips for g-pairs in lower dimensions, if $(X, B + M)$ has a minimal model in the sense of Birkar-Shokurov over Z, then $(X, B + M)$ has a minimal model over Z by Proposition 3.9 and by Theorem 6.14.

2.2.5 Existence of Mori Fiber Spaces

We conclude this section with a theorem that establishes the existence of Mori fiber spaces for non-pseudo-effective g-pairs essentially in full generality.

Theorem 2.40. *Assuming that all varieties below are quasi-projective, the following statements hold.*

(i) *If $(X/Z, B)$ is a non-pseudo-effective lc pair, then it has a Mori fiber space over Z.*

(ii) *If $(X/Z, B + M)$ is a non-pseudo-effective NQC lc g-pair such that $(X, 0)$ is $\mathbb{Q}$-factorial klt, then it has a Mori fiber space over Z.*

(iii) *If $(X/Z, B+M)$ is a non-pseudo-effective NQC lc g-pair, then it has a Mori fiber space in the sense of Birkar-Shokurov over Z.*

Proof.
(i) Follows immediately from [HH19, Theorem 1.7].

(ii) Follows immediately from [BZ16, Lemma 4.4(1)].

(iii) Let $h\colon (T, B_T + M_T) \to (X, B + M)$ be a dlt blow-up of $(X, B + M)$. By (ii), $(T, B_T + M_T)$ has a Mori fiber space $(Y, B_Y + M_Y)$ over Z, and it follows now from Remark 2.32 that $(Y, B_Y + M_Y)$ is a Mori fiber space in the sense of Birkar-Shokurov of $(X, B + M)$ over Z. □

Lastly, we comment on the above theorem. First, as far as (i) is concerned, the klt case was originally established by [BCHM10, Corollary 1.3.3], while the dlt case follows from [Bir12a, Theorem 4.1(ii)], which also relies on [BCHM10]. Second, the argument for the proof of (ii) is actually contained in the first lines of the proof of Lemma 2.51. Third, regarding (iii), we stress that our main goal is, of course, to deduce the existence of Mori fiber spaces *in the usual sense* (not just in the sense of Birkar-Shokurov) for non-pseudo-effective NQC lc g-pairs, but no analog of [HH19, Theorem 1.7] in the setting of g-pairs is currently available.

2.3 The Minimal Model Program for Generalized Pairs

In this section we discuss thoroughly the MMP in the setting of g-pairs. Specifically, we recall the definitions of divisorial contractions and flips for g-pairs and we prove analogs of [KM98, Proposition 3.36(1), Proposition 3.37(1), Corollary 3.42, Corollary 3.43 and Corollary 3.44] in this general setting. Furthermore, we explain exhaustively when and how one can run an MMP (with scaling of an ample divisor) in the context of g-pairs. Note that such MMPs have already been used in previous subsections of the thesis and we provide here the missing details. Additionally, we reproduce several results from our papers, namely [LT19, Lemma 2.17, Corollary 2.18, Lemma 2.19, Lemma 2.20 and Lemma 2.23], [LMT20, Lemma 3.1 and Lemma 3.2] and [CT20, Lemma 2.15], which concern or whose proofs involve the MMP with scaling of an ample divisor for g-pairs. In this regard, the current section complements and significantly expands [BZ16, Section 4] and [HL18, Section 3.1]. Its contents will be of fundamental importance for the remainder of the thesis.

2.3.1 Divisorial Contractions and Flips

We begin with the definition of a divisorial contraction and then continue with the definition of a flip in the context of g-pairs.

Definition 2.41. Let $(X/Z, B + M)$ be a g-pair. A projective birational morphism $\pi\colon X \to W$ over Z to a normal variety W such that

- $\operatorname{codim}_X \operatorname{Exc}(\pi) = 1$,
- the relative Picard number is $\rho(X/W) = 1$, and
- $-(K_X + B + M)$ is ample over W,

is called a *divisorial contraction over* Z.

Definition 2.42. A diagram

$$\begin{array}{ccc} (X, B+M) & \overset{\pi}{\dashrightarrow} & (X^+, B^+ + M^+) \\ & {\scriptstyle\theta}\searrow \quad \swarrow{\scriptstyle\theta^+} & \\ & W & \end{array}$$

over Z, where

- $(X/Z, B + M)$ and $(X^+/Z, B^+ + M^+)$ are g-pairs and W is a normal variety,
- θ and θ^+ are projective birational morphisms and π is a birational map,
- $\theta_* B \geq \theta^+_* B^+$,
- M and M^+ are pushforwards of the same nef $\mathbb{R}$-divisor on a common higher model of X and X^+,
- $-(K_X + B + M)$ is nef over W and $K_{X^+} + B^+ + M^+$ is nef over W,

is called a *quasi-flip over* Z.

If, moreover,

- π is an isomorphism in codimension one,
- $\pi_* B = B^+$ (or, equivalently, $\theta_* B = \theta^+_* B^+$),
- $-(K_X + B + M)$ is ample over W and $K_{X^+} + B^+ + M^+$ is ample over W,

then the given diagram is called an *ample small quasi-flip over* Z.

If, in addition to the above,

- θ and θ^+ are isomorphisms in codimension one, and
- the relative Picard numbers are $\rho(X/W) = \rho(X^+/W) = 1$,

then the above diagram is called a $(K_X + B + M)$-*flip over* Z. In this case the map θ (resp. θ^+) is called *the flipping contraction over* Z (resp. *the flipped contraction over* Z) and the set $\operatorname{Exc}(\theta)$ (resp. $\operatorname{Exc}(\theta^+)$) is called *the flipping locus* (resp. *the flipped locus*).

The definition of quasi-flips given above is [HM18, Definition 1.8], while the one of ample small quasi-flips is taken from our paper [LMT20], cf. [HM18, Definition 1.9]. The latter will play a significant role in Chapter 5, where we will deal with *the special termination for NQC lc g-pairs*, though some results regarding ample small quasi-flips will be discussed in Subsection 2.3.5.

Remark 2.43.

(1) If

$$\begin{array}{ccc} (X, B+M) & \overset{\pi}{\dashrightarrow} & (X^+, B^+ + M^+) \\ & \underset{\theta}{\searrow} \quad W \quad \underset{\theta^+}{\swarrow} & \end{array}$$

is an ample small quasi-flip (over Z), then $(X^+, B^+ + M^+)$ is the canonical model of $(X, B+M)$ over W by definition.

In conclusion, according to Lemma 2.38, an ample small quasi-flip, hence a flip, is unique if it exists.

(2) Flips for klt pairs exist by [BCHM10, Corollary 1.4.1], whereas flips for lc pairs exist by [Bir12a, Corollary 1.2] or [HX13, Corollary 1.8], see also [Fuj17, Corollaries 4.8.12 and 4.8.14]. On the other hand, the existence of flips for g-pairs is still unknown in full generality, but it can nonetheless be derived under some rather mild conditions, see Subsection 2.3.2 for the details.

Now, we investigate the behavior of discrepancies under a divisorial contraction or a flip.

Lemma 2.44. *Let $(X/Z, B+M)$ be a g-pair and let $\pi\colon (X, B+M) \to (W, B_W + M_W)$ be a divisorial contraction over Z[5]. Then for any divisorial valuation F over X we have*

$$a(F, X, B+M) \leq a(F, W, B_W + M_W),$$

and the strict inequality holds if and only if $c_X(F)$ is contained in $\operatorname{Exc}(\pi)$*. Moreover, the following hold:*

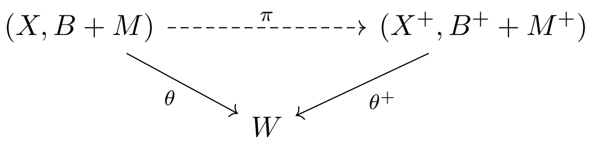
(i) *If $(X, B+M)$ is terminal (resp. canonical) and $E \wedge B = 0$, where E is the sum of the π-exceptional prime divisors, then $(W, B_W + M_W)$ is also terminal (resp. canonical).*

(ii) *If $(X, B+M)$ is klt (resp. plt, dlt, lc), then $(W, B_W + M_W)$ is also klt (resp. plt, dlt, lc).*

Proof. The first part of the statement follows immediately from Lemma 2.18(iii) and yields the second part of the statement in case $(X, B+M)$ is terminal, canonical, klt or lc. In particular, regarding (i), the condition $E \wedge B = 0$ means that the divisors E and B have no common components, so if F is an exceptional divisorial valuation over W which is not exceptional over X, then $c_X(F)$ is an irreducible component of E and we therefore have $0 = a(F, X, B+M) < a(F, W, B_W + M_W)$. Finally, if $(X, B+M)$ is dlt, then $(W, B_W + M_W)$ is also dlt by [HL18, Lemma 3.7], and if, additionally, $(X, B+M)$ is plt, then $(W, B_W + M_W)$ is also plt by Proposition 2.20. □

[5] We refer to Subsection 2.3.2 for the precise definition of the g-pair structure on W. Here, we have implicitly assumed that $K_W + B_W + M_W$ is $\mathbb{R}$-Cartier, since this is actually the only condition required in order for the statement of the lemma to make sense, and, of course, that the map π exists. As we will see below, both of these assumptions are satisfied when $(X, B+M)$ is lc and $(X, 0)$ is $\mathbb{Q}$-factorial klt.

Lemma 2.45. *Let $(X/Z, B + M)$ be a g-pair and let*



be a $(K_X + B + M)$-flip over Z. Then for any divisorial valuation F over X we have

$$a(F, X, B + M) \leq a(F, X^+, B^+ + M^+),$$

and the strict inequality holds if and only if either $c_X(F)$ is contained in the flipping locus $\mathrm{Exc}(\theta)$ *or $c_{X^+}(F)$ is contained in the flipped locus* $\mathrm{Exc}(\theta^+)$.

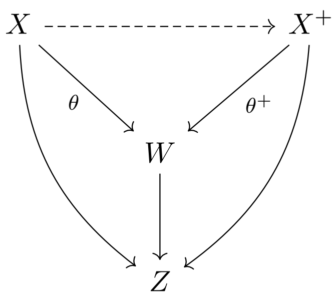
In particular, if $(X, B + M)$ is terminal (resp. canonical, klt, plt, dlt, lc), then $(X^+, B^+ + M^+)$ is also terminal (resp. canonical, klt, plt, dlt, lc).

Proof. The first part of the statement follows immediately from Lemma 2.18(iii) and yields the remaining assertions in case $(X, B + M)$ is terminal, canonical, klt or lc. If $(X, B + M)$ is dlt, then $(X^+, B^+ + M^+)$ is also dlt by [HL18, Lemma 3.7], and if, additionally, $(X, B+M)$ is plt, then $(X^+, B^+ + M^+)$ is also plt by Proposition 2.20. □

Remark 2.46. There is an obvious analog of Lemma 2.45 for ample small quasi-flips, which is also obtained using Lemma 2.18(iii). This observation will be used without explicit mention in Subsection 2.3.5.

We conclude this subsection by recalling [Bir07, Definition 2.3]. This definition will only be needed in Chapter 6, where we will be concerned with *the termination of flips conjecture*.

Definition 2.47. Let $X \to Z$ be a projective morphism of normal varieties and let D be an $\mathbb{R}$-Cartier $\mathbb{R}$-divisor on X. A *D-flip over Z* is a diagram



where

- X^+ and W are normal varieties, which are projective over Z,
- θ and θ^+ are projective birational morphisms with

  $$\mathrm{codim}_X \mathrm{Exc}(\theta) \geq 2 \ \text{ and } \ \mathrm{codim}_{X^+} \mathrm{Exc}(\theta^+) \geq 2,$$

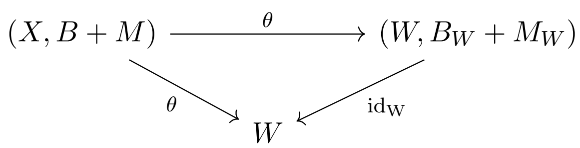
  that is, θ and θ^+ are isomorphisms in codimension one,
- the relative Picard number is $\rho(X/W) = 1$,
- $-D$ is ample over W, and the strict transform D^+ of D on X^+ is $\mathbb{R}$-Cartier and ample over W.

In particular, if $(X/Z, B)$ is a usual pair, then we recover the notion of a $(K_X + B)$-*flip over Z* by taking $D = K_X + B$ in the above definition.

2.3.2 Running an MMP

Let $(X, B+M)$ be a g-pair with data $X' \xrightarrow{f} X \to Z$ and M'. Assume that $(X, B+M)$ is lc and that $(X, 0)$ is klt – note that the following discussion remains valid if we assume instead that $(X, B+M)$ is klt due to [HL18, Lemma 3.5]. Assume also that $K_X + B + M$ is not nef over Z and consider a $(K_X + B + M)$-negative extremal ray R over Z. By [HL18, Lemma 3.5] there is a boundary $\mathbb{R}$-divisor Δ on X such that (X, Δ) is klt and R is also a $(K_X + \Delta)$-negative extremal ray over Z. It follows from [KM98, Theorem 3.25(3)] that R can be contracted and we denote by $\theta\colon X \to W$ the associated contraction over Z. We consider now the following two cases.

Case I: Assume that $\theta\colon X \to W$ is a divisorial contraction. Setting $B_W := \theta_* B$ and $M_W := (\theta \circ f)_* M'$, we may induce on W the structure of a g-pair $(W, B_W + M_W)$ with data $X' \xrightarrow{\theta \circ f} W \to Z$ and M', provided that $K_W + B_W + M_W$ is $\mathbb{R}$-Cartier. Under this condition, $(W, B_W + M_W)$ is lc by Lemma 2.44(ii) and $(W, 0)$ is klt by [KM98, Corollary 3.43(1) and Corollary 2.35(1)]. We obtain the following diagram over Z:

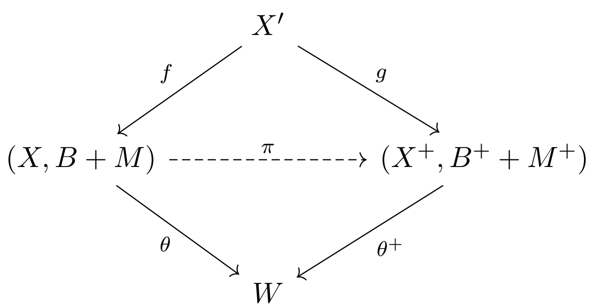


Note that if X is $\mathbb{Q}$-factorial, then W is also $\mathbb{Q}$-factorial by [KM98, Corollary 3.18], hence W can be endowed with the aforementioned g-pair structure. Additionally, the exceptional locus $\mathrm{Exc}(\theta)$ of θ is an irreducible divisor on X by [KMM87, Proposition 5-1-6], and a uniruled variety by [KMM87, Proposition 5-1-8]. Furthermore, by [Fuj17, Proposition 4.8.18] we have $\rho(W/Z) = \rho(X/Z) - 1$. Lastly, we remark that these properties of $\mathrm{Exc}(\theta)$ need not hold without the assumption that X is $\mathbb{Q}$-factorial.

Case II: Assume that $\theta\colon X \to W$ is a flipping contraction. Then the $(K_X + \Delta)$-flip of θ exists by [BCHM10, Corollary 1.4.1]; we denote by $\theta^+\colon X^+ \to W$ the corresponding flipped contraction and by π the induced map $(\theta^+)^{-1} \circ \theta\colon X \dashrightarrow X^+$, which is an isomorphism in codimension one. We remark in passing that

$$\dim \mathrm{Exc}(\theta) + \dim \mathrm{Exc}(\theta^+) \geq \dim X - 1$$

by [KMM87, Lemma 5-1-17] and by [HK10, Remark 5.20]. We may also assume that X' is a sufficiently high birational model of X so that the induced map $g\colon X' \dashrightarrow X^+$ is a morphism. Setting $B^+ := \pi_* B$ and $M^+ := g_* M'$, we may induce on X^+ the structure of a g-pair $(X^+, B^+ + M^+)$ with data $X' \xrightarrow{g} X^+ \to Z$ and M', provided that $K_{X^+} + B^+ + M^+$ is $\mathbb{R}$-Cartier. Under this condition, $(X^+, B^+ + M^+)$ is lc by Lemma 2.45 and $(X^+, 0)$ is klt by [KM98, Corollary 3.42 and Corollary 2.35(1)]. Thus, we obtain the following diagram over Z:



which is clearly a $(K_X + B + M)$-flip over Z.

Note that if X is $\mathbb{Q}$-factorial, then X^+ is also $\mathbb{Q}$-factorial by [KM98, Proposition 3.37(1)], hence X^+ can be endowed with the aforementioned g-pair structure. Furthermore, by [Fuj17, Proposition 4.8.20] we have $\rho(X^+/Z) = \rho(X/Z)$.

In conclusion, divisorial contractions and flips exist for lc g-pairs whose underlying variety is $\mathbb{Q}$-factorial klt. However, without these extra assumptions on the underlying variety of a given lc g-pair, the existence of flips (or actually even an analog of [Fuj17, Theorem 4.5.2]) is currently an open problem in birational geometry.

Taking the above into account, we infer that we may always run an MMP (with scaling of an ample divisor) over Z for any lc g-pair $(X/Z, B + M)$ such that $(X, 0)$ is $\mathbb{Q}$-factorial klt, and in particular for any $\mathbb{Q}$-factorial dlt g-pair $(X/Z, B + M)$, see Remark 2.7. We elaborate on this below. We also stress that (as part of the definition of the MMP) the nef part of every g-pair occuring in a given MMP is the same.

MMP for Generalized Pairs

Let $(X, B+M)$ be a g-pair with data $X' \xrightarrow{f} X \to Z$ and M'. Assume that $(X, B+M)$ is lc and that $(X, 0)$ is $\mathbb{Q}$-factorial klt. Then we may run a $(K_X + B + M)$-MMP over Z as follows (but we do not know whether it terminates). If $K_X + B + M$ is nef over Z, then we stop. Otherwise, there exists a $(K_X + B + M)$-negative extremal ray R over Z, which can be contracted, and we denote by $\theta \colon X \to W$ the associated extremal contraction over Z. We distinguish now three cases. First, if θ is a Mori fiber space, then we stop. Second, if θ is a divisorial contraction, then W admits a g-pair structure $(W, B_W + M_W)$ with data $X' \to W \to Z$ and M', and $(W, B_W + M_W)$ satisfies the same conditions as $(X, B + M)$, so we may now continue the process with $(W, B_W + M_W)$. Third, if θ is a flipping contraction, then its flip $\pi \colon X \dashrightarrow X^+$ exists, the variety X^+ admits a g-pair structure $(X^+, B^+ + M^+)$ with data $X' \to X^+ \to Z$ and M', and $(X^+, B^+ + M^+)$ satisfies the same conditions as $(X, B + M)$, so we may now continue the process with $(X^+, B^+ + M^+)$. Therefore, by repeating this procedure, we obtain a $(K_X + B + M)$-MMP over Z. This MMP was first outlined in [HL18, Section 3.1].

Comment. If $(X/Z, B + M)$ is an lc g-pair of dimension 2, then any $(K_X + B + M)$-MMP over Z terminates. Indeed, by Corollary 1.17 there are no flipping contractions in dimension 2, hence any $(K_X + B + M)$-MMP over Z consists only of divisorial contractions and we know that at each step the relative Picard number over Z drops by one. This yields the assertion.

It is worth mentioning that the additional assumption "$(X, 0)$ is $\mathbb{Q}$-factorial klt", which is currently indispensable in order to run MMPs in the setting of g-pairs, is reduntant in this case due to [HL20b, Lemma 2.4].

Last but not least, the above fact, together with [KMM87, Theorem 5-1-15], Corollary 3.10 and Theorem 3.17, allow us to derive from Theorem 6.14 a new proof of the termination of flips (hence of any MMP) for pseudo-effective NQC lc g-pairs of dimension 3, cf. Theorem 6.10.

MMP with Scaling for Generalized Pairs

Let $(X, B+M)$ be a g-pair with data $X' \xrightarrow{f} X \to Z$ and M'. Assume that $(X, B+M)$ is lc and that $(X, 0)$ is $\mathbb{Q}$-factorial klt. Then we may run a $(K_X + B + M)$-MMP with scaling of an ample divisor over Z as follows (but we do not know whether it terminates). Let A be an effective general ample over Z $\mathbb{R}$-Cartier $\mathbb{R}$-divisor on X such that $(X, B+A+M)$ is lc (see Lemma 2.4 and Remark 2.11(2)) and $K_X+B+M+A$ is nef over Z. Set

$$\lambda := \inf\big\{t \in \mathbb{R}_{\geq 0} \mid K_X+B+M+tA \text{ is nef over } Z\big\}.$$

If $\lambda = 0$ (or, equivalently, K_X+B+M is nef over Z), then we stop. Otherwise, $\lambda \in (0,1]$, and we may choose $0<\varepsilon \ll 1$ such that $0<\lambda(1-\varepsilon)<\lambda$ and $0<\lambda\epsilon \ll 1$. By [HL18, Lemma 3.5] there exists a boundary Δ on X such that (X,Δ) is klt and $\Delta \sim_{\mathbb{R},Z} B+M+\lambda(1-\varepsilon)A$. Therefore,

$$K_X+\Delta \sim_{\mathbb{R},Z} K_X+B+M+\lambda(1-\varepsilon)A$$

is not nef over Z, while

$$K_X+\Delta+\lambda\varepsilon A \sim_{\mathbb{R},Z} K_X+B+M+\lambda A$$

is nef over Z. In addition, $(X, \Delta+\lambda\varepsilon A)$ is klt and clearly

$$\lambda\epsilon = \inf\big\{s \in \mathbb{R}_{\geq 0} \mid K_X+\Delta+sA \text{ is nef over } Z\big\}.$$

By [Bir10a, Lemma 3.1] there exists an extremal ray R over Z such that

$$(K_X+\Delta)\cdot R<0 \quad\text{and}\quad (K_X+\Delta+\lambda\varepsilon A)\cdot R = 0.$$

In particular, $A\cdot R>0$[6] and therefore

$$(K_X+B+M)\cdot R<0 \quad\text{and}\quad (K_X+B+M+\lambda A)\cdot R=0.$$

By [KM98, Theorem 3.25(3)], R can be contracted and we denote by $\theta\colon X\to W$ the associated extremal contraction over Z. Now, we distinguish three cases.

1) If θ is a Mori fiber space, then we stop.

2) If θ is a divisorial contraction, then W admits a g-pair structure (W, B_W+M_W) with data $X'\to W\to Z$ and M', (W, B_W+M_W) is lc and $(W,0)$ is $\mathbb{Q}$-factorial klt. Additionally, $K_W+B_W+M_W+\lambda A_W$ is nef over Z by [KM98, Theorem 3.25(3)] and $(W, B_W+A_W+M_W)$ is lc by Lemma 2.18(iii), where A_W is the strict transform of A on W. Hence, we may now repeat the process with (W, B_W+M_W) and A_W in place of $(X, B+M)$ and A, respectively.

3) If θ is a flipping contraction, then its flip $\pi\colon X \dashrightarrow X^+$ exists, the variety X^+ admits a g-pair structure $(X^+, B^+ + M^+)$ with data $X'\to X^+\to Z$ and M', $(X^+, B^+ + M^+)$ is lc and $(X^+, 0)$ is $\mathbb{Q}$-factorial klt. Additionally (and as above), $(X^+, B^+ + A^+ + M^+)$ is lc and $K_{X^+}+B^+ + M^+ + \lambda A^+$ is nef over Z, where A^+ is the strict transform of A on X^+. Hence, we may now repeat the process with $(X^+, B^+ + M^+)$ and A^+ in place of $(X, B+M)$ and A, respectively.

Therefore, by continuing in the same fashion, we obtain a (K_X+B+M)-MMP with scaling of A over Z. This MMP was originally defined in [BZ16, Section 4].

[6] We emphasize that it is not necessary to invoke the ampleness of A over Z in order to show that $A\cdot R>0$. This observation is vital for the repetition of the procedure in question, since the ampleness of A over Z is not preserved along the MMP.

Comment. Let $(X/Z, B+M)$ be an NQC g-pair with data $X' \xrightarrow{f} X \to Z$ and M'. Assume that $(X, B+M)$ is lc and that $(X,0)$ is $\mathbb{Q}$-factorial klt. Let P' be an NQC divisor on X', let N be an effective $\mathbb{R}$-divisor on X, set $P := f_*P'$ and assume that $\big(X, (B+N)+(M+P)\big)$ is lc and that $K_X+B+N+M+P$ is nef over Z. According to [HL18, Section 3.4], we may run a (K_X+B+M)-MMP with scaling of $N+P$ over Z, which Han and Li call *MMP with scaling of an NQC divisor* and whose termination is also not known. This MMP is actually the proper analog of [Bir12a, Definition 2.4] in the setting of g-pairs and includes the MMP with scaling of an ample divisor discussed above as a special case. We will not reproduce here its construction.

Remark 2.48. Let $(X_1/Z, B_1+M_1)$ be an lc g-pair such that $(X_1, 0)$ is $\mathbb{Q}$-factorial klt. Run a $(K_{X_1}+B_1+M_1)$-MMP with scaling of an ample divisor A_1 over Z. Denote by $(X_i/Z, B_i+M_i)$ the g-pairs occuring in this MMP and by A_i the strict transform of A_1 on X_i. For each $i \geq 1$ let

$$\lambda_i := \inf\{t \in \mathbb{R}_{\geq 0} \mid K_{X_i}+B_i+M_i+tA_i \text{ is nef over } Z\}$$

be *the nef threshold of A_i with respect to (X_i, B_i+M_i)*. By construction the sequence $\{\lambda_i\}_{i=1}^{\infty}$ is non-increasing. In fact, we have $1 \geq \lambda_i \geq \lambda_{i+1} \geq 0$ for all $i \geq 1$.

Additionally, it is worthwhile to mention that if

$$\lambda := \lim_{i\to\infty} \lambda_i > 0,$$

then the $(K_{X_1}+B_1+M_1)$-MMP with scaling of A_1 over Z terminates, see the second paragraph of the proof of Lemma 2.49 for the details.

In general, the termination of an MMP with scaling of an ample divisor is still an open problem. However, several cases have already been settled. In the setting of usual pairs, sufficient conditions for its termination are provided by [BCHM10, Corollary 1.4.2], [Bir12a, Theorem 4.1] (see also [Has19, Theorem 2.11]), [Hu17, Theorem 1.5], [HH19, Theorem 1.7] and Theorem 4.15. Analogous termination results in the context of g-pairs are [BZ16, Lemma 4.4], [HL18, Theorem 4.1] and Theorem 4.16 (see also Subsection 3.1.2 for the results that led to Theorems 4.15 and 4.16).

Finally, note that the termination of an MMP with scaling of an ample divisor for a g-pair usually relies on [BCHM10, Corollary 1.4.2]. This is actually the main reason why we have to impose quasi-projectivity assumptions when running such MMPs if we want to deduce their termination.

Convention. From this point forward we assume that all varieties considered are normal and quasi-projective and that a variety X over a variety Z is projective over Z.

The next result generalizes [Dru11, Théorème 3.3] to the context of g-pairs as well as to the relative setting. Its proof is analogous to the proof of [Fuj11c, Theorem 2.3] and has already appeared implicitly in the proof of [HL18, Proposition 3.8]. Nevertheless, we provide all the details for the benefit of the reader.

Lemma 2.49. *Let $(X/Z, B+M)$ be an lc g-pair such that $(X,0)$ is $\mathbb{Q}$-factorial klt and K_X+B+M is pseudo-effective over Z. Assume that we have a (K_X+B+M)-MMP with scaling of an ample divisor A over Z. Then, on some variety X_i in this MMP, the strict transform of K_X+B+M becomes movable over Z. In particular, the restriction of this strict transform to a very general fiber of the induced morphism $X_i \to Z$ is movable.*

Proof. Let $(X_i/Z, B_i+M_i)$ be the g-pairs occuring in the MMP, where $(X_1, B_1+M_1) := (X, B+M)$, let A_i be the strict transform of A on X_i, and set

$$\lambda_i := \inf\{t \in \mathbb{R}_{\geq 0} \mid K_{X_i} + B_i + M_i + tA_i \text{ is nef over } Z\},\ i \geq 1$$

and $\lambda := \lim_{i\to\infty} \lambda_i$. We distinguish two cases.

Assume first that $\lambda > 0$. Then the given MMP is also a $(K_X+B+M+\frac{\lambda}{2}A)$-MMP. Thus, by [HL18, Lemma 3.5] there exists a boundary Δ on X such that $K_X+\Delta \sim_{\mathbb{R},Z} K_X+B+M+\frac{\lambda}{2}A$, the pair (X,Δ) is klt and Δ is big over Z. By [BCHM10, Corollary 1.4.2], the $(K_X+\Delta)$-MMP with scaling of A over Z terminates, and therefore the original MMP terminates.

Assume now that $\lambda = 0$. Then we may assume that the MMP does not terminate (since otherwise the assertion is clear), and that the strict transform of K_X+B+M never becomes movable over Z. We may also assume that the MMP consists only of flips. For each $i \geq 1$, let H_i be a divisor on X_i which is ample over Z and such that, if $H_{i,X}$ is the strict transform of H_i on X, then $\lim_{i\to\infty}[H_{i,X}] = 0$ in $N^1(X/Z)$. Since $K_{X_i} + B_i + M_i + \lambda_i A_i + H_i$ is ample over Z for every $i \geq 1$, the strict transform $K_X+B+M+\lambda_i A+H_{i,X}$ is movable over Z for every $i \geq 1$, and therefore K_X+B+M is also movable over Z, a contradiction.

Finally, the last assertion of the lemma follows readily from the first one. □

2.3.3 Two Key Applications

We prove below a criterion for a g-pair to be relatively effective. This result plays a fundamental role in the proofs of Theorems 3.13 and 3.14.

Lemma 2.50. *Let $(X, B+M)$ be a $\mathbb{Q}$-factorial dlt g-pair such that X is quasi-projective and let $f\colon X \to Y$ be a projective surjective morphism to a normal quasi-projective variety Y. Assume that*

$$\nu\big(F, (K_X+B+M)|_F\big) = 0 \quad \textit{and} \quad h^1(F, \mathcal{O}_F) = 0$$

for a very general fiber F of f. Then K_X+B+M is effective over Y.

Proof. We run a (K_X+B+M)-MMP with scaling of an ample divisor over Y. By Lemma 2.49, after finitely many steps of this MMP we obtain a (K_X+B+M)-negative birational contraction $\theta\colon X \dashrightarrow X'$ such that $(K_{X'}+B'+M')|_{F'}$ is movable, where $B' := \theta_*B$, M' is a pushforward of M and F' is a very general fiber of the induced morphism $f'\colon X' \to Y$. Additionally, since $\nu\big(F, (K_X+B+M)|_F\big) = 0$ by assumption, it follows from [LP20a, Section 2.2] that $\nu\big(F', (K_{X'}+B'+M')|_{F'}\big) = 0$, and hence $(K_{X'}+B'+M')|_{F'} \equiv 0$ by [Nak04, Propositions III.1.14(1) and V.2.7(8)].

By Lemma 1.18 there exists an open subset $U \subseteq Y$ such that

$$(K_{X'}+B'+M')|_{U'} \equiv_U 0,$$

where $U' = (f')^{-1}(U)$. Note that the natural projection $\mathrm{Div}_{\mathbb{R}}(U') \to N^1(U'/U)$ is defined over $\mathbb{Q}$, so there exist $\mathbb{Q}$-divisors $D_1, \dots, D_m$ on U' such that $D_i \equiv_U 0$ for each $i \in \{1, \dots, m\}$ and real numbers $r_1, \dots, r_m$ such that

$$(K_{X'}+B'+M')|_{U'} = \sum_{i=1}^{m} r_i D_i. \tag{2.9}$$

Since X' is a klt variety, so is F', hence F' has rational singularities, see Remark 2.7. Since $h^1(F, \mathcal{O}_F) = 0$ by assumption, we deduce by [Har77, Exercise III.8.1] that $h^1(F', \mathcal{O}_{F'}) = 0$. Therefore, $D_i|_{F'} \sim_{\mathbb{Q}} 0$ for every $i \in \{1, \dots, m\}$, and, after possibly shrinking U, by Lemma 1.19 we have $D_i \sim_{\mathbb{Q},U} 0$. But then by (2.9) we obtain

$$(K_{X'} + B' + M')|_{U'} \sim_{\mathbb{R},U} 0.$$

It follows now from [BCHM10, Lemma 3.2.1] that $K_{X'} + B' + M'$ is effective over Y, and therefore so is $K_X + B + M$. □

The following result also plays a crucial role in the proof of Theorem 3.14. It is a special case of [HL20b, Lemma 4.3] and builds on [DHP13, Proposition 8.7], [Gon15, Lemma 3.1] and [DL15, Theorem 3.3]. Nevertheless, we provide a detailed proof below.

Lemma 2.51. *Let $(X, B + M)$ be an NQC $\mathbb{Q}$-factorial dlt g-pair of dimension n with data $T \to X \to Z$ and L. Assume that the divisor $K_X + B + M$ is pseudo-effective over Z and that for each $\varepsilon > 0$ the divisor $K_X + B + (1 - \varepsilon)M$ is not pseudo-effective over Z. Then there exist a birational contraction $\varphi \colon X \dashrightarrow X'$ over Z and a fibration $f \colon X' \to Y$ over Z such that:*

(i) *$(X', B' + M')$ is an NQC $\mathbb{Q}$-factorial lc g-pair, where $B' := \varphi_* B$ and M' is the pushforward of L,*

(ii) *$K_{X'} + B' + M' \sim_{\mathbb{R},Y} 0$,*

(iii) *φ is a $\big(K_X + B + (1 - \varepsilon)M\big)$-MMP over Z for some $0 < \varepsilon \ll 1$ and f is the corresponding Mori fiber space.*

Proof. Fix a general ample over Z divisor $A \geq 0$ on X. Consider a decreasing sequence $\{\varepsilon_i\}_{i=1}^{\infty}$ of positive real numbers such that $\lim_{i \to \infty} \varepsilon_i = 0$. For each $i \geq 1$ let

$$y_i = \inf \big\{ t \in \mathbb{R}_{\geq 0} \mid K_X + B + (1 - \varepsilon_i)M + tA \text{ is pseudo-effective over } Z \big\}$$

be *the pseudo-effective threshold of A with respect to $\big(X, B + (1 - \varepsilon_i)M\big)$*. Observe that $y_i > 0$ for all $i \geq 1$.

Fix $i \geq 1$. We may run a $\big(K_X + B + (1 - \varepsilon_i)M\big)$-MMP with scaling of A over Z. Note that this MMP is also a $(K_X + B + (1 - \varepsilon_i)M + \nu A)$-MMP for some (actually any) $0 < \nu < y_i$. By [HL18, Lemma 3.5] there exists a boundary divisor Δ such that $K_X + B + (1 - \varepsilon_i)M + \nu A \sim_{\mathbb{R},Z} K_X + \Delta$ and the pair (X, Δ) is klt. Therefore, this MMP is clearly also a $(K_X + \Delta)$-MMP with scaling of A over Z, which terminates with a Mori fiber space $g_i \colon X_i \to Y_i$ over Z by [BCHM10, Corollary 1.3.3]. Let $f_i \colon X \dashrightarrow X_i$ be the resulting birational contraction over Z. Denote by B_i the strict transform of B on X_i and by M_i the pushforward of L on X_i. Since $K_X + B + M$ is pseudo-effective over Z and f_i is a birational contraction, it follows that $K_{X_i} + B_i + M_i$ is pseudo-effective over Z, hence there exist effective $\mathbb{R}$-divisors E_j on X_i such that $\lim_{j \to \infty} [E_j] = [K_{X_i} + B_i + M_i]$ in $N^1(X_i/Z)$. Let C be a curve on X_i which does not belong to $\bigcup \operatorname{Supp} E_j$ and is contracted by g_i. Then

$$\big(K_{X_i} + B_i + (1 - \varepsilon_i)M_i\big) \cdot C < 0$$

by the definition of the MMP, and

$$(K_{X_i} + B_i + M_i) \cdot C \geq 0$$

by the choice of C. Therefore, there exists $\eta_i \in (1-\varepsilon_i, 1]$ such that

$$(K_{X_i} + B_i + \eta_i M_i) \cdot C = 0,$$

and since all contracted curves are numerically proportional, we obtain

$$K_{X_i} + B_i + \eta_i M_i \equiv_{Y_i} 0.$$

In particular, if F_i is a very general fiber of g_i and if $B_{F_i} := B_i|_{F_i}$ and $M_{F_i} := M_i|_{F_i}$, then we have

$$K_{F_i} + B_{F_i} + \eta_i M_{F_i} \equiv 0.$$

Now, for every $i \geq 1$, set

$$\tau_i = \sup\big\{t \in \mathbb{R}_{\geq 0} \mid (X_i, B_i + tM_i) \text{ is lc}\big\}$$

and note that $1-\varepsilon_i \leq \tau_i$, since $(X_i, B_i + (1-\varepsilon_i)M_i)$ is an lc g-pair. If $(X_i, B_i + M_i)$ is not lc for infinitely many i, then after passing to a subsequence we can assume that $\tau_i < 1$ for all $i \geq 1$, and since $1-\varepsilon_i \leq \tau_i$ and $\lim_{i\to\infty}(1-\varepsilon_i) = 1$, we can also assume that the sequence $\{\tau_i\}_{i=1}^{\infty}$ of lc thresholds is strictly increasing, but this contradicts *the ascending chain condition for lc thresholds* [BZ16, Theorem 1.5]. Therefore, the g-pairs $(X_i, B_i + M_i)$ are lc for $i \gg 0$, and thus by Lemma 2.14 the g-pairs $(X_i, B_i + \eta_i M_i)$ are lc for $i \gg 0$. In particular, the g-pairs $(F_i, B_{F_i} + \eta_i M_{F_i})$ are lc for $i \gg 0$. Since $K_{F_i} + B_{F_i} + \eta_i M_{F_i} \equiv 0$, it follows from *the global ascending chain condition* [BZ16, Theorem 1.6] that the sequence $\{\eta_i\}_{i=1}^{\infty}$ is eventually constant, hence $\eta_i = 1$ for $i \gg 0$, since $1-\varepsilon_i < \eta_i \leq 1$ and $\lim_{i\to\infty}(1-\varepsilon_i) = 1$.

Consequently, we choose φ to be any of the f_i for $i \gg 0$, we set $X' := X_i$ and $Y := Y_i$, and we also take f to be the corresponding Mori fiber space g_i. This completes the proof. □

2.3.4 MMPs as Sequences of Flops

The following two results exploit *the boundedness of the length of extremal rays*. The first one is a special case of [Bir11, Proposition 3.2(5)]. The second one is an analog of the first one in the context of g-pairs. We remark that similar statements (for g-pairs) have also been observed in [HM18] and [HL18, Section 3.3]. Lastly, we refer to [Kaw91, Theorem 1], [Fuj17, Section 4.6] and [HL18, Section 3.2] for further details regarding the length of extremal rays.

Lemma 2.52. *Let $(X/Z, B)$ be an lc pair such that $K_X + B$ is nef over Z. Then there exists $\varepsilon_0 > 0$ such that for every $\varepsilon \in (0, \varepsilon_0)$ any $\big(K_X + (1-\varepsilon)B\big)$-MMP over Z is $(K_X + B)$-trivial.*

Proof. According to [Fuj11a, Remark 18.8] the arguments in [Bir11, Section 3] work also for lc pairs. Hence, the lemma follows from [Bir11, Proposition 3.2(5)]. □

Lemma 2.53. *Let $(X/Z, B+M)$ be an NQC lc g-pair such that $(X, 0)$ is $\mathbb{Q}$-factorial klt and $K_X + B + M$ is nef over Z. Then there exists $\varepsilon_0 > 0$ such that for every $\varepsilon \in (0, \varepsilon_0)$ any $\big(K_X + B + (1-\varepsilon)M\big)$-MMP with scaling of an ample divisor over Z is $(K_X + B + M)$-trivial.*

Proof. By [HL18, Proposition 3.16] there exist $\mathbb{Q}$-divisors $B_1, \dots, B_m$ and $M_1, \dots, M_m$ such that each g-pair $(X/Z, B_i + M_i)$ is lc with $K_X + B_i + M_i$ nef over Z, and there exist positive real numbers $\alpha_1, \dots, \alpha_m$ such that $\sum \alpha_i = 1$ and

$$K_X + B + M = \sum_{i=1}^{m} \alpha_i (K_X + B_i + M_i). \tag{2.10}$$

Fix a positive integer r such that $r(K_X + B_i + M_i)$ is Cartier for each $i \in \{1, \dots, m\}$. Consider the set

$$\mathcal{S} = \big\{ \sum \alpha_i n_i > 0 \mid n_i \in \mathbb{N},\ 1 \le i \le m \big\}.$$

Then clearly there exists $\beta > 0$ such that $s > \beta$ for all $s \in \mathcal{S}$. Set

$$\varepsilon_0 := \frac{\beta}{\beta + 2r \dim X}$$

and fix any $0 < \varepsilon < \varepsilon_0$.

We run a $\big(K_X + B + (1-\varepsilon)M\big)$-MMP with scaling of an ample divisor over Z. It suffices to show that this MMP is $(K_X + B_i + M_i)$-trivial at the first step for each $i \in \{1, \dots, m\}$, since then the strict transform of $K_X + B_i + M_i$ stays nef and $r(K_X + B_i + M_i)$ stays Cartier along the MMP due to [KM98, Theorem 3.25(4)].

Let R be a $\big(K_X + B + (1-\varepsilon)M\big)$-negative extremal ray over Z. Since by assumption we have $(K_X + B + M) \cdot R \ge 0$, we infer

$$(K_X + B) \cdot R < 0 \quad \text{and} \quad M \cdot R > 0.$$

By the boundedness of the length of extremal rays [HL18, Proposition 3.13] we may find a curve C on X whose class belongs to R such that

$$-2 \dim X \le (K_X + B) \cdot C < 0 \tag{2.11}$$

and

$$-2 \dim X \le \big(K_X + B + (1-\varepsilon)M\big) \cdot C < 0. \tag{2.12}$$

From (2.11) and (2.12) we obtain $(1-\varepsilon) M \cdot C \le 2 \dim X$. Therefore

$$0 < M \cdot C \le \frac{2 \dim X}{1 - \varepsilon}.$$

This implies

$$(K_X + B + M) \cdot C = \big(K_X + B + (1-\varepsilon)M\big) \cdot C + \varepsilon M \cdot C < \frac{2\varepsilon \dim X}{1-\varepsilon} < \frac{\beta}{r}.$$

If $(K_X + B + M) \cdot C > 0$, then $r(K_X + B + M) \cdot C \in \mathcal{S}$ by (2.10), a contradiction to the choice of β. Hence $(K_X + B + M) \cdot C = 0$, and by (2.10) we obtain $(K_X + B_i + M_i) \cdot C = 0$. This finishes the proof. □

2.3.5 Lifting a Sequence of Ample Small Quasi-Flips

The next result is a slight generalization of [CT20, Lemma 2.15] and constitutes one of the main ingredients for the proof of Theorem 6.9. It allows us to pass from a sequence of (ample small quasi-)flips for klt g-pairs to a sequence of flips for terminal g-pairs under certain conditions (which, actually, may not be possible to achieve in general, but can at least be achieved with serious effort in the setting of Theorem 6.9).

Lemma 2.54. *Let $(X_1/Z, B_1+M_1)$ be a klt g-pair. Consider a sequence of ample small quasi-flips over Z:*

$$\begin{array}{ccccccc} (X_1, B_1+M_1) & \overset{\pi_1}{\dashrightarrow} & (X_2, B_2+M_2) & \overset{\pi_2}{\dashrightarrow} & (X_3, B_3+M_3) & \overset{\pi_3}{\dashrightarrow} & \cdots \\ & \searrow \; Z_1 \; \swarrow & & \searrow \; Z_2 \; \swarrow & & & \end{array}$$

Assume that for each $i \geq 1$ there exists a $\mathbb{Q}$-factorial terminalization $(Y_i', \Delta_i' + N_i')$ of $(X_i, B_i + M_i)$ such that each Y_{i+1}' is isomorphic in codimension one to Y_i' and each Δ_{i+1}' is the strict transform of Δ_i'. Then there exists a diagram

$$\begin{array}{ccccccc} (Y_1, \Delta_1+N_1) & \overset{\rho_1}{\dashrightarrow} & (Y_2, \Delta_2+N_2) & \overset{\rho_2}{\dashrightarrow} & (Y_3, \Delta_3+N_3) & \overset{\rho_3}{\dashrightarrow} & \cdots \\ \downarrow h_1 & & \downarrow h_2 & & \downarrow h_3 & & \\ (X_1, B_1+M_1) & \overset{\pi_1}{\dashrightarrow} & (X_2, B_2+M_2) & \overset{\pi_2}{\dashrightarrow} & (X_3, B_3+M_3) & \overset{\pi_3}{\dashrightarrow} & \cdots \\ & \searrow \; Z_1 \; \swarrow & & \searrow \; Z_2 \; \swarrow & & & \end{array}$$

where, for each $i \geq 1$, the map ρ_i is a sequence of $(K_{Y_i} + \Delta_i + N_i)$-flips over Z_i and the map h_i is a $\mathbb{Q}$-factorial terminalization of $(X_i, B_i + M_i)$.

In particular, the sequence on top of the above diagram is a sequence of flips for a $\mathbb{Q}$-factorial terminal g-pair $(Y_1, \Delta_1 + N_1)$.

Proof. Fix $i \geq 1$. Since $(X_{i+1}, B_{i+1}+M_{i+1})$ is the canonical model of (X_i, B_i+M_i) over Z_i by Remark 2.43, by assumption and by construction of a $\mathbb{Q}$-factorial terminalization we deduce that $(Y_{i+1}', \Delta_{i+1}' + N_{i+1}')$ is a minimal model of $(Y_i', \Delta_i' + N_i')$ over Z_i.

Set $(Y_1, \Delta_1+N_1) := (Y_1', \Delta_1'+N_1')$ and denote by $h_1\colon (Y_1, \Delta_1+N_1) \to (X_1, B_1+M_1)$ the corresponding morphism. By [BZ16, Lemma 4.4(2)] there exists a $(K_{Y_1}+\Delta_1+N_1)$-MMP with scaling of an ample divisor over Z_1 which terminates with a minimal model $(Y_2, \Delta_2 + N_2)$ of $(Y_1, \Delta_1 + N_1)$ over Z_1. Since $(X_2, B_2 + M_2)$ is the canonical model of (Y_1, Δ_1+N_1) over Z_1, by Proposition 2.38 there exists a projective birational morphism $h_2\colon Y_2 \to X_2$ such that $K_{Y_2} + \Delta_2 + N_2 \sim_{\mathbb{R}} h_2^*(K_{X_2} + B_2 + M_2)$. Since $(Y_2', \Delta_2' + N_2')$ is a minimal model of $(Y_1, \Delta_1 + N_1)$ over Z_1, by Proposition 2.37 we deduce that this MMP with scaling over Z_1 consists only of flips, and therefore $h_2\colon (Y_2, \Delta_2 + N_2) \to (X_2, B_2 + M_2)$ is a $\mathbb{Q}$-factorial terminalization of $(X_2, B_2 + M_2)$. By continuing this process analogously, we obtain the required diagram. □

With a similar argument and by invoking Lemma 2.24(ii) instead of Lemma 2.24(i) we obtain the following result, which will not be used elsewhere in the thesis, but is included here for the sake of completeness nonetheless. Note that the corresponding statement for usual pairs appears, for instance, in Step 1 of the proof of [Bir10b, Theorem 1.2] and its proof uses [Bir09, Remark 2.3]. Lastly, observe that (in contrast to the other results of this subsection) we consider a sequence of flips instead of a sequence of ample small quasi-flips over Z. The reason why we do so will become clear from the proof of this result.

Lemma 2.55. *Let $(X_1/Z, B_1 + M_1)$ be a klt g-pair. Consider a sequence of flips over Z:*

$$\begin{array}{ccccccc} (X_1, B_1 + M_1) & \overset{\pi_1}{\dashrightarrow} & (X_2, B_2 + M_2) & \overset{\pi_2}{\dashrightarrow} & (X_3, B_3 + M_3) & \overset{\pi_3}{\dashrightarrow} & \cdots \\ & {\scriptstyle\theta_1}\searrow \quad \swarrow{\scriptstyle\theta_1^+} & & {\scriptstyle\theta_2}\searrow \quad \swarrow{\scriptstyle\theta_2^+} & & & \\ & Z_1 & & Z_2 & & & \end{array}$$

Then there exists a diagram

$$\begin{array}{ccccccc} (Y_1, \Delta_1 + N_1) & \overset{\rho_1}{\dashrightarrow} & (Y_2, \Delta_2 + N_2) & \overset{\rho_2}{\dashrightarrow} & (Y_3, \Delta_3 + N_3) & \overset{\rho_3}{\dashrightarrow} & \cdots \\ {\scriptstyle h_1}\downarrow & & {\scriptstyle h_2}\downarrow & & {\scriptstyle h_3}\downarrow & & \\ (X_1, B_1 + M_1) & \overset{\pi_1}{\dashrightarrow} & (X_2, B_2 + M_2) & \overset{\pi_2}{\dashrightarrow} & (X_3, B_3 + M_3) & \overset{\pi_3}{\dashrightarrow} & \cdots \\ & {\scriptstyle\theta_1}\searrow \quad \swarrow{\scriptstyle\theta_1^+} & & {\scriptstyle\theta_2}\searrow \quad \swarrow{\scriptstyle\theta_2^+} & & & \\ & Z_1 & & Z_2 & & & \end{array}$$

where, for each $i \geq 1$, the map $\rho_i \colon Y_i \dashrightarrow Y_{i+1}$ is a sequence of $(K_{Y_i} + \Delta_i + N_i)$-flips over Z_i and the map h_i is a small $\mathbb{Q}$-factorialization of the g-pair $(X_i, B_i + M_i)$.

In particular, the sequence on top of the above diagram is a sequence of flips for a $\mathbb{Q}$-factorial klt g-pair $(Y_1, \Delta_1 + N_1)$.

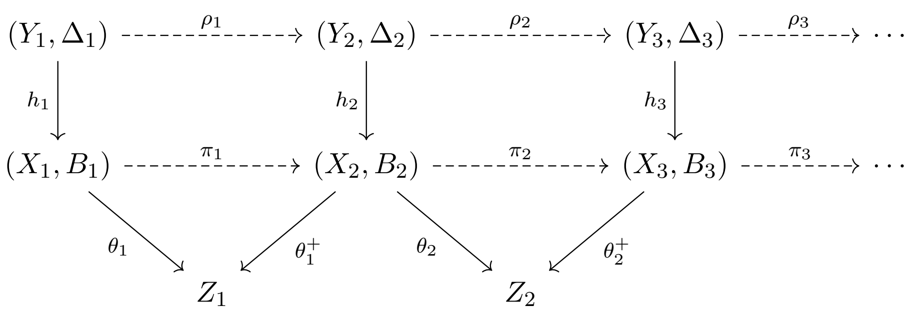
Proof. Let $h_1 \colon (Y_1, \Delta_1 + N_1) \to (X_1, B_1 + M_1)$ be a small $\mathbb{Q}$-factorialization of the g-pair $(X_1, B_1 + M_1)$, see Lemma 2.24(ii). By [BZ16, Lemma 4.4(2)] there exists a $(K_{Y_1} + \Delta_1 + N_1)$-MMP with scaling of an ample divisor over Z_1 which terminates with a minimal model $(Y_2, \Delta_2 + N_2)$ of $(Y_1, \Delta_1 + N_1)$ over Z_1. We claim that this MMP consists only of flips. Indeed, if a divisorial contraction with exceptional prime divisor E appears at some step of this MMP, then the strict transform of E on Y_1 must be a $(\theta_1 \circ h_1)$-exceptional divisor, which is absurd, since both θ_1 and h_1 are small contractions. Now, since $(X_2, B_2 + M_2)$ is the canonical model of $(Y_1, \Delta_1 + \Xi_1)$ over Z_1, by Proposition 2.38 there exists a projective birational morphism $h_2 \colon Y_2 \to X_2$ such that $K_{Y_2} + \Delta_2 + \Xi_2 \sim_{\mathbb{R}} h_2^*(K_{X_2} + B_2 + M_2)$. Note that h_2 is a small contraction. Consequently, the g-pair $(Y_2, \Delta_2 + M_2)$ is a small $\mathbb{Q}$-factorialization of the g-pair $(X_2, B_2 + M_2)$. By continuing this process analogously, we obtain the required diagram. □

The following result allows us to pass from a sequence of (small ample quasi-)flips for lc pairs to a sequence of flips for dlt pairs.

Lemma 2.56. *Let $(X_1/Z, B_1)$ be an lc pair. Consider a sequence of ample small quasi-flips over Z:*

$$\begin{array}{ccccccc} (X_1, B_1) & \overset{\pi_1}{\dashrightarrow} & (X_2, B_2) & \overset{\pi_2}{\dashrightarrow} & (X_3, B_3) & \overset{\pi_3}{\dashrightarrow} & \cdots \\ & {\scriptstyle\theta_1}\searrow \quad \swarrow{\scriptstyle\theta_1^+} & & {\scriptstyle\theta_2}\searrow \quad \swarrow{\scriptstyle\theta_2^+} & & & \\ & Z_1 & & Z_2 & & & \end{array}$$

Then there exists a diagram



where, for each $i \geq 1$, the map $\rho_i \colon Y_i \dashrightarrow Y_{i+1}$ is a $(K_{Y_i} + \Delta_i)$-MMP over Z_i and the map h_i is a dlt blow-up of the pair (X_i, B_i).

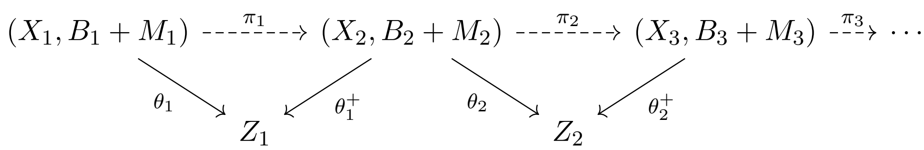
In particular, the sequence on top of the above diagram is an MMP for a $\mathbb{Q}$-factorial dlt pair (Y_1, Δ_1).

Proof. Let $h_1 \colon (Y_1, \Delta_1) \to (X_1, B_1)$ be a dlt blow-up of (X_1, B_1). It follows from Remark 2.43(1) that the pair (X_1, B_1) has a minimal model in the sense of Birkar-Shokurov over Z_1, hence (Y_1, Δ_1) has a minimal model in the sense of Birkar-Shokurov over Z_1 by [Has19, Lemma 2.15]. Therefore, by [Bir12a, Theorem 1.9(ii),(iii)] there exists a $(K_{Y_1} + \Delta_1)$-MMP with scaling of an ample divisor over Z_1 which terminates with a minimal model (Y_2, Δ_2) of (Y_1, Δ_1) over Z_1. Since (X_2, B_2) is the canonical model of (Y_1, Δ_1) over Z_1, by Proposition 2.38 there exists a morphism $h_2 \colon Y_2 \to X_2$ such that $K_{Y_2} + \Delta_2 \sim_{\mathbb{R}} h_2^*(K_{X_2} + B_2)$. In particular, the pair (Y_2, Δ_2) is a dlt blow-up of (X_2, B_2). By continuing this process analogously, we obtain the required diagram. □

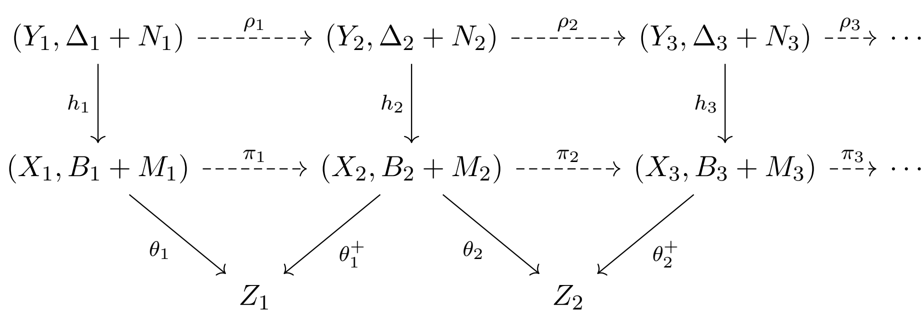
Finally, we derive the analog of Lemma 2.56 in the context of g-pairs. For its proof we need some notions and results from Chapters 3 and 4, but, despite that, we incorporate this result here in order to render this subsection complete. Hence, we emphasize that Lemma 2.57 will be applied only in Chapters 5 and 6.

Lemma 2.57. *Assume the existence of minimal models for smooth varieties of dimension $n - 1$.*

Let $(X_1/Z, B_1 + M_1)$ be an NQC lc g-pair of dimension n. Consider a sequence of ample small quasi-flips over Z:



Then there exists a diagram



where, for each $i \geq 1$*, the map* $\rho_i \colon Y_i \dashrightarrow Y_{i+1}$ *is a* $(K_{Y_i} + \Delta_i + N_i)$*-MMP over* Z_i *and the map* h_i *is a dlt blow-up of the g-pair* $(X_i, B_i + M_i)$*.*

In particular, the sequence on top of the above diagram is an MMP for an NQC $\mathbb{Q}$*-factorial dlt g-pair* $(Y_1, \Delta_1 + N_1)$*.*

Proof. Let $h_1 \colon (Y_1, \Delta_1 + N_1) \to (X_1, B_1 + M_1)$ be a dlt blow-up of $(X_1, B_1 + M_1)$. It follows from Remark 2.43(1) that the g-pair $(X_2, B_2 + M_2)$ is a minimal model of $(X_1, B_1 + M_1)$ over Z_1. Hence, $(X_1, B_1 + M_1)$ admits an NQC weak Zariski decomposition over Z_1 by Corollary 3.10, and it follows now from Remark 3.6 that $(Y_1, \Delta_1 + N_1)$ admits an NQC weak Zariski decomposition over Z_1. Therefore, by Theorem 4.16(ii) there exists a $(K_{Y_1} + \Delta_1 + N_1)$-MMP with scaling of an ample divisor over Z_1 which terminates with a minimal model $(Y_2, \Delta_2 + N_2)$ of $(Y_1, \Delta_1 + N_1)$ over Z_1. Since $(X_2, B_2 + M_2)$ is the canonical model of $(Y_1, \Delta_1 + N_1)$ over Z_1, by Proposition 2.38 there exists a morphism $h_2 \colon Y_2 \to X_2$ such that $K_{Y_2} + \Delta_2 + N_2 \sim_{\mathbb{R}} h_2^*(K_{X_2} + B_2 + M_2)$. In particular, the g-pair $(Y_2, \Delta_2 + N_2)$ is a dlt blow-up of the g-pair $(X_2, B_2 + M_2)$. By continuing this process analogously, we obtain the required diagram. □

3

Zariski Decompositions

This chapter is devoted to the study of two distinct types of Zariski decompositions, which can be regarded as higher-dimensional analogs of the classical Zariski decomposition on surfaces [Laz04, Theorem 2.3.19]. More precisely, in Section 3.1 we deal with *NQC weak Zariski decompositions*, which play a fundamental role in the thesis and emerge repeatedly in the sequel, while in Section 3.2 we discuss *NQC Nakayama-Zariski decompositions*. We establish the basic properties of these decompositions and we are especially concerned with the problem of their existence, linking it to the existence of minimal models.

The contents of Section 3.1 are mainly taken from our joint paper [LT19] with Vladimir Lazić (specifically, we reproduce everything from [LT19, Sections 2 and 3] that is related to NQC weak Zariski decompositions), while the contents of Section 3.2 are new and the results appearing there parallel those of Subsection 3.1.1.

Applications of the theory of higher-dimensional Zariski decompositions developed in this chapter towards the existence of minimal models conjecture will be presented in the next chapter of the thesis. Furthermore, for an overview of the various types of Zariski decompositions that have appeared in the literature, including the classical Zariski decomposition of an effective divisor on smooth projective surface, we refer to [KMM87, Section 7.3], [Pro03], [Laz04, Section 2.3.E], [Bir12b, Section 1] and [BH14].

Throughout this chapter, unless otherwise stated, we assume that varieties are normal and quasi-projective and that a variety X over a variety Z is projective over Z.

3.1 NQC Weak Zariski Decompositions

Weak Zariski decompositions were introduced by Birkar [Bir12b], who showed that there is a basic relation between the existence of such decompositions and the existence of minimal models. Specifically, assuming the termination of flips in lower dimensions, an lc pair has a minimal model in the sense of Birkar-Shokurov if and only if it admits a weak Zariski decomposition, see [Bir12b, Theorem 1.5]. The NQC condition in the definition of weak Zariski decompositions was added later by Han and Li [HL18], who proved that for NQC $\mathbb{Q}$-factorial dlt g-pairs the existence of NQC weak Zariski decompositions in dimension $\leq n$ is equivalent to the existence of minimal models in dimension $\leq n$, see [HL18, Theorem 1.5].

In this section we recall the definition of NQC weak Zariski decompositions and we study their fundamental properties. Our investigation culminates in Theorem 3.17, which constitutes one of the main results of the thesis and which improves considerably on [HM18, Theorem 2]. It shows that the existence of NQC weak Zariski decompositions for NQC lc g-pairs follows from the existence of NQC weak Zariski decompositions for smooth varieties, rendering thus these two statements equivalent.

Last but not least, with the aid of Theorem 3.17 we will give in Chapter 4 the state-of-the-art statements concerning the relation between the existence of minimal models and the existence of NQC weak Zariski decompositions, which refine the aforementioned results of Birkar [Bir12b] and Han and Li [HL18].

3.1.1 Definition and Basic Properties

Definition 3.1. Let $\pi\colon X \to Z$ be a projective morphism of normal varieties and let D be an $\mathbb{R}$-Cartier $\mathbb{R}$-divisor on X. An *NQC weak Zariski decomposition of D over Z* consists of a projective birational morphism $f\colon W \to X$ from a normal variety W and a numerical equivalence $f^*D \equiv_Z P + N$, where P is an NQC divisor (over Z) on W and N is an effective $\mathbb{R}$-Cartier $\mathbb{R}$-divisor on W.

Note that if D admits an NQC weak Zariski decomposition over Z, then D is necessarily pseudo-effective over Z. On the other hand, it has been shown that there exists a pseudo-effective divisor that does not admit an NQC weak Zariski decomposition, see [Les14, Theorem 1.1].

Remark 3.2. With the same notation as in Definition 3.1, assume that D has an NQC weak Zariski decomposition over Z and consider a projective surjective morphism $g\colon Y \to X$ from a normal variety Y. Then g^*D has an NQC weak Zariski decomposition over Z. Indeed, by considering a resolution of indeterminacies $(p,q)\colon T \to Y \times W$ of $f^{-1} \circ g\colon Y \dashrightarrow W$ such that T is normal, we have

$$p^*g^*D = q^*f^*D \equiv_Z q^*P + q^*N,$$

which proves the assertion.

If, additionally, g is birational, then the converse is also clear, namely, if g^*D has an NQC weak Zariski decomposition over Z, then D has an NQC weak Zariski decomposition over Z.

Remark 3.3. With the same notation as in Definition 3.1, assume that D has an NQC weak Zariski decomposition over Z and let M be an $\mathbb{R}$-Cartier $\mathbb{R}$-divisor on X which is the pushforward of an NQC divisor on some higher model of X. Then $D + M$ has an NQC weak Zariski decomposition over Z. Indeed, we may assume that $M = f_*M'$, where M' is an NQC divisor on W, and then by the Negativity lemma [KM98, Lemma 3.39(1)] we have $f^*M = M' + E$, where E is an effective f-exceptional $\mathbb{R}$-Cartier $\mathbb{R}$-divisor on W. Hence,

$$f^*(D + M) \equiv_Z (P + M') + (N + E),$$

which proves the assertion.

Lemma 3.4. *Let $f\colon X \dashrightarrow Y$ be a birational contraction between $\mathbb{Q}$-factorial varieties which are projective over Z. Let D be an $\mathbb{R}$-divisor on X such that the map f is D-non-positive. Then D has an NQC weak Zariski decomposition over Z if and only if f_*D has an NQC weak Zariski decomposition over Z.*

Proof. Set $G = f_*D$. Since f is D-non-positive, if $(p, q)\colon W \to X \times Y$ is a resolution of indeterminacies of f, then there exists an effective q-exceptional $\mathbb{R}$-Cartier $\mathbb{R}$-divisor E on W such that $p^*D \equiv_Z q^*G + E$. By Remark 3.2 we may replace X with W, and hence we may assume that f is a morphism and

$$D \equiv_Z f^*G + E. \tag{3.1}$$

Now, assume that D has an NQC weak Zariski decomposition over Z. Then there exists a projective birational morphism $\pi\colon T \to X$ from a normal variety T such that $\pi^*D \equiv_Z P + N$, where P is an NQC divisor on T and N is an effective $\mathbb{R}$-Cartier $\mathbb{R}$-divisor on T. If we set $g := f \circ \pi$, then $G \equiv_Z g_*P + g_*N$ by (3.1). By the Negativity lemma [KM98, Lemma 3.39(1)] we have $g^*g_*P = P + F$ for some effective g-exceptional $\mathbb{R}$-Cartier $\mathbb{R}$-divisor F on T, and thus

$$g^*G \equiv_Z g^*g_*P + g^*g_*N = P + (F + g^*g_*N),$$

which shows that G has an NQC weak Zariski decomposition over Z.

Conversely, if G has an NQC weak Zariski decomposition over Z, then we conclude by arguing as in Remark 3.2 and by taking (3.1) into account. □

Definition 3.5. Let $(X/Z, B+M)$ be an NQC g-pair. We say that $(X, B+M)$ *admits an NQC weak Zariski decomposition over* Z if the divisor $K_X + B + M$ admits an NQC weak Zariski decomposition over Z.

Remark 3.6. Let $(X/Z, B + M)$ be an NQC lc g-pair. It follows from Remark 3.2 that $(X, B + M)$ admits an NQC weak Zariski decomposition over Z if and only if any dlt blow-up of $(X, B + M)$ admits an NQC weak Zariski decomposition over Z.

The next observation appears also in Step 1 of the proof of [HL18, Theorem 5.4] and generalizes (the second part of) [Bir12b, Remark 2.4(i)] to the setting of g-pairs (see Lemma 2.31 for the generalization of the first part of that Remark). Nevertheless, we provide here the details for the convenience of the reader.

Remark 3.7. Let $(X/Z, B + M)$ be an NQC lc g-pair with data $X' \xrightarrow{f} X \to Z$ and M'. Assume that $(X, B + M)$ admits an NQC weak Zariski decomposition over Z. By replacing X' with a higher model, we may assume that f is a sufficiently high log resolution of (X, B) and that

$$f^*(K_X + B + M) \equiv_Z P + N,$$

where P is an NQC divisor on X' and N is an effective $\mathbb{R}$-Cartier $\mathbb{R}$-divisor on X', see Remark 3.2. If $B' := f_*^{-1}B + E$, where E is the sum of the f-exceptional prime divisors on X', then $(X', B' + M')$ is a log smooth model of $(X, B + M)$ and we may write

$$K_{X'} + B' + M' \sim_{\mathbb{R}} f^*(K_X + B + M) + F,$$

where F is an effective f-exceptional $\mathbb{R}$-Cartier $\mathbb{R}$-divisor on X'. Consequently,

$$K_{X'} + B' + M' \equiv_Z P + (N + F)$$

is an NQC weak Zariski decomposition of $(X', B' + M')$ over Z.

The next lemma plays a key role in the proof of Theorem 3.17 and it will also be used repeatedly in Section 4.1. We remark that its proof is based on the idea of the proof of [Has18b, Lemma 3.2].

Lemma 3.8. *Let $n, k \in \mathbb{Z}_{\geq 1}$ with $n \geq k$. Assume that every pseudo-effective smooth pair $(X/Z, 0)$ of dimension n admits an NQC weak Zariski decomposition over Z. Then every pseudo-effective smooth pair $(Y/Z, 0)$ of dimension k admits an NQC weak Zariski decomposition over Z.*

Proof. Let $(Y/Z, 0)$ be a k-dimensional pair as in the statement of the lemma. Set $X := Y \times A$, where A is any $(n-k)$-dimensional abelian variety, and let $p\colon X \to Y$ and $q\colon X \to A$ be the projection maps. Then $K_X \sim p^*K_Y$; in particular, K_X is pseudo-effective. By assumption there exists a projective birational morphism $f\colon W \to X$ from a normal variety W and a numerical equivalence

$$f^*K_X \equiv_Z P + N, \tag{3.2}$$

where P is an NQC divisor on W and N is an effective $\mathbb{R}$-Cartier $\mathbb{R}$-divisor on W. Now, let F be a very general fiber of q, observe that $F \simeq Y$, and set $F_W := f^{-1}(F)$. By restricting (3.2) to F_W we obtain

$$\left(f|_{F_W}\right)^* K_F \equiv_Z P|_{F_W} + N|_{F_W},$$

which is an NQC weak Zariski decomposition of $(Y, 0)$ over Z. □

A basic result concerning the existence of NQC weak Zariski decompositions for NQC lc g-pairs is [HL18, Proposition 5.1]. We reproduce here this result and we also provide its proof for the sake of completeness.

Proposition 3.9. *Let $(X/Z, B+M)$ be an NQC lc g-pair. If $(X, B+M)$ has a minimal model in the sense of Birkar-Shokurov over Z, then it admits an NQC weak Zariski decomposition over Z.*

Proof. Let $\varphi\colon (X, B+M) \dashrightarrow (Y, B_Y + M_Y)$ be a minimal model in the sense of Birkar-Shokurov of $(X, B+M)$ over Z. By [HL18, Proposition 3.16] there exist positive real numbers $\mu_1, \dots, \mu_m$ and $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisors $P_1, \dots, P_m$ on Y which are nef over Z such that

$$K_Y + B_Y + M_Y = \sum_{i=1}^{m} \mu_i P_i. \tag{3.3}$$

Let $(p, q)\colon W \to X \times Y$ be a resolution of indeterminacies of φ. By Lemma 2.18(i) we may write

$$p^*(K_X + B + M) \sim_{\mathbb{R}} q^*(K_Y + B_Y + M_Y) + E, \tag{3.4}$$

where E is an effective q-exceptional $\mathbb{R}$-Cartier $\mathbb{R}$-divisor on W. Hence, by (3.3) and (3.4) we obtain

$$p^*(K_X + B + M) \equiv_Z \sum_{i=1}^{m} \mu_i q^* P_i + E,$$

which is an NQC weak Zariski decomposition of $(X, B+M)$ over Z. □

Corollary 3.10. *Let $(X/Z, B+M)$ be an NQC lc g-pair. If $(X, B+M)$ has a minimal model over Z, then it admits an NQC weak Zariski decomposition over Z.*

Proof. If $(Y, B_Y + M_Y)$ is a minimal model of $(X, B+M)$ over Z, then any dlt blow-up of $(Y, B_Y + M_Y)$ is a minimal model in the sense of Birkar-Shokurov of $(X, B + M)$ over Z, and now we conclude by Proposition 3.9. Alternatively, we repeat verbatim the proof of Proposition 3.9, but this time we invoke [HL20b, Proposition 2.6] instead of [HL18, Proposition 3.16]; note that the former is a refinement of the latter. □

3.1.2 On the Termination of MMPs with Scaling

The following result is a slightly reformulated version of [HL18, Theorem 1.7]. Hence, we only give a sketch of its proof, referring to [HL18] and [Bir12b, Proof of Theorem 1.5, (1) $\Longrightarrow$ (4)] for the details.

Theorem 3.11. *Assume the existence of NQC weak Zariski decompositions for NQC $\mathbb{Q}$-factorial dlt g-pairs of dimension at most $n-1$.*

Let $(X/Z, B+M)$ be an NQC lc g-pair of dimension n which admits an NQC weak Zariski decomposition over Z. Then the following statements hold.

(i) *The g-pair $(X, B+M)$ has a minimal model in the sense of Birkar-Shokurov over Z.*

(ii) *If, additionally, $(X, 0)$ is $\mathbb{Q}$-factorial klt, then any $(K_X + B + M)$-MMP with scaling of an ample divisor over Z terminates. In particular, $(X, B + M)$ has a minimal model over Z.*

Proof.
(i) Follows immediately from (ii) and Lemma 2.31.

(ii) By the proof of [HL18, Theorem 1.7], to prove (ii) it suffices to show that $(X, B+M)$ has a minimal model in the sense of Birkar-Shokurov over Z.

Let $\mathcal{S}$ be the set of all NQC lc g-pairs of dimension n whose underlying variety is $\mathbb{Q}$-factorial klt and which admit an NQC weak Zariski decomposition over Z but do not have a minimal model in the sense of Birkar-Shokurov over Z. If $\mathcal{S} = \emptyset$, then we are done. Otherwise, to each element $(Y/Z, \Delta_Y + M_Y) \in \mathcal{S}$ with an NQC weak Zariski decomposition $g_Y^*(K_Y + \Delta_Y + M_Y) \equiv_Z P_Y + N_Y$, where $g_Y \colon Y' \to Y$ is a birational model, we can associate an invariant $\theta(Y/Z, \Delta_Y + M_Y, N_Y)$ as in [HL18, Definition 5.2]. Next, as in Step 1 of the proof of [HL18, Theorem 5.4], we may assume that we chose an element $(Y/Z, \Delta_Y + M_Y) \in \mathcal{S}$ which is an NQC log smooth g-pair with an NQC weak Zariski decomposition $K_Y + \Delta_Y + M_Y \equiv_Z P_Y + N_Y$ that minimizes this invariant in the set $\mathcal{S}$. If $\theta(Y/Z, \Delta_Y + M_Y, N_Y) = 0$, then a contradiction follows from Step 2 of the proof of [HL18, Theorem 5.4], while if $\theta(Y/Z, \Delta_Y + M_Y, N_Y) > 0$, then a contradiction follows from Step 3 of the proof of [HL18, Theorem 5.4]. □

Theorem 3.12. *Assume the existence of NQC weak Zariski decompositions for NQC $\mathbb{Q}$-factorial dlt g-pairs of dimension at most $n-1$.*

Let $(X/Z, B)$ be an lc pair of dimension n. If (X, B) admits an NQC weak Zariski decomposition over Z, then there exists a $(K_X + B)$-MMP with scaling of an ample divisor over Z which terminates. In particular, (X, B) has a minimal model over Z.

Proof. By [HH19, Theorem 1.7] it suffices to show that (X, B) has a minimal model in the sense of Birkar-Shokurov over Z (see also the Comment in Subsection 2.2.1). We conclude by Theorem 3.11(i). □

3.1.3 On the Existence of NQC Weak Zariski Decompositions

First, we prove two technical results, Theorems 3.13 and 3.14, which show that, modulo suitable assumptions in lower dimensions, we may infer the existence of NQC weak Zariski decompositions in specific cases. Their proofs are similar and follow the same strategy as that of [HM18, Theorem 2]. Next, we use these results in order to deduce Theorem 3.17, the main result of this chapter, as well as two complementary results, Corollaries 3.18 and 3.19. The latter two concern the existence of NQC weak Zariski decompositions for a certain class of pairs.

Theorem 3.13. *Assume the existence of NQC weak Zariski decompositions for NQC lc g-pairs of dimension at most* $n-1$*.*

Let $(X/Z, \Delta)$ *be a pseudo-effective* $\mathbb{Q}$*-factorial dlt pair of dimension* n *such that for each* $\varepsilon > 0$ *the divisor* $K_X + (1-\varepsilon)\Delta$ *is not pseudo-effective over* Z*. Then* (X, Δ) *admits an NQC weak Zariski decomposition over* Z*.*

Proof. We proceed in four steps.

Step 1: In this step we show that we may assume the following:

Assumption 1. There exists a fibration $\xi\colon X \to Y$ over Z to a normal quasi-projective variety Y such that $\dim Y < \dim X$ and such that:

(a_1) $\nu\big(F, (K_X + \Delta)|_F\big) = 0$ and $h^1(F, \mathcal{O}_F) = 0$ for a very general fiber F of ξ,

(b_1) $K_X + (1-\varepsilon)\Delta$ is not ξ-pseudo-effective for any $\varepsilon > 0$.

To this end, pick a decreasing sequence $\{\varepsilon_i\}_{i=1}^{\infty}$ of positive real numbers such that $\lim_{i\to\infty} \varepsilon_i = 0$. By [Gon15, Lemma 3.1][1] applied to the divisors $\Delta_i := (1-\varepsilon_i)\Delta$, there exists a birational contraction $\varphi\colon X \dashrightarrow S$ over Z and a fibration $f\colon S \to Y$ over Z such that, if we set $\Delta_S := \varphi_*\Delta$, then:

(a) (S, Δ_S) is a $\mathbb{Q}$-factorial lc pair,

(b) Y is a normal quasi-projective variety with $\dim Y < \dim X$,

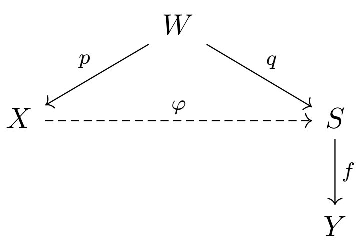
(c) $K_S + \Delta_S \equiv_Y 0$,

(d) φ is a $\big(K_X + (1-\varepsilon_i)\Delta\big)$-MMP for some $i \gg 0$ and f is the corresponding Mori fiber space.

Let $(p, q)\colon W \to X \times S$ be a resolution of indeterminacies of φ such that W is smooth. We may write

$$K_W + \Delta_W \sim_{\mathbb{R}} p^*(K_X + \Delta) + E, \tag{3.5}$$

where the divisors Δ_W and E are effective and have no common components. By passing to a higher model we may assume that the pair (W, Δ_W) is log smooth.



[1]This lemma works in the relative setting.

Let F be a very general fiber of f and set $F_W = q^{-1}(F) \subseteq W$, $\Delta_F := \Delta_S|_F$ and $\Delta_{F_W} := \Delta_W|_{F_W}$. Then the divisors $K_F + \Delta_F$ and $K_{F_W} + \Delta_{F_W}$ are pseudo-effective and we have $(q|_{F_W})_*(K_{F_W} + \Delta_{F_W}) = K_F + \Delta_F$, see (3.5). By [DL15, Lemma 3.1] and by (c), we obtain

$$\nu(F_W, K_{F_W} + \Delta_{F_W}) \leq \nu(F, K_F + \Delta_F) = 0,$$

hence $\nu(F_W, K_{F_W} + \Delta_{F_W}) = 0$, see [Nak04, Remark V.2.6(5) and Proposition V.2.7(6)].

Moreover, for every $\varepsilon > 0$, the divisor $K_{F_W} + (1-\varepsilon)\Delta_{F_W}$ is not pseudo-effective, since otherwise the divisor $K_F + (1-\varepsilon)\Delta_F = (q_{F_W})_*\big(K_{F_W} + (1-\varepsilon)\Delta_{F_W}\big)$ would be pseudo-effective for some $\varepsilon > 0$, a contradiction to (c) and (d).

Furthermore, since S is a klt variety by (d), so is F, hence F has rational singularities. Additionally, $h^1(F, \mathcal{O}_F) = 0$ by (d) and by the Kodaira vanishing theorem. It follows that $h^1(F_W, \mathcal{O}_{F_W}) = 0$.

If $K_W + \Delta_W$ has an NQC weak Zariski decomposition over Z, then $K_X + \Delta$ has an NQC weak Zariski decomposition over Z by (3.5) and by Lemma 3.4.

Therefore, by replacing (X, Δ) with (W, Δ_W) and by setting $\xi := f \circ q$, we achieve Assumption 1.

Step 2: If $\dim Y = 0$ (and thus necessarily $\dim Z = 0$), then $K_X + \Delta \equiv N_\sigma(K_X + \Delta)$ by [Nak04, Proposition V.2.7(8)]. Hence, $K_X + \Delta$ has an NQC weak Zariski decomposition, and we are done.

Step 3: Assume from now on that $\dim Y > 0$. In this step we show that we may assume the following:

Assumption 2. There exists a fibration $g\colon X \to T$ to a normal quasi-projective variety T such that:

(a_2) $0 < \dim T < \dim X$,

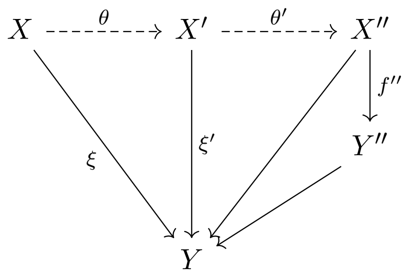
(b_2) $K_X + \Delta \equiv_T 0$,

(c_2) the numerical equivalence over T coincides with the $\mathbb{R}$-linear equivalence over T.

However, instead of the pair (X, Δ) being $\mathbb{Q}$-factorial dlt, we may only assume that it is an lc pair such that $(X, 0)$ is $\mathbb{Q}$-factorial klt.

To this end, by Assumption 1 (a_1) and by Lemma 2.50 the divisor $K_X + \Delta$ is effective over Y. Hence, by assumptions of the theorem and by Theorem 3.11, we may run a $(K_X + \Delta)$-MMP with scaling of an ample divisor over Y which terminates, and we obtain a birational contraction $\theta\colon X \dashrightarrow X'$ over Y. Set $\Delta' := \theta_*\Delta$ and let $\xi'\colon X' \to Y$ be the induced morphism.

By Lemma 2.52 there exists $\delta > 0$ such that, if we run a $\big(K_{X'} + (1-\delta)\Delta'\big)$-MMP with scaling of an ample divisor over Y, then this MMP is $(K_{X'} + \Delta')$-trivial. Note that $K_{X'} + (1-\delta)\Delta'$ is not ξ'-pseudo-effective: indeed, by possibly choosing δ smaller, we may assume that the map θ is $\big(K_X + (1-\delta)\Delta\big)$-negative, and the claim follows since $K_X + (1-\delta)\Delta$ is not ξ-pseudo-effective by (b_1). Therefore, this relative $\big(K_{X'} + (1-\delta)\Delta'\big)$-MMP terminates with a Mori fiber space $f''\colon X'' \to Y''$ over Y by [BCHM10, Corollary 1.3.3]. Let $\theta'\colon X' \dashrightarrow X''$ denote that MMP and set $\Delta'' := \theta'_*\Delta'$.



Then the variety X'' is $\mathbb{Q}$-factorial, the pair (X'', Δ'') is lc by construction and by Lemma 2.18(iii), the pair $(X'', 0)$ is klt since the pair $(X'', (1-\delta)\Delta'')$ is dlt, and by Lemma 2.52 we have

$$K_{X''} + \Delta'' \equiv_{Y''} 0.$$

Furthermore, the numerical equivalence over Y'' coincides with the $\mathbb{R}$-linear equivalence over Y'', since f'' is an extremal contraction, see [KM98, Theorem 3.25(4)].

If $K_{X''} + \Delta''$ has an NQC weak Zariski decomposition over Z, then $K_X + \Delta$ has an NQC weak Zariski decomposition over Z by Lemma 3.4.

Therefore, by replacing (X, Δ) with (X'', Δ'') and by setting $T := Y''$ and $g := f''$, we achieve Assumption 2.

Step 4: By [Bir11, Proposition 3.2(3)] there exist $\mathbb{Q}$-divisors $\Delta_1, \dots, \Delta_m$ such that each pair (X, Δ_i) is lc with $K_X + \Delta_i$ nef over T, and there exist positive real numbers $r_1, \dots, r_m$ such that $\sum r_i = 1$ and

$$K_X + \Delta = \sum_{i=1}^{m} r_i (K_X + \Delta_i). \tag{3.6}$$

Then $K_X + \Delta_i \equiv_T 0$ for all $i \in \{1, \dots, m\}$ by (b_2) and by (3.6), so $K_X + \Delta_i \sim_{\mathbb{Q},T} 0$ for all $i \in \{1, \dots, m\}$ by (c_2). Hence, by [FG14, Theorem 3.6] for each $i \in \{1, \dots, m\}$ there exists an lc g-pair $(T/Z, B_i + M_i)$ on T such that the B_i and M_i are $\mathbb{Q}$-divisors and

$$K_X + \Delta_i \sim_{\mathbb{Q}} g^*(K_T + B_i + M_i). \tag{3.7}$$

By (3.6) and (3.7) we obtain

$$K_X + \Delta = g^*(K_T + B_T + M_T), \tag{3.8}$$

where $B_T := \sum r_i B_i$ and $M_T := \sum r_i M_i$. By construction, the resulting g-pair $(T/Z, B_T + M_T)$ is NQC lc, and $K_T + B_T + M_T$ is pseudo-effective over Z by (3.8). Hence, by assumptions of the theorem and by (a_2), the g-pair $(T/Z, B_T + M_T)$ has an NQC weak Zariski decomposition over Z, which induces an NQC weak Zariski decomposition over Z for $(X/Z, \Delta)$ due to (3.8) and Remark 3.2, as desired. □

Theorem 3.14. *Assume the existence of NQC weak Zariski decompositions for NQC lc g-pairs of dimension at most $n-1$.*

Let $(X/Z, \Delta + M)$ be an NQC $\mathbb{Q}$-factorial dlt g-pair of dimension n with data $V \to X \to Z$ and L. Assume that the divisor $K_X + \Delta + M$ is pseudo-effective over Z and that for each $\varepsilon > 0$ the divisor $K_X + \Delta + (1-\varepsilon)M$ is not pseudo-effective over Z. Then $(X, \Delta + M)$ admits an NQC weak Zariski decomposition over Z.

Proof. By Lemma 2.51 there exist a birational contraction $\varphi\colon X \dashrightarrow S$ over Z and a fibration $f\colon S \to Y$ over Z such that:

(a) $(S, \Delta_S + M_S)$ is an NQC $\mathbb{Q}$-factorial lc g-pair, where $\Delta_S := \varphi_*\Delta$ and M_S is a pushforward of L,

(b) Y is a normal quasi-projective variety with $\dim Y < \dim X$,

(c) $K_S + \Delta_S + M_S \sim_{\mathbb{R},Y} 0$,

(d) φ is a $\big(K_X + \Delta + (1-\zeta)M\big)$-MMP for some $0 < \zeta \ll 1$ and f is the corresponding Mori fiber space.

As in Step 1 of the proof of Theorem 3.13, by replacing X with a higher model we may assume the following:

Assumption 1. There exists a fibration $\xi\colon X \to Y$ over Z to a normal quasi-projective variety Y such that $\dim Y < \dim X$ and such that:

(a_1) $\nu\big(F, (K_X + \Delta + M)|_F\big) = 0$ and $h^1(F, \mathcal{O}_F) = 0$ for a very general fiber F of ξ,

(b_1) $K_X + \Delta + (1-\varepsilon)M$ is not ξ-pseudo-effective for any $\varepsilon > 0$.

If $\dim Y = 0$ (and thus necessarily $\dim Z = 0$), then $K_X + \Delta + M$ has an NQC weak Zariski decomposition as in Step 2 of the proof of Theorem 3.13, and we are done.

Assume from now on that $\dim Y > 0$. Note that $(X, \Delta + M)$ is also a g-pair over Y. It follows from (a_1) and from Lemma 2.50 that the divisor $K_X + \Delta + M$ is effective over Y. Hence, by assumptions of the theorem and by Theorem 3.11, we may run a $(K_X + \Delta + M)$-MMP with scaling of an ample divisor over Y which terminates. We obtain a birational contraction $\theta\colon X \dashrightarrow X'$ over Y and a g-pair $(X', \Delta' + M')$, where $\Delta' := \theta_*\Delta$ and M' is a pushforward of L, and we denote by $\xi'\colon X' \to Y$ the induced morphism.

By Lemma 2.53 there exists $\delta > 0$ such that, if we run a $\big(K_{X'} + \Delta' + (1-\delta)M'\big)$-MMP with scaling of an ample divisor A over Y, then this MMP is $(K_{X'} + \Delta' + M')$-trivial. Note that $K_{X'} + \Delta' + (1-\delta)M'$ is not ξ'-pseudo-effective by (b_1) as in Step 3 of the proof of Theorem 3.13. Therefore, this relative $\big(K_{X'} + \Delta' + (1-\delta)M'\big)$-MMP terminates with a Mori fiber space $f''\colon X'' \to Y''$ over Y as in the proof of Lemma 2.51. We obtain a birational contraction $\theta'\colon X' \dashrightarrow X''$ and a g-pair $(X'', \Delta'' + M'')$, where $\Delta'' := \theta'_*\Delta'$ and M'' is a pushforward of L.

Then the variety X'' is $\mathbb{Q}$-factorial, the NQC g-pair $(X'', \Delta'' + M'')$ is lc, the pair $(X'', 0)$ is klt by Remark 2.7 since the g-pair $(X'', \Delta'' + (1-\delta)M'')$ is dlt, and by Lemma 2.53 we have

$$K_{X''} + \Delta'' + M'' \equiv_{Y''} 0. \tag{3.9}$$

Furthermore, the numerical equivalence over Y'' coincides with the $\mathbb{R}$-linear equivalence over Y'', since f'' is an extremal contraction, see [KM98, Theorem 3.25(4)]. Moreover, the divisor M'' is ample over Y'' by (3.9) and since $-(K_{X''} + \Delta'' + (1-\delta)M'')$ is f''-ample.

Then, as in Step 3 of the proof of Theorem 3.13, we may replace $(X, \Delta + M)$ with $(X'', \Delta'' + M'')$ and set $T := Y''$, and thus we may assume the following:

Assumption 2. There exists a fibration $g\colon X \to T$ to a normal quasi-projective variety T/Z such that:

(a_2) $0 < \dim T < \dim X$,

(b$_2$) $K_X + \Delta + M \equiv_T 0$,

(c$_2$) the numerical equivalence over T coincides with the $\mathbb{R}$-linear equivalence over T,

(d$_2$) $\rho(X/T) = 1$,

(e$_2$) M is ample over T.

However, instead of the g-pair $(X, \Delta + M)$ being $\mathbb{Q}$-factorial dlt, we may only assume that it is an NQC lc g-pair such that $(X, 0)$ is $\mathbb{Q}$-factorial klt.

The g-pair $(X/Z, \Delta + M)$ is NQC lc and such that $(X, 0)$ is $\mathbb{Q}$-factorial klt and $K_X + \Delta + M$ is nef over T. Since $\overline{\mathrm{NE}}(X/T)$ is extremal in $\overline{\mathrm{NE}}(X/Z)$ according to [Deb01, p. 12] (see also [KMM87, Lemma 4-2-2]), by applying [HL18, Proposition 3.16] to the collection of extremal rays of $\overline{\mathrm{NE}}(X/Z)$ corresponding to $\overline{\mathrm{NE}}(X/T)$, we deduce that there exist $\mathbb{Q}$-divisors $\Delta_1, \dots, \Delta_m$ and $M_1, \dots, M_m$ such that each g-pair $(X/Z, \Delta_i + M_i)$ is lc with $K_X + \Delta_i + M_i$ nef over T, and there exist positive real numbers $r_1, \dots, r_m$ such that $\sum r_i = 1$ and

$$K_X + \Delta + M = \sum_{i=1}^{m} r_i (K_X + \Delta_i + M_i). \tag{3.10}$$

Then $K_X + \Delta_i + M_i \equiv_T 0$ for all $i \in \{1, \dots, m\}$ by (b$_2$) and by (3.10), and thus $K_X + \Delta_i + M_i \sim_{\mathbb{Q},T} 0$ for all $i \in \{1, \dots, m\}$ by (c$_2$). Hence, by [Fil19, Theorem 6][2] for each $i \in \{1, \dots, m\}$ there exists an lc g-pair $(T/Z, B_i + N_i)$ on T such that the B_i and N_i are $\mathbb{Q}$-divisors and

$$K_X + \Delta_i + M_i \sim_{\mathbb{Q}} g^*(K_T + B_i + N_i). \tag{3.11}$$

By (3.10) and (3.11) we obtain

$$K_X + \Delta + M = g^*(K_T + B_T + N_T), \tag{3.12}$$

where $B_T := \sum r_i B_i$ and $N_T := \sum r_i N_i$. By construction, the g-pair $(T/Z, B_T + N_T)$ is NQC lc, and $K_T + B_T + N_T$ is pseudo-effective over Z by (3.12). Hence, by assumptions of the theorem and by (a$_2$), the g-pair $(T/Z, B_T + N_T)$ admits an NQC weak Zariski decomposition over Z, which induces an NQC weak Zariski decomposition over Z for $(X/Z, \Delta + M)$ due to (3.12) and Remark 3.2, as desired. □

Remark 3.15. The canonical bundle formula from [Fil19] that was implemented above has been recently extended to NQC g-pairs by Han and Liu [HL20c]. In particular, [HL20c, Theorem 1.2] can be invoked in order to shorten significantly the last paragraph of the previous proof.

For further information around *the canonical bundle formula*, which obviously plays a crucial role in the proof of both Theorem 3.13 and Theorem 3.14, we refer to [FG12, FG14, FL19, HL20c] and the relevant references therein. A gentle introduction to this topic is [FL20].

Corollary 3.16. *Assume that lc pairs of dimension at most n admit NQC weak Zariski decompositions. Then NQC lc g-pairs of dimension n admit NQC weak Zariski decompositions.*

[2]This is a generalisation of [Fil20, Theorem 1.4] to the context of projective morphisms of quasi-projective g-pairs with rational boundary parts and rational nef parts.

Proof. The proof is by induction on the dimension n. We may therefore assume that NQC lc g-pairs of dimension at most $n-1$ have NQC weak Zariski decompositions.

Let $(X/Z, B+M)$ be a pseudo-effective NQC lc g-pair of dimension n. By passing to a dlt blow-up and by Remark 3.6 we may assume that it is a pseudo-effective NQC $\mathbb{Q}$-factorial dlt g-pair. Set

$$\tau := \inf\big\{t \in \mathbb{R}_{\geq 0} \mid K_X + B + tM \text{ is pseudo-effective over } Z\big\}.$$

We distinguish two cases.

Assume first that $\tau = 0$. By Remark 2.7 the pseudo-effective pair (X, B) is lc, hence it has an NQC weak Zariski decomposition over Z by assumption, and therefore so does the g-pair $(X, B+M)$ by Remark 3.3.

Assume now that $0 < \tau \leq 1$. By Theorem 3.14 the g-pair $(X, B+\tau M)$ has an NQC weak Zariski decomposition over Z, and therefore so does the g-pair $(X, B+M)$ by Remark 3.3. □

We are now ready to state and prove the main result of this chapter.

Theorem 3.17. *The existence of NQC weak Zariski decompositions for smooth varieties of dimension n implies the existence of NQC weak Zariski decompositions for NQC lc g-pairs of dimension n.*

Proof. By assumption and by Lemma 3.8 we may assume the existence of NQC weak Zariski decompositions for smooth varieties of dimension at most n. By induction on the dimension we may assume the existence of NQC weak Zariski decompositions for NQC lc g-pairs of dimension at most $n-1$. Thus, by Corollary 3.16 it suffices to show the existence of NQC weak Zariski decompositions for lc pairs of dimension n.

Let $(X/Z, B)$ be a pseudo-effective lc pair of dimension n. By passing to a dlt blow-up and by Remark 3.6 we may assume that (X, B) is $\mathbb{Q}$-factorial dlt. Next, by passing to a log resolution and by Lemma 3.4 we may assume that (X, B) is log smooth. Set

$$\tau := \inf\{t \in \mathbb{R}_{\geq 0} \mid K_X + tB \text{ is pseudo-effective over } Z\}.$$

We distinguish two cases.

Assume first that $\tau = 0$. Then the pseudo-effective smooth pair $(X, 0)$ has an NQC weak Zariski decomposition over Z by assumption, and therefore so does the pair (X, B).

Assume now that $0 < \tau \leq 1$. It follows from Theorem 3.13 that the pair $(X, \tau B)$ has an NQC weak Zariski decomposition over Z, and therefore so does the pair (X, B). □

Using Theorems 3.13 and 3.14 we may also deduce the existence of NQC weak Zariski decompositions for pairs and g-pairs such that a general fiber of the structure morphism is covered by rational curves. Note that these results are not stated in [LT19], but they appear implicitly in the proofs of [LT19, Theorems C and 4.3].

Corollary 3.18. *Assume the existence of NQC weak Zariski decompositions for NQC lc g-pairs of dimension at most $n-1$.*

Let $(X/Z, B)$ be a pseudo-effective lc pair of dimension n such that a general fiber of the morphism $X \to Z$ is uniruled. Then (X, B) admits an NQC weak Zariski decomposition over Z.

Proof. As in the second paragraph of the proof of Theorem 3.17 we may assume that the pair (X, B) is log smooth. Then K_X is not pseudo-effective over Z by assumption and by [BDPP13, Corollary 0.3]. Set

$$\tau := \inf \big\{ t \in \mathbb{R}_{\geq 0} \mid K_X + tB \text{ is pseudo-effective over } Z \big\}$$

and observe that $0 < \tau \leq 1$. It follows from Theorem 3.13 that the pair $(X, \tau B)$ has an NQC weak Zariski decomposition over Z, and therefore so does the pair (X, B). □

Corollary 3.19. *Assume the existence of NQC weak Zariski decompositions for NQC lc g-pairs of dimension at most $n-1$.*

Let $(X/Z, B+M)$ be a pseudo-effective NQC lc g-pair of dimension n such that a general fiber of the morphism $X \to Z$ is uniruled. Then $(X, B+M)$ admits an NQC weak Zariski decomposition over Z.

Proof. As in the proof of Corollary 3.18, we may assume that the g-pair $(X, B+M)$ is log smooth and that K_X is not pseudo-effective over Z. Set

$$\tau := \inf \big\{ t \in \mathbb{R}_{\geq 0} \mid K_X + t(B+M) \text{ is pseudo-effective over } Z \big\}$$

and observe that $0 < \tau \leq 1$. We distinguish two cases.

Assume first that $K_X + \tau B$ is pseudo-effective over Z. Then it follows from Remark 2.7 and Corollary 3.18 that the pair $(X, \tau B)$ has an NQC weak Zariski decomposition over Z, and therefore so does the g-pair $(X, B+M)$ by Remark 3.3.

Assume now that $K_X + \tau B$ is not pseudo-effective over Z. Set

$$\mu := \inf \big\{ t \in \mathbb{R}_{\geq 0} \mid K_X + \tau B + tM \text{ is pseudo-effective over } Z \big\}$$

and observe that $0 < \mu \leq \tau$. It follows from Lemma 2.14 and from Theorem 3.14 that the g-pair $(X, \tau B + \mu M)$ has an NQC weak Zariski decomposition over Z, and therefore so does the g-pair $(X, B+M)$ by Remark 3.3. □

3.2 NQC Nakayama-Zariski Decompositions

Nakayama [Nak04] defined a decomposition

$$D = P_\sigma(D) + N_\sigma(D)$$

for any pseudo-effective $\mathbb{R}$-divisor D on a smooth projective variety X, where $P_\sigma(D)$ is movable by [Nak04, Lemma III.1.8 and Proposition III.1.14(1)] and $N_\sigma(D)$ is effective by construction. This decomposition is called *the Nakayama-Zariski decomposition of D*. The divisors $P_\sigma(D)$ and $N_\sigma(D)$ are called *the positive part* and *the negative part*, respectively, of the Nakayama-Zariski decomposition of D.

Birkar and Hu [BH14, Section 4] extended the above decomposition to the singular setting as follows: If D is a pseudo-effective $\mathbb{R}$-Cartier $\mathbb{R}$-divisor on a normal projective variety X, then we consider a resolution $f : W \to X$ of X and we set

$$P_\sigma(D) := f_* P_\sigma(f^* D).$$

Note that the definition of $P_\sigma(D)$ does not depend on the choice of the resolution by [Nak04, Theorem III.5.16].

For various properties of the Nakayama-Zariski decomposition we refer to [Nak04, Chapter III], [Dru11], [LP20a, Section 2.2], and in particular to [BH14, Lemma 4.1], which will be used repeatedly below and in Section 4.2.

In this section, imitating [HL18, Definition 2.13], we introduce NQC Nakayama-Zariski decompositions, adding essentially the NQC condition to the corresponding definition of Birkar and Hu from [BH14, Section 4]. We emphasize that we work here exclusively in the absolute setting, that is, we assume that $Z = \operatorname{Spec} \mathbb{C}$. Additionally, all the results appearing below are completely analogous to the ones in Subsection 3.1.1 and their proofs are similar as well.

Definition 3.20. Let X be a normal projective variety and let D be a pseudo-effective $\mathbb{R}$-Cartier $\mathbb{R}$-divisor on X. We say that D *admits an NQC Nakayama-Zariski decomposition* if there exists a resolution $f\colon W \to X$ such that $P_\sigma(f^*D)$ is NQC.

Remark 3.21. With the same notation as in Definition 3.20, assume that D has an NQC Nakayama-Zariski decomposition and let $g\colon Y \to X$ be a surjective morphism from a normal projective variety Y. Then g^*D has an NQC Nakayama-Zariski decomposition. Indeed, by considering a resolution of indeterminacies $(p,q)\colon T \to Y \times W$ of $f^{-1} \circ g\colon Y \dashrightarrow W$ such that T is smooth, we have

$$P_\sigma\big(p^*g^*D\big) = P_\sigma\big(q^*f^*D\big) = q^*P_\sigma\big(f^*D\big)$$

due to [Nak04, Corollary III.5.17], which proves the assertion.

If, additionally, g is birational, then the converse is also clear, namely, if g^*D has an NQC Nakayama-Zariski decomposition, then D has an NQC Nakayama-Zariski decomposition.

Lemma 3.22. *Let $f\colon X \dashrightarrow Y$ be a birational contraction between normal projective varieties. Let D be an $\mathbb{R}$-Cartier $\mathbb{R}$-divisor on X such that the map f is D-non-positive. Then D has an NQC Nakayama-Zariski decomposition if and only if f_*D an NQC Nakayama-Zariski decomposition.*

Proof. Set $G = f_*D$. Since f is D-non-positive, if $(p,q)\colon W \to X \times Y$ is a resolution of indeterminacies of f such that W is smooth, then there exists an effective q-exceptional $\mathbb{R}$-divisor E on W such that $p^*D \sim_{\mathbb{R}} q^*G + E$. By Remark 3.21 we may replace X with W and thus we may assume that f is a morphism and

$$D \sim_{\mathbb{R}} f^*G + E. \tag{3.13}$$

Assume that D has an NQC Nakayama-Zariski decomposition. Then there exists a resolution $g\colon V \to X$ such that $P_\sigma(g^*D)$ is NQC. By [BH14, Lemma 4.1(2)] and by (3.13) we obtain

$$P_\sigma(g^*D) = P_\sigma(g^*f^*G + g^*E) = P_\sigma\big((f \circ g)^*G\big),$$

which shows that G has an NQC Nakayama-Zariski decomposition.

Conversely, if G has an NQC Nakayama-Zariski decomposition, then we conclude by arguing as in Remark 3.21 and by taking (3.13) into account. □

Definition 3.23. Let $(X, B+M)$ be an NQC g-pair. We say that $(X, B+M)$ *admits an NQC Nakayama-Zariski decomposition* if the divisor $K_X + B + M$ has an NQC Nakayama-Zariski decomposition.

Remark 3.24. Let $(X, B+M)$ be an NQC lc g-pair. It follows from Remark 3.21 that $(X, B+M)$ admits an NQC Nakayama-Zariski decomposition if and only if any dlt blow-up of $(X, B+M)$ admits an NQC Nakayama-Zariski decomposition.

Remark 3.25. Let $(X, B+M)$ be an NQC lc g-pair. Assume that $(X, B+M)$ admits an NQC Nakayama-Zariski decomposition. We may take a sufficiently high log smooth model $f\colon (W, B_W+M_W) \to (X, B+M)$ of $(X, B+M)$ (see Notation 2.29) such that $P_\sigma\big(f^*(K_X+B+M)\big)$ is NQC, see the proof of Remark 3.21. By Remark 2.30 we may write

$$K_W + B_W + M_W \sim_{\mathbb{R}} f^*(K_X+B+M) + F,$$

where F is an effective f-exceptional $\mathbb{R}$-divisor on W. It follows now from [BH14, Lemma 4.1(2)] that

$$P_\sigma(K_W+B_W+M_W) = P_\sigma\big(f^*(K_X+B+M)\big),$$

which implies that (W, B_W+M_W) admits an NQC Nakayama-Zariski decomposition.

Proposition 3.26. *Let $(X, B+M)$ be an NQC lc g-pair. If $(X, B+M)$ has a minimal model in the sense of Birkar-Shokurov, then it admits an NQC Nakayama-Zariski decomposition.*

Proof. Let $\varphi\colon (X, B+M) \dashrightarrow (Y, B_Y+M_Y)$ be a minimal model in the sense of Birkar-Shokurov of $(X, B+M)$. Consider a resolution of indeterminacies $(p, q)\colon W \to X \times Y$ of φ such that W is smooth. By Lemma 2.18(i) we may write

$$p^*(K_X+B+M) \sim_{\mathbb{R}} q^*(K_Y+B_Y+M_Y) + E,$$

where E is an effective q-exceptional $\mathbb{R}$-divisor on W. Moreover, by [HL18, Proposition 3.16] we infer that $K_Y+B_Y+M_Y$ is NQC. Hence, by [BH14, Lemma 4.1(2)] we obtain

$$P_\sigma\big(p^*(K_X+B+M)\big) = P_\sigma\big(q^*(K_Y+B_Y+M_Y)\big) = q^*(K_Y+B_Y+M_Y),$$

which proves the assertion. □

Corollary 3.27. *Let $(X, B+M)$ be an NQC lc g-pair. If $(X, B+M)$ has a minimal model, then it admits an NQC Nakayama-Zariski decomposition.*

Proof. The proof is identical to the proof of Corollary 3.10, replacing Proposition 3.9 with Proposition 3.26. □

4

On the Existence of Minimal Models for Log Canonical Generalized Pairs

The aim of this chapter is to address the following conjecture of the MMP.

Existence of Minimal Models Conjecture. *Let $(X/Z, B+M)$ be a pseudo-effective NQC lc g-pair. Then $(X, B+M)$ has a minimal model over Z.*

For a (non-exhaustive but) fairly complete overview of the currently known results regarding the existence of minimal models conjecture for usual pairs we refer to the introduction of the papers [Bir10a, Lai11, GL13, Has18a]. In particular, we remark that the papers [BCHM10, CL12, CL13] establish the existence of (good) minimal models for klt pairs *of general type* and of arbitrary dimension.

In this chapter we obtain the following results. First, we reduce the above conjecture to the problem of the existence of minimal models for smooth varieties. We also show that the existence of minimal models is essentially equivalent to the existence of NQC weak Zariski decompositions and, in the absolute setting, to the existence of NQC Nakayama-Zariski decompositions, building on previous works of Birkar [Bir12b], Birkar and Hu [BH14], and Han and Li [HL18]. Additionally, we prove that minimal models exist for a certain class of g-pairs under mild assumptions in lower dimensions. Finally, we present some immediate corollaries of our results in dimensions 4 and 5.

The contents of Section 4.1 are taken from our joint paper [LT19] with Vladimir Lazić. Specifically, we reproduce everything from [LT19, Sections 1 and 4] that is related to the existence of minimal models, but we also include some results that were only hinted in [LT19]. In Section 4.2 we generalize [BH14, Theorem 1.1] to the context of g-pairs using the theory of NQC Nakayama-Zariski decompositions that we developed in Section 3.2. To the best of our knowledge, the main result of this section, namely Theorem 4.18, is new and has not appeared elsewhere in the literature.

Throughout this chapter we assume that varieties are normal and quasi-projective and that a variety X over a variety Z is projective over Z.

4.1 Minimal Models and NQC Weak Zariski Decompositions

The first result in this section is similar to Lemma 3.8, but concerns minimal models instead of weak Zariski decompositions. It will not be used elsewhere in the thesis.

Lemma 4.1. *Let $n, k \in \mathbb{Z}_{\geq 1}$ with $n \geq k$. Assume that every pseudo-effective smooth pair $(X/Z, 0)$ of dimension n has a minimal model over Z. Then every pseudo-effective smooth pair $(Y/Z, 0)$ of dimension k has a minimal model over Z.*

Proof. Let $(Y/Z, 0)$ be a k-dimensional pair as in the statement of the lemma. By assumption and by Corollary 3.10 every pseudo-effective smooth pair $(X/Z, 0)$ of dimension n admits an NQC weak Zariski decomposition over Z. By Lemma 3.8 and by Theorem 3.17 every NQC lc g-pair over Z of dimension at most n admits an NQC weak Zariski decomposition over Z. In particular, $(Y, 0)$ admits an NQC weak Zariski decomposition over Z, and hence it has a minimal model over Z by Theorem 3.12. □

From this point forward our presentation follows a specific pattern, namely we first prove a statement for usual pairs and then we deduce its analog for g-pairs. We begin with one of the main theorems of the thesis, followed by its generalization to the setting of g-pairs.

Theorem 4.2. *The existence of minimal models for smooth varieties of dimension n implies the existence of minimal models for lc pairs of dimension n.*

Proof. Let $(X/Z, B)$ be a pseudo-effective lc pair of dimension n. By assumption, Lemma 3.8 and Corollary 3.10, we may assume the existence of NQC weak Zariski decompositions for smooth varieties of dimension at most n. Thus, by Theorem 3.17 we may assume the existence of NQC weak Zariski decompositions for NQC lc g-pairs of dimension at most $n - 1$, and that the pair (X, B) has an NQC weak Zariski decomposition over Z. We conclude by Theorem 3.12. □

Theorem 4.3. *Assume the existence of minimal models for smooth varieties of dimension n.*

Let $(X/Z, B + M)$ be a pseudo-effective NQC lc g-pair of dimension n such that $(X, 0)$ is $\mathbb{Q}$-factorial klt. Then $(X, B + M)$ has a minimal model over Z.

Proof. The proof is identical to the proof of Theorem 4.2, replacing Theorem 3.12 with Theorem 3.11(ii). □

Remark 4.4. If in Theorem 4.3 we drop the assumption that $(X, 0)$ is $\mathbb{Q}$-factorial klt, then by Theorem 3.11(i) we infer that the NQC lc g-pair $(X, B + M)$ has a minimal model in the sense of Birkar-Shokurov over Z.

The next result improves both the assumptions in lower dimensions and the conclusions of [Bir11, Corollary 1.7] and [Bir12b, Theorem 1.5]. Its analog in the context of g-pairs, which is discussed afterwards, refines [HL18, Theorem 1.5] in a similar way.

Theorem 4.5. *Assume the existence of minimal models for smooth varieties of dimension $n - 1$.*

Let $(X/Z, B)$ be an lc pair of dimension n. Then the following are equivalent:

(i) *(X, B) admits an NQC weak Zariski decomposition over Z,*

(ii) *(X, B) has a minimal model over Z.*

Proof. Note that (i) follows from (ii) by Corollary 3.10 (observe that the assumptions in lower dimensions are reduntant for this implication). Conversely, assume that (X, B) has an NQC weak Zariski decomposition over Z. As in the proof of Theorem 4.2 we may assume the existence of NQC weak Zariski decompositions for NQC lc g-pairs of dimension at most $n - 1$. We conclude by Theorem 3.12. □

Theorem 4.6. *Assume the existence of minimal models for smooth varieties of dimension $n-1$.*

Let $(X/Z, B+M)$ be an NQC lc g-pair of dimension n such that $(X,0)$ is $\mathbb{Q}$-factorial klt. Then the following are equivalent:

(i) *$(X, B+M)$ admits an NQC weak Zariski decomposition over Z,*

(ii) *$(X, B+M)$ has a minimal model over Z.*

Proof. The proof is identical to the proof of Theorem 4.5, replacing Theorem 3.12 with Theorem 3.11(ii). □

Remark 4.7. If in Theorem 4.6 we drop the assumption that $(X,0)$ is $\mathbb{Q}$-factorial klt, then by Proposition 3.9 and by Theorem 3.11(i) we infer that the NQC lc g-pair $(X, B+M)$ has a minimal model in the sense of Birkar-Shokurov over Z if and only if it admits an NQC weak Zariski decomposition over Z.

Now, it is worth mentioning that Theorem 4.5 leads to a refinement of [Bir12b, Corollary 1.6], while by invoking Theorem 4.6 we also generalize [Bir12b, Corollary 1.6] to the context of g-pairs accordingly.

Corollary 4.8. *Assume the existence of minimal models for smooth varieties of dimension $n-1$.*

Let $(X/Z, B_1)$ and $(X/Z, B_2)$ be lc pairs of dimension n such that $B_1 \leq B_2$. If (X, B_1) has a minimal model over Z, then (X, B_2) also has a minimal model over Z.

Proof. By Corollary 3.10 the pair (X, B_1) admits an NQC weak Zariski decomposition over Z. Since $B_1 \leq B_2$, we may write $B_2 = B_1 + G$ for some effective divisor G on X, hence the pair (X, B_2) also admits an NQC weak Zariski decomposition over Z. We conclude by Theorem 4.5. □

Corollary 4.9. *Assume the existence of minimal models for smooth varieties of dimension $n-1$.*

Let $(X/Z, B_1+M_1)$ and $(X/Z, B_2+M_2)$ be NQC lc g-pairs of dimension n such that $(X,0)$ is $\mathbb{Q}$-factorial klt, $B_1 \leq B_2$ and $M_2 - M_1$ is the pushforward of an NQC divisor on some higher model of X. If (X, B_1+M_1) has a minimal model over Z, then (X, B_2+M_2) also has a minimal model over Z.

Proof. If the given g-pairs come with data $X_1' \to X \to Z$ and M_1', and $X_2' \to X \to Z$ and M_2', respectively, then, by possibly passing to a sufficiently high birational model $f\colon X' \to X$, by the assumptions we know that there exists an NQC divisor Q' on X' such that

$$Q := f_*Q' = M_2 - M_1, \tag{4.1}$$

and also that we have

$$f^*(K_X + B_1 + M_1) \equiv_Z P + N, \tag{4.2}$$

where P is an NQC divisor on X' and N is an effective $\mathbb{R}$-Cartier $\mathbb{R}$-divisor on X'. Now, by the Negativity lemma [KM98, Lemma 3.39(1)] we obtain

$$f^*Q = Q' + E, \tag{4.3}$$

where E is an effective f-exceptional $\mathbb{R}$-Cartier $\mathbb{R}$-divisor on X'. In addition, since $B_1 \leq B_2$, we may write

$$B_2 = B_1 + G, \tag{4.4}$$

where G is an effective $\mathbb{R}$-divisor on X. Consequently, by (4.1), (4.2), (4.3) and (4.4) we obtain

$$f^*(K_X + B_2 + M_2) \equiv_Z (P + Q') + (N + f^*G + E),$$

which is an NQC weak Zariski decomposition of $(X, B_2 + M_2)$ over Z. We conclude by Theorem 4.6. □

Theorem 4.10. *Assume the existence of minimal models for smooth varieties of dimension $n-1$.*

Let $(X/Z, B)$ be a pseudo-effective lc pair of dimension n such that a general fiber of the morphism $X \to Z$ is uniruled. Then (X, B) has a minimal model over Z.

Proof. As in the proof of Theorem 4.2 we may assume the existence of NQC weak Zariski decompositions for NQC lc g-pairs of dimension at most $n-1$. Hence, by Corollary 3.18 we infer that (X, B) admits an NQC weak Zariski decomposition over Z. We conclude by Theorem 4.5. □

Theorem 4.11. *Assume the existence of minimal models for smooth varieties of dimension $n-1$.*

Let $(X/Z, B + M)$ be a pseudo-effective NQC lc g-pair of dimension n such that $(X, 0)$ is $\mathbb{Q}$-factorial klt and a general fiber of the morphism $X \to Z$ is uniruled. Then $(X, B + M)$ has a minimal model over Z.

Proof. The proof is identical to the proof of Theorem 4.10, replacing Corollary 3.18 with Corollary 3.19 and Theorem 4.5 with Theorem 4.6. □

Remark 4.12. If in Theorem 4.11 we drop the assumption that $(X, 0)$ is $\mathbb{Q}$-factorial klt, then by Theorem 3.11(i) we infer that the NQC lc g-pair $(X, B+M)$ has a minimal model in the sense of Birkar-Shokurov over Z.

4.1.1 Corollaries in Dimensions 4 and 5

Recall that the existence of minimal models for terminal 4-folds follows from [KMM87, Theorem 5-1-15]. Consequently, we derive immediately the following two corollaries in dimensions 4 and 5 from the previous results. As above, the first one concerns usual pairs, while the second one concerns g-pairs. Afterwards, we also comment on these two corollaries.

Corollary 4.13. *Let $(X/Z, B)$ be a pseudo-effective lc pair.*

(i) *If $\dim X = 4$, then (X, B) has a minimal model over Z.*

(ii) *If $\dim X = 5$, then (X, B) has a minimal model over Z if and only if it admits an NQC weak Zariski decomposition over Z. In particular, if $K_X + B$ is effective over Z, then (X, B) has a minimal model over Z.*

(iii) *If $\dim X = 5$ and a general fiber of the morphism $X \to Z$ is uniruled, then (X, B) has a minimal model over Z.*

Proof. Due to [KMM87, Theorem 5-1-15], (i) follows from Theorem 4.2, (ii) follows from Theorem 4.5, and (iii) follows from Theorem 4.10. □

Corollary 4.14. *Let $(X/Z, B+M)$ be a pseudo-effective NQC lc g-pair such that $(X, 0)$ is $\mathbb{Q}$-factorial klt.*

(i) *If $\dim X = 4$, then $(X, B+M)$ has a minimal model over Z.*

(ii) *If $\dim X = 5$, then $(X, B+M)$ has a minimal model over Z if and only if it admits an NQC weak Zariski decomposition over Z. In particular, if $K_X + B + M$ is effective over Z, then $(X, B+M)$ has a minimal model over Z.*

(iii) *If $\dim X = 5$ and a general fiber of the morphism $X \to Z$ is uniruled, then $(X, B+M)$ has a minimal model over Z.*

Proof. Due to [KMM87, Theorem 5-1-15], (i) follows from Theorem 4.3, (ii) follows from Theorem 4.6, and (iii) follows from Theorem 4.11. □

As far as Corollary 4.13 is concerned, (i) was first proved in [Sho09] and another proof was given later in [Bir12b], while (ii) was established in [Bir10a]; however, we stress that, unlike the results in these references, we deduce the existence of minimal models in the usual sense. Finally, (iii) was originally obtained in [LT19].

Furthermore, Corollary 4.14 was also obtained in [LT19], but it was simply not stated there. Note that if we drop the assumption that $(X, 0)$ is $\mathbb{Q}$-factorial klt, then, as mentioned above, we shall replace everywhere the phrase "$(X, B+M)$ has a minimal model over Z" with the phrase "$(X, B+M)$ has a minimal model in the sense of Birkar-Shokurov over Z".

4.1.2 On the Termination of MMPs with Scaling - Revisited

Theorem 4.15. *Assume the existence of minimal models for smooth varieties of dimension $n-1$.*

Let $(X/Z, B)$ be an lc pair of dimension n. If (X, B) admits an NQC weak Zariski decomposition over Z, then there exists a $(K_X + B)$-MMP with scaling of an ample divisor over Z which terminates. In particular, (X, B) has a minimal model over Z.

Proof. As in the proof of Theorem 4.2 we may assume the existence of NQC weak Zariski decompositions for NQC lc g-pairs of dimension at most $n-1$. We conclude by Theorem 3.12. □

Theorem 4.16. *Assume the existence of minimal models for smooth varieties of dimension $n-1$.*

Let $(X/Z, B+M)$ be an NQC lc g-pair of dimension n which admits an NQC weak Zariski decomposition over Z. Then the following statements hold.

(i) *The g-pair $(X, B+M)$ has a minimal model in the sense of Birkar-Shokurov over Z.*

(ii) *If, additionally, $(X, 0)$ is $\mathbb{Q}$-factorial klt, then any $(K_X + B + M)$-MMP with scaling of an ample divisor over Z terminates. In particular, $(X, B+M)$ has a minimal model over Z.*

Proof. The proof is identical to the proof of Theorem 4.15, replacing Theorem 3.12 with Theorem 3.11. □

4.2 Minimal Models and NQC Nakayama-Zariski Decompositions

Lemma 4.17. *Let $(X, B+M)$ be a log smooth g-pair such that M is NQC and*

$$K_X + B + M = P + N,$$

where $P := P_\sigma(K_X + B + M)$ is NQC and $N := N_\sigma(K_X + B + M)$. Then $(X, B+M)$ has a minimal model.

Proof. Note that $(X, B + M + \alpha P)$ is a $\mathbb{Q}$-factorial dlt polarized pair for any $\alpha \in \mathbb{R}_{\geq 0}$. Hence, if we choose a sufficiently large α, then by [HL18, Lemma 3.18] we may run a $(K_X + B + M + \alpha P)$-MMP with scaling of an ample divisor which is P-trivial. In particular, the nefness of P is preserved along the MMP due to [KM98, Theorem 3.25(4)] and the MMP is also a $(K_X + B + M)$-MMP. In addition, by Lemma 2.49 we reach a model Y on which $K_Y + B_Y + M_Y + \alpha P_Y$ is a movable $\mathbb{R}$-divisor. We denote by $\varphi\colon X \dashrightarrow Y$ the induced map and by N_Y the strict transform of N on Y.

Since

$$\begin{aligned} N_\sigma\big(K_X + B + M + \alpha P + \alpha N\big) &= N_\sigma\big((1+\alpha)(P+N)\big) \\ &= N_\sigma\big((1+\alpha)(K_X + B + M)\big) \\ &= (1+\alpha)N, \end{aligned}$$

it follows from [BH14, Lemma 4.1(4)] and from the above equalities that

$$\begin{aligned} P_\sigma\big(K_X + B + M + \alpha P\big) &= P_\sigma\big((1+\alpha)(P+N) - \alpha N\big) \\ &= P_\sigma\big((1+\alpha)(P+N)\big) \\ &= (1+\alpha)P. \end{aligned}$$

Furthermore, by [BH14, Lemma 4.1(5)] and by the above equalities we obtain

$$\begin{aligned} P_\sigma\big(K_Y + B_Y + M_Y + \alpha P_Y\big) &= \varphi_* P_\sigma\big(K_X + B + M + \alpha P\big) \\ &= (1+\alpha)P_Y, \end{aligned}$$

whereas [Nak04, Proposition III.1.14(1)] yields

$$\begin{aligned} P_\sigma\big(K_Y + B_Y + M_Y + \alpha P_Y\big) &= K_Y + B_Y + M_Y + \alpha P_Y \\ &= (1+\alpha)P_Y + N_Y, \end{aligned}$$

since $K_Y + B_Y + M_Y + \alpha P_Y$ is movable. Consequently, $N_Y = 0$. It follows now from the above equalities that $K_Y + B_Y + M_Y = P_Y$ is nef. Hence, the MMP terminates and the resulting g-pair $(Y, B_Y + M_Y)$ is a minimal model of $(X, B+M)$. □

Theorem 4.18. *Let $(X, B+M)$ be an NQC lc g-pair. If $(X, B+M)$ admits an NQC Nakayama-Zariski decomposition, then $(X, B+M)$ has a minimal model in the sense of Birkar-Shokurov.*

Proof. The statement follows immediately by combining Lemma 2.31, Remark 3.25 and Lemma 4.17. □

5

Special Termination

Special termination is a termination statement with a strong geometric flavor. Roughly speaking, it predicts the following: given a sequence of flips with respect to an lc pair, after finitely many steps the flipping locus avoids the non-klt locus. Even though this statement was originally conceived by Shokurov [Sho92, Sho03, Sho04], the first complete proof of special termination was given by Fujino [Fuj07] for dlt pairs, while the most general statement for lc pairs was missing from the literature until recently; we refer to [Fuj07, Section 4.2] and [Fuj11b, Section 5.2] for a relevant discussion.

The purpose of this chapter is to provide this missing statement. More precisely, we establish the special termination for lc pairs and we also prove its analog for NQC lc g-pairs. These results were obtained in our joint paper [LMT20] with Vladimir Lazić and Joaquín Moraga, thus the material presented below is taken primarily from [LMT20].

Although not immediately apparent, special termination plays a central role towards the termination of flips conjecture (see Chapter 6). Indeed, one of the main applications of special termination for dlt (g-)pairs is the reduction of the termination of flips for lc (g-)pairs to the termination of flips for klt (g-)pairs. Furthermore, special termination has also been utilized sometimes in order to deduce the termination of specific MMPs, see [Bir10a, Bir11, Bir12a, Bir12b] for such instances. Finally, it is worthwhile to mention that special termination for lc pairs was invoked by Birkar [Bir07] in order to prove the termination of flips for *effective* lc pairs. In the next and final chapter of the thesis we will deduce an analog of Birkar's aforementioned result in the context of g-pairs with the aid of special termination for NQC lc g-pairs.

Throughout this chapter, unless otherwise stated, we assume that varieties are normal and quasi-projective and that a variety X over a variety Z is projective over Z. We also assume the existence of flips for NQC lc g-pairs, cf. Remark 2.43(2).

5.1 What is Special Termination?

Let

$$\begin{array}{ccccccc}
(X_1, B_1 + M_1) & \overset{\pi_1}{\dashrightarrow} & (X_2, B_2 + M_2) & \overset{\pi_2}{\dashrightarrow} & (X_3, B_3 + M_3) & \overset{\pi_3}{\dashrightarrow} & \cdots \\
\theta_1 \searrow & & \swarrow \theta_1^+ \quad \theta_2 \searrow & & \swarrow \theta_2^+ & & \\
& Z_1 & & Z_2 & & &
\end{array}$$

be a sequence of flips over Z starting from an lc g-pair $(X_1/Z, B_1 + M_1)$. *Special termination* of the given sequence of flips means that after finitely many steps the flipping locus $\operatorname{Exc}(\theta_i)$ does not intersect the non-klt locus of the g-pair $(X_i/Z, B_i+M_i)$, i.e., there exists $N \in \mathbb{Z}_{\geq 1}$ such that

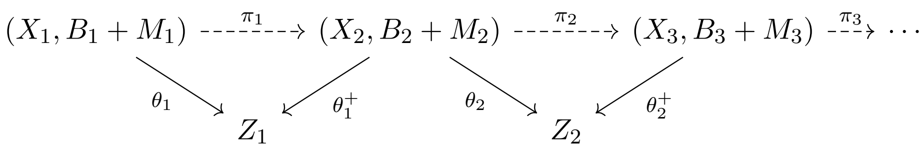
$$\operatorname{Exc}(\theta_i) \cap \operatorname{Nklt}(X_i, B_i + M_i) = \emptyset \text{ for all } i \geq N.$$

In particular, if $(X_1, B_1 + M_1)$ is a $\mathbb{Q}$-factorial dlt g-pair, then it follows from Remark 2.11(1) that special termination of the given sequence of flips means that eventually the flipping locus $\operatorname{Exc}(\theta_i)$ avoids $\operatorname{Supp}\lfloor B_i \rfloor$.

Furthermore, the phrase *special termination with scaling (of an ample divisor)* refers to sequences of flips with scaling (of an ample divisor) and its meaning is apparent from the above.

Last but not least, we prove below an immediate consequence of special termination, namely, if the flipping locus $\operatorname{Exc}(\theta_i)$ does not intersect the non-klt locus of (X_i, B_i+M_i) for all $i \gg 0$, then the flipped locus $\operatorname{Exc}(\theta_i^+)$ does not intersect the non-klt locus of $(X_{i+1}, B_{i+1} + M_{i+1})$ for all $i \gg 0$ as well.

Lemma 5.1. *Let*



be a sequence of flips over Z starting from an lc g-pair $(X_1/Z, B_1+M_1)$. If there exists $N \in \mathbb{Z}_{\geq 1}$ such that

$$\operatorname{Exc}(\theta_i) \cap \operatorname{Nklt}(X_i, B_i + M_i) = \emptyset \textit{ for all } i \geq N,$$

then also

$$\operatorname{Exc}(\theta_i^+) \cap \operatorname{Nklt}(X_{i+1}, B_{i+1} + M_{i+1}) = \emptyset \textit{ for all } i \geq N.$$

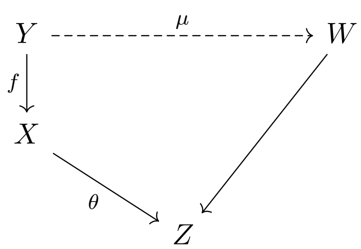
Proof. Fix $i \geq N$ and let S_{i+1} be an lc center of $(X_{i+1}, B_{i+1} + M_{i+1})$. Then there is a divisorial valuation E over X with $c_{X_{i+1}}(E) = S_{i+1}$ and $a(E, X_{i+1}, B_{i+1}+M_{i+1}) = -1$. Note first that $S_{i+1} \nsubseteq \operatorname{Exc}(\theta_i^+)$, since otherwise by Lemma 2.45 we would have

$$-1 \leq a(E, X_i, B_i + M_i) < a(E, X_{i+1}, B_{i+1} + M_{i+1}) = -1,$$

which is impossible. Hence, $\pi_i \colon X_i \dashrightarrow X_{i+1}$ is an isomorphism at the generic point of S_{i+1}. Moreover, if $S_{i+1} \cap \operatorname{Exc}(\theta_i^+) \neq \emptyset$, then it follows from assumption and Lemma 2.45 that the lc center $S_i := c_{X_i}(E)$ of X_i does not intersect $\operatorname{Exc}(\theta_i)$. Thus, $S_i \subseteq X_i \smallsetminus \operatorname{Exc}(\theta_i)$, which implies that $S_{i+1} \subseteq X_{i+1} \smallsetminus \operatorname{Exc}(\theta_i^+)$, a contradiction. Consequently, $S_{i+1} \cap \operatorname{Exc}(\theta_i^+) = \emptyset$. This finishes the proof. □

5.2 The Key Result

Lemma 5.2. *Let $f \colon Y \to X$ be a projective birational morphism between normal varieties. Assume that we have a diagram*



where θ is birational and μ is an isomorphism in codimension one. Let D_X be an $\mathbb{R}$-Cartier $\mathbb{R}$-divisor on X, set $D_Y := f^*D_X$ and $D_W := \mu_*D_Y$, and assume that D_W is $\mathbb{R}$-Cartier. Let $V_X \subseteq X$ and $V_Y \subseteq Y$ be closed subsets such that $f(V_Y) = V_X$. Assume that:

(i) *μ is D_Y-non-positive,*

(ii) *D_W is semi-ample over Z,*

(iii) *V_Y is contained in the locus in Y where the map μ is an isomorphism, and*

(iv) *$\operatorname{Exc}(\theta)$ is covered by curves γ which are contracted by θ and satisfy $D_X \cdot \gamma < 0$.*

Then $\operatorname{Exc}(\theta) \cap V_X = \emptyset$.

Proof. Arguing by contradiction, assume that there exists $x \in \operatorname{Exc}(\theta) \cap V_X$ and set $F := f^{-1}(x)$. We first claim that

$$F \subseteq \mathbf{B}(D_Y/Z). \tag{5.1}$$

To this end, by (iv) we may find a curve $\gamma \subseteq \operatorname{Exc}(\theta)$ passing through x, contracted by θ and such that $D_X \cdot \gamma < 0$. But then for each $H \in |D_X/Z|_{\mathbb{R}}$ we have $H \cdot \gamma < 0$, and thus $x \in \gamma \subseteq \operatorname{Supp} H$. This implies that $x \in \mathbf{B}(D_X/Z)$, and by Lemma 1.20 we infer that

$$F \subseteq f^{-1}\big(\mathbf{B}(D_X/Z)\big) = \mathbf{B}(D_Y/Z),$$

as desired.

Now, since $f(V_Y) = V_X$, we have $F \cap V_Y \neq \emptyset$, and by (5.1) we obtain

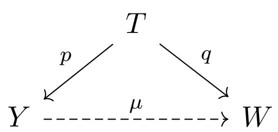
$$V_Y \cap \mathbf{B}(D_Y/Z) \neq \emptyset. \tag{5.2}$$

Define $V_W := \mu(V_Y)$ and note that V_W is well-defined by (iii). We claim that

$$V_W \cap \mathbf{B}(D_W/Z) \neq \emptyset,$$

which would then contradict (ii) and finish the proof.

To this end, by (iii) there exists a resolution of indeterminacies $(p,q)\colon T \to Y \times W$ of the map μ such that p and q are isomorphisms over some neighbourhoods of V_Y and V_W, respectively.



Then by (i) there exists an effective q-exceptional $\mathbb{R}$-divisor E_T on T such that

$$p^*D_Y \sim_{\mathbb{R}} q^*D_W + E_T.$$

Fix $G_W \in |D_W/Z|_{\mathbb{R}}$, note that $q^*G_W + E_T \sim_{\mathbb{R}} p^*\big(D_Y + (\theta \circ f)^*Q_Z\big)$ for some $\mathbb{R}$-Cartier $\mathbb{R}$-divisor Q_Z on Z and set

$$G_Y := p_*(q^*G_W + E_T) \in |D_Y/Z|_{\mathbb{R}}.$$

It follows from the Negativity lemma [KM98, Lemma 3.39(1)] (as in the proof of Lemma 1.20) that

$$p^*G_Y = q^*G_W + E_T.$$

Since $V_Y \cap \operatorname{Supp} G_Y \neq \emptyset$ by (5.2), we obtain

$$\emptyset \neq p^{-1}(V_Y) \cap \operatorname{Supp}(p^*G_Y) = p^{-1}(V_Y) \cap \operatorname{Supp}(q^*G_W + E_T),$$

and hence $p^{-1}(V_Y) \cap q^{-1}(\operatorname{Supp} G_W) \neq \emptyset$, as $p^{-1}(V_Y)$ does not intersect $\operatorname{Supp} E_T$ by construction. Thus, as $V_W = q\big(p^{-1}(V_Y)\big)$, we have

$$V_W \cap \operatorname{Supp} G_W \neq \emptyset,$$

and the claim follows. □

5.3 Special Termination for Log Canonical Pairs

In this section we reduce the special termination for lc pairs of dimension n to the termination of flips for klt pairs of dimension at most $n-1$. But first, for the sake of completeness of the presentation, we gather below all the known results concerning the special termination for dlt pairs:

(1) [Fuj07, Theorem 4.2.1]: The termination of flips for $\mathbb{Q}$-factorial dlt pairs of dimension at most $n-1$ implies the special termination for $\mathbb{Q}$-factorial dlt pairs of dimension n.

(2) [Fuj11b, Theorem 29] (see also [Bir10a, Lemma 3.6], [Bir11, Proof of Corollary 1.7], [Bir12a, Remark 2.10]): The termination of flips with scaling (of an ample divisor) for $\mathbb{Q}$-factorial dlt pairs of dimension at most $n-1$ implies the special termination with scaling (of an ample divisor) for $\mathbb{Q}$-factorial dlt pairs of dimension n.

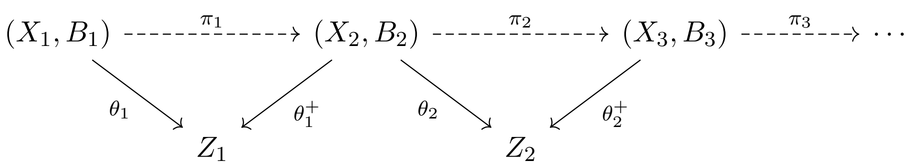
Now, we state one of the most notable applications of (1), which was also mentioned in the introduction, but we omit its proof and refer to [Fuj17, Lemmas 4.3.8 and 4.9.3] for the details. However, in Section 5.5 we will give a detailed proof of its analog in the setting of g-pairs and we stress that these proofs are essentially identical to each other.

Lemma 5.3. *The termination of flips for $\mathbb{Q}$-factorial klt pairs of dimension at most n implies the termination of flips for lc pairs of dimension n.*

Finally, as announced above, we generalize (1) to lc pairs.

Theorem 5.4. *Assume the termination of flips for $\mathbb{Q}$-factorial klt pairs of dimension at most $n-1$.*

Let $(X_1/Z, B_1)$ be an lc pair of dimension n. Consider a sequence of flips over Z:



Then there exists $N \in \mathbb{Z}_{\geq 1}$ such that

$$\operatorname{Exc}(\theta_i) \cap \operatorname{Nklt}(X_i, B_i) = \emptyset \text{ for all } i \geq N.$$

Proof. By Lemma 2.56 there exists a diagram

$$
\begin{array}{ccccccccc}
(Y_1,\Delta_1) & \overset{\rho_1}{\dashrightarrow} & & (Y_2,\Delta_2) & \overset{\rho_2}{\dashrightarrow} & & (Y_3,\Delta_3) & \overset{\rho_3}{\dashrightarrow} & \cdots \\
\downarrow h_1 & & & \downarrow h_2 & & & \downarrow h_3 & & \\
(X_1,B_1) & \overset{\pi_1}{\dashrightarrow} & & (X_2,B_2) & \overset{\pi_2}{\dashrightarrow} & & (X_3,B_3) & \overset{\pi_3}{\dashrightarrow} & \cdots \\
& \searrow^{\theta_1} & \swarrow_{\theta_1^+} & & \searrow^{\theta_2} & \swarrow_{\theta_2^+} & & & \\
& & Z_1 & & & Z_2 & & &
\end{array}
$$

where (Y_1, Δ_1) is a $\mathbb{Q}$-factorial dlt pair and the sequence of rational maps ρ_i is a composition of steps of a $(K_{Y_1} + \Delta_1)$-MMP. By relabelling, we may assume that this MMP is a sequence of flips. Moreover, by Lemma 5.3 we may assume the termination of flips for $\mathbb{Q}$-factorial dlt pairs of dimension at most $n-1$, so by [Fuj07, Theorem 4.2.1] we may also assume that the flipping locus avoids the non-klt locus at each step in this MMP. We conclude by applying Lemma 5.2 for $X = X_1$, $Y = Y_1$, $D_X = K_{X_1} + B_1$, $D_Y = K_{Y_1} + \Delta_1$, $V_X = \operatorname{Nklt}(X_1, B_1)$ and $V_Y = \operatorname{Nklt}(Y_1, \Delta_1)$, taking Remark 2.11(2) into account. □

Our goal in the remainder of this chapter is to generalize Theorem 5.4 to the context of g-pairs. But before we start pursuing this goal, we also remark that Han and Li have obtained a version of (2) for g-pairs, which concerns the MMP with scaling of an NQC divisor, see [HL18, Theorem 4.5]. Furthermore, we highlight that from this point forward we will often invoke adjunction for dlt g-pairs; in other words, if $(X, B + M)$ is a dlt g-pair and if S is an lc center of $(X, B + M)$, then we will define a dlt g-pair $(S, B_S + M_S)$ by the formula $K_S + B_S + M_S = (K_X + B + M)|_S$, see Notation 2.17.

5.4 The Difficulty of an NQC DLT Generalized Pair Obtained by Adjunction

The *difficulty* is a quantity associated with a pair, which counts divisorial valuations with certain discrepancies. It is typically used in termination arguments, usually as follows: one first shows that it decreases after a flip and one can thus conclude the termination of flips, provided, for instance, that the difficulty is a non-negative integer.

The first version of the difficulty was introduced by Shokurov [Sho85] in order to prove the termination of flips for terminal 3-folds. Since then several variants of the difficulty have appeared in the literature [KMM87, Kol89, Mat91, Kaw92, K$^+$92, Fuj04, Fuj05, AHK07, Mor18, CT20] and have been employed for similar purposes, as the main idea to tackle the termination of flips conjecture has remained the same ever since. In particular, a version of the difficulty in the setting of g-pairs will be introduced in Section 6.1 and will be used in order to establish the termination of flips for terminal g-pairs of dimension 3.

Moreover, other variants of the difficulty have been introduced in [K$^+$92, Fuj07, HL18] in order to address the special termination. Here, we recall Han and Li's definition of the difficulty of a dlt g-pair obtained from another dlt g-pair by adjunction [HL18]. Afterwards, we prove some basic properties of the difficulty that play a crucial role in the proof of Theorem 5.9 which concerns the special termination for dlt g-pairs.

Definition 5.5. Let $(X, B+M)$ be an NQC dlt g-pair with data $X' \to X \to Z$ and M'. We may write $B = \sum_{i=1}^{k} b_i B_i$ with distinct prime divisors B_i and $b_i \in (0,1]$, and $M' = \sum_{j=1}^{l} \mu_j M'_j$ with M'_j Cartier divisors on X' which are nef over Z and $\mu_j \in (0, +\infty)$. Set $b = \{b_1, \dots, b_k\}$, $\mu = \{\mu_1, \dots, \mu_l\}$ and

$$\mathcal{S}(b,\mu) = \left\{1 - \frac{1}{m} + \sum_{i=1}^{k} \frac{r_i b_i}{m} + \sum_{j=1}^{l} \frac{s_j \mu_j}{m} \leq 1 \;\middle|\; m \in \mathbb{Z}_{\geq 1},\ r_i, s_j \in \mathbb{Z}_{\geq 0}\right\}.$$

Let S be an lc center of $(X, B+M)$ and define a dlt g-pair $(S, B_S + M_S)$ by adjunction. Note that the coefficients of B_S belong to the set $\mathcal{S}(b,\mu)$ by the proof of [BZ16, Proposition 4.9]. For each $\alpha \in \mathcal{S}(b,\mu)$, set

$$d_{<-\alpha}(S, B_S + M_S) = \#\big\{E \mid a(E, S, B_S + M_S) < -\alpha,\ c_S(E) \not\subseteq \operatorname{Supp}\lfloor B_S \rfloor\big\}$$

and

$$d_{\leq-\alpha}(S, B_S + M_S) = \#\big\{E \mid a(E, S, B_S + M_S) \leq -\alpha,\ c_S(E) \not\subseteq \operatorname{Supp}\lfloor B_S \rfloor\big\}.$$

The *difficulty* of the g-pair $(S, B_S + M_S)$ is defined as

$$d_{b,\mu}(S, B_S + M_S) = \sum_{\alpha \in \mathcal{S}(b,\mu)} \Big(d_{<-\alpha}(S, B_S + M_S) + d_{\leq-\alpha}(S, B_S + M_S)\Big).$$

Lemma 5.6. *With the same notation as in Definition 5.5, the following statements hold:*

(i) *There exists $\gamma \in (0,1)$ such that $a(E, S, B_S + M_S) \geq -\gamma$ for each divisorial valuation E over S such that $c_S(E) \not\subseteq \operatorname{Supp}\lfloor B_S \rfloor$.*

(ii) *The set $\mathcal{S}(b,\mu) \cap [0,\gamma]$ is finite.*

(iii) *We have*
$$0 \leq d_{b,\mu}(S, B_S + M_S) < +\infty.$$

Proof. Let $S' \xrightarrow{\sigma} S \to Z$ and M'_S be the data of the dlt g-pair $(S, B_S + M_S)$. Consider the set $U := S \setminus \operatorname{Supp}\lfloor B_S \rfloor$ and let $U' := \sigma^{-1}(U)$. Then we obtain the klt g-pair $(U, B_S|_U + M_S|_U)$ with data $U' \to U \xrightarrow{\text{id}} U$ and $M'_S|_{U'}$. Define B'_S by the equation

$$K_{S'} + B'_S + M'_S \sim_{\mathbb{R}} \sigma^*(K_S + B_S + M_S).$$

Then for each divisorial valuation E over S such that $c_S(E) \not\subseteq \operatorname{Supp}\lfloor B_S \rfloor$ we have

$$\begin{aligned} a(E, S, B_S + M_S) &= a(E, U, B_S|_U + M_S|_U) \\ &= a(E, U', B'_S|_{U'} + M'_S|_{U'}) = a(E, U', B'_S|_{U'}), \end{aligned}$$

hence

$$d_{<-\alpha}(S, B_S + M_S) = \#\big\{E \mid a(E, U', B'_S|_{U'}) < -\alpha\big\} \tag{5.3}$$

and

$$d_{\leq-\alpha}(S, B_S + M_S) = \#\big\{E \mid a(E, U', B'_S|_{U'}) \leq -\alpha\big\}. \tag{5.4}$$

Since by Remark 2.7 the pair $(U', B'_S|_{U'})$ is klt, by (5.3) and (5.4) there exists $\gamma \in (0,1)$ such that (i) holds, and in particular:

$$d_{<-\alpha}(S, B_S + M_S) = d_{\leq -\alpha}(S, B_S + M_S) = 0 \quad \text{if } \alpha > \gamma.$$

On the other hand, observe that $d_{<-\alpha}(S, B_S + M_S)$ and $d_{\leq -\alpha}(S, B_S + M_S)$ are finite for any $\alpha \in \mathcal{S}(b, \mu)$ by (5.3) and (5.4) and by [KM98, Proposition 2.36(2)]. Since the set $\mathcal{S}(b, \mu) \cap [0, \gamma]$ is finite by [K+92, Lemma 7.4.4], (ii) and (iii) follow. $\square$

Proposition 5.7. *Assume the notation of Definition 5.5. Consider a $(K_X + B + M)$-flip over Z:*

$$\begin{array}{ccc} (X, B+M) & \overset{\pi}{\dashrightarrow} & (X^+, B^+ + M^+) \\ & {\scriptstyle\theta}\searrow \quad \swarrow{\scriptstyle\theta^+} & \\ & W & \end{array}$$

Assume that π is an isomorphism at the generic point of S and define S^+ as the strict transform of S on X^+. Moreover, assume that $\pi|_S$ is an isomorphism along $\operatorname{Supp}\lfloor B_S \rfloor$. *Then the following statements hold.*

(i) *We have*

$$d_{b,\mu}(S, B_S + M_S) \geq d_{b,\mu}(S^+, B_{S^+} + M_{S^+}).$$

(ii) *If there exists a divisorial valuation E over S such that $c_S(E)$ is a divisor but $c_{S^+}(E)$ is not a divisor, then there exists $\alpha_0 \in \mathcal{S}(b, \mu) \setminus \{1\}$ such that*

$$d_{\leq -\alpha_0}(S, B_S + M_S) > d_{\leq -\alpha_0}(S^+, B_{S^+} + M_{S^+}).$$

(iii) *If there exists a divisorial valuation E over S such that $c_S(E)$ is not a divisor but $c_{S^+}(E)$ is a divisor, then there exists $\alpha_0 \in \mathcal{S}(b, \mu) \setminus \{1\}$ such that*

$$d_{<-\alpha_0}(S, B_S + M_S) > d_{<-\alpha_0}(S^+, B_{S^+} + M_{S^+}).$$

(iv) *If $\pi|_S$ is not an isomorphism in codimension one, then*

$$d_{b,\mu}(S, B_S + M_S) > d_{b,\mu}(S^+, B_{S^+} + M_{S^+}).$$

Proof.
(i) Follows immediately from Lemma 2.18(ii).

(ii) First, observe that $c_S(E) \not\subseteq \operatorname{Supp}\lfloor B_S \rfloor$ and $c_{S^+}(E) \not\subseteq \operatorname{Supp}\lfloor B_{S^+} \rfloor$, since $\pi|_S$ is an isomorphism along $\operatorname{Supp}\lfloor B_S \rfloor$. Recall also that the coefficients of B_S (and B_{S^+}) belong to the set $\mathcal{S}(b, \mu)$. Thus, there exists $\alpha_0 \in \mathcal{S}(b, \mu) \setminus \{1\}$ such that, by Lemma 2.18(iv),

$$-\alpha_0 = a(E, S, B_S + M_S) < a(E, S^+, B_{S^+} + M_{S^+}),$$

and (ii) follows.

(iii) As in (ii), it holds again that $c_S(E) \not\subseteq \operatorname{Supp}\lfloor B_S \rfloor$ and $c_{S^+}(E) \not\subseteq \operatorname{Supp}\lfloor B_{S^+} \rfloor$. Then there exists $\alpha_0 \in \mathcal{S}(b, \mu) \setminus \{1\}$ such that, by Lemma 2.18(iv),

$$a(E, S, B_S + M_S) < a(E, S^+, B_{S^+} + M_{S^+}) = -\alpha_0,$$

and (iii) follows.

(iv) Follows immediately from (i), (ii) and (iii). $\square$

5.5 Special Termination for NQC Log Canonical Generalized Pairs

In this section we reduce the special termination for NQC lc g-pairs of dimension n to the termination of flips for NQC klt g-pairs of dimension at most $n-1$. We begin with the following lemma.

Lemma 5.8. *Let*

$$\begin{array}{ccccccccc} (X_1, B_1+M_1) & \overset{\pi_1}{\dashrightarrow} & & (X_2, B_2+M_2) & \overset{\pi_2}{\dashrightarrow} & & (X_3, B_3+M_3) & \overset{\pi_3}{\dashrightarrow} & \cdots \\ & \searrow^{\theta_1} & & \swarrow_{\theta_1^+} & \searrow^{\theta_2} & & \swarrow_{\theta_2^+} & & \\ & & Z_1 & & & Z_2 & & & \end{array}$$

be a sequence of flips over Z starting from an lc g-pair $(X_1/Z, B_1+M_1)$. Then there exists $N \in \mathbb{Z}_{\geq 1}$ such that for each $i \geq N$ the flipping locus $\mathrm{Exc}(\theta_i)$ does not contain any lc center of (X_i, B_i+M_i).

Proof. By Remark 2.11(2) the number of lc centers of any lc g-pair is finite. Moreover, by Lemma 2.18(iii), at the i-th step of the given sequence of flips, every lc center of $(X_{i+1}, B_{i+1}+M_{i+1})$ is also an lc center of (X_i, B_i+M_i), and if an lc center of (X_i, B_i+M_i) is contained in the flipping locus $\mathrm{Exc}(\theta_i)$, then the number of lc centers of $(X_{i+1}, B_{i+1}+M_{i+1})$ is smaller than the number of lc centers of (X_i, B_i+M_i).

Now, for every $i \geq 1$ we denote by ν_i the number of lc centers of (X_i, B_i+M_i). It follows by the above that $\{\nu_i\}_{i=1}^{\infty}$ is a non-increasing sequence of non-negative integers with $\nu_1 < +\infty$ and such that $\nu_i > \nu_{i+1}$ whenever an lc center of (X_i, B_i+M_i) is contained in $\mathrm{Exc}(\theta_i)$. Hence, $\{\nu_i\}_{i=1}^{\infty}$ must eventually stabilize. This yields the statement. □

Now, we generalize [Fuj07, Theorem 4.2.1] to the context of g-pairs, following closely the proofs of [Fuj07, Theorem 4.2.1] and [HL18, Theorem 4.5].

Theorem 5.9. *Assume the termination of flips for NQC $\mathbb{Q}$-factorial dlt g-pairs of dimension at most $n-1$.*

Let $(X_1/Z, B_1+M_1)$ be an NQC $\mathbb{Q}$-factorial dlt g-pair of dimension n. Consider a sequence of flips over Z:

$$\begin{array}{ccccccccc} (X_1, B_1+M_1) & \overset{\pi_1}{\dashrightarrow} & & (X_2, B_2+M_2) & \overset{\pi_2}{\dashrightarrow} & & (X_3, B_3+M_3) & \overset{\pi_3}{\dashrightarrow} & \cdots \\ & \searrow^{\theta_1} & & \swarrow_{\theta_1^+} & \searrow^{\theta_2} & & \swarrow_{\theta_2^+} & & \\ & & Z_1 & & & Z_2 & & & \end{array}$$

Then there exists $N \in \mathbb{Z}_{\geq 1}$ such that

$$\mathrm{Exc}(\theta_i) \cap \mathrm{Nklt}(X_i, B_i+M_i) = \emptyset \text{ for all } i \geq N.$$

Proof. We prove by induction on d the following claim, which implies the theorem.

Claim 5.10. *For each non-negative integer d there exists a positive integer N_d such that $\mathrm{Exc}(\theta_i)$ is disjoint from each lc center of dimension at most d for all $i \geq N_d$.*

To this end, by Lemma 5.8 there exists a positive integer N_0 such that the set $\operatorname{Exc}(\theta_i)$ does not contain any lc center of $(X_i, B_i + M_i)$ for $i \geq N_0$. This proves Claim 5.10 for $d = 0$. Moreover, by relabelling, we may assume that $N_0 = 1$, and therefore for each $i \geq 1$ the map $\pi_i \colon X_i \dashrightarrow X_{i+1}$ is an isomorphism at the generic point of each lc center of $(X_i, B_i + M_i)$.

Let d be a positive integer. By induction and by relabelling, we may assume that each map π_i is an isomorphism along every lc center of dimension at most $d - 1$.

Now, we consider an lc center S_1 of $(X_1, B_1 + M_1)$ of dimension d. We obtain a sequence of birational maps $\pi_i|_{S_i} \colon S_i \dashrightarrow S_{i+1}$, where S_i is the strict transform of S_1 on X_i. Note that every lc center of the NQC dlt g-pair $(S_i, B_{S_i} + M_{S_i})$ is an lc center of $(X_i, B_i + M_i)$, and hence, by induction, each map π_i is an isomorphism along $\operatorname{Supp}\lfloor B_{S_i} \rfloor$. Then by Proposition 5.7 and since the difficulty takes values in $\mathbb{N}$, after relabelling the indices we may assume that S_i and S_{i+1} are isomorphic in codimension one for every $i \geq 1$.

Moreover, by relabelling, we may assume that $(\pi_i|_{S_i})_* B_{S_i} = B_{S_{i+1}}$ for every $i \geq 1$. Indeed, this is equivalent to having

$$a(E, S_i, B_{S_i} + M_{S_i}) = a(E, S_{i+1}, B_{S_{i+1}} + M_{S_{i+1}}) \tag{5.5}$$

for each component of B_{S_i} and $B_{S_{i+1}}$, and we verify these equalities below. To this end, fix an irreducible component E of B_{S_1}. Observe that the strict transform of E on S_i is a divisor on S_i, denoted again by E (abusing notation). Since each map π_i is an isomorphism along $\operatorname{Supp}\lfloor B_{S_i} \rfloor$, if $E \subseteq \operatorname{Supp}\lfloor B_{S_1} \rfloor$, then $E \subseteq \operatorname{Supp}\lfloor B_{S_i} \rfloor$ and clearly E satisfies (5.5). For the same reason, if $E \not\subseteq \operatorname{Supp}\lfloor B_{S_1} \rfloor$, then $E \not\subseteq \operatorname{Supp}\lfloor B_{S_i} \rfloor$, and we also note that the coefficient of E in B_{S_i} is $-a(E, S_i, B_{S_i} + M_{S_i}) \in \mathcal{S}(b, \mu) \cap [0, 1)$. It follows from Lemma 2.18(ii) that $\big\{-a(E, S_i, B_{S_i} + M_{S_i})\big\}_{i=1}^{\infty}$ is a non-increasing sequence of elements of the set $\mathcal{S}(b, \mu)$, which is bounded below by 0 and bounded above by γ_1, where the constant $\gamma_1 \in (0, 1)$ occurs from Lemma 5.6(i). By Lemma 5.6(ii) we infer that this sequence must eventually stabilize; in other words, there exists $N_E \in \mathbb{Z}_{\geq 1}$ such that E satisfies (5.5) for every $i \geq N_E$. (However, note that E may no longer be a component of B_{S_i} for $i \geq N_E$, that is, we may have $a(E, S_i, B_{S_i} + M_{S_i}) = 0$ for every $i \geq N_E$.) In conclusion, after relabelling the indices we may assume that (5.5) holds; this concludes the proof of our assertion that $(\pi_i|_{S_i})_* B_{S_i} = B_{S_{i+1}}$.

For every $i \geq 1$ denote by T_i the normalization of $\theta_i(S_i)$. By Lemma 2.57 there exists a diagram

$$\begin{array}{ccccccc}
(W_1, \Delta_1 + N_1) & \overset{\rho_1}{\dashrightarrow} & (W_2, \Delta_2 + N_2) & \overset{\rho_2}{\dashrightarrow} & (W_3, \Delta_3 + N_3) & \overset{\rho_3}{\dashrightarrow} & \cdots \\
\downarrow{\scriptstyle h_1} & & \downarrow{\scriptstyle h_2} & & \downarrow{\scriptstyle h_3} & & \\
(S_1, B_{S_1} + M_{S_1}) & \overset{\pi_1|_{S_1}}{\dashrightarrow} & (S_2, B_{S_2} + M_{S_2}) & \overset{\pi_2|_{S_2}}{\dashrightarrow} & (S_3, B_{S_3} + M_{S_3}) & \dashrightarrow & \cdots \\
{\scriptstyle \theta_1|_{S_1}} \searrow & & \swarrow {\scriptstyle \theta_1^+|_{S_1}} \quad {\scriptstyle \theta_2|_{S_2}} \searrow & & \swarrow {\scriptstyle \theta_2^+|_{S_2}} & & \\
& T_1 & & T_2 & & &
\end{array}$$

where the sequence of rational maps ρ_i yields an MMP for the NQC $\mathbb{Q}$-factorial dlt g-pair $(W_1, \Delta_1 + N_1)$ and the g-pairs $(W_i, \Delta_i + N_i)$ are dlt blow-ups of the g-pairs $(S_i, B_{S_i} + M_{S_i})$. By the assumptions of the theorem, this $(K_{W_1} + \Delta_1 + N_1)$-MMP terminates, so after relabelling, we may assume that

$$(W_i, \Delta_i + N_i) = (W_{i+1}, \Delta_{i+1} + N_{i+1}) \quad \text{for all } i \geq 1.$$

Since $-(K_{W_i} + \Delta_i + N_i)$ is nef over T_i and $K_{W_{i+1}} + \Delta_{i+1} + N_{i+1}$ is nef over T_i by construction, we deduce that $K_{W_i} + \Delta_i + N_i$ is numerically trivial over T_i for each $i \geq 1$. In particular, $K_{S_i} + B_{S_i} + M_{S_i}$ and $K_{S_{i+1}} + B_{S_{i+1}} + M_{S_{i+1}}$ are numerically trivial over T_i for each $i \geq 1$. Thus, $\theta_i|_{S_i}$ and $\theta_i^+|_{S_{i+1}}$ contract no curves. Indeed, if there were a curve $C_i \subseteq S_i$ which were contracted by $\theta_i|_{S_i}$, then we would have

$$0 = (K_{S_i} + B_{S_i} + M_{S_i}) \cdot C_i = (K_{X_i} + B_i + M_i)|_{S_i} \cdot C_i = (K_{X_i} + B_i + M_i) \cdot C_i < 0,$$

which is a contradiction; we argue similarly for $\theta_i^+|_{S_{i+1}}$. Hence, the maps $\theta_i|_{S_i}$ and $\theta_i^+|_{S_{i+1}}$ are quasi-finite, and since they are both projective, they are actually finite, and it follows now from Lemma 1.16 that $\theta_i|_{S_i}$ and $\theta_i^+|_{S_{i+1}}$ are isomorphisms. Consequently, all maps $\pi_i|_{S_i}$ are isomorphisms. This finishes the inductive proof of Claim 5.10, hence the proof of the theorem. □

Next, we obtain the analog of Lemma 5.3 in the setting of g-pairs.

Lemma 5.11. *The termination of flips for NQC $\mathbb{Q}$-factorial klt g-pairs of dimension at most n implies the termination of flips for NQC lc g-pairs of dimension n.*

Proof. By induction we may assume the termination of flips for NQC $\mathbb{Q}$-factorial dlt g-pairs of dimension at most $n-1$.

Let

$$\begin{array}{ccccccc}
(X_1, B_1 + M_1) & \overset{\pi_1}{\dashrightarrow} & (X_2, B_2 + M_2) & \overset{\pi_2}{\dashrightarrow} & (X_3, B_3 + M_3) & \overset{\pi_3}{\dashrightarrow} & \cdots \\
 & {\scriptstyle\theta_1}\searrow \quad \swarrow{\scriptstyle\theta_1^+} & & {\scriptstyle\theta_2}\searrow \quad \swarrow{\scriptstyle\theta_2^+} & & & \\
 & Z_1 & & Z_2 & & &
\end{array}$$

be a sequence of flips over Z starting from an NQC lc g-pair $(X_1/Z, B_1 + M_1)$ of dimension n. By Lemma 2.57 there exists a diagram

$$\begin{array}{ccccccc}
(Y_1, \Delta_1 + N_1) & \overset{\rho_1}{\dashrightarrow} & (Y_2, \Delta_2 + N_2) & \overset{\rho_2}{\dashrightarrow} & (Y_3, \Delta_3 + N_3) & \overset{\rho_3}{\dashrightarrow} & \cdots \\
\downarrow{\scriptstyle h_1} & & \downarrow{\scriptstyle h_2} & & \downarrow{\scriptstyle h_3} & & \\
(X_1, B_1 + M_1) & \overset{\pi_1}{\dashrightarrow} & (X_2, B_2 + M_2) & \overset{\pi_2}{\dashrightarrow} & (X_3, B_3 + M_3) & \overset{\pi_3}{\dashrightarrow} & \cdots \\
 & {\scriptstyle\theta_1}\searrow \quad \swarrow{\scriptstyle\theta_1^+} & & {\scriptstyle\theta_2}\searrow \quad \swarrow{\scriptstyle\theta_2^+} & & & \\
 & Z_1 & & Z_2 & & &
\end{array}$$

where the sequence of rational maps ρ_i is a composition of steps in an MMP for an NQC $\mathbb{Q}$-factorial dlt g-pair $(Y_1, \Delta_1 + N_1)$. To prove the statement, it suffices to show that this MMP terminates. By relabelling, we may assume that this MMP consists only of flips. By Theorem 5.9 and by relabelling again, we may also assume that at each step the flipping locus avoids the non-klt locus. Consequently, this sequence of flips is also a sequence of flips for the NQC $\mathbb{Q}$-factorial klt g-pair $\big(Y_1, (\Delta_1 - \lfloor \Delta_1 \rfloor) + N_1\big)$, and this sequence terminates by assumption. This concludes the proof. □

Finally, we prove the analog of Theorem 5.4 in the context of g-pairs.

Theorem 5.12. *Assume the termination of flips for NQC $\mathbb{Q}$-factorial klt g-pairs of dimension at most $n-1$.*

Let $(X_1/Z, B_1 + M_1)$ be an NQC lc g-pair of dimension n. Consider a sequence of flips over Z:

$$\begin{array}{ccccccccc}
(X_1, B_1 + M_1) & & \overset{\pi_1}{\dashrightarrow} & & (X_2, B_2 + M_2) & & \overset{\pi_2}{\dashrightarrow} & & (X_3, B_3 + M_3) \overset{\pi_3}{\dashrightarrow} \cdots \\
& \underset{\theta_1}{\searrow} & & \underset{\theta_1^+}{\swarrow} & & \underset{\theta_2}{\searrow} & & \underset{\theta_2^+}{\swarrow} & \\
& & Z_1 & & & & Z_2 & &
\end{array}$$

Then there exists $N \in \mathbb{Z}_{\geq 1}$ such that

$$\operatorname{Exc}(\theta_i) \cap \operatorname{Nklt}(X_i, B_i + M_i) = \emptyset \textit{ for all } i \geq N.$$

Proof. The proof is identical to the proof of Theorem 5.4, replacing Lemma 5.3 with Lemma 5.11, Lemma 2.56 with Lemma 2.57, and [Fuj07, Theorem 4.2.1] with Theorem 5.9. □

6

On the Termination of Flips for Log Canonical Generalized Pairs

The purpose of this chapter is to address one of the main open problems in the MMP, the termination of flips conjecture, in the context of g-pairs.

Termination of Flips Conjecture. *Let $(X/Z, B+M)$ be an NQC lc g-pair. Then any sequence of flips over Z starting from $(X, B+M)$ terminates.*

For an overview of all the currently known results regarding the termination of flips conjecture we refer to the introduction of our paper [CT20]. In particular, we recall here that Kawamata [Kaw92] and Shokurov [Sho85, Sho96] established the termination of flips for lc pairs of dimension 3; Kawamata, Matsuda and Matsuki [KMM87] proved the case of terminal 4-folds; Fujino [Fuj04, Fuj05] settled the case of 4-dimensional canonical pairs with rational coefficients; Alexeev, Hacon and Kawamata [AHK07] verified the termination of flips conjecture for klt pairs (X, B) such that $\dim X = 4$ and $-(K_X + B)$ is effective, while Birkar [Bir10b] treated the case of 4-dimensional klt pairs with rational coefficients and Iitaka dimension at least 2. It is worthwhile to mention here that these papers constitute the main body of work towards the termination of flips conjecture until recently and also that the central idea in all of them is to introduce an appropriate difficulty and to study its behavior under flips in order to conclude.

In this chapter we confirm the termination of flips conjecture in dimension 3, utilizing several ideas from [K^+92, Kaw92, Sho96] and exploiting a suitable version of the difficulty for g-pairs, as well as for pseudo-effective g-pairs in dimension 4, following Birkar's strategy from [Bir07], which differs remarkably from the classical approach. Along the way we obtain an analog of Birkar's inductive termination result [Bir07, Theorem 1.3] in the context of g-pairs by invoking *the special termination for NQC lc g-pairs* (see Theorem 5.12) and *the ascending chain condition for lc thresholds* [BZ16, Theorem 1.5]. In particular, our result extends [HM18, Theorem 1] to the setting of $\mathbb{R}$-divisors, following a different approach.

The contents of this chapter are taken exclusively from our joint paper [CT20] with Guodu Chen. However, our presentation here is slightly different from the original one. Specifically, even though we use *log discrepancies* in [CT20] – which is in general quite common in recent works in birational geometry – we have reformulated here everything in terms of *discrepancies* instead in order to be consistent with the terminology used in all previous chapters of the thesis.

Throughout this chapter we assume that varieties are normal and quasi-projective and that a variety X over a variety Z is projective over Z. We also assume the existence of flips for NQC lc g-pairs and we recall that flips exist for NQC lc g-pairs whose underlying variety is $\mathbb{Q}$-factorial klt, see Remark 2.43(2).

6.1 The Difficulty of an NQC Terminal Generalized Pair

For a brief and general discussion about the concept of the *difficulty* of a (g-)pair we refer to Section 5.4. Here, we recall a version of the difficulty, which was introduced in [CT20] and which generalizes [KM98, Definition 6.20] to the setting of NQC g-pairs, and we demonstrate subsequently that it is finite and non-increasing under flips. In the next section we will use the difficulty crucially in order to deduce the termination of flips for NQC $\mathbb{Q}$-factorial terminal g-pairs of dimension 3.

Definition 6.1. Let $(X, B+M)$ be an NQC terminal g-pair with data $X' \xrightarrow{f} X \to Z$ and $M' = \sum_{j=1}^{l} \mu_j M'_j$, where $B = \sum_{i=1}^{s} b_i B_i$, $b_i \in [0,1)$ and the B_i are distinct prime divisors, $\mu_j \geq 0$ and the M'_j are $\mathbb{Q}$-Cartier divisors which are nef over Z. Set $b := \max\{b_1, \dots, b_s\}$ ($b := 0$ if $B = 0$), consider the set

$$\mathcal{S}_b := \left\{ \sum_{i=1}^{s} n_i b_i + \sum_{j=1}^{l} m_j \mu_j \geq b \;\middle|\; n_i, m_j \in \mathbb{Z}_{\geq 0} \right\}$$

($S_0 = \{0\}$ if $B = M = 0$), and for every $\xi \in \mathcal{S}_b$, set

$$d_\xi(X, B+M) := \#\big\{E \mid E \text{ is an exceptional divisorial valuation over } X \text{ with } a(E, X, B+M) < 1-\xi\big\}.$$

The *difficulty* of the given g-pair $(X, B+M)$ is defined as

$$d(X, B+M) := \sum_{\xi \in \mathcal{S}_b} d_\xi(X, B+M).$$

Lemma 6.2. *Assume the notation of Definition 6.1. Then*

$$0 \leq d(X, B+M) < +\infty.$$

Moreover, if

$$\begin{array}{ccc} (X, B+M) & \overset{\pi}{\dashrightarrow} & (X^+, B^+ + M^+) \\ & {\scriptstyle\theta}\searrow \quad \swarrow{\scriptstyle\theta^+} & \\ & W & \end{array}$$

is a $(K_X + B + M)$-flip over Z, then

$$d(X, B+M) \geq d(X^+, B^+ + M^+).$$

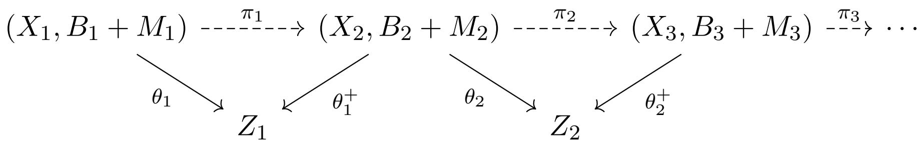
Proof. The difficulty is obviously non-negative, and it follows from Lemma 2.45 that it is non-increasing under a flip, so we deal with its finiteness below. For every $\xi \in \mathcal{S}_b \cap [1, +\infty)$ we clearly have $d_\xi(X, B+M) = 0$. On the other hand, there are only finitely many $\xi \in \mathcal{S}_b \cap [0,1)$ and for each such ξ it holds that $d_\xi(X, B+M) < +\infty$ by Proposition 2.15; indeed, every such ξ necessarily satisfies $1 - \xi \leq 1 - b$ and we invoke Remark 2.16 to conclude. Hence, $d(X, B+M) < +\infty$. □

6.2 The Termination of Flips in Dimension 3

In this section we prove the termination of flips conjecture in dimension 3 (see Theorem 6.10). We proceed as follows. First, we prove the termination of flips for 3-dimensional NQC $\mathbb{Q}$-factorial terminal g-pairs (see Theorem 6.3) by invoking Lemma 6.2. Second, we prove the termination of flips for 3-dimensional NQC $\mathbb{Q}$-factorial klt g-pairs (see Theorem 6.9). In order to do so, we obtain analogs of [Sho96, Proposition 4.4 and Lemma 4.4.1] in the context of g-pairs (see Propositions 6.6 and 6.8), which enable us to apply Lemma 2.54 and thus reduce Theorem 6.9 to Theorem 6.3. Third, we deduce Theorem 6.10 as an immediate consequence of Lemma 5.11 and Theorem 6.9.

Theorem 6.3. *Let $(X_1/Z, B_1 + M_1)$ be a 3-dimensional NQC $\mathbb{Q}$-factorial terminal g-pair. Then any sequence of flips over Z starting from $(X_1, B_1 + M_1)$ terminates.*

Proof. Let



be a sequence of flips over Z starting from the g-pair $(X_1, B_1 + M_1)$ which comes with data $X_1' \xrightarrow{f_1} X_1 \to Z$ and $M_1' = \sum_{j=1}^{l} \mu_j M_{1,j}'$, where $\mu_j \geq 0$ and the $M_{1,j}'$ are $\mathbb{Q}$-Cartier divisors which are nef over Z. Note that each g-pair $(X_i, B_i + M_i)$ in this sequence of flips is terminal by Lemma 2.45 and comes with data $X_1' \xrightarrow{f_i} X_i \to Z$ and M_1'. We will prove the statement by induction on the number of components of B_1.

First, consider the case when $B_1 = 0$. Since X_{i+1} is terminal, it is smooth at the generic point of every flipped curve. If C_{i+1} is a flipped curve with the generic point $\eta_{i+1} \in X_{i+1}$ and if E_{i+1} is the prime divisor obtained by blowing up X_{i+1} at η_{i+1}, then by Lemmas 2.12 and 2.45 we have

$$a(E_{i+1}, X_i, M_i) < a(E_{i+1}, X_{i+1}, M_{i+1}) = 1 - \sum m_j \mu_j$$

some non-negative integers m_j, and therefore $d(X_i, M_i) > d(X_{i+1}, M_{i+1})$. Consequently, if the given sequence of flips did not terminate, then we would obtain a strictly decreasing sequence $\{d(X_i, M_i)\}_{i=1}^{+\infty}$ of non-negative integers with $d(X_1, M_1) < +\infty$, which is impossible. Hence the given sequence of flips over Z terminates.

Now, assume that the statement holds when the number of components of B_1 is $\leq s - 1$, and suppose that the number of components of B_1 is $s \geq 1$. Write $B_1 = \sum_{k=1}^{s} b_k B_{1,k}$, where $b_k \in (0, 1)$ and the $B_{1,k}$ are distinct prime divisors. Set $b := \max\{b_1, \dots, b_s\}$, let $D_1 := \sum_{k : b_k = b} B_{1,k}$, and denote by $B_{i,k}$ (resp. D_i) the strict transforms of $B_{1,k}$ (resp. D_1) on X_i for any i, k.

Claim 6.4. *After finitely many flips, no flipping curve and no flipped curve is contained in the strict transforms D_i of D_1.*

Grant the above claim for a moment and relabel the given sequence of flips so that no flipping curve is contained in the strict transforms of D_1. Fix $i \geq 1$. Then $D_i \cdot C \geq 0$ for all flipping curves $C \subseteq X_i$, and thus the $(K_{X_i} + B_i + M_i)$-flip over Z is also a $\big(K_{X_i} + (B_i - bD_i) + M_i\big)$-flip over Z. Consequently, the given sequence of $(K_{X_i} + B_i + M_i)$-flips over Z is also a sequence of $\big(K_{X_i} + (B_i - bD_i) + M_i\big)$-flips over Z, where $B_i - bD_i = \sum_{k : b_k < b} b_k B_{i,k}$, and hence it terminates by induction. It remains to prove Claim 6.4.

Proof of Claim 6.4. We first show that eventually no flipped curve is contained in the strict transforms D_i of D_1. Note that whenever a flipped curve $C_{i+1} \subseteq X_{i+1}$ is contained in $\operatorname{Supp} D_{i+1}$, then it holds that

$$d(X_i, B_i + M_i) > d(X_{i+1}, B_{i+1} + M_{i+1}).$$

Indeed, using notation and arguing as in the second paragraph of the proof, we have

$$\begin{aligned} a(E_{i+1}, X_i, B_i + M_i) &< a(E_{i+1}, X_{i+1}, B_{i+1} + M_{i+1}) \\ &= 1 - \sum n_k b_k - \sum m_j \mu_j \end{aligned}$$

for some non-negative integers $n_k = \operatorname{mult}_{C_{i+1}} B_{i+1,k}$ and m_j such that

$$\xi := \sum_k n_k b_k + \sum_j m_j \mu_j \geq \sum_{k: b_k = b} n_k b_k \geq \sum_{k: b_k = b} b \geq b,$$

which yields the assertion. Since $d(X_1, B_1 + M_1) < +\infty$ and since the difficulty takes values in $\mathbb{N}$, this situation can occur only finitely many times in the given sequence of flips over Z. In other words, after finitely many flips, no flipped curve is contained in the strict transforms D_i of D_1; by relabelling the given sequence of flips, we may assume that this holds for all $i \geq 1$.

We prove now that eventually no flipping curve is contained in the strict transforms D_i of D_1. To this end, fix $i \geq 1$ and consider the corresponding step

$$\begin{array}{ccc} (X_i, B_i + M_i) & \overset{\pi_i}{\dashrightarrow} & (X_{i+1}, B_{i+1} + M_{i+1}) \\ & {\scriptstyle \theta_i} \searrow \quad \swarrow {\scriptstyle \theta_i^+} & \\ & Z_i & \end{array}$$

of the given sequence of flips over Z. We will construct below the following diagram:

$$\begin{array}{ccccc} D_i^\nu & & \xrightarrow{\; g_i = \psi_{i+1}^{-1} \circ \varphi_i \;} & & D_{i+1}^\nu \\ \downarrow {\scriptstyle \nu_i} & \searrow {\scriptstyle \varphi_i} & & \swarrow {\scriptstyle \psi_{i+1}} & \downarrow {\scriptstyle \nu_{i+1}} \\ D_i & & \left(\theta_i(D_i)\right)^\nu = \left(\theta_i^+(D_{i+1})\right)^\nu & & D_{i+1} \\ & {\scriptstyle \theta_i|_{D_i}} \searrow & \downarrow & \swarrow {\scriptstyle \theta_i^+|_{D_{i+1}}} & \\ & & \theta_i(D_i) = \theta_i^+(D_{i+1}) & & \end{array}$$

Let $\nu_i \colon D_i^\nu \to D_i$ be the normalization of D_i and let $\varphi_i \colon D_i^\nu \to \left(\theta_i(D_i)\right)^\nu$ be the morphism between the normalizations induced by $\nu_i \circ \theta_i|_{D_i}$. Since D_{i+1} contains no flipped curves, the restriction $\theta_i^+|_{D_{i+1}} \colon D_{i+1} \to \theta_i^+(D_{i+1})$ of the flipped contraction θ_i^+ to D_{i+1} is an isomorphism away from finitely many points, so it is a finite birational morphism, and thus $\nu_{i+1} \circ \theta_i^+|_{D_{i+1}}$ induces an isomorphism $\psi_{i+1} \colon D_{i+1}^\nu \to \left(\theta_i^+(D_{i+1})\right)^\nu$ between the normalizations by Lemma 1.16. Observe now that $\theta_i(D_i) = \theta_i^+(D_{i+1})$, since π_i is an isomorphism in codimension one. Hence, we obtain a projective birational morphism $g_i := \psi_{i+1}^{-1} \circ \varphi_i \colon D_i^\nu \to D_{i+1}^\nu$, which, by construction, can only contract curves $\gamma \subseteq D_i^\nu$ whose image $\nu_i(\gamma) \subseteq D_i$ is a flipping curve contained in $\operatorname{Supp} D_i$.

Consider the minimal resolution $q_1 \colon D_1^\mu \to D_1^\nu$ of D_1^ν and for each $i \geq 1$ the induced map $\zeta_i := g_{i-1} \circ \cdots \circ g_1 \circ q_1 \colon D_1^\mu \to D_i^\nu$, which is in particular a resolution of D_i^ν. By [AHK07, Lemma 1.6(2)] we obtain a non-increasing sequence $\{\rho_i\}_{i=1}^{+\infty}$ of positive integers, where

$$\rho_i := \rho\big(D_1^\mu / Z\big) - \#\{\text{exceptional prime divisors of } \zeta_i\}, \ i \geq 1.$$

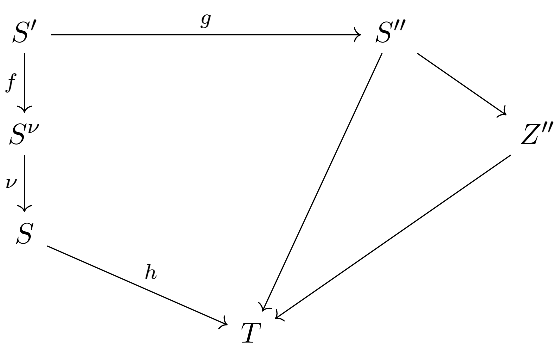
Since $\rho_1 < +\infty$, the sequence $\{\rho_i\}_{i=1}^{+\infty}$ must eventually stabilize. By construction of the maps g_i, this implies that eventually D_i contains no flipping curves. In other words, after finitely many flips no flipping curve is contained in the strict transforms of D_1, as claimed. □

This completes the proof. □

The author and Guodu Chen would like to thank Jingjun Han for showing them the following result. We provide a detailed proof for the benefit of the reader. Note also that we use crucially [Mat02, Theorem 1-4-8] together with [KMM87, Lemma 4-2-2].

Lemma 6.5. *Let X be a normal quasi-projective variety of dimension 3 and let S be a prime divisor on X such that $K_X + S$ is $\mathbb{Q}$-Cartier. If there exists a fibration $h \colon S \to T$ to a variety T with $\dim T \leq 1$, then there exists a curve $C \subseteq S$ which is contracted by h and such that $(K_X + S) \cdot C \geq -3$.*

Proof. Let $\nu \colon S^\nu \to S$ be the normalization of S and let $f \colon S' \to S^\nu$ be the minimal resolution of S^ν. We may run a $K_{S'}$-MMP over T which terminates either with a good minimal model S'' of S' over T and associated Iitaka fibration $S'' \to Z''$ over T or with a Mori fiber space $S'' \to Z''$ over T; we denote by $g \colon S' \to S''$ the induced morphism over T. Note that in the first case S'' is covered by curves C'' which are contracted over T and satisfy $K_{S''} \cdot C'' = 0$, while in the second case S'' is covered by curves C'' which are contracted over T and satisfy $-3 \leq K_{S''} \cdot C'' < 0$. Overall, we obtain the following diagram:



Furthermore, by adjunction there exists an effective $\mathbb{Q}$-divisor $\operatorname{Diff}_{S^\nu}(0)$ on S^ν such that $\nu^*(K_X + S) \sim_{\mathbb{Q}} K_{S^\nu} + \operatorname{Diff}_{S^\nu}(0)$ (see [Kol13, Section 4.1]). Consider the divisor $f^*\big(K_{S^\nu} + \operatorname{Diff}_{S^\nu}(0)\big)$, where the *pullback* is taken *in the sense of Mumford* (see [Mat02, Remark 4-6-3(i)]). Since this pullback is linear and respects effectivity (see [Mum61, Section II(b)]), we have $f^*\big(K_{S^\nu} + \operatorname{Diff}_{S^\nu}(0)\big) \geq f^*(K_{S^\nu})$. Moreover, by [Mat02, Theorem 4-6-2] we deduce that $f^*(K_{S^\nu}) \geq K_{S'}$. Therefore, the divisor $E := f^*\big(K_{S^\nu} + \operatorname{Diff}_{S^\nu}(0)\big) - K_{S'}$ is effective.

Now, pick a point $s' \in S' \setminus \big(\operatorname{Supp} E \cup \operatorname{Exc}(\nu\circ f) \cup \operatorname{Exc}(g)\big)$ and note that S, S' and S'' are isomorphic to each other near s'. By the above we may find a curve $C'' \subseteq S''$ passing through $s'' := g(s')$ which is contracted over T and satisfies $K_{S''} \cdot C'' \geq -3$. Therefore, there exists a curve $C' \subseteq S'$ passing through s' such that $g(C') = C''$ and whose image $C := (\nu \circ f)(C')$ is a curve on S which is contracted by h, since $s' \notin \operatorname{Exc}(\nu \circ f)$ and C'' is contracted over T. By construction and by the projection formula we obtain

$$\begin{aligned}(K_X + S) \cdot C &= (K_X + S) \cdot (\nu \circ f)_* C' = (\nu \circ f)^*(K_X + S) \cdot C' \\ &= (K_{S'} + E) \cdot C' \geq K_{S'} \cdot C'.\end{aligned}$$

Additionally, by [Mat02, Definition-Proposition 1-1-1(v) and Theorem 1-1-6] we may write $K_{S'} \sim g^* K_{S''} + F$, where F is an effective g-exceptional divisor. Note that $C' \nsubseteq \operatorname{Supp} F$, since $s' \notin \operatorname{Exc}(g)$. Thus, by construction and by the projection formula we obtain

$$K_{S'} \cdot C' = (g^* K_{S''} + F) \cdot C' \geq g^* K_{S''} \cdot C' = K_{S''} \cdot C'' \geq -3.$$

In conclusion, the curve C has the desired properties. □

Proposition 6.6. *Let N be a non-negative integer and let ε be a positive real number. Then there exists a positive integer I depending only on N, ε and satisfying the following.*

Assume that $(X, B + M)$ is a 3-dimensional $\mathbb{Q}$-factorial klt g-pair with data $X' \to X \to Z$ and M' such that

(i) *there are exactly N exceptional divisorial valuations E over X with discrepancy $a(E, X, B + M) \leq 0$, and*

(ii) *for any exceptional divisorial valuation E over X with $a(E, X, B + M) > 0$ it holds that $a(E, X, B + M) \geq \varepsilon$.*

Then ID is Cartier for any Weil divisor D on X.

Proof. We will prove the statement by induction on N.

Assume first that $N = 0$ and let $(X, B + M)$ be a $\mathbb{Q}$-factorial terminal g-pair such that $a(E, X, B + M) \geq \varepsilon$ for any exceptional divisorial valuation E over X. Then $(X, 0)$ is terminal. Let p be the *index* of X, i.e., p is the smallest positive integer such that pK_X is Cartier, and let D be a Weil divisor on X. By [Kaw88, Corollary 5.2], pD is Cartier. Since there is nothing to prove if $p = 1$, we assume that $p > 1$. By [Sho92, Appendix by Y. Kawamata, Theorem] we may find an exceptional divisorial valuation E over X such that $a(E, X, 0) = \frac{1}{p}$, and by assumption we infer that $p \leq \frac{1}{\varepsilon}$. We conclude that $I := \lfloor \frac{1}{\varepsilon} \rfloor !$ has the required property.

Assume now that the statement holds for all integers k with $0 \leq k \leq N - 1$ and let $(X, B + M)$ be a $\mathbb{Q}$-factorial klt g-pair such that there are exactly $N \geq 1$ exceptional divisorial valuations E over X with $a(E, X, B + M) \leq 0$, and for any exceptional divisorial valuation E over X with $a(E, X, B + M) > 0$ we actually have $a(E, X, B + M) \geq \varepsilon$. Let D be a Weil divisor on X. To prove the statement, we distinguish two cases.

Case I: There exists an exceptional divisorial valuation E over X with discrepancy $a := a(E, X, B + M) \in (-1, 0)$.

By [BZ16, Lemmas 4.5 and 4.6] there exist a $\mathbb{Q}$-factorial klt g-pair $(Y, \Delta + \Xi)$ with data $X' \xrightarrow{g} Y \to Z$ and M' and an extremal contraction $h\colon Y \to X$ with exceptional prime divisor E such that

$$K_Y + \Delta + \Xi \sim_{\mathbb{R}} h^*(K_X + B + M). \tag{6.1}$$

Note that $\Delta = B_Y - aE$, where B_Y is the strict transform of B on Y. Additionally, possibly replacing X' with a higher model, we may assume that $f = h \circ g$ and that g is a log resolution of (Y, Δ). Also, we denote by E' the strict transform of E on X' and we note that E' is an f-exceptional prime divisor. Next, we may write

$$h^*D \sim_{\mathbb{Q}} D_Y + qE \;\text{ for some } q \in \mathbb{Q}_{\geq 0}, \tag{6.2}$$

where D_Y is the strict transform of D on Y, and

$$h^*M \sim_{\mathbb{R}} \Xi + rE \;\text{ for some } r \in \mathbb{R}_{\geq 0}. \tag{6.3}$$

Indeed, by Remark 2.7 and since $f = h \circ g$, we know that $f^*M \sim_{\mathbb{R}} M' + E_f$, where E_f is an effective f-exceptional $\mathbb{R}$-divisor which can be expressed as $E_f = \sum q_k E'_k + rE'$, where q_k, $r \in \mathbb{R}_{\geq 0}$ and the E'_k are g-exceptional prime divisors, so we obtain (6.3) simply pushing this relation down to Y by g. Now, by applying the inductive hypothesis to the g-pair $(Y, \Delta + \Xi)$, we deduce that there exists a positive integer I'_1 depending only on N, ε and such that $I'_1 D_Y$ and $I'_1 E$ are Cartier.

Claim 6.7. *There exists a positive integer I_1 depending on N, ε and such that $I_1 \in I'_1\mathbb{Z}$ and $qI_1 \in I'_1\mathbb{Z}$.*

Grant this for the time being. Then $h^*(I_1 D) \sim_{\mathbb{Q}} I_1 D_Y + (qI_1)E$ is Cartier, and thus $I_1 D$ itself is Cartier by [KM98, Theorem 3.25(4)]. Hence, I_1 has the required property. It remains to prove Claim 6.7 in order to complete the proof in this case.

Proof of Claim 6.7. First, we show that $1 + a \geq \varepsilon$. Indeed, we may write

$$K_{X'} + B' + M' \sim_{\mathbb{R}} f^*(K_X + B + M)$$

for some $\mathbb{R}$-divisor B' on X'. Note that $\operatorname{mult}_{E'} B' = -a$, where E' is the strict transform of the h-exceptional prime divisor E on X'. If $\phi\colon W \to X'$ is the blow-up of X' along a general curve γ on E' and if E_W is the irreducible component of the exceptional divisor of ϕ which dominates γ, then by [KM98, Lemma 2.29] and by construction we obtain

$$a(E_W, X, B + M) = a(E_W, X', B' + M') = a(E_W, X', B') = 1 - (-a) = 1 + a.$$

Since $a(E_W, X, B + M) = 1 + a > 0$, we infer that $1 + a \geq \varepsilon$.

Furthermore, by Lemma 6.5 there exists a curve $C \subseteq E$ which is contracted by h and such that $(K_Y + E) \cdot C \geq -3$. Note also that $E \cdot C < 0$, see the proof of [KM98, Proposition 2.5]. Moreover, (6.2) and (6.3) yield

$$q = -\frac{D_Y \cdot C}{E \cdot C} \quad \text{and} \quad \Xi \cdot C = -rE \cdot C \geq 0.$$

Recall now that $K_Y + B_Y - aE + \Xi \sim_{\mathbb{R}} h^*(K_X + B + M)$, see (6.1). Since $C \not\subseteq \operatorname{Supp} B_Y$, we have $(K_Y + B_Y + E + \Xi) \cdot C \geq -3$, and since $1 + a \geq \varepsilon > 0$, we obtain

$$-E \cdot C = \frac{-(K_Y + B_Y + E + \Xi) \cdot C}{1 + a} \leq \frac{3}{\varepsilon}.$$

Since

$$q = \frac{D_Y \cdot C}{-E \cdot C} = \frac{I_1'(D_Y \cdot C)}{-I_1'(E \cdot C)},$$

where $I_1'(D_Y \cdot C)$ and $-I_1'(E \cdot C)$ are integers, and since $\varepsilon \leq 1 + a < 1$, it follows readily that the integer $I_1 := \lfloor \frac{3I_1'}{\varepsilon} \rfloor !$ satisfies the required properties. □

Case II: Each one of the N exceptional divisorial valuations $E_1, \dots, E_N$ over X (with non-positive discrepancy) has discrepancy $a(E_j, X, B+M) = 0$, where $1 \leq j \leq N$.

Recall that for any exceptional divisorial valuation E over X with positive discrepancy with respect to $(X, B+M)$ it holds that $a(E, X, B+M) \geq \varepsilon$. Consider an effective ample over Z $\mathbb{R}$-divisor H on X with sufficiently small coefficients which contains the center on X of every valuation E_j so that $a\big(E_j, X, (B+H)+M\big) < 0$ for every $1 \leq j \leq N$ and any other exceptional divisorial valuation E over X has $a\big(E, X, (B+H)+M\big) \geq \frac{\varepsilon}{2}$, see Lemma 2.4. Moreover, by [BZ16, Remark 4.2(2)] the g-pair $\big(X, (B+H)+M\big)$ is klt. We may now repeat verbatim the proof from Case I, working with the g-pair $\big(X, (B+H)+M\big)$ instead, and derive thus an integer I_2 with the desired properties.

Finally, note that the integer I_2 obtained in Case II is possibly different from the integer I_1 obtained in Case I. Then the integer $I := I_1 \cdot I_2$ satisfies the conclusion of the statement. This concludes the proof. □

Proposition 6.8. *Let N be a non-negative integer, let ε be a positive real number and let Γ be a finite set of non-negative real numbers. Then there exists a finite set $\mathcal{D} \subseteq [-1, 0]$ depending only on N, ε, Γ and satisfying the following.*

Assume that $(X, B+M)$ is a 3-dimensional NQC $\mathbb{Q}$-factorial klt g-pair with data $X' \to X \to Z$ and $M' = \sum \mu_j M_j'$ such that

(i) *$B \in \Gamma$,*

(ii) *$\mu_j \in \Gamma$ and M_j' is a Cartier divisor which is nef over Z for any j,*

(iii) *there are exactly N exceptional divisorial valuations E over X with discrepancy $a(E, X, B+M) \leq 0$, and*

(iv) *for any exceptional divisorial valuation E over X with $a(E, X, B+M) > 0$ it holds that $a(E, X, B+M) \geq \varepsilon$.*

Then for any divisorial valuation F over X with $a(F, X, B+M) \leq 0$ it holds that $a(F, X, B+M) \in \mathcal{D}$.

Proof. Let $(X, B+M)$ be a g-pair satisfying all the assumptions of the proposition. By Proposition 6.6 there exists a positive integer I_0 depending only on N, ε such that $I_0 D$ is Cartier for any Weil divisor D on X. Furthermore, by [Che20, Theorem 1.4] there exist a finite set Γ_1 of positive real numbers and a finite set Γ_2 of non-negative rational numbers which depend only on Γ and such that

$$K_X + B + M = \sum \alpha_i (K_X + B^i + M^i),$$

for some $\alpha_i \in \Gamma_1$ with $\sum \alpha_i = 1$, some $B^i \in \Gamma_2$ and some $(M^i)' = \sum_j \mu_{ij} M_j'$ with $M^i = f_*(M^i)'$ and $\mu_{ij} \in \Gamma_2$ for any i, j. Therefore, we may find a positive integer I

such that $I\Gamma_2 \subseteq I_0\mathbb{Z}$; in particular, $I(K_X + B^i + M^i)$ is Cartier for any i. Since for any divisorial valuation F over X we now have

$$a(F, X, B + M) = \sum \alpha_i \, a(F, X, B^i + M^i),$$

we conclude that there exists a subset $\mathcal{D}$ of the finite set

$$\left\{ -\frac{1}{I} \sum \alpha_i n_i \;\middle|\; \alpha_i \in \Gamma_1,\ n_i \in \mathbb{Z}_{\geq 0} \right\} \cap [-1, 0]$$

with the desired properties. □

Theorem 6.9. *Let $(X_1/Z, B_1 + M_1)$ be a 3-dimensional NQC $\mathbb{Q}$-factorial klt g-pair. Then any sequence of flips over Z starting from $(X_1, B_1 + M_1)$ terminates.*

Proof. Let

$$\begin{array}{ccccccc} (X_1, B_1 + M_1) & \overset{\pi_1}{\dashrightarrow} & (X_2, B_2 + M_2) & \overset{\pi_2}{\dashrightarrow} & (X_3, B_3 + M_3) & \overset{\pi_3}{\dashrightarrow} & \cdots \\ & \searrow \quad \swarrow & & \searrow \quad \swarrow & & & \\ & Z_1 & & Z_2 & & & \end{array}$$

be a sequence of flips over Z starting from $(X_1, B_1 + M_1)$. For every $i \geq 1$ consider a $\mathbb{Q}$-factorial terminalization $h_i' \colon (Y_i', \Delta_i' + \Xi_i') \to (X_i, B_i + M_i)$ of $(X_i, B_i + M_i)$, see Lemma 2.24(i). To show that this sequence of flips terminates we proceed in three steps.

Step 1: By Proposition 2.15 there exists a positive real number ε_0 such that there are $\nu_0 \in \mathbb{N}$ exceptional divisorial valuations E over X_1 with $a(E, X_1, B_1 + M_1) < \varepsilon_0$; in particular, N_0 of them, where $0 \leq N_0 \leq \nu_0$, have discrepancy $a(E, X_1, B_1 + M_1) \leq 0$. It follows from Lemma 2.45 that for every $i \geq 1$ there are at most ν_0 exceptional divisorial valuations E over X_i with $a(E, X_i, B_i + M_i) < \varepsilon_0$, and at most N_0 of them with $a(E, X_i, B_i + M_i) \leq 0$. In particular, there exist a positive integer k and a non-negative integer $N \leq N_0$ such that for every $i \geq k$ there are exactly N exceptional divisorial valuations $E_1, \dots, E_N$ over X_i with discrepancy $a(E_j, X_i, B_i + M_i) \leq 0$ for every $1 \leq j \leq N$, and thus at most $\nu_0 - N$ exceptional divisorial valuations E over X_i with $a(E, X_i, B_i + M_i) \in (0, \varepsilon_0)$; note that k is chosen as the smallest positive integer with this property. In particular, there are exactly ν, where $0 \leq \nu \leq \nu_0 - N$, exceptional divisorial valuations $F_1, \dots, F_\nu$ over X_k with $a(F_s, X_k, B_k + M_k) \in (0, \varepsilon_0)$ for every $1 \leq s \leq \nu$. Set

$$\varepsilon := \min \big\{ a(F_1, X_k, B_k + M_k), \dots, a(F_\nu, X_k, B_k + M_k), \varepsilon_0 \big\}$$

and note that $\varepsilon = \varepsilon_0$ if and only if $\nu = 0$. By Lemma 2.45 we infer that for every $i \geq k$ and for every exceptional divisorial valuation E over X_i with $a(E, X_i, B_i + M_i) > 0$ it actually holds that $a(E, X_i, B_i + M_i) \geq \varepsilon$. By relabelling the indices of the given sequence of flips, we may assume that $k = 1$.

Step 2: By Step 1 and by construction of a $\mathbb{Q}$-factorial terminalization, each $h_i' \colon Y_i' \to X_i$ extracts the exceptional divisorial valuations $E_1, \dots, E_N$ and each Δ_i' is given by

$$\Delta_i' = (h_i')_*^{-1} B_i + \sum_{j=1}^{N} \big(- a(E_j, X_i, B_i + M_i) \big) E_j. \tag{6.4}$$

Hence, each Y'_{i+1} is isomorphic in codimension one to Y'_i. Additionally, we claim that after finitely many flips $\pi_i \colon X_i \dashrightarrow X_{i+1}$, each Δ'_{i+1} is the strict transform of Δ'_i. Indeed, by Step 1 we may apply Proposition 6.8 and deduce that there exists a finite set $\mathcal{V} \subseteq [-1, 0]$ such that for every $i \geq 1$ and for every $1 \leq j \leq N$ it holds that $a(E_j, X_i, B_i + M_i) \in \mathcal{V}$. Thus, by Lemma 2.45, for every $1 \leq j \leq N$ we obtain a non-decreasing sequence $\{a(E_j, X_i, B_i + M_i)\}_{i=1}^{+\infty}$ of elements of the finite set $\mathcal{V}$, which must therefore eventually stabilize. Hence, there exists a positive integer r such that $a(E_j, X_r, B_r + M_r) = a(E_j, X_i, B_i + M_i)$ for every $i \geq r$ and for every $1 \leq j \leq N$. Consequently, as asserted, Δ'_{i+1} is the strict transform of Δ'_i for every $i \geq r$, see (6.4). By relabelling the indices of the given sequence of flips, we may assume that $r = 1$.

Step 3: By Step 2 we may apply Lemma 2.54 and obtain thus a diagram

$$\begin{array}{ccccccc} (Y_1, \Delta_1 + \Xi_1) & \overset{\rho_1}{\dashrightarrow} & (Y_2, \Delta_2 + \Xi_2) & \overset{\rho_2}{\dashrightarrow} & (Y_3, \Delta_3 + \Xi_3) & \overset{\rho_3}{\dashrightarrow} & \cdots \\ \Big\downarrow{\scriptstyle h_1} & & \Big\downarrow{\scriptstyle h_2} & & \Big\downarrow{\scriptstyle h_3} & & \\ (X_1, B_1 + M_1) & \overset{\pi_1}{\dashrightarrow} & (X_2, B_2 + M_2) & \overset{\pi_2}{\dashrightarrow} & (X_3, B_3 + M_3) & \overset{\pi_3}{\dashrightarrow} & \cdots \\ & \searrow \quad \swarrow & & \searrow \quad \swarrow & & & \\ & Z_1 & & Z_2 & & & \end{array}$$

where the top row yields a sequence of flips over Z starting from the NQC $\mathbb{Q}$-factorial terminal g-pair $(Y_1, \Delta_1 + \Xi_1)$. By Theorem 6.3 the sequence $Y_i \dashrightarrow Y_{i+1}$ of flips over Z terminates, and thus the sequence $X_i \dashrightarrow X_{i+1}$ of flips over Z terminates. This concludes the proof. □

Theorem 6.10. *Let $(X/Z, B + M)$ be a 3-dimensional NQC lc g-pair. Then any sequence of flips over Z starting from $(X, B + M)$ terminates.*

Proof. Follows immediately from Lemma 5.11 and Theorem 6.9. □

6.3 On the Termination of Flips for Pseudo-Effective NQC Log Canonical Generalized Pairs

In this section we prove two of the main results of the thesis. First, we obtain an analog of Birkar's inductive termination result [Bir07, Theorem 1.3] in the setting of g-pairs (see Theorem 6.14). We note that Birkar's theorem regards effective pairs, whereas our result concerns pseudo-effective g-pairs admitting NQC weak Zariski decompositions, and we stress that the existence of NQC weak Zariski decompositions is a weaker condition than the numerical non-vanishing (see also [Has18b, Theorem 1.4]). Second, we verify the termination of flips conjecture for pseudo-effective NQC lc g-pairs of dimension 4 (see Theorem 6.15). Note that the proof of the former occupies almost the whole section, while the proof of the latter is given at the very end and it is actually an immediate corollary of Theorems 6.10 and 6.14.

The phrase *special termination for NQC lc g-pairs of dimension n* encountered in the sequel means that special termination (as explained in Section 5.1) applies to any sequence of flips starting from any NQC lc g-pair of dimension n. Below we will also use frequently the definition of a flip with respect to an arbitrary $\mathbb{R}$-Cartier divisor, see Definition 2.47. Last but not least, before we turn our attention to the proofs of the aforementioned results, we make the following comments about Theorems 6.14 and 6.15 and we also outline our strategy for the proof of the former.

Note that the termination of flips for pseudo-effective lc pairs of dimension 4 with real coefficients is a special case of Theorem 6.15. The rational coefficients case of the aforementioned statement was announced previously by Moraga [Mor18] in the second version of his preprint, following a radically different approach from ours. After we posted our joint preprint [CT20] with Guodu Chen on the *arXiv*, Moraga informed us that in the third version of his preprint [Mor18] he has also obtained Theorem 6.15 independently, using methods similar to ours. Besides, it is worthwhile to mention that, presently, there is no proof of the aforementioned statement that does not utilize the theory of g-pairs.

Note that Theorem 6.14 extends [HM18, Theorem 1] to the setting of $\mathbb{R}$-divisors, and, combined with Theorem 3.17, yields a different proof of [LT19, Corollary G], which does not depend on [HM18]. Moreover, to establish Theorem 6.14, we proceed as follows. First, we prove an analog of [Bir07, Lemma 3.2] in the context of g-pairs (see Theorem 6.12). We emphasize that the ascending chain condition for lc thresholds [BZ16, Theorem 1.5], which plays a fundamental role in the proofs of Theorem 6.12, [Mor18, Theorem 1] and [HM18, Theorem 1], is invoked here without passing to some open subset of the varieties involved (as is the case in [Bir07]). Second, we obtain Theorem 6.14 essentially as an easy consequence of Theorems 5.12 and 6.12, using the intermediate Corollary 6.13 as a bridge in order to highlight the significance of special termination in our proof.

After this rather lengthy introduction, we now concentrate on the proof of Theorem 6.14. We begin with a somewhat technical result that plays a fundamental role in the proof of Theorem 6.12.

Lemma 6.11. *Let $(X, B+M)$ be an lc g-pair with data $X' \xrightarrow{f} X \to Z$ and M'. Let*

$$\begin{array}{ccc} (X, B+M) & \overset{\pi}{\dashrightarrow} & (X^+, B^+ + M^+) \\ & {\scriptstyle\theta}\searrow \quad \swarrow{\scriptstyle\theta^+} & \\ & W & \end{array}$$

be a flip over Z such that the flipping locus $\operatorname{Exc}(\theta)$ does not intersect the non-klt locus of $(X, B+M)$, that is, $\operatorname{Exc}(\theta) \cap \operatorname{Nklt}(X, B+M) = \emptyset$. Let $h\colon (Y, \Delta+\Xi) \to (X, B+M)$ be a dlt blow-up of $(X, B+M)$, where the g-pair $(Y, \Delta+\Xi)$ comes with data $X' \xrightarrow{g} Y \to Z$ and M'; we may assume that X' is a sufficiently high model so that $f = h \circ g$. Run a $(K_Y + \Delta + \Xi)$-MMP with scaling of an ample divisor over W and let

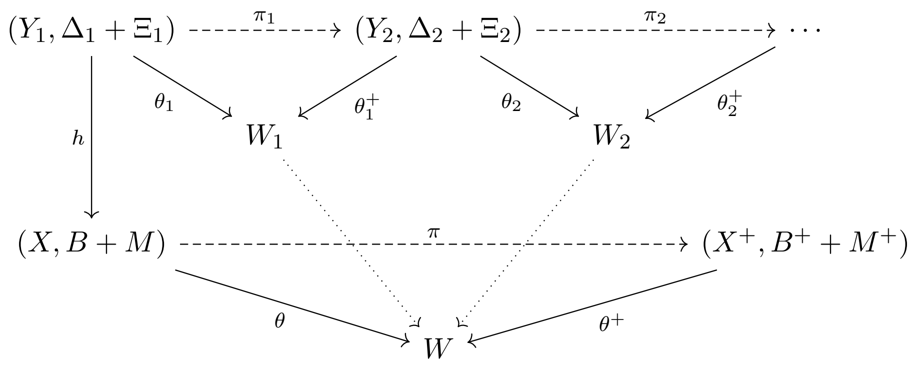
$$\begin{array}{ccc} (Y_i, \Delta_i + \Xi_i) & \overset{\pi_i}{\dashrightarrow} & (Y_{i+1}, \Delta_{i+1} + \Xi_{i+1}) \\ & {\scriptstyle\theta_i}\searrow \quad \swarrow{\scriptstyle\theta_i^+} & \\ & W_i & \end{array}$$

be its intermediate steps, where $(Y_1, \Delta_1 + \Xi_1) := (Y, \Delta + \Xi)$. Then:

(i) *This $(K_Y + \Delta + \Xi)$-MMP over W consists only of flips, at each step the flipping locus avoids the non-klt locus of $(Y_i, \Delta_i + \Xi_i)$, that is, $\operatorname{Exc}(\theta_i) \cap \operatorname{Supp}\lfloor \Delta_i \rfloor = \emptyset$ for every $i \geq 1$, and thus the sequence of $(K_Y + \Delta + \Xi)$-flips over W is also a sequence of $\big(K_Y + \Delta^{<1} + \Xi\big)$-flips over W, where $\big(Y, \Delta^{<1} + \Xi\big)$ is a $\mathbb{Q}$-factorial klt g-pair.*

(ii) *Assume now that there exist an $\mathbb{R}$-divisor P' on X' which is nef over Z and an effective $\mathbb{R}$-divisor N on X such that $P+N$ is $\mathbb{R}$-Cartier, $-(P+N)$ is ample over W and $h^*(P+N) = Q + \Lambda + F$, where $P := f_*P'$, $Q := g_*P'$, $\Lambda \geq 0$ and* $\operatorname{Supp} F \subseteq \operatorname{Supp}\lfloor\Delta\rfloor$*. Then the sequence of $(K_Y + \Delta + \Xi)$-flips over W is also a sequence of $(Q + \Lambda)$-flips over W.*

Proof. The following diagram describes the $(K_Y + \Delta + \Xi)$-MMP with scaling over W:



and we remark that $\pi_i = \theta_i$, $Y_{i+1} = W_i$ and $\theta_i^+ = \mathrm{Id}_{W_i}$ whenever θ_i is a divisorial contraction.

(i) First, we show that

$$\operatorname{Exc}(\theta_i) \cap \operatorname{Supp}\lfloor\Delta_i\rfloor = \emptyset \quad \text{for every } i \geq 1. \tag{6.5}$$

To this end, set $U := X \setminus \operatorname{Exc}(\theta)$, $V_1 := h^{-1}(U)$, $T = \theta(U)$, and note that T is an open subset of W. Since $\pi|_U \colon U \to \pi(U)$ is an isomorphism, $(K_X + B + M)|_U$ is trivially semi-ample over T, and thus $(K_{Y_1} + \Delta_1 + \Xi_1)|_{V_1} = (h|_{V_1})^*\big((K_X + B + M)|_U\big)$ is also semi-ample over T. Therefore, $V_1 \cap \operatorname{Exc}(\theta_1) = \emptyset$, so V_1 is contained in the locus where π_1 is an isomorphism, and we may thus consider its isomorphic image $V_2 := \pi_1(V_1)$ and infer that $(K_{Y_2} + \Delta_2 + \Xi_2)|_{V_2}$ is semi-ample over T, hence $V_2 \cap \operatorname{Exc}(\theta_2) = \emptyset$. By proceeding analogously, if we set $V_i := \pi_{i-1}(V_{i-1})$, then we obtain that

$$V_i \cap \operatorname{Exc}(\theta_i) = \emptyset \quad \text{for all } i \geq 1. \tag{6.6}$$

We can now readily derive (6.5) from (6.6) as follows. We first note that

$$V_1 \supseteq \operatorname{Nklt}(Y_1, \Delta_1 + \Xi_1) = \operatorname{Supp}\lfloor\Delta_1\rfloor,$$

since $\operatorname{Nklt}(X, B+M) \subseteq U$ by assumption, and subsequently, by construction and by the above relation, we deduce inductively that

$$V_i \supseteq \operatorname{Nklt}(Y_i, \Delta_i + \Xi_i) = \operatorname{Supp}\lfloor\Delta_i\rfloor \quad \text{for all } i \geq 2.$$

Consequently, we obtain (6.5) by combining the above two relations with (6.6).

Next, to prove that the $(K_{Y_1} + \Delta_1 + \Xi_1)$-MMP with scaling over W consists only of flips, we argue by contradiction. If a divisorial contraction $\theta_i \colon Y_i \to W_i$ with exceptional prime divisor $E_i := \operatorname{Exc}(\theta_i)$ appears at the i-th step of this MMP (we may assume that i is the smallest such index), then by (6.5) we have $E_i \cap \operatorname{Supp}\lfloor\Delta_i\rfloor = \emptyset$, so the strict transform E_1 of E_i on Y_1 cannot be a component of $\lfloor\Delta_1\rfloor$. On the other hand, E_1 must be contracted over W, and since θ is a small map, E_1 is an h-exceptional prime divisor, and thus a component of $\lfloor\Delta_1\rfloor$ by construction of a dlt blow-up, a contradiction.

Finally, if we denote by R_i the $(K_{Y_i}+\Delta_i+\Xi_i)$-negative extremal ray contracted at the i-th step of the given sequence of flips, then (6.5) implies

$$\big(K_{Y_i}+\Delta_i^{<1}+\Delta_i^{=1}+\Xi_i\big)\cdot R_i=\big(K_{Y_i}+\Delta_i^{<1}+\Xi_i\big)\cdot R_i,$$

whence the last assertion of (i).

(ii) For every $i\geq 1$ we denote by Q_i, Λ_i and F_i the strict transforms on Y_i of $Q_1:=Q$, $\Lambda_1:=\Lambda$ and $F_1:=F$, respectively. Since $-(P+N)$ is ample over W and since $\rho(X/W)=1$, we have $K_X+B+M\equiv_W \alpha(P+N)$ for some $\alpha>0$, and thus

$$K_Y+\Delta+\Xi=h^*(K_X+B+M)\equiv_W \alpha h^*(P+N)=\alpha(Q+\Lambda+F).$$

Therefore,

$$K_{Y_i}+\Delta_i+\Xi_i\equiv_W \alpha(Q_i+\Lambda_i+F_i)\quad\text{for every } i\geq 1. \tag{6.7}$$

Moreover, by assumption and by (i) we infer that

$$\operatorname{Supp} F_i\subseteq \operatorname{Supp}\lfloor\Delta_i\rfloor\quad\text{for every } i\geq 1. \tag{6.8}$$

Hence, if $C_i\subseteq Y_i$ is a flipping curve at the i-th step of the sequence of $(K_{Y_1}+\Delta_1+\Xi_1)$-flips over W, then by (6.5), (6.7) and (6.8) we obtain

$$(Q_i+\Lambda_i)\cdot C_i=(Q_i+\Lambda_i+F_i)\cdot C_i=\frac{1}{\alpha}(K_{Y_i}+\Delta_i+\Xi_i)\cdot C_i<0.$$

In other words, $(Q_i+\Lambda_i)\cdot R_i<0$ for every $i\geq 1$, as desired. □

Theorem 6.12. *Assume the existence of minimal models for smooth varieties of dimension $n-1$ and the special termination for NQC lc g-pairs of dimension n.*

Let

$$\begin{array}{ccccccccc}
(X_1,B_1+M_1) & & \overset{\pi_1}{\dashrightarrow} & & (X_2,B_2+M_2) & & \overset{\pi_2}{\dashrightarrow} & & (X_3,B_3+M_3) \overset{\pi_3}{\dashrightarrow}\cdots\\
& \searrow^{\theta_1} & & \swarrow_{\theta_1^+} & & \searrow^{\theta_2} & & \swarrow_{\theta_2^+} & \\
& & Z_1 & & & & Z_2 & &
\end{array}$$

be a sequence of flips over Z starting from an NQC lc g-pair (X_1,B_1+M_1) of dimension n, where each g-pair (X_i,B_i+M_i) in the above sequence of flips comes with data $X_1'\xrightarrow{f_i}X_i\to Z$ and M_1'. Assume that there exist an NQC divisor P_1' on X_1' and an effective $\mathbb{R}$-divisor N_1 on X_1 such that P_1+N_1 is $\mathbb{R}$-Cartier and $-(P_i+N_i)$ is ample over Z_i for every $i\geq 1$, where $P_i:=(f_i)_*P_1'$ and N_i is the strict transform of N_1 on X_i. Then the given sequence of flips over Z terminates.

Proof. Assume that the given sequence of flips over Z does not terminate. We derive a contradiction by violating the ascending chain condition for lc thresholds [BZ16, Theorem 1.5].

Step 1: Let $t_1\geq 0$ be the lc threshold of P_1+N_1 with respect to (X_1,B_1+M_1). For each $i\geq 1$, since $-(P_i+N_i)$ is ample over Z_i and since $\rho(X_i/Z_i)=1$, we have $K_{X_i}+B_i+M_i\equiv_{Z_i}\alpha_i(P_i+N_i)$ for some $\alpha_i>0$. Therefore, the given sequence of $(K_{X_1}+B_1+M_1)$-flips over Z is also a sequence of $\big(K_{X_1}+(B_1+t_1N_1)+(M_1+t_1P_1)\big)$-flips over Z. By the special termination there exists an integer $\ell\geq 1$ such that

$$\operatorname{Exc}(\theta_i)\cap \operatorname{Nklt}\big(X_i,(B_i+t_1N_i)+(M_i+t_1P_i)\big)=\emptyset\quad\text{for every } i\geq\ell. \tag{6.9}$$

By relabelling the given sequence of flips, we may assume that $\ell = 1$.

Step 2: Consider a dlt blow-up

$$h_1 \colon (Y_1, \Delta_1 + \Xi_1) \to \big(X_1, (B_1 + t_1 N_1) + (M_1 + t_1 P_1)\big)$$

of the NQC lc g-pair $\big(X_1, (B_1 + t_1 N_1) + (M_1 + t_1 P_1)\big)$ such that $f_1 = h_1 \circ g_1$, where the g-pair $(Y_1, \Delta_1 + \Xi_1)$ comes with data $X_1' \xrightarrow{g_1} Y_1 \to Z$ and $\Xi_1' := M_1' + t_1 P_1'$. Set $L_1 := (g_1)_* M_1'$ and $Q_1 := (g_1)_* P_1'$ and note that

$$\Xi_1 = (g_1)_* \Xi_1' = L_1 + t_1 Q_1, \; (h_1)_* L_1 = M_1 \; \text{ and } \; (h_1)_* Q_1 = P_1.$$

By Lemma 2.57 there exists some $(K_{Y_1} + \Delta_1 + \Xi_1)$-MMP with scaling of an ample divisor over Z_1 which terminates with a dlt blow-up

$$h_2 \colon (Y_2, \Delta_2 + \Xi_2) \to \big(X_2, (B_2 + t_1 N_2) + (M_2 + t_1 P_2)\big)$$

of the NQC lc g-pair $\big(X_2, (B_2 + t_1 N_2) + (M_2 + t_1 P_2)\big)$ such that $f_2 = h_2 \circ g_2$, where the g-pair $(Y_2, \Delta_2 + \Xi_2)$ comes with data $X_1' \xrightarrow{g_2} Y_2 \to Z$ and Ξ_1'. As above, set $L_2 := (g_2)_* M_1'$ and $Q_2 := (g_2)_* P_1'$ and note that

$$\Xi_2 = (g_2)_* \Xi_1' = L_2 + t_1 Q_2, \; (h_2)_* L_2 = M_2 \; \text{ and } \; (h_2)_* Q_2 = P_2.$$

Thus, we obtain the following diagram:

$$\begin{array}{ccccc}
 & & X_1' & & \\
 & \swarrow^{g_1} & & \searrow^{g_2} & \\
(Y_1, \Delta_1 + \Xi_1) & & \dashrightarrow & & (Y_2, \Delta_2 + \Xi_2) \\
\downarrow{h_1} & & & & \downarrow{h_2} \\
\big(X_1, (B_1 + t_1 N_1) + (M_1 + t_1 P_1)\big) & & \overset{\pi_1}{\dashrightarrow} & & \big(X_2, (B_2 + t_1 N_2) + (M_2 + t_1 P_2)\big) \\
 & \searrow_{\theta_1} & & \swarrow_{\theta_1^+} & \\
 & & Z_1 & &
\end{array}$$

By Step 1 and by Lemma 6.11(i) this $(K_{Y_1} + \Delta_1 + \Xi_1)$-MMP with scaling over Z_1 consists only of flips and it is also a sequence of $(K_{Y_1} + \Delta_1^{<1} + \Xi_1)$-flips over Z_1, where the g-pair $(Y_1, \Delta_1^{<1} + \Xi_1)$ is $\mathbb{Q}$-factorial klt. Additionally, we will demonstrate below that Lemma 6.11(ii) may also be applied in this setting.

To this end, we first write trivially $\Delta_1 = \Delta_1^{<1} + \Delta_1^{=1}$. More precisely, if E denotes the sum of the h_1-exceptional prime divisors, then by construction of a dlt blow-up (see Lemma 2.24(iii)) we have

$$\Delta_1 = (h_1)_*^{-1} \big((B_1 + t_1 N_1)^{<1}\big) + (h_1)_*^{-1} \big((B_1 + t_1 N_1)^{=1}\big) + E. \tag{6.10}$$

We define now three effective $\mathbb{R}$-divisors Γ_1, Δ_1' and Δ_1'' on Y_1 such that $\operatorname{Supp} \Gamma_1 \subseteq \operatorname{Supp} \Delta_1^{=1}$ and

$$\Delta_1^{<1} = \Delta_1' + t_1 \Delta_1'' \quad \text{and} \quad (h_1)_*^{-1} N_1 = \Gamma_1 + \Delta_1'' \tag{6.11}$$

We write $B_1 = \sum \beta_k G_k$ and $N_1 = \sum \nu_k G_k$, where $\beta_k \in [0,1]$ and $\nu_k \in [0,+\infty)$, so that $B_1 + t_1 N_1 = \sum (\beta_k + t_1\nu_k) G_k$, where $\beta_k + t_1\nu_k \in [0,1]$ by construction. We set

$$\Delta_1' := \sum_{k:\beta_k+t_1\nu_k<1} \beta_k (h_1)_*^{-1} G_k \quad \text{so that} \quad \operatorname{Supp}\big((h_1)_*\Delta_1'\big) \subseteq \operatorname{Supp} B_1,$$

$$\Delta_1'' := \sum_{k:\beta_k+t_1\nu_k<1} \nu_k (h_1)_*^{-1} G_k \quad \text{so that} \quad \operatorname{Supp}\big((h_1)_*\Delta_1''\big) \subseteq \operatorname{Supp} N_1,$$

$$\Gamma_1 := \sum_{k:\beta_k+t_1\nu_k=1} \nu_k (h_1)_*^{-1} G_k \quad \text{so that} \quad \operatorname{Supp}\big((h_1)_*\Gamma_1\big) \subseteq \operatorname{Supp} N_1.$$

Using (6.10), it is easy to see that (6.11) holds, and we also observe that

$$\operatorname{Supp}\Gamma_1 \subseteq \operatorname{Supp}\Big((h_1)_*^{-1}\big((B_1+t_1N_1)^{=1}\big)\Big) \subseteq \operatorname{Supp}\Delta_1^{=1}.$$

Since $f_1 = h_1 \circ g_1$, $P_1 = (f_1)_* P_1'$ and $Q_1 = (g_1)_* P_1'$, by (6.10) and (6.11) we may write

$$(h_1)^*(P_1 + N_1) = Q_1 + (h_1)_*^{-1} N_1 + F_1 = Q_1 + \Delta_1'' + \big(\Gamma_1 + F_1\big),$$

where F_1 is an h_1-exceptional $\mathbb{R}$-divisor and $\operatorname{Supp}\big(\Gamma_1 + F_1\big) \subseteq \operatorname{Supp}\Delta_1^{=1}$ by construction. Therefore, we may apply Lemma 6.11(ii), as claimed. We infer that the sequence of $(K_{Y_1} + \Delta_1 + \Xi_1)$-flips over Z_1 is also a sequence of $(Q_1 + \Delta_1'')$-flips over Z_1.

Note that, due to (6.9), we may repeat the above procedure for every NQC lc g-pair $\big(X_i, (B_i + t_1 N_i) + (M_i + t_1 P_i)\big)$ of the sequence $X_i \dashrightarrow X_{i+1}$ of flips over Z in order to produce a diagram as above (case $i = 1$), starting every time (case $i \geq 2$) with the dlt blow-up $h_i\colon (Y_i, \Delta_i + \Xi_i) \to \big(X_i, (B_i + t_1N_i) + (M_i + t_1P_i)\big)$ that was obtained from the previous repetition of the procedure, where each g-pair $(Y_i, \Delta_i + \Xi_i)$ comes with data $X_1' \xrightarrow{g_i} Y_i \to Z$ and $\Xi_1' = M_1' + t_1 P_1'$. Specifically, each such repetition produces a sequence of $(K_{Y_i} + \Delta_i + \Xi_i)$-flips over Z_i, which is also a sequence of $(K_{Y_i} + \Delta_i^{<1} + \Xi_i)$-flips over Z_i starting from the NQC $\mathbb{Q}$-factorial klt g-pair $(Y_i, \Delta_i^{<1} + \Xi_i)$, as well as a sequence of $(Q_i + \Delta_i'')$-flips over Z_i, where (as in the case $i = 1$ we have) $\Delta_i^{<1} = \Delta_i' + t_1\Delta_i''$ and $\Xi_i = L_i + t_1 Q_i$ with $L_i = (g_i)_* M_1'$ and $Q_i = (g_i)_* P_1'$. Moreover, note that each repetition of the procedure in question is clearly compatible with the previous one.

Since we may consider every flip over Z_i as a flip over Z, we obtain a sequence of $\big(K_{Y_1} + \Delta_1^{<1} + \Xi_1 = K_{Y_1} + (\Delta_1' + t_1\Delta_1'') + (L_1 + t_1Q_1)\big)$-flips over Z starting from the NQC $\mathbb{Q}$-factorial klt g-pair $(Y_1, \Delta_1^{<1} + \Xi_1) = \big(Y_1, (\Delta_1' + t_1\Delta_1'') + (L_1 + t_1Q_1)\big)$, which is also a sequence of $(Q_1 + \Delta_1'')$-flips over Z. This sequence of flips does not terminate, since by construction it is also a sequence of $(K_{Y_1} + \Delta_1 + \Xi_1)$-flips over Z and if it terminated, then the given sequence of $(K_{X_1} + B_1 + M_1)$-flips over Z would also terminate, but this contradicts our initial assumption. Moreover, it follows from Lemma 2.14 that $(Y_1, \Delta_1' + L_1)$ is an NQC $\mathbb{Q}$-factorial klt g-pair.

Step 3: Let t_2 be the lc threshold of $Q_1 + \Delta_1''$ with respect to $(Y_1, \Delta_1' + L_1)$. By the previous paragraph we deduce that $t_2 > t_1$ and that the sequence of $(K_{Y_1} + (\Delta_1' + t_1\Delta_1'') + (L_1 + t_1Q_1))$-flips over Z is also a sequence of $(K_{Y_1} + (\Delta_1' + t_2\Delta_1'') + (L_1 + t_2Q_1))$-flips over Z with respect to the NQC $\mathbb{Q}$-factorial lc g-pair $\big(Y_1, (\Delta_1' + t_2\Delta_1'') + (L_1 + t_2Q_1)\big)$. We may now apply the special termination as in Step 1 and continue as in Step 2.

Step 4: By repeating Steps 2 and 3, we obtain a strictly increasing sequence $\{t_i\}_{i=1}^{\infty}$ of lc thresholds. However, this contradicts [BZ16, Theorem 1.5]. □

Corollary 6.13. *Assume the existence of minimal models for smooth varieties of dimension $n-1$ and the special termination for NQC lc g-pairs of dimension n.*

Let $(X_1, B_1 + M_1)$ be an NQC lc g-pair of dimension n with data $X_1' \xrightarrow{f_1} X_1 \to Z$ and M_1'. If $(X_1, B_1 + M_1)$ admits an NQC weak Zariski decomposition over Z, then any sequence of flips over Z starting from $(X_1, B_1 + M_1)$ terminates.

Proof. Let

$$\begin{array}{ccccccc} (X_1, B_1 + M_1) & \overset{\pi_1}{\dashrightarrow} & (X_2, B_2 + M_2) & \overset{\pi_2}{\dashrightarrow} & (X_3, B_3 + M_3) & \overset{\pi_3}{\dashrightarrow} & \cdots \\ & \searrow \quad \swarrow & & \searrow \quad \swarrow & & & \\ & Z_1 & & Z_2 & & & \end{array}$$

be a sequence of flips over Z starting from the g-pair $(X_1, B_1 + M_1)$. By Lemma 2.57 we obtain a diagram

$$\begin{array}{ccccccc} (Y_1, \Delta_1 + \Xi_1) & \overset{\rho_1}{\dashrightarrow} & (Y_2, \Delta_2 + \Xi_2) & \overset{\rho_2}{\dashrightarrow} & (Y_3, \Delta_3 + \Xi_3) & \overset{\rho_3}{\dashrightarrow} & \cdots \\ \downarrow{\scriptstyle h_1} & & \downarrow{\scriptstyle h_2} & & \downarrow{\scriptstyle h_3} & & \\ (X_1, B_1 + M_1) & \overset{\pi_1}{\dashrightarrow} & (X_2, B_2 + M_2) & \overset{\pi_2}{\dashrightarrow} & (X_3, B_3 + M_3) & \overset{\pi_3}{\dashrightarrow} & \cdots \\ & \searrow \quad \swarrow & & \searrow \quad \swarrow & & & \\ & Z_1 & & Z_2 & & & \end{array}$$

where each map $h_i \colon (Y_i, \Delta_i + \Xi_i) \to (X_i, B_i + M_i)$ is a dlt blow-up of the NQC lc g-pair $(X_i, B_i + M_i)$ and the top row yields a $(K_{Y_1} + \Delta_1 + \Xi_1)$-MMP over Z for the NQC $\mathbb{Q}$-factorial dlt g-pair $(Y_1, \Delta_1 + \Xi_1)$ with data $X_1' \xrightarrow{g_1} Y_1 \to Z$ and M_1'.

To prove the statement, it suffices to show that this $(K_{Y_1} + \Delta_1 + \Xi_1)$-MMP terminates. By relabelling, we may assume that it consists only of flips. Moreover, since $(X, B+M)$ admits an NQC weak Zariski decomposition over Z, by replacing X_1' with a higher model we may assume that $f_1 = h_1 \circ g_1$ and that $(g_1)^*(K_{Y_1} + \Delta_1 + \Xi_1) \equiv_Z P_1' + N_1'$, where P_1' is an NQC divisor on X_1' and N_1' is an effective $\mathbb{R}$-Cartier $\mathbb{R}$-divisor on X_1', see Remark 3.2. Set $P_1 := (g_1)_* P_1'$ and $N_1 := (g_1)_* N_1'$ and observe that $P_1 + N_1$ is $\mathbb{R}$-Cartier and satisfies $K_{Y_1} + \Delta_1 + \Xi_1 \equiv_Z P_1 + N_1$. We conclude by Theorem 6.12. □

Theorem 6.14. *Assume the termination of flips for NQC $\mathbb{Q}$-factorial klt g-pairs of dimension at most $n-1$.*

Let $(X/Z, B+M)$ be a pseudo-effective NQC lc g-pair of dimension n. If $(X, B+M)$ admits an NQC weak Zariski decomposition over Z, then any sequence of flips over Z starting from $(X, B+M)$ terminates.

Proof. Follows immediately from Theorem 5.12 and Corollary 6.13. □

Theorem 6.15. *Let $(X/Z, B+M)$ be a 4-dimensional NQC lc g-pair. If $K_X + B + M$ is pseudo-effective over Z, then any sequence of flips over Z starting from $(X, B+M)$ terminates.*

Proof. By Theorem 6.10 we know that any sequence of flips over Z starting from any NQC lc g-pair of dimension 3 terminates. Moreover, by [KMM87, Theorem 5-1-15], Corollary 3.10 and Theorem 3.17, we infer that the given g-pair $(X/Z, B+M)$ admits an NQC weak Zariski decomposition over Z. We conclude by Theorem 6.14. □

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