Finding Fair and Efficient Allocations

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Abstract

Abstract. We study the problem of *fair division*, where the goal is to allocate a set of *items* among a set of *agents* in a "fair" manner. In particular, we focus on settings in which the items to be divided are either *indivisible goods* or *divisible bads*. Despite their practical significance, both these settings have been much less investigated than the *divisible goods* setting.

In the first part of the dissertation, we focus on the fair division of indivisible goods. Our fairness criterion is *envy-freeness up to any good* (EFX). An allocation is EFX if no agent envies another agent following the removal of a single good from the other agent's bundle. Despite significant investment by the research community, the existence of EFX allocations remains open and is considered one of the most important open problems in fair division. In this thesis, we make significant progress on this question. First, we show that when agents have general valuations, we can determine an EFX allocation with a small number of unallocated goods (*almost EFX allocation*). Second, we demonstrate that when agents have structured valuations, we can determine an almost EFX allocation that is also *efficient* in terms of *Nash welfare*. Third, we prove that EFX allocations exist when there are three agents with additive valuations. Finally, we reduce the problem of finding improved guarantees on EFX allocations to a novel problem in extremal graph theory.

In the second part of this dissertation, we turn to the fair division of divisible bads. Like in the setting of divisible goods, *competitive equilibrium with equal incomes (CEEI)* has emerged as the best mechanism for allocating divisible bads. However, neither a polynomial time algorithm nor any hardness result is known for the computation of CEEI with bads. We study the problem of dividing bads in the classic *Arrow-Debreu* setting (a setting that generalizes CEEI). We show that in sharp contrast to the Arrow-Debreu setting with goods, determining whether a competitive equilibrium exists, is NP-hard in the case of divisible bads. Furthermore, we prove the existence of equilibrium under a simple and natural sufficiency condition. Finally, we show that even on instances that satisfy this sufficiency condition, determining a competitive equilibrium is PPAD-hard. Thus, we settle the complexity of finding a competitive equilibrium in the Arrow-Debreu setting with divisible bads.

Zusammenfassung. Die Arbeit untersucht das Problem der gerechten Verteilung (*fair division*), welches zum Ziel hat, eine Menge von Gegenständen (*items*) einer Menge von Akteuren (*agents*) "zuzuordnen". Dabei liegt der Schwerpunkt der Arbeit auf Szenarien, in denen die zu verteilenden Gegenstände entweder unteilbare Güter (*indivisible goods*) oder teilbare Pflichten (*divisible bads*) sind. Trotz ihrer praktischen Relevanz haben diese Szenarien in der Forschung bislang bedeutend weniger Aufmerksamkeit erfahren als das Szenario mit teilbaren Gütern (*divisible goods*).

Der erste Teil der Arbeit konzentriert sich auf die gerechte Verteilung unteilbarer Güter. Unser Gerechtigkeitskriterium ist Neid-Freiheit bis auf irgendein Gut (*envy-freeness up to any good*, EFX). Eine Zuordnung ist EFX, wenn kein Akteur einen anderen Akteur beneidet, nachdem ein einzelnes Gut aus dem Bündel des anderen Akteurs entfernt wurde. Die Existenz von EFX-Zuordnungen ist trotz ausgeprägter Bemühungen der Forschungsgemeinschaft ungeklärt und wird gemeinhin als eine der wichtigsten offenen Fragen des Feldes angesehen. Wir unternehmen wesentliche Schritte hin zu einer Klärung dieser Frage. Erstens zeigen wir, dass wir für Akteure mit allgemeinen Bewertungsfunktionen stets eine EFX-Zuordnung finden können, bei der nur eine kleine Anzahl von Gütern unallokiert bleibt (partielle EFX-Zuordnung, *almost EFX allocation*). Zweitens demonstrieren wir, dass wir für Akteure mit strukturierten Bewertungsfunktionen eine partielle EFX-Zuordnung bestimmen können, die zusätzlich effizient im Sinne der Nash-Wohlfahrtsfunktion ist. Drittens beweisen wir, dass EFX-Zuordnungen für drei Akteure mit additiven Bewertungsfunktionen immer existieren. Schließlich reduzieren wir das Problem, verbesserte Garantien für EFX-Zuordnungen zu finden, auf ein neuartiges Problem in der extremalen Graphentheorie.

Der zweite Teil der Arbeit widmet sich der gerechten Verteilung teilbarer Pflichten. Wie im Szenario mit teilbaren Gütern hat sich auch hier das Wettbewerbsgleichgewicht bei gleichem Einkommen (*competitive equilibrium with equal incomes*, CEEI) als der beste Allokationsmechanismus zur Verteilung teilbarer Pflichten erwiesen. Gleichzeitig sind weder polynomielle Algorithmen noch Schwere-Resultate für die Berechnung von CEEI mit Pflichten bekannt. Die Arbeit untersucht das Problem der Verteilung von Pflichten im klassischen Arrow-Debreu-Modell (einer Generalisierung von CEEI). Wir zeigen, dass es NP-hart ist, zu entscheiden, ob es im Arrow-Debreu-Modell mit Pflichten ein Wettbewerbsgleichgewicht gibt – im scharfen Gegensatz zum Arrow-Debreu-Modell mit Gütern. Ferner beweisen wir die Existenz eines Gleichgewichts unter der Annahme einer einfachen und natürlichen hinreichenden Bedingung. Schließlich zeigen wir, dass die Bestimmung eines Wettbewerbsgleichgewichts sogar für Eingaben, die unsere hinreichende Bedingung erfüllen, PPAD-hart ist. Damit klären wir die Komplexität des Auffindens eines Wettbewerbsgleichgewichts im Arrow-Debreu-Modell mit teilbaren Pflichten.

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Dedication

Dedicated to my parents, Amitabha Ray Chaudhury and Susan Ray Chaudhury.

Contents

CHAPTER 1
Introduction

Fair division has developed into a fundamental branch of mathematical economics over the last seven decades (since the seminal work of Hugo Steinhaus in the 1940s [90]). In a classical fair division problem, the goal is to "fairly" allocate a set items among a set of agents. Early mentions of such problems date back to the Bible and ancient Greek mythology. Even today, several real-life scenarios are paradigmatic of the problems in this domain, e.g., division of family inheritance [85], divorce settlements [24], spectrum allocation [55], air traffic management [93], course allocation [13] and many more[1]. For the past two decades, the computer science community has developed concrete formulations and tractable solutions to fair division problems and thus contributing substantially to the development of the field. With the advent of the Internet and the rise of centralized electronic platforms that intend to impose fairness constraints on their decisions (e.g., Airbnb would like to fairly matching hosts and guests, and Uber would like to fairly match drivers and riders etc..), there has been an increasing demand for computationally tractable protocols to solve fair division problems.

In an instance of a fair division problem, we have a set of *agents* and a set of *items*, and the goal is to determine an allocation of the items among the agents that makes every agent content i.e., is "fair" and achieves high welfare, i.e., is "efficient". The items to be divided can be *divisible* or *indivisible*, and they can be *desirable (goods)* or *undesirable (bads or chores)*. Motivated by applications, there are several notions of *fairness* and *efficiency*, which lead to several distinct problems.

The most extensively studied setting is that of divisible goods. In this setting, determining a *competitive equilibrium with equal incomes (CEEI)* is a canonical way of getting a fair and efficient allocation. In a CEEI, one creates a virtual market with the agents and the goods and gives each agent the same purchasing power (say we give every agent 1 USD). Thereafter, one determines prices for the goods, and an allocation of the goods to the agents, such that under the given prices and the spending constraints (each agent may spend up to 1 USD), each agent is allocated the goods that maximize her utility. Such an allocation is envy-free (fair), i.e., no agent prefers another agent's bundle to her own, and Pareto-optimal (efficient), i.e., there is no way to give an agent a better bundle without giving another agent a worse bundle. At first glance, it is not clear why such prices and allocations exist. However, there is an extensive line of work on *competitive equilibrium* (also referred to as market equilibrium) that not only shows the existence of such prices and allocations but also describes several fast algorithms to compute them. In fact, competitive equilibrium theory has a long history going back to the works of Léon Walras in 1874 [94]. However, the emphasis always was on determining prices at which demand equals supply and it is not until quite recently the techniques and concepts from competitive equilibrium theory have been leveraged to find fair and efficient allocation of items that go beyond divisible goods. Most notably, the following two settings have

[1] Check [1] and [2] for more detailed explanation of fair division protocols used in day to day problems.

received growing attention in the last decade:

(1) fair and efficient division of *indivisible goods*, and

(2) fair and efficient division of *divisible bads.* (In particular, competitive equilibrium with divisible bads.)

Both of these settings are practically relevant: For instance, jewellery, artworks, estates, and electronics are indivisible goods that frequently require allocation, and household chores, teaching loads, and job shifts are divisible bads that must often be split up in everyday life. Despite the similarity in the nature of the problems, both settings pose far more challenges than the setting with divisible goods. In this thesis, after introducing the basic concepts and notations (Chapter 2), we investigate these difficulties and answer fundamental questions in both settings (Chapters 3-7), before discussing avenues for future research (Chapter 8). In summary, our main contributions are as follows:

Fair and Efficient Allocation of Indivisible Goods.

Fair division problems involving indivisible goods have been relatively understudied, primarily because classic fairness notions such as envy-freeness and proportionality cannot be guaranteed even in trivial instances, such as a setting with two agents and a single indivisible good that both agents find valuable. However, over the last decade, several relaxations of envy-freeness and proportionality have been proposed and studied. In this thesis, we consider one of the most important relaxations of envy-freeness: envy-freeness up to any good (EFX).

Envy-freeness up to any good (EFX). "The closest analogue of envy-freeness" in the context of indivisible goods is that of *envy-freeness up to any good* (EFX) [28]. An allocation is said to be EFX if no agent envies another agent following the removal of *any* single good from the other agent's bundle. Until now, it is not known whether EFX allocations exist even when agents have *additive valuations*, despite "significant effort" by the research community ([28, 78]). Ariel Procaccia, in an editorial note in Communications of the ACM [87], refers to the question as

"fair division's biggest open problem".

In this thesis, we take significant steps towards solving this problem. In Chapter 3, we show that even when agents have much more general valuations than additive valuations[2], an EFX allocation always exists if we allow a small number of goods to remain unallocated. To be precise, the number of goods not allocated is less than the number of agents[3], and for each agent, the value of the unallocated goods is smaller than the value of the bundle allocated to the agent. This result makes substantial progress towards understanding the existence of EFX allocations, given that prior to this result, the only settings in which EFX allocations were known to exist for general valuations were the setting with only two

[2]A valuation v is additive if $v(S) = \sum_{s \in S} v(\{s\})$ for all S.

[3]Number of agents is significantly smaller than the number of goods.

agents or the setting in which all agents have the same valuation [84]. In Chapter 6, we improve the bound on the number of unallocated goods to be sublinear in the number of agents using further novel ideas and techniques. In particular, we establish a connection between the number of unallocated goods and a problem in extremal graph theory.

Despite the above result, the existence of *complete* EFX allocations, where no goods remain unallocated, remained a hard problem even for three agents with additive valuations ("highly non-trivial problem" according to [84]). In Chapter 5, we show that EFX allocations always exist when there are three agents with additive valuations. This result is surprising as there are many other fairness notions in discrete fair division (such as maximin-share [MMS]) that do not exist even when there only three agents with additive valuations. We introduce several novel ideas and techniques in this work and also highlight the drawbacks of the existing techniques by disproving a conjecture made about EFX.

Nash welfare and approximations. Alongside fairness, another desirable property of an allocation is "efficiency": a measure of the overall welfare that the allocation achieves. One of the most common measures of economic efficiency is *Nash welfare*[4]– defined as the geometric mean of the valuations of the agents. It is intuitive that an allocation having high Nash welfare will have less skew in the valuation functions of the agents. At a high-level, Nash welfare captures the natural balance between fairness and efficiency and therefore is widely regarded as a direct indicator of the fairness and efficiency of an allocation. As a result, the problem of maximizing Nash welfare has independently received a great deal of attention from the research community [42, 6, 7, 18, 59]. In Chapter 4, we show that in polynomial-time, we can determine an allocation that maintains all the fairness guarantees achieved in Chapter 3, and achieves a good approximation of Nash welfare. In fact, when agents have more general valuations (e.g., submodular or subadditive), our algorithm improves on the best known approximations for Nash welfare.

Fair and Efficient Allocation of Divisible Bads.

Several real life scenarios may involve the fair division of undesirable bads, also known as chores. As in the case of divisible goods, finding a *competitive equilibrium* is again the best mechanism for dividing chores. However, there are no algorithmic or hardness results for determining a competitive equilibrium with chores in any fundamental economic model, even when the agents have *linear disutility functions.*

Hardness of finding a CE, even when agents have linear disutilities. In Chapter 7, we study competitive equilibria in the classic *linear Arrow-Debreu* setting with chores, where agents divide their chores amongst themselves while minimizing their disutility. This setting generalizes the CEEI setting with chores. The Arrow-Debreu setting with chores is significantly more complex than the Arrow-Debreu setting with goods. To start with, in the setting with goods, there are simple polynomial-time verifiable necessary and sufficient conditions for the existence of a competitive equilibrium.

[4]Maximum Nash welfare implies some other efficiency measures such as Pareto-optimality.

In contrast, the problem of determining the existence of competitive equilibrium with chores is NP-hard. Thus, we can only hope for polynomial-time verifiable sufficient (not necessary and sufficient) conditions that capture interesting instances. To this end, we formulate polynomial-time verifiable simple and natural sufficiency conditions, and then prove the existence of competitive equilibrium under these conditions using a novel fixed-point formulation. However, surprisingly, even under these sufficiency conditions, we show that determining a competitive equilibrium is PPAD-hard. These results come in sharp contrast to the setting with goods, in which there are strongly polynomial-time algorithms [63].

Bibliographic Notes.

The contributions presented in this thesis are based on the following publications and drafts:

Chapter 3: B. R. Chaudhury, T. Kavitha, K. Mehlhorn, and A. Sgouritsa. A little charity guarantees almost envy-freeness. In *Proceedings of the 31st Symposium on Discrete Algorithms (SODA)*, pages 2658–2672, 2020

The full version of this paper will appear in *SIAM Journal on Computing (SICOMP)*.

Chapter 4: B. R. Chaudhury, J. Garg, and R. Mehta. Fair and efficient allocations under subadditive valuations. In *AAAI*, 2021 (To appear)

Chapter 5: B. R. Chaudhury, J. Garg, and K. Mehlhorn. EFX exists for three agents. In *EC*, pages 1–19. ACM, 2020

Chapter 6: B. R. Chaudhury, J. Garg, K. Mehlhorn, R. Mehta, and P. Misra. Improving EFX guarantees through rainbow cycle number. *CoRR*, abs/2103.01628, 2021

This paper will appear in the Proceedings of the 22nd ACM Conference on *Economics and Computation (EC 2021).*

Chapter 7: B. R. Chaudhury, J. Garg, P. McGlaughlin, and R. Mehta. Dividing bads is harder than dividing goods: On the complexity of fair and efficient division of chores. *CoRR*, abs/2008.00285, 2020

CHAPTER 2
Background and Preliminaries

In this chapter, we introduce the fundamental concepts and techniques used in fair division and competitive equilibrium theory. An instance of fair division is given by a set of agents, a set of *items*, and each agent has a valuation function that captures her utility for the set of bundles that can be allocated to her. The goal is to determine an allocation that is *fair* and *efficient*. Throughout this thesis, we will focus on *envy-freeness and its relaxations* as the fairness measure and the *Nash welfare* (for indivisible goods) and *Pareto-optimality* (for divisible bads) as the measure of efficiency [1]. There are other notions of fairness in the fair division literature and we will discuss them briefly in this chapter.

We now elaborate on fair and efficient allocation of divisible goods. We remark that the following discussion focuses on the fair and efficient allocation of *homogeneous* goods. There is impressive and extensive research done on Cake-cutting [51, 24, 91, 86, 11, 10] which is beyond the scope of this thesis and will not be discussed.

2.1 Fair and Efficient Allocation of Divisible Goods.

We are given a set of n agents $[n]$ and a set of m divisible goods M. Without loss of generality, we assume that there is one unit of each good. Each agent $i \in [n]$ has a valuation function $v_i \colon \mathbb{R}^m_{\geq 0} \to \mathbb{R}_{\geq 0}$ that quantifies her utility for the bundles that can be allocated to her. In an allocation X, we refer to agent i's bundle as X_i. Each $X_i \in \mathbb{R}^m_{\geq 0}$ and is expressed as $\langle X_{i1}, X_{i2}, \ldots, X_{im} \rangle$, where X_{ij} is the amount of good j allocated to agent i. Agent i's utility from X_i is $v_i(X_i)$. A general assumption made about the valuation functions of all agents is that they are *locally non-satiable* (generalizes monotonicity), i.e, for all $i \in [n]$, for all bundles X_i and for all $\varepsilon > 0$, there exists some X'_i such that $\|X'_i - X_i\|_2 \leq \varepsilon$ and $v_i(X'_i) > v_i(X_i)$. We now define envy-freeness and Pareto-optimality.

Envy-free allocation. An allocation X is envy-free if and only if for all agent i and j, we have $v_i(X_i) \geq v_i(X_j)$, i.e, no agent strictly prefers the bundle of any other agent to her own.

One may wonder that why determining an allocation that is envy-free or proportional (fair) is not good enough, i.e., why do we care for efficiency in addition to fairness. We address this concern. Consider an instance I with two agents and two goods in Table 2.1. Agent 1's valuation $v_1(X_1) = 1 \cdot X_{11} + 0 \cdot X_{12}$ and agent 2's valuation $v_2(X_2) = 0 \cdot X_{21} + 1 \cdot X_{22}$. Consider an allocation Y, where $Y_{11} = Y_{12} = Y_{21} = Y_{22} = 1/2$, meaning we give every agent $1/2$ units of both the goods. Observe that Y is envy-free and therefore fair. However, there is another allocation Z where $Z_{11} = Z_{22} = 1$ and

[1] We will explain envy-freeness and its relaxations, Nash welfare and Pareto-optimality shortly.

	good 1	good 2
Agent 1	1	0
Agent 2	0	1

Table 2.1: Instance I. Agent 1 has a valuation of 1 for one unit of good 1, and has a valuation of 0 for one unit of good 2. Valuations are linear and separable across the goods, meaning that we have $v_1(X_1) = 1 \cdot X_{11} + 0 \cdot X_{12}$. Similarly, agent 2 has a valuation of 0 for one unit of good 1, and has a valuation of 1 for one unit of good 2, and $v_2(X_2) = 0 \cdot X_{21} + 1 \cdot X_{22}$.

$Z_{12} = Z_{21} = 0$, meaning we give g_1 entirely to 1 and g_2 entirely to 2. Notice that Z is also envy free, but both agents are strictly better off than they were in Y. This suggests that some fair allocations are stricly preferred over the other as they achieve better welfare on a total as well as an individual basis. Therefore, we should find fair allocations with high overall welfare. This is why we impose efficiency requirements on our desired allocation. Now we define Pareto-optimality, one of the most well accepted notions of economic efficiency.

Pareto-optimal allocation. An allocation X is Pareto-optimal if and only if there exists no other allocation Y such that $v_i(Y_i) \geq v_i(X_i)$ for all $i \in [n]$, with a strict inequality for at least one i. Intuitively, an allocation X is Pareto-optimal if there is no way to give an agent a strictly better bundle without giving any other agent a worse bundle.

Observe that envy-free allocations are easy to find: give $1/n$ fraction of each good to every agent. However, it is non-trivial to show the existence of envy-free allocations that are Pareto-optimal. It is not at all immediate that such allocations (that are envy-free and Pareto-optimal) exist. A great line of seminal works in mathematical economics answer this question positively, i.e., in all instances where agents have *concave valuations*, there exists allocations that satisfy envy-freeness and Pareto-optimality simultaneously. The proof of existence follows from the existence of a competitive equilibrium with equal incomes (CEEI) in the same setting with the agents $[n]$ and the goods M. In a CEEI setting, we design a virtual market with the agents $[n]$ and the goods M. We equip each agent with 1 dollar of money and determine the prices of the goods in M and the allocation of the goods to the agents in $[n]$ at which the *market clears* (when demand equals supply). We now give a formal description of CEEI.

Competitive Equilibrium with Equal Incomes (CEEI). Given $[n]$ and M, in a CEEI, we determine a non-negative price p_j for each good $j \in M$ and an allocation X such that

- Each agent is allocated the bundle X_i^* which maximizes her utility subject to a spending constraint of 1 unit, i.e.,

$$X_i^* \in \mathit{argmax}_{X_i \in \mathbb{R}^m_{\geq 0}} \{v_i(X_i) \mid X_{ij} \geq 0 \ \ \forall i, j \text{ and } \sum_{j \in M} X_{ij} \cdot p_j \leq 1\}$$

- all the goods are completely allocated, i.e., $\sum_{i\in[n]} X^*_{ij} = 1$ (as we assumed that there is one unit of each good).

We briefly mention why X^* is envy-free and Pareto-optimal: The bundle allocated each agent satisfies the spending constraint. Since each agent receives the bundle that maximizes her utility under the spending constraint, the bundle of any other agent (that also satisfies the spending constraint) will not give her more utility. Thus, X^* is envy-free. We now show that X^* is Pareto-optimal. Consider any allocation Y which allocates all the goods and $v_\ell(Y_\ell) > v_i(X^*_\ell)$ for some $\ell \in [n]$. First note that $\sum_{i\in[n]}\sum_{j\in M} Y_{ij}\cdot p_j = \sum_{i\in[n]}\sum_{j\in M} X^*_{ij}\cdot p_j \leq n$:

$$\begin{aligned}
\sum_{i\in[n]}\sum_{j\in M} Y_{ij}\cdot p_j &= \sum_{j\in M} p_j \cdot \sum_{i\in[n]} Y_{ij} \\
&= \sum_{j\in M} p_j \\
&= \sum_{j\in M} p_j \cdot \sum_{i\in[n]} X^*_{ij} \\
&= \sum_{i\in[n]}\sum_{j\in M} X^*_{ij}\cdot p_j \\
&\leq n \qquad\qquad\qquad (\text{as } \sum_{j\in M} X^*_{ij}\cdot p_j \leq 1 \text{ for all } i \in [n]).
\end{aligned}$$

Since every agent is allocated a bundle that maximizes her utility under the spending constraint, if an agent gets a strictly better bundle in any other allocation, then the spending constraint must be violated. Since $v_\ell(Y_\ell) > v_\ell(X^*_\ell)$, we have $\sum_{j\in M} Y_{\ell j}\cdot p_j > 1$. Again, since $\sum_{i\in[n]}\sum_{j\in M} Y_{ij}\cdot p_j = \sum_{i\in[n]}\sum_{j\in M} X^*_{ij}\cdot p_j \leq n$, there must be an ℓ' such that $\sum_{j\in M} Y_{\ell' j}\cdot p_j < 1$. Since $v_{\ell'}(\cdot)$ is locally non-satiable, there exists a bundle $X_{\ell'}$ and a sufficiently small scalar $\varepsilon > 0$ such that $\|X_{\ell'} - Y_{\ell'}\|_2 \leq \varepsilon$, and $\sum_{j\in M} X_{\ell' j}\cdot p_j \leq 1$, and $v_{\ell'}(X_{\ell'}) > v_{\ell'}(Y_{\ell'})$. Note that by the definition of $X^*_{\ell'}$ we have that $v_{\ell'}(X^*_{\ell'}) \geq v_{\ell'}(X_{\ell'}) > v_{\ell'}(Y_{\ell'})$. Therefore, for any allocation Y such that $v_\ell(Y_\ell) > v_\ell(X^*_\ell)$, there exists an agent $\ell' \in [n]$ such that $v_{\ell'}(Y_{\ell'}) < v_{\ell'}(X^*_{\ell'})$. Thus X^* is also Pareto-optimal.

Now the crucial question is whether such set of prices and allocation satisfying the properties of CEEI exist and can they be determined efficiently. To this end, we briefly elaborate the history of markets and market equilibirum.

2.1.1 Existence and Computation of CEEI – Market Equilibrium Theory

A market is a fundamental system in economics which involves a set of consumers and resources. The market mechanisms involve the allocation of these resources to the consumers through determination of prices for these resources, which again depends on the supply and demand interactions. In particular, the prices and the allocation are determined in a way where demand equals supply; often referred to as a *competitive equilibrium.* Study of existence of competitive equilibrium dates back to Léon Walras in 1874 [94]. However, a formal proof to the existence of equilibrium in a very general economic model, was provided by Kenneth Arrow and Gérard Debreu in 1954 [8] and also independently by Lionel W. Mckenzie in 1954 [76][2]. This result is widely regarded as the crown jewel of mathematical economics.

Finding fair allocations through market mechanisms (like CEEI) has a very unique trait of being natural and intuitive as the market mechanisms are very natural real-world mechanisms. At the same time, this is also surprising, given that a competitive equilibrium is an "inherently decentralized concept", where each agent acts according to her best interest (and thereby defining the demand for the resources), while fairness and efficiency are "inherently centralized concepts" aimed at the welfare of the society. We now discuss the *Arrow-Debreu market*, which is one of the most fundamental and general market models. We will see later in this subsection that CEEI is a special case of a *competitive equilibrium* in a Arrow-Debreu market.

Arrow-Debreu Markets. An Arrow-Debreu market or equivalently an *exchange market* was introduced by Léon Walras in 1874 [94]. The market comprises of the agents $[n]$ and goods M. Each agent has an initial endowment of the goods. Formally, agent i brings $w_{i,j}$ amounts of good j to the market. At a competitive equilibrium, we determine a non-negative price p_j for each good $j \in M$ and an allocation X^* such that,

- Each agent purchases the most preferred bundle of goods in exchange of her initial endowment, i.e.,

$$X_i^* \in argmax_{X_i \in \mathbb{R}^m_{\geq 0}} \{v_i(X_i) \mid X_{ij} \geq 0 \;\; \forall i, j \text{ and } \sum_{j \in M} X_{ij} \cdot p_j \leq \sum_{j \in M} w_{i,j} \cdot p_j\}$$

- all the goods are completely allocated, i.e., $\sum_{i \in [n]} X^*_{ij} = 1$.

Notice that the equilibrium prices are also scale-invariant. The Arrow-Debreu market generalizes the markets where agents have fixed budgets/ money instead of initial endowment of the goods. The latter markets are called *Fisher markets* and they were introduced by Irving Fisher in 1891 [57]. We now formally define the Fisher markets.

Fisher Markets. The market comprises of the agents $[n]$ and the goods M. Each agent i has an initial budget of $m_i > 0$. At a competitive equilibrium, we determine a non-negative price p_j for each good $j \in M$ and an allocation X^* such that,

- Each agent purchases the most preferred bundle of goods by spending at most m_i dollars, i.e.,

$$X_i^* \in argmax_{X_i \in \mathbb{R}^m_{\geq 0}} \{v_i(X_i) \mid X_{ij} \geq 0 \;\; \forall i, j \text{ and } \sum_{j \in M} X_{ij} \cdot p_j \leq m_i\}$$

- all the goods are completely allocated, i.e., $\sum_{i \in [n]} X^*_{ij} = 1$.

A Fisher market can be also seen as a special case of the Arrow-Debreu market where for all $i \in [n]$ and $j \in M$ we have $w_{i,j} = m_i$[3].

[2] Mckenzie also made improvements later in 1959 [77].

[3] We also scale down the price vector by $\sum_{j \in M} p_j$.

Now, observe that a CEEI is a special case of a competitive equilibrium in a Fisher market where $m_i = 1$ for all $i \in [n]$. Therefore, we have the following containment relation,

$$\text{CEEI} \subset \text{Competitive equilibrium in Fisher markets} \subset \text{Competitive equilibrium in Arrow-Debreu markets.}$$

Arrow and Debreu [8], and Mckenzie [76, 77] had shown the existence of competitive equilibrium in Arrow-Debreu markets under some mild conditions, when agents have concave valuation functions. These conditions are satisfied by a Fisher market and thus the existence result immediately implies the existence of a CEEI when agents have concave valuation functions. Unfortunately, all of these results suffer the same drawback of being non-constructive. Since the past two decades, the computer science community has contributed substantially to coming up with algorithms to determine the competitive equilibrium in all of these market models. We now highlight some significant contributions along this direction.

Computational aspects of determining a market equilibrium. There have been substantial algorithmic studies on both Fisher markets and Arrow-Debreu markets over the last twenty years. The full coverage of all these results is well beyond the scope of the thesis. We refer the reader to [41] for a detailed survey. We highlight some of the important algorithms. One of the most fundamental valuation functions that has been extensively studied are *linear valuation functions*, where for all $i \in [n]$, we have $v_i(X_i) = \sum_{j \in M} X_{ij} \cdot v_{ij}$ where v_{ij} is the utility derived by agent i from consuming one unit of good j.

Fisher markets: The first polynomial-time algorithm for *linear Fisher markets* (where agents have linear valuation functions) was given by Devanur, Papadimitriou, Saberi and Vazirani [47]. The algorithm is a primal-dual algorithm and is weakly polynomial.

There are some gradient descent [40] and ellipsoid method based approaches to compute a competitive equilibrium and *approximate competitive equilibrium* (one in which almost all the goods are sold) respectively in linear Fisher markets. Most of such approaches either build on or are inspired from the following fascinating convex program formulation of the competitive equilibrium, introduced by Eisenberg and Gale [52].

$$\begin{aligned}
\text{maximize} \quad & \sum_{i \in [n]} m_i \log(v_i(X_i)) \\
\text{subject to} \quad & v_i(X_i) = \sum_{j \in M} v_{ij} \cdot X_{ij}, && \forall i \in [n] \\
& \sum_{i \in [n]} X_{ij} = 1, && \forall j \in M \\
& X_{ij} \geq 0, && \forall i \in [n], \forall j \in M
\end{aligned}$$

Any allocation that maximizes the weighted sum of logarithms of the valuation of the agents or equivalently the weighted product of the valuations, is an allocation corresponding to a competitive equilibrium. To verify the claim, one needs to apply the KKT conditions on the above convex program where the prices of the goods correspond to the dual variables for the set of inequalities $\{\sum_{i \in [n]} X_{ij} = 1 \mid j \in M\}$; in particular, the dual variable λ_j corresponding to the inequality $\sum_{i \in [n]} X_{ij} = 1$ represents the price of the good j. *Furthermore, the KKT conditions of the convex program also imply that the prices at a competitive equilibrium are unique.* Note that this is a different proof of existence (the proof by Arrow-Debreu and Mckenzie made use of Kakutani's fixed point theorem) of competitive equilibrium in linear Fisher markets. Jain and Vazirani further explored the potential of the above convex program and show that it captures competitive equilibrium in a wide variety of markets - all of these markets are now coined as *Eisenberg Gale markets* [67].

In 2010, James Orlin [82] provided a strongly polynomial-time algorithm[4] for determining a competitive equilibrium in linear Fisher markets.

Both polynomial-time algorithms and hardness results are known for determining a competitive equilibrium in Fisher markets when agents have more general valuation functions than linear valuation functions. When agents have budget-additive valuation functions, then there exists polynomial-time algorithms to determine a competitive equilibrium [20]. However, when agents have separable piecewise linear concave (SPLC) valuations, then determining a competitive equilibrium is PPAD-hard [39].

Arrow-Debreu market: Similar to the linear Fisher markets, there have been several algorithms for *linear Arrow-Debreu* markets. The first polynomial-time algorithms were the ellipsoid algorithm by Jain [66] and the interior point algorithm by Ye [96]. Similar to the Eisenberg-Gale convex program, there are several convex program formulations that capture the competitive equilibrium in linear Arrow-Debreu markets [81, 46]. The first combinatorial algorithm was given by Duan and Mehlhorn [50] (later improved in [49]). However, all of the aforementioned algorithms were weakly polynomial and the existence of a strongly polynomial-time algorithm remained an enigmatic open problem and this was quite recently settled by Garg and Vegh [63]. Similar to Fisher markets, the Arrow-Debreu markets have also been studied when agents have more general valuations [37, 38].

To summarize, a competitive equilibrium exists under mild assumptions in a Arrow-Debreu market and these assumptions are always satisfied by Fisher markets. Therefore, CEEI always exists, implying that envy-free and Pareto-optimal allocations always exist when the goods to be allocated are divisible. There are several polynomial-time algorithms to determine a competitive equilibrium in linear Fisher markets (also for CEEI) and linear Arrow-Debreu markets. There are also algorithms that run in polynomial-time when agents have valuations that go beyond additive. There are also known barriers (PPAD-hardness) to come up with polynomial-time algorithms when agents have more general valuations.

2.2 Fair and Efficient Allocation of Indivisible Goods.

Fair division of indivisible goods is a natural setting for combinatorial and algorithmic analysis and thus has received substantial contributions from the computer science community. The fundamental problems in this domain pose challenges and barriers of a different flavor than their counterparts in fair division of divisible goods. In this section,

[4]The algorithm is in PSPACE and the total number of arithmetic operations performed by the algorithm depends only on the size of the input and not on the bitlength of the input

we briefly cover some of important concepts and techniques that are prerequisites for the Chapters in Part 1 of the thesis.

Classical fairness notions like envy-freeness do not exist even in trivial instances: consider a simple setting of two agents and one indivisible good that they both find valuable. In any feasible allocation, one agent will be left with an empty bundle and will thus not get her fair share. Therefore, people have proposed relaxations of envy-freeness. In the majority of this thesis, we deal with relaxations of envy-freeness and thus will discuss them in more detail in this section. Although we relax the notions of fairness, the notions of efficiency remains the same. We still use Pareto-optimality as a measure of efficiency. We will also talk about an alternative measure of efficiency called *Nash welfare* in this section.

We now briefly describe the setup. Similar to the case of allocating divisible goods, we have a set of n agents $[n]$ and m goods M. An allocation X is a partition of M into n bundles $\langle X_1, X_2, \dots, X_n \rangle$, where each agent is allocated the bundle X_i. We now define the relaxed fairness notions. Each agent i has a valuation function $v_i \colon 2^M \to \mathbb{R}_{\geq 0}$ that captures her utility for each subset of goods. We assume that the valuations functions are *normalized* ($v_i(\emptyset) = 0$) and *monotone* ($v_i(S \cup \{g\}) \geq v_i(S)$). Most of the studies, make stronger assumptions on the valuation functions. While some of the main results in this thesis work when agents have general valuation functions, others crucially make stronger assumptions on the valuation functions. In this light, we briefly define the three well studied and fundamental valuation classes:

- **Additve valuation functions:** These are the most well studied class of valuations and they are the analogue of linear valuations in discrete fair division. A valuation function $v \colon 2^M \to \mathbb{R}_{\geq 0}$ is an additive valuation function if the valuation on any set of goods is the sum of valuation of the individual goods in the set, i.e., for all $S \subseteq M$, we have $v(S) = \sum_{s \in S} v(\{s\})$.
- **Submodular valuation functions:** These valuation functions are more general than additive valuation functions and they capture the property of *diminishing marginal returns.* A valuation function $v \colon 2^M \to \mathbb{R}_{\geq 0}$ is a submodular valuation function if for all $A \subseteq S \subseteq M$ and $g \notin S$, we have $v(A \cup \{g\}) \geq v(S \cup \{g\})$.
- **Subadditive valuation functions:** These are more general than submodular valuation functions and capture the concept of *natural monopoly* in economics. Formally, a valuation function $v \colon 2^M \to \mathbb{R}_{\geq 0}$ is subadditive if for all $A, B \subseteq M$ we have $v(A) + v(B) \geq v(A \cup B)$.

2.2.1 Envy-freeness up to *one* good (EF1)

Envy-freeness up to *one* good (EF1) was introduced by Budish [26]. An allocation X is said to be EF1 if no agent i envies another agent j after the removal of *some* good in j's bundle, i.e., $v_i(X_i) \geq v_i(X_j \setminus \{g\})$ for *some* $g \in X_j$. So we allow i to envy j, but the envy must disappear after the removal of some valuable good (according to agent i) from j's bundle. Note that there is no actual removal: This is simply to assess how agent i values her own bundle when compared to j's bundle.

We give a small example: Consider an instance comprising of two agents and three goods, namely a laptop, iPad and a phone. Both agents have *additive valuation functions*

	laptop	iPad	phone
Agent 1	10	9	3
Agent 2	10	9	3

Table 2.2: Instance used to illustrate EF1. Both agents have identical additive valuation functions, i.e., a valuation of 10 for the laptop, 9 for the iPad and 3 for the phone.

(the analogue of linear valuations in discrete fair division) where the valuation on any set is the sum of valuations of the individual goods in the set. The valuation that each agent has for each good is captured in Table 2.2. Notice that an allocation where agent 1 gets the laptop and agent 2 gets the iPad and the phone seems intuitively fair. Also, note that this allocation is EF1: agent 2 has a total valuation of $9 + 3 = 12$ which is larger than her valuation for agent 1's bundle (which is 10) and therefore she does not envy agent 1. Agent 1 on the other hand, indeed envies agent 2 as her valuation for her own bundle is 10 and her valuation for agent 2's bundle is 12; however, following the removal of the phone from agent 2's bundle, agent 1 does not envy agent 2 anymore as agent 1 values the laptop more than the iPad. Therefore the allocation is EF1. Now, consider the allocation where agent 1 gets the laptop and iPad, while agent 2 gets only the phone. Intuitively, it seems to be an unfair allocation and we observe that the allocation is indeed not EF1: agent 2 values both the laptop and the iPad more than the phone, implying that following removal of any good (irrespective of whether it is the laptop or the iPad) from agent 1's bundle, the envy does not disappear.

Existence and computation of EF1 allocations. Lipton et al. [72] show that EF1 allocations exist when agents have monotone valuations. Moreover, such allocations can be determined with strongly polynomial number of *value queries* - queries that given an agent i and a good set $S \subseteq M$, outputs $v_i(S)$. We briefly discuss this algorithm. A simple yet crucial concept introduced by this algorithm is that of an envy-graph: Given an allocation X, an envy-graph E_X has vertices corresponding to the agents and there is an edge from agent i to agent j in E_X if and only if i envies j, i.e., $v_i(X_i) < v_i(X_j)$. The invariant maintained is that the envy-graph is a DAG: a cycle corresponds to a cycle of envy and by swapping bundles along a cycle, every agent becomes better-off and the number of envy edges decreases. More precisely, if $i_0 \to i_1 \to i_2 \to \ldots \to i_{\ell-1} \to i_0$ is a cycle in the envy graph, then reassigning $X_{i_{j+1}}$ to agent i_j for $0 \le j < \ell$ (indices are to be read modulo ℓ) will increase the valuation of every agent in the cycle. Also if there was an edge from s to some i_k where s is not a part of the cycle, this edge just gets directed now from s to i_{k+1} after we exchange bundles along the cycle. Thus the number of envy edges in the graph does not increase and the valuations of the agents in the cycle goes up. Thus cycles can be eliminated. The algorithm in [72] runs in rounds and always maintains an allocation that is also EF1 - it starts with an empty allocation in the first round which is trivially EF1. At the beginning of every round, an unenvied agent s (this is a source vertex in this DAG) is identified and an unallocated good g is allocated to s. The new allocation is also EF1, as nobody will envy the bundle of s after removing the good g. All the set of operations can be implemented with strongly polynomial many value queries.

Recall that the goal is to compute fair "and efficient" allocations. Unfortunately, the algorithm in Lipton [72] does not give us any guarantee on Pareto-optimality. This brings us to the question whether EF1 and Pareto-optimality can be guaranteed at the same time and if yes, how fast can we determine such allocations.

Existence and computation of efficient EF1 allocations. Unfortunately, there are instances where no EF1 allocation is Pareto-optimal. However, when agents have simpler and more structured valuations, there are some positive results. The most well studied class of valuations is that of *linear valuations* also known as *additive valuations* in this context, where for each agent i, we have that $v_i(S) = \sum_{g \in S} v_i(\{g\})$ for all $S \subseteq M$. When agents have additive valuations, Caragiannis et al. [28] show that an integral version of the Eisenberg Gale program will ensure the existence of EF1 allocations that are also Pareto-optimal, further showing the power of this program.

$$
\begin{aligned}
\text{maximize} \quad & \sum_{i \in [n]} \log(v_i(X_i)) & \\
\text{subject to} \quad & v_i(X_i) = \sum_{j \in M} v_{ij} \cdot X_{ij}, & \forall i \in [n] \\
& \sum_{i \in [n]} X_{ij} = 1, & \forall j \in M \\
& X_{ij} \in \{0, 1\}, & \forall i \in [n], \forall j \in M
\end{aligned}
$$

Equivalently, any allocation that maximizes the sum of logarithms of the valuations of the agents or the product of the valuations of the agents is EF1 and Pareto-optimal. The geometric mean of the valuations of the agents $(\prod_{i \in [n]} v_i(X_i))^{\frac{1}{n}}$ is also called the *Nash welfare* of the allocation X. Observe that for all monotone valuation functions, any allocation that maximizes the Nash welfare is also Pareto-optimal, as if there is another allocation that Pareto dominates the Nash welfare maximizing allocation, then the same allocation has higher Nash welfare also, which is a contradiction. Also, due to its *non-binary* nature, *Nash welfare is also considered a measure of economic efficiency* [27]. For additive valuation functions, it turns out that the Nash welfare maximizing allocation is also EF1 and thus the Nash welfare of an allocation is a direct indicator of the fairness of the allocation when agents have additive valuation functions.

We now address the algorithmic question regarding determining allocations that are EF1 and Pareto-optimal. Unfortunately, maximizing Nash welfare is APX-hard [70], ruling out the existence of any polynomial-time approximations scheme (PTAS). Barman et al. [18] provide a pseudo-polynomial-time algorithm that determines an allocation that is EF1 and Pareto-optimal. In the same paper, Barman et al. [18] show that the EF1 and Pareto-optimal allocation computed by their algorithm also achieves a 1.445 approximation of the optimum Nash welfare (Nash welfare maximization has received a great deal of independent interest– see [42], [6], [7], [59]). The existence of a polynomial-time algorithm to determine an allocation that is EF1 and Pareto-optimal still remains an enigmatic open problem in discrete fair division.

However, an EF1 allocation may be unsatisfactory, even from a fairness point of view: Intuitively, EF1 insists that the envy disappears after the removal of the *most valuable* *good* according to the envying agent from the envied agent's bundle—however, in many cases, the most valuable good might be the primary reason for very large envy to exist in the first place. Therefore, stronger notions of fairness are desirable in many circumstances. This leads us to the next relaxation of envy-freeness, namely, envy-freeness up to any good (EFX).

2.2.2 Envy-freeness up to *any* good (EFX)

This relaxation was introduced by Caragiannis et al. [28]. An allocation X is said to be EFX if no agent i envies another agent j after the removal of *any* good in j's bundle, i.e., $v_i(X_i) \geq v_i(X_j \setminus \{g\})$ for *all* $g \in X_j$. Unlike EF1, in an EFX allocation, the envy between any pair of agents disappears after the removal of the *least valuable good* (according to agent i) from j's bundle. Note that every EFX allocation is an EF1 allocation, but not *vice-versa*. Consider a simple example of two agents with additive valuations and three goods $\{a, b, c\}$ from [36], where the agents valuation for individual goods are as follows,

	g_1	g_2	g_3
Agent 1	1	1	2
Agent 2	1	1	2

Observe that g_3 is twice as valuable than g_1 or g_2 for both agents. An allocation where one agent gets $\{g_1\}$ and the other gets $\{g_2, g_3\}$ is EF1 but not EFX. The only possible EFX allocation is where one agent gets $\{g_3\}$ and the other gets $\{g_1, g_2\}$, which is clearly fairer than the given EF1 allocation. This example also shows how EFX helps to rule out some unsatisfactory EF1 allocations. Caragiannis et al. [27] remark that

> "*Arguably, EFX is the best fairness analog of envy-freeness of indivisible items.*"

Existence and Computation of EFX allocations. While an EF1 allocation is always guaranteed to exist, very little is known about the existence of EFX allocations. Caragiannis et al. [28] state that

> "*Despite significant effort, we were not able to settle the question of whether an EFX allocation always exists (assuming all items must be allocated), and leave it as an enigmatic open question.*"

Plaut and Roughgarden [84] show two scenarios for which EFX allocations are guaranteed to exist: (*i*) All agents have identical valuations (i.e., $v_1 = v_2 = \cdots = v_n$), and (*ii*) Two agents (i.e., $n = 2$). We briefly elaborate their proofs.

First consider the case where all agents have the same valuation v. For simplicity, we only consider only *non-degenerate instances*, i.e., instances where $v(S) \neq v(S')$ for all $S \neq S'$[5]. For all such instances, any allocation that maximizes the valuation of the agent with the smallest valuation is EFX: Assume otherwise and let X be an allocation that

[5] In Chapter 3, we show that it suffices to only consider non-degenerate instances.

maximizes the valuation of the agent with the smallest valuation, and there are agents i and j such that $v(X_i) < v(X_j \setminus \{g\})$ for some $g \in X_j$. Note that since agents have the same valuation function, if $v(X_i) < v(X_j \setminus \{g\})$, then we have $v(X_{i_{min}}) < v(X_j \setminus \{g\})$ where i_{min} is the agent with the smallest valuation in X. Consider the allocation $X' = \langle X_1, \ldots, X_{i_{min}} \cup \{g\}, \ldots, X_j \setminus \{g\}, \ldots X_n \rangle$. Note that the only changed bundles are that of i_{min} and j, and both of them have valuations still higher than i_{min}'s initial valuation, i.e., $v(X_{i_{min}} \cup \{g\}) > v(X_{i_{min}})$ (strict inequality follows because of our non-degeneracy assumption) and $v(X_j \setminus \{g\}) > v(X_{i_{min}})$. This implies that all agents in X' have a valuation strictly larger than the valuation of i_{min} in X, further implying that X is not the allocation that maximizes the valuation of the agent with smallest utility, which is a contradiction.

Now consider the scenario when there are only two agents. Then EFX allocation can be guaranteed by the classic *cut and choose* protocol. Agent 1 divides the goods into two bundles X_1 and X_2 such that $v_1(X_1) > v_1(X_2) > v_1(X_1 \setminus \{g\})$ for all $g \in X_1$. Note that such a division is possible as we just proved that when agents have identical valuations then EFX allocations exist; thus agent 1 can divide the good set into two bundles assuming that both agents have the same valuation as she has and thus we have the aforementioned inequalities. Now, agent 2 picks her favorite bundle out of X_1 and X_2 and the other bundle is allocated to agent 1. Note that agent 2 will not envy agent 1 as she chose the better bundle out of the two and agent 1 will not envy agent 2 as she divided the goods in a way that she is fine with either of the bundles.

In the same paper, Plaut and Roughgarden [84] show that determining an EFX allocation with two agents with identical submodular valuations requires exponential number of value queries. However, when agents have additive valuations, then in polynomial-time, one can determine an EFX allocation in both the special cases mentioned above. Unfortunately, starting from three agents, even for the well studied class of additive valuations, it was open whether EFX allocations exist. Plaut and Roughgarden [84] also remark that:

> "*The problem seems highly non-trivial even for three players with different additive valuations.*"

In Chapter 5 of this thesis we resolve this question by showing that EFX allocations exist when there are three agents with additive valuations. However, starting from four agents, the question still remains open.

Approximate EFX allocations. Since very little is known about the existence of EFX allocations, there have been studies on approximations of the same. An allocation $X = \langle X_1, X_2, \ldots, X_n \rangle$ is an α-EFX allocation (an α-approximate-EFX allocation) for some scalar $\alpha \in (0, 1]$, if for all pairs of agents i and j, we have $v_i(X_i) \geq \alpha \cdot v_i(X_j \setminus \{g\})$ for all $g \in X_j$. Plaut and Roughgarden [84] show the existence of 1/2-EFX allocations when agents have subadditive valuations. Amanatadis et al. show the existence of a $(\phi - 1)$-EFX allocation [5] where ϕ is the golden ratio, when agents have additive valuations.

Existence and Computation of efficient EFX allocations. Plaut and Roughgarden [84] show that even when agents have additive valuations, there are instances where no EFX allocation is Pareto-optimal. We show the example here. Consider an instance with two agents and three goods g_1, g_2 and g_3. The agents valuations for the goods are as below,

	g_1	g_2	g_3
Agent 1	1	2	0
Agent 2	0	2	1

Notice that in any EFX allocation, the agent that gets g_2, cannot get any other good , as this will lead to the other agent envying it following the removal of a single good. Therefore, in any EFX allocation, one agent gets g_2 and the other agent gets g_1 and g_3. Consider the allocation where agent 2 gets g_2 and agent 1 gets g_1 and g_3. Note that there is a pareto-dominating allocation: agent 2 gets g_2 and g_3 and agent 1 gets g_1. We can also argue symmetrically for the other EFX allocation. Therefore, we cannot find allocations that are EFX and Pareto-optimal. In fact, changing $v_1(\{g_1\})$ and $v_2(\{g_3\})$ to some small $\delta > 0$, will ensure that no δ-EFX allocation is Pareto-optimal.

Since we cannot guarantee Pareto-optimality for any good approximation on EFX, it is natural to ask whether we can find EFX allocations that have high Nash welfare[6] To this end, Caragiannis et al. [27] show that when agents have additive valuations, we can determine *partial EFX allocations* that have high Nash welfare. An allocation $X = \langle X_1, X_2, \ldots, X_n \rangle$ is called a partial-EFX allocation if X is EFX and not all goods are necessarily allocated, i.e., $\cup_{i \in [n]} X_i \subseteq M$. There is always a trivial partial EFX allocation where each X_i is empty. Therefore, a good partial EFX allocation is the one which has good *qualitative* and *quantitative* guarantees on the unallocated goods. Caragiannis et al. [27] show that there exists a partial EFX allocation where every agent gets a bundle which she values at least as much as half of her value for the bundle she receives in a *Nash welfare* maximizing allocation. This implies that there exists a partial EFX allocation that achieves a 1/2-approximation of the optimum Nash welfare. This result served as an initiation study for several results that followed on finding good relaxations of EFX allocations (this includes good approximate EFX allocations also) that have high Nash welfare. Parts of Chapter 3 and Chapter 6 and all of Chapter 4 of this thesis is dedicated to finding efficient relaxed EFX allocations.

2.2.3 Other Fairness Notions (for Goods)

In this section, we briefly mention some other notions of fairness. Note that envy-freeness and its relaxations are highly “non-local” fairness notions i.e., whether an agent receives her fair share or not depends on how she values the bundles of all the other agents. In the fair division literature, there is another fairness notion which is inherently “local”. This fairness notion is *proportionality*. An allocation is proportional if and only if every agent receives a bundle that she values a $1/n$ fraction of the entire good set. Envy-freeness and proportionality are independent fairness notions when agents have general monotone valuations. However, when agents have linear valuations (for divisible goods) and *subadditive valuations*[7] (for indivisible goods), envy-freeness implies proportionality.

[6]Recall that Nash welfare is also a measure of efficiency.

[7]A valuation v is subadditive if for any two sets A and B, we have $v(A \cup B) \leq v(A) + v(B)$.

Thus, for divisible goods, when agents have linear valuations a CEEI allocation is envy-free, Pareto-optimal and proportional.

Similar to envy-freeness, a proportional allocation does not exist when the goods to be divided are indivisible. Therefore relaxations of the same have been proposed and studied. We mention two of the well studied relaxations:

Proportionality up to *one* good (PROP1). Similar to EF1, an allocation X is said to be proportional up to *one* good or equivalently PROP1 if for each agent $i \in [n]$, there exists a good g such that $v_i(X_i \cup \{g\}) \geq v_i(M)/n$. This notion was introduced by Conitzer et al. [43] as relaxations of envy-freeness are not adaptable for fairness in public decision makings. For additive valuations, an EF1 allocation is also a PROP1 allocation: in an EF1 allocation, for each agent $i \in [n]$, we have that $v_i(X_i) \geq v_i(X_j \setminus \{g\})$ for some $g \in X_j$. Let h be such that $h \notin X_i$ and $v_i(\{h\})$ is maximum. Then, for all $j \in [n]$ we have

$$\begin{aligned}
v_i(X_i \cup \{h\}) &= v_i(X_i) + v_i(\{h\}) \\
&\geq v_i(X_j \setminus \{g\}) + v_i(\{h\}) && \text{(as } X \text{ is EF1)} \\
&= v_i(X_j) - v_i(\{g\}) + v_i(\{h\}) \\
&\geq v_i(X_j) && \text{(as } v_i(\{h\}) \geq v_i(\{g\}) \text{ by definition of } h\text{)}
\end{aligned}$$

Therefore we have that $v_i(X_i \cup \{h\}) \geq v_i(X_j)$ for all $j \in [n]$. Summing the inequality over all $j \in [n]$ we have that $v_i(X_i) \geq (1/n) \cdot \sum_{j \in [n]} v_i(X_j) = (1/n) \cdot v_i(M)$. Thus, when agents have additive valuations PROP1 allocations exist (as EF1 allocations exist) and can be determined in polynomial-time. Similarly any allocation that maximizes the Nash welfare is also PROP1 and Pareto-optimal. However, in contrast to the algorithmic results on finding allocations that are EF1 and Pareto-optimal, there is a strongly polynomial-time algorithm that determines PROP1 and Pareto-optimality [17].

One could also define the notion of *proportionality up to* any *good (PROPX)* where for each agent i and for all $g \notin X_i$ we have $v_i(X_i \cup \{g\}) \geq 1/n \cdot v_i(M)$. However, even when agents have additive valuations, there are instances where no allocations are PROPX [12].

Maximin share (MMS). Suppose agent i has to partition M into n *bundles* (or sets) knowing that she would receive the worst bundle with respect to her valuation. Then i will choose a partition of M that maximizes the valuation of the worst bundle (wrt her valuation). The value of this worst bundle is the maximin share of agent i. An important question here is: does there always exist an allocation of M where every agent gets a bundle worth at least her maximin share?

Formally, let $[n]$ and M be the sets of n agents and m goods, respectively. We define the maximin share of an agent (say, i) as follows: (here $\mathcal{X}$ is the set of all complete allocations)

$$MMS_i(n, M) = \max_{\langle X_1, \ldots, X_n \rangle \in \mathcal{X}} \min_{j \in [n]} v_i(X_j).$$

The goal is to determine an allocation $\langle X_1, X_2, \ldots, X_n \rangle$ of M such that for every i we have $v_i(X_i) \geq MMS_i(n, M)$. This question was first posed by Budish [26]. Procaccia and Wang [88] showed that such an allocation need not exist, even in the restricted setting of only three agents with additive valuations! Thereafter, *approximate-MMS* allocations were studied [88, 61, 64, 62] mostly in the setting where agents have additive valuations[8] and we know polynomial-time algorithms to find allocations where for all i, agent i gets a bundle of value at least $\alpha \cdot MMS_i(n, M)$; the current best guarantee for α is $3/4 - \varepsilon$ by Ghodsi et al. [64] (for any $\varepsilon > 0$) and this was very recently improved to $3/4$ by Garg and Taki [62].

2.3 Fair and Efficient Allocation of Divisible Bads

Fair and efficient allocation of divisible bads has received far less attention than its goods counterpart, although many real life scenario involves division of non-disposable bads (chores). Similar to the setup with divisible goods, we have a set of agents $[n]$, a set of bads M, and each agent i has a *disutility* function $d_i \colon \mathbb{R}^m_{\geq 0} \to \mathbb{R}_{\geq 0}$ capturing the agent's pain/ inconvenience for an allocated bad bundle. Similar to the goods case, a CEEI is the best mechanism for a fair and efficient allocation of divisible bads. A CEEI allocation guarantees envy-freeness and Pareto-optimality. We now formally define the notion of CEEI for bads. Similar to the case with goods, in a CEEI, we determine a non-negative price p_j for each bad $j \in M$ and an allocation X such that

- each agent is allocated the bundle X^*_i which minimizes her disutility subject to a earning constraint of 1 unit, i.e.,

$$X^*_i \in \mathit{argmin}_{X_i \in \mathbb{R}^m_{\geq 0}} \{d_i(X_i) \mid X_{ij} \geq 0 \;\; \forall i, j \text{ and } \sum_{j \in M} X_{ij} \cdot p_j \geq 1\}$$

- all the bads are completely allocated, i.e., $\sum_{i \in [n]} X^*_{ij} = 1$ (w.l.o.g. assume that there is one unit of each bad).

While dividing bads, a CEEI exists when agents have continuous, satiable and *convex* valuations[9].

Although CEEI is the best mechanism for dividing divisible goods and bads, CEEI for bads exhibit far less structure, causing many algorithmic challenges. Bogomolnaia et al. [22] show that there are several disconnected CEEI, even when agents have linear valuations. This is in sharp contrast to CEEI with divisible goods, where all the competitive equilibria are captured by the solutions to the Eisenberg-Gale convex program. We elaborate the issue of several disconnected equilibria briefly: Consider an instance with two agents a_1 and a_2 and two bads b_1 and b_2. Both agents have linear disutility values, which is captured from the matrix below: a_1 has a disutility of 1 for one unit of b_1 and 3 for one unit of b_2, while a_2 has a disutility of 3 for one unit of b_1 and 1 for one unit of b_2.

	b_1	b_2
a_1	1	3
a_2	3	1

[8] Ghodsi et al [64] also talk about the setting when agents have submodular valuations and subadditive valuations.

[9] Note that CEEI exists in the context of goods even when valuations are non-convex

Let $p = \langle p(b_1), p(b_2) \rangle$ be an equilibrium price vector. In a CEEI, each agent is allocated the bad that has minimum disutility to price ratio, i.e., minimum *pain per buck* (since both agents have linear disutilities and the bads are divisible). Let MPB_a denote the *minimum pain per buck* bundle for agent a at prices p: a bad $b \in MPB_a$ if and only if $\frac{d(a,b)}{p(b)} \leq \frac{d(a,b')}{p(b')}$ for all other bads b' in the instance. Observe that this small instance exhibits exactly three competitive equilibria:

- The first competitive equilibrium is when both $p(b_1)$ and $p(b_2)$ are set to 1. Note that only $MPB_{a_1} = \{b_1\}$ and $MPB_{a_2} = \{b_2\}$. Thus a_1 earns her entire one unit of money from b_1 and a_2 earns her entire one unit of money from b_2.
- The second competitive equilibrium is when a_1 earns from both b_1 and b_2. For this we set $p(b_1)$ to $1/2$ and $p(b_2)$ to $3/2$. Note that $MPB_{a_1} = \{b_1, b_2\}$ and $MPB_{a_2} = \{b_2\}$. Under these prices, a_2 earns her entire money by doing $2/3$ of b_2, and a_1 earns her money by doing all of b_1 and $1/3$ of b_2.
- The third competitive equilibria is symmetric to the second: a_2 earns from both b_1 and b_2. We set $p(b_2)$ to $1/2$ and $p(b_1)$ to $3/2$.

The instance exhibits no other competitive equilibrium. Thus, the set of prices and allocations at competitive equilibria are disjoint. In fact, [22] show that there exists exponentially many disconnected competitive equilibrium in fair division of divisible bads.

There are polynomial-time *enumerative* algorithms ([60]) known that work when the number of agents is $\mathcal{O}(1)$ or the number of goods is $\mathcal{O}(1)$. These algorithms exhaustively search all competitive equilirbira, which turns out to be polynomial if $n \in \mathcal{O}(1)$ or $m \in \mathcal{O}(1)$. In [32], Chaudhury et al. give an LCP formulation (implying a simplex like algorithm) for determining a CEEI with a *mixed manna* which includes goods and bads[10]. This is the only non-enumerative algorithm known for the setting with divisible bads.

Despite all the algorithmic challenges outlined above, no hardness result is also known for CEEI with division bads. We initiate this hardness study with our results in Chapter 7, where we study the existence and computational complexity of determining a competitive equilibrium in the Arrow-Debreu model with chores. Similar to the case with divisible goods, competitive equilibrium in the Arrow-Debreu model is a generalization of CEEI. In particular, we show that when agents can have infinite disutilities for bads (signifying that the agent is incapable of completing the task/ chore) a competitive equilibrium may not always exist. Furthermore, it is NP-hard to determine the existence of a competitive equilibrium. Thereafter, we propose simple and natural, polynomial-time verifiable sufficiency conditions, under which competitive equilibrium exists. Then, we show that even under the said sufficiency conditions, it is PPAD-hard to determine a competitive equilibrium. Unfortunately, our hardness proof does not extend to the CEEI setting with chores and we leave this as an enigmatic problem for future research.

[10] so the LCP formulation also works for the all bads setting

PART I

Fair and Efficient Allocation of Indivisible Goods

CHAPTER 3

EFX Allocations with Bounded Charity

In this chapter, we show the existence of a relaxation of EFX. Since the existence of EFX allocations remains open despite significant effort by the research community, it is natural to look into its relaxations. As mentioned in Chapter 2, one natural relaxation is that of approximate-EFX: we say an allocation $X = \langle X_1, X_2, \ldots, X_n \rangle$ is α-EFX (or equivalently α-approximate EFX) for some $\alpha \in [0, 1]$, if and only if for all pairs of agents i and j, we have $v_i(X_i) \geq \alpha \cdot v_i(X_j \setminus \{g\})$ for all $g \in X_j$. Plaut and Roughgarden [84] showed the existence of 1/2-EFX allocations when agents have sub-additive valuations. Amanatadis et al. showed the existence of $(\phi - 1)$-EFX allocations when agents have additive valuations, where ϕ is the golden ratio.

Quite recently, Caragiannis et al. [27] introduced another interesting relaxation of EFX, called EFX with charity, where the goal is to determine a good *partial EFX allocation* (where some of the goods are unallocated). Observe that there exists a trivial partial EFX allocation: the one where every agent gets an emptyset and the entire set of goods remain unallocated. This partial EFX allocation is clearly not interesting. Therefore, the goal is to obtain a partial EFX allocation with some "qualitative" and "quantitative" bounds on the goods that are allocated. With this goal in mind, Caragiannis et al. [27] show that when agents have additive valuations, then there exists a partial EFX allocation $X = \langle X_1, X_2, \ldots, X_n \rangle$, such that each agent's valuation for her bundle is at least half of her valuation for the bundle she receives in an allocation that maximizes Nash welfare[1], i.e., for all $i \in [n]$, we have $v_i(X_i) \geq v_i(X_i^*)/2$, where $X^* = \langle X_1^*, X_2^*, \ldots, X_n^* \rangle$ is an allocation that has highest Nash welfare. However, whenever there are some unallocated goods, it is only natural to consider the valuation the agents have for this set of goods ("quality") and also how many or what fraction of the goods are being unallocated ("quantity"). There are no such guarantees in the partial EFX allocation determined by the algorithm in [27]. In this chapter, we wish to address this issue, even when agents have more general valuation functions!

3.1 EFX with Bounded-Charity.

Our goal is to determine a partial EFX allocation such that no agent values the set of unallocated goods too highly and also a significant fraction of the goods get allocated. To this end, we state the main result of this chapter.

Theorem 3.1. *There exists a partition of the good set M into $n+1$ bundles, $X_1, X_2, \ldots, X_n$ and P (pool of unallocated goods) such that*

- *X is EFX.*

[1] Recall that an allocation that maximizes Nash welfare has fairness and efficiency properties when agents have additive valuations.

- *$v_i(X_i) \geq v_i(P)$ for all $i \in [n]$, i.e., no agent envies the set of unallocated goods, and*
- *$|P| < n$, i.e., less than n goods remain unallocated ($n \ll m$).*

We call an allocation X that satisfies the conditions in Theorem 3.1 an *EFX allocation X with bounded charity P*. We present a simple algorithmic proof to Theorem 3.1. At a high level, our algorithm is as follows: we iteratively maintain a partial EFX allocation, and as long as

- there is some agent who envies the pool, i.e., for some $i \in [n]$, we have $v_i(X_i) < v_i(P)$,
- or if the number of goods in the pool P are larger than n,

we determine (constructively) another partial EFX allocation X' that Pareto-dominates X , i.e., each agent is at least as happy in X' as she was in X and one agent is strictly happier: $v_i(X'_i) \geq v_i(X_i)$ for all $i \in [n]$ with at least one strict inequality. Since the valuations are integral and upper-bounded, we cannot keep updating the partial EFX allocation. Thus, the update process halts after finite iterations, and our final partial allocation satisfies the conditions in Theorem 3.1.
Most envious agent. The crucial step of our algorithm is obtaining an EFX allocation X' from X that Pareto dominates X when all the conditions in Theorem 3.1 are not satisfied. For this crucial step, we introduce the concept of a *most envious agent*. Given an allocation X, and any set $S \subseteq M$, we denote the set of most envious agents of S as $A_X(S)$.

Definition 3.2. *Given a set $S \subseteq M$ and an allocation X, an agent i is a most envious agent of the set S or $i \in A_X(S)$ if and only if there exists $Z_i \subseteq S$ such that $v_i(Z_i) > v_i(X_i)$, and for any agent j (including i), we have $v_j(Z') \leq v_j(X_j)$ for all $Z' \subset Z_i$ (no agent envies a strict subset of Z_i).*

Intuitively, an agent is a most envious agent for a set S, if no other agent envies a strict subset of the *inclusion-wise minimal subset* of S that i envies. Now, we state a necessary and sufficient condition for $A_X(S)$ to be non-empty.

Observation 3.3. *$A_X(S) \neq \emptyset$ if and only if there is some agent i such that $v_i(S) > v_i(X_i)$. Also, in $\mathcal{O}(n \cdot |S|^2)$ time, one can find an agent $t \in A_X(S)$ and a set $Z \subseteq S$ such that $v_t(Z) > v_t(X_t)$ and no agent envies a strict subset of Z.*

Proof. We first show the "only if" direction. For any agent $t \in A_X(S)$, we have $v_t(S) > v_t(X_t)$. Thus, if for all agents $i \in [n]$, we have $v_i(S) \leq v_i(X_i)$, then $A_X(S) = \emptyset$.

Now, we show the "if" direction. Let i be the agent such that $v_i(X_i) < v_i(S)$. We construct a sequence (t_ℓ, Z_ℓ) as follows: initially we set t_1 to i and Z_1 to S. Assume that $(t_{\ell-1}, Z_{\ell-1})$ are defined. If no agent (including $t_{\ell-1}$) envies $Z_{\ell-1}$ following the removal of any good from $Z_{\ell-1}$, then we stop, otherwise let i' be the agent that envies $Z_{\ell-1} \setminus \{g\}$ for some $g \in Z_{\ell-1}$. We set t_ℓ to i' and Z_ℓ to $Z_{\ell-1} \setminus \{g\}$ and continue. We will eventually stop, as with every next pair in the sequence, the size of the set is reducing by one. Say we stop at ℓ^*. Then, we have that agent t_{ℓ^*} envies $Z_{\ell^*} \subseteq S$ and no agent envies a strict subset of Z_{ℓ^*}. Thus $t_{\ell^*} \in A_X(S)$ and therefore $A_X(S) \neq \emptyset$.

From the proof of the "if direction", it is clear that we can determine the agent $t_{\ell^*} \in A_X(S)$ and the set Z_{ℓ^*} in $\mathcal{O}(n \cdot |S|^2)$ time: the maximum length of the sequence (t_ℓ, Z_ℓ) is $|S|+1$ as the size of the set $Z_\ell = |S| + 1 - \ell$. For each value of ℓ, it takes $\mathcal{O}(n \cdot |Z_\ell|) \in \mathcal{O}(n \cdot |S|)$ time to find $(t_{\ell+1}, Z_{\ell+1})$. Thus the total time needed to find an arbitrary $t \in A_X(S)$ is $\mathcal{O}(n \cdot |S|^2)$. ☐

We now state the exact construction of X' from X, where we will see that this natural and simple concept of a most envious agent is very crucial.

Case 1: $v_i(X_i) < v_i(P)$ for some $i \in [n]$. We first look into the case where there exists an agent $i \in [n]$ that envies the pool. Observe that since there is at least one agent that envies the pool, $A_X(P)$ is not empty (Observation 3.3). We first determine an agent $i' \in A_X(P)$ and the set $Z_{i'}$ such that $v_{i'}(Z_{i'}) > v_{i'}(X_{i'})$ and no agent envies a strict subset of $Z_{i'}$ in $\mathcal{O}(n \cdot |P|^2)$ time (by Observation 3.3) . Then, we update our allocation X to X' as follows,

$$
\begin{aligned}
X'_{i'} &= Z_{i'}, & \\
X'_i &= X_i & \text{for all } i \neq i'.
\end{aligned}
$$

First, observe that X' is also EFX: The only new bundle is that of i'. Nobody envies i' up to any good as nobody envies a strict subset of $Z_{i'}$ (by definition of a most envious agent of P). Also agent i' has a strictly better bundle than before as $v_{i'}(Z_{i'}) > v_{i'}(X_{i'})$ and all the other agents retain their previous bundles. Therefore, if agent i' did not envy any bundle up to any good in X, she will not envy any bundle up to any good after she gets a better bundle in X'. Thus X' is also EFX. Now observe that X' Pareto-dominates X: all agents other than i' retain their previous bundles and thus their valuations remain the same and agent i''s valuation strictly increases as she gets a better bundle. Therefore, we obtain a partial EFX allocation X' which Pareto dominates X. We call this entire update procedure as U_1 and it is outlined in Algorithm 1.

function $U_1(X, P)$
    Precondition: $v_i(X_i) < v_i(P)$ for some $i \in [n]$.
    Find $t \in A_X(P)$ and $Z \subseteq P$ as in Observation 3.3.
    $X'_t \leftarrow Z$ and $X'_i \leftarrow X_i$ for all $i \neq t$.
    $P' \leftarrow (P \setminus Z) \cup X_t$.
    return (X', P').
end function

Algorithm 1: Update Rule U_1

Case 2: $|P| \geq n$. We now look into the second case, where there are at least n goods in the pool. First, observe that if there exists any good $g \in P$ and an agent $i \in [n]$ such that no agent envies a strict subset of $X_i \cup \{g\}$, then we trivially get a partial EFX allocation $X' = \langle X_1, X_2, \ldots, X_i \cup \{g\}, \ldots, X_n \rangle$ in which all agents are at least as happy as they were in X, i.e., $v_i(X'_i) \geq v_i(X_i)$ for all $i \in [n]$[2]. However, with each such update,

[2] X' may not Pareto-dominate X as $v_i(X_i \cup \{g\})$ may be equal to $v_i(X_i)$.

the number of unallocated goods decreases. Thus, we may have at most m consecutive such updates. *From here on, we assume that for each good $g \in P$ and and for each agent $i \in [n]$, there is an agent $j \neq i$ who likes a strict subset of the bundle $X_i \cup \{g\}$.*

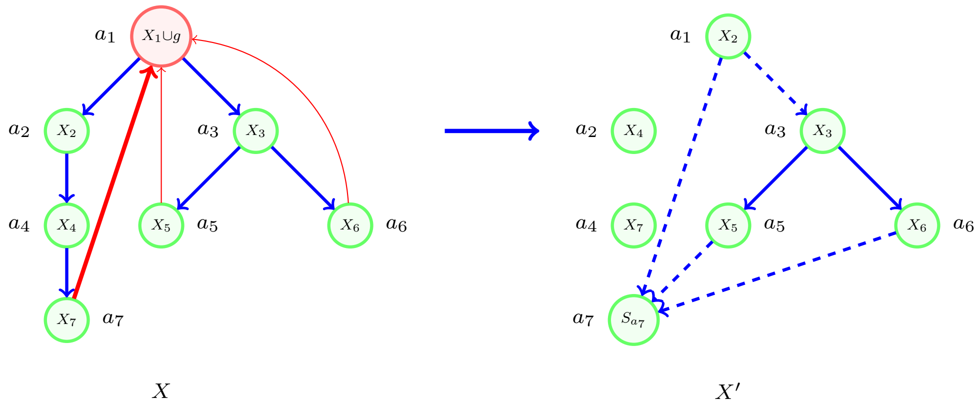


Figure 3.1: Illustration of the update rule. The figure on the left indicates the envy-graph corresponding to the allocation X. Blue edges indicate *weak envy* edges: the blue edge from a_i to a_j signifies that a_i envies a_j, however a_i does not envy any strict subset of a_j's bundle. The red edges indicate *strong envy* edges: a red edge from a_i to a_j indicates that a_i also envies some strict subset of a_j's bundle. Initially the envy-graph has only blue edges with a_1 as the only source. Agents a_7, a_5 and a_6 *strongly envy* a_1 when we give a_1 the good g and a_7 is a most envious agent (hence the thicker red edge). The figure on the right indicates the envy-graph of the allocation X' that we obtain after applying the update rule. We shift the bundles along the path $a_1 \to a_2 \to a_4 \to a_7$ and give a_7, the subset Z_{a_7} that she still envies and no other agent envies strongly. All the agents along the path (a_1, a_2, a_4, a_7) strictly improve their valuations. All the red edges disappear as a_7 is a most envious agent by definition, no other agent envies a strict subset of Z_{a_7} (there may or may not be blue edges directed towards a_7, depending on whether a_1, a_5 or a_6 were also most envious agents).

To complete the description of the update rule, we recall the notion of an *envy-graph* defined in Chapter 2. Given an allocation X, an envy-graph E_X has vertices corresponding to the agents and there is an edge from agent i to agent j in E_X if and only if i envies j, i.e., $v_i(X_i) < v_i(X_j)$. We can assume without loss of generality that E_X is acyclic.

Fact 3.4 ([72])**.** *Let $X = \langle X_1, X_2, \ldots, X_n \rangle$ be an EFX allocation such that E_X is cyclic. Then there exists another EFX allocation $Y = \langle Y_1, Y_2, \ldots, Y_n \rangle$, where for all $i \in [n]$, $Y_i = X_j$ for some $j \in [n]$, such that E_Y is acyclic and Y Pareto dominates X.*[3]

Our update rule relies on the configuration of E_X. For ease of explanation, we start by describing the update rule in the simple scenario where E_X has a single source.

Warmup: E_X has only one source. For ease of explanation, we first discuss a simple case: when the envy-graph E_X has only a single source, namely s. We pick a good g arbitrarily

[3]The allocation Y is obtained by exchanging the bundles along the cycles in E_X. Thus, the bundles remain the same, but the owners of the bundles may change.

from P. Recall that we are in the case where there is some agent $i \neq s$ that envies a strict subset of $X_s \cup \{g\}$. Therefore, the set $A_X(X_s \cup \{g\}) \neq \emptyset$. Let $t \in A_X(X_s \cup \{g\})$ and let $Z_t \subseteq X_s \cup \{g\}$ such that $v_t(Z_t) > v_t(X_t)$ and no other agent envies a strict subset of Z_t (we can determine t and Z_t in polynomial-time by Observation 3.3). Since E_X has only a single source, t is reachable from s by a path $s \to i_1 \to \cdots \to i_{k-1} \to t$ in E_X. We do a leftwise shift of bundles along this path: so s gets i_1's bundle, and for $1 \leq r \leq k-1$: i_r gets i_{r+1}'s bundle (where $i_k = t$), and finally t gets Z_t. The goods in $X_s \cup \{g\} \setminus Z_t$ are thrown back into the pool of unallocated goods. (See Figure 3.1).

We now show that X' is also EFX and Pareto dominates X. To this end, first observe that the agents along the path from s to t have got better bundles than what they had in X, and the rest of the agents retain their previous bundles. Therefore, X' Pareto-dominates X. Now, we show that X' is also EFX. Note that for every agent $i \neq t$, we have $X'_i = X_{i'}$ for some $i' \in [n]$: if i was in the path from s to t, then i gets the bundle of the agent next to it in the path from s to t in E_X, and if i was not in the path, then i retains its previous bundle. Also, we have $v_i(X'_i) \geq v_i(X_i)$ for all $i \in [n]$. Now consider any two agents i and j. We show that $v_i(X'_i) \geq v_i(X'_j \setminus \{h\})$ for all $h \in X'_j$, by distinguishing between the following two cases,

- $j = t$: In this case, we have $v_i(X'_i) \geq v_i(X_i) \geq v_i(Z')$ for all $Z' \subset Z_t = X'_t$ (by the definition of Z_t from Observation 3.3). Thus we have that $v_i(X'_i) \geq v_i(X'_t \setminus \{h\})$ for all $h \in X'_t$.
- $j \neq t$: In this case, we have $v_i(X'_i) \geq v_i(X_i) \geq v_i(X_{j'} \setminus \{h\})$ for all $j' \in [n]$ and all $h \in X_{j'}$ as X was EFX. Since $j \neq t$, we have that $X_j = X_{j'}$ for some $j' \in [n]$. Therefore, we have $v_i(X'_i) \geq v_i(X'_j \setminus \{h\})$ for all $h \in X'_j$.

Therefore X' is EFX and Pareto-dominates X.

E_X has multiple sources. The crucial idea used so far is that given a source s and a good $g \in P$, we get the *strongly envied bundle* $X_s \cup \{g\}$ to a most envious agent without decreasing the valuations of the other agents. When E_X has only a single source s, this is easy as the most envious agent is reachable from the source s. However, this is not true when the envy-graph has multiple sources, and the most envious agent of $X_s \cup \{g\}$ determined by Observation 3.3 is not reachable from s. However, we show that there is a fix to this problem, provided that there are enough goods in the pool. In particular when $|P| \geq n$.

Given a source s in the envy-graph E_X, let $C(s)$ denote all the set of nodes that are reachable from s in the envy-graph. Now, we make a small observation.

Observation 3.5. *In polynomial-time , for some $\ell \geq 1$, we can determine distinct goods $g_0, g_1, \ldots, g_{\ell-1}$ in P, distinct sources $s_0, s_1, \ldots, s_{\ell-1}$ in E_X, distinct agents $t_1, t_2, \ldots, t_\ell$ and sets $Z_i \subseteq X_{s_i} \cup \{g_i\}$ for all $i \in \{0, 1, \ldots, \ell-1\}$ such that $t_i \in C(s_i)$, t_i is a most envious agent of $X_{s_{i-1}} \cup \{g_{i-1}\}$ for $i \in \{0, \ldots, \ell-1\}$, (indices are modulo ℓ) and $v_{t_i}(Z_{i-1}) > v_{t_i}(X_{t_i})$ and no agent envies a strict subset of Z_{i-1}.*

Proof. By assumption, for every source s in E_X and every good $g \in P$, there exists an agent $j \in [n]$ and some subset $S' \subseteq X_s \cup \{g\}$ such that we have $v_j(S') > v_j(X_j)$. Therefore $A_X(X_s \cup \{g\})$ is not empty for all sources s in E_X and all $g \in P$. Construct a sequence of triples (s_i, g_i, t_{i+1}), $i \geq 0$ defined as follows: let s_0 be an arbitrary source in E_X and g_0 be an arbitrary good in P. Assume we have defined s_{i-1} and g_{i-1}. Let t_i be a most envious agent of $X_{s_{i-1}} \cup \{g_{i-1}\}$, and $Z_{i-1} \subseteq X_{s_{i-1}} \cup \{g_{i-1}\}$ such that $v_{t_i}(Z_{i-1}) > v_{t_i}(X_{t_i})$ and no agent envies a strict subset of Z_{i-1} (indices are modulo ℓ)–note that this can be determined by the polynomial-time procedure in Observation 3.3. If $t_i \notin C(s_0) \cup \cdots \cup C(s_{i-1})$, let s_i be such that $t_i \in C(s_i)$. Also, let g_i be a good in P distinct from g_0 to g_{i-1}. If $t_i \in C(s_0) \cup \cdots \cup C(s_{i-1})$, then stop the construction of the sequence and let j be the maximum index such that $t_i \in C(s_j)$. Set $\ell = i - j$ and return $s_j, \ldots, s_{i-1}$, $g_j, \ldots, g_{i-1}$, $t_{j+1}, \ldots, t_i$ and $Z_j, \ldots, Z_{i-1}$. See Figure 3.2 for an illustration.

Since $|P| \geq n$, the construction is well defined and we cannot run out of goods. The sources and goods are pairwise distinct by construction. The agents t_1 to t_{i-1} are also distinct by construction. Finally, agent t_i is distinct from any agent t_k for $j < k < i$ as $t_i \in C(s_j)$ and $t_k \notin (C(s_0) \cup \cdots \cup C(s_{k-1}))$ for all $k < i$, implying that $t_k \notin C(s_j)$ for all $j < k < i$. ☐

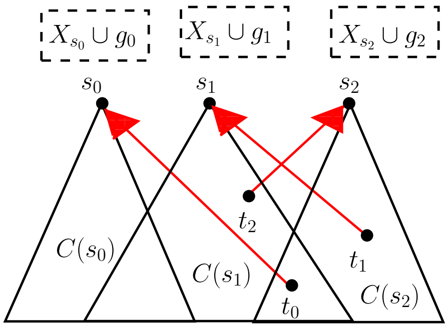


Figure 3.2: Illustration of the proof of Observation 3.5. We have t_i as a most envious agent of $X_{s_{i-1}} \cup \{g_{i-1}\}$. Moreover, $t_i \notin C(s_0) \cup \cdots \cup C(s_{i-1})$ for $i = 1, 2$ and $t_3 \in C(s_0) \cup \cdots \cup C(s_2)$. Note that $j = 1$ is the largest index such that $t_3 \in C(s_j)$. The cycle is defined by s_1, s_2, g_1, g_2, t_2 and t_3.

Let $g_0, g_1, \ldots, g_{\ell-1}$ and $s_0, s_1, \ldots, s_{\ell-1}$, and $t_1, t_2, \ldots, t_\ell$ and $Z_0, Z_1, \ldots, Z_{\ell-1}$ be the goods, sources, most envious agents and valuable subsets respectively that satisfy the conditions in Observation 3.5. Also, for each $i \in \{0, 1, \ldots, \ell-1\}$, let $u_0^i \to u_1^i \to \cdots \to u_{m_i}^i$ be the path of length m_i from $s_i = u_0^i$ to $t_i = u_{m_i}^i$ in $C(s_i)$. We define X' as follows,

$$\begin{aligned} X'_{u_k^i} &= X_{u_{k+1}^i} && \forall k \in \{0, \ldots, m_i - 1\}, \forall i \in \{0, \ldots, \ell - 1\}, \\ X'_{t_i} &= Z_{i-1} && \forall i \in \{1, \ldots, \ell\} \end{aligned}$$

We now show that our allocation X' is EFX and Pareto-dominates X. To this end, we first observe that the valuations of the agents for their bundles have either increased or remained the same (since either the agents are left with their old bundles or assigned bundles that they envied). In particular, the valuations of all the agents in $\bigcup_{i=0}^{\ell-1} \bigcup_{k=0}^{m_i} \{u_k^i\}$ are strictly larger, where the vertices u_k^i are defined above. Thus X' Pareto-dominates X.

It remains to show that the allocation X' is EFX. To this end, let $T = \{t_1, t_2, \ldots, t_\ell\}$. Observe that for any agent $i \notin T$, we have that $X'_r = X_{r'}$ for some $r' \in [n]$: if r was in the path from some s_i to t_i, then r gets the bundle of the agent next to it in the path, and if r was not part of any path from s_i to t_i for all $i \in \{0, 1, \ldots, \ell - 1\}$, then r retains its bundle in X. Also, since none of the agents valuations went down in X', we have that $v_i(X'_i) \geq v_i(X_i)$ for all $i \in [n]$. Now consider any two agents j and k in $[n]$. We show that $v_j(X'_j) \geq v_j(X'_k \setminus \{g\})$ for all $g \in X'_k$, distinguishing between two cases depending on k:

- $k \in T$: Let $k = t_r$ for some $r \in [\ell]$. In this case, we have $v_i(X'_i) \geq v_i(X_i) \geq v_i(Z')$ for all $Z' \subset Z_{r-1} = X'_{t_r}$ (by the definition of Z_{r-1} and t_r). Thus we have $v_i(X'_i) \geq v_i(X'_{t_r} \setminus \{g\})$ for all $g \in X'_{t_r}$.

- $k \notin T$: In this case, we have $v_i(X'_i) \geq v_i(X_i) \geq v_i(X_{j'} \setminus \{g\})$ for all $j' \in [n]$ and all $g \in X_{j'}$ as X was EFX. Since $k \notin T$, we have that $X'_k = X_{k'}$ for some $k' \in [n]$. Therefore, we have $v_i(X'_i) \geq v_i(X_{k'} \setminus \{g\}) = v_i(X'_k \setminus \{g\})$ for all $g \in X'_k$.

Therefore X' is EFX and Pareto-dominates X. We call this entire update procedure U_2 and it is outlined in Algorithm 2

function $U_2(X, P)$
  Precondition: $|P| \geq n$.
  if $\exists i \in [n]$, $\exists g \in P$, such that $\langle X_1, \ldots, X_i \cup \{g\}, \ldots X_n \rangle$ is EFX **then**
    $X' \leftarrow \langle X_1, X_2, \ldots, X_i \cup \{g\}, \ldots X_n \rangle$.
    $P' \leftarrow P \setminus \{g\}$.
  else
    $(s_0, \ldots s_{\ell-1}, g_0, \ldots, g_{\ell-1}, t_1, \ldots, t_\ell, Z_0, \ldots, Z_{\ell-1}) \leftarrow \text{FindCycle}(X, P)$.
    for all $i \in \{0, 1, \ldots, \ell - 1\}$ **do**
      Let $u_0^i \to \cdots \to u_{m_i}^i$ be the path from $s_i = u_0^i$ to $t_i = u_{m_i}^i$.
    end for
    $P' \leftarrow \left(P \setminus \cup_{i=0}^{\ell-1}\{g_i\}\right) \bigcup_{i=0}^{\ell-1} ((X_{s_i} \cup g_i) \setminus Z_i)$.
    $X'_{u_k^i} \leftarrow X_{u_{k+1}^i}$ for all $k \in \{0, \ldots, m_i - 1\}$ and all $i \in \{0, \ldots, \ell - 1\}$.
    $X'_{t_i} \leftarrow Z_{i-1}$ for all $i \in \{1, \ldots, \ell\}$.
    $X'_j \leftarrow X_j$ for all other j.
  end if
  return (X', P').
end function

Algorithm 2: Update Rule U_2

function FINDCYCLE(X, P)
  $i \leftarrow 0$
  $s_i \leftarrow$arbitrary source in E_X and $g_i \leftarrow$arbitrary good in P.
  $t_{i+1} \leftarrow$ arbitrary agent in $A_X(X_{s_i} \cup \{g_i\})$.
  while $t_{i+1} \notin C(s_0) \cup C(s_1) \cup \cdots \cup C(s_i)$ **do**
    $s_{i+1} \leftarrow s$ where $t_{i+1} \in C(s)$.
    $g_{i+1} \leftarrow$ arbitrary good in $P \setminus \{g_0, g_1, \dots, g_i\}$.
    Find $t_{i+2} \in A_X(X_{s_{i+1}} \cup \{g_{i+1}\})$ and Z_{i+1} as in Observation 3.3.
    $i \leftarrow i + 1$.
  end while
  $j \leftarrow$ largest index less than $i + 1$ such that $t_{i+1} \in C(s_j)$.
  return $(s_j, \dots, s_i, g_j, \dots, g_i, t_{j+1}, \dots, t_{i+1}, Z_j, \dots, Z_i)$.
end function

Algorithm 3: The subroutine FindCycle used by update rule U_2 (Algorithm 2).

Thus, we showed that given any partial EFX allocation X and a set of unallocated goods P, if $v_i(X_i) < v_i(P)$ for some agent $i \in [n]$ or if $|P| \geq n$, then we can constructively find another partial EFX allocation X' that Pareto-dominates X. Since the valuations functions can be assumed to be integral without loss of generality, after finite number of updates, there should be no agent that envies the pool, and the number of goods in the pool should be at most the number of agents. This completes the proof of Theorem 3.1. The overall algorithm is outlined in Algorithm 4.

$X_i \leftarrow \emptyset$ for all $i \in [n]$ and $P \leftarrow M$.
while $v_i(X_i) < v_i(P)$ for some agent i or $|P| \geq n$ **do**
  if $v_i(X_i) < v_i(P)$ for some agent i **then**
    $(X, P) \leftarrow U_1(X, P)$.
  else
    $(X, P) \leftarrow U_2(X, P)$.
  end if
  Decyclify E_X.
end while
return (X, P).

Algorithm 4: Algorithm to determine EFX with bounded charity

Running time of Algorithm 4. By Observation 3.3, in $\mathcal{O}(n \cdot |S|^2)$, for any $S \subseteq M$ and an allocation X, one can find an agent $t \in A_X(S)$ and $Z \subseteq S$ s.t. $v_t(Z_t) > v_t(X_t)$ no agent envies a strict subset of Z. Thus every step in the update rues U_1 (Algorithm 1) and U_2 (Algorithm 2) can be implemented in poly(n,m) time. Thus each iteration in Algorithm 4 can be implemented in poly(n, m) time. Since the valuations are integral, and in each iteration the valuation of at least one agent increases by unity, the total number of iterations are $n \cdot V$, where $V = max_{i \in [n]} v_i(M)$. Thus, Algorithm 4 runs in time poly(n, m, V) time. The running time is unfortunately, pseudo-polynomial and cannot be improved, since the increase in individual valuations of the agents when we perform the update rules could be very small. However, if we just wanted an "almost" EFX allocation with bounded charity, i.e., for every pair of agents i and j, we are happy to ensure that $(1+\varepsilon)\cdot v_i(X_i) \geq v_i(X_j)$, and $(1+\varepsilon)\cdot v_i(X_i) \geq v_i(P)$ for every i, and $|P| < n$, then we have an algorithm that runs in $\text{poly}(n, m, \frac{1}{\varepsilon}, \log V)$ time and finds the desired allocation. We formalize this statement in the following lemma:

Theorem 3.6. *For normalized and monotone valuations, given any $\varepsilon > 0$, using $\text{poly}(n, m, \frac{1}{\varepsilon}, \log V)$ value queries, we can find an allocation $X = \langle X_1, X_2, \ldots, X_n\rangle$ and a pool of unallocated goods P such that*

- *for any pair of agents i and j, we have $(1+\varepsilon)\cdot v_i(X_i) \geq v_i(X_j \setminus \{g\})$ for all $g \in X_j$,*
- *for any agent i, we have $(1+\varepsilon)\cdot v_i(X_i) \geq v_i(P)$, and*
- *$|P| < n$.*[4]

The proof follows in a straightforward manner from the proof of Theorem 3.1. The key idea is that the "almost" EFX property is violated if and only if $(1+\varepsilon)\cdot v_i(X_i) < v_i(X_j \setminus g)$ for some $i, j \in [n]$ or $(1+\varepsilon)\cdot v_i(X_i) < v_i(P)$ for some $i \in [n]$. So every time we apply update rules U_1 or U_2, there is a multiplicative improvement (by a factor of $1+\varepsilon$) in the valuation of some agents. Since these valuations are upper-bounded by V we get a bound of $\text{poly}(n, m, \log_{(1+\varepsilon)} V) = \text{poly}(n, m, \frac{1}{\varepsilon}, \log V)$ on the number of iterations and consequently also on the number of value queries.

Robustness of Algorithm 3.1. We make a small remark about the robustness of our main algorithm. Observe crucially, that the algorithm can start with any partial EFX allocation Y, and return a final EFX allocation X with bounded charity, such that $v_i(X_i) \geq v_i(Y_i)$ for all $i \in [n]$. Therefore, if our initial EFX allocation has good welfare (Nash welfare or social welfare), the welfare of the final EFX allocation can only be more. This small observation will be very crucially used when we want to give efficiency Guarantees for our EFX allocations in Chapter 4. On a high-level, all it takes to get some efficiency guarantees is to start off with a clever initial EFX allocation. In Chapter 4, we show that for additive valuations, we can determine $(1-\varepsilon)$-EFX allocation with bounded charity and a 2-approximation to the maximum Nash welfare and when agents have subadditive valuations we can determine a $(1-\varepsilon)$-EFX allocation with bounded charity and an $\mathcal{O}(n)$ approximation of Nash welfare (in fact, on a much larger class of efficiency functions, namely, the generalized p-mean welfare function)[5] in polynomial-time . Barman et al. [15] showed that it requires an exponential number of value queries to provide any sublinear approximation to Nash welfare under subadditive valuations. Therefore, by choosing the initial partial EFX allocation carefully, in polynomial-time , our algorithm yields an $(1-\varepsilon)$-EFX allocation with bounded charity that also achieves the best possible approximation of the Nash welfare in polynomial-time .

[4]Observe that $(1+\varepsilon)\cdot v_i(X_i) \geq v_i(X_j \setminus \{g\})$ for all $g \in X_j$, implies that $v_i(X_i) \geq (1-\varepsilon)\cdot v_i(X_j \setminus \{g\})$ for all $g \in X_j$ and $(1+\varepsilon)\cdot v_i(X_i) \geq v_i(P)$ implies that $v_i(X_i) \geq (1-\varepsilon)\cdot v_i(P)$ (as $\frac{1}{1+\varepsilon} \geq 1-\varepsilon$). Therefore, Theorem 3.6 implies that we can determine a $(1-\varepsilon)$-EFX allocation with bounded charity with $\text{poly}(n, m, \frac{1}{\varepsilon}, \log V)$ value queries.

[5]Independently, Barman et al. [15] also showed how to achieve an $\mathcal{O}(n)$ approximation for the generalized p-mean welfare.

3.2 Additive Valuations: Implications for Other Notions of Fairness

The most well-understood class of valuation functions is the set of additive valuations. We consider the case when all agents have additive valuations and show that our allocation or very minor variants of our allocation can guarantee several other notions of fairness. In this section, we discuss the implications of our result (Theorem 3.1) on coming up with allocations that satisfy some approximation of other fairness notions, namely Maximin Share (MMS) and Groupwise Maximin Share (GMMS).

3.2.1 Number of Unallocated Goods and MMS Guarantee.

Another interesting and well-studied notion of fairness is *maximin share*, a popular relaxation of *Proportionality* in discrete fair division. Recall the definition of MMS from Chapter 2. Given a set $[n]$ of agents and a set M of m indivisible goods the maximin share of an agent (say, i) $MMS_i(n, M)$ is defined as follows: (here $\mathcal{X}$ is the set of all complete allocations)

$$MMS_i(n, M) = \max_{\langle X_1,\dots,X_n\rangle \in \mathcal{X}} \min_{j\in[n]} v_i(X_j).$$

The goal is to determine an allocation $\langle X_1, X_2, \dots, X_n\rangle$ of M such that for every i, we have $v_i(X_i) \geq MMS_i(n, M)$. This question was first posed by Budish [26]. Procaccia and Wang [88] showed that such an allocation need not exist, even in the restricted setting of only three agents! Thereafter, *approximate-MMS* allocations were studied [88, 61, 64, 62] and there are polynomial-time algorithms to find allocations where for all i, agent i gets a bundle of value at least $\alpha \cdot MMS_i(n, M)$; the current best guarantee for α is $3/4 - \varepsilon$ by Ghodsi et al. [64] (for any $\varepsilon > 0$) and this was very recently improved to $3/4$ by Garg and Taki [62]. Amanatidis et al. [4] showed that any complete EFX allocation is also a $\frac{4}{7}$-MMS allocation.

We show that our allocation promises better MMS guarantees when the number of unallocated goods is large: Recall that our algorithm continues till $|P|$ is smaller than the number of sources in the envy-graph E_X and also recall that the sources are *unenvied* agents. In particular, if $|P| = n - 1$, then the number of sources in E_X is n; so no agent envies another. That is, for each i, we have $v_i(X_i) \geq v_i(X_j)$ for all $j \in [n]$. Moreover, $v_i(X_i) \geq v_i(P)$. So we have

$$\begin{aligned} v_i(X_i) &\geq \frac{v_i(M)}{n+1} \\ &\geq \left(1 + \frac{1}{n}\right)^{-1} \cdot \frac{v_i(M)}{n} \\ &\geq \left(1 + \frac{1}{n}\right)^{-1} \cdot MMS_i(n, M), \end{aligned}$$

where for every agent i, the inequality $MMS_i(n, M) \leq v_i(M)/n$ holds for additive valuations. Thus larger the size of P, higher is the number of unenvied agents, and since no agent envies the pool, the allocation is "closer" to a proportional allocation[6], which gives the desired MMS guarantees. Formally, we show the following theorem.

Theorem 3.7. *Given a set of $[n]$ agents with additive valuations and a set of M of indivisible goods, there exists a partial EFX allocation $X = \langle X_1, X_2, \ldots, X_n \rangle$ and a set of unallocated goods P, such that (X, P) satisfies the guarantees in Theorem 3.1 and $v_i(X_i) \geq MMS_i(n, M)/(2 - \frac{|P|}{n})$.*

Hence, larger the number of unallocated goods, better are the guarantees that we get on MMS. The extreme values are $|P| = 0$ and $|P| = n - 1$. When $|P| = 0$, we have a complete EFX allocation and when $|P| = n - 1$, we have an EFX allocation that is an *almost-MMS* allocation: $v_i(X_i) \geq (1 - 1/n) \cdot MMS_i(n, M)$ for all i. *This, in fact shows that having large number of unallocated goods is not necessarily a weakness of the fair allocation in Theorem 3.1, as this gives good guarantees in other fairness notions.*

We now show the proof of Theorem 3.7. The following proposition from [61] will be useful. It states that if we exclude any set of agents and at most the same number of any goods from N and M, respectively, the maximin share of any remaining agent can only increase.

Proposition 3.8 (Garg and Taki [61])**.** *Let N be a set of n agents with additive valuations and M be a set of m goods. If $N' \subseteq N$ and $M' \subseteq M$ are such that $|N \setminus N'| \geq |M \setminus M'|$, then for any agent $i \in N'$, we have $MMS_i(n', M') \geq MMS_i(n, M)$ where $n' = |N'|$.*

Proof of Theorem 3.7. Let (X, P) be the allocation guaranteed by Theorem 3.1.

Let k be the number of unallocated goods ($k = |P|$). We fix some agent i and let $N' \subseteq N$ be the set of agents consisting of all sources in E_X, agent i and all other agents j with $|X_j| \geq 2$. Let M' be the set of goods allocated to the agents in N'. Observe that every agent in $N \setminus N'$ is allocated at most one good and so $|N \setminus N'| \geq |M \setminus (M' \cup P)|$. By Proposition 3.8, it holds that $MMS_i(n', M' \cup P) \geq MMS_i(n, M)$ where $n' = |N'|$. Thus, it suffices to show that $v_i(X_i) \geq MMS_i(n', M' \cup P)/\left(2 - \frac{k}{n}\right)$.

Consider any agent $j \in N'$ with $|X_j| \geq 2$. Because X is EFX, it holds that $v_i(X_i) \geq v_i(X_j \setminus \{g\})$ for all $g \in X_j$. Since the valuations are additive, we have

$$v_i(X_i) \ \geq \ \left(1 - \frac{1}{|X_j|}\right) \cdot v_i(X_j) \ \geq \ \frac{1}{2} \cdot v_i(X_j).$$

We know the following inequalities hold:

$$\begin{aligned} v_i(X_i) &\geq v_i(P), &(3.1)\\ v_i(X_i) &\geq v_i(X_j) \text{ for all } j \text{ that were sources in } G_X, &(3.2)\\ 2v_i(X_i) &\geq v_i(X_j) \text{ for all other } j \in N'. &(3.3) \end{aligned}$$

Recall that the number of sources is at least $|P| + 1 = k + 1$. Summing up all inequalities in (7.16)-(3.3) and using the fact that v_i is additive, we have

$$(2(n' - (k + 1)) + k + 2) \cdot v_i(X_i) \ \geq \ v_i(M' \cup P).$$

[6] each agent i gets a bundle that she values $v_i(M)/n$

Hence we have

$$\begin{aligned}
v_i(X_i) &\geq \frac{v_i(M' \cup P)}{2n' - k} \\
&\geq \frac{v_i(M' \cup P)}{n'} \cdot \frac{n'}{2n' - k} \\
&\geq MMS_i(n', M' \cup P) \cdot \frac{n'}{2n' - k} \quad \text{(since } v_i \text{ is additive)} \\
&= MMS_i(n', M' \cup P) / \left(2 - \tfrac{k}{n'}\right) \\
&\geq MMS_i(n', M' \cup P) / \left(2 - \tfrac{k}{n}\right) \quad \text{(since } n' \leq n\text{)}. \quad \square
\end{aligned}$$

3.2.2 An Improved Bound for Approximate-GMMS

Barman et al. [16] recently introduced a notion of fairness called *groupwise maximin share* (GMMS) which is stronger than MMS and EFX. An allocation is said to be GMMS, if the MMS condition is satisfied for *every subgroup* of agents and the union of the sets of goods allocated to them. Formally,

Definition 3.9. *Given a set N of n agents and a set M of m goods, an allocation $X = \langle X_1, X_2, \ldots, X_n \rangle$ is α-GMMS if for every $N' \subseteq N$ and all $i \in N'$, we have:*

$$v_i(X_i) \;\geq\; \alpha \cdot MMS_i(n', \bigcup_{j \in N'} X_j) \text{ where } n' = |N'|.$$

Observe that every GMMS, i.e. $\alpha = 1$, allocation is an MMS allocation. Additionally, every GMMS allocation, is also a complete EFX allocation [16]. It is known [16] that GMMS strictly generalizes MMS. In particular, it was shown in [16] that GMMS allocations rule out some very unsatisfactory allocations that have MMS guarantees. For example, consider an instance with n agents with additive valuations and a set M of $n-1$ goods and every agent has a valuation of one for each good. Since the number of goods is less than the number of agents, we have $MMS_i(n, M) = 0$ for every agent i. So any allocation has MMS guarantees. It is not hard to see that the only allocation with a GMMS guarantee is one where $n-1$ agents get one good each and one agent is left without any goods. See Subsection 2.1 in [16] for more discussion.

Naturally, it is a harder problem to approximate GMMS than MMS. While $\frac{3}{4}$-MMS allocations always exist, the largest α for which α-GMMS allocations are known to exist is $\frac{1}{2}$ [16]. We now describe how to modify our allocation so that the resulting allocation is $\frac{4}{7}$-GMMS.

Let $X = \langle X_1, \ldots, X_n \rangle$ be the allocation and P be the pool of unallocated goods that satisfy the conditions of Theorem 3.1. Without loss of generality, assume that agent 1 is a source in the envy-graph E_X. Define the complete allocation $Y = \langle Y_1, \ldots, Y_n \rangle$ as follows:

$$\begin{aligned}
Y_1 &= X_1 \cup P, \text{ and} \\
Y_i &= X_i \text{ for all } i \neq 1.
\end{aligned}$$

We show in Theorem 3.10 that Y is our desired allocation. The proof of Theorem 3.10 is similar to [4, Proposition 3.4]. We also remark that one can use the proof of [4, Proposition 3.4] to show that any EFX allocation is a 4/7-GMMS. However note that Y is not necessarily an EFX allocation. But it has sufficiently nice properties so that we can still show that it is 4/7-GMMS. For the sake of convenience, we will refer to goods in the proof of Theorem 3.10 as *items*.

Theorem 3.10. *Given a set N of n agents with additive valuations and a set M of m items, the allocation Y defined above satisfies is $\frac{4}{7}$-GMMS.*

Proof. We need to show that for every $\widetilde{N} \subseteq N$ and all $i \in \widetilde{N}$, we have $v_i(Y_i) \geq \frac{4}{7} MMS_i(\widetilde{n}, \widetilde{M})$ where $\widetilde{n} = |\widetilde{N}|$ and $\widetilde{M} = \bigcup_{j \in \widetilde{N}} Y_j$.

Fix some $i \in \widetilde{N}$. Define N' as the subset of $\widetilde{N}$ that contains i and all agents that have been allocated at least two items in Y, i.e., $j \in N'$ if and only if $j = i$ or $|Y_j| \geq 2$. Let $M' = \bigcup_{j \in N'} Y_j$.

Note that Y allocates all items of $\widetilde{M} \setminus M'$ to agents in $\widetilde{N} \setminus N'$. Since every agent in $\widetilde{N} \setminus N'$ has been allocated at most one item, we have $|\widetilde{N} \setminus N'| \geq |\widetilde{M} \setminus M'|$. Proposition 3.8 tells us that $MMS_i(n', M') \geq MMS_i(\widetilde{n}, \widetilde{M})$ where $n' = |N'|$. Thus, it suffices to show $v_i(Y_i) \geq 4/7 \cdot MMS_i(n', M')$.

We now classify the items in M' as good or bad. An item is *good* if it is contained in a Y_j of cardinality at least three or is contained in Y_1 or Y_i. All other items are *bad*, i.e., a bad item is contained in a bundle of cardinality two different from Y_1 and Y_i. A bundle in Y containing good items is good.

We will next reduce the problem further. Let x be the number of bad items in M'. Since the bad items are contained in bundles of cardinality two, the good items in M' come from $n' - x/2$ good bundles of Y. As long as $x > n'$, we will apply a reduction step. Each reduction step will reduce the number of bad items in M' by two, the number of agents by one, will not decrease the MMS_i-value, and will leave the quantity $n' - x/2$ and set of good items in M' unchanged.

Let $Z = \langle Z_1, Z_2, \dots Z_{n'} \rangle$ be an optimal MMS partition for agent i of the set M' of items. If there are more than n' bad items in M', then there is a set Z_k with at least *two* bad items, say g_1 and g_2. We distribute the items in $Z_k \setminus \{g_1, g_2\}$ arbitrarily among the other sets in Z. So we have a partition of the set $M' \setminus \{g_1, g_2\}$ of items into $n' - 1$ many bundles. The value for agent i of any remaining bundles did not decrease. We set M' to $M' \setminus \{g_1, g_2\}$ and decrement n'. Note that we reduced the number of bad items by two, the number of agents by one, did not decrease $MMS_i(n', M')$ and the set of good items in M' still come from the $n' - x/2$ good bundles in Y. We keep repeating this reduction until M' contains at most n' bad items. At this point, we have a set M' of items and an integer n' with the following properties:

(1) The number x of bad items in M' is at most n'.

(2) $MMS_i(n', M') \geq MMS_i(\widetilde{n}, \widetilde{M})$, and

(3) The set of good items in M' has not changed. They come from $n' - x/2$ good bundles in Y.

We will next relate the value of good and bad items to the value of Y_i.

Lemma 3.11. *We have*

(a) For any bad item g, $v_i(g) \leq v_i(Y_i)$.

(b) $v_i(Y_1) \leq 2v_i(Y_i)$.

(c) If $j \neq 1$ and Y_j is a good bundle then $v_i(Y_j) \leq 3/2 \cdot v_i(Y_i)$.

Proof. First observe that by the construction of the complete allocation Y, we have $v_i(Y_i) \geq v_i(X_i)$ for all $i \in [n]$. Now, we prove each of the statements in the lemma.

(a) Since g is a bad item, it must belong to a bundle that has exactly two goods. Let $Y_j = \{g, g'\}$ be the bundle containing g. Since $j \neq 1$ (by definition of a bad item), we have $v_i(Y_i) \geq v_i(X_i) \geq v_i(X_j \setminus \{g'\}) = v_i(Y_j \setminus \{g'\}) = v_i(g)$.

(b) Since agent 1 is a source, $v_i(X_i) \geq v_i(X_1)$. By Theorem 3.1, $v_i(X_i) \geq v_i(P)$. Therefore, $v_i(Y_1) = v_i(X_1 \cup P) = v_i(X_1) + v_i(P) \leq v_i(X_i) + v_i(X_i) = 2v_i(X_i) = 2v_i(Y_i)$.

(c) Let $g \in Y_j$ be such that $v_i(\{g\})$ is minimal. Then $v_i(Y_j \setminus \{g\}) \leq v_i(Y_i)$ and $v_i(\{g\}) \leq v_i(Y_j)/|Y_j|$. Thus,

$$\begin{aligned} v_i(Y_i) &\geq \left(1 - \frac{1}{|Y_j|}\right) \cdot v_i(Y_j) \\ &\geq \left(1 - \frac{1}{3}\right) \cdot v_i(Y_j) \\ &= \frac{2}{3} \cdot v_i(Y_j). \end{aligned}$$

□

Now we are ready to show the bound on GMMS. We have x bad items in M'. The good items in M' come from $n' - x/2$ good bundles. Also $x \leq n'$. The total value of the good items for agent i is at most

$$\frac{3}{2}(n' - \tfrac{x}{2} - 2) \cdot v_i(Y_i) + v_i(Y_i) + 2v_i(Y_i) = \frac{3}{2}(n' - \tfrac{x}{2}) \cdot v_i(Y_i),$$

since there are at most $n' - x/2 - |\{1, i\}|$ good bundles different from Y_1 and Y_i: each of value at most $(3/2 \cdot v_i(Y_i))$, and Y_1 has value at most $2v_i(Y_i)$. Also, the total value of the bad items for agent i is at most $x \cdot v_i(Y_i)$, since there are x many bad items and each bad item is worth at most $v_i(Y_i)$. Therefore,

$$\begin{aligned} v_i(M') &= v_i(\text{bad items in } M') + v_i(\text{good items in } M') \\ &\leq x \cdot v_i(Y_i) + \frac{3}{2}(n' - \frac{x}{2}) \cdot v_i(Y_i) \\ &= \frac{6n' + x}{4} \cdot v_i(Y_i) \leq \frac{7n'}{4} \cdot v_i(Y_i). \end{aligned}$$

Therefore, we have $v_i(Y_i) \geq (4/7) \cdot (v_i(M')/n') \geq (4/7) \cdot MMS_i(n', M')$ □

This concludes this chapter. We present a graphical view of our main results and techniques so far, that will help us to build more results in the upcoming chapters and also enable the reader to have a global picture of the thesis. See Figure 3.3.

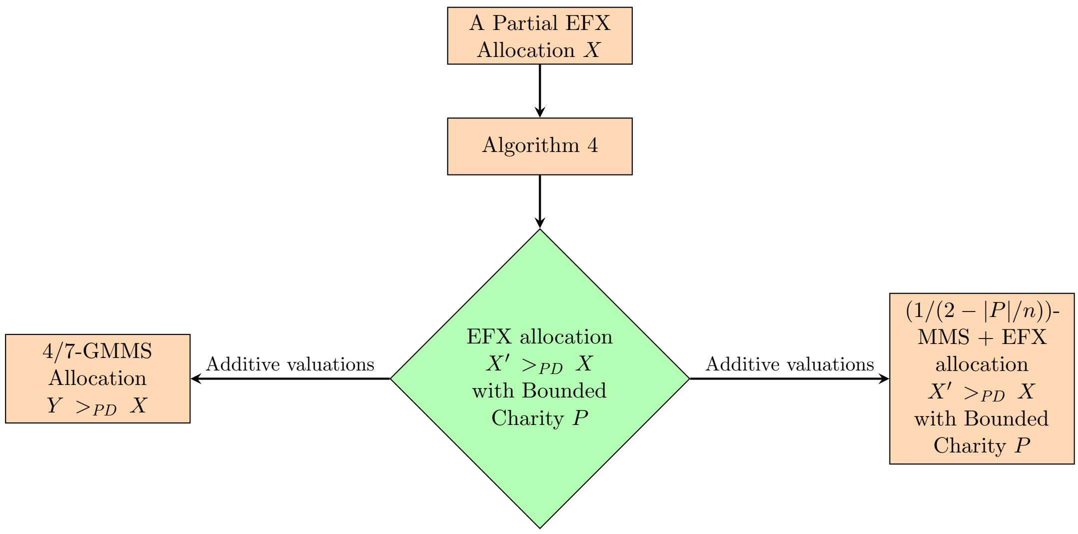


Figure 3.3: We start with any partial EFX allocation X (trivially one could start with an allocation where every agent receives no good). Algorithm 4 outputs an EFX allocation X' with some bounded charity P, where $X' >_{PD} X$ (X' Pareto-dominates X). For additive valuations we are able to give more fairness guarantees in terms of MMS and GMMS, but each time our final allocation Pareto-dominates X. On an intuitive level, all welfare guarantees of the first partial EFX allocation is maintained in all of our final allocations, and we will see how by choosing a clever partial EFX allocation at the beginning, we get efficiency guarantees to all of our final allocations in Chapter 4.

CHAPTER 4

Efficient EFX Allocations

In this chapter, we focus on finding fair and *efficient* allocations. As mentioned in Chapter 1, efficiency is a measure of the overall welfare that an allocation achieves. Even when the underlying notion of fairness is EFX, there maybe some EFX allocations that are highly unsatisfactory: consider a scenario with two agents a_1 and a_2 and two goods g_1 and g_2. We define $v_i(\{g_j\}) = 1$ if $i = j$ and $v_i(\{g_j\}) = 0$ if $i \neq j$. The allocation $a_1 \leftarrow \{g_2\}$ and $a_2 \leftarrow \{g_1\}$ is an EFX allocation, as following the removal of any single good, each agents bundle is empty. However, there is clearly, a more satisfactory allocation, such as $a_1 \leftarrow \{g_1\}$ and $a_2 \leftarrow \{g_2\}$. Thus, in this chapter, we aim to find EFX allocations with bounded charity that are also efficient.

As mentioned in Chapter 1, there are several measures of economic efficiency - one of the common measures being Pareto-optimality. However, there are trivial instances where there exists no EFX allocation that is Pareto-optimal [84]: for instance two agents a_1 and a_2 and three goods g_1, g_2 and g_3. We have $v_1(g_2) = v_2(g_2) = 2$ and $v_1(g_1) = v_2(g_3) = 1.9$ and $v_1(g_3) = v_2(g_1) = 0$.

However, an alternative and a *non-binary* measure of efficiency is that of Nash welfare. Recall that the Nash welfare of an allocation is defined as the geometric mean of the valuations $NW(X) = \Big(\prod_{i \in [n]} v_i(X_i)\Big)^{1/n}$. As mentioned in Chapter 1, an allocation that has highest Nash welfare is also Pareto-optimal. When agents have additive valuations, an allocation with highest Nash welfare is also envy-free up to one good (EF1). Thus, a natural question is can we find EFX allocations with bounded charity that have high Nash welfare. We answer this question in two scenarios: when agents have additive valuations and when agents have subadditive valuations. The main result of this chapter is captured by the following theorem,

Theorem 4.1. *Given an instance comprising of a set of agents $[n]$, a set of goods M, let X^* be the allocation with highest Nash welfare. Then, in polynomial-time , we can determine a $(1-\varepsilon)$-EFX allocation X with bounded charity P[1], such that*

- $NW(X) \geq (1/2.89) \cdot NW(X^*)$, *when agents have additive valuations, and*
- $NW(X) \geq ((1-\varepsilon)/4(n+1)) \cdot NW(X^*)$, *when agents have subadditive valuations.*

Discussion on Theorem 4.1. Apart from the main takeaway that there exists EFX allocations with bounded charity and high Nash welfare, we highlight certain other significance of Theorem 4.1. In particular, this result makes substantial progress in the studies dedicated to approximating Nash welfare for valuations more general than additive valuations[2] in polynomial-time . Although there exists a polynomial-time algorithm that

[1] Allocation X is $(1-\varepsilon)$-EFX, $|P| < n$, and for all $i \in [n]$, we have $v_i(X_i) \geq (1-\varepsilon) \cdot v_i(P)$.

[2] More details on Nash welfare approximation for additive valuations can be found in Chapter 1

achieves a 1.445 approximation on Nash welfare when agents have budget-additive SPLC valuations [30], the best approximation that can be achieved in polynomial-time when agents have subadditive valuations, prior to this work is $\mathcal{O}(m)$, where m is the number of goods. In fact, our algorithm also improves the best polynomial-time approximation of $\mathcal{O}(n\log(n))$ for Nash welfare when agents have submodular valuations [59] as well.

Our technique is also fairly general and our approximation guaranteed hold for a much broader class of welfare functions such as the *generalized p-mean welfare* $GM_p(X) = \Big((1/n)\cdot\sum_{i\in[n]} v_i(X_i)^p\Big)^{1/p}$. Maximizing this welfare function captures a wide range of fairness and efficiency measures that have been used frequently in the literature: maximizing Nash welfare for $p=0$, max-min welfare (also known as the egalitarian welfare) for $p=-\infty$ and maximizing social welfare for $p=1$. Barman and Sundaram [19] also mention that,

> "*generalized means with* $p \in (-\infty, 1]$ *exactly constitute the family of welfare functions that satisfy the Pigou-Dalton transfer principle and a few other key axioms.*"

In the same paper, they show that when agents have identical valuations, there is an algorithm that provides an $\mathcal{O}(1)$ factor approximation to the p-mean welfare. In this chapter, we show that we can determine an EFX allocation X with bounded charity P such that $GM_p(X) \geq 1/(5n)\cdot GM_p(X^*)$, where X^* is the allocation that maximizes $GM_p(X)$. Our approximation also asymptotically matches the current best approximation ratio for special cases like $p=-\infty$ [69], while also retaining the remarkable fairness properties of EFX with bounded charity.

We now present the proof of Theorem 4.1. We consider the cases when agents have additive valuations and when agents have subadditive valuations separately. At a high level, our procedures for both cases look similar: carefully select a partial EFX allocation with high welfare and then run Algorithm 4 (in Chapter 3) on the goods that remain.

4.1 Additive Valuations

The guarantee for Nash welfare when agents have additive valuations, follows almost immediately from the seminal work of Caragiannis et al [27]. They show that given an allocation X^*, that is a ρ-approximation of the maximum Nash welfare, in polynomial-time , one can determine a partial EFX allocation $X = \langle X_1, X_2, \ldots, X_n\rangle$, such that for all $i \in [n]$, we have $X_i \subseteq X_i^*$ and $v_i(X_i) \geq (1/2\rho)\cdot v_i(X_i^*)$. Observe that $NW(X) \geq (1/2\rho)\cdot NW(X^*)$. Recall that Algorithm 3.1 can start with any partial EFX allocation and output a final EFX allocation with bounded charity that Pareto-dominates the initial partial EFX allocation. Thus, if we start Algorithm 3.1 with allocation X, then the for the final allocation X', we have $v_i(X') \geq v_i(X_i) \geq (1/2\rho)\cdot v_i(X_i^*)$ for all $i\in[n]$. Thus $NW(X') \geq (1/2\rho)\cdot NW(X^*)$. Note that this implies that *there exists EFX allocations with bounded charity that achieve a (1/2) approximation of the Nash welfare (by setting ρ to* 1*)*, when agents have additve valuations (setting ρ to 1). Unfortunately, determining an allocation with highest Nash welfare is APX-hard and Algorithm 3.1 is pseudo-polynomial. As a fix, we use the algorithm in Barman et al. [18] that determines an allocation with $e^{\frac{1}{e}}$-approximation of the optimum Nash welfare in polynomial-time , and Algorithm 3.1

to determine a $(1-\varepsilon)$-EFX allocation (this runs in polynomial-time). With this, we get an algorithm that runs in polynomial-time , and returns a $(1-\varepsilon)$-EFX allocation with bounded charity that achieves $1/(2e^{\frac{1}{e}}) = (1/2.89)$-approximation of the optimum Nash welfare.

4.2 Subadditive Valuations

One of the primary motivation of our techniques used here, is to show that certain fair allocations can give us reasonably efficient allocations. While, an arbitrary EFX allocation does not give us any guarantee on the generalized p-mean welfare, even in the context of additive valuations, we outline that certain EFX allocations with bounded charity can help us get good approximations to a broad class of welfare measures like the generalized p-mean welfare.

We first give an intuitive overview of the Algorithm: Let us consider the scenario that a given instance admits an envy-free allocation, i.e., there is a partition of the goods into n bundles $X_1, X_2, \ldots, X_n$ such that for all pairs of agents i and j we have $v_i(X_i) \geq v_i(X_j)$. In that case, for each agent i we have

$$\begin{aligned}
n \cdot v_i(X_i) &\geq \sum_{j \in [n]} v_i(X_j) \\
&\geq v_i(\cup_{j \in [n]} X_j) && \text{(by subadditivity)} \\
&= v_i(M)
\end{aligned}$$

This implies that $v_i(X_i) \geq (1/n) \cdot v_i(M)$. Since in any optimal allocation no agent can get a valuation more than $v_i(M)$, we conclude that each agent has a bundle worth $1/n$ times her bundle in the p-mean welfare maximizing allocation. This would immediately give us an n-approximation for generalized p-mean welfare. However, most instances may not admit an envy-free allocation. Naturally, we then look into the close relaxation of envy-freeness that is known to exist in the context of indivisible goods, say EFX with bounded charity [3]. So let us consider the EFX allocation X with bounded charity P: Here we can partition the given instance into $n+1$ bundles $X_1, X_2, \ldots, X_n, P$ such that for all pairs of agents i and j we have $v_i(X_i) \geq v_i(X_j \setminus \{g\})$ for all $g \in X_j$ and for all agents $i \in [n]$, we have $v_i(X_i) \geq v_i(P)$. Let us first look into all the bundles X_j that are not singleton, i.e., $|X_j| \geq 2$: We have that $v_i(X_i) \geq v_i(X_j \setminus \{g\})$ for *all* $g \in X_j$, implying that $v_i(X_i) \geq (1/2) \cdot max\Big(v_i(X_j \setminus \{g\}), v_i(\{g\})\Big)$ (as $|X_j| \geq 2$). Thus,

[3] In order to get a polynomial-time algorithm, we need to work with the notion of $(1-\varepsilon)$-EFX allocation with bounded charity. However for the sake of clarity here, we stick to EFX allocation with bounded charity.

$$\begin{aligned}(n+1)\cdot v_i(X_i) &\geq \sum_{|X_j|\geq 2} \frac{1}{2}\cdot\Big(v_i(X_j\setminus\{g\})+v_i(\{g\})\Big)+v_i(P) \\ &\geq \frac{1}{2}\cdot\sum_{|X_j|\geq 2} v_i(X_j)+v_i(P) && \text{(by subadditivity)} \\ &\geq \frac{1}{2}\cdot v_i(\cup_{|X_j|\geq 2}X_j\cup P) && \text{(by subadditivity)}\end{aligned} \tag{4.1}$$

Let S be the set of all the goods in singleton bundles in X, i.e., $S=\{g\mid\exists j,\ X_j=\{g\}\}$. Then, from (4.1) we have the guarantee that for every agent $v_i(X_i)\geq(1/2(n+1))\cdot v_i(M\setminus S)$. Therefore, in any EFX allocation with bounded charity, every agent has an $1/2(n+1)$ fraction of her valuation on the goods she receives from $M\setminus S$ in the optimal allocation, i.e., $v_i\big(X_i^*\cap(M\setminus S)\big)$ where $X^*=(X_1^*,\ldots,X_n^*)$ is the allocation that has the highest generalized p-mean welfare. The only problem is how to allocate the goods in the set S appropriately.

The only scenario where an incorrect allocation of the goods in S causes a significant decrease in the p-mean welfare is when there are agents who have a substantially high valuation for some goods in S. However, we could be in a scenario where there are only a few goods in S (say less than $n/3$) which are very valuable to many agents and then we may not be able to give every agent a bundle that she values $1/n$ times the whole set S[4]. Therefore, we need to compare our allocation with the allocation that maximizes the p-mean welfare.

We briefly sketch how we compare the allocation with the allocation that maximizes the p-mean welfare. The good aspect of the situation is that the number of goods in S is small, i.e., $|S|\leq n$. Let H_i denote the set of n goods that are valued by agent i the most, *i.e.*, all goods in H_i are more valuable than any good outside H_i. Now, we find a *single good allocation* (where each agent gets exactly one good) of the *high valued goods*, namely the set $\mathbf{H}=\cup_{i\in[n]}H_i$, optimally to the agents assuming that we can give each agent at least $1/n$ times their valuation for the *low valued goods*, namely the set $M\setminus H_i$, i.e., we find a single good allocation, where every agent i gets exactly one high valued good $h_i\in H_i$, that maximizes $\sum_{i\in[n]}\big(v_i(\{h_i\})+\frac{1}{n}v_i(M\setminus H_i)\big)^p$ (such allocations can be found efficiently by *matching algorithms*). Let us call the current single good allocation Y. Note that Y is trivially EFX as every agent has exactly one good. We then run the Algorithm 4 from Chapter 3, starting with Y as the initial partial EFX allocation. *The intuition being that the low valued goods appear in non-singleton bundles and the high valued goods occur in singleton bundles* in the final EFX allocation X. Since the low valued goods occur in non-singleton bundles, we are indeed able to give every agent $1/n$ times their valuation for the low valued goods. Also, we have allocated the high valued goods correctly (up to a factor of $1/n$, as we computed a single good allocation, while the optimum need not necessarily give every agent exactly one high valued good) as we started out with an optimum allocation of the high valued goods.

[4]A very simple scenario is to divide n goods among n agents with identical additive valuations, where all agents have a valuation of 1 for a single good and $\varepsilon\ll 1/n$ for the rest of the goods. In any division there will be $n-1$ agents who do not get $1/n$ of their valuation on the set of n goods

Now we elaborate our algorithm. Our algorithm primarily has two steps: first, it allocates the high valued goods carefully and then runs Algorithm 4 (Chapter 3) on the remaining set of goods. We outline the details of both these procedures and analyze the guarantees that can be ensured.

Allocating the high valued goods Y. We first formally define the notion of high valued goods for an agent. For each agent i, we order the goods in M as $\left\{g_1^i, g_2^i, \ldots, g_m^i\right\}$ such that $v_i(g_1^i) \geq v_i(g_2^i) \geq \cdots \geq v_i(g_m^i)$. Let $H_i = \left\{g_1^i, g_2^i, \ldots, g_n^i\right\}$. We refer to H_i as the set of high valued goods for agent i. Also for each good g_k^i, and an agent i, we define $rank_i(g_k^i) = k$. Notice that if for any agent i, if $rank_i(g) < rank_i(g')$, then $v_i(g) \geq v_i(g')$.

We now outline how we compute the initial allocation Y. We construct the complete bipartite graph $G = ([n] \cup M, [n] \times M)$ with the weight of the edge from agent i to good g, w_{ig} being

- $n \cdot v_i(\{g\}) + v_i(M \setminus H_i)$ if $p = -\infty$,
- $\log\Big(n \cdot v_i(\{g\}) + v_i(M \setminus H_i)\Big)$ if $p = 0$ and
- $\Big(n \cdot v_i(\{g\}) + v_i(M \setminus H_i)\Big)^p$ otherwise.

Depending on the value of p, we choose an appropriate matching mechanism to determine Y. Y is determined such that $\cup_{i\in[n]}(i, Y_i)$ is

- a max-min matching[5] in G if $p = -\infty$,
- a maximum weight matching in G if $p \geq 0$, and,
- a minimum weight perfect matching in G if $p < 0$ and $p \neq -\infty$.

Let Y be the allocation outputed by the corresponding matching subroutine. Let $\mathbf{Y} = \cup_{i\in[n]} Y_i$. We modify the allocation Y slightly such that $\cup_{i\in[n]}(i, Y_i)$ still remains the optimum matching, but no agent ranks a good outside $\mathbf{Y}$ lower than she ranks the good allocated to her in Y (Y_i), i.e., we wish to determine an allocation Y such that for all agent $i \in [n]$ and all $g \notin \mathbf{Y}$, we have $rank_i(Y_i) < rank_i(g)$. To achieve this, as long as there is an agent $i \in [n]$ and a good $g \notin \mathbf{Y}$ such that $rank_i(g) < rank_i(Y_i)$ we set $Y_i \leftarrow \{g\}$. Note that such an operation does not worsen the objective function of the matching: $v_i(\{g\}) \geq v_i(Y_i)$ (as $rank_i(g) < rank_i(Y_i)$) and hence $w_{ig} \geq w_{iY_i}$ for $p = -\infty$ and $p \in [0, 1]$, while $w_{ig} \leq w_{iY_i}$ for $p < 0$ and $p \neq -\infty$. This implies that the objective value of the matching does not decrease when $p \in [0, 1]$ and $p = -\infty$ and the objective value of the matching does not increase when $p < 0$ and $p \neq -\infty$. Therefore, $\cup_{i\in[n]}(i, Y_i)$ still stays an optimum matching, but $\sum_{i\in[n]} rank_i(Y_i)$ strictly decreases. Since $n \leq \sum_{i\in[n]} rank_i(Y_i) \leq nm$, after $\mathcal{O}(nm)$ iterations we will have an allocation Y such that $\cup_{i\in[n]}(i, Y_i)$ is still an optimum matching, but for all agents $i \in [n]$ and for all goods $g \notin \mathbf{Y}$ we have $rank_i(Y_i) < rank_i(g)$.

The complete algorithm is outlined in Algorithm 5 (Selection of the allocation Y is captured in steps 1 to 5). We now make some helpful observations about the allocation Y.

[5] This is a matching that maximizes the weight of the smallest edge in the matching.

Lemma 4.2. *For all $i \in [n]$, we have $Y_i \subset H_i$. Furthermore, for all*

- $g \notin H_i$, *and*
- $g \notin \mathbf{Y}$,

we have $v_i(Y_i) \geq v_i(\{g\})$.

Proof. We first show that $Y_i \subset H_i$. We prove the same by contradiction. Assume otherwise, i.e., $Y_i \not\subset H_i$. In that case, note that there is always a good $g \in H_i \setminus \mathbf{Y}$ (as $|H_i| = |\mathbf{Y}| = n$ and there is a good in $\mathbf{Y}$ (namely Y_i) which is not in H_i). From the definition of H_i, it is clear that $rank_i(g) < rank_i(g')$ for all $g' \notin H_i$. Thus, we have $rank_i(g) < rank_i(Y_i)$ when $g \notin \mathbf{Y}$, which is a contradiction. Therefore $Y_i \subset H_i$. This also shows that for all $g \notin H_i$ we have $v_i(\{g\}) \leq v_i(Y_i)$ (as $Y_i \subset H_i$ and any good in H_i is at least as valuable as any good outside H_i to agent i). We have that $rank_i(Y_i) < rank_i(g)$ for all $g \notin \mathbf{Y}$, immediately implying that $v_i(Y_i) \geq v_i(\{g\})$. Thus, $v_i(Y_i) \geq v_(\{g\})$ for all $g \notin H_i$ and for all $g \notin \mathbf{Y}$. □

Run Algorithm 4 (Chapter 3) on the Remaining Goods. Once we determined the initial allocation Y, we run Algorithm 4 (Chapter 3) on the remaining goods starting with Y as the initial allocation (Y is a feasible initial allocation as it is trivially an EFX allocation as every agent has exactly a single good). Let Z be the final $(1-\varepsilon)$-EFX allocation with bounded charity P. As mentioned earlier, the singleton sets allocated to the agents are the barriers to proving our desired p-mean welfare approximation for any $(1-\varepsilon)$-EFX allocation with bounded charity. However, since we started with a good initial allocation (namely Y), we first show that we have some nice properties about the singleton sets in the final allocation Z.

Observation 4.3. *If $|Z_\ell| = 1$ for any $\ell \in [n]$, then we have $Z_\ell \subset \mathbf{Y}$.*

Proof. Since Z is obtained by running Algorithm 4 starting with Y as the initial allocation, we have for every agent i that $v_i(Z_i) \geq v_i(Y_i)$ (Algorithm 4 returns a final allocation Z that Pareto-dominates the initial allocation Y (see Figure 3.3 in Chapter 3)). In particular, Algorithm 4 ensures that for any agent ℓ, if the final bundle Z_ℓ is not the same as the initial bundle Y_ℓ, then $v_\ell(Z_\ell) > v_\ell(Y_\ell)$. Now consider an agent ℓ such that $|Z_\ell| = 1$. If $Z_\ell = Y_\ell$, then we immediately have $Z_\ell \subset \mathbf{Y}$. So now consider the case when $Z_\ell \neq Y_\ell$. Then we have $v_\ell(Z_\ell) > v_\ell(Y_\ell)$. By Lemma 4.2, we know that no good outside $\mathbf{Y}$ can be more valuable to agent ℓ than Y_ℓ. Therefore $Z_\ell \subset \mathbf{Y}$. □

Now we show a lower bound on the final valuation of an agent in terms of the low valued goods.

Observation 4.4. *We have $v_i(Z_i) \geq (1-\varepsilon) \cdot \frac{v_i(M \setminus \mathbf{Y})}{2(n+1)}$ for all $i \in [n]$.*

Proof. Fix an agent i. Consider any agent j such that Z_j is not a singleton. Since Z is a $(1-\varepsilon)$-EFX allocation with bounded charity P, we have that $v_i(Z_i) \geq (1-\varepsilon) \cdot v_i(Z_j \setminus \{g\})$ for all $g \in Z_j$ and $v_i(Z_i) \geq (1-\varepsilon) \cdot v_i(P)$. Since $|Z_j| \geq 2$, we can say that $v_i(Z_i) \geq (1-\varepsilon) \cdot max(v_i(Z_j \setminus \{g\}), v_i(\{g\}))$. Therefore we have,

$$\begin{aligned}
v_i(Z_i) &\geq (1-\varepsilon) \cdot \frac{(v_i(Z_j \setminus \{g\}) + v_i(\{g\}))}{2} \\
&\geq (1-\varepsilon) \cdot \frac{v_i(Z_j)}{2} && \text{(by subadditivity)}
\end{aligned}$$

Let $S = \cup_{|Z_\ell|=1} Z_\ell$. By Observation 4.3, we know that $S \subseteq \mathbf{Y}$. We have,

$$\begin{aligned}
(n+1-|S|)v_i(Z_i) &\geq (1-\varepsilon) \cdot \frac{1}{2} \sum_{|Z_j|\geq 2} v_i(Z_j) + (1-\varepsilon) \cdot v_i(P) \\
&\geq (1-\varepsilon) \cdot \frac{1}{2} v_i\Big(\bigcup_{|Z_j|\geq 2} Z_j \cup P \Big) && \text{(by subadditivity)} \\
&= \frac{(1-\varepsilon)}{2} v_i(M \setminus S) \\
&\geq \frac{(1-\varepsilon)}{2} v_i(M \setminus Y) && \text{(since } S \subseteq \mathbf{Y})
\end{aligned}$$

Therefore, we have $v_i(Z_i) \geq ((1-\varepsilon)/(2(n+1-|S|))) \cdot v_i(M \setminus \mathbf{Y}) \geq (1-\varepsilon) \cdot v_i(M \setminus \mathbf{Y})/2(n+1)$. □

Now we prove a lower bound on $v_i(Z_i)$ in terms of the initial allocation Y and the set of low valuable goods for agent i, i.e., $M \setminus H_i$.

Lemma 4.5. *For all $i \in [n]$, we have $v_i(Z_i) \geq \frac{(1-\varepsilon)}{4(n+1)} \cdot \Big(n \cdot v_i(Y_i) + v_i\Big(M \setminus H_i\Big)\Big)$.*

Proof. We have $v_i(Z_i) \geq v_i(Y_i)$ (since Z is an allocation determined by Algorithm 4 with Y as the initial allocation) and from Observation 4.4 we have $v_i(Z_i) \geq (1-\varepsilon) \cdot \frac{v_i(M \setminus \mathbf{Y})}{2(n+1)}$. Therefore, for all $i \in [n]$, we have

$$\begin{aligned}
v_i(Z_i) &\geq \frac{1}{2} \cdot \Big(v_i(Y_i) + \frac{(1-\varepsilon)}{2(n+1)} \cdot v_i(M \setminus \mathbf{Y}) \Big) \\
&= \frac{1}{2} \cdot \Big(v_i(Y_i) + \frac{(1-\varepsilon)}{2(n+1)} \cdot v_i\Big((M \setminus (\mathbf{Y} \cap H_i)) \setminus (\mathbf{Y} \setminus H_i) \Big) \Big) \\
&\geq \frac{1}{2} \cdot \Big(v_i(Y_i) + \frac{(1-\varepsilon)}{2(n+1)} \cdot v_i\Big(M \setminus (\mathbf{Y} \cap H_i) \Big) - \frac{(1-\varepsilon)}{2(n+1)} \cdot v_i\Big(\mathbf{Y} \setminus H_i \Big) \Big) \\
&\geq \frac{1}{2} \cdot \Big(v_i(Y_i) + \frac{(1-\varepsilon)}{2(n+1)} \cdot v_i\Big(M \setminus H_i \Big) - \frac{(1-\varepsilon)}{2(n+1)} \cdot v_i\Big(\mathbf{Y} \setminus H_i \Big) \Big), \qquad (4.2)
\end{aligned}$$

where the second last inequality follows from subadditivity and the last inequality follows from the fact that $\mathbf{Y} \cap H_i \subseteq H_i$. By Lemma 4.2, we know that $v_i(Y_i) \geq v_i(\{g\})$

1: **Construct** $G = \langle [n] \cup M, [n] \times M \rangle$ with

$$w_{ig} = \begin{cases} n \cdot v_i(\{g\}) + v_i(M \setminus H_i) & \text{if } p = -\infty \\ \log\Big(n \cdot v_i(\{g\}) + v_i(M \setminus H_i)\Big) & \text{if } p = 0 \\ \Big(n \cdot v_i(\{g\}) + v_i(M \setminus H_i)\Big)^p & \text{otherwise} \end{cases}$$

2: **Set** Y such that

$$\cup_{i \in [n]} (i, Y_i) = \begin{cases} \textbf{Max-Min-Matching}(G) & \text{if } p = -\infty \\ \textbf{Min-Weight-Perfect-Matching}(G) & \text{if } p < 0 \text{ and } p \neq -\infty \\ \textbf{Max-Weight-Matching}(G) & \text{otherwise} \end{cases}$$

3: **while** $\exists i \in [n]$ and $\exists g \notin \mathbf{Y}$ such that $rank_i(g) < rank_i(Y_i)$ **do**
4: $\quad Y_i \leftarrow \{g\}$.
5: **end while**
6: **Set** $(Z, P) \leftarrow$ Run Algorithm 4 for $(1-\varepsilon)$-*EFX allocation with bounded charity*, with initial allocation $\langle Y_1, \ldots, Y_n \rangle$.

Algorithm 5: Determining a $(1-\varepsilon)$-EFX allocation with bounded charity and $((1-\varepsilon)/4(n+1))$-approximation on optimum p-mean.

for all $g \notin H_i$. In particular, $v_i(Y_i) \geq v_i(\{g\})$ for all $g \in \mathbf{Y} \setminus H_i$. Thus,

$$\begin{aligned} v_i(\mathbf{Y} \setminus H_i) &\leq \sum_{g \in \mathbf{Y} \setminus H_i} v_i(\{g\}) && \text{(by subadditivity)} \\ &\leq \sum_{g \in \mathbf{Y} \setminus H_i} v_i(Y_i) \\ &= |\mathbf{Y} \setminus H_i| \cdot v_i(Y_i) \\ &\leq n \cdot v_i(Y_i) && \text{(as } |\mathbf{Y}| = n) \\ &\leq (n+1) \cdot v_i(Y_i). \end{aligned}$$

Substituting the upper bound for $v_i(\mathbf{Y} \setminus H_i)$ in (4.2) we have

$$\begin{aligned} v_i(Z_i) &\geq \frac{1}{2} \cdot \left((1 - \frac{(1-\varepsilon)}{2}) \cdot v_i(Y_i) + \frac{(1-\varepsilon)}{2(n+1)} \cdot v_i\Big(M \setminus H_i\Big) \right) \\ &\geq \frac{1}{2} \cdot \left(\frac{1}{2} \cdot v_i(Y_i) + \frac{(1-\varepsilon)}{2(n+1)} \cdot v_i\Big(M \setminus H_i\Big) \right) \\ &\geq \frac{(1-\varepsilon)}{4(n+1)} \cdot \left((n+1) \cdot v_i(Y_i) + v_i\Big(M \setminus H_i\Big) \right) \\ &\geq \frac{(1-\varepsilon)}{4(n+1)} \cdot \left(n \cdot v_i(Y_i) + v_i\Big(M \setminus H_i\Big) \right) \end{aligned}$$

□

The final allocation is the set Z which is obtained by running Algorithm 4 starting with Y as the initial allocation. Therefore, our final allocation is a $(1-\varepsilon)$-EFX allocation with bounded charity. We would now show the approximation guarantees that the algorithm achieves. The sections that follow prove that the allocation Z has a p-welfare that is $\frac{(1-\varepsilon)}{4(n+1)}$ times p-mean welfare achieved by the optimal allocation. Each section from here on presents the proof for particular value or a range of values of p.

4.2.1 Case $p = -\infty$

This is the case where $GM_p(X) = min_{i\in[n]} v_i(X_i)$. Let X^* be the allocation with the highest p-mean value and let g_i^* be agent i's most valuable good in X_i^*. We will show in this subsection that $GM_p(Z) \geq \frac{(1-\varepsilon)}{4(n+1)} \cdot GM_p(X^*)$. First observe that by Lemma 4.5, we have that for all $i \in [n]$, $v_i(Z_i) \geq \frac{(1-\varepsilon)}{4(n+1)} \cdot \Big(n \cdot v_i(Y_i) + v_i\Big(M \setminus H_i\Big)\Big)$. Therefore,

$$min_{i\in[n]} v_i(Z_i) \geq min_{i\in[n]} \frac{(1-\varepsilon)}{4(n+1)} \cdot \Big(n \cdot v_i(Y_i) + v_i\Big(M \setminus H_i\Big)\Big)$$

Recall that Y is chosen such that $\cup_{i\in[n]}(i, Y_i)$ is a max-min matching in the bipartite graph $G = ([n] \cup M, [n] \times M)$ where the weight of an edge from agent i to good g, $w_{ig} = n \cdot v_i(\{g\}) + v_i(M \setminus H_i)$. Also note that $\cup_{i\in[n]}(i, g_i^*)$ is a feasible matching in G. Thus we have

$$min_{i\in[n]}\Big(n \cdot v_i(Y_i) + v_i(M \setminus H_i)\Big) \geq min_{i\in[n]}\Big(n \cdot v_i(\{g_i^*\}) + v_i(M \setminus H_i)\Big)$$

Therefore we have,

$$\begin{aligned}
min_{i\in[n]} v_i(Z_i) &\geq \frac{(1-\varepsilon)}{4(n+1)} \cdot min_{i\in[n]}\Big(n \cdot v_i(\{g_i^*\}) + v_i(M \setminus H_i)\Big) \\
&\geq \frac{(1-\varepsilon)}{4(n+1)} \cdot min_{i\in[n]}\Big(n \cdot v_i(\{g_i^*\}) + v_i\Big(X_i^* \cap (M \setminus H_i)\Big)\Big) \\
&\geq \frac{(1-\varepsilon)}{4(n+1)} \cdot min_{i\in[n]}\Big(v_i(X_i^* \cap H_i) + v_i\Big(X_i^* \cap (M \setminus H_i)\Big)\Big) && \text{(as } |H_i| = n\text{)} \\
&\geq \frac{(1-\varepsilon)}{4(n+1)} \cdot min_{i\in[n]} v_i(X_i^*) && \text{(by subadditivity)}
\end{aligned}$$

This shows that $GM_p(Z) \geq \frac{(1-\varepsilon)}{4(n+1)} \cdot GM_p(X^*)$ when $p = -\infty$.

4.2.2 Case $p < 0$ and $p \neq -\infty$

The proof in this subsection is very similar to the proof when $p = -\infty$. Still for completeness, we sketch the whole proof. Let X^* be the allocation with the highest p-mean value and let g_i^* be agent i's most valuable good in X_i^*. Similar to the case $p = -\infty$, will show in this section that $GM_p(Z) \geq \frac{(1-\varepsilon)}{4(n+1)} \cdot GM_p(X^*)$. We now define

$$R(Z) = \sum_{i\in[n]} v_i(Z_i)^p$$

Note that $GM_p(Z) = \Big(\frac{1}{n} \cdot R(Z)\Big)^{\frac{1}{p}}$. We now prove an upper bound on $R(Z)$.

Lemma 4.6. *We have* $R(Z) \leq \frac{(1-\varepsilon)^p}{\left(4(n+1)\right)^p} \cdot \left(\sum_{i\in[n]} v_i(X_i^*)^p\right)$.

Proof. By Lemma 4.5, we have that for all $i \in [n]$, $v_i(Z_i) \geq \frac{(1-\varepsilon)}{4(n+1)} \cdot \left(n \cdot v_i(Y_i) + v_i\left(M \setminus H_i\right)\right)$. Therefore,

$$R(Z) \leq \sum_{i\in[n]} \left(\frac{(1-\varepsilon)}{4(n+1)} \cdot \left(n \cdot v_i(Y_i) + v_i(M \setminus H_i)\right)\right)^p \qquad \text{(as } p \text{ is negative)}$$

Recall that Y is chosen such that $\cup_{i\in[n]}(i, Y_i)$ is a minimum weight perfect matching in the bipartite graph $G = ([n] \cup M, [n] \times M)$ where the weight of an edge from agent i to good g, $w_{ig} = \left(n \cdot v_i(\{g\}) + v_i(M \setminus H_i)\right)^p$. Note that $\cup_{i\in[n]}(i, g_i^*)$ is a feasible matching in G. Thus we have,

$$\sum_{i\in[n]} \left(n \cdot v_i(Y_i) + v_i(M \setminus H_i)\right)^p \leq \sum_{i\in[n]} \left(n \cdot v_i(\{g_i^*\}) + v_i(M \setminus H_i)\right)^p$$

Therefore we have[6]

$$\begin{aligned}
R(Z) &\leq \sum_{i\in[n]} \left(\frac{(1-\varepsilon)}{4(n+1)} \cdot \left(n \cdot v_i(\{g_i^*\}) + v_i(M \setminus H_i)\right)\right)^p \\
&= \frac{(1-\varepsilon)^p}{\left(4(n+1)\right)^p} \cdot \sum_{i\in[n]} \left(n \cdot v_i(\{g_i^*\}) + v_i(M \setminus H_i)\right)^p \\
&\leq \frac{(1-\varepsilon)^p}{\left(4(n+1)\right)^p} \cdot \sum_{i\in[n]} \left(n \cdot v_i(\{g_i^*\}) + v_i\left(X_i^* \cap (M \setminus H_i)\right)\right)^p \\
&\leq \frac{(1-\varepsilon)^p}{\left(4(n+1)\right)^p} \cdot \sum_{i\in[n]} \left(v_i(X_i^* \cap H_i) + v_i\left(X_i^* \cap (M \setminus H_i)\right)\right)^p && \text{(as } |H_i| = n\text{)} \\
&\leq \frac{(1-\varepsilon)^p}{\left(4(n+1)\right)^p} \cdot \sum_{i\in[n]} v_i(X_i^*)^p && \text{(by subadditivity)}
\end{aligned}$$

□

Now we are ready to prove the guarantee on the p-mean welfare. We have,

$$\begin{aligned}
GM_p(Z) &= \left(\frac{1}{n} \cdot R(Z)\right)^{\frac{1}{p}} \\
&\geq \left(\frac{1}{n} \cdot \frac{(1-\varepsilon)^p}{\left(4(n+1)\right)^p} \cdot \left(\sum_{i\in[n]} v_i(X_i^*)^p\right)\right)^{\frac{1}{p}} && \text{(by Lemma 6.3, and also p is negative)} \\
&\geq \frac{(1-\varepsilon)^p}{4(n+1)} \cdot GM_p(X^*)
\end{aligned}$$

This shows that $GM_p(Z) \geq \frac{1}{4(n+1)} \cdot GM_p(X^*)$ when $p \in (-\infty, 0)$.

[6] For the set of inequalities that follow the reader is reminded that we are in the case where $p < 0$.

4.2.3 Case $p = 0$: Nash Welfare

This is the case where $GM_p(X) = \Big(\prod_{i\in[n]} v_i(X_i)\Big)^{\frac{1}{n}}$. Let X^* be the allocation with the highest p-mean value and let g_i^* be agent i's most valuable good in X_i^*. Like in the earlier subsections, we will show in this subsection that $GM_p(Z) \geq \frac{(1-\varepsilon)}{4(n+1)} \cdot GM_p(X^*)$. First observe that by Lemma 4.5, we have that for all $i \in [n]$, $v_i(Z_i) \geq \frac{(1-\varepsilon)}{4(n+1)} \cdot \Big(n \cdot v_i(Y_i) + v_i\Big(M \setminus H_i\Big)\Big)$. Therefore,

$$\Big(\prod_{i\in[n]} v_i(Z_i)\Big)^{\frac{1}{n}} \geq \Big(\prod_{i\in[n]} \frac{(1-\varepsilon)}{4(n+1)} \cdot \Big(n \cdot v_i(Y_i) + v_i\Big(M \setminus H_i\Big)\Big)\Big)^{\frac{1}{n}}$$
$$= \frac{(1-\varepsilon)}{4(n+1)} \cdot \Big(\prod_{i\in[n]} \Big(n \cdot v_i(Y_i) + v_i\Big(M \setminus H_i\Big)\Big)\Big)^{\frac{1}{n}}$$

Recall that Y was chosen such that (i, Y_i) is a maximum weight matching in the bipartite graph $G = ([n] \cup M, [n] \times M)$ where the weight of an edge from agent i to good g, $w_{ig} = \log\Big(n \cdot v_i(\{g\}) + v_i(M \setminus H_i)\Big)$. Note that $\cup_{i\in[n]}(i, g_i^*)$ is a feasible matching in G. Thus we have

$$\sum_{i\in[n]} \log\Big(n \cdot v_i(Y_i) + v_i(M \setminus H_i)\Big) \geq \sum_{i\in[n]} \log\Big(n \cdot v_i(\{g_i^*\}) + v_i(M \setminus H_i)\Big)$$
$$\implies \prod_{i\in[n]} \Big(n \cdot v_i(Y_i) + v_i(M \setminus H_i)\Big) \geq \prod_{i\in[n]} \Big(n \cdot v_i(\{g_i^*\}) + v_i(M \setminus H_i)\Big)$$

Therefore we have,

$$\Big(\prod_{i\in[n]} v_i(Z_i)\Big)^{\frac{1}{n}} \geq \frac{(1-\varepsilon)}{4(n+1)} \cdot \Big(\prod_{i\in[n]} \Big(n \cdot v_i(\{g_i^*\}) + v_i(M \setminus H_i)\Big)\Big)^{\frac{1}{n}}$$
$$\geq \frac{(1-\varepsilon)}{4(n+1)} \cdot \Big(\prod_{i\in[n]} \Big(n \cdot v_i(\{g_i^*\}) + v_i\big(X_i^* \cap (M \setminus H_i)\big)\Big)\Big)^{\frac{1}{n}}$$
$$\geq \frac{(1-\varepsilon)}{4(n+1)} \cdot \Big(\prod_{i\in[n]} \Big(v_i(X_i^* \cap H_i) + v_i\big(X_i^* \cap (M \setminus H_i)\big)\Big)\Big)^{\frac{1}{n}} \qquad \text{(as } |H_i| = n\text{)}$$
$$\geq \frac{(1-\varepsilon)}{4(n+1)} \cdot \Big(\prod_{i\in[n]} v_i(X_i^*)\Big)^{\frac{1}{n}} \qquad \text{(by subadditivity)}$$

This shows that $GM_p(Z) \geq \frac{(1-\varepsilon)}{4(n+1)} \cdot GM_p(X^*)$ when $p = 0$.

4.2.4 Case $p \in (0, 1]$

The proof of the approximation guarantee in this case follows almost the same proof in the Subsection 4.2.2, with the only difference that since p is positive and we compute a Maximum weight matching in the bipartite graph $G = ([n] \cup M, [n] \times M)$ where the weight of an edge from agent i to good g, $w_{ig} = \Big(n \cdot v_i(\{g\}) + v_i(M \setminus H_i)\Big)^p$ and we will have lower bounds on $R(Z)$ and consequently also lower bounds on $GM_p(Z)$.

Therefore, our algorithm computes an EFX allocation Z with bounded charity, which is also an $(1-\varepsilon)/(4(n+1))$-approximation of the optimum generalized p-mean welfare.

Proof of Theorem 4.1. We showed that the allocation Z computed by Algorithm 5 is a $(1-\varepsilon)$-EFX allocation with bounded charity P and $GM_p(Z) \geq \frac{1-\varepsilon}{4(n+1)} \cdot GM_p(X^*)$. It suffices to show that Algorithm 5 runs in polynomial-time . Note that steps 1 of the algorithm can be implemented in poly(n, m) time. Step 2 can also be realized in polynomial-time as all the matching subroutines run in poly(n, m). The while loop in step 3 runs for poly(n, m) iterations as with each iterations $\sum_{i \in [n]} rank_i(Y_i)$ decreases by 1 and $n < \sum_{i \in [n]} rank_i(Y_i) \leq nm$. In step 4, we run the Algorithm 3.1 with Y as the initial allocation. Since we are interested in $(1-\varepsilon)$-EFX allocation with bounded charity, Algorithm 3.1 runs in polynomial-time by Theorem 3.6 in Chapter 3. Therefore, the entire algorithm runs in polynomial-time .

We remark that a minor variant of our approach (changing the weights of the edges of the complete bipartite graph $G([n] \cup B, [n] \times B)$ appropriately - step 1 of Algorithm 5) gives a $\mathcal{O}(n)$ approximation on weighted generalized p-mean, defined as $WGM_p(X) = \big(\sum_{i \in [n]} \eta_i \cdot v_i(X_i)^p\big)^{\frac{1}{p}}$. In particular, we also get an $\mathcal{O}(n)$ approximation algorithm for asymmetric Nash welfare when agents have submodular valuations (improving the current best bound of $\mathcal{O}(n \cdot \log n)$ by Garg et al. [59]).

In a very recent work, Barman et al. [15], show that for all $p \in (-\infty, 1]$ and all $\varepsilon > 0$, getting an $\mathcal{O}(n^{1-\varepsilon})$-approximation of the generalized p-mean welfare under subadditive valuations requires exponential value queries. Thus, our algorithm is able to achieve the *best approximation of the generalized p-mean welfare in polynomial-time , when agents have subadditive valuations, while still having remarkable fairness properties.*

Figure 4.1 summarizes the fairness and efficiency guarantees we are able to ensure when agents have different valuation functions.

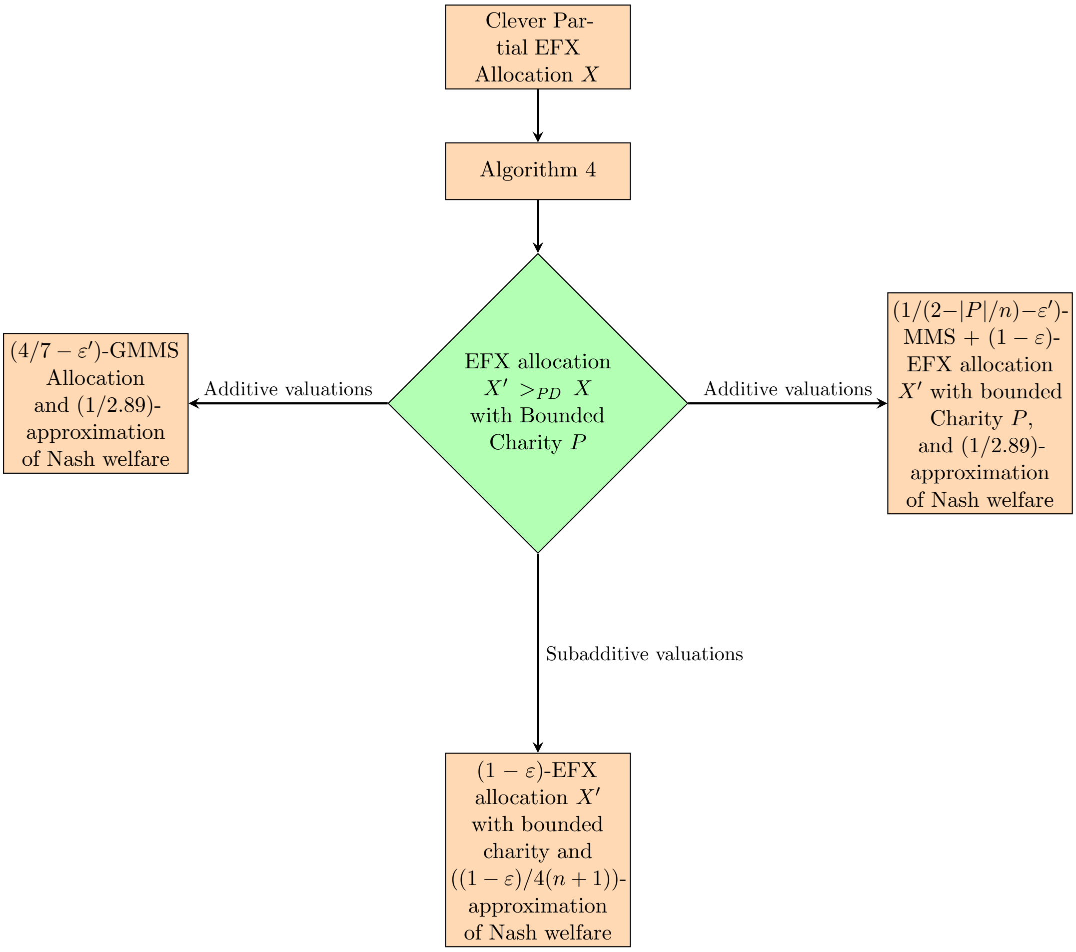


Figure 4.1: Illustration of the efficiency guarantees that we get on all the fair allocations we determined in Chapter 3 (Figure 3.3) in polynomial-time . When agents have additive valuations, we are able to get an approximation of $1/2.89$ of the optimum Nash welfare and when agents have subadditive valuations we are able to get an approximation of $1/4(n+1)$ of the optimum Nash welfare.

CHAPTER 5

EFX Allocations for Three Agents

In this chapter we show that EFX allocations exist when there are *three agents* with *additive valuations*. This is the first result on *complete* EFX allocations, i.e, EFX allocations in which all the goods are allocated. Observe that Theorem 3.1 from Chapter 3 implies that there exists an EFX allocation with at most two goods (as $n = 3$) unallocated. However, allocating the remaining two goods seems to be highly non-trivial. We elaborate this point briefly. Recall that Algorithm 4 determines a partial EFX allocation with only at most two goods unallocated, by proving that whenever we have a partial EFX allocation and three or more goods left unallocated, then there is another partial EFX allocation that Pareto dominates the existing partial EFX allocation. It is natural to ask whether this approach works when there is even one unallocated good. In this chapter, we highlight the limitation of this approach. We exhibit an instance with three agents, seven goods, and a partial EFX allocation on six goods. We show that there is no complete EFX allocation that Pareto dominates the partial EFX allocation on the six goods. In fact, we can further show that no complete EFX allocation has higher Nash welfare than the partial EFX allocation on the six goods, thereby falsifying the *monotonicity conjecture on EFX* by Caragiannis et al [27]. Thus, we need a different approach to find complete EFX allocations and this chapter takes the first step towards the same.

Theorem 5.1. *EFX allocations always exist for three agents with additive valuations.*

We first briefly explain our overall approach. We start by sketching the simple algorithm of Plaut and Roughgarden [84] that determines an EFX allocation when all agents have the same valuation function, say v. Note that since agents have the same valuation function, if $v(X_i) < v(X_j \setminus \{g\})$ for two agents i and j for some $g \in X_j$, then we have $v(X_{i_{min}}) < v(X_j \setminus \{g\})$ where i_{min} is the agent with the lowest valuation. The algorithm in [84] starts off with an arbitrary allocation (not necessarily EFX), and as long as there are agents i and j such that $v(X_i) < v(X_j \setminus \{g\})$ for some $g \in X_j$, the algorithm takes the good g away from j (j's new bundle is $X_j \setminus \{g\}$) and adds it to i_{min}'s bundle (i_{min}'s new bundle is $X_{i_{min}} \cup \{g\}$). Also, note that after re-allocation, the only changed bundles are that of i_{min} and j, and both of them have valuations still higher than i_{min}'s initial valuation, i.e., $v(X_{i_{min}} \cup \{g\}) > v(X_{i_{min}})$ and $v(X_j \setminus \{g\}) > v(X_{i_{min}})$. Observe that such an operation increases the valuation of an agent with the lowest valuation. Thus, after finitely many applications of this re-allocation, we must arrive at an EFX allocation. Note that this crucially uses the fact that the agents have identical valuations. In the general case, the valuation of agent j may drop significantly after removing g and j's current valuation may be less than i_{min}'s initial valuation. *Therefore, it is important to understand how agents value good(s) that we move across the bundles.* To this end, we carefully split every bundle into *upper* and *lower* half bundles (see (5.1) in Section 5.1). We systematically quantify the agent's relative valuations agents have for these upper and lower half bundles, and in most cases, we are able to move these bundles from one agent to the other, improve the valuation of some of the agents, and while still guaranteeing EFX property. This idea is detailed in Sections 5.2 and 5.3.

We need to show that there is progress after every swap of half bundles. The typical method here is to show improvement of the valuation vector on the Pareto front (like Algorithm 4 in Chapter 3 and the algorithm that returns a 1/2-EFX allocation in Plaut Roughgarden [84]). However, as mentioned earlier, there are limitations to this approach: In particular, we show an instance and a partial EFX allocation such that the valuation vector of any complete EFX allocation does not Pareto dominate the valuation vector of the existing partial EFX allocation. To overcome this barrier, we first pick an arbitrary agent a at the beginning and show that whenever we are unable to improve the valuation vector on the Pareto front, we can strictly increase a's valuation. In other words, the valuation of a particular agent a never decreases throughout re-allocations, and it improves after finitely many re-allocations, showing convergence. A more elaborate discussion on this technique is presented in Section 5.1.

5.1 Notation and Tools

We introduce some new notations and definitions in this chapter for convenience. We write $v_i(g)$ for $v_i(\{g\})$, $v_i(X_i \cup g)$ for $v_i(X_i \cup \{g\})$, and $v_i(X_j \setminus g)$ for $v_i(X_j \setminus \{g\})$. Further, we write $S \oplus_i T$ for $v_i(S) \oplus v_i(T)$ with $\oplus \in \{\leq, \geq, <, >\}$. Given an allocation $X = \langle X_1, X_2, \dots, X_n \rangle$ we say that i *strongly envies* a bundle $S \subseteq M$ if $X_i <_i S \setminus g$ for some $g \in S$, and we say that i *weakly envies* S if $X_i <_i S$ but $X_i \geq_i S \setminus g$ for all $g \in S$. From this perspective an allocation is an EFX allocation if and only if no agent strongly envies another agent.

Non-degenerate instances: We call an instance $I = \langle [3], M, \mathcal{V} \rangle$ non-degenerate if and only if no agent values two different sets equally, i.e., $\forall i \in [3]$ we have $v_i(S) \neq v_i(T)$ for all $S \neq T$. We first show that it suffices to deal with non-degenerate instances. Let $M = \{g_1, g_2, \dots, g_m\}$. We perturb any instance I to $I(\varepsilon) = \langle [3], M, \mathcal{V}(\varepsilon) \rangle$, where for every $v_i \in \mathcal{V}$ we define $v'_i \in \mathcal{V}(\varepsilon)$, as

$$v'_i(g_j) = v_i(g_j) + \varepsilon 2^j.$$

Lemma 5.2. *Let $\delta = \min_{i \in [3]} \min_{S,T \colon v_i(S) \neq v_i(T)} |v_i(S) - v_i(T)|$ and let $\varepsilon > 0$ be such that $\varepsilon \cdot 2^{m+1} < \delta$. Then*

(1) For any agent i and $S, T \subseteq M$ such that $v_i(S) > v_i(T)$, we have $v'_i(S) > v'_i(T)$.

(2) $I(\varepsilon)$ is a non-degenerate instance. Furthermore, if $X = \langle X_1, X_2, X_3 \rangle$ is an EFX allocation for $I(\varepsilon)$ then X is also an EFX allocation for I.

Proof. For the first statement of the lemma, observe that

$$\begin{aligned} v_i'(S) - v_i'(T) &= v_i(S) - v_i(T) + \varepsilon(\sum_{g_j \in S \setminus T} 2^j - \sum_{g_j \in T \setminus S} 2^j) \\ &\geq \delta - \varepsilon \sum_{g_j \in T \setminus S} 2^j \\ &\geq \delta - \varepsilon \cdot (2^{m+1} - 1) \\ &> 0 \end{aligned}$$

For the second statement of the lemma, consider any two sets $S, T \subseteq M$ such that $S \neq T$. Now, for any $i \in [3]$, if $v_i(S) \neq v_i(T)$, we have $v_i'(S) \neq v_i'(T)$ by the first statement of the lemma. If $v_i(S) = v_i(T)$, we have $v_i'(S) - v_i'(T) = \varepsilon(\sum_{g_j \in S \setminus T} 2^j - \sum_{g_j \in T \setminus S} 2^j) \neq 0$ (as $S \neq T$). Therefore, $I(\varepsilon)$ is non-degenerate.

For the final claim, let us assume that X is an EFX allocation in $I(\varepsilon)$ and not an EFX allocation in I. Then there exist i, j, and $g \in X_j$ such that $v_i(X_j \setminus g) > v_i(X_i)$. In that case, we have $v_i'(X_j \setminus g) > v_i'(X_i)$ by the first statement of the lemma, implying that X is not an EFX allocation in $I(\varepsilon)$ as well, which is a contradiction. $\square$

From now on we only deal with non-degenerate instances. In non-degenerate instances, all agents have positive valuation for all goods.

New Potential ϕ. An allocation X' *Pareto dominates* an allocation X if $v_i(X_i) \leq v_i(X_i')$ for all i with strict inequality for at least one i. The existing algorithms for "EFX with bounded charity" (Chapter 3) or "approximate EFX allocations" [84] construct a sequence of EFX allocations in which each allocation Pareto dominates its predecessor. However we exhibit in Section 5.4 a partial EFX allocation that is not Pareto dominated by any complete EFX allocation. Thus we need a more flexible approach in the search for a complete EFX allocation.

We name the agents a, b, and c arbitrarily and consider the lexicographic ordering of the triples

$$\phi(X) = (v_a(X_a), v_b(X_b), v_c(X_c)),$$

i.e., $\phi(X) \prec_{lex} \phi(X')$ (X' *dominates* X) if (i) $v_a(X_a) < v_a(X_a')$ or (ii) $v_a(X_a) = v_a(X_a')$ and $v_b(X_b) < v_b(X_b')$ or (iii) $v_a(X_a) = v_a(X_a')$ and $v_b(X_b) = v_b(X_b')$ and $v_c(X_c) < v_c(X_c')$. We construct a sequence of allocations in which each allocation dominates its predecessor. Of course, if X' Pareto dominates X, then it also dominates X, so we can use all the update rules in [36].

Our goal then is to iteratively construct a sequence of EFX allocations such that each EFX allocation dominates its predecessor. To this end, we first recollect some standard tools that we defined in Chapter 3.

Most envious agent. We crucially use the notion of a most envious agent, introduced in Chapter 3. Recall the definition of a most envious agent.

Definition. *Given a set $S \subseteq M$ and an allocation X, an agent i is a most envious agent of the set S or $i \in A_X(S)$ if and only if there exists $Z_i \subseteq S$ such that $v_i(Z_i) > v_i(X_i)$,* *and for any agent j*[1]*, we have $v_j(Z') \leq v_j(X_j)$ for all $Z' \subset Z_i$ (no agent envies a strict subset of Z_i).*

We also recall the observation made in Chapter 3, about a sufficient condition when the set $A_X(S)$ is not an $\emptyset$.

Observation 5.3. *$A_X(S) \neq \emptyset$ if and only if there is some agent i such that $v_i(S) > v_i(X_i)$. Also, in $\mathcal{O}(n \cdot |S|^2)$ time, one can find an agent $t \in A_X(S)$ and a set $Z \subseteq S$ such that $v_t(Z_t) > v_t(X_t)$ and no agent envies a strict subset of Z.*

Throughout this chapter, we will often be referring to the most envious agents of the sets $X_i \cup g$ for an agent $i \in [n]$ and an unallocated good g. Thus, we now state an immediate consequence of Observation 5.3.

Observation 5.4. *Given an allocation X, and an unallocated good g, for any $i \in [3]$, $A_X(X_i \cup g) \neq \emptyset$.*

Proof. It suffices to prove that there exists at least one agent who strictly prefers $X_i \cup g$ over her own bundle in allocation X. This is guaranteed since we are dealing with non-degenerate instances, in which $X_i \cup g >_i X_i$. □

Champions and Champion Graph M_X: Let X be the partial EFX allocation at any stage in our algorithm, and let g be an unallocated good. We say that i *champions* j (w.r.t g) if i is a most envious agent for $X_j \cup g$, i.e., $i \in A_X(X_j \cup g)$. We define the *champion graph M_X*, where each vertex corresponds to an agent and there is a directed edge $(i, j) \in M_X$ if and only if i champions j.

Observation 5.5. *The champion graph M_X is cyclic.*

Proof. By Observation 5.4, we have that the set of champions of any agent is never empty. Therefore, every vertex in M_X has at least one incoming edge. Thus M_X is cyclic. □

If i champions j, we define G_{ij} as the subset of $X_j \cup g$ such that $(X_j \cup g) \setminus G_{ij} >_i X_i$, and for all agents k (including i) we have $Z \leq_k X_k$ where $Z \subset (X_j \cup g) \setminus G_{ij}$. Since the valuations are additive, note that such a subset can be identified as the set K of the k least valuable goods for i in $X_j \cup g$ such that $(X_j \cup g) \setminus K >_i X_i$ and k is maximum. Now we make some small observations.

Observation 5.6. *Assume i champions j.*

(1) We have $((X_j \cup g) \setminus G_{ij}) \setminus h \leq_k X_k$ for all $h \in (X_j \cup g) \setminus G_{ij}$ and all agents k including i.

(2) If agent k does not champion j, we have $(X_j \cup g) \setminus G_{ij} \leq_k X_k$.

Proof. No agent envies a strict subset of $(X_j \cup g) \setminus G_{ij}$ by the definition of G_{ij}. Therefore for all agents k (including i), we have $((X_j \cup g) \setminus G_{ij}) \setminus h \leq_k X_k$.

Now we prove the second statement of the observation by contradiction. Assume that k does not champion j. However, we have $(X_j \cup g) \setminus G_{ij} >_k X_k$. By definition of G_{ij} we know that no agent envies a strict subset of $(X_j \cup g) \setminus G_{ij}$. Then $k \in A_X(X_j \cup g)$ (by definition of a most envious agent) and k champions j (w.r.t g) which is a contradiction. □

[1]Note that j can also be i.

We next mention two cases where it is known how to obtain a Pareto dominating EFX allocation from an existing EFX allocation. For an allocation X, let E_X be the *envy-graph*, in which vertices represent agents, and in which there is a directed edge from i to j if i envies j, i.e., $X_j >_i X_i$. Recall that we can assume without loss of generality that E_X is acyclic.

Fact 5.7 ([72]). *Let $X = \langle X_1, X_2, X_3 \rangle$ be an EFX allocation. Then there exists another EFX allocation $Y = \langle Y_1, Y_2, Y_3 \rangle$, where for all $i \in [3]$, $Y_i = X_j$ for some $j \in [3]$, such that E_Y is acyclic and $\phi(Y) \succeq_{lex} \phi(X)$ (because Y Pareto-dominates X).*

We generalize an observation made in Chapter 3 about the existence of a suitable update rule when E_X has a single source.

Observation 5.8 (Chapter 3). *Consider an EFX allocation X. Let s be any agent and let g be an unallocated good. If i champions s and i is reachable from s in E_X, then there is an EFX allocation Y Pareto dominating X. Additionally, agent s is strictly better off in Y, i.e., $Y_s >_s X_s$.*

Proof. We have that i is reachable from s in E_X. Let $t_1 \to t_2 \to \cdots \to t_k$ be the path from $t_1 = s$ to $t_k = i$ in E_X. We determine a new allocation Y as follows:

$$\begin{aligned} Y_{t_j} &= X_{t_{j+1}} && \text{for } j \in [k-1] \\ Y_i &= (X_s \cup g) \setminus G_{is}) && \\ Y_\ell &= X_\ell && \text{for all other } \ell \end{aligned}$$

Note that every agent along the path has strictly improved her valuation: Agents t_1 to t_{k-1} got bundles they envied in E_X and agent i championed s and got $(X_s \cup g) \setminus G_{is}$, which is more valuable to i than X_i (by definition of G_{is}). Also, every other agent retained their previous bundles and thus their valuations are not lower than before. Thus $\phi(Y) \succ_{lex} \phi(X)$ and also $Y_s >_s X_s$ (s was an agent along the path). It only remains to argue that Y is EFX. To this end, consider any two agents j and j'. We wish to show that j does not strongly envy j' in Y.

Case $j' \neq i$: Note that $Y_{j'} = X_\ell$ for some $\ell \in [3]$ (j' either received a bundle of another agent when we shifted the bundles along the path or retained the previous bundle). Also, note that $Y_j \geq_j X_j$ (no agent is worse off in Y). Therefore, $Y_j \geq_j X_j \geq_j X_\ell \setminus h =_j Y_{j'} \setminus h$ for all $h \in Y_{j'}$ (j did not strongly envy ℓ in X as X was EFX).

Case $j' = i$: We have $Y_{j'} = (X_s \cup g) \setminus G_{is}$. Since i championed s, by Observation 5.6 (part 1) we have that $((X_s \cup g) \setminus G_{is}) \setminus h \leq_j X_j$. Like earlier, $Y_j \geq_j X_j$ (no agent is worse off in Y). Thus j does not strongly envy i. □

Observation 5.8 implies that if there is some unallocated good and (i) if the envy-graph E_X has a single source[2] or (ii) any agent champions himself then there is a strictly Pareto dominating EFX allocation.

Corollary 5.9. *Let X be an EFX allocation, and g be an unallocated good. If E_X has a single source s, or M_X has a 1-cycle involving agent s, then there is an EFX allocation Y that Pareto dominates X in which $Y_s >_s X_s$.*

[2]A source is a vertex in E_X with in-degree zero.

Proof. If E_X has a single source s, the champion of s (which always exist, by Observation 5.4) is reachable from s. If M_X has a 1-cycle involving agent s then again the champion of s (which is s itself) is reachable from s. In both cases, since the champion of s is reachable from s in the envy-graph E_X, there is a Pareto dominating allocation Y such that $Y_s >_s X_s$ by Observation 5.8. $\square$

Hence, starting from Section 5.2, we only discuss the cases where the envy-graph has more than one source and there are no self-champions.

We start with some simple yet crucial observations.

Observation 5.10. *If i champions j and $X_i \geq_i X_j$, then $g \notin G_{ij}$, $G_{ij} \subseteq X_j$, and $G_{ij} <_i g$.*

Proof. We have $i \in A_X(X_j \cup g)$. Since $g \notin X_j$, $G_{ij} \subseteq X_j \cup g$, and valuations are additive and we have that $v_i((X_j \cup g) \setminus G_{ij}) = v_i(X_j) + v_i(g) - v_i(G_{ij})$. Again since $i \in A_X(X_j \cup g)$, by the definition of G_{ij}, $(X_j \cup g) \setminus G_{ij} >_i X_i$, and hence, $v_i(X_i) < v_i(X_j) + v_i(g) - v_i(G_{ij})$. Now we have $X_i \geq_i X_j$, implying that $G_{ij} <_i g$, and therefore, $g \notin G_{ij}$. $\square$

Observation 5.10 tells us that if i champions j, and i does not envy j, then $G_{ij} \subseteq X_j$. Therefore, we can split the bundle of agent j into two parts G_{ij} and $X_j \setminus G_{ij}$. We refer to G_{ij} as the *lower-half bundle* of j, and to $X_j \setminus G_{ij}$ as the *upper-half bundle* of j, and visualize the bundle of agent j as

$$X_j = \underset{(j)}{\begin{array}{|c|}\hline X_j \setminus G_{ij} \\ \hline G_{ij} \\ \hline \end{array}} \qquad \text{if } i \text{ champions } j \text{ and } i \text{ does not envy } j. \tag{5.1}$$

We collect some more useful facts about the values of lower and upper half bundles.

Observation 5.11. *If i champions j and j does not champion himself (self-champion), then we have $G_{ij} \neq \emptyset$ and $G_{ij} \geq_j g$.*

Proof. Since j does not self-champion, by Observation 5.6 (part 2), we have that $(X_j \cup g) \setminus G_{ij} \leq_j X_j$. Since $g \notin X_j$ and $G_{ij} \subseteq X_j \cup g$ we have $v_j((X_j \cup g) \setminus G_{ij}) = v_j(X_j) + v_j(g) - v_j(G_{ij}) \leq v_j(X_j)$, implying that $G_{ij} \geq_j g$. Since the value of g for j is non-zero, G_{ij} is non-empty. $\square$

Observation 5.12. *Let i champion j, and $X_i \geq_i X_j$. Let i' champion k and $X_{i'} \geq_{i'} X_k$. If i does not champion k, then $X_j \setminus G_{ij} >_i X_k \setminus G_{i'k}$.*

Proof. Since $i \in A_X(X_j \cup g)$ and $X_i \geq_i X_j$, by Observation 5.10, we have $g \notin G_{ij}$. Thus, $G_{ij} \subseteq X_j$. By the same reasoning, $g \notin G_{i'k}$ and $G_{i'k} \subseteq X_k$. Therefore, $(X_j \cup g) \setminus G_{ij} = (X_j \setminus G_{ij}) \cup g$, and $(X_k \cup g) \setminus G_{i'k} = (X_k \setminus G_{i'k}) \cup g$. By the definition of G_{ij}, we have $(X_j \setminus G_{ij}) \cup g >_i X_i$. Since $i \notin A_X(X_k \cup g)$, we have $X_i \geq_i (X_k \setminus G_{i'k}) \cup g$ by Observation 5.6 (part 2). Combining the two inequalities, we have $(X_j \setminus G_{ij}) \cup g >_i (X_k \setminus G_{i'k}) \cup g$, which implies $X_j \setminus G_{ij} >_i X_k \setminus G_{i'k}$. $\square$

In the upcoming sections, we show how to derive a dominating EFX allocation from an existing EFX allocation. Corollary 5.9 already deals with the cases that E_X has a single source or M_X has a 1-cycle. *We proceed under the following general assumptions: E_X is cycle-free and has at least two sources and there is no 1-cycle in M_X.* We distinguish the remaining cases by the number of sources in E_X.

5.2 Existence of EFX: Three sources in the Envy-Graph

If E_X has three sources, the allocation X is envy-free, i.e., $X_i \geq_i X_j$ for all i and j. We make a case distinction by whether or not M_X contains a 2-cycle.

5.2.1 2-cycle in M_X

Assume without loss of generality that agent 2 champions agent 1 and agent 1 champions agent 2. Since $X_1 \geq_1 X_2$ and $X_2 \geq_2 X_1$, the bundles X_1 and X_2 decompose according to 5.1. Since neither 1 nor 2 self-champion (as M_X has no 1-cycle), by Observation 5.12, we have $X_2 \setminus G_{12} >_1 X_1 \setminus G_{21}$ and $X_1 \setminus G_{21} >_2 X_1 \setminus G_{12}$. We swap the upper-halves of X_1 and X_2 to obtain

$$X' = \underset{(1)}{\begin{array}{|c|}\hline X_2 \setminus G_{12} \\ \hline G_{21} \\ \hline \end{array}} \qquad \underset{(2)}{\begin{array}{|c|}\hline X_1 \setminus G_{21} \\ \hline G_{12} \\ \hline \end{array}} \qquad \underset{(3)}{\boxed{X_3}}.$$

Note that agent 3 has the same valuation as before, while 1 and 2 are strictly better off. If X' is EFX we are done. So assume otherwise. We first determine the potential strong envy edges.

- *From 1*: We replaced the more valuable (according to 1) $X_2 \setminus G_{12}$ in X_2 with the less valuable $X_1 \setminus G_{21}$ and left X_3 unchanged. Thus 1 is strictly better off and according to him, the valuations of the bundles of 2 and 3 in X' is at most the valuation of their bundles in X. As 1 did not envy 2 and 3 before in X, 1 does not envy 2 and 3 in X'.
- *From 2*: A symmetrical argument shows that 2 does not envy 1 and 3.
- *From 3*: For agent 3, the sum of the valuations of agents 1 and 2 has not changed by the swap and 3 envied neither 1 nor 2 before the swap. Thus 3 envies at most one of the agents 1 and 2 after the swap. Assume without loss of generality that she envies agent 2. We then replace the lower-half bundle of agent 2 (G_{12}) with g to obtain

  $$X'' = \underset{(1)}{\begin{array}{|c|}\hline X_2 \setminus G_{12} \\ \hline G_{21} \\ \hline \end{array}} \qquad \underset{(2)}{\begin{array}{|c|}\hline X_1 \setminus G_{21} \\ \hline g \\ \hline \end{array}} \qquad \underset{(3)}{\boxed{X_3}}.$$

  In X'', agent 2 is still strictly better off than in X since by the definition of G_{21}, we have $(X_1 \setminus G_{21}) \cup g >_2 X_2$. Thus, X'' Pareto dominates X. We still need to show that X'' is EFX. To this end, observe that as we have not changed the bundles of agents 1 and 3, there is no strong envy between them. So we only need to exclude strong envy edges to and from agent 2.

  - *Nobody strongly envies agent 2*: Note that 2 championed 1. Thus, $((X_1 \setminus G_{21}) \cup g) \setminus h \leq_1 X_1$ and $((X_1 \setminus G_{21}) \cup g) \setminus h \leq_3 X_3$ for all $h \in (X_1 \setminus G_{21}) \cup g$ by Observation 5.6 (part 1). Since both 1 and 3 are not worse off than before, they do not strongly envy 2.

- *Agent 2 does not envy anyone*: We have that $(X_1 \setminus G_{21}) \cup g >_2 X_2$. Also according to 2, the valuation of the current bundles of 1 and 3 is at most their previous one, and 2 did not envy them before (when she had X_2). Hence, 2 does not envy 1 and 3.

We have thus shown that X'' is EFX and Pareto dominates X. Actually, the strategy described above handles a more general situation. It yields a Pareto dominating EFX allocation as long as 3 envies neither 1 nor 2 initially, even if 1 and 2 envied (*not strongly envied*) 3 initially:

Remark 5.13. *Let X be an EFX allocation, and let g be an unallocated good. If M_X has a 2-cycle, say involving agents 1 and 2, and agent 3 envies neither 1 nor 2, then there exists an EFX allocation Y Pareto dominating X.*

Remark 5.13 will be helpful when we deal with certain instances where E_X has two sources later in Section 5.3.

5.2.2 No 2-cycle in M_X

We now consider the case when M_X has no two cycle. Since M_X is cyclic and we neither have a 1-cycle nor a 2-cycle, we must have a 3-cycle. Let us assume w.l.o.g. that agent $i + 1$ is the unique champion of agent i (indices are modulo 3, so $i + 1$ corresponds to $(i \bmod 3) + 1$). Since, in addition, $i + 1$ does not envy i, all three bundles decompose according to (5.1) and the current allocation can be written as

$$X = \underset{(1)}{\begin{array}{|c|}\hline X_1 \setminus G_{21} \\ \hline G_{21} \\ \hline \end{array}} \quad \underset{(2)}{\begin{array}{|c|}\hline X_2 \setminus G_{32} \\ \hline G_{32} \\ \hline \end{array}} \quad \underset{(3)}{\begin{array}{|c|}\hline X_3 \setminus G_{13} \\ \hline G_{13} \\ \hline \end{array}}.$$

Let us collect what we know for agent 1's valuation of the upper-half bundles: 1 uniquely champions 3, while 2 and 3 uniquely champion 1 and 2, respectively. Also, the current allocation is envy-free. Thus $X_i \geq X_j$ for all $i, j \in [3]$. By Observation 5.12, we know that $X_3 \setminus G_{13} >_1 \max_1(X_1 \setminus G_{21}, X_2 \setminus G_{32})$[3] ($X_3 \setminus G_{13}$ is 1's favorite upper-half bundle).

Now, let us collect what we know for agent 1's valuation of the lower-half bundles: 1 champions 3 and does not envy 3's bundle. Thus, by Observation 5.10, $G_{13} <_1 g$ and $g \notin G_{13}$. Also, 1 does not champion himself, and 3 champions 1. Thus, by Observation 5.11, $g \leq_1 G_{21}$. We can make similar statements about agents 2 and 3. Since $g \notin G_{21}$, and our instance is assumed to be non-degenerate, we even have $g <_1 G_{21}$. Tables 5.1 and 5.2 summarize this information.

We first move to an allocation where everyone gets their favorite upper-half bundle (we achieve this by performing a cyclic shift of the upper-half bundles). Thus, the new allocation is:

$$X' = \underset{(1)}{\begin{array}{|c|}\hline X_3 \setminus G_{13} \\ \hline G_{21} \\ \hline \end{array}} \quad \underset{(2)}{\begin{array}{|c|}\hline X_1 \setminus G_{21} \\ \hline G_{32} \\ \hline \end{array}} \quad \underset{(3)}{\begin{array}{|c|}\hline X_2 \setminus G_{32} \\ \hline G_{13} \\ \hline \end{array}}$$

[3] $\max_1(X_1 \setminus G_{21}, X_2 \setminus G_{32})$ is 1's favorite bundle out of $X_1 \setminus G_{21}$ and $X_2 \setminus G_{32}$

Agent 1	$X_3 \setminus G_{13} >_1 \max_1(X_1 \setminus G_{21}, X_2 \setminus G_{32})$
Agent 2	$X_1 \setminus G_{21} >_2 \max_2(X_2 \setminus G_{32}, X_3 \setminus G_{13})$
Agent 3	$X_2 \setminus G_{32} >_3 \max_3(X_3 \setminus G_{13}, X_1 \setminus G_{21})$

Table 5.1: No 2-cycle in M_X: Ordering for the upper half bundles.

Agent 1	$G_{21} >_1 g >_1 G_{13}$
Agent 2	$G_{32} >_2 g >_2 G_{21}$
Agent 3	$G_{13} >_3 g >_3 G_{32}$

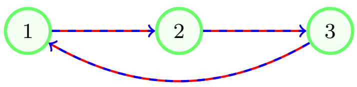
Table 5.2: No 2-cycle in M_X: Ordering for the lower half bundles. Furthermore, $g \notin G_{13}$, $g \notin G_{21}$, and $g \notin G_{32}$.

Clearly, every agent is strictly better off, and thus, X' Pareto dominates X. If X' is EFX, we are done. So we assume otherwise. What envy edges could exist? We first observe that no agent will envy the agent from whom it took its upper-half during the cyclic shift.

Observation 5.14. *In X', agent $i+1$ does not envy agent i for all $i \in [3]$ (indices are modulo 3).*

Proof. We just show the proof for $i = 1$, and the other cases follow symmetrically. Note that 2 values its current upper-half more than 1's upper-half (it has its favorite upper-half): $X_1 \setminus G_{21} >_2 X_3 \setminus G_{13}$. Similarly 2's also values its lower-half more than 1's lower-half: $G_{32} \geq_2 g >_2 G_{21}$. Therefore, 2 values her entire bundle more than 1's bundle, and hence does not envy 1. □

Therefore, the only envy edges (and hence *strong* envy edges) can be from agent i to agent $i+1$ as shown in the following figure.[4]

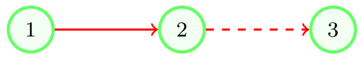


We now distinguish two cases depending on the number of such strong envy edges.

Three strong envy edges: In this case, the envy-graph is a 3-cycle. We perform a cyclic shift of the bundles and obtain an EFX allocation Pareto dominating the initial allocation X.

At most two strong envy edges: Note that in this case, there is a strong envy edge from at least one agent $i \in [3]$ to $i+1$ and there is no strong envy edge from at least one agent $j \in [3]$ to $j+1$. Let us assume without loss of generality that there is a strong envy edge from 1 to 2 , there may or may not be a strong envy edge from 2 to 3, and there is no strong envy edge from 3 to 1.

[4]In the figures that follow, we use red edges to indicate strong envy, and blue edges to indicate weak envy.



Note that 1 is strictly better off in X' than in X. The existence of envy from 1 and 2, despite this improvement, allows us to say more about the preference ordering of the upper-half and the lower-half bundles.

Observation 5.15. *If 1 envies 2 in X', $X_1 \setminus G_{21} >_1 X_2 \setminus G_{32}$, and $G_{32} >_1 G_{21}$.*

Proof. We argue by contradiction. Therefore, assume that i.e. $X_1 \setminus G_{21} \leq_1 X_2 \setminus G_{32}$ or $G_{32} \leq_1 G_{21}$. If $X_1 \setminus G_{21} \leq_1 X_2 \setminus G_{32}$, then

$$\begin{aligned}
(X_1 \setminus G_{21}) \cup G_{32} &\leq_1 (X_2 \setminus G_{32}) \cup G_{32} \\
&= X_2 \\
&\leq_1 X_1 && \text{(since 1 did not envy 2 before)} \\
&<_1 (X_3 \setminus G_{13}) \cup G_{21} && \text{(since 1 is better off than before)}
\end{aligned}$$

implying that 1 does not envy 2, a contradiction. If $G_{32} \leq_1 G_{21}$, then

$$\begin{aligned}
(X_1 \setminus G_{21}) \cup G_{32} &\leq_1 (X_1 \setminus G_{21}) \cup G_{21} \\
&= X_1 \\
&<_1 (X_3 \setminus G_{13}) \cup G_{21} && \text{(since 1 is better off than before)}
\end{aligned}$$

again implying that 1 does not envy 2, a contradiction. □

So we now have

$$X_2 \setminus G_{32} <_1 X_1 \setminus G_{21} <_1 X_3 \setminus G_{13} \quad \text{and} \quad G_{13} <_1 g <_1 G_{21} <_1 G_{32}. \tag{5.2}$$

We replace the lower-half bundle of 2 (G_{32}) by g to obtain

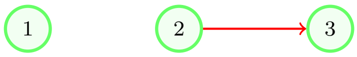
$$X'' = \begin{array}{ccc} \begin{array}{|c|}\hline X_3 \setminus G_{13} \\ \hline G_{21} \\ \hline \end{array} & \begin{array}{|c|}\hline X_1 \setminus G_{21} \\ \hline g \\ \hline \end{array} & \begin{array}{|c|}\hline X_2 \setminus G_{32} \\ \hline G_{13} \\ \hline \end{array} \\ (1) & (2) & (3) \end{array}.$$

Note that agents 1 and 3 are still strictly better off (as we have not changed their bundles after the cyclic shift of the upper-half bundles) than in X. Agent 2 championed 1, thus, $X_1 \setminus G_{21} \cup g >_2 X_2$, and agent 2 is also strictly better off. Hence, X'' Pareto dominates X. If there are no strong envy edges, we are done. So assume otherwise. We first note that the only possible strong envy edge is from 2 to 3:

- *Agent 1 does not envy anyone*: 1 did not envy 3 in X' and the bundles of 1 and 3 are the same in X' and X''. 1 does not envy 2 anymore as she prefers her own upper-half bundle and lower-half bundle to 2's upper-half bundle and lower-half bundle respectively, i.e., $X_3 \setminus G_{13} >_1 X_1 \setminus G_{21}$ (from Table 5.1) and $G_{21} \geq_1 g$ (from Table 5.2).

- *Agent 3 does not envy anyone*: We use a similar argument. 3 did not envy 1 in X' and the bundles of 1 and 3 are the same in X' and X''. 3 does not envy 2 as well as she prefers her own upper-half bundle and lower-half bundle to 2's upper-half bundle and lower-half bundle respectively, namely $X_2 \setminus G_{32} >_3 X_1 \setminus G_{21}$ (from Table 5.1) and $G_{13} \geq_3 g$ (from Table 5.2).
- *Agent 2 does not envy 1:* Note that agent 2 has her favorite upper-half bundle and values it more than 1's upper-half bundle: $X_1 \setminus G_{21} >_2 X_3 \setminus G_{13}$ (from Table 5.1) and 2 also values her lower-half bundle more than 1's lower-half bundle: $g >_2 G_{21}$ (from Table 5.2).

Therefore, the only possible strong envy edge is from 2 to 3 as shown below.



Similar to Observation 5.15, we can now infer more about 2's preference ordering for the bundles:

Observation 5.16. *If 2 strongly envies 3 in X'', we have $X_2 \setminus G_{32} >_2 X_3 \setminus G_{13}$ and $G_{13} >_2 G_{32}$.*

Proof. As in Observation 5.15, we argue by contradiction. Therefore, assume that i.e. $X_2 \setminus G_{32} \leq_2 X_3 \setminus G_{13}$ or $G_{13} \leq_2 G_{32}$. If $X_2 \setminus G_{32} \leq_2 X_3 \setminus G_{13}$, then

$$\begin{aligned}(X_2 \setminus G_{32}) \cup G_{13} &\leq_2 (X_3 \setminus G_{13}) \cup G_{13} \\ &= X_3 \\ &\leq_2 X_2 && \text{(since 2 did not envy 3 before)} \\ &<_2 (X_1 \setminus G_{21}) \cup g && \text{(as 2 is better off than before)}\end{aligned}$$

implying that 2 does not envy 3, a contradiction. If $G_{13} \leq_2 G_{32}$, then

$$\begin{aligned}(X_2 \setminus G_{32}) \cup G_{13} &\leq_2 (X_2 \setminus G_{32}) \cup G_{32} \\ &= X_2 \\ &<_1 (X_1 \setminus G_{21}) \cup g && \text{(as 2 is better off than before)}\end{aligned}$$

again implying that 2 does not envy 3, a contradiction. ☐

So we now have

$$X_3 \setminus G_{13} <_2 X_2 \setminus G_{32} <_2 X_1 \setminus G_{21} \quad \text{and} \quad G_{21} <_2 g <_2 G_{32} < G_{13}. \tag{5.3}$$

We are ready to construct the final allocation. To this end, consider the bundle $(X_1 \setminus G_{21}) \cup G_{13}$. Note that,

$$\begin{aligned}(X_1 \setminus G_{21}) \cup G_{13} &>_2 (X_1 \setminus G_{21}) \cup G_{32} && \text{(as } G_{13} >_2 G_{32} \text{ from Observation 5.16)} \\ &\geq_2 (X_1 \setminus G_{21}) \cup g && \text{(as } G_{32} \geq_2 g \text{ from Table 5.2)} \\ &>_2 X_2 && \text{(as 2 championed 1)}\end{aligned}$$

Let Z be a smallest cardinality subset of $(X_1 \setminus G_{21}) \cup G_{13}$ such that $Z >_2 X_2$. Since $g \notin X_1$ and $g \notin G_{13}$, $g \notin Z$. We now give two allocations, depending on how much 3 values Z.

Case $Z >_3 X_3$: Consider

$$X''' = \underset{(1)}{\begin{array}{|c|}\hline X_3 \setminus G_{13} \\ \hline g \\ \hline \end{array}} \quad \underset{(2)}{\begin{array}{|c|}\hline X_2 \setminus G_{32} \\ \hline G_{32} \\ \hline \end{array}} \quad \underset{(3)}{\begin{array}{|c|}\hline Z \\ \hline \end{array}}.$$

Since 1 was the champion of 3, we have $(X_3 \setminus G_{13}) \cup g >_1 X_1$. Thus, 1 and 3 are strictly better off, and 2 has the same bundle as in X. Therefore, X''' Pareto dominates X. We still need to show that X''' is EFX.

- *Nobody strongly envies agent 1*: Since 1 is the champion of 3, we have that $((X_3 \setminus G_{13}) \cup g) \setminus h <_2 X_2$ and $((X_3 \setminus G_{13}) \cup g) \setminus h <_3 X_3$ for all $h \in (X_3 \setminus G_{13}) \cup g$ by Observation 5.6 (part 1). As both 2 and 3 are not worse off than in X, neither of them strongly envies $(X_3 \setminus G_{13}) \cup g$.
- *Nobody envies agent 2*: Both 1 and 3 are strictly better off than in X and they did not envy X_2 in X. Thus they do not envy X_2 now.
- *Nobody strongly envies agent 3*: We first show that 1 does not envy $(X_1 \setminus G_{21}) \cup G_{13}$. This follows from the observation that 1 prefers her own upper-half bundle to $X_1 \setminus G_{21}$ and lower-half bundle to G_{13}: $X_3 \setminus G_{13} >_1 X_1 \setminus G_{21}$ (from Table 5.1) and $g >_1 G_{13}$ (from Table 5.2). Thus $(X_3 \setminus G_{13}) \cup g >_1 (X_1 \setminus G_{21}) \cup G_{13}$. Therefore, 1 does not envy Z either, as $Z \subseteq (X_1 \setminus G_{21}) \cup G_{13}$. Agent 2 does not strongly envy Z since Z is a smallest cardinality subset of $(X_1 \setminus G_{21}) \cup G_{13}$ that 2 values more than X_2. Thus $Z \setminus h \leq_2 X_2$ for all $h \in Z$.

Case $Z \leq_3 X_3$: Consider

$$X''' = \underset{(1)}{\begin{array}{|c|}\hline X_3 \setminus G_{13} \\ \hline G_{32} \\ \hline \end{array}} \quad \underset{(2)}{\begin{array}{|c|}\hline Z \\ \hline \end{array}} \quad \underset{(3)}{\begin{array}{|c|}\hline X_2 \setminus G_{32} \\ \hline g \\ \hline \end{array}}.$$

We first show that 1 is strictly better off in X''' than in X. Observe that

$$\begin{aligned} (X_3 \setminus G_{13}) \cup G_{32} &>_1 (X_3 \setminus G_{13}) \cup G_{21} && \text{(by Observation 5.15)} \\ &\geq_1 (X_3 \setminus G_{13}) \cup g && (G_{21} \geq_1 g \text{ from Table 5.2}) \\ &>_1 X_1 && \text{(as 1 championed 3)} \end{aligned}$$

2 is better off as $Z >_2 X_2$ by definition of Z. 3 is also better off than in X as it championed 2 and thus $X_2 \setminus G_{32} \cup g >_3 X_3$. Thus, all agents are strictly better off, and hence X''' Pareto dominates X. We next show that X''' is EFX.

- *Nobody envies agent 1*: Agent 2 does not envy 1 since

$$\begin{aligned} (X_3 \setminus G_{13}) \cup G_{32} &<_2 (X_2 \setminus G_{32}) \cup G_{32} && \text{(by Observation 5.16)} \\ &= X_2 \\ &<_2 Z && \text{(by definition of Z).} \end{aligned}$$

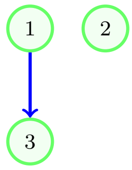


Figure 5.1: Envy-Graph for two sources when $(2,3) \notin E_X$: Green nodes correspond to the agents. Blue edges are the edges in E_X.

Agent 3 does not envy 1 either since she prefers her current upper-half bundle to and lower-half bundle to 1's upper-half bundle and lower-half bundle, respectively, i.e., $X_2 \setminus G_{32} >_3 X_3 \setminus G_{13}$ (from Table 5.1) and $g >_3 G_{32}$ (from Table 5.2).

- *Nobody envies agent 2*: Observe that 1 does not envy $(X_1 \setminus G_{21}) \cup G_{13}$ since 1 is strictly better off, $G_{21} \geq_1 g >_1 G_{13}$ from Table 5.2, and $G_{32} >_1 G_{21}$ by Observation 5.15. Thus $(X_3 \setminus G_{13}) \cup G_{32} >_1 (X_1 \setminus G_{21}) \cup G_{21} >_1 (X_1 \setminus G_{21}) \cup G_{13}$. Therefore, 1 does not envy Z either as $Z \subseteq (X_1 \setminus G_{21}) \cup G_{13}$.

  Agent 3 does not envy 2 since $(X_2 \setminus G_{32}) \cup g >_3 X_3$ (see above) and $X_3 \geq_3 Z$.
- *Nobody strongly envies agent 3*: Since 3 is the champion of 2, we have $((X_2 \setminus G_{32}) \cup g) \setminus h <_2 X_2$ and $((X_2 \setminus G_{32}) \cup g) \setminus h <_1 X_1$ for all $h \in (X_2 \setminus G_{32}) \cup g$ by Observation 5.6 (part 1). As both 1 and 2 are strictly better off (in X''') than in X, neither of them strongly envies $(X_2 \setminus G_{32}) \cup g$.

We have thus shown that given an allocation X such that E_X has three sources and M_X has a 3-cycle, there exists an EFX allocation Y Pareto dominating X. We summarize our main result for this section:

Lemma 5.17. *Let X be a partial EFX allocation and g be an unallocated good. If E_X has three sources, then there is an EFX allocation Y Pareto dominating X.*

5.3 Existence of EFX: Two sources in the Envy-Graph

Let us assume that agents 1 and 2 are the sources, and let $(1,3) \in E_X$. We have two configurations for E_X now, depending on whether or not $(2,3) \in E_X$. If $(2,3) \in E_X$, it is relatively straightforward to determine a new EFX allocation Pareto dominating X. Agent 3 is reachable from both 1 and 2 in E_X, and hence, if 3 champions either 1 or 2, we have a Pareto dominating EFX allocation by Observation 5.8. If 3 champions neither 1 nor 2, 1 and 2 must be champions of each other (Recall that no agent self-champions). Also note that 3 envies neither 1 nor 2. Therefore, by Remark 5.13, we have a Pareto dominating EFX allocation.

From now on, we assume that $(2,3) \notin E_X$.

The envy-graph of the scenario is now as shown in Figure 5.1. Next, we discuss the possible configurations of the champion graph M_X. We show that most configurations are easily handled. If 3 champions 1, then by Observation 5.8, there is a Pareto dominating EFX allocation. If 3 does not champion 1, and since 1 does not self-champion, agent 2 champions 1. If now 1 champions 2, we have a 2-cycle in M_X involving 1 and 2, and 3 envies neither of them. Therefore by Remark 5.13, there is a Pareto dominating EFX allocation. Thus, we may assume that 1 does not champion 2. Since 2 does not self-champion, agent 3 champions 2. *There are only three possible configurations for M_X now, depending on who champions 3 (only 1, only 2, both 1 and 2 as 3 does not self-champion) (see Figure 5.2).*

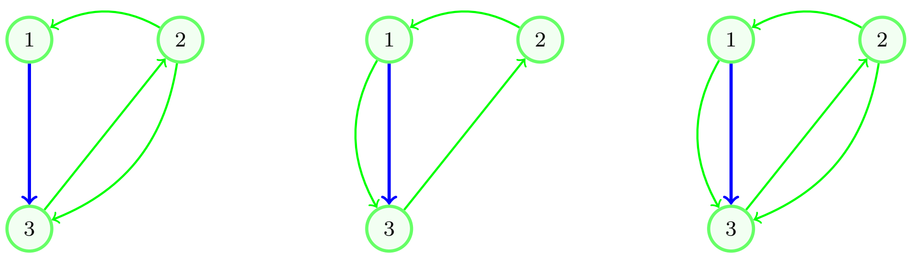


Figure 5.2: The possible states of M_X that require further discussion: Green nodes correspond to the agents. Blue edges are the edges in E_X and green edges are the edges in M_X. There is a unique configuration of E_X and three different configurations of M_X.

We now show how to deal with these configurations of M_X. In Section 5.2, we showed how to move from the current allocation X to an allocation that Pareto dominates X. In Section 5.4, we show that this is impossible in this particular configuration of E_X and M_X. More specifically, we exhibit an EFX allocation X that is not Pareto dominated by any complete EFX allocation. We also show that there is no complete EFX allocation with higher Nash welfare than X, thereby falsifying a conjecture of Caragiannis et al. [27].

Recall that our potential is $\phi(X) = (v_a(X_a), v_b(X_b), v_c(X_c))$. We move to an allocation in which agent a is strictly better off. We distinguish the cases: $a = 1$, $a = 2$, and $a = 3$.

Also, recall that we are in the scenario where 2 champions 1 and 2 does not envy 1. Similarly 3 champions 2 and 3 does not envy 2. Therefore, by Observation 5.10, we have that $g \notin G_{21}$ and $g \notin G_{32}$, and hence, the bundles X_1 and X_2 decompose according to (5.1). Also, since 2 champions 1 and 1 does not self-champion, by Observation 5.11, we have that $G_{21} \neq \emptyset$, and a similar argument also shows that $G_{32} \neq \emptyset$.

5.3.1 Agent a is agent 1 or 3

We start from the allocation

$$X = \underset{(1)}{\boxed{\begin{array}{c} X_1 \setminus G_{21} \\ \hline G_{21} \end{array}}} \quad \underset{(2)}{\boxed{\begin{array}{c} X_2 \setminus G_{32} \\ \hline G_{32} \end{array}}} \quad \underset{(3)}{\boxed{X_3}}.$$

Our goal is to determine an EFX allocation in which 1 and 3 are strictly better off (2 may be worse off). To this end, we consider

$$X' = \underset{(1)}{\boxed{X_3}} \quad \underset{(2)}{\boxed{\begin{array}{c} X_1 \setminus G_{21} \\ \hline G_{32} \end{array}}} \quad \underset{(3)}{\boxed{\begin{array}{c} X_2 \setminus G_{32} \\ \hline g \end{array}}}.$$

In X', every agent is better off than in X: 1 is better off because $X_3 >_1 X_1$ (1 envied 3 in E_X). We now show that 2 is better off: 2 championed 1 and 3 championed 2. Also, 2 did not self-champion, 2 did not envy 1 and 3 did not envy 2 . Therefore, by Observation 5.12, (setting $i = k = 2$, $j = 1$, $i' = 3$), we have that $X_1 \setminus G_{21} >_2 X_2 \setminus G_{32}$. Hence, $(X_1 \setminus G_{21}) \cup G_{32} >_2 (X_2 \setminus G_{32}) \cup G_{32} = X_2$. Thus 2 is also better off. Agent 3 is better off as 3 championed 2, and by the definition of G_{32}, we have $(X_2 \setminus G_{32} \cup g) >_3 X_3$. Thus X' Pareto dominates X. If X' is EFX, we are done. So assume otherwise. We show that the only possible strong envy edge will be from 1 to 2.

- *Nobody envies 1*: Note that 1 has X_3 and neither 2 nor 3 envied X_3 earlier (3 had X_3 and 2 did not envy 3). Since both 2 and 3 are better off than before, they do not envy 1.
- *Nobody strongly envies 3: 1 does not strongly envy 3 and 2 does not envy 3:* 3 championed 2 and 1 did not. Therefore, by Observation 5.6 (part 1) we have $((X_2 \setminus G_{32}) \cup g) \setminus h \leq_1 X_1$ for all $h \in (X_2 \setminus G_{32}) \cup g$. Since 1 is better off than in X, it does not strongly envy 3. Agent 2 does not envy 3 since its prefers both of its parts over the corresponding part of agent 3. This was argued above for the top part and follows from Observation 5.11
- *3 does not envy 2*: 3 championed 2 and 3 did not envy 2 earlier. Therefore by Observation 5.10 we have that $G_{32} <_3 g$. Therefore $(X_1 \setminus G_{21}) \cup G_{32} <_3 (X_1 \setminus G_{21}) \cup g$. Since 2 championed 1 and 3 did not, by Observation 5.6 (part 2), we have $((X_1 \setminus G_{21}) \cup g) \leq_3 X_3$. Since 3 is better off than in X, 3 does not envy 2.

Thus, the only strong envy edge is from 1 to 2. The current state of the envy-graph is depicted below:

Let Z be a smallest cardinality subset of $(X_1 \setminus G_{21}) \cup G_{32}$ that 2 values more than $\max_2((X_2 \setminus G_{32}) \cup g, X_3)$, where $\max_2((X_2 \setminus G_{32}) \cup g, X_3)$ is defined as the more valuable bundle out of $(X_2 \setminus G_{32}) \cup g$ and X_3 according to 2. Note that $\max_2((X_2 \setminus G_{32}) \cup g, X_3) \leq_2 (X_1 \setminus G_{21}) \cup G_{32}$ since 2 does not envy neither 1 nor 3 in X'. Since the instance is non-degenerate, the inequality is strict, and hence Z exists. We now consider two allocations depending on 1's value for Z.

Case $Z \leq_1 X_3$: We replace 2's current bundle with Z and obtain

$$X'' = \underset{(1)}{\boxed{X_3}} \quad \underset{(2)}{\boxed{Z}} \quad \underset{(3)}{\boxed{\begin{array}{c} X_2 \setminus G_{32} \\ \hline g \end{array}}}$$

Agents 1 and 3 have the same bundles as in X' and hence are strictly better off than in X. Thus, X'' dominates X, as $a = 1$ or $a = 3$ and we improve a strictly. We next show that X'' is EFX. Since the only bundle we have changed is that of 2, and there were no strong envy edges between 1 and 3 earlier, it suffices to show that there are no strong envy edges to and from 2.

- *Nobody envies 2*: 3 did not envy the set $(X_1 \setminus G_{21}) \cup G_{32}$. As $Z \subseteq (X_1 \setminus G_{21}) \cup G_{32}$, agent 3 does not envy Z either . 1 does not envy Z because we are in the case where $Z \leq_1 X_3$.
- *2 does not envy anyone*: This follows from the definition of Z itself since $Z >_2 \max_2((X_2 \setminus G_{32}) \cup g, X_3)$.

Case $Z >_1 X_3$: In this case, we consider

$$X'' = \begin{array}{ccc} \boxed{Z} & \boxed{max_2((X_2 \setminus G_{32}) \cup g, X_3)} & \boxed{min_2((X_2 \setminus G_{32}) \cup g, X_3)} \\ (1) & (2) & (3) \end{array}$$

Agent 1 is still strictly better off than in X as we are in the case $Z >_1 X_3 >_1 X_1$, and agent 3 is not worse off than before as both X_3 and $(X_2 \setminus G_{32}) \cup g$ are at least as valuable to him as her previous bundle X_3. We first show that X'' is EFX.

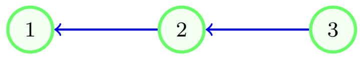
- *1 does not envy anyone*: We are in the case where $Z >_1 X_3$ and 1 did not envy $(X_2 \setminus G_{32}) \cup g$ when she had X_3 itself (and now 1 is better off than with X_3). Thus, 1 does not envy anyone.
- *2 does not strongly envy anyone*: Since 2 chooses the better bundle out of X_3 and $(X_2 \setminus G_{32}) \cup g$, 2 does not envy 3. Agent 2 does not strongly envy 1 since by the definition of Z, we have $Z \setminus h \leq_2 max_2((X_2 \setminus G_{32}) \cup g, X_3)$ for all $h \in Z$. However, note that 2 envies 1. Thus, 2 does not envy 3 and does not strongly envy 1 (but envies 1).
- *3 does not strongly envy anyone*: 3 did not envy the set $(X_1 \setminus G_{21}) \cup G_{32}$, [5] and $X_3 \leq X_3''$ as we argued above. Thus, 3 will not envy Z either as $Z \subseteq (X_1 \setminus G_{21}) \cup G_{32}$. We next show that 3 does not strongly envy 2, observe that $(X_2 \setminus G_{32}) \cup g >_3 X_3$. Therefore, if $\min_2((X_2 \setminus G_{32}) \cup g, X_3) = (X_2 \setminus G_{32}) \cup g$, we are done. So assume $\min_2((X_2 \setminus G_{32}) \cup g, X_3) = X_3$. Since 3 championed 2 and from Observation 5.6 (part 1), we have that $((X_2 \setminus G_{32}) \cup g) \setminus h \leq_3 X_3$ for all $h \in (X_2 \setminus G_{32}) \cup g$: Thus 3 does not strongly envy 2.

Now if $a = 1$, we are done, as X'' is EFX and agent 1 strictly improved. So assume $a = 3$. If $\min_2((X_2 \setminus G_{32}) \cup g, X_3) = (X_2 \setminus G_{32}) \cup g$, then agent 3 is strictly better off and we are done. This leaves the case that agent 3 gets X_3, and hence

$$X'' = \begin{array}{ccc} \boxed{Z} & \begin{array}{|c|}\hline X_2 \setminus G_{32} \\ \hline g \\ \hline \end{array} & \boxed{X_3} \\ (1) & (2) & (3) \end{array}$$

The envy-graph $E_{X''}$ with respect to allocation X'' is a path (shown below): 1 does not envy anyone, 2 envies 1 (not strongly) and does not envy 3, and 3 envies 2.

[5]We repeat the argument made earlier: 3 championed 2 and 3 did not envy 2 earlier. Therefore, by Observation 5.10 we have that $G_{32} <_3 g$. Hence, $(X_1 \setminus G_{21}) \cup G_{32} <_3 (X_1 \setminus G_{21}) \cup g$. Since 2 championed 1 and 3 did not, by Observation 5.6 (part 2), we have $((X_1 \setminus G_{21}) \cup g) \leq_3 X_3$.



Also, note that we have some unallocated goods, e.g., the goods in G_{21}. Recall that we argued $G_{21} \neq \emptyset$ in the paragraph just before Section 5.3.1. Consider any good $g' \in G_{21}$. Since 3 is the only source in $E_{X''}$, by Corollary 5.9, there is an EFX allocation X''' Pareto dominating X'', where $X_3''' >_3 X_3'' = X_3$. Thus, we have an EFX allocation X''' that dominates X (as agent 3 is strictly better off and $a = 3$).

5.3.2 Agent a is agent 2

Recall that we argued just before the beginning of Section 5.3.1 that $g \notin G_{21}$ and $g \notin G_{32}$. Thus, the current EFX allocation X is

$$X = \underset{(1)}{\boxed{\begin{array}{c} X_1 \setminus G_{21} \\ \hline G_{21} \end{array}}} \quad \underset{(2)}{\boxed{\begin{array}{c} X_2 \setminus G_{32} \\ \hline G_{32} \end{array}}} \quad \underset{(3)}{\boxed{X_3}}$$

Our aim is to determine an EFX allocation, in which agent 2 has a bundle more valuable than X_2. First, observe that $(X_1 \setminus G_{21}) \cup g$ is such a bundle. As 2 championed 1, we have $(X_1 \setminus G_{21}) \cup g >_2 X_2$ by the definition of G_{21}. We also observe that both agents 1 and 3 value X_3 as least as much as X_2 and $(X_1 \setminus G_{21}) \cup g$.

Observation 5.18. $X_3 >_i \max_i(X_2, ((X_1 \setminus G_{21}) \cup g)$ *for* $i \in \{1, 3\}$.

Proof. We argue $\geq_i$; strict inequality then follows from non-degeneracy.

Nobody envies 2 in X. Thus, $X_2 \leq_3 X_3$, and $X_2 \leq_1 X_1 <_1 X_3$ (the strict inequality holds as 1 envies 3 in X).

2 is the unique champion of 1 in X (both 1 and 3 do not champion 1). Therefore, by Observation 5.6 (part 2), we have $(X_1 \setminus G_{21}) \cup g \leq_3 X_3$ and $(X_1 \setminus G_{21}) \cup g \leq_1 X_1 <_1 X_3$ (the strict inequality holds as 1 envies 3 in X). □

For $i \in \{1, 3\}$, let κ_i be the size of a smallest subset Z_i of X_3 such that $Z_i >_i \max_i((X_1 \setminus G_{21}) \cup g, X_2)$. We use the relative size of κ_1 and κ_3 to differentiate between agents 1 and 3. We use w (winner) to denote the agent with the smaller value of κ_i, i.e., $w = 1$ if $\kappa_1 \leq \kappa_3$ and $w = 3$ if $\kappa_1 > \kappa_3$. We use ℓ (loser) for the other agent. Consider

$$X' = \underset{(w)}{\boxed{X_3}} \quad \underset{(\ell)}{\boxed{max_\ell(X_2, (X_1 \setminus G_{21}) \cup g)}} \quad \underset{(2)}{\boxed{min_\ell(X_2, (X_1 \setminus G_{21}) \cup g)}}$$

In X', the only possible strong envy edge is from ℓ to w. By Observation 5.18, w envies neither ℓ nor 2. Note that 2 championed 1 and therefore, $(X_1 \setminus G_{21}) \cup g >_2 X_2$, but by Observation 5.6 (part 1), we have $((X_1 \setminus G_{21}) \cup g) \setminus h \leq_2 X_2$ for all $h \in (X_1 \setminus G_{21}) \cup g$. Thus, 2 gets a bundle worth at least X_2 and does not strongly envy ℓ. 2 also does not envy w (as she did not envy X_3 when she had X_2). ℓ does not envy 2 as she chooses the better bundle out of X_2 and $X_1 \setminus G_{21} \cup g$. Thus, the only possible strong envy edge is from ℓ to w. How we proceed then depends on whether or not ℓ strongly envies w.

ℓ does not strongly envy w: Then X' is EFX. If $\min_\ell(X_2, (X_1 \setminus G_{21}) \cup g) = (X_1 \setminus G_{21}) \cup g$, we are done as X' dominates X (2 is strictly better off and $a = 2$). So assume otherwise. Then

$$X' = \begin{array}{ccc} \boxed{X_3} & \boxed{X_1 \setminus G_{21} \cup g} & \boxed{X_2} \\ (w) & (\ell) & (2) \end{array}$$

By Observation 5.18, ℓ envies w. Since 2 only envies ℓ, ℓ only envies w, and w does not envy anyone, the envy-graph $E_{X'}$ is a path with source 2.

$$2 \longrightarrow \ell \longrightarrow w$$

Also, note that there are unallocated goods, namely the goods in G_{21} (we argued just before the beginning of Section 5.3.1 that $G_{21} \neq \emptyset$). Therefore, by Corollary 5.9, there is an EFX allocation X'', in which 2 is strictly better off. Thus, X'' dominates X as 2 is strictly better off and $a = 2$.

ℓ strongly envies w: We keep removing the least valuable good *according to w* from w's bundle, until ℓ does not strongly envy w anymore. Let Z be the bundle obtained in this way. Consider

$$X' = \begin{array}{ccc} \boxed{Z} & \boxed{max_\ell(X_2, (X_1 \setminus G_{21}) \cup g)} & \boxed{min_\ell(X_2, (X_1 \setminus G_{21}) \cup g)} \\ (w) & (\ell) & (2) \end{array}$$

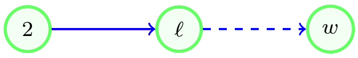
Claim 5.19. *w does not envy 2 and ℓ.*

Proof. Recall that κ_w is the smallest cardinality of a subset of X_3 that w still values more than $\max_w(X_2, (X_1 \setminus G_{21}) \cup g)$; κ_w was defined just after Observation 5.18. Such a set can be obtained by removing w's $|X_3| - \kappa_w$ least valuable goods from X_3. Observe that Z is obtained by removing $|X_3| - |Z|$ of w's least valuable goods from X_3. If $|Z| \geq \kappa_w$, w will envy neither 2 nor ℓ. If $|Z| < \kappa_w \leq \kappa_\ell$ (recall that $\kappa_w \leq \kappa_\ell$), let h be the last good removed. Then ℓ strongly envies $Z \cup h$ (otherwise we would not have removed h), meaning that there exists an $h' \in Z \cup h$ such that $(Z \cup h) \setminus h' >_\ell \max_\ell(X_2, (X_1 \setminus G_{21}) \cup g)$. Thus, there is a subset of X_3 of size $|(Z \cup h) \setminus h'| < \kappa_w + 1 - 1 = \kappa_w$ that ℓ values more than $\max_\ell(X_2, (X_1 \setminus G_{21}) \cup g)$, a contradiction to $\kappa_w \leq \kappa_\ell$. $\square$

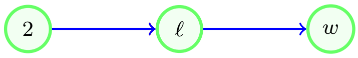
The allocation X' is EFX: w envies neither 2 nor ℓ, ℓ does not strongly envy w, ℓ does not envy 2, and 2 envies neither ℓ nor w. If $\min_\ell(X_2, (X_1 \setminus G_{21}) \cup g)$ is $X_1 \setminus G_{21} \cup g$, then we are done as X' dominates X (2 is strictly better off and $a = 2$). So assume otherwise. Then

$$X' = \begin{array}{ccc} \boxed{Z} & \boxed{X_1 \setminus G_{21} \cup g} & \boxed{X_2} \\ (w) & (\ell) & (2) \end{array}$$

In X', w envies nobody (by Claim 5.19), 2 envies ℓ, and ℓ may or may not envy w. We distinguish cases according to whether or not ℓ envies w.

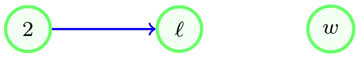


Case ℓ envies w: Then, the current envy-graph is a path with 2 as the source.



Since there are unallocated goods, namely the goods in G_{21} (we argued just before the beginning of Section 5.3.1 that $G_{21} \neq \emptyset$), by Corollary 5.9, there is an EFX allocation X'' in which agent 2 is strictly better off. The allocation X'' dominates X (as 2 is strictly better off and $a = 2$).

Case ℓ does not envy w: Then the current envy-graph has two sources, namely w and 2, and one envy edge from 2 to ℓ.



There are at least two unallocated goods, the goods in G_{21} (we argued just before the beginning of Section 5.3.1 that $G_{21} \neq \emptyset$) and the goods in $X_3 \setminus Z$ (note that this set is not empty; we definitely have removed at least one good from X_3 as ℓ strongly envied w when w had X_3). Now consider the allocation X' and some $g' \in G_{21}$. If the champion of 2 is 2 itself or ℓ (definition of champion based on allocation X' and the unallocated good g'), by Observation 5.8 there is an EFX allocation Y where the source, namely 2, is strictly better off and hence Y will dominate X. So assume that the champion of 2 is w, i.e., $w \in A_{X'}(X'_2 \cup g')$. Let $g'' \in X_3 \setminus Z$ be the last element that we removed from X_3 when we constructed Z from X_3. Then ℓ strongly envies $Z \cup g''$ and, according to w, g'' is the least valuable good in $Z \cup g''$. We observe that ℓ is the unique champion of w (definition of champion based on allocation X' and the unallocated good g'') ,i.e., $A_{X'}(X'_w \cup g'') = \{\ell\}$.

Observation 5.20. *For any good $g'' \in X_3 \setminus Z$, we have $A_{X'}(X'_w \cup g'') = \{\ell\}$.*

Proof. We have $X'_w = Z$. First we show that $2 \notin A_{X'}(Z \cup g'')$. Note that $Z \cup g'' \subseteq X_3$. Since $X_2 \geq_2 X_3$ (as 2 did not envy 3 in X), 2 will not envy $Z \cup g''$ either.

Note that ℓ strongly envies $Z \cup g''$. Hence, there exists $h \in Z \cup g''$ such that $(Z \cup g'') \setminus h >_\ell X'_\ell$. However, $(Z \cup g'') \setminus h \leq_w X'_w = Z$: By the construction of Z, g'' is w's least valuable good in $Z \cup g''$. Thus, the removal of any good from $Z \cup g''$ will result in a bundle whose value for w is no more than the value of Z for w Therefore, ℓ envies a subset $(Z \cup g'') \setminus h$ of $Z \cup g''$ that no other agent (agent 2 and w) strongly envies. Thus, $\ell \in A_{X'}(Z \cup g'')$. $\square$

Consider

$$X'' = \begin{array}{ccc} \boxed{(X'_2 \cup g') \setminus G_{w2}} & \boxed{(X'_w \cup g'') \setminus G_{\ell w}} & \boxed{X'_\ell} \\ (w) & (\ell) & (2) \end{array}$$

or equivalently

$$X'' = \underset{(w)}{\boxed{(X_2 \cup g') \setminus G_{w2}}} \quad \underset{(\ell)}{\boxed{(Z \cup g'') \setminus G_{\ell w}}} \quad \underset{(2)}{\boxed{(X_1 \setminus G_{21}) \cup g}}.$$

Note that every agent is strictly better off than in X'. w championed 2, and by the definition of G_{w2}, we have $(X'_2 \cup g') \setminus G_{w2} >_w X'_w$. Similarly, ℓ championed w, and by the definition of $G_{\ell w}$, we have $(X'_w \cup g'') \setminus G_{\ell w} >_\ell X'_\ell$. 2 is better off as 2 envied ℓ in X' i.e. $X'_2 <_2 X'_\ell$. Now we have an allocation X'' in which agent 2 is strictly better off than it was in X. Thus, X'' dominates X (as $a = 2$). It suffices to show that X'' is EFX now. To this end, observe that,

- *Nobody strongly envies w:* w championed 2. Thus, by Observation 5.6 (part 1), we have that $((X'_2 \cup g') \setminus G_{w2}) \setminus h \leq_2 X'_2$ and $((X'_2 \cup g') \setminus G_{w2}) \setminus h \leq_\ell X'_\ell$ for all $h \in ((X'_2 \cup g') \setminus G_{w2})$. Since both 2 and ℓ are better off than before (in X'), they do not strongly envy w.
- *Nobody strongly envies ℓ:* The argument is very similar to the previous case. ℓ championed 2. Thus, by Observation 5.6 (part 1), we have that $((X'_w \cup g'') \setminus G_{\ell w}) \setminus h \leq_2 X'_2$ and $((X'_w \cup g'') \setminus G_{\ell w}) \setminus h \leq_w X'_w$ for all $h \in ((X'_w \cup g'') \setminus G_{\ell w})$. Since both 2 and w are better off than before (than they were in X'), they do not strongly envy w.
- *Nobody strongly envies 2:* Both w and ℓ did not envy X'_ℓ (ℓ had X'_ℓ and w did not envy ℓ) when they had X'_w and X'_ℓ itself. Both w and ℓ are strictly better off than they were in X'. Therefore, they also do not envy 2.

We conclude that there is an EFX allocation dominating X in the case, $a = 2$ as well.

This allows us to summarize our main result for this section as follows,

Lemma 5.21. *Let X be a partial EFX allocation, and let g be an unallocated good, where the envy-graph E_X has two sources. Then there is an EFX allocation Y dominating X.*

Having covered all the cases, we arrive at our main result:

Theorem 5.22. *For any instance $I = \langle [3], M, \mathcal{V} \rangle$ where all $v_i \in \mathcal{V}$ are additive, an EFX allocation always exists.*

Proof. We start off with an empty allocation ($X_i = \emptyset$ for all $i \in [3]$), which is trivially EFX. As long as X is not a complete EFX allocation, there is an allocation Y that dominates X: If E_X has a single source or M_X has a 1-cycle, there is a dominating EFX allocation Y by Corollary 5.9. Lemmas 5.17 and 5.21 establish the existence of Y when E_X has multiple sources and M_X does not have a 1-cycle. Since ϕ is bounded from above, the process must stop. When it stops, we have arrived at a complete EFX allocation. $\square$

	g_1	g_2	g_3	g_4	g_5	g_6	g_7
$\mathbf{a_1}$	8	2	12	2	0	17	1
$\mathbf{a_2}$	5	0	9	4	10	0	3
$\mathbf{a_3}$	0	0	0	0	9	10	2

Table 5.3: An instance where no complete EFX allocation dominates the EFX allocation X for the first six goods defined in the text. The valuations are assumed to be additive and the entry in row i and column j is the value of good j for agent i.

5.4 Limitations of the Approach from Chapter 3

In this section, we highlight some barriers to the techniques developed in Chapter 3 for computin an EFX allocation with bounded charity. We give an instance with three agents and seven goods such that there is a partial EFX allocation for six of the goods that is not Pareto-dominated by any complete EFX allocation for the full set of goods. We also generalize this example and give an instance with a partial EFX allocation which has a Nash welfare larger than the Nash welfare of any complete EFX allocation. These examples make it unlikely that there is an iterative algorithm towards a complete EFX allocation that improves the current EFX allocation in each iteration either in the sense of Pareto domination or in the sense of Nash welfare (like the algorithms in [84] and Chapter 3). The second example also falsifies the EFX monotonicity conjecture (see Conjecture 5.24) by Caragiannis et al. [27].

Theorem 5.23. *For the instance given in Table 6.1, the partial allocation $X = \langle X_1, X_2, X_3 \rangle$, where*

$$X_1 = \{g_2, g_3, g_4\} \qquad X_2 = \{g_1, g_5\} \qquad X_3 = \{g_6\},$$

is an EFX allocation of the first six goods. No complete EFX allocation Pareto dominates X.

Proof. Note that $v_1(X_1) = 16$, $v_2(X_2) = 15$, and $v_3(X_3) = 10$. We will show that there is no complete EFX allocation X' with $v_1(X'_1) \geq 16$, $v_2(X'_2) \geq 15$ and $v_3(X'_3) \geq 10$. To this end, we systematically consider potential bundles X'_1 that can keep a_1's valuation at or above 16.

Let us first assume $g_6 \in X'_1$, and hence, $v_1(X'_1) \geq 17$. Now, to ensure $v_3(X'_3) \geq 10$, we need to allocate g_5 and g_7 to a_3. We are left with goods g_1, g_2, g_3 and g_4. In order to ensure $v_2(X'_2) \geq 15$, we definitely need to allocate g_1, g_3 and g_4 to a_2. Now even if we allocate the remaining good g_2 to a_1, we will have $v_1(X'_1) = v_1(\{g_2, g_6\}) = 19 < 20 = v_1(\{g_1, g_3\}) \leq v_1(X'_2 \setminus g_4)$. Therefore, a_1 will strongly envy a_2. Thus $g_6 \notin X'_1$.

If $g_6 \notin X'_1$ and $v_1(X'_1) \geq 16$, X'_1 must contain g_3 (the total valuation for a_1 of all the goods other than g_3 and g_7 is less than 16). We need to consider several subcases.

Assume $g_1 \in X'_1$ first. Since X'_1 already contains g_1 and g_3, the goods that can be allocated to a_2 and a_3 are g_2, g_4, g_5, g_6, and g_7. In order to ensure $v_2(X'_2) \geq 15$ we need to allocate g_4, g_5, and g_7 to a_2. Even if we allocate all the remaining goods (g_2 and g_6) to a_3, we have $v_3(X'_3) = v_3(\{g_3, g_6\}) = 10 < 11 = v_3(\{g_5, g_7\}) \leq v_3(X'_2 \setminus g_4)$. Therefore, a_3 will strongly envy a_2.

Thus $g_1 \notin X'_1$. Since neither g_1 nor g_6 belongs to X'_1, the only way to ensure $v_1(X'_1) \geq 16$ is to at least allocate g_2, g_3, and g_4 to a_1(we can allocate more). Similarly, given that the goods not allocated yet are g_1, g_5, g_6, and g_7, the only way to ensure $v_1(X'_2) \geq 15$ is to allocate at least g_1 and g_5 to a_2. Similarly, the only way to ensure $v_3(X'_3) \geq 10$ now is to allocate at least g_6 to a_3. We next show that adding g_7 to any one of the existing bundles will cause a violation of the EFX property.

- Adding g_7 to X'_1: a_2 strongly envies a_1 as $v_2(X'_2) = 15 < 16 = v_2(\{g_3, g_4, g_7\}) = v_2(X'_1 \setminus g_2)$.
- Adding g_7 to X'_2: a_3 strongly envies a_2 as $v_3(X'_3) = 10 < 11 = v_3(\{g_5, g_7\}) = v_3(X'_2 \setminus g_1)$.
- Adding g_7 to X'_3: a_1 strongly envies a_3 as $v_1(X'_1) = 16 < 17 = v_1(g_6) = v_1(X'_3 \setminus g_7)$.

Thus, there exists no complete EFX allocations Pareto dominating X. □

We now move on to the second example. We will modify the example in Table 6.1 to highlight some barriers in the existence of "efficient" EFX allocations, i.e., complete EFX allocations with high Nash welfare. We have seen the existence of EFX allocations with bounded charity that have high Nash welfare in Chapter 4. However, we have limited knowledge on the existence of complete EFX allocations with high Nash welfare, even when there are three agents with additive valuations. To this end, Caragiannis et al. [27] mention the following conjecture, which if true, would imply the existence of complete EFX allocations with high Nash welfare.[6]

Conjecture 5.24. *Adding an good to an instance that admits an EFX allocation results in another instance that admits an EFX allocation with Nash welfare at least as high as that of the partial allocation before.*

We will now show that this conjecture is false, which suggests that EFX demands "too much fairness" and some "trade-offs with efficiency" may be necessary. In particular, we construct an instance I', such that there exists a partial EFX allocation X with Nash welfare $NW(X)$ strictly larger than the Nash welfare $NW(X')$ of any complete EFX allocation X'. From the example in Table 6.1, it is clear that in any complete EFX allocation, we need to decrease the valuation of one of the agents. The high level idea is to modify I to I' such that the decrease in valuation of one of the agents is significantly more than the increase in valuation of the other agents.

Theorem 5.25. *For the instance I' with three agents and seven goods given in Table 5.4, the allocation $X = \langle X_1, X_2, X_3 \rangle$, where*

$$X_1 = \{g_2, g_3, g_4\} \qquad X_2 = \{g_1, g_5\} \qquad X_3 = \{g_6\},$$

is an EFX allocation of the first six goods whose Nash welfare is larger than the Nash welfare of any complete EFX allocation.[7]

[6] In their talk at EC'19 they explicitly mention this as the "Monotonicity Conjecture".

[7] The reader is encouraged to keep an eye on Table 5.4 for the entire proof of Theorem 5.25.

	g_1	g_2	g_3	g_4	g_5	g_6	g_7
$\mathbf{a_1}$	$\varepsilon^3 + 6\varepsilon^5$	$2\varepsilon^5$	$10 - \varepsilon^3$	ε^3	$10 - 2\varepsilon^3$	$10 + 3\varepsilon^5$	ε^5
$\mathbf{a_2}$	ε	0	$10 - \varepsilon^2 + \varepsilon^6$	$2\varepsilon^2$	10	0	$\varepsilon - \varepsilon^2$
$\mathbf{a_3}$	0	0	0	0	$10 - \varepsilon^4$	10	$2\varepsilon^4$

Table 5.4: An instance where no complete EFX allocation has larger Nash welfare than the EFX allocation X for the first six goods defined in the text. The valuations are assumed to be additive and the entry in row i and column j is the value of good j for agent i; ε is positive, but infinitesimally small.

Proof. Observe that $NW(X) = ((10 + 2\varepsilon^5) \cdot (10 + \varepsilon) \cdot (10))^{1/3}$. Let X' be a complete EFX allocation with maximum Nash welfare.

Lemma 5.26. *X' allocates the goods g_3, g_5 and g_6 to distinct agents. Additionally,*

- *X'_2 contains exactly one good from $\{g_3, g_5\}$.*
- *X'_3 contains exactly one good from $\{g_5, g_6\}$.*

Proof. Consider the following complete EFX allocation $\hat{X} = \langle \hat{X}_1, \hat{X}_2, \hat{X}_3 \rangle$:

$$\hat{X}_1 = \{g_6\} \qquad \hat{X}_2 = \{g_3, g_4, g_7\} \qquad \hat{X}_3 = \{g_1, g_2, g_5\}$$

It is easy to verify that $\hat{X}$ is EFX and $NW(\hat{X}) = ((10 + 3\varepsilon^5)(10 + \varepsilon + \varepsilon^6)(10 - \varepsilon^4))^{1/3}$. Since X' is a complete EFX allocation with maximum Nash welfare, we have $NW(X') \geq NW(\hat{X})$. If g_3, g_5, and g_6 are not allocated to distinct agents, there is an agent a_i who does not get any of these goods. The valuation of this agent is at most 4ε (since ε is the maximum valuation of any agent for any good outside the set $\{g_3, g_5, g_6\}$). The valuation of the other two agents can be at most $3 \cdot (10 + \varepsilon) + 4\varepsilon = 30 + 7\varepsilon$ (since ε is the maximum valuation of any agent for any good outside the set $\{g_3, g_5, g_6\}$, and $10 + \varepsilon$ upper bounds the maximum valuation of any good in $\{g_3, g_5, g_6\}$). Thus $NW(X') \leq ((4\varepsilon) \cdot (30 + 7\varepsilon)^2)^{1/3} < NW(\hat{X})$ for sufficiently small ε.

A similar argument shows that X'_2 contains at least one good from $\{g_3, g_5\}$ and X'_3 contains at least one good from $\{g_5, g_6\}$ (since these are the only goods that the agents value close to 10). Since the goods g_3, g_5, and g_6 are allocated to distinct agents, a_2 will get exactly one good from $\{g_3, g_5\}$ and a_3 will get exactly one good from $\{g_5, g_6\}$. ☐

Let us denote the set $\{g_5, g_6, g_7\}$ as VAL_3, the goods valuable for agent a_3. Note that $v_3(X'_3) = v_3(X'_3 \cap VAL_3)$. We will now prove our claim by studying the cases that arise depending on $X'_3 \cap VAL_3$. By Lemma 5.26, $X'_3 \cap VAL_3$ is non-empty and contains exactly one of g_5 and g_6. Thus, $X'_3 \cap VAL_3$ can be $\{g_5\}$, $\{g_6\}$, $\{g_5, g_7\}$, or $\{g_6, g_7\}$ only.

Lemma 5.27. *If $X'_3 \cap VAL_3 = \{g_5\}$, then $NW(X') < NW(X)$.*

Proof. We have that $v_3(X'_3) = v_3(X'_3 \cap VAL_3) = 10 - \varepsilon^4$. Lemma 5.26 implies that X'_2 contains g_3 and X'_1 contains g_6. Note that X'_1 cannot contain any additional good other than g_6 as this would lead to a_3 strongly envying a_1 (note that $v_3(g_6) = 10 > 10 - \varepsilon^4 = v_3(X'_3)$). Therefore $v_1(X'_1) = 10 + 3\varepsilon^5$. Now we distinguish two cases depending on whether or not X'_2 contains g_1.

- $g_1 \in X'_2$: In this case, $X'_2 = \{g_1, g_3\}$, as otherwise a_1 strongly envies a_2 (note that $v_1(X'_1) = 10 + 3\varepsilon^5 < 10 + 6\varepsilon^5 = v_1(\{g_1, g_3\})$, and hence, $v_2(X'_2) = v_2(\{g_1, g_3\}) = 10 + \varepsilon + \varepsilon^6 - \varepsilon^2$. Thus,
$$\frac{v_1(X'_1)}{v_1(X_1)} = 1 + \frac{\varepsilon^5}{10 + 2\varepsilon^5}, \qquad \frac{v_2(X'_2)}{v_2(X_2)} = 1 - \frac{\varepsilon^2 - \varepsilon^6}{10 + \varepsilon}, \text{ and } \qquad \frac{v_3(X'_3)}{v_3(X_3)} \leq 1,$$
and hence, $NW(X')/NW(X) < 1$.

- $g_1 \notin X'_2$: Then $v_2(X'_2) \leq v_2(\text{remaining goods}) = v_2(\{g_2, g_3, g_4, g_7\}) = 10 + \varepsilon + \varepsilon^6$, and hence,
$$\frac{NW(X')}{NW(X)} = ((1 + \frac{\varepsilon^5}{10 + 2\varepsilon^5})(1 + \frac{\varepsilon^6}{10 + \varepsilon})(1 - \frac{\varepsilon^4}{10}))^{1/3} < 1$$
. ☐

Lemma 5.28. *If $X'_3 \cap VAL_3 = \{g_5, g_7\}$, then $NW(X') < NW(X)$.*

Proof. This proof follows the proof of Lemma 5.27 closely. We have $v_3(X'_3) = v_3(X'_3 \cap VAL_3) = 10 + \varepsilon^4$. Lemma 5.26 implies that X'_2 contains g_3 and X'_1 contains g_6. We now distinguish two cases depending on whether or not $\{g_1, g_4\} \subseteq X'_2$.

- $\{g_1, g_4\} \subseteq X'_2$: Then a_1 strongly envies a_2 as $v_1(X'_1) \leq v_1(\text{remaining goods}) = v_1(\{g_2, g_6\}) = 10 + 5\varepsilon^5 < 10 + 6\varepsilon^5 = v_1(\{g_1, g_3\}) \leq v_1(X'_2 \setminus g_4)$.

- $\{g_1, g_4\} \not\subseteq X'_2$. Then $v_2(X'_2) \leq v_2(\{g_1, g_2, g_3\}) = 10 + \varepsilon - \varepsilon^2 + \varepsilon^6$ (not giving the less valuable g_4 and giving everything else that remains). Also, $v_1(X'_1) \leq v_1(\{g_1, g_2, g_4, g_6\}) = 10 + 2\varepsilon^3 + 11\varepsilon^5$. Thus,
$$\frac{v_1(X'_1)}{v_1(X_1)} = 1 + \frac{2\varepsilon^3 + 9\varepsilon^5}{10 + 2\varepsilon^5}, \qquad \frac{v_2(X'_2)}{v_2(X_2)} = 1 - \frac{\varepsilon^2 - \varepsilon^6}{10 + \varepsilon}, \text{ and } \qquad \frac{v_3(X'_3)}{v_3(X_3)} = 1 + \frac{\varepsilon^4}{10}$$
, and hence, $NW(X') < NW(X)$. ☐

Lemma 5.29. *If $X'_3 \cap VAL_3 = \{g_6, g_7\}$, then $NW(X') < NW(X)$.*

Proof. We have $v_3(X'_3) = v_3(X'_3 \cap VAL_3) = 10 + 2\varepsilon^4$. By Lemma 5.26, one of g_3 and g_5 will be allocated to each of a_2 and a_1. We argue that $g_1 \in X'_1$. If $g_1 \notin X'_1$, then
$$\begin{aligned} v_1(X'_1) &\leq max(v_1(g_3), v_1(g_5)) + v_1(\{g_2, g_4\}) \\ &= (10 - \varepsilon^3) + \varepsilon^3 + 2\varepsilon^5 \\ &< 10 + 3\varepsilon^5 \\ &= v_1(g_6) \\ &= v_1(X'_3 \setminus g_7), \end{aligned}$$
and hence, a_1 strongly envies a_3.

Therefore $g_1 \in X'_1$. But we still have $v_1(X'_1) \leq max(v_1(g_3), v_1(g_5)) + v_1(\{g_1, g_2, g_4\}) = (10 - \varepsilon^3) + (2\varepsilon^3 + 8\varepsilon^5) = 10 + \varepsilon^3 + 8\varepsilon^5$. However, since $g_1 \in X'_1$, we have that $v_2(X'_2) \leq max(v_2(g_3), v_2(g_5)) + v_2(\{g_2, g_4\}) = 10 + 2\varepsilon^2$. Thus,
$$\frac{v_1(X'_1)}{v_1(X_1)} = 1 + \frac{\varepsilon^3 + 6\varepsilon^5}{10 + 2\varepsilon^5}, \qquad \frac{v_2(X'_2)}{v_2(X_2)} \leq 1 - \frac{\varepsilon - 2\varepsilon^2}{10 + \varepsilon}, \text{ and } \qquad \frac{v_3(X'_3)}{v_3(X_3)} = 1 + \frac{2\varepsilon^4}{10}$$
, and hence, $NW(X') < NW(X)$. ☐

Lemma 5.30. *If $X'_3 \cap VAL_3 = \{g_6\}$ and $g_3 \in X'_2$, then $NW(X') < NW(X)$.*

Proof. We have $v_3(X'_3) = v_3(X'_3 \cap VAL_3) = 10$. Since g_3 and g_5 are allocated to a_1 and a_2, respectively, and $g_3 \in X'_2$, we have $g_5 \in X'_1$ by Lemma 5.26. We now distinguish two cases depending, on whether or not $g_1 \in X'_2$.

- $g_1 \in X'_2$: Then X'_2 cannot contain any other goods than g_1 and g_3, else a_1 will strongly envy a_2: $v_1(X'_1) \leq v_1(\text{remaining goods}) \leq v_1(\{g_2, g_4, g_5, g_7\}) = 10 - \varepsilon^3 + 3\varepsilon^5 < 10 + 6\varepsilon^5 = v_1(\{g_1, g_3\})$. Therefore $v_2(X'_2) = v_2(\{g_1, g_3\}) = 10 + \varepsilon - \varepsilon^2 + \varepsilon^6$. Also, note that $v_1(X'_1) \leq v_1(\{g_2, g_4, g_5, g_7\}) = 10 - \varepsilon^3 + 3\varepsilon^5$. In that case, the valuations of both a_1 and a_2 decrease, and that of a_3 does not increase. Thus $NW(X') < NW(X)$.

- $g_1 \notin X'_2$: Then X'_2 cannot contain both of g_4 and g_7, else a_1 will strongly envy a_2: $v_1(X'_1) \leq v_1(\text{remaining goods}) = v_1(\{g_1, g_2, g_5\}) = 10 - \varepsilon^3 + 8\varepsilon^5 < 10 = v_1(\{g_3, g_4\}) = v_1(X'_2 \backslash g_7)$. Therefore, $v_2(X'_2) \leq max(v_2(g_4), v_2(g_7)) + v_2(\text{remaining goods}) \leq max(v_2(g_4), v_2(g_7)) + v_2(\{g_2, g_3\}) = 10 + \varepsilon - 2\varepsilon^2 + \varepsilon^6$ and $v_1(X'_1) \leq v_1(\{g_1, g_2, g_4, g_5, g_7\}) = 10 + 9\varepsilon^5$. Thus,

$$\frac{v_1(X'_1)}{v_1(X_1)} = 1 + \frac{7\varepsilon^5}{10 + 2\varepsilon^5}, \qquad \frac{v_2(X'_2)}{v_2(X_2)} \leq 1 - \frac{2\varepsilon^2 - \varepsilon^6}{10 + \varepsilon}, \text{ and } \qquad \frac{v_3(X'_3)}{v_3(X_3)} = 1$$

  , and hence, $NW(X') < NW(X)$. □

Lemma 5.31. *If $X'_3 \cap VAL_3 = \{g_6\}$ and $g_3 \notin X'_2$, then $NW(X') < NW(X)$.*

Proof. We have $v_3(X'_3) = v_3(X'_3 \cap VAL_3) = 10$. Since $g_3 \notin X'_2$, we have $g_5 \in X'_2$ and $g_3 \in X'_1$ by Lemma 5.26. We now distinguish two cases depending on whether or not $g_7 \in X'_2$.

- $g_7 \in X'_2$: Then X'_2 cannot contain any other goods than g_5 and g_7, else a_3 will strongly envy a_2: $v_3(X'_3) = 10 < 10 + \varepsilon^4 = v_3(\{g_5, g_7\})$. Therefore, $v_2(X'_2) = v_2(\{g_5, g_7\}) = 10 + \varepsilon - \varepsilon^2$ and $v_1(X'_1) \leq v_1(\text{remaining goods}) = v_1(\{g_1, g_2, g_3, g_4\}) = 10 + \varepsilon^3 + 8\varepsilon^5$. Thus,

$$\frac{v_1(X'_1)}{v_1(X_1)} = 1 + \frac{\varepsilon^3 + 6\varepsilon^5}{10 + 2\varepsilon^5}, \qquad \frac{v_2(X'_2)}{v_2(X_2)} \leq 1 - \frac{\varepsilon^2}{10 + \varepsilon}, \text{ and } \qquad \frac{v_3(X'_3)}{v_3(X_3)} = 1$$

  , and hence, $NW(X') < NW(X)$.

- $g_7 \notin X'_2$: Then X'_2 cannot contain both of g_1 and g_4 else a_1 will strongly envy a_2: $v_1(X'_1) \leq v_1(\text{remaining goods}) = v_1(\{g_2, g_3, g_7\}) = 10 - \varepsilon^3 + 3\varepsilon^5 < 10 - \varepsilon^3 + 6\varepsilon^5 = v_1(\{g_1, g_5\}) = v_1(X'_2 \setminus g_4)$. Now we consider two cases depending on whether or not $g_1 \in X'_2$.

  - $g_1 \in X'_2$: Then X'_2 cannot have g_4. Thus $v_2(X'_2) \leq v_2(g_1) + v_2(\text{remaining goods}) = v_2(g_1) + v_2(\{g_2, g_5\}) = 10 + \varepsilon = v_2(X_2)$. Note that X'_1 cannot have all of the remaining goods g_2, g_3, g_4, g_7, else a_2 will strongly envy a_1: $v_2(X'_2) \leq 10 + \varepsilon < 10 + \varepsilon + \varepsilon^6 = (10 - \varepsilon^2 + \varepsilon^6) + (2\varepsilon^2) + (\varepsilon - \varepsilon^2) = v_2(\{g_3, g_4, g_7\}) = v_2(\{g_2, g_3, g_4, g_7\} \backslash g_2)$. Therefore, X'_1 is a strict subset of $\{g_2, g_3, g_4, g_7\}$, and it should contain g_7 (as we are in the case where neither X_2' nor X_3' can have g_7). Since a_1's valuation for g_7 is strictly less than her valuation for any of g_2, g_3, and g_4, we have that $v_1(X_1') < v_1(\{g_2, g_3, g_4\}) = v_1(X_1)$. Since we are in the case where $v_2(X_2') \leq v_2(X_2)$ and $v_3(X_3') = v_3(X_3)$, we have $NW(X') < NW(X)$.

– $g_1 \notin X_2'$: Then $v_2(X_2') \leq v_2(\text{remaining goods}) = v_2(\{g_2, g_4, g_5\}) = 10 + 2\varepsilon^2$ and $v_1(X_1') \leq v_1(\{g_1, g_2, g_3, g_4, g_7\}) = 10 + \varepsilon^3 + 9\varepsilon^5$. Thus,

$$\frac{v_1(X_1')}{v_1(X_1)} = 1 + \frac{\varepsilon^3 + 7\varepsilon^5}{10 + 2\varepsilon^5}, \qquad \frac{v_2(X_2')}{v_2(X_2)} \leq 1 - \frac{\varepsilon - 2\varepsilon^2}{10 + \varepsilon}, \text{ and } \qquad \frac{v_3(X_3')}{v_3(X_3)} = 1$$

, and hence, $NW(X') < NW(X)$. □

Lemmas 5.30 and 5.31 immediately imply the following:

Lemma 5.32. *If $X_3' \cap VAL_3 = \{g_6\}$, then $NW(X') < NW(X)$.*

We are now ready to complete the proof. Lemma 5.26 implies that a_3 gets exactly one good from $\{g_5, g_6\}$. Thus, $X_3' \cap VAL_3 \neq \emptyset$, and $\{g_5, g_6\} \not\subseteq X_3' \cap VAL_3$. So $X_3' \cap VAL_3 \in \{\{g_5\}, \{g_6\}, \{g_5, g_7\}, \{g_6, g_7\}\}$. However, Lemmas 5.27, 5.28, 5.29, and 5.32 imply that in all of these cases, $NW(X') < NW(X)$. □

CHAPTER 6

Almost EFX Allocations with Sublinear Charity

In this Chapter, we show the existence of improved relaxations of EFX allocations. Given the hardness of this problem, we believe that studying weaker relaxations (of EFX allocations) is a systematic and promising direction to find the answer regarding the existence of EFX allocations. We elaborate this point briefly. It has been suspected in Plaut and Roughgarden [84] that EFX allocations may not exist in the general setting:

> "*We suspect that at least for general valuations, there exist instances where no EFX allocation exists, and it may be easier to find a counterexample in that setting.*"

However, finding counter-examples, at least in the additive setting, would also be a very challenging task: Quite recently Manurangsi and Suksompong [73] show that when agents valuations for individual goods are drawn at random from a probability distribution, then EFX allocations exist with high probability. This demands a non brute-force approach to find counter-examples, if any. Thus finding better relaxations (improving the approximation factor or reducing the number of unallocated goods in a partial EFX allocation) is a systematic way to find the right answer to this open problem. We achieve exactly this by the first main result of this chapter,

Theorem 6.1. *For all $\varepsilon \in (0, 1/2]$ we can determine a partial allocation X and a set of unallocated goods P in polynomial-time such that*

- *X is $(1-\varepsilon)$-EFX,*
- *$|P| \leq 64(n/\varepsilon)^{4/5}$.*

We remark that reducing the number of unallocated goods could be quite challenging: Indeed, a corollary from the main result (Theorem 3.1) in Chapter 3 already establishes that there exists a partial EFX allocation with at most two goods unallocated when there are three agents. However, removing the last two goods to obtain an EFX allocation for three agents turned out to be highly non-trivial task and the proof in Chapter 5 requires careful and cumbersome case analysis. Furthermore, in Section 6.5 of this chapter, we show that the technique in Chapter 5 does not extend to four agents with additive valuations for finding a $(1-\varepsilon)$-EFX allocation.

In this chapter, we show a novel method that reduces the problem of determining good relaxations of EFX allocations to a combinatorial problem in graph theory. We now briefly elaborate the combinatorial graph problem . We define the *rainbow cycle number* of an integer as follows.

Definition 6.2. *For any positive integer d, the rainbow cycle number or $R(d)$ is the largest k such that there exists a directed k-partite graph $G = (\cup_{i\in[k]} V_i, E)$ such that*

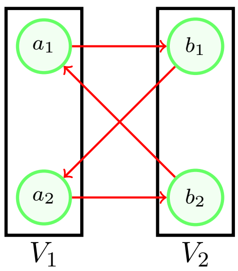
(1) $|V_i| \leq d$ *for all* $i \in [k]$,

(2) for any two distinct parts V_i *and* V_j *in* G, *every vertex in* V_i *has an incoming edge from a vertex in* V_j, *and*

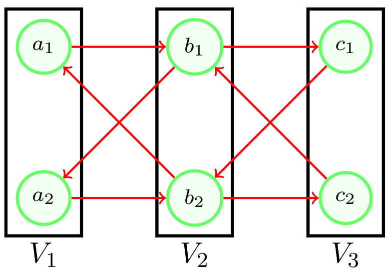
(3) there exists no cycle in G *that intersects each part at most once.*

To give the reader a clearer understanding of Definition 6.2, we show that $R(1) = 1$ and $R(2) = 2$. Let us deduce that $R(1) = 1$: It is clear that G can be a single vertex and satisfy all the conditions in Definition 6.2 and thus $R(1) \geq 1$. However, $R(1)$ cannot be larger than one, as otherwise we have two parts V_1 and V_2 in a graph G, where there is exactly one vertex each in V_1 and V_2. So let $V_1 = \{a_1\}$ and $V_2 = \{a_2\}$. By condition 2 in Definition 6.2, we must have an edge from a_1 to a_2 and an edge from a_2 to a_1. This gives a 2-cycle $a_1 \to a_2 \to a_1$. However, this cycle contains exactly one vertex from each V_1 and V_2, which contradicts condition 3 in Definition 6.2.

Now, using a more involved argument we show that $R(2) = 2$. We first show that $R(2) \leq 2$ by contradiction. Let us assume otherwise and let V_1, V_2 and V_3 be any three parts of G. We first look into the edges of the induced bipartite graph $G[V_1 \cup V_2]$. Without loss of generality, let us assume that vertex b_1 in V_2 has an incoming edge from vertex a_1 in V_1. By condition 2 in Definition 6.2, a_1 has an incoming edge from some vertex in V_2. However, this vertex cannot be b_1 as this will violate condition 3 in Definition 6.2. This implies that there must be another vertex in V_2, say b_2 that has an edge to a_1. Again, by a similar argument, b_2 cannot have an incoming edge from a_1 and therefore has an incoming edge from another vertex in V_1, say a_2 and a_2 has the incoming edge from b_1 and not b_2 (since there can be no other vertices in V_2). Thus, the induced bipartite graph $G[V_1 \cup V_2]$ is a four-cycle as shown below



Note that the induced bipartite graph $G[V_2 \cup V_3]$ will be isomorphic to $G[V_1 \cup V_2]$. Thus, so far we have the following edges in $G[V_1 \cup V_2 \cup V_3]$,



We now look at the edges between the parts V_1 and V_3. Since $G[V_1 \cup V_3]$ is isomorphic to $G[V_1 \cup V_2]$, it must also be a four-cycle and hence in $G[V_1 \cup V_3]$, there is either an edge from a_1 to c_1 or from c_1 to a_1. If there is an edge from a_1 to c_1, then we have a 3-cycle $a_1 \to c_1 \to b_2 \to a_1$, which visits each part of G at most once and thus this is a contradiction. Similarly, if there is an edge from c_1 to a_1, then also we have a 3-cycle $a_1 \to b_1 \to c_1 \to a_1$, which visits each part of G at most once and thus this is also a contradiction. Thus, $R(2) \leq 2$. Also we have that $R(2) \geq 2$: the bipartite graph $G[V_1 \cup V_2]$. Therefore, we have $R(2) = 2$.

However, it is not at all clear what values $R(d)$ takes, or if it is finite for all integers d. As the second key result of this chapter, we show that any upper-bound on $R(d)$ implies the existence of $(1-\varepsilon)$-EFX allocations with sublinear many unallocated goods and better upper bounds on $R(d)$ give better upper bounds on the number of unallocated goods. Formally,

Theorem 6.3. *Let $h(d) = d \cdot R(d)$ and $\varepsilon \in (0, 1/2]$. Let $h^{-1}(n/\varepsilon)$ be the smallest integer such that $h(d) \geq n/\varepsilon$. Then, there is a $(1-\varepsilon)$-EFX allocation X and a set of unallocated goods P such that $|P| \leq (4n/(\varepsilon \cdot h^{-1}(2n/\varepsilon))$.*

The beneficial aspect of Theorem 6.3 is that it gives a clean graph theoretic framework for determining improved relaxations of EFX allocations. However, Theorem 6.3 is meaningless is $R(d)$ is not finite for all d. We briefly sketch the proof that this is indeed the case:

$R(d)$ **is finite.** We briefly show that for any $d \in \mathbb{N}$, $R(d)$ is finite. Consider a k-partite graph $G = (\cup_{i \in [k]} V_i, E)$ in Definition 6.2. For all $i \in [k]$, let $V_i = \{(i,1), (i,2), \dots, (i, |V_i|)\}$. For all $i < j$ and $i' < j'$, we say that the directed bipartite graphs $G[V_i \cup V_j]$ and $G[V_{i'} \cup V_{j'}]$ have the same configuration if and only if for each directed edge from vertex (i,a) to (j,b) (and equivalently from (j,b') to (i,a')) in $G[V_i \cup V_j]$, there is an edge from (i',a) to (j',b) (and equivalently from (j',b') to (i',a')) in $G[V_{i'} \cup V_{j'}]$ and vice-versa. We first show that if there are $4d$ parts in G, say w.l.o.g. $V_1, V_2, \dots, V_{4d}$, such that the induced directed bipartite graph $G[V_i \cup V_j]$ has the same configuration for all $1 \leq i < j \leq 4d$, then there exists a cycle in G that visits each part at most once.

Consider the parts V_1 and V_2, and the induced directed bipartite graph $G[V_1 \cup V_2]$. Since every vertex in one part has an incoming edge from a vertex in the other part, $G[V_1 \cup V_2]$ is cyclic. Let the simple cycle be $C = (1, i_1) \to (2, i_2) \to (1, i_3) \to \cdots \to (2, i_{2\beta}) \to (1, i_1)$ for some $\beta \leq d$. Since all the induced bipartite graphs $G[V_i \cup V_j]$ have the same configuration for all $1 \leq i < j \leq 4d$, we can claim that for all $\ell \in [\beta]$, for each edge $(1, i_{2\ell-1}) \to (2, i_{2\ell})$ in C, there is an edge from $(2\ell-1, i_{2\ell-1})$ to $(4d-\ell, i_{2\ell})$ in $G[V_{2\ell-1}, V_{4d-\ell}]$ (note that $2\ell - 1 < 4d - \ell$ as $\ell \leq \beta \leq d$). Similarly for all $\ell \in [\beta]$, for each edge $(2, i_{2\ell}) \to (1, i_{2\ell+1})$ in C ($2\beta+1$ is to interpreted as 1), there is an edge from $(4d-\ell, i_{2\ell})$ to $(2\ell+1, i_{2\ell+1})$ in $G[V_{2\ell+1}, V_{4d-\ell}]$ (again, note that $2\ell+1 < 4d-\ell$ as $\ell \leq \beta \leq d$). This implies that there is a cycle $C' = (1, i_1) \to (4d-1, i_2) \to (3, i_3) \to (4d-2, i_4) \to \cdots \to (4d-\beta, i_{2\beta}) \to (1, i_1)$ in G. Clearly, C visits each part of G at most once. Therefore, there cannot be $4d$ parts in G such that the induced directed bipartite graph $G[V_i \cup V_j]$ has the same configuration for all $1 \leq i < j \leq 4d$.

We now rephrase the question about an upper bound on $R(d)$. Let $\mathcal{D}$ be the set of all configurations of a directed bipartite graph, where the number of vertices in each part is at most d and every vertex has an incoming edge. We treat $\mathcal{D}$ as a set of *colors* and note that $|\mathcal{D}| \in 2^{\mathcal{O}(d^2)}$. Now consider a complete graph K_k with vertex set $[k]$, where the vertex $\ell \in [k]$ corresponds to part V_ℓ in G. For all $1 \leq i < j \leq k$, we color/label the edge (i, j) in K_k with a color from $\mathcal{D}$. The color on the edge (i, j) corresponds to the configuration of the directed bipartite graph $G[V_i \cup V_j]$. Clearly, $R(d)$ must be strictly smaller than the largest k such that every coloring of the edges of K_k with colors from $\mathcal{D}$ contains a monochromatic clique of size $4d$. This value of k corresponds to the *(multicolor) Ramsey number* [48] $\mathcal{R}(n_1, n_2, \dots n_{|\mathcal{D}|})$ in which $n_i = 4d$ for all $i \in [|\mathcal{D}|]$. This number is finite and the current best known upper bounds on it are exponential in $|\mathcal{D}|$ and d [53, 71, 48, 44]. Therefore, $R(d)$ is also bounded. However, this upper-bound is very large and only provides a weak version of Theorem 7.3. This necessitates the study of finding "good" upper bounds on $R(d)$; in particular, upper bounds that are polynomial in d. We address this in Section 6.3 by showing that $R(d) \in \mathcal{O}(d^4)$. This brings us to the third main result of this chapter.

Theorem 6.4. *For all $d \geq 1$, we have $R(d) \leq d^4 + d$. Furthermore, let G be a k-partite digraph with $k > d^4 + d$ parts of cardinality at most d each, such that for every vertex v and any part W not containing v, there is an edge from W to v. Then, there exists a cycle in G visiting each part at most once, and it can be found in time polynomial in k.*

Theorems 6.4 and 6.3 imply Theorem 6.1. We remark that, although we give a polynomial upper bound on $R(d)$, we believe that there is further room for improvement. As an illustration, our upper bound in Theorem 6.4 for $d = 2$ is $2^4 + 2 = 18$, while we showed that $R(2) \leq 2$. We suspect that $R(d) \in \mathcal{O}(d)$ (implying existence of $(1-\varepsilon)$-EFX allocations with $\mathcal{O}(\sqrt{n/\varepsilon})$ many unallocated goods). We believe that finding better upper bounds on $R(d)$ is a natural combinatorial question and better upper-bounds to $R(d)$ imply the existence of better relaxations of EFX allocations. Therefore, investigating better upper bounds on $R(\cdot)$ is of interest in its own right and we leave this as an interesting open problem.

Finding $(1-\varepsilon)$-EFX allocations with high Nash welfare. We have mentioned throughout the thesis that *efficiency* is also a desirable property of the final allocation. Similar to the algorithm in Chapter 3, we show that with minor modifications to our main algorithm, we can determine an allocation that satisfies the conditions in Theorem 6.1, and achieves a $2e^{1/e} \approx 2.89$ approximation of the Nash welfare, i.e., in polynomial-time we can find efficient $(1-\varepsilon)$-EFX allocation with sublinear charity (Section 6.4).

6.1 Notation and Tools

Similar to Chapter 5, we write $v_i(g)$ instead of $v_i(\{g\})$ and $v_i(S \cup g)$ for $v_i(S \cup \{g\})$. For a fixed $0 < \varepsilon < 1$, we say that an agent i

- *envies* a set S of goods if $v_i(X_i) < v_i(S)$,
- *heavily envies* a set S of goods if $v_i(X_i) < (1-\varepsilon)v_i(S)$,
- *strongly envies* a set S of goods if it heavily envies a proper subset of S, and

An agent envies (heavily envies, strongly envies) an agent j if it has these feelings for the set X_j. Clearly, strong envy implies heavy envy implies envy. An allocation X' *strongly*

Pareto-dominates an allocation X, or equivalently $X' >_{PD} X$, if and only if $v_i(X'_i) \geq v_i(X_i)$ for all $i \in [n]$ and for some agent $i' \in [n]$ we have $(1-\varepsilon) \cdot v_{i'}(X'_{i'}) \geq v_{i'}(X_{i'})$.

At a high level, our algorithm is similar to previous algorithms used to prove the existence of relaxations of EFX allocations [36, 84, 33]. Our algorithm always maintains a $(1-\varepsilon)$-EFX allocation on the set of allocated goods and as long as the current allocation and the set of unallocated goods P satisfies "some properties", it determines another $(1-\varepsilon)$-EFX allocation that strongly Pareto-dominates the previous $(1-\varepsilon)$-EFX allocation. Since the valuation of an agent for the entire good set is bounded, this procedure will eventually converge to a $(1-\varepsilon)$-EFX allocation, where the current allocation and the set of unallocated goods do not satisfy these properties. The bulk of the effort goes into determining the right properties under which one can come up with update rules that transform one $(1-\varepsilon)$-EFX allocation into a "better" $(1-\varepsilon)$-EFX allocation. We briefly recollect the update rules used in [72] and [36].

Envy cycle elimination [72]. The *envy-graph* E_X of an $(1-\varepsilon)$-EFX allocation X has the agents as its vertex set and there is an edge from vertex i to vertex j in E_X if agent i envies agent j, i.e., $v_i(X_i) < v_i(X_j)$. The paper [72] shows that whenever E_X has a cycle, then one can determine another $(1-\varepsilon)$-EFX allocation X' in which no agent has a worse bundle and $E_{X'}$ is acyclic. Formally,

Lemma 6.5 ([72])**.** *Consider a $(1-\varepsilon)$-EFX allocation X. If there is a cycle in E_X, then in polynomial-time , we can determine a $(1-\varepsilon)$-EFX allocation X' such that $v_i(X'_i) \geq v_i(X_i)$ for all $i \in [n]$, and $E_{X'}$ is acyclic.*[1]

Update rules in Chapter 3. We modify the update rules in Chapter 3 slightly, as we are dealing with $(1-\varepsilon)$-EFX allocations and not EFX allocations. These rules are more involved and make essential use of the concept of a *a most envious agent*. We adjust the definition of a most envious agent from Chapter 3: Given an allocation X, and any set $S \subseteq M$, we denote the set of most envious agents of S as $A_X(S)$.

Definition 6.6. *Given a set $S \subseteq M$ and an allocation X, an agent i is a most envious agent of the set S or $i \in A_X(S)$ if and only if there exists $Z_i \subseteq S$ such that $(1-\varepsilon) \cdot v_i(Z_i) > v_i(X_i)$ (agent i heavily envies Z_i), and for all agents j, we have $(1-\varepsilon) \cdot v_j(Z') \leq v_j(X_j)$ for all $Z' \subset Z_i$ (no agent strongly envies Z_i).*

Also, we can re-adjust the necessary and sufficient condition (Observation 3.3 from Chapter 3) for $A_X(S)$ to be non-empty.

Observation 6.7. *$A_X(S) \neq \emptyset$ if and only if there is some agent i that heavily envies S. Also, in $\mathcal{O}(n \cdot |S|^2)$ time, one can find an agent $t \in A_X(S)$ and a set $Z \subseteq S$ such that t heavily envies Z and no agent strongly envies Z.*

Following the definition of a *champion* in Chapter 5, we say that given an allocation X, an agent i champions j w.r.t an unallocated good g if and only if $i \in A_X(X_j \cup g)$.

[1] Let C be an envy cycle. For each edge (i,j) of the cycle one assigns in X' the bundle X_j to i. One continues in this way as long as there is a cycle in the envy-graph.

We now recall and subtly change the update rules. The first rule, is a modification of the update rule U_1(Algorithm 1) in Chapter 3. This is applicable whenever there is an agent that heavily envies the set of unallocated goods[2].

Lemma 6.8 (U_1)**.** *Consider a $(1-\varepsilon)$-EFX allocation X and let P be the set of unallocated goods. If there is an agent $i \in [n]$ that heavily envies P, then in polynomial-time , we can determine[3] a $(1-\varepsilon)$-EFX allocation $X' >_{PD} X$.*

The second update rule U_2 (Algorithm 2) in Chapter 3 is applicable whenever there are more than n unallocated goods. However, we decompose U_2 in Chapter 3 into two update rules in this chapter: U_2 and U_3. Rule U_2 is applicable whenever we can allocate an unallocated good to an unenvied agent (a source in E_X), without creating any strong envy. In this case, we simply allocate this good to the corresponding source. This creates another $(1-\varepsilon)$-EFX allocation where no agent gets a worse bundle and the number of unallocated goods decreases.

Lemma 6.9 (U_2)**.** *Consider a $(1-\varepsilon)$-EFX allocation X. If there is a source s in E_X and an unallocated good g such that no agent strongly envies $X_s \cup g$, then $X' = \langle X_1, X_2, \dots, X_s \cup g, \dots, X_n \rangle$ is a $(1-\varepsilon)$-EFX allocation and $v_i(X'_i) \geq v_i(X_i)$ for all $i \in [n]$.*

Note that there can be at most m consecutive applications of this rule as the number of unallocated goods decreases by one every time we apply this update rule. Update rule U_3 is a refinement of envy-cycle elimination.

Lemma 6.10 (U_3)**.** *Consider a $(1-\varepsilon)$-EFX allocation X. If there exists a set of sources $s_1, s_2, \dots s_\ell$ in E_X, a set of unallocated goods $g_1, g_2, \dots, g_\ell$, and a set of agents $t_1, t_2, \dots, t_\ell$, such that each t_i is reachable from s_i in E_X and t_i is the champion of $X_{s_{i+1}}$ w.r.t g_{i+1} (indices are modulo ℓ), then in polynomial-time , we can determine[4] a $(1-\varepsilon)$-EFX allocation $X' >_{PD} X$.*

6.2 Relating the Number of Unallocated Goods to the Rainbow Cycle Number

In this section, we give the proof of Theorem 6.3, i.e, we show how any upper bound on $R(d)$ allows us to obtain a $(1-\varepsilon)$-EFX with sublinear many goods unallocated. More precisely, we show that given a $(1-\varepsilon)$-EFX allocation X, if E_X is acyclic, and the update rules U_1 and U_2 are not applicable, and the number of unallocated goods is larger than $4n/(\varepsilon \cdot h^{-1}(2n/\varepsilon))$, then rule U_3 is applicable. Therefore, for most of this section, we

[2] In Chapter 3, this is applicable whenever there is an agent that envies (instead of heavily envies) the set of unallocated goods

[3] Let t be the most envious agent of P and $Z \subseteq P$ be such that t heavily envies Z and no agent strongly envies Z. In X', one assigns Z to t and changes the pool to $X_t \cup (P \setminus Z)$.

[4] Let $Z_{i+1} \subseteq X_{s_{i+1}} \cup g_{i+1}$ be the smallest subset of $X_{s_{i+1}} \cup g_{i+1}$ such that t_i heavily envies Z_{i+1} and no agent strongly envies Z_{i+1}. One then essentially proceeds as in cycle elimination. For each i, one assigns Z_{i+1} to t_i and to each agent on the path from s_i to t_i except for t_i one assigns the bundle owned by the successor on the path.

proceed under the assumption

E_X is acyclic and the update rules U_1(Lemma 6.8) and U_2 (Lemma 6.9) are not applicable. (*)

We start with some definitions. Given a partial allocation X, we call an unallocated good g *valuable* to an agent i if $v_i(g) > \varepsilon \cdot v_i(X_i)$. We first make an observation about the agents that could potentially strongly envy $X_s \cup g$, where s is a source in E_X and g is an unallocated good.

Observation 6.11. *Consider an unallocated good g and any source s in E_X. If agent i heavily envies $X_s \cup g$, then g is valuable to agent i.*

Proof. We have $v_i(X_s) \leq v_i(X_i)$ since s is a source of E_X and $v_i(X_i) < (1-\varepsilon)v_i(X_s \cup g)$ since i heavily envies $X_s \cup g$. Thus $v_i(X_i) < (1-\varepsilon)(v_i(X_i) + v_i(g))$ and hence $(1-\varepsilon)v_i(g) > \varepsilon v_i(X_i)$. □

Note that under assumption (*), for each unallocated good g, and each source s in the envy-graph, there is an agent that strongly envies $X_s \cup g$ (since the conditions of the update rule U_2 (Lemma 6.9) are not satisfied). Thus, each unallocated good is valuable to some agent. Now, we make a classification of the unallocated goods based on the number of agents that find them valuable. To be precise, given an allocation X, we classify the unallocated goods into two categories: *high-demand goods* H_X and *low-demand goods* L_X. A good g belongs to H_X, if it is valuable to at least $d+1$ agents and to L_X if it is valuable to at most d agents. We will choose the exact value of d later (right now, just think of it as any integer less than n). Observe that the set of unallocated goods $P = H_X \cup L_X$. To prove our claim, it suffices to show that when $|H_X| + |L_X| > 4n/(\varepsilon \cdot h^{-1}(2n/\varepsilon))$, the rule U_3 is applicable. To this end, we first make a simple observation about $|H_X|$.

Observation 6.12. *Under assumption (*), we have $|H_X| < 2n/(\varepsilon \cdot d)$.*

Proof. For each good $g \in H_X$, let η_g be the number of agents that find g valuable. By definition of H_X, we have that $\eta_g > d$ and hence $\sum_g \eta_g > |H_X|d$. We next upper bound $\sum_g \eta_g$ by $n \cdot (2/\varepsilon)$ by showing that at most $2/\varepsilon$ unallocated goods are valuable to any agent.

Consider any agent i. By assumption (*), rule U_1 is not applicable and hence the value of the unallocated goods to i is at most $1/(1-\varepsilon)v_i(X_i)$. This is at most $2v_i(X_i)$ since $\varepsilon \leq 1/2$. Any valuable good has value at least $\varepsilon v_i(X_i)$ for i. Thus the number of unallocated goods valuable to i is at most $2/\varepsilon$. □

We next bound $|L_X|$. In particular, we show that $|L_X| \leq R(d)$. To this end, we introduce the notion of *group champion graph* G.

Group champion graph. To each agent a, we assign a source $s(a)$, such that a is reachable from $s(a)$ in the envy-graph E_X. Recall that we operating under assumption (*) and hence E_X is acyclic. If a is reachable from multiple sources, we pick $s(a)$ arbitrarily from these sources. Let $k := |L_X|$. For each $g \in L_X$, let Q_g be the set of all agents that find g valuable. By definition of L_X, we have $|Q_g| \leq d$ for all $g \in L_X$. We now define a k-partite graph $G = (\cup_{g \in L_X} V_g, E)$, in which the part V_g corresponding to g consists of

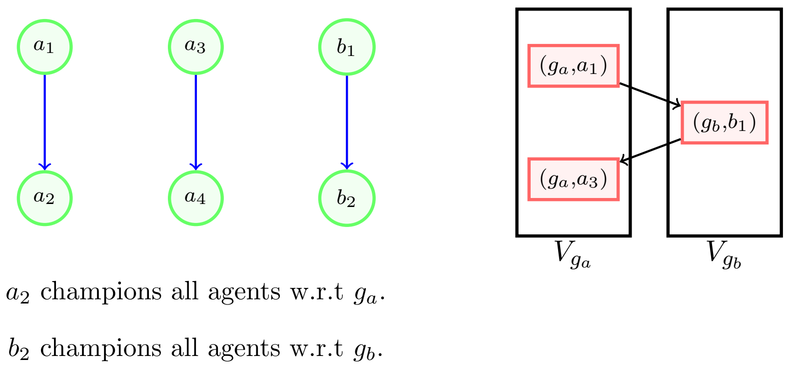


Figure 6.1: Illustration of a group champion graph. We have an instance with six agents $\cup_{i\in[4]}a_i$ and $\cup_{i\in[2]}b_i$ and two unallocated goods, namely g_a and g_b. The agents $\cup_{i\in[4]}a_i$ find g_a valuable and the agents $\cup_{i\in[2]}b_i$ find g_b valuable. The envy-graph E_X of the instance is shown on the left side. E_X shows that $s(a_2) = a_1$, $s(a_4) = a_3$, and $s(b_2) = b_1$. Also, we have that agent a_2 champions all the agents w.r.t g_a and b_2 champions all the agents w.r.t g_b. The group champion graph (right) has two parts, V_{g_a} corresponding to g_a and V_{g_b} corresponding to g_b. V_{g_a} contains the copies of the sources of all the agents that find g_a valuable, namely (g_a, a_1) and (g_a, a_3). Similarly, V_{g_b} contains (g_b, b_1). There is an edge from (g_a, a_1) to (g_b, b_1) as a_2 (which is reachable from a_1 in E_X) champions b_1 w.r.t to g_a. Similarly, there is an edge from (g_b, b_1) to (g_a, a_3) as b_2 (which is reachable from b_1 in E_X) champions a_3 w.r.t to g_b.

copies of the sources assigned to the agents in Q_g, formally, $V_g = \{(g, s(a)) \mid a \in Q_g\}$. For any goods g and h and agents $a \in Q_g$ and $b \in Q_h$, there is an edge from $(g, s(a))$ in V_g to $(h, s(b))$ in V_h if and only if a is the champion of $X_{s(b)} \cup g$ (see Figure 6.1 for an illustration). We now make an observation about the set of edges between V_g and V_h in G for any $g, h \in L_X$.

Observation 6.13. *Under assumption (*): Consider any $g, h \in L_X$. Then each vertex in V_h, has an incoming edge from a vertex in V_g.*

Proof. Consider any vertex $(h, s(b)) \in V_h$. By assumption (*), there is an agent that strongly envies the bundle $X_{s(b)} \cup g$. Otherwise, rule U_1 would be applicable. By Observation 6.11, all agents that strongly envy $X_{s(b)} \cup g$, consider g valuable and hence belong to Q_g. Let a be the champion of $X_{s(b)} \cup g$. By Observation 6.7, a strongly envies $X_{s(b)} \cup g$ and hence belongs to Q_g. Thus there is an edge from $(g, s(a))$ in V_g to $(h, s(b))$ in V_h (by the construction of G). $\square$

Now we claim that the existence of a cycle that visits each part of G at most once, would imply the existence of a $(1-\varepsilon)$-EFX allocation that Pareto-dominates the existing $(1-\varepsilon)$-EFX allocation.

Lemma 6.14. *Given a cycle C in G that contains at most one vertex from each V_g, for all $g \in L_X$, we can determine a $(1-\varepsilon)$-EFX allocation $X' >_{PD} X$ in polynomial-time .*

Proof. Let $C = (g_{i+1}, s_i) \to (g_{i+2}, s_{i+1}) \to \cdots \to (g_{j+1}, s_j) \to (g_{i+1}, s_i)$ be a cycle in G that visits each part at most once. It will become clear below, why we index the g's starting at $i+1$. Consider the sequence $s_i, s_{i+1}, \ldots, s_j$. If all the sources in this sequence are not distinct, there exists a contiguous subsequence $s_{i'}, s_{i'+1}, \ldots, s_{j'}$ where all the sources are distinct and $s_{j'+1} = s_{i'}$ with $i \leq i' < j' \leq j$ (index $j+1$ is to be interpreted as i).

We now work with the sequence $s_{i'}, s_{i'+1}, \ldots, s_{j'}$ where all the sources are distinct and $s_{j'+1} = s_{i'}$. For all $\ell \in [i'+1, j'+1]$, the existence of the edge $(g_\ell, s_{\ell-1}) \to (g_{\ell+1}, s_\ell)$ implies the existence of an agent $t_{\ell-1}$ such that $t_{\ell-1}$ is the champion of $X_{s_\ell} \cup g_\ell$ and $s(t_{\ell-1}) = s_{\ell-1}$, i.e., $t_{\ell-1}$ is reachable from $s_{\ell-1}$ in E_X. Since the sources $s_{i'}, s_{i'+1}, \ldots, s_{j'}$ are distinct, the agents $a_{i'}, a_{i'+1}, \ldots, a_{j'}$ are also distinct (as each agent has a unique source assigned). Therefore, we have distinct sources $s_{i'}, \ldots, s_{j'}$ in E_X, distinct goods $g_{j'+1}, g_{i'+1}, \ldots, g_{j'}$ and distinct agents $t_{i'}, \ldots t_{j'}$ that satisfy the conditions under which the update rule U_3 (Lemma 6.10) is applicable. By applying U_3 we can get a $(1-\varepsilon)$-EFX allocation $X' >_{PD} X$. □

With Lemma 6.14, we are now ready to give an upper bound on $|L_X|$. Observe that $|L_X|$ equals the number of parts in G. Now the question is how many parts can G have such that it does not admit a cycle that visits each part at most once. This is where we upper bound $|L_X|$ with the rainbow cycle number.

Lemma 6.15. *Consider a $(1-\varepsilon)$-EFX allocation X. If $|L_X| > R(d)$, there is a $(1-\varepsilon)$-EFX allocation $X' >_{PD} X$.*

Proof. Recall that $|L_X| = k$, where k is the number of parts in G. Note that each part of G corresponds to the sources assigned to the agents that find a particular good in L_X valuable (Q_g for some $g \in L_X$). By definition of L_X, there are at most d agents that find a good in L_X valuable. Thus each part has at most d vertices. Again, by Observation 6.13, between any two parts V_g and V_h of G, each vertex in V_h has an incoming edge from a vertex in V_g. Therefore, by Definition 6.2, we have that if $k > R(d)$, then there exists a cycle C in G that visits each part at most once. Once we have C, by Lemma 6.14, we can determine a $(1-\varepsilon)$-EFX allocation $X' >_{PD} X$. □

Given a $(1-\varepsilon)$-EFX allocation X such that $|L_X| > R(d)$, Lemma 6.15 only gives the existence of a $(1-\varepsilon)$-EFX allocation $X' >_{PD} X$. However, to determine X' in polynomial-time , one needs to find a cycle C in G which visits each part at most once when $|L_X| > R(d)$, in polynomial-time . Let us remark that this is a non-trivial problem in general, reminiscent of the well-known k-Path and k-Cycle problems which are NP-complete [45]. Here, the input is a (di)graph G and an integer k, and the objective is to determine of there is a path (cycle) on at least k-distinct vertices of the graph. These problems can be solved in $2^{\mathcal{O}(k)} \cdot \mathrm{poly}(\mathsf{n})$ time using techniques based on color-coding, hash-functions and splitters [45, 3, 79]. In particular, we can reduce k-Path to the following problem in polynomial-time : find a k-path in a *colorful* graph on n vertices, whose vertices have been colored with $\mathcal{O}(\mathrm{poly}(k) \cdot \log n)$ colors, such that every vertex of the k-path has a distinct color. However, for our purposes the construction of the cycle C in G is a part of the proof of Theorem 6.21 (described in Section 6.3: we show that in polynomial-time , one can find a cycle in a $(d^4 + d)$-partite digraph, in which each part has at most d vertices and for any two parts V and V' in the digraph, every vertex in V' has an incoming edge from some vertex in V and vice-versa. This implies that if $|L_X| > d^4 + d$, then in polynomial-time , we can determine a cycle C in G that visits each part at most once and then determine a $(1-\varepsilon)$-EFX allocation $X' >_{PD} X$ by applying U_3. This also implies that $R(d) \le d^4 + d$. Therefore,

Lemma 6.16. *Consider a $(1-\varepsilon)$-EFX allocation X. If $|L_X| > d^4+d$, then in polynomial-time , we can determine a $(1-\varepsilon)$-EFX allocation $X' >_{PD} X$.*

Putting it together. We give the existence proof and indicate in brackets the changes required for the polynomial-time algorithm. We start with an empty allocation, which is trivially a $(1-\varepsilon)$-EFX. Then, our algorithm iteratively maintains a $(1-\varepsilon)$-EFX allocation X and a pool of unallocated goods. In each iteration, the algorithm first makes E_X acyclic in polynomial-time (Lemma 6.5). Thereafter, our algorithm checks whether any one of the update rules U_1 and U_2 is applicable. If U_1 is applicable, then our algorithm determines a $(1-\varepsilon)$-EFX allocation $X' >_{PD} X$. If U_2 is applicable, then our algorithm determines an allocation a $(1-\varepsilon)$-EFX allocation X' where $v_i(X'_i) \ge v_i(X_i)$ for all $i \in [n]$ and the number of unallocated goods reduces. If neither U_1 nor U_2 is applicable, then it determines the sets H_X and L_X. By Lemma 6.12, we have $|H_X| \le 2n/(\varepsilon \cdot d)$. If $|L_X| \le R(d)$ $(|L_X| \le d^4+d)$, then it returns the allocation X. Otherwise it determines a cycle that visits each part of G at most once and then determines $(1-\varepsilon)$-EFX allocation $X' >_{PD} X$ by applying update rule U_3 (by Lemma 6.15). If $|L_X| > d^4 + d$, the cycle can be determined in polynomial-time (Theorem 6.21 in Section 6.3). Therefore, when the algorithm terminates, we have that $|H_X| \le 2n/(\varepsilon \cdot d)$ and $|L_X| \le R(d)$, $(|L_X| \le d^4 + d)$ implying that the total number of unallocated goods is $|H_X| + |L_X| \le 2 \cdot \max(2n/(\varepsilon \cdot d), R(d))$ $(2 \cdot \max(2n/(\varepsilon \cdot d), 2d^4))$.

We now state the explicit value of d, first for the existence proof. We choose d as the smallest integer such that $2n/(\varepsilon d) \le R(d)$, i.e, $d = h^{-1}(2n/\varepsilon)$. [5] Therefore, the number of unallocated goods is at most $4n/(\varepsilon \cdot h^{-1}(2n/\varepsilon))$.

For the algorithmic result, we choose d as the smallest integer such that $2n/(\varepsilon \cdot d) \le 2d^4$. Then $d = \lceil (n/\varepsilon)^{1/5} \rceil$ and the number of unallocated goods is at most $4\lceil (n/\varepsilon)^{1/5} \rceil^4$. This is less than $64(n/\varepsilon)^{4/5}$.

It only remains to show that the algorithm will terminate. We prove a polynomial bound on the number of iterations. The bound applies to the existence and the algorithmic version. To this end, note that in each iteration, after removing cycles from E_X, our algorithm determines a new $(1-\varepsilon)$-EFX allocation X' through one of the following procedures:

- applying U_1,
- applying U_2,
- determining a cycle C that visits each part in G at most once and then applying U_3.

Note that the initial envy-cycle elimination and subsequent application of all of the above procedures ensure that $v_i(X'_i) \ge v_i(X_i)$ for all $i \in [n]$ (Lemmas 6.5, 6.8, 6.9, 6.10). Thus, throughout the algorithm the valuation of an agent never decreases. Note that there cannot be more than m *consecutive applications* of U_2, as the number of unallocated goods decreases with each application of U_2. Every time we apply U_1 or U_3, we ensure that $X' >_{PD}$

[5] Recall that $h(d) = d \cdot R(d)$ in Definition 6.2 and that $h^{-1}(2n/\varepsilon)$ is defined as the smallest integer such that $h(d) \ge 2n/\varepsilon$.

X, implying that the valuation of some agent improves by a factor of at least $(1+\varepsilon)$. Since each agent's valuation is bounded by $W = max_{i\in[n]} v_i(M)$, and the valuation of an agent never decreases throughout the algorithm, we can have at most $\text{poly}(n, m, W, 1/\varepsilon)$ many iterations that involve applications of U_1 and U_3. Therefore, the total number of iterations of our algorithm is $m \cdot$ (iterations involving application of U_2 or U_3) which is also $\text{poly}(n, m, \log W, 1/\varepsilon)$. Notice that in the algorithmic case, each of the iterations can also be implemented in polynomial-time : U_1 and U_2 can be implemented in polynomial-time (Lemmas 6.8 and 6.9). When $|L_X| \geq 2d^4 \geq d^4 + d$, then in polynomial-time we can determine the cycle C and apply U_3 (Lemma 6.16). We can now state the main result of this section.

Theorem 6.17. *Let $h(d) = d \cdot R(d)$. Then there is a $(1-\varepsilon)$-EFX allocation X and a set of unallocated goods P such that $|P| \leq (4n/(\varepsilon \cdot h^{-1}(n/(\varepsilon)))$. In polynomial-time , one can find a $(1-\varepsilon)$-EFX allocation and a set P of unallocated goods such that $|P| \leq 64(n/\varepsilon)^{4/5}$.*

Note that any upper bound on the rainbow cycle number will imply an upper bound on the number of unallocated goods.

6.3 Bounds on the Rainbow Cycle Number

In this section, we give the proof of Theorem 6.4. We briefly recall the setup: There is a k-partite digraph $G = (\cup_{i\in[k]} V_i, E_G)$ such that each part has at most d vertices. For every distinct parts V_i and V_j, every vertex in V_j has an incoming edge from some vertex in V_i. There is no cycle in G that visits each part at most once. Our goal is to establish an upper bound on k.

We now introduce some helpful notations and concepts. For each $i \in [k]$, we represent the vertices in the part V_i as $(i, \text{vertex id})$, i.e, $V_i = \{(i, 1), (i, 2), \ldots, (i, |V_i|)\}$. For any positive integer d and $a, b \in [d]$, we use $\sigma_d(a, b)$ to denote $(a-1)\cdot d + b$. Note that $1 \leq \sigma_d(a, b) \leq d^2$. The $\sigma_d(a, b)$ captures the lexicographic ordering among the pairs $\cup_{a\in[d]} \cup_{b\in[d]} (a, b)$. For any Boolean vector $u \in \{0, 1\}^r$, we use $u[k]$ to refer to the k^{th} coordinate of the vector u. We introduce the simple yet crucial notion of *representative set for a set of Boolean vectors*. Given a set D of r-dimensional Boolean vectors, the set $B \subseteq D$ is a *representative set* of D, if $\{\ell \mid a[\ell] = 1 \text{ for some } a \in D\} = \{\ell \mid b[\ell] = 1 \text{ for some } b \in B\}$. We first make an observation about the size of B.

Observation 6.18. *Given any set D of r-dimensional Boolean vectors, there exists a representative set $B \subseteq D$ of size at most r.*

Proof. For each coordinate $\ell \in [r]$ we do: if there is a vector $a \in D$ with $a[\ell] = 1$, we put one such vector into B. Clearly, $|B| \leq r$. □

We prove Theorem 6.4 by contradiction. To be precise, we show that if $k > d^4 + d$, then there exists a cycle in G that visits every part at most once. Moreover, this cycle can be found in time polynomial in k.

We construct the cycle in two steps. We first show the existence of a part $V_{\tilde{\ell}}$ such that there is a directed cycle that visits only the parts $V_{\tilde{\ell}}$, V_1, V_2, $\ldots$, V_d and moreover each of the parts V_1, V_2, $\ldots$, V_d at most once. In the second step we replace the vertices in $V_{\tilde{\ell}}$ in this cycle by vertices in distinct parts.

For each ordered pair $(i, j) \in [d] \times [d]$, and $\ell \in [k] \setminus [d]$, we define a d^2-dimensional vector $u_{i,j,\ell}$ as follows: for all $x \in [d]$ and $y \in [d]$, we set $u_{i,j,\ell}[\sigma_d(x, y)] = 1$ if and only if there exists a path $(i, x) \to (\ell, z) \to (j, y)$ in G for some $(\ell, z) \in V_\ell$, i.e., if there exists a path from vertex (i, x) in V_i to vertex (j, y) in V_j through some vertex in V_ℓ. Otherwise, we set $u_{i,j,\ell}[\sigma_d(x, y)] = 0$.

Let $\mathcal{L} = [k] \setminus [d]$. For each ordered pair $(i, j) \in [d] \times [d]$, we construct the sets $B^{i,j}$ and $\mathcal{L}^{i,j}$ as follows: For each (i, j) taken in the increasing order of $\sigma_d(i, j)$, define $\mathcal{L}^{i,j} = \mathcal{L}$ and $B^{i,j}$ as a representative vector set of $\{u_{i,j,\ell} \mid \ell \in \mathcal{L}^{i,j}\}$ of size at most d^2. A set $B^{i,j}$ of this size exists because our vectors have dimension d^2. Then we set $\mathcal{L} = \mathcal{L} \setminus \{\ell \mid u_{i,j,\ell} \in B^{i,j}\}$. At most d^2 elements are removed from $\mathcal{L}$ in each iteration.

For clarity, we write $\mathcal{L}^f$ to denote the set $\mathcal{L}$ at the end of the construction. Observe that $|\mathcal{L}^f| \geq 1$. This holds since we start with a set of size larger than d^4 and removed at most d^2 elements in each of the d^2 iterations.

Observation 6.19. *Consider distinct ordered pairs $(i, j) \in [d] \times [d]$ and $(i', j') \in [d] \times [d]$. The sets $\{\ell \mid u_{i,j,\ell} \in B^{i,j}\}$ and $\{\ell \mid u_{i',j',\ell} \in B^{i',j'}\}$ are disjoint.*

Proof. Let us assume without loss of generality that $\sigma_d(i, j) < \sigma_d(i', j')$. Consider any ℓ such that $u_{i,j,\ell} \in B^{i,j}$. Then ℓ is removed from $\mathcal{L}$ at the end of the iteration for the pair (i, j) and hence does not belong to $\mathcal{L}$ at the beginning of the iteration for the pair (i', j'). Consequently $u_{i',j',\ell} \notin B^{i',j'}$ (by definition of $B^{i',j'}$, if $u_{i',j',\ell} \in B^{i',j'}$, then $\ell \in \mathcal{L}^{i',j'}$). $\square$

At the end of the construction, we arbitrarily pick a $\tilde{\ell} \in \mathcal{L}^f$ (this is possible as $\mathcal{L}^f \neq \emptyset$). Now, we make a small observation about the vector $u_{i,j,\tilde{\ell}}$ for all $i, j \in [d]$.

Observation 6.20. *For all $i, j \in [d]$, if $u_{i,j,\tilde{\ell}}[q] = 1$ for some $q \in [d^2]$, then there exists a vector $u_{i,j,l'} \in B^{i,j}$ such that $u_{i,j,l'}[q] = 1$.*

Proof. Observe that $\mathcal{L}^f \subseteq \mathcal{L}^{i,j}$. Therefore, $\tilde{l} \in \mathcal{L}^{i,j}$. By definition, $B^{i,j}$ is a representative vector set of $\{u_{i,j,\ell} \mid \ell \in \mathcal{L}^{i,j}\}$. Therefore, by the definition of representative set, there exists a vector $u_{i,j,\ell'} \in B^{i,j}$ such that $u_{i,j,\ell'}[q] = 1$. $\square$

We are now ready for the construction of a cycle that visits each part at most once. We first show that there exists a cycle C in G that visits only the parts $V_{\tilde{\ell}}$, V_1, ..., V_d and each of the parts $V_1, \ldots V_d$ at most once, i.e, the only part it may visit more than once is $V_{\tilde{\ell}}$. See Figure 6.2 for an illustration.

Let $(\tilde{\ell}, w_d)$ be an arbitrary vertex in $V_{\tilde{\ell}}$. We construct a path

$$(\tilde{\ell}, w_0) \to (1, v_1) \to \ldots \to (i{-}1, v_{i-1}) \to (\tilde{\ell}, w_{i-1}) \to (i, v_i) \to (\tilde{\ell}, w_i) \to \ldots \to (d, v_d) \to (\tilde{\ell}, w_d)$$

by starting at $(\tilde{\ell}, w_d)$ and tracing backwards: We start in $(\tilde{\ell}, w_d)$. Assume that we already traced back to $(\tilde{\ell}, w_i)$ with $i = d$ initially. By the construction of G, there must be an edge from some vertex (i, v_i) in V_i to $(\tilde{\ell}, w_i)$ in $V_{\tilde{\ell}}$, and there must be an edge from some vertex $(\tilde{\ell}, w_{i-1})$ in $V_{\tilde{\ell}}$ to (i, v_i) in V_i. Thus there is the path $(\tilde{\ell}, w_{i-1}) \to (i, v_i) \to (\tilde{\ell}, w_i)$ in G. We keep continuing this procedure until we reach $(\tilde{\ell}, w_0)$.

Since the part $V_{\tilde{\ell}}$ can have at most d vertices, by the pigeonhole principle, there must be i and j with $0 \leq i < j \leq d$ such that $w_i = w_j$. Let C be the subpath from $(\tilde{\ell}, w_i)$ to $(\tilde{\ell}, w_j)$, i.e.,

$$C = (\tilde{\ell}, w_i) \to (i + 1, v_{i+1}) \to (\tilde{\ell}, w_{i+1}) \to \ldots \to (\tilde{\ell}, w_{j-1}) \to (j, v_j) \to (\tilde{\ell}, w_j).$$

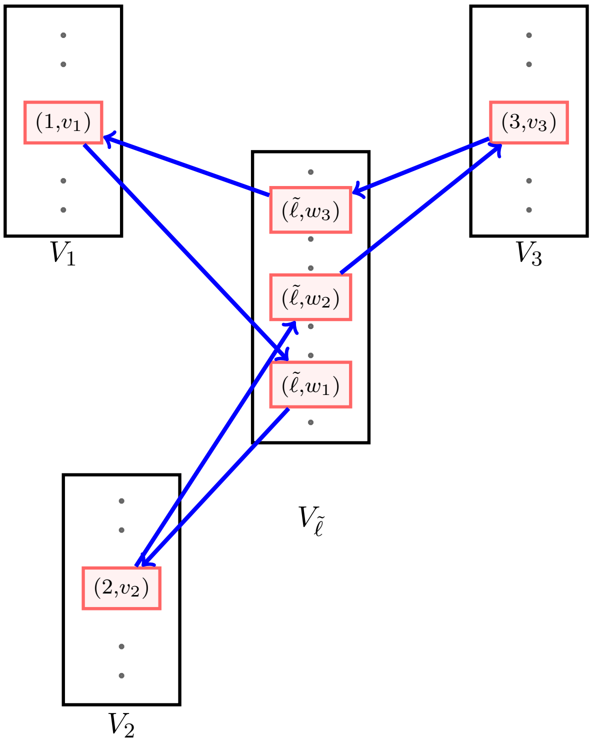


Figure 6.2: Illustration of the first part of the construction. The cycle in the figure visits the parts V_1, V_2 and V_3 exactly once and the part $V_{\tilde{\ell}}$ three times. It is given by $(\tilde{\ell}, w_3) \to (1, v_1) \to (\tilde{\ell}, w_1) \to (2, v_2) \to (\tilde{\ell}, w_2) \to (3, v_3) \to (\tilde{\ell}, w_3)$.

Observe that C visits all the parts of G except $V_{\tilde{\ell}}$ at most once. We now show that by using "bypass" parts we can make the cycle simple. For clarity, we rewrite C as

$$C = (i+1, v_{i+1}) \to (\tilde{\ell}, w_{i+1}) \to \ldots \to (\tilde{\ell}, w_{j-1}) \to (j, v_j) \to (\tilde{\ell}, w_j) \to (i+1, v_{i+1}).$$

Making the Cycle Simple. For all $q \in [i+1, j]$ consider the subpath

$$(q, v_q) \to (\tilde{\ell}, w_q) \to (q+1, v_{q+1})$$

of C (index $j+1$ is to be interpreted as $i+1$). The existence of such a subpath in G implies that $u_{q,q+1,\tilde{\ell}}[\sigma_d(v_q, v_{q+1})] = 1$. By Observation 6.20, we know that there is a vector $u_{q,q+1,\ell_q} \in B^{q,q+1}$ such that $u_{q,q+1,\ell_q}[\sigma_d(v_q, v_{q+1})] = 1$. This implies that there exists a part V_{ℓ_q}, and a vertex (ℓ_q, y_q) in part V_{ℓ_q}, such that there is a subpath

$$(q, v_q) \to (\ell_q, y_q) \to (q+1, v_{q+1}) \ .$$

By Observation 6.19, we have that $\ell_q \neq \ell_{q'}$ for all $q \neq q'$. Therefore we have a simple cycle C' in G that visits each part in G at most once, namely,

$$C' = (i+1, v_{i+1}) \to (\ell_{i+1}, y_{i+1}) \to \cdots \to (\ell_{j-1}, y_{j-1}) \to (j, v_j) \to (\ell_j, y_j) \to (i+1, v_{i+1}).$$

See Figure 6.3 for an illustration of this entire procedure.

Therefore if $k > d^4 + d$, then there exists a cycle in G that visits each part at most once. Moreover, this cycle can be found in time polynomial in k. With this we arrive at the main result of this section.

Theorem 6.21. *For all $d \geq 1$, we have $R(d) \leq d^4 + d$. Furthermore, Let G be a k-partite digraph with $k > d^4 + d$ parts of cardinality at most d each, such that for every vertex v and any part W not containing v, there is an edge from W to v. Then, there exists a cycle in G visiting each part at most once, and it can be found in time polynomial in k.*

An improved upper bound on $R(d)$ would imply a better bound on the number of unallocated goods. However, we show that an exponential improvement (e.g. $R(d) \in \text{poly}(\log(d))$) is not possible by showing a *linear* lower bound, i.e., $R(d) \geq d$. However, this still leaves room for polynomial improvement and we suspect that $R(d) \in \mathcal{O}(d)$. This would imply the existence of a $(1-\varepsilon)$-EFX allocation with $\mathcal{O}(\sqrt{n/\varepsilon})$ many goods unallocated. For a polynomial-time algorithm, the construction of a cycle as in Theorem 6.21 would have to be polynomial-time . However, we remark that this is an initiation study for determining $(1-\varepsilon)$-EFX allocations with sublinear number of unallocated goods and we use concepts like the *group champion graph* that are natural extensions of the champion graph. We believe that this still leaves room for developing more sophisticated concepts and techniques that may reduce the number of unallocated goods to $o(\sqrt{n/\varepsilon})$.

Lower bound on $R(d)$. We show that $R(d) \geq d$. We construct a d-partite graph $G = (\cup_{i\in[d]} V_i, E)$ such that each part V_i has d vertices, for all pairs of parts V_i and V_j, every vertex in V_j has an incoming edge from a vertex in V_i and vice-versa, and there exists no cycle that visits each part at most once.

We now define the edges in G. Let $V_i = \{(i, 0), (i, 1), \ldots, (i, d-1)\}$. Consider any i and j such that $i < j$. For each $0 \leq \ell \leq d-1$, we have an edge from (i, ℓ) in V_i to (j, ℓ) in V_j and there is an edge from (j, ℓ) in V_j to $(i, (\ell+1) \bmod d)$ in V_i (see Figure 6.4 for an illustration). One can easily verify that for all parts V_i and V_j, every vertex in part V_j has an incoming edge from part V_i and vice-versa. It suffices to show that G admits no cycle that visits each part at most once.

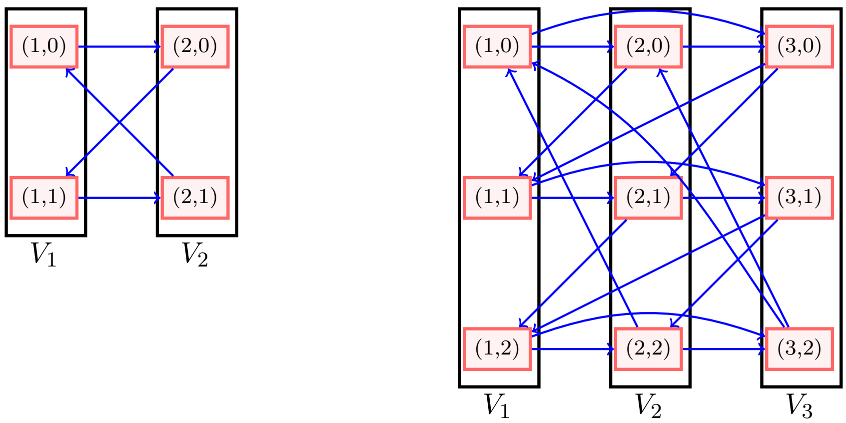


Figure 6.4: Illustration of the construction of d-partite graph G that satisfies all the conditions in Definition 6.2, for $d = 2$ (left) and $d = 3$ (right).

Lemma 6.22. *There exists no cycle in G that visits each part at most once.*

Proof. We prove by contradiction. Assume that there is a cycle $C = (i_1, \ell_1) \to (i_2, \ell_2) \to \cdots \to (i_r, \ell_r) \to (i_1, \ell_1)$ that visits each part at most once, i.e., $i_1 \neq i_2 \neq \cdots \neq i_r$. From here on, all the indices are modulo r. Note that by the construction of the edges of G, for all $q \in [r]$, we have $\ell_{q+1} = \ell_q$ if $i_q < i_{q+1}$ and $\ell_{q+1} = (\ell_q + 1) \bmod d$ if $i_q > i_{q+1}$. Let $\#_1 = \{q \in [r] \mid i_q > i_{q+1}\}$ (recall that $r+1$ is 1). The existence of the cycle C in G implies that $\ell_1 = (\ell_1 + \#_1) \bmod d$.

Since $i_1 \neq i_2 \neq \cdots \neq i_r$ and there exists the cycle C in G, there are indices q' and q'' such that $i_{q'} > i_{q'+1}$ and $i_{q''} < i_{q''+1}$, further implying that $1 \leq \#_1 \leq r-1$. Since G has d parts, we have $r \leq d$, implying that $1 \leq \#_1 \leq d-1$. However this implies that $(\ell_1 + \#_1) \mod d \neq \ell_1$, which is a contradiction. □

6.4 Finding Efficient $(1-\varepsilon)$-EFX Allocations with Sublinear Charity

We note that like the algorithms in [36, 84], our algorithm is flexible with the initialization, i.e., starting with any initial $(1-\varepsilon)$-EFX allocation X, it can determine a final $(1-\varepsilon)$-EFX allocation Y with at most $\mathcal{O}((n/\varepsilon)^{\frac{4}{5}})$ many goods unallocated and $v_i(Y_i) \geq v_i(X_i)$ for all $i \in [n]$. This is consequence of the fact that the valuation of an agent never decreases throughout our algorithm. Therefore, our algorithm maintains the welfare of the initial allocation. Thus, if we choose the initial $(1-\varepsilon)$-EFX allocation carefully, we can also guarantee high Nash welfare for our final $(1-\varepsilon)$-EFX allocation with sublinear many goods unallocated. To this end, we use an important result from Caragiannis et al. [27] about determining partial EFX allocations with high Nash welfare in polynomial-time .

Theorem 6.23 ([27])**.** *In polynomial-time , we can determine a partial EFX allocation X such that $NW(X) \geq 1/(2.88) \cdot NW(X^*)$ where X^* is the Nash welfare maximizing allocation.*[6]

Let X be the partial EFX allocation that achieves a 2.88 approximation of the Nash welfare. We run our algorithm starting with X as the initial allocation. The final $(1-\varepsilon)$-EFX allocation with sublinear many unallocated goods is also a 2.88 approximation of the Nash welfare as the valuations of the agents in the final allocation is at least their valuations in X. Therefore, we have the following theorem,

Theorem 6.24. *In polynomial-time , we can determine a $(1-\varepsilon)$-EFX allocation with $\mathcal{O}((n/\varepsilon)^{\frac{4}{5}})$ goods unallocated. Furthermore, $NW(X) \geq 1/(2.88) \cdot NW(X^*)$.*[7]

[6] In fact, the result in [27] show the existence of partial EFX allocations that achieve a 1/2 approximation of the Nash welfare. However, in polynomial-time , one can only find a partial EFX allocation with a 1/2.88 approximation of the Nash welfare.

[7] Note that using the existence of partial EFX allocations with 1/2 approximation to Nash welfare, one can also claim the existence of a $(1-\varepsilon)$-EFX allocation X with $\mathcal{O}((n/\varepsilon)^{\frac{4}{5}})$ goods unallocated such that $NW(X) \geq 1/2 \cdot NW(X^*)$.

6.5 Limitations of the Approach from Chapter 5

In Chapter 5, an algorithmic proof to the existence of EFX allocations is shown for three agents with additive valuations. We briefly sketch the proof technique in Chapter 5 and then highlight why it does not work for determining a $(1-\varepsilon)$-EFX allocations with just four agents. Let the three agents be a, b and c and for any allocation X, let $\phi(X)$ be the vector $\langle v_a(X_a), v_b(X_b), v_c(X_c)\rangle$. The algorithm starts with an empty allocation which is trivially EFX and as long as there is an unallocated good, the algorithm determines another EFX allocation X' such that $\phi(X')$ is lexicographically larger than $\phi(X)$, i.e., either $v_a(X'_a) > v_a(X_a)$ or $v_a(X'_a) = v_a(X_a)$ and $v_b(X'_b) > v_b(X_b)$ or $v_a(X'_a) = v_a(X_a)$, $v_b(X'_b) = v_b(X_b)$ and $v_c(X'_c) > v_c(X_c)$. We now show that such a technique cannot be used to show the existence of $(1-\varepsilon)$-EFX allocations for four agents.

Theorem 6.25. *There exists an instance I with four agents, $\{a, b, c, d\}$ with additive valuations, nine goods $\{g_i \mid i \in [9]\}$ and a partial $(1-\varepsilon)$-EFX allocation X on the goods $\cup_{i\in[8]} g_i$, such that in all complete $(1-\varepsilon)$-EFX allocation, the valuation of agent a will be strictly less than her valuation in X. This shows that for any complete $(1-\varepsilon)$-EFX allocation Y, we have $\phi(X)$ is lexicographically larger than $\phi(Y)$.*

We now elaborate the instance used in Theorem 6.25. We remark that our instance builds on the instance in Chapter 5, that is used to show the existence of a partial EFX allocation which is not Pareto-dominated by any complete EFX allocation. The full description of our instance is captured by Table 6.1. We choose our $\varepsilon \ll 1$. The sub-instance defined by the agents b, c and d, and the goods $\cup_{i\in[6]} g_i \cup g_9$ is the instance in Chapter 5 used to show the existence of a partial EFX allocation which is not Pareto-dominated by any complete EFX allocation. We now specify the allocation X.

$$X_a = \{g_7, g_8\} \qquad X_b = \{g_2, g_3, g_4\}$$
$$X_c = \{g_1, g_5\} \qquad X_d = \{g_6\}$$

The good g_9 is unallocated. We will show that in any complete $(1-\varepsilon)$-EFX allocation, agent a cannot have both g_7 and g_8. This would imply that agent a's valuation in any final $(1-\varepsilon)$-EFX allocation is strictly less than her valuation in X (as agent a's valuation for all goods other than g_7 and g_8 is zero). We prove this claim by contradiction. So assume that Y is a complete $(1-\varepsilon)$-EFX allocation and $\{g_7, g_8\} \subseteq Y_a$. Note that $v_b(g_7) = 31$, $v_c(g_7) = 29$, and $v_d(g_7) = 19$. Since Y_a contains at least one other good namely g_8, each of the agents b, c and d need to be allocated bundles that they value at least 31, 29 and 19 respectively.

First, consider the case that $g_6 \in Y_b$. Then we have $v_b(Y_b) \geq 34$. Now, to ensure $v_d(Y_d) \geq 19$, we need to allocate g_5 and g_9 to d, as d values all the other goods zero. We are left with goods g_1, g_2, g_3 and g_4. In order to ensure $v_c(Y_c) \geq 29$, we definitely need to allocate g_1, g_3 and g_4 to c. Now, even if we allocate the remaining good g_2 to b, we have $v_b(Y_b) = v_b(\{g_2, g_6\}) = 38 < (1-\varepsilon)\cdot 40 = (1-\varepsilon)\cdot v_b(\{g_1, g_3\}) \leq (1-\varepsilon)\cdot v_b(Y_c \setminus g_4)$. Therefore, b will strongly envy c. Thus $g_6 \notin Y_b$.

If $g_6 \notin Y_b$ and $v_b(Y_b) \geq 31$, Y_b must contain g_3 (the total valuation for b of all the goods other than g_3, g_6, g_7 and g_8 is less than 31). Now we consider some more subcases.

Let us first assume that $g_1 \in Y_b$. Since Y_b already contains g_1 and g_3, the goods that can be allocated to c and d are g_2, g_4, g_5, g_6, and g_9. In order to ensure $v_c(Y_c) \geq 29$ we need to allocate g_4, g_5, and g_9 to c. Now, even if we allocate all the remaining goods (g_2 and g_6) to d, we have $v_d(Y_d) = v_d(\{g_3, g_6\}) = 20 < (1-\varepsilon) \cdot 22 = (1-\varepsilon) \cdot v_d(\{g_5, g_7\}) \leq (1-\varepsilon) \cdot v_d(Y_c \setminus g_4)$. Therefore, d will strongly envy c.

Thus $g_1 \notin Y_b$. Since neither g_1 nor g_6 belongs to Y_b, the only way to ensure $v_b(Y_b) \geq 31$ is to at least allocate g_2, g_3, and g_4 to b (we can allocate more). Similarly, given that the goods not allocated yet are g_1, g_5, g_6, and g_9, the only way to ensure $v_c(Y_c) \geq 29$ is to allocate at least g_1 and g_5 to c. Similarly, the only way to ensure $v_d(Y_d) \geq 19$ now is to allocate at least g_6 to d. Now we only have to allocate g_9. We show that adding g_9 to any one of the existing bundles will cause a violation of the $(1-\varepsilon)$-EFX property.

- Adding g_9 to Y_a: b, c and d strongly envies a as $v_b(Y_b) = 32 < (1-\varepsilon) \cdot 33 = (1-\varepsilon) \cdot v_b(\{g_7, g_9\}) \leq (1-\varepsilon) \cdot v_b(Y_a \setminus g_8)$. Similarly we have $v_c(Y_c) = 30 < (1-\varepsilon) \cdot 35 = (1-\varepsilon) \cdot v_c(\{g_7, g_9\}) \leq (1-\varepsilon) \cdot v_c(Y_a \setminus g_8)$ and $v_d(Y_d) = 20 < (1-\varepsilon) \cdot 23 = (1-\varepsilon) \cdot v_d(\{g_7, g_9\}) \leq (1-\varepsilon) \cdot v_d(Y_a \setminus g_8)$.
- Adding g_9 to Y_b: c strongly envies b as $v_c(Y_c) = 30 < (1-\varepsilon) \cdot 32 = (1-\varepsilon) \cdot v_c(\{g_3, g_4, g_7\}) = (1-\varepsilon) \cdot v_c(Y_b \setminus g_2)$.
- Adding g_9 to Y_c: d strongly envies c as $v_d(Y_d) = 20 < (1-\varepsilon) \cdot 22 = (1-\varepsilon) \cdot v_d(\{g_5, g_9\}) = (1-\varepsilon) \cdot v_d(Y_c \setminus g_1)$.
- Adding g_9 to Y_d: b strongly envies d as $v_b(Y_a) = 32 < (1-\varepsilon) \cdot 34 = (1-\varepsilon) \cdot v_b(g_6) = (1-\varepsilon) \cdot v_b(Y_d \setminus g_9)$.

This shows that $\{g_7, g_8\} \not\subseteq Y_a$ for any complete $(1-\varepsilon)$-EFX allocation Y. This implies that agent a's valuation in Y is strictly less than her valuation in X, implying that $\phi(X)$ is lexicographically larger than $\phi(Y)$. This shows that the approach from Chapter 5 cannot be generalized to guarantee $(1-\varepsilon)$-EFX allocation when there are four or more agents.

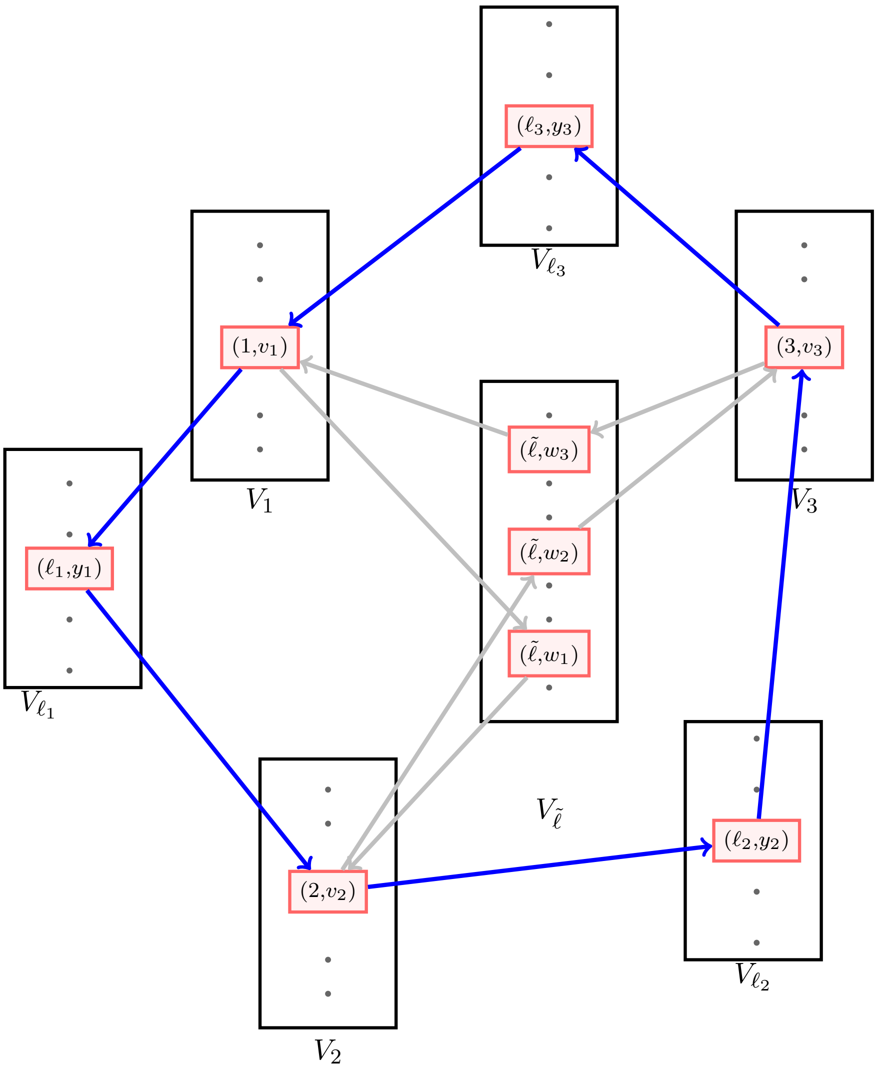


Figure 6.3: Illustration of the existence of a cycle that visits every part at most once. We take the instance in Figure 6.2, where there exists a cycle C that visits every part other than $V_{\tilde{\ell}}$ at most once. The edges of the cycle C are light gray color in color. The figure shows how to obtain a cycle C' that visits every part at most once from C. The edges of C' are blue in color. For all $i \in [3]$, we replace the subpath in C of the form $(i, v_i) \to (\tilde{\ell}, w_i) \to (i+1, v_{i+1})$ $(3+1$ is to be interpreted as $1)$ by $(i, v_i) \to (\ell_i, y_i) \to (i+1, v_{i+1})$ to get C'.

	g_1	g_2	g_3	g_4	g_5	g_6	g_7	g_8	g_9
a	0	0	0	0	0	0	6	4	0
b	16	4	24	4	0	34	31	0	2
c	10	0	18	8	20	0	29	0	6
d	0	0	0	0	18	20	19	0	4

Table 6.1: An instance where showing that the technique in Chapter 5 cannot be used to determine $(1-\varepsilon)$-EFX allocations with four agents. In particular, given a $(1-\varepsilon)$-EFX allocation X and the unallocated good g_9, there is no complete $(1-\varepsilon)$-EFX allocation where the valuation of agent a does not strictly decrease, i.e., in any complete $(1-\varepsilon)$-EFX allocations Y, we have $v_a(Y_a) < v_a(X_a)$.

PART II

Fair and Efficient Allocation of Divisible Bads

CHAPTER 7

Competitive Equilibrium with Divisible Bads

In this Chapter, we study the existence and computational complexity of competitive equilibrium with *divisible* chores (bads) in one of the most fundamental economic model–the exchange model. The exchange model is like a barter system, where each agent brings a bundle of chores that needs to be completed and exchanges them with others to optimize their (dis)utility. For example, a set of university students teaching each other in a group study, to optimize the time and effort required. A competitive equilibrium is a set of prices (one for each chore) and an allocation of the chores to the agents where all chores are completely assigned and each agent gets its most preferred bundle subject to the price of her initial bundle.

We assume that agents have linear disutility (cost) functions, i.e., the disutility of an agent is $\sum_j d_{ij} X_{ij}$, where d_{ij} is the disutility agent i gets from doing a unit amount of chore j, and X_{ij} indicates the amount of chore j that agent i does. Clearly, an agent can do a chore within a reasonable amount of time only if she has the skill set required for it. For example, a professor trained in computer science (CS) can teach a CS course in the upcoming semester, but may not have skill set to teach a course in music. This essentially boils down to not allocating certain chores to certain agents. In case of goods, this is achieved by specifying *zero* utility values to some items, and its analogue for chores is specifying *infinite* disutility. Thus, we allow an agent to have infinite disutility for some chores, which is unlike the model in [22], where every agent is assumed to have finite disutility for every chore. We now formally define the Arrow-Debreu model with chores.

7.0.1 Model and Notations

A chore division problem consists of a set of m divisible chores (bads), namely $B = \{b_1, \ldots, b_m\}$, and a set of n agents $A = \{a_1, \ldots, a_n\}$. Each agent a_i has $d(a_i, b_j) \in (0, \infty]$ disutility (pain) for doing unit amount of chore b_j.[1] Here, infinite disutility implies that the agent does not have required skill set to do the chore in a reasonable amount of time. If agent a_i is assigned bundle $X_i = \langle X_{i1}, \ldots, X_{im} \rangle \in \mathbb{R}^m_{\geq 0}$ where X_{ij} is the amount of chore b_j she gets, then her total disutility is $d_i(X_i) = \sum_{j \in [m]} d(a_i, b_j) \cdot X_{ij}$. We study the problem under exchange model, where agent a_i brings $w(a_i, b_j)$ amount of chore b_j to be done (by herself or other agents).

Given prices $p = \langle p(b_1), p(b_2), \ldots, p(b_m) \rangle \in \mathbb{R}^m_{\geq 0}$ for chores, where $p(b_j)$ denotes the payment for doing unit amount of chore b_j, agent a_i needs to earn $\sum_{j \in [m]} w(a_i, b_j) \cdot p(b_j)$ in order to pay to get her own chores done. In this light, we define the feasible set of chores that can be allocated to agent a_i, at the price vector p, as $F_i(p) =$

[1] If $d(a_i, b_j)$ is zero, then chore b_j can be safely assigned to agent a_i and can be removed from the instance.

$\left\{X_i \in \mathbb{R}^m_{\geq 0} \mid \sum_{j\in[m]} X_{ij} \cdot p(b_j) \geq \sum_{j\in[m]} w(a_i, b_j) \cdot p(b_j)\right\}$. She can earn this amount by doing other chores, while minimizing her disutility – this defines her *optimal bundle* (or optimal chore set).

$$OB_i(p) = \underset{X_i \in F_i(p)}{\arg\max}\ \ d_i(X_i). \tag{7.1}$$

It is easy to see that in her optimal bundle agent a_i gets assigned only those chores that minimizes her disutility per dollar earned and agent i earns money exactly equal to the total price of her endowments. Formally, if $X_i \in OB_i(p)$, then,

$$\forall j \in [m], \quad X_{ij} > 0 \quad \Rightarrow \quad \frac{d(a_i, b_j)}{p(b_j)} \leq \frac{d(a_i, b_{j'})}{p(b_{j'})} \quad \forall j' \in [m],$$

and

$$\sum_{j\in[m]} X_{ij} \cdot p(b_j) = \sum_{j\in[m]} w(a_i, b_j) \cdot p(b_j).$$

In the above ratios, to deal with zero prices and infinite disutilities we assume that $\infty/a > b/0$ for any $a, b \geq 0$. Clearly, an optimal bundle of an agent contains only those chores for which she has finite disutility.

Price vector p is said to be at *competitive equilibrium (CE)* if all chores are completely assigned when every agent gets her optimal bundle, *i.e.*, $X_i \in OB_i(p)$ for all $i \in [n]$ and $\sum_{i\in[n]} X_{ij} = \sum_{i\in[n]} w(a_i, b_j)$, for all $j \in [m]$. It is without loss of generality to assume that each chore is available in one unit total, i.e. for each $b_j \in B$, $\sum_{i\in[m]} w(a_i, b_j) = 1$ (through appropriate scaling of the disutility values). We now formally describe our problem.

Definition 7.1 (Chore Division in the Arrow-Debreu Model)**.** *Given a set of agents $A = \{a_1, a_2, \ldots, a_n\}$, chores $B = \{b_1, b_2, \ldots, b_m\}$, disutilities $d(\cdot,\cdot)$ and endowments $w(\cdot,\cdot)$, our goal is to find a price vector $p = \langle p(b_1), p(b_2), \ldots, p(b_m)\rangle \in \mathbb{R}^m_{\geq 0}$ and allocation $X = \langle X_1, X_2, \ldots, X_n\rangle$, such that*

- *Every agent gets their optimal bundle: $X_i \in OB_i(p)$.*
- *All chores are completely allocated: $\sum_{i\in[n]} X_{ij} = \sum_{i\in[n]} w(a_i, b_j)$ for all $b_j \in B$.*

Observe that the equilibrium prices are scale invariant: if p is an equilibrium price vector then so is $\alpha \cdot p$ for any positive scalar α. Furthermore, at equilibrium $p(b_j) > 0$ for each chore j, otherwise no agent would be willing to do it. A competitive equilibrium $\langle p, X\rangle$ has many desirable properties like envy-freeness and Pareto optimality in the chore division with equal income [22]. Similarly, competitive equilibrium for the exchange model too satisfies Pareto optimality and *weighted* envy-freeness[2].

Fisher Model and CEEI. The Fisher model is a special case of exchange model, where instead of the endowment of chores, each agent a_i has a requirement of earning a fixed amount of money $e(a_i) \geq 0$, i.e., the only change is in the definition of the feasible set of chores that can be allocated to an agent at a given price vector p,

[2]Weight of an agent at given prices is the total monetary cost of the chores she brings. Naturally, higher the cost of her chores (more money she has to earn), larger is her share of disutility.

$F_i(p) = \left\{ X_i \in \mathbb{R}^m_{\geq 0} \mid \sum_{j\in[m]} X_{ij} \cdot p(b_j) \geq e(a_i) \right\}$. If $e(a_i) = 1$ for all $a_i \in A$ then resulting equilibrium is called *competitive equilibrium with equal income*. Clearly, CEEI is a special case of the Fisher model. Observe that determining competitive equilibrium in the Fisher model, can be modeled as determining competitive equilibrium in the exchange model, by setting $w(a_i, b_j) = e(a_i)$ for each $a_i \in A$ and $b_j \in B$, while keeping the disutility values as is. Therefore, similar to the case with divisible goods, we have that

CEEI $\subset$ competitive equilibrium in the Fisher model $\subset$ competitive equilibrium in the Arrow-Debreu model.

The existence and computational complexity of competitive equilibrium is well-understood in the case of goods for different utility functions (check Chapter 2). In case of linear utilities, there is a simple (polynomial-time verifiable) necessary and sufficient condition for checking existence [58, 46], the set of equilibria is convex, and there are many (strongly) polynomial-time algorithms [66, 95, 50, 49, 63]. For the case of chores, [22, 23] study the Fisher model (special case of exchange), and they show that the set of competitive equilibrium could be non-convex. However, despite all these fundamental differences to the setting with divisible goods, neither polynomial-time algorithms nor hardness results have been obtained so far. The first question we consider is:

Question 1. Like in the case of goods, are there polynomial-time verifiable necessary and sufficient conditions for the existence of a competitive equilibrium in the Arrow-Debreu model with chores?

Our first result answers the above question negatively. We show that the problem of checking the existence of a competitive equilibrium is strongly NP-hard.[3] This rules out obtaining polynomial-time verifiable necessary and sufficient condition unless P=NP.[4] Therefore, the next best hope is to obtain weakest possible sufficiency conditions that also capture interesting instances, leading to the next question.

Question 2. Are there polynomial-time verifiable sufficient conditions that guarantee the existence of a competitive equilibrium with chores? And, can we compute a competitive division in polynomial-time under them?

Our next set of results addresses the above question. First we show existence under two conditions. The first condition, known as *strong connectivity of the exchange graph*, is an artifact of the exchange model, and is required in case of goods as well [75]; if a set of agents can consume only a strict subset of the chores that they cummulatively bring then no prices can ensure demand equals supply. The second sufficiency condition relies on the structure of the disutility matrix. The second condition dictates that for any two agents, the sets of chores towards which they have finite disutility is either identical or disjoint. While this condition is specific to chores, it is simple, polynomial-time verifiable, and seems to be unavoidable (see Example 7.6).

Next we show that for instances satisfying the sufficiency conditions computing a competitive equilibrium is PPAD-hard. This comes as a surprise given (strongly)

[3] We note that our NP-hardness result also holds for the competitive equilibrium with equal incomes (CEEI) model, which is a special case of Fisher. And, additionally it holds even for constant-approximate CEEI.

[4] In turn, the condition for the existence that is polynomial-time checkable may not be unique.

polynomial algorithms in the case of goods, since both the problems are very similar. At a high-level, our proof builds on the approaches of [37, 38] that show hardness for the goods case under more general utility functions exhibiting *non-monotonicity.* In our case, we deal with linear disutility functions, which do not have non-monotonicity. We need to use properties of the chore division problem intricately to construct gadgets and make them work together, e.g., higher priced chores are more valuable; clearly this is not possible with goods. To the best of our knowledge, these are the first hardness results for the chore division problem, and in fact for any economic model under *linear* preferences. We now formally describe the results.

7.0.2 Overview of Our Results and Techniques

In this section we discuss the high-level ideas and techniques used to prove our main results. We first note that in general, a chore division instance may not admit a competitive equilibrium as demonstrated by the following example.

Example 7.2. *There are two agents a_1 and a_2, and two chores b_1 and b_2. We have $w(a_i, b_j) = 1$ for all $i, j \in [2]$, and $d(a_1, b_1) = d(a_2, b_1) = 1$, and $d(a_1, b_2) = \infty$ and $d(a_2, b_2) = 2$. Let $p(b_1)$ and $p(b_2)$ be the prices of the chores at a competitive equilibrium.*

Observe that since $d(a_1, b_2) = \infty$, a_1 earns her entire money of $w(a_1, b_1) \cdot p(b_1) + w(a_1, b_2) \cdot p(b_2)$ from b_1. Therefore, at a competitive equilibrium, the total price of the chore b_1 is at least the total money earned by a_1: $(w(a_1, b_1)+w(a_2, b_1))\cdot p(b_1) \geq (w(a_1, b_1)\cdot p(b_1) + w(a_1, b_2) \cdot p(b_2))$. This implies that $2 \cdot p(b_1) \geq p(b_1) + p(b_2)$, further implying that $p(b_1) \geq p(b_2)$. In that case observe that the disutility to price ratio of b_2 is strictly less than that of b_1 for a_2: $d(a_2, b_1)/p(b_1) = 1/p(b_1) < 2/p(b_1) \leq 2/p(b_2) = d(a_2, b_2)/p(b_2)$. Thus, none of the agents are willing to do chore b_2, and therefore it remains unassigned, a contradiction.

It is well known that in the Arrow-Debreu model, a competitive equilibrium may not exist while dividing goods as well. In fact, there are *polynomial-time checkable* necessary and sufficient conditions for existence of competitive equilibrium with goods. The next natural question is to obtain similar conditions for the chore division as well. However, in Section 7.1 we prove the following theorem.

Theorem 7.3. *Determining whether an instance I of chore division in the Fisher model admits a competitive equilibrium is strongly NP-hard.*

The above theorem rules out obtaining polynomial-time checkable necessary and sufficient conditions for existence of competitive equilibrium unless P=NP.[5] The next best hope is to design weakest possible conditions that ensures competitive equilibrium and captures interesting class of instances. To this end, we derive two conditions.

The first condition is an artifact of the exchange setting, and is required for dividing goods as well [75, 92]: if a set of agents are interested to consume only a strict subset of the endowment that they cumulatively own, then no prices can ensure demand equals supply. We now define a condition that helps us resolve this issue [6]. To define the condition, we first define the *economy graph* of a given instance of chore division.

[5] In turn there is no unique condition that ensures CE.

[6] In fact, Condition 1 is the analogue of the necessary and sufficient condition required for competitive equilibrium to exist in exchange markets with goods

Definition 7.4 (Economy Graph [75]). *Given an instance $I = \langle A, B, d(\cdot,\cdot), w(\cdot,\cdot)\rangle$, an Economy Graph $G = (A, E)$ is a graph, with vertices corresponding to the agents and there exists an edge from a_i to a_j if and only if there exist a chore $c \in B$, such that $w(a_i, c) > 0$ and $d(a_j, c) \neq \infty$.*

Now we define the first condition.

Definition 7.5 (Condition 1 [75]). *The economy graph of the instance is strongly connected.*

Observe that our instance in 7.2 does satisfy Condition 1, yet does not admit a competitive equilibrium. The primary reason for non-existence of equilibrium in 7.2 is that sets $\{b \in B \mid d(a_1, b) \neq \infty\}$ and $\{b \in B \mid d(a_2, b) \neq \infty\}$ are neither same nor disjoint. Next by generalizing this example we demonstrate that unless the finite disutility chore sets of any two agents are either identical or disjoint, a competitive equilibrium may not exist. In particular, given any integer $n > 1$ and $m > 1$, we create a chore division instance with n agents and m chores that satisfies Condition 1, has exactly one agent-chore pair with infinite disutility, and does not admit a competitive equilibrium.

Example 7.6. *There are n agents $a_1, a_2, \ldots, a_n$, and m chores $b_1, b_2, \ldots, b_m$. We set $w(a_i, b_j) = 1$ for all $i \in [n]$ and $j \in [m]$. So there is a total of n units of each chore b_j, for all $j \in [m]$. Now, we set $d(a_i, b_j) = 1$ for all $i \in [n]$ and $j \in [m-1]$. We set $d(a_i, b_m) = nm$ for all $i \in [n-1]$ and $d(a_n, b_m) = \infty$.*

Since $w(a_i, b_j) = 1$, for all $i \in [n]$ and $j \in [m]$, the instance in Example 7.6 does satisfy Condition 1 (the economy graph of the instance is a clique). Observe that since all the agents have the same disutility for the chores $\cup_{j\in[m-1]} b_j$, the prices of all these chores will be the same at a competitive equilibrium (otherwise some of the chores will remain unallocated). Therefore, let p be the price of a chore b_j for $j \in [m-1]$, and p' be the price of the chore b_m at a competitive equilibrium. Since a_n has infinite disutility for b_m, she will earn her entire money of $\sum_{j\in[m]} w(a_n, b_j) \cdot p(b_j) = (m-1)\cdot p + p'$ from the chores in $\cup_{j\in[m-1]} b_j$. Therefore, at a competitive equilibrium, the total price of the chores in $\cup_{j\in[m-1]} b_j$ is at least the total money earned by a_n, i.e., total prices of the chores owned by agent a_n, implying that $\sum_{j\in[m-1]} \sum_{i\in[n]} w(a_i, b_j)\cdot p(b_j) \geq \sum_{j\in[m]} w(a_n, b_j)\cdot p(b_j)$. This implies that $(m-1)\cdot n \cdot p \geq (m-1)\cdot p + p'$, further implying that $(m-1)\cdot(n-1)\cdot p \geq p'$. In that case observe that the disutility to price ratio of b_m is strictly less than that of b_1 for any agent a_i, for $i \in [n-1]$: $d(a_i, b_1)/p(b_1) = 1/p \leq ((n-1)\cdot(m-1))/p' < nm/p' = d(a_i, b_m)/p(b_m)$. Thus, none of the agents are willing to do chore b_m, and it remains unallocated, a contradiction.

Our next condition is to circumvent the primary issue in Example 7.6 that results in the non-existence of a competitive equilibrium. To this end, we define the disutility graph $D = (A \cup B, E_D)$ as the bipartite graph with the set of agents A and the set of chores B forming the independent sets and there is an edge from an $a \in A$ to a $b \in B$ when $d(a, b) \neq \infty$. Examples 7.2 and 7.6 demonstrate that whenever there is a connected component D' of D which is not a biclique, there exists disutility values for which the instance will not admit a competitive equilibrium. This brings us to our second sufficiency condition.

Definition 7.7 (Condition 2). *The disutility graph is a disjoint union of bicliques.*

We show as the second main result of this chapter, that Conditions 1 and 2 guarantees the existence of a competitive equilibrium.

Theorem 7.8. *A chore division instance satisfying Conditions 1 and 2 admits a competitive equilibrium.*

The proof of existence of a competitive equilibrium under these two conditions makes use of Kakutani as well as Brower fixed-point theorems in non-trivial ways. To start with, unlike the known existence proofs for the goods setting, the simplex domain of prices does not seem to suffice here. Secondly, zero priced chores pose the issue of un-defined demand sets for the agents. We fix this issue by introducing a more complicated domain of the prices and while maintaining the new price-domain we come up with a novel Kakutani's fixed point formulation that invokes Brower's fixed-point within it. This tool may be of independent interest to show the existence of equilibrium for other related settings.

Observe that both conditions 1 and 2 are simple and polynomial-time verifiable. However, we show that computing a competitive equilibrium for instances that satisfy Conditions 1 and 2, is PPAD-hard. This is in sharp contrast to the competitive equilibrium with goods, where there exists a strongly polynomial algorithm [63] to compute a competitive equilibrium for instances that satisfy the necessary and sufficient conditions[7].

Theorem 7.9. *Finding a competitive equilibrium in a chore division instance satisfying Conditions 1 and 2, is PPAD-hard.*

Roadmap of the Chapter. The rest of the chapter is dedicated to proving Theorems 7.3, 7.8 and 7.9. In Section 7.1, we show that the determining whether an arbitrary instance of chore division admits a competitive equilibrium is NP-hard (proof of Theorem 7.3). Then, in Section 7.2, we show the proof of existence of a competitive equilibrium in instances satisfying the sufficiency conditions (proof of Theorem 7.8). Finally, in Section 7.3, we show the PPAD-hardness of determining a competitive equilibrium on instances that satisfy the said sufficiency conditions (proof of Theorem 7.9).

7.1 Complexity of Determining the Existence of a Competitive Equilibrium

In this section, we show that determining whether an instance of chore division admits a competitive equilibrium or not is strongly NP-hard. In fact, we show that even determining whether the instance admits a good approximation of a competitive equilibrium in the Fisher model is strongly NP-hard. Now, we formally define the problem of determining an α-approximate competitive equilibrium in the Fisher model.

Definition 7.10 (α-Competitive Equilibrium in Chore Division in the Fisher Model)**.** *Given a set of agents $A = \{a_1, a_2, \ldots, a_n\}$, chores $B = \{b_1, b_2, \ldots, b_m\}$, disutilities $d(\cdot, \cdot)$ and fixed earnings $e(\cdot)$, our goal is to find a price vector $p = \langle p(b_1), p(b_2), \ldots, p(b_m)\rangle \in \mathbb{R}^m_{\geq 0}$ and allocation $X = \langle X_1, X_2, \ldots, X_n\rangle$, such that*

[7] the condition analogous to our Condition 1

- *Every agent gets their optimal bundle:* $X_i \in OB_i(p)$ [8].
- *All chores are* almost *completely allocated:* $\alpha \cdot \sum_{i\in[n]} w(a_i, b_j) \leq \sum_{i\in[n]} X_{ij} \leq \frac{1}{\alpha} \cdot \sum_{i\in[n]} w(a_i, b_j)$ *for all* $b_j \in B$.

We show that finding a $(\frac{11}{12} + \delta)$-competitive equilibrium with chores for any $\delta > 0$ in the Fisher model is strongly NP-hard. This will imply that determining a $(\frac{11}{12} + \delta)$-competitive equilibrium in the exchange setting is also strongly NP-hard. Later, in this section we also extend the method to show NP-hardness for finding a $(\frac{11}{12}+\delta)$-competitive equilibrium even in the CEEI setting. In particular, any polynomial-time algorithm that determines whether an instance admits a $(\frac{11}{12} + \delta)$-competitive equilibrium in the Fisher model will give a polynomial-time algorithm for 3-SAT.

We quickly recall the 3-SAT problem:

Problem 7.11. ***(3-SAT)***
Given*: A set of variables* $X = \{x_1, x_2, \ldots, x_n\}$ *and a set of clauses* $\mathbf{C} = \{C_1, C_2, \ldots, C_m\}$ *where each clause is a disjunction of exactly three* literals[9]*.*
Find*: An assignment* $A : X \to \{T, F\}$ *such that all the clauses are* satisfied[10] *or output that no such assignment exists.*

Given any instance $I = \langle X, \mathbf{C}\rangle$ of 3-SAT, we will create an instance $E(I)$ of chore division such that for any $\delta > 0$, there exists an $(\frac{11}{12} + \delta)$-competitive equilibrium in $E(I)$ if and only if there exists an assignment A that satisfies all the clauses in $\mathbf{C}$ in I. We first briefly sketch the intuition, before we move to the construction of the gadgets required for our reduction.

Several Disconnected Equilibria. We sketch a very simple scenario that could arise in chore division in the Fisher model: Consider an instance with two agents a_1 and a_2 with a fixed earning of one unit each and two chores b_1 and b_2. The disutility values are given below where a_1 has a disutility of 1 for b_1 and 3 for b_2, while a_2 has a disutility of ∞ for b_1 and 1 for b_2.

	b_1	b_2
a_1	1	3
a_2	∞	1

Let $p = \langle p(b_1), p(b_2)\rangle$ be an equilibrium price vector. Also, throughout this section we use the notation MPB_a to denote the *minimum pain per buck* bundle for agent a at the prices p: a chore $b \in MPB_a$ if and only if $\frac{d(a,b)}{p(b)} \leq \frac{d(a,b')}{p(b')}$ for all other chores b' in the instance. Observe that this small instance exhibits exactly two competitive equilibria:

[8] Recall that in the Fisher model, we have a subtle difference in the definition of $F_i(p)$ (and consequently $OB_i(p)$). We have $OB_i(p) = \underset{X_i \in F_i(p)}{\arg\max}\ d_i(X_i)$, where $F_i(p) = \left\{X_i \in \mathbb{R}^m_{\geq 0} \mid \sum_{j\in[m]} X_{ij} \cdot p(b_j) \geq e(a_i)\right\}$

[9] A literal is a variable or the negation of a variable

[10] A clause $C_r = \ell_1 \vee \ell_2 \vee \ell_3$, where each ℓ_i is a literal, is satisfied if and only if $A(\ell_i) = T$ for at least one $i \in [3]$.

- The first competitive equilibrium is when both $p(b_1)$ and $p(b_2)$ are set to 1. Note that only $MPB_{a_1} = \{b_1\}$ and $MPB_{a_2} = \{b_2\}$. Thus a_1 earns her entire one unit of money from b_1 and a_2 earns her entire one unit of money from b_2.

- The second competitive equilibrium is when a_1 earns from both b_1 and b_2. For this we set $p(b_1)$ to 1/2 and $p(b_2)$ to 3/2. Note that $MPB_{a_1} = \{b_1, b_2\}$ and $MPB_{a_2} = \{b_2\}$. Under these prices, a_2 earns her entire money by doing 2/3 of b_2, and a_1 earns her money by doing all of b_1 and 1/3 of b_2.

Also, observe that there exists no competitive equilibrium at any other set of prices. This is a striking difference to the scenario with only goods to divide, where all competitive equilibrium exists at a unique price vector. Now, let us introduce another agent a_3 and another chore b_3 in the instance. Let us say that a_3 has a fixed earning of one unit, and both agents a_1 and a_2 have a disutility of ∞ towards b_3. We now discuss two scenarios that may arise depending on a_3's disutility towards the chores b_1, b_2 and b_3:

(1) a_3 has a disutility of 1 towards b_3 and b_2, and ∞ towards b_1.

(2) a_3 has a disutility of 1 towards b_3, $\frac{1}{2}$ towards b_1 and ∞ towards b_2.

We will now show that, at a competitive equilibrium, in scenario (1), $b_2 \notin MPB_{a_1}$ and in scenario (2), $b_2 \in MPB_{a_1}$, suggesting that depending on the valuation of a_3, only one *local* equilibrium among the agents a_1, a_2 and chores b_1 and b_2 is admissible at a competitive equilibrium. Let $p(b_1)$, $p(b_2)$ and $p(b_3)$ denote the prices of chores at an equilibrium. Note that since both a_1 and a_2 have a disutility of ∞ for b_3, they only earn money from b_1 and b_2. Thus $p(b_1) + p(b_2) \geq 2$. Note that in both scenarios b_3 should be in MPB_{a_3} as a_3 is the only agent with finite disutility towards it. Now,

- In scenario (1): Since $b_3 \in MPB_{a_3}$, we have $\frac{d(a_3,b_3)}{p(b_3)} \leq \frac{d(a_3,b_2)}{p(b_2)}$ or equivalently $\frac{1}{p(b_3)} \leq \frac{1}{p(b_2)}$, implying that $p(b_3) \geq p(b_2)$. This in turn implies that

$$\begin{aligned} p(b_2) + 2 &\leq p(b_2) + (p(b_1) + p(b_2)) && (\text{as } p(b_1) + p(b_2) \geq 2) \\ &\leq p(b_1) + p(b_2) + p(b_3) && (\text{as } p(b_2) \leq p(b_3)) \\ &= 3. \end{aligned}$$

  Thus we have $p(b_2) \leq 1$, implying that $p(b_1) \geq 1$ (as $p(b_1) + p(b_2) \geq 2$). Therefore, we can conclude that $b_2 \notin MPB_{a_1}$ as the disutility to price ratio of b_1 is strictly less than that of b_2 for agent a_1.

- In scenario (2): Since $b_3 \in MPB_{a_3}$, we have $\frac{d(a_3,b_3)}{p(b_3)} \leq \frac{d(a_3,b_1)}{p(b_1)}$, we have $\frac{1}{p(b_3)} \leq \frac{1}{2p(b_1)}$, implying that $p(b_3) \geq 2p(b_1)$. This in turn implies that

$$\begin{aligned} 2p(b_1) + 2 &\leq 2p(b_1) + (p(b_1) + p(b_2)) && (\text{as } p(b_1) + p(b_2) \geq 2) \\ &\leq p(b_1) + p(b_2) + p(b_3) && (\text{as } 2p(b_1) \leq p(b_3)) \\ &= 3. \end{aligned}$$

  Thus we have $p(b_1) \leq \frac{1}{2}$, implying that $p(b_2) \geq \frac{3}{2}$. Therefore, the disutility to price ratio of b_2 is at most that of b_1 for agent a_1 and thus we conclude that $b_2 \in MPB_{a_1}$.

Thus, as mentioned earlier, the valuations of the agents outside the local sub-instance formed by agents a_1, a_2 and chores b_1, b_2, impose a specific local equilibrium (among the two disjoint local equilibria) in the sub-instance. We will now show that when there are n such local sub-instances (resulting in 2^n disjoint equilibria), finding the correct local equilibria becomes intractable.

7.1.1 Variable Gadgets

For each variable x_i, we introduce two agents a_1^i and a_2^i and two chores b_1^i and b_2^i. We set

$$d(a_1^i, b_1^i) = 1, \qquad d(a_1^i, b_2^i) = 3,$$
$$d(a_2^i, b_1^i) = \infty, \qquad d(a_2^i, b_2^i) = 1.$$

See Figure 7.1 for an illustration. We set the earnings of both a_1^i and a_2^i to be one, i.e., $e(a_1^i) = e(a_2^i) = 1$. Also, for all $i \in [n]$ agents a_1^i and a_2^i have a disutility of ∞ for all other goods in the instance (that have been introduced and will be introduced by clause gadgets in the next subsection).

7.1.2 Clause Gadgets

For each clause $C_r = (\ell_i \vee \ell_j \vee \ell_k)$, where ℓ_i is either the variable x_i or its negation $\neg x_i$, we introduce four agents n_i^r, n_j^r, n_k^r and $\mathbf{n}^r$, and three chores m_i^r, m_j^r, and m_k^r. We define the disutility of the agents as follows: For each literal ℓ_i, if

- $\ell_i = x_i$, then,

$$d(n_i^r, b_2^i) = 1 \quad \text{and} \quad d(n_i^r, m_i^r) = \varepsilon$$
$$d(\mathbf{n}^r, b_2^i) = 1 \quad \text{and} \quad d(\mathbf{n}^r, m_i^r) = \varepsilon.$$

  for some $0 < \varepsilon \ll 1$, but $\frac{1}{\varepsilon} \in \mathcal{O}(1)$.

- $\ell_i = \neg x_i$, then,

$$d(n_i^r, b_1^i) = \tfrac{2}{3} \quad \text{and} \quad d(n_i^r, m_i^r) = \tfrac{4\varepsilon}{3}$$
$$d(\mathbf{n}^r, b_1^i) = \tfrac{2}{3} \quad \text{and} \quad d(\mathbf{n}^r, m_i^r) = \tfrac{4\varepsilon}{3}.$$

For all other agents and chores pair, the disutility is ∞. See Figure 7.1 for an illustration. We set $e(n_i^r) = e(n_j^r) = e(n_k^r) = \varepsilon$ and $e(\mathbf{n}^r) = \#(C_r) \cdot (\frac{\varepsilon}{2}) + \overline{\#}(C_r) \cdot (\varepsilon) - \varepsilon'$, where $\#(C_r)$ is the number of literals in C_r that are not negations of variables and $\overline{\#}(C_r)$ is the number of literals in C_r that are negations of variables [11], and $\varepsilon' < \frac{\varepsilon}{2}$ (the exact value of ε' will depend on δ [12] and will be made clear in the proof of Lemma 7.14). We make a small claim about the total earning requirements for the agents n_i^r, n_j^r, n_k^r and $\mathbf{n}^r$.

[11] This implies that $\#(C_r) + \overline{\#}(C_r) = 3$

[12] Reminder to what δ is: recall that we are trying to show the hardness of determining whether an instance admits a $(\frac{11}{12} + \delta)$-competitive equilibrium or not.

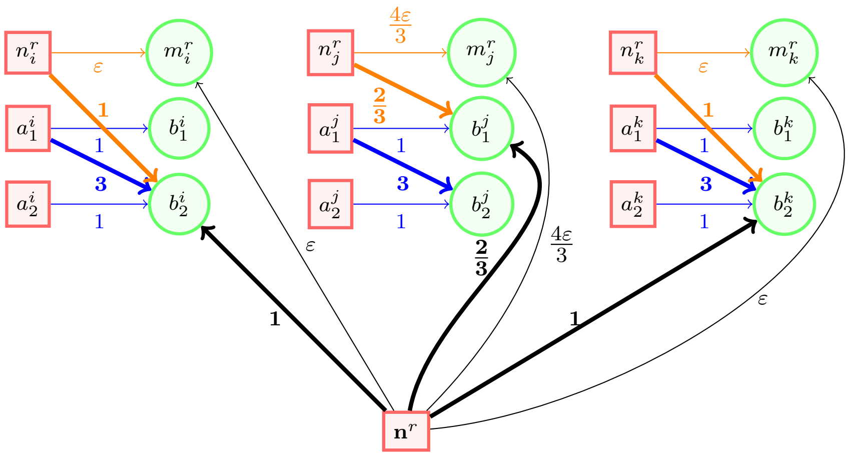


Figure 7.1: Illustration of the variable gadgets corresponding to x_i, x_j and x_k, and the clause gadget $C_r = (x_i \vee \neg x_j \vee x_k)$. The red squared nodes represent the agents and the green circle nodes represent the chores. Only finite disutility values have been indicated. The disutility edges from agents in the variable gadgets are outlined by blue edges. The disutility edges from agents n_ℓ^r for $\ell \in \{i, j, k\}$ are outlined by orange edges and the disutility edges from agent $\mathbf{n}^r$ are outlined by black edges. Thicker disutility edges have a higher disutility than the thinner disutility edges of the same color.

Claim 7.12. *For each clause $C_r = \ell_i \vee \ell_j \vee \ell_k$ in I, we have $e(n_i^r) + e(n_j^r) + e(n_k^r) + e(\mathbf{n}^r) = \#(C_r) \cdot (\frac{3\varepsilon}{2}) + \overline{\#}(C_r) \cdot (2\varepsilon) - \varepsilon'$*

Proof. We have,

$$\begin{aligned} e(n_i^r) + e(n_j^r) + e(n_k^r) + e(\mathbf{n}^r) &= 3\varepsilon + \#(C_r) \cdot (\tfrac{\varepsilon}{2}) + \overline{\#}(C_r) \cdot (\varepsilon) - \varepsilon' \\ &= (\#(C_r) + \overline{\#}(C_r))\varepsilon + \#(C_r) \cdot (\tfrac{\varepsilon}{2}) + \overline{\#}(C_r) \cdot (\varepsilon) - \varepsilon' \\ &= \#(C_r) \cdot (\tfrac{3\varepsilon}{2}) + \overline{\#}(C_r) \cdot (2\varepsilon) - \varepsilon' \end{aligned}$$ ☐

We now show how to map any allocation in $E(I)$ to an assignment of variables in I. Consider any *money allocation* f under some prices p in $E(I)$, i.e., if agent i does X_{ij} amount of chore j, then $f(i, j) = X_{ij} \cdot p(j)$.

If agent a_1^i does some of chore b_2^i, i.e., $f(a_1^i, b_2^i) > 0$, then we set x_i to F and if $f(a_1^i, b_2^i) = 0$, then we set x_i to T.

We now make some basic observations.

Observation 7.13. *Let p be the prices of chores and f be the money allocation corresponding to a competitive equilibrium in $E(I)$. Consider any clause $C_r = (\ell_i \vee \ell_j \vee \ell_k)$. Then,*

(1) if $\ell_i = x_i$ and $f(a_1^i, b_2^i) > 0$ then $p(m_i^r) \geq \frac{3\varepsilon}{2}$, and

(2) if $\ell_i = \neg x_i$ and $f(a_1^i, b_2^i) = 0$, then $p(m_i^r) \geq 2\varepsilon$.

Proof. We first prove part 1. If $f(a_1^i, b_2^i) > 0$, then $b_2^i \in MPB_{a_1^i}$, implying that $\frac{d(a_1^i,b_2^i)}{p(b_2^i)} \leq \frac{d(a_1^i,b_1^i)}{p(b_1^i)}$. Therefore, we have that $p(b_2^i) \geq \frac{d(a_1^i,b_2^i)}{d(a_1^i,b_1^i)} \cdot p(b_1^i) = 3p(b_1^i)$. Also, note that since agents a_1^i and a_2^i have finite disutility only for chores b_1^i and b_2^i, they will only earn from chores b_1^i and b_2^i. This implies that $p(b_1^i) + p(b_2^i) \geq e(a_1^i) + e(a_2^i) = 2$. Also, since $p(b_2^i) \geq 3p(b_1^i)$ we have that $p(b_2^i) \geq 3/2$. Now, observe that the only agents who have finite disutility towards m_i^r are the agents n_i^r and $\mathbf{n}^r$. Since $\ell_i = x_i$, both n_i^r and $\mathbf{n}^r$ have a disutility of 1 towards b_2^i and ε towards m_i^r. Therefore, for m_i^r to be in either $MPB_{n_i^r}$ or $MPB_{\mathbf{n}^r}$, we need $\frac{\varepsilon}{p(m_i^r)} \leq \frac{1}{p(b_2^i)} \leq \frac{2}{3}$. This implies that $p(m_i^r) \geq \frac{3\varepsilon}{2}$.

The proof of part 2 is very similar. Note that agent a_1^i has finite disutility only for chores b_1^i and b_2^i. If $f(a_1^i, b_2^i) = 0$, then she only earns her required money of $e(a_1^i)$ by doing chore b_1^i, implying that $p(b_1^i) \geq e(a_1^i) = 1$. Similar to the proof in part 1, observe that the only agents who have finite disutility towards m_i^r are the agents n_i^r and $\mathbf{n}^r$. Since $\ell_i = \neg x_i$, both n_i^r and $\mathbf{n}^r$ have a disutility of $\frac{2}{3}$ towards b_1^i and $\frac{4\varepsilon}{3}$ towards m_i^r. Therefore, for m_i^r to be in either $MPB_{n_i^r}$ or $MPB_{\mathbf{n}^r}$, we need $\frac{4\varepsilon}{3p(m_i^r)} \leq \frac{2}{3p(b_1^i)} \leq \frac{2}{3}$ (as $p(b_i^1) \geq 1$). This implies that $p(m_i^r) \geq 2\varepsilon$. □

Lemma 7.14. *If there is no satisfying assignment to the instance $I = \langle X, \mathbf{C} \rangle$ of 3-SAT, then $E(I)$ does not admit any $(\frac{11}{12} + \delta)$-competitive equilibrium for any $\delta > 0$.*

Proof. We prove by contradiction. Assume otherwise and let p be the equilibrium prices of chores and f be the corresponding money allocation. Recall the mapping from an equilibrium allocation to the assignment of variables: For each $i \in [n]$, if $f(a_1^i, b_2^i) > 0$, then we set x_i to F and if $f(a_1^i, b_2^i) = 0$, then we set x_i to T. Since I admits no satisfying assignment, there exists a clause $C_r = \ell_i \vee \ell_j \vee \ell_k$ which is unsatisfied. For every literal $\ell_i \in C_r$ such that $\ell_i = x_i$, note that x_i is F. Therefore, we have that $f(a_1^i, b_2^i) > 0$. This implies that $p(m_i^r) \geq \frac{3\varepsilon}{2}$ (by Observation 7.13). Similarly, for every literal ℓ_i in C_r such that $\ell_i = \neg x_i$, note that x_i is T. Therefore, we have that $f(a_1^i, b_2^i) = 0$, implying that $p(m_i^r) \geq 2\varepsilon$ (by Observation 7.13). We write the price of chore m_i^r, $p(m_i^r)$ as $\frac{3\varepsilon}{2} + \delta(m_i^r)$ if $\ell_i = x_i$ and $2\varepsilon + \delta(m_i^r)$ if $\ell_i = \neg x_i$, where $\delta(m_i^r)$ is the deviation of the price of m_i^r from its lower bound. Therefore, we have $p(m_i^r) + p(m_j^r) + p(m_k^r) = \#(C_r) \cdot (\frac{3\varepsilon}{2}) + \overline{\#}(C_r) \cdot (2\varepsilon) + \delta(m_i^r) + \delta(m_j^r) + \delta(m_k^r)$. Note that the only agents who have finite disutility for chores m_i^r, m_j^r and m_k^r are the agents n_i^r, n_j^r, n_k^r and $\mathbf{n}^r$. However, by Claim 7.12, we have that $e(n_i^r) + e(n_j^r) + e(n_k^r) + e(\mathbf{n}^r) = \#(C_r) \cdot (\frac{3\varepsilon}{2}) + \overline{\#}(C_r) \cdot (2\varepsilon) - \varepsilon'$ which is strictly less that the sum of prices of chores m_i^r, m_j^r and m_k^r. In particular we have, $\sum_{h \in \{i,j,k\}} p(m_h^r) - (\sum_{h \in \{i,j,k\}} e(n_h^r) + e(\mathbf{n}^r)) = \varepsilon' + \sum_{h \in \{i,j,k\}} \delta(m_h^r)$. Therefore, there exists at least one chore $m_{h'}^r$ with $h' \in \{i,j,k\}$ such that the difference between the total price of the chore and the total money earned from the chore by the agents is $\frac{\varepsilon' + \sum_{h \in \{i,j,k\}} \delta(m_h^r)}{3} \geq \frac{\varepsilon' + \delta(m_{h'}^r)}{3}$. Thus, the portion of chore $m_{h'}^r$ left undone is at least,

$$
\begin{aligned}
&= \frac{\varepsilon' + \delta(m_{h'}^r)}{3 \cdot p(m_{h'}^r)} \\
&\geq \frac{\varepsilon' + \delta(m_{h'}^r)}{3 \cdot (2\varepsilon + \delta(m_{h'}^r))} && \text{(as } p(m_{h'}^r) \text{ is either } \tfrac{3\varepsilon}{2} + \delta(m_{h'}^r) \text{ or } 2\varepsilon + \delta(m_{h'}^r)) \\
&\geq \frac{\varepsilon'}{3 \cdot (2\varepsilon)} && \text{(as } \varepsilon' < \frac{\varepsilon}{2}).
\end{aligned}
$$

Since our reduction works for any choice of $\varepsilon' < \frac{\varepsilon}{2}$, we can choose an ε' such that $\frac{\varepsilon'}{(6\varepsilon)} > \frac{1}{12} - \delta$, implying that we do not have a $(\frac{11}{12} + \delta)$-competitive equilibrium, which is a contradiction. □

Lemma 7.15. *If there exists a satisfying assignment to the instance $I = \langle X, \mathbf{C} \rangle$ of 3-SAT, then $E(I)$ admits a competitive equilibrium.*

Proof. Consider any satisfying assignment in I. We now show how to construct the prices p and the money allocation f corresponding to a competitive allocation. We will ensure that the agents in the variable gadgets earn only from the chores in the variable gadgets and the agents in the clause gadgets earn only from the chores in the clause gadgets.

Prices and Allocation of Chores in Variable Gadgets. For each variable x_i,

- If $x_i = T$, then we set $p(b_1^i) = 1$ and $p(b_2^i) = 1$.
- If $x_i = F$, then we set $p(b_1^i) = \frac{1}{2}$ and $p(b_2^i) = \frac{3}{2}$.

Since the agents in the variable gadgets have finite disutility only for some goods in the variable gadgets (and have disutility of ∞ for every good in the clause gadget) we can already define their optimal bundles (*MPB* bundles). If $x_i = T$, then observe that $MPB_{a_1^i} = \{b_1^i\}$ and $MPB_{a_2^i} = \{b_2^i\}$. Thus, agent a_1^i earns 1 unit of money from doing chore b_1^i entirely and agent a_2^i earns 1 unit of money by doing chore b_2^i entirely. When $x_i = F$, then observe that $MPB_{a_1^i} = \{b_1^i, b_2^i\}$ and $MPB_{a_2^i} = \{b_2^i\}$. Thus, agent a_1^i earns 1 unit of money from doing chore b_1^i entirely and b_2^i partly (1/3 of chore b_2^i) and agent a_2^i earns 1 unit of money by doing chore b_2^i partly (2/3 of chore b_2^i). Now we make an immediate, simple observation:

Observation 7.16. *When $x_i = T$, then $f(a_1^i, b_2^i) = 0$ and when $x_i = F$, we have $f(a_1^i, b_2^i) > 0$.*

Observe that all the local sub-instances corresponding to the variable gadgets have cleared. It suffices to show that there exists a competitive equilibrium for local sub-instances corresponding to the clause gadgets. We now look into the agents and chores in the clause gadget.

Prices and Allocation of Chores in Clause Gadgets. Consider a clause $C_r = \ell_i \vee \ell_j \vee \ell_k$. Let $S_r \subseteq \{\ell_i, \ell_j, \ell_k\}$ be the literals that evaluate to T [13] and $U_r \subseteq \{\ell_i, \ell_j, \ell_k\}$ be the set of literals that evaluate to F under the assignment X. Since X is a satisfying assignment, at least one of the literals will evaluate to T and thus $|S_r| \geq 1$ and $|U_r| \leq 2$. Let $\#(S_r)$ and $\#(U_r)$ be the number of literals in S_r and U_r respectively that are not negations of variables and similarly let $\overline{\#}(S_r)$ and $\overline{\#}(U_r)$ be the number of literals that are negations of variables in S_r and U_r respectively. Let α_r be a scalar such that

$$\alpha_r \cdot (\#(U_r) \cdot \tfrac{3\varepsilon}{2} + \overline{\#}(U_r) \cdot (2\varepsilon)) = |U_r| \cdot \varepsilon + e(\mathbf{n}^r) \tag{7.2}$$

[13] A literal $\ell_i = x_i$ evaluates to T if x_i is set to T and the literal $\ell_i = \neg x_i$ evaluates to T when x_i is set to F.

We now set the prices of the chores in the clause gadgets. Consider any clause $C_r = \ell_i \vee \ell_j \vee \ell_k$ in I (with S_r and U_r defined appropriately). For every literal $\ell_\theta \in S_r$, set,

$$p(m_\theta^r) = \begin{cases} \varepsilon & \text{if } \ell_\theta = \neg x_\theta, \\ \varepsilon & \text{if } \ell_\theta = x_\theta \text{ and } U_r \neq \emptyset, \\ \varepsilon + \frac{e(n^r)}{\#(S_r)} & \text{if } \ell_\theta = x_\theta \text{ and } U_r = \emptyset. \end{cases}$$

For every $\ell_\theta \in U_r$, set

$$p(m_\theta^r) = \begin{cases} \alpha_r \cdot (\frac{3\varepsilon}{2}) & \text{if } \ell_\theta = x_\theta \\ \alpha_r \cdot (2\varepsilon) & \text{if } \ell_\theta = \neg x_\theta. \end{cases}$$

We will now show that under the above prices for the chores in the clause gadgets, we can determine a money flow where all the clause agents earn all of their money from their optimal bundles and all the clause chores will be completed. We distinguish two cases, depending on whether $U_r = \emptyset$ or not,

Case $U_r \neq \emptyset$: In this case, we first observe that α_r is strictly larger than 1:

Observation 7.17. *We have well defined scalar $\alpha_r > 1$.*

Proof. Since we are in the case where $U_r \neq \emptyset$, we have $\#(U_r) \cdot \frac{3\varepsilon}{2} + \overline{\#}(U_r) \cdot (2\varepsilon) > 0$, thus α_r is well defined. For the claim of the lemma, it suffices to show that $|U_r| \cdot \varepsilon + e(\mathbf{n}^r) > \#(U_r) \cdot \frac{3\varepsilon}{2} + \overline{\#}(U_r) \cdot (2\varepsilon)$. To this end,

$$\begin{aligned} |U_r| \cdot \varepsilon + e(\mathbf{n}^r) &= (\#(U_r) + \overline{\#}(U_r)) \cdot \varepsilon + e(\mathbf{n}^r) \\ &= (\#(U_r) + \overline{\#}(U_r)) \cdot \varepsilon + \#(C_r) \cdot \tfrac{\varepsilon}{2} + \overline{\#}(C_r) \cdot (\varepsilon) - \varepsilon' \ . \end{aligned} \tag{7.3}$$

Since the literals that are not negations of variables in U_r are also not negations of variables in C_r we have $\#(U_r) \leq \#(C_r)$. By a similar argument we also have $\overline{\#}(U_r) \leq \overline{\#}(C_r)$. Since $|U_r| \leq 2$ we also have $\#(U_r)+\overline{\#}(U_r) < \#(C_r)+\overline{\#}(C_r)$, implying that either $\#(U_r) < \#(C_r)$ or $\overline{\#}(U_r) < \overline{\#}(C_r)$. Therefore, we have that $\#(C_r) \cdot \frac{\varepsilon}{2} + \overline{\#}(C_r) \cdot (\varepsilon) \geq \#(U_r) \cdot \frac{\varepsilon}{2} + \overline{\#}(U_r) \cdot (\varepsilon) + \frac{\varepsilon}{2}$. Plugging this inequality in (7.3), we have

$$\begin{aligned} |U_r| \cdot \varepsilon + e(\mathbf{n}^r) &\geq (\#(U_r) + \overline{\#}(U_r)) \cdot \varepsilon + \#(U_r) \cdot \tfrac{\varepsilon}{2} + \overline{\#}(U_r) \cdot (\varepsilon) + \tfrac{\varepsilon}{2} - \varepsilon' \\ &> (\#(U_r) + \overline{\#}(U_r)) \cdot \varepsilon + \#(U_r) \cdot \tfrac{\varepsilon}{2} + \overline{\#}(U_r) \cdot (\varepsilon) \qquad (\text{as } \varepsilon' < \tfrac{\varepsilon}{2}) \\ &= \#(U_r) \cdot \tfrac{3\varepsilon}{2} + \overline{\#}(U_r) \cdot (2\varepsilon) \ . \end{aligned}$$ □

We now characterize the optimal bundles (*MPB* chores) for each agent in the clause gadget under the set prices.

Observation 7.18. *For each literal $\ell_\theta \in S_r$, we have $m_\theta^r \in MPB_{n_\theta^r}$.*

Proof. We consider the cases, whether the $\ell_\theta = x_\theta$ or $\ell_\theta = \neg x_\theta$.

- $\ell_\theta = x_\theta$: Note that the only other chore (other than m_θ^r) for which agent n_θ^r has finite disutility is chore b_2^θ. Since $\ell_\theta \in S_r$, this means that $x_\theta = T$ and therefore we have $p(b_2^\theta) = 1$ (the way we assigned the prices to the chores in the variable gadgets). Now observe that,

$$\begin{aligned}\frac{d(n_\theta^r, m_\theta^r)}{p(m_\theta^r)} &= \frac{\varepsilon}{\varepsilon} \\ &= 1 \\ &= \frac{d(n_\theta^r, b_2^\theta)}{p(b_2^\theta)} \ .\end{aligned}$$

Therefore $m_\theta^r \in MPB_{n_\theta^r}$.

- $\ell_\theta = \neg x_\theta$: Note that the only other chore (other than m_θ^r) for which agent n_θ^r has finite disutility is chore b_1^θ. Since $\ell_\theta \in S_r$, this means that $x_\theta = F$ and therefore we have $p(b_1^\theta) = \frac{1}{2}$ (the way we assigned the prices to the chores in the variable gadgets). Now observe that,

$$\begin{aligned}\frac{d(n_\theta^r, m_\theta^r)}{p(m_\theta^r)} &= \frac{4\varepsilon}{3\varepsilon} \\ &= \frac{4}{3} \\ &= \frac{2}{3 \cdot \frac{1}{2}} \\ &= \frac{d(n_\theta^r, b_1^\theta)}{p(b_1^\theta)} \ .\end{aligned}$$

Therefore, $m_\theta^r \in MPB_{n_\theta^r}$. □

This implies that for all literals ℓ_θ in S_r, the agent n_θ^r will earn her entire money of ε by doing the chore ℓ_θ entirely. Therefore, now we only need to look at the agents n_θ^r and chores m_θ^r where $\ell_\theta \in U_r$. To this end we observe that,

Observation 7.19. *For each literal $\ell_\theta \in U_r$, we have $m_\theta^r \in MPB_{n_\theta^r}$ and $m_\theta^r \in MPB_{\mathbf{n}^r}$.*

Proof. We first show that $m_\theta^r \in MPB_{n_\theta^r}$. We make a distinction based on whether $\ell_\theta = x_\theta$ or $\ell_\theta = \neg x_\theta$.

- $\ell_\theta = x_\theta$: In this case we have $p(m_\theta^r) = \alpha_r \cdot (\frac{3\varepsilon}{2})$. Note that the only other chore (other than m_θ^r) for which agent n_θ^r has finite disutility is chore b_2^θ. Since $\ell_\theta \in U_r$, this means that $x_\theta = F$ and therefore we have $p(b_2^\theta) = \frac{3}{2}$ (the way we assigned the prices to the chores in the variable gadgets). Now observe that,

$$\begin{aligned}\frac{d(n_\theta^r, m_\theta^r)}{p(m_\theta^r)} &= \frac{1}{\alpha_r} \cdot \frac{\varepsilon}{\frac{3\varepsilon}{2}} \\ &= \frac{1}{\alpha_r} \cdot \frac{2}{3} \\ &= \frac{1}{\alpha_r} \cdot \frac{d(n_\theta^r, b_2^\theta)}{p(b_2^\theta)} \\ &< \frac{d(n_\theta^r, b_2^\theta)}{p(b_2^\theta)} \ . && \text{(as } \alpha_r > 1 \text{ by Observation 7.17)}\end{aligned} \tag{7.4}$$

- $\ell_\theta = \neg x_\theta$: In this case we have $p(m_\theta^r) = \alpha_r \cdot (2\varepsilon)$. Note that the only other chore (other than m_θ^r) for which agent n_θ^r has finite disutility is chore b_1^θ (the way we assigned the prices to the chores in the variable gadgets). Since $\ell_\theta \in U_r$, this means that $x_\theta = T$ and therefore we have $p(b_1^\theta) = 1$. Now observe that,

$$\begin{aligned}
\frac{d(n_\theta^r, m_\theta^r)}{p(m_\theta^r)} &= \frac{1}{\alpha_r} \cdot \frac{4\varepsilon}{3 \cdot 2\varepsilon} \\
&= \frac{1}{\alpha_r} \cdot \frac{2}{3} \\
&= \frac{1}{\alpha_r} \cdot \frac{d(n_\theta^r, b_1^\theta)}{p(b_1^\theta)} \\
&< \frac{d(n_\theta^r, b_1^\theta)}{p(b_1^\theta)} \quad . \qquad \text{(as } \alpha_r > 1 \text{ by Observation 7.17)}
\end{aligned} \tag{7.5}$$

Thus, in both cases we have $m_\theta^r \in MPB_{n_\theta^r}$.

We will now show that $m_\theta^r \in MPB_{\mathbf{n}^r}$ as well. We do this by showing that the disutility to price ratio of the chores m_θ^r, when $\ell_\theta \in U_r$, is minimum for the agent $\mathbf{n}^r$. *To this end, first crucially observe that from* (7.4) *and* (7.5)*, irrespective of whether* $\ell_\theta = x_\theta$ *or* $\ell_\theta = \neg x_\theta$*, we have* $\frac{d(n_\theta^r, m_\theta^r)}{p(m_\theta^r)} = \frac{1}{\alpha_r} \cdot \frac{2}{3}$. Also, note that the disutility profile agent $\mathbf{n}^r$ has for chore m_θ^r and the chores in the variable gadget of x_θ (b_1^θ and b_2^θ) is identical to the disutility profile of agent n_θ^r for the same set of chores. Therefore, for all $\ell_\theta \in U_r$ we have $\frac{d(\mathbf{n}^r, m_\theta^r)}{p(m_\theta^r)} = \frac{1}{\alpha_r} \cdot \frac{2}{3}$ (irrespective of whether $\ell_\theta = x_\theta$ or $\ell_\theta = \neg x_\theta$) which is also strictly less than both $\frac{d(\mathbf{n}^r, b_2^\theta)}{p(b_2^\theta)}$ and $\frac{d(\mathbf{n}^r, b_1^\theta)}{p(b_1^\theta)}$. We now look at disutility to price ratio that agent $\mathbf{n}^r$ has for chores in S_r. Observe that for all $\ell_\beta \in S_r$ we have $p(m_\beta^r) = \varepsilon$ and $d(\mathbf{n}^r, m_\beta^r) \geq \varepsilon$ (as the disutility is ε if $\ell_\beta = x_\beta$ and is $\frac{4\varepsilon}{3}$ if $\ell_\beta = \neg x_\beta$). This implies that for all $\ell_\beta \in S_r$ we have $\frac{d(\mathbf{n}^r, m_\beta^r)}{p(m_\beta^r)} \geq 1 > \frac{2}{3} > \frac{1}{\alpha_r} \cdot \frac{2}{3}$ (as $\alpha_r > 1$ by Observation 7.17). Therefore, the disutility to price ratio of the chores m_θ^r, when $\ell_\theta \in U_r$, for agent $\mathbf{n}^r$ is $\frac{1}{\alpha_r} \cdot \frac{2}{3}$ which is at most the disutility to price ratio of all the chores for which $\mathbf{n}^r$ has finite disutility . Therefore, we have $\bigcup_{\ell_\theta \in U_r} m_\theta^r \subseteq MPB_{\mathbf{n}^r}$. ☐

Now that we have identified the *MPB* chores for all the agents in the clause gadgets, we are ready to show the money flow allocation. We set

$$\begin{aligned}
f(n_\theta^r, m_\theta^r) &= \varepsilon && \text{(for all } \ell_\theta \in C_r) \\
f(\mathbf{n}^r, m_\theta^r) &= p(m_\theta^r) - \varepsilon \ . && \text{(for all } \ell_\theta \in U_r)
\end{aligned}$$

All agents spend on their corresponding *MPB* chores. Observe that for all $\ell_\theta \in S_r$, the agents n_θ^r earn their money of ε by doing chore m_θ^r completely. Now, for all $\ell_\theta \in U_r$, the agents n_θ^r earn their money of ε by doing chore m_θ^r partially. The agent $\mathbf{n}^r$ earns her entire money by completing whatever is left of the chores in $\bigcup_{\ell_\theta \in U_r} m_\theta^r$ after agents $\cup_{\ell_\theta \in U_r} n_\theta^r$ do their share. It only suffices to show that agent $\mathbf{n}^r$ earns exactly $e(\mathbf{n}^r)$. To this end, we observe that the total money earned by $\mathbf{n}^r$ is

$$\begin{aligned}
\sum_{\ell_\theta \in U_r} f(\mathbf{n}^r, m_\theta^r) &= \sum_{\ell_\theta \in U_r} (p(m_\theta^r) - \varepsilon) \\
&= \alpha_r \cdot (\#(U_r) \cdot \tfrac{3\varepsilon}{2} + \overline{\#}(U_r) \cdot (2\varepsilon)) - |U_r| \cdot \varepsilon \\
&= e(\mathbf{n}^r)\ . && \text{(by (7.2))}
\end{aligned}$$

Therefore, we have an allocation where the agents in the corresponding variable gadgets earn their money by completing the chores in the variable gadgets and the agents in the clause gadget earn their entire money by completing the chores in the clause gadgets. This concludes the proof for the case $U_r \neq \emptyset$.

Case $U_r = \emptyset$: In this case, we have that all the literals in the clause C_r belongs to the set S_r. Therefore, for all the literals ℓ_θ occurring in C_r, we have,

$$p(m_\theta^r) = \begin{cases} \varepsilon & \text{if } \ell_\theta = \neg x_\theta, \\ \varepsilon + \frac{e(n^r)}{\#(S_r)} & \text{if } \ell_\theta = x_\theta \end{cases}$$

Like earlier, we will identify the *MPB* chores for all the clause gadget agents and then will outline a money flow allocation where every agent earns all her money and all the chores are completed. We first look into the agents n_θ^r. Very similar to Observation 7.18, we can claim that $m_\theta^r \in MPB_{n_\theta^r}$ with a very similar argument as the one used in the proof of Observation 7.18: The agent n_θ^r has finite disutility only for chores m_θ^r and b_2^θ if $\ell_\theta = x_\theta$, and only for chores m_θ^r and b_1^θ if $\ell_\theta = \neg x_\theta$, and the price of the chore $p(m_\theta^r)$ is at least ε (it is more if $\ell_\theta = x_\theta$), while the prices of chores b_θ^1 and b_θ^2 are the same as in Observation 7.18.

Now, let us consider agent $\mathbf{n}^r$. Since the disutility profile of agent $\mathbf{n}^r$ is identical to that of n_θ^r, when restricted to chores b_1^θ, b_2^θ and m_θ^r, we can conclude that the disutility to price ratio of m_θ^r for $\mathbf{n}^r$ is at most that of chores b_1^θ and b_2^θ. Now observe that the disutility to price ratio of all chores m_θ^r for $\mathbf{n}^r$ where $\ell_\theta = x_\theta$ is $\frac{d(\mathbf{n}^r, m_\theta^r)}{p(m_\theta^r)} = \frac{\varepsilon}{p(m_\theta^r)} \leq 1$ (as $p(m_\theta^r) = \varepsilon + \frac{e(\mathbf{n}^r)}{\#(S_r)}$), while the disutility to price ratio all chores m_θ^r for $\mathbf{n}^r$ where $\ell_\theta = \neg x_\theta$ is $\frac{d(\mathbf{n}^r, m_\theta^r)}{p(m_\theta^r)} = \frac{4\varepsilon}{3p(m_\theta^r)} > 1$ (as $p(m_\theta^r) = \varepsilon$). Since $\mathbf{n}^r$ has finite disutility only for the chores in the clause gadget of C_r and the chores in the corresponding variable gadgets, we can claim that $\bigcup_{\{\theta | \ell_\theta = x_\theta\}} m_\theta^r \subseteq MPB_{\mathbf{n}^r}$. Now, that we have identified the *MPB* chores for the agents in the clause gadget, we outline a money flow,

$$\begin{aligned}
f(n_\theta^r, m_\theta^r) &= \varepsilon && \text{(for all } \ell_\theta) \\
f(\mathbf{n}^r, m_\theta^r) &= p(m_\theta^r) - \varepsilon\ . && \text{(for all } \ell_\theta = x_\theta)
\end{aligned}$$

All the agents spend on their corresponding *MPB* chores. Observe that for all ℓ_θ, the agents n_θ^r earn their entire money of ε by doing chore m_θ^r (partially if $\ell_\theta = x_\theta$ and completely when $\ell_\theta = \neg x_\theta$). The agent $\mathbf{n}^r$ earns her entire money by completing whatever is left of the chores in $\bigcup_{\{\theta | \ell_\theta = x_\theta\}} m_\theta^r$. It only suffices to show that agent $\mathbf{n}^r$ earns exactly

$e(\mathbf{n}^r)$. To this end, we observe that the total money earned by $\mathbf{n}^r$ is

$$\begin{aligned}\sum_{\{\theta|\ell_\theta = x_\theta\}} f(\mathbf{n}^r, m^r_\theta) &= \sum_{\{\theta|\ell_\theta = x_\theta\}} (p(m^r_\theta) - \varepsilon)\\ &= \#(S_r)\cdot\left(\varepsilon + \frac{e(\mathbf{n}^r)}{\#(S_r)} - \varepsilon\right)\\ &= e(\mathbf{n}^r).\end{aligned}$$

Therefore, we have an allocation where the agents in the variable gadgets earn their money by completing the chores in the variable gadgets and the agents in the clause gadgets earn their entire money by completing the chores in the clause gadgets. This concludes the proof for the case $U_r = \emptyset$. □

This brings us to the main result of this section.

Theorem 7.20. *Determining an $(\frac{11}{12} + \delta)$-competitive equilibrium, for any $\delta > 0$, in chore division in the Fisher model is strongly NP-hard.*

Proof. Given any instance $I = \langle X, \mathbf{C}\rangle$ of 3-SAT, in polynomial-time we can construct an instance $E(I)$ of chore division comprising of all variable gadgets and clause gadgets. Also, observe all the entries in the disutility matrix $d(\cdot,\cdot)$ and the money vector $e(\cdot)$ are constants (Thus all input parameters can be expressed with polynomial bit size in unary notation). Lemma 7.14 implies that we have a $(\frac{11}{12} + \delta)$-competitive equilibrium only if I is satisfiable and Lemma 7.15 implies that if I is satisfiable, then $E(I)$ admits a competitive equilibrium (and thus also a $(\frac{11}{12} + \delta)$-competitive equilibrium). □

Remark 7.21. *Note that every instance of chore division in the Fisher model $\langle A, B, d(\cdot,\cdot), e(\cdot)\rangle$, where $e(a)$ is an integer for all $a \in A$, can be transformed into an instance $I' = \langle A', B, d'(\cdot,\cdot)\rangle$ of chore division in the CEEI model (where $e(a) = 1$ for all $a \in A'$) by creating $e(a)$ many identical copies (having the exact same disutility profile) of the agent $a \in A$ (the good set remains unchanged): Every α-competitive equilibrium in I' will also be an α-competitive equilibrium in I. Observe that in our instance $E(I)$, we can scale the earning functions of all the agents by some large scalar $\gamma(\varepsilon, \varepsilon')$ to make the earnings of the agents integral. Again, since $e(a) \in \mathcal{O}(1)$ and $\frac{1}{\varepsilon}, \frac{1}{\varepsilon'} \in \mathcal{O}(1)$, we have $|A'| = \mathcal{O}(|A|)$ and all the input parameters of A' (all entries in the disutility matrix $d'(\cdot,\cdot)$) can be expressed with polynomial bit size in unary notation.* Therefore, finding an $(\frac{11}{12} + \delta)$-competitive equilibrium, for any $\delta > 0$, in chore division in the CEEI model is also strongly NP-hard.

7.2 Sufficiency Conditions for the Existence of a Competitive Equilibrium

In this section, we show the existence of a competitive equilibrium under the conditions mentioned in Section 7.0.2. Recall the sufficiency conditions.

Condition 1: The economy graph G of the instance is strongly connected, and

Condition 2: D is a disjoint union of bicliques $D_1, D_2, \ldots, D_d$ for some $d \geq 1$.

Let $\mathcal{I}$ denote all the instances of chore division that satisfy Condition 1 and Condition 2. We now show that all instances in $\mathcal{I}$ admit a competitive equilibrium. The proof of existence is very involved and thus we first give a brief overview of the same.

7.2.1 Overview of the Proof

Most equilibrium existence results [80, 8] are based on either Brouwer's or Kakutani fixed-point theorems. The Brouwer's (Kakutani's) fixed-point theorem says that given a function (correspondence) ϕ from D to itself, there exists an $x \in D$ such that $f(x) = x$ ($x \in f(x)$), if f is continuous (has closed graph) and D is convex and compact [25, 68]. Our proof invokes both Brouwer's and Kakutani's fixed-point theorems, the former nested inside the later. This approach may be of independent interest to prove existence in other settings.

We first briefly discuss why existence proofs for determining a competitive equilibrium with goods do not easily extend to chores, and this will eventually lead us to the new approach. Most existence proofs for determining a competitive equilibrium with goods define a fixed-point formulation on the domain of prices that forms a simplex [8, 75], i.e., if there are m goods, then the domain is the simplex $\Delta_m = \{p \in \mathbb{R}^m_{\geq 0} \mid \sum_{j=1}^m p_j = 1\}$. Given the prices, it computes optimal bundles of agents and adjusts prices based on *excess demand*. At a fixed-point, no change in prices will imply no excess demand, leading to a competitive equilibrium.

This approach immediately fails for the chore division problem due to the issue of *infeasible optimal bundle*: Given a price vector from the simplex domain, if agent a_i's chore endowment has positive total monetary cost ($\sum_{j\in[m]} w(a_i, b_j) \cdot p(b_j) > 0$) while the chores she is able to do have zero prices (all $p(b_j) = 0$ for all b_j s.t. $d(a_i, b_j)$ is finite), then there is no way she can earn enough money to pay for her chores, in turn making the set $F_i(p)$ in (7.1) empty. The reason why this issue does not arise in case of goods is that, there, the agents are allowed to spend *at most* the total price of their endowments (for bads it is *at least*), thereby reversing the inequality in the definition of the set $F_i(p)$, which ensures that the all zero vector in $\mathbb{R}^m_{\geq 0}$ is always a feasible vector.

To resolve the above issue, first we need to work with more involved price domain that ensures that total monetary cost of the chores and endowments is the same inside every component of the disutility graph. Recall the bipartite disutility graph $D = (A \cup B, E_D)$ where there is an edge $(a, b) \in E_D$ if and only if $d(a, b) \neq \infty$. Let $D_1 = (A_1 \cup B_1, E_{D_1}), D_2 = (A_2 \cup B_2, E_{D_2}), \ldots, D_d = (A_d \cup B_d, E_{D_d})$ be the connected components of D. Then, our new price domain is,

$$\mathbf{P} = \left\{ p \in \mathbb{R}^m_{\geq 0} \mid \sum_{j\in[m]} p(b_j) = 1 \text{ and} \sum_{b\in B_k} p(b) = \sum_{a\in A_k} \sum_{j\in[m]} w(a, b_j)p(b_j) \quad \forall k \in [d] \right\} \tag{7.6}$$

Now observe that if for any agent $a \in A_k$, for some $k \in [d]$, the chores she is interested in (the set B_k), have zero prices, then the total price of her endowment is also zero as $p \in \mathbf{P}$. In this case, agent a need not earn anything. As a result, she does not need to do any chore and the all zero vector in $\mathbb{R}^m_{\geq 0}$ is a feasible optimal chore set for agent a. Therefore, for any price vector $p \in \mathbf{P}$, for any agent i, we have that the set $F_i(p)$ is not empty and neither is the optimal bundle set in (7.1). However, there is still an issue with zero prices, a different one: It can be the case that for some component $(A_k \cup B_k, E_{D_k})$, the prices of all the chores in B_k are zero, and prices of all the chores that agents in A_k bring are also zero. In that case, the optimal bundle of any agent $a \in A_k$ consists of only the all zero vector because none of them have to earn anything! However, this will make the optimal bundle set change *non-continuously* with respect to prices, which is a major roadblock in proving continuity like property (the closed graph property) for the fixed-point formulation: for instance consider a simple scenario where there is a component D_k in the disutility graph comprising of just one agent a and one chore b. Agent a has some positive endowment of only one chore $b' \neq b$, say $w(a, b') = 1$ and $w(a, j) = 0$ for all other $j \in B$. Now, consider a sequence of price-vectors $(p_n)_{n \in \mathbb{N}}$ in $\mathbf{P}$, such that $p_n(b') = p_n(b) = 1/n$. Observe that for every $n \in \mathbb{N}$, the optimal bundle of agent a is $X_{ab} = 1$ and $X_{at} = 0$ for all other $t \in B$, as the only chore a is interested in is b, and she has to do one unit of b, to earn her money of $w(a, b') \cdot p(b') = 1 \cdot (1/n) = 1/n$. However, at the limit of the sequence $(p_n)_{n \in \mathbb{N}}$, say p_*, we have $p_*(b) = p_*(b') = 0$ and the only unique optimal bundle for agent a is the all zero vector in $\mathbb{R}^m_{\geq 0}$. Thus, the optimal bundle may not change continuously with the price-vectors in $\mathbf{P}$.

To fix the above issue, we define *extended optimal bundle* set, which is same as the optimal bundle set of an agent $a_i \in A_k$, if the total price of the chores in B_k is strictly positive, otherwise it is the set of all feasible allocations of chores in B_k. This will help us ensure *continuity* of the final correspondence. However, we will have to make sure that at the fixed-point, the *extended optimal bundle* is the *optimal bundle* for every agent (one way to do this is to ensure that there are no zero prices at the fixed point). For the allocations, we will work with the following domain: for some sufficiently large constant C, we define

$$\mathbf{X} = \{X \in \mathbb{R}^{mn}_{\geq 0} \mid 0 \leq X_{ij} \leq C, \ \forall a_i \in A, \forall b_j \in B\} \tag{7.7}$$

Then the set of extended optimal bundles of an agent $a_i \in A_k$ is:

$$EOB_i(p) = \begin{cases} \{X_i \in \mathbf{X} \mid X_{ij} > 0 \text{ only if } d(a_i, b_j) \neq \infty\} & \text{if } \sum_{b \in B_k} p(b) = 0, \\ OB_i(p) & \text{otherwise.} \end{cases} \tag{7.8}$$

Fixed-point formulation. The domain of our fixed point formulation is $\mathbf{S} = \mathbf{P} \times \mathbf{X}$. Next, we define a correspondence $\phi : \mathbf{S} \to 2^{\mathbf{S}}$ that is the product of two correspondences $\phi_1 : \mathbf{S} \to 2^{\mathbf{P}}$ and $\phi_2 : \mathbf{S} \to 2^{\mathbf{X}}$. For a given $(p, X) \in \mathbf{S}$, $\phi(p, X) = \phi_1(p, X) \times \phi_2(p, X)$. Out of these, $\phi_2(p, X)$ is the set of extended optimal bundles at prices p. Formally,

$$\phi_2(p, X) = \{X \in \mathbf{X} \mid X_i \in EOB_i(p), \ \forall a_i \in A\}$$

The exact formulation of ϕ_1 is involved and requires to invoke Brouwer's fixed-point theorem. Therefore, let us first state the properties of ϕ_1 that we need to ensure, and discuss how they help us map fixed-points of ϕ to the competitive equilibria of the chore division instance. For a given $(p, X) \in \mathbf{S}$, if $p' \in \phi_1(p, X)$, then it must be that

- $p' \in P$ and for all components $D_k = (A_k \cup B_k, E_{D_k})$ of the disutility graph, and chores b_j and $b_{j'}$ in B_k, where $p(b_{j'}) > 0$, we have

$$\frac{p'(b_j)}{p'(b_{j'})} = \frac{p(b_j) + max(\sum_{i \in [n]} w(a_i, b_j) - \sum_{i \in [n]} X_{ij}, 0)}{p(b_{j'}) + max(\sum_{i \in [n]} w(a_i, b_{j'}) - \sum_{i \in [n]} X_{ij'}, 0)}. \tag{7.9}$$

Fixed-points to a competitive equilibrium. Let (p, X) be a fixed-point of ϕ, i.e., $(p, X) \in \phi(p, X)$. We first show that at any fixed-point, the prices of all the chores are strictly positive. To the contrary, suppose $p(b_j) = 0$ for some $b_j \in B$, and let b_j belong to component $D_k = (A_k \cup B_k, E_{D_k})$ of the disutility graph D. We claim that some component of D has chores with both zero and positive prices: Either it is D_k itself, or if all the chores in D_k have zero prices, then using the fact that $p \in \mathbf{P}$, we have $\sum_{a_i \in A_k} \sum_{j \in [m]} w(a_i, b_j) \cdot p(b_j) = \sum_{b_j \in B_k} p(b_j) = 0$. This implies that the prices of all the chores owned by agents in D_k are zero, and some of them must belong to other components due to the *strong connectivity* of the economy graph (Condition 1). Recursing this argument, and also using the fact that sum of all the prices is 1, there must be a component with a zero priced chore, but the sum of prices of the chores in the component is positive, say component $D_\ell = (A_\ell \cup B_\ell, E_{D_\ell})$.

Let b^0 and b^+ be the chores in D_ℓ with $p(b^0) = 0$ and $p(b^+) > 0$. For every agent in $a_i \in A_\ell$, their $EOB_i(p) = OB_i(p)$, since total price of the chores in B_ℓ is positive (by (7.8)). Since every $a_i \in D_\ell$ has finite disutility for both b^0 and b^+ (due to sufficiency Condition 2), her disutility-per-buck for b^0 is strictly more than that for b^1. Due to (7.1), if $X_i \in OB_i(p)$ then $X_{ib^0} = 0$ for all $i \in A_\ell$. Since, every agent $a \notin A_\ell$ have infinite disutility for b^0, we have that $X_{ib^0} = 0$ for all $i \in [n]$. Since our correspondence ϕ satisfies (7.9), and $p(b^0) = 0$ and $p(b^+) > 0$, we have,

$$
\begin{aligned}
0 = \frac{p(b^0)}{p(b^+)} &= \frac{p(b^0) + max(\sum_{i\in[n]} w(a_i, b^0) - \sum_{i\in[n]} X_{ib^0}, 0)}{p(b^+) + max(\sum_{i\in[n]} w(a_i, b^+) - \sum_{i\in[n]} X_{ib^+}, 0)} \\
&= \frac{0 + \sum_{i\in[n]} w(a_i, b^0)}{p(b^+) + max(\sum_{i\in[n]} w(a_i, b^+) - \sum_{i\in[n]} X_{ib^+}, 0)} \\
&> 0, \text{ a contradiction.}
\end{aligned}
$$

Therefore, at a fixed point, there is no chore with a zero price. Now, we briefly describe why fixed-point (p, X) correspond to the prices and allocation at a competitive equilibrium. Let $r_j(X)$ denote the amount of the chore b_j left undone under X, i.e., $r_j(X) = max(\sum_{i\in[n]} w(a_i, b_j) - \sum_{i\in[n]} X_{ij}, 0)$. Since all chores have positive price at p, extended optimal bundle set of every agent is her optimal bundle set (by (7.8)) and thereby $X \in \phi_2(p, X)$ ensures that $X_i \in OB_i(p)$ for every agent $a_i \in A$. Now we only need to ensure demand meets supply for every chore. If not, then some chore b_j in component D_k, which is not completed, i.e., $r_j(X) > 0$. Since $p \in \mathbf{P}$, we have that the cumulative price of the endowments of the agents in a component of the disutility graph equals the total price of the chores in the same component. Since every agent spends on their optimal bundle, the cumulative price of the endowments of the agents equals the total earning of that agents in A_k from B_k. Therefore, if one chore b_j is underdone, i.e., $r_j(X) > 0$, there there exists some other chore $b_{j'}$ is overdone, i.e., $r_{j'}(X) = 0$. Again using (7.9), we have $\frac{p(b_j)}{p(b_{j'})} = \frac{p(b_j)+r_j(X)}{p(b_{j'})+r_{j'}(X)} > \frac{p(b_j)}{p(b_{j'})+r_{j'}(X)} = \frac{p(b_j)}{p(b_{j'})}$, a contradiction.

Our next task is to define the correspondence ϕ_1, so that for any given $(p, X) \in \mathbf{S}$, (7.9) holds for every $p' \in \phi_1(p, X)$, and $p' \in \mathbf{P}$. This in fact is the trickiest part of our proof and constitutes the main bulk of our efforts.

To get $p' \in \mathbf{P}$, we need to make ensure that the $p' \in \Delta_m$, and for every component D_k of the disutility graph D, total prices of the chores in D_k equals total cost of endowments of agents in D_k. To this end, for every chore b_j in component D_k, let $q(b_j) = p(b_j)+r_j(X)$, where $r_j(X)$ is the amount of chore b_j left undone as defined above, and $\beta_j = \frac{q(b_j)}{\sum_{b\in D_k} q(b)}$. Note that for (7.9), we want that for any $b_j, b_{j'} \in D_k$ with $p(b_{j'}) > 0$, $\frac{p'(b_j)}{p'(b_{j'})} = \frac{q(b_j)}{q(b_{j'})} = \frac{\beta_j}{\beta_{j'}}$. Thus, if $\tilde{p}_k = \sum_{b\in D_k} p(b)$ then $p'(b_j)$ must be $\beta_j\tilde{p}_k$. This reduces to one unknown per component of D, namely $\tilde{p}_k$ for each $k \in [d]$.

Next, we write a system of linear equations to compute $\tilde{p}_k$'s such that all the constraints of domain $\mathbf{P}$ are satisfied. The simplex constraints for the prices in $\mathbf{P}$ can be encoded by ensuring $\tilde{p} \in \Delta_d$. Next, for each component D_k, the following constraint imposes total endowment costs of agents in D_k equals total prices of chores in D_k.

$$\sum_{a_i\in A_k}\sum_{k'\in[d]}\sum_{b_{j'}\in B_{k'}} w(a_i, b_{j'}) \cdot (\beta_{j'}\tilde{p}_{k'}) = \sum_{b_j\in B_k} (\beta_j\tilde{p}_k)$$

Let $M(\beta) \in \mathbb{R}^{d\times d}$ denote the matrix of this linear system. Then, our goal becomes to find a vector $v \in \Delta_d$, in the null space of $M(\beta)$. It is not obvious why such a vector should exist. Our high-level approach to show the same is as follows: We can equivalently express the linear system of equations $M(\beta)\cdot v = 0$ as $M'(\beta)\cdot v = v$, where $M'(\beta) = M(\beta) + I$, where I is the identity matrix. *We show that if we define a function $f : \mathbb{R}^d \to \mathbb{R}^d$ as $f(v) = M'(\beta)\cdot v$, then f maps the d-dimensional simplex Δ_d to itself (this is non-trivial).* Restricting f to only the simplex, we get a continuous map $f : \Delta_d \to \Delta_d$ and therefore it has a fixed-point by the Brouwer's fixed-point theorem. At every fixed-point v we have $M'(\beta)\cdot v = v$ implying $M(\beta)\cdot v = 0$. Since $v \in \Delta_d$ we get the vector we needed.

The above scheme will work if the β_js are well defined. However, for a component D_k if $\sum_{b\in D_k} q(b)$ turns out to be zero, then β_js are ill-defined and causes issues with proving continuity like properties of ϕ. To handle this, we define a set of permissible βs, namely,

$$\mathcal{B} = \left\{ \beta \in \mathbb{R}^m_{\geq 0} \mid \forall k \in [d], \begin{array}{ll} \sum_{b_j\in D_k} \beta_j = 1 & \text{if } \sum_{b\in D_k} t(b) = 0 \\ \forall b_j \in D_k,\ \beta_j = \frac{q(b_j)}{\sum_{b\in D_k} q(b)} & \text{otherwise} \end{array} \right\}.$$

And for each $\beta \in \mathcal{B}$, the above process will compute a $p' \in \phi_1(p, X)$. By construction, each of these p''s will satisfy, $p' \in \mathbf{P}$ and equation (7.9), as needed. However, it is not immediate why such a set of p's will form a convex set, as required to apply the Kakutani's fixed point theorem.

In fact, to apply the Kakutani's fixed-point theorem, we need to show that the above complex process creates a ϕ, that has closed graph (continuity-like property), and $\phi(p, X)$ is convex for each $(p, X) \in \mathbf{S}$. This again requires involved argument and is formally proved in Lemmas 7.35 and 7.36 of Subsection 7.2.2. Then, ϕ is sure to have a fixed-point which maps to competitive equilibrium as discussed above.

Our proof technique extends to show existence of a competitive equilibrium for chore division with general convex disutility functions where an *agent can do only a subset of chores* and with arbitrary endowments, under similar sufficiency conditions. Thereby, it extends the existence results of [89, 74] that requires that every agent has finite convex disutility for all the chores. Thus, our overall approach may be of independent interest to handle more general problems involving chores.

We now elaborate the proof in the next Subsection.

7.2.2 Elaborate Proof

Consider any instance $I = \langle G, D\rangle \in \mathcal{I}$ such that G is the economy graph of the instance and $D = \cup_{i\in[d]} D_i$, where each $D_i = (A_i \cup B_i, E_{D_i})$ is a complete bipartite graph, disjoint from $D_{i'}$ ($i' \neq i$). For ease of notation,

- we represent our set A of n agents as $[n]$ (we write a_i as i) and the set B of m chores as $[m]$ (we write chore b_j as j),
- we also write p_j to denote the price of chore b_j (instead of $p(b_j)$) and $w_{i,j}$ to represent the agent a_i's initial endowment of chore b_j (instead of $w(a_i, b_j)$), and
- lastly, we also assume without loss of generality that the total endowment of each chore is one: $\sum_{i\in[n]} w_{i,j} = 1$.

Now, we briefly introduce some basic definitions and concepts required to prove the existence of a competitive equilibrium.

Normalized Prices and Bounded Allocations. A price vector $p = \langle p_1, p_2, \dots, p_m\rangle$ is called a *normalized price vector* if

- $p_j \geq 0$ for all $j \in [m]$,
- $\sum_{j\in[m]} p_j = 1$, and
- $\sum_{i\in A_k} \sum_{j\in[m]} w_{i,j} \cdot p_j = \sum_{j\in B_k} p_j$ for each component D_k in the disutility graph, i.e., sum of prices of chores in D_k equals the sum of total money of the agents in D_k.

Let P be the set of all normalized price vectors. We first show that the set P is non-empty.

Observation 7.22. *We have $P \neq \emptyset$.*

Proof. Here we will make use of a general fact that will be useful for a proof later as well.

Fact 7.23. *Let $Z \in \mathbb{R}^{n\times n}$ be a square matrix such that $Z_{ij} \geq 0$ for all $j \neq i$ (all the non-diagonal entries of Z are non-negative) and $\sum_{i\in[n]} Z_{ij} = 0$ for all $j \in [n]$ (column sums are zero), then there exists a vector $t \in \mathbb{R}^n_{\geq 0}$ such that $\sum_{i\in[n]} t_i = 1$ and $Z \cdot t = 0$.*

The proof of this fact can be found at the end of this section. Using this fact, we will outline a proof that P is non-empty. For each component D_k of the disutility matrix, we pick a chore $b_k \in B_k$ and we set $p_j = 0$ for all $j \in B_k \setminus \{b_k\}$. Note that to show that P is non-empty, it suffices to show that there exists a vector $p' = \langle p'_1, p'_2, \dots, p'_d\rangle$ (intuitively each p'_k corresponds to the price of chore $b_k \in B_k$, i.e., p_{b_k}) such that $p'_k \geq 0$ for all $k \in [d]$, $\sum_{k\in[d]} p'_k = 1$ and we have,

$$\sum_{i\in A_k} \sum_{k'\in[d]} w_{i,b_{k'}} \cdot p'_{k'} - p'_k = 0 \qquad \text{for all } k \in [d] \tag{7.10}$$

Let W be the coefficient matrix of the system of equations in (7.10), i.e., $W \cdot p' = 0$ represents the system of equations in (7.10). Observe that $W_{kk'} = \sum_{i\in A_k} w_{i,b_{k'}}$ if $k \neq k'$

and $W_{kk} = \sum_{i \in A_k} w_{i,b_k} - 1$. Therefore the non-diagonal entries of W are non-negative and also note that the column sum is zero:

$$\begin{aligned}
\sum_{k \in [d]} W_{kk'} &= \sum_{k \in [d]} \sum_{i \in A_k} w_{i,b_{k'}} - 1 \\
&= \sum_{i \in [n]} w_{i,b_{k'}} - 1 \\
&= 1 - 1 && \text{(total endowment of chore } b_{k'} \text{ is one)} \\
&= 0.
\end{aligned}$$

Therefore W satisfies all the conditions in Fact 7.23. Therefore, by Fact 7.23 there exists a $p' \in \mathbb{R}^d_{\geq 0}$, such that $\sum_{k \in [d]} p'_d = 1$ and $W \cdot p' = 0$. Therefore, P is non-empty. □

Since P is defined by a set of linear equalities and inequalities, P is closed and convex too. Additionally, since $p \in \mathbb{R}^m_{\geq 0}$ and $\sum_{j \in [m]} p_j = 1$ for all $p \in P$, P is compact.

An allocation $X \in \mathbb{R}^{n \times m}_{\geq 0}$, is called a *bounded allocation* if each X_{ij} (quantifies the amount of chore j allocated to agent i) is non-negative and is at most $m \cdot \frac{d_{max}}{d_{min}}$, where d_{max} and d_{min} refer to the largest and smallest finite entry in the disutility matrix. Let $\mathbf{X}$ be the set of all bounded allocations. Observe that the set $\mathbf{X}$ is non-empty, convex and compact. Also, we have that P is non-empty, convex and compact. We define a compact, convex and non-empty subset of $\mathbb{R}^{(m+nm)}$, $S = \bigcup_{p \in P} \bigcup_{X \in \mathbf{X}} \langle p, X \rangle$ [14].

Correspondence ϕ. Our goal is to define a *correspondence* or equivalently a *set valued function* $\phi : S \to 2^S$, such that ϕ has at least one fixed point and any fixed point of ϕ will correspond to competitive equilibrium. We will first show some properties that if satisfied by ϕ, then ϕ will have at least one fixed point and any fixed point of ϕ will correspond to a competitive equilibrium. Then, we will define a ϕ that satisfies these properties.

Properties. We first make some basic definitions that will help us to state the properties. We call a bounded allocation $Y \in \mathbf{X}$ an *extended optimal allocation at the price vector* p if and only if,

- for all $i \in A_k$, we have $Y_{ij} > 0$ only if $d(i,j) \neq \infty$, and
- for all $i \in A_k$, where $\sum_{j \in B_k} p_j > 0$, we have $Y_{ij} > 0$ only if $\frac{d(i,j)}{p_j} \leq \frac{d(i,\ell)}{p_\ell}$ for all $\ell \in [m]$, and
- for all $i \in A_k$, where $\sum_{j \in B_k} p_j > 0$, we have $\sum_{j \in [m]} Y_{ij} \cdot p_j = \sum_{j \in [m]} w_{i,j} \cdot p_j$.

Let $\mathbf{X}^p \subseteq \mathbf{X}$ denote the set of all extended optimal allocations at the price vector p. Note that in an extended optimal allocation, the only agents that may not get their optimal bundles (defined in Definition 7.1) are the ones that belong to a component where the sum of prices of all the chores in the component are zero, as in an extended

[14] We abuse notation slightly here: $\langle p, X \rangle$ refers to the $(m + nm)$-dimensional vector $\langle p_1, p_2, \ldots, p_m, X_{11}, X_{12}, \ldots, X_{nm} \rangle$.

optimal allocation, an agent that belongs to a component where the sum of prices of all the chores is zero, can be allocated any bundle that does not involve her earning from a chore with infinite disutility (and not necessarily her optimal bundle). However, if $p_j > 0$ for all $j \in [m]$, then every extended optimal allocation is also an optimal allocation (where every agent receives their respective optimal bundles). Right now, it may not be immediate that $\mathbf{X}^p$ is non-empty. However, we show that this is indeed the case, as agents are allowed to consume goods to a significant extent (Y_{ij} is allowed to be as large as $m \cdot \frac{d_{max}}{d_{min}}$).

Lemma 7.24. *For all $p \in P$, we have $\mathbf{X}^p \subseteq \mathbf{X}$ and $\mathbf{X}^p \neq \emptyset$.*

Proof. By definition $\mathbf{X}^p \subseteq \mathbf{X}$. Therefore, it suffices to show that it is non-empty. Consider any $p \in P$. Consider an agent a in the component D_k. Let $\mathbf{w}(a) = \sum_{j \in [m]} w_{a,j} \cdot p_j$. If $\mathbf{w}(a) = 0$, then we set $Y_{aj} = 0$ for all $j \in [m]$ and we trivially have $\sum_{j \in [m]} Y_{aj} \cdot p_j = \sum_{j \in [m]} w_{a,j} \cdot p_j = 0$ and $\langle Y_{a1}, \ldots, Y_{am} \rangle$ is an extended optimal bundle for agent a at p (irrespective of whether $\sum_{j \in B_k} p_j > 0$ or not). So assume that $\mathbf{w}(a) > 0$. Since $p \in P$, we have that the sum of prices of the chores in D_k, $\sum_{j \in B_k} p_j = \sum_{i \in A_k} \sum_{j \in [m]} w_{i,j} \cdot p_j \geq \sum_{j \in [m]} w_{aj} \cdot p_j = \mathbf{w}(a) > 0$. This implies that there is at least one chore b in the component D_k such that $p_b \geq \frac{\mathbf{w}(a)}{m}$. Let b' be a chore such that $d(a, b') \neq \infty$, and $\frac{d(a,b')}{p_{b'}} \leq \frac{d(a,\ell)}{p_\ell}$ for all $\ell \in [m]$. This implies that $\frac{d(a,b')}{p_{b'}} \leq \frac{d(a,b)}{p_b}$. Therefore, we have that

$$
\begin{aligned}
p_{b'} &\geq \frac{d(a,b')}{d(a,b)} \cdot p_b \\
&\geq \frac{d_{min}}{d_{max}} \cdot p_b \\
&\geq \frac{d_{min}}{m d_{max}} \cdot \mathbf{w}(a).
\end{aligned}
$$

We set $Y_{ab'} = \frac{\mathbf{w}(a)}{p_{b'}}$. Observe that $Y_{ab'} \leq m \cdot \frac{d_{max}}{d_{min}}$. Therefore, Y is a bounded allocation, i.e., $Y \in \mathbf{X}$. Also, note that agent a earns her entire money of $\mathbf{w}(a)$ by doing $Y_{ab'} = \frac{\mathbf{w}(a)}{p_{b'}}$ amount of chore b' such that $d(a, b') \neq \infty$, $\frac{d(a,b')}{p_{b'}} \leq \frac{d(a,\ell)}{p_\ell}$ for all $\ell \in [m]$. Thus, Y is an extended optimal bundle also. Therefore, $\mathbf{X}^p \neq \emptyset$. □

We are now ready to define the properties of ϕ. For any point $\langle p, X \rangle \in S$, consider any point $\langle p', X' \rangle \in \phi(\langle p, X \rangle)$. Then,

- Property $\mathbf{P}_1$: $X' \in \mathbf{X}^p$ and $p' \in P$.
- Property $\mathbf{P}_2$: For any two agents i and j that belong to the same component D_k of the disutility graph D (say i, $j \in A_k$), such that $p_j > 0$, we have

$$
\frac{p'_i}{p'_j} = \frac{p_i + max(1 - \sum_{\ell \in [n]} X_{\ell i}, 0)}{p_j + max(1 - \sum_{\ell \in [n]} X_{\ell j}, 0)}.
$$

  .

- Property $\mathbf{P}_3$: $\phi(\langle p, X \rangle)$ is non-empty and convex.

- Property $\mathbf{P}_4$: ϕ has a *closed graph*[15].

We will now show that any correspondence ϕ that satisfies $\mathbf{P}_1, \mathbf{P}_2, \mathbf{P}_3$ and $\mathbf{P}_4$ will have at least one fixed point and any fixed point will correspond to a competitive equilibrium. We first show that ϕ has a fixed point.

Lemma 7.25. *Consider any correspondence ϕ that satisfies properties $\mathbf{P}_1$, $\mathbf{P}_2$, $\mathbf{P}_3$ and $\mathbf{P}_4$. ϕ has a fixed point.*

Proof. By property $\mathbf{P}_1$ we have that if $\langle p', X' \rangle \in \phi(\langle p, X \rangle)$, then $\langle p', X' \rangle \in S$ (as $p' \in P$ and $X' \in \mathbf{X}^p \subseteq \mathbf{X}$). Therefore, $\phi : S \to 2^S$. The set S is non-empty, compact and convex. Furthermore, by properties $\mathbf{P}_3$ and $\mathbf{P}_4$, we have that $\phi(\langle p, X \rangle)$ is non-empty and convex, and ϕ has a closed graph. Therefore, by Kakutani's fixed point theorem, ϕ has a fixed point. $\square$

Now we show that any fixed point of a correspondence ϕ that satisfies properties $\mathbf{P}_1$, $\mathbf{P}_2$, $\mathbf{P}_3$ and $\mathbf{P}_4$ gives a competitive equilibrium.

Lemma 7.26. *Consider any correspondence ϕ that satisfies properties $\mathbf{P}_1$, $\mathbf{P}_2$, $\mathbf{P}_3$ and $\mathbf{P}_4$. Consider any fixed point $\langle p, X \rangle$ of ϕ. Then $\langle p, X \rangle$ is a competitive equilibrium.*

Proof. Consider any fixed point $\langle p, X \rangle \in \phi(\langle p, X \rangle)$. By property $\mathbf{P}_1$, it follows that $X \in \mathbf{X}^p$. As mentioned after that the definition of the extended optimal bundle, if we have $p_j > 0$ for all $j \in [m]$, then each agent gets her optimal bundle in $\mathbf{X}^p$. Therefore, to show that p and X correspond to a competitive equilibrium, it suffices to show that $p_j > 0$ for all $j \in [m]$ and $\sum_{i \in [n]} X_{ij} = 1$ for all chores $j \in [m]$. We first show that $p_j > 0$ for all $j \in [m]$. We prove this by contradiction. Let us assume that there are some chores with zero prices. But first, we make an observation that if there are some chores with zero prices, one of the chores will belong to a component, where the sum of prices of all the chores in that component is non-zero.

Claim 7.27. *Let p be any price vector in P. If there exists some chore j such that $p_j = 0$, then there exists a chore b in the component D_ℓ of the disutility graph such that $p_b = 0$ and $\sum_{j \in B_\ell} p_j > 0$.*

Proof. We prove this claim by contradiction. Assume otherwise: All chores with zero prices only occur in components where the sum of prices of the chores in the component is zero. Let $D_{\ell_1}, D_{\ell_2}, \ldots, D_{\ell_r}$ be the components of the disutility graph where the sum of prices of all the chores in the component are zero, and there are no chores with zero prices in the components $\bigcup_{k \in [d] \setminus \{\ell_1, \ldots, \ell_r\}} D_k$. Since the economy graph G is strongly connected (by Condition 1), there is an edge from some agent in $\bigcup_{k \in [r]} A_{\ell_k}$ to some agent in $\bigcup_{k \in [d] \setminus \{\ell_1, \ldots, \ell_r\}} A_{\ell_k}$, say from an agent $b' \in A_{\ell_{r'}}$ for some $\ell_{r'} \in \{\ell_1, \ell_2, \ldots, \ell_r\}$, to an agent $\tilde{b} \in A_{\ell_{\tilde{r}}}$ for $\ell_{\tilde{r}} \in [d] \setminus \{\ell_1, \ldots, \ell_r\}$. Since the agent $\tilde{b}$ has finite disutility only for the chores in $B_{\ell_{\tilde{r}}}$, we can conclude that there exists a chore $\tilde{c} \in B_{\ell_{\tilde{r}}}$, such that $w_{b', \tilde{c}} > 0$. Since $\tilde{c} \in B_{\ell_{\tilde{r}}}$, and there are no chores with zero prices in $D_{\ell_{\tilde{r}}}$ (by assumption), and therefore we also have $p_{\tilde{c}} > 0$. Then, we have $\sum_{j \in [m]} w_{b', j} \cdot p_j \geq w_{b', \tilde{c}} \cdot p_{\tilde{c}} > 0$, implying

[15] A correspondence $\phi : X \to 2^Y$ has a *closed graph* if for all sequences $(x_n)_{n \in \mathbb{N}}$ and $(y_n)_{n \in \mathbb{N}}$, with $(x_n)_{n \in \mathbb{N}}$ converging to x and $(y_n)_{n \in \mathbb{N}}$ converging to y, such that $x_n \in X$ and $y_n \in \phi(x_n)$ for all n, we have $y \in \phi(x)$.

that $\sum_{i\in A_{\ell_{r'}}}\sum_{j\in[m]} w_{i,j}\cdot p_j > 0$. However, since $p\in P$, we have that for component $D_{\ell_{r'}}$ of the disutility graph, the sum of prices of the chores in the component equals the sum of prices of the chores owned by the agents in the same component, implying $\sum_{j\in B_{\ell_{r'}}} p_j = \sum_{i\in A_{\ell_{r'}}}\sum_{j\in[m]} w_{i,j}\cdot p_j > 0$, which is a contradiction. □

Thus, let b be a chore in the component D_k of the disutility graph such that $p_b = 0$ and $\sum_{j\in B_k} p_j > 0$. Then, there is at least one chore $b'\in B_k$ such that $p_{b'} > 0$. Since D_k is biclique (by Condition 2), we have that $d(i,b')\neq\infty$ for all $i\in A_k$. This implies that for all agents $i\in A_k$, we have $\frac{d(i,b')}{p_{b'}} < \frac{d(i,b)}{p_b}$. Since $X\in\mathbf{X}^p$, we have that $X_{ib}=0$, for all $i\in A_k$ and also for all $i\in[n]$ (as $X\in\mathbf{X}^p$ and $X_{ib}>0$ only if $d(i,b)\neq\infty$ and for all agents in $[n]\setminus A_k$ we have $d(i,b)=\infty$), implying $\sum_{\ell\in[n]} X_{\ell b}=0$. Since b and b' both belong to the same component D_k, and $p_{b'}>0$, by Property $\mathbf{P}_2$, we have,

$$\begin{aligned}\frac{p_b}{p_{b'}} &= \frac{p_b + max(1-\sum_{\ell\in[n]} X_{\ell b},0)}{p_{b'} + max(1-\sum_{\ell\in[n]} X_{\ell b'},0)}\\ &= \frac{0+1}{p_{b'} + max(1-\sum_{\ell\in[n]} X_{\ell b'},0)}\\ &\neq 0\\ &= \frac{p_b}{p_{b'}},\end{aligned}$$

which is a contradiction. Thus, none of the chores can have zero prices and therefore, we have $p_j>0$ for all $j\in[m]$.

We now show that $\sum_{i\in[n]} X_{ij}=1$ for all $j\in[m]$. We prove this also by contradiction. So assume otherwise and for some chore $b\in B_k$ we have $\sum_{i\in[n]} X_{ib}>1$ (or $\sum_{i\in[n]} X_{ib}<1$). Note that, since $p\in P$, for the component D_k of the disutility graph, we have,

$$\sum_{j\in B_k} p_j = \sum_{i\in A_k}\sum_{j\in[m]} w_{i,j}\cdot p_j. \tag{7.11}$$

Also, since $X\in\mathbf{X}^p$ and every component of the disutility graph has non-zero total price of the chores in it, for every agent $i\in A_k$, we have $\sum_{j\in[m]} w_{i,j}\cdot p_j = \sum_{j\in[m]} X_{ij}\cdot p_j = \sum_{j\in B_k} X_{ij}\cdot p_j$. Substituting $\sum_{j\in[m]} w_{i,j}\cdot p_j$ as $\sum_{j\in B_k} X_{ij}\cdot p_j$ in (7.11) we have,

$$\begin{aligned}\sum_{j\in B_k} p_j &= \sum_{i\in A_k}\sum_{j\in B_k} X_{ij}\cdot p_j\\ &= \sum_{i\in[n]}\sum_{j\in B_k} X_{ij}\cdot p_j\\ &= \sum_{j\in B_k} p_j\cdot(\sum_{i\in[n]} X_{ij})\ .\end{aligned}$$

Therefore, if $\sum_{i\in[n]} X_{ib}>1$ (or $\sum_{i\in[n]} X_{ib}<1$) for some $b\in B_k$, then there exists a $b'\in B_k$ such that $\sum_{i\in[n]} X_{ib'}<1$ (or $\sum_{i\in[n]} X_{ib'}>1$). This would imply that $\frac{p_b+max(1-\sum_{\ell\in[n]} X_{\ell b},0)}{p_{b'}+max(1-\sum_{\ell\in[n]} X_{\ell b'},0)} < \frac{p_b}{p_{b'}}$ when $\sum_{i\in[n]} X_{ib}>1$ and $\frac{p_b+max(1-\sum_{\ell\in[n]} X_{\ell b},0)}{p_{b'}+max(1-\sum_{\ell\in[n]} X_{\ell b'},0)} > \frac{p_b}{p_{b'}}$ when $\sum_{i\in[n]} X_{ib}<1$, which is a contradiction (as $\frac{p_b+max(1-\sum_{\ell\in[n]} X_{\ell b},0)}{p_{b'}+max(1-\sum_{\ell\in[n]} X_{\ell b'},0)} = \frac{p_b}{p_{b'}}$ if $\langle p,X\rangle$ is a fixed point by property $\mathbf{P}_2$). □

Now, it suffices to show that there exists a correspondence ϕ that satisfies all the four properties to show the existence of competitive equilibrium for every instance $I \in \mathcal{I}$. To this end, we first define a correspondence ϕ and show that it satisfies all the four properties.

Finding a Correspondence ϕ that Satisfies all the Properties. Given a $p \in P$ and $X \in \mathbf{X}$, we define the vector $q(p, X) = \langle q_1(p, X), q_2(p, X), \ldots, q_m(p, X)\rangle$ such that

$$q_j(p, X) = p_j + max(1 - \sum_{i \in [n]} X_{ij}, 0) \tag{7.12}$$

We now introduce a variable $\beta_j(p, X)$ for each chore $j \in [m]$. Let $\beta(p, X) = \langle \beta_1(p, X), \beta_2(p, X), \ldots, \beta_m(p, X)\rangle$. We now outline some constraints that $\beta(p, X)$ must satisfy. We have,

$$\beta_j(p, X) \geq 0 \qquad \forall\ j \in [m], \tag{7.13}$$

$$\sum_{j \in B_k} \beta_j(p, X) = 1, \qquad \forall\ k \in [d], \tag{7.14}$$

$$\beta_j(p, X) = \frac{q_j(p, X)}{\sum_{j' \in B_k} q_{j'}(p, X)} \qquad \forall\ j, k, \text{ s.t. } j \in B_k, \text{ and } \sum_{j' \in B_k} q_{j'}(p, X) > 0. \tag{7.15}$$

Let $\mathcal{B}(p, X)$ be set of all $\beta(p, X)$ that satisfy the system of linear equalities and inequalities in (7.13), (7.14) and (7.15). We now show that $\mathcal{B}(p, X)$ is non-empty, convex and compact.

For each $\beta(p, X) \in \mathcal{B}(p, X)$, we introduce a system of linear equations with a variable $\tilde{p}_k$ for each component D_k of the disutility graph D. Let $\tilde{p} = \langle \tilde{p}_1, \tilde{p}_2, \ldots, \tilde{p}_d\rangle$ (recall that d is the number of components of the disutility graph). We now outline a system of linear equations that needs to be satisfied by a vector $\tilde{p}$. As of now, let us think of each $\tilde{p}_k$ as the sum of prices of the chores in the component D_k and $\beta_j(p, X) \cdot \tilde{p}_k$ as the price of each chore $j \in B_k$. With these price meanings in mind, for each component D_k of D, we write the equation (variables being $\bigcup_{k \in [d]} \tilde{p}_k$) that represents the price of the cumulative endowments of the agents of the component equals the total prices of the chores in the same component.

$$\sum_{i \in A_k} \sum_{k' \in [d]} \sum_{j \in B_{k'}} w_{i,j} \cdot \beta_j(p, X) \cdot \tilde{p}_k - \tilde{p}_k = 0. \tag{7.16}$$

We represent the system of equations in (7.16) as

$$M(\beta(p, X)) \cdot \tilde{p} = \mathbf{0}\ . \tag{7.17}$$

We now make some observation about the non-negativity of the non-diagonal entries and the zero column sums of the matrix $M(\beta(p, X))$.

Observation 7.28. *We have $M(\beta(p, X))_{kk'} \geq 0$ as long as $k \neq k'$ (every non-diagonal entry of $M(\beta, X)$ is non-zero) and $\sum_{k \in [d]} M(\beta(p, X))_{kk'} = 0$ for all $k' \in [d]$ (column sums are zero).*

Proof. We first carefully look at any column $M(\beta(p,X))_{*k'}$ of $M(\beta(p,X))$. Note that for all $k \neq k'$, we have, $M(\beta(p,X))_{kk'} = \sum_{i \in A_k} \sum_{j \in B_{k'}} w_{i,j} \cdot \beta_j(p,X)$. We have $M_{kk} = \sum_{i \in A_k} \sum_{j \in B_k} w_{i,j} \cdot \beta_j(p,X) - 1$. Therefore, every non-diagonal entry in $M(\beta(p,X))$ is non-negative. Now we just need to show that $\mathbf{1}^T \cdot M(\beta(p,X))_{*k'} = 0$. Observe,

$$\begin{aligned}
\mathbf{1}^T \cdot M(\beta(p,X))_{*k'} &= \sum_{k \in [d]} \sum_{i \in A_k} \sum_{j \in B_{k'}} w_{i,j} \cdot \beta_j(p,X) - 1 \\
&= \sum_{j \in B_{k'}} \beta_j(p,X) \cdot \sum_{k \in [d]} \sum_{i \in A_k} w_{i,j} - 1 \\
&= \sum_{j \in B_{k'}} \beta_j(p,X) \cdot \sum_{i \in [n]} w_{i,j} - 1 \\
&= \sum_{j \in B_{k'}} \beta_j(p,X) - 1 \\
&= 0.
\end{aligned}$$

This shows that $\mathbf{1}^T \cdot M(\beta(p,X)) = \mathbf{0}^T$. □

We first make some observations about the solution to the system of equations in (7.17) (and consequently (7.16)). Observe that $M(\beta(p,X))$ satisfies all the conditions in Fact 7.23. Therefore, we have the following Observation.

Observation 7.29. *For each $\beta(p,X) \in \mathcal{B}(p,X)$, there exists a vector $\tilde{p} \in \mathbb{R}^d_{\geq 0}$, such that $\sum_{j \in [d]} \tilde{p}_j = 1$ and $M(\beta(p,X)) \cdot \tilde{p} = 0$.*

We are now ready to define the correspondence. Given any $\langle p, X \rangle \in S$, we determine the vector $q(p,X)$ as in (7.12). Let $\mathcal{B}(p,X)$ be the set of all $\beta(p,X)$ that satisfy the set of linear equalities and inequalities in 7.13, 7.14 and 7.15. For each $\beta(p,X) \in \mathcal{B}(p,X)$, let $\tilde{P}(\beta(p,X)) \subseteq R^d_{\geq 0}$ be the set of all vectors that satisfy the conditions in Observation 7.29. We now define the set $\overline{P}(\beta(p,X)) \subseteq \mathbb{R}^m_{\geq 0}$ as,

$$\overline{P}(\beta(p,X)) = \left\{ \overline{p} \in \mathbb{R}^m_{\geq 0} \mid \overline{p}_j = \beta_j(p,X) \cdot \tilde{p}_k \text{ where chore } j \in B_k \text{ and } \tilde{p} \in \tilde{P}(\beta(p,X)) \right\} \tag{7.18}$$

Given any $\langle p, X \rangle \in S$, we define

$$\phi(\langle p, X \rangle) = \left\{ \langle \overline{p}, X' \rangle \mid \overline{p} \in \overline{P}(\beta(p,X)) \text{ and } \beta(p,X) \in \mathcal{B}(p,X) \text{ and } X' \in \mathbf{X}^p \right\} \quad . \tag{7.19}$$

For the rest of this section, we will now show that ϕ satisfies properties $\mathbf{P}_1$, $\mathbf{P}_2$, $\mathbf{P}_3$ and $\mathbf{P}_4$.

ϕ satisfies properties $\mathbf{P}_1$, $\mathbf{P}_2$, $\mathbf{P}_3$ and $\mathbf{P}_4$. Now that we have defined the correspondence, we prove that it satisfies all the necessary properties. To this end consider a point $\langle p', X' \rangle \in \phi(\langle p, X \rangle)$.

Lemma 7.30 (Property $\mathbf{P}_1$)**.** *Let $\langle p', X' \rangle \in \phi(\langle p, X \rangle)$. We have $X' \in \mathbf{X}^p$ and $p' \in P$.*

Proof. We need to show that $p' \in P$ and $X' \in \mathbf{X}^p \subseteq \mathbf{X}$. Note that by the definition of ϕ we have $X' \in \mathbf{X}^p \subseteq \mathbf{X}$. Therefore, we only need to show that $p' \in P$. Given p and X, let $q(p, X)$ be the vector obtained as in (7.12) and let $\mathcal{B}(p, X)$ be the set of all $\beta(p, X) \in \mathbb{R}^m$ that satisfy the set of linear inequalities and equalities in 7.13, 7.14 and 7.15. By the definition of the correspondence ϕ (Equation 7.19), we have that $p' \in \overline{P}(\beta'(p, X))$ for some $\beta'(p, X) \in \mathcal{B}(p, X)$. Equation 7.18 implies that for each chore $j \in B_k$, we have $p'_j = \beta'_j(p, X) \cdot \tilde{p}_k$, where $\tilde{p} \in \tilde{P}(\beta'(p, X))$. Now we make three claims which show that $p' \in P$.

Claim 7.31. *We have $p'_j \geq 0$ for all $j \in [m]$.*

Proof. Let us consider any chore j that belongs to the component D_k of the disutility graph. We have,

$$p'_j = \beta'_j(p, X) \cdot \tilde{p}_k.$$

$\beta'(p, X)$ satisfies the system of linear inequalities in 7.13 and thus $\beta'_j(p, X) \geq 0$. Also, $\tilde{p} \in \tilde{P}(\beta'_j(p, X))$ and by the definition of $\tilde{P}(\beta'(p, X))$ and Observation 7.29 we have that $\tilde{p}_k \geq 0$. Thus $p'_j \geq 0$. □

Claim 7.32. *We have $\sum_{j \in [m]} p'_j = 1$.*

Proof. We have,

$$\begin{aligned}\sum_{j \in [m]} p'_j &= \sum_{k \in [d]} \sum_{j \in B_k} \beta'_j(p, X) \cdot \tilde{p}_k \\ &= \sum_{k \in [d]} \tilde{p}_k \cdot \sum_{j \in B_k} \beta'_j(p, X).\end{aligned}$$

Since $\beta'(p, X)$ satisfies the set of linear equalities in (7.14), we have that $\sum_{j \in B_k} \beta'_j(p, X) = 1$. Therefore, we have $\sum_{j \in [m]} p'_j = \sum_{k \in [d]} \tilde{p}_k$. Since $\tilde{p} \in \tilde{P}(\beta'(p, X))$, by definition of $\tilde{P}(\beta'(p, X))$ and Observation 7.29, we have that $\sum_{k \in [d]} \tilde{p}_k = 1$ and thus $\sum_{j \in m} p'_j = 1$. □

Claim 7.33. *For each component D_k of the disutility graph, we have $\sum_{i \in A_k} \sum_{j \in [m]} w_{i,j} \cdot p'_j = \sum_{j \in B_k} p'_j$.*

Proof. We have,

$$\sum_{i \in A_k} \sum_{j \in [m]} w_{i,j} \cdot p'_j = \sum_{i \in A_k} \sum_{k' \in [d]} \sum_{j \in B_{k'}} w_{i,j} \cdot \beta'_j(p, X) \cdot \tilde{p}_{k'} \ .$$

$\tilde{p} \in \tilde{P}(\beta'(p, X))$, and by definition of $\tilde{P}(\beta'(p, X))$, $\tilde{p}$ satisfies (7.17) and therefore also (7.16). Thus, we have $\sum_{i \in A_k} \sum_{k' \in [d]} \sum_{j \in B_{k'}} w_{i,j} \cdot \beta'_j(p, X) \cdot \tilde{p}_{k'} = \tilde{p}_k$. Therefore, we have

$$\begin{aligned}\sum_{i \in A_k} \sum_{j \in [m]} w_{i,j} \cdot p'_j &= \tilde{p}_k \\ &= \sum_{j \in B_k} \beta'_j(p, X) \cdot \tilde{p}_k \qquad \left(\text{as } \sum_{j \in B_k} \beta'_j(p, X) = 1 \text{ by (7.14)}\right) \\ &= \sum_{j \in B_k} p'_j.\end{aligned}$$

□

This shows that $p' \in P$ and completes the proof. $\square$

Lemma 7.34 (Property $\mathbf{P}_2$). *Let $\langle p', X'\rangle \in \phi(\langle p, X\rangle)$. For any two agents i and j that belong to the same component D_k of the disutility graph D, such that $p_j > 0$, we have $p'_i/p'_j = \big(p_i + max(1 - \sum_{\ell\in[n]} X_{\ell i}, 0)\big)/\big(p_j + max(1 - \sum_{\ell\in[n]} X_{\ell j}, 0)\big)$.*

Proof. Consider any $\langle p', X'\rangle \in \phi(\langle p, X\rangle)$. By the definition of the correspondence ϕ (Equation 7.19), we have that $p' \in \overline{P}(\beta'(p, X))$ for some $\beta'(p, X) \in \mathcal{B}(p, X)$. Equation 7.18 implies that for each chore $j \in B_k$, we have $p'_j = \beta'_j(p, X) \cdot \tilde{p}_k$, where $\tilde{p} \in \tilde{P}(\beta'(p, X))$.

Let i, j be two chores in the component D_k of the disutility graph such that $p_j > 0$. Since $p_j > 0$, we have that $q_j(p, X) = p_j + max(1 - \sum_{\ell\in[n]} X_{\ell j}, 0) > 0$. Therefore, $\sum_{j'\in B_k} q_{j'}(p, X) > 0$. This implies that for all $j \in B_k$, we have $\beta'_j(p, X) = q_j(p, X)/(\sum_{j'\in B_k} q_{j'}(p, X))$. Therefore we have,

$$\begin{aligned}
\frac{p'_i}{p'_j} &= \frac{\beta'_i(p, X)\cdot \tilde{p}_k}{\beta'_j(p, X)\cdot \tilde{p}_k} \\
&= \frac{\beta'_i(p, X)}{\beta'_j(p, X)} \\
&= \frac{q_i(p, X)}{q_j(p, X)} \\
&= \frac{p_i + max(1 - \sum_{\ell\in[n]} X_{\ell i}, 0)}{p_j + max(1 - \sum_{\ell\in[n]} X_{\ell j}, 0)}\,. \qquad \text{(by definition of } q(p, X) \text{ in (7.12))}
\end{aligned}$$ $\square$

Lemma 7.35 (Property $\mathbf{P}_3$). *$\phi(\langle p, X\rangle)$ is non-empty and convex.*

Proof. We first show that $\mathbf{X}^p$ is convex. Consider $Y \in \mathbf{X}^p$ and $Y' \in \mathbf{X}^p$. Let $Y'' = \lambda \cdot Y + (1-\lambda)\cdot Y'$ for some $\lambda \in [0, 1]$. First observe that $0 \le min(Y_{ij}, Y'_{ij}) \le Y''_{ij} \le max(Y_{ij}, Y'_{ij}) \le m \cdot \frac{d_{max}}{d_{min}}$. Therefore, $Y'' \in \mathbf{X}$. Now to show that $Y'' \in \mathbf{X}^p$, we need to show that,

(1) for all $i \in A_k$, we have $Y''_{ij} > 0$ only if $d(i, j) \neq \infty$, and

(2) for all $i \in A_k$, where $\sum_{j\in B_k} p_j > 0$, we have $Y''_{ij} > 0$ only if $\frac{d(i,j)}{p_j} \le \frac{d(i,\ell)}{p_\ell}$ for all $\ell \in [m]$, and

(3) for all $i \in A_k$, where $\sum_{j\in B_k} p_j > 0$, we have $\sum_{j\in[m]} Y''_{ij}\cdot p_j = \sum_{j\in[m]} w_{i,j}\cdot p_j$ for all $i \in [n]$.

To this end, note that for all $i \in A_k$, both Y_{ij} and Y'_{ij} are positive, only if $d(i, j) \neq \infty$. Therefore, $Y''_{ij} > 0$ only if $d(i, j) \neq \infty$. Similarly, for all $i \in A_k$, where $\sum_{j\in B_k} p_j > 0$, both Y_{ij} and Y'_{ij} are positive, only if $\frac{d(i,j)}{p_j} \le \frac{d(i,\ell)}{p_\ell}$ for all $\ell \in [m]$. Therefore $Y''_{ij} > 0$ only if $\frac{d(i,j)}{p_j} \le \frac{d(i,\ell)}{p_\ell}$ for all $\ell \in [m]$.

Lastly, for all $i \in A_k$, where $\sum_{j\in B_k} p_j > 0$, we have,

$$\begin{aligned}\sum_{j\in[m]} Y''_{ij} \cdot p_j &= \sum_{j\in[m]} (\lambda \cdot Y_{ij} + (1-\lambda)\cdot Y'_{ij}) \cdot p_j \\ &= \lambda \cdot (\sum_{j\in[m]} Y_{ij} \cdot p_j) + (1-\lambda)\cdot(\sum_{j\in[m]} Y'_{ij} \cdot p_j) \\ &= \lambda \cdot \sum_{j\in[m]} w_{i,j} \cdot p_j + (1-\lambda) \cdot \sum_{j\in[m]} w_{i,j}\cdot p_j \\ &= \sum_{j\in[m]} w_{i,j} \cdot p_j \ .\end{aligned}$$

Thus, $Y'' \in \mathbf{X}^p$. Therefore, $\mathbf{X}^p$ is convex. By Lemma 7.24, we have that $\mathbf{X}^p$ is non-empty as well. Therefore $\mathbf{X}^p$ is convex and non-empty.

Let $P' = \{p \mid p \in \overline{P}(\beta(p,X)) \text{ for } \beta(p,X) \in \mathcal{B}(p,X)\}$. We now show that P' is convex and non-empty. By Observation 7.29, we have that for each $\beta(p,X) \in \mathcal{B}(p,X)$, $\tilde{P}(\beta(p,X)) \neq \emptyset$ and by definition of $\overline{P}(\beta(p,X))$ (Equation 7.18), we have that $\overline{P}(\beta(p,X))$ is also non-empty. Therefore, P' is also non-empty. Now we show that P' is convex as well. To this end, consider two price vectors t and t' in P' or equivalently $t \in \overline{P}(\beta(p,X))$ and $t' \in \overline{P}(\beta'(p,X))$. To show convexity of P', it suffices to show that $\lambda\cdot t + (1-\lambda)\cdot t' \in P'$ for all $\lambda \in [0,1]$ or equivalently $\lambda \cdot t + (1-\lambda)\cdot t' \in \overline{P}(\beta''(p,X))$ for some $\beta''(p,X) \in \mathcal{B}(p,X)$. To this end, we observe that for each chore j in the component D_k of the disutility graph, we have $t_j = \beta_j(p,X)\cdot s_k$, where $s \in \tilde{P}(\beta(p,X))$, and $t'_j = \beta'_j(p,X)\cdot s'_k$, where $s' \in \tilde{P}(\beta'(p,X))$. We now define the vectors $\beta''(p,X)$ and $t'' \in \mathbb{R}^m$ as follows: For each chore j in component D_k of the disutility graph, we define

$$\beta''_j(p,X) = \begin{cases} \frac{\lambda\cdot\beta_j(p,X)\cdot s_k + (1-\lambda)\cdot\beta'_j(p,X)\cdot s'_k}{\lambda s_k + (1-\lambda)\cdot s'_k} & \text{if } s_k \neq 0 \text{ or } s'_k \neq 0 \\ \beta_j(p,X) & \text{otherwise,}\end{cases}$$

and

$$t''_j = \beta''_j(p,X)\cdot s''_k,$$

where $s'' = (\lambda\cdot s + (1-\lambda)\cdot s')$. We first prove that $t'' = \lambda\cdot t + (1-\lambda)\cdot t'$: Consider any $j \in B_k$. If both s_k and s'_k are zero, then $s''_k = \lambda \cdot s_k + (1-\lambda)\cdot s''_k = 0$. Therefore, we have

$$\begin{aligned} t''_j &= \beta''_j(p,X)\cdot s''_k \\ &= 0 \\ &= \beta_j(p,X)\cdot s_k + \beta'_j(p,X)\cdot s'_k && (\text{as } s_k = s'_k = 0) \\ &= \lambda\cdot t_j + (1-\lambda)t'_j \end{aligned}$$

When at least one of s_k or s'_k is non-zero, then $s''_k = \lambda\cdot s_k + (1-\lambda)\cdot s'_k \neq 0$ and we have,

$$\begin{aligned} t''_j &= \beta''(p,X)\cdot s''_k \\ &= \frac{\lambda\cdot\beta_j(p,X)\cdot s_k + (1-\lambda)\cdot\beta'_j(p,X)\cdot s'_k}{\lambda s_k + (1-\lambda)\cdot s'_k}\cdot(\lambda s_k + (1-\lambda)\cdot s'_k) \\ &= \lambda\cdot\beta_j(p,X)\cdot s_k + (1-\lambda)\cdot\beta'_j(p,X)\cdot s'_k \\ &= \lambda\cdot t_j + (1-\lambda)\cdot t'_j. \end{aligned}$$

Now it suffices to show that $\beta''(p,X) \in \mathcal{B}(p,X)$ and $s''_k \in \tilde{P}(\beta''(p,X))$ as this will imply that $\lambda t + (1-\lambda)t' = t'' \in \overline{P}(\beta''(p,X))$ for some $\beta''(p,X) \in \mathcal{B}(p,X)$. We first show that $\beta''(p,X) \in \mathcal{B}(p,X)$. Since s_k, s'_k, $\beta_j(p,X)$, and $\beta'_j(p,X)$ are non-negative and $\lambda \in [0,1]$, we have that $\beta''_j(p,X) \geq 0$ for all $j \in [m]$ and thus $\beta''(p,X)$ satisfies the linear inequalities in 7.13.

Now we show that $\beta''(p,X)$ satisfies the linear equalities in 7.14. To this end, for any component D_k of the disutility graph. If $s_k = s'_k = 0$, then we have $\beta''_j(p,X) = \beta_j(p,X)$ for all $j \in B_k$. Therefore, we have $\sum_{j\in B_k} \beta''_j(p,X) = \sum_{j\in B_k} \beta_j(p,X) = 1$ (as $\beta(p,X) \in \mathcal{B}(p,X)$) and we are done. If one of s_k or s'_k is non-zero, we have,

$$\begin{aligned}
\sum_{j\in B_k} \beta''_j(p,X) &= \sum_{j\in B_k} \frac{\lambda\cdot\beta_j(p,X)\cdot s_k + (1-\lambda)\cdot\beta'_j(p,X)\cdot s'_k}{\lambda\cdot s_k + (1-\lambda)\cdot s'_k} \\
&= \frac{\lambda\cdot s_k\cdot\sum_{j\in B_k}\beta_j(p,X) + (1-\lambda)\cdot s'_k\cdot\sum_{j\in B_k}\beta'_j(p,X)}{\lambda\cdot s_k + (1-\lambda)\cdot s'_k} \\
&= \frac{\lambda\cdot s_k + (1-\lambda)\cdot s'_k}{\lambda\cdot s_k + (1-\lambda)\cdot s'_k} \\
&= 1
\end{aligned}$$

Finally, we show that $\beta''(p,X)$ satisfies the linear equalities in 7.15. To this end, consider any component D_k such that $\sum_{j'\in B_k} q_j(p,X) > 0$. In this case, we have $\beta_j(p,X) = \beta'_j(p,X) = q_j(p,X)/(\sum_{j'\in B_k} q_{j'}(p,X))$. Now, if $s_k = s'_k = 0$, then we have $\beta''_j(p,X) = \beta_j(p,X) = q_j(p,X)/\ (\sum_{j'\in B_k} q_{j'}(p,X))$ and we are done. If one of s_k or s'_k is non-zero we have,

$$\begin{aligned}
\beta''_j(p,X) &= \frac{\lambda\cdot\beta_j(p,X)\cdot s_k + (1-\lambda)\cdot\beta'_j(p,X)\cdot s'_k}{\lambda\cdot s_k + (1-\lambda)\cdot s'_k} \\
&= \frac{\lambda\cdot\frac{q_j(p,X)}{\sum_{j'\in B_k} q_{j'}(p,X)}\cdot s_k + (1-\lambda)\cdot\frac{q_j(p,X)}{\sum_{j'\in B_k} q_{j'}(p,X)}\cdot s'_k}{\lambda\cdot s_k + (1-\lambda)\cdot s'_k} \\
&= \frac{q_j(p,X)}{\sum_{j'\in B_k} q_{j'}(p,X)}\cdot\frac{\lambda\cdot s_k + (1-\lambda)\cdot s'_k}{\lambda\cdot s_k + (1-\lambda)\cdot s'_k} \\
&= \frac{q_j(p,X)}{\sum_{j'\in B_k} q_{j'}(p,X)}.
\end{aligned}$$

Thus $\beta''(p,X) \in \mathcal{B}(p,X)$. Now, it only suffices to show that $s'' \in \tilde{P}(\beta''(p,X))$. Recall that to show that $s'' \in \tilde{P}(\beta''(p,X))$, we need to show that $s''_k \geq 0$ for all $k \in [d]$, $\sum_{k\in[d]} s''_k = 1$, and for all $k \in [d]$, we have,

$$\sum_{i\in A_k}\sum_{k'\in[d]}\sum_{j\in B_{k'}} w_{i,j}\cdot\beta''_j(p,X)\cdot s''_{k'} - s''_k = 0$$

To this end, we first note that since $s \in \tilde{P}(\beta(p,X))$, we have that

- $s_k \geq 0$ for all $k \in [d]$,
- $\sum_{k\in[d]} s_k = 1$, and

- $\sum_{i\in A_k}\sum_{k'\in[d]}\sum_{j\in B_{k'}} w_{i,j}\cdot\beta_j(p,X)\cdot s_{k'} - s_k = 0$ for all $k\in[d]$.

Analogous conditions are also satisfied by s' as it belongs to $\tilde{P}(\beta'(p,X))$. Now, observe that $s''_k = \lambda\cdot s_k + (1-\lambda)\cdot s'_k \geq 0$ as both s_k and s'_k are non-negative. Similarly,

$$\begin{aligned}\sum_{k\in[d]} s''_k &= \lambda\cdot\sum_{k\in[d]} s_k + (1-\lambda)\cdot\sum_{k\in[d]} s'_k\\ &= \lambda\cdot 1 + (1-\lambda)\cdot 1 = 1.\end{aligned}$$

Finally, we show that $\sum_{i\in A_k}\sum_{k'\in[d]}\sum_{j\in B_{k'}} w_{i,j}\cdot\beta''_j(p,X)\cdot s''_{k'} - s''_k = 0$. To this end, let $K = \{k \mid k\in[d] \text{ and } s''_k > 0\}$. Note that it suffices to show $\sum_{i\in A_k}\sum_{k'\in K}\sum_{j\in B_{k'}} w_{i,j}\cdot\beta''_j(p,X)\cdot s''_{k'} - s''_k = 0$ for all $k\in[d]$. Also, for all $k\in K$, we have $s_k = s'_k = 0$ as well. Therefore, we also have $\sum_{i\in A_k}\sum_{k'\in K}\sum_{j\in B_{k'}} w_{i,j}\cdot\beta_j(p,X)\cdot s_{k'} - s_k = 0$ and $\sum_{i\in A_k}\sum_{k'\in K}\sum_{j\in B_{k'}} w_{i,j}\cdot\beta'_j(p,X)\cdot s'_{k'} - s'_k = 0$ for all $k\in[d]$. Now note that for all $k\in[d]$, we have,

$$\begin{aligned}&\sum_{i\in A_k}\sum_{k'\in K}\sum_{j\in B_{k'}} w_{i,j}\cdot\beta''_j(p,X)\cdot s''_{k'} - s''_k\\ &= \sum_{i\in A_k}\sum_{k'\in K}\sum_{j\in B_{k'}} w_{i,j}\cdot\frac{\lambda\cdot\beta_j(p,X)\cdot s_k + (1-\lambda)\cdot\beta'_j(p,X)\cdot s'_k}{\lambda s_k + (1-\lambda)\cdot s'_k}\cdot s''_{k'} - s''_k\\ &= \sum_{i\in A_k}\sum_{k'\in K}\sum_{j\in B_{k'}} w_{i,j}\cdot\frac{\lambda\cdot\beta_j(p,X)\cdot s_k + (1-\lambda)\cdot\beta'_j(p,X)\cdot s'_k}{s''_k}\cdot s''_k - s''_k\\ &= \sum_{i\in A_k}\sum_{k'\in K}\sum_{j\in B_{k'}} w_{i,j}\cdot(\lambda\cdot\beta_j(p,X)\cdot s_k + (1-\lambda)\cdot\beta'_j(p,X)\cdot s'_k) - s''_k\\ &= \sum_{i\in A_k}\sum_{k'\in K}\sum_{j\in B_{k'}} w_{i,j}\cdot(\lambda\cdot\beta_j(p,X)\cdot s_k + (1-\lambda)\cdot\beta'_j(p,X)\cdot s'_k) - (\lambda\cdot s_k + (1-\lambda)\cdot s'_k)\\ &= \lambda\cdot\Big(\sum_{i\in A_k}\sum_{k'\in K}\sum_{j\in B_{k'}} w_{i,j}\cdot\beta_j(p,X)\cdot s_k - s_k\Big) + (1-\lambda)\cdot\Big(\sum_{i\in A_k}\sum_{k'\in K}\sum_{j\in B_{k'}} w_{i,j}\cdot\beta'_j(p,X)\cdot s'_k - s'_k\Big)\\ &= 0+0=0.\end{aligned}$$

Therefore $s''\in\tilde{P}(\beta''(p,X))$.

Therefore, we have that both the sets P' and $\mathbf{X}^p$ are non-empty and convex. thus, $\phi(\langle p,X\rangle)$ is also non-empty and convex as it is the Cartesian product of P' and $\mathbf{X}^p$. □

Lemma 7.36 (Property $\mathbf{P}_4$)**.** *ϕ has a closed graph.*

Proof. Consider a sequence $(\langle p^n,X^n\rangle)_{n\in\mathbb{N}}$ that converges to $\langle p^*,X^*\rangle$ and $\langle p^n,X^n\rangle\in S$ for all n. Similarly, consider the sequence $(\langle r^n,Y^n\rangle)_{n\in\mathbb{N}}$ that converges to $\langle r^*,Y^*\rangle$, such that $\langle r^n,Y^n\rangle\in\phi(\langle p^n,X^n\rangle)$ for all n. To show that ϕ has a closed graph, we need to show that $\langle r^*,Y^*\rangle\in\phi(\langle p^*,X^*\rangle)$. To show that, $\langle r^*,Y^*\rangle\in\phi(\langle p^*,X^*\rangle)$, we need to show,

(1) $r^*\in\overline{P}(\beta(p^*,X^*))$, for some $\beta(p^*,X^*)\in\mathcal{B}(p^*,X^*)$, and

(2) $Y^*\in\mathbf{X}^{p^*}$.

Proving $r^* \in \overline{P}(\beta(p^*, X^*))$, for some $\beta(p^*, X^*) \in \mathcal{B}(p^*, X^*)$: We first outline the necessary and sufficient condition for any vector p' to be in $\overline{P}(\beta(p, X))$, as this will be useful for our proof.

Observation 7.37. *$p' \in \overline{P}(\beta(p, X))$ if and only if*

(1) $p' \in P$, and

(2) for each chore j in component D_k, we have $p'_j = \beta_j(p, X) \cdot \sum_{j \in B_k} p'_j$.

Proof. We first show the "if" direction. To show that $p' \in \overline{P}(\beta(p, X))$, it suffices to show that for each chore $j \in B_k$, we have $p'_j = \beta_j(p, X) \cdot \tilde{p}_k$, such that $\tilde{p} \in \tilde{P}(\beta(p, X))$. For each component D_k of the disutility graph, let $\tilde{p}_k = \sum_{j \in B_k} p'_j$. Observe that for each chore $j \in B_k$ we have $p'_j = \beta_j(p, X) \cdot \tilde{p}_k$. It now suffices to show that $\tilde{p} = \langle \tilde{p}_1, \tilde{p}_2, \ldots, \tilde{p}_d \rangle \in \tilde{P}(\beta(p, X))$. To this end, observe that $\tilde{p}_k = \sum_{j \in B_k} p'_j \geq 0$ as $p'_j \geq 0$ for all $j \in [m]$ (as $p' \in P$). Furthermore, $\sum_{k \in [d]} \tilde{p}_k = \sum_{j \in [m]} p'_j = 1$ (as $p' \in P$). Now, to show $\tilde{p} \in \tilde{P}(\beta(p, X))$, it suffices to show that $\tilde{p}$ satisfies the system of equations in (7.17) or equivalently those in (7.16). To this end, since $p' \in P$, for each component D_k we have,

$$\sum_{i \in A_k} \sum_{j \in [m]} w_{i,j} \cdot p'_j = \sum_{j \in B_k} p'_j \ .$$

Or equivalently,

$$\sum_{i \in A_k} \sum_{k' \in [d]} \sum_{j \in B_{k'}} w_{i,j} \cdot p'_j = \sum_{j \in B_k} p'_j \ .$$

Substituting every p'_j as $\beta_j(p, X) \cdot \tilde{p}_k$ where chore j is in the component D_k we have,

$$\sum_{i \in A_k} \sum_{k' \in [d]} \sum_{j \in B_{k'}} w_{i,j} \cdot \beta_j(p, X) \cdot \tilde{p}_{k'} = \tilde{p}_k \ .$$

Therefore, $\tilde{p}_k$ satisfies (7.16). Thus $\tilde{p} \in \tilde{P}(\beta(p, X))$.

Now we show the "only if" direction. So assume $p' \in \overline{P}(\beta(p, X))$. Then, by Claims 7.31, 7.32 and 7.33 we have that $p' \in P$. Also by the definition of $\overline{P}(\beta(p, X))$, we also have that there exists a vector $\tilde{p} = \langle \tilde{p}_1, \ldots, \tilde{p}_d \rangle$ such that for all $j \in [m]$ we have $p'_j = \beta_j(p, X) \cdot \tilde{p}_k$ where D_k is the component in the disutility graph containing chore j. So it just suffices to show that $\tilde{p}_k = \sum_{j \in B_k} p'_j$ for all $k \in [d]$. To this end, observe that,

$$\begin{aligned}
\sum_{j \in B_k} p'_j &= \sum_{j \in B_k} \beta_j(p, X) \cdot \tilde{p}_k \\
&= \tilde{p}_k \cdot \sum_{j \in B_k} \beta_j(p, X) \\
&= \tilde{p}_k. && \text{(as } \beta(p, X) \text{ satisfies 7.14)}
\end{aligned}$$

□

We are now ready to show that $r^* \in \overline{P}(\beta(p^*, X^*))$ for some $\beta(p^*, X^*) \in \mathcal{B}(p^*, X^*)$. r^* is the limit of the sequence $(r^n)_{n \in \mathbb{N}}$, p^* is the limit of the sequence $(p^n)_{n \in \mathbb{N}}$, and X^* is the limit of the sequence $(X^n)_{n \in \mathbb{N}}$. Since for all n, $\langle r^n, Y^n \rangle \in \phi(\langle p^n, X^n \rangle)$, we can conclude that each $r^n \in P$. Since the set P is compact (and therefore closed), we have that $r^* \in P$ as well. Now, by Observation 7.37, it suffices to show that for each chore j in component D_k, we have $r^*_j = \beta_j(p^*, X^*) \cdot \sum_{j' \in B_k} r^*_{j'}$ for some $\beta(p^*, X^*) \in \mathcal{B}(p^*, X^*)$. To this end, we first define a vector $\beta(p^*, X^*) \in \mathcal{B}(p^*, X^*)$ and then we show that indeed $r^*_j = \beta_j(p^*, X^*) \cdot \sum_{j' \in B_k} r^*_{j'}$.

- For all chores $j \in B_k$ such that $\sum_{j' \in B_k} q_{j'}(p^*, X^*) > 0$, we set $\beta_j(p^*, X^*) = q_j(p^*, X^*)/(\sum_{j' \in B_k} q_{j'}(p^*, X^*))$.
- For all chores $j \in B_k$ such that $\sum_{j' \in B_k} q_{j'}(p^*, X^*) = 0$ and $\sum_{j' \in B_k} r^*_{j'} > 0$, we set $\beta_j(p^*, X^*) = r^*_j/(\sum_{j' \in B_k} r^*_{j'})$.
- For all chores $j \in B_k$ such that $\sum_{j' \in B_k} q_{j'}(p^*, X^*) = 0$ and $\sum_{j' \in B_k} r^*_{j'} = 0$, we set $\beta_j(p^*, X^*) = 1/|B_k|$.

It can be verified that $\beta(p^*, X^*)$ satisfies all the linear inequalities and equalities in 7.13, 7.14 and 7.15. Therefore, we have $\beta(p^*, X^*) \in \mathcal{B}(p^*, X^*)$. Now it just suffices to show that $r^*_j = \beta_j(p^*, X^*) \cdot \sum_{j' \in B_k} r^*_{j'}$. To this end, observe that for all chores $j \in B_k$ such that $\sum_{j' \in B_k} q_{j'}(p^*, X^*) = 0$, we already have that $r^*_j = \beta_j(p^*, X^*) \cdot \sum_{j' \in B_k} r^*_{j'}$: For a chore $j \in B_k$, where $\sum_{j' \in B_k} r^*_{j'} > 0$, we have $r^*_{j'} = \left(r^*_j/(\sum_{j' \in B_k} r^*_{j'})\right) \cdot (\sum_{j' \in B_k} r^*_{j'}) = \beta_j(p^*, X^*) \cdot (\sum_{j' \in B_k} r^*_{j'})$. Similarly, for a chore $j \in B_k$ where $\sum_{j' \in B_k} r^*_{j'} = 0$, we have $r^*_{j'} = 0 = (1/|B_k|) \cdot 0 = \beta_j(p^*, X^*) \cdot (\sum_{j' \in B_k} r^*_{j'})$.

Therefore, it only suffices to show $r^*_j = \beta_j(p^*, X^*) \cdot \sum_{j' \in B_k} r^*_{j'}$ for chores j, such that $j \in B_k$ and $\sum_{j' \in B_k} q_{j'}(p^*, X^*) > 0$. To this end, consider any chore $j \in B_k$, such that $\sum_{j' \in B_k} q_{j'}(p^*, X^*) > 0$. Let $\delta = \sum_{j' \in B_k} q_{j'}(p^*, X^*) > 0$ and let $0 < \varepsilon \ll \delta/(2n \cdot m)$. Let $S^* \subseteq S$ be the set of all points $\langle p', X' \rangle \in S$, that have a distance of at most ε from $\langle p^*, X^* \rangle$. Observe that for any $\langle p', X' \rangle \in S^*$ we have,

$$
\begin{aligned}
\sum_{j' \in B_k} q_{j'}(p', X') &= \sum_{j' \in B_k} \left(p'_{j'} + max(1 - \sum_{i \in [n]} X'_{ij'}, 0)\right) \\
&\geq \sum_{j \in B_k} \left((p^*_{j'} - \varepsilon) + (max(1 - \sum_{i \in [n]} X^*_{ij'}, 0) - n\varepsilon)\right) \\
&= \sum_{j' \in B_k} \left(p^*_{j'} + max(1 - \sum_{i \in [n]} X^*_{ij'}, 0)\right) - \sum_{j' \in B_k} (n+1)\varepsilon \\
&\geq \sum_{j' \in B_k} q_{j'}(p^*, X^*) - 2nm\varepsilon \\
&= \delta - 2nm\varepsilon \\
&> 0.
\end{aligned}
$$

Thus, for all $\langle p', X' \rangle \in S^*$, we have $\sum_{j' \in B_k} q_{j'}(p', X') > 0$, implying that for all $\beta(p', X') \in \mathcal{B}(p', X')$ we have $\beta_j(p', X') = q_j(p', X')/(\sum_{j' \in B_k} q_{j'}(p', X'))$. Since, $\sum_{j' \in B_k} q_{j'}(p', X') > 0$ for all $\langle p', X' \rangle \in S^*$, we have that $\beta_j(p', X')$ is well defined and continuous for all $\langle p', X' \rangle \in S^*$. We define $f_j(r, p, X) = r_j - \beta_j(p, X) \cdot \sum_{j' \in B_k} r_{j'}$. Since $\beta_j(p, X)$ is well defined and continuous for all $\langle p, X \rangle \in S^*$, we have that $f_j(r, p, X)$ is well defined and continuous for all $\langle p, X \rangle \in S^*$ and $r \in P$.

Now, consider any $0 < \varepsilon \ll \delta/(2nm)$. Since the sequences $(r^n)_{n \in \mathbb{N}}$ and $(\langle p^n, X^n \rangle)_{n \in \mathbb{N}}$ converge to r^* and $\langle p^*, X^* \rangle$ respectively, there exists a $n'(\varepsilon) \in \mathbb{N}$ such that for all $n > n'(\varepsilon)$, we have $|r^* - r^n| < \varepsilon$ and $|\langle p^*, X^*\rangle - \langle p^n, X^n\rangle| < \varepsilon$. In that case, for all $n > n'(\varepsilon)$, we have $\langle p^n, X^n\rangle \in S^*$. Therefore, $f_j(r^{n'(\varepsilon)+n}, p^{n'(\varepsilon)+n}, X^{n'(\varepsilon)+n})$ is well defined for all $n \in \mathbb{N}$. We define a new sequence $(h^n)_{n\in\mathbb{N}}$, such that $h^n = f_j(r^{n'(\varepsilon)+n}, p^{n'(\varepsilon)+n}, X^{n'(\varepsilon)+n})$. Since $f_j(r, p, X)$ is well defined and continuous for all $\langle p, X\rangle \in S^*$ and $r \in P$, and $\langle p^{n'(\varepsilon)+n}, X^{n'(\varepsilon)+n}\rangle \in S^*$ and $r^{n'(\varepsilon)+n} \in P$ for all $n \in \mathbb{N}$, we have that the limit of the sequence $(h^n)_{n\in\mathbb{N}}$ is $h^* = f_j(r^*, p^*, X^*)$. Again, since $r^n \in \overline{P}(\beta(p^n, X^n))$ for all $n \in \mathbb{N}$, we have by Observation 7.37 that $h^n = f_j(r^{n'(\varepsilon)+n}, p^{n'(\varepsilon)+n}, X^{n'(\varepsilon)+n}) = r_j^{n'(\varepsilon)+n} - \beta_j(p^{n'(\varepsilon)+n}, X^{n'(\varepsilon)+n}) \cdot \sum_{j'\in B_k} r_{j'}^{n'(\varepsilon)+n} = 0$ for all $n \in \mathbb{N}$. Therefore, the limit of the sequence $(h^n)_{n\in\mathbb{N}}$ is $h^* = 0$. This implies that $f_j(r^*, p^*, X^*) = 0$, further implying that $r_j^* - \beta_j(p^*, X^*) \cdot \sum_{j'\in B_k} r_{j'}^* = 0$. Thus, we have $r_j^* = \beta_j(p^*, X^*) \cdot \sum_{j'\in B_k} r_{j'}^*$ for all chores j, such that $j \in B_k$, where $\sum_{j'\in B_k} q_{j'}(p^*, X^*) > 0$.

Proving $Y^* \in \mathbf{X}^{p^*}$: To show $Y^* \in X^{p^*}$, we need to show that

(1) $Y^* \in \mathbf{X}$,

(2) for all $i \in A_k$, we have $Y_{ij} > 0$ only if $d(i,j) \neq \infty$,

(3) for all $i \in A_k$, where $\sum_{j\in B_k} p_j^* > 0$, we have $Y_{ij} > 0$ only if $\frac{d(i,j)}{p_j^*} \leq \frac{d(i,\ell)}{p_\ell^*}$ for all $\ell \in [m]$, and

(4) for all $i \in A_k$, where $\sum_{j\in B_k} p_j^* > 0$, we have $\sum_{j\in[m]} Y_{ij} \cdot p_j^* = \sum_{j\in[m]} w_{i,j} \cdot p_j^*$ for all $i \in [n]$.

Since $\langle r^n, Y^n\rangle \in \phi(\langle p^n, X^n\rangle)$ for all n, we have that $Y^n \in \mathbf{X}$ for all n. Since $\mathbf{X}$ is compact (and therefore closed), we have that $Y^* \in \mathbf{X}$ as well.

We show part 2 by contradiction. Assume that there exists an $i \in A_k$, where $Y_{ij}^* = \delta > 0$, and $d(i,j) = \infty$. Since the sequence $(Y^n)_{n\in\mathbb{N}}$ converges to Y^*, we know that for every $\varepsilon > 0$, there exists an $n'(\varepsilon) \in \mathbb{N}$ be such that for $n > n'(\varepsilon)$ we have $|Y_{ij}^* - Y_{ij}^n| < \varepsilon$. Choosing a $\varepsilon \ll \delta$, we can ensure that $|Y_{ij}^* - Y_{ij}^n| < \varepsilon$, implying that $Y_{ij}^n \geq \delta - \varepsilon > 0$ for all $n > n'(\varepsilon)$. Therefore $Y_{ij}^n > 0$ for all $n > n'(\varepsilon)$ (while $d(i,j) = \infty$) which contradicts the fact that $Y^n \in \mathbf{X}$.

We show part 3 by contradiction. Consider any agent $i \in A_k$, where $\sum_{\ell\in B_k} p_\ell^* > 0$. Since $\sum_{\ell\in B_k} p_\ell^* > 0$, the chore j such that $\frac{d(i,j)}{p_j^*}$ is minimum has price $p_j^* > 0$. So for contradiction, let us assume that $Y_{ij'}^* = \beta > 0$, and $\frac{d(i,j')}{p_{j'}^*} > \frac{d(i,j)}{p_j^*}(1 + \delta)$ for some $\delta > 0$. Since the sequence $(Y^n)_{n\in\mathbb{N}}$ converges to Y^* and p^n converges to p^*, we know that for every $\varepsilon > 0$, there exists an $n'(\varepsilon) \in \mathbb{N}$ be such that for $n > n'(\varepsilon)$ we have $|Y_{ij}^* - Y_{ij}^n| < \varepsilon$ and $|p_j^* - p_j^n| < \varepsilon$ for all $j \in [m]$. For a sufficiently small $\varepsilon > 0$, we can ensure that $Y_{ij'}^* \geq \beta - \varepsilon > 0$ and $\frac{d(i,j')}{p_{j'}^n} > \frac{d(i,j)}{p_j^n}$, contradicting the fact that $Y^n \in \mathbf{X}^{p^n}$ for all $n > n'(\varepsilon)$.

Finally, we prove part 3 by contradiction. Assume that $\sum_{j\in[m]} w_{i,j} \cdot p_j^* - \sum_{j\in[m]} Y_{ij}^* \cdot p_j^* = \delta$ for some non-zero δ. Since the sequence $(Y^n)_{n\in\mathbb{N}}$ converges to Y^* and p^n converges to p^*, we know that for every $\varepsilon > 0$, there exists an $n'(\varepsilon) \in \mathbb{N}$ be such that for $n > n'(\varepsilon)$ we have $|Y_{ij}^* - Y_{ij}^n| < \varepsilon$ and $|p_j^* - p_j^n| < \varepsilon$ for all $j \in [m]$. Therefore, by choosing a sufficiently small ε we can ensure that $\sum_{j\in[m]} w_{i,j} \cdot p_j^n - \sum_{j\in[m]} Y_{ij}^n \cdot p_j^n \neq 0$, for all $n > n'(\varepsilon)$, which contradicts the fact that $Y^n \in \mathbf{X}^{p^n}$ for all $n > n'(\varepsilon)$. $\square$

We are now ready to state the main result of this section

Theorem 7.38. *Every instance $I \in \mathcal{I}$ admits a competitive equilibrium.*

Proof. We defined a correspondence ϕ that satisfies properties $\mathbf{P}_1$, $\mathbf{P}_2$, $\mathbf{P}_3$ and $\mathbf{P}_4$ by Lemmas 7.30, 7.34, 7.35 and 7.36. By Lemma 7.25 we have that any correspondence that satisfies the properties $\mathbf{P}_1$, $\mathbf{P}_2$, $\mathbf{P}_3$ and $\mathbf{P}_4$ has a fixed point. Finally, by Lemma 7.26, any fixed point of this correspondence will correspond to a competitive equilibrium in I. Therefore, our correspondence ϕ has at least one fixed point and this fixed point corresponds to a competitive equilibrium. □

Proof of Fact 7.23: Recall Fact 7.23.

Fact. *Let $Z \in \mathbb{R}^{n\times n}$ be a square matrix such that $Z_{ij} \geq 0$ for all $j \neq i$ (all the non-diagonal entries of Z are non-negative) and $\sum_{i\in[n]} Z_{ij} = 0$ (column sums are zero), then there exists a vector $t \in \mathbb{R}^n_{\geq 0}$ such that $\sum_{i\in[n]} t_i = 1$ and $Z \cdot t = 0$.*

Proof. Let $\lambda \gg max_{i,j\in[n]}(|Z_{ij}|)$. Let $Z' = \frac{1}{\lambda}Z$. Observe that every t that satisfies $Z' \cdot t = 0$, also satisfies $Z \cdot t = 0$ and vice versa. Also, each entry in the matrix Z' has absolute value is less than one. Let $Z'' = (Z' + I)$ where I is the identity matrix. Note that every entry in the matrix Z'' is non-negative. Also every t that satisfies $Z'' \cdot t = t$, also satisfies $Z' \cdot t = 0$ and therefore also satisfies $Z \cdot t = 0$ and vice versa. From here on, we will be dealing with the following system of equations

$$Z'' \cdot t = t \ . \tag{7.20}$$

We first observe that the matrix Z'' is column stochastic: For all $j \in [n]$, we have

$$\begin{aligned}
\sum_{i\in[n]} Z''_{ij} &= \sum_{i\in[n]} (\frac{1}{\lambda} \cdot Z_{ij} + I_{ij}) \\
&= \sum_{i\in[n]} \frac{1}{\lambda} \cdot Z_{ij} + 1 \\
&= 0 + 1 && \text{(Column sums are zero in } Z) \\
&= 1 \ .
\end{aligned}$$

Now, let $\Delta_n = \left\{ r \in \mathbb{R}^n_{\geq 0} \mid \sum_{j\in[n]} r_j = 1 \right\}$ be the n dimensional simplex. Observe that the set Δ_n is non-empty, convex and compact. We first make a small claim.

Claim 7.39. *Let $r' = Z'' \cdot r$. If $r \in \Delta_n$ then $r' \in \Delta_n$.*

Proof. Since every entry in the matrix Z'' and every component of the vector r is non-negative, we also have that every component of r' is also non-negative: $r'_j \geq 0$ for all

$j \in [d]$. Now observe that

$$\begin{aligned}
\sum_{j\in[n]} r'_j &= \mathbf{1}^T \cdot r' \\
&= \mathbf{1}^T \cdot Z'' \cdot r \\
&= \mathbf{1}^T \cdot r && \text{(as } Z'' \text{ is column stochastic)} \\
&= \sum_{j\in[m]} r_j = 1.
\end{aligned}$$

Thus, $r' \in \Delta_n$. □

We define $f : \Delta_n \to \Delta_n$ such that $f(r) = Z'' \cdot r$. Observe that f is also continuous. Thus, by Brouwer's fixed point theorem there is a $t \in \Delta_n$, such that $f(t) = t$ or equivalently $Z'' \cdot t = t$. □

7.3 PPAD-Hardness of Determining a Competitive Equilibrium

In this section, we show that chore division is intractable even for the instances that satisfy Conditions 1 and 2 mentioned in Section 7.2. We show that it is PPAD-hard to find a competitive equilibrium on instances that satisfy Conditions 1 and 2 in Section 7.2. We will show that any polynomial-time algorithm that determines a competitive equilibrium on instances that satisfy Conditions 1 and 2, will yield a polynomial-time algorithm to find an equilibrium in a *normalized polymatrix game*. The normalized polymatrix game is known to be PPAD-hard [38].

A polymatrix game is represented by a game graph where each node is a player who plays a two-player game with each of her neighbors. She has to play the same strategy with each of her neighbors and her payoff is the sum of the payoffs on each of her incident edges. If there are n players and each of them has exactly two strategies to choose from, then such a game can be represented by $2n \times 2n$ matrix. When thought of it as $n \times n$ block matrix, where each block is 2×2 matrix, then $(i, j)^{th}$ block is the payoff matrix of player i for the game on edge (i, j). Formally,

Problem. ***(Normalized Polymatrix Game) [38]***
Given: *A $2n \times 2n$ rational matrix $\mathbf{M}$ with every entry in $[0, 1]$ and $\mathbf{M}_{i,2j-1} + \mathbf{M}_{i,2j} = 1$ for all $i \in [2n]$ and $j \in [n]$.*
Find: *Equilibrium strategy vector $x \in \mathbb{R}^{2n}_{\geq 0}$ such that $x_{2i-1} + x_{2i} = 1$ and*

$$\begin{aligned}
x^T \cdot \mathbf{M}_{*,2i-1} > x^T \cdot \mathbf{M}_{*,2i} + \tfrac{1}{n} &\implies x_{2i} = 0. \\
x^T \cdot \mathbf{M}_{*,2i} > x^T \cdot \mathbf{M}_{*,2i-1} + \tfrac{1}{n} &\implies x_{2i-1} = 0.
\end{aligned}$$

where $M_{*,k}$ represents the k^{th} column of the matrix $\mathbf{M}$.

Given an instance $I = \langle M \rangle$ of the polymatrix game, we construct an instance $E(I)$ of chore division. We show that $E(I)$ satisfies Conditions 1 and 2 in Section 7.2 and therefore admits a competitive equilibrium. Then, we show how to obtain an equilibrium vector $x \in \mathbb{R}^{2n}_{\geq 0}$ from the prices of the chores in $E(I)$ at a competitive equilibrium in polynomial-time . This would imply the PPAD-hardness of finding a competitive equilibrium even under the sufficiency conditions.

The key properties that our hard instance $E(I)$ exhibits are *pairwise equal endowments*, *fixed earning*, *price equality*, *price regulation* and *reverse ratio amplification* (we will give a precise definition of these properties in Subsection 7.3.1). These techniques (constructing hard instances exhibiting these properties) have been used earlier to prove PPAD-hardness for determining a competitive equilibrium in the exchange model with goods when agents have constant elasticity of substitution (CES) utilities [38] and even for the Fisher model with goods when agents have separable piecewise linear concave (SPLC) utilities [39]. However, the challenge is to construct these gadgets and make them work together only using linear disutility functions; as clearly this is not possible in case of goods, when agents have linear utility functions.

7.3.1 Creating the Instance $E(I)$

We elaborate our construction and proof of reduction: We first introduce all agents and chores. Thereafter, we define the disutility matrix and endowment matrix and show that our instance satisfies the sufficiency conditions mentioned in Section 7.2, and therefore admits a competitive equilibrium. Then, we show that our instance exhibits the four properties of *pairwise equal endowments*, *fixed earning*, *price equality*, *price regulation* and *reverse ratio amplification*, and thus in polynomial-time we can construct the equilibrium strategy vector x for I from any competitive equilibrium in $E(I)$.

7.3.2 Agent and Chore Sets

We define the set of $K = 2c \cdot \lceil \log(n) \rceil$ many sets of chores, where $c = 4$ (observe crucially that K is even),

$$B_k = \left\{ \cup_{i \in [2n]} b^k_i \right\} \qquad\qquad \text{for all } k \in [K],$$

and K many sets of agents

$$A_k = \begin{cases} \left\{ \cup_{i \in [2n]} a^1_i \right\} \cup \left\{ \cup_{i \in [2n]} a'_i \right\} & \text{when } k = 1, \\ \left\{ \cup_{i \in [2n]} a^k_i \right\} \cup \left\{ \cup_{i \in [n]} \overline{a}^k_i \right\} & \text{when } 2 \leq k \leq K-1, \\ \left\{ \cup_{i,j \in [2n]} a^K_{i,j} \right\} \cup \left\{ \cup_{i \in [n]} \overline{a}^K_i \right\} & \text{when } k = K. \end{cases}$$

We now define the disutility matrix and the endowment matrix of the instance.

Disutility Matrix and the Disutility Graph. The disutility graph of our instance will be a disjoint union of complete bipartite graphs. We now describe the disutility matrix: We define only the disutility values in the matrix that are finite (the disutility of all agent-chore pair not mentioned should be assumed to be ∞). *For all $k \in [K]$, for each pair of chores b^k_{2i-1} and b^k_{2i}, there are a set of agents that have finite disutility towards them and have infinite disutility towards all other chores; Additionally, these agents also happen to be either in A_k or A_{k-1} (indices are modulo K). Thus, each component in the disutility graph comprises of the chores b^k_{2i-1} and b^k_{2i} and the agents that have* *finite disutility towards them.* We now outline these agents and their disutilities for every $k \in [K]$. To define the finite entries in the disutility matrix, we introduce the scalars $\frac{1}{n^{3c}} = \alpha_1, \alpha_2, \ldots, \alpha_K$ such that each $\alpha_{i+1} = \frac{3}{2} \cdot \alpha_i$ for all $i \in [K-1]$. Before we define the disutility matrix, we make an obvious claim about the scalars α_i for all $i \in [K]$, which will be useful later,

Claim 7.40. *We have $n^c \cdot \alpha_1 < \alpha_K \leq \frac{1}{n^c}$.*

Proof. We first show the lower bound. We have $\alpha_K = (\frac{3}{2})^{K-1} \cdot \alpha_1 = (\frac{3}{2})^{2c\lceil \log(n) \rceil - 1} \cdot \alpha_1 > 2^{c\log(n)} \cdot \alpha_1 = n^c \cdot \alpha_1$. Similarly, for the upper bound, we have, $\alpha_K = (\frac{3}{2})^{K-1} \cdot \alpha_1 = (\frac{3}{2})^{2c\lceil \log(n) \rceil - 1} \cdot \alpha_1 < 2^{2c\log(n)} \cdot \alpha_1 = n^{2c} \cdot \alpha_1 = \frac{1}{n^c}$ (as $\alpha_1 = \frac{1}{n^{3c}}$). $\square$

We now define the disutility matrix:

- $k = 1$: For each $i \in [n]$, we first define the disutilities of the agents that have finite disutility for chores b^1_{2i-1} and b^1_{2i}. For each $i \in [n]$ we have,

$$\begin{aligned}
d(a^K_{i',2i-1}, b^1_{2i-1}) &= (1-\alpha_1) \quad \text{and} \quad d(a^K_{i',2i-1}, b^1_{2i}) = (1+\alpha_1) \quad \text{for all } i' \in [2n] \\
d(a^K_{i',2i}, b^1_{2i-1}) &= (1+\alpha_1) \quad \text{and} \quad d(a^K_{i',2i}, b^1_{2i}) = (1-\alpha_1) \quad \text{for all } i' \in [2n] \\
d(a'_{2i-1}, b^1_{2i-1}) &= (1-\alpha_1) \quad \text{and} \quad d(a'_{2i-1}, b^1_{2i}) = (1+\alpha_1) \\
d(a'_{2i}, b^1_{2i-1}) &= (1+\alpha_1) \quad \text{and} \quad d(a'_{2i}, b^1_{2i}) = (1-\alpha_1).
\end{aligned}$$

  Therefore, for each $i \in [n]$, we have a component D^1_i in the disutility graph which is a complete bipartite graph comprising of agents $\left\{\cup_{i' \in [2n]} a^K_{i',2i-1}\right\} \bigcup \left\{\cup_{i' \in [2n]} a^K_{i',2i}\right\}$ $\bigcup \left\{a'_{2i-1}, a'_{2i}\right\}$ and chores $\left\{b^1_{2i-1}, b^1_{2i}\right\}$ (see Figure 7.2 (left subfigure) for an illustration).

- $2 \leq k \leq K$: For each $i \in [n]$ we have,

$$\begin{aligned}
d(a^{k-1}_{2i-1}, b^k_{2i-1}) &= (1-\alpha_k) \qquad \text{and} \qquad d(a^{k-1}_{2i-1}, b^k_{2i}) = (1+\alpha_k) \\
d(a^{k-1}_{2i}, b^k_{2i-1}) &= (1+\alpha_k) \qquad \text{and} \qquad d(a^{k-1}_{2i}, b^k_{2i}) = (1-\alpha_k) \\
d(\overline{a}^k_i, b^k_{2i-1}) &= (1-\alpha_k) \qquad \text{and} \qquad d(\overline{a}^k_i, b^k_{2i}) = (1-\alpha_k) \ .
\end{aligned}$$

  Therefore, for every k such that $2 \leq k \leq K$, for each $i \in [n]$, we have a connected component D^k_i in the disutility graph which is a complete bipartite graph comprising of agents $\left\{a^{k-1}_{2i-1}, a^{k-1}_{2i}, \overline{a}^k_i\right\}$ and chores $\left\{b^k_{2i-1}, b^k_{2i}\right\}$ (see Figure 7.2 (right subfigure) for an illustration).

It is clear that the disutility graph is a disjoint union of complete bipartite graphs, namely, the union of D^k_i for all $i \in [n]$ and $k \in [K]$. Therefore,

$$E(I) \text{ satisfies Condition 2 in Section 7.2.}$$

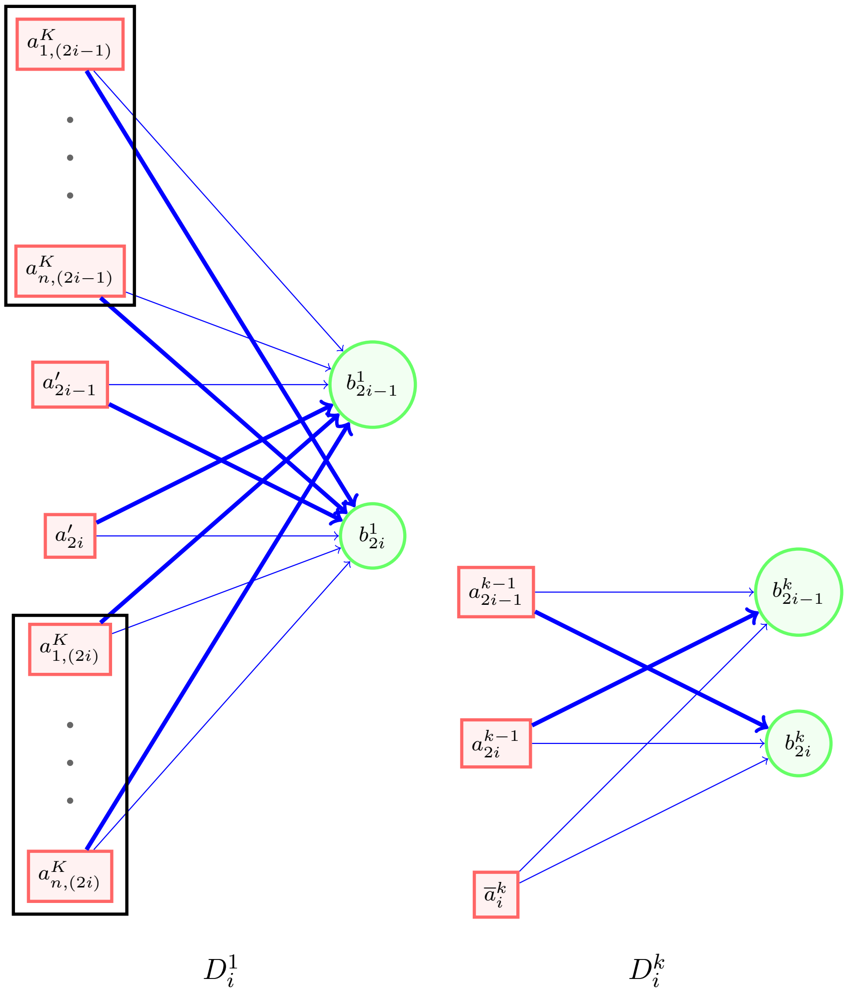


Figure 7.2: Illustration of the disutility graph corresponding to the disutility matrix: On the left, we have the component D^1_i, and on the right we have D^k_i when $2 \leq k \leq K$. The edges are colored in order to also encode the disutility matrix. The thin blue edges from agents to chores depict a disutility of $1 - \alpha_1$ for D^1_i (left), and $1 - \alpha_k$ for D^k_i when $2 \leq k \leq K$ (right). Similarly, the thick blue edges from agents to chores depict a disutility of $1 + \alpha_1$ for D^1_i (left) and $1 + \alpha_k$ for D^k_i (right).

Endowment Matrix. All agents in A_k have endowments of chores only in B_k for all $k \in [K]$. We only mention the non-zero agent-chore endowments (all agent-chore endowments not mentioned are zero).

- $k = 1$: For each $i \in [2n]$ we have,

$$w(a_i^1, b_i^1) = n.$$

  Also, for each $i \in [n]$ we have

$$w(a'_{2i-1}, b^1_{2i-1}) = w(a'_{2i-1}, b^1_{2i}) = \frac{1}{2} \cdot (1 - \alpha_K) \cdot (2n - \sum_{j \in [2n]} \mathbf{M}_{j,2i-1})$$
$$w(a'_{2i}, b^1_{2i-1}) = w(a'_{2i}, b^1_{2i}) = \frac{1}{2} \cdot (1 - \alpha_K) \cdot (2n - \sum_{j \in [2n]} \mathbf{M}_{j,2i}).$$

- $2 \le k \le K - 1$: For each $i \in [n]$, we have,

$$w(a^k_{2i-1}, b^k_{2i-1}) = n \quad \text{and} \quad w(a^k_{2i}, b^k_{2i}) = n$$
$$w(\overline{a}^k_i, b^k_{2i-1}) = \delta_k \quad \text{and} \quad w(\overline{a}^k_i, b^k_{2i}) = \delta_k \ ,$$

  where $\delta_k = \frac{n \cdot \alpha_k}{2}$. The reason behind the exact choice of the value of δ_k will become explicit when we show that our instance satisfies the reverse ratio amplification property in Section 7.3.3. As of now, the reader is encouraged to think of it just as a small scalar.

- $k = K$: For each $i \in [n]$ we have,

$$w(a^K_{2i-1,j}, b^K_{2i-1}) = \mathbf{M}_{2i-1,j} \quad \text{and} \quad w(a^K_{2i,j}, b^K_{2i}) = \mathbf{M}_{2i,j} \quad \text{for all } j \in [2n]$$
$$w(\overline{a}^K_i, b^K_{2i-1}) = \delta_K \quad \text{and} \quad w(\overline{a}^K_i, b^K_{2i}) = \delta_K \ ,$$

  where $\delta_K = \frac{n \cdot \alpha_K}{2}$ (the reason behind the choice of value will become explicit in Section 7.3.3).

Strongly Connected Economy Graph. We now show that the economy graph G of our instance is strongly connected. For ease of explanation, we introduce the notion of *economy graph of components* $W = ([d], E_W)$, where there is an edge from $i \in [d]$ to $j \in [d]$, if and only if, there is an agent $a \in D_i$ that has a positive endowment of some chore in $b \in D_j$. We now make a claim that strong connectivity of W implies strong connectivity of the economy graph G.

Claim 7.41. *If W is strongly connected then G is also strongly connected.*

Proof. Consider any two agents a and a'. Let $a \in D_i$ and $a' \in D_j$.[16] Consider any chore b that agent a has a positive endowment of and let $D_{i'}$ be the component in the disutility graph that contains b.[17] Then since $D_{i'}$ is a biclique in our instance, every agent in $D_{i'}$

[16] Note that j could also be equal to i

[17] Again, i' could also be equal to i

has finite disutility for the chore b. Therefore, every agent in $D_{i'}$ is reachable from a with an edge in the economy graph G. Now, since W is strongly connected, there is a path from $\ell_1 \to \ell_2 \to \cdots \to \ell_k$ from $i' = \ell_1$ to $j = \ell_k$. Let a_{ℓ_r} be the agent in the component D_{ℓ_r}, that has a positive endowment of some chore in the component $D_{\ell_{r+1}}$ for all $r \in [k-1]$. Again, since each D_{ℓ_r} is a biclique, every agent in D_{ℓ_r} has a finite disutility for every chore in D_{ℓ_r}. Thus, there is an edge in the economy graph G from a_{ℓ_r} to every agent in D_{ℓ_r}, in particular there is an edge between a_{ℓ_r} and $a_{\ell_{r+1}}$ in G. Thus, we have a path $a \to a_{\ell_1} \to \cdots \to a_{\ell_{k-1}} \to a'$ in G. Therefore, if W is strongly connected, then there is a path between any two agents in G, implying that G is also strongly connected. □

Form here on, we show that W is strongly connected. Observe that the disutility graph consists of connected components D_i^k for $k \in [K]$ and $i \in [n]$. Also observe that every component D_i^k in the disutility graph comprises of exactly two chores b_{2i-1}^k and b_{2i}^k. Therefore, to show that there exists an edge from component $D_{i'}^{k'}$ to D_i^k in W, it suffices to show that $D_{i'}^{k'}$ contains agents that own parts of chores b_{2i-1}^k and b_{2i}^k. We now outline the edges in our exchange graph (see Figure 7.3):

- For all $i \in [n]$, and $2 \leq k \leq K$ there is an edge in W from D_i^k to D_i^{k-1}: D_i^k contains the agents a_{2i-1}^{k-1} and a_{2i}^{k-1} that own parts of chores b_{2i-1}^{k-1} and b_{2i}^{k-1} respectively (see Figure 7.3).
- For all $i \in [n]$, there is an edge in W from D_i^1 to D_j^K for all $j \in [n]$: Consider any $j \in [n]$. Observe that the component D_i^1 contains the agents $a_{2j-1,2i}^K$ and $a_{2j,2i}^K$ and the agents $a_{2j-1,2i}^K$ and $a_{2j,2i}^K$ own parts of chores b_{2j-1}^K and b_{2j}^K respectively (see Figure 7.3).

Observe that all nodes are reachable from any D_i^1 ($i \in [n]$). Also, from any arbitrary $D_{i'}^{k'}$, the node $D_{i'}^1$ is reachable and since every node is reachable from $D_{i'}^1$, every node is also reachable from $D_{i'}^{k'}$ as well. Therefore, the economy graph of components W, is strongly connected. Therefore, by Claim 7.41 we have that,

$$E(I) \text{ satisfies Condition 1 in Section 7.2.}$$

Thus, $E(I)$ satisfies Conditions 1 and 2 in Section 7.2 and therefore admits a competitive equilibrium. Let $p(b_i^k)$ denote the price of any chore b_i^k at a competitive equilibrium. We now prove that our instance satisfies the required properties of *pairwise equal endowments*, *price equality*, *fixed earning*, *price regulation* and *reverse ratio amplification*.

7.3.3 $E(I)$ Satisfies All the Properties

Pairwise Equal Endowments. Here, we show that for all $i \in [n]$ and for all $k \in [K]$ the total endowment of b_{2i-1}^k equals the total endowment of b_{2i}^k and the total endowments of each chore in $E(I)$ is $\mathcal{O}(n)$.

Lemma 7.42. *For all $i \in [2n]$, the total endowments of chores b_{2i-1}^k and b_{2i}^k is*

(1) $n + n \cdot (1 - \alpha_K)$, if $k = 1$. In particular, a'_{2i-1} and a'_{2i} together*, own $n \cdot (1 - \alpha_K)$ units of chores b_{2i-1}^k and b_{2i}^k each.*

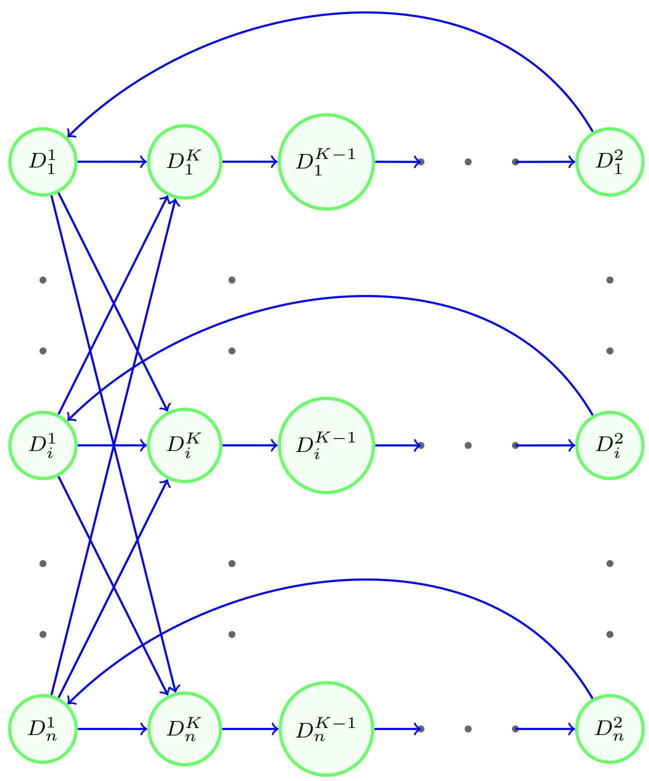


Figure 7.3: Illustration of the strong connectivity of the economy graph of components of our instance. Observe that all nodes are reachable from any D_i^1 $(i \in [n])$. Also, from any arbitrary $D_{i'}^{k'}$, the node $D_{i'}^1$ is reachable and since every node is reachable from $D_{i'}^1$, every node is also reachable from $D_{i'}^{k'}$ as well. Therefore, the economy graph of components is strongly connected.

(2) $n + \delta_k$, *if* $2 \leq k \leq K$.

Proof. When $k = 1$, the only agents that have positive endowments of b^1_{2i} are a^1_{2i} (has an endowment of n) , a'_{2i} (has an endowment of $\frac{1}{2} \cdot (1 - \alpha_K) \cdot (2n - \sum_{j \in [2n]} \mathbf{M}_{j,2i})$) and a'_{2i-1}(has an endowment of $\frac{1}{2} \cdot (1 - \alpha_K) \cdot (2n - \sum_{j \in [2n]} \mathbf{M}_{j,2i-1})$). Therefore, the total endowment of b^1_{2i} from the agents a'_{2i} and a'_{2i-1} is

$$\begin{aligned}
&= \frac{1}{2} \cdot (1 - \alpha_K) \cdot (2n - \sum_{j \in [2n]} \mathbf{M}_{j,2i}) + \frac{1}{2} \cdot (1 - \alpha_K) \cdot (2n - \sum_{j \in [2n]} \mathbf{M}_{j,2i-1}) \\
&= \frac{1}{2} \cdot (1 - \alpha_K) \cdot (4n - \sum_{j \in [2n]} (\mathbf{M}_{j,2i} + \mathbf{M}_{j,2i-1})).
\end{aligned}$$

Recall that $\mathbf{M}_{j,2i} + \mathbf{M}_{j,2i-1} = 1$. Therefore, the total endowment of b^1_{2i} from the agents a'_{2i} and a'_{2i-1} is

$$\begin{aligned}
&= \frac{1}{2} \cdot (1 - \alpha_K) \cdot (4n - 2n) \\
&= (1 - \alpha_K) \cdot n.
\end{aligned}$$

Therefore, the total endowment of chore b^1_{2i} is $n + n \cdot (1 - \alpha_K)$. A similar argument will show that the total endowment of chore b^1_{2i-1} is also $n + n \cdot (1 - \alpha_K)$ and that agents a'_{2i-1} and a'_{2i} *together*, own $n \cdot (1 - \alpha_K)$ units of it.

When $2 \leq k \leq K - 1$, the only agents that have positive endowments of b^k_{2i} are a^k_{2i} (has an endowment of n) and $\overline{a}^k_i$ (has an endowment of δ_k). Therefore, the total endowment is $n + \delta_k$. A similar argument will show that the total endowment of chore b^k_{2i-1} is also $n + \delta_k$.

When $k = K$, the only agents that have positive endowments of b^K_{2i} are the agents $a^K_{2i,j}$ (has an endowment of $\mathbf{M}_{2i,j}$) for all $j \in [2n]$ and the agent $\overline{a}^K_i$ (has an endowment of δ_K). Therefore, the total endowment of chore b^K_{2i} is

$$\begin{aligned}
&= \sum_{j \in [2n]} \mathbf{M}_{2i,j} + \delta_K \\
&= \sum_{j \in [n]} (\mathbf{M}_{2i,2j-1} + \mathbf{M}_{2i,2j}) + \delta_K \\
&= \sum_{j \in [n]} 1 + \delta_K \\
&= n + \delta_K
\end{aligned}$$

A similar argument will show that the total endowment of chore b^K_{2i-1} is also $n + \delta_K$. $\square$

Price Equality. Here we will show that the sum of prices of chores b^1_{2i-1} and b^1_{2i} equals that of b^K_{2i-1} and b^K_{2i}. Let us define

$$\pi^k_i = p(b^k_{2i-1}) + p(b^k_{2i}), \quad \forall i \in [2n], k \in [K] \ .$$

Since the prices corresponding to a competitive equilibrium is scale-invariant, we can assume without loss of generality that $\pi^1_1 = 2$. We now state the main lemma of price equality:

Lemma 7.43. *For all $i \in [n]$ and for all $k \in [K]$, we have $\pi_i^k = 2$.*

Proof. We show this in two steps: First we show that $\pi_i^1 = \pi_i^k$ for all $k \in [K]$. Then we show that $\pi_i^1 = \frac{1}{n}\sum_{j\in[n]} \pi_j^K$ for all $i \in [n]$, implying that $\pi_i^1 = \pi_j^1$ for all $i, j \in [n]$. Since for all $i \in [n]$ and $k \in [K]$, $\pi_i^k = \pi_i^1$ and $\pi_i^1 = \pi_1^1$, we will have that $\pi_i^k = \pi_1^1 = 2$.

We first show $\pi_i^1 = \pi_i^k$ for all $k \in [K]$: Consider any $k \in [2, K]$ and any $i \in [n]$. Observe that the agents a_{2i-1}^{k-1}, a_{2i}^{k-1}, $\overline{a}_i^k$ and chores b_{2i-1}^k, b_{2i}^k form the connected component D_i^k in the disutility graph. This implies that the agents a_{2i-1}^{k-1}, a_{2i}^{k-1} and $\overline{a}_i^k$ earn all of their money at a competitive equilibrium from chores b_{2i-1}^k and b_{2i}^k. Now, note that $\overline{a}_i^k$ owns δ_k units of both b_{2i-1}^k and b_{2i}^k only and has finite disutility only for chores b_{2i-1}^k and b_{2i}^k. Therefore, at a competitive equilibrium, $\overline{a}_i^k$ has to earn $\delta_k \cdot \pi_i^k$ money from chores b_{2i-1}^k and b_{2i}^k, to pay for her endowments. Thus, the total money the agents a_{2i-1}^{k-1} and a_{2i}^{k-1} earn from chores b_{2i-1}^k and b_{2i}^k is the total price of these chores remaining after $\overline{a}_i^k$ earns her share of $\delta_k \cdot \pi_i^k$, which is $(n+\delta_k)\cdot \pi_i^k - \delta_k \cdot \pi_i^k = n \cdot \pi_i^k$ (as the total endowment of each b_{2i}^k and b_{2i-1}^k is $n + \delta_k$ for all $k \in [2, K]$ by Lemma 7.42). At a competitive equilibrium, the total money earned by the agents a_{2i-1}^{k-1} and a_{2i}^{k-1} should be equal to the total prices of chores they own (n units of b_{2i-1}^{k-1} and b_{2i}^{k-1} respectively). Thus we have, $n \cdot \pi_i^{k-1} = n \cdot \pi_i^k$. This implies that,

$$\pi_i^k = \pi_i^{k-1} = \cdots = \pi_i^1.$$

We now show that $\pi_i^1 = \frac{1}{n}\sum_{j\in[n]} \pi_j^K$: This time, we look into the connected component D_i^1 of the disutility graph. We can claim that the agents $\cup_{j\in[2n]} a_{j,2i-1}^K$, $\cup_{j\in[2n]} a_{j,2i}^K$ and the agents a'_{2i-1} and a'_{2i} earn all of their money at a competitive equilibrium from chores b_{2i-1}^1 and b_{2i}^1. Observe that both agents a'_{2i-1} and a'_{2i} own some units of chores b_{2i-1}^1 and b_{2i}^1 only. Since the only chores towards which a'_{2i-1} and a'_{2i} have finite disutility are also b_{2i-1}^1 and b_{2i}^1, we can conclude that at a competitive equilibrium, to pay for their endowments, agents a'_{2i-1} and a'_{2i}, together earn $n\cdot(1-\alpha_K)\cdot\pi_i^1$ amount of money from chores b_{2i-1}^1 and b_{2i}^1 (as from Lemma 7.42, statement 1, we have that a'_{2i-1} and a'_{2i} together own $n\cdot(1-\alpha_K)$ units of both chores b_{2i-1}^1 and b_{2i}^1). Thus, the total money agents $\cup_{j\in[2n]} a_{j,2i-1}^K$ and $\cup_{j\in[2n]} a_{j,2i}^K$ earn at a competitive equilibrium is the total prices of chores b_{2i-1}^1 and b_{2i}^1 minus the total money earned by agents a'_{2i-1} and a'_{2i}: $(n+n\cdot(1-\alpha_K))\cdot\pi_i^1 - n\cdot(1-\alpha_K)\cdot\pi_i^1 = n\cdot\pi_i^1$. At a competitive equilibrium, the total money that these agents earn must equal the total prices of chores they own. Recall that each agent $a_{\ell,\ell'}^K$ owns $\mathbf{M}_{\ell,\ell'}$ units of b_ℓ^K. Therefore, we have

$$\begin{aligned}
n \cdot \pi_i^1 &= \sum_{j\in[2n]} \mathbf{M}_{j,2i} \cdot p(b_j^K) + \sum_{j\in[2n]} \mathbf{M}_{j,2i-1} \cdot p(b_j^K) \\
&= \sum_{j\in[2n]} (\mathbf{M}_{j,2i} + \mathbf{M}_{j,2i-1}) \cdot p(b_j^K) \\
&= \sum_{j\in[2n]} p(b_j^K) && \text{(using } \mathbf{M}_{j,2i-1} + \mathbf{M}_{j,2i} = 1) \\
&= \sum_{j\in[n]} \pi_j^K
\end{aligned}$$

This implies that $\pi_i^1 = \frac{1}{n}\sum_{j\in[n]} \pi_j^K$. □

Fixed Earning. Here, we show that in every competitive equilibrium, the earning of each agent a'_i for $i \in [2n]$ is fixed.

Lemma 7.44. *For all $i \in [2n]$, we have that the earning of agent a'_i is $(1-\alpha_K) \cdot (2n - \sum_{j\in[2n]} \mathbf{M}_{j,i})$.*

Proof. Let $i = 2i'$. Then agent $a'_{2i'}$ owns $\frac{1}{2} \cdot (1-\alpha_K) \cdot (2n - \sum_{j\in[2n]} \mathbf{M}_{j,2i'})$ units of both chores $b^1_{2i'-1}$ and $b^1_{2i'}$. Since the earning of any agent at a competitive equilibrium equals the sum of prices of chores she owns, we have that the earning of agent $2i'$ is

$$
\begin{aligned}
&= \frac{1}{2} \cdot (1-\alpha_K) \cdot (2n - \sum_{j\in[2n]} \mathbf{M}_{j,2i'}) \cdot (p(b^1_{2i'-1}) + p(b^1_{2i'})) \\
&= \frac{1}{2} \cdot (1-\alpha_K) \cdot (2n - \sum_{j\in[2n]} \mathbf{M}_{j,2i'}) \cdot \pi^1_{i'} \\
&= \frac{1}{2} \cdot (1-\alpha_K) \cdot (2n - \sum_{j\in[2n]} \mathbf{M}_{j,2i'}) \cdot 2 && \text{(by Lemma 7.43).} \\
&= (1-\alpha_K) \cdot (2n - \sum_{j\in[2n]} \mathbf{M}_{j,2i'}).
\end{aligned}
$$

Similarly, when $i = 2i' - 1$ we can show that the total earning of agent $a'_{2i'-1}$ is $(1 - \alpha_K) \cdot (2n - \sum_{j\in[2n]} \mathbf{M}_{j,2i'-1})$. Thus the total earning of any agent a'_i in a competitive equilibrium is $(1-\alpha_K) \cdot (2n - \sum_{j\in[2n]} \mathbf{M}_{j,i})$. □

Price Regulation. Here, we show that for all $k \in [K]$ and $i \in [2n]$ the ratio of the prices of chores b^k_{2i-1} and b^k_{2i} is bounded.

Lemma 7.45. *For all $k \in [K]$ and for all $i \in [n]$, we have $\frac{1-\alpha_k}{1+\alpha_k} \leq \frac{p(b^k_{2i-1})}{p(b^k_{2i})} \leq \frac{1+\alpha_k}{1-\alpha_k}$.*

Proof. We prove the lower bound ($\frac{1-\alpha_k}{1+\alpha_k} \leq \frac{p(b^k_{2i-1})}{p(b^k_{2i})}$) by contradiction. The proof for the upper bound is symmetric. So assume that $\frac{1-\alpha_k}{1+\alpha_k} > \frac{p(b^k_{2i-1})}{p(b^k_{2i})}$. In that case, none of the agents in the connected component D^k_i will do any part of chore b^k_{2i-1} (as the disutility to price ratio of b^k_{2i-1} will be strictly more than that of b^k_{2i}). Since all the other agents have a disutility of ∞ for b^k_{2i-1}, it will remain unallocated. Therefore, the current prices for chores are not the prices corresponding to a competitive equilibrium, which is a contradiction. □

Reverse Ratio Amplification. Lastly, we show the property that when the price of chore b^k_i is at a limit, then the price of chore b^{k+1}_i is at the opposite limit, i.e., when $p(b^k_i) = 1 + \alpha_k$, then we have $p(b^{k+1}_i) = 1 - \alpha_{k+1}$ and similarly when $p(b^k_i) = 1 - \alpha_k$, then we have $p(b^{k+1}_i) = 1 + \alpha_{k+1}$.

Lemma 7.46. *For all $1 \leq k < K$ and $i \in [n]$, we have that,*

(1) if $\frac{p(b^k_{2i-1})}{p(b^k_{2i})} = \frac{1-\alpha_k}{1+\alpha_k}$, then $\frac{p(b^{k+1}_{2i-1})}{p(b^{k+1}_{2i})} = \frac{1+\alpha_{k+1}}{1-\alpha_{k+1}}$, and

(2) if $\frac{p(b^k_{2i-1})}{p(b^k_{2i})} = \frac{1+\alpha_k}{1-\alpha_k}$*, then* $\frac{p(b^{k+1}_{2i-1})}{p(b^{k+1}_{2i})} = \frac{1-\alpha_{k+1}}{1+\alpha_{k+1}}$.

Proof. We just show the proof of part 1. The proof for part 2 is symmetric. Let us assume that $\frac{p(b^k_{2i-1})}{p(b^k_{2i})} = \frac{1-\alpha_k}{1+\alpha_k}$. By Lemma 7.43, we have that $\pi^k_i = p(b^k_{2i-1}) + p(b^k_{2i}) = 2$. Therefore, $p(b^k_{2i-1}) = 1 - \alpha_k$ and $p(b^k_{2i}) = 1 + \alpha_k$. Observe that agent a^k_{2i} owns n units of chore b^k_{2i} and has finite disutility only for the chores b^{k+1}_{2i-1} and b^{k+1}_{2i} (a^k_{2i} belongs in the connected component D^{k+1}_i). Since at a competitive equilibrium, the total earning of agent a^k_{2i} equals the sum of prices of chores she owns, we have that a^k_{2i} earns $n \cdot p(b^k_{2i}) = n(1+\alpha_k)$ amount of money from chores b^{k+1}_{2i-1} and b^{k+1}_{2i}. We claim that it suffices to show that a^k_{2i} earns some of her money from the chore b^{k+1}_{2i-1}: This would immediately imply that $\frac{d(a^k_{2i}, b^{k+1}_{2i-1})}{p(b^{k+1}_{2i-1})} \leq \frac{d(a^k_{2i}, b^{k+1}_{2i})}{p(b^{k+1}_{2i})}$, further implying that $\frac{p(b^{k+1}_{2i-1})}{p(b^{k+1}_{2i})} \geq \frac{1+\alpha_{k+1}}{1-\alpha_{k+1}}$. However, by Lemma 7.45, we have that $\frac{p(b^{k+1}_{2i-1})}{p(b^{k+1}_{2i})} \leq \frac{1+\alpha_{k+1}}{1-\alpha_{k+1}}$, and thus we can conclude that $\frac{p(b^{k+1}_{2i-1})}{p(b^{k+1}_{2i})} = \frac{1+\alpha_{k+1}}{1-\alpha_{k+1}}$.

For the rest of the proof, we show that a^k_{2i} earns positive amount of money from chore b^{k+1}_{2i-1}. We prove this by contradiction. So let us assume that a^k_{2i} earns all her money of $n \cdot p(b^k_{2i}) = n \cdot (1 + \alpha_k)$, only from chore b^{k+1}_{2i}. We will now show that the current prices of chores are not the prices corresponding to a competitive equilibrium by distinguishing between two cases,

- $p(b^{k+1}_{2i}) = 1 + x$ **for some** $x > 0$**:** In this case, we have $p(b^{k+1}_{2i-1}) = 1 - x$ (as $\pi^{k+1}_i = 2$) and therefore $p(b^{k+1}_{2i}) > p(b^{k+1}_{2i-1})$. Observe that in this case, agent $\overline{a}^{k+1}_i$ will also earn all of her money of $\delta_{k+1} \cdot (p(b^{k+1}_{2i-1}) + p(b^{k+1}_{2i-1})) = 2\delta_{k+1}$ from b^{k+1}_{2i} only (as the disutility to price ratio of b^{k+1}_{2i} is strictly smaller than that of b^{k+1}_{2i-1}). Therefore, we have that the total money agents a^k_{2i} and $\overline{a}^{k+1}_i$ earn from b^{k+1}_{2i} is,

$$
\begin{aligned}
&= 2\delta_{k+1} + n \cdot (1 + \alpha_k) \\
&= 2\delta_{k+1} + n \cdot (1 + \frac{2}{3} \cdot \alpha_{k+1}) && (\text{as } \alpha_{k+1} = \frac{3}{2} \cdot \alpha_k) \\
&= n \cdot (1 + \frac{2}{3} \cdot \alpha_{k+1} + \frac{2\delta_{k+1}}{n}) \\
&= n \cdot (1 + \frac{2}{3} \cdot \alpha_{k+1} + \alpha_{k+1}) && (\text{as } \delta_{k+1} = \frac{n}{2} \cdot \alpha_{k+1}) \\
&> n \cdot (1 + \frac{3}{2} \cdot \alpha_{k+1} + \frac{\alpha^2_{k+1}}{2}) && (\text{as } \alpha_{k+1} \ll \frac{1}{3} \text{ by Claim 7.40}) \\
&= n \cdot (1 + \frac{\alpha_{k+1}}{2}) \cdot (1 + \alpha_{k+1}) \\
&= (n + \delta_{k+1}) \cdot (1 + \alpha_{k+1}) && (\text{as } \delta_{k+1} = \frac{n}{2} \cdot \alpha_{k+1}),
\end{aligned}
$$

  which is a contradiction, as the total price of b^{k+1}_{2i} is at most $(n + \delta_{k+1}) \cdot (1 + \alpha_{k+1})$ (there is a total endowment of $n + \delta_{k+1}$ for chore b^{k+1}_{2i} by Lemma 7.42, and $p(b^{k+1}_{2i}) \leq 1 + \alpha_{k+1}$).

- $p(b^{k+1}_{2i}) = 1 - x$ **for** $0 \leq x < \alpha_{k+1}$**:** Since the total endowment of b^{k+1}_{2i} is $n + \delta_{k+1}$

by Lemma 7.42 and $p(b_{2i}^{k+1}) = 1 - x$, the total price of chore b_{2i}^{k+1} is,

$$\begin{aligned}
&= (n + \delta_{k+1}) \cdot (1 - x) \\
&\leq (n + \delta_{k+1}) \\
&< (n + \frac{4}{3}\delta_{k+1}) \\
&= n \cdot (1 + \frac{4\delta_{k+1}}{3n}) \\
&= n \cdot (1 + \frac{2\alpha_{k+1}}{3}) && (\text{as } \delta_{k+1} = \frac{n}{2} \cdot \alpha_{k+1}) \\
&= n \cdot (1 + \alpha_k) && (\text{as } \alpha_{k+1} = \frac{3}{2} \cdot \alpha_k),
\end{aligned}$$

which is the total money that agent a_{2i}^k earns from b_{2i}^{k+1}, which is a contradiction. $\square$

Since K is even, a repeated application of Lemma 7.46 will yield the following lemma,

Lemma 7.47. *We have,*

(1) if $\frac{p(b^1_{2i-1})}{p(b^1_{2i})} = \frac{1-\alpha_1}{1+\alpha_1}$, *then* $\frac{p(b^K_{2i-1})}{p(b^K_{2i})} = \frac{1+\alpha_K}{1-\alpha_K}$, *and*

(2) if $\frac{p(b^1_{2i-1})}{p(b^1_{2i})} = \frac{1+\alpha_1}{1-\alpha_1}$, *then* $\frac{p(b^K_{2i-1})}{p(b^K_{2i})} = \frac{1-\alpha_K}{1+\alpha_K}$.

Now that we have shown that our instance satisfies the desired properties of price equality, fixed earning, price regulation and reverse ratio amplification, we are ready to outline how to determine the equilibrium strategy vector x for the instance I of the polymatrix game, given the competitive equilibrium prices of the instance $E(I)$ of chore division:

$$x_i = \frac{p(b^K_i) - (1-\alpha_K)}{2 \cdot \alpha_K}$$

It is clear that given the prices of chores at a competitive equilibrium, the equilibrium strategy vector can be obtained in linear time. We will now show that x is the desired equilibrium strategy vector for instance I of the polymatrix game.

Lemma 7.48. $x = \langle x_1, x_2, \dots, x_{2n}\rangle$ *is an equilibrium strategy vector for the polymatrix game instance* I.

Proof. First, observe that since our instance satisfies the price equality (Lemma 7.43) and price regulation (Lemma 7.45) we have that for all $i \in [2n]$, $1-\alpha_K \le p(b^K_i) \le 1+\alpha_K$. Therefore, for all $i \in [2n]$ $x_i \ge 0$. Furthermore, for all $i \in [n]$ we have $x_{2i-1} + x_{2i} = \frac{p(b^K_{2i-1})+p(b^K_{2i})-2(1-\alpha_K)}{2\cdot\alpha_K} = \frac{2\alpha_K}{2\alpha_K} = 1$ (as our instance satisfies price equality: by Lemma 7.43 we have $p(b^K_{2i-1})+p(b^K_{2i}) = 2$). Now we will show that if $x^T \cdot \mathbf{M}_{*,2i} > x^T \cdot \mathbf{M}_{*,2i-1} + \frac{1}{n}$, then $x_{2i-1} = 0$. The proof for the other symmetric condition will be similar. So let us assume that $x^T \cdot \mathbf{M}_{*,2i} > x^T \cdot \mathbf{M}_{*,2i-1} + \frac{1}{n}$. Observe that the agents that have a disutility of $1-\alpha_1$ towards chore b^1_{2i} are $\left\{\cup_{j\in[2n]} a^K_{j,2i}\right\} \cup a'_{2i}$. Observe that at a competitive equilibrium, the total earning of the agents $\left\{\cup_{j\in[2n]} a^K_{j,2i}\right\} \cup a'_{2i}$ equals the sum of prices of chores they own, which is,

$$\begin{aligned}
&= \sum_{j\in[2n]} \mathbf{M}_{j,2i} \cdot p(b^K_j) + (1-\alpha_K)\cdot(2n - \sum_{j\in[2n]} \mathbf{M}_{j,2i}) && \text{(by Lemma 7.44)}\\
&= \sum_{j\in[2n]} \mathbf{M}_{j,2i} \cdot (2\alpha_K \cdot x_j + (1-\alpha_K)) + (1-\alpha_K)\cdot(2n - \sum_{j\in[2n]} \mathbf{M}_{j,2i}) && \text{(substituting } p(b^K_j))\\
&= \sum_{j\in[2n]} 2\alpha_K \cdot x_j \cdot \mathbf{M}_{j,2i} + (1-\alpha_K)\cdot \sum_{j\in[2n]} \mathbf{M}_{j,2i} + (1-\alpha_K)\cdot(2n - \sum_{j\in[2n]} \mathbf{M}_{j,2i})\\
&= 2\alpha_K x^T \cdot \mathbf{M}_{*,2i} + 2n\cdot(1-\alpha_K).
\end{aligned}$$

Similarly, the total earning of the agents that have a disutility of $1-\alpha_1$ towards b^1_{2i-1} is $2\alpha_K x^T \cdot \mathbf{M}_{*,2i-1} + 2n\cdot(1-\alpha_K)$. Observe that the agents with disutility $1-\alpha_1$ towards b^1_{2i} can earn all of their money only from the chores b^1_{2i} or b^1_{2i-1} (as these are the only chores towards which they have finite disutility). Also note that both chores b^1_{2i-1} and b^1_{2i} have the same total endowment which is $n + n \cdot (1 - \alpha_K)$ by Lemma 7.42(part 1). Now if, the agents with disutility $1 - \alpha_1$ towards b^1_{2i} earn all of their money, entirely from b^1_{2i} (they earn nothing from b^1_{2i-1}), then we will have $p(b^1_{2i}) \geq \frac{2\alpha_K x^T \cdot \mathbf{M}_{*,2i} + 2n \cdot (1-\alpha_K)}{n + n \cdot (1-\alpha_K)}$ and $p(b^1_{2i-1}) \leq \frac{2\alpha_K x^T \cdot \mathbf{M}_{*,2i-1} + 2n \cdot (1-\alpha_K)}{n + n \cdot (1-\alpha_K)}$. Since, $x^T \cdot \mathbf{M}_{*,2i} > x^T \cdot \mathbf{M}_{*,2i-1} + \frac{1}{n}$. we have $p(b^1_{2i}) > p(b^1_{2i-1}) + \frac{1}{n} \cdot \frac{2\alpha_K}{n + n \cdot (1-\alpha_K)} > p(b^1_{2i-1}) + \frac{\alpha_K}{n^2}$. Again, since $\frac{\alpha_K}{n^2} \gg \alpha_1$ (by Claim 7.40), we have that $\frac{p(b^1_{2i})}{p(b^1_{2i-1})} > \frac{1+\alpha_1}{1-\alpha_1}$, which is a contradiction as our instance satisfies price-regulation property (by Lemma 7.45). Therefore, the agents that have a disutility of $1 - \alpha_1$ towards b^1_{2i} should also earn their money from b^1_{2i-1}. But this is only possible if $\frac{p(b^1_{2i})}{p(b^1_{2i-1})} = \frac{1-\alpha_1}{1+\alpha_1}$. Since our instance also satisfies the reverse ratio amplification, by Lemma 7.47 we have that $\frac{p(b^K_{2i})}{p(b^K_{2i-1})} = \frac{1+\alpha_K}{1-\alpha_K}$. Since $p(b^K_{2i}) + p(b^K_{2i-1}) = 2$ by price equality property (Lemma 7.43), we have that $p(b^K_{2i-1}) = 1 - \alpha_K$. Therefore, we have

$$\begin{aligned} x_{2i-1} &= \frac{(1-\alpha_K) - (1-\alpha_K)}{2 \cdot \alpha_K} \\ &= 0. \end{aligned}$$

A very similar argument will show that when $x^T \cdot \mathbf{M}_{*,2i-1} > x^T \cdot \mathbf{M}_{*,2i} + \frac{1}{n}$, then $x_{2i} = 0$. Thus, $x = \langle x_1, x_2, \ldots, x_n \rangle$ is an equilibrium strategy vector for the polymatrix game I. □

Thus, this immediately implies the main result of this section.

Theorem 7.49. *Let $\mathcal{I}$ be the set of all instances that satisfy Conditions 1 and 2 in Section 7.2. Chore division is PPAD-hard even when restricted to the set of instances $\mathcal{I}$.*

Proof. We bring all the points together. Normalized polymatrix game is PPAD-hard [38]. Given an instance I of the normalized polymatrix game, in polynomial-time we can determine the instance $E(I)$. $E(I)$ satisfies the sufficiency conditions mentioned in Section 7.2 and therefore admits a competitive equilibrium. Given the equilibrium prices for $E(I)$, in polynomial-time we can determine the equilibrium strategy vector for the polymatrix game. Therefore, chore division is PPAD-hard even on instances that satisfy the sufficiency conditions in Section 7.2. □

CHAPTER 8
Outlook

In this thesis, we studied fair and efficient allocation of *indivisible goods* and *divisible bads*.

While studying the setting of indivisible goods, our focus was to determine the existence of good relaxations of EFX allocations with high Nash welfare. In Chapter 3, we showed that even when agents have general valuation functions, we can determine a partial EFX allocation such that no agent envies the set of unallocated goods and less than n goods remain unallocated. Our algorithm starts with any partial EFX allocation and iteratively transforms the existing partial EFX allocation into a *better* EFX allocation, namely a partial EFX allocation that Pareto-dominates the previous partial EFX allocation until the aforementioned properties are satisfied. Furthermore, in Chapter 4, we show that by cleverly choosing the first partial EFX allocation, our algorithm gives all the aforementioned guarantees with high Nash welfare. Although, the results in Chapters 3 and 4 showed the existence of almost EFX allocations with high Nash welfare, the problem of determining *complete* EFX allocations still remained highly non-trivial. In Chapter 5, we were able to prove the existence of EFX allocations when there are three agents with additive valuations through a very involved procedure. Quite recently, Berger et al. [21] built on the techniques developed in Chapter 5 to show the existence of EFX allocations with at most one unallocated good when there are four agents with additive valuations. Thus, the recent works on complete EFX allocations when there is a small number of agents show that substantially improving the upper bounds on the number of unallocated goods in Chapter 3 will be non-trivial. In Chapter 6, we showed that when agents have additive valuations, we can determine a $(1-\varepsilon)$-EFX allocation for any $\varepsilon \in (0, 1/2]$ with sublinearly many unallocated goods and high Nash welfare. We achieved this by reducing the problem of finding EFX allocations with sublinear charity to a combinatorial graph problem, which might be of independent interest.

Despite all the results presented in this thesis, the EFX eistence problem still remains open, even when there are just four agents. In Chapters 5 and 6, we also showed that the techniques developed in this thesis are not sufficient (at least in their current form) to answer the EFX existence question for arbitrary agents. However, we believe that our positive results on the existence of EFX allocations when there are three agents and the existence of good relaxations of EFX allocations may allow us to hope that EFX allocations always exist, at least when agents have structured valuation functions like additive valuation functions. A rudimentary stepping stone would be to give a simpler proof to the existence of EFX allocations when there are three agents with additive valuations (the proof in Chapter 5). To be more precise, given a partial EFX allocation X and an unallocated good g, is there a simpler way to determine an EFX allocation X' such that $\phi(X') > \phi(X)$ for some integral and upper-bounded function ϕ? It might help to impose further constraints on our allocation space: For instance, in all the existence proofs in this thesis, we transformed a partial EFX allocation X to a "better" partial

EFX allocation X' when some properties were not satisfied. It might help to maintain a Pareto-optimal partial EFX allocation (instead of any arbitrary partial EFX allocation), and whenever there is an unallocated good, we transform the current Pareto-optimal partial EFX allocation into a "better" Pareto-optimal partial EFX allocation. Note that if we could maintain this invariant, then none of the update rules from Section 5.2 of Chapter 5 would not be required, as they crucially use the fact that the current partial EFX allocation is not Pareto-optimal. However, the stronger invariant also requires update rules different from those used in Section 5.3 of Chapter 5. There is a subtlety here that may need clarification: In Chapter 2, we show the instance described by Plaut and Roughgarden [84], where no EFX allocation is Pareto-optimal. However, the instance is *degenerate*, i.e., there are two distinct sets of goods that are valued equally by an agent. However, in Chapter 5, we show that w.l.o.g., we can assume that our instances are *non-degenerate*. Therefore, we believe that investigating the existence of Pareto-optimal EFX allocations in non-degenerate instances could be a concrete starting point to answer the EFX existence question for four or more agents.

The existence of EFX allocations remains the primary open problem in the study of discrete fair division. However, there are some other problems which may be more tractable. One avenue may be to show the existence of EFX allocations when agents have more general valuation functions than additive valuation functions. Quite recently, Berger et al. [21] show that with subtle changes, our proof can be used to show the existence of EFX allocations when there are three agents with *nice cancellable valuations*. A valuation v is a nice cancellable valuation if it is non-degenerate ($v(A) \neq v(B)$ for all $A \neq B$) and for all sets $A, B \subseteq M$ and $g \in M \setminus (A \cup B)$, $v(A \cup \{g\}) > v(B \cup \{g\})$ implies $v(A) > v(B)$. It would be interesting to find an existence proof or a counterexample in the three-agent setting when agents have submodular or subadditive valuation functions. Another direction for further research is to seek better approximation guarantees for Nash welfare maximization when agents have submodular valuations. In Chapter 4, we improve the approximation ratio from $\mathcal{O}(n \log(n))$ to $\mathcal{O}(n)$. However, the best known lower bound is $e/(e-1) \approx 1.58$ [59], i.e., there is a gap of $\Omega(n)$ between the upper and lower bounds, that can be closed by future work.

In the fair and efficient allocation of divisible bads, the best division for chores, just like for goods, is arguably the one based on competitive equilibrium with equal incomes (CEEI). Although both settings (goods and bads) seem similar at a high level, the seminal work of Bogomolnoia et al. [22] shows that CEEI with bads exhibits far less structure than that with goods (several disconnected equilibria, exponentially many equilibrium prices). Despite all these algorithmic challenges, neither any hardness result nor a polynomial time algorithm is known. In Chapter 7, we study the chore division problem in the classic linear Arrow-Debreu setting (or equivalently the linear exchange setting), where a set of agents want to divide their divisible chores (bads) amongst themselves to minimize their disutilities (costs). The Arrow-Debreu setting is a generalization of the CEEI setting. Our results on the computational complexity of the Arrow-Debreu setting with chores are in sharp contrast to the results known for goods. We proved that determining whether an arbitrary instance admits a competitive equilibrium is NP-hard, while in the Arrow-Debreu setting with goods, there is a necessary and sufficient condition for the existence of a competitive equilibrium, which is polynomial time verifiable. Furthermore, we formulated simple and natural polynomial-time verifiable sufficiency conditions and show the existence of competitive equilibrium under these conditions. We explained why the fixed-point formulation used to show the existence of competitive equilibrium in the goods setting does not extend to our setting, and we overcame this obstacle through a novel fixed-point formulation which is significantly more involved. Finally, we proved that even for instances that satisfy our sufficiency conditions, determining a competitive equilibrium is PPAD-hard. To the best of our knowledge, these were the first hardness results for any economic model under linear preferences.

Although our hardness results were the first hardness results for any economic model with chores, the complexity of determining a CEEI with chores is still open, which should be of interest to the fair division community. In a very recent paper [32], we were able to show the PPAD-hardness of determining a CEEI with chores, when agents have more general disutility functions (separable, piecewise linear, convex disutilities). However, the problem remains unsolved for agents with linear disutility functions. Additionally, our results only showed PPAD-hardness and we do not see an immediate way to adapt our fixed-point formulation to show PPAD-membership[1]. A PPAD-membership result would settle the complexity of chore division in the Arrow-Debreu setting and therefore provides another interesting opportunity for further research.

Further Avenues. Most of the questions discussed in this thesis assume that all agents are equally entitled. However, several real-world scenarios require division of items among agents with unequal entitlements. Thus, *asymmetric fair division* is a relevant direction for future research. While there has been some work on asymmetric fair division [56, 29, 14, 12, 9], the asymmetric variants of EFX have not been studied to the best of our knowledge. We believe that study of asymmetric EFX and relaxations would require significantly different techniques than the ones developed in this thesis. The crucial differnce lies in the meaning of *envy*: in the symmetric case, an agent i envies j if i prefers j's bundle to her own, while in the asymmetric case, an agent i envies j because she feels that j gets more than what she (agent i) believes is j's fair share, meaning that i may not prefer j's bundle to her own but still envy j. This makes several basic protocols like envy-cycle-elimination that are used heavily in symmetric fair division unusable. We suspect that asymmetric EFX may not exist in general, but good approximations of the asymmetric EFX should be feasible and is a fruitful direction for further investigation.

Another interesting but understudied avenue is *two sided fairness*. A *two sided market* consists of two sets of agents and the agents in one set have preferences over the agents in the other. There are several real-life two sided markets that require fair division. For instance, hosts and guests on Airbnb, drivers and riders in Uber etc.. In contrast to the one-sided markets (the ones we have considered in this dissertation) that are inherently *decentralized*, most of the two sided markets are operated by centralized electronic platforms (Airbnb matching hosts and guests, Uber matching drivers and riders) and thus provide more scope of employing the fair division protocols. We are aware of only two papers [65, 83] that consider two sided fairness and we believe that there is substantial room for future research.

Finally, we feel that it would be great to investigate the applications and significance

[1] Computation of fixed-points of *polynomial piecewise linear functions* is in PPAD [54]

of the classic fairness concepts and tehniques in all domains that aim for collective decision making involving a set of agents that have preferences.

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