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Non-hyperoctahedral categories of two-colored partitions part I: new categories

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Abstract

Compact quantum groups can be studied by investigating their representation categories in analogy to the Schur–Weyl/Tannaka–Krein approach. For the special class of (unitary) "easy" quantum groups, these categories arise from a combinatorial structure: rows of two-colored points form the objects, partitions of two such rows the morphisms. Vertical/horizontal concatenation and reflection give composition, monoidal product and involution. Of the four possible classes $\mathcal{O}$, $\mathcal{B}$, $\mathcal{S}$ and $\mathcal{H}$ of such categories (inspired, respectively, by the classical orthogonal, bistochastic, symmetric and hyperoctahedral groups), we treat the first three—the non-hyperoctahedral ones. We introduce many new examples of such categories. They are defined in terms of subtle combinations of block size, coloring and non-crossing conditions. This article is part of an effort to classify all non-hyperoctahedral categories of two-colored partitions. It is purely combinatorial in nature. The quantum group aspects are left out.

Keywords Quantum group · Unitary easy quantum group · Unitary group · Half-liberation · Tensor category · Two-colored partition · Partition of a set · Category of partitions · Brauer algebra

Mathematics Subject Classification Primary 05A18; Secondary 20G42

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1 Introduction

In Woronowicz's approach [15–17], (compact) quantum groups are understood as certain non-commutative spaces, the formal duals of C^*-algebras, carrying a special Hopf algebra structure, for which a non-commutative version of Pontryagin duality can be proven. Reminiscent of the theorems of Tannaka–Krein and Schur–Weyl, a duality exists between the class of compact quantum groups and a particular class of involutive monoidal linear categories. The finite-dimensional unitary representations of a given compact quantum group form such a category, and, conversely, a unique maximal compact quantum group can be reconstructed from any such tensor category [16].

Banica and Speicher [3] showed that sets of points as objects and partitions of finite sets as morphisms with vertical concatenation as composition, horizontal concatenation as monoidal product and reflection as involution provide concrete, combinatorial initial data for such representation categories. Their construction yields compact quantum subgroups of the free orthogonal quantum group O_n^+ introduced by Wang [13] as a non-commutative counterpart of the classical group O_n of orthogonal real-valued matrices. All examples of compact quantum groups arising in this fashion, the so-called easy quantum groups, have since been classified [1,3,8,10,14].

As Freslon, Tarrago and the second author demonstrated [4,11,12], Banica and Speicher's approach can be generalized to categories of partitions of sets of *two-colored* points. In contrast to the uncolored case, here, vertical concatenation of partitions, i.e., the composition of morphisms, is restricted to such partitions with matching colorings of their points. This construction yields combinatorial compact quantum subgroups of the free unitary quantum group U_n^+, a quantum analogue of the classical unitary group U_n also introduced by Wang [13]. A collective endeavor to find all such "unitary easy" quantum groups was initiated by Tarrago and the second author in [12] and has since been advanced by Gromada in [5] as well as by the authors in [7] and [6].

The classification of all unitary easy quantum groups has been approached from several different angles of attack. Tarrago and the second author classified in [12] all *non-crossing* categories $\mathcal{C} \subseteq \mathcal{P}^{\circ\bullet}$ of two-colored partitions, i.e., $\mathcal{C} \subseteq \mathcal{NC}^{\circ\bullet}$, and all categories $\mathcal{C}$ of two-colored partitions with $\in \mathcal{C}$, the so-called *group case*. In contrast, in [5] Gromada determined all categories $\mathcal{C}$ with the property of being *globally colorized*, meaning $\otimes$ $\in \mathcal{C}$. Lastly, the authors of the present article found all categories $\mathcal{C}$ with $\langle \emptyset \rangle \subseteq \mathcal{C} \subseteq \langle \rangle$, i.e., categories of *neutral pair partitions*, corresponding to easy compact quantum groups G with $U_n^+ \supseteq G \supseteq U_n$, the "unitary half-liberations", in [7] and [6].

The present article is concerned with *non-hyperoctahedral* categories of two-colored partitions, i.e., categories $\mathcal{C} \subseteq \mathcal{P}^{\circ\bullet}$ with $\in \mathcal{C}$ or $\notin \mathcal{C}$. We define explicitly certain sets of partitions and show that each of them constitutes a non-hyperoctahedral category. See the next section for an overview.

This article is part of an effort to classify all non-hyperoctahedral categories of two-colored partitions. In subsequent articles it will be shown that the categories found in the present article are pairwise distinct and actually constitute *all* possible non-hyperoctahedral categories. Furthermore, a set of generating partitions for each non-hyperoctahedral category will be determined.

About *hyperoctahedral* categories of two-colored partitions, very little is known for the moment. Note that in the uncolored case categories $\mathcal{C} \subseteq \mathcal{P}$ with $\uparrow\otimes\uparrow \notin \mathcal{C}$ and $\sqcap\!\sqcap\!\sqcap \in \mathcal{C}$ give rise to quantum subgroups of the free hyperoctahedral quantum group H_n^+ [2], hence the name.

2 Main result

Many new examples of non-hyperoctahedral categories of two-colored partitions are provided. Roughly, they are determined by combinations of constraints on the

(i) sizes of blocks,
(ii) coloring of the points,
(iii) allowed crossings between blocks.

A bit more precisely: The coloring of any two-colored partition $p \in \mathcal{P}^{\circ\bullet}$ induces on the set of points a measure-like structure, the *color sum* σ_p, and a metric-like one, the *color distance* δ_p. Measuring the set of all points yields the *total color sum* $\Sigma(p)$.

Let now $\mathcal{S} \subseteq \mathcal{P}^{\circ\bullet}$ be an arbitrary set of two-colored partitions and consider the following data:

(1) The set of block sizes:

$$F(\mathcal{S}) := \{\, |B| \mid p \in \mathcal{S},\ B \text{ block of } p\}.$$

(2) The set of block color sums:

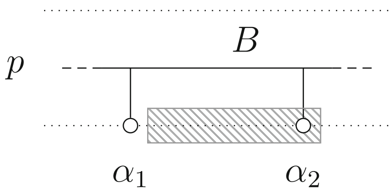
$$V(\mathcal{S}) := \{\sigma_p(B) \mid p \in \mathcal{S},\ B \text{ block of } p\}.$$

(3) The set of total color sums:

$$\Sigma(\mathcal{S}) := \{\, \Sigma(p) \mid p \in \mathcal{S}\}.$$

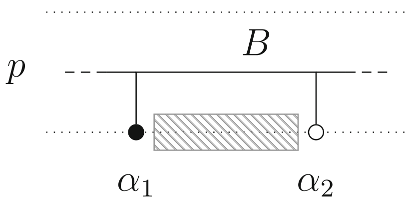
(4) The set of color distances between any two subsequent legs of the same block having the *same* normalized color:

$$L(\mathcal{S}) := \{\, \delta_p(\alpha_1, \alpha_2) \mid p \in \mathcal{S},\ B \text{ block of } p,\ \alpha_1, \alpha_2 \in B,\ \alpha_1 \neq \alpha_2, \\ ]\alpha_1, \alpha_2[_p \cap B = \emptyset,\ \sigma_p(\{\alpha_1, \alpha_2\}) \neq 0\}.$$



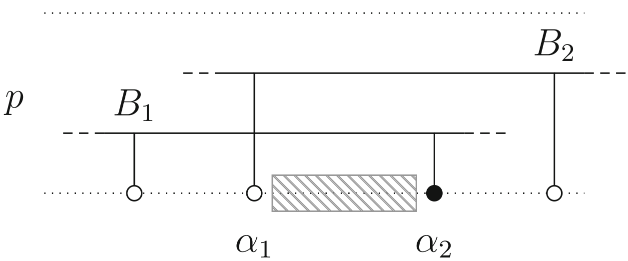
(5) The set of color distances between any two subsequent legs of the same block having *different* normalized colors:

$$K(\mathcal{S}) := \{\,\delta_p(\alpha_1, \alpha_2) \mid p \in \mathcal{S},\ B \text{ block of } p,\ \alpha_1, \alpha_2 \in B,\ \alpha_1 \neq \alpha_2,\\ ]\alpha_1, \alpha_2[_p \cap B = \emptyset,\ \sigma_p(\{\alpha_1, \alpha_2\}) = 0\}.$$



(6) The set of color distances between any two legs belonging to two crossing blocks

$$X(\mathcal{S}) := \{\,\delta_p(\alpha_1, \alpha_2) \mid p \in \mathcal{S},\ B_1, B_2 \text{ blocks of } p,\ B_1 \text{ crosses } B_2,\\ \alpha_1 \in B_1,\ \alpha_2 \in B_2\}.$$



It is a subtle question which combinations of conditions on these six quantities define categories of partitions. We clarify it and we obtain a huge variety of new categories of partitions.

Main Theorem (Theorem 6.20) *For any tuple* (f, v, s, l, k, x) *listed in the table below, where* $u \in \{0\} \cup \mathbb{N}$, $m \in \mathbb{N}$, $D \subseteq \{0, \ldots, \lfloor \frac{m}{2} \rfloor\}$, $E \subseteq \{0\} \cup \mathbb{N}$ *and where* N *is any subsemigroup of* $(\mathbb{N}, +)$*, a non-hyperoctahedral category of two-colored partitions is given by*

$$\mathcal{R}_{(f,v,s,l,k,x)} := \{p \in \mathcal{P}^{\circ\bullet} \mid F(\{p\}) \subseteq f,\ V(\{p\}) \subseteq v,\ \Sigma(\{p\}) \subseteq s,\\ L(\{p\}) \subseteq l,\ K(\{p\}) \subseteq k,\ X(\{p\}) \subseteq x\}.$$

no.	f	v	s	l	k	x
(1)	$\{2\}$	$\pm\{0,2\}$	$2um\mathbb{Z}$	$m\mathbb{Z}$	$m\mathbb{Z}$	$\mathbb{Z}$
(2)	$\{2\}$	$\pm\{0,2\}$	$2um\mathbb{Z}$	$m+2m\mathbb{Z}$	$2m\mathbb{Z}$	$\mathbb{Z}$
(3)	$\{2\}$	$\pm\{0,2\}$	$2um\mathbb{Z}$	$m+2m\mathbb{Z}$	$2m\mathbb{Z}$	$\mathbb{Z}\backslash m\mathbb{Z}$
(4)	$\{2\}$	$\{0\}$	$\{0\}$	$\emptyset$	$m\mathbb{Z}$	$\mathbb{Z}$
(5)	$\{2\}$	$\pm\{0,2\}$	$\{0\}$	$\{0\}$	$\{0\}$	$\mathbb{Z}\backslash\pm N$
(6)	$\{2\}$	$\{0\}$	$\{0\}$	$\emptyset$	$\{0\}$	$\mathbb{Z}\backslash\pm N$
(7)	$\{2\}$	$\{0\}$	$\{0\}$	$\emptyset$	$\{0\}$	$\mathbb{Z}\backslash\{0\}\backslash\pm N$
(8)	$\{1,2\}$	$\pm\{0,1,2\}$	$um\mathbb{Z}$	$m\mathbb{Z}$	$m\mathbb{Z}$	$\mathbb{Z}\backslash(\pm D+m\mathbb{Z})$
(9)	$\{1,2\}$	$\pm\{0,1,2\}$	$2um\mathbb{Z}$	$m+2m\mathbb{Z}$	$2m\mathbb{Z}$	$\mathbb{Z}\backslash(\pm D+m\mathbb{Z})$
(10)	$\{1,2\}$	$\pm\{0,1\}$	$um\mathbb{Z}$	$\emptyset$	$m\mathbb{Z}$	$\mathbb{Z}\backslash(\pm D+m\mathbb{Z})$
(11)	$\{1,2\}$	$\pm\{0,1,2\}$	$\{0\}$	$\{0\}$	$\{0\}$	$\mathbb{Z}\backslash\pm E$
(12)	$\{1,2\}$	$\pm\{0,1\}$	$\{0\}$	$\emptyset$	$\{0\}$	$\mathbb{Z}\backslash\pm E$
(13)	$\mathbb{N}$	$\mathbb{Z}$	$um\mathbb{Z}$	$m\mathbb{Z}$	$m\mathbb{Z}$	$\mathbb{Z}\backslash(\pm D+m\mathbb{Z})$
(14)	$\mathbb{N}$	$\mathbb{Z}$	$\{0\}$	$\{0\}$	$\{0\}$	$\mathbb{Z}\backslash\pm E$

Here, $\pm S := S\cup(-S)$ *for any set* $S\subseteq\mathbb{Z}$*.*

Some of these correspond to the non-hyperoctahedral categories previously discovered in [12], [5], [7] and [6] (see Sect. 7).

A lot about the order structure of the set of the categories from the Main Theorem can be read off of the description given there: Whenever each of the six entries of one tuple $Q=(f,v,s,l,k,x)$ is a subset of the corresponding entry of another Q', then $\mathcal{R}_Q\subseteq\mathcal{R}_{Q'}$ (see Lemma 6.2).

There are many ways of arranging the categories $\mathcal{R}_{(f,v,s,l,k,x)}$ into families. The particular presentation in the Main Theorem was chosen mainly because it was the shortest to write down which the authors could come up with.

Despite the complex order structure of the set of categories given in the Main Theorem, a few salient patterns shall be pointed out: The first entry f of a tuple (f,v,s,l,k,x) decides whether the category $\mathcal{R}_{(f,v,s,l,k,x)}$ belongs to case $\mathcal{O}$ ("orthogonal", corresponding to $f=\{2\}$), to case $\mathcal{B}$ ("bistochastic", $f=\{1,2\}$) or to case $\mathcal{S}$ ("symmetric", $f=\mathbb{N}$). The $\mathcal{O}$-$\mathcal{B}$-$\mathcal{S}$-$\mathcal{H}$ distinction (where $\mathcal{H}$ stands for "hyperoctahedral") originates on the (quantum) group side of the theory and refers to the four classical matrix groups O_N, B_N, S_N and H_N of the respective names.

Another eye-catching division is that into those categories $\mathcal{R}_{(f,v,s,l,k,x)}$ with $k\neq\{0\}$ and those with $k=\{0\}$. The former have a "periodical" nature: The parameters $\mathbb{Z}\backslash x$ are congruence classes with respect to some positive integer m, the "period". In contrast, categories with $k=\{0\}$ know no such constraint, they are "aperiodical".

Among the periodical categories, distinguishing between the different values of the "period" m yields another coarse-graining. This systematization is similar to the distinction between globally and locally colorized categories [12, Definition 2.3], i.e., between the categories which contain $\sqcap_{\circ\circ}\otimes\sqcap_{\bullet\bullet}$ and those which do not: $\mathcal{R}_{(f,v,s,l,k,x)}$ is globally colorized if and only if $1\in l$. In all these cases the period is of course 1. And the only way m may be equal to 1 without $\mathcal{R}_{(f,v,s,l,k,x)}$ being globally colorized is that $l=\emptyset$.

With that, the third and final major distinction to be pointed out here has been touched upon. It consists in separating the family into those categories $\mathcal{R}_{(f,v,s,l,k,x)}$ with $l = \emptyset$ and those with $l \neq \emptyset$. In the former case, any two subsequent legs of the same block must alternate in color. In the special case of $f = \{2\}$ this means that the category must consist of neutral pair partitions, i.e., be a subcategory of $\langle\ \rangle$.

3 Basic definitions: partitions

Details on as well as examples and illustrations of two-colored partitions and their categories can be found in [12]. However, we quickly recall the basics. In addition, certain definitions from [6,7] are given here in greater generality.

After revisiting the fundamental definition of two-colored partitions, specialized language is introduced to describe them more easily, in particular the concepts of *orientation*, *normalized color*, *color sum* and *color distance*.

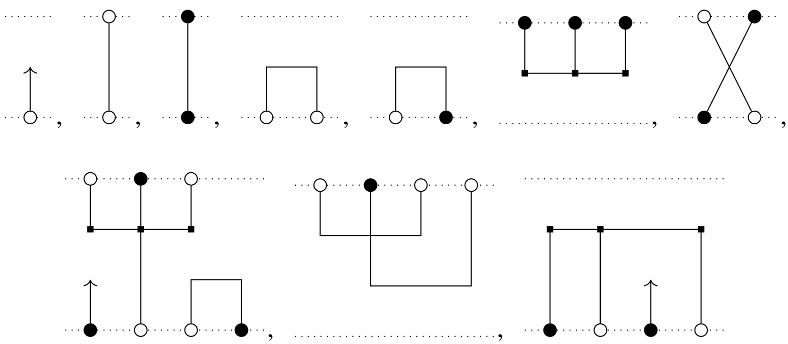
Notation 3.1

(a) For the remainder of the article fix a pair of two (arbitrarily chosen) injections ${}_{\blacksquare}(\,\cdot\,)$ and ${}^{\blacksquare}(\,\cdot\,)$ with common domain $\mathbb{N}$ and disjoint images.
(b) For any $\{k, \ell\} \subseteq \{0\} \cup \mathbb{N}$, let $\mathrm{L}_\ell^k := \{{}_{\blacksquare}i \mid i \in \mathbb{N},\ i \leq \ell\} \cup \{{}^{\blacksquare}j \mid j \in \mathbb{N},\ j \leq k\}$.
(c) Finally, let $\circ$ and $\bullet$ be any sets with $\circ \neq \bullet$.

3.1 Two-colored partitions

Recall that a *set-theoretical partition* of any set X is any set of pairwise disjoint non-empty subsets of X whose union is X. In this article we will strictly distinguish between "partitions" in this sense and the one we define next.

For any $\{k, \ell\} \subseteq \{0\} \cup \mathbb{N}$ a *(two-colored)* (k,ℓ)*-partition*, p is specified by the following data: 1) a set-theoretical partition, the *blocks* of p, of the set L_ℓ^k, and 2) a mapping $\mathrm{L}_\ell^k \to \{\circ, \bullet\}$, the *coloring* of p. The *empty partition* $\emptyset$, given by the empty set, is the only $(0,0)$-partition. The set of all (k,ℓ)-partitions is denoted by $\mathcal{P}^{\circ\bullet}(k,\ell)$. Moreover, let $\mathcal{P}^{\circ\bullet} := \bigcup_{\{k,\ell\}\subseteq\{0\}\cup\mathbb{N}} \mathcal{P}^{\circ\bullet}(k,\ell)$. We represent partitions pictorially as follows:



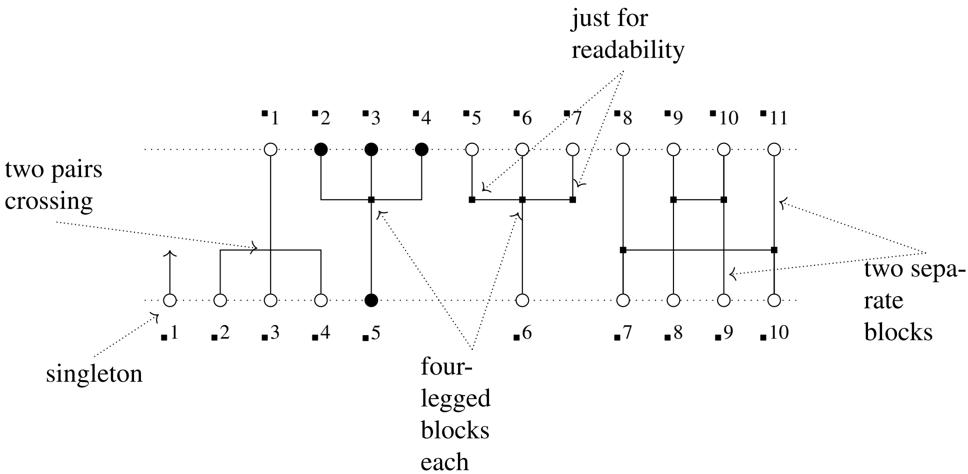
Given $p \in \mathcal{P}^{\circ\bullet}(k,\ell)$, we say that L^0_ℓ is the *lower row* of p, containing its *lower points*, and that L^k_0 is its *upper row*, given by the set of its *upper points*. Altogether, we speak of the elements of L^k_ℓ as the *points* of p. We consider each non-empty row of p to be equipped with the total order induced by $\mathbb{N}$, its *linear order*. If we speak of the *rank* of a point, then in reference to these orders, in each row we say that lower ranks are to the *left* of higher ranks (and, correspondingly, higher ranks to the *right* of lower ranks). Moreover, an upper point and a lower one of the same rank are said to be *opposites*. The same goes for any pair of sets covering the same ranks on different rows.

If a set B is a block of p, we write $B \in p$. The elements of a block are called its *legs*. If a block contains both upper and lower points, we speak of a *through block*. If on the other hand a block is confined to one row, it is said to be *non-through*. And we speak of *lower* and *upper non-through blocks*, respectively, depending on which row it is.

The number of legs of a block is its *size*. Blocks of size one are called *singletons*, blocks of size two *pairs*. A partition with only pairs is called a *pair partition*. We denote by $\mathcal{P}^{\circ\bullet}_2$ the set of all pair partitions and by $\mathcal{P}^{\circ\bullet}_{\leq 2}$ the set of all partitions all of whose blocks are singletons or pairs.

The coloring of a partition is said to assign to each point its *color*, either $\bullet$ or $\circ$. We call the two colors $\bullet$ and $\circ$ *inverse* to each other and write $\overline{\circ} := \bullet$ and $\overline{\bullet} := \circ$, accordingly.

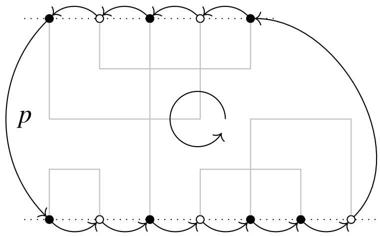
Here is an overview of the graphical representation of partitions:



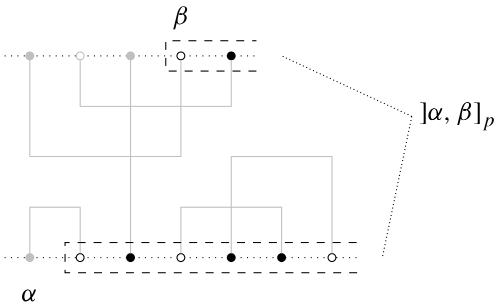
3.2 Order and intervals

For any $\{k,\ell\} \subseteq \{0\} \cup \mathbb{N}$ and any $p \in \mathcal{P}^{\circ\bullet}(k,\ell)$, each row of p is already equipped with a natural total linear order, namely the one given by ascending rank. Now, we endow the entire set of points of p with a total *cyclic order* by specifying its successor function:

It is defined by, on the one hand, ${}_{\blacksquare}i \mapsto {}_{\blacksquare}(i+1)$ for any $i < \ell$ and ${}^{\blacksquare}j \mapsto {}^{\blacksquare}(j-1)$ for any $1 < j$, and, on the other hand, by ${}^{\blacksquare}1 \mapsto {}_{\blacksquare}1$ and ${}_{\blacksquare}\ell \mapsto {}^{\blacksquare}k$ if $k > 0$ and $\ell > 0$, by ${}^{\blacksquare}1 \mapsto {}^{\blacksquare}k$ if $\ell = 0 < k$ and by ${}_{\blacksquare}\ell \mapsto {}_{\blacksquare}1$ if $k = 0 < \ell$. That means p carries the counter-clockwise orientation.



The terms *successor*, *predecessor* and *neighbor* always refer to the cyclic order. With respect to the cyclic order on the points of $p \in \mathcal{P}^{\circ\bullet}(k, \ell)$ it makes sense to speak of *intervals* $]\alpha, \beta[_p$, $]\alpha, \beta]_p$, $[\alpha, \beta[_p$ and $[\alpha, \beta]_p$ for any two points α and β of p with, importantly, $\alpha \neq \beta$.

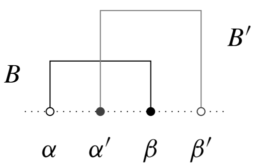


We call a set S of points *consecutive* if S is empty, an interval, a singleton set or the complement of a singleton set.

3.3 Ordered tuples and crossings

We can extend the notion of intervals to tuples of more than two points: For $n \geq 3$ many pairwise distinct points $\alpha_1, \ldots, \alpha_n$ in $p \in \mathcal{P}^{\circ\bullet}$ we say that the tuple $(\alpha_1, \ldots, \alpha_n)$ is *ordered* in p if for all $\{i, j, k\} \subseteq \{1, \ldots, n\}$ with $i < j$ the set $]\alpha_i, \alpha_j[_p$ contains α_k if and only if $i < k < j$. In fact, we can even talk about tuples of pairwise disjoint consecutive sets being ordered.

We say that two distinct blocks B and B' of p *cross* each other if there exist points α, β, α' and β' in p such that the tuple $(\alpha, \alpha', \beta, \beta')$ is ordered and such that $\{\alpha, \beta\} \subseteq B$ and $\{\alpha', \beta'\} \subseteq B'$.



If p has no crossing blocks, we call p a *non-crossing* partition. The set of all non-crossing partitions is denoted by $\mathcal{NC}^{\circ\bullet}$.

3.4 Normalized color and color sum

Just like the cyclic order is often more convenient than the linear orderings of the rows, it is useful to consider besides the original, *native* coloring of the points a second one: By the *normalized color* of a point α in $p \in \mathcal{P}^{\circ\bullet}$ we mean simply its native color in case α is a lower point, but the inverse of its native color if α is an upper point.

If P_p is the set of all points of p, we call the signed measure σ_p on P_p which assigns 1 to normalized $\circ$ and -1 to normalized $\bullet$ the *color sum* of p. Null sets of σ_p are also referred to as *neutral*.

The color sum $\Sigma(p) := \sigma_p(P_p)$ of the set P_p of all points of p is called the *total color sum* of p.

3.5 Color distance

Besides the color sum, a measure-like structure on the points of a partition, the coloring of the partition also induces a metric-like one: Given two points α and β in $p \in \mathcal{P}^{\circ\bullet}$, we call

$$\delta_p(\alpha, \beta) := \begin{cases} \Sigma(p) & \text{if } \alpha = \beta, \\ \sigma_p(]\alpha, \beta[_p) & \text{if } \alpha \neq \beta \text{ and } \alpha, \beta \text{ have different normalized colors,} \\ \sigma_p(]\alpha, \beta]_p) & \text{if } \alpha \neq \beta \text{ and } \alpha, \beta \text{ have the same normalized color,} \end{cases}$$

the *color distance* from α to β in p. For $\alpha \neq \beta$ this means

$$\delta_p(\alpha, \beta) = \tfrac{1}{2}\sigma_p(\{\alpha\}) + \sigma_p(]\alpha, \beta[_p) + \tfrac{1}{2}\sigma_p(\{\beta\}).$$

The map δ_p indeed has properties of a “distance”.

Lemma 3.2 *For any points α, β and γ in any $p \in \mathcal{P}^{\circ\bullet}$,*

(a) $\delta_p(\alpha, \alpha) \equiv 0 \mod \Sigma(p)$,
(b) $\delta_p(\alpha, \beta) \equiv -\delta_p(\beta, \alpha) \mod \Sigma(p)$, *and*
(c) $\delta_p(\alpha, \gamma) \equiv \delta_p(\alpha, \beta) + \delta_p(\beta, \gamma) \mod \Sigma(p)$.

Proof.

(a) The first claim is part of the definition of δ_p.
(b) By Part (a) we can assume $\alpha \neq \beta$. Rewrite the definition of δ_p as

$$\delta_p(\alpha, \beta) = \sigma_p(]\alpha, \beta]_p) + \tfrac{1}{2}\left(\sigma_p(\{\alpha\}) - \sigma_p(\{\beta\})\right).$$

Using $\sigma_p(]\alpha, \beta]_p) \equiv -\sigma_p(]\beta, \alpha]_p) \mod \Sigma(p)$ now proves the claim.

(c) Again, we can suppose α, β and γ are pairwise distinct. Otherwise, Parts (a) and (b) already prove the claim. Now compute, employing the formula for δ_p from the proof of Claim (b),

$$\begin{aligned}\delta_p(\alpha,\beta)+\delta_p(\beta,\gamma) &= \sigma_p(]\alpha,\beta]_p)+\sigma_p(]\beta,\gamma]_p)\\ &\quad+\tfrac{1}{2}\left(\sigma_p(\{\alpha\})-\sigma_p(\{\beta\})\right)+\tfrac{1}{2}\left(\sigma_p(\{\beta\})-\sigma_p(\{\gamma\})\right).\end{aligned}$$

The claim then follows from $\sigma_p(]\alpha,\beta]_p)+\sigma_p(]\beta,\gamma]_p) \equiv \sigma_p(]\alpha,\gamma]_p) \mod \Sigma(p)$. □

4 Basic definitions: categories of partitions

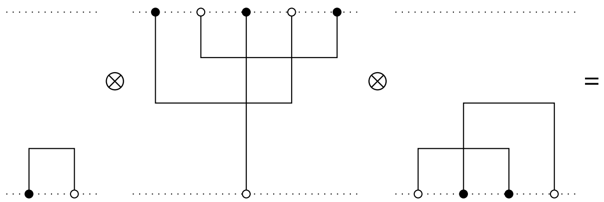
The definition of two-colored partitions recalled, we recapitulate the definitions of *operations* for partitions and of *categories*. Again, see [12] for more.

4.1 Fundamental operations on partitions

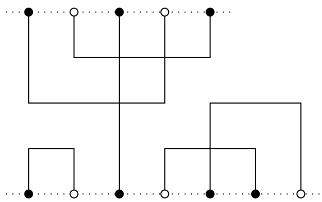
There are three basic operations we execute on two-colored partitions: tensor product, involution and composition.

4.1.1 Tensor product

For any $\{k_1, k_2, \ell_1, \ell_2\} \subseteq \{0\} \cup \mathbb{N}$ and any partitions $p_1 \in \mathcal{P}^{\circ\bullet}(k_1, \ell_1)$ and $p_2 \in \mathcal{P}^{\circ\bullet}(k_2, \ell_2)$, if τ is the back-shift $\mathrm{L}_{\ell_1+\ell_2}^{k_1+k_2} \backslash \mathrm{L}_{\ell_1}^{k_1} \to \mathrm{L}_{\ell_2}^{k_2}$ given by ${}_{\blacksquare}(\ell_1+i) \mapsto {}_{\blacksquare}i$ and ${}^{\blacksquare}(k_1+j) \mapsto {}^{\blacksquare}j$, then the *tensor product* of (p_1, p_2) is defined as the partition $p_1 \otimes p_2 \in \mathcal{P}^{\circ\bullet}(k_1+k_2, \ell_1+\ell_2)$ whose blocks are $\{B_1 \mid B_1 \in p_1\} \cup \{\tau^{-1}(B_2) \mid B_2 \in p_2\}$ and whose coloring is $c_1 \cup (c_2 \circ \tau)$, where c_1 is the coloring of p_1 and c_2 that of p_2. Less formally, we append p_2 to the right of p_1. This is an associative operation. Especially, we can write tensor powers like $p^{\otimes n}$ given by $p \otimes \ldots \otimes p$ with n factors. And we define $p^{\otimes 0} := \emptyset$.



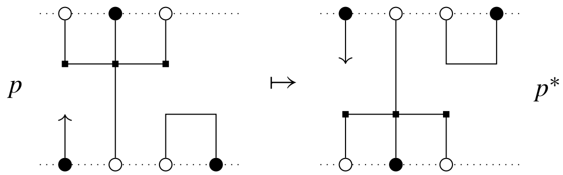
p_1 p_2 p_3



$p_1 \otimes p_2 \otimes p_3$

4.1.2 Involution

If $\{k, \ell\} \subseteq \{0\} \cup \mathbb{N}$, if $p \in \mathcal{P}^{\circ\bullet}(k, \ell)$ has coloring c and if ϱ is the exchange $\mathrm{L}_\ell^k \to \mathrm{L}_k^\ell$ given by ${}_{\blacksquare} j \mapsto {}^{\blacksquare} j$ and ${}^{\blacksquare} i \mapsto {}_{\blacksquare} i$, then the *involution* or *adjoint* of p is defined as the partition $p^* \in \mathcal{P}^{\circ\bullet}(\ell, k)$ with blocks $\{\varrho^{-1}(B) \mid B \in p\}$ and colors $c \circ \varrho$. In other words, the operation swaps the roles of upper and lower rows. Since, obviously, $(p^*)^* = p$ the name "involution" is justified.



4.1.3 Composition

Recall that the set of set-theoretical partitions of any given set forms a complete lattice with respect to the partial order defined by $s \leq s'$ if and only if for any $D \in s$ there exists $D' \in s'$ such that $D \subseteq D'$. In particular, it makes sense to speak of *joins* (least upper bounds).

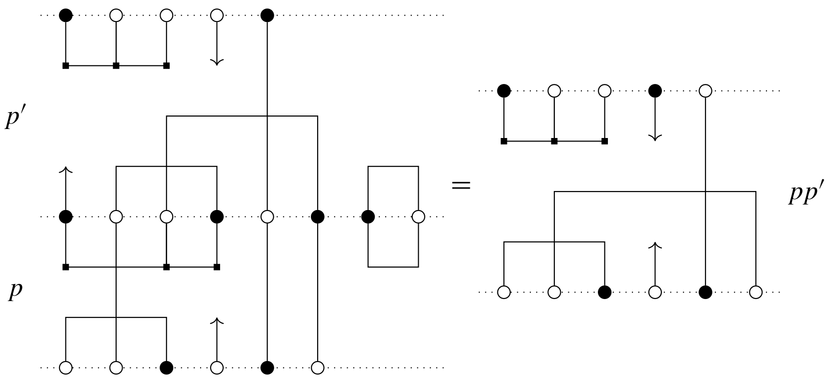
For any two partitions p with coloring c and p' with coloring c', we say that the pairing (p, p') is *composable* if there are $\{k, \ell, m\} \subseteq \{0\} \cup \mathbb{N}$ such that $p \in \mathcal{P}^{\circ\bullet}(\ell, m)$ and $p' \in \mathcal{P}^{\circ\bullet}(k, \ell)$ and such that $c({}^{\blacksquare} j) = c'({}_{\blacksquare} j)$ for each $j \in \mathbb{N}$ with $j \leq \ell$.

If so, then the *composition* $pp' \in \mathcal{P}^{\circ\bullet}(k, m)$ of (p, p') is the partition whose coloring is given by the union of c restricted to L_m^0 and c' restricted to L_0^k and whose blocks are given by

$$\{B \mid B \in p \text{ and } B \subseteq \mathrm{L}_m^0\} \cup \{B' \mid B' \in p' \text{ and } B' \subseteq \mathrm{L}_0^k\}$$

$$\cup \left\{ \left(\left(\mathrm{L}_m^0 \cap \bigcup_{\substack{B \in p \\ B \cap \chi^{-1}(D) \neq \emptyset}} B \right) \cup \left(\mathrm{L}_0^k \cap \bigcup_{\substack{B' \in p' \\ B' \cap D \neq \emptyset}} B' \right) \right) \,\middle|\, D \in s \right\} \backslash \{\emptyset\},$$

where $\chi : \mathrm{L}_\ell^0 \to \mathrm{L}_0^\ell,\ {}_{\blacksquare}j \mapsto {}^{\blacksquare}j$ and where s is the join of the set-theoretical partitions $\{\chi^{-1}(B \cap \mathrm{L}_0^\ell) \mid B \in p\} \backslash \{\emptyset\}$ and $\{B' \cap \mathrm{L}_\ell^0 \mid B' \in p'\} \backslash \{\emptyset\}$ of L_ℓ^0. In the graphical representation, composition roughly translates as "continuing" the block-representing strings across the two diagrams.



Since forming joins of set-theoretical partitions is associative, the composition of two-colored partitions is an associative (partial) operation as well.

4.2 Categories of two-colored partitions

We call any set $\mathcal{C} \subseteq \mathcal{P}^{\circ\bullet}$ a *category* if it contains the partitions $\emptyset$, $\uparrow^{\circ}$, $\uparrow^{\bullet}$, $\sqcap^{\bullet\circ}$ and $\sqcap^{\circ\bullet}$ and is closed under tensor products, involution and composition of composable pairings [12, Section 1.3], building on [3, Definition 2.2]. The set of all categories of two-colored partitions is denoted by $\mathsf{PCat}^{\circ\bullet}$.

For every set $\mathcal{G} \subseteq \mathcal{P}^{\circ\bullet}$ we write $\langle \mathcal{G} \rangle$ for the smallest category (with respect to $\subseteq$) which contains $\mathcal{G}$. We say that $\mathcal{G}$ *generates* $\langle \mathcal{G} \rangle$. If $\mathcal{G} = \{p\}$ for some $p \in \mathcal{P}^{\circ\bullet}$, we slightly abuse notation by writing $\langle p \rangle$ instead of $\langle \{p\} \rangle$ for $\langle \mathcal{G} \rangle$. Also, we mix the two, writing, $\langle \mathcal{G}, q \rangle$ for $q \in \mathcal{P}^{\circ\bullet}$ instead of $\langle \mathcal{G} \cup \{p\} \rangle$.

Definition 4.1 [12, Definition 2.2] We say that a category $\mathcal{C} \subseteq \mathcal{P}^{\circ\bullet}$ is in *case*

(a) $\mathcal{O}$ if $\uparrow^{\circ} \otimes \uparrow^{\bullet} \notin \mathcal{C}$ and $\sqcap\sqcap\sqcap^{\circ\bullet\circ\bullet} \notin \mathcal{C}$,
(b) $\mathcal{B}$ if $\uparrow^{\circ} \otimes \uparrow^{\bullet} \in \mathcal{C}$ and $\sqcap\sqcap\sqcap^{\circ\bullet\circ\bullet} \notin \mathcal{C}$,
(c) $\mathcal{S}$ if $\uparrow^{\circ} \otimes \uparrow^{\bullet} \in \mathcal{C}$ and $\sqcap\sqcap\sqcap^{\circ\bullet\circ\bullet} \in \mathcal{C}$,
(d) $\mathcal{H}$ if $\uparrow^{\circ} \otimes \uparrow^{\bullet} \notin \mathcal{C}$ and $\sqcap\sqcap\sqcap^{\circ\bullet\circ\bullet} \in \mathcal{C}$.

In this series of articles, we are only interested in the first three cases. The hyperoctahedral case seems to be the most complex of the four, similarly to the classification in the uncolored case, where the same is true of the corresponding case $\mathcal{H}$ defined by $\uparrow \otimes \uparrow \notin \mathcal{C}$ or $\sqcap\sqcap\sqcap \in \mathcal{C}$ for categories $\mathcal{C} \subseteq \mathcal{P}$ (see [8,10]).

Definition 4.2 Let $\mathcal{C} \subseteq \mathcal{P}^{\circ\bullet}$ be a category.

(a) Equivalently to saying $\mathcal{C}$ is in case $\mathcal{H}$ we also call $\mathcal{C}$ *hyperoctahedral*.
(b) Hence, if instead $\mathcal{C}$ is in case $\mathcal{O}$, $\mathcal{B}$ or $\mathcal{S}$, we say that $\mathcal{C}$ is *non-hyperoctahedral*.
(c) The set of all non-hyperoctahedral categories is denoted by $\mathsf{PCat}^{\circ\bullet}_{\mathrm{NHO}}$.

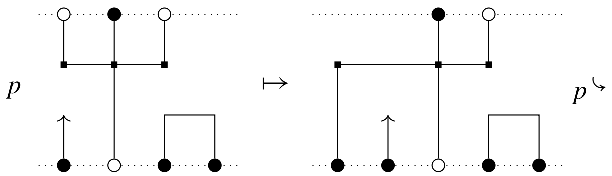
4.3 Composite category operations

The basic category operations can be used to construct further generic transformations.

4.3.1 Rotation

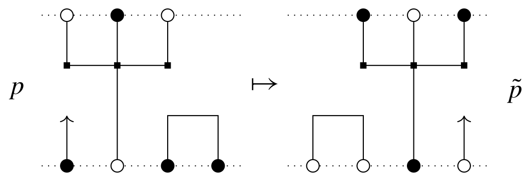
Categories are closed under four *basic rotations*: Let $\{k, \ell\} \subseteq \{0\}\cup\mathbb{N}$ satisfy $k+\ell > 0$. If $0 < k$, define two mappings $\mathrm{L}^{k-1}_{\ell+1} \to \mathrm{L}^{k}_{\ell}$: $\rho_{\hookrightarrow}$ is given by ${}_{\blacksquare}(i+1) \mapsto {}_{\blacksquare}i$ and ${}_{\blacksquare}1 \mapsto {}^{\blacksquare}1$ and ${}^{\blacksquare}j \mapsto {}^{\blacksquare}(j+1)$ and $\rho_{\rcurvearrowdown}$ extends the identity by ${}_{\blacksquare}(\ell+1) \mapsto {}^{\blacksquare}k$. Similarly, provided $0 < \ell$, two mappings $\mathrm{L}^{k+1}_{\ell-1} \to \mathrm{L}^{k}_{\ell}$ can be defined: $\rho_{\curvearrowright}$ is given by ${}^{\blacksquare}(j+1) \mapsto {}^{\blacksquare}j$ and ${}^{\blacksquare}1 \mapsto {}_{\blacksquare}1$ and ${}_{\blacksquare}(i-1) \mapsto {}_{\blacksquare}i$ and $\rho_{\curvearrowleft}$ extends the identity by ${}^{\blacksquare}(k+1) \mapsto {}_{\blacksquare}\ell$.

For any $p \in \mathcal{P}^{\circ\bullet}(k, \ell)$ with coloring c and any $r \in \{\hookrightarrow, \rcurvearrowdown, \curvearrowright, \curvearrowleft\}$, whenever ρ_r is defined, let the *rotation* p^r be the partition with blocks $\{\rho_r^{-1}(B) \mid B \in p\}$ and colors $c \circ \rho_r$ except that, importantly, x_r is instead assigned the inverse of $c(\rho_r(x_r))$, where $x_{\hookrightarrow} = {}_{\blacksquare}1$ and $x_{\rcurvearrowdown} = {}_{\blacksquare}(\ell+1)$ and $x_{\curvearrowright} = {}^{\blacksquare}1$ and $x_{\curvearrowleft} = {}^{\blacksquare}(k+1)$. Moreover, we call $p^{\circlearrowright}$ defined by $(p^{\curvearrowright})^{\rcurvearrowdown}$ if $\ell > 0$ and by $(p^{\rcurvearrowdown})^{\curvearrowright}$ if $k > 0$ (which is compatible) the *clockwise cyclic rotation* and, likewise, $p^{\circlearrowleft}$ defined by $(p^{\hookrightarrow})^{\curvearrowleft}$ if $k > 0$ and by $(p^{\curvearrowleft})^{\hookrightarrow}$ if $\ell > 0$ the *counter-clockwise cyclic rotation* of p.



4.3.2 Verticolor reflection

Given $\{k, \ell\} \subseteq \{0\}\cup\mathbb{N}$ and $p \in \mathcal{P}^{\circ\bullet}(k, \ell)$ with coloring c, the *reflection* $\hat{p} \in \mathcal{P}^{\circ\bullet}(k, \ell)$ is defined as the partition with blocks $\{\kappa^{-1}(B) \mid B \in p\}$ and coloring $c\circ\kappa$ where κ is the self-mapping of L^{k}_{ℓ} with ${}_{\blacksquare}i \mapsto {}_{\blacksquare}(\ell-i+1)$ and ${}^{\blacksquare}j \mapsto {}^{\blacksquare}(k-j+1)$. The *color inversion* $\overline{p}$ of p is constructed by inverting the coloring of p pointwise. And the *verticolor reflection* $\tilde{p}$ is the color inversion of the reflection of p. Categories are closed under verticolor reflection but generally neither under reflection nor color inversion.

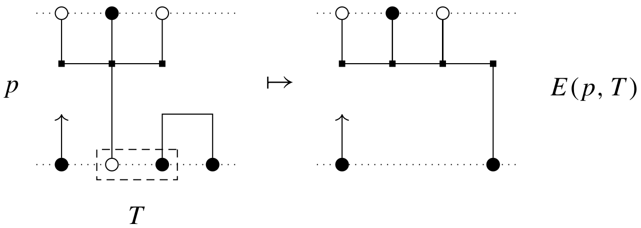


4.3.3 Erasing

Lastly, categories are closed under erasing turns: Let $\{k, l\} \subseteq \{0\} \cup \mathbb{N}$, let $p \in \mathcal{P}^{\circ\bullet}(k, \ell)$ and let p have coloring c. A *turn* in p is any set T of two points which is neutral and consecutive in p. If so and if $\ell' = \ell - |\mathrm{L}_\ell^0 \cap T|$ and $k' = k - |\mathrm{L}_0^k \cap T|$, then let ϵ be the unique mapping $\mathrm{L}_{\ell'}^{k'} \to \mathrm{L}_\ell^k$ which maps upper points to upper points and lower points to lower points, which is increasing with respect to the linear orders, and whose image is $\mathrm{L}_\ell^k \backslash T$. The *erasing* of T from p is the partition $E(p, T)$ with coloring $c \circ \epsilon$ and with blocks

$$\left\{\epsilon^{-1}(B) \mid B \in p \text{ and } B \cap T = \emptyset\right\} \cup \left\{\bigcup \left\{\epsilon^{-1}(B\backslash T) \mid B \in p \text{ and } B \cap T \neq \emptyset\right\}\right\} \backslash \{\emptyset\}$$

In particular, because $|T| = 2$, any set B is a block of $E(p, T)$ if and only if $\epsilon(B) \in p$ or there exist (not necessarily distinct) blocks B_1 and B_2 of p with $B_1 \cap T \neq \emptyset$ and $B_2 \cap T \neq \emptyset$ such that $\epsilon(B) = (B_1 \cup B_2)\backslash T$.



4.4 Alternative characterization of categories

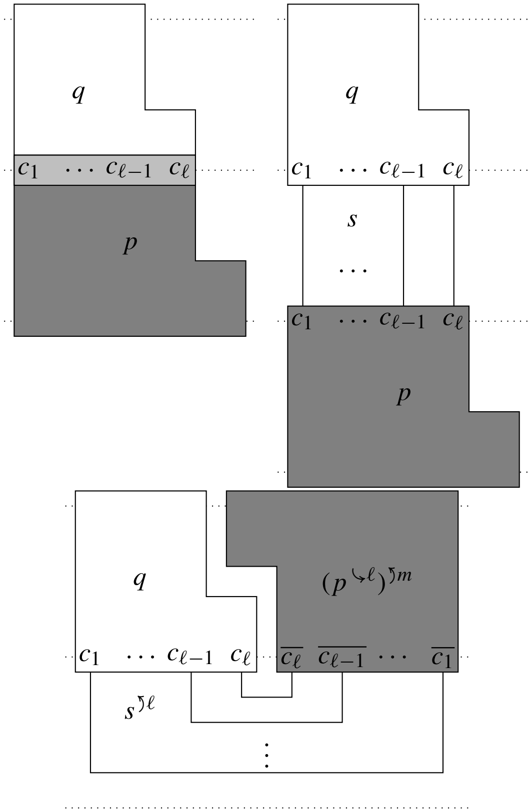
Using the composite operations, we can give a helpful characterization of categories based on the idea in the proof of [9, Lemma 3.6].

Lemma 4.3 (Alternative Characterization) *Any set $\mathcal{C} \subseteq \mathcal{P}^{\circ\bullet}$ is a category if and only if $\overset{\circ}{\underset{\circ}{|}} \in \mathcal{C}$ and $\mathcal{C}$ is closed under tensor products, basic rotations, verticolor reflection and erasing turns.*

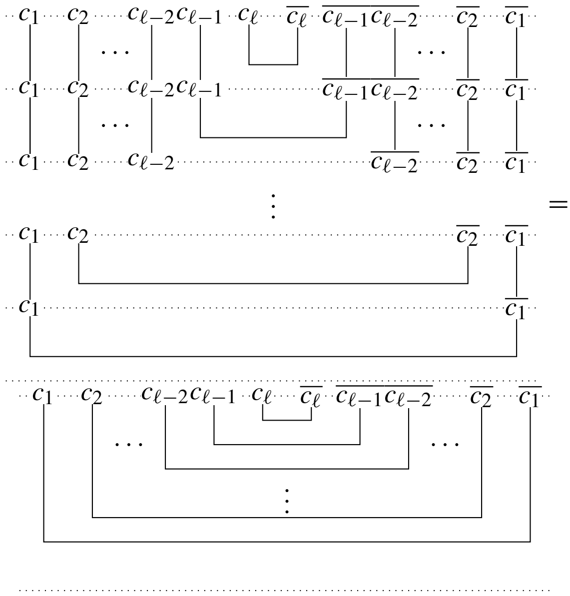
Proof Suppose $\mathcal{C}$ meets the conditions in the claim. We show it to be a category. Since $\overset{\circ}{\underset{\circ}{|}}^{\rotatebox{90}{$\curvearrowleft$}} = \sqcap_{\circ\bullet}$, since $\sqcap_{\circ\bullet}^{\curvearrowright} = \overset{\bullet}{\underset{\bullet}{|}}$ and since $\overset{\bullet}{\underset{\bullet}{|}}^{\rotatebox{90}{$\curvearrowleft$}} = \sqcap_{\bullet\circ}$ and because $\mathcal{C}$ is closed under rotations, we find $\{\overset{\bullet}{\underset{\bullet}{|}}, \sqcap_{\circ\bullet}, \sqcap_{\bullet\circ}\} \subseteq \mathcal{C}$. Erasing the only turn in $\sqcap_{\circ\bullet} \in \mathcal{C}$, under which $\mathcal{C}$ is invariant, produces $\emptyset \in \mathcal{C}$.

For any $\{\ell, m\} \subseteq \{0\} \cup \mathbb{N}$ and any partition $p \in \mathcal{C}(\ell, m)$ the identity $p^* = ((\tilde{p})^{\curvearrowright m})^{\rotatebox{90}{$\curvearrowleft$}\ell}$ and the assumptions that $\mathcal{C}$ is stable under rotations and verticolor reflection proves that $p^* \in \mathcal{C}$. Hence, $\mathcal{C}$ is also involution-invariant.

Lastly, let $\{k, \ell, m\} \subseteq \{0\} \cup \mathbb{N}$, let $p \in \mathcal{C}(\ell, m)$, let $q \in \mathcal{C}(k, \ell)$ and let (p, q) be composable. We want to show $r := pq \in \mathcal{C}$. Since $\mathcal{C}$ is closed under rotations, it suffices to prove $r^{\rotatebox{90}{$\curvearrowright$}m} \in \mathcal{C}$. Let $(c_1, \ldots, c_\ell)$ for $\{c_1, \ldots, c_\ell\} \subseteq \{\circ, \bullet\}$ be the colors of the lower points of q left to right. Let s be the tensor product of partitions from $\{\overset{\circ}{\underset{\circ}{|}}, \overset{\bullet}{\underset{\bullet}{|}}\}$ with lower row of coloring $(c_1, \ldots, c_\ell)$. Then, (p, s) and (s, q) are composable and $psq = pq = r$. The diagram below illustrates that the pairing $(s^{\rotatebox{90}{$\curvearrowright$}\ell}, q \otimes ((p^{\hookrightarrow\ell})^{\rotatebox{90}{$\curvearrowright$}m}))$ is composable as well and that its composition yields the partition $r^{\rotatebox{90}{$\curvearrowright$}m}$.



Our assumptions guarantee $e_0 := q \otimes ((p^{\hookrightarrow\ell})^{\circlearrowleft m}) \in \mathcal{C}$. Because $\mathcal{C}$ is assumed invariant under erasing turns, if we define the turn $T_0 := \{{\scriptstyle\blacksquare}\ell, {\scriptstyle\blacksquare}(\ell+1)\}$ in $e_0 \in \mathcal{C}$ and then for every $j \in [\![\ell-1]\!]$ the turn $T_j := \{{\scriptstyle\blacksquare}(\ell-j), {\scriptstyle\blacksquare}(\ell-j+1)\}$ in $e_j := E(e_{j-1}, T_{j-1}) \in \mathcal{C}$, then the partition $e_\ell := E(e_{\ell-1}, T_{\ell-1}) \in \mathcal{C}$ is identical with $r^{\circlearrowleft m}$ as the diagram below shows.



Thus we have deduced $r^{\rceil m} \in \mathcal{C}$ as claimed. This proves that $\mathcal{C}$ is closed under composition of composable pairs and thus a category. □

5 The sets $\mathcal{R}_Q$

In the following, we will define in several steps an index set Q and for each $Q \in \mathsf{Q}$ a set $\mathcal{R}_Q \subseteq \mathcal{P}^{\circ\bullet}$ of partitions. The aim will be to show that each of these constitutes a non-hyperoctahedral category (see Theorem 6.20, the main result of this article). Auxiliary objects L and Z aid in defining Q and $(\mathcal{R}_Q)_{Q\in\mathsf{Q}}$.

Notation 5.1 For every set S denote its power set by $\mathfrak{P}(S)$.

Definition 5.2 We define the *parameter domain* L as the six-fold Cartesian product of $\mathfrak{P}(\mathbb{Z})$:

$$\mathsf{L} := \mathfrak{P}(\mathbb{Z}) \times \mathfrak{P}(\mathbb{Z}) \times \mathfrak{P}(\mathbb{Z}) \times \mathfrak{P}(\mathbb{Z}) \times \mathfrak{P}(\mathbb{Z}) \times \mathfrak{P}(\mathbb{Z}).$$

Definition 5.3 Define the *analyzer* $Z : \mathfrak{P}(\mathcal{P}^{\circ\bullet}) \to \mathsf{L}$ by

$$Z := (\, F,\ V,\ \Sigma,\ L,\ K,\ X\,)$$

where, for all $\mathcal{S} \subseteq \mathcal{P}^{\circ\bullet}$,

(a) $F(\mathcal{S}) := \{\, |B| \mid p \in \mathcal{S},\ B \in p\}$ is the set of block sizes,
(b) $V(\mathcal{S}) := \{\, \sigma_p(B) \mid p \in \mathcal{S},\ B \in p\}$ is the set of block color sums,
(c) $\Sigma(\mathcal{S}) := \{\, \Sigma(p) \mid p \in \mathcal{S}\}$ is the set of total color sums,

(d)

$$L(\mathcal{S}) := \{\, \delta_p(\alpha_1, \alpha_2) \mid p \in \mathcal{S},\ B \in p,\ \{\alpha_1, \alpha_2\} \subseteq B,\ \alpha_1 \neq \alpha_2, \\ ]\alpha_1, \alpha_2[_p \cap B = \emptyset,\ \sigma_p(\{\alpha_1, \alpha_2\}) \neq 0\}$$

is the set of color distances between any two subsequent legs of the *same* block having the *same* normalized color,

(e)

$$K(\mathcal{S}) := \{\, \delta_p(\alpha_1, \alpha_2) \mid p \in \mathcal{S},\ B \in p,\ \{\alpha_1, \alpha_2\} \subseteq B,\ \alpha_1 \neq \alpha_2, \\ ]\alpha_1, \alpha_2[_p \cap B = \emptyset,\ \sigma_p(\{\alpha_1, \alpha_2\}) = 0\}$$

is the set of color distances between any two subsequent legs of the *same* block having *different* normalized colors and

(f)

$$X(\mathcal{S}) := \{\, \delta_p(\alpha_1, \alpha_2) \mid p \in \mathcal{S},\ \{B_1, B_2\} \subseteq p,\ B_1 \text{ crosses } B_2 \text{ in } p, \\ \alpha_1 \in B_1,\ \alpha_2 \in B_2\}$$

is the set of color distances between any two legs belonging to two *crossing* blocks.

The parameter domain L and the analyzer Z allow us to now define the following map which will later induce the announced family $(\mathcal{R}_Q)_{Q\in\mathsf{Q}}$.

Notation 5.4 Given any family $(S_i)_{i\in I}$ of sets, we write $\leq$ for the product order on the Cartesian product $\bigtimes_{i\in I} \mathfrak{P}(S_i)$ induced by the partial orders $\subseteq$ on the factors.

Definition 5.5 Define the *parameterization* as the mapping

$$\mathcal{R}:\ \mathsf{L} \to \mathfrak{P}(\mathcal{P}^{\circ\bullet}),\ L \mapsto \mathcal{R}_L := \{p \in \mathcal{P}^{\circ\bullet} \mid Z(\{p\}) \leq L\}.$$

With $\mathcal{R}$ we can single out sets of partitions by placing constraints on the six aggregated combinatorial features of partitions listed above.

Notation 5.6

(a) For all $\{x, y\} \subseteq \mathbb{Z}$ and $\{A, B\} \subseteq \mathfrak{P}(\mathbb{Z})$ write

$$xA+yB := \{xa + yb \mid a \in A,\ b \in B\}.$$

Moreover, put then $xA - yB := xA + (-y)B$. Per $A = \{1\}$ expressions like $x + yB$ are defined as well, and per $x = 1$ so are such like $A + yB$.

(b) Let $\pm S := S \cup (-S)$ for all sets $S \subseteq \mathbb{Z}$.

(c) For all $m \in \mathbb{Z}$ and $D \subseteq \mathbb{Z}$ define

$$D_m := \pm D + m\mathbb{Z} \quad \text{and} \quad D'_m := \pm(D \cup \{0\}) + m\mathbb{Z}.$$

(d) Use the abbreviations $[\![0]\!] := \emptyset$ and $[\![k]\!] := \{1, \ldots, k\}$ for all $k \in \mathbb{N}$.

Definition 5.7 Define the *parameter range* Q as the subset of L comprising exactly all tuples

$$(f, v, s, l, k, x)$$

listed below, where $u \in \{0\} \cup \mathbb{N}$, $m \in \mathbb{N}$, $D \subseteq \{0\} \cup [\![\lfloor \frac{m}{2} \rfloor]\!]$, where $E \subseteq \{0\} \cup \mathbb{N}$ and where N is a subsemigroup of $(\mathbb{N}, +)$:

f	v	s	l	k	x
$\{2\}$	$\pm\{0, 2\}$	$2um\mathbb{Z}$	$m\mathbb{Z}$	$m\mathbb{Z}$	$\mathbb{Z}$
$\{2\}$	$\pm\{0, 2\}$	$2um\mathbb{Z}$	$m{+}2m\mathbb{Z}$	$2m\mathbb{Z}$	$\mathbb{Z}$
$\{2\}$	$\pm\{0, 2\}$	$2um\mathbb{Z}$	$m{+}2m\mathbb{Z}$	$2m\mathbb{Z}$	$\mathbb{Z} \backslash m\mathbb{Z}$
$\{2\}$	$\{0\}$	$\{0\}$	$\emptyset$	$m\mathbb{Z}$	$\mathbb{Z}$
$\{2\}$	$\pm\{0, 2\}$	$\{0\}$	$\{0\}$	$\{0\}$	$\mathbb{Z} \backslash N_0$
$\{2\}$	$\{0\}$	$\{0\}$	$\emptyset$	$\{0\}$	$\mathbb{Z} \backslash N_0$
$\{2\}$	$\{0\}$	$\{0\}$	$\emptyset$	$\{0\}$	$\mathbb{Z} \backslash N_0'$
$\{1, 2\}$	$\pm\{0, 1, 2\}$	$um\mathbb{Z}$	$m\mathbb{Z}$	$m\mathbb{Z}$	$\mathbb{Z} \backslash D_m$
$\{1, 2\}$	$\pm\{0, 1, 2\}$	$2um\mathbb{Z}$	$m{+}2m\mathbb{Z}$	$2m\mathbb{Z}$	$\mathbb{Z} \backslash D_m$
$\{1, 2\}$	$\pm\{0, 1\}$	$um\mathbb{Z}$	$\emptyset$	$m\mathbb{Z}$	$\mathbb{Z} \backslash D_m$
$\{1, 2\}$	$\pm\{0, 1, 2\}$	$\{0\}$	$\{0\}$	$\{0\}$	$\mathbb{Z} \backslash E_0$
$\{1, 2\}$	$\pm\{0, 1\}$	$\{0\}$	$\emptyset$	$\{0\}$	$\mathbb{Z} \backslash E_0$
$\mathbb{N}$	$\mathbb{Z}$	$um\mathbb{Z}$	$m\mathbb{Z}$	$m\mathbb{Z}$	$\mathbb{Z} \backslash D_m$
$\mathbb{N}$	$\mathbb{Z}$	$\{0\}$	$\{0\}$	$\{0\}$	$\mathbb{Z} \backslash E_0$

With Q and $\mathcal{R}$ defined, so has been the family $(\mathcal{R}_Q)_{Q \in \mathsf{Q}}$. The characterizing conditions $Z(\{\cdot\}) \leq Q$ of the sets $\mathcal{R}_Q$ for $Q \in \mathsf{Q}$ will be successively explained in Sect. 6 in the process of proving their invariance under the category operations.

6 Invariance of $\mathcal{R}_Q$ under the category operations

The strategy for proving that the sets defined in the preceding Sect. 5 are actually categories of partitions is the following: We choose the most convenient elements Q of Q, the ones for which it is easiest to prove that $\mathcal{R}_Q$ is a category. Once we have verified that these sets $\mathcal{R}_Q$ are categories, we show that any other $Q \in \mathsf{Q}$ can be written as a meet of a suitable family $\mathsf{Q}' \subseteq \mathsf{Q}$ of convenient ones in the complete lattice L. Then, Lemma 6.2 will allow us to conclude that $\mathcal{R}_Q$ is a category for each $Q \in \mathsf{Q}$.

Notation 6.1 Given a family $(S_i)_{i \in I}$ of sets, we use the symbols $\bigcap_\times$ for the meet and $\bigcup_\times$ for the join operator of the product order $\leq$ with respect to $\subseteq$ on $\times_{i \in I} \mathfrak{P}(S_i)$.

Lemma 6.2 *The mapping $\mathcal{R} : \mathsf{L} \to \mathfrak{P}(\mathcal{P}^{\circ\bullet})$ is monotonic and preserves meets.*

Proof. For any $\{L, L'\} \subseteq \mathsf{L}$ with $L \leq L'$ and any $p \in \mathcal{P}^{\circ\bullet}$ the condition $p \in \mathcal{R}_L$, i.e., $Z(\{p\}) \leq L$, implies $Z(\{p\}) \leq L'$ and thus $p \in \mathcal{R}_{L'}$. Hence, $\mathcal{R}$ is monotonic. And for all subsets $\mathsf{L}' \subseteq \mathsf{L}$,

$$\begin{aligned}\bigcap\{\mathcal{R}_L \mid L \in \mathsf{L}'\} &= \bigcap\{\{p \in \mathcal{P}^{\circ\bullet} \mid Z(\{p\}) \leq L\} \mid L \in \mathsf{L}'\} \\ &= \{p \in \mathcal{P}^{\circ\bullet} \mid \forall L \in \mathsf{L}' : Z(\{p\}) \leq L\} \\ &= \{p \in \mathcal{P}^{\circ\bullet} \mid Z(\{p\}) \leq \textstyle\bigcap_{\times} \mathsf{L}'\} \\ &= \mathcal{R}_{\bigcap_{\times} \mathsf{L}'},\end{aligned}$$

where we have only used the definition of $\mathcal{R}$. □

6.1 Behavior of *Z* under category operations

In order to show that $\mathcal{R}_Q$ is a category for the convenient values $Q \in \mathsf{Q}$, we will use the Alternative Characterization (Lemma 4.3). It pays to consider abstractly how Z behaves under the (alternative) category operations beforehand.

6.1.1 Behavior of *Z* under rotation

The first set of operations are the four basic rotations defined in Sect. 4.3.1.

Lemma 6.3 *Let $p \in \mathcal{P}^{\circ\bullet}$ and let $r \in \{↳, ⤹, \curvearrowright, ⤵\}$ be such that p^r is defined and let ρ_r be the bijection which rotates the points of p^r back to their original positions in p.*

(a) $\sigma_{p^r}(S) = \sigma_p(\rho_r(S))$ for any set S of points in p^r.
(b) $]\alpha, \beta[_{p^r} = \rho_r^{-1}(]\rho_r(\alpha), \rho_r(\beta)[_p)$ for any points α and β of p^r with $\alpha \neq \beta$.
(c) $\delta_{p^r}(\alpha, \beta) = \delta_p(\rho_r(\alpha), \rho_r(\beta))$ for any points α and β of p^r.

Proof. Resume the notation from the definition of the rotation operation in Sect. 4.3.1.

(a) The coloring of p^r is given by $c \circ \rho_r$ except that the point x_r has the inverse color of $c(\rho_r(x_r))$ instead. However, because x_r is located on a different row in p^r than $\rho_r(x_r)$ is in p, the *normalized* colors of x_r in p^r and of $\rho_r(x_r)$ in p do concur. Since no other point changes rows during the rotation operation, that means that each point γ of p^r has the same normalized color in p^r as the point $\rho_r(\gamma)$ has in p. And this conclusion is equivalent to the claim.
(b) By distinguishing a multitude of cases, it can be checked that the definition of ρ_r ensures that any triple of pairwise distinct points $(\gamma_1, \gamma_2, \gamma_3)$ is ordered in p^r if and only if the triple $(\rho_r(\gamma_1), \rho_r(\gamma_2), \rho_r(\gamma_3))$ is ordered in p. And from that the claim follows.
(c) If $\alpha = \beta$, then Part (a) proves $\delta_{p^r}(\alpha, \beta) = \Sigma(p^r) = \Sigma(p) = \delta_p(\rho(\alpha), \rho(\beta))$. Should $\alpha \neq \beta$, then Parts (a) and (b) let us infer

$$\begin{aligned}\delta_{p^r}(\alpha, \beta) &= \tfrac{1}{2}\sigma_{p^r}(\{\alpha\}) + \sigma_{p^r}(]\alpha, \beta[_{p^r}) + \tfrac{1}{2}\sigma_{p^r}(\{\beta\}) \\ &= \tfrac{1}{2}\sigma_p(\{\rho_r(\alpha)\}) + \sigma_p(]\rho_r(\alpha), \rho_r(\beta)[_p) + \tfrac{1}{2}\sigma_p(\{\rho_r(\beta)\}) \\ &= \delta_p(\rho_r(\alpha), \rho_r(\beta)),\end{aligned}$$

which is what we needed to see. □

Lemma 6.4 *For any* $r \in \{ \hookrightarrow, \rcurvearrowdown, \curvearrowright, \circlearrowright \}$ *and any* $p \in \mathcal{P}^{\circ\bullet}$ *such that* p^r *is defined,*

$$Z(\{p^r\}) \leq Z(\{p\}).$$

Proof Denote by ρ_r map sending the points of p^r back to their former locations in p. Recall that, by definition, for any B, saying $B \in p^r$ and $\rho_r(B) \in p$ are equivalent.

Step 1: *Component* F. The fact that ρ_r is bijective implies $F(\{p^r\}) = \{\, |B| \mid B \in p^r\} = \{\, |\rho_r(B)| \mid \rho_r(B) \in p\} \subseteq F(\{p\})$.

Step 2: *Component* V. By Lemma 6.3 (a), similarly, $V(\{p^r\}) = \{\sigma_{p^r}(B) \mid B \in p^r\} = \{\sigma_p(\rho_r(B)) \mid \rho_r(B) \in p\} \subseteq V(\{p\})$.

Step 3: *Component* Σ. The same lemma yields $\Sigma(p^r) = \Sigma(p)$ in particular.

Step 4: *Components* L *and* K. For any $B \in p^r$ and any $\{\alpha_1, \alpha_2\} \subseteq B$ with $\alpha_1 \neq \alpha_2$, (and thus also $\rho_r(B) \in p$ and $\{\rho_r(\alpha_1), \rho_r(\alpha_2)\} \subseteq \rho_r(B)$ and $\rho_r(\alpha_1) \neq \rho_r(\alpha_2)$), if $]\alpha_1, \alpha_2[_{p^r} \cap B = \emptyset$, then Lemma 6.3 (b) guarantees $]\rho_r(\alpha_1), \rho_r(\alpha_2)[_p \cap \rho_r(B) = \rho_r(]\alpha_1, \alpha_2[_{p^r} \cap B) = \emptyset$. Therefore the integer $\delta_{p^r}(\alpha_1, \alpha_2) = \delta_p(\rho_r(\alpha_1), \rho_r(\alpha_2))$ (by Lemma 6.3 (c)) is an element of $L(\{p\})$ if $\sigma_p(\{\rho_r(\alpha_1), \rho_r(\alpha_2)\} \neq 0$ and of $K(\{p\})$ otherwise. Because the statements $\sigma_{p^r}(\{\alpha_1, \alpha_2\}) = 0$ and $\sigma_p(\{\epsilon(\alpha_1), \epsilon(\alpha_2)\}) = 0$ are equivalent by Lemma 6.3 (a) we have thus verified $(L, K)(\{p^r\}) \leq (L, K)(\{p\})$.

Step 5: *Component* X. Let $\{B_1, B_2\} \subseteq p^r$, let $\alpha_1 \in B_1$ and $\alpha_2 \in B_2$ and let B_1 and B_2 cross in p^r. The last assumption can be equivalently expressed as there existing $\{\eta, \theta\} \subseteq B_1$ such that $]\eta, \theta[_{p^r} \cap B_2 \neq \emptyset$ and $]\theta, \eta[_{p^r} \cap B_2 \neq \emptyset$. Lemma 6.3 (b) lets us infer $]\rho_r(\eta), \rho_r(\theta)[_p \cap \rho_r(B_2) = \rho_r(]\eta, \theta[_{p^r} \cap B_2) \neq \emptyset$ and also $]\rho_r(\theta), \rho_r(\eta)[_p \cap \rho_r(B_2) = \rho_r(]\theta, \eta[_{p^r} \cap B_2) \neq \emptyset$. Hence, $\rho_r(B_1)$ and $\rho_r(B_2)$ cross in p. In consequence, Lemma 6.3 (c) allows us to conclude $\delta_{p^r}(\alpha_1, \alpha_2) = \delta_p(\rho_r(\alpha_1), \rho_r(\alpha_2)) \in X(\{p\})$ by $\rho_r(\alpha_1) \in \rho_r(B_1)$ and $\rho_r(\alpha_2) \in \rho_r(B_2)$. With $X(\{p^r\}) \subseteq X(\{p\})$ thus confirmed, the full assertion is now clear. $\square$

6.1.2 Behavior of *Z* under verticolor reflection

Next, we treat the operation of verticolor reflection defined in Sect. 4.3.2.

Lemma 6.5 *Let* $p \in \mathcal{P}^{\circ\bullet}$ *be arbitrary and let* κ *be the bijection which reflects the points of* $\tilde{p}$ *at the vertical axis.*

(a) $\sigma_{\tilde{p}}(S) = -\sigma_p(\kappa(S))$ *for any set* S *of points in* $\tilde{p}$.
(b) $]\alpha, \beta[_{\tilde{p}} = \kappa^{-1}(]\kappa(\beta), \kappa(\alpha)[_p)$ *for any points* α *and* β *of* $\tilde{p}$ *with* $\alpha \neq \beta$.
(c) $\delta_{\tilde{p}}(\alpha, \beta) = -\delta_p(\kappa(\beta), \kappa(\alpha))$ *for any points* α *and* β *of* $\tilde{p}$.

Proof.

(a) If c is the coloring of p, then the reflection $\hat{p}$ by definition has the coloring $c \circ \kappa$. Since κ maps lower points to lower points and upper points to upper points, the normalized color of any point γ of $\hat{p}$ in $\hat{p}$ is hence the same as the normalized color of $\kappa(\gamma)$ in p. Since the verticolor reflection $\tilde{p}$ is obtained from $\hat{p}$ by inverting all colors (and thus also normalized colors), the normalized colors of γ in $\tilde{p}$ and of $\kappa(\gamma)$ in p are thus exactly inverse to each other. And that yields the claim.

(b) The definition of κ and case distinctions between upper and lower points show that any triple $(\gamma_1, \gamma_2, \gamma_3)$ of pairwise distinct points of $\hat{p}$ is ordered there if and only if the triple $(\kappa(\gamma_3), \kappa(\gamma_2), \kappa(\gamma_1))$ is ordered in p. In other words, reflection inverts cyclic order. Since $\hat{p}$ and $\tilde{p}$ differ only in their colorings, the claim follows from that.

(c) If $\alpha = \beta$, then $\delta_{\tilde{p}}(\alpha, \beta) = \Sigma(\tilde{p}) = -\Sigma(p) = \delta_p(\kappa)$ by Part (a). Hence, by the definitions of $\delta_{\tilde{p}}$ and δ_p and by Parts (a) and (b), whenever $\alpha \neq \beta$, then

$$\begin{aligned}\delta_{\tilde{p}}(\alpha, \beta) &= \tfrac{1}{2}\sigma_{\tilde{p}}(\{\alpha\}) + \sigma_{\tilde{p}}(]\alpha, \beta[_{\tilde{p}}) + \tfrac{1}{2}\sigma_{\tilde{p}}(\{\beta\})\\ &= -\tfrac{1}{2}\sigma_p(\{\kappa(\alpha)\}) - \sigma_p(]\kappa(\beta), \kappa(\alpha)[_p) - \tfrac{1}{2}\sigma_p(\{\kappa(\beta)\})\\ &= -\delta_p(\kappa(\beta), \kappa(\alpha)),\end{aligned}$$

as claimed. □

Lemma 6.6 *For all* $p \in \mathcal{P}^{\circ\bullet}$,

$$Z(\{\tilde{p}\}) \leq (F, -V, -\Sigma, -L, -K, -X)(\{p\}).$$

Proof Let κ be the map which sends the points of $\tilde{p}$ to their former places in p and recognize that for any B the definition of $\tilde{p}$ means that $B \in \tilde{p}$ if and only if $\kappa(B) \in p$.

Step 1: *Component* F. Since κ is a bijective mapping, $F(\{\tilde{p}\}) = \{\,|B| \mid B \in \tilde{p}\} = \{\,|\kappa(B)| \mid \kappa(B) \in p\} \subseteq F(\{p\})$.

Step 2: *Component* V. By applying Lemma 6.5 (a) we can similarly conclude $V(\{p\}) = \{\sigma_{\tilde{p}}(B) \mid B \in \tilde{p}\} = \{-\sigma_p(\kappa(B)) \mid \kappa(B) \in p\} \subseteq -V(\{p\})$.

Step 3: *Component* Σ. As a special case of Lemma 6.5 (a) we obtain $\Sigma(\tilde{p}) = -\Sigma(p)$, which proves $\Sigma(\{\tilde{p}\}) \subseteq -\Sigma(\{p\})$.

Step 4: *Components* L and K. Let $B \in \tilde{p}$ and $\{\alpha_1, \alpha_2\} \subseteq B$ with $\alpha_1 \neq \alpha_2$ and $]\alpha_1, \alpha_2[_{\tilde{p}} \cap B = \emptyset$ be arbitrary. It follows that not only $\kappa(B) \in p$ and $\{\kappa(\alpha_2), \kappa(\alpha_1)\} \subseteq \kappa(B)$ and $\kappa(\alpha_2) \neq \kappa(\alpha_1)$ but also, by Lemma 6.5 (b), that $]\kappa(\alpha_2), \kappa(\alpha_1)[_p \cap \kappa(B) = \kappa(]\alpha_1, \alpha_2[_{\tilde{p}} \cap B) = \emptyset$. Thus, according to Lemma 6.5 (c) we have shown $\delta_{\tilde{p}}(\alpha_1, \alpha_2) = -\delta_p(\kappa(\alpha_2), \kappa(\alpha_1))$ to be an element of $-L(\{p\})$ if $\sigma_p(\{\kappa(\alpha_2), \kappa(\alpha_1)\}) \neq 0$ and one of $-K(\{p\})$ otherwise. Because $\sigma_{\tilde{p}}(\{\alpha_1, \alpha_2\}) = 0$ if and only if $\sigma_p(\{\kappa(\alpha_2), \kappa(\alpha_1)\}) = 0$ by Lemma 6.5 (a), that is all we needed to prove in order to see $(L, K)(\{\tilde{p}\}) \leq (-L, -K)(\{p\})$.

Step 5: *Component* X. If $\{B_1, B_2\} \subseteq \tilde{p}$ and if B_1 and B_2 cross in $\tilde{p}$ and if $\alpha_1 \in B_1$ and $\alpha_2 \in B_2$, then there exist $\{\eta, \theta\} \subseteq B_1$ with $\eta \neq \theta$ such that $]\eta, \theta[_{\tilde{p}} \cap B_2 \neq \emptyset$ and $]\theta, \eta[_{\tilde{p}} \cap B_2 \neq \emptyset$. Lemma 6.5 (b) then proves $]\kappa(\theta), \kappa(\eta)[_p \cap \kappa(B_2) = \kappa(]\eta, \theta[_{\tilde{p}} \cap B_2) \neq \emptyset$ and, likewise, $]\kappa(\eta), \kappa(\theta)[_p \cap \kappa(B_2) = \kappa(]\theta, \eta[_{\tilde{p}} \cap B_2) \neq \emptyset$. Thus, $\kappa(B_1)$ and $\kappa(B_2)$ cross in p. Since crossing each other is a symmetric relation, $\kappa(B_2)$ also crosses $\kappa(B_1)$. With the help of Lemma 6.5 (c), we can therefore conclude $\delta_{\tilde{p}}(\alpha_1, \alpha_2) = -\delta_p(\kappa(\alpha_2), \kappa(\alpha_1)) \in -X(\{p\})$. Thus, $X(\{\tilde{p}\}) \subseteq -X(\{p\})$, completing the proof. □

6.1.3 Behavior of *Z* under tensor products

The third is the fundamental operation of tensor product defined in Sect. 4.1.1.

Lemma 6.7 *Let $\{p_1, p_2\} \subseteq \mathcal{P}^{\circ\bullet}$ be arbitrary and for each $i \in \{1, 2\}$ let S_i be the set of points of $p_1 \otimes p_2$ coming from p_i and let τ_i be the bijection sending the points of S_i to their original positions in p_i.*

(a) $\sigma_{p_1\otimes p_2}(S) = \sum_{i=1}^{2} \sigma_{p_i}(\tau_i(S \cap S_i))$ *for any set S of points in $p_1 \otimes p_2$.*
(b) For any $i \in \{1, 2\}$ and any $\{\alpha, \beta\} \subseteq S_i$ with $\alpha \neq \beta$ the set $]\alpha, \beta[_{p_1\otimes p_2}$ is given by $\tau_i^{-1}(]\tau_i(\alpha), \tau_i(\beta)[_{p_i})$ or $\tau_i^{-1}(]\tau_i(\alpha), \tau_i(\beta)[_{p_i}) \cup S_{3-i}$.
(c) For any $i \in \{1, 2\}$ and any $\{\alpha, \beta\} \subseteq S_i$ with $\alpha \neq \beta$ the integer $\delta_{p_1\otimes p_2}(\alpha, \beta)$ is given by $\delta_{p_i}(\tau_i(\alpha), \tau_i(\beta))$ or $\delta_{p_i}(\tau_i(\alpha), \tau_i(\beta)) + \Sigma(p_{3-i})$.

Proof.

(a) If c_1 and c_2 are the colorings of p_1 and p_2, respectively, then $p_1 \otimes p_2$ by definition has the coloring $\bigcup_{i=1}^{2}(c_i \circ \tau_i)$. Because τ_1 is a restriction of the identity and τ_2 the back-shift from the definition of $\otimes$ in Sect. 4.1.1, both these maps send lower points to lower points and upper points to upper points. In consequence, for any $i \in \{1, 2\}$ the normalized color of any point $\gamma \in S_i$ in $p_1 \otimes p_2$ is the same as that of $\tau_i(\gamma)$ in p_i. Now, the assertion is clear.
(b) The claim follows from the definition of $p_1 \otimes p_2$ via a large number of case distinctions.
(c) In case $\alpha = \beta$ we can conclude from Part (a) that $\delta_{p_1\otimes p_2}(\alpha, \beta) = \Sigma(p_1 \otimes p_2) = \sigma_{p_1\otimes p_2}(S_1 \cup S_2) = \sum_{j=1}^{2} \sigma_{p_j}(\tau_j(S_j)) = \Sigma(p_1) + \Sigma(p_2) = \delta_{p_i}(\tau_i(\alpha), \tau_i(\beta)) + \Sigma(p_{3-i})$. Hence, let $\alpha \neq \beta$. By Part (a) we know

$$\sigma_{p_1\otimes p_2}(\tau_i^{-1}(]\tau_i(\alpha), \tau_i(\beta)[_{p_i})) = \sigma_{p_i}(]\tau_i(\alpha), \tau_i(\beta)[_{p_i})$$

and

$$\begin{aligned}\sigma_{p_1\otimes p_2}(\tau_i^{-1}(]\tau_i(\alpha), \tau_i(\beta)[_{p_i}) \cup S_{3-i}) &= \sigma_{p_i}(]\tau_i(\alpha), \tau_i(\beta)[_{p_i}) + \sigma_{p_{3-i}}(S_{3-i})\\ &= \sigma_{p_i}(]\tau_i(\alpha), \tau_i(\beta)[_{p_i}) + \Sigma(p_{3-i}).\end{aligned}$$

Hence, $\sigma_{p_1\otimes p_2}(]\alpha, \beta[_{p_1\otimes p_2})$ is given by $\sigma_{p_i}(]\tau_i(\alpha), \tau_i(\beta)[_{p_i})$ or $\sigma_{p_i}(]\tau_i(\alpha), \tau_i(\beta)[_{p_i})+ \Sigma(p_{3-i})$ by Part (b). And of, course, $\sigma_{p_1\otimes p_2}(\{\alpha\}) = \sigma_{p_i}(\{\tau_i(\alpha)\})$ and, likewise, $\sigma_{p_1\otimes p_2}(\{\beta\}) = \sigma_{p_i}(\{\tau_i(\beta)\})$ by Part (a). The definitions of $\delta_{p_1\otimes p_2}$ and δ_{p_i} therefore imply

$$\begin{aligned}\delta_{p_1\otimes p_2}(\alpha, \beta) &= \tfrac{1}{2}\sigma_{p_1\otimes p_2}(\{\alpha\}) + \sigma_{p_1\otimes p_2}(]\alpha, \beta[_{p_1\otimes p_2}) + \tfrac{1}{2}\sigma_{p_1\otimes p_2}(\{\beta\})\\ &= \tfrac{1}{2}\sigma_{p_i}(\{\tau_i(\alpha)\}) + \sigma_{p_i}(]\tau_i(\alpha), \tau_i(\beta)[_{p_i}) + \tfrac{1}{2}\sigma_{p_i}(\{\tau_i(\beta)\}) + x\Sigma(p_{3-i})\\ &= \delta_{p_i}(\tau_i(\alpha), \tau_i(\beta)) + x\Sigma(p_{3-i})\end{aligned}$$

for some $x \in \{0, 1\}$, which is what we needed to prove. □

Lemma 6.8 *Let $\{p_1, p_2\} \subseteq \mathcal{P}^{\circ\bullet}$ be arbitrary.*

(a) $F(\{p_1 \otimes p_2\}) \subseteq F(\{p_1, p_2\})$.
(b) $V(\{p_1 \otimes p_2\}) \subseteq V(\{p_1, p_2\})$.
(c) $\Sigma(\{p_1 \otimes p_2\}) \subseteq \gcd(\Sigma(\{p_1, p_2\}))\mathbb{Z}$.

(d) $Y(\{p_1 \otimes p_2\}) \subseteq Y(\{p_1, p_2\}) + \gcd(\Sigma(\{p_1, p_2\}))\mathbb{Z}$ *for all* $Y \in \{L, K, X\}$.

Proof. For each $i \in \{1, 2\}$ let S_i be the set of points of $p_1 \otimes p_2$ stemming from p_i and let τ_i be the mapping which moves them to their original locations in p_i. Recall that the blocks of $p_1 \otimes p_2$ are given by $\bigcup_{i=1}^{2}\{B \subseteq S_i \mid \tau_i(B) \in p_i\}$.

(a) As τ_1 and τ_2 are bijections, $F(\{p_1 \otimes p_2\}) = \{|B| \mid B \in p_1 \otimes p_2\} = \bigcup_{i=1}^{2}\{|B| \mid B \subseteq S_i,\ \tau_i(B) \in p_i\} = \bigcup_{i=1}^{2}\{|\tau_i(B)| \mid B \subseteq S_i,\ \tau_i(B) \in p_i\} \subseteq F(\{p_1, p_2\})$.
(b) By Lemma 6.7 (a), quite similarly, $V(\{p_1 \otimes p_2\}) = \{\sigma_{p_1\otimes p_2}(B) \mid B \in p_1 \otimes p_2\} = \bigcup_{i=1}^{2}\{\sigma_{p_1\otimes p_2}(B) \mid B \subseteq S_i,\ \tau_i(B) \in p_i\} = \bigcup_{i=1}^{2}\{\sigma_{p_i}(B) \mid B \subseteq S_i,\ \tau_i(B) \in p_i\} \subseteq V(\{p_1, p_2\})$.
(c) Applying Lemma 6.7 (a) once more shows $\Sigma(p_1 \otimes p_2) = \sigma_{p_1\otimes p_2}(S_1 \cup S_2) = \sum_{i=1}^{2} \sigma_{p_i}(S_i) = \Sigma(p_1) + \Sigma(p_2) \subseteq \gcd(\Sigma(\{p_1, p_2\}))\mathbb{Z}$.
(d) We distinguish two cases based on the value of Y.
Case 1: *Parameters* L *and* K. Let $B \in p_1 \otimes p_2$ and $\{\alpha_1, \alpha_2\} \subseteq B$ be arbitrary with $\alpha_1 \neq \alpha_2$ and $]\alpha_1, \alpha_2[_p \cap B = \emptyset$. Then, we find $i \in \{1, 2\}$ with $B \subseteq S_i$ and $\tau_i(B) \in p_i$. Of course, $\{\tau_i(\alpha_1), \tau_i(\alpha_2)\} \subseteq \tau_i(B)$ and $\tau_i(\alpha_1) \neq \tau_i(\alpha_2)$. Moreover, since by Lemma 6.7 (b) the set $]\alpha_1, \alpha_2[_{p_1\otimes p_2}$ can only be given by $\tau_i^{-1}(]\tau_i(\alpha), \tau_i(\beta)[_{p_i})$ or $\tau_i^{-1}(]\tau_i(\alpha), \tau_i(\beta)[_{p_i}) \cup S_{3-i}$ we can conclude from $B \subseteq S_i$ that $]\alpha_1, \alpha_2[_{p_1\otimes p_2} \cap B = \tau_i^{-1}(]\tau_i(\alpha), \tau_i(\beta)[_{p_i}) \cap B$. It follows $\emptyset = \tau_i(]\alpha_1, \alpha_2[_{p_1\otimes p_2} \cap B) =]\tau_i(\alpha_1), \tau_i(\alpha_2)[_{p_i} \cap \tau_i(B)$. Finally, Lemma 6.7 (c) tells us $\delta_{p_1\otimes p_2}(\alpha_1, \alpha_2) \in \delta_{p_i}(\tau_i(\alpha_1), \tau_i(\alpha_2)) \in \Sigma(p_{3-i})\mathbb{Z}$. In light of the fact that $\sigma_{p_1\otimes p_2}(\{\alpha_1, \alpha_2\}) = 0$ if and only if $\sigma_{p_i}(\{\tau_i(\alpha_1), \tau_i(\alpha_2)\}) = 0$ by Lemma 6.7 (a), we have thus shown $Y(\{p_1 \otimes p_2\}) \subseteq Y(\{p_1, p_2\}) + \gcd(\Sigma(\{p_1, p_2\}))\mathbb{Z}$.
Case 2: Let $\{B_1, B_2\} \subseteq p_1 \otimes p_2$, let B_1 and B_2 cross each other in $p_1 \otimes p_2$ and let $\alpha_1 \in B_1$ and $\alpha_2 \in B_2$ be arbitrary. Then, there exists $i \in \{1, 2\}$ such that $B_1 \subseteq S_i$. As an intermediate step we show $B_2 \subseteq S_i$. The crossing implies the existence of $\{\eta, \theta\} \subseteq B_1$ such that $\eta \neq \theta$ and $]\eta, \theta[_{p_1\otimes p_2} \cap B_2 \neq \emptyset$ and $]\theta, \eta[_{p_1\otimes p_2} \cap B_2 \neq \emptyset$. By Lemma 6.7 (b) there are $\{O_{\eta,\theta}, O_{\theta,\eta}\} \subseteq \{\emptyset, S_{3-i}\}$ such that $]\eta, \theta[_{p_1\otimes p_2} = \tau_i^{-1}(]\tau_i(\eta), \tau_i(\theta)[_{p_i}) \cup O_{\eta,\theta}$ and $]\theta, \eta[_{p_1\otimes p_2} = \tau_i^{-1}(]\tau_i(\theta), \tau_i(\eta)[_{p_i}) \cup O_{\theta,\eta}$.
If $B_2 \subseteq S_{3-i}$ were true, then the consequence would be $\emptyset \neq]\eta, \theta[_{p_1\otimes p_2} \cap B_2 = O_{\eta,\theta} \cap B_2$ and $\emptyset \neq]\theta, \eta[_{p_1\otimes p_2} \cap B_2 = O_{\theta,\eta} \cap B_2$. Because moreover $\{O_{\eta,\theta} \cap B_2, O_{\theta,\eta} \cap B_2\} \subseteq \{\emptyset, B_2\}$ under this assumption, we would be able to conclude that the non-empty set B_2 would simultaneously be a subset of each of the two disjoint sets $]\eta, \theta[_{p_1\otimes p_2}$ and $]\theta, \eta[_{p_1\otimes p_2}$, which is impossible. Hence, $B_2 \subseteq S_i$ is true instead.
From $B_2 \subseteq S_i$ it follows $\emptyset \neq]\eta, \theta[_{p_1\otimes p_2} \cap B_2 = \tau_i^{-1}(]\tau_i(\eta), \tau_i(\theta)[_{p_i}) \cap B_2$ and, likewise, $]\theta, \eta[_{p_1\otimes p_2} \cap B_2 = \tau_i^{-1}(]\tau_i(\theta), \tau_i(\eta)[_{p_i}) \cap B_2$. Therefore, $\emptyset \neq]\tau_i(\eta), \tau_i(\theta)[_{p_i} \cap \tau_i(B_2)$ and $\emptyset \neq]\tau_i(\theta), \tau_i(\eta)[_{p_i} \cap \tau_i(B_2)$. Hence, $\tau_i(B_1)$ and $\tau_i(B_2)$ cross in p_i. Because $\delta_{p_1\otimes p_2}(\alpha_1, \alpha_2) \in \delta_{p_i}(\tau_i(\alpha_1), \tau_i(\alpha_2)) \in \Sigma(p_{3-i})\mathbb{Z}$ by Lemma 6.7 (c), that proves $X(\{p_1 \otimes p_2\}) \subseteq X(\{p_1, p_2\}) + \gcd(\Sigma(\{p_1, p_2\}))\mathbb{Z}$. □

6.1.4 Behavior of Z under erasing turns

The effect of the erasing operation (defined in Sect. 4.3.3) on Z is too complicated to treat well abstractly beyond the following observations.

Lemma 6.9 *Let $p \in \mathcal{P}^{\circ\bullet}$ be any partition, let T be any turn in p and let ϵ be the map which sends the points of $E(p, T)$ to their former positions in p.*

(a) $\sigma_{E(p,T)}(S) = \sigma_p(\epsilon(S))$ for any set S of points in $E(p, T)$.
(b) $]\alpha, \beta[_{E(p,T)} = \epsilon^{-1}(]\epsilon(\alpha), \epsilon(\beta)[_p \backslash T)$ for any points $\alpha \neq \beta$ of $E(p, T)$.
(c) $\delta_{E(p,T)}(\alpha, \beta) = \delta_p(\epsilon(\alpha), \epsilon(\beta))$ for any points α and β of $E(p, T)$.

Proof.

(a) If c is the coloring of p, then by definition $E(p, T)$ has the coloring $c \circ \epsilon$. Since ϵ maps lower points to lower points and upper ones to upper ones, the normalized color of any point γ in $E(p, T)$ is hence the same as the normalized color of $\epsilon(\gamma)$ in p.
(b) By distinguishing cases depending on where T is located and which combination of lower points and upper points one is considering, one can show that any triple $(\gamma_1, \gamma_2, \gamma_3)$ of pairwise distinct points of $E(p, T)$ is ordered in $E(p, T)$ if and only if $(\epsilon(\gamma_1), \epsilon(\gamma_2), \epsilon(\gamma_3))$ is ordered in p. If one keeps in mind that the image of ϵ is the complement of T, the claim follows from that.
(c) If P_p denotes the set of all points of p, then $\epsilon^{-1}(P_p \backslash T)$ is that of $E(p, T)$. Hence, by Part (a), if $\alpha = \beta$, then T being neutral in p implies

$$\begin{aligned}\delta_{E(p,T)}(\alpha, \beta) &= \Sigma(E(p, T)) = \sigma_{E(p,T)}(\epsilon^{-1}(P_p \backslash T)) \\ &= \sigma_p(P_p \backslash T) = \sigma_p(P_p) - \sigma_p(T) = \Sigma(p) \\ &= \delta_p(\alpha, \beta).\end{aligned}$$

Now, let $\alpha \neq \beta$ instead. Because $\{\epsilon(\alpha), \epsilon(\beta)\} \cap T = \emptyset$, exactly one of the triples $(\{\epsilon(\alpha)\}, T, \{\epsilon(\beta)\})$ and $(\{\epsilon(\beta)\}, T, \{\epsilon(\alpha)\})$ of pairwise disjoint consecutive sets in p is ordered in p. In the first case, $T \subseteq]\epsilon(\alpha), \epsilon(\beta)[_p$ proves $\sigma_p(]\epsilon(\alpha), \epsilon(\beta)[_p \backslash T) = \sigma_p(]\epsilon(\alpha), \epsilon(\beta)[_p) - \sigma_p(T) = \sigma_p(]\epsilon(\alpha), \epsilon(\beta)[_p)$. And, in the second one, the same result $\sigma_p(]\epsilon(\alpha), \epsilon(\beta)[_p \backslash T) = \sigma_p(]\epsilon(\alpha), \epsilon(\beta)[_p)$ follows by $]\epsilon(\alpha), \epsilon(\beta)[_p \cap T = \emptyset$. Thus, Parts (a) and (b) let us infer $\sigma_{E(p,T)}(]\alpha, \beta[_{E(p,T)}) = \sigma_p(]\epsilon(\alpha), \epsilon(\beta))[_p)$ either way. Hence, Part (a) and the definitions of $\delta_{E(p,T)}$ and δ_p show

$$\begin{aligned}\delta_{E(p,T)}(\alpha, \beta) &= \tfrac{1}{2}\sigma_{E(p,T)}(\{\alpha\}) + \sigma_{E(p,T)}(]\alpha, \beta[_{E(p,T)}) + \tfrac{1}{2}\sigma_{E(p,T)}(\{\beta\}) \\ &= \tfrac{1}{2}\sigma_p(\{\epsilon(\alpha)\}) + \sigma_p(]\epsilon(\alpha), \epsilon(\beta)[_p) + \tfrac{1}{2}\sigma_p(\{\epsilon(\beta)\}) \\ &= \delta_p(\epsilon(\alpha), \epsilon(\beta)),\end{aligned}$$

as we had claimed. □

6.2 Total color sum

The first family of elements $Q \in \mathsf{Q}$ is chosen such that all components except the one for Σ are trivial. That means we restrict the allowed total color sums of partitions.

Remark 6.10 For any $m \in \{0\} \cup \mathbb{N}$ the set $\mathcal{R}_Q$ with $Q = (\mathbb{N}, \mathbb{Z}, m\mathbb{Z}, \mathbb{Z}, \mathbb{Z}, \mathbb{Z})$ is exactly *the set of all partitions whose total color sum is a multiple of* m.

No further constraints on $p \in \mathcal{P}^{\circ\bullet}$ are imposed by the other components of Q via $Z(\{p\}) \leq Q$ since these conditions are satisfied by *any* partition.

Lemma 6.11 [12, Lemma 2.6] *For any* $m \in \{0\} \cup \mathbb{N}$ *the set* $\mathcal{R}_Q$*, where*

$$Q = (\mathbb{N}, \mathbb{Z}, m\mathbb{Z}, \mathbb{Z}, \mathbb{Z}, \mathbb{Z}),$$

is a category of two-colored partitions.

Proof We use the Alternative Characterization (Lemma 4.3) to show that $\mathcal{R}_Q$ is a category. By Remark 6.10 it suffices to show that $\Sigma(\{\cdot\}) \subseteq m\mathbb{Z}$ is stable under the alternative category operations.

Rotation: According to Lemma 6.4 for any $r \in \{$ ⤷, ⤸, ↷, ⤹ $\}$ and $p \in \mathcal{R}_Q$, it holds that $\Sigma(\{p^r\}) \subseteq \Sigma(\{p\}) \subseteq m\mathbb{Z}$. Thus, $\Sigma(\{\cdot\}) \subseteq m\mathbb{Z}$ is stable under rotations.

Verticolor reflection: Lemma 6.6 and $-m\mathbb{Z} = m\mathbb{Z}$ show $\Sigma(\{\tilde{p}\}) \subseteq -\Sigma(\{p\}) \subseteq m\mathbb{Z}$ for all $p \in \mathcal{R}_Q$, proving that $\Sigma(\{\cdot\}) \subseteq m\mathbb{Z}$ is preserved by verticolor reflection

Tensor product: Given $\{p_1, p_2\} \subseteq \mathcal{R}_Q$, we can employ Lemma 6.8 (c) to infer $\Sigma(\{p_1 \otimes p_2\}) \subseteq \gcd(\Sigma(\{p_1, p_2\}))\mathbb{Z} \subseteq m\mathbb{Z}$ since $\Sigma(\{p_1, p_2\}) \subseteq m\mathbb{Z}$. In conclusion, $\Sigma(\{\cdot\}) \subseteq m\mathbb{Z}$ is invariant under tensor products.

Erasing: Lastly, for any $p \in \mathcal{R}_Q$ with total set of points P_p and any turn T in p, if ϵ is the map which sends the points of $E(p, T)$ to their original locations in p, then $\Sigma(E(p, T)) = \sigma_{E(p,T)}(\epsilon^{-1}(P_p \backslash T)) = \sigma_p(P_p \backslash T) = \sigma_p(P_p) - \sigma_p(T) = \Sigma(p) \in m\mathbb{Z}$ by Lemma 6.9 (a) because $\sigma_p(T) = 0$. That concludes the proof. $\square$

6.3 Block size and color sum

A second subset of Q gathers elements where only the components for F and V are non-trivial. So, we only impose (interdependent) conditions on the sizes and color sums of blocks.

Remark 6.12

(a) For $Q = (\{1, 2\}, \pm\{0, 1, 2\}, \mathbb{Z}, \mathbb{Z}, \mathbb{Z}, \mathbb{Z})$ (Part (a) of the ensuing Lemma 6.13) the set $\mathcal{R}_Q$ is simply $\mathcal{P}^{\circ\bullet}_{\leq 2}$, *the set of all partitions all of whose blocks have at most two legs*.
Only the F-condition in $Z(\{\cdot\}) \leq Q$ is important; The apparent V-constraint is merely a reflection of the one for F: Any singleton block must have color sum 1 ($\circ$) or -1 ($\bullet$), and any pair block can only ever be neutral ($\circ\bullet$ or $\bullet\circ$) or have color sums 2 ($\circ\circ$) or -2 ($\bullet\bullet$). And all the remaining conditions imposed by $Z(\{\cdot\}) \leq Q$ are trivially satisfied by *any* partition.

(b) If $Q = (\{2\}, \pm\{0, 2\}, 2\mathbb{Z}, \mathbb{Z}, \mathbb{Z}, \mathbb{Z})$ (Part (b) of Lemma 6.13), then $\mathcal{R}_Q = \mathcal{P}_2^{\circ\bullet}$ is simply *the set of all pair partitions*. Again, the V-condition merely reflects the F-constraint.
Likewise, the non-trivial value $2\mathbb{Z}$ of the Σ-component of Q does *not* impose any restriction on $\mathcal{R}_Q$ not already placed by the F-constraints: For any partition $p \in \mathcal{P}^{\circ\bullet}$,

$$\Sigma(p) = \sigma_p\left(\bigcup_{B\in p} B\right) = \sum_{B\in p} \sigma_p(B). \tag{1}$$

Hence, $\Sigma(p) = 2\mathbb{Z}$ is a necessary consequence of $V(\{p\}) \subseteq \pm\{0, 2\}$.
The conditions induced by the L-, K- and X-components of Q remain redundant.

(c) The set $\mathcal{R}_Q$ with $Q = (\{2\}, \{0\}, \{0\}, \emptyset, \mathbb{Z}, \mathbb{Z})$ (Part (c) of Lemma 6.13) is identical to $\mathcal{P}_{2,\mathrm{nb}}^{\circ\bullet}$, *the set of all pair partitions with neutral blocks*. (See [7] and [6] for a classification of all its subcategories.)
In contrast to the two previous examples, the constraint on block color sums is *not* merely a consequence of the range of allowed block sizes. The F-constraint restricts $\mathcal{R}_Q$ to a subset of $\mathcal{P}_2^{\circ\bullet}$ and the V-constraint prohibits non-neutral blocks.
However, the non-trivial value $\{0\}$ of the Σ-component of Q imposes *no* additional restrictions: $\Sigma(p) = 0$ is, by Eq. (1), a necessary consequence of $V(\{p\}) = \{0\}$ for any $p \in \mathcal{P}^{\circ\bullet}$.
And also the non-trivial L-component $\emptyset$ of Q provides *no* additional constraints beyond the ones induced by the F- and V-components: A pair block which is neutral, i.e., has normalized coloring $\circ\bullet$ or $\bullet\circ$, cannot have subsequent legs of the same normalized color. Thus, for any partition $p \in \mathcal{P}_2^{\circ\bullet}$ with $V(\{p\}) = \{0\}$ it automatically holds that $L(\{p\}) = \emptyset$.

(d) Lastly, $\mathcal{R}_Q$ where $Q = (\{1, 2\}, \pm\{0, 1\}, \mathbb{Z}, \emptyset, \mathbb{Z}, \mathbb{Z})$ (Part (d) of Lemma 6.13) is given exactly by *the set of all partitions all of whose blocks have at most two legs and all of whose pair blocks are neutral*.
As said before, restricting F to $\{1, 2\}$ reduces the allowed set of block color sums to $\pm\{0, 1, 2\}$ and the values -2 and 2 can then only stem from pair blocks. Excluding these two values then forces all pair blocks to be neutral. So, as in Part (c), the F- and V-components induce constraints in their own right.
However, the non-trivial value $\emptyset$ of the L-component of Q is merely a reflection of the F- and V-constraints because neutral pair blocks cannot have subsequent legs of the same normalized color.

Lemma 6.13 *The set $\mathcal{R}_Q$ is a category of partitions if*

(a) $Q = (\{1, 2\}, \pm\{0, 1, 2\}, \mathbb{Z}, \mathbb{Z}, \mathbb{Z}, \mathbb{Z})$,
(b) $Q = (\{2\}, \pm\{0, 2\}, 2\mathbb{Z}, \mathbb{Z}, \mathbb{Z}, \mathbb{Z})$,
(c) $Q = (\{2\}, \{0\}, \{0\}, \emptyset, \mathbb{Z}, \mathbb{Z})$, *or*
(d) $Q = (\{1, 2\}, \pm\{0, 1\}, \mathbb{Z}, \emptyset, \mathbb{Z}, \mathbb{Z})$.

Proof We treat the four claims largely simultaneously. Let Q be one of the four elements named in Claims (a)–(d) and let f_Q be the first and v_Q the second component of Q. Remark 6.12 showed that, if $(f_Q, v_Q) = (\{1, 2\}, \pm\{0, 1, 2\})$ (Case (a)) or $(f_Q, v_Q) = (\{2\}, \pm\{0, 2\})$ (Case (b)), then

$$\mathcal{R}_Q = \{p \in \mathcal{P}^{\circ\bullet} \mid F(\{p\}) \subseteq f_Q\},$$

and that, if $(f_Q, v_Q) = (\{2\}, \{0\})$ (Case (c)) or $(f_Q, v_Q) = (\{1, 2\}, \pm\{0, 1\})$ (Case (d)), then

$$\mathcal{R}_Q = \{p \in \mathcal{P}^{\circ\bullet} \mid (F, V)(\{p\}) \leq (f_Q, v_Q)\}.$$

We use the Alternative Characterization (Lemma 4.3) to show that $\mathcal{R}_Q$ is a category. As $\overset{\circ}{\underset{\circ}{|}} \in \mathcal{R}_Q$ is clearly true, that means it suffices to prove that in Cases (a) and (b) the condition $F(\{\cdot\}) \subseteq f_Q$ and in Cases (c) and (d) the condition $(F, V)(\{\cdot\}) \leq (f_Q, v_Q)$ are stable under rotations, tensor products, verticolor reflection and erasing turns.

Rotation: For any $p \in \mathcal{R}_Q$ and $r \in \{\hookleftarrow, \hookrightarrow, \curvearrowright, \curvearrowleft\}$ Lemma 6.4 implies $(F, V)(\{p^r\}) \leq (F, V)(\{p\}) \leq (f_Q, v_Q)$. Thus, $(F, V)(\{\cdot\}) \leq (f_Q, v_Q)$ is stable under rotations.

Verticolor reflection: Given arbitrary $p \in \mathcal{R}_Q$, we are guaranteed by Lemma 6.6 that $(F, V)(\{\tilde{p}\}) \leq (F, -V)(\{p\}) \leq (f_Q, -v_Q)$. Since $-v_Q = v_Q$ this proves that the constraint $(F, V)(\{\cdot\}) \leq (f_Q, v_Q)$ is preserved by verticolor reflection.

Tensor product: According to Lemma 6.8 (a) and (b), for any $\{p_1, p_2\} \subseteq \mathcal{R}_Q$, it holds that $(F, V)(\{p_1 \otimes p_2\}) \leq (F, V)(\{p_1, p_2\}) \leq (f_Q, v_Q)$. Hence, tensor products respect $(F, V)(\{\cdot\}) \leq (f_Q, v_Q)$.

Erasing: Lastly, let $p \in \mathcal{R}_Q$ be arbitrary and let T be a turn in p. We have to show $F(\{E(p, T)\}) \subseteq f_Q$ and, but just in Cases (c) and (d), also $V(\{E(p, T)\}) \subseteq v_Q$. Let ϵ be the map which sends the points of $E(p, T)$ back to their former places in p.

Let B be an arbitrary block of $E(p, T)$. What we need to prove is $|B| \in f_Q$ as well as $\sigma_{E(p,T)}(B) \in v_Q$, the latter however just in Cases (c) and (d). If $\epsilon(B) \in p$, then $|B| \in f_Q$ and, by Lemma 6.9 (a), also $\sigma_{E(p,T)}(B) = \sigma_p(\epsilon(B)) \in v_Q$ since we have assumed $p \in \mathcal{R}_Q$. Thus, suppose $\epsilon(B) \notin p$. Then, there are (not necessarily distinct) blocks B_1 and B_2 of p with $B_1 \cap T \neq \emptyset$ and $B_2 \cap T \neq \emptyset$, with $T \subseteq B_1 \cup B_2$ and with $\epsilon(B) = (B_1 \cup B_2) \backslash T \neq \emptyset$.

Step 1: *Block size.* We prove $|B| \in f_Q$. Since ϵ is injective, since $|T| = 2$ and since $T \subseteq B_1 \cup B_2$,

$$|B| = |\epsilon(B)| = |B_1 \cup B_2| - |T| = \begin{cases} |B_1| + |B_2| - 2 & \text{if } B_1 \neq B_2, \\ |B_1| - 2 & \text{if } B_1 = B_2. \end{cases} \tag{2}$$

By definition, $f_Q = \{1, 2\}$ or $f_Q = \{2\}$. We treat the two cases individually.

Case 1.1: *Parts (a) and (d).* If $f_Q = \{1, 2\}$, then in particular both $|B_1| \leq 2$ and $|B_2| \leq 2$. Hence, no matter whether $B_1 = B_2$ or $B_1 \neq B_2$, Eq. (2) implies $|B| \leq 2$, meaning $|B| \in f_Q$, as was claimed.

Case 1.2: *Parts (b) and (c).* If we assume $f_Q = \{2\}$ instead, then $|B_1| = |B_2| = 2$. Now, $B_1 = B_2$ is impossible since otherwise Eq. (2) would imply $|B| = 0$, contradicting $B \neq \emptyset$. Rather, $B_1 \neq B_2$ must be true. Then, $|B| = 2 \in f_Q$ by Eq. (2).

Step 2: *Block color sum.* We show $\sigma_p(B) \in v_Q$ in Cases (c) and (d). Since $\sigma_p(T) = 0$ and $T \subseteq B_1 \cup B_2$, we infer by Lemma 6.9 (a):

$$\begin{aligned}\sigma_{E(p,T)}(B) &= \sigma_p(\epsilon(B)) \\ &= \sigma_p((B_1 \cup B_2)\backslash T) \\ &= \sigma_p((B_1 \cup B_2)\backslash T) + \sigma_p(T) \\ &= \sigma_p(B_1 \cup B_2) \\ &= \begin{cases} \sigma_p(B_1) + \sigma_p(B_2) & \text{if } B_1 \neq B_2, \\ \sigma_p(B_1) & \text{if } B_1 = B_2. \end{cases}\end{aligned} \tag{3}$$

Cases (c) and (d) correspond to the possibilities $(f_Q, v_Q) = (\{2\}, \{0\})$ and $(f_Q, v_Q) = (\{1, 2\}, \pm\{0, 1\})$. We treat them individually.

Case 2.1: *Part (c).* If we assume $(f_Q, v_Q) = (\{2\}, \{0\})$, i.e., in particular $\sigma_p(B_1) = \sigma_p(B_2) = 0$, then, irrespective of whether $B_1 = B_2$ or $B_1 \neq B_2$, Eq. (3) proves $\sigma_{E(p,T)}(B) = 0$.

Case 2.2: *Part (d).* Now, let $(f_Q, v_Q) = (\{1, 2\}, \pm\{0, 1\})$. Then, B_1 and B_2 are singletons or pair blocks and $\{\sigma_p(B_1), \sigma_p(B_2)\} \subseteq \pm\{0, 1\}$. If both B_1 and B_2 were singletons, then $B_1 \cap T \neq \emptyset$ and $B_2 \cap T \neq \emptyset$ and $T \subseteq B_1 \cup B_2$ would imply $B_1 \cup B_2 = T$ and thus the contradiction $B = (B_1 \cup B_2)\backslash T = \emptyset$. Hence, we know that at least one of the blocks B_1 and B_2 must be a pair. The assumptions $p \in \mathcal{R}_Q$ and $v_Q = \pm\{0, 1\}$ force the pair blocks of p to be neutral (since non-neutral pair blocks have color sums 2 or -2). It follows $\sigma_p(B_1) = 0$ or $\sigma_p(B_2) = 0$. Now, no matter whether $B_1 = B_2$ or $B_1 \neq B_2$, Eq. (3) proves $\sigma_{E(p,T)}(B) \in \pm\{0, 1\} = v_Q$, which concludes the proof. □

Remark 6.14 Although we are only interested in non-hyperoctahedral categories in this article, it deserves pointing out that the proof of Lemma 6.13 also shows the existence of certain hyperoctahedral categories. Namely, the set $\{p \in \mathcal{P}^{\circ\bullet} \mid V(\{p\}) \subseteq g\mathbb{Z}\}$ is a category for every $g \in \{0\} \cup \mathbb{N}$.

In order to see this, we again use the Alternative Characterization (Lemma 4.3) of categories. Lemmata 6.4, 6.6 and 6.8 (b) imply that $V(\{\cdot\}) \subseteq g\mathbb{Z}$ is stable under rotations, verticolor reflection and tensor products. While verifying Lemma 6.13, we showed that for any $p \in \mathcal{P}^{\circ\bullet}$, any turn T in p and any $B \in E(p, T)$ the following is true if ϵ is the map from the definition of the erasing operation: Either $\epsilon(B) \in p$ and then $\sigma_{E(p,T)}(B) = \sigma_p(B)$, or $\epsilon(B) \notin p$ and then there are $\{B_1, B_2\} \subseteq p$ such that $\sigma_{E(p,T)}(B) = \sigma_p(B_1) + \sigma_p(B_2)$ or $\sigma_{E(p,T)}(B) = \sigma_p(B_1)$. Thus, $V(\{E(p, T)\}) \subseteq V(\{p\}) \cup (V(\{p\}) + V(\{p\}))$, which proves the claim.

6.4 Color distances between legs of the same block

For our third family of elements of Q, the L- and K-components are non-trivial, implying constraints on the color distances between subsequent legs of the same block. In part, this translates to a condition on the color distances between *arbitrary*—not just subsequent—legs.

Lemma 6.15 *Let $m \in \{0\} \cup \mathbb{N}$ and*

$$Q = (\mathbb{N}, \mathbb{Z}, m\mathbb{Z}, m\mathbb{Z}, m\mathbb{Z}, \mathbb{Z})$$

and let $p \in \mathcal{P}^{\circ\bullet}$ be arbitrary. Then, $Z(\{p\}) \le Q$ if and only if $\delta_p(\alpha, \beta) \in m\mathbb{Z}$. for any $B \in p$ and any $\{\alpha, \beta\} \subseteq B$.

Proof Suppose $\delta_p(\alpha, \beta) \in m\mathbb{Z}$ for each block B of p and any two legs α and β of B. Since we can especially choose $\alpha \neq \beta$ and $]\alpha, \beta[_p \cap B = \emptyset$ it follows $(L, K)(\{p\}) \le (m\mathbb{Z}, m\mathbb{Z})$. On the other hand, picking $\alpha = \beta$ shows $\Sigma(p) = \delta_p(\alpha, \alpha) \in m\mathbb{Z}$, which lets us conclude $\Sigma(\{p\}) \subseteq m\mathbb{Z}$. As $(F, V, X)(\{p\}) \le (\mathbb{N}, \mathbb{Z}, \mathbb{Z})$ is trivially true, that proves one implication.

To show the converse, let $Z(\{p\}) \le Q$, let $B \in p$ and let $\{\alpha, \beta\} \subseteq B$. If $\alpha = \beta$, then $\delta_p(\alpha, \beta) = \Sigma(p) \in m\mathbb{Z}$ as $\Sigma(\{p\}) \subseteq m\mathbb{Z}$. Thus, we can assume $\alpha \neq \beta$. We can further suppose that $]\alpha, \beta[_p \cap B \neq \emptyset$ since otherwise $\delta_p(\alpha, \beta) \in (L \cup K)(\{p\}) \subseteq m\mathbb{Z}$ by $Z(\{p\}) \le Q$. Then, p has at least three points. If so, then we find iteratively by moving from α in direction of β in accordance with the cyclic order a number $n \in \mathbb{N}$ of legs $\gamma_1, \ldots, \gamma_n$ of B such that, writing $\gamma_0 := \alpha$ and $\gamma_{n+1} := \beta$, the points $\gamma_0, \ldots, \gamma_{n+1}$ are pairwise distinct, such that $(\gamma_0, \ldots, \gamma_{n+1})$ is ordered and such that $]\alpha, \beta[_p \cap B = \{\gamma_1, \ldots, \gamma_n\}$. Now, Lemma 3.2 (c) implies $\delta_p(\alpha, \beta) \equiv \sum_{k=0}^{n} \delta_p(\gamma_k, \gamma_{k+1}) \mod m$. Because $]\gamma_k, \gamma_{k+1}[\cap B = \emptyset$ and thus $\delta_p(\gamma_k, \gamma_{k+1}) \in (L \cup K)(\{p\}) \subseteq m\mathbb{Z}$ by $Z(\{p\}) \le Q$ for any $k \in \{0\} \cup [\![n]\!]$, that proves the claim. □

Remark 6.16

(a) For any $m \in \{0\} \cup \mathbb{N}$, if $Q = (\mathbb{N}, \mathbb{Z}, m\mathbb{Z}, m\mathbb{Z}, m\mathbb{Z}, \mathbb{Z})$ (Part (a) of Lemma 6.17 below), then $\mathcal{R}_Q$ is given by *the set of all partitions such that the color distance between any two legs of the same block is a multiple of m*, as seen in Lemma 6.15. Mostly, it is the L- and K-components of Q inducing the characteristic constraints via $Z(\{\cdot\}) \le Q$. The F-, V- and X-conditions are redundant. However, the Σ-condition generally is *not*. And it is generally not implied by the L- and K-constraints either. (For example, if $p = \updownarrow^{\otimes(m+1)}$, then $L(\{p\}) = K(\{p\}) = \emptyset \subseteq m\mathbb{Z}$ but $\Sigma(\{p\}) = \{m + 1\} \not\subseteq m\mathbb{Z}$ if $m \ge 2$).

(b) If $Q = (\{1, 2\}, \pm\{0, 1, 2\}, 2m\mathbb{Z}, m{+}2m\mathbb{Z}, 2m\mathbb{Z}, \mathbb{Z})$ (Part (b) of Lemma 6.17) for some $m \in \mathbb{N}$, then the set $\mathcal{R}_Q$ is a subset of the set from Part (a): Still, any partition $p \in \mathcal{P}^{\circ\bullet}$ is required to have color distances in $m\mathbb{Z}$ between any two legs of the same block if it is to be an element of $\mathcal{R}_Q$. However, now, additionally, all blocks of p must have sizes one or two, the total color sum $\Sigma(p)$ must be a multiple of $2m$ (and not just m) and, most importantly, the color distances between subsequent legs of the same block must satisfy two different conditions, depending on their normalized colors. Since blocks can have size two at most, two legs of the same block are subsequent if and only if they are distinct. Hence, $\mathcal{R}_Q$ is *the set of all partitions such that*

- *every block has at most two legs,*
- *the total color sum is an* even *multiple of m,*
- *the color distance between any two* distinct *legs of the same block is*

- *an* odd *multiple of m if they have identical normalized colors and*
- *an* even *multiple of m if they have* different *normalized colors.*

Also, note that, in contrast to Part (a), the parameter $m = 0$ is *not* allowed.
For points α and β in $p \in \mathcal{P}^{\circ\bullet}$, saying

$$\delta_p(\alpha, \beta) \in \frac{\sigma_p(\{\alpha, \beta\})}{2} m + 2m\mathbb{Z}.$$

is a helpful technical way of expressing that the distance of the points is an odd multiple of m if they have the same normalized color and an even multiple otherwise.
The constraints induced via $Z(\{\cdot\}) \leq Q$ by the F-, V- and Σ-components are non-trivial and *not* implied by the L- and K-restrictions (for the same reason as before). Of course, the X-condition is still redundant.

Lemma 6.17 *For any $m \in \{0\} \cup \mathbb{N}$ the set $\mathcal{R}_Q$ is a category of partitions if*

(a) $Q = (\mathbb{N}, \mathbb{Z}, m\mathbb{Z}, m\mathbb{Z}, m\mathbb{Z}, \mathbb{Z})$, *or*
(b) $Q = (\{1, 2\}, \pm\{0, 1, 2\}, 2m\mathbb{Z}, m{+}2m\mathbb{Z}, 2m\mathbb{Z}, \mathbb{Z})$.

Proof. The two claims can be verified largely simultaneously. Let Q be one of the tuples given in Claims (a) and (b) and abbreviate $Q = (f_Q, \{0\} \cup (\pm f_Q), k_Q, l_Q, k_Q, \mathbb{Z})$. We prove that $\mathcal{R}_Q$ is a category by means of the Alternative Characterization (Lemma 4.3). Clearly, $\mathord{\uparrow} \in \mathcal{R}_Q$. By Lemma 6.11 the condition $\Sigma(\{\cdot\}) \subseteq k_Q$ is stable under rotation, tensor products, verticolor reflection and erasing turns. The constraint $(F, V)(\{\cdot\}) \leq (f_Q, \{0\} \cup (\pm f_Q))$ is trivially preserved in Case (a), and it is preserved in Case (b) as well according to Lemma 6.13 (a). Hence, it is sufficient to show that $(L, K)(\{\cdot\}) \leq (l_Q, k_Q)$ is invariant under the alternative category operations.

Rotation: By Lemma 6.4 it holds $(L, K)(\{p^r\}) \leq (L, K)(\{p\}) \leq (l_Q, k_Q)$ for any $p \in \mathcal{R}_Q$ and any $r \in \{\hookrightarrow, ⤸, \curvearrowright, ⤹\}$. Consequently, $(L, K)(\{\cdot\}) \leq (l_Q, k_Q)$ is stable under rotations.

Verticolor reflection: Given $p \in \mathcal{R}_Q$, Lemma 6.6 lets us infer that $(L, K)(\{\tilde{p}\}) \leq (-L, -K)(\{\tilde{p}\}) \leq (-l_Q, -k_Q)$. Because $-l_Q = l_Q$ and $-k_Q = k_Q$ we have thus proven that verticolor reflection preserves $(L, K)(\{\cdot\}) \leq (l_Q, k_Q)$.

Tensor product: Let $\{p_1, p_2\} \subseteq \mathcal{R}_Q$ be arbitrary. Then, $\Sigma(\{p_1, p_2\}) \subseteq k_Q$ in particular. Since k_Q is a subgroup of $\mathbb{Z}$ we conclude $\gcd(\Sigma(\{p_1, p_2\}))\mathbb{Z} \subseteq k_Q$. Therefore, Lemma 6.8 (d) implies $L(\{p_1 \otimes p_2\}) \subseteq L(\{p_1, p_2\}) + \gcd(\Sigma(\{p_1, p_2\}))\mathbb{Z} \subseteq l_Q + k_Q$ and $K(\{p_1 \otimes p_2\}) \subseteq K(\{p_1, p_2\}) + \gcd(\Sigma(\{p_1, p_2\}))\mathbb{Z} \subseteq k_Q + k_Q$. Because $l_Q + k_Q \subseteq l_Q$ and $k_Q + k_Q \subseteq k_Q$ by definition of l_Q and k_Q this proves $(L, K)(\{p_1 \otimes p_2\}) \leq (l_Q, k_Q)$. In conclusion, $(L, K)(\{\cdot\}) \leq (l_Q, k_Q)$ is respected by tensor products.

Erasing: Lastly, let T be any turn in any $p \in \mathcal{R}_Q$ and let ϵ be the map which sends the points of $E(p, T)$ to their original locations in p. We show $(L, K)(\{E(p, T)\}) \leq (l_Q, k_Q)$. Let $B \in E(p, T)$ and $\{\alpha, \beta\} \subseteq B$ be arbitrary with $\alpha \neq \beta$ and $]\alpha, \beta[_{E(p,T)} \cap B = \emptyset$. What we then have to prove is that $\delta_{E(p,T)}(\alpha, \beta) \in l_Q$ if $\sigma_{E(p,T)}(\{\alpha, \beta\}) \neq 0$ and that $\delta_{E(p,T)}(\alpha, \beta) \in k_Q$ if $\sigma_{E(p,T)}(\{\alpha, \beta\}) = 0$. By

Lemma 6.9 (a) and (c) it suffices to show

$$\delta_p(\epsilon(\alpha), \epsilon(\beta)) \overset{!}{\in} \begin{cases} l_Q & \text{if } \sigma_p(\{\epsilon(\alpha), \epsilon(\beta)\}) \neq 0, \\ k_Q & \text{otherwise.} \end{cases} \tag{4}$$

To prove (4) we now distinguish the two Cases (a) and (b).

Case 1: *Part (a).* Since $l_Q = k_Q = m\mathbb{Z}$, Assertion (4) simplifies to the claim

$$\delta_p(\epsilon(\alpha), \epsilon(\beta)) \overset{!}{\in} m\mathbb{Z}. \tag{5}$$

Because Q is as required by Lemma 6.15, we can immediately deduce (5) from $p \in \mathcal{R}_Q$ if $\epsilon(\alpha)$ and $\epsilon(\beta)$ belong to the same block in p. Thus, it remains to assume that the blocks B_1 of $\epsilon(\alpha)$ and B_2 of $\epsilon(\beta)$ in p are distinct and prove (5) under this premise.

Step 1.1: *Decomposing* $\delta_p(\epsilon(\alpha), \epsilon(\beta))$. Because $B_1 \neq B_2$ the definition of $E(p, T)$ requires the existence of $\alpha' \in B_1 \cap T$ and $\beta' \in B_2 \cap T$. Lemma 3.2 (c) yields

$$\delta_p(\epsilon(\alpha), \epsilon(\beta)) \equiv \delta_p(\epsilon(\alpha), \alpha') + \delta_p(\alpha', \beta') + \delta_p(\beta', \epsilon(\beta)) \mod m,$$

where we have used the consequence $\Sigma(p) \in m\mathbb{Z}$ of $p \in \mathcal{R}_Q$. By this congruence, Assertion (5) can be seen by proving that all summands on the right-hand side are multiples of m. Hence, that is what we now show.

Step 1.2: *Color distances of* $\epsilon(\alpha)$ *and* α' *and of* $\epsilon(\beta)$ *and* β'. Because $\{\epsilon(\alpha), \alpha'\} \subseteq B_1$ are legs of the same block of p and because $p \in \mathcal{R}_Q$ Lemma 6.15 guarantees $\delta_p(\epsilon(\alpha), \alpha') \in m\mathbb{Z}$. Likewise, $\{\epsilon(\beta), \beta'\} \subseteq B_2$ implies $\delta_p(\beta', \epsilon(\beta)) \in m\mathbb{Z}$.

Step 1.3: *Color distance of* α' *and* β'. Because $B_1 \neq B_2$, necessarily $\alpha' \neq \beta'$ and $T = \{\alpha', \beta'\}$. As T is a turn the neighboring points α' and β' have inverse normalized colors. We infer $\delta_p(\alpha', \beta') = 0$ or $\delta_p(\alpha', \beta') = \Sigma(p)$, depending on whether α' precedes β' or the other way around. As $\Sigma(p) \in m\mathbb{Z}$ due to $p \in \mathcal{R}_Q$ it thus follows $\delta_p(\alpha', \beta') \in m\mathbb{Z}$ in any case. And that is what we needed to see.

Case 2: *Part (b).* In this case, $l_Q = m+2m\mathbb{Z}$ and $k_Q = 2m\mathbb{Z}$ mean that Assertion (4) can be expressed equivalently as

$$\delta_p(\epsilon(\alpha), \epsilon(\beta)) \overset{!}{\in} \frac{\sigma_p(\{\epsilon(\alpha), \epsilon(\beta)\})}{2} m + 2m\mathbb{Z}. \tag{6}$$

If $\epsilon(B) \in p$, then $p \in \mathcal{R}_Q$ implies (6) immediately. Hence, we only need to prove (6) for the case that $\epsilon(B) \notin p$. We want to use Lemma 3.2 (c) again. Hence, we must find a suitable choice of points α' and β' to decompose $\delta_p(\epsilon(\alpha), \epsilon(\beta))$ with.

Step 2.1: *Finding points* α' *and* β'. As $\epsilon(B) \notin p$ there exist $\{B_1, B_2\} \subseteq p$ with $B_1 \cap T \neq \emptyset$, with $B_2 \cap T \neq \emptyset$, with $T \subseteq B_1 \cup B_2$ and with $\epsilon(B) = (B_1 \cup B_2) \backslash T$. In particular, $\{\epsilon(\alpha), \epsilon(\beta)\} \subseteq B_1 \cup B_2$.

The points $\epsilon(\alpha)$ and $\epsilon(\beta)$ belong to different blocks in p: If there existed $i \in \{1, 2\}$ with $\{\epsilon(\alpha), \epsilon(\beta)\} \in B_i$, then it would follow $B_i = \{\epsilon(\alpha), \epsilon(\beta)\}$ since $\alpha \neq \beta$, since ϵ is injective and since $|B_i| \leq 2$ by $p \in \mathcal{P}^{\circ\bullet}_{\leq 2}$. As $\{\epsilon(\alpha), \epsilon(\beta)\} \cap T = \emptyset$ that would contradict $B_i \cap T \neq \emptyset$.

By renaming B_1 and B_2, we can assume $\epsilon(\alpha) \in B_1$ and $\epsilon(\beta) \in B_2$. Since distinct blocks are disjoint, $B_1 \cap B_2 = \emptyset$. Because $\{\epsilon(\alpha), \epsilon(\beta)\} \cap T = \emptyset$ and $B_1 \cap T \neq \emptyset$ and $B_2 \cap T \neq \emptyset$ and because $|B_1| \leq 2$ and $|B_2| \leq 2$ we find $\alpha' \in B_1 \cap T$ and $\beta' \in B_2 \cap T$ such that $B_1 = \{\epsilon(\alpha), \alpha'\}$ and $B_2 = \{\epsilon(\beta), \beta'\}$.

Step 2.2: *Relating the normalized colors of* $\epsilon(\alpha), \alpha', \epsilon(\beta)$ *and* β'. Since $B_1 \neq B_2$, since $\epsilon(B) = (B_1 \cup B_2)\backslash T$ and since $\sigma_p(T) = 0$,

$$\begin{aligned}\sigma_p(\{\epsilon(\alpha), \epsilon(\beta)\}) &= \sigma_p(\epsilon(B)) = \sigma_p(B_1) + \sigma_p(B_2)\\ &= \sigma_p(\{\epsilon(\alpha), \alpha'\}) + \sigma_p(\{\beta', \epsilon(\beta)\}), \end{aligned} \tag{7}$$

as in the proof of Lemma 6.13.

Step 2.3: *Color distances of* $\epsilon(\alpha)$ *and* α' *and of* β' *and* $\epsilon(\beta)$. The assumption $p \in \mathcal{R}_Q$ furthermore guarantees

$$\begin{aligned} &\delta_p(\epsilon(\alpha), \alpha') \in \frac{\sigma_p(\{\epsilon(\alpha), \alpha'\})}{2} m + 2m\mathbb{Z}\\ \text{and}\quad &\delta_p(\beta', \epsilon(\beta)) \in \frac{\sigma_p(\{\beta', \epsilon(\beta)\})}{2} m + 2m\mathbb{Z} \end{aligned} \tag{8}$$

because $\{\epsilon(\alpha), \alpha'\} \subseteq B_1$ and $\alpha \neq \alpha'$ on the one hand and $\{\beta', \epsilon(\beta)\} \subseteq B_2$ and $\beta' \neq \beta$ on the other hand.

Step 2.4: *Color distance of* α' *and* β'. Because $T = \{\alpha', \beta'\}$ is a turn, $\delta_p(\alpha', \beta') = 0$ or $\delta_p(\alpha', \beta') = \Sigma(p)$. From $p \in \mathcal{R}_Q$ it follows $\Sigma(p) \in k_Q = 2m\mathbb{Z}$ and thus

$$\delta_p(\alpha', \beta') \in 2m\mathbb{Z}. \tag{9}$$

We now have all ingredients to prove (6).

Step 2.5: *Synthesis.* Lemma 3.2 (c) yields

$$\begin{aligned} \delta_p(\epsilon(\alpha), \epsilon(\beta)) &\equiv \delta_p(\epsilon(\alpha), \alpha') + \delta_p(\alpha', \beta') + \delta_p(\beta', \epsilon(\beta)) \mod 2m,\\ &\overset{(9)}{\equiv} \delta_p(\epsilon(\alpha), \alpha') + \delta_p(\beta', \epsilon(\beta)) \mod 2m.\\ &\overset{(8)}{\equiv} \frac{\sigma_p(\{\epsilon(\alpha), \alpha'\})}{2} m + \frac{\sigma_p(\{\beta', \epsilon(\beta)\})}{2} m \mod 2m\\ &\overset{(7)}{\equiv} \frac{\sigma_p(\{\epsilon(\alpha), \epsilon(\beta)\})}{2} m \mod 2m. \end{aligned}$$

With the proof of (6) thus complete, so is the proof overall. □

6.5 Color distances between legs of crossing blocks

The last family of elements of Q we treat exhibits non-trivial X-components.

Remark 6.18 If $Q = (\mathbb{N}, \mathbb{Z}, m\mathbb{Z}, m\mathbb{Z}, m\mathbb{Z}, \mathbb{Z}\backslash E)$ for any $m \in \{0\} \cup \mathbb{N}$ and any $E \subseteq \mathbb{Z}$ with $E = -E = E + m\mathbb{Z}$ (as in Lemma 6.19 below), we can employ Lemma 6.15

to understand the set $\mathcal{R}_Q$. It is given by *the set of all partitions such that the color distance between any two legs of the same block is a multiple of* m *and such that, whenever the color distance between any two points is an element of* E, *they belong to non-crossing blocks.*

While the F- and V-components of Q, of course, effectively induce no conditions at all via $Z(\{\cdot\}) \leq Q$, the Σ-, L- and K-conditions are non-trivial and they are *not* implied by the X-constraint.

Lemma 6.19 *For any* $m \in \{0\} \cup \mathbb{N}$ *and* $E \subseteq \mathbb{Z}$ *with* $E = -E = E + m\mathbb{Z}$ *the set* $\mathcal{R}_Q$ *is a category if*

$$Q = (\mathbb{N}, \mathbb{Z}, m\mathbb{Z}, m\mathbb{Z}, m\mathbb{Z}, \mathbb{Z}\backslash E).$$

Proof Let Q be of the kind described in the claim. Once more, we use the Alternative Characterization (Lemma 4.3) to show that $\mathcal{R}_Q$ is a category. By Lemma 6.17 (a) it suffices to consider the X-component of Z, i.e., to prove that the constraint $X(\{\cdot\}) \subseteq \mathbb{Z}\backslash E$ is invariant under rotation, tensor products, verticolor reflection and erasing turns.

Rotation: With the help of Lemma 6.4 we can infer $X(\{p^r\}) \subseteq X(\{p\}) \subseteq \mathbb{Z}\backslash E$ for any $p \in \mathcal{R}_Q$ and any $r \in \{↪, ⤸, \curvearrowright, ⤹\}$. Thus, $X(\{\cdot\}) \subseteq \mathbb{Z}\backslash E$ is preserved by rotations.

Verticolor reflection: For any $p \in \mathcal{R}_Q$ it holds $X(\{\tilde{p}\}) \subseteq -X(\{p\}) \subseteq -\mathbb{Z}\backslash E$ by Lemma 6.6. Since our assumption $E = -E$ implies $-\mathbb{Z}\backslash E = \mathbb{Z}\backslash E$, this proves the condition $X(\{\cdot\}) \subseteq \mathbb{Z}\backslash E$ stable under verticolor reflection.

Tensor product: Let $\{p_1, p_2\} \subseteq \mathcal{R}_Q$ be arbitrary. Then, $\Sigma(\{p_1, p_2\}) \subseteq m\mathbb{Z}$ implies $\gcd(\Sigma(\{p_1, p_2\}))\mathbb{Z} \subseteq m\mathbb{Z}$. Consequently, Lemma 6.8 (d) yields $X(\{p_1 \otimes p_2\}) \subseteq X(\{p_1, p_2\}) + \gcd(\Sigma(\{p_1, p_2\}))\mathbb{Z} \subseteq \mathbb{Z}\backslash E + m\mathbb{Z}$. As we have assumed $E = E + m\mathbb{Z}$, also $\mathbb{Z}\backslash E = \mathbb{Z}\backslash E + m\mathbb{Z}$ and thus $X(\{p_1 \otimes p_2\}) \subseteq \mathbb{Z}\backslash E$. In conclusion, tensor products respect $X(\{\cdot\}) \subseteq \mathbb{Z}\backslash E$.

Erasing: Let $p \in \mathcal{R}_Q$ be arbitrary, let T be any turn in p and let ϵ be the mapping which moves the points of $E(p, T)$ back to their original positions in p. We show $X(\{E(p, T)\}) \subseteq \mathbb{Z}\backslash E$. Let α_1 and α_2 be any points in $E(p, T)$ whose respective blocks B_1 and B_2 cross in $E(p, T)$. By Lemma 6.9 (c) all we have to show is

$$\delta_p(\epsilon(\alpha_1), \epsilon(\alpha_2)) \notin E. \tag{10}$$

If both $\epsilon(B_1) \in p$ and $\epsilon(B_2) \in p$, then (10) is true by the assumption $p \in \mathcal{R}_Q$. Thus, we only need to show (10) in the opposite case. If so, then, by nature of the erasing operation, exactly one of the sets $\epsilon(B_1)$ and $\epsilon(B_2)$ is a block of p.

Step 1: *Reduction to the case* $\epsilon(B_1) \notin p$. The assumption $p \in \mathcal{R}_Q$ ensures $\Sigma(p) \in m\mathbb{Z}$. Hence, Lemma 3.2 (b) implies $\delta_p(\epsilon(\alpha_1), \epsilon(\alpha_2)) \equiv -\delta_p(\epsilon(\alpha_2), \epsilon(\alpha_1)) \mod m$. Because we assume $\mathbb{Z}\backslash E = -(\mathbb{Z}\backslash E) + m\mathbb{Z}$, that means $\delta_p(\epsilon(\alpha_1), \epsilon(\alpha_2)) \in \mathbb{Z}\backslash E$ if and only if $\delta_p(\epsilon(\alpha_2), \epsilon(\alpha_1)) \in \mathbb{Z}\backslash E$. Hence, it suffices to prove (10) for the case $\epsilon(B_1) \notin p$.

Step 2: *Expressing* $\delta_p(\epsilon(\alpha_1), \epsilon(\alpha_2))$ *as the color distance between some blocks* $B_{1,i}$ *and* $\epsilon(B_2)$ *of* p. Since $\epsilon(B_1) \notin p$ there are (not necessarily distinct) blocks $B_{1,1}$ and $B_{1,2}$ of p with $B_{1,1} \cap T \neq \emptyset$, with $B_{1,2} \cap T \neq \emptyset$, with $T \subseteq B_{1,1} \cup B_{1,2}$ and with $\epsilon(B_1) = (B_{1,1} \cup B_{1,2})\backslash T$. By renaming if necessary, we can always achieve $\epsilon(\alpha_1) \in B_{1,1}$. We show that, for any $\beta_1 \in B_{1,1} \cup B_{1,2}$ and any $\beta_2 \in \epsilon(B_2)$,

$$\delta_p(\epsilon(\alpha_1), \epsilon(\alpha_2)) \equiv \delta_p(\beta_1, \beta_2) \mod m. \tag{11}$$

Two cases must be distinguished.

Case 2.1: *β_1 and $\epsilon(\alpha_1)$ belong to the same block.* Let $\beta_1 \in B_{1,1}$. By Lemma 3.2 (c),

$$\delta_p(\epsilon(\alpha_1), \epsilon(\alpha_2)) \equiv \delta_p(\epsilon(\alpha_1), \beta_1) + \delta_p(\beta_1, \beta_2) + \delta_p(\beta_2, \epsilon(\alpha_2)) \mod m. \tag{12}$$

According to Lemma 6.15, the assumption $p \in \mathcal{R}_Q$ implies $\delta_p(\epsilon(\alpha_1), \beta_1) \in m\mathbb{Z}$ and $\delta_p(\beta_2, \epsilon(\alpha_2)) \in m\mathbb{Z}$ since $\{\epsilon(\alpha_1), \beta_1\} \subseteq B_{1,1}$ and $\{\beta_2, \epsilon(\alpha_2)\} \subseteq \epsilon(B_2)$. Hence, the only term on the right side of (12) possibly surviving is $\delta_p(\beta_1, \beta_2)$. That proves (11) in this case.

Case 2.2: *β_1 and $\epsilon(\alpha_1)$ belong to different blocks.* Now, suppose $\beta_1 \in B_{1,2}$ instead. Since we have assumed $B_{1,1} \cap T \neq \emptyset$ and $B_{1,2} \cap T \neq \emptyset$ we can infer the existence of $\gamma_{1,1} \in B_{1,1}$ and $\gamma_{1,2} \in B_{1,2}$ with $\gamma_{1,1} \neq \gamma_{1,2}$ and $T = \{\gamma_{1,1}, \gamma_{1,2}\}$. Again, we use Lemma 3.2 (c) to derive

$$\begin{aligned}\delta_p(\epsilon(\alpha_1), \epsilon(\alpha_2)) \equiv \delta_p(\epsilon(\alpha_1), \gamma_{1,1}) + \delta_p(\gamma_{1,1}, \gamma_{1,2}) + \delta_p(\gamma_{1,2}, \beta_1) \\ + \delta_p(\beta_1, \beta_2) + \delta_p(\beta_2, \epsilon(\alpha_2)) \mod m.\end{aligned} \tag{13}$$

All summands on the right-hand side of (13) except for possibly $\delta_p(\beta_1, \beta_2)$ are multiples of m:

Because T is a turn, $\gamma_{1,1}$ and $\gamma_{1,2}$ are neighbors of different normalized colors, meaning $\delta_p(\gamma_{1,1}, \gamma_{1,2}) = 0$ or $\delta_p(\gamma_{1,1}, \gamma_{1,2}) = \Sigma(p) \in m\mathbb{Z}$ and thus $\delta_p(\gamma_{1,1}, \gamma_{1,2}) \in m\mathbb{Z}$ in any case.

And, thanks to $p \in \mathcal{R}_Q$, from $\{\epsilon(\alpha_1), \gamma_{1,1}\} \subseteq B_{1,1}$, from $\{\gamma_{1,2}, \beta_1\} \subseteq B_{1,2}$ and from $\{\beta_2, \epsilon(\alpha_2)\} \subseteq \epsilon(B_2)$ follow $\delta_p(\epsilon(\alpha_1), \gamma_{1,1}) \in m\mathbb{Z}$ and $\delta_p(\gamma_{1,2}, \beta_1) \in m\mathbb{Z}$ and $\delta_p(\beta_2, \epsilon(\alpha_2)) \in m\mathbb{Z}$ according to Lemma 6.15.

Thus (13) verifies (11) in this case.

Step 3: *Showing that one $B_{1,i}$ and $\epsilon(B_2)$ cross in p.* If we can establish that $B_{1,1}$ and $\epsilon(B_2)$ or $B_{1,2}$ and $\epsilon(B_2)$ cross each other in p, then Eq. (11) proves (10) as $\mathbb{Z}\backslash E = (\mathbb{Z}\backslash E) + m\mathbb{Z}$. So, this is all we have to show.

Because the blocks B_1 and B_2 cross in $E(p, T)$ we find points $\{\chi_{1,1}, \chi_{1,2}\} \subseteq B_1$ and $\{\eta_2, \theta_2\} \subseteq B_2$ such that $(\chi_{1,1}, \eta_2, \chi_{1,2}, \theta_2)$ is ordered in $E(p, T)$ (and thus also the tuple $(\epsilon(\chi_{1,1}), \epsilon(\eta_2), \epsilon(\chi_{1,2}), \epsilon(\theta_2))$ in p). If $\epsilon(\chi_{1,1})$ and $\epsilon(\chi_{1,2})$ (each of which is contained in $B_{1,1} \cup B_{1,2}$) both belong to $B_{1,1}$ or both to $B_{1,2}$, then we have already found the desired crossing with $\epsilon(B_2)$. Hence, we can assume that neither is the case, i.e., that $B_{1,1} \neq B_{1,2}$ and that $\epsilon(\chi_{1,1})$ and $\epsilon(\chi_{1,2})$ belong to different blocks in p. If $\epsilon(\chi_{1,1}) \notin B_{1,1}$, then we rename $\chi_{1,1} \leftrightarrow \chi_{1,2}$ and $\eta_2 \leftrightarrow \theta_2$. Thus, we can achieve that $\epsilon(\chi_{1,1}) \in B_{1,1}$ and $\epsilon(\chi_{1,2}) \in B_{1,2}$ while maintaining $\{\eta_2, \theta_2\} \subseteq B_2$ and while keeping $(\chi_{1,1}, \eta_2, \chi_{1,2}, \theta_2)$ ordered.

Let $\gamma_{1,1}$ and $\gamma_{1,2}$ be as before, i.e., $\gamma_{1,1} \in B_{1,1} \cap T$ and $\gamma_{1,2} \in B_{1,2} \cap T$. Since they are neighbors, they have the same position in the cyclic order of p with respect to the tuple $(\epsilon(\chi_{1,1}), \epsilon(\eta_2), \epsilon(\chi_{1,2}), \epsilon(\theta_2))$. Let $i \in \{1, 2\}$ be such that $\gamma_{1,i}$ precedes $\gamma_{1,3-i}$ in p. Then, the below table shows that, no matter how the points are arranged, we can find a crossing between $B_{1,1}$ and $\epsilon(B_2)$ or between $B_{1,2}$ and $\epsilon(B_2)$:

ordered tuple	crossing	between
$(\epsilon(\chi_{1,1}), \epsilon(\eta_2), \epsilon(\chi_{1,2}), \epsilon(\theta_2), \underline{\gamma_{1,i}, \gamma_{1,3-i}})$	$(\epsilon(\eta_2), \epsilon(\chi_{1,2}), \epsilon(\theta_2), \gamma_{1,2})$	$B_{1,2}$ and $\epsilon(B_2)$
$(\epsilon(\chi_{1,1}), \epsilon(\eta_2), \epsilon(\chi_{1,2}), \underline{\gamma_{1,i}, \gamma_{1,3-i}}, \epsilon(\theta_2))$	$(\epsilon(\chi_{1,1}), \epsilon(\eta_2), \gamma_{1,1}, \epsilon(\theta_2))$	$B_{1,1}$ and $\epsilon(B_2)$
$(\epsilon(\chi_{1,1}), \epsilon(\eta_2), \underline{\gamma_{1,i}, \gamma_{1,3-i}}, \epsilon(\chi_{1,2}), \epsilon(\theta_2))$	$(\epsilon(\chi_{1,1}), \epsilon(\eta_2), \gamma_{1,1}, \epsilon(\theta_2))$	$B_{1,1}$ and $\epsilon(B_2)$
$(\epsilon(\chi_{1,1}), \underline{\gamma_{1,i}, \gamma_{1,3-i}}, \epsilon(\eta_2), \epsilon(\chi_{1,2}), \epsilon(\theta_2))$	$(\gamma_{1,2}, \epsilon(\eta_2), \epsilon(\chi_{1,2}), \epsilon(\theta_2))$	$B_{1,2}$ and $\epsilon(B_2)$

So, in combination with what we showed in Step 2, that completes the proof. $\square$

6.6 Synthesis

As announced we can combine the previous results to prove that all sets defined in Sect. 5 are categories. Recall Notation 5.6.

Theorem 6.20 *For every $Q \in \mathsf{Q}$ the set $\mathcal{R}_Q$ is a non-hyperoctahedral category:*

$$\mathcal{R} : \mathsf{Q} \to \mathsf{PCat}^{\circ\bullet}_{NHO}.$$

Proof For any $\mathsf{Q}' \subseteq \mathsf{Q}$, by Lemma 6.2,

$$\bigcap\{\mathcal{R}_Q \mid Q \in \mathsf{Q}'\} = \mathcal{R}_{\bigcap_\times \mathsf{Q}'}.$$

We show that for each $Q \in \mathsf{Q}$ there exists a set $\mathsf{Q}' \subseteq \mathsf{Q}$ such that $Q = \bigcap_\times \mathsf{Q}'$ and such that $\mathcal{R}_{Q'}$ is a category for each $Q' \in \mathsf{Q}'$. As $\mathsf{PCat}^{\circ\bullet}$ is a complete lattice, that then proves that $\mathcal{R}_Q$ is a category of two-colored partitions for every $Q \in \mathsf{Q}$. It is straightforward to check that indeed $\updownarrow_\circ \otimes \updownarrow_\bullet \in \mathcal{R}_Q$ or $\sqcap\sqcap_{\circ\bullet\circ\bullet} \notin \mathcal{R}_Q$ for every $Q \in \mathsf{Q}$.

For this proof only, we use names for specific elements of the set Q: The below table applies for any $m \in \{0\} \cup \mathbb{N}$ and $E \subseteq \mathbb{Z}$ with $E = -E = E + m\mathbb{Z}$.

	F	V	Σ	L	K	X	
F_2	$\{2\}$	$\pm\{0, 2\}$	$2\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	6.13 (b)
$\mathsf{F}_{\leq 2}$	$\{1, 2\}$	$\pm\{0, 1, 2\}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	6.13 (a)
V_0	$\{2\}$	$\{0\}$	$\{0\}$	$\emptyset$	$\mathbb{Z}$	$\mathbb{Z}$	6.13 (c)
V_{01}	$\{1, 2\}$	$\pm\{0, 1\}$	$\mathbb{Z}$	$\emptyset$	$\mathbb{Z}$	$\mathbb{Z}$	6.13 (d)
S_m	$\mathbb{N}$	$\mathbb{Z}$	$m\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	6.11
K_m	$\mathbb{N}$	$\mathbb{Z}$	$m\mathbb{Z}$	$m\mathbb{Z}$	$m\mathbb{Z}$	$\mathbb{Z}$	6.17 (a)
$\mathsf{K}^{\curlywedge}_m$	$\{1, 2\}$	$\pm\{0, 1, 2\}$	$2m\mathbb{Z}$	$m+2m\mathbb{Z}$	$2m\mathbb{Z}$	$\mathbb{Z}$	6.17 (b)
$\mathsf{X}_{m,E}$	$\mathbb{N}$	$\mathbb{Z}$	$m\mathbb{Z}$	$m\mathbb{Z}$	$m\mathbb{Z}$	$\mathbb{Z}\backslash E$	6.19

The corresponding sets $\mathcal{R}_Q$ for all these elements $Q \in \mathsf{Q}$ are categories of partitions, as shown by the respective lemma cited in the last column.

The ensuing table lists how each element of Q can be written as a meet in L of the specific elements defined above. Here, $u \in \{0\} \cup \mathbb{N}$, $m \in \mathbb{N}$, $D \subseteq \{0\} \cup [\![\lfloor \frac{m}{2} \rfloor]\!]$, $E \subseteq \{0\} \cup \mathbb{N}$ and N is a subsemigroup of $(\mathbb{N}, +)$. Recall $N_0 = \pm N$ and $N_0' = \pm N \cup \{0\}$.

F	V	S	L	K	X	$\bigcap_\times$
$\{2\}$	$\pm\{0,2\}$	$2um\mathbb{Z}$	$m\mathbb{Z}$	$m\mathbb{Z}$	$\mathbb{Z}$	$\mathrm{F}_2, \mathrm{S}_{2um}, \mathrm{K}_m$
$\{2\}$	$\pm\{0,2\}$	$2um\mathbb{Z}$	$m{+}2m\mathbb{Z}$	$2m\mathbb{Z}$	$\mathbb{Z}$	$\mathrm{F}_2, \mathrm{S}_{2um}, \mathrm{K}_m^\wedge$
$\{2\}$	$\pm\{0,2\}$	$2um\mathbb{Z}$	$m{+}2m\mathbb{Z}$	$2m\mathbb{Z}$	$\mathbb{Z}\backslash m\mathbb{Z}$	$\mathrm{F}_2, \mathrm{S}_{2um}, \mathrm{K}_m^\wedge, X_{m,m\mathbb{Z}}$
$\{2\}$	$\{0\}$	$\{0\}$	$\emptyset$	$m\mathbb{Z}$	$\mathbb{Z}$	$\mathrm{V}_0, \mathrm{K}_m$
$\{2\}$	$\pm\{0,2\}$	$\{0\}$	$\{0\}$	$\{0\}$	$\mathbb{Z}\backslash N_0$	$\mathrm{F}_2, \mathrm{X}_{0,N_0}$
$\{2\}$	$\{0\}$	$\{0\}$	$\emptyset$	$\{0\}$	$\mathbb{Z}\backslash N_0$	$\mathrm{V}_0, \mathrm{X}_{0,N_0}$
$\{2\}$	$\{0\}$	$\{0\}$	$\emptyset$	$\{0\}$	$\mathbb{Z}\backslash N_0'$	$\mathrm{V}_0, \mathrm{X}_{0,N_0'}$
$\{1,2\}$	$\pm\{0,1,2\}$	$um\mathbb{Z}$	$m\mathbb{Z}$	$m\mathbb{Z}$	$\mathbb{Z}\backslash D_m$	$\mathrm{F}_{\leq 2}, \mathrm{S}_{um}, \mathrm{X}_{m,D_m}$
$\{1,2\}$	$\pm\{0,1,2\}$	$2um\mathbb{Z}$	$m{+}2m\mathbb{Z}$	$2m\mathbb{Z}$	$\mathbb{Z}\backslash D_m$	$\mathrm{S}_{2um}, \mathrm{K}_m^\wedge, \mathrm{X}_{m,D_m}$
$\{1,2\}$	$\pm\{0,1\}$	$um\mathbb{Z}$	$\emptyset$	$m\mathbb{Z}$	$\mathbb{Z}\backslash D_m$	$\mathrm{V}_{01}, \mathrm{S}_{um}, \mathrm{X}_{m,D_m}$
$\{1,2\}$	$\pm\{0,1,2\}$	$\{0\}$	$\{0\}$	$\{0\}$	$\mathbb{Z}\backslash E_0$	$\mathrm{F}_{\leq 2}, \mathrm{X}_{0,E_0}$
$\{1,2\}$	$\pm\{0,1\}$	$\{0\}$	$\emptyset$	$\{0\}$	$\mathbb{Z}\backslash E_0$	$\mathrm{V}_{01}, \mathrm{X}_{0,E_0}$
$\mathbb{N}$	$\mathbb{Z}$	$um\mathbb{Z}$	$m\mathbb{Z}$	$m\mathbb{Z}$	$\mathbb{Z}\backslash D_m$	$\mathrm{S}_{um}, \mathrm{X}_{m,D_m}$
$\mathbb{N}$	$\mathbb{Z}$	$\{0\}$	$\{0\}$	$\{0\}$	$\mathbb{Z}\backslash E_0$	X_{0,E_0}

That concludes the proof. □

7 Concluding remarks

The present article defined a family of sets $\mathcal{R}_Q$ of two-colored partitions indexed by elements Q of an index set Q and confirmed each such set to be a non-hyperoctahedral category. This finding extends the previous classification results by Tarrago and the second author from [12], by Gromada from [5] and by the authors from [7] and [6]. Most of those prior results describe the categories they classify both in terms of their elements and their generators. The only exception is the so-called *group case* of categories, where only generators are known.

In the following, for each non-hyperoctahedral category $\mathcal{C} \subseteq \mathcal{P}^{\circ\bullet}$ from those articles an index $Q \in \mathsf{Q}$ with $\mathcal{C} = \mathcal{R}_Q$ will be provided—except for the group case, where only $\mathcal{C} \subseteq \mathcal{R}_Q$ will be clear at the moment and where $\mathcal{C} \supseteq \mathcal{R}_Q$ will be shown in a future part of the article series.

Each category $\mathcal{C}$ from the literature will be referenced by, firstly, its name, i.e., symbol, from the respective article and, secondly, by the same graphical depictions of the generating partitions employed there. However, the definition of $\mathcal{C}$ in terms of its elements given in the article in question is not repeated here. That is because in providing an index Q with $\mathcal{C} = \mathcal{R}_Q$ a full description of the elements of $\mathcal{C}$ is obtained via the definition $\mathcal{R}_Q = \{p \in \mathcal{P}^{\circ\bullet} \mid Z(\{p\}) \leq Q\}$.

7.1 Globally colorized non-crossing case

In [12, Theorem 7.1] Tarrago and the second author described all globally colorized non-crossing categories. A category $\mathcal{C} \subseteq \mathcal{P}^{\circ\bullet}$ is said to be *non-crossing* if $\mathcal{C} \subseteq \mathcal{NC}^{\circ\bullet}$ and *globally colorized* if $\sqcap \otimes \sqcap \in \mathcal{C}$. Five families of such categories exist. The non-hyperoctahedral ones among those are tabled below, where $k \in \{0\} \cup \mathbb{N}$ arbitrary. Here, as in the following, the first column gives the name of the category, the second a generator and columns three to eight a corresponding element of Q.

		F	V	Σ	L	K	X
$\mathcal{O}_{\mathrm{glob}}(2k)$	$\sqcap^{\otimes k}$, $\sqcap \otimes \sqcap$	$\{2\}$	$\pm\{0,2\}$	$2k\mathbb{Z}$	$1+2\mathbb{Z}$	$2\mathbb{Z}$	$\emptyset$
$\mathcal{B}'_{\mathrm{glob}}(k)$	$\uparrow^{\otimes k}$, $\uparrow\sqcap\uparrow$, $\uparrow\otimes\uparrow$, $\sqcap\otimes\sqcap$	$\{1,2\}$	$\pm\{0,1,2\}$	$k\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$\emptyset$
$\mathcal{B}_{\mathrm{glob}}(2k)$	$\uparrow^{\otimes 2k}$, $\uparrow\otimes\uparrow$, $\sqcap\otimes\sqcap$	$\{1,2\}$	$\pm\{0,1,2\}$	$2k\mathbb{Z}$	$1+2\mathbb{Z}$	$2\mathbb{Z}$	$\emptyset$
$\mathcal{S}_{\mathrm{glob}}(k)$	$\uparrow^{\otimes k}$, $\sqcap\sqcap\sqcap$, $\uparrow\otimes\uparrow$, $\sqcap\otimes\sqcap$	$\mathbb{N}$	$\mathbb{Z}$	$k\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$\emptyset$

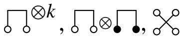
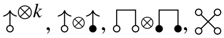
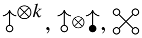
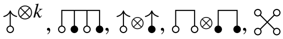
7.2 Locally colorized non-crossing case

A category $\mathcal{C} \subseteq \mathcal{P}^{\circ\bullet}$ is said to be *locally colorized* if it is not globally colorized, i.e., if $\sqcap\otimes\sqcap \notin \mathcal{C}$. All non-crossing locally colorized categories were found by Tarrago and the second author in [12, Theorem 7.2]. The ensuing table, where $r \in \mathbb{N}\backslash\{1\}$ and $u \in \{0\} \cup \mathbb{N}$ can be arbitrary, provides the correspondence for the non-hyperoctahedral case.

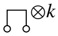
		F	V	Σ	L	K	X
$\mathcal{O}_{\mathrm{loc}}$	$\emptyset$	$\{2\}$	$\{0\}$	$\{0\}$	$\emptyset$	$\{0\}$	$\emptyset$
$\mathcal{B}'_{\mathrm{loc}}(ur, r, 0)$	$\uparrow^{\otimes ur}$, $\uparrow^{\otimes r}\,\boxed{\uparrow^{\otimes r}}$, $\uparrow\sqcap\uparrow$, $\uparrow\otimes\uparrow$	$\{1,2\}$	$\pm\{0,1,2\}$	$ur\mathbb{Z}$	$r\mathbb{Z}$	$r\mathbb{Z}$	$\emptyset$
$\mathcal{B}'_{\mathrm{loc}}(2ur, 2r, r)$	$\uparrow^{\otimes 2ur}$, $\uparrow^{\otimes r+1}\,\boxed{\uparrow^{\otimes r-1}}$, $\uparrow^{\otimes 2r}\,\boxed{\uparrow^{\otimes 2r}}$, $\uparrow\otimes\uparrow$	$\{1,2\}$	$\pm\{0,1,2\}$	$2ur\mathbb{Z}$	$r+2r\mathbb{Z}$	$2r\mathbb{Z}$	$\emptyset$
$\mathcal{B}_{\mathrm{loc}}(ur, r)$	$\uparrow^{\otimes ur}$, $\uparrow^{\otimes r}\,\boxed{\uparrow^{\otimes r}}$, $\uparrow\otimes\uparrow$	$\{1,2\}$	$\pm\{0,1\}$	$ur\mathbb{Z}$	$\emptyset$	$r\mathbb{Z}$	$\emptyset$
$\mathcal{B}_{\mathrm{loc}}(u, 1)$	$\uparrow^{\otimes u}$, $\uparrow\sqcap\uparrow$, $\uparrow\otimes\uparrow$	$\{1,2\}$	$\pm\{0,1\}$	$u\mathbb{Z}$	$\emptyset$	$\mathbb{Z}$	$\emptyset$
$\mathcal{B}'_{\mathrm{loc}}(0, 0, 0)$	$\uparrow\sqcap\uparrow$, $\uparrow\otimes\uparrow$	$\{1,2\}$	$\pm\{0,1,2\}$	$\{0\}$	$\{0\}$	$\{0\}$	$\emptyset$
$\mathcal{B}_{\mathrm{loc}}(0, 0)$	$\uparrow\otimes\uparrow$	$\{1,2\}$	$\pm\{0,1\}$	$\{0\}$	$\emptyset$	$\{0\}$	$\emptyset$
$\mathcal{S}_{\mathrm{loc}}(ur, r)$	$\uparrow^{\otimes ur}$, $\uparrow^{\otimes r}\,\boxed{\uparrow^{\otimes r}}$, $\sqcap\sqcap\sqcap$, $\uparrow\otimes\uparrow$	$\mathbb{N}$	$\mathbb{Z}$	$ur\mathbb{Z}$	$r\mathbb{Z}$	$r\mathbb{Z}$	$\emptyset$
$\mathcal{S}_{\mathrm{loc}}(0, 0)$	$\sqcap\sqcap\sqcap$, $\uparrow\otimes\uparrow$	$\mathbb{N}$	$\mathbb{Z}$	$\{0\}$	$\{0\}$	$\{0\}$	$\emptyset$

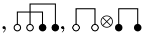
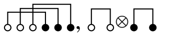
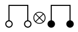
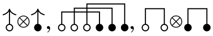
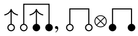
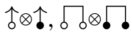
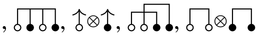
7.3 Group case

If $\times \in \mathcal{C}$, a category $\mathcal{C} \subseteq \mathcal{P}^{\circ\bullet}$ belongs to the *group case*. All such categories were classified by Tarrago and the second author in [12, Theorem 8.3] in terms of their generators. For each non-hyperoctahedral group case category $\mathcal{C}$ the table below, where k runs through all of $\{0\} \cup \mathbb{N}$, gives an element $Q \in \mathsf{Q}$ with $\mathcal{C} \subseteq \mathcal{R}_Q$. That much can be readily verified via the generators. In a future part of the present article series, it will be shown that $\mathcal{C} \supseteq \mathcal{R}_Q$ holds as well.

		F	V	Σ	L	K	X
$\mathcal{O}_{\mathrm{grp,glob}}(2k)$		$\{2\}$	$\pm\{0,2\}$	$2k\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$
$\mathcal{O}_{\mathrm{grp,loc}}$		$\{2\}$	$\pm\{0,2\}$	$\{0\}$	$\emptyset$	$\mathbb{Z}$	$\mathbb{Z}$
$\mathcal{B}_{\mathrm{grp,glob}}(k)$		$\{1,2\}$	$\pm\{0,1,2\}$	$k\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$
$\mathcal{B}_{\mathrm{grp,loc}}(k)$		$\{1,2\}$	$\pm\{0,1,2\}$	$k\mathbb{Z}$	$\emptyset$	$\mathbb{Z}$	$\mathbb{Z}$
$\mathcal{S}_{\mathrm{grp,glob}}(k)$		$\mathbb{N}$	$\mathbb{Z}$	$k\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$

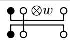
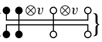
7.4 Globally colorized case

Extending [12, Theorem 7.1], Gromada determined in [5, Theorem 3.10] all globally colorized categories. Write $u_{2k} :=$  for each $k \in \mathbb{N}$ with $u_0 := \emptyset$ and $s_k :=$  for each $k \in \mathbb{N}$ with $s_0 := \emptyset$. For every $k \in \{0\} \cup \mathbb{N}$ the following table lists the correspondences in the globally colorized non-hyperoctahedral case:

		F	V	Σ	L	K	X
$\mathcal{O}_{\mathrm{grp,glob}}(2k)$	u_{2k}, 	$\{2\}$	$\pm\{0,2\}$	$2k\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$
$\mathcal{O}_{\mathrm{hl,glob}}(2k)$	u_{2k}, 	$\{2\}$	$\pm\{0,2\}$	$2k\mathbb{Z}$	$1+2\mathbb{Z}$	$2\mathbb{Z}$	$\mathbb{Z}$
$\mathcal{O}_{\mathrm{glob}}(2k)$	u_{2k}, 	$\{2\}$	$\pm\{0,2\}$	$2k\mathbb{Z}$	$1+2\mathbb{Z}$	$2\mathbb{Z}$	$\emptyset$
$\mathcal{B}_{\mathrm{grp,glob}}(k)$	s_k, 	$\{1,2\}$	$\pm\{0,1,2\}$	$k\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$
$\mathcal{B}_{\mathrm{hl,glob}}(2k)$	s_{2k}, 	$\{1,2\}$	$\pm\{0,1,2\}$	$2k\mathbb{Z}$	$1+2\mathbb{Z}$	$2\mathbb{Z}$	$\mathbb{Z}$
$\mathcal{B}'_{\mathrm{glob}}(k)$	s_k, 	$\{1,2\}$	$\pm\{0,1,2\}$	$k\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$\emptyset$
$\mathcal{B}_{\mathrm{glob}}(2k)$	s_{2k}, 	$\{1,2\}$	$\pm\{0,1,2\}$	$2k\mathbb{Z}$	$1+2\mathbb{Z}$	$2\mathbb{Z}$	$\emptyset$
$\mathcal{S}_{\mathrm{grp,glob}}(k)$	s_k, 	$\mathbb{N}$	$\mathbb{Z}$	$k\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$
$\mathcal{S}_{\mathrm{glob}}(k)$	s_k, 	$\mathbb{N}$	$\mathbb{Z}$	$k\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$\emptyset$

7.5 Neutral pair partitions

For every $w \in \mathbb{N}$ and all subsemigroups D and E of $(\mathbb{N}, +)$ such that $\mathbb{N}\backslash D$ is finite but non-empty and such that $\mathbb{N}\backslash E$ is infinite, the following identities are valid:

		F	V	Σ	L	K	X
$\mathcal{S}_w$		$\{2\}$	$\{0\}$	$\{0\}$	$\emptyset$	$w\mathbb{Z}$	$\mathbb{Z}$
$\mathcal{S}_0$		$\{2\}$	$\{0\}$	$\{0\}$	$\emptyset$	$\{0\}$	$\mathbb{Z}$
$\mathcal{I}_E$	$\{\mathrm{Br}_\bullet([\![k]\!]\backslash E)\}_{k\in\mathbb{N}}$, 	$\{2\}$	$\{0\}$	$\{0\}$	$\emptyset$	$\{0\}$	$\mathbb{Z}\backslash E_0$
$\mathcal{I}_{E\cup\{0\}}$	$\{\mathrm{Br}_\bullet([\![k]\!]\backslash E)\}_{k\in\mathbb{N}}$	$\{2\}$	$\{0\}$	$\{0\}$	$\emptyset$	$\{0\}$	$\mathbb{Z}\backslash E'_0$
$\mathcal{I}_D$	$\mathrm{Br}_\bullet(\mathbb{N}\backslash D)$, 	$\{2\}$	$\{0\}$	$\{0\}$	$\emptyset$	$\{0\}$	$\mathbb{Z}\backslash D_0$
$\mathcal{I}_{D\cup\{0\}}$	$\mathrm{Br}_\bullet(\mathbb{N}\backslash D)$	$\{2\}$	$\{0\}$	$\{0\}$	$\emptyset$	$\{0\}$	$\mathbb{Z}\backslash D'_0$
$\mathcal{I}_{\mathbb{N}}$		$\{2\}$	$\{0\}$	$\{0\}$	$\emptyset$	$\{0\}$	$\{0\}$
$\mathcal{I}_{\mathbb{N}_0}$	$\emptyset$	$\{2\}$	$\{0\}$	$\{0\}$	$\emptyset$	$\{0\}$	$\emptyset$

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