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# On the stable dynamical spectrum of complex surfaces

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**Abstract**
We characterize Salem numbers which have some power arising as dynamical degree of an automorphism on a complex (projective) 2-Torus, K3 or Enriques surface.

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## 1 Introduction

To a bimeromorphic transformation $F : X \dashrightarrow X$ of a Kähler surface one can associate its *dynamical degree*

$$\lambda(F) = \limsup_{n\to\infty}(||(F^n)^*||)^{1/n},$$

where $F^*$ denotes the action on $H^2(X,\mathbb{Z})$ and $||\cdot||$ is any norm on $\mathrm{End}(H^2(X,\mathbb{Z}))$. The dynamical degree is a bimeromorphic invariant of $(X, F)$ which measures the dynamical complexity of $F$. In the projective case, it describes the asymptotic degree growth of defining equations for $F$. If $F$ is an automorphism, then the dynamical degree $\lambda(F)$ is given by the spectral radius of $F^*$. In fact it is either 1 or a Salem number, that is, an algebraic integer $\lambda > 1$ which is Galois conjugate to $1/\lambda$ and all whose other conjugates lie on the unit circle.

The question this paper is dealing with is: which Salem numbers are dynamical degrees of surface automorphisms and which ones are coming from automorphisms of projective surfaces?

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If the dynamical degree of an automorphism $F$ of a surface $X$ is $\lambda(F) > 1$, then $X$ is a blow up of the projective plane in at least 10 points, or a blow up of a 2-Torus, a K3 or an Enriques surface [6]. For rational surfaces the contribution to the dynamical spectrum coming from automorphisms is described in terms of Weyl groups in [12] and the case of complex 2-Tori (respectively Abelian surfaces) is completely described in [11]. The complete determination of the dynamical spectrum of complex projective K3 surfaces seems out of reach.

However, we can relax the problem by asking for stable realizations of dynamical degrees instead, that is: which Salem numbers have some power that is realized as a dynamical degree of a (projective) surface? Indeed this question is more tractable, and the answer is rather simple:

**Theorem 1.1** *Let $\lambda$ be a Salem number of degree $d$. Then there is an $n \in \mathbb{N}$, a projective K3 surface $X$ and an automorphism $F : X \to X$ with dynamical degree $\lambda(F) = \lambda^n$ if and only if $d \leq 20$.*

Recall that the degree of a Salem number is the degree of its minimal polynomial. We illustrate the theorem with case of the minimal Salem number $\lambda_d$ of degree $d$. Conjecturally, the smallest Salem number is Lehmer's number $\lambda_{10} \approx 1.17628$. In [9] the author gives a strategy to decide whether a single given Salem number $\lambda$ is the dynamical degree of an automorphism of a complex projective K3 surface. This strategy is then applied in [5,9] to show that $\lambda_d$ is a dynamical degree of an automorphism of a *projective* K3 surface if and only if $14, 16 \neq d \leq 18$. Using the strategy in [9] and the improved positivity test from [5] one obtains that

$$\lambda_{14}^9, \lambda_{16}^7, \lambda_{20}^{11}$$

are realized on projective K3 surfaces. In the non-projective realm even $\lambda_{14}, \lambda_{16}, \lambda_{20}$ and $\lambda_{22}$ are realized.

Considering the stable dynamical degrees for complex tori and Enriques surfaces, we obtain a uniform answer which depends only on the Betti and Hodge numbers of Tori, Enriques and K3 surfaces given by $b_2 = 6, 10, 22$ and $h^{1,1} = 4, 10, 20$.

**Theorem 1.2** *Let $\lambda$ be a Salem number of degree $d$ with minimal polynomial $s(x)$. Then there is some power $\lambda^n, n \in \mathbb{N}$, that is realized as dynamical degree of an automorphism on a complex 2-torus (resp. an Enriques, a K3 surface) if and only if*

(1) $d < 6$ *(resp.* 10, 22*) or*
(2) $d = 6$ *(resp.* 10, 22*) and $-s(1)s(-1)$ is a square integer.*

*If additionally $d \leq 4$ (resp. 10, 20) then some for some $n' \in \mathbb{N}$, the power $\lambda^{n'}$ is realized on a* projective *Abelian (resp. Enriques, K3) surface.*

The proof proceeds as follows. All Tori (resp. Enriques, K3 surfaces) are diffeomorphic. Hence, the isometry class of the lattice $H^2(X, \mathbb{Z})$ is independent of which Torus (resp. Enriques, K3 surface) $X$ we have chosen. It is abstractly isomorphic to some known lattice $L$. Given an isometry $f \in O(L)$ it is possible, using some Torelli theorem, to decide whether $f$ is in the image of the natural representation

$\mathrm{Aut}(X) \to O(H^2(X,\mathbb{Z})) \cong O(L)$ for some $X$. This is the case if $f$ preserves some extra linear data such as a Hodge structure or has trivial mod 2 reduction. The most intricate case is that of *projective* K3 surfaces. There $f$ has to preserve a chamber of the positive cone-corresponding to the ample cone. Since it is usually infinite sided, it is notoriously difficult to control. For a given concrete $f$ it is now algorithmically possible to decide whether a chamber is preserved [5,9]. However, the algorithm can only deal with a single isometry at a time. In Proposition 4.3, we give a sufficient condition for a chamber to be preserved. We expect that it will be useful to study the stable dynamical spectrum of supersingular K3 surfaces (as in [5]) and IHSM manifolds (as in [1]) as well.

## 2 Preliminaries

In this section we review the necessary material from [9,10] concerning the theory of lattices, their isometries, discriminant forms, gluings and twists.

### 2.1 Lattices

A *lattice* is a finitely generated free abelian group $L$ equipped with a non-degenerate integer valued bilinear form

$$\langle \cdot, \cdot \rangle \colon L \times L \to \mathbb{Z}.$$

It is called *even* if $x^2 := \langle x, x\rangle \in 2\mathbb{Z}$ for all $x \in L$ and *unimodular* if it is of determinant $\pm 1$. An *isometry* $M \to L$ of lattices is an isomorphism of $\mathbb{Z}$-modules preserving the bilinear forms. The orthogonal group $O(L)$ consists of the self isometries of the lattice $L$. The *signature* (pair) of a lattice is denoted by $(s_+, s_-)$ where $s_+$ (respectively $s_-$) is the number of positive (respectively negative) eigenvalues of the Gram matrix. A lattice is called *indefinite* if both $s_+$ and $s_-$ are non-zero and *hyperbolic* if it is indefinite and $s_+ = 1$. Indefinite, even unimodular lattices are classified up to isometry by their signature pair $(s_+, s_-)$. We denote such a lattice by $\mathrm{II}_{s_+,s_-}$. The *dual lattice* $L^\vee$ of $L$ is given by

$$L^\vee = \{x \in L \otimes \mathbb{Q} \mid \langle x, L\rangle \subseteq \mathbb{Z}\}.$$

The *discriminant group* $D_L = L^\vee/L$ is a finite abelian group of cardinality $|\det L|$. If $L$ is an even lattice, then its discriminant group carries the *discriminant form*, given by

$$q_L \colon D_L \to \mathbb{Q}/2\mathbb{Z} \quad x \mapsto \langle x, x\rangle \mod 2\mathbb{Z}.$$

By definition of the dual lattice $q_L$ is non-degenerate. The *length* $l(D)$ of an abelian group $D$ is its minimal number of generators. If $M \subseteq L$ are lattices of the same rank, then we call $L$ an *overlattice* of $M$. Even overlattices $L$ of a lattice $M$ correspond bijectively to totally isotropic subgroups

$$L/M = H \subseteq D_M$$

of the discriminant group, i.e., with $q_M|H = 0$. For a prime number $p$ we denote by $\mathbb{Z}_p$ the p-adic integers and by $\mathbb{Q}_p$ the p-adic numbers. The discriminant form (and group) has an orthogonal decomposition into its $p$-primary parts $(q_L)_p$

$$q_L = \bigoplus_p \left((q_L)_p \colon (D_L)_p \to \mathbb{Q}_p/2\mathbb{Z}_p\right)$$

where $(q_L)_p$ is the discriminant form of $L \otimes \mathbb{Z}_p$ (defined analogously).

## 2.2 Embeddings and gluing

An embedding of lattices $M \hookrightarrow L$ is called *primitive* if the cokernel $L/M$ is torsion free. Let $M \hookrightarrow L$ be a primitive embedding into an even unimodular lattice $L$ and $N = M^\perp$ the orthogonal complement. It is primitive as well. We have a chain of inclusions

$$M \oplus N \hookrightarrow L = L^\vee \hookrightarrow M^\vee \oplus N^\vee.$$

The orthogonal projections provide us with isomorphisms

$$D_M = M^\vee/M \cong L/(M \oplus N) \cong N^\vee/N = D_N.$$

The composite map $\phi\colon D_M \to D_N$ is called the glue map. It satisfies $q_N(\phi(x)) = -q_M(x)$. Conversely, given such a glue map, its graph

$$\Gamma_\phi = \{x + \phi(x) \in D_M \oplus D_N \mid x \in D_M\}$$

is isotropic with respect to the discriminant quadratic form $q_{M\oplus N}$. Hence it defines an overlattice $L_\phi$ via $L_\phi/\,(M \oplus N) := \Gamma_\phi$. This overlattice turns out to be unimodular. Let $f \in O(M)$ and $g \in O(N)$ be isometries. Then $f \oplus g \in O(M \oplus N)$ extends to the overlattice $L_\phi$ if and only if $\phi \circ \bar{f} = \bar{g} \circ \phi$ where $\bar{f} \in O(q_M)$ and $\bar{g} \in O(q_N)$ are the induced actions. Note that we can always arrange this condition for some power $(f \oplus g)^n$ with $n \in \mathbb{N}$ such that $\bar{f}^n = \mathrm{id}_{D_M}$ and $\bar{g}^n = \mathrm{id}_{D_N}$.

The following easy to use criterion for the existence of a primitive embedding will be useful for us later.

**Proposition 2.1** [10, Corollary 1.12.3] *Let $S$ be an even lattice of signature $(s_+, s_-)$. Then $S$ has a primitive embedding into an even unimodular lattice $L$ of signature $(l_+, l_-)$ if $l_+ \geq s_+$, $l_- \geq s_-$ and $l(D_S) + 3 \leq \operatorname{rk} L - \operatorname{rk} S$.*

In order to use the proposition one can reduce the length of the discriminant group by taking a maximal overlattice.

**Lemma 2.2** *Let $L$ be an even lattice and $M$ a maximal even overlattice. Then $l(D_M) \leq 3$.*

***Proof*** Using the classification of finite quadratic forms given in [10, Prop. 1.8.1], one sees that a finite quadratic form represents 0 if the group has length at least four. In particular, if a lattice $M$ has $l(D_M) \geq 4$, then we can find an element $x \neq 0$ of $D_M$ with $q(x) = 0$. This element provides us with a proper even overlattice of $M$. Thus $M$ cannot be maximal. $\square$

### 2.3 Twists

A pair $(L, f)$ of a lattice $L$ and an isometry $f$ of $L$ with characteristic polynomial $s(x) \in \mathbb{Z}[x]$ is called an $s(x)$-lattice. Given an $s(x)$-lattice $(L, f)$ and $t \in \mathbb{Z}[f + f^{-1}]$, we obtain a new symmetric bilinear form on $L$ by setting

$$\langle g_1, g_2 \rangle_t = \langle t g_1, g_2 \rangle .$$

The lattice $L$ equipped with this new product is called the *twist* of $L$ by $t$ and is denoted by $(L(t), f)$. Note that the twist of an even lattice stays even [8, Prop 4.1]. Twisting may change the signature and determinant of a lattice. If $t^2 \in \mathbb{Z}[f + f^{-1}]$ is a square, then $L(t^2)$ is isomorphic, via $x \mapsto tx$, to the sublattice $tL$ of $L$. In particular, the signatures of $L(t^2)$ and $L$ coincide. Of particular interest is the case when the characteristic polynomial $s(x)$ of $f$ is irreducible. Then $K = \mathbb{Q}[f]$ is a degree two extension of the field $k = \mathbb{Q}[f + f^{-1}]$ with Galois group generated by $\sigma$ with $f^\sigma = f^{-1}$. We denote by $\mathcal{O}_k$ and $\mathcal{O}_K$ the rings of integers of $k$ and $K$.

**Lemma 2.3** *Let $(L, f)$ be an $s(x)$-lattice with $s(x)$ irreducible. Suppose that $t \in \mathbb{Z}[f + f^{-1}]$ is a prime of $k$ split in $K$ of norm $p$ not dividing* $\det L \cdot \operatorname{discr} s(x)$. *Then the $p$-primary part $(q_{L(t^n)})_p$ of the discriminant quadratic form of the twisted $s(x)$-lattice $L(t^n)$ is isomorphic to*

$$q : \left(\mathbb{Z}/p^n\mathbb{Z}\right)^2 \to \mathbb{Q}_p/2\mathbb{Z}_p \quad (x_1, x_2) \mapsto \frac{2x_1x_2}{p^n}.$$

***Proof*** Since $p$ does not divide $\operatorname{discr} s(x)$ and $\det L$, we may without loss of generality assume that $\mathbb{Z}[f]$ is the full ring of integers of $K$ and $L$ is unimodular. Then it is easy to see that $D_{L(t^n)} \cong \mathbb{Z}[f]/(t^n)$. Since $t$ in $k$ is split in $K$, we find an ideal $\mathfrak{p} = (p, f - a)$ of $\mathbb{Z}[f]$ with $t = \mathfrak{p}\mathfrak{p}^\sigma$. Then $\mathfrak{p}^\sigma = (p, f - a')$ with $a, a' \in \mathbb{Z}$, $aa' \equiv 1 \mod p$ and $a \not\equiv a' \mod p$. Now,

$$\mathbb{Z}[f]/(t)^n \cong \mathbb{Z}[f]/\mathfrak{p}^n \times \mathbb{Z}[f]/(\mathfrak{p}^\sigma)^n \cong \left(\mathbb{Z}/p^n\mathbb{Z}\right)^2 .$$

Let $v$ be a generator of $\mathbb{Z}[f]/(p, f - a)^n$ then $fv = \alpha v$ for some $\alpha \in \mathbb{Z}$ with $\alpha \equiv a \mod p$. Since $f$ induces an isometry of the discriminant form, we obtain

$$q(v) = q(fv) = q(\alpha v) = \alpha^2 q(v).$$

But $\alpha^2 \equiv a^2 \neq 1 \mod p$, hence $q(v) = 0$. The same argument works for a generator $w$ of $\mathbb{Z}[f]/(p, f - a')^n$. Since the quadratic form is non-degenerate, we can rescale $w$ to obtain the desired result. $\square$

### 2.4 Positivity

Let $L$ be an even lattice. A *root* of $L$ is $r \in L$ with $\langle r, r\rangle = -2$. We denote the set of roots by $\Delta_L$. If $L$ is hyperbolic, we set

$$V_L = \{x \in L \mid \langle x, x\rangle > 0, \forall r \in \Delta_L : \langle r, x\rangle \neq 0\}$$

which is an open set. If $L$ is negative definite, we define

$$V_L = \{x \in L \mid x \neq 0, \forall r \in \Delta_L : \langle r, x\rangle \neq 0\}.$$

In both cases the connected components of $V_L$ are called the *chambers* of $V_L$. An isometry $f \in O(L)$ is called *positive* if it preserves a chamber. We denote by $O^+(L)$ the subgroup stabilizing each connected component of the light cone $V^0 = \{x \in L \otimes \mathbb{R} \mid x \neq 0,\ \langle x, x\rangle = 0\}$. A dual perspective on positivity is that of obstructing roots. An *obstructing root* for $f$ is $r \in \Delta_L$ such that there is no $h \in L$ with $h^\perp$ negative definite and $h.f^i(r) > 0$ for all $i \in \mathbb{Z}$. We call $r$ a *cyclic root* for $f$ if

$$r + f(r) + f^2(r) + \cdots + f^i(r) = 0$$

for some $i > 0$. Cyclic roots are obstructing. If $L$ is negative definite, then every obstructing root is cyclic. We have the following

**Theorem 2.4** [9, 2.1] *A map $f \in O^+(L)$ is positive if and only if it has no obstructing roots. The set of obstructing roots, modulo the action of $f$, is finite.*

Let $L$ be hyperbolic and $f \in O(L)$ with spectral radius a Salem number $\lambda$. Denote by $\gamma$ the real plane spanned by the eigenspaces for $\lambda$ and $\lambda^{-1}$. Then the obstructing roots for $f$ are the cyclic roots together with the roots $r$ such that $r^\perp \cap \gamma$ is positive definite. To see this, note that the closure of any $f$-invariant chamber contains the intersection of $\gamma$ with the positive cone. First suppose that $\gamma \subseteq r^\perp$. Then $r \in \gamma^\perp \cap L$ which is negative definite. In particular, if $r$ is obstructing, then it is cyclic. On the other hand, if $r^\perp$ intersects $\gamma$ in the positive cone, then $\gamma$ crosses the wall $r^\perp$ of a camber. Then $\gamma$ is not contained in the closure of a single chamber.

## 3 Surfaces and their automorphisms

Let $X$ be a either a 2-Torus, a K3 surface or an Enriques surface. Its second singular cohomology group modulo torsion $H^2(X, \mathbb{Z})/tors$ equipped with the cup product is a unimodular lattice isomorphic to

$$H^2(X, \mathbb{Z})/tors \cong \begin{cases} \mathrm{II}_{3,3} & \text{for } X \text{ a 2-Torus} \\ \mathrm{II}_{1,9} & \text{for } X \text{ an Enriques surface} \\ \mathrm{II}_{3,19} & \text{for } X \text{ a K3 surface.} \end{cases}$$

It admits a Hodge decomposition

$$H^2(X,\mathbb{Z})\otimes\mathbb{C}\cong H^2(X,\mathbb{C})=H^{2,0}(X)\oplus H^{1,1}(X)\oplus H^{0,2}(X)$$

where $H^{i,j}(X)\cong H^j(X,\Omega^i_X)$, $H^{i,j}(X)=\overline{H^{j,i}(X)}$ and $H^{1,1}(X)=(H^{2,0}(X)\oplus H^{0,2}(X))^\perp$ is hyperbolic. By Lefschetz' Theorem on $(1,1)$ classes we can recover the numerical divisor classes from the Hodge structure as

$$\operatorname{Num}(X)=H^{1,1}(X)\cap H^2(X,\mathbb{Z})/tors.$$

We note that a (compact) Kähler surface $X$ is projective if and only if there is a divisor of positive square, i.e. $\operatorname{Num}(X)$ has signature $(1,\operatorname{rk}\operatorname{Num}(X)-1)$. The transcendental lattice is defined as the smallest primitive sublattice $T(X)\subseteq H^2(X,\mathbb{Z})$ whose complexification contains $H^{2,0}(X)\subseteq T(X)\otimes\mathbb{C}$. Since in our case $h^{2,0}(X)\in\{0,1\}$, the Hodge structure on $T(X)$ is irreducible.

Let $F\in\operatorname{Aut}(X)$ be an automorphism with dynamical degree $\lambda>1$. The minimal polynomial $s(x)\in\mathbb{Z}[x]$ of $\lambda$ is called a *Salem polynomial*. Then $F^*$ is semisimple with characteristic polynomial $s(x)c(x)$ where $c(x)$ is a product of cyclotomic polynomials [7, Thm. 3.2]. Set

$$S=\ker s(F^*)\subseteq H^2(X,\mathbb{Z})/tors.$$

By irreducibility of $T(X)$, the minimal polynomial of $F^*|T(X)$ must be irreducible in $\mathbb{Q}[x]$. Hence, either $T(X)=S$ or $S\subseteq\operatorname{Num}(X)$. Since the real eigenvectors for $\lambda$ and $\lambda^{-1}$ span a hyperbolic plane, the signature of $S$ is either $(1,\deg s(x)-1)$ if $S\subseteq\operatorname{Num}(X)$ or in case $S=T(X)$ it is $(3,\deg s(x)-3)$. In the first case $X$ is projective and in the second not.

In the following three Lemmas we collect criteria for an isometry of the cohomology lattice to come from an automorphism of a surface.

**Lemma 3.1** [2] *The automorphism group of a very general Enriques surface $X$ is the 2-congruence subgroup given by the kernel of*

$$O^+(H^2(X,\mathbb{Z}))\to O(H^2(X,\mathbb{Z})\otimes\mathbb{F}_2).$$

*It is of finite index in the orthogonal group of $H^2(X,\mathbb{Z})/tors\cong \mathrm{II}_{1,9}$.*

**Lemma 3.2** *Let $s(x)$ be a Salem polynomial of degree $d$ and $f\in O(\mathrm{II}_{3,3})$ an isometry with characteristic polynomial $s(x)(x-1)^{6-d}$ which acts trivially on $\mathrm{II}_{3,3}\otimes\mathbb{F}_2$. Then one can find a complex 2-torus $X$ with $\mathrm{II}_{3,3}=H^2(X,\mathbb{Z})$ and $F\in\operatorname{Aut}(X)$ such that $F^*=f$. The 2-torus $X$ is projective if and only if $S=\ker s(f)$ has signature $(1,d-1)$.*

***Proof*** Let $L$ be a free $\mathbb{Z}$ module of rank 4. An orientation on $L$ is an isomorphism $\det\colon \bigwedge^4 L\xrightarrow{\sim}\mathbb{Z}$. It equips $\bigwedge^2 L$ with the structure of an even unimodular lattice which we identify with $\mathrm{II}_{3,3}$. We can choose an eigenvector $\eta\in\mathrm{II}_{3,3}\otimes\mathbb{C}$ of $f\otimes\mathbb{C}$

such that $\langle \eta, \eta \rangle = 0$ and $\langle \eta, \bar{\eta} \rangle > 0$. Since $\mathbb{C}\eta$ is isotropic, it is in the image of the Plücker embedding $Gr(2, L \otimes \mathbb{C}) \to \bigwedge^2 L \cong \mathrm{II}_{3,3}$. Hence, we get a complex 2-plane $S$ with $\bigwedge^2 S = \mathbb{C}\eta$ and $S \oplus \bar{S} = L \otimes \mathbb{C}$. This defines a weight one Hodge structure on $L$, i.e., a complex 2-torus $X$. We can view $f$ as an isometry of $H^2(X, \mathbb{Z}) = \bigwedge^2 L \cong \mathrm{II}_{3,3}$ which, by construction, preserves the Hodge structure on $H^2(X, \mathbb{Z})$. Since $f \otimes \mathbb{F}_2$ is the identity, we can apply, [3, V (3.2)] to get an automorphism $F$ of $X$ with $F^* = \pm f$. However, both $F^*$ and $f$ stabilize each connected component of the positive cone of $H^{1,1}(X)$. Hence, they are equal. □

**Lemma 3.3** *Let $f \in O(\mathrm{II}_{3,19})$ be an isometry with characteristic polynomial $s(x)(x-1)^{22-d}$ where $s(x)$ is a Salem polynomial. Then one can find a K3 surface $X$, $F \in \mathrm{Aut}(X)$ and an isometry $\phi : \mathrm{II}_{3,19} \to H^2(X, \mathbb{Z})$ such that $F^* = \phi \circ f \circ \phi^{-1}$ if and only if*

(1) *$S = \ker s(f)$ has signature $(3, d-3)$ or*
(2) *$S$ has signature $(1, d-1)$ and $f|S$ is positive.*

*In case (2) $X$ is projective and in case (1) not.*

***Proof*** The lemma follows once we check the conditions of [9, 6.1]. If the signature of $S$ is $(3, d-3)$, then we take as period an eigenvector $\eta \in S \otimes \mathbb{C}$ of $f$ with $\langle \eta, \bar{\eta} \rangle > 0$. Since $f$ is the identity on $S^\perp$ there are no cyclic roots, and $f|S^\perp$ is positive. If the signature of $S$ is $(1, d-1)$, then we take as period $\eta$ in $S^\perp \otimes \mathbb{C}$ such that $\langle \eta, \bar{\eta} \rangle > 0$ and $\eta^\perp \cap \mathrm{II}_{3,19} = S$. □

## 4 Proof of Theorem 1.2

In order to prove the main Theorem 1.2, we need to produce isometries of certain lattices with given spectral radius. In general this can be difficult. Hence, we simplify the problem by asking for *rational* isometries first. Indeed, here the answer is known as is displayed by the following Lemma 4.1. We postpone its proof till the end of this paper.

**Lemma 4.1** *Let $L \in \{\mathrm{II}_{3,3}, \mathrm{II}_{1,9}, \mathrm{II}_{3,19}\}$ and $s(x)$ be a Salem polynomial of degree $d$. Then there exists a rational isometry $f \in O(L \otimes \mathbb{Q})$ with characteristic polynomial $\det(x \cdot \mathrm{Id} - f) = s(x)(x-1)^{\mathrm{rk}\, L - d}$ if and only if either*

(1) *$d \leq \mathrm{rk}\, L - 2$ or*
(2) *$d = \mathrm{rk}\, L$ and $-s(1)s(-1)$ is a square.*

*In case (1) we can find $f$ such that $\ker s(f)$ is hyperbolic. If the signature of $L$ is $(3, \mathrm{rk}\, L - 3)$, then we can find $f$ such that $\ker s(f)$ has signature $(3, d-3)$.*

Typically, a rational isometry $f \in O(L \otimes \mathbb{Q})$ does not preserve $L$. Since we are only considering the *stable* dynamical spectrum, we may replace $f$ by some power $f^n$.

**Lemma 4.2** *Let $L$ be a lattice and $f \in O(L \otimes \mathbb{Q})$ a rational isometry with*

$$\det(x \cdot \mathrm{Id} - f) \in \mathbb{Z}[x].$$

*Then one can find $n \in \mathbb{N}$ such that $f^n \in O(L)$.*

***Proof*** Since the characteristic polynomial of $f$ is integral, the $\mathbb{Z}$-module $\mathbb{Z}[f]L$ is finitely generated and of the same rank as $L$. Consequently, for the index $k = [\mathbb{Z}[f]L : L]$ we get the chain of inclusions

$$k\mathbb{Z}[f]L \subseteq L \subseteq \mathbb{Z}[f]L.$$

Conclude by taking $n \in \mathbb{N}$ such that $f^n$ acts as the identity on the finite quotient $\mathbb{Z}[f]L/k\mathbb{Z}[f]L$. □

We now have all the ingredients for the

***Proof of Theorem 1.2 for Tori and Enriques surfaces*** A combination of Lemmas 4.1 and 4.2 provides us with isometries of $L \in \{\mathrm{II}_{3,3}, \mathrm{II}_{1,9}\}$ with spectral radius some power of the desired Salem number. After raising the isometries to some sufficiently divisible power, we can assume that they satisfy the conditions of Lemmas 3.1 and 3.2. □

The same argument completes the proof of the main theorem for non-projective K3 surfaces. It remains for us to control the positivity of the isometries to prove the result for projective K3 surfaces as well.

**Proposition 4.3** *Let $f \in O(S)$ be an isometry of a hyperbolic lattice $S$ with characteristic polynomial a Salem polynomial $s(x)$. If*

$$|\det S| > 4 \operatorname{discr} s(x),$$

*then $f$ preserves a chamber of the positive cone.*

***Proof*** Let $L$ and $f$ be as in the proposition and denote by

$$\pi : S \otimes \mathbb{R} \to \ker(f + f^{-1} - \lambda - \lambda^{-1})$$

the orthogonal projection where $\lambda > 1$ is a root of $s(x)$. Suppose that $f$ does not preserve a chamber. Then, by Theorem 2.4, there is an obstructing root $r \in L$. This means that $\langle r, r\rangle = -2$ and $r^{\perp}$ crosses the geodesic $\gamma$ of $f$, i.e. $\langle \pi(r), \pi(r)\rangle < 0$. Since $s(x)$ is irreducible over $\mathbb{Q}$, $\mathbb{Z}[f]r$ is a sublattice of full rank, and hence

$$|\det S| \leq |\det \mathbb{Z}[f]r|.$$

The basic idea at this point is that the obstructing roots modulo the action of $f$ lie in some compact fundamental domain in $S \otimes \mathbb{R}$ depending only on $s(x)$. Then we can maximize $|\det \mathbb{Z}[f]r|$ over all $r$ in this fundamental domain. We extend the bilinear form to a $\mathbb{C}$-linear form on $S\otimes\mathbb{C}$ and compute the determinant of $\mathbb{Z}[f]r$ in an eigenbasis of $f$. We can find $u_1, u_2 \in S \otimes \mathbb{R}$ and $v_i \in S \otimes \mathbb{C}$ such that

$$f(u_1) = \lambda u_1, \quad f(u_2) = 1/\lambda u_2, \quad f(v_i) = \alpha_i v_i, \quad i \in \{1, \dots, k\}$$

where $\lambda > 1$, $1/\lambda$, $\alpha_i, \overline{\alpha}_i$ are the complex roots of $s(x)$ and $\deg s(x) = 2k + 2$. After rescaling, we may assume that $\langle u_1, u_2\rangle = 1$ and $\langle v_i, \overline{v_i}\rangle = -1$ for $i \in \{1, \dots, k\}$. All other inner products vanish. Now, write

$$r = x_1 u_1 + x_2 u_2 + \sum_{i=1}^{k} (y_i v_i + \overline{y_i v_i})$$

for $x_1, x_2 \in \mathbb{R}$, $y_i \in \mathbb{C}$. The Van-der-Monde determinant yields that

$$|\det\langle f^i(r), f^j(r)\rangle| = |x_1 x_2|^2 \prod_{i=1}^{k} |y_i|^4 \operatorname{discr} s.$$

Since $r$ is obstructing, $x_1 x_2 = \langle x_1 u_1, x_2 u_2\rangle \in [-2, 0)$ and in these coordinates $\langle r, r\rangle = x_1 x_2 - \sum_{i=1}^{k} |y_i|^2 = -2$, i.e. the coordinates of the root $r$ lie in the set

$$K = \left\{(x, y) \in \mathbb{R}^2 \times \mathbb{C}^k : x_1 x_2 \in [-2, 0),\ x_1 x_2 - \sum_{i=1}^{k} |y_i|^2 = -2\right\}.$$

Then, assuming $k \neq 0$,

$$\begin{aligned}
|\det S| &\le |\det\langle f^i(r), f^j(r)\rangle| && (1)\\
&\le \sup\left\{|x_1 x_2| \prod_{i=1}^{k} |y_i|^2 : (x, y) \in K\right\}^2 \operatorname{discr} s && (2)\\
&= \sup_{c\in(0,2]} \left(c^2 \cdot \sup\left\{\prod_{i=1}^{k} |y_i|^2 : \sum_i |y_i|^2 = 2 - c\right\}^2\right) \operatorname{discr} s && (3)\\
&= \left[\sup_{c\in(0,2]} c(2-c)^k \sup\left\{\prod_{i=1}^{k} |y_i|^2 : \sum_i |y_i|^2 = 1\right\}\right]^2 \operatorname{discr} s && (4)\\
&= \left[\frac{2}{1+k}\left(\frac{2}{k}\right)^k \left(1 - \frac{1}{1+k}\right)^k\right]^2 \operatorname{discr} s && (5)\\
&\le \operatorname{discr} s. && (6)
\end{aligned}$$

In line (5) we have used that that $c(2-c)^k$ $(0 \le c \le 2)$ takes its maximum in $c = 2/(k+1)$ and that

$$1/k^k = \max\left\{\prod_{i=1}^{k} y_i^2 \mid (y_1, \dots, y_k) \in \mathbb{R}^k, \sum_{i=1}^{k} y_i^2 = 1\right\}.$$

Here the maxima lie at $|y_i| = 1/\sqrt{k}$, $i \in \{1, \dots, k\}$. For the case of $k = 0$ one obtains $4 \operatorname{discr} s$. □

***Proof of Theorem 1.1*** For any natural number $n$ let $s_n(x)$ denote the minimal polynomial of $\lambda^n$. Suppose that $\deg s_1(x) \le 18$. There exists an even hyperbolic $s_1(x)$-lattice

$(S, f)$. After replacing $f$ by $f^n$ for some sufficiently divisible $n \in \mathbb{N}$, we may assume that $f$ acts trivially on the discriminant group. In particular, $f$ preserves any integral overlattice. Then, we can replace $S$ by a maximal even integral overlattice. By Lemma 2.2 this assures that the length $l(D_S) \leq 3$. Let $\mathfrak{p} < \mathcal{O}_k$ be a prime ideal of degree 1 over a prime number $p$ not dividing $\det S$. By finiteness of the class group, there is $l \in \mathbb{N}$ with $\mathfrak{p}^l = t\mathcal{O}_k$ principal, where $t \in \mathcal{O}_k$. Replace $S$ by its twist $S(t^{2r})$ for some large $r \in \mathbb{N}$ such that

$$4 \operatorname{discr} s_n(x) < p^{2lr}.$$

Then $S$ is hyperbolic as $t^{2r}$ is a square, and $f^n$ preserves a chamber by Proposition 4.3. Since $\mathfrak{p}$ is of degree 1, we still have $l(D_S) \leq 3$. Hence, we may apply Proposition 2.1 to obtain a primitive embedding $i : S \hookrightarrow \mathrm{II}_{3,19}$. After replacing $f$ by some power which acts trivially on $D_S$, we can continue $f$ by the identity on $i(S)^{\perp}$ to get an isometry of $\mathrm{II}_{3,19}$. Finally, use Lemma 3.3 (2) to get a projective K3 surface $X$ and $F \in \operatorname{Aut}(X)$ with dynamical degree some power of $\lambda$.

In the case that $\deg s_1(x) = 20$, we have to work a little harder to obtain a primitive embedding. By Lemmas 4.1 and 4.2, for some $n \in \mathbb{N}$ we get an isometry $f \in O(\mathrm{II}_{3,19})$ with characteristic polynomial $s_n(x)(x-1)^2$ and such that $S = \ker s_n(f)$ is hyperbolic. Set $C = S^{\perp} = \ker(f - id)$. The primitive extension $S \oplus C \hookrightarrow \mathrm{II}_{3,19}$, provides us with an isomorphism of discriminant quadratic forms $q_S \cong q_C(-1)$. By Chebotarev's density theorem there are infinitely many prime ideals $\mathfrak{p} < \mathcal{O}_k$ of degree one, split in $K/k$ such that

$$p \equiv 1 \mod (8 \det C)$$

where $(p) = \mathbb{Z} \cap \mathfrak{p}$. Choose one such $\mathfrak{p}$ with $p > \operatorname{discr} s_n(x)$ and which does not divide the determinant of $S$. We can find $l \in \mathbb{N}$ and $t \in \mathcal{O}_k$ with $\mathfrak{p}^l = t\mathcal{O}_k$, and then

$$|\det S(t^2)| = |\det S| p^{2l} > \operatorname{discr} s_n(x).$$

For primes $p' \neq p$, $S(t^2) \otimes \mathbb{Z}_{p'} \to S \otimes \mathbb{Z}_{p'}$, $x \mapsto tx$ is an isometry. Hence,

$$(q_{S(t^2)})_{p'} \cong (q_S)_{p'} \cong (q_C)_{p'}(-1) \cong (q_{C(p^{2l})})_{p'}(-1).$$

By Lemma 2.3

$$(q_{S(t^2)})_p \text{ is represented by } p^{-2l} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Further, since $C$ is of rank 2 and $p$ does not divide $\det C$,

$$(q_{C(p^{2l})})_p(-1) \text{ is represented by } p^{-2l} \begin{pmatrix} 1 & 0 \\ 0 & \det C \end{pmatrix}.$$

The two forms are isomorphic if and only if both $-1$ and $\det C$ are of the same square class in $\mathbb{Z}_p^{\times} / \left(\mathbb{Z}_p^{\times}\right)^2$. This is computed by the Legendre symbols $\left(\frac{-1}{p}\right)$ and $\left(\frac{\det C}{p}\right)$. Since $p \equiv 1 \mod (8 \det C)$, we have that

$$\left(\frac{\det C}{p}\right) = \left(\frac{p}{\det C}\right) = 1 = \left(\frac{-1}{p}\right).$$

By assembling the isometries from different primes, we can construct a glue map $q_{S(t^2)} \cong q_{C(p^{2t})}(-1)$. This provides a primitive embedding of $S(t^2)$ into $\mathrm{II}_{3,19}$ with orthogonal complement $C(p^{2l})$. Proceed as before to obtain the projective K3 surface with its automorphism. □

### 4.1 Proof of Lemma 4.1

Except for the signature condition, Lemma 4.1 is a special case of [4, Cor. 9.2, Prop. 11.9]. In what follows, we inspect the original proof. For the readers convenience, we recall some of the notation involved. We need only the following special case: $k = \mathbb{Q}$ and $\Sigma_k$ is the set of its places. Then $q$ is the quadratic form on $L \otimes \mathbb{Q}$. An isometry $t \in O(q)$ induces the structure of a self-dual torsion $\mathbb{Q}[x]$-module on $L_\mathbb{Q}$ via $p(x).v = p(t)v$ for $v \in L_\mathbb{Q}$ and $p(x) \in \mathbb{Q}[x]$. We set

$$M_0 = (\mathbb{Q}[x]/(x-1))^{22-r}, \quad M_1 = \mathbb{Q}[x]/s(x), \quad \text{and} \quad M = M_0 \oplus M_1.$$

Then $\mathcal{C}_{M,q}$ is the set of all collections of forms $C = \{q_i^\nu\}$ for $i \in I_0 = \{0, 1\}$ and $\nu \in \Sigma_k$, such that $q_i^\nu$ has an isometry with module $M_i \otimes \mathbb{Q}_\nu$ and $q_0^\nu \oplus q_1^\nu$ is isomorphic to the localization $q^\nu = q \otimes \mathbb{Q}_\nu$ of $q$ at $\nu$. For $i \in I_0$, we set

$$T_i(C) = \{\nu \in \Sigma_k \mid w(q_i^\nu) = 1\}$$

where $w(q_i^\nu)$ is the Hasse invariant of $q_i^\nu$. Let $\mathcal{F}_{M,q}$ be the subset of $\mathcal{C}_{M,q}$ such that for all $i \in I_0$, $T_i(C)$ is a finite set.

The main step involved is

**Theorem 4.4** [4, Thm. 10.8] *Let $M$ be a self-dual torsion $k[x]$-module which is finite dimensional as a $k$-vector space. Suppose that the quadratic form $q$ over the global field $k$ has an isometry with module $M$ over $k_\nu$ for all places $\nu$ of $k$. Then $q$ has an isometry with module $M$ if and only if there exists a collection $C = \{q_i^\nu\} \in \mathcal{F}_{M,q}$ such that for all $i \in I_0$, the cardinality of $T_i(C)$ is even. In this case $q_i^\nu = q_i \otimes \kappa_\nu$.*

***Proof of Lemma 4.1*** By [4, Prop. 11.9], we find $f \in O(q)$ with module $M$. Hence its characteristic polynomial has the desired form

$$\det(x \cdot \mathrm{Id} - f) = s(x)(x-1)^{22-r}.$$

If $q_1 = q|S_\mathbb{Q}$ has signature $(1, r-1)$, we are done, else $q_1$ has signature $(3, r-3)$. By Theorem 4.4, this provides us with a collection $C = \{q_i^\nu\} \in \mathcal{F}_{M,q}$ with $q_i^\nu = q_i \otimes \kappa_\nu$ such that $|T_i(C)|$ is even for $i \in \{0, 1\}$. We set

$$d_1 \equiv \det q_1 \equiv s(1)s(-1) \mod \mathbb{Q}^{\times 2}$$

and

$$d_0 \equiv \det q_0 \equiv d_1 \det q = -s(1)s(-1) \mod \mathbb{Q}^{\times 2}.$$

Denote by $\Omega(M_i, d_i)$ the set of finite places $\nu$ of $\mathbb{Q}$ such that for any $\epsilon \in \{0, 1\}$ there is a quadratic space $Q$ over $\mathbb{Q}_\nu$ with determinant $d_i$, Hasse invariant $w(Q) = \epsilon$ and which has an isometry with module $M_i$. By [4, Prop. 11.9] every finite place is in $\Omega(M_0, d_0)$. Hence

$$\Omega_{0,1} = \Omega(M_0, d_0) \cap \Omega(M_1, d_1) = \Omega(M_1, d_1),$$

which is non-empty by [4, Lem. 9.4, 9.6] and Chebotarev's density theorem for the degree two extension $\mathbb{Q}[\lambda]$ of $\mathbb{Q}[\lambda + \lambda^{-1}]$. Choose a place $p \in \Omega_{0,1}$. We can define a new collection $\tilde{C} = \{\tilde{q}_i^\nu\}$ by $\tilde{q}_i^\nu = q_i^\nu$ for $i \in \{0, 1\}$ and $\nu \neq p, \infty$, For $\tilde{q}_i^\infty$, we take forms of signature $(1, d-1)$, respectively $(2, \operatorname{rk} L - 2 - d)$. At $p$ we just switch the Hasse invariants of $q_i^p$. By [4, Lem. 9.6], $\tilde{C} \in \mathcal{C}_{M,q}$, and moreover $\tilde{C} \in \mathcal{F}_{M,q}$. Since the Hasse invariants have changed at two places, $|T_i(\tilde{C})|$ is still even. Finally, $\tilde{C}$ meets the conditions of Theorem 4.4 and the claim follows. □

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