ORIGINAL ARTICLE

An Existence Theory for Gravity–Capillary Solitary Water Waves

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Abstract

In the applied mathematics literature solitary gravity–capillary water waves are modelled by approximating the standard governing equations for water waves by a Korteweg-de Vries equation (for strong surface tension) or a nonlinear Schrödinger equation (for weak surface tension). These formal arguments have been justified by sophisticated techniques such as spatial dynamics and centre-manifold reduction methods on the one hand and variational methods on the other. This article presents a complete, self-contained account of an alternative, simpler approach in which one works directly with the Zakharov–Craig–Sulem formulation of the water-wave problem and uses only rudimentary fixed-point arguments and Fourier analysis.

Keywords Solitary waves · Dirichlet–Neumann operator · Korteweg-de Vries equation · Nonlinear Schrödinger equation

1 Introduction

1.1 The Main Results

The classical water-wave problem concerns the two-dimensional, irrotational flow of a perfect fluid of unit density subject to the forces of gravity and surface tension. We use dimensionless variables, choosing h as length scale, $(h/g)^{\frac{1}{2}}$ as time scale and introducing the Bond number $\beta = \sigma/gh^2$, where h is the depth of the water in its undisturbed state, g is the acceleration due to gravity and $\sigma > 0$ is the coefficient of surface tension. The fluid thus occupies the domain $D_\eta = \{(x, y) : x \in \mathbb{R}, y \in (0, 1 + \eta(x, t))\}$, where (x, y) are the usual Cartesian coordinates and $\eta > -1$ is a function of the spatial coordinate x and time t, and the mathematical problem is

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formulated in terms of an Eulerian velocity potential $\varphi(x, y, t)$ which solves Laplace's equation

$$\varphi_{xx} + \varphi_{yy} = 0, \qquad 0 < y < 1 + \eta, \qquad (1.1)$$

and the boundary conditions

$$\varphi_y = 0, \qquad\qquad y = 0, \qquad (1.2)$$
$$\eta_t = \varphi_y - \eta_x \varphi_x, \qquad\qquad y = 1 + \eta, \qquad (1.3)$$
$$\varphi_t = -\frac{1}{2}\varphi_x^2 - \frac{1}{2}\varphi_y^2 - \eta + \beta \left[\frac{\eta_x}{\sqrt{1 + \eta_x^2}}\right]_x, \qquad y = 1 + \eta. \qquad (1.4)$$

Travelling waves are solutions of (1.1)–(1.4) of the form $\eta(x, t) = \eta(x - ct)$, $\varphi(x, y, t) = \varphi(x - ct, y)$, while *solitary waves* are non-trivial travelling waves which satisfy the asymptotic conditions $\eta(x - ct) \to 0$ as $|x - ct| \to \infty$; they correspond to localised disturbances of permanent form which move from left to right with constant speed c.

It is instructive to review the formal weakly nonlinear theory for travelling waves. We begin with the linear dispersion relation for a two-dimensional periodic travelling wave train of wave number $k \geq 0$ and speed $c > 0$, namely

$$c^2 = \frac{1 + \beta k^2}{f(k)}, \qquad f(k) = k \coth k$$

(see Fig. 1). The function $k \mapsto c(k)$ has a unique global minimum at $k = \omega$, and one finds that $\omega = 0$ (with $c(0) = 1$) for $\beta > \frac{1}{3}$ and $\omega > 0$ for $\beta < \frac{1}{3}$. We denote the minimum value of c by c_0, so that $c_0^2 = 1$ for $\beta > \frac{1}{3}$ and $c_0^2 = 2\omega/(2\omega f(\omega) - \omega^2 f'(\omega))$ for $\beta < \frac{1}{3}$ (the formula $\beta = f'(\omega)/(2\omega f(\omega) - \omega^2 f'(\omega))$ defines a bijection between the values of $\beta \in (0, \frac{1}{3})$ and $\omega \in (0, \infty)$). Using c as a bifurcation parameter, we expect branches of small-amplitude solitary waves to bifurcate at $c = c_0$ (where the
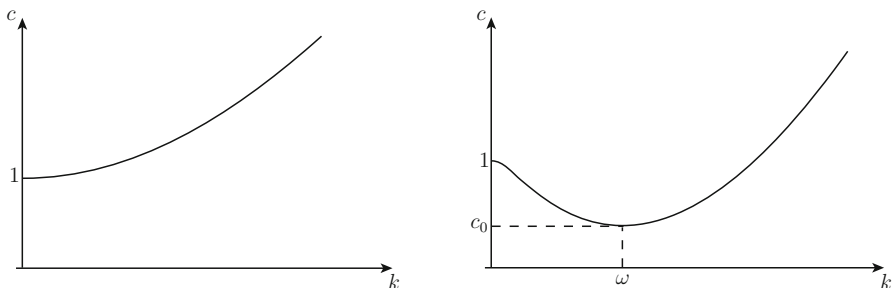

Fig. 1 Dispersion relation for a travelling wave train of wave number $k \geq 0$ and speed $c > 0$ with strong surface tension (left) and weak surface tension (right)
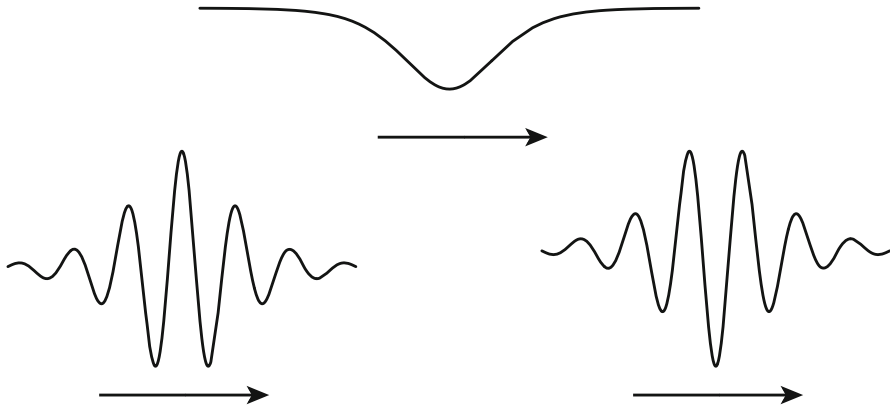
Fig. 2 Solitary wave of depression predicted by the Korteweg-de Vries equation for strong surface tension (above) and the symmetric solitary waves predicted by the nonlinear Schrödinger equation for weak surface tension (below)

linear group and phase speeds are equal) into the region $\{c < c_0\}$ where linear periodic wave trains are not supported (see Dias and Kharif [8, Sect. 3]).

In the case $\beta > \frac{1}{3}$ ('strong surface tension'), one writes $c^2 = 1 - \varepsilon^2$, where ε is a small positive number, substitutes the Ansatz

$$\eta(x) = \varepsilon^2 \rho_1(X) + \varepsilon^2 \rho_2(X) + \cdots, \tag{1.5}$$

where $X = \varepsilon x$, into the travelling-wave version of Eqs. (1.1)–(1.4), and finds that ρ_1 satisfies the stationary Korteweg-de Vries equation

$$\rho - (\beta - \tfrac{1}{3})\rho_{XX} + \tfrac{3}{2}\rho^2 = 0; \tag{1.6}$$

this equation admits an explicit solitary wave of depression given by the formula

$$\rho^\star(X) = -\operatorname{sech}^2\left(\frac{X}{2(\beta - \frac{1}{3})^{1/2}}\right) \tag{1.7}$$

(see Benjamin [3]). In the case $\beta < \frac{1}{3}$ ('weak surface tension'), one writes $c^2 = c_0^2(1 - \varepsilon^2)$, uses the Ansatz

$$\eta(x) = \tfrac{1}{2}\varepsilon\big(\zeta_1(X)e^{i\omega x} + \overline{\zeta_1(X)}e^{-i\omega x}\big) + \varepsilon^2 \zeta_0(X) \tag{1.8}$$
$$+ \tfrac{1}{2}\varepsilon^2\big(\zeta_2(X)e^{2i\omega x} + \overline{\zeta_2(X)}e^{-2i\omega x}\big) + \cdots,$$

and finds that ζ_1 satisfies the stationary nonlinear Schrödinger equation

$$-a_1\zeta_{XX} + a_2\zeta - a_3|\zeta|^2\zeta = 0, \tag{1.9}$$

where

$$a_1 = \tfrac{1}{2} g''(\omega), \qquad a_2 = c_0^2 f(\omega)$$

and

$$a_3 = \tfrac{1}{4} \left(2g(2\omega)^{-1} A(\omega)^2 c_0^4 + 2g(0)^{-1} B(\omega)^2 c_0^4 + C(\omega) c_0^2 + \tfrac{3}{2} \beta \omega^4 \right)$$

with

$$A(\omega) = \tfrac{3}{2}\omega^2 - \tfrac{1}{2} f(\omega)^2 - f(\omega) f(2\omega), \qquad B(\omega) = \omega^2 - 2f(\omega) - f(\omega)^2,$$
$$C(\omega) = -6\omega^2 f(\omega) + 4f(\omega)^2 + 2f(\omega)^2 f(2\omega)$$

and

$$g(k) = 1 + \beta k^2 - c_0^2 f(k)$$

(see Ablowitz and Segur [1]). This equation admits a family $\{e^{i\theta_0} \zeta^\star\}_{\theta_0 \in [0, 2\pi)}$ of solitary-wave solutions, where

$$\zeta^\star(X) = \left(\frac{2a_2}{a_3} \right)^{1/2} \operatorname{sech} \left(\frac{a_2}{a_1} \right)^{1/2} X, \qquad (1.10)$$

two of which, namely $\pm\zeta^\star$ (corresponding to $\theta_0 = 0$ and π), are symmetric. (The positivity of a_3 follows by elementary arguments after substituting the expressions for β and c_0^2 as functions of ω.) The corresponding free-surface profiles are sketched in Fig. 2.

The results of these formal calculations have been rigorously confirmed by spatial dynamics and centre-manifold methods on the one hand (Kirchgässner [17], Amick and Kirchgässner [2], Sachs [20], Iooss and Kirchgässner [15], Iooss and Pérouème [16]) and variational techniques on the other (Buffoni [4,5], Groves and Wahlén [11, 12]), the results of which are summarised in the following theorem.

Theorem 1.1

(i) *Suppose that $\beta > \tfrac{1}{3}$ and $c^2 = 1 - \varepsilon^2$. For each sufficiently small value of $\varepsilon > 0$ there exists a symmetric solitary-wave solution of (1.1)–(1.4) whose free surface is given by*

$$\eta(x) = \varepsilon^2 \rho^\star(\varepsilon x) + o(\varepsilon^2)$$

uniformly over $x \in \mathbb{R}$.

(ii) *Suppose that $\beta < \tfrac{1}{3}$ and $c^2 = c_0^2(1 - \varepsilon^2)$, where $c_0 = c(\omega)$ is the global minimum of the linear dispersion relation (see Fig. 1 (right)). For each sufficiently small*

value of $\varepsilon > 0$ there exist two symmetric solitary-wave solutions of (1.1)–(1.4) *whose free surfaces are given by*

$$\eta(x) = \pm\varepsilon\zeta^{\star}(\varepsilon x) \cos \omega x + o(\varepsilon)$$

uniformly over $x \in \mathbb{R}$.

This article presents an alternative, simpler proof of Theorem 1.1 in which one works directly with the Zakharov–Craig–Sulem formulation of the travelling water-wave equations (see below) and uses only rudimentary fixed-point arguments and Fourier analysis. Some intermediate results are special cases of more general theorems available elsewhere; their proofs have been included here for the sake of a complete, self-contained exposition.

1.2 Methodology

We proceed by formulating the water-wave problem (1.1)–(1.4) in terms of the variables η and $\Phi = \varphi|_{y=1+\eta}$ (see Zakharov [22] and Craig and Sulem [7]). The Zakharov–Craig–Sulem formulation of the water-wave problem is

$$\eta_t - G(\eta)\Phi = 0,$$

$$\Phi_t + \eta + \frac{1}{2}\Phi_x^2 - \frac{(G(\eta)\Phi + \eta_x\Phi_x)^2}{2(1+\eta_x^2)} - \beta\left[\frac{\eta_x}{\sqrt{1+\eta_x^2}}\right]_x = 0,$$

where the velocity potential φ is recovered as the (unique) solution of the boundary-value problem

$$\begin{aligned}
\varphi_{xx} + \varphi_{yy} &= 0, & 0 < y < 1+\eta, \\
\varphi_y &= 0, & y = 0, \\
\varphi &= \Phi, & y = 1+\eta,
\end{aligned}$$

and the *Dirichlet–Neumann operator* $G(\eta)$ is given by $G(\eta)\Phi = \varphi_y - \eta_x\varphi_x|_{y=1+\eta}$. Travelling waves are solutions of the form $\eta(x,t) = \eta(x-ct)$, $\Phi(x,t) = \Phi(x-ct)$; they satisfy

$$-c\eta_x - G(\eta)\Phi = 0, \tag{1.11}$$

$$-c\Phi_x + \eta + \frac{1}{2}\Phi_x^2 - \frac{(G(\eta)\Phi + \eta_x\Phi_x)^2}{2(1+\eta_x^2)} - \beta\left[\frac{\eta_x}{\sqrt{1+\eta_x^2}}\right]_x = 0. \tag{1.12}$$

It is possible to reduce Eqs. (1.11), (1.12) to a single equation for η. Using (1.11), one finds that $\Phi = -cG(\eta)^{-1}\eta_x$, and inserting this formula into (1.12) yields the equation

$$\mathcal{K}(\eta) - c^2\mathcal{L}(\eta) = 0, \tag{1.13}$$

where

$$K(\eta) = \eta - \beta \left[\frac{\eta_x}{\sqrt{1 + \eta_x^2}} \right]_x, \tag{1.14}$$

$$\mathcal{L}(\eta) = -\frac{1}{2}(K(\eta)\eta)^2 + \frac{(\eta_x - \eta_x K(\eta)\eta)^2}{2(1 + \eta_x^2)} + K(\eta)\eta \tag{1.15}$$

and

$$K(\eta)\xi = -(G(\eta)^{-1}\xi_x)_x.$$

Note the equivalent definition

$$K(\eta)\xi = -(\varphi|_{y=1+\eta})_x, \tag{1.16}$$

where φ is the solution of the boundary-value problem

$$\varphi_{xx} + \varphi_{yy} = 0, \qquad 0 < y < 1 + \eta, \tag{1.17}$$
$$\varphi_y = 0, \qquad y = 0, \tag{1.18}$$
$$\varphi_y - \eta_x \varphi_x = \xi_x, \qquad y = 1 + \eta \tag{1.19}$$

(which is unique up to an additive constant).

We proceed by defining the Fourier transform $\hat{u} = \mathcal{F}[u]$ of a function u of a real variable by the formula

$$\hat{u}(k) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} u(x) e^{-ikx} \, dx$$

and using the notation $m(D)$ with $D = -i\partial_x$ for the Fourier multiplier-operator with symbol m, so that $m(D)u = \mathcal{F}^{-1}[m\hat{u}]$. The Ansätze (1.5) and (1.9) suggest that the Fourier transform of a solitary wave is concentrated near the points $k = \pm\omega$ (which coincide at $k = 0$ when $\beta > \frac{1}{3}$). Indeed, writing $c^2 = c_0^2(1 - \varepsilon^2)$, one finds that the linearisation of (1.13) at $\varepsilon = 0$ is

$$g(D)\eta = 0,$$

where

$$g(k) = 1 + \beta k^2 - c_0^2 k \coth k \geq 0, \qquad k \in \mathbb{R},$$

with equality precisely when $k = \pm\omega$ (so that $g(\omega) = g'(\omega) = 0$ and $g''(\omega) > 0$). We therefore decompose η into the sum of functions η_1 and η_2 whose Fourier transforms $\hat{\eta}_1$ and $\hat{\eta}_2$ are supported in the region $S = (-\omega - \delta, -\omega + \delta) \cup (\omega - \delta, \omega + \delta)$ (with

Fig. 3 (a) The support of $\hat{\eta}_1$ is contained in the set S, where $S = (-\delta, \delta)$ for $\beta > \frac{1}{3}$ (left) and $S = (-\omega - \delta, -\omega + \delta) \cup (\omega - \delta, \omega + \delta)$ for $\beta < \frac{1}{3}$ (right)

$\delta \in (0, \frac{1}{3}))$ and its complement (see Fig. 3), so that $\eta_1 = \chi(D)\eta$, $\eta_2 = (1 - \chi(D))\eta$, where χ is the characteristic function of the set S. Decomposing (1.13) into

$$\chi(D)\left(\mathcal{K}(\eta_1 + \eta_2) - c_0^2(1 - \varepsilon^2)\mathcal{L}(\eta_1 + \eta_2)\right) = 0,$$

$$(1 - \chi(D))\left(\mathcal{K}(\eta_1 + \eta_2) - c_0^2(1 - \varepsilon^2)\mathcal{L}(\eta_1 + \eta_2)\right) = 0,$$

one finds that the second equation can be solved for η_2 as a function of η_1 for sufficiently small values of $\varepsilon > 0$; substituting $\eta_2 = \eta_2(\eta_1)$ into the first yields the reduced equation

$$\chi(D)\left(\mathcal{K}(\eta_1 + \eta_2(\eta_1)) - c_0^2(1 - \varepsilon^2)\mathcal{L}(\eta_1 + \eta_2(\eta_1))\right) = 0$$

for η_1 (see Sect. 3).

Finally, the scaling

$$\eta_1(x) = \varepsilon^2 \rho(X), \qquad X = \varepsilon x, \tag{1.20}$$

transforms the reduced equation into

$$\varepsilon^{-2} g(\varepsilon D)\rho + f(\varepsilon D)\rho + \tfrac{3}{2}\chi_0(\varepsilon D)\rho^2 + O(\varepsilon^{1/2}) = 0 \tag{1.21}$$

for $\beta > \frac{1}{3}$, while the scaling

$$\eta_1(x) = \tfrac{1}{2}\varepsilon\zeta(X)e^{i\omega x} + \tfrac{1}{2}\varepsilon\overline{\zeta(X)}e^{-i\omega x}, \qquad X = \varepsilon x, \tag{1.22}$$

transforms the reduced equation into

$$\varepsilon^{-2} g(\omega + \varepsilon D)\zeta + c_0^2 f(\omega + \varepsilon D)\zeta - a_3 \chi_0(\varepsilon D)(|\zeta|^2\zeta) + O(\varepsilon^{1/2}) = 0 \tag{1.23}$$

for $\beta < \frac{1}{3}$; here χ_0 is the characteristic function of the set $(-\delta, \delta)$, the symbol D now means $-i\partial_X$ and precise estimates for the remainder terms are given in Sect. 4. Eqs. (1.21) and (1.23) are termed *full dispersion* versions of (perturbed) stationary Korteweg-de Vries and nonlinear Schrödinger equations since they retain the linear part of the original equation (1.13); the fully reduced model equations (1.6) and (1.9) are recovered from them in the formal limit $\varepsilon \to 0$.

Variational versions of this reduction procedure have previously been given by Groves and Wahlén [12]. Starting with the observation that (1.13) is the Euler–

Lagrange equation for the functional

$$\mathcal{J}(\eta) = \int_{\mathbb{R}^2} \left(\frac{1}{2}\eta^2 + \beta(\sqrt{1 + \eta_x^2 + \eta_z^2} - 1) - \frac{1}{2}c^2\eta\, K(\eta)\eta \right) \mathrm{d}x\,\mathrm{d}z,$$

they use the decomposition $\eta = \eta_1 + \eta_2(\eta_1)$ and scaling of η_1 described above to derive reduced variational functionals for ρ and ζ whose Euler–Lagrange equations are given to leading order by (1.6) and (1.9). Critical points of the reduced functionals (and hence solitary-wave solutions of the reduced equations) are found by the direct methods of the calculus of variations. In the present paper we apply a more direct perturbative approach introduced by Stefanov and Wright [21] for another full dispersion Korteweg-de Vries equation, namely, the Whitham equation (see Ehrnström et al. [9] for a variational treatment of this equation).

The travelling-wave Whitham equation is

$$cu - m(D)u - u^2 = 0, \qquad m(k) = \sqrt{\frac{\tanh k}{k}}.$$

Noting that its linear dispersion relation has a unique global maximum at $k = 0$ (with $c(0) = 1$), one writes $c = 1 + \varepsilon^2$ and seeks solitary waves of the form

$$u = \varepsilon^2 w(X), \qquad X = \varepsilon x,$$

so that

$$\varepsilon^{-2}(1 + \varepsilon^2 - m(\varepsilon D))w - w^2 = 0,$$

which can be rewritten as a fixed-point equation of the form

$$w = \left(1 - \tfrac{1}{6}\partial_X^2 + O(\varepsilon^2) \right)^{-1} w^2. \tag{1.24}$$

In the formal limit $\varepsilon \to 0$ we recover the stationary Korteweg-de Vries equation

$$w - \tfrac{1}{6}w_{XX} - w^2 = 0;$$

its (unique, symmetric) solitary-wave solution $w^\star$ is nondegenerate in the sense that the only bounded solution of its linearisation at $w^\star$ is $w_X^\star$. Restricting to spaces of symmetric functions eliminates this antisymmetric solution of the linearised equation and a solution to (1.24) can be constructed as a perturbation of $w^\star$ using the implicit-function theorem.

In Sect. 5 we apply the above argument to (1.21) and (1.23), first reformulating them as fixed-point equations. The functions $\rho^\star$ and $\pm\zeta^\star$ are nondegenerate solutions of (1.6) and (1.9) in the sense that the only bounded solutions of their linearisations at $\rho^\star$ and $\pm\zeta^\star$ are respectively $\rho_X^\star$ and $\pm\zeta_X^\star$, $\pm\mathrm{i}\zeta^\star$. Observe that equation (1.13) is invariant under the reflection $\eta(x) \mapsto \eta(-x)$, and the reduction procedure preserves this property: the reduced equation for η_1 is invariant under the reflection

$\eta_1(x) \mapsto \eta_1(-x)$, so that (1.21) and (1.23) are invariant under respectively $\rho(x) \mapsto \rho(-x)$ and $\zeta(x) \mapsto \overline{\zeta(-x)}$. Restricting to spaces of symmetric functions thus eliminates the antisymmetric solutions $\rho^{\star\prime}$ and $\pm \zeta_X^\star, \pm i\zeta^\star$ of the linearised equations, and solutions to (1.21) and (1.23) can be constructed as perturbations of $\rho^\star$ and $\pm\zeta^\star$ using an appropriate version of the implicit-function theorem.

1.3 Function Spaces

We study the equation

$$\mathcal{K}(\eta) - c_0^2(1 - \varepsilon^2)\mathcal{L}(\eta) = 0$$

in the basic space $\mathcal{X} = H^2(\mathbb{R})$, where

$$H^s(\mathbb{R}) = \{\eta \in \mathcal{S}'(\mathbb{R}) : \|\eta\|_s := \|(1 + D^2)^{s/2}\eta\|_{L^2(\mathbb{R})} < \infty\}, \qquad s \geq 0,$$

are the usual Bessel-potential spaces. The decomposition $\eta = \eta_1 + \eta_2$, where $\eta_1 = \chi(D)\eta$, $\eta_2 = (1 - \chi(D))\eta$, is accommodated by writing $\mathcal{X}$ as the direct sum of $\mathcal{X}_1 = \chi(D)\mathcal{X}$ and $\mathcal{X}_2 = (1 - \chi(D))\mathcal{X}$, where $\mathcal{X}_1$ and $\mathcal{X}_2$ are equipped with respectively the scaled norm

$$\|\!|\eta_1|\!\| := \left(\int_{\mathbb{R}} (1 + \varepsilon^{-2}(|k| - \omega)^2)|\hat{\eta}_1(k)|^2 \, dk \right)^{1/2}$$

and the usual norm for $H^2(\mathbb{R})$. The norm for $\mathcal{X}_1$ is so chosen because the final scalings (1.20) and (1.22) transform $\|\!|\eta_1|\!\|$ into a multiple of the standard norm for $H^1(\mathbb{R})$, namely, $\|\!|\eta_1|\!\| = \varepsilon^{3/2}\|\rho\|_1$ and $\|\!|\eta_1|\!\| = \varepsilon^{1/2}\|\zeta\|_1$, and the reduced equations (1.21) and (1.23) are discussed in this space.

The following proposition yields in particular the estimate

$$\|\eta_1\|_\infty \lesssim \varepsilon^{1/2}\|\!|\eta_1|\!\|$$

for the supremum norm of $\eta_1 \in \mathcal{X}_1$. We can also estimate higher-order derivatives of $\eta_1 \in \mathcal{X}_1$ using the fact that the support of $\hat{\eta}_1$ is contained in the fixed bounded set S, so that, for example

$$\|\eta_1\|_n \lesssim \|\eta_1\|_0, \qquad \|\eta_1\|_{n,\infty} \lesssim \varepsilon^{1/2}\|\!|\eta_1|\!\|$$

for each $n \in \mathbb{N}_0$.

Proposition 1.2 *The estimate*

$$\|\hat{\eta}_1\|_{L^1(\mathbb{R})} \lesssim \varepsilon^{1/2}\|\!|\eta_1|\!\|$$

holds for each $\eta_1 \in \mathcal{X}_1$.

Proof This estimate follows from the calculation

$$\int_{\mathbb{R}} |\hat{\eta}_1(k)| \, dk = \int_{\mathbb{R}} \frac{(1 + \varepsilon^{-2}(|k| - \omega)^2)^{1/2}}{(1 + \varepsilon^{-2}(|k| - \omega)^2)^{1/2}} |\hat{\eta}_1(k)| \, dk$$

$$\lesssim \|\|\eta\|\| \left(\int_{\omega-\delta}^{\omega+\delta} \frac{1}{1 + \varepsilon^{-2}(k - \omega)^2} \, dk \right)^{1/2}$$

$$= \tfrac{1}{2} \pi^{1/2} \varepsilon^{1/2} \|\|\eta\|\|.$$

$\square$

It is also helpful to use the larger space

$$\mathcal{Z} = \{ \eta \in \mathcal{S}'(\mathbb{R}) : \|\eta\|_{\mathcal{Z}} := \|\hat{\eta}_1\|_{L^1(\mathbb{R})} + \|\eta_2\|_2 < \infty \},$$

into which $H^2(\mathbb{R})$ is continuously embedded. In Sect. 2 we demonstrate that $K(\cdot): \mathcal{Z} \to \mathcal{L}(H^{3/2}(\mathbb{R}), H^{1/2}(\mathbb{R}))$ is analytic at the origin and deduce that $\mathcal{K}, \mathcal{L}$ map the open neighbourhood

$$U = \{ \eta \in H^2(\mathbb{R}) : \|\eta\|_{\mathcal{Z}} < M \}$$

of the origin in $H^2(\mathbb{R})$ analytically into $L^2(\mathbb{R})$ for sufficiently small values of M. Moreover, we take advantage of the estimate

$$\|\eta\|_{\mathcal{Z}} \lesssim \varepsilon^{1/2} \|\|\eta_1\|\| + \|\eta_2\|_2 \tag{1.25}$$

for $\eta \in H^2(\mathbb{R})$ to obtain estimates for $\mathcal{K}$ and $\mathcal{L}$ which are necessary for the reduction procedure described above (see Sect. 3). Note, however, that in the entirety of the existence theory we work in the fixed subset U of $H^2(\mathbb{R})$ (whose elements are 'well-behaved' functions).

2 Analyticity

In this section, we study the operator K given by (1.16) using basic results from the theory of analytic functions in Banach spaces (see the treatise by Buffoni and Toland [6] for a complete account). In particular, we present an elementary proof that $K(\cdot): \mathcal{Z} \to \mathcal{L}(H^{3/2}(\mathbb{R}), H^{1/2}(\mathbb{R}))$, and hence $\mathcal{K}, \mathcal{L}: U \to L^2(\mathbb{R})$, are analytic at the origin (see Sect. 1.3 above). A more comprehensive treatment of the analyticity of operators of Dirichlet–Neumann and Neumann–Dirichlet type in water-wave problems is given by Lannes [18, Ch. 3 and Appendix A] (see also Nicholls and Reitich [19] and Hu and Nicholls [14]).

We begin with the boundary-value problem (1.17)–(1.19), which is handled using the change of variable

$$y' = \frac{y}{1 + \eta}, \qquad u(x, y') = \varphi(x, y),$$

to map $\Sigma_\eta = \{(x, y): x \in \mathbb{R}, 0 < y < 1 + \eta(x)\}$ to the strip $\Sigma = \mathbb{R} \times (0, 1)$. Dropping the primes, one finds that (1.17)–(1.19) are transformed into

$$
\begin{aligned}
u_{xx} + u_{yy} &= \partial_x F_1(\eta, u) + \partial_y F_2(\eta, u), & 0 < y < 1, && (2.1) \\
u_y &= 0, & y = 0, && (2.2) \\
u_y &= F_2(\eta, u) + \xi_x, & y = 1, && (2.3)
\end{aligned}
$$

where

$$
F_1(\eta, u) = -\eta u_x + y \eta_x u_y, \qquad F_2(\eta, u) = \frac{\eta u_y}{1 + \eta} + y \eta_x u_x - \frac{y^2}{1 + \eta} \eta_x^2 u_y,
$$

and

$$
K(\eta)\xi = -u_x|_{y=1}.
$$

We discuss (2.1)–(2.3) using the standard Sobolev spaces $H^n(\Sigma)$, $n \in \mathbb{N}$, together with $H_\star^{n+1}(\Sigma)$, $n \in \mathbb{N}$, which is defined as the completion of

$$
\mathcal{S}(\Sigma, \mathbb{R}) = \{u \in C^\infty(\bar{\Sigma}) : |x|^m |\partial_x^{\alpha_1} \partial_y^{\alpha_2} u| \text{ is bounded for all } m, \alpha_1, \alpha_2 \in \mathbb{N}_0\}
$$

with respect to the norm

$$
\|u\|_{n+1,\star}^2 := \|u_x\|_{H^n(\Sigma)}^2 + \|u_y\|_{H^n(\Sigma)}^2.
$$

Proposition 2.1 *For each $F_1, F_2 \in H^n(\Sigma)$ and $\xi \in H^{n+1/2}(\mathbb{R})$, $n \in \mathbb{N}$, the boundary-value problem*

$$
\begin{aligned}
u_{xx} + u_{yy} &= \partial_x F_1 + \partial_y F_2, & 0 < y < 1, \\
u_y &= 0, & y = 0, \\
u_y &= F_2(\eta, u) + \xi_x, & y = 1,
\end{aligned}
$$

admits a unique solution $u = S(F_1, F_2, \xi)$ in $H_\star^{n+1}(\Sigma)$ given (with a slight abuse of notation, in that derivatives should be taken) by the explicit formula

$$
S(F_1, F_2, \xi) = \mathcal{F}^{-1}\left[\int_0^1 \left(\mathrm{i}k G(y, \tilde{y}) \hat{F}_1 - G_{\tilde{y}}(y, \tilde{y}) \hat{F}_2\right) \mathrm{d}\tilde{y} - \mathrm{i}k G(y, 1)\hat{\xi}\right],
$$

in which

$$
G(y, \tilde{y}) =
\begin{cases}
-\dfrac{\cosh |k| y \cosh |k|(1 - \tilde{y})}{|k| \sinh |k|}, & 0 \le y \le \tilde{y} \le 1, \\[3mm]
-\dfrac{\cosh |k| \tilde{y} \cosh |k|(1 - y)}{|k| \sinh |k|}, & 0 \le \tilde{y} \le y \le 1,
\end{cases}
$$

so that

$$\|S(F_1, F_2, \xi)\|_{n+1,\star} \lesssim \|F_1\|_{H^n(\Sigma)} + \|F_2\|_{H^n(\Sigma)} + \|\xi\|_{n+1/2}.$$

Lemma 2.2 *For each $\xi \in H^{3/2}(\mathbb{R})$ and each sufficiently small $\eta \in \mathcal{Z}$ the boundary-value problem* (2.1)–(2.3) *admits a unique solution $u \in H^2_\star(\Sigma)$. Furthermore, the mapping $\mathcal{Z} \to \mathcal{L}(H^{3/2}(\mathbb{R}), H^2_\star(\Sigma))$ given by $\eta \mapsto (\xi \mapsto u)$ is analytic at the origin.*

Proof Define

$$T : H^2_\star(\Sigma) \times \mathcal{Z} \times H^{3/2}(\mathbb{R}) \to H^2_\star(\Sigma)$$

by

$$T(u, \eta, \xi) = u - S(F_1(\eta, u), F_2(\eta, u), \xi),$$

and note that the solutions of (2.1)–(2.3) are precisely the zeros of $T(\cdot, \eta, \xi)$. Using the estimates

$$\|\eta w\|_{H^1(\Sigma)} \lesssim \|\eta\|_{1,\infty} \|w\|_{H^1(\Sigma)}$$
$$\lesssim (\|\eta_1\|_{1,\infty} + \|\eta_2\|_2)\|w\|_{H^1(\Sigma)},$$

$$\|y\eta_x w\|_{H^1(\Sigma)} \lesssim (\|\eta_{1x}\|_{1,\infty} \|w\|_{H^1(\Sigma)}$$
$$+ \|\eta_{2x} w\|_{L^2(\Sigma)} + \|\eta_{2x} w_x\|_{L^2(\Sigma)} + \|\eta_{2xx} w\|_{L^2(\Sigma)})$$
$$\lesssim (\|\eta_{1x}\|_{1,\infty} + \|\eta_{2x}\|_\infty)\|w\|_{H^1(\Sigma)} + \|\eta_{2xx}\|_0 \|w\|_{H^1(\Sigma)}$$
$$\lesssim (\|\eta_{1x}\|_{1,\infty} + \|\eta_2\|_2)\|w\|_{H^1(\Sigma)},$$

$$\|y^2\eta^n \eta_x^2 w\|_{H^1(\Sigma)} \lesssim \|\eta\|_{1,\infty}^n (\|\eta_{1x}\|_{1,\infty}^2 \|w\|_{H^1(\Sigma)}$$
$$+ \|\eta_{2x}^2 w\|_{L^2(\Sigma)} + \|\eta_{2x}^2 w_x\|_{L^2(\Sigma)} + \|\eta_{2x}\eta_{2xx} w\|_{L^2(\Sigma)})$$
$$\lesssim \|\eta\|_{1,\infty}^n ((\|\eta_{1x}\|_{1,\infty} + \|\eta_{2x}\|_\infty)^2 \|w\|_{H^1(\Sigma)}$$
$$+ \|\eta_{2x}\|_\infty \|\eta_{2xx}\|_0 \|w\|_{H^1(\Sigma)})$$
$$\lesssim (\|\eta_1\|_{2,\infty} + \|\eta_2\|_2)^{n+2} \|w\|_{H^1(\Sigma)}$$

(uniformly in n) and

$$\|\eta_1\|_{2,\infty} + \|\eta_2\|_2 \lesssim \|\hat{\eta}_1\|_{L^1(\mathbb{R})} + \|\eta_2\|_2 = \|\eta\|_{\mathcal{Z}},$$

one finds that the mappings $\mathcal{Z} \times H^2_\star(\Sigma) \to H^1(\Sigma)$ given by $(\eta, u) \mapsto F_1(\eta, u)$ and $(\eta, u) \mapsto F_2(\eta, u)$ are analytic at the origin; it follows that T is also analytic at the origin. Furthermore $T(0, 0, 0) = 0$ and

$$d_1 T[0, 0, 0](u) = u - S(F_1(0, u), F_2(0, u), 0) = u - S(0, 0, 0) = u$$

(because S is linear and F_1, F_2 are linear in their second arguments), so that $d_1 T[0, 0, 0] = I$ is an isomorphism. By the analytic implicit-function theorem there

exist open neighbourhoods N_1 and N_2 of the origin in $\mathcal{Z}$ and $H^{3/2}(\mathbb{R})$ and an analytic function $v : N_1 \times N_2 \to H^2_\star(\Sigma)$, such that

$$T(v(\eta, \xi), \eta, \xi) = 0.$$

Since v is linear in ξ one can take N_2 to be the whole space $H^{3/2}(\mathbb{R})$. $\qquad \square$

Corollary 2.3 *The mapping* $K(\cdot) \colon \mathcal{Z} \to \mathcal{L}(H^{3/2}(\mathbb{R}), H^{1/2}(\mathbb{R}))$ *is analytic at the origin.*

Corollary 2.4 *The formulae* (1.14)*,* (1.15) *define functions* $U \to L^2(\mathbb{R})$ *which are analytic at the origin and satisfy* $\mathcal{K}(0) = \mathcal{L}(0) = 0$.

Proof This result follows from Corollary 2.3 and the facts that $H^1(\mathbb{R})$ is a Banach algebra and $(u_1, u_2) \mapsto u_1 u_2$ is a bounded bilinear mapping $H^{1/2}(\mathbb{R}) \times H^{1/2}(\mathbb{R}) \to L^2(\mathbb{R})$ (see Hörmander [13, Theorem 8.3.1]). $\qquad \square$

In keeping with Lemma 2.2 and Corollaries 2.3 and 2.4 we write

$$u(\eta, \xi) = \sum_{j=0}^{\infty} u_j(\eta, \xi), \tag{2.4}$$

where u_j is homogeneous of degree j in η and linear in ξ, and

$$K(\eta) = \sum_{j=0}^{\infty} K_j(\eta), \qquad \mathcal{K}(\eta) = \sum_{j=1}^{\infty} \mathcal{K}_j(\eta), \qquad \mathcal{L}(\eta) = \sum_{j=1}^{\infty} \mathcal{L}_j(\eta),$$

where $K_j(\eta), \mathcal{K}_j(\eta)$ and $\mathcal{L}_j(\eta)$ are homogeneous of degree j in η (and we accordingly abbreviate $K_0(\eta)$ to K_0).

Remark 2.5 Note that $K_j(\eta) = m_j(\{\eta\}^{(n)})$, where m_j is a bounded, symmetric, j-linear mapping $\mathcal{Z}^j \to \mathcal{L}(H^{3/2}(\mathbb{R}), H^{1/2}(\mathbb{R}))$.

We examine the first few terms

$$\begin{aligned}
\mathcal{K}_1(\eta) &= \eta - \beta \eta_{xx}, \\
\mathcal{K}_2(\eta) &= 0, \\
\mathcal{K}_3(\eta) &= \tfrac{3}{2} \beta \eta_x^2 \eta_{xx}
\end{aligned} \tag{2.5}$$

and

$$\begin{aligned}
\mathcal{L}_1(\eta) &= K_0 \eta, \\
\mathcal{L}_2(\eta) &= \tfrac{1}{2} \eta_x^2 - \tfrac{1}{2}(K_0 \eta)^2 + K_1(\eta)\eta, \\
\mathcal{L}_3(\eta) &= -K_0 \eta K_1(\eta)\eta - \eta_x^2 K_0 \eta + K_2(\eta)\eta
\end{aligned} \tag{2.6}$$

in the Maclaurin expansions of $\mathcal{K}$ and $\mathcal{L}$ in more detail since they play a prominent role in our subsequent calculations. We begin by computing explicit expressions for K_0, K_1 and K_2.

Lemma 2.6

(i) *The operators K_0 and K_1 are given by the formulae*

$$K_0\xi = f(D)\xi, \qquad K_1(\eta)\xi = -(\eta\xi_x)_x - K_0(\eta K_0\xi)$$

for each $\eta \in H^2(\mathbb{R})$ and $\xi \in H^{3/2}(\mathbb{R})$.

(ii) *The operator K_2 is given by the formula*

$$K_2(\eta)\xi = \tfrac{1}{2}(\eta^2 K_0\xi)_{xx} + \tfrac{1}{2}K_0(\eta^2\xi_{xx}) + K_0(\eta K_0(\eta K_0\xi))$$

under the additional regularity hypothesis that $\eta \in H^3(\mathbb{R})$ and $\xi \in H^{5/2}(\mathbb{R})$.

Proof (i) The solution to the boundary-value problem

$$
\begin{aligned}
u_{0xx} + u_{0yy} &= 0, & 0 < y < 1, \\
u_{0y} &= 0, & y &= 0, \\
u_{0y} &= \xi_x, & y &= 1,
\end{aligned}
$$

is

$$u_0 = \mathcal{F}^{-1}\left[\frac{ik}{|k|}\frac{\cosh|k|y}{\sinh|k|}\hat{\xi}\right],$$

while the solution to the boundary-value problem

$$
\begin{aligned}
u_{1xx} + u_{1yy} &= (-\eta u_{0x} + y\eta_x u_{0y})_x + (\eta u_{0y} + y\eta_x u_{0x})_y, & 0 < y < 1, \\
u_{1y} &= 0, & y &= 0, \\
u_{1y} &= \eta u_{0y} + \eta_x u_{0x}, & y &= 1,
\end{aligned}
$$

is

$$u_1 = y\eta u_{0y} + \mathcal{F}^{-1}\left[\frac{ik}{|k|}\frac{\cosh|k|y}{\sinh|k|}\mathcal{F}[\eta u_{0x}|_{y=1}]\right],$$

whence

$$K_0\xi = -u_{0x}|_{y=0} = f(D)\xi, \qquad K_1(\eta)\xi = -u_{1x}|_{y=0} = -(\eta\xi_x)_x - K_0(\eta K_0\xi).$$

(ii) Supposing that $\xi \in H^{5/2}(\mathbb{R})$, so that $u^0 \in H^3_\star(\Sigma)$, and $\eta \in H^3(\mathbb{R})$, so that $u^1 \in H^3_\star(\Sigma)$ (see Proposition 2.1), we find that the solution $u^2 \in H^4_\star(\Sigma)$ to the boundary-value problem

$$u_{2xx} + u_{2yy} = (-\eta u_{1x} + y\eta_x u_{1y})_x$$
$$+ (\eta u_{1y} + y\eta_x u_{1x})_y - \eta^2 u_{0yy} - (y^2 \eta_x^2 u_{0y})_y, \qquad 0 < y < 1,$$
$$u_{2y} = 0, \qquad\qquad\qquad y = 0,$$
$$u_{2y} = \eta u_{1y} + \eta_x u_{1x} - \eta^2 u_{0y} - \eta_x^2 u_{0y}, \qquad\qquad y = 1,$$

is

$$u_2 = -\tfrac{1}{2}\eta^2 (y^2 u_{0y})_y + \eta u_{1y} - \mathcal{F}^{-1}\left[\frac{ik}{|k|}\frac{\cosh|k|y}{\sinh|k|}\mathcal{F}[\tfrac{1}{2}(\eta^2 u_{0y})_x\big|_{y=1}]\right]$$
$$+ \mathcal{F}^{-1}\left[\frac{ik}{|k|}\frac{\cosh|k|y}{\sinh|k|}\mathcal{F}[\eta u_{1x}\big|_{y=1}]\right].$$

It follows that

$$K_2(\eta)\xi = -u_{2x}\big|_{y=0}$$
$$= -\tfrac{1}{2}(\eta^2 u_{0xx})_x + (\eta^2 u_{0y})_x$$
$$\quad - (\eta(\eta u_{0y} + \eta_x u_{0x}))_x - \tfrac{1}{2}K_0((\eta^2 \xi_x)_x) - K_0(\eta K_1(\eta)\xi),$$
$$= \tfrac{1}{2}(\eta^2 K_0 \xi_x)_x + (\eta\eta_x K_0 \xi)_x$$
$$\quad - \tfrac{1}{2}K_0((\eta^2 \xi_x)_x) + K_0(\eta(\eta\xi_x)_x) + K_0(\eta K_0(\eta K_0 \xi))$$
$$= \tfrac{1}{2}(\eta^2 K_0 \xi)_{xx} + \tfrac{1}{2}K_0(\eta^2 \xi_{xx}) + K_0(\eta K_0(\eta K_0 \xi)).$$

□

Remark 2.7 Explicit expressions for $K_3, K_4, \ldots$ can be computed in a similar fashion. However, computing an expansion in terms of Fourier-multiplier operators in this fashion leads a loss of one derivative at each order. It is therefore necessary to compensate by increasing the regularity of ξ and η by one derivative at each order.

Corollary 2.8

(i) *The function $\mathcal{L}_2$ is given by the formula*

$$\mathcal{L}_2(\eta) = \tfrac{1}{2}\left(\eta_x^2 - (K_0\eta)^2 - (\eta^2)_{xx} - 2K_0(\eta K_0\eta)\right)$$

for each $\eta \in H^2(\mathbb{R})$.

(ii) *The function $\mathcal{L}_3$ is given by the formula*

$$\mathcal{L}_3(\eta) = K_0\eta\, K_0(\eta K_0\eta) + K_0(\eta K_0(\eta K_0\eta)) + \eta(K_0\eta)\eta_{xx}$$
$$+ \tfrac{1}{2}K_0(\eta^2 \eta_{xx}) + \tfrac{1}{2}(\eta^2 K_0\eta)_{xx}$$

under the additional regularity hypothesis that $\eta \in H^3(\mathbb{R})$.

Finally, we record the representation $\mathcal{L}_2(\eta) = m(\eta, \eta)$, where

$$m(u, v) = \tfrac{1}{2} \left(u_x v_x - (K_0 u)(K_0 v) - (uv)_{xx} - K_0(u K_0 v + v K_0 u) \right), \quad (2.7)$$

which is helpful when performing calculations, and some straightforward estimates for the higher-order parts of $\mathcal{K}$ and $\mathcal{L}$.

Proposition 2.9 *The estimate* $\|m(u, v)\|_0 \lesssim \|u\|_{\mathcal{Z}} \|v\|_2$ *holds for each* $u, v \in H^2(\mathbb{R})$.

Proposition 2.10

(i) *The quantities*

$$\mathcal{K}_c(\eta) := \sum_{j=3}^{\infty} \mathcal{K}_j(\eta), \qquad \mathcal{L}_c(\eta) := \sum_{j=3}^{\infty} \mathcal{L}_j(\eta)$$

satisfy the estimates

$$\|\mathcal{K}_c(\eta)\|_0 \lesssim \|\eta\|_{\mathcal{Z}}^2 \|\eta\|_2, \qquad \|d\mathcal{K}_c[\eta](v)\|_0 \lesssim \|\eta\|_{\mathcal{Z}}^2 \|v\|_2 + \|\eta\|_{\mathcal{Z}} \|\eta\|_2 \|v\|_{\mathcal{Z}},$$
$$\|\mathcal{L}_c(\eta)\|_0 \lesssim \|\eta\|_{\mathcal{Z}}^2 \|\eta\|_2, \qquad \|d\mathcal{L}_c[\eta](v)\|_0 \lesssim \|\eta\|_{\mathcal{Z}}^2 \|v\|_2 + \|\eta\|_{\mathcal{Z}} \|\eta\|_2 \|v\|_{\mathcal{Z}}$$

for each $\eta \in U$ *and* $v \in H^2(\mathbb{R})$.

(ii) *The quantities*

$$\mathcal{K}_r(\eta) := \sum_{j=4}^{\infty} \mathcal{K}_j(\eta), \qquad \mathcal{L}_r(\eta) := \sum_{j=4}^{\infty} \mathcal{L}_j(\eta) + \tfrac{1}{2}(K_1(\eta)\eta)^2$$

satisfy the estimates

$$\|\mathcal{K}_r(\eta)\|_0 \lesssim \|\eta\|_{\mathcal{Z}}^4 \|\eta\|_2, \qquad \|d\mathcal{K}_r[\eta](v)\|_0 \lesssim \|\eta\|_{\mathcal{Z}}^4 \|v\|_2 + \|\eta\|_{\mathcal{Z}}^3 \|\eta\|_2 \|v\|_{\mathcal{Z}},$$
$$\|\mathcal{L}_r(\eta)\|_0 \lesssim \|\eta\|_{\mathcal{Z}}^3 \|\eta\|_2, \qquad \|d\mathcal{L}_r[\eta](v)\|_0 \lesssim \|\eta\|_{\mathcal{Z}}^3 \|v\|_2 + \|\eta\|_{\mathcal{Z}}^2 \|\eta\|_2 \|v\|_{\mathcal{Z}}$$

for each $\eta \in U$ *and* $v \in H^2(\mathbb{R})$.

Proof These estimates follow from the explicit formulae (2.5), (2.6), together with the calculations

$$\mathcal{K}(\eta) - \mathcal{K}_1(\eta) - \mathcal{K}_2(\eta) - \mathcal{K}_3(\eta) = \beta \left(1 - \tfrac{3}{2}\eta_x^2 - \frac{1}{(1 + \eta_x^2)^{3/2}} \right) \eta_{xx}$$

and

$$\mathcal{L}(\eta) - \mathcal{L}_1(\eta) - \mathcal{L}_2(\eta) - \mathcal{L}_3(\eta)$$
$$= -\tfrac{1}{2}(K_1(\eta)\eta)^2 - \tfrac{1}{2}(K_r(\eta)\eta)^2 - (K_0\eta + K_1(\eta))K_r(\eta)\eta - \eta_x^2(K_1(\eta)\eta + K_r(\eta)\eta)$$
$$+ \tfrac{1}{2}\eta_x^2(K(\eta)\eta)^2 - \frac{\eta_x^2}{2(1+\eta_x^2)}(\eta_x - \eta_x K(\eta)\eta)^2 + (K_r(\eta) - K_2(\eta))\eta,$$

where

$$K_r(\eta) = \sum_{j=2}^{\infty} K_j(\eta).$$

$\square$

3 Reduction

In this section, we reduce the equation

$$\mathcal{K}(\eta) - c_0^2(1 - \varepsilon^2)\mathcal{L}(\eta) = 0 \qquad (3.1)$$

to a locally equivalent equation for η_1. Clearly $\eta \in U$ satisfies (3.1) if and only if

$$\chi(D)\left(\mathcal{K}(\eta_1 + \eta_2) - c_0^2(1 - \varepsilon^2)\mathcal{L}(\eta_1 + \eta_2)\right) = 0,$$
$$(1 - \chi(D))\left(\mathcal{K}(\eta_1 + \eta_2) - c_0^2(1 - \varepsilon^2)\mathcal{L}(\eta_1 + \eta_2)\right) = 0,$$

and these equations can be rewritten as

$$g(D)\eta_1 + c_0^2\varepsilon^2 K_0\eta_1 + \chi(D)\mathcal{N}(\eta_1 + \eta_2) = 0, \qquad (3.2)$$
$$g(D)\eta_2 + c_0^2\varepsilon^2 K_0\eta_2 + (1 - \chi(D))\mathcal{N}(\eta_1 + \eta_2) = 0, \qquad (3.3)$$

in which

$$\mathcal{N}(\eta) = \mathcal{K}_c(\eta) - c_0^2(1 - \varepsilon^2)(\mathcal{L}_2(\eta) + \mathcal{L}_c(\eta)).$$

We proceed by writing (3.3) as a fixed-point equation for η_2 using Proposition 3.1, which follows from the fact that $g(k) \gtrsim |k|^2$ for $k \notin S$, and solving it for η_2 as a function of η_1 using Theorem 3.2, which is proved by a straightforward application of the contraction mapping principle. Substituting $\eta_2 = \eta_2(\eta_1)$ into (3.2) yields a reduced equation for η_1. Note that the reduced equation is invariant under the reflection $\eta_1(x) \mapsto \eta_1(-x)$, which is inherited from the invariance of (3.1) under the reflection $\eta(x) \mapsto \eta(-x)$ (see below).

Proposition 3.1 *The mapping* $(1 - \chi(D))g(D)^{-1}$ *is a bounded linear operator* $L^2(\mathbb{R}) \to \mathcal{X}_2$.

Theorem 3.2 *Let $\mathcal{X}_1$, $\mathcal{X}_2$ be Banach spaces, X_1, X_2 be closed, convex sets in, respectively, $\mathcal{X}_1$, $\mathcal{X}_2$ containing the origin and $\mathcal{G}\colon X_1 \times X_2 \to \mathcal{X}_2$ be a smooth function. Suppose that there exists a continuous function $r\colon X_1 \to [0, \infty)$, such that*

$$\|\mathcal{G}(x_1, 0)\| \leq \tfrac{1}{2}r, \quad \|\mathrm{d}_2\mathcal{G}[x_1, x_2]\| \leq \tfrac{1}{3}$$

for each $x_2 \in \bar{B}_r(0) \subseteq X_2$ and each $x_1 \in X_1$.

Under these hypotheses there exists for each $x_1 \in X_1$ a unique solution $x_2 = x_2(x_1)$ of the fixed-point equation $x_2 = \mathcal{G}(x_1, x_2)$ satisfying $x_2(x_1) \in \bar{B}_r(0)$. Moreover $x_2(x_1)$ is a smooth function of $x_1 \in X_1$ and in particular satisfies the estimate

$$\|\mathrm{d}x_2[x_1]\| \leq 2\|\mathrm{d}_1\mathcal{G}[x_1, x_2(x_1)]\|.$$

3.1 Strong Surface Tension

Suppose that $\beta > \tfrac{1}{3}$. We write (3.3) in the form

$$\eta_2 = -(1 - \chi(D))g(D)^{-1}\left(\varepsilon^2 K_0\eta_2 + \mathcal{N}(\eta_1 + \eta_2)\right) \tag{3.4}$$

and apply Theorem 3.2 with

$$X_1 = \{\eta_1 \in \mathcal{X}_1\colon |\!|\!|\eta_1|\!|\!| \leq R_1\}, \qquad X_2 = \{\eta_2 \in \mathcal{X}_2\colon \|\eta_2\|_2 \leq R_2\};$$

the function $\mathcal{G}$ is given by the right-hand side of (3.4). Using Proposition 1.2 one can guarantee that $\|\hat{\eta}_1\|_{L^1(\mathbb{R}^2)} < \tfrac{1}{2}M$ for all $\eta_1 \in X_1$ for an arbitrarily large value of R_1; the value of R_2 is constrained by the requirement that $\|\eta_2\|_2 < \tfrac{1}{2}M$ for all $\eta_2 \in X_2$.

Lemma 3.3 *The estimates*

(i) $\|\mathcal{G}(\eta_1, \eta_2)\|_2 \lesssim \varepsilon^{1/2}|\!|\!|\eta_1|\!|\!|^2 + \varepsilon^{1/2}|\!|\!|\eta_1|\!|\!|\,\|\eta_2\|_2 + |\!|\!|\eta_1|\!|\!|\,\|\eta_2\|_2^2 + \|\eta_2\|_2^2 + \varepsilon^2\|\eta_2\|_2$,

(ii) $\|\mathrm{d}_1\mathcal{G}[\eta_1, \eta_2]\|_{\mathcal{L}(\mathcal{X}_1, \mathcal{X}_2)} \lesssim \varepsilon^{1/2}|\!|\!|\eta_1|\!|\!| + \varepsilon^{1/2}\|\eta_2\|_2 + \|\eta_2\|_2^2$,

(iii) $\|\mathrm{d}_2\mathcal{G}[\eta_1, \eta_2]\|_{\mathcal{L}(\mathcal{X}_1, \mathcal{X}_2)} \lesssim \varepsilon^{1/2}|\!|\!|\eta_1|\!|\!| + |\!|\!|\eta_1|\!|\!|\,\|\eta_2\|_2 + \|\eta_2\|_2 + \varepsilon^2$

hold for each $\eta_1 \in X_1$ and $\eta_2 \in X_2$.

Proof Observe that

$$\|K_0\eta_2\|_0 \lesssim \|\eta_2\|_2,$$

and using Propositions 2.9 and 2.10(i), one finds that

$$\|\mathcal{L}_2(\eta_1 + \eta_2)\|_0 = \|m(\{\eta_1 + \eta_2\}^{(2)})\|_0 \lesssim \|\eta_1\|_{\mathcal{Z}}\|\eta_1\|_2 + \|\eta_1\|_{\mathcal{Z}}\|\eta_2\|_2 + \|\eta_2\|_{\mathcal{Z}}\|\eta_2\|_2$$

and

$$\|\mathcal{K}_{\mathrm{c}}(\eta_1 + \eta_2)\|_0, \ \|\mathcal{L}_{\mathrm{c}}(\eta_1 + \eta_2)\|_0 \lesssim \|\eta_1 + \eta_2\|_{\mathcal{Z}}^2\|\eta_1 + \eta_2\|_2;$$

part (i) follows from these estimates and inequality (1.25). Parts (ii) and (iii) are obtained in a similar fashion. □

Theorem 3.4 *Equation* (3.4) *has a unique solution* $\eta_2 \in X_2$ *which depends smoothly upon* $\eta_1 \in X_1$ *and satisfies the estimates*

$$\|\eta_2(\eta_1)\|_2 \lesssim \varepsilon^{1/2} \||\eta_1\||^2, \qquad \|\mathrm{d}\eta_2[\eta_1]\|_{\mathcal{L}(X_1,X_2)} \lesssim \varepsilon^{1/2} \||\eta_1\||.$$

Proof Choosing R_2 and ε sufficiently small and setting $r(\eta_1) = \sigma \varepsilon^{1/2} \||\eta_1\||^2$ for a sufficiently large value of $\sigma > 0$, one finds that

$$\|\mathcal{G}(\eta_1, 0)\|_2 \lesssim \tfrac{1}{2} r(\eta_1), \qquad \|\mathrm{d}_2 \mathcal{G}[\eta_1, \eta_3]\|_{\mathcal{L}(X_2,X_2)} \lesssim \varepsilon^{1/2}$$

for $\eta_1 \in X_1$ and $\eta_2 \in \overline{B}_{r(\eta_1)}(0) \subset X_2$ (Lemma 3.3(i), (iii)). Theorem 3.2 asserts that equation (3.4) has a unique solution η_2 in $\overline{B}_{r(\eta_1)}(0) \subset X_2$ which depends smoothly upon $\eta_1 \in X_1$, and the estimate for its derivative follows from Lemma 3.3(ii). □

Substituting $\eta_2 = \eta_2(\eta_1)$ into (3.2) yields the reduced equation

$$g(D)\eta_1 + c_0^2 \varepsilon^2 K_0 \eta_1 + \chi(D)\mathcal{N}(\eta_1 + \eta_2(\eta_1)) = 0 \tag{3.5}$$

for $\eta_1 \in X_1$. Observe that this equation is invariant under the reflection $\eta_1(x) \mapsto \eta_1(-x)$; a familiar argument shows that it is inherited from the corresponding invariance of (3.2), (3.4) under $\eta_1(x) \mapsto \eta_1(-x)$, $\eta_2(x) \mapsto \eta_2(-x)$ when applying Theorem 3.2.

3.2 Weak Surface Tension

Suppose that $\beta < \tfrac{1}{3}$. Since $\chi(D)\mathcal{L}_2(\eta_1) = 0$ the nonlinear term in (3.2) is at leading order cubic in η_1, so that this equation may be rewritten as

$$g(D)\eta_1 + c_0^2 \varepsilon^2 K_0 \eta_1 + \chi(D)\left(\mathcal{N}(\eta_1 + \eta_2) + c_0^2(1 - \varepsilon^2)\mathcal{L}_2(\eta_1)\right) = 0. \tag{3.6}$$

To compute the reduced equation for η_1 we need an explicit formula for the leading-order quadratic part of $\eta_2(\eta_1)$, which is evidently given by

$$F(\eta_1) := c_0^2(1 - \varepsilon^2)(1 - \chi(D))g(D)^{-1}\mathcal{L}_2(\eta_1). \tag{3.7}$$

It is convenient to write $\eta_2 = F(\eta_1) + \eta_3$ and (3.3) in the form

$$\eta_3 = -(1 - \chi(D))g(D)^{-1}$$
$$\left(c_0^2(1 - \varepsilon^2)\mathcal{L}_2(\eta_1) + \mathcal{N}(\eta_1 + F(\eta_1) + \eta_3) + c_0^2 \varepsilon^2 K_0(F(\eta_1) + \eta_3)\right) \tag{3.8}$$

(with the requirement that $\eta_1 + F(\eta_1) + \eta_3 \in U$). We apply Theorem 3.2 to equation (3.8) with

$$X_1 = \{\eta_1 \in \mathcal{X}_1 : \|\|\eta_1\|\| \leq R_1\}, \qquad X_3 = \{\eta_3 \in \mathcal{X}_2 : \|\eta_3\|_3 \leq R_3\};$$

the function $\mathcal{G}$ is given by the right-hand side of (3.8). (Here we write X_3 rather than X_2 for notational clarity.) Using Proposition 1.2 one can guarantee that $\|\hat{\eta}_1\|_{L^1(\mathbb{R})} < \frac{1}{2}M$ for all $\eta_1 \in X_1$ for an arbitrarily large value of R_1; the value of R_3 is constrained by the requirement that $\|F(\eta_1) + \eta_3\|_2 < \frac{1}{2}M$ for all $\eta_1 \in X_1$ and $\eta_3 \in X_3$, so that $\eta_1 + F(\eta_1) + \eta_3 \in U$ (Proposition 3.5 below asserts that $\|F(\eta_1)\|_2 = O(\varepsilon^{1/2})$ uniformly over $\eta_1 \in X_1$).

Proposition 3.5 *The estimates*

$$\|F(\eta_1)\|_2 \lesssim \varepsilon^{1/2} \|\|\eta_1\|\|^2, \qquad \|dF[\eta_1]\|_{\mathcal{L}(\mathcal{X}_1, \mathcal{X}_2)} \lesssim \varepsilon^{1/2} \|\|\eta_1\|\|$$

hold for each $\eta_1 \in X_1$.

Proof This result follows from the formula

$$F(\eta_1) = c_0^2 (1 - \varepsilon^2)(1 - \chi(D))g(D)^{-1} m_2(\eta_1, \eta_1),$$

Proposition 2.9 and inequality (1.25). □

Remark 3.6 Noting that

$$K_0 F(\eta_1) = c_0^2 (1 - \varepsilon^2)(1 - \chi(D))g(D)^{-1} f(D)m_2(\eta_1, \eta_1)$$

and that $m(u_1, v_1)$ has compact support for all $u_1, v_1 \in \mathcal{X}_1$, one finds that $K_0 F(\eta_1)$ satisfies the same estimates as $F(\eta_1)$.

Lemma 3.7 *The quantity*

$$\mathcal{N}_1(\eta_1, \eta_3) = \mathcal{L}_2(\eta_1 + F(\eta_1) + \eta_3) - \mathcal{L}_2(\eta_1) \qquad (3.9)$$

satisfies the estimates

(i) $\|\mathcal{N}_1(\eta_1, \eta_3)\|_0 \lesssim \varepsilon \|\|\eta_1\|\|^3 + \varepsilon^{1/2} \|\|\eta_1\|\|^2 \|\eta_3\|_2 + \varepsilon^{1/2} \|\|\eta_1\|\| \|\eta_3\|_2 + \|\eta_3\|_2^2$,
(ii) $\|d_1 \mathcal{N}_1[\eta_1, \eta_3]\|_{\mathcal{L}(\mathcal{X}_1, L^2(\mathbb{R}))} \lesssim \varepsilon \|\|\eta_1\|\|^2 + \varepsilon^{1/2} \|\|\eta_1\|\| \|\eta_3\|_2 + \varepsilon^{1/2} \|\eta_3\|_2$,
(iii) $\|d_2 \mathcal{N}_1[\eta_1, \eta_3]\|_{\mathcal{L}(\mathcal{X}_2, L^2(\mathbb{R}))} \lesssim \varepsilon^{1/2} \|\|\eta_1\|\| + \|\eta_3\|_2$

for each $\eta_1 \in X_1$ and $\eta_3 \in X_3$.

Proof We estimate

$$\mathcal{N}_1(\eta_1, \eta_3) = 2m(\eta_1, F(\eta_1) + \eta_3) + m(F(\eta_1) + \eta_3, F(\eta_1) + \eta_3)$$

and its derivatives, which are computed using the chain rule, using Propositions 2.9 and 3.5 and inequality (1.25). □

Lemma 3.8 *The quantity*

$$\mathcal{N}_2(\eta_1, \eta_3) = \mathcal{K}_c(\eta_1 + F(\eta_1) + \eta_3) - c_0^2(1 - \varepsilon^2)(\mathcal{L}_c(\eta_1 + F(\eta_1) + \eta_3) \quad (3.10)$$

satisfies the estimates

(i) $\|\mathcal{N}_2(\eta_1, \eta_3)\|_0 \lesssim (\varepsilon^{1/2}\|\|\eta_1\|\| + \|\eta_3\|_2)^2(\|\|\eta_1\|\| + \|\eta_3\|_2),$
(ii) $\|d_1\mathcal{N}_2[\eta_1, \eta_3]\|_{\mathcal{L}(\mathcal{X}_1, L^2(\mathbb{R}))} \lesssim (\varepsilon^{1/2}\|\|\eta_1\|\| + \|\eta_3\|_2)^2,$
(iii) $\|d_2\mathcal{N}_2[\eta_1, \eta_3]\|_{\mathcal{L}(\mathcal{X}_2, L^2(\mathbb{R}))} \lesssim (\varepsilon^{1/2}\|\|\eta_1\|\| + \|\eta_3\|_2)(\|\|\eta_1\|\| + \|\eta_3\|_2)$

for each $\eta_1 \in X_1$ and $\eta_3 \in X_3$.

Proof We estimate $\mathcal{N}_2$ and its derivatives, which are computed using the chain rule, using Propositions 2.10(i) and 3.5 and inequality (1.25). □

Altogether we have established the following estimates for $\mathcal{G}$ and its derivatives (see Remark 3.6 and Lemmata 3.7 and 3.8).

Lemma 3.9 *The estimates*

(i) $\|\mathcal{G}(\eta_1, \eta_3)\|_2 \lesssim (\varepsilon^{1/2}\|\|\eta_1\|\| + \|\eta_3\|_2)^2(1 + \|\|\eta_1\|\| + \|\eta_3\|_2) + \varepsilon^2\|\eta_3\|_2,$
(ii) $\|d_1\mathcal{G}[\eta_1, \eta_3]\|_{\mathcal{L}(\mathcal{X}_1, \mathcal{X}_2)} \lesssim (\varepsilon^{1/2}\|\|\eta_1\|\| + \|\eta_3\|_2)(\varepsilon^{1/2} + \varepsilon^{1/2}\|\|\eta_1\|\| + \|\eta_3\|_2),$
(iii) $\|d_2\mathcal{G}[\eta_1, \eta_3]\|_{\mathcal{L}(\mathcal{X}_2, \mathcal{X}_2)} \lesssim (\varepsilon^{1/2}\|\|\eta_1\|\| + \|\eta_3\|_2)(1 + \|\|\eta_1\|\| + \|\eta_3\|_2) + \varepsilon^2$

hold for each $\eta_1 \in X_1$ and $\eta_3 \in X_3$.

Theorem 3.10 *Equation* (3.8) *has a unique solution $\eta_3 \in X_3$ which depends smoothly upon $\eta_1 \in X_1$ and satisfies the estimates*

$$\|\eta_3(\eta_1)\|_2 \lesssim \varepsilon\|\|\eta_1\|\|^2, \qquad \|d\eta_3[\eta_1]\|_{\mathcal{L}(\mathcal{X}_1, \mathcal{X}_2)} \lesssim \varepsilon\|\|\eta_1\|\|.$$

Proof Choosing R_3 and ε sufficiently small and setting $r(\eta_1) = \sigma\varepsilon\|\|\eta_1\|\|^2$ for a sufficiently large value of $\sigma > 0$, one finds that

$$\|\mathcal{G}(\eta_1, 0)\|_2 \lesssim \tfrac{1}{2}r(\eta_1), \qquad \|d_2\mathcal{G}[\eta_1, \eta_3]\|_{\mathcal{L}(\mathcal{X}_2, \mathcal{X}_2)} \lesssim \varepsilon^{1/2}$$

for $\eta_1 \in X_1$ and $\eta_3 \in \overline{B}_{r(\eta_1)}(0) \subset X_3$ (Lemma 3.9(i), (iii)). Theorem 3.2 asserts that equation (3.8) has a unique solution η_3 in $\overline{B}_{r(\eta_1)}(0) \subset X_3$ which depends smoothly upon $\eta_1 \in X_1$, and the estimate for its derivative follows from Lemma 3.9(ii). □

Substituting $\eta_2 = F(\eta_1) + \eta_3(\eta_1)$ into (3.6) yields the reduced equation

$$g(D)\eta_1 + c_0^2\varepsilon^2 K_0\eta_1$$
$$+ \chi(D)\left(\mathcal{N}(\eta_1 + F(\eta_1) + \eta_3(\eta_2)) + c_0^2(1 - \varepsilon^2)\mathcal{L}_2(\eta_1)\right) = 0 \quad (3.11)$$

for $\eta_1 \in X_1$. This equation is also invariant under the reflection $\eta_1(x) \mapsto \eta_1(-x)$; it is inherited from the invariance of (3.6), (3.8) under $\eta_1(x) \mapsto \eta_1(-x)$, $\eta_3(x) \mapsto \eta_3(-x)$ when applying Theorem 3.2.

4 Derivation of the Reduced Equation

In this section we compute the leading-order terms in the reduced equations (3.5) and (3.11) and hence derive the perturbed full dispersion Korteweg-de Vries and nonlinear Schrödinger equations announced in Sect. 1. The main steps are approximating the Fourier-multiplier operators appearing in lower-order terms by constants, estimating higher-order terms and performing the scalings (1.20) and (1.22).

It is convenient to introduce some additional notation to estimate higher-order 'remainder' terms.

Definition 4.1

(i) The symbol $\mathcal{O}(\varepsilon^\gamma \||\eta_1\||^r)$ denotes a smooth function $\mathcal{R} : X_1 \to L^2(\mathbb{R})$ which satisfies the estimates

$$\|\mathcal{R}(\eta_1)\|_0 \lesssim \varepsilon^\gamma \||\eta_1\||^r, \qquad \|\mathrm{d}\mathcal{R}[\eta_1]\|_{\mathcal{L}(\mathcal{X}_1, L^2(\mathbb{R}))} \lesssim \varepsilon^\gamma \||\eta_1\||^{r-1}$$

for each $\eta_1 \in X_1$ (where $\gamma \geq 0$, $r \geq 1$), and the underscored notation $\underline{\mathcal{O}}(\varepsilon^\gamma \||\eta_1\||^r)$ indicates additionally that the Fourier transform of $\mathcal{R}(\eta_1)$ lies in a fixed compact set (independently of ε and uniformly over $\eta_1 \in X_1$). Furthermore

$$\underline{\mathcal{O}}_0(\varepsilon^\gamma \||\eta_1\||^r) := \chi_0(D)\mathcal{O}(\varepsilon^\gamma \||\eta_1\||^r),$$
$$\underline{\mathcal{O}}_+(\varepsilon^\gamma \||\eta_1\||^r) := \chi^+(D)\mathcal{O}(\varepsilon^\gamma \||\eta_1\||^r),$$

where χ_0 and χ^+ are the characteristic functions of the sets $(-\delta, \delta)$ and $(\omega - \delta, \omega + \delta)$ (for $\omega > 0$).

(ii) The symbol $\underline{\mathcal{O}}_n^\varepsilon(\|u\|_1^r)$ denotes $\chi_0(\varepsilon D)\mathcal{R}(u)$, where $\mathcal{R}$ is a smooth function $B_R(0) \subseteq \chi_0(\varepsilon D)H^1(\mathbb{R}) \to H^n(\mathbb{R})$ or $B_R(0) \subseteq H^1(\mathbb{R}) \to H^n(\mathbb{R})$ which satisfies the estimates

$$\|\mathcal{R}(u)\|_n \lesssim \|u\|_1^r, \quad \|\mathrm{d}\mathcal{R}[u]\|_{\mathcal{L}(H^1(\mathbb{R}), H^n(\mathbb{R}))} \lesssim \|u\|_1^{r-1}$$

for each $u \in B_R(0)$ (with $r \geq 1$, $n \geq 0$).

4.1 Strong Surface Tension

The leading-order terms in the reduced equation

$$g(D)\eta_1 + c_0^2 \varepsilon^2 K_0 \eta_1 + \chi(D)\mathcal{N}(\eta_1 + \eta_2(\eta_1)) = 0$$

derived in Sect. 3.1 are computed by approximating the operators ∂_x and K_0 in the quadratic part of the equation by constants.

Proposition 4.2 *The estimates*

(i) $\eta_{1x} = \mathcal{Q}_0(\varepsilon \|\!|\eta_1|\!\|)$,
(ii) $K_0\eta_1 = \eta_1 + \mathcal{Q}_0(\varepsilon \|\!|\eta_1|\!\|)$,
(iii) $K_0\eta_1^2 = \eta_1^2 + \mathcal{Q}(\varepsilon^{3/2} \|\!|\eta_1|\!\|)$

hold for each $\eta_1 \in X_1$.

Proof Note that

$$\|\eta_{1x}\|_0^2 = \|\,|k|\hat{\eta}_1\|_0^2 \leq \varepsilon^2 \|\!|\eta_1|\!\|^2$$

and

$$\|(K_0 - I)\eta_1\|_0^2 = \left\|\,(|k|\coth|k| - 1)\,\hat{\eta}_1\right\|_0^2 \lesssim \|\,|k|^2\hat{\eta}_1\|_0^2 \leq \varepsilon^2 \|\!|\eta_1|\!\|^2;$$

the corresponding estimates for their derivatives are trivially satisfied since the operators are linear. The quantity to be estimated in (iii) is quadratic in η_1; it therefore suffices to estimate the corresponding bilinear operator. The argument used above yields

$$\|(K_0 - I)(u_1 v_1)\|_0$$
$$\lesssim \left\|\,|k|\int_{\mathbb{R}}|\hat{u}_1(k - s)||\hat{v}_1(s)|\,\mathrm{d}s\right\|_0$$
$$\lesssim \left\|\int_{\mathbb{R}}|k - s||\hat{u}_1(k - s)||\hat{v}_1(s)|\,\mathrm{d}s + \int_{\mathbb{R}}|s||\hat{u}_1(k - s)||\hat{v}_1(s)|\,\mathrm{d}s\right\|_0$$
$$\lesssim \|\,|k|\hat{u}_1\,\|_0\|\hat{v}_1\|_{L^1(\mathbb{R})} + \|\hat{u}_1\|_{L^1(\mathbb{R})}\|\,|k|\hat{v}_1\,\|_0$$
$$\lesssim \varepsilon^{3/2}\|\!|u_1|\!\|\,\|\!|v_1|\!\|$$

for each $u_1, v_1 \in X_1$, where we have also used Young's inequality. □

Lemma 4.3 *The estimate*

$$\mathcal{L}_2(\eta_1 + \eta_2(\eta_1)) = -\tfrac{3}{2}\eta_1^2 + \mathcal{O}(\varepsilon \|\!|\eta_1|\!\|^2)$$

holds for each $\eta_1 \in X_1$.

Proof Using Proposition 2.9 and Theorem 3.4, one finds that

$$\mathcal{L}_2(\eta_1 + \eta_2(\eta_1)) = m(\eta_1, \eta_1) + 2m(\eta_1, \eta_2(\eta_1)) + m(\eta_2(\eta_1), \eta_2(\eta_1))$$
$$= \mathcal{L}_2(\eta_1) + \mathcal{O}(\varepsilon \|\!|\eta_1|\!\|^3),$$

and

$$\mathcal{L}_2(\eta_1) = -\tfrac{3}{2}\eta_1^2 + \mathcal{Q}(\varepsilon^{3/2}\|\!|\eta_1|\!\|^2)$$

because of (2.7) and Proposition 4.2. □

Lemma 4.4 *The estimate*

$$\mathcal{N}(\eta_1 + \eta_2(\eta_1)) = \mathcal{O}(\varepsilon \|\|\eta_1\|\|^3)$$

holds for each $\eta_1 \in X_1$.

Proof This result follows from Proposition 2.10(i) and Theorem 3.4. □

We conclude that the reduced equation for η_1 is the *perturbed full dispersion Korteweg-de Vries equation*

$$g(D)\eta_1 + \varepsilon^2 K_0 \eta_1 + \tfrac{3}{2}\chi_0(D)\eta_1^2 + \underline{\mathcal{Q}}_0(\varepsilon \|\|\eta_1\|\|^2) = 0,$$

and applying Proposition 4.2, one can further simplify it to

$$g(D)\eta_1 + \varepsilon^2 \eta_1 + \tfrac{3}{2}\chi_0(D)\eta_1^2 + \underline{\mathcal{Q}}_0(\varepsilon \|\|\eta_1\|\|^2) + \underline{\mathcal{Q}}_0(\varepsilon^3 \|\|\eta_1\|\|) = 0.$$

Finally, we introduce the Korteweg-de Vries scaling

$$\eta_1(x) = \varepsilon^2 \rho(\varepsilon x),$$

so that $\rho \in B_R(0) \subseteq \chi(\varepsilon D)H^1(\mathbb{R})$, where $R > 0$ and ε is chosen small enough that $\varepsilon^{3/2}R \leq R_1$, solves the equation

$$\varepsilon^{-2}g(\varepsilon D)\rho + \rho + \tfrac{3}{2}\chi_0(\varepsilon D)\rho^2 + \varepsilon^{1/2}\underline{\mathcal{Q}}_0^\varepsilon(\|\rho\|_1) = 0 \tag{4.1}$$

(note that $\|\|\eta\|\| = \varepsilon^{3/2}\|\rho\|_1$, the change of variable from x to $X = \varepsilon x$ introduces an additional factor of $\varepsilon^{1/2}$ in the remainder term and the symbol D now means $-\mathrm{i}\partial_X$). The invariance of the reduced equation under $\eta_1(x) \mapsto \eta_1(-x)$ is inherited by (4.1), which is invariant under the reflection $\rho(X) \mapsto \rho(-X)$.

4.2 Weak Surface Tension

In this section we compute the leading-order terms in the reduced equation (3.11) derived in Sect. 3.2. To this end, we write

$$\eta_1 = \eta_1^+ + \eta_1^-,$$

where $\eta_1^+ = \chi^+(D)\eta_1$ and $\eta_1^- = \overline{\eta_1^+}$, so that η_1^+ satisfies the equation

$$g(D)\eta_1^+ + c_0^2 \varepsilon^2 K_0 \eta_1^+$$
$$+ \chi^+(D)\left(\mathcal{N}(\eta_1 + F(\eta_1) + \eta_3(\eta_1)) + c_0^2(1 - \varepsilon^2)\mathcal{L}_2(\eta_1)\right) = 0 \tag{4.2}$$

(and η_1^- satisfies its complex conjugate). We again begin by showing how Fourier-multiplier operators acting upon the function η_1 may be approximated by constants.

Lemma 4.5 *The estimates*

(i) $\partial_x \eta_1^+ = +i\omega\eta_1^+ + \mathcal{O}_+(\varepsilon\|\eta_1\|)$,

(ii) $\partial_x^2 \eta_1^+ = -\omega^2\eta_1^+ + \mathcal{O}_+(\varepsilon\|\eta_1\|)$,

(iii) $K_0\eta_1^+ = f(\omega)\eta_1^+ + \mathcal{O}_+(\varepsilon\|\eta_1\|)$,

(iv) $K_0((\eta_1^+)^2) = f(2\omega)(\eta_1^+)^2 + \mathcal{O}(\varepsilon^{3/2}\|\eta_1\|^2)$,

(v) $K_0(\eta_1^+\eta_1^-) = \eta_1^+\eta_1^- + \mathcal{O}(\varepsilon^{3/2}\|\eta_1\|^2)$,

(vi) $\mathcal{F}^{-1}[g(k)^{-1}\mathcal{F}[(\eta_1^+)^2]] = g(2\omega)^{-1}(\eta_1^+)^2 + \mathcal{O}(\varepsilon^{3/2}\|\eta_1\|^2)$,

(vii) $\mathcal{F}^{-1}[g(k)^{-1}\mathcal{F}[\eta_1^+\eta_1^-]] = g(0)^{-1}\eta_1^+\eta_1^- + \mathcal{O}(\varepsilon^{3/2}\|\eta_1\|^2)$,

(viii) $K_0((\eta_1^+)^2\eta_1^-) = f(\omega)(\eta_1^+)^2\eta_1^- + \mathcal{O}(\varepsilon^2\|\eta_1\|^3)$

hold for each $\eta_1 \in X_1$.

Proof Note that

$$\|\partial_x \eta_1^+ - i\omega\eta_1^+\|_0^2 \le \int_{\mathbb{R}} (k-\omega)^2|\hat{\eta}_1(k)|^2 \, dk \le \varepsilon^2\|\eta_1\|^2,$$

and iterating this argument yields (ii); moreover

$$\|K_0\eta_1^+ - f(\omega)\eta_1^+\|_0^2 = \|(f(k) - f(\omega))\hat{\eta}_1^+\|_0 \lesssim \int_{\mathbb{R}} (k-\omega)^2|\hat{\eta}_1(k)|^2 \, dk \le \varepsilon^2\|\eta_1\|^2.$$

The corresponding estimates for their derivatives are trivially satisfied since the operators are linear.

Notice that the quantities to be estimated in (iv)–(vii) are quadratic in η_1; it therefore suffices to estimate the corresponding bilinear operators. To this end we take u_1, $v_1 \in \mathcal{X}_1$. The argument used for (iii) above yields

$$\|\mathcal{F}[K_0(u_1^+ v_1^+) - f(2\omega)u_1^+ v_1^+]\|_0$$
$$\lesssim \left\| |k - 2\omega| \int_{\mathbb{R}} |\hat{u}_1^+(k-s)||\hat{v}_1^+(s)| \, ds \right\|_0$$
$$\lesssim \left\| \int_{\mathbb{R}} |k - s - \omega||\hat{u}_1^+(k-s)||\hat{v}_1^+(s)| \, ds \right.$$
$$\left. + \int_{\mathbb{R}} |s - \omega||\hat{u}_1^+(k-s)||\hat{v}_1^+(s)| \, ds \right\|_0$$
$$\lesssim \| \, |k-\omega|\hat{u}_1^+ \, \|_0\|\hat{v}_1^+\|_{L^1(\mathbb{R})} + \|\hat{u}_1^+\|_{L^1(\mathbb{R})}\| \, |k-\omega, 0)|\hat{v}_1^+ \, \|_0$$
$$\lesssim \varepsilon^{3/2}\|u_1\| \, \|v_1\|,$$

where we have also used Young's inequality. Turning to (v), we note that

$$\|\mathcal{F}[K_0(u_1^+ v_1^-) - \underbrace{f(0)}_{=1} u_1^+ v_1^-]\|_0$$
$$\lesssim \left\| |k| \int_{\mathbb{R}} |\hat{u}_1^+(k-s)||\hat{v}_1^-(s)| \, ds \right\|_0$$

$$\lesssim \left\| \int_{\mathbb{R}} |k - s - \omega| |\hat{u}_1^+(k-s)| |\hat{v}_1^-(s)| \, ds \right.$$
$$\left. + \int_{\mathbb{R}} |s + \omega| |\hat{u}_1^+(k-s)| |\hat{v}_1^-(s)| \, ds \right\|_0$$
$$\lesssim \| \, |k - \omega| \hat{u}_1^+ \, \|_0 \|\hat{v}_1^+\|_{L^1(\mathbb{R})} + \|\hat{u}_1^+\|_{L^1(\mathbb{R})} \| \, |k + \omega| \hat{v}_1^+ \, \|_0$$
$$\lesssim \varepsilon^{3/2} \|\!|u_1|\!\| \, \|\!|v_1|\!\|.$$

Estimates (vi) and (vii) are obtained in the same fashion.

To establish (viii) we similarly estimate the relevant trilinear operator. Take u_1, v_1, $w_1 \in \mathcal{X}_1$ and observe that

$$\|\mathcal{F}[K_0(u_1^+ v_1^+ w_1^-) - f(\omega)u_1^+ v_1^+ w_1^-]\|_0$$
$$\lesssim \left\| |k - \omega| \int_{\mathbb{R}} |\hat{u}_1^+(k-s)| |\mathcal{F}[v_1^+ w_1^+](s)| \, ds_1 \, ds_2 \right\|_0$$
$$\lesssim \left\| \int_{\mathbb{R}} |k - s + \omega| |\hat{u}_1^+(k-s)| |\mathcal{F}[v_1^+ w_1^+](s)| \, ds \right.$$
$$\left. + \int_{\mathbb{R}} |(s - 2\omega| |\hat{u}_1^+(k-s)| |\mathcal{F}[v_1^+ w_1^+](s)| \, ds \right\|_0$$
$$\lesssim \| \, |k + \omega| \hat{u}_1^+ \, \|_{L^1(\mathbb{R})} \|v_1^+ w_1^+\|_0 + \|\hat{u}_1^+\|_{L^1(\mathbb{R})} \| \, |k - 2\omega| \mathcal{F}[v_1^+ w_1^+] \|_0$$
$$\lesssim \| \, |k + \omega| \hat{u}_1^+ \, \|_0 \|v_1^+\|_\infty \|w_1^+\|_0 + \|\hat{u}_1^+\|_{L^1(\mathbb{R})} \| \, |k - 2\omega| \mathcal{F}[v_1^+ w_1^+] \|_0$$
$$\lesssim \varepsilon^2 \|\!|u_1|\!\| \, \|\!|v_1|\!\| \, \|\!|w_1|\!\|.$$

$\square$

We proceed by approximating each term in the quadratic and cubic parts of equation (4.2) according to the rules established in Lemma 4.5, recalling that

$$\mathcal{N}(\eta_1 + F(\eta_1) + \eta_3) = \mathcal{N}_2(\eta_1, \eta_3) - c_0^2(1 - \varepsilon^2)\mathcal{N}_1(\eta_1, \eta_3),$$

where $F(\eta_1)$, $\mathcal{N}_1(\eta_1, \eta_3)$ and $\mathcal{N}_1(\eta_1, \eta_3)$ are defined by respectively (3.7), (3.9) and (3.10).

Proposition 4.6 *The estimate*

$$F(\eta_1) = g(2\omega)^{-1} c_0^2 A(\omega) \left((\eta_1^+)^2 + (\eta_1^-)^2 \right) + g(0)^{-1} c_0^2 B(\omega)\eta_1^+ \eta_1^- + \underline{\mathcal{O}}(\varepsilon^{3/2}\|\!|\eta_1|\!\|^2),$$

where

$$A(\omega) = \tfrac{3}{2}\omega^2 - \tfrac{1}{2}f(\omega)^2 - f(\omega)f(2\omega), \qquad B(\omega) = \omega^2 - 2f(\omega) - f(\omega)^2,$$

holds for each $\eta_1 \in X_1$.

Proof Using equation (2.7) and the expansions given in Lemma 4.5, we find that

$$\mathcal{L}_2(\eta_1) = m(\eta_1, \eta_1) = A(\omega)\left((\eta_1^+)^2 + (\eta_1^-)^2\right) + B(\omega)\eta_1^+\eta_1^- + \underline{\mathcal{O}}(\varepsilon^{3/2}\|\eta_1\|^2).$$

It follows that

$$(1 - \chi(D))g(D)^{-1}\mathcal{L}_2(\eta_1)$$
$$= g(2\omega)^{-1}A(\omega)\left((\eta_1^+)^2 + (\eta_1^-)^2\right) + g(0)^{-1}B(\omega)\eta_1^+\eta_1^- + \underline{\mathcal{O}}(\varepsilon^{3/2}\|\eta_1\|^2)$$

because of Lemma 4.5(vi), (vii) and the facts that $\chi(D)\mathcal{L}_2(\eta_1) = 0$ and

$$(1 - \chi(D))g(D)^{-1}\underline{\mathcal{O}}(\varepsilon^{3/2}\|\eta_1\|^2) = \underline{\mathcal{O}}(\varepsilon^{3/2}\|\eta_1\|^2)$$

(since $(1 - \chi(k))g(k)^{-1}$ is bounded). We conclude that

$$F(\eta_1) = c_0^2(1 - \varepsilon^2)(1 - \chi(D))g(D)^{-1}\mathcal{L}_2(\eta_1)$$
$$= g(2\omega)^{-1}c_0^2 A(\omega)\left((\eta_1^+)^2 + (\eta_1^-)^2\right) + g(0)^{-1}c_0^2 B(\omega)\eta_1^+\eta_1^- + \underline{\mathcal{O}}(\varepsilon^{3/2}\|\eta_1\|^2).$$

$\square$

Proposition 4.7 *The estimate*

$$\chi^+(D)\mathcal{N}_1(\eta_1, \eta_3(\eta_1))$$
$$= \left(2g(2\omega)^{-1}A(\omega)^2 c_0^2 + g(0)^{-1}B(\omega)^2 c_0^2\right)\chi^+(D)\left((\eta_1^+)^2\eta_1^-\right) + \underline{\mathcal{O}}_+(\varepsilon^{3/2}\|\eta_1\|^3)$$

holds for each $\eta_1 \in X_1$.

Proof Observe that

$$\mathcal{N}_1(\eta_1, \eta_3(\eta_1)) = 2m(\eta_1, F(\eta_1) + \eta_3(\eta_1)) + m(F(\eta_1) + \eta_3(\eta_1), F(\eta_1) + \eta_3(\eta_1))$$
$$= 2m(\eta_1, F(\eta_1)) + \mathcal{O}(\varepsilon^{3/2}\|\eta_1\|^3),$$

in which we have used the calculations

$$m(\eta_1, \eta_3(\eta_1)) = \mathcal{O}(\varepsilon^{3/2}\|\eta_1\|^3), \quad m(F(\eta_1), \eta_3(\eta_1)) = \mathcal{O}(\varepsilon^{3/2}\|\eta_1\|^4)$$

(see Propositions 2.9 and 3.5 and Theorem 3.10) and

$$m(F(\eta_1), F(\eta_1)) = \underline{\mathcal{O}}(\varepsilon^{3/2}\|\eta_1\|^4)$$

(because of Propositions 2.9 and 4.6). Furthermore

$$\chi^+(D)m(\eta_1, F(\eta_1)) = g(2\omega)^{-1}c_0^2 A(\omega)\chi^+(D)m(\eta_1^-, (\eta_1^+)^2)$$
$$+ g(0)^{-1}c_0^2 B(\omega)\chi^+(D)m(\eta_1^+, \eta_1^+\eta_1^-) + \underline{\mathcal{O}}_+(\varepsilon^{3/2}\|\eta_1\|^3),$$

and it follows from (2.7) and Lemma 4.5 that

$$m(\eta_1^-, (\eta_1^+)^2)) = A(\omega)(\eta_1^+)^2\eta_1^- + \underline{\mathcal{O}}(\varepsilon^{3/2}\|\eta_1\|^3),$$
$$m(\eta_1^+, \eta_1^+\eta_1^-) = \tfrac{1}{2}B(\omega)(\eta_1^+)^2\eta_1^- + \underline{\mathcal{O}}(\varepsilon^{3/2}\|\eta_1\|^3).$$

$\square$

Proposition 4.8 *The estimates*

$$\chi^+(D)\mathcal{K}_3(\eta_1 + F(\eta_1) + \eta_3(\eta_1)) = -\tfrac{3}{2}\beta\omega^4\chi^+(D)\big((\eta_1^+)^2\eta_1^-\big) + \underline{\mathcal{O}}_+(\varepsilon^{3/2}\|\eta_1\|^3),$$
$$\chi^+(D)\mathcal{L}_3(\eta_1 + F(\eta_1) + \eta_3(\eta_1)) = C(\omega)\chi^+(D)\big((\eta_1^+)^2\eta_1^-\big) + \underline{\mathcal{O}}_+(\varepsilon^{3/2}\|\eta_1\|^3),$$

where

$$C(\omega) = -6\omega^2 f(\omega) + 4f(\omega)^2 + 2f(\omega)^2 f(2\omega),$$

hold for each $\eta_1 \in X_1$.

Proof Using the estimates for $F(\eta_1)$ and $\eta_3(\eta_1)$ given in Proposition 3.5 and Theorem 3.10, we find that

$$\mathcal{K}_3(\eta_1 + F(\eta_1) + \eta_3(\eta_1)) = \mathcal{K}_3(\eta_1) + \mathcal{O}(\varepsilon^{3/2}\|\eta_1\|^4)$$

and

$$\chi^+(D)\mathcal{K}_3(\eta_1) = -\tfrac{3}{2}\beta\omega^4\chi^+(D)\big((\eta_1^+)^2\eta_1^-\big) + \underline{\mathcal{O}}_+(\varepsilon^{3/2}\|\eta_1\|^3)$$

(because of equation (2.5)). It similarly follows from the formula

$$\mathcal{L}_3(\eta) = -K_0\eta K_1(\eta)\eta - \eta_x^2 K_0\eta + m_2(\eta, \eta)\eta$$

(see equation (2.6) and Remark 2.5) that

$$\mathcal{L}_3(\eta_1 + F(\eta_1) + \eta_3(\eta_1)) = \mathcal{L}_3(F(\eta_1) + \eta_1) + \mathcal{O}(\varepsilon^{3/2}\|\eta_1\|^4),$$

and using Corollary 2.8(ii) twice yields

$$\mathcal{L}_3(F(\eta_1) + \eta_1) = \mathcal{L}_3(\eta_1) + \underline{\mathcal{O}}(\varepsilon^{3/2}\|\eta_1\|^3)$$

and

$$\chi^+(D)\mathcal{L}_3(\eta_1) = C(\omega)\chi^+(D)\big((\eta_1^+)^2\eta_1^-\big) + \underline{\mathcal{O}}_+(\varepsilon^{3/2}\|\eta_1\|^3).$$

$\square$

Proposition 4.9 *The estimates*

$$\mathcal{K}_{\mathrm{r}}(\eta_1 + F(\eta_1) + \eta_3(\eta_1)) = \mathcal{O}(\varepsilon^2 \|\!|\eta_1|\!\|^5),$$
$$\mathcal{L}_{\mathrm{r}}(\eta_1 + F(\eta_1) + \eta_3(\eta_1)) = \mathcal{O}(\varepsilon^{3/2} \|\!|\eta_1|\!\|^4)$$

hold for each $\eta_1 \in X_1$.

Proof This result follows from Propositions 2.10(ii) and 3.5 and Theorem 3.10. □

Proposition 4.10 *The estimate*

$$-\tfrac{1}{2}\chi^+(D)\big(K_1(\eta_1 + F(\eta_1) + \eta_3(\eta_1))(\eta_1 + F(\eta_1) + \eta_3(\eta_1))\big)^2 = \underline{\mathcal{Q}}_+(\varepsilon^{3/2}\|\!|\eta_1|\!\|^4)$$

holds for each $\eta_1 \in X_1$.

Proof Using Proposition 3.5 and Theorem 3.10 we find that

$$-\tfrac{1}{2}\big(K_1(\eta_1 + F(\eta_1) + \eta_3(\eta_1))(\eta_1 + F(\eta_1) + \eta_3(\eta_1))\big)^2$$
$$= -\tfrac{1}{2}(K_1(\eta_1)\eta_1)^2 + \mathcal{O}(\varepsilon^{3/2}\|\!|\eta_1|\!\|^4),$$

and furthermore

$$-\tfrac{1}{2}\chi^+(D)(K_1(\eta_1)\eta_1)^2 = -\tfrac{1}{2}\chi^+(D)\big((\eta_1\eta_{1x})_x + K_0(\eta_1 K_0\eta_1)\big)^2 = 0$$

(see Lemma 2.6(i)). □

Corollary 4.11 *The estimate*

$$\chi^+(D)\mathcal{N}_2(\eta_1, \eta_3(\eta_1)) = -\tfrac{3}{2}\beta\omega^4 - C(\omega)c_0^2 + \underline{\mathcal{Q}}_+(\varepsilon^{3/2}\|\!|\eta_1|\!\|^3)$$

holds for each $\eta_1 \in X_1$.

We conclude that the reduced equation for η_1 is the *perturbed full dispersion nonlinear Schrödinger equation*

$$g(D)\eta_1^+ + c_0^2\varepsilon^2 K_0\eta_1^+ - D(\omega)\chi^+(|\eta_1^+|^2\eta_1^+) + \underline{\mathcal{Q}}_+(\varepsilon^{3/2}\|\!|\eta_1|\!\|^3) = 0,$$

where

$$D(\omega) = 2g(2\omega)^{-1}A(\omega)^2 c_0^4 + 2g(0)^{-1}B(\omega)^2 c_0^4 + C(\omega)c_0^2 + \tfrac{3}{2}\beta\omega^4,$$

and applying Lemma 4.5(iii), one can further simplify it to

$$g(D)\eta_1^+ + c_0^2 f(\omega)\varepsilon^2\eta_1^+$$
$$- D(\omega)\chi^+(|\eta_1^+|^2\eta_1^+) + \underline{\mathcal{Q}}_+(\varepsilon^{3/2}\|\!|\eta_1|\!\|^3) + \underline{\mathcal{Q}}_+(\varepsilon^3\|\!|\eta_1|\!\|) = 0.$$

Finally, we introduce the nonlinear Schrödinger scaling

$$\eta_1^+(x) = \tfrac{1}{2}\varepsilon\zeta(\varepsilon x)e^{i\omega x},$$

so that $\zeta \in B_R(0) \subseteq \chi_0(\varepsilon D)H^1(\mathbb{R})$, where $R > 0$ and ε is chosen small enough that $\varepsilon^{1/2}R \leq 2R_1$, solves the equation

$$\varepsilon^{-2}g(\omega + \varepsilon D)\zeta + c_0^2 f(\omega)\zeta - \tfrac{1}{4}D(\omega)\chi_0(\varepsilon D)(|\zeta|^2\zeta) + \varepsilon^{1/2}\underline{\mathcal{O}}_0^\varepsilon(\|\zeta\|_1) = 0 \quad (4.3)$$

(note that $\|\!|\eta_1|\!\| = \varepsilon^{1/2}\|\zeta\|_1$, the change of variable from x to $X = \varepsilon x$ introduces an additional factor of $\varepsilon^{1/2}$ in the remainder term and the symbol D now means $-i\partial_X$). The invariance of the reduced equation under $\eta_1(x) \mapsto \eta_1(-x)$ is inherited by (4.3), which is invariant under the reflection $\zeta(X) \mapsto \overline{\zeta(-X)}$.

5 Solution of the Reduced Equation

In this section, we find solitary-wave solutions of the reduced equations

$$\varepsilon^{-2}g(\varepsilon D)\rho + \rho + \tfrac{3}{2}\chi_0(\varepsilon D)\rho^2 + \varepsilon^{1/2}\underline{\mathcal{O}}_0^\varepsilon(\|\rho\|_1) = 0 \qquad (5.1)$$

and

$$\varepsilon^{-2}g(\omega + \varepsilon D)\zeta + c_0^2 f(\omega)\zeta - a_3\chi_0(\varepsilon D)(|\zeta|^2\zeta) + \varepsilon^{1/2}\underline{\mathcal{O}}_0^\varepsilon(\|\zeta\|_1) = 0, \quad (5.2)$$

noting that in the formal limit $\varepsilon \to 0$ they reduce to respectively the stationary Korteweg-de Vries equation

$$-(\beta - \tfrac{1}{3})\rho_{XX} + \rho + \tfrac{3}{2}\rho^2 = 0 \qquad (5.3)$$

and the stationary nonlinear Schrödinger equation

$$-a_1\zeta_{XX} + a_2\zeta - a_3|\zeta|^2\zeta = 0, \qquad (5.4)$$

which have explicit (symmetric) solitary-wave solutions $\rho^\star$ and $\pm\zeta^\star$ (Eqs. (1.7) and (1.10)). For this purpose we use a perturbation argument, rewriting (5.1) and (5.2) as fixed-point equations and applying the following version of the implicit-function theorem.

Theorem 5.1 *Let $\mathcal{X}$ be a Banach space, X_0 and Λ_0 be open neighbourhoods of respectively $x^\star$ in $\mathcal{X}$ and the origin in $\mathbb{R}$ and $\mathcal{G} : X_0 \times \Lambda_0 \to \mathcal{X}$ be a function which is differentiable with respect to $x \in X_0$ for each $\lambda \in \Lambda_0$. Furthermore, suppose that $\mathcal{G}(x^\star, 0) = 0$, $d_1\mathcal{G}[x^\star, 0] : \mathcal{X} \to \mathcal{X}$ is an isomorphism,*

$$\lim_{x \to x^\star} \|d_1\mathcal{G}[x, 0] - d_1\mathcal{G}[x^\star, 0]\|_{\mathcal{L}(\mathcal{X})} = 0$$

and

$$\lim_{\lambda \to 0} \|\mathcal{G}(x, \lambda) - \mathcal{G}(x, 0)\|_{\mathcal{X}} = 0, \quad \lim_{\lambda \to 0} \|d_1\mathcal{G}[x, \lambda] - d_1\mathcal{G}[x, 0]\|_{\mathcal{L}(\mathcal{X})} = 0$$

uniformly over $x \in X_0$.

There exist open neighbourhoods X of $x^\star$ in $\mathcal{X}$ and Λ of 0 in $\mathbb{R}$ (with $X \subseteq X_0$, $\Lambda \subseteq \Lambda_0$) and a uniquely determined mapping $h : \Lambda \to X$ with the properties that

(i) *h is continuous at the origin (with $h(0) = x^\star$),*
(ii) *$\mathcal{G}(h(\lambda), \lambda) = 0$ for all $\lambda \in \Lambda$,*
(iii) *$x = h(\lambda)$ whenever $(x, \lambda) \in X \times \Lambda$ satisfies $\mathcal{G}(x, \lambda) = 0$.*

5.1 Strong Surface Tension

Theorem 5.2 *For each sufficiently small value of $\varepsilon > 0$ equation (5.1) has a small-amplitude, symmetric solution ρ_ε in $\chi_0(\varepsilon D)H^1(\mathbb{R})$ with $\|\rho_\varepsilon - \rho^\star\|_1 \to 0$ as $\varepsilon \to 0$.*

The first step in the proof of Theorem 5.2 is to write (5.1) as the fixed-point equation

$$\rho + \varepsilon^2\big(\varepsilon^2 + g(\varepsilon D)\big)^{-1}\left(\tfrac{3}{2}\chi_0(\varepsilon D)\rho^2 + \varepsilon^{1/2}\underline{\mathcal{O}}_0^\varepsilon(\|\rho\|_1)\right) = 0 \tag{5.5}$$

and use the following result to 'replace' the nonlocal operator with a differential operator.

Proposition 5.3 *The inequality*

$$\left|\frac{\varepsilon^2}{\varepsilon^2 + g(\varepsilon k)} - \frac{1}{1 + (\beta - \tfrac{1}{3})k^2}\right| \lesssim \frac{\varepsilon}{(1 + k^2)^{1/2}}$$

holds uniformly over $|k| < \delta/\varepsilon$.

Proof Clearly

$$\left|\frac{\varepsilon^2}{\varepsilon^2 + g(\varepsilon k)} - \frac{1}{1 + (\beta - \tfrac{1}{3})k^2}\right| = \frac{g(\varepsilon k) - (\beta - \tfrac{1}{3})k^2\varepsilon^2}{(\varepsilon^2 + g(\varepsilon k))(1 + (\beta - \tfrac{1}{3})k^2)};$$

furthermore

$$g(s) - (\beta - \tfrac{1}{3})s^2 \lesssim |s|^3, \qquad |s| \le \delta,$$

and

$$g(s) \gtrsim s^2, \qquad s \in \mathbb{R}.$$

It follows that

$$\left| \frac{\varepsilon^2}{\varepsilon^2 + g(\varepsilon k)} - \frac{1}{1 + (\beta - \frac{1}{3})k^2} \right| \lesssim \frac{\varepsilon |k|^3}{(1 + |k|^2)^2}, \qquad |k| < \delta/\varepsilon.$$

$\square$

Using the above proposition, one can write equation (5.5) as

$$\rho + F_\varepsilon(\rho) = 0,$$

where

$$F_\varepsilon(\rho) = \frac{3}{2} \left(1 - (\beta - \tfrac{1}{3})\partial_X^2 \right)^{-1} \chi_0(\varepsilon D)\rho^2 + \varepsilon^{1/2}\underline{\mathcal{O}}_1^\varepsilon(\|\rho\|_1).$$

It is convenient to replace this equation with

$$\rho + \tilde{F}_\varepsilon(\rho) = 0,$$

where $\tilde{F}_\varepsilon(\rho) = F_\varepsilon(\chi_0(\varepsilon D)\rho)$ and study it in the fixed space $H^1(\mathbb{R})$ (the solution sets of the two equations evidently coincide).

We establish Theorem 5.6 by applying Theorem 5.1 with

$$\mathcal{X} = H_e^1(\mathbb{R}) := \{u \in H^1(\mathbb{R}) : u(X) = u(-X) \text{ for all } X \in \mathbb{R}\},$$

$X = B_R(0)$, $\Lambda_0 = (-\varepsilon_0, \varepsilon_0)$ for a sufficiently small value of ε_0, and

$$\mathcal{G}(\rho, \varepsilon) := \rho + \tilde{F}_{|\varepsilon|}(\rho)$$

(here ε is replaced by $|\varepsilon|$ so that $\mathcal{G}(\rho, \varepsilon)$ is defined for ε in a full neighbourhood of the origin in $\mathbb{R}$). Observe that

$$\mathcal{G}(\rho, \varepsilon) - \mathcal{G}(\rho, 0)$$
$$= \frac{3}{2} \left(1 - (\beta - \tfrac{1}{3})\partial_X^2 \right)^{-1} [\chi_0(|\varepsilon|D)(\chi_0(|\varepsilon|D)\rho)^2 - \rho^2] + |\varepsilon|^{1/2}\underline{\mathcal{O}}_1^{|\varepsilon|}(\|\rho\|_1),$$

and noting that

$$\lim_{\varepsilon \to 0} \|\chi_0(|\varepsilon|D) - I\|_{\mathcal{L}(H^1(\mathbb{R}), H^{3/4}(\mathbb{R}))} = 0$$

because

$$\|\chi_0(|\varepsilon|D)u - u\|_{3/4}^2 = \int_{|k| > \frac{\delta}{|\varepsilon|}} (1 + |k|^2)^{3/4} |\hat{u}|^2 \, dk$$

$$\leq \sup_{|k| > \frac{\delta}{|\varepsilon|}} (1 + |k|^2)^{-1/4} \int_{|k| > \frac{\delta}{|\varepsilon|}} (1 + |k|^2)|\hat{u}|^2 dk$$

$$\leq \left(1 + \frac{\delta^2}{|\varepsilon|^2}\right)^{-1/4} \|u\|_1^2,$$

that

$$\chi_0(|\varepsilon|D)(\chi_0(|\varepsilon|D)\rho)^2 - \rho^2$$
$$= \chi_0(|\varepsilon|D)(\chi_0(|\varepsilon|D)\rho + \rho)(\chi_0(|\varepsilon|D) - I)\rho + (\chi_0(|\varepsilon|D) - I)\rho^2$$

and that $H^{3/4}(\mathbb{R})$ is a Banach algebra, we find that

$$\lim_{\varepsilon \to 0} \|\mathcal{G}(\rho, \varepsilon) - \mathcal{G}(\rho, 0)\|_1 = 0, \quad \lim_{\varepsilon \to 0} \|d_1\mathcal{G}[\rho, \varepsilon] - d_1\mathcal{G}[\rho, 0]\|_{\mathcal{L}(H^1(\mathbb{R}))} = 0$$

uniformly over $\rho \in B_R(0)$. The equation

$$\mathcal{G}(\rho, 0) = \rho + \tfrac{3}{2}\big(1 - (\beta - \tfrac{1}{3})\partial_X^2\big)^{-1}\rho^2 = 0$$

has the (unique) nontrivial solution $\rho^\star$ in $H_e^1(\mathbb{R})$ and it remains to show that

$$d_1\mathcal{G}[\rho^\star, 0] = I + 3\big(1 - (\beta - \tfrac{1}{3})\partial_X^2\big)^{-1}(\rho^\star\cdot)$$

is an isomorphism.

Noting that $\rho^\star \in \mathcal{S}(\mathbb{R})$, we obtain the following result by a familiar argument (see Kirchgässner [17, Proposition 5.1] or Friesecke & Pego [10, §4]).

Proposition 5.4 *The formula* $\rho \mapsto 3\big(1 - (\beta - \tfrac{1}{3})\partial_X^2\big)^{-1}(\rho^\star\rho)$ *defines a compact linear operator* $H^1(\mathbb{R}) \to H^1(\mathbb{R})$ *and* $H_e^1(\mathbb{R}) \to H_e^1(\mathbb{R})$.

This proposition implies in particular that $d_1\mathcal{G}[\rho^\star, 0]$ is a Fredholm operator with index 0. Its kernel coincides with the set of symmetric bounded solutions of the ordinary differential equation

$$-(\beta - \tfrac{1}{3})\rho_{XX} + \rho + 3\rho^\star\rho = 0, \tag{5.6}$$

and the next proposition shows that this set consists of only the trivial solution, so that $d_1\mathcal{G}[\rho^\star, 0]$ is an isomorphism.

Proposition 5.5 *Every bounded solution to the equation* (5.6) *is a multiple of* $\rho_X^\star$ *and is therefore antisymmetric.*

Proof Define

$$\rho_1(\check{X}) = \operatorname{sech}^2(\tfrac{1}{2}\check{X}) \tanh(\tfrac{1}{2}\check{X}),$$
$$\rho_2(\check{X}) = \cosh \check{X} + \tfrac{3}{2} \operatorname{sech}^2(\tfrac{1}{2}\check{X}) \left(-8 + 2\cosh \check{X} + 5\check{X} \tanh(\tfrac{1}{2}\check{X})\right),$$

where $\check{X} = (\beta - \tfrac{1}{3})^{-1/2} X$, and observe that $\{\rho_1, \rho_2\}$ is a fundamental solution set for (5.6). Its bounded solutions are therefore precisely the multiples of ρ_1. $\qquad\square$

5.2 Weak Surface Tension

Theorem 5.6 *For each sufficiently small value of $\varepsilon > 0$ equation (5.2) has two small-amplitude, symmetric solutions $\zeta_\varepsilon^\pm$ in $\chi_0(\varepsilon D) H^1(\mathbb{R})$ with $\|\zeta_\varepsilon^\pm \mp \zeta^\star\|_1 \to 0$ as $\varepsilon \to 0$.*

We again begin the proof of Theorem 5.6 by 'replacing' the nonlocal operator in the fixed-point formulation

$$\zeta + \varepsilon^2 \big(\varepsilon^2 c_0^2 f(\omega) + g(\omega + \varepsilon D)\big)^{-1} \left(-a_3 \chi_0(\varepsilon D)(|\zeta|^2 \zeta) + \varepsilon^{1/2} \underline{\mathcal{O}}_0^\varepsilon(\|\zeta\|_1)\right) = 0$$

of equation (5.2) with a differential operator.

Proposition 5.7 *The inequality*

$$\left| \frac{\varepsilon^2}{c_0^2 f(\omega)\varepsilon^2 + g(\omega + \varepsilon k)} - \frac{1}{a_1 + a_2 k^2} \right| \lesssim \frac{\varepsilon}{(1 + k^2)^{1/2}}$$

holds uniformly over $|k| < \delta/\varepsilon$.

Proof Clearly

$$\left| \frac{\varepsilon^2}{c_0^2 f(\omega)\varepsilon^2 + g(\omega + \varepsilon k)} - \frac{1}{a_1 + a_2 k^2} \right|$$
$$= \frac{|g(\omega + \varepsilon k) - \varepsilon^2(c_0^2 f(\omega) + \tfrac{1}{2} g''(\omega)k^2)|}{(c_0^2 f(\omega)\varepsilon^2 + g(\omega + \varepsilon k))(c_0^2 f(\omega) + \tfrac{1}{2} g''(\omega)k^2)},$$

while

$$g(\omega + s) - \tfrac{1}{2} g''(\omega)s^2 \lesssim |s|^3, \qquad |s| \le \delta,$$

and

$$g(\omega + s) \gtrsim s^2, \qquad s \in \mathbb{R}.$$

It follows that

$$\left| \frac{\varepsilon^2}{c_0^2 f(\omega)\varepsilon^2 + g(\omega + \varepsilon k)} - \frac{1}{a_1 + a_2 k^2} \right| \lesssim \frac{\varepsilon |k|^3}{(1 + |k|^2)^2}, \qquad |k| < \delta/\varepsilon.$$

□

Using the above proposition, one can write equation (5.2) as

$$\zeta + F_\varepsilon(\zeta) = 0,$$

where

$$F_\varepsilon(\zeta) = -a_3 \left(a_1 - a_2 \partial_X^2 \right)^{-1} |\zeta|^2 \zeta + \varepsilon^{1/2} \underline{\mathcal{O}}_1^\varepsilon(\|\zeta\|_1),$$

or equivalently with

$$\zeta + \tilde{F}_\varepsilon(\zeta) = 0,$$

where $\tilde{F}_\varepsilon(\zeta) = F_\varepsilon(\chi_0(\varepsilon D)\zeta)$, and studying it in the fixed space $H^1(\mathbb{R}, \mathbb{C})$. We establish Theorem 5.2 by applying Theorem 5.1 with

$$\mathcal{X} = H_e^1(\mathbb{R}, \mathbb{C}) = \{\zeta \in H^1(\mathbb{R}) : \zeta(X) = \overline{\zeta(-X)} \text{ for all } X \in \mathbb{R}\},$$

$X = B_R(0)$, $\Lambda_0 = (-\varepsilon_0, \varepsilon_0)$ for a sufficiently small value of ε_0 and

$$\mathcal{G}(\zeta, \varepsilon) := \zeta + \tilde{F}_{|\varepsilon|}(\zeta).$$

Observe that

$$\begin{aligned}
\mathcal{G}(\zeta, \varepsilon) &- \mathcal{G}(\zeta, 0) \\
&= -a_3 \left(a_1 - a_2 \partial_X^2 \right)^{-1} \left[\chi_0(|\varepsilon|D)\left(|\chi_0(|\varepsilon|D)\zeta|^2 \chi_0(|\varepsilon|D)\zeta \right) - |\zeta|^2 \zeta| \right) \right] \\
&\quad + |\varepsilon|^{\frac{1}{2}} \underline{\mathcal{O}}_1^{|\varepsilon|}(\|\zeta\|_1),
\end{aligned}$$

so that

$$\lim_{\varepsilon \to 0} \|\mathcal{G}(\zeta, \varepsilon) - \mathcal{G}(\zeta, 0)\|_1 = 0, \quad \lim_{\varepsilon \to 0} \|d_1 \mathcal{G}[\zeta, \varepsilon] - d_1 \mathcal{G}[\zeta, 0]\|_{\mathcal{L}(H^1(\mathbb{R}, \mathbb{C}))} = 0$$

uniformly over $\zeta \in B_R(0)$.

The equation

$$\mathcal{G}(\zeta, 0) = \zeta - a_3 \left(a_1 - a_2 \partial_X^2 \right)^{-1} |\zeta|^2 \zeta = 0$$

has (precisely two) nontrivial solutions $\pm\zeta^\star$ in $H^1_e(\mathbb{R}, \mathbb{C})$, which are both real, and the fact that $d_1\mathcal{G}[\pm\zeta^\star, 0]$ is an isomorphism is conveniently established by using real coordinates. Define $\zeta_1 = \text{Re}\,\zeta$ and $\zeta_2 = \text{Im}\,\zeta$, so that

$$d_1\mathcal{G}[\pm\zeta^\star, 0](\zeta_1 + i\zeta_2) = \mathcal{G}_1(\zeta_1) + i\mathcal{G}_2(\zeta_2),$$

where $\mathcal{G}_1 : H^1_e(\mathbb{R}) \to H^1_e(\mathbb{R})$ and $\mathcal{G}_2 : H^1_o(\mathbb{R}) \to H^1_o(\mathbb{R})$ are given by

$$\mathcal{G}_1(\zeta_1) = \zeta_1 - 3a_3\left(a_1 - a_2\partial^2_X\right)^{-1}\zeta^{\star 2}\zeta_1,$$

$$\mathcal{G}_2(\zeta_2) = \zeta_2 - a_3\left(a_1 - a_2\partial^2_X\right)^{-1}\zeta^{\star 2}\zeta_2$$

and

$$H^1_e(\mathbb{R}) := \{u \in H^1(\mathbb{R}) : u(X) = u(-X) \text{ for all } X \in \mathbb{R}\},$$
$$H^1_o(\mathbb{R}) := \{u \in H^1(\mathbb{R}) : u(X) = -u(-X) \text{ for all } X \in \mathbb{R}\}.$$

Proposition 5.8 *The formulae*

$$\zeta_1 \mapsto -3a_3\left(a_1 - a_2\partial^2_X\right)^{-1}\zeta^{\star 2}\zeta_1, \qquad \zeta_2 \mapsto -a_3\left(a_1 - a_2\partial^2_X\right)^{-1}\zeta^{\star 2}\zeta_2$$

define compact linear operators $H^1(\mathbb{R}) \to H^1(\mathbb{R})$, $H^1_e(\mathbb{R}) \to H^1_e(\mathbb{R})$ *and* $H^1_o(\mathbb{R}) \to H^1_o(\mathbb{R})$.

The previous proposition implies in particular that $\mathcal{G}_1$, $\mathcal{G}_2$ are Fredholm operators with index 0. The kernel of $\mathcal{G}_1$ coincides with the set of symmetric bounded solutions of the ordinary differential equation

$$-a_1\zeta_{1XX} + a_2\zeta_1 - 3a_3\zeta^{\star 2}\zeta_1 = 0, \tag{5.7}$$

while the kernel of $\mathcal{G}_2$ coincides with the set of antisymmetric bounded solutions of the ordinary differential equation

$$-a_1\zeta_{1XX} + a_2\zeta_1 - a_3\zeta^{\star 2}\zeta_1 = 0, \tag{5.8}$$

and the next proposition shows that these sets consists of only the trivial solution, so that $\mathcal{G}_1$, $\mathcal{G}_2$ and hence $d_1\mathcal{G}[\zeta^\star, 0]$ are isomorphisms.

Proposition 5.9

(i) *Every bounded solution to the equation* (5.7) *is a multiple of* $\zeta^\star_X$ *and is therefore antisymmetric.*

(ii) *Every bounded solution to the equation* (5.8) *is a multiple of* $\zeta^\star$ *and is therefore symmetric.*

Proof Introducing the scaled variables

$$\check{\zeta}_1(\check{X}) = (a_3/a_2)^{1/2}\zeta_1(X), \qquad \check{\zeta}_2(\check{X}) = (a_3/a_2)^{1/2}\zeta_2(X), \qquad \check{X} = (a_2/a_1)^{1/2}X$$

transforms Eqs. (5.7), (5.8) into

$$-\check{\zeta}_{1\check{X}\check{X}} + \check{\zeta}_1 - 3\check{\zeta}^{\star 2}\check{\zeta}_1 = 0, \tag{5.9}$$

$$-\check{\zeta}_{2\check{X}\check{X}} + \check{\zeta}_2 - \check{\zeta}^{\star 2}\check{\zeta}_2 = 0, \tag{5.10}$$

where $\check{\zeta}^{\star}(\check{X}) = \sqrt{2}\,\text{sech}\,\check{X}$, and it obviously suffices to establish the corresponding results for these equations.

Define

$$\check{\zeta}_{1,1}(\check{X}) = \text{sech}\,\check{X}\,\tanh\check{X},$$

$$\check{\zeta}_{1,2}(\check{X}) = \text{sech}\,\check{X}\left(-3 + \cosh^2\check{X} + 3\check{X}\tanh\check{X}\right),$$

$$\check{\zeta}_{2,1}(\check{X}) = \text{sech}\,\check{X},$$

$$\check{\zeta}_{2,2}(\check{X}) = \text{sech}\,\check{X}\left(2\check{X} + \sinh 2\check{X}\right)$$

and observe that $\{\zeta_{1,1}, \zeta_{1,2}\}$ is a fundamental solution set for (5.9), whose bounded solutions are therefore precisely the multiplies of $\check{\zeta}^{\star}_{\check{X}}$, while $\{\zeta_{2,1}, \zeta_{2,2}\}$ is a fundamental solution set for (5.10), whose bounded solutions are therefore precisely the multiples of $\check{\zeta}^{\star}$. □

References

1. Ablowitz, M.J., Segur, H.: On the evolution of packets of water waves. J. Fluid Mech. **92**, 691–715 (1979)
2. Amick, C.J., Kirchgässner, K.: A theory of solitary water waves in the presence of surface tension. Arch. Rat. Mech. Anal. **105**, 1–49 (1989)
3. Benjamin, T.B.: The solitary wave with surface tension. Q. Appl. Math. **40**, 231–234 (1982)
4. Buffoni, B.: Existence and conditional energetic stability of capillary-gravity solitary water waves by minimisation. Arch. Rat. Mech. Anal. **173**, 25–68 (2004)
5. Buffoni, B.: Conditional energetic stability of gravity solitary waves in the presence of weak surface tension. Topol. Meth. Nonlinear Anal. **25**, 41–68 (2005)

6. Buffoni, B., Toland, J.F.: Analytic theory of global bifurcation. Princeton University Press, Princeton (2003)
7. Craig, W., Sulem, C.: Numerical simulation of gravity waves. J. Comp. Phys. **108**, 73–83 (1993)
8. Dias, F., Kharif, C.: Nonlinear gravity and capillary-gravity waves. Ann. Rev. Fluid Mech. **31**, 301–346 (1999)
9. Ehrnström, M., Groves, M.D., Wahlén, E.: On the existence and stability of solitary-wave solutions to a class of evolution equations of Whitham type. Nonlinearity **25**, 2903–2936 (2012)
10. Friesecke, G., Pego, R.L.: Solitary waves on FPU lattices: I. Qualitative properties, renormalization and continuum limit. Nonlinearity **12**, 1601–1627 (1999)
11. Groves, M.D., Wahlén, E.: On the existence and conditional energetic stability of solitary water waves with weak surface tension. C. R. Math. Acad. Sci. Paris **348**, 397–402 (2010)
12. Groves, M.D., Wahlén, E.: Existence and conditional energetic stability of solitary gravity-capillary water waves with constant vorticity. Proc. R. Soc. Edin. A **145**, 791–883 (2015)
13. Hörmander, L.: Lectures on nonlinear hyperbolic differential equations. Springer, Heidelberg (1997)
14. Hu, B., Nicholls, D.P.: Analyticity of Dirichlet–Neumann operators on Hölder and Lipschitz domains. SIAM J. Math. Anal. **37**, 302–320 (2006)
15. Iooss, G., Kirchgässner, K.: Bifurcation d'ondes solitaires en présence d'une faible tension superficielle. C. R. Acad. Sci. Paris, Sér. 1 **311**, 265–268 (1990)
16. Iooss, G., Pérouème, M.C.: Perturbed homoclinic solutions in reversible 1:1 resonance vector fields. J. Diff. Equ. **102**, 62–88 (1993)
17. Kirchgässner, K.: Nonlinearly resonant surface waves and homoclinic bifurcation. Adv. Appl. Mech. **26**, 135–181 (1988)
18. Lannes, D.: The water waves problem: mathematical analysis and asymptotics. Number 188 in Mathematical surveys and monographs. American Mathematical Society, Providence (2013)
19. Nicholls, D.P., Reitich, F.: A new approach to analyticity of Dirichlet–Neumann operators. Proc. R. Soc. Edin. A **131**, 1411–1433 (2001)
20. Sachs, R.L.: On the existence of small amplitude solitary waves with strong surface-tension. J. Diff. Equ. **90**, 31–51 (1991)
21. Stefanov, A., Wright, J.D.: Small amplitude traveling waves in the full-dispersion Whitham equation. J. Dyn. Diff. Equ. **32**, 85–99 (2020)
22. Zakharov, V.E.: Stability of periodic waves of finite amplitude on the surface of a deep fluid. Zh. Prikl. Mekh. Tekh. Fiz. **9**, 86–94 (1968). (English translation J. Appl. Mech. Tech. Phys. **9**, 190–194.)