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ORIGINAL PAPER

On the Borisov–Nuer conjecture and the image of the Enriques-to-$K3$ map

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Abstract

We discuss the Borisov–Nuer conjecture in connection with the canonical maps from the moduli spaces $\mathcal{M}^a_{En,h}$ of polarized Enriques surfaces with fixed $h \in U \oplus E_8(-1)$ to the moduli space $\mathcal{F}_g$ of polarized $K3$ surfaces of genus g with $g = h^2 + 1$, and we exhibit a naturally defined locus $\Sigma_g \subset \mathcal{F}_g$. One direct consequence of the Borisov–Nuer conjecture is that Σ_g would be contained in a particular Noether–Lefschetz divisor in $\mathcal{F}_g$, which we call the Borisov–Nuer divisor and we denote by $\mathcal{BN}_g$. In this short note, we prove that $\Sigma_g \cap \mathcal{BN}_g$ is non-empty whenever $(g-1)$ is divisible by 4. To this end, we construct polarized Enriques surfaces (Y, H_Y), with H_Y^2 divisible by 4, which verify the conjecture. In particular, when we consider the moduli space of (numerically) polarized Enriques surfaces which contains such (Y, H_Y), the conjecture also holds for any other polarized Enriques surface (Y', H'_Y) contained in the same moduli.

KEYWORDS

Borisov–Nuer conjecture, Enriques surface, Jacobian Kummer surface, numerically polarized Enriques surface

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1 | INTRODUCTION

Let Y be an Enriques surface over $\mathbb{C}$, that is, a smooth projective surface with $p_g(Y) = q(Y) = 0$ and $2K_Y = \mathcal{O}_Y$. The universal covering of Y is given by an étale double cover map $\sigma_Y : X_Y \to Y$ where X is a $K3$ surface. Hence, an Enriques surface Y determines a pair (X_Y, θ_Y), where X_Y is its $K3$ cover, and θ_Y is a fixed-point-free involution on X_Y so that σ_Y coincides with the quotient map $X_Y \to X_Y/\theta_Y$. In particular, studying Enriques surfaces Y is equivalent to studying pairs (X, θ) of $K3$ surfaces X and fixed-point-free involutions θ on X.

A *polarized Enriques surface* is a pair (Y, H_Y), where Y is an Enriques surface and $H_Y \in \mathrm{Pic}(Y)$ is an ample line bundle. A *numerically polarized Enriques surface* is a pair $(Y, [H_Y])$, where $[H_Y] \in \mathrm{Num}(Y)$ denotes the numerical class of an ample line bundle H_Y on Y. Note that two line bundles $L \ncong L'$ on an Enriques surface Y have the same numerical class if and only if $L \cong L' \otimes \omega_Y$.

Recall that $\mathrm{Num}(Y) \cong U \oplus E_8(-1)$ for arbitrary Enriques surface Y, realized by an isometry (usually called *marking*) $\varphi : \mathrm{Num}(Y) = H^2(Y, \mathbb{Z})_f \to U \oplus E_8(-1)$. Let $h \in U \oplus E_8(-1)$ be a primitive element of positive degree $h^2 > 0$. Thanks to lattice theory, Gritsenko and Hulek were able to give a construction of the moduli space $\mathcal{M}^a_{En,h}$ of numerically polarized Enriques surfaces $(Y, [H_Y])$, where $h = \varphi([H_Y])$ for some marking φ on Y, as an open subvariety of a modular variety $\mathcal{M}_{En,h}$. Indeed, its points are in $1:1$ correspondence with the isomorphism classes of numerically polarized Enriques surfaces [9, Theorem 3.2]. Since this correspondence does not depend on the choice of a marking φ sending H_Y to h, we may choose and fix a marking φ for each Y, and simply write $h = [H_Y]$ instead of $\varphi([H_Y])$. We refer to [9] for the construction and more details.

The moduli space $\mathcal{M}^a_{En,h}$ is a 10-dimensional quasi-projective variety, and the locus $\mathcal{M}^{nn}_{En,h}$ corresponding to unnodal surfaces (i.e., with no smooth (−2)-curves) is open. For an alternate approach to moduli spaces $\widehat{\mathcal{E}}_{g,\phi}$ using the invariant ϕ, parametrizing pairs of smooth Enriques surfaces Y and ample numerical classes $[H] \in \mathrm{Num}(Y)$ with $H^2 = 2g - 2$ and $\phi(H) := \min\{H.E \mid E^2 = 0, E > 0\} = \phi$, we refer to [7].

Let us consider $\mathcal{F}_g$ the moduli space of polarized $K3$ surfaces of genus $g = h^2 + 1$. Note that g is odd and $g \geq 5$. For any ample numerical class h, we have a natural map

$$\begin{array}{rcl} \eta_h : \mathcal{M}^a_{En,h} & \to & \mathcal{F}_g \\ (Y, h = [H_Y]) & \mapsto & (X_Y, \sigma_Y^* H_Y). \end{array}$$

Then the locus

$$\Sigma_g := \bigcup_{h^2 = g-1} im(\eta_h) \subseteq \mathcal{F}_g \tag{1.1}$$

consists of polarized $K3$ surfaces (X, H_X) which appear as pullbacks of polarized Enriques surfaces (Y, H_Y). Notice that, for any fixed degree $g - 1$, there are only finitely many numerical classes $h \in \mathrm{Num}(Y)$ with $h^2 = g - 1$. Indeed, from [7, Proposition 4.16] it follows that the number of irreducible components of the moduli space $\widehat{\mathcal{E}}_{g',\phi}$ coincides with the number of possible simple decomposition types for h for fixed values $h^2 = 2g' - 2$ and $\phi(h) = \phi$. Since $0 < \phi^2 \leq h^2$ by [8, Corollary 2.7.1], there are only finitely many possible choices of ϕ, which implies the claim. Alternatively, we can argue as follows. (This was brought to our attention by A. Knutsen.) The space $\mathcal{M}^a_{En,h}$ is a quotient of similarly defined spaces of polarized Enriques surfaces, which exist by the theory of Viehweg as quasi-projective varieties, and therefore have finitely many components.

In this note, we discuss a conjecture of Borisov and Nuer on the Enriques lattice $\mathrm{Num}(Y) \cong U \oplus E_8(-1)$, motivated by the Ulrich bundle existence problem, and connect it to the maps η_h. Let us briefly recall what are Ulrich bundles. Let $X \subset \mathbb{P}^N$ be a smooth projective variety of dimension n, and let $H = \mathcal{O}_X(1)$ be a very ample line bundle on X. A vector bundle $\mathcal{E}$ on X which satisfies the following cohomology vanishing condition

$$H^i(X, \mathcal{E}(-j)) = 0 \text{ for all } i \text{ and } 1 \leq j \leq n \tag{1.2}$$

is called an Ulrich bundle on X (see [6]). They have many interesting applications, in particular, they connect several different topics in algebra and geometry, see [3, 6]. One important problem within this topic is to find an Ulrich bundle of smallest possible rank on a given variety. For an Enriques surface Y, together with a very ample line bundle $H_Y = \mathcal{O}_Y(1)$, it is known that Y always carries an Ulrich bundle of rank 2 (see [2, 4]). On the other hand, Borisov and Nuer observed that the existence of an Ulrich line bundle N on a polarized unnodal Enriques surface (Y, H_Y) is equivalent to the numerical condition

$$(N - H_Y)^2 = (N - 2H_Y)^2 = -2, \tag{1.3}$$

that is, H_Y can be written as a difference of two (−2)-line bundles. Here, the unnodal assumption is required only to assure the vanishing of certain cohomology groups. Thus, it is natural to focus only on Equation (1.3). They conjectured that it is always possible to find such a line bundle N for any choice of polarization H_Y, or even more, for any line bundle:

Conjecture 1.1 ([5, Conjecture 2.2]). *For any line bundle H on an Enriques surface Y, there is a line bundle $N \in \mathrm{Pic}(Y)$ such that $(N - H)^2 = (N - 2H)^2 = -2$.*

Suppose that (Y, H_Y) verifies the Borisov–Nuer conjecture; we have a line bundle N on Y which satisfies the above Equation (1.3). We translate the conjecture in terms of line bundles on its $K3$ covers by observing the image under η_h defined above. Let $\sigma : X \to Y = X/\theta$ be the universal cover, $H_X := \sigma^* H_Y$, and let $M := \sigma^* N$. Equation (1.3) is equivalent to

$$\begin{cases} H_X^2 = 2g-2, \\ M^2 = 4g-8, \\ H_X \cdot M = 3g-3, \end{cases}$$

where $g = H_Y^2 + 1 \geq 5$ is an odd integer. It is natural to consider polarized K3 surfaces (X, H_X) equipped with such a line bundle M:

Definition 1.2. We define the *Borisov–Nuer* divisor $\mathcal{BN}_g$ as the Noether–Lefschetz divisor $\mathcal{NL}_{2g-3,3g-3} \subset \mathcal{F}_g$ for an odd integer $g \geq 5$ (here, the subscript stands for the numbers $\left(\frac{1}{2}M^2 + 1\right)$ and $H_X \cdot M$, respectively), i.e.,

$$\mathcal{BN}_g := \{(X, H_X) \in \mathcal{F}_g \mid \text{there exists } M, M^2 = 4g-8, H_X \cdot M = 3g-3\}.$$

Hence, if (Y, H_Y) and N satisfy Equation (1.3), then the image (X, H_X) must lie in the Borisov–Nuer divisor $\mathcal{BN}_g$.

Note that a line bundle M on the $K3$ cover X is contained in $\sigma^* \operatorname{Pic}(Y)$ if and only if $\theta^* M \cong M$ (see [11]). In the case, the pushforward $\sigma_* M$ splits as a direct sum of two line bundles $N \oplus (N \otimes K_Y)$, where $M \cong \sigma^* N$. We consider the sublocus

$$\Xi_g = \left\{ \exists\, \theta : X \to X \;\middle|\; \begin{array}{l} \text{fixed-point-free involution such that } \theta^* H_X \cong H_X, \\ (X, H_X) \in \mathcal{F}_g \; \exists\, M \in \operatorname{Pic}(X) \text{ such that } \theta^* M \cong M, \\ \text{and } (X, H_X, M) \in \mathcal{BN}_g. \end{array} \right\}$$

consisting of polarized $K3$ surfaces of genus g which can be obtained by pullback of some polarized Enriques surface (Y, H_Y) together with a line bundle N so that the triple (Y, H_Y, N) verifies the Conjecture 1.1. In particular, we immediately have $\Xi_g \subseteq \Sigma_g \subset \mathcal{F}_g$, cf. (1.1). Since the Picard number $\rho(Y)$ of an Enriques surface Y is 10, both loci Ξ_g and Σ_g have high codimensions in $\mathcal{F}_g$. With this notation, Conjecture 1.1 implies:

Conjecture 1.3. *The two loci Ξ_g and Σ_g coincide.*

Since the locus Ξ_g is contained in the Borisov–Nuer divisor $\mathcal{BN}_g$ by definition, this conjecture admits the following much weaker version:

Question. Is Σ_g contained in $\mathcal{BN}_g$?

At the moment, the Borisov–Nuer conjecture is known for only a few examples: Fano polarization Δ and its multiple $k\Delta$ by Borisov and Nuer themselves [5, Theorem 2.4], and a degree 4 polarization [1, Theorem 13]. In particular, Ξ_g is nonempty when $g = 5$ or $g = 11$. To have a better understanding, it is worthwhile to observe Ξ_g, and to collect more evidences for the Borisov–Nuer conjecture.

In this paper, we construct examples of points in Ξ_g for various values of g. Suppose that Conjecture 1.1 holds for a single polarized Enriques surface (Y, H_Y). Let $h = [H_Y]$ denote its numerical class. Since all the Enriques surfaces Y have the same lattice structure $\operatorname{Num}(Y) \cong U \oplus E_8(-1)$, we immediately have that Conjecture 1.1 holds for every numerically polarized Enriques surface $(Y', [H'_Y]) \in \mathcal{M}^a_{En,h}$, i.e., H'_Y and $H'_Y \otimes \omega_{Y'}$ verifies the conjecture. Hence, it suffices to construct only one numerically polarized Enriques surface $(Y, [H_Y])$ from the moduli space $\mathcal{M}^a_{En,h}$ which makes Conjecture 1.1 hold. The key ingredient is a Jacobian Kummer surface $X = Km(C)$ of a general curve C of genus 2, similar as in [1]. Such a Jacobian Kummer surface has plenty of technical merits, for instance:

- X has a fixed-point-free involution θ, that is, X is the $K3$ cover of some Enriques surface Y;
- intersection theory of X is well-understood;
- the pullback homomorphism $\theta^* : \operatorname{Pic}(X) \to \operatorname{Pic}(X)$ is well-understood;
- the Picard number $\rho(X)$ is quite big, so there are more chances to find a certain line bundle.

The main result of this paper is the nonemptiness of the locus Ξ_g for various values g as follows, see Theorem 3.7:

Theorem 1.4. *When $g-1$ is divisible by 4, the locus Ξ_g is nonempty. In other words, for any given $k>0$ and any Enriques surface Y, there is an ample and globally generated line bundle H_Y and a line bundle N on Y such that $H_Y^2 = 4k$ and $\left(N - H_Y\right)^2 = \left(N - 2H_Y\right)^2 = -2$.*

The outline of the paper is the following. In Section 2, we review some basic facts on Enriques surfaces, Jacobian Kummer surfaces as $K3$ covers of Enriques surfaces, and line bundles. We also fix the notation we use. In Section 3, we describe a construction of a polarized Enriques surface which verifies the Borisov–Nuer conjecture using a Jacobian Kummer surface and we provide a few more examples in the case when $(g-1)$ is not divisible by 4.

2 | PRELIMINARIES

We recall some basic facts on Enriques surfaces and Jacobian Kummer surfaces. As the above discussion indicates, we translate the Borisov–Nuer conjecture and Equation (1.3) on an Enriques surface Y in terms of line bundles on its $K3$ cover X. To construct an Enriques surface from its $K3$ cover, we need a $K3$ surface X together with a fixed-point-free involution θ so that the quotient X/θ becomes an Enriques surface. Thanks to the following theorem of Keum, we pick algebraic Kummer surfaces as candidates:

Lemma 2.1 ([12, Theorem 2])**.** *An algebraic Kummer surface is a K3 cover of some Enriques surface.*

When the covering map $\sigma : X \to X/\theta = Y$ of an Enriques surface is fixed, we also need to ask which line bundles on X are pullbacks of some line bundles on Y. The answer is also well-known, thanks to Horikawa.

Lemma 2.2 ([11, Theorem 5.1])**.** *Let X be a $K3$ surface, let $\theta : X \to X$ be a fixed-point-free involution, and let $\sigma : X \to Y = X/\theta$ be the $2:1$ étale cover. Then the image of the map $\sigma^* : \mathrm{Pic}(Y) \to \mathrm{Pic}(X)$ is the set of line bundles M in X such that $\theta^* M \cong M$.*

Next, we recall the construction of a Jacobian Kummer surface and intersection theory over it. Let C be a generic curve of genus 2. Its Jacobian variety $\mathcal{A} = J(C)$ is an Abelian surface with Néron–Severi group $NS(\mathcal{A}) = \mathbb{Z}\cdot[\Theta]$ with $\Theta^2 = 2$. Note that $\mathcal{A}$ has a natural involution ι with 16 fixed points. The complete linear system $|2\Theta|$ defines a morphism to $\mathbb{P}^3$, which factors through the singular quartic $\mathcal{A}/\iota$ (Kummer quartic) with 16 ordinary double points. The Kummer surface $X = Km(\mathcal{A})$ is defined as the minimal desingularization of $\mathcal{A}/\iota$. Throughout the rest of the paper, we fix the notations as follows.

Notation 2.3. We follow the notation as in [1].

- C : a generic curve of genus 2 with 6 Weierstrass points $p_1, \dots, p_6 \in C$;
- $X = Km(C)$: Jacobian Kummer surface associated to C, which is the minimal desingularization of $J(C)/\iota$;
- $\theta : X \to X$: a fixed-point-free involution, so-called "switch", induced by the even theta characteristic $\left[p_4 + p_5 - p_6\right]$;
- $\sigma : X \to Y = X/\theta$: the quotient map so that Y is an Enriques surface;
- L : the line bundle induced by the hyperplane section of the singular quartic $J(C)/\iota \subseteq \mathbb{P}^3$;
- E_0, E_{ij} $(1 \le i < j \le 6)$: sixteen (-2)-curves called *nodes*;
- T_i $(1 \le i \le 6), T_{ij6}$ $(1 \le i < j \le 5)$: sixteen (-2)-curves called *tropes*, which satisfy the following relations

$$\begin{aligned} T_i &= \frac{1}{2}\left(L - E_0 - \sum_{k \ne i} E_{ik}\right) && \text{for } 1 \le i \le 6, \text{ and} \\ T_{ij6} &= \frac{1}{2}\left(L - E_{i6} - E_{j6} - E_{ij} - E_{\ell m} - E_{mn} - E_{\ell n}\right) && \text{for } 1 \le i < j \le 5, \end{aligned}$$

where $\{l, m, n\}$ is the complement of $\{i, j\}$ in $\{1,2,3,4,5\}$ (see [17, Lemma 4.1]).

Note that $L^2 = 4$, $L \cdot E_0 = L \cdot E_{ij} = 0$, and two distinct nodes do not intersect.

Let us describe the nodes and the tropes more precisely. Following the notation in [17], the 16 nodes are labeled by the corresponding 2-torsion points in the Jacobian $\mathcal{A} = J(C)$:

$$\begin{aligned} E_0 &= \text{ node corresponding to } [0] \in \mathcal{A}; \\ E_{ij} &= E_{[p_i - p_j]} = \text{ node corresponding to } \left[p_i - p_j\right] \in \mathcal{A}, 1 \le i < j \le 6. \end{aligned}$$

The tropes are labeled using their associated theta–characteristics of C [17], e.g. $T_i = T_{[p_i]}$ corresponds to $[p_i]$ and $T_{ijk} = T_{[p_i+p_j-p_k]}$ corresponds to $[p_i + p_j - p_k]$ for any $i < j < k$. Note that $T_{ijk} = T_{\ell mn}$ if $\{i,j,k\} \cup \{\ell, m, n\} = \{1, \dots, 6\}$.

Also note that the pullback θ^* swaps the nodes E_α and the tropes $T_{\alpha+\beta}$ in the following way, cf. [16] and [17, Section 4, Section 5]:

Nodes		Tropes	Nodes		Tropes
E_0	$\leftrightarrow$	T_{456}	E_{25}	$\leftrightarrow$	T_{246}
E_{12}	$\leftrightarrow$	T_3	E_{26}	$\leftrightarrow$	T_{136}
E_{13}	$\leftrightarrow$	T_2	E_{34}	$\leftrightarrow$	T_{356}
E_{14}	$\leftrightarrow$	T_{156}	E_{35}	$\leftrightarrow$	T_{346}
E_{15}	$\leftrightarrow$	T_{146}	E_{36}	$\leftrightarrow$	T_{126}
E_{16}	$\leftrightarrow$	T_{236}	E_{45}	$\leftrightarrow$	T_6
E_{23}	$\leftrightarrow$	T_1	E_{46}	$\leftrightarrow$	T_5
E_{24}	$\leftrightarrow$	T_{256}	E_{56}	$\leftrightarrow$	T_4

It is well-known that $\{E_0, E_{ij}, T_i, T_{ij6}\}$ spans $\mathrm{Pic}(X)$ [13, Lemma 3.1], and hence $\{L, E_0, E_{ij}\}$ spans $\mathrm{Pic}(X) \otimes \frac{1}{2}\mathbb{Z}$ if we allow $\frac{1}{2}\mathbb{Z}$ coefficients. For simplicity, we mostly consider a linear combination of L, E_0, E_{ij} in $\frac{1}{2}\mathbb{Z}$ coefficients, however, we have to carefully choose the coefficients so that the linear combination gives an element in $\mathrm{Pic}(X)$.

3 | CONSTRUCTION USING K3 COVERS

Let (Y, H_Y) be a polarized Enriques surface, and let $\sigma : X \to Y = X/\theta$ be its $K3$ cover. Suppose it verifies Conjecture 1.3, that is, Y has a line bundle N which fits into Equation (1.3). Equation (1.3) can be completely translated into the numerical conditions on its $K3$ cover. Namely, we are interested in line bundles $M \in \sigma^* \mathrm{Pic}(Y) \subseteq \mathrm{Pic}(X)$ which verifies the equation

$$(M - H_X)^2 = (M - 2H_X)^2 = -4 \tag{3.1}$$

where $H_X := \sigma^* H_Y$. Note that if H_Y is ample and globally generated, then H_X is also ample and globally generated, and vice versa.

Now let X be a Jacobian Kummer surface associated to a generic curve C of genus 2. As mentioned in the previous section, some line bundles in $\mathrm{Pic}(X)$ require rational coefficients in $\frac{1}{2}\mathbb{Z}$ when we write it as linear combinations of L and nodes E_{ij}. One typical example is called an even eight:

Lemma 3.1. *The set of 8 nodes $\{E_0, E_{16}, E_{23}, E_{24}, E_{25}, E_{34}, E_{35}, E_{45}\}$ forms an even eight, that is,*

$$(E_0 + E_{16} + E_{23} + E_{24} + E_{25} + E_{34} + E_{35} + E_{45})$$

is divisible by 2 in $\mathrm{Pic}(X)$.

Proof. It is straightforward from a direct computation

$$\begin{aligned} L - T_4 - E_{14} - T_{146} - E_{46} &= L - \frac{1}{2}(L - E_0 - E_{14} - E_{24} - E_{34} - E_{45} - E_{46}) - E_{14} \\ &\quad - \frac{1}{2}(L - E_{14} - E_{16} - E_{46} - E_{23} - E_{25} - E_{35}) - E_{46} \\ &= \frac{1}{2}(E_0 + E_{16} + E_{23} + E_{24} + E_{25} + E_{34} + E_{35} + E_{45}). \end{aligned}$$

□

Also note that the complementary set of nodes $\{E_{12}, E_{13}, E_{14}, E_{15}, E_{26}, E_{36}, E_{46}, E_{56}\}$ also forms an even eight. Since

$$\theta^*(E_{12} + E_{15} + E_{26} + E_{56}) = T_3 + T_{146} + T_{136} + T_4 = 2L - \sum_{i,j}^{16} E_{ij} + (E_{12} + E_{15} + E_{26} + E_{56})$$

and by similar computations, grouping them by those 4 line bundles makes the problem easier. Let $F_\bullet$ be the sum of four nodes E_{ij}, namely,

$$\begin{cases} F_1 = E_{12} & + & E_{15} & + & E_{26} & + & E_{56}, \\ F_2 = E_{13} & + & E_{14} & + & E_{36} & + & E_{46}, \\ F_3 = E_{23} & + & E_{25} & + & E_{34} & + & E_{45}, \\ F_4 = E_0 & + & E_{16} & + & E_{24} & + & E_{35}. \end{cases}$$

We have

$$\theta^* L \cong 3L - \sum_{i,j}^{16} E_{ij},$$

$$\theta^* F_k \cong 2L - \sum_{i,j}^{16} E_{ij} + F_k \text{ for each } 1 \le k \le 4.$$

Consider a linear combination of the form $M = \alpha L - \beta_1 F_1 - \beta_2 F_2 - \beta_3 F_3 - \beta_4 F_4$ as a special case. First, we need to check when M becomes a θ^*-invariant line bundle on X.

Lemma 3.2. *A linear combination $M = \alpha L - \beta_1 F_1 - \beta_2 F_2 - \beta_3 F_3 - \beta_4 F_4$ is a line bundle in* Pic(X) *such that $\theta^* M \cong M$ if and only if $\beta_i \in \frac{1}{2}\mathbb{Z}$, $\beta_1 + \beta_2 \in \mathbb{Z}$, $\beta_3 + \beta_4 \in \mathbb{Z}$, and $\alpha = \beta_1 + \beta_2 + \beta_3 + \beta_4$.*

Proof. Recall that Pic(X) is spanned by integral linear combinations of nodes E_{ij} and tropes T_i, T_{ij6}. In particular, $\alpha, \beta_i \in \frac{1}{2}\mathbb{Z}$. We first check the condition $\theta^* M \cong M$. A direct computation shows that $\theta^* M \cong M$ if and only if $\alpha = \beta_1 + \beta_2 + \beta_3 + \beta_4$.

We still need to show that $M \in$ Pic(X). Since $F_1 + F_2$ and $F_3 + F_4$ are divisible by 2 in Pic(X), but no other $F_i + F_j$ are divisible by 2 [15, Proposition V.6]. Therefore, the coefficients β_i are elements in $\frac{1}{2}\mathbb{Z}$ such that $\beta_1 + \beta_2 \in \mathbb{Z}$ and $\beta_3 + \beta_4 \in \mathbb{Z}$. □

Example 3.3. Let $\beta_1 = \beta_2 = \beta_3 = \beta_4 = \frac{1}{2}$, and $\alpha = \sum \beta_i = 2$. The line bundle $H_X = 2L - \frac{1}{2}\left(F_1 + F_2 + F_3 + F_4\right)$ satisfies the assumptions in Lemma 3.2, and defines an embedding of X into $\mathbb{P}^5$ as the intersection of 3 quadrics [18, Theorem 2.5]. Such a Kummer surface $\left(X, H_X\right)$ carries a line bundle M such that $\theta^* M \cong M$, namely,

$$M = 3L - F_1 - F_2 - F_4$$

as in [1, proof of Theorem 13]. Furthermore, H_X and M satisfies Equation (3.1) as desired.

Let $H_X = \alpha L - \sum_{k=1}^4 \beta_k F_k$, and let $M = \alpha' L - \sum_{k=1}^4 \beta'_k F_k$. Suppose that both H_X and M satisfy the assumptions in Lemma 3.2. Now our question becomes:

Question 3.4. For a given ample polarization $H_X = \alpha L - \sum_{k=1}^4 \beta_k F_k$, find values β'_i so that the line bundles H_X and M verify Equation (3.1).

By taking the substitutions

$$\begin{cases} S = \beta'_1 - \beta_1, \\ T = \beta'_2 - \beta_2, \\ U = \beta'_3 - \beta_3, \\ V = \beta'_4 - \beta_4, \end{cases}$$

Equation (3.1) gives the system of two quadratic Diophantine equations, namely:

$$4(S + T + U + V)^2 - 8S^2 - 8T^2 - 8U^2 - 8V^2 = -4,$$
$$4(\alpha - (S + T + U + V))^2 - 8\left(\beta_1 - S\right)^2 - 8\left(\beta_2 - T\right)^2 - 8\left(\beta_3 - U\right)^2 - 8\left(\beta_4 - V\right)^2 = -4.$$

Dividing both equations by 4 and taking their difference, we have

$$(S + T + U + V)^2 - 2S^2 - 2T^2 - 2U^2 - 2V^2 = -1, \tag{3.2}$$

$$2\alpha(S+T+U+V)-4\beta_1 S-4\beta_2 T-4\beta_3 U-4\beta_4 V-\frac{d}{4}=0, \tag{3.3}$$

where $\alpha=\beta_1+\beta_2+\beta_3+\beta_4$ and $d=H_X^2=4\alpha^2-8\left(\beta_1^2+\beta_2^2+\beta_3^2+\beta_4^2\right)$. Therefore, finding M is equivalent to finding a solution (S,T,U,V) of this system of Diophantine equations (3.2), (3.3), where the corresponding M satisfies the assumptions in Lemma 3.2.

In most cases, finding integral solutions of a system of Diophantine equations is extremely hard even though it has rationally parametrized solutions. Instead, we provide a sufficient condition on β_i's so that the system has a solution (S,T,U,V) which fits into all the conditions we need.

Proposition 3.5. *Let $\beta_1,\beta_2,\beta_3,\beta_4\in\frac{1}{2}\mathbb{Z}$ such that $\beta_1+\beta_2\in\mathbb{Z}$, $\beta_3+\beta_4\in\mathbb{Z}$, and*

$$2S=\frac{1}{2\left(\beta_3+\beta_4\right)}\left[\left(\beta_1+\beta_2+\beta_3+\beta_4\right)^2-2\left(\beta_1^2+\beta_2^2+\beta_3^2+\beta_4^2\right)+2\left(\beta_3-\beta_4\right)\right]\in\mathbb{Z}.$$

Then the above system of Diophantine equations has a solution $(S,T,U,V)=\left(S,S,\frac{1}{2},-\frac{1}{2}\right)$ so that $\beta_1',\dots,\beta_4'$ satisfy the assumptions in Lemma 3.2.

Proof. It is clear that $\left(S,S,\frac{1}{2},-\frac{1}{2}\right)$ is a solution for Equation (3.2). Substitute into Equation (3.3), we have a univariable linear equation

$$4\left(\beta_3+\beta_4\right)S-\left(\beta_1+\beta_2+\beta_3+\beta_4\right)^2+2\left(\beta_1^2+\beta_2^2+\beta_3^2+\beta_4^2\right)-2\beta_3+2\beta_4=0.$$

It is straightforward that such a solution $(S,T,U,V)=\left(S,S,\frac{1}{2},-\frac{1}{2}\right)$ provides $\beta_1',\beta_2',\beta_3',\beta_4'$ which satisfies the assumptions in Lemma 3.2. □

By taking suitable quadruples $\left(\beta_1,\beta_2,\beta_3,\beta_4\right)$, we obtain a number of polarized Enriques surfaces establishing the Borisov–Nuer conjecture as follows.

Proposition 3.6. *Suppose that $H_X=\left(\beta_1+\beta_2+\beta_3+\beta_4\right)L-\beta_1F_1-\beta_2F_2-\beta_3F_3-\beta_4F_4$ is an ample and globally generated line bundle on X such that $\beta_1,\dots,\beta_4$ satisfy the assumptions in Proposition 3.5. Then there is a polarized Enriques surface $\left(Y,H_Y\right)$ and a line bundle N on Y such that $H_X^2=2H_Y^2$ and $\left(N-H_Y\right)^2=\left(N-2H_Y\right)^2=-2$. In particular, the Borisov–Nuer conjecture holds for $\left(Y,H_Y\right)$.*

Proof. Let $S=\frac{1}{4\left(\beta_3+\beta_4\right)}\left[\left(\beta_1+\beta_2+\beta_3+\beta_4\right)^2-2\left(\beta_1^2+\beta_2^2+\beta_3^2+\beta_4^2\right)+2\left(\beta_3-\beta_4\right)\right]$. Proposition 3.5 implies that $\left(S,S,\frac{1}{2},-\frac{1}{2}\right)$ is a solution of the system of Diophantine equations (3.2), (3.3). Hence, the line bundle

$$M:=\left(\beta_1+\beta_2+\beta_3+\beta_4+2S\right)L-\left(\beta_1+S\right)F_1-\left(\beta_2+S\right)F_2+\left(\beta_3+\frac{1}{2}\right)F_3-\left(\beta_4-\frac{1}{2}\right)F_4$$

verifies the conditions $\theta^*M\cong M$ and $\left(M-H_X\right)^2=\left(2H_X-M\right)^2=-4$.

By Lemma 2.2, there are line bundles H_Y and N on an Enriques surface $Y=X/\theta$ such that $\sigma^*H_Y=H_X$, $\sigma^*N=M$ where $\sigma:X\to Y=X/\theta$ is the quotient map. Since $\sigma_*H_X=H_Y\oplus\left(H_Y\otimes K_Y\right)$ and $\sigma_*M=N\oplus\left(N\otimes K_Y\right)$, we conclude that $\left(N-H_Y\right)^2=\left(2H_Y-N\right)^2=-2$. □

Together with a discussion on the moduli of (numerically) polarized Enriques surfaces, we get the following nonemptiness.

Theorem 3.7. *The locus Ξ_g contained in the Borisov–Nuer divisor $\mathcal{BN}_g\subset\mathcal{F}_g$ of polarized $K3$ surfaces of degree $H_X^2=2g-2$ is nonempty when $2g-2$ is divisible by 8. In particular, there is a numerically polarized Enriques surface $\left(Y,h=\left[H_Y\right]\right)\in\mathcal{M}_{En,h}^a$ which verifies the Borisov–Nuer conjecture when $h^2=g-1$ is divisible by 4. Moreover, the conjecture also holds for every $\left(Y',\left[H_{Y'}\right]\right)\in\mathcal{M}_{En,h}^a$.*

Proof. Let X be a general Jacobian Kummer surface as above. It suffices to construct a pair $\left(H_X,M\right)$ of line bundles on X determined by the values β_i's and β_i''s satisfying Proposition 3.5. Suppose $g=4k+1$ so that $H_X^2=8k$ is divisible by 8. We pick $H_X=(k+1)L-\frac{k}{2}\left(F_1+F_2\right)-\frac{1}{2}\left(F_3+F_4\right)$ so that $H_X^2=8k,\beta_1=\beta_2=\frac{k}{2},\beta_3=\beta_4=\frac{1}{2}$. Note that $H_X=\left[2L-\frac{1}{2}\left(F_1+F_2+F_3+F_4\right)\right]+(k-1)\left[L-\frac{1}{2}\left(F_1+F_2\right)\right]$ is a sum of two line bundles. Since the former one is very ample, and the later one is a multiple of a line bundle which induces an elliptic fibration over $\mathbb{P}^1$ (see [14, Fibration 7] and [10, Section 5.1]), their sum H_X is indeed ample and globally generated.

Moreover, the value

$$\frac{1}{2(\beta_3+\beta_4)}\left[(\beta_1+\beta_2+\beta_3+\beta_4)^2-2(\beta_1^2+\beta_2^2+\beta_3^2+\beta_4^2)+2(\beta_3-\beta_4)\right]=k$$

is an integer, we conclude that there is a line bundle M which verifies the equation

$$(M-H_X)^2=(M-2H_X)^2=-4$$

by Proposition 3.5. For instance, we may take $M=(2k+1)L-k(F_1+F_2)-F_3$. □

Corollary 3.8. *Let $(Y, h=[H_Y])\in\mathcal{M}^a_{En,h}$ be a numerically polarized Enriques surface appearing in Theorem 3.7. Let $(Y',[H'_Y])\in\mathcal{M}^a_{En,h}$ be any numerically polarized unnodal Enriques surface. Then the Enriques surface Y' has an H'_Y-Ulrich line bundle, in the sense of [1, Definition 1].*

Proof. Note that any $(Y',[H'_Y])\in\mathcal{M}^a_{En,h}$ carries a line bundle $N'\in\mathrm{Pic}(Y')$ such that $(N'-H'_Y)^2=(N'-2H'_Y)^2=-2$. Since Y' is general, it is unnodal; it does not contain any smooth (-2)-curves. By [5, Proposition 2.1], N' is an H'_Y-Ulrich line bundle as desired. □

Example 3.9. There are several possible choices of H_X satisfying the assumptions of Proposition 3.5 and Theorem 3.7 when we fix the degree H_X^2. For instance, take $\beta_1=\beta_2=\frac{m}{2}$, $\beta_3=\beta_4=\frac{n}{2}$ where m,n are positive integers. The line bundle $H_X:=(m+n)L-\frac{m}{2}(F_1+F_2)-\frac{n}{2}(F_3+F_4)$ is ample and globally generated with the self-intersection number $H_X^2=8mn$. Furthermore, the value

$$\frac{1}{2(\beta_3+\beta_4)}\left[(\beta_1+\beta_2+\beta_3+\beta_4)^2-2(\beta_1^2+\beta_2^2+\beta_3^2+\beta_4^2)+2(\beta_3-\beta_4)\right]=m$$

is always an integer, so we are able to find a solution of Diophantine equations (3.2), (3.3).

Remark 3.10. The system of Diophantine equations (3.2), (3.3) needs not to have a desired solution. For example, let $\beta_1=1$, $\beta_2=\beta_3=\beta_4=0$. Then the second equation (3.3) becomes

$$-2S+2T+2U+2V=-1.$$

Since $-S+T=(\beta'_2-\beta'_1)-(\beta_2-\beta_1)$ and $U+V$ are integers, the left-hand side must be an even integer. Hence, there is no solution which satisfies the assumptions. In general, by a simple parity argument, one can easily check that the system (3.2), (3.3) does not have a solution (S,T,U,V) such that the corresponding M satisfies the assumptions of Lemma 3.2 when the number $\frac{d}{4}$ (which stands for $\frac{1}{4}H_X^2$ in the context) is not an even integer. This is the reason why it is not easy to verify the nonemptiness of Ξ_g when $g-1$ is not divisible by 4. For instance, we cannot verify that Borisov–Nuer conjecture holds for a Fano polarized Enriques surface (Y,Δ) in the above arguments, since $g-1=\Delta^2=10$ is not divisible by 4.

However, there might be plenty of chances to find a solution of Equation (3.1) using the same Jacobian Kummer surface. We only address a few more examples as evidence. We cannot guarantee that the following bundles H_X are ample and/or globally generated, however, this aspect is not very important from the viewpoint of the original Borisov–Nuer conjecture.

(i) Let $H_X=4L-2F_1-F_2-\frac{1}{2}F_3-\frac{1}{2}F_4$ so that $\theta^*H_X\cong H_X$ and $H_X^2=20$. We take M as

$$M=6L-3F_1-\frac{3}{2}(E_0+E_{13}+E_{14}+E_{16}+E_{25}+E_{34}+E_{36}+E_{46}).$$

Since

$$L-T_1-T_{346}+E_{12}+E_{15}=\frac{1}{2}(E_0+E_{13}+E_{14}+E_{16}+E_{25}+E_{34}+E_{36}+E_{46}),$$

M is a line bundle on X. Furthermore, M satisfies $\theta^*M\cong M$ and $(M-H_X)^2=(M-2H_X)^2=-4$. Hence, there is an Enriques surface Y and two line bundles H_Y,N with $H_Y^2=10$ such that $(N-H_Y)^2=(2H_Y-N)^2=-2$.

(ii) Let $H_X = 6L - 3F_1 - 2F_2 - \frac{1}{2}F_3 - \frac{1}{2}F_4$ so that $\theta^* H_X \cong H_X$ and $H_X^2 = 36$. We take M as

$$M = 8L - \frac{7}{2}F_1 - \frac{3}{2}F_2 - \frac{3}{2}\left(E_0 + E_{13} + E_{14} + E_{16} + E_{25} + E_{34} + E_{36} + E_{46}\right).$$

We have $M \in \mathrm{Pic}(X)$, $\theta^* M \cong M$, and H_X, M satisfy Equation (3.1).

(iii) Let $H_X = 8L - 4F_1 - 3F_2 - \frac{1}{2}F_3 - \frac{1}{2}F_4$. We have $\theta^* H_X \cong H_X$ and $H_X^2 = 52$. We take M as

$$M = 10L - 4\left(F_1 + F_2\right) - \frac{1}{2}\left(E_0 + E_{13} + E_{14} + E_{16} + E_{25} + E_{34} + E_{36} + E_{46}\right) - \left(E_{23} + E_{24} + E_{35} + E_{45}\right).$$

We have $M \in \mathrm{Pic}(X)$, $\theta^* M \cong M$, and H_X, M satisfy Equation (3.1).

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