Compact Matrix Quantum Groups and Their Representation Categories

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Abstrakt

Im Zentrum dieser Dissertation steht die Untersuchung kompakter Quantengruppen und ihrer Darstellungskategorien sowie die Weiterentwicklung des Diagrammkalküls, der für Berechnungen in diesen Strukturen benutzt wird.

Ein effektives Werkzeug für die Untersuchung jener Darstellungskategorien sind so genannte *Partitionskategorien*. Eine der Hauptaufgaben ist hier die Lösung von Klassifikationsproblemen. Im Falle der ursprünglichen Definition von Banica und Speicher wurde die Klassifikation von Raum und Weber vollständig erreicht. Nichtsdestotrotz werden heute viele Verallgemeinerungen von Partitionskategorien eingeführt, untersucht, klassifiziert – oder warten noch auf ihre Entdeckung. Diese Doktorarbeit trägt zu diesem Gebiet auf folgende Weise bei.

(a) Wir klassifizieren *global gefärbte* Kategorien zweifarbiger Partitionen.
(b) Wir führen *Partitionen mir extra Singletons* und deren Kategorien ein. Wir konstruieren einen Funktor, der diese neuen Kategorien mit zweifarbigen Partitionskategorien in Beziehung setzt. Dieser Funktor erlaubt es uns insbesondere Klassifikationsresultate von einer Struktur auf die andere zu übertragen.
(c) Wir untersuchen *lineare Partitionskategorien*, bei denen man auch Linearkombinationen bilden darf. Wir zeigen erste echte (so genannte *non-easy*) Beispiele solcher linearen Partitionskategorien auf; zuvor waren keine solchen bekannt. Diese Beispiele wurden mit Hilfe von Computerexperimenten entdeckt. Wir interpretieren diese dann als Bilder klassischer *easy* Partitionskategorien unter bestimmten Funktoren.

Das Studium von Partitionskategorien ist von Quantengruppen her motiviert, wie bereits erwähnt. Die wesentliche Anwendung ist die Konstruktion von Beispielen – jede Partitionskategorie induziert eine kompakte Matrixquantengruppe. Insofern liefern Erkenntnisse über die Struktur der Partitionskategorien auch Einblicke in die Struktur der entsprechenden Quantengruppen. Der Versuch letztere genauer zu beschreiben kann mitunter die Definition interessanter Konstruktionen von Quantengruppen nach sich ziehen. In dieser Dissertation leisten wir dazu folgende Beiträge.

(d) Wir untersuchen Tensorkomplexifizierungen von Quantengruppen und interpretieren insbesondere die Ergebnisse von (a).
(e) Wir untersuchen freie Komplexizifizierungen von Quantengruppen.
(f) Wir definieren das *Verkleben* und das *Entkleben* für Quantengruppen, das die Tensor- und die freie Komplexifizierung verallgemeinert. Im Zuge dessen interpretieren wir auch die Resultate aus (b).
(g) Wir definieren neue Produktkonstruktionen für kompakte Matrixquantengruppen, die das duale freie Produkt und das Tensorprodukt interpolieren. Dadurch können wir auch einige Kategorien von (b) interpretieren.
(h) Wir studieren homogene kompakte Matrixquantengruppen, deren Fundamentaldarstellung reduzibel ist. Dadurch geben wir insbesondere eine Interpretation für die Objekte aus (c) an.
(i) Schließlich wenden wir uns noch antikommutativen Verdrehungen der orthogonalen Gruppe zu, um die restlichen in (c) gefundenen Kategorien zu erklären.

Die Doktorarbeit basiert auf den Veröffentlichungen [Gro18, GW20, GW19a, GW19b] des Autors.

Abstract

The main topic of this thesis is the investigation of compact matrix quantum groups and their representation categories and the further development of a diagrammatic calculus used for computations in those structures.

An efficient tool for studying representation categories of quantum groups are so-called *categories of partitions*. The main goal here is to solve classification problems. In the case of the original Banica–Speicher categories of partitions, this was solved a few years ago by Raum and Weber. Nevertheless, many generalizations of categories of partitions are being introduced, studied, classified or still waiting to be discovered. This thesis contributes to this area by the following achievements:

(a) We classify *globally colourized* categories of two-coloured partitions.
(b) We introduce *categories of partitions with extra singletons*. We construct a functor linking this structure with categories of two-coloured partitions. In particular, this functor allows to transfer classification results from one structure to the other.
(c) We study *linear categories of partitions*, where linear combinations of partitions are allowed. We bring first proper (*non-easy*) examples of these linear categories as no examples were known before this project. These were obtained by performing some computer experiments. We interpret these categories as images of classical (*easy*) categories of partitions by some functors.

As we mentioned above, the motivation for studying categories of partitions is to study quantum groups. The main application is to construct examples – every category of partitions induces a compact matrix quantum group. Nevertheless, understanding the structure of partition categories also gives us insight into the structure of the corresponding quantum groups. Trying to describe the associated quantum groups may motivate the definition of some interesting quantum group constructions. In this thesis, we do the following:

(d) We study the tensor complexification of quantum groups. In particular, we interpret the result of (a).
(e) We study the free complexification of quantum groups.
(f) We introduce certain *gluing* and *ungluing* procedures generalizing the tensor and free complexification. In particular, we interpret the result of (b).
(g) We introduce new product constructions for compact matrix quantum groups that interpolate the dual free and tensor product. Those also interpret some results of (b).
(h) We study homogeneous compact matrix quantum groups with reducible fundamental representation. In particular, we interpret many results of (c).
(i) We study certain anticommutative twists of the orthogonal group. This interprets the rest of the categories obtained in (c).

The thesis is built on the author's publications [Gro18, GW20, GW19a, GW19b].

Acknowledgement

I wish to express my deepest gratitude to my supervisor Moritz Weber. He introduced me to this exciting field of mathematics and proposed a PhD project, which initiated a fruitful collaboration going beyond the original plans. He was carefully reading all my work and giving me lots of valuable advices regarding the mathematical work itself as well as its presentation in research articles and talks. I greatly appreciate also his outstanding relationship with us – his PhD students –, which created an excellent atmosphere free of any kind of stress or negative emotions.

The great atmosphere is surely also a contribution of all the members of the *Free probability* research group, the members of the SFB graduate school as well as the whole compact quantum group research community.

During my study, I had the opportunity to participate in many scientific conferences and summer schools and to present my own results on some of them. I am grateful to the organizers of all these events for inviting me and providing an opportunity to present myself. I am thankful to the scientific community for interesting lectures, inspiring conversations and helpful reviews of my articles. I would like to pay my special regards to Adam Skalski for hosting me at the Mathematical Institute during my research stay in Warsaw.

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I am indebted to my parents, who were supporting me for my whole life. Without them, my studies would not be possible at all. Another carrying person lives with me in Germany and promised to stay by me for the rest of our lives – my wife Alena. Thanks to her, I did not have to perform my studies six hundred kilometres from my home – I found my new home here together with her.

My last thanks are addressed to all of my old friends who visited me in Germany as well as all the new friends I made here. Also they made the life in a foreign country much easier.

Contents

List of used symbols

$\simeq$	Isomorphism between structures	
A'	Commutant of A	12
$A \odot B$	Algebraic tensor product	13
$A \otimes_{\max} B$	Maximal tensor product of C*-algebras	14
$A \otimes_{\min} B$	Minimal tensor product of C*-algebras	13
$A * B$	Free product of C*-algebras	16
$\mathrm{Alt}\,\mathscr{C}$	Two-coloured category generated by alternating coloured partitions	75
A^t	Transposition of a matrix A	
$\mathcal{B}_{(\delta)}$	Mapping $b_k \mapsto b_k + (-1)^k {\uparrow}^{\otimes k}$	99
$\mathscr{B}(H)$	Algebra of bounded operators on Hilbert space H	
b_k	Block partition	66, 96, 99
B_N	Bistochastic group	108
B_N^+	Free bistochastic quantum group	108
B_N'	Modified bistochastic group $B_N \tilde{\times} \mathbb{Z}_2$	108
$B_N'^+$	Free modified bistochastic quantum group $B_N^+ \tilde{\times} \mathbb{Z}_2$	108
$B_N^{\#+}$	Freely modified free bistochastic quantum group $B_N^+ \tilde{*} \mathbb{Z}_2$	108
$\mathbb{C}$	Set of complex numbers	
$\mathbb{C}\Gamma$	Group algebra of a discrete group Γ	16, 31
$C(G)$	C*-algebra associated to quantum group G	20
$C(X)$	C*-algebra of continuous functions over a compact space X	11
$C_\mathrm{r}(G)$	Reduced C*-algebra of G	27
$C_\mathrm{u}(G)$	Full C*-algebra of G	27
$C^*(\cdots \mid \cdots)$	Universal C*-algebra	15
$C^*(A_0)$	C*-envelope of a $*$-algebra A_0	15
$C^*(\Gamma)$	Group C*-algebra	16, 31
$C^*_\mathrm{r}(\Gamma)$	Reduced group C*-algebra	16, 31
$\mathscr{C}^{\vert}$	Category $\{p \in \mathscr{C} \mid p$ does not contain any extra singleton$\}$	72
$\mathscr{C}(k)$	The set $\mathscr{C}(0,k)$	59
$\mathscr{C}(k,l)$	Morphism spaces of a category of partitions $\mathscr{C}$	57
$\mathscr{C}_0$	Subcategory $\{p \in \mathscr{C} \mid c(p) = 0\}$	67
$\mathscr{C}_0^{\vartriangle}$	Extra-singleton category $\langle \mathscr{C}, {}_{\vartriangle}\!\setminus\!\setminus^{\triangledown} \rangle^{\vartriangle}$	76
$\mathscr{C}_k^{\vartriangle}$	Extra-singleton category $\langle \mathscr{C}, {}_{\vartriangle}\!\setminus\!\setminus^{\triangledown}, (\uparrow \otimes {\vartriangle})^{\otimes k} \rangle^{\vartriangle}$	76
$c(w), c(p)$	#(white points) – #(black points) in a word w or a partition p	66
$\mathcal{D}$	Disjoining isomorphism	91
d_α	Degree of the irreducible representation u^α	150
E	Trivial group	
e_i	The i-th vector of the standard basis of $\mathbb{C}^N$	
F	Certain functor $\mathscr{P}^{\vartriangle} \to \mathscr{P}^{\circ\bullet}$	73
$\mathsf{FundRep}_G$	Category generated by the fundamental representation of G	45
$\mathsf{FundRep}_G^{k\text{-ext}}$	$\mathbb{Z}_k$-extended representation category corresponding to G	149
$\mathsf{FundRep}_G^{\mathscr{O}}$	$\mathscr{O}$-coloured representation category corresponding to G	146
$\hat{G}$	Discrete dual of a compact quantum group Γ	31
$G_\pm^{\mathrm{irr}}$	Quantum group $V_{(N,\pm)} G V_{(N,\pm)}^*$	128
G^σ	Deformation of G corresponding to a 2-cocycle σ	138
$G \hat{*} H$	Dual free product of quantum groups G and H	36
$G \tilde{*} H$	Glued free product of quantum groups G and H	37
$G\, {}^{s}\hat{*}^{r}_{k}\, H$	Quantum subgroup of $G \hat{*} H$ given by the relation $(sr)^k = 1$	122, 164

$G \cap H$	Intersection of quantum groups G and H	38
$G \times H$	Tensor product of quantum groups G and H	35
$G \tilde{\times} H$	Glued tensor product of quantum groups G and H	37
$G \times_k H$	Certain product of quantum groups G and H	117, 122, 163
$G \mathbin{\times\!\!\!/} H$	Certain product of quantum groups G and H	117, 122, 163
$G \mathbin{\times\!\!\!\backslash} H$	Certain product of quantum groups G and H	163
$\langle G, H\rangle$	Compact quantum group topologically generated by G and H	39
$\gcd(k,l)$	Greatest common divisor of k and l	
GL_N	General linear group of $N \times N$ invertible matrices	
H_N	Hyperoctahedral group	108
H_N^+	Free hyperoctahedral quantum group	108
h_s	Partition represented by the word $(\mathsf{ab})^s$	60
I_G	Ideal associated to a CMQG G	30
id	Identity mapping	
id_a	Identity mapping/morphism $a \to a$	40
$\mathrm{Irr}\, G$	Set of classes of irreducible representations of G	24
$\mathcal{J}$	Joining isomorphism	92
$k(\mathscr{C})$	Degree of reflection of $\mathscr{C}$	66
$\mathscr{L}(V,W)$	Linear maps $V \to W$	
$\mathrm{lcm}(k,l)$	Least common multiple of k and l	
$\mathrm{Lrot}\, T$	Left rotation of a morphism T	46, 58, 63
$L^2(G)$	GNS Hilbert space corresponding to $C(G)$	27
$l^2(\Gamma)$	Hilbert space of l^2-sequences on a discrete group Γ	16
$l^\infty(\Gamma)$	Algebra of l^∞-sequences on a discrete (quantum) group Γ	33
$M_n(A)$	Algebra of $n \times n$ matrices with elements in algebra A	
Mat	Category of matrices	41
$\mathrm{Mor}(u,v)$	Space of intertwiners between representations u and v	23
$\mathbb{N}$	Set of natural numbers $\{1,2,\ldots\}$	
$\mathbb{N}_0$	Set of natural numbers with zero $\{0,1,2,\ldots\}$	
n_α	The size of the irreducible representation u^α, $\alpha \in \mathrm{Irr}\, G$	24
NC	Set of non-crossing partitions	61
NCPart_δ	Linear category of all non-crossing partitions	81, 96
NCPart'_δ	Linear category of non-crossing partitions of even length	96
$\mathscr{O}^*$	Monoid of all words over the alphabet $\mathscr{O}$	45, 63
$\mathscr{O}^k$	Set of all words of length k over the alphabet $\mathscr{O}$	45, 63
$\mathscr{O}_{\circ\bullet}$	Set $\{\circ, \bullet\}$	45
$\mathscr{O}_{\triangle}$	Set $\{\triangle, \mid\}$	71
$\mathscr{O}_{\vdots}$	Set $\{\vdots, \uparrow\}$	133
$O(G)$	Coordinate ring (Hopf algebra) of a matrix (quantum) group G	22
$O^+(F)$	Universal unitary quantum group	29
O_N^+	Free orthogonal quantum group	21, 108
O_N	Orthogonal group of $N \times N$ orthogonal matrices	
O_N^{-1}	The q-deformation of O_N at $q=-1$	141
$\lvert p\rvert$	Length of a partition p	56
$p^\star$	Reflection of a partition p	58
$\dot{p}$	Dotted partition corresponding to the ordinary partition p	134
$\mathscr{P}$	Category of all partitions	56
$\mathscr{P}(k,l)$	Set of all partitions with k upper and l lower points	56
$\mathscr{P}^{\circ\bullet}$	Category of all two-coloured partitions	64
$\mathscr{P}^{\circ\bullet}(k,l)$	Category of all two-col. partitions with k upper and l lower points	64

$\mathscr{P}^{\triangle}$	•	Category of all partitions with extra singletons	71
$\mathscr{P}^{\triangle}_{\text{even}}$	•	Partitions with extra singletons having even length	73
$\mathcal{P}_{(\delta)}$	•	Mapping $p \mapsto \pi_{(\delta)}^{\otimes l} p \pi_{(\delta)}^{\otimes k}$ for $p \in \mathsf{Part}_\delta(k,l)$	94
$P_{(N)}$	•	Orthogonal projection $\mathbb{C}^N \to \mathbb{C}^N$ onto the subspace $\operatorname{span}\{\xi\}^\perp$	127, 130
Pair_δ	•	Linear category of all pairings	81
Part_δ	•	Linear category of all partitions	80
$\mathsf{Part}_\delta(k,l)$	•	Formal linear combinations of partitions in $\mathscr{P}(k,l)$	80
$\mathsf{Part}_N^{\vdots}$	•	Linear category of all partitions with $\mathscr{O}_{\vdots}$ as a dotted category	134
$\mathsf{PartRed}_{(\delta)}$	•	Reduced category $\mathcal{P}_{(\delta)}\mathsf{Part}_\delta$	94
$\operatorname{Pol} G$	•	Hopf $*$-algebra associated to G	26
$\mathbb{R}$	•	Set of real numbers	
Rp	•	Rotation of a partition p	59, 63
$R^{a_k}\xi$	•	Rotation of a morphism ξ	148
Rep_G	•	Representation category of a quantum group G	43
$\operatorname{rl}(p,q)$	•	The number of deleted remaining loops when composing q and p	80
$\operatorname{Rrot} T$	•	Right rotation of a morphism T	46, 58, 63
S_N^+	•	Free symmetric quantum group	21, 108
S_N	•	Symmetric group on N points (represented by permutation matrices)	
S_N'	•	Modified symmetric group $S_N \tilde{\times} \mathbb{Z}_2$	108
$S_N'^+$	•	Free modified symmetric group $S_N^+ \tilde{\times} \mathbb{Z}_2$	108
span	•	Linear span of a set of vectors	
$\mathcal{T}_{(\delta)}$	•	The mapping $p \mapsto \tau_{(\delta)}^{\otimes l} p \tau_{(\delta)}^{\otimes k}$ for $p \in \mathsf{Part}_\delta(k,l)$	91
T_p	•	Intertwiner associated to an element $p \in \mathsf{Part}_N$	104
T_p^σ	•	Intertwiner associated to p under deformation by σ	139
u°	•	Synonym for u	45
$u^\bullet$	•	Representation $F\bar{u}F^{-1}$ dual to u	45
u^α	•	Unitary irreducible representation corresponding to some class $\alpha \in \operatorname{Irr} G$	24
$U^+(F)$	•	Universal unitary quantum group	29
U_N^+	•	Free unitary quantum group	21
U_N	•	Unitary group of $N \times N$ unitary matrices	
$U_{(N,\pm)}$	•	Certain $N \times N$ unitary matrix	128
$\mathcal{U}_{(N,\pm)}$	•	Functor $\mathsf{Part}_N^{\vdots} \to \mathsf{Part}_{N-1}^{\triangle}$	135
$\mathcal{V}_{(\delta,\pm)}$	•	The mapping $p \mapsto v_{(\delta,\pm)}^{\otimes l} p v_{(\delta,\pm)}^{\otimes k}$	99
$V_{(N,\pm)}$	•	Certain $N \times N$ coisometry matrix	128
$\mathscr{W}_k$	•	Certain monoid	149
$\lvert w \rvert$	•	Length of a word w	45
$[w]$	•	Number of squares in w	149
$\bar{w}$	•	Colour inversion on a word w	45, 63, 149
w^*	•	Colour inversion composed with reflection on a word w	45, 63, 149
$w^\star$	•	Reflection of a word w	45
$\mathbb{Z}$	•	Group of all integers	33
$\mathbb{Z}_k$	•	Cyclic group of order k	33
$\hat{\Gamma}$	•	Compact dual of a discrete quantum group Γ	31
Δ	•	Comultiplication	19
δ_p	•	Blockwise Kronecker delta associated to a partition p	104
$\xi^\star$	•	Reflection of a morphism ξ	148
ξ_p	•	Synonym for T_p assuming $p \in \mathsf{Part}(0,k)$	106
$\pi_{(\delta)}$	•	The linear combination $\lvert - \frac{1}{\delta} \vdots$	94
$\Pi_i p$	•	Contraction of a partition p on the i-th position	59

$\Pi_i^{a_i}\xi$	• Contraction of a morphism ξ	147
π_s	• Partition represented by the word $\mathsf{a}_1\mathsf{a}_2\cdots\mathsf{a}_s\mathsf{a}_s\cdots\mathsf{a}_2\mathsf{a}_1\mathsf{a}_1\mathsf{a}_2\cdots\mathsf{a}_s\mathsf{a}_s\cdots\mathsf{a}_2\mathsf{a}_1$.	60
π_u	• Representation of $\mathbb{C}^{\hat{G}}$ associated to $u \in M_n(\operatorname{Pol} G)$	32
$\tau_{(\delta)}$	• The linear combination $\mid - \frac{2}{\delta} ¦$	91
$\upsilon_{(\delta,\pm)}$	• The linear combination $\mid - \frac{1}{\delta-1}\left(1 \pm \frac{1}{\sqrt{\delta}}\right) \in \mathsf{Part}_{\delta-1}(1,1)$	99
Ψ	• Functor $\mathscr{P}^{\circ\bullet} \to \mathscr{P}$	65
$\uparrow$	• Singleton partition	58
$\emptyset$	• Empty set; monoid identity (empty word or identity object)	42, 45
$1, 1_A$	• Unit of an algebra (of an algebra A)	10
1_N	• Identity matrix $N \times N$	

Introduction

In the following paragraphs, we would like to introduce the reader to the contents of this thesis, which constitutes a mixture of algebra, combinatorics and functional analysis.

First of all, we briefly summarize the main mathematical structures studied in the thesis. That is, compact quantum groups and categories of partitions. Using the latter to study the former is the main motive of this thesis.

Then we report on the main results of the thesis. Those include: obtaining new classification results for coloured categories of partitions, investigating linear categories of partitions and bringing the first examples of those structures, and introducing new product constructions for compact matrix quantum groups.

We conclude this introduction by some comments on how the thesis is structured.

Quantum groups

Quantum groups are a natural generalization of the concept of a *group* in non-commutative geometry. We will mainly be interested in *compact* quantum groups that are conveniently described using C*-algebras.

Given a compact topological space X, we can introduce the algebra $C(X)$ of complex-valued continuous functions over X. Conversely, by Gelfand duality, it holds that any commutative unital C*-algebra is of this form. So, commutative unital C*-algebras can be seen as an equivalent description of compact topological spaces. The basic idea of non-commutative geometry is that non-commutative C*-algebras should then be interpreted as some *non-commutative topology*.

Now, we can generalize the concept of a group. A compact group is a compact space G with a continuous multiplication map $m\colon G\times G\to G$. We can dualize this map to the associated algebra $C(G)$ as a *comultiplication* $\Delta\colon C(G)\to C(G\times G)=C(G)\otimes C(G)$ putting $\Delta(f)(x,y):=f(xy)$. Any non-commutative C*-algebra A with such a comultiplication map $\Delta\colon A\to A\otimes A$ (satisfying some additional axioms) is then called a *compact quantum group*. Compact groups then can be considered as a special case of compact quantum groups.

The history of quantum groups may be traced back to the work of Hopf [Hop41] in 1941, who noticed that the cohomology ring of a compact group G has a homomorphism $H^*(G)\to H^*(G)\otimes H^*(G)$. An algebraic structure today known as a *Hopf algebra* was defined by Pierre Cartier in [Car56]. Michio Jimbo [Jim85] and Vladimir Drinfeld [Dri88] initiated a rapid development in the area of quantum groups introducing certain Hopf algebras by deforming semisimple Lie groups. It was also Drinfeld, who suggested the term *quantum group* for those deformations. Finally, in the work of Stanisław Woronowicz [Wor87, Wor98] the definition of compact quantum groups was introduced in terms of C*-algebras as we indicated above.

There are many applications and connections of quantum groups to many areas of mathematics. Quantum groups are also seen as a promising tool for describing symmetries in modern physics – this also gave them their name. Let us mention a few applications focusing mainly on compact matrix quantum groups presented in this thesis. First of all, compact quantum groups are closely connected to the theory of operator algebras. The C*-algebras and von Neumann algebras associated to compact quantum groups are interesting objects on their own and can be studied within the operator algebra theory. Secondly, there are some applications of compact quantum groups to *free probability* theory. In particular, one can generalize de Finetti theorems by considering quantum symmetries of random variables. See [[Web16]] for an overview of those results. As for the connections to physics, let us mention two recent results constituting a very exciting connection between compact quantum groups and the theory of *quantum information theory*. First, the construction of highly entangled subspaces using the representation theory of quantum groups [BC18]. Second, the connection between non-local games and quantum automorphism groups of graphs [LMR20].

Categories of partitions

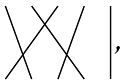
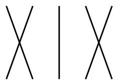
Categories of partitions are used to model the *intertwiner spaces* of (quantum) group representations. Given a group G (or a quantum group in general) and two representations $\varphi\colon G \to \mathrm{GL}_n$ and $\psi\colon G \to \mathrm{GL}_m$, we define the space of intertwiners

$$\mathrm{Mor}(\varphi,\psi) := \{T \in \mathscr{L}(\mathbb{C}^n,\mathbb{C}^m) \mid T\varphi(g) = \psi(g)T \ \forall g \in G\}.$$

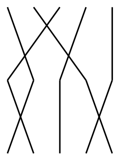
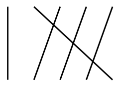
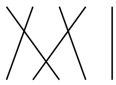
As an example, take the general linear group GL_N and denote by φ its fundamental representation acting on $V := \mathbb{C}^N$, that is, the identity map $\varphi := \mathrm{id}\colon \mathrm{GL}_N \to \mathrm{GL}_N$. For any $k \in \mathbb{N}_0$, the symmetric group S_k acts on $V^{\otimes k} = V \otimes \cdots \otimes V$ by permuting the factors. For $\pi \in S_k$, denote by $T_\pi\colon V^{\otimes k} \to V^{\otimes k}$ the corresponding linear map $x_1 \otimes x_2 \otimes \cdots \otimes x_k \mapsto x_{\pi^{-1}(1)} \otimes x_{\pi^{-1}(2)} \otimes \cdots \otimes x_{\pi^{-1}(k)}$. The famous Schur–Weyl duality then says that

$$\mathrm{Mor}(\varphi^{\otimes k},\varphi^{\otimes k}) = \mathrm{span}\{T_\pi \mid \pi \in S_k\} \simeq \mathbb{C}S_k.$$

Such permutations $\pi \in S_k$ have a nice diagrammatic representation as pairings. We draw two parallel lines of k points and draw edges between those two lines in a way such that the i-th point of the first line is connected with the j-th point of the second line if and only if $j = \pi(i)$. We give an example of two permutations on five points together with the corresponding diagrammatic representations:

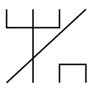
$$\pi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 4 & 1 & 3 & 5 \end{pmatrix} =$$  , $$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 1 & 3 & 5 & 4 \end{pmatrix} =$$  .

We can express the group operations on S_k in terms of these diagrams. Composition of two permutations $\sigma \circ \pi$ is computed as a vertical concatenation. The group inverse is given by flipping the diagram with respect to the horizontal axis. Considering the above given examples, we have

$\sigma\pi =$  $=$  , $\pi^{-1} =$  .

The diagram representing a given partition $\pi \in S_k$ can be interpreted also in terms of the map T_π as follows. Each vertex in the graph symbolises one copy of the vector space V. Each line symbolises an identity map $V \to V$. Finally, the diagram as a whole just shows, how to combine those identity maps to obtain a map $V^{\otimes k} \to V^{\otimes k}$.

Suppose that the vector space V is equipped with a bilinear symmetric form $T_\sqcup\colon V \otimes V \to \mathbb{C}$. For simplicity, consider the standard one $T_\sqcup(e_i \otimes e_j) = \delta_{ij}$. Its adjoint is a symmetric tensor $T_\sqcap\colon \mathbb{C} \to V \otimes V$ mapping $\alpha \mapsto \alpha \sum_i e_i \otimes e_i$. Now, we can associate linear maps to any diagram describing a pairing of k points on one line and l points on a second line. Again we associate the identity $V \to V$ to the vertical line $|$, then we associate the map $T_\sqcup$ to the pairing of two upper points $\sqcup$ and the tensor $\xi_\sqcap = \sum_i e_i \otimes e_i$ to the pairing of two lower points $\sqcap$. Then any pairing p with k upper and l lower points is associated a map $T_p\colon V^{\otimes k} \to V^{\otimes l}$. For example,

$p =$ 

is associated a map $T_p\colon V^{\otimes 4} \to V^{\otimes 4}$ mapping

$$T_p(x_1 \otimes x_2 \otimes x_3 \otimes x_4) = T_\sqcup(x_1,x_3)\, x_4 \otimes x_2 \otimes \xi_\sqcap.$$

By the result of Brauer [Bra37], this extends the Schur–Weyl duality to the case of the orthogonal group: Again, denote by $\varphi\colon O_N \to \mathrm{GL}_N$ the fundamental representation of O_N.

Denote by $\mathsf{Pair}_N(k,l)$ the set of all pairings of k points in the upper line and l points in the lower line. Then

$$\mathrm{Mor}(\varphi^{\otimes k},\varphi^{\otimes l}) = \mathrm{span}\{T_p \mid p \in \mathsf{Pair}_N(k,l)\}.$$

To extend the structure even more, instead of considering the *category of all pairings*, we can consider the *category of all partitions*. That is, the k upper and l lower points are not partitioned into pairs but into arbitrary non-empty disjoint subsets. For such a category, we can again define an appropriate *functor* $p \mapsto T_p$. The image of such a functor is then a category of linear maps. The key result for our work is the generalization of so-called Tannaka–Krein duality formulated by Woronowicz [Wor88].

Theorem (Woronowicz–Tannaka–Krein)**.** Let $\mathfrak{C}(k,l) \subseteq \mathscr{L}((\mathbb{C}^N)^{\otimes k},(\mathbb{C}^N)^{\otimes l})$ be a rigid monoidal $*$-category. Then there is a unique orthogonal compact matrix quantum group G with fundamental representation u such that $\mathrm{Mor}(u^{\otimes k},u^{\otimes l}) = \mathfrak{C}(k,l)$.

Banica and Speicher noticed in [BS09] that using this result, we can establish the following correspondence:

Categories of partitions	$\xrightarrow{\text{[BS09]}}$	Categories of intertwiners	$\xleftrightarrow{\text{[Wor88]}}$	Compact matrix quantum groups

An important questions in the field of compact quantum groups is to look for examples. The above considerations give us a tool how to construct a lot of them. Finding instances of categories of partitions, we immediately obtain instances of compact matrix quantum groups. However, not every quantum group can be described by a category of partitions. Therefore, another important task is to generalize the concept of partition categories and to construct examples of quantum groups in this more general framework.

This programme started with the work of Banica and Speicher [BS09] describing this correspondence and bringing first examples. This was followed by a series of articles culminating in the complete classification of Banica–Speicher partition categories by Raum and Weber [RW16]. The study of generalizations of partition categories started by introducing coloured partitions by Freslon, Tarrago, and Weber [Fre17, TW17], for which partial classification results are also available. Contributing to this programme constitutes one of the main goals of this thesis.

Main results

The author's original results are presented in Sections 4.5, 4.6, 6.2.3, 6.4 and the whole chapters 5, 7, 8. The text of those parts is based on the articles [Gro18, GW20, GW19a, GW19b]. Let us here briefly summarize the results.

As we already mentioned, the classification of categories of partitions in the sense of the original definition of Banica and Speicher is already done [RW16]. So, we need to study some generalizations of categories of partitions in order to find new examples of quantum groups. We focused on two approaches – (1) colouring of the partitioned points building on preliminary work by [TW17, TW18, Fre17, Fre19] and (2) introducing linear combinations of partitions.

Nevertheless, let us stress here that this thesis does not merely bring new examples of quantum groups. Obtaining classification results for partition categories provides us deep understanding of the structure of representation categories and quantum groups themselves. It motivates us to define new quantum group constructions interpreting these classification results. These findings can be then formulated for any compact matrix quantum groups without the need of referring to partitions.

Global colourization and tensor complexification

Let us start with coloured partitions. There are several possibilities to define a colouring of the partitioned points. One of them is to choose two colours dual to each other. As ordinary non-coloured partitions correspond to orthogonal quantum groups $G \subseteq O_N^+$, partitions with two colours dual to each other allow us to describe unitary quantum groups $G \subseteq U_N^+$. This approach was already studied before and there are some classification results. In particular, the classification of non-crossing two-coloured partitions was obtained by Tarrago and Weber in [TW18]. In Section 4.5, we obtain another classification result, namely the classification of all *globally colourized* categories. In Section 6.2.3, we interpret the results within the theory of compact quantum groups relating it to *tensor complexifications* $H \tilde{\times} \hat{\mathbb{Z}}_k$.

Theorem (4.5.10)**.** Every globally colourized category $\mathscr{C}$ is determined by a number $k(\mathscr{C})$ and a non-coloured category of partitions not containing the singleton partition. The full classification of globally colourized categories is summarized in Table 4.1 (p. 70).

Theorem (6.2.4, 6.2.7)**.** Let $\mathscr{C}$ be a globally colourized category of partitions, denote $k := k(\mathscr{C})$. Denote $H \subseteq O_N^+$ the quantum group corresponding to the non-coloured category of partitions $\langle \mathscr{C}, \sqcap_{\circ\circ} \rangle$. Then $\mathscr{C}$ corresponds to the quantum group $H \tilde{\times} \hat{\mathbb{Z}}_k \subseteq U_N^+$. In the case $k = 0$, we replace $\hat{\mathbb{Z}}_k$ by $\hat{\mathbb{Z}}$. Conversely, given an orthogonal quantum group $H \subseteq O_N^+$ corresponding to some category of partitions, then any quantum group of the form $H \tilde{\times} \hat{\mathbb{Z}}_k$, $k \in \mathbb{N}_0$ is a unitary quantum group corresponding to some globally colourized category of partitions.

In Section 8.2, we generalize the above result studying tensor complexifications without reference to partitions. We define the notion of *globally colourized quantum groups* (Def. 8.2.9) and *degree of reflection of a quantum group* (Def. 8.2.4). The following two theorems generalize the theorem above. The first one aims at characterizing the global colourization in terms of the tensor complexification construction $H \tilde{\times} \hat{\mathbb{Z}}_k$, while the second analyses the tensor complexification in terms of relations, representation categories, and topological generation.

Theorem (8.2.11)**.** Consider $G \subseteq U^+(F)$ with $F\bar{F} = c\,1_N$, $c \in \mathbb{R}$. Then G is globally colourized with zero degree of reflection if and only if $G = H \tilde{\times} \hat{\mathbb{Z}}$, where $H = G \cap O^+(F)$.

Theorem (8.2.13)**.** Consider a compact matrix quantum group $G = (C(G), v)$, $k \in \mathbb{N}_0$. Denote by z the generator of $C^*(\mathbb{Z}_k)$ and by $u := vz$ the fundamental representation of $G \tilde{\times} \hat{\mathbb{Z}}_k$. We have the following characterizations of $G \tilde{\times} \hat{\mathbb{Z}}_k$.

(1) The ideal $I_{G \tilde{\times} \hat{\mathbb{Z}}_k}$ of algebraic relations in $C(G \tilde{\times} \hat{\mathbb{Z}}_k)$ is the $\mathbb{Z}_k$-homogeneous part of the ideal I_G corresponding to G.

(2) The representation category of $G \tilde{\times} \hat{\mathbb{Z}}_k$ looks as follows

$$\mathrm{Mor}(u^{\otimes w_1}, u^{\otimes w_2}) = \begin{cases} \mathrm{Mor}(v^{\otimes w_1}, v^{\otimes w_2}) & \text{if } c(w_2) - c(w_1) \text{ is a multiple of } k, \\ \{0\} & \text{otherwise.} \end{cases}$$

(3) The quantum group $G \tilde{\times} \hat{\mathbb{Z}}_k$ is topologically generated by G and $\hat{\mathbb{Z}}_k$.

Alternating colourization and free complexification

In Section 8.2.6, we study representation categories of the *free complexification* $H \tilde{*} \hat{\mathbb{Z}}_k$. To interpret the result in terms of partitions, we introduce the category $\mathrm{Alt}\,\mathscr{C}$ generated by alternating coloured partitions.

First, we characterize the relations and representation categories of free complexifications.

Theorem (8.2.24)**.** Let H be a compact matrix quantum group with degree of reflection $k \neq 1$. Then all $H \tilde{*} \hat{\mathbb{Z}}_l$ coincide for all $l \in \mathbb{N}_0 \setminus \{1\}$. The ideal $I_{H \tilde{*} \hat{\mathbb{Z}}_l}$ is generated by the alternating polynomials in I_H. The representation category of $H \tilde{*} \hat{\mathbb{Z}}_l$ is a (wide) subcategory of the representation category of H generated by $\mathrm{Mor}(1, v^{\otimes (\circ\bullet)^j})$, $j \in \mathbb{Z}$. This also holds if $k = 1$ and $l = 0$.

Then we show the converse – that any quantum group given by alternating relations is a free complexification.

Theorem (8.2.29)**.** Consider $G \subseteq U^+(F)$ with $F\bar{F} = c\,1_N$. Then G is alternating and invariant with respect to the colour inversion if and only if it is of the form $G = H \mathbin{\tilde{*}} \hat{\mathbb{Z}}$, where $H = G \cap O^+(F)$.

Finally, we reformulate the results for partitions.

Proposition (6.2.8)**.** Let $\mathscr{C} \subseteq \mathscr{P}$ be a category of partitions and denote by $H \subseteq O_N^+$ the corresponding easy quantum group. Then $H \mathbin{\tilde{*}} \hat{\mathbb{Z}}$ is a unitary easy quantum group corresponding to the category $\operatorname{Alt}\mathscr{C}$.

(1) If $\uparrow \notin \mathscr{C}$, then $H \mathbin{\tilde{*}} \hat{\mathbb{Z}}_k = G \mathbin{\tilde{*}} \hat{\mathbb{Z}}$ for all $k \in \mathbb{N}$ and it corresponds to the category $\operatorname{Alt}\mathscr{C}$.
(2) If $\uparrow \in \mathscr{C}$, then $H \mathbin{\tilde{*}} \hat{\mathbb{Z}}_k$ corresponds to the category $\langle \operatorname{Alt}\mathscr{C}, \uparrow^{\otimes k} \rangle$.

Extra singletons and the gluing procedure

In Section 4.6, we introduce another possibility of colouring of partitions, which we call *partitions with extra singletons*. Categories of partitions with extra singletons are designed to describe quantum groups $G \subseteq O_N^+ \mathbin{\hat{*}} \hat{\mathbb{Z}}_2$. This interpretation is presented in Section 6.4. The main result regarding those categories is that their classification is essentially equivalent to the above mentioned two-coloured partition categories. This correspondence also reveals the quantum group interpretation.

Theorem (4.6.8)**.** There is a functor F providing a one-to-one correspondence:

Categories of partitions with extra singletons having an even length $\mathscr{C} \subseteq \mathscr{P}^{\triangle}_{\text{even}}$	$\overset{F}{\longleftrightarrow}$	Categories of two-coloured partitions $\tilde{\mathscr{C}} \subseteq \mathscr{P}^{\circ\bullet}$ that are invariant with respect to the colour inversion

Theorem (6.4.13)**.** Let $\mathscr{C}$ be a category of partitions with extra singletons containing only partitions of even length and $G \subseteq O_N \mathbin{\hat{*}} \hat{\mathbb{Z}}_2$ the associated quantum group. Let $\tilde{\mathscr{C}} := F(\mathscr{C})$ be the corresponding two-coloured category and $\tilde{G} \subseteq U_N^+$ the associated quantum group. Then $\tilde{G}$ is the so-called *glued version* of G.

The glued version can be defined for arbitrary quantum group $G \subseteq O_N^+ \mathbin{\hat{*}} \hat{\mathbb{Z}}_2$ or even for quantum groups $G \subseteq U^+(F) \mathbin{\hat{*}} \hat{\mathbb{Z}}_k$. However, the above theorems suggest that in some cases, we could be able also to reverse this process by defining an *ungluing* procedure. We study this problem in Section 8.4 with the following result

Theorem (8.4.13)**.** There is a one-to-one correspondence provided by gluing and *canonical* $\mathbb{Z}_2$*-ungluing*.

Quantum groups $G \subseteq O^+(F) * \hat{\mathbb{Z}}_2$ with degree of reflection two	$\longleftrightarrow$	Quantum groups $\tilde{G} \subseteq U^+(F)$ that are invariant with respect to the colour inversion

These considerations motivated us to define new $\mathbb{Z}_2$-extensions $H \times_k \hat{\mathbb{Z}}_2$ as $\mathbb{Z}_2$-ungluings of the tensor complexification $H \times \hat{\mathbb{Z}}_k$ in Section 6.4.5. In Section 8.3, we generalize this construction defining new products denoted by $G \mathbin{\hat{\times}} H$, $G \mathbin{\check{\times}} H$, $G \times_{2k} H$. An important result is showing that the relations defining those quantum groups really define something new that lies strictly between the dual free product and the tensor product.

Theorem (8.3.2)**.** Consider quantum groups G, H. Then the products $G \mathbin{\hat{\times}} H$, $G \mathbin{\check{\times}} H$, $G \times_{2k} H$ are indeed well-defined quantum groups. We have the following inclusions

$$G * H \begin{matrix} \supseteq & G \mathbin{\hat{\times}} H & \supseteq \\ \supseteq & G \mathbin{\check{\times}} H & \supseteq \end{matrix} G \times_0 H \supseteq G \times_{2k} G \supseteq G \times_{2l} H \supseteq G \times_2 H = G \times H,$$

where we assume $k, l \in \mathbb{N}$ such that l divides k. The last three inclusions are strict if and only if the degree of reflection of both G and H is different from one.

Linear categories of partitions

Categories of partitions as defined by Banica and Speicher are associated with compact matrix quantum groups G such that $S_N \subseteq G \subseteq O_N^+$. As we already mentioned, not all such quantum groups are described by categories of partitions. If we want to describe the remaining ones by some partition categories, we have to work not only with partitions, but also with their linear combinations. Categories that do not require working with linear combinations are called *easy*, whereas those where linear combinations are needed are called *non-easy*.

Basically nothing was known about linear categories of partitions before the start of this PhD project. So, our first goal was to construct some non-easy examples. In order to do so, we implemented an algorithm that takes a given linear combination of partitions and computes the category it generates. Then we can look whether this category can also be described just by partitions – then it is easy and we already know this category – or whether we need linear combinations. This algorithm always computes just an approximation of the whole category, so it may happen that this test is false positive. However, we obtain at least a list of candidates for new linear categories of partitions. The algorithm is described in Section 5.2, which is followed by Section 5.3 with concrete applications of it and a list of many new candidates for non-easy categories (see also Table 5.1 on page 79).

Our second goal is naturally to interpret the new linear categories of partitions, that is, to find the corresponding quantum groups. Besides finding the quantum groups, the interpretation also serves as a proof that the categories we found are really new and non-easy. Secondly, it allows us to understand the structure of the categories. Applying similar constructions, we can discover even more new non-easy categories. In contrast with the original papers [GW20, GW19a], we decided to separate these considerations. In Section 5.4, we analyse the new categories as such and prove that they are indeed new. In Chapter 7, we give the quantum group interpretation.

To be more concrete, let us summarize the most important results regarding linear categories of partitions. We introduce some linear maps $\mathcal{V}_{(N,\pm)}$ and $\mathcal{P}_{(N)}$ acting on partitions such that given a category $\mathscr{K}$, it might happen that $\mathcal{V}_{(N,\pm)}\mathscr{K}$ or $\langle \mathcal{P}_{(N)}\mathscr{K}\rangle$ are non-easy although $\mathscr{K}$ is easy. Such categories constitute one class of our non-easy examples.

Theorem (7.1.9)**.** Let G be a quantum group with $S_N \subseteq G \subseteq B_N^{\#+}$ corresponding to a category of partitions $\mathscr{K}$. Then the category $\mathcal{V}_{(N,\pm)}\mathscr{K}$ corresponds to the quantum group $V_{(N,\pm)} G V_{(N,\pm)}^*$, where $V_{(N,\pm)}$ is a certain coisometry.

Theorem (7.1.12)**.** Let $\mathscr{K}$ be a category of partitions. Denote by H the quantum group corresponding to the category $\mathcal{V}_{(N,\pm)}\mathscr{K}$. Then we can construct the quantum group corresponding to the following categories:

$$\begin{array}{lll} \langle \mathcal{P}_{(N)}\mathscr{K}\rangle_N & \text{corresponds to} & U_{(N,\pm)}^*(H * \hat{\mathbb{Z}}_2)U_{(N,\pm)}, \\ \langle \mathcal{P}_{(N)}\mathscr{K}, \sqcap\!\!\sqcap\, \rangle_N & \text{corresponds to} & U_{(N,\pm)}^*(H \times \hat{\mathbb{Z}}_2)U_{(N,\pm)}, \\ \langle \mathcal{P}_{(N)}\mathscr{K}, \uparrow\rangle_N & \text{corresponds to} & U_{(N,\pm)}^*(H \times E)U_{(N,\pm)}, \end{array}$$

where $E = (\mathbb{C}, 1)$ is the trivial (quantum) group and $U_{(N,\pm)}$ is some unitary.

As a different example, let us mention some anticommutative twists of the orthogonal group.

Theorem (5.4.6, 5.4.8, 7.4.11, 7.4.13)**.** The following two categories are both non-easy, but isomorphic to the category of all pairings. They correspond to some 2-cycle twists of the orthogonal group O_N.

$$\langle \times - 2\,\text{⌶}\rangle_N, \qquad \langle \times - \frac{2}{N}(\diagup\!\!\!\mid + \mid\!\!\!\diagdown) + \frac{4}{N}\,\mid\,\mid\rangle\rangle_N$$

Structure and notation

The thesis is divided into three parts – preliminaries, partition categories, and results on quantum groups. The preliminary part contains all the essential foundations of C*-algebras, quantum groups and monoidal categories. The aim is to make the thesis self-contained and provide a clear summary of the essential results. We recommend some reading in the introductory text of the corresponding chapters.

The second part of the thesis is on purpose written without any reference to quantum groups. We believe that this viewpoint can make this area interesting also for researchers from different fields of mathematics such as category theory or combinatorics. This part contains two chapters – Chapter 4 about categories of partitions without the linear structure, which is essentially pure combinatorics, and Chapter 5 on linear categories of partitions.

The last part applies the results on partition categories to the theory of quantum groups. Chapter 6 describes the basic correspondence between partitions and quantum groups and focuses on easy quantum groups. Chapter 7 then analyses the non-easy categories of partitions from the quantum group perspective. Lastly, Chapter 8 generalizes some of the previous results eliminating the usage of categories of partitions completely.

The thesis is concluded by Chapter 9, where we discuss some open problems and potential directions for further research.

Finally, let us have some remarks on the notation. All new terms that are being defined are highlighted with **boldface**. Most of them are listed in the index at the end of the thesis. The preliminary definitions are usually contained in text, while original and not commonly known definitions are usually mentioned in numbered paragraphs. Mathematical notation may also be introduced either in the text or in numbered paragraphs. For an easier orientation, we provide a thorough list of the notation on page xii. The bibliography is contained at the end of the thesis. For an easier orientation within the references, we divide them into *primary sources*, i.e. original research articles, and *secondary sources* such as textbooks, monographs or surveys. While primary sources are cited using normal brackets (such as [Gro18]), the secondary sources are cited using double brackets (such as ⟦Tim08⟧).

Part I

Preliminaries

In the preliminary part, we give a brief introduction to the theory of compact quantum groups and summarize all the necessary results from the theory of C*-algebras, quantum groups and monoidal categories. We assume that the reader has a basic knowledge of algebra and functional analysis.

Chapter 1
C*-algebras

The underlying structure for quantum groups are not groups, but algebras. The idea is that we work with analogues of algebras of certain functions defined on groups. Compact quantum groups are conveniently described using C*-algebras representing the algebras of continuous functions. This chapter summarizes all the necessary elements of the theory of C*-algebras. In particular, we review the so-called *Gelfand duality*, which plays a big role from the philosophical perspective showing that our approach makes sense, and the construction of universal C*-algebras, which plays a big role from the practical perspective since this will be our tool to construct compact quantum groups. The main references for this chapter are the books ⟦Mur90, Bla06, Tak79⟧ and short summaries provided in ⟦Tim08, Web17⟧.

1.1 Basic definitions around C*-algebras

In this section, we summarize the basic definitions regarding normed algebras. In particular, we introduce the notion of a C*-algebra.

A vector space A together with a bilinear map $A \times A \to A$ called and denoted as multiplication $(a,b) \mapsto ab$ is called an **algebra**. In this work, we will always assume that the multiplication is associative, so A is also a ring.

A number of properties of algebras are defined in a natural way. If the multiplication has a neutral element, it is called the **unit** and denoted by 1 or 1_A. A subset $B \subseteq A$ is called a **subalgebra** if it is an algebra with respect to the same operations. A subalgebra invariant with respect to left/right/both multiplication(s) with elements of A is called a left/right/two-sided **ideal**. A two-sided ideal will be called just *ideal*.

An **involution** on an algebra A is an antilinear mapping $*\colon A \to A$ satisfying

$$(a^*)^* = a, \quad (ab)^* = b^*a^* \quad \text{for all } a \in A.$$

An algebra with involution is called an **involutive algebra** or a **$*$-algebra**. Similarly we define **$*$-subalgebra** and **$*$-ideal**.

The space of bounded operators on some Hilbert space H will be denoted by $\mathscr{B}(H)$. The composition of operators defines a multiplication on this vector space and Hermitian conjugation defines there an involution. Thus, it is a $*$-algebra.

We adopt many notions from bounded operators to general $*$-algebras. An element a^* is called the **adjoint** to a. An element a is called

- **self-adjoint** if $a^* = a$,
- **normal** if $aa^* = a^*a$,
- **projection** if $a^* = a = a^2$,
- **isometry** if $a^*a = 1$,
- **unitary** if $a^*a = aa^* = 1$.

For algebras A, B, a map $\varphi\colon A \to B$ preserving the algebraic structure is called a **homomorphism**. If it is bijective, it is called an **isomorphism**. If A and B are $*$-algebras and φ preserves the involution, it is called a $*$-homomorphism.

A **($*$-)representation** of a ($*$-)algebra is a ($*$-)homomorphism $\pi\colon A \to \mathscr{B}(H)$. It is **faithful** if it is injective. It is called **irreducible** if H does not contain any non-trivial invariant subspace with respect to operators in $\pi(A)$. A vector $x \in H$ is called **cyclic vector** of π if the set $\pi(A)x$ is dense in H. Two representations $\pi_j\colon A \to \mathscr{B}(H_j)$ for $j = 1,2$ are **(unitarily) equivalent** if there is a (unitary) bijection $U \in \mathscr{B}(H_1, H_2)$ such that $\pi_2(a)U = U\pi_1(a)$ for all $a \in A$.

An algebra A equipped with a norm $\|\cdot\|$ is called a **normed algebra** if the norm is **submultiplicative**, i.e.

$$\|ab\| \le \|a\|\|b\| \quad \text{for all } a, b \in A.$$

A normed algebra A with a unit 1_A is called a **unital normed algebra** if $\|1_A\| = 1$. A complete (unital) normed algebra is called a **(unital) Banach algebra**.

A **(unital) Banach $*$-algebra** A is a (unital) Banach algebra equipped with an involution $*$ satisfying $\|a^*\| = \|a\|$ for all $a \in A$. A **C*-algebra** is a Banach $*$-algebra satisfying

$$\|aa^*\| = \|a\|^2 \quad \text{for all } a \in A. \tag{1.1}$$

Any norm or seminorm satisfying the condition (1.1) is called a **C*-norm** or **C*-seminorm**.

1.1.1 Theorem. Let A be a Banach algebra and B a C*-algebra. Then every $*$-homomorphism $\varphi: A \to B$ is norm-decreasing, i.e. $\|\varphi(a)\| \le \|a\|$.

As a simple corollary, we have that every $*$-homomorphism of C*-algebras is continuous and every $*$-isomorphism of C*-algebras is an isometry.

An element of a C*-algebra $a \in A$ is called **positive** if it is of the form $a = b^*b$ for some $b \in A$.

A linear functional f on a $*$-algebra A is called **positive** if $f(a^*a) \ge 0$ for all $a \in A$. If A is a Banach $*$-algebra and a positive functional f on A satisfies $\|f\| = 1$, then it is called a **state** on A. A state is called **faithful** if $f(a^*a) \ne 0$ for every $a \ne 0$.

1.2 Basic results on C*-algebras

In this section, we summarize the fundamental results of operator algebras that we are going to use in this thesis. Namely the Gelfand duality, the GNS construction, and the double commutant theorem.

1.2.1 Algebra of continuous functions

Let us now turn to the special case of abelian C*-algebras. A canonical example here is the algebra of continuous functions $C(X)$ defined on some compact topological space X together with the supremum norm. Or, more generally, the algebra $C_0(X)$ of continuous functions vanishing at infinity over a locally compact space X.

One of the very important results of mathematical analysis is the Stone–Weierstrass theorem, which can be formulated as follows.

Let A be an algebra of functions $f: X \to \mathbb{C}$ for some set X. We say that A **separates points** in X if, for all $x \in X$, there are two functions $f, g \in A$ such that $f(x) \ne g(x)$. We say that A **vanishes nowhere** on X if, for all $x \in X$, there is a function $f \in A$ such that $f(x) \ne 0$.

1.2.1 Theorem (Stone–Weierstrass)**.** Let X be a compact Hausdorff space. Consider a $*$-subalgebra $A \subseteq C(X)$. If A separates points of X and vanishes nowhere in X, then A is dense in $C(X)$.

A second important fact is so-called *Gelfand duality*, which says that all abelian C*-algebras are isomorphic to algebras of continuous functions.

Let A be an abelian algebra. A **character** on A is a non-zero homomorphism $\tau: A \to \mathbb{C}$. We denote by $\Omega(A)$ the set of characters on A. We consider the relative weak* topology on $\Omega(A)$.

1.2.2 Theorem. If A is an abelian Banach algebra, then $\Omega(A)$ is a locally compact Hausdorff space. If, in addition, A is unital, then $\Omega(A)$ is compact.

For any $a \in A$, we can define a function $\hat{a}: \Omega(A) \to \mathbb{C}$ mapping $\tau \mapsto \tau(a)$. The function $\hat{a}$ is called the **Gelfand transform** of a. It holds that $\hat{a}$ is continuous and vanishes at infinity. The map $\varphi: A \to C_0(\Omega(A))$ mapping $a \mapsto \hat{a}$ is called the **Gelfand representation**.

1.2.3 Theorem (Gelfand duality)**.** Let A be an abelian Banach algebra such that $\Omega(A)$ is non-empty. Then the Gelfand representation is a norm-decreasing homomorphism. If A is a C*-algebra, then the Gelfand representation is an isometric $*$-isomorphism. That is, every abelian C*-algebra A is isomorphic to the algebra of continuous functions $C(X)$, where $X = \Omega(A)$.

1.2.4 Theorem. Let X, Y be compact Hausdorff spaces. Then there is a bijection between continuous maps $\Phi\colon X \to Y$ and unital $*$-homomorphisms $\hat{\Phi}\colon C(Y) \to C(X)$ defined as $\hat{\Phi}(f) = f \circ \Phi$. The homomorphism $\hat{\Phi}$ is injective, resp. surjective if and only if Φ is surjective, resp. injective.

To summarize, Gelfand duality states that there are the following mutual correspondences:

Compact topological spaces X	$\xleftrightarrow{A=C(X)}$	Commutative C*-algebras A
Injective/surjective continuous maps $\Phi\colon X \to Y$	$\xleftrightarrow{\hat{\Phi}(f)=f\circ\Phi}$	Surjective/injective unital $*$-homomorphisms $\hat{\Phi}\colon C(Y) \to C(X)$

1.2.2 GNS construction

This section describes a way how to construct representation of C*-algebras formulated by Gelfand, Naimark, and Segal.

1.2.5 Theorem (GNS construction)**.** Let A be a unital Banach $*$-algebra and f a positive functional on A. Then there exists a Hilbert space H and a representation $\pi\colon A \to \mathscr{B}(H)$ with a cyclic vector x_0 such that for all $a \in A$ we have $f(a) = \langle x_0, \pi(a)x_0\rangle$. Any other representation π' satisfying this property is unitarily equivalent to π.

If f is a faithful state, then we have $\|\pi(a)x_0\| = \langle x_0, \pi(a)^*\pi(a)x_0\rangle = f(a^*a) \neq 0$ for every $a \neq 0$, so the GNS representation is faithful.

A general GNS representation need not be faithful. However, for C*-algebras, there is always a faithful representation. Denote by $\pi_f\colon A \to \mathscr{B}(H_f)$ the GNS-representation corresponding to functional f on A. We can compute the direct sum of all representations $\pi := \bigoplus_{f\geq 0} \pi_f$, which maps $A \to \mathscr{B}(H)$, where $H = \bigoplus_{f\geq 0} H_f$. This representation is called **universal**. It can be shown that it is faithful.

1.2.6 Theorem (Gelfand–Naimark)**.** Every C*-algebra has a faithful representation. In particular, its universal representation is faithful.

This theorem provides an alternative concrete definition of C*-algebras. A C*-algebra is a norm-closed $*$-subalgebra $A \subseteq \mathscr{B}(H)$ for some Hilbert space H. The definition provided in Section 1.1 can be seen as an abstract axiomatization of this operator-algebraic concept.

1.2.3 Double commutant theorem

In this section, we will formulate the finite-dimensional version of so-called double commutant theorem. The general infinite-dimensional version of the theorem constitutes the basis of the theory of so-called von Neumann algebras. Nevertheless, we will not use those in our thesis and the finite-dimensional version will be sufficient for our purposes.

Let $A \subseteq M_n(\mathbb{C})$ be a set of matrices (or, more generally, a set of bounded operator on some Hilbert space). We define the **commutant** of A to be the set

$$A' := \{S \in M_n(\mathbb{C}) \mid TS = ST \text{ for all } T \in A\}.$$

It is easy to see that A' is actually an algebra. If A is closed with respect to the involution, then A' is a $*$-algebra. Finally, it is worth noticing that

$$B \subseteq A \quad \Rightarrow \quad A' \subseteq B'.$$

Now, let us formulate the theorem:

1.2.7 Theorem (double commutant theorem)**.** Let $A \subseteq M_n(\mathbb{C})$ be a $*$-algebra of matrices containing the identity matrix. Then $A'' = A$.

As a consequence, taking $*$-algebras $A, B \subseteq M_n(\mathbb{C})$ containing the identity, we can formulate the above observation as an equivalence:

$$B \subseteq A \quad \Leftrightarrow \quad A' \subseteq B'.$$

1.3 Constructing C*-algebras

We discuss several ways of constructing C*-algebras.

1.3.1 Quotients

Let A be a ($*$-)algebra and $I \subseteq A$ a ($*$-)ideal. Then we can define the quotient ($*$-)algebra A/I.

1.3.1 Theorem. Let I be a closed ideal in a normed algebra A. Then the quotient A/I is a normed algebra with respect to the norm

$$\|a\| = \inf_{b \in I} \|a - b\|.$$

If A is a Banach algebra or a C*-algebra, then A/I is a Banach algebra resp. C*-algebra.

1.3.2 Direct sums

Let $\{A_i\}_{i \in \Omega}$ be a set of Banach algebras. We define the **direct sum** of the Banach algebras $\{A_i\}$ as the direct sum of Banach spaces

$$\bigoplus_{i \in \Omega} A_i := \left\{ (a_i)_{i \in \Omega} \mid a_i \in A_i,\ \|(a_i)\|_\infty := \sup_{i \in \Omega} \|a_i\| < \infty \right\}$$

together with pointwise multiplication and involution $(a_\lambda)(b_\lambda) = (a_\lambda b_\lambda)$, $(a_\lambda)^* = (a_\lambda^*)$.

It is clear that if the system $\{A_\lambda\}$ consists of C*-algebras, then their direct sum is also a C*-algebra.

1.3.3 Tensor products

Now, let us have a look on a tensor product of C*-algebras. Here, the situation is a bit complicated since there are more possibilities how to define a C*-norm on the tensor product.

Let A_1, A_2 be $*$-algebras. The **algebraic tensor product** of A_1 and A_2 will be denoted by $A_1 \odot A_2$. It is defined as the algebraic tensor product of the vector spaces A_1 and A_2 together with multiplication and involution given as

$$(a_1 \otimes a_2)(b_1 \otimes b_2) = (a_1 b_1 \otimes a_2 b_2), \quad (a_1 \otimes a_2)^* = a_1^* \otimes a_2^*.$$

Given two $*$-homomorphisms $\varphi_1 : A_1 \to B_1$ and $\varphi_2 : A_2 \to B_2$, we can define their tensor product $\varphi_1 \otimes \varphi_2 : A_1 \odot A_2 \to B_1 \odot B_2$ simply by $(\varphi_1 \otimes \varphi_2)(a_1 \otimes a_2) = \varphi_1(a_1) \otimes \varphi_2(a_2)$. If the homomorphisms are injective, then also their tensor product is injective.

In particular, for any two C*-algebras A_1 and A_2, we can consider their (faithful) representation $\pi_i : A_i \to \mathscr{B}(H_i)$ and construct a (faithful) representation $\pi_1 \otimes \pi_2$ of the algebraic tensor product $A_1 \odot A_2$.

Consequently, there exists at least one C*-norm on the algebraic tensor product $A_1 \odot A_2$ defined as $\|x\| = \|(\pi_1 \otimes \pi_2)(x)\|$, where π_1 and π_2 are some faithful representation of A_1 and A_2. We define the **minimal norm** also known as the **spatial norm** by

$$\|x\|_{\min} := \sup\{\|(\pi_1 \otimes \pi_2)(x)\| \mid \pi_i : A_i \to \mathscr{B}(H_i) \text{ a representation}\}.$$

The completion of $A \odot B$ with respect to this norm is called the **minimal tensor product** and denoted by $A \otimes_{\min} B$. Actually, it indeed holds that this C*-norm is the minimal one on $A_1 \odot A_2$; therefore, it is equal to $\|(\pi_1 \otimes \pi_2)(x)\|$ for any pair of faithful representations.

1.3.2 Proposition. Let A_1, A_2, B_1, and B_2 be C*-algebras. Let $\varphi_1: A_1 \to B_1$ and $\varphi_2: A_2 \to B_2$ be homomorphisms. Then $\varphi_1 \otimes \varphi_2: A_1 \odot A_2 \to B_1 \odot B_2$ can be extended to a homomorphism $A_1 \otimes_{\min} A_2 \to B_1 \otimes_{\min} B_2$.

For a general C*-norm γ on $A_1 \odot A_2$, we denote by $A_1 \otimes_\gamma A_2$ the completion of $A_1 \odot A_2$ with respect to this norm.

1.3.3 Theorem. Let A_1 and A_2 be C*-algebras, γ a C*-seminorm on $A_1 \odot A_2$. Then

$$\gamma(a_1 \otimes a_2) \leq \|a_1\| \|a_2\|.$$

We define the **maximal norm** on $A_1 \odot A_2$ as

$$\|x\|_{\max} := \sup\{\gamma(a_1 \otimes a_2) \mid \gamma \text{ a C*-norm on } A_1 \odot A_2\}.$$

This indeed defines a C*-norm thanks to the preceding theorem as $\|a_1 \otimes a_2\|_{\max} \leq \|a_1\| \|a_2\| < \infty$. Again, we denote the completion of the algebraic tensor product by $A \otimes_{\max} B$ and call it the **maximal tensor product**.

A C*-algebra A is called **nuclear** if, for every C*-algebra B, there is a unique C*-norm on $A \odot B$. If A is nuclear, we write just $A \otimes B := A \otimes_{\min} B = A \otimes_{\max} B$ for the completion with respect to the unique C*-norm.

One particular example is the algebra of matrices $M_n(\mathbb{C})$, which is nuclear and it holds that for any C*-algebra A we have

$$M_n(\mathbb{C}) \otimes A = M_n(\mathbb{C}) \odot A \simeq M_n(A),$$

where $M_n(A)$ is the algebra of matrices with entries in A.

1.3.4 Theorem (Takesaki)**.** Every abelian C*-algebra is nuclear.

In particular, we have that

$$C(X) \otimes C(Y) \simeq C(X \times Y) \tag{1.2}$$

for any compact spaces X, Y.

1.3.4 Universal C*-algebras

The idea behind the construction of universal C*-algebras is that we want to define a C*-algebra by a set of *generators* and *relations*. For many algebraic structures, this is the most basic and also very simple construction. For instance, constructing *universal ∗-algebras*, we need to first construct a *free ∗-algebra* and then *quotient out* the desired relations. That is, take a quotient by a ∗-ideal generated by those relations.

To be more precise, let $E = \{x_i\}_{i\in\Omega}$ be a set of symbols. The ∗-algebra $\mathbb{C}\langle x_i, x_i^*\rangle_{i\in\Omega}$ of non-commutative polynomials in variables x_i and x_i^* together with involution defined as $(x_i)^* = x_i^*$ is called the **free ∗-algebra** generated by $E = \{x_i\}$.

Let $R = \{p_j\}_{j\in\Lambda} \subseteq \mathbb{C}\langle x_i, x_i^*\rangle$ be a set of polynomials and let I be the ideal generated by this set. Then we define the **universal ∗-algebra** generated by E and R, i.e. by generators x_i and relations $p_j(x_i, x_i^*) = 0$, as the quotient

$$*\text{-}(E \mid R) = *\text{-}\Big(x_i \mid p_j(x_i, x_i^*) = 0\Big) := \mathbb{C}\langle x_i, x_i^*\rangle / I. \tag{1.3}$$

Universal ∗-algebras have by definition the following *universal property*. Let B be a ∗-algebra generated by some elements $\{\tilde{x}_i\}_{i\in\Omega} \subseteq B$ and suppose that those elements satisfy all the relations in R, so $p(\tilde{x}_i, \tilde{x}_i^*) = 0$ in B. Then there exists a surjective $*$-homomorphism $\varphi\colon A \to B$ mapping $x_i \mapsto \tilde{x}_i$.

For C*-algebras, the situation is a bit more complicated since we do not only have the algebraic structure, but also the analytic structure – the C*-norm. We first need to construct the corresponding universal $*$-algebra A_0, then find an appropriate C*-norm here and finally construct a completion A of the algebra A_0 with respect to this form. To preserve also the universal property, the C*-norm should be the maximal one.

Consider a $*$-algebra A_0. For every $a \in A_0$, let us define

$$\|a\| := \sup\{\|\pi(a)\| \mid \pi \text{ is a representation of } A_0\} = \sup\{\|a\|_\gamma \mid \gamma \text{ is a C*-seminorm on } A_0\}.$$

Suppose that $\|a\| < \infty$ for all $a \in A$. Then $\|\cdot\|$ is a C*-seminorm on A_0. Thus, it induces a C*-norm on the quotient algebra A_0/I, where I is the ideal of A generated by $a \in A$ such that $\|a\| = 0$. Then the completion of A_0/I with respect to the C*-norm is called the **enveloping C*-algebra** of A_0 and denoted by $C^*(A_0)$.

By definition, the enveloping algebra has the following universal property. Denote by q the natural map $A_0 \to C^*(A_0)$. Let φ be a $*$-homomorphism $A_0 \to B$, where B is a C*-algebra. Then there exists a unique $*$-homomorphism $\psi\colon C^*(A_0) \to B$, such that $\varphi = \psi \circ q$.

An example of such a construction is the maximal tensor product: $A \otimes_{\max} B = C^*(A \odot B)$.

Let $E = \{x_i\}_{i\in\Omega}$ be a set of symbols, $R = \{p_j\}_{j\in\Lambda} \subseteq \mathbb{C}\langle x_i, x_i^*\rangle$ set of relations, $I \subseteq \mathbb{C}\langle x_i, x_i^*\rangle$ the $*$-ideal generated by R. Denote by $A_0 := \mathbb{C}\langle x_i, x_i^*\rangle/I = *\text{-}(E \mid R)$ the universal $*$-algebra generated by E and R. Then we define the **universal C*-algebra** generated by E and R, i.e. by generators x_i and relations $p_j(x_i, x_i^*) = 0$, as the enveloping algebra

$$C^*(E \mid R) = C^*\Big(x_i \mid p_j(x_i, x_i^*) = 0\Big) := C^*(A_0) \tag{1.4}$$

if this enveloping algebra exists.

Now, two problems may occur in the last step constructing the C*-envelope. First of all, the seminorm we construct may not be a norm. As we said, this then requires to quotient out all the elements $a \in A_0$ with $\|a\| = 0$. This should, however, not be seen as an actual problem, but rather as a natural consequence of enriching the algebraic structure. For instance, the relation $xx^* = 0$ does not imply $x = 0$ for $*$-algebras, but it obviously must imply $x = 0$ for C*-algebras as $\|x\|^2 = \|xx^*\| = 0$. Second problem, which may occur, is that the norm is not bounded. That is, $\|a\| = \infty$ for some $a \in A_0$, so we again do not obtain a proper C*-norm. This is, in contrast, an actual problem. Some relations simply cannot be realized within C*-algebras. A classical example is the relation $xy - yx = 1$, which has no faithful representation by bounded operators.

We can make the construction of universal C*-algebras even more general. Instead of relations of the form $p_j(x_i, x_i^*) = 0$, we can consider the following ones

$$\left\|p_j(x_i, x_i^*)\right\| \leq \eta_j.$$

So, consider $R = \{(p_j, \eta_j)\}_{j\in\Lambda}$, where $p_j \in A$ and $\eta_j \in \mathbb{R}^+$. A $*$-homomorphism $\pi\colon A \to \mathscr{B}(H)$, where H is some Hilbert space is called a **representation** of the pair $(E \mid R)$ if all the relations $\left\|\pi(p_j(x_i, x_i^*))\right\| \leq \eta_j$ are satisfied. Again, we can define

$$\|a\| := \sup\{\|\pi(a)\| \mid \pi \text{ is a representation of } (E \mid R)\}$$

for all $a \in A$. Let I be the ideal generated by all $a \in A$ such that $\|a\| = 0$. If we have $\|a\| < \infty$ for all $a \in A$, then we can define the universal C*-algebra generated by E and R again using equation (1.4).

1.3.5 Free product

An important application of the universal C*-algebra construction is the so called *free product*. Let A, B be C*-algebras. The **free product** $A * B$ is a universal algebra generated by A and B with no other relations. So, we take $E = \{x_a\}_{a\in A} \cup \{y_b\}_{b\in B}$ as the generators and define

$$A * B := C^*(x_a, y_b;\ a \in A, b \in B \mid x_{a_1}x_{a_2} = x_{a_1a_2},\ y_{b_1}y_{b_2} = y_{b_1b_2}).$$

If A and B are unital, we define the **unital free product** $A *_{\mathbb{C}} B$ by identifying the unities

$$A *_{\mathbb{C}} B := C^*(x_a, y_b;\ a \in A, b \in B \mid x_{a_1}x_{a_2} = x_{a_1a_2},\ y_{b_1}y_{b_2} = y_{b_1b_2},\ x_1 = y_1).$$

In general, if D is a C*-algebra embedded into both A and B via some $*$-homomorphisms $\varphi\colon D \to A$, $\psi\colon D \to B$, we can define the **amalgamated free product** as

$$A *_D B := C^*(x_a, y_b;\ a \in A, b \in B \mid x_{a_1}x_{a_2} = x_{a_1a_2},\ y_{b_1}y_{b_2} = y_{b_1b_2},\ x_{\varphi(d)} = y_{\psi(d)}).$$

Note that the maximal tensor product can also be constructed in a similar way. We have

$$A \otimes_{\max} B = C^*(x_a, y_b;\ a \in A, b \in B \mid x_{a_1}x_{a_2} = x_{a_1a_2},\ y_{b_1}y_{b_2} = y_{b_1b_2},\ x_a y_b = y_b x_a).$$

We can also define the *free product of $*$-algebras*, which has basically the same definition except that we use universal $*$-algebras instead of universal C*-algebras. The free product of C*-algebras can be then understood as the C*-envelope of the algebraic free product. There is also a notion of a *reduced* free product. The idea is that having C*-algebras A_1, A_2 equipped with faithful states φ_1, φ_2, we are able to construct a free product $(A, \varphi) = (A_1, \varphi_1) * (A_2, \varphi_2)$, where A is a C*-algebra being a completion of the algebraic free product $A_1 * A_2$ with respect to some alternative C*-norm that allow to extend φ_1 and φ_2 to a faithful state φ on A. This construction is very important especially for the theory of free probability. See [[Dyk16]] for more details.

1.3.6 C*-algebras associated to discrete groups

Now we present an important class of algebras and C*-algebras, namely the group (C*-)algebras.

Let Γ be a discrete group. A $*$-algebra of finite formal linear combinations of elements in Γ together with the multiplication being a linear extension of the group multiplication and the involution defined by taking the group inverse is called the **group algebra** and denoted by $\mathbb{C}\Gamma$. One can view this algebra as the universal $*$-algebra generated by some elements u_g, $g \in \Gamma$ together with relations $u_g^* u_g = u_g u_g^* = 1$ and $u_g u_h = u_{gh}$.

An alternative approach to define this algebra is to take the set of all finitely supported functions $f\colon \Gamma \to \mathbb{C}$ and define their product by convolution

$$(f_1 * f_2)(g) := \sum_{h\in\Gamma} f_1(h) f_2(h^{-1}g), \qquad f^*(g) := \overline{f(g^{-1})}.$$

Indeed, such an algebra is obviously generated by the delta functions $u_g(h) = \delta_{g,h}$ being one only if $h = g$ and zero otherwise. The convolution then satisfies $u_g * u_h = u_{gh}$ and the involution $u_g^* = u_{g^{-1}}$.

The **full group C*-algebra** $C^*(\Gamma)$ is then defined as the C*-envelope of $\mathbb{C}\Gamma$. That is,

$$C^*(\Gamma) := C^*(u_g,\ g \in \Gamma \mid u_g^* u_g = u_g u_g^* = 1,\ u_g u_h = u_{gh}).$$

Consider now the Hilbert space $l^2(\Gamma)$ with orthonormal basis $\{\delta_g\}_{g\in\Gamma}$. There is a **left regular representation** λ of the group Γ acting on $l^2(\Gamma)$ again simply by group multiplication or convolution $\lambda_g \delta_h := \delta_{gh}$. The **reduced group C*-algebra** $C^*_{\mathrm{r}}(\Gamma)$ is then defined as the closed linear span of the maps $\lambda_g \in \mathscr{B}(l^2(\Gamma))$.

Equivalently, $C^*_{\mathrm{r}}(\Gamma)$ is the completion of $\mathbb{C}\Gamma$ with respect to the C*-norm defined as

$$\|x\|_{\mathrm{r}} := \sup_{\|y\|_2 = 1} \left\|xy\right\|_2,$$

where $\|\cdot\|_2$ is the l^2-norm.

Note that there are indeed examples of discrete groups, where the full and the reduced C*-algebras do not coincide. For instance, every discrete group containing a free subgroup on two generators has this property. This question is closely connected to the notion of *amenability*.

A group Γ is **amenable** if there exists a state μ on $l^\infty(\Gamma)$ (the C*-algebra of bounded functions on Γ) that is invariant under the left translation action. That is, $\mu(sf) = \mu(f)$ for all $s \in \Gamma$ and $f \in l^\infty(\Gamma)$. There are very many equivalent statements characterizing the amenability. Let us mention two of them. A discrete group Γ is amenable if and only if the full and the reduced group C*-algebras coincide $C^*(\Gamma) = C^*_{\mathrm{r}}(\Gamma)$. This holds if and only if $C^*_{\mathrm{r}}(\Gamma)$ is nuclear. For more information, see [[BO08]].

Chapter 2
Compact quantum groups

In this chapter, we provide a brief introduction to the theory of compact quantum groups. We focus mainly on compact *matrix* quantum groups. We try to explain the most basic facts in detail; however, more advanced theorems are presented without a proof.

The mathematical structure that is often used to describe quantum groups is the so-called *Hopf algebra*. The term *quantum group* was first used by Drinfeld to name some particular examples of Hopf algebras, which he constructed as deformations of some Lie groups. Although the notion of a quantum group is nowadays widely used among mathematicians, it still does not have any universal definition. In this thesis, we will work with the notion of *compact quantum groups*. Compact quantum groups, in contrast with plain quantum groups, have a commonly accepted clear definition within the framework of C*-algebras. They form a straightforward generalization of compact groups sharing many important properties with them regarding the representation theory or the Haar integration.

Although it is clear, what is a compact quantum group, there are still many different approaches, how to describe this structure. For this thesis, the oldest approach of *compact matrix quantum groups* developed by Woronowicz in [Wor87] will be the central one (we formulate the definition in Sect. 2.1.2). As the name suggest, compact matrix quantum groups generalize compact matrix groups. Later, Woronowicz came up with a straightforward generalization of this concept [Wor98] – the compact quantum groups as we mentioned above (Definition in Sect. 2.1.1). However, in the meantime also several other approaches were developed by other authors. In particular, Dijkhuizen and Koornwinder [DK94] showed that compact (matrix) quantum groups can be equivalently described using Hopf algebras instead of C*-algebras (we will formulate this correspondence in Sect. 2.3.1). Another approach is to generalize discrete groups instead of compact groups as those structures are supposed to be dual to each other through the Pontryagin duality. This started with the work [PW90] and was further developed in [ER94, VDa96] (we formulate basics of this duality in Sect. 2.4). Finally, a further generalization, which is possible, but very hard to formulate, are locally compact quantum groups, whose foundation were set in [KV00] (we will not touch those in the thesis at all).

Let us mention some literature that serve as a reference for this section as well as a recommendation for a reader who is not familiar with the concept of compact quantum groups. In the first place, we we would like to mention the lecture notes ⟦Web17, Fre19⟧, whose aim is the same as ours – introducing compact quantum groups and their connection with partition categories. Compact quantum groups from the perspective of representation categories is also the topic of the monograph ⟦NT13⟧. Compact quantum groups from slightly broader perspective are presented in the textbook ⟦Tim08⟧. Finally, we also refer to another survey on compact quantum groups ⟦MVD98⟧.

2.1 Definitions and examples

There are several different approaches of how to define (compact) quantum groups. We present them in this section. The way, how the different definitions relate to each other, is explained in the subsequent text. For our work, the second approach of *compact matrix quantum groups* will play the key role.

2.1.1 Compact quantum groups

As described in Section 1.2.1, there is a one-to-one correspondence between compact topological spaces X and commutative C*-algebras consisting of continuous functions over those spaces.

In this sense, we can understand non-commutative C*-algebras as a description of some *non-commutative topological spaces*. Such a non-commutative algebra would again play the role of the algebra of continuous functions over this non-commutative space.

An important property of topological spaces are their symmetries described by groups. In order to describe symmetries of a non-commutative topology, we need to generalize the concept of a group.

A **compact quantum group** is a pair $G = (A, \Delta)$, where A is a C*-algebra and Δ is a unital $*$-homomorphism mapping $A \to A \otimes_{\min} A$ called **comultiplication**, which is *coassociative*, i.e.

$$(\Delta \otimes \mathrm{id}) \circ \Delta = (\mathrm{id} \otimes \Delta) \circ \Delta,$$

and satisfies the so called *cancellation property*, i.e. the spaces

$$\Delta(A)(1 \otimes A) = \operatorname{span}\{\Delta(a)(1 \otimes b) \mid a, b \in A\},$$
$$\Delta(A)(A \otimes 1) = \operatorname{span}\{\Delta(a)(b \otimes 1) \mid a, b \in A\}$$

are both dense in $A \otimes_{\min} A$.

Such a definition was first introduced by Woronowicz in [Wor98].

Compact quantum groups indeed generalize compact groups since, as we are going to show in Proposition 2.1.2, there is a one-to-one correspondence between groups and quantum groups with underlying commutative C*-algebras. Here, by a **compact group** we mean a group with Hausdorff topology such that the multiplication and taking inverses are continuous maps. To prove this correspondence, we first need the following lemma.

2.1.1 Lemma. A compact semigroup with the cancellation property is a group.

Proof. Consider a compact semigroup G and an element $h \in G$. Let H be the closed subsemigroup generated by h. Since it is generated by just one element, it is abelian. Considering two ideals I_1 and I_2 of H, their intersection must be non-empty since $I_1 \cap I_2 \supseteq I_1 I_2 \neq \emptyset$. Thus, since G is compact, the intersection of all ideals of H must be non-empty. Let us denote it by $I \neq \emptyset$. Consider an element $i \in I$. Since I is an ideal, we have $iI \subseteq I$. Since it is the minimal one, we have an equality. Therefore, there must exist an element $e \in I$ such that $ie = i$. Now, we can multiply this equality by any element $g \in G$ from the right and then cancel i from the left to obtain $eg = g$ for every $g \in G$. Afterwards, we can do the same from the left with some $g' \in G$ to see that $g'e = g'$ for every $g' \in G$. Thus, e is the group identity of G. This also proves that $h = he \in I$ (actually $I = H$) and since I is the minimal ideal of H, we have $hI = I$. In particular, there exists an element $k \in I$ such that $hk = e$, which is the inverse of h.

Since G is compact, the continuity of the multiplication already implies the continuity of taking inversions. Indeed, consider the map $G \times G \to G \times G$ mapping $(g, h) \mapsto (g, gh)$ which is continuous and bijective. From compactness, it follows that also the inverse $(g, h) \mapsto (g, g^{-1}h)$ is continuous, so the assignment $g \mapsto g^{-1}$ must also be continuous. □

2.1.2 Proposition. For every compact group G, we can define a unital $*$-homomorphism $\Delta\colon C(G) \to C(G) \otimes C(G) \simeq C(G \times G)$ as

$$\Delta(f)(x, y) = f(xy) \quad \text{for all } f \in C(G) \text{ and } x, y \in G. \tag{2.1}$$

Then $(C(G), \Delta)$ forms a quantum group. Conversely, for every quantum group (A, Δ), where A is commutative, there is a group G such that $A \simeq C(G)$ and Δ is given by equation (2.1).

Proof. Consider a group G and let us prove that Δ is a comultiplication. The fact that Δ is a unital $*$-homomorphism is easily checked directly. Alternatively, it follows from Theorem 1.2.4

since Δ acts as a pullback by the group multiplication $\Delta = \hat{m}$. The coassociativity follows from the associativity of the group multiplication: Denote $\Delta(f) =: \sum_n g_n \otimes h_n$. Then

$$((\Delta \otimes \mathrm{id}) \circ \Delta)(f)(x,y,z) = \sum_n (\Delta(g_n) \otimes h_n)(x,y,z) = \sum_n g_n(xy)h_n(z) = \Delta(f)(xy,z) = f(xyz)$$

Similarly, we can show that $((\mathrm{id} \otimes \Delta) \circ \Delta)(f)(x,y,z) = f(xyz)$.

To show that the space $\Delta(C(G))(1 \otimes C(G))$ is dense in $C(G) \otimes C(G) \simeq C(G \times G)$, we use the Stone–Weierstrass theorem (Thm. 1.2.1). We can easily see that it is a $*$-subalgebra and it contains the constant function, so it is nowhere vanishing. It consists of linear combinations of functions of the form $(x,y) \mapsto f(xy)g(y)$. The fact that it separates points in $G \times G$ follows from the fact that $C(G) \odot C(G)$ consisting of functions $(z,y) \mapsto f(z)g(y)$ is also dense in $C(G \times G)$ and that since G is a group, we have $(x_1y_1, y_1) \neq (x_2y_2, y_2)$ if and only if $(x_1, y_1) \neq (x_2, y_2)$.

Similarly, we can show that $\Delta(C(G))(C(G) \otimes 1)$ is dense in $C(G) \otimes C(G)$.

Now, consider a quantum group (A, Δ) with commutative A. According to Gelfand theorem (Thm. 1.2.3), there is a compact topological space G, such that $A = C(G)$. According to Theorem 1.2.4, the comultiplication $\Delta: C(G) \to C(G) \otimes C(G) \simeq C(G \times G)$ defines a continuous map $m: G \times G \to G$ satisfying $\Delta(f)(x,y) = f(m(x,y))$. Using this theorem we can also see that the map m defines an associative multiplication, i.e. $m(m(x,y),z) = m(x, m(y,z))$ if and only if

$$\Delta(f)(m(x,y),z) = f(m(m(x,y),z)) = f(m(x,m(y,z))) = \Delta(f)(x,m(y,z)),$$

which is, as we indicated in the beginning of the proof, equivalent to the coassociativity condition.

Thus, G is a compact semigroup such that $A = C(G)$ with multiplication satisfying equation (2.1). Now, it suffices to prove that G is a group.

We are going to prove that it has the cancellation property, i.e., for all $x_1, x_2, y \in G$, we have that $x_1y = x_2y$ implies $x_1 = x_2$ and $yx_1 = yx_2$ implies $x_1 = x_2$. Let us prove the first one, the second is proven similarly. It is equivalent to the statement that the map $(x,y) \mapsto (xy,y)$ is injective. According to Theorem 1.2.4, this is equivalent to the statement that the map $C(G \times G) \to C(G \times G)$ defined as

$$f \mapsto f', \quad f'(x,y) = f(xy,y)$$

is surjective. This is true thanks to the quantum group cancellation property stating that $\Delta(A)(1 \otimes A)$, which consists of linear combinations of functions of the form $g(xy)h(y)$, $g,h \in C(G)$, is dense in $C(G \times G)$. □

In the spirit of what was mentioned in the beginning of this section, we should interpret A as the space of continuous functions over some possibly non-commutative topological space G. Therefore, we denote $C(G) := A$ even in the case when A is not commutative.

A (C*-)algebra equipped with a comultiplication Δ is called **cocommutative** if $\Delta = \tau \circ \Delta$, where τ is a homomorphism swapping the tensor factors. It is easy to see that a group G is abelian (commutative) if and only if $C(G)$ is cocommutative. In the field of quantum groups, the term *commutativity* usually refers to the underlying algebra. Stating that some (quantum) group is or is not commutative might hence be a bit confusing. Therefore, we will avoid using the term commutativity when referring to (quantum) groups. We will say that a quantum group *is a group* if $C(G)$ is commutative. We will say that a (quantum) group is **abelian** if $C(G)$ is cocommutative.

2.1.2 Compact matrix quantum groups

An important class of groups are matrix groups, i.e. subgroups of the group of all invertible matrices GL_N for some N. A quantum counterpart was defined by Woronowicz in [Wor87] even before the general definition of compact quantum groups.

Let A be a C*-algebra, $u_{ij} \in A$ for $i, j \in \{1, \ldots, N\}$ for some $N \in \mathbb{N}$. Denote $u := (u_{ij})_{i,j=1}^{N} \in M_n(A)$. The pair (A, u) is called a **compact matrix quantum group** (abbreviated CMQG) if

(1) the elements u_{ij}, $i, j = 1, \ldots, N$, generate A,
(2) the matrices u and u^t are invertible,
(3) the assignment

$$u_{ij} \mapsto \sum_{k=1}^{n} u_{ik} \otimes u_{kj}$$

extends to a $*$-homomorphism $\Delta: A \to A \otimes_{\min} A$ called the **comultiplication**.

The matrix u is called the **fundamental representation** of the CMQG.

Note that we should prove that the pair (A, Δ) is indeed a quantum group. This will be proven as Theorem 2.2.6.

Note also that compact matrix groups can indeed be treated as a special case of compact matrix quantum groups. Let $G \subseteq \mathrm{GL}_N$ be a compact matrix group. Take $A := C(G)$ the algebra of continuous functions over G and define u_{ij} to be the functions assigning each matrix $g \in G$ its (i, j)-th element $u_{ij}(g) = g_{ij}$. Using Stone–Weierstrass theorem (Thm. 1.2.1) one can easily show that u_{ij} generate A. The matrix u (and hence u^t) is invertible since $u^{-1}(g) = u(g^{-1})$. Finally, the map Δ coincides with the comultiplication on G as a quantum group since

$$\Delta(u_{ij})(g, h) = u_{ij}(gh) = \sum_{k} g_{ik} h_{kj} = \sum_{k} u_{ik}(g) u_{kj}(h) = \left(\sum_{k} u_{ik} \otimes u_{kj} \right)(g, h).$$

2.1.3 Examples. One of the most important examples are the quantum generalizations of the orthogonal and unitary group. The orthogonal group can be described as

$$O_N = \{B \in M_N(\mathbb{C}) \mid B_{ij} = \bar{B}_{ij}, BB^t = B^t B = 1\}.$$

So, it can be treated also as a compact matrix quantum group $(C(O_N), u)$, where $C(O_N)$ can be described as a universal C*-algebra

$$C(O_N) = C^*(u_{ij},\ i, j = 1, \ldots, N \mid u_{ij} = u_{ij}^*, uu^t = u^t u = 1, u_{ij} u_{kl} = u_{kl} u_{ij}).$$

Such an algebra can be *quantized* by dropping the commutativity relation. This was done by Wang in [Wan95a] and the result $O_N^+ = (C(O_N^+), u)$ with

$$C(O_N^+) := C^*(u_{ij},\ i, j = 1, \ldots, N \mid u_{ij} = u_{ij}^*, uu^t = u^t u = 1)$$

is called the **free orthogonal quantum group**.

In the same article, Wang also defined the **free unitary quantum group**

$$C(U_N^+) := C^*(u_{ij},\ i, j = 1, \ldots, N \mid u,\ u^t \text{ unitary}).$$

Here, by u being unitary, we mean the condition $uu^* = u^* u = 1$, where $u^* = (u_{ji}^*)$.

Finally, let us mention the quantization of the symmetric group S_N. This group can be also treated as a matrix group, where each permutation is represented by a permutation matrix consisting of *zeros* and *ones* such that in each row and column there is precisely one entry equal to *one* and the rest are *zeros*. In [Wan98] Wang defined the **free symmetric quantum group** as a compact matrix quantum group S_N^+ with

$$C(S_N^+) = C^*(u_{ij},\ i, j = 1, \ldots, N \mid u_{ij}^2 = u_{ij} = u_{ij}^*,\ \sum_{k} u_{ik} = \sum_{k} u_{kj} = 1),$$

while

$$C(S_N) = C^*(u_{ij},\ i, j = 1, \ldots, N \mid u_{ij}^2 = u_{ij} = u_{ij}^*,\ \sum_{k} u_{ik} = \sum_{k} u_{kj} = 1,\ u_{ij} u_{kl} = u_{kl} u_{ij}).$$

2.1.3 Hopf algebras

The last approach to quantum groups presented in this thesis is more algebraic. In order to describe compact quantum groups, we may use the theory of Hopf algebras. Hopf algebras were first formally defined by Cartier in [Car56] under the name *hyperalgebras*. The precise connection with compact quantum groups will be described in Section 2.3.1.

A **Hopf $*$-algebra** is a unital $*$-algebra A equipped with the following maps.

(a) A unital $*$-homomorphism $\Delta: A \to A \odot A$ called the **comultiplication** satisfying the *coassociativity* condition

$$(\Delta \otimes \mathrm{id}) \circ \Delta = (\mathrm{id} \otimes \Delta) \circ \Delta.$$

(b) A linear map $\varepsilon: A \to \mathbb{C}$ called the **counit** satisfying

$$(\varepsilon \otimes \mathrm{id}) \circ \Delta = \mathrm{id} = (\mathrm{id} \otimes \varepsilon) \circ \Delta.$$

(c) A linear map $S: A \to A$ called the **antipode** satisfying

$$m \circ (S \otimes \mathrm{id}) \circ \Delta = \eta \circ \varepsilon = m \circ (\mathrm{id} \otimes S) \circ \Delta.$$

Here, we denote by $m: A \odot A \to A$ the multiplication on A and by $\eta: \mathbb{C} \to A$ the embedding $\lambda \mapsto \lambda \cdot 1_A$. It actually automatically holds that ε is a $*$-homomorphism and S is an antihomomorphism.

As the names of the operations suggest, they dualize the group multiplication, the group identity, and the group inverse. To be more concrete, let G be a matrix group. Denote by $O(G)$ the $*$-algebra generated by the coordinate functions $u_{ij}(g) = g_{ij}$ and the coordinates of the matrix inverse $u'_{ij}(g) := [g^{-1}]_{ij}$. Then $O(G)$ is a Hopf $*$-algebra with respect to the following operations.

$$\begin{aligned} (\Delta(f))(x,y) &= f(xy), &\quad \text{that is,} \quad & \Delta(u_{ij}) = \sum_k u_{ik} \otimes u_{kj}, \\ \varepsilon(f) &= f(e), &\quad \text{that is,} \quad & \varepsilon(u_{ij}) = \delta_{ij}, \\ (S(f))(x) &= f(x^{-1}), &\quad \text{that is,} \quad & S(u_{ij}) = u'_{ij}. \end{aligned}$$

Let us stress that Hopf algebras are not the same thing as compact quantum groups. In a sense they are more general. We are lacking the analytic structure here, so they are rather generalizing any groups, not only the compact ones. But as we mentioned at the beginning of this chapter, not even the term *quantum group* is used as a synonym for Hopf algebras. Quantum groups are usually considered to be some special Hopf algebras that are in some sense close to groups.

As we already mentioned, we are going to describe the connection between compact quantum groups and Hopf algebras in Section 2.3.1. In particular, we are going to show how to express the *compactness* in terms of Hopf algebras.

Regarding the comultiplication, let us also mention here the *Sweedler notation*, which is very common especially in the literature on Hopf algebras. Considering a tensor product of algebras or vector spaces, every element can be expressed as some linear combination (possibly infinite in the case of C*-algebras) of elementary tensors. In particular, consider a Hopf algebra A and an element $a \in A$. Then it might be useful to express the comultiplication $\Delta(a) \in A \odot A$ as a linear combination $\Delta(a) = \sum_{i=1}^n b_i \otimes c_i$, where b_i, c_i are some elements of A (we already did this in the proof of Proposition 2.1.2). In order to simplify the notation, we write just

$$\Delta(a) =: \sum_{(a)} a_{(1)} \otimes a_{(2)},$$

where the symbols $a_{(1)}$ and $a_{(2)}$ stand for the elements b_i and c_i as we just described. Note that some authors even omit the summation sign. The coassociativity axiom then allows us also to introduce this for repeated application of Δ such as

$$(\Delta \otimes \mathrm{id})(\Delta(a)) = (\mathrm{id} \otimes \Delta)(\Delta(a)) =: \sum_{(a)} a_{(1)} \otimes a_{(2)} \otimes a_{(3)}.$$

As an application, we can formulate the cocommutativity condition introduced at the end of Section 2.1.1 for Hopf algebras. A Hopf algebra A is cocommutative if and only if $\sum_{(x)} x_{(1)} \otimes x_{(2)} = \sum_{(x)} x_{(2)} \otimes x_{(1)}$. The only place in this thesis, where we use the Sweedler notation is Section 7.4.

2.2 Representation theory and fundamental properties

Compact groups are distinguished among groups by several fundamental properties they satisfy. In particular, we mean the invariant *Haar* integration and the results regarding their representation theory. Those properties are shared also by compact quantum groups as we are going to describe in this section. All those properties were obtained by Woronowicz already in the first articles [Wor87, Wor98], where he introduced compact matrix quantum groups, resp. general compact quantum groups.

We would also like to mention the article [[MVD98]]. Its character is rather expository and does not contain many new results, but it was the first survey on the newly developed topic of compact quantum groups, which brought more order to the theory and provided more convenient proofs of some results.

2.2.1 Representations of quantum groups

In this section, we are going to define the concept of a *representation* for quantum groups.

For a classical compact group G, an n-dimensional representation is a continuous homomorphism $\varphi\colon G \to M_n(\mathbb{C})$. It can be represented by functions $u_{ij} \in C(G)$ mapping elements $g \in G$ to the (i,j)-th entry of their representation matrices $u_{ij}(g) = [\varphi(g)]_{ij}$. The homomorphism property $\varphi(gh) = \varphi(g)\varphi(h)$ can be written as

$$\Delta(u_{ij})(g,h) = u_{ij}(gh) = \sum_k \varphi(g)_{ik}\,\varphi(h)_{kj} = \sum_k u_{ik}(g)u_{kj}(h) = \left(\sum_k u_{ik} \otimes u_{kj}\right)(g,h), \tag{2.2}$$

where Δ is the comultiplication on $C(G)$. This motivates the following definition.

Let A be a unital $*$-algebra equipped with a unital $*$-homomorphism $\Delta\colon A \to A \odot A$ (for example, a Hopf $*$-algebra). An ***n*-dimensional corepresentation** of A is a matrix $u \in M_n(A)$ such that

$$\Delta(u_{ij}) = \sum_k u_{ik} \otimes u_{kj}.$$

The same definition works for a unital C*-algebra with $\Delta\colon A \to A \otimes_{\min} A$. In particular, if $G = (A,\Delta)$ is a compact quantum group, we call u a **representation** of G. (Following the idea that the quantum group G should be the *dual* object to $C(G) = A$.)

If a representation u has a matrix inverse, we call it a **non-degenerate** representation; if it is unitary, we call it a **unitary** representation.

Let $u \in M_n(A)$ and $v \in M_m(A)$. We can consider the direct sum of matrices $u \oplus v \in M_{n+m}(A)$, the matrix tensor product (meaning the Kronecker product) $u \otimes v \in M_{nm}(A)$ or the complex conjugate $\bar{u} = (u_{ij}^*) \in M_n(A)$. It is easy to check that if u and v are representations of some compact quantum group (A,Δ), then those operations define new representations of (A,Δ).

Let $u \in M_n(A)$ and $v \in M_m(A)$ be representations of a compact quantum group (A,Δ). A linear map $T\colon \mathbb{C}^n \to \mathbb{C}^m$ is called an **intertwiner** if $Tu = vT$. The space of all such maps is denoted by $\mathrm{Mor}(u,v)$. The representations u and v are called **(unitarily) equivalent** if there exists an invertible (unitary) operator T intertwining those representations.

Any corepresentation $u \in M_n(A)$ indeed *coacts* on the space $\mathbb{C}^n$ in the following sense. For any $x \in \mathbb{C}^n$ performing the matrix multiplication, we get $ux \in \mathbb{C}^n \otimes A$. So, u can be thought of as a map $\mathbb{C}^n \to \mathbb{C}^n \otimes A$ or an element of $u \in M_n(\mathbb{C}) \otimes A$. In general, a corepresentation on a vector space V can be described by a map $V \to V \otimes A$.

A subspace $V \subseteq \mathbb{C}^n$ is called **invariant** if for all $x \in V$ we have $ux \in V \otimes A$. Equivalently, we can say that V is invariant if $uP = PuP$, where P is the projection $\mathbb{C}^n \to V$. A representation $u \in M_n(A)$ is called **irreducible** if the only invariant subspaces are 0 and $\mathbb{C}^n$.

For a compact quantum group G, we denote by $\mathrm{Irr}\, G$ the set of classes of irreducible representations of G up to equivalence. For $\alpha \in \mathrm{Irr}\, G$, we denote by u^α some unitary representative of the class α. We denote by n_α the size of the representation, so $u^\alpha \in M_{n_\alpha}(C(G))$.

2.2.2 Haar state

Another feature of compact groups that can be generalized to the quantum world is the so called *Haar integration*. Let us first recall the classical case. For any locally compact group G, there exists an (up to scalar multiplication) unique measure μ that is *left-invariant*, which means that $\mu(tS) = \mu(S)$ for any Borel set $S \subseteq G$ and any element $t \in G$. Equivalently, we may write this condition as

$$\int_G f(st)\,\mathrm{d}\mu(s) = \int_G f(s)\,\mathrm{d}\mu(s)$$

for every $f \in C(G)$ and every $t \in G$. If G is compact, we may express the integration with respect to the Haar measure as a positive linear functional $h\colon C(G) \to \mathbb{C}$ mapping $f \mapsto \int_G f\,\mathrm{d}\mu$. The left-invariance can be then expressed as

$$(h \otimes \mathrm{id})(\Delta(f))(t) = h(f),$$

that is,

$$(h \otimes \mathrm{id})(\Delta(f)) = h(f)\,1_{C(G)}.$$

In addition, compact groups are known to be *unimodular*, which means that the Haar measure is also right-invariant.

This is generalized to compact quantum groups in the following theorem.

2.2.1 Theorem. For any compact quantum group G, there is a unique state h on $C(G)$ such that

$$(h \otimes \mathrm{id})(\Delta(a)) = h(a)\,1_{C(G)}, \qquad (\mathrm{id} \otimes h)(\Delta(a)) = h(a)\,1_{C(G)}$$

for every $a \in C(G)$. This state is called the **Haar state** on G.

2.2.3 Fundamental results of the representation theory

The following statements generalize the fundamental results from the representation theory of compact groups.

2.2.2 Proposition (Schur's lemma)**.** Let u and v be (unitary) irreducible representations of some quantum group. Then either u and v are (unitarily) equivalent and $\mathrm{Mor}(u,v)$ is one-dimensional or $\mathrm{Mor}(u,v) = 0$.

Proof. One can check that the kernel resp. image of any intertwiner $T \in \mathrm{Mor}(u,v)$ are invariant subspaces of u, resp. v. Hence, if u and v are irreducible, any non-zero intertwiner T is invertible. Suppose now, we have a second intertwiner $S \in \mathrm{Mor}(u,v)$. Then $\det(S - \lambda T)$, considered as a polynomial in λ, must have a zero at some λ_0. Since $S-\lambda_0 T$ is a singular intertwiner, it must be zero, so $S=\lambda_0 T$. This proves that the dimension of $\mathrm{Mor}(u,v)$ is either one or zero.

Finally, if u and v are unitary and $\mathrm{Mor}(u,v)\ni T\neq 0$, then $T^*\in\mathrm{Mor}(v,u)$. So, $T^*T\in\mathrm{Mor}(u,u)$ and $TT^*\in\mathrm{Mor}(v,v)$ and by what was already proven, both must be multiples of the identity. Therefore, T is unitary up to scaling. □

Schur's lemma is actually valid for any group, not only for the compact ones. Also our version is valid for any Hopf algebra, not only for compact quantum groups. The following statements, however, already use the compactness.

2.2.3 Proposition. Every non-degenerate representation is equivalent to a unitary representation.

Proof. Let $u\in M_n(C(G))$ be a non-degenerate representation of a compact quantum group G. Denote by h the Haar state on G and define $Q_{ij}:=h([u^*u]_{ij})=\sum_k h(u^*_{ki}u_{kj})$. Since u is invertible, we have that u^*u is positive definite and hence $Q\in M_n(\mathbb{C})$ is positive definite. Now, using the left-invariance of the Haar state, we have

$$Q_{ij}=\sum_k h(u^*_{ki}u_{kj})=\sum_k (h\otimes\mathrm{id})\Delta(u^*_{ki}u_{kj})=\sum_{k,l,m} h(u^*_{kl}u_{km})u^*_{li}u_{mj}=\sum_{l,m} Q_{lm}u^*_{li}u_{mj}=[u^*Qu]_{ij}.$$

Finally, we can define $v:=Q^{1/2}uQ^{-1/2}$ an equivalent representation to u, which is unitary since $v^*v=Q^{-1/2}u^*QuQ^{-1/2}=1_N$. □

2.2.4 Theorem. Every representation of a compact quantum group is equivalent to a direct sum of irreducible ones.

Proof. Let G be a compact quantum group and $u\in M_n(C(G))$ its representation. From Proposition 2.2.3, we can assume that u is unitary. Suppose $V\subseteq\mathbb{C}^n$ is an invariant subspace of u. Then we can choose an orthonormal basis for V and complete it to an orthonormal basis for $\mathbb{C}^n$. Let U be the corresponding unitary transformation matrix $\mathbb{C}^n\to\mathbb{C}^n$ such that $U(V)=\mathbb{C}^m\subseteq\mathbb{C}^n$. Then since $\mathbb{C}^m$ is invariant subspace of $v:=UuU^*$, it must be of the block form

$$v=\begin{pmatrix} v_{11} & v_{12}\\ 0 & v_{22}\end{pmatrix}.$$

But since v is unitary, we must have $v_{12}=0$, so $u=v_{11}\oplus v_{22}$. Since one-dimensional representations are always irreducible, the statement follows by mathematical induction. □

Finally, we present the proof of the fact that compact matrix quantum groups are compact quantum groups. This is based on the following lemma.

2.2.5 Lemma. [[MVD98]] Let A be a unital C*-algebra and let $\Delta\colon A\to A\otimes_{\min}A$ be a unital $*$-homomorphism. If A, as a Banach algebra, is generated by the matrix elements of non-degenerate finite-dimensional corepresentations, then $G=(A,\Delta)$ is a compact quantum group.

Proof. First, we prove the coassociativity. It is sufficient to prove it for the generators, for which we have

$$((\Delta\otimes\mathrm{id})\circ\Delta)(u_{ij})=\sum_{k,l}u_{ik}\otimes u_{kl}\otimes u_{lj}=((\mathrm{id}\otimes\Delta)\circ\Delta)(u_{ij}).$$

Now, let us prove that $\Delta(A)(1\otimes A)$ is dense in $A\otimes_{\min}A$. Take u a non-degenerate finite-dimensional representation and denote $v:=u^{-1}$ its inverse (as a matrix). Then

$$\sum_j \Delta(u_{ij})(1\otimes v_{jl})=\sum_{j,k}u_{ik}\otimes u_{kj}v_{jl}=\sum_k u_{ik}\otimes\delta_{kl}=u_{il}\otimes 1\in\Delta(A)(1\otimes A).$$

Now, consider $a \otimes 1, b \otimes 1 \in \Delta(A)(1 \otimes A)$. Denote

$$a \otimes 1 =: \sum_\alpha \Delta(a_{1,\alpha})(1 \otimes a_{2,\alpha}), \quad b \otimes 1 =: \sum_\beta \Delta(b_{1,\beta})(1 \otimes b_{2,\beta}).$$

Then,

$$ab \otimes 1 = \sum_\alpha \Delta(a_{1,\alpha})(1 \otimes a_{2,\alpha})(b \otimes 1) = \sum_\alpha \Delta(a_{1,\alpha})(b \otimes 1)(1 \otimes a_{2,\alpha}) = \\ = \sum_{\alpha,\beta} \Delta(a_{1,\alpha})\Delta(b_{1,\beta})(1 \otimes b_{2,\beta})(1 \otimes a_{1,\alpha}) = \sum_{\alpha,\beta} \Delta(a_{1,\alpha} b_{1,\beta})(1 \otimes b_{2,\beta} a_{1,\alpha}),$$

so $ab \otimes 1 \in \Delta(A)(1 \otimes A)$.

Denote by A_0 the algebra generated by the matrix elements of non-degenerate finite-dimensional representations, so $\bar{A}_0 = A$. We have proven that $A_0 \odot A = (A_0 \otimes 1)(1 \otimes A) \subseteq \Delta(A)(1 \otimes A)$. Thus, $A \otimes_{\min} A = \overline{A_0 \odot A} = \overline{\Delta(A)(1 \otimes A)}$.

The second cancellation property is proven similarly. □

2.2.6 Theorem. Every compact matrix quantum group is a compact quantum group.

Proof. Let (A, u) be a CMQG. The fundamental representation u is obviously a representation. As we already mentioned, any representation $u = (u_{ij})$ induces a representation $\bar{u} = (u_{ij}^*)$. Since u^t is invertible, it follows that $\bar{u} = (u^t)^*$ is invertible. Since A is, as a C*-algebra, generated by the elements u_{ij}, it follows that A, as a Banach algebra, is generated by the elements u_{ij} and u_{ij}^*. Thus the previous theorem applies. □

2.3 Further definitions and properties

2.3.1 Various algebras associated to quantum groups

In this section, we explain the link between compact quantum groups and Hopf algebras. Then we define additional C*-algebras associated to a given compact quantum group.

Let G be a compact quantum group. We define $\operatorname{Pol} G$ to be the span of matrix coefficients of all representations of G. It is actually a unital $*$-algebra since, for any two representations u and v, the element $u_{ij} v_{kl}^*$ is a matrix element of $u \otimes \bar{v}$. Moreover, the comultiplication on $C(G)$ restricts to $\operatorname{Pol} G$ since $\Delta(u_{ij}) = \sum_k u_{ik} \otimes u_{kj} \subseteq \operatorname{Pol} G \odot \operatorname{Pol} G$. In addition, we can define a counit and an antipode by

$$\varepsilon(u_{ij}) = \delta_{ij}, \qquad S(u_{ij}) = (u^{-1})_{ij}$$

to turn it into a Hopf $*$-algebra.

2.3.1 Proposition. The set $\{u_{ij}^\alpha \mid \alpha \in \operatorname{Irr} G;\ i, j = 1, \ldots, n_\alpha\}$, forms a vector space basis of $\operatorname{Pol} G$.

Proof. From Theorem 2.2.4 we have that every representation is a direct sum of irreducibles, so it is a generating set. From the Schur's lemma (Prop. 2.2.2), it follows that this set is linearly independent. □

As we already mentioned, Hopf algebras provide an alternative way for describing quantum groups. We make this idea more precise in the following. A Hopf $*$-algebra A is called **compact** if it is spanned by the matrix entries of its finite-dimensional unitary corepresentations.

2.3.2 Theorem. $\operatorname{Pol} G$ is a unique dense compact Hopf $*$-algebra in $C(G)$. The Haar state is faithful on $\operatorname{Pol} G$.

We leave this theorem without proof. See ⟦Tim08, Theorem 5.4.1⟧.

2.3.3 Theorem. Let A_0 be a compact Hopf algebra. Then there exists a compact quantum group G such that $A_0 = \operatorname{Pol} G$.

Proof. We provide only parts of the proof. We are going to construct $C(G)$ as the universal C*-envelope of A_0. So, first of all it is necessary to show that the universal envelope $C^*(A_0)$ exists. This follows from the fact that the generators of A_0 being matrix entries of unitary matrices can have norm at most one. Indeed, let u be a corepresentation of A_0, then unitarity of u means in particular that $\sum_k u_{ik} u_{ik}^* = 1$. Since all the summands are positive, it follows that $u_{ik} u_{ik}^* \le 1$ for every k and hence $\|u_{ik}\|^2 = \|u_{ik} u_{ik}^*\| \le 1$ for any C*-seminorm $\|\cdot\|$ on A_0.

Now, we can also extend the comultiplication of A_0 to the C*-completion $A := C^*(A_0)$ and obtain $\Delta\colon A \to A \otimes_{\min} A$. Then it follows from Lemma 2.2.5 that $G = (A, \Delta_{\mathrm{u}})$ is a compact quantum group.

The most complicated part, which we are going to skip here, is to show that A_0 has a *faithful* $*$-representation on some Hilbert space and hence A_0 is contained in A. See ⟦Tim08, Theorem 5.4.3⟧ or ⟦NT13, Theorem 1.6.7⟧ for a complete proof.

Consequently, since every corepresentation of A_0 is a representation of G, we also have $A_0 \subseteq \operatorname{Pol} G$. The fact that $A_0 = \operatorname{Pol} G$ then follows from the uniqueness in Theorem 2.3.2. □

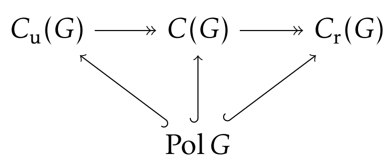
For a compact matrix quantum group G, we can denote by $O(G)$ the $*$-subalgebra of $C(G)$ generated by the matrix entries of the fundamental representation u_{ij}. By definition of a compact matrix quantum group, we have that $O(G) \subseteq \operatorname{Pol} G$ is dense in $C(G)$. It is actually easy to see that the operations from $\operatorname{Pol} G$ restrict to $O(G)$ and hence $O(G)$ is a Hopf $*$-algebra. From uniqueness in Theorem 2.3.2, it follows that $O(G) = \operatorname{Pol} G$.

Now, let G be an arbitrary compact quantum group. We denote by $C_{\mathrm{u}}(G) := C^*(\operatorname{Pol} G)$ the C*-envelope of $\operatorname{Pol} G$ and call it the **full C*-algebra** of G. As follows from the proof of Theorem 2.3.3, this defines a compact quantum group $(C_{\mathrm{u}}(G), \Delta_{\mathrm{u}})$ called the **full version** of G. Note that by the universal property of the C*-envelope, the counit ε extends to a $*$-homomorphism $\varepsilon\colon C_{\mathrm{u}}(G) \to \mathbb{C}$.

Consider π the GNS representation of the C*-algebra $C(G)$ with respect to the Haar state h and denote by $L^2(G)$ the corresponding Hilbert space. Since h is faithful on $\operatorname{Pol} G$, we have that π represents faithfully the $*$-algebra $\operatorname{Pol} G$ on $\mathscr{B}(L^2(G))$.

We denote by $C_{\mathrm{r}}(G) := \pi(C(G)) \subseteq \mathscr{B}(L^2(G))$ the image of $C(G)$ by the GNS-representation associated to the Haar state. We can define a comultiplication on $C_{\mathrm{r}}(G)$ by $\Delta_{\mathrm{r}}(\pi(a)) := (\pi \otimes \pi)(\Delta(a))$. It has to be checked that this is a correct definition by showing $\Delta_{\mathrm{r}}(\ker \pi) \subseteq \ker(\pi \otimes \pi)$. One can also prove that $h_{\mathrm{r}}(\pi(a)) := h(a)$ defines a faithful Haar state on $C_{\mathrm{r}}(G)$. The C*-algebra $C_{\mathrm{r}}(G)$ is called the **reduced C*-algebra** of G and the quantum group $(C_{\mathrm{r}}(G), \Delta_{\mathrm{r}})$ is called the **reduced version** of G.

To summarize, we have the following algebras associated to a given quantum group $G = (C(G), \Delta)$.



As a side remark, note that we can also define the algebra $L^\infty(G)$ as the enveloping von Neumann algebra of $C_{\mathrm{r}}(G)$.

Strictly speaking, the quantum group G, its full version, and its reduced version may be different quantum groups. However, this viewpoint turns out not to be the right one. Note, for example, that all those quantum groups have exactly the same representation theory since this depends only on the dense Hopf $*$-algebra $\operatorname{Pol} G$. We should rather view the different versions of G as different possibilities how to describe the same abstract object – the single compact quantum group G. The various C*-algebras are then just various ways how to construct a C*-completion of the Hopf $*$-algebra $\operatorname{Pol} G$.

This is analogical to the situation with discrete groups, where a single group Γ is associated several different group C*-algebras. Following this analogy, we say that a compact quantum group G is **coamenable** if the surjections in the diagram are isomorphisms, so $C_{\mathrm{u}}(G) = C(G) = C_{\mathrm{r}}(G)$. This holds, for example, for all compact groups, but not in general for compact quantum groups. The analogy with discrete groups becomes even more fitting in Section 2.4, where we interpret the algebras $C_{\mathrm{u}}(G)$ and $C_{\mathrm{r}}(G)$ as group C*-algebras of the discrete dual quantum group $\Gamma = \hat{G}$. Let us mention an alternative characterization of coamenability from [BMT01].

2.3.4 Theorem. A compact quantum group G is coamenable if and only if the Haar state h is faithful on $C(G)$ and the counit ε extends to a bounded functional on $C(G)$. (Note that the first condition holds automatically if G is in its reduced version and the second one holds automatically if G is in its full version.)

It may seem that introducing compact quantum groups using the theory of C*-algebras is an unnecessary complication since we can simply work with the associated Hopf $*$-algebras. Although it is enough to use Hopf algebras for defining compact quantum groups, it may be convenient to consider the C*-algebras to study them. Similarly as in the case of discrete groups, where we certainly do not need C*-algebras to define them, but the associated C*-algebras are useful to study discrete groups (e.g. their amenability). In addition, some may consider the Hopf algebraic approach less satisfying since we must put into the definition the fact that the algebra is spanned by the coefficients of the finite-dimensional unitary representations (equivalently, the existence of the Haar state). In the C*-algebraic approach, one obtains these results as properties (as in the classical case), which may seem more natural.

The situation gets a bit simpler in the matrix case, where we require the algebra to be generated by the coefficients of the fundamental representation anyway. So, we can formulate the following alternative definition: A compact matrix quantum group G is a pair (A, u), where A is a Hopf $*$-algebra (usually denoted $O(G)$ or $\operatorname{Pol} G$) and u is a unitarizable matrix with coefficients in A such that

$$\Delta(u_{ij}) = \sum_k u_{ik} \otimes u_{kj}, \qquad \varepsilon(u_{ij}) = \delta_{ij}, \qquad S(u_{ij}) = (u^{-1})_{ij}.$$

However, we will stick to the C*-algebraic notation from Section 2.1.2 in our thesis. Moreover, all compact quantum groups will appear in the full version (that is, we assume $C(G) = C_{\mathrm{u}}(G)$). Nevertheless, a reader that is not familiar with C*-algebras can think of $\operatorname{Pol} G$ instead of $C(G)$ and obtain an analogous purely algebraic statement in most cases.

Finally, let us mention some references for the statements presented in this section. The fact that every compact (matrix) quantum group induces a dense Hopf $*$-algebra was mentioned already by Woronowicz in the original works [Wor87, Wor98]. The algebraic approach was formulated in the paper [DK94], which also contains a detailed comparison of their approach with other authors (in particular, with Woronowicz). The definition of the full and reduced C*-algebras associated with quantum groups was first mentioned in [BS93].

2.3.2 Morphisms of quantum groups, quantum subgroups and quotients

As we just mentioned, given a compact quantum group G, the C*-algebras $C(G)$, $C_{\mathrm{r}}(G)$, and $C_{\mathrm{u}}(G)$ as well as the Hopf $*$-algebra $\operatorname{Pol} G$ should be seen as equivalent descriptions of a single quantum group G. In this spirit, we define the notion of *isomorphism* for compact quantum groups. Closely connected are also the notions of *quantum subgroup* and *quotient quantum group*, which we also define below. Those notions were originally introduced by Wang in [Wan95a]. In [Wan97], Wang remarked that those notions should be modified to respect the fact that a single quantum group can be represented by different (non-isomorphic) algebras.

Let G and H be compact quantum groups. We say that they are **isomorphic**, denoted by $G \simeq H$, if there is a $*$-isomorphism $\varphi\colon \mathrm{Pol}\, G \to \mathrm{Pol}\, H$ (equivalently $\varphi\colon C_{\mathrm{u}}(G) \to C_{\mathrm{u}}(H)$) satisfying $(\varphi \otimes \varphi) \circ \Delta_G = \Delta_H \circ \varphi$. We say that H is an **(embedded) quantum subgroup** of G if there exists a surjective $*$-homomorphism $\mathrm{Pol}\, G \to \mathrm{Pol}\, H$ satisfying the same property.

For matrix groups, in contrast with abstract groups, it makes in addition also sense to ask, whether they are *equal*, that is, represented by the same matrices. We can generalize this notion also to the quantum group case.

Let G and H be compact matrix quantum groups with fundamental representations $u \in M_N(C(G))$ and $v \in M_M(C(H))$. We consider them to be **identical**, denoted by $G = H$, if $N = M$ and the map $u_{ij} \mapsto v_{ij}$ extends to a $*$-isomorphism $\mathrm{Pol}\, G \to \mathrm{Pol}\, H$. They are called **similar** if $N = M$ and there exists an invertible matrix $T \in M_N(\mathbb{C})$ such that $u_{ij} \mapsto [TvT^{-1}]_{ij}$ extends to a $*$-isomorphism. Then we write $G = THT^{-1}$. The quantum group H is called a **quantum subgroup** of G, denoted by $H \subseteq G$, if $N = M$ and $u_{ij} \mapsto v_{ij}$ extends to a surjective $*$-homomorphism.

A related question is the following: Given a quantum group G, how to construct its quantum subgroups? Following the duality principle and what was mentioned above, we need to form some kind of a quotient of the corresponding algebra.

Let A be a Hopf $*$-algebra. A set $I \subseteq A$ is called a **coideal** if

$$\Delta(I) \subseteq I \odot A + A \odot I \quad \text{and} \quad \varepsilon(I) = 0.$$

A coideal that is also a ($*$-)ideal is called a **($*$-)biideal**. A **Hopf ($*$-)ideal** is a ($*$-)biideal that is invariant under the antipode, that is, $S(I) \subseteq I$. For any Hopf $*$-algebra A and a Hopf $*$-ideal $I \subseteq A$ the quotient A/I becomes a Hopf $*$-algebra with respect to the operations induced from A.

It is easy to check that, given compact quantum groups $H \subseteq G$, the corresponding surjection $\varphi\colon \mathrm{Pol}\, G \to \mathrm{Pol}\, H$ defines a Hopf $*$-ideal $I = \ker \varphi$ and that $\mathrm{Pol}\, H \simeq \mathrm{Pol}\, G/I$. Conversely, given a compact quantum group G, any Hopf $*$-ideal $I \subseteq \mathrm{Pol}\, G$ defines a compact quantum subgroup $H \subseteq G$ such that $\mathrm{Pol}\, H = \mathrm{Pol}\, G/I$. Indeed, as we mentioned above, $\mathrm{Pol}\, G/I$ is a Hopf $*$-algebra and according to Theorem 2.3.3, it is enough to check that it is generated by its unitary corepresentations. But we know that $\mathrm{Pol}\, G$ is generated by the unitary corepresentations and it is easy to see that any corepresentation of $\mathrm{Pol}\, G$ serves also as a corepresentation of $\mathrm{Pol}\, G/I$.

Given a coideal I_0 invariant under the antipode, then the $*$-ideal $I = \{ab, ba, a^*b, ba^* \mid a \in I_0, b \in A\}$ generated by I_0 keeps the property of being a coideal and invariant under S. Hence I is a Hopf $*$-ideal. Let G be a compact quantum group. By saying, *let H be the quantum subgroup of G given by relations R*, we implicitly assume that the $*$-ideal I generated by R is a Hopf $*$-ideal and then define H to be the subgroup of G given by I.

Alternatively, one can formulate those considerations also in terms of C*-algebras. Let G be a compact quantum group. A **Woronowicz C*-ideal** of $C(G)$ is a C*-ideal $I \subseteq C(G)$ such that $\Delta(I) \subseteq \ker(\pi \otimes \pi)$, where π is the quotient map $C(G) \to C(G)/I$. Again, we have a one-to-one correspondence between quantum subgroups $H \subseteq G$ and Woronowicz C*-ideals $I \subseteq C(G)$.

In the spirit of the duality between (quantum) groups and the associated algebras, we can define also quotients of quantum groups. A compact quantum group H is said to be a **quotient quantum group** of a compact quantum group G if $\mathrm{Pol}\, H$ is a Hopf $*$-subalgebra of $\mathrm{Pol}\, G$.

2.3.3 Universal compact matrix quantum groups

Let G be a compact matrix quantum group and $u \in M_N(C(G))$ its fundamental representation. Suppose that u is unitary. In contrast with the classical case, this does not imply that $\bar{u}$ or u^t are unitary. Nevertheless, according to Proposition 2.2.3, they are unitarizable. This means that there exists an invertible matrix $F \in M_N(\mathbb{C})$ such that $F\bar{u}F^{-1}$ is unitary. Hence, G is a quantum subgroup of the **universal unitary quantum group** $U^+(F)$ defined by the C*-algebra

$$C(U^+(F)) := C^*(u_{ij},\ i, j = 1, \ldots, N \mid u,\ F\bar{u}F^{-1} \text{ unitary}).$$

We also define the **universal orthogonal quantum groups** $O^+(F)$ for any $F \in M_N(\mathbb{C})$ satisfying $F\bar{F} = c\,1_N$, $c \in \mathbb{R}$ through the C*-algebra

$$C(O^+(F)) := C^*(u_{ij},\ i,j = 1,\ldots,N \mid u = F\bar{u}F^{-1} \text{ unitary}).$$

We get the standard free unitary and orthogonal quantum groups U_N^+ and O_N^+ by choosing $F = 1_N$.

2.3.5 Remark. The condition $F\bar{u}F^{-1}$ being unitary can be written as

$$\bar{u}(F^*F)^{-1}u^t(F^*F) = 1_N = u^t(F^*F)\bar{u}(F^*F)^{-1}.$$

So, the definition of $U^+(F)$ actually depends only on the matrix F^*F, not on the particular choice of F.

2.3.6 Remark. In the case of O_N^+, we have that $\bar{u} = u$, so the unitarity of u, that is, the relation $uu^* = u^*u = 1_N$ can be replaced by $uu^t = u^tu = 1_N$. Similar thing can be done in the case of $O^+(F)$. Using the relation $u = F\bar{u}F^{-1}$ we can see that the unitarity of u is equivalent to

$$u(F^*)^{-1}u^tF^* = (F^*)^{-1}u^tF^*u = 1_N, \qquad \text{or, equivalently,} \qquad uF^tu^t(F^t)^{-1} = F^tu^t(F^t)^{-1}u = 1_N.$$

2.3.7 Remark. The reason why we require $F\bar{F} = c\,1_N$ for the definition of $O^+(F)$ is that otherwise we get some kind of degeneracy. Applying the equality $u = F\bar{u}F^{-1}$ recursively to itself, we obtain $u = F\bar{F}u(F\bar{F})^{-1}$, which is an additional "unwanted" relation. Another viewpoint is that we require the fundamental representation u to be irreducible. This can hold only if $F\bar{F} = c\,1_N$ since, as we just computed, we have $F\bar{F} \in \mathrm{Mor}(u,u)$. From the Schur's lemma (Prop. 2.2.2), we have that u is irreducible if and only if $\mathrm{Mor}(u,u) = \{c\,1_N\}_{c\in\mathbb{C}}$. The constant c must be real by its definition since $\bar{F} = cF^{-1}$, so $c = F\bar{F} = \bar{F}F = \bar{c}$.

The universal unitary quantum group was introduced by Van Daele and Wang [VDW96]. The definition of the universal orthogonal quantum group comes from [Ban96]. Note that there have been several alternative definitions for the orthogonal case (see [VDW96, Wan98]) and also for the unitary case the notation may vary (some authors use the matrix $Q = F^*F$ to characterize the quantum groups as in the original paper [VDW96]).

A compact quantum group G is said to be of **Kac type** if the antipode on $\mathrm{Pol}\,G$ satisfies $S^2 = \mathrm{id}$. Recall that the antipode is defined by $S(u_{ij}) = [u^{-1}]_{ij}$ for any non-degenerate representation u of G. In particular, taking the universal unitary quantum group $U^+(F)$ and denoting by u its fundamental representation, we have

$$S(u_{ij}) = u_{ji}^*, \qquad S(u_{ij}^*) = [(F^*F)^{-1}u^t(F^*F)]_{ij}.$$

So, it is of Kac type only if $F^*F = 1_N$. That is, U_N^+ is the only Kac type quantum group among all $U^+(F)$. In general, a compact matrix quantum group G with unitary fundamental representation u is of Kac type if and only if $G \subseteq U_N^+$ for some N.

There are several other conditions characterizing Kac type quantum groups, which we will not discuss here. See for example ⟦NT13, Prop. 1.7.9⟧.

2.3.4 Compact matrix quantum groups determined by algebraic relations

For a fixed $N \in \mathbb{N}$, we denote by $\mathbb{C}\langle x_{ij}, x_{ij}^*\rangle = \mathbb{C}\langle x_{ij}, x_{ij}^* \mid i,j = 1,\ldots,N\rangle$ the free $*$-algebra generated by N^2 elements x_{ij}. To every compact matrix quantum group G with fundamental representation $u \in M_N(C(G))$, we associate an ideal

$$I_G := \{f \in \mathbb{C}\langle x_{ij}, x_{ij}^*\rangle \mid f(u_{ij}, u_{ij}^*) = 0\}.$$

This ideal determines the compact quantum group since

$$\mathrm{Pol}\,G = O(G) = \mathbb{C}\langle x_{ij}, x_{ij}^*\rangle/I, \quad C_{\mathrm{u}}(G) = C^*(\mathrm{Pol}\,G).$$

2.3.8 Lemma. Let G and H be compact matrix quantum groups. Then $H \subseteq G$ if and only if $I_G \subseteq I_H$.

Proof. Denote by u and v the fundamental representations of G and H, respectively. Suppose $H \subseteq G$, so there is a surjective $*$-homomorphism $\varphi\colon \operatorname{Pol} G \to \operatorname{Pol} H$ mapping $u_{ij} \mapsto v_{ij}$. Denote by $q\colon \mathbb{C}\langle x_{ij}, x_{ij}^* \rangle \to \operatorname{Pol} G$ the surjective $*$-homomorphism $x_{ij} \mapsto u_{ij}$. Then we have $I_G = \ker q \subseteq \ker(\varphi \circ q) = I_H$.

For the converse, suppose $I_G \subseteq I_H$. This directly implies that $u_{ij} \mapsto v_{ij}$ extends to a $*$-homomorphism since all relations that hold in $\operatorname{Pol} G$ must also hold in $\operatorname{Pol} H$. □

An extended version of this lemma will be provided in Section 3.4.5 as Proposition 3.4.15.

2.4 Discrete quantum groups

In this section, we introduce the notion of a *discrete quantum group*. The idea is to interpret the C*-algebra $C(G)$ associated to some compact quantum group G as a group C*-algebra associated to some discrete quantum group Γ. This generalizes the so-called Pontryagin duality. In the compact quantum group setting, this idea first appeared in [PW90]. The theory of discrete quantum groups bypassing compact quantum groups was then developed in [ER94, VDa96]. The most general framework, where the Pontryagin duality can be formulated are locally compact quantum groups introduced in [KV00]. Compact quantum groups and discrete quantum groups can be then considered as a special case of locally compact quantum groups. The basics of the duality between compact and discrete quantum groups is sketched in the lecture notes ⟦FSS17⟧.

Let Γ be a discrete group. The full group C*-algebra $C^*(\Gamma)$ or the reduced group C*-algebra $C^*_{\mathrm{r}}(\Gamma)$ together with the comultiplication defined as

$$\Delta(u_g) = u_g \otimes u_g$$

is a compact quantum group. We will denote this quantum group by $\hat{\Gamma}$ and call it the **compact dual** of Γ. The group C*-algebras contain the dense Hopf $*$-algebra $\mathbb{C}\Gamma$ with counit and antipode defined as

$$\varepsilon(u_g) = \delta_{g,e}, \qquad S(u_g) = u_{g^{-1}} = u_g^*.$$

Let G be a compact quantum group. An element $a \in C(G)$ is called **group-like** if $\Delta(a) = a \otimes a$. Equivalently, it means that a defines a one-dimensional representation of G. This implies that all group-likes are actually contained in $\operatorname{Pol} G$.

2.4.1 Lemma. All group-like elements of $C(G)$ are unitary. Together they form a group.

Proof. As we just mentioned, a group-like $a \in C(G)$ is a one-dimensional representation. According to Proposition 2.2.3, it must be equivalent to a unitary representation. But since a is one-dimensional, it must itself be unitary. Given two group-likes $a, b \in C(G)$, we have that $\Delta(ab) = \Delta(a)\Delta(b) = ab \otimes ab$, so ab is also group-like. Alternatively, note that the algebra multiplication here coincides with the representation tensor product, so $ab = a \otimes b$ must be a one-dimensional representation. The group inversion is given by the $*$-operation. Also in this case, we already know that the complex conjugate of a representation is again a representation, so a^* must be group-like. □

2.4.2 Theorem. Let G be an abelian compact quantum group. Then all irreducible representations of G are one-dimensional. So, $\operatorname{Irr} G$ coincides with the group of group-like elements. Let us denote it by Γ:

$$\Gamma := \operatorname{Irr} G = \{u \in C(G) \mid \Delta(u) = u \otimes u\}.$$

Then $G = \hat{\Gamma}$. That is,

$$C^*(\Gamma) = C_{\mathrm{u}}(G), \quad C^*_{\mathrm{r}}(\Gamma) = C_{\mathrm{r}}(G), \quad \mathbb{C}\Gamma = \operatorname{Pol} G.$$

We give a proof of this theorem in the following subsection.

Taking an arbitrary compact quantum group G, we can formally define its *discrete dual* $\hat{G}$ as a quantum group characterized by the algebras

$$C^*(\hat{G}) := C_{\mathrm{u}}(G), \quad C^*_{\mathrm{r}}(\hat{G}) := C_{\mathrm{r}}(G), \quad \mathbb{C}\hat{G} := \operatorname{Pol} G.$$

To be more precise, we define formally **discrete quantum groups** by exactly the same definition as compact quantum groups. What changes is the notation. The C*-algebra underlying a discrete quantum group Γ is denoted by $C^*(\Gamma)$ and interpreted as the group C*-algebra of Γ. The associated Hopf $*$-algebra is denoted by $\mathbb{C}\Gamma$. Considering a compact quantum group G, we can interpret it as a discrete quantum group and call it the **discrete dual** of G denoted by $\hat{G}$. Conversely, taking a discrete quantum group Γ, we can consider it as a compact quantum group and call it the **compact dual** of Γ denoted by $\hat{\Gamma}$.

This idea generalizes the so-called *Pontryagin duality*. For every abelian locally compact group G, we can construct its dual $\hat{G}$ as the group of all continuous homomorphisms $G \to \mathbb{T}$, where $\mathbb{T} \subseteq \mathbb{C}$ is the unit circle. This is again an abelian locally compact group. Repeating the construction we arrive at the original group G.

Nevertheless, the duality formulated above in the quantum case is so far more or less just an empty formal statement. We give the discrete duals more concrete meaning in the following subsection. In its full generality, the Pontryagin duality can be formulated in the setting of locally compact quantum groups.

2.4.1 Dual algebras

Let G be a compact quantum group and denote by $\Gamma := \hat{G}$ its discrete dual. We denote by $\mathbb{C}^\Gamma$ the vector space dual of $\operatorname{Pol} G$. This is a $*$-algebra with respect to the following operations

$$\omega\nu := \omega * \nu := (\omega \otimes \nu) \circ \Delta, \qquad \omega^*(a) := \overline{\omega(S(a)^*)},$$

where $\omega, \nu \in \mathbb{C}^\Gamma$, $a, b \in \operatorname{Pol} G$. This algebra plays the role of the algebra of all functions (sequences) $\Gamma \to \mathbb{C}$.

Given $u \in M_n(\operatorname{Pol} G)$ a (unitary) representation of G, that is, a corepresentation of $\operatorname{Pol} G$, we define a ($*$-)representation $\pi_u : \mathbb{C}^\Gamma \to M_n(\mathbb{C})$ as $[\pi_u(\omega)]_{ij} = \omega(u_{ij})$. It is indeed a ($*$-)representation since

$$[\pi_u(\omega\nu)]_{ij} = (\omega\nu)(u_{ij}) = (\omega \otimes \nu)(\Delta(u_{ij})) = \sum_k \omega(u_{ik})\nu(u_{kj}) = [\pi_u(\omega)\pi_u(\nu)]_{ij},$$
$$[\pi_u(\omega^*)]_{ij} = \omega^*(u_{ij}) = \overline{\omega(u_{ji})} = [\pi_u(\omega)^*]_{ij}.$$

In the second row, we used the fact that for unitary u, we have $S(u_{ij}) = u^*_{ji}$.

2.4.3 Lemma. Let G be a compact quantum group and $u \in M_n(C(G))$ its representation. Then the image of π_u equals to the commutant of $\operatorname{Mor}(u, u)$. That is,

$$\{\pi_u(\omega) \mid \omega \in \mathbb{C}^{\hat{G}}\} = \{S \in M_n(\mathbb{C}) \mid TS = ST \text{ for all } T \in \operatorname{Mor}(u, u)\}.$$

Proof. To prove the inclusion $\subseteq$, take an arbitrary $\omega \in \mathbb{C}^{\hat{G}}$ and an arbitrary $T \in \operatorname{Mor}(u, u)$. We need to prove that $\pi_u(\omega)$ commutes with T. This indeed holds since

$$[\pi_u(\omega)T]_{ij} = \sum_k \omega(u_{ik})T_{kj} = \sum_k T_{ik}\omega(u_{kj}) = [T\pi_u(\omega)]_{ij}.$$

To prove the converse, we can use the double commutant theorem (Thm. 1.2.7). It is enough to prove that $(\pi_u(\mathbb{C}^{\hat{G}}))' \subseteq \operatorname{Mor}(u, u)$. Taking any T commuting with $[\pi_u(\omega)]_{ij}$, we can use the same computation as above to show that $\sum_k \omega(u_{ik})T_{kj} = \sum_k T_{ik}\omega(u_{kj})$ for every $\omega \in \mathbb{C}^{\hat{G}}$. But since it holds for every ω, we must have $uT = Tu$, so $T \in \operatorname{Mor}(u, u)$. □

Since $\{u^\alpha_{ij}\}$ with $\alpha \in \operatorname{Irr} G$ form a vector space basis, we have that any $\omega \in \mathbb{C}^\Gamma$ is determined by the numbers $\omega(u^\alpha_{ij}) = [\pi_{u^\alpha}(\omega)]_{ij}$. Consequently, we have

$$\mathbb{C}^\Gamma \simeq \prod_{\alpha\in\operatorname{Irr} G} M_{n_\alpha}(\mathbb{C}),$$

where the isomorphism is provided by $\prod_{\alpha\in\operatorname{Irr} G} \pi_{u^\alpha}$.

Replacing the direct product by the algebraic direct sum, we obtain an algebra denoted by $c_{00}(\Gamma)$ corresponding to finitely supported sequences on Γ. Taking the c_0 direct sum or l^∞ direct sum, we can define also the algebras $c_0(\Gamma)$ or $l^\infty(\Gamma)$.

The algebra $c_{00}(\Gamma)$ is actually a Hopf $*$-algebra with respect to the following operations

$$(\hat{\Delta}(\omega))(a\otimes b) = \omega(ab), \qquad \hat{\varepsilon}(\omega) = \omega(1), \qquad \hat{S}\omega = \omega\circ S.$$

where $\omega, \nu \in c_{00}(\Gamma)$, $a, b \in \operatorname{Pol} G$. Note that these operations can actually be defined also on $\mathbb{C}^\Gamma$, but the comultiplication would map $\mathbb{C}^\Gamma \to \mathbb{C}^{\Gamma\times\Gamma} := (\operatorname{Pol} G \odot \operatorname{Pol} G)^* \supsetneq \mathbb{C}^\Gamma \odot \mathbb{C}^\Gamma$ with the inclusion being strict whenever $\operatorname{Pol} G$ is infinite-dimensional.

The multiplication on $\operatorname{Pol} G$ is transformed into the comultiplication on $c_{00}(\Gamma)$ and the comultiplication on $\operatorname{Pol} G$ is transformed into the multiplication on $c_{00}(\Gamma)$. In particular, $c_{00}(\Gamma)$ is commutative, resp. cocommutative if and only if $\operatorname{Pol} G$ is cocommutative, resp. commutative.

Now, we give the proof of Theorem 2.4.2.

Proof of Theorem 2.4.2. Since G is abelian, we have that $\mathbb{C}^{\hat{G}}$ must be commutative. Consequently $n_\alpha = 1$ for every $\alpha \in \operatorname{Irr} G$. Finally, from Proposition 2.3.1, we know that $\operatorname{Pol} G = \operatorname{span}\{u^\alpha \mid \alpha \in \operatorname{Irr} G\} = \mathbb{C}\Gamma$. □

2.4.2 Representations, corepresentations, finitely generated groups

Let Γ be a discrete quantum group and $G := \hat{\Gamma}$ its compact dual. A **corepresentation** of Γ is a representation of G, that is, a matrix u with entries in $\mathbb{C}\Gamma \subseteq C^*(\Gamma)$ such that $\Delta(u_{ij}) = \sum_k u_{ik}\otimes u_{kj}$. If the algebra $\mathbb{C}\Gamma$ (or $C^*(\Gamma)$) is generated by the matrix entries of some representation u, that is, if $(C^*(\Gamma), u)$ is a compact matrix quantum group, we say that Γ is **finitely generated**.

2.4.4 Proposition. Let Γ be a finitely generated discrete group and denote by $g_1,\dots,g_n$ its generators. Then $(C^*(\Gamma), u)$, where $u = \operatorname{diag}(u_{g_1},\dots,u_{g_n})$ is a compact matrix quantum group. Conversely, let $G = (C(G), u)$ be a compact matrix quantum group such that u is diagonal and unitary. Denote by Γ the group of all group-like elements in $C(G)$. Then $G = \hat{\Gamma}$.

Proof. Let $\Gamma = \langle g_1,\dots,g_n\rangle$ be a discrete group. Since g_i generate all $g\in G$, we surely have that the unitaries $u_{g_i} \in \mathbb{C}\Gamma \subseteq C^*(\Gamma)$ generate all the u_g, $g\in G$, which by definition generate the whole algebras $\mathbb{C}\Gamma$ and $C^*(\Gamma)$. The matrix u is obviously a corepresentation of Γ since we have $\Delta(u_{g_i}) = u_{g_i}\otimes u_{g_i}$ by definition of the comultiplication on Γ. In addition, it is unitary and therefore also invertible.

The converse already follows from Theorem 2.4.2 since any compact matrix quantum group with diagonal fundamental representation is obviously abelian. But we can also give a direct proof: Denote by Γ the group of unitaries in $C(G)$ generated by the elements u_{ii}. Its elements are all representations of G and linearly span the whole $\operatorname{Pol} G$, so from Proposition 2.3.1, we must have that $\Gamma = \operatorname{Irr} G$ contains all the irreducibles and, in particular, all the group-likes. Consequently, we have that $\operatorname{Pol} G = \mathbb{C}\Gamma$ and hence $G = \hat{\Gamma}$. □

2.4.5 Example. We will often work with the cyclic groups $\Gamma = \mathbb{Z}_k$ or $\Gamma = \mathbb{Z}$. They are generated by a single element g satisfying $g^k = e$ in the case of $\mathbb{Z}_k$ or by a single element g with no relation in the case of $\mathbb{Z}$. Hence, the associated group C*-algebras can be written as

$$C^*(\mathbb{Z}_k) = C^*(z \mid z^k = 1,\ zz^* = z^*z = 1), \qquad C^*(\mathbb{Z}) = C^*(z \mid zz^* = z^*z = 1).$$

This defines compact matrix quantum groups $\hat{\mathbb{Z}}_k = (C^*(\mathbb{Z}_k), (z))$ and $\hat{\mathbb{Z}} = (C^*(\mathbb{Z}), (z))$, where the fundamental representation is a 1×1 matrix with a single entry z.

Note that since the cyclic groups are abelian, the associated group algebras are commutative and hence the compact duals are also groups. Therefore, this also serves as the most simple example of the classical Pontryagin duality. We have that $\hat{\mathbb{Z}} = \mathbb{T}$, where $\mathbb{T} = U_1 \subseteq \mathbb{C}$ is the unit circle (indeed, we may define an isomorphism $C^*(\mathbb{Z}) \to C(\mathbb{T})$ mapping $z^k \mapsto f_k$ with $f_k(t) = t^k$, whose inverse is given by the Fourier series decomposition). For finite cyclic groups, we have $\hat{\mathbb{Z}}_k = \mathbb{Z}_k$ (the isomorphism $C^*(\mathbb{Z}_k) \to C(\mathbb{Z}_k)$ given by a similar formula).

Let $G = (C(G), u)$ be a compact matrix quantum group with the fundamental representation u being unitary. Let $\hat{\Gamma}$ be the quantum subgroup of G given by the relation $u_{ij} = 0$ for all $i \neq j$. We call $\hat{\Gamma}$ the **diagonal subgroup** of G. It is a compact matrix quantum group with diagonal unitary fundamental representation and hence it is a dual of a finitely generated group Γ. We call this group the **dual diagonal subgroup** of G.

2.4.6 Example. If G is a compact matrix group, then the above construction defines a diagonal subgroup $\hat{\Gamma} \subseteq G$ of all matrices in G that are diagonal. In this case, $\hat{\Gamma}$ is also a group. The dual diagonal subgroup Γ is then the Pontryagin dual of $\hat{\Gamma}$. For example, the diagonal subgroup $\hat{\Gamma}$ of the orthogonal group O_N consists of all diagonal orthogonal matrices. Those are exactly all diagonal matrices with ± 1 on the diagonal – that is $\mathbb{Z}_2^N$. Its Pontryagin dual Γ is isomorphic to $\hat{\Gamma}$ and hence also equals to $\mathbb{Z}_2^N$. For the unitary group U_N, we get the subgroup of all diagonal unitary matrices, that is matrices with complex units on the diagonal: $\hat{\Gamma} = \mathbb{T}^N \subsetneq U_N$. Its Pontryagin dual is then $\Gamma = \mathbb{Z}^N$.

2.4.7 Example. Recall the definition of the free orthogonal quantum group $O_N^+ = (C(O_N^+), u)$ with

$$C(O_N^+) = C^*(u_{ij} \mid u_{ij} = u_{ij}^*,\ uu^t = u^t u = 1).$$

Then the diagonal subgroup $\hat{\Gamma} = (C^*(\Gamma), \tilde{u})$ is determined by

$$C^*(\Gamma) = C^*(g_i \mid g_i = g_i^*,\ g_i^2 = 1) = C^*(\mathbb{Z}_2^{*N}),$$

so $\Gamma = \mathbb{Z}_2 * \cdots * \mathbb{Z}_2 = \mathbb{Z}_2^{*N}$. Similarly, for the free unitary quantum group U_N^+, we easily see that the associated dual diagonal subgroup is the N-fold free product $\mathbb{Z}^{*N}$.

Finally, let us just mention that it is possible to define also a *representation* of a given discrete quantum group Γ on some Hilbert space H. It is defined as an element $U \in l^\infty(\Gamma) \bar{\otimes} \mathscr{B}(H)$ satisfying some property analogous to Eq. (2.2). Here, $\bar{\otimes}$ denotes the *von-Neumann-algebraic tensor product*. Such a representation then induces a representation of the algebra $\mathbb{C}\Gamma$. We will not use this notion in our thesis.

2.5 Quantum group constructions

In this section, we present some constructions that allow us to produce new quantum groups from old ones. In particular, we are going to generalize the group *direct product* to the quantum case, then we are going to present a related construction of *glued products* and finally we generalize the construction of an *intersection* of two matrix groups and a matrix group *generated* by two given ones.

2.5.1 Direct and free product of groups

Let us first recall some product constructions for groups. Let G and H be groups. Then we can construct their **direct product**

$$G \times H = \{(g, h) \mid g \in G, h \in H\}$$

with group operation $(g_1, h_1)(g_2, h_2) = (g_1 g_2, h_1 h_2)$. We can then identify the group G with a subgroup $\{(g, e_H)\}_{g \in G} \subseteq G \times H$ and similarly $H \simeq (e_G, H) \subseteq G \times H$. Then we can say that the elements of G *commute* with the elements of H in $G\times H$ in the sense that $gh = (g,e)(e,h) = (g,h) = (e,h)(g,e) = hg$. In addition, the groups G and H can also be obtained as quotient groups of $G\times H$.

If G and H are compact matrix groups with fundamental representations u and v, then $G\times H$ is also a compact group that can be represented by the direct sum $u\oplus v$. For the associated C*-algebra of continuous functions, we have $C(G\times H) = C(G)\otimes C(H)$ as this holds for any compact spaces (see Eq. (1.2)). We can also have a look on the comultiplication on $C(G\times H)$. Take $f_1\in C(G)$ and $f_2\in C(H)$ and take $(g_1,h_1),(g_2,h_2)\in G\times H$, then

$$\begin{aligned}\Delta_{G\times H}(f_1\otimes f_2)((g_1,h_1),(g_2,h_2)) &= (f_1\otimes f_2)((g_1g_2),(h_1h_2))\\ &= f_1(g_1g_2)f_2(h_1h_2) = \Delta_G(f_1)(g_1,g_2)\Delta_H(f_2)(h_1,h_2),\end{aligned}$$

so $\Delta_{G\times H}(f_1\otimes f_2) = \Delta_G(f_1)\Delta_H(f_2)$.

Nevertheless, the direct product is defined for any pair of groups, not only the compact ones. In particular, if Γ_1 and Γ_2 are discrete groups, then $\Gamma_1\times\Gamma_2$ defines a discrete group. For the associated group algebras, we obviously have $\mathbb{C}(\Gamma_1\times\Gamma_2) = \mathbb{C}\Gamma_1\odot\mathbb{C}\Gamma_2$ and hence $C^*(\Gamma_1\times\Gamma_2) = C^*(\Gamma_1)\otimes_{\max} C^*(\Gamma_2)$.

Finally, let us mention the group free product. Let Γ_1 and Γ_2 be discrete groups. The definition of the **free product** $\Gamma_1 * \Gamma_2$ is similar to free product of C*-algebras. Formally, it is a set of *reduced words* over $\Gamma_1\sqcup\Gamma_2$, that is, words where elements of Γ_1 and Γ_2 alternate and the group identity never appears. If Γ_1 is presented by generators $g_1,\dots,g_m$ and Γ_2 by generators $h_1,\dots,h_n$, then $\Gamma_1 * \Gamma_2$ is presented by $g_1,\dots,g_m,h_1,\dots,h_n$ and the union of the corresponding relations. The group algebra of a group free product is simply the free product of algebras $C^*(\Gamma_1 * \Gamma_2) = C^*(\Gamma_1) *_{\mathbb{C}} C^*(\Gamma_2)$. The free product of two non-trivial groups is never finite nor compact.

2.5.2 Direct products of compact quantum groups

In this section, we present the constructions of Wang [Wan95a, Wan95b], who defined the tensor product and the dual free product of compact quantum groups. There are two viewpoints on those constructions – either from the compact quantum group perspective or the discrete quantum group perspective. From the compact quantum group viewpoint, both constructions generalize the direct product of compact groups.

Let G and H be compact quantum groups. We define their **tensor product** $G\times H$ to be the quantum group with underlying C*-algebra $C(G\times H) := C(G)\otimes_{\max} C(H)$ and comultiplication defined as

$$\Delta_{G\times H}(a\otimes b) = \Delta_G(a)\Delta_H(b) \qquad \text{for all } a\in C(G),\ b\in C(H). \tag{2.3}$$

Formally, we should rather write $\Delta_{G\times H}(\iota_G(a)\otimes\iota_H(b)) = (\iota_G\otimes\iota_G)(\Delta_G(a))(\iota_H\otimes\iota_H)(\Delta_H(b))$, where $\iota_G\colon C(G)\to C(G)\otimes_{\max} C(H)$ and $\iota_H\colon C(H)\to C(G)\otimes_{\max} C(H)$ are the canonical inclusions.

2.5.1 Theorem. The above mentioned construction indeed defines a compact quantum group $G\times H$.

From the considerations in Section 2.5.1 it is now clear that taking two compact groups G and H, their quantum group tensor product coincides with the group direct product.

Considering an element $a\otimes b\in C(G)\otimes_{\max} C(H)$, we will usually omit the sign $\otimes$. One way to view this is to consider $C(G)\otimes_{\max} C(H)$ as a quotient of $C(G) *_{\mathbb{C}} C(H)$ with respect to the relations $ab = ba$ for $a\in C(G)$ and $b\in C(H)$. The other viewpoint is to consider $C(G)$ and $C(H)$ as subalgebras of $C(G)\otimes_{\max} C(H)$. This also shows that G and H can be considered as quotient quantum groups of $G\times H$. In addition, we also have that $C(G)$ and $C(H)$ are quotients of $C(G)\otimes_{\max} C(H)$, so G and H are quantum subgroups of $G\times H$.

If G and H are compact matrix quantum groups, we can define the structure of a compact matrix quantum group on $G\times H$.

2.5.2 Proposition. Let $G = (C(G), u)$ and $H = (C(H), v)$ be compact matrix quantum groups. Then

$$G \times H = (C(G) \otimes_{\max} C(H), u \oplus v)$$

is also a compact matrix quantum group. It is a matrix realization of the tensor product of G and H as defined above.

Proof. It is straightforward to see that $(C(G \otimes_{\max} H), u \oplus v)$ is indeed a compact matrix quantum group. Now let us denote by $\Delta_{G\times H}$ the comultiplication on the tensor product of G and H defined by Eq. (2.3). The only remaining point is to show that $\Delta_{G\times H}$ coincides with the matrix comultiplication defined by $u \oplus v$. This is also clear. □

Now, we may want to generalize this construction by modifying the C*-algebra multiplication, but keeping the comultiplication (that plays here the role of the group multiplication). In particular, if we are dealing with *free* compact quantum groups, we may want to *liberate* the C*-algebra multiplication. So, we define the following.

Let G and H be compact quantum groups. We define their **dual free product** $G \hat{*} H$ to be the quantum group with underlying C*-algebra $C(G \hat{*} H) := C(G) *_{\mathbb{C}} C(H)$ and comultiplication the unique unital $*$-homomorphism satisfying

$$\Delta_{G\hat{*}H}(a) = \Delta_G(a), \quad \Delta_{G\hat{*}H}(b) = \Delta_H(b) \qquad \text{for all } a \in C(G),\, b \in C(H). \tag{2.4}$$

Again, formally, we mean $\Delta_{G\hat{*}H}(\iota_G(a)) = (\iota_G \otimes \iota_G)(\Delta_G(a))$, $\Delta_{G\hat{*}H}(\iota_H(b)) = (\iota_H \otimes \iota_H)(\Delta_H(b))$, where ι_G and ι_H are the canonical inclusions into $C(G) *_{\mathbb{C}} C(H)$.

2.5.3 Theorem. The above mentioned construction indeed defines a compact quantum group $G \hat{*} H$.

2.5.4 Proposition. Let $G = (C(G), u)$ and $H = (C(H), v)$ be compact matrix quantum groups. Then

$$G \hat{*} H = (C(G) *_{\mathbb{C}} C(H), u \oplus v)$$

is also a compact matrix quantum group. It is a matrix realization of the dual free product of G and H as defined above.

Proof. Same as Proposition 2.5.2. □

For a pair of compact *matrix* quantum groups G and H, considering the tensor product $G \times H$ or the dual free product $G \hat{*} H$, we will always mean those particular matrix constructions presented in Propositions 2.5.2, 2.5.4.

Let us now present the discrete quantum group viewpoint on those constructions.

Let Γ_1 and Γ_2 be discrete quantum groups and denote by $G_1 = \hat{\Gamma}_1$ and $G_2 = \hat{\Gamma}_2$ their compact duals. We define their **direct product** to be the discrete quantum group $\Gamma_1 \times \Gamma_2 := \widehat{G_1 \times G_2}$. That is, the discrete quantum group with underlying group C*-algebra $C^*(\Gamma_1 \times \Gamma_2) = C^*(\Gamma_1) \otimes_{\max} C^*(\Gamma_2)$ and comultiplication given by Eq. (2.3).

We define the **free product** $\Gamma_1 * \Gamma_2 := \widehat{G_1 \hat{*} G_2}$. That is, the discrete quantum group with underlying group C*-algebra $C^*(\Gamma_1 * \Gamma_2) = C^*(\Gamma_1) *_{\mathbb{C}} C^*(\Gamma_2)$ and comultiplication given by Eq. (2.4).

In case when Γ_1 and Γ_2 are discrete groups, the above defined direct and free product exactly corresponds to the classical construction for groups.

This also explains the name *dual free product*. Since we are freeing the algebra multiplication, not the comultiplication, the quantum group $G \hat{*} H$ has nothing to do with the group free product $G * H$. It is rather the *dual* of a free product construction as we just explained. In the literature (in particular, in the original work of Wang [Wan95a]), the dual free product is often called just the *free product* referring to the free product of the C*-algebras.

Finally, let us mention, how the irreducible representations of free and tensor products look like.

2.5.5 Theorem. Let G and H be compact quantum groups. Let $\{u^\alpha\}_{\alpha\in\operatorname{Irr}G}$ and $\{v^\beta\}_{\beta\in\operatorname{Irr}H}$ be complete sets of irreducible representation of G and H. Denote by ι_G and ι_H the embeddings of $C(G)$ and $C(H)$ into $C(G \hat{*} H)$, respectively, and denote $w^\alpha_{ij} := \iota_G(u^\alpha_{ij})$ and $w^\beta_{ij} := \iota_H(v^\beta_{ij})$. Then a complete set of irreducible representations of $G \hat{*} H$ is formed by the trivial representation together with

$$w^{\gamma_1} \otimes w^{\gamma_2} \otimes \cdots \otimes w^{\gamma_n},$$

where $\gamma_i \in \operatorname{Irr}G \cup \operatorname{Irr}H$ are non-trivial representations such that the sets $\operatorname{Irr}G$ and $\operatorname{Irr}H$ alternate, so if $\gamma_i \in \operatorname{Irr}G$, then $\gamma_{i+1} \in \operatorname{Irr}H$ and vice versa.

The similarity with the definition of the group free product is no coincidence. If $G = \hat{\Gamma}_1$ and $H = \hat{\Gamma}_2$ for some discrete groups Γ_1, Γ_2. Then $\operatorname{Irr}G = \Gamma_1$ and $\operatorname{Irr}H = \Gamma_2$ and $\operatorname{Irr}(G \hat{*} H) = \Gamma_1 * \Gamma_2$.

2.5.6 Theorem. Let G and H be compact quantum groups. Let $\{u^\alpha\}_{\alpha\in\operatorname{Irr}G}$ and $\{v^\beta\}_{\beta\in\operatorname{Irr}H}$ be complete sets of irreducible representation of G and H. Denote by ι_G and ι_H the embeddings of $C(G)$ and $C(H)$ into $C(G \times H)$, respectively, and denote $w^\alpha_{ij} := \iota_G(u^\alpha_{ij})$ and $w^\beta_{ij} := \iota_H(v^\beta_{ij})$. Then a complete set of irreducible representations of $G \times H$ is formed by $w^\alpha \otimes w^\beta$ with $\alpha \in \operatorname{Irr}G$, $\beta \in \operatorname{Irr}H$.

Let us finish with a well known result about stability of coamenability under the tensor product. (Unfortunately, we did not find a reference, see [Cra17] for a more general result.)

2.5.7 Proposition. Let G and H be compact quantum groups. Then G and H are coamenable if and only if $G \times H$ is coamenable.

Proof. We use the characterization of coamenability from Theorem 2.3.4. The right-left implication is trivial since $C(G), C(H) \subseteq C(G \times H)$. For the left-right implication, take $A := C(G) \otimes_{\min} C(H)$. The minimal tensor product can alternatively also be used to describe $G \times H$ as it surely contains the dense subalgebra $\operatorname{Pol}(G \times H) = \operatorname{Pol}G \odot \operatorname{Pol}H$. If ε_G, ε_H are counits on G and H, we can extend $\varepsilon_G \otimes \varepsilon_H$ to A by Proposition 1.3.2. Similarly, the Haar state h on A can be expressed as $h = h_G \otimes h_H$ [Wan95b, Proposition 2.7]. If both h_A and h_B are faithful states, then also $h_A \otimes h_B$ is a faithful state by [Avi82, Appendix]. □

2.5.3 Glued products

In this section, we present a less standard product construction, which is defined only for matrix quantum groups. It was formally defined in [TW17] to interpret some coloured categories of partitions in terms of compact matrix quantum groups. It will play a crucial role also in this thesis.

Let $G = (C(G), u)$ and $H = (C(H), v)$ be compact matrix quantum groups. We define the **glued tensor product**

$$G \tilde{\times} H := (C(G \tilde{\times} H), u \otimes v),$$

where $C(G \tilde{\times} H)$ is the C*-subalgebra of $C(G) \otimes_{\max} C(H)$ generated by $u_{ij} v_{kl}$ – the elements of the tensor product $u \otimes v$.

Similarly, we define the **glued free product**

$$G \tilde{*} H := (C(G \tilde{*} H), u \otimes v),$$

where $C(G \tilde{*} H)$ is the C*-subalgebra of $C(G) *_{\mathbb{C}} C(H)$ generated by $u_{ij} v_{kl}$.

The glued versions of the tensor and free products $G \tilde{\times} H$ and $G \tilde{*} H$ are by definition quotient quantum groups of the standard constructions $G \times H$ and $G \hat{*} H$. Often it happens that the elements $u_{ij} v_{kl}$ already generate the whole C*-algebra, so actually $G \tilde{\times} H \simeq G \times H$ or $G \tilde{*} H \simeq H \hat{*} H$. Even in this case, however, we should not put the equality sign here. Although the quantum groups can have the same underlying C*-algebra and hence be isomorphic, they are never identical as compact matrix quantum groups since their fundamental representations are always different – $u \oplus v$ in the standard case and $u \otimes v$ in the glued case.

2.5.8 Example. Let us have a look on how the definition of glued tensor product looks like for groups. Let G and H be two matrix groups, then we have

$$G \tilde{\times} H = \{A \otimes B \mid A \in G, B \in H\},$$

where $\otimes$ denotes the Kronecker product.

As a concrete example, consider the symmetric group S_N represented by the permutation matrices and consider the cyclic group of order two $\hat{\mathbb{Z}}_2 = \mathbb{Z}_2$ represented by a single complex number ± 1. Then $S_N \tilde{\times} \mathbb{Z}_2$ consists of $N \times N$ permutation matrices multiplied by a global sign. Thus, $S_N \tilde{\times} \mathbb{Z}_2$ is actually isomorphic to $S_N \times \mathbb{Z}_2$. Nevertheless, by $S_N \times \mathbb{Z}_2$ we mean a different matrix realization. The ordinary product $S_N \times \mathbb{Z}_2$ consists of $(N+1) \times (N+1)$ matrices with block diagonal structure, where one block is formed by an $N \times N$ permutation matrix and the second block is the single number ± 1.

In general, take any cyclic group $\hat{\mathbb{Z}}_k = \mathbb{Z}_k$ with $k \in \mathbb{N}$ represented by the k-th roots of unity. Then for any matrix group G, we have

$$G \tilde{\times} \mathbb{Z}_k = \{\mathrm{e}^{2\pi i j/k} A \mid j = 0, \ldots, k-1;\ A \in G\}.$$

We can do the same for the whole unit disk $\hat{\mathbb{Z}} = \mathbb{T} \subseteq \mathbb{C}$

$$G \tilde{\times} \mathbb{T} = \{zA \mid z \in \mathbb{T};\ A \in G\}.$$

Given a compact matrix quantum group G, we call $G \tilde{\times} \hat{\mathbb{Z}}$ the **tensor complexification** of G, $G \tilde{\times} \hat{\mathbb{Z}}_k$ is the **tensor *k*-complexification**, $G \tilde{*} \hat{\mathbb{Z}}$ is the **free complexification** and $G \tilde{*} \hat{\mathbb{Z}}_k$ is the **free *k*-complexification** of G. The free complexification was studied already by Banica in [Ban99a, Ban08].

2.5.4 Intersection of compact matrix quantum groups

Given two groups H_1, H_2 embedded into a larger one, we can compute their intersection $H_1 \cap H_2$, which is again a group. In particular, we can take compact matrix groups H_1, H_2 represented by matrices of the same size and ask, what is their intersection $H_1 \cap H_2$ – the largest subgroup of both. This concept can be generalized to the case of compact matrix quantum groups.

Let H_1 and H_2 be compact matrix quantum groups with fundamental representations v_1 and v_2 of the same size. We denote by $H_1 \cap H_2$ the **intersection** of H_1 and H_2 defined as the largest quantum subgroup of both H_1 and H_2. That is $H_1 \cap H_2 =: G = (C(G), u)$ is defined by the fact that

(1) $G \subseteq H_1, H_2$, so the size of u coincides with the size of v_1 and v_2 and there are surjective $*$-homomorphisms $\varphi_k \colon C(H_k) \to C(G)$ mapping $[v_k]_{ij} \mapsto u_{ij}$ for $k = 1, 2$.
(2) For every compact quantum group $\tilde{G}$ such that $G \subseteq \tilde{G} \subseteq H_1, H_2$, we have $G = \tilde{G}$.

The quantum group $H_1 \cap H_2$ is unique and always exists as follows from the following proposition.

2.5.9 Proposition. Let H_1 and H_2 be compact matrix quantum groups with fundamental representations of the same size. Then the intersection $H_1 \cap H_2$ is defined by the ideal $I_{H_1 \cap H_2} = I_{H_1} + I_{H_2}$. Conversely, the ideal $I_{H_1} + I_{H_2}$ always defines a compact matrix quantum group – namely the intersection $H_1 \cap H_2$.

We will prove this proposition as a part of Proposition 3.4.17, see Remark 3.4.18.

2.5.5 Topological generation

Given two compact matrix groups represented by matrices of the same size $H_1, H_2 \subseteq \mathrm{GL}_N$, we may ask, what compact matrix group they generate. That is, find the smallest compact subgroup $\langle H_1, H_2\rangle \subseteq \mathrm{GL}_N$ containing both H_1 and H_2. We may ask the same question also for compact matrix quantum groups. The idea goes back to [Chi15, BCV17].

Let H_1 and H_2 be compact matrix quantum groups with fundamental representations v_1 and v_2 of the same size. We define $G := \langle H_1, H_2\rangle$ to be the smallest compact matrix quantum group containing H_1 and H_2. We say that G is **topologically generated** by H_1 and H_2. That is $G = (C(G), u)$ is a quantum group satisfying the following.

(1) $H_1, H_2 \subseteq G$, so the size of u coincides with the size of v_1 and v_2 and there are surjective $*$-homomorphisms $\varphi_k \colon C(G) \to C(H_k)$ mapping $u_{ij} \mapsto [v_k]_{ij}$ for $k = 1, 2$.
(2) For every compact quantum group $\tilde{G}$ such that $H_1, H_2 \subseteq \tilde{G} \subseteq G$, we have $G = \tilde{G}$.

Note that already in the case of groups it may happen that for two compact matrix groups H_1, H_2, the group they generate $\langle H_1, H_2\rangle$ is not compact. We can fix this issue assuming that the compact groups are unitary $H_1, H_2 \subseteq U_N$. Then surely $\langle H_1, H_2\rangle \subseteq U_N$ and hence it is compact.

Thus, also for compact matrix quantum groups H_1 and H_2, the quantum group $\langle H_1, H_2\rangle$ may not exist unless we assume $H_1, H_2 \subseteq U^+(F)$ for some common $F \in \mathrm{GL}_N$ (this can be formulated as an equivalence, see Proposition 3.4.19). Nevertheless, if the quantum group $\langle H_1, H_2\rangle$ exists, then it is unique. One way to see this is that if two quantum groups G_1 and G_2 both contain H_1 and H_2, then also $G_1 \cap G_2$ contains both H_1 and H_2. Alternatively, it also follows from the following characterization.

2.5.10 Proposition. Let H_1 and H_2 be compact matrix quantum groups with fundamental representations of the same size. If the quantum group $\langle H_1, H_2\rangle$ exists, then it corresponds to the ideal $I_{\langle H_1, H_2\rangle} = I_{H_1} \cap I_{H_2}$. Conversely, suppose $H_1, H_2 \subseteq U^+(F)$. Then $I_G := I_{H_1} \cap I_{H_2}$ defines a compact matrix quantum group $G = \langle H_1, H_2\rangle$.

Proof. Let us start with the first part. We know that $H_1, H_2 \subseteq \langle H_1, H_2\rangle$, so, according to Lemma 2.3.8, we have that $I_{\langle H_1, H_2\rangle} \subseteq I_{H_1}, I_{H_2}$ and hence $I_{\langle H_1, H_2\rangle} \subseteq I_{H_1} \cap I_{H_2}$. Denote $I_G := I_{H_1} \cap I_{H_2}$. Since H_1 and H_2 are quantum subgroups of $\langle H_1, H_2\rangle$, we have that $I_{H_k}/I_{\langle H_1, H_2\rangle}$ are Hopf $*$-ideals. Consequently, also $I_G/I_{\langle H_1, H_2\rangle}$ is a Hopf $*$-ideal, so I_G defines a compact quantum group G such that $H_1, H_2 \subseteq G \subseteq \langle H_1, H_2\rangle$. So, $G = \langle H_1, H_2\rangle$.

For the converse, suppose that $H_1, H_2 \subseteq U^+(F)$ and denote $I_G := I_{H_1} \cap I_{H_2}$. Then again $I_{H_k}/I_{U^+(F)}$ are Hopf $*$-ideals, so $I_G/I_{U^+(F)}$ is also a Hopf $*$-ideal and hence I_G defines a compact matrix quantum group G. Obviously $G \subseteq H_1, H_2$ and also for any other $\tilde{G} \subseteq H_1, H_2$, we have $\tilde{G} \subseteq G$. □

Chapter 3
Monoidal categories and Tannaka–Krein duality

In this chapter, we bring some definitions for the abstract algebraic structure that provides the connection between set partitions and compact quantum groups. The structure is a monoidal category and the connection is via Tannaka–Krein duality.

Let us stress that the category theory occurs in this work not because of any obsession of the author with formulating everything in terms of categories. Quite the contrary, the categories appear here since it is the most fitting algebraic structure for our purpose. We will try to formulate everything as simple and concrete as possible avoiding unnecessary abstractions typical for the theory of categories.

Since our use of the abstract category theory will be very restricted, we believe that the short summary provided in the following sections is enough for everybody without any background in category theory to understand the concept of partition categories and Tannaka–Krein duality. Nevertheless, if this thesis was a textbook, the crucial last section 3.4 would probably be shifted somewhere to the back to Chapter 8 or so. Readers not familiar with monoidal categories and Tannaka–Krein duality are advised to read Section 3.4 very briefly for the first time and get back to the proofs later since the rest of the thesis will provide a concrete illustration of the abstract concepts presented in Section 3.4.

As a general reference for the category theory, we mention the classical book ⟦McL98⟧. The main subject of our study are the categories of representations of quantum groups. Those categories are described in the monography ⟦NT13⟧. Their study begun by the work of Woronowicz in [Wor88], who used the W*-categories defined in [GLR85] to study representations of quantum groups. Although it is not a proper reference, we would like to acknowledge the helpfulness of the project *n*Lab.[1]

3.1 General introduction to category theory

Categories constitute a natural generalization of the classical algebraic structures, such as a group or an algebra, to the case where the multiplication is not defined for every pair of elements. As a motivating example, take a topological space X and consider the set $\pi(X)$ of homotopy classes of all paths in X. This set looks almost like a group – if we define the multiplication as concatenation of paths then we can find (sort of) a unit and an inverse for every element. The only problem is that two paths can be concatenated only if one starts in the same point as the other ends.

A **category** is a triple $C = (\operatorname{Obj} C, \{\operatorname{Mor}(a,b)\}_{a,b\in\operatorname{Obj} C}, \cdot)$, where

(a) $\operatorname{Obj} C$ is a set,[2] whose elements are called **objects**,
(b) $\operatorname{Mor}(a,b)$ for $a,b\in\operatorname{Obj} C$ are sets, whose elements are called **morphisms**,
(c) for every triple $a,b,c\in\operatorname{Obj} C$, there is an operation $\cdot\colon \operatorname{Mor}(b,c)\times\operatorname{Mor}(a,b)\to\operatorname{Mor}(a,c)$.

Those must satisfy that for every object $a\in\operatorname{Obj} C$ there is an **identity morphism** $\mathrm{id}_a\in\operatorname{Mor}(a,a)$ such that $\mathrm{id}_a\cdot T = T$ and $S\cdot\mathrm{id}_a = S$ for every $T\in\operatorname{Mor}(b,a)$ and every $S\in\operatorname{Mor}(a,c)$, where $b,c\in\operatorname{Obj} C$.

In the example above, the topological space X considered as a set of objects and, for any two points $x,y\in X$, the set of homotopy classes of paths between x and y as a set of morphisms

[1] See ncatlab.org.
[2] Usually, the collections of objects and morphisms are not required to be sets, but generally any classes. A category, where those classes are indeed sets is then called a *small* category. Here, we will work with small categories only and call them simply *categories*.

between x and y give rise to a category. This category is, in addition, a *groupoid* since it satisfies the "group property" that every morphism $\varphi \in \mathrm{Mor}(x,y)$ has its inverse, that is, a path $\varphi^{-1} \in \mathrm{Mor}(y,x)$ such that $\varphi \cdot \varphi^{-1} = \mathrm{id}_y$ and $\varphi^{-1} \cdot \varphi = \mathrm{id}_x$. It is the so-called *fundamental groupoid* of X.

In our case, the following example will be more important. Note that the set of all square matrices $M_N(\mathbb{C})$ over $\mathbb{C}$ forms an algebra. If we want to define an algebraic structure that describes not only square matrices but any matrices, we have to give up the condition that every pair of elements is composable. We are looking for a category that generalises algebras – an *algebroid*.

A **linear category** or an **algebroid** is a category C, where the sets of morphisms $\mathrm{Mor}(a,b)$ form a vector space for every $a,b \in \mathrm{Obj}\, C$.

Consider the set of natural numbers including zero $\mathbb{N}_0$ as a set of objects and, for any $n,m \in \mathbb{N}_0$, the vector space of all complex $m \times n$ matrices as a space of morphisms from n to m. Then we can define the composition of morphisms via the matrix multiplication. Such a category is called the **category of all matrices** over $\mathbb{C}$ and denoted by Mat.

Morphisms are called morphisms simply because the objects of a given category can often be identified with instances of some algebraic structure and then the morphisms are indeed homomorphisms between them. For example, in the category of complex matrices, we can identify any object $n \in \mathbb{N}_0$ with the complex vector space $\mathbb{C}^n$. Then the morphisms from n to m (i.e. the $m \times n$ matrices) indeed play the role of linear maps (i.e. vector space homomorphisms) from $\mathbb{C}^n$ to $\mathbb{C}^m$. We will denote those morphism spaces $\mathsf{Mat}(n,m)$.

Note that instead of considering the category of matrices Mat, where the objects play the role of $\mathbb{C}^n$ vector spaces, we could consider the category of *all* finite-dimensional vector spaces $\mathsf{Vect}_{\mathrm{fin}}$. Here, the objects would be all vector spaces and the morphisms would be linear maps between those vector spaces. Note however that in this case the class of objects is not a set any more. Nevertheless, it is one of the fundamental results of linear algebra that finite-dimensional vector spaces are determined by their dimension. In terms of the category theory, we can say that the categories Mat and $\mathsf{Vect}_{\mathrm{fin}}$ are *equivalent*.

The above example also illustrates one confusing aspect of the common notation. Sometimes the name of the category refers to its morphism spaces (such as the category of matrices) and sometimes it refers to the set of objects (such as the category of vector spaces). In correspondence with this distinction, we sometimes identify the category with its set of objects (we write $V \in \mathsf{Vect}_{\mathrm{fin}}$ for a finite-dimensional vector space V) or with its set of morphisms. However, let us stress that, regardless of the notation, the most important data in a given category are always the morphisms, not the objects. (As we just mentioned, the categories Mat and $\mathsf{Vect}_{\mathrm{fin}}$ are considered to be equivalent although the associated classes of objects are completely different.)

As in the case of any other algebraic structure, we need to define the notion of a homomorphism. Let C_1 and C_2 be categories. A **functor** $F\colon C_1 \to C_2$ is a collection of maps

(a) $F\colon \mathrm{Obj}\, C_1 \to \mathrm{Obj}\, C_2$,
(b) $F\colon \mathrm{Mor}(a,b) \to \mathrm{Mor}(F(a),F(b))$ for all $a,b \in \mathrm{Obj}\, C_1$

such that

$$F(ST) = F(S)F(T), \quad F(\mathrm{id}_a) = \mathrm{id}_{F(a)} \tag{3.1}$$

for every $T \in \mathrm{Mor}(a,b)$, $S \in \mathrm{Mor}(b,c)$, $a,b,c \in \mathrm{Obj}\, C_1$. Note that condition (3.1) is a kind of a homomorphism property for the morphisms. So, a functor is a *morphism of morphisms*. In the case of linear categories, we assume functors to be linear maps.

A functor is called

- **faithful** if it is injective on morphisms,

- **embedding** if it is injective on both objects and morphisms,
- **full** if it is surjective on morphisms,
- **fully faithful** if it is bijective on morphisms,
- **isomorphism** if it is bijective on both objects and morphisms.

For a given category C, its **subcategory** is a category D with $\mathrm{Obj}\, D \subseteq \mathrm{Obj}\, C$, $\mathrm{Mor}_D(a,b) \subseteq \mathrm{Mor}_C(a,b)$, $\mathrm{id}_a^D = \mathrm{id}_a^C$ for $a, b \in D$ and composition given by restriction. This induces a functor embedding $D \to C$. We say that D is a **full subcategory** if the embedding is full. Equivalently, this means that D is given by restricting to a subset of objects $\mathrm{Obj}\, D \subseteq \mathrm{Obj}\, C$ and then taking $\mathrm{Mor}_D(a,b) = \mathrm{Mor}_C(a,b)$ for all $a,b \in \mathrm{Obj}\, D$. In contrast, D is called a **wide subcategory** if $\mathrm{Obj}\, D = \mathrm{Obj}\, C$, but $\mathrm{Mor}_D(a,b) \subseteq \mathrm{Mor}_C(a,b)$.

As an example, notice that Mat can be embedded into $\mathsf{Vect}_{\mathrm{fin}}$. Conversely, by choosing some vector space basis for each finite-dimensional vector space, we may construct a faithful functor $\mathsf{Vect}_{\mathrm{fin}} \to \mathsf{Mat}$.

3.2 Monoidal $*$-categories

Let us now continue with the example of the category of matrices. Within this category, we can actually define additional operations to the composition. First, given two matrices $A \in \mathsf{Mat}(n_1, m_1)$, $B \in \mathsf{Mat}(n_2, m_2)$ we can compute their *tensor product*, which is a matrix $A \otimes B \in \mathsf{Mat}(n_1 n_2, m_1 m_2)$.

A **monoidal category** is a category C equipped with the following additional operations called and denoted usually as a **tensor product**

(a) $\otimes\colon \mathrm{Obj}\, C \times \mathrm{Obj}\, C \to \mathrm{Obj}\, C$,
(b) $\otimes\colon \mathrm{Mor}(a_1, b_1) \times \mathrm{Mor}(a_2, b_2) \to \mathrm{Mor}(a_1 \otimes a_2, b_1 \otimes b_2)$ for all $a_1, a_2, b_1, b_2 \in \mathrm{Obj}\, C$

such that

(1) the operations form a *bifunctor*, that is, $(R \otimes T)(S \otimes U) = RS \otimes TU$ for appropriate morphisms R, S, T, U,
(2) all the operations are associative,[1]
(3) there is an identity object $\emptyset \in \mathrm{Obj}\, C$ such that $\emptyset \otimes a = a \otimes \emptyset = a$ for every object a and $\mathrm{id}_\emptyset \otimes T = T \otimes \mathrm{id}_\emptyset = T$ for every morphism T.[2]

Thus, the tensor product defines the structure of a monoid on the set of objects $\mathrm{Obj}\, C$, hence the name. In the case of linear categories, we assume the tensor product of morphisms to be bilinear.

A functor $F\colon C \to D$ between two monoidal categories is called **monoidal** if it preserves the monoidal structure. That is, $F(a \otimes b) = F(a) \otimes F(b)$ and $F(T \otimes S) = F(T) \otimes F(S)$.

Before we move to the second operation, let us introduce a new category $\mathsf{FinHilb}$ of finite-dimensional Hilbert spaces. Its objects are all finite-dimensional Hilbert spaces. To assure that it indeed is a set, we assume that those are realized as $\mathbb{C}^n$. So, objects of $\mathsf{FinHilb}$ are pairs $(\mathbb{C}^n, \langle\cdot,\cdot\rangle)$, where $n \in \mathbb{N}_0$ and $\langle\cdot,\cdot\rangle$ is some inner product on $\mathbb{C}^n$ (possibly non-standard). The set of morphisms between $H, K \in \mathsf{FinHilb}$ is again the linear space $\mathscr{L}(H,K)$ of all linear maps $H \to K$.[3]

[1] Here, we again do not bring the definition in its full generality. Usually, a weaker condition of associativity "up to an isomorphism" is assumed. Monoidal categories, where the operations $\otimes$ are strictly associative are then called *strict*. All monoidal categories in this thesis will be strict.

[2] Also these equalities may hold only up to some isomorphism in non-strict categories.

[3] In the infinite-dimensional case, bounded operators must be considered. This choice of morphism spaces does not seem to fit into the categorical philosophy since general linear maps do not preserve the inner product and hence are not the proper morphisms. However, considering isometries only would be rather inconvenient (for example, the morphism space $\mathbb{C}^n \to \mathbb{C}^m$ for $n > m$ would be always empty), so it is better in our case to consider indeed all linear maps and encode the Hilbert space structure of the objects into the $*$-structure of morphisms as described below.

The inner product induces a new operation on the morphism spaces. For any linear map $A \in \mathsf{FinHilb}(H,K)$, we can compute its adjoint $A^* \in \mathsf{FinHilb}(K,H)$.

An **involutive category** is a category C equipped with a map $*\colon \mathrm{Mor}(a,b) \to \mathrm{Mor}(b,a)$ for every $a,b \in \mathrm{Obj}\, C$ called the **involution**, which is

(1) *involutive*, i.e. $T^{**} = T$,
(2) *contravariant*, i.e. $(ST)^* = T^*S^*$.

A **$*$-category** is an involutive linear category over $\mathbb{C}$ (or any field with involution), where the involution is an antilinear map. A functor between two involutive categories is called **unitary** if it preserves the involution. In the case of **monoidal involutive categories**, we also require $(T \otimes S)^* = T^* \otimes S^*$ for every pair of morphisms T, S.

Thus, $\mathsf{FinHilb}$ is a monoidal $*$-category. We can make also Mat to be a monoidal $*$-category introducing the involution as the Hermitian conjugation (i.e. taking adjoint with respect to the standard inner product).

Let C be a monoidal involutive category and take an object $a \in \mathrm{Obj}\, C$. We say that $\bar{a} \in \mathrm{Obj}\, C$ is a **dual object** to a if there exist **duality morphisms** $R \in \mathrm{Mor}(1, \bar{a} \otimes a)$ and $\bar{R} \in \mathrm{Mor}(1, a \otimes \bar{a})$ such that

$$(\bar{R}^* \otimes \mathrm{id}_a)(\mathrm{id}_a \otimes R) = \mathrm{id}_a, \qquad (R^* \otimes \mathrm{id}_{\bar{a}})(\mathrm{id}_{\bar{a}} \otimes \bar{R}) = \mathrm{id}_{\bar{a}}.$$

It holds that dual objects are determined uniquely up to an isomorphism. If every object in C has a dual, then we call C a **rigid** category.

3.3 Representation categories

In this section, we bring another example of a monoidal $*$-category. Consider a compact quantum group G. (In particular, one can have some group in mind.) We define Rep_G to be the category of unitary representations of G as follows. The set of objects is the set of all unitary representations of G denoted by Rep_G as well. For a representation $u \in \mathsf{Rep}_G$, denote by $n(u)$ its size (i.e. $u \in M_{n(u)}(C(G))$). The set of morphisms between representations u and v is the set of all linear operators intertwining u and v. Recall that given two representations $u, v \in \mathsf{Rep}_G$, we define the space of intertwiners

$$\mathrm{Mor}(u,v) := \{T\colon \mathbb{C}^{n(u)} \to \mathbb{C}^{n(v)} \mid Tu = vT\},$$

which now plays the role of the morphism space between u and v in Rep_G.

The fact that this indeed defines a monoidal $*$-category results from the following six simple statements about quantum group representations. Suppose u, u', v, v', $w \in \mathsf{Rep}_G$.

(1) The identity matrix $1_{n(u)} \in \mathrm{Mor}(u,u)$.
(2) If $T_1 \in \mathrm{Mor}(u,v)$ and $T_2 \in \mathrm{Mor}(v,w)$, then $T_2T_1 \in \mathrm{Mor}(u,w)$.
(3) If $T \in \mathrm{Mor}(u,v)$, then $T^* \in \mathrm{Mor}(v,u)$.
(4) If $T_1 \in \mathrm{Mor}(u,u')$ and $T_2 \in \mathrm{Mor}(v,v')$, then $T_1 \otimes T_2 \in \mathrm{Mor}(u \otimes u', v \otimes v')$.
(5) It holds that $(u \otimes v) \otimes w = u \otimes (v \otimes w)$.
(6) The trivial representation $1 \in \mathsf{Rep}_G$ satisfies $n(1) = 1$ and $1 \otimes u = u \otimes 1 = u$.

The fact that the morphisms are realized as linear operators, i.e. $\mathrm{Mor}(u,v) \subseteq \mathsf{Mat}(n(u), n(v))$ means that the function n defines a unitary faithful functor $\mathsf{Rep}_G \to \mathsf{Mat}$, so called *fibre functor*. Such categories are sometimes called *concrete*. Note also that the category Rep_G is, in addition, closed under taking subrepresentations and direct sums of representations. This can also be axiomatized within the category theory.

The representation categories of groups and quantum groups were studied by many researchers. The history starts with the work of Tannaka [Tan39] and Krein [Kre49], who showed how to reconstruct compact groups from their representation categories and how to characterize such categories. Today, the result is known as the *Tannaka–Krein duality*. Later many researchers such as Cartier (who defined Hopf algebras for this purpose [Car56]), Saaerda Rivano, or Deligne studied similar duality for algebraic groups introducing so-called *Tannakian categories*. See ⟦EGNO15⟧ for more information about tensor categories and Tannaka–Krein duality in this more algebraic context. The crucial result for our thesis is the work of Woronowicz [Wor88], who generalized the Tannaka–Krein duality for compact quantum groups.

There are many different formulations of the Tannaka–Krein duality for compact quantum groups. In the original paper [Wor88], Woronowicz had compact matrix quantum groups in mind and he used the category Rep_G with some distinguished element playing the role of the fundamental representation. In the appendix of [Wan97], Wang remarks that a similar formulation and the same proof works also for general compact quantum groups. Below, we give a formulation from ⟦NT13⟧, which is more abstract and uses more categorical language. For the matrix case, we will show in Section 3.4 that it may be more convenient to use a bit different category, which is smaller and easier to work with.

3.3.1 Theorem (Woronowicz–Tannaka–Krein for CQGs)**.** Let C be a rigid monoidal $*$-category with finite direct sums and subobjects, $F\colon C \to \mathsf{FinHilb}$ a unitary fibre functor. Then there exists a unique up to isomorphism compact quantum group G and a unitary monoidal equivalence $E\colon C \to \mathsf{Rep}_G$ such that F is naturally unitarily monoidally isomorphic to the composition of the canonical fibre functor $\mathsf{Rep}_G \to \mathsf{FinHilb}$ with E.

This may sound a bit complicated; however, the main idea is quite simple. We already know that we can associate the monoidal $*$-category Rep_G to any compact quantum group G. The Tannaka–Krein duality claims the converse: given a monoidal $*$-category (with some additional axioms), we can interpret it as a representation category of some compact quantum group G and reconstruct G from these data.

We will not explain this formulation of the theorem any further nor give a proof of it. In the following section, we reformulate Tannaka–Krein duality to the special setting of compact matrix quantum groups and give a proof there.

Before we move to the next section, let us mention a few additional remarks.

3.3.2 Proposition. Let G be a compact quantum group and H its quotient. Then Rep_H is a full subcategory of Rep_G. More concretely, Rep_H consists of all representations $u \in \mathsf{Rep}_G$ such that $u_{ij} \in \operatorname{Pol} H \subseteq \operatorname{Pol} G$.

Proof. The proof of this statement becomes clear if we understand, what it precisely means. First, recall that by H being a quotient of G, we mean that $\operatorname{Pol} H$ is a Hopf $*$-subalgebra of $\operatorname{Pol} G$. Now, if u is a representation of G and it has entries in $\operatorname{Pol} H$, it must also be a representation of H. Conversely, any representation of H is also a representation of G. Finally, the definition of the morphism spaces $\operatorname{Mor}(u,v)$ for $u,v \in \mathsf{Rep}_H$ is the same also for G, so the morphism spaces in Rep_H and Rep_G coincide. This is precisely what a full subcategory means – restricting the set of objects while keeping the morphisms spaces. □

3.3.3 Corollary. Let H be a quotient quantum group of G. Then $\operatorname{Irr} H \subseteq \operatorname{Irr} G$. More precisely,

$$\operatorname{Irr} H = \{\alpha \in \operatorname{Irr} G \mid [u^\alpha]_{ij} \in \operatorname{Pol} H \ \forall i,j\} \subseteq \operatorname{Irr} G.$$

Proof. In Proposition 3.3.2, we showed that the representations of H are just a subset of representations of G and that the morphism spaces remain the same. In particular, the notion of irreducibility does not change. □

Finally, we mention another algebraic structure linked to the set of all representations.

3.3.4 Definition. Let G be a compact quantum group. Let R be the set of all equivalence classes of finite-dimensional representations of G. The operation of direct sum $\oplus$ and tensor product $\otimes$

of representations is well defined also for the equivalence classes. We call $(R,\oplus,\otimes)$ the **fusion semiring** of G.

As we already mentioned, the concept of direct sums can be formalized also in the theory of monoidal categories. In that case, the structure of the fusion semiring is already contained in Rep_G as the underlying set (semiring) of objects. This algebraic structure allows to formalize one of the basic questions of the representation theory, namely characterizing the *fusion rules*. That is, given two irreducible representations u and v, how does their tensor product $u\otimes v$ decompose into irreducibles?

3.4 Tannaka–Krein for compact matrix quantum groups

In this section, we reformulate the Woronowicz–Tannaka–Krein duality focusing more on the concrete matrix realization of the quantum groups. Instead of working with the whole category Rep_G of all representations, it is enough, in the matrix case, to use the category $\mathsf{FundRep}_G$ of representations that can be constructed from the fundamental representation just by taking tensor products and complex conjugations. This approach is more convenient for practical computations. Also the proof of the duality is more accessible in this case since it does not require any advanced category theory.

This approach was essentially used from the beginning to produce examples of compact quantum groups. Already in the work [Wor88], Woronowicz constructed the quantum deformation of SU_2 by specifying the associated category generated by the fundamental representation. An extensive use of such an approach began with the discovery of easy quantum groups in [BS09]. However, researchers started to formalize this approach more concretely only recently. We refer to the works [Fre17, Mal18] and the surveys [[Ban19, Fre19]]. Nevertheless, the definitions and notation are still not completely fixed, so a lot of the notation and formulations in this section are the invention of the author although the statements are well-known to the experts.

3.4.1 Two-coloured representation categories

In this section, we formalize the monoidal $*$-categories that will appear in the matrix version of the Tannaka–Krein duality.

Let $G=(C(G),u)$ be a compact matrix quantum group and denote by N the size of the fundamental representation $u\in M_N(C(G))$. Suppose that u is unitary, so we can consider $G\subseteq U^+(F)$ for some $F\in GL_N$. Denote $u^\circ:=u$ and $u^\bullet:=F\bar{u}F^{-1}$.

We define $\mathscr{O}_{\circ\bullet}:=\{\circ,\bullet\}$ a set consisting of a white and a black point. The letter $\mathscr{O}$ stands for *object* since those points will serve as objects in some category. Denote by $\mathscr{O}_{\circ\bullet}^*$ the free monoid over the alphabet $\mathscr{O}_{\circ\bullet}$ and by $\mathscr{O}_{\circ\bullet}^k$ the set of all words of length k. We denote by $\emptyset\in\mathscr{O}_{\circ\bullet}$ the empty word. For any word $w\in\mathscr{O}_{\circ\bullet}^*$, we denote by $|w|$ its length. We define a homomorphism $w\mapsto\bar{w}$ by $\bar{\bullet}=\circ$, $\bar{\circ}=\bullet$ called the **colour inversion**. We denote by $w^\star$ the **reflection** of a word, that is, reading it backwards. We define an **involution** on $\mathscr{O}_{\circ\bullet}^*$ by reversing the word *and* inverting the colours $w^*:=\bar{w}^\star$. For example, $(\bullet\bullet\circ)^*=\bullet\circ\circ$. For any word $w\in\mathscr{O}_{\circ\bullet}^*$, we denote by $u^{\otimes w}$ the corresponding tensor product of representations u° and $u^\bullet$. For example, taking $w=\bullet\bullet\circ$, we have $u^{\otimes w}=u^\bullet\otimes u^\bullet\otimes u^\circ$.

For a pair of words $w_1,w_2\in\mathscr{O}_{\circ\bullet}^*$, denote

$$\mathsf{FundRep}_G(w_1,w_2):=\mathrm{Mor}(u^{\otimes w_1},u^{\otimes w_2})=\{T\colon(\mathbb{C}^N)^{\otimes|w_1|}\to(\mathbb{C}^N)^{\otimes|w_2|}\mid Tu^{\otimes w_1}=u^{\otimes w_2}T\}. \tag{3.2}$$

Such a collection of vector spaces forms a rigid monoidal $*$-category in the following sense.

3.4.1 Definition. Consider a natural number $N\in\mathbb{N}$. A **two-coloured representation category** is a rigid monoidal $*$-category $\mathfrak{C}$ with $\mathscr{O}_{\circ\bullet}^*$ being the monoid of objects (tensor product of objects obtained via the monoid operation, dual objects by involution) and morphisms realized as linear maps

$$\mathfrak{C}(w_1,w_2)\subseteq\mathscr{L}((\mathbb{C}^N)^{\otimes|w_1|},(\mathbb{C}^N)^{\otimes|w_2|}).$$

Equivalently, without referring to the categorical definitions, we can say that a two coloured representation category is a collection of subspaces $\mathfrak{C}(w_1,w_2)\subseteq\mathscr{L}((\mathbb{C}^N)^{\otimes|w_1|},(\mathbb{C}^N)^{\otimes|w_2|})$ satisfying the following five axioms

(1) For $T\in\mathfrak{C}(w_1,w_2)$, $T'\in\mathfrak{C}(w_1',w_2')$, we have $T\otimes T'\in\mathfrak{C}(w_1w_1',w_2w_2')$.
(2) For $T\in\mathfrak{C}(w_1,w_2)$, $S\in\mathfrak{C}(w_2,w_3)$, we have $ST\in\mathfrak{C}(w_1,w_3)$.
(3) For $T\in\mathfrak{C}(w_1,w_2)$, we have $T^*\in\mathfrak{C}(w_2,w_1)$.
(4) For every word $w\in\mathscr{O}_{\circ\bullet}^*$, we have $1_N^{\otimes|w|}\in\mathfrak{C}(w,w)$.
(5) There exist vectors $\xi_{\sqcap_{\circ\bullet}}\in\mathfrak{C}(\emptyset,\circ\bullet)$ and $\xi_{\sqcap_{\bullet\circ}}\in\mathfrak{C}(\emptyset,\bullet\circ)$ such that

$$(\xi^*_{\sqcap_{\circ\bullet}}\otimes 1_N)(1_N\otimes\xi_{\sqcap_{\bullet\circ}})=1_N,\qquad(\xi^*_{\sqcap_{\bullet\circ}}\otimes 1_N)(1_N\otimes\xi_{\sqcap_{\circ\bullet}})=1_N.\tag{3.3}$$

3.4.2 Remark. The axiom (5) says that the objects $\circ$ and $\bullet$ are dual to each other. This already implies that every object $w\in\mathscr{O}_{\circ\bullet}^*$ has a dual w^*. Indeed, taking any $w=c_1\cdots c_k\in\mathscr{O}_{\circ\bullet}^*$, we can construct

$$\xi_w:=\xi_{c_1}\cdot(1_N\otimes\xi_{c_2}\otimes 1_N)\cdots(1_N^{\otimes(k-1)}\otimes\xi_{c_k}\otimes 1_N^{\otimes(k-1)})\in\mathfrak{C}(\emptyset,ww^*),$$

where $\xi_\circ:=\xi_{\sqcap_{\circ\bullet}}$, $\xi_\bullet:=\xi_{\sqcap_{\bullet\circ}}$. Then we can check that the conjugation equations

$$(\xi_w^*\otimes 1_N)(1_N\otimes\xi_{w^*})=1_N,\qquad(\xi_{w^*}^*\otimes 1_N)(1_N\otimes\xi_w)=1_N$$

hold and hence w and w^* are indeed dual to each other.

3.4.3 Remark (Frobenius reciprocity)**.** The rigidity of a category induces isomorphisms between certain morphism spaces. We define a map $\mathrm{Rrot}\colon\mathfrak{C}(w_1,w_2a)\to\mathfrak{C}(w_1\bar{a},w_2)$ for $a\in\{\circ,\bullet\}$ by $T\mapsto(1_N^{\otimes|w_2|}\otimes\xi_a^*)(T\otimes 1_N)$. We call this map the **right rotation**. This map is invertible with $\mathrm{Rrot}^{-1}\colon\mathfrak{C}(w_1a,w_2)\to\mathfrak{C}(w_1,w_2\bar{a})$ defined as $T\mapsto(T\otimes 1_N)(1_N^{\otimes|w_1|}\otimes\xi_a)$. Similarly, we can define the **left rotation** $\mathrm{Lrot}\colon\mathfrak{C}(aw_1,w_2)\to\mathfrak{C}(w_1,\bar{a}w_2)$.

Consequently, this means that the spaces $\mathfrak{C}(w_1,w_2)$ are already determined by the spaces $\mathfrak{C}(\emptyset,w)$ since

$$\mathfrak{C}(w_1,w_2)=\mathrm{Rrot}^{|w_1|}\mathfrak{C}(\emptyset,w_2w_1^*)=\mathrm{Lrot}^{-|w_1|}\mathfrak{C}(\emptyset,w_1^*w_2).$$

The spaces $\mathsf{FundRep}_G(\emptyset,w)=\mathrm{Mor}(1,u^{\otimes w})=\{\xi\in(\mathbb{C}^N)^{\otimes|w|}\mid u^{\otimes w}\xi=\xi\}$ are sometimes denoted by $\mathrm{Fix}(u^{\otimes w})$ and its elements are called the **fixed points** of $u^{\otimes w}$.

Note that the words *left* and *right* do not refer to the direction of the rotation, but to the place, where the rotation acts (it moves either the left-most letter or the right-most letter). In particular, the inverse of the left rotation is not the right rotation and vice versa.

3.4.4 Proposition. For any compact matrix quantum group G, $\mathsf{FundRep}_G$ is a two-coloured representation category.

Proof. The axioms (1)–(4) are obvious. For the duality morphisms, we can write explicit formulae

$$[\xi_{\sqcap_{\circ\bullet}}]_{ij}=F_{ji},\qquad[\xi_{\sqcap_{\bullet\circ}}]_{ij}=[\bar{F}^{-1}]_{ji}.\tag{3.4}$$

We indeed have $\xi_{\sqcap_{\circ\bullet}}\in\mathsf{FundRep}_G(\emptyset,\circ\bullet)$ since the relation $(u^\circ\otimes u^\bullet)\xi_{\sqcap_{\circ\bullet}}=\xi_{\sqcap_{\circ\bullet}}$ says nothing else but $uu^*=1_N$. Similarly, we have $\xi_{\sqcap_{\bullet\circ}}\in\mathsf{FundRep}_G(\emptyset,\bullet\circ)$ since the relation $(u^\bullet\otimes u^\circ)\xi_{\sqcap_{\bullet\circ}}=\xi_{\sqcap_{\bullet\circ}}$ is equivalent to $u^\bullet u^{\bullet *}=1_N$. Checking the conjugation equations (3.3) is also straightforward. □

3.4.5 Remark. The category $\mathsf{FundRep}_G$ already contains all the information of the whole category Rep_G. Actually, Rep_G can be reconstructed from $\mathsf{FundRep}_G$ by a certain completion procedure, so-called *Karoubi envelope*, adding direct sums of representations, subrepresentations and equivalent representations. In particular, note the following facts.

Every irreducible representation of G is contained in $u^{\otimes w}$ for some w. Indeed, let $\mathscr{I}$ be the set of all irreducible representations of G that are subrepresentations of $u^{\otimes w}$ for any $w \in \mathscr{O}_{\circ\bullet}^*$. Since the matrix entries of all the representations generate $O(G) = \operatorname{Pol} G$ and since every representation is a direct sum of irreducibles, we know that the matrix entries of the irreducibles in $\mathscr{I}$ generate $\operatorname{Pol} G$. From Proposition 2.3.1 we then have $\mathscr{I} = \operatorname{Irr} G$.

Given some $u^{\otimes w}$, all its subrepresentations are in one-to-one correspondence with projections in $\operatorname{Mor}(u^{\otimes w}, u^{\otimes w}) = \mathsf{FundRep}(w,w)$. This follows directly from the definition of a subrepresentation. In particular, the irreducible subrepresentations correspond to the minimal projections. Given two irreducibles $u^{\otimes w_1} P_1$ and $u^{\otimes w_2} P_2$, they are equivalent if and only if $P_2 \mathsf{FundRep}_G(w_1, w_2) P_1 \neq \{0\}$.

3.4.2 The duality theorem for CMQGs

The following formulation of Woronowicz–Tannaka–Krein duality basically coincides with [Fre17], where it is formulated without a proof. A similar statement, however only in the *orthogonal* case (that is, with objects indexed by natural numbers instead of words), was formulated and proven also in [Mal18]. We follow here the idea of proof from [Mal18]. The orthogonal version of the statement is presented in Section 3.4.3. Another, even more simplified version, is proven in ⟦Fre19⟧.

3.4.6 Theorem (Woronowicz–Tannaka–Krein for CMQGs)**.** Let $\mathfrak{C}$ be a two-coloured representation category. Then there exists a unique compact matrix quantum group G such that $\mathsf{FundRep}_G = \mathfrak{C}$. It is determined by the ideal

$$I_G = \operatorname{span}\left\{ [T x^{\otimes w_1} - x^{\otimes w_2} T]_{ji} \;\middle|\; \begin{array}{c} T \in \mathfrak{C}(w_1, w_2);\ w_1, w_2 \in \mathscr{O}_{\circ\bullet}^*; \\ i_1, \dots, i_{|w_1|}, j_1, \dots, j_{|w_2|} = 1, \dots, N \end{array} \right\} \subseteq \mathbb{C}\langle x_{ij}, x_{ij}^* \rangle.$$

In order to give a sense to the formula for I_G and to prepare the proof of Theorem 3.4.6, we need to specify the matrix F, so that the matrix $u^\bullet$ is well defined. We fix the duality morphisms satisfying Eqs. (3.3) and define F according to Eqs. (3.4). Note that the duality morphisms are not defined uniquely by Eqs. (3.3). Part of the "uniqueness" statement is that the resulting quantum group does not depend on the particular choice of F.

3.4.7 Lemma. With the notation of Thm. 3.4.6, $I_G / I_{U^+(F)}$ is a Hopf $*$-ideal.

Proof. We denote by v the fundamental representation of $U^+(F)$, so

$$I := I_G / I_{U^+(F)} = \operatorname{span}\{[T v^{\otimes w_1} - v^{\otimes w_2} T]_{ji} \mid T \in \mathfrak{C}(w_1, w_2)\}.$$

(I is an ideal): Take any $T \in \mathfrak{C}(w_1, w_2)$, $\boldsymbol{i} = (i_1, \dots, i_k)$, $\boldsymbol{j} = (j_1, \dots, j_l)$, so $[T v^{\otimes w_1} - v^{\otimes w_2} T]_{ji} \in I$. Then taking any v_{mn}^a with $m, n = 1, \dots, N$ and $a \in \mathscr{O}_{\circ\bullet}$, we have

$$[T v^{\otimes w_1} - v^{\otimes w_2} T]_{ji} v_{mn}^a = [T' v^{\otimes w_1'} - v^{\otimes w_2'} T']_{j'i'} \in I,$$

where $T' := T \otimes 1_N \in \mathfrak{C}(w_1', w_2')$ denoting $w_1' = w_1 a$, $w_2' = w_2 a$, $\boldsymbol{i}' = (\boldsymbol{i}, m)$, $\boldsymbol{j}' = (\boldsymbol{j}, n)$. So, I is a right-ideal; similarly, we prove that it is also a left ideal.

(I is an coideal): Again, take arbitrary $T \in \mathfrak{C}(w_1, w_2)$, $w_1 = (c_1, \ldots, c_k)$, $w_2 = (d_1, \ldots, d_l)$, $\boldsymbol{i}$, $\boldsymbol{j}$. Then

$$\begin{aligned}
\Delta([Tv^{\otimes w_1} - v^{\otimes w_2}T]_{\boldsymbol{ji}}) &= \sum_{\boldsymbol{m}} T_{\boldsymbol{jm}} \Delta(v^{c_1}_{m_1 i_1}) \cdots \Delta(v^{c_k}_{m_k i_k}) - \sum_{\boldsymbol{n}} T_{\boldsymbol{ni}} \Delta(v^{d_1}_{j_1 n_1}) \cdots \Delta(v^{d_l}_{j_l n_l}) \\
&= \sum_{\boldsymbol{ms}} T_{\boldsymbol{jm}} (v^{c_1}_{m_1 s_1} \cdots v^{c_k}_{m_k s_k}) \otimes (v^{c_1}_{s_1 i_1} \cdots v^{c_k}_{s_k i_k}) - \sum_{\boldsymbol{nt}} T_{\boldsymbol{ni}} (v^{d_1}_{j_1 t_1} \cdots v^{d_l}_{j_l t_l}) \otimes (v^{d_1}_{t_1 n_1} \cdots v^{d_l}_{t_l n_l}) \\
&= \sum_{\boldsymbol{s}} [Tv^{\otimes w_1} - v^{\otimes w_2}T]_{\boldsymbol{js}} \otimes (v^{c_1}_{s_1 i_1} \cdots v^{c_k}_{s_k i_k}) + \sum_{\boldsymbol{t}} (v^{d_1}_{j_1 t_1} \cdots v^{d_l}_{j_l t_l}) \otimes [Tv^{\otimes w_1} - v^{\otimes w_2}T]_{\boldsymbol{ti}} \\
&\in I \odot \operatorname{Pol} U^+(F) + \operatorname{Pol} U^+(F) \odot I.
\end{aligned}$$

($S(I) \subseteq I$): This time we define T' as a rotation of T

$$T' := \operatorname{Lrot}^{-l} \operatorname{Rrot}^{-k} T = (\xi^*_{w_2^*} \otimes 1_N^{\otimes k})(1_N^{\otimes l} \otimes T \otimes 1_N^{\otimes k})(1_N^{\otimes l} \otimes \xi_{w_1}) \in \mathfrak{C}(w_2^*, w_1^*).$$

Using Equations (3.4), we can compute that $T' = (\bar{F}^{-1})^{\otimes w_1^*} T'' \bar{F}^{\otimes w_2^*}$, where $[T'']_{(a_1,\ldots,a_k),(b_1,\ldots,b_l)} = T_{(b_l,\ldots,b_1),(a_k,\ldots,a_1)}$, $F^\circ = F$, $F^\bullet = \bar{F}^{-1}$. Since $T' \in \mathfrak{C}(w_2^*, w_1^*)$, we have

$$I \ni [(\bar{F}^{-1})^{\otimes w_1^*} T'' \bar{F}^{\otimes w_2^*} v^{\otimes w_2^*} - v^{\otimes w_1^*} (\bar{F}^{-1})^{\otimes w_1^*} T'' \bar{F}^{\otimes w_2^*}]_{\boldsymbol{ij}}$$

for every $\boldsymbol{i}$, $\boldsymbol{j}$. Since we have already proven that I is an ideal, we can multiply the above matrix by $\bar{F}^{\otimes w_1^*}$ from left and by $(\bar{F}^{-1})^{\otimes w_2^*}$ from right to obtain

$$\begin{aligned}
I &\ni [T'' \bar{F}^{\otimes w_2^*} v^{\otimes w_2^*} (\bar{F}^{-1})^{\otimes w_2^*} - \bar{F}^{\otimes w_1^*} v^{\otimes w_1^*} (\bar{F}^{-1})^{\otimes w_1^*} T'']_{(i_k,\ldots,i_1),(j_l,\ldots,j_1)} \\
&= [T'' \bar{v}^{\otimes \tilde{w}_2} - \bar{v}^{\otimes \tilde{w}_1} T'']_{(i_k,\ldots,i_1),(j_l,\ldots,j_1)} = \sum_{\boldsymbol{n}} T_{\boldsymbol{ni}} (v^{d_l}_{n_l j_l})^* \cdots (v^{d_1}_{n_1 j_1})^* - \sum_{\boldsymbol{m}} T_{\boldsymbol{jm}} (v^{c_k}_{i_k m_k})^* \cdots (v^{c_1}_{i_1 m_1})^* \\
&= -S[Tv^{\otimes w_1} - v^{\otimes w_2}T]_{\boldsymbol{ji}}.
\end{aligned}$$

($I^* \subseteq I$): Take any $T \in \mathfrak{C}(w_1, w_2)$, $\boldsymbol{i} = (i_1, \ldots, i_k)$, $\boldsymbol{j} = (j_1, \ldots, j_l)$, so $[Tv^{\otimes w_1} - v^{\otimes w_2}T]_{\boldsymbol{ji}} \in I$. Then also $T^* \in \mathfrak{C}(w_2, w_1)$ and hence $[v^{\otimes w_1}T^* - T^*v^{\otimes w_2}]_{\boldsymbol{ij}} \in I$, so

$$I \ni S[v^{\otimes w_1}T^* - T^*v^{\otimes w_2}]_{\boldsymbol{ij}} = ([Tv^{\otimes w_1} - v^{\otimes w_2}T]_{\boldsymbol{ji}})^*. \qquad \square$$

We are now ready to prove Woronowicz–Tannaka–Krein duality for compact matrix quantum groups.

Proof of Theorem 3.4.6. As we have proven in Lemma 3.4.7, $I_G/I_{U^+(F)}$ is a Hopf $*$-ideal, so I_G indeed defines a compact quantum group $G \subseteq U^+(F)$ with

$$\operatorname{Pol} G = \mathbb{C}\langle x_{ij}, x_{ij}^* \rangle / I_G = \operatorname{Pol} U^+(F) / (I_G/I_{U^+(F)}).$$

As we already mentioned, this definition may depend on the duality morphisms we pick to determine the matrix F, which defines the representation $x^\bullet$ and gives sense to the symbols $x^{\otimes w}$. So, let $\tilde{\xi}_{\sqcap_{\circ\bullet}}$, $\tilde{\xi}_{\sqcap_{\bullet\circ}}$ be alternative solutions of the conjugation equations (3.3) and let $\tilde{F}$ be the alternative matrix and $\tilde{G}$ the alternative resulting quantum group. Then we have

$$(\tilde{\xi}^*_{\sqcap_{\bullet\circ}} \otimes 1_N)(1_N \otimes \xi_{\sqcap_{\circ\bullet}}) = F\tilde{F}^{-1} \in \mathsf{FundRep}_G(\bullet, \bullet) = \operatorname{Mor}(u^\bullet, u^\bullet).$$

This means that $u^\bullet = \tilde{F}F^{-1}u^\bullet F\tilde{F}^{-1} = \tilde{F}\bar{u}\tilde{F}^{-1}$ and hence $G \subseteq \tilde{G}$. From symmetry, we have $G = \tilde{G}$.

It remains to prove that we indeed have $\mathsf{FundRep}_G = \mathfrak{C}$ (from construction, we can actually easily see that $\mathsf{FundRep}_G \supseteq \mathfrak{C}$) and that G is a unique quantum group with this property, that is, if $\mathsf{FundRep}_{\tilde{G}} = \mathfrak{C}$ for some quantum group $\tilde{G} \subseteq U^+(F)$, then surely $G = \tilde{G}$ (again, from construction, we obviously have $G \supseteq \tilde{G}$).

We need some preparation. We are going to use the double commutant theorem, so let us define the following finite-dimensional objects.

$$\begin{aligned} A_n &:= \{f \in \mathbb{C}\langle x_{ij}, x_{ij}^*\rangle \mid \deg f \leq n\} \subseteq \mathbb{C}\langle x_{ij}, x_{ij}^*\rangle \quad \text{(considering } \deg x_{ij} = \deg x_{ij}^* = 1\text{)},\\ I_n &:= I_G \cap A_n = \{f \in I_G \mid \deg f \leq n\} = \operatorname{span}\{[Tx^{\otimes w_1} - xT^{\otimes w_2}] \mid |w_1|, |w_2| \leq n\},\\ V_w &:= (\mathbb{C}^N)^{\otimes k}, \quad w \in \mathscr{O}^k_{\circ\bullet},\\ V_n &:= \bigoplus_{\substack{w\in\mathscr{O}^k_{\circ\bullet}\\ k\leq n}} V_w,\\ B_n &:= \bigoplus_{\substack{w_1,w_2\in\mathscr{O}^*_{\circ\bullet}\\ |w_1|,|w_2|\leq n}} \mathfrak{C}(w_1, w_2) \subseteq \mathscr{L}(V_n),\\ U_n &:= \bigoplus_{\substack{w\in\mathscr{O}^k_{\circ\bullet}\\ k\leq n}} u^{\otimes w} \in \mathscr{L}(V_n) \otimes C(G). \end{aligned}$$

Now, the equality $\mathsf{FundRep}_G = \mathfrak{C}$ is equivalent to saying $\operatorname{Mor}(U_n, U_n) = B_n$ for every n. By the double commutant theorem (Thm. 1.2.7), this equality can be equivalently expressed by applying the commutant to both sides. By Lemma 2.4.3, we can express the left-hand side as

$$\operatorname{Mor}(U_n, U_n)' = \{\pi_{U_n}(\omega) \mid \omega \in (\operatorname{Pol} G)^*\} = \{\pi_{U_n}(\omega) \mid \omega \in (A_n/I_n)^*\}, \tag{3.5}$$

where, in the second equality, we used the fact that taking $\omega \in (\operatorname{Pol} G)^*$, the result $\pi_{U_n}(\omega)$ is a matrix with entries $\omega([u^{\otimes w}]_{ji})$, which only depends on how ω acts on the subspace $A_n/I_n \subseteq \operatorname{Pol} G$.

On the right hand side, we have the commutant

$$B_n' = \{S \in \mathscr{L}(V_n) \mid TS = ST \text{ for all } T \in B_n\}.$$

For operators $S \in \mathscr{L}(V_n)$, let us denote by $S_{\mathbf{j}\mathbf{i}}^{w_2w_1}$ their matrix entries. Then we can write the condition above as

$$S_{\mathbf{j}\mathbf{i}}^{w_2w_1} = 0 \qquad \text{if } w_1 \neq w_2, \tag{3.6}$$

$$\sum_{\mathbf{k}} T_{\mathbf{j}\mathbf{k}} S_{\mathbf{k}\mathbf{i}}^{w_1w_1} = \sum_{\mathbf{k}} S_{\mathbf{j}\mathbf{k}}^{w_2w_2} T_{\mathbf{k}\mathbf{i}} \qquad \text{for all } T \in \mathfrak{C}(w_1, w_2) \text{ with } |w_1|, |w_2| \leq n. \tag{3.7}$$

For every $S \in \mathscr{L}(V_n)$, let us denote by $\omega_S \in A_n^*$ the functional mapping $[x^{\otimes w}]_{\mathbf{j}\mathbf{i}} \mapsto S_{\mathbf{j}\mathbf{i}}^{ww}$. Then we claim that

$$I_n = \{f \in A_n \mid \omega_S(f) = 0 \text{ for all } S \in B_n'\}.$$

This is indeed true since the defining relations $[Tx^{\otimes w_1} - x^{\otimes w_2}T]_{\mathbf{j}\mathbf{i}}$ exactly map to Relation (3.7) after applying ω_S.

Now, let us analyse the commutant from Eq. (3.5). We have

$$(A_n/I_n)^* = \{\omega \in A_n^* \mid \omega(f) = 0 \text{ for all } f \in I_n\} = \{\omega_S \mid S \in B_n'\},$$

so the functionals ω_S are well-defined also on A_n/I_n. Moreover, we have $[\pi_{U_n}(\omega_S)]_{\mathbf{j}\mathbf{i}}^{w_1w_2} = \delta_{w_1w_2}\omega_S([u^{\otimes w_1}]_{\mathbf{j}\mathbf{i}}) = S_{\mathbf{j}\mathbf{i}}^{w_1w_2}$, so $\pi_{U_n}(\omega_S) = S$. This proves the desired equality $\operatorname{Mor}(U_n, U_n)' = B_n'$.

So, let us move to the uniqueness. We need to prove that the ideal determining the quantum group is already uniquely determined by the associated category. So, suppose there is some other $\tilde{G} = (C(\tilde{G}), \tilde{u})$ with $\mathsf{FundRep}_{\tilde{G}} = \mathfrak{C}$. Denote, as above, $\tilde{U}_n := \bigoplus_{|w|\leq n} \tilde{u}^{\otimes w}$ and $\tilde{I}_n = I_{\tilde{G}} \cap A_n$. The equality $\mathsf{FundRep}_{\tilde{G}} = \mathfrak{C}$ can then be equivalently described as

$$\{\pi_{\tilde{U}_n}(\omega) \mid \omega \in (A_n/\tilde{I}_n)^*\} = B_n'.$$

We need to prove that $\tilde{I}_n = I_n$. The space $\tilde{I}_n$ is equivalently described by the set of functionals vanishing on $\tilde{I}_n$

$$(A_n/\tilde{I}_n)^* = \{\omega \in A_n^* \mid \omega(f) = 0 \text{ for all } f \in \tilde{I}_n\}.$$

Those functionals are determined by their coordinates $\omega([x^{\otimes w}]_{\mathbf{j}\mathbf{i}}) = \omega([\tilde{u}^{\otimes w}]_{\mathbf{j}\mathbf{i}}) = \pi_{\tilde{U}_n}(\omega)$. Thus, $(A_n/\tilde{I}_n)^*$ is determined by the algebra B_n', which is determined by the category $\mathfrak{C}$. □

3.4.8 Remark. The standard Tannaka–Krein duality formulated as Theorem 3.3.1 for the whole representation category Rep_G associates a quantum group up to isomorphism. In contrast, the category $\mathsf{FundRep}_G$ used in the formulation of Theorem 3.4.6 determines also the fundamental representation. So, two quantum groups G_1 and G_2 have the same representation category $\mathsf{FundRep}_{G_1} = \mathsf{FundRep}_{G_2}$ if and only if they are identical according to the definition formulated in Section 2.3.2.

3.4.3 Orthogonal version

In this section, we focus on more specialized version of the duality, where we assume that $G \subseteq O^+(F)$ for some F. This allows to simplify even more the set of objects of the associated category. In the unitary version, we needed to consider words over an alphabet with two elements $\mathscr{O}_{\circ\bullet} = \{\circ, \bullet\}$ that stand for the representation u° and $u^\bullet$. Now, we have $u^\circ = u^\bullet$, so one letter is enough. Words over an alphabet with one letter are nothing else but natural numbers.

3.4.9 Definition. Consider a natural number $N \in \mathbb{N}$. A **non-coloured representation category** is a collection of spaces

$$\mathfrak{C}(k,l) \subseteq \mathscr{L}((\mathbb{C}^N)^{\otimes k}, (\mathbb{C}^N)^{\otimes l})$$

with $k, l \in \mathbb{N}_0$ such that

(1) For $T \in \mathfrak{C}(k,l)$, $T' \in \mathfrak{C}(k',l')$, we have $T \otimes T' \in \mathfrak{C}(k+k', l+l')$.
(2) For $T \in \mathfrak{C}(k,l)$, $S \in \mathfrak{C}(l,m)$, we have $ST \in \mathfrak{C}(k,m)$.
(3) For $T \in \mathfrak{C}(k,l)$, we have $T^* \in \mathfrak{C}(l,k)$.
(4) For every $k \in \mathbb{N}_0$, we have $1_N^{\otimes k} \in \mathfrak{C}(k,k)$.
(5) There exists a vector $\xi_{\sqcap} \in \mathfrak{C}(0,2)$ such that $(\xi_{\sqcap}^* \otimes 1_N)(1_N \otimes \xi_{\sqcap}) = c\,1_N$, $c \in \mathbb{R}$.

It is a rigid monoidal $*$-category, where the monoid of objects are the natural numbers $\mathbb{N}_0$ and every object is self-dual.

3.4.10 Proposition. Let G be a compact matrix quantum group. We have $G \subseteq O^+(F)$ for some $F \in \mathrm{GL}_N$ such that $F\bar{F} = c\,1_N$ if and only if $\mathsf{FundRep}(w_1, w_2) = \mathsf{FundRep}(w_1', w_2')$ whenever $|w_1| = |w_1'|$ and $|w_2| = |w_2'|$.

Proof. The inclusion $G \subseteq O^+(F)$ is equivalent to saying that $u^\circ = u^\bullet$, which is equivalent to saying that $1_N \in \mathrm{Mor}(u^\circ, u^\bullet) = \mathsf{FundRep}(\circ, \bullet)$. So, if $\mathsf{FundRep}(w_1, w_2) = \mathsf{FundRep}(w_1', w_2')$ whenever $|w_1| = |w_1'|$ and $|w_2| = |w_2'|$, then in particular $1_N \in \mathrm{Mor}(u,u) = \mathrm{Mor}(u^\circ, u^\bullet)$, which proves the right-left implication.

Now consider $G \subseteq O^+(F)$, so $1_N \in \mathsf{FundRep}(\circ, \bullet)$. This implies that $1_N^{\otimes k} \in \mathsf{FundRep}(w, w')$ for every $w, w' \in \mathscr{O}_{\circ\bullet}^k$. So, $\mathsf{FundRep}(w_1', w_2') = 1_N^{\otimes |w_2|} \mathsf{FundRep}(w_1, w_2) 1_N^{\otimes |w_2|} = \mathsf{FundRep}(w_1, w_2)$. □

For $G \subseteq O^+(F)$, we can denote

$$\mathsf{FundRep}(k,l) := \mathsf{FundRep}(\circ^k, \circ^l) = \mathrm{Mor}(u^{\otimes k}, u^{\otimes l}).$$

This gives $\mathsf{FundRep}$ the structure of a non-coloured representation category. In particular, we have $\xi_{\sqcap} := \xi_{\sqcap_{\circ\bullet}} \in \mathsf{FundRep}(0,2)$, satisfying $(\xi_{\sqcap}^* \otimes 1_N)(1_N \otimes \xi_{\sqcap}) = F\bar{F} = c\,1_N$.

3.4.11 Corollary (Non-coloured Tannaka–Krein for CMQGs)**.** Let $\mathfrak{C}$ be a non-coloured representation category. Then there exists a unique compact matrix quantum group $G = (C(G), u) \subseteq O^+(F)$ with $F_{ji} = [\xi_{\sqcap}]_{ij}$ (hence $F\bar{F} = c\,1_N$) such that $\mathsf{FundRep}_G = \mathfrak{C}$, that is, $\mathrm{Mor}(u^{\otimes k}, u^{\otimes l}) = \mathfrak{C}(k,l)$.

Proof. This follows directly from Theorem 3.4.6. We can make $\mathfrak{C}$ a two-coloured representation category by setting $\mathfrak{C}(w_1, w_2) := \mathfrak{C}(|w_1|, |w_2|)$ for every $w_1, w_2 \in \mathscr{O}_{\circ\bullet}^*$. □

3.4.4 Generators of a representation category

Quantum groups are usually characterized not by the whole ideal I_G, but only by certain relations that generate the ideal. We would like to formulate such a notion also for the representation categories.

For any collection of sets $C(w_1,w_2)$ of linear maps $(\mathbb{C}^N)^{\otimes|w_1|}\to(\mathbb{C}^N)^{\otimes|w_2|}$ satisfying the axiom (5) of two-coloured representation categories, we denote by $\langle C\rangle$ the smallest category containing C. We say that C **generates** this category.

3.4.12 Proposition. Let $G\subseteq U^+(F)$ be a compact matrix quantum group. Suppose that the associated category $\mathsf{FundRep}_G$ is generated by C. Then $I_G\subseteq\mathbb{C}\langle x_{ij},x_{ij}^*\rangle$ is an ideal generated by

$$\{[Tx^{\otimes w_1}-x^{\otimes w_2}T]_{ji}\mid T\in C(w_1,w_2)\},$$

so

$$C(G)=C^*(u_{ij}\mid[Tu^{\otimes w_1}]_{ji}=[u^{\otimes w_2}T]_{ji};\ T\in C(w_1,w_2)).$$

Proof. Denote by I the ideal generated by C as formulated above. Obviously, we have $I\subseteq I_G$. To prove the opposite inclusion, it is enough to prove that

$$\mathfrak{C}(w_1,w_2):=\{T\colon(\mathbb{C}^N)^{\otimes|w_1|}\to(\mathbb{C}^N)^{\otimes|w_2|}\mid Tx^{\otimes w_1}-x^{\otimes w_2}T\in I\}$$

form a category. Then, since obviously $C\subseteq\mathfrak{C}$ and hence $\mathsf{FundRep}_G\subseteq\mathfrak{C}$, we must have $I_G\subseteq I$.

So, denote $A:=\mathbb{C}\langle x_{ij},x_{ij}^*\rangle/I$ and by v_{ij} denote the images of x_{ij} by the natural homomorphism. Take $T_1\in\mathfrak{C}(w_1,w_2)$, $T_2\in\mathfrak{C}(w_2,w_3)$. Then

$$T_2T_1v^{\otimes w_1}=T_2v^{\otimes w_2}T_1=v^{\otimes w_3}T_2T_1,$$

so $T_2T_1\in\mathfrak{C}(w_1,w_3)$. For tensor product and involution, the proof is similar. □

3.4.13 Corollary. Let $G\subseteq U^+(F)$ be a compact matrix quantum group. Then the associated ideal I_G is generated by the relations $u^{\otimes w}\xi=\xi$ for $\xi\in\mathsf{FundRep}(\emptyset,w)$, $w\in\mathscr{O}_{\circ\bullet}^*$.

Proof. By Frobenius reciprocity (Remark 3.4.3), any two-coloured representation category $\mathfrak{C}$ is generated by the spaces $\mathfrak{C}(\emptyset,w)$, $w\in\mathscr{O}_{\circ\bullet}^*$. Then we apply Proposition 3.4.12. □

Let us also formulate a non-coloured version of the proposition.

3.4.14 Proposition. Let $G\subseteq O^+(F)$ be a compact matrix quantum group. Suppose that the associated category $\mathsf{FundRep}_G$ is generated by C. Then $I_G\subseteq\mathbb{C}\langle x_{ij},x_{ij}^*\rangle$ is an ideal generated by the relation $x_{ij}^\circ=x_{ij}^\bullet$ and

$$\{[Tx^{\otimes k}-x^{\otimes l}T]_{ji}\mid T\in C(k,l)\},$$

so

$$C(G)=C^*(u_{ij}\mid u=F\bar{u}F^{-1},[Tu^{\otimes k}]_{ji}=[u^{\otimes l}T]_{ji};\ T\in C(k,l)).$$

3.4.5 Quantum subgroups, intersections, topological generation

In this section, we link the notions of quantum subgroup, intersection of quantum groups and quantum group topological generation with the corresponding representation categories. In Chapter 2, we linked these notions with the associated ideal I_G. However, in the case of intersection, we skipped the proof, which we provide now.

3.4.15 Proposition. Suppose G,H are compact matrix quantum groups with unitary fundamental representations. The following are equivalent.

(1) $H\subseteq G$,
(2) $I_H\supseteq I_G$,
(3) $\mathsf{FundRep}_H(w_1,w_2)\supseteq\mathsf{FundRep}_G(w_1,w_2)$ for all $w_1,w_2\in\mathscr{O}_{\circ\bullet}^*$.

Proof. The equivalence (1) ⇔ (2) was proven as Lemma 2.3.8. The equivalence (2) ⇔ (3) follows from the Tannaka–Krein duality (Thm. 3.4.6). □

3.4.16 Remark. In Proposition 3.3.2, we showed that, if K is a quotient quantum group of G, then Rep_K is a full subcategory of Rep_G. So, Proposition 3.4.15 might sound confusing as subcategories should rather correspond to quotient groups.

The explanation lies in the fact that we are not only working with a different category, but we are using also a different notion of a *subcategory*. For K a quotient of G, Rep_K is a *full* subcategory of Rep_G. That is, considering all possible representations of G, we choose just some of them and we keep all the intertwiners. In contrast, if H is a quantum subgroup of G, then $\mathsf{FundRep}_H$ is a *wide* subcategory of $\mathsf{FundRep}_G$. We are not picking a subset of representations here. The set of objects remains always the same – it is the monoid $\mathscr{O}^*_{\circ\bullet}$ representing tensor products of the fundamental representation and its dual. In contrast, we are restricting the sets of morphisms here.

In Chapter 8, we are going to enrich the monoid of objects a bit. Then it will also be interesting to study the full subcategories and those will indeed correspond to quotient quantum groups, see Proposition 8.2.3.

3.4.17 Proposition. Suppose G, H_1, and H_2 are compact matrix quantum groups with unitary fundamental representations of the same size. The following are equivalent.

(1) $G = H_1 \cap H_2$,
(2) $I_G = I_{H_1} + I_{H_2}$,
(3) $\mathsf{FundRep}_G$ is generated by $\mathsf{FundRep}_{H_1} \cup \mathsf{FundRep}_{H_2}$.

Proof. The equivalence (2) $\Leftrightarrow$ (3) follows from Proposition 3.4.12: The equality $I_G = I_{H_1} + I_{H_2}$ is equivalent to saying the I_G is the ideal generated by $I_{H_1} \cup I_{H_2}$. So, if $\mathsf{FundRep}_G$ is generated by $\mathsf{FundRep}_{H_1} \cup \mathsf{FundRep}_{H_2}$, we have indeed (2) directly by Proposition 3.4.12. For the converse, suppose that $I_G = I_{H_1} + I_{H_2}$. Let $\mathfrak{C}$ be the category generated by $\mathsf{FundRep}_{H_1} \cup \mathsf{FundRep}_{H_2}$. Then by Proposition 3.4.12, $\mathfrak{C}$ corresponds to a quantum group $\tilde{G}$ with $I_{\tilde{G}} = I_{H_1} + I_{H_2} = I_G$, so $\tilde{G} = G$.

Now, we prove (1) $\Rightarrow$ (3). Denote by $\tilde{G}$ the compact matrix quantum group corresponding to the category $\mathsf{FundRep}_{\tilde{G}} := \langle \mathsf{FundRep}_{H_1}, \mathsf{FundRep}_{H_2} \rangle$. From Proposition 3.4.15, we know that $\tilde{G} \subseteq H_1, H_2$. We also know that $G \subseteq H_1, H_2$, so from Proposition 3.4.15, we have that $\mathsf{FundRep}_{H_k} \subseteq \mathsf{FundRep}_G$ for $k = 1, 2$ and hence $\mathsf{FundRep}_{\tilde{G}} \subseteq \mathsf{FundRep}_G$. This, however, means that $G \subseteq \tilde{G}$ and hence $G = \tilde{G}$ by the definition of $H_1 \cap H_2$.

The converse (3) $\Rightarrow$ (1) is proven similarly. We surely have $G \subseteq H_1, H_2$ and, in addition, G is obviously the largest possible with this property, so $G = H_1 \cap H_2$. □

3.4.18 Remark. As a consequence of this proposition, we have that the intersection is defined uniquely. In Section 2.5.4, we promised to prove a similar statement without the assumption on the fundamental representations v_1 and v_2 of H_1 and H_2 being unitary (see Prop. 2.5.9). From Proposition 2.2.3, we know that they can be unitarized. So, suppose $T_1 v_1 T_1^{-1}$ and $T_2 v_2 T_2^{-1}$ are unitary. Then we can study the representation category corresponding to the unitary quantum group $G := T_1(H_1 \cap H_2)T_1^{-1}$, which is given by

$$\mathsf{FundRep}_G = \langle \mathsf{FundRep}_{T_1 H_1 T_1^{-1}}, T_1 T_2^{-1} \mathsf{FundRep}_{T_2 H_2 T_2^{-1}} T_2 T_1^{-1} \rangle,$$

so $I_G = T_1(I_{H_1} + I_{H_2})T_1^{-1}$ and consequently $I_{H_1 \cap H_2} = I_{H_1} + I_{H_2}$.

3.4.19 Proposition. Suppose G, H_1, and H_2 are compact matrix quantum groups with unitary fundamental representations of the same size. The following are equivalent.

(1) $G = \langle H_1, H_2 \rangle$,
(2) $I_G = I_{H_1} \cap I_{H_2}$,
(3) $\mathsf{FundRep}_G(w_1, w_2) = \mathsf{FundRep}_{H_1}(w_1, w_2) \cap \mathsf{FundRep}_{H_2}(w_1, w_2)$ for all $w_1, w_2 \in \mathscr{O}^*_{\circ\bullet}$.

In particular, $\langle H_1, H_2 \rangle$ exists if and only if there is a matrix F such that $H_1, H_2 \subseteq U^+(F)$.

Proof. The equivalence (1) $\Leftrightarrow$ (2) was proven in Proposition 2.5.10. The equivalence (2) $\Leftrightarrow$ (3) follows from the Tannaka–Krein duality (Thm. 3.4.6).

The collection of spaces $\mathsf{FundRep}_{H_1}(k,l) \cap \mathsf{FundRep}_{H_2}(k,l)$ defines a compact matrix quantum group if and only if it is a two-coloured representation category. All axioms are automatically satisfied except for (5) – the existence of duals. We have (5) if and only if $\mathsf{FundRep}_{H_1}$ and $\mathsf{FundRep}_{H_2}$ contain common duality morphisms $\xi_{\sqcap_{\circ\bullet}}, \xi_{\sqcap_{\bullet\circ}}$. This is equivalent to saying $H_1, H_2 \subseteq U^+(F)$, where $F_{ji} = [\xi_{\sqcap_{\circ\bullet}}]_{ij}$. □

Part II

Partitions

This part is basically independent of the rest of the thesis. We do not refer at all to the theory of quantum groups here. However, the motivation for the problems comes from the theory of compact matrix quantum groups and will be described in Part III. For the readers who are interested in applications of categories of partitions to quantum groups, it may be convenient to read every chapter or section of this part parallel to the corresponding text in Part III.

Categories of partitions in the context of compact quantum groups were defined by Banica and Speicher in [BS09]. Afterwards, a lot of effort was made to classify those structures. The classification was successfully completed in [RW16]. At the same time, the structure of quantum groups corresponding to categories of partitions (so called *easy* quantum groups) were also studied. The results of this theory were summarized in the survey ⟦Web17⟧.

Although much work has been already done in the theory of categories of partitions, the research in this area is surely not exhausted. In particular, categories of partitions in the sense of the original definition of Banica and Speicher are far away from describing *all* quantum groups. Thus, a wide range of possibilities to generalize the concept of categories of partitions opens up to be able to describe more quantum groups. In this part, we describe several possibilities of generalizing categories of partitions, where the author contributed also by his research.

In order to simplify the orientation in the results, we summarize all the sections of Part II and Part III in the following table. In the first column, we list sections of Part II, in the second column, we state the corresponding section of Part III that analyses the associated quantum groups. In the third column, we briefly describe the content and in the last column we list the main references the sections are based on. The numbers of sections that are mainly based on the author's results are highlighted in italics.

Section in Part II	Section in Part III	Content	References
4.1–4.2, 5.1	6.1	Introduction and basic results for the non-coloured case	[BS09, ...]
4.3	6.3	Coloured partitions in general	[Fre19, GW19b]
4.4–*4.5*	*6.2*	Two-coloured partitions	[TW18, TW17, Gro18]
4.6	*6.4*	Partitions with extra singletons	[GW19b]
5.2–*5.3*	—	Computer experiments with linear categories	[GW19a]
5.4	*7.1*, *7.3*, *7.4*	Interpreting non-easy categories	[GW20, GW19a]
—	*7.2*	Link between linear and extra-singleton categories	[GW19b]

Chapter 4
Categories of partitions

This chapter is essentially pure combinatorics. We study categories of partitions as defined by Banica and Speicher and their coloured generalizations. We focus in this chapter on classification problems of those categories. The complete classification is already available for the original categories of partitions, but still an open problem for the categories of coloured partitions. The basic underlying elements of those structures are *set partitions*, that is, combinatorial objects. The classification problems are, therefore, very much of a combinatorial nature.

Let us now summarize the structure and main results of this chapter. Section 4.1 introduces the concept of categories of partitions as it was defined by Banica and Speicher in [BS09]. In Section 4.2, we summarize the classification of categories of partitions and we give an overview of the basic classification techniques. Regarding these classical categories of partitions, we refer to the survey ⟦Web17⟧, where the classification result as well as the link to the theory of quantum groups is explained. Section 4.3 introduces the concept of colouring for partitions in general. In Section 4.4, we study the particular example of *two-coloured* partitions that were defined in [TW18, Fre17]. We also summarize the classification result of [TW18]. After this, two sections with the author's original results follow.

In Section 4.5, the classification of *globally colourized* categories of partitions is presented. Globally colourized categories form a special class of two-coloured categories containing a specific partition that allows to do colour permutations. We solve the classification problem by showing that every globally colourized category is characterized by its degree of reflection and by a non-coloured category.

Theorem (4.5.10)**.** Every globally colourized category $\mathscr{C}$ is determined by a number $k(\mathscr{C})$ and a non-coloured category of partitions not containing the singleton partition. The full classification of globally colourized categories is summarized in Table 4.1.

In Section 6.4, we introduce another generalization of categories of partitions, namely categories of *partitions with extra singletons*. The motivation for such a definition will be explained in Part III. Here, we attack the classification problem by showing that it is essentially equivalent to classification of two-coloured categories. Moreover, the functor appearing in the theorem will play an important role later when we interpret the categories.

Theorem (4.6.8)**.** There is a functor providing a one-to-one correspondence between categories of partitions with extra singletons of even length and categories of two-coloured partitions that are invariant with respect to the colour inversion.

4.1 Categories of non-coloured partitions

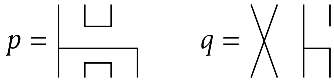
This section introduces the categories of partitions as defined in [BS09].

4.1.1 Definition of categories of partitions

Consider $k, l \in \mathbb{N}_0$; by a **partition** of k upper and l lower points we mean a partition of the set $\{1,\ldots,k\} \sqcup \{1,\ldots,l\} \approx \{1,\ldots,k+l\}$. That is, a decomposition of the set of $k+l$ points into non-empty disjoint subsets called **blocks**. The first k points are called **upper** and the last l points are called **lower**. The set of all partitions on k upper and l lower points is denoted by $\mathscr{P}(k,l)$. The number $|p| := k+l$ for $p \in \mathscr{P}(k,l)$ is called the **length** of p.

We illustrate partitions graphically by putting k points in one row and l points in another row below and connecting by lines those points that are grouped in one block. All lines are drawn between those two rows.

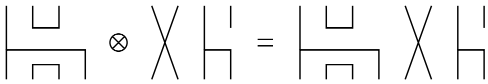
Below, we give an example of two partitions $p \in \mathscr{P}(3,4)$ and $q \in \mathscr{P}(4,4)$ including their graphical representation. The first set of points is decomposed into three blocks, whereas the second one splits into four blocks. In addition, the first one is an example of a **non-crossing** partition, i.e. a partition that can be drawn in a way that lines connecting different blocks do not intersect (following the rule that all lines are between the two rows of points). In contrast, the second partition has one crossing.


(4.1)

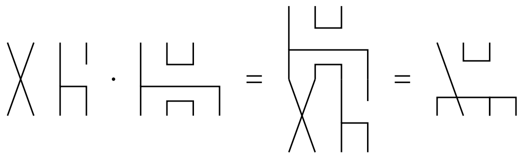
In our graphical notation, if two or more strings cross each other, we never assume they are connected. On the other hand, if three strings meet at one point (typically like this ⊤), we of course assume they are connected. Thus, a partition on two upper and two lower points where all points are in a single block is denoted like this , whereas the diagram ╳ stands for a partition consisting of two blocks.

We define the following operations on the set of all partitions. Every operation is illustrated on an example using the partitions (4.1).

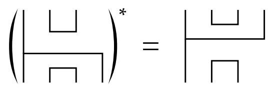
For $p \in \mathscr{P}(k,l)$, $q \in \mathscr{P}(k',l')$, we define their **tensor product** $p \otimes q \in \mathscr{P}(k+k', l+l')$ by putting their graphical representations side by side.



For $p \in \mathscr{P}(k,l)$, $q \in \mathscr{P}(l,m)$, we define their **composition** $qp \in \mathscr{P}(k,m)$ by putting the graphical representation of q below p. We interpret the upper row of p with k points as the upper row of qp and the lower row of q with m points as the lower row of qp. The potential extra cycles occurring in the middle should be deleted.

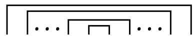


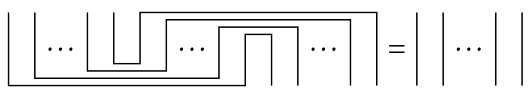
For $p \in \mathscr{P}(k,l)$, we define its **involution** $p^* \in \mathscr{P}(l,k)$ by reversing its graphical representation with respect to the horizontal axis.



The above defined operations give partitions the structure of a rigid monoidal involutive category (see Sect. 3.2) in the following sense. The set of objects here is the set of natural numbers $\mathbb{N}_0$, on which the tensor product acts as addition. Partitions $p \in \mathscr{P}(k,l)$ are then morphisms between k and l. For every $k \in \mathbb{N}$ we have the identity morphism

$$\mathrm{id}_k = |^{\otimes k} = \underbrace{|\;|\cdots|}_{k\times} \in \mathscr{P}(k,k).$$

The **empty partition** $\emptyset \in \mathscr{P}(0,0)$ is then the identity associated to the object 0. Every object $k \in \mathbb{N}_0$ is dual to itself. This follows from the fact that, for every $k \in \mathbb{N}_0$, we have the partition $r =$  $\in \mathscr{P}(0,2k)$ satisfying

$(r^* \otimes \mathrm{id}_k)(\mathrm{id}_k \otimes r) =$  $= \mathrm{id}_k$

We will call this category the **category of all partitions** and denote it by $\mathscr{P}$ (the same as we denote the set of all partitions).

A collection of sets $\mathscr{C}(k,l) \subseteq \mathscr{P}(k,l)$, $k,l \in \mathbb{N}_0$ containing the **identity partition** $| \in \mathscr{C}(1,1)$ and the **pair partition** $\sqcap \in \mathscr{C}(0,2)$ that is closed under the category operations defined above will be called a **category of partitions**. That is, a category of partitions is any rigid wide subcategory $\mathscr{C} \subseteq \mathscr{P}$ (recall the definition from the end of Sect. 3.1). Note that all the duality morphisms $\sqcap\!\!\!\sqcap \ldots \sqcap \ldots$ are generated by the pair partition and hence are always contained in any category of partitions.

For any partition p, its blocks of size one are called **singletons**. In particular, a partition consisting of a single point is called a **singleton**. For clarity, we denote it by an arrow $\uparrow \in \mathscr{P}(0,1)$, $\downarrow \in \mathscr{P}(1,0)$.

4.1.2 Frobenius reciprocity and partitions on one line

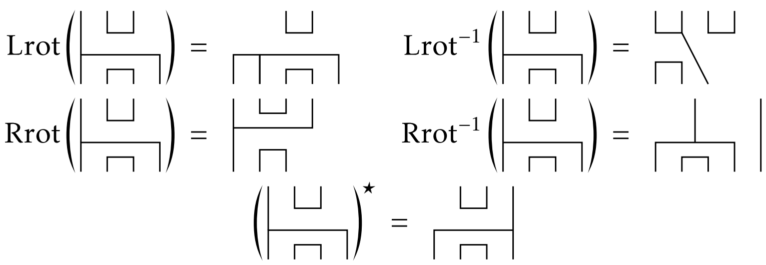
In this section, we define additional operations on the categories of partitions that can be performed thanks to the presence of the duality morphism. The main purpose is to show that given a category of partitions $\mathscr{C}$, the sets of partitions with lower points only $\mathscr{C}(0,k)$ already determine the whole category. This will simplify some proofs in the future in two ways: First, if we need to prove something for all partitions $p \in \mathscr{C}$, it is usually enough to prove it for partitions with lower points only. Secondly, we will define some new operations on partitions that will allow us to formulate an alternative definition of a category of partitions, which works with partitions on one line. If we then need to prove that some set is a category, it will often be easier to work with those one-line operations than with the standard ones.

Similar statements (also known as the *Frobenius reciprocity*) are actually true for any rigid monoidal category – we already formulated it for representation categories in Remark 3.4.3 and we will mention it in subsequent sections also for coloured generalizations of categories of partitions. In particular, the idea of *rotating* partitions was already mentioned in [BS09] and used extensively in the classification programme. Nevertheless, some of the definitions and statements were first formulated in [Gro18, GW19a] (in particular the alternative definition of a category of partitions).

For $p \in \mathscr{P}(k,l)$, $k > 0$, its **left rotation** is a partition $\mathrm{Lrot}\, p \in \mathscr{P}(k-1,l+1)$ obtained by moving the leftmost point of the upper row to the beginning of the lower row. Similarly, for $p \in \mathscr{P}(k,l)$, $l > 0$, we can define its **right rotation** $\mathrm{Rrot}\, p \in \mathscr{P}(k+1,l-1)$ by moving the last point of the lower row to the end of the upper row. Both operations are obviously invertible.

For $p \in \mathscr{P}(k,l)$, we define its **vertical reflection** $p^\star \in \mathscr{P}(k,l)$ by vertically reflecting its graphical representation. Note that $p^\star$ differs from p^*, which was defined as the *horizontal* reflection.

Borrowing again the partition p from Eq. (4.1), we can illustrate those definitions on this example



4.1.1 Lemma. [Web13, RW14] Let $\mathscr{C}$ be a category of partitions. Then it is closed under rotations (left and right including inverses) and vertical reflection.

Proof. For all the rotations, the proof is similar. For $p \in \mathscr{P}(k,l)$ we have

$$\mathrm{Lrot}\, p = (| \otimes p)(\sqcap \otimes |^{\otimes k-1}), \qquad \mathrm{Rrot}^{-1}\, p = (p \otimes |)(|^{\otimes k-1} \otimes \sqcap), \qquad \text{if } k > 0,$$
$$\mathrm{Lrot}^{-1}\, p = (\sqcup \otimes |^{\otimes l-1})(| \otimes p), \qquad \mathrm{Rrot}\, p = (|^{\otimes l-1} \otimes \sqcup)(p \otimes |), \qquad \text{if } l > 0.$$

Finally, we have

$$p^\star = \mathrm{Lrot}^l \, \mathrm{Rrot}^k \, p^*. \qquad \square$$

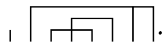
Since any category of partitions $\mathscr{C}$ is closed under rotations, it means that it is completely described by the collection of spaces $\mathscr{C}(k) := \mathscr{C}(0,k)$ of partitions with lower points only. To provide an equivalent definition of a category of partitions using only partitions with lower points, we can define the following operations on $\mathscr{P}(k)$.

For $p \in \mathscr{P}(k) = \mathscr{P}(0,k)$, we define its **rotation** $Rp \in \mathscr{P}(k)$ as the partition obtained by shifting the last point to the front. We can write it using left and right rotation as $Rp = (\mathrm{Lrot} \circ \mathrm{Rrot})p$.

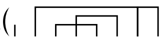
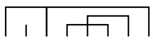
For $p \in \mathscr{P}(k)$, by **reflection** of p, we mean its vertical reflection $p^\star \in \mathscr{P}(k)$.

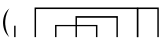
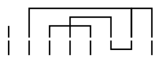
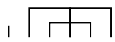
Let $p \in \mathscr{P}(k)$, $k > 2$ and let $i \in \{1, \ldots, k\}$. By **contraction** of p on the i-th point, we mean $\Pi_i p := R^{-i+1} \Pi_1 R^{i-1} p$, where $\Pi_1 q = (\sqcup \otimes |^{\otimes k-2})q$. We essentially take the i-th and the $(i+1)$-st point, connect the corresponding blocks, and then remove the two points.

Again, we illustrate those constructions on examples. Taking p and q from Eq. (4.1), we can define their rotated versions

$p' := \mathrm{Lrot}^3 \, p =$ , $\qquad q' := \mathrm{Lrot}^4 \, q =$ . (4.2)

Now, we illustrate the operations on q':

$R($  $) =$ ,

$\Pi_6($  $) =$  $=$ .

4.1.2 Proposition. [Gro18, GW19a] For any category of partitions $\mathscr{C}$, the collection of sets $\mathscr{C}(k) = \mathscr{C}(0,k)$, $k \in \mathbb{N}_0$ is closed under tensor products, rotations, reflections, and contractions. Conversely, for any collection of sets of partitions with lower points $\mathscr{C}(k) \subseteq \mathscr{P}(0,k)$ that contains $\sqcap \in \mathscr{C}(0,2)$ and is closed under tensor products, rotations, reflections, and contractions, the sets

$$\mathscr{C}(k,l) := \{\mathrm{Rrot}^k \, p \mid p \in \mathscr{C}(0, k+l)\} = \{\mathrm{Lrot}^{-k} \, p \mid p \in \mathscr{C}(k+l)\}$$

form a category of partitions.

Proof. The first part of the proposition follows from the fact that all the four operations for partitions on one line are composed of category operations from Section 4.1.1.

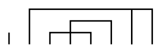
The converse statement is proved by the following:

$$\begin{aligned}
\mathrm{Lrot}^{-k} \, p \otimes \mathrm{Rrot}^{k'} \, q &= \mathrm{Lrot}^{-k} \, \mathrm{Rrot}^{k'} (p \otimes q), \\
(\mathrm{Rrot}^k \, p)^* &= \mathrm{Rrot}^l \, p^\star, \\
(\mathrm{Rrot}^l \, q)(\mathrm{Rrot}^k \, p) &= \mathrm{Rrot}^k \, \Pi_{m+1} \Pi_{m+2} \cdots \Pi_{m+l} (q \otimes p),
\end{aligned}$$

where we assume that $p \in \mathscr{P}(k+l)$ and $q \in \mathscr{P}(k'+l')$ for the first row and $q \in \mathscr{P}(l+m)$ for the last row. $\square$

The partitions with lower points only can be conveniently represented not only graphically but also using words. We can assign a letter to each block of a given partition $p \in \mathscr{P}(k)$. Then p is represented by a word $a_1 a_2 \cdots a_k$, where a_i is the letter representing the block of the point i.

As an example, we mention the word representation of partitions p' and q' from Eq. (4.2):

$p' =$  $=$ aabbccb, $\quad q' =$  $=$ abcdcdbb.

4.2 Classification of categories of partitions

In this section, we summarize the known classification results for categories of partitions and present the basic classification tools. Those results and similar ideas are then used also in the coloured case.

4.2.1 Full classification summary

The categories of partitions were studied in [BS09, BCS10, Web13, RW14, RW15] and their full classification was completed in [RW16].

The classification is summarized in the following table. It is divided into four cases. The **non-crossing categories** consisting of non-crossing partitions, the **group categories** containing the **crossing partition** $\times$, the **half-liberated categories** containing the **half-liberating partition** $\ast$, but not the crossing partition, and the rest.

Non-crossing	$\langle\rangle, \langle \sqcap\!\sqcap\!\sqcap \rangle, \langle \sqcap\!\sqcap\!\sqcap, \uparrow\otimes\uparrow\rangle, \langle\uparrow\otimes\uparrow\rangle, \langle \sqcap_{\shortmid}\ \shortmid\rangle, \langle\uparrow\rangle, \langle\uparrow, \sqcap\!\sqcap\!\sqcap\rangle$	
Group	$\langle\times\rangle, \langle\times,\uparrow\rangle, \langle\times,\uparrow\otimes\uparrow\rangle$	
	$\langle\times, \sqcap\!\sqcap\!\sqcap\rangle, \langle\times, \sqcap\!\sqcap\!\sqcap,\uparrow\rangle, \langle\times, \sqcap\!\sqcap\!\sqcap, \uparrow\otimes\uparrow\rangle$	$(\ast)$
Half-liberated	$\langle\ast\rangle, \langle\ast\uparrow\otimes\uparrow\rangle$	
	$\langle\ast, \sqcap\!\sqcap\!\sqcap\rangle, \langle\ast, \sqcap\!\sqcap\!\sqcap, h_s\rangle, s\geq 3$	$(\ast)$
The rest	$\langle\pi_s\rangle, s\geq 2, \langle\pi_l \mid l\in\mathbb{N}\rangle, A\trianglelefteq\mathbb{Z}_2^{\infty}$	

Here, we denote by h_s, $s\in\mathbb{N}_0$ (here $s\geq 3$) the partition of length $2s$ consisting of two blocks, where the first block connects all odd points and the second block connects all even points. We can represent this partition by the word $(\mathsf{ab})^s = \underbrace{\mathsf{ababab}\cdots\mathsf{ab}}_{s\times}$.

By π_s, we denote the partition represented by the following word of length $4s$

$$\pi_s = \mathsf{a}_1\mathsf{a}_2\cdots\mathsf{a}_s\mathsf{a}_s\cdots\mathsf{a}_2\mathsf{a}_1\mathsf{a}_1\mathsf{a}_2\cdots\mathsf{a}_s\mathsf{a}_s\cdots\mathsf{a}_2\mathsf{a}_1.$$

Finally, consider the infinite free product $\mathbb{Z}_2^{*\infty}$ and denote its generators by a_i, $i\in\mathbb{N}$. Then the elements of $\mathbb{Z}_2^{*\infty}$ are words over the alphabet $\{\mathsf{a}_i\}_{i\in\mathbb{N}}$ and therefore stand for partitions. It can be shown that if a normal subgroup $A\trianglelefteq\mathbb{Z}_2^{*\infty}$ is so called sS_∞-invariant, i.e. invariant with respect to the homomorphisms of the form $\mathsf{a}_i\mapsto\mathsf{a}_{\varphi(i)}$, where $\varphi\colon\mathbb{N}\to\mathbb{N}$ is any map that coincides with the identity outside of a finite set, then the corresponding partitions form a category. Such categories are called **group-theoretical**. This is indicated in the last entry of the summary.

The group categories and half-liberated categories in the rows marked by asterisk $(\ast)$ are special instances of group-theoretical categories. Except for this fact, all the categories in the summary are pairwise distinct.

In the following, we summarize the basic classification techniques.

4.2.2 Important partitions for classification

We already mentioned two important partitions contained in every category – the identity partition $|$ and the pair partition $\sqcap$. The latter allowed us to perform the rotation operation. There are some other distinguished partitions, whose presence in the category allows us to formulate some invariance principles. Such considerations play a key role in the classification of categories of partitions.

The ideas presented here were, of course, used through the whole classification programme. Nevertheless, it is hard to give some reference since the ideas were never formulated as explicit statements in the case of non-coloured partitions. The formulation we use here mostly coincides with [TW18], where it was formulated for categories of two-coloured partitions.

4.2.1 Lemma. Let $\mathscr{C}$ be a category of partitions.

(1) If $\sqcap\!\sqcap\!\sqcap \in \mathscr{C}$, then $\mathscr{C}$ is closed under connecting neighbouring blocks.
(2) If $\uparrow \otimes \uparrow \in \mathscr{C}$, then $\mathscr{C}$ is closed under disconnecting an arbitrary point from a block.
(3) If $\sqcap\!\!\!\mid \; \mid \in \mathscr{C}$, then $\uparrow \otimes \uparrow \in \mathscr{C}$, so (2) applies. In addition, $\mathscr{C}$ is closed under shifting singletons.
(4) If $\times \in \mathscr{C}$, then $\mathscr{C}$ is closed under permutations of points.
(5) If $\ast \in \mathscr{C}$, then $\mathscr{C}$ is closed under permutations of points on even positions and odd positions separately.

Proof. The proofs for all the cases are basically the same. So, let us illustrate the proof of the lemma on the item (1).

Consider the partition $q = \times \; \sqcup\!\!\!\mid$. Choose two neighbouring points, say the second and the third one in the bottom row. Now, we want to unite their blocks, i.e. to construct the partition $q' = \times\!\!\!-\!\!\!\sqcup\!\!\!\mid$.

The assumption $\sqcap\!\sqcap\!\sqcap \in \mathscr{C}$ implies that $\sqcup\!\!\!\sqcap \in \mathscr{C}$ by rotation. Since also $| \in \mathscr{C}$, we have $p := | \otimes \sqcup\!\!\!\sqcap \otimes | \in \mathscr{C}$. One can easily check that $q' = pq$. □

4.2.2 Lemma. Let $\mathscr{C}$ be a category of partitions

(1) The category $\mathscr{C}$ contains a partition with a block of size one if and only if $\mathscr{C}$ contains $\uparrow \otimes \uparrow$.
(2) The category $\mathscr{C}$ contains a partition with a block of size greater than two if and only if $\mathscr{C}$ contains $\sqcap\!\sqcap\!\sqcap$.
(3) The category $\mathscr{C}$ contains a partition of odd length if and only if $\uparrow \in \mathscr{C}$.

Proof. Suppose $p \in \mathscr{C}$ contains a singleton. Without loss of generality we can assume that p is of the form $p = \uparrow \otimes p'$, where $p' \in \mathscr{P}(k-1)$ (otherwise use rotations). We have

$$\begin{aligned} \left(| \otimes (\sqcup)^{\otimes (k-1)/2}\right) p &= \uparrow && \text{if } k \text{ is odd,} \\ \left(| \otimes | \otimes (\sqcup)^{\otimes (k-2)/2}\right) p &= \uparrow \otimes \uparrow && \text{if } k \text{ is even.} \end{aligned}$$

The second statement is a bit complicated. Take some $p \in \mathscr{C}$ containing a block of size greater than two. By rotation, we can assume that p has lower points only and that the first point is incident with the mentioned block. Then we do contractions in the middle of $p \otimes p^\star$ until we contract some point incident with the mentioned block. The resulting partition q then again has a block with size greater than two (the size must be even, so at least four actually) and the first and the last point of q are incident to this block. Finally, we can construct $\sqcap\!\sqcap\!\sqcap$ by contracting $Rq \otimes (Rq)^\star$. See [TW18, Lemma 2.1] for more details on the proof.

The last statement is the easiest one. Just take a partition $p \in \mathscr{P}(0, k)$ for k odd and contract until we get the singleton $\uparrow$. □

4.2.3 Sketch of proof for non-crossing and group categories

To illustrate the work with categories of partitions, let us sketch the proof of the classification for the simplest cases [Web13] – the non-crossing case and the group case. Note however, that the rest of the classification is much harder to obtain.

First, it is easy to see the following.

4.2.3 Proposition. The set of non-crossing partitions $NC \subseteq \mathscr{P}$ forms a category.

It is also easy to check that the intersection of any two categories of partitions is again a category of partitions. In particular, we can construct the category $\mathscr{C}_{NC} := \mathscr{C} \cap NC$ for any given $\mathscr{C}$.

4.2.4 Proposition. Let $\mathscr{C}$ be a group category. Then $\mathscr{C} = \langle \mathscr{C}_{NC}, ╳ \rangle$.

Proof. Every element $p \in \mathscr{C}$ can be transformed by permuting points to a non-crossing partition $p_{NC} \in \mathscr{C}_{NC} \subseteq \mathscr{C}$ (using the crossing partition ╳, see Lemma 4.2.1). Reversing this process, we can obtain back the original partition p, so $p \in \langle \mathscr{C}_{NC}, ╳ \rangle$. □

As a consequence, the classification of group categories can be obtained from the classification of non-crossing categories simply by adding the crossing partition ╳ to them. We only have to check, whether the group categories are mutually different. In the end, we find out that seven non-crossing categories induce six group categories since $\langle$┌╷┐ ╷, ╳$\rangle = \langle \uparrow \otimes \uparrow$, ╳$\rangle$.

Now, let us sketch the proof of the classification in the non-crossing case.

One can check that for the smallest category we have

$$\langle \rangle = NC_2 := \{p \in NC \mid \text{all blocks of } p \text{ are of size two}\}.$$

Suppose we have a category $\mathscr{C}$ containing a partition with a block of size one, but not a partition with a block of size larger than two. This is equivalent to stating $\uparrow \otimes \uparrow \in \mathscr{C}$ and ┌┬┬┐ $\notin \mathscr{C}$. Then either $\mathscr{C}$ is equal to

$$\langle \uparrow \rangle = \{p \in NC \mid \text{all blocks of } p \text{ are of size} \leq 2\}$$

or $\mathscr{C}$ contains only partitions of even length. We can check that

$$\langle \uparrow \otimes \uparrow \rangle = \left\{ p \in NC \;\middle|\; \begin{array}{c} |p| \text{ is even; all blocks of } p \text{ are of size} \leq 2; \\ \text{every block of size 2 has an even number of points between its legs} \end{array} \right\}.$$

The number of points *between* two given is counted with respect to the cyclical order. That is, the points in the upper row are ordered from right to left, then the leftmost point in the lower row follows and the rest of the lower points ordered from left to right. Since p has even length, it does not matter from which side we count.

One can also check that a partition $p \in \langle \uparrow \rangle \setminus \langle \uparrow \otimes \uparrow \rangle$ of even length, that is, a partition containing a block connecting two odd points or two even points, must generate ┌╷┐ ╷. So, the last possibility is that $\mathscr{C}$ is equal to

$$\langle ┌╷┐\,╷ \rangle = \{p \in NC \mid |p| \text{ is even; all blocks of } p \text{ are of size} \leq 2\}.$$

In the case when ┌┬┬┐ $\in \mathscr{C}$, but $\uparrow \otimes \uparrow \notin \mathscr{C}$, the situation is simple, since

$$\langle ┌┬┬┐ \rangle = \{p \in NC \mid \text{all blocks of } p \text{ are of even size}\}.$$

Any larger category would contain $\uparrow \otimes \uparrow$.

Finally suppose $\uparrow \otimes \uparrow \in \mathscr{C}$ and ┌┬┬┐ $\in \mathscr{C}$. In this case, the category is closed under connecting neighbouring blocks and removing points from blocks, so it is irrelevant, which point is in which block. The only relevant thing is the number of points in the partition. Since the category contains the pair partitions of length two, it is closed under adding or subtracting an even number of points. Thus there are only two possibilities

$$\langle ┌┬┬┐, \uparrow \otimes \uparrow \rangle = \{p \in NC \mid |p| \text{ is even}\},$$
$$\langle ┌┬┬┐, \uparrow \rangle = NC.$$

4.3 Coloured partitions

In this section, we introduce in general the concept of colouring partitions. As a reference, see [Fre19, GW19b].

Let $\mathscr{O}$ be a set with involution $x \mapsto \bar{x}$. We denote by $\mathscr{O}^*$ the monoid of words over $\mathscr{O}$ and by $\mathscr{O}^k$ the set of all words of length $k \in \mathbb{N}_0$. The involution is extended as a homomorphism $w \mapsto \bar{w}$ to the whole $\mathscr{O}^*$. We call it the **colour inversion**. We denote by w^* the word $\bar{w}$ read backwards.

An $\mathscr{O}$**-coloured partition** on k upper and l lower points is a triple (p, w_1, w_2), where $p \in \mathscr{P}(k,l)$, $w_1 \in \mathscr{O}^k$, $w_2 \in \mathscr{O}^l$. We denote $\mathscr{P}^{\mathscr{O}}(w_1,w_2) := \{(p,w_1,w_2) \mid p \in \mathscr{P}(k,l)\}$ the set of all partitions with **upper colour pattern** w_1 and **lower colour pattern** w_2. The set of all $\mathscr{O}$-coloured partitions with k upper and l lower points will be denoted by $\mathscr{P}^{\mathscr{O}}(k,l) := \bigcup_{w_1 \in \mathscr{O}^k, w_2 \in \mathscr{O}^l} \mathscr{P}^{\mathscr{O}}(w_1,w_2)$. We denote by $\mathscr{P}^{\mathscr{O}}$ the set of all $\mathscr{O}$-coloured partitions.

We define the category operations induced by category operations of ordinary partitions. That is,

- **tensor product** $(p,w_1,w_2) \otimes (q,w_3,w_4) = (p \otimes q, w_1w_3, w_2w_4)$,
- **composition** $(q,w_2,w_3) \cdot (p,w_1,w_2) = (qp, w_1, w_3)$,
- **involution** $(p,w_1,w_2)^* = (p^*, w_2, w_1)$.

This gives $\mathscr{P}^{\mathscr{O}}$ the structure of a monoidal involutive category (see Sect. 3.2) with $\mathscr{O}^*$ being the monoid of objects and $\mathscr{P}^{\mathscr{O}}(w_1,w_2)$ the morphism spaces. Let us stress that the composition is defined only if the colour patterns match (not just the number of points). We denote by $\emptyset \in \mathscr{O}^*$ the *empty word*, which also plays the role of the *identity object* of the monoidal category.

Until now, we did not use the involution on the set $\mathscr{O}$, this comes only in the following definition. A **category of** $\mathscr{O}$**-coloured partitions** is a collection of subsets $\mathscr{C}(w_1,w_2) \subseteq \mathscr{P}^{\mathscr{O}}(w_1,w_2)$ closed under the category operations and containing the identity partitions $(|, x, x)$ and pair partitions $(\sqcap, x, \bar{x})$ for every $x \in \mathscr{O}$. In other words, $\mathscr{C}$ is any rigid wide subcategory of $\mathscr{P}^{\mathscr{O}}$ such that the duality of objects is given by the involution $w \mapsto w^*$ on $\mathscr{O}^*$.

Again, any $\mathscr{O}$-coloured category is also closed under **left** and **right rotations**

$$\mathrm{Lrot}(p, xw_1, w_2) = (\mathrm{Lrot}\, p, w_1, \bar{x}w_2), \qquad \mathrm{Rrot}(p, w_1, w_2x) = (\mathrm{Rrot}\, p, w_1\bar{x}, w_2).$$

and **(verticolour) reflections** $(p,w_1,w_2)^\star = (p^\star, w_1^*, w_2^*)$. For $\mathscr{O}$-coloured partitions with lower points only, we define in addition the **rotation** $R(p, \emptyset, wx) = (Rp, \emptyset, xw)$ and **contraction**, which is defined in a similar manner, but only if the two points that are contracted have colours dual to each other. That is, $\Pi_1(p, \emptyset, x\bar{x}w) = (\Pi_1 p, \emptyset, w)$.

Again, a category $\mathscr{C} \subseteq \mathscr{P}^{\mathscr{O}}$ is completely described by the collection of sets $\mathscr{C}(k) := \mathscr{C}(0,k)$. Conversely, such a collection of sets defines a category if and only if it is closed under tensor products, contractions, rotations and reflections. (See Prop. 4.1.2.)

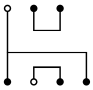
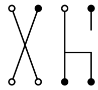
4.4 Two colours dual to each other

Now, we introduce two-coloured categories as defined in [TW18, Fre17]. In this case, the full classification is still an open problem. However, there has been a rapid development regarding this question in the recent time. The results known so far are available in [TW18, Gro18, MW19a, MW19b, MW19c, MW20]. After formulating basic definitions, we mention some tools for solving classification problems developed in [TW18] and we summarize classification results obtained in [TW18]. In the following section (Sect. 4.5), the author's contribution to the classification programme will be presented.

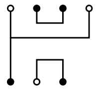
4.4.1 Definition

By a **two-coloured partition**, we mean an $\mathcal{O}_{\circ\bullet}$-coloured partition, where $\mathcal{O}_{\circ\bullet} = \{\circ, \bullet\}$ with $\bar{\circ} = \bullet$, $\bar{\bullet} = \circ$. We also use the term **category of two-coloured partitions** for a category of $\mathcal{O}_{\circ\bullet}$-coloured partitions. Sometimes, we also say two-coloured *unitary* partitions to avoid confusion with partitions on two self-dual colours.

So, a two-coloured partition is a partition, where, in addition, we assign to each point a white or black colour. Again, we will represent such objects pictorially. Below, we show an example of a possible colouring of the partitions in Equation (4.1).

$p =$  $\in \mathscr{P}^{\circ\bullet}(\circ\bullet\bullet, \bullet\circ\bullet\bullet) \subseteq \mathscr{P}^{\circ\bullet}(3,4)$ $\qquad q =$  $\in \mathscr{P}^{\circ\bullet}(\circ\bullet\circ\bullet, \circ\circ\bullet\bullet) \subseteq \mathscr{P}^{\circ\bullet}(4,4)$ (4.3)

Recall that also in the coloured case, we have left and right rotations, but in contrast with the non-coloured case we must change the colour of a point when moving it from one row to the other. For example, taking p from (4.3), we have

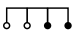
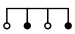
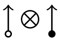
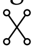
$\operatorname{Rrot} p =$ .

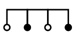
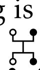
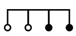
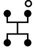
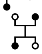
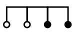
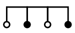
We also have the verticolour reflections, which is a vertical reflection composed with reversing all colours.

4.4.2 Important two-coloured partitions for classification

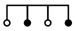
In this section, we formulate analogues of Lemmata 4.2.1, 4.2.2 and then introduce some additional notation. Another important partitions and corresponding invariant principles that are specific for the two-coloured framework will be discussed in Sections 4.4.3–4.4.4.

4.4.1 Lemma. Let $\mathscr{C}$ be a category of two-coloured partitions.

(1) If  $\in \mathscr{C}$, then $\mathscr{C}$ is closed under connecting neighbouring blocks.
(2) If  $\in \mathscr{C}$, then $\mathscr{C}$ is closed under connecting neighbouring blocks if they meet at two points with inverse colours.
(3) If  $\in \mathscr{C}$, then $\mathscr{C}$ is closed under disconnecting an arbitrary point from a block.
(4) If  $\in \mathscr{C}$, then  $\in \mathscr{C}$, so (b) applies. In addition, $\mathscr{C}$ is closed under shifting singletons.
(5) If  $\in \mathscr{C}$, then $\mathscr{C}$ is closed under permutations of points (carrying their colour).

Proof. The proof is similar as in Lemma 4.2.1. One subtlety worth mentioning is the difference between item (1) and (2). The possible rotations of  contain only  and , but not  and . In contrast, the possible rotations of  contain all , , , . Using composition, we can generate also  and  from the last two. This also proves the implication  $\in \mathscr{C} \Rightarrow$  $\in \mathscr{C}$. □

4.4.2 Lemma. Let $\mathscr{C}$ be a category of partitions.

(1) The category $\mathscr{C}$ contains a partition with a block of size one if and only if $\mathscr{C}$ contains .
(2) The category $\mathscr{C}$ contains a partition with a block of size greater than two if and only if $\mathscr{C}$ contains .

Proof. As in Lemma 4.2.2. □

As a consequence, the categories of partitions behave quite differently depending on whether those mentioned partitions are elements or not. For that reason, we define the following four cases for categories of partitions.

- $\mathscr{C}$ is in **case $\mathcal{O}$**, if $\uparrow_{\circ} \otimes \uparrow_{\bullet} \notin \mathscr{C}$ and $\sqcap\!\sqcap\!\sqcap_{\circ\bullet\circ\bullet} \notin \mathscr{C}$,
- $\mathscr{C}$ is in **case $\mathcal{B}$**, if $\uparrow_{\circ} \otimes \uparrow_{\bullet} \in \mathscr{C}$ and $\sqcap\!\sqcap\!\sqcap_{\circ\bullet\circ\bullet} \notin \mathscr{C}$,
- $\mathscr{C}$ is in **case $\mathcal{H}$**, if $\uparrow_{\circ} \otimes \uparrow_{\bullet} \notin \mathscr{C}$ and $\sqcap\!\sqcap\!\sqcap_{\circ\bullet\circ\bullet} \in \mathscr{C}$,
- $\mathscr{C}$ is in **case $\mathcal{S}$**, if $\uparrow_{\circ} \otimes \uparrow_{\bullet} \in \mathscr{C}$ and $\sqcap\!\sqcap\!\sqcap_{\circ\bullet\circ\bullet} \in \mathscr{C}$.

The letters $\mathcal{O}$, $\mathcal{B}$, $\mathcal{H}$, and $\mathcal{S}$ stand for orthogonal, bistochastic, hyperoctahedral, and symmetric group, which are groups corresponding to particular instances of those categories, see Chapter 6.

4.4.3 Relation to non-coloured categories

In this section, we study a functor $\Psi \colon \mathscr{P}^{\circ\bullet} \to \mathscr{P}$ forgetting the colour patterns [TW18]. Taking its preimage allows us to embed non-coloured categories into the framework of the two-coloured ones.

4.4.3 Lemma. Let $\mathscr{C}$ be a category of two-coloured partitions containing the unicoloured pair partition $\sqcap_{\circ\circ} \in \mathscr{C}$. Then $\mathscr{C}$ is closed under changing colours arbitrarily. That is, if $p \in \mathscr{C}$, then $p' \in \mathscr{C}$, where p' is obtained from p by making an arbitrary choice for the colours of the points (keeping all blocks the same).

Proof. The category $\mathscr{C}$ contains $\sqcap_{\circ\circ}$ if and only if it contains $|^{\circ}_{\bullet}$ and $|^{\bullet}_{\circ}$. Changing the colour pattern of a given partition $p \in \mathscr{C}$ can be made by composing it with appropriate tensor product of $|^{\circ}_{\circ}$, $|^{\bullet}_{\bullet}$, $|^{\circ}_{\bullet}$, and $|^{\bullet}_{\circ}$. □

This means that if a category contains the unicoloured pair, then the colouring of its elements is irrelevant. Hence, such a category can be identified with a category of non-coloured partitions in the following way.

4.4.4 Definition. Let $\Psi \colon \mathscr{P}^{\circ\bullet} \to \mathscr{P}$ be the map given by forgetting the colours of a two-coloured partition. For $\mathscr{C} \subseteq \mathscr{P}$, denote $\Psi^{-1}(\mathscr{C}) \subseteq \mathscr{P}^{\circ\bullet}$ its preimage under Ψ.

4.4.5 Proposition.

(1) Let $\mathscr{C} \subseteq \mathscr{P}$ be a category of non-coloured partitions. Then $\Psi^{-1}(\mathscr{C}) \subseteq \mathscr{P}^{\circ\bullet}$ is a category of two-coloured partitions containing the unicoloured pair partitions $\sqcap_{\circ\circ}$ and $\sqcap_{\bullet\bullet}$.

(2) Let $\mathscr{C} \subseteq \mathscr{P}^{\circ\bullet}$ be a category of two-coloured partitions containing the unicoloured pair partition $\sqcap_{\circ\circ}$ (or equivalently $\sqcap_{\bullet\bullet}$). Then $\Psi(\mathscr{C}) \subseteq \mathscr{P}$ is a category of non-coloured partitions and $\Psi^{-1}(\Psi(\mathscr{C})) = \mathscr{C}$.

Hence, there is a one-to-one correspondence between categories of non-coloured partitions and categories of two-coloured partitions containing $\sqcap_{\circ\circ}$.

In the following we will often not distinguish between categories of non-coloured partitions and categories of two-coloured partitions containing $\sqcap_{\circ\circ}$. In particular, a (two-coloured) category will be called **non-coloured** if it contains $\sqcap_{\circ\circ}$.

4.4.4 Global colourization and degree of reflection

The tensor product $\sqcap_{\circ\circ} \otimes \sqcap_{\bullet\bullet}$ possesses a similar, but weaker property. The following again comes from [TW18].

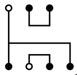
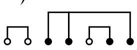
4.4.6 Lemma. Let $\mathscr{C}$ be a category of partitions such that $\sqcap_{\circ\circ} \otimes \sqcap_{\bullet\bullet} \in \mathscr{C}$. Then the sets $\mathscr{C}(k)$ are closed under permutation of colours. That is, if $p \in \mathscr{C}(k)$, then $p' \in \mathscr{C}(k)$, where p' is obtained from p by changing colours in such a way that the number of white points (and black points) in p' is the same as in p.

Proof. Having $\sqcap_{\circ\circ} \otimes \sqcap_{\bullet\bullet} \in \mathscr{C}$ implies having also $\overset{\circ}{\underset{\bullet}{|}} \otimes \overset{\bullet}{\underset{\circ}{|}} \in \mathscr{C}$ and $\overset{\bullet}{\underset{\circ}{|}} \otimes \overset{\circ}{\underset{\bullet}{|}} \in \mathscr{C}$. Using one of those two partitions tensored with appropriate choice of $\overset{\circ}{\underset{\circ}{|}}$ and $\overset{\bullet}{\underset{\bullet}{|}}$ allows us to transpose colours of two neighbouring points. This generates arbitrary permutations of points. □

So, if $\sqcap_{\circ\circ} \otimes \sqcap_{\bullet\bullet} \in \mathscr{C}$, the only thing that matters (for partitions with lower points only) is the number of white and black points. The actual distribution of colours in a partition is irrelevant. If we want to formulate this also for partitions that have both upper and lower points, we must remember that by rotation, the colour changes.

4.4.7 Definition. A category of two-coloured partitions is called **globally colourized** if it contains $\sqcap_{\circ\circ} \otimes \sqcap_{\bullet\bullet}$. Otherwise, we call the category **locally colourized**.

4.4.8 Definition. For a word $w \in \mathscr{O}^*_{\circ\bullet}$, we denote by $c(w)$ the number of white points in w minus the number of black points in w. For a partition $p \in \mathscr{P}^{\circ\bullet}(w_1, w_2)$, we denote $c(p) := c(w_2) - c(w_1)$.

As an example, consider the partition $p =$  from Eq. (4.3). For its upper colour pattern, we have $c(\circ\bullet\bullet) = 1 - 2 = -1$, its lower colour pattern has $c(\bullet\circ\bullet\bullet) = 1 - 3 = -2$. Thus, for the whole partition, $c(p) = -2 - (-1) = -1$. Equivalently, we can have a look on the one-line version of p, which is $p' := \mathrm{Lrot}^3 p =$ , where we see straight away $c(p') = c(\circ\circ\bullet\bullet\circ\bullet\bullet) = -1 = c(p)$.

4.4.9 Definition. For a category $\mathscr{C}$, we define $k(\mathscr{C})$ to be the minimum of all numbers $c(p)$ such that $c(p) > 0$ and $p \in \mathscr{C}$ if such a partition exists in $\mathscr{C}$. Otherwise we set $k(\mathscr{C}) := 0$. The parameter $k(\mathscr{C})$ is called the **degree of reflection** of $\mathscr{C}$.

4.4.10 Lemma. For the map $c : \mathscr{P}^{\circ\bullet} \to \mathbb{Z}$, the following holds true.

(1) $c(p \otimes q) = c(p) + c(q)$,
(2) $c(pq) = c(p) + c(q)$,
(3) $c(p^*) = c(p^\star) = -c(p)$,
(4) $c(\mathrm{Lrot}\, p) = c(\mathrm{Rrot}\, p) = c(p)$.

If p has lower points only, then also

(5) $c(\Pi_i p) = c(p)$,
(6) $c(Rp) = c(p)$.

Here, we suppose that the assumptions for the composition in (2) and contraction in (5) are satisfied.

Proof. Follows directly from the definition of the operations. See [TW18, Lemma 2.6]. □

4.4.11 Lemma. Let $\mathscr{C}$ be a category of partitions. For every $p \in \mathscr{P}^{\circ\bullet}$, the number $c(p)$ is a multiple of $k(\mathscr{C})$.

Proof. From Lemma 4.4.10, it follows that the set $\{c(p) \mid p \in \mathscr{C}\}$ is a subgroup of $\mathbb{Z}$. Hence, it must be equal to $k\mathbb{Z}$ for some $k \in \mathbb{N}_0$. But obviously $k = k(\mathscr{C})$ since it is indeed the minimal positive $c(p)$. □

4.4.5 Classification of non-crossing two-coloured partitions

Now, let us mention the classification theorems of [TW18]. We denote $b_k = \sqcap\!\cdots\!\sqcap_{\circ\circ\cdots\circ} \in \mathscr{P}^{\circ\bullet}(0, k)$ the so-called **block partition** consisting of a single block of size k.

4.4.12 Theorem. Let $\mathscr{C} \subseteq NC^{\circ\bullet}$ be a globally colourized category of non-crossing partitions. Then it coincides with one of the following categories.

$$
\begin{aligned}
\mathcal{O}_{\mathrm{glob}}(k) &= \langle \sqcap_{\circ\circ}^{\otimes k/2}, \sqcap_{\circ\circ} \otimes \sqcap_{\bullet\bullet} \rangle && \text{for } k \in 2\mathbb{N}_0 \\
\mathcal{H}_{\mathrm{glob}}(k) &= \langle b_k, \sqcap\!\sqcap\!\sqcap_{\circ\bullet\circ\bullet}, \sqcap_{\circ\circ} \otimes \sqcap_{\bullet\bullet} \rangle && \text{for } k \in 2\mathbb{N}_0 \\
\mathcal{S}_{\mathrm{glob}}(k) &= \langle \uparrow_{\circ}^{\otimes k}, \sqcap\!\sqcap\!\sqcap_{\circ\bullet\circ\bullet}, \uparrow_{\circ} \otimes \uparrow_{\bullet}, \sqcap_{\circ\circ} \otimes \sqcap_{\bullet\bullet} \rangle && \text{for } k \in \mathbb{N}_0 \\
\mathcal{B}_{\mathrm{glob}}(k) &= \langle \uparrow_{\circ}^{\otimes k}, \uparrow_{\circ} \otimes \uparrow_{\bullet}, \sqcap_{\circ\circ} \otimes \sqcap_{\bullet\bullet} \rangle && \text{for } k \in 2\mathbb{N}_0 \\
\mathcal{B}'_{\mathrm{glob}}(k) &= \langle \uparrow_{\circ}^{\otimes k}, \sqcap\!\!\!\!\uparrow\!\sqcap_{\circ\circ\bullet} \uparrow_{\bullet}, \uparrow_{\circ} \otimes \uparrow_{\bullet}, \sqcap_{\circ\circ} \otimes \sqcap_{\bullet\bullet} \rangle && \text{for } k \in \mathbb{N}_0
\end{aligned}
$$

4.4.13 Theorem. Let $\mathscr{C} \subseteq NC^{\circ\bullet}$ be a locally colourized category of non-crossing partitions. Then it coincides with one of the following categories.

$$\begin{aligned}
\mathcal{O}_{\rm loc} &= \langle\rangle\\
\mathcal{H}'_{\rm loc} &= \langle \sqcap\!\sqcap\!\sqcap_{\circ\bullet\circ\bullet} \rangle\\
\mathcal{H}_{\rm loc}(k,d) &= \langle b_k, b_d \otimes b_d^{\star}, \sqcap\!\sqcap\!\sqcap_{\circ\circ\bullet\bullet}, \sqcap\!\sqcap\!\sqcap_{\circ\bullet\bullet\circ}, \sqcap_{\circ\circ} \otimes \sqcap_{\bullet\bullet} \rangle && \text{for } k,d \in \mathbb{N}_0 \setminus \{1,2\},\ d \mid k\\
\mathcal{S}_{\rm loc}(k,d) &= \langle \uparrow^{\otimes k}, \ulcorner \uparrow^{\otimes d} \urcorner \uparrow^{\otimes d}, \sqcap\!\sqcap\!\sqcap_{\circ\bullet\circ\bullet}, \uparrow \otimes \uparrow \rangle && \text{for } k,d \in \mathbb{N}_0 \setminus \{1\},\ d \mid k\\
\mathcal{B}_{\rm loc}(k,d) &= \langle \uparrow^{\otimes k}, \uparrow \otimes \uparrow, \ulcorner \uparrow^{\otimes d} \urcorner \uparrow^{\otimes d} \rangle && \text{for } k,d \in \mathbb{N}_0,\ d \mid k\\
\mathcal{B}'_{\rm loc}(k,d,0) &= \langle \uparrow^{\otimes k}, \sqcap \downarrow, \uparrow \otimes \uparrow, \ulcorner \uparrow^{\otimes d} \urcorner \uparrow^{\otimes d}, \sqcap \downarrow \rangle && \text{for } k \in \mathbb{N}_0 \setminus \{1\},\ d \mid k\\
\mathcal{B}'_{\rm loc}(k,d,d/2) &= \langle \uparrow^{\otimes k}, \ulcorner \uparrow^{\otimes r} \urcorner \uparrow^{\otimes r}, \uparrow \otimes \uparrow, \ulcorner \uparrow^{\otimes d} \urcorner \uparrow^{\otimes d}, \sqcap \downarrow \rangle\\
& && \text{for } k \in \mathbb{N}_0 \setminus \{1\},\ d \in 2\mathbb{N} \setminus \{2\},\ d \mid k,\ r = d/2
\end{aligned}$$

4.5 Classification of globally colourized categories

In this section we present some of the author's original results – the classification of globally colourized categories. The section essentially coincides with [Gro18, Sect. 3].

As we described in Section 4.4.4, for partitions in globally colourized categories, the only relevant things are the (non-coloured) partition structure and the number $c(p)$. The actual distribution of the colours is not important. The classification result says a similar thing for the whole categories – the only relevant data are some non-coloured category and the degree of reflection $k(\mathscr{C})$.

We obtain the classification result in two steps. In Section 4.5.1, we study categories $\mathscr{C}$ with $k(\mathscr{C}) = 0$. Those behave exactly the same way as non-coloured categories – indeed, taking $p, q \in \mathscr{C}$, we can always compose them as if they were non-coloured if we do necessary colour permutations beforehand. This will lead to a one-to-one correspondence between some non-coloured categories and categories with degree of reflection zero. A category $\mathscr{C}$ with an arbitrary degree of reflection induces a category $\mathscr{C}_0$ with degree of reflection zero. In Section 4.5.2, we show how to reconstruct $\mathscr{C}$ from $\mathscr{C}_0$.

4.5.1 Categories with zero degree of reflection

4.5.1 Lemma. Let $\mathscr{C}$ be a category of partitions.

(1) If $k(\mathscr{C}) = 0$, then all partitions in $\mathscr{C}$ have even length.
(2) Let $\sqcap_{\circ\circ} \in \mathscr{C}$. Then all partitions in $\mathscr{C}$ have even length if and only if $\uparrow \notin \mathscr{C}$.

Proof. The first part is obvious: If a partition $p \in \mathscr{C}$ has odd length then it cannot have the same amount of white and black points, so $c(p) \neq 0$ and hence $k(\mathscr{C}) \neq 0$.

The second part essentially follows from Lemma 4.2.2 as the assumption of (2) means that $\mathscr{C}$ is a non-coloured category. Take $p \in \mathscr{C}(k)$ with k odd. Since the colouring of p does not matter, we can perform contractions (and necessary colour changes) on p until we get the singleton $\uparrow$. □

4.5.2 Definition. Let $\mathscr{C}$ be a category of partitions. Denote $\mathscr{C}_0 := \{p \in \mathscr{C} \mid c(p) = 0\}$.

4.5.3 Lemma. Let $\mathscr{C}$ be a category of partitions. Then $\mathscr{C}_0$ is a category of partitions with $k(\mathscr{C}_0) = 0$.

Proof. Using Lemma 4.4.10, we see that the category operations applied on partitions with $c(p) = 0$ again produce partitions with $c(p) = 0$. Thus, the subset $\mathscr{C}_0 \subseteq \mathscr{C}$ is closed under the category operations. Moreover, $c(\updownarrow_{\circ\bullet}) = c(\updownarrow_{\bullet\circ}) = c(\sqcap_{\circ\bullet}) = c(\sqcap_{\bullet\circ}) = 0$, so all of them belong to $\mathscr{C}_0$.

The fact that $k(\mathscr{C}_0) = 0$ follows directly from the definition of $\mathscr{C}_0$. □

Recall the definition of the map $\Psi\colon \mathscr{P}^{\circ\bullet} \to \mathscr{P}$ from Definition 4.4.4.

4.5.4 Lemma. Let $\mathscr{C}$ be a globally colourized category with $k(\mathscr{C})=0$ (or equivalently $\mathscr{C}=\mathscr{C}_0$). Then

(1) $\Psi(\mathscr{C})$ is a category of non-coloured partitions.
(2) In terms of two-coloured partitions, it corresponds to the non-coloured category

$$\Psi^{-1}(\Psi(\mathscr{C}))=\langle\mathscr{C},\sqcap_{\circ\circ}\rangle.$$

(3) Moreover, $\mathscr{C}=\langle\mathscr{C},\sqcap_{\circ\circ}\rangle_0$.

Proof. To prove the first statement, we have to show that $\Psi(\mathscr{C})$ contains pair partitions and it is closed under the category operations. We have that $\sqcap_{\circ\bullet}\in\mathscr{C}$, so $\sqcap\in\Psi(\mathscr{C})$. For $p,q\in\mathscr{C}$ we have that $\Psi(p)\otimes\Psi(q)=\Psi(p\otimes q)$. We also have that $\Psi(p)^\star=\Psi(p^\star)$. Finally, for any $p\in\mathscr{C}(k)$ and any $i\in\{1,\ldots,k\}$, we can shift colours in p in a way that the i-th and $(i+1)$-st point have different colours. Then $\Pi_i(\Psi(p))=\Psi(\Pi_i p)$.

The second statement is obvious. Surely both $\mathscr{C}\subseteq\Psi^{-1}(\Psi(\mathscr{C}))$ and $\sqcap_{\circ\circ}\in\Psi^{-1}(\Psi(\mathscr{C}))$, so we have the inclusion $\supseteq$. For the other one, we use Proposition 4.4.5 to see that $\Psi^{-1}(\Psi(\mathscr{C}))\subseteq\Psi^{-1}(\Psi(\langle\mathscr{C},\sqcap_{\circ\circ}\rangle))=\langle\mathscr{C},\sqcap_{\circ\circ}\rangle$.

In the third statement, the inclusion $\subseteq$ is obvious. For the opposite inclusion, take $p\in\langle\mathscr{C},\sqcap_{\circ\circ}\rangle_0$. This means that we are taking an arbitrary element of $\langle\mathscr{C},\sqcap_{\circ\circ}\rangle$ such that $c(p)=0$. Applying Ψ on the equality in (2), we get $\Psi(\mathscr{C})=\Psi(\langle\mathscr{C},\sqcap_{\circ\circ}\rangle)$. This means that there is $p'\in\mathscr{C}$ such that $\Psi(p')=\Psi(p)$. Since we have $c(p')=c(p)=0$ and since $\mathscr{C}$ is globally colourized, we can shift colours in p' to obtain $p\in\mathscr{C}$. □

4.5.5 Lemma. Let $\tilde{\mathscr{C}}\subseteq\mathscr{P}$ be a category of non-coloured partitions such that $\uparrow\notin\mathscr{C}$. Denote by $\bar{\mathscr{C}}:=\Psi^{-1}(\tilde{\mathscr{C}})\subseteq\mathscr{P}^{\circ\bullet}$ the corresponding category in terms of two-coloured partitions.

(1) It holds that $\tilde{\mathscr{C}}=\Psi(\bar{\mathscr{C}}_0)$ or, equivalently, $\bar{\mathscr{C}}=\langle\bar{\mathscr{C}}_0,\sqcap_{\circ\circ}\rangle$.
(2) If $\tilde{\mathscr{C}}=\langle p_1,\ldots,p_n\rangle$ for some $p_1,\ldots,p_n\in\mathscr{P}$, then $\bar{\mathscr{C}}_0=\langle p_1',\ldots,p_n',\sqcap_{\circ\circ}\otimes\sqcap_{\bullet\bullet}\rangle$, where p_i' is a colouring of p_i with $c(p_i')=0$ for every $i\in\{1,\ldots,n\}$.

Proof. The inclusion $\tilde{\mathscr{C}}\supseteq\Psi(\bar{\mathscr{C}}_0)$ is obvious. For the opposite, take $p\in\tilde{\mathscr{C}}$. Since $\uparrow\notin\mathscr{C}$, p must have even length. Thus, it has a colouring p' such that $c(p')=0$, so $p'\in\bar{\mathscr{C}}_0$ and hence $p\in\Psi(\bar{\mathscr{C}}_0)$. The equivalent description of this equality follows from Lemma 4.5.4.

For the second statement, the existence of appropriate $p_1',\ldots,p_n'$ again follows from the fact that $\uparrow\notin\mathscr{C}$, so all the partitions $p_1,\ldots,p_n$ have even length. Then it is easy to see that all the generators $p_1',\ldots,p_n'$ and $\sqcap_{\circ\circ}\otimes\sqcap_{\bullet\bullet}$ are elements of $\bar{\mathscr{C}}_0$. Finally, take arbitrary $p\in\bar{\mathscr{C}}_0$. Then

$$\Psi(p)\in\Psi(\bar{\mathscr{C}}_0)=\tilde{\mathscr{C}}=\langle p_1,\ldots,p_n\rangle\subseteq\Psi(\langle p_1',\ldots,p_n',\sqcap_{\circ\circ}\otimes\sqcap_{\bullet\bullet}\rangle),$$

where we used that $\langle p_1',\ldots,p_n',\sqcap_{\circ\circ}\otimes\sqcap_{\bullet\bullet}\rangle$ satisfies the assumptions of Lemma 4.5.4, so its image under Ψ is a category containing $p_1,\ldots,p_n$. So, there is $p'\in\langle p_1',\ldots,p_n',\sqcap_{\circ\circ}\otimes\sqcap_{\bullet\bullet}\rangle$ such that $\Psi(p)=\Psi(p')$. Since $c(p)=c(p')=0$ and since the category $\langle p_1',\ldots,p_n',\sqcap_{\circ\circ}\otimes\sqcap_{\bullet\bullet}\rangle$ is globally colourized, it must contain also p. □

4.5.6 Proposition. There is a bijection between globally colourized categories $\mathscr{C}$ with $k(\mathscr{C})=0$ and non-coloured categories $\bar{\mathscr{C}}$ with $\uparrow_\circ\notin\bar{\mathscr{C}}$ given by $\mathscr{C}\mapsto\langle\mathscr{C},\sqcap_{\circ\circ}\rangle$ with inverse $\bar{\mathscr{C}}\mapsto\bar{\mathscr{C}}_0$.

Proof. Denote by Φ_1 the mentioned map and by Φ_2 the alleged inverse. Lemma 4.5.4 says that $\Phi_2\Phi_1$ is the identity and Lemma 4.5.5 says that $\Phi_1\Phi_2$ is the identity. □

4.5.2 Categories with non-zero degrees of reflection

4.5.7 Lemma. Let $\mathscr{C}$ be a globally colourized category.

(1) If $\uparrow_\circ\otimes\uparrow_\bullet\notin\mathscr{C}$, then $\sqcap_{\circ\uparrow_{\circ}\bullet}\,\uparrow_{\bullet}\notin\mathscr{C}$.
(2) If $\sqcap_{\circ\uparrow_{\circ}\bullet}\,\uparrow_{\bullet}\notin\mathscr{C}$, then all partitions have even length.

Proof. The first part follows from the fact that $\uparrow_{\circ}\otimes\uparrow_{\bullet}$ is a contraction of $\sqcap_{\circ\,\uparrow_{\circ}\,\bullet}\uparrow_{\bullet}$.

Now, consider $p\in\mathscr{C}(k)$ with k odd. Then take

$$p' := R(p\otimes\sqcap_{\bullet\circ})\otimes p^{\star} = \sqcap_{\circ\, p\,\bullet}\, p^{\star}$$

Since there is the same amount of black and white points in p', we can use Lemma 4.4.6 to permute colours in such a way that we can contract the partition to $\sqcap_{\circ\,\uparrow_{\circ}\,\bullet}\uparrow_{\bullet}$. □

Recall the definition of $\mathscr{C}_0$ from Def. 4.5.2.

4.5.8 Lemma. Let $\mathscr{C}$ be a globally colourized category, denote $k := k(\mathscr{C})$.

(1) If $\uparrow_{\circ}\otimes\uparrow_{\bullet}\in\mathscr{C}$, then $\mathscr{C}=\langle\mathscr{C}_0,\uparrow_{\circ}^{\otimes k}\rangle$.
(2) If $\mathscr{C}$ contains only partitions of even length (in particular, if $\sqcap_{\circ\,\uparrow_{\circ}\,\bullet}\uparrow_{\bullet}\notin\mathscr{C}$ or $\uparrow_{\circ}\otimes\uparrow_{\bullet}\notin\mathscr{C}$), then k is even and $\mathscr{C}=\langle\mathscr{C}_0,\sqcap_{\circ\circ}^{\otimes k}\rangle$.

Proof. We begin with the statement of (2). Suppose that every partition $p\in\mathscr{C}$ has even length. Since $k(\mathscr{C})=k$, there is a partition $p\in\mathscr{C}$ such that $c(p)=k$. We can repeatedly perform contractions on p until we get p' containing only k white points. Since every partition is of even length, we have that k is even.

Now, let us prove the inclusion $\supseteq$ in (2). Again, take $p\in\mathscr{C}$ such that $c(p)=k$. Using Lemma 4.4.6 we can permute colours in $\sqcap_{\circ\bullet}^{\otimes k/2}\otimes p\in\mathscr{C}$ in such a way that the first k points are white. Then we can perform contractions on the rest of the points and obtain $\sqcap_{\circ\circ}^{\otimes k/2}\in\mathscr{C}$. Note that this also implies that $\sqcap_{\circ\circ}^{\otimes nk},\sqcap_{\bullet\bullet}^{\otimes nk}\in\mathscr{C}$ for any $n\in\mathbb{N}$.

Now, we prove the inclusion $\supseteq$ in (1). Again, take $p\in\mathscr{C}$ such that $c(p)=k$. Perform contractions until we get p' consisting of k white points only. Now, according to Lemma 4.4.2, we can disconnect all points in p' using $\uparrow_{\circ}\otimes\uparrow_{\bullet}$ to obtain $\uparrow_{\circ}^{\otimes k}\in\mathscr{C}$. Again, this already implies that $\uparrow_{\circ}^{\otimes nk},\uparrow_{\bullet}^{\otimes nk}\in\mathscr{C}$ for any $n\in\mathbb{N}$.

The proof of the inclusion $\subseteq$ is the same in both cases. Denote either $q_k:=\uparrow_{\circ}^{\otimes k}$, $q_{-k}:=\uparrow_{\bullet}^{\otimes k}$ in case (1) or $q_k:=\sqcap_{\circ\circ}^{\otimes k/2}$, $q_{-k}:=\sqcap_{\bullet\bullet}^{\otimes k/2}$ in case (2). We have already proven that $q_{nk}\in\mathscr{C}$ for all $n\in\mathbb{Z}$. Take an arbitrary $p\in\mathscr{C}$. From Lemma 4.4.11, we see that $c(p)$ is a multiple of k, so $q_{-c(p)}\in\mathscr{C}$ and hence $p':=p\otimes q_{-c(p)}\in\mathscr{C}$. Since $c(p')=0$, we have $p'\in\mathscr{C}_0$. Finally, p can be obtained by repeated contraction of $p\otimes q_{-c(p)}\otimes q_{c(p)}=p'\otimes q_{c(p)}\in\langle\mathscr{C}_0,q_k\rangle$. □

4.5.9 Lemma. Let $\mathscr{C}_1,\mathscr{C}_2$ be non-coloured categories such that $\uparrow_{\circ}\notin\mathscr{C}_1,\mathscr{C}_2$. Suppose one of the following is true

(a) $\uparrow_{\circ}\otimes\uparrow_{\bullet}\in\mathscr{C}_1,\mathscr{C}_2$ and $\langle(\mathscr{C}_1)_0,\uparrow_{\circ}^{\otimes k_1}\rangle=\langle(\mathscr{C}_2)_0,\uparrow_{\circ}^{\otimes k_2}\rangle$ for some $k_1,k_2\in\mathbb{N}_0$ or
(b) $\langle(\mathscr{C}_1)_0,\sqcap_{\circ\circ}^{\otimes k_1/2}\rangle=\langle(\mathscr{C}_2)_0,\sqcap_{\circ\circ}^{\otimes k_2/2}\rangle$ for $k_1,k_2\in 2\mathbb{N}_0$.

Then $k:=k_1=k_2$ and

(1) $\mathscr{C}_1=\mathscr{C}_2$ if k is even,
(2) $\langle\mathscr{C}_1,\uparrow_{\circ}\rangle=\langle\mathscr{C}_2,\uparrow_{\circ}\rangle$ if k is odd.

Proof. Suppose $\langle(\mathscr{C}_1)_0,q_{k_1}\rangle=\langle(\mathscr{C}_2)_0,q_{k_2}\rangle$, where q_j denotes $\uparrow_{\circ}^{\otimes j}$ in case (a) or $\sqcap_{\circ\circ}^{\otimes j}$ in case (b). Then $k_1=k_2$ follows from the fact that $k(\langle\mathscr{C}_0,q_j\rangle)=j$. If k is even, we can use Lemma 4.5.4 to obtain

$$\mathscr{C}_1=\langle(\mathscr{C}_1)_0,\sqcap_{\circ\circ}\rangle=\langle(\mathscr{C}_1)_0,q_{k_1},\sqcap_{\circ\circ}\rangle=\langle(\mathscr{C}_2)_0,q_{k_2},\sqcap_{\circ\circ}\rangle=\langle(\mathscr{C}_2)_0,\sqcap_{\circ\circ}\rangle=\mathscr{C}_2.$$

In the second and fourth equality, we used the fact that (a) $\sqcap^{\otimes k/2}\in\mathscr{C}$ for every category $\mathscr{C}$ and $k\in 2\mathbb{N}_0$, so, in particular, $\sqcap_{\circ\circ}^{\otimes k_i/2}\in\langle(\mathscr{C}_i)_0,\sqcap_{\circ\circ}\rangle$; (b) $\uparrow^{\otimes k}\in\mathscr{C}$ for every category $\mathscr{C}$ containing $\uparrow\otimes\uparrow$ and $k\in 2\mathbb{N}_0$, so, in particular, $\uparrow_{\circ}^{\otimes k_i}\in\langle(\mathscr{C}_i)_0,\sqcap_{\circ\circ}\rangle$.

If k is odd, we have

$$\langle\mathscr{C}_1,\uparrow_{\circ}\rangle=\langle(\mathscr{C}_1)_0,\sqcap_{\circ\circ},\uparrow_{\circ}\rangle=\langle(\mathscr{C}_1)_0,\uparrow_{\circ}^{\otimes k_1},\sqcap_{\circ\circ}\rangle=\langle(\mathscr{C}_2)_0,\uparrow_{\circ}^{\otimes k_2},\sqcap_{\circ\circ}\rangle=\langle(\mathscr{C}_2)_0,\sqcap_{\circ\circ},\uparrow_{\circ}\rangle=\langle\mathscr{C}_2,\uparrow_{\circ}\rangle.$$ □

4.5.3 The classification theorem

4.5.10 Theorem. [Gro18, Theorem 3.1] Every globally colourized category $\mathscr{C}$ is determined by the number $k(\mathscr{C})$ and a non-coloured category of partitions not containing the singleton. Therefore, the right column of Table 4.1 forms a complete classification of globally colourized categories. All of them are pairwise inequivalent except for the rows denoted by asterisk $(*)$, which are special instances of the last family parametrized by normal subgroups A of $\mathbb{Z}_2^{*\infty}$.

$$\langle\rangle \longrightarrow \mathcal{O}_{\mathrm{glob}}(k) = \langle \sqcap^{\otimes k/2}, \sqcap \otimes \sqcap \rangle,\ k \in 2\mathbb{N}_0$$
$$\langle \sqcap\sqcap\sqcap \rangle \longrightarrow \mathcal{H}_{\mathrm{glob}}(k) = \langle \sqcap^{\otimes k/2}, \sqcap\sqcap\sqcap, \sqcap \otimes \sqcap \rangle,\ k \in 2\mathbb{N}_0$$
$$\langle \sqcap\sqcap\sqcap, \uparrow\otimes\uparrow \rangle \longrightarrow \mathcal{S}_{\mathrm{glob}}(k) = \langle \uparrow^{\otimes k}, \sqcap\sqcap\sqcap, \uparrow\otimes\uparrow, \sqcap \otimes \sqcap \rangle,\ k \in \mathbb{N}_0$$
$$\langle \uparrow\otimes\uparrow \rangle \longrightarrow \mathcal{B}_{\mathrm{glob}}(k) = \langle \uparrow^{\otimes k}, \uparrow\otimes\uparrow, \sqcap \otimes \sqcap \rangle,\ k \in 2\mathbb{N}_0$$
$$\langle \sqcap\,\mid \rangle \longrightarrow \mathcal{B}'_{\mathrm{glob}}(k) = \langle \uparrow^{\otimes k}, \sqcap\,\downarrow, \sqcap \otimes \sqcap \rangle,\ k \in \mathbb{N}_0$$
$$\langle \times \rangle \longrightarrow \mathcal{O}_{\mathrm{grp,glob}}(k) = \langle \sqcap^{\otimes k/2}, \times, \sqcap \otimes \sqcap \rangle,\ k \in 2\mathbb{N}_0$$
$$\langle \sqcap\sqcap\sqcap, \times \rangle \longrightarrow \mathcal{H}_{\mathrm{grp,glob}}(k) = \langle \sqcap^{\otimes k/2}, \sqcap\sqcap\sqcap, \times, \sqcap \otimes \sqcap \rangle,\ k \in 2\mathbb{N}_0 \quad (*)$$
$$\langle \sqcap\sqcap\sqcap, \uparrow\otimes\uparrow, \times \rangle \longrightarrow \mathcal{S}_{\mathrm{grp,glob}}(k) = \langle \uparrow^{\otimes k}, \sqcap\sqcap\sqcap, \uparrow\otimes\uparrow, \times, \sqcap \otimes \sqcap \rangle,\ k \in \mathbb{N}_0$$
$$\langle \uparrow\otimes\uparrow, \times \rangle \longrightarrow \mathcal{B}_{\mathrm{grp,glob}}(k) = \langle \uparrow^{\otimes k}, \uparrow\otimes\uparrow, \times, \sqcap \otimes \sqcap \rangle,\ k \in \mathbb{N}_0$$
$$\langle \divideontimes \rangle \longrightarrow \mathcal{O}_{\mathrm{hl,glob}}(k) = \langle \sqcap^{\otimes k/2}, \divideontimes, \sqcap \otimes \sqcap \rangle,\ k \in 2\mathbb{N}_0$$
$$\langle \sqcap\sqcap\sqcap, \divideontimes \rangle \longrightarrow \mathcal{H}_{\mathrm{hl,glob}}(k,0) = \langle \sqcap^{\otimes k/2}, \sqcap\sqcap\sqcap, \divideontimes, \sqcap \otimes \sqcap \rangle,\ k \in 2\mathbb{N}_0 \quad (*)$$
$$\langle \sqcap\sqcap\sqcap, \divideontimes, h_s \rangle \longrightarrow \mathcal{H}_{\mathrm{hl,glob}}(k,s) = \langle \sqcap^{\otimes k/2}, h_s^0, \sqcap\sqcap\sqcap, \divideontimes, \sqcap \otimes \sqcap \rangle, k \in 2\mathbb{N}_0, s \geq 3 \quad (*)$$
$$\langle \uparrow\otimes\uparrow, \divideontimes \rangle \longrightarrow \mathcal{B}_{\mathrm{hl,glob}}(k) = \langle \uparrow^{\otimes k}, \uparrow\otimes\uparrow, \divideontimes, \sqcap \otimes \sqcap \rangle,\ k \in 2\mathbb{N}_0$$
$$\langle \pi_s \rangle \longrightarrow \mathcal{H}_{\pi}(k,s) = \langle \sqcap^{\otimes k/2}, \pi_s^0, \sqcap \otimes \sqcap \rangle,\ k \in 2\mathbb{N}_0, s \geq 2$$
$$\langle \pi_l \mid l \in \mathbb{N} \rangle \longrightarrow \mathcal{H}_{\pi}(k,\infty) = \langle \sqcap^{\otimes k/2}, \pi_l^0, \sqcap \otimes \sqcap \mid l \in \mathbb{N} \rangle,\ k \in 2\mathbb{N}_0$$
$$A \trianglelefteq \mathbb{Z}_2^{*\infty} \mathrm{sS}_\infty \mathrm{inv.}, \uparrow\otimes\uparrow \notin A \longrightarrow \mathcal{H}_A(k) = \langle \sqcap^{\otimes k/2}, A_0, \sqcap \otimes \sqcap \rangle,\ k \in 2\mathbb{N}_0$$

Table 4.1 Complete classification of globally colourized categories

Before proving the theorem, let us comment on Table 4.1. Most of its notation is explained in Sections 4.2 and 4.4.2.

The first column of the table lists all categories of non-coloured partitions that do not contain the singleton $\uparrow$. The summary of the full classification is provided in Section 4.2.

Note that there are two instances of group-theoretical categories that contain $\uparrow\otimes\uparrow$ and hence they have to be treated separately. Namely, it is $\langle \sqcap\sqcap\sqcap, \uparrow, \times \rangle$ (which should not be considered at all since it contains the singleton) and $\langle \sqcap\sqcap\sqcap, \uparrow\otimes\uparrow, \times \rangle$.

Besides that, we also treat separately those group-theoretical categories that contain the crossing partition $\times$ or the half-liberating partition $\divideontimes$. The corresponding rows are denoted by asterisk $(*)$.

The right column lists the corresponding globally colourized categories. We use the following notation. By h_s^0 we denote a coloured counterpart of h_s for which $c(h_s^0) = 0$. Thanks to the global colourization, it does not matter which particular colourization we choose. For definiteness, we can say, for example, that the colours in h_s^0 alternate beginning with white. In the same way we define π_s^0. For A a sS_∞-invariant normal subgroup of $\mathbb{Z}_2^{*\infty}$, we denote by A_0 the category of partitions arising from the non-coloured category of partitions corresponding to A in the sense of Definition 4.5.2.

Note that the first five rows describe the classification of globally colourized categories of non-crossing partitions, the following four lines classify the globally colourized group categories, i.e. those containing the crossing partition $\times$ (both already obtained in [TW18], see Theorem 4.4.12), and the following four rows classify all globally colourized half-liberated categories of partitions, i.e. those containing the half-liberating partition $\ast$, but not the crossing partition $\times$.

Proof of Theorem 4.5.10. Proposition 4.5.6 tells us how to obtain all globally colourized categories with $k = 0$ from non-coloured ones (containing partitions of even length). Lemma 4.5.8 tells us how to obtain all globally colourized categories from those with $k = 0$.

The left column of Table 4.1 contains all non-coloured categories of partitions of even length. All of them are pairwise distinct (except for the ones denoted by asterisk being special instances of the last one). In the right column, we construct the corresponding globally colourized categories. For $k = 0$, the generators of the category are given by Lemma 4.5.5. For $k \neq 0$, we just have to add the partition $\sqcap^{\otimes k/2}$ or $\uparrow^{\otimes k}$ according to Lemma 4.5.8.

To prove that all categories we have constructed are pairwise distinct, we use Lemma 4.5.9. According to this lemma, distinct non-coloured categories $\mathscr{C}_1 \neq \mathscr{C}_2$ can lead to equal globally colourized categories only in the case $\uparrow \otimes \uparrow \in \mathscr{C}_1, \mathscr{C}_2$ if $\langle \mathscr{C}_1, \updownarrow \rangle = \langle \mathscr{C}_2, \updownarrow \rangle$. The only pairs $(\mathscr{C}_1, \mathscr{C}_2)$ of non-coloured categories containing $\uparrow \otimes \uparrow$ and satisfying $\langle \mathscr{C}_1, \uparrow \rangle = \langle \mathscr{C}_2, \uparrow \rangle$ are $(\langle \uparrow \otimes \uparrow \rangle, \langle ⊓\!\!\!\!\!\sqcap \rangle)$ and $(\langle \uparrow \otimes \uparrow, \times \rangle, \langle \uparrow \otimes \uparrow, \ast \rangle)$. For the corresponding globally colourized categories we may easily see that indeed $\mathcal{B}_{\mathrm{glob}}(k) = \mathcal{B}'_{\mathrm{glob}}(k)$ and $\mathcal{B}_{\mathrm{grp,glob}}(k) = \mathcal{B}_{\mathrm{hl,glob}}(k)$ for k odd. □

4.5.11 Remark. The set of generators for the categories is, of course, not unique. We used the partition $\updownarrow^{\otimes k}$ for the cases $\updownarrow \otimes \updownarrow \in \mathscr{C}$ and the partition $\sqcap^{\otimes k/2}$ for the rest. Nevertheless, as we see from Lemma 4.5.8, we could have used $\sqcap^{\otimes k/2}$ for all cases when $\mathscr{C}$ consists of partitions of even length and $\updownarrow^{\otimes k}$ for the rest. In addition, we could reformulate Lemmata 4.5.8 and 4.5.9 and use the white block partition $b_k = \sqcap\cdots$ in all cases when $\sqcap\sqcap \in \mathscr{C}$.

4.5.12 Remark. The results in the non-crossing case indeed match with Theorem 4.4.12 taken from [TW18]. The only difference is that, for the hyperoctahedral categories, the generator b_k is used instead of $\sqcap^{\otimes k/2}$ in Theorem 4.4.12.

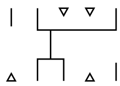
4.6 Partitions with extra singletons

In this section, we study partitions with extra singletons as they were defined in [GW19b]. The motivation for the definition will be explained in Part III of the thesis. The main goal of this section is to present the classification obtained in [GW19b]. We do not solve the classification problem explicitly. Instead, we show that it is essentially equivalent to classification of two-coloured categories of partitions. This correspondence is based on some functor F, which will play an important role also later in the thesis.

4.6.1 Definition of partitions with extra singletons

Consider the set $\mathscr{O}_\vartriangle = \{\vartriangle, \mid\}$ with the trivial involution $\vartriangle \mapsto \vartriangle$, $\mid \mapsto \mid$. A **partition with extra singletons** is an $\mathscr{O}_\vartriangle$-coloured partition p, where all points of the colour $\vartriangle$ are singletons. Any set $\mathscr{C}$ of partitions with extra singletons that is closed under the category operations and contains the partitions $|$, $\substack{\triangledown \\ \vartriangle}$, $\sqcap$, and $\vartriangle \otimes \vartriangle$ is called a **category of partitions with extra singletons**. The category of all partitions with extra singletons is denoted by $\mathscr{P}^\vartriangle$. The points with colour $\vartriangle$ are called **extra singletons**. The smallest category of partitions with extra singletons containing given $p_1, \ldots, p_n \in \mathscr{P}^\vartriangle$ is denoted by $\langle p_1, \ldots, p_n \rangle^\vartriangle$.

A typical example of partition with extra singletons looks as follows

$p =$  . (4.4)

Note that we have now two kinds of singletons. We still have the “ordinary” singletons – i.e. blocks of the colour $|$ containing a single point – and the extra singletons depicted by the triangle $\triangle$.

Although the identity morphism $\substack{\triangledown\\ \triangle}$ and the duality morphism $\triangle\otimes\triangle$ look a bit differently in this case, the rotations and one-line operations work in the same manner as with any other coloured partitions.

In the following text, we attack the classification problem for categories of partitions with extra singletons. We are not going to solve it explicitly. Instead, we will show that the classification problem is equivalent to classification of two-coloured unitary partitions introduced in Section 4.4. However, we need to treat some special cases first.

4.6.2 Full subcategory of non-coloured partitions

This section is analogical to Section 4.4.3. We are going to describe the connection between non-coloured categories and categories with extra singletons.

The set of all ordinary partitions $\mathscr{P}$ can be viewed as a subset of $\mathscr{P}^{\triangle}$. Moreover, $\mathscr{P}$ is a full subcategory of $\mathscr{P}^{\triangle}$ given by restricting to objects with no extra singletons. In this sense, every category of partitions with extra singletons $\mathscr{C}$ induces a full subcategory of one-coloured partitions:

$$\mathscr{C}^{|} := \{p \in \mathscr{C} \mid p \text{ does not contain any extra singleton}\}.$$

4.6.1 Lemma. Let $\mathscr{C}$ be a category of partitions with extra singletons.

(1) If $\triangle \in \mathscr{C}$, then $\mathscr{C} = \langle \mathscr{C}^{|}, \triangle\rangle^{\triangle}$.
(2) If $\triangle \notin \mathscr{C}$, $\uparrow\otimes\triangle \notin \mathscr{C}$, but ${}_{\triangle}\!\diagdown^{\triangledown} \in \mathscr{C}$, then $\mathscr{C} = \langle \mathscr{C}^{|}, {}_{\triangle}\!\diagdown^{\triangledown}\rangle^{\triangle}$.

Proof. Suppose $\triangle \in \mathscr{C}$ and take any $p \in \mathscr{C}$. Then since $\triangle \in \mathscr{C}$, we can remove all extra singletons in p by composition and obtain some $q \in \mathscr{C}^{|}$. The partition p can be obtained by reversing this process, i.e. taking $q \in \mathscr{C}^{|}$ and tensoring it with extra singletons $\triangle$, which proves that $\mathscr{C}$ is generated by $\mathscr{C}^{|}$ and $\triangle$.

Similarly for the second case. The conditions $\triangle \notin \mathscr{C}$ and $\uparrow\otimes\triangle \notin \mathscr{C}$ are equivalent to assuming that any partition $p \in \mathscr{C}$ contains an even number of extra singletons. This allows to reconstruct any partition $p \in \mathscr{C}$ from some non-coloured version $q \in \mathscr{C}^{|}$ using ${}_{\triangle}\!\diagdown^{\triangledown}$ (using composition with partitions of the form $|\cdots|{}_{\triangle}\!\diagdown^{\triangledown}|\cdots|$ we can move any extra singleton to any position). □

4.6.3 Partitions of odd length

We show that the case of partitions of odd length can be reduced to the case of partitions of even length.

4.6.2 Lemma. Let $\mathscr{C}$ be a category of partitions with extra singletons. Suppose $\mathscr{C}$ contains a partition of odd length. Then $\triangle \in \mathscr{C}$ or $\uparrow \in \mathscr{C}$.

Proof. Suppose $p \in \mathscr{C}$ has odd length $k > 1$. Without loss of generality, suppose that p has lower points only, i.e. $p \in \mathscr{C}(k)$. Then because of the odd length of p, there must be two neighbouring points in p (alternatively the first and the last point) of the same colour, so they can be contracted. By induction, we can contract any partition of odd length to a partition of length one, i.e. a singleton or an extra singleton. □

For the case $\triangle \in \mathscr{C}$, recall Lemma 4.6.1 saying that the category $\mathscr{C}$ is determined by the one-coloured category $\mathscr{C}^{|}$. Thus, the classification of such categories reduces to the one-coloured case.

The case when the singleton $\uparrow$ is contained in the category can be transformed to the case when it is not.

4.6.3 Lemma. Let $\mathscr{C}$ be a category of partitions with extra singletons such that $\uparrow \in \mathscr{C}$. Then $\mathscr{C} = \langle \tilde{\mathscr{C}}, \uparrow \rangle^{\vartriangle}$, where

$$\tilde{\mathscr{C}} = \{p \in \mathscr{C} \mid |p| \text{ is even}\}.$$

Proof. The inclusion $\supseteq$ is obvious. For the converse, consider $p \in \mathscr{C}$ with odd length. Then we have $p \otimes \uparrow \in \tilde{\mathscr{C}}$, so $p \in \langle p \otimes \uparrow, \uparrow \rangle^{\vartriangle} \subseteq \langle \tilde{\mathscr{C}}, \uparrow \rangle^{\vartriangle}$. □

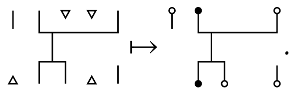
4.6.4 From extra-singleton categories to two-coloured categories

This section contains the main result regarding categories of partitions with extra singletons. We construct a functor that links the extra-singleton categories with categories of two-coloured partitions.

4.6.4 Definition. We define a functor $F\colon \mathscr{P}^{\vartriangle} \to \mathscr{P}^{\circ\bullet}$ as follows.

- Consider an object in $\mathscr{P}^{\vartriangle}$, that is, a word $w \in \mathscr{O}_{\vartriangle}^{*}$. Then $F(w)$ is obtained by colouring all the points in w with alternating white and black colour starting with white and then deleting all extra singletons. In particular, two neighbouring points in $F(w)$ have the same colour if and only if the corresponding points in w are separated by an odd number of $\vartriangle$.
- Consider a partition $p \in \mathscr{P}^{\vartriangle}(w_1, w_2)$. Then $F(p)$ is a two-coloured partition with upper colour pattern $F(w_1)$ and lower colour pattern $F(w_2)$ with the same block structure as p (ignoring the extra singletons).

4.6.5 Example. As a typical example, take the partition from Equation (4.4). We map it as follows



That is, we colour the odd points (i.e. first, third and fifth on both rows) with white colour and the even points (the second and fourth on both rows) with black. Then we erase all the triangles. Further examples are the following

Note, in particular, that we have $F(\substack{\triangledown\\ \vartriangle} \otimes p) = \overline{F(p)}$, where the bar denotes the colour inversion $\circ \leftrightarrow \bullet$. Note also that the image of a partition $p \in \mathscr{P}^{\vartriangle}$ is invariant with respect to adding a pair of consecutive extra singletons to p and adding arbitrary amount of extra singletons to the end of the upper or lower row. Conversely, for any $\tilde{p} \in \mathscr{P}^{\circ\bullet}$, its preimages differ only by such changes. In particular, any word $w \in \mathscr{O}_{\circ\bullet}$ and any partition $\tilde{p} \in \mathscr{P}^{\circ\bullet}$ have a unique shortest preimage.

4.6.6 Proposition. The map F satisfies:

(1) $F(w_1 \otimes w_2) = F(w_1) \otimes F(w_2)$ or $F(w_1 \otimes w_2) = F(w_1) \otimes \overline{F(w_2)}$.
(2) For p, q of even length, we have $F(p \otimes q) = F(p) \otimes F(q)$ or $F(p \otimes q) = F(p) \otimes \overline{F(q)}$.
(3) If p and q are composable, then $F(p)$ and $F(q)$ are composable and $F(qp) = F(q)F(p)$,
(4) $F(p^*) = F(p)^*$.

Thus, F is a unitary functor (by (3) and (4)), but not a monoidal functor (by (1) and (2)).

Proof. The proof is straightforward. Note that in (1) we apply the colour inversion if and only if the length of w_1 is odd. In (2) we use the fact that, for $p \in \mathscr{P}^{\vartriangle}(k,l)$ of even length, we have that either both k and l are even and we do not have to apply the colour inversion for $F(q)$ or both k and l are odd and then we apply the colour inversion for both the upper and the lower row of $F(q)$. □

4.6.7 Definition. We denote by $\mathscr{P}^{\vartriangle}_{\text{even}}$ the set of all partitions with extra singletons having even length. From now on we will consider F to be defined only on $\mathscr{P}^{\vartriangle}_{\text{even}}$. In particular, given a category $\tilde{\mathscr{C}} \subseteq \mathscr{P}^{\circ\bullet}$, we denote by $F^{-1}(\tilde{\mathscr{C}})$ its preimage inside $\mathscr{P}^{\vartriangle}_{\text{even}}$.

4.6.8 Theorem. [GW19b, Theorem 4.10] The map F defines a one-to-one correspondence

Categories of partitions with extra singletons having an even length $\mathscr{C} \subseteq \mathscr{P}^{\vartriangle}_{\text{even}}$	$\overset{F}{\longleftrightarrow}$	Categories of two-coloured partitions $\tilde{\mathscr{C}} \subseteq \mathscr{P}^{\circ\bullet}$ that are invariant with respect to colour inversions

To be more precise, the following holds.

(1) Let $\mathscr{C}$ be a category of partitions with extra singletons of even length. Then $F(\mathscr{C})$ is a category of two-coloured partitions, which is invariant with respect to colour inversions.
(2) Let $\tilde{\mathscr{C}}$ be a category of two-coloured partitions invariant with respect to colour inversions. Then $F^{-1}(\tilde{\mathscr{C}})$ is a category of partitions with extra singletons of even length.

It holds that $F^{-1}(F(\mathscr{C})) = \mathscr{C}$ and $F(F^{-1}(\tilde{\mathscr{C}})) = \tilde{\mathscr{C}}$.

Proof. Consider a category $\mathscr{C} \subseteq \mathscr{P}^{\vartriangle}_{\text{even}}$. As mentioned in Example 4.6.5, we have that $F(\overset{\triangledown}{\vartriangle} \otimes p)$ is the colour inversion of $F(p)$, so $F(\mathscr{C})$ is indeed closed under colour inversions. From Proposition 4.6.6, it directly follows that $F(\mathscr{C})$ is closed under involution. It is also closed under tensor products since we have that either $F(p)\otimes F(q) = F(p\otimes q)$ or $F(p)\otimes F(q) = F(p\otimes \overset{\triangledown}{\vartriangle} \otimes q)$. To check that $F(\mathscr{C})$ is closed under compositions, it is enough to prove that, for any composable pair $\tilde{p}, \tilde{q} \in F(\mathscr{C})$, there exist $p, q \in \mathscr{C}$ composable such that $\tilde{p} = F(p)$ and $\tilde{q} = F(q)$. It suffices to take p with the shortest possible lower row (with no extra singletons at the end and neighbouring extra singletons anywhere) and q with the shortest possible upper row.

The part (2) is proven similarly.

The equality $F(F^{-1}(\tilde{\mathscr{C}})) = \tilde{\mathscr{C}}$ is surely satisfied since it holds for any map F.

Since $\vartriangle \otimes \vartriangle \in \mathscr{C}$ for any category $\mathscr{C}$, we have that any category is closed under adding or removing pairs of neighbouring extra singletons. Since also $\overset{\triangledown}{\vartriangle} \in \mathscr{C}$, we can also add arbitrary amount of extra singletons to the end of the lower and the upper row. Consequently, any category $\mathscr{C}$ contains with any element $p \in \mathscr{C}$ the whole preimage $F^{-1}(F(p))$. Therefore, we also have $F^{-1}(F(\mathscr{C})) = \mathscr{C}$. This also proves that the described relationship is indeed a one-to-one correspondence. □

4.6.9 Proposition. Let $S \subseteq \mathscr{P}^{\vartriangle}$ be a set of partitions with extra singletons. Then $F(\langle S\rangle^{\vartriangle}) = \langle F(S)\rangle$.

Proof. The assertion follows from Theorem 4.6.8, namely from the fact that both F and F^{-1} map a category to a category. We surely have the inclusion $\supseteq$ since obviously $F(S) \subseteq F(\langle S\rangle^{\vartriangle})$ and $F(\langle S\rangle^{\vartriangle})$ is a category, so it must contain the category generated by $F(S)$. For the converse inclusion, we surely have $S \subseteq F^{-1}(\langle F(S)\rangle)$. Since we have a category on the right-hand side, it must contain $\langle S\rangle^{\vartriangle}$ and then we just apply F to both sides. □

4.6.5 An application to the theory of two-coloured partitions

This correspondence not only brings classification results for categories of partitions with extra singletons, but also conversely it brings new insight to the theory of two-coloured unitary partitions. In this section, we will define a new construction of a two-coloured category $\mathrm{Alt}\,\mathscr{C}$ induced by some non-coloured category $\mathscr{C}$.

Recall the forgetful functor $\Psi\colon \mathscr{P}^{\circ\bullet} \to \mathscr{P}$ acting on two-coloured partitions by forgetting the colour patterns from Definition 4.4.4.

4.6.10 Lemma. Let $\mathscr{C} \subseteq \mathscr{P}$ be a category of partitions such that $\uparrow \notin \mathscr{C}$. Then

$$F(\langle \mathscr{C}, {}_{\vartriangle}\backslash^{\triangledown} \rangle^{\vartriangle}) = \Psi^{-1}(\mathscr{C}).$$

Proof. The left-hand side equals to $\langle F(\mathscr{C}), \overset{\circ}{\underset{\bullet}{|}} \rangle$ by Proposition 4.6.9. The image $F(\mathscr{C})$ contains some colourization of partitions in $\mathscr{C}$. Thanks to the partition $\overset{\circ}{\underset{\bullet}{|}}$, the category $\langle F(\mathscr{C}), \overset{\circ}{\underset{\bullet}{|}} \rangle$ actually contains all the colourizations of all partitions in $\mathscr{C}$ and therefore equals to $\Psi^{-1}(\mathscr{C})$. Note that $F(\mathscr{C})$ is defined only if $\mathscr{C}$ contains only partitions of even length, which is equivalent to the assumption $\uparrow \notin \mathscr{C}$. □

4.6.11 Definition. We say that a two-coloured partition $p \in \mathscr{P}^{\circ\bullet}$ has an **alternating colouring** if the colour pattern of both upper and lower points alternates (between white and black), the colour of the first points of both rows coincide, and the colour of the last point of both rows coincide (consequently, p is of even length). For a two-coloured category $\mathscr{C} \subseteq \mathscr{P}^{\circ\bullet}$, we denote by $\mathrm{Alt}\,\mathscr{C}$ the category generated by elements of $\mathscr{C}$ that have an alternating colouring. For an ordinary category $\mathscr{C} \subseteq \mathscr{P}$, we denote $\mathrm{Alt}\,\mathscr{C} := \mathrm{Alt}\,\Psi^{-1}(\mathscr{C})$ the category generated by alternating coloured partitions in $\mathscr{C}$.

4.6.12 Lemma. Let $\mathscr{C} \subseteq \mathscr{P}$ be a category such that $\uparrow \notin \mathscr{C}$. Then

$$F(\langle \mathscr{C} \rangle^{\vartriangle}) = \mathrm{Alt}\,\mathscr{C}.$$

Proof. Follows directly from the definition of F and Alt. □

4.6.13 Remark. The operation Alt for two-coloured categories in general corresponds to the operation $\mathscr{C} \mapsto \langle \mathscr{C}^{|} \rangle^{\vartriangle}$ for categories with extra singletons. More precisely, we have $\mathrm{Alt}\,\mathscr{C} = F\langle (F^{-1}(\mathscr{C}))^{|} \rangle^{\vartriangle}$.

4.6.14 Lemma. Let $\mathscr{C} \subseteq \mathscr{P}$ be a category of partitions with $\uparrow \notin \mathscr{C}$. Then ${}_{\vartriangle}\backslash\backslash^{\triangledown} \notin \langle \mathscr{C} \rangle^{\vartriangle}$.

Proof. Let us define the set $\bar{\mathscr{C}}$ of all partitions with extra singletons $p \in \mathscr{P}^{\vartriangle}$ such that (1) if we remove all extra singletons from p then it is an element of $\mathscr{C}$, (2) between every pair of points incident to a common block, there is an even number of extra singletons. We prove that $\bar{\mathscr{C}}$ is a category. This is clear if we consider the operations on one line. Indeed, one can easily see that $\bar{\mathscr{C}}$ is closed under tensor product, contractions, rotations and reflections. Since $\mathscr{C} \subseteq \bar{\mathscr{C}}$, we must have $\langle \mathscr{C} \rangle^{\vartriangle} \subseteq \bar{\mathscr{C}}$. Obviously, ${}_{\vartriangle}\backslash\backslash^{\triangledown} \notin \bar{\mathscr{C}}$. □

4.6.15 Proposition. Let $\mathscr{C} \subseteq \mathscr{P}$ be a category of partitions with $\uparrow \notin \mathscr{C}$. Then $\mathrm{Alt}\,\mathscr{C}$ is a locally colourized category of partitions. In addition, we have that $\Psi(\mathrm{Alt}\,\mathscr{C}) = \mathscr{C}$, so distinct non-coloured categories $\mathscr{C}$ induce distinct two-coloured categories $\mathrm{Alt}\,\mathscr{C}$.

Proof. The fact that $\mathrm{Alt}\,\mathscr{C}$ is locally colourized is a consequence of the preceding lemma. The second statement is obvious: Since $\uparrow \notin \mathscr{C}$, all partitions $p \in \mathscr{C}$ have even length. Hence, all $p \in \mathscr{C}$ can be coloured alternating yielding $\tilde{p} \in \mathrm{Alt}\,\mathscr{C}$. Forgetting the colouring by Ψ get us back to the original category $\mathscr{C}$. □

We can put this section into the context of the classification of globally-colourized categories from Section 4.5. Taking a category $\mathscr{C} \subseteq \mathscr{P}$ with $\uparrow \notin \mathscr{C}$, we can construct the following sequence of mutually distinct two-coloured categories

$$\mathrm{Alt}\,\mathscr{C} \subsetneq \mathscr{C}_0 \subseteq \mathscr{C}_k := \langle \mathscr{C}_0, \sqcap_{\circ\,\circ}^{\otimes k/2} \rangle \subseteq \mathscr{C}_1 = \Psi^{-1}(\mathscr{C}),$$

where $k \in 2\mathbb{N}_0$. If $\uparrow \otimes \uparrow \in \mathscr{C}$, we can replace $\sqcap_{\circ\,\circ}^{\otimes k/2}$ by $\uparrow_{\circ}^{\otimes k}$ and take $k \in \mathbb{N}_0$.

Finally, we can restrict ourselves to non-crossing categories and compare this result with the classification from [TW18], which was summarized in Section 4.4.5.

$$\begin{aligned}
\mathrm{Alt}\langle\rangle &= \langle\rangle = \mathcal{O}_{\mathrm{loc}}\\
\mathrm{Alt}\langle \sqcap\!\sqcap\!\sqcap \rangle &= \langle \sqcap\!\sqcap\!\sqcap_{\circ\bullet\circ\bullet} \rangle = \mathcal{H}'_{\mathrm{loc}}\\
\mathrm{Alt}\langle \sqcap\!\sqcap\!\sqcap, \uparrow\otimes\uparrow \rangle &= \langle \sqcap\!\sqcap\!\sqcap_{\circ\bullet\circ\bullet}, \uparrow_\circ\otimes\uparrow_\bullet \rangle = \mathcal{S}_{\mathrm{loc}}(0,0)\\
\mathrm{Alt}\langle \uparrow\otimes\uparrow \rangle &= \langle \uparrow_\circ\otimes\uparrow_\bullet \rangle = \mathcal{B}_{\mathrm{loc}}(0,0)\\
\mathrm{Alt}\langle \sqcap\!\sqcap\;| \rangle &= \langle \sqcap\!\sqcap_{\circ\bullet\circ}\;|_\bullet \rangle = \mathcal{B}'_{\mathrm{loc}}(0,0,0)
\end{aligned}$$

4.6.6 Concrete classification results

In this section, we use Theorem 4.6.8 to transfer the available classification results for unitary two-coloured partitions to the case of categories of partitions with extra singletons. We focus mainly on the globally-colourized case classified in Section 4.5.

Recall that a category of two-coloured partitions $\mathscr{C} \in \mathscr{P}^{\circ\bullet}$ is globally-colourized if $\sqcap_{\circ\circ}\otimes\sqcap_{\bullet\bullet} \in \mathscr{C}$ or, equivalently, $\uparrow_\circ\otimes\uparrow_\bullet \in \mathscr{C}$. This holds if and only if the category $F^{-1}(\mathscr{C}) \subseteq \mathscr{P}^{\vartriangle}$ contains the partition ${}_{\vartriangle}\backslash\backslash^{\triangledown}$ (see Example 4.6.5). Given a two-coloured category of partitions $\mathscr{C} \subseteq \mathscr{P}^{\circ\bullet}$, recall also the definition of the subcategory $\mathscr{C}_0$ containing partitions $p \in \mathscr{C}(w_1, w_2)$ with zero colour sum, that is, $c(p) := c(w_2) - c(w_1) = 0$.

4.6.16 Lemma. Let $\mathscr{C} \subseteq \mathscr{P}$ be a category of partitions. Then

$$(\Psi^{-1}(\mathscr{C}))_0 = \langle \mathrm{Alt}\,\mathscr{C}, \uparrow_\circ\otimes\uparrow_\bullet \rangle.$$

Proof. The category Alt$\mathscr{C}$ contains some particular zero-sum colourings of partitions in $\mathscr{C}$. Adding the globally-colourizing partition $\uparrow_\circ\otimes\uparrow_\bullet$ we have that $\langle \mathrm{Alt}\,\mathscr{C}, \uparrow_\circ\otimes\uparrow_\bullet \rangle$ contains all the zero-sum colourings and hence the category coincides with $(\Psi^{-1}(\mathscr{C}))_0$. □

4.6.17 Definition. Let $\mathscr{C} \subseteq \mathscr{P}$ be a category of partitions such that $\uparrow \notin \mathscr{C}$. We define the following categories of partitions with extra singletons.

$$\mathscr{C}_0^{\vartriangle} := \langle \mathscr{C}, {}_{\vartriangle}\backslash\backslash^{\triangledown} \rangle^{\vartriangle}, \qquad \mathscr{C}_{2k}^{\vartriangle} := \langle \mathscr{C}, ({}_{\vartriangle}\backslash^{\triangledown})^{\otimes k} \rangle^{\vartriangle}$$

for any $k \in \mathbb{N}$. If $\uparrow\otimes\uparrow \in \mathscr{C}$, then we also define

$$\mathscr{C}_k^{\vartriangle} := \langle \mathscr{C}, {}_{\vartriangle}\backslash\backslash^{\triangledown}, (\uparrow\otimes{\scriptstyle\vartriangle})^{\otimes k} \rangle^{\vartriangle}.$$

Note that if $\uparrow\otimes\uparrow \in \mathscr{C}$, then $\langle \mathscr{C}, ({}_{\vartriangle}\backslash^{\triangledown})^{\otimes k} \rangle^{\vartriangle} = \langle \mathscr{C}, {}_{\vartriangle}\backslash\backslash^{\triangledown}, (\uparrow\otimes{\scriptstyle\vartriangle})^{\otimes 2k} \rangle^{\vartriangle}$, so the two above mentioned definitions of $\mathscr{C}_{2k}^{\vartriangle}$ coincide. This can be proven by applying the functor F from Definition 4.6.4. Thanks to Proposition 4.6.9, we can write the two-coloured version of the equality as

$$\langle \uparrow_\circ\otimes\uparrow_\bullet, \uparrow_\bullet\otimes\uparrow_\circ, \uparrow_\circ^{\otimes 2k} \rangle = \langle \uparrow_\circ\otimes\uparrow_\bullet, \uparrow_\bullet\otimes\uparrow_\circ, \uparrow_\bullet^{\otimes k} \rangle,$$

which then follows from Lemma 4.5.8.

4.6.18 Lemma. Let $\mathscr{C} \subseteq \mathscr{P}$ be a category of partitions such that $\uparrow \notin \mathscr{C}$. Then

$$\Psi^{-1}(\mathscr{C})_0 = F(\mathscr{C}_0^{\vartriangle}), \qquad \langle \Psi^{-1}(\mathscr{C})_0, \uparrow_\bullet^{\otimes k} \rangle = F(\mathscr{C}_{2k}^{\vartriangle}), \qquad \langle \Psi^{-1}(\mathscr{C})_0, \uparrow_\circ^{\otimes k} \rangle = F(\mathscr{C}_k^{\vartriangle}),$$

where $k \in \mathbb{N}$ and the last equality makes sense only if $\uparrow\otimes\uparrow \in \mathscr{C}$.

Proof. The proof is similar in all three cases using Proposition 4.6.9 and Lemma 4.6.16. Take, for example, the middle one. We have

$$F(\mathscr{C}_{2k}^{\vartriangle}) = \langle F(\mathscr{C}), F(({}_{\vartriangle}\backslash^{\triangledown})^{\otimes k}) \rangle = \langle \mathrm{Alt}\,\mathscr{C}, \uparrow_\bullet^{\otimes k} \rangle = \langle \Psi^{-1}(\mathscr{C})_0, \uparrow_\bullet^{\otimes k} \rangle. \qquad □$$

Non-crossing:	$\langle {}_{\vartriangle}\backslash\backslash^{\triangledown}, ({}_{\vartriangle}\backslash^{\triangledown})^{\otimes k/2}\rangle$, $\langle \sqcap\!\sqcap\!\sqcap, {}_{\vartriangle}\backslash\backslash^{\triangledown}, ({}_{\vartriangle}\backslash^{\triangledown})^{\otimes k/2}\rangle$, $\langle \uparrow\otimes\uparrow, {}_{\vartriangle}\backslash\backslash^{\triangledown}, (\uparrow\otimes{\vartriangle})^{\otimes k}\rangle$, $k\in 2\mathbb{N}_0$
	$\langle {}_{\shortmid}\backslash^{\shortmid}, {}_{\vartriangle}\backslash\backslash^{\triangledown}, (\uparrow\otimes{\vartriangle})^{\otimes k}\rangle$, $\langle \sqcap\!\sqcap\!\sqcap, \uparrow\otimes\uparrow, {}_{\vartriangle}\backslash\backslash^{\triangledown}, (\uparrow\otimes{\vartriangle})^{\otimes k}\rangle$, $k\in\mathbb{N}_0$
$\times\in\mathscr{C}$:	$\langle \times, {}_{\vartriangle}\backslash\backslash^{\triangledown}, ({}_{\vartriangle}\backslash^{\triangledown})^{\otimes k/2}\rangle$, $k\in 2\mathbb{N}_0$
	$\langle \times, \uparrow\otimes\uparrow, {}_{\vartriangle}\backslash\backslash^{\triangledown}, (\uparrow\otimes{\vartriangle})^{\otimes k}\rangle$, $\langle \times, \sqcap\!\sqcap\!\sqcap, \uparrow\otimes\uparrow, {}_{\vartriangle}\backslash\backslash^{\triangledown}, (\uparrow\otimes{\vartriangle})^{\otimes k}\rangle$, $k\in\mathbb{N}_0$
	$\langle \times, \sqcap\!\sqcap\!\sqcap, {}_{\vartriangle}\backslash\backslash^{\triangledown}, ({}_{\vartriangle}\backslash^{\triangledown})^{\otimes k/2}\rangle$, $k\in 2\mathbb{N}_0$ (∗)
$\divideontimes\in\mathscr{C}$:	$\langle \divideontimes, {}_{\vartriangle}\backslash\backslash^{\triangledown}, ({}_{\vartriangle}\backslash^{\triangledown})^{\otimes k/2}\rangle$, $\langle \divideontimes, \uparrow\otimes\uparrow, {}_{\vartriangle}\backslash\backslash^{\triangledown}, (\uparrow\otimes{\vartriangle})^{\otimes k}\rangle$, $k\in 2\mathbb{N}_0$
	$\langle \divideontimes, \sqcap\!\sqcap\!\sqcap, {}_{\vartriangle}\backslash\backslash^{\triangledown}, ({}_{\vartriangle}\backslash^{\triangledown})^{\otimes k/2}\rangle$, $k\in 2\mathbb{N}_0$ (∗)
	$\langle \divideontimes, \sqcap\!\sqcap\!\sqcap, h_s, {}_{\vartriangle}\backslash\backslash^{\triangledown}, ({}_{\vartriangle}\backslash^{\triangledown})^{\otimes k/2}\rangle$, $k\in 2\mathbb{N}_0, s\geq 3$ (∗)
The rest:	$\langle \pi_s, {}_{\vartriangle}\backslash\backslash^{\triangledown}, ({}_{\vartriangle}\backslash^{\triangledown})^{\otimes k/2}\rangle, s\geq 2$, $\langle \pi_l, {}_{\vartriangle}\backslash\backslash^{\triangledown}, ({}_{\vartriangle}\backslash^{\triangledown})^{\otimes k/2} \mid l\in\mathbb{N}\rangle$, $k\in 2\mathbb{N}_0$
	$\langle A, {}_{\vartriangle}\backslash\backslash^{\triangledown}, ({}_{\vartriangle}\backslash^{\triangledown})^{\otimes k/2}\rangle$, $\uparrow\otimes\uparrow\notin A\trianglelefteq\mathbb{Z}_2^{\infty}$, $k\in 2\mathbb{N}_0$

Table 4.2 Classification of categories with extra singletons containing ${}_{\vartriangle}\backslash\backslash^{\triangledown}$. Categories in the rows marked by the asterisk (∗) are special cases of the group-theoretical categories.

4.6.19 Theorem. [GW19b, Theorem 6.1] Every category of partitions with extra singletons containing only partitions of even length is of the form $\mathscr{C}_k^{\vartriangle}$, where $\mathscr{C}\subseteq\mathscr{P}$ is some category of partitions such that $\uparrow\notin\mathscr{C}$ and $k\in\mathbb{N}_0$ is even unless $\uparrow\otimes\uparrow\in\mathscr{C}$. Distinct pairs $(\mathscr{C},k)$ define distinct categories $\mathscr{C}_k^{\vartriangle}$ with the exception that

$$\langle \uparrow\otimes\uparrow, {}_{\vartriangle}\backslash\backslash^{\triangledown}, (\uparrow\otimes{\vartriangle})^{\otimes k}\rangle = \langle {}_{\shortmid}\backslash^{\shortmid}, {}_{\vartriangle}\backslash\backslash^{\triangledown}, (\uparrow\otimes{\vartriangle})^{\otimes k}\rangle$$
$$\langle \divideontimes, \uparrow\otimes\uparrow, {}_{\vartriangle}\backslash\backslash^{\triangledown}, (\uparrow\otimes{\vartriangle})^{\otimes k}\rangle = \langle \times, \uparrow\otimes\uparrow, {}_{\vartriangle}\backslash\backslash^{\triangledown}, (\uparrow\otimes{\vartriangle})^{\otimes k}\rangle$$

for all k odd.

Proof. As already mentioned, we just apply Theorem 4.6.8 to the classification from Section 4.5. The statement is then just reformulation of Theorem 4.5.10. Using Lemma 4.6.18, we find the preimage of the categories mentioned in Lemmata 4.5.8, 4.5.9 by the functor F (all of the categories are invariant with respect to the colour inversion). The mutual inequality is discussed in the proof of Theorem 4.5.10. □

So, if we want to obtain a list of all categories with partitions with extra singletons, we just need to take the classification of all categories of partitions $\mathscr{C}$ and construct the categories $\mathscr{C}_k^{\vartriangle}$. We already did similar work in the unitary two-coloured case – the classification is summarized in Table 4.1 (p. 70). Now, we can just copy the result applying the functor F on all categories in Table 4.1. The result is listed in Table 4.2.

Let us also mention the case when the category with extra singletons $\mathscr{C}$ contains also partitions of odd length. Recall from Subsection 4.6.3 that this can happen only if $\uparrow\in\mathscr{C}$ or ${\vartriangle}\in\mathscr{C}$. In the latter case, the classification is equivalent to the classification of ordinary categories.

4.6.20 Proposition. Let $\mathscr{C}$ be a category of partitions with extra singletons such that $\uparrow\in\mathscr{C}$, ${\vartriangle}\notin\mathscr{C}$, and ${}_{\vartriangle}\backslash\backslash^{\triangledown}\in\mathscr{C}$. Then $\mathscr{C}$ is one of the following categories

$$\langle\uparrow, {}_{\vartriangle}\backslash^{\triangledown}\rangle^{\vartriangle},\quad \langle\sqcap\!\sqcap\!\sqcap, \uparrow, {}_{\vartriangle}\backslash^{\triangledown}\rangle^{\vartriangle},\quad \langle\uparrow, \times, {}_{\vartriangle}\backslash^{\triangledown}\rangle^{\vartriangle},\quad \langle\sqcap\!\sqcap\!\sqcap, \uparrow, \times, {}_{\vartriangle}\backslash^{\triangledown}\rangle^{\vartriangle}.$$

Proof. According to Lemma 4.6.3, we have that $\mathscr{C}=\langle\tilde{\mathscr{C}},\uparrow\rangle^{\vartriangle}$, where $\tilde{\mathscr{C}}$ is some category of partitions with extra singletons with elements of even length. Thus, we get all possible categories $\mathscr{C}$ by adding the singleton $\uparrow$ to the categories listed in Table 4.2. We find out that there are only the above mentioned four distinct instances. □

We can do the same thing with the classification of non-crossing two-coloured categories that was obtained in [TW18] and summarized in Section 4.4.5. We mention only the locally colourized categories since the globally colourized ones were handled above.

$\langle\rangle^{\vartriangle}$, $\langle ⊓⊓⊓ \rangle^{\vartriangle}$, $\langle ⊓⊓⊓, \uparrow\rangle^{\vartriangle}$, $\langle\uparrow\rangle^{\vartriangle}$,	
$\langle b_k^{\vartriangle}, b_d^{\vartriangle} \otimes b_d^{\vartriangle *}, ⊓_{\vartriangle}⊓⊓_{\vartriangle}⊓, ⊓⊓⊓ \rangle^{\vartriangle}$	$k, d \in \mathbb{N}_0 \setminus \{1,2\}$, $d \mid k$
$\langle (\uparrow \otimes \vartriangle)^{\otimes k}, (\uparrow \otimes \vartriangle)^{\otimes d} \otimes \mid \otimes ({}^{\triangledown} \otimes \downarrow)^{\otimes d}, ⊓⊓⊓, \uparrow \otimes \uparrow \rangle$	$k, d \in \mathbb{N}_0 \setminus \{1\}$, $d \mid k$
$\langle (\uparrow \otimes \vartriangle)^{\otimes k}, (\uparrow \otimes \vartriangle)^{\otimes d} \otimes \mid \otimes ({}^{\triangledown} \otimes \downarrow)^{\otimes d}, \uparrow \otimes \uparrow \rangle$	$k, d \in \mathbb{N}_0$, $d \mid k$
$\langle (\uparrow \otimes \vartriangle)^{\otimes k}, (\uparrow \otimes \vartriangle)^{\otimes d} \otimes \mid \otimes ({}^{\triangledown} \otimes \downarrow)^{\otimes d}, {}_{\shortmid}\!\!\setminus\!{}^{\shortmid} \rangle$	$k, d \in \mathbb{N}_0 \setminus \{1\}$, $d \mid k$
$\langle (\uparrow \otimes \vartriangle)^{\otimes k}, (\uparrow \otimes \vartriangle)^{\otimes d} \otimes \mid \otimes ({}^{\triangledown} \otimes \downarrow)^{\otimes d}$,	
$\quad (\uparrow \otimes \vartriangle)^{d/2+1} \otimes {}^{\triangledown}_{\vartriangle} \otimes \mid \otimes ({}^{\triangledown} \otimes \downarrow)^{\otimes d/2-1} \rangle$	$k \in \mathbb{N}_0 \setminus \{1\}$, $d \in 2\mathbb{N}_0 \setminus \{0,2\}$, $d \mid k$

Table 4.3 Classification of non-crossing categories with extra singletons that do not contain ${}_{\vartriangle}\!\setminus\!\setminus^{\triangledown}$

4.6.21 Proposition. Let $\mathscr{C} \subseteq \mathscr{P}^{\circ\bullet}$ be a category of non-crossing partitions with extra singletons such that ${}_{\vartriangle}\!\setminus\!\setminus^{\triangledown} \notin \mathscr{C}$. Then $\mathscr{C}$ equals to one of the categories in Table 4.3.

In the table, we denote

$$b_k^{\vartriangle} := ⊓_{\vartriangle}⊓_{\vartriangle}⊓_{\vartriangle} \cdots ⊓_{\vartriangle} \in \mathscr{P}^{\vartriangle}(0, 2k)$$

the partition consisting of a block of length k and k extra singletons.

Proof. The classification of locally colourized non-crossing two-coloured categories is summarized in Theorem 4.4.13. All of the categories are invariant with respect to the colour inversion. According to Theorem 4.6.8, we obtain all non-crossing categories of partitions with extra singletons of even length by applying F^{-1} to this classification. Categories containing the extra singleton $\vartriangle$ contain also the partition ${}_{\vartriangle}\!\setminus\!\setminus^{\triangledown}$, so they are not listed. Categories containing the singleton $\uparrow$ are obtained by adding the singleton to the categories with partitions of even length. □

Finally, let us review the recent development in the classification programme of two-coloured categories and its relationship with categories with extra singletons. In [MW19a, MW19b], Mang and Weber classified all categories of two-coloured partitions with neutral pairings, that is, all categories $\tilde{\mathscr{C}} \subseteq \mathscr{P}^{\circ\bullet}$ with $\langle\rangle \subseteq \tilde{\mathscr{C}} \subseteq \langle \times \rangle$. Applying the functor F to these results, we get classification of all categories of partitions with extra singletons $\mathscr{C} \subseteq \mathscr{P}^{\vartriangle}$ with $\langle\rangle^{\vartriangle} \subseteq \mathscr{C} \subseteq \langle \times \rangle^{\vartriangle}$. Classification of all non-hyperoctahedral two-coloured categories (that is, $\uparrow \otimes \uparrow \in \tilde{\mathscr{C}}$ or $⊓⊓⊓ \notin \tilde{\mathscr{C}}$) is a work in progress. Preliminary results are available in [MW19c, MW20], which provides another rich source for categories of partitions with extra singletons.

Chapter 5

Linear categories of partitions

In this chapter, we study *linear categories of partitions*, which are obtained by adding the linear structure to the categories of partitions and modifying the composition rule by some scalar factor. The linear categories are actually much more appropriate structures to describe representation categories of quantum groups. Besides that, they are interesting examples of monoidal $*$-categories as such and are studied within the category theory even in cases when they do not correspond to any quantum group [Del07, CO11, CH17].

The reason why we were studying the categories without the linear structure in the previous chapter is that they are much easier to work with. Their classification is essentially a combinatorial problem, which makes it easier to solve. In contrast, there was basically nothing known about *non-easy* linear categories of partitions when we started the PhD project. A linear category of partitions is called *non-easy* whenever working with non-trivial linear combinations of partitions is essential to describe it – it is not induced by any ordinary category of partitions in the sense of the previous chapter.

The main result of this chapter is twofold. As no examples of non-easy categories were known before, the first goal was to obtain such examples. This was done using some computer experiments.[1] We describe the idea in Section 5.2 and provide the concrete computations in Section 5.3. Secondly, we study the categories by theoretical means and prove that they are indeed new and non-easy. This is done in Section 5.4.

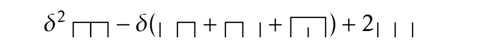
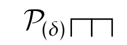
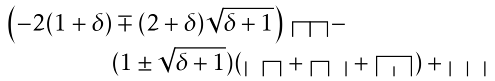
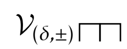
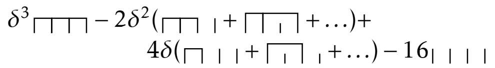
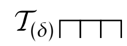
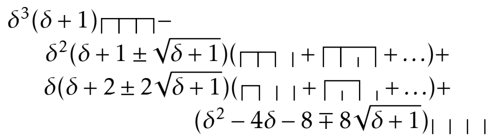
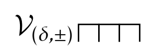
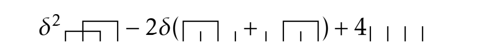
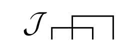
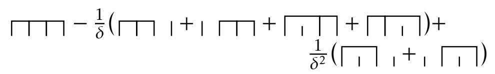
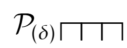
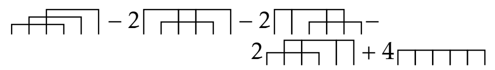
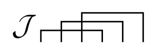
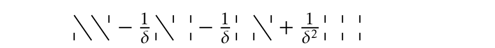
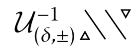
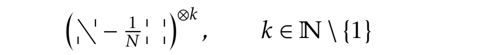
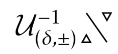
Paragraph w/ candid.	Generator as a full linear combination	Section where it is studied	Systematical description
5.3.2		5.4.1, 5.4.5	
5.3.2		5.4.6	
5.3.5		5.4.2	
5.3.5		5.4.6	
5.3.6		5.4.3	
5.3.6		5.4.4	
5.3.9		5.4.5	
—		5.4.4	
—		7.2.5	
—		7.2.5	

Table 5.1 Summary of all generators of non-easy linear categories of partitions studied in this thesis

[1] This is also the main part of the thesis that profited of the integration in the collaborative research centre SFB-TRR 195 *Symbolic Tools in Mathematics and their Application*.

To summarize the results of Section 5.3 and give an overview of Section 5.4, we list in Table 5.1 all the linear combinations of partitions appearing in this thesis that generate non-easy categories. In the first column, we give a reference to the corresponding paragraph in Section 5.3, where the linear combination was discovered. In the second column, we explicitly write down the linear combination of partitions. In the third column, we refer to the corresponding section, where the linear combination was studied. We interpret those linear combinations usually as images of some mappings and we give this interpretation in the last column. Note that the expressions in the second and the last column may not be equal; however, they generate the same category.

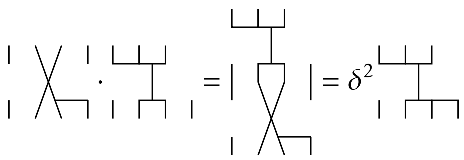
The study of the generators inspired us to define additional linear combinations that were not discovered by our experiments (since they are too big). We list also those at the end of the table.

Note that Table 5.1 does not yet provide an exhaustive summary of all non-easy categories we found. We list here only the generators. Those generators can be further combined with other partitions to define additional non-easy categories. A list of all non-easy categories together with the associated quantum groups appearing in this thesis will be provided in Table 7.1 on page 125.

5.1 Definition of linear categories of partitions

We fix a complex number $\delta \in \mathbb{C}$ and define $\mathrm{Part}_\delta(k,l) = \mathrm{span}\,\mathscr{P}(k,l)$ to be the vector space of formal linear combinations of partitions with k upper and l lower points. Let us stress that we do not assume any relations between the partitions, so $\mathscr{P}(k,l)$ is a basis of $\mathrm{Part}_\delta(k,l)$.

Recall the definition of the category operations (tensor product, composition and involution) from Section 4.1.1. For two partitions $p \in \mathscr{P}(k,l)$, $q \in \mathscr{P}(l,m)$, we denote by $\mathrm{rl}(p,q)$ the number of *remaining loops* that emerge when putting q below p in order to compute the composition qp. Those are the connected components of the diagram that are not connected to any of the upper or the lower points. We modify the definition of composition on Part_δ by adding a multiplicative factor δ for each such deleted loop. Thus, in total, we have the factor $\delta^{\mathrm{rl}(p,q)}$. An example follows:



We extend the composition as well as the tensor product linearly to the whole vector space. We extend the involution antilinearly. This gives Part_δ the structure of a monoidal $*$-category. Any collection of vector subspaces $\mathscr{K}(k,l) \subseteq \mathrm{Part}_\delta(k,l)$ containing the identity partition $| \in \mathscr{K}(1,1)$ and the pair partition $\sqcap \in \mathscr{K}(0,2)$ that is closed under the category operations is called a **linear category of partitions**.

Similarly as in the case of partitions, by $p \in \mathrm{Part}_\delta$ or $p \in \mathscr{K}$, we mean that p is element of one of the spaces $\mathrm{Part}_\delta(k,l)$ or $\mathscr{K}(k,l)$, respectively. For given $p_1,\ldots,p_n \in \mathrm{Part}_\delta$, we denote by $\langle p_1,\ldots,p_n\rangle_\delta$ the smallest linear category of partitions containing $p_1,\ldots,p_n$. Any element in $\langle p_1,\ldots,p_n\rangle_\delta$ can be obtained from the generators $p_1,\ldots,p_n$, the identity partition $|$, and the pair partition $\sqcap$ by performing a finite amount of category operations and linear combinations.

Instead of having different categories for different parameters δ, we can consider "all of them at once". That is, define a category Part, where the morphism spaces $\mathrm{Part}(k,l)$ are modules over the polynomial ring $R := \mathbb{C}[\delta]$.

We can also extend linearly or antilinearly the additional operations that were defined in Section 4.1.2 such as rotations, reflections and contractions. Also here we denote $\mathrm{Part}_\delta(k) := \mathrm{Part}_\delta(0,k)$.

Let us now link the linear categories of partitions with the categories without the linear structure that were studied in Chapter 4.

5.1.1 Proposition. Let $\mathscr{C} \subseteq \mathscr{P}$ be a category of partitions. Put $\mathscr{K}(k,l) := \operatorname{span}\mathscr{C}(k,l)$ for every $k,l \in \mathbb{N}_0$. Then $\mathscr{K} \subseteq \mathsf{Part}_\delta$ is a linear category of partitions for every $\delta \in \mathbb{C}$. Conversely, suppose that $\mathscr{K} \subseteq \mathsf{Part}_\delta$, $\delta \neq 0$ is a linear category of partitions such that $\mathscr{K}(k,l) = \operatorname{span}\mathscr{C}(k,l)$ for some collection of sets $\mathscr{C}(k,l) \subseteq \mathscr{P}(k,l)$. Then $\mathscr{C}$ is a category of partitions.

Proof. The proof is straightforward. The spaces $\mathscr{K}(k,l)$ are closed under the category operations if and only if the bases of the spaces $\mathscr{C}(k,l)$ are. The category operations on $\mathscr{K}$ and $\mathscr{C}$ coincide up to scalar factor δ, hence the equivalence (for $\delta \neq 0$). □

Such a linear category $\mathscr{K}$ linearly generated by some ordinary category $\mathscr{C}$ is called **easy** and we write $\mathscr{K} = \operatorname{span}\mathscr{C}$ as a shorthand for $\mathscr{K}(k,l) = \operatorname{span}\mathscr{C}(k,l)$. If this is not the case, then $\mathscr{K}$ is called **non-easy**.

5.1.2 Remark. Consider partitions $p_1,\ldots,p_n \in \mathscr{P}$, $\delta \neq 0$. Then $\langle p_1,\ldots,p_n\rangle_\delta = \operatorname{span}\langle p_1,\ldots,p_n\rangle$. In particular, such a category is always easy. This again follows directly from the fact that the category operations on linear categories and ordinary categories coincide up to a scalar factor.

We will use special notation for some easy linear categories of partitions. For example, we denote $\mathsf{Pair}_\delta := \langle \times \rangle_\delta$ the linear category of all pairings. We denote $\mathsf{NCPart}_\delta := \operatorname{span} NC = \langle ⊓⊓⊓, \uparrow\rangle_\delta$ the linear category of all non-crossing partitions.

5.2 Algorithmic search for linear categories of partitions

Since we wanted to find some first examples of non-easy linear categories of partitions – and it was not a priori clear how to construct them –, we decided to choose an algorithmic approach to reach this goal. The following text, i.e. Sections 5.2 and 5.3, is based on the article [GW19a].

The idea of using a computer to find examples of non-easy categories is very simple. Consider a linear combination of partitions $p \in \mathsf{Part}_\delta(k,l)$ and try to generate the whole category $\mathscr{K} := \langle p\rangle_\delta$ by iterating the category operations on the set $\{|, ⊓, p\}$. Unfortunately, there is no theoretical result that would assure that, after performing a given amount of category operations on the generators, we get all elements of $\mathscr{K}(i,j)$ for some $i,j \in \mathbb{N}_0$. Thus, we cannot directly use the computer to prove non-easiness of a category. However, we are able to prove *easiness* of a category and hence, excluding the easy cases, we obtain at least candidates for the non-easy categories. The precise way, how we use this to look for non-easy categories is described in Section 5.3.

We describe the algorithm in the following sections. In Section 5.2.1, we list some additional simple observations that will simplify the search. In Section 5.2.2, we describe some preprocessing, that is, some preparatory computations we have to perform on our computer before we can run the algorithm. In Section 5.2.3, we discuss the actual procedures we have to program in the computer and, in Section 5.2.4, we summarize the whole algorithm. Finally, in Section 5.2.5, we discuss the limits of the algorithm.

5.2.1 Observations on how to detect non-easy categories

We formulate a series of additional simple observations that should simplify the search for non-easy categories.

First of all, when looking for examples of non-easy categories, it makes sense to look just for the categories generated by one element.

5.2.1 Observation. Let $p_1,\ldots,p_n \in \mathsf{Part}_\delta(k,l)$. If $\langle p_1,\ldots,p_n\rangle_\delta$ is non-easy, then at least one of the categories $\langle p_1\rangle_\delta,\ldots,\langle p_n\rangle_\delta$ is non-easy.

Proof. If all the categories $\langle p_i\rangle_\delta$ are easy, then $\langle p_1,\ldots,p_n\rangle_\delta$ is generated by partitions, which, according to Remark 5.1.2, implies that it is easy. □

The following observation describes how to prove easiness of the category $\langle p\rangle_\delta$.

5.2.2 Observation. Consider $p\in\mathsf{Part}_\delta(k,l)$ and express it in the basis of partitions as $p=\sum_i\alpha_i p_i$, where $\alpha_i\in\mathbb{C}$ are non-zero numbers and $p_i\in\mathscr{P}(k,l)$ are mutually different partitions. Then the category $\langle p\rangle_\delta$ is easy if and only if it contains all the partitions p_i.

Proof. Left-right implication follows from uniqueness of coordinates with respect to a given basis. Right-left multiplication follows from Remark 5.1.2. □

The following result further reduces the computational complexity. In particular, it allows to avoid using the antilinear operation of reflection.

5.2.3 Observation. Let S be a set of linear combinations of partitions on one line which is closed under the operation of reflection and contains the pair partition $\sqcap$. Then any element of $\langle S\rangle$ can be obtained by performing a finite amount of tensor products, contractions and rotations and taking linear combinations. It is automatically closed under reflections.

Proof. We have

$$(p\otimes q)^\star=q^\star\otimes p^\star,$$
$$(\Pi p)^\star=(R^{-2}\circ\Pi\circ R^2)p^\star,$$
$$(Rp)^\star=R^{-1}p^\star.$$

Hence, any set S remains closed under reflections after applying arbitrary amount of tensor product, contractions, and rotations. □

Finally, the following proposition will be useful to reduce the amount of generators p we have to consider. Recall that by $R\colon\mathsf{Part}_\delta(k)\to\mathsf{Part}_\delta(k)$ we denote the linear operator of rotation on one line.

5.2.4 Proposition. Consider $p,q\in\mathsf{Part}_\delta(k)$ and let f be a polynomial of degree less than k. Then $\langle f(R)p+q\rangle=\langle g(R)p+\tilde q\rangle$, where $g(x)=\gcd(f(x),x^k-1)$ and $\tilde q\in\mathsf{Part}_\delta(k)$.

Proof. Consider $f(x)$ as an element of the algebra $A:=\mathbb{C}[x]/I$, where I is the ideal generated by x^k-1. Since $R^k=I$, the evaluation $h(R)$ for $h\in A$ does not depend on the particular representative.

There certainly exists $\tilde f\in A$ such that $f=\tilde f g$ and $\tilde f$ is coprime to x^k-1 (just take $\tilde f(x)=(f(x)+j(x^k-1))/g(x)$ for appropriate $j\in\mathbb{N}$). Then $\tilde f$, as an element of A, is not a divisor of zero and hence, since A is finite-dimensional, it is invertible. Therefore, there exists $h\in A$ such that $hf=h\tilde f g=g$ and we have that $\langle f(R)p+q\rangle\supseteq\langle h(R)(f(R)p+q)\rangle=\langle g(R)p+\tilde q\rangle$, where $\tilde q:=h(R)q$.

The opposite inclusion is easy $\langle g(R)p+\tilde q\rangle\supseteq\langle\tilde f(R)(g(R)p+\tilde q)\rangle=\langle f(R)p+q\rangle$. □

5.2.2 Preprocessing

First, we need to compute the matrices of the operations of tensor product, contraction and rotation as linear maps. Note that the number of partitions of l points is given by *Bell numbers* B_l (see Table 5.2). So, the dimension of $\mathsf{Part}_\delta(l)$ is B_l. Thus, we can identify $\mathsf{Part}_\delta(l)\simeq\mathbb{C}^{B_l}$ identifying the partitions $p\in\mathscr{P}(l)$ with the standard basis in $\mathbb{C}^{B_l}$. Actually, it is more convenient not to specify the value of δ and consider rather $\mathsf{Part}(l)\simeq R^{B_l}$ for $R:=\mathbb{C}[\delta]$.

The tensor product $\otimes\colon\mathsf{Part}(k)\times\mathsf{Part}(l)\to\mathsf{Part}(k+l)$ of partitions can be viewed as a linear map

$$\mathrm{tens}\colon R^{B_kB_l}\to R^{B_{k+l}}.$$

Similarly, we can define the matrices corresponding to contraction and rotation

$$\mathrm{contr}\colon R^{B_l}\to R^{B_{l-2}},\qquad \mathrm{rot}\colon R^{B_l}\to R^{B_l}.$$

We fix a *length bound* $l_0\in\mathbb{N}_0$ and compute all those matrices for $l\le l_0$ (resp. $k+l\le l_0$ in the case of the tensor product).

5.2.3 Adding procedures

We define modules $K_l \subseteq R^{B_l}$ for $l \leq l_0$ that are going to approximate the spaces $\mathscr{K}(l)$ of some category $\mathscr{K} \subseteq \mathsf{Part}$. To fill the modules, we define the following procedures.

The procedure AddParts takes a set $S \in R^{B_l}$ representing a set of linear combinations of partitions from $\mathscr{K}(l)$ and adds it to the module K_l. In addition, it adds all the rotations of the partitions to K_l and all their contractions to the corresponding K_{l-2i}. Thus, we end up with an approximation of $\mathscr{K}$, which contains the set S and is closed under taking rotations and contractions.

```
procedure AddParts(l ∈ {1,...,l_0}, S ⊆ R^{B_l})
    if l ≥ 2 then AddParts(l − 2, contr(S))
    K_l := K_l + S
    for j ∈ {1,...,l − 1} do
        S := rot(S)
        if l ≥ 2 then AddParts(l − 2, contr(S))
        K_l := K_l + S
```

The procedure AddTensors takes all pairs $x \in K_k$ and $y \in K_l$ such that $k + l \leq l_0$ and computes the vector corresponding to the partition tensor product $\mathrm{tens}(x \otimes y)$. Note that we can assume $k \leq l$ since we have $q \otimes p = R^l(p \otimes q)$ for $p \in \mathsf{Part}(k)$ and $q \in \mathsf{Part}(l)$. To add the results to the category approximation, we use the procedure AddParts, so we add also all the rotations and contractions of the tensor products.

```
procedure AddTensors
for k ∈ {1,...,⌊l_0/2⌋} do
    for l ∈ {k,...,l_0 − k} do
        AddParts(k + l, tens(K_k ⊗ K_l))
```

5.2.4 The algorithm determining candidates for non-easy categories

Suppose we have generators $p_1, \ldots, p_n \in \mathsf{Part}(l_i)$, $l_i \leq l_0$. Then we can compute an approximation of $\mathscr{K} := \langle p_1, \ldots, p_n \rangle$ by performing the following algorithm.

(1) AddParts$(2, \sqcap)$; AddParts$(l_i, \{p_i, p_i^\star\})$ for every $i = 1, \ldots, n$.
(2) Repeat AddTensors until this procedure leaves all the modules K_l unchanged.

At this stage, our category approximation is closed under contractions, rotations, reflections, and tensor products whose result has length lower or equal to the length bound l_0. (Note that the closedness with respect to reflections follows from Observation 5.2.3.)

Now we can use Observation 5.2.2 to see, whether there is a chance for $\mathscr{K}$ to be non-easy. If all the non-trivial summands of all the generators $p_1, \ldots, p_n$ are contained in $\mathscr{K}$, then it is surely easy. Otherwise, it may be non-easy.

As follows from Observation 5.2.1, we should first look for singly-generated categories. This way, we obtain a list of linear combinations of partitions that (maybe) generate non-easy categories. Afterwards, we can try to combine those generators or add some partitions as generators and test the easiness of categories generated by more elements.

In order to be able to really search through all possible generators and not just pick some randomly, we may add some extra variables $a_1, \ldots, a_m$ to the ring R. Then we can start with a generator p depending on $a_1, \ldots, a_m$ as parameters. The precise strategy for the practical computations will be more clear in Section 5.3.

l	1	2	3	4	5	6	7	8	9	10	11	12
B_l	1	2	5	15	52	203	877	4 140	21 147	115 975	678 570	4 213 597

Table 5.2 Bell numbers

5.2.5 Limits of the algorithm

The fact that the category approximation is closed under the category operations in the above sense, however, does not mean that our approximation is faithful. It may happen that, in order to obtain a partition on l points for $l \leq l_0$, we need to compute an intermediate result with length greater than the length bound l_0 first.

If we need a more reliable approximation, we need to increase the length bound l_0. We should always choose l_0 to be at least $2l_1$ since otherwise we cannot even compute $p \otimes p$ and the results will be completely unreliable.

The value of the length bound l_0 has, of course, its limits. The Bell numbers B_l grow exponentially with l, so the module dimensions become huge very quickly. In Table 5.2, we list the Bell numbers for some small l. We see that the maximal value of l_0, which can be achieved for a usual computer, is about $l_0 = 10$. In Section 5.3, we will discuss results for generators of length $l_1 \leq 4$, which is pretty much close to the maximum that can be achieved without further assumptions.

5.3 Concrete computations

In this section, we are going to present concretely how our algorithm is applied. The algorithm was implemented in Singular [DGPS18]. We also used Maple[1] [Map17] for solving systems of polynomial equations. As a result of these computations we obtain a list of linear categories of partitions that are candidates for non-easy categories. The actual non-easiness will be proven in Section 5.4.

In all computations that follow, we use the length bound $l_0 := 8$.

Let us also comment on some restrictions on the parameter δ we are going to make here. One should not bother too much about those restrictions. It is not that our computation would not work for all $\delta \in \mathbb{C}$. It is rather that we are going to throw away some candidates that are not so interesting for us. If we then want to make the statement of the form "the following are *all* possible candidates", we have to restrict the δ to keep this true. The reason why we are ignoring some solutions is that we are interested mainly in applications to compact quantum groups and this means that we focus on solutions that work for (almost) all $\delta \in \mathbb{N}$. To preserve peace of mind, the reader may assume for the whole computation that $\delta \in (4, \infty)$ and then no complications will appear.

Nevertheless, the reader that is interested in the presented solutions as examples of abstract monoidal categories may want to go through the whole computation again and analyse the cases that we skipped here. In addition, we should mention that we skipped also some solutions that work for $\delta = 4$ and hence may be relevant also for the theory of compact quantum groups.

5.3.1 Generator of length one and two

The space Part(1) is one-dimensional being the span of the singleton partition. Therefore, any category generated by an element of length one is easy.

Similarly for the length two. We have Part(2) = span{⊓, | |}. Since ⊓ is in any category by definition, we again have that any category generated by an element of length two is easy.

5.3.2 Generator of length three

For $l = 3$, we have the following partitions

$$\mathscr{P}(3) = \{⊓⊓, ⊓\ |, ⊓|⊓, |\ ⊓, |\ |\ |\}.$$

[1] Maple is a trademark of Waterloo Maple Inc.

So, a general element $p \in \mathsf{Part}_\delta(3)$ can be expressed as follows

$$p = a\,\text{┌┬┐} + b_1\,\text{┌┐ |} + b_2\,\text{| ┌┐} + b_3\,\text{┌╷┐} + c\,\text{| | |},$$

where $a, b_1, b_2, b_3, c \in \mathbb{C}$. Now, our goal is to exclude such values of those parameters, for which $\mathscr{K} := \langle p \rangle_\delta$ is easy.

5.3.1 Lemma. A linear category $\mathscr{K} = \langle p \rangle_\delta$ with $p \in \mathsf{Part}_\delta(3)$ is easy if and only if $\uparrow \in \mathscr{K}$. Hence, $\mathscr{K}$ is non-easy if and only if $\mathscr{K}(1)$ is empty.

Proof. If $\uparrow \in \mathscr{K}$, then all the partitions ┌┐ |, | ┌┐, ┌╷┐, and | | | are in $\mathscr{K}$. If p contains also ┌┬┐ as a summand, then also ┌┬┐ $\in \mathscr{K}$. So, we have either $\mathscr{K} = \langle \uparrow \rangle_\delta$ or $\mathscr{K} = \langle \text{┌┬┐}, \uparrow \rangle_\delta$. In both cases $\mathscr{K}$ is easy. Conversely, if $\mathscr{K}$ is easy, then it must contain at least one of the partitions in $\mathscr{P}(3)$. Each of them generate the singleton. □

Running AddParts(p) (over the ring $\mathbb{C}[\delta, a, b_1, b_2, b_3, c]$), we get immediately that $\mathscr{K}(1)$ contains the following elements

$$(a + b_1 + b_2 + \delta b_3 + \delta c)\uparrow, \quad (a + b_1 + \delta b_2 + b_3 + \delta c)\uparrow, \quad (a + \delta b_1 + b_2 + b_3 + \delta c)\uparrow.$$

If $\mathscr{K}$ is non-easy, then $\mathscr{K}(1)$ must be empty, which leads to equations

$$\begin{aligned} a + b_1 + b_2 + \delta b_3 + \delta c &= 0 \\ a + b_1 + \delta b_2 + b_3 + \delta c &= 0 \\ a + \delta b_1 + b_2 + b_3 + \delta c &= 0. \end{aligned}$$

By subtracting the equations one from each other, we get $b_i(1-\delta) = b_j(1-\delta)$ for $i, j = 1, 2, 3$. Suppose $\delta \neq 1$, then non-easiness implies that $b := b_1 = b_2 = b_3$. Substituting this to one of the equations, we get an additional condition

$$a + (2+\delta)b + \delta c = 0.$$

So, we can put $a := -(2+\delta)b - \delta c$. Now, we can run our algorithm again over $\mathbb{C}[\delta, b, c]$. After one iteration of AddTensors, we get that $\mathscr{K}$ contains

$$(\delta-1)(\delta-2)(\delta c + 2b)(\delta c^2 + 2bc - b^2)\uparrow.$$

Thus, excluding the case $\delta = 1, 2$, the category can be non-easy only if

$$b = -c\delta/2 \quad \text{or} \quad b = (1 \pm \sqrt{\delta+1})c.$$

For $c = 0$, we have also $b = 0$, so the category is easy. For $c \neq 0$, we can normalize p dividing by c.

5.3.2 Candidates. Assuming $\delta \in \mathbb{C} \setminus \{0, 1, 2\}$, the following are the only candidates on non-easy linear categories of partitions that are generated by a single element $p \in \mathsf{Part}_\delta(3)$:

$$\langle \delta^2\,\text{┌┬┐} - \delta(\text{| ┌┐} + \text{┌┐ |} + \text{┌╷┐}) + 2\,\text{| | |} \rangle_\delta,$$
$$\Big\langle \Big(-2(1+\delta) - (2+\delta)\sqrt{\delta+1}\Big)\,\text{┌┬┐} - (1+\sqrt{\delta+1})(\text{| ┌┐} + \text{┌┐ |} + \text{┌╷┐}) + \text{| | |} \Big\rangle_\delta,$$
$$\Big\langle \Big(-2(1+\delta) + (2+\delta)\sqrt{\delta+1}\Big)\,\text{┌┬┐} - (1-\sqrt{\delta+1})(\text{| ┌┐} + \text{┌┐ |} + \text{┌╷┐}) + \text{| | |} \Big\rangle_\delta.$$

Note that we could have derived the equations providing the conditions for non-easiness even without our algorithm. Indeed, the linear ones can be written as $\Pi R^i p = 0$ for $i = 1, 2$ and the quadratic one as

$$\Pi_2 \Pi_3 (p \otimes p) = \text{| └─└┘─┘ |}(p \otimes p) = 0.$$

The algorithm was useful first for providing the idea to solve such equations and secondly for checking (although not proving) that the categories remain non-easy even after more iterations of the tensor product.

5.3.3 Generator of length four, case of no singletons

A generator $p \in \mathrm{Part}_\delta(4)$ can be parametrized as follows

$$p = a_1 \sqcap\!\sqcap\!\sqcap + a_2 \sqcap\!\!\!\sqcap\!\!\!\sqcap + b_1 \sqcap\!\sqcap\,| + b_2 |\,\sqcap\!\sqcap + b_3 \sqcap_{|}\!\sqcap + b_4 \sqcap\!\sqcap_{|} + c_1 \sqcap\,|\,| + c_2 |\,\sqcap\,| + c_3 |\,|\,\sqcap + c_4 \sqcap_{|\,|} + d_1 \sqcap_{|}\,| + d_2 |\,\sqcap_{|} + e\,|\,|\,|\,|.$$

We omit the non-crossing pair partitions $\sqcap\,\sqcap$ and $\sqcap_{\sqcap}$ since they are contained in every category.

Again, we want to exclude those parameters for which $\mathscr{K} := \langle p \rangle_\delta$ is easy. Here, the situation is a bit more complicated because we do not have a criterion for easiness analogous to Lemma 5.3.1. So, we divide the situation in different cases. In this section, we assume $\uparrow\otimes\uparrow \notin \mathscr{K}$. We subdivide our computation even further:

Case 1. Generator not being rotationally symmetric.

First, let us briefly discuss the case, when p is not rotationally symmetric. This means that

$$0 \neq (R-1)p =: \tilde{p} = \tilde{b}_1 \sqcap\!\sqcap\,| + \tilde{b}_2 |\,\sqcap\!\sqcap + \tilde{b}_3 \sqcap_{|}\!\sqcap + \tilde{b}_4 \sqcap\!\sqcap_{|} + \tilde{c}_1 \sqcap\,|\,| + \tilde{c}_2 |\,\sqcap\,| + \tilde{c}_3 |\,|\,\sqcap + \tilde{c}_4 \sqcap_{|\,|} + d(\sqcap_{|}\,| - |\,\sqcap_{|}),$$

where we denote $\tilde{b}_1 = b_4 - b_1$, $\tilde{b}_2 = b_1 - b_2$ and so on, so

$$\tilde{b}_1 + \tilde{b}_2 + \tilde{b}_3 + \tilde{b}_4 = 0,$$
$$\tilde{c}_1 + \tilde{c}_2 + \tilde{c}_3 + \tilde{c}_4 = 0.$$

Denote by $\beta\colon \mathrm{Part}_\delta(2) \to \mathbb{C}$ the linear functional giving the coefficient of $\uparrow\otimes\uparrow$ for a given linear combination $q \in \mathrm{Part}_\delta(2)$, i.e. mapping $\alpha \sqcap + \beta \uparrow\otimes\uparrow \mapsto \beta$. Since $\uparrow\otimes\uparrow \notin \langle \tilde{p} \rangle_\delta$, we have four linear equations of the form $\beta(\Pi(R^i \tilde{p})) = 0$, which read

$$\tilde{b}_1 + \tilde{b}_4 + \delta\tilde{c}_1 + \tilde{c}_2 + \tilde{c}_4 = 0,$$
$$\tilde{b}_2 + \tilde{b}_1 + \delta\tilde{c}_2 + \tilde{c}_3 + \tilde{c}_1 = 0,$$
$$\tilde{b}_3 + \tilde{b}_2 + \delta\tilde{c}_3 + \tilde{c}_4 + \tilde{c}_2 = 0,$$
$$\tilde{b}_4 + \tilde{b}_3 + \delta\tilde{c}_4 + \tilde{c}_1 + \tilde{c}_3 = 0.$$

Together with the equations above, this leads to

$$\tilde{c}_3 = -\tilde{c}_1, \quad \tilde{c}_4 = -\tilde{c}_2, \quad \tilde{b}_2 = -\tilde{b}_1 - \delta\tilde{c}_2, \quad \tilde{b}_3 = \tilde{b}_1 + \delta(\tilde{c}_1 + \tilde{c}_2), \quad \tilde{b}_4 = -\tilde{b}_1 - \delta\tilde{c}_1.$$

We can write $\tilde{p} = f(R) \sqcap\,|\,| + \tilde{q}$, where $f(x) = \tilde{c}_1 + \tilde{c}_2 x + \tilde{c}_3 x^2 + \tilde{c}_4 x^3$. According to Proposition 5.2.4, we can assume that f is a divisor of $x^4 - 1$. Thanks to the first two equations above, we see that $f(1) = 0$ and $f(-1) = 0$, so $f(x)$ is a multiple of $x^2 - 1$. For $f(x) \neq 0$ (that is, either $f(x) = x^2 - 1$ or $f(x) = (x^2 - 1)(x \pm i)$), running one iteration of AddTensors shows that assuming $\delta \neq 2, 4$ we have $\uparrow\otimes\uparrow \in \langle \tilde{p} \rangle_\delta$, which is a contradiction.

In the case $f(x) = 0$, we have

$$\tilde{p} = \tilde{b}(\sqcap\!\sqcap\,| - |\,\sqcap\!\sqcap + \sqcap_{|}\!\sqcap - \sqcap\!\sqcap_{|}) + \tilde{d}(\sqcap_{|}\,| - |\,\sqcap_{|}).$$

One iteration of AddTensors yields $\tilde{b} = (-2 \pm \sqrt{4 - \delta})\tilde{d}$. Note that the involution acts on p by exchanging $\tilde{b} \mapsto -\bar{\tilde{b}}$ and $\tilde{d} \mapsto -\bar{\tilde{d}}$. Thus, both $\tilde{b}$ and $\tilde{d}$ must be real up to scaling by a complex number. This can be achieved only for $\delta \leq 4$.

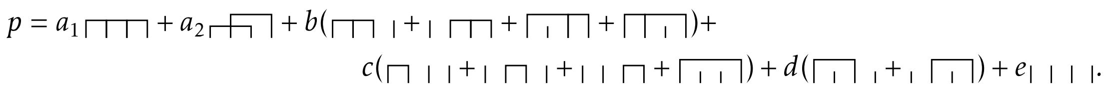
5.3.3 Proposition. Consider $\delta \in \mathbb{C} \setminus (-\infty, 4]$. Let $p \in \mathsf{Part}_\delta(4)$ such that $\uparrow \otimes \uparrow \notin \mathscr{K} := \langle p \rangle_\delta$ is non-easy. Then p is rotationally symmetric.

Proof. Follows from the considerations above. □

To conclude, Case 1 is not relevant assuming δ is a large natural number.

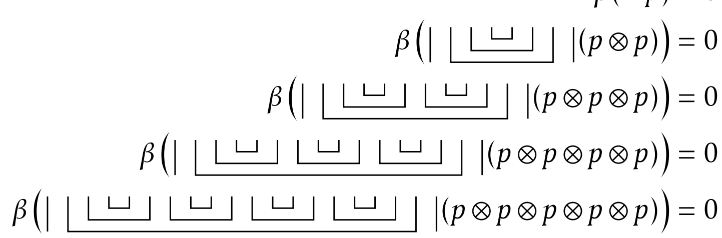
Case 2. Rotationally symmetric generator.

Now, suppose p is of the form



Recall the notation $\beta\colon \mathsf{Part}_\delta(2) \to \mathbb{C}$ for the linear functional giving the coefficient of $\uparrow \otimes \uparrow$. As $\uparrow \otimes \uparrow \notin \mathscr{K}$, we must have $\beta(q) = 0$ for all $q \in \mathscr{K}(2)$. So, our idea for computing concrete coefficients providing a candidate for a non-easy category is to solve the following equations.

$$\beta(\Pi p) = 0 \tag{5.1}$$



(5.2)

(5.3)

(5.4)

(5.5)

5.3.4 Remarks.

(a) All the equations are homogeneous. (Their solution is obviously invariant with respect to scaling.)

(b) The first equation containing one copy of p is linear, the second one is quadratic and so on.

(c) The rotational symmetry reduces the number of variables and equations. Note for example that there is essentially just one way how to construct a tensor product of two copies of p and then contract it to size two. Similarly for three copies of p. For four copies, there are two additional ways, but it turns out that the corresponding equations already follow from (5.1)–(5.4).

(d) The reflection acts on p by complex conjugating all the parameters. If it turns out that the system of equations has discrete solutions only (up to scaling), then the assumption of non-easiness implies that all the coefficients are up to scaling real. (Otherwise p and $p^\star$ are linearly independent, so $p + \alpha p^\star \in \langle p \rangle$ would be a one-parameter set of solutions.)

We were not able to solve those equations in full generality. So, let us focus on some special cases.

Case 2a. $a_2 = 0$, i.e. p is non-crossing.

In this case, unless $b = c = d = e = 0$, we have that $\uparrow \otimes \uparrow \notin \langle p \rangle_\delta$ already implies that $\langle p \rangle_\delta$ is non-easy. Since we have only five variables, four homogeneous equations (5.1)–(5.4) are already enough to obtain a list of discrete solutions (up to scaling). Using Maple, we found the following eight solutions:

$$a_1 = 1, \quad b = 0, \quad c = 0, \quad d = 0, \quad e = 0, \tag{5.6}$$

$$a_1 = \delta^3, \quad b = -2\delta^2, \quad c = 4\delta, \quad d = 4\delta, \quad e = -16, \tag{5.7}$$

$$a_1 = \delta(\delta+1)(\delta+2\mp 2\sqrt{\delta+1}), \quad b = \delta(-\delta-1\pm\sqrt{\delta+1}), \quad c=\delta, \quad d=\delta, \quad e=\delta-2\mp 2\sqrt{\delta+1}, \tag{5.8}$$

$$a_1 = 2\delta^2(2\pm\sqrt{4-\delta}), \quad b=-\delta^2, \quad c=2\delta, \quad d=\mp\delta\sqrt{4-\delta}, \quad e=2(-2\pm\sqrt{4-\delta}), \tag{5.9}$$

$$\begin{aligned} a_1 &= 2\delta^3(3\mp 2\sqrt{3-\delta}), \quad b=\delta^2(-2\delta\pm\sqrt{3-\delta}), \quad c=\delta(4\delta-3), \\ d &= \delta(\delta\pm 2(\delta-1)\sqrt{3-\delta}), \quad e=\pm 2(3\delta-2)\sqrt{3-\delta}+7\delta-6. \end{aligned} \tag{5.10}$$

There are also some additional solutions for $\delta=-1,\,3,\,4,\,(3\pm\sqrt{5})/2$, which we will not mention here. The first solution of the list above is the easy one. The following two lines – (5.7) and (5.8) – are interesting for us. Their non-easiness will be proven as Proposition 5.4.5 and Proposition 5.4.34, respectively. Solutions (5.9) and (5.10) are real only for $\delta\le 4$, resp. $\delta\le 3$, so we will ignore them here.

We can summarize the results in the following proposition.

5.3.5 Candidates. Consider $\delta\in\mathbb{C}\setminus(-\infty,4]$. Let $p\in\mathsf{Part}_\delta(4)$ be non-crossing such that $\mathscr{K}:=\langle p\rangle_\delta$ is non-easy and $\uparrow\otimes\uparrow\notin\mathscr{K}$. Then $\mathscr{K}$ is equal to one of the following three categories

$$\langle\delta^3\,\sqcap\!\sqcap\!\sqcap - 2\delta^2(\sqcap\!\sqcap\,| + \sqcap\!|\!\sqcap + \dots) + 4\delta(\sqcap\,|\,| + \sqcap\!|\!\sqcap\,| + \dots) - 16\,|\,|\,|\,|\rangle_\delta,$$

$$\begin{aligned}\langle\delta^3(\delta+1)\,\sqcap\!\sqcap\!\sqcap &- \delta^2(\delta+1\pm\sqrt{\delta+1})(\sqcap\!\sqcap\,| + \sqcap\!|\!\sqcap + \dots) + \\ &\delta(\delta+2\pm 2\sqrt{\delta+1})(\sqcap\,|\,| + \sqcap\!|\!\sqcap\,| + \dots) + (\delta^2-4\delta-8\mp 8\sqrt{\delta+1})\,|\,|\,|\,|\rangle_\delta.\end{aligned}$$

Note that the two categories on the second line can be easy only for $\delta\in[-1,\infty)$ since otherwise the generator does not have real coefficients.

Case 2b. $c=0\neq a_2$.

We again use Maple to obtain the solutions. One of the solutions is a very complicated one that can be expressed in terms of roots of some polynomial equation of degree nine. We will not study it further. Then we have a solution of the form

$$a_1=0, \quad a_2=\delta^2, \quad b=0, \quad d=-2\delta, \quad e=4. \tag{5.11}$$

Finally, there is a solution where a_1 and a_2 are arbitrary and $b=d=e=0$. This solution is somehow obvious – the category $\langle a_1\,\sqcap\!\sqcap\!\sqcap + a_2\,\sqcap\!\!\sqcap\!\!\sqcap\rangle_\delta$ can never contain $\uparrow\otimes\uparrow$ since all blocks of both $\sqcap\!\sqcap\!\sqcap$ and $\sqcap\!\!\sqcap\!\!\sqcap$ have even size. This, however, says nothing about its non-easiness, so let us use our algorithm to investigate the category.

For simplicity, we can divide the generator by a_2 (for $a_2=0$ is the category obviously easy), that is, consider $p:=\sqcap\!\!\sqcap\!\!\sqcap + a\,\sqcap\!\sqcap\!\sqcap$. After one iteration of AddTensors, we see that $\langle p\rangle_\delta$ may be non-easy only if $a=-2$.

5.3.6 Candidates. We have two new candidates for non-easy categories

$$\langle\delta^2\,\sqcap\!\!\sqcap\!\!\sqcap - 2\delta(\sqcap\!|\!\sqcap\,| + |\,\sqcap\!|\!\sqcap) + 4\,|\,|\,|\,|\rangle_\delta, \quad\text{and}\quad \langle\sqcap\!\!\sqcap\!\!\sqcap - 2\,\sqcap\!\sqcap\!\sqcap\rangle_\delta.$$

The non-easiness of both is proven by Proposition 5.4.6 and 5.4.8, respectively. Moreover, we will prove that both are actually isomorphic to the category of all pairings $\mathsf{Pair}_\delta=\langle\times\rangle_\delta$.

5.3.4 Generator of length four, case with singletons

In this subsection, we assume $\uparrow\otimes\uparrow\in\mathscr{K}$, so we can assume p is of the form

$$p = a_1\,\sqcap\!\sqcap\!\sqcap + a_2\,\sqcap\!\!\sqcap\!\!\sqcap + b_1\,\sqcap\!\sqcap\,| + b_2\,|\,\sqcap\!\sqcap + b_3\,\sqcap\!|\!\sqcap\!\sqcap + b_4\,\sqcap\!\sqcap\!|\!\sqcap + d_1\,\sqcap\!|\!\sqcap\,| + d_2\,|\,\sqcap\!|\!\sqcap. \tag{5.12}$$

We do not include $|\,|\,|\,|$ and rotations of $\sqcap\,|\,|$ in the linear combination since those are generated by $\uparrow\otimes\uparrow$.

5.3.7 Proposition. Consider $\delta \in \mathbb{C} \setminus \{0,2\}$. Let p be of the form (5.12). Suppose $\mathscr{K} := \langle \uparrow\otimes\uparrow, p\rangle_\delta$ is non-easy and $\{13\}\{2\}\{4\} \notin \mathscr{K}$. Then p is rotationally symmetric.

Proof. Assume

$$0 \neq (R-1)p =: \tilde{p} = \tilde{b}_1 \{123\}\{4\} + \tilde{b}_2 \{1\}\{234\} + \tilde{b}_3 \{134\}\{2\} - (\tilde{b}_1+\tilde{b}_2+\tilde{b}_3)\{124\}\{3\} + d(\{13\}\{2\}\{4\} - \{1\}\{24\}\{3\}).$$

We will prove that $\langle \tilde{p}, \uparrow\otimes\uparrow\rangle_\delta = \langle \{123\}\{4\}, \{13\}\{2\}\{4\}\rangle_\delta$ (which contains all partitions on four points except for $\{13\}\{24\}$). This already implies that $\langle p, \uparrow\otimes\uparrow\rangle_\delta$ either equals to $\langle \{123\}\{4\}, \{13\}\{2\}\{4\}\rangle_\delta$ or to $\langle \{123\}\{4\}, \{13\}\{2\}\{4\}, \{13\}\{24\}\rangle_\delta$, so it is easy.

After one iteration of AddTensor on $\langle \uparrow\otimes\uparrow, \tilde{p}\rangle_\delta$, we see that $\{13\}\{2\}\{4\} \in \langle \uparrow\otimes\uparrow, \tilde{p}\rangle_\delta$ assuming $\delta \neq 2$. Hence, we can set $d = 0$ and repeat the algorithm for $\langle \tilde{p}, \{13\}\{2\}\{4\}\rangle_\delta$. After one iteration of AddTensor, we generate $\{123\}\{4\}$ assuming $\delta \neq 0$. □

Case 1. Assuming $\{13\}\{2\}\{4\} \notin \mathscr{K}$.

Take

$$p = a_1 \{1234\} + a_2 \{13\}\{24\} + b(\{123\}\{4\} + \{1\}\{234\} + \{134\}\{2\} + \{124\}\{3\}) + d(\{13\}\{2\}\{4\} + \{1\}\{24\}\{3\}). \tag{5.13}$$

Running one iteration of AddTensor on $\langle \uparrow\otimes\uparrow, p\rangle_\delta$, we compute $a_1 = -b\delta$, $a_2 = -b - d\delta$. Further iterations of AddTensor suggest that this category indeed does not contain $\{13\}\{2\}\{4\}$ and is indeed non-easy for any $b, d \in \mathbb{C}$.

We can write $p = a_1 p_1 + a_2 p_2$, where (assuming $\delta \neq 0$)

$$p_1 = \{1234\} - \frac{1}{\delta}(\{123\}\{4\} + \{1\}\{234\} + \{134\}\{2\} + \{124\}\{3\}) + \frac{1}{\delta^2}(\{13\}\{2\}\{4\} + \{1\}\{24\}\{3\}), \tag{5.14}$$

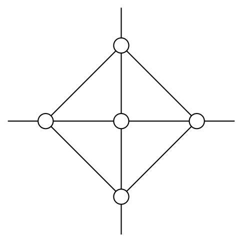
$$p_2 = \{13\}\{24\} + \{13\}\{2\}\{4\} + \{1\}\{24\}\{3\}. \tag{5.15}$$

In Propositions 5.4.24 and 5.4.28, we will show that the categories $\langle p_1\rangle_\delta$, $\langle p_2\rangle_\delta$ and $\langle p_1, p_2\rangle_\delta$ are indeed noneasy (note that p_1 essentially coincides with $\mathcal{P}_{(\delta)}\{1234\}$ and p_2 essentially coincides with $\mathcal{P}_{(\delta)}\{13\}\{24\}$, where $\mathcal{P}_{(\delta)}$ will be defined in Def. 5.4.11).

5.3.8 Remark. It actually holds that $\langle p\rangle_\delta = \langle p_1, p_2\rangle_\delta$ for any non-trivial linear combination $p = a_1p_1 + a_2p_2$. For most choices of a_1, a_2, this can be computed with our algorithm. However, choosing

$$p = 2\delta p_1 + (2-\delta)p_2 = 2\delta\{1234\} + (2-\delta)\{13\}\{24\} - 2(\{123\}\{4\} + \{1\}\{234\} + \{134\}\{2\} + \{124\}\{3\}) + \{13\}\{2\}\{4\} + \{1\}\{24\}\{3\},$$

the category $\langle p\rangle_\delta$ appears to be new. That is, even if we iterate AddTensor until the modules become stable, we do not obtain p_1 and p_2. As we mentioned at the beginning of this remark, in reality, the p_1 and p_2 are elements of the category. We can compute it "by hand" (preferably, again with the help of computer), if we do the computation described by the following graph.



Here the vertices stand for copies of the generator p – each vertex has degree four as p is a linear combination of partitions of four points – and the edges describe contractions (free edges connected just to one vertex are the outputs). One can check that it is indeed possible to perform this computation using the category operations. The key point is that this graph is planar.

So, if we do such a computation, the result is

$$(2\delta^5 - 18\delta^4 + 48\delta^3 - 48\delta^2 + 96\delta - 64)\delta p_1 - (\delta^6 - 11\delta^5 + 50\delta^4 - 144\delta^3 + 304\delta^2 - 320\delta + 64)p_2 + \ldots,$$

where the dots stand for some partitions that are already generated by $\uparrow\otimes\uparrow$. For $\delta \neq 0, 2, 3, 4$, this is a different linear combination than we started with, so we can indeed generate p_1 and p_2.

Case 2b. Assuming $\sqcap\!\sqcap \in \mathscr{K}$

In this case, we are interested in categories of the form $\mathscr{K} := \langle \sqcap\!\sqcap, p\rangle_\delta$, where

$$p = a_1 \sqcap\sqcap\sqcap + a_2 \sqcap\!\!\sqcap\!\!\sqcap + b(\sqcap\sqcap\,| + |\,\sqcap\sqcap + \sqcap|\sqcap + \sqcap\sqcap|).$$

Using our algorithm, it can be again proven that non-easiness implies $a_1 = -b\delta$. So, our candidates are quantum groups of the form $\langle a_1p_1 + a_2p_2, \sqcap\!\sqcap\rangle_\delta = \langle a_1p_1 + a_2 \sqcap\!\!\sqcap\!\!\sqcap, \sqcap\!\sqcap\rangle_\delta$, where p_1 and p_2 are given by Eqs. (5.14), (5.15) (this time, we can also ignore the summands $\sqcap\!\sqcap$, $\sqcap\!\sqcap$ in the formulae (5.14), (5.15)).

Again, our algorithm shows, that actually $\langle a_1p_1 + a_2 \sqcap\!\!\sqcap\!\!\sqcap, \sqcap\!\sqcap\rangle_\delta = \langle p_1, \sqcap\!\!\sqcap\!\!\sqcap, \sqcap\!\sqcap\rangle$ for most choices of a_1, a_2. From Remark 5.3.8, it actually follows that we have this for all $a_1, a_2 \neq 0$ if we assume $\delta \neq 0, 2, 3, 4$.

Finally, let us mention that we can, in addition, construct the non-easy categories of the form $\langle \uparrow, p\rangle$. Again, see Proposition 5.4.28.

5.3.9 Candidates. Consider the following candidates for non-easy categories.

$$\langle p_1, \uparrow\otimes\uparrow\rangle_\delta \qquad \langle p_1, p_2, \uparrow\otimes\uparrow\rangle_\delta \tag{5.16}$$

$$\langle p_1, \sqcap\!\sqcap\rangle_\delta \qquad \langle p_1, \sqcap\!\sqcap, \sqcap\!\!\sqcap\!\!\sqcap\rangle_\delta \tag{5.17}$$

$$\langle p_1, \uparrow\rangle_\delta \qquad \langle p_1, \uparrow, \sqcap\!\!\sqcap\!\!\sqcap\rangle_\delta \tag{5.18}$$

Here, p_1, p_2 are given by Eqs. (5.14), (5.15). Assuming $\delta \neq 0, 2, 3, 4$, those on lines (5.16), (5.17) are the only non-easy categories containing $\uparrow\otimes\uparrow$ generated by a single element of $\mathsf{Part}_\delta(4)$.

5.4 Direct proofs of non-easiness

In this section, we provide proofs of non-easiness of the categories discovered by computer experiments as described in the previous section. The section is based on results obtained in [GW19a, GW20]. However, we formulate them in more detail than in [GW19a] and bring them from a different perspective not refering to the quantum group picture in contrast with [GW20].

We formulate this section not only to really prove the statements, but also to show different kinds of proof techniques connected with non-easy quantum groups and to show interesting isomorphisms between different linear categories of partitions.

5.4.1 General contractions

In this section, we present a proof that was not published in any article. The reason is that it works only for one specific category, whose non-easiness is possible to proof also by other means (see Sect. 5.4.5). Nevertheless, we consider the proof technique to be quite interesting, so we decided to include it here.

The basic idea of proof is the following. Suppose p is rotationally symmetric. If $p \in \mathsf{Part}_\delta(l)$ generates $p' \in \mathsf{Part}_\delta(l')$, this means that p' was made from p by a series of tensor products, contractions and rotations. We can simplify this process a bit. First, we produce a k-fold tensor product $p^{\otimes k}$ and then perform some more general contractions. Namely, we can express $p' = qp^{\otimes k}$, where $q \in \mathsf{Pair}_\delta(l'k, l)$ is some pairing. In fact, we can generate in such a way any element of $\langle p, \times\rangle_\delta$.

5.4.1 Proposition. The category

$$\langle \delta^2 \sqcap\sqcap - \delta(|\,\sqcap + \sqcap\,| + \sqcap\!\sqcap) + 2\,|\;|\;|\rangle_\delta$$

is non-easy for every $\delta \in \mathbb{C}$.

Proof. Denote $p := \delta^2 \sqcap\!\sqcap - \delta(|\,\sqcap + \sqcap\,| + \sqcap\!\!\!\sqcap) + 2|\,|\,|$, $\mathscr{K} := \langle p\rangle_\delta$. We showed in Lemma 5.3.1 that $\mathscr{K}$ is non-easy if and only if $\uparrow \notin \langle p\rangle_\delta$. If we had $\uparrow \in \mathscr{K}$, it would mean that by a series of tensor products, contractions and rotations, we can produce a non-zero multiple of $\uparrow$ from p. Thanks to p being rotationally invariant, this would imply that there exists $k \in \mathbb{N}$ and $q' \in \mathsf{Pair}_\delta(3k-1,0)$ such that $(|\otimes q')p^{\otimes k} = \alpha\uparrow$ for some $\alpha \neq 0$. Consequently, $(|\otimes q)p = \alpha\uparrow$ for $q := q'(|\otimes|\otimes p^{\otimes(k-1)}) \in \mathsf{Part}_\delta(2,0)$.

Hence, it is enough to prove that $qp = 0$ for every $q \in \mathsf{Part}_\delta(2,0)$. In Section 5.3.2, we checked that $(|\otimes\sqcup)p = \Pi_2 p = 0$. It is straightforward to check that also $(|\otimes\downarrow\otimes\downarrow)p = 0$. □

Actually, this also proves non-easiness of the category $\langle p, \times\rangle_\delta$.

We could formulate a more general statement such as: Let $p \in \mathsf{Part}_\delta(l)$ be rotational symmetric with l odd. Suppose $(|\otimes q)p = 0$ for every $q \in \mathsf{Part}_\delta(l-1,0)$. Then $\langle p\rangle_\delta$ and $\langle p,\times\rangle_\delta$ are non-easy categories.

This might sound like a promising way of constructing new non-easy categories. We only have to solve some system of linear equations. For dimension reasons, we actually surely will have a plenty of solutions. However, it might happen that all the discovered non-easy categories are actually equal to the above mentioned one. This is at least the case for $l = 5$.

5.4.2 Isomorphism by conjugation

In this section, we assume $\delta \neq 0$.

5.4.2 Definition. We define the linear combination $\tau_{(\delta)} := | - \frac{2}{\delta}\,{}^{\shortmid}_{\shortmid} \in \mathsf{Part}_\delta(1,1)$. For any $p \in \mathsf{Part}_\delta(k,l)$, we set $\mathcal{T}_{(\delta)}p := \tau_{(\delta)}^{\otimes l} p \tau_{(\delta)}^{\otimes k}$.

It holds that $\tau_{(\delta)}\cdot\tau_{(\delta)} = |$ and $\tau_{(\delta)}^* = \tau_{(\delta)}$. In operator language, we would say that $\tau_{(\delta)}$ is a self-adjoint unitary. Consequently, conjugation by $\tau_{(\delta)}$ defines a category isomorphism.

5.4.3 Proposition. $\mathcal{T}_{(\delta)}$ is a monoidal $*$-isomorphism $\mathsf{Part}_\delta \to \mathsf{Part}_\delta$.

Proof. The fact that $\mathcal{T}_{(\delta)}$ is a monoidal unitary functor follows from the above mentioned properties of $\tau_{(\delta)}$. Finally, we also have $\mathcal{T}_{(\delta)}^2 = \mathrm{id}$, which proves that it is an isomorphism. □

5.4.4 Remark. If $\uparrow\otimes\uparrow \in \mathscr{K} \subseteq \mathsf{Part}_\delta$, then $\mathcal{T}_{(\delta)}\mathscr{K} = \mathscr{K}$. Indeed, $\uparrow\otimes\uparrow \in \mathscr{K}$ implies $\tau_{(\delta)} \in \mathscr{K}$ and hence $\mathcal{T}_{(\delta)}\mathscr{K} \subseteq \mathscr{K}$. From the isomorphism property, we have the equality. This implication cannot be reversed. For example, we have $\mathcal{T}_{(\delta)}\times = \times$, so $\mathcal{T}_{(\delta)}\langle\times\rangle_\delta = \langle\times\rangle_\delta$ although $\uparrow\otimes\uparrow \notin \langle\times\rangle_\delta$.

As a non-trivial example, we can compute that

$$\mathcal{T}_{(\delta)}\sqcap\!\sqcap\!\sqcap = \sqcap\!\sqcap\!\sqcap - \frac{2}{\delta}(\sqcap\!\sqcap\,| + \sqcap\!{}_{\shortmid}\!\sqcap + \sqcap\!{}_{\shortmid}\!\sqcap + |\,\sqcap\!\sqcap)$$
$$+\frac{4}{\delta^2}(\sqcap\,|\,| + \sqcap_{\shortmid}\,{}_{\shortmid} + |\,\sqcap\,| + \sqcap_{\shortmid\,\shortmid} + {}_{\shortmid}\,\sqcap_{\shortmid} + |\,|\,\sqcap) - \frac{16}{\delta^3}|\,|\,|\,|.$$

5.4.5 Proposition. [GW20, Example 7.6] The category $\langle\mathcal{T}_{(\delta)}\sqcap\!\sqcap\!\sqcap\rangle_\delta$ is non-easy. In particular, $\langle\mathcal{T}_{(\delta)}\sqcap\!\sqcap\!\sqcap\rangle_\delta \neq \langle\sqcap\!\sqcap\!\sqcap\rangle_\delta$.

Proof. The inequality follows simply from the fact that $\mathcal{T}_{(\delta)}\sqcap\!\sqcap\!\sqcap \notin \langle\sqcap\!\sqcap\!\sqcap\rangle_\delta$. If the category $\langle\mathcal{T}_{(\delta)}\sqcap\!\sqcap\!\sqcap\rangle_\delta$ was easy, then it would contain $\sqcap\!\sqcap\!\sqcap$ and be strictly larger than $\langle\sqcap\!\sqcap\!\sqcap\rangle_\delta$, which would contradict $\mathcal{T}_{(\delta)}$ being a category isomorphism. □

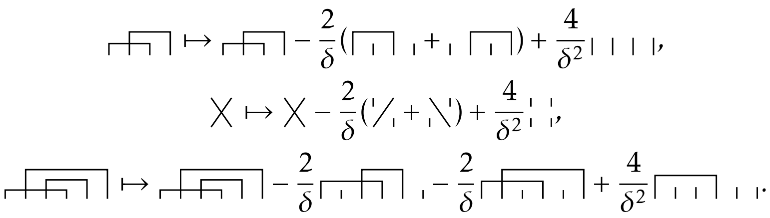
5.4.3 The disjoining isomorphism

The following proposition was mentioned as [GW19a, Proposition 4.5]. However, we gave only a sketch of proof there. Here, we explain it more in detail.

5.4.6 Proposition. The category $\mathscr{K} := \langle\times - \frac{2}{\delta}({}^{\shortmid}\!/_{\shortmid} + {}_{\shortmid}\backslash^{\shortmid}) + \frac{4}{\delta^2}\,{}^{\shortmid}_{\shortmid}\,{}^{\shortmid}_{\shortmid}\rangle_\delta$ is isomorphic to $\langle\times\rangle_\delta$ for every $\delta \neq 0$. Consequently, it is a non-easy category.

Proof. We give an explicit formula for the isomorphism $\mathcal{D}\colon \mathsf{Pair}_\delta \to \mathscr{K}$ acting on any $p \in \mathsf{Pair}_\delta$ as follows. Every pair block that has between its legs an odd number of points is replaced by

⟨pair⟩ $-\frac{2}{\delta}$⟨singletons⟩. (We use cyclical order of points in the partition. Since all pair partitions have even number of points, it does not matter from which side we count.) For example,



Now, we only have to prove that it indeed is a monoidal $*$-isomorphism.

The proof becomes more clear if we formulate it for partitions with lower points only. In order to check that $\mathcal{D}$ is indeed a monoidal unitary functor, we have to prove that it commutes with the one-line operations. This is clear for the tensor product, rotation, and reflection. Now, let us prove that for any $p \in \mathsf{Pair}_\delta(k)$, we have $\mathcal{D}(\Pi_1 p) = \Pi_1(\mathcal{D}p)$. We can assume that p is a partition, not a linear combination. If $p = ⊓ \otimes q$, then the statement is clear, so assume that the first two points of p belong to different blocks. We call a pair block *even* if it has an even number of points between its legs, otherwise it is *odd*.

If the blocks corresponding to the first two points of p are even, then by contracting them, we get an even block. The mapping $\mathcal{D}$ acts on even blocks as the identity, so it clearly commutes with the contraction. When contracting an even block with an odd block, we get an odd block. Odd blocks are mapped to $⊓ - \frac{2}{\delta}\uparrow\otimes\uparrow = \mathrm{Lrot}\,\tau_{(\delta)}$ by $\mathcal{D}$. When contracting $\mathrm{Lrot}\,\tau_{(\delta)}$ with a normal block ⊓, we get $\mathrm{Lrot}\,\tau_{(\delta)}$, so everything is fine also in this case. Finally if both the blocks are odd, then by contracting them, we get an even block. Also when contracting $\mathrm{Lrot}\,\tau_{(\delta)}$ with another copy of $\mathrm{Lrot}\,\tau_{(\delta)}$, we get simply ⊓.

Finally, non-easiness of the category follows directly from the explicit description of its elements: if the category was easy, then it would be equal to $\langle ╳, {}^{'}\!/_{,}, {}_{,}\backslash^{'}, {}^{'}_{,}\,{}^{'}_{,}\rangle_\delta = \langle ╳, \uparrow\otimes\uparrow\rangle_\delta$, which is surely larger and hence non-isomorphic with $\langle ╳\rangle_\delta$. □

5.4.7 Remark. At first sight, it might appear a bit confusing that we prove the non-easiness of a category by showing that it is isomorphic to an easy category. But note that the *easiness* and *non-easiness* are by no means some fundamental abstract characterizations of the categories. It just says whether we chose a convenient or an inconvenient way how do describe them. The whole point of Section 5.4 is to express non-easy categories in terms of the easy ones. That is, to find a convenient description of categories that were defined inconveniently using linear combinations of partitions.

5.4.4 The joining isomorphism

Also the following proposition was mentioned as [GW19a, Proposition 4.5]. We again explain the proof more in detail here.

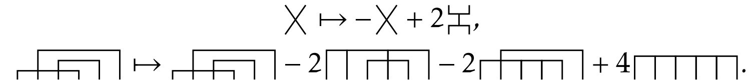
5.4.8 Proposition. The category $\mathscr{K} := \langle 2⊠ - ╳\rangle_\delta$ is isomorphic to $\langle ╳\rangle_\delta$ for every $\delta \in \mathbb{C}$. Consequently, it is a non-easy category.

Proof. We give an explicit formula for the isomorphism $\mathsf{Pair}_\delta \to \mathscr{K}$ acting on a pair partition p as follows. Every crossing in p is replaced by $-$⟨crossing⟩ $+\,2$⟨a single block⟩ (by a single block we mean, that the two blocks that were crossing are united). To be more precise, let $\mathsf{a}_1, \ldots, \mathsf{a}_k$ be the set of blocks of p and denote by X_p the set of pairs $\{\mathsf{a}_i, \mathsf{a}_j\}$ that cross each other. Then we define

$$\mathcal{J}p := (-1)^{|X_p|} \sum_{\Xi \subseteq X_p} (-2)^{|\Xi|} p_\Xi,$$

where p_Ξ is created from p by unifying the pairs of blocks in Ξ.

For example, we map



The second example in word representation reads

$$\mathcal{J}\,\mathsf{abcacb} = p - 2p_{\{\{\mathsf{a},\mathsf{b}\}\}} - 2p_{\{\{\mathsf{a},\mathsf{c}\}\}} + 4p_{\{\{\mathsf{a},\mathsf{b}\},\{\mathsf{a},\mathsf{c}\}\}} = \mathsf{abcacb} - 2\,\mathsf{aacaca} - 2\,\mathsf{abaaab} + 4\,\mathsf{aaaaaa}.$$

Now, we only have to prove that it indeed is a monoidal $*$-isomorphism. We will do this working with partitions on one line. It is clear that the mapping commutes with the tensor product, rotation and reflection. We need to prove this for contraction.

Take a pair partition p on $k+2$ points. If $p = \sqcap \otimes q$, then it is easy to see that indeed $\Pi_1(\mathcal{J}p) = \mathcal{J}(\Pi_1 p) = \mathcal{J}q$. Now, suppose that the first two points of p belong to different blocks. Denote the first block by letter a and the second block by letter b, so the word representation of p is $p = \mathsf{ab}x_1x_2\cdots x_k$. Denote $q := \Pi_1 p$ and its word representation $q = \tilde{x}_1\tilde{x}_2\cdots\tilde{x}_k$, where $\tilde{x}_i = x_i$ if $x_i \neq \mathsf{a},\mathsf{b}$ and $\tilde{x}_i = \mathsf{c}$ if $x_i = \mathsf{a}$ or $x_i = \mathsf{b}$. Then it holds that

$$\begin{aligned} X_q = &\Big\{\{x,y\} \in X_p \;\Big|\; \mathsf{a},\mathsf{b} \notin \{x,y\}\Big\} \cup \\ &\Big\{\{\mathsf{c},x\} \;\Big|\; \{\mathsf{a},x\} \in X_p \text{ and } \{\mathsf{b},x\} \notin X_p\Big\} \cup \\ &\Big\{\{\mathsf{c},x\} \;\Big|\; \{\mathsf{a},x\} \notin X_p \text{ and } \{\mathsf{b},x\} \in X_p\Big\}. \end{aligned}$$

Denote by π the embedding $X_q \to X_p$. We will prove that $\Pi_1(\mathcal{J}p) = \mathcal{J}(\Pi_1 p) = \mathcal{J}q$.

It is easy to see that

$$\mathcal{J}q = (-1)^{|X_q|}\,\Pi_1\left(\sum_{\Xi\subseteq\pi(X_q)\subseteq X_p} (-2)^{|\Xi|} p_\Xi\right).$$

In case when $\{\mathsf{a},\mathsf{b}\} \notin X_p$, we have $|X_q| = |X_p|$, so we can exchange this in the formula above. In case when $\{\mathsf{a},\mathsf{b}\} \in X_p$, we have $\Pi_1 p_\Xi = \Pi_1 p_{\Xi\cup\{\mathsf{a},\mathsf{b}\}}$, so

$$\mathcal{J}q = (-1)^{|X_p|}\,\Pi_1\left(\sum_{\Xi\subseteq\pi(X_q)\subseteq X_p} (-2)^{|\Xi|} p_\Xi + \sum_{\Xi\subseteq\pi(X_q)\subseteq X_p} (-2)^{|\Xi|+1} p_{\Xi\cup\{\mathsf{a},\mathsf{b}\}}\right).$$

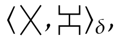
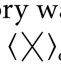
It suffices to prove that the rest of the sum is zero. Choose a block x of p such that $\{\mathsf{a},x\} \in X_p$ and $\{\mathsf{b},x\} \in X_p$. Then

$$\Pi_1\left(p_{\{\{\mathsf{a},x\}\}} + p_{\{\{\mathsf{b},x\}\}} - 2p_{\{\{\mathsf{a},x\},\{\mathsf{b},x\}\}}\right) = 0.$$

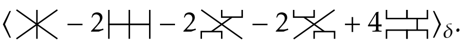
Consequently, for any $\Xi \subseteq X_p$,

$$\Pi_1\left((-2)^{|\Xi\cup\{\{\mathsf{a},x\}\}|} p_{\Xi\cup\{\{\mathsf{a},x\}\}} + (-2)^{|\Xi\cup\{\{\mathsf{b},x\}\}|} p_{\Xi\cup\{\{\mathsf{b},x\}\}} + (-2)^{|\Xi\cup\{\{\mathsf{a},x\},\{\mathsf{b},x\}\}|} p_{\Xi\cup\{\{\mathsf{a},x\},\{\mathsf{b},x\}\}}\right) = 0.$$

The missing part of the sum above is a sum of such terms, so this proves the statement.

Finally, the non-easiness of the category again follows directly from the explicit description of its elements: if the category was easy, then it would be equal to $_\delta$, which is larger and hence non-isomorphic with $_\delta$. □

5.4.9 Remark. We can apply this isomorphism also on subcategories of Pair_δ. The only easy subcategories are the following two. The category of all non-crossing pairings $\langle\rangle_\delta$, where the isomorphism acts as the identity since there is no crossing, and the category $_\delta$ that is mapped onto the following non-easy category



This leads to an additional new non-easy category that was not discovered by our computer experiments since it is generated by a partition of six points.

5.4.5 Projective morphism

In this section, we assume that $\delta \neq 0$. The text is based on [GW20].

5.4.10 Definition. We define $\pi_{(\delta)} := | - \frac{1}{\delta} {}^{\uparrow}_{\downarrow} \in \mathsf{Part}_\delta(1,1)$.

It satisfies $\pi_{(\delta)} \cdot \pi_{(\delta)} = \pi_{(\delta)}$ and $\pi^*_{(\delta)} = \pi_{(\delta)}$. In operator language, $\pi_{(\delta)}$ is an orthogonal projection. This allows us to define the following

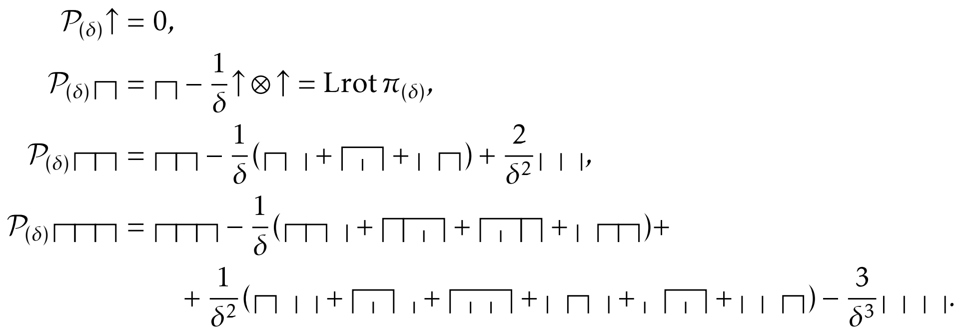
5.4.11 Definition. For any $p \in \mathsf{Part}_\delta(k,l)$ we denote $\mathcal{P}_{(\delta)}p := \pi^{\otimes l}_{(\delta)} p \pi^{\otimes k}_{(\delta)}$. We denote

$$\mathsf{PartRed}_\delta(k,l) := \mathcal{P}_{(\delta)}\mathsf{Part}_\delta(k,l) = \{\pi^{\otimes l}_{(\delta)} p \pi^{\otimes k}_{(\delta)} \mid p \in \mathsf{Part}_\delta(k,l)\}.$$

5.4.12 Proposition. The collection of vector spaces $\mathsf{PartRed}_\delta(k,l)$ is closed under the category operations. It forms a monoidal $*$-category with identity morphism $\pi^{\otimes k}_{(\delta)} \in \mathsf{PartRed}_\delta(k,k)$ and duality morphisms $\mathrm{Lrot}^k \pi^{\otimes k}_{(\delta)} \in \mathsf{PartRed}(0,2k)$.

Proof. Straightforward, see also the proof of Proposition 5.4.16. □

5.4.13 Example. As an example, let us compute the action of $\mathcal{P}_{(\delta)}$ on small block partitions:



5.4.14 Definition. Any collection of spaces $\mathscr{K}(k,l) \subseteq \mathsf{PartRed}(k,l)$ containing $\pi_{(\delta)}$ and $\mathrm{Lrot}\,\pi_{(\delta)}$ that is closed under the category operations will be called a **reduced linear category of partitions**. For given $p_1,\ldots,p_n \in \mathsf{Part}_\delta$, we denote by $\langle p_1,\ldots,p_n\rangle_{\delta\text{-red}}$ the smallest reduced category containing those *generators*.

5.4.15 Remarks.

(a) Although we have $\mathsf{PartRed}_\delta(k,l) \subseteq \mathsf{Part}_\delta(k,l)$ as vector spaces, $\mathsf{PartRed}_\delta$ is not a subcategory of Part_δ as it has different identity morphisms. That is, the embedding $\mathsf{PartRed}_\delta \to \mathsf{Part}_\delta$ is not a functor.

(b) Likewise, the mapping $\mathcal{P}_{(\delta)}\colon \mathsf{Part}_\delta \to \mathsf{PartRed}_\delta$ is not a functor. Indeed, note that we have $\mathcal{P}_{(\delta)}\uparrow = 0$, so

$$\mathcal{P}_{(\delta)}\downarrow \cdot \mathcal{P}_{(\delta)}\uparrow = 0 \neq \delta = \mathcal{P}_{(\delta)}(\downarrow \cdot \uparrow).$$

(c) The operator $\mathcal{P}_{(\delta)}$ acts by cutting legs from blocks. In particular, for any $p \in \mathsf{Part}_\delta$, we have that $\mathcal{P}_{(\delta)}p = p + q$, where q is a linear combination of partitions containing at least one singleton.

(d) For any linear combination p of partitions containing a singleton, we have $\mathcal{P}_{(\delta)}p = 0$. This follows from the fact that $\pi_{(\delta)}\uparrow = 0$.

5.4.16 Proposition. Let $\mathscr{K}$ be a linear category of partitions such that $\uparrow \otimes \uparrow \in \mathscr{K}$. Then $\mathcal{P}_{(\delta)}\mathscr{K}$ is a reduced category.

Proof. Since $| \in \mathscr{K}$, we have $\pi_{(\delta)} = \mathcal{P}_{(\delta)}| \in \mathcal{P}_{(\delta)}\mathscr{K}$ and similarly for its rotations.

Thanks to the projective property of $\pi_{(\delta)}$, we have $\pi^{\otimes l}_{(\delta)} p = p$ and $p\pi^{\otimes k}_{(\delta)} = p$ for $p \in \mathcal{P}_{(\delta)}\mathscr{K}(k,l)$, which can be used to prove closedness under the category operations. For example, taking any $p \in \mathcal{P}_{(\delta)}\mathscr{K}(k,l)$ and $q \in \mathcal{P}_{(\delta)}\mathscr{K}(l,m)$, we have $qp \in \mathscr{K}(k,m)$ since $\mathcal{P}_{(\delta)}\mathscr{K} \subseteq \mathscr{K}$ and hence also

$$qp = \pi^{\otimes m}_{(\delta)} qp \pi^{\otimes k}_{(\delta)} = \mathcal{P}_{(\delta)}(qp) \in \mathcal{P}_{(\delta)}\mathscr{K}(k,m).$$ □

5.4.17 Remark. For every reduced category $\mathscr{K} \subseteq \mathsf{PartRed}_\delta$, we have $\uparrow\otimes\uparrow \in \langle\mathscr{K}\rangle_\delta$. Indeed, $\uparrow\otimes\uparrow$ is a rotation of ¦, which is a linear combination of $|$ and $\pi_{(\delta)}$.

5.4.18 Lemma. Let $p_1,\ldots,p_n \in \mathsf{PartRed}_\delta$. Then

$$\langle\langle p_1,\ldots,p_n\rangle_{\delta\text{-red}}\rangle_\delta = \langle p_1,\ldots,p_n,\uparrow\otimes\uparrow\rangle_\delta.$$

Proof. On the left-hand side, there is a linear category containing $p_1,\ldots,p_n$, and $\uparrow\otimes\uparrow$, which implies the inclusion $\supseteq$. The category on the right hand side contains $p_1,\ldots,p_n$, and rotations of $\pi_{(\delta)}$ and it is, of course, closed under the category operations. So, $\langle p_1,\ldots,p_n\rangle_{\delta\text{-red}} \subseteq \langle p_1,\ldots,p_n,\uparrow\otimes\uparrow\rangle_\delta$, which implies the opposite inclusion $\subseteq$. □

We have proven that $\mathcal{P}_{(\delta)}\mathscr{K}$ is a reduced category for any linear category of partitions $\mathscr{K}$ containing $\uparrow\otimes\uparrow$. Now, we prove that all reduced categories are of this form.

5.4.19 Proposition. Let $\mathscr{K}$ be a reduced category. Then

$$\mathscr{K} = \mathcal{P}_{(\delta)}\langle\mathscr{K}\rangle = \mathcal{P}_{(\delta)}\langle\mathscr{K},\sqcap\!\sqcap\,\uparrow\rangle_\delta = \mathcal{P}_{(\delta)}\langle\mathscr{K},\uparrow\rangle_\delta.$$

Proof. Denote by K the set of all $p' \in \mathsf{Part}_\delta$ such that p' was made by adding singletons to some $p \in \mathscr{K}$. To be more precise, we can formulate this condition recursively: for any $p \in K$, it holds that either $p \in \mathscr{K}$ or there is $q \in K$ such that p is some rotation of $q\otimes\uparrow$ (including the possibility that q is a multiple of the empty partition, so $p = \alpha\uparrow \in K$ and $p = \alpha\downarrow \in K$).

Now, let us prove that the collection of spaces $\operatorname{span} K(k,l)$ is a linear category of partitions (we will denote it just by $\operatorname{span} K$ in the following text). The identity partition is a linear combination of $\pi_{(\delta)} \in \mathscr{K} \subseteq K$ and ¦ $\in K$, so it is contained in $\operatorname{span} K$. Similarly the pair partitions are also contained in $\operatorname{span} K$. It is clear that K is closed under tensor product and involution, so let us prove it for the composition. Take arbitrary composable $p',q' \in K$, which were made from $p,q \in \mathscr{K}$ by adding singletons. If the added singletons in the lower row of p' do not exactly match the singletons in the upper row of q', we have $q'p' = 0$ since $\pi_{(\delta)}\uparrow = 0$. Otherwise it is easy to see that $q'p'$ can be made from qp by adding singletons and multiplying by some power of δ, so $q'p' \in \operatorname{span} K$.

This implies that

$$\mathscr{K} \subseteq \langle\mathscr{K}\rangle_\delta \subseteq \langle\mathscr{K},\sqcap\!\sqcap\,\uparrow\rangle_\delta \subseteq \langle\mathscr{K},\uparrow\rangle_\delta \subseteq \operatorname{span} K.$$

Now, since $\mathcal{P}_{(\delta)}p = 0$ for any p containing a singleton, we have $\mathcal{P}_{(\delta)}K = \mathcal{P}_{(\delta)}\mathscr{K} = \mathscr{K}$. So, applying $\mathcal{P}_{(\delta)}$ on the chain of containments above, we have

$$\mathscr{K} \subseteq \mathcal{P}_{(\delta)}\langle\mathscr{K}\rangle_\delta \subseteq \mathcal{P}_{(\delta)}\langle\mathscr{K},\sqcap\!\sqcap\,\uparrow\rangle_\delta \subseteq \mathcal{P}_{(\delta)}\langle\mathscr{K},\uparrow\rangle_\delta \subseteq \mathscr{K},$$

which implies the statement of the proposition. □

5.4.20 Corollary. For any $p_1,\ldots,p_n \in \mathcal{P}_{(\delta)}\mathsf{Part}_\delta$, we have

$$\langle p_1,\ldots,p_n\rangle_{\delta\text{-red}} = \mathcal{P}_{(\delta)}\langle p_1,\ldots,p_n,\uparrow\otimes\uparrow\rangle_\delta.$$

Proof. Combining Proposition 5.4.19 and Lemma 5.4.18, we have

$$\langle p_1,\ldots,p_n\rangle_{\delta\text{-red}} = \mathcal{P}_{(\delta)}\langle\langle p_1,\ldots,p_n\rangle_{\delta\text{-red}}\rangle_\delta = \mathcal{P}_{(\delta)}\langle p_1,\ldots,p_n,\uparrow\otimes\uparrow\rangle_\delta.$$ □

5.4.21 Proposition. Let $\mathscr{K}$ be a reduced category. Then the following categories are mutually different

$$\langle\mathscr{K}\rangle_\delta \subsetneq \langle\mathscr{K}, \sqcap\!\!\!\!\!{\scriptstyle|}\;\;{\scriptstyle|}\rangle_\delta \subsetneq \langle\mathscr{K},\uparrow\rangle_\delta.$$

Proof. We are able to explicitly describe the categories in terms of the reduced category $\mathscr{K}$.

First of all, it is easy to see that the last category $\langle\mathscr{K},\uparrow\rangle_\delta$ coincides with $\operatorname{span} K$, where K is defined as in the proof of Proposition 5.4.19.

We can define $K' \subseteq K$ by choosing only those elements, where we added just an even number of singletons. With similar argumentation, we can show that $\operatorname{span} K'$ is a category and hence that $\operatorname{span} K' = \langle\mathscr{K}, \sqcap\!\!\!\!\!{\scriptstyle|}\;\;{\scriptstyle|}\rangle_\delta$. Obviously $\uparrow \notin \operatorname{span} K'$, so we have just proven strictness of the second inclusion.

Finally, we define K'' inductively as follows: $K''(k,l)$ contains all elements of $\mathscr{K}(k,l)$ and appropriate rotations of $p\otimes\uparrow\otimes q\otimes\uparrow$ with $p\in\mathscr{K}(0,m)$, $q\in\mathscr{K}(0,k+l-m-2)$. Again, we can prove that $\operatorname{span} K''$ is a category equal to $\langle\mathscr{K}\rangle_\delta$, which surely does not contain $\sqcap\!\!\!\!\!{\scriptstyle|}\;\;{\scriptstyle|}$. □

This was for the preparation, now let us go for the concrete categories. Recall from Section 4.2.3 the following categories.

$$\mathsf{NCPart}_\delta := \langle \sqcap\!\sqcap\!\sqcap, \uparrow\rangle_\delta = \langle\sqcap\!\sqcap\rangle_\delta = \operatorname{span}\{\text{all non-crossing partitions}\},$$
$$\mathsf{NCPart}'_\delta := \langle \sqcap\!\sqcap\!\sqcap, \uparrow\otimes\uparrow\rangle_\delta = \operatorname{span}\{\text{all non-crossing partitions of even length}\}.$$

5.4.22 Lemma. It holds that

$$\mathcal{P}_{(\delta)}\mathsf{NCPart}_\delta = \langle\mathcal{P}_{(\delta)}\sqcap\!\sqcap\rangle_{\delta\text{-red}}.$$

Proof. It is easy to check that $\mathsf{NCPart}_\delta = \langle\sqcap\!\sqcap\rangle_\delta = \langle\mathcal{P}_{(\delta)}\sqcap\!\sqcap,\uparrow\rangle_\delta$ (see also Example 5.4.13). From Lemma 5.4.18, we have $\langle\langle\mathcal{P}_{(\delta)}\sqcap\!\sqcap\rangle_{\delta\text{-red}}\rangle_\delta = \langle\mathcal{P}_{(\delta)}\sqcap\!\sqcap,\uparrow\otimes\uparrow\rangle_\delta$. Adding the singleton to the category on both sides, we have

$$\mathsf{NCPart}_\delta = \langle\mathcal{P}_{(\delta)}\sqcap\!\sqcap,\uparrow\rangle_\delta = \langle\langle\mathcal{P}_{(\delta)}\sqcap\!\sqcap\rangle_{\delta\text{-red}},\uparrow\rangle_\delta.$$

Finally, we use Proposition 5.4.19 to derive

$$\mathcal{P}_{(\delta)}\mathsf{NCPart}_\delta = \mathcal{P}_{(\delta)}\langle\langle\mathcal{P}_{(\delta)}\sqcap\!\sqcap\rangle_{\delta\text{-red}},\uparrow\rangle_\delta = \langle\mathcal{P}_{(\delta)}\sqcap\!\sqcap\rangle_{\delta\text{-red}}.$$ □

5.4.23 Lemma. Suppose $\delta \neq 3$. It holds that

$$\mathcal{P}_{(\delta)}\mathsf{NCPart}'_\delta = \langle\mathcal{P}_{(\delta)}\sqcap\!\sqcap\!\sqcap\rangle_{\delta\text{-red}}.$$

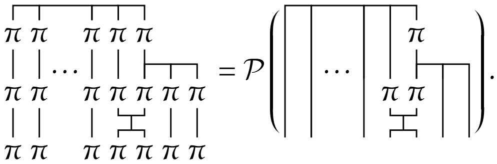
Proof. In this case, the inclusion $\supseteq$ follows from Proposition 5.4.16. For the converse, it is enough to show that $\mathcal{P}_\delta p \in \langle\mathcal{P}_{(\delta)}\sqcap\!\sqcap\!\sqcap\rangle_{\delta\text{-red}}$ for every non-crossing partition p of even length. We will show it in four steps.

Step 1. $\mathcal{P}_{(\delta)}b_k \in \langle\mathcal{P}_{(\delta)}\sqcap\!\sqcap\!\sqcap\rangle_{\delta\text{-red}}$ for all k even.

Here $b_k \in \mathscr{P}(0,k)$ is the block partition, that is, a partition where all points are contained in one block. We will show the statement by induction. It holds for $k=2$ by the definition of reduced categories and for $k=4$ since $\mathcal{P}_{(\delta)}b_4$ is the generator. Now, consider $k>4$ and suppose $\mathcal{P}_{(\delta)}b_i \in \mathcal{P}_{(\delta)}\langle\sqcap\!\sqcap\!\sqcap\rangle_{\delta\text{-red}}$ for all $i<k$ even. We compute that

$$\begin{aligned}
&(\pi^{\otimes(k-4)}\otimes\mathcal{P}_{(\delta)}\overset{\sqcup}{\sqcap}\otimes\pi^{\otimes 2})(\pi^{\otimes(k-3)}\otimes\mathcal{P}_{(\delta)}{\vdash\!\sqcap\!\sqcap})\mathcal{P}_{(\delta)}b_{k-2} = \\
&\qquad= (\pi^{\otimes(k-4)}\otimes\mathcal{P}_{(\delta)}\overset{\sqcup}{\sqcap}\otimes\pi^{\otimes 2})\Big(\mathcal{P}_{(\delta)}b_k - \frac{1}{\delta}\mathcal{P}_{(\delta)}b_{k-3}\otimes\mathcal{P}_{(\delta)}\sqcap\!\sqcap\Big) = \\
&\qquad= \left(1-\frac{3}{\delta}\right)\mathcal{P}_{(\delta)}b_k + \frac{1}{\delta^2}\Big(\mathcal{P}_{(\delta)}b_{k-2}\otimes\mathcal{P}_{(\delta)}\sqcap + R^{-2}(\mathcal{P}_{(\delta)}b_{k-2}\otimes\mathcal{P}_{(\delta)}\sqcap) \\
&\qquad\qquad + \mathcal{P}_{(\delta)}b_{k-4}\otimes\mathcal{P}_{(\delta)}b_4\Big) - \frac{1}{\delta^3}\mathcal{P}_{(\delta)}b_{k-4}\otimes\mathcal{P}_{(\delta)}\sqcap\otimes\mathcal{P}_{(\delta)}\sqcap.
\end{aligned}$$

All the terms except for $\mathcal{P}_{(\delta)}b_k$ are surely elements of the category $\langle\mathcal{P}_{(\delta)}\sqcap\!\sqcap\!\sqcap\rangle_{\delta\text{-red}}$, so $\mathcal{P}_{(\delta)}b_k$ must be as well. The idea of the computation is maybe more clear using the pictorial representation for partitions. Using the definition $\mathcal{P}_{(\delta)}p = \pi_{(\delta)}^{\otimes l} p \pi_{(\delta)}^{\otimes k}$ for $p \in \mathrm{Part}_\delta(k,l)$ and the projective property $\pi_{(\delta)}\pi_{(\delta)} = \pi_{(\delta)}$, we can express the left hand side in the following form



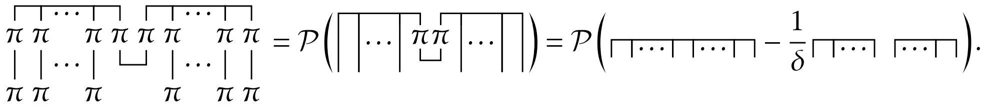
We obtain the result on the right-hand side simply by substituting $\pi = | - \frac{1}{\delta}{}^{\shortmid}_{\shortmid}$.

Step 2. $\mathcal{P}_{(\delta)}b_k \otimes \mathcal{P}_{(\delta)}b_l \in \langle \mathcal{P}_{(\delta)}\sqcap\!\sqcap\!\sqcap \rangle_{\delta\text{-red}}$ for any $k, l \geq 2$ such that $k+l$ is even.

This can be seen inductively from the following

$$(\pi^{\otimes k} \otimes \mathcal{P}_{(\delta)}\sqcup \otimes \pi^{\otimes l})(\mathcal{P}_{(\delta)}b_{k+1} \otimes \mathcal{P}_{(\delta)}b_{l+1}) = \mathcal{P}_{(\delta)}b_{k+l} - \frac{1}{\delta}(\mathcal{P}_{(\delta)}b_k \otimes \mathcal{P}_{(\delta)}b_l).$$

Pictorially,

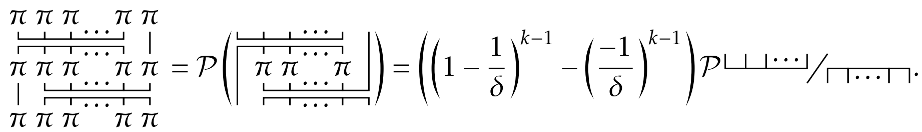


Step 3. $(\mathcal{P}_{(\delta)}b_k)^* \otimes | \otimes \mathcal{P}_{(\delta)}b_k \in \langle \mathcal{P}_{(\delta)}\sqcap\!\sqcap\!\sqcap \rangle_{\delta\text{-red}}$ for any $k \in \mathbb{N}$ (including the odd ones).

For $k = 1$ we have $\mathcal{P}_{(\delta)}b_k = 0$. Considering $k \geq 2$, the assertion follows from

$$(\pi \otimes (\mathcal{P}_{(\delta)}b_k)^* \otimes \mathcal{P}_{(\delta)}b_k)((\mathcal{P}_{(\delta)}b_k)^* \otimes \mathcal{P}_{(\delta)}b_k \otimes \pi) = \left(\left(1 - \frac{1}{\delta}\right)^{k-1} - \left(\frac{-1}{\delta}\right)^{k-1}\right)(\mathcal{P}_{(\delta)}b_k)^* \otimes | \otimes \mathcal{P}_{(\delta)}b_k.$$

Pictorially,



Step 4. $\mathcal{P}_{(\delta)}p \in \langle \mathcal{P}_{(\delta)}\sqcap\!\sqcap\!\sqcap \rangle_{\delta\text{-red}}$ for every non-crossing partition p of even length.

Without loss of generality, we can assume that p has lower points only since reduced categories are closed under rotations. Now, denote by $l_1, \ldots, l_n$ the sizes of the blocks in p ordered in such a way that first come all even numbers and then all odd numbers. Since p is of even length, we have that $\sum l_i$ is even, so there is an even number of odd numbers in the tuple (l_i). We can construct $\mathcal{P}_{(\delta)}p \in \mathcal{P}_{(\delta)}\langle \mathcal{P}_{(\delta)}\sqcap\!\sqcap\!\sqcap \rangle_{\delta\text{-red}}$ by computing $\mathcal{P}_{(\delta)}b_{l_1} \otimes \cdots \otimes \mathcal{P}_{(\delta)}b_{l_n} \in \langle \mathcal{P}_{(\delta)}\sqcap\!\sqcap\!\sqcap \rangle_{\delta\text{-red}}$ and then using compositions with $(\mathcal{P}_{(\delta)}b_k)^* \otimes \pi \otimes \mathcal{P}_{(\delta)}b_k = \mathcal{P}_{(\delta)}\sqcup\!\sqcup\cdots\sqcup / \sqcap\!\cdots\!\sqcap \in \langle \mathcal{P}_{(\delta)}\sqcap\!\sqcap\!\sqcap \rangle_{\delta\text{-red}}$ to move the blocks to their positions. □

5.4.24 Proposition. The categories

$$\begin{array}{ccccccc} \langle \mathcal{P}_{(\delta)}\sqcap\!\sqcap \rangle = \langle \mathcal{P}_{(\delta)}\sqcap\!\sqcap, \uparrow \otimes \uparrow \rangle_\delta & \subsetneq & \langle \mathcal{P}_{(\delta)}\sqcap\!\sqcap, \sqcap\!\shortmid\!\sqcap\ {}_{\shortmid} \rangle_\delta & \subsetneq & \langle \mathcal{P}_{(\delta)}\sqcap\!\sqcap, \uparrow \rangle_\delta = \mathsf{NCPart}_\delta \\ \cup\!\!\!\!\!\shortmid & & \cup\!\!\!\!\!\shortmid & & \cup\!\!\!\!\!\shortmid \\ \langle \mathcal{P}_{(\delta)}\sqcap\!\sqcap\!\sqcap \rangle = \langle \mathcal{P}_{(\delta)}\sqcap\!\sqcap\!\sqcap, \uparrow \otimes \uparrow \rangle_\delta & \subsetneq & \langle \mathcal{P}_{(\delta)}\sqcap\!\sqcap\!\sqcap, \sqcap\!\shortmid\!\sqcap\ {}_{\shortmid} \rangle_\delta & \subsetneq & \langle \mathcal{P}_{(\delta)}\sqcap\!\sqcap\!\sqcap, \uparrow \rangle_\delta \end{array}$$

are mutually distinct and, except for the top right one, are all non-easy.

Proof. Thanks to Lemmata 5.4.22 and 5.4.23, we can replace $\mathcal{P}_{(\delta)}\sqcap\!\sqcap$ and $\mathcal{P}_{(\delta)}\sqcap\!\sqcap\!\sqcap$ by $\mathscr{K}_1 := \mathcal{P}_{(\delta)}\langle \sqcap\!\sqcap \rangle_\delta$ and $\mathscr{K}_2 := \mathcal{P}_{(\delta)}\langle \sqcap\!\sqcap\!\sqcap \rangle_\delta$, respectively. Surely $\mathscr{K}_1 \neq \mathscr{K}_2$ since $\mathscr{K}_2$ contains no partition of odd length. From Proposition 5.4.19, it follows that we have $\mathcal{P}_{(\delta)}\mathscr{K} = \mathscr{K}_1$ for categories $\mathscr{K}$ in the first line, whereas $\mathcal{P}_{(\delta)}\mathscr{K} = \mathscr{K}_2$ for categories $\mathscr{K}$ from the second line. Consequently, no category from the first line can be equal to any category from the second line. The strictness of the horizontal inclusions follows from Proposition 5.4.21. Finally, if the first or the second category from the first line was easy, then it would contain the singleton $\uparrow$ and hence be equal to the last one. Also if the last category of the second row was easy, then it would be equal to NCPart_δ. If one of the first two categories was easy then surely adding the singleton would preserve the easiness and hence the last one would be easy. □

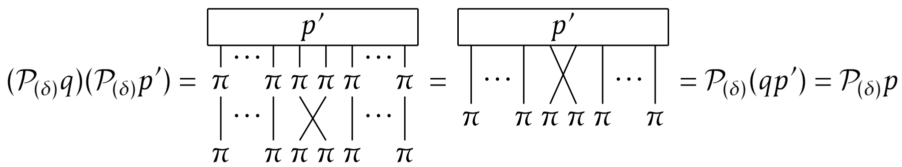
5.4.25 Lemma. We have the following equalities.

(1) $\langle \mathcal{P}_{(\delta)} \crosspart \rangle_{\delta\text{-red}} = \mathcal{P}_{(\delta)} \langle \crosspart, \uparrow\otimes\uparrow \rangle_\delta = \mathcal{P}_{(\delta)} \langle \crosspart, \uparrow \rangle_\delta$.
(2) $\langle \mathcal{P}_{(\delta)} \halflibpart \rangle_{\delta\text{-red}} = \mathcal{P}_{(\delta)} \langle \halflibpart, \uparrow\otimes\uparrow \rangle_\delta$.
(3) $\langle \mathcal{P}_{(\delta)} \crosspart, \mathcal{P}_{(\delta)} \sqcap\!\sqcap \rangle_{\delta\text{-red}} = \mathcal{P}_{(\delta)} \langle \crosspart, \sqcap\!\sqcap \rangle_\delta = \mathcal{P}_{(\delta)} \mathsf{Part}_\delta = \mathsf{PartRed}_\delta$.
(4) $\langle \mathcal{P}_{(\delta)} \crosspart, \mathcal{P}_{(\delta)} \sqcap\!\sqcap\!\sqcap \rangle_{\delta\text{-red}} = \mathcal{P}_{(\delta)} \langle \crosspart, \sqcap\!\sqcap\!\sqcap, \uparrow\otimes\uparrow \rangle_\delta$.

These four reduced categories are mutually distinct.

Proof. In all cases, the inclusion $\subseteq$ is obvious. Below, we explain the inclusions $\supseteq$. The proof below also explicitly describes elements of the reduced categories, from which it follows that they are indeed mutually distinct.

In case (1), denote $\mathscr{K} := \langle \crosspart, \uparrow \rangle_\delta$. We need to prove that $\mathcal{P}_{(\delta)} p \in \langle \mathcal{P}_{(\delta)} \crosspart \rangle_\delta$ for every $p \in \mathscr{K}$. Since $\mathscr{K}$ is easy, it is enough to prove this for partitions p, which linearly generate $\mathscr{K}$. In addition, we have $\mathcal{P}_{(\delta)} p = 0$ whenever p contains a singleton. Thus, it is enough to consider partitions p not containing singletons. Those are exactly all pairings, i.e. partitions $p \in \langle \crosspart \rangle$. Without loss of generality, we can assume that p has lower points only, so $p \in \mathsf{Pair}_\delta(k)$. Such a pairing p was made from some non-crossing pairing (such as $\sqcap^{\otimes k/2}$) by permuting its points. The reduced category $\langle \mathcal{P}_{(\delta)} \crosspart \rangle_{\delta\text{-red}}$ contains $\mathcal{P}_{(\delta)} \sqcap^{\otimes k/2} = \pi_{(\delta)}^{\otimes k/2}$. By induction, it is enough to prove that if $\mathcal{P}_{(\delta)} p' \in \langle \mathcal{P}_{(\delta)} \crosspart \rangle_{\delta\text{-red}}$, then $\mathcal{P}_{(\delta)} p \in \langle \mathcal{P}_{(\delta)} \crosspart \rangle_{\delta\text{-red}}$, where $p, p' \in \mathsf{Pair}_\delta(k)$ such that p was made from p' by transposing neighbouring points. This transposition can be realized as $p = qp'$, where $q = |\cdots| \crosspart |\cdots|$. The proof is finished by observing that $\pi^{\otimes k} q \pi^{\otimes k} = \pi^{\otimes k} q$, so $\mathcal{P}_{(\delta)} p = \pi^{\otimes k} q p' = \pi^{\otimes k} q \pi^{\otimes k} p' = (\mathcal{P}_{(\delta)} q)(\mathcal{P}_{(\delta)} p')$. Let us illustrate this pictorially:



The proof in all the remaining cases is similar. The key part is to determine the set of all partitions that are elements of the easy category on the right-hand side and do not contain singletons. In case (2), the category $\langle \halflibpart, \uparrow\otimes\uparrow \rangle$ contains all partitions with blocks of size one and two such that both legs of all blocks of size two are either on even position or on odd position [Web13, Prop. 3.5]. So, excluding partitions with singletons, we get exactly elements of $\langle \halflibpart \rangle$. Those can be obtained applying permutations that map even points to even points on the non-crossing pairings. For the induction, we then may use the transposition $|\cdots| \halflibpart |\cdots|$. In both cases (1) and (2), it is maybe worth mentioning that the corresponding reduced category is actually isomorphic to $\langle \crosspart \rangle_{\delta-1}$ and $\langle \halflibpart \rangle_{\delta-1}$, respectively; the isomorphism is provided by $\mathcal{V}_{(\delta,\pm)}$ defined in the following section.

For case (3), we need to prove that $\mathcal{P}_{(\delta)} p \in \langle \mathcal{P}_{(\delta)} \crosspart, \mathcal{P}_{(\delta)} \sqcap\!\sqcap \rangle_\delta$ for any partition p not containing a singleton. This is again a permutation $p = qp'$, where $q \in \mathsf{Pair}_\delta(k,k)$ and $p' \in \mathsf{Part}_\delta(k)$ is non-crossing partition not containing a singleton. From Lemma 5.4.22, we know that $\mathcal{P}_{(\delta)} p' \in \langle \mathcal{P}_{(\delta)} \sqcap\!\sqcap \rangle_\delta \subseteq \langle \mathcal{P}_{(\delta)} \crosspart, \mathcal{P}_{(\delta)} \sqcap\!\sqcap \rangle_\delta$ and from item (1), we have that $\mathcal{P}_{(\delta)} q \in \langle \mathcal{P}_{(\delta)} \crosspart \rangle_{\delta\text{-red}} \subseteq \langle \mathcal{P}_{(\delta)} \crosspart, \mathcal{P}_{(\delta)} \sqcap\!\sqcap \rangle_{\delta\text{-red}}$. Finally, again $\mathcal{P}_{(\delta)} p = \mathcal{P}_{(\delta)} q \mathcal{P}_{(\delta)} p$.

Case (4) is basically the same as case (3) except that we work with partitions of even size and we need to use Lemma 5.4.23. □

5.4.26 Lemma. We have the following inclusions.

$$\begin{array}{ccccc}
\langle \crosspart, \uparrow\otimes\uparrow \rangle_\delta & = & \langle \crosspart, \sqcap\!\sqcap_{\shortmid\shortmid} \rangle_\delta & \subsetneq & \langle \crosspart, \uparrow \rangle_\delta \\
\cup\!\!\!\!\shortmid & & \| & & \| \\
\langle \mathcal{P}_{(\delta)} \crosspart, \uparrow\otimes\uparrow \rangle_\delta & \subsetneq & \langle \mathcal{P}_{(\delta)} \crosspart, \sqcap\!\sqcap_{\shortmid\shortmid} \rangle_\delta & \subsetneq & \langle \mathcal{P}_{(\delta)} \crosspart, \uparrow \rangle_\delta
\end{array}$$

Proof. The first row is known from classification of the easy categories, see Section 4.2. In the second row, we can replace $\mathcal{P}_{(\delta)}\times$ by $\langle \mathcal{P}_{(\delta)}\times\rangle_{\delta\text{-red}}$ thanks to Lemma 5.4.25. The inclusions then follow from Proposition 5.4.21. The vertical equalities are then easy to see if we write

$$\mathcal{P}_{(\delta)}\times = \times - \frac{1}{\delta}\,{}^{\shortmid}\!/_{\shortmid} - \frac{1}{\delta}\,{}_{\shortmid}\!\backslash^{\shortmid} + \frac{1}{\delta^2}\,{}^{\shortmid}_{\shortmid}\,{}^{\shortmid}_{\shortmid}. \qquad \square$$

5.4.27 Lemma. We have the following inclusions.

$$\begin{array}{ccccc}
\langle \times,\uparrow\otimes\uparrow\rangle_\delta & = & \langle \times,\sqcap\!\!\sqcap\rangle_\delta & \subsetneq & \langle \times,\uparrow\rangle_\delta \\
\cup\!\!\!\!\!\not\;\; & & \| & & \| \\
\langle \ast,\uparrow\otimes\uparrow\rangle_\delta & \subsetneq & \langle \ast,\sqcap\!\!\sqcap\rangle_\delta & \subsetneq & \langle \ast,\uparrow\rangle_\delta \\
\cup\!\!\!\!\!\not\;\; & & \cup\!\!\!\!\!\not\;\; & & \cup\!\!\!\!\!\not\;\; \\
\langle \mathcal{P}_{(\delta)}\ast,\uparrow\otimes\uparrow\rangle_\delta & \subsetneq & \langle \mathcal{P}_{(\delta)}\ast,\sqcap\!\!\sqcap\rangle_\delta & \subsetneq & \langle \mathcal{P}_{(\delta)}\ast,\uparrow\rangle_\delta
\end{array}$$

Proof. The first two rows are again known from classification of easy categories, see Section 4.2 and also [Web13]. The last row follows again from Lemma 5.4.25 and Proposition 5.4.21. Now we explain the strictness of the vertical inclusions between the second and the last row. For the second and third column, we simply apply $\mathcal{P}_{(\delta)}$. We get $\langle \mathcal{P}_{(\delta)}\times\rangle_{\delta\text{-red}}$ for the second row, but $\langle \mathcal{P}_{(\delta)}\ast\rangle_{\delta\text{-red}}$ for the third row, which proves the inequality.

As for the first column, we can see that $({}^{\shortmid}_{\shortmid}\otimes\pi_{(\delta)}\otimes\pi_{(\delta)})\ast \in \langle \ast,\uparrow\otimes\uparrow\rangle_\delta$. We prove that this element cannot be contained in $\langle \mathcal{P}_{(\delta)}\ast,\uparrow\otimes\uparrow\rangle_\delta$. If it was there, then it would be contained also in $\langle \mathcal{P}_{(\delta)}\ast,\uparrow\rangle_\delta$. Composing with $\downarrow\otimes|\otimes|$ from left and with $|\otimes|\otimes\uparrow$ from right, we get $\mathcal{P}_{(\delta)}\times \in \langle \mathcal{P}_{(\delta)}\ast,\uparrow\rangle_\delta$ and hence $\mathcal{P}_{(\delta)}\times \in \langle \mathcal{P}_{(\delta)}\ast\rangle_{\delta\text{-red}} \subsetneq \langle \mathcal{P}_{(\delta)}\times\rangle_{\delta\text{-red}}$, which is a contradiction. $\square$

Let us summarize all the non-easy categories we obtained above.

5.4.28 Proposition. The categories

$$\begin{array}{ccccc}
\langle \mathcal{P}_{(\delta)}\times,\mathcal{P}_{(\delta)}\sqcap\sqcap,\uparrow\otimes\uparrow\rangle_\delta & \subsetneq & \langle \mathcal{P}_{(\delta)}\times,\mathcal{P}_{(\delta)}\sqcap\sqcap,\sqcap\!\!\sqcap\rangle_\delta & \subsetneq & \langle \mathcal{P}_{(\delta)}\times,\mathcal{P}_{(\delta)}\sqcap\sqcap,\uparrow\rangle_\delta = \mathsf{Part}_\delta \\
\cup\!\!\!\!\!\not\;\; & & \cup\!\!\!\!\!\not\;\; & & \cup\!\!\!\!\!\not\;\; \\
\langle \mathcal{P}_{(\delta)}\times,\mathcal{P}_{(\delta)}\sqcap\sqcap\sqcap,\uparrow\otimes\uparrow\rangle_\delta & \subsetneq & \langle \mathcal{P}_{(\delta)}\times,\mathcal{P}_{(\delta)}\sqcap\sqcap\sqcap,\sqcap\!\!\sqcap\rangle_\delta & \subsetneq & \langle \mathcal{P}_{(\delta)}\times,\mathcal{P}_{(\delta)}\sqcap\sqcap\sqcap,\uparrow\rangle_\delta \\
\cup\!\!\!\!\!\not\;\; & & \cup\!\!\!\!\!\not\;\; & & \cup\!\!\!\!\!\not\;\; \\
\langle \mathcal{P}_{(\delta)}\times,\uparrow\otimes\uparrow\rangle_\delta & \subsetneq & \langle \mathcal{P}_{(\delta)}\times,\sqcap\!\!\sqcap\rangle_\delta & \subsetneq & \langle \mathcal{P}_{(\delta)}\times,\uparrow\rangle_\delta \\
\cup\!\!\!\!\!\not\;\; & & \cup\!\!\!\!\!\not\;\; & & \cup\!\!\!\!\!\not\;\; \\
\langle \mathcal{P}_{(\delta)}\ast,\uparrow\otimes\uparrow\rangle_\delta & \subsetneq & \langle \mathcal{P}_{(\delta)}\ast,\sqcap\!\!\sqcap\rangle_\delta & \subsetneq & \langle \mathcal{P}_{(\delta)}\ast,\uparrow\rangle_\delta
\end{array}$$

are mutually distinct and, except for the top right one, are all non-easy. They are also distinct from the categories of Proposition 5.4.24.

Proof. Follows directly from all the results above. $\square$

5.4.6 Category coisometry

In this section, we assume $\delta > 0$, $\delta \neq 1$.

5.4.29 Definition. We define $\upsilon_{(\delta-1,\pm)} := | - \frac{1}{\delta-1}\left(1 \pm \frac{1}{\sqrt{\delta}}\right) \in \mathsf{Part}_{\delta-1}(1,1)$.[1] For every $p \in \mathscr{P}(k,l)$, we define $\mathcal{B}_{(\delta)}p \in \mathsf{Part}_{\delta-1}(k,l)$ the linear combination that was made from p by replacing every block of k points by $\langle\text{block}\rangle + (-1)^k\langle\text{singletons}\rangle$. We extend this definition linearly to define a map $\mathcal{B}_{(\delta)}\colon \mathsf{Part}_\delta \to \mathsf{Part}_{\delta-1}$. Finally, for every $p \in \mathsf{Part}_\delta(k,l)$, we define $\mathcal{V}_{(\delta,\pm)}p := \upsilon_{(\delta-1,\pm)}^{\otimes l}(\mathcal{B}_{(\delta)}p)\upsilon_{(\delta-1,\pm)}^{\otimes k}$.

[1] As a remark on consistency with the previous text, where we denoted the special partitions by small Greek letters, let us note that here υ is supposed to be the Greek letter *upsilon*, not the Latin v.

5.4.30 Example. As an example, let us mention how $\mathcal{V}_{(\delta,\pm)}$ acts on the smallest block partitions.

$$\begin{aligned}
\mathcal{V}_{(\delta,\pm)}\uparrow &= 0,\\
\mathcal{V}_{(\delta,\pm)}\sqcap &= \sqcap,\\
\mathcal{V}_{(\delta,\pm)}\sqcap\!\!\sqcap &= \sqcap\!\!\sqcap - \frac{1}{\delta-1}\left(1 \pm \frac{1}{\sqrt{\delta}}\right)(\sqcap\ | + \overline{|\ \,|\,|} + |\ \sqcap) + \frac{1}{(\delta-1)^2}\left(2 \pm \frac{\delta+1}{\sqrt{\delta}}\right)|\ |\ |,\\
\mathcal{V}_{(\delta,\pm)}\sqcap\!\!\sqcap\!\!\sqcap &= \sqcap\!\!\sqcap\!\!\sqcap - \frac{1}{\delta-1}\left(1 \pm \frac{1}{\sqrt{\delta}}\right)(\sqcap\!\!\sqcap\ | + \sqcap\!\!\sqcap\!\!\sqcap' + \sqcap'\!\!\sqcap\!\!\sqcap + |\ \sqcap\!\!\sqcap)+\\
&\quad + \frac{1}{(\delta-1)^2}\left(\frac{\delta+1}{\delta} \pm \frac{2}{\sqrt{\delta}}\right)(\sqcap\ |\ | + \overline{|\ \,|}\ | + \overline{|\ \,|\ \,|\ \,|}+\\
&\quad + |\ \sqcap\ | + |\ \overline{|\ \,|\ \,|} + |\ |\ \sqcap) + \frac{1}{(\delta-1)^3}\left(\frac{\delta^2-6\delta-5}{\delta} \mp \frac{8}{\sqrt{\delta}}\right)|\ |\ |\ |.
\end{aligned}$$

5.4.31 Remarks.

(a) We have $\mathcal{B}_{(\delta)}\uparrow = 0$. Consequently, $\mathcal{B}_{(\delta)}p = 0 = \mathcal{V}_{(\delta,\pm)}p$ for every $p \in \mathscr{P}$ containing a singleton.

(b) As a consequence of the preceding point and Remark 5.4.15.c, $\mathcal{V}_{(\delta,\pm)} = \mathcal{V}_{(\delta,\pm)} \circ \mathcal{P}_{(\delta)}$.

(c) The mapping $\mathcal{V}_{(\delta,\pm)}\colon \mathsf{Part}_\delta \to \mathsf{Part}_{\delta-1}$ is not a functor. The same counterexample as in the case of $\mathcal{P}_{(\delta)}$ works here (see Remark 5.4.15.b).

(d) The operator $\mathcal{V}_{(\delta,\pm)}$ acts by cutting legs from blocks. In particular, for any $p \in \mathsf{Part}_\delta$, we have that $\mathcal{V}_{(\delta,\pm)}p = p + q$, where q is a linear combination of partitions containing at least one singleton (cf. Remark 5.4.15.c).

(e) Consequently, $\mathcal{V}_{(\delta,\pm)}$ acts injectively on partitions with no singletons. For the same reason, it also acts injectively on $\mathsf{PartRed}_\delta = \mathcal{P}_{(\delta)}\mathsf{Part}_\delta$.

(f) $\mathcal{V}_{(\delta,\pm)}$ acts *blockwise* on partitions $p \in \mathscr{P}$. That is, we may map all the blocks constituting a given partition $p \in \mathscr{P}$ an then "assemble" the image of p from the images of the blocks. More formally, we could say that $\mathcal{V}_{(\delta)}$ commutes with tensor products and arbitrary permutations of points. It follows from the fact that $\mathcal{B}_{(\delta)}$ acts blockwise by definition and conjugating by a given partition is also a blockwise operation.

(g) We have $v_{(\delta-1,\pm)}\uparrow = \mp\frac{1}{\sqrt{\delta}}\uparrow$. So, if $b_k \in \mathscr{P}(k)$ is a partition consisting of a single block, we have
$$\mathcal{V}_{(\delta,\pm)}b_k = v^{\otimes k}(b_k + (-1)^k\uparrow^{\otimes k}) = v^{\otimes k}b_k + \left(\frac{\pm 1}{\sqrt{\delta}}\right)^k \uparrow^{\otimes k}.$$

5.4.32 Proposition. The mapping $\mathcal{V}_{(\delta,\pm)}$ acts on $\mathsf{PartRed}_\delta$ as a faithful monoidal unitary functor.

Proof. The injectivity of $\mathcal{V}_{(\delta,\pm)}$ was mentioned in Remark 5.4.31.e. It remains to prove the functorial property. We will work with partitions with lower points only. So, we need to prove that $\mathcal{V}_{(\delta,\pm)}$ commutes with tensor product, contractions, rotations and reflections. The only non-trivial part are the contractions, the other operations follow from the fact that $\mathcal{V}_{(\delta,\pm)}$ acts blockwise as mentioned in Remark 5.4.31.f.

Since $\mathcal{V}_{(\delta,\pm)}$ acts blockwise, it is enough to prove it for blocks. Denote by $b_k \in \mathscr{P}(k)$ the partition consisting of a single block. Then we have to prove that
$$\Pi_1\mathcal{V}_{(\delta,\pm)}\mathcal{P}_{(\delta)}b_k = \mathcal{V}_{(\delta,\pm)}\Pi_1\mathcal{P}_{(\delta)}b_k \quad\text{and}\quad \Pi_{k-1}\mathcal{V}_{(\delta,\pm)}\mathcal{P}_{(\delta)}(b_k\otimes b_l) = \mathcal{V}_{(\delta,\pm)}\Pi_{k-1}\mathcal{P}_{(\delta)}(b_k\otimes b_l).$$
To do that, first note that $\sqcup(v_{(\delta-1,\pm)}\otimes v_{(\delta-1,\pm)}) = \sqcup - \frac{1}{\delta}\downarrow\otimes\downarrow$. So, assuming $k > 2$, we have[1]
$$\begin{aligned}
\Pi_1\mathcal{V}_{(\delta,\pm)}b_k &= \left(\left(\sqcup - \frac{1}{\delta}\downarrow\otimes\downarrow\right)\otimes v^{\otimes(k-2)}\right)(b_k + (-1)^k\uparrow^{\otimes k})\\
&= \left(1 - \frac{1}{\delta}\right)v^{\otimes(k-2)}(b_{k-2} + (-1)^k\uparrow^{\otimes(k-2)}) = \mathcal{V}_{(\delta,\pm)}\Pi_1\mathcal{P}_{(\delta)}b_k.
\end{aligned}$$

[1] Pay attention to the fact that the computations take place mostly in $\mathsf{Part}_{\delta-1}$, not Part_δ!

For $k = 2$, this equality holds as well since $\Pi_1 \mathcal{V}_{(\delta,\pm)} \sqcap = \Pi_1 \sqcap = \delta = \mathcal{V}_{(\delta,\pm)} \Pi_1 \mathcal{P}_{(\delta)} \sqcap$. Now, assuming $k, l > 1$, we have

$$\begin{aligned}
&\Pi_{k-1} \mathcal{V}_{(\delta,\pm)}(b_k \otimes b_l) \\
&= \left(v^{\otimes(k-1)} \otimes \left(\sqcup - \frac{1}{\delta} \downarrow \otimes \downarrow \right) \otimes v^{\otimes(l-1)} \right) (b_k \otimes b_l + (-1)^k \uparrow^{\otimes k} \otimes b_l + (-1)^l b_k \otimes \uparrow^{\otimes l} + (-1)^{k+l} \uparrow^{\otimes(k+l)}) \\
&= v^{\otimes(k+l-2)} \Bigg(b_{k+l-2} - \frac{1}{\delta} b_{k-1} \otimes b_{l-1} + \frac{(-1)^k}{\delta} \uparrow^{\otimes(k-1)} \otimes b_{l-1} + \\
&\qquad\qquad + \frac{(-1)^l}{\delta} b_{k-1} \otimes \uparrow^{\otimes(l-1)} + \left(1 - \frac{1}{\delta}\right) (-1)^{k+l} \uparrow^{\otimes(k+l-2)} \Bigg) \\
&= \mathcal{V}_{(\delta,\pm)} \left(b_{k+l-2} - \frac{1}{\delta} (b_{k-1} \otimes b_{l-1}) \right) = \mathcal{V}_{(\delta,\pm)} \Pi_{k-1} \mathcal{P}_{(\delta)} (b_k \otimes b_l).
\end{aligned}$$

For $k = 1$ or $l = 1$ both sides are obviously equal to zero. □

5.4.33 Remark. Consequently, $\mathcal{V}_{(\delta,\pm)}$ defines an isomorphism between any reduced category of partitions $\mathscr{K} \subseteq \mathsf{PartRed}_\delta$ and its image $\mathcal{V}_{(\delta,\pm)} \mathscr{K} \subseteq \mathsf{Part}_{\delta-1}$. So, also for any ordinary linear category of partitions $\mathscr{K} \subseteq \mathsf{Part}_\delta$, we have an isomorphism between $\mathcal{P}_{(\delta)} \mathscr{K}$ and $\mathcal{V}_{(\delta,\pm)} \mathscr{K}$.

5.4.34 Proposition. The categories

$$\langle \mathcal{V}_{(\delta,\pm)} \sqcap\sqcap \rangle_{\delta-1}, \qquad \langle \mathcal{V}_{(\delta,\pm)} \sqcap\sqcap\sqcap \rangle_{\delta-1}$$

are both non-easy.

Proof. From Proposition 5.4.32, it follows that the above mentioned categories are isomorphic to $\langle \mathcal{P}_{(\delta)} \sqcap\sqcap \rangle_{\delta\text{-red}}$ and $\langle \mathcal{P}_{(\delta)} \sqcap\sqcap\sqcap \rangle_{\delta\text{-red}}$, respectively. If they were easy, they would contain all the summands of $\mathcal{V}_{(\delta,\pm)} \sqcap\sqcap$, resp. $\mathcal{V}_{(\delta,\pm)} \sqcap\sqcap\sqcap$ (Observation 5.2.2) and hence be equal to $\mathsf{NCPart}_{\delta-1}$, resp. $\mathsf{NCPart}'_{\delta-1}$,. This cannot happen since

$$\begin{aligned}
\dim \langle \mathcal{V}_{(\delta,\pm)} \sqcap\sqcap \rangle_{\delta-1}(0,3) &= \dim \langle \mathcal{P}_{(\delta)} \sqcap\sqcap \rangle_{\delta\text{-red}}(0,3) < \dim \mathsf{NCPart}_\delta(0,3) = \dim \mathsf{NCPart}_{\delta-1}(0,3), \\
\dim \langle \mathcal{V}_{(\delta,\pm)} \sqcap\sqcap\sqcap \rangle_{\delta-1}(0,4) &= \dim \langle \mathcal{P}_{(\delta)} \sqcap\sqcap\sqcap \rangle_{\delta\text{-red}}(0,4) < \dim \mathsf{NCPart}'_\delta(0,4) = \dim \mathsf{NCPart}'_{\delta-1}(0,4).
\end{aligned}$$

□

Part III

Quantum groups

This part is about applications of partitions on quantum groups. Categories of partitions were introduced in [BS09] in order to model the representation theory of compact matrix quantum groups. We study categories of partitions in this thesis with the same motivation. We describe briefly the link between partition categories and representation categories of quantum groups in the beginning of Chapter 6.

The main goal of Part III is to apply the results we obtained in Part II to the theory of quantum groups.

In Chapter 4, we classified globally colourized categories of partitions, we introduced partitions with extra singletons, we defined a functor F linking them with two-coloured partitions, and we used this functor to obtain classification results. We interpret all those outcomes in Chapter 6. Globally colourized categories precisely correspond to tensor complexifications of easy quantum groups, the functor F corresponds to some *gluing procedure* and hence classification results for partitions with extra singletons can be interpreted as some *ungluings* of unitary easy quantum groups. In particular, this leads to definition of new $\mathbb{Z}_2$-extensions of quantum groups.

In Chapter 5, we presented first examples of non-easy linear categories of partitions. Those linear categories were studied in Section 5.4 using certain mappings acting on partitions. In Chapter 7, we interpret those mappings, which allows us to describe the quantum groups corresponding to the new non-easy categories. Additional non-easy categories are also defined in Chapter 7.

Finally, Chapter 8 has a slightly different character than then chapters before. We generalize some statements obtained in the previous chapters without the need to refer to partitions. For example, we study the representation categories of tensor and free complexifications. We study the gluing procedure and reverse it to obtain some *ungluing* procedure. The notion of *degree of reflection* defined originally in [TW18] for categories of two-coloured partitions is generalized to a notion for arbitrary compact matrix quantum groups. The $\mathbb{Z}_2$-extensions are generalized to product constructions for arbitrary pairs of compact matrix quantum groups.

Chapter 6

Partition quantum groups and easy examples

In this chapter, we describe the connection between categories of partitions and quantum groups. This connection was established in [BS09], where the first examples of categories of partitions appeared. Those examples were connected with quantum groups that were already known. However, subsequent research in the theory of categories of partitions led to the discovery of new categories of partitions corresponding to new quantum groups that were not known before.

In the beginning of this chapter, we explain the connection between partitions and quantum groups. The main results of this chapter are then the interpretation of the coloured categories and classification results obtained in Chapter 4.

In Section 6.2.3, we interpret the globally-colourized categories of partitions. We show that those exactly correspond to tensor complexifications of orthogonal easy quantum groups (that is, quantum groups corresponding to non-coloured categories).

Theorem (6.2.4, 6.2.7)**.** Let $\mathscr{C}$ be a globally colourized category of partitions, denote $k := k(\mathscr{C})$. Denote $H \subseteq O_N^+$ the quantum group corresponding to the non-coloured category of partitions $\langle \mathscr{C}, \sqcap \rangle$. Then $\mathscr{C}$ corresponds to the quantum group $H \tilde{\times} \hat{\mathbb{Z}}_k \subseteq U_N^+$. In the case $k = 0$, we replace $\hat{\mathbb{Z}}_k$ by $\hat{\mathbb{Z}}$. Conversely, any quantum group of the form $H \tilde{\times} \hat{\mathbb{Z}}_k$, where H is an orthogonal easy quantum group and $k \in \mathbb{N}_0$, is a unitary easy quantum group corresponding to a globally colourized category.

In Section 6.4, we interpret categories of partitions with extra singletons. The main result here is interpreting the functor F from Definition 4.6.4 and consequently the resulting classification from Theorem 4.6.8. We can formulate the result as follows.

Theorem (6.4.13)**.** Let $\mathscr{C}$ be a category of partitions with extra singletons containing only partitions of even length and $G \subseteq O_N \,\hat{*}\, \hat{\mathbb{Z}}_2$ the associated quantum group. Let $\tilde{\mathscr{C}}$ be the corresponding two-coloured category and $\tilde{G} \subseteq U_N^+$ the associated quantum group. Then $\tilde{G}$ is the so-called *glued version* of G.

Non-easy linear categories of partitions that were obtained in Chapter 5 will be treated separately in Chapter 7.

6.1 Non-coloured case and orthogonal CMQGs

In this section, we present the idea of [BS09] to use categories of partitions to describe representation categories of certain quantum groups. In contrast with [BS09], we use *linear* categories of partitions to describe this correspondence. Nevertheless, then we are going to focus on the *easy* categories.

6.1.1 Defining a functor $\mathsf{Part}_N \to \mathsf{Mat}$

Consider a partition $p \in \mathscr{P}(k,l)$ and an integer $N \in \mathbb{N}$. We define the linear map $T_p^{(N)}$: $(\mathbb{C}^N)^{\otimes k} \to (\mathbb{C}^N)^{\otimes l}$ by its entries $[T_p^{(N)}]_{ji} := \delta_p(\boldsymbol{i},\boldsymbol{j})$, so

$$T_p^{(N)}(e_{i_1} \otimes \cdots \otimes e_{i_k}) := \sum_{j_1,\ldots,j_l=1}^{N} \delta_p(\boldsymbol{i},\boldsymbol{j})(e_{j_1} \otimes \cdots \otimes e_{j_l}), \tag{6.1}$$

where we denote $\boldsymbol{i} = (i_1,\ldots,i_k)$ and $\boldsymbol{j} = (j_1,\ldots,j_l)$ and the symbol $\delta_p(\boldsymbol{i},\boldsymbol{j})$ is defined as "blockwise Kronecker delta". That is, assign the k points in the upper row of p the numbers $i_1,\ldots,i_k$ (from left to right) and the l points in the lower row $j_1, \dots, j_l$ (again from left to right). Then $\delta_p(\boldsymbol{i}, \boldsymbol{j}) = 1$ if the points belonging to the same block are assigned the same numbers. Otherwise $\delta_p(\boldsymbol{i}, \boldsymbol{j}) = 0$.

To bring an example, recall the partitions p and q from Equation (4.1).

In this case, we have

$$[T_p^{(N)}]_{\boldsymbol{j}\boldsymbol{i}} = \delta_{i_1 j_1 j_4} \delta_{i_2 i_3} \delta_{j_2 j_3}, \qquad [T_q^{(N)}]_{\boldsymbol{j}\boldsymbol{i}} = \delta_{i_1 j_2} \delta_{i_2 j_1} \delta_{i_3 j_3 j_4}.$$

We extend the definition of the map $p \mapsto T_p^{(N)}$ linearly to the vector spaces $\mathrm{Part}_N(k,l)$. Usually, we will suppress the upper index (N) as the dimension N should be clear from the context.

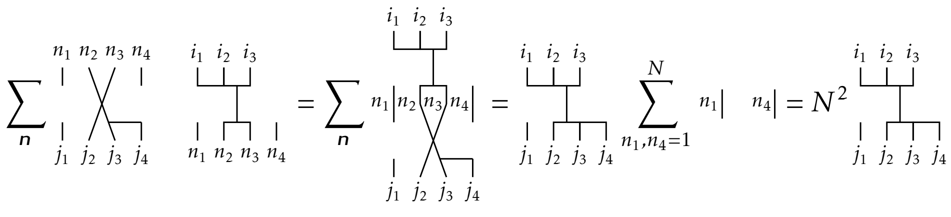
6.1.1 Theorem. [BS09] The map $p \mapsto T_p$ defines a monoidal unitary functor $\mathrm{Part}_N \to \mathsf{Mat}$. That is, we have

(1) $T_{p \otimes q} = T_p \otimes T_q$,
(2) $T_{qp} = T_q T_p$
(3) $T_{p^*} = T_p^*$.

As a consequence, $p \mapsto T_p$ does *not* define a functor $\mathscr{P} \to \mathsf{Mat}$ unless $N = 1$. For the category of partitions $\mathscr{P}$ without the linear structure, the functorial property (2) holds only up to a scalar factor: $T_{qp} = N^{\mathrm{rl}(p,q)} T_q T_p$.

Proof. For the tensor product, take $p \in \mathscr{P}(k,l)$, $q \in \mathscr{P}(k',l')$. Then one can easily see that $\delta_{p \otimes q}(\boldsymbol{i}\boldsymbol{i}', \boldsymbol{j}\boldsymbol{j}') = \delta_p(\boldsymbol{i}, \boldsymbol{j}) \delta_q(\boldsymbol{i}', \boldsymbol{j}')$. The involution is also simple since we immediately see that $\delta_{p^*}(\boldsymbol{j}, \boldsymbol{i}) = \delta_p(\boldsymbol{i}, \boldsymbol{j})$.

The composition is a bit more complicated. Take $p \in \mathscr{P}(k,l)$, $q \in \mathscr{P}(l,m)$ and denote by r their composition qp as partitions, that is, without the scalar factor. So, their composition as elements of Part_N is given by $qp = N^{\mathrm{rl}(p,q)} r$. Now, we are interested in the sum $\sum_{\boldsymbol{n}} \delta_p(\boldsymbol{i}, \boldsymbol{n}) \delta_q(\boldsymbol{n}, \boldsymbol{j})$. For each point $\alpha \in \{1, \dots, l\}$ of the lower row of p or the upper row of q that is connected to some point in upper row of p or lower row of q, the summation over n_α disappears because of the deltas. We are left with $\delta_r(\boldsymbol{i}, \boldsymbol{k})$ and summation over n_α with α belonging to the remaining loops. The summation over each remaining loop gives the factor N. The procedure is probably better to understand from the following illustration.



□

6.1.2 Remark. The functor $p \mapsto T_p$ is not injective. Indeed, consider, for example, $N = 2$. Then we have

$$\delta_{|\,|\,|} = 1 = \delta_{|\,\sqcap} + \delta_{\sqcap\,|} + \delta_{\sqcap\!\!\!\sqcap} - 2\delta_{\sqcap\sqcap},$$

so

$$T_{|\,|\,|} = T_{|\,\sqcap} + T_{\sqcap\,|} + T_{\sqcap\!\!\!\sqcap} - 2T_{\sqcap\sqcap}.$$

The non-injectivity will not play a role in this thesis. Nevertheless, it is a problem that was already studied by many researchers since it may become important in some situations. So, let us mention some results results proving the injectivity in special situations. Already in [BS09, Thm. 1.10], it is proven that $\{T_p \mid p \in \mathscr{P}(k,l)\}$ are linearly independent for $k + l \leq N$. A slightly generalized result is that the set $\{T_p \mid p \in \mathscr{P}(k,l);\ \#\{\text{blocks of } p\} \leq N\}$ is linearly independent. This was proven as [GW20, Prop. 3.3] using the results from [Maa18]. Already in [BS09, Theorem 3.8], it was also mentioned that the intertwiners corresponding to non-crossing partitions are linearly independent for $N \geq 4$ (the authors refer to [Ban99b]; however, the result is contained essentially already in [Tut93]). For non-crossing pairings, the same can be proven for $N \geq 2$ (follows from [BC10]). See [Jun19] for more detailed discussion.

6.1.3 Corollary. Let $\mathscr{K} \subseteq \mathsf{Part}_N$ be a linear category of partitions. Suppose $\mathscr{K}$ is generated by a set K. Let $\mathfrak{C}$ be the image of $\mathscr{K}$ under the functor $p \mapsto T_p$, that is

$$\mathfrak{C}(k,l) = \{T_p \mid p \in \mathscr{K}(k,l)\} \subseteq \mathscr{L}((\mathbb{C}^N)^{\otimes k}, (\mathbb{C}^N)^{\otimes l}).$$

Then $\mathfrak{C}$ is a non-coloured representation category with duality morphism $\xi_{\sqcap} = T_{\sqcap}$. It is generated by $C = \{T_p \mid p \in K\} \cup \xi_{\sqcap}$.

Proof. Follows directly from $p \mapsto T_p$ being a functor. Note that every category of partitions contains the pair partition $\sqcap$ by definition. In the case of representation categories, we have to list it as a generator explicitly. □

Considering a linear combination of partitions with lower points only $p \in \mathsf{Part}_N(0,k)$, the associated map $T_p\colon \mathbb{C} \to (\mathbb{C}^N)^{\otimes k}$ can be identified with a vector in $(\mathbb{C}^N)^{\otimes k}$. We will sometimes denote this vector by ξ_p. In particular, we will often use vectors

$$\xi_{\sqcap} = \sum_{i=1}^{N} e_i \otimes e_i \in \mathbb{C}^N \otimes \mathbb{C}^N,$$

$$\xi_{\uparrow} = \sum_{i=1}^{N} e_i \in \mathbb{C}^N.$$

6.1.2 Using Tannaka–Krein duality

As we just showed in Corollary 6.1.3, the image of a linear category of partitions by the functor $p \mapsto T_p$ is a non-coloured representation category. Hence, we can use it as an input for the Tannaka–Krein duality (the orthogonal version formulated as Corollary 3.4.11).

6.1.4 Lemma. We have $\{T_p \mid p \in \mathsf{Part}_N(k,l)\} = \mathsf{FundRep}_{S_N}(k,l)$.

Proof. Denote by $\mathfrak{C}$ the image of Part_N under the functor $p \mapsto T_p$. We mentioned in Section 4.2.3 that the category of all partitions is generated by $\times$, $\sqcap\!\sqcap\!\sqcap$, and $\uparrow$. Instead of $\sqcap\!\sqcap\!\sqcap$, we may use $\substack{\sqcup\\ \sqcap} = \mathrm{Rrot}^2\, \sqcap\!\sqcap\!\sqcap$. Therefore, $\mathfrak{C}$ is generated by $T_{\sqcap}$, $T_{\times}$, $T_{\substack{\sqcup\\ \sqcap}}$, $T_{\uparrow}$. Let G be the quantum group corresponding to $\mathfrak{C}$. According to Proposition 3.4.14, we have

$$C(G) = C^*(u_{ij} \mid u_{ij} = u_{ij}^*, u^{\otimes 2} T_{\sqcap} = T_{\sqcap}, u^{\otimes 2} T_{\times} = T_{\times} u^{\otimes 2}, u^{\otimes 2} T_{\substack{\sqcup\\ \sqcap}} = T_{\substack{\sqcup\\ \sqcap}} u^{\otimes 2}, u T_{\uparrow} = T_{\uparrow}).$$

Now we only need to check that $C(G) = C(S_N)$, so $\mathfrak{C} = \mathsf{FundRep}_{S_N}$. The relation $u^{\otimes 2} T_{\times} = T_{\times} u^{\otimes 2}$ implies that $C(G)$ is commutative, so G is a matrix group. The relation corresponding to $\sqcap$ says that the matrices are orthogonal, $\substack{\sqcup\\ \sqcap}$ says that they consist of *zeros* and *ones* and $\uparrow$ says that there is exactly one *one* per each row (see Table 6.1). Those are exactly the defining properties of permutation matrices representing the symmetric group S_N. □

This lemma can be interpreted as a generalization of the Schur–Weyl duality for partition categories (or partition algebras). The result was first formulated in [HR05]. As we can see, proving the result at this point was quite simple thanks to all the theory developed until now, in particular, the concrete formulation of the Tannaka–Krein duality from Corollary 3.4.11. Moreover, we did not only proved the duality, we rather *derived* the form of the associated group. It was explicitly constructed translating the partitions generating the category into some relations. And we can do it not only for the category of all partitions, but for any (linear) category of partitions. This is the main idea of the whole theory we are trying to describe in this thesis. Another classical result illustrating this approach is the Brauer's duality associating the orthogonal group to the category of all pairings as we mentioned in the introduction of the thesis. Additional examples are listed in Table 6.2, which still forms just a tiny part of possible applications this approach has.

Now, we formulate the considerations above into a proposition. Let us stress that within the whole Chapters 6 and 7, we will only consider quantum groups of Kac type (see Sect. 2.3.3). In particular, a compact matrix quantum group G with $N\times N$ fundamental representation will be called **orthogonal** if $G\subseteq O_N^+$. In addition, we say that G is **homogeneous** if $S_N\subseteq G$.

6.1.5 Proposition. For every linear category of partitions $\mathscr{K}\subseteq\mathsf{Part}_N$ there exists a unique orthogonal compact matrix quantum group $G=(C(G),u)$ such that

$$\mathsf{FundRep}_G(k,l)=\mathrm{Mor}(u^{\otimes k},u^{\otimes l})=\{T_p\mid p\in\mathscr{K}(k,l)\}. \tag{6.2}$$

Such a quantum group G is automatically homogeneous. Conversely, for every orthogonal homogeneous compact matrix quantum group G there exists a (*not* unique) linear category of partitions $\mathscr{K}\subseteq\mathsf{Part}_N$ such that (6.2) holds.

Proof. The first part follows from the non-coloured version of Tannaka–Krein duality (Corollary 3.4.11) applied to the image of $\mathscr{K}$ by the functor $p\mapsto T_p$. The associated quantum group is indeed orthogonal in the sense of the above mentioned definition as Tannaka–Krein gives us $G\subseteq O^+(F)$ with $F_{ji}=[\xi_{\sqcap}]_{ij}=[T_{\sqcap}]_{ij}=\delta_{ij}$.

For the converse, take a compact quantum group G with $S_N\subseteq G\subseteq O_N^+$. According to Proposition 3.4.15, we have $\mathsf{FundRep}_{S_N}\supseteq\mathsf{FundRep}_G\supseteq\mathsf{FundRep}_{O_N^+}$. Hence, we can take an inverse image of those inclusions under the functor $p\mapsto T_p$ and obtain a collection of spaces $\mathscr{K}(k,l)$ such that $\mathsf{Part}_N\supseteq\mathscr{K}$. $\mathscr{K}$ must be closed under the category operations since it is a preimage of a category. It also certainly contains $\sqcap$ and $|$ since $T_{\sqcap},T_|\in\mathsf{FundRep}_{O_N^+}\subseteq\mathsf{FundRep}_G$. Hence, $\mathscr{K}$ is a linear category of partitions. □

6.1.6 Remark. The reason why the linear category of partitions associated to a given quantum group is not unique is that the functor $p\mapsto T_p$ is not injective. For the sake of uniqueness, we could say that we will always work with the *maximal* category, that is, with the preimage of the functor as we did in the proof of Proposition 6.1.5. However, this would be very inconvenient since for many quantum groups there will be much better choice such as some easy category.

Compact matrix quantum groups corresponding to an easy category of partitions are called **orthogonal easy quantum groups**. In other words, G is an orthogonal easy quantum group if there exists a category of partitions $\mathscr{C}\subseteq\mathscr{P}$ such that $\mathrm{Mor}(u^{\otimes k},u^{\otimes l})=\operatorname{span}\mathscr{C}(k,l)$. This is the original definition of easy quantum groups from [BS09]. Generalizations of partitions using colours allow to use an *easy* description also for larger classes of quantum groups. We will use the term *easy quantum group* without the adjective *orthogonal* for any quantum group, whose representation category can be described by some category of partitions without the need of using linear combinations of partitions.

6.1.7 Proposition. Let $\mathscr{K}$ be a linear category of partitions generated by a set K. Then the associated quantum group is given by $G=(C(G),u)$, where

$$C(G)=C^*(u_{ij}\mid u_{ij}=u_{ij}^*,\ u\text{ orthogonal},\ T_pu^{\otimes k}=u^{\otimes l}T_p\ \forall p\in K),$$

where the numbers k and l always denote the number of upper and lower points of p.

Proof. Follows from Corollary 6.1.3 and Proposition 3.4.14. □

So, to summarize, we have a functor $p \mapsto T_p$. The linear map T_p can be interpreted as an intertwiner in a quantum group, where it also plays the role of an algebraic relation (as in Table 6.1). A category of partitions then defines a unique quantum group thanks to the Tannaka–Krein duality using precisely those relations. Again, in a more concise way:

lin. comb. of partitions	$p \in \mathsf{Part}_N(k,l)$
$\mapsto$ intertwiner	$T_p \colon (\mathbb{C}^N)^{\otimes k} \to (\mathbb{C}^N)^{\otimes l}$
$\mapsto$ relation	$T_p u^{\otimes k} = u^{\otimes l} T_p$
lin. category of partitions	$\mathscr{K} \subseteq \mathsf{Part}_N(k,l)$
$\mapsto$ representation category	$\mathsf{FundRep}_G(k,l) = \{T_p \mid p \in \mathscr{K}(k,l)\}$
$\mapsto$ quantum group	$G = (C(G),u);\ C(G) = C^*(u_{ij} \mid T_p u^{\otimes k} = u^{\otimes l} T_p;\ p \in \mathscr{K}(k,l))$

6.1.3 Examples

In Table 6.1, we list relations implied by certain important partitions. Then, in Table 6.2, we list all possible categories of partitions that can be generated from those partitions and name the corresponding quantum groups. One should consider this as the *definition* of the corresponding quantum groups (at least in the cases that were not defined before). To give an idea of the meaning of those quantum groups, we describe their classical counterparts. Recall Remark 3.4.8 stating that the category defines the compact matrix quantum group including its fundamental representation, so including the matrix structure, not only as an abstract object. Hence, we must determine also every group of Table 6.2 as a *matrix* group.

The orthogonal group O_N is simply the group of all orthogonal matrices. The symmetric group S_N will always be represented by permutation matrices. By $S_N' = S_N \tilde{\times} \mathbb{Z}_2$, we denote the **modified symmetric group** consisting of permutation matrices with a global sign (recall Example 2.5.8). In contrast, H_N is the **hyperoctahedral group** represented by *signed permutation matrices*, that is, every entry has an independent sign. We can write this group also in terms of a wreath product as $H_N = \mathbb{Z}_2 \wr S_N$. By B_N we denote the **bistochastic group** consisting of *bistochastic matrices*, that is, orthogonal matrices whose rows and columns sum up to one. The **modified bistochastic group** $B_N' = B_N \tilde{\times} \mathbb{Z}_2$ has again an additional global sign.

In the middle column of Table 6.2, we have the *free quantum* counterparts of the groups on the left hand side. Those correspond to the non-crossing categories. Recall that we have one non-crossing category more in comparison with the group categories. Consequently, we have two different *modifications* of the free bistochastic quantum group. In the last column, the *half-liberated* quantum groups are listed. Those are characterized by the *half-liberated commutativity* of the form $abc = cba$ for $a,b,c \in \{u_{ij}\}$ (cf. Table 6.1).

The quantum groups corresponding to six of the non-crossing categories were studied already in [BS09]. The missing category was discovered in [Web13]. In [Rau12, Web13], the modifications of the free symmetric and bistochastic quantum groups were interpreted. (Those $\mathbb{Z}_2$ factors are actually an interesting phenomenon that will be studied also in this thesis – in

¦ $\sim \uparrow\otimes\uparrow \sim \downarrow\otimes\downarrow$	$s := \sum_k u_{ik} = \sum_k u_{kj}\ \forall i,j;\ s^2 = 1$
╲ (implies $\uparrow\otimes\uparrow$)	$u_{ij}s = su_{ij}\ \forall i,j$
$\uparrow \sim \downarrow$ (implies $\uparrow\otimes\uparrow$)	$\sum_k u_{ik} = 1 = \sum_k u_{kj} (= s)\ \forall i,j$
⊓ $\sim$ ⊔	$uu^t = 1_N = u^t u$ (assumed by default)
⌶	$u_{ik}u_{jk} = 0 = u_{ki}u_{kj}\ \forall i,j,k,\ i \neq j$
╳	$u_{ij}u_{kl} = u_{kl}u_{ij}\ \forall i,j,k,l$
⚹	$u_{ij}u_{kl}u_{mn} = u_{mn}u_{kl}u_{ij}\ \forall i,j,k,l,m,n$

Table 6.1 Relations corresponding to certain partitions

Groups		Free QGs		Half-liberated QGs	
$\langle \times \rangle_N$	O_N	$\langle\rangle_N$	O_N^+	$\langle ⋇ \rangle_N$	O_N^*
$\langle \sqcap\!\sqcap\!\sqcap, \times \rangle_N$	H_N	$\langle \sqcap\!\sqcap\!\sqcap \rangle_N$	H_N^+	$\langle \sqcap\!\sqcap\!\sqcap, ⋇ \rangle_N$	H_N^*
$\langle \uparrow\otimes\uparrow, \times \rangle_N$		$\langle \uparrow\otimes\uparrow \rangle_N$	$B_N^{\#+} \simeq B_N * \hat{\mathbb{Z}}_2$		
$= \langle \backslash, \times \rangle_N$	$B_N' = B_N \tilde{\times} \hat{\mathbb{Z}}_2$	$\langle \backslash \rangle_N$	$B_N'^+ = B_N \tilde{\times} \hat{\mathbb{Z}}_2$		
$\langle \uparrow, \times \rangle_N$	$B_N \simeq O_{N-1}$	$\langle \uparrow \rangle_N$	$B_N^+ \simeq O_{N-1}^+$		
$\langle \sqcap\!\sqcap\!\sqcap, \uparrow\otimes\uparrow, \times \rangle_N$	$S_N' = S_N \tilde{\times} \hat{\mathbb{Z}}_2$	$\langle \sqcap\!\sqcap\!\sqcap, \uparrow\otimes\uparrow \rangle_N$	$S_N'^+ = S_N \tilde{\times} \hat{\mathbb{Z}}_2$		
$\langle \sqcap\!\sqcap\!\sqcap, \uparrow, \times \rangle_N$	S_N	$\langle \sqcap\!\sqcap\!\sqcap, \uparrow \rangle_N$	S_N^+		

Table 6.2 Quantum groups corresponding to certain easy categories

Section 7.1 using linear categories of ordinary partitions and in Section 6.4 using partitions with extra singletons that were introduced precisely for this purpose.) The half-liberated quantum groups were studied in [BCS10].

6.2 Two-coloured case and unitary CMQGs

Two-coloured partitions are used to describe a larger class of quantum groups – instead of restricting to the orthogonal ones $G \subseteq O_N^+$, we are able to describe unitary quantum groups $G \subseteq U_N^+$. In the first two subsections of this section, we briefly present this correspondence. However, the main point of this section is to present the results of [Gro18] interpreting the globally-colourized categories of two-coloured partitions, which were classified in Section 4.5.

For an interpretation of categories of two-coloured partitions, we use the same map $p \mapsto T_p$. What changes is that we have different sets of objects. Categories of two-coloured partitions with $\mathscr{O}_{\circ\bullet}^*$ being the monoid of objects are mapped to two-coloured representation categories with the same monoid of objects. This makes the assignment $p \mapsto T_p$ again a functor. Actually, in the spirit of Section 3.4, using two-coloured partitions may seem more natural than non-coloured partitions.

6.2.1 Tannaka–Krein with two colours

Recall again that we focus on Kac type quantum groups in this chapter. In particular, we say that G is a **unitary** quantum group if $G \subseteq U_N^+$.

In Chapter 5, we added the linear structure to the category of all partitions and defined the category Part_N. Similarly, we can add the linear structure to the category of all two-coloured partitions and define $\mathsf{Part}_N^{\circ\bullet}$. Then, we can formulate the following proposition.

6.2.1 Proposition. For every linear category of two-coloured partitions $\mathscr{K} \subseteq \mathsf{Part}_N^{\circ\bullet}$, there exists a unique compact matrix quantum group $G = (C(G), u) \subseteq U_N^+$ such that

$$\mathsf{FundRep}_G(w_1, w_2) = \mathrm{Mor}(u^{\otimes w_1}, u^{\otimes w_2}) = \{T_p \mid p \in \mathscr{K}(w_1, w_2)\}. \tag{6.3}$$

Such a quantum group G is automatically homogeneous. Conversely, for every unitary homogeneous compact matrix quantum group G, there exists a (*not* unique) linear category of two-coloured partitions $\mathscr{K} \subseteq \mathsf{Part}_N^{\circ\bullet}$ such that (6.3) holds.

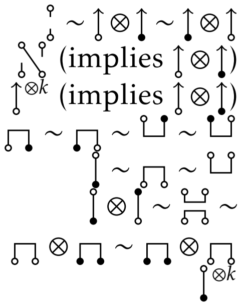
Proof. Same as in the case of Proposition 6.1.5. Again, follows from the Tannaka–Krein duality (Theorem 3.4.6). □

Since we are going to study only the *easy* case for two-coloured partitions, let us reformulate the last proposition for ordinary categories.

6.2.2 Proposition. For every category of two-coloured partitions $\mathscr{C} \subseteq \mathscr{P}^{\circ\bullet}$ and for every $N \in \mathbb{N}$, there exists a unique compact matrix quantum group $G = (C(G), u) \subseteq U_N^+$ such that

$$\mathsf{FundRep}_G(w_1, w_2) = \mathrm{Mor}(u^{\otimes w_1}, u^{\otimes w_2}) = \mathrm{span}\{T_p \mid p \in \mathscr{C}(w_1, w_2)\}. \tag{6.4}$$

Proof. Follows directly from Proposition 6.2.1 if we set $\mathscr{K}(k,l) := \mathrm{span}\,\mathscr{C}(k,l)$. □



$z := \sum_k u_{ik} = \sum_k u_{kj}\ \forall i,j;\ zz^* = z^*z = 1$

$u_{ij}z = zu_{ij}\ \forall i,j$

$z^k = 1\ \forall i,j$

$uu^* = \bar{u}u^t = u^t\bar{u} = u^*u = 1_N$ (assumed by default)

$u = \bar{u} \Leftrightarrow uu^t = u^tu = 1_N$

$u_{ij}u_{kl}^* = u_{ij}^*u_{kl}\ \forall i,j,k,l \Leftrightarrow w := \sum_k u_{ik}^2 = \sum_k u_{kj}^2\ \forall i,j;$

$ww^* = w^*w = 1$

$u_{i_1j_1}\cdots u_{i_kj_k} = u_{i_1j_1}^*\cdots u_{i_kj_k}^*\ \forall i_1,\ldots,i_k,j_1,\ldots,j_k$

Table 6.3 Relations corresponding to certain two-coloured partitions

Any quantum group G associated to a two-coloured category of partitions $\mathscr{C} \subseteq \mathscr{P}^{\circ\bullet}$ is called a **unitary easy quantum group** [TW17].

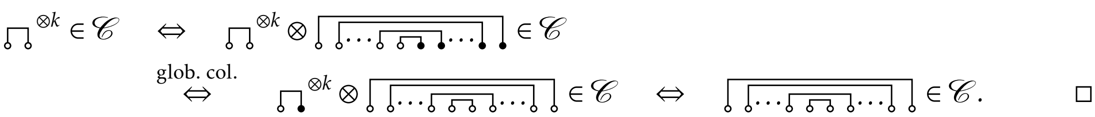
6.2.2 Relations associated to two-coloured partitions

Given a two-coloured partition $p \in \mathscr{P}^{\circ\bullet}(w_1,w_2)$, we assign to it the same intertwiner T_p considered as a linear map $(\mathbb{C}^N)^{\otimes k} \to (\mathbb{C}^N)^{\otimes l}$ independently of w_1,w_2; however, its interpretation differs depending on the colour patterns w_1,w_2. Consequently, also the relation induced by the partition $T_pu^{\otimes w_1} = u^{\otimes w_2}T_p$ depends on the colour pattern. In Table 6.3, we list the relations corresponding to the most important two-coloured partitions.

Since we will be interested in interpreting the classification of globally colourized categories of partitions in the following text, we should list in Table 6.3 the relations corresponding to the generators $^{\otimes k}$ and $^{\otimes k}$ (recall Theorem 4.5.10). However, instead of the latter partition, we decided to include $^{\otimes k}$ as the corresponding relation is much simpler. The following lemma says that the relations are equivalent.

6.2.3 Lemma. Let $\mathscr{C} \subseteq \mathscr{P}^{\circ\bullet}(k,l)$ be a category of two-coloured partitions. Then $^{\otimes k} \in \mathscr{C}$ if and only if $^{\otimes k} \in \mathscr{C}$.

Proof. First of all, note that $^{\otimes k}$ is a rotated version of nested pairings. Now, one can easily check by induction that it is contained in any category (it is, after all, the duality morphism corresponding to $\circ^k$). Obviously, also $^{\otimes k}$ is in any category. Finally note that both $^{\otimes k}$ and $^{\otimes k}$ imply global colourization (is obtained by contraction from $^{\otimes k} \otimes {}^{\otimes k}$; similarly, is obtained from $^{\otimes k} \otimes {}^{\otimes k}$). Hence, we have the following series of equivalences



6.2.3 Interpretation of globally-colourized categories

In this section, we present the results of [Gro18] supplementing the classification of globally-colourized categories of partitions with the quantum group interpretation.

The following theorem was actually already implicitly contained in [TW17] for the case of non-crossing and group categories. In [Gro18], we applied the same technique of proof on all of the globally colourized categories.

Recall the definition of the glued tensor product and tensor complexification from Section 2.5.3.

6.2.4 Theorem. Let $\mathscr{C}$ be a globally colourized category of partitions, denote $k := k(\mathscr{C})$. Denote by $H \subseteq O_N^+$ the quantum group corresponding to the non-coloured category of partitions

$\langle \mathscr{C}, \sqcap_{\circ\circ} \rangle$. Then $\mathscr{C}$ corresponds to the quantum group $H \tilde{\times} \hat{\mathbb{Z}}_k \subseteq U_N^+$. In the case $k=0$, we replace $\hat{\mathbb{Z}}_k$ by $\hat{\mathbb{Z}}$.

Proof. We will divide the proof into two cases: Case (a) $\sqcap_{\circ\circ\bullet}\,\uparrow_{\bullet} \in \mathscr{C}$ and case (b) $\sqcap_{\circ\circ\bullet}\,\uparrow_{\bullet} \notin \mathscr{C}$. According to Lemma 4.5.8, we have $\mathscr{C} = \langle \mathscr{C}_0, \uparrow_{\circ}^{\otimes k} \rangle$ in case (a) and $\mathscr{C} = \langle \mathscr{C}_0, \sqcap_{\circ\circ}^{\otimes k/2} \rangle$ in case (b). Since large parts of the proof will be identical for both cases, let us denote $\mathscr{C} = \langle \mathscr{C}_0, q_k \rangle$, where q_k is either $\uparrow_{\circ}^{\otimes k}$ or $\sqcap_{\circ\circ}^{\otimes k/2}$. Denote by G the quantum group associated to the category $\mathscr{C}$.

Denote by R_{q_k} the relation corresponding to q_k and by $\mathscr{R}(u)$ all the relations associated to the category $\mathscr{C}_0$. Note that the unicoloured pair $\sqcap_{\circ\circ} = u_1$ corresponds to the relation making the matrix u orthogonal. So, we have

$$\begin{aligned}
C(G) &= C^*(u_{ij} \mid \mathscr{R}(u),\ R_{q_k}(u),\ u \text{ and } u^t \text{ unitary}),\\
C(H) &= C^*(v_{ij} \mid \mathscr{R}(v),\ R_{q_k}(v),\ v = \bar{v},\ vv^t = v^t v = 1_N),\\
C(H) \otimes_{\max} C^*(\mathbb{Z}_k) &= C^*\left(v_{ij}, z \;\middle|\; \begin{matrix} \mathscr{R}(v),\ R_{q_k}(v),\ v = \bar{v},\ vv^t = v^t v = 1_N, \\ v_{ij} z = z v_{ij},\ zz^* = z^* z = 1,\ z^k = 1 \end{matrix} \right),\\
C(H \tilde{\times} \hat{\mathbb{Z}}_k) &\subseteq C(H) \otimes_{\max} C^*(\mathbb{Z}_k) \quad \text{generated by } u'_{ij} := v_{ij} z.
\end{aligned}$$

To show that G is *identical* with $H \tilde{\times} \hat{\mathbb{Z}}_k$, we have to prove that there exists a $*$-isomorphism $\alpha\colon C(G) \to C(H \tilde{\times} \hat{\mathbb{Z}}_k)$ mapping $u_{ij} \mapsto u'_{ij}$. We will show it in two steps. First, we show that this assignment extends to a surjective $*$-homomorphism α. Secondly, we find a $*$-homomorphism $\beta\colon C(H) \otimes_{\max} C^*(\mathbb{Z}_k) \to M_t(C(G))$ such that $\beta \circ \alpha = \iota$, where $t = 1$ in case (a) and $t = 2$ in case (b) and $\iota\colon C(G) \to M_t(C(G))$ is the embedding $x \mapsto x \cdot 1_t$. This will prove the injectivity of α.

$$\begin{array}{ccc}
M_t(C(G)) & \xleftarrow{\ \beta\ } & C(H) \otimes_{\max} C^*(\mathbb{Z}_k) \\
\uparrow{\scriptstyle \iota} & & \cup\!\vert \\
C(G) & \xrightarrow[\ \alpha\ ]{} & C(H \tilde{\times} \hat{\mathbb{Z}}_k)
\end{array}$$

Step 1. There is a surjective $*$-homomorphism $\alpha\colon C(G) \to C(H \tilde{\times} \hat{\mathbb{Z}}_k)$ mapping $u_{ij} \mapsto u'_{ij}$.

We show that the elements $u'_{ij} \in C(H \tilde{\times} \hat{\mathbb{Z}}_k) \subseteq C(H) \otimes_{\max} C^*(\mathbb{Z}_k)$ satisfy the relations of u_{ij} in $C(G)$.

Indeed, the unitarity is clear since $u'u'^* = vzz^*v^* = vv^* = 1$ and similarly for u^*u and for u^t.

Moreover, all elements $p \in \mathscr{C}_0$ satisfy $c(p) = 0$, so they have the same amount of white and black points. Thus, in the relations $\mathscr{R}(u)$, there is the same amount of conjugated u_{ij}^* as not conjugated u_{ij}. If we put vz instead of u, there is the same amount of z and z^* in every relation. So, since z and z^* commute with everything, they all cancel out. Therefore, the relations $\mathscr{R}(vz) = \mathscr{R}(v)$ are also satisfied in $C(H) \otimes_{\max} C^*(\mathbb{Z}_k)$.

Similarly for the case R_{q_k}. We know that $c(q_k) = k$ and since $z^k = 1$, it follows that $R_{q_k}(vz) = R_{q_k}(v)$, which is satisfied in $C(H) \otimes_{\max} C^*(\mathbb{Z}_k)$.

Finally, from the universal property of $C(G)$, we know that there is a $*$-homomorphism $\alpha\colon C(G) \to C(H \tilde{\times} \hat{\mathbb{Z}}_k)$ mapping $u_{ij} \mapsto u'_{ij}$. Since $C(H \tilde{\times} \hat{\mathbb{Z}}_k)$ is generated by u'_{ij}, the homomorphism must be surjective.

Step 2a. Suppose $\sqcap_{\circ\circ\bullet}\,\uparrow_{\bullet} \in \mathscr{C}$. Define in $C(G)$

$$z' := \sum_l u_{il} = \sum_l u_{lj}, \qquad v'_{ij} := u_{ij} z'^*.$$

There is a $*$-homomorphism $\beta\colon C(H) \otimes_{\max} C^*(\mathbb{Z}_k) \to C(G)$ mapping $v_{ij} \mapsto v'_{ij}$, $z \mapsto z'$ satisfying $\beta \circ \alpha = \mathrm{id}$.

We will again show that v'_{ij} and z' in $C(G)$ satisfy the relations of v_{ij} and z in $C(H) \otimes_{\max} C^*(\mathbb{Z}_k)$. Then the existence of β will follow from the universal property of $C(H) \otimes_{\max} C^*(\mathbb{Z}_k)$.

First of all note that $\sqcap\!\sqcap\,\downarrow \in \mathscr{C}$ implies that $\uparrow \otimes \uparrow \in \mathscr{C}$, so we indeed have that $\sum_l u_{il} = \sum_l u_{lj}$ for any i,j and that z is unitary. Since we are in case (a), we have $\uparrow^{\otimes k} \in \mathscr{C}$, which corresponds to the relation $z'^k = 1$ (see Table 6.3).

The relation for $\sqcap\!\sqcap\,\downarrow \sim \diagdown$ means that $u_{ij} z' = z' u_{ij}$, which implies that

$$v'_{ij} z' = u_{ij} z'^* z' = u_{ij} = z' u_{ij} z'^* = z' v'_{ij}.$$

The relation for $\sqcap \otimes \sqcap \sim \uparrow \otimes \uparrow$ implies that

$$u_{ij} z'^* = \sum_l u_{ij} u_{kl}^* = \sum_l u_{ij}^* u_{kl} = u_{ij}^* z',$$

from which we deduce that

$$v'_{ij} = u_{ij} z'^* = u_{ij}^* z' = z' u_{ij}^* = v'^*_{ij}.$$

The orthogonality of v' follows simply from the equality $v'v'^* = uu^*$ and $v'^*v' = u^*u$, which follows from the fact that u commutes with z'. Similarly, all relations $\mathscr{R}(v')$ are satisfied since they are equivalent to $\mathscr{R}(u)$ thanks to the fact that u commutes with z' and hence all occurrences of z' cancel with z'^*. Finally, if $k \neq 0$, we can use the relation $z'^k = 1$ to show that the relation $R_{\uparrow^{\otimes k}}(v')$ is equivalent to $R_{\uparrow^{\otimes k}}(u)$.

Step 2b. Suppose $\sqcap\!\sqcap\,\downarrow \notin \mathscr{C}$. Denote $w := \sum_l u_{il}^2 = \sum_l u_{lj}^2 \in C(G)$. Define in $M_2(C(G))$

$$z' := \begin{pmatrix} 0 & w \\ 1 & 0 \end{pmatrix}, \quad v'_{ij} := \begin{pmatrix} 0 & u_{ij} \\ u_{ij}^* & 0 \end{pmatrix}.$$

There is a $*$-homomorphism $\beta\colon C(H) \otimes_{\max} C^*(\mathbb{Z}_k) \to M_2(C(G))$ mapping $v_{ij} \mapsto v'_{ij}$, $z \mapsto z'$ satisfying $\beta \circ \alpha = \iota$.

Again, we will prove that v'_{ij} and z' satisfy the appropriate relations and the statement will follow from the universal property of the algebra $C(H) \otimes_{\max} C^*(\mathbb{Z}_k)$.

The fact that $\sum_l u_{il}^2 = \sum_l u_{lj}^2$ for every i,j corresponds to the relation of $\sqcup\!\sqcap \sim \sqcap \otimes \sqcap$. We also have that $ww^* = w^*w = 1$ (again, see Table 6.3). Finally,

$$u_{ij}^* w = \sum_l u_{ij}^* u_{il}^2 = \sum_l u_{ij} u_{il}^* u_{il} = u_{ij}$$

and similarly $wu_{ij}^* = u_{ij}$. And using the relation for $\uparrow^{\otimes k/2} \sim \sqcap^{\otimes k/2}$ we prove that

$$w^{k/2} = \sum_{l_1} u_{il_1} u_{il_1} \cdots \sum_{l_{k/2}} u_{il_{k/2}} u_{il_{k/2}} = \sum_{l_1} u_{il_1} u_{il_1}^* \cdots \sum_{l_{k/2}} u_{il_{k/2}} u_{il_{k/2}}^* = 1.$$

One can now easily check the validity of all the relations. In particular, we have that

$$v'_{ij} z' = z' v'_{ij} = u_{ij} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

so $v'_{ij} = u_{ij} z'^* 1_2$. Therefore, for the relations $\mathscr{R}(v)$ we can apply the same argument as in the case (a). □

In the last theorem, we constructed precisely all quantum groups of the form $H \tilde{\times} \hat{\mathbb{Z}}_k$, where either

(1) $H \subseteq O_N^+$ corresponds to a category $\tilde{\mathscr{C}}$ such that $\uparrow \notin \tilde{\mathscr{C}}$ and k is even or
(2) $H \subseteq O_N^+$ corresponds to a category $\tilde{\mathscr{C}}$ such that $\uparrow \in \tilde{\mathscr{C}}$ and k is odd.

Indeed, consider a globally colourized category $\mathscr{C}$. If $k := k(\mathscr{C})$ is odd, then $\mathscr{C}$ must contain a partition of odd length. According to Lemmata 4.5.7 and 4.5.8, we have that $\uparrow_\circ^{\otimes k} \in \mathscr{C}$ and hence $\uparrow_\circ \in \bar{\mathscr{C}} := \langle \mathscr{C}, \sqcap_{\circ\circ} \rangle$. On the other hand, if k is even, then $\mathscr{C}$ does not contain any partition of odd length and hence $\uparrow_\circ \notin \bar{\mathscr{C}}$.

Conversely, let $\tilde{\mathscr{C}}$ be any category of non-coloured partitions corresponding to an orthogonal quantum group $H \subseteq O_n^+$ and denote $\bar{\mathscr{C}} := \Psi^{-1}(\tilde{\mathscr{C}})$ the corresponding category in terms of two-coloured partitions. If $\uparrow \in \tilde{\mathscr{C}}$, then we can construct $\mathscr{C} := \langle \bar{\mathscr{C}}_0, \uparrow_\circ^{\otimes k} \rangle$ for k odd and prove by the same arguments as in Lemma 4.5.5 that $\bar{\mathscr{C}} = \langle \mathscr{C}, \sqcap_{\circ\circ} \rangle$. Thus, Theorem 6.2.4 implies that $H \tilde{\times} \hat{\mathbb{Z}}_k$ is a unitary easy quantum group corresponding to the category $\mathscr{C}$. If $\uparrow \notin \tilde{\mathscr{C}}$, then we can similarly construct $\mathscr{C} := \langle \bar{\mathscr{C}}_0, \sqcap_{\circ\circ}^{\otimes k/2} \rangle$ for k even and use Theorem 6.2.4 to see that $H \tilde{\times} \hat{\mathbb{Z}}_k$ is the quantum group corresponding to $\mathscr{C}$.

Now, we would like to characterize the product $H \tilde{\times} \mathbb{Z}_k$ for an arbitrary orthogonal easy quantum group H and an arbitrary number $k \in \mathbb{N}_0$.

There are only four non-coloured categories of partitions containing the singleton. Namely the following:

$\langle \uparrow, \sqcap\sqcap\sqcap \rangle$	quantum group S_N^+,
$\langle \uparrow \rangle$	quantum group B_N^+,
$\langle \uparrow, \sqcap\sqcap\sqcap, \times \rangle$	(quantum) group S_N,
$\langle \uparrow, \times \rangle$	(quantum) group B_N.

Note that they can be formed by adding the singleton to the following categories:

$\langle \uparrow \otimes \uparrow, \sqcap\sqcap\sqcap \rangle$	quantum group $S_N'^+ = S_N^+ \tilde{\times} \hat{\mathbb{Z}}_2$,
$\langle \sqcap\!\!\sqcap \rangle$	quantum group $B_N'^+ = B_N^+ \tilde{\times} \hat{\mathbb{Z}}_2$,
$\langle \uparrow \otimes \uparrow, \sqcap\sqcap\sqcap, \times \rangle$	(quantum) group $S_N' = S_N \tilde{\times} \hat{\mathbb{Z}}_2$,
$\langle \uparrow \otimes \uparrow, \times \rangle$	(quantum) group $B_N' = B_N \tilde{\times} \hat{\mathbb{Z}}_2$.

Their k-tensor complexifications for k even are described by the following proposition.

6.2.5 Proposition. For any $k \in 2\mathbb{N}_0$ we have the following

$$\begin{aligned} S_N^+ \tilde{\times} \hat{\mathbb{Z}}_k &= (S_N^+ \tilde{\times} \hat{\mathbb{Z}}_2) \tilde{\times} \hat{\mathbb{Z}}_k && (\text{category } \mathcal{S}_{\mathrm{glob}}(k)),\\ B_N^+ \tilde{\times} \hat{\mathbb{Z}}_k &= (B_N^+ \tilde{\times} \mathbb{Z}_2) \tilde{\times} \hat{\mathbb{Z}}_k && (\text{category } \mathcal{B}'_{\mathrm{glob}}(k)),\\ S_N \tilde{\times} \hat{\mathbb{Z}}_k &= (S_N \tilde{\times} \hat{\mathbb{Z}}_2) \tilde{\times} \hat{\mathbb{Z}}_k && (\text{category } \mathcal{S}_{\mathrm{grp,glob}}(k)),\\ B_N \tilde{\times} \hat{\mathbb{Z}}_k &= (B_N \tilde{\times} \hat{\mathbb{Z}}_2) \tilde{\times} \hat{\mathbb{Z}}_k && (\text{category } \mathcal{B}_{\mathrm{grp,glob}}(k)). \end{aligned}$$

Proof. From Theorem 6.2.4, it follows that, for each row, the quantum group on the right hand side corresponds to the category in parentheses. Repeating the proof of Theorem 6.2.4 for the quantum group on the left hand side proves the equality. In particular, note that all the categories are in case (a) and actually the only thing we have to check is that $v'_{ij} \in C(G)$ satisfies the relation corresponding to the singleton $\sum_l v'_{il} = \sum_l v'_{lj} = 1$ for any i, j. This follows from the fact that

$$\sum_l v'_{il} = \sum_l u_{il} z'^* = z' z'^* = 1$$

and similarly for the other relation. □

6.2.6 Proposition. Let H be an orthogonal easy quantum group corresponding to a category $\mathscr{C}$ such that $\uparrow \notin \mathscr{C}$ and let k be odd. Then $H \mathbin{\tilde\times} \mathbb{Z}_k = H \mathbin{\tilde\times} \mathbb{Z}_{2k}$.

Proof. Denote the fundamental representation of H by v, the generator of $C^*(\mathbb{Z}_{2k})$ by z, and the generator of $C^*(\mathbb{Z}_k)$ by z'. Then we can denote the fundamental representation of $C(H \mathbin{\tilde\times} \hat{\mathbb{Z}}_{2k})$ by $u_{ij} := v_{ij}z$ and the fundamental representation of $C(H \mathbin{\tilde\times} \hat{\mathbb{Z}}_k)$ by $u'_{ij} := v_{ij}z'$. We have to find an isomorphism $C(H \mathbin{\tilde\times} \hat{\mathbb{Z}}_{2k}) \to C(H \mathbin{\tilde\times} \hat{\mathbb{Z}}_k)$ mapping $u_{ij} \mapsto u'_{ij}$.

Let us define a homomorphism $\alpha\colon C(H) \otimes C^*(\mathbb{Z}_{2k}) \to C(H) \otimes C^*(\mathbb{Z}_k)$ by $\alpha(v_{ij}) = v_{ij}$ and $\alpha(z) = z'$. The existence of such a homomorphism follows from the universal property of $C(H) \otimes C^*(\mathbb{Z}_{2k})$ since we have $z'^{2k} = 1$.

We would like to define a homomorphism $\beta\colon C(H) \otimes C^*(\mathbb{Z}_k) \to C(H) \otimes C^*(\mathbb{Z}_{2k})$ by $\beta(v_{ij}) = v_{ij}z^{*k}$ and $\beta(z') = z^{k+1}$. Obviously, $\beta(z')$ satisfies the relation of z' since $\beta(z')^k = z^{(k+1)k} = 1$. Since $\uparrow \notin \mathscr{C}$, we know that $\mathscr{C}$ contains only partitions with even length, so all the relations of $C(H)$ contain monomials in v_{ij} of even length. Thus, $\beta(v_{ij})$ satisfy the relations of v_{ij} since all the z^{*k} cancel out. Therefore, such a homomorphism β exists from a universal property of $C(H) \otimes C^*(\mathbb{Z}_k)$.

Now since we have

$$\alpha(u_{ij}) = \alpha(v_{ij}z) = v_{ij}z' = u'_{ij},$$
$$\beta(u'_{ij}) = \beta(v_{ij}z') = v_{ij}z^{*k}z^{k+1} = v_{ij}z = u_{ij},$$

it follows that α restricted to $C(H \mathbin{\tilde\times} \mathbb{Z}_{2k})$ is a surjective homomorphism onto $C(H \mathbin{\tilde\times} \mathbb{Z}_k)$ mapping $u_{ij} \mapsto u'_{ij}$, it has an inverse provided by the map β, and hence it is the desired isomorphism. □

6.2.7 Corollary. Any quantum group of the form $H \mathbin{\tilde\times} \hat{\mathbb{Z}}_k$, where H is an orthogonal easy quantum group and $k \in \mathbb{N}_0$, is a unitary easy quantum group corresponding to a globally colourized category.

Proof. Denote $\bar{\mathscr{C}}$ the category corresponding to the orthogonal quantum group H in terms of two-coloured partitions. If $\uparrow \notin \bar{\mathscr{C}}$ and k is even, then, as we already mentioned, $H \mathbin{\tilde\times} \hat{\mathbb{Z}}_k$ is a unitary easy quantum group corresponding to the category $\langle \bar{\mathscr{C}}_0, \sqcap^{\otimes k/2} \rangle$. If k is odd, then, according to Proposition 6.2.6, $H \mathbin{\tilde\times} \hat{\mathbb{Z}}_k = H \mathbin{\tilde\times} \hat{\mathbb{Z}}_{2k}$, so it reduces to the previous case and $H \mathbin{\tilde\times} \hat{\mathbb{Z}}_k$ corresponds to the category $\langle \bar{\mathscr{C}}_0, u_{2k} \rangle$.

If $\uparrow \in \bar{\mathscr{C}}$ and k is odd, then again we already mentioned that $H \mathbin{\tilde\times} \hat{\mathbb{Z}}_k$ is a unitary easy quantum group corresponding to the category $\langle \bar{\mathscr{C}}_0, \uparrow^{\otimes k} \rangle$. If k is even, then, according to Proposition 6.2.5, $H \mathbin{\tilde\times} \hat{\mathbb{Z}}_k = (H \mathbin{\tilde\times} \hat{\mathbb{Z}}_2) \mathbin{\tilde\times} \hat{\mathbb{Z}}_k$, where $H \mathbin{\tilde\times} \hat{\mathbb{Z}}_2$ is an orthogonal easy quantum group corresponding to a category that does not contain the singleton, so the situation reduces to the first case. In fact, we again have that $H \mathbin{\tilde\times} \hat{\mathbb{Z}}_k$ corresponds to the category $\langle \bar{\mathscr{C}}_0, \uparrow^{\otimes k} \rangle = \langle \bar{\mathscr{C}}_0, \sqcap^{\otimes k/2} \rangle$. □

6.2.4 Interpretation of the alternating colouring

Consider a non-coloured category of partitions $\mathscr{C} \subseteq \mathscr{P}$. In Section 4.6.5, we brought a definition of a two-coloured category $\mathrm{Alt}\,\mathscr{C} \subseteq \mathscr{P}^{\circ\bullet}$ generated by alternating colourized partitions in $\mathscr{C}$. Below, we bring a quantum group interpretation of those categories. We are able to provide a complete proof of the proposition only in Chapter 8. Nevertheless, we are stating it here since it logically belongs to this chapter.

6.2.8 Proposition. Let $\mathscr{C} \subseteq \mathscr{P}$ be a category of partitions and denote by $G \subseteq O_N^+$ the corresponding easy quantum group. Then $G \mathbin{\tilde{*}} \hat{\mathbb{Z}}$ is a unitary easy quantum group corresponding to the category $\mathrm{Alt}\,\mathscr{C}$.

(1) If $\uparrow \notin \mathscr{C}$, then $G \mathbin{\tilde{*}} \hat{\mathbb{Z}}_k = G \mathbin{\tilde{*}} \hat{\mathbb{Z}}$ for all $k \in \mathbb{N}$ and it corresponds to the category $\mathrm{Alt}\,\mathscr{C}$.

(2) If $\uparrow \in \mathscr{C}$, then $G \mathbin{\tilde{*}} \hat{\mathbb{Z}}_k$ corresponds to the category $\langle \mathrm{Alt}\mathscr{C}, \uparrow^{\otimes k}\rangle$.

This proposition will follow as a consequence of Proposition 8.2.31, which is formulated in terms of linear categories of partitions. Note that $\mathscr{C}$ contains some partition of odd length if and only if it contains the singleton $\uparrow$. See also Proposition 6.4.15 providing a proof in a special case (part (1) for $k = 2$).

6.3 Coloured partitions in general

The idea of using coloured partitions for describing the representation categories of quantum groups is the following. Consider a set of colours $\mathscr{O}$. Then, to every $x \in \mathscr{O}$, we associate a representation u^x of some quantum group G. We should also define an involution $x \mapsto \bar{x}$ on $\mathscr{O}$, which translates to taking a dual representation $u^{\bar{x}}$. Then a word $w = x_1 \cdots x_k \in \mathscr{O}^k$ translates to taking the corresponding k-fold tensor product $u^{x_1} \otimes \cdots \otimes u^{x_k}$. Applying then the mapping $p \mapsto T_p$ to some $\mathscr{O}$-coloured category of partitions, we obtain an *$\mathscr{O}$-coloured representation category* (see also Section 8.1.1). Now it may not be clear, how to assign a quantum group to such a structure.

Consider $p \in \mathscr{P}^{\mathscr{O}}$. We say that two colours of $\mathscr{O}$ are **independent** in p if no block of p contains both of these colours.

For simplicity, consider two self-dual colours $\mathscr{O} = \{\circ, \square\}$ and fix two natural numbers $N_\circ, N_\square \in \mathbb{N}$. Then any category $\mathscr{C} \subseteq \mathscr{P}^{\mathscr{O}}$ containing only partitions where the two colours are independent can be assigned a quantum group $G = (C(G), u)$, where $S_{N_\circ} \times S_{N_\square} \subseteq G \subseteq O^+_{N_\circ} \mathbin{\hat{*}} O^+_{N_\square}$.

The colour $\circ$ corresponds to some representation u° and the colour $\square$ corresponds to some representation $u^\square$. Thus, the words w over $\mathscr{O}$ correspond to tensor products $u^{\otimes w}$. A partition $p \in \mathscr{C}(w_1, w_2)$, where $w_1 = a_1 \cdots a_k$ and $w_2 = b_1 \cdots b_l$ then corresponds to a map $T_p : \mathbb{C}^{N_{a_1}} \otimes \cdots \otimes \mathbb{C}^{N_{a_k}} \to \mathbb{C}^{N_{b_1}} \otimes \cdots \otimes \mathbb{C}^{N_{a_l}}$ given again by Equation (6.1) (where the summation for each j_n goes from 1 to N_{a_n}). Then, we can construct a quantum group $G = (C(G), u)$, where $u = u^\circ \oplus u^\square$ and

$$C(G) = C^*(u^\circ_{ij}, u^\square_{ij} \mid u^\circ = \bar{u}^\circ,\ u^\square = \bar{u}^\square,\ T_p u^{\otimes w_1} = u^{\otimes w_2} T_p\ \forall p \in \mathscr{C}(w_1, w_2)).$$

To assure that such a quantum group indeed exists, we again use the Tannaka–Krein theorem for quantum groups. Let us denote $N := N_\circ + N_\square$. Using the canonical projections $\mathbb{C}^N \to \mathbb{C}^{N_\circ}$, $\mathbb{C}^N \to \mathbb{C}^{N_\square}$ and the corresponding embeddings, we can interpret the maps T_p as a mapping $(\mathbb{C}^N)^{\otimes k} \to (\mathbb{C}^N)^{\otimes l}$ and hence as intertwiners for $u = \bar{u}$.

We have $G = O^+_{N_\circ} \mathbin{\hat{*}} O^+_{N_\square}$ for $\mathscr{C}$ consisting of all non-crossing pair partitions with $\circ$ and $\square$ being independent. Likewise, $G = S_{N_\circ} \times S_{N_\square}$ for $\mathscr{C}$ consisting of all partitions with $\circ$ and $\square$ independent.

In the special case $N := N_\circ = N_\square$, we can consider really all partitions over $\mathscr{O}$ without assuming that the colours are independent. This allows us to somehow *amalgamate* the two factors. The resulting quantum group G will then sit between $S_N \subseteq G \subseteq O_N \mathbin{\hat{*}} O_N$, where S_N is taken as a subgroup of $S_N \times S_N$ by identifying the two factors. This is the approach originally formulated by Freslon in [Fre17]. Non-crossing categories on two self-dual colours were classified in [Fre19].

6.4 Extra-singleton case and $\mathbb{Z}_2$-extensions

In this section, we closely follow our article [GW19b]. The goal is to describe quantum groups G, whose fundamental representation u decomposes as a direct sum $v \oplus r$ of an N-dimensional representation $v \in M_N((C(G))$ and a one-dimensional representation $r \in C(G)$. In order to do so, we consider a two-letter alphabet $\mathscr{O}_\triangle = \{\triangle, |\}$, where the triangle $\triangle$ corresponds to the representation r and the line $|$ corresponds to the representation v. Since the representation r

is one-dimensional, one can see that the block structure of the colour $\vartriangle$ is irrelevant, because it does not affect the corresponding map T_p at all. Consequently, the appropriate structure for describing such quantum groups are *categories of partitions with extra singletons*, which we introduced in Section 4.6. The motivation for this work is to answer the following question.

6.4.1 Question. Given a quantum group H, are there any quantum groups G with

$$H \,\hat{*}\, \hat{\mathbb{Z}}_2 \supsetneq G \supsetneq H \times \hat{\mathbb{Z}}_2?$$

6.4.1 The corresponding quantum groups

Let us summarize here the meaning of categories of partitions with extra singletons by applying the general considerations mentioned in Section 6.3. A category of partitions with extra singletons, being a coloured category with independent colours, corresponds to some quantum group G with $S_N \times S_{N'} \subseteq G \subseteq O_N^+ \,\hat{*}\, O_{N'}^+$. As we mentioned above, we will always assume $N' = 1$, so actually we have

$$S_N \times E \subseteq G \subseteq O_N^+ \,\hat{*}\, \hat{\mathbb{Z}}_2,$$

where $E = (\mathbb{C}, 1)$ is the trivial (compact matrix quantum) group. So, G is a matrix quantum group with matrix of size $(N+1)\times(N+1)$ having a block structure with one block of size N and second block of size one. We will usually denote the fundamental representation of G by

$$u = v \oplus r = \begin{pmatrix} v & 0 \\ 0 & r \end{pmatrix}.$$

The colour $\shortmid$ then corresponds to v and the extra singleton $\vartriangle$ correspond to r, that is, we have $u^{\shortmid} = v$ and $u^{\vartriangle} = r$.

As usual, we are going to define the quantum group G associated to a category $\mathscr{C}$ using the relations $T_p u^{\otimes w_1} = u^{\otimes w_2} T_p$. First, we have to describe the T_p maps. In our case, they can be defined as follows. Consider a partition with extra singletons $p \in \mathscr{P}^{\vartriangle}$. Denote by k' resp. l' the number of upper resp. lower points of the colour $\shortmid$ (i.e. not being extra singletons). Then we define $T_p \colon (\mathbb{C}^N)^{\otimes k'} \to (\mathbb{C}^N)^{\otimes l'}$ by Equation (6.1) ignoring all the extra singletons in p. The extra singletons become important when we interpret the partition p as an intertwiner $T_p u^{\otimes w_1} = u^{\otimes w_2} T_p$, where w_1 and w_2 are the upper and the lower colour pattern of p, respectively.

6.4.2 Example. Given a partition ${}_{\vartriangle}\!\diagdown^{\triangledown}$, it is associated a map $T_{{}_{\vartriangle}\!\diagdown^{\triangledown}} \colon \mathbb{C}^N \to \mathbb{C}^N$, which coincides with the map associated to the identity partition $|$. It is the identity $\mathbb{C}^N \to \mathbb{C}^N$. However, the interpretation of those partitions are different. While the identity partition gives us just the trivial relation

$$v = vT_| = T_| v = v,$$

the relation associated to the partition ${}_{\vartriangle}\!\diagdown^{\triangledown}$ reads

$$vr = v \otimes r = T_{{}_{\vartriangle}\!\diagdown^{\triangledown}}(v \otimes r) = (r \otimes v) T_{{}_{\vartriangle}\!\diagdown^{\triangledown}} = r \otimes v = rv.$$

See Example 6.4.9 for another instance of a T_p map and a relation associated to a partition with extra singletons.

Now a quantum group $G = (C(G), v \oplus r)$ corresponding to a given category of partitions with extra singletons $\mathscr{C} \subseteq \mathscr{P}^{\vartriangle}$ can be defined by

$$C(G) = C^*(v_{ij}, r \mid v_{ij} = v_{ij}^*,\ r = r^*,\ T_p u^{\otimes w_1} = u^{\otimes w_2} T_p \ \forall p \in \mathscr{C}(w_1, w_2)),$$

where $u^{\shortmid} = v$ and $u^{\vartriangle} = r$.

As we mentioned in Proposition 6.1.7, we do not have to consider the relations corresponding to all the partitions in $\mathscr{C}$, but only to some generating set of $\mathscr{C}$. So, suppose $\mathscr{C} = \langle C \rangle^{\vartriangle}$, then we have

$$C(G) = C^*\left(v_{ij}, r \;\middle|\; \begin{matrix} v = \bar{v},\ vv^t = v^t v = 1_N, r = r^*,\ r^2 = 1 \\ T_p u^{\otimes w_1} = u^{\otimes w_2} T_p \ \forall p \in C(w_1, w_2) \end{matrix}\right).$$

Note that the orthogonality relations $vv^t = v^t v = 1$ and $r^2 = 1$ correspond to the partitions $\sqcap$ and $\triangle\otimes\triangle$, which are contained in any category by definition, but they are usually not explicitly listed as generators.

6.4.2 Relations associated to partitions with extra singletons

We give a few examples of partitions with extra singletons and the corresponding quantum group relations in Table 6.4. Recall that we must assume that all the generators v_{ij} and r are self-adjoint – this does not follow from any partition relation.

The first two partitions correspond to orthogonality of u and r and are by definition present in all categories with extra singletons. The following partitions allow us to construct the most basic instances of extra-singleton categories and the corresponding quantum groups.

$\sqcap$	$vv^t = 1$ (assumed by default)
$\triangle\otimes\triangle$	$r^2 = 1$ (assumed by default)
$\triangle$	$r = 1$
$_{\triangle}\!\diagdown^{\triangledown}$	$v_{ij} r = r v_{ij}$
$_{\triangle}\!\diagdown\!\diagdown^{\triangledown}$	$v_{ij} v_{kl} r = r v_{ij} v_{kl}$

Table 6.4 Relations corresponding to certain partitions with extra singletons

6.4.3 Proposition. Let $\mathscr{C} \subseteq \mathscr{P}$ be an ordinary category of partitions corresponding to a quantum group $H \subseteq O_N^+$. Then

(1) $\langle \mathscr{C} \rangle^{\vartriangle}$ corresponds to $H \mathbin{\hat{*}} \hat{\mathbb{Z}}_2$,
(2) $\langle \mathscr{C}, {_{\triangle}\!\diagdown^{\triangledown}} \rangle^{\vartriangle}$ corresponds to $H \times \hat{\mathbb{Z}}_2$,
(3) $\langle \mathscr{C}, \triangle \rangle^{\vartriangle}$ corresponds to $H \times E$,

where $E = (\mathbb{C}, 1)$ is the trivial (quantum) group.

Proof. We just need to look at the relations implied by the generators of the categories and find out which quantum subgroup $G \subseteq O_N^+ \mathbin{\hat{*}} \hat{\mathbb{Z}}_2$ they determine. In the first case, we only have partitions without extra singletons, so in the corresponding relations only the elements v_{ij} appear (not the subrepresentation r). In particular, those relations correspond to the subgroup $H \subseteq O_N^+$, so, taken as generators of a category with extra singletons, they define the subgroup $H \mathbin{\hat{*}} \hat{\mathbb{Z}}_2 \subseteq O_N^+ \mathbin{\hat{*}} \hat{\mathbb{Z}}_2$. Indeed, since we do not add any new relations on r, the r remains free from v_{ij} and keeps representing the factor $\hat{\mathbb{Z}}_2$.

In the second case, we have, in addition, the generator $_{\triangle}\!\diagdown^{\triangledown}$, which corresponds to the relation $v_{ij} r = r v_{ij}$. Thus, the category corresponds to the quantum subgroup of $H \mathbin{\hat{*}} \hat{\mathbb{Z}}_2$ given by this relation. The relation is simply commutativity of the factors $C(H)$ and $C^*(\mathbb{Z}_2)$, so the corresponding quantum group is the tensor product $H \times \hat{\mathbb{Z}}_2$.

Finally, the last instance corresponds to the subgroup of $H \mathbin{\hat{*}} \hat{\mathbb{Z}}_2$ with respect to the relation $r = 1$. This relation corresponds to taking just the trivial subgroup of $\hat{\mathbb{Z}}_2$. □

The example $_{\triangle}\!\diagdown\!\diagdown^{\triangledown}$ corresponding to some weaker kind of commutativity gives us already an answer to Question 6.4.1. Recall the classification result from Section 4.6.6. Considering any category of partitions $\mathscr{C} \subseteq \mathscr{P}$ such that $\uparrow \notin \mathscr{C}$, we have that

$$\langle \mathscr{C} \rangle^{\vartriangle} \subsetneq \mathscr{C}_0^{\vartriangle} \subsetneq \mathscr{C}_k^{\vartriangle} \subsetneq \mathscr{C}_l^{\vartriangle} \subsetneq \mathscr{C}_2^{\vartriangle} = \langle \mathscr{C}, {_{\triangle}\!\diagdown^{\triangledown}} \rangle^{\vartriangle}.$$

Here, we assume that $2 < k, l \in \mathbb{N}$ are even unless $\uparrow \otimes \uparrow \in \mathscr{C}$ and that k divides l.

We can supplement the list in Proposition 6.4.3 with the following definitions.

(4) The quantum group corresponding to $\mathscr{C}_0^{\vartriangle} = \langle \mathscr{C}, {}_{\vartriangle}\!\setminus\!\setminus^{\triangledown} \rangle^{\vartriangle}$ is denoted by $H \mathbin{\times\!\!\!\!\times} \hat{\mathbb{Z}}_2$,
(5) The quantum group corresponding to $\mathscr{C}_k^{\vartriangle}$ is denoted by $H \times_k \hat{\mathbb{Z}}_2$,

Then we can say that for any easy quantum group H, whose category of partitions does not contain the singleton $\uparrow$, we have

$$H \mathbin{\hat{*}} \hat{\mathbb{Z}}_2 \supsetneq H \mathbin{\times\!\!\!\!\times} \hat{\mathbb{Z}}_2 \supsetneq H \times_k \hat{\mathbb{Z}}_2 \supsetneq H \times_l \hat{\mathbb{Z}}_2 \supsetneq H \times \hat{\mathbb{Z}}_2.$$

We get back to those $\mathbb{Z}_2$-extensions in Section 6.4.5. A more general definition of those products is provided in Section 8.3.

6.4.3 Induced one-coloured categories

In the preceding subsection, we studied the most simple constructions of how to get an extra-singleton category from an ordinary category and what are the corresponding quantum groups. Here, we are going the opposite direction and study non-coloured categories induced by the extra-singleton categories.

Recall from Section 4.6.2 the full subcategory $\mathscr{C}^{|} \subseteq \mathscr{C}$ defined for every extra-singleton category $\mathscr{C}$.

6.4.4 Lemma. Let $\mathscr{C}$ be a category of partitions with extra singletons corresponding to a quantum group $G = (C(G), v \oplus r)$. Then the one-coloured category $\mathscr{C}^{|}$ corresponds to the quantum group $H = (A, v)$, where $A \subseteq C(G)$ is the subalgebra generated by $\{v_{ij}\}_{i,j=1}^{N}$.

Proof. H is clearly a compact quantum group. Following the definition of a CMQG from Sect. 2.1.2, we see that (1) A is generated by v_{ij} by definition, (2) if a block diagonal matrix has an inverse, then the blocks must also be invertible, (3) the comultiplication is given simply by restriction to A. Directly from the construction of G, we have that the intertwiner spaces $\mathrm{Mor}(v^{\otimes k}, v^{\otimes l})$ are precisely given by partitions in $\mathscr{C}^{|}$. □

Secondly, given a category of partitions with extra singletons $\mathscr{C}$, we can somehow ignore the extra singletons. Let us define the following

$$\mathscr{C}^{\vartriangle=1} := (\bar{\mathscr{C}})^{|},$$

where $\bar{\mathscr{C}} = \langle \mathscr{C}, \vartriangle \rangle^{\vartriangle}$. Obviously, we have $\mathscr{C}^{|} \subseteq \mathscr{C}^{\vartriangle=1}$.

6.4.5 Lemma. Let $\mathscr{C}$ be a category of partitions with extra singletons corresponding to a quantum group $G = (C(G), v \oplus r)$. Then the one-coloured category $\mathscr{C}^{\vartriangle=1}$ corresponds to the quantum group $\tilde{H} = (A, \tilde{v})$, where A is the quotient of $C(G)$ by the relation $r = 1$ and $\tilde{v}_{ij}$ are the images of v_{ij} under the natural homomorphism.

Proof. The partition $\vartriangle$ corresponds to the relation $r = 1$, so $\bar{\mathscr{C}}$ is the category corresponding to the quantum subgroup $\bar{G} \subseteq G$ defined by imposing the relation $r = 1$. That is, $\bar{G} = (A, \tilde{v} \oplus 1)$. Using the preceding lemma, we get that $\mathscr{C}^{\vartriangle=1}$ corresponds to the quantum group $\tilde{H}$. □

6.4.6 Remark. Recall Lemma 4.6.1. Thanks to Proposition 6.4.3, we can complement it with the quantum group picture. Let $\mathscr{C}$ be a category of partitions with extra singletons. Denote by G the quantum group corresponding to $\mathscr{C}$, and by H the quantum group corresponding to $\mathscr{C}^{|}$. Then we have the following.

(1) If $\vartriangle \in \mathscr{C}$, then $\mathscr{C} = \langle \mathscr{C}^{|}, \vartriangle \rangle^{\vartriangle}$ and $G = H \times E$.
(2) If $\vartriangle \notin \mathscr{C}$, $\uparrow \otimes \vartriangle \notin \mathscr{C}$, but ${}_{\vartriangle}\!\setminus^{\triangledown} \in \mathscr{C}$, then $\mathscr{C} = \langle \mathscr{C}^{|}, {}_{\vartriangle}\!\setminus^{\triangledown} \rangle^{\vartriangle}$ and $G = H \times \hat{\mathbb{Z}}_2$.

6.4.7 Proposition. Let $\mathscr{C}$ be a category of partitions with extra singletons. Denote by G the quantum group corresponding to $\mathscr{C}$, by H the quantum group corresponding to $\mathscr{C}^{|}$ and by $\tilde{H}$ the quantum group corresponding to $\mathscr{C}^{\vartriangle=1}$. Then

$$\tilde{H} \times E \subseteq G \subseteq H \mathbin{\tilde{*}} \hat{\mathbb{Z}}_2.$$

Proof. According to Lemma 6.4.4 and Proposition 6.4.3, the quantum group $H \mathbin{\tilde{*}} \hat{\mathbb{Z}}_2$ corresponds to the category $\langle \mathscr{C}^{|} \rangle^{\vartriangle}$. According to Lemma 6.4.5 and Proposition 6.4.3, the quantum group $H \times E$ corresponds to the category $\langle \mathscr{C}^{\vartriangle=1}, \vartriangle \rangle^{\vartriangle} = \langle \mathscr{C}, \vartriangle \rangle^{\vartriangle}$. We indeed have

$$\langle \mathscr{C}, \vartriangle \rangle^{\vartriangle} \supseteq \mathscr{C} \supseteq \langle \mathscr{C}^{|} \rangle^{\vartriangle}. \qquad \square$$

Note that the quantum groups $\tilde{H}$ and H may be different. That is, given a quantum group $G \subseteq O_N^+ \mathbin{\tilde{*}} \hat{\mathbb{Z}}_2$, we do not always have $H \times E \subseteq G \subseteq H \mathbin{\tilde{*}} \hat{\mathbb{Z}}_2$ for some $H \subseteq O_N^+$. We present an example in the following paragraph.

6.4.8 Example. Consider the category $\mathscr{C} := \langle \overset{\triangledown}{\shortmid} \rangle^{\vartriangle}$. It holds that $\mathscr{C}^{|} = \langle \uparrow \otimes \uparrow \rangle$. Indeed, one can easily see that $\uparrow \otimes \uparrow$ is generated by $\overset{\triangledown}{\shortmid}$ (compose $(\overset{\triangledown}{\shortmid} \otimes \overset{\triangledown}{\shortmid}) \cdot (\vartriangle \otimes \vartriangle)$), which proves the inclusion $\supseteq$. Conversely, one can see that all partitions in $\mathscr{C}$ have blocks of size at most two and we can also prove that $\setminus^{|}_{|} \notin \mathscr{C}$ (otherwise we would have $\setminus^{\triangledown}_{\vartriangle} \in \mathscr{C}$ and hence $\setminus\setminus^{\triangledown}_{\vartriangle} \in \mathscr{C}$, which is not the case according to the classification in Proposition 4.6.21). Hence, we have the inclusion $\subseteq$. Similarly, one can prove that $\mathscr{C}^{\vartriangle=1} = \langle \uparrow \rangle$.

The category $\langle \uparrow \otimes \uparrow \rangle$ corresponds to the quantum group $B_N^{\#+}$, which is a subgroup of O_N^+ given by the relation $s := \sum_k v_{ik} = \sum_k v_{kj}$ for all $i, j = 1, \ldots, N$ (see Table 6.2). The category $\langle \uparrow \rangle$ corresponds to the quantum group B_N^+, which is a quantum subgroup of $B_N^{\#+}$ given by $s = 1$. According to Proposition 6.4.7 we have that $\mathscr{C}$ corresponds to a quantum group G with

$$B_N^+ \times E \subseteq G \subseteq B_N^{\#+} \mathbin{\tilde{*}} \hat{\mathbb{Z}}_2.$$

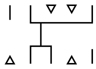
More concretely, as a subgroup of $B_N^{\#+} \mathbin{\tilde{*}} \hat{\mathbb{Z}}_2$, it is given by the relation $r = s$ arising from the partition $\overset{\triangledown}{\shortmid}$.

In fact, G is isomorphic to $B_N^{\#+}$ (just take the $*$-isomorphism $C(G) \to C(B_N^{\#+})$ mapping $v_{ij} \mapsto v_{ij}$ and $r \mapsto s$). Nevertheless, B_N^+ is the maximal compact matrix quantum group H that can be embedded in G in the form $H \times E \subseteq G$ (that is, having a surjective $*$-homomorphism $C(G) \to C(H)$ mapping $r \mapsto 1$).

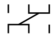
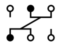
For those considerations, note the important distinction between two quantum groups being *isomorphic* (existence of a C*-algebra isomorphism that preserves the comultiplication) and being *identical* as matrix quantum groups (the fundamental representations must coincide as well; in particular, they must have the same size).

6.4.4 The gluing functor

Recall the definition of the functor $F\colon \mathscr{P}^{\vartriangle} \to \mathscr{P}^{\circ\bullet}$ from Section 4.6.4. In principle, the functor was mapping $|\vartriangle \mapsto \circ$ and $\vartriangle| \mapsto \bullet$. The partition structure does not change under F (if we ignore the extra singletons). Consequently, the maps T_p and $T_{F(p)}$ for a given $p \in \mathscr{P}^{\vartriangle}(w_1, w_2)$ are exactly the same maps $(\mathbb{C}^N)^{\otimes k'} \to (\mathbb{C}^N)^{\otimes l'}$, where k' and l' are the lengths of the words $F(w_1)$ and $F(w_2)$. The only thing that changes is the interpretation of those maps. The map T_p is considered as an intertwiner in $\mathrm{Mor}(u^{\otimes w_1}, u^{\otimes w_2})$ with $u^{|} = v$ and $u^{\vartriangle} = r$ for some quantum group $G = (C(G), v \oplus r) \subseteq O_N^+ \mathbin{\tilde{*}} \hat{\mathbb{Z}}_2$. In contrast, the map $T_{F(p)}$ is interpreted as an intertwiner in $\mathrm{Mor}(\tilde{v}^{\otimes F(w_1)}, \tilde{v}^{\otimes F(w_2)})$, where $\tilde{v}^{\circ} = \tilde{v}$ and $\tilde{v}^{\bullet} = \bar{\tilde{v}}$, for some quantum group $\tilde{G} = (C(\tilde{G}), \tilde{v}) \subseteq U_N^+$.

6.4.9 Example. As an example, let us take the partition $p =$ . We associate to it a map $T_p\colon \mathbb{C}^N \otimes \mathbb{C}^N \otimes \mathbb{C}^N \to \mathbb{C}^N \otimes \mathbb{C}^N \otimes \mathbb{C}^N$ by Equation (6.1) ignoring the extra singletons. That is, we have

$$T_p(e_{i_1} \otimes e_{i_2} \otimes e_{i_3}) = \delta_{i_2 i_3}\, e_{i_2} \otimes e_{i_2} \otimes \sum_{k=1}^{N} e_k.$$

As we just mentioned, it coincides with the T_p map associated to the partition  and hence also with the map associated to $F(p) =$ . The meaning of the partition p is the relation

$$T_p(v \otimes v \otimes r \otimes r \otimes v) = (r \otimes v \otimes v \otimes r \otimes v)T_p,$$

which can also be written as

$$\delta_{i_1 i_2} \sum_{l=1}^{N} v_{l j_1} v_{i_1 j_2} r^2 v_{i_1 j_3} = \delta_{j_2 j_3} \sum_{k=1}^{N} r v_{i_1 j_2} v_{i_2 j_2} r v_{i_3 k}.$$

The meaning of the partition $F(p)$ is the relation

$$T_p(\tilde{v} \otimes \bar{\tilde{v}} \otimes \tilde{v}) = (\bar{\tilde{v}} \otimes \tilde{v} \otimes \tilde{v})T_p,$$

which can also be written as

$$\delta_{i_1 i_2} \sum_{l=1}^{N} v_{l j_1} v_{i_1 j_2}^* v_{i_1 j_3} = \delta_{j_2 j_3} \sum_{k=1}^{N} v_{i_1 j_2}^* v_{i_2 j_2} v_{i_3 k}.$$

We can say that the functor F changes the corresponding relations by mapping $v_{ij} r \mapsto \tilde{v}_{ij}$ and $r v_{ij} \mapsto \tilde{v}_{ij}^*$. See also Theorem 6.4.13.

6.4.10 Definition. Consider a quantum group $G \subseteq O_N^+ \hat{*} \hat{\mathbb{Z}}_2$ with fundamental representation $v \oplus r$. Denote $\tilde{v}_{ij} := v_{ij} r$ and let A be the C*-subalgebra of $C(G)$ generated by $\tilde{v}_{ij}$. Then $\tilde{G} := (A, u)$ is called the **glued version** of G.

6.4.11 Remark. It is easy to check that the comultiplication on G satisfies $\Delta(\tilde{v}_{ij}) = \sum_k \tilde{v}_{ik} \otimes \tilde{v}_{kj}$, so its restriction provides a comultiplication on $\tilde{G}$. Thus, $\tilde{G}$ is a compact matrix quantum group. It is a quantum quotient of G.

6.4.12 Remark. The definition generalizes the glued product construction from Sect. 2.5.3. It is easy to see that $G \tilde{*} \hat{\mathbb{Z}}_2$ is the glued version of $G \hat{*} \hat{\mathbb{Z}}_2$ and $G \tilde{\times} \hat{\mathbb{Z}}_2$ is the glued version of $G \times \hat{\mathbb{Z}}_2$.

Now we present the main result of Section 6.4 interpreting the classification of partitions with extra singletons. We show that the functor F from Definition 4.6.4 corresponds to the gluing procedure defined above.

6.4.13 Theorem. Consider a category $\mathscr{C} \subseteq \mathscr{P}^{\vartriangle}_{\mathrm{even}}$. Denote by $G = (C(G), v \oplus r)$ the quantum group corresponding to $\mathscr{C}$ and by $\tilde{G} = (C(\tilde{G}), \tilde{v})$ the quantum group corresponding to the category $\tilde{\mathscr{C}} := F(\mathscr{C})$. Then there is an injective $*$-homomorphism $\iota\colon C(\tilde{G}) \to C(G)$ mapping $\tilde{v}_{ij} \mapsto v_{ij} r$. In other words, $\tilde{G}$ is the glued version of G.

Proof. To prove the existence of a $*$-homomorphism $\iota\colon C(\tilde{G}) \to C(G)$ mapping $\tilde{v}_{ij} \mapsto v_{ij} r$, we need to show that the elements $\tilde{v}_{ij}' := v_{ij} r \in C(G)$ satisfy all the relations of the generators $\tilde{v}_{ij} \in C(\tilde{G})$. We observed this already in the beginning of this subsection. Indeed, all relations in $C(\tilde{G})$ are of the form $T_{\tilde{p}} \tilde{v}^{\otimes \tilde{w}_1} = \tilde{v}^{\otimes \tilde{w}_2} T_{\tilde{p}}$ for some $\tilde{p} \in \tilde{\mathscr{C}}(\tilde{w}_1, \tilde{w}_2)$. Take any preimage $p \in \mathscr{C}(w_1, w_2)$,

$p \in F^{-1}(\tilde{p})$. We already showed that $T_p = T_{\tilde{p}}$. One can also check that $\tilde{v}'^{\otimes\tilde{w}_1} = u^{\otimes w_1}$ and $\tilde{v}'^{\otimes\tilde{w}_2} = u^{\otimes w_2}$ (as usual, we take $u = v \oplus r$). So, we have

$$T_{\tilde{p}}\tilde{v}'^{\otimes\tilde{w}_1} = T_p u^{\otimes w_1} = u^{\otimes w_2} T_p = \tilde{v}'^{\otimes\tilde{w}_2} T_{\tilde{p}}.$$

To prove the injectivity, we are going to use a similar trick as in the proof of Theorem 6.2.4. We will show that there is a $*$-homomorphism $\beta\colon C(G) \to M_2(C(\tilde{G}))$ mapping

$$r \mapsto r' := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad v_{ij} \mapsto v'_{ij} := \begin{pmatrix} 0 & \tilde{v}_{ij} \\ \tilde{v}^*_{ij} & 0 \end{pmatrix}.$$

If we prove that such a homomorphism exists, then it is easy to check that

$$(\beta \circ \iota)\tilde{v}_{ij} = \tilde{v}''_{ij} := \begin{pmatrix} \tilde{v}_{ij} & 0 \\ 0 & \tilde{v}^*_{ij} \end{pmatrix},$$

so $\beta \circ \iota$ is obviously injective, which implies the injectivity of ι.

The proof of existence of such a homomorphism β is similar to the proof of existence of ι. We have to prove that the elements r' and v'_{ij} satisfy the same relations as the generators r and v_{ij}. Again, we have that all the relations for r and v_{ij} are of the form $T_p u^{\otimes w_1} = u^{\otimes w_2} T_p$ for $p \in \mathscr{C}(w_1, w_2)$. Since we assume $\mathscr{C} \subseteq \mathscr{P}^{\vartriangle}_{\rm even}$, we have that p is of even length. Without loss of generality, we can assume that both w_1 and w_2 have even length (otherwise, consider $p \otimes \substack{\blacktriangledown \\ \vartriangle}$, which induces obviously an equivalent relation). Any monomial in v'_{ij} and r' of even length can be expressed in terms of $\tilde{v}''_{ij}$ and $\tilde{v}''^*_{ij}$. Indeed, notice that $v'_{ij} r' = \tilde{v}''_{ij}$, so $r' v'_{ij} = (v'_{ij} r')^* = \tilde{v}''^*_{ij}$ and $v'_{ij} v'_{kl} = v'_{ij} r' r' v'_{kl} = \tilde{v}''_{ij} \tilde{v}''^*_{kl}$. Consequently, one can see that $u'^{\otimes w_1} = \tilde{v}''^{\otimes F(w_1)}$ and also $u'^{\otimes w_2} = \tilde{v}''^{\otimes F(w_2)}$ (denoting $u' = v' \oplus r'$). Thus, using also the equality $T_p = T_{F(p)} = T_{\overline{F(p)}}$, we have

$$T_p u'^{\otimes w_1} = T_p \tilde{v}''^{\otimes F(w_1)}_{ij} = \tilde{v}''^{\otimes F(w_2)}_{ij} T_p = u'^{\otimes w_2} T_p. \qquad \square$$

6.4.14 Example. In [Ban97], it was proven that $U_N^+ = O_N^+ \tilde{*} \hat{\mathbb{Z}}$. In [TW17, Proposition 6.20], it was proven that we can actually exchange $\mathbb{Z}$ for $\mathbb{Z}_2$, so we have $U_N^+ = O_N^+ \tilde{*} \hat{\mathbb{Z}}_2$. The latter is a simple consequence of Theorem 6.4.13. Indeed, the quantum group $O_N^+ \hat{*} \mathbb{Z}_2$ corresponds to the smallest category with extra singletons $\mathscr{C} := \langle\rangle^{\vartriangle}$. The quantum group U_N^+ corresponds to the smallest two-coloured category $\tilde{\mathscr{C}} := \langle\rangle^{\circ\bullet}$, which is the image of $\mathscr{C}$ under F. So, U_N^+ is a glued version of $O_N^+ \hat{*} \hat{\mathbb{Z}}_2$.

Note that the theorem above agrees with the spirit of Proposition 3.3.2 (see also the discussion in Remark 3.4.16) that full subcategories correspond to quotient quantum groups since by restricting to a smaller underlying algebra, we are taking away some representations.

Finally, recall the definition of $\mathrm{Alt}\mathscr{C} \subseteq \mathscr{P}^{\circ\bullet}$ for a given $\mathscr{C} \subseteq \mathscr{P}$ from Section 4.6.5. As an application of Theorem 6.4.13, we can prove a special case of Proposition 6.2.8 interpreting the category $\mathrm{Alt}\mathscr{C}$.

6.4.15 Proposition. Let $\mathscr{C} \subseteq \mathscr{P}$ be a category of partitions with $\uparrow \notin \mathscr{C}$ and denote by $H \subseteq O_N^+$ the corresponding quantum group. Then $\mathrm{Alt}\mathscr{C}$ corresponds to $H \tilde{*} \hat{\mathbb{Z}}_2$.

Proof. From Proposition 6.4.3, it follows that $\langle\mathscr{C}\rangle^{\vartriangle}$ corresponds to the quantum group $H \hat{*} \hat{\mathbb{Z}}_2$. From Lemma 4.6.12, it follows that $\mathrm{Alt}\mathscr{C}$ is its image under F. By Theorem 6.4.13 this implies that it corresponds to the glued version of $H \hat{*} \hat{\mathbb{Z}}_2$, which is $H \tilde{*} \hat{\mathbb{Z}}_2$. $\square$

We will study glued and tensor complexifications more detail in detail in Chapter 8, where the full version of Proposition 6.2.8 will be proven.

6.4.5 Summary of the new $\mathbb{Z}_2$-extensions

In this section, we are going to look more in detail on some quantum groups coming out from the classification of categories of partitions with extra singletons. First of all, let us again formally define the quantum groups $H \mathbin{\tilde{\times}} \hat{\mathbb{Z}}_2$ and $H \times_k \hat{\mathbb{Z}}_2$ that we mentioned in Section 6.4.2. We generalize those products for arbitrary pairs of quantum groups in Section 8.3.

6.4.16 Definition. Consider a compact matrix quantum group $H = (C(H), v) \subseteq O_N^+$. Denote by r the generator of $\mathbb{Z}_2$. We define the product $H \mathbin{\tilde{\times}} \hat{\mathbb{Z}}_2 = H \times_0 \hat{\mathbb{Z}}_2$ to be the quantum subgroup of $H * \hat{\mathbb{Z}}_2$ given by the relations

$$v_{ij} v_{kl} r = r v_{ij} v_{kl}, \tag{6.5}$$

that is, the relations corresponding to the partition $\triangle\!\setminus\!\setminus^{\triangledown}$. We define quantum groups $G \times_{2k} \hat{\mathbb{Z}}_2$, $k \in \mathbb{N}$ by imposing additional relations

$$v_{i_1 j_1} r v_{i_2 j_2} r \cdots v_{i_k j_k} r = r v_{i_1 j_1} r v_{i_2 j_2} \cdots r v_{i_k j_k}, \tag{6.6}$$

which correspond to the partition $(\triangle\!\setminus^{\triangledown})^{\otimes k}$.

6.4.17 Definition. Consider a compact matrix quantum group $H = (C(H), v) \subseteq O_N^+$. Denote by r the generator of $\mathbb{Z}_2$. Assume H has a one-dimensional representation s (typically, we consider $s = \sum_k v_{ik}$). We define the product $H \,{}^s\!\hat{*}_k\, \hat{\mathbb{Z}}_2$ given by the relation $(sr)^k = 1$ (if $s = \sum_k v_{ik}$, then it corresponds to the partition $(\uparrow \otimes \triangle)^{\otimes k}$). We also define the product $H \,{}^s\!\times_k \hat{\mathbb{Z}}_2$ combining the relation $(sr)^k = 1$ with Relations (6.5).

6.4.18 Remark. Since all the relations mentioned here are intertwiner relations, that is, they are of the form $T u^{\otimes w} = u^{\otimes w} T$, they must surely define compact matrix quantum groups even if H does not correspond to any category of partitions.

6.4.19 Proposition. Let $H \subseteq O_N^+$ be a compact matrix quantum group with a one-dimensional representation s. Then

$$\begin{aligned} H * \hat{\mathbb{Z}}_2 \supseteq H \,{}^s\!\hat{*}_k\, \hat{\mathbb{Z}}_2 \supseteq H \,{}^s\!\times_k \hat{\mathbb{Z}}_2 \supseteq H \times \hat{\mathbb{Z}}_2 \quad & \text{for any } k \in \mathbb{N}, \\ H \,{}^s\!\hat{*}_k\, \hat{\mathbb{Z}}_2 \supseteq H \,{}^s\!\hat{*}_l\, \hat{\mathbb{Z}}_2 \quad & \text{if } l \text{ divides } k, \\ H \,{}^s\!\times_k \hat{\mathbb{Z}}_2 \supseteq H \,{}^s\!\times_l \hat{\mathbb{Z}}_2 \quad & \text{if } l \text{ divides } k. \end{aligned}$$

Proof. Straightforward from the definition of the products. □

6.4.20 Proposition. Consider $H \subseteq O_N^+$ with a representation $s = \sum_k v_{ik}$. Then $H \,{}^s\!\times_{2k} \hat{\mathbb{Z}}_2 = H \times_{2k} \hat{\mathbb{Z}}_2$.

Proof. We need to prove that the set of relations $(\sum_k v_{ik})^2 = 1$, $(sr)^{2k} = 1$, and Relations (6.5) is equivalent to the relation $\left(\sum_k v_{ij}\right)^2 = 1$ together with Relations (6.6). We can do this in terms of partition calculus. That is, we need to prove the following

$$\langle \uparrow \otimes \uparrow, \triangle\!\setminus\!\setminus^{\triangledown}, (\uparrow \otimes \triangle)^{\otimes 2k} \rangle^{\triangle} = \langle \uparrow \otimes \uparrow, (\triangle\!\setminus^{\triangledown})^{\otimes k} \rangle^{\triangle}.$$

This is equivalent to the two-coloured version of the equality obtained by applying the functor F form Def. 4.6.4:

$$\langle \uparrow \otimes \uparrow, \mid \otimes \mid, \uparrow^{\otimes 2k} \rangle = \langle \uparrow \otimes \uparrow, \mid \otimes \mid, \uparrow^{\otimes k} \rangle.$$

This can be proven using Lemma 4.5.8. □

Let us now summarize the $\mathbb{Z}_2$-extensions in connection with the functor F and the gluing procedure in a form of a table.

two-coloured category $\tilde{\mathscr{C}}$	$\mathrm{Alt}\,\mathscr{C}$	$\Psi^{-1}(\mathscr{C})_0$	$\langle \Psi^{-1}(\mathscr{C})_0, \mathord{\uparrow}^{\otimes k}\rangle$	$\Psi^{-1}(\mathscr{C})$
corresp. quantum group	$H \mathbin{\tilde{*}} \hat{\mathbb{Z}}_2$	$H \mathbin{\tilde{\times}} \hat{\mathbb{Z}}$	$H \mathbin{\tilde{\times}} \hat{\mathbb{Z}}_{2k}$	$H = H \mathbin{\tilde{\times}} \hat{\mathbb{Z}}_2$
the preimage $F^{-1}(\tilde{\mathscr{C}})$	$\langle \mathscr{C} \rangle^{\vartriangle}$	$\langle \mathscr{C}, {}_{\vartriangle}\backslash\backslash^{\triangledown} \rangle^{\vartriangle}$	$\langle \mathscr{C}, ({}_{\vartriangle}\backslash^{\triangledown})^{\otimes k} \rangle^{\vartriangle}$	$\langle \mathscr{C}, {}_{\vartriangle}\backslash^{\triangledown} \rangle^{\vartriangle}$
corresp. quantum group	$H * \hat{\mathbb{Z}}_2$	$H \mathbin{\times\!\!\!\times} \hat{\mathbb{Z}}_2$	$H \times_{2k} \hat{\mathbb{Z}}_2$	$H \times \hat{\mathbb{Z}}_2$

Table 6.5 Categories of partitions corresponding to various glued products and their "$\mathbb{Z}_2$-unglued" versions.

6.4.21 Proposition. Let $\mathscr{C} \subseteq \mathscr{P}$ be a category of partitions such that $\uparrow \notin \mathscr{C}$ corresponding to a quantum group $H \subseteq O_N^+$. Then Table 6.5 shows the quantum groups corresponding to the various categories constructed from $\mathscr{C}$. All the categories are mutually distinct (and hence also the quantum groups for large enough N).

Proof. First, let us check that the first row indeed maps to the third row under F^{-1}. For the first column it follows from Lemma 4.6.12. For the last column, it follows from Lemma 4.6.10. For the rest, it follows from Lemma 4.6.18.

Now, let us check the quantum group picture. Let us start with the upper part of the table. The first column was proven in Proposition 6.4.15. The rest follows from Theorem 6.2.4 (see also the discussion after the theorem). For the lower part of the table, the first and last column follow from Proposition 6.4.3 and the rest follows directly from the definitions of the products.

The categories in the last three columns are mutually unequal from Lemma 4.5.9. Thanks to the obvious inclusions, it remains only to prove inequality between the first two columns. It can be seen that $\langle \mathscr{C} \rangle^{\vartriangle}$ contains only those partitions with extra singletons where we can find a pairing of the extra singletons that does not cross the blocks of colour ı. Since ${}_{\vartriangle}\backslash\backslash^{\triangledown}$ does not satisfy this property, we have $\langle \mathscr{C} \rangle^{\vartriangle} \subsetneq \langle \mathscr{C}, {}_{\vartriangle}\backslash\backslash^{\triangledown} \rangle^{\vartriangle}$. □

6.4.22 Remark (Comparison with [Fre19]). Our classification problem is closely related to the classification of partition categories with two self-dual colours, which was solved in the non-crossing case by Freslon in [Fre19]. Let us state here explicitly the relation to our work. Strictly speaking we are solving two different problems as Freslon looks for quantum groups G' with $S_N \subseteq G' \subseteq O_N^+ \mathbin{\hat{*}} O_N^+$ while we are looking for quantum groups G with $S_N \times E \subseteq G \subseteq O_N^+ \mathbin{\hat{*}} \mathbb{Z}_2$. Nevertheless, any category of partitions with extra singletons can be also considered as a category with two self-dual colours. Hence, the classification of non-crossing categories of extra-singletons (summarized in Section 4.6.6) was already intrinsically contained in [Fre19] as well as many quantum group relations discussed in this section. On the other hand, in our work, we do not restrict to the non-crossing case and even here we state the results in much more explicit way.

We can compare our $\mathbb{Z}_2$-extensions with [Fre19, Section 5]. The difference is that instead of studying quantum subgroups of $G \mathbin{\hat{*}} H$ determined by relations involving some one-dimensional subrepresentation r of H, we set $H := \hat{\mathbb{Z}}_2$ and work with its one-dimensional fundamental representation. As a particular example, note that the quantum group $BO_N^{+\#} \subseteq O_N^+ * B_N^{+\#}$ from [Fre19, Definition 5.2] is essentially defined by Relations (6.5) if we interpret r as the one-dimensional representation of $B_N^{+\#}$ given by $r = \sum_k w_{ik}$ (w being the fundamental representation of $B_N^{+\#}$).

Chapter 7

Non-easy quantum groups

In this chapter, we interpret non-easy linear categories of partitions that were discovered by computer experiments as described in Chapter 5. We call the associated quantum groups **non-easy quantum groups**. More or less, we will follow Section 5.4 extending the results with the quantum group interpretation.

We provide a summary of all the non-easy categories and their interpretation in Table 7.1 on the following page. In the first column of the table, we refer to the paragraph, where the corresponding categories were proven to be non-easy. The category itself is then written in the third column. We use some special notation to represent the generators. The expanded form of those generators as linear combinations of partitions is summarized in Table 5.1 (p. 79). In the second column of the table, we refer to a paragraph, where the interpretation of the category is presented. The actual quantum group is then listed in the last column.

Let us now mention the main theorems of this chapter, on which the interpretation of the non-easy categories is based on.

In Section 7.1, we study quantum groups, whose fundamental representation is irreducible. We define a unitary matrix $U_{(N,\pm)}$, a coisometry $V_{(N,\pm)}$, and a projection $P_{(N)}$, which allow to study those quantum groups and the corresponding invariant subspaces. In particular, this interprets the non-easy generators that were constructed using the mappings $\mathcal{P}_{(N,\pm)}$ and $\mathcal{V}_{(N,\pm)}$ from Sections 5.4.5 and 5.4.6.

Theorem (7.1.9)**.** Let G be a quantum group such that $S_N \subseteq G \subseteq B_N^{\#+}$. This means that its representation category is described by a linear category of partitions $\mathscr{K}$ containing $\uparrow\otimes\uparrow$. Then it holds that $S_{N-1} \subseteq (S_N)_\pm^{\mathrm{irr}} \subseteq G_\pm^{\mathrm{irr}} \subseteq O_{N-1}^+$ and $G_\pm^{\mathrm{irr}} := V_{(N,\pm)} G V_{(N,\pm)}^*$ corresponds to the category $\mathcal{V}_{(N,\pm)}\mathscr{K} \subseteq \mathsf{Part}_{N-1}$.

Theorem (7.1.12)**.** Let $\mathscr{K} \subseteq \mathsf{PartRed}_N$ be a reduced category. Denote by H the quantum group $S_{N-1} \subseteq H \subseteq O_{N-1}^+$ corresponding to the category $\mathcal{V}_{(N,\pm)}\mathscr{K} \subseteq \mathsf{Part}_{N-1}$. Then we can construct the quantum group corresponding to the following categories:

$$\begin{aligned} \langle\mathscr{K}\rangle_N &\quad \text{corresponds to} \quad U_{(N,\pm)}^*(H \hat{*} \hat{\mathbb{Z}}_2)U_{(N,\pm)}, \\ \langle\mathscr{K}, {}^{\shortmid}\!/\!{}_{\shortmid}\rangle_N &\quad \text{corresponds to} \quad U_{(N,\pm)}^*(H \times \hat{\mathbb{Z}}_2)U_{(N,\pm)}, \\ \langle\mathscr{K}, \uparrow\rangle_N &\quad \text{corresponds to} \quad U_{(N,\pm)}^*(H \times E)U_{(N,\pm)}, \end{aligned}$$

where $E = (\mathbb{C}, 1)$ is the trivial (quantum) group.

As we already showed in Section 6.4, compact matrix quantum groups where the fundamental representation has a one-dimensional subrepresentation are conveniently described by partitions with extra singletons. The considerations from Section 7.1 are generalized in Section 7.2, where the link to extra-singleton categories is formulated. In particular, we define a functor $\mathcal{U}_{(N,\pm)}$ such that the following holds.

Theorem (7.2.9)**.** It holds that

$$T_{\mathcal{U}_{(N,\pm)}p} = U_{(N,\pm)}^{\otimes l} T_p U_{(N,\pm)}^{*\otimes k}$$

for any $p \in \mathsf{Part}_N(w_1, w_2)$, $w_1 \in \mathscr{O}_{\vdots}^k$, $w_2 \in \mathscr{O}_{\vdots}^l$. Thus, considering a linear category of partitions $\mathscr{K} \subseteq \mathsf{Part}_N$ containing $\uparrow\otimes\uparrow$ and the corresponding quantum group G, it holds that the linear category with extra singletons $\mathcal{U}_{(N,\pm)}\mathscr{K}$ corresponds to the quantum group $U_{(N,\pm)} G U_{(N,\pm)}^*$.

We also interpret the category isomorphisms described in Section 5.4. First of all, the isomorphism $\mathcal{T}_{(N)}$ from Section 5.4.2 corresponds just to a quantum group similarity.

Paragraph w/ categ.	Paragraph w/ interp.	Category	Quantum group
5.4.6	7.4.13	$\langle \mathcal{D} \times \rangle$	O_N^{σ}
5.4.8	7.4.11	$\langle \mathcal{J} \times \rangle$	O_N^{-1}
5.4.9	7.4.11	$\langle \mathcal{J} \ast \rangle$	O_N^{*-1}
5.4.24	7.1.13	$\langle \mathcal{P}_{(\delta)} \sqcap\!\sqcap, \uparrow \otimes \uparrow \rangle$	$U^*_{(N,\pm)} \left((S_N^+)^{\mathrm{irr}}_{\pm} \,\hat{*}\, \hat{\mathbb{Z}}_2 \right) U_{(N,\pm)}$
5.4.24	7.1.13	$\langle \mathcal{P}_{(\delta)} \sqcap\!\sqcap, \sqcap\,\shortmid\;\shortmid \rangle$	$U^*_{(N,\pm)} \left((S_N^+)^{\mathrm{irr}}_{\pm} \times \hat{\mathbb{Z}}_2 \right) U_{(N,\pm)}$
5.4.24	7.1.13	$\langle \mathcal{P}_{(\delta)} \sqcap\!\sqcap\!\sqcap, \uparrow \otimes \uparrow \rangle$	$U^*_{(N,\pm)} \left(((S_N^+)^{\mathrm{irr}}_{\pm} \,\tilde{\times}\, \hat{\mathbb{Z}}_2) \,\hat{*}\, \hat{\mathbb{Z}}_2 \right) U_{(N,\pm)}$
5.4.24	7.1.13	$\langle \mathcal{P}_{(\delta)} \sqcap\!\sqcap\!\sqcap, \sqcap\,\shortmid\;\shortmid \rangle$	$U^*_{(N,\pm)} \left(((S_N^+)^{\mathrm{irr}}_{\pm} \,\tilde{\times}\, \hat{\mathbb{Z}}_2) \times \hat{\mathbb{Z}}_2 \right) U_{(N,\pm)}$
5.4.24	7.1.13	$\langle \mathcal{P}_{(\delta)} \sqcap\!\sqcap\!\sqcap, \uparrow \rangle$	$U^*_{(N,\pm)} \left(((S_N^+)^{\mathrm{irr}}_{\pm} \,\tilde{\times}\, \hat{\mathbb{Z}}_2) \times E \right) U_{(N,\pm)}$
5.4.28	7.1.14	$\langle \mathcal{P}_{(\delta)} \times, \mathcal{P}_{(\delta)} \sqcap\!\sqcap, \uparrow \otimes \uparrow \rangle$	$U^*_{(N,\pm)} \left((S_N)^{\mathrm{irr}}_{\pm} \,\hat{*}\, \hat{\mathbb{Z}}_2 \right) U_{(N,\pm)}$
5.4.28	7.1.14	$\langle \mathcal{P}_{(\delta)} \times, \mathcal{P}_{(\delta)} \sqcap\!\sqcap, \sqcap\,\shortmid\;\shortmid \rangle$	$U^*_{(N,\pm)} \left((S_N)^{\mathrm{irr}}_{\pm} \times \hat{\mathbb{Z}}_2 \right) U_{(N,\pm)}$
5.4.28	7.1.14	$\langle \mathcal{P}_{(\delta)} \times, \mathcal{P}_{(\delta)} \sqcap\!\sqcap\!\sqcap, \uparrow \otimes \uparrow \rangle$	$U^*_{(N,\pm)} \left(((S_N)^{\mathrm{irr}}_{\pm} \,\tilde{\times}\, \hat{\mathbb{Z}}_2) \,\hat{*}\, \hat{\mathbb{Z}}_2 \right) U_{(N,\pm)}$
5.4.28	7.1.14	$\langle \mathcal{P}_{(\delta)} \times, \mathcal{P}_{(\delta)} \sqcap\!\sqcap\!\sqcap, \sqcap\,\shortmid\;\shortmid \rangle$	$U^*_{(N,\pm)} \left(((S_N)^{\mathrm{irr}}_{\pm} \,\tilde{\times}\, \hat{\mathbb{Z}}_2) \times \hat{\mathbb{Z}}_2 \right) U_{(N,\pm)}$
5.4.28	7.1.14	$\langle \mathcal{P}_{(\delta)} \times, \mathcal{P}_{(\delta)} \sqcap\!\sqcap\!\sqcap, \uparrow \rangle$	$U^*_{(N,\pm)} \left(((S_N)^{\mathrm{irr}}_{\pm} \,\tilde{\times}\, \hat{\mathbb{Z}}_2) \times E \right) U_{(N,\pm)}$
5.4.28	7.1.14	$\langle \mathcal{P}_{(\delta)} \times, \uparrow \otimes \uparrow \rangle$	$U^*_{(N,\pm)} \left(O_{N-1} \,\hat{*}\, \hat{\mathbb{Z}}_2 \right) U_{(N,\pm)}$
5.4.28	7.1.14	$\langle \mathcal{P}_{(\delta)} \times, \sqcap\,\shortmid\;\shortmid \rangle$	$U^*_{(N,\pm)} \left(O_{N-1} \times \hat{\mathbb{Z}}_2 \right) U_{(N,\pm)}$
5.4.28	7.1.14	$\langle \mathcal{P}_{(\delta)} \times, \uparrow \rangle$	$U^*_{(N,\pm)} \left(O_{N-1} \times E \right) U_{(N,\pm)}$
5.4.28	7.1.14	$\langle \mathcal{P}_{(\delta)} \ast, \uparrow \otimes \uparrow \rangle$	$U^*_{(N,\pm)} \left(O^*_{N-1} \,\hat{*}\, \hat{\mathbb{Z}}_2 \right) U_{(N,\pm)}$
5.4.28	7.1.14	$\langle \mathcal{P}_{(\delta)} \ast, \sqcap\,\shortmid\;\shortmid \rangle$	$U^*_{(N,\pm)} \left(O^*_{N-1} \times \hat{\mathbb{Z}}_2 \right) U_{(N,\pm)}$
5.4.28	7.1.14	$\langle \mathcal{P}_{(\delta)} \ast, \uparrow \rangle$	$U^*_{(N,\pm)} \left(O^*_{N-1} \times E \right) U_{(N,\pm)}$
5.4.34	7.1.10	$\langle \mathcal{V}_{(\delta)} \sqcap\!\sqcap \rangle$	$(S_N^+)^{\mathrm{irr}}_{\pm}$
5.4.34	7.1.10	$\langle \mathcal{V}_{(\delta)} \sqcap\!\sqcap\!\sqcap \rangle$	$(S_N^+)^{\mathrm{irr}}_{\pm} \,\tilde{\times}\, \hat{\mathbb{Z}}_2$
7.2.10	7.2.10	$\langle \mathcal{U}^{-1}_{(\delta)} {}_{\vartriangle}\!\diagdown\!\diagdown^{\triangledown} \rangle$	$U^*_{(N,\pm)} \left(O_N^+ \times\!\!\!\times \hat{\mathbb{Z}}_2 \right) U_{(N,\pm)}$
7.2.10	7.2.10	$\langle \mathcal{U}^{-1}_{(\delta)} ({}_{\vartriangle}\!\diagdown^{\triangledown})^{\otimes k} \rangle$	$U^*_{(N,\pm)} \left(O_N^+ \times_{2k} \hat{\mathbb{Z}}_2 \right) U_{(N,\pm)}$
7.2.11	7.2.11	$\langle \mathcal{P}_{(\delta)} \ast, \mathcal{U}^{-1}_{(\delta)} {}_{\vartriangle}\!\diagdown\!\diagdown^{\triangledown} \rangle$	$U^*_{(N,\pm)} \left(O_N^* \times\!\!\!\times \hat{\mathbb{Z}}_2 \right) U_{(N,\pm)}$
7.2.11	7.2.11	$\langle \mathcal{P}_{(\delta)} \ast, \mathcal{U}^{-1}_{(\delta)} ({}_{\vartriangle}\!\diagdown^{\triangledown})^{\otimes k} \rangle$	$U^*_{(N,\pm)} \left(O_N^* \times_{2k} \hat{\mathbb{Z}}_2 \right) U_{(N,\pm)}$
7.2.11	7.2.11	$\langle \mathcal{P}_{(\delta)} \times, \mathcal{U}^{-1}_{(\delta)} {}_{\vartriangle}\!\diagdown\!\diagdown^{\triangledown} \rangle$	$U^*_{(N,\pm)} \left(O_N \times\!\!\!\times \hat{\mathbb{Z}}_2 \right) U_{(N,\pm)}$
7.2.11	7.2.11	$\langle \mathcal{P}_{(\delta)} \times, \mathcal{U}^{-1}_{(\delta)} ({}_{\vartriangle}\!\diagdown^{\triangledown})^{\otimes k} \rangle$	$U^*_{(N,\pm)} \left(O_N \times_{2k} \hat{\mathbb{Z}}_2 \right) U_{(N,\pm)}$

Table 7.1 Summary of all non-easy categories of partitions studied in this thesis and the corresponding quantum group interpretation

Proposition (7.3.1)**.** Let $G = (C(G), u)$ be a homogeneous orthogonal quantum group corresponding to a category $\mathscr{K} \subseteq \mathsf{Part}_N$. Then $\tilde{G} := T_{\tau_{(N)}} G T_{\tau_{(N)}}^{-1}$ is also homogeneous and orthogonal. It corresponds to the category $\tilde{\mathscr{K}} := \mathcal{T}_{(N)}\mathscr{K}$.

Secondly, the disjoining and joining isomorphisms from Sections 5.4.3 and 5.4.4 correspond to some twisting. Given a quantum group G, we define some twist G^σ. In some special cases, one can describe G^σ by twisting the associated category $\mathscr{K}$. This is the case of the mentioned isomorphisms. However, this might not be possible in general. Nevertheless, one can always twist the functor $p \mapsto T_p$.

Theorem (7.4.5)**.** Let G be a quantum group group with $H_N \subseteq G \subseteq O_N^+$ corresponding to some linear category of partitions $\mathscr{K}$. Then the representation category of G^σ is described by the same partition category $\mathscr{K}$ if one uses the functor T^σ instead of T. That is,

$$\mathrm{Mor}(\hat{u}^{\otimes k}, \hat{u}^{\otimes l}) = \{T_p^\sigma \mid p \in \mathscr{K}(k,l)\}.$$

Based on all these theorems, we formulate the propositions listed in the second column of Table 7.1 that provide an interpretation of all the non-easy categories discovered in Chapter 5.

7.1 Quantum group subrepresentations

In this section we interpret constructions from Sections 5.4.5, 5.4.6. For easy quantum groups, whose fundamental representation is reducible, we study the projection and coisometry onto the $(N-1)$-dimensional invariant subspace and link it with the operations $\mathcal{P}$ and $\mathcal{V}$ acting on partitions. The section is based on the article [GW20].

7.1.1 Warm up example

Recall the bistochastic quantum group B_N^+ corresponding to the category $\langle \uparrow \rangle_N$. It is given by relations turning its fundamental representation $u = (u_{ij})_{i,j=1}^N$ into an orthogonal matrix, (that is, $u = \bar{u}$, $uu^t = u^t u = 1_N$) together with the bistochastic relation $u\xi_\uparrow = \xi_\uparrow$, where $\xi_\uparrow \in \mathbb{C}^N$ is the vector filled with entries all equal to one. Now, for any orthogonal matrix $U \in M_N(\mathbb{C})$ mapping $U\xi_\uparrow = \alpha e_N$ for some $\alpha \in \mathbb{C}$, we have

$$UuU^t = \begin{pmatrix} v & 0 \\ 0 & 1 \end{pmatrix} = v \oplus 1$$

with $v \in M_{N-1}(C(B_N^+))$. In addition, v is orthogonal, so we have just proved the isomorphism $B_N^+ \simeq O_{N-1}^+$ that was first formulated in [Rau12].

Now, any quantum subgroup $G \subseteq B_N^+$ has the same invariant subspace – in particular S_N or S_N^+. A natural question is then: To which quantum subgroup of O_{N-1}^+ is S_N^+ isomorphic? How to describe its intertwiner spaces? Surprisingly, this question is much harder for S_N^+ than for B_N^+ and depends on the choice of the unitary U.

To make this more precise, let $G = (C(G), u)$ be any CMQG with $S_N \subseteq G \subseteq B_N^+$. Then, as the category of B_N^+ is generated by the singleton $\uparrow$, we have again $u\xi_\uparrow = \xi_\uparrow$. Hence, for any orthogonal matrix U as above, we infer that G is isomorphic to some quantum group G^{irr} with $S_{N-1} \subseteq G^{\mathrm{irr}} \subseteq O_{N-1}^+$. Although G and G^{irr} are isomorphic (as compact quantum groups), they have different fundamental representations (so they are not identical compact matrix quantum groups), so we have no information about the category $\mathsf{FundRep}_{G^{\mathrm{irr}}}$ in general – and it may be very different from $\mathsf{FundRep}_G$ as in the example $B_N^+ \simeq O_{N-1}^+$.

We can make the statement even more general and consider a CMQG $S_N \subseteq G \subseteq B_N^{\#+}$, whose fundamental representation decomposes as $UuU^* = v \oplus r$, where $r \in C(G)$ is a one-dimensional representation of G. Then, the quantum group G^{irr} determined by the subrepresentation v might not be isomorphic to G itself, but essentially to some quotient $G/\hat{\mathbb{Z}}_2$.

7.1.2 Quantum groups with reducible subrepresentation

The fundamental representation of S_N decomposes into a direct sum of two irreducible representations: the trivial representation acting on the invariant subspace spanned by the vector $\xi_\uparrow = \sum_{i=1}^N e_i$ and the *standard representation*, which acts faithfully on the orthogonal complement of $\operatorname{span}\{\xi_\uparrow\}$. Therefore, the fundamental representation of any quantum group $G \supseteq S_N$ has at most those two invariant subspaces.

Recall that any compact matrix quantum group G with $S_N \subseteq G \subseteq O_N^+$ can be described by some linear category of partitions $\mathscr{K}$.

7.1.1 Lemma. Let G be a compact matrix quantum group with $S_N \subseteq G \subseteq O_N^+$ corresponding to a linear category of partitions $\mathscr{K}$. The fundamental representation of G is reducible if and only if $\uparrow \otimes \uparrow \in \mathscr{K}$, which holds if and only if $G \subseteq B_N^{\#+}$.

Proof. As mentioned above, the fundamental representation u of a quantum group $G \supseteq S_N$ is reducible if and only if $\operatorname{span}\{\xi_\uparrow\}$ is an invariant subspace. The projection onto $\operatorname{span}\{\xi_\uparrow\}$ can be written as $\frac{1}{N} T_{¦}$. Thus, u is reducible if and only if $T_{¦} \in \operatorname{Mor}(u,u)$, which holds if and only if $¦ \in \mathscr{K}$, which holds if and only if $\uparrow \otimes \uparrow \in \mathscr{K}$. (Recall that $B_N^{\#+}$ is the easy quantum group, whose category is generated by the partition $\uparrow \otimes \uparrow$.) □

Consider a quantum group G such that $S_N \subseteq G \subseteq B_N^{\#+}$, so its fundamental representation u has two invariant subspaces – $\operatorname{span}\{\xi_\uparrow\}$ and its orthogonal complement $\operatorname{span}\{\xi_\uparrow\}^\perp$. This means that taking any linear map $U: \mathbb{C}^N \to \mathbb{C}^N$ such that $\operatorname{span}\{\xi_\uparrow\}^\perp$ is mapped onto the space spanned by the first $N-1$ basis vectors $\operatorname{span}\{e_1, \ldots, e_{N-1}\}$ and $\xi_\uparrow$ is mapped onto (a multiple of) e_N, we get that $UuU^{-1} = v \oplus r$, where $v \in M_{N-1}(C(G))$ and $r \in C(G)$.

If, in addition, the matrix U is orthogonal, then UuU^{-1} is orthogonal, which means that v is orthogonal and r is a self-adjoint unitary (i.e. $r = r^*$ and $r^2 = 1$). In particular, both v and r are unitary representations of G. To extract just the subrepresentation v, we can define an $(N-1) \times N$ matrix V by taking the first $N-1$ rows of U. Then we have $v = VuV^*$.

Note that in the condition $U\xi_\uparrow = \alpha e_N$, the orthogonality implies $\alpha = \pm\sqrt{N}$. The condition $U(\operatorname{span}\{\xi_\uparrow\}^\perp) \subseteq \operatorname{span}\{e_1, \ldots, e_{N-1}\}$ is then satisfied automatically. Equivalently, we may require that the last row of U equals to $\frac{\pm 1}{\sqrt{N}} \xi_\uparrow^*$.

For the rest of this subsection, suppose that $U \in M_N(\mathbb{C})$ is an orthogonal matrix such that $U\xi_\uparrow = \pm\sqrt{N} e_N$ and V is the $(N-1) \times N$ matrix obtained by taking the first $N-1$ rows of U.

7.1.2 Lemma. V^* is an isometry and $\ker V = \operatorname{span}\{\xi_\uparrow\}$. That is, $VV^* = 1_{N-1}$ and $V^*V = P_{(N)}$, where $P_{(N)}$ is the orthogonal projection onto $\operatorname{span}\{\xi_\uparrow\}^\perp$.

Proof. The matrix V can be expressed as $V = EU$, where E is the "standard" coisometry $\mathbb{C}^N \to \mathbb{C}^{N-1}$ mapping $e_i \mapsto e_i$ for $i < N$ and $e_N \mapsto 0$. So, we have

$$VV^* = EUU^*E^* = EE^* = 1_{N-1}.$$

From this, it already follows that V^*V is a projection. Its range is $V^*V\mathbb{C}^N = V^*\mathbb{C}^{N-1}$, so it is spanned by the rows of V and hence it is indeed the orthogonal complement of the last row of U, which is a multiple of $\xi_\uparrow$. □

From the block structure $UuU^* = v \oplus r$, it follows that $\Delta(v_{ij}) = \sum_k v_{ik} \otimes v_{kj}$, so we can define the following.

7.1.3 Definition. Let $G = (C(G), u)$, $u = (u_{ij})_{i,j=1}^N$ be a compact matrix quantum group such that $S_N \subseteq G \subseteq B_N^{\#+}$. Then we denote $VGV^* := (A, v)$, where $v = VuV^* \in M_{N-1}(C(G))$ and A is the C*-subalgebra of $C(G)$ generated by $\{v_{ij}\}_{i,j=1}^{N-1}$.

7.1.4 Lemma. Let $G = (C(G), u)$, $u = (u_{ij})_{i,j=1}^N$ be a compact matrix quantum group such that $S_N \subseteq G \subseteq B_N^{\#+}$ and denote by v the fundamental representation of VGV^*. Then its intertwiner spaces are

$$\mathrm{Mor}(v^{\otimes k}, v^{\otimes l}) = \{V^{\otimes l} T V^{*\otimes k} \mid T \in \mathrm{Mor}(u^{\otimes k}, u^{\otimes l})\}.$$

Proof. For the inclusion $\supseteq$, take an arbitrary $T \in \mathrm{Mor}(u^{\otimes k}, u^{\otimes l})$, so $u^{\otimes l} T = T u^{\otimes k}$. Now, we have

$$(V^{\otimes l} u^{\otimes l} V^{*\otimes l})(V^{\otimes l} T V^{*\otimes k}) = V^{\otimes l} u^{\otimes l} T V^{*\otimes k} = V^{\otimes l} T u^{\otimes k} V^{*\otimes k} = (V^{\otimes l} T V^{*\otimes k})(V^{\otimes k} u^{\otimes k} V^{*\otimes k}),$$

where we used that V^*V is the projection onto the $(N-1)$-dimensional invariant subspace, so it commutes with u. For the inclusion $\subseteq$, we can similarly prove that given an intertwiner $T \in \mathrm{Mor}(v^{\otimes k}, v^{\otimes l})$, we have that $V^{*\otimes l} T V^{\otimes k} \in \mathrm{Mor}(u^{\otimes k}, u^{\otimes l})$ and we can express $T = V^{\otimes k}(V^{*\otimes k} T V^{\otimes l}) V^{*\otimes l}$ using that VV^* is the identity. □

So, considering G to be a quantum group corresponding to some linear category of partitions $\mathscr{K}$, we have that the intertwiners are given by T_p with $p \in \mathscr{K}$. The intertwiners of VGV^* are then of the form $V^{\otimes l} T_p V^{\otimes k*}$. In the following, we are going to find an explicit linear combination of partitions q such that $T_q = V^{\otimes l} T_p V^{\otimes k*}$. In order to do so, we have to make a special choice on the matrix V.

7.1.3 Quantum groups G_+^{irr} and G_-^{irr} and associated partition categories

Now, we define such an orthogonal matrix U explicitly. Recall that we require the last row of U to be $\frac{\pm 1}{\sqrt{N}} \xi_\uparrow^*$. For the rest of the matrix U, we can choose arbitrary rows that complete the last one to an orthonormal basis. Nevertheless, we choose a very specific symmetric form, where the matrix elements U_{ij} can be written as a combination $a\delta_{ij} + b$. The motivation will be clear in the following text (compare also with Section 7.3).

7.1.5 Definition. Let us define two orthogonal matrices $U_{(N,+)}, U_{(N,-)} \in M_N(\mathbb{C})$ as follows

$$[U_{(N,\pm)}]_{ij} = \delta_{ij} - \frac{1}{N-1}\left(1 \pm \frac{1}{\sqrt{N}}\right),$$
$$[U_{(N,\pm)}]_{iN} = [U_{(N,\pm)}]_{Nj} = [U_{(N,\pm)}]_{NN} = \pm\frac{1}{\sqrt{N}}$$

for $i, j \in \{1, \ldots, N-1\}$. Let us also denote by $V_{(N,\pm)}$ the $(N-1) \times N$ matrix formed by the first $N-1$ rows of $U_{(N,\pm)}$.

7.1.6 Definition. Let $G = (C(G), u)$ be a compact matrix quantum group such that $S_N \subseteq G \subseteq B_N^{\#+}$. We define the quantum groups G_+^{irr} and G_-^{irr} by $G_\pm^{\mathrm{irr}} := V_{(N,\pm)} G V_{(N,\pm)}^*$ in the sense of Definition 7.1.3.

7.1.7 Remark. The quantum groups G_+^{irr} and G_-^{irr} are by construction similar. We study the similarity in Section 7.3, see Proposition 7.3.2.

Recall Definition 5.4.29 introducing the following notation

- $v_{(N-1,\pm)} = | - \frac{1}{N-1}\left(1 \pm \frac{1}{\sqrt{N}}\right) \substack{\shortmid \\ \shortmid} \in \mathsf{Part}_{N-1}(1,1)$,
- $\mathcal{B}_{(N)}\colon \mathsf{Part}_N \to \mathsf{Part}_{N-1}$ acts blockwise (see Remark 5.4.31) sending $b_k \mapsto b_k + (-1)^k \uparrow^{\otimes k}$, where b_k is the block of k points,
- $\mathcal{V}_{(N,\pm)}\colon \mathsf{Part}_N \to \mathsf{Part}_{N-1}$ sends $p \mapsto v_{(N-1,\pm)}^{\otimes l}(\mathcal{B}_{(N)} p) v_{(N-1,\pm)}^{\otimes k}$ for $p \in \mathsf{Part}_N(k,l)$.

Let us also denote by $B_{(N)}$ the $(N-1) \times N$ matrix with entries $[B_{(N)}]_{ij} = \delta_{ij} - \delta_{Nj}$.

7.1.8 Lemma. Consider $N \in \mathbb{N} \setminus \{1\}$ and $p \in \mathsf{Part}_N(k,l)$, $k, l \in \mathbb{N}_0$. Then the following holds:

(1) $V_{(N,\pm)} = T_{v_{(N-1),\pm}} B_{(N)}$,
(2) $T_{\mathcal{B}_{(N)} p} = B_{(N)} p B_{(N)}^*$,
(3) $T_{\mathcal{V}_{(N,\pm)} p} = V_{(N,\pm)}^{\otimes l} T_p V_{(N,\pm)}^{*\otimes k}$.

Proof. To show item (1), note that the first $N-1$ columns of the product $T_{v_{(N-1,\pm)}}B_{(N)}$ is equal to the matrix $T_{v_{(N-1,\pm)}}$ itself and those are exactly the first $N-1$ columns of $V_{(N,\pm)}$. All the entries of the last column of the product are mutually equal and can be computed as

$$-\sum_{j=1}^{N-1}\left(\delta_{ij}-\frac{1}{N-1}\left(1\pm\frac{1}{\sqrt{N}}\right)\right)=-1+\left(1\pm\frac{1}{\sqrt{N}}\right)=\pm\frac{1}{\sqrt{N}},$$

which indeed exactly coincides with the last column of $V_{(N,\pm)}$.

It is enough to show item (2) for p being a partition. Since $\mathcal{B}_{(N)}$ acts blockwise, it is enough to show it for block partitions, i.e. partitions $b_k\in\mathscr{P}(0,k)$ consisting of a single block. So take any $i_1,\dots,i_k\in\{1,\dots,N-1\}$, then

$$[B_{(N)}^{\otimes k}T_{b_k}]_{i_1,\dots,i_k}=\sum_{j_1,\dots,j_k=1}^{N}(\delta_{i_1j_1}-\delta_{Nj_1})\cdots(\delta_{i_kj_k}-\delta_{Nj_k})\delta_{j_1,\dots,j_k}=\delta_{i_1,\dots,i_k}+(-1)^k=[T_{\mathcal{B}b_k}]_{i_1,\dots,i_k}.$$

Finally, item (3) is proven by the following

$$V_{(N,\pm)}^{\otimes l}T_pV_{(N,\pm)}^{*\otimes k}=T_{v_{(N-1,\pm)}}^{\otimes l}B_{(N)}^{\otimes l}T_pB_{(N)}^{*\otimes k}T_{v_{(N-1,\pm)}}^{*\otimes k}=T_{v_{(N-1,\pm)}^{\otimes l}}T_{\mathcal{B}_{(N)}p}T_{v_{(N-1,\pm)}^{\otimes l}}=T_{\mathcal{V}_{(N,\pm)}p}.\qquad\square$$

7.1.9 Theorem. [GW20, Theorems 4.8, 4.13] Let G be a quantum group such that $S_N\subseteq G\subseteq B_N^{\#+}$. This means that its representation category is described by a linear category of partitions $\mathscr{K}$ containing $\uparrow\otimes\uparrow$. Then it holds that $S_{N-1}\subseteq(S_N)_\pm^{\rm irr}\subseteq G_\pm^{\rm irr}\subseteq O_{N-1}^+$ and $G_\pm^{\rm irr}$ corresponds to the category $\mathcal{V}_{(N,\pm)}\mathscr{K}\subseteq\mathsf{Part}_{N-1}$.

Proof. Denote by u the fundamental representation of G and by v the fundamental representation of $G_\pm^{\rm irr}$. It follows directly from Lemmata 7.1.4, 7.1.8(3) that

$$\mathrm{Mor}(v^{\otimes k},v^{\otimes l})=\{V_{(N,\pm)}^{\otimes l}T_pV_{(N,\pm)}^{*\otimes k}\mid p\in\mathscr{K}(k,l)\}=\{T_{\mathcal{V}_{(\mathbb{N},\pm)}p}\mid p\in\mathscr{K}(k,l)\}=\{T_p\mid p\in\mathcal{V}_{(N,\pm)}\mathscr{K}\}.$$

So, $G^{\rm irr}$ indeed corresponds to $\mathcal{V}_{(N,\pm)}\mathscr{K}$. The fact that $S_{N-1}\subseteq G_\pm^{\rm irr}\subseteq O_{N-1}^+$ follows from the fact that $G_\pm^{\rm irr}$ corresponds to a partition category. In particular, surely $\mathcal{V}_{(N,\pm)}\mathscr{K}\subseteq\mathcal{V}_{(N,\pm)}\mathsf{Part}_N$, so $(S_N)_\pm^{\rm irr}\subseteq G_\pm^{\rm irr}$. $\square$

Note that the first part of Theorem 7.1.9 could be proven also directly, see [GW20, Theorem 4.8].

As an application, we can interpret the non-easy categories from Proposition 5.4.34.

7.1.10 Proposition. [GW20, Proposition 6.3] Suppose $N\geq 4$. It holds that

$$\langle\mathcal{V}_{(N,\pm)}\sqcap\!\sqcap\!\sqcap\rangle_{N-1}\text{ corresponds to }(S_N^+\tilde{\times}\hat{\mathbb{Z}}_2)_\pm^{\rm irr}=(S_N^+)_\pm^{\rm irr}\tilde{\times}\hat{\mathbb{Z}}_2,$$
$$\langle\mathcal{V}_{(N,\pm)}\sqcap\!\sqcap\rangle_{N-1}\text{ corresponds to }(S_N^+)_\pm^{\rm irr}.$$

Proof. Using Lemma 5.4.22 and Proposition 5.4.32, we derive

$$\mathcal{V}_{(N,\pm)}\mathsf{NCPart}_N=\mathcal{V}_{(N,\pm)}\mathcal{P}_{(N)}\mathsf{NCPart}_N=\mathcal{V}_{(N,\pm)}\langle\mathcal{P}_{(N)}\sqcap\!\sqcap\rangle_{N\text{-red}}=\langle\mathcal{V}_{(N,\pm)}\sqcap\!\sqcap\rangle_{N-1},$$

so, according to Theorem 7.1.9, $\langle\mathcal{V}_{(N,\pm)}\sqcap\!\sqcap\rangle_{N-1}$ corresponds to $(S_N^+)_\pm^{\rm irr}$.

Similarly, we deduce that $\langle\mathcal{V}_{(N,\pm)}\sqcap\!\sqcap\!\sqcap\rangle_{N-1}$ corresponds to $(S_N^+\tilde{\times}\hat{\mathbb{Z}}_2)_\pm^{\rm irr}=V_{(N,\pm)}(S_N^+\tilde{\times}\hat{\mathbb{Z}}_2)V_{(N,\pm)}^*$. The quantum group $S_N^+\tilde{\times}\hat{\mathbb{Z}}_2$ is determined by the fundamental representation of the form $u'=su$, where u is the fundamental representation of S_N^+, s generates $C^*(\mathbb{Z}_2)$ and $su_{ij}=u_{ij}s$. We see that $V_{(N,\pm)}u'V_{(N,\pm)}^*=s(V_{(N,\pm)}uV_{(N,\pm)}^*)=sv$, where v generates the quantum group $(S_N^+)_\pm^{\rm irr}$, so indeed $(S_N^+\tilde{\times}\hat{\mathbb{Z}}_2)_\pm^{\rm irr}=(S_N^+)_\pm^{\rm irr}\tilde{\times}\hat{\mathbb{Z}}_2$. $\square$

7.1.4 The subrepresentation within the N-dimensional setting

The last section interpreted the non-easy categories constructed using the map $\mathcal{V}_{(N,\pm)}$ defined in Sect. 5.4.6. Now, we would like to interpret the categories from Sect. 5.4.5 constructed using the map $\mathcal{P}_{(N)}$. Recall that this map sends $p \mapsto \pi_{(N)}^{\otimes l} p \pi_{(N)}^{\otimes k}$ for $p \in \mathrm{Part}_N(k,l)$. First of all, note the following observation.

7.1.11 Lemma. $T_{\pi_{(N)}}$ equals to $P_{(N)}$ – the orthogonal projection onto $\mathrm{span}\{\xi_\uparrow\}^\perp$.

Proof. We have that $\frac{1}{N} T_{\vdots} = \frac{1}{N} T_\uparrow T_\downarrow = \frac{1}{N} \xi_\uparrow \xi_\uparrow^*$ is the orthogonal projection onto $\mathrm{span}\{\xi_\uparrow\}$. Therefore, $T_{\pi_{(N)}} = 1 - \frac{1}{N} T_{\vdots}$ is the projection onto the orthogonal complement. □

In Section 5.4.6, we defined the concept of *reduced categories*. In particular, given a linear category of partitions $\mathscr{K}$, then $\mathcal{P}_{(N)}\mathscr{K}$ defines a reduced category. Suppose $\mathscr{K}$ corresponds to a quantum group $G = (C(G), u)$. Then the meaning of the reduced category $\mathcal{P}_{(N)}\mathscr{K}$ is that it describes the intertwiner spaces of a quantum group $P_{(N)} G P_{(N)}$ acting on the invariant subspace $P_{(N)}\mathbb{C}^N$. This is actually not a compact matrix quantum group according to our definition since its fundamental representation $P_{(N)} u P_{(N)}$ is not invertible. The answer to the question how to describe this as a compact matrix quantum group was given in the previous text – we constructed an isometry $V_{(N,\pm)}\colon \mathbb{C}^N \to \mathbb{C}^{N-1}$ mapping isomorphically $P_{(N)}\mathbb{C}^N \to \mathbb{C}^{N-1}$ and worked with the compact matrix quantum group $V_{(N,\pm)} G V_{(N,\pm)}^*$.

Now let us ask a different question. Given a reduced category $\mathscr{K}$, we want to interpret the categories

$$\langle \mathscr{K} \rangle_\delta \subsetneq \langle \mathscr{K}, \sqcap\!\sqcap\,{\scriptstyle|}\rangle_\delta \subsetneq \langle \mathscr{K}, \uparrow \rangle_\delta$$

constructed in Proposition 5.4.21.

7.1.12 Theorem. [GW20, Theorem 5.19] Let $\mathscr{K} \subseteq \mathrm{PartRed}_N$ be a reduced category. Denote by H the quantum group $S_{N-1} \subseteq H \subseteq O_{N-1}^+$ corresponding to the category $\mathcal{V}_{(N,\pm)}\mathscr{K} \subseteq \mathrm{Part}_{N-1}$. Then we can construct the quantum group corresponding to the following categories:

$$\begin{aligned} &\langle \mathscr{K} \rangle_N && \text{corresponds to} && U_{(N,\pm)}^* (H \,\hat{*}\, \hat{\mathbb{Z}}_2) U_{(N,\pm)}, \\ &\langle \mathscr{K}, {}^{\shortmid}\!/\!{}_{\shortmid} \rangle_N && \text{corresponds to} && U_{(N,\pm)}^* (H \times \hat{\mathbb{Z}}_2) U_{(N,\pm)}, \\ &\langle \mathscr{K}, \uparrow \rangle_N && \text{corresponds to} && U_{(N,\pm)}^* (H \times E) U_{(N,\pm)}, \end{aligned}$$

where $E = (\mathbb{C}, 1)$ is the trivial (quantum) group.

Proof. In order to simplify the notation, denote $U := U_{(N,\pm)}$, $V := V_{(N,\pm)}$. Denote by G the quantum group corresponding to $\langle \mathscr{K} \rangle_N$ and by $(u_{ij})_{i,j=1}^n$ its fundamental representation.

As was shown in Remark 5.4.17, $\uparrow \otimes \uparrow \in \langle \mathscr{K} \rangle_N$, so $UuU^* = v \oplus r$, where $v = VuV^* \in M_{N-1}(C(G))$ and $r = \sum_k u_{ik} \in C(G)$ such that $r^2 = 1$. Using Proposition 5.4.19 and Remark 5.4.31.b, we derive $\mathcal{V}\langle \mathscr{K} \rangle_N = \mathcal{V}\mathcal{P}\langle \mathscr{K} \rangle_N = \mathcal{V}\mathscr{K}$, so we see that v is the fundamental representation of H, i.e. $H = G_\pm^{\mathrm{irr}}$ according to Theorem 7.1.9. To prove that $G = U^*(H * \hat{\mathbb{Z}}_2)U$, it remains to show that there are no additional relations in $C(G)$ apart from the relations for v and the relations $r = r^*$, $r^2 = 1$.

The relations in $C(G)$ are precisely those corresponding to partitions in the category $\langle \mathscr{K} \rangle_N$, which is generated by $\mathscr{K}$ (and the pair partition $\sqcap$ of course). So, the relations are the orthogonality of u, which is equivalent to orthogonality of v and the relations $r = r^*$, $r^2 = 1$, and the relations implied by the partitions $p \in \mathscr{K}$. Taking any $p \in \mathscr{K}$, the relation $T_p u^{\otimes k} = u^{\otimes l} T_p$ is equivalent to

$$U^{\otimes l} T_p U^{*\otimes k} (v \oplus r)^{\otimes k} = U^{\otimes l} T_p u^{\otimes k} U^{*\otimes k} = U^{\otimes l} u^{\otimes l} T_p U^{*\otimes k} = (v \oplus r)^{\otimes l} U^{\otimes l} T_p U^{*\otimes k}.$$

Noticing that $T_p = T_{\pi^{\otimes k} p \pi^{\otimes l}} = P^{\otimes k} T_p P^{\otimes l}$ and that $UP = EU$, where E is the orthogonal projection onto the first $N-1$ basis vectors, we see that those relations only contain the subrepresentation v and hence are equivalent to the relations in $C(H)$.

The partitions ${}^{\shortmid}\!/\!{}_{\shortmid}$ and $\uparrow$ correspond to additional relations $ru_{ij} = u_{ij}r$ and $r = 1$, respectively. From this, the rest of the theorem follows. □

One can interpret the meaning of the presented theorem also from the other side. Consider a quantum group G corresponding to some category $\mathscr{K}$. In Section 7.1.3, we described the categories corresponding to $G_\pm^{\rm irr} = V_{(N,\pm)} G V_{(N,\pm)}^*$ removing the one-dimensional subrepresentation. Now the natural question is: how can we go back to the quantum group G and the category $\mathscr{K}$? Can we somehow reconstruct G from $G_\pm^{\rm irr}$? The most canonical way to extend the quantum group $G_\pm^{\rm irr}$ by some one-dimensional factor is to construct the products $H \times E$, $H \times \hat{\mathbb{Z}}_2$, and $H \mathbin{\hat{*}} \hat{\mathbb{Z}}_2$. The associated categories are then given by Theorem 7.1.12.

In Section 6.4, we introduced some alternative $\mathbb{Z}_2$-extensions $H \mathbin{\tilde{\times}} \hat{\mathbb{Z}}_2$ and $H \times_{2k} \hat{\mathbb{Z}}_2$ based on partitions with extra singletons. The question whether we can reconstruct also those within the setting of linear combinations of partitions is the motivation for Section 7.2.

Now, as an application for Theorem 7.1.12, let us interpret all the non-easy quantum groups from Propositions 5.4.24 and 5.4.28.

7.1.13 Proposition. The categories from Proposition 5.4.24

$$\begin{array}{ccccc}
\langle \mathcal{P}_{(N)}\sqcap\!\sqcap, \uparrow\otimes\uparrow\rangle_N & \subsetneq & \langle \mathcal{P}_{(N)}\sqcap\!\sqcap, \sqcap_{\!\shortmid}\,{}_{\shortmid}\rangle_N & \subsetneq & NC_N \\
\cup\!\!\!\!\!\not\;\; & & \cup\!\!\!\!\!\not\;\; & & \cup\!\!\!\!\!\not\;\; \\
\langle \mathcal{P}_{(N)}\sqcap\!\sqcap\!\sqcap, \uparrow\otimes\uparrow\rangle_N & \subsetneq & \langle \mathcal{P}_{(N)}\sqcap\!\sqcap\!\sqcap, \sqcap_{\!\shortmid}\,{}_{\shortmid}\rangle_N & \subsetneq & \langle \mathcal{P}_{(N)}\sqcap\!\sqcap\!\sqcap, \uparrow\rangle_N \\
\cup\!\!\!\!\!\not\;\; & & \cup\!\!\!\!\!\not\;\; & & \cup\!\!\!\!\!\not\;\; \\
\langle \uparrow\otimes\uparrow\rangle_N & \subsetneq & \langle \sqcap_{\!\shortmid}\,{}_{\shortmid}\rangle_N & \subsetneq & \langle\uparrow\rangle_N
\end{array}$$

correspond to quantum groups $G := U_{(N,\pm)}^* G' U_{(N,\pm)}$, where G' equals to

$$\begin{array}{ccccc}
(S_N^+)_\pm^{\rm irr} \mathbin{\hat{*}} \hat{\mathbb{Z}}_2 & \supsetneq & (S_N^+)_\pm^{\rm irr} \times \hat{\mathbb{Z}}_2 & \supsetneq & (S_N^+)_\pm^{\rm irr} \times E \\
\cap\!\!\!\!\!\not\;\; & & \cap\!\!\!\!\!\not\;\; & & \cap\!\!\!\!\!\not\;\; \\
((S_N^+)_\pm^{\rm irr} \mathbin{\tilde{\times}} \hat{\mathbb{Z}}_2) \mathbin{\hat{*}} \hat{\mathbb{Z}}_2 & \supsetneq & ((S_N^+)_\pm^{\rm irr} \mathbin{\tilde{\times}} \hat{\mathbb{Z}}_2) \times \hat{\mathbb{Z}}_2 & \supsetneq & ((S_N^+)_\pm^{\rm irr} \mathbin{\tilde{\times}} \hat{\mathbb{Z}}_2) \times E \\
\cap\!\!\!\!\!\not\;\; & & \cap\!\!\!\!\!\not\;\; & & \cap\!\!\!\!\!\not\;\; \\
O_{N-1}^+ \mathbin{\hat{*}} \hat{\mathbb{Z}}_2 & \supsetneq & O_{N-1}^+ \times \hat{\mathbb{Z}}_2 & \supsetneq & O_{N-1}^+ \times E
\end{array}$$

and $E = (\mathbb{C}, (1))$ is the trivial quantum group.

Proof. To prove this proposition, we just use Theorem 7.1.12 for each row. Let us have a look on the first row in more detail. Here, we take $\mathscr{K} := \mathcal{P}_{(N)}\mathsf{NCPart}_N = \langle \mathcal{P}_{(N)}\sqcap\!\sqcap\rangle_{N\text{-red}}$ (see Lemma 5.4.22). According to Proposition 5.4.34, the linear category $\mathcal{V}_{(N,\pm)}\mathscr{K} = \langle \mathcal{V}_{(N,\pm)}\sqcap\!\sqcap\rangle_{N-1}$ corresponds to the quantum group $H := (S_N^+)_\pm^{\rm irr}$. From Theorem 7.1.12 it follows that the quantum groups

$$\langle \mathscr{K}, \uparrow\otimes\uparrow\rangle_N \subsetneq \langle \mathscr{K}, \sqcap_{\!\shortmid}\,{}_{\shortmid}\rangle_N \subsetneq \langle \mathscr{K}, \uparrow\rangle_N$$

indeed correspond to the quantum groups given by the first row of the second table. Now, using Lemma 5.4.18 we see that $\langle \mathscr{K}, \uparrow\otimes\uparrow\rangle_N = \langle \mathcal{P}_{(N)}\sqcap\!\sqcap, \uparrow\otimes\uparrow\rangle_N$. Noticing that both $\sqcap_{\!\shortmid}\,{}_{\shortmid}$ and $\uparrow$ generate $\uparrow\otimes\uparrow$, we can use Lemma 5.4.18 to prove also $\langle \mathscr{K}, \sqcap_{\!\shortmid}\,{}_{\shortmid}\rangle_N = \langle \mathcal{P}_{(N)}\sqcap\!\sqcap, \sqcap_{\!\shortmid}\,{}_{\shortmid}\rangle_N$ and $\langle \mathscr{K}, \uparrow\rangle_N = \langle \mathcal{P}_{(N)}\sqcap\!\sqcap, \uparrow\rangle_N$. Finally, it is easy to see that the latter category equals to the linear category spanned by all non-crossing partitions NCPart_N. Indeed, note that NCPart_N is generated by $\sqcap\!\sqcap$, which is clearly contained in $\langle \mathcal{P}_{(N)}\sqcap\!\sqcap, \uparrow\rangle_N$.

The second and third line can be proven by exactly the same argumentation as the first one. For the second line, we use Lemma 5.4.23; for the third line, note that $\mathcal{V}_{(N,\pm)}$ acts on pair blocks as identity and singletons sends to zero, so $\mathcal{V}_{(N,\pm)}\langle\uparrow\rangle_N = \langle\rangle_{N-1}$. □

Note that the lower lines of the diagrams in Proposition 7.1.13 reveal the well-known isomorphisms $B_N^+ \simeq O_{N-1}^+$, $B_N'^+ \simeq O_{N-1} \times \hat{\mathbb{Z}}_2$, and $B_N^{\#+} \simeq O_{N-1} * \hat{\mathbb{Z}}_2$ discovered in [Rau12, Theorem 4.1] and [Web13, Proposition 5.2].

7.1.14 Proposition. The categories form Proposition 5.4.28

$$\begin{array}{ccccc}
\langle \mathcal{P}_{(\delta)}\diagup\!\!\!\!\diagdown, \mathcal{P}_{(\delta)}\sqcap\!\sqcap, \uparrow\otimes\uparrow\rangle_\delta & \subsetneq & \langle \mathcal{P}_{(\delta)}\diagup\!\!\!\!\diagdown, \mathcal{P}_{(\delta)}\sqcap\!\sqcap, \sqcap\!\shortmid\;\shortmid\rangle_\delta & \subsetneq & \langle \mathcal{P}_{(\delta)}\diagup\!\!\!\!\diagdown, \mathcal{P}_{(\delta)}\sqcap\!\sqcap, \uparrow\rangle_\delta = \mathsf{Part}_\delta \\
\cup\!\!\!\shortmid & & \cup\!\!\!\shortmid & & \cup\!\!\!\shortmid \\
\langle \mathcal{P}_{(\delta)}\diagup\!\!\!\!\diagdown, \mathcal{P}_{(\delta)}\sqcap\!\sqcap\!\sqcap, \uparrow\otimes\uparrow\rangle_\delta & \subsetneq & \langle \mathcal{P}_{(\delta)}\diagup\!\!\!\!\diagdown, \mathcal{P}_{(\delta)}\sqcap\!\sqcap\!\sqcap, \sqcap\!\shortmid\;\shortmid\rangle_\delta & \subsetneq & \langle \mathcal{P}_{(\delta)}\diagup\!\!\!\!\diagdown, \mathcal{P}_{(\delta)}\sqcap\!\sqcap\!\sqcap, \uparrow\rangle_\delta \\
\cup\!\!\!\shortmid & & \cup\!\!\!\shortmid & & \cup\!\!\!\shortmid \\
\langle \mathcal{P}_{(\delta)}\diagup\!\!\!\!\diagdown, \uparrow\otimes\uparrow\rangle_\delta & \subsetneq & \langle \mathcal{P}_{(\delta)}\diagup\!\!\!\!\diagdown, \sqcap\!\shortmid\;\shortmid\rangle_\delta & \subsetneq & \langle \mathcal{P}_{(\delta)}\diagup\!\!\!\!\diagdown, \uparrow\rangle_\delta \\
\cup\!\!\!\shortmid & & \cup\!\!\!\shortmid & & \cup\!\!\!\shortmid \\
\langle \mathcal{P}_{(\delta)}\ast, \uparrow\otimes\uparrow\rangle_\delta & \subsetneq & \langle \mathcal{P}_{(\delta)}\ast, \sqcap\!\shortmid\;\shortmid\rangle_\delta & \subsetneq & \langle \mathcal{P}_{(\delta)}\ast, \uparrow\rangle_\delta
\end{array}$$

correspond to quantum groups $G := U^*_{(N,\pm)} G' U_{(N,\pm)}$, where G' equals to

$$\begin{array}{ccccc}
(S_N)^{\mathrm{irr}}_\pm \mathbin{\hat{*}} \hat{\mathbb{Z}}_2 & \supsetneq & (S_N)^{\mathrm{irr}}_\pm \times \hat{\mathbb{Z}}_2 & \supsetneq & (S_N)^{\mathrm{irr}}_\pm \times E \\
\cap\!\!\!\shortmid & & \cap\!\!\!\shortmid & & \cap\!\!\!\shortmid \\
((S_N)^{\mathrm{irr}}_\pm \mathbin{\tilde{\times}} \hat{\mathbb{Z}}_2) \mathbin{\hat{*}} \hat{\mathbb{Z}}_2 & \supsetneq & ((S_N)^{\mathrm{irr}}_\pm \mathbin{\tilde{\times}} \hat{\mathbb{Z}}_2) \times \hat{\mathbb{Z}}_2 & \supsetneq & ((S_N)^{\mathrm{irr}}_\pm \mathbin{\tilde{\times}} \hat{\mathbb{Z}}_2) \times E \\
\cap\!\!\!\shortmid & & \cap\!\!\!\shortmid & & \cap\!\!\!\shortmid \\
O_{N-1} \mathbin{\hat{*}} \hat{\mathbb{Z}}_2 & \supsetneq & O_{N-1} \times \hat{\mathbb{Z}}_2 & \supsetneq & O_{N-1} \times E \\
\cap\!\!\!\shortmid & & \cap\!\!\!\shortmid & & \cap\!\!\!\shortmid \\
O^*_{N-1} \mathbin{\hat{*}} \hat{\mathbb{Z}}_2 & \supsetneq & O^*_{N-1} \times \hat{\mathbb{Z}}_2 & \supsetneq & O^*_{N-1} \times E
\end{array}$$

and $E = (\mathbb{C}, (1))$ is the trivial quantum group.

Proof. The proof is again the same. This time, we use Lemma 5.4.25 to find the image of the categories under $\mathcal{V}_{(N,\pm)}$. □

The category $\langle \mathcal{P}_{(N)}\ast, \uparrow\rangle_N$ and the corresponding quantum group was independently discovered by Banica [Ban18].

7.2 Reducible CMQGs in terms of extra singletons

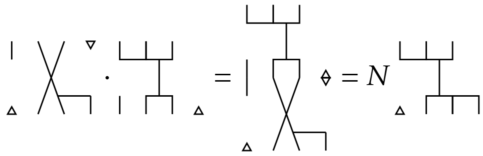
In Section 7.1, we interpreted many linear categories of partitions by decomposing the associated fundamental representation as $u = v \oplus r$ (up to similarity given by the unitary $U_{(N,\pm)}$). To describe such quantum groups, it may be much more convenient to use the framework of *partitions with extra singletons* – this was actually our original motivation to introduce extra-singleton categories. Now we can ask how those categories are related. As an answer, we will construct a functor $\mathcal{U}_{(N,\pm)}\colon \mathsf{Part}^{\shortmid}_N \to \mathsf{Part}^{\vartriangle}_{N-1}$. Here, $\mathsf{Part}^{\shortmid}_N$ is a slight modification of Part_N and $\mathsf{Part}^{\vartriangle}_{N-1}$ is a linear category of formal linear combinations of partitions with extra singletons.

But we can also go the converse direction. For example, in Theorem 7.1.12, we described the partition categories corresponding to the quantum groups $U^*_{(N,\pm)} G U_{(N,\pm)}$, where G is of the form $H \mathbin{\hat{*}} \hat{\mathbb{Z}}_2$, $H \times \hat{\mathbb{Z}}_2$, or $H \times E$ and $S_{N-1} \subseteq S_N^{\mathrm{irr}} \subseteq H \subseteq O^+_{N-1}$. In Section 6.4.2, we introduced additional products $G = H \mathbin{\hat{\times}} \hat{\mathbb{Z}}_2$ and $G = H \times_k \hat{\mathbb{Z}}_2$ satisfying $H \mathbin{\hat{*}} \hat{\mathbb{Z}}_2 \supseteq G \supseteq H \times \hat{\mathbb{Z}}_2$. Those are described by certain categories of partitions with extra singletons. Now we can apply the similarity $U_{(N,\pm)}$ and ask what linear categories of ordinary partitions describe the quantum group $U^*_{(N,\pm)} G U_{(N,\pm)}$ in the case of those new products. We obtain those as preimages by the functor $\mathcal{U}_{(N,\pm)}$.

This section is based on [GW19b, Section 7].

7.2.1 Linear categories of partitions with extra singletons

Similarly as with ordinary partitions, we can introduce the linear structure also for partitions with extra singletons. So, consider $N \in \mathbb{N}$ and define $\mathsf{Part}^{\vartriangle}_N(k,l) := \operatorname{span} \mathscr{P}^{\vartriangle}(k,l)$ the vector space of formal linear combinations of partitions with extra singletons. We modify the composition by introducing additional multiplicative factor N for each remaining loop *of the colour* $\shortmid$. If two extra singletons $\vartriangle$ disappear by composition, no additional factor is added. An example:



Again, we extend the tensor product and composition linearly and involution antilinearly to the whole vector spaces.

7.2.2 Separating linear combinations of partitions

One can already notice that the categories Part_N and $\mathsf{Part}_{N-1}^{\vartriangle}$ cannot be related in a straightforward way since the sets of objects are different. Another issue to notice is the following: As we remarked in Section 4.6.2, $\mathscr{P}$ can be considered as a full subcategory of $\mathscr{P}^{\vartriangle}$. In the linear case, the same holds: Part_{N-1} is a full subcategory of $\mathsf{Part}_{N-1}^{\vartriangle}$. An analogous statement for ordinary partitions is not true: $\mathsf{PartRed}_N$ is not a subcategory of Part_N. To fix this, we need to reconsider the set of objects on Part_N.

Consider the alphabet $\mathscr{O}_{\vdots} = \{\vdots, \uparrow\}$. Put $\pi^{\uparrow} := \frac{1}{N} {}^{\shortmid}_{\shortmid}$, $\pi^{\vdots} := \pi_{(N)} = | - \frac{1}{N} {}^{\shortmid}_{\shortmid}$ and similarly define the orthogonal projections $P^{\vdots} := T_{\pi^{\vdots}}$, $P^{\uparrow} := T_{\pi^{\uparrow}}$. For $w = a_1 \cdots a_k \in \mathscr{O}_{\vdots}^k$, we denote $\pi^{\otimes w} := \pi^{a_1} \otimes \cdots \otimes \pi^{a_k}$ and similarly $P^{\otimes w} := P^{a_1} \otimes \cdots \otimes P^{a_k}$.

For any $k \in \mathbb{N}_0$, the set of all $\pi^{\otimes w}$, $w \in \mathscr{O}_{\vdots}^k$ forms a complete set of mutually orthogonal projections in the sense that

$$\pi^{\otimes w} \pi^{\otimes w} = \pi^{\otimes w}, \quad (\pi^{\otimes w})^* = \pi^{\otimes w},$$
$$\pi^{\otimes w_1} \pi^{\otimes w_2} = 0 \quad \text{for } w_1 \neq w_2, |w_1| = |w_2|,$$
$$\sum_{w \in \mathscr{O}_{\vdots}^k} \pi^{\otimes w} = |^{\otimes k}.$$

Thus, any $p \in \mathsf{Part}_N(k,l)$ can be uniquely decomposed as

$$p = \sum_{\substack{w_1 \in \mathscr{O}_{\vdots}^k \\ w_2 \in \mathscr{O}_{\vdots}^l}} p_{w_2}^{w_1} \qquad \text{with} \qquad p_{w_2}^{w_1} = \pi^{\otimes w_2} p \pi^{\otimes w_1}. \tag{7.1}$$

We will say that $p \in \mathsf{Part}_N(k,l)$ is **separated** if there is $w_1 \in \mathscr{O}_{\vdots}^k$ and $w_2 \in \mathscr{O}_{\vdots}^l$ such that $p = p_{w_2}^{w_1}$. For example, for any $p \in \mathsf{Part}_N(k,l)$, all summands $p_{w_2}^{w_1}$ in the decomposition $p = \sum p_{w_2}^{w_1}$ are separated.

7.2.1 Definition. Any collection of vector spaces $\mathscr{K}(w_1, w_2) \subseteq \{p_{w_2}^{w_1} \mid p \in \mathsf{Part}_N(|w_1|, |w_2|)\}$ closed under the category operations and containing $\pi_{(N)}$, $\mathrm{Lrot}\,\pi_{(N)}$, ${}^{\shortmid}_{\shortmid}$, and $\uparrow \otimes \uparrow$ is called a **linear category of dotted partitions** or a **dotted category** for short.

7.2.2 Remark. In the spirit of category theory, the composition of $p \in \mathscr{K}(w_1, w_2)$ and $q \in \mathscr{K}(w_3, w_4)$ should be allowed only if the objects match, that is, $w_2 = w_3$. Nevertheless, note that if $w_2 \neq w_3$ then we have anyway $qp = 0$.

7.2.3 Proposition. There is the following one-to-one correspondence.

(1) Let $\mathscr{K}$ be a linear category of partitions containing $\uparrow \otimes \uparrow$. Then the collection of vector spaces

$$\mathscr{K}(w_1, w_2) := \{p \in \mathscr{K}(|w_1|, |w_2|) \mid p = p_{w_2}^{w_1}\} \tag{7.2}$$

forms a linear category of dotted partitions.

(2) Let $\mathscr{K}$ be a linear category of dotted partitions. Then

$$\mathscr{K}(k,l) = \bigoplus_{\substack{w_1 \in \mathscr{O}_{\vdots}^{k} \\ w_2 \in \mathscr{O}_{\vdots}^{l}}} \mathscr{K}(w_1, w_2) \tag{7.3}$$

define a linear category of partitions containing $\uparrow \otimes \uparrow$.

Proof. It is clear that if $\mathscr{K}$ is closed under the category operations as an ordinary category, then it must be closed under the category operations as a dotted category. Given that $\uparrow \otimes \uparrow \in \mathscr{K}$, then also $\,^{|}_{|}$, $\pi_{(N)}$, $\mathrm{Lrot}\,\pi_{(N)} \in \mathscr{K}$. So, $\mathscr{K}$ is a dotted category. The converse direction (2) is also clear.

To prove that it is a one-to-one correspondence, we have to check that for any ordinary category $\mathscr{K}$ containing $\uparrow \otimes \uparrow$, the morphism spaces are of the form (7.3). This is indeed true. Since $\uparrow \otimes \uparrow \in \mathscr{K}$, we have also $\pi^{\otimes w} \in \mathscr{K}$ for any any word w. Hence also $p_{w_2}^{w_1} = \pi^{\otimes w_2} p \pi^{\otimes w_1} \in \mathscr{K}$ for any $p \in \mathscr{K}(k,l)$. Therefore all the summands in the decomposition (7.1) of any $p \in \mathscr{K}$ are contained in $\mathscr{K}$. □

7.2.4 Definition. We denote by $\mathrm{Part}_N^{\vdots}$ the category of all partitions Part_N taken as a dotted category.

7.2.3 Basis for separated partitions

Take a partition $p \in \mathscr{P}(k,l)$. Define a word $w_1 \in \mathscr{O}_{\vdots}^{k}$ in such a way that on the i-th position there is the letter $\uparrow$ if p has a singleton on the i-th position in the upper row. Otherwise, we put the letter $\vdots$. Similarly we define the word $w_2 \in \mathscr{O}_{\vdots}^{l}$ corresponding to the lower row of p. Then we define $\dot{p} := p_{w_2}^{w_1}$. We depict the linear combination $\dot{p}$ pictorially using the graphical representation of p and replacing all the non-singleton blocks by dotted lines. The linear combinations $\dot{p}$ for any $p \in \mathscr{P}$ are called **dotted partitions**.

For example, taking $p := {}_{|}\backslash\backslash^{|}$, we denote

$$\dot{p} := {}_{|}\dot{\backslash}\dot{\backslash}^{|} := (\pi^{\vdots} \otimes \pi^{\vdots} \otimes \pi^{\uparrow})\,{}_{|}\backslash\backslash^{|}(\pi^{\uparrow} \otimes \pi^{\vdots} \otimes \pi^{\vdots}) = {}_{|}\backslash\backslash^{|} - \frac{1}{N}\,{}_{|}\backslash^{|}\;{}^{|}_{|} - \frac{1}{N}\,{}^{|}_{|}\;{}_{|}\backslash^{|} + \frac{1}{N^2}\,{}^{|}_{|}\;{}^{|}_{|}\;{}^{|}_{|}.$$

7.2.5 Lemma. The set $\{\dot{p} \mid p \in \mathscr{P}(k,l)\}$ forms a basis of the vector space $\mathrm{Part}_N(k,l)$ for any $k, l \in \mathbb{N}_0$.

Proof. For a partition $p \in \mathscr{P}$, we have that $\dot{p} = p + q$, where q is a linear combination of partitions having strictly more blocks than p. Consequently, if we order the dotted partition with respect to the number of blocks, then the matrix of coefficients of $\dot{p}$ with respect to the basis of standard partitions is triangular with non-zero entries on the diagonal. □

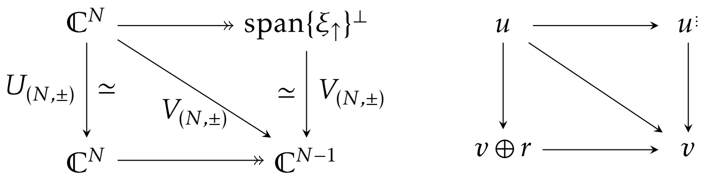
Consequently, the category of all dotted partitions $\mathrm{Part}_N^{\vdots}$ is spanned by dotted partitions.

7.2.4 Introducing the functor $\mathrm{Part}_N^{\vdots} \to \mathrm{Part}_{N-1}^{\blacktriangle}$

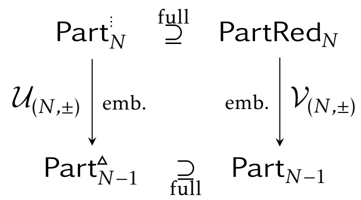
As one can already see, the categories of dotted partitions are somehow similar to categories of partitions with extra singletons. In this section, we are going to construct a functor mapping one structure to the other. However, this functor is a bit more complicated than just "mapping dotted blocks to normal blocks and singletons to extra singletons" as one might expect. In Section 7.2.5, we are going to restrict ourselves to pair partitions, where the functor indeed acts in this simple way.

Consider a linear category of partitions $\mathscr{K}$ such that $\uparrow \otimes \uparrow \in \mathscr{K}$, so it corresponds to a quantum group $G = (C(G), u)$, $S_N \subseteq G \subseteq B_N^{+\#}$, where the fundamental representation is reducible having a one-dimensional invariant subspace $\mathrm{span}\{\xi_{\uparrow}\}$. In Section 7.1, we defined an orthogonal matrix $U_{(N,\pm)} \in M_N(\mathbb{C})$ such that $U_{(N,\pm)} u U_{(N,\pm)}^*$ has a block structure $v \oplus r$ separating the two subrepresentations of u.

We studied which quantum group is generated by the $(N-1)$-dimensional subrepresentation of u. This can be done in two ways: either we consider the projection $P^{\vdots}\colon\mathbb{C}^N\to\mathbb{C}^N$ onto $\mathrm{span}\{\xi_\uparrow\}^\perp$ and study $u^{\vdots}=P^{\vdots}uP^{\vdots}$ or we first apply the map $U_{(N,\pm)}$ and then project onto the subspace generated by the first $N-1$ basis vectors and study $v=V_{(N,\pm)}uV^*_{(N,\pm)}$, where $V_{(N,\pm)}$ is the coisometry formed by the first $N-1$ rows of $U_{(N,\pm)}$. To summarize, we have the following maps.



In Section 7.1, we studied representation categories of u, $u^{\vdots}$ and v using linear categories of partitions. Categories of partitions with extra singletons allow us to study the representation category of $v\oplus r$. Now we are going to define and study the category isomorphism $\mathcal{U}_{(N,\pm)}$ that completes the following commutative diagram.



7.2.6 Remark. As indicated in the diagram above, $\mathsf{PartRed}_N$ is a full subcategory of $\mathsf{Part}^{\vdots}_N$. Indeed, we have $\mathsf{PartRed}_N(k,l)=\mathsf{Part}^{\vdots}(\vdots^k,\vdots^l)$. The category operations are defined the same way and the identity morphisms $\pi^{\otimes k}_{(N)}$ for $\mathsf{PartRed}_N$ coincide with the identity morphisms corresponding to the objects $\vdots^k$ in $\mathsf{Part}^{\vdots}_N$.

7.2.7 Definition. We define a functor $\mathcal{U}_{(N,\pm)}\colon\mathsf{Part}^{\vdots}_N\to\mathsf{Part}^{\vartriangle}_{N-1}$ as follows.

- On objects, $\mathcal{U}_{(N,\pm)}$ acts as a word isomorphism mapping $\vdots\mapsto|$ and $\uparrow\mapsto\vartriangle$.
- For morphisms, we describe the action on the basis of dotted partitions. Taking a dotted partition $\dot{p}\in\mathsf{Part}_N(w_1,w_2)$, the functor $\mathcal{U}_{(N,\pm)}$ acts blockwise. All singletons $\uparrow$ are mapped to $\sqrt{N}\vartriangle$. Any dotted block is mapped using the map $\mathcal{V}_{(N,\pm)}$, so

  $$\mathcal{U}_{(N,\pm)}\dot{b}_l:=\mathcal{V}_{(N,\pm)}\dot{b}_l=\mathcal{V}_{(N,\pm)}b_l,$$

  where b_l is a partition consisting of a single block of size $l>1$.

7.2.8 Proposition. $\mathcal{U}_{(N,\pm)}$ is indeed a faithful monoidal unitary functor. That is, we have

(1) $\mathcal{U}_{(N,\pm)}p\otimes\mathcal{U}_{(N,\pm)}q=\mathcal{U}_{(N,\pm)}(p\otimes q)$,
(2) $\mathcal{U}_{(N,\pm)}q\cdot\mathcal{U}_{(N,\pm)}p=\mathcal{U}_{(N,\pm)}(qp)$ if p and q are composable,
(3) $(\mathcal{U}_{(N,\pm)}p)^*=\mathcal{U}_{(N,\pm)}p^*$

for any $p,q\in\mathsf{Part}_N$.

Proof. Since $\mathcal{U}_{(N,\pm)}$ acts blockwise, it is clear that it behaves well with respect to the tensor product and involution. It is enough to show the functorial property (2) for dotted partitions. Here, we have to check that it behaves well in the case of singletons and dotted blocks. For singletons, it is easy to see it directly. For dotted blocks, it follows from $\mathcal{V}_{(N,\pm)}$ being a functor (Prop. 5.4.32). Also the faithfulness follows from $\mathcal{V}_{(N,\pm)}$ being faithful. □

Finally, we come to the main result of Section 7.2 – the interpretation of the functor $\mathcal{U}_{(N,\pm)}$. Recall Theorem 7.1.9 that gave a link between the coisometry $V_{(N,\pm)}: \mathbb{C}^N \to \mathbb{C}^{N-1}$ and the mapping $\mathcal{V}_{(N,\pm)}: \mathsf{Part}_N \to \mathsf{Part}_{N-1}$. (We can actually consider $\mathcal{V}_{(N,\pm)}$ as a functor $\mathsf{Part}^{\vdots}_N \to \mathsf{Part}_{N-1}$.) More precisely, the assignment $G \mapsto V_{(N,\pm)} G V^*_{(N,\pm)}$ can be described in terms of the linear categories of partitions as $\mathscr{K} \mapsto \mathcal{V}_{(N,\pm)}\mathscr{K}$. The following theorem extends this result. Instead of the coisometry $V_{(N,\pm)}$, whose kernel is the one-dimensional invariant subspace, we take the unitary $U_{(N,\pm)}$. Instead of the functor $\mathcal{V}_{(N,\pm)}$, which sends all singletons to zero, we take the faithful functor $\mathcal{U}_{(N,\pm)}$.

7.2.9 Theorem. It holds that

$$T_{\mathcal{U}_{(N,\pm)}p} = U^{\otimes l}_{(N,\pm)} T_p U^{*\otimes k}_{(N,\pm)}$$

for any $p \in \mathsf{Part}_N(w_1, w_2)$, $w_1 \in \mathscr{O}^k_{\vdots}$, $w_2 \in \mathscr{O}^l_{\vdots}$. Thus, considering a linear category of partitions $\mathscr{K} \subseteq \mathsf{Part}_N$ containing $\uparrow \otimes \uparrow$ and the corresponding quantum group G, it holds that the linear category with extra singletons $\mathcal{U}_{(N,\pm)}\mathscr{K}$ corresponds to the quantum group $U_{(N,\pm)} G U^*_{(N,\pm)}$.

Proof. Compare with Lemma 7.1.8, Theorem 7.1.9 and their proofs. Again, it is enough to show the equality for block partitions. To be more precise, in this case we check it for the singleton $p = \uparrow$ and for the dotted blocks $\dot{b}_l \in \mathsf{Part}_N(\emptyset, \vdots^l)$.

For the singleton, we have

$$T_{\mathcal{U}_{(N,\pm)}\uparrow} = \sqrt{N}\, T_{\vartriangle} = \sqrt{N}\, e_N = U_{(N,\pm)}\xi_\uparrow = U_{(N,\pm)} T_\uparrow.$$

For the dotted blocks it follows directly from Lemma 7.1.8(3). □

7.2.5 Dotted pairings: new source of non-easy categories

In this subsection, we present the main application of Theorem 7.2.9. Let us focus on categories $\mathscr{K} \subseteq \langle \uparrow, \times \rangle$, where all blocks have size at most two. Since $\mathcal{U}_{(N,\pm)}$ acts blockwise, its action is described by the image of a dotted pairing and the singleton. We have

$$\begin{array}{rcl} \vdots & \mapsto & | \\ \uparrow & \mapsto & \sqrt{N}\,\vartriangle \\ \langle \uparrow, \times \rangle \supseteq \mathscr{K} \supseteq \langle \uparrow \otimes \uparrow \rangle & \to & \langle \vartriangle, \times \rangle^{\vartriangle} \supseteq \mathscr{C} \supseteq \langle \rangle^{\vartriangle} \\ O_N \times E \subseteq G \subseteq O_N^+ \mathbin{\hat{*}} \hat{\mathbb{Z}}_2 & \to & B_N \subseteq \tilde{G} \subseteq B_N^{\#+} \end{array}$$

Thanks to this simple action of $\mathcal{U}_{(N,\pm)}$ it is easy to compute its inverse restricted to the category of all pair partitions with extra singletons $\langle \vartriangle, \times \rangle^{\vartriangle}$. Namely, we map all pair blocks to dotted blocks $| \mapsto \vdots$ and all extra singletons to normalized ordinary singletons $\vartriangle \mapsto \frac{1}{\sqrt{N}}\uparrow$.

Since we have a partial classification of the extra singleton pair categories in the easy case, this induces a large class of new examples of non-easy linear categories of partitions corresponding to quantum groups $B_N \subseteq G \subseteq B_N^{\#+}$. We can take the classification result [MW19a, MW19b, MW19c, MW20], apply the functor F from Section 4.6.4, obtain categories with extra singletons and apply the functor $\mathcal{U}$ to them.

As an example, we can compute

$$\mathcal{U}^{-1}_{(N,\pm)} {}_{\vartriangle}\setminus\!\setminus^{\triangledown} = \frac{1}{N}\, {}^{\shortmid}\!\!\setminus\!\setminus\!\!{}^{\shortmid} = \setminus\!\setminus' - \frac{1}{N^2}\, {}^{\shortmid}\!\setminus' \,\vdots - \frac{1}{N^2}\, \vdots\, {}^{\shortmid}\!\setminus' + \frac{1}{N^3}\, \vdots\,\vdots\,\vdots\,\vdots,$$

$$\mathcal{U}^{-1}_{(N,\pm)} ({}_{\vartriangle}\setminus^{\triangledown})^{\otimes k} = \left(\frac{1}{N}\, {}^{\shortmid}\!\setminus'\right)^{\otimes k} = \left(\frac{1}{N}\, {}^{\shortmid}\!\setminus' - \frac{1}{N^2}\, \vdots\,\vdots\right)^{\otimes k}.$$

Moreover, we can apply $\mathcal{U}^{-1}_{(N,\pm)}$ to the whole categories $\langle {}_{\vartriangle}\setminus\!\setminus^{\triangledown} \rangle^{\vartriangle}$ and $\langle ({}_{\vartriangle}\setminus^{\triangledown})^{\otimes k} \rangle^{\vartriangle}$ corresponding to the quantum groups $O_N^+ \ltimes \hat{\mathbb{Z}}_2$ and $O_N^+ \times_{2k} \hat{\mathbb{Z}}_2$. This way, we obtain non-easy categories corresponding to new non-easy quantum groups that are isomorphic to the original ones (see also Remark 5.4.7).

7.2.10 Proposition. The following are non-easy and mutually distinct linear categories of partitions including their interpretation

$$\left\langle \text{(partition)} - \frac{1}{N}\text{(partition)} - \frac{1}{N}\text{(partition)} + \frac{1}{N^2}\text{(partition)} \right\rangle_N \quad\leftrightarrow\quad U^*_{(N,\pm)} O_N^+ \times \hat{\mathbb{Z}}_2 U_{(N,\pm)},$$

$$\left\langle \left(\text{(partition)} - \frac{1}{N}\text{(partition)}\right)^{\otimes k} \right\rangle_N \quad\leftrightarrow\quad U^*_{(N,\pm)} O_N^+ \times_{2k} \hat{\mathbb{Z}}_2 U_{(N,\pm)}, \quad k \in \mathbb{N} \setminus \{1\}.$$

Proof. As we already mentioned, the generators actually equal to $\text{(partition)} = N\,\mathcal{U}^{-1}_{(N,\pm)}\text{(partition)}$, resp. $N^k(\text{(partition)})^{\otimes k} = N^k \mathcal{U}^{-1}_{(N,\pm)}(\text{(partition)})^{\otimes k}$. Strict inclusions for the categories with extra singletons induce corresponding inclusions in our case. In particular, this proves the mutual inequality of the categories and their non-easiness. For the latter, note that the smallest easy category containing any of those above must be $\langle \text{(partition)} \rangle_N = \langle \text{(partition)} \rangle_N$. $\square$

It is easy to write down the relations corresponding to those categories. Recall the relations for the partitions with extra singletons

$$\text{(partition)} \quad\leftrightarrow\quad r v_{ij} v_{kl} = v_{ij} v_{kl} r,$$

$$(\text{(partition)})^{\otimes k} \quad\leftrightarrow\quad r v_{i_1 j_1} r v_{i_2 j_2} \cdots r v_{i_k j_k} = v_{i_1 j_1} r v_{i_2 j_2} r \cdots v_{i_k j_k} r,$$

where $v \oplus r$ is the fundamental representation of the quantum group. The quantum groups corresponding to the categories from Proposition 7.2.10, i.e. defined by the dotted partitions (partition), resp. $(\text{(partition)})^{\otimes k}$ are quantum subgroups of $B_N^{\#+} = (C(B_N^{\#+}), u)$ defined by precisely the same relations if we interpret r as the one-dimensional subrepresentation $r := \sum_k u_{ik} = \sum_k u_{kj}$ and $v := u^{\text{(partition)}} = u - \frac{1}{N} r$.

7.2.11 Remark. Additional non-easy categories can be obtained by adding the linear combinations $\text{(partition)} = \mathcal{P}_{(N)}\text{(partition)}$ or $\text{(partition)} = \mathcal{P}_{(N)}\text{(partition)}$ to any of the two categories from Proposition 7.2.10. The interpretation is then obtained by replacing O_N^+ with O_N^* or O_N, respectively.

7.3 Quantum group similarity induced by τ

In this section, we interpret the isomorphism $\mathcal{T}_{(N)}$ defined in Section 5.4.2. Recall the definition of the partition $\tau_{(N)} = | - \frac{2}{N}\text{(partition)} \in \mathsf{Part}_N(1,1)$.

7.3.1 Proposition. Let $G = (C(G), u)$ be a homogeneous orthogonal quantum group corresponding to a category $\mathscr{K} \subseteq \mathsf{Part}_N$. Then $\tilde{G} := T_{\tau_{(N)}} G T^{-1}_{\tau_{(N)}}$ is also homogeneous and orthogonal. It corresponds to the category $\tilde{\mathscr{K}} := \mathcal{T}_{(N)}\mathscr{K}$.

Proof. We assume that G is homogeneous and orthogonal, so $S_N \subseteq G \subseteq O_N^+$. The matrix $T_{\tau_{(N)}}$ is an intertwiner of S_N, so $S_N \subseteq \tilde{G}$. We mentioned in Sect. 5.4.2 that $\tau_{(N)}$ behaves as a self-adjoint unitary, so $\tau^*_{(N)} = \tau_{(N)}$ and $\tau_{(N)}\tau_{(N)} = |$. The same relations must hence hold also for $T_{\tau_{(N)}}$, so, in particular, it has to be orthogonal. Consequently, $\tilde{G} \subseteq O_N^+$.

Let $\tilde{u} = T_{\tau_{(N)}} u T^{-1}_{\tau_{(N)}} = T_{\tau_{(N)}} u T_{\tau_{(N)}}$ be the fundamental representation of $\tilde{G}$. One can easily see that $T \in \mathrm{Mor}(u^{\otimes k}, u^{\otimes l})$ if and only if $T^{\otimes l}_{\tau_{(N)}} T T^{\otimes l}_{\tau_{(N)}} \in \mathrm{Mor}(\tilde{u}^{\otimes k}, \tilde{u}^{\otimes l})$. $\square$

Moreover the orthogonal matrix $T_{\tau_{(N)}}$ provide the similarity between the quantum groups G_+^{irr} and G_-^{irr} introduced in Def. 7.1.6.

7.3.2 Proposition. Consider a compact matrix quantum group G such that $S_N \subseteq G \subseteq B_N^{\#+}$. Then $T_{\tau_{(N-1)}} G_\pm^{\mathrm{irr}} T^{-1}_{\tau_{(N-1)}} = G_\mp^{\mathrm{irr}}$, i.e. G_+^{irr} and G_-^{irr} are similar.

Proof. First, we check that $V_{(N,\mp)} V^*_{(N,\pm)} = T_{\tau_{(N)}}$. Recall from Lemma 7.1.8(1) that $V_{(N,\pm)} = T_{v(N-1,\pm)} B_{(N)}$. Easy computation shows that $B_{(N)} B^*_{(N)} = T_{|+\text{(partition)}}$. So, $V_{(N,\mp)} V^*_{(N,\pm)} = T_p$ with $p = v_{(N-1,\pm)}(| + \text{(partition)}) v_{(N-1,\pm)}$, which indeed equals $\tau_{(N)}$. Finally, it is easy to see that this operator indeed defines the similarity between G_+^{irr} and G_-^{irr} since

$$V_{(N,\mp)} V^*_{(N,\pm)} G_\pm^{\mathrm{irr}} V_{(N,\pm)} V^*_{(N,\mp)} = V_{(N,\mp)} V^*_{(N,\pm)} V_{(N,\pm)} G V^*_{(N,\pm)} V_{(N,\pm)} V^*_{(N,\mp)} = V_{(N,\mp)} G V^*_{(N,\mp)} = G_\mp^{\mathrm{irr}}. \quad \square$$

7.4 Anticommutative twists

In this section, we interpret the category isomorphisms $\mathcal{D}$ and $\mathcal{J}$ described in Sections 5.4.3, 5.4.4. As a consequence, we are going to interpret the non-easy categories

$$\left\langle \times - \frac{2}{N}(\,{}^{\shortmid}\!/_{\shortmid} + {}_{\shortmid}\backslash^{\shortmid}) + \frac{4}{N^2}\,{}^{\shortmid}_{\shortmid}\,{}^{\shortmid}_{\shortmid} \right\rangle_N \quad \text{and} \quad \langle 2\,\sqcap\!\!\sqcup - \times \rangle_N.$$

As we showed in the above mentioned sections, these categories are both isomorphic to the category of all pairings Pair_N. The idea is that instead of studying the image of these categories under the standard functor $p \mapsto T_p$, we study some alternative functors $p \mapsto \tilde{T}_p = T_{\mathcal{D}p}$, resp. $p \mapsto \tilde{T}_p = T_{\mathcal{J}p}$ acting on pair partitions $p \in \mathsf{Pair}_N$. Changing this functor also changes the interpretation of the partitions in terms of relations. In particular, the crossing partition $\times$, which generates the whole category Pair_N, will no longer imply commutativity. We get some deformed commutativity instead. More concretely, some minus signs will appear, so the commutativity will partially change to anticommutativity.

This section basically coincides with [GW19a, Section 6].

7.4.1 2-cocycle deformations

We briefly describe a construction from [Doi93, Sch96, BY14].

Let A be a Hopf $*$-algebra. A **unitary 2-cocycle** on A is a convolution invertible linear map $\sigma\colon A \otimes A \to \mathbb{C}$ satisfying

$$\sum_{(x),(y)} \sigma(x_{(1)}, y_{(1)})\sigma(x_{(2)}y_{(2)}, z) = \sum_{(y),(z)} \sigma(y_{(1)}, z_{(1)})\sigma(x, y_{(2)}z_{(2)}),$$
$$\sigma^{-1}(x,y) = \overline{\sigma(S(x)^*, S(y)^*)},$$

and $\sigma(x,1) = \sigma(1,x) = \varepsilon(x)$ for $x,y,z \in A$, where σ^{-1} denotes the convolution inverse of σ and we use the Sweedler notation $\Delta(x) = \sum_{(x)} x_{(1)} \otimes x_{(2)}$.

Let G be a compact quantum group and σ a 2-cocycle on the associated Hopf $*$-algebra $\operatorname{Pol} G$. Then we can define its deformation G^σ, where $\operatorname{Pol} G^\sigma$ coincides with $\operatorname{Pol} G$ as a coalgebra (i.e. a vector space with comultiplication and a counit) and the $*$-algebra structure is defined as follows

$$\hat{x}\hat{y} = \sum_{(x),(y)} \sigma(x_{(1)}, y_{(1)})\sigma^{-1}(x_{(3)}, y_{(3)})\widehat{x_{(2)}y_{(2)}}, \tag{7.4}$$

$$\hat{x}^* = \sum_{(x)} \sigma(S(x_{(5)})^*, x^*_{(4)})\sigma^{-1}(x^*_{(2)}, S(x_{(1)})^*)\widehat{x^*_{(3)}}, \tag{7.5}$$

where $\hat{x}$ denotes $x \in \operatorname{Pol} G$ viewed as an element of $\operatorname{Pol} G^\sigma$.

In particular, if G is a compact matrix quantum group with fundamental representation u, then G^σ is again a compact matrix quantum group with fundamental representation $\hat{u} = (\hat{u}_{ij})_{i,j}$.

It holds that the quantum groups G and G^σ have monoidally equivalent representation categories [Sch96].

Consider compact quantum groups $H \subseteq G$, so there is a surjection $q\colon \operatorname{Pol} G \to \operatorname{Pol} H$. Then a 2-cocycle σ on H induces a 2-cocycle $\sigma_q := \sigma \circ (q \otimes q)$ on G. We will construct 2-cocycles on quantum groups induced by bicharacters on dual discrete quantum subgroups $\hat{\Gamma} \subseteq G$.

Let Γ be a group. A **unitary bicharacter** on Γ is a map $\varphi\colon \Gamma \times \Gamma \to \mathbb{T}$ (here $\mathbb{T}$ denotes the complex unit circle) satisfying

$$\varphi(xy,z) = \varphi(x,z)\varphi(y,z), \qquad \varphi(x,yz) = \varphi(x,y)\varphi(x,z).$$

In particular, we have $\varphi(x,e) = \varphi(e,x) = 1$. It is easy to see that any unitary bicharacter φ on a discrete group Γ extends to a unitary 2-cocycle on $\mathbb{C}\Gamma = \operatorname{Pol} \hat{\Gamma}$.

7.4.2 Anticommutative twists

We now make a special choice for σ. Consider any $\sigma \in M_N(\{\pm 1\})$. One can easily check that the map $(t_i, t_j) \mapsto \sigma_{ij}$, where $t_1, \dots, t_N$ are generators of $\mathbb{Z}_2^N$, uniquely extends to a bicharacter on $\mathbb{Z}_2^N$. This induces a 2-cocycle on any quantum group G containing $\hat{\mathbb{Z}}_2^N$ as a quantum subgroup.

So, suppose G is a compact matrix quantum group with fundamental representation $u \in M_N(\operatorname{Pol} G)$ and $q\colon \operatorname{Pol} G \to \mathbb{C}\mathbb{Z}_2^N$ maps $u_{ij} \mapsto t_i \delta_{ij}$. Let us, for simplicity, restrict to the case $G \subseteq O_N^+$.

For a multi-index $\boldsymbol{i} = (i_1, \dots, i_k)$, denote $\sigma_{\boldsymbol{i}} := \prod_{1 \le m < n \le k} \sigma_{i_m i_n}$.

7.4.1 Lemma. Suppose, $\bar{u} = u$, i.e. $u_{ij}^* = u_{ij}$. Then

$$\hat{u}_{ij}^* = \sigma_{ii}\sigma_{jj}\hat{u}_{ij}, \tag{7.6}$$

$$\hat{u}_{i_1 j_1} \cdots \hat{u}_{i_k j_k} = \sigma_{\boldsymbol{i}}\sigma_{\boldsymbol{j}}\,\widehat{(u_{i_1 j_1} \cdots u_{i_k j_k})}. \tag{7.7}$$

Proof. Both formulae are obtained simply by using the defining formulae (7.4), (7.5). For the second one, we need to apply induction on k. □

Recall the definition of the universal unitary and orthogonal quantum groups from Section 2.3.3.

7.4.2 Proposition. Suppose $G \subseteq O_N^+$. Then $G^\sigma \subseteq O^+(F) \subseteq U^+(F) = U_N^+$ with $F_{ij} = \delta_{ij}\sigma_{ii}$.

Proof. All the relations are checked using Lemma 7.4.1. The relation $\bar{\hat{u}} = F^{-1}\hat{u}F$ is just a matrix version of Eq. (7.6). Checking the unitarity of $\hat{u}$ is also straightforward. As an example, let us check the relation $\hat{u}\hat{u}^* = 1_N$:

$$\sum_k \hat{u}_{ik}\hat{u}_{jk}^* = \sum_k \sigma_{jj}\sigma_{kk}\hat{u}_{ik}\hat{u}_{jk} = \sum_k \sigma_{ij}\sigma_{jj}\widehat{u_{ik}u_{jk}} = \delta_{ij}.$$

Finally, the fact that $U^+(F) = U_N^+$ follows from $F^*F = 1_N$. Indeed, $F\bar{\hat{u}}F^{-1}$ being unitary can be written as $F\bar{\hat{u}}F^{-1}(F^*)^{-1}\hat{u}^t F^* = 1_N$ and $(F^*)^{-1}\hat{u}^t F^* F \bar{\hat{u}} F^{-1} = 1_N$. These relations are obviously equivalent to $\bar{\hat{u}}\hat{u}^t = 1_N = \hat{u}^t\bar{\hat{u}}$. □

Now we analyse the intertwiner spaces for the twisted quantum group G^σ. This will also prove the equivalence of the representation categories for our special choice of the 2-cocycle.

7.4.3 Proposition. Consider $G = (C(G), u) \subseteq O_N^+$. Then

$$\operatorname{Mor}(\hat{u}^{\otimes k}, \hat{u}^{\otimes l}) = \{T^\sigma \mid T \in \operatorname{Mor}(u^{\otimes k}, u^{\otimes l})\}$$

with $T^\sigma_{\boldsymbol{ij}} = T_{\boldsymbol{ij}}\sigma_{\boldsymbol{i}}\sigma_{\boldsymbol{j}}$.

Proof. If $T \in \operatorname{Mor}(u^{\otimes k}, u^{\otimes l})$, it means that $Tu^{\otimes k} = u^{\otimes l}T$, which is certainly equivalent to $T\widehat{u^{\otimes k}} = \widehat{u^{\otimes l}}T$. We can rewrite this in matrix entries as

$$\sum_{\boldsymbol{m}} T_{\boldsymbol{im}}\,\widehat{(u_{m_1 j_1} \cdots u_{m_k j_k})} = \sum_{\boldsymbol{n}} \widehat{(u_{i_1 n_1} \cdots u_{i_l n_l})}\,T_{\boldsymbol{nj}}.$$

Now, applying Lemma 7.4.1, we can rewrite this as

$$\sum_{\boldsymbol{m}} \frac{T_{\boldsymbol{im}}}{\sigma_{\boldsymbol{m}}\sigma_{\boldsymbol{j}}}\hat{u}_{m_1 j_1} \cdots \hat{u}_{m_k j_k} = \sum_{\boldsymbol{n}} \frac{T_{\boldsymbol{nj}}}{\sigma_{\boldsymbol{i}}\sigma_{\boldsymbol{n}}}\hat{u}_{i_1 n_1} \cdots \hat{u}_{i_l n_l}.$$

Finally, using the fact that $\sigma_{\boldsymbol{i}}, \sigma_{\boldsymbol{j}} = \pm 1$, we can see that this is equivalent to $T^\sigma\hat{u}^{\otimes k} = \hat{u}^{\otimes l}T^\sigma$. □

In connection with partition categories, we can interpret this result as follows. Consider $G := H_N$ the hyperoctahedral group, which corresponds to the category $\mathsf{EvenPart}_N := \langle \times, \sqcap\!\sqcap \rangle_N$ spanned by partitions with blocks of even length. It is the smallest homogeneous quantum group having $\hat{\mathbb{Z}}_2^N$ as a quantum subgroup. The matrix σ then defines an alternative functor $T^\sigma\colon \mathsf{EvenPart}_N \to \mathsf{Mat}$ mapping $p \mapsto T_p^\sigma$ with $[T_p^\sigma]_{\boldsymbol{ij}} = [T_p]_{\boldsymbol{ij}}\sigma_{\boldsymbol{i}}\sigma_{\boldsymbol{j}} = \delta_p(\boldsymbol{j}, \boldsymbol{i})\sigma_{\boldsymbol{i}}\sigma_{\boldsymbol{j}}$.

7.4.4 Lemma. The map $T^\sigma\colon \mathsf{EvenPart}_N \to \mathsf{Mat}$ is indeed a monoidal unitary functor.

Proof. Checking that T^σ behaves well with respect to composition and involution is straightforward using the fact that $p \mapsto T_p$ is a monoidal unitary functor. Let us do it for the composition.

$$[T_q^\sigma T_p^\sigma]_{\boldsymbol{ac}} = \sum_{\boldsymbol{b}} [T_q^\sigma]_{\boldsymbol{ab}} [T_p^\sigma]_{\boldsymbol{bc}} = \sum_{\boldsymbol{b}} [T_q^\sigma]_{\boldsymbol{ab}} [T_p^\sigma]_{\boldsymbol{bc}} \sigma_{\boldsymbol{a}} \sigma_{\boldsymbol{b}} \sigma_{\boldsymbol{b}} \sigma_{\boldsymbol{c}}$$
$$= \sigma_{\boldsymbol{a}} \sigma_{\boldsymbol{c}} \sum_{\boldsymbol{b}} [T_q^\sigma]_{\boldsymbol{ab}} [T_p^\sigma]_{\boldsymbol{bc}} = [T_{qp}]_{\boldsymbol{ac}} \sigma_{\boldsymbol{a}} \sigma_{\boldsymbol{c}} = [T_{qp}^\sigma]_{\boldsymbol{ac}}$$

The tensor product is a bit more complicated. We need to check that

$$\sigma_{\boldsymbol{ac}} \sigma_{\boldsymbol{bd}} \delta_{p\otimes q}(\boldsymbol{ac}, \boldsymbol{bd}) = \sigma_{\boldsymbol{a}} \sigma_{\boldsymbol{b}} \sigma_{\boldsymbol{c}} \sigma_{\boldsymbol{d}} \delta_p(\boldsymbol{a}, \boldsymbol{b}) \delta_q(\boldsymbol{c}, \boldsymbol{d})$$

for any two partitions $p \in \mathscr{P}(k,l)$, $q \in \mathscr{P}(m,n)$ with blocks of even length. We know that $p \mapsto T_p$ is a monoidal functor, so $\delta_{p\otimes q}(\boldsymbol{ac}, \boldsymbol{bd}) = \delta_p(\boldsymbol{a}, \boldsymbol{b}) \delta_q(\boldsymbol{c}, \boldsymbol{d})$. Take any $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$, $\boldsymbol{d}$ such that $\delta_{p\otimes q}(\boldsymbol{ac}, \boldsymbol{bd}) = 1$. We need to show that $\sigma_{\boldsymbol{ac}} \sigma_{\boldsymbol{bd}} = \sigma_{\boldsymbol{a}} \sigma_{\boldsymbol{b}} \sigma_{\boldsymbol{c}} \sigma_{\boldsymbol{d}}$. Equivalently, we need to show that

$$\prod_{i=1}^{k} \prod_{j=1}^{l} \prod_{s=1}^{m} \prod_{t=1}^{n} \sigma_{a_i c_s} \sigma_{b_j d_t} = 1.$$

Recall that we assume that all blocks of p and q have even size. Consequently, one can check that, for every block V of p and every block W of q, there is an even amount of terms $\sigma_{a_i c_s}$ or $\sigma_{b_j d_t}$ of the product with $i \in V$ and $s \in W$ resp. $j \in V$ and $t \in W$. Since we assume $\delta_{p\otimes q}(\boldsymbol{ac}, \boldsymbol{bd}) = 1$, the multiindices $\boldsymbol{ab}$ and $\boldsymbol{cd}$ are constant on the blocks. As a consequence, the product of those terms always equals one. □

7.4.5 Corollary. Let G be a quantum group group with $H_N \subseteq G \subseteq O_N^+$ corresponding to some linear category of partitions $\mathscr{K}$. Then the representation category of G^σ is described by the same partition category $\mathscr{K}$ if one uses the functor T^σ instead of T. That is,

$$\mathrm{Mor}(\hat{u}^{\otimes k}, \hat{u}^{\otimes l}) = \{T_p^\sigma \mid p \in \mathscr{K}(k,l)\}.$$

Proof. Follows directly from Proposition 7.4.3 and the definition of T_p^σ. □

7.4.6 Proposition. For any $p \in \mathsf{EvenPart}_N \cap \mathsf{NCPart}_N$, we have $T_p^\sigma = T_p$. In particular, twisting by σ leads to a new quantum group only for categories with crossings.

Proof. It is enough to prove the statement for partitions. Then by linearity of T and T^σ, it must hold also for linear combinations.

So, let p be a non-crossing partition with blocks of even size. It is known that non-crossing partitions are always of the form of some nested blocks. That is, up to rotation, we have $p = q \otimes b$, where b is a partition consisting of a single block. Since both T and T^σ are monoidal functors, it is enough to check the statement for block partitions. So, let $b_{2l} \in \mathscr{P}(0,2l)$ be a partition with a single block of $2l$ points. Then indeed

$$[T_{b_2 l}^\sigma]_{\boldsymbol{i}} = \delta_{\boldsymbol{i}} \sigma_{\boldsymbol{i}} = \delta_{\boldsymbol{i}} \sigma_{i_1}^{2l} = \delta_{\boldsymbol{i}} = [T_p]_{\boldsymbol{i}}.$$ □

Crossing partitions correspond to some commutativity relations. The cocycle twist corresponding to the matrix σ then puts some extra signs to the relations, which may make them anticommutative. In particular, it may be interesting to study the relation corresponding to the simple crossing ╳, which then reads

$$\sigma_{ik} \sigma_{jl} \hat{u}_{ij} \hat{u}_{kl} = \sigma_{ki} \sigma_{lj} \hat{u}_{kl} \hat{u}_{ij}. \tag{7.8}$$

7.4.3 Examples

In the theory of quantum groups constructed by *deforming* classical groups, the deformations are usually parametrized by some number q. One speaks about *q-deformations* and *q-commutativity*. The idea is that instead of removing the commutativity relation $ab = ba$, one can replace it by something like $ab = q\,ba$. We will not go too much into details here, for more information, see e.g. [[KS97]]. Let us just make the following definition of a q-deformation of the orthogonal group O_N at $q = -1$. In the compact quantum group context, that is, using the language developed in this thesis, this definition can be found for example in [BBC07].

7.4.7 Definition. We define the quantum group O_N^{-1} as a quantum subgroup of O_N^+ given by the following relations:

$$\begin{aligned} &u_{ij}u_{ik} = -u_{ik}u_{ij}, \quad u_{ji}u_{ki} = -u_{ki}u_{ji} && \text{for } i \neq j, \\ &u_{ij}u_{kl} = u_{kl}u_{ij} && \text{for } i \neq k, j \neq l. \end{aligned} \tag{7.9}$$

7.4.8 Example. [BBC07, Theorem 4.3] If we choose

$$\sigma_{ij} = \begin{cases} -1 & i < j, \\ +1 & i \geq j, \end{cases}$$

we get q-commutativity for $q = -1$.

Indeed, substituting into Eq. (7.8), we get exactly Relations (7.9).

7.4.9 Example. If we choose

$$\sigma_{ij} = \begin{cases} \sigma_i\sigma_j & i < j, \\ +1 & i \geq j \end{cases} \qquad \text{with} \qquad \sigma_i = \begin{cases} +1 & i \leq n, \\ -1 & i > n \end{cases}$$

for some fixed $n < N$, we get some kind of graded commutativity. The commutativity relation (7.8) becomes

$$u_{ij}u_{kl} = \sigma_i\sigma_j\sigma_k\sigma_l\, u_{kl}u_{ij}.$$

7.4.4 Constructing a partition category isomorphism

In certain cases, it may happen that given a compact matrix quantum group G such that $O_N^+ \supseteq G \supseteq H_N$ corresponding to some partition category $\mathscr{K}$, the deformation also satisfies $O_N^+ \supseteq G^\sigma \supseteq H_N$ and hence is again described by a linear category of partitions $\tilde{\mathscr{K}}$ using the standard functor $p \mapsto T_p$ rather than $p \mapsto T_p^\sigma$.

This happens in the case of the (-1)-deformations. Indeed, taking σ as in Example 7.4.8 and $O_N^+ \supseteq G \supseteq H_N$, we have

$$O_N^+ = O_N^{+\sigma} \supseteq G^\sigma \supseteq H_N^\sigma = H_N.$$

It is easy to check the following

$$T^\sigma_{\times} = -T_{\times} + 2T_{\sqcup\!\sqcap}, \qquad T^\sigma_{\sqcup\!\sqcap} = T_{\sqcup\!\sqcap}. \tag{7.10}$$

As a consequence, we have that O_N^{-1} is a quantum group determined by the category of all pairings $\mathsf{Pair}_N = \langle \times \rangle_N$ using the functor $p \mapsto T_p^\sigma$ or, equivalently, by the category $\langle \times - 2\sqcup\!\sqcap \rangle_N$ (which is isomorphic to Pair_N by Prop. 5.4.8) using the standard functor $p \mapsto T_p$.

To put it in a different way: O_N^{-1} is a twist of O_N. In order to describe its representation category using partitions, we have to either twist the functor $p \mapsto T_p$ or to twist the partition category itself.

7.4.10 Remark. In general, it is possible to show that there is an isomorphism of monoidal $*$-categories $\varphi\colon \mathsf{EvenPart}_\delta \to \mathsf{EvenPart}_\delta$ mapping

$$\times \mapsto -\times + 2\,\text{⊓⊔}, \qquad \text{⊓⊔} \mapsto \text{⊓⊔}.$$

Taking $\delta = N \in \mathbb{N}$, it holds that $T_p^\sigma = T_{\varphi(p)}$ with σ as in Example 7.4.8.

However, most easy categories are stable under this isomorphism. The interesting examples are the following ones.

7.4.11 Proposition. The following non-easy linear categories of partitions

$$\langle \times - 2\,\text{⊓⊔} \rangle_N, \qquad \langle \text{⋇} - 2\,\text{|⊓|⊔|} - 2\,\text{⋇}' - 2\,\text{⋇}'' + 4\,\text{⊓⊔⊓⊔} \rangle_N$$

correspond to the quantum groups $O_N^{-1} = O_N^\sigma$ and $O_N^{*-1} = O_N^{*\sigma}$, respectively, where σ comes from Example 7.4.8. That is, those are (-1)-deformations of the quantum groups O_N and O_N^*.

Proof. It follows from the fact that the categories are images of $\mathsf{Pair}_N = \langle \times \rangle_N$, resp. $\langle \text{⋇} \rangle_N$ by the above defined isomorphism φ. See Section 5.4.4. □

7.4.12 Remark. One could obtain many examples of non-easy two-coloured categories by reformulating these results to the unitary case and applying them to the half-liberated two-coloured categories recently obtained in [MW19a, MW19b].

Now, consider σ as in Example 7.4.9. Here, we can see that

$$O_N^+ \supseteq O_N^\sigma \supseteq O_n \times O_{N-n}.$$

In particular, choosing $n = N - 1$, we have

$$O_N^+ \supseteq O_N^\sigma \supseteq O_n \times \hat{\mathbb{Z}}_2 \simeq B_N'.$$

7.4.13 Proposition. The non-easy category

$$\mathscr{K} = \langle \times - \frac{2}{N}(\text{'/,} + \text{,\\'}) + \frac{4}{N}\,\text{¦ ¦} \rangle_N$$

corresponds to the quantum group $G = U_{(N,\pm)}^* O_N^\sigma U_{(N,\pm)}$, where σ is defined as in Example 7.4.9 for $n = N - 1$ and $U_{(N,\pm)}$ is a unitary matrix defined in Def. 7.1.5.

Proof. It is straightforward to check that $T_p = U_{(N,\pm)}^* T_\times^\sigma U_{(N,\pm)}$ for $p = \times - \frac{2}{N}(\text{'/,} + \text{,\\'}) + \frac{4}{N}\,\text{¦ ¦}$. □

Recall that we already showed in Proposition 5.4.6 that this category is isomorphic to the category of pairings $\mathsf{Pair}_N = \langle \times \rangle_N$.

Chapter 8
From partition categories to general CMQGs

The aim of this chapter is to generalize notions introduced for categories of partitions or easy quantum groups to general compact matrix quantum groups. Studying categories of partitions brings not only examples of new quantum groups, but allows us to better understand the structure of the representation categories and the quantum groups themselves. The classification programme results in many new interesting constructions that can be generalized also beyond the scope of categories of partitions and easy quantum groups.

As an example, recall the glued products that were defined in [TW17]. The motivation for introducing them was to interpret some quantum groups resulting from classification of categories of non-crossing two-coloured partitions. However, the construction works in general for any compact matrix quantum groups. In a similar manner, we generalize here some concepts that appeared in the previous chapters of this thesis in a way that we can leave out the partitions.

We summarize the main results of this chapter below. The links with previous chapters are illustrated in Table 8.1.

Concept	Definition and main results for partitions	easy CMQGs	general CMQGs
Degree of reflection	4.4.9	—	8.2.4, 8.2.6
Global colourization, tensor complexification	4.4.7, 4.5.10	6.2.4	8.2.9, 8.2.11, 8.2.13
Alternating colourization, free complexification	4.6.11, 4.6.15	6.2.8	8.2.20, 8.2.24, 8.2.29
Interpolating products	—	6.4.16	8.3.1, 8.3.2
Gluing and ungluing	4.6.4, 4.6.8	6.4.10, 6.4.13	8.2.1, 8.2.3, 8.4.11, 8.4.13

Table 8.1 Overview of the concepts introduced in the thesis that are generalized in Chapter 8

We are going to study the concept of the *degree of reflection* defined for two-coloured categories in Def. 4.4.9. We define it for arbitrary quantum group by the following equivalent conditions.

Proposition (8.2.4, 8.2.6)**.** Let $G \subseteq U^+(F)$ be a compact matrix quantum group. The following are equivalent.

(1) The quantum subgroup of G given by relations $u_{ij} = 0$, $u_{ii} = u_{jj}$ for $i \neq j$ is $\hat{\mathbb{Z}}_k$.
(2) We have $\{c(w_2) - c(w_1) \mid \mathsf{FundRep}_G(w_1, w_2) \neq \{0\}\} = k\mathbb{Z}$.
(3) The number k is the largest such that I_G is $\mathbb{Z}_k$-homogeneous.
(4) The number k is the largest such that $G = G \tilde{\times} \hat{\mathbb{Z}}_k$.

In items (3) and (4), we consider zero to be larger than every natural number (equivalently, consider the order defined by "is a multiple of"). The number k is then called the *degree of reflection* of the quantum group G.

We will study the representation categories of tensor complexifications generalizing Theorem 6.2.4.

Theorem (8.2.11)**.** Consider $G \subseteq U^+(F)$ with $F\bar{F} = c\,1_N$, $c \in \mathbb{R}$. Then G is globally colourized with zero degree of reflection if and only if $G = H \tilde{\times} \hat{\mathbb{Z}}$, where $H = G \cap O^+(F)$.

Theorem (8.2.13)**.** Consider a compact matrix quantum group $G = (C(G), v)$, $k \in \mathbb{N}_0$. Denote by z the generator of $C^*(\mathbb{Z}_k)$ and by $u := vz$ the fundamental representation of $G \tilde{\times} \hat{\mathbb{Z}}_k$. We have the following characterizations of $G \tilde{\times} \hat{\mathbb{Z}}_k$.

(1) The ideal $I_{G\tilde{\times}\hat{\mathbb{Z}}_k}$ is the $\mathbb{Z}_k$-homogeneous part of I_G. That is,

$$I_{G\tilde{\times}\hat{\mathbb{Z}}_k} = \{f \in I_G \mid f_l \in I_G \text{ for every } l \in \mathbb{Z}_k\}.$$

(2) The representation category of $G \tilde{\times} \hat{\mathbb{Z}}_k$ looks as follows:

$$\mathrm{Mor}(u^{\otimes w_1}, u^{\otimes w_2}) = \begin{cases} \mathrm{Mor}(v^{\otimes w_1}, v^{\otimes w_2}) & \text{if } c(w_2) - c(w_1) \text{ is a multiple of } k, \\ \{0\} & \text{otherwise.} \end{cases}$$

(3) We have $G \tilde{\times} \hat{\mathbb{Z}}_k = \langle G, E \tilde{\times} \hat{\mathbb{Z}}_k \rangle$. (Here, E denotes the trivial group of the appropriate size, so $E \tilde{\times} \hat{\mathbb{Z}}_k$ is the quantum group $\hat{\mathbb{Z}}_k$ with the representation $z \oplus \cdots \oplus z = z1_N$.)

We will also study free complexifications, whose representation categories were not yet studied even in the partition formalism.

Theorem (8.2.24)**.** Let H be a compact matrix quantum group with degree of reflection $k \neq 1$. Then all $H \tilde{*} \hat{\mathbb{Z}}_l$ coincide for all $l \in \mathbb{N}_0 \setminus \{1\}$. The ideal $I_{H\tilde{*}\hat{\mathbb{Z}}_l}$ is generated by the alternating polynomials in I_H. The representation category of $H \tilde{*} \hat{\mathbb{Z}}_l$ is a (wide) subcategory of the representation category of H generated by $\mathrm{Mor}(1, v^{\otimes(\circ\bullet)^j})$, $j \in \mathbb{N}_0$. This also holds if $k = 1$ and $l = 0$.

Theorem (8.2.29)**.** Consider $G \subseteq U^+(F)$ with $F\bar{F} = c\,1_N$. Then G is alternating and invariant with respect to the colour inversion if and only if it is of the form $G = H \tilde{*} \hat{\mathbb{Z}}$, where $H = G \cap O^+(F)$.

In Section 8.3, we are going to define new products denoted by $G \mathbin{\dot{\times}} H$, $G \mathbin{\times\!\!\!\times} H$, $G \times_k H$ generalizing the quantum groups introduced in Definition 6.4.16. An important result is showing that the relations defining those quantum groups really define something new that lies strictly between the dual free product and the tensor product.

Theorem (8.3.2)**.** Consider quantum groups G, H. Then the products from Definition 8.3.1 are indeed well-defined quantum groups. We have the following inclusions

$$G \hat{*} H \begin{matrix} \supseteq & G \mathbin{\dot{\times}} H & \supseteq \\ \supseteq & G \mathbin{\times\!\!\!\times} H & \supseteq \end{matrix} G \times_0 H \supseteq G \times_{2k} G \supseteq G \times_{2l} H \supseteq G \times_2 H = G \times H,$$

where we assume $k, l \in \mathbb{N}$ such that l divides k. The last three inclusions are strict if and only if the degree of reflection of both G and H is different from one.

Finally, in Section 8.4, we study the gluing procedure (Def. 6.4.10) and discuss how we can reverse this process and define some *ungluing procedure*. In particular, we aim to generalize the one-to-one correspondence from Theorem 4.6.8.

Theorem (8.4.13)**.** There is a one-to-one correspondence between

(1) quantum groups $G \subseteq O^+(F) \hat{*} \hat{\mathbb{Z}}_2$ with degree of reflection two and
(2) quantum groups $\tilde{G} \subseteq U^+(F)$ that are invariant with respect to colour the inversion.

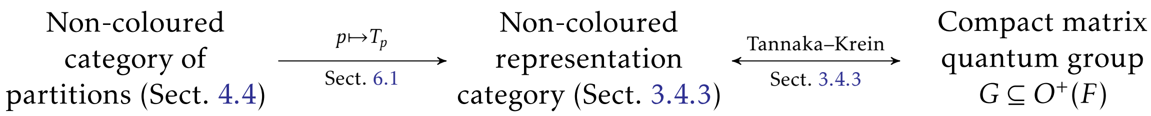
This correspondence is provided by gluing and canonical $\mathbb{Z}_2$-ungluing.

The results presented here will be a part of an article in preparation [Gro20a]. Some results of [GW19b] are also included here. In the whole chapter, we assume that the quantum groups appear in the full version (see Sect. 2.3.1). That is, we denote $C(G) := C_{\mathrm{u}}(G)$ for a compact quantum group G.

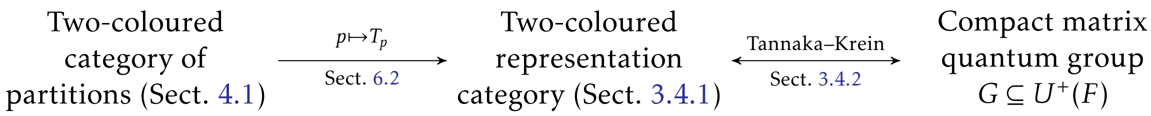
8.1 Coloured representation categories

In this section, we look more in detail on coloured representation categories. First of all, let us link the problem with what was already presented.

Recall the summary in the end of Section 6.1.2. We introduced the following:

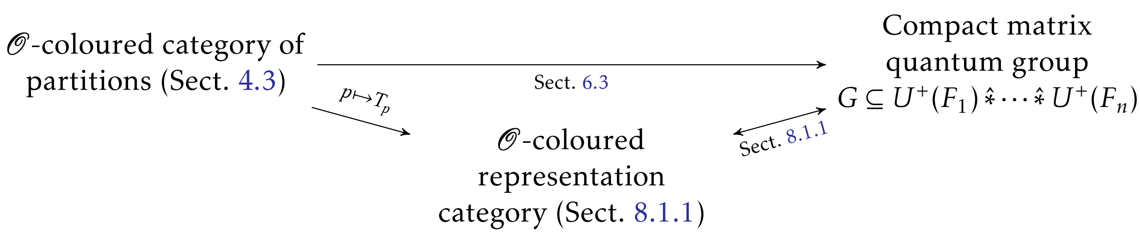


Note that if we start with a category of partitions, the quantum group at the end must actually satisfy $S_N \subseteq G \subseteq O_N^+$. For two-coloured structures, we have a similar correspondence



Again, starting with a two-coloured category of partitions, we actually get a quantum group satisfying $S_N \subseteq G \subseteq U_N^+$.

In Section 4.3, we introduced partitions with arbitrary colouring. Those were interpreted within the theory of quantum groups in Section 6.3. However, we skipped the middle step in Section 6.3 – we did not give a proper definition of an *$\mathscr{O}$-coloured representation category*. We give this definition and formulate the corresponding Tannaka–Krein duality in Section 8.1.1 to complete the following diagram.



In Section 8.1.2, we look in more detail on the Frobenius reciprocity mentioned in Remark 3.4.3. In particular, we generalize the operations on partitions with lower points from Section 4.1.2. Finally, in Section 8.1.3, we focus on a special case of coloured representation categories describing quantum groups $G \subseteq U^+(F) \hat{*} \hat{\mathbb{Z}}_k$.

8.1.1 $\mathscr{O}$-coloured representation categories

In Section 3.4, we presented the Woronowicz–Tannaka–Krein duality formulated for compact matrix quantum groups, which used the notion of *two-coloured representation categories*. Those structures describe arbitrary compact matrix quantum groups $G \subseteq U^+(F)$ for arbitrary F. Since any representation of a compact quantum group can be unitarized, we are literally able to describe any compact matrix quantum group by this structure.

From this point of view, it may seem that it makes no sense to try to generalize the concept of two-coloured representation categories. On the other hand, we have seen in Section 6.4 that it may be convenient for practical reasons to consider a different setting with a different monoid of objects than $\mathscr{O}_{\circ\bullet}^*$.

Suppose, for example, that we are interested in quantum subgroups $G \subseteq U^+(F_1) \hat{*} U^+(F_2)$. Equivalently, those are quantum groups with fundamental representation of the form $u = v_1 \oplus v_2$ such that $v^\circ := v_1$, $v^\bullet := F_1 \bar{v}_1 F_1^{-1}$, $v^\square := v_2$, $v^\blacksquare := F_2 \bar{v}_2 F_2^{-1}$ are all unitary. This implies that also u and $u^\bullet := (F_1 \oplus F_2)\bar{u}(F_1 \oplus F_2)^{-1}$ are unitary, so it is perfectly fine to describe such a quantum group G by the associated two-coloured category $\mathsf{FundRep}(w_1,w_2) = \mathrm{Mor}(u^{\otimes w_1}, u^{\otimes w_2})$ with $w_1, w_2 \in \mathscr{O}^*_{\circ\bullet}$. However, it might also be interesting to introduce the alphabet $\mathscr{O} := \{\circ,\bullet,\square,\blacksquare\}$ and study the intertwiners $\mathrm{Mor}(v^{\otimes w_1}, v^{\otimes w_2})$ with $w_1, w_2 \in \mathscr{O}^*$. These two approaches are absolutely equivalent, the latter might be, however, more convenient to work with.

8.1.1 Definition. Let $\mathscr{O}$ be a set with involution $x \mapsto \bar{x}$. Recall the notation of Section 4.3. Consider a set of integers $N_x \in \mathbb{N}$ for every $x \in \mathscr{O}$ such that $N_{\bar{x}} = N_x$. An **$\mathscr{O}$-coloured representation category** is a rigid monoidal $*$-category $\mathfrak{C}$ with $\mathscr{O}^*$ being the monoid of objects (tensor product of objects is the monoid operation, dual objects are obtained by involution) and morphisms realized as linear maps

$$\mathfrak{C}(x_1\cdots x_k, y_1\cdots y_l) \subseteq \mathscr{L}(\mathbb{C}^{N_{x_1}} \otimes \cdots \otimes \mathbb{C}^{N_{x_k}}, \mathbb{C}^{N_{y_1}} \otimes \cdots \otimes \mathbb{C}^{N_{y_l}}). \tag{8.1}$$

Equivalently, without referring to the categorical definitions, we can say that an $\mathscr{O}$-coloured representation category is a collection of subspaces (8.1) satisfying the following five axioms

(1) For $T \in \mathfrak{C}(w_1,w_2)$, $T' \in \mathfrak{C}(w_1',w_2')$, we have $T \otimes T' \in \mathfrak{C}(w_1w_1', w_2w_2')$.
(2) For $T \in \mathfrak{C}(w_1,w_2)$, $S \in \mathfrak{C}(w_2,w_3)$, we have $ST \in \mathfrak{C}(w_1,w_3)$.
(3) For $T \in \mathfrak{C}(w_1,w_2)$, we have $T^* \in \mathfrak{C}(w_2,w_1)$.
(4) For every word $w = x_1\cdots x_k \in \mathscr{O}^*$, we have $1_{\sum_i N_{x_i}} \in \mathfrak{C}(w,w)$.
(5) There exist vectors $\xi_x \in \mathfrak{C}(\emptyset, x\bar{x})$ such that $(\xi_x^* \otimes 1_{N_x})(1_{N_x} \otimes \xi_{\bar{x}}) = 1_{N_x}$ for every $x \in \mathscr{O}$.

In order to formulate the correspondence with compact matrix quantum groups, one should, in addition, divide the set of objects $\mathscr{O}$ into *white* and *black* points $\mathscr{O} = \mathscr{O}_\mathrm{w} \cup \mathscr{O}_\mathrm{b}$ such that $\mathscr{O}_\mathrm{b} = \{\bar{x} \mid x \in \mathscr{O}_\mathrm{w}\}$. The union may not be disjoint as there may be self-dual elements in $\mathscr{O}$.

Now, we formulate one direction of the correspondence between $\mathscr{O}$-coloured representation categories and compact matrix quantum groups.

8.1.2 Proposition. Let G be a compact matrix quantum group with a unitary fundamental representation u that decomposes as a direct sum $u = \bigoplus_{x\in\mathscr{O}_\mathrm{w}} u^x$. Denote the dual unitary representations $u^{\bar{x}} := F_x \bar{u}^x F_x^{-1}$, so

$$u^\bullet = FuF^{-1} = \bigoplus_{x\in\mathscr{O}_\mathrm{w}} u^{\bar{x}}.$$

Put $\mathscr{O} := \{x, \bar{x} \mid x \in \mathscr{O}_\mathrm{w}\}$. Then

$$\mathsf{FundRep}^{\mathscr{O}}_G(w_1,w_2) := \mathrm{Mor}(u^{\otimes w_1}, u^{\otimes w_2})$$

forms an $\mathscr{O}$-coloured representation category.

Proof. Same as Prop. 3.4.4. □

The converse direction, that is, Tannaka–Krein duality for $\mathscr{O}$-coloured categories, can be formulated as follows.

8.1.3 Proposition. Let $\mathscr{O}$ be a set with involution decomposed into white and black points as indicated above $\mathscr{O} = \mathscr{O}_\mathrm{w} \cup \mathscr{O}_\mathrm{b}$. Let $\mathfrak{C}$ be an $\mathscr{O}$-coloured representation category. Then there exists a unique compact matrix quantum group G such that its fundamental representation decomposes as $u = \bigoplus_{x\in\mathscr{O}_\mathrm{w}} u^x$ and we have

$$\mathsf{FundRep}^{\mathscr{O}}_G(w_1,w_2) = \mathrm{Mor}(u^{\otimes w_1}, u^{\otimes w_2}) = \mathfrak{C}(w_1,w_2).$$

Proof. Let us denote $N := \sum_{x\in\mathscr{O}_\mathrm{w}} N_x$, so we can write $\mathbb{C}^N = \bigoplus_{x\in\mathscr{O}_\mathrm{w}} \mathbb{C}^{N_x}$. Denote by $V^x = V^{\bar{x}}$ the canonical isometries $\mathbb{C}^{N_x} \to \mathbb{C}^N$ respecting this decomposition. For a word $w \in \mathscr{O}^*$, we denote by $V^{\otimes w}$ the corresponding tensor product of those isometries.

We can define a two-coloured representation category $\tilde{\mathfrak{C}}$ as follows. Let $\tilde{w} \in \mathscr{O}_{\circ\bullet}^k$ be a two-coloured word. Denote by $\mathscr{O}^{\tilde{w}}$ the set of all words $w \in \mathscr{O}^k$ such that the i-th letter of w is white (i.e. belongs to $\mathscr{O}_{\rm w}$) if and only if the i-th letter of $\tilde{w}$ is white. Then define

$$\tilde{\mathfrak{C}}(\tilde{w}_1, \tilde{w}_2) := \operatorname{span}\{V^{\otimes w_2} T V^{\otimes w_1 *} \mid w_1 \in \mathscr{O}^{\tilde{w}_1}, w_2 \in \mathscr{O}^{\tilde{w}_2}, T \in \mathfrak{C}(w_1, w_2)\}.$$

It is straightforward to check that $\tilde{\mathfrak{C}}$ is indeed a two-coloured category. Then we can use the Woronowicz–Tannaka–Krein duality (Theorem 3.4.6) to construct the quantum group G. Finally, projecting the morphism spaces of $\tilde{\mathfrak{C}} = \mathsf{FundRep}_G$ back to the subspaces $\mathbb{C}^{N_x} \subseteq \mathbb{C}^N$, we get back the $\mathscr{O}$-coloured category $\mathfrak{C}$, which proves that we indeed have $\mathfrak{C} = \mathsf{FundRep}_G^{\mathscr{O}}$. □

To illustrate the statements above, let us draw some concrete matrices. Suppose that we have $G \subseteq U_{N_1}^+ \hat{*} \cdots \hat{*}\, U_{N_n}^+$. This means that the fundamental representation u is of the block-diagonal form

$$u = \bigoplus_{i=1}^n u^i = \begin{pmatrix} u^1 & & & \\ & u^2 & & \\ & & \ddots & \\ & & & u^n \end{pmatrix},$$

where the blocks u^i are of size $N_i \times N_i$. One way to describe this quantum group is using the two-coloured representation category $\tilde{\mathfrak{C}} = \mathsf{FundRep}_G$, where the objects are words $w \in \mathscr{O}_{\circ\bullet}$. The other way is to use the $\mathscr{O}$-coloured representation category $\mathfrak{C} = \mathsf{FundRep}_G^{\mathscr{O}}$, where $\mathscr{O} = \{x_1, \ldots, x_n\}$. The letters, $x_1, \ldots, x_n$ of course correspond to the blocks of u, so $u^{x_i} = u^i$. Now, the categories $\mathfrak{C}$ and $\tilde{\mathfrak{C}}$ are related in the way described in the proof of Proposition 8.1.2. Namely, the intertwiners from $\tilde{\mathfrak{C}}$ have a block structure with the blocks coming from $\mathfrak{C}$. For example, the morphism space $\tilde{\mathfrak{C}}(\bullet, \circ) = \operatorname{Mor}(\bar{u}, u)$ is linearly spanned by the intertwiners

$$V^{x_i} T V^{x_j *} = \begin{pmatrix} & & j & \\ i & \cdots\cdots & T & \\ & & & \\ & & & \end{pmatrix}, \qquad \text{where } T \in \operatorname{Mor}(\bar{x}_j, x_i).$$

8.1.2 One-line operations for representation categories

We have already mentioned in Remark 3.4.3 that representation categories $\mathfrak{C}$ are determined just by the spaces $\mathfrak{C}(\emptyset, w)$ since the elements of $\mathfrak{C}(w_1, w_2)$ can then be obtained by some *rotations*. The same holds also for categories of partitions (and in general for any monoidal categories) as we described in Section 4.1.2. Moreover, we defined in Section 4.1.2 some operations for partitions on one-line that allow to formulate an alternative definition for a category of partitions using only partitions on one line, that is, elements $p \in \mathscr{P}(0, k)$. In this section, we do the same thing for representation categories. We formulate the results in the general framework of $\mathscr{O}$-coloured representation categories, but one can, of course, specialize also to the two-coloured or non-coloured case as well.

Consider an $\mathscr{O}$-coloured representation category $\mathfrak{C}$ for some set of colours $\mathscr{O}$. We use the notation of the previous section. In particular, given a colour $x \in \mathscr{O}$, we denote by $N_x \in \mathbb{N}$ the associated dimension, by $\xi_x \in \mathfrak{C}(\emptyset, x\bar{x})$ the associated duality morphisms and by $F^x \in M_{N_x}(\mathbb{C})$ the matrix with entries $F^x_{ij} = [\xi_x]_{ji}$. Recall also the definition of left and right rotations, which we defined in Remark 3.4.3 for two-coloured representation categories. Their definition can be reformulated for $\mathscr{O}$-coloured categories in a straightforward way. Now, we define the following operations on the sets $\mathfrak{C}(\emptyset, w)$.

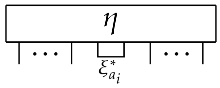
- If $a_{i+1} = \bar{a}_i$, we define the **contraction**:

$$\Pi_i^{a_i}\colon \mathfrak{C}(\emptyset, a_1 \cdots a_k) \to \mathfrak{C}(\emptyset, a_1 \cdots a_{i-1} a_{i+2} \cdots a_k),$$
$$\Pi_i^{a_i}\eta := (1_{N_{a_1}} \otimes \cdots \otimes 1_{N_{a_{i-1}}} \otimes \xi_{a_i}^* \otimes 1_{N_{a_{i+2}}} \otimes \cdots \otimes 1_{N_{a_k}})\eta.$$

On elementary tensors, it acts as

$$\Pi_i^{a_i}(\eta_1 \otimes \cdots \otimes \eta_k) = (\eta_{i+1}^t F^{a_i} \eta_i)\, \eta_1 \otimes \cdots \otimes \eta_{i-1} \otimes \eta_{i+2} \otimes \cdots \otimes \eta_k.$$

Pictorially,

$\Pi_i^{a_i}\eta =$ 

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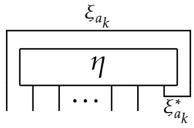
- We define the **rotation**:

$$R^{a_k}\colon \mathfrak{C}(\emptyset, a_1 \cdots a_k) \to \mathfrak{C}(\emptyset, a_k a_1 \cdots a_{k-1}), \qquad R^{a_k} := \mathrm{Lrot} \circ \mathrm{Rrot}, \quad \text{so}$$
$$R^{a_k}\eta = (1_{N_{a_1}} \otimes \cdots \otimes 1_{N_{a_{k-1}}} \otimes \xi_{a_k}^*)(1_{N_{a_k}} \otimes \eta \otimes 1_{N_{a_k}})\xi_{a_k}.$$

On elementary tensors, it acts as

$$R^{a_k}(\eta_1 \otimes \cdots \otimes \eta_k) = (F^{a_k}\bar{F}^{a_k}\eta_k) \otimes \eta_1 \otimes \cdots \otimes \eta_{k-1}.$$

Pictorially,

$R^{a_k}\eta =$ 

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For a word $w' = b_1 \cdots b_l$, we define $R^{w'}\colon \mathfrak{C}(\emptyset, ww') \to \mathfrak{C}(\emptyset, w'w)$ by $R^{w'} := R^{b_1} \circ \cdots \circ R^{b_l}$. We denote by $R^{-w'}$ its inverse.

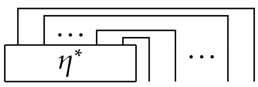
- We define the **reflection**:

$$\star\colon \mathfrak{C}(\emptyset, a_1 \cdots a_k) \to \mathfrak{C}(\emptyset, a_k \cdots a_1)$$
$$\eta^\star := \mathrm{Rrot}^{-k}\,\eta^* = (\eta^* \otimes 1_{N_{a_k}} \otimes \cdots \otimes 1_{N_{a_1}})\xi_{a_1 \cdots a_k}.$$

On elementary tensors, it acts as

$$(\eta_1 \otimes \cdots \otimes \eta_k)^\star = (F^{a_k}\bar{\eta}_k) \otimes \cdots \otimes (F^{a_1}\bar{\eta}_1).$$

Pictorially,

$R\eta =$ 

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8.1.4 Proposition. For any $\mathscr{O}$-coloured representation category $\mathfrak{C}$, the collection of sets $\mathfrak{C}(0, w)$, $w \in \mathscr{O}^*$ is closed under tensor products, contractions, rotations, inverse rotations, and reflections. Conversely, for any collection of vector spaces $\mathfrak{C}(w) \subseteq \mathbb{C}^{N_w}$, $N_{a_1 \cdots a_k} := \sum_i N_{a_i}$, that is closed under tensor products, contractions, rotations, inverse rotations, and reflections and satisfies axiom (5) of $\mathscr{O}$-coloured representation categories, the sets

$$\mathfrak{C}(w_1, w_2) := \{\mathrm{Rrot}^{w_1^*}\,\xi \mid \xi \in \mathfrak{C}(w_2 w_1^*)\} = \{\mathrm{Lrot}^{-w_1^*}\,p \mid p \in \mathfrak{C}(w_1^* w_2)\}$$

form an $\mathscr{O}$-coloured representation category.

Proof. Same as with Proposition 4.1.2. □

8.1.3 $\mathbb{Z}_k$-extended representation categories

Now, let us focus on the situation, when we have $G \subseteq U^+(F) \mathbin{\hat{*}} \hat{\mathbb{Z}}_k$ for some $k \in \mathbb{N}_0$ (considering $\mathbb{Z}_k = \mathbb{Z}$ for $k = 0$). That is, the fundamental representation of G decomposes as $u = v \oplus z$, where v is N-dimensional and z is one-dimensional. We use the alphabet $\mathscr{O} = \{\square, \blacksquare, \triangle, \blacktriangle\}$, with $u^\square = v$, $u^\blacksquare = F\bar{v}F^{-1}$, $u^\triangle = z$, $u^\blacktriangle = z^*$.

Note that if $k \neq 0$, we have $z^* = z^{k-1} = z^{\otimes(k-1)}$, so we actually do not need to consider the representation z^* and we could omit $\blacktriangle$ in our alphabet. This makes sense, in particular, for $k = 2$, where $z^* = z$, so in a sense we have $\blacktriangle = \triangle$. This is what we did when we introduced categories of partitions with extra singletons in Sections 4.6 and 6.4. In this chapter, we will choose a bit different approach by imposing the relation $\blacktriangle = \triangle^{k-1}$.

8.1.5 Definition. We define $\mathscr{W}_0$ to be the monoid with generators $\square$, $\blacksquare$, $\triangle$, $\blacktriangle$ satisfying the relations $\triangle\blacktriangle = \blacktriangle\triangle = \emptyset$. We define $\mathscr{W}_k$ to be the quotient of $\mathscr{W}_0$ with respect to the additional relation $\triangle^k = \emptyset$ (which implies $\blacktriangle^k = \emptyset$). We define an involution $w \to \bar{w}$ on $\mathscr{W}_k$ as a homomorphism inverting the colours. Another involution $w \mapsto w^*$ inverts the colours and reverses the word. For any element $w \in \mathscr{W}_k$ we define $[w]$ to be the number of squares in w (which is well-defined in contrast with the overall length of w).

As an example, consider $k = 3$ and take the word $w = \square\blacksquare\triangle\square$ as an element of $\mathscr{W}_3$. We then have

$$\bar{w} = \square\triangle\blacksquare\square,$$
$$w^* = \blacksquare\blacktriangle\square\blacksquare = \blacksquare\triangle\triangle\square\blacksquare,$$
$$[w] = 3 = [\bar{w}] = [w^*].$$

8.1.6 Definition. A **$\mathbb{Z}_k$-extended representation category** is a rigid monoidal $*$-category $\mathfrak{C}$ with $\mathscr{W}_k$ being the monoid of objects (again, tensor product is the monoid operation, duality maps $w \mapsto \bar{w}$) and morphisms realized as linear maps

$$\mathfrak{C}(w_1, w_2) \subseteq \mathscr{L}((\mathbb{C}^N)^{\otimes[w_1]}, (\mathbb{C}^N)^{\otimes[w_2]}).$$

8.1.7 Proposition. Consider a compact matrix quantum group $G \subseteq U^+(F) \mathbin{\hat{*}} \hat{\mathbb{Z}}_k$. Denote its fundamental representation $u = v \oplus z$ as above. Then

$$\mathsf{FundRep}_G^{k\text{-ext}}(w_1, w_2) := \mathrm{Mor}(u^{\otimes w_1}, u^{\otimes w_2})$$

forms a $\mathbb{Z}_k$-extended representation category. Conversely, for any $\mathbb{Z}_k$-extended representation category $\mathfrak{C}$, there exists a unique compact matrix quantum group $G \subseteq U^+(F) \mathbin{\hat{*}} \hat{\mathbb{Z}}_k$ for some F such that $\mathfrak{C} = \mathsf{FundRep}_G^{k\text{-ext}}$.

Proof. The proposition is a special case of Props. 8.1.2, 8.1.3. The only thing to notice is that the above definition of $\mathsf{FundRep}_G^{k\text{-ext}}$ makes sense since given two words $w, w' \in \mathscr{O}^*$ representing the same element of $\mathscr{W}_k$, we have $u^{\otimes w} = u^{\otimes w'}$. □

8.2 Representation categories of glued products

In this section, we study the gluing procedure – in particular the representation categories of the glued products $G \mathbin{\tilde{\times}} \hat{\mathbb{Z}}_k$ and $H \mathbin{\tilde{*}} \hat{\mathbb{Z}}_k$ – and the degree of reflection.

8.2.1 Grading on algebras associated to quantum groups

The purpose of this subsection is to fix some notation regarding grading of algebras.

A $\mathbb{Z}_k$-grading of a $*$-algebra A is a decomposition of the algebra into a vector space direct sum

$$A = \bigoplus_{i \in \mathbb{Z}_k} A_i$$

such that the multiplication and involution of the algebra respect the operation on $\mathbb{Z}_k$, that is,

$$A_i A_j \subseteq A_{ij}, \quad A_i^* \subseteq A_{-i}.$$

An element of the i-th part $a \in A_i$ is called **$\mathbb{Z}_k$-homogeneous of degree i**.

By definition, every element $f \in A$ uniquely decomposes as $f = \sum_{i \in \mathbb{Z}_k} f_i$ with $f_i \in A_i$. We call the elements f_i the **homogeneous components** of f.

An ideal $I \subseteq A$ is called **$\mathbb{Z}_k$-homogeneous** if it contains with every element f all its homogeneous components f_i. A quotient of the algebra with respect to a homogeneous ideal inherits the grading.

There is a natural structure of a $\mathbb{Z}_k$-grading on the algebra $\mathbb{C}\langle x_{ij}, x_{ij}^* \rangle$ given by associating degree one to the variables x_{ij}, and associating degree minus one to the variables x_{ij}^*. In this chapter, by a $\mathbb{Z}_k$-grading, we will always mean this particular grading.

Consider a compact matrix quantum group G. If the ideal I_G is homogeneous, then the $*$-algebra $\operatorname{Pol} G$ inherits this grading. By definition, the entries u_{ij} of the fundamental representation are then $\mathbb{Z}_k$-homogeneous of degree one. For any $w \in \mathscr{O}_{\circ\bullet}$, the tensor product $u^{\otimes w}$ is $\mathbb{Z}_k$-homogeneous of degree $c(w)$.

As we mentioned in Remark 3.4.5, any irreducible representation of G is a subrepresentation of $u^{\otimes w}$ for some w. So, the grading passes also to the fusion semiring of irreducible representations (see Definition 3.3.4) in the following sense. For any $\alpha \in \operatorname{Irr} G$, there is $d_\alpha \in \mathbb{Z}_k$ such that all the matrix entries of u^α are $\mathbb{Z}_k$-homogeneous of degree d_α. We will call d_α the **degree** of the irreducible u^α.

The definition of a $\mathbb{Z}_k$-grading for C*-algebras is quite simple for $k \in \mathbb{N}$. In the case of the group $\mathbb{Z}$ or other groups, it gets a bit complicated and we will not mention it here. Let A be a C*-algebra. A $\mathbb{Z}_k$-grading on A is defined by a **grading automorphism**, that is, an automorphism $\alpha : A \to A$ satisfying α^k. Its spectrum consists of k-th roots of unity and the corresponding eigenspaces can be identified with the homogeneous parts of A satisfying the properties of the algebraic definition above.

If A is a $\mathbb{Z}_k$-graded $*$-algebra by the algebraic definition, we can define the grading automorphism by setting $\alpha(x) = e^{2\pi i j/k} x$ for $x \in A_k$. The grading automorphism can be then extended to the C*-envelope $C^*(A)$ by the universal property. In particular, a grading on $\operatorname{Pol} G$ can be extended to $C(G)$.

8.2.2 Gluing procedure

Recall the construction of a glued version associated to some quantum group $G \subseteq O_N^+ \hat{*} \hat{\mathbb{Z}}_2$ from Definition 6.4.10. The same definition can be actually formulated for any compact matrix quantum group $G \subseteq U^+(F) \hat{*} \hat{\mathbb{Z}}_k$.

8.2.1 Definition. Consider a quantum group $G \subseteq U^+(F) \hat{*} \hat{\mathbb{Z}}_k$ for some $F \in \mathrm{GL}_N$, $k \in \mathbb{N}_0$. Denote by $v \oplus r$ the fundamental representation of G. Consider $\tilde{v}_{ij} := v_{ij} r$ and let A be the C*-subalgebra of $C(G)$ generated by $\tilde{v}_{ij}$. Then $\tilde{G} := (A, u)$ is called the **glued version** of G.

Recall the definition of the $\mathbb{Z}_k$-extended category $\mathsf{FundRep}_G^{k\text{-ext}}$ with the monoid of objects $\mathscr{W}_k$ from Section 8.1.3. We now aim to describe the representation category corresponding to the glued version $\tilde{G}$. We do this by generalizing the functor F from Def. 4.6.4 (actually its inverse) to the framework of two-coloured and $\mathbb{Z}_k$-extended representation categories and reformulating Theorem 6.4.13 to this setting.

8.2.2 Definition. Let us fix $k \in \mathbb{N}_0$. Then for any word $w \in \mathscr{O}_{\circ\bullet}^*$ we associate its **glued version** $\tilde{w} \in \mathscr{W}_k$ mapping $\circ \mapsto \square\triangle$, $\bullet \mapsto \blacktriangle\blacksquare$.

8.2.3 Proposition. Consider $G \subseteq U^+(F) \tilde{*} \hat{\mathbb{Z}}_k$ with fundamental representation $u = v \oplus z$. Let $\tilde{G}$ be the glued version of G. Then

$$\mathsf{FundRep}_{\tilde{G}}(w_1, w_2) = \mathsf{FundRep}_G^{k\text{-ext}}(\tilde{w}_1, \tilde{w}_2)$$

for every $w_1, w_2 \in \mathscr{O}_{\circ\bullet}^*$ and its glued versions $\tilde{w}_1, \tilde{w}_2 \in \mathscr{W}_k$. That is $\mathsf{FundRep}_{\tilde{G}}$ is a full subcategory of $\mathsf{FundRep}_G^{k\text{-ext}}$ given by considering the glued words only. The ideal associated to $\tilde{G}$ can be described as

$$I_{\tilde{G}} = \{f \in \mathbb{C}\langle \tilde{v}_{ij}, \tilde{v}_{ij}^* \rangle \mid f(v_{ij}z, z^*v_{ij}^*) \in I_G\} \simeq I_G \cap \mathbb{C}\langle v_{ij}z, z^*v_{ij}^* \rangle.$$

Proof. Denote by $\tilde{v} = vz$ the fundamental representation of $\tilde{G}$. Consider a word w and its glued version $\tilde{w}$. Directly from the definitions of $\tilde{v}$ and $\tilde{w}$, we have $\tilde{v}^{\otimes w} = u^{\otimes \tilde{w}}$. So,

$$\mathsf{FundRep}_{\tilde{G}}(w_1, w_2) = \mathrm{Mor}(\tilde{v}^{\otimes w_1}, \tilde{v}^{\otimes w_2}) = \mathrm{Mor}(u^{\otimes \tilde{w}_1}, u^{\otimes \tilde{w}_2}) = \mathsf{FundRep}_G^{k\text{-ext}}(\tilde{w}_1, \tilde{w}_2).$$

For the ideal, we have

$$\begin{aligned} I_{\tilde{G}} &= \{f \in \mathbb{C}\langle \tilde{v}_{ij}, \tilde{v}_{ij}^* \rangle \mid 0 = f(\tilde{v}_{ij}, \tilde{v}_{ij}^*) = f(v_{ij}z, z^*v_{ij}^*) \text{ in } C(\tilde{G}) \subseteq C(G)\} \\ &= \{f \in \mathbb{C}\langle \tilde{v}_{ij}, \tilde{v}_{ij}^* \rangle \mid f(v_{ij}z, z^*v_{ij}^*) \in I_G\}. \end{aligned}$$ □

8.2.3 Degree of reflection

In Definition 4.4.9, we defined the *degree of reflection* characterizing some partition categories. Here, we associate a degree of reflection to arbitrary quantum groups and give several equivalent formulations of this definition that do not refer to the representation category.

Recall from Section 2.4.2 that given a quantum group $G = (C(G), u)$, we can construct a quantum subgroup of G – so called *diagonal subgroup* – imposing the relation $u_{ij} = 0$ for all $i \neq j$. If we, in addition, impose the relation $u_{ii} = u_{jj}$ for all i and j, we get a quantum group corresponding to a C*-algebra generated by a single unitary. Therefore, it must be a dual of some cyclic group.

8.2.4 Definition. Let G be a quantum group and denote by $\hat{\Gamma}$ the quantum subgroup of G given by $u_{ij} = 0$, $u_{ii} = u_{jj}$ for all $i \neq j$. The order of the cyclic group Γ is called the **degree of reflection** of G. If the order is infinite, we set the degree of reflection to zero.

8.2.5 Lemma. Let $G = (C(G), u)$ be a compact matrix quantum group, $k \in \mathbb{N}_0$. The following are equivalent.

(0) The number k divides the degree of reflection of G.
(1) The mapping $u_{ij} \mapsto \delta_{ij} z$ extends to a $*$-homomorphism $\varphi: C(G) \to C^*(\hat{\mathbb{Z}}_k)$.
(2) For any $w \in \mathscr{O}_{\circ\bullet}^*$, $\mathrm{Mor}(1, u^{\otimes w}) \neq \{0\}$ only if $c(w)$ is a multiple of k.
(3) The ideal I_G is $\mathbb{Z}_k$-homogeneous.
(4) We have $G = G \tilde{\times} \hat{\mathbb{Z}}_k$.

Proof. (0) ⇔ (1): If k_0 is the degree of reflection of G, then directly by the definition there is a $*$-homomorphism $\varphi: C(G) \to C^*(\hat{\mathbb{Z}}_{k_0})$. Such an homomorphism obviously exists also if k is a divisor of k_0. By definition, $\mathbb{Z}_{k_0}$ is the largest cyclic group with this property, so k must be a divisor of k_0.

(1) ⇒ (2): Take any $\xi \in \mathrm{Mor}(1, u^{\otimes w})$, so $u^{\otimes w}\xi = \xi$. Applying the homomorphism φ, we get $z^{c(w)}\xi = \xi$. If $\xi \neq 0$, we must have $z^{c(w)} = 1$, so $c(w)$ is a multiple of k.

(2) ⇒ (3): By Corollary 3.4.13, I_G is generated by the relations $u^{\otimes w}\xi = \xi$, $\xi \in \mathrm{Mor}(1, u^{\otimes w})$. Since the entries of $u^{\otimes w}$ are monomials of degree $c(w)$, the relations $u^{\otimes w}\xi = \xi$ are $\mathbb{Z}_{c(w)}$-homogeneous (of degree zero). Consequently, they are also $\mathbb{Z}_k$-homogeneous and hence generate a $\mathbb{Z}_k$-homogeneous ideal.

(3) ⇒ (4): We need to show that $u_{ij} \mapsto u_{ij}z$ extends to a $*$-isomorphism $C(G) \to C(G \tilde{\times} \hat{\mathbb{Z}}_k)$. To prove that it extends to a homomorphism, take any $f \in I_G$. Suppose f is $\mathbb{Z}_k$-homogeneous of degree l. Then, since u_{ij} and z commute, we have $f(u_{ij}z) = f(u_{ij})z^l = 0$. It is surjective directly from definition. For injectivity, note that the projection to the first tensor component $C(G) \otimes_{\max} C^*(\mathbb{Z}_k) \to C(G)$ restricts to the inverse of α.

(4) ⇒ (1): We define $\varphi := (\varepsilon \otimes \mathrm{id})\alpha$, where ε is the counit of G and α is the $*$-homomorphism $C(G) \to C(G) \otimes C^*(\hat{\mathbb{Z}}_k)$. Then indeed $\varphi(u_{ij}) = \varepsilon(u_{ij})z = \delta_{ij}z$. □

As a consequence, we have four equivalent formulations of the degree of reflection.

8.2.6 Proposition. Let G be a compact matrix quantum group, $k \in \mathbb{N}_0$. The following are equivalent.

(1) The number k is the degree of reflection of G.
(2) We have $\{c(w_2) - c(w_1) \mid \mathsf{FundRep}_G(w_1, w_2) \neq \{0\}\} = k\mathbb{Z}$.
(3) The number k is the largest such that I_G is $\mathbb{Z}_k$-homogeneous.
(4) The number k is the largest such that $G = G \tilde{\times} \hat{\mathbb{Z}}_k$.

In items (3) and (4), we consider zero to be larger than every natural number (equivalently, consider the order defined by "is a multiple of").

Proof. We just take the maximal k (in the above mentioned sense) satisfying the equivalent conditions in Lemma 8.2.5. For (2) note that the set $\{c(w_2) - c(w_1) \mid \mathsf{FundRep}_G(w_1, w_2) \neq \{0\}\}$ is indeed a subgroup of $\mathbb{Z}$. The fact that $\{c(w_2) - c(w_1)\}$ is closed under addition follows from $\mathsf{FundRep}_G$ being closed under the tensor product. The fact that $\{c(w_2) - c(w_1)\}$ is closed under subtraction follows from $\mathsf{FundRep}_G$ being closed under the involution. The statement (2) in Lemma 8.2.5 can be formulated as $\{c(w_2) - c(w_1) \mid \mathsf{FundRep}_G(w_1, w_2) \neq \{0\}\} \subseteq k\mathbb{Z}$. Taking the maximal k, we gain the equality. □

8.2.7 Remark. The item (2) from the previous proposition corresponds to the partition-categorical definition Def. 4.4.9. So, considering a unitary easy quantum group G corresponding to a category $\mathscr{C} \subseteq \mathscr{P}^{\circ\bullet}$, we can say that the degree of reflection of G (by Def. 8.2.4) coincides with the degree of reflection of $\mathscr{C}$ (by Def. 4.4.9).

8.2.8 Proposition. Consider $G \subseteq O^+(F)$. Then one of the following is true.

(1) The degree of reflection of G is one and $\mathsf{FundRep}_G(0, k) \neq \{0\}$ for some odd $k \in \mathbb{N}_0$.
(2) The degree of reflection of G is two and $\mathsf{FundRep}_G(k, l) = \{0\}$ for every $k + l$ odd.

Proof. Recall from Proposition 3.4.10 that the intertwiner spaces $\mathsf{FundRep}_G(w_1, w_2)$ depend only on the length of the words w_1 and w_2 for $G \subseteq O^+(F)$. This allowed us to introduce the notation $\mathsf{FundRep}_G(k, l)$. Now the proposition follows from Proposition 8.2.6. First of all, the degree of reflection must be a divisor of two (that is, either one or two) since we have $\mathsf{FundRep}_G(\emptyset, \circ\circ) = \mathsf{FundRep}_G(\emptyset, \circ\bullet) \ni \xi_{\sqcap} \neq 0$. Then G has degree of reflection one if and only if $\mathsf{FundRep}_G(\emptyset, w) = \mathsf{FundRep}_G(0, |w|) \neq \{0\}$ for some word w with $c(w) = 1$. Such a word with $c(w) = 1$ must be of odd length. □

8.2.4 Global colourization

We introduced the notion of a globally-colourized category of partitions in Definition 4.4.7. Here, we reformulate the definition and some results for arbitrary compact matrix quantum groups.

8.2.9 Definition. A compact matrix quantum group $G = (C(G), u)$ is called **globally colourized** if the following holds

$$u_{ij} u_{kl}^* = u_{kl}^* u_{ij}. \tag{8.2}$$

Assuming $G \subseteq U^+(F)$, this can be equivalently expressed using the entries of the unitary representations $u^\circ = u$ and $u^\bullet = F\bar{u}F^{-1}$ as

$$u^\circ_{ij} u^\bullet_{kl} = u^\bullet_{ij} u^\circ_{kl}. \tag{8.3}$$

The following proposition essentially reformulates Lemma 4.4.6 to the setting of representation categories. Compare also with Proposition 3.4.10, which is related to Lemma 4.4.3 in a similar way.

8.2.10 Proposition. A compact matrix quantum group $G = (C(G), u)$ is globally colourized if and only if, for every w_1, w_2, w_1', $w_2' \in \mathscr{O}^*_{\circ\bullet}$ satisfying $|w_1'| = |w_1|$, $|w_2'| = |w_2|$, $c(w_2') - c(w_1') = c(w_2) - c(w_1)$, we have

$$\mathrm{Mor}(u^{\otimes w_1'}, u^{\otimes w_2'}) = \mathrm{Mor}(u^{\otimes w_1}, u^{\otimes w_2}).$$

Proof. The equality (8.3) can be also expressed as $u^\circ \otimes u^\bullet = u^\bullet \otimes u^\circ$, so it is equivalent to saying that the identity is an intertwiner between $u^\circ \otimes u^\bullet$ and $u^\bullet \otimes u^\circ$. From this, the right-left implication follows directly.

For the left-right implication, from Frobenius reciprocity, it is enough to show the equality for $w_1 = w_1' = \emptyset$. It is easy to infer that if the identity is in $\mathrm{Mor}(u^\circ \otimes u^\bullet, u^\bullet \otimes u^\circ)$, we must also have the identity in $\mathrm{Mor}(u^\bullet \otimes u^\circ, u^\circ \otimes u^\bullet)$ and hence also in $\mathrm{Mor}(u^{\otimes w_2}, u^{\otimes w_2'})$, and $\mathrm{Mor}(u^{\otimes w_2'}, u^{\otimes w_2})$, which implies the desired equality. □

Consider $H \subseteq O^+_N(F)$. It is easy to check that the tensor complexification $H \tilde{\times} \hat{\mathbb{Z}}_k$ is a globally colourized quantum group with degree of reflection k for every $k \in \mathbb{N}_0$. In the following theorem, we prove the converse for $k = 0$.

8.2.11 Theorem. Consider $G \subseteq U^+(F)$ with $F\bar{F} = c\,1_N$, $c \in \mathbb{R}$. Then G is globally colourized with zero degree of reflection if and only if $G = H \tilde{\times} \hat{\mathbb{Z}}$, where $H = G \cap O^+(F)$.

Proof. We denote by u, v, z the fundamental representations of G, H, and $\hat{\mathbb{Z}}_k$, respectively. The quantum group H is the quantum subgroup of G defined by the relation $v^\circ = v^\bullet$. As mentioned above, the right-left implication is clear since v_{ij} commute with z, so

$$u^\circ_{ij} u^\bullet_{kl} = v_{ij} z z^* v_{kl} = z^* v_{ij} v_{kl} z = u^\bullet_{ij} u^\circ_{kl}.$$

Now, let us prove the left-right implication. First, we show that there is a surjective $*$-homomorphism

$$\alpha \colon C(G) \to C(H \tilde{\times} \hat{\mathbb{Z}}) \subseteq C(H) \otimes_{\max} C^*(\mathbb{Z})$$

mapping $u_{ij} \mapsto u'_{ij} := v_{ij} z$. To show this, take any element $f \in I_G$. Since I_G is $\mathbb{Z}$-homogeneous, we can assume that f is also $\mathbb{Z}$-homogeneous of some degree l. Then $f(u'_{ij}) = f(v_{ij} z) = f(v_{ij}) z^l = 0$. This proves the existence of such a homomorphism. Its surjectivity is obvious.

Now it remains to prove that α is injective and hence is a $*$-isomorphism. Denote by $\xi \in \mathrm{Mor}(1, u^\circ \otimes u^\bullet) \subseteq \mathbb{C}^N \otimes \mathbb{C}^N$ the tensor with entries $\xi_{ij} = \frac{1}{\sqrt{\mathrm{Tr}(F^*F)}} F_{ji}$, which is normalized so that $\xi^* \xi = 1$. We construct a $*$-homomorphism

$$\beta \colon C(H) \otimes_{\max} C^*(\mathbb{Z}) \to M_2(C(G))$$

mapping

$$z \mapsto z' := \begin{pmatrix} 0 & y \\ 1 & 0 \end{pmatrix}, \quad v_{ij} \mapsto v'_{ij} := \begin{pmatrix} 0 & u^\circ_{ij} \\ u^\bullet_{ij} & 0 \end{pmatrix}, \quad y := \xi^* (u \otimes u) \xi.$$

To prove the existence of such a homomorphism, we need the following.

Using the fact that $\xi \xi^* \in \mathrm{Mor}(u^\circ \otimes u^\bullet, u^\circ \otimes u^\bullet) = \mathrm{Mor}(u \otimes u, u \otimes u)$ (the equality follows from global colourization thanks to Proposition 8.2.10), we derive

$$y y^* = \xi^* (u \otimes u) \xi \xi^* (u^* \otimes u^*) \xi = \xi^* \xi \xi^* (u u^* \otimes u u^*) \xi = 1$$

and similarly $y^* y = 1$. From this, we can also deduce $z' z'^* = z'^* z' = 1$.

Using the fact that $1_N \otimes \xi^* \in \mathrm{Mor}(u^\circ \otimes u^\circ \otimes u^\bullet, u^\circ) = \mathrm{Mor}(u^\bullet \otimes u^\circ \otimes u^\circ, u^\circ)$, we derive

$$u^\bullet y = (1_N \otimes \xi^*)(u^\bullet \otimes u^\circ \otimes u^\circ)(1_N \otimes \xi) = u^\circ$$

and similarly $yu^\bullet = u^\circ$. This allows us to see that $v'_{ij} z' = z' v'_{ij} = u_{ij} 1_2$.

Now, it only remains to show that all relations of the generators v_{ij} are satisfied by v'_{ij}. For this, note that I_H is generated by the relations $v^\circ = v^\bullet$ and the ideal I_G. For the first part, we use the assumption $F\bar{F} = c\,1_N$ to derive

$$v'^\bullet := (1_2 \otimes F)\begin{pmatrix} 0 & \bar{u}^\bullet \\ \bar{u}^\circ & 0 \end{pmatrix}(1_2 \otimes F^{-1}) = \begin{pmatrix} 0 & F\bar{F}u\bar{F}^{-1}F^{-1} \\ F\bar{u}F^{-1} & 0 \end{pmatrix} = \begin{pmatrix} 0 & u^\circ \\ u^\bullet & 0 \end{pmatrix} = v'.$$

For the second part, take any $f \in I_G$. Assume it is $\mathbb{Z}$-homogeneous of degree i. Then we have

$$f(v'_{ij}) = f(u_{ij} z'^*) = f(u_{ij}) z'^{-i} = 0.$$

This concludes the proof of existence of β. Now, noticing that $\beta \circ \alpha$ is the embedding of $C(G)$ into diagonal matrices over $C(G)$, we see that α must be injective. □

8.2.12 Remark. We leave the situation for general degree of reflection $k \in \mathbb{N}$ open. Modifying the proof, it is actually easy to show that, for any globally colourized G with degree of reflection k, we have

$$H \tilde{\times} \hat{\mathbb{Z}}_k \subseteq G \subseteq H \tilde{\times} \hat{\mathbb{Z}}.$$

However, we were unable to prove the inclusion $G \subseteq H \tilde{\times} \hat{\mathbb{Z}}_k$. This problem is actually equivalent to proving a stronger version of Proposition 8.2.10: Consider G globally colourized with degree of reflection k. Taking $w_1, w_2 \in \mathscr{O}^l_{\circ\bullet}$, does it hold that $\mathrm{Mor}(1, u^{\otimes w_1}) = \mathrm{Mor}(1, u^{\otimes w_2})$ whenever $c(w_1) \equiv c(w_2)$ modulo k?

8.2.5 Tensor complexification

In this section, we will study the tensor complexification (recall Sect. 2.5.3) and its representation categories.

8.2.13 Theorem. Consider a compact matrix quantum group $G = (C(G), v)$, $k \in \mathbb{N}_0$. Denote by z the generator of $C^*(\mathbb{Z}_k)$ and by $u := vz$ the fundamental representation of $G \tilde{\times} \hat{\mathbb{Z}}_k$. We have the following characterizations of $G \tilde{\times} \hat{\mathbb{Z}}_k$.

(1) The ideal $I_{G\tilde{\times}\hat{\mathbb{Z}}_k}$ is the $\mathbb{Z}_k$-homogeneous part of I_G. That is,

$$I_{G\tilde{\times}\hat{\mathbb{Z}}_k} = \{f \in I_G \mid f_l \in I_G \text{ for every } l \in \mathbb{Z}_k\}.$$

(2) The representation category of $G \tilde{\times} \hat{\mathbb{Z}}_k$ looks as follows:

$$\mathrm{Mor}(u^{\otimes w_1}, u^{\otimes w_2}) = \begin{cases} \mathrm{Mor}(v^{\otimes w_1}, v^{\otimes w_2}) & \text{if } c(w_2) - c(w_1) \text{ is a multiple of } k, \\ \{0\} & \text{otherwise.} \end{cases}$$

(3) We have $G \tilde{\times} \hat{\mathbb{Z}}_k = \langle G, E \tilde{\times} \hat{\mathbb{Z}}_k \rangle$. (Here, E denotes the trivial group of the appropriate size, so $E \tilde{\times} \hat{\mathbb{Z}}_k$ is the quantum group $\hat{\mathbb{Z}}_k$ with the representation $z \oplus \cdots \oplus z = z1_N$.)

Proof. For (1), we can express

$$f(u_{ij}, u^*_{ij}) = f(v_{ij} z, z^* v^*_{ij}) = \sum_{l \in \mathbb{Z}_k} f_l(v_{ij} z, z^* v^*_{ij}) = \sum_{l \in \mathbb{Z}_k} f_l(v_{ij}, v^*_{ij}) z^l,$$

where $f = \sum_l f_l$ is the decomposition of into the homogeneous components f_l of degree l. If all $f_l \in I_G$, so $f_l(v_{ij}, v^*_{ij}) = 0$, we have $f(v_{ij} z, z^* v^*_{ij}) = 0$, so $f \in I_{G\tilde{\times}\hat{\mathbb{Z}}_k}$. Conversely, if there is some $l \in \mathbb{Z}_k$ such that $f_l \notin I_G$, then $f(v_{ij} z, z^* v^*_{ij}) \neq 0$ and hence $f \notin I_{G\tilde{\times}\hat{\mathbb{Z}}_k}$.

For (2), first we prove that $\mathrm{Mor}(1, u^{\otimes w}) = \mathrm{Mor}(1, v^{\otimes w})$ if $c(w) \in k\mathbb{Z}$. Indeed, we have $u^{\otimes w} = (vz)^{\otimes w} = z^{c(w)} v^{\otimes w} = v^{\otimes w}$. Secondly, the fact that $\mathrm{Mor}(1, u^{\otimes w}) = \{0\}$ if $c(w) \notin k\mathbb{Z}$ follows from Lemma 8.2.5.

We prove (3) using Proposition 3.4.19. The category corresponding to $G \tilde{\times} \hat{\mathbb{Z}}_k$ (given by (2)) is indeed the intersection of the category $\mathsf{FundRep}_G$ and the category $\mathsf{FundRep}_{E \tilde{\times} \hat{\mathbb{Z}}_k}$, whose morphism spaces are given by

$$\mathrm{Mor}((z 1_N)^{\otimes w_1}, (z 1_N)^{\otimes w_2}) = \begin{cases} \mathbb{C}^N & \text{if } c(w_2) - c(w_1) \text{ is a multiple of } k, \\ \{0\} & \text{otherwise.} \end{cases} \qquad \square$$

8.2.14 Remark. An alternative proof of the proposition above could go as follows. One can easily see that the $\mathbb{Z}_k$-extended category associated to $G \times \hat{\mathbb{Z}}_k$ looks as follows

$$\mathsf{FundRep}^{k\text{-ext}}_{G \times \hat{\mathbb{Z}}_k}(w_1, w_2) = \begin{cases} \mathsf{FundRep}_G(w_1', w_2') & \text{if } t(w_2) - t(w_1) \text{ is a multiple of } k, \\ \{0\} & \text{otherwise,} \end{cases}$$

where $w_1', w_2' \in \mathscr{O}^*_{\circ\bullet}$ are created from $w_1, w_2 \in \mathscr{W}_k$ mapping $\square \mapsto \circ$, $\blacksquare \mapsto \bullet$, $\vartriangle \mapsto \emptyset$, $\blacktriangle \mapsto \emptyset$ and by $t(w)$ we mean the number of white triangles $\vartriangle$ minus the number of black triangles $\blacktriangle$ in w (which is a well-defined element of $\mathbb{Z}_k$). The item (2) of the proposition then follows from Proposition 8.2.3.

8.2.15 Remark. As a consequence of Theorem 8.2.13, we have that

$$(G \tilde{\times} \hat{\mathbb{Z}}_k) \tilde{\times} \hat{\mathbb{Z}}_l = G \tilde{\times} \hat{\mathbb{Z}}_{\mathrm{lcm}(k,l)}$$

A direct proof of this statement was formulated already in [TW17, Proposition 8.2].

Already in Section 2.5.3, we mentioned that the glued tensor product is often isomorphic to the standard one. We characterize this situation for products with $\hat{\mathbb{Z}}_k$ in Proposition 8.2.17. Before formulating it, we prove a lemma.

8.2.16 Lemma. Let $G \subseteq U^+(F)$ be a quantum group with degree of reflection k. Denote by v its fundamental representation. Consider $l \in \mathbb{N}_0$ and denote by z the generator of $C^*(\mathbb{Z}_l)$. Then $z^k \in C(G \tilde{\times} \hat{\mathbb{Z}}_l)$ for every l. Consequently, $z^{nk_0} \in C(G \tilde{\times} \hat{\mathbb{Z}}_l)$ for every $n \in \mathbb{N}_0$, where $k_0 := \gcd(k, l)$.

Proof. From Proposition 8.2.6, we can find a vector $\xi \in \mathrm{Mor}(1, v^{\otimes w})$ with $c(w) = k$ and $\|\xi\| = 1$. Recall that $C(G \tilde{\times} \hat{\mathbb{Z}}_l)$ is generated by the elements $v_{ij} z$ and that v_{ij} commute with z, so

$$C(G \tilde{\times} \hat{\mathbb{Z}}_l) \ni \xi^* (vz)^{\otimes w} \xi = \xi^* v^{\otimes w} \xi \, z^{c(w)} = z^k.$$

Consequently, $z^{nk} \in C(G \tilde{\times} \hat{\mathbb{Z}}_k)$ for every n and obviously $\{z^{nk}\}_{n \in \mathbb{N}_0} = \{z^{nk_0}\}_{n \in \mathbb{N}_0}$. $\square$

8.2.17 Proposition. Let $G \subseteq U^+(F)$ be a quantum group with degree of reflection k. Consider a number $l \in \mathbb{N}_0$. Then $G \tilde{\times} \mathbb{Z}_l \simeq G \times \mathbb{Z}_l$ if and only if k is coprime with l.

Proof. Assume we have $G \tilde{\times} \hat{\mathbb{Z}}_l \simeq G \times \hat{\mathbb{Z}}_l$. Suppose d is a divisor of both k and l. Then we must have also $G \tilde{\times} \hat{\mathbb{Z}}_d \simeq G \times \hat{\mathbb{Z}}_d$. But from Lemma 8.2.5, we have that $G \tilde{\times} \hat{\mathbb{Z}}_d = G$, which is a contradiction unless $d = 1$.

For the converse, denote by v the fundamental representation of G and by z the generator of $C^*(\mathbb{Z}_l)$. It is enough to show that we have $z \in C(G \tilde{\times} \hat{\mathbb{Z}}_l) \subseteq C(G) \otimes_{\max} C^*(\mathbb{Z}_l)$ since this already implies the equality of the C*-algebras. This follows directly from Lemma 8.2.16. $\square$

8.2.18 Remark. If l is not coprime with k, but $l_0 := l / \gcd(k, l)$ is coprime with k, we can use Remark 8.2.15, Lemma 8.2.5, and Proposition 8.2.17 to obtain

$$G \tilde{\times} \mathbb{Z}_l = (G \tilde{\times} \mathbb{Z}_{\gcd(k,l)}) \tilde{\times} \mathbb{Z}_{l_0} = G \tilde{\times} \mathbb{Z}_{l_0} \simeq G \times \mathbb{Z}_{l_0}.$$

Finally, we are going to characterize irreducible representations of the tensor complexification. Recall Theorem 2.5.6 characterizing irreducibles of the standard tensor product.

8.2.19 Proposition. Let $G \subseteq U^+(F)$ be a quantum group with degree of reflection k. Consider arbitrary $l \in \mathbb{N}_0$. Then $G \tilde{\times} \hat{\mathbb{Z}}_l$ has the following complete set of mutually inequivalent irreducible representations

$$\{u^\alpha z^{ki+d_\alpha} \mid \alpha \in \operatorname{Irr} G,\ i = 0,\ldots,l_0 - 1\}, \tag{8.4}$$

where $l_0 = l/\gcd(k,l)$ and z is the generator of $C^*(\mathbb{Z}_k)$.

Proof. Since G has degree of reflection k, the ideal I_G is $\mathbb{Z}_k$-homogeneous by Proposition 8.2.6. This means that the algebra $\operatorname{Pol} G$ is $\mathbb{Z}_k$-graded assigning v_{ij} degree one and v_{ij}^* degree minus one, where v is the fundamental representation of G. Consequently, the entries of any irreducible representation u^α, $\alpha \in \operatorname{Irr} G$ are $\mathbb{Z}_k$-homogeneous of some degree d_α (recall Sect. 8.2.1).

By Theorem 8.2.13, $I_{G\tilde{\times}\hat{\mathbb{Z}}_l}$ is the $\mathbb{Z}_l$-homogeneous part of I_G. Consequently, $I_{G\tilde{\times}\hat{\mathbb{Z}}_l}$ is $\mathbb{Z}_{\operatorname{lcm}(k,l)}$-homogeneous and $\operatorname{Pol}(G \tilde{\times} \hat{\mathbb{Z}}_l)$ is $\mathbb{Z}_{\operatorname{lcm}(k,l)}$-graded. However, this time the degree is computed with respect to the variables $u_{ij} := v_{ij} z$.

The irreducible representation of the standard tensor product $G \times \hat{\mathbb{Z}}_l$ are described by Theorem 2.5.6. Namely, those are exactly all $u^\alpha z^n$ with $\alpha \in \operatorname{Irr} G$, $n = 0,\ldots,l-1$. Recall from Corollary 3.3.3, that the irreducibles of $G \tilde{\times} \hat{\mathbb{Z}}_l$ form a subset of irreducibles of $G \times \hat{\mathbb{Z}}_l$. Hence, we need to determine all the pairs (α, n) such that $u^\alpha z^n$ is a matrix with entries in $C(G \tilde{\times} \hat{\mathbb{Z}}_l) \subseteq C(G \times \hat{\mathbb{Z}}_l)$.

We first prove that every irreducible of $G \tilde{\times} \hat{\mathbb{Z}}_l$ is equivalent to one from Eq. (8.4). As we just mentioned, it must be of the form $u^\alpha z^n$ for some α, n. Since it is a representation of $G \tilde{\times} \hat{\mathbb{Z}}_l$, it must be a subrepresentation of $u^{\otimes w} = v^{\otimes w} z^{c(w)}$ for some $w \in \mathscr{O}_{\circ\bullet}^*$. Consequently, u^α is a subrepresentation of $v^{\otimes w}$, so $d_\alpha \equiv c(w)$ modulo k. In addition, we must also have $n \equiv c(w)$ modulo l. As a consequence, $n \equiv d_\alpha$ modulo $k_0 := \gcd(k,l)$. Thus, we must have $n = k_0 i + d_\alpha$ for some $i \in \mathbb{Z}$. Obviously, $\{z^{k_0 i + d_\alpha}\}_{i \in \mathbb{Z}} = \{z^{ki+d_\alpha}\}_{i=1}^{l_0}$.

For the converse inclusion, we need to show that the entries of $u^\alpha z^{ki+d_\alpha}$ are elements of $C(G \tilde{\times} \hat{\mathbb{Z}}_l)$ for every α, i. Since u^α is an irreducible representation of G, it must be a subrepresentation of $v^{\otimes w}$ for some $w \in \mathscr{O}_{\circ\bullet}^*$. Consequently, $u^\alpha z^{c(w)}$ is a subrepresentation of $u^{\otimes w} = v^{\otimes w} z^{c(w)}$. Hence, it is a representation of $G \tilde{\times} \hat{\mathbb{Z}}_l$. From Lemma 8.2.16, it follows that also $u^\alpha z^{ki+c(w)}$ is a representation of $G \tilde{\times} \hat{\mathbb{Z}}_l$. Since $d_\alpha \equiv c(w)$ modulo k, this is equivalent to considering representations $u^\alpha u^{ki+d_\alpha}$. □

8.2.6 Free complexification

The goal of this section is to characterize the representation categories of the free complexifications, that is, the quantum groups $H \tilde{*} \hat{\mathbb{Z}}_l$. For the free complexification, we do not have many results yet even in the easy case. In [TW17], the two-coloured categories corresponding to free complexifications of free orthogonal easy quantum groups are provided. For us, the motivating result is Proposition 6.4.15 linking the free complexification by $\mathbb{Z}_2$ with the category $\operatorname{Alt}\mathscr{C}$ generated by alternating coloured partitions. This proposition was proven with the help of categories of partitions with extra singletons describing the dual free product with $\mathbb{Z}_2$ and the functor F describing the gluing procedure. Also here, we will make use of the $\mathbb{Z}_l$-extended representation categories describing the dual free product $H \hat{*} \hat{\mathbb{Z}}_l$ and then we will glue the factors and apply Proposition 8.2.3 to find the corresponding representation category. An interesting result is that the free complexification $H \tilde{*} \hat{\mathbb{Z}}_l$ actually does not depend on the number l unless the degree of reflection of H equals to one.

8.2.20 Definition. A monomial of even length of the form $x_{i_1 j_1} x_{i_2 j_2}^* x_{i_3 j_3} x_{i_4 j_4}^* \cdots \in \mathbb{C}\langle x_{ij}, x_{ij}^* \rangle$, where the variables with and without star alternate, is called **alternating**. A linear combination of alternating monomials, where either all start with a non-star variable or all start with a star variable, is called an **alternating polynomial**. A quantum group G is called **alternating** if I_G is generated by alternating polynomials.

Considering a compact matrix quantum group $G \subseteq U^+(F)$ with unitary fundamental representation u, recall the notation $u^\circ := u$, $u^\bullet := F\bar{u}F^{-1}$. So, the relations of G can be alternatively expressed by polynomials in variables u°_{ij}, $u^\bullet_{ij}$ instead of u_{ij} and u^*_{ij}. Since the transformation between those two sets of variables is linear, the definition of an alternating quantum group can be stated in unchanged form also using the alternative ideal.

8.2.21 Lemma. Let H be a compact matrix quantum group, $k,l \in \mathbb{N}_0$ such that $\gcd(k,l) \neq 1$ (using the convention $\gcd(0,k) = k$). Then we have

$$H \tilde{*} \hat{\mathbb{Z}} = (H \tilde{\times} \hat{\mathbb{Z}}_k) \tilde{*} \hat{\mathbb{Z}}_l.$$

Proof. We denote by v, z, r, s the fundamental representations of H, $\hat{\mathbb{Z}}$, $\hat{\mathbb{Z}}_k$, and $\hat{\mathbb{Z}}_l$, respectively. We need to find a $*$-isomorphism $C(H \tilde{*} \hat{\mathbb{Z}}) \to C((H \tilde{\times} \hat{\mathbb{Z}}_k) \tilde{*} \hat{\mathbb{Z}}_l)$ mapping $v_{ij}z \mapsto v_{ij}sr$.

First, we see that there exists a $*$-homomorphism

$$\alpha\colon C(H) *_{\mathbb{C}} C^*(\mathbb{Z}) \to (C(H) \otimes_{\max} C^*(\mathbb{Z}_k)) *_{\mathbb{C}} C^*(\mathbb{Z}_l)$$

mapping

$$v_{ij} \mapsto v_{ij}, \qquad z \mapsto sr.$$

since sr is a unitary.

This $*$-homomorphism then restricts to a surjective $*$-homomorphism of the form we are looking for. It remains to prove that it is injective. To prove this, we construct a $*$-homomorphism

$$\beta\colon (C(H) \otimes_{\max} C^*(\mathbb{Z}_k)) *_{\mathbb{C}} C^*(\mathbb{Z}_l) \to M_d\Big(C(H) *_{\mathbb{C}} C^*(\mathbb{Z})\Big)$$

mapping

$$v_{ij} \mapsto \begin{pmatrix} v_{ij} & & 0 \\ & \ddots & \\ 0 & & v_{ij} \end{pmatrix}, \qquad s \mapsto \begin{pmatrix} & & & 1 \\ 1 & & & \\ & \ddots & & \\ & & 1 & \end{pmatrix}, \qquad r \mapsto \begin{pmatrix} & z^* & & & \\ & & 1 & & \\ & & & \ddots & \\ & & & & 1 \\ z & & & & \end{pmatrix},$$

where $d \neq 1$ is some common divisor of k and l.

We can check that the images satisfy all the defining relations between the generators, so such a homomorphism indeed exists. Now, we can see that $\beta \circ \alpha$ is injective, so α must be injective. Thus, the restriction of α we are interested in is also injective. □

Now, we describe the representation category of $H \tilde{*} \hat{\mathbb{Z}}_l$ for arbitrary H and l. As we indicated in the beginning of this section, the strategy is to first describe the $\mathbb{Z}_l$-extended representation category associated to the dual free product $H \hat{*} \hat{\mathbb{Z}}_l$ and then to obtain the representation category of its glued version – the glued free product $H \tilde{*} \hat{\mathbb{Z}}_l$ – using Proposition 8.2.3. Recall the notation for $\mathbb{Z}_l$-extended categories from Section 8.1.3. Recall also the alternative definition of $\mathscr{O}$-coloured representation category from Section 8.1.2.

8.2.22 Proposition. Let H be a compact matrix quantum group and $l \in \mathbb{N}$. Then the $\mathbb{Z}_l$-extended category $\mathsf{FundRep}^{l\text{-ext}}_{H \hat{*} \hat{\mathbb{Z}}_l}$ is generated by the collection $C(\iota(w_1), \iota(w_2)) := \mathsf{FundRep}_H(w_1, w_2)$, where $\iota\colon \mathscr{O}^*_{\circ\bullet} \to \mathscr{W}_l$ is the injective homomorphism mapping $\circ \mapsto \square$, $\bullet \mapsto \blacksquare$. Moreover, we have the following inductive description. If $w \in \mathscr{W}_k$ contains no triangles, i.e. $w = \iota(w')$ for some $w' \in \mathscr{O}^*_{\circ\bullet}$, then

$$\mathsf{FundRep}^{l\text{-ext}}_{H \hat{*} \hat{\mathbb{Z}}_l}(\emptyset, w) = \mathsf{FundRep}_H(\emptyset, w').$$

Otherwise,

$$\mathsf{FundRep}^{l\text{-ext}}_{H \hat{*} \hat{\mathbb{Z}}_l}(\emptyset, w) = \left\{ R^{w_0}(\xi_1 \otimes \cdots \otimes \xi_l) \;\middle|\; \begin{array}{c} w = w_0 \vartriangle w_1 \vartriangle \cdots \vartriangle w_l \\ \xi_i \in \mathsf{FundRep}^{l\text{-ext}}_{H \hat{*} \hat{\mathbb{Z}}_l}(\emptyset, w_i),\ i = 1, \ldots, l-1 \\ \xi_l \in \mathsf{FundRep}^{l\text{-ext}}_{H \hat{*} \hat{\mathbb{Z}}_l}(\emptyset, w_l w_0) \end{array} \right\}.$$

Proof. Let $\mathfrak{C}$ be the $\mathbb{Z}_l$-extended category generated by C. Then the associated quantum group $G = (C(G), v \oplus z)$ is a quantum subgroup of $U^+(F) \hat{*} \hat{\mathbb{Z}}_l$ defined by the relations of H for v and no relations for z (except for $zz^* = z^*z = 1 = z^l$). But this is exactly the free product $H \hat{*} \hat{\mathbb{Z}}_l$. Now, it remains to prove that $\mathfrak{C}$ is given by the above described recursion.

The inclusion $\supseteq$ follows from the fact that $\mathfrak{C}$ has to be closed under the category operations. To check the inclusion $\subseteq$, it is enough to check that the right-hand side defines a category. That is, we need to check that it is closed under tensor products, contractions, rotations, inverse rotations, and reflections as defined in Sect. 8.1.2. Checking this is straightforward using induction. Nevertheless it may become a bit lengthy to check all the details. We will do it here for the rotation and tensor product.

So, denote the sets given by the inductive description by $\tilde{\mathfrak{C}}$. If we take words w without triangles, that is, $w = \iota(w')$, then the sets $\tilde{\mathfrak{C}}(\emptyset, w) = \mathsf{FundRep}_H(\emptyset, w')$ are closed under all the operations since $\mathsf{FundRep}_H$ is a category. To show closedness under rotations in general, we do an induction on the length of the word w. So, consider an element $\xi \in \tilde{\mathfrak{C}}(\emptyset, w)$ with $w = w_0 {\vartriangle} w_1 {\vartriangle} \cdots {\vartriangle} w_l$, so it is of the form $\xi = R^{w_0}(\xi_1 \otimes \cdots \otimes \xi_l)$. First, suppose that w_l is not empty and denote by x its last letter. Then we directly have $R^x \xi = R^{xw_0}(\xi_1 \otimes \cdots \otimes \xi_l) \in \tilde{\mathfrak{C}}(\emptyset, Rw)$. For the case $w_l = \emptyset$, note that $R^{\vartriangle} = \mathrm{id}$. So, we need to check that $\tilde{\mathfrak{C}}(\emptyset, {\vartriangle} w_0 {\vartriangle} w_1 {\vartriangle} \cdots {\vartriangle} w_{l-1}) \ni \xi = (R^{w_0}\xi_l) \otimes \xi_1 \otimes \cdots \otimes \xi_{l-1}$. This is true thanks to the fact that $R^{w_0}\xi_l \in \tilde{\mathfrak{C}}(\emptyset, w_0)$ by induction. For the inverse rotations, the proof goes exactly the same way.

Now, we can also prove closedness under the tensor product. Take $\xi \in \tilde{\mathfrak{C}}(\emptyset, w)$, $\eta \in \tilde{\mathfrak{C}}(\emptyset, w')$. We will do the induction on the length of w. Actually, we can assume that $|w| \geq |w'|$ since we can swap the factors by rotation: $\eta \otimes \xi = R^{w'}(\xi \otimes R^{-w'}\eta)$. So, assume $w = w_0 {\vartriangle} w_1 {\vartriangle} \cdots {\vartriangle} w_l$, so ξ is of the form $\xi = R^{w_0}(\xi_1 \otimes \cdots \otimes \xi_l) \in \tilde{\mathfrak{C}}(\emptyset, w_0 {\vartriangle} w_1 {\vartriangle} \cdots {\vartriangle} w_l)$. Then we have

$$\begin{aligned}
\xi_l &\in \tilde{\mathfrak{C}}(\emptyset, w_l w_0) && \text{by assumption,}\\
R^{w_0}\xi_l &\in \tilde{\mathfrak{C}}(\emptyset, w_0 w_l) && \text{by closedness under rotations,}\\
R^{w_0}\xi_l \otimes \eta &\in \tilde{\mathfrak{C}}(\emptyset, w_0 w_l w') && \text{by induction hypothesis,}\\
\tilde{\xi}_l = R^{-w_0}(R^{w_0}\xi_l \otimes \eta) &\in \tilde{\mathfrak{C}}(\emptyset, w_l w' w_0) && \text{by closedness under inverse rotations,}\\
\xi \otimes \eta = R^{w_0}(\xi_1 \otimes \ldots \otimes \xi_{l-1} \otimes \tilde{\xi}_l) &\in \tilde{\mathfrak{C}}(\emptyset, w_0 {\vartriangle} w_1 {\vartriangle} \cdots {\vartriangle} w_l w') && \text{by definition of } \tilde{\mathfrak{C}}.
\end{aligned}$$

□

8.2.23 Lemma. We can arrange the recursion of the above proposition in such a way that the words $w_1, \ldots, w_{l-1}$ contain no triangles, so we have

$$\mathsf{FundRep}^{l\text{-ext}}_{H \hat{*} \hat{\mathbb{Z}}_l}(\emptyset, w) = \left\{ R^{w_0}(\xi_1 \otimes \cdots \otimes \xi_l) \;\middle|\; \begin{array}{c} w = w_0 {\vartriangle} w_1 {\vartriangle} \cdots {\vartriangle} w_l \\ \xi_i \in \mathsf{FundRep}_H(\emptyset, w_i'),\ i = 1, \ldots, l-1 \\ \xi_l \in \mathsf{FundRep}^{l\text{-ext}}_{H \hat{*} \hat{\mathbb{Z}}_l}(\emptyset, w_l w_0) \end{array} \right\}.$$

Proof. We prove this by induction. Take an arbitrary word $w \in \mathscr{W}_l$ and suppose that the above description works for any shorter word. Now consider an element $\xi \in \mathsf{FundRep}^{l\text{-ext}}_{H \hat{*} \hat{\mathbb{Z}}_l}(\emptyset, w)$, so it is of the form $\xi = R^{w_0}(\xi_1 \otimes \cdots \otimes \xi_l)$ corresponding to the decomposition $w = w_0 {\vartriangle} w_1 {\vartriangle} \cdots {\vartriangle} w_l$. Suppose now that w_i contains some triangles for some $i \in \{1, \ldots, l-1\}$. By induction hypothesis, we can write $\xi_i = R^{a_0}(\eta_1 \otimes \cdots \otimes \eta_l)$ corresponding to $w_i = a_0 {\vartriangle} a_1 {\vartriangle} \cdots {\vartriangle} a_l$, where $a_1, \ldots, a_{l-1}$ contain no triangles, so $\eta_i \in \mathsf{FundRep}_H(\emptyset, a_i')$. But this means that we can write also

$$\xi = R^{w_0 \cdots w_{i-1} a_0}(\eta_1 \otimes \cdots \otimes \eta_{l-1} \otimes \tilde{\eta}_l),$$

where

$$\begin{aligned}
\tilde{\eta}_l &= R^{-a_0}(R^{a_0}\eta_l \otimes \xi_{i+1} \otimes \cdots \otimes \xi_l \otimes R^{-w_1}\xi_1 \otimes \cdots \otimes R^{-w_{i-1}}\xi_{i-1})\\
&\qquad \in \mathsf{FundRep}^{l\text{-ext}}_{H \hat{*} \hat{\mathbb{Z}}_l}(\emptyset, a_l {\vartriangle} w_{i+1} {\vartriangle} \cdots {\vartriangle} w_l w_0 {\vartriangle} w_1 {\vartriangle} \cdots {\vartriangle} w_{i-1} {\vartriangle} a_0).
\end{aligned}$$

□

In the following theorem, we describe the representation category of the free complexification. In the formulation, we use the following notation. Given an element $w \in \mathscr{O}^*_{\circ\bullet}$ or $w \in \mathscr{W}_l$, we use negative powers to indicate the colour inversion, that is, $w^{-j} = \bar{w}^j$. For example, $(\circ\bullet)^{-2} = (\bullet\circ)^2 = \bullet\circ\bullet\circ$.

8.2.24 Theorem. Let H be a compact matrix quantum group with degree of reflection $k \neq 1$. Then all $H \tilde{*} \hat{\mathbb{Z}}_l$ coincide for all $l \in \mathbb{N}_0 \setminus \{1\}$. The ideal $I_{H\tilde{*}\hat{\mathbb{Z}}_l}$ is generated by the alternating polynomials in I_H. The representation category $\mathsf{FundRep}_{H\tilde{*}\hat{\mathbb{Z}}_l}$ is a (wide) subcategory of the representation category $\mathsf{FundRep}_H$ generated by the sets $C(\emptyset,(\circ\bullet)^j) := \mathsf{FundRep}_H(\emptyset,(\circ\bullet)^j)$, $j \in \mathbb{Z}$. This also holds if $k=1$ and $l=0$.

Proof. Let $I \subseteq \mathbb{C}\langle x_{ij}, x_{ij}^*\rangle$ be the ideal generated by the alternating polynomials in I_H. Denote by $u_{ij} = v_{ij}z$ the fundamental representation of $H \tilde{*} \hat{\mathbb{Z}}_l$. To prove that $I \subseteq I_{H\tilde{*}\hat{\mathbb{Z}}_l}$, take any alternating polynomial $f \in I_H$. If all monomials in f start with a non-star variable, we have $f(v_{ij}z) = f(v_{ij}) = 0$; if all monomials start with a star variable, then $f(v_{ij}z) = z^* f(v_{ij})z = 0$. In both cases, we have proven that $f \in I_{H\tilde{*}\hat{\mathbb{Z}}_l}$. The opposite inclusion $I \supseteq I_{H\tilde{*}\hat{\mathbb{Z}}_l}$ will follow from the statement about representation categories as all the relations corresponding to the elements of C are alternating.

Note that it is enough to prove the statement for $k \neq 1$ and $l \neq 0$. Indeed, for $k \neq 1$ and $l = 0$, we have by Lemma 8.2.21 that $H \tilde{*} \hat{\mathbb{Z}} = H \tilde{*} \hat{\mathbb{Z}}_k$. For $k=1$, $l=0$ we use Lemma 8.2.21 to express $H \tilde{*} \hat{\mathbb{Z}} = (H \tilde{\times} \hat{\mathbb{Z}}_2) \tilde{*} \mathbb{Z}_l$. Since $c((\circ\bullet)^j) = 0 \in 2\mathbb{Z}$ for every j, we have $\mathsf{FundRep}_H(\emptyset,(\circ\bullet)^j) = \mathsf{FundRep}_{H\tilde{\times}\hat{\mathbb{Z}}_2}(\emptyset,(\circ\bullet)^j)$.

So, let $\mathfrak{C}$ be the two-coloured representation category generated by C. We need to prove that $\mathsf{FundRep}_{H\tilde{*}\hat{\mathbb{Z}}_l}(\emptyset, w) = \mathfrak{C}(\emptyset, w)$ for every $w \in \mathscr{O}^*_{\circ\bullet}$. In order to do that, we will use Proposition 8.2.3, whose statement can be, in this case, formulated as

$$\mathsf{FundRep}_{H\tilde{*}\hat{\mathbb{Z}}_l}(\emptyset, w) = \mathsf{FundRep}^{l\text{-ext}}_{H\hat{*}\hat{\mathbb{Z}}_l}(\emptyset, \tilde{w}), \tag{8.5}$$

where $\tilde{w} \in \mathscr{W}_l$ is the glued version of $w \in \mathscr{O}^*_{\circ\bullet}$.

Let us start with the easier inclusion $\supseteq$. Since $\mathsf{FundRep}_{H\tilde{*}\hat{\mathbb{Z}}_l}$ is a category, it is enough to show that $\mathsf{FundRep}_{H\tilde{*}\hat{\mathbb{Z}}_l}(\emptyset, w) \supseteq C(\emptyset, w)$ for every $w = (\circ\bullet)^j$, $j \in \mathbb{Z}$. Note that the glued version of w is in this case $\tilde{w} = (\square\vartriangle\blacktriangle\blacksquare)^j = (\square\blacksquare)^j$. Combining Proposition 8.2.22 and Equation (8.5), we have

$$C(\emptyset,(\circ\bullet)^j) = \mathsf{FundRep}_H(\emptyset,(\circ\bullet)^j) = \mathsf{FundRep}_{H\hat{*}\hat{\mathbb{Z}}_k}(\emptyset,(\square\blacksquare)^j) = \mathsf{FundRep}_{H\tilde{*}\hat{\mathbb{Z}}_l}(\emptyset,(\circ\bullet)^j).$$

We will prove the opposite inclusion $\subseteq$ by induction on the length of w. Take some

$$\xi \in \mathsf{FundRep}_{H\tilde{*}\hat{\mathbb{Z}}_l}(\emptyset, w) = \mathsf{FundRep}^{l\text{-ext}}_{H\hat{*}\hat{\mathbb{Z}}_l}(\emptyset, \tilde{w}).$$

Suppose $\xi \neq 0$. According to Lemma 8.2.23, we can assume that $\tilde{w} = w_0 \vartriangle w_1 \vartriangle \cdots \vartriangle w_l$, where $w_1, \ldots, w_{l-1}$ contain no triangles, and then $\xi = R^{w_0}(\xi_1 \otimes \cdots \otimes \xi_l)$ with $\xi_i \in \mathsf{FundRep}_H(\emptyset, w_i')$ and $\xi_l \in \mathsf{FundRep}_{H\hat{*}\hat{\mathbb{Z}}_l}(\emptyset, w_l w_0)$. Since $\tilde{w}$ is the glued version of w, this means that in all the words $w_1, \ldots, w_{l-1}$ the colours alternate (two consecutive white squares would necessarily have a white triangle $\vartriangle$ between them, two consecutive black squares would have $\blacktriangle = \vartriangle^{l-1}$ between them). Moreover, since we assume $\xi_i \neq 0$, we must have $c(w_i') \in k\mathbb{Z}$ and, since $k \neq 1$, this means that the w_i's are of even length. So, $w_i = (\square\blacksquare)^{j_i}$, $w_i' = (\circ\bullet)^{j_i}$.

Finally note that if we delete $\circ\bullet$ or $\bullet\circ$ from some word w, its glued version will be given by deleting $\square\blacksquare$ resp. $\blacksquare\square$. In particular, denote by $\hat{w}$ the element w after deleting all the subwords $w_1', \ldots, w_{l-1}'$. Its glued version is then $w_0 \vartriangle^l w_l = w_0 w_l$. Using the induction hypothesis, this finishes the proof as we have

$$\xi = R^{w_0}(\xi_1 \otimes \cdots \otimes \xi_l)$$

with

$$\xi_i \in \mathsf{FundRep}_H(\emptyset, w_i') = \mathsf{FundRep}_H(\emptyset, (\circ\bullet)^{j_i}) = C(\emptyset, (\circ\bullet)^{j_i}) \qquad \text{for } i = 1, \ldots, l-1$$

$$R^{w_0}\xi_l \in \mathsf{FundRep}_{H\hat{*}\hat{\mathbb{Z}}_2}(\emptyset, w_0 w_l) = \mathsf{FundRep}_{H\tilde{*}\hat{\mathbb{Z}}_2}(\emptyset, \hat{w}) = \mathfrak{C}(\emptyset, \hat{w}). \qquad \square$$

We may ask what happens if we iterate those free complexifications. The following statement was again already formulated in [TW17]; however, without a proof. (Note that it generalizes Lemma 8.2.21 dropping the assumption $\gcd(k,l) \neq 1$.)

8.2.25 Proposition. Let H be a compact matrix quantum group, $k,l\in\mathbb{N}_0\setminus\{1\}$. Then

$$(H\mathbin{\tilde{\times}}\hat{\mathbb{Z}}_k)\mathbin{\tilde{*}}\hat{\mathbb{Z}}_l=(H\mathbin{\tilde{*}}\hat{\mathbb{Z}}_k)\mathbin{\tilde{*}}\hat{\mathbb{Z}}_l=H\mathbin{\tilde{*}}\mathbb{Z}.$$

Proof. The second equality follows directly from Theorem 8.2.24 – we see that iterating the operation on the categories for the second time cannot change it since $\mathsf{FundRep}_H(\emptyset,(\circ\bullet)^j)=\mathsf{FundRep}_{H\tilde{*}\hat{\mathbb{Z}}_k}(\emptyset,(\circ\bullet)^j)$. For the first equality, we use, in addition, Theorem 8.2.13. Since $c((\circ\bullet)^j)=0$, we have $\mathsf{FundRep}_{H\tilde{\times}\hat{\mathbb{Z}}_k}(\emptyset,(\circ\bullet)^j)=\mathsf{FundRep}_H(\emptyset,(\circ\bullet)^j)$. □

Again, we can ask in what situations does it happen that the glued free product $H\mathbin{\tilde{*}}\hat{\mathbb{Z}}_l$ is isomorphic to the standard one. Obviously, the necessary condition is that H has degree of reflection one since $H\tilde{*}\hat{\mathbb{Z}}_l\simeq H\hat{*}\hat{\mathbb{Z}}_l$ implies $H\tilde{\times}\hat{\mathbb{Z}}_l\simeq H\times\hat{\mathbb{Z}}_l$ and here we can use Proposition 8.2.17. We can formulate the converse in the case of globally-colourized quantum groups H (in particular, if $H\subseteq O^+(F)$).

8.2.26 Proposition. Let H be a globally colourized compact matrix quantum group with degree of reflection one. Then $H\mathbin{\tilde{*}}\hat{\mathbb{Z}}_k\simeq H\mathbin{\hat{*}}\hat{\mathbb{Z}}_k$ for every $k\in\mathbb{N}_0$.

Proof. Denote by v the fundamental representation of H and by z the generator of $C^*(\mathbb{Z}_k)$. Again, it is enough to show that we have $z\in C(H\mathbin{\tilde{\times}}\hat{\mathbb{Z}}_k)\subseteq C(H)\otimes_{\max}C^*(\mathbb{Z}_k)$ since this already implies the equality of the C*-algebras. From Proposition 8.2.6, we can find a vector $\xi\in\mathrm{Mor}(1,v^{\otimes w})$ with $c(w)=1$ and $\|\xi\|=1$. Since H is globally colourized, we have $\mathrm{Mor}(1,v^{\otimes w})=\mathrm{Mor}(1,v^{\otimes\tilde{w}})$, where $\tilde{w}=\circ\bullet\circ\bullet\ldots\circ$, $|\tilde{w}|=|w|$. For such a word, we have $(vz)^{\otimes\tilde{w}}=(v^\circ z)(z^*v^\bullet)(v^\circ z)\cdots(v^\circ z)=v^{\otimes\tilde{w}}z$, so

$$\xi^*(vz)^{\otimes\tilde{w}}\xi=\xi^*(v^{\otimes\tilde{w}}z)\xi=\xi^*\xi\,z=z.$$ □

8.2.7 Free complexification of orthogonal quantum groups

In this section, we will study more in detail the free complexification $H\mathbin{\hat{*}}\hat{\mathbb{Z}}_k$ with $H\subseteq O^+(F)$. Recall from Section 2.3.3 that we define $O^+(F)\subseteq U^+(F)$ only for F satisfying $F\bar{F}=c\,1_N$ for some $c\in\mathbb{R}$. We will use this assumption in the whole section.

8.2.27 Definition. A quantum group $G=(C(G),u)\subseteq U^+(F)$ with $F\bar{F}=cI$ is called **invariant with respect to the colour inversion** if the map $u_{ij}\mapsto[F\bar{u}F^{-1}]_{ij}$ extends to a $*$-isomorphism.

Let us explain a bit this definition. First of all, note that the required $*$-homomorphism maps

$$u^\circ_{ij}\mapsto u^\bullet_{ij},\qquad u^\bullet_{ij}\mapsto u^\circ_{ij}.$$

Indeed, the first assignment is exactly the definition. For the second one, we have

$$u^\bullet_{ij}=[F\bar{u}F^{-1}]_{ij}\mapsto[F\bar{F}u\bar{F}^{-1}F^{-1}]_{ij}=u_{ij}$$

thanks to the assumption $F\bar{F}=c\,1_N$. In the Kac case $F=1_N$, the homomorphism maps $u_{ij}\mapsto u^*_{ij}$. But let us stress that for general elements of $C(G)$ the homomorphism does not coincide with the $*$-operation (simply because the $*$ is not a homomorphism).

Secondly, we have the following alternative formulations.

8.2.28 Proposition. Consider $G=(C(G),u)\subseteq U^+(F)$ with $F\bar{F}=c\,1_N$. Then the following are equivalent.

(1) $C(G)$ has an automorphism $u^\circ_{ij}\leftrightarrow u^\bullet_{ij}$. That is, G is invariant w.r.t. the colour inversion.
(2) I_G is invariant w.r.t. $x^\circ_{ij}\leftrightarrow x^\bullet_{ij}$. More precisely, we mean one of the following equivalent conditions.
    (a) I_G is invariant w.r.t. the $*$-homomorphism mapping $x^\circ_{ij}\mapsto x^\bullet_{ij}$
    (b) I_G is invariant w.r.t. the homomorphism mapping $x^\circ_{ij}\mapsto x^\bullet_{ij}$ and $x^\bullet_{ij}\mapsto x^\circ_{ij}$.

(3) $\mathsf{FundRep}_G$ is invariant w.r.t. $\circ \leftrightarrow \bullet$. That is, $\mathsf{FundRep}_G(\bar{w}_1, \bar{w}_2) = \mathsf{FundRep}_G(w_1, w_2)$.

Proof. The equivalence (1) $\Leftrightarrow$ (2a) follows from the universal property of $C(G)$.

For (1) $\Rightarrow$ (3), take $T \in \mathsf{FundRep}_G(w_1, w_2)$, so $Tu^{\otimes w_1} = u^{\otimes w_2} T$. Applying the automorphism, we get $Tu^{\otimes \bar{w}_1} = u^{\otimes \bar{w}_2} T$, so $T \in \mathsf{FundRep}_G(\bar{w}_1, \bar{w}_2)$.

For (3) $\Rightarrow$ (2b), we use the Tannaka–Krein, namely the fact that I_G is spanned by the relations of the form $Tx^{\otimes w_1} = x^{\otimes w_2} T$. Those relations are invariant with respect to the homomorphism $x^\circ \mapsto x^\bullet$, $x^\bullet \mapsto x^\circ$ since this homomorphism maps $x^w \mapsto x^{\bar{w}}$. Consequently, the whole ideal I_G must be invariant with respect to this homomorphism.

The implication (2b) $\Rightarrow$ (1) again follows from the universal property of $C(G)$. We get that $u^\circ_{ij} \mapsto u^\bullet_{ij}$, $u^\bullet_{ij} \mapsto u^\circ_{ij}$ extends to a homomorphism $C(G) \to C(G)$. Using the assumption $F\bar{F} = c1_N$, we can show this actually must be a $*$-homomorphism. □

As an example, note that all the universal unitary quantum groups $U^+(F)$ with $F\bar{F} = c1_N$ have this property. In addition, any quantum group $G \subseteq O^+(F)$ has this property.

8.2.29 Theorem. Consider $G \subseteq U^+(F)$ with $F\bar{F} = c1_N$. Then G is alternating and invariant with respect to the colour inversion if and only if it is of the form $G = H \tilde{*} \hat{\mathbb{Z}}$, where $H = G \cap O^+(F)$.

Proof. The right-left implication follows from Theorem 8.2.24: The fact that $H \tilde{*} \hat{\mathbb{Z}}$ is alternating is precisely the statement of Theorem 8.2.24. As we mentioned above, $H \subseteq O^+(F)$ is surely invariant with respect to the colour inversion. According to Proposition 8.2.28, this is equivalent to saying that the associated category $\mathsf{FundRep}_H$ is invariant with respect to the colour inversion. In particular, we must have $\mathsf{FundRep}_H(\emptyset, (\circ\bullet)^j) = \mathsf{FundRep}_H(\emptyset, (\bullet\circ)^j)$, which are the generators of $\mathsf{FundRep}_{H \tilde{*} \hat{\mathbb{Z}}}$ according to Theorem 8.2.24. Consequently, also $H \tilde{*} \hat{\mathbb{Z}}$ must be invariant with respect to the colour inversion.

In order to prove the left-right implication, we construct a surjective $*$-homomorphism

$$\alpha\colon C(G) \to C(H \tilde{*} \mathbb{Z})$$

mapping $u_{ij} \mapsto u'_{ij} := v_{ij} z$. To prove that such a homomorphism exists, take any alternating element $f \in I_G$. Since $H \subseteq G$, we have $f(v_{ij}) = 0$. We need to prove that $f(u'_{ij}) = 0$. If all terms of f start with a non-star variable, then $f(u'_{ij}) = f(v_{ij} z) = f(v_{ij}) = 0$; if all terms start with a star variable, then $f(u'_{ij}) = z^* f(v_{ij}) z = 0$.

It remains to prove that α is injective. To do that, we define a $*$-homomorphism

$$\beta\colon C(H) *_{\mathbb{C}} C^*(\mathbb{Z}) \to M_2(C(G))$$

mapping

$$v_{ij} \mapsto v'_{ij} := \begin{pmatrix} 0 & u^\circ_{ij} \\ u^\bullet_{ij} & 0 \end{pmatrix}, \quad z \mapsto z' := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

We immediately see that indeed $z'z'^* = z'^* z' = 1$. In exactly the same way as in the proof of Theorem 8.2.11, we also prove that $v'^\bullet := (1_2 \otimes F)\bar{v}'(1_2 \otimes F^{-1}) = v' =: v'^\circ$. Finally, take $f \in I_G$ and, for convenience, use the representation in variables u°_{ij} and $u^\bullet_{ij}$. Suppose $f(x^\circ_{ij}, x^\bullet_{ij})$ is alternating such that all variables start with x°_{ij}. We have

$$f(v'_{ij}, v'_{ij}) = \begin{pmatrix} f(u^\circ_{ij}, u^\bullet_{ij}) & 0 \\ 0 & f(u^\bullet_{ij}, u^\circ_{ij}) \end{pmatrix} = 0,$$

where $f(u^\circ_{ij}, u^\bullet_{ij}) = 0$ directly by $f \in I_G$ and $f(u^\bullet_{ij}, u^\circ_{ij})$ by invariance under the colour inversion.

Since obviously $\beta \circ \alpha$ is injective, we have proven that α is a $*$-isomorphism. □

Considering a quantum group $H = (C(H), v) \subseteq O^+(F)$, we have $H \tilde{\times} \hat{\mathbb{Z}}_k \subseteq H \tilde{\times} \hat{\mathbb{Z}} \subseteq H \tilde{*} \hat{\mathbb{Z}}$. We express those subgroups in terms of relations in the variables $u_{ij} = v_{ij} z$. Of course, those subgroups are given by the relations $v_{ij} z = z v_{ij}$ and $z^k = 1$, but those may not be well-defined in $C(H \tilde{*} \hat{\mathbb{Z}})$ as we may not have $z \in C(H \tilde{*} \hat{\mathbb{Z}})$.

8.2.30 Proposition. Consider $H \subseteq O^{+}(F)$. Then $H \mathbin{\tilde\times} \hat{\mathbb{Z}}$ is a quantum subgroup of $H \mathbin{\tilde *} \hat{\mathbb{Z}}$ given by the relation

$$u^{\circ}_{ij} u^{\bullet}_{kl} = u^{\bullet}_{ij} u^{\circ}_{kl}. \tag{8.6}$$

For $k \in \mathbb{N}$, $H \mathbin{\tilde\times} \hat{\mathbb{Z}}_{2k}$ is a quantum subgroup of $H \mathbin{\tilde\times} \hat{\mathbb{Z}}$ with respect to the relation

$$u^{\circ}_{i_1 j_1} \cdots u^{\circ}_{i_k j_k} = u^{\bullet}_{i_1 j_1} \cdots u^{\bullet}_{i_k j_k}. \tag{8.7}$$

Before proving the statement, note that Relations (8.6) and (8.7) correspond to the two-coloured partitions $\overset{\circ}{\underset{\bullet}{|}} \otimes \overset{\bullet}{\underset{\circ}{|}}$ and $\overset{\circ}{\underset{\bullet}{|}}^{\otimes k}$, respectively. Hence, those are exactly the same relations that were used to construct the quantum groups $H \mathbin{\tilde\times} \hat{\mathbb{Z}}$ and $H \mathbin{\tilde\times} \hat{\mathbb{Z}}_{2k}$ in Theorem 6.2.4.

Proof. Relation (8.6) is the relation of global colourization (see Def. 8.2.9) and it is obviously satisfied in $H \mathbin{\tilde\times} \hat{\mathbb{Z}}$. We just need to show that imposing this relation is enough. By Corollary 3.4.13 and Theorem 8.2.13, the ideal $I_{H \tilde\times \hat{\mathbb{Z}}}$ is generated by relations of the form $u^{\otimes w}\xi = \xi$, where $\xi \in \mathsf{FundRep}_{H \tilde\times \hat{\mathbb{Z}}}(\emptyset, w) = \mathsf{FundRep}_{H}(\emptyset, 2l)$, $c(w) = 0$, $l := |w|/2$. In $H \mathbin{\tilde *} \hat{\mathbb{Z}}$, we have a relation of the form $u^{\otimes(\circ\bullet)^l}\xi = \xi$. The former relation can surely be derived from the latter one and Relation (8.6) since it is obtained just by permuting the white and black circles.

The second statement is proven in a similar way. If we denote $u_{ij} = v_{ij} z$, then Relations (8.7) say $v_{i_1 j_1} \cdots v_{i_k j_k} z^k = v_{i_1 j_1} \cdots v_{i_k j_k} z^{-k}$, which is surely satisfied in $H \mathbin{\tilde\times} \hat{\mathbb{Z}}_{2k}$. For the converse, the ideal $I_{H \tilde\times \hat{\mathbb{Z}}_k}$ is generated by relations of the form $u^{\otimes w}\xi = \xi$, where $\xi \in \mathsf{FundRep}_{H \tilde\times \hat{\mathbb{Z}}}(\emptyset, w) = \mathsf{FundRep}_{H}(0, 2l)$, where $c(w)$ is a multiple of $2k$ and $l := |w|/2$. Again, this relation can be derived from $u^{\otimes(\circ\bullet)^l}\xi = \xi$ using Rel. (8.6) to permute colours and Rel. (8.7) to swap colours of k consecutive white points to black or vice versa. □

Finally, since Theorem 8.2.24 is rather new – it is not just a straightforward generalization of what we know from partition categories – let us formulate it now also in the language of partitions. The following proposition generalizes Proposition 6.4.15. We formulate it now in terms of linear combinations of partitions. The formulation for ordinary partitions was mentioned already as Proposition 6.2.8.

8.2.31 Proposition. Let $\mathscr{K} \subseteq \mathsf{Part}_N$ be a linear category of partitions and denote by $H \subseteq O^{+}_N$ the corresponding quantum group. Then $H \mathbin{\tilde *} \hat{\mathbb{Z}} \subseteq U^{+}_N$ corresponds to the category $\mathrm{Alt}\,\mathscr{K}$. Moreover, the following holds.

(1) If $0 \neq p \in \mathscr{K}(0, l)$ for some l odd, then $H \mathbin{\tilde *} \hat{\mathbb{Z}}_k$ corresponds to the category $\langle \mathrm{Alt}\,\mathscr{K}, \tilde{p}^{\otimes k} \rangle$, where $\tilde{p}$ is the partition p with colour pattern $\circ\bullet\circ\bullet\cdots\circ$.

(2) If $\mathscr{K}(0, l) = \{0\}$ for all l odd, then $H \mathbin{\tilde *} \hat{\mathbb{Z}}_k = H \mathbin{\tilde *} \hat{\mathbb{Z}}$ for all $k \in \mathbb{N}$.

Proof. The base statement that $H \mathbin{\tilde *} \hat{\mathbb{Z}}$ corresponds to $\mathrm{Alt}\,\mathscr{K}$ follows directly from Theorem 8.2.24. By Proposition 8.2.8, the distinction of the cases correspond to the situation that either (1) H has degree of reflection one or (2) H has degree of reflection two. So, item (2) is also contained in Theorem 8.2.24.

For item (1), denote by $w := \circ\bullet\cdots\circ$ the word of length l with alternating colours. Since H has degree of reflection one, we have $H \mathbin{\tilde *} \hat{\mathbb{Z}}_m \simeq H \mathbin{\hat *} \hat{\mathbb{Z}}_m$ for every $m \in \mathbb{N}_0$ by Proposition 8.2.26. We can actually prove this directly repeating the proof of Prop. 8.2.26: Denote by v the fundamental representation of H and by z the fundamental representation of $\hat{\mathbb{Z}}_m$. Then since we have $v^{\otimes l}\xi_p = \xi_p$, we must have

$$C(H \mathbin{\tilde\times} \hat{\mathbb{Z}}_m) \ni \xi_p^* u^{\otimes w} \xi_p = \xi_p^* (v^{\otimes l} z) \xi_p = z \left\| \xi_p \right\|.$$

Now, $H \mathbin{\tilde *} \hat{\mathbb{Z}}_k$ is just a quantum subgroup of $H \mathbin{\tilde *} \hat{\mathbb{Z}}$ with respect to the relation $z^k = 1$. Note that

$$u^{\otimes w^k} \xi_p^{\otimes k} = (v^{\otimes kl} z^k) \xi_p^{\otimes k} = \xi_p^{\otimes k} z^k.$$

So, the relation $z^k = 1$ can be written as $u^{\otimes w^k} \xi_p^{\otimes k} = \xi_p^{\otimes k}$, which is exactly the relation corresponding to $\tilde{p}$. □

8.3 Interpolating products

In this section, we define quantum group product constructions interpolating the free product $G \hat{*} H$ and the tensor product $G \times H$, for any given pair of quantum groups G and H. It is a generalization of the construction of the $\mathbb{Z}_2$-extensions introduced in Section 6.4.5.

8.3.1 Definition. Let G and H be compact matrix quantum groups and denote by u and v their respective fundamental representations. We define the following quantum subgroups of $G \hat{*} H$. The product $G \ltimes H$ is defined by taking the quotient of $C(G \hat{*} H)$ by the relations

$$ab^*x = xab^*, \qquad a^*bx = xa^*b. \tag{8.8}$$

The product $G \rtimes H$ is defined by the relations

$$ax^*y = x^*ya, \qquad axy^* = xy^*a. \tag{8.9}$$

The product $G \times_0 H$ by the combination of all Relations (8.8) and (8.9). Finally, given $k \in \mathbb{N}$, the product $G \times_{2k} H$ is defined by the relations

$$a_1x_1 \cdots a_kx_k = x_1a_1 \cdots x_ka_k, \tag{8.10}$$

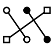
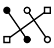
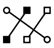
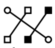
where $a, b, a_1, \ldots, a_k \in \{u_{ij}\}$ and $x, y, x_1, \ldots, x_k \in \{v_{ij}\}$. (Equivalently, we can assume $a, b, a_1, \ldots, a_k \in \operatorname{span}\{u_{ij}\}$ and $x, y, x_1, \ldots, x_k \in \operatorname{span}\{v_{ij}\}$.)

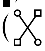
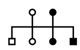
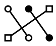
8.3.2 Theorem. Consider quantum groups G, H. Then the products from Definition 8.3.1 are indeed well-defined quantum groups. We have the following inclusions

$$G \hat{*} H \begin{matrix} \supseteq & G \ltimes H & \supseteq \\ \supseteq & G \rtimes H & \supseteq \end{matrix} G \times_0 H \supseteq G \times_{2k} G \supseteq G \times_{2l} H \supseteq G \times_2 H = G \times H,$$

where we assume $k, l \in \mathbb{N}$ such that l divides k. The last three inclusions are strict if and only if the degree of reflection of both G and H is different from one (assuming, of course, $k > l > 1$).

Proof. It is a direct verification that in all cases the relations generate a Hopf $*$-ideal, so they provide a good definition of quantum subgroups. Alternatively, it follows from the fact that all the relations come from some intertwiners as described below.

Denote by u the fundamental representation of G and by v the fundamental representation of H. Without loss of generality, we can assume that both u and v are unitary representations since any representation of a quantum group is similar to a unitary one. Let us use the white circle $\circ$ as the symbol for the representation u and the black circle $\bullet$ for the unitarization $u^\bullet = F_1\bar{u}F_1^{-1}$. We use the white square $\square$ for v and the black square $\blacksquare$ for the unitary $v^\blacksquare = F_2\bar{v}F_2^{-1}$. Then the relations are actually partition relations corresponding respectively to the partitions , , , , and ()$^{\otimes k}$.

We can use the partition calculus to show that Relations (8.10) imply both (8.8) and (8.9) for any $k \in \mathbb{N}$. Indeed, rotating ()$^{\otimes k}$, we get ()$^{\otimes k}$. Then, using compositions with the pair partitions, one can contract the tensor product ()$^{\otimes k} \otimes$ ()$^{\otimes k}$ to the partition , which can then be rotated to . The other partitions can be obtained similarly. All the remaining inclusions are clear. Note that the partition calculus is just a shorthand for manipulating with the corresponding intertwiners and hence with the corresponding relations. This works also if the quantum group is not fully described by a category of partitions, in which case we may be able to handle just a part of its representation theory by partitions. All of the arguments here can be translated into direct manipulations with the relations themselves (using the unitarity relations as well).

It remains to prove the statement about strictness. Denote by m the degree of reflection of G and by n the degree of reflection of H (recall Definition 8.2.4). First, suppose that m and n

are both different from one. Then it is sufficient to prove the strictness for the corresponding subgroups $\hat{\mathbb{Z}}_m$ and $\hat{\mathbb{Z}}_n$. So, we need to prove the strictness of the following inclusions

$$\hat{\mathbb{Z}}_m \times_0 \hat{\mathbb{Z}}_n \supseteq \hat{\mathbb{Z}}_m \times_{2k} \hat{\mathbb{Z}}_n \supseteq \hat{\mathbb{Z}}_m \times_{2l} \hat{\mathbb{Z}}_n,$$

Directly from the definition, we have $\hat{\mathbb{Z}}_m \times_0 \hat{\mathbb{Z}}_n = \hat{\mathbb{Z}}_m \hat{*} \hat{\mathbb{Z}}_n = \widehat{\mathbb{Z}_m * \mathbb{Z}_n}$. Indeed, the matrices u and v in this case have only one entry, say a and x. The Relations (8.8) then become trivial:

$$aa^*x = x = xaa^*, \qquad a^*ax = x = xa^*a$$

and likewise the relations (8.9).

For $m = n = 2$, we have that $\hat{\mathbb{Z}}_2 \times_{2k} \hat{\mathbb{Z}}_2$ is the dual of the dihedral group of order $4k$, so we indeed have the strictness here. For general m and n, let us just briefly sketch the proof. From the definition, we have that $\hat{\mathbb{Z}}_m \times_{2k} \hat{\mathbb{Z}}_n$ is the dual of the finitely presented group $\langle a, b \mid a^n = 1 = b^m,\ (ab)^k = (ba)^k \rangle$. We need to prove that $(ab)^l \neq (ba)^l$. To do so, let us further divide the relation $(ab)^k = 1$. We obtain the so-called von Dyck group $D(m,n,k)$, which has an action on a (possibly non-Euclidean) plane. From this action, we can see that $(ab)^l$ and $(ba)^l$ are indeed different for $l < k$ (unless $m = n = 2$).

Now, assuming $m = 1$, we are going to show that $G \ltimes\!\!\rtimes H = G \times H$. Consider a $\mathbb{Z}$-grading on the polynomials $\mathbb{C}\langle x_{ij}, x_{ij}^* \rangle$ assigning the degree one to the variables x_{ij} and degree minus one to the variables x_{ij}^*. Then Relations (8.8) are equivalent to $f(u_{ij}, u_{ij}^*)x = xf(u_{ij}, u_{ij}^*)$ for any $x \in \{v_{ij}\}$ and f a homogeneous polynomial of degree zero. From Proposition 8.2.6, we have that there exists a non-zero intertwiner $T \in \mathrm{Mor}(u^{\otimes w}, 1)$ with $c(w) = 1$. This means that there is a polynomial g of degree minus one such that $g(u_{ij}, u_{ij}^*) = 1$. Taking any $a \in \{u_{ij}\}$ and $x \in \{v_{ij}\}$, we have that $ag(u_{ij}, u_{ij}^*)$ is a polynomial in u_{ij} of degree zero. Hence, we have

$$ax = a\,g(u_{ij}, u_{ij}^*)\,x = xa\,g(u_{ij}, u_{ij}^*) = xa. \qquad \square$$

We could continue inventing other relations coupling somehow the factors $C(G)$ and $C(H)$ using partitions. We believe however, that the above mentioned definition is the most natural. Nevertheless, as an example of a different possibility, let us define the following.

8.3.3 Definition. Let G and H be quantum groups. Suppose G has a one-dimensional representation s and H has a one-dimensional representation r. Then we define $G\,{}^{s}\hat{*}^{r}_{k}\,H$ to be a quantum subgroup of $G \hat{*} H$ given by the relation $(sr)^k = 1$.

It is easy to check that this relation indeed defines a quantum subgroup. One way to see that this subgroup should not coincide with the tensor product (at least if G and H are "non-trivial enough") is to notice that the relation is non-crossing in the following sense. Consider $T_1 \in \mathrm{Mor}(u^{\otimes k_1}, s)$ and $T_2 \in \mathrm{Mor}(u^{\otimes k_2}, r)$. Then imposing the relation means adding the intertwiner $(T_1 \otimes T_2)^{\otimes k}$ to $\mathrm{Mor}((u \oplus v)^{\otimes 2k}, 1)$, which is a tensor product of intertwiners acting non-trivially either just on u or just on v (compare with the definition of non-crossing partitions). In particular, if $G \subseteq B_{N_1}^{\#+}$, so we can consider $s := \sum_k u_{ik}$, and $H \subseteq B_{N_2}^{\#+}$, so we can consider $r := \sum_k v_{ik}$, then the relation $(sr)^k = 1$ corresponds to $(\uparrow \otimes \uparrow)^{\otimes k}$.

This particular construction and many other relations that couple some one-dimensional subrepresentations of the factors G and H were already described in [Fre19, Section 5].

8.4 Ungluing

The purpose of this section is to reverse the gluing procedure from Definitions 6.4.10, 8.2.1. The motivating result is the one-to-one correspondence formulated in terms of partition categories in Theorem 4.6.8. The functor F providing this correspondence translates to the quantum group language exactly in terms of gluing. In this section, we aim to generalize the result outside easy quantum groups.

Recall from Def. 8.2.1 that given a quantum group $G \subseteq U^+(F) \mathbin{\hat{*}} \hat{\mathbb{Z}}_k$ with fundamental representation $u = v \oplus z$, we define its glued version to be the quantum group $\tilde{G} \subseteq U^+(F)$ with fundamentar representation $\tilde{u} := vz$ and underlying C*-algebra $C(\tilde{G}) \subseteq C(G)$ generated by the elements $v_{ij}z$.

8.4.1 Definition. Consider $\tilde{G} \subseteq U^+(F)$, $k \in \mathbb{N}_0$. Then any $G \subseteq U^+(F) \mathbin{\hat{*}} \hat{\mathbb{Z}}_k$ such that $\tilde{G}$ is a glued version of G is called a $\mathbb{Z}_k$**-ungluing** of $\tilde{G}$.

In Section 8.4.1, we are going to study the ungluings in general and show that they always exist. Unsurprisingly, an ungluing of a quantum group is not defined uniquely. The ungluings introduced in Section 8.4.1 are universal, but not particularly interesting. In Section 8.4.2, we are going to study more interesting ungluings of the form $G \subseteq O^+(F) \mathbin{\hat{*}} \hat{\mathbb{Z}}_2$, which allow us to generalize the one-to-one correspondence from Theorem 4.6.8. We formulate the result as Theorem 8.4.13, which constitutes the main result of this section.

8.4.1 General ungluings

8.4.2 Proposition. There exists a $\mathbb{Z}_k$-ungluing for every quantum group $\tilde{G}$ and for every $k \in \mathbb{N}_0$. Namely, we have the *trivial* $\mathbb{Z}_k$-ungluing $\tilde{G} \times E$, where $E \subseteq \hat{\mathbb{Z}}_k$ is the trivial group. Moreover, $\tilde{G} \times \hat{\mathbb{Z}}_k$ is an ungluing of $\tilde{G}$ whenever k divides the degree of reflection of G.

Proof. The first statement is obvious as we have $\tilde{G} \mathbin{\tilde{\times}} E = \tilde{G}$. The second follows from Lemma 8.2.5 as we have $\tilde{G} \mathbin{\tilde{\times}} \hat{\mathbb{Z}}_k = \tilde{G}$. □

Let us denote by $\iota \colon \mathbb{C}\langle \tilde{x}_{ij}, \tilde{x}_{ij}^* \rangle \to \mathbb{C}\langle x_{ij}, x_{ij}^*, z, z^* \rangle$ the embedding $\tilde{x}_{ij} \mapsto x_{ij}z$. Consider $\tilde{G} \subseteq U^+(F)$ and G its ungluing. The fact that $\tilde{G}$ is a glued version of G is, according to Proposition 8.2.3, characterized by the equality $\tilde{I}_{\tilde{G}} := \iota(I_{\tilde{G}}) = I_G \cap \iota(\mathbb{C}\langle \tilde{x}_{ij}, \tilde{x}_{ij}^* \rangle)$. Consequently, we have $I_G \supseteq \tilde{I}_{\tilde{G}}$ for every ungluing G of a quantum group $\tilde{G}$.

8.4.3 Definition. Consider $\tilde{G} \subseteq U^+(F)$, $k \in \mathbb{N}_0$. Let $I_G \subseteq \mathbb{C}\langle x_{ij}, x_{ij}^*, z, z^* \rangle$ be the $*$-ideal generated by $\tilde{I}_{\tilde{G}}$. Put $C(G) := C^*(\mathbb{C}\langle x_{ij}, x_{ij}^*, z, z^* \rangle / I_G)$. Then $G := (C(G), v \oplus z)$ is called the **maximal** $\mathbb{Z}_k$**-ungluing** of $\tilde{G}$.

8.4.4 Proposition. The maximal $\mathbb{Z}_k$-ungluing always exists. That is, considering the notation of Definition 8.4.3, G is indeed a compact matrix quantum group and $\tilde{G}$ is indeed its glued version.

Proof. First of all, note that $\tilde{I}_{\tilde{G}}$ contains the relations $vv^* = v^*v = 1_N$ and $v^\bullet v^{\bullet *} = v^{\bullet *} v^\bullet = 1_N$, where $v^\bullet = F\bar{v}F^{-1}$. So, if G is well defined, then we must have $G \subseteq U^+(F) \mathbin{\hat{*}} \hat{\mathbb{Z}}_k$.

To prove that G is well defined, we need to check that $I_G / I_{U^+(F)}$ is a Hopf $*$-ideal. Since $I_{\tilde{G}} / I_{U^+(F)}$ is a Hopf $*$-ideal, we have that $\tilde{I}_{\tilde{G}} / I_{U^+(F)}$ is a coideal invariant under the antipode. Consequently, the ideal generated by $\tilde{I}_{\tilde{G}} / I_{U^+(F)}$ must be a Hopf $*$-ideal. (See Sect. 2.3.2.)

From the construction, it is clear that, if $\tilde{G}$ has some $\mathbb{Z}_k$-ungluing, then G must be the maximal one (since we take the smallest possible ideal I_G). But every quantum group has the trivial ungluing as mentioned in Prop. 8.4.2. □

8.4.5 Remark. We do not have to know explicitly the whole ideal $I_{\tilde{G}}$ to compute the maximal ungluing. Consider $\tilde{G} \subseteq U^+(F)$ and suppose that it is determined by a set of relations $\tilde{R}$. That is, $I_{\tilde{G}}$ is generated by the coideal $\tilde{R}$. Then the maximal $\mathbb{Z}_k$-ungluing G of $\tilde{G}$ is defined by the relations $R := \iota(\tilde{R})$. That is, taking the generating relations for $\tilde{G}$ and exchanging $\tilde{v}_{ij}$ for $v_{ij}z$ and $\tilde{v}_{ij}^*$ for $z^* v_{ij}^*$.

Alternatively, we can describe the maximal ungluing by its representation category. Recall the definition of the gluing homomorphism $\mathscr{O}_{\circ\bullet}^* \to \mathscr{W}_k$ mapping $\circ \mapsto \square\triangle$, $\bullet \mapsto \blacktriangle\blacksquare$. Given a word $w \in \mathscr{O}_{\circ\bullet}^*$, the image $\tilde{w}$ under this homomorphism is called the *glued version* of w by

Definition 8.2.2. If G is a quantum group and $\tilde{G}$ its glued version, then $\mathsf{FundRep}_{\tilde{G}}$ is a full subcategory of $\mathsf{FundRep}_G^{k\text{-ext}}$ according to Proposition 8.2.3. The full embeding is given exactly by the above mentioned gluing homomorphism. Consequently, the maximal ungluing G of some $\tilde{G}$ should be a quantum group with the minimal representation category containing $\mathsf{FundRep}_{\tilde{G}}$ as a full subcategory.

8.4.6 Proposition. Consider $\tilde{G} \subseteq U^+(F)$ and $G \subseteq U^+(F) \hat{*} \hat{\mathbb{Z}}_k$ its maximal $\mathbb{Z}_k$-ungluing. Then the $\mathbb{Z}_k$-extended representation category $\mathsf{FundRep}_G^{k\text{-ext}}$ is generated by the sets $C(\tilde{w}_1, \tilde{w}_2) = \mathsf{FundRep}_{\tilde{G}}(w_1, w_2)$, where $\tilde{w}_1, \tilde{w}_2$ are glued versions of $w_1, w_2 \in \mathscr{O}^*_{\circ\bullet}$. In addition, if $\mathsf{FundRep}_G$ is generated by some $\tilde{C}_0$, then $\mathsf{FundRep}_G^{k\text{-ext}}$ is generated by $C_0(\tilde{w}_1, \tilde{w}_2) = \tilde{C}_0(w_1, w_2)$.

Proof. By Tannaka–Krein duality, (Thm. 3.4.6), $I_{\tilde{G}}$ is linearly spanned by relations of the form $[T\tilde{v}^{\otimes w_1} - v^{\otimes w_2} T]_{ji}$. By definition of the maximal ungluing, the ideal I_G is generated by elements of $\tilde{I}_{\tilde{G}}$, which are exactly the relations $[Tu^{\otimes \tilde{w}_1} - u^{\otimes \tilde{w}_2} T]_{ji}$ corresponding to the set C.

For the second stament, notice that if $\tilde{C}_0$ generates $\mathsf{FundRep}_{\tilde{G}}$, then C_0 must generate C. This follows simply from the fact that gluing of words is a monoid homomorphism (see also Prop. 8.2.3). Consequently, by what was already proven, C_0 generates $\mathsf{FundRep}_G$. □

8.4.7 Example. As an example, consider the quantum group $\tilde{G}_k := O_N^+ \tilde{\times} \hat{\mathbb{Z}}_{2k}$, $k \in \mathbb{N}$. By Theorem 6.2.4, alternatively by Proposition 8.2.30, it corresponds to the two-coloured category of partitions $\langle \overset{\circ}{\underset{\bullet}{|}}{}^{\otimes k} \rangle$, so it can be also defined as a quantum subgroup of U_N^+ given by the relations

$$\tilde{v}_{i_1 j_1} \tilde{v}_{i_2 j_2} \cdots \tilde{v}_{i_k j_k} = \tilde{v}^*_{i_1 j_1} \tilde{v}^*_{i_2 j_2} \cdots \tilde{v}^*_{i_k j_k},$$

where $\tilde{v}$ denotes the fundamental representation of $\tilde{G}$. We can also take $\tilde{G}_0 := O_N^+ \tilde{\times} \hat{\mathbb{Z}}$, which corresponds to the category $\langle \overset{\circ}{\underset{\bullet}{|}} \otimes \overset{\bullet}{\underset{\circ}{|}} \rangle$ and hence is a quantum subgroup of U_N^+ defined by the relation

$$\tilde{v}_{ij} \tilde{v}^*_{kl} = \tilde{v}^*_{ij} \tilde{v}_{kl}.$$

Now, take arbitrary $l \in \mathbb{N}_0$. The maximal $\mathbb{Z}_l$-ungluing of $\tilde{G}_k$ is a quantum group $G_k \subseteq U_N^+ \hat{*} \hat{\mathbb{Z}}_l$ with fundamental representation of the form $v \oplus z$ given by the same relations if we substitute $\tilde{v}_{ij}$ by $v_{ij} z$, that is,

$$v_{i_1 j_1} z v_{i_2 j_2} z \cdots v_{i_k j_k} z = z^* v^*_{i_1 j_1} z^* v^*_{i_2 j_2} \cdots z^* v^*_{i_k j_k}.$$

The ungluing G_0 is defined by the relation

$$v_{ij} v^*_{kl} = z^* v^*_{ij} v_{kl} z.$$

Diagrammatically, we can just put a white triangle $\vartriangle$ after every white circle $\circ$ and a black triangle $\blacktriangle$ in front of every black circle $\bullet$. So, the first relation corresponds to the partition $(\overset{\circ\ \triangledown}{\underset{\blacktriangle\ \bullet}{\diagdown}})^{\otimes k}$ and the second one to $\blacktriangle \overset{\circ}{\underset{\bullet}{|}} \overset{\bullet}{\underset{\circ}{|}} \vartriangle$.

We may have hoped to split the original quantum group $\tilde{G}_k$ into smaller pieces – namely to obtain $O_N^+ \times \hat{\mathbb{Z}}_{2k}$ as the ungluing. However, this is not what happens here. The ungluing G_k seems to be rather more complicated than the original $\tilde{G}_k$.

This will actually happen always. The maximal ungluing never provides any useful decomposition into smaller pieces since we always have the trivial decomposition inside the maximal one $\tilde{G} \times E \subseteq G$. It makes much more sense to look for "small" ungluings rather than for the maximal one. In the following section, we are going to study $\mathbb{Z}_2$-ungluings of the form $G \subseteq O^+(F) \hat{*} \hat{\mathbb{Z}}_2$, which surprisingly exist for a large class of quantum groups $\tilde{G} \subseteq U^+(F)$.

8.4.2 Orthogonal ungluings

As just mentioned, constructing large unitary ungluings $G \subseteq U^+(F) \hat{*} \hat{\mathbb{Z}}_k$ may not be very useful. In this subsection, we study ungluings that are orthogonal, that is, of the form $G \subseteq O^+(F) \hat{*} \hat{\mathbb{Z}}_2$. For the rest of this subsection, we assume $F\bar{F} = c\,1_N$.

For a given quantum group $G \subseteq O^+(F) \hat{*} \hat{\mathbb{Z}}_2$, we will denote by I_G the corresponding ideal inside $A := \mathbb{C}\langle x_{ij}\rangle * \mathbb{C}\mathbb{Z}_2$ (instead of taking $\mathbb{C}\langle x_{ij}, r\rangle$ or $\mathbb{C}\langle x_{ij}, x_{ij}^*, r, r^*\rangle$). The generator of $\mathbb{C}\mathbb{Z}_2$ will be denoted by r. Note that we have to consider the non-standard involution $x_{ij}^* = [F^{-1}xF]_{ij}$ on A. The algebra A is $\mathbb{Z}_2$-graded (assigning all variables x_{ij} and r degree one). We will denote by $\tilde{A}$ the even part of A. Then $\tilde{A}r$ is the odd part of A.

8.4.8 Lemma. The mapping $x_{ij} \mapsto x_{ij}r$ extends to an injective $*$-homomorphism $\iota_A\colon \mathbb{C}\langle x_{ij}, x_{ij}^*\rangle \to A$. Its image $\iota_A(A)$ equals to $\tilde{A}$ – the even part of A.

Proof. The existence of the $*$-homomorphism ι_A follows from the fact that its domain is a free algebra. The injectivity can be proven the same way as in Theorem 6.4.13. Alternatively, we can also prove it directly together with the statement about the image.

Obviously for any monomial $f \in \mathbb{C}\langle x_{ij}, x_{ij}^*\rangle$, the image $\iota_A(f)$ has even degree. Conversely, we need to show that, for any monomial of even degree $\tilde{f} \in \tilde{A}$, there exists a unique monomial $f \in \mathbb{C}\langle x_{ij}, x_{ij}^*\rangle$ such that $\tilde{f} = \iota_A(f)$. This is done easily by induction on the "length" of $\tilde{f}$ measured by the number of variables x_{ij} or x_{ij}^* (ignoring the r's). Suppose $\tilde{f}$ is in the reduced form, that is, the variable r does not appear twice consecutively. If $\tilde{f}$ starts with the variable x_{ij}, we can write $\tilde{f}(x_{ij}, r) = x_{ij}r\tilde{f}_0(x_{ij}, r)$ for some $f_0 \in \tilde{A}$, so $\iota_A^{-1}(\tilde{f})(x_{ij}, x_{ij}^*) = x_{ij}\iota_A^{-1}(\tilde{f}_0)(x_{ij}, x_{ij}^*)$. If $\tilde{f}$ starts with r followed by x_{ij}, so $\tilde{f}(x_{ij}, r) = rx_{ij}\tilde{f}_0(x_{ij}, r)$ for some $f_0 \in \tilde{A}$, then $\iota_A^{-1}(\tilde{f})(x_{ij}, x_{ij}^*) = x_{ij}^*\iota_A^{-1}(\tilde{f}_0)(x_{ij}, x_{ij}^*)$. □

8.4.9 Remark. $\tilde{A}$ is the $*$-subalgebra of A generated by the elements $x_{ij}r$. Consequently, for any $G \subseteq O^+(F) \hat{*} \hat{\mathbb{Z}}_2$, we can express the coordinate algebra associated to its glued version $\tilde{G}$ as

$$\operatorname{Pol}\tilde{G} = \{f(v_{ij}, r) \mid f \in \tilde{A}\} \subseteq \operatorname{Pol} G.$$

In addition, we can rephrase Proposition 8.2.3 by saying that a quantum group $\tilde{G} \subseteq U^+(F)$ is a glued version of $G \subseteq O^+(F) \hat{*} \hat{\mathbb{Z}}_2$ if and only if we have $\tilde{I}_{\tilde{G}} := \iota_A(I_{\tilde{G}}) = I_G \cap \tilde{A}$.

Recall the definition of quantum groups $\tilde{G} \subseteq U^+(F)$ invariant with respect to the colour inversion from Def. 8.2.27.

8.4.10 Proposition. Consider a compact matrix quantum group $\tilde{G} \subseteq U^+(F)$ with $F\bar{F} = c\,1_N$ invariant with respect to the colour inversion. Let G' be its maximal $\mathbb{Z}_2$-ungluing. Then $G := G' \cap (O^+(F) \hat{*} \hat{\mathbb{Z}}_2)$ is also a $\mathbb{Z}_2$-ungluing.

8.4.11 Definition. The quantum group G from Proposition 8.4.10 will be called the **canonical $\mathbb{Z}_2$-ungluing** of $\tilde{G}$.

Before proving the proposition, recall that $I_{G'} \subseteq \mathbb{C}\langle x_{ij}, x_{ij}^*, r, r^*\rangle$ is defined as the smallest ideal containing $\iota(I_{\tilde{G}})$. Consequently, $I_G = I_{G'}/(x^\circ = x^\bullet, r^2 = 1)$ is the smallest ideal of the algebra $A = \mathbb{C}\langle x_{ij}, x_{ij}^*, r, r^*\rangle/(x^\circ = x^\bullet, r^2 = 1)$ containing $\tilde{I}_{\tilde{G}} = \iota(I_{\tilde{G}})/(x^\circ = x^\bullet, r^2 = 1)$. In other words, the canonical $\mathbb{Z}_2$-ungluing is determined by relations of the form $f(x_{ij}r, rx_{ij})$ with $f \in I_{\tilde{G}} \subseteq \mathbb{C}\langle x_{ij}, x_{ij}^*\rangle$.

Proof. Adopting the notation introduced above, we need to prove that $I_G \cap \tilde{A} = \tilde{I}_{\tilde{G}}$. We prove that

$$I_G = \tilde{I}_{\tilde{G}} + \tilde{I}_{\tilde{G}}r = \operatorname{span}\{f, fr \mid f \in \tilde{I}_{\tilde{G}}\}.$$

Then it will be clear that $I_G \cap \tilde{A} = (\tilde{I}_{\tilde{G}} + \tilde{I}_{\tilde{G}}r) \cap \tilde{A} = \tilde{I}_{\tilde{G}}$ since $\tilde{I}_{\tilde{G}} \subseteq \tilde{A}$, so $\tilde{I}_{\tilde{G}}r \cap \tilde{A} = \emptyset$.

To prove the equality, it is enough to show that the right-hand side is an ideal since then it must be the smallest one containing $\tilde{I}_{\tilde{G}}$, which is exactly I_G. So, denote the right-hand side by I. By Proposition 8.2.28, G being invariant with respect to the colour inversion means that I_G is invariant with respect to interchanging $x_{ij}^\circ \leftrightarrow x_{ij}^\bullet$. Applying ι_A, we get that $\tilde{I}_{\tilde{G}}$ is invariant with respect to $x_{ij}r \leftrightarrow rx_{ij}$, so $\tilde{I}_{\tilde{G}}$ is invariant with respect to conjugation by r, that is, $x \mapsto rxr$.

We use that to prove that I is an ideal. The subspace I is obviously invariant under right multiplication by r. For the left multiplication, we can write $rx = (rxr)r$. For multiplication by x_{ij}, we can write $xx_{ij} = (xx_{ij}r)r$ and $x_{ij}x = r((x_{ij}r)^{\bullet}x)$. □

8.4.12 Proposition. Consider $G \subseteq O^+(F) \,\tilde{*}\, \hat{\mathbb{Z}}_2$ and denote by k its degree of reflection. Then exactly one of the following situations occurs.

(1) If $k = 1$, then $C(\tilde{G}) = C(G)$ and hence $\tilde{G} \simeq G$.
(2) If $k = 2$, then $C(G)$ is $\mathbb{Z}_2$-graded and $C(\tilde{G})$ is its even part.

Proof. Since G is orthogonal, its degree of reflection must be either one or two by Proposition 8.2.8. First, let us assume that $k = 1$. To show that $C(\tilde{G}) = C(G)$, it is enough to prove that $r \in C(\tilde{G})$. The assumption $k = 1$ means that there is a vector $\xi \in \mathrm{Mor}(1, u^{\otimes k})$, $\|\xi\| = 1$ for some k odd, so we have $\xi^* u^{\otimes k} \xi = 1$ in $C(G)$. Consequently, $r = \xi^* u^{\otimes k} \xi r$ holds in $C(G)$, and, since it is an even polynomial, it must be an element of $C(\tilde{G})$ (see Remark 8.4.9).

If the degree of reflection equals two, then by Proposition 8.2.6 the $\mathbb{Z}_2$-grading of A passes to $\mathrm{Pol}\, G$ and hence also $C(G)$. As mentioned in Remark 8.4.9, $\mathrm{Pol}\, \tilde{G}$ consists of even polynomials in the generators v_{ij} and r and hence is the even part of $\mathrm{Pol}\, G$. □

The following theorem provides a non-easy counterpart of Theorems 4.6.8, 6.4.13.

8.4.13 Theorem. There is a one-to-one correspondence between

(1) quantum groups $G \subseteq O^+(F) \,\tilde{*}\, \hat{\mathbb{Z}}_2$ with degree of reflection two and
(2) quantum groups $\tilde{G} \subseteq U^+(F)$ that are invariant with respect to the colour inversion.

This correspondence is provided by gluing and canonical $\mathbb{Z}_2$-ungluing.

Proof. Almost everything follows from Proposition 8.4.10. The only remaining thing to prove is that, given $G \subseteq O^+(F) \,\tilde{*}\, \hat{\mathbb{Z}}_2$ and $\tilde{G}$ its glued version, then $\tilde{G}$ is invariant with respect to the colour inversion and G is its canonical $\mathbb{Z}_2$-ungluing. The first assertion follows from the fact that I_G and hence also $\tilde{I} := I_G \cap \tilde{A} = \tilde{I}_{\tilde{G}}$ is invariant with respect to conjugation by r. For the second assertion, we need to prove that $I_G = \tilde{I} + \tilde{I}r$ (see the proof of Proposition 8.4.10). Since G has degree of reflection two, we have that I_G is $\mathbb{Z}_2$-graded and hence it decomposes into an even and odd part, which is precisely $\tilde{I}$ and $\tilde{I}r$. □

Recall now Theorem 6.4.13 saying that, given a compact quantum group $G \subseteq O_N^+ \,\tilde{*}\, \hat{\mathbb{Z}}_2$ corresponding to a category of partitions with extra singletons $\mathscr{C} \subseteq \mathscr{P}^{\Delta}_{\mathrm{even}}$, its glued version $\tilde{G} \subseteq U_N^+$ is described by the category $\tilde{\mathscr{C}} := F(\mathscr{C})$. Note that the assumption that $\mathscr{C}$ contains only partitions of even length is equivalent to assuming that G has degree of reflection two. Reversing this process, we can say that, given a unitary easy quantum group $\tilde{G} \subseteq U_N^+$ corresponding to some category $\tilde{\mathscr{C}}$, its canonical $\mathbb{Z}_2$-ungluing is an easy quantum group $G \subseteq O_N^+ \,\tilde{*}\, \hat{\mathbb{Z}}_2$ corresponding to the category $\mathscr{C} := F^{-1}(\mathscr{C})$. In this way, Theorem 8.4.13 generalizes the one-to-one correspondence from Theorem 4.6.8.

As a particular example, recall from Section 6.4.5 that we introduced the $\mathbb{Z}_2$-extensions $H \times_{2k} \hat{\mathbb{Z}}_2$ exactly to be the canonical $\mathbb{Z}_2$-ungluings of $H \,\tilde{\times}\, \hat{\mathbb{Z}}_{2k}$. In the following proposition, we give a direct proof for arbitrary $H \subseteq O^+(F)$.

8.4.14 Proposition. Consider $H \subseteq O^+(F)$ with degree of reflection two, $k \in \mathbb{N}_0$. Then the canonical $\mathbb{Z}_2$-ungluing of $H \,\tilde{\times}\, \hat{\mathbb{Z}}_{2k}$ is $H \times_{2k} \hat{\mathbb{Z}}_2$.

Proof. From Proposition 8.2.30, we can express $H \,\tilde{\times}\, \hat{\mathbb{Z}}_{2k}$ as a quantum subgroup of $H \,\tilde{*}\, \hat{\mathbb{Z}} = H \,\tilde{*}\, \hat{\mathbb{Z}}_2$ given by certain relations. Denoting by $\tilde{v}$ the fundamental representation of $H \,\tilde{\times}\, \hat{\mathbb{Z}}_{2k}$, we just have to "unglue" the relations substituting $\tilde{v}_{ij}$ by $v_{ij}r$. For $H \,\tilde{\times}\, \hat{\mathbb{Z}}$, we get

$$v_{ij}v_{kl} = rv_{ij}v_{ij}r,$$

which is obviously equivalent to $v_{ij}v_{kl}r = rv_{ij}v_{kl}$ – the defining relation for $H \mathbin{\tilde{\times}} \hat{\mathbb{Z}}_2 = H \times_0 \hat{\mathbb{Z}}_2$. For $H \mathbin{\tilde{\times}} \hat{\mathbb{Z}}_{2k}$, $k > 0$, we get

$$v_{i_1 j_1} z v_{i_2 j_2} z \cdots v_{i_k j_k} z = z v_{i_1 j_1} z v_{i_2 j_2} \cdots z v_{i_k j_k},$$

which is exactly the defining relation for $H \times_{2k} \hat{\mathbb{Z}}_2$. □

8.4.3 Irreducibles and coamenability for ungluings and $\mathbb{Z}_2$-extensions

In this section, we will study properties of the gluing procedure and canonical $\mathbb{Z}_2$-ungluing. In many cases, we can view the $\mathbb{Z}_2$-extensions $H \times_{2k} \hat{\mathbb{Z}}_2$ as a motivating example. It remains an open question, whether one can generalize the statements for arbitrary products $H_1 \times_{2k} H_2$.

For this section, we will assume that $G \subseteq O_N^+ \mathbin{\hat{*}} \hat{\mathbb{Z}}_2$ has degree of reflection two.

For quantum groups G and H, the property $\operatorname{Pol} H \subseteq \operatorname{Pol} G$ can be understood either as H being a quotient of G or the discrete dual $\hat{H}$ being a quantum subgroup of $\hat{G}$. In that case, we can study the homogeneous space $\hat{G}/\hat{H}$ by defining

$$l^\infty(\hat{G}/\hat{H}) := \{x \in l^\infty(\hat{G}) \mid x(ab) = x(b) \text{ for all } a \in \operatorname{Pol} H \text{ and } b \in \operatorname{Pol} G\},$$

where $l^\infty(\hat{G})$ is the space of all bounded functionals on $\operatorname{Pol} G$.

8.4.15 Proposition. Consider $G \subseteq O_N^+ \mathbin{\hat{*}} \mathbb{Z}_2$ with degree of reflection two and denote by $\tilde{G}$ its glued version. Then $\hat{G}/\hat{\tilde{G}}$ consists of two points. More precisely,

$$l^\infty(\hat{G}/\hat{\tilde{G}}) = \{x \in l^\infty(\hat{G}) \text{ constant on the } \mathbb{Z}_2\text{-homogeneous parts of } \operatorname{Pol} G\} \simeq \mathbb{C}^2.$$

Proof. Consider $x \in l^\infty(\hat{G})$. By Proposition 8.4.12, $\operatorname{Pol} G$ is $\mathbb{Z}_2$-graded. Putting $b := 1$ in the equality $x(ab) = x(b)$, we get that x is constant on the even part. Putting $b := r$, we get that x is constant on the odd part. □

Recall that a compact quantum group G is called *coamenable* if $C(G) = C_{\mathrm{r}}(G)$ (recall that we denote $C(G) = C_{\mathrm{u}}(G)$ in this chapter). This notion dualizes the *amenability* for discrete (quantum) groups.

8.4.16 Proposition. Consider $G \subseteq O_N^+ \mathbin{\hat{*}} \hat{\mathbb{Z}}_2$ and $\tilde{G} \subseteq U_N^+$ its glued version. Then G is coamenable if and only if $\tilde{G}$ is coamenable.

Proof. Let us denote the surjections $q\colon C(G) \to C_{\mathrm{r}}(G)$ and $\tilde{q}\colon C(\tilde{G}) \to C_{\mathrm{r}}(\tilde{G})$. Taking $a \in C(\tilde{G})$, we have

$$q(a) = \tilde{q}(a) \qquad \text{and} \qquad q(ar) = \tilde{q}(a)r.$$

Since $C(G) = C(\tilde{G}) + C(\tilde{G})r$ from Proposition 8.4.12, we obviously have that q is an isomorphism if and only if $\tilde{q}$ is. □

Coamenability is not preserved by dual free products (since amenability of discrete groups is not preserved by free products). On the other hand, it is preserved by the tensor product by Proposition 2.5.7. An interesting question is whether the new interpolating products preserve coamenability. We answer this question in the special case of the $\mathbb{Z}_2$-extensions.

First, let us formulate the following well-known lemma.

8.4.17 Lemma. If a compact quantum group G is coamenable, then any its quotient is coamenable.

Proof. If H is a quotient of G, so $C(H) \subseteq C(G)$, then the coamenability of H directly follows from Theorem 2.3.4. □

8.4.18 Proposition. Consider $H \subseteq O_N^+$, $k \in \mathbb{N}_0$. Then $H \times_{2k} \hat{\mathbb{Z}}_2$ is coamenable if and only if H is coamenable.

Proof. The left-right implication follows directly from Lemma 8.4.17. Now, let us prove the right-left implication. If the degree of reflection of H is one, then $H \times_{2k} \hat{\mathbb{Z}}_2 = H \times \mathbb{Z}_2$ by Theorem 8.3.2, so we can apply Proposition 2.5.7.

Now suppose H has degree of reflection two. If H is coamenable, then $H \times \hat{\mathbb{Z}}_{2k}$ is coamenable by Theorem 8.3.2, so $H \tilde{\times} \hat{\mathbb{Z}}_{2k}$ is coamenable by Lemma 8.4.17 and $H \times_{2k} \hat{\mathbb{Z}}_2$ is coamenable by Proposition 8.4.16. □

Now, we are going to look on the irreducible representations of the ungluings.

8.4.19 Proposition. Consider $G \subseteq O^+(F) \hat{*} \hat{\mathbb{Z}}_2$ with degree of reflection two and fundamental representation $v \oplus r$. Let $\tilde{G} \subseteq U_N^+$ be its glued version. Then the irreducibles of G are given by

$$\{u^\alpha, u^\alpha r \mid \alpha \in \operatorname{Irr} \tilde{G}\}.$$

Proof. First, we prove that all the matrices are indeed representations of G. Surely all u^α are representations. The r is also a representation. Hence $u^\alpha r = u^\alpha \otimes r$ must also be representations.

Secondly, we prove that the representations are mutually inequivalent. The representations u^α are mutually inequivalent by definition. From this, it follows that the representations $u^\alpha r$ are mutually inequivalent. Since G has degree of reflection two, we have that $\operatorname{Pol} G$ is graded with $\operatorname{Pol} \tilde{G}$ being its even part. So, the entries of u^α are even, whereas the entries of $u^\alpha r$ are odd, so they cannot be equivalent.

Finally, we need to prove that those are all the representations. This can be proven using the fact that entries of irreducible representations form a basis of the polynomial algebra (Prop. 2.3.1). If we prove that the entries of the representations span the whole $\operatorname{Pol} G$, we are sure that we have all the irreducibles. This is indeed true:

$$\operatorname{span}\{u_{ij}^\alpha, u_{ij}^\alpha r \mid \alpha \in \operatorname{Irr} \tilde{G}\} = \operatorname{Pol} \tilde{G} + \operatorname{Pol} \tilde{G} r = \operatorname{Pol} G.$$ □

8.4.20 Proposition. Consider $H \subseteq O^+(F)$ with degree of reflection two, $k \in \mathbb{N}_0$. Then the complete set of mutually inequivalent irreducible representations of $H \times_{2k} \hat{\mathbb{Z}}_2$ is given by

$$u^{\alpha,i,\eta} = u^\alpha s^i r^\eta, \qquad \alpha \in \operatorname{Irr} H,\ i \in \{0,\ldots,k-1\},\ \eta \in \{0,1\}, \tag{8.11}$$

where

$$s = \sum_l v_{il} r v_{il}^* r = \sum_k v_{kj}^* r v_{kj} r \quad \text{for any } i,j = 1,\ldots,N. \tag{8.12}$$

Here $v \oplus r$ denotes the fundamental representation of $H \times_{2k} \hat{\mathbb{Z}}_2$.

Proof. Denote $\tilde{v} := vr$ the fundamental representation of the glued version of $H \times_{2k} \hat{\mathbb{Z}}_2$, which can be identified with $H \tilde{\times} \hat{\mathbb{Z}}_{2k}$ by Proposition 8.4.14. Thus, we can also write $\tilde{v}_{ij} = v_{ij} z \in C(H \tilde{\times} \hat{\mathbb{Z}}_{2k}) \subseteq C(H \times \hat{\mathbb{Z}}_{2k})$. We can also express

$$s = \sum_l v_{il} r v_{il}^* r = [vrvr^*]_{ii} = [vr(F^{-1} vrF)^t]_{ii} = [\tilde{v}(F^{-1}\tilde{v}F)^t]_{ii} = [v(F^{-1}vF)^t z^2]_{ii} = [vv^*]_{ii} z^2 = z^2$$

and similarly for the second expression in Eq. (8.12). In particular, we have that $s = z^2$ is a representation of $H \tilde{\times} \hat{\mathbb{Z}}_{2k}$ and hence also of $H \times_{2k} \mathbb{Z}_2$.

According to Proposition 8.2.19, we know that irreducibles of $H \tilde{\times} \hat{\mathbb{Z}}_{2k}$ are of the form $u^{\alpha,i} := u^\alpha z^{2i+d_\alpha}$, $\alpha \in \operatorname{Irr} H$, $i = 0,\ldots,k-1$ and $d_\alpha \in \{0,1\}$ is the degree of α. According to Proposition 8.4.19, we have that the set of irreducibles of $H \times_{2k} \hat{\mathbb{Z}}_{2k}$ is $u^{\alpha,i} r^\eta$. Now the only point is to express these in terms of u^α, s, and r.

Suppose first that $d_\alpha = 0$. In this case, the situation is simple since we can use the above mentioned fact that $z^2 = s$ to derive

$$u^{\alpha,i} r^\eta = u^\alpha z^{2i} r^\eta = u^\alpha s^i r^\eta.$$

In the situation $d_\alpha = 1$, we need to prove that $u^\alpha z = u^\alpha r$. The left hand side is a representation of $H \tilde{\times} \hat{\mathbb{Z}}_{2k}$ – the glued version of $H \times \hat{\mathbb{Z}}_{2k}$ – and the right-hand side is a representation of the glued version of $H \times_{2k} \hat{\mathbb{Z}}_2$. As we already mentioned, these quantum groups coincide, so the equality makes sense. (Note that it is not possible to show that $z = r$. Not only that this is not true, the equality does not even make sense since s and r are not elements of a common algebra.) Since $u^\alpha z$ is a representation of $H \tilde{\times} \hat{\mathbb{Z}}_{2k}$, we have, in particular, that the entries of the representation $u^\alpha_{ab} z$ are elements of $\mathrm{Pol}(H \tilde{\times} \hat{\mathbb{Z}}_{2k})$. That is, there are polynomials $f^\alpha_{ab} \in \mathbb{C}\langle x^\circ_{ij}, x^\bullet_{ij}\rangle$ of degree one such that $f^\alpha_{ab}(\tilde{v}^\circ_{ij}, \tilde{v}^\bullet_{ij}) = f^\alpha_{ab}(v_{ij} z, z^* v_{ij}) = u^\alpha_{ab} z$. Since v_{ij} commute with z, we can arrange the "$\circ\bullet$-pattern" of the monomials in f^α_{ab} in an arbitrary way if we keep the property that they have degree one. In particular, we can say that every monomials of f^α_{ab} have an alternating colour pattern, that is, they are of the form $x^\circ_{i_1 j_1} x^\bullet_{i_2 j_2} \cdots x^\circ_{i_n j_n}$. Then we can express

$$u^\alpha_{ab} z = f^\alpha_{ab}(v_{ij} z, z^* v_{ij}) = f^\alpha_{ab}(\tilde{v}^\circ_{ij}, \tilde{v}^\bullet_{ij}) = f^\alpha_{ab}(v_{ij} r, r v_{ij}) = u^\alpha_{ab} r.$$

This is exactly what we wanted to prove. Now we have

$$u^{\alpha,i} r^\eta = u^\alpha z z^i r^\eta = u^\alpha r s^i r^\eta$$

To obtain the form in the statement, note just that $rsr = s^* = s^{k-1}$. □

Chapter 9

Conclusion

The original goal for the PhD project was to come up with the first examples of non-easy categories of partitions using some computer experiments. In the end, the thesis surpasses this goal significantly as the study of the linear categories of partitions forms just a part of the results.

In this concluding chapter, we would like to set our research into some more general framework and to show what are the possible directions for further research.

From quantum groups to partitions and back

The research presented in our thesis follows some kind of general philosophy (see also ⟦Web17, Sect. 7.5.5⟧)

9.1 Philosophy. How to produce results on quantum groups using partitions:

Input: Take (some result on) some (quantum) group.

(1) Describe it using partitions.
(2) Generalize it in the framework of partitions.
(3) Generalize it for a larger class of quantum groups without the need of using partitions.

Of course, one does not always have to perform all the three steps.

As an example, we might consider the concept of tensor complexification and global colourization.

Input: Non-crossing globally colourized categories of two-coloured partitions correspond to tensor complexification of free easy quantum groups. [TW17]

(1) The input is already formulated using partitions.
(2) We generalize it by dropping the non-crossing condition (Thm. 6.2.4)
(3) We generalize it further by dropping the easiness condition. On the other hand, we must assume a degree of reflection zero. (Thm. 8.2.11).

The last step of Philosophy 9.1 is probably the most interesting one for further applications. In this thesis, we devoted Chapter 8 to it. Let us mention here a few open problems and directions for further research that emerged in this chapter.

As we just mentioned, we were able to prove Theorem 8.2.11 only for degree of reflection zero. However, we have no counterexample for other degrees of reflection.

9.2 Open problem. Prove or disprove Theorem 8.2.11 for degree of reflection $k \neq 0$. (See also Remark 8.2.12.)

In Section 8.4.3, we formulated some properties of the $\mathbb{Z}_2$-extensions $H \times_{2k} \hat{\mathbb{Z}}_2$ using the fact that we can describe those as ungluings of $H \mathbin{\tilde{\times}} \hat{\mathbb{Z}}_{2k}$. Another direction for further research would be to generalize those results for the interpolating products $G \times_{2k} H$ considering arbitrary quantum groups G and H.

9.3 Further research direction. Study properties of the interpolating products $G \ltimes H$, $G \rtimes H$, $G \times_{2k} H$. What are the irreducible representations? Do the products preserve amenability? Can one give a formula for L^2-Betti numbers?

Linear combinations of partitions

The results of our thesis regarding linear categories of partitions may be seen as a contribution to achieve the following.

9.4 Ultimate goal. Classify all linear categories of partitions.

Solving this problem would solve also the classification of all quantum groups G with $S_N \subseteq G \subseteq O_N^+$.

One can ask how reasonable this classification problem is and whether it is possible to solve it anyhow. From our experience, working with linear combinations of partitions is much harder than working with plain partitions because one is losing the combinatorial nature. Nevertheless, the research in this direction can bring very interesting results even if we do not achieve the Ultimate goal. Let us review what our thesis brought in this direction and mention a few possible directions for further research.

In our thesis, we found several examples of non-easy quantum groups and then we essentially used them as an input to Philosophy 9.1.

9.5 Philosophy. Using Philosophy 9.1 to interpret non-easy categories of partitions.

(0) Find a non-easy linear category of partitions and interpret the corresponding quantum group. Use it as an input for Philosophy 9.1, that is, continue as follows:
(1) Find a combinatorial description of the representation category (generalizing the partition approach, e.g. using coloured points). That is, reformulate the description of the representation category trying to avoid linear combinations.
(2) Study this new combinatorial structure (e.g. obtain some classification).
(3) Generalize the result for arbitrary quantum groups.

In our case, the research ran as follows: (0) We used computer experiments to find examples of non-easy categories of partitions (Sects. 5.2, 5.3). Then we were trying to interpret the quantum groups and many of them appeared to be somehow connected with quantum groups with reducible fundamental representation having a one-dimensional subrepresentation (Sect. 7.1). (1) This motivated us to study such quantum groups in a more systematic way and to invent a combinatorial structure for this purpose – categories of partitions with extra singletons (Sects. 4.6, 6.4). (2) We obtained many classification results here. Some of them were interpreted as some new $\mathbb{Z}_2$-extensions of the form $H \times_k \hat{\mathbb{Z}}_2$. (3) Finally, we generalized those extensions as products of two arbitrary compact quantum groups (Section 8.3).

9.6 Further research direction. Continuing in search for non-easy categories

Since linear categories of partitions are far from being classified, continuing in the work we began is one of the possible directions for further research. Note however that we essentially reached the limit of our naive computer algorithm here. For continuing in the search using computer, one would have to introduce some optimization. For example, note that all the non-easy generators we discovered are rotationally symmetric. So, we may focus on such generators only.

9.7 Further research direction. Studying the categories we omitted in Section 5.3.

In Section 5.3, we omitted some non-easy categories because they were not defined for all $\delta \in \{5,6,7,\ldots\}$. Nevertheless, categories defined for $\delta = 4$ might actually also induce some non-trivial quantum groups and hence be interesting for us. Moreover, modifying the functor $p \mapsto T_p$, we might even be able to interpret categories defined for a larger class of the parameter δ (i.e. not necessarily natural numbers). Note for example that the category of all non-crossing pairings NCPair_δ (also known as the Temperley–Lieb category) can be interpreted as a quantum group representation category for all $\delta \in [2,\infty)$ (see ⟦NT13, Sect. 2.5⟧).

9.8 Further research direction. Testing concrete hypotheses.

The framework of linear categories of partitions may be convenient for solving some concrete hypotheses such as the following.

9.9 Open problem. Is there a quantum group G with $S_N \subsetneq G \subsetneq S_N^+$ for some N?

9.10 Open problem. Is there a quantum group G with $O_N^* \subsetneq G \subsetneq O_N^+$ for some N?

See [BBCC13, Ban18] for a more detailed discussion on such maximality results. For the first problem, we know that there is no such quantum group for $N \leq 5$ (for $N = 1, 2, 3$, it is trivial as $S_N = S_N^+$ here; for $N = 4$, it was proven in [BB09]; for $N = 5$, it was proven in [Ban18]). The article [BBCC13] proves that there is no quantum group between O_N and O_N^*.

Those questions can be reformulated in terms of linear combinations of partitions as follows.

9.11 Open problem. Is there a linear category of partitions $\mathscr{K}$ with $\mathsf{Part}_N \supsetneq \mathscr{K} \supsetneq \mathsf{NCPart}_N$ for some N?

9.12 Open problem. Is there a linear category of partitions $\mathscr{K}$ with $\langle \ast \rangle_N \supsetneq \mathscr{K} \supsetneq \langle \rangle_N$ for some N?

Other diagrammatic categories

We may try to generalize the partition categories even further not only by introducing some colourings. A very interesting concept appeared recently in [MR19] – so called *graph categories*. Graph categories are also categories equipped with some diagrammatic calculus that can be used to model representation categories of quantum groups. They are particularly useful to describe quantum groups connected to symmetries of graphs.

We can do similar stuff with graph categories as we did with partition categories – we can classify them or we can incorporate them into Philosophy 9.1 or 9.5.

Let us also mention the work [VV19] that introduces yet another diagrammatic calculus for describing representation categories of certain quantum groups (including an interesting combinatorial open problem, see [VV19, Question 7.1]).

Current project: the free Coxeter D_4

As a concrete application of many of the above mentioned ideas, let us mention a concrete project the author is working on [Gro20b].

The motivation is the following. There are three infinite series of Coxeter groups – series A formed by the symmetric groups, series B formed by the hyperoctahedral groups, and series D. While there is a free counterpart for the first two series, no free counterpart for Coxeter groups of type D was introduced yet.

In our project, we aim to define the free counterpart of the Coxeter group of type D in a special case $N = 4$. We do this first by introducing some new non-easy linear category of partitions. Then we look for an alternative description of this category in the spirit of Philosophy 9.5. Namely the graph categories appear to be a convenient framework for this purpose. The associated graph category turns out to be very interesting and we study it in more detail.

References

For an easier orientation in the bibliography, we divide the references into *primary sources*, i.e. original research articles, and *secondary sources* such as textbooks, monographs or surveys.

Secondary sources

[[Ban19]] Teodor Banica. Quantum groups, from a functional analysis perspective. *Advances in Operator Theory* **4**(1), 164–196 (2019). doi:10.15352/aot.1804-1342

[[Bla06]] Bruce Blackadar. *Operator Algebras: Theory of C*-Algebras and von Neumann Algebras*, Berlin: Springer-Verlag (2006). ISBN 978-3-540-28486-4

[[BO08]] Nathanial P. Brown and Narutaka Ozawa. *C*-Algebras and Finite Dimensional Approximations*, Providence: American Mathematical Society (2008). ISBN 978-0-8218-4381-9

[[Dyk16]] Ken Dykema. Free group factors. In: D. Voiculescu, N. Stammeier and M. Weber, eds., *Free Probability and Operator Algebras*, Zürich: European Mathematical Society (2016). ISBN 978-3-03719-165-1

[[EGNO15]] Pavel Etingof, Shlomo Gelaki, Dmitri Nikshych and Victor Ostrik. *Tensor Categories*, Providence: American Mathematical Society (2015). ISBN 978-1-4704-3441-0

[[FSS17]] Uwe Franz, Adam Skalski and Piotr M. Sołtan. Introduction to compact and discrete quantum groups (2017). arXiv:1703.10766

[[Fre19]] Amaury Freslon. Applications of non-crossing partitions to quantum groups (2019). `https://www.imo.universite-paris-saclay.fr/~freslon/Documents/Talks/Copenhaguen_2019.pdf`

[[KS97]] Anatoli Klimyk and Konrad Schmüdgen. *Quantum Groups and Their Representations*, Berlin: Springer-Verlag (1997). ISBN 978-3-642-64601-0

[[McL98]] Saunders Mac Lane. *Categories for the Working Mathematician*, New York: Springer (1998). ISBN 0-387-98403-8

[[MVD98]] Ann Maes and Alfons Van Daele. Notes on Compact Quantum Groups. *Nieuw Archief voor Wiskunde (4)* **16**(1–2), 73–112 (1998).

[[Mur90]] Gerard J. Murphy. *C*-algebras and Operator Theory*, London: Academic Press (1990). ISBN 0-12-511360-9

[[NT13]] Sergey Neshveyev and Lars Tuset. *Compact Quantum Groups and Their Representation Categories*, Paris: Société Mathématique de France (2013). ISBN 978-2-85629-777-3

[[Rud76]] Walter Rudin. *Principles of Mathematical Analysis*, New York: McGraw-Hill (1976). ISBN 0-07-054235-X

[[Tak79]] Masamichi Takesaki. *Theory of Operator Algebras I*, New York: Springer-Verlag (1979). ISBN 978-1-4612-6190-2

[[Tim08]] Thomas Timmermann. *An Invitation to Quantum Groups and Duality*, Zürich: European Mathematical Society (2008). ISBN 978-3-03719-043-2

[[Web16]] Moritz Weber. Easy quantum groups. In: D. Voiculescu, N. Stammeier and M. Weber, eds., *Free Probability and Operator Algebras*, Zürich: European Mathematical Society (2016). ISBN 978-3-03719-165-1

[[Web17]] Moritz Weber. Introduction to compact (matrix) quantum groups and Banica–Speicher (easy) quantum groups. *Proceedings – Mathematical Sciences* **127**(5), 881–933 (2017). doi:10.1007/s12044-017-0362-3

Primary sources

[Avi82] Daniel Avitzour. Free Products of C^*-Algebras. *Transactions of the American Mathematical Society* **271**(2), 423–435 (1982). jstor.org/stable/1998890

[BS93] Saad Baaj and Georges Skandalis. Unitaires multiplicatifs et dualité pour les produits croisés de C*-algébres. *Annales scientifiques de l'École Normale Supérieure* 4e série, **26**(4), 425–488 (1993). doi:10.24033/asens.1677

[Ban96] Teodor Banica. Théorie des représentations du groupe quantique compact libre $O(n)$. *Comptes rendus de l'Académie des sciences. Série 1, Mathématique* **322**(3), 241–244 (1996). ark:12148/bpt6k5766451w/f49

[Ban97] Teodor Banica. Le Groupe Quantique Compact Libre $U(n)$. *Communications in Mathematical Physics* **190**(1), 143–172 (1997). doi:10.1007/s002200050237

[Ban99a] Teodor Banica. Representations of compact quantum groups and subfactors. *Journal für die reine und angewandte Mathematik* **509**, 167–198 (1999). doi:10.1515/crll.1999.509.167

[Ban99b] Teodor Banica. Symmetries of a generic coaction. *Mathematische Annalen* **314**, 763–780 (1999). doi:10.1515/crll.1999.509.167

[Ban08] Teodor Banica. A Note on Free Quantum Groups. *Annales Mathématiques Blaise Pascal* **15**(2), 135–146 (2008). doi:10.5802/ambp.243

[Ban18] Teodor Banica. Homogeneous quantum groups and their easiness level (2018). arXiv:1806.06368

[BB09] Teodor Banica and Julien Bichon. Quantum groups acting on 4 points. *Journal für die reine und angewandte Mathematik* **626**, 75–114 (2009). doi:10.1515/CRELLE.2009.003

[BBC07] Teodor Banica, Julien Bichon and Benoît Collins. The hyperoctahedral quantum group. *Journal of the Ramanujan Mathematical Society* **22**, 345–384 (2007).

[BBCC13] Teodor Banica, Julien Bichon, Benoît Collins and Stephen Curran. A Maximality Result for Orthogonal Quantum Groups. *Communications in Algebra* **41**(2), 656–665 (2013). doi:10.1080/00927872.2011.633138

[BC10] Teodor Banica and Stephen Curran. Decomposition results for Gram matrix determinants. *Journal of Mathematical Physics* **51**(11), 113503 (2010). doi:10.1063/1.3511332

[BCS10] Teodor Banica, Stephen Curran and Roland Speicher. Classification results for easy quantum groups. *Pacific Journal of Mathematics* **247**(1), 1–26 (2010). doi:10.2140/pjm.2010.247.1

[BS09] Teodor Banica and Roland Speicher. Liberation of orthogonal Lie groups. *Advances in Mathematics* **222**(4), 1461–1501 (2009). doi:10.1016/j.aim.2009.06.009

[BY14] Julien Bichon and Robert Yuncken. Quantum subgroups of the compact quantum group $SU_{-1}(3)$. *Bulletin of the London Mathematical Society* **46**(2), 315-328 (2014). doi:10.1112/blms/bdt105

[BC18] Michael Brannan and Benoît Collins. Highly Entangled, Non-random Subspaces of Tensor Products from Quantum Groups. *Communications in Mathematical Physics* **358**, 1007–1025 (2018). doi:10.1007/s00220-017-3023-6

[BCV17] Michael Brannan, Benoît Collins and Roland Vergnioux. The Connes embedding property for quantum group von Neumann algebras. *Transactions of the American Mathematical Society* **369**, 3799–3819 (2017). doi:10.1090/tran/6752

[Bra37] Richard Brauer. On Algebras Which are Connected with the Semisimple Continuous Groups. *Annals of Mathematics* **38**(4), 857–872 (1937). doi:10.2307/1968843

[BMT01] Erik Bédos, Gerard J. Murphy and Lars Tuset. Co-amenability of compact quantum groups. *Journal of Geometry and Physics* **40**(2), 129–153 (2001). doi:10.1016/S0393-0440(01)00024-9

[Car56] Pierre Cartier. Dualité de Tannaka des groupes et des algèbres de Lie. *Comptes rendus hebdomadaires des séances de l'Académie des sciences* **242**(1), 322–325 (1956). ark:12148/bpt6k3194j/f322

[Chi15] Alexandru Chirvasitu. Residually finite quantum group algebras. *Journal of Functional Analysis* **268**(11), 3508–3533 (2015). doi:10.1016/j.jfa.2015.01.013

[CH17] Jonathan Comes and Thorsten Heidersdorf. Thick ideals in Deligne's category $\mathrm{Rep}(O_\delta)$. *Journal of Algebra* **480**, 237–265 (2017). doi:10.1016/j.jalgebra.2017.01.050

[CO11] Jonathan Comes and Victor Ostrik. On blocks of Deligne's category $\mathrm{Rep}(S_t)$. *Advances in Mathematics* **226**(2), 1331–1377 (2011). doi:10.1016/j.aim.2010.08.010

[Cra17] Jason Crann. On hereditary properties of quantum group amenability. *Proceedings of the American Mathematical Society* **145**, 627–635 (2017). doi:10.1090/proc/13365

[DGPS18] Wolfram Decker, Gert-Martin Greuel, Gerhard Pfister and Hans Schönemann. SINGULAR *4-1-1 — A computer algebra system for polynomial computations* (2018). http://www.singular.uni-kl.de

[Del07] Pierre Deligne. La catégorie des représentations du groupe symétrique S_t, lorsque t n'est pas un entier naturel. In: V. B. Mehta, eds., *Algebraic groups and homogeneous spaces*, (2007). ISBN 978-8173198021

[DK94] Mathijs S. Dijkhuizen and Tom H. Koornwinder. CQG algebras: A direct algebraic approach to compact quantum groups. *Letters in Mathematical Physics* **32**(4), 315–330 (1994). doi:10.1007/BF00761142

[Doi93] Yukio Doi. Braided bialgebras and quadratic blalgebras. *Communications in Algebra* **21**(5), 1731-1749 (1993). doi:10.1080/00927879308824649

[Dri88] Vladimir G. Drinfeld. Quantum groups. *Journal of Soviet Mathematics* **41**(2), 898–915 (1988). doi:10.1007/BF01247086

[ER94] Edward G. Effros and Zhong-Jin Ruan. Discrete quantum groups I: The Haar measure. *International Journal of Mathematics* **05**(05), 681–723 (1994). doi:10.1142/S0129167X94000358

[Fre17] Amaury Freslon. On the partition approach to Schur–Weyl duality and free quantum groups. *Transformation Groups* **22**(3), 707–751 (2017). doi:10.1007/s00031-016-9410-9

[Fre19] Amaury Freslon. On two-coloured noncrossing partition quantum groups. *Transactions of the American Mathematical Society* **372**, 4471–4508 (2019). doi:10.1090/tran/7846

[GLR85] Paul Ghez, Ricardo Lima and John E. Roberts. W*-categories. *Pacific Journal of Mathematics* **120**(1), 79–109 (1985). projecteuclid.org/euclid.pjm/1102703884

[Gro18] Daniel Gromada. Classification of globally colorized categories of partitions. *Infinite Dimensional Analysis, Quantum Probability and Related Topics* **21**(04), 1850029 (2018). doi:10.1142/S0219025718500297

[Gro20a] Daniel Gromada. Gluing compact matrix quantum groups. *In preparation* (2020).

[Gro20b] Daniel Gromada. Free Coxeter quantum group D_4. *In preparation* (2020).

[GW19a] Daniel Gromada and Moritz Weber. Generating linear categories of partitions (2019). arXiv:1904.00166

[GW19b] Daniel Gromada and Moritz Weber. New products and $\mathbb{Z}_2$-extensions of compact matrix quantum groups (2019). arXiv:1907.08462

[GW20] Daniel Gromada and Moritz Weber. Intertwiner spaces of quantum group subrepresentations. *Communications in Mathematical Physics* **376**, 81–115 (2020). doi:10.1007/s00220-019-03463-y

[HR05] Tom Halverson and Arun Ram. Partition algebras. *European Journal of Combinatorics* **26**(6), 869–921 (2005). doi:10.1016/j.ejc.2004.06.005

[Hop41] Heinz Hopf. Über die Topologie der Gruppen-Mannigfaltigkeiten und ihre Verallgemeinerungen. *Annals of Mathematics* **42**(1), 22–52 (1941). jstor.org/stable/1968985

[Jim85] Michio Jimbo. A q-difference analogue of $U(\mathfrak{g})$ and the Yang-Baxter equation. *Letters in Mathematical Physics* **10**, 63–69 (1985). doi:10.1007/BF00704588

[Jun19] Stefan Jung. Linear independences of maps associated to partitions (2019). arXiv:1906.10533

[Kre49] Mark G. Krein. A principle of duality for bicompact groups and quadratic block algebras. *Doklady Akademii Nauk SSSR* **69**, 725–728 (1949).

[KV00] Johan Kustermans and Stefaan Vaes. Locally compact quantum groups. *Annales scientifiques de l'École Normale Supérieure* **33**(6), 837–934 (2000). doi:10.1016/s0012-9593(00)01055-7

[LMR20] Martino Lupini, Laura Mančinska and David E. Roberson. Nonlocal games and quantum permutation groups. *Journal of Functional Analysis* **279**(5), 108592 (2020). doi:10.1016/j.jfa.2020.108592

[Maa18] Laura Maassen. The intertwiner spaces of non-easy group-theoretical quantum groups. *Journal of Noncommutative Geometry*, to appear (2018). arXiv:1808.07693

[Mal18] Sara Malacarne. Woronowicz Tannaka–Krein duality and free orthogonal quantum groups. *Mathematica Scandinavica* **122**(1), 151-160 (2018). doi:10.7146/math.scand.a-97320

[MW19a] Alexander Mang and Moritz Weber. Categories of Two-Colored Pair Partitions, Part I: Categories Indexed by Cyclic Groups. *The Ramanujan Journal*, to appear (2019). doi:10.1007/s11139-019-00149-w

[MW19b] Alexander Mang and Moritz Weber. Categories of Two-Colored Pair Partitions, Part II: Categories Indexed by Semigroups (2019). arXiv:1901.03266

[MW19c] Alexander Mang and Moritz Weber. Non-Hyperoctahedral Categories of Two-Colored Partitions, Part I: New Categories (2019). arXiv:1907.11417

[MW20] Alexander Mang and Moritz Weber. Non-Hyperoctahedral Categories of Two-Colored Partitions, Part II: All Possible Parameter Values (2020). arXiv:2003.00569

[MR19] Laura Mančinska and David E. Roberson. Quantum isomorphism is equivalent to equality of homomorphism counts from planar graphs (2019). arXiv:1910.06958

[Map17] Maplesoft; a division of Waterloo Maple Inc. *Maple 2017*.

[PW90] Piotr Podleś and Stanisław L. Woronowicz. Quantum deformation of Lorentz group. *Communications in Mathematical Physics* **130**(2), 381–431 (1990). projecteuclid.org:443/euclid.cmp/1104200517

[Rau12] Sven Raum. Isomorphisms and fusion rules of orthogonal free quantum groups and their free complexifications. *Proceedings of the American Mathematical Society* **120**, 3207–3218 (2012). doi:10.1090/S0002-9939-2012-11264-X

[RW14] Sven Raum and Moritz Weber. The combinatorics of an algebraic class of easy quantum groups. *Infinite Dimensional Analysis, Quantum Probability and Related Topics* **17**(03), 1450016 (2014). doi:10.1142/S0219025714500167

[RW15] Sven Raum and Moritz Weber. Easy quantum groups and quantum subgroups of a semi-direct product quantum group. *Journal of Noncommutative Geometry* **9**(4), 1261–1293 (2015). doi:10.4171/JNCG/223

[RW16] Sven Raum and Moritz Weber. The Full Classification of Orthogonal Easy Quantum Groups. *Communications in Mathematical Physics* **341**(3), 751–779 (2016). doi:10.1007/s00220-015-2537-z

[Sch96] Peter Schauenburg. Hopf bigalois extensions. *Communications in Algebra* **24**(12), 3797-3825 (1996). doi:10.1080/00927879608825788

[Tan39] Tadao Tannaka. Über den Dualitätssatz der nichtkommutativen topologischen Gruppen. *Tohoku Mathematical Journal, First Series* **45**, 1–12 (1939). `https://www.jstage.jst.go.jp/article/tmj1911/45/0/45_0_1/_article`

[TW17] Pierre Tarrago and Moritz Weber. Unitary Easy Quantum Groups: The Free Case and the Group Case. *International Mathematics Research Notices* **2017**(18), 5710–5750 (2017). doi:10.1093/imrn/rnw185

[TW18] Pierre Tarrago and Moritz Weber. The classification of tensor categories of two-colored noncrossing partitions. *Journal of Combinatorial Theory, Series A* **154**, 464–506 (2018). doi:10.1016/j.jcta.2017.09.003

[Tut93] William T. Tutte. The Matrix of Chromatic Joins. *Journal of Combinatorial Theory, Series B* **57**(2), 269–288 (1993). doi:10.1006/jctb.1993.1021

[VV19] Stefaan Vaes and Matthias Valvekens. Property (T) discrete quantum groups and subfactors with triangle presentations. *Advances in Mathematics* **345**, 382–428 (2019). doi:10.1016/j.aim.2019.01.023

[VDa96] Alfons Van Daele. Discrete Quantum Groups. *Journal of Algebra* **180**(2), 431–444 (1996). doi:10.1006/jabr.1996.0075

[VDW96] Alfons Van Daele and Shuzhou Wang. Universal quantum groups. *International Journal of Mathematics* **07**(02), 255–263 (1996). doi:10.1142/S0129167X96000153

[Wan95a] Shuzhou Wang. Free products of compact quantum groups. *Communications in Mathematical Physics* **167**(3), 671–692 (1995). doi:10.1007/BF02101540

[Wan95b] Shuzhou Wang. Tensor Products and Crossed Products of Compact Quantum Groups. *Proceedings of the London Mathematical Society* **s3-71**(3), 695–720 (1995). doi:10.1112/plms/s3-71.3.695

[Wan97] Shuzhou Wang. Krein duality for compact quantum groups. *Journal of Mathematical Physics* **38**(1), 524–534 (1997). doi:10.1063/1.531832

[Wan98] Shuzhou Wang. Quantum Symmetry Groups of Finite Spaces. *Communications in Mathematical Physics* **195**(1), 195–211 (1998). doi:10.1007/s002200050385

[Web13] Moritz Weber. On the classification of easy quantum groups. *Advances in Mathematics* **245**, 500–533 (2013). doi:10.1016/j.aim.2013.06.019

[Wor87] Stanisław L. Woronowicz. Compact matrix pseudogroups. *Communications in Mathematical Physics* **111**(4), 613–665 (1987). doi:10.1007/BF01219077

[Wor88] Stanisław L. Woronowicz. Tannaka–Krein duality for compact matrix pseudogroups. Twisted $SU(N)$ groups. *Inventiones mathematicae* **93**(1), 35–76 (1988). doi:10.1007/BF01393687

[Wor98] Stanisław L. Woronowicz. Compact quantum groups. In: A. Connes, K. Gawedzki and J. Zinn-Justin, eds., *Symétries quantiques (Les Houches, 1995)*, Amsterdam: North-Holland (1998).

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