Quantum automorphism groups of finite graphs

Dissertation
zur Erlangung des Grades
des Doktors der Naturwissenschaften
der Fakultät für Mathematik und Informatik
der Universität des Saarlandes

Simon Schmidt

Saarbrücken, 2020

Datum der Verteidigung:	2. Juli 2020
Dekan:	Prof. Dr. Thomas Schuster
Prüfungsausschuss:	Vorsitzende:
	Prof. Dr. Gabriela Weitze-Schmithüsen
	Berichterstatter:
	Prof. Dr. Moritz Weber
	Prof. Dr. Roland Speicher
	Prof. Dr. Julien Bichon
	Akademischer Mitarbeiter:
	Dr. Tobias Mai

Abstract

The present work contributes to the theory of quantum permutation groups. More specifically, we develop techniques for computing quantum automorphism groups of finite graphs and apply those to several examples.

Amongst the results, we give a criterion on when a graph has quantum symmetry. By definition, a graph has quantum symmetry if its quantum automorphism group does not coincide with its classical automorphism group. We show that this is the case if the classical automorphism group contains a pair of disjoint automorphisms. Furthermore, we prove that several families of distance-transitive graphs do not have quantum symmetry. This includes the odd graphs, the Hamming graphs $H(n,3)$, the Johnson graphs $J(n,2)$, the Kneser graphs $K(n,2)$ and all cubic distance-transitive graphs of order ≥ 10. In particular, this implies that the Petersen graph does not have quantum symmetry, answering a question asked by Banica and Bichon in 2007. Moreover, we show that the Clebsch graph does have quantum symmetry and prove that its quantum automorphism group is equal to SO_5^{-1} answering a question asked by Banica, Bichon and Collins. More generally, for odd n, the quantum automorphism group of the folded n-cube graph is SO_n^{-1}. With one graph missing, we can now decide whether or not a distance-regular graph of order ≤ 20 does have quantum symmetry. We present a table including those results. As a byproduct, we obtain a pair of distance-regular graphs with the same intersection array, where one of them does have quantum symmetry and the other one does not.

Additionally, we discuss connections of quantum automorphism groups of finite graphs to planar algebras associated to group actions and quantum isomorphisms of graphs. Using those connections, we give two examples of graphs with quantum symmetry, whose automorphism groups do not contain any pair of disjoint automorphisms. Those are the Higman-Sims graph and a graph obtained from a linear binary constraint system.

This work contains the results of the three research articles [48], [49] and [50] by the author. The other articles ([25], [35], [41], [51], [52]) by the author are mentioned in the margin.

Zusammenfassung

Die vorliegende Arbeit trägt zur Theorie der Quantenpermutationsgruppen bei. Genauer gesagt entwickeln wir Techniken zur Berechnung von Quantenautomorphismengruppen endlicher Graphen und wenden diese auf mehrere Beispiele an.

Unter anderem geben wir ein Kriterium an, wann ein Graph Quantensymmetrie hat. Per Definition hat ein Graph Quantensymmetrie, wenn seine Quantenautomorphismengruppe nicht mit der klassischen Automorphismengruppe übereinstimmt. Wir zeigen dass dies der Fall ist, wenn die klassische Automorphismengruppe ein Paar von disjunkten Automorphismen enthält. Außerdem beweisen wir, dass mehrere Familien von distanz-transitiven Graphen keine Quantensymmetrie haben. Dazu gehören die Odd-Graphen, die Hamming-Graphen $H(n,3)$, die Johnson-Graphen $J(n,2)$, die Kneser-Graphen $K(n,2)$ und alle kubischen distanz-transitiven Graphen der Ordnung ≥ 10. Dies zeigt insbesondere, dass der Petersen-Graph keine Quantensymmetrie aufweist, was eine Frage von Banica und Bichon aus dem Jahr 2007 beantwortet. Darüber hinaus zeigen wir, dass der Clebsch-Graph Quantensymmetrie besitzt und beweisen, dass seine Quantenautomorphismengruppe gleich SO_5^{-1} ist, was eine Frage von Banica, Bichon und Collins beantwortet. Allgemeiner ist die Quantenautomorphismengruppe des gefalteten n-Würfel-Graphen SO_n^{-1}, für ungerade n. Mit einer Ausnahme können wir jetzt entscheiden, ob ein distanz-regulärer Graph der Ordnung ≤ 20 Quantensymmetrie hat oder nicht. Wir präsentieren eine Tabelle mit diesen Ergebnissen. Als Nebenprodukt erhalten wir ein Paar distanz-regulärer Graphen mit demselben Intersection array, wobei einer von ihnen Quantensymmetrie aufweist und der andere nicht.

Zusätzlich diskutieren wir Zusammenhänge von Quantenautomorphismengruppen endlicher Graphen mit planaren Algebren, die von Gruppenwirkungen kommen, und Quantenisomorphismen von Graphen. Mit Hilfe dieser Zusammenhänge geben wir zwei Beispiele für Graphen mit Quantensymmetrie an, deren Automorphismengruppe jeweils kein Paar disjunkter Automorphismen enthält. Es handelt sich um den Higman-Sims-Graphen und einen Graphen, der aus einem linearen binären Gleichungssystem konstruiert wurde.

Diese Arbeit enthält die Ergebnisse der drei Forschungsartikel [48], [49] und [50] des Autors. Die anderen Arbeiten ([25], [35], [41], [51], [52]) des Autors finden am Rande Erwähnung.

Acknowledgements

First and foremost, I want to thank Moritz Weber for his superb supervision. He always cared about my well-being and supported me whenever I needed it. I am grateful for learning a lot about operator algebras and quantum groups from him. This includes his lectures, which got me interested in those topics, but also the numerous discussions and chats in between. Also, he came up with several small research questions during my PhD studies, which always kept me motivated. Furthermore, I want to thank him for the opportunity to take part in many conferences. This allowed me to come in contact with many people from the quantum group community.

I want to thank all members of the research group of Roland Speicher for creating a pleasant environment during my PhD studies. Furthermore, I am grateful to Roland Speicher for giving me the opportunity to attend conferences such as the one-month program in Montreal. My thanks also go to Daniel Gromada and Laura Maaßen, with whom I had a lot of fun during several conference meetings.

I am thankful to Julien Bichon for kindly agreeing to review this thesis. Moreover, I want to thank him for inspiring discussions on quantum automorphism groups of graphs. Furthermore, I am grateful to Ion Nechita for inviting me to Timisoara and to Toulouse. He was a great host for both of those stays and I really enjoyed working with him. I thank David Roberson for many nice discussions on quantum automorphisms and quantum isomorphisms.

Sincere thanks go to my friend and office mate Felix Leid. We had a lot of nice chats in our office, but also knew when to focus on our work. Moreover, I really enjoyed travelling with him. Furthermore, I thank my colleagues and friends Rami Ahmad, Nils Gutheil, Benedikt Hewer, Pascal Kattler and Andreas Widenka for all the time we spent together at Saarland University.

Ich bedanke mich bei meiner Familie, die mich immer unterstützt hat. Außerdem geht mein tiefster Dank an meine Freunde, auf die ich immer zählen kann.

Contents

Introduction

The present work concerns quantum automorphism groups of finite graphs. Those generalize classical automorphism groups of a graph in the framework of Woronowicz's compact matrix quantum groups.

The origin of quantum automorphism groups of graphs

Quantum groups were first introduced by Drinfeld and Jimbo in 1986. Shortly after, Woronowicz [60] gave a definition of compact quantum groups based on C^*-algebras. The idea is the following: Instead of studying a compact group G we can look at the algebra of continuous functions $C(G)$ over G and the map $\Delta : C(G) \to C(G \times G) \cong C(G) \otimes C(G)$ which we get by dualizing the group multiplication. To generalize the notion of a compact group, we use (not necessarily commutative) unital C^*-algebras as underlying algebras instead of just considering function algebras. In the spirit of non-commutative geometry, we think of the C^*-algebra $A = C(G^+)$ as continuous functions on some quantum group G^+, replacing the classical group G. We also need a map $\Delta : C(G^+) \to C(G^+) \otimes C(G^+)$, where finding the right compatibility conditions was part of Woronowicz's achievements.

In 1997, Wang [58] characterized quantum symmetries of finite spaces by compact quantum groups. Especially, he defined the quantum symmetric group S_n^+, a quantum analogue of the symmetric group S_n. He showed that S_n^+ is the quantum automorphism group of the space consisting of n points. Instead of considering n points, one can take a graph on n vertices and ask for its quantum automorphism group. This has been initiated by Banica and Bichon. In 2003, Bichon [13] gave a definition of quantum automorphism groups of graphs which we will denote $G^*_{aut}(\Gamma)$ in this thesis. Two years later, Banica [3] defined the quantum automorphism group $G^+_{aut}(\Gamma)$ which slightly differs from $G^*_{aut}(\Gamma)$. Nowadays, Banica's version is used more frequently in the literature.

Motivation

The concept of quantum automorphisms originates from the notion of graph automorphisms. The automorphisms of a finite graph capture its symmetries. For a finite graph Γ, a graph automorphism is a bijection σ on the vertices, where vertices i and j are adjacent if and only if $\sigma(i)$ and $\sigma(j)$ are adjacent. Composition gives a group structure on the set of graph automorphisms and we get the automorphism group of the graph, which we denote by $\mathrm{Aut}(\Gamma)$. The quantum automorphism group $G^+_{aut}(\Gamma)$ of a graph generalizes this concept, we study the quantum symmetries of the graph. In general, the quantum group $G^+_{aut}(\Gamma)$ strengthens the symmetry of a graph in a non-commutative context: It holds

$$\mathrm{Aut}(\Gamma) \subseteq G^+_{aut}(\Gamma)$$

in a certain sense. Therefore, we have the natural question: When do $\mathrm{Aut}(\Gamma)$ and $G^+_{aut}(\Gamma)$ coincide? In virtue of this question, we say that a graph Γ has no quantum symmetry if $\mathrm{Aut}(\Gamma) = G^+_{aut}(\Gamma)$. Otherwise, we say that the graph Γ has quantum symmetry. The obvious followup question is: What does $G^+_{aut}(\Gamma)$ look like, if $\mathrm{Aut}(\Gamma) \neq G^+_{aut}(\Gamma)$? Note that similar to $\mathrm{Aut}(\Gamma) \subseteq S_n$, we have $G^+_{aut}(\Gamma) \subseteq S_n^+$. Thus, it holds

$$\mathrm{Aut}(\Gamma) \subseteq G^+_{aut}(\Gamma) \subseteq S_n^+$$

in general. This shows that computing quantum automorphisms groups of graphs with $G^+_{aut}(\Gamma) \neq \mathrm{Aut}(\Gamma)$ yields (potentially new) quantum subgroups of the quantum symmetric group S_n^+.

Furthermore, Banica associates a planar algebra to the quantum automorphism group of a graph in [3]. Planar algebras were first introduced by Jones [34] to study subfactors. One may ask whether or not a planar algebra is generated by its 2-box space. This is for example studied by Curtin [24] and Ren ([45], [46]) for certain planar algebras related to graphs. Using the connection to quantum automorphism groups of graphs, one can obtain further planar algebras that are generated by their 2-box space.

More recently, a connection to quantum information theory gave another motivation for studying quantum automorphism groups of finite graphs. The quantum isomorphism game, introduced by Aterias, Mančinska, Roberson, Šámal, Severini and Varvitsiotis in [1], is a nonlocal game in which two players try to convince a referee that they know an isomorphism between two graphs Γ_1 and Γ_2. With classical strategies, the players succeed with probability one if and only if the two graphs are isomorphic. Allowing quantum strategies, it is possible to win the game with probability one if and only if Γ_1 and Γ_2 are quantum isomorphic. It is important to note that there exist graphs that are quantum isomorphic but not isomorphic, see for example [1]. Also, quantum isomorphisms are in deep connection to quantum automorphism groups. For example, it was shown in [36] that two connected graphs are quantum isomorphic if and only if the quantum automorphism group of the disjoint union of Γ_1 and Γ_2 has an orbit that intersects the vertices of Γ_1 and Γ_2. Thus, by studying quantum automorphism groups of graphs, we get more insight on quantum isomorphisms and quantum strategies of the isomorphism game.

Known results on quantum automorphism groups of graphs

The theory of quantum automorphism groups of graphs is a quite young topic. At the moment there are only few articles about quantum automorphism groups of graphs. We review some further results. Regarding explicit examples, Banica and Bichon computed the quantum automorphism group $G^+_{aut}(\Gamma)$ of all vertex-transitive graphs on less or equal to eleven vertices, except the Petersen graph, in [6]. Furthermore, together with Collins, they computed the quantum automorphism group of the n-cube graphs in [9]. In [8], Banica, Bichon and Chenevier considered circulant graphs on p vertices, for p prime. They showed that if the graph has some further properties, then it has no quantum symmetry. Moreover, quantum automorphism groups of graph products have been studied by Banica and Bichon [6] and also by Chassaniol in [21]. Also, Fulton [27] investigated the quantum automorphism groups of trees with certain automorphism groups. The intertwiner spaces of $G^+_{aut}(\Gamma)$ are investigated by Chassaniol in [20], [22] and Mančinska and Roberson in [38]. Lupini, Mančinska and Roberson showed in [36] that almost all graphs have trivial quantum automorphism group. Especially, we get that almost all graphs have no quantum symmetry. This is the quantum analogue of the fact that almost all graphs have no symmetry, which was proven by Erdős and Rényi in [26].

Outline and main results

The outline of this thesis is the following. In Chapter 1 we establish basic definitions and notions we need later on. Those are mostly related to compact matrix quantum groups and finite graphs. **In this thesis, all graphs are finite and have no multiple edges.**

We give Banica's and Bichon's definitions of quantum automorphism groups in Chapter 2. Furthermore, we review the work on the quantum automorphism group of graph products and give some examples. We discuss the intertwiner spaces associated to quantum automorphism groups of graphs and then survey the quantum orbital algebra. Lastly, we obtain the first result of this thesis: An example of a graph, where we get strict inequalities between the automorphism group and the two definitions of quantum automorphism group simultaneously (Example 2.5.3).

In Chapter 3, we develop tools for computing the quantum automorphism group. At first, we show that a graph has quantum symmetry if its automorphism group contains a pair of disjoint automorphisms. We say that automorphisms σ and τ are disjoint if they have disjoint support, i.e. $\tau(i) \neq i$ implies $\sigma(i) = i$ and vice versa.

Theorem A (Theorem 3.1.2)**.** *Let $\Gamma = (V, E)$ be a finite graph without multiple edges. If there exist two non-trivial, disjoint automorphisms $\sigma, \tau \in \mathrm{Aut}(\Gamma)$, $\mathrm{ord}(\sigma) = n, \mathrm{ord}(\tau) = m$, we get a surjective *-homomorphism $\varphi : C(G^+_{aut}(\Gamma)) \to C^*(\mathbb{Z}_n * \mathbb{Z}_m)$. In particular, Γ has quantum symmetry.*

To see whether a given graph Γ has quantum symmetry, we may check this criterion first. If we find two non-trivial, disjoint automorphisms of Γ, then the graph has quantum symmetry. Otherwise, we do not know whether or not the graph has quantum symmetry. We may try to prove that Γ has no quantum symmetry. Some tools for proving that a graph has no quantum symmetry can be found in Section 3.2. Amongst these tools we want to highlight the following two.

Theorem B (Lemma 3.2.2 & Lemma 3.2.4)**.** *Let Γ be a finite, undirected graph, let $(u_{ij})_{1 \leq i,j \leq n}$ be the generators of $C(G^+_{aut}(\Gamma))$ and let $d(s,t)$ be the distance of $s, t \in V$.*

(i) If we have $d(i,k) \neq d(j,l)$, then $u_{ij}u_{kl} = 0$.

(ii) Let Γ be distance-transitive. Let $j_1, l_1 \in V$ and put $m := d(j_1, l_1)$. If $u_{aj_1}u_{bl_1} = u_{bl_1}u_{aj_1}$ for all a, b with $d(a,b) = m$, then we have $u_{ij}u_{kl} = u_{kl}u_{ij}$ for all i, k, j, l with $d(j,l) = m = d(i,k)$.

Note that a distance-transitive graph is a graph such that for any given pair of vertices i, j in distance a and any other pair of vertices k, l with $d(k,l) = a$ there is a graph automorphism $\sigma : V \to V$ with $\sigma(i) = k$ and $\sigma(j) = l$. For distance-transitive graphs, the previous theorem yields that we only have to consider one specific pair of vertices in a certain distance to show that all generators associated to all pairs in this distance commute. Doing this for every distance suffices to show that this graph has no quantum symmetry. Therefore, the number of cases we have to prove that some generators commute equals the diameter of the distance-transitive graph. Especially, it does not depend on the number of vertices or edges of the graph, which is one of the reasons why we can prove that certain families of graphs have no quantum symmetry, despite including graphs of arbitrary size.

We use the results of Chapter 3 in Chapter 4 to compute the quantum automorphism group of several families of distance-transitive graphs. We summarize the results in the next theorem.

Theorem C (Theorems 4.1.1, 4.1.7, 4.1.13, 4.1.18, 4.1.20 & Section 4.2)**.** *We have the following results for distance-transitive graphs.*

(i) The Petersen graph has no quantum symmetry.

(ii) The odd graphs O_k have no quantum symmetry.

(iii) The Hamming graphs $H(n,3)$, $n \in \mathbb{N}$, $H(1,2)$ and $H(m,1)$, $m = 1,2,3$, have no quantum symmetry. For all other values, the graph $H(n,k)$ has quantum symmetry.

(iv) For $n \geq 5$, the Johnson graphs $J(n,2)$ and the Kneser graphs $K(n,2)$ do not have quantum symmetry.

(v) Moore graphs of diameter two have no quantum symmetry.

(vi) Let Γ be a cubic distance-transitive graph of order ≥ 10. Then Γ has no quantum symmetry.

In 2007, Banica and Bichon ([6]) asked whether or not the Petersen graph has quantum symmetry. Theorem C (i) answers this question. We remark that the Petersen graph is isomorphic to the odd graph O_3 and the Kneser graph $K(5,2)$. It is furthermore a Moore graph of diameter two and a cubic distance-transitive graph. Therefore, Theorem C (i) follows from Theorem C (ii), (iv), (v) and (vi).

Building on Theorem C (vi), we may present a table with the quantum automorphism groups of all cubic distance-transitive graphs. There are twelve such graphs as shown by Biggs and Smith in [16]. For three of them, namely the complete graph on four points K_4, the complete bipartite graph on six points $K_{3,3}$ as well as the cube Q_3, it was known before that they have quantum symmetry and their quantum automorphism groups are given in [6]. The remaining ones have no quantum symmetry by Theorem C (vi). We also give the intersection arrays (see Definition 1.2.21) of the graphs in the table. We obtain the following table for the cubic distance-transitive graphs.

Name of Γ	Order	$\mathrm{Aut}(\Gamma)$	$G^+_{aut}(\Gamma)$	Intersection array
K_4 ([6])	4	S_4	S_4^+	{3;1}
$K_{3,3}$ ([6])	6	$S_3 \wr \mathbb{Z}_2$	$S_3 \wr_* \mathbb{Z}_2$	{3,2;1,3}
Cube Q_3 ([6])	8	$S_4 \times \mathbb{Z}_2$	$S_4^+ \times \mathbb{Z}_2$	{3,2,1;1,2,3}
Petersen graph	10	S_5	$\mathrm{Aut}(\Gamma)$	{3,2;1,1}
Heawood graph	14	$PGL(2,7)$	$\mathrm{Aut}(\Gamma)$	{3,2,2;1,1,3}
Pappus graph	18	ord 216	$\mathrm{Aut}(\Gamma)$	{3,2,2,1;1,1,2,3}
Desargues graph	20	$S_5 \times \mathbb{Z}_2$	$\mathrm{Aut}(\Gamma)$	{3,2,2,1,1;1,1,2,2,3}
Dodecahedron	20	$A_5 \times \mathbb{Z}_2$	$\mathrm{Aut}(\Gamma)$	{3,2,1,1,1;1,1,1,2,3}
Coxeter graph	28	$PGL(2,7)$	$\mathrm{Aut}(\Gamma)$	{3,2,2,1;1,1,1,2}
Tutte 8-cage	30	$\mathrm{Aut}(S_6)$	$\mathrm{Aut}(\Gamma)$	{3,2,2,2;1,1,1,3}
Foster graph	90	ord 4320	$\mathrm{Aut}(\Gamma)$	{3,2,2,2,2,1,1,1;1,1,1,1,2,2,2,3}
Biggs-Smith graph	102	$PSL(2,17)$	$\mathrm{Aut}(\Gamma)$	{3,2,2,2,1,1,1;1,1,1,1,1,1,3}

Table 1: Quantum automorphism groups of all cubic distance-transitive graphs.

In addition to the previous table, we study further distance-transitive graphs, preferably of order ≤ 20 in Section 4.3. Distance-transitive graphs are especially distance-regular. We say that a graph is distance-regular if for two vertices v, w, the number of vertices in distance k to v and in distance l to w only depend on k, l and $d(v,w)$. There is only one distance-regular graph of order ≤ 20 that is not distance-transitive, namely the Shrikhande graph. We also show that this graph has no quantum symmetry in Section 4.3.

Except for the Johnson graph $J(6,3)$, we know for all distance-regular graphs with up to 20 vertices whether or not they have quantum symmetry. Distance-regular graphs of order $11 \leq n \leq 20$ that have quantum symmetry are

(i) the 4×4 rook's graph (Proposition 4.1.12),

(ii) the 4-cube ([9]),

(iii) the Clebsch graph (Corollary 5.1.2)

(iv) the complete graphs K_m,

(v) the cycles C_m,

(vi) the complete bipartite graphs $K_{m,m}$,

(vii) the crown graphs $(K_m \square K_2)^c$,

where we choose suitable m for the families (iv)–(vii). The quantum automorphism groups of those four families can be found in [6]. The distance-regular graphs of order $n \leq 11$ can be found in [6], since all distance-regular graphs of order ≤ 11 are vertex-transitive.

For some graphs, for example the Petersen graph, we only need the values of the intersection array to show that this graph has no quantum symmetry. Thus one might guess that the intersection array contains all information about the quantum symmetry of a graph. But this is not the case because of the following. The Shrikhande graph and the 4×4 rook's graph have the same intersection array, see Table 2. We know that the 4×4 rook's graph has quantum symmetry by Proposition 4.1.12, whereas the Shrikhande graph has no quantum symmetry, see Subsection 4.3. This shows that in general one needs to use further graph properties to decide whether or not a graph has quantum symmetry, the intersection array is not enough.

Besides deciding whether or not certain graphs have quantum symmetry, we furthermore compute the quantum automorphism group of a family of graphs that have quantum symmetry. Recall that if a graph has no quantum symmetry, then the quantum automorphism group coincides with its classical automorphism group. Otherwise, for graphs that have quantum symmetry, we have to additionally compute the quantum automorphism group. The family of graphs we are dealing with are the folded n-cube graphs whose quantum automorphism groups we are studying in Chapter 5. Our techniques are similar to those of [9], where it was shown that the quantum automorphism group of the n-cube graph is equal to O_n^{-1}. The folded n-cube graphs are Cayley graphs and we study their eigenvalues and eigenspaces to get more insight on the quantum automorphism group. We have the following result.

Theorem D (Theorem 5.4.3)**.** *For n odd, the quantum automorphism group of the folded n-cube graph FQ_n is SO_n^{-1}.*

The folded 5-cube graph is isomorphic to the Clebsch graph. Therefore, we get that the quantum automorphism group of the Clebsch graph is SO_5^{-1} which answers a question asked by Banica, Bichon and Collins in [10].

Building on the results of Chapter 4 and Chapter 5, we get the following table. It contains all distance-regular graphs with up to 20 vertices and the distance-transitive graphs of Theorem C (i)–(vi) and Theorem D. Similar to Table 1, we include the intersection array of the graphs.

Name of Γ	Order	$\mathrm{Aut}(\Gamma)$	$G^+_{aut}(\Gamma)$	Intersection array
Octahedron $J(4,2)$ ([6])	6	$Z_2 \wr S_3$	$\mathbb{Z}_2 \wr_* S_3$	{4,1;1,4}
Cube Q_3 ([6])	8	$S_4 \times \mathbb{Z}_2$	$S_4^+ \times \mathbb{Z}_2$	{3,2,1;1,2,3}
Paley graph P_9 ([6])	9	$S_3 \wr \mathbb{Z}_2$	$\mathrm{Aut}(\Gamma)$	{4,2;1,2}
Petersen graph	10	S_5	$\mathrm{Aut}(\Gamma)$	{3,2;1,1}
Icosahedron	12	$A_5 \times \mathbb{Z}_2$	$\mathrm{Aut}(\Gamma)$	{5,2,1;1,2,5}
Paley graph P_{13} ([22])	13	$\mathbb{Z}_{13} \rtimes \mathbb{Z}_6$	$\mathrm{Aut}(\Gamma)$	{6,3;1,3}
Heawood graph	14	$PGL(2,7)$	$\mathrm{Aut}(\Gamma)$	{3,2,2;1,1,3}
co-Heawood graph	14	$PGL(2,7)$	$\mathrm{Aut}(\Gamma)$	{4,3,2;1,2,4}
Line graph of Petersen graph	15	S_5	$\mathrm{Aut}(\Gamma)$	{4,2,1;1,1,4}
Cube Q_4 ([9])	16	H_4	O_4^{-1}	{4,3,2,1;1,2,3,4}
4×4 rook's graph $H(2,4)$	16	$S_4 \wr \mathbb{Z}_2$	?(has qsym)	{6,3;1,2}
Shrikhande graph	16	$\mathbb{Z}_4^2 \rtimes D_6$	$\mathrm{Aut}(\Gamma)$	{6,3;1,2}
Clebsch graph	16	$\mathbb{Z}_2^4 \rtimes S_5$	SO_5^{-1}	{5,4;1,2}
Paley graph P_{17} ([22])	17	$\mathbb{Z}_{17} \rtimes \mathbb{Z}_8$	$\mathrm{Aut}(\Gamma)$	{8,4;1,4}
Pappus graph	18	ord 216	$\mathrm{Aut}(\Gamma)$	{3,2,2,1;1,1,2,3}
Johnson graph $J(6,3)$	20	$S_6 \times \mathbb{Z}_2$	?	{9,4,1;1,4,9}
Desargues graph	20	$S_5 \times \mathbb{Z}_2$	$\mathrm{Aut}(\Gamma)$	{3,2,2,1,1;1,1,2,2,3}
Dodecahedron	20	$A_5 \times \mathbb{Z}_2$	$\mathrm{Aut}(\Gamma)$	{3,2,1,1,1;1,1,1,2,3}
Hoffman-Singleton graph	50	$PSU(3,5^2)$	$\mathrm{Aut}(\Gamma)$	{7,6;1,1}
K_n ([6])	n	S_n	S_n^+	$\{n-1;1\}$
C_n, $n \neq 4$ ([6])	n	D_n	$\mathrm{Aut}(\Gamma)$	$(\star^1)$
$K_{n,n}$ ([6])	$2n$	$S_n \wr \mathbb{Z}_2$	$S_n^+ \wr_* \mathbb{Z}_2$	$\{n, n; 1, n\}$
$(K_n \Box K_2)^c$ ([6])	$2n$	$S_n \times \mathbb{Z}_2$	$S_n^+ \times \mathbb{Z}_2$	$(\star^2)$
Johnson graph $J(n,2)$,$n \geq 5$	$\binom{n}{2}$	S_n	$\mathrm{Aut}(\Gamma)$	$\{2n-4, n-3;1,4\}$
Kneser graph $K(n,2)$,$n \geq 5$	$\binom{n}{2}$	S_n	$\mathrm{Aut}(\Gamma)$	$(\star^3)$
Odd graphs O_k	$\binom{2k-1}{k-1}$	S_{2k-1}	$\mathrm{Aut}(\Gamma)$	$(\star^4)$
Hamming graphs $H(n,3)$	3^n	$S_3 \wr S_n$	$\mathrm{Aut}(\Gamma)$	$(\star^5)$

Table 2: Quantum automorphism groups of distance-regular graphs up to 20 vertices and some additional graphs.

Here

$$(\star^1) = \{2,1,...,1;1,...,1,2\} \text{ for } n \text{ even}, \{2,1,...,1;1,...,1,1\} \text{ for } n \text{ odd},$$

$$(\star^2) = \{n-1, n-2, 1; 1, n-1, n-2\},$$

$$(\star^3) = \{(n-2)(n-3)/2, 2n-8; 1, (n-3)(n-4)/2\},$$

$$(\star^4) = \{k, k-1, k-1 \dots, l+1, l+1, l; 1, 1, 2, 2, \dots, l, l\} \text{ for } k = 2l-1,$$
$$\{k, k-1, k-1, \dots l+1, l+1; 1, 1, 2, 2, \dots, l-1, l-1, l\} \text{ for } k = 2l,$$

$$(\star^5) = \{2n, 2n-2, \dots, 2; 1, 2, \dots, n\}.$$

In Chapter 6, we review a generating property of planar algebras that is related to quantum automorpism groups of finite graphs. This generating property was introduced by Ren in [45]. Therein, he briefly mentions that the generating property is connected to graphs having no quantum symmetry. We discuss this connection in detail in Chapter 6. We will use the results of the Chapter 4 to get many examples of graphs having this generating property. Furthermore, it was asked in [10] whether the Higman-Sims graph has quantum symmetry. We use the generating property to get the following theorem.

Theorem E (Theorem 6.3.3)**.** *The Higman-Sims graph has quantum symmetry.*

We use Sage [53] to check that the automorphism group of the Higman-Sims graph does not contain disjoint automorphisms. This shows that the converse direction of Theorem A is not true.

We study quantum isomorphisms in Chapter 7. Those were first defined in [1] via a nonlocal game, called the isomorphism game. We review this nonlocal game and give equivalent definitions of quantum isomorphisms. Furthermore, using monoidal equivalence, we prove the next theorem.

Theorem F (Theorem 7.2.6)**.** *Let Γ_1, Γ_2 be quantum isomorphic graphs. If one of the graphs Γ_1 or Γ_2 has disjoint automorphisms, then both graphs have quantum symmetry.*

Using this theorem, we obtain another example of a graph having quantum symmetry and no disjoint automorphisms. This example originates from a construction used in [36] to get pairs of quantum isomorphic graphs that are not isomorphic.

Finally, we collect open questions that came up during the author's research and discuss ways to tackle them in Chapter 8. For example, we ask to investigate the two missing cases in Table 2 and further graphs for which the quantum automorphism groups are not known. Moreover, we ask the question whether or not there is a graph with trivial automorphism group that also has quantum symmetry.

Chapter 1

Preliminaries

In this chapter, we review basic notions and definitions we need in this thesis. This includes a short introduction to compact matrix quantum groups, which generalize compact groups $G \subseteq \mathrm{GL}_n(\mathbb{C})$. Then, we discuss the quantum symmetric group S_n^+, which constitutes the quantum analogue of the symmetric group S_n. In Section 1.2 we review basic definitions and properties of finite graphs. Note that the automorphism group of a graph is the classical counterpart of the quantum automorphism group. The latter is important for this work and will be introduced in Chapter 2.

1.1 Compact matrix quantum groups

We start with compact matrix quantum groups which were defined by Woronowicz [60, 62] in 1987. See [42, 54] for recent books on compact quantum groups.

Definition 1.1.1. A *compact matrix quantum group* G is a pair $(C(G), u)$, where $C(G)$ is a unital (not necessarily commutative) C^*-algebra which is generated by u_{ij}, $1 \leq i, j \leq n$, the entries of a matrix $u \in M_n(C(G))$. Moreover, the *-homomorphism $\Delta : C(G) \to C(G) \otimes C(G)$, $u_{ij} \mapsto \sum_{k=1}^n u_{ik} \otimes u_{kj}$ must exist, and u and its transpose u^t must be invertible. The matrix u is usually called *fundamental representation*.

Remark 1.1.2. In the previous definition, the symbol $\otimes$ denotes the minimal tensor product of C^*-algebras. Unless stated otherwise, we write $A \otimes B$ for the minimal tensor product of the C^*-algebras A and B, throughout this thesis.

We now introduce the notion of a quantum subgroup.

Definition 1.1.3. Let $(C(G), u)$ and $(C(H), v)$ be compact matrix quantum groups.

(i) We say that $(C(G), u)$ is a *quantum subgroup* of $(C(H), v)$, if there is a surjective *-homomorphism $\varphi : C(H) \to C(G)$ such that $\Delta_G \circ \varphi = (\varphi \otimes \varphi) \circ \Delta_H$.

(ii) Two compact matrix quantum groups $(C(G), u)$ and $(C(H), v)$ are isomorphic as compact quantum groups, if the *-homomorphism φ from above is a *-isomorphism. We denote this by $(C(G), u) = (C(H), v)$.

Compact matrix quantum groups generalize compact matrix groups in the following sense.

Theorem 1.1.4 ([60])**.** *Let $G = (C(G), u)$ be a compact matrix quantum group where $C(G)$ is a commutative C^*-algebra. Then there exists a compact matrix group $\tilde{G}$ such that $(C(\tilde{G}), \tilde{u}) = (C(G), u)$. Here $\tilde{u} = (\tilde{u}_{ij})$ is the matrix with entries $\tilde{u}_{ij} : \tilde{G} \to \mathbb{C}$, $g \mapsto g_{ij}$.*

There are several products of quantum groups. We define the tensor product and the free product of quantum groups.

Proposition 1.1.5 ([57])**.** *Let $G = (C(G), u)$ and $H = (C(H), v)$ be compact matrix quantum groups. Then $G \times H := (C(G) \otimes_{\max} C(H), u \oplus v)$ is a compact matrix quantum group, where*

$$u \oplus v = \begin{pmatrix} u & 0 \\ 0 & v \end{pmatrix}.$$

Here $C(G) \otimes_{\max} C(H)$ is the universal C^*-algebra with generators u_{ij} and v_{kl} such that $u_{ij}v_{kl} = v_{kl}u_{ij}$ for all i, j, k, l and u_{ij}, v_{kl} fulfill the relations of $C(G)$ and $C(H)$, respectively, where additionally $1_{C(G)} = 1_{C(H)}$. The quantum group $G \times H$ is called the tensor product *of G and H.*

Proposition 1.1.6 ([56])**.** *Let $G = (C(G), u)$ and $H = (C(H), v)$ be compact matrix quantum groups. Then $G * H := (C(G) * C(H), u \oplus v)$ is a compact matrix quantum group. Here $C(G) * C(H)$ is the universal C^*-algebra with generators u_{ij} and v_{kl} such that u_{ij} and v_{kl} fulfill the relations of $C(G)$ and $C(H)$, respectively, where additionally $1_{C(G)} = 1_{C(H)}$. We call $G * H$ the* free product *of G and H.*

An action of a compact matrix quantum group on a C^*-algebra is defined as follows ([43, 58]).

Definition 1.1.7. Let $G = (C(G), u)$ be a compact matrix quantum group and let B be a C^*-algebra. A *(left) action* of G on B is a unital *-homomorphism $\alpha : B \to B \otimes C(G)$ such that

(i) $(\mathrm{id} \otimes \Delta) \circ \alpha = (\alpha \otimes \mathrm{id}) \circ \alpha$

(ii) $\alpha(B)(1 \otimes C(G))$ is linearly dense in $B \otimes C(G)$.

The following important example of a compact matrix quantum group is due to Wang [58]. It is the quantum analogue of the symmetric group S_n.

Definition 1.1.8. The *quantum symmetric group* $S_n^+ = (C(S_n^+), u)$ is the compact matrix quantum group, where

$$C(S_n^+) := C^*(u_{ij},\, 1 \leq i,j \leq n \,|\, u_{ij} = u_{ij}^* = u_{ij}^2,\, \sum_{k=1}^{n} u_{ik} = \sum_{k=1}^{n} u_{ki} = 1).$$

A matrix $u = (u_{ij})_{1 \leq i,j \leq n}$, where u_{ij} are elements of some unital C^*-algebra is called *magic unitary* if $u_{ij} = u_{ij}^* = u_{ij}^2$ for all i, j and $\sum_{k=1}^{n} u_{ik} = \sum_{k=1}^{n} u_{ki} = 1$ for all i. Thus, $C(S_n^+)$ is the universal C^*-algebra generated by the entries of a magic unitary $u = (u_{ij})_{1 \leq i,j \leq n}$.

It is worked out in [59] for example, that S_n^+ is a compact matrix quantum group. We remark that $C(S_n)$ is the abelization of $C(S_n^+)$, i.e.

$$C(S_n^+)/\langle u_{ij}u_{kl} - u_{kl}u_{ij} \rangle = C(S_n).$$

In [58], Wang showed that S_n^+ is the universal compact matrix quantum group acting on $X_n = \{1, \dots, n\}$. This action is of the form $\alpha : C(X_n) \to C(X_n) \otimes C(S_n^+)$,

$$\alpha(e_i) = \sum_j e_j \otimes u_{ji}.$$

This is the quantum analogue of the fact that the symmetric group is the universal group acting on n points.

Remark 1.1.9. Let A be a C^*-algebra. It is known (see for example [44, Corollary A.3]) that projections $p_i \in A$ (i.e. $p_i = p_i^2 = p_i^*$), $1 \leq i \leq n$, with $\sum_{i=1}^{n} p_i = 1$ are orthogonal (i.e. $p_i p_j = 0$ for $i \neq j$). This is true because of the following arguments. Choose $j \in \{1, \dots, n\}$ and consider the equation $\sum_{i=1}^{n} p_i = 1$. Multiplying by p_j from the left and the right yields

$$\sum_{i=1}^{n} p_j p_i p_j = p_j.$$

By subtracting p_j on both sides, we obtain

$$\sum_{i \neq j} p_j p_i p_j = 0.$$

Since $p_j p_i p_j$ is a positive element for all i, we get $p_j p_i p_j = 0$. Using the C^*-norm, we finally obtain $p_i p_j = 0$.

This has the following consequence: Let u_{ij}, $1 \leq i, j \leq n$, be the generators of $C(S_n^+)$. Then, we have

$$u_{ij}u_{ik} = \delta_{jk}u_{ij} \qquad \text{and} \qquad u_{ji}u_{ki} = \delta_{jk}u_{ji}$$

for the generators of $C(S_n^+)$.

Remark 1.1.10. The C^*-algebra $C(S_n^+)$ is commutative for $n = 1, 2, 3$, it is non-commutative for $n \geq 4$. This can be seen by the following arguments. It is immediate for $n = 1$. Regarding $n = 2$, the matrix u is of the form

$$u = \begin{pmatrix} p & 1-p \\ 1-p & p \end{pmatrix}.$$

Thus $C(S_2^+)$ is generated by p and $1 - p$ which obviously commute. For the case $n = 3$, we use the argument of [36]. If $i = k$ or $j = l$, the generators u_{ij} and u_{kl} commute by Remark 1.1.9. Let $i \neq k, j \neq l$. Then

$$u_{ij}u_{kl} = u_{ij}u_{kl}\sum_{s=1}^{3} u_{is} = u_{ij}u_{kl}(u_{ij} + u_{il} + u_{it}),$$

where we choose t such that $\{j, l, t\} = \{1, 2, 3\}$. By Remark 1.1.9, it holds $u_{ij}u_{kl}u_{il} = 0$ and

$$u_{ij}u_{kl}u_{it} = u_{ij}(1 - u_{kj} - u_{kt})u_{it} = u_{ij}u_{it} = 0.$$

We conclude $u_{ij}u_{kl} = u_{ij}u_{kl}u_{ij}$ and applying the involution yields $u_{ij}u_{kl} = u_{kl}u_{ij}$. Therefore $C(S_3^+)$ is commutative. In a more sophisticated manner, we will use similar arguments later to show that generators of certain C^*-algebras commute.

For $n = 4$, the surjective *-homomorphism

$$\varphi : C(S_4^+) \to C^*(p, q \,|\, p = p^* = p^2, q = q^* = q^2),$$
$$u \mapsto \begin{pmatrix} p & 1-p & 0 & 0 \\ 1-p & p & 0 & 0 \\ 0 & 0 & q & 1-q \\ 0 & 0 & 1-q & q \end{pmatrix}$$

yields the non-commutativity of $C(S_4^+)$, since $C^*(p, q \,|\, p = p^* = p^2, q = q^* = q^2)$ is a non-commutative C^*-algebra. Regarding $n > 4$, we can use the *-homomorphism φ from above, where we furthermore put $\varphi(u_{ii}) = 1$ and $\varphi(u_{ij}) = 0$ for $i, j > 4$, $i \neq j$, to see that $C(S_n^+)$ is non-commutative.

Quantum subgroups of the quantum symmetric group are called *quantum permutation groups.* The next proposition deals with another product of quantum groups, where we restrict ourselves to quantum permutation groups.

Proposition 1.1.11 ([14])**.** *Let $G = (C(G), u)$, $H = (C(H), v)$ be quantum permutation groups with $u \in M_n(C(G))$, $v \in M_m(C(H)$. The* free wreath product *$G \wr_* H := (C(G) *_w C(H), w)$ is a quantum permutation group with $w = (w_{ia,jb}) = (u_{ij}^{(a)} v_{ab}) \in M_{nm}(C(G) *_w C(H))$, where $u^{(a)} = (u_{ij}^{(a)})$ are copies of u. Here $C(G) *_w C(H)$ is the universal C^*-algebra generated by $u_{ij}^{(a)}, v_{ab}$ with the relations of the C^*-algebra $C(G)^{*m} * C(H)$, where additionally $u_{ij}^{(a)} v_{ab} = v_{ab} u_{ij}^{(a)}$.*

1.2 Finite graphs

The definitions in this section are well-known and can for example be found in the books [18], [32].

Let $\Gamma = (V, E)$ be a *finite graph without multiple edges*, i.e. finite sets of *vertices* V and *edges* $E \subseteq V \times V$. A *loop* in a graph is an edge $(i, i) \in E$. A graph is called *undirected*, if for all $(i, j) \in E$, we also have $(j, i) \in E$. The *order* of a graph denotes the number of elements in V, i.e. the number of vertices in the graph.

For the rest of this work, we assume that Γ is a finite graph without multiple edges.

The following definitions are for undirected graphs Γ. Let $v \in V$. The vertex $u \in V$ is called a *neighbor* of v, if $(v, u) \in E$. A *path* of length m joining two vertices $i, k \in V$ is a sequence of vertices $a_0, a_1, \ldots, a_m$ with $i = a_0$, $a_m = k$ such that $(a_n, a_{n+1}) \in E$ for $0 \leq n \leq m - 1$. A *cycle* of length m is a path of length m where $a_0 = a_m$ and all other vertices in the sequence are distinct. The *degree* $\deg v$ of a vertex $v \in V$ denotes the number of edges in Γ incident with v. We say that a graph Γ is *k-regular* or a *regular graph of degree k* for some $k \in \mathbb{N}_0$, if $\deg v = k$ for all $v \in V$. The 3-regular graphs are also called *cubic* graphs.

Example 1.2.1. The Petersen graph is a finite, undirected graph on ten vertices and is defined by the drawing in Figure 1.1. We will denote the Petersen graph by P in this work. It is an important example with interesting properties, see for example [32].

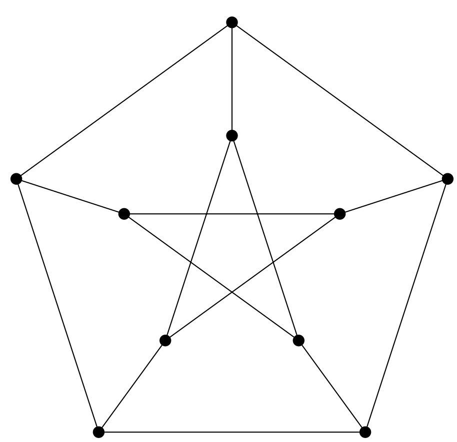


Figure 1.1: The Petersen Graph

Definition 1.2.2. Let Γ be an undirected graph. We define the *girth* $g(\Gamma)$ of a graph to be the length of a smallest cycle it contains.

For example, we have $g(\mathrm{P}) = 5$, where P denotes the Petersen Graph (see Figure 1.1).

Definition 1.2.3. Let Γ be an undirected graph. A *clique* is a subset of vertices $W_1 \subseteq V$ such that any vertices in W_1 are adjacent. A clique, such that there is no clique with more vertices is called *maximal clique*. The *clique number* of Γ is the number of vertices of a maximal clique. On the other hand, an *independent set* is a subset $W_2 \subseteq V$ such that no vertices in W_2 are adjacent.

Definition 1.2.4. Let $\Gamma = (V, E)$ be an undirected graph without multiple edges and without loops. The *complement* of Γ is a graph $\Gamma^c = (V, E')$, where $E' = \{(i, j) \in V \times V \mid (i, j) \notin E, i \neq j\}$.

We recall the definition of line graphs and incidence graphs since those constructions will be used explicitely in this thesis.

Definition 1.2.5. Let Γ be an undirected graph. The *line graph* $L(\Gamma)$ of Γ is the graph whose vertices correspond to edges of Γ and whose vertices are connected if and only if the corresponding edges are incident in Γ (i.e. are connected by a vertex).

See for example [23] for the next definition.

Definition 1.2.6. Given $c, d \in \mathbb{N}$, a *configuration* (P, L) consists of points $P = \{p_1, \ldots, p_a\}$ and lines $L = \{L_1, \ldots, L_b\}$ in a plane, such that

(i) there are c points on each line and d lines through each point,

(ii) two different lines intersect each other at most once,

(iii) two different points are connected by one line at most.

Definition 1.2.7. Let (P, L) be a configuration with points $P = \{p_1, \ldots, p_a\}$ and lines $L = \{L_1, \ldots, L_b\}$. Then the *incidence graph* of the configuration is a bipartite graph consisting of vertices $P \cup L$. Here P and L are independent sets (in the sense of Definition 1.2.3) and we have an edge between p_j and L_k if and only if p_j is adjacent to L_k in the configuration.

Example 1.2.8. The Fano plane is a well-known configuration with $c = 3, d = 3$. The Heawood graph is the incidence graph of the Fano plane (Figure 1.2), see [23]. We denote by L_{ijk} the line of the Fano plane that goes through the points i, j and k.

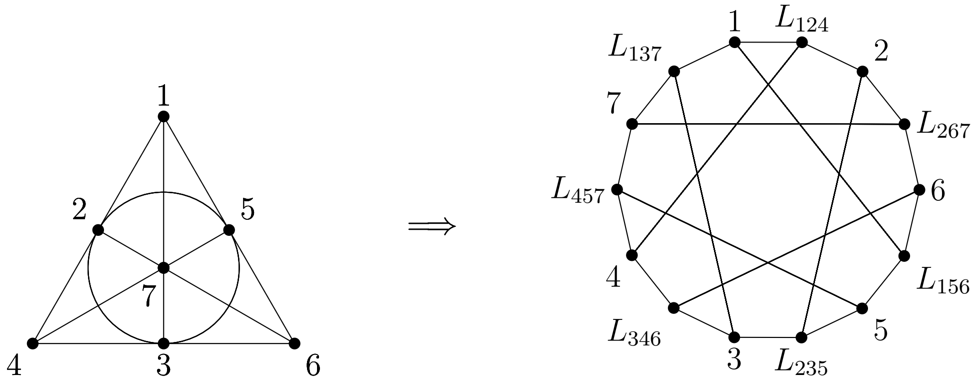


Figure 1.2: The Heawood graph is the incidence graph of the Fano plane

The following definitions concern the automorphism group of a graph. The automorphism group will be generalized to the quantum setting in the next chapter. We start with the definition of the adjacency matrix.

Definition 1.2.9. Let Γ be a finite graph of order n, without multiple edges. The *adjacency matrix* $\varepsilon \in M_n(\{0,1\})$ is the matrix where $\varepsilon_{ij} = 1$ if $(i,j) \in E$ and $\varepsilon_{ij} = 0$ otherwise.

Definition 1.2.10. Let $\Gamma = (V, E)$ be a finite graph without multiple edges. A *graph automorphism* is a bijection $\sigma : V \to V$ such that $(i,j) \in E$ if and only if $(\sigma(i), \sigma(j)) \in E$. The set of all graph automorphisms of Γ forms a group, the *automorphism group* $\mathrm{Aut}(\Gamma)$. If Γ has n vertices, we can view $\mathrm{Aut}(\Gamma)$ as a subgroup of the symmetric group S_n, in the following way.

$$\mathrm{Aut}(\Gamma) = \{\sigma \in S_n \mid \sigma\varepsilon = \varepsilon\sigma\} \subseteq S_n.$$

Here ε denotes the adjacency matrix of the graph.

Vertex-transitive graphs are graphs such that their automorphism groups acts transitively on their vertex set.

Definition 1.2.11. Let $\Gamma = (V, E)$ be a finite graph without multiple edges. We say that Γ is *vertex-transitive*, if for all $v, w \in V$ there exists $\varphi \in \mathrm{Aut}(\Gamma)$ with $\varphi(v) = w$.

Remark 1.2.12. Together with Vogeli and Weber, the author introduced the notion *uniformly vertex-transitive*, which is a stronger version of vertex-transitivity. See [51] for more on this.

A way to get more examples of graphs is to consider graph products. In Section 2.2, we will discuss the quantum automorphism groups of the products given in the next definition.

Definition 1.2.13. Let $\Gamma_1 = (V_{\Gamma_1}, E_{\Gamma_1})$, $H = (V_{\Gamma_2}, E_{,\Gamma_2})$ be finite graphs. Denote by $\varepsilon_{\Gamma_1} \in \mathrm{M}_n(\mathbb{C})$, $\varepsilon_{\Gamma_2} \in \mathrm{M}_m(\mathbb{C})$ the adjacency matrices of Γ_1 and Γ_2, respectively. We have the following graph products.

(i) The *cartesian product* $\Gamma_1 \square \Gamma_2$ is the graph with vertex set $V_{\Gamma_1} \times V_{\Gamma_2}$, where (u_1, u_2) and (v_1, v_2) are connected if and only if $(u_1 = v_1$ and $(u_2, v_2) \in E_{\Gamma_2})$ or $((u_1, v_1) \in E_{\Gamma_1}$ and $u_2 = v_2)$. For the adjacency matrix, we get

$$\varepsilon_{\Gamma_1 \square \Gamma_2} = \varepsilon_{\Gamma_1} \otimes \mathrm{id}_{\mathrm{M}_m(\mathbb{C})} + \mathrm{id}_{\mathrm{M}_n(\mathbb{C})} \otimes \varepsilon_{\Gamma_2}.$$

(ii) The *tensor product* $\Gamma_1 \times \Gamma_2$ is the graph with vertex set $V_{\Gamma_1} \times V_{\Gamma_2}$, where (u_1, u_2) and (v_1, v_2) are connected if and only if $(u_1, v_1) \in E_{\Gamma_1}$ and $(u_2, v_2) \in E_{\Gamma_2}$. Therefore

$$\varepsilon_{\Gamma_1 \times \Gamma_2} = \varepsilon_{\Gamma_1} \otimes \varepsilon_{\Gamma_2}.$$

(iii) The *strong product* $\Gamma_1 \boxtimes \Gamma_2$ is the graph with vertex set $V_{\Gamma_1} \times V_{\Gamma_2}$, where (u_1, u_2) and (v_1, v_2) are connected if and only if $(u_1 = v_1$ or $(u_2, v_2) \in E_{\Gamma_2})$ and $(u_2 = v_2$ or $(u_1, v_1) \in E_{\Gamma_1})$. This yields

$$\varepsilon_{\Gamma_1 \boxtimes \Gamma_2} = (\varepsilon_{\Gamma_1} + \mathrm{id}_{\mathrm{M}_n(\mathbb{C})}) \otimes (\varepsilon_{\Gamma_2} + \mathrm{id}_{\mathrm{M}_m(\mathbb{C})}) - \mathrm{id}_{\mathrm{M}_{nm}(\mathbb{C})}.$$

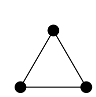
(iv) The *lexicographic product* $\Gamma_1 \circ \Gamma_2$ is the graph with vertex set $V_{\Gamma_1} \times V_{\Gamma_2}$, where (u_1, u_2) and (v_1, v_2) are connected if and only if $(u_2, v_2) \in E_{\Gamma_2}$ or $((u_1, v_1) \in E_{\Gamma_1}$ and $u_2 = v_2)$. We obtain

$$\varepsilon_{\Gamma_1 \circ \Gamma_2} = \varepsilon_{\Gamma_1} \otimes \mathrm{id}_{\mathrm{M}_m(\mathbb{C})} + J \otimes \varepsilon_{\Gamma_2},$$

where J denotes the all-ones matrix.

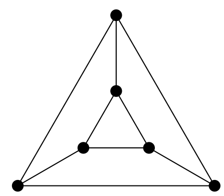
We give some examples of graphs that we obtain by taking products of complete graphs

Example 1.2.14. Take the complete graphs K_2 and K_3, i.e.

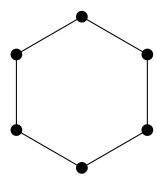
K_2 = , K_3 =  .

Then, we have

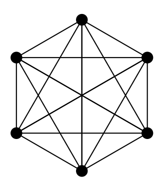
(i) the cartesian product

$$K_2 \square K_3 \quad = \quad$$  ,

(ii) the tensor product

$$K_2 \times K_3 \quad = \quad$$  ,

(iii) and the strong product

$$K_2 \boxtimes K_3 \quad = \quad$$  .

An important case for the lexicographic product is the disjoint union of n copies of a graph Γ. One gets the disjoint union of n copies of Γ by considering the lexicographic product of Γ with the graph on n points with no edges, i.e.

$$\Gamma \circ K_n^c = n\Gamma.$$

Next, we introduce distance-regular and distance-transitive graphs, see for example [18]. Those will be important in Chapters 3 and 4.

Definition 1.2.15. Let Γ be an undirected graph and $v, w \in V$.

(a) The *distance* $d(v, w)$ of two vertices is the length of a shortest path connecting v and w.

(b) The *diameter* of Γ is the greatest distance between any two vertices v, w.

Definition 1.2.16. Let Γ be a regular graph. We say that Γ is *distance-regular* if for two vertices v, w, the number of vertices at distance k to v and at distance l to w only depend on k, l and $d(v, w)$.

Example 1.2.17. The Petersen graph (Figure 1.1) and the Heawood graph (Figure 1.2) are distance-regular, see for example [32].

Definition 1.2.18. Let $\Gamma = (V, E)$ be a k-regular graph on n vertices. We say that Γ is *strongly regular* if there exist λ, $\mu \in \mathbb{N}_0$ such that

(i) adjacent vertices have λ common neighbors,

(ii) non-adjacent vertices have μ common neighbors.

In this case, we say that Γ has parameters (n, k, λ, μ).

Example 1.2.19. The Petersen graph (Figure 1.1) is strongly regular with parameters $(10, 3, 0, 1)$, see for example [32].

Remark 1.2.20. Strongly regular graphs are exactly the distance-regular graphs with diameter two.

The next definition introduces the intersection array. The intersection array is important to understand the structure of a distance-regular graph.

Definition 1.2.21. Let Γ be a distance-regular graph with diameter d. The *intersection array* of Γ is a sequence of integers $\{b_0, b_1, \dots, b_{d-1}; c_1, c_2, \dots, c_d\}$, such that for any two vertices v, w at distance $d(v, w) = i$, there are exactly b_i neighbors of w at distance $i + 1$ to v and exactly c_i neighbors of w at distance $i - 1$ to v.

Example 1.2.22. The Petersen graph (Figure 1.1) has intersection array $\{3, 2; 1, 1\}$ and the Heawood graph (Figure 1.2) has intersection array $\{3, 2, 2; 1, 1, 3\}$, see [32].

We will now give the definition of distance-transitive graphs.

Definition 1.2.23. Let Γ be a regular graph. We say that Γ is *distance-transitive* if for all $(i, k), (j, l) \in V \times V$ with $d(i, k) = d(j, l)$, there is an automorphism $\varphi \in \mathrm{Aut}(\Gamma)$ with $\varphi(i) = j, \varphi(k) = l$.

Example 1.2.24. The Petersen graph (Figure 1.1) and the Heawood graph (Figure 1.2) are distance-transitive, see [32].

Remark 1.2.25. Let Γ be a distance-transitive graph and let $v, w \in V$. Since we have an automorphism $\varphi \in \mathrm{Aut}(\Gamma)$ with $\varphi(v) = x, \varphi(w) = y$ for every pair of vertices x, y with $d(x, y) = d(v, w)$, we see that the number of vertices at distance k to v and at distance l to w only depend on k, l and $d(v, w)$. Thus, we see that every distance-transitive graph is distance-regular.

Chapter 2

The quantum automorphism group of a graph

This chapter concerns the definition of quantum automorphism groups of graphs and related objects such as the quantum orbital algebra. The quantum automorphism group of a graph constitutes the main object of the thesis. There are actually two definitions of quantum automorphism groups, denoted $G^+_{aut}(\Gamma)$ and $G^*_{aut}(\Gamma)$. We start with those definitions in Section 2.1 and see how $G^+_{aut}(\Gamma)$ behaves when taking graph products in Section 2.2. We review the intertwiner spaces of quantum automorphism groups of graphs in Section 2.3. In the subsequent section, we use the quantum orbital algebra to get some further relations on the generators of $G^+_{aut}(\Gamma)$ if the graph has certain properties. At last, in Section 2.5, we compare $G^+_{aut}(\Gamma)$ and $G^*_{aut}(\Gamma)$ and give an example where $\mathrm{Aut}(\Gamma) \neq G^*_{aut}(\Gamma) \neq G^+_{aut}(\Gamma)$.

2.1 Definitions and basic properties

In 2005, Banica [3] gave the following definition of a quantum automorphism group of a finite graph.

Definition 2.1.1. Let $\Gamma = (V, E)$ be a finite graph on n vertices $V = \{1, ..., n\}$. The *quantum automorphism group* $G^+_{aut}(\Gamma)$ is the compact matrix quantum group $(C(G^+_{aut}(\Gamma)), u)$, where $C(G^+_{aut}(\Gamma))$ is the universal C^*-algebra with generators u_{ij}, $1 \leq i, j \leq n$ and relations

$$u_{ij} = u_{ij}^* = u_{ij}^2, \qquad 1 \leq i, j, k \leq n, \tag{2.1.1}$$

$$\sum_{l=1}^n u_{il} = 1 = \sum_{l=1}^n u_{li}, \qquad 1 \leq i \leq n, \tag{2.1.2}$$

$$u\varepsilon = \varepsilon u, \tag{2.1.3}$$

where (2.1.3) is nothing but $\sum_k u_{ik}\varepsilon_{kj} = \sum_k \varepsilon_{ik}u_{kj}$.

To justify the definition, we have to show that $G^+_{aut}(\Gamma)$ is a compact matrix quantum group. For this, we need u, u^t to be invertible. This is true by Relations (2.1.1), (2.1.2), since we know $u^{-1} = u^t$ for the quantum symmetric group S_n^+. It remains to show that $\Delta : C(G^+_{aut}(\Gamma)) \to C(G^+_{aut}(\Gamma)) \otimes C(G^+_{aut}(\Gamma))$, $u_{ij} \mapsto \sum_k u_{ik} \otimes u_{kj}$ is a *-homomorphism. For this, it is enough to prove $\Delta(\sum_k u_{ik}\varepsilon_{kj}) = \Delta(\sum_k \varepsilon_{ik}u_{kj})$, since we know that S_n^+ is a compact matrix quantum group.

Lemma 2.1.2 ([3])**.** *Let $\Gamma = (V, E)$ be a finite graph on n vertices, let u_{ij}, $1 \leq i, j \leq n$, be the generators of $C(G^+_{aut}(\Gamma))$ and let Δ be the map*

$$\Delta : C(G^+_{aut}(\Gamma)) \to C(G^+_{aut}(\Gamma)) \otimes C(G^+_{aut}(\Gamma)),$$
$$u_{ij} \mapsto \sum_k u_{ik} \otimes u_{kj}.$$

Then, we have $\Delta(\sum_k u_{ik}\varepsilon_{kj}) = \Delta(\sum_k \varepsilon_{ik}u_{kj})$.

Proof. It holds

$$\begin{aligned}
\Delta\left(\sum_k u_{ik}\varepsilon_{kj}\right) &= \sum_k \varepsilon_{kj} \sum_s u_{is} \otimes u_{sk} \\
&= \sum_s u_{is} \otimes \left(\sum_k u_{sk}\varepsilon_{kj}\right) \\
&= \sum_s u_{is} \otimes \sum_k \varepsilon_{sk}u_{kj} \\
&= \sum_{s,k} \varepsilon_{sk}(u_{is} \otimes u_{kj})
\end{aligned}$$

and

$$\begin{aligned}
\Delta\left(\sum_k \varepsilon_{ik}u_{kj}\right) &= \sum_k \varepsilon_{ik} \sum_s u_{ks} \otimes u_{sj} \\
&= \sum_s \left(\sum_k \varepsilon_{ik}u_{ks}\right) \otimes u_{sj} \\
&= \sum_s \left(\sum_k u_{ik}\varepsilon_{ks}\right) \otimes u_{sj} \\
&= \sum_{k,s} \varepsilon_{ks}(u_{ik} \otimes u_{sj}).
\end{aligned}$$

Thus, we have $\Delta(\sum_k u_{ik}\varepsilon_{kj}) = \Delta(\sum_k \varepsilon_{ik}u_{kj})$. □

The previous lemma now yields that $G^+_{aut}(\Gamma)$ is a compact matrix quantum group. The following proposition can for example be found in [21].

Proposition 2.1.3. *Let* Γ *be a finite graph. In* $C(S_n^+)$*, Relation* (2.1.3) *is equivalent to the relations*

$$u_{ij}u_{kl} = u_{kl}u_{ij} = 0, \qquad (i,k) \notin E, (j,l) \in E, \tag{2.1.4}$$

$$u_{ij}u_{kl} = u_{kl}u_{ij} = 0, \qquad (i,k) \in E, (j,l) \notin E. \tag{2.1.5}$$

Proof. Assume that Relation (2.1.3) holds. Then, we have

$$\sum_{t;(t,l)\in E} u_{it} = \sum_{s;(i,s)\in E} u_{sl}$$

for all $1 \leq i, l \leq n$. Let $(i,k) \notin E, (j,l) \in E$. It holds

$$u_{ij}u_{kl} = u_{ij}\left(\sum_{t;(t,l)\in E} u_{it}\right)u_{kl},$$

since $u_{ij}u_{it} = \delta_{jt}u_{ij}$, where we know that j is part of the sum since we have $(j,l) \in E$. By assumption, we obtain

$$u_{ij}u_{kl} = u_{ij}\left(\sum_{t;(t,l)\in E} u_{it}\right)u_{kl} = u_{ij}\left(\sum_{s;(i,s)\in E} u_{sl}\right)u_{kl}.$$

Because of $(i,k) \notin E$, we see that k is not part of the sum above and since it holds $u_{sl}u_{kl} = \delta_{sk}u_{kl}$, we deduce

$$u_{ij}u_{kl} = u_{ij}\left(\sum_{s;(i,s)\in E} u_{sl}\right)u_{kl} = 0.$$

Applying the involution to $u_{ij}u_{kl} = 0$ yields $u_{kl}u_{ij} = 0$ and we get Relation (2.1.4). Deriving Relation (2.1.5) is completely analogous.

Now, assume that Relations (2.1.4) and (2.1.5) hold. Then we get

$$\begin{aligned}\sum_{t;(t,l)\in E} u_{it} &= \left(\sum_{t;(t,l)\in E} u_{it}\right)\left(\sum_{s=1}^{n} u_{sl}\right)\\ &= \left(\sum_{t;(t,l)\in E} u_{it}\right)\left(\sum_{s;(s,i)\in E} u_{sl}\right)\\ &= \sum_{s;(s,i)\in E}\left(\sum_{t;(t,l)\in E} u_{it}\right) u_{sl}\\ &= \sum_{s;(s,i)\in E}\left(\sum_{t=1}^{n} u_{it}\right) u_{sl}\\ &= \sum_{s;(s,i)\in E} u_{sl}\end{aligned}$$

for all $1 \leq i, l \leq n$. Here we used Relations (2.1.2), (2.1.4) and (2.1.5). □

When working with the generators and relations of $G^+_{aut}(\Gamma)$, it is usually more convenient to consider Relations (2.1.4), (2.1.5) instead of Relation (2.1.3). These relations are frequently used in Chapters 3 and 4. We also immediately get the following lemma.

Lemma 2.1.4. *Let $\Gamma = (V, E)$ be a finite graph without multiple edges and without loops. Then $G^+_{aut}(\Gamma) = G^+_{aut}(\Gamma^c)$.*

Proof. This follows from Proposition 2.1.3, since the Relations (2.1.4), (2.1.5) stay the same if we go over to the complement Γ^c. □

There is another definition of quantum automorphism groups by Bichon [13] in 2003. This is a quantum subgroup of the one defined by Banica.

Definition 2.1.5. Let Γ be a finite graph on n vertices $V = \{1, ..., n\}$. The *quantum automorphism group* $G^*_{aut}(\Gamma)$ is the compact matrix quantum group $(C(G^*_{aut}(\Gamma)), u)$, where $C(G^*_{aut}(\Gamma))$ is the universal C^*-algebra with generators u_{ij}, $1 \leq i, j \leq n$, Relations (2.1.1) – (2.1.3) and

$$u_{ij}u_{kl} = u_{kl}u_{ij}, \qquad (i,k),(j,l) \in E. \tag{2.1.6}$$

We compare the Definitions 2.1.1 and 2.1.5 in Section 2.5. The quantum group $G^+_{aut}(\Gamma)$ is used more often nowadays, for example because Lemma 2.1.4 holds for $G^+_{aut}(\Gamma)$ and $\mathrm{Aut}(\Gamma)$, but not for $G^*_{aut}(\Gamma)$. Furthermore, the relations of $G^+_{aut}(\Gamma)$ match the ones of quantum isomorphisms of graphs, see Section 7.

The next lemma justifies that both definitions are generalizations of the automorphism group of a graph.

Lemma 2.1.6. *Let* Γ *be a finite graph with adjacency matrix* ε*. Let* (A, u) *be the compact matrix quantum group, where*

$$\begin{aligned} A &:= C^*(u_{ij} \mid u_{ij} = u_{ij}^* = u_{ij}^2,\ \sum_{k=1}^n u_{ik} = 1 = \sum_{k=1}^n u_{ki},\ u\varepsilon = \varepsilon u,\ u_{ij}u_{kl} = u_{kl}u_{ij}) \\ &= C(G^+_{aut}(\Gamma))/\langle u_{ij}u_{kl} = u_{kl}u_{ij}\rangle. \end{aligned}$$

Then

$$C(\mathrm{Aut}(\Gamma)) = A.$$

Proof. Since the C^*-algebra A is commutative, the Gelfand-Naimark Theorem yields $A = C(\mathrm{Spec}(A))$. By Timmermann's book [54, Proposition 5.1.3] we know that $\mathrm{Spec}(A)$ is a group with group law $m : \mathrm{Spec}(A) \times \mathrm{Spec}(A) \to \mathrm{Spec}(A)$, $(\varphi_1, \varphi_2) \mapsto (\varphi_1 \otimes \varphi_2) \circ \Delta$ and by definition we have

$$\mathrm{Spec}(A) = \{\varphi : A \to \mathbb{C} \text{ unital *-homomorphism}\}.$$

Let $\sigma \in \mathrm{Aut}(\Gamma) \subseteq M_n(\{0, 1\})$ and define $\varphi_\sigma(u_{ij}) := \sigma_{ij}$ for all $1 \le i, j \le n$. We show $\mathrm{Spec}(A) = \{\varphi_\sigma \mid \sigma \in \mathrm{Aut}(\Gamma)\}$. The entries σ_{ij} obviously commute, and since σ is a permutation matrix, we have

$$\sum_{k=1}^n \sigma_{ik} = 1 = \sum_{k=1}^n \sigma_{ki} \quad \text{and} \quad \sigma_{ij} = \sigma_{ij}^* = \sigma_{ij}^2.$$

We also have $\sigma\varepsilon = \varepsilon\sigma$ by Definition 1.2.10, and therefore φ_σ is a *-homomorphism. Because of

$$\varphi_\sigma(1) = \varphi_\sigma\left(\sum_{k=1}^n u_{ik}\right) = \sum_{k=1}^n \varphi_\sigma(u_{ik}) = \sum_{k=1}^n \sigma_{ik} = 1,$$

φ_σ is also unital. Hence $\varphi_\sigma \in \mathrm{Spec}(A)$.

Now, let $\varphi \in \mathrm{Spec}(A)$. Define $\sigma_{ij} := \varphi(u_{ij})$. We then have $\varphi_\sigma = \varphi$. Since φ is a *-homomorphism, it holds $\sigma_{ij}^2 = \sigma_{ij}^* = \sigma_{ij}$ and therefore $\sigma_{ij} \in \{0, 1\}$. We also get

$$\sum_{k=1}^n \sigma_{ik} = 1 = \sum_{k=1}^n \sigma_{ki} \quad \text{and} \quad \sigma\varepsilon = \varepsilon\sigma,$$

which means that σ is a permutation matrix with $\sigma\varepsilon = \varepsilon\sigma$. Thus $\sigma \in \mathrm{Aut}(\Gamma)$. In summary, we obtain $\mathrm{Spec}(A) = \{\varphi_\sigma \mid \sigma \in \mathrm{Aut}(\Gamma)\}$.

Now, we have to check that $f : \text{Spec}(A) \to \text{Aut}(\Gamma)$, $\varphi_\sigma \mapsto \sigma$ is a group isomorphism. The map is obviously a bijection, it remains to show $f(m(\varphi_{\sigma_1}, \varphi_{\sigma_2})) = f(\varphi_{\sigma_1}) \circ f(\varphi_{\sigma_2})$. It holds

$$\begin{aligned} m((\varphi_{\sigma_1}, \varphi_{\sigma_2}))(u_{ij}) &= (\varphi_{\sigma_1} \otimes \varphi_{\sigma_2})\Delta(u_{ij}) \\ &= \sum_{k=1}^{n} \varphi_{\sigma_1}(u_{ik}) \otimes \varphi_{\sigma_2}(u_{kj}) \\ &= \sum_{k=1}^{n} (\sigma_1)_{ik}(\sigma_2)_{kj} \\ &= (\sigma_1 \circ \sigma_2)_{ij} \\ &= \varphi_{\sigma_1 \circ \sigma_2}(u_{ij}) \end{aligned}$$

and therefore $m((\varphi_{\sigma_1}, \varphi_{\sigma_2})) = \varphi_{\sigma_1 \circ \sigma_2}$. Thus

$$f(m(\varphi_{\sigma_1}, \varphi_{\sigma_2})) = f(\varphi_{\sigma_1 \circ \sigma_2}) = \sigma_1 \circ \sigma_2 = f(\varphi_{\sigma_1}) \circ f(\varphi_{\sigma_2}).$$

To complete the proof, we have to show $C(\text{Aut}(\Gamma)) = C(\text{Spec}(A))$. By the previous arguments, we know that the map $C(\text{Spec}(A)) \to C(\text{Aut}(\Gamma))$, $g \mapsto g \circ f$ is a group isomorphism. Since both groups have finitely many elements, we directly get that this map is also an isomorphism of compact groups. This yields $C(\text{Aut}(\Gamma)) = C(\text{Spec}(A))$ as compact groups. $\square$

By the previous lemma and Definitions 2.1.1, 2.1.5, we see that

$$\text{Aut}(\Gamma) \subseteq G^*_{aut}(\Gamma) \subseteq G^+_{aut}(\Gamma),$$

i.e. we have surjective *-homomorphisms

$$\begin{array}{ccccc} C(G^+_{aut}(\Gamma)) & \to & C(G^*_{aut}(\Gamma)) & \to & C(\text{Aut}(\Gamma)) \\ u_{ij} & \mapsto & u_{ij} & \mapsto & \varphi_{ij}. \end{array}$$

Here φ_{ij} is the function $\varphi_{ij} : \text{Aut}(\Gamma) \to \mathbb{C}$, $\varphi(\sigma) = \sigma_{ij}$.

The next definition is due to Banica and Bichon [6]. The terminology is used frequently in this thesis.

Definition 2.1.7. Let $\Gamma = (V, E)$ be a finite graph. We say that Γ has *no quantum symmetry* if one of the following, obviously equivalent, conditions hold:

(i) $C(G^+_{aut}(\Gamma))$ is commutative,

(ii) $C(G^+_{aut}(\Gamma)) = C(\text{Aut}(\Gamma))$,

(iii) $\mathrm{Aut}(\Gamma) = G^+_{aut}(\Gamma)$, i.e. the surjection $u_{ij} \mapsto \varphi_{ij}$ above is in fact an isomorphism.

If $C(G^+_{aut}(\Gamma))$ is non-commutative, we say that Γ has *quantum symmetry.*

We give the following first example.

Example 2.1.8. Let $K_n = (V_{K_n}, E_{K_n})$ be the complete graph on n vertices, i.e. $V_{K_n} = \{1, \dots n\}$ and $E_{K_n} = (V \times V) \setminus \{(i,i) \,|\, i \in V\}$. We see that $(i,k) \notin E$ implies $i = k$ and thus Relations (2.1.4), (2.1.5) translate to $u_{ij}u_{il} = 0$, $u_{ji}u_{li} = 0$ for $j \neq l$. Those relations follow from Relations (2.1.1), (2.1.2) by Remark 1.1.9. Therefore, we have $G^+_{aut}(K_n) = S^+_n$. By Remark 1.1.10, we know that $C(S^+_n)$ is commutative for $n \leq 3$ and non-commutative for $n \geq 4$. Therefore, K_n does not have quantum symmetry for $n \leq 3$ whereas for $n \geq 4$, it does have quantum symmetry. Regarding $G^*_{aut}(K_n)$, Relation (2.1.6) together with Remark 1.1.9 shows that the generators of $G^*_{aut}(K_n)$ commute. Thus $G^*_{aut}(K_n) = S_n$.

We give more examples of quantum automorphism groups, now for the graphs on four vertices.

Example 2.1.9. In [52], Weber and the author computed the quantum automorphism groups of all undirected graphs on four vertices. We have the following table, see also [52]. Note that H^+_n is the hyperoctahedral quantum group, see [9].

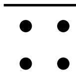
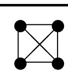
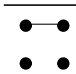
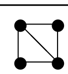
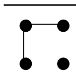
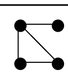
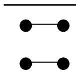
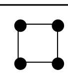
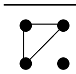
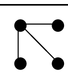
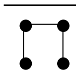
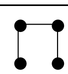
Γ	Γ^c	$\mathrm{Aut}(\Gamma)$	$G^*_{aut}(\Gamma^c)$	$G^*_{aut}(\Gamma)$	$G^+_{aut}(\Gamma)$
		S_4	S_4	S_4^+	S_4^+
		$\mathbb{Z}_2 \times \mathbb{Z}_2$	$\mathbb{Z}_2 \times \mathbb{Z}_2$	$\widehat{\mathbb{Z}_2 * \mathbb{Z}_2}$	$\widehat{\mathbb{Z}_2 * \mathbb{Z}_2}$
		$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$
		D_4	D_4	H_2^+	H_2^+
		S_3	S_3	S_3	S_3
		$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$

Table 2.1: Quantum automorphism groups of all undirected graphs on four vertices.

An action of a compact matrix quantum group on a graph is an action on the functions on the vertices, but with additional structure. This concept was introduced by Banica and Bichon [3, 13]. It is used in Chapter 5 to compute the quantum automorphism group of folded cube graphs.

Definition 2.1.10. Let $\Gamma = (V, E)$ be a finite graph and G be a compact matrix quantum group. Recall that

$$C(V) = C^*(e_1, \dots, e_{|V|} \,|\, e_i = e_i^* = e_i^2, \sum_{j=1}^{|V|} e_j = 1).$$

An *action of* G *on* Γ is an action of G on $C(V)$ such that the magic unitary matrix $(v_{ij})_{1 \le i,j \le |V|}$ associated to the formular

$$\alpha(e_i) = \sum_{j=1}^{|V|} e_j \otimes v_{ji}$$

commutes with the adjacency matrix, i.e $v\varepsilon = \varepsilon v$.

Remark 2.1.11. If G acts on a graph Γ, then we have a surjective *-homomorphism $\varphi : C(G^+_{aut}(\Gamma)) \to C(G)$, $u \mapsto v$.

The following theorem shows that commutation with the magic unitary u yields invariant subspaces.

Theorem 2.1.12 (Theorem 2.3 of [3])**.** *Let* $\alpha : C(X_n) \to C(X_n) \otimes C(G), \alpha(e_i) = \sum_j e_j \otimes v_{ji}$ *be an action, where* G *is a compact matrix quantum group and let* K *be a linear subspace of* $C(X_n)$*. The matrix* (v_{ij}) *commutes with the projection onto* K *if and only if* $\alpha(K) \subseteq K \otimes C(G)$*.*

Looking at the spectral decomposition of the adjacency matrix, we see that this action preserves the eigenspaces of the adjacency matrix.

Corollary 2.1.13. *Let* $\Gamma = (V, E)$ *be an undirected finite graph with adjacency matrix* ε*. The action* $\alpha : C(V) \to C(V) \otimes C(G^+_{aut}(\Gamma))$, $\alpha(e_i) = \sum_j e_j \otimes u_{ji}$*, preserves the eigenspaces of* ε*, i.e.* $\alpha(E_\lambda) \subseteq E_\lambda \otimes C(G^+_{aut}(\Gamma))$ *for all eigenspaces* E_λ*.*

Proof. It follows from the spectral decomposition that every projection P_{E_λ} onto E_λ is a polynomial in ε. Thus it commutes with the fundamental representation u and Theorem 2.1.12 yields the assertion. □

2.2 Quantum automorphism groups of graph products

We give an overview of the results regarding the quantum automorphism group of graph products. The products are defined in Definition 1.2.13. Most of this was worked out in [6]. We added the strong product (item (*iii*)), where the proofs are similar to those appearing in [6].

Proposition 2.2.1. *Let Γ_1 and Γ_2 be finite graphs without multiple edges. There are surjective *-homomorphisms between the following C^*-algebras*

(i) $C(G^+_{aut}(\Gamma_1 \Box \Gamma_2)) \to C(G^+_{aut}(\Gamma_1)) \otimes_{\max} C(G^+_{aut}(\Gamma_2))$,

(ii) $C(G^+_{aut}(\Gamma_1 \times \Gamma_2)) \to C(G^+_{aut}(\Gamma_1)) \otimes_{\max} C(G^+_{aut}(\Gamma_2))$,

(iii) $C(G^+_{aut}(\Gamma_1 \boxtimes \Gamma_2)) \to C(G^+_{aut}(\Gamma_1)) \otimes_{\max} C(G^+_{aut}(\Gamma_2))$,

(iv) $C(G^+_{aut}(\Gamma_1 \circ \Gamma_2)) \to C(G^+_{aut}(\Gamma_1)) *_w C(G^+_{aut}(\Gamma_2))$.

Proof. As mentioned in Definition 1.2.13, we have the following adjacency matrices for those graph products

$$\begin{aligned}
\varepsilon_{\Gamma_1 \Box \Gamma_2} &= \varepsilon_{\Gamma_1} \otimes \mathrm{id}_{\mathrm{M}_m(\mathbb{C})} + \mathrm{id}_{\mathrm{M}_n(\mathbb{C})} \otimes \varepsilon_{\Gamma_2},\\
\varepsilon_{\Gamma_1 \times \Gamma_2} &= \varepsilon_{\Gamma_1} \otimes \varepsilon_{\Gamma_2},\\
\varepsilon_{\Gamma_1 \boxtimes \Gamma_2} &= (\varepsilon_{\Gamma_1} + \mathrm{id}_{\mathrm{M}_n(\mathbb{C})}) \otimes (\varepsilon_{\Gamma_2} + \mathrm{id}_{\mathrm{M}_m(\mathbb{C})}) - \mathrm{id}_{\mathrm{M}_{nm}(\mathbb{C})},\\
\varepsilon_{\Gamma_1 \circ \Gamma_2} &= \varepsilon_{\Gamma_1} \otimes \mathrm{id}_{\mathrm{M}_m(\mathbb{C})} + J \otimes \varepsilon_{\Gamma_2}.
\end{aligned}$$

Here $\varepsilon_{\Gamma_1} \in \mathrm{M}_n(\mathbb{C})$, $\varepsilon_{\Gamma_2} \in \mathrm{M}_m(\mathbb{C})$ denote the adjacency matrices of Γ_1, Γ_2, respectively, $J \in \mathrm{M}_n(\mathbb{C})$ denotes the all-ones matrix. Now, let u_{ij}, v_{kl} be as in Definition 1.1.5, i.e. $u_{ij}v_{kl} = v_{kl}u_{ij}$ for all i, j, k, l, u_{ij} fulfill the relations of $C(G^+_{aut}(\Gamma_1))$ and v_{kl} fulfill the relations of $C(G^+_{aut}(\Gamma_2))$. The matrix

$$u \otimes v = (u_{ij}v_{kl})_{(ik,jl)}$$

is a magic unitary that commutes with $\varepsilon_{\Gamma_1 \Box \Gamma_2}, \varepsilon_{\Gamma_1 \times \Gamma_2}$ and $\varepsilon_{\Gamma_1 \boxtimes \Gamma_2}$, since u commutes with ε_{Γ_1} and v commutes with ε_{Γ_2}. Therefore, we obtain the required *-homomorphisms for (i)–(iii). Summing over i and over k, respectively, shows that they are surjective. Similarly, the matrix $(w_{ia,jb})_{(ia,jb)} = (u^{(a)}_{ij} v_{ab})_{(ia,jb)}$ from Proposition 1.1.11 is magic unitary and commutes with $\varepsilon_{\Gamma_1 \circ \Gamma_2}$, since u commutes with ε_{Γ_1} and J, v commute with ε_{Γ_2}. We conclude that there is a *-homomorphism $C(G^+_{aut}(\Gamma_1 \circ \Gamma_2)) \to C(G^+_{aut}(\Gamma_1)) *_w C(G^+_{aut}(\Gamma_2))$, where summing over i and over b, respectively, yields surjectivity. □

The next theorem shows that if the spectra of our graphs behave in a certain way, the surjections from above are actually isomorphisms. Recall the quantum group products from Proposition 1.1.5 and Proposition 1.1.11. Parts (i), (ii) and (iv) can be found in [6].

Theorem 2.2.2. *Let $\Gamma_1 = (V_1, E_1)$ and $\Gamma_2 = (V_2, E_2)$ be finite, connected, undirected, regular graphs. Let $\sigma_{\Gamma_1} = \{\lambda_i \,|\, i = 1, \dots, n\}$ be the set of distinct eigenvalues of ε_{Γ_1} and $\sigma_{\Gamma_2} = \{\mu_j \,|\, j = 1, \dots, m\}$ be the set of distinct eigenvalues of ε_{Γ_2}. Then we have* *(i)* $G^+_{aut}(\Gamma_1 \Box \Gamma_2) = G^+_{aut}(\Gamma_1) \times G^+_{aut}(\Gamma_2)$ *if* $\{\lambda_i - \lambda_j \,|\, i,j = 1,\dots,n\} \cap \{\mu_k - \mu_l \,|\, k,l = 1,\dots,m\} = \{0\}$.

(ii) $G^+_{aut}(\Gamma_1 \times \Gamma_2) = G^+_{aut}(\Gamma_1) \times G^+_{aut}(\Gamma_2)$ *if* $\sigma_{\Gamma_1}, \sigma_{\Gamma_2}$ *do not contain* 0 *and* $\{\frac{\lambda_i}{\lambda_j} \,|\, i,j = 1,\dots,n\} \cap \{\frac{\mu_k}{\mu_l} \,|\, k,l = 1,\dots,m\} = \{1\}$.

(iii) $G^+_{aut}(\Gamma_1 \boxtimes \Gamma_2) = G^+_{aut}(\Gamma_1) \times G^+_{aut}(\Gamma_2)$ *if* $\sigma_{\Gamma_1}, \sigma_{\Gamma_2}$ *do not contain* -1 *and* $\{\frac{\lambda_i+1}{\lambda_j+1} \,|\, i,j = 1,\dots,n\} \cap \{\frac{\mu_k+1}{\mu_l+1} \,|\, k,l = 1,\dots,m\} = \{1\}$.

(iv) $G^+_{aut}(\Gamma_1 \circ \Gamma_2) = G^+_{aut}(\Gamma_1) \wr_* G^+_{aut}(\Gamma_2)$ *if* $\{\lambda_1 - \lambda_i \;\,|\, i = 2,\dots,n\} \cap \{-s\mu_j \,|\, j = 1,\dots,m\} = \emptyset$*, where* s *is the order (i.e. number of vertices) and* λ_1 *is the degree of the regular graph* Γ_1 *(i.e.* λ_1 *is the degree one of the vertices, which is the same for all vertices, since* Γ_1 *is regular). Here* Γ_2 *is not necessarily connected.*

Proof. We show (iii), the proof of the other statements is similar (see [6]). Let $\varepsilon_{\Gamma_1} = \sum_i \lambda_i P_i$ be the spectral decomposition of ε_{Γ_1}, where we choose λ_1 to be the degree of Γ_1. Since Γ_1 is connected, λ_1 has multiplicity 1 and thus P_1 is an orthogonal projection onto $\mathbb{C}1$. Similarly, we have $\varepsilon_{\Gamma_2} = \sum_j \mu_j Q_j$, μ_1 being the degree of Γ_2, Q_1 a projection onto $\mathbb{C}1$. We obtain

$$\begin{aligned}\varepsilon_{\Gamma_1 \boxtimes \Gamma_2} &= (\varepsilon_{\Gamma_1} + \mathrm{id}_{\mathrm{M}_n(\mathbb{C})}) \otimes (\varepsilon_{\Gamma_2} + \mathrm{id}_{\mathrm{M}_m(\mathbb{C})}) - \mathrm{id}_{\mathrm{M}_{nm}(\mathbb{C})} \\ &= \left(\sum_i (\lambda_i + 1)P_i\right) \otimes \left(\sum_j (\mu_j + 1)Q_j\right) - \sum_{i,j} P_i \otimes Q_j \\ &= \sum_{i,j} [(\lambda_i + 1)(\mu_j + 1) - 1](P_i \otimes Q_j),\end{aligned}$$

where we used $\sum_i P_i = \mathrm{id}_{\mathrm{M}_n(\mathbb{C})}$, $\sum_j Q_j = \mathrm{id}_{\mathrm{M}_m(\mathbb{C})}$. It holds $\sum_{i,j} P_i \otimes Q_j = \mathrm{id}_{\mathrm{M}_{nm}(\mathbb{C})}$ and $(\lambda_i + 1)(\mu_j + 1) - 1$ are distinct scalars for different tuples (i,j) by assumption. Therefore, $\sum_{i,j}[(\lambda_i+1)(\mu_j+1)-1](P_i \otimes Q_j)$ is the spectral decomposition of $\varepsilon_{\Gamma_1 \boxtimes \Gamma_2}$. Let $v = (v_{ik,jl})_{(ik,jl)}$ be the fundamental representation of $G^+_{aut}(\Gamma_1 \boxtimes \Gamma_2)$. Since $(P_i \otimes Q_j)$ is a polynomial in $\varepsilon_{\Gamma_1 \boxtimes \Gamma_2}$, it commutes with v. Summing over i and over j, respectively, we get that v also commutes with $1 \otimes Q_j$ and $P_i \otimes 1$, respectively. Especially, it commutes with $1 \otimes Q_1$ and $P_1 \otimes 1$. Those are projections onto $C(V_1) \otimes \mathbb{C}1$ and $\mathbb{C}1 \otimes C(V_2)$. Theorem 2.1.12 shows that the action $\alpha : (C(V_1) \otimes C(V_2)) \to (C(V_1) \otimes C(V_2)) \otimes C(G^+_{aut}(\Gamma_1 \boxtimes \Gamma_2))$, $e_x \otimes e_p \mapsto \sum_{y,q} (e_y \otimes e_q) \otimes v_{yx,qp}$ fulfills

$$\alpha(C(V_1) \otimes \mathbb{C}1) \subseteq (C(V_1) \otimes \mathbb{C}1) \otimes C(G^+_{aut}(\Gamma_1 \boxtimes \Gamma_2))$$

and

$$\alpha(\mathbb{C}1 \otimes C(V_2)) \subseteq (\mathbb{C}1 \otimes C(V_2)) \otimes C(G^+_{aut}(\Gamma_1 \boxtimes \Gamma_2)).$$

Therefore, we get

$$\alpha(e_x \otimes 1) = \sum_y (e_y \otimes 1) \otimes w_{yx}$$
$$\alpha(1 \otimes e_p) = \sum_q (1 \otimes e_q) \otimes u_{qp},$$

where $w = (w_{xy})$, $u = (u_{pq})$ are magic unitaries. We deduce

$$\alpha(e_x \otimes e_p) = \sum_{y,q} (e_y \otimes e_q) \otimes u_{qp} w_{yx} = \sum_{y,q} (e_y \otimes e_q) \otimes w_{yx} u_{qp},$$

which shows that w_{xy} and u_{pq} commute. Furthermore, we have $v = u \otimes w$, i.e. $v_{xp,yq} = w_{xy} u_{pq}$. We now show that w_{xy} commutes with ε_{Γ_1} and u_{pq} commutes with ε_{Γ_2}, then we get the surjective *-homomorphism $C(G^+_{aut}(\Gamma_1)) \otimes_{\max} C(G^+_{aut}(\Gamma_2)) \to C(G^+_{aut}(\Gamma_1 \boxtimes \Gamma_2))$, $w'_{xy} \mapsto w_{xy}$, $u'_{pq} \mapsto u_{pq}$. For this, recall

$$(\varepsilon_{\Gamma_1 \boxtimes \Gamma_2})_{ia,jb} = [(\varepsilon_{\Gamma_1})_{ij} + \delta_{ij}][(\varepsilon_{\Gamma_2})_{ab} + \delta_{ab}] - \delta_{ij}\delta_{ab}.$$

We have

$$\begin{aligned}
(v\varepsilon_{\Gamma_1 \boxtimes \Gamma_2})_{ia,jb} &= \sum_{c,d} v_{ia,cd} (\varepsilon_{\Gamma_1 \boxtimes \Gamma_2})_{cd,jb} \\
&= \sum_{c,d} v_{ia,cd} ([(\varepsilon_{\Gamma_1})_{cj} + \delta_{cj}][(\varepsilon_{\Gamma_2})_{db} + \delta_{db}] - \delta_{cj}\delta_{db}) \\
&= \sum_{c,d} w_{ic} u_{ad} ([(\varepsilon_{\Gamma_1})_{cj} + \delta_{cj}][(\varepsilon_{\Gamma_2})_{db} + \delta_{db}] - \delta_{cj}\delta_{db})
\end{aligned}$$

and

$$\begin{aligned}
(\varepsilon_{\Gamma_1 \boxtimes \Gamma_2} v)_{ia,jb} &= \sum_{c,d} (\varepsilon_{\Gamma_1 \boxtimes \Gamma_2})_{ia,cd} v_{cd,jb} \\
&= \sum_{c,d} ([(\varepsilon_{\Gamma_1})_{ic} + \delta_{ic}][(\varepsilon_{\Gamma_2})_{ad} + \delta_{ad}] - \delta_{ic}\delta_{ad}) v_{cd,jb} \\
&= \sum_{c,d} ([(\varepsilon_{\Gamma_1})_{ic} + \delta_{ic}][(\varepsilon_{\Gamma_2})_{ad} + \delta_{ad}] - \delta_{ic}\delta_{ad}) w_{cj} u_{db}.
\end{aligned}$$

Recall that λ_1 is the degree of Γ_1. Therefore

$$\sum_i (\varepsilon_{\Gamma_1})_{ic} = \lambda_1 = \sum_j (\varepsilon_{\Gamma_1})_{cj} \qquad \text{and} \qquad \sum_j \delta_{cj} = 1.$$

We also know $(v\varepsilon_{\Gamma_1\boxtimes\Gamma_2})_{ia,jb} = (\varepsilon_{\Gamma_1\boxtimes\Gamma_2}v)_{ia,jb}$ and summing over i and j yields

$$\sum_d u_{ad}((\varepsilon_{\Gamma_2})_{db} + \delta_{db})(\lambda_1 + 1) - \delta_{db}) = \sum_d((\varepsilon_{\Gamma_2})_{ad} + \delta_{ad})(\lambda_1 + 1) - \delta_{ad})u_{db}.$$

This is equivalent to $\sum_d u_{ad}(\varepsilon_{\Gamma_2})_{db} = \sum_d(\varepsilon_{\Gamma_2})_{ad}u_{db}$, which means that u commutes with ε_{Γ_2}. Similarly, by summing over a and b, we obtain that w commutes with ε_{Γ_1}. Summarizing, we get the surjective *-homomorphism $C(G^+_{aut}(\Gamma_1)) \otimes_{\max} C(G^+_{aut}(\Gamma_2)) \to C(G^+_{aut}(\Gamma_1 \boxtimes \Gamma_2))$, $w'_{xy} \mapsto w_{xy}$, $u'_{pq} \mapsto u_{pq}$, which is inverse to the map in the previous proposition. □

Using this theorem, we can often compute the quantum automorphism group of graph products.

Example 2.2.3. We compute the quantum automorphism groups of the products in Example 1.2.14. For the graphs K_2 and K_3, it holds $\sigma(K_2) = \{-1, 1\}$ and $\sigma(K_3) = \{-1, 2\}$. By Theorem 2.2.2 (i) and (ii), we obtain

(i) $G^+_{aut}(K_2 \square K_3) = S_2 \times S_3$,

(ii) $G^+_{aut}(K_2 \times K_3) = S_2 \times S_3$.

Recall that $K_2 \boxtimes K_3 = K_6$. We see that it is an important that the spectra of the graphs do not contain -1 for the strong product (Theorem 2.2.2 (iii)) as

$$G^+_{aut}(K_2 \boxtimes K_3) = G^+_{aut}(K_6) = S_6^+ \neq S_2 \times S_3.$$

Remark that the conditions of Theorem 2.2.2, (i) – (iii), are not fulfilled if we choose $\Gamma_1 = \Gamma_2$. For example, we do not know the quantum automorphism group of the 4×4 rook's graph which is the cartesian product of K_4 with itself. Still, Proposition 2.2.1 is useful to see whether or not such a graph has quantum symmetry, see for example Proposition 4.1.12. The next proposition shows what the quantum automorphism group of the disjoint union of n copies of a graph looks like.

Proposition 2.2.4 ([5])**.** *Let Γ be a finite graph and let $n\Gamma$ be the disjoint union of n copies of Γ. Then*

$$G^+_{aut}(n\Gamma) = G^+_{aut}(\Gamma) \wr_* S_n^+.$$

One could now ask what happens for the disjoint union of non-isomorphic graphs. For a partial answer, we need the notion of quantum isomorphism. This concept is introduced in Section 7. Using this, we get an answer for non-quantum-isomorphic graphs, see Corollary 7.1.4.

2.3 Intertwiner spaces of quantum automorphism groups of graphs

An alternative way of studying quantum automorphism groups of graphs is by looking at their intertwiner spaces. By Woronowicz's Tannaka-Krein duality [61], there is a one-to-one correspondence of compact matrix quantum groups and tensor categories with duals. The intertwiner spaces of quantum automorphism groups of graphs were first studied by Chassaniol in [22]. Then Mančinska and Roberson gave a full description of the intertwiner spaces in [38]. We give a brief overview in this section. We start with the classical case, see [22].

Definition 2.3.1. Let $G \subseteq S_n$ be a permutation group and identify the elements $g \in G$ with their associated permutation matrices. For $k, l \in \mathbb{N}$, let $C(k,l) := \mathrm{Hom}((\mathbb{C}^n)^{\otimes k}, (\mathbb{C}^n)^{\otimes l})$. The *intertwiner space* $C_G(k,l)$ is defined as follows:

$$C_G(k,l) = \{T \in C(k,l) \,|\, Tg^{\otimes k} = g^{\otimes l}T \text{ for all } g \in G\}.$$

Here $g^{\otimes k}$ is the $n^k \times n^k$ matrix $g^{\otimes k} = (g_{i_1 j_1} \dots g_{i_k j_k})_{i_1 \dots i_k j_1 \dots j_k}$.

Those intertwiners form a tensor category with duals, allowing us to use Tannaka-Krein duality later.

Proposition 2.3.2. *The collection of vector spaces $C_G(k,l)$ is a* tensor category with duals*, in the sense that*

(i) if $T, T' \in C_G(k,l)$, then $\alpha T + \beta T' \in C_G(k,l)$ for all $\alpha, \beta \in \mathbb{C}$,

(ii) if $T \in C_G(k,l), T' \in C_G(s,t)$, then $T \otimes T' \in C_G(k+s, l+t)$,

(iii) if $T \in C_G(k,l), T' \in C_G(s,k)$, then $TT' \in C_G(s,l)$,

(iv) if $T \in C_G(k,l)$, then $T^* \in C_G(k,l)$,

(v) and we have $\mathrm{id}_n \in C_G(1,1)$.

For permutation groups, we have the following important intertwiners.

(1) $U \in C_G(0,1)$, where $U(1) = \sum_{s=1}^n e_s$,

(2) $M \in C_G(2,1)$, where $M(e_i \otimes e_j) = \delta_{ij} e_i$,

(3) $S \in C_G(2,2)$, where $S(e_i \otimes e_j) = e_j \otimes e_i$.

Those are intertwiners because of the properties of permutation matrices. By definition of the automorphism group $\mathrm{Aut}(\Gamma)$ of a graph Γ, we also have

(4) $\varepsilon_\Gamma \in C_{\mathrm{Aut}(\Gamma)}(1,1)$.

Here ε_Γ is the adjacency matrix of Γ, i.e. $\varepsilon_\Gamma(e_i) = \sum_s \varepsilon_{is} e_s$.

It is shown in [22] that the tensor category is generated by the previously mentioned intertwiners, i.e. one can express every intertwiner in the generators using the operations $+, \circ, \otimes, *$.

Proposition 2.3.3. *Let Γ be a finite graph. Then*

$$C_{\mathrm{Aut}(\Gamma)} = \langle U, M, S, \varepsilon_\Gamma \rangle_{+,\circ,\otimes,*}.$$

Now, we come to the intertwiner spaces of quantum permutation groups.

Definition 2.3.4. Let $G^+ \subseteq S_n^+$ be a quantum permutation group with fundamental representation $u \in M_n(C(G^+))$. Let $u^{\otimes k}$ be the $n^k \times n^k$ matrix $u^{\otimes k} = (u_{i_1 j_1} \dots u_{i_k j_k})_{i_1 \dots i_k j_1 \dots j_k}$. Then, the intertwiner spaces $C_{G^+}(k,l)$ are

$$C_{G^+}(k,l) = \{T \in C(k,l) \,|\, Tu^{\otimes k} = u^{\otimes l} T\}$$

for $k, l \in \mathbb{N}$.

Those also form a tensor category with duals.

Proposition 2.3.5. *The collection of vector spaces $C_{G^+}(k,l)$ is a tensor category with duals.*

This tensor category is also generated by certain intertwiners. Here $S(e_i \otimes e_j) = e_j \otimes e_i$ is not necessarily an intertwiner.

Theorem 2.3.6 ([22])**.** *Let Γ be a finite graph. Then*

$$C_{G^+_{aut}(\Gamma)} = \langle U, M, \varepsilon_\Gamma \rangle_{+,\circ,\otimes,*}.$$

The important consequence of the theorem, also using Tannaka-Krein duality, is the following.

Corollary 2.3.7. *Let Γ be a finite graph. Then Γ has no quantum symmetry if and only if $S \in C_{G^+_{aut}(\Gamma)}(2,2)$.*

This corollary is used in [22] to show that the Paley graphs P_{13} and P_{17} have no quantum symmetry. We give an alternative proof of this in Proposition 4.1.25.

The subsequent theorem gives an explicit description of the intertwiner spaces of the quantum automorphism group of a graph. It extends Theorem 2.3.6 substantially: Instead of just knowing the generators, we see what a general intertwiner of $G^+_{aut}(\Gamma)$ looks like.

Theorem 2.3.8 ([38])**.** *The intertwiner spaces of quantum automorphism groups of graphs are the span of matrices whose entries count homomorphisms from planar graphs to Γ, partitioned according to the images of certain labelled vertices of the planar graph.*

This description is used in [38] to give a nice criterion for proving that two graphs are not quantum isomorphic, see Theorem 7.1.5.

2.4 The quantum orbital algebra

In this section, we review results of [27] and [36] which will be used in later chapters. Let G be a group acting on a finite set V via the action $G \times V \to V$, $(g, x) \mapsto gx$. Then $i \in V$ and $j \in V$ are in the same *orbit* of G if there exists $g \in G$ such that $gi = j$. One can also define a diagonal action $G \times (V \times V) \to (V \times V)$, $(g, (x, y)) \mapsto (gx, gy)$, where $(i, j) \in V \times V$ and $(k, l) \in V \times V$ are defined to be in the same *orbital* of G if there exists $g \in G$ with $(gi, gj) = (k, l)$. There is a similar concept for quantum permutation groups, which was introduced by Lupini, Mančinska and Roberson in [36]. Remark that this notion was also defined by Banica and Freslon in [11].

Definition 2.4.1. Let V be a finite set and let u be the fundamental representation of a quantum permutation group G^+. Define the relations $\sim_1$ and $\sim_2$ on V and $V \times V$, respectively, as follows

(i) $i \sim_1 j$ if $u_{ij} \neq 0$,

(ii) $(i, j) \sim_2 (k, l)$ if $u_{ik}u_{jl} \neq 0$.

Those are equivalence relations by [36, Lemma 3.2 & 3.4]. The *orbits* and *orbitals* of G^+ are the equivalence classes of these relations, respectively. In the case where $G^+ = G^+_{aut}(\Gamma)$ for some graph Γ, we refer to its orbits as *quantum orbits* of the graph Γ. Similarly, we call the orbitals of $G^+_{aut}(\Gamma)$ the *quantum orbitals* of Γ.

The next definition is due to Higman [30].

Definition 2.4.2. Let V be a finite set. A *coherent configuration* is a partition $\mathcal{R} = \{R_i \,|\, i \in I\}$ of $V \times V$ that satisfies the following.

(i) There is a subset $J \subseteq I$ such that $\{R_j \,|\, j \in J\}$ is a partition of the diagonal $\{(x, x) \,|\, x \in V\}$.

(ii) For $R_i \in \mathcal{R}$, we also have $\{(y, x) \,|\, (x, y) \in R_i\} \in \mathcal{R}$.

(iii) For all $i, j, k \in I$ and any $(x, z) \in R_k$, the number of $y \in V$ such that $(x, y) \in R_i$ and $(y, z) \in R_j$ is a constant p^k_{ij} that does not depend on x and z.

We call the matrices $A^{(i)}$, $i \in I$, where

$$A^{(i)}_{xy} := \begin{cases} 1, (x, y) \in R_i \\ 0, \text{ otherwise,} \end{cases}$$

the *characteristic matrices* of $\mathcal{R}$.

It is well-known that orbitals of groups form a coherent configuration. This is also true for orbitals of quantum permutation groups, see for instance Theorem 3.10 of [36].

Definition 2.4.3. We say that a subset $\mathcal{A} \subseteq \mathrm{M}_n(\mathbb{C})$ is a *coherent algebra* if

(i) $\mathcal{A}^* = \mathcal{A}$,

(ii) $\mathcal{A}$ is a unital algebra with respect to matrix multiplication,

(iii) $\mathcal{A}$ is a unital algebra with respect to entrywise matrix multiplication.

Here the identity matrix I is the unit with respect to matrix multiplication, the all-ones matrix J is the unit with respect to entrywise matrix multiplication.

There is the following one-to-one correspondence between coherent configurations and coherent algebras. On the one hand, the linear span of the characteristic matrices of a coherent configuration $\mathcal{R}$ is a coherent algebra. On the other hand, every coherent algebra $\mathcal{A}$ has a basis of zero-one matrices which are characteristic matrices of a coherent configuration. Therefore, we obtain coherent algebras from orbitals and quantum orbitals.

Definition 2.4.4 ([36])**.** Let $\Gamma = (V, E)$ be a graph. We associate the following three coherent algebras to the graph.

(i) The *coherent algebra of* Γ, denoted $\mathcal{CA}(\Gamma)$ is the smallest coherent algebra containing the adjacency matrix.

(ii) The automorphism group $\mathrm{Aut}(\Gamma)$ induces an action on $V \times V$. As stated before, the orbitals of $\mathrm{Aut}(\Gamma)$ on V form a coherent configuration. The corresponding coherent algebra is called the *orbital algebra* $\mathcal{O}(\Gamma)$.

(iii) The quantum orbitals of $G^+_{aut}(\Gamma)$ on V form a coherent configuration. The corresponding coherent algebra is called the *quantum orbital algebra* $\mathcal{QO}(\Gamma)$.

We remark that the coherent algebra $\mathcal{CA}(\Gamma)$ can be computed in polynomial time via the Weisfeiler-Leman algorithm, see [28] for more on this. The subsequent proposition relates the three coherent algebras from Definition 2.4.4.

Proposition 2.4.5 ([36])**.** *Let Γ be a finite graph. We have the following chain of inclusions*

$$\mathcal{CA}(\Gamma) \subseteq \mathcal{QO}(\Gamma) \subseteq \mathcal{O}(\Gamma).$$

Now, we come to equivalent characterizations of elements of the quantum orbital algebra. The next proposition shows that the intertwiner space $C_{G^+_{aut}(\Gamma)}(1, 1)$ is equal to the quantum orbital algebra.

Proposition 2.4.6 ([36]). *Let Γ be a finite graph, let $u = (u_{ij})_{1 \leq i,j \leq n}$ be the fundamental representation of $G^+_{aut}(\Gamma)$ and let $M \in \mathrm{M}_n(\mathbb{C})$. Then $uM = Mu$ if and only if M is in the quantum orbital algebra $\mathcal{QO}(\Gamma)$.*

Recall from Definition 2.1.10 that there is a natural action $\alpha : C(V) \to C(V) \otimes C(G^+_{aut}(\Gamma))$,

$$\alpha(e_i) = \sum_{j=1}^{|V|} e_j \otimes u_{ji}.$$

Such an action can be extended diagonally to $\alpha^{\otimes 2} : (C(V) \otimes C(V)) \to (C(V) \otimes C(V)) \otimes C(G^+_{aut}(\Gamma))$,

$$\alpha^{\otimes 2}(e_i \otimes e_j) = \sum_{k,l=1}^{|V|} e_k \otimes e_l \otimes u_{ik} u_{jl}.$$

The following lemma will be important in Chapter 6. It connects the quantum orbital algebra to the 2-boxes of the quantum-group-action planar algebra.

Lemma 2.4.7 ([36]). *Let Γ be a graph and α the action from above. Then $\alpha^{\otimes 2}(f) = f \otimes 1$ if and only if f is constant on the quantum orbitals of Γ.*

For some cases, the coherent algebra $\mathcal{QO}(\Gamma)$ also determines the quantum automorphism group of the graph.

Proposition 2.4.8 ([36]). *Let Γ be a finite graph. It holds $\mathcal{QO}(\Gamma) = \mathrm{M}_n(\mathbb{C})$ if and only if $G^+_{aut}(\Gamma) = \{e\}$.*

Combining Propositions 2.4.5 and 2.4.8, we see that knowledge on $\mathcal{CA}(\Gamma)$ can be useful to obtain that a graph has trivial quantum automorphism group.

Corollary 2.4.9 ([36]). *Let Γ be a finite graph. If $\mathcal{CA}(\Gamma) = \mathrm{M}_n(\mathbb{C})$, then $G^+_{aut}(\Gamma) = \mathrm{Aut}(\Gamma) = \{e\}$.*

It is known that $\mathcal{CA}(\Gamma) = \mathrm{M}_n(\mathbb{C})$ holds for almost all graphs ([2, Theorem 4.1]), therefore we obtain the next theorem.

Theorem 2.4.10 ([36]). *Almost all graphs have trivial quantum automorphism group.*

Erdős and Renyi showed in [26] that almost all graphs have trivial automorphism group. Thus, the previous theorem constitutes the quantum analogue of this fact. Also in [26], it was shown that almost all trees do have symmetry. The quantum analogue of this is also true and was worked out by Junk, Weber and the author in [35]. The key incredients of the proof are Theorem 3.1.2 and the fact that almost all trees have two cherries.

Theorem 2.4.11. *Almost all trees do have quantum symmetry.*

Summarizing, we have the following theorem.

Theorem 2.4.12.

(i) Almost all graphs have no symmetry [26].

(ii) Almost all graphs have no quantum symmetry [36].

(iii) Almost all trees have symmetry [26].

(iv) Almost all trees have quantum symmetry [35].

We turn our attention back to coherent algebras. Sometimes the coherent algebra $\mathcal{CA}(\Gamma)$ does not help to compute the quantum automorphism group of Γ. Considering distance-regular graphs of diameter d, the algebra $\mathcal{CA}(\Gamma)$ is always $(d+1)$-dimensional, whereas $\mathcal{QO}(\Gamma) = \mathrm{M}_n(\mathbb{C})$ is still possible. We restrict to distance-transitive graphs in the following example.

Example 2.4.13. It is well-known that the coherent algebra $\mathcal{CA}(\Gamma)$ of a distance-transitive graph Γ of diameter d is $(d+1)$-dimensional. Furthermore, if Γ is a distance-transitive graph, then also $\mathcal{O}(\Gamma)$ is $(d+1)$-dimensional, which implies

$$\mathcal{CA}(\Gamma) = \mathcal{QO}(\Gamma) = \mathcal{O}(\Gamma).$$

In Section 4.3, we will see that the Shrikhande graph is an example of a graph where

$$\mathcal{CA}(\Gamma) \neq \mathcal{QO}(\Gamma).$$

The following proposition can also be found in [27]. We give an alternative proof here, using the quantum orbital algebra.

Proposition 2.4.14. *Let $\Gamma = (V, E)$ be a graph, let ε be its adjacency matrix and let $(u_{ij})_{1\leq i,j\leq n}$ be the generators of $C(G^+_{aut}(\Gamma))$. If $(\varepsilon^l)_{ii} \neq (\varepsilon^l)_{jj}$ for some $i, j \in V$, $l \geq 1$, then $u_{ij} = 0$.*

Proof. By definition, we have $\varepsilon \in \mathcal{CA}(\Gamma)$ and thus $\varepsilon^l \in \mathcal{CA}(\Gamma)$ for all $l \geq 1$. By Proposition 2.4.5, we also have $\varepsilon^l \in \mathcal{QO}(\Gamma)$. Since ε^l lies in the span of the characteristic matrices of the quantum orbitals, we know that $(\varepsilon^l)_{ii} \neq (\varepsilon^l)_{jj}$ implies that (i, i) and (j, j) are not in the same quantum orbital. Therefore we get $u_{ij} = u_{ij}u_{ij} = 0$ by the definition of quantum orbitals. □

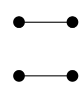
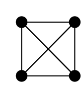
Remark 2.4.15. Note that it is only possible to get $u_{ij} = 0$ if there is no automorphism that sends i to j. This is true because of the following. Let $i, j \in V$ and assume there exists $\tau \in \mathrm{Aut}(\Gamma)$ such that $\tau(i) = j$. We get $\tau_{ij} = 1$ for the corresponding permutation matrix. For the function $\varphi_{ij} : \mathrm{Aut}(\Gamma) \to \mathbb{C}$, $\varphi(\sigma) = \sigma_{ij}$, we deduce $\varphi_{ij} \neq 0$, since $\varphi_{ij}(\tau) = 1$. We infer $u_{ij} \neq 0$, because we have the *-homomorphism $C(G^+_{aut}(\Gamma)) \to C(\mathrm{Aut}(\Gamma))$, $u_{ij} \mapsto \varphi_{ij}$. Especially, if Γ is vertex-transitive, then $u_{ij} \neq 0$ for all $i, j \in V$. We say that Γ is *quantum vertex-transitive*, if $u_{ij} \neq 0$ for all $i, j \in V$. By the previous argument, we immediately see that every vertex-transitive graph is quantum vertex-transitive. The other direction is not true, a counterexample is given in [36].

2.5 Comparing $G^*_{aut}(\Gamma)$ and $G^+_{aut}(\Gamma)$

In this section, we compare the two definitions of quantum automorphism groups of graphs. We will see later on, that they often coincide, for example if the graph does not contain any quadrangles (Lemma 3.2.5). Recall from Section 2.1 that

$$\mathrm{Aut}(\Gamma) \subseteq G^*_{aut}(\Gamma) \subseteq G^+_{aut}(\Gamma).$$

We already know graphs Γ_1, Γ_2 with $\mathrm{Aut}(\Gamma_1) \neq G^*_{aut}(\Gamma_1)$ and $G^*_{aut}(\Gamma_2) \neq G^+_{aut}(\Gamma_2)$. For example, take

$\Gamma_1 =$ , $\Gamma_2 =$ .

We know $\mathrm{Aut}(\Gamma_1) = D_4 \neq H_2^+ = G^*_{aut}(\Gamma_1)$ from Table in 2.1 and $G^*_{aut}(\Gamma_2) = S_4 \neq S_4^+ = G^+_{aut}(\Gamma_2)$ from Example 2.1.8. Still, we have $G^*_{aut}(\Gamma_1) = G^+_{aut}(\Gamma_1)$ and $\mathrm{Aut}(\Gamma_2) = G^*_{aut}(\Gamma_2)$ for those graphs. Therefore, one may ask whether or not there are graphs where $G^+_{aut}(\Gamma)$ and $G^*_{aut}(\Gamma)$ are not the same and also differ from $\mathrm{Aut}(\Gamma)$. We obtain some examples throughout this section.

The following lemma shows how the quantum automorphism groups behave, if we add points to a given graph without connecting them to anything.

Lemma 2.5.1. *Let $\Gamma' = (V', E')$ be a finite graph without multiple edges with $V' = \{1, \dots, n\}$, where every $v \in V'$ has at least one neighbor in V'. Now, consider $\Gamma = (V, E)$, where we add k disconnected vertices, i.e. $V = \{1, \dots, n+k\}$, $E = E'$. Then*

$$G^+_{aut}(\Gamma) = G^+_{aut}(\Gamma') * S_k^+ \qquad \text{and} \qquad G^*_{aut}(\Gamma) = G^*_{aut}(\Gamma') * S_k^+.$$

Proof. We label the vertices of Γ as follows.

$$\begin{array}{ccccc} 1 \text{ to } n & & n+1 & & n+k \\ \Gamma' & & \bullet & \cdots & \bullet \end{array}$$

Then the adjacency matrix of Γ is of the form

$$\varepsilon = \begin{pmatrix} \varepsilon' & 0_{k,n} \\ 0_{n,k} & 0_{k,k} \end{pmatrix},$$

where ε' denotes the adjacency matrix of Γ'. For $n+1 \leq v \leq n+k$, we have $(\varepsilon^2)_{vv} = 0$, because those vertices do not have any neighbors in Γ. Since every $v \in V'$ has a neighbor, we get $(\varepsilon^2)_{vv} \geq 1$ for $1 \leq v \leq n$. Thus, using Proposition 2.4.14 we get $u_{ij} = 0$ for $1 \leq i \leq n$, $n+1 \leq j \leq n+k$ and $u_{ij} = 0$ for $1 \leq j \leq n$, $n+1 \leq i \leq n+k$. Therefore

$$u = \begin{pmatrix} u' & 0_{k,n} \\ 0_{n,k} & u'' \end{pmatrix},$$

where $u' \in M_n(C(G^+_{aut}(\Gamma))$, $u'' \in M_k(C(G^+_{aut}(\Gamma))$. It holds

$$u\varepsilon = \begin{pmatrix} u' & 0_{k,n} \\ 0_{n,k} & u'' \end{pmatrix} \begin{pmatrix} \varepsilon' & 0_{k,n} \\ 0_{n,k} & 0_{k,k} \end{pmatrix} = \begin{pmatrix} u'\varepsilon' & 0_{k,n} \\ 0_{n,k} & 0_{n,n} \end{pmatrix}$$

and

$$\varepsilon u = \begin{pmatrix} \varepsilon' & 0_{k,n} \\ 0_{n,k} & 0_{k,k} \end{pmatrix} \begin{pmatrix} u' & 0_{k,n} \\ 0_{n,k} & u'' \end{pmatrix} = \begin{pmatrix} \varepsilon' u' & 0_{k,n} \\ 0_{n,k} & 0_{n,n} \end{pmatrix}.$$

We deduce that $u\varepsilon = \varepsilon u$ implies $u'\varepsilon' = \varepsilon' u'$, but no further relation on u''. Summarising, we get that u' fulfills the relations of $C(G^+_{aut}(\Gamma'))$ and u'' fulfills the relations of $C(S_k^+)$. Now, Proposition 1.1.6 yields the assertion. The result for $G^*_{aut}(\Gamma)$ follows in the same way. □

With this result, we can produce examples, where we get strict inequalities between the automorphism group and the two quantum automorphism groups simultaneously.

Proposition 2.5.2. *Take Γ, Γ' as in Lemma 2.5.1. For $k \geq 2$ and $\{e\} \neq G^*_{aut}(\Gamma') \neq G^+_{aut}(\Gamma')$, we get*

$$\mathrm{Aut}(\Gamma) \neq G^*_{aut}(\Gamma) \neq G^+_{aut}(\Gamma).$$

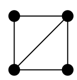
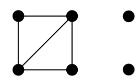
Proof. By Lemma 2.5.1, we know

$$G^*_{aut}(\Gamma) = S_k^+ * G^*_{aut}(\Gamma'),$$
$$G^+_{aut}(\Gamma) = S_k^+ * G^+_{aut}(\Gamma').$$

Since we have $G^*_{aut}(\Gamma') \neq G^+_{aut}(\Gamma')$ by assumption, we get $G^*_{aut}(\Gamma) \neq G^+_{aut}(\Gamma)$. As $C(S_k^+) * C(G^*_{aut}(\Gamma))$ is non-commutative for $k \geq 2$, we get the assertion. □

We give an explicit example in the following.

Example 2.5.3. Consider the graphs

$\Gamma' =$  , $\Gamma =$  .

Looking at Table of 2.1, we see

$$G^*_{aut}(\Gamma') = \mathbb{Z}_2 \times \mathbb{Z}_2 \qquad \text{and} \qquad G^+_{aut}(\Gamma') = \widehat{\mathbb{Z}_2 * \mathbb{Z}_2}.$$

Now, using the previous lemma yields

$$G^+_{aut}(\Gamma) = \widehat{\mathbb{Z}_2 * \mathbb{Z}_2 * \mathbb{Z}_2}, \qquad G^*_{aut}(\Gamma) = \widehat{(\mathbb{Z}_2 \times \mathbb{Z}_2) * \mathbb{Z}_2},$$

$$\mathrm{Aut}(\Gamma) = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2.$$

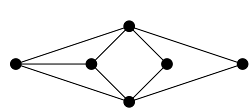
and thus

$$\mathrm{Aut}(\Gamma) \neq G^*_{aut}(\Gamma) \neq G^+_{aut}(\Gamma).$$

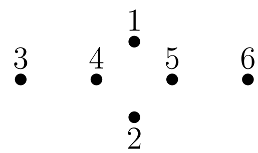
Let us even go further and show

$$\begin{array}{ccccc} \mathrm{Aut}(\Gamma) & \neq & G^*_{aut}(\Gamma) & \neq & G^+_{aut}(\Gamma) \\ \Vert & & \nparallel & & \Vert \\ \mathrm{Aut}(\Gamma^c) & \neq & G^*_{aut}(\Gamma^c) & \neq & G^+_{aut}(\Gamma^c). \end{array}$$

The complement of Γ looks as follows.

$\Gamma^c =$ 

We know that $G^+_{aut}(\Gamma^c) = G^+_{aut}(\Gamma)$ and $\mathrm{Aut}(\Gamma^c) = \mathrm{Aut}(\Gamma)$. For $G^*_{aut}(\Gamma^c)$, label the graph like this.



We directly get

$$u = \begin{pmatrix} u_{11} & 1-u_{11} & 0 & 0 & 0 & 0 \\ 1-u_{11} & u_{11} & 0 & 0 & 0 & 0 \\ 0 & 0 & u_{33} & 1-u_{33} & 0 & 0 \\ 0 & 0 & 1-u_{33} & u_{33} & 0 & 0 \\ 0 & 0 & 0 & 0 & u_{55} & 1-u_{55} \\ 0 & 0 & 0 & 0 & 1-u_{55} & u_{55} \end{pmatrix}$$

by Proposition 2.4.14. Since we have $(1,3),(1,5) \in E^c$, we get $u_{11}u_{33} = u_{33}u_{11}$ and $u_{11}u_{55} = u_{55}u_{33}$ in $G^*_{aut}(\Gamma^c)$. Because $(i,j) \notin E^c$ for $i \in \{3,4\}$ and $j \in \{5,6\}$, we do not get any commutation relation between u_{33} and u_{55}. Thus

$$G^*_{aut}(\Gamma^c) = \widehat{(\mathbb{Z}_2 * \mathbb{Z}_2) \times \mathbb{Z}_2}.$$

Therefore, Γ is an example of a graph where

$$\begin{array}{ccccc} \mathrm{Aut}(\Gamma) & \neq & G^*_{aut}(\Gamma) & \neq & G^+_{aut}(\Gamma) \\ \parallel & & \nparallel & & \parallel \\ \mathrm{Aut}(\Gamma^c) & \neq & G^*_{aut}(\Gamma^c) & \neq & G^+_{aut}(\Gamma^c). \end{array}$$

Chapter 3

Tools for computing the quantum automorphism group of a graph

We start this chapter by giving a sufficient criterion for a graph to have quantum symmetry. For this, one has to find a certain pair of automorphisms of the graph, which can be checked by hand or using the computer. This criterion is not necessary (see Chapters 6 and 7), but in practice it is hard to find graphs that have quantum symmetry while not having such a pair of automorphisms. Then we develop tools for proving the commutativity of the generators of the quantum automorphism group. These tools are used frequently in the next chapter. Note that in contrast to our previous criterion, the tools are used to prove that a graph has *no* quantum symmetry. At the end of this chapter, we present a strategy how to tackle the problem of computing the quantum automorphism group of a given graph. This chapter relies on parts of the articles [49, Section 2] and [50, Section 3] by the author.

3.1 A criterion for a graph to have quantum symmetry

This section is based on [49, Section 2]. We show that a graph has quantum symmetry if the automorphism group of the graph contains a certain pair of permutations. For this we need the following definition.

Definition 3.1.1. Let $V = \{1, \dots, r\}$. We say that two permutations $\sigma : V \to V$ and $\tau : V \to V$ are *disjoint*, if $\sigma(i) \neq i$ implies $\tau(i) = i$ and vice versa, for all $i \in V$.

We can now prove Theorem A.

Theorem 3.1.2 (Theorem A)**.** *Let $\Gamma = (V, E)$ be a finite graph without multiple edges. If there exist two non-trivial, disjoint automorphisms $\sigma, \tau \in \mathrm{Aut}(\Gamma)$, $\mathrm{ord}(\sigma) =$*

*$n, \mathrm{ord}(\tau) = m$, then we get a surjective *-homomorphism $\varphi : C(G^+_{aut}(\Gamma)) \to C^*(\mathbb{Z}_n * \mathbb{Z}_m)$. In particular, Γ has quantum symmetry.*

Proof. Let $\sigma, \tau \in \mathrm{Aut}(\Gamma)$ be non-trivial disjoint automorphisms with $\mathrm{ord}(\sigma) = n, \mathrm{ord}(\tau) = m$. Define

$$\begin{aligned}A &:= C^*(p_1, \dots, p_n, q_1, \dots, q_m | p_k = p_k^* = p_k^2, q_l = q_l^* = q_l^2, \sum_{k=1}^{n} p_k = 1 = \sum_{l=1}^{m} q_l) \\ &\cong C^*(\mathbb{Z}_n * \mathbb{Z}_m).\end{aligned}$$

We want to use the universal property to get a surjective *-homomorphism $\varphi : C(G^+_{aut}(\Gamma)) \to A$. This yields the non-commutativity of $C(G^+_{aut}(\Gamma))$, since it holds $p_i q_j \neq q_j p_i$ in $C^*(\mathbb{Z}_n * \mathbb{Z}_m)$. In order to do so, define

$$u' := \sum_{l=1}^{m} \tau^l \otimes q_l + \sum_{k=1}^{n} \sigma^k \otimes p_k - \mathrm{id}_{\mathrm{M_r}(\mathbb{C}) \otimes A} \in \mathrm{M_r}(\mathbb{C}) \otimes A \cong \mathrm{M_r}(A)$$

for $V = \{1, \dots, r\}$, where τ^l, σ^k denote the permutation matrices corresponding to $\tau^l, \sigma^k \in \mathrm{Aut}(\Gamma)$. This yields

$$u'_{ij} = \sum_{l=1}^{m} \delta_{j\tau^l(i)} q_l + \sum_{k=1}^{n} \delta_{j\sigma^k(i)} p_k - \delta_{ij} \in A.$$

Now, we show that u' does fulfill the relations of $u \in \mathrm{M_r}(\mathbb{C}) \otimes A$, the fundamental representation of $G^+_{aut}(\Gamma)$. Since we have $\tau^l, \sigma^k \in \mathrm{Aut}(\Gamma)$, it holds $\tau^l \varepsilon = \varepsilon \tau^l$ and $\sigma^k \varepsilon = \varepsilon \sigma^k$ for all $1 \leq l \leq m$, $1 \leq k \leq n$, where ε denotes the adjacency matrix of Γ. Therefore, we have

$$\begin{aligned}u'(\varepsilon \otimes 1) &= \left(\sum_{l=1}^{m} \tau^l \otimes q_l + \sum_{k=1}^{n} \sigma^k \otimes p_k - \mathrm{id}_{\mathrm{M_r}(\mathbb{C}) \otimes A} \right) (\varepsilon \otimes 1) \\ &= \sum_{l=1}^{m} \tau^l \varepsilon \otimes q_l + \sum_{k=1}^{n} \sigma^k \varepsilon \otimes p_k - (\varepsilon \otimes 1) \\ &= \sum_{l=1}^{m} \varepsilon \tau^l \otimes q_l + \sum_{k=1}^{n} \varepsilon \sigma^k \otimes p_k - (\varepsilon \otimes 1) \\ &= (\varepsilon \otimes 1) \left(\sum_{l=1}^{m} \tau^l \otimes q_l + \sum_{k=1}^{n} \sigma^k \otimes p_k - \mathrm{id}_{\mathrm{M_r}(\mathbb{C}) \otimes A} \right) \\ &= (\varepsilon \otimes 1) u'.\end{aligned}$$

Furthermore, it holds

$$\begin{aligned}\sum_{i=1}^{r} u'_{ji} &= \sum_{i=1}^{r}\left(\sum_{l=1}^{m}\delta_{i\tau^l(j)}q_l + \sum_{k=1}^{n}\delta_{i\sigma^k(j)}p_k\right) - 1\\ &= \left(\sum_{l=1}^{m} q_l\right) + \left(\sum_{k=1}^{n} p_k\right) - 1\\ &= 1.\end{aligned}$$

A similar computation shows $\sum_{i=1}^{r} u'_{ij} = 1$. Since τ and σ are disjoint, we have

$$u'_{ij} = \sum_{l=1}^{m}\delta_{j\tau^l(i)}q_l + \sum_{k=1}^{n}\delta_{j\sigma^k(i)}p_k - \delta_{ij} = \begin{cases}\sum_{k\in N_{ij}} p_k, & \text{if } \sigma(i)\neq i\\ \sum_{l\in M_{ij}} q_l, & \text{if } \tau(i)\neq i\\ \delta_{ij}, & \text{otherwise,}\end{cases}$$

where $N_{ij} = \{k \in \{1,\dots,n\};\, \sigma^k(i) = j\}, M_{ij} = \{l \in \{1,\dots,m\};\, \tau^l(i) = j\}$. Thus, all entries of u' are projections. By the universal property, we get a *-homomorphism $\varphi : C(G^+_{aut}(\Gamma)) \to A, u \mapsto u'$.

It remains to show that φ is surjective. We know $\operatorname{ord}(\sigma) = n$. For all $k_1 \neq k_2, k_1, k_2 \in \{1,\dots,n\}$, there exists $s \in V$ such that

$$\sigma^{k_1}(s) \neq \sigma^{k_2}(s),$$

as otherwise $\sigma^{k_1} = \sigma^{k_2}$. By similar considerations, there exist $t \in V$ such that

$$\tau^{l_1}(t) \neq \tau^{l_2}(t)$$

for $l_1 \neq l_2, l_1, l_2 \in \{1,\dots,m\}$. Therefore, we have

$$\begin{aligned}\varphi(u_{1\sigma^k(1)}\dots u_{r\sigma^k(r)}) &= u'_{1\sigma^k(1)}\dots u'_{r\sigma^k(r)} = p_k,\\ \varphi(u_{1\tau^l(1)}\dots u_{r\tau^l(r)}) &= u'_{1\tau^l(1)}\dots u'_{r\tau^l(r)} = q_l\end{aligned}$$

for all $k \in \{1,\dots,n\}, l \in \{1,\dots,m\}$ and since A is generated by p_k and q_l, φ is surjective. □

Remark 3.1.3. Let K_4 be the full graph on 4 points. By Example 2.1.8, we know that $\operatorname{Aut}(K_4) = S_4$ and $G^+_{aut}(K_4) = S_4^+$. We have disjoint automorphisms in S_4: For example $\sigma = (12), \tau = (34) \in S_4$ give us the well-known surjective *-homomorphism

$$\begin{aligned}\varphi : C(S_4^+) &\to C^*(p, q \,|\, p = p^* = p^2, q = q^* = q^2),\\ u &\mapsto \begin{pmatrix} p & 1-p & 0 & 0\\ 1-p & p & 0 & 0\\ 0 & 0 & q & 1-q\\ 0 & 0 & 1-q & q\end{pmatrix},\end{aligned}$$

yielding the non-commutativity of $C(S_4^+)$ (see Remark 1.1.10).

Remark 3.1.4. Let $\Gamma = (V, E)$ be a finite graph without multiple edges, where there exist two non-trivial, disjoint automorphisms $\sigma, \tau \in \mathrm{Aut}(\Gamma)$. To show that Γ has quantum symmetry it is enough to see that we have the surjective *-homomorphism

$$\varphi' : C(G^+_{aut}(\Gamma)) \to C^*(p, q \,|\, p = p^* = p^2, q = q^* = q^2),$$
$$u \mapsto \sigma \otimes p + \tau \otimes q + \mathrm{id}_{M_r(\mathbb{C})} \otimes (1 - q - p).$$

Remark 3.1.5. In Theorem 6.3.3, we will see that the Higman-Sims graph is an example of a graph that has quantum symmetry but no disjoint automorphisms. Thus, the converse direction of Theorem 3.1.2 is not true. Besides the Higman-Sims graph, one of the graphs appearing in Example 7.2.7 also has those properties. On the other hand, for graphs on a small number of vertices ($n \leq 6$), having disjoint automorphims is equivalent to having quantum symmetry, see the author's joint work with Eder, Levandovskyy, Schanz, Steenpass, and Weber [25].

3.2 Tools for proving commutativity of the generators

In this section, we develop tools to obtain commutation relations between the generators of the quantum automorphism group. In contrast to the previous section, these tools are used to prove that a graph does *not* have quantum symmetry. We are guided by Section 3 of [50] by the author. The following, trivial fact can be found for example in [48].

Lemma 3.2.1. *Let $(u_{ij})_{1 \leq i,j \leq n}$ be the generators of $C(G^+_{aut}(\Gamma))$. If we have*

$$u_{ij} u_{kl} = u_{ij} u_{kl} u_{ij}$$

then u_{ij} and u_{kl} commute.

Proof. Since $u_{ij} u_{kl} u_{ij}$ is selfadjoint, we infer the result. □

3.2.1 Path length comparison

Looking at Definition 2.1.1 and especially Relations (2.1.4), (2.1.5), we have orthogonality for generators u_{ij} and u_{kl} of $C(G^+_{aut}(\Gamma))$ if the vertices i, k are adjacent and j, l are non-adjacent or vice versa. The next lemma shows that u_{ij} and u_{kl} are also orthogonal if $d(i, k) \neq d(j, l)$, where $d(i, j)$ denotes the distance between vertices i and j, see Definition 1.2.15. This yields that, if we want to show that a graph has no quantum symmetry, it suffices to look at words $u_{ij} u_{kl}$, where $d(i, k) = d(j, l)$.

Lemma 3.2.2. *Let Γ be a finite, undirected graph and let $(u_{ij})_{1 \leq i,j \leq n}$ be the generators of $C(G^+_{aut}(\Gamma))$. If we have $d(i, k) \neq d(j, l)$, then $u_{ij} u_{kl} = 0$.*

Proof. We may assume $m := d(i,k) < d(j,l)$. For $m = 0$, we get $i = k$ and hence $u_{ij}u_{il} = 0$ since u_{ij}, u_{il} are orthogonal projections by Relations (2.1.1), (2.1.2). If $m = 1$, then $(i,k) \in E$ while $(j,l) \notin E$, so Relation (2.1.5) yields the assertion. Otherwise, there is a path of length $m \geq 2$ from i to k, say $i, a_1, a_2, \dots, a_{m-1}, k$. By Relation (2.1.2), we get

$$\begin{aligned} u_{ij}u_{kl} &= u_{ij}\left(\sum_{b_1} u_{a_1b_1}\right)\left(\sum_{b_2} u_{a_2b_2}\right)\dots\left(\sum_{b_{m-1}} u_{a_{m-1}b_{m-1}}\right)u_{kl} \\ &= \sum_{b_1,\dots,b_{m-1}} u_{ij}u_{a_1b_1}u_{a_2b_2}\dots u_{a_{m-1}b_{m-1}}u_{kl}. \end{aligned}$$

Since $d(j,l) > m$, there is no path of length m between j and l. Thus, for all $b_0 := j, b_1, \dots, b_{m-1}, b_m := l$, there are two vertices b_x, b_{x+1} with $(b_x, b_{x+1}) \notin E$. We get $u_{a_xb_x}u_{a_{x+1}b_{x+1}} = 0$ by Relation (2.1.5) and therefore

$$u_{ij}u_{a_1b_1}u_{a_2b_2}\dots u_{a_{m-1}b_{m-1}}u_{kl} = 0$$

for all $b_1, \dots, b_{m-1}$. We conclude

$$u_{ij}u_{kl} = \sum_{b_1,\dots,b_{m-1}} u_{ij}u_{a_1b_1}u_{a_2b_2}\dots u_{a_{m-1}b_{m-1}}u_{kl} = 0.$$

□

Remark 3.2.3. Lemma 3.2.2 also follows from the fact that pairs of vertices at different distances are not in the same quantum orbital, see Section 2.4.

3.2.2 Pairs of vertices in the same distance

Because of the previous lemma, we are interested in showing $u_{ij}u_{kl} = u_{kl}u_{ij}$ for $d(i,k) = d(j,l)$, since this is enough to prove that a graph has no quantum symmetry. We will see that for distance-transitive graphs it suffices to study one pair of vertices (j_1, l_1) in distance m to obtain commutativity of all u_{ij}, u_{kl} with $d(i,k) = d(j,l) = m$.

Lemma 3.2.4. *Let Γ be a distance-transitive graph and let $(u_{ij})_{1\leq i,j\leq n}$ be the generators of $C(G^+_{aut}(\Gamma))$. Let $j_1, l_1 \in V$ and put $m := d(j_1, l_1)$. If $u_{aj_1}u_{bl_1} = u_{bl_1}u_{aj_1}$ for all a, b with $d(a,b) = m$, then we have $u_{ij}u_{kl} = u_{kl}u_{ij}$ for all i,k,j,l with $d(j,l) = m = d(i,k)$.*

Proof. Let $j_1, l_1 \in V$ and $u_{aj_1}u_{bl_1} = u_{bl_1}u_{aj_1}$ for all a, b with $d(a,b) = m$. Furthermore, let $\varphi \in \mathrm{Aut}(\Gamma)$. We have $\varepsilon(\varphi u \varphi^{-1}) = (\varphi u \varphi^{-1})\varepsilon$ since φ and φ^{-1} commute with the adjacency matrix ε of Γ. Therefore, the map $\tilde{\varphi} : C(G^+_{aut}(\Gamma)) \to$

$C(G^+_{aut}(\Gamma)), u_{ij} \mapsto (\varphi u \varphi^{-1})_{ij} = u_{\varphi(i)\varphi(j)}$ is a *-isomorphism for all $\varphi \in \mathrm{Aut}(\Gamma)$. For all pairs j, l with $d(j,l) = m$, there is a graph automorphism $\varphi_{j,l}$ with $\varphi_{j,l}(j_1) = j$, $\varphi_{j,l}(l_1) = l$, since Γ is distance-transitive (see Definition 1.2.23). Let $\tilde{\varphi}_{j,l}$ be the $*$-isomorphism corresponding to $\varphi_{j,l}$. We obtain

$$u_{ij}u_{kl} = \tilde{\varphi}_{j,l}(u_{\varphi^{-1}_{j,l}(i),j_1}u_{\varphi^{-1}_{j,l}(k),l_1}) = \tilde{\varphi}_{j,l}(u_{\varphi^{-1}_{j,l}(k),l_1}u_{\varphi^{-1}_{j,l}(i),j_1}) = u_{kl}u_{ij},$$

for all i, j, k, l with $d(j,l) = m = d(i,k)$, since we know $d(\varphi^{-1}_{j,l}(i), \varphi^{-1}_{j,l}(k)) = m$ and thus have $u_{\varphi^{-1}_{j,l}(i),j_1}u_{\varphi^{-1}_{j,l}(k),l_1} = u_{\varphi^{-1}_{j,l}(k),l_1}u_{\varphi^{-1}_{j,l}(i),j_1}$ by assumption. □

3.2.3 No common neighbors

For the rest of this section, we give criteria on properties of the graph Γ (for example containing quadrangles or having certain values in the intersection array) that allow us to say that certain generators of $G^+_{aut}(\Gamma)$ commute. The following theorem generalizes Theorem 3.2 of [48], which shows that $G^+_{aut}(\mathrm{P}) = G^*_{aut}(\mathrm{P})$ for the Petersen graph P (see Figure 1.1). Recall that a quadrangle is a cycle on four vertices, see Section 1.2.

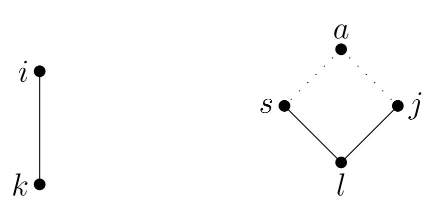
Lemma 3.2.5. *Let Γ be an undirected graph that does not contain any quadrangle. Then $G^+_{aut}(\Gamma) = G^*_{aut}(\Gamma)$.*

Proof. Let $(u_{ij})_{1 \leq i,j \leq n}$ be the generators of $C(G^+_{aut}(\Gamma))$ and let $(i,k) \in E$, $(j,l) \in E$. It holds

$$u_{ij}u_{kl} = u_{ij}u_{kl}\left(\sum_{s;(l,s)\in E} u_{is}\right)$$

by Relations (2.1.2) and (2.1.5).

Take $s \neq j$ with $(l,s) \in E$. Since we also have $(j,l) \in E$, the only common neighbor of j and s is l as otherwise we would get a quadrangle in Γ, contradicting our assumption. Hence, for all $a \neq l$, we have $(a,s) \notin E$ or $(a,j) \notin E$. The vertices are related as follows.



Therefore, Relation (2.1.5) implies $u_{ka}u_{is} = 0$ or $u_{ij}u_{ka} = 0$ for all $a \neq l$. By also using Relations (2.1.1) and (2.1.2), we get

$$u_{ij}u_{kl}u_{is} = u_{ij}\left(\sum_{a=1}^{n} u_{ka}\right)u_{is} = u_{ij}u_{is} = 0.$$

Therefore, we obtain

$$u_{ij}u_{kl} = u_{ij}u_{kl}\left(\sum_{s;(l,s)\in E} u_{is}\right) = u_{ij}u_{kl}u_{ij}$$

and we conclude $u_{ij}u_{kl} = u_{kl}u_{ij}$ by Lemma 3.2.1. This yields $G^+_{aut}(\Gamma) = G^*_{aut}(\Gamma)$. $\square$

3.2.4 One common neighbor

The upcoming lemma deals with graphs where adjacent vertices have one common neighbor. This common neighbor yields orthogonality of certain products of generators, which implies $G^+_{aut}(\Gamma) = G^*_{aut}(\Gamma)$.

Lemma 3.2.6. *Let Γ be an undirected graph such that adjacent vertices have exactly one common neighbor. Then $G^+_{aut}(\Gamma) = G^*_{aut}(\Gamma)$. In particular, we have $G^+_{aut}(\Gamma) = G^*_{aut}(\Gamma)$ for distance-regular graphs with $b_0 = b_1 + 2$ in the intersection array.*

Proof. Let $(u_{ij})_{1\leq i,j\leq n}$ be the generators of $C(G^+_{aut}(\Gamma))$ and let $(i,k),(j,l) \in E$. Using Relations (2.1.2), (2.1.5) we get

$$u_{ij}u_{kl} = u_{ij}u_{kl}\left(\sum_{p;(l,p)\in E} u_{ip}\right).$$

Let us now take a closer look at the products $u_{ij}u_{kl}u_{ip}$. We want to show $u_{ij}u_{kl}u_{ip} = 0$ for $p \neq j, (l,p) \in E$. Firstly, there is exactly one $p_1 \neq j, (l,p_1) \in E$ such that $(p_1, j) \in E$ since adjacent vertices have exactly one neighbor and we have $(j,l) \in E$. In this case we have $(j,a) \notin E$ or $(a,p_1) \notin E$ for $a \neq l$, because l is the only common neighbor of j and p_1. We deduce

$$u_{ij}u_{kl}u_{ip_1} = u_{ij}\left(\sum_{a} u_{ka}\right)u_{ip_1} = u_{ij}u_{ip_1} = 0$$

by Relations (2.1.5) and (2.1.2).

Secondly, let $p \notin \{j, p_1\}$, $(l,p) \in E$ and let s be the only common neighbor of i and k. The vertices are related like this:

It holds

$$u_{ij}u_{kl} = u_{ij}\left(\sum_b u_{sb}\right)u_{kl} = u_{ij}u_{sp_1}u_{kl} \tag{3.2.1}$$

by Relations (2.1.2), (2.1.5), since p_1 is the only common neighbor of j and l. We also know that j is the only common neighbor of p_1 and l and since we have $(l,p) \in E$, we deduce $(p_1,p) \notin E$. Relations (2.1.2), (2.1.4) now yield

$$0 = u_{sp_1}u_{ip} = u_{sp_1}\left(\sum_c u_{cl}\right)u_{ip} = u_{sp_1}u_{kl}u_{ip} \tag{3.2.2}$$

because k is the only common neighbor of s and i. Using Equations (3.2.1) and (3.2.2), we obtain

$$u_{ij}u_{kl}u_{ip} = u_{ij}u_{sp_1}u_{kl}u_{ip} = 0.$$

Summarizing, we have for all $p \neq j$:

$$u_{ij}u_{kl}u_{ip} = 0.$$

Finally, we get

$$u_{ij}u_{kl} = u_{ij}u_{kl}\left(\sum_p u_{ip}\right) = u_{ij}u_{kl}u_{ij}$$

and by Lemma 3.2.1 we conclude $u_{ij}u_{kl} = u_{kl}u_{ij}$ for $(i,k),(j,l) \in E$. □

Remark 3.2.7. By Lemma 3.2.5 and Lemma 3.2.6, we see that $G^*_{aut}(\Gamma) \neq G^+_{aut}(\Gamma)$ can only hold for graphs that contain quadrangles where additionally adjacent vertices do not have exactly one common neighbor.

3.2.5 A technical lemma

The next lemma is technical and mostly used to shorten the upcoming proofs.

Lemma 3.2.8. *Let* Γ *be a finite, undirected graph and let* $(u_{ij})_{1\leq i,j\leq n}$ *be the generators of* $C(G^+_{aut}(\Gamma))$. *Let* $d(i,k) = d(j,l) = m$. *Let* q *be a vertex with* $d(j,q) = s$, $d(q,l) = t$ *and* $u_{kl}u_{aq} = u_{aq}u_{kl}$ *for all* a *with* $d(a,k) = t$. *Then*

$$u_{ij}u_{kl} = u_{ij}u_{kl} \sum_{\substack{p;d(l,p)=m,\\ d(p,q)=s}} u_{ip}.$$

In particular, if we have $m = 2$ *and if* $G^+_{aut}(\Gamma) = G^*_{aut}(\Gamma)$ *holds, then choosing* $s = t = 1$ *implies*

$$u_{ij}u_{kl} = u_{ij}u_{kl} \sum_{\substack{p;d(l,p)=2,\\ (p,q)\in E}} u_{ip}.$$

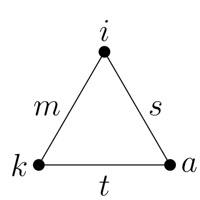
Proof. Using Lemma 3.2.2 and Relation (2.1.2), we know

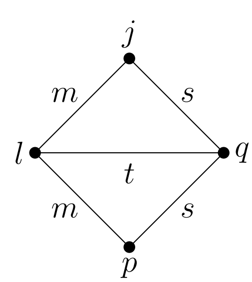
$$u_{ij}u_{kl} = u_{ij}u_{kl} \sum_{p;d(l,p)=m} u_{ip}.$$

Additionally, we want to obtain

$$u_{ij}u_{kl} = u_{ij}u_{kl} \sum_{\substack{p;d(l,p)=m,\\ d(p,q)=s}} u_{ip}.$$

The idea is to insert the condition on the distance s at some position in the monomials and then move it next to the sum $\sum_{p;d(l,p)=m} u_{ip}$ using $u_{kl}u_{aq} = u_{aq}u_{kl}$. The sum will then inherit it, in a way. The vertices are related as follows, where an edge labelled by m denotes a path of length m between the vertices.





By Relation (2.1.2) and Lemma 3.2.2, we have

$$u_{ij}u_{kl} = u_{ij}\left(\sum_{a} u_{aq}\right)u_{kl} = u_{ij}\left(\sum_{\substack{a;d(k,a)=t,\\ d(i,a)=s}} u_{aq}\right)u_{kl}.$$

Furthermore, also by Relation (2.1.2) and Lemma 3.2.2, it holds

$$\begin{aligned} u_{ij}u_{kl} &= u_{ij}\left(\sum_{\substack{a;d(k,a)=t,\\ d(i,a)=s}} u_{aq}\right)u_{kl} \\ &= u_{ij}\left(\sum_{\substack{a;d(k,a)=t,\\ d(i,a)=s}} u_{aq}\right)u_{kl}\sum_{p} u_{ip} \\ &= u_{ij}\left(\sum_{\substack{a;d(k,a)=t,\\ d(i,a)=s}} u_{aq}\right)u_{kl}\sum_{p;d(l,p)=m} u_{ip}. \end{aligned}$$

Now comes the crucial step, shifting the inserted sum next to $\sum_{p;d(l,p)=m} u_{ip}$. Since we have $u_{kl}u_{aq} = u_{aq}u_{kl}$ for all a with $d(a,k) = t$ and by Lemma 3.2.2, we get

$$\begin{aligned} u_{ij}u_{kl} &= u_{ij}\left(\sum_{\substack{a;d(k,a)=t,\\ d(i,a)=s}} u_{aq}\right)u_{kl}\sum_{p;d(l,p)=m} u_{ip} \\ &= u_{ij}u_{kl}\left(\sum_{\substack{a;d(k,a)=t,\\ d(i,a)=s}} u_{aq}\right)\sum_{p;d(l,p)=m} u_{ip} \\ &= u_{ij}u_{kl}\left(\sum_{\substack{a;d(k,a)=t,\\ d(i,a)=s}} u_{aq}\right)\sum_{\substack{p;d(l,p)=m,\\ d(p,q)=s}} u_{ip}. \end{aligned}$$

We reverse the shifting by using $u_{kl}u_{aq} = u_{aq}u_{kl}$ for all a with $d(a,k) = t$ again. We obtain

$$u_{ij}u_{kl} = u_{ij}u_{kl}\left(\sum_{\substack{a;d(k,a)=t,\\ d(i,a)=s}} u_{aq}\right)\sum_{\substack{p;d(l,p)=m,\\ d(p,q)=s}} u_{ip}$$

$$= u_{ij}\left(\sum_{a;d(k,a)=t,d(i,a)=s} u_{aq}\right)u_{kl}\sum_{\substack{p;d(l,p)=m,\\ d(p,q)=s}} u_{ip}.$$

By Lemma 3.2.2 and Relation (2.1.2), we get

$$u_{ij}u_{kl} = u_{ij}\left(\sum_{a} u_{aq}\right)u_{kl}\sum_{\substack{p;d(l,p)=m,\\ d(p,q)=s}} u_{ip}$$

$$= u_{ij}u_{kl}\sum_{\substack{p;d(l,p)=m,\\ d(p,q)=s}} u_{ip}$$

and this completes the proof. □

3.2.6 Relations between vertices in certain distances

The following result is helpful, if one has a specific labelling of the vertices and if it is not too hard to see which vertices are in distance m to the given ones. It is a more sophisticated version of Lemma 3.2.6, now considering vertices in distance m instead of adjacent vertices.

Lemma 3.2.9. *Let Γ be a finite, undirected graph and let $(u_{ij})_{1\leq i,j\leq n}$ be the generators of $C(G^+_{aut}(\Gamma))$. Let $d(i,k) = d(j,l) = m$ and let $p \neq j$ be a vertex with $d(p,l) = m$. Let q be a vertex with $d(q,l) = s$ and $d(j,q) \neq d(q,p)$. Then*

$$u_{ij}\left(\sum_{\substack{t;d(t,j)=d(t,p)=m,\\ d(t,q)=s}} u_{kt}\right)u_{ip} = 0.$$

Especially, if l is the only vertex satisfying $d(l,q) = s$, $d(l,j) = m$ and $d(l,p) = m$, we obtain $u_{ij}u_{kl}u_{ip} = 0$.

Proof. By Relation (2.1.2) and Lemma 3.2.2, it holds

$$u_{ij}\left(\sum_{\substack{t;d(t,j)=d(t,p)=m,\\ d(t,q)=s}} u_{kt}\right)u_{ip} = u_{ij}\left(\sum_{\substack{a;d(i,a)=d(j,q),\\ d(k,a)=s}} u_{aq}\right)\left(\sum_{\substack{t;d(t,j)=d(t,p)=m,\\ d(t,q)=s}} u_{kt}\right)u_{ip}.$$

Now, let $b \neq t$, for all t appearing in the above sum. We prove

$$u_{ij}\left(\sum_{\substack{a;d(i,a)=d(j,q),\\ d(k,a)=s}} u_{aq}\right)u_{kb}u_{ip} = 0.$$

Indeed, if $d(b,q) \neq s$ or $d(b,p) \neq m$, then we get

$$u_{ij}\left(\sum_{\substack{a;d(i,a)=d(j,q),\\ d(k,a)=s}} u_{aq}\right)u_{kb}u_{ip} = 0,$$

by Lemma 3.2.2. On the other hand, if $d(b,q) = s$ and $d(b,p) = m$, then we have $d(b,j) \neq m$ by assumption. This yields

$$u_{ij}\left(\sum_{\substack{a;d(i,a)=d(j,q),\\ d(k,a)=s}} u_{aq}\right)u_{kb}u_{ip} = u_{ij}u_{kb}u_{ip} = 0,$$

also by Relation (2.1.2) and Lemma 3.2.2. Therefore, we have

$$\begin{aligned} u_{ij}\left(\sum_{\substack{t;d(t,j)=d(t,p)=m,\\ d(t,q)=s}} u_{kt}\right)u_{ip} &= u_{ij}\left(\sum_{\substack{a;d(i,a)=d(j,q),\\ d(k,a)=s}} u_{aq}\right)\left(\sum_{\substack{t;d(t,j)=d(t,p)=m,\\ d(t,q)=s}} u_{kt}\right)u_{ip} \\ &= u_{ij}\left(\sum_{\substack{a;d(i,a)=d(j,q),\\ d(k,a)=s}} u_{aq}\right)\left(\sum_{b} u_{kb}\right)u_{ip}. \end{aligned}$$

Using $\sum_b u_{kb} = 1$, we deduce

$$u_{ij}\left(\sum_{\substack{t;d(t,j)=d(t,p)=m,\\ d(t,q)=s}} u_{kt}\right)u_{ip} = u_{ij}\left(\sum_{\substack{a;d(i,a)=d(j,q),\\ d(k,a)=s}} u_{aq}\right)u_{ip}.$$

Since we assumed $d(j,q) \neq d(q,p)$, the condition $d(i,a) = d(j,q)$ implies $d(i,a) \neq d(q,p)$. Thus, Lemma 3.2.2 yields $u_{aq}u_{ip} = 0$ for all such a and we get

$$u_{ij}\left(\sum_{\substack{t;d(t,j)=d(t,p)=m,\\ d(t,q)=s}} u_{kt}\right)u_{ip} = u_{ij}\left(\sum_{\substack{a;d(i,a)=d(j,q),\\ d(k,a)=s}} u_{aq}\right)u_{ip} = 0.$$

□

3.2.7 Values in the intersection array

In the subsequent lemma, we see that certain values in the intersection array of a graph Γ give commutation relations of the generators of $G^+_{aut}(\Gamma)$. Recall Definition 1.2.21 of the intersection array.

Lemma 3.2.10. *Let Γ be a distance-regular graph with intersection array $\{b_0, b_1, \dots, b_{d-1}; c_1, c_2, \dots, c_d\}$ and let $(u_{ij})_{1\leq i,j\leq n}$ be the generators of $C(G^+_{aut}(\Gamma))$. Let $c_m \geq 2$ for some $m \geq 2$ and assume*

$$u_{ij}u_{kl} = u_{kl}u_{ij}$$

for all vertices i, j, k, l with $d(i, k) = d(j, l) = m - 1$. If

(a) $c_2 = 1$ and $b_1 + 1 = b_0$,

(b) $c_2 = 1$ and $b_1 + 2 = b_0$,

(c) or $c_2 = 2$, $m = 2$ and $b_1 + 3 = b_0$,

then we have $u_{ij}u_{kl} = u_{kl}u_{ij}$ for all i, j, k, l with $d(i, k) = d(j, k) = m$.

Proof. Let $d(i, k) = d(j, l) = m$. Since $c_m \geq 2$, there are two neighbors t, τ of j in distance $m - 1$ to l. Since we have $u_{ac}u_{bd} = u_{bd}u_{ac}$ for $d(a, b) = d(c, d) = m - 1$ by assumption, we get

$$u_{ij}u_{kl} = u_{ij}u_{kl}\left(\sum_{\substack{p;d(p,l)=m,\\(t,p)\in E}} u_{ip}\right) \quad \text{and} \quad u_{ij}u_{kl} = u_{ij}u_{kl}\left(\sum_{\substack{p;d(p,l)=m,\\(\tau,p)\in E}} u_{ip}\right)$$

by Lemma 3.2.8. We deduce

$$\begin{aligned} u_{ij}u_{kl} &= u_{ij}u_{kl}\left(\sum_{\substack{p;d(p,l)=m,\\(t,p)\in E}} u_{ip}\right) \\ &= u_{ij}u_{kl}\left(\sum_{\substack{p;d(p,l)=m,\\(\tau,p)\in E}} u_{ip}\right)\left(\sum_{\substack{p;d(p,l)=m,\\(t,p)\in E}} u_{ip}\right) \end{aligned}$$

$$= u_{ij}u_{kl}\left(\sum_{\substack{p;d(p,l)=m,\\(\tau,p)\in E,(t,p)\in E}} u_{ip}\right).$$

In case (a), we know from $b_1 + 1 = b_0$ that Γ does not contain a triangle. Therefore we have $d(t,\tau) = 2$, since they have a common neighbor j and they are not connected, because otherwise there would be a triangle in Γ. Then $c_2 = 1$ implies that j is the only common neighbor of t and τ. Thus only j satisfies $d(j,l) = m$, $(\tau, j) \in E$, $(t, j) \in E$.

In case (b), we either have $(t,\tau) \in E$ or $d(t,\tau) = 2$. If $(t,\tau) \in E$, then $b_1+2 = b_0$ implies that j is the only common neighbor of t and τ. If $d(t,\tau) = 2$, we get that j is the only common neighbor of t and τ because $c_2 = 1$.

In case (c), we get that j and l are the only common neighbors of t and τ by similar considerations as in case (b). Thus, j is the only vertex satisfying the above conditions.

Summarizing, in all three cases

$$u_{ij}u_{kl} = u_{ij}u_{kl}\left(\sum_{\substack{p;d(p,l)=m,\\(\tau,p)\in E,(t,p)\in E}} u_{ip}\right) = u_{ij}u_{kl}u_{ij}$$

and then Lemma 3.2.1 completes the proof. □

3.3 A strategy to compute the quantum automorphism group of a given graph

In this section, we give a checklist to follow if one wants to compute the quantum automorphism group of a graph.

First, one should check if the automorphism group of the graph has disjoint automorphisms.

(1) If Γ has disjoint automorphisms, we get that the graph has quantum symmetry by Theorem 3.1.2. Now, we want to know $G^+_{aut}(\Gamma)$ precisely. This may be a complicated task, one can try to proceed as follows.

(1.1) Check if Γ is a graph product that fulfills the conditions appearing in Theorem 2.2.2. If Γ is such a product, we obtain the quantum automorphism group and we are done if we know the quantum automorphism groups of the graphs it is constructed from.

(1.2) Otherwise, try to use the coherent algebra (see Section 2.4) to get some insight about the generators and their products. Sometimes this is enough to obtain the quantum automorphism group, see for example Corollary 2.4.9.

(1.3) If Γ is a Cayley graph, one can use its eigenvalues and eigenvectors to compute $G^+_{aut}(\Gamma)$. This works for example for the cube graphs ([9]) and the folded cube graphs (Chapter 5). Still, if the generating set is not nice, the relations get complicated really quickly.

(1.4) Apart from that, one has to invent new methods to compute the quantum automorphism group of the graph.

(2) If Γ has no disjoint automorphisms, then it is not clear whether or not Γ has quantum symmetry. Still, it seems to be a good idea to first try to prove that it has no quantum symmetry. We have to show that the generators of $G^+_{aut}(\Gamma)$ commute. The strategy may be as follows.

(2.1) By Lemma 3.2.2, we know that it suffices to show $u_{ij}u_{kl} = u_{kl}u_{ij}$ for $d(i,k) = d(j,l)$.

(2.2) Choose a distance $d(i,k) = d(j,l) = m$ (usually, one starts with $m = 1$, then $m = 2$ and so on).

(2.3) First check if Lemma 3.2.5, Lemma 3.2.6 or Lemma 3.2.10 applies. If this is the case, then we know $u_{ij}u_{kl} = u_{kl}u_{ij}$ at least for $d(i,k) = d(j,l) = m$ for this fixed m.

(2.4) If Γ is distance-transitive, we know that it is enough to show $u_{ij_1}u_{kl_1} = u_{kl_1}u_{ij_1}$ for one pair (j_1, l_1) and all (i,k) with $d(i,k) = m$ by using Lemma 3.2.4. Otherwise one has to consider several cases.

(2.5) If we know the neighbors of l_1 (for example because of a known construction of the graph), one can apply Lemma 3.2.9 and use the equations to deduce $u_{ij}u_{kl} = u_{kl}u_{ij}$ for $d(i,k) = d(j,l) = m$.

(2.6) If this does not work, we have to treat this distance m in the graph as a special case and try to get $u_{ij}u_{kl} = u_{kl}u_{ij}$ by other methods.

Chapter 4

Some families of graphs that have no quantum symmetry

This chapter constituted the centerpiece of the thesis in hand. Using the results of the previous chapter, we prove that several families of graphs do not have quantum symmetry, where we focus on distance-transitive graphs. Recall that a distance-transitive graph is a graph such that for any given pair of vertices i, j in distance a and any other pair of vertices k, l with $d(k,l)=a$ there is a graph automorphism $\sigma : V \to V$ with $\sigma(i)=k$ and $\sigma(j)=l$. Distance-transitive graphs usually have large automorphism groups. Furthermore, the class of distance-transitive graph contains many well-known graphs such as the Petersen graph. We start by showing in Section 4.1 that the odd graphs, the Hamming graphs $H(d,3)$, the Johnson graphs $J(n,2)$ and the Kneser graphs $K(n,2)$ have no quantum symmetry. Furthermore, we prove that the Moore graphs of diameter two and the Paley graphs P_9, P_{13} and P_{17} have no quantum symmetry. We proceed with showing in Section 4.2 that all cubic distance-transitive graphs of order ≤ 10 have no quantum symmetry. Note that the Petersen graph (see Figure 1.1) is isomorphic to the odd graph O_3 and the Kneser graph $K(5,2)$, while it is also a Moore graph of diameter two as well as a cubic distance-transitive graph. Thus, we especially show that the Petersen graph has no quantum symmetry, i.e. $G^+_{aut}(\mathrm{P}) = S_5$. This answers a question asked by Banica and Bichon ([6]) in 2007. In Section 4.3, we investigate more quantum automorphism groups of distance-regular graphs of order ≤ 20. The graphs appearing in Sections 4.1 – 4.3 and their quantum automorphism groups are listed in the Tables 1 and 2 in the introduction. In the beginning of every section, we will also give small tables with the graphs appearing therein. Sections 4.1 – 4.3 are based on the articles [48] and [50] by the author. Finally, we give an example of a graph Γ with $\mathrm{Aut}(\Gamma) = \mathbb{Z}_2 \times \mathbb{Z}_2$ that has no quantum symmetry in Section 4.4.

4.1 Families of distance-transitive graphs

In this section we deal with families of distance-transitive graphs. Those are graphs with large automorphism groups and Lemma 3.2.4 allows us to tackle their quantum automorphism groups. The considered families are well-known and can for example be found in [18]. Their quantum automorphism groups have not been computed before. The families are mostly constructed from systems of sets, which is useful in order to keep track of how certain vertices are related or what vertices in a certain distance look like. This allows us to use tools like Lemma 3.2.9 effectively. We refer to [50, Section 4] by the author. We obtain the following table from the results of Section 4.1.

Name of Γ	Order	$\mathrm{Aut}(\Gamma)$	$G^+_{aut}(\Gamma)$	Intersection array
Paley graph P_9 ([6])	9	$S_3 \wr \mathbb{Z}_2$	$\mathrm{Aut}(\Gamma)$	{4,2;1,2}
Petersen graph	10	S_5	$\mathrm{Aut}(\Gamma)$	{3,2;1,1}
Paley graph P_{13} ([22])	13	$\mathbb{Z}_{13} \rtimes \mathbb{Z}_6$	$\mathrm{Aut}(\Gamma)$	{6,3;1,3}
Paley graph P_{17} ([22])	17	$\mathbb{Z}_{17} \rtimes \mathbb{Z}_8$	$\mathrm{Aut}(\Gamma)$	{8,4;1,4}
Hoffman-Singleton graph	50	$PSU(3, 5^2)$	$\mathrm{Aut}(\Gamma)$	{7,6;1,1}
Johnson graph $J(n, 2)$,$n \geq 5$	$\binom{n}{2}$	S_n	$\mathrm{Aut}(\Gamma)$	$\{2n-4,n-3;1,4\}$
Kneser graph $K(n, 2)$,$n \geq 5$	$\binom{n}{2}$	S_n	$\mathrm{Aut}(\Gamma)$	$(\star^1)$
Odd graphs O_k	$\binom{2k-1}{k-1}$	S_{2k-1}	$\mathrm{Aut}(\Gamma)$	$(\star^2)$
Hamming graphs $H(n, 3)$	3^n	$S_3 \wr S_n$	$\mathrm{Aut}(\Gamma)$	$(\star^3)$

Table 4.1: The graphs appearing in Section 4.1

Here

$$(\star^1) = \{(n-2)(n-3)/2, 2n-8; 1, (n-3)(n-4)/2\},$$

$$(\star^2) = \{k, k-1, k-1 \dots, l+1, l+1, l; 1, 1, 2, 2, \dots, l, l\} \text{ for } k = 2l-1,$$
$$\{k, k-1, k-1, \dots l+1, l+1; 1, 1, 2, 2, \dots, l-1, l-1, l\} \text{ for } k = 2l,$$

$$(\star^3) = \{2n, 2n-2, \dots, 2; 1, 2, \dots, n\}.$$

Looking at Table 4.1, we especially read off the next theorem (Theorem C (i)).

Theorem 4.1.1. *The Petersen graph has no quantum symmetry.*

It was asked in 2007 by Banica and Bichon ([6]) whether or not the Petersen graph has quantum symmetry. Theorem 4.1.1 was proven first by the author in [48]. Then, the author generalized the techniques of [48] in [50]. Therefore, the previous theorem is a special case of Theorem 4.1.7, Theorem 4.1.18 and of Theorem 4.1.20.

4.1.1 The odd graphs

We show that the odd graphs do not have quantum symmetry. The odd graphs generalize the Petersen graph P in the sense that $O_3 = \mathrm{P}$. We use the strategy described in Section 3.3.

Definition 4.1.2. Let $k \geq 2$. The graph O_k with vertices corresponding to $(k-1)$-subsets of $\{1, \dots, 2k-1\}$, where two vertices are connected if and only if the corresponding subsets are disjoint is called *odd graph*.

The odd graphs have the following properties, see for example [18, Proposition 9.1.7]. Recall the definitions of diameter, girth and intersection array from Definition 1.2.15, Definition 1.2.2 and Definition 1.2.21, respectively.

Remark 4.1.3. Odd graphs are distance-transitive with $\mathrm{Aut}(O_k) = S_{2k-1}$, diameter $k-1$ and girth $g(O_k) \geq 5$ for $k \geq 3$. They have the intersection array

$$\{k, k-1, k-1 \dots, l+1, l+1, l; 1, 1, 2, 2, \dots, l, l\} \qquad \text{for } k = 2l-1,$$
$$\{k, k-1, k-1, \dots, l+1, l+1; 1, 1, 2, 2, \dots, l-1, l-1, l\} \qquad \text{for } k = 2l.$$

The following easy lemma is given for the convenience of the reader and is used in the upcoming proofs.

Lemma 4.1.4. *Let Γ be a graph with girth $g(\Gamma) \geq 5$. Then vertices i, j with $d(i, j) = 2$ have exactly one common neighbor.*

Proof. By definition, vertices i, j with $d(i, j) = 2$ have a common neighbor, say s. If there exists another common neighbor $t \neq s$, then we get a quadrangle i, s, j, t, i in the graph which contradicts $g(\Gamma) \geq 5$. □

The next lemma describes for the odd graphs how subsets corresponding to vertices in distance two are related. This will be used freely in the upcoming theorem.

Lemma 4.1.5. *Let O_k be an odd graph, $k \geq 3$ and consider two vertices $v, w \in V_{O_k}$. Then $d(v, w) = 2$ if and only if the corresponding $(k-1)$-subsets of $\{1, \dots, 2k-1\}$ have exactly $k-2$ elements in common.*

Proof. By definition, it holds $d(v, w) = 2$ if and only if v, w are distinct, non-adjacent and they have a common neighbor. Looking at the definition of O_k, we see that v, w have a common neighbor if and only if there is a $(k-1)$-subset of $\{1, \dots, 2k-1\}$ that is disjoint to the subsets corresponding to v and w. But this is true exactly if the union of the corresponding subsets has equal or less than k elements, which also shows that v and w are not adjacent if they have a common neighbor. Since we are dealing with $(k-1)$-subsets, having less than k elements already implies $v = w$, therefore the union of the subsets has exactly k elements. It is easy to see that the union of $(k-1)$-subsets has k elements if and only if the subsets have $k-2$ elements in common. □

Example 4.1.6. The graph O_2 is the triangle and O_3 is the Petersen graph (see Figure 1.1). We know that O_2 has no quantum symmetry by Example 2.1.8. We will see that the Petersen graph also has no quantum symmetry.

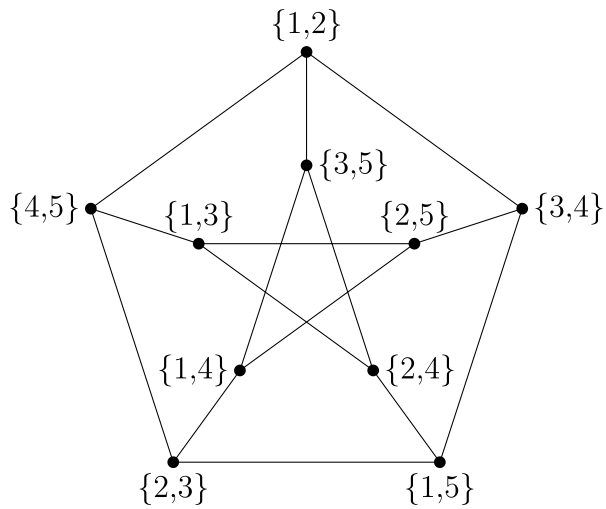


Figure 4.1: The odd graph O_3 is isomorphic to the Petersen graph

We are now ready to prove Theorem C (*ii*).

Theorem 4.1.7. *The odd graphs have no quantum symmetry.*

Proof. Since we know that O_2 has no quantum symmetry, we can assume $k \geq 3$. Then we know that O_k has girth $g(O_k) \geq 5$ and thus we get $G^+_{aut}(O_k) = G^*_{aut}(O_k)$ by Lemma 3.2.5, i.e. Relation (2.1.6) holds.

The odd graph O_k has diameter $k-1$. Taking this and Lemma 3.2.2 into account, it remains to show $u_{ij}u_{pq} = u_{pq}u_{ij}$ for $2 \leq d(i,p) = d(j,q) \leq k-1$.

Take $d(i,p) = d(j,q) = 2$. We want to show $u_{ij}u_{pq} = u_{pq}u_{ij}$. Since O_k is distance-transitive, it is enough to show $u_{ij}u_{pq} = u_{pq}u_{ij}$ for the vertices

- $j = \{1, \dots, k-1\}$,
- $q = \{1, \dots, k-2, k\}$

by Lemma 3.2.4.

Step 1: It holds $u_{ij}u_{pq} = u_{ij}u_{pq} \sum_{\substack{s=1,\\ s\neq k-1}}^{k} u_{id_s}$, *where* d_s *is the vertex* $\{1, \dots, k\} \backslash \{s\}$.

By Lemma 4.1.4, the vertices j and q have exactly one common neighbor. This is

$a = \{k+1, \ldots, 2k-1\}$. Since $G^+_{aut}(O_k) = G^*_{aut}(O_k)$, we get

$$u_{ij}u_{pq} = u_{ij}u_{pq} \sum_{\substack{b;d(b,q)=2,\\(a,b)\in E}} u_{ib}$$

by Lemma 3.2.8. Let us now take a look at neighbors b of a in distance 2 to q. Neighbors of $a = \{k+1, \ldots, 2k-1\}$ are $d_s = \{1, \ldots, k\}\backslash\{s\}$, where $s = 1, \ldots, k$. Those are in distance two to q if $s \neq k-1$ and we have $d_k = j$. Thus

$$u_{ij}u_{pq} = u_{ij}u_{pq} \sum_{\substack{b;d(b,q)=2,\\(a,b)\in E}} u_{ib} = u_{ij}u_{pq} \sum_{\substack{s=1,\\s\neq k-1}}^{k} u_{id_s}.$$

Step 2: It holds $u_{ij}u_{pq}u_{id_s} = 0$ for all $s \in \{1, \ldots, k-2\}$.
Take d_s with $s \in \{1, \ldots, k-2\}$. Let $t = \{1, \ldots, k-2, k+1\}$. We get that $d(j,t) = 2$ and $d(q,t) = 2$ since they have the common neighbor $\{k, k+2, \ldots, 2k-1\}$ and $\{k-1, k+2, \ldots 2k-1\}$, respectively. Because $t \cup d_s = \{1, \ldots, k+1\}$, we see that there is no $(k-1)$-subset of $\{1, \ldots, 2k-1\}$ disjoint to both t and d_s and we deduce $d(t, d_s) \neq 2$. Furthermore, we get that q and $r_s = \{1, \ldots, k-1, k+1\}\backslash\{s\}$ are the only vertices in distance two to j, d_s and t. This holds since the only $(k-1)$-subsets of $\{1, \ldots, 2k-1\}$ that have $k-2$ elements in common with $\{1, \ldots, k-1\}$, $\{1, \ldots, k\}\backslash\{s\}$, $s \neq k-1, k$ and $\{1, \ldots, k-2, k+1\}$ are $\{1, \ldots, k-2, k\}$ and $\{1, \ldots, k-1, k+1\}\backslash\{s\}$. Now, Lemma 3.2.9 yields

$$u_{ij}(u_{pq} + u_{pr_s})u_{id_s} = 0. \tag{4.1.1}$$

Since we have $g(O_k) \geq 5$, i and p have exactly one common neighbor by Lemma 4.1.4. We denote the common neighbor by c. Recall that a is the only common neighbor of j and q. Using Equation (4.1.1), we get

$$u_{ca}u_{ij}(u_{pq} + u_{pr_s})u_{id_s} = 0$$

and because of Relation (2.1.6), we obtain

$$u_{ij}u_{ca}(u_{pq} + u_{pr_s})u_{id_s} = 0.$$

Since the sets $\{k+1, \ldots, 2k-1\}$ and $\{1, \ldots k-1, k+1\}\backslash\{s\}$ are not disjoint, we have $(a, r_s) \notin E$. But we know $(c, p) \in E$ by the choice of c, thus we get $u_{ij}u_{ca}u_{pr_s}u_{id_s} = 0$ by Relation (2.1.5). This yields

$$u_{ij}u_{ca}u_{pq}u_{id_s} = 0. \tag{4.1.2}$$

The vertex a is the only common neighbor of j and p, therefore it holds

$$u_{ij}u_{ca}u_{pq} = u_{ij}\left(\sum_e u_{ce}\right)u_{pq} = u_{ij}u_{pq}$$

by Relations (2.1.2) and (2.1.5). We deduce

$$u_{ij}u_{pq}u_{id_s} = u_{ij}u_{ca}u_{pq}u_{id_s} = 0$$

from Equation (4.1.2).

Step 3: It holds $u_{ij}u_{pq} = u_{pq}u_{ij}$.
Recall that $d_k = j$. From previous steps, we get

$$u_{ij}u_{pq} = u_{ij}u_{pq}\sum_{\substack{s=1,\\ s\neq k-1}}^{k} u_{id_s} = u_{ij}u_{pq}u_{ij}$$

and Lemma 3.2.1 yields $u_{ij}u_{pq} = u_{pq}u_{ij}$.

By the previous considerations, it remains to show $u_{ij}u_{pq} = u_{pq}u_{ij}$ for $3 \leq d(i,p) = d(j,q) \leq k-1$. We have $c_2 = 1$, $b_1 + 1 = b_0$ and $c_d \geq 2$ for all $d \geq 3$ in the intersection array of O_k and thus we obtain the desired equations by using Lemma 3.2.10 (a) $(k-3)$-times. □

4.1.2 Hamming graphs

In this subsection, we give a precise description for which values $d, q \in \mathbb{N}$ the Hamming graph $H(d,q)$ has quantum symmetry and for which it does not. Hamming graphs include the n-cube graphs, for which the quantum automorphism groups are already known from [9].

Definition 4.1.8. Let $S = \{1, \dots, q\}$ for $q \in \mathbb{N}$ and let $d \in \mathbb{N}$. The *Hamming graph* $H(d,q)$ is the graph with vertex set S^d, where vertices are adjacent if they differ in exactly one coordinate.

Remark 4.1.9. Note that vertices in $H(d,q)$ are in distance m to each other if and only if they differ in exactly m coordinates.

We state some properties of the Hamming graphs in the following remark, see for example [18, Theorem 9.2.1] and [19, Subsection 12.4.1].

Remark 4.1.10. The Hamming graphs are distance-transitive and we have $H(d,q) = K_q^{\square d}$, where $\square$ denotes the Cartesian product of graphs (see Definition 1.2.13).

Hamming graphs include the following families of graphs.

Example 4.1.11.

(i) The Hamming graphs $H(d,1)$ are the complete graphs K_d.

(ii) For $q = 2$, we obtain the n-cube graphs.

(iii) The Hamming graphs $H(2,q)$ are the $q \times q$ rook's graphs.

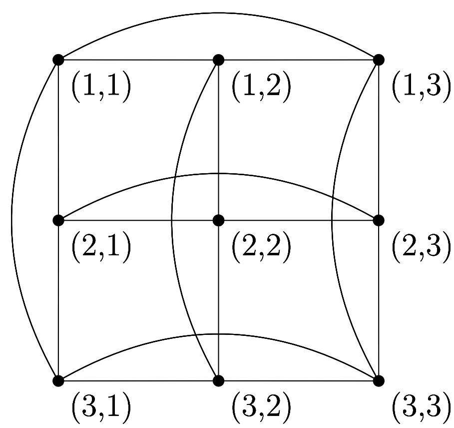


Figure 4.2: The Hamming graph $H(2,3)$

The following proposition is an easy consequence of Proposition 2.2.1.

Proposition 4.1.12. *Let $q \geq 4$, $d \in \mathbb{N}$ or $q = 2$, $d \geq 2$. Then $H(d,q)$ has quantum symmetry.*

Proof. Let $q \geq 4$, $d \in \mathbb{N}$. We know $H(d,q) = K_q^{\square d}$ and by Proposition 2.2.1, we have a surjective *-homomorphism $\varphi : C(G^+_{aut}(H(d,q)) \to C(S_q^+) \otimes C(G^+_{aut}(K_q^{\square d-1}))$. Thus, if $q \geq 4$, this yields that $C(G^+_{aut}(H(d,q)))$ is non-commutative, because $C(S_q^+)$ is non-commutative.

Let $q = 2$, $d \geq 2$. We get a surjective *-homomorphism $\varphi : C(G^+_{aut}(H(d,2)) \to C(H_2^+) \otimes C(G^+_{aut}(K_2^{\square d-2}))$ by Proposition 2.2.1. Thus the C^*-algebra $C(G^+_{aut}(H(d,2))$ is non-commutative, since $C(H_2^+)$ is non-commutative. □

Theorem C (iii) says that we know whether or not $H(d,q)$ has quantum symmetry for all possible values of $d, q \in \mathbb{N}$. By the previous proposition, we know that $H(d,q)$ has quantum symmetry for $q \geq 4$, $d \in \mathbb{N}$ or $q = 2$, $d \geq 2$. For $q = 2$ and $d = 1$, we have $H(1,2) = K_2$, which has no quantum symmetry by Example 2.1.8. We also know that $H(d,1) = K_d$ by Example 4.1.11, where we know that those have quantum symmetry for $d \geq 4$ and do not have quantum symmetry for $d \leq 3$ by Example 2.1.8. Therefore, the cases $q = 3, d \in \mathbb{N}$ remain. The following theorem completes the proof of Theorem C *(iii)*.

Theorem 4.1.13. *The Hamming graphs $H(d,3)$ do not have quantum symmetry, for $d \in \mathbb{N}$.*

Proof. Let i,j be adjacent vertices. Thus they differ in exactly one coordinate $i_s \neq j_s$. Since we have $q = 3$, this means that there is only one vertex that differs in exactly one coordinate to i and j, namely k with $k_a = i_a = j_a$ for all $a \neq s$ and $k_s \neq i_s$, $k_s \neq j_s$. Therefore, adjacent vertices have exactly one neighbor and we get $G^+_{aut}(H(d,3)) = G^*_{aut}(H(d,3))$ by Lemma 3.2.6. Hence Relation (2.1.6) holds.

The Hamming graph $H(d,3)$ has diameter d. Using Lemma 3.2.2, it remains to show $u_{ij}u_{kl} = u_{kl}u_{ij}$ for all i,j,k,l with $2 \leq d(i,k) = d(j,l) \leq d$ to obtain $G^+_{aut}(H(d,3)) = \mathrm{Aut}(H(d,3))$. Consider

- $s = (1, \dots, 1)$,
- $t^{(m)} = (t_1^{(m)}, \dots t_d^{(m)})$, where $t_1^{(m)} = \dots = t_m^{(m)} = 2$, $t_{m+1}^{(m)} = \dots = t_d^{(m)} = 1$ for $2 \leq m \leq d$,
- $p_1 = (2, 1, \dots, 1)$ and $p_2 = (1, 2, 1, \dots, 1)$.

Step 1: The only common neighbor of p_1 and p_2 in distance m to $t^{(m)}$ is s.
The only common neighbors of p_1 and p_2 are s and $(2, 2, 1, \dots, 1)$. We obtain that s is the only common neighbor of p_1, p_2 in distance m to $t^{(m)}$, since it holds $d(t^{(m)}, (2, 2, 1, \dots, 1)) = m - 2$.

Step 2: We have $u_{ij}u_{kl} = u_{kl}u_{ij}$ for $d(i,k) = d(j,l) = 2$.
Let $d(i,k) = d(j,l) = 2$. By Lemma 3.2.4, it is enough to consider $j = s$ and $l = t^{(2)}$. Since we know $(s, p_1), (s, p_2) \in E$, $(t^{(2)}, p_1), (t^{(2)}, p_2) \in E$ and since we have $G^+_{aut}(H(d,3)) = G^*_{aut}(H(d,3))$, we get

$$u_{is}u_{kt^{(2)}} = u_{is}u_{kt^{(2)}} \sum_{\substack{q; d(q,t^{(2)})=2, \\ (q,p_1) \in E}} u_{iq} \quad \text{and} \quad u_{is}u_{kt^{(2)}} = u_{is}u_{kt^{(2)}} \sum_{\substack{q; d(q,t^{(2)})=2, \\ (q,p_2) \in E}} u_{iq}$$

by Lemma 3.2.8. We deduce

$$u_{is}u_{kt^{(2)}} = u_{is}u_{kt^{(2)}} \sum_{\substack{q; d(q,t^{(2)})=2, \\ (q,p_1) \in E, (q,p_2) \in E}} u_{iq}.$$

By *Step 1*, we know that s is the only common neighbor of p_1, p_2 at distance two to $t^{(2)}$. Therefore we obtain $u_{is}u_{kt^{(2)}} = u_{is}u_{kt^{(2)}}u_{is}$. Then Lemma 3.2.1 yields $u_{is}u_{kt^{(2)}} = u_{kt^{(2)}}u_{is}$.

Step 3: We have $u_{ij}u_{kl} = u_{kl}u_{ij}$ for $d(i,k) = d(j,l) = 3$.
Now, let $d(i,k) = d(j,l) = 3$. By Lemma 3.2.4, we can choose $j = s$ and $l = t^{(3)}$. Since we know $(s,p_1),(s,p_2) \in E$, $d(t^{(3)},p_1) = (t^{(3)},p_2) = 2$ and have $u_{ac}u_{bd} = u_{bd}u_{ac}$ for all a,b,c,d with $d(a,b) = d(c,d) = 2$ by *Step 2*, we get

$$u_{is}u_{kt^{(3)}} = u_{is}u_{kt^{(3)}} \sum_{\substack{q;d(q,t^{(3)})=3,\\ (q,p_1)\in E}} u_{iq} \quad \text{and} \quad u_{is}u_{kt^{(3)}} = u_{is}u_{kt^{(3)}} \sum_{\substack{q;d(q,t^{(3)})=3,\\ (q,p_2)\in E}} u_{iq}$$

by using Lemma 3.2.8. We obtain

$$u_{is}u_{kt^{(3)}} = u_{is}u_{kt^{(3)}} \sum_{\substack{q;d(q,t^{(3)})=3,\\ (q,p_1)\in E,(q,p_2)\in E}} u_{iq}$$

and get $u_{is}u_{kt^{(3)}} = u_{is}u_{kt^{(3)}}u_{is}$, since the only common neighbor of p_1, p_2 at distance three to $t^{(3)}$ is s by *Step 1*. Then Lemma 3.2.1 yields $u_{is}u_{kt^{(3)}} = u_{kt^{(3)}}u_{is}$.

Step 4: We have $u_{ij}u_{kl} = u_{kl}u_{ij}$ for $d(i,k) = d(j,l) \geq 4$.
Repeating the argument above $(d-3)$-times yields the assertion. □

Remark 4.1.14. In a sense, the values for which the Hamming graphs do have quantum symmetry are not surprising. The Hamming graphs have quantum symmetry if and only if Theorem 3.1.2 applies, i.e. the automorphism group of $H(d,q)$ contains disjoint automorphisms if and only if $H(d,q)$ has quantum symmetry.

4.1.3 The Johnson graphs $J(n,2)$ and the Kneser graphs $K(n,2)$

In the following we show that $J(n,2)$ and $K(n,2)$ have no quantum symmetry for $n \geq 5$. For $n < 5$, the quantum automorphism groups of $J(n,2)$ and $K(n,2)$ are already known from [6]. More generally, there are Johnson graphs $J(n,k)$ and Kneser graphs $K(n,k)$ for $k \in \mathbb{N}$, where we do not know the quantum automorphism groups for the cases $k > 2$. Since we know that the odd graphs O_k are the Kneser graphs $K(2k-1,k-1)$, we dealt with some special case in Subsection 4.1.1. We have to leave the other cases open, see also Chapter 8.

Definition 4.1.15. Let $n,k \in \mathbb{N}$.

(i) The *Johnson graph* $J(n,k)$ is the graph with vertices corresponding to k-subsets of $\{1,\dots,n\}$, where two vertices are connected if and only if the intersection of the corresponding subsets has $(k-1)$ elements.

(ii) The *Kneser graph* $K(n,k)$ is the graph with vertices corresponding to k-subsets of $\{1,\dots,n\}$, where two vertices are connected if and only if the corresponding subsets are disjoint.

Example 4.1.16.

(i) The Kneser graphs $K(n,1)$ are the complete graphs K_n.

(ii) The Johnson graphs $J(n,2)$ are the line graphs of the complete graphs K_n.

(iii) The Kneser graphs $K(2k-1,k-1)$ are the odd graphs O_k.

Remark 4.1.17. The Kneser graphs $K(n,2)$ are distance-transitive with diameter 2, see [18, Theorem 9.1.2]. For $n \leq 4$, the quantum automorphism groups of $K(n,2)$ are known, since $K(4,2) = 3K_2$ (the disjoint union of 3 copies of K_2). Note that $K(5,2) = \mathrm{P}$, where P denotes the Petersen graph (see Figure 1.1).

The following gives a proof of Theorem C (*iv*).

Theorem 4.1.18. *For $n \geq 5$, the Johnson graph $J(n,2)$ and the Kneser graph $K(n,2)$ do not have quantum symmetry.*

Proof. We show that $J(n,2)$ has no quantum symmetry for $n \geq 5$. This suffices because $K(n,2)$ is the complement of $J(n,2)$.

Let $(i,k),(j,l) \in E$. We want to prove $u_{ij}u_{kl} = u_{kl}u_{ij}$. Since $J(n,2)$ is distance-transitive, it suffices to show this for

- $j = \{1,2\}$,
- $l = \{1,3\}$

by Lemma 3.2.4.

Step 1: We have $u_{ij}u_{kl} = u_{ij}u_{kl}\left(\sum_{\substack{a=2,\\ a\neq 3}}^{n} u_{i\{1,a\}} + \sum_{\substack{b=2,\\ b\neq 3}}^{n} u_{i\{3,b\}}\right)$.

By Relations (2.1.2), (2.1.5), it holds

$$u_{ij}u_{kl} = u_{ij}u_{kl}\sum_{p;(l,p)\in E} u_{ip}.$$

Since $l = \{1,3\}$, it has neighbors $\{1,a\}$, $\{3,b\}$, $a,b \neq 1,3$ and thus

$$u_{ij}u_{kl} = u_{ij}u_{kl}\sum_{p;(l,p)\in E} u_{ip} = u_{ij}u_{kl}\left(\sum_{\substack{a=2,\\ a\neq 3}}^{n} u_{i\{1,a\}} + \sum_{\substack{b=2,\\ b\neq 3}}^{n} u_{i\{3,b\}}\right).$$

Step 2: It holds $u_{ij}u_{kl}u_{i\{1,a\}} = 0$ for $a \in \{4, \dots, n\}$ and $u_{ij}u_{kl}u_{i\{2,3\}} = 0$.
Let $p = \{1, a\}$. The common neighbors of p and $j = \{1, 2\}$ are $\{2, a\}$ and $\{1, c\}$ for $c \notin \{1, 2, a\}$. Therefore

$$u_{ij}\left(u_{k\{2,a\}} + \sum_{\substack{c=3,\\ c\neq a}}^{n} u_{k\{1,c\}}\right) u_{ip} = u_{ij}\left(\sum_{s;(s,j)\in E,(s,p)\in E} u_{ks}\right) u_{ip} = u_{ij}u_{ip} = 0 \quad (4.1.3)$$

as $j \neq p$. The only common neighbors of j, p and $\{2, d\}$, where $d \notin \{1, 2, a\}$, are $\{2, a\}$ and $\{1, d\}$. We also know $d(j, \{2, d\}) = 1 \neq 2 = d(p, \{2, d\})$ and thus we obtain

$$u_{ij}(u_{k\{2,a\}} + u_{k\{1,d\}})u_{ip} = 0 \quad (4.1.4)$$

for all such d by Lemma 3.2.9. This yields

$$u_{ij}(u_{k\{2,a\}} + u_{k\{1,d\}})u_{ip} = 0 = u_{ij}(u_{k\{2,a\}} + u_{k\{1,3\}})u_{ip}$$

and we deduce

$$u_{ij}u_{k\{1,d\}}u_{ip} = u_{ij}u_{k\{1,3\}}u_{ip}$$

for $d \notin \{1, 2, a\}$. Putting this into Equation (4.1.3), we infer

$$u_{ij}(u_{k\{2,a\}} + (n - 3)u_{k\{1,3\}})u_{ip} = 0.$$

Using Equation (4.1.4) with $d = 3$, we get

$$(n - 4)u_{ij}u_{k\{1,3\}}u_{ip} = 0.$$

Since we assumed $n \geq 5$, we obtain $u_{ij}u_{k\{1,3\}}u_{ip} = u_{ij}u_{kl}u_{ip} = 0$. Furthermore, we also get $u_{ij}u_{k\{2,a\}}u_{i\{1,a\}} = 0$ by Equation (4.1.4). By repeating the arguments for $p = \{2, 3\}$, one obtains $u_{ij}u_{k\{1,3\}}u_{i\{2,3\}} = u_{ij}u_{kl}u_{i\{2,3\}} = 0$.

Step 3: It holds $u_{ij}u_{kl}u_{i\{3,b\}} = 0$ for $b \in \{4, \dots, n\}$.
Let $p = \{3, b\}$, $b \in \{4, \dots, n\}$. Since $l = \{1, 3\}$ and $\{1, b\}$ are the only common neighbors of $j = \{1, 2\}$, p and $\{1, e\}$, where $e \notin \{1, 2, 3, b\}$, we have

$$u_{ij}(u_{kl} + u_{k\{1,b\}})u_{ip} = 0 \quad (4.1.5)$$

by Lemma 3.2.9, because $(j, \{1, e\}) \in E, (p, \{1, e\}) \notin E$. Now, multiplying Equation (4.1.5) by $u_{ip}u_{kl}$ from the left, we obtain

$$u_{ip}u_{kl}u_{ij}(u_{kl} + u_{k\{1,b\}})u_{ip} = u_{ip}u_{kl}u_{ij}u_{kl}u_{ip} + u_{ip}u_{kl}u_{ij}u_{k\{1,b\}}u_{ip} = 0.$$

Similar to $u_{ij}u_{kl}u_{i\{1,a\}} = 0$ (see Step *2*), we obtain $u_{kl}u_{ij}u_{k\{1,b\}} = 0$. Thus, we get

$$u_{ip}u_{kl}u_{ij}u_{kl}u_{ip} = 0,$$

which implies $u_{ij}u_{kl}u_{ip} = 0$.

Step 4: We have $u_{ij}u_{kl} = u_{kl}u_{ij}$ for $(i,k),(j,l) \in E$.
From *Steps 1–3*, we deduce that all but one summand give a zero contribution to the term. We get

$$u_{ij}u_{kl} = u_{ij}u_{kl}\left(\sum_{\substack{a=2,\\ a\neq 3}}^{n} u_{i\{1,a\}} + \sum_{\substack{b=2,\\ b\neq 3}}^{n} u_{i\{3,b\}}\right) = u_{ij}u_{kl}u_{ij}$$

and therefore obtain $u_{ij}u_{kl} = u_{kl}u_{ij}$ for $(i,k),(j,l) \in E$ by Lemma 3.2.1.

Step 5: We have $u_{ij}u_{kl} = u_{kl}u_{ij}$ for $d(i,k) = d(j,l) = 2$.
Let $d(i,k) = 2 = d(j,l)$. We show $u_{ij}u_{kl} = u_{kl}u_{ij}$, where we can choose

- $j = \{1,2\}$,
- $l = \{3,4\}$

by Lemma 3.2.4. The vertices $\{1,3\}$, $\{2,4\}$ are common neighbors of j and l. By Lemma 3.2.8, we get

$$u_{ij}u_{kl} = u_{ij}u_{kl} \sum_{\substack{\{a,b\};\{a,b\}\cap\{3,4\}=\emptyset,\\ (\{a,b\},\{1,3\})\in E}} u_{i\{a,b\}} = u_{ij}u_{kl} \sum_{\substack{\{a,b\};\{a,b\}\cap\{3,4\}=\emptyset,\\ 1\in\{a,b\}}} u_{i\{a,b\}}$$

and

$$u_{ij}u_{kl} = u_{ij}u_{kl} \sum_{\substack{\{c,d\};\{c,d\}\cap\{3,4\}=\emptyset,\\ (\{c,d\},\{2,4\})\in E}} u_{i\{c,d\}} = u_{ij}u_{kl} \sum_{\substack{\{c,d\};\{c,d\}\cap\{3,4\}=\emptyset,\\ 2\in\{c,d\}}} u_{i\{c,d\}}.$$

We deduce

$$u_{ij}u_{kl} = u_{ij}u_{kl}\left(\sum_{\substack{\{a,b\};\{a,b\}\cap\{3,4\}=\emptyset,\\ 1\in\{a,b\}}} u_{i\{a,b\}}\right)\left(\sum_{\substack{\{c,d\};\{c,d\}\cap\{3,4\}=\emptyset,\\ 2\in\{c,d\}}} u_{i\{c,d\}}\right).$$

Since we know $u_{i\{a,b\}}u_{i\{c,d\}} = 0$ for $\{a,b\} \neq \{c,d\}$, we obtain

$$\begin{aligned}
u_{ij}u_{kl} &= u_{ij}u_{kl}\left(\sum_{\substack{\{a,b\};\{a,b\}\cap\{3,4\}=\emptyset,\\ 1\in\{a,b\}}} u_{i\{a,b\}}\right)\left(\sum_{\substack{\{c,d\};\{c,d\}\cap\{3,4\}=\emptyset,\\ 2\in\{c,d\}}} u_{i\{c,d\}}\right)\\
&= u_{ij}u_{kl}\left(\sum_{\substack{\{a,b\};\{a,b\}\cap\{3,4\}=\emptyset,\\ 1\in\{a,b\},2\in\{a,b\}}} u_{i\{a,b\}}\right)\\
&= u_{ij}u_{kl}u_{ij},
\end{aligned}$$

since $\{1,2\}$ is the only subset containing 1 and 2. Then Lemma 3.2.1 completes the proof, since $J(n,2)$ has diameter 2. □

Remark 4.1.19. The author did not succeed in generalizing the proof above to the Johnson graphs $J(n,k)$, $k \geq 3$. The problem is the following. Consider for example the Johnson graph $J(6,3)$. As in the previous proof, we choose some vertices, say $j = \{1,2,3\}$ and $l = \{1,2,4\}$. Similar to *Step 2* and *Step 3*, we want to show $u_{ij}u_{kl}u_{ip} = 0$ for all neighbors p of l. Take for example $p = \{1,2,5\}$. Then $j = \{1,2,3\}$ and $p = \{1,2,5\}$ have the common neighbors $\{1,2,4\}$, $\{1,2,6\}$, $\{1,3,5\}$ and $\{2,3,5\}$. Similar to Equation (4.1.3), we know

$$u_{ij}(u_{kl} + u_{k\{1,2,6\}} + u_{k\{1,3,5\}} + u_{k\{2,3,5\}})u_{ip} = 0.$$

But we are not able to deduce $u_{ij}u_{kl}u_{ip} = 0$ from this.

Therefore, we need another idea to prove the theorem in full generality. One should also keep in mind that it is possible that the Johnson graphs do have quantum symmetry for some parameters n, k with $k \geq 3$.

4.1.4 Moore graphs of diameter two

We show that the Moore graphs of diameter two have no quantum symmetry. Those are precisely the strongly regular graphs with girth five or equivalently all strongly regular graphs with $\mu = 0$, $\lambda = 1$. Recall from Definition 1.2.18 that μ denotes the number of common neighbors of adjacent vertices, while λ is the number of common neighbors of non-adjacent vertices. Hoffman and Singleton showed in [31] that the only possible degrees for those graphs are 2, 3, 7 and 57. For the degrees 2, 3 and 7 there exist unique strongly regular graphs with girth five: the 5-cycle, the Petersen graph and the Hoffman-Singleton graph. The existence of such a graph of degree 57 is still an open problem, see for example [18, Section 6.7]. The next theorem proves Theorem C (v).

Theorem 4.1.20. *Strongly regular graphs with girth five have no quantum symmetry.*

Proof. Since the graph Γ has girth five, we get $G^+_{aut}(\Gamma) = G^*_{aut}(\Gamma)$ by Lemma 3.2.5.

Let $d(i,k) = d(j,l) = 2$. It remains to show $u_{ij}u_{kl} = u_{kl}u_{ij}$ by Lemma 3.2.2, since strongly regular graphs have diameter two.

Step 1: We have $u_{ij}u_{kl} = u_{ij}u_{kl} \sum_{\substack{p; d(p,l)=2, \\ (p,t)\in E}} u_{ip}$, *where t is the only common neighbor of j and l.*

By Lemma 4.1.4, there exists exactly one vertex s such that $(i,s) \in E, (k,s) \in E$ and exactly one vertex t such that $(j,t) \in E, (l,t) \in E$. We get

$$u_{ij}u_{kl} = u_{ij}u_{st}u_{kl},$$

by Relations (2.1.2), (2.1.5) and it holds

$$u_{ij}u_{kl} = u_{ij}u_{kl} \sum_{\substack{p; d(p,l)=2, \\ (p,t)\in E}} u_{ip},$$

because we have $G^+_{aut}(\Gamma) = G^*_{aut}(\Gamma)$ and thus we can use Lemma 3.2.8.

Step 2: We have $u_{ij}u_{kl}u_{ip} = 0$ for $p \neq j$ with $d(p,l) = 2$, $(p,t) \in E$.

If Γ is 2-regular (the 5-cycle) then we are done, because the only vertex p with $d(p,l) = 2$, $(p,t) \in E$ is j. Since strongly regular graphs are especially regular, we can assume that Γ is n-regular with $n \geq 3$ in the remaining part of the proof. Take $p \neq j$ with $d(p,l) = 2, (p,t) \in E$. It holds

$$u_{ij}u_{kl}u_{ip} = u_{ij}u_{kl}u_{st}u_{ip} = u_{ij}\left(\sum_{\substack{a; (a,k)\in E, \\ d(a,i)=2}} u_{ab}\right) u_{kl}u_{st}u_{ip},$$

where we choose $b \neq t$ with $(b,l) \in E$, which implies $d(b,j) = d(b,p) = 2$. Because of Lemma 4.1.4, we know that l is the only common neighbor of b and t. We deduce

$$u_{ij}\left(\sum_{\substack{a; (a,k)\in E, \\ d(a,i)=2}} u_{ab}\right) u_{kl}u_{st}u_{ip} = u_{ij}\left(\sum_{\substack{a; (a,k)\in E, \\ d(a,i)=2}} u_{ab}\right) u_{st}u_{ip}$$

by Relations (2.1.2), (2.1.5). By Lemma 4.1.4, there exist exactly one vertex e such that $(i,e) \in E, (e,a) \in E$ for all a with $d(a,i) = 2$ and exactly one vertex f such that $(j,f) \in E, (b,f) \in E$. This yields

$$u_{ij}u_{ab}u_{st}u_{ip} = u_{ij}u_{ef}u_{ab}u_{st}u_{ip} = u_{ij}u_{ef}u_{ab}u_{ip}u_{st},$$

where we also used $u_{ip}u_{st} = u_{st}u_{ip}$. Because of $u_{ef}u_{ab} = u_{ab}u_{ef}$, we get

$$u_{ij}u_{ab}u_{st}u_{ip} = u_{ij}u_{ef}u_{ab}u_{ip}u_{st} = u_{ij}u_{ab}u_{ef}u_{ip}u_{st}.$$

It holds $(f,p) \notin E$, because otherwise j and p would have two common neighbors, t and f, where we know $t \neq f$ since we have $(b,f) \in E$ whereas $d(b,t) = 2$. Because we know $(i,e) \in E$, we obtain

$$u_{ij}u_{ab}u_{st}u_{ip} = u_{ij}u_{ab}u_{ef}u_{ip}u_{st} = 0,$$

by Relation (2.1.5). Summarizing, we get

$$u_{ij}u_{kl}u_{ip} = u_{ij}\left(\sum_{\substack{a;(a,k)\in E,\\ d(a,i)=2}} u_{ab}\right)u_{st}u_{ip} = \sum_{\substack{a;(a,k)\in E,\\ d(a,i)=2}} u_{ij}u_{ab}u_{st}u_{ip} = 0.$$

Step 3: We have $u_{ij}u_{kl} = u_{kl}u_{ij}$.
By the previous steps, we conclude

$$u_{ij}u_{kl} = u_{ij}u_{kl}\sum_{\substack{p;d(p,l)=2\\ (p,t)\in E}} u_{ip} = u_{ij}u_{kl}u_{ij},$$

which implies that u_{ij} and u_{kl} commute by Lemma 3.2.1. This completes the proof. □

Remark 4.1.21. Taking the previous theorems in account, the only new insight we get from Theorem 4.1.20 is that the Hoffman-Singleton graph has no quantum symmetry. Also, if the strongly regular graph with parameters $(3250, 57, 0, 1)$ exists, then it has no quantum symmetry by the previous theorem.

4.1.5 Paley graphs P_9, P_{13} and P_{17}

Paley graphs were introduced by Erdős and Renyi in [26], where they studied their symmetries. The Paley graphs are constructed using finite fields. We use this construction to show that P_9, P_{13} and P_{17} have no quantum symmetry. The quantum symmetry of Paley graphs was also studied by Chassaniol in [22], see Remark 4.1.26.

Definition 4.1.22. Let q be a prime power with $q = 1 \bmod 4$ and let $\mathbb{F}_q$ be the finite field with q elements. The *Paley graph* P_q is the graph with vertex set $\mathbb{F}_q$, where vertices are connected if and only if their difference is a square in $\mathbb{F}_q$.

Remark 4.1.23. The Paley graphs are distance-transitive and self-complementary.

Theorem 4.1.24. *Let P_q be the Paley graph on q vertices. Then P_q does not have quantum symmetry if there exist adjacent vertices $v, w \in V_{P_q}$ with the following property: For every neighbor $s \neq v$ of w, there exists a vertex t that is either*

(a) adjacent to w with $d(v,s) \neq d(s,t)$ such that w is the only common neighbor of v and s that is adjacent to t, or

(b) non-adjacent to w with $d(v,s) \neq d(s,t)$ such that w is the only common neighbor of v and s that is non-adjacent to t.

Proof. Let u_{ij}, $i,j \in \mathbb{F}_q$ be the generators of $C(G^+_{aut}(\Gamma))$. Let $(i,k),(j,l) \in E$. We want to show $u_{ij}u_{kl} = u_{kl}u_{ij}$. By Lemma 3.2.4, it is enough to prove this for $j = v$, $l = w$, where v, w are the vertices with the property from above. We have

$$u_{iv}u_{kw} = u_{iv}u_{kw} \sum_{s;(s,w)\in E} u_{is},$$

by Relations (2.1.2), (2.1.5). By the assumptions on v and w, we get $u_{ij}u_{kl}u_{is} = 0$ for all $s \neq v$ by Lemma 3.2.9. This yields $u_{iv}u_{kw} = u_{iv}u_{kw}u_{iv}$. We conclude $u_{iv}u_{kw} = u_{kw}u_{iv}$ by Lemma 3.2.1.

The Paley graphs have diameter two, therefore it remains to show $u_{ij}u_{kl} = u_{kl}u_{ij}$ for $(i,k),(j,l) \notin E$. Since P_q is self-complementary, the same arguments as above work for $(i,k),(j,l) \notin E$ and we get that P_q has no quantum symmetry. □

Corollary 4.1.25. *The Paley graphs P_9, P_{13} and P_{17} have no quantum symmetry.*

Proof. Note that the Paley graph P_9 is strongly regular with parameters (9,4,1,2). Thus, Lemma 3.2.6 yields $G^+_{aut}(P_9) = G^*_{aut}(P_9)$. Since P_9 is self-complementary, the arguments in the proof of Lemma 3.2.6 also work for $(i,k),(j,l) \notin E$. Thus $C(G^+_{aut}(P_9))$ is commutative.

Consider the Paley graph P_{13}. To determine the neighbors of the vertices of P_{13}, observe that $0,1,3,4,9,10,12$ are the squares in $\mathbb{F}_{13}$. Let $(i,k),(j,l) \in E$. To apply Theorem 4.1.24, we choose $v = 1$ and $w = 2$. The neighbors of 2 are $1,3,5,6,11,12$. The task is now to find for every neighbor $p \neq 1$ of 2 a vertex q, $d(q,2) = a$ with $d(1,q) \neq d(q,p)$, such that 2 is the only common neighbor of $1,p$ in distance a to q. We find the following vertices that fulfill these properties: 11 for 3, 11 for 5, 3 for 6, 5 for 11 and 5 for 12. We get that P_{13} has no quantum symmetry by Theorem 4.1.24.

Concerning the Paley graph P_{17}, observe that $0,2,4,8,9,13,15,16$ are the squares in $\mathbb{F}_{17}$. Let $(i,k),(j,l) \in E$. In virtue of Theorem 4.1.24, we choose $v = 1$ and $w = 2$. The neighbors of 2 are $1,3,4,6,10,11,15,17$. As for P_{13}, the task is to find for every neighbors $p \neq 1$ of 2 a vertex q, $d(q,2) = a$ with $d(1,q) \neq d(q,p)$, such that 2 is the only common neighbor of $1, p$ in distance a to q. We have the following vertices that fulfill these properties: 10 for 3, 10 for 4, 17 for 6, 4 for 10, 15 for 11, 11 for 15, 6 for 17. We deduce from Theorem 4.1.24 that P_{17} has no quantum symmetry. □

Remark 4.1.26. Note that it was already shown in [6] that P_9 has no quantum symmetry. In [22], it was proven that P_{13} and P_{17} have no quantum symmetry. Thus, we just give alternative proofs of those facts. One could try to get similar results for other Paley graphs P_q, $q > 17$. But using our method one has to treat them case by case, we do not get a general statement for all Paley graphs in this way. Still, note that for checking if Theorem 4.1.24 applies, one just has to check properties of the graph, there is no need in working with $G^+_{aut}(P_q)$ itself.

4.2 Quantum automorphism groups of cubic distance-transitive graphs

This section is based on [50, Section 5]. We study the quantum automorphism groups of all cubic distance-transitive graphs. Those quantum automorphism groups are known for the complete graph K_4, the complete bipartite graph $K_{3,3}$ and the cube Q_3 from [6]. The following result was established by Biggs and Smith in [16].

Theorem 4.2.1 (Biggs, Smith)**.** *There are exactly twelve cubic distance-transitive graphs.*

Thus, there are nine remaining graphs. We treat them case by case. From the results of this section, we get the following table, coinciding with Table 1.

Name of Γ	Order	$\mathrm{Aut}(\Gamma)$	$G^+_{aut}(\Gamma)$	Intersection array
K_4 ([6])	4	S_4	S_4^+	{3;1}
$K_{3,3}$ ([6])	6	$S_3 \wr \mathbb{Z}_2$	$S_3 \wr_* \mathbb{Z}_2$	{3,2;1,3}
Cube Q_3 ([6])	8	$S_4 \times \mathbb{Z}_2$	$S_4^+ \times \mathbb{Z}_2$	{3,2,1;1,2,3}
Petersen graph	10	S_5	$\mathrm{Aut}(\Gamma)$	{3,2;1,1}
Heawood graph	14	$PGL(2,7)$	$\mathrm{Aut}(\Gamma)$	{3,2,2;1,1,3}
Pappus graph	18	ord 216	$\mathrm{Aut}(\Gamma)$	{3,2,2,1;1,1,2,3}
Desargues graph	20	$S_5 \times \mathbb{Z}_2$	$\mathrm{Aut}(\Gamma)$	{3,2,2,1,1;1,1,2,2,3}
Dodecahedron	20	$A_5 \times \mathbb{Z}_2$	$\mathrm{Aut}(\Gamma)$	{3,2,1,1,1;1,1,1,2,3}
Coxeter graph	28	$PGL(2,7)$	$\mathrm{Aut}(\Gamma)$	{3,2,2,1;1,1,1,2}
Tutte 8-cage	30	$\mathrm{Aut}(S_6)$	$\mathrm{Aut}(\Gamma)$	{3,2,2,2;1,1,1,3}
Foster graph	90	ord 4320	$\mathrm{Aut}(\Gamma)$	{3,2,2,2,2,1,1,1;1,1,1,1,2,2,2,3}
Biggs-Smith graph	102	$PSL(2,17)$	$\mathrm{Aut}(\Gamma)$	{3,2,2,2,1,1,1;1,1,1,1,1,1,3}

Table 4.2: Quantum automorphism groups of all cubic distance-transitive graphs.

We first start with a useful lemma. The proof is similar to the one of Theorem 3.3 in [48].

Lemma 4.2.2. *Let* Γ *be a cubic graph with girth* $g(\Gamma) \geq 5$ *and let* $(u_{ij})_{1\leq i,j\leq n}$ *be the generators of* $C(G^+_{aut}(\Gamma))$*. Then we have* $u_{ij}u_{kl} = u_{kl}u_{ij}$ *for* $d(i,k) = 2 = d(j,l)$*.*

Proof. Let $d(i,k) = 2 = d(j,l)$. There exist exactly one $s \in V$ such that $(i,s) \in E, (k,s) \in E$ and exactly one $t \in V$ such that $(j,t) \in E, (l,t) \in E$ by Lemma 4.1.4. We know $u_{st}u_{kl} = u_{kl}u_{st}$ by Lemma 3.2.5 and therefore we get

$$u_{ij}u_{kl} = u_{ij}u_{kl}\left(\sum_{\substack{p;d(l,p)=2,\\(t,p)\in E}} u_{ip}\right)$$

by Lemma 3.2.8. The graph Γ is 3-regular, thus we know that t has three neighbors. Those are j, l and a third neighbor which we denote by q. Only j and q are in distance two to l. We deduce

$$u_{ij}u_{kl} = u_{ij}u_{kl}(u_{ij} + u_{iq}).$$

It holds

$$u_{ij}u_{kl} = u_{ij}\left(\sum_a u_{sa}\right)u_{kl} = u_{ij}\left(\sum_{a;(j,a)\in E,(a,l)\in E} u_{sa}\right)u_{kl} = u_{ij}u_{st}u_{kl},$$

since we know that t is the only common neighbor of j and l. Observe that

$$u_{ij}u_{st}u_{kq}u_{iq} = 0 \quad \text{and} \quad u_{ij}u_{st}u_{kj}u_{iq} = u_{ij}u_{kj}u_{st}u_{iq} = 0$$

by Relations (2.1.1) and $u_{st}u_{kj} = u_{kj}u_{st}$ from Lemma 3.2.5. We therefore get

$$\begin{aligned}
u_{ij}u_{kl}u_{iq} &= u_{ij}u_{st}u_{kl}u_{iq}\\
&= u_{ij}u_{st}(u_{kl} + u_{kj} + u_{kq})u_{iq}\\
&= u_{ij}u_{st}\left(\sum_{a;(t,a)\in E} u_{ka}\right)u_{iq}\\
&= u_{ij}u_{st}\left(\sum_{a=1}^{n} u_{ka}\right)u_{iq}\\
&= u_{ij}u_{st}u_{iq},
\end{aligned}$$

where we also used Relations (2.1.2), (2.1.5). By Relation (2.1.1) and using $u_{ij}u_{st} = u_{st}u_{ij}$, we obtain

$$u_{ij}u_{kl}u_{iq} = u_{ij}u_{st}u_{iq} = u_{st}u_{ij}u_{iq} = 0,$$

since $j \neq q$.

We conclude

$$u_{ij}u_{kl} = u_{ij}u_{kl}u_{ij}.$$

Then Lemma 3.2.1 yields $u_{ij}u_{kl} = u_{kl}u_{ij}$ and this completes the proof. □

Lemma 4.2.3. *Let Γ be a cubic distance-regular graph of order ≥ 10 and let $(u_{ij})_{1\leq i,j\leq n}$ be the generators of $C(G^+_{aut}(\Gamma))$. Then we have $u_{ij}u_{kl} = u_{kl}u_{ij}$ for $d(i,k) = d(j,l) \leq 2$.*

Proof. For the intersection array, we have $b_1 = 2$ and $c_2 = 1$ for all cubic distance-regular graphs of order ≥ 10. It follows that all those graphs have girth ≥ 5: Because Γ is 3-regular and $b_1 = 2$, adjacent vertices have no common neighbor, which means that there is no triangle in Γ. Since $c_2 = 1$, vertices in distance two only have one common neighbor. We deduce that there is no quadrangle in Γ.

Since we know that Γ has girth ≥ 5, using Lemma 3.2.5 and Lemma 4.2.2 yields $u_{ij}u_{kl} = u_{kl}u_{ij}$ for $d(i,k) = d(j,l) \leq 2$. □

In the following we study the quantum automorphism groups of the remaining nine cubic distance-transitive graphs and prove Theorem C (vi). Our strategy for proving that those graphs do not have quantum symmetry includes that previous lemma and looks like this:

(1) By Lemma 4.2.3, we know $u_{ij}u_{kl} = u_{kl}u_{ij}$ for $d(i,k) = d(j,l) \leq 2$. Using Lemma 3.2.2, it remains to prove $u_{ij}u_{kl} = u_{kl}u_{ij}$ for $d(i,k) = d(j,l) \geq 3$.

(2) Choose $m \geq 3$. First, check the intersection array of the graph to see whether or not Lemma 3.2.10 or Lemma 4.2.4 applies. If one of them can be used, we get $u_{ij}u_{kl} = u_{kl}u_{ij}$ for $d(i,k) = d(j,l) = m$. Note that Lemma 4.2.4 is specific to cubic distance-regular graphs and will be discussed when it is needed.

(3) If (2) does not work, we use the structure of the graph and tools like Lemma 4.2.5 to obtain $u_{ij}u_{kl} = u_{kl}u_{ij}$ for $d(i,k) = d(j,l) = m$.

As a reminder we write the intersection array in parantheses to the graph. We always write u_{ij}, $1 \leq i,j \leq n$, for the generators of $C(G^+_{aut}(\Gamma))$.

The Petersen graph ($\{3, 2; 1, 1\}$)

The Petersen graph (see Figure 1.1) has no quantum symmetry by Lemma 4.2.3 since it is a cubic distance-regular graph with diameter two. As already mentioned in the previous section, the result also follows from Subsections 4.1.1, 4.1.3 and 4.1.4, because the Petersen graph is isomorphic to the odd graph O_3, the Kneser graph $K(5, 2)$ and is a Moore graph of diameter two.

The Heawood graph ($\{3, 2, 2; 1, 1, 3\}$)

Since the Heawood graph (see Figure 4.3) has diameter three, we have $d(i, k)$, $d(j, l) \leq 3$ for all $i, j, k, l \in V$.

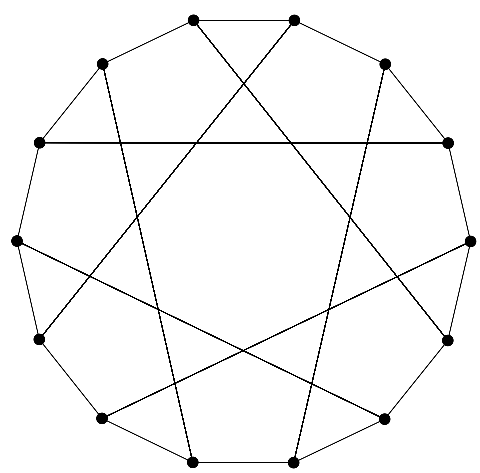


Figure 4.3: The Heawood graph

By Lemma 4.2.3, we know $u_{ij}u_{kl} = u_{kl}u_{ij}$ for $d(i, k) = d(j, l) \leq 2$. Because of Lemma 3.2.2, it just remains to prove $u_{ij}u_{kl} = u_{kl}u_{ij}$ for $d(i, k) = d(j, l) = 3$, to get that the Heawood graph has no quantum symmetry. But this follows from Lemma 3.2.10 (a), because we have $c_2 = 1$, $c_3 = 3$ and $b_1 + 1 = b_0$.

The Pappus graph ($\{3, 2, 2, 1; 1, 1, 2, 3\}$)

The Pappus graph (see Figure 4.4) has diameter four, thus $d(i, k), d(j, l) \leq 4$ for all $i, j, k, l \in V$. We know $u_{ij}u_{kl} = u_{kl}u_{ij}$ for $d(i, k) = d(j, l) \leq 2$ because of Lemma 4.2.3. It holds $c_3 = 2, c_4 = 3$, $b_1 + 1 = b_0$ and we can use Lemma 3.2.10 (a) two times to get $u_{ij}u_{kl} = u_{kl}u_{ij}$ for $3 \leq d(i, k) = d(j, l) \leq 4$. Using Lemma 3.2.2, we conclude that the Pappus graph has no quantum symmetry.

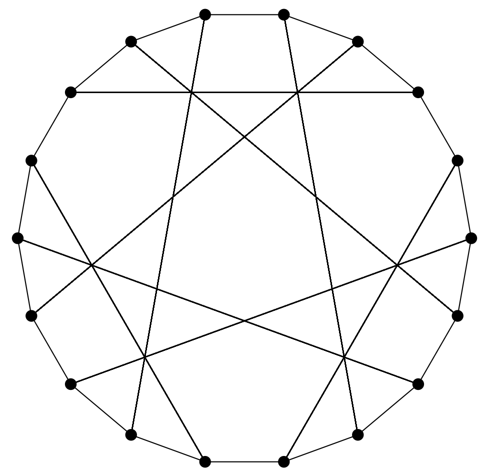


Figure 4.4: The Pappus graph

The Desargues graph ($\{3, 2, 2, 1, 1; 1, 1, 2, 2, 3\}$)

The Desargues graph (see Figure 4.5) has diameter five. Therefore, we have $d(i,k)$, $d(j,l) \leq 5$ for all $i,j,k,l \in V$. We know $u_{ij}u_{kl} = u_{kl}u_{ij}$ for $d(i,k) = d(j,l) \leq 2$ by Lemma 4.2.3. Using Lemma 3.2.2, it remains to show $u_{ij}u_{kl} = u_{kl}u_{ij}$ for $3 \leq d(i,k) = d(j,l) \leq 5$. This follows from applying Lemma 3.2.10 (a) three times, since $c_3 = 2$, $c_4 = 2$, $c_5 = 3$ and $b_1 + 1 = b_0$. Thus, the Desargues graph has no quantum symmetry.

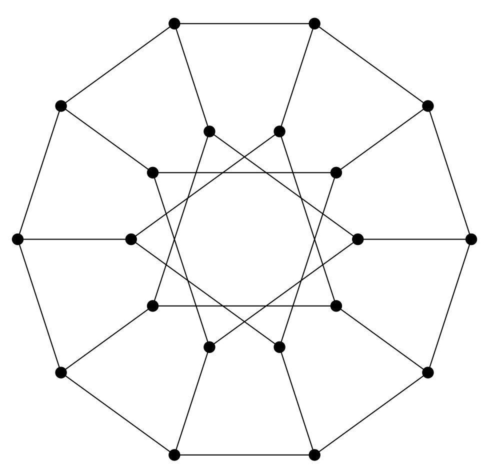


Figure 4.5: The Desargues graph

To deal with more graphs we need an additional lemma.

Lemma 4.2.4. *Let Γ be a cubic distance regular graph. If we know that $u_{ij}u_{kl} = u_{kl}u_{ij}$ for $d(i,k) = d(j,l) \leq m-1$ and it either holds*

(i) $b_{m-1} = 1$ or

(ii) $b_{m-1} = 2$ and $b_m = c_m = 1$, girth $g(\Gamma) \geq 2m$,

then we get $u_{ij}u_{kl} = u_{kl}u_{ij}$ for $d(i,k) = d(j,l) = m$.

Proof. Let $d(i,k) = d(j,l) = m$. Let t be a neighbor of j in distance $m-1$ to l. Since we know that $u_{ac}u_{bd} = u_{bd}u_{ac}$ for $d(a,b) = d(c,d) = m-1$, we get

$$u_{ij}u_{kl} = u_{ij}u_{kl} \sum_{\substack{p;d(l,p)=m,\\(t,p)\in E}} u_{ip},$$

by Lemma 3.2.8.

For (i), we have $b_{m-1} = 1$ and we deduce that j is the only neighbor of t with $d(l,j) = m$, since $d(t,l) = m-1$. Therefore

$$u_{ij}u_{kl} = u_{ij}u_{kl} \sum_{\substack{p;d(l,p)=m,\\(t,p)\in E}} u_{ip} = u_{ij}u_{kl}u_{ij},$$

and Lemma 3.2.1 yields $u_{ij}u_{kl} = u_{kl}u_{ij}$ for $d(i,k) = d(j,l) = m$.

Regarding (ii), we have $b_{m-1} = 2$. Thus there are two neighbors of t with $d(l,j) = m$, since $d(t,l) = m-1$. Those are j and another vertex q. Therefore

$$u_{ij}u_{kl} = u_{ij}u_{kl} \sum_{\substack{p;d(l,p)=m,\\(t,p)\in E}} u_{ip} = u_{ij}u_{kl}(u_{ij} + u_{jq}).$$

It holds $b_m = c_m = 1$ and since Γ is a cubic graph, this implies that there is exactly one neighbor, say s, of k at distance m to i. Similarly, we have neighbors a, b of l at distance m to j, q respectively. We deduce

$$u_{ij}u_{kl}u_{iq} = u_{ij}u_{sa}u_{kl}u_{sb}u_{iq} = u_{ij}u_{sa}u_{sb}u_{kl}u_{iq}$$

by Relations (2.1.2), (2.1.5) and since $u_{kl}u_{sb} = u_{sb}u_{kl}$. Assume $a = b$. Then we know $d(a,j) = m = d(a,q)$. We also have $d(a,t) = m$, since we know $d(l,t) = m-1$, $(l,a) \in E$ and Γ has girth $g(\Gamma) \geq 2m$. But then t has two neighbors at distance m to a, namely j and q. This contradicts the fact that there is exactly one neighbor of t at distance m to a. This yields $a \neq b$ and therefore

$$u_{ij}u_{kl}u_{iq} = u_{ij}u_{sa}u_{sb}u_{kl}u_{iq} = 0.$$

Summarizing, we get

$$u_{ij}u_{kl} = u_{ij}u_{kl} \sum_{\substack{p;d(l,p)=3,\\(t,p)\in E}} u_{ip} = u_{ij}u_{kl}(u_{ij} + u_{jq}) = u_{ij}u_{kl}u_{ij}$$

and now Lemma 3.2.1 yields $u_{ij}u_{kl} = u_{kl}u_{ij}$ for $d(i,k) = d(j,l) = m$. □

The Dodecahedron ($\{3, 2, 1, 1, 1; 1, 1, 1, 2, 3\}$)

The Dodecahedron (see Figure 4.6) has diameter five, thus we have $d(i,k), d(j,l) \leq 5$ for all $i, j, k, l \in V$. Lemma 4.2.3 yields $u_{ij}u_{kl} = u_{kl}u_{ij}$ for $d(i,k) = d(j,l) \leq 2$. We get $u_{ij}u_{kl} = u_{kl}u_{ij}$ for $d(i,k) = d(j,l) = 3$ by Lemma 4.2.4 (i), since the Dodecahedron is a cubic distance regular graph with $b_2 = 1$. Now, since we have $c_4 = 2$, $c_5 = 3$ and $b_1 + 1 = b_0$, we can use Lemma 3.2.10 (a) two times to get $u_{ij}u_{kl} = u_{kl}u_{ij}$ for $4 \leq d(i,k) = d(j,l) \leq 5$. We conclude that the Dodecahedron has no quantum symmetry by Lemma 3.2.2.

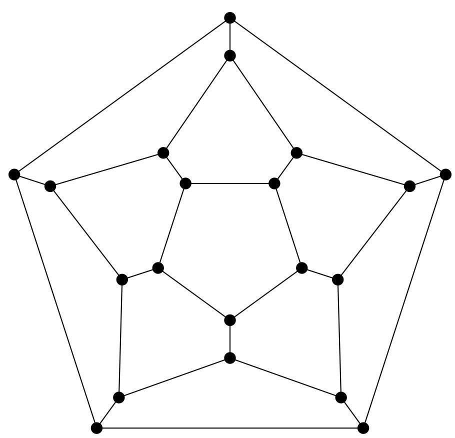


Figure 4.6: The Dodecahedral graph

The Coxeter graph ($\{3, 2, 2, 1; 1, 1, 1, 2\}$)

Since the Coxeter graph (see Figure 4.7) has diameter four, we have $d(i,k), d(j,l) \leq 4$ for all $i, j, k, l \in V$. By Lemma 4.2.3, we know $u_{ij}u_{kl} = u_{kl}u_{ij}$ for $d(i,k) = d(j,l) \leq 2$. We have $b_2 = 2, b_3 = 1, c_3 = 1$ in the intersection array of the Coxeter graph, where the Coxeter graph has girth 7. Thus, we can use Lemma 4.2.4 (ii) to get $u_{ij}u_{kl} = u_{kl}u_{ij}$ for $d(i,k) = d(j,l) = 3$. We obtain $u_{ij}u_{kl} = u_{kl}u_{ij}$ for $d(i,k) = d(j,l) = 4$ by Lemma 3.2.10 (a), since we have $c_4 = 2$ and $b_1 + 1 = b_0$. Then Lemma 3.2.2 yields that the Coxeter graph has no quantum symmetry.

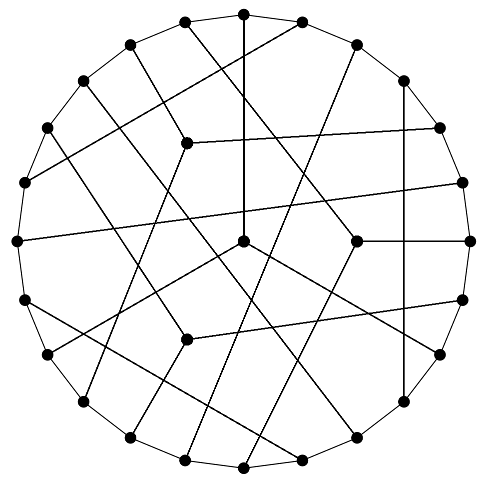


Figure 4.7: The Coxeter graph

We give the following technical lemma because it applies to the three remaining graphs.

Lemma 4.2.5. *Let Γ be a cubic distance regular graph of order ≥ 10 with $b_2 = 2$, $g(\Gamma) \geq 7$ and let $d(i,k) = d(j,l) = 3$. Then*

$$u_{ij}u_{kl} = u_{ij}u_{kl}(u_{ij} + u_{iq}),$$

where q is the unique vertex adjacent to the neighbor x of j, $d(x,l) = 2$ with $d(q,j) = 2$ and $d(l,q) = 3$.

Proof. Let $d(i,k) = d(j,l) = 3$. Let x be the unique vertex with $(j,x) \in E, d(x,l) = 2$. It is unique because we assumed $g(\Gamma) \geq 7$. By Lemma 4.2.3, we get $u_{kl}u_{yx} = u_{yx}u_{kl}$ for all $y \in V$ with $d(k,y) = 2$. We obtain

$$u_{ij}u_{kl} = u_{ij}u_{kl} \sum_{\substack{p;(x,p)\in E, \\ d(p,l)=3}} u_{ip}$$

by Lemma 3.2.8. We conclude

$$u_{ij}u_{kl} = u_{ij}u_{kl}(u_{ij} + u_{iq}),$$

because x has three neighbors where two of them are at distance three to l, since $b_2 = 2$. □

The Tutte 8-cage ($\{3, 2, 2, 2; 1, 1, 1, 3\}$)

The Tutte 8-cage (see Figure 4.8) has diameter four, thus we have $d(i,k), d(j,l) \leq 4$ for $i,j,k,l \in V$. Lemma 4.2.3 yields $u_{ij}u_{kl} = u_{kl}u_{ij}$ for $d(i,k) = d(j,l) \leq 2$. Let $d(i,k) = d(j,l) = 4$. The Tutte 8-cage is the incidence graph of the Cremona-Richmond configuration, see [23]. Therefore we can label one of the maximal independent sets by unordered 2-subsets of $\{1,\dots,6\}$, where vertices at distance two to the vertex $\{a,b\}$ are exactly those corresponding to a 2-subset that does not contain a or b (see Figure 4.8). The remaining vertices in the maximal independent set are exactly the vertices in distance four to $\{a,b\}$.

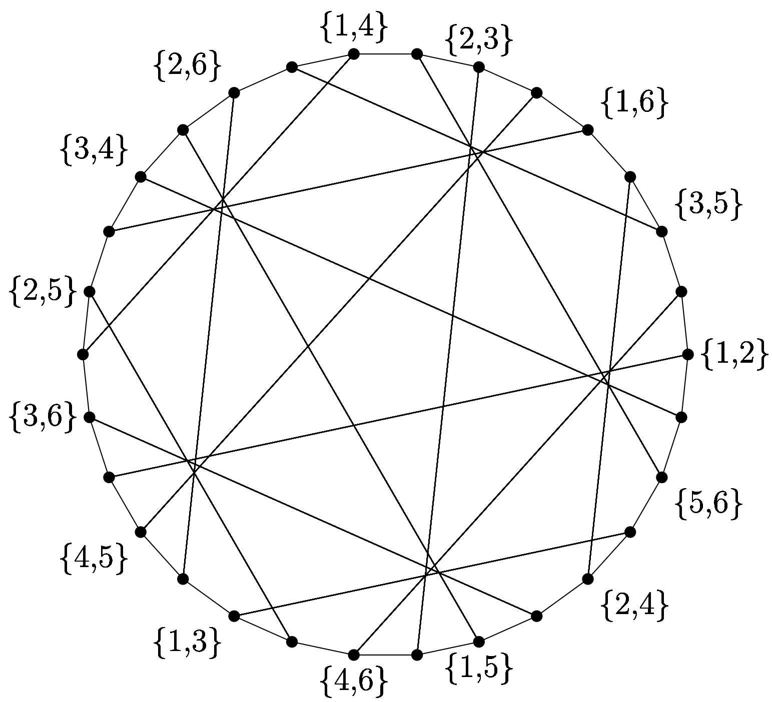


Figure 4.8: The Tutte 8-cage

Using this labelling we write $j = \{1,2\}, l = \{1,3\}$ and show that

$$u_{i\{1,2\}}u_{k\{1,3\}} = u_{k\{1,3\}}u_{i\{1,2\}}.$$

This suffices to get $u_{ij}u_{kl} = u_{kl}u_{ij}$ for $d(i,k) = d(j,l) = 4$ by Lemma 3.2.4, because the Tutte 8-cage is distance-transitive.

Step 1: We have $u_{i\{1,2\}}u_{k\{1,3\}} = u_{i\{1,2\}}u_{k\{1,3\}}(u_{i\{1,2\}} + u_{i\{2,3\}})$.
There are three vertices t_a, $a \in \{1,2,3\}$, such that $d(j,t_a) = d(t_a,l) = 2$, because

we have $c_4 = 3$ and $c_3 = 1$. Since we know $u_{st_a} u_{kl} = u_{kl} u_{st_a}$ by Lemma 4.2.2, we get

$$u_{ij} u_{kl} = u_{ij} u_{kl} \left(\sum_{\substack{p; d(p,l)=4, \\ d(p,t_a)=2}} u_{ip} \right)$$

by Lemma 3.2.8. Using this, we obtain

$$\begin{aligned} u_{ij} u_{kl} &= u_{ij} u_{kl} \left(\sum_{\substack{p; d(p,l)=4, \\ d(p,t_1)=2}} u_{ip} \right) \left(\sum_{\substack{p; d(p,l)=4, \\ d(p,t_2)=2}} u_{ip} \right) \left(\sum_{\substack{p; d(p,l)=4, \\ d(p,t_3)=2}} u_{ip} \right) \\ &= u_{ij} u_{kl} \left(\sum_{\substack{p; d(p,l)=4, \\ d(p,t_a)=2,\, a=1,2,3}} u_{ip} \right). \end{aligned} \tag{4.2.1}$$

The vertices in distance two to $\{1,2\}$ and $\{1,3\}$ are $t_1 = \{4,5\}$, $t_2 = \{4,6\}$ and $t_3 = \{5,6\}$. Looking at Equation (4.2.1), we only have to consider vertices that are in distance two to those three vertices. The only 2-subset of $\{1,\ldots,6\}$ besides $\{1,2\}$ and $\{1,3\}$ that does not contain $4, 5$ or 6 is $\{2,3\}$. Thus we get

$$u_{i\{1,2\}} u_{k\{1,3\}} = u_{i\{1,2\}} u_{k\{1,3\}} (u_{i\{1,2\}} + u_{i\{2,3\}}).$$

Step 2: We have $u_{i\{1,2\}} u_{k\{1,3\}} u_{i\{2,3\}} = 0$.
Using Relations (2.1.2) and (2.1.4), we obtain

$$u_{i\{1,2\}} u_{k\{1,3\}} u_{i\{2,3\}} = u_{i\{1,2\}} \left(\sum_{\substack{s; d(s,i)=2, \\ d(k,s)=4}} u_{s\{3,4\}} \right) u_{k\{1,3\}} \left(\sum_{\substack{t; d(t,i)=2, \\ d(k,t)=4}} u_{t\{1,5\}} \right) u_{i\{2,3\}}.$$

The vertex $\{1,3\}$ is the only one in distance four to $\{1,2\}, \{2,3\}, \{3,4\}$ and $\{1,5\}$, because the only pair of numbers where at least one of them is contained in those subsets are 1 and 3. Let $q \neq \{1,3\}$. If $d(q,\{1,5\}) \neq 4$ or $d(q,\{3,4\}) \neq 4$, then

$$u_{i\{1,2\}} \left(\sum_{\substack{s; d(s,i)=2, \\ d(k,s)=4}} u_{s\{3,4\}} \right) u_{kq} \left(\sum_{\substack{t; d(t,i)=2, \\ d(k,t)=4}} u_{t\{1,5\}} \right) u_{i\{2,3\}} = 0,$$

by Lemma 3.2.2. If we have $d(q,\{1,5\}) = 4$ and $d(q,\{3,4\}) = 4$, but $d(q,\{1,2\}) \neq 4$, we get

$$\begin{aligned} u_{i\{1,2\}} \left(\sum_{\substack{s; d(s,i)=2,\\ d(k,s)=4}} u_{s\{3,4\}} \right) u_{kq} \left(\sum_{\substack{t; d(t,i)=2,\\ d(k,t)=4}} u_{t\{1,5\}} \right) u_{i\{2,3\}} \\ = u_{i\{1,2\}} u_{kq} \left(\sum_{\substack{t; d(t,i)=2,\\ d(k,t)=4}} u_{t\{1,5\}} \right) u_{i\{2,3\}} = 0, \end{aligned}$$

by using Relations (2.1.2), (2.1.4) and Lemma 3.2.2. A similar argument shows

$$u_{i\{1,2\}} \left(\sum_{\substack{s; d(s,i)=2,\\ d(k,s)=4}} u_{s\{3,4\}} \right) u_{kq} \left(\sum_{\substack{t; d(t,i)=2,\\ d(k,t)=4}} u_{t\{1,5\}} \right) u_{i\{2,3\}} = 0$$

for q with $d(q,\{1,5\}) = 4$, $d(q,\{3,4\}) = 4$ and $d(q,\{2,3\}) \neq 4$. This yields

$$\begin{aligned} u_{i\{1,2\}} u_{k\{1,3\}} u_{i\{2,3\}} &= u_{i\{1,2\}} \left(\sum_{\substack{s; d(s,i)=2,\\ d(k,s)=4}} u_{s\{3,4\}} \right) u_{k\{1,3\}} \left(\sum_{\substack{t; d(t,i)=2,\\ d(k,t)=4}} u_{t\{1,5\}} \right) u_{i\{2,3\}} \\ &= u_{i\{1,2\}} \left(\sum_{\substack{s; d(s,i)=2,\\ d(k,s)=4}} u_{s\{3,4\}} \right) \left(\sum_a u_{ka} \right) \left(\sum_{\substack{t; d(t,i)=2,\\ d(k,t)=4}} u_{t\{1,5\}} \right) u_{i\{2,3\}} \\ &= u_{i\{1,2\}} \left(\sum_{\substack{s; d(s,i)=2,\\ d(k,s)=4}} u_{s\{3,4\}} \right) \left(\sum_{\substack{t; d(t,i)=2,\\ d(k,t)=4}} u_{t\{1,5\}} \right) u_{i\{2,3\}}. \end{aligned}$$

Since $d(\{1,2\},\{3,4\}) = d(\{2,3\},\{1,5\}) = 2$, we know that $u_{i\{1,2\}}$ commutes with $u_{s\{3,4\}}$ and $u_{i\{2,3\}}$ commutes with $u_{t\{1,5\}}$ by Lemma 4.2.2. We deduce

$$u_{i\{1,2\}} u_{k\{1,3\}} u_{i\{2,3\}} = u_{i\{1,2\}} \left(\sum_{\substack{s; d(s,i)=2,\\ d(k,s)=4}} u_{s\{3,4\}} \right) \left(\sum_{\substack{t; d(t,i)=2,\\ d(k,t)=4}} u_{t\{1,5\}} \right) u_{i\{2,3\}}$$

$$= \left(\sum_{\substack{s;d(s,i)=2,\\ d(k,s)=4}} u_{s\{3,4\}} \right) u_{i\{1,2\}} u_{i\{2,3\}} \left(\sum_{\substack{t;d(t,i)=2,\\ d(k,t)=4}} u_{t\{1,5\}} \right)$$
$$= 0,$$

since $u_{i\{1,2\}}u_{i\{2,3\}} = 0$.

Step 3: It holds $u_{i\{1,2\}}u_{k\{1,3\}} = u_{k\{1,3\}}u_{i\{1,2\}}$.
Using *Step 1* and *Step 2*, we obtain $u_{i\{1,2\}}u_{k\{1,3\}} = u_{i\{1,2\}}u_{k\{1,3\}}u_{i\{1,2\}}$. By Lemma 3.2.1, we get that $u_{i\{1,2\}}$ and $u_{k\{1,3\}}$ commute.

Step 4: It holds $u_{ij}u_{kl} = u_{kl}u_{ij}$ for $d(i,k) = d(j,l) = 3$.
Let $d(i,k) = d(j,l) = 3$. We prove that $u_{ij}u_{kl} = u_{kl}u_{ij}$ for $d(i,k) = d(j,l) = 3$. Let x be the unique vertex adjacent to j and in distance two to l. This vertex is unique because the Tutte 8-cage has girth eight. By Lemma 4.2.5, we get

$$u_{ij}u_{kl} = u_{ij}u_{kl}(u_{ij} + u_{iq}),$$

where q is the unique vertex adjacent to x with $d(q,j) = 2, d(l,q) = 3$. Take a neighbor t of j at distance four to l. Then we have

$$u_{ij}u_{kl}u_{iq} = u_{ij} \left(\sum_{\substack{s;(s,i)\in E,\\ d(k,s)=4}} u_{st} \right) u_{kl}u_{iq} = u_{ij}u_{kl} \left(\sum_{\substack{s;(s,i)\in E,\\ d(k,s)=4}} u_{st} \right) u_{iq},$$

because of Relations (2.1.2), (2.1.5) and $u_{kl}u_{st} = u_{st}u_{kl}$ for all such s since $d(t,l) = d(s,k) = 4$. Assume that t is connected to q. Then j and q have two common neighbors, x and t, where we know that $x \neq t$ because we have $d(x,l) = 2$ whereas $d(t,l) = 4$. But then we get the quadrangle j, x, q, t, j and this contradicts the fact that the Tutte 8-cage has girth eight. Thus, t and q are not adjacent. We deduce

$$u_{ij}u_{kl}u_{iq} = u_{ij}u_{kl} \left(\sum_{\substack{s;(s,i)\in E,\\ d(k,s)=4}} u_{st} \right) u_{iq} = 0$$

by Relation (2.1.5). Thus we get $u_{ij}u_{kl} = u_{ij}u_{kl}u_{ij}$ and we obtain $u_{ij}u_{kl} = u_{kl}u_{ij}$ for $d(i,k) = d(j,l) = 3$ by Lemma 3.2.1.

Summarizing, we have $u_{ij}u_{kl} = u_{kl}u_{ij}$ for $d(i,k) = d(j,l) \leq 4$. Using Lemma 3.2.2, we conclude that the Tutte 8-cage has no quantum symmetry.

The Foster graph $(\{3,2,2,2,2,1,1,1;1,1,1,1,2,2,2,3\})$

The Foster graph has diameter eight. Therefore, we have $d(i,k), d(j,l) \leq 8$ for $i,j,k,l \in V$. By Lemma 4.2.3, we know $u_{ij}u_{kl} = u_{kl}u_{ij}$ for $d(i,k) = d(j,l) \leq 2$.

Let $d(i,k) = 3$, $d(j,l) = 3$. We want to show $u_{ij}u_{kl} = u_{kl}u_{ij}$.

Step 1: It holds $u_{ij}u_{kl} = u_{ij}u_{kl}(u_{ij} + u_{iq})$, where q is the unique vertex adjacent to the neighbor x of j, $d(x,l) = 2$ with $d(q,j) = 2, d(l,q) = 3$.
The Foster graph has girth ten. Thus, by Lemma 4.2.5, we get

$$u_{ij}u_{kl} = u_{ij}u_{kl}(u_{ij} + u_{iq}),$$

for q as above.

Step 2: It holds $u_{ij}u_{kl}u_{iq} = u_{ij}\left(\sum_{\substack{s;(s,k)\in E,\\ d(s,i)=4}} u_{sz}\right)u_{iq}$ for $z \in V$ with $(z,l) \in E$ and $d(z,j) = 4$.
Take z with $(z,l) \in E, d(z,j) = 4$. Using Relations (2.1.2) and (2.1.4), we obtain

$$u_{ij}u_{kl}u_{iq} = u_{ij}\left(\sum_{\substack{s;(s,k)\in E,\\ d(s,i)=4}} u_{sz}\right)u_{kl}u_{iq}.$$

We know $(z,l) \in E$ and $d(l,q) = 3$. Since we have $b_3 = 2$ and $c_3 = 1$, it either holds $d(z,q) = 2$ or $d(z,q) = 4$. Assume $d(z,q) = 2$. Then we get a cycle of length ≤ 6, since we have $d(z,q) = 2$, $d(q,l) = 3$ and $(z,l) \in E$. But this contradicts the fact that the Foster graph has girth ten and we conclude $d(z,q) = 4$. It holds $c_4 = 1$, thus l is the only neighbor of z at distance three to q. This yields

$$\begin{aligned} u_{ij}\left(\sum_{\substack{s;(s,k)\in E,\\ d(s,i)=4}} u_{sz}\right)u_{kl}u_{iq} &= u_{ij}\left(\sum_{\substack{s;(s,k)\in E,\\ d(s,i)=4}} u_{sz}\right)\left(\sum_{\substack{a;(a,z)\in E,\\ d(a,q)=3}} u_{ka}\right)u_{iq} \\ &= u_{ij}\left(\sum_{\substack{s;(s,k)\in E,\\ d(s,i)=4}} u_{sz}\right)\left(\sum_{a} u_{ka}\right)u_{iq} \\ &= u_{ij}\left(\sum_{\substack{s;(s,k)\in E,\\ d(s,i)=4}} u_{sz}\right)u_{iq}, \end{aligned}$$

by using Lemma 3.2.2 and Relations (2.1.2), (2.1.5).

Step 3: It holds $u_{ij}u_{sz}u_{iq} = 0$ *for* $s \in V$ *with* $(s,k) \in E$ *and* $d(i,s) = 4$.
For every such s, take t with $d(i,t) = 4, d(s,t) = 2$ (exists because for a neighbor of s at distance five to i there is a neighbor $t \neq s$ with $d(i,t) = 4$, since $c_5 = 2$). We get

$$u_{ij}u_{sz}u_{iq} = u_{ij}\left(\sum_{\substack{p;d(p,z)=2,\\ d(p,j)=4}} u_{tp}\right)u_{sz}u_{iq} = u_{ij}u_{sz}\left(\sum_{\substack{p;d(p,z)=2,\\ d(p,j)=4}} u_{tp}\right)u_{iq} \tag{4.2.2}$$

by Lemma 3.2.2 and because we have $u_{tp}u_{sz} = u_{sz}u_{tp}$ by Lemma 4.2.2. The Foster graph has girth ten, therefore there is exactly one neighbor of j at distance two to l. This is the vertex x from *Step 1*. We know that q is also a neighbor of x. Take p with $d(p,z) = 2$, $d(p,j) = 4$. We want to show $d(p,q) \neq 4$. For this, we assume $d(p,q) = 4$ and prove that then x has three neighbors in distance four to p, contradicting $c_5 = 2$. We have $d(p,j) = 4$ by the choice of p. Let y be the third neighbor of x. We have $d(x,l) = 2$ and know $d(j,l) = d(q,l) = 3$, therefore the remaining neighbor y of x has to be adjacent to l as otherwise $d(x,l) \neq 2$. We also have $(z,l) \in E$ and $d(z,p) = 2$, where we know $z \neq y$ since $d(x,j) = 2$, but $d(z,j) = 4$. It holds $d(l,p) = 3$ and we have $c_3 = 1$, $b_3 = 2$. Thus, l has one neighbor at distance two to p and two neighbors at distance four to p. Since we know that z is the neighbor of l with $d(z,p) = 2$, we conclude $d(y,p) = 4$ because y is another neighbor of l. Thus x has the three neighbors j, q and y in distance four to p contradicting $c_5 = 2$. We conclude $d(p,q) \neq 4$ for all p with $d(p,z) = 2$, $d(p,j) = 4$. By Lemma 3.2.2, we deduce

$$u_{ij}u_{sz}u_{iq} = u_{ij}u_{sz}\left(\sum_{\substack{p;d(p,z)=2,\\ d(p,j)=4}} u_{tp}\right)u_{iq} = 0$$

for all s with $(s,k) \in E, d(s,i) = 4$.

Step 4: It holds $u_{ij}u_{kl} = u_{kl}u_{ij}$ *for* $d(i,k) = d(j,l) = 3$.
The Steps *2* and *3* yield

$$u_{ij}u_{kl}u_{iq} = u_{ij}\left(\sum_{\substack{s;(s,k)\in E,\\ d(s,i)=4}} u_{sz}\right)u_{iq} = 0.$$

Using *Step 1*, we get $u_{ij}u_{kl} = u_{ij}u_{kl}u_{ij}$ for $d(i,k) = d(j,l) = 3$ and we obtain $u_{ij}u_{kl} = u_{kl}u_{ij}$ by Lemma 3.2.1.

Step 5: We have $u_{ij}u_{kl} = u_{kl}u_{ij}$ for $4 \leq d(i,k) = d(j,l) \leq 8$.
Let $d(i,k) = d(j,l) = 4$. There is exactly one a with $(a,i) \in E$, $d(k,a) = 3$, and exactly one b with $(b,j) \in E$, $d(l,b) = 3$ since $c_4 = 1$. It holds

$$u_{ij}u_{kl} = u_{ij}u_{kl}\left(\sum_{\substack{p;d(p,l)=4,\\(b,p)\in E}} u_{ip}\right)$$

by Lemma 3.2.8 since $u_{kl}u_{ab} = u_{ab}u_{kl}$. There are exactly two vertices adjacent to b and at distance four to l, since $d(b,l) = 3$ and $b_3 = 2$. One of them is j, so we get

$$u_{ij}u_{kl} = u_{ij}u_{kl}(u_{ij} + u_{iq}),$$

where q is the other neighbor of b in distance four to l. We have

$$u_{ij}u_{kl}u_{iq} = u_{ij}\left(\sum_{\substack{p;d(p,l)=2,\\d(p,j)=4}} u_{tp}\right)u_{kl}u_{iq} = u_{ij}u_{kl}\left(\sum_{\substack{p;d(p,l)=2,\\d(p,j)=4}} u_{tp}\right)u_{iq}$$

by Relations (2.1.2), (2.1.5) and $u_{kl}u_{tp} = u_{tp}u_{kl}$. But now we are in the same situation as in Equation (4.2.2), thus by the same argument we get $d(p,q) \neq 4$. By Lemma 3.2.2, we deduce $u_{ij}u_{kl}u_{iq} = 0$. This implies $u_{ij}u_{kl} = u_{ij}u_{kl}u_{ij}$ and Lemma 3.2.1 yields $u_{ij}u_{kl} = u_{kl}u_{ij}$ for $d(i,k) = d(j,l) = 4$.

Since we now know $u_{ij}u_{kl} = u_{kl}u_{ij}$ for $d(i,k), d(j,l) \leq 4$ and it holds $c_2 = 1$, $b_1 + 1 = b_0$ and $c_n \geq 2$ for $5 \leq n \leq 8$, we can use Lemma 3.2.10 (a) four times to get $u_{ij}u_{kl} = u_{kl}u_{ij}$ for $d(i,k) = d(j,l) \leq 8$. Then Lemma 3.2.2 yields that the Foster graph has no quantum symmetry.

The Biggs-Smith graph ($\{3,2,2,2,1,1,1;1,1,1,1,1,1,3\}$)

Since the Biggs-Smith graph has diameter seven, we have $d(i,k), d(j,l) \leq 7$ for $i,j,k,l \in V$. By Lemma 4.2.3, we get $u_{ij}u_{kl} = u_{kl}u_{ij}$ for $d(i,k) = d(j,l) \leq 2$.

Let $d(i,k) = d(j,l) = 4$. We show $u_{ij}u_{kl} = u_{kl}u_{ij}$ for $d(i,k) = d(j,l) = 4$.

Step 1: It holds $u_{ij}u_{kl} = u_{ij}u_{kl}(u_{ij} + u_{iq_1} + u_{iq_2} + u_{iq_3})$, where $d(j,q_1) = 2$ and $d(j,q_2) = d(j,q_3) = 4$.

Since the Biggs-Smith graph has girth nine, there is exactly one vertex t with $d(t,j)=d(t,l)=2$ and exactly one vertex s with $d(i,s)=d(s,k)=2$. It holds

$$u_{ij}u_{kl}=u_{ij}u_{kl}\left(\sum_{\substack{p;d(p,l)=4,\\ d(p,t)=2}}u_{ip}\right)$$

by Lemma 3.2.8, since we know $u_{kl}u_{at}=u_{at}u_{kl}$ for a with $d(a,l)=2$. There are exactly four vertices that are at distance four to l and at distance two to t, where one of them is j (There are six vertices at distance two to t, where one of them is l and another one is a vertex at distance two to l. The rest is at distance four to l). We deduce

$$u_{ij}u_{kl}=u_{ij}u_{kl}(u_{ij}+u_{iq_1}+u_{iq_2}+u_{iq_3}),$$

where $d(j,q_1)=2$ and $d(j,q_2)=d(j,q_3)=4$.

Step 2: We have $u_{ij}u_{kl}u_{iq_1}=0$.
We know that the Biggs-Smith is 3-regular and it holds $b_4=c_4=1$. Thus, if we have $d(a,b)=4$ for vertices a,b, there is exactly one neighbor of a in distance four to b. Therefore we have exactly one neighbor x of k with $d(x,i)=4$ and exactly one neighbor y of l with $d(y,j)=4$. By Relations (2.1.2), (2.1.5), we deduce

$$u_{ij}u_{kl}u_{iq_1}=u_{ij}u_{xy}u_{kl}u_{iq_1}.$$

Denote by z_1 the common neighbor of j and q_1. We know that j and q_1 are two neighbors of z_1 at distance four to l. Thus $d(z_1,l)=4$ contradicts $c_4=b_4=1$ and $d(z_1,l)=5$ contradicts $c_5=1$. But it holds $d(z_1,l)\in\{3,4,5\}$ since z_1 has neighbors in distance four to l. We deduce $d(z_1,l)=3$. We have $b_3=2$, $c_3=1$ and know $d(z_1,l)=3$, thus l has two neighbors at distance four to z_1 and one neighbor at distance two to z_1. If $d(y,z_1)=2$, then we get $d(y,j)\leq 3$ since $(j,z_1)\in E$ contradicting $d(y,j)=4$. We conclude $d(y,z_1)=4$ as y is a neighbor of l not in distance two to z_1. Furthermore it holds $d(q_1,y)\neq 4$, as otherwise z_1 would have the two neighbors j and q_1 at distance four to y contradicting $b_4=c_4=1$, since $d(z_1,y)=4$. But this yields

$$u_{ij}u_{kl}u_{iq_1}=u_{ij}u_{xy}u_{kl}u_{iq_1}=u_{ij}u_{kl}u_{xy}u_{iq_1}=0,$$

by using Relation (2.1.5) and $u_{xy}u_{kl}=u_{kl}u_{xy}$.

Step 3: We have $u_{ij}u_{kl}u_{iq_2}=0$.
We know that q_2 and q_3 are in distance two to t. Thus, they have to be adjacent to

one of the neighbors of t. They cannot be adjacent to z_1, because z_1 has neighbors j, q_1 and t and the Biggs-Smith graph is 3-regular. It holds $b_2 = 2$, $c_2 = 1$ and we know $d(t, l) = 2$, which means that t has one neighbor, say z_2, adjacent to l and two neighbors (z_1 and one more) in distance three to l. Denote the third neighbor of t by z_3. Recall $d(q_2, l) = d(q_3, l) = 4$. Thus q_2, q_3 cannot be adjacent to z_2 as otherwise $d(q_2, l) = d(q_3, l) \leq 2$. We conclude that q_2, q_3 are both neighbors of z_3, where $d(z_3, l) = 3$. The vertices z_2 and y are neighbors of l. We have $d(l, t) = 2$ and $(t, z_2) \in E$, where we know that the other neighbors of l are in distance three to t because we have $b_2 = 2$. We deduce $d(y, t) = 3$. It also holds $d(z_2, y) = 2$ since they have the common neighbor l and the Biggs-Smith graph has girth nine. Thus we get $d(z_3, y) = 4$ by $b_3 = 2$, since $d(t, y) = 3$ and we know that z_1 is the neighbor of t in distance two to y. Because of $d(z_3, y) = 4$ and $c_4 = b_4 = 1$, we see that only one of the vertices q_2, q_3 is in distance four to y, say this is q_3. We obtain

$$u_{ij}u_{kl}u_{iq_2} = u_{ij}u_{xy}u_{kl}u_{iq_2} = u_{ij}u_{kl}u_{xy}u_{iq_2} = 0,$$

by Relations (2.1.2), (2.1.5), using $u_{xy}u_{kl} = u_{kl}u_{xy}$ and Lemma 3.2.2.

Step 4: It holds $u_{ij}u_{kl}u_{iq_3} = 0$.
We have $d(q_3, y) = 4$ and since l is a neighbor of y at distance four to q_3, we know that the two neighbors $c, d \neq l$ of y are not in distance four to q_3 because $c_4 = b_4 = 1$. Therefore

$$u_{ij}u_{xy}u_{kc}u_{iq_3} = 0 = u_{ij}u_{xy}u_{kd}u_{iq_3}$$

by Lemma 3.2.2. We deduce

$$\begin{aligned} u_{ij}u_{kl}u_{iq_3} &= u_{ij}u_{xy}u_{kl}u_{iq_3} \\ &= u_{ij}u_{xy}(u_{kl} + u_{kc} + u_{kd})u_{iq_3} \\ &= u_{ij}u_{xy}\left(\sum_{a;(y,a)\in E} u_{ka}\right)u_{iq_3} \\ &= u_{ij}u_{xy}\left(\sum_{a} u_{ka}\right)u_{iq_3} \\ &= u_{ij}u_{xy}u_{iq_3}, \end{aligned}$$

by also using Relations (2.1.2), (2.1.5). Now, take e, f to be the vertices with $d(e, i) = d(e, x) = 2$, $d(f, j) = d(f, y) = 2$ (those are unique, since $d(i, x) = d(j, y) = 4$ and the Biggs-Smith graph has girth nine). It holds

$$u_{ij}u_{xy}u_{iq_3} = u_{ij}u_{ef}u_{xy}u_{iq_3} = u_{ij}u_{xy}u_{ef}u_{iq_3},$$

by Relations (2.1.2), (2.1.5) and since we know $u_{xy}u_{ef} = u_{ef}u_{xy}$ by Lemma 4.2.3. We have $d(f, q_3) \neq 2$ because otherwise there would be two vertices, f and t, in distance two to j and q_3, so we would get an cycle of length ≤ 8 in the Biggs-Smith graph($f \neq t$, since $d(y, f) = 2, d(y, t) = 3$). Thus, by Lemma 3.2.2, we get

$$u_{ij}u_{kl}u_{iq_3} = u_{ij}u_{xy}u_{iq_3} = u_{ij}u_{xy}u_{ef}u_{iq_3} = 0.$$

Step 5: It holds $u_{ij}u_{kl} = u_{kl}u_{ij}$ for $d(i, k) = d(j, l) = 4$.
From Steps *1–4*, we deduce $u_{ij}u_{kl} = u_{ij}u_{kl}u_{ij}$ for $d(i, k) = d(j, l) = 4$ and we get $u_{ij}u_{kl} = u_{kl}u_{ij}$ by Lemma 3.2.1.

Step 6: We have $u_{ij}u_{kl} = u_{kl}u_{ij}$ for $d(i, k) = d(j, l) = 3$.
Let $d(i, k) = d(j, l) = 3$. We obtain

$$u_{ij}u_{kl} = u_{ij}u_{kl}(u_{ij} + u_{iq}),$$

by Lemma 4.2.5, where $d(q, j) = 2, d(l, q) = 3$. We have $b_3 = 2$ and therefore there are two neighbors t_1, t_2 of j in distance four to l and two neighbors s_1, s_2 of i in distance four to k. At least one of them, say t_1, is not connected to q, since otherwise we would get the quadrangle j, t_1, q, t_2, j. By Lemma 3.2.2 we get $u_{s_a t_1}u_{iq} = 0$, $a = 1, 2$. Because we know $u_{s_a t_1}u_{kl} = u_{kl}u_{s_a t_1}$, since $d(s_a, k) = 4 = d(t_1, l)$, we deduce

$$u_{ij}u_{kl}u_{iq} = u_{ij}(u_{s_1t_1} + u_{s_2t_1})u_{kl}u_{iq} = u_{ij}u_{kl}(u_{s_1t_1} + u_{s_2t_1})u_{iq} = 0.$$

This yields $u_{ij}u_{kl} = u_{ij}u_{kl}u_{ij}$ for $d(i, k) = d(j, l) = 3$ and we obtain $u_{ij}u_{kl} = u_{kl}u_{ij}$ by Lemma 3.2.1.

Step 7: We have $u_{ij}u_{kl} = u_{kl}u_{ij}$ for $5 \leq d(i, k) = d(j, l) \leq 7$.
We now have $u_{ij}u_{kl} = u_{kl}u_{ij}$ for $d(i, k) = d(j, l) \leq 4$ and since $b_4 = 1$, we get $u_{ij}u_{kl} = u_{kl}u_{ij}$ for $d(i, k) = d(j, l) = 5$ by Lemma 4.2.4 (i). We have $b_5 = 1$ and thus, using Lemma 4.2.4 (i) again, we obtain $u_{ij}u_{kl} = u_{kl}u_{ij}$ for $d(i, k) = d(j, l) = 6$. Lemma 3.2.10 (a) now yields $u_{ij}u_{kl} = u_{kl}u_{ij}$ for $d(i, k) = d(j, l) = 7$, because $c_2 = 1$, $b_1 + 1 = b_0$, $c_7 = 3$. Using Lemma 3.2.2, we conclude that the Biggs-Smith graph has no quantum symmetry.

Remark 4.2.6. There is only one cubic distance-regular graph that is not distance-transitive. This is the Tutte 12-cage. We do not know whether or not this graph has quantum symmetry.

4.3 Further distance-regular graphs with no quantum symmetry

In this section, we study further distance-regular graphs of order ≤ 20. We follow [50, Section 6]. Our strategy is similar to the one in Section 4.2.4. We assume

$(u_{ij})_{1\leq i,j\leq n}$ to be the generators of $C(G^+_{aut}(\Gamma))$ and show that the graph Γ has no quantum symmetry in the corresponding subsection. We obtain the following table from the results in this section.

Name of Γ	Order	$\mathrm{Aut}(\Gamma)$	$G^+_{aut}(\Gamma)$	Intersection array
Icosahedron	12	$A_5 \times \mathbb{Z}_2$	$\mathrm{Aut}(\Gamma)$	{5,2,1;1,2,5}
co-Heawood graph	14	$PGL(2,7)$	$\mathrm{Aut}(\Gamma)$	{4,3,2;1,2,4}
Line graph of Petersen graph	15	S_5	$\mathrm{Aut}(\Gamma)$	{4,2,1;1,1,4}
Shrikhande graph	16	$\mathbb{Z}_4^2 \rtimes D_6$	$\mathrm{Aut}(\Gamma)$	{6,3;1,2}

Table 4.3: Quantum automorphism groups of some distance-regular graphs on a small number of vertices

The first graph we are considering is the co-Heawood graph, which is the bipartite complement of the Heawood graph with respect to the complete bipartite graph $K_{7,7}$ and thus closely related to the Heawood graph.

The co-Heawood graph ($\{4, 3, 2; 1, 2, 4\}$)

The co-Heawood graph has diameter three. Therefore we have $d(i, k), d(j, l) \leq 3$ for $i, j, k, l \in V$. Since the co-Heawood graph is the bipartite complement of the Heawood graph with respect to $K_{7,7}$, we see that vertices at distance three to a vertex i are exactly those that are connected to i in the Heawood graph. Vertices at distance two are the same ones in both graphs, since those are the six other vertices in the same maximal independent set as i. And finally the vertices that are connected to i in the co-Heawood graph are those at distance three to i in the Heawood graph. Therefore, we can use the same arguments as in Lemma 3.2.5 to obtain that $u_{ij}u_{kl} = u_{kl}u_{ij}$ for $d(i, k) = d(j, l) = 3$ for the co-Heawood graph. Also arguments of the proof of Lemma 4.2.2 work similarly to show $u_{ij}u_{kl} = u_{kl}u_{ij}$ for $d(i, k) = d(j, l) = 2$ by replacing neighbors with vertices at distance three. Then using the same approach as in Lemma 3.2.10 (a), also replacing neighbors with vertices at distance three, we get $u_{ij}u_{kl} = u_{kl}u_{ij}$ for $(i, k), (j, l) \in E$. We obtain that the co-Heawood graph has no quantum symmetry by Lemma 3.2.2.

The line graph of the Petersen graph ($\{4, 2, 1; 1, 1, 4\}$)

The line graph of the Petersen graph (see Figure 4.9) has diameter three and thus we have $d(i, k), d(j, l) \leq 3$ for $i, j, k, l \in V$. Since adjacent vertices have exactly one common neighbor, Lemma 3.2.6 yields $G^+_{aut}(L(\mathrm{P})) = G^*_{aut}(L(\mathrm{P}))$. Therefore Relation (2.1.6) holds.

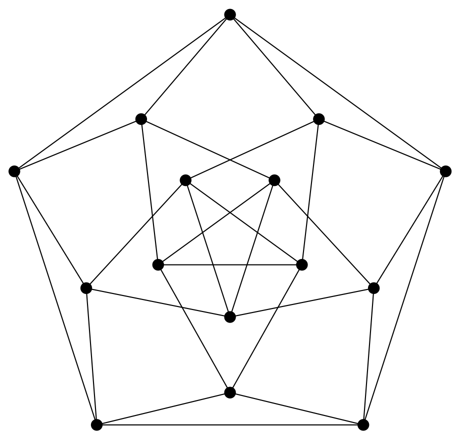


Figure 4.9: The line graph of the Petersen graph

Now, let $d(i,k) = d(j,l) = 2$. We want to prove $u_{ij}u_{kl} = u_{kl}u_{ij}$. We know that the Petersen graph is the Kneser graph $K(5,2)$. Thus, vertices in the line graph of the Petersen graph are of the form $\{\{a,b\},\{c,d\}\}$, where $\{a,b\}$, $\{c,d\}$ are disjoint 2-subsets of $\{1,\dots,5\}$. Two vertices are connected if and only if they have exactly one 2-subset in common. The line graph of the Petersen graph is distance-transitive, therefore it suffices to show $u_{ij}u_{kl} = u_{kl}u_{ij}$ for

- $j = \{\{1,2\},\{3,4\}\}$,
- $l = \{\{1,3\},\{4,5\}\}$

by Lemma 3.2.4. The only common neighbor of j and l is $t = \{\{1,2\},\{4,5\}\}$. Since we know $G^+_{aut}(L(\mathrm{P})) = G^*_{aut}(L(\mathrm{P}))$, we get

$$u_{ij}u_{kl} = u_{ij}u_{kl} \sum_{\substack{p;d(p,l)=2,\\(p,t)\in E}} u_{ip}$$

by Lemma 3.2.8. Besides j, the vertex $q = \{\{1,2\},\{3,5\}\}$ is the only other vertex in distance two to l which is also adjacent to t. This yields

$$u_{ij}u_{kl} = u_{ij}u_{kl}(u_{ij} + u_{iq}).$$

The vertex $b = \{\{1,3\},\{2,4\}\}$ is adjacent to l and in distance three to j. Using Relations (2.1.2) and (2.1.4), we deduce

$$u_{ij}u_{kl}u_{iq} = u_{ij}\left(\sum_{\substack{a;d(a,i)=3,\\(a,k)\in E}} u_{ab}\right)u_{kl}u_{iq}.$$

Because of Relation (2.1.6), we get

$$u_{ij}u_{kl}u_{iq} = u_{ij}u_{kl}\left(\sum_{\substack{a;d(a,i)=3,\\(a,k)\in E}} u_{ab}\right)u_{iq}.$$

We see that $\{\{2,4\},\{3,5\}\}$ is a neighbor of b and q. This yields $d(b,q) \leq 2$ and since we have $d(a,i) = 3$, we obtain

$$u_{ij}u_{kl}u_{iq} = u_{ij}u_{kl}\left(\sum_{\substack{a;d(a,i)=3,\\(a,k)\in E}} u_{ab}\right)u_{iq} = 0,$$

by Lemma 3.2.2. Summarizing, it holds $u_{ij}u_{kl} = u_{ij}u_{kl}u_{ij}$. By Lemma 3.2.1, we see that u_{ij} and u_{kl} commute.

For $d(i,k) = d(j,l) = 3$, all conditions for Lemma 3.2.10 (b) are fulfilled and we get $u_{ij}u_{kl} = u_{kl}u_{ij}$ for $d(i,k) = d(j,l) = 3$. Using Lemma 3.2.2, we deduce that the line graph of the Petersen graph has no quantum symmetry.

The upcoming lemma allows us to deal with more distance-regular graphs. Recall that the clique number of a graph is the number of vertices of a maximal clique (see Definition 1.2.3).

Lemma 4.3.1. *Let Γ be an undirected graph with clique number three, where adjacent vertices and vertices at distance two have exactly two common neighbors. Then we have $G^+_{aut}(\Gamma) = G^*_{aut}(\Gamma)$.*

Proof. Let $(i,k),(j,l) \in E$. By Relations (2.1.2), (2.1.5) we have

$$u_{ij}u_{kl} = u_{ij}u_{kl}\left(\sum_{p;(l,p)\in E} u_{ip}\right).$$

Denote the two common neighbors of j and l by p_1, p_2.

We have $(p_1,p_2) \notin E$, since otherwise we get a clique of size four, but we know that the clique number of Γ is three. Also p_1, p_2 have two common neighbors since $d(p_1,p_2) = 2$, where we know that those are l and j. This yields that l is the only common neighbor of p_1, p_2 and j. We also have $(j,p_1) \in E$, $(p_1,p_2) \notin E$ by previous considerations and deduce

$$u_{ij}u_{kl}u_{ip_a} = 0, \quad a = 1,2$$

by Lemma 3.2.9, where we choose $q = p_1$ for p_2 and vice versa.

Now, let $p \notin \{j, p_1, p_2\}$ and $(l, p) \in E$ (this implies $(p, j) \notin E$). We know that we have $(p_1, p) \notin E$ or $(p_2, p) \notin E$ since otherwise p_1 and p_2 have three common neighbors: j, l and p. Choose p_x, $x \in \{1, 2\}$ such that $(p_x, p) \notin E$. Since p_x, p have l as common neighbor and we know $d(p_x, p) = 2$, there is exactly one other common neighbor $q \neq l$ of p_x, p. It holds $(j, q) \notin E$, because otherwise j, p_x and p would be common neighbors of l and q, but we know that they can only have two common neighbors since $d(l, q) \leq 2$. Therefore l is the only common neighbor of j, p_x and p. We also have $(j, p_x) \in E$, $(p_x, p) \notin E$ and we obtain

$$u_{ij}u_{kl}u_{ip} = 0$$

by Lemma 3.2.9, where we choose $q = p_x$.

Summarizing, we get

$$u_{ij}u_{kl} = u_{ij}u_{kl}\left(\sum_{p;(l,p)\in E} u_{ip}\right) = u_{ij}u_{kl}u_{ij}$$

and by Lemma 3.2.1 we get $u_{ij}u_{kl} = u_{kl}u_{ij}$ for $(i, k), (j, l) \in E$. □

The Icosahedron ($\{5, 2, 1; 1, 2, 5\}$)

The Icosahedron (see Figure 4.10) has diameter three and therefore we have $d(i, k)$, $d(j, l) \leq 3$ for $i, j, k, l \in V$. Since adjacent vertices and vertices at distance two have exactly two common neighbors and since it is known that the clique number is three, we get $u_{ij}u_{kl} = u_{kl}u_{ij}$ for $(i, k), (j, l) \in E$ by Lemma 4.3.1 and $u_{ij}u_{kl} = u_{kl}u_{ij}$ for $d(i, k) = 2 = d(j, l)$ by Lemma 3.2.10 (c).

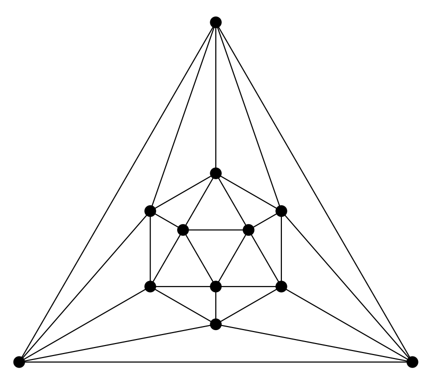


Figure 4.10: The Icosahedral graph

By Lemma 3.2.2 we know that u_{ij} and u_{kl} commute if $d(i, k) \neq d(j, l)$. Thus it remains to show $u_{ij}u_{kl} = u_{kl}u_{ij}$ for $d(i, k) = 3 = d(j, l)$. Note that for every vertex

x, there is exactly one other vertex at distance three to x. Let $d(i,k) = 3 = d(j,l)$. By Lemma 3.2.2, we get

$$u_{ij}u_{kl} = u_{ij}u_{kl} \sum_{p;d(l,p)=3} u_{ip}.$$

Since j is the only vertex in distance three to l, we conclude

$$u_{ij}u_{kl} = u_{ij}u_{kl}u_{ij}.$$

Then Lemma 3.2.1 yields $u_{ij}u_{kl} = u_{kl}u_{ij}$ and we get that the Icosahedron has no quantum symmetry.

The Shrikhande graph ($\{6, 3; 1, 2\}$)

First note that the Shrikhande graph (see Figure 4.11) is strongly regular with parameters $(16, 6, 2, 2)$. Thus it has diameter two and we know $d(i,k), d(j,l) \leq 2$ for $i, j, k, l \in V$. Since $\lambda = \mu = 2$, we know that every two vertices have exactly two common neighbors. It is also known that the clique number is three. By Lemma 4.3.1, we obtain $u_{ij}u_{kl} = u_{kl}u_{ij}$ for $(i,k),(j,l) \in E$. Then all the conditions of Lemma 3.2.10 (c) are met, we get $u_{ij}u_{kl} = u_{kl}u_{ij}$ for $(i,k),(j,l) \notin E$. We conclude that the Shrikhande graph has no quantum symmetry.

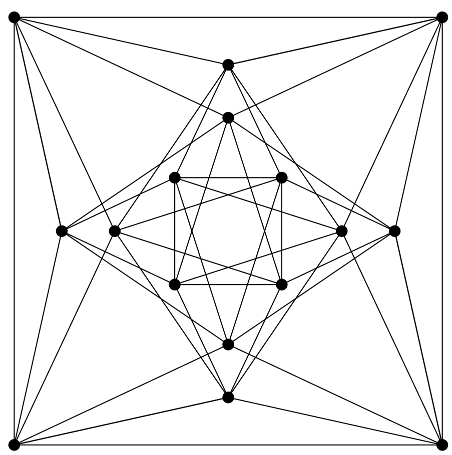


Figure 4.11: The Shrikhande graph

Corollary 4.3.2. *Let Γ be a distance-regular graph. The intersection array of Γ does not determine whether or not Γ has quantum symmetry.*

Proof. Looking at Table 2, we see that the Shrikhande graph and the 4×4-rook's graph have the same intersection array. We know by Proposition 4.1.12 that the

4×4-rook's graphs has quantum symmetry whereas the Shrikhande graph has no quantum symmetry by the previous arguments. □

The next corollary deals with the quantum orbital algebra of the Shrikhande graph. See Section 2.4 for the definition of the quantum orbital algebra and the coherent algebra of a graph. The Shrikhande graph is a nice example of a graph whose quantum orbital algebra is different from the coherent algebra of the graph.

Corollary 4.3.3. *The quantum orbital algebra of the Shrikhande graph does not coincide with its coherent algebra.*

Proof. Since the Shrikhande graph has no quantum symmetry, we get that the quantum orbital algebra and the classical orbital algebra are the same. It is known that the coherent algebra of the Shrikhande graph does not coincide with its orbital algebra. □

4.4 An example of a graph with automorphism group $\mathbb{Z}_2 \times \mathbb{Z}_2$ that has no quantum symmetry

In this section, we focus on graphs with automorphism group $\mathbb{Z}_2 \times \mathbb{Z}_2$. It was shown in [27, Theorem 6.4.1] that all trees with automorphism group $\mathbb{Z}_2 \times \mathbb{Z}_2$ do have quantum symmetry. Furthermore, this is also true for many examples appearing in the literature, for example [25], [52]. Therefore, one might ask whether there is a graph Γ with $\mathrm{Aut}(\Gamma) = G^+_{aut}(\Gamma) = \mathbb{Z}_2 \times \mathbb{Z}_2$.

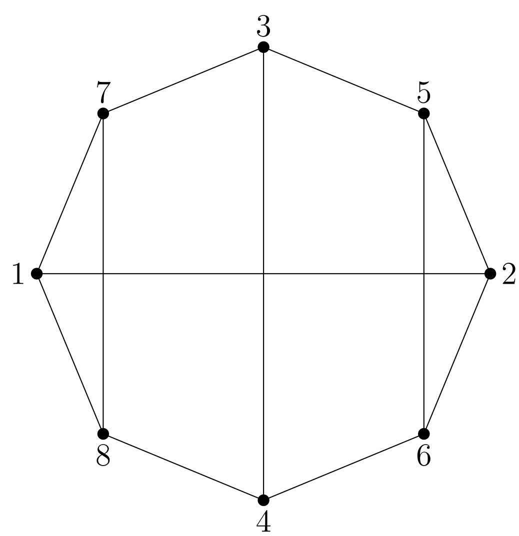


Figure 4.12: A graph Γ with automorphism group $\mathbb{Z}_2 \times \mathbb{Z}_2$

Consider the graph in Figure 4.12. Obviously, it holds $\mathrm{Aut}(\Gamma) = \mathbb{Z}_2 \times \mathbb{Z}_2$, because $\sigma = (12)(57)(68)$ and $\tau = (34)(56)(78)$ are automorphisms of Γ with $\sigma^2 = \mathrm{id} = \tau^2$ and $\tau\sigma = \sigma\tau$ that generate $\mathrm{Aut}(\Gamma)$. Note that σ and τ are not disjoint, so Theorem 3.1.2 does not apply. In fact, we may even prove the next theorem.

Theorem 4.4.1. *The graph in Figure 4.12 has no quantum symmetry.*

Proof. Let ε be the adjacency matrix corresponding to the graph in Figure 4.12. Then the diagonal of ε^4 is the following

$$(15, 15, 19, 19, 17, 17, 17, 17).$$

By Proposition 2.4.14, we obtain that the fundamental representation is of the form

$$u = \begin{pmatrix} p & 1-p & 0 & 0 & 0 & 0 & 0 & 0 \\ 1-p & p & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & q & 1-q & 0 & 0 & 0 & 0 \\ 0 & 0 & 1-q & q & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & u_{55} & u_{56} & u_{57} & u_{58} \\ 0 & 0 & 0 & 0 & u_{65} & u_{66} & u_{67} & u_{68} \\ 0 & 0 & 0 & 0 & u_{75} & u_{76} & u_{77} & u_{78} \\ 0 & 0 & 0 & 0 & u_{85} & u_{86} & u_{87} & u_{88} \end{pmatrix}. \tag{4.4.1}$$

Every vertex $v \in \{5, \dots, 8\}$ is adjacent to exactly one $x \in \{1, 2\}$ and one $y \in \{3, 4\}$. For $v_1, v_2 \in \{5, \dots, 8\}$ with corresponding $x_1, x_2 \in \{1, 2\}$ and $y_1, y_2 \in \{3, 4\}$, we get

$$u_{v_1v_2} = u_{v_1v_2} \sum_k u_{x_1k} = u_{v_1v_2} \sum_{k;(k,v_2)\in E} u_{x_1k} = u_{v_1v_2} u_{x_1x_2}$$

by using Relations (2.1.2), (2.1.5), where we know $u_{x_1k} = 0$ for $(k, v_2) \in E$, $k \in V \backslash \{x_2\}$ by (4.4.1). Similarly, we get

$$u_{v_1v_2} = u_{v_1v_2} u_{y_1y_2}$$

as well as

$$u_{v_1v_2} = u_{x_1x_2} u_{v_1v_2}.$$

From this we deduce

$$u_{v_1v_2} = u_{x_1x_2} u_{v_1v_2} = u_{x_1x_2} u_{v_1v_2} u_{y_1y_2}.$$

The only common neighbor of x_2 and y_2 is v_2, which yields

$$u_{v_1v_2} = u_{x_1x_2} u_{v_1v_2} u_{y_1y_2} = u_{x_1x_2} \left(\sum_k u_{v_1k} \right) u_{y_1y_2} = u_{x_1x_2} u_{y_1y_2}. \tag{4.4.2}$$

Since $u_{v_1v_2}$ is selfadjoint, we see that $u_{y_1y_2}$ and $u_{x_1x_2}$ commute. Especially, p and q commute (choose for example $v_1 = 5, v_2 = 5$). Using Equation (4.4.2), we obtain

$$u = \begin{pmatrix} p & 1-p & 0 & 0 & 0 & 0 & 0 & 0 \\ 1-p & p & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & q & 1-q & 0 & 0 & 0 & 0 \\ 0 & 0 & 1-q & q & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & pq & p(1-q) & (1-p)q & (1-p)(1-q) \\ 0 & 0 & 0 & 0 & p(1-q) & pq & (1-p)(1-q) & (1-p)q \\ 0 & 0 & 0 & 0 & (1-p)q & (1-p)(1-q) & pq & p(1-q) \\ 0 & 0 & 0 & 0 & (1-p)(1-q) & (1-p)q & p(1-q) & pq \end{pmatrix}$$

and since p and q commute, $C(G^+_{aut}(\Gamma))$ is commutative. □

Remark 4.4.2. We remark that this graph was also considered by Fulton in [27, Section 6.5]. It is mentioned therein that the graph in Figure 4.12 *does* have quantum symmetry, which is not true due to our previous theorem.

Chapter 5

Quantum automorphism groups of folded cube graphs

In this chapter, we show that folded cube graphs have quantum symmetry and compute their quantum automorphism group in the odd case. We remark that the folded 5-cube graph is isomorphic to the Clebsch graph (see Figure 5.1) and we get $G^+_{aut}(\Gamma_{Clebsch}) = SO_5^{-1}$. It was asked to investigate the quantum automorphism group of the Clebsch graph and to show whether or not it has quantum symmetry in [10]. We start by proving that the Clebsch graph has quantum symmetry in Section 5.1. For this, we find disjoint automorphisms of the Clebsch graph and then use Theorem 3.1.2. In Section 5.2, we take a closer look at the compact matrix quantum group SO_n^{-1}. We especially study relations between the generators of the quantum group. Then, we review folded n-cube graphs FQ_n in Section 5.3. We use the fact that those graphs are Cayley graphs to get more insight about their eigenvalues and eigenspaces. Finally, in Section 5.4, we compute the quantum automorphism group of the folded n-cube graphs, where we see $G^+_{aut}(FQ_n) = SO_n^{-1}$. For this, we use similar techniques as in [9], where Banica, Bichon and Collins computed the quantum automorphism group of the n-cube graphs. We proceed as in Sections 3 – 5 of [49] by the author.

5.1 The Clebsch graph has quantum symmetry

As an application of Theorem 3.1.2, we show that the Clebsch graph does have quantum symmetry in this section. Later on, we will study the quantum automorphism group of this graph. To show that the Clebsch graph has quantum symmetry, we have to get a pair of non-trivial, disjoint automorphisms of the Clebsch graph.

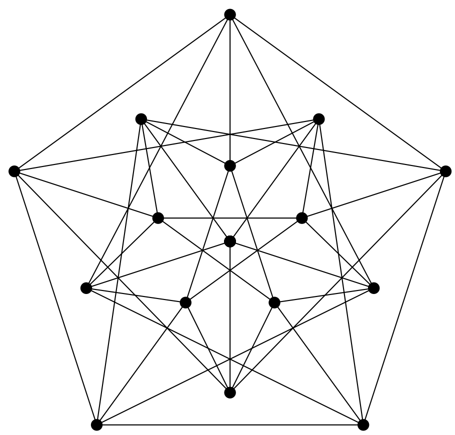
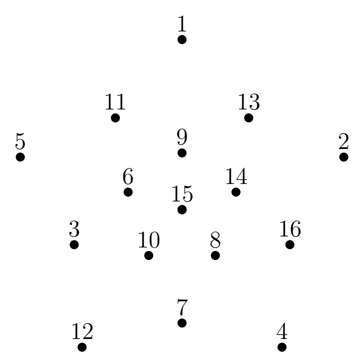


Figure 5.1: The Clebsch graph

Proposition 5.1.1. *The Clebsch graph has disjoint automorphisms.*

Proof. We label the graph as follows



Then we get two non-trivial disjoint automorphisms of this graph

$$\sigma = (2\,3)(6\,7)(10\,11)(14\,15),$$
$$\tau = (1\,4)(5\,8)(9\,12)(13\,16).$$

□

Now, Theorem 3.1.2 yields the following.

Corollary 5.1.2. *The Clebsch graph has quantum symmetry.*

Proof. By Theorem 3.1.2, we get that the C^*-algebra $C(G^+_{aut}(\Gamma_{Clebsch}))$ is non-commutative. Looking at the proof of Theorem 3.1.2, we get the surjective *-homomorphism $\varphi : C(G^+_{aut}(\Gamma_{Clebsch})) \to C^*(p, q \,|\, p = p^* = p^2, q = q^* = q^2)$,

$$u \mapsto u' = \begin{pmatrix} u'' & 0 & 0 & 0 \\ 0 & u'' & 0 & 0 \\ 0 & 0 & u'' & 0 \\ 0 & 0 & 0 & u'' \end{pmatrix},$$

where

$$u'' = \begin{pmatrix} q & 0 & 0 & 1-q \\ 0 & p & 1-p & 0 \\ 0 & 1-p & p & 0 \\ 1-q & 0 & 0 & q \end{pmatrix}.$$

□

Remark 5.1.3.

(i) The Clebsch graph is the folded 5-cube graph, which will be introduced in Section 5.3. There we will study the quantum automorphism group for $(2m+1)$-folded cube graphs going beyond Corollary 5.1.2.

(ii) Using Theorem 3.1.2, it is also easy to see that the folded cube graphs have quantum symmetry, but this will also follow from Theorem 5.4.3.

5.2 The quantum group SO_n^{-1}

Now, we will have a closer look at the quantum group SO_n^{-1}, but first we define O_n^{-1}, which appeared in [9] as the quantum automorphism group of the hypercube graph. For both it is immediate to check that the comultiplication Δ is a *-homomorphism.

Definition 5.2.1. We define $O_n^{-1} = (C(O_n^{-1}), u)$ to be the compact matrix quantum group, where $C(O_n^{-1})$ is the universal C^*-algebra with generators u_{ij}, $1 \leq i, j \leq n$ and relations

$$u_{ij} = u_{ij}^*, \qquad 1 \leq i, j \leq n, \tag{5.2.1}$$

$$\sum_{k=1}^{n} u_{ik}u_{jk} = \sum_{k=1}^{n} u_{ki}u_{kj} = \delta_{ij}, \qquad 1 \leq i, j \leq n, \tag{5.2.2}$$

$$u_{ij}u_{ik} = -u_{ik}u_{ij}, \, u_{ji}u_{ki} = -u_{ki}u_{ji}, \qquad k \neq j, \tag{5.2.3}$$

$$u_{ij}u_{kl} = u_{kl}u_{ij}, \qquad i \neq k, j \neq l. \tag{5.2.4}$$

For $n = 3$, SO_n^{-1} appeared in [7], where Banica and Bichon showed $SO_3^{-1} = S_4^+$. Our main result in this paper is that for n odd, SO_n^{-1} is the quantum automorphism group of the folded n-cube graph.

Definition 5.2.2. We define $SO_n^{-1} = (C(SO_n^{-1}), u)$ to be the compact matrix quantum group, where $C(SO_n^{-1})$ is the universal C^*-algebra with generators u_{ij}, $1 \leq i, j \leq n$, Relations (5.2.1) – (5.2.4) and

$$\sum_{\sigma \in S_n} u_{\sigma(1)1} \dots u_{\sigma(n)n} = 1. \tag{5.2.5}$$

Lemma 5.2.3. *Let $(u_{ij})_{1 \leq i,j \leq n}$ be the generators of $C(SO_n^{-1})$. Then*

$$\sum_{\sigma \in S_n} u_{\sigma(1)1} \dots u_{\sigma(n-1)n-1} u_{\sigma(n)k} = 0$$

for $k \neq n$.

Proof. Let $1 \leq k \leq n-1$. Using Relations (5.2.3) and (5.2.4) we get

$$\begin{aligned} u_{\sigma(1)1} \dots u_{\sigma(k)k} \dots u_{\sigma(n-1)n-1} u_{\sigma(n)k} &= -u_{\sigma(1)1} \dots u_{\sigma(n)k} \dots u_{\sigma(n-1)n-1} u_{\sigma(k)k} \\ &= -u_{\tau(1)1} \dots u_{\tau(k)k} \dots u_{\tau(n-1)n-1} u_{\tau(n)k} \end{aligned}$$

for $\tau = \sigma \circ (k\, n) \in S_n$. Therefore the summands corresponding to σ and τ sum up to zero. The result is then clear. □

The next lemma gives an equivalent formulation of Relation (5.2.5). One direction is a special case of [55, Lemma 4.6].

Lemma 5.2.4. *Let A be a C^*-algebra and let $u_{ij} \in A$ be elements that fulfill Relations (5.2.1) – (5.2.4). Let $j \in \{1, \dots, n\}$ and define*

$$I_j = \{(i_1, \dots, i_{n-1}) \in \{1, \dots, n\}^{n-1} \,|\, i_a \neq i_b \text{ for } a \neq b, i_s \neq j \text{ for all } s\}.$$

The following are equivalent

(i) We have

$$1 = \sum_{\sigma \in S_n} u_{\sigma(1)1} \dots u_{\sigma(n)n}.$$

(ii) It holds

$$u_{jn} = \sum_{(i_1,\dots,i_{n-1}) \in I_j} u_{i_1 1} \dots u_{i_{n-1} n-1}, \quad 1 \leq j \leq n.$$

Proof. We first show that (ii) implies (i). It holds

$$1 = \sum_{j=1}^{n} u_{jn}^2 = \sum_{j=1}^{n} \sum_{(i_1,\dots,i_{n-1})\in I_j} u_{i_1 1} \dots u_{i_{n-1} n-1} u_{jn},$$

where we used Relation (5.2.2) and (ii). Furthermore, we have

$$\begin{aligned}\sum_{j=1}^{n} \sum_{(i_1,\dots,i_{n-1})\in I_j} u_{i_1 1} \dots u_{i_{n-1} n-1} u_{jn} &= \sum_{\substack{i_1,\dots,i_n;\\ i_a\neq i_b \text{ for } a\neq b}} u_{i_1 1} \dots u_{i_n n}\\ &= \sum_{\sigma\in S_n} u_{\sigma(1)1} \dots u_{\sigma(n)n}\end{aligned}$$

and thus (ii) implies (i).

Now we show that (i) implies (ii). We have

$$u_{jn} = \sum_{\sigma\in S_n} u_{\sigma(1)1} \dots u_{\sigma(n)n} u_{jn} = \sum_{k=1}^{n} \sum_{\sigma\in S_n} u_{\sigma(1)1} \dots u_{\sigma(n-1)n-1} u_{\sigma(n)k} u_{jk},$$

since $\sum_{\sigma\in S_n} u_{\sigma(1)1} \dots u_{\sigma(n-1)n-1} u_{\sigma(n)k} u_{jk} = 0$ for $k \neq n$ by Lemma 5.2.3. We get

$$\begin{aligned}\sum_{k=1}^{n} \sum_{\sigma\in S_n} u_{\sigma(1)1} \dots u_{\sigma(n-1)n-1} u_{\sigma(n)k} u_{jk} &= \sum_{\sigma\in S_n} u_{\sigma(1)1} \dots u_{\sigma(n-1)n-1} \sum_{k=1}^{n} u_{\sigma(n)k} u_{jk}\\ &= \sum_{\sigma\in S_n} u_{\sigma(1)1} \dots u_{\sigma(n-1)n-1} \delta_{\sigma(n)j}\\ &= \sum_{(i_1,\dots,i_{n-1})\in I_j} u_{i_1 1} \dots u_{i_{n-1} n-1},\end{aligned}$$

where we used Relation (5.2.2) and we obtain $u_{jn} = \sum_{(i_1,\dots,i_{n-1})\in I_j} u_{i_1 1} \dots u_{i_{n-1} n-1}$. □

We now discuss representations of SO_{2m+1}^{-1}. For definitions and background for this proposition, we refer to [12, 15, 47].

Proposition 5.2.5. *The category of representations of SO_{2m+1}^{-1} is tensor equivalent to the category of representations of SO_{2m+1}.*

Proof. We first show that $C(SO_{2m+1}^{-1})$ is a cocycle twist of $C(SO_{2m+1})$ by proceeding like in [15, Section 4]. Take the unique bicharacter $\sigma : \mathbb{Z}_2^{2m} \times \mathbb{Z}_2^{2m} \to \{\pm 1\}$ with

$$\begin{aligned}&\sigma(t_i, t_j) = -1 = -\sigma(t_j, t_i), &&\text{for } 1 \leq i < j \leq 2m,\\ &\sigma(t_i, t_i) = (-1)^m, &&\text{for } 1 \leq i \leq 2m+1,\end{aligned}$$

$$\sigma(t_i, t_{2m+1}) = (-1)^{m-i} = -\sigma(t_{2m+1}, t_i), \qquad \text{for } 1 \leq i \leq 2m,$$

where we use the identification $\mathbb{Z}_2^{2m} = \langle t_1, \ldots, t_{2m+1} \,|\, t_i^2 = 1, t_i t_j = t_j t_i, t_{2m+1} = t_1 \ldots t_{2m} \rangle$. Let H be the subgroup of diagonal matrices in SO_{2m+1} having ± 1 entries. We get a surjective *-homomorphism

$$\pi : C(SO_{2m+1}) \to C^*(\mathbb{Z}_2^{2m})$$
$$u_{ij} \mapsto \delta_{ij} t_i$$

by restricting the functions on SO_{2m+1} to H and using Fourier transform. Thus we can form the twisted algebra $C(SO_{2m+1})^\sigma$, where we have the multiplication

$$[u_{ij}][u_{kl}] = \sigma(t_i, t_k)\sigma^{-1}(t_j, t_l)[u_{ij} u_{kl}] = \sigma(t_i, t_k)\sigma(t_j, t_l)[u_{ij} u_{kl}].$$

We see that the generators $[u_{ij}]$ of $C(SO_{2m+1})^\sigma$ fulfill the same relations as the generators of $C(SO_{2m+1}^{-1})$ and therefore we get an surjective *-homomorphism $\varphi : C(SO_{2m+1}^{-1}) \to C(SO_{2m+1})^\sigma$, $u_{ij} \mapsto [u_{ij}]$. This is an isomorphism for example by using Theorem 3.5 of [29]. Now, Corollary 1.4 and Proposition 2.1 of [12] yield the assertion. □

5.3 The folded n-cube graph FQ_n

In what follows, we will review folded cube graphs FQ_n and show that for odd n, the quantum automorphism group of FQ_n is SO_n^{-1}. Folded cube graphs are for example discussed in [18, Section 9.2].

Definition 5.3.1. The folded n-cube graph FQ_n is the graph with vertex set $V = \{(x_1, \ldots, x_{n-1}) \,|\, x_i \in \{0, 1\}\}$, where two vertices $(x_1, \ldots, x_{n-1})$ and $(y_1, \ldots, y_{n-1})$ are connected if they differ at exactly one position or if $(y_1, \ldots, y_{n-1}) = (1 - x_1, \ldots, 1 - x_{n-1})$.

Remark 5.3.2. To justify the name, one can obtain the folded n-cube graph by identifing every opposite pair of vertices from the n-hypercube graph.

It is known that the folded cube graphs are Cayley graphs, we recall this fact in the next lemma.

Lemma 5.3.3. *The folded n-cube graph FQ_n is the Cayley graph of the group $\mathbb{Z}_2^{n-1} = \langle t_1, \ldots t_n \rangle$, where the generators t_i fulfill the relations $t_i^2 = 1, t_i t_j = t_j t_i, t_n = t_1 \ldots t_{n-1}$.*

Proof. Consider the Cayley graph of $\mathbb{Z}_2^{n-1} = \langle t_1, \ldots t_n \rangle$. The vertices are elements of $\mathbb{Z}_2^{n-1}$, which are products of the form $g = t_1^{i_1} \ldots t_{n-1}^{i_{n-1}}$. The exponents are in one to one correspondence to $(x_1, \dots, x_{n-1})$, $x_i \in \{0,1\}$, thus the vertices of the Cayley graph are the vertices of the folded n-cube graph. The edges of the Cayley graph are drawn between vertices g, h, where $g = ht_i$ for some i. For $k \in \{1, \dots, n-1\}$, the operation $h \to ht_k$ changes the k-th exponent to $1 - i_k$, so we get edges between vertices that differ at exactly one exponent. The operation $h \to ht_n$ takes $t_1^{j_1} \dots t_{n-1}^{j_{n-1}}$ to $t_1^{1-j_1} \dots t_{n-1}^{1-j_{n-1}}$, thus we get the remaining edges of FQ_n. □

We will now discuss the eigenvalues and eigenvectors of the adjacency matrix of FQ_n. We write

$$\begin{aligned} \mathbb{Z}_2^n &= \{t_1^{i_1} \dots t_n^{i_n} | i_1, \dots, i_n \in \{0,1\}\}, \\ C(\mathbb{Z}_2^n) &= \operatorname{span}(e_{t_1^{i_1} \dots t_n^{i_n}} \mid t_1^{i_1} \dots t_n^{i_n} \in \mathbb{Z}_2^n), \end{aligned}$$

where

$$e_{t_1^{i_1} \dots t_n^{i_n}} : \mathbb{Z}_2^n \to \mathbb{C}, \qquad e_{t_1^{i_1} \dots t_n^{i_n}}(t_1^{j_1} \dots t_n^{j_n}) = \delta_{i_1 j_1} \dots \delta_{i_n j_n}.$$

Lemma 5.3.4. *The eigenvectors and corresponding eigenvalues of FQ_n are given by*

$$\begin{aligned} w_{i_1 \dots i_{n-1}} &= \sum_{j_1, \dots, j_{n-1}=0}^{1} (-1)^{i_1 j_1 + \dots + i_{n-1} j_{n-1}} e_{t_1^{j_1} \dots t_{n-1}^{j_{n-1}}} \\ \lambda_{i_1 \dots i_{n-1}} &= (-1)^{i_1} + \dots + (-1)^{i_{n-1}} + (-1)^{i_1 + \dots + i_{n-1}}, \end{aligned}$$

when the vector space spanned by the vertices of FQ_n is identified with $C(\mathbb{Z}_2^{n-1})$.

Proof. Let ε be the adjacency matrix of FQ_n. Then we know for a vertex p and a function f on the vertices that

$$\varepsilon f(p) = \sum_{q;(q,p) \in E} f(q).$$

This yields

$$\varepsilon e_{t_1^{j_1} \dots t_{n-1}^{j_{n-1}}} = \sum_{k=1}^{n} e_{t_k t_1^{j_1} \dots t_{n-1}^{j_{n-1}}} = e_{t_1^{j_1+1} \dots t_{n-1}^{j_{n-1}}} + \dots + e_{t_1^{j_1} \dots t_{n-1}^{j_{n-1}+1}} + e_{t_1^{j_1+1} \dots t_{n-1}^{j_{n-1}+1}}.$$

For the vectors in the statement we get

$$\varepsilon w_{i_1 \dots i_{n-1}} = \sum_{j_1, \dots, j_{n-1}} (-1)^{i_1 j_1 + \dots + i_{n-1} j_{n-1}} \varepsilon e_{t_1^{j_1} \dots t_{n-1}^{j_{n-1}}}$$

$$
\begin{aligned}
&= \sum_{s=1}^{n-1} \sum_{j_1,\dots,j_{n-1}} (-1)^{i_1 j_1 + \dots + i_{n-1} j_{n-1}} e_{t_1^{j_1} \dots t_s^{j_s+1} \dots t_{n-1}^{j_{n-1}}} \\
&\quad + \sum_{j_1,\dots,j_{n-1}} (-1)^{i_1 j_1 + \dots + i_{n-1} j_{n-1}} e_{t_1^{j_1+1} \dots t_{n-1}^{j_{n-1}+1}}.
\end{aligned}
$$

Using the index shift $j'_s = j_s + 1 \bmod 2$, for $s \in \{1, \dots n-1\}$, we get

$$
\begin{aligned}
\varepsilon w_{i_1 \dots i_{n-1}} &= \sum_{s=1}^{n-1} \sum_{j_1,\dots j'_s,\dots j_{n-1}} (-1)^{i_1 j_1 + \dots + i_s (j'_s+1) + \dots + i_{n-1} j_{n-1}} e_{t_1^{j_1} \dots t_s^{j'_s} \dots t_{n-1}^{j_{n-1}}} \\
&\quad + \sum_{j'_1,\dots,j'_{n-1}} (-1)^{i_1 (j'_1+1) + \dots + i_{n-1}(j'_{n-1}+1)} e_{t_1^{j'_1} \dots t_{n-1}^{j'_{n-1}}} \\
&= \sum_{s=1}^{n-1} \sum_{j_1,\dots,j'_s,\dots,j_{n-1}} (-1)^{i_s} (-1)^{i_1 j_1 + \dots + i_{n-1} j_{n-1}} e_{t_1^{j_1} \dots t_s^{j'_s} \dots t_{n-1}^{j_{n-1}}} \\
&\quad + \sum_{j'_1,\dots,j'_{n-1}} (-1)^{i_1 + \dots + i_{n-1}} (-1)^{i_1 j'_1 + \dots + i_{n-1} j'_{n-1}} e_{t_1^{j'_1} \dots t_{n-1}^{j'_{n-1}}} \\
&= ((-1)^{i_1} + \ldots + (-1)^{i_{n-1}} + (-1)^{i_1 + \ldots + i_{n-1}}) w_{i_1 \dots i_{n-1}} \\
&= \lambda_{i_1 \dots i_{n-1}} w_{i_1 \dots i_{n-1}}.
\end{aligned}
$$

Since those are 2^{n-1} vectors that are linearly independent, the assertion follows. □

One can obtain a C^*-algebra from the group $\mathbb{Z}_2^n$ by either considering the continuous functions $C(\mathbb{Z}_2^n)$ over the group or the group C^*-algebra $C^*(\mathbb{Z}_2^n)$. Since $\mathbb{Z}_2^n$ is abelian, we know that $C(\mathbb{Z}_2^n) \cong C^*(\mathbb{Z}_2^n)$ by Pontryagin duality. This isomorphism is given by the Fourier transform and its inverse. Here

$$
C^*(\mathbb{Z}_2^n) = C^*(t_1, \dots, t_n \,|\, t_i = t_i^*, t_i^2 = 1, t_i t_j = t_j t_i).
$$

The proof of the following proposition can be found in [4] for example.

Proposition 5.3.5. *The *-homomorphisms*

$$
\varphi : C(\mathbb{Z}_2^n) \to C^*(\mathbb{Z}_2^n), \qquad e_{t_1^{i_1} \dots t_n^{i_n}} \to \frac{1}{2^n} \sum_{j_1,\dots,j_n=0}^{1} (-1)^{i_1 j_1 + \dots + i_n j_n} t_1^{j_1} \dots t_n^{j_n}
$$

and

$$
\psi : C^*(\mathbb{Z}_2^n) \to C(\mathbb{Z}_2^n), \qquad t_1^{i_1} \dots t_n^{i_n} \to \sum_{j_1,\dots,j_n=0}^{1} (-1)^{i_1 j_1 + \dots + i_n j_n} e_{t_1^{j_1} \dots t_n^{j_n}},
$$

where $i_1, \dots, i_n \in \{0, 1\}$, are inverse to each other. The map φ is called Fourier transform, the map ψ is called inverse Fourier transform.

The following lemma shows what the eigenvectors look like if we identify the vector space spanned by the vertices of FQ_n with $C^*(\mathbb{Z}_2^{n-1})$.

Lemma 5.3.6. *In $C^*(\mathbb{Z}_2^{n-1}) = C^*(t_1, \dots, t_n \,|\, t_i^2 = 1, t_it_j = t_jt_i, t_n = t_1 \dots t_{n-1})$ the eigenvectors of FQ_n are*

$$\hat{w}_{i_1 \dots i_{n-1}} = t_1^{i_1} \dots t_{n-1}^{i_{n-1}}$$

corresponding to the eigenvalues $\lambda_{i_1 \dots i_{n-1}}$ from Lemma 5.3.4.

Proof. We obtain $\hat{w}_{i_1 \dots i_{n-1}}$ by using the Fourier transform on $w_{i_1 \dots i_{n-1}}$ from Lemma 5.3.4. □

Note that certain eigenvalues in Lemma 5.3.4 coincide. We get a better description of the eigenvalues and eigenspaces of FQ_n in the next lemma.

Lemma 5.3.7. *The eigenvalues of FQ_n are given by $\lambda_k = n - 2k$ for $k \in 2\mathbb{Z} \cap \{0, \dots, n\}$. The eigenvectors $t_1^{i_1} \dots t_{n-1}^{i_{n-1}}$ corresponding to λ_k have word lengths k or $k-1$ and form a basis of E_{λ_k}. Here E_{λ_k} denotes the eigenspace to the eigenvalue λ_k.*

Proof. Let $k \in 2\mathbb{Z} \cap \{0, \dots, n\}$. By Lemma 5.3.4 and Lemma 5.3.6, we get that an eigenvector $t_1^{i_1} \dots t_{n-1}^{i_{n-1}}$ of word length k (here $k \neq n$, if n is even) with respect to $t_1, \dots, t_{n-1}$ corresponds to the eigenvalue

$$(-1)^{i_1} + \dots + (-1)^{i_{n-1}} + (-1)^{i_1 + \dots + i_{n-1}} = -k + (n-1-k) + 1 = n - 2k.$$

Now consider an eigenvector $t_1^{i_1} \dots t_{n-1}^{i_{n-1}}$ of word length $k-1$. Then we get the eigenvalue

$$(-1)^{i_1} + \dots + (-1)^{i_{n-1}} + (-1)^{i_1 + \dots + i_{n-1}} = -(k-1) + (n-k) - 1 = n - 2k.$$

We go through all the eigenvectors of Lemma 5.3.6 in this way and we obtain exactly the eigenvalues $\lambda_k = n - 2k$. Since the eigenvectors of word lengths k or $k-1$ are exactly those corresponding to λ_k, they form a basis of E_{λ_k}. □

5.4 The quantum automorphism group of FQ_{2m+1}

From now on, we restrict to the folded n-cube graphs, where $n = 2m+1$ is odd. We show that in this case, the quantum automorphism group is SO_n^{-1}. It was asked in [10] by Banica, Bichon and Collins to investigate the quantum automorphism group of the Clebsch graph. Since the folded 5-cube graph is the Clebsch graph we get $G_{aut}^+(\Gamma_{Clebsch}) = SO_5^{-1}$. At first, we need the following lemma.

Lemma 5.4.1. *Let $\tau_1, \ldots, \tau_n$ be generators of $C^*(\mathbb{Z}_2^{n-1})$ with $\tau_i^2 = 1, \tau_i\tau_j = \tau_j\tau_i$, $\tau_n = \tau_1 \ldots \tau_{n-1}$ and let A be a C^*-algebra with elements $u_{ij} \in A$ fulfilling Relations (5.2.1) – (5.2.4). Let $(i_1, \ldots, i_l) \in \{1, \ldots, n\}^l$ with $i_a \neq i_b$ for $a \neq b$, where $1 \leq l \leq n$. Then*

$$\sum_{j_1,\ldots,j_l=1}^{n} \tau_{j_1} \ldots \tau_{j_l} \otimes u_{j_1 i_1} \ldots u_{j_l i_l} = \sum_{\substack{j_1,\ldots,j_l;\\ j_a \neq j_b \text{ for } a\neq b}} \tau_{j_1} \ldots \tau_{j_l} \otimes u_{j_1 i_1} \ldots u_{j_l i_l}.$$

Proof. Let $j_s = j_{s+1} = k$ and let the remaining j_l be arbitrary. Summing over k, we get

$$\begin{aligned}\sum_{k=1}^{n} &\tau_{j_1} \ldots \tau_{j_{s-1}} \tau_k^2 \tau_{j_{s+2}} \ldots \tau_{j_l} \otimes u_{j_1 i_1} \ldots u_{k i_s} u_{k i_{s+1}} \ldots u_{j_l i_l} \\ &= \tau_{j_1} \ldots \tau_{j_{s-1}} \tau_{j_{s+2}} \ldots \tau_{j_l} \otimes u_{j_1 i_1} \ldots \left(\sum_{k=1}^{n} u_{k i_s} u_{k i_{s+1}} \right) \ldots u_{j_l i_l} \\ &= 0\end{aligned}$$

by Relation (5.2.2) since $i_s \neq i_{s+1}$. Doing this for all $s \in \{1, \ldots, l-1\}$ we get

$$\sum_{j_1,\ldots,j_l=1}^{n} \tau_{j_1} \ldots \tau_{j_l} \otimes u_{j_1 i_1} \ldots u_{j_l i_l} = \sum_{j_1 \neq \cdots \neq j_l} \tau_{j_1} \ldots \tau_{j_l} \otimes u_{j_1 i_1} \ldots u_{j_l i_l}.$$

Now, let $j_s = j_{s+2} = k$ and let $j_1 \neq \cdots \neq j_l$. Since $k = j_s \neq j_{s+1}$ and $i_a \neq i_b$ for $a \neq b$, we have $u_{k i_s} u_{j_{s+1} i_{s+1}} = u_{j_{s+1} i_{s+1}} u_{k i_s}$. We also know that $\tau_k \tau_{j_{s+1}} = \tau_{j_{s+1}} \tau_k$ and thus

$$\begin{aligned}\sum_{k=1}^{n} &\tau_{j_1} \ldots \tau_{j_{s-1}} \tau_k \tau_{j_{s+1}} \tau_k \tau_{j_{s+3}} \ldots \tau_{j_l} \otimes u_{j_1 i_1} \ldots u_{k i_s} u_{j_{s+1} i_{s+1}} u_{k i_{s+2}} \ldots u_{j_l i_l} \\ &= \tau_{j_1} \ldots \tau_{j_{s-1}} \tau_{j_{s+1}} \tau_{j_{s+3}} \ldots \tau_{j_l} \otimes u_{j_1 i_1} \ldots u_{j_{s+1} i_{s+1}} \left(\sum_{k=1}^{n} u_{k i_s} u_{k i_{s+2}} \right) \ldots u_{j_l i_l} \\ &= 0\end{aligned}$$

by Relation (5.2.2) since $i_s \neq i_{s+2}$. This yields

$$\sum_{j_1,\ldots,j_l=1}^{n} \tau_{j_1} \ldots \tau_{j_l} \otimes u_{j_1 i_1} \ldots u_{j_l i_l} = \sum_{\substack{j_1,\ldots,j_l;\\ j_a \neq j_b \text{ for } 0<|a-b|\leq 2}} \tau_{j_1} \ldots \tau_{j_l} \otimes u_{j_1 i_1} \ldots u_{j_l i_l}$$

The assertion follows after iterating this argument l times. □

We first show that SO_n^{-1} acts on the folded n-cube graph.

Lemma 5.4.2. *For n odd, the quantum group SO_n^{-1} acts on FQ_n.*

Proof. We need to show that there exists an action

$$\alpha : C(V_{FQ_n}) \to C(V_{FQ_n}) \otimes C(SO_n^{-1}), \qquad \alpha(e_i) = \sum_{j=1}^{|V_{FQ_n}|} e_j \otimes v_{ji}$$

such that (v_{ij}) commutes with the adjacency matrix of FQ_n. By Fourier transform, this is the same as getting an action

$$\alpha : C^*(\mathbb{Z}_2^{n-1}) \to C^*(\mathbb{Z}_2^{n-1}) \otimes C(SO_n^{-1}),$$

where we identify the functions on the vertex set of FQ_n with $C^*(\mathbb{Z}_2^{n-1})$. We claim that

$$\alpha(\tau_i) = \sum_{j=1}^{n} \tau_j \otimes u_{ji}$$

gives the answer, where

$$\tau_i = t_1 \dots \check{t}_i \dots t_{n-1} \text{ for } 1 \leq i \leq n-1, \qquad \tau_n = t_n$$

for t_i as in Lemma 5.3.6 and (u_{ij}) is the fundamental representation of SO_n^{-1}. Here $\check{t}_i$ means that t_i is not part of the product. These τ_i generate $C^*(\mathbb{Z}_2^{n-1})$, with relations $\tau_i = \tau_i^*, \tau_i^2 = 1, \tau_i\tau_j = \tau_j\tau_i$ and $\tau_n = \tau_1 \dots \tau_{n-1}$. Define

$$\tau_i' = \sum_{j=1}^{n} \tau_j \otimes u_{ji}.$$

To show that α defines a *-homomorphism, we have to show that the relations of the generators τ_i also hold for τ_i'. It is obvious that $(\tau_i')^* = \tau_i'$. Using Relations (5.2.2)–(5.2.4) it is straightforward to check that $(\tau_i')^2 = 1$ and $\tau_i'\tau_j' = \tau_j'\tau_i'$. Now, we show $\tau_n' = \tau_1' \dots \tau_{n-1}'$. By Lemma 5.4.1, it holds

$$\begin{aligned}\tau_1' \dots \tau_{n-1}' &= \sum_{\substack{i_1,\dots,i_{n-1};\\ i_a \neq i_b \text{ for } a \neq b}} \tau_{i_1} \dots \tau_{i_{n-1}} \otimes u_{i_1 1} \dots u_{i_{n-1} n-1} \\ &= \sum_{j=1}^{n} \sum_{(i_1,\dots,i_{n-1}) \in I_j} \tau_{i_1} \dots \tau_{i_{n-1}} \otimes u_{i_1 1} \dots u_{i_{n-1} n-1},\end{aligned}$$

where $I_j = \{(i_1, \ldots, i_{n-1}) \in \{1, \ldots, n\}^{n-1} \,|\, i_a \neq i_b \text{ for } a \neq b, i_s \neq j \text{ for all } s\}$ like in Lemma 5.2.4. For all $(i_1, \ldots, i_{n-1}) \in I_j$, we know that $\tau_{i_1} \ldots \tau_{i_{n-1}} = \tau_1 \ldots \check{\tau}_j \ldots \tau_n$. Using $\tau_n = \tau_1 \ldots \tau_{n-1}$ and $\tau_i^2 = 1$, we get $\tau_1 \ldots \check{\tau}_j \ldots \tau_n = \tau_j$ and thus

$$\sum_{j=1}^{n} \sum_{(i_1,\ldots,i_{n-1}) \in I_j} \tau_{i_1} \ldots \tau_{i_{n-1}} \otimes u_{i_1 1} \ldots u_{i_{n-1} n-1}$$
$$= \sum_{j=1}^{n} \left(\tau_j \otimes \sum_{(i_1,\ldots,i_{n-1}) \in I_j} u_{i_1 1} \ldots u_{i_{n-1} n-1} \right).$$

The equivalent formulation of Relation (5.2.5) in Lemma 5.2.4 yields

$$\tau_1' \ldots \tau_{n-1}' = \sum_{j=1}^{n} \left(\tau_j \otimes \sum_{(i_1,\ldots,i_{n-1}) \in I_j} u_{i_1 1} \ldots u_{i_{n-1} n-1} \right) = \sum_{j=1}^{n} \tau_j \otimes u_{jn} = \tau_n'.$$

Summarising, the map α exists and is a *-homomorphism. It is straightforward to check that α is unital and since u is a representation, α is coassociative.

Now, we show that $\alpha(C^*(\mathbb{Z}_2^{n-1}))(1 \otimes C(SO_n^{-1}))$ is linearly dense in $C^*(\mathbb{Z}_2^{n-1}) \otimes C(SO_n^{-1})$. It holds

$$\sum_{i=1}^{n} \alpha(\tau_i)(1 \otimes u_{ki}) = \sum_{j=1}^{n} \left(\tau_j \otimes \sum_{i=1}^{n} u_{ji} u_{ki} \right) = \sum_{j=1}^{n} \tau_j \otimes \delta_{jk} = \tau_k \otimes 1,$$

thus $(\tau_k \otimes 1) \in \alpha(C^*(\mathbb{Z}_2^{n-1}))(1 \otimes C(SO_n^{-1}))$ for $1 \leq k \leq n$. Since α is unital, we also get $1 \otimes C(SO_n^{-1}) \subseteq \alpha(C^*(\mathbb{Z}_2^{n-1}))(1 \otimes C(SO_n^{-1}))$. By a standard argument, see for example [52, Section 4.2], we get that $\alpha(C^*(\mathbb{Z}_2^{n-1}))(1 \otimes C(SO_n^{-1}))$ is linearly dense in $C^*(\mathbb{Z}_2^{n-1}) \otimes C(SO_n^{-1})$.

It remains to show that the magic unitary matrix associated to α commutes with the adjacency matrix of FQ_n. We want to show that α preserves the eigenspaces of the adjacency matrix, i.e. $\alpha(E_\lambda) \subseteq E_\lambda \otimes C(SO_n^{-1})$ for all eigenspaces E_λ, then Theorem 2.1.12 yields the assertion. Since it holds $t_j = \tau_j \tau_n$, by Lemma 5.3.6 we have eigenvectors

$$\hat{w}_{i_1 \ldots i_{n-1}} = t_1^{i_1} \ldots t_{n-1}^{i_{n-1}} = \begin{cases} \tau_1^{i_1} \ldots \tau_{n-1}^{i_{n-1}} & \text{for } \sum_{k=1}^{n-1} i_k \text{ even} \\ \tau_1^{1-i_1} \ldots \tau_{n-1}^{1-i_{n-1}} & \text{for } \sum_{k=1}^{n-1} i_k \text{ odd} \end{cases}$$

corresponding to the eigenvalues $\lambda_{i_1 \ldots i_{n-1}}$ as in Lemma 5.3.4. Using Lemma 5.3.7, we see that the eigenspaces E_{λ_k} are spanned by eigenvectors $\tau_1^{i_1} \ldots \tau_{n-1}^{i_{n-1}}$ of word lengths k or $n-k$, where we consider the word length with respect to $\tau_1, \ldots, \tau_{n-1}$.

Let $1 \leq l \leq n-1$. By Lemma 5.4.1, we have for $i_1, \dots, i_l$, $i_a \neq i_b$ for $a \neq b$:

$$\alpha(\tau_{i_1} \dots \tau_{i_l}) = \sum_{\substack{j_1,\dots,j_l; \\ j_a \neq j_b \text{ for } a \neq b}} \tau_{j_1} \dots \tau_{j_l} \otimes u_{j_1 i_1} \dots u_{j_l i_l}.$$

For $\tau_{j_1} \dots \tau_{j_l}$, where $j_s \neq n$ for all s, we immediately get that this is in the same eigenspace as $\tau_{i_1} \dots \tau_{i_l}$ since $\tau_{j_1} \dots \tau_{j_l}$ has the same word length as $\tau_{i_1} \dots \tau_{i_l}$. Take now $\tau_{j_1} \dots \tau_{j_l}$, where we have $j_s = n$ for some s. We get

$$\begin{aligned} \tau_{j_1} \dots \tau_{j_l} &= \tau_{j_1} \dots \check{\tau_{j_s}} \dots \tau_{j_l} \tau_n \\ &= \tau_{j_1} \dots \check{\tau_{j_s}} \dots \tau_{j_l} \tau_1 \dots \tau_{n-1}, \end{aligned}$$

which has word length $n-1-(l-1) = n-l$, thus it is in the same eigenspace as $\tau_{i_1} \dots \tau_{i_l}$. This yields

$$\alpha(E_\lambda) \subseteq E_\lambda \otimes C(SO_n^{-1}),$$

for all eigenspaces E_λ and thus SO_n^{-1} acts on FQ_n by Theorem 2.1.12. □

Now, we can prove the Theorem D.

Theorem 5.4.3. *For n odd, the quantum automorphism group of the folded n-cube graph FQ_n is SO_n^{-1}.*

Proof. By Lemma 5.4.2 we get a surjective map $C(G^+_{aut}(FQ_n)) \to C(SO_n^{-1})$. We have to show that this is an isomorphism between $C(SO_n^{-1})$ and $C(G^+_{aut}(FQ_n))$. Consider the universal action on FQ_n

$$\beta : C^*(\mathbb{Z}_2^{n-1}) \to C^*(\mathbb{Z}_2^{n-1}) \otimes C(G^+_{aut}(FQ_n)).$$

Consider $\tau_1, \dots, \tau_n$ like in Lemma 5.4.2. They have word length $n-2$ or $n-1$ with respect to $t_1, \dots, t_{n-1}$ and they form a basis of E_{-n+2} by Lemma 5.3.7. Therefore, we get elements x_{ij} such that

$$\beta(\tau_i) = \sum_{j=1}^{n} \tau_j \otimes x_{ji}$$

by Corollary 2.1.13. Similar to [9] one shows that x_{ij} fulfill Relations (5.2.1)–(5.2.4). It remains to show that Relation (5.2.5) holds. Applying β to $\tau_n = \tau_1 \dots \tau_{n-1}$ and using Lemma 5.4.1 yields

$$\sum_j \tau_j \otimes x_{jn} = \beta(\tau_n) = \sum_{\substack{i_1,\dots,i_{n-1}; \\ i_a \neq i_b \text{ for } a \neq b}} \tau_{i_1} \dots \tau_{i_{n-1}} \otimes x_{i_1 1} \dots x_{i_{n-1} n-1}$$

$$= \sum_{j=1}^{n} \sum_{(i_1,\dots,i_{n-1}) \in I_j} \tau_{i_1} \dots \tau_{i_{n-1}} \otimes x_{i_1 1} \dots x_{i_{n-1} n-1}.$$

As in the proof of Lemma 5.4.2, we have $\tau_{i_1} \dots \tau_{i_{n-1}} = \tau_j$ for $(i_1, \dots, i_{n-1}) \in I_j$ and we get

$$\sum_{j=1}^{n} \tau_j \otimes x_{jn} = \sum_{j=1}^{n} \left(\tau_j \otimes \sum_{(i_1,\dots,i_{n-1}) \in I_j} x_{i_1 1} \dots x_{i_{n-1} n-1} \right).$$

We deduce

$$x_{jn} = \sum_{(i_1,\dots,i_{n-1}) \in I_j} x_{i_1 1} \dots x_{i_{n-1} n-1},$$

which is equivalent to Relation (5.2.5) by Lemma 5.2.4. Thus, we also get a surjective map $C(SO_n^{-1}) \to C(G_{aut}^+(FQ_n))$ which is inverse to the map $C(G_{aut}^+(FQ_n)) \to C(SO_n^{-1})$. □

Remark 5.4.4. The folded 3-cube graph is the full graph on four points, thus our theorem yields $S_4^+ = SO_3^{-1}$, as already shown in [7].

Remark 5.4.5. We do not have a similar theorem for folded cube graphs FQ_n with n even, since the eigenspace associated to the smallest eigenvalue behaves different in the odd case. Recall from Lemma 5.3.7 that the smallest eigenvalue in the odd case is $-n+2$, whereas it is $-n$ in the even case. In the even case, the eigenspace E_{-n} is one dimensional. In contrast, for n odd, the eigenspace E_{-n+2} is n dimensional. The author did not succeed in finding an eigenspace for which a similar strategy as in Lemma 5.4.2 and Theorem 5.4.3 could be applied, for n even.

Chapter 6

A generating property for planar algebras

In this chapter we review a generating property for planar algebras which is closely related to quantum automorphism groups of graphs. This generating property was introduced by Ren in [45] and the connection to quantum automorphism groups is mentioned therein. We discuss in detail that the group-action planar algebra associated to certain graphs is generated by its 2-boxes if and only if the graph has no quantum symmetry. This especially holds for distance-transitive graphs. Using the results of Chapter 4, we obtain many new examples of graphs having this generating property.

Moreover, we use the equivalence to show that the Higman-Sims graph has quantum symmetry (Theorem 6.3.3), where it was asked in [10] to investigate its quantum automorphism group and to show whether or not it has quantum symmetry. We will see that the Higman-Sims graph is another example of a graph that has quantum symmetry and no disjoint automorphisms besides the graph in Example 7.2.7, which yields that the converse direction of Theorem 3.1.2 does not hold.

6.1 Group-action planar algebra and a generating property

At first, we briefly recall the definition of a planar algebra. For the definition of planar tangles and more details see [24], [34].

Definition 6.1.1. A *planar algebra* $\mathcal{P}$ is a collection of finite-dimensional vector spaces $(\mathcal{P}_n)_{n\in\mathbb{N}\cup\{-,+\}}$ such that each planar tangle T of degree k with n internal disks $D_1, \dots, D_n$ of degree $k_1, \dots, k_n$, respectively, yields a multilinear map

$$Z_T : \bigotimes_{1 \le i \le n} \mathcal{P}_{k_i} \to \mathcal{P}_k.$$

The composition of those maps has to be compatible with the composition of planar tangles.

The *spin planar algebra* $\mathcal{P}$ denotes the planar algebra of [24, Section 3.1]. Here $\mathcal{P}_k$ is the vector space of all linear functionals $f : W^k \to \mathbb{C}$ for all $k \in \mathbb{N}$ and $\mathcal{P}_+ = \mathcal{P}_- = \mathbb{C}$, where W denotes a finite-dimensional vector space. We do not need the definition of the map Z_T for our purposes, it is given in [24, Section 3.1].

In the following, we identify linear functionals $f : W^k \to \mathbb{C}$ with elements in $W^{\otimes k}$ via

$$f : W^k \to \mathbb{C} \qquad \longleftrightarrow \qquad \sum_{i_1, \dots, i_k} f(e_{i_1}, \dots, e_{i_k}) e_{i_1} \otimes \cdots \otimes e_{i_k} \in W^{\otimes k}.$$

Definition 6.1.2. Let W be a d-dimensional vector space with basis $\{e_1, \dots, e_d\}$ and let $\mathcal{P}$ be the associated spin planar algebra. Let G be a subgroup of the symmetric group S_d. Then G has the natural action $\alpha : W \times G \to V$, $(e_i, g) \mapsto e_{g(i)}$ which can be extended diagonally to the action

$$\alpha^{\otimes n} : W^{\otimes n} \times G \to W^{\otimes n},$$
$$(e_{i_1} \otimes \cdots \otimes e_{i_n}, g) \mapsto e_{g(i_1)} \otimes \cdots \otimes e_{g(i_n)}.$$

This induces an action of G on $\mathcal{P}$. The *group-action planar algebra* $\mathcal{P}^G$ is the fixed point algebra of the group action, that is

$$\mathcal{P}_n^G = \{x \in \mathcal{P}_n \,|\, \alpha^{\otimes n}(x, g) = x \text{ for all } g \in G\}.$$

We furthermore need the notion of a planar algebra generated by a set of elements.

Definition 6.1.3. A planar algebra $\mathcal{P}$ is generated by a set of elements S, if for every $x \in \mathcal{P}_n$, there exists a planar tangle T, such that $Z_T(s_1, \dots, s_k) = x$ for some $s_i \in S$.

Definition 6.1.4. We denote by $\mathcal{A}^G \subseteq \mathcal{P}^G$ the planar subalgebra generated by $\mathcal{P}_2^G$.

A natural question is now to ask for which groups G the planar algebras $\mathcal{A}^G$ and $\mathcal{P}^G$ coincide. For a graph Γ and $G = \mathrm{Aut}(\Gamma)$, we have the following definition.

Definition 6.1.5. Let Γ be a finite graph. Then Γ has *the generating property* if the group-action planar algebra $\mathcal{P}^{\mathrm{Aut}(\Gamma)}$ is generated by $\mathcal{P}_2^{\mathrm{Aut}(\Gamma)}$, i.e. $\mathcal{P}^{\mathrm{Aut}(\Gamma)} = \mathcal{A}^{\mathrm{Aut}(\Gamma)}$.

Our definition differs from the one in [45], where a graph Γ has the generating property if $\mathcal{P}^{\mathrm{Aut}(\Gamma)}$ is generated by the adjacency matrix ε of Γ. We use the previous definition to stress the importance of orbitals and quantum orbitals of Γ (see Definition 2.4.1). Note that both definitions coincide if and only if the orbitals and the quantum orbitals are the same by Theorem 6.2.3. Moreover, remark that the generating property is only defined for strongly regular graphs in [45], both definitions are still valid for other graphs. The next definition is a quantum analogue of the group-action planar algebra, see for example [3].

Definition 6.1.6. Let W be a d-dimensional vector space with basis $\{e_1, \dots, e_d\}$ and let $\mathcal{P}$ be the associated spin planar algebra. Let G^+ be a quantum subgroup of the quantum symmetric group S_d^+ with fundamental respresentation $v \in M_d(C(G^+))$. Then G^+ has the natural action $\beta : W \to W \otimes C(G^+)$, $e_i \mapsto \sum_j e_j \otimes v_{ji}$ which can be extended to the action

$$\beta^{\otimes n} : W^{\otimes n} \to W^{\otimes n} \otimes C(G^+),$$
$$e_{i_1} \otimes \dots \otimes e_{i_n} \mapsto \sum_{j_1,\dots,j_n=1}^{d} e_{j_1} \otimes \dots \otimes e_{j_n} \otimes u_{j_1 i_1} \dots u_{j_n i_n}.$$

This induces an action of G^+ on $\mathcal{P}$. The *quantum-group-action planar algebra* $\mathcal{P}^{G^+}$ is the fixed point planar algebra of the quantum group action, that is

$$\mathcal{P}_n^{G^+} = \{x \in \mathcal{P}_n \mid \beta^{\otimes n}(x) = x \otimes 1\}.$$

For quantum automorphism groups of graphs, we have a nice description of $\mathcal{P}^{G^+_{aut}(\Gamma)}$.

Theorem 6.1.7 ([3], Theorem 6.1)**.** *The planar algebra $\mathcal{P}^{G^+_{aut}(\Gamma)}$ is equal to the planar algebra generated by ε, where ε denotes the adjacency matrix of Γ.*

6.2 Connection to graphs having no quantum symmetry

We show that, for certain graphs, having the generating property is equivalent to having no quantum symmetry. Note that one can also get this connection by using the intertwiner spaces of $G^+_{aut}(\Gamma)$ (see Section 2.3), we give more direct arguments. The following proposition is mentioned in [10, Section 13], we give a proof here.

Proposition 6.2.1. *Let Γ be a graph. The following are equivalent:*

(i) it holds $\mathcal{P}^{\mathrm{Aut}(\Gamma)} = \mathcal{P}^{G^+_{aut}(\Gamma)}$,

(ii) we have $R \in \mathcal{P}_4^{G^+_{aut}(\Gamma)}$, where $R = \sum_{i,j} e_i \otimes e_j \otimes e_i \otimes e_j$,

(iii) the graph Γ has no quantum symmetry.

Proof. The equivalence of (i) and (ii) follows from Theorem 6.1.7, [24, Theorem 5.10] and [24, Theorem 4.1]. We see the equivalence of (ii) and (iii) by the following. If $R \in \mathcal{P}_4^{G^+_{aut}(\Gamma)}$, then

$$\begin{aligned}\sum_{i,j}\sum_{t_1,\dots,t_4} e_{t_1} \otimes e_{t_2} \otimes e_{t_3} \otimes e_{t_4} \otimes u_{t_1 i}u_{t_2 j}u_{t_3 i}u_{t_4 j} &= \beta^{\otimes 4}(R)\\ &= R \otimes 1\\ &= \sum_{t_1,t_2} e_{t_1} \otimes e_{t_2} \otimes e_{t_1} \otimes e_{t_2} \otimes 1\end{aligned}$$

which is equivalent to

$$\sum_{i,j} u_{t_1 i}u_{t_2 j}u_{t_3 i}u_{t_4 j} = \delta_{t_1 t_3}\delta_{t_2 t_4} 1.$$

Choosing $i', j' \in V$ and multiplying by $u_{t_1 i'}$ from the left and by $u_{t_4 j'}$ from the right yields

$$u_{t_1 i'}u_{t_2 j'}u_{t_3 i'}u_{t_4 j'} = \delta_{t_1 t_3}\delta_{t_2 t_4} u_{t_1 i'}u_{t_4 j'}. \tag{6.2.1}$$

Thus, we get

$$u_{t_1 i'}u_{t_4 j'} = u_{t_1 i'}u_{t_4 j'}u_{t_1 i'}u_{t_4 j'} = \left(\sum_{s_1} u_{s_1 i'}\right) u_{t_4 j'}u_{t_1 i'} \left(\sum_{s_2} u_{s_2 j'}\right) = u_{t_4 j'}u_{t_1 i'}$$

by using Equation (6.2.1) and Relation (2.1.2). Hence the generators of $C(G^+_{aut}(\Gamma))$ commute which means that Γ has no quantum symmetry.

For the other direction, let $C(G^+_{aut}(\Gamma))$ be commutative. Then

$$\begin{aligned}\beta^{\otimes 4}(R) &= \sum_{i,j}\sum_{t_1,\dots,t_4} e_{t_1} \otimes e_{t_2} \otimes e_{t_3} \otimes e_{t_4} \otimes u_{t_1 i}u_{t_2 j}u_{t_3 i}u_{t_4 j}\\ &= \sum_{i,j}\sum_{t_1,\dots,t_4} e_{t_1} \otimes e_{t_2} \otimes e_{t_3} \otimes e_{t_4} \otimes u_{t_1 i}u_{t_3 i}u_{t_2 j}u_{t_4 j}\\ &= \sum_{i,j}\sum_{t_1,t_2} e_{t_1} \otimes e_{t_2} \otimes e_{t_1} \otimes e_{t_2} \otimes u_{t_1 i}u_{t_2 j}\\ &= \sum_{t_1,t_2} e_{t_1} \otimes e_{t_2} \otimes e_{t_1} \otimes e_{t_2} \otimes 1\\ &= R \otimes 1,\end{aligned}$$

by using Relation (2.1.2), i.e. $R \in \mathcal{P}_4^{G^+_{aut}(\Gamma)}$. □

The next lemma shows a connection of the 2-box space of $\mathcal{P}^{\mathrm{Aut}(\Gamma)}$ and $\mathcal{P}^{G^+_{aut}(\Gamma)}$ with the orbitals and quantum orbitals of Γ, respectively.

Lemma 6.2.2. *Let $x = \sum_{i,j} a_{ij}(e_i \otimes e_j)$ and let $a = (a_{ij})_{i,j}$ be the matrix with entries a_{ij}. We have*

(i) $x \in \mathcal{P}_2^{\mathrm{Aut}(\Gamma)}$ if and only if $a \in \mathcal{O}(\Gamma)$,

(ii) and $x \in \mathcal{P}_2^{G^+_{aut}(\Gamma)}$ if and only if $a \in \mathcal{QO}(\Gamma)$.

Proof. For (i), we have that $x \in \mathcal{P}_2^{\mathrm{Aut}(\Gamma)}$ if and only if

$$\sum_{i,j} a_{ij}(e_i \otimes e_j) = \sum_{i,j} a_{ij}(e_{g(i)} \otimes e_{g(j)})$$

for all $g \in \mathrm{Aut}(\Gamma)$. But this is equivalent to a being constant on the orbitals, i.e. $a \in \mathcal{O}(\Gamma)$. Statement (ii) follows from Lemma 2.4.7. □

We can now relate the equality of $\mathcal{A}^{\mathrm{Aut}(\Gamma)}$ and $\mathcal{P}^{G^+_{aut}(\Gamma)}$ to the equality of the orbitals and quantum orbitals of Γ.

Theorem 6.2.3. *Let Γ be a finite graph. The following are equivalent:*

(i) $\mathcal{A}^{\mathrm{Aut}(\Gamma)} = \mathcal{P}^{G^+_{aut}(\Gamma)}$, i.e. $\mathcal{A}_n^{\mathrm{Aut}(\Gamma)} = \mathcal{P}_n^{G^+_{aut}(\Gamma)}$ for all n,

(ii) $\mathcal{A}_2^{\mathrm{Aut}(\Gamma)} = \mathcal{P}_2^{G^+_{aut}(\Gamma)}$,

(iii) $\mathcal{QO}(\Gamma) = \mathcal{O}(\Gamma)$.

Proof. We first proof the equivalence of (i) and (ii). If $\mathcal{A}^{\mathrm{Aut}(\Gamma)} = \mathcal{P}^{G^+_{aut}(\Gamma)}$, then we particularly have $\mathcal{A}_2^{\mathrm{Aut}(\Gamma)} = \mathcal{P}_2^{G^+_{aut}(\Gamma)}$. For the other direction, we know by Theorem 6.1.7 that $\mathcal{P}^{G^+_{aut}(\Gamma)}$ is generated by $\varepsilon \in \mathcal{P}_2^{\mathrm{Aut}(\Gamma)}$ and by definition $\mathcal{A}^{\mathrm{Aut}(\Gamma)}$ is generated by all elements of $\mathcal{P}_2^{\mathrm{Aut}(\Gamma)}$. Thus, we get $\mathcal{P}^{G^+_{aut}(\Gamma)} \subseteq \mathcal{A}^{\mathrm{Aut}(\Gamma)}$. Since we have $\mathcal{A}_2^{\mathrm{Aut}(\Gamma)} = \mathcal{P}_2^{G^+_{aut}(\Gamma)}$, the generators of $\mathcal{A}^{\mathrm{Aut}(\Gamma)}$ are also contained in $\mathcal{P}^{G^+_{aut}(\Gamma)}$ and thus $\mathcal{A}^{\mathrm{Aut}(\Gamma)} = \mathcal{P}^{G^+_{aut}(\Gamma)}$. The equivalence of (ii) and (iii) follows from Lemma 6.2.2. □

Combining the previous results, we get the following corollaries.

Corollary 6.2.4. *Let Γ be a graph with $\mathcal{QO}(\Gamma) = \mathcal{O}(\Gamma)$. Then Γ has the generating property if and only if Γ has no quantum symmetry.*

Proof. Since we have $\mathcal{QO}(\Gamma) = \mathcal{O}(\Gamma)$, we know $\mathcal{A}^{\mathrm{Aut}(\Gamma)} = \mathcal{P}^{G^+_{aut}(\Gamma)}$ by Theorem 6.2.3. By Proposition 6.2.1, we get $\mathcal{A}^{\mathrm{Aut}(\Gamma)} = \mathcal{P}^{\mathrm{Aut}(\Gamma)}$ if and only if Γ has no quantum symmetry. □

Corollary 6.2.5. *Let Γ be a graph. If Γ has no quantum symmetry, then it also has the generating property.*

Proof. If Γ has no quantum symmetry, then we especially have $\mathcal{QO}(\Gamma) = \mathcal{O}(\Gamma)$. Then Corollary 6.2.4 completes the proof. □

Corollary 6.2.6. *Let Γ be a graph with $\mathcal{QO}(\Gamma) = \mathcal{O}(\Gamma)$. If* $\mathrm{Aut}(\Gamma)$ *contains disjoint automorphisms, then Γ does not have the generating property.*

Proof. This follows from Corollary 6.2.4 and Theorem 3.1.2. □

6.3 Applications

Looking at Example 2.4.13, we get a class of graphs with $\mathcal{QO}(\Gamma) = \mathcal{O}(\Gamma)$.

Example 6.3.1. Let Γ be a distance-transitive graph. Then Γ has the generating property if and only it has no quantum symmetry.

By the previous example, we find many graphs in Chapter 4 that have the generating property. We see that the Kneser graphs $K(n, 2)$ for $n \geq 5$ have the generating property, which is also discussed in [45]. Furthermore, restricting to strongly regular graphs, we have the following.

Corollary 6.3.2.

(i) The Paley graphs P_9, P_{13} and P_{17}, the Shrikhande graph and the Hoffman-Singleton graph do have the generating property,

(ii) The Clebsch graph and the 4×4 rook's graph do not have the generating property.

It is mentioned in [45] and briefly discussed in [34, Example 2.8] that the Higman-Sims graph does not have the generating property. We give an explicit proof here to show that the Higman-Sims graph has quantum symmetry.

Theorem 6.3.3. *The Higman-Sims graph* HS *has quantum symmetry.*

Proof. We show that the Higman-Sims graph does not have the generating property (see Definition 6.1.5). Then the statement follows from Corollary 6.2.4, since we know that the Higman-Sims graph is distance-transitive, which yields $\mathcal{QO}(\mathrm{HS}) = \mathcal{O}(\mathrm{HS})$. We will show that $\dim(\mathcal{A}_6^{\mathrm{Aut(HS)}}) \neq \dim(\mathcal{P}_6^{\mathrm{Aut(HS)}})$, deducing $\dim(\mathcal{A}^{\mathrm{Aut(HS)}}) \neq \dim(\mathcal{P}^{\mathrm{Aut(HS)}})$.

The dimension of $\mathcal{P}_k^{\mathrm{Aut(HS)}}$ can be computed by the Cauchy-Frobenius-Burnside formula, see [24, Section 3.2]. It holds

$$\dim(\mathcal{P}_k^{\mathrm{Aut(HS)}}) = \frac{1}{|\,\mathrm{Aut(HS)}|} \sum_{g \in \mathrm{Aut(HS)}} (\pi(g))^k,$$

where $\pi : \mathrm{Aut(HS)} \to \mathbb{C}$ maps $g \in \mathrm{Aut(HS)}$ to the number of fixed points of g. We have $|\,\mathrm{Aut(HS)}| = 88.704.000$ and

$$\dim(\mathcal{P}_k^{\mathrm{Aut(HS)}}) \geq \frac{100^k}{|\,\mathrm{Aut(HS)}|},$$

since $\mathrm{id} \in \mathrm{Aut(HS)}$ and $\pi(\mathrm{id}) = 100$. We deduce

$$\dim(\mathcal{P}_6^{\mathrm{Aut(HS)}}) \geq \frac{100^6}{|\,\mathrm{Aut(HS)}|} > 11000.$$

Regarding $\dim(\mathcal{A}_6^{\mathrm{Aut(HS)}})$, we know the following. Due to Jaeger [33], one can define a spin model (X, R, Q), where both matrices R, Q are linear combinations of the adjacency matrix $\varepsilon_{\mathrm{HS}}$ of the Higman-Sims graph, the all-ones matrix J and the identity matrix I. Especially, $\{I, R, Q\}$ forms a basis of $\mathcal{A}_2^{\mathrm{Aut(HS)}}$, since $\varepsilon_{\mathrm{HS}}$ and J can be obtained by linear combinations of those matrices and therefore R, Q generate $\mathcal{A}^{\mathrm{Aut(HS)}}$. Since R, Q are obtained from a spin model, they fulfill Relations $(i)-(v)$ from [34, Example 2.4] and thus $\mathcal{A}^{\mathrm{Aut(HS)}}$ is a planar subalgebra of the $\mathcal{BMW}$ planar algebra. As shown in [34, Example 2.4], it holds $\dim(\mathcal{BMW}_n) = (2n-1)!!$ and therefore we have $\dim(\mathcal{A}_n^{\mathrm{Aut(HS)}}) \leq (2n-1)!!$. In particular, we get

$$\dim(\mathcal{A}_6^{\mathrm{Aut(HS)}}) \leq 11!! = 10395.$$

Summarizing, we have

$$\dim(\mathcal{A}_6^{\mathrm{Aut(HS)}}) < 11000 < \dim(\mathcal{P}_6^{\mathrm{Aut(HS)}}).$$

Thus $\mathcal{A}^{\mathrm{Aut(HS)}} \neq \mathcal{P}^{\mathrm{Aut(HS)}}$, which means that the Higman-Sims graph does not have the generating property. □

Remark 6.3.4. Using Sage [53], one can check that the Higman-Sims graph has no disjoint automorphisms. Thus this gives another example of a graph that has quantum symmetry but no disjoint automorphisms besides Example 7.2.7, which shows that the converse direction of Theorem 3.1.2 is not true.

Remark 6.3.5. It would be nice to have an explicit surjective *-homomorphism $\varphi : G_{aut}^+(\mathrm{HS}) \to A$, where A is a non-commutative C*-algebra. We have to leave this as an open question.

Chapter 7

Quantum isomorphisms

Quantum isomorphisms constitute the main topic of this chapter. Similar to quantum automorphisms being quantum analogues of graph automorphisms, quantum isomorphisms are quantum versions of graph isomorphisms. Those quantum isomorphisms were first defined in [1]. In Section 7.1, we review the definition of quantum isomorphisms and the nonlocal game associated to them. In Section 7.2, we start by giving some basics in representation theory of compact quantum groups which are needed at the end of this section.

By Theorem 3.1.2, we know that a graph with disjoint automorphisms has quantum symmetry. Using monoidal equivalence, we prove that if a graph Γ has disjoint automorphisms, then any graph quantum isomorphic to Γ also has quantum symmetry (see Theorem 7.2.6). This constitutes the main result of this chapter. We obtain another example of a graph having quantum symmetry and no disjoint automorphisms from this theorem. The graph comes from a construction used in [36] to obtain quantum isomorphic but non-isomorphic graphs.

7.1 Equivalent definitions and the isomorphism game

Recall that for graphs $\Gamma_1 = (V_1, E_1)$, $\Gamma_2 = (V_2, E_2)$ a *graph isomorphism* is a bijection $\varphi : V_1 \to V_2$ such that $(i, j) \in E_1$ if and only if $(\varphi(i), \varphi(j)) \in E_2$. The permutation matrix σ with $\sigma_{ij} = \delta_{\varphi(i)j}$ then fulfills $\varepsilon^{(1)}\sigma = \sigma\varepsilon^{(2)}$, where $\varepsilon^{(1)}$, $\varepsilon^{(2)}$ are the respective adjacency matrices of Γ_1, Γ_2. We get back a graph isomorphism from such a permutation matrix by putting $\varphi(i) = j$ if $\sigma_{ij} = 1$. We have the following generalization, see [1].

Definition 7.1.1. Let $\Gamma_1 = (V_1, E_1)$, $\Gamma_2 = (V_2, E_2)$ be finite graphs without multiple edges and let $\varepsilon^{(1)}$, $\varepsilon^{(2)}$ be their respective adjacency matrices. We say that Γ_1 and

Γ_2 are *quantum isomorphic*, denoted $\Gamma_1 \cong_q \Gamma_2$, if there exists a unital C^*-algebra A with elements $u_{ij} \in A$, $i \in V_1, j \in V_2$, such that

$$u_{ij} = u_{ij}^* = u_{ij}^2, \qquad i \in V_1, j \in V_2, \tag{7.1.1}$$

$$\sum_{k\in V_2} u_{ik} = 1 = \sum_{k \in V_1} u_{kj}, \qquad i \in V_1, j \in V_2, \tag{7.1.2}$$

$$\varepsilon^{(1)} u = u\varepsilon^{(2)}. \tag{7.1.3}$$

Here Relation (7.1.3) means $\sum_{k\in V_1} \varepsilon_{ik}^{(1)} u_{kj} = \sum_{k \in V_2} u_{ik}\varepsilon_{kj}^{(2)}$.

The definition originates from a nonlocal game which we will discuss briefly. For more details, see [1]. The nonlocal game is called *isomorphism game*. Given two graphs Γ_1 and Γ_2, the (Γ_1, Γ_2)-isomorphism game is played as follows.

A referee sends each of the two players, Alice and Bob, a vertex of Γ_1 or Γ_2. Each player must respond to the referee with a vertex of Γ_1 or Γ_2. Alice and Bob win the game if two condition are met. The first condition is the following.

(1) If a player receives a vertex of Γ_1, he must respond with a vertex of Γ_2 and vice versa.

If (1) holds, Alice receives or responds with a vertex of Γ_1, which we denote by $\gamma_{1,A}$, and responds or receives a vertex of Γ_2, which we will call $\gamma_{2,A}$. Define $\gamma_{1,A}$ and $\gamma_{2,B}$ similarly for Bob. For two vertices v_1 and v_2, we use $\mathrm{rel}(v_1, v_2)$ to denote the *relationship* of the vertices, that is, whether they are equal, adjacent, or non-adjacent (and distinct). Then we can express the second condition easily.

(2) It holds $\mathrm{rel}(\gamma_{1,A}, \gamma_{1,B}) = \mathrm{rel}(\gamma_{2,A}, \gamma_{2,B})$.

Alice and Bob know the graphs before the game starts and can agree on a strategy. They cannot communicate during the game and one round of the game is played. A strategy is called *perfect* if it has winning probability 1, i.e. if Alice and Bob always win the game. It is shown in [1] that there is a perfect strategy for the (Γ_1, Γ_2)-isomorphism game if and only if Γ_1 and Γ_2 are isomorphic.

To get to quantum isomorphisms, we have to consider quantum strategies of the isomorphism game. In a quantum strategy, players share a quantum state $\psi \in H$, which is a unit vector ψ in a Hilbert space H. Both players perform a quantum measurement. Upon receiving a vertex $x \in V_{\Gamma_1} \cup V_{\Gamma_2}$, this is modelled by positive-operator valued measurements, $\mathcal{E}_{xy} = \{E_{xy} \in B(H) \,|\, y \in V_{\Gamma_1} \cup V_{\Gamma_2}\}$ for Alice and $\mathcal{F}_{xy} = \{F_{xy} \in B(H) \,|\, y \in V_{\Gamma_1} \cup V_{\Gamma_2}\}$ for Bob. Here $E_{xy}, F_{x'y'}$ are positive operators, where

$$\sum_y E_{xy} = 1 = \sum_{y'} F_{x'y'} \qquad \text{and} \qquad E_{xy}F_{x'y'} = F_{x'y'}E_{xy}$$

for all $x, y, x', y' \in V_{\Gamma_1} \cup V_{\Gamma_2}$. The probability of Alice and Bob responding y, y' upon receiving x and x', respectively, is given by the following formula:

$$p(y, y' \,|\, x, x') = \psi^* E_{xy} F_{x'y'} \psi.$$

The quantum strategy is perfect if and only if $p(y, y' \,|\, x, x') = 1$ for all $x, y, x', y' \in V_{\Gamma_1} \cup V_{\Gamma_2}$. Similar to classical isomorphisms, we have the next theorem, due to [36].

Theorem 7.1.2. *Let $\Gamma_1 = (V_1, E_1)$, $\Gamma_2 = (V_2, E_2)$ be finite graphs without multiple edges. Then Γ_1 and Γ_2 are quantum isomorphic if and only if there is a perfect quantum strategy for the (Γ_1, Γ_2)-isomorphism game.*

Now, we come to an equivalent formulation of quantum isomorphism for connected graphs. To check if two connected graphs are isomorphic, one can also look at their disjoint union and then consider the orbits of its automorphism group. Then the graphs are isomorphic if and only if one has vertices from different graphs in the same orbit. The next theorem shows that there is a similar way to check if two graphs are quantum isomorphic, now using quantum orbits (see Definition 2.4.1).

Theorem 7.1.3 (Theorem 4.5 in [36])**.** *Let $\Gamma_1 = (V_1, E_1)$ and $\Gamma_2 = (V_2, E_2)$ be connected graphs. Then Γ_1 and Γ_2 are quantum isomorphic if and only if there exist $i \in V_1$ and $j \in V_2$ that are in the same quantum orbit of the disjoint union $\Gamma_1 \cup \Gamma_2$.*

We use this to show that the quantum automorphism group of non-quantum-isomorphic graphs is the free product of their quantum automorphism groups.

Corollary 7.1.4. *Let $\Gamma_1 = (V_1, E_1)$, $\Gamma_2 = (V_2, E_2)$ be connected graphs that are not quantum isomorphic. Then*

$$G^+_{aut}(\Gamma_1 \cup \Gamma_2) = G^+_{aut}(\Gamma_1) * G^+_{aut}(\Gamma_2).$$

Proof. Let $\varepsilon_{\Gamma_1 \cup \Gamma_2}$, ε_{Γ_1} and ε_{Γ_2} be the adjacency matrices of the corresponding graphs. Label $\Gamma_1 \cup \Gamma_2$ such that

$$\varepsilon_{\Gamma_1 \cup \Gamma_2} = \begin{pmatrix} \varepsilon_{\Gamma_1} & 0 \\ 0 & \varepsilon_{\Gamma_2} \end{pmatrix}.$$

Since Γ_1 and Γ_2 are not quantum isomorphic, we know by the previous theorem that there are no $i \in V_1$, $j \in V_2$ in the same quantum orbit of $\Gamma_1 \cup \Gamma_2$. Therefore, we have $u_{ij} = 0$ for all $i \in V_1$, $j \in V_2$ or $i \in V_2$, $j \in V_1$, where u is the fundamental representation of $G^+_{aut}(\Gamma_1 \cup \Gamma_2)$. This yields

$$u = \begin{pmatrix} v & 0 \\ 0 & w \end{pmatrix},$$

where $v \in \mathrm{M}_{|V_1|}(C(G^+_{aut}(\Gamma_1 \cup \Gamma_2)))$, $w \in \mathrm{M}_{|V_2|}(C(G^+_{aut}(\Gamma_1 \cup \Gamma_2)))$. We also see that $u\varepsilon_{\Gamma_1 \cup \Gamma_2} = \varepsilon_{\Gamma_1 \cup \Gamma_2} u$ is equivalent to $v\varepsilon_{\Gamma_1} = \varepsilon_{\Gamma_1} v$ and $w\varepsilon_{\Gamma_2} = \varepsilon_{\Gamma_2} w$. Now, looking at the definition of the free product (Proposition 1.1.6), we get the desired *-homomorphisms in both directions by using the respective universal properties. □

The subsequent theorem is due to [38].

Theorem 7.1.5. *Let $\Gamma_1 = (V_1, E_1)$, $\Gamma_2 = (V_2, E_2)$ be finite graphs without multiple edges. Then Γ_1 and Γ_2 are quantum isomorphic if and only if they admit the same number of homomorphisms from any planar graph.*

This theorem is particularly useful for showing that two graphs are not quantum isomorphic. Indeed, if one finds a planar graph, where the number of homomorphisms to Γ_1 and Γ_2 differ, then the theorem shows that Γ_1 and Γ_2 are not quantum isomorphic.

For ruling out that two graphs are quantum isomorphic, we have the following corollary ([36, Corollary 4.7]). Recall the definition of the coherent algebra of a graph from Section 2.4. See [36] for more on isomorphisms of coherent algebras.

Corollary 7.1.6. *Suppose that Γ_1 and Γ_2 are quantum isomorphic graphs with adjacency matrices ε_1 and ε_2, respectively. Then there exists an isomorphism Φ of their coherent algebras such that $\Phi(\varepsilon_1) = \varepsilon_2$.*

The next corollary is a direct consequence of the previous corollary and [39, Theorem 6.12]. It is specific to distance-regular graphs.

Corollary 7.1.7. *Let Γ_1 be a distance-regular graph. If Γ_2 is a graph that is quantum isomorphic to Γ_1, then Γ_2 is distance-regular and cospectral to Γ_1.*

Note that for connected distance-regular graphs, being cospectral is equal to having the same intersection array. Therefore, we get that connected distance-regular graphs which are quantum isomorphic have the same intersection array. It is furthermore known that quantum isomorphic graphs have the same number of connected components (this follows for example from [39, Corollary 6.2]). Thus, for a connected distance-regular graph Γ with unique intersection array, there are no quantum isomorphic graphs that are not isomorphic to Γ.

7.2 Quantum automorphism groups of quantum isomorphic graphs

For the rest of this chapter, we work towards finding a graph that has quantum symmetry but no disjoint automorphisms, giving another example that the converse direction of Theorem 3.1.2 is not true. We get such an example by Theorem 7.2.6, which says that if a graph Γ has disjoint automorphisms, then any graph quantum isomorphic to Γ also has quantum symmetry. This is the main result of Chapter 7. To prove it, we need some background in representation theory of quantum groups. We recall some basics here, for more see [37], [42].

Lemma 7.2.1. *Any irreducible unitary representation of a compact quantum group is finite-dimensional.*

The following proposition can for example be found in [37, Propositions 7.1&7.3].

Proposition 7.2.2. *Let $G = (C(G), v)$ be a compact matrix quantum group and let $\{u^\alpha \,|\, \alpha \in I\}$ be a complete set of mutually inequivalent, irreducible unitary representations. The subspace $C(G)_0$ of $C(G)$ spanned by the matrix coefficients of all finite-dimensional unitary representations is a dense *-subalgebra of $C(G)$ with basis $\{u^\alpha_{pq} \,|\, \alpha \in I, 1 \le p, q \le n(\alpha)\}$.*

We did not find a source for our next proposition. However, it looks like a known fact.

Proposition 7.2.3. *Let $G = (C(G), u)$ be a compact matrix quantum group. Then $C(G)$ is infinite-dimensional if and only if G has infinitely many mutually inequivalent, irreducible unitary representations.*

Proof. The C^*-algebra $C(G)$ is infinite-dimensional if and only if the dense *-subalgebra $C(G)_0$ (see Proposition 7.2.2) is infinite-dimensional. But this is true if and only if $\{u^\alpha_{pq} \,|\, \alpha \in I, 1 \le p, q \le n(\alpha)\}$ has infinitely many elements and since $n(\alpha)$ is finite for all α by Lemma 7.2.1, this is equivalent to G having infinitely many mutually inequivalent, irreducible unitary representations. □

For the subsequent definition, denote by $\mathrm{Irr}(G)$ the set of equivalence classes of irreducible, unitary representations of G and by $\mathrm{Mor}(u, v)$ the intertwiner space of u, v.

Definition 7.2.4. Let G_1, G_2 be compact matrix quantum groups. We say that G_1, G_2 are *monoidally equivalent* if there exists a bijection $\varphi : \mathrm{Irr}(G_1) \to \mathrm{Irr}(G_2)$ together with linear isomorphisms

$$\varphi : \mathrm{Mor}(u_1 \otimes \cdots \otimes u_n, v_1 \otimes \cdots \otimes v_m) \to \mathrm{Mor}(\varphi(u_1) \otimes \cdots \otimes \varphi(u_n), \varphi(v_1) \otimes \cdots \otimes \varphi(v_m))$$

such that $\varphi(1_{G_1}) = \varphi(1_{G_2})$ and such that $\varphi(S \circ T) = \varphi(S) \circ \varphi(T)$ whenever $S \circ T$ is well-defined, $\varphi(S^*) = \varphi(S)^*$, $\varphi(S \otimes T) = \varphi(S) \otimes \varphi(T)$ for morphisms S, T.

Now, we connect this to quantum isomorphic graphs and their quantum automorphism groups.

Theorem 7.2.5 ([17])**.** *Let Γ_1, Γ_2 be quantum isomorphic graphs. Then $G^+_{aut}(\Gamma_1)$ and $G^+_{aut}(\Gamma_2)$ are monoidally equivalent.*

Our previous considerations yield the following.

Theorem 7.2.6. *Let Γ_1, Γ_2 be quantum isomorphic graphs. If one of the graphs Γ_1 or Γ_2 has disjoint automorphisms, then both graphs have quantum symmetry.*

Proof. Assume that Γ_1 has disjoint automorphisms. By Theorem 3.1.2 there exists a surjective *-homomorphism $\varphi : C(G^+_{aut}(\Gamma_1)) \to C^*(\mathbb{Z}_n * \mathbb{Z}_m)$. This shows that Γ_1 has quantum symmetry and it especially yields that $C(G^+_{aut}(\Gamma_1))$ is infinite-dimensional. Therefore, using Proposition 7.2.3, we see that $G^+_{aut}(\Gamma_1)$ has infinitely many mutually inequivalent, irreducible unitary representations. Since Γ_1 and Γ_2 are quantum isomorphic, $G^+_{aut}(\Gamma_1)$ and $G^+_{aut}(\Gamma_2)$ are monoidally equivalent by Theorem 7.2.5. We deduce, that $G^+_{aut}(\Gamma_2)$ also has infinitely many mutually inequivalent, irreducible unitary representations. Again using Proposition 7.2.3, we obtain that $C(G^+_{aut}(\Gamma_2))$ is infinite-dimensional. But this implies that $C(G^+_{aut}(\Gamma_2))$ is non-commutative, as otherwise $C(G^+_{aut}(\Gamma_2)) = C(\mathrm{Aut}(\Gamma_2))$, where $C(\mathrm{Aut}(\Gamma_2))$ is finite-dimensional. □

The task is now to find quantum isomorphic graphs where one of them has disjoint automorphisms, but the other one does not. Then our previous theorem yields that both graphs have quantum symmetry, especially the one without disjoint automorphisms. We give an explicit graph in the next example. David Roberson and the author previously worked out another way to see that this graph has quantum symmetry (unpublished). However, we use Theorem 7.2.6 now.

Example 7.2.7. Consider the complete bipartite graph $K_{3,4}$. We obtain two linear binary constraint systems (LBCS) in the following way (see [36, Section 4.4 & 4.5]). For each $i \in V$ we will have a constraint C_i and for each $j \in E$ we will have a variable $x_j \in \mathbb{F}_2$. Let $S_i := \{j \mid \text{ the edge } j \text{ is incident to the vertex } i\}$ and define C_i to be the constraint $\sum_{j \in S_i} x_j = 0$ over $\mathbb{F}_2$. For $K_{3,4}$, this yields

$$\begin{aligned} &x_1 + x_2 + x_3 + x_4 = 0, && x_1 + x_5 + x_9 = 0,\\ &x_5 + x_6 + x_7 + x_8 = 0, && x_2 + x_6 + x_{10} = 0,\\ &x_9 + x_{10} + x_{11} + x_{12} = 0, && x_3 + x_7 + x_{11} = 0,\\ & && x_4 + x_8 + x_{12} = 0. \end{aligned} \tag{7.2.1}$$

We get the second LBCS by using the same construction as before, but we choose a vertex $i^* \in V$, where we put $\sum_{j \in S^*_i} x_j = 1$ over $\mathbb{F}_2$. Besides this, we leave the LBCS unchanged from the first one. For example one can get the following system

$$\begin{aligned} &x_1 + x_2 + x_3 + x_4 = 0, && x_1 + x_5 + x_9 = 0,\\ &x_5 + x_6 + x_7 + x_8 = 0, && x_2 + x_6 + x_{10} = 0,\\ &x_9 + x_{10} + x_{11} + x_{12} = 0, && x_3 + x_7 + x_{11} = 0,\\ & && x_4 + x_8 + x_{12} = 1. \end{aligned} \tag{7.2.2}$$

We construct a graph from every LBCS as follows. Consider as vertices the partial solutions over $\mathbb{F}_2$ of the constraints C_i for every $i \in V$, for example we have the solutions $(0,0,0), (1,1,0), (1,0,1), (0,1,1)$ for $x_1 + x_5 + x_9 = 0$. Two vertices in the graph are connected either if they are solutions of the same constraint or if they are solutions of constraints that share one the common variable, where the solutions have different values on this variable. An example for the latter would be the solution $(0, 0, 0, 0)$ of $x_1 + x_2 + x_3 + x_4 = 0$ and the solution $(1, 1, 0)$ of $x_1 + x_5 + x_9 = 0$ (Here the constraints share the variable x_1, where the value of x_1 differs for the solutions).

By [36, Theorem 4.9], the graphs assigned to the LBCS (7.2.1) and (7.2.2) are quantum isomorphic but not isomorphic. Using Sage [53] one can show that the graph associated to the LBCS (7.2.1) does not have disjoint automorphisms, whereas the graph associated to the LBCS (7.2.2) has disjoint automorphisms. By Theorem 7.2.6, both graphs have quantum symmetry. In particular, the graph associated to the LBCS (7.2.1) has quantum symmetry and no disjoint automorphisms.

Chapter 8

Open questions

This chapter concerns open questions that occured during the research of the author or are questions of the community of quantum automorphism groups of graphs. We present those questions and give some insight and ideas on them.

8.1 Existence of graphs with quantum symmetry that have trivial automorphism group

In [26], it was shown by Erdős and Rényi that almost every graph has trivial automorphism group. The quantum counterpart has been proven by Lupini, Mančinska and Roberson in [36] (see Theorem 2.4.10): Almost all graphs have trivial quantum automorphism group. Therefore, it seems natural to have that $\mathrm{Aut}(\Gamma) = \{e\}$ implies $G^+_{aut}(\Gamma) = \{e\}$. Until now, there is no proof of such a result. Also, we do not have any examples where $\mathrm{Aut}(\Gamma) = \{e\}$, but $G^+_{aut}(\Gamma) \neq \{e\}$. The existence of such an example would mean that there are symmetry phenomena in the quantum situation, which cannot be seen from the classical viewpoint. We ask the following question.

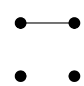
Question 8.1.1. *Is there a graph with quantum symmetry that has trivial automorphism group?*

We review an idea how one could get candidates for graphs with quantum symmetry that have trivial automorphism group. If one finds a pair of quantum isomorphic, but not isomorphic graphs that both have trivial automorphism group, then their disjoint union would give an example with the desired properties. By Corollary 7.1.6, we know that quantum isomorphic graphs have equivalent coherent algebras. On the search for candidates, one should first make sure that this is true. For example, strongly regular graphs with the same parameters have equivalent coherent algebras. Thus, strongly regular graphs that have the same parameters and trivial automorphism group seem to be good candidates for a pair of graphs we are searching for. There are many such graphs for the parameters $(36, 15, 6, 6)$, but we do not know if there is a quantum isomorphic pair with such parameters.

More generally, one can ask how much information the automorphism group $\mathrm{Aut}(\Gamma)$ contains on the quantum automorphism group $G^+_{aut}(\Gamma)$. For example, we know that if there is a pair of disjoint automorphisms in $\mathrm{Aut}(\Gamma)$, then we know $G^+_{aut}(\Gamma) \neq \mathrm{Aut}(\Gamma)$ by Theorem 3.1.2. The other way around, one may ask whether there are groups G where

$$\mathrm{Aut}(\Gamma) = G \qquad \Longrightarrow \qquad G^+_{aut}(\Gamma) = G.$$

This question can especially be asked for small groups, for example $\{e\}$ (which is equivalent to Question 8.1.1), $\mathbb{Z}_2$ and $\mathbb{Z}_3$. Here, we also do not have any example of graphs that have quantum symmetry and automorphism group $\mathbb{Z}_2$ or $\mathbb{Z}_3$. The smallest group where we know that there are graphs with the same automorphism group and different quantum automorphism group is $\mathbb{Z}_2 \times \mathbb{Z}_2$. For example, the graph in Section 4.4 has automorphism group and quantum automorphism groups $\mathbb{Z}_2 \times \mathbb{Z}_2$, whereas the graph

Γ = 

fulfills $\mathrm{Aut}(\Gamma) = \mathbb{Z}_2 \times \mathbb{Z}_2$ and $G^+_{aut}(\Gamma) = \widehat{\mathbb{Z}_2 * \mathbb{Z}_2}$ (see Table 2.1).

8.2 Unknown quantum automorphism groups of some specific graphs

There are several cases of graphs, where the author does not how to compute the quantum automorphism group and there are no results about them. Some of those graphs already appeared in Table 2. We want to know the quantum automorphism group of the following graphs.

(i) By Theorem 3.1.2, we know that the 4×4 rook's graph does have quantum symmetry. What is the corresponding quantum automorphism group?

(ii) Does the Johnson graph $J(6, 3)$, or more generally the graph $J(n, k)$ for $k \geq 3$, have quantum symmetry? The author did not succeed in finding a similar proof as for $J(n, 2)$.

(iii) Regarding Table 1, it is known that there is only one cubic distance-regular graph that is not distance-transitive. This is the Tutte 12-cage. We do not know whether or not this graph has quantum symmetry.

(iv) In Theorem 5.4.3, we proved that the quantum automorphism group of the folded n-cube graphs is SO_n^{-1} for n odd. What happens for the case where n is even?

(v) On can obtain a pair of quantum isomorphic graphs by the same construction as in Example 7.2.7, but choosing $K_{3,3}$ instead of $K_{3,4}$. Do those graphs have quantum symmetry?

(vi) We know by Theorem 6.3.3 that the Higman-Sims graph has quantum symmetry. What is its quantum automorphism group?

8.3 The quantum automorphism groups of vertex-transitive graphs on 12 vertices

As already mentioned in this thesis, the quantum automorphism groups of all vertex-transitive graphs up to eleven vertices are known from [6] and Theorem 4.1.1. More recently, Chassaniol [22] did the same for all vertex-transitive graphs on 13 vertices. Note that there are way more vertex-transitive graphs on 12 vertices than on 13 vertices. The obvious question to ask is:

Question 8.3.1. *What are the quantum automorphism groups of all vertex-transitive graphs on* 12 *vertices?*

There are 74 vertex-transitive graphs on 12 vertices. Up to complements, we end up with 38 graphs we have to study. Using the strategy discussed in Section 3.3, one should be able to get most, if not all, of the quantum automorphism groups of those graphs. There are easy cases, for example, if the graph is a product of graphs where the quantum automorphism group is known and where Theorem 2.2.2 applies. For some instances, it is more difficult to compute the quantum automorphism group, for example if one has to show that the graph has no quantum symmetry. As a first test, one can use the Sinkhorn type algorithm by Nechita, Weber and the author [41] to see whether one should try to show that some graph does not have quantum symmetry. Also in [41], there is a table including four vertex-transitive graphs on 12 vertices, where the algorithm predicts that they do not have quantum symmetry. Giving an explicit proof for those predictions would be a first step. Computing the quantum automorphism group for all vertex-transitive graphs on 12 vertices could be a lot of work, but it also looks like a doable task to the author.

8.4 Graphs with quantum symmetry and finite quantum automorphism group

For the quantum symmetric group S_n^+, it is known that $C(S_n^+)$ is infinite-dimensional exactly when $S_n \neq S_n^+$, i.e. for $n \geq 4$. For the complete graphs on n points, this means that if those graphs have quantum symmetry, then the C^*-algebra associated to the quantum automorphism group is infinite-dimensional. One might ask whether this is true for arbitrary graphs or whether there is a graph Γ with quantum symmetry, where $C(G_{aut}^+(\Gamma))$ is finite-dimensional. Furthermore, we remark that graphs with infinite-dimensional quantum automorphism group have quantum symmetry, since otherwise $C(G_{aut}^+(\Gamma)) = C(\mathrm{Aut}(\Gamma))$, where $C(\mathrm{Aut}(\Gamma))$ is finite-dimensional. Thus one might ask whether the converse direction is true. We have the following question.

Question 8.4.1. *Is there a graph Γ with quantum symmetry, while $C(G_{aut}^+(\Gamma))$ is finite-dimensional?*

Note that for graphs Γ with a pair of disjoint automorphisms, the C^*-algebra $C(G_{aut}^+(\Gamma))$ is infinite-dimensional. This is the case, because there is a surjective *-homomorphism to $C^*(\mathbb{Z}_n * \mathbb{Z}_m)$ by Theorem 5.4.3. Therefore, the automorphism group of an example we are searching for does not contain any disjoint automorphisms. At the moment, we only know two graphs with quantum symmetry and no disjoint automorphisms, the Higman-Sims graph (see Theorem 6.3.3) and the graph appearing in Example 7.2.7. By looking at the proof of Theorem 7.2.6, the C^*-algebra corresponding to the graph of Example 7.2.7 is infinite-dimensional. We do not know if this is also the case for the Higman-Sims graph.

8.5 Quantum isomorphic but not isomorphic graphs on a small number of vertices

In Example 7.2.7, we used the construction appearing in [36, Section 4] to obtain quantum isomorphic but not isomorphic graphs. The smallest known pair of such graphs is on 24 vertices, see also [36, Section 4]. One can get this pair by the same construction as in Example 7.2.7, but choosing $K_{3,3}$ instead of $K_{3,4}$. This yields the following natural question.

Question 8.5.1. *Is there a pair of quantum isomorphic, not isomorphic graphs on less than* 24 *vertices ?*

The fact that quantum isomorphic graphs have equivalent coherent algebras (Corollary 7.1.6) shows that there are no graph on strictly less than 16 vertices that are quantum isomorphic but not isomorphic. Thus, for getting a new smallest example, one should consider pairs of graphs on 16 to 23 vertices and try to show the desired properties. The following theorem ([40, Corollary 4.15/5.4]) could be helpful for those purposes.

Theorem 8.5.2. *Let Γ be a graph with no quantum symmetry. Then there is a bijective correspondence between the following two sets:*

(i) Isomorphism classes of graphs Γ' such that Γ and Γ' are quantum isomorphic, but not isomorphic,

(ii) Non-trivial subgroups of central type (L, ψ) of $\mathrm{Aut}(\Gamma)$ with coisotropic stabilizer up to some equivalence relation.

Using the theorem our strategy of finding pairs of quantum isomorphic graphs could be the following. We choose a graph on 16 to 23 vertices, where we know that it has no quantum symmetry and then look for specific subgroups of its classical automorphism group. If we find such subgroups, we know that there is a pair of quantum isomorphic graphs and we are done.

Since this thesis contains many new examples of graphs that have no quantum symmetry, it would not only be interesting to find a smallest pair of graph that are quantum isomorphic but not isomorphic, but also search for graphs that are quantum isomorphic to some known distance-regular graphs. For distance-regular graphs, we know from Corollary 7.1.7 that quantum isomorphic graphs are also distance-regular with the same parameters. Especially, if there is a distance-regular graph with unique parameters, then we know that there is no graph quantum isomorphic to this one. Still, there are lots of distance-regular graphs whose parameters are not unique.

Bibliography

[1] Albert Atserias, Laura Mančinska, David E. Roberson, Robert Šámal, Simone Severini, and Antonios Varvitsiotis. Quantum and non-signalling graph isomorphisms. *J. Combin. Theory Ser. B*, 136:289–328, 2019. 2, 3, 9, 123, 124

[2] László Babai and Ludek Kucera. Canonical labelling of graphs in linear average time. In *20th Annual Symposium on Foundations of Computer Science, San Juan, Puerto Rico, 29-31 October 1979*, pages 39–46. IEEE Computer Society, 1979. 37

[3] Teodor Banica. Quantum automorphism groups of homogeneous graphs. *J. Funct. Anal.*, 224(2):243–280, 2005. 1, 2, 21, 22, 27, 28, 117

[4] Teodor Banica. Higher orbitals of quizzy quantum group actions. *Adv. in Appl. Math.*, 109:1–37, 2019. 108

[5] Teodor Banica and Julien Bichon. Free product formulae for quantum permutation groups. *J. Inst. Math. Jussieu*, 6(3):381–414, 2007. 32

[6] Teodor Banica and Julien Bichon. Quantum automorphism groups of vertex-transitive graphs of order ≤ 11. *J. Algebraic Combin.*, 26(1):83–105, 2007. 3, 5, 6, 7, 8, 26, 28, 29, 30, 59, 60, 67, 75, 133

[7] Teodor Banica and Julien Bichon. Quantum groups acting on 4 points. *J. Reine Angew. Math.*, 626:75–114, 2009. 104, 114

[8] Teodor Banica, Julien Bichon, and Gaetan Chenevier. Graphs having no quantum symmetry. *Ann. Inst. Fourier*, pages 955–971, 2007. 3

[9] Teodor Banica, Julien Bichon, and Benoît Collins. The hyperoctahedral quantum group. *J. Ramanujan Math. Soc.*, 22(4):345–384, 2007. 3, 6, 7, 8, 27, 57, 64, 101, 103, 113

[10] Teodor Banica, Julien Bichon, and Benoît Collins. Quantum permutation groups: a survey. In *Noncommutative harmonic analysis with applications to*

probability, volume 78 of *Banach Center Publ.*, pages 13–34. Polish Acad. Sci. Inst. Math., Warsaw, 2007. 7, 9, 101, 109, 115, 117

[11] Teodor Banica and Amaury Freslon. Modeling questions for quantum permutations. *Infin. Dimens. Anal. Quantum Probab. Relat. Top.*, 21(2):1850009, 26, 2018. 35

[12] Julien Bichon. Hopf-Galois systems. *J. Algebra*, 264(2):565–581, 2003. 105, 106

[13] Julien Bichon. Quantum automorphism groups of finite graphs. *Proc. Amer. Math. Soc.*, 131(3):665–673, 2003. 1, 24, 27

[14] Julien Bichon. Free wreath product by the quantum permutation group. *Algebr. Represent. Theory*, 7(4):343–362, 2004. 14

[15] Julien Bichon and Robert Yuncken. Quantum subgroups of the compact quantum group $SU_{-1}(3)$. *Bull. Lond. Math. Soc.*, 46(2):315–328, 2014. 105

[16] N. L. Biggs and D. H. Smith. On trivalent graphs. *Bull. London Math. Soc.*, 3:155–158, 1971. 5, 75

[17] Michael Brannan, Alexandru Chirvasitu, Kari Eifler, Samuel Harris, Vern Paulsen, Xiaoyu Su, and Mateusz Wasilewski. Bigalois extensions and the graph isomorphism game. *arXiv:1812.11474*, 2018. 127

[18] A. E. Brouwer, A. M. Cohen, and A. Neumaier. *Distance-regular graphs*, volume 18 of *Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)]*. Springer-Verlag, Berlin, 1989. 15, 19, 60, 61, 64, 68, 71, 106

[19] Andries E. Brouwer and Willem H. Haemers. *Spectra of graphs*. Universitext. Springer, New York, 2012. 64

[20] Arthur Chassaniol. Contributions à l'étude des groupes quantiques de permutations (contributions to the study of quantum permutation groups). *PhD Thesis, Clermont-Ferrand*, 2016. 3

[21] Arthur Chassaniol. Quantum automorphism group of the lexicographic product of finite regular graphs. *J. Algebra*, pages 23–45, 2016. 3, 23

[22] Arthur Chassaniol. Study of quantum symmetries for vertex-transitive graphs using intertwiner spaces. *arXiv:1904.00455*, 2019. 3, 8, 33, 34, 60, 73, 75, 133

[23] H. S. M. Coxeter. Self-dual configurations and regular graphs. *Bull. Amer. Math. Soc.*, 56:413–455, 1950. 16, 83

[24] Brian Curtin. Some planar algebras related to graphs. *Pacific J. Math.*, 209(2):231–248, 2003. 2, 115, 116, 118, 121

[25] Christian Eder, Viktor Levandovskyy, Julien Schanz, Simon Schmidt, Andreas Steenpass, and Moritz Weber. Existence of quantum symmetries for graphs on up to seven vertices: a computer based approach. *arXiv:1906.12097*, 2019. iii, v, 46, 98

[26] P. Erdős and A. Rényi. Asymmetric graphs. *Acta Math. Acad. Sci. Hungar.*, 14:295–315, 1963. 3, 37, 38, 73, 131

[27] Melanie Fulton. The quantum automorphism group and undirected trees. *PhD Thesis, Virginia*, 2006. 3, 35, 38, 98, 100

[28] Martin Fürer. On the combinatorial power of the Weisfeiler-Lehman algorithm. In *Algorithms and complexity*, volume 10236 of *Lecture Notes in Comput. Sci.*, pages 260–271. Springer, Cham, 2017. 36

[29] Pierre Guillot, Christian Kassel, and Akira Masuoka. Twisting algebras using non-commutative torsors: explicit computations. *Math. Z.*, 271(3-4):789–818, 2012. 106

[30] D. G. Higman. Coherent configurations. I. *Rend. Sem. Mat. Univ. Padova*, 44:1–25, 1970. 35

[31] Alan J Hoffman and Robert R Singleton. On Moore graphs with diameters 2 and 3. *IBM Journal of Research and Development*, 4(5):497–504, 1960. 71

[32] D. A. Holton and J. Sheehan. *The Petersen graph*, volume 7 of *Australian Mathematical Society Lecture Series*. Cambridge University Press, Cambridge, 1993. 15, 20

[33] François Jaeger. Strongly regular graphs and spin models for the Kauffman polynomial. *Geom. Dedicata*, 44(1):23–52, 1992. 121

[34] V.F.R. Jones. Planar algebras. *arXiv:math/9909027*, 1999. 2, 115, 120, 121

[35] Luca Junk, Simon Schmidt, and Moritz Weber. Almost all trees have quantum symmetry. *To appear in Archiv der Mathematik, arXiv:1911.02952*, 2019. iii, v, 37, 38

[36] Martino Lupini, Laura Mančinska, and David Roberson. Nonlocal Games and Quantum Permutation Groups. *arXiv:1712.01820*, 2017. 3, 9, 14, 35, 36, 37, 38, 39, 123, 125, 126, 128, 129, 131, 134

[37] Ann Maes and Alfons Van Daele. Notes on compact quantum groups. *Nieuw Arch. Wisk. (4)*, 16(1-2):73–112, 1998. 126, 127

[38] Laura Mančinska and David E. Roberson. Quantum isomorphism is equivalent to equality of homomorphism counts from planar graphs. *arXiv:1910.06958*, 2019. 3, 33, 34, 126

[39] Laura Mančinska, David E. Roberson, and Antonios Varvitsiotis. Semidefinite relaxations of quantum isomorphism. *In progress*, 2017. 126

[40] Benjamin Musto, David Reutter, and Dominic Verdon. The Morita theory of quantum graph isomorphisms. *Comm. Math. Phys.*, 365(2):797–845, 2019. 135

[41] Ion Nechita, Simon Schmidt, and Moritz Weber. Sinkhorn algorithm for quantum permutation groups. *arXiv:1911.04912*, 2019. iii, v, 133

[42] Sergey Neshveyev and Lars Tuset. *Compact quantum groups and their representation categories*, volume 20 of *Cours Spécialisés [Specialized Courses]*. Société Mathématique de France, Paris, 2013. 11, 126

[43] Piotr Podleś. Symmetries of quantum spaces. Subgroups and quotient spaces of quantum SU(2) and SO(3) groups. *Comm. Math. Phys.*, 170(1):1–20, 1995. 12

[44] Iain Raeburn. *Graph algebras*, volume 103 of *CBMS Regional Conference Series in Mathematics*. Published for the Conference Board of the Mathematical Sciences, Washington, DC; by the American Mathematical Society, Providence, RI, 2005. 13

[45] Yunxiang Ren. A generating problem for subfactors. *arXiv:1910.09723*, 2019. 2, 9, 115, 117, 120

[46] Yunxiang Ren. Universal skein theory for group actions. *Advances in Mathematics*, 356:106804, 2019. 2

[47] Peter Schauenburg. Hopf bi-Galois extensions. *Comm. Algebra*, 24(12):3797–3825, 1996. 105

[48] Simon Schmidt. The Petersen graph has no quantum symmetry. *Bull. Lond. Math. Soc.*, 50(3):395–400, 2018. iii, v, 46, 48, 59, 60, 76

[49] Simon Schmidt. Quantum automorphisms of folded cube graphs. *To appear in Annales de l'Institut Fourier, arXiv:1810.11284*, 2018. iii, v, 43, 101

[50] Simon Schmidt. On the quantum symmetry of distance-transitive graphs. *Advances in Mathematics*, 368:107150, 2020. iii, v, 43, 46, 59, 60, 75, 92

[51] Simon Schmidt, Chase Vogeli, and Moritz Weber. Uniformly vertex-transitive graphs. *arXiv:1912.00060*, 2019. iii, v, 17

[52] Simon Schmidt and Moritz Weber. Quantum symmetries of graph C*-algebras. *Canadian Mathematical Bulletin*, 61(4):848–864, 2018. iii, v, 27, 98, 112

[53] The Sage Developers. *SageMath, the Sage Mathematics Software System (Version 8.9)*, 2019. `https://www.sagemath.org`. 9, 121, 129

[54] Thomas Timmermann. *An invitation to quantum groups and duality.* EMS Textbooks in Mathematics. European Mathematical Society (EMS), Zürich, 2008. 11, 25

[55] Alfons Van Daele. Dual pairs of Hopf ∗-algebras. *Bull. London Math. Soc.*, 25(3):209–230, 1993. 104

[56] Shuzhou Wang. Free products of compact quantum groups. *Comm. Math. Phys.*, 167(3):671–692, 1995. 12

[57] Shuzhou Wang. Tensor products and crossed products of compact quantum groups. *Proc. London Math. Soc. (3)*, 71(3):695–720, 1995. 12

[58] Shuzhou Wang. Quantum symmetry groups of finite spaces. *Comm. Math. Phys.*, 195(1):195–211, 1998. 1, 12, 13

[59] Moritz Weber. Introduction to compact (matrix) quantum groups and Banica-Speicher (easy) quantum groups. *Proc. Indian Acad. Sci. Math. Sci.*, 127(5):881–933, 2017. 13

[60] S. L. Woronowicz. Compact matrix pseudogroups. *Comm. Math. Phys.*, 111(4):613–665, 1987. 1, 11, 12

[61] S. L. Woronowicz. Tannaka-Krein duality for compact matrix pseudogroups. Twisted SU(N) groups. *Inventiones mathematicae*, 93(1):35–76, 1988. 33

[62] S. L. Woronowicz. A remark on compact matrix quantum groups. *Lett. Math. Phys.*, 21(1):35–39, 1991. 11