The SYK model and matrices with q-Gaussian entries

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Abstract

The subject of this thesis are q-Gaussian variables and its relations with two kind of matrix models: the Sachdev-Ye-Kitaev (SYK) model and Wigner matrices with q-Gaussian entries. The results are divided into two sections.

In the first part, we extend the results of Feng-Tian-Wei [12, 13] to the multivariate setting. As a consequence, we show that the dynamical version of the SYK model approximates the q-Brownian motion. We also characterize the asymptotic limit of fluctuations for the SYK model.

In the second part, we characterize the fluctuations for Wigner matrices with q-Gaussian entries, and present a generalization of fluctuations for the case of block matrices. This is connected with the recent work of Belinschi-Diaz-Mingo [1]. Wigner matrices with q-Gaussian entries and the SYK model, under some parameter assumptions, approximate semicircular variables. We compare the results for fluctuations obtained in this thesis with the fluctuations of the Gaussian Unitary Ensemble, which is the canonical matrix model that approximate semicircular variables.

Zusammenfassung

Das Thema dieser Doktorarbeit sind q-Gaußsche Variablen und ihre Beziehung zu zwei Zufallsmatrixenmodellen: dem Sachdev-Ye-Kitaev-Modell (SYK) und Wignermatrizen mit q-Gaußschen Einträgen. Die Arbeit ist dementsprechend in zwei Teile unterteilt.

Im ersten Teil erweitern wir die Resultate von Feng-Tian-Wei [12, 13] auf den multivariaten Fall. Insbesondere erhalten wir daraus, dass die dynamische Version des SYK-Modells die q-Brownsche Bewegung approximiert. Wir charakterisieren weiterhin den asymptotischen Limes von Fluktuationen des SYK-Modells.

Im zweiten Teil charakterisieren wir die Fluktuationen von Wignermatrizen mit q-Gaußschen Einträgen, sowie eine Verallgemeinerung auf den Fall von Blockmatrizen; letzteres stellt eine Beziehungen zu jüngeren Arbeiten von Belinschi-Diaz-Mingo [1] dar. Wignermatrizen mit q-Gaußschen Einträgen sowie das SYK-Modell für gewisse Parameter approximieren Halbkreiselemente. Wir vergleichen unsere Resultate über Fluktuationen mit den Fluktuationen des GUE Ensembles, welches das kanonische Matrizenmodell zur Approximation von Halbkreiselementen ist.

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Contents

Chapter 1

Introduction

The main topic of this thesis are q-Gaussian variables and their relations with the asymptotic distribution of two kinds of matrix models: the Sachdev-Ye-Kitaev (SYK) model and Wigner matrices with q-Gaussian entries. We showed that a dynamical version of the SYK model converges towards the q-Brownian motion. Also, we studied the convergence of fluctuation moments and higher cumulants of these models. For Wigner matrices with q-Gaussian entries we also extended these results to block matrices. This part is connected with the work of Belinschi-Diaz-Mingo [1] on the asymptotic fluctuations for block matrices with Gaussian entries.

The q-Gaussian distribution was introduced in [6, 8] by Bozejko, Kümmerer and Speicher as the finite dimensional distribution of a generalized Brownian motion (GBM). A GBM $\mathcal{G} = (G(f))_{f\in\mathcal{H}}$ is a family of self-adjoint operators $G(f)$ where $f \in \mathcal{H}$ for some real Hilbert space $\mathcal{H}$, together with a state φ_t in the algebra generated by the $G(f)$ and is given by

$$\varphi_t\left(G(f_1)\cdots G(f_n)\right) = \begin{cases} 0 & \text{if } n \text{ is odd} \\ \displaystyle\sum_{\pi\in\mathcal{P}_2(n)} t(\pi) \prod_{(l,r)\in\pi} \langle f_l, f_r\rangle & \text{if } n \text{ even,} \end{cases} \tag{1.1}$$

where $\mathcal{P}_2(n)$ denotes the set of pair partitions and $t(\pi)$ is a weight function. The mapping $G(f)$ is the non-commutative analog of the function

$$W(f) = \int_{\mathbb{R}^+} f(t)dB_t, \qquad \text{for} \quad f \in L^2(\mathbb{R}^+),$$

where $(B_t)_{t\geq 0}$ is the classical Brownian motion. The random variables $W(f)$ are also known as "generalized increments". They also turn out to have Gaussian distribution, so their mixed moments are given by the Wick formula

$$\mathbb{E}\left[W(f_1)\cdots W(f_{2n})\right] = \sum_{\pi\in\mathcal{P}_2(2n)} \prod_{(l,r)\in\pi} \langle f_l, f_r\rangle.$$

The Wick formula characterizes Gaussian distribution. So, Equation (1.1) can be interpreted as: the increments $G(f)$ of the GBM are Gaussian with respect to φ_t. The concrete structure of a GBM depends on the weight function $t(\pi)$. However, we also require the linear functional φ_t to be positive definite. It is still an open problem to characterize the weight functions $t(\pi)$ that give rise to positive linear functionals φ_t. The general theory for this kind of functions was started in [7].

The q-Brownian motion $\mathcal{B}_q = (B_q(f))_{f\in L^2(\mathbb{R}^+)}$ is defined as a GBM with weight function

$$t_q(\pi) = q^{cr(\pi)},$$

where $cr(\pi)$ is the number of crossings of π. The positivity of the linear functional $\varphi_q := \varphi_{t_q}$ was proven in [6] for q in the interval $[-1, 1]$. The increments $B_q(f)$ of this process are called q-Gaussian variables. The construction of the q-Brownian motion is presented in Section 2.7. Other examples of GBM can be found in [7, 5].

One can see from Equation (1.1) that for $q = 1$ the q-Brownian motion reduces to the classical Brownian motion. Another important subcase is the value $q = 0$. In this case the process $\mathcal{B}_0$ is the free Brownian motion and its increments form a semicircular family in the sense of free probability. Concretely this means

$$\varphi_0\left(B_0(f_1)\cdots B_0(f_{2n})\right) = \sum_{\pi\in NC_2(2n)} \prod_{(l,r)\in\pi} \langle f_l, f_r\rangle,$$

where $NC_2(2n)$ is the set of non-crossing pair partitions. In both cases $q = 0, 1$ the orthogonality of $f_1, \ldots, f_n$ translate into in classical and free independence, respectively.

The distribution of one q-Gaussian variable $x = B_q(f)$ is well understood. For simplicity let us assume $||f|| = 1$. From the spectral theory for self-adjoint operators we know that there exists a probability measure μ_q on $\mathbb{R}$ such that for $n \geq 0$

$$\varphi_q(x^n) = \int_{\mathbb{R}} t^n d\mu_q(t).$$

The existence of such a probability measure relies on the positivity of the linear functional φ_q. Actually, this is why we need the linear functional to be positive. Many things are known for the measure μ_q. For example μ_q has a continuous density for $-1 < q \leq 1$ supported on $\left[\frac{-2}{\sqrt{1-q}}, \frac{2}{\sqrt{1-q}}\right]$ and is given by

$$f_q(t) = \frac{\sqrt{1-q}}{2\pi}\sqrt{4-(1-q)t^2}\sum_{k=1}^{\infty}(-1)^{k-1}q^{\frac{k(k-1)}{2}}\mathcal{U}_{2k-2}\left(\frac{x\sqrt{1-q}}{2}\right),$$

where $\mathcal{U}_k$ is the k^{th} Chebyschev polynomial of the second kind. In particular for the cases $q = -1, 0, 1$ the above formula reduces to

$$\mu_{-1} = \frac{1}{2}(\delta_{-1} + \delta_1), \qquad \mu_0(t) = \frac{\sqrt{4-t^2}}{2\pi}dt, \qquad \mu_1(t) = \frac{e^{-\frac{t^2}{2}}}{\sqrt{2\pi}}dt,$$

that is symmetric Bernoulli, Semicircular and Gaussian distributions. A more convenient description of μ_q is given by its Cauchy transform

$$G_q(z) = \int_{\mathbb{R}} \frac{1}{z-t}d\mu_q(t) = \cfrac{1}{z - \cfrac{1}{z - \cfrac{1+q}{z - \cfrac{1+q+q^2}{z - \dots}}}} \qquad \text{for} \quad q \in [-1, 1].$$

However, the analytic description of several q-Gaussians remains to be an open problem. See for example [29, Section 6]. The only joint description is given by Equation (1.1).

One of the most remarkable properties of the Gaussian distribution and of the Brownian motion is that it appears as the limit object in the Central Limit Theorem (CLT). It is natural to ask whether the q-Gaussian distribution and the q-Brownian motion also appear as a CLT.

Before discussing the question, let us have a look at the classical theory. In classical probability, a CLT is a statement about the convergence of the distribution of $\frac{X_1+\cdots+X_n}{\sqrt{n}}$ to the distribution of a random variable X, where the $(X_n)_{n\geq 1}$ is a family of independent random variables. In general, the distribution of X is either trivial or an infinite divisible distribution. Since we are interested in Gaussian variables, we assume the random variables $(X_n)_{n\geq 1}$ to be characterized by their moments, then X has Gaussian distribution.

The crucial requirement for a non-commutative version of the CLT is to replace the independence of the variables, by some other kind of property. This can be tensor, boolean, free independence, or other kind of rule for computing mixed-moments, for instance, the factorization of naturally ordered products [37] or the factorization of pyramidally ordered products [23, 6]. Independence can be also replaced by exchangeability plus the singleton factorization property [22].

A non-commutative CLT for the q-Brownian motion was provided by Speicher in [37]. The statement is a non-commutative version of the passage from random walks to Brownian motion. He assumes some technical assumptions on a sequence of selfadjoint variables $(X_n)_{n\geq 1}$ and showed that

$$\frac{X_1 + \cdots + X_n}{\sqrt{n}} \xrightarrow{n\to\infty} x,$$

where x is a q-Gaussian variable. The property that we want to highlight here is that the variables $(X_n)_{n\geq 1}$ commute or anti-commute. Thus, the q-Gaussian distribution arise as a CLT for a mixture of commuting and anti-commuting variables. This observation is important for us because the SYK model can be also seen as a sum of variables that commute or anti-commute. See Lemma 1. Then up to some point it is natural to expect that the limit distribution of the SYK model is the q-Gaussian distribution.

The theory of free probability was introduced by Voiculescu in a series of papers [38, 39, 40]. It can be described as non-commutative probability plus a notion of independence called freeness. The characteristic that makes free probability special among other approaches to non-commutative probability is the connection with random matrices. Voiculescu showed [41] that classical independent Wigner matrices converge in distribution to free semicircular variables. This result is known as asymptotic freeness. Thus, random matrix theory appears as a bridge between classical and free probability. For the purpose of this thesis, we will consider Wigner matrices with Gaussian entries. This kind of random matrix model is commonly known as Gaussian Unitary Ensemble (GUE). In particular, the result of asymptotic freeness says that independent GUE approximate free semicirculars.

Now the natural question is: is there a random matrix model that approximates q-Gaussian variables? Śniady provides an affirmative answer to this question in [35] by constructing a sparse random matrix model that approximates orthonormal q-Gaussian variables. An alternative exposition to this matrix model can be found in [9].

Another matrix model that approximates q-Gaussians for small q are the so called Gibbs measures. They were introduced by Guionnet-Shlyakhtenko [17] in the context of free transport theory. To the knowledge of the author, this is the only non-sparse random matrix model that approximates q-Gaussian variables.

There is also a random matrix model for q-circular systems. The model was introduced by Mingo-Nica [25]. The q-circular system is a "complex version" of q-Gaussian variables. An interesting feature of the model is that it can be used to approximate z-circular systems, where the parameter z is a complex number with $|z| < 1$.

Another matrix model that approximates q-Gaussian variables is the Sachdev-Ye-Kitaev (SYK) model. The model was proposed in 1993 by Sachdev-Ye [33] as a model of quantum random spin system. Later on, in 2015, it was promoted by Kitaev [20] as a model for quantum holography. It is interesting to us that the origins of the SYK model are completely independent from the development of GBM and q-deformations.

The SYK model has two parameters n and q_n, where n is a positive even integer and $1 \leq q_n \leq n$. The model is defied by

$$H_{n,q_n} := \frac{\sqrt{-1}^{\lfloor q_n/2 \rfloor}}{\binom{n}{q_n}^{1/2}} \sum_{1 \leq i_1 < \cdots < i_{q_n} \leq n} J_{i_1,\ldots,i_{q_n}} \psi_{i_1} \cdots \psi_{i_{q_n}}, \tag{1.2}$$

where the coefficients $J_{i_1,\ldots,i_{q_n}}$ are random variables and the $\psi_1, \ldots, \psi_n$ are anti-commuting variables known as Majorana fermions. See Definition 3.2 for precise details. A similar model related to density states in quantum spin glasses was also considered by Erdös-Schröder [11].

It seems to be of great interest in the analysis of the model when $q_n = 4$ for all $n \geq 1$. For instance, the SYK model proposed by Kitaev in [20] had $q_n = 4$. Other authors [14, 15, 10] consider the double scaled limit. This means $n, q_n \to \infty$ and assume the existence of the limit

$$\frac{q_n^2}{n} \longrightarrow \lambda \in [0, \infty]. \tag{1.3}$$

The authors of [14, 15, 10] observe that under the hypothesis (1.3) we have

$$\mathbb{E}\left[\frac{1}{2^{n/2}} \operatorname{Tr}(H_{n,q_n}^{2k})\right] \xrightarrow{n,q_n \to \infty} \sum_{\pi \in \mathcal{P}_2(2k)} e^{-2\lambda cr(\pi)}. \tag{1.4}$$

Based on the work of Erdös-Schröder [11], Feng-Tian-Wei [12] presented a rigorous version of (1.4). Observe that on the right side of (1.4) are the moments of the q-Gaussian variable for $q = e^{-2\lambda}$. One can also choose the sign of q by fixing the parity of the sequence q_n. We extend this result to the multivariate situation.

The main result of this thesis is.

Theorem 1. *The joint distribution of independent copies of the SYK model $(H_1, \cdots, H_p)$ converges in distribution to the joint distribution of orthonormal q-Gaussian variables $(x_1, \cdots, x_p)$ under the hypothesis of double scaled limit (1.3).*

As a consequence we obtain

Corollary 1. *Consider the dynamical version of the SYK model*

$$H_{n,q_n}(t) := \frac{\sqrt{-1}^{\lfloor q_n/2 \rfloor}}{\binom{n}{q_n}^{1/2}} \sum_{1 \leq i_1 < \cdots < i_{q_n} \leq n} B_{i_1,\ldots,i_{q_n}}(t) \psi_{i_1} \cdots \psi_{i_{q_n}}, \qquad \textit{for} \quad t \geq 0,$$

where $B_{i_1,\dots,i_{q_n}}(t)$ are independent classical Brownian motions. The processes $(H_{n,q_n}(t))_{t\geq 0}$ converge to the q-Brownian motion.

We also investigate the asymptotics of classical cumulants for the SYK model. These quantities also appear in the physics literature [16, 2, 3] under the name of n-point correlator functions. In particular, Berkooz and collaborators computed the 2-point and 4-point correlator functions [2, 3]. The second cumulant had also been considered in a second paper by Feng-Tian-Wei [13]. We extended their result and characterized the asymptotic limit of classical cumulants $(c_m)_{m\geq 1}$ of all orders.

Theorem 2. *Assume the random coefficients $J_{i_1,\dots,i_{q_n}}$ of the SYK model (3.2) to be independent copies of a centered Gaussian random variables with variance one. Then*

$$\binom{n}{q_n}^{m-1} c_m\left(\mathrm{tr}(H^{k_1},\dots,\mathrm{tr}(H^{k_m})\right) \xrightarrow{n\to\infty} \sum_{\substack{\pi'\in\mathcal{P}_2(k)\\ \pi'\leq\theta}} \sum_{\substack{\pi\in\mathcal{P}_2(k)\\ \pi\vee\theta=1_k\\ |\pi\vee\pi'|=\frac{k}{2}-m+1}} q^{cr(\pi')}.$$

Theorem 2 is a direct consequence of Theorem 9, where we consider the most general case of cumulants of traces of products of independent copies of the SYK model. See section 3.3 for more details.

Our result about the convergence of independent SYK matrices to orthonormal q-Gaussian variables resembles the well known result of Voiculescu of asymptotic freeness. Actually, for $q = 0$ we obtained the asymptotic freeness of independent SYK matrices. The motivation for our study of fluctuation moments and higher order cumulants of the SYK model was to compare the former with the canonical fluctuations for semicircular variables, i.e., fluctuations coming from independent copies of the Gaussian Unitary Ensemble (GUE). In spite of the fact that under some parameter assumptions, namely

$$\frac{q_n^2}{n} \xrightarrow{n,q_n\to\infty} \infty, \tag{1.5}$$

the SYK model has the same asymptotic distribution as independent GUE matrices, they have different fluctuation moments. The fluctuations for the SYK model under assumption (1.5), are described by two pair partitions one of them is non-crossing but the another may have crossigns. See the Figure 1.1.

We also analyze the fluctuations of another matrix model that approximates semicircular variables and that is connected to q-Gaussian variables, namely, Wigner matrices with orthogonal q-Gaussian entries. See Section 4.1 for detais of the matrix model. It was shown by Voiculescu in the beginnings of free

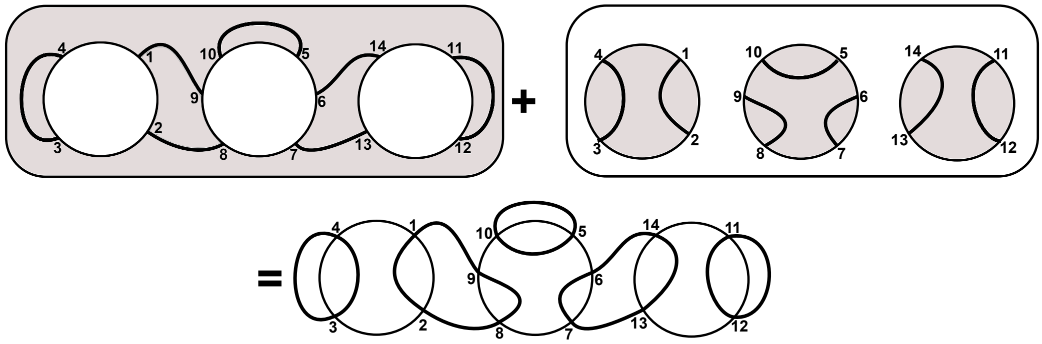

Figure 1.1: An example of the kind of partitions that appears in the description of $c_3\,(\mathrm{tr}(H^4, \mathrm{tr}(H^6), \mathrm{tr}(H^4))$. On top there are two pair-partitions, the external pair partition π on the left and the internal pair-partition π' on the right. The diagram below represents $\pi \vee \pi'$. Observe that $|\pi \vee \pi'| \leq \frac{14}{2} - 3 + 1 = 5$.

probability that Wigner matrices with free semicircular entries are themselves semicirculars. Later on, it was observed by Shlyakhtenko that the same result is true, asymptotically, if we replace free semicircular by orthogonal q-Gaussian variables. In this thesis we study the fluctuations for Wigner matrices with q-Gaussian entries.

The kind of fluctuations that show up in the case of Wigner matrices with q-Gaussian entries are similar to the ones of GUE matrices. Both fluctuation moments are described in terms of annular pairings but in the former case, we also have to consider the information of the number of crossings of each annular pairing. See Figure 1.2.

For the sake of simplicity, we present here the result we obtain for fluctuations of Wigner matrices with q-Gaussian entries in the case of one matrix. The multivariate version is fully developed in Section 4.2.

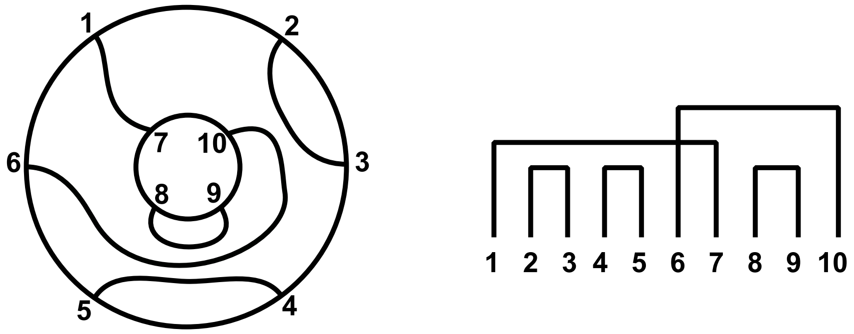

Figure 1.2: Annular non-crossing pair partitions can have crossings when are drawn in a linear way.

Theorem 3. *Let $k_1, \dots, k_m$ be positive integers and X_N a Wigner matrix with q-Gaussian entries. The asymptotics for cumulants in powers of X_N are given by*

$$N^{2-m} c_m \left(\mathrm{Tr}(X_N^{k_1}), \dots, \mathrm{Tr}(X_N^{k_m}) \right) \xrightarrow{N \to \infty} \sum_{\pi \in NC_2(k_1,\dots,k_m)} q^{cr(\pi)}$$

where $cr(\pi)$ stands for the number of crossings of π when is consider as an ordinary pair partition.

The above statement can be also lifted to the case of block matrices with correlated Wigner blocks. This can be done by choosing carefully the notation. See Section 4.3 for more details regarding to this extension. In particular, for the case $q = 1$ we recovered some of the results of Belinschi-Diaz-Mingo [1] about the asymptotics for the second cumulant of Block Gaussian matrices. See Remark 7 for more details about the connection with the work of Belinschi-Diaz-Mingo [1].

The thesis is organized as follows. In the following section we present the necessary background for the thesis. In Chapter 3 we present the construction of the SYK model and analyze the asymptotic distribution and fluctuations for the model. The Chapter 4 is devoted to the study of Wigner matrices with q-Gaussian entries. The chapter ends with a final comment about the connections with the work of Belinschi-Diaz-Mingo [1].

Chapter 2

Preliminaries

In this chapter we give the basic definitions of non-commutative probability, and use them as a common framework to present some results of random matrix theory, Free Probability and its extensions.

2.1 Non-commutative probability

Let us review the basic definitions of classical probability theory.

Definition 1. *A σ-algebra $\mathcal{F}$ on a non-empty set Ω is a collection of subsets of Ω such that*

(1) $\Omega \in \mathcal{F}$.

(2) If $A \in \mathcal{F}$, then $A^c \in \mathcal{F}$.

(3) If $A_n \in \mathcal{F}$ for $n \geq 1$, then $\bigcup_{n=1}^{\infty} A_n \in \mathcal{F}$.

The pair $(\Omega, \mathcal{F})$ is also known as a measure space.

Elements in a σ-algebra are commonly known as events. If the set Ω is equipped with a topology τ, then we define the Borel σ-algebra on Ω as the smallest σ-algebra that contains τ. We denote it by $\mathcal{B}(\Omega)$. In particular, the Borel σ-algebra on $\mathbb{R}$ with respect to the usual topology is denoted by $\mathcal{B}(\mathbb{R})$.

Definition 2. *A function $f : \Omega_1 \to \Omega_2$ between measure spaces $(\Omega_1, \mathcal{F}_1)$ and $(\Omega_2, \mathcal{F}_2)$ is said to be measurable if for every $A \in \mathcal{F}_2$ we have $X^{-1}(A) \in \mathcal{F}_1$.*

A σ-algebra is also the natural domain of definition for a probability measure.

Definition 3. *A probability measure* $\mathbb{P}$ *on a measure space* $(\Omega, \mathcal{F})$ *is a function* $\mathbb{P} : \mathcal{F} \to [0,1]$ *such that*

(1) $\mathbb{P}(\Omega) = 1$.

(2) Let $A_n \in \mathcal{F}$ *be such that* $A_n \cap \mathcal{A}_m = \emptyset$ *for* $n \neq m$*, then*

$$\mathbb{P}\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \mathbb{P}(A_n).$$

Property (2) *is known as* σ*-additivity.*

Definition 4. *A classical probability space is a triplet* $(\Omega, \mathcal{F}, \mathbb{P})$*, where* Ω *is a non-empty set,* $\mathcal{F}$ *is a* σ*-algebra of subsets of* Ω *and* $\mathbb{P}$ *is a probability measure on* $(\Omega, \mathcal{F})$*.*

The non-commutative version of a classical probability space is obtained by looking at a special class of functions.

Definition 5. *Let* $(\Omega, \mathcal{F}, \mathbb{P})$ *be a probability space.*

(1) A function $X : \Omega \to \mathbb{R}$ *is said to be a random variable if for every* $A \in \mathcal{B}(\mathbb{R})$ *we have that* $X^{-1}(A) \in \mathcal{F}$*.*

(2) The expected value of a random variable X *is defined as the Lebesgue integral*

$$\mathbb{E}[X] := \int_{\Omega} X(\omega) d\mathbb{P}(\omega) \tag{2.1}$$

whenever it exists.

(3) For $n \geq 1$ *the* n*-th moment of a random variable* X *is defined by*

$$\mathbb{E}\left[X^n\right] = \int_{\Omega} X^n(\omega) d\mathbb{P}(\omega)$$

whenever it exists.

Conceptually, there is no difference between measurable functions and random variables. However, it is common to use the name "random variable" when we want to emphasize that we are in a probability space.
The expected value defined in (2.1) can be defined as a linear functional on a large class of random variables.

Notation 1. *(1) Let* $L^p(\Omega, \mathcal{F}, \mathbb{P})$ *be the set of random variables in* $(\Omega, \mathcal{F}, \mathbb{P})$ *such that* $E[|X|^n] < \infty$ *for* $n = 1, \ldots, p$*.*

(2) Denoted by

$$L^{-\infty}(\Omega, \mathcal{F}, \mathbb{P}) := \bigcap_{1 \le p < \infty} L^p(\Omega, \mathcal{F}, \mathbb{P}),$$

the set of random variables with moments of all orders.

Then the expected value defined in (5(3)) can be seen as a linear functional $\mathbb{E} : L^{-\infty}(\Omega, \mathcal{F}, \mathbb{P}) \to \mathbb{R}$. The pair $(L^{-\infty}(\Omega, \mathcal{F}, \mathbb{P}), \mathbb{E})$ is the canonical example of a non-commutative probability space.

Definition 6. *A non-commutative probability space (ncps) is a tuple $(\mathcal{A}, \varphi)$, where $\mathcal{A}$ is a unital algebra and $\varphi : \mathcal{A} \to \mathbb{C}$ is linear functional such that $\varphi(1) = 1$.*

A very important class of random variables are the so called characteristic functions.

Definition 7. *Let $(\Omega, \mathcal{F})$ be a measure space. For every element $A \in \mathcal{F}$ we define the characteristic function of A by*

$$\mathbb{1}_A(\omega) = \begin{cases} 1 & \text{if} \quad \omega \in \mathcal{A} \\ 0 & \text{if} \quad \omega \notin \mathcal{A}. \end{cases}$$

In some sense, characteristic functions encode the information of $\mathcal{F}$. We can also encode the probability measure, by taking expected values of this class of functions. Namely, for every $A \in \mathcal{F}$

$$\mathbb{P}(A) = \mathbb{E}[\mathbb{1}_A].$$

This leads us to the definition of distribution for a random variable.

Definition 8. *Let X be a random variable on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, then the distribution μ_X of X is a probability measure on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ given by*

$$\mu_X(A) := \mathbb{P}(X^{-1}(A)) = \mathbb{E}[\mathbb{1}_{X^{-1}(A)}], \tag{2.2}$$

where A is an element in the Borel σ-algebra $\mathcal{B}(\mathbb{R})$.

A closer look at (2.2) shows that

$$\int_{\mathbb{R}} \mathbb{1}_A(t) d\mu_X(t) = \int_{\Omega} \mathbb{1}_{X^{-1}(A)}(\omega) d\mathbb{P}(\omega), \tag{2.3}$$

where A is a Borel set in $\mathbb{R}$. Observe that by canonical methods (2.3) can be extended to

$$\int_{\mathbb{R}} f(t) d\mu_X(t) = \int_{\Omega} Y(\omega) d\mathbb{P}(\omega), \tag{2.4}$$

where $Y = f(X)$, for some Borel measurable function $f : \mathbb{R} \to \mathbb{R}$ such that the integrals exist. Equation (2.4) is also known as the Image measure Theorem [21, Theorem 4.10] and can be extended to $L^{-\infty}(\mathbb{R}, \mathcal{B}(\mathbb{R}), \mu_X)$. In this way our definition of distribution extends to

Definition 9. *The (extended) distribution of a random variable $X : \Omega \to \mathbb{R}$ on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is the linear functional*

$$\overline{\mu}_X : L^{-\infty}(\mathbb{R}, \mathcal{B}(\mathbb{R}), \mu_X) \to \mathbb{R}$$

defined by

$$\overline{\mu}_X(f) := \mathbb{E}[f(X)] = \int_{\mathbb{R}} f(t) d\mu_X(t).$$

Remark 1. *The pair $(L^{-\infty}(\mathbb{R}, \mathcal{B}(\mathbb{R}), \mu_X), \overline{\mu}_X)$ fits in the definition of ncps. The linear functional $\overline{\mu}_X$ is positive, namely, for any $f \in L^{-\infty}(\mathbb{R}, \mathcal{B}(\mathbb{R}), \mu_X)$ with $f \geq 0$ we have $\overline{\mu}_X(f) \geq 0$.*

There are examples of ncps that do not come from classical probability.

Example 1. *Let d be a positive integer and let $M_d(\mathbb{C})$ be the set of square matrices. For a matrix $A = (a_{i,j})_{i,j=1}^d$ denoted by*

$$\mathrm{Tr}(A) = a_{1,1} + \cdots + a_{d,d} \qquad \text{and} \qquad \mathrm{tr}(A) = \frac{\mathrm{Tr}(A)}{d}.$$

Then the pair $(\mathbb{M}_d(\mathbb{C}), \mathrm{tr})$ is a ncps.

In the previous example we can observe some additional structure. The first is positivity. For any matrix $A = (a_{i,j})_{i,j}^d$ define the canonical involution $A^* = (\overline{a_{j,i}})_{i,j}^d$. Then observe that $\mathrm{tr}(AA^*) = \sum_{i,j=1}^d |a_{i,j}|^2 \geq 0$. Second, the trace has the so called trace property, namely, $\mathrm{tr}(AB) = \mathrm{tr}(BA)$. This property is also presented in Remark 1. However, it is hidden in the commutativity of $\mathbb{R}$. This motivates the following definition.

Definition 10. *(1) A non-commutative distribution is a linear functional $\mu : \mathbb{C}\langle X_1, \cdots, X_n\rangle \to \mathbb{C}$ such that*

(i) $\mu(1) = 1$.

(ii) $\mu(PP^*) \geq 0$ *for any* $P \in \mathbb{C}\langle X_1, \cdots, X_n\rangle$.

(iii) $\mu(PQ) = \mu(QP)$ *for any* $P, Q \in \mathbb{C}\langle X_1, \cdots, X_n\rangle$.

(2) Let $(\mathcal{A}, \varphi)$ be a ncps and $a_1, \ldots, a_n \in \mathcal{A}$. The joint distribution of $(a_1, \ldots, a_n)$ is given by

$$\begin{array}{rccl} \mu_a : & \mathbb{C}\langle X_1, \cdots, X_n\rangle & \longrightarrow & \mathbb{C} \\ & P & \longmapsto & \mu_a(P) := \varphi(P(a_1, \ldots, a_n)). \end{array}$$

Example 2. *(1) Let X be a random variable in a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. The functional $\overline{\mu}_X$ defined in Definition 9 satisfies the axioms of non-commutative distribution.*

(2) Given the matrices $A_1, \ldots, A_n \in \mathbb{M}_d(\mathbb{C})$, their joint distribution μ is given by

$$\begin{array}{rccl} \mu: & \mathbb{C}\langle X_1, \cdots, X_n\rangle & \longrightarrow & \mathbb{C} \\ & P & \longmapsto & \mu(P) := \operatorname{tr}(P(A_1, \ldots, A_n)). \end{array}$$

It follows from the properties of the trace that μ satisfy conditions $(i)-(iii)$ in Definition 10.

2.2 Gaussian matrices and Wigner Law

In the previous section we presented two of the fundamental examples of ncps, namely, $(L^{-\infty}(\Omega, \mathcal{F}, \mathbb{P}), \mathbb{E})$ and $(\mathbb{M}_d(\mathbb{C}), \operatorname{tr})$. Now we put these two examples together and present random matrices from the point of view of non-commutative probability.

Example 3. *Consider the algebra*

$$\mathcal{M} = \mathbb{M}_d\left(L^{-\infty}(\Omega, \mathcal{F}, \mathbb{P})\right)$$

and the linear functional

$$\psi = \mathbb{E} \otimes \operatorname{tr}.$$

The pair $(\mathcal{M}, \psi)$ is the ncps of random matrices of size $d \times d$.

Not every random matrix fits in the previous example. However, the random matrix models that we study in this thesis can be seen as elements in $\mathcal{M}$ for some probability space and positive integer d.
One of the random matrix models that has been extensively studied in the literature is the Gaussian Unitary Ensemble (GUE). We use the results known for GUE matrices to guide the research in this thesis.

Definition 11. *The Gaussian unitary ensemble (GUE) is a sequence of random matrices $(X_N)_{N\geq 1}$, such that for every $N \geq 1$ the entries of $X_N = \left(x_{i,j}^{(N)}\right)_{i,j=1}^{N}$ are specified according to*

(1) $x_{i,j}^{(N)} = \overline{x_{j,i}^{(N)}}$.

(2) The diagonal entries $x_{i,i}^{(N)}$ are Gaussian random variables with mean zero and variance one.

(3) The off-diagonal entries are complex random variables whose real $\Re\left(x_{i,j}^{(N)}\right)$ *and imaginary* $\Im\left(x_{i,j}^{(N)}\right)$ *parts have Gaussian distribution and*

$$\mathbb{E}\left[x_{i,j}^{(N)}\right] = 0 \qquad and \qquad \mathbb{E}\left[\left|x_{i,j}^{(N)}\right|^2\right] = \frac{1}{N},$$

for any $1 \leq i < j \leq N$.

(4) For any $N \geq 1$ *the variables* $\left\{x_{i,i}^{(N)}\right\}_{1\leq i\leq N} \cup \left\{\Re\left(x_{i,j}^{(N)}\right), \Im\left(x_{i,j}^{(N)}\right)\right\}_{1\leq i<j\leq N}$ *are independent.*

One of the first results in random matrices was the identification of the moments of GUE matrices.

Theorem 4. ***(Wigner Law, Wigner 1955).*** *Let* X_N *be a GUE. We have that*

$$\mathbb{E}\left[\frac{1}{N}\operatorname{Tr}\left(X_N^n\right)\right] \xrightarrow{N\to\infty} \begin{cases} C_{n/2} & \text{if } n \text{ is even} \\ 0 & \text{if } n \text{ is odd,} \end{cases}$$

where $C_0 = C_1 = 1$ *and for* $n \geq 2$

$$C_n = \frac{1}{n+1}\binom{2n}{n}.$$

The elements in the sequence $(C_n)_{n\geq 1}$ *are known as Catalan numbers.*

It was also noted by Wigner that the Catalan numbers are the moments of the semicircular distribution. For that reason Theorem 4 is also known as Wigner semicircular law.

Notation 2. *The centered semicircular distribution of radius* r *is a a probability measure* μ_r *on* $\mathbb{R}$*, supported in the interval* $[-r, r]$ *and defined by*

$$\mu_r(dx) = \frac{2}{\pi r^2}\sqrt{r^2 - (dx)^2}.$$

The standard semicircular distribution is the centered semicircular distribution with radius $r = 2$*. Unless stated otherwise, we will use the name semicircular or semicircular distribution for the standard semicircular distribution. In particular for* $n \geq 1$

$$\int_{-2}^{2} \frac{x^{2n}}{\pi}\sqrt{4 - x^2}dx = C_n.$$

2.3 Free Probability

The theory of Free Probability started in the 1980s with the work of Voiculescu [38, 39, 40]. Free Probability can be thought of as non-commutative probability plus a notion of independence called freeness. One of the key concepts in classical probability is the notion of independence. The simplest formulation of classical independence goes as follows

Definition 12. *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a classical probability space and J an index set with cardinality bigger than two.*

(1) A family of events $\{A_i\}_{i\in J}$ in $\mathcal{F}$ is said to be classical independent or independent if for any finite subset $F \subset J$ we have

$$\mathbb{P}\left(\bigcap_{i\in F} A_i\right) = \prod_{i\in F} \mathbb{P}(A_i).$$

(2) A family of σ-algebras $\{\mathcal{F}_i\}_{i\in I}$ contained in $\mathcal{F}$ is said to be independent if for every choice $A_i \in \mathcal{F}_i$ $(i \in J)$ the family $\{A_i\}_{i\in J}$ is independent.

(3) A family $\{X_i\}_{i\in J}$ of random variables is independent if the σ-algebras $\{X_i^{-1}(\mathcal{B}(\mathbb{R}))\}_{i\in J}$ are independent.

This notion of independence can be also encoded in the ncps $(L^{-\infty}(\Omega, \mathcal{F}, \mathbb{P}), \mathbb{E})$. The encoding is also better suited for an algebraic generalization.

Proposition 1. *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a classical probability space and let $\{X_i\}_{i\in I}$ be a family of random variables. The following are equivalent:*

(1) The family $\{X_i\}_{i\in I}$ is independent.

(2) For any bounded Borel measurable functions $f_i : \mathbb{R} \to \mathbb{R}$ and finite subset $F \subset I$ we have that

$$\mathbb{E}\left[\prod_{i\in F} f_i(X_i)\right] = \prod_{i\in F} \mathbb{E}\left[f_i(X_i)\right].$$

Proposition 1 is a general characterization of classical independence. It remains true when we consider general Borel measurable functions if the random variables have bounded support. This equivalent formulation of independence can be abstracted in the framework of ncps.

Definition 13. *(1) Let $(\mathcal{A}, \varphi)$ be a ncps and and $\mathcal{A}_i \subset \mathcal{A}$ for $i \in I$ a family of unital subalgebras of $\mathcal{A}$. The algebras $\{A_i\}_{i \in I}$ are said to be tensor independent if for any choice $a_i \in \mathcal{A}_i$*

$$\varphi\left(\prod_{i \in F} a_i\right) = \prod_{i \in F} \varphi(a_i),$$

where $F \subset I$ is any finite subset.

(2) A family of variables $a_i \in \mathcal{A}$ for $i \in I$ is called tensor independent if the unital algebras $\mathcal{A}_i := alg\{1, a_i\}$ are tensor independent.

The canonical way to construct independent classical random variables is by taking Cartesian products of classical probability spaces. In the same way we can construct tensor independent random variables by tensoring ncps. The concept of free independence arises when we change the tensor product for a new operation on algebras: the free product. In [38] Voiculescu introduced the notion of free product for C^*-algebras. This operation on C^*-algebras can be also performed on unital algebras over $\mathbb{C}$. See [28, Definition 6.1]. The precise construction and definition of free product can be consulted in [28, Lectures 6 and 7]. Similar to classical and tensor independence, the definition of freeness does not involve directly the definition of free product.

Definition 14. *Let $(\mathcal{A}, \varphi)$ be a ncps and $\mathcal{A}_i \subset \mathcal{A}$ for $i \in I$ a family of unital subalgebras of $\mathcal{A}$. The algebras $(A_i)_{i \in I}$ are called free independent if for any positive integer n and $a_1 \in \mathcal{A}_{i_1}, \dots, a_n \in \mathcal{A}_{i_n}$ with $i_1 \leq i_2 \neq \cdots \neq i_n$ we have that:*

$$\varphi(a_1) = \cdots = \varphi(a_n) = 0$$

implies

$$\varphi(a_1 \cdots a_n) = 0.$$

In classical probability, the central limit theorem (CLT) appears when we take sums of independent random variables with finite moments. The same distribution appears when we use tensor independence. The situation is different when we replace tensor by free independence. In the latter case we obtain the semicircular distribution. A discussion on the free and tensor CLTs can be consulted in [28, Lecture 8].
Non-commutative variables with semicircular distribution play a central role in free probability. They are the free analogs of Gaussian variables.

Definition 15. *Let $(\mathcal{A}, \varphi)$ be a ncps.*

(1) *An element in $S \in \mathcal{A}$ is said to be a centered semicircular element of variance σ^2 if*

$$\varphi\left(s^n\right) = \begin{cases} C_k\, \sigma^{2k} & \text{if } n = 2k \text{ for } k \geq 1 \\ 0 & \text{otherwise.} \end{cases}$$

For short we will say that S is semicircular or a semicircular variable if S is a centered semicircular element of variance one.

(2)

In 1991, Voiculescu [41] showed the connection between free probability and random matrices. This result provides a deep insight on Wigner Law. More precisely, Wigner Law says that GUE matrices approximate semicircular variables. A big achievement of Voiculescu was that he identified that independent GUE matrices approximate free independent semicircular s.

Theorem 5. ***(Asymptotic freeness, Voiculescu 1991).*** *Let $X_N^{(p)}$ for $p \in I$ be a family of independent copies of a GUE and $(S^{(q)})_{p\in I}$ a family of free independent semicircular variables in a ncps $(\mathcal{A}, \varphi)$. Then for any $p_1, \ldots, p_n \in I$*

$$\mathbb{E}\left[\frac{1}{N}\operatorname{Tr}\left(X_N^{(p_1)} \cdots X_N^{(p_n)}\right)\right] \xrightarrow{N\to\infty} \varphi\left(S^{(p_1)} \cdots S^{(p_n)}\right).$$

Also, the convergence holds almost surely.

The statement of asymptotic freeness that we present in this thesis is definitely not the most general. We emphasize the use of GUE matrices for the purpose of this work. In Chapter 4 we present a proof of asymptotic freeness for a matrix model that is a generalization of the GUE.

2.4 Cumulants

Free cumulants are the key concept in the combinatorial side of free probability. The combinatorial approach to freeness was inspired by the work of Rota [31, 32] about cumulants in classical probability. Free and classical cumulants are multilinear functionals that can be defined in any ncps. Their key property is that free (respectively classical) independence can be characterized via free (respectively classical) cumulants. The theory of cumulants is groundend in the theory of Möbius functions on lattices of partitions. We primarily follow the monographs [28, 36].

Notation 3. *(1) For a positive integer $k < n$ we denote $[l, n] = \{k, k + 1, \ldots, n\}$. In the particular case $k = 1$ we also use the notation $[1, n] = [n]$.*

(2) *The set of partitions of* $[n]$ *is denoted by* $\mathcal{P}(n)$. *This means that if* $\pi = \{V_1, \ldots, V_r\} \in \mathcal{P}(n)$, *then* $V_1, \ldots, V_r$ *are non-empty pairwise disjoint sets whose union is* $[k]$. *The sets* $V_1, \ldots, V_r$ *are called the blocks of* π.

(3) *The set of pair partitions* $\mathcal{P}_2(k)$ *is defined for*

$$\mathcal{P}_2(n) = \{\pi \in \mathcal{P}(n) | \quad \forall\, V \in \pi, \quad |V| = 2\},$$

where $|V|$ *denotes the size of the set* V. *If* n *is odd then* $\mathcal{P}_2(n) = \emptyset$.

(4) *We say that a partition* $\pi \in \mathcal{P}(n)$ *has a crossing if there exist two different blocks* $V_1, V_2 \in \pi$ *and* $1 \leq a < b < c < d \leq n$, *such that* $a, c \in V_1$ *and* $b, d \in V_2$. *A partition* π *is said to be non-crossing if it has no crossings. The set of non-crossing partitions of* $[n]$ *is denoted by* $NC(n)$. *In particular, the set of non-crossing pair partitions is defined by*

$$NC_2(n) = NC(n) \cap \mathcal{P}_2(n).$$

(5) *Given partitions* $\pi, \sigma \in \mathcal{P}(n)$ *we write* $\pi \leq \sigma$ *if every block of* π *is contained in a block of* σ.

The language of partitions is quite useful when dealing with non-commutative variables. For this purpose we introduce the following notations that will be often used in the next chapter.

Notation 4. *(1) Consider a family* $\{X_s\}_{s \in A}$ *of non-commutative variables. Given* $\alpha : [k] \to A$ *we denote*

$$X_\alpha := X_{\alpha(1)} \cdots X_{\alpha(k)}. \tag{2.5}$$

(2) *In case we have several families of non-commutative variables* $\{X_s^{(r)}\}_{s \in A}$ *for* $r \in B$ *we will also use similar notations. That is, given* $\alpha : [k] \to A$ *and* $\varepsilon : [k] \to B$ *we denote*

$$X_\alpha^\varepsilon := X_{\alpha(1)}^{(\varepsilon(1))} \cdots X_{\alpha(k)}^{(\varepsilon(k))}. \tag{2.6}$$

(3) *It will be useful to specify the functions* $\alpha : [k] \to A$ *via partitions. For this purpose we define for every function* $\alpha : [k] \to A$ *between discrete spaces*

$$\ker \alpha := \{\alpha^{-1}(a) | \, a \in A \text{ and } \alpha^{-1}(a) \neq \emptyset\}.$$

(3) *Given* $B \in \ker \alpha$, *we will denote the common value of* α *in* B *by*

$$\alpha(B) := \alpha(b_1) = \cdots = \alpha(b_r), \tag{2.7}$$

where $B = \{b_1, \ldots, b_r\}$.

Lattices are particular cases of partially ordered sets where a notion of "maximum" and "minimum" exists. The order structure introduced in Notation 3 (5) turns the set of partitions (as well as the set of non-crossing partitions) into a lattice.

Proposition 2. *(1) The relation $\leq$ is a partial order that turns $\mathcal{P}(n)$ into a lattice. This means that for every $\pi, \sigma \in \mathcal{P}(n)$ there exist unique partitions $\pi \wedge \sigma$ and $\pi \vee \sigma$ in $\mathcal{P}(n)$ with the properties:*

(i) For every partition $\rho \in \mathcal{P}(n)$, with $\pi \leq \rho$ and $\sigma \leq \rho$, we have $\pi \wedge \sigma \leq \rho$.

(ii) For every partition $\rho \in \mathcal{P}(n)$, with $\rho \leq \pi$ and $\rho \leq \sigma$, we have $\rho \leq \pi \vee \sigma$.

We call $\pi \wedge \sigma$ and $\pi \vee \sigma$ the maximum and minimum of π and σ, respectively.

(2) The relation $\leq$ restricted to $NC(n)$ is also a partial order that turns $NC(n)$ into a lattice. In particular this means that the maximum and minimum can be taken as well in $NC(n)$.

There is a Möbius function attached to every partially ordered set. This function is defined in a recursive way.

Definition 16. *The Möbius function on a lattice $(P, \leq)$ is a function*

$$\mu_P : \{(\pi, \sigma) \in P^2 | \quad \pi \leq \sigma\} \longrightarrow \mathbb{C}$$

defined recursively by the relation

$$\sum_{\substack{\rho \in P \\ \pi \leq \rho \leq \sigma}} \mu_P(\pi, \rho) = \begin{cases} 1 & \text{if} \quad \pi = \sigma \\ 0 & \text{if} \quad \pi < \sigma. \end{cases}$$

The partially ordered sets $\mathcal{P}(n)$ and $NC(n)$ play a central role in the definitions of cumulants. They are also characterized by their Möbius function.

Example 4. *The Möbius functions on $(\mathcal{P}(n), \leq)$ and $(NC(n), \leq)$ are given by*

$$\mu_{\mathcal{P}(n)}(\pi, 1_n) = (-1)^{|\pi|-1}(|\pi| - 1)!,$$

and

$$\mu_{NC(n)}(0_n, 1_n) = (-1)^{n-1} C_{n-1}.$$

For the definition of cumulants we need some additional notation.

Notation 5. *Let* $(\mathcal{A}, \varphi)$ *be a ncps. For every partition* $\pi \in \mathcal{P}(n)$ *denote*

$$\varphi_\pi(a_1, \ldots, a_n) = \prod_{\substack{V \in \pi \\ V = \{i_1 < \cdots < i_r\}}} \varphi(a_{i_1} \cdots a_{i_r}).$$

In Chapter 4 we present a matrix valued generalization of the functionals φ_π. See Notation 9.

Definition 17. *Let* $(\mathcal{A}, \varphi)$ *be a ncps.*

(1) Classical cumulants are the multilinear functionals

$$c_n(a_1, \ldots, a_n) = \sum_{\pi \in \mathcal{P}(n)} \varphi_\pi(a_i, \ldots, a_n) \mu_{\mathcal{P}(n)}(\pi, 1_n).$$

(2) Free cumulants are the multilinear functionals

$$\kappa_n(a_1, \ldots, a_n) = \sum_{\pi \in \mathcal{P}(n)} \varphi_\pi(a_i, \ldots, a_n) \mu_{NC(n)}(\pi, 1_n).$$

The key property of free and classical cumulants is that they characterize free and classical independence. The proof of the following result can be consulted in [28, Theorems 11.16 and 11.32].

Theorem 6. *Let* $(\mathcal{A}, \varphi)$ *be a ncps and* $(A_i)_{i \in I}$ *be subalgebras of* $\mathcal{A}$*.*

(1) The algebras $(A_i)_{i \in I}$ *are free independent in* $(\mathcal{A}, \varphi)$*, iff for all* $n \geq 2$ *and* $a_1 \in \mathcal{A}_{i_1}, \ldots, a_n \in \mathcal{A}_{i_n}$ *with* $i_1, \ldots, i_n \in I$ *we have that*

$$\kappa_n(a_1, \ldots, a_n) = 0,$$

whenever there exists $1 \leq r < s \leq n$ *such that* $i_r \neq i_s$*.*

(2) The commutative algebras $(A_i)_{i \in I}$ *are tensor independent in* $(\mathcal{A}, \varphi)$*, iff for all* $n \geq 2$ *and* $a_1 \in \mathcal{A}_{i_1}, \ldots, a_n \in \mathcal{A}_{i_n}$ *with* $i_1, \ldots, i_n \in I$ *we have that*

$$c_n(a_1, \ldots, a_n) = 0,$$

whenever there exists $1 \leq r < s \leq n$ *such that* $i_r \neq i_s$*.*

2.5 Fluctuations for GUE matrices

Now we want to have a closer look at the result of asymptotic freeness for GUE matrices. Let $X_N^{(p)}$ for $p \in I$ be independent GUE matrices. We have seen in Theorem 5 that

$$\frac{1}{N} \operatorname{Tr}\left(X_N^{(p_1)} \cdots X_N^{(p_n)}\right) \xrightarrow{N \to \infty} \varphi\left(S^{(p_1)} \cdots S^{(p_n)}\right),$$

almost surely.

Following ideas from Johansson [18] it was shown by Mingo-Speicher [27] that the random variables

$$F(X_N^{(p_1)} \cdots X_N^{(p_n)}) := \operatorname{Tr}\left(X_N^{(p_1)} \cdots X_N^{(p_n)}\right) - N\varphi\left(S^{(p_1)} \cdots S^{(p_n)}\right)$$

converge in distribution to centered Gaussian random variables. They also characterize the covariance between $F(X_N^{(p_1)} \cdots X_N^{(p_n)})$ and $F(X_N^{(r_1)} \cdots X_N^{(r_m)})$ for arbitrary choices of $p_1, \ldots, p_n, r_1, \ldots, r_m \in I$. This gives rise to the definition of fluctuation moments.

Definition 18. *Let I be an index set. For every $p \in I$ consider the sequence of matrices $(Y_N^{(p)})_{N\geq 1}$ where $Y_N^{(p)}$ is and $N \times N$ matrix for every $N \geq 1$. The fluctuation moments of the family $(Y_N^{(p)})_{N\geq 1, p\in I}$ are defined by*

$$c_2\left(\operatorname{Tr}\left(Y_N^{(Y_1)} \cdots Y_N^{(p_n)}\right), \operatorname{Tr}\left(Y_N^{(r_1)} \cdots Y_N^{(r_m)}\right)\right),$$

where $p_1, \ldots, p_n, r_1, \ldots, r_m \in I$.

The characterization of covariances for the random variables $F(X_N^{(p_1)} \cdots X_N^{(p_n)})$ was given in terms of annular permutations.

Annular permutations can be seen as an extension of the concept of non-crossing partitions. There is a natural way to embed the set of non-crossing partitions $NC(n)$ into the set of permutations S_n. For the embedding we need to consider for every $r \geq 1$ the one cycle permutations $\gamma_r = (1, 2, \ldots, r)$. We identify $\pi = \{V_1, \ldots, V_s\} \in NC(n)$ with the permutation $\tilde{\pi} = \gamma_{|V1|} \cdots \gamma_{|V_s|}$, where every cycle $\gamma_{|V_r|}$ acts on the corresponding block V_r. Since π and $\tilde{\pi}$ are quite similar we will use the same notation for both objects. Also, we use the same notation $NC(n)$ for the set of non-crossing partitions and for its embedding in S_n.

Given a permutation $\sigma \in S_n$ let us denote by $\#(\sigma)$ the number of cycles in σ. It was observed by Biane [4] that the subset $NC(n) \subset S_n$ can be also described as the set of permutations $\pi \in S_n$ that satisfy

$$\#(\pi) + \#(\pi^{-1}\gamma_n) = n + 1. \tag{2.8}$$

The above equations has a nice "geodesic interpretation" in term of the distance function

$$d(\sigma, \tau) = n - \#(\sigma^{-1}\tau), \qquad \text{for} \quad \sigma, \tau \in S_n.$$

The proof that d is symmetric and satisfy $d(\sigma, \sigma) = 0$ follows directly from definition. The triangle inequality follows from the observation that $n - \#(\sigma)$ equals the minimum non-negative integer k such that σ can be written as a product of k transpositions. Using the triangle inequality we get that for every $\pi \in S_n$ we have that

$$n - 1 = d(e, \gamma_n) \leq d(e, \pi) + d(\pi, \gamma_n) = n - \#(\pi) + n - \#(\pi^{-1}\gamma_n),$$

where e is the identity in S_n. As a consequence

$$\#(\pi) + \#(\pi^{-1}\gamma_n) \leq n + 1, \qquad \text{for every} \quad \pi \in S_n. \tag{2.9}$$

For this reason Equation (2.8) is also known as Biane's geodesic condition.

Later on, inequality (2.9) was generalized by Mingo-Nica [24] to the case of two permutations

$$\#(\pi) + \#(\pi^{-1}\sigma) + \#(\sigma) \leq n + 2 \cdot \#(\pi \vee \sigma), \quad \text{for every} \quad \pi, \sigma \in S_n, \tag{2.10}$$

where $\#(\pi \vee \sigma)$ is the number of orbits into which $[n]$ is split by the joint action of π and σ. They use this inequality to define the annular non-crossing permutations.

Definition 19. *Let $k_1, \ldots, k_m$ be positive integers and set $k = k_1 + \cdots + k_m$. Let γ be the permutation in S_k with cycles $(1, \ldots, k_1) \cdots (k_1 + \cdots + k_{m-1} + 1, \ldots, k)$.*

(1) We say that a permutation $\pi \in S_k$ is a $(k_1, \ldots, k_m)$-annular permutation if satisfy the following condition

$$\#(\pi) + \#(\pi^{-1}\gamma) = k - m + 2 \cdot \#(\pi \vee \gamma).$$

(2) We say that a permutation $\pi \in S_k$ is $(k_1, \ldots, k_m)$-connected if

$$\#(\pi \vee \gamma) = 1.$$

(3) The set of $(k_1, \ldots, k_m)$-annular permutations that are also $(k_1, \ldots, k_m)$-connected is denoted by $NC(k_1, \ldots, k_m)$.

(4) The set of $(k_1, \ldots, k_m)$-annular pair permutations is defined by

$$NC_2(k_1, \ldots, k_m) = \{\pi \in NC(k_1, \ldots, k_m) \mid \pi^2 = e\}.$$

The characterization for fluctuations and higher cumulants of GUE matrices is the following.

Theorem 7. *Let $X_N^{(p)}$ for $p \in I$ be a family of independent copies of a GUE and $S^{(p)}$ for $p \in I$ is a family of free independent semicircular variables in a ncps $(\mathcal{A}, \varphi)$. Let γ be the permutations with cycles $(1, \dots, k_1) \cdots (k_1 + \cdots + k_{m-1} + 1, \dots, k)$, where $k = k_1 + \cdots + k_m$. Consider $\varepsilon : [k] \to I$ and ε_1 the restriction of ε to $[1, k_1]$ and for $i = 2, \dots, m$ let ε_i be the restriction of ε to $[k_1 + \cdots + k_m + 1, k]$. Then we have*

$$\frac{c_m\left(\mathrm{Tr}\left(X_N^{\varepsilon_1}\right), \dots, \mathrm{Tr}\left(X_N^{(\varepsilon_m)}\right)\right)}{N^{2-m}} \xrightarrow{N\to\infty} \sum_{\pi \in NC_2(k_1,\dots,k_m)} \prod_{(l,r)\in\pi} \varphi\left(S^{(\varepsilon(l))} S^{(\varepsilon(r))}\right),$$

where

$$X_N^{\varepsilon_1} := X_N^{\varepsilon_1(1)} \cdots X_N^{(\varepsilon_1(k_1))},$$

as in Notation 4.

In this thesis we present analogue results for the SYK model and for Wigner matrices with q-Gaussian entries.

2.6 Operator-valued Free Probability

Another concept that play a crucial role in classical probability is the one of conditional independence. It turns out that this important concept can be also defined in an algebraic way.

Definition 20. *Let $\mathcal{A}$ be a unital algebra and $\mathcal{B} \subset \mathcal{A}$ be a subalgebra with $1 \in \mathcal{B}$. A linear map $E : \mathcal{A} \to \mathcal{B}$ is a conditional expectation if*

$$E[b] = b \qquad \forall b \in \mathcal{B}$$

and

$$E[b_1, ab_2] = b_1 E[a] b_2 \qquad \forall a \in \mathcal{A}, \quad \forall b_1, b_2 \in \mathcal{B}.$$

An operator-valued ncps is then a triplet $(\mathcal{A}, E, \mathcal{B})$ where $E : \mathcal{A} \to \mathcal{B}$ is a conditional expectation. In this setting an operator-valued distribution of $x \in \mathcal{A}$ is the collection of operator-valued moments

$$E[b_0 x b_1 \cdots b_{n-1} x b_n] = b_0 E[x b_1, \cdots b_{n-1} x] b_n \in \mathcal{B}.$$

Many results and definitions from Free Probability can be lifted to operator-valued versions. In particular the combinatorial approach to Free Probability displays its whole power in the operator-value setting. See the monograph [36] for concrete details. We present here the definition that we use in Chapter 4.

Definition 21. *Let $(\mathcal{A}, E, \mathcal{B})$ be an operator-valued ncps.*

(1) The operator-valued free cumulants

$$\kappa_n^{\mathcal{B}} : \mathcal{A}^n \longrightarrow \mathcal{B}, \qquad n \geq 1$$

are define by the moment-cumulant formula

$$E(a_1 \cdots a_n) = \sum_{\pi \in NC(n)} \kappa_\pi^{\mathcal{B}}(a_1, \cdots, a_n),$$

where the arguments of $\kappa_\pi^{\mathcal{B}}$ are distributed according to the blocks of π respecting its nested structure.

(2) The multiplicative family $(\kappa_\pi^{\mathcal{B}})_{n\geq 1, \pi \in NC(n)}$ is defined by

$$\kappa_\pi^{\mathcal{B}}(a_1, \dots, a_n) = \sum_{\substack{\sigma \in NC(n) \\ \sigma \leq \pi}} E_\sigma(a_1, \dots, a_n),$$

where the arguments of E_σ are distributed according to the blocks of σ respecting its nested structure.

(3) We say that $S \in \mathcal{A}$ is an $\mathcal{B}$-values semicircular if

$$\kappa_n^{\mathcal{B}}(S, b_1 S, \dots, S b_{n-1}, S) = 0$$

for all $n \neq 2$ and all $b_1, \dots, b_{n-1} \in \mathcal{B}$.

2.7 q-Gaussian variables

Consider a real Hilbert space $\mathcal{H}_{\mathbb{R}}$ and its complexification $\mathcal{H} := \mathcal{H} \oplus i\mathcal{H}_{\mathbb{R}}$, for us the inner product of $\mathcal{H}$ is linear in the first entry. In this thesis we will fix a unit vector $\Omega \in \mathcal{H}$, also called the vacuum vector. On the algebraic Fock space

$$\mathcal{F}_{alg}(\mathcal{H}) := \mathbb{C}\Omega \oplus \bigoplus_{n=1}^{\infty} \mathcal{H}^{\otimes n},$$

consider for $q \in [-1, 1]$ and $f_1, \cdots, f_n, g_1, \cdots, g_m \in \mathcal{H}$ the sesquilinear form

$$\langle f_1 \otimes \cdots \otimes f_n, g_1 \otimes \cdots \otimes g_m \rangle_q := \delta_{nm} \sum_{\sigma \in S_n} q^{i(\sigma)} \prod_{k=1}^{n} \langle f_k, g_{\sigma(k)} \rangle, \tag{2.11}$$

where $i(\sigma)$ is the number of inversions of σ, i.e.,

$$i(\sigma) := \#\{(i,j) \in \{1, \cdots, n\}^2 | i < j, \ \ \sigma(i) > \sigma(j)\}.$$

The non trivial part is to show that (2.11) is positive definite. This is equivalent to the fact that the function $\sigma \mapsto q^{i(\sigma)}$ is positive definite, i.e. for every $r : S_n \to \mathbb{C}$

$$\sum_{\pi,\sigma \in S_n} q^{i(\pi^{-1}\sigma)} r(\sigma)\overline{r(\pi)} \geq 0.$$

The Full Fock is defined as the completion of the algebraic Fock space with respect to the inner product (2.11), that is

$$\mathcal{F}_q(\mathcal{H}) := \overline{\mathcal{F}_{alg}(\mathcal{H})}^{\langle\, ,\, \rangle_q}.$$

On the algebra of bounded operators over $\mathcal{F}_q(\mathcal{H})$ consider the following linear functional:

$$\begin{array}{rcl} \varphi : B(\mathcal{F}_q(\mathcal{H})) & \longrightarrow & \mathbb{C} \\ T & \longmapsto & \langle T\Omega, \Omega \rangle_q. \end{array}$$

In particular $\varphi(1) = 1$, then the tuple $(B(\mathcal{F}_q(\mathcal{H}), \varphi))$ is a non-commutative probability space. Now for $h \in \mathcal{H}$ we define the q-creation operator $a^*(h)$, given by

$$a^*(h)\Omega = h,$$
$$a^*(h) h_1 \otimes \cdots \otimes h_n = h \otimes h_1 \otimes \cdots \otimes h_n.$$

Its adjoint (with respect to the q-inner product), the q-annihilation operator $a(h)$, is given by

$$a(h)\Omega = 0,$$
$$a(h) h_1 \otimes \cdots \otimes h_n = \sum_{r=1}^{n} q^{r-1} \langle h_r, h \rangle h_1 \otimes \cdots \otimes h_{r-1} \otimes h_{r+1} \otimes \cdots \otimes h_n.$$

These operators satisfy the q-commutations relations

$$a(f)a^*(g) - q a^*(g) a(f) = \langle f, g \rangle \cdot 1 \qquad (f, g \in \mathcal{H}).$$

For $q = 1$, $q = 0$ and $q = -1$ this reduces to the CCR-relations, the Cuntz relations, and the CAR-relations, respectively. With the exception of the case $q = 1$, the operators $a^*(f)$ are bounded. Operators of the form

$$s_q(f) := a(f) + a^*(f)$$

for $f \in \mathcal{H}$ are called q-Gaussian (or q-semicircular) elements.We will say that the q-Gaussian variables $s_q(f_1), \cdots, s_q(f_k)$ are orthogonal (orthonormal) if the corresponding vectors $f_1, \cdots, f_k \in \mathcal{H}$ are orthogonal (orthonormal) in $\mathcal{H}$.

Definition 22. *The (multivariate) q-Gaussian distribution is defined as the non commutative distribution of a collection of q-Gaussians with respect to the vacuum expectation state.*

It was shown in [6], for orthonormal $h_1, \ldots, h_p \in \mathcal{H}$ the joint distribution of $s_q(h_1), \ldots, s_q(h_p)$ with respect to τ can be described in the following way: for any $\varepsilon : \{1, \ldots, k\} \to \{1, \ldots, p\}$ we have

$$\tau\left(s_q(h_{\varepsilon(1)}) \cdots s_q(h_{\varepsilon(k)})\right) = \sum_{\substack{\pi \in \mathcal{P}_2(k) \\ \pi \leq \ker \varepsilon}} q^{cr(\pi)}.$$

For $p = 1$, the q-Gaussian distribution is a probability measure on the interval $[\frac{-2}{\sqrt{1-q}}, \frac{2}{\sqrt{1-q}}]$ and is given by

$$f_q(t) = \frac{\sqrt{1-q}}{2\pi} \sqrt{4 - (1-q)t^2} \sum_{k=1}^{\infty} (-1)^{k-1} q^{\frac{k(k-1)}{2}} \mathcal{U}_{2k-2}\left(\frac{x\sqrt{1-q}}{2}\right),$$

For the special cases $q = 1$, $q = 0$, and $q = -1$, this reduces to the classical Gaussian distribution, the semicircular distribution, and the symmetric Bernoulli distribution on ± 1, respectively.

The multivariate q-Gaussian distribution can actually be seen as the increments of a q-version of a Brownian motion. Namely, if we take as our underlying Hilbert space $\mathcal{H} = L^2(\mathbb{R}_0^+)$ and as indexing vectors the family $1_{[0,t]}$ $(t \geq 0)$ of characteristic functions of intervals $[0, t]$, then the process $((B_q(t))_{t \geq 0}$ with $B_q(t) = s_q(1_{[0,t]})$ is called q-Brownian motion. In the case $q = 1$ it is indeed classical Brownian motion (in the sense that it has the same expectation values as classical Brownian motion), and in the case $q = 0$ it is free Brownian motion.

Chapter 3

The Sachdev-Ye-Kitaev model

The Sachdev-Ye-Kitaev model (SYK) was introduced in 1993 by Sachdev and Ye [33] as a model for quantum spin systems. Later on, in 2015, it was promoted by Kitaev [20] as a model for quantum holography.

In 2014, a similar model for quantum spin systems was analyzed in the mathematical literature by Keating [19] and extended in the same year by Erdös-Schröder [11]. In 2018, Feng-Tian-Wei realized that the model used by Erdös-Schröder was closely related to the SYK model.

The generalization made by Erdös-Schröder exhibits a phase transition between the Gaussian and Semicircular distributions. Feng-Tian-Wei [12] in turn used similar methods as in [11] to show that the SYK model presents a phase transition between Gaussian, Semicircular and symmetric Bernoulli distributions. Some of the results in [11, 12] had been proven in the physics community. See for example [14] and the references given there. However, the mathematical proofs were given in [11, 12].

In the context of non-commutative probability the transition between Gaussian, Semicircular and symmetric Bernoulli appeared in 1991 in the work of Bozejko, Kümmerer and Speicher [6, 8] under the name of q-Gaussian distribution. The q-Gaussian distribution depends on a parameter q that can take on any value $[-1, 1]$. The Gaussian, Semicircular and symmetric Bernoulli distributions appear as particular cases of the q-Gaussian variables when q equals 1, 0 and -1, respectively.

In [6] Bozejko and Speicher provided a combinatorial description of several orthonormal q-Gaussian variables. The same combinatorial description was used by Feng-Tian-Wei in [12] to describe the asymptotic empirical eigenvalue distribution of one copy of the SYK model.

We were motivated by the work of Feng-Tian-Wei and extended it to the miltivariate situation. The main result of our work, presented in this chapter, is that we have shown that the asymptotic eigenvalue distribution of several independent copies of the SYK model can be described by orthonormal q-Gaussian variables.

The main tool used in this chapter is the moment method. We adapt the computations made in [11, 12] and provide a concrete description of how the parameter q arises from a "double scaling" in the SYK model. See Lemma 4 below. As a biproduct, we also compute mix cumulants of traces of products for the multivarite SYK model. See Theorem 9 below. This extends a result from Feng-Tian-Wei [13] about fluctuations for the SYK model.

3.1 The SYK model

In this section we present the random matrix model that will be the central object in this chapter. For this purpose we need the following definition.

Definition 23. *The Majorana fermions are a family of anti-commuting variables $\psi_1, \cdots, \psi_n$ that are also roots of the identity, i.e., they satisfy the following equations*

$$\psi_i\psi_j + \psi_j\psi_i = 2\delta_{ij} \qquad \textit{for} \quad 1 \leq i, j \leq n. \tag{3.1}$$

For an even number n, there is a way to construct n Majorana ferminons with square matrices of size $2^{n/2}$. The construction uses Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

in the following fashion: for $n = 2r$ each Majorana fermion is constructed as an r-fold tensor product

$$\begin{array}{lcl} \psi_1 = \sigma_1 \otimes 1 \otimes \cdots \otimes 1 & & \psi_{r+1} = \sigma_2 \otimes 1 \otimes \cdots \otimes 1 \\ \psi_2 = \sigma_3 \otimes \sigma_1 \otimes \cdots \otimes 1 & & \psi_{r+2} = \sigma_3 \otimes \sigma_2 \otimes \cdots \otimes 1 \\ \quad\vdots & , & \quad\vdots \\ \psi_r = \sigma_3 \otimes \sigma_3 \otimes \cdots \otimes \sigma_1 & & \psi_{2r} = \sigma_3 \otimes \sigma_3 \otimes \cdots \otimes \sigma_2 \end{array}$$

where the 1 in the tensor products represents the 2×2 identity matrix. In particular, for $n = 2$ the above expressions are reduce to $\psi_1 = \sigma_1$ and $\psi_2 = \sigma_2$. In this way the $\psi_1, \cdots, \psi_n$ Majorana fermions are realized as square matrices of size $2^{n/2}$.

The SYK model is a random matrix model, constructed as a random linear combination of products of Majorana fermions

Definition 24. *The SYK model H_{n,q_n} depends on two parameters n and q_n, where n is an even number and q_n is a integer number between 1 and $\frac{n}{2}$, and is given by*

$$H_{n,q_n} := \frac{i^{\lfloor q_n/2 \rfloor}}{\binom{n}{q_n}^{1/2}} \sum_{1 \le i_1 < \cdots < i_{q_n} \le n} J_{i_1,\dots,i_{q_n}} \psi_{i_1} \cdots \psi_{i_{q_n}}, \tag{3.2}$$

where the random coefficients $J_{i_1,\dots,i_{q_n}}$ are independent real random variables with moments of all orders and

$$\mathbb{E}[J_{i_1,\dots,i_{q_n}}] = 0, \quad \mathbb{E}[J^2_{i_1,\dots,i_{q_n}}] = 1.$$

In the main theorem we do not assume the variables $J_{i_1,\dots,i_{q_n}}$ to be identically distributed, but we require uniformly bounded moments. For the result about fluctuations we require identical distribution. It will be important to distinguish the parity of q_n, see Theorem 8.

We are interested in products of independent copies of the SYK-model. For this purpose it is convenient to have a compact notation for (3.2). This motivates the following notation: for $1 \le q_n \le \frac{n}{2}$ consider the set of tuples

$$I_n := \{(i_1, \cdots, i_{q_n}) | 1 \le i_1 < \cdots < i_{q_n} \le n\},$$

and for each $R = (i_1, \cdots, i_{q_n}) \in I_n$ denote $J_R := J_{i_1,\cdots,i_{q_n}}$ and consider the new variables

$$\mathbf{\Psi}_R := \psi_{i_1} \cdots \psi_{i_{q_n}} i^{\lfloor q_n/2 \rfloor}. \tag{3.3}$$

Then for $1 \le q_n \le \frac{n}{2}$ we rewrite the SYK-model as

$$H_{n,q_n} := \frac{1}{|I_n|^{1/2}} \sum_{R \in I_n} J_R \mathbf{\Psi}_R.$$

We collect some properties of the variables (3.3) in the following lemma. See Section 3.4 for the proof.

Lemma 1. *For every $R, Q \in I_n$ with $R \neq Q$ we have the identities*

$$\mathbf{\Psi}_R^2 = I, \tag{3.4}$$

and

$$\mathbf{\Psi}_Q \mathbf{\Psi}_R = (-1)^{q_n + |Q \cap R|} \mathbf{\Psi}_R \mathbf{\Psi}_Q. \tag{3.5}$$

So, for two different multi-indices Q and R the variables $\mathbf{\Psi}_Q$ and $\mathbf{\Psi}_R$ commute or anti commute depending on the parity of q_n and on the size of the intersection of the multi-indices. The variables (3.3) also behave well with respect to the trace, see Lemma 2.

3.2 Asymptotic distribution

In this section we present a multi-variable as well as a dynamical version of a result from [12] and [11].

Theorem 8. *Consider p independent and identically distributed copies* $H_1, \cdots, H_p$ *of the SYK model* H_{n,q_n}*, with uniformly bounded random coefficients (3.2). We assume the existence of the limit*

$$\frac{q_n^2}{n} \to \lambda \in [0, \infty], \quad \text{as } n \to \infty,$$

and describe this in terms of a number $q \in [-1, 1]$ *in the following form:*

i) If $(q_n)_{n \geq 1}$ *is a sequence of even positive integers, then* $q = e^{-2\lambda}$.

ii) If $(q_n)_{n \geq 1}$ *is a sequence of odd positive integers, then* $q = -e^{-2\lambda}$.

Then $(H_1, \cdots, H_p)$ *converges in distribution to a tuple of q-Gaussian variables* $(s_q(h_1), \ldots, s_q(h_p))$ *for an orthonormal system* $h_1, \ldots, h_p$*. Concretely, this means that for every positive integer k and for every* $\varepsilon : [k] \to [p]$*, we have that*

$$\lim_{n\to\infty} \mathbb{E}\left[\operatorname{tr}(H_{\varepsilon(1)} \cdots H_{\varepsilon(k)})\right] = \sum_{\substack{\pi \in \mathcal{P}_2(k) \\ \pi \leq \ker \varepsilon}} q^{cr(\pi)} = \tau\left(s_q(h_{\varepsilon(1)}) \cdots s_q(h_{\varepsilon(k)})\right). \tag{3.6}$$

The right side of (3.6) should be understood as zero when k is odd.

Corollary 2. *Consider the following dynamical version of the SYK model:*

$$H(t) := \frac{i^{\lfloor q_n/2 \rfloor}}{\binom{n}{q_n}^{1/2}} \sum_{1 \leq i_1 < \cdots < i_{q_n} \leq n} J_{i_1,\ldots,i_{q_n}}(t) \psi_{i_1} \cdots \psi_{i_{q_n}}, \tag{3.7}$$

where the $J_{i_1,\ldots,i_{q_n}}(t)$ *(with* $n \in \mathbb{N}$*,* $1 \leq i_1 < \cdots < i_{q_n} \leq n$*) are independent classical Brownian motions, and the* q_n *and q are as in Theorem 8. Then, the process* $(H(t))_{t\geq 0}$ *converges, for* $n \to \infty$*, to the q-Brownian motion* $(B_q(t))_{t\geq 0}$*, in the sense that we have for all* $0 \leq t_1, \ldots, t_k$ *that*

$$\lim_{n\to\infty} \mathbb{E}\left[\operatorname{tr}(H(t_1) \cdots H(t_k))\right] = \tau\left(B_q(t_1) \cdots B_q(t_k)\right). \tag{3.8}$$

Proof. In order to see that the processes $(H(t))_{t\geq 0}$ converges towards the q-Brownian motion, we have to check that for $0 = t_0 < t_1 < \cdots < t_p$, the finite dimensional distribution $(H(t_1), \ldots, H(t_p))$ of the process $(H(t))_{t\geq 0}$, converges towards the finite dimensional distribution $(B(t_1), \ldots, B(t_p))$ of the q-Brownian motion $(B(t))_{t\geq 0}$. Since we can find a matrix A such that

$$\begin{pmatrix} H(t_p) \\ \vdots \\ H(t_1) \end{pmatrix} = A \begin{pmatrix} \frac{H(t_p)-H(t_{p-1})}{\sqrt{t_p-t_{p-1}}} \\ \vdots \\ \frac{H(t_1)-H(t_0)}{\sqrt{t_1-t_0}} \end{pmatrix},$$

it is enough to show the convergence of normalized increments:

$$\begin{pmatrix} \frac{H(t_p)-H(t_{p-1})}{\sqrt{t_p-t_{p-1}}} \\ \vdots \\ \frac{H(t_1)-H(t_0)}{\sqrt{t_1-t_0}} \end{pmatrix} \xrightarrow{n\to\infty} \begin{pmatrix} \frac{B(t_p)-B(t_{p-1})}{\sqrt{t_p-t_{p-1}}} \\ \vdots \\ \frac{B(t_1)-B(t_0)}{\sqrt{t_1-t_0}} \end{pmatrix}. \tag{3.9}$$

The left side above has the same distribution as a tuple $(H_1, \ldots, H_p)$ of independent copies of $H(1)$. The right side has the distribution of orthogonal q-Gaussian variables. The convergence in (3.9) follows from Theorem 8 □

Proof. (Theorem 8)
For this consider the following expansion for the left side of (3.6)

$$\mathbb{E}\left[\operatorname{tr}(H_\varepsilon)\right] = \frac{1}{|I_n|^{k/2}} \sum_{\alpha:[k]\to I_n} \mathbb{E}\left[J_\alpha^\varepsilon\right] \operatorname{tr}(\boldsymbol{\Psi}_\alpha) \tag{3.10}$$

The variables $\boldsymbol{\Psi}_R$, for $R \in I_n$, were introduced in (3.3). We are also using notation for products of non-commutative variables introduced in (2.5) and (2.6). We can split the sum in (3.10) as

$$\sum_{\alpha:[k]\to I_n} = \sum_{\substack{\alpha:[k]\to I_n \\ |\ker\alpha|<k/2}} + \sum_{\substack{\alpha:[k]\to I_n \\ |\ker\alpha|=k/2}} + \sum_{\substack{\alpha:[k]\to I_n \\ |\ker\alpha|>k/2}}.$$

If $|\ker\alpha| > k/2$ then $\ker\alpha$ has a block of size one, then $\mathbb{E}\left[J_\alpha^\varepsilon\right] = 0$.
For the other cases we need the following lemma

Lemma 2. *For every $\alpha : [k] \to I_n$ we have the following*

i) If $\ker\alpha$ has a block of odd size, then $\operatorname{Tr}(\boldsymbol{\Psi}_{\alpha(1)} \cdots \boldsymbol{\Psi}_{\alpha(k)}) = 0$.

ii) If every block in $\ker\alpha$ has even size, then $\boldsymbol{\Psi}_{\alpha(1)} \cdots \boldsymbol{\Psi}_{\alpha(k)} = \pm I$, where I is the identity matrix.

iii) For $\pi \in \mathcal{P}_2(k)$ with $\ker\alpha = \pi$ we have the identity

$$\operatorname{tr}(\boldsymbol{\Psi}_\alpha) = (-1)^{q_n cr(\pi) + \sum |\alpha(V)\cap\alpha(W)|},$$

where the sum is taken over all pairs $\{V, W\}$ of crossing blocks in π. We are using here notation (2.5) and (2.7). Also, for $Q, R \in I_n$ we denote by $Q \cap R$, the set of indices that Q and R have in common.

The proof for Lemma 2 can be found in Section 3.4.
For the case $|\ker\alpha| < k/2$, by Lemma 2 get the bound

$$\left| \frac{1}{|I_n|^{k/2}} \sum_{\substack{\alpha:[k]\to I_n \\ |\ker\alpha|<k/2}} \mathbb{E}\left[J_\alpha^\varepsilon\right] \operatorname{tr}\left(\boldsymbol{\Psi}_\alpha\right) \right| \le c_k \frac{|I_n|^{k/2-1}}{|I_n|^{k/2}} = \frac{c_k}{|I_n|}. \tag{3.11}$$

The constant c_k comes from the uniform bound condition on the random coefficients in (3.2). For the last part we will consider a random variable X_n with hypergeometric distribution, i.e. for every non negative integer s

$$\mathbb{P}(X_n = s) = \frac{\binom{q_n}{s}\binom{n-q_n}{q_n-s}}{\binom{n}{q_n}}, \text{ for } 0 \le s \le q_n. \tag{3.12}$$

Lemma 3. *For $\pi \in \mathcal{P}_2(k)$ we have the following identity*

$$\left((-1)^{q_n}\mathbb{E}\left[(-1)^{X_n}\right]\right)^{cr(\pi)} = \frac{1}{|I_n|^{k/2}} \sum_{\substack{\alpha:[k]\to I_n \\ \ker\alpha \ge \pi}} \operatorname{tr}(\boldsymbol{\Psi}_\alpha).$$

The proof for Lemma 3 can be found in Section 3.4.
For the case $|\ker\alpha| = k/2$ we can assume $\ker\alpha \in \mathcal{P}_2(k)$, otherwise $\ker\alpha$ has a block of size one, then $\mathbb{E}\left[J_\alpha^\varepsilon\right] = 0$. Also the condition $\ker\alpha \in \mathcal{P}_2(k)$ implies

$$\mathbb{E}\left[J_\alpha^\varepsilon\right] = \begin{cases} 1 & \text{if} \quad \ker\alpha \le \ker\varepsilon, \\ 0 & \quad\text{otherwise.} \end{cases}$$

This together with Lemma 3 yields

$$\begin{aligned} &\frac{1}{|I_n|^{k/2}} \sum_{\pi\in\mathcal{P}_2(k)} \sum_{\substack{\alpha:[k]\to I_n \\ \ker\alpha=\pi}} \mathbb{E}\left[J_\alpha^\varepsilon\right] \operatorname{tr}\left(\boldsymbol{\Psi}_\alpha\right) \\ = &\sum_{\substack{\pi\in\mathcal{P}_2(k) \\ \ker\alpha\le\ker\epsilon}} \left((-1)^{q_n}\mathbb{E}\left[(-1)^{X_n}\right]\right)^{cr(\pi)} - \frac{1}{|I_n|^{k/2}} \sum_{\substack{\pi\in\mathcal{P}_2(k) \\ \ker\alpha\le\ker\epsilon}} \sum_{\substack{\alpha:[k]\to I_n \\ \ker\alpha>\pi}} \operatorname{tr}\left(\boldsymbol{\Psi}_\alpha\right). \end{aligned} \tag{3.13}$$

With Lemma 2 we find a bound for the correction term

$$\left| \frac{1}{|I_n|^{k/2}} \sum_{\substack{\pi \in \mathcal{P}_2(k) \\ \ker \alpha \leq \ker \epsilon}} \sum_{\substack{\alpha:[k]\to I_n \\ \ker \alpha > \pi}} \operatorname{tr}(\mathbf{\Psi}_\alpha) \right| \leq \sum_{\substack{\pi \in \mathcal{P}_2(k) \\ \ker \alpha \leq \ker \epsilon}} \frac{|I_n|^{|\pi|-1}}{|I_n|^{k/2}} = \frac{(k-1)!!}{|I_n|}.$$

Combining equations (3.10), (3.11) and (3.13) we obtain

$$\mathbb{E}\left[\operatorname{tr}(H_\varepsilon)\right] = \sum_{\substack{\pi \in \mathcal{P}_2(k) \\ \ker \alpha \leq \ker \epsilon}} \left((-1)^{q_n}\mathbb{E}\left[(-1)^{X_n}\right]\right)^{cr(\pi)} + O(|I_n|^{-1})$$

It remains to show that $\mathbb{E}\left[(-1)^{X_n}\right]$ converges to a real number $q \in [-1,1]$, when n and q_n go to infinity. The next lemma shows that this is the case under some condition on $\mathbb{E}(X_n)$.

Lemma 4. *For a random variable X_n with hypergeometric distribution, as in (3.12), we have the following:*

i) The first moment of X_n is

$$\mathbb{E}\left[X_n\right] = \frac{q_n^2}{n}.$$

ii) If $\mathbb{E}\left[X_n\right] \to 0$ then $X_n \to \delta_0$ in distribution and then

$$\mathbb{E}\left[(-1)^{X_n}\right] \to 1.$$

iii) If $\mathbb{E}\left[X_n\right] \to \lambda < \infty$ then X_n converge in distribution to the Poisson distribution with parameter λ and then

$$\mathbb{E}\left[(-1)^{X_n}\right] \to e^{-2\lambda}.$$

iv) If $\mathbb{E}\left[X_n\right] \to \infty$ then

$$\mathbb{E}\left[(-1)^{X_n}\right] \to 0.$$

The proof of (i) is well known result, which can be consulted for example in [30]. For the proof of (ii) see Section 3.4. The proofs of (iii) and (iv) can be found in [11] and [12], respectively. □

3.3 Fluctuations

The classical cumulants are a family $(c_m)_{m\in\mathbb{N}}$ of multilinear functionals, given by

$$c_m(a_1,\dots,a_m) = \sum_{\sigma\in\mathcal{P}(m)} \mathbb{E}_\sigma\left[a_1,\dots,a_m\right]\mu(\sigma,1_m), \tag{3.14}$$

where $\mathbb{E}_\sigma$ stands for

$$\mathbb{E}_\sigma\left[a_1,\dots,a_m\right] = \prod_{\substack{B\in\sigma \\ B=\{i_1,\dots,i_r\}}} \mathbb{E}\left[a_{i_1}\cdots a_{i_r}\right],$$

and $\mu(\sigma,1_m) = (-1)^{|\sigma|-1}(|\sigma|-1)!$ is the Möbius function. This family of functionals characterizes tensor independence. See for example [28] for more details on the characterization of tensor independence, an also for background on Möbius functions.

In this section we will identify the convergence of $c_m(H_{\varepsilon_1},\dots,H_{\varepsilon_m})$, in a similar way as in Theorem 8. Theorem 9 is an extension of a result that originally appeared in [13].

Theorem 9. *Let $(H_k)_{k\in\mathbb{N}}$ be independent copies of the SYK model H_{n,q_n} with centered Gaussian random coefficients of variance one. For positive integers $m,k_1,\dots,k_m$ denote $T_1=[1,k_1], T_2=[1+k_1,k_1+k_2],\dots,T_m=[1+k_1+\cdots+k_{m-1},k_1+\cdots+k_m]$ and set*

$$k := k_1+\cdots+k_m \qquad \text{and} \qquad \theta=\{T_1,\dots,T_m\}.$$

Given a function $\varepsilon:[1,k]\to\mathbb{N}$, let us denote by ε_i the restriction of ε to T_i. Under the same assumptions on n and q_n as in Theorem 8, we have

$$\binom{n}{q_n}^{m-1} c_m\left(\mathrm{tr}(H_{\varepsilon_1}),\dots,\mathrm{tr}(H_{\varepsilon_m})\right) \xrightarrow{n\to\infty} \sum_{\substack{\pi'\in\mathcal{P}_2(k)\\ \pi'\le\theta}} \sum_{\substack{\pi\in\mathcal{P}_2(k)\\ \pi\vee\theta=1_k \\ \pi\le\ker\varepsilon \\ |\pi\vee\pi'|=\frac{k}{2}-m+1}} q^{cr(\pi')}.$$

The parameter q is obtained in the same way as in Theorem 8.

Proof. First expand the expression for the cumulant

$$c_m\left(\mathrm{tr}(H_{\varepsilon_1}),\dots,\mathrm{tr}(H_{\varepsilon_m})\right) = \frac{1}{|I_n|^{\frac{k}{2}}} \sum_{\substack{\alpha_i:T_i\to I_n\\ 1\le i\le m}} c_m(J^{\varepsilon_1}_{\alpha_1},\dots,J^{\varepsilon_m}_{\alpha_m})\,\mathrm{tr}(\boldsymbol{\Psi}_{\alpha_1})\cdots\mathrm{tr}(\boldsymbol{\Psi}_{\alpha_m}). \tag{3.15}$$

Using the formula for cumulants with products as entries and the Gaussianity of the random variables J we get

$$c_m(J_{\alpha_1}^{\varepsilon_1}, \ldots, J_{\alpha_m}^{\varepsilon_m}) = \sum_{\substack{\pi \in \mathcal{P}_2(k) \\ \pi \vee \theta = 1_k}} c_\pi \left(J_{\alpha(1)}^{(\varepsilon(1))}, \ldots, J_{\alpha(k)}^{(\varepsilon(k))} \right), \tag{3.16}$$

where for every collection $(\alpha_i : T_i \to I_n)_{1 \le i \le m}$ we define $\alpha : [1, k] \to I_n$ in the same way as ε.
If in the right side of equation (3.16) we have that $\pi \wedge \ker \alpha$ has a block of size one, then by independence of the J's we have

$$c_\pi \left(J_{\alpha(1)}^{(\varepsilon(1))}, \ldots, J_{\alpha(k)}^{(\varepsilon(k))} \right) = 0.$$

Then we can assume that every block in $\pi \wedge \ker \alpha$ have size bigger that one. Since π is a pair-partition this implies that $\pi = \pi \wedge \ker \alpha$ and then

$$\pi \le \ker \alpha. \tag{3.17}$$

In particular this implies that the value of $c_\pi \left(J_{\alpha(1)}^{(\varepsilon(1))}, \ldots, J_{\alpha(k)}^{(\varepsilon(k))} \right)$ do not depend on α

$$c_\pi \left(J_{\alpha(1)}^{(\varepsilon(1))}, \ldots, J_{\alpha(k)}^{(\varepsilon(k))} \right) = \begin{cases} 1 & \text{if} \quad \pi \le \ker \varepsilon \\ 0 & \text{if} \quad \pi \not\le \ker \varepsilon \end{cases}. \tag{3.18}$$

Now observe that from Lemma 2 we have that $\mathrm{tr}(\mathbf{\Psi}_{\alpha_1}) \cdots \mathrm{tr}(\mathbf{\Psi}_{\alpha_m}) \neq 0$, if and only if all blocks in $\ker \alpha_1, \ldots, \ker \alpha_m$ have even size. This implies that there exist pair-partitions $\pi_i' \in \mathcal{P}_2(T_i)$ such that $\pi_i' \le \ker \alpha_i$ for every $1 \le i \le m$ and as a consequence

$$\pi' := \bigcup_{i=1}^{m} \pi_i' \le \ker \alpha \wedge \theta. \tag{3.19}$$

Observe that π' is also an element of $\mathcal{P}_2(k)$. From Equations (3.17) and (3.19) we get that

$$\pi \vee \pi' \le \ker \alpha. \tag{3.20}$$

Conditions $\pi' \le \theta$ and $\pi \vee \theta = 1_k$ implies that

$$|\pi \vee \pi'| \le \frac{k}{2} - m + 1. \tag{3.21}$$

To see this we have to observe that π has at least $m - 1$ blocks such that each of then connect two different blocks from π'.

Using Equations (3.16) and (3.18) we can simplify Equation (3.15) in the following way:

$$\binom{n}{q_n}^{m-1} c_m\left(\operatorname{tr}(H_{\varepsilon_1}),\ldots,\operatorname{tr}(H_{\varepsilon_m})\right)$$

$$= \sum_{\alpha:[k]\to I_n} \sum_{\substack{\pi\in\mathcal{P}_2(k)\\ \pi\vee\theta=1_k}} c_\pi\left(J^{(\varepsilon(1))}_{\alpha(1)},\ldots,J^{(\varepsilon(k))}_{\alpha(k)}\right)\frac{\operatorname{tr}(\mathbf{\Psi}_{\alpha_1})\cdots\operatorname{tr}(\mathbf{\Psi}_{\alpha_m})}{|I_n|^{\frac{k}{2}-m+1}}$$

$$= \sum_{\substack{\pi\in\mathcal{P}_2(k)\\ \pi\vee\theta=1_k\\ \pi\le\ker\varepsilon}} \sum_{\alpha:[k]\to I_n} \frac{\operatorname{tr}(\mathbf{\Psi}_{\alpha_1})\cdots\operatorname{tr}(\mathbf{\Psi}_{\alpha_m})}{|I_n|^{\frac{k}{2}-m+1}}.$$

Now using Equations (3.19) and (3.20) we express the second sum above as

$$= \sum_{\substack{\pi\in\mathcal{P}_2(k)\\ \pi\vee\theta=1_k\\ \pi\le\ker\varepsilon}} \sum_{\substack{\pi'\in\mathcal{P}_2(k)\\ \pi'\le\theta}} \sum_{\substack{\alpha:[k]\to I_n\\ \ker\alpha\ge\pi\vee\pi'}} \frac{\operatorname{tr}(\mathbf{\Psi}_{\alpha_1})\cdots\operatorname{tr}(\mathbf{\Psi}_{\alpha_m})}{|I_n|^{\frac{k}{2}-m+1}}.$$

Using the restriction (3.21) we obtain that

$$= \sum_{\substack{\pi\in\mathcal{P}_2(k)\\ \pi\vee\theta=1_k\\ \pi\le\ker\varepsilon}} \sum_{\substack{\pi'\in\mathcal{P}_2(k)\\ \pi'\le\theta\\ |\pi\vee\pi'|=\frac{k}{2}-m+1}} \sum_{\substack{\alpha:[k]\to I_n\\ \ker\alpha\ge\pi\vee\pi'}} \frac{\operatorname{tr}(\mathbf{\Psi}_{\alpha_1})\cdots\operatorname{tr}(\mathbf{\Psi}_{\alpha_m})}{|I_n|^{\frac{k}{2}-m+1}} + O(|I_n|^{-1})$$

$$= \sum_{\substack{\pi\in\mathcal{P}_2(k)\\ \pi\vee\theta=1_k\\ \pi\le\ker\varepsilon}} \sum_{\substack{\pi'\in\mathcal{P}_2(k)\\ \pi'\le\theta\\ |\pi\vee\pi'|=\frac{k}{2}-m+1}} \sum_{\substack{\alpha:[k]\to I_n\\ \ker\alpha\ge\pi'}} \frac{\operatorname{tr}(\mathbf{\Psi}_{\alpha_1})\cdots\operatorname{tr}(\mathbf{\Psi}_{\alpha_m})}{|I_n|^{\frac{k}{2}}} + O(|I_n|^{-1}).$$

Now we can use Lemma 3 and rewrite the third sum in the following way

$$= \sum_{\substack{\pi\in\mathcal{P}_2(k)\\ \pi\vee\theta=1_k\\ \pi\le\ker\varepsilon}} \sum_{\substack{\pi'\in\mathcal{P}_2(k)\\ \pi'\le\theta\\ |\pi\vee\pi'|=\frac{k}{2}-m+1}} \left((-1)^{q_n}\mathbb{E}\left[(-1)^{X_n}\right]\right)^{cr(\pi')} + O(|I_n|^{-1})$$

Lemma 4 completes the proof.

$$\xrightarrow{n,q_n\to\infty} \sum_{\substack{\pi'\in\mathcal{P}_2(k)\\ \pi'\le\theta}} \sum_{\substack{\pi\in\mathcal{P}_2(k)\\ \pi\vee\theta=1_k\\ \pi\le\ker\varepsilon\\ |\pi\vee\pi'|=\frac{k}{2}-m+1}} q^{cr(\pi')}.$$

□

Remark 2. *In this remark we provide a description of the pair* (π', π) *of pair-partitions that appears in statement of Theorem 9. This description is close to the description of higher order freeness.*
First we observe that given $\pi', \pi \in \mathcal{P}_2(k)$ *with* $\pi' \leq \theta$ *and* $\pi \vee \theta = 1_k$ *we have that*

$$|\pi' \vee \pi| \leq \frac{k}{2} - m + 1.$$

An example of $\pi' \vee \pi$ *is represented in Figure 3.1.*

The partitions π' *and* π *can be also read from the diagram. For this we have*

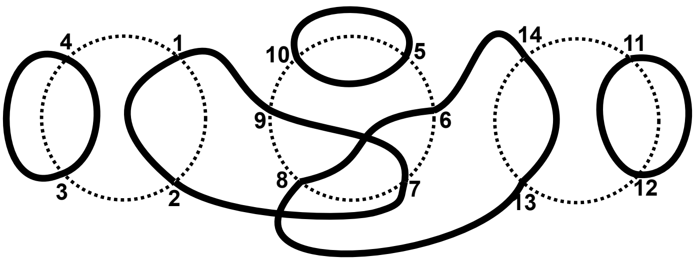

Figure 3.1: The doted circles represent the partition $\theta = \{(1,2,3,4),(5,6,7,8,9,10),(11,12,13,14)\}$. The loops represent the blocks of $\pi' \vee \pi$ where $\pi', \pi \in \mathcal{P}_2(k)$ with $\pi' \leq \theta$, $\pi \vee \theta = 1_k$ and $|\pi' \vee \pi| = \frac{14}{2} - 3 + 1 = 5$.

to differentiate between the internal and the external part. See Figure 3.2. The

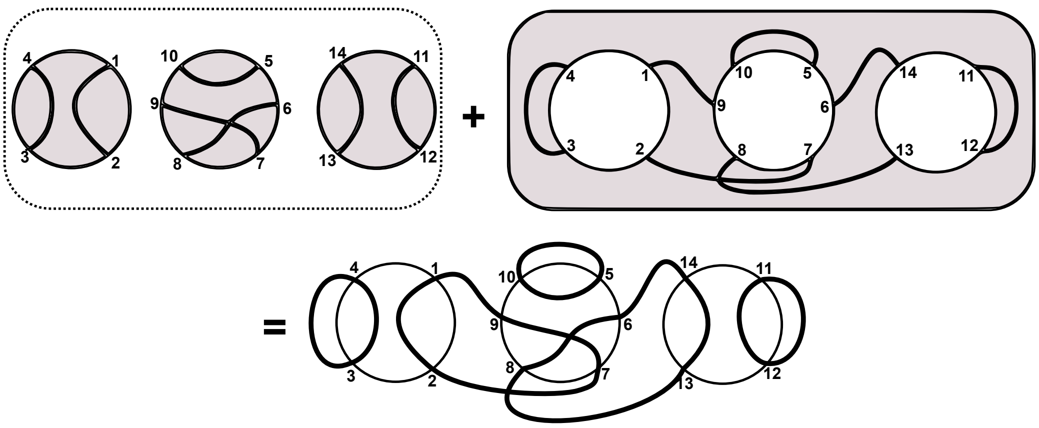

Figure 3.2: Separation of $\pi' \vee \pi$ into its internal π' and external π structures.

contribution of the pair (π', π) *depends only on the internal part, i.e., in the case*

of Figure 3.1 the contribution is $q^{cr(\pi')} = q$. *Observe that the representation of* π *also has a crossing but this is not considered in Theorem 9. We actually do not know how to give a mathematical definition of the external crossings in Figure 3.1.*
The description of the fluctuations for the case of independent copies of the SYK model is similar, we just have to add the information of the function $\varepsilon : [k] \to \mathbb{N}$ *by coloring the points on the circles. See Figure 3.3 for an example of the kind of partitions that appear in the description of* $c_m\left(\mathrm{tr}(H_1^2 H_2^2), \mathrm{tr}(H_2 H_1^2 H_2 H_1 H_2), \mathrm{tr}(H_2^3 H_1)\right)$.

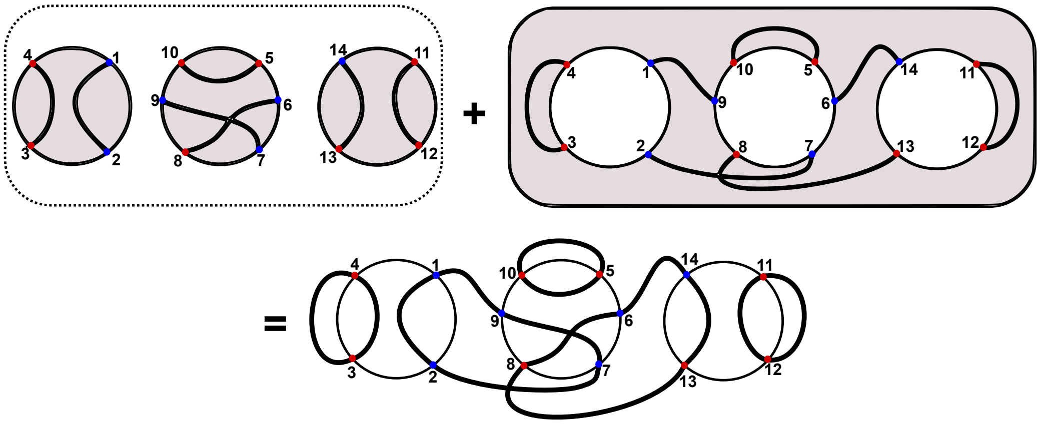

Figure 3.3: The external structure must connect blocks of the same color. For the internal structure there are no restrictions.

3.4 Proof of the lemmas

Proof of Lemma 1. For $R = (i_1, \ldots, i_{q_n}) \in I_n$, a direct computation yields

$$\begin{aligned}\mathbf{\Psi}_R^2 &= (\psi_{i_{q_1}} \cdots \psi_{i_{q_n}})(\psi_{i_{q_1}} \cdots \psi_{i_{q_n}}) i^{2\lfloor \frac{q_n}{2} \rfloor} \\ &= (-1)^{\lfloor \frac{q_n}{2} \rfloor}(-1)^{q_n-1}(-1)^{q_n-2} \cdots (-1)^{q_n-q_n} I \\ &= (-1)^{\lfloor \frac{q_n}{2} \rfloor}(-1)^{\frac{q_n(q_n-1)}{2}} \\ &= I,\end{aligned}$$

where I is the identity matrix. In the last equation we used that $\lfloor \frac{q_n}{2} \rfloor$ and $\frac{q_n(q_n-1)}{2}$ have the same parity.
Now let $R = (i_1, \ldots, i_{q_n})$ and $Q = (j_1, \ldots, j_{q_n})$ be in I_n, observe that

$$(\psi_{i_{q_1}} \cdots \psi_{i_{q_n}})\psi_{j_1} = \begin{cases} \psi_{j_1}(\psi_{i_1} \cdots \psi_{i_{q_n}})(-1)^{q_n} & \text{if} \quad j_1 \notin \{i_1, \ldots, i_{q_n}\} \\ \psi_{j_1}(\psi_{i_1} \cdots \psi_{i_{q_n}})(-1)^{q_n+1} & \text{if} \quad j_1 \in \{i_1, \ldots, i_{q_n}\} \end{cases}$$

Then by iteration we get

$$\begin{aligned}\mathbf{\Psi}_R\mathbf{\Psi}_Q &= i^{2\lfloor \frac{q_n}{2} \rfloor}(\psi_{i_1} \cdots \psi_{i_{q_n}})(\psi_{j_1} \cdots \psi_{j_{q_n}}) \\ &= i^{2\lfloor \frac{q_n}{2} \rfloor}(-1)^{q^2+|Q\cap R|}(\psi_{j_1} \cdots \psi_{j_{q_n}})(\psi_{i_1} \cdots \psi_{i_{q_n}}) \\ &= (-1)^{q_n+|Q\cap R|}\mathbf{\Psi}_Q\mathbf{\Psi}_R,\end{aligned}$$

where $|Q \cap R|$ stands for the number of common indices in Q and R. □

Proof of Lemma 2. For $\alpha : [k] \to I_n$ with $\ker \alpha = \{V_1, \cdots, V_r\}$, it follows from the anti-commutation relation (3.5) that

$$\mathbf{\Psi}_{\alpha(1)} \cdots \mathbf{\Psi}_{\alpha(k)} = \pm \mathbf{\Psi}_{\alpha(V_1)}^{|V_1|} \cdots \mathbf{\Psi}_{\alpha(V_r)}^{|V_r|}. \tag{3.22}$$

Notation $\alpha(V)$ was introduced in (2.7).

i) Because of property (3.4), we can assume without loss of generality, that the $|V_1|, \ldots, |V_r|$ are all odd, then it follows from (3.22) that

$$\mathbf{\Psi}_{\alpha(1)} \cdots \mathbf{\Psi}_{\alpha(k)} = \pm \mathbf{\Psi}_{\alpha(V_1)} \cdots \mathbf{\Psi}_{\alpha(V_r)}. \tag{3.23}$$

Form the definition of the variables $\mathbf{\Psi}_R$ and the relation $\psi_i\psi_j + \psi_j\psi_i = 2\delta_{ij}$, we get that

$$\mathbf{\Psi}_{\alpha(V_1)} \cdots \mathbf{\Psi}_{\alpha(V_r)} = \pm \psi_{i_1} \cdots \psi_{i_l}\, i^{r\lfloor \frac{q_n}{2} \rfloor},$$

for some different indices $1 \le i_1, \ldots, i_l \le n$. So, it is enough to check the product of different $\psi_1, \ldots, \psi_l$. For l even we have $\psi_1 \cdots \psi_l = -\psi_l\psi_1 \cdots \psi_{l-1}$, then applying the trace and using the trace property we get the result. For l odd we take ψ_x different from all $\psi_1, \dots, \psi_l$, this element always exist because n is always even. Then by the anti-commutation relations $\psi_1 \cdots \psi_l = -\psi_x \psi_1 \cdots \psi_l \psi_x$. Evaluating the trace in the last equation and applying the trace property we get the result.

ii) It follows from (3.4) and (3.22) that, if the $|V_1|, \dots, |V_r|$ are all even, then $\mathbf{\Psi}_{\alpha(V_i)}^{|V_i|} = I$.

iii) We now assume $\ker \alpha \in \mathcal{P}_2(k)$ and we want to determine the sign in (3.22). If $\ker \alpha \in NC_2(k) := \{\pi \in \mathcal{P}_2(k) | cr(\pi) = 0\}$, then $\mathbf{\Psi}_\alpha = I$. This comes from the iterative characterization of elements in $NC_2(k)$; see, for example, [28, Remark: 9.2]. If $\ker \alpha \notin NC_2(k)$, then we need to apply the relation (3.5) to each crossing in $\ker \alpha$, and reduce $\mathbf{\Psi}_{\alpha(1)} \cdots \mathbf{\Psi}_{\alpha(k)}$ to the identity. In this processes we obtain $(-1)^{q_n + |\alpha(V) \cap \alpha(W)|}$ for each pair $\{V, W\}$ of crossing blocks in $\ker \alpha$.

□

Proof of Lemma 3. Consider the classical probability space $(\Omega_n, \mathcal{F}_n, \mathbb{P}_n)$, where

$$\Omega_n : = \{\omega : [k] \to I_n | \ \ker \omega \geq \pi\},$$

$\mathcal{F}_n$ is the power set of Ω_n, and $\mathbb{P}_n$ the counting measure. For each pair of different blocks $\{V, W\}$ in π define the random variable

$$X_{VW}(\omega) = |\omega(V) \cap \omega(W)|.$$

Then from Lemma 2 we get

$$\begin{aligned} \frac{1}{|I_n|^{k/2}} \sum_{\substack{\alpha:[k]\to I_n \\ \ker \alpha \geq \pi}} \frac{\mathrm{Tr}(\mathbf{\Psi}_{\alpha(1)} \cdots \mathbf{\Psi}_{\alpha(k)})}{2^{n/2}} &= \frac{(-1)^{q_n cr(\pi)}}{|I_n|^{k/2}} \sum_{\substack{\alpha:[k]\to I_n \\ \ker \alpha \geq \pi}} (-1)^{\sum X_{VW}(\alpha)} \\ &= (-1)^{q_n cr(\pi)} \mathbb{E}\left[(-1)^{\sum X_{VW}}\right], \end{aligned} \tag{3.24}$$

where the sum $\sum X_{VW}$ is taken over all crossing pairs $\{V, W\}$ of blocks in π. For each block $V \in \pi$ define the random variable $X_V(\omega) := \omega(V)$. Notice that $\{X_V\}_{V \in \pi}$ is a family of independent random variables with uniform distribution on I_n, and $X_{VW} = |X_V \cap X_W|$. It follows from the symmetric definition of X_{VW} that these variables are identically distributed for different blocks $V \neq W$. For

$r \in \{0, 1, 2, \cdots, q_n\}$ and different blocks V, W we have

$$\begin{aligned} \mathbb{P}(X_{VW} = r) &= \frac{1}{|I_n|} \sum_{R \in I_n} \mathbb{P}(X_{VW} = r | X_V = R) \\ &= \frac{1}{|I_n|} \sum_{R \in I_n} \frac{\binom{q_n}{r}\binom{n-q_n}{q_n-r}}{\binom{n}{q_n}} \\ &= \frac{\binom{q_n}{r}\binom{n-q_n}{q_n-r}}{\binom{n}{q_n}}, \end{aligned} \tag{3.25}$$

so the variables X_{VW} have hypergeometric distribution (3.25). Now from the independence of the X_V, it follows that for different blocks $V_1, \ldots, V_4$ the variables $X_{V_1V_2}$ and $X_{V_3V_4}$ are independent. It also follows from the independence of the X_V that X_{VW} and X_{WZ} are independent given $\{X_W = R\}$ for some $R \in I_n$.Then we have

$$\begin{aligned} \mathbb{P}(X_{VW} = r, X_{WZ} = s) &= \frac{1}{|I_n|} \sum_{R \in I_n} \mathbb{P}(X_{VW} = r, X_{WZ} = s | X_W = R) \\ &= \frac{1}{|I_n|} \sum_{R \in I_n} \frac{\binom{q_n}{r}\binom{n-q_n}{q_n-r}}{\binom{n}{q_n}} \frac{\binom{q_n}{s}\binom{n-q_n}{q_n-s}}{\binom{n}{q_n}} \\ &= \mathbb{P}(X_{VW} = r)\mathbb{P}(X_{WZ} = s). \end{aligned}$$

So, the variables $\{X_{VW}\}_{V,W \in \pi,\, V \neq W}$ are independent. The statement of the lemma follows now from (3.24). □

Proof of Lemma 4. We only have to prove part (ii). For this observe that

$$\mathbb{E}\left[(-1)^{X_n}\right] = 1 + \sum_{s=1}^{q_n} (-1)^s \mathbb{P}(X_n = s) \to 1.$$

The convergence follows from the estimate

$$\begin{aligned} \left| \sum_{s=1}^{q_n} (-1)^s \mathbb{P}(X_n = s) \right| &\leq \sum_{s=1}^{q_n} \mathbb{P}(X_n = s) \\ &= \mathbb{P}(X_n > 0) \leq \mathbb{E}[X_n] \to 0. \end{aligned}$$

Also $\mathbb{P}(X_n > 0) \to 0$ implies $X_n \to \delta_0$ in distribution. □

Chapter 4

Fluctuation for matrices with q-Gaussian entries

In this chapter we study fluctuations of another matrix model that approximate semicircular variables, namely, hermitian matrices with q-Gaussian entries. Also, motivated by the recent work of Diaz et al [1] we extend the results for global fluctuations in this model, to global fluctuations for block matrices with q-Gaussian entries.

Hermitian matrices with semicircular entries were considered by Voiculescu [38] in the beginnings of free probability. He showed that hermitian matrices with free semicircular entries are themselves semicirculars. It was observed later by Shlyakhtenko [34] that this result is still true when we replace free semicircular entries by orthonormal q-Gaussian (or q-semicircular) entries. In particular, the asymptotic limit distribution of this kind of matrices is also semicircular.

Our motivation for the study of hermitian matrices with q-Gaussian entries comes from the description of the fluctuations of GUE matrices in terms of second order freeness. This description was given by Mingo-Nica [24] in terms of annular non-crossing permutations. The annular non-crossing permutations involved in the description can be pictured as non-crossing annular pair partitions. In spite of the fact that annular non-crossing pair partitions have no crossings when they are drawn in the annulus, they have crossings when they are drawn in a linear way. See figure 4.1 for an example.

We have seen so far that q-Gaussian variables are described in terms of pair partitions and their crossings. This suggests that the global fluctuations for hermitian matrices with q-Gaussian entries may depend on the parameter q. This situation differs from the result of Shlyakhtenko [34] about the asymptotic distribution of matrices with q-Gaussian entries, where the parameter q did not play any role in the limit distribution. We clarify this dependency in Theorem 11. This result is used to understand global fluctuations for block matrices with

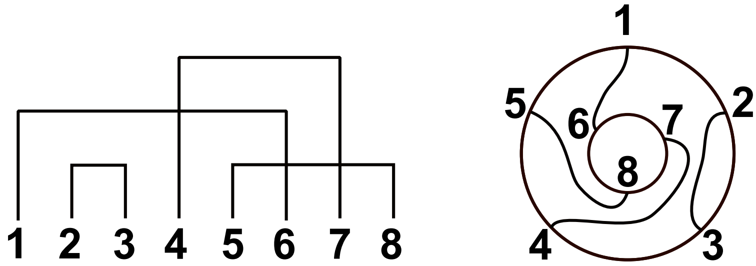

Figure 4.1: Annular $(5,3)$ and linear representations of the partition $\pi = \{(1,6),(2,3),(4,7),(5,8)\}$. The linear representation (right) has three crossings but the annular representation (left) has none.

q-Gaussian entries in Theorem 12.

4.1 Matrix model

Let $(\mathcal{A},\varphi)$ and $(\mathcal{S},\phi)$ be non-commutative probability spaces.

Notation 6. *(1) For $p, N \geq 1$ and $1 \leq i,j \leq N$ let $x_{i,j}^{(p)} \in \mathcal{A}$ be a family of q-Gaussian variables with covariance given by*

$$\varphi(x_{ij}^{(p)} x_{kl}^{(r)}) = \delta_{il}\delta_{jk}\sigma(p,r), \tag{4.1}$$

for some fixed covariance function

$$\sigma : \mathbb{N}^2 \to \mathbb{C}. \tag{4.2}$$

(2) The object of study in this chapter is the following matrix model

$$X^{(p)} := \frac{1}{\sqrt{N}} \left(x_{i,j}^{(p)}\right)_{i,j=1}^{N}, \qquad p \geq 1. \tag{4.3}$$

(3) For $p \geq 1$ let $S_p \in \mathcal{S}$ be a semicircular family covariance σ as in (4.2), i.e.

$$\phi(S_p S_r) = \sigma(p,r), \qquad \textit{for} \qquad p,r \geq 1. \tag{4.4}$$

The matrices $X^{(p)}$ for $p \geq 1$ are considered in the non-commutative probaility space

$$\left(\mathbb{M}_N(\mathcal{A}), \varphi \otimes \frac{\mathrm{Tr}}{N}\right).$$

Observe that for $q = 1$ the distribution of the matrix model defined in (4.3), coincides with the distribution of the GUE. The standard way to construct matrices as in (4.3) is by imitating the construction of GUE matrices. See Definition 11. But replacing independent Gaussian by orthogonal q-Gaussian. In particular the matrices (4.3) are selfadjoint.
It was shown by Shlyakhtenko [34], that the matrices defined in (4.3) converge to $(S_p)_{p\geq 1}$. For completeness, we present here an elementary proof, which we use it as a guideline for the generalization in the next section.

Theorem 10. *The joint distribution of $X^{(p)}$ for $p \geq 1$, converges to the semi-circular family $(S_p)_{p\geq 1}$ with covariance as in (4.4), i.e.,*

$$\varphi\left(\frac{\mathrm{Tr}\left(X^{(p_1)}\cdots X^{(p_n)}\right)}{N}\right) \xrightarrow{N\to\infty} \sum_{\pi\in NC_2(n)} \phi_\pi(S_{p_1},\ldots,S_{p_n}),$$

where ϕ_π is the multiplicative extension of ϕ, more precisely

$$\phi_\pi(S_{p_1},\ldots,S_{p_n}) = \prod_{(l,r)\in\pi} \sigma(p_l,p_r).$$

Proof. Let γ be the cycle permutation $(1,2,\ldots,n)$. Observe that

$$\begin{aligned}
\varphi\left(\frac{\mathrm{Tr}\left(X^{(p_1)}\cdots X^{(p_n)}\right)}{N}\right) &= \frac{1}{N^{\frac{n}{2}+1}} \sum_{i_1,\ldots,i_n=1}^{N} \varphi\left(x^{(p_1)}_{i_1,i_2}\cdots x^{(p_n)}_{i_n,i_1}\right) \\
&= \frac{1}{N^{\frac{n}{2}+1}} \sum_{i_1,\ldots,i_n=1}^{N} \sum_{\pi\in\mathcal{P}_2(n)} q^{cr(\pi)} \prod_{(l,r)\in\pi} \varphi\left(x^{(p_1)}_{i_l,i_{\gamma(l)}} x^{(p_r)}_{i_r,i_{\gamma(r)}}\right) \\
&= \frac{1}{N^{\frac{n}{2}+1}} \sum_{\pi\in\mathcal{P}_2(n)} q^{cr(\pi)} \sum_{i_1,\ldots,i_n=1}^{N} \prod_{(l,r)\in\pi} \delta_{i_l,i_{\gamma(r)}}\delta_{i_{\gamma(l)},i_r}\sigma\left(p_l,p_r\right) \\
&= \frac{1}{N^{\frac{n}{2}+1}} \sum_{\pi\in\mathcal{P}_2(n)} q^{cr(\pi)} N^{\#(\gamma\pi)} \prod_{(l,r)\in\pi} \sigma(p_l,p_r) \qquad (4.5) \\
&\xrightarrow{N\to\infty} \sum_{\pi\in NC_2(n)} \prod_{(l,r)\in\pi} \sigma(p_l,p_r).
\end{aligned}$$

In equation 4.5 we used the identity

$$\sum_{i_1,\ldots,i_n=1}^{N} \prod_{(l,r)\in\pi} \delta_{i_l,i_{\gamma(r)}}\delta_{i_r,i_{\gamma(l)}} = N^{\#(\gamma\pi)}.$$

Here we are using the canonical identification of pair partitions with idempotent permutations in $[1, n]$. $\#(\pi\gamma)$ stands for the number of cycles in the permutation $\pi\gamma$. The equation says: the product is non zero only when the indices are constant on the cycles of $\gamma\pi$, i.e., for every $(l, r) = (l, \pi(l)) \in \pi$

$$\delta_{i_l, i_{\gamma(r)}} = \delta_{i_l, i_{\gamma(\pi(l))}} \Rightarrow i_l = i_{\gamma\pi(l)}.$$

For taking the limit we used the triangle inequality which asserts that

$$\#(\pi) + \#(\pi^{-1}\gamma) \leq n + 1,$$

for all permutations $\pi \in S_n$. See Equation (2.9) In our case $\#(\pi) = \frac{n}{2}$ and $\pi^{-1} = \pi$, then

$$\#(\gamma\pi) \leq \frac{n}{2} + 1.$$

So in (4.5) the only terms that survive in the limit $N \to \infty$ are those pair partitions π that satisfy $\#(\gamma\pi) = \frac{n}{2} + 1$. Thus according to Biane's geodesic condition (2.8), only non crossing pair partitions survive in the limit.

□

Remark 3. *In particular, if the covariance is diagonal, i.e.,* $\sigma(p, r) = \delta_{p,r}$*, equation (4.5) simplifies according to*

$$\prod_{(l,r)\in\pi} \sigma(h(l), h(r)) = \begin{cases} 1 & \text{if} \quad \pi \leq \ker h \\ 0 & \text{otherwise} \end{cases}.$$

In this case we have convergence to a free semicircular family.

4.2 Fluctuations

In this section we analyze the fluctuations of the matrix model introduced in 4.3. Theorem 11 resembles the result from Mingo-Nica [24] of the fluctuations for GUE matrices.

The main difference between fluctuations of GUE matrices and fluctuations of matrices with q-semicircular entries is that, in the later case, we have to consider annular and linear representations of pair partitions. More precisely, in Theorem 11 fluctuations are expressed as a sum over non-crossing annular pair partitions as in [24] and weigh each term in the sum according to the number of crossings in its linear representation.

We illustrate this with an example. In Figure 4.2 we showed three different representations of pair partitions of four elements. We see that the partition represented in the third column has a crossing when it is represented in a linear

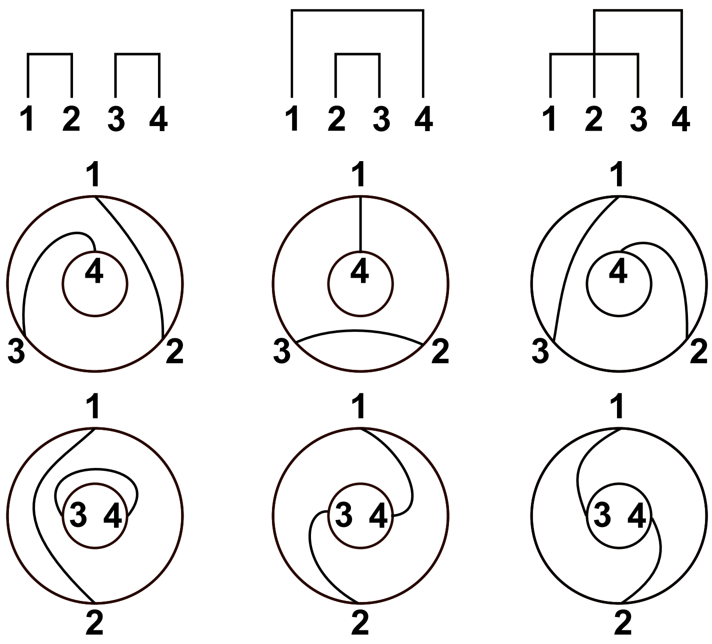

Figure 4.2: Linear, $NC_2(3,1)$ and $NC_2(2,2)$ representations for $\mathcal{P}_2(4)$. Observe that in the third column the annular representations have no crossings whereas the linear have one crossing.

way (firs row), but has no crossings when it is represented as an element in $NC_2(3,1)$ or $NC_2(2,2)$.
In general, not every pair partition in $\mathcal{P}_2(k)$ can be represented as an element in $NC_2(k_1,k_2)$ with $k_1+k_2=k$. For an example see Figure 4.3. In this work, we deal only with non-crossing annular pair partitions. However, the description of annular crossings and their potential relations with non-commutative probability remain to be an open question.

In this section we use the classical cumulants in $(\mathcal{A},\varphi)$

$$c_m(a_1,\ldots,a_m)=\sum_{\sigma\in\mathcal{P}(m)}\varphi_\sigma(a_1,\ldots,a_m)\mu(\sigma,1_m) \tag{4.6}$$

where $\mu(\sigma,1_m)=(-1)^{|\sigma|-1}(|\sigma|-1)!$ is the Möbius function in the lattice $\mathcal{P}(m)$, to analyze traces of products of the matrices $X^{(p)}$, defined in (4.3).

Notation 7. *(1) For positive integers $m,k_1,\ldots,k_m$ we set $k:=k_i+\cdots+k_m$ and $k_0=0$. Consider the following partition*

$$\theta=\{T_1,\ldots,T_m\}, \tag{4.7}$$

where $T_1=[1,k_1],T_2=[1+k_1,k_1+k_2],\ldots,T_m=[1+k_1+\cdots+k_{m-1},k]$.

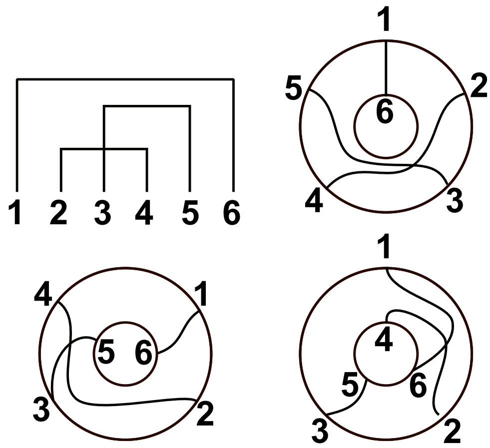

Figure 4.3: Different representations for the same pair partition. The four representations have crossings.

(2) Given a function

$$\varepsilon : [1, k] \to \mathbb{N}$$

and $1 \leq i \leq m$, we denote by ε_i the restriction of ε to T_i.

(3) The shorthand notation for products

$$\begin{aligned} X^{\varepsilon} &= X^{(\varepsilon(1))} \cdots X^{(\varepsilon(k))} \\ &= X^{\varepsilon_1} \cdots X^{\varepsilon_m} \end{aligned}$$

where

$$X^{\varepsilon_i} := X^{(\varepsilon_i(k_1+\cdots+k_{i-1}+1))} \cdots X^{(\varepsilon_i(k_1+\cdots+k_i))},$$

is also used in the rest of this chapter.

The following Theorem follows ideas from second order free probability. What we are doing here is identifying the second order distribution of the matrices $X^{(p)}$ ($p \in \mathbb{N}$).

Theorem 11. *Let m be a positive integer and $k_1, \ldots, k_m, k$ and $\varepsilon : [i, k] \to \mathbb{N}$ as above. The asymptotics for cumulants in the variables $X^{(p)}$ for $p \geq 1$ are given by*

$$N^{2-m} c_m \left(\mathrm{Tr}(X^{\varepsilon_1}) \cdots \mathrm{Tr}(X^{\varepsilon_m}) \right) \xrightarrow{N \to \infty} \sum_{\pi \in NC_2(k_1, \ldots, k_m)} q^{cr(\pi)} \phi_\pi(S_{\varepsilon(1)}, , \ldots, S_{\varepsilon(k)}),$$

where

$$\phi_\pi(S_{p_1}, \ldots, S_{p_n}) = \prod_{(l,r) \in \pi} \sigma(\varepsilon(l), \varepsilon(r)),$$

and $(S_p)_{p\geq 1}$ *is a semicircular family with covariance* σ. *See Notation 6.*

Proof. Let γ be the permutation $\gamma = (1, \cdots, k_1)(k_1+1, \cdots, k_1+k_2)\cdots(k_1+\cdots+k_{m-1}+1, \cdots, k)$. First we find an expression for the moment

$$\varphi\left(\mathrm{Tr}(X^{\varepsilon_1})\cdots\mathrm{Tr}(X^{\varepsilon_m})\right) = \frac{1}{N^{k/2}}\sum_{i_1,\ldots,i_k=1}^{N}\varphi(x^{(\varepsilon(1))}_{i_1 i_{\gamma(1)}}\cdots x^{(\varepsilon(k))}_{i_k i_{\gamma(k)}})$$
$$= \sum_{\pi\in\mathcal{P}_2(k)} q^{cr(\pi)} N^{\#(\gamma\pi)-k/2}\prod_{(l,r)\in\pi}\sigma(\varepsilon(l),\varepsilon(r)) \qquad (4.8)$$

Now for every partition $\sigma\in\mathcal{P}(m)$ we define $\hat{\sigma}\in\mathcal{P}(k)$ by

$$\hat{\sigma} = \{\bigcup_{s\in B} T_s | B\in\sigma\}. \qquad (4.9)$$

The sets T_i were defined in (4.3). Then it follows directly from equation (4.8) that for $\sigma\in\mathcal{P}(m)$

$$\varphi_\sigma\left(\mathrm{Tr}(X^{\varepsilon_1}),\ldots,\mathrm{Tr}(X^{\varepsilon_m})\right) = \prod_{B\in\sigma}\varphi\left(\prod_{b\in B} X^{\varepsilon_b}\right) \qquad (4.10)$$
$$= \sum_{\substack{\pi\in\mathcal{P}_2(k)\\ \pi\leq\hat{\sigma}}} q^{cr(\pi)} N^{\#(\gamma\pi)-k/2}\prod_{(l,r)\in\pi}\sigma(\varepsilon(l),\varepsilon(r)).$$

The product $\prod_{b\in B} X^{\varepsilon_b}$ in the right side of equation (4.10) has to be understood as an ordered product according to the order in block B. Then from the definition of cumulants we have

$$c_m\left(\mathrm{Tr}(X^{\varepsilon_1}),\ldots,\mathrm{Tr}(X^{\varepsilon_m})\right) = \sum_{\sigma\in\mathcal{P}(m)}\varphi_\sigma\left(\mathrm{Tr}(X^{\varepsilon_1}),\ldots,\mathrm{Tr}(X^{\varepsilon_m})\right)\mu(\sigma,1_m)$$
$$= \sum_{\sigma\in\mathcal{P}(m)}\mu(\sigma,1_m)\sum_{\substack{\pi\in\mathcal{P}_2(k)\\ \pi\leq\hat{\sigma}}} q^{cr(\pi)} N^{\#(\gamma\pi)-k/2}\prod_{(l,r)\in\pi}\sigma(\varepsilon(l),\varepsilon(r))$$
$$= \sum_{\pi\in\mathcal{P}_2(k)} q^{cr(\pi)} N^{\#(\gamma\pi)-k/2}\prod_{(l,r)\in\pi}\sigma(\varepsilon(l),\varepsilon(r))\sum_{\substack{\sigma\in\mathcal{P}(m)\\ \pi\leq\hat{\sigma}}}\mu(\sigma,1_m) \qquad (4.11)$$

For every partition $\pi\in\mathcal{P}(k)$ we define a partition in $\tilde{\pi}\in\mathcal{P}(m)$ in the following way: The indices $i,j\in[m]$ lie in the same block of $\tilde{\pi}$ if and only if there exists a block of π that connects $[1+k_1+\cdots,+k_{i-1},k_1+\cdots+k_i]$ with $[1+k_1+\cdots,+k_{j_1},k_1+\cdots+k_j]$. Notice that this is the inverse of the mapping $\sigma\mapsto\hat{\sigma}$. Now observe that for $\pi\in\mathcal{P}_2(k)$ and $\sigma\in\mathcal{P}(m)$ we have

$$\pi\leq\hat{\sigma} \text{ iff } \tilde{\pi}\leq\sigma.$$

Now we can use the fundamental property of Möbius functions:

$$\sum_{\substack{\sigma\in\mathcal{P}(m)\\ \tilde{\pi}\le\sigma}} \mu(\sigma,1_m) = \begin{cases} 1 & \text{if} \quad \tilde{\pi}=1_m, \\ 0 & \text{otherwise.} \end{cases}$$

Then equation (4.11) becomes

$$c_m\left(\mathrm{Tr}(X^{\varepsilon_1}),\dots,\mathrm{Tr}(X^{\varepsilon_m})\right) = \sum_{\substack{\pi\in\mathcal{P}_2(k)\\ \tilde{\pi}=1_m}} q^{cr(\pi)} N^{\#(\gamma\pi)-k/2} \prod_{(l,r)\in\pi} \sigma(h(l),h(r)). \quad (4.12)$$

From [24] we take the following inequality as valid for every permutation $\tau\in S_k$

$$\#(\tau)+\#(\gamma\tau^{-1}) \le k-m+2\#(\gamma\vee\tau), \quad (4.13)$$

where $\#(\gamma\vee\tau)$ is the number of orbits into which $\{1,\dots,k\}$ is split under the joint action of τ and γ. In the context of (4.12), inequality (4.13) means that

$$\#(\gamma\pi)-\frac{k}{2} \le 2\,\#(\gamma\vee\pi)-m.$$

The condition that $\tilde{\pi}=1_m$ means that γ and π act transitively on $[k]$, then $\#(\gamma\vee\pi)=1$, so we get

$$\#(\gamma\pi)-\frac{k}{2} \le 2-m. \quad (4.14)$$

It is possible to interpret the set of pair partitions with the set of permutations of order two

$$\mathcal{P}_2(k) \cong \{\pi\in S_k \mid \pi^2=e\}.$$

According with inequality (4.14). If we multiplying both sides of (4.12) by $N^{-(2-m)}$ then, the only permutations that remain in the large N limit are those permutations $\pi\in\mathcal{P}(k)$ such that

$$\#(\gamma\pi)-\frac{k}{2} = 2-m.$$

Those are the annular non-crossing permutations $NC_2(k_1,\dots,k_m)$. □

Remark 4. *(1) For $q=1$, the Theorem 11 recovers the result of Mingo-Nica [24] for fluctuations of GUE matrices.*

(2) In the case $q=0$. The partitions in $NC_2(k_1,\dots,k_m)$ that contribute in the limit are those linear and annular non-crossing. This kind of partitions where identified by Diaz et al [1] under the name of " double line pairings".

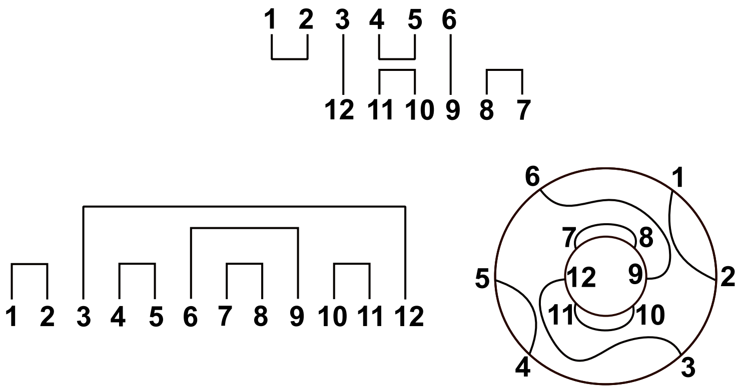

Figure 4.4: On top is display a double line pair partition in twelve points. On the second row, the same pair partition with its linear and annular representations. Observe that all representations have no crossings.

In particular the second order Cauchy transform for the matrix model $X^{(p)}$ defined in Notation 6

$$G_{X^{(p)}}(z,w) = \sum_{k,l\geq 1}^{\infty} \frac{\alpha_{k,l}}{z^{k+1}w^{l+1}},$$

where

$$\begin{aligned}\alpha_{k,l} &= \lim_{N\to\infty} Cov\left(\left(X^{(p)}\right)^k, \left(X^{(p)}\right)^l\right) \\ &= \left| NC_2(k+l) \bigcap NC_2(k,l) \right|.\end{aligned}$$

is the number of double line pairings in two lines with k and l points. Figure 4 for an example of a double line pair partition.

4.3 Block matrices with q-Gaussian entries

In this section we use Theorems 10 and 11 to give a description for the global fluctuations of block matrices with q-Gaussian entries. The goal is to obtain a block matrix version of Theorem 11. At the moment, we do not have a theory of operator-valued second order freeness. We hope that this work may give some insight on this line of research.

We keep working with the ncpss $(\mathcal{A}, \varphi)$ and $(\mathcal{S}, \phi)$. But we relabel the q-Gaussian variables and the semicircular family, in a way that is more convenient to work with block matrices.

Notation 8. *(1) Let $x_{i,j}^{(k,l)} \in \mathcal{A}$ for $(i, j, k, l \in \mathbb{N})$ be a family of q-Gaussian variables with covariance given by*

$$\varphi(x_{i,j}^{(k,l)} x_{v,w}^{(r,s)}) = \delta_{iw}\delta_{jv}\sigma(k, l; r, s), \tag{4.15}$$

for some covariance function σ.

(2) Let d be a positive integer. For each pair $1 \leq k, l \leq d$ we define the matrices

$$X^{(N)}(k, l) = \frac{1}{\sqrt{N}} \left(x_{i,j}^{(k,l)}\right)_{i,j=1}^{N},$$

and the $d \times d$-block matrix $X \in \mathbb{M}_{Nd}(\mathcal{A})$ with blocks $X^{(N)}(k, l) \in \mathbb{M}_d(\mathcal{A})$, i.e., for matrix units $e_{i,j}$ in $\mathbb{M}_d(\mathbb{C})$ we have that

$$X_N = \sum_{k,l=1}^{d} e_{k,l} \otimes X^{(N)}(k, l). \tag{4.16}$$

(3) Let $S_{p,r} \in \mathcal{S}$ for $p, r \geq 1$ be a semicircular family with covariance

$$\phi(S_{k,l}S_{r,s}) = \sigma(k, l; r, s). \tag{4.17}$$

for the same covariance function σ as in (4.15).

(4) Consider the operator-valued ncps $(\mathbb{M}_d(\mathcal{S}), E, \mathbb{M}_d(\mathbb{C}))$, where we identified

$$\mathbb{M}_d(\mathcal{S}) \cong \mathbb{M}_d(\mathbb{C}) \otimes \mathcal{S}$$

and defined

$$E := id_{\mathbb{M}_d(\mathbb{C})} \otimes \phi.$$

Let $S \in \mathbb{M}_d(\mathcal{S})$ be the operator-valued semicircular element

$$S := \begin{pmatrix} S_{1,1} & \cdots & S_{1,d} \\ \vdots & \ddots & \vdots \\ S_{1,1} & \cdots & S_{dd} \end{pmatrix}. \tag{4.18}$$

We have seen in Theorem 10 that for every $1 \leq k, l \leq d$

$$X^{(N)}(k,l) \xrightarrow{N\to\infty} S_{k,l}$$

when the matrices $X^{(N)}(k,l)$ are considered as elements in $\big(\mathbb{M}_N(\mathcal{A}), \varphi \otimes \frac{\mathrm{Tr}}{N}\big)$. From here we can conclude that

$$X_N = \begin{pmatrix} X_N(1,1) & \dots & X_N(1,d) \\ \vdots & \ddots & \vdots \\ X_N(d,1) & \dots & X_N(d,d) \end{pmatrix} \xrightarrow{N\to\infty} \begin{pmatrix} S_{1,1} & \cdots & S_{1,d} \\ \vdots & \ddots & \vdots \\ S_{1,1} & \cdots & S_{d,d} \end{pmatrix} = S,$$

in distribution, when X_N is considered as an element in the operator-valued ncps

$$\left(\mathbb{M}_d\left(\,\mathbb{M}_N(\mathcal{A})\,\right),\ E_\varphi \circ \frac{1}{N} E_{\mathrm{Tr}},\ \mathbb{M}_d(\mathbb{C})\right).$$

Where we identify

$$\mathbb{M}_d\left(\,\mathbb{M}_N(\mathcal{A})\,\right) \cong \mathbb{M}_d(\mathbb{C}) \otimes \mathbb{M}_N(\mathbb{C}) \otimes \mathcal{A},$$

and define the conditional expectations

$$\begin{aligned} E_{\mathrm{Tr}} &:= id_{\mathbb{M}_d(\mathbb{C})} \otimes \mathrm{Tr} \otimes 1_{\mathcal{A}}, \\ E_\varphi &:= id_{\mathbb{M}_d(\mathbb{C})} \otimes \varphi. \end{aligned}$$

The following diagram summarizes the operator-valued ncps that we use in this section

$$\begin{array}{ccc} \mathbb{M}_d(\mathbb{M}_N(\mathcal{A})) & & M_d(\mathcal{S}) \\ {\scriptstyle \frac{1}{N}E_{\mathrm{Tr}}}\big\downarrow & & \big\downarrow{\scriptstyle E} \\ \mathbb{M}_d(\mathcal{A}) & \xrightarrow[E_\varphi]{} & M_d(\mathbb{C}) \end{array}$$

In some sense, there is also an horizontal arrow from $\mathbb{M}_d(\mathbb{M}_N(\mathcal{A}))$ to $M_d(\mathcal{S})$, given by the convergence $X_N \xrightarrow{N\to\infty} S$.
Now we turn our attention to the operator-valued fluctuations of X_N. First we have to define classical cumulants in an operator-valued ncps. For this purpose we follow [28] and introduce the following notation.

Notation 9. *(1) For every $n \geq 1$ and for $\pi \in \mathcal{P}(n)$*

$$\begin{array}{rccc} \varphi_\pi : & \mathcal{A}^n & \longrightarrow & \mathbb{C} \\ & (a_1, \dots, a_n) & \longmapsto & \displaystyle\prod_{V\in\pi} \varphi(V)\,[a_1, \dots, a_n]\,, \end{array}$$

where for $V = \{i_1 < \cdots < i_s\}$ we denote

$$\varphi(V)\,[a_1, \dots, a_n] := \varphi(a_{i_1} \cdots a_{i_s}).$$

(2) We introduce here a matrix-valued version of (9), in the following way: for every $n \geq 1$ and $\pi \in \mathcal{P}(n)$ we define

$$\begin{array}{rccc} E_{\varphi_\pi}: & \mathbb{M}_d(\mathcal{A})^n & \longrightarrow & \mathbb{M}_d(\mathbb{C}) \\ & (Y^{(1)},\dots,Y^{(n)}) & \longmapsto & \left(\sum_{i_1,\dots,i_{n-1}=1} \varphi_\pi \left(Y^{(1)}_{i,i_1}, \cdots, Y^{(n)}_{i_{n-1},j} \right) \right)_{i,j=1}^{d} \end{array}$$

Remark 5. *(1) Observe that E_{φ_π} is the natural generalization of*

$$E_\varphi \left(Y^{(1)} \cdots Y^{(n)} \right) = \left(\sum_{i_1,\dots,i_{n-1}=1} \varphi \left(Y^{(1)}_{i,i_1} \cdots Y^{(n)}_{i_{n-1},j} \right) \right)_{i,j=1}^{d}.$$

(2) In the case that π is a non crossing partition, it is possible to write $E_\pi \left(Y^{(1)} \cdots Y^{(n)} \right)$ in a simpler form using products and compositions, e.g., for $\pi = \{(1,2),(3,6),(4,5)\}$ we have

$$E_\pi \left(Y^{(1)},\dots,Y^{(5)} \right) = E\left(Y^{(1)} \right) Y^{(2)} E\left(Y^{(3)} E\left(Y^{(4)} \right) Y^{(5)} \right).$$

For a general $\pi \in \mathcal{P}(n)$ we do not know whether is possible to write E_π using products and compositions. The advantage of Notation 9 (2) is that it makes sense for an arbitrary partition π.

We now use Notation 9 to define matrix-valued classical cumulants.

Definition 25 (Matrix-valued classical cumulats). *For matrices $Y^{(1)}, \dots Y^{(n)} \in \mathbb{M}_d(\mathcal{A})$, we define the matrix-valued classical cumulants in $(\mathbb{M}_d(\mathcal{A}), E_\varphi, \mathbb{M}_d(\mathbb{C}))$ by*

$$c_n^{E_\varphi}(Y^{(1)},\dots,Y^{(n)}) = \sum_{\pi \in \mathcal{P}(n)} \mu(\pi, 1_n) E_{\varphi_\pi} \left[Y^{(1)},\dots,Y^{(n)} \right] \tag{4.19}$$

where $\mu(\pi, 1_n)$ is the Möbius function in $\mathcal{P}(n)$.

This notation has appeared before in the free probability literature. See for example [26, Section 9.3].

Remark 6. *(1) Definition 25 is a compact notation for cumulants in the entries of $Y^{(1)},\dots,Y^{(n)}$. We can see this by looking at one of the entries in the matrix defined in (4.19).*

$$\begin{aligned} \left[c_n^{E_\varphi}(Y^{(1)},\dots,Y^{(n)}) \right]_{i,j} &= \sum_{\pi \in \mathcal{P}(n)} \mu(\pi, 1_n) \sum_{i_1,\dots,i_{n-1}=1} \varphi_\pi \left(Y^{(1)}_{i,i_1}, \cdots, Y^{(n)}_{i_{n-1},j} \right) \\ &= \sum_{i_1,\dots,i_{n-1}=1} c_n \left(Y^{(1)}_{i,i_1}, \cdots, Y^{(n)}_{i_{n-1},j} \right). \end{aligned}$$

(2) A justification for the name matrix-valued classical cumulant, is that the moment-cumulant formula can be lifted to this matrix-valued version.

$$\begin{aligned}
E_\varphi\left(Y^{(1)}\cdots Y^{(n)}\right) &= \left(\sum_{i_1,\dots,i_{n-1}=1}\varphi\left(Y^{(1)}_{i,i_1}\cdots Y^{(n)}_{i_{n-1},j}\right)\right)_{i,j=1}^{d}\\
&= \left(\sum_{i_1,\dots,i_{n-1}=1}\sum_{\pi\in\mathcal{P}(n)}c_\pi\left(Y^{(1)}_{i,i_1},\cdots,Y^{(n)}_{i_{n-1},j}\right)\right)_{i,j=1}^{d}\\
&= \sum_{\pi\in\mathcal{P}(n)}\left(\sum_{i_1,\dots,i_{n-1}=1}c_\pi\left(Y^{(1)}_{i,i_1},\cdots,Y^{(n)}_{i_{n-1},j}\right)\right)_{i,j=1}^{d}\\
&= \sum_{\pi\in\mathcal{P}(n)}c_n^{E_\varphi\pi}(Y^{(1)},\dots,Y^{(n)}).
\end{aligned}$$

The main result of this section is a matrix-valued version of Theorem 11. Before stating the result let us introduce the following notation.

Notation 10. *(1) We identify $\mathbb{M}_d(\mathbb{C})$ with the subalgebra $\mathbb{M}_d(\mathbb{C})\otimes I_n\subset\mathbb{M}_d(\mathcal{A})$. For $A\in\mathbb{M}_d(\mathbb{C})$ we denote $\tilde{A}:=A\otimes I_N$.*

(2) Let $m,k_1,\dots,k_m$ be positive integers and set $k:=k_1+\cdots+k_m$. For matrices $A^{(1)},\dots,A^{(k)}\in\mathbb{M}_d(\mathbb{C})$ denote

$$\begin{aligned}
P_1 :=& X_N\tilde{A}^{(1)}\cdots X_N\tilde{A}^{(k_1)},\\
&\vdots\\
P_m :=& X_N\tilde{A}^{(1+k_1+\cdots+k_{m-1})}\cdots X_N\tilde{A}^{(k)}.
\end{aligned}$$

The following is the main result of this section:

Theorem 12. *Let X_N be as in equation (4.16) with covariance (4.15). For $m\in\mathbb{N}$ and $P_1,\dots,P_m$ as above, the asymptotic behavior of matrix-valued cumulants satisfy*

$$\frac{c_m^{E_\varphi}\left(E_{\mathrm{Tr}}[P_1],\dots,E_{\mathrm{Tr}}[P_m]\right)}{N^{m-2}}\xrightarrow{N\to\infty}\sum_{\pi\in NC_2(k_1,\dots,k_m)}q^{cr(\pi)}E_\pi\left(SA^{(1)},\dots,SA^{(k)}\right),$$

where E_π is the multiplicative extension of $E:\mathbb{M}_d(\mathcal{S})\to\mathbb{M}_d(\mathbb{C})$ in the sense of Notation 9, and S is the operator-valued semicircular element defined in (4.18).

For the proof of Theorem 12 we need the analogous of the genus expansion. Observe that this is a matrix-valued version of Equation (4.5).

Proposition 3 (Genus expansion)**.** *For matrices* $A^{(1)}, \ldots, A^{(k)} \in \mathbb{M}_d(\mathbb{C})$ *we have that*

$$E_\varphi\left[E_{\mathrm{Tr}}\left[X_N\tilde{A}^{(1)}\cdots X_N\tilde{A}^{(k)}\right]\right] = \sum_{\pi\in\mathcal{P}_2(k)} q^{cr(\pi)} E_\pi\left[SA^{(1)},\ldots,SA^{(k)}\right] N^{\#(\gamma\pi)-\frac{k}{2}},$$

where $\gamma = (1,2,\ldots,k)$ *is the one-cycle permutation, and* $\pi \in \mathcal{P}_2(k)$ *can be seen as a product of commuting transpositions (one transposition for each block) in* S_k*. The expression* $\#(\gamma\pi)$ *denotes the number of cycles in* $\gamma\pi$*.*

Proof. First observe that $\tilde{A}^{(i)} = \sum_{s,t=1}^{d} e_{s,t} \otimes A^{(i)}_{s,t} I_N$, then

$$\begin{aligned} X_N\tilde{A}^{(i)} &= \sum_{p,r=1}^{d}\sum_{s,t=1}^{d} e_{p,r}e_{s,t} \otimes X^{(N)}(p,r)A^{(i)}_{s,t} \\ &= \sum_{p,r,t=1}^{d} e_{p,t} \otimes X^{(N)}(p,r)A^{(i)}_{r,t}. \end{aligned}$$

Observe that $X^{(N)}(p,r)$ is an $N \times N$ matrix and $A^{(i)}_{s,t}$ is a scalar.
Iterating the above procedure we get

$$X_N\tilde{A}^{(1)}\cdots X_N\tilde{A}^{(k)} = \sum_{i_1,\ldots,i_{2k+1}=1}^{d} e_{i_1,i_{2k+1}} \otimes X^{(N)}(i_1,i_2)A^{(1)}_{i_2,i_3}\cdots X^{(N)}(i_{2k-1},i_{2k})A^{(k)}_{i_{2k},i_{2k+1}}. \quad (4.20)$$

Now consider a fixed tuple $1 \leq i_1, \ldots, i_{2k+1} \leq d$, and observe that

$$E_\varphi \circ E_{\mathrm{Tr}} \left[e_{i_1,i_{2n+1}} \otimes X^{(N)}(i_1,i_2) A^{(1)}_{i_2,i_3} \cdots X^{(N)}(i_{2k-1},i_{2k}) A^{(k)}_{i_{2k},i_{2k+1}} \right]$$

$$E_\varphi \left[e_{i_1,i_{2n+1}} \otimes \mathrm{Tr} \left(X^{(N)}(i_1,i_2) A^{(1)}_{i_2,i_3} \cdots X^{(N)}(i_{2k-1},i_{2k}) A^{(k)}_{i_{2k},i_{2k+1}} \right) \right]$$

$$= \frac{1}{N^{k/2}} \sum_{t_1,\ldots,t_k=1}^{N} e_{i_1,i_{2n+1}} \otimes \varphi \left(x^{(i_1,i_2)}_{t_1,t_2} A^{(1)}_{i_2,i_3} \cdots x^{(i_{2k-1},i_{2k})}_{t_k,t_1} A^{(k)}_{i_{2k},i_{2k+1}} \right)$$

$$= \frac{1}{N^{k/2}} \sum_{t_1,\ldots,t_k=1}^{N} \sum_{\pi \in \mathcal{P}_2(k)} q^{cr(\pi)} \left[\prod_{(l,r)\in\pi} \delta_{l,\gamma(r)} \gamma_{r,\gamma(l)} \right]$$

$$e_{i_1,i_{2n+1}} \otimes \varphi_\pi \left(x^{(i_1,i_2)}_{t_1,t_2} A^{(1)}_{i_2,i_3},\ \ldots\ , x^{(i_{2k-1},i_{2k})}_{t_k,t_1} A^{(k)}_{i_{2k},i_{2k+1}} \right)$$

$$= \sum_{\pi \in \mathcal{P}_2(k)} q^{cr(\pi)} N^{\#(\gamma\pi)-\frac{k}{2}}$$

$$e_{i_1,i_{2n+1}} \otimes \phi_\pi \left(S_{i_1,i_2} A^{(1)}_{i_2,i_3},\ \ldots\ , S_{i_{2k-1},i_{2k}} A^{(k)}_{i_{2k},i_{2k+1}} \right)$$

In the last equation we are using the entries of S that have the same covariance as the blocks of X_N.
Taking the sum over all possible tuples $1 \leq i_1, \ldots, i_{2k+1} \leq d$ we obtain

$$E_\varphi \circ E_{\mathrm{Tr}} \left[X_N \tilde{A}^{(1)} \cdots X_N \tilde{A}^{(k)} \right] = \sum_{\pi \in \mathcal{P}_2(k)} q^{cr(\pi)} E_\pi \left[SA^{(1)}, \ldots, SA^{(k)} \right] N^{\#(\gamma\pi)-\frac{k}{2}}.$$

□

Proof. (Proof for Theorem 12)
Given a partition $\sigma \in \mathcal{P}(m)$ we have to compute

$$E_{\varphi_\sigma} \left(E_{\mathrm{Tr}}[P_1], \ldots, E_{\mathrm{Tr}}[P_m] \right)$$

in the same way as we did in Equation (4.10). For this purpose we denote $\hat{\sigma}$ the canonical embedding of $\mathcal{P}(m)$ in $\mathcal{P}(k)$. For a precise definition see Equation (4.9). Observe that we are using a uniform notation for the positive integers $m, k_1, \ldots, k_m$ and $k = k_1 + \cdots + k_m$.
It follows from Equation (4.20) that for $1 \leq r \leq m$

$$E_{\mathrm{Tr}}[P_r] = \sum_{i_1^{(r)},\ldots,i_{2k_r+1}^{(r)}=1}^{d} e_{i_1^{(r)},i_{2k_r+1}^{(r)}} \otimes \mathrm{Tr} \left(X^{(N)}(i_1^{(r)}, i_2^{(r)}) A^{(1)}_{i_2,i_3} \cdots X^{(N)}(i_{2k_r-1}^{(r)}, i_{2k_r}^{(r)}) A^{(k_r)}_{i_{2k_r}^{(r)},i_{2k_r+1}^{(r)}} \right).$$

Then by a relabeling of subscripts we get

$$E_{\mathrm{Tr}}[P_1]\cdots E_{\mathrm{Tr}}[P_m]$$
$$\sum_{i_1,\ldots,i_{2k+1}=1}^{d} e_{i_1,i_{2k+1}} \otimes \mathrm{Tr}_\theta \left(X^{(N)}(i_1,i_2)A^{(1)}_{i_2,i_3}, \cdots, X^{(N)}(i_{2k_r-1},i_{2k})A^{(k)}_{i_{2k},i_{2k+1}} \right),$$

where θ is the partition

$$\theta = \{T_1, \ldots, T_m\},$$

with $T_1 = [1,k_1], T_2 = [1+k_1, k_1+k_2], \ldots, T_m = [1+k_1+\cdots+k_{m-1}, k]$ and Tr_θ is the multiplicative extension of Tr as defined in Notation 9.
Let γ be the permutation with cycles $\gamma = (1,\ldots,k_1)\cdots(1+k_1+\cdots+k_{m-1},k)$.
For convenience, let us denote by Z the vector

$$Z_{i_1,\ldots,i_{2k+1}} := \left(X^{(N)}(i_1,i_2)A^{(1)}_{i_2,i_3}, \cdots, X^{(N)}(i_{2k_r-1},i_{2k})A^{(k)}_{i_{2k},i_{2k+1}} \right) \in \mathbb{M}_N(\mathcal{A})^m.$$

Then we have

$$\begin{aligned} &E_{\varphi_\sigma}\left(E_{\mathrm{Tr}}[P_1],\ldots,E_{\mathrm{Tr}}[P_m]\right) \\ &\quad = \sum_{i_1,\ldots,i_{2k+1}=1}^{d} e_{i_1,i_{2k+1}} \otimes \varphi_\sigma\left(\mathrm{Tr}(T_1)(Z_{i_1,\ldots,i_{2k+1}}),\ldots,\mathrm{Tr}(T_m)(Z_{i_1,\ldots,i_{2k+1}})\right). \end{aligned} \tag{4.21}$$

We refer to Notation 9.(1) for a precise definition of $\mathrm{Tr}(T_1)(Z_{i_1,\ldots,i_{2k+1}})$.
Now using the same argument as in Equation (4.10) we get that for a fixed tuple $1 \le i_1,\ldots,i_{2k+1} \le d$

$$\begin{aligned} &\varphi_\sigma\left(\mathrm{Tr}(T_1)(Z_{i_1,\ldots,i_{2k+1}}),\ldots,\mathrm{Tr}(T_m)(Z_{i_1,\ldots,i_{2k+1}})\right) \\ &\qquad = \sum_{\substack{\pi\in\mathcal{P}_2(k)\\ \pi\le\hat{\sigma}}} q^{cr(\pi)} N^{\#(\gamma\pi)-\frac{k}{2}}\, \phi_\pi\left(W_{i_1,\ldots,i_{2k+1}}\right) \end{aligned} \tag{4.22}$$

where

$$W_{i_1,\ldots,i_{2k+1}} := \left(S_{i_1,i_2}A^{(1)}_{i_2,i_3}, \cdots, S_{i_{2k_r-1},i_{2k}}A^{(k)}_{i_{2k},i_{2k+1}} \right) \in \mathcal{S}^m.$$

When we replace $Z_{i_1,\ldots,i_{2k+1}}$ by $W_{i_1,\ldots,i_{2k+1}}$ we also replace φ by ϕ, this is because the blocks of X_N have the same covariance structure that the entries of S.
Now from the definition of matrix-valued cumulant, Definition 25, and from Equations (4.21) and (4.22), we have that taking the sum over all partitions in

$\mathcal{P}(m)$ produce

$$c_m^{E_\varphi}\left(E_{\text{Tr}}[P_1],\ldots,E_{\text{Tr}}[P_m]\right)=\sum_{\sigma\in\mathcal{P}(m)}\mu(\sigma,1_m)E_{\varphi_\sigma}\left(E_{\text{Tr}}[P_1],\ldots,E_{\text{Tr}}[P_m]\right)$$

$$=\sum_{\sigma\in\mathcal{P}(m)}\mu(\sigma,1_m)\sum_{i_1,\ldots,i_{2k+1}=1}^{d}$$

$$e_{i_1,i_{2k+1}}\otimes\varphi_\sigma\left(\text{Tr}(T_1)(Z_{i_1,\ldots,i_{2k+1}}),\ldots,\text{Tr}(T_m)(Z_{i_1,\ldots,i_{2k+1}})\right)$$

$$=\sum_{\sigma\in\mathcal{P}(m)}\mu(\sigma,1_m)\sum_{i_1,\ldots,i_{2k+1}=1}^{d}\sum_{\substack{\pi\in\mathcal{P}_2(k)\\ \pi\leq\hat{\sigma}}}q^{cr(\pi)}N^{\#(\gamma\pi)-\frac{k}{2}}e_{i_1,i_{2k+1}}\otimes\phi_\pi\left(W_{i_1,\ldots,i_{2k+1}}\right)$$

$$=\left(\sum_{i_2,\ldots,i_{2k}=1}^{d}\sum_{\sigma\in\mathcal{P}(m)}\mu(\sigma,1_m)\sum_{\substack{\pi\in\mathcal{P}_2(k)\\ \pi\leq\hat{\sigma}}}q^{cr(\pi)}N^{\#(\gamma\pi)-\frac{k}{2}}\,\phi_\pi\left(W_{i_1,\ldots,i_{2k+1}}\right)\right)_{i_1,i_{2k+1}=1}^{d}$$

$$=\sum_{\sigma\in\mathcal{P}(m)}\mu(\sigma,1_m)\sum_{\substack{\pi\in\mathcal{P}_2(k)\\ \pi\leq\hat{\sigma}}}q^{cr(\pi)}N^{\#(\gamma\pi)-\frac{k}{2}}\left(\sum_{i_2,\ldots,i_{2k}=1}^{d}\phi_\pi\left(W_{i_1,\ldots,i_{2k+1}}\right)\right)_{i_1,i_{2k+1}=1}^{d}$$

$$=\sum_{\substack{\pi\in\mathcal{P}_2(k)\\ \tilde{\pi}=1_m}}q^{cr(\pi)}N^{\#(\gamma\pi)-\frac{k}{2}}E_\pi\left(SA^{(1)},\ldots,SA^{(k)}\right)\tag{4.23}$$

The last equation follows from the same argument as in Equation (4.12). Multiplying both sides in Equation (4.23) by $N^{-(2-m)}$, and using inequality (4.14) we get that in the large N limit, the only partitions that contributes in the limit are those $\pi\in\mathcal{P}_2(k)$ such that

$$\#(\gamma\pi)-\frac{k}{2}=2-m.$$

Those are the annular non-crossing permutations $NC_2(k_1,\ldots,k_m)$. □

Remark 7. *(1) The matrix model X_N defined in Equation (4.16) possesses four parameters: the parameter $q\in[-1,1]$ for q-Gaussian entries, a positive integer d that denotes the number of blocks in X_N, a positive integer N that denotes the size of each block and a covariance function $\sigma:[d]\times[d]\to\mathbb{C}$ that codifies the joint distribution of the entries in different blocks of X_N.*
Theorem 10 says that the limit distribution of X_N when N goes to infinity converges to the distribution of an operator-valued semicircular element S,

defined in (4.18). The matrix model S depends only of two parameters: the size of the matrix d and the covariance function σ that describes the joint distribution of the entries of S.
The information of S is not enough if we are interested in asymptotic limits of classical cumulants. As Theorems 10 and 11 shows, we need to add the information of the parameter q.

(2) *The quantity*
$$c_m^{E_\varphi}\left(E_{\mathrm{Tr}}[P_1],\ldots,E_{\mathrm{Tr}}[P_m]\right)$$
contains additional parameters: a positive integer m for the degree of the cumulant $c_m^{E_\varphi}$ and positive integers $k_1,\ldots,k_m$ that denote the degree of the monomials
$$\begin{aligned} P_1 :=& X_N\tilde{A}^{(1)}\cdots X_N\tilde{A}^{(k_1)}, \\ &\vdots \\ P_m :=& X_N\tilde{A}^{(1+k_1+\cdots+k_{m-1})}\cdots X_N\tilde{A}^{(k_1+\cdots+k_m)}. \end{aligned}$$
We do not consider the matrices $A^{(1)},\ldots,A^{(k)}\in\mathbb{M}_d(\mathbb{C})$ where $k:=k_1+\cdots+k_m$, as parameters because Theorem 12 remains true for an arbitrary choice of these matrices. Remember that we denote $\tilde{A}=A\otimes I_N$.

(3) *Theorem 12 contains Theorems 10 and 11. Theorem 11 is recovered when we set $d=1$ and Theorem 10 is recovered when we set $d=1$ and $m=1$. The relevance of Theorem 10 is that it is itself an extension of the result of asymptotic freeness for GUE matrices. In the same way, Theorem 11 is an extension of the results for the second order distribution of GUE matrices.*

(4) *Theorem 12 for $q=1$ and $m=2$ was shown by Belinschi-Diaz-Mingo in [1, Proposition 10]. The key point to perform the extension of these results to the case $m>2$ is the introduction of Notation 9 and the matrix-valued "multiplicative extension" of the conditional expectation $E:\mathbb{M}_d(\mathcal{S})\to\mathbb{M}_d(\mathbb{C})$.*

(5) *The matrix-valued classical cumulants $c_m^{E_\varphi}$ introduced in Definition 25 are similar to the cumulants defined by Belinschi-Diaz-Mingo in [1, Definition 24] and [1, Proposition 7]. However the ones defined in this thesis satisfy a moment-cumulant formula. The proof of this fact is an easy computation presented in Remark 6 (2).*

(6) *The main result obtained by Diaz et al in [1] is that they found a functional equation for the second order Cauchy transform in the case $q=1$. This*

means that they put all the fluctuation moments

$$\alpha_{n,m} := c_2^{E_\varphi}\left(E_{\mathrm{Tr}}\left[(X_N\tilde{A})^n\right], E_{\mathrm{Tr}}\left[(X_N\tilde{B})^m\right]\right)$$

into one power series

$$G(A,B) = \sum_{r,s\geq 1} \frac{\alpha_{n,m}}{A^{r+1}B^{s+1}}$$

and obtained a functional equation for $G(zI_N, wI_N)$, where I_N is the identity matrix. We showed the connection between the "double-line pairings" and pair partitions that are annular and linear non-crossing. See Figure 4 for an example.

There are still two open questions related to the matrix-valued cumulants introduced in this thesis. First, whether it is possible to use the matrix-valued cumulants to simplify the result of Belinschi-Diaz-Mingo [1]. Second, whether it is possible to extend their results to the case $q \neq 1$. The case $q = 0$ is also considered in [1, Definition 28] under the name of "double-line Cauchy transform".

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