On unitarily invariant spaces and Cowen-Douglas theory

Characterization of Toeplitz operators, Wold decomposition type theorems and fiber dimension for invariant subspaces

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Abstract

Due to a classical result by Brown and Halmos, a bounded linear operator T on the Hardy space $H^2(\mathbb{D})$ is a Toeplitz operator T with bounded measurable symbol if and only if it satisfies the identity $M_z^* T M_z = T$. Olofsson and Louchichi proved that Toeplitz operators with harmonic symbol on the generalized Bergman spaces $A_m^2(\mathbb{D})$ can be characterized by a more general algebraic operator identity. We extend this result to the multidimensional setting of Toeplitz operators with pluriharmonic symbol on a unitarily invariant functional Hilbert space with appropriate kernel on the unit ball $\mathbb{B}_d \subseteq \mathbb{C}^d$. In the same setting, we use a multivariable extension of an analytic operator model constructed by Shimorin for single left invertible operators to deduce a variant of the classical Wold decomposition theorem. In doing so, we extend a Wold-type decomposition theorem on generalized Bergman spaces by Olofsson and Giselsson. Finally, we study the fiber dimension, an invariant of $\mathbb{C}[z]$-submodules of the space $\mathcal{O}(\Omega, D)$ of analytic functions on a complex submanifold $\Omega \subseteq \mathbb{C}^d$ with values in a finite-dimensional vector space D. We use model theorems for (weak) Cowen-Douglas tuples on Banach spaces to extend the definition of fiber dimension to the setting of (invariant) subspaces for such tuples. Here, we extend results by Chang, Chen and Fang for single Cowen-Douglas operators on a Hilbert space and deduce certain natural properties of this invariant.

Zusammenfassung

Nach einem klassischen Ergebnis von Brown und Halmos ist ein beschränkter, linearer Operator T auf dem Hardyraum $H^2(\mathbb{D})$ ein Toeplitz-Operator mit beschränktem messbarem Symbol genau dann, wenn er die algebraische Identität $M_z^* T M_z = T$ erfüllt. Olofsson und Louchichi haben gezeigt, dass Toeplitz-Operatoren mit harmonischem Symbol auf den verallgemeinerten Bergmanräumen $A^2_m(\mathbb{D})$ auf ähnliche Art durch eine algebraische Identität charakterisiert werden können. Wir beweisen ein entsprechendes Resultat für Toeplitz-Operatoren mit pluriharmonischem Symbol auf unitär invarianten funktionalen Hilberträumen mit geeignetem Kern über der Einheitskugel $\mathbb{B}_d \subseteq \mathbb{C}^d$. Wir benutzen eine mehrdimensionale Verallgemeinerung eines Modellsatzes von Shimorin, um reguläre Operatortupel zu charakterisieren, die eine Wold-Zerlegung in einen koisometrischen Teil und einen Shiftteil besitzen. Ein entsprechender Satz für einzelne Operatoren geht zurück auf Olofsson und Giselsson. Abschließend betrachten wir die Faserdimension, eine Invariante von $\mathbb{C}[z]$-Untermoduln des Raums $\mathcal{O}(\Omega, D)$ der analytischen Funktionen auf einer komplexen Untermannigfaltigkeit $\Omega \subseteq \mathbb{C}^d$ mit Werten in einem endlichdimensionalen Vektorraum D. Wir nutzen ähnliche Modellsätze für (schwache) Cowen-Douglas-Tupel auf Banachräumen, um eine Faserdimension für (invariante) Teilräume solcher Tupel zu definieren. Dabei verallgemeinern wir Ergebnisse von Chang, Chen und Fang für einzelne Cowen-Douglas-Operatoren auf Hilberträumen. Wir beenden diese Arbeit, indem wir einige Eigenschaften dieser Invariante herleiten.

Contents

1 Introduction

A bounded linear operator $T \in L(H^2(\mathbb{T}))$ on the Hardy space $H^2(\mathbb{T})$ is called a Toeplitz operator with symbol $f \in L^\infty(\mathbb{T})$ if it is the compression

$$T = P_{H^2(\mathbb{T})} M_f|_{H^2(\mathbb{T})}$$

of the multiplication operator $M_f : L^2(\mathbb{T}) \to L^2(\mathbb{T}), M_f g = fg$. A classical result by Brown and Halmos from [BH64] characterizes Toeplitz operators on $H^2(\mathbb{T})$ as those operators $T \in L(H^2(\mathbb{T}))$ which satisfy the algebraic identity

$$M_z^* T M_z = T,$$

where $M_z : H^2(\mathbb{T}) \to H^2(\mathbb{T}), M_z f = zf$, denotes the multiplication operator with the independent variable. Via Poisson extension the Hardy space $H^2(\mathbb{T}) \subseteq L^2(\mathbb{T})$ can be interpreted as an analytic functional Hilbert space $H^2(\mathbb{D})$ on the open unit disc. A natural question is whether a similar characterization of Toeplitz operators is still true on other standard functional Hilbert spaces on the unit disc. To answer this question in the particular case of the Bergman space $A^2(\mathbb{D})$, Englis observed in [Eng92] that the Toeplitz operators actually form a dense subset of $L(A^2(\mathbb{D}))$ in the strong operator topology. Thus an algebraic characterization of Toeplitz operators would extend to the whole space and in turn, every bounded linear operator on $A^2(\mathbb{D})$ would be a Toeplitz operator which is easily seen to be false. However, somewhat surprisingly, in the setting of the generalized Bergman spaces $A^2_m(\mathbb{D})$ $(m \in \mathbb{N})$ which encompasses the classical case $A^2_2(\mathbb{D}) = A^2(\mathbb{D})$, Olofsson and Louhichi were able to characterize the Toeplitz operators with harmonic symbol as those operators $T \in L(A^2_m(\mathbb{D}))$ which satisfy the algebraic identity

$$M_z'^* T M_z' = \sum_{k=0}^{m-1} (-1)^k \binom{m}{k+1} M_z^k T M_z^{*k}. \tag{1.1}$$

Here $M_z' = M_z(M_z^* M_z)^{-1}$ is the Cauchy dual of the left-invertible multiplication operator $M_z \in L(A^2_m(\mathbb{D}))$ and the generalized Bergman spaces $A^2_m(\mathbb{D})$ are the functional Hilbert spaces with reproducing kernels

$$K_m^{\mathbb{D}} : \mathbb{D} \times \mathbb{D} \to \mathbb{C}, (z, w) \mapsto \frac{1}{(1 - z\overline{w})^m} \quad (m \in \mathbb{N}).$$

In the recent paper [EL19], this result was extended to the generalized Bergman spaces $A^2_m(\mathbb{B}_d)$ on the unit ball $\mathbb{B}_d \subseteq \mathbb{C}^d$ which are defined as the functional Hilbert spaces with reproducing kernels

$$K_m : \mathbb{B}_d \times \mathbb{B}_d \to \mathbb{C}, K_m(z,w) = \frac{1}{(1-\langle z,w\rangle)^m} \quad (m \in \mathbb{N}).$$

To obtain an adequate generalization of the identity (1.1), one needs to define a suitable multivariable version of the Cauchy dual. Since the row operator $M_z : A^2_m(\mathbb{B}_d)^d \to A^2_m(\mathbb{B}_d)$ has closed range, the operator $M_z^* M_z$: $\operatorname{ran} M_z^* \to \operatorname{ran} M_z^*$ is invertible and it turns out that there is a natural extension $M_z' : A^2_m(\mathbb{B}_d)^d \to A^2_m(\mathbb{B}_d)$ of the operator $M_z(M_z^* M_z)^{-1}$ which forms a perfect substitute for the one-variable Cauchy dual on the unit ball. Another problem in the multidimensional setting, arising when $m < d$, concerns the definition of Toeplitz operators. In the absence of a nice normal extension of $M_z \in L(A^2_m(\mathbb{B}_d))^d$, it proves convenient to define multiplication operators T_f with symbol $f \in A^2_m(\mathbb{B}_d)$ only densely on

$$\{g \in A^2_m(\mathbb{B}_d); fg \in A^2_m(\mathbb{B}_d)\}$$

and to call an operator $T \in L(A^2_m(\mathbb{B}_d))$ a Toeplitz operator with pluriharmonic symbol $f = g + \overline{h}$ $(g, h \in A^2_m(\mathbb{B}_d))$ if it acts as

$$Tp = T_g p + T_h^* p$$

on all polynomials $p \in \mathbb{C}[z]$. This definition coincides with the classical definition for $m \geq d$. The aforementioned result from [EL19] then takes the following form.

Theorem 1.0.1 (Theorems 4 and 5 in [EL19])**.** Let $T \in L(A^2_m(\mathbb{B}_d))$. Then T satisfies the identity

$$M_z'^* T M_z' = P_{\operatorname{ran} M_z^*}\Big(\oplus \sum_{k=0}^{m-1} (-1)^k \binom{m}{k+1} \sigma^k_{M_z}(T)\Big) P_{\operatorname{ran} M_z^*}. \tag{1.2}$$

if and only if $T = T_f$ is a Toeplitz operator with pluriharmonic symbol f.

Examining the right-hand side of (1.2), one notices a close similarity to the power-series expansion of the reciprocal of the corresponding kernel, namely

$$\frac{1}{K_m}(z,w) = \sum_{k=0}^{m} (-1)^k \binom{m}{k} \langle z,w\rangle^k \qquad (z,w \in \mathbb{B}_d).$$

With this in mind, it is possible to extend Theorem 1.0.1 to more general situations. Let $k : \mathbb{D} \to \mathbb{C}, k(z) = \sum_{k=0}^{\infty} a_k z^k$, be an analytic function without zeroes such that $a_0 = 1$, $a_k > 0$ for all k,

$$\sup_{k \in \mathbb{N}} \frac{a_k}{a_{k+1}} < \infty, \quad \inf_{k \in \mathbb{N}} \frac{a_k}{a_{k+1}} > 0$$

and such that almost all coefficients c_k in the power series representation $\frac{1}{k}(z) = \sum_{k=0}^{\infty} c_k z^k$ of the reciprocal function have the same sign. Then the kernel $K : \mathbb{B}_d \times \mathbb{B}_d \to \mathbb{C}, K(z, w) = k(\langle z, w \rangle)$ defines an analytic functional Hilbert space H_K such that the row operator $M_z : H_K^d \to H_K$ has closed range and the same idea as in the case $H_K = A_m^2(\mathbb{B}_d)$ can be used to define a Cauchy dual M_z' of $M_z \in L(H_K)^d$. Replacing the sum on the right-hand side of (1.2) by the SOT-convergent operator series

$$\Delta_{M_z;T} = \sum_{k=0}^{\infty} (-c_{k+1}) \sigma_{M_z}^k(T) \quad \left(\sigma_{M_z}(T) = \sum_{i=1}^{d} M_{z_i} T M_{z_i}^*\right)$$

one obtains the following more general version of Theorem 1.0.1.

Theorem 1.0.2 (Theorem 3.3.6)**.** Let $T \in L(H_K)$. Then the identity

$$M_z'^* T M_z' = P_{\operatorname{ran} M_z^*} (\oplus \Delta_{M_z;T}) P_{\operatorname{ran} M_z^*}.$$

is satisfied if and only if T is a Toeplitz operator with pluriharmonic symbol.

Here Toeplitz operators in $L(H_K)$ are defined exactly as in the case $K = K_m$ explained above. Let us give some more details about the structure of the proof of Theorem 1.0.2. Let

$$H_K = \bigoplus_{k=0}^{\infty} \mathbb{H}_k$$

be the orthogonal decomposition of H_K into the spaces $\mathbb{H}_k$ consisting of all homogeneous polynomials of degree k. In a first step, we consider only linear operators $T \in L(H_K)$ which are homogeneous of positive degree l with respect to this orthogonal decomposition, that is, operators with $T\mathbb{H}_k \subseteq \mathbb{H}_{k+l}$ for all k. For homogeneous operators of positive degree, it is easy to show that they are multiplication operators with polynomial symbol if they fulfill the operator identity from Theorem 1.0.2. An arbitrary operator $T \in L(H_K)$ can be written as the SOT-limit

$$T = \text{SOT} - \lim_{N \to \infty} \sum_{|k| \leq N} \left(1 - \frac{|k|}{N+1}\right) T_k,$$

where the operators T_k are homogeneous of degree $k \in \mathbb{Z}$. Using the fact that $T_k^* = (T^*)_{-k}$ for all $k \in \mathbb{Z}$ one can use the aforementioned special case to prove the general result stated in Theorem 1.0.2.
We conclude this part of the thesis by showing that the pluriharmonic symbol f of a Toeplitz operator T is given by its Berezin transform $f = \tilde{T} : \mathbb{B}_d \to \mathbb{C}, \tilde{T}(z) = \langle TK_z, K_z \rangle$ where $K_z = \frac{K(\cdot,z)}{\|K(\cdot,z)\|}$ is the normalized kernel at $z \in \mathbb{B}_d$. Furthermore, we prove that a given operator $T \in L(H_K)$ is a Toeplitz operator with pluriharmonic symbol if and only if its Berezin transform $\tilde{T} : \mathbb{B}_d \to \mathbb{C}$ is a pluriharmonic function.

In the second part of this thesis, we use similar methods to extend the classical Wold decomposition theorem, to the setting of generalized Bergman spaces $A^2_\nu(\mathbb{B}_d)$ ($\nu \in [1, \infty[$). A given isometry T on a Hilbert space H can be decomposed into the direct sum

$$T = T^{(0)} \oplus T^{(1)} \in L(H_0 \oplus H_1)$$

of a unitary operator $T^{(0)} \in L(H_0)$ and an operator $T^{(1)} \in L(H_1)$ which is unitarily equivalent to a Hardy space shift $M_z \in L(H^2(\mathbb{D}, D))$ with multiplicity. The Hardy space $H^2(\mathbb{D}, D)$ is given by the $L(D)$-valued analytic reproducing kernel

$$K : \mathbb{D} \times \mathbb{D} \to L(D), K(z,w) = \frac{1_D}{1 - z\overline{w}}.$$

The operator identity $1_H - T^*T = 0$ characterizes isometries and is related to the coefficients occurring in the reciprocal of the kernel K. In [GO12], Giselsson and Olofsson considered similar operator identities related to the reciprocal of the reproducing kernel $K_m^{\mathbb{D}}$ of the generalized Bergman space $A^2_m(\mathbb{D})$ from above. In [GO12] it is shown that a left-invertible operator $T \in L(H)$ that satisfies a higher order operator identity associated with the Bergman space $A^2_m(\mathbb{D})$ admits a decomposition

$$T = T^{(0)} \oplus T^{(1)} \in L(H_0 \oplus H_1)$$

into an m-coisometry $T^{(0)} \in L(H_0)$ and an operator $T^{(1)} \in L(H_1)$ which is unitarily equivalent to a Bergman shift $M_z \in L(A^2_m(\mathbb{D}, D))$. We indicate a possible extension of this result to the case of multidimensional generalized Bergman spaces $A^2_\nu(\mathbb{B}_d)$ given by the reproducing kernels

$$K_\nu : \mathbb{B}_d \times \mathbb{B}_d \to \mathbb{C}, (z,w) \mapsto \frac{1}{(1 - \langle z, w \rangle)^\nu} \quad (\nu > 0).$$

Theorem 1.0.3 (Theorem 5.12)**.** Let $T \in L(H)^d$ be a commuting row contraction and let $\nu \geq 1$. Then the following conditions for T are equivalent:

(i) The tuple T is regular at $z = 0$, the SOT-limit

$$\Delta_T = \mathrm{SOT} - \sum_{k=0}^{\infty} (-1)^k \binom{\nu}{k+1} \sigma_T^k(1_H)$$

exists and T satisfies the identity $(T^*T)^{-1} = (\oplus \Delta_T)|_{\operatorname{ran} T^*}$,

(ii) $T = T^{(0)} \oplus T^{(1)} \in L(H^{(0)} \oplus H^{(1)})^d$ is the direct sum of a spherical coisometry $T^{(0)} \in L(H^{(0)})^d$ and a tuple $T^{(1)} \in L(H^{(1)})^d$ which is unitarily equivalent to $M_z \in L(A^2_\nu(\mathbb{B}_d, D))^d$ for some Hilbert space D.

Here the regularity condition for T replaces the condition of left invertibility demanded by Olofsson and Giselsson in the single variable case. Extending the one-variable case we define $H_\infty = \bigcap_{k=0}^{\infty} \sum_{|\alpha|=k} T^\alpha H$ and call a commuting tuple T analytic if $H_\infty = \{0\}$. A major step in the proof of Theorem 1.0.3 is to show that an analytic tuple $T \in L(H)^d$ which is regular at 0 and satisfies the operator identities from above is unitarily equivalent to a shift tuple $M_z \in L(A^2_\nu(\mathbb{B}_d, W(T)))^d$, where $W(T)$ denotes the wandering subspace of T. This result even holds for more general functional Hilbert spaces H_K on the unit ball $\mathbb{B}_d \subseteq \mathbb{C}^d$. Extending a corresponding one-dimensional analytic model of Shimorin for left invertible analytic operators, we show that each commuting tuple $T \in L(H)^d$ that is analytic and regular at 0 is unitarily equivalent to the multiplication tuple M_z on an analytic functional Hilbert space $\hat{H} \subseteq \mathcal{O}(\Omega_T, W(T))$ where $\Omega_T \subseteq \mathbb{C}^d$ is a suitable open ball with center 0. This is the main result of Chapter 4.2.

Theorem 1.0.4 (Theorem 4.2.5)**.** Let $T \in L(H)^d$ be regular at 0. Then there is a continuous linear map

$$V : H \to \mathcal{O}(\Omega_T, W(T))$$

with $Vx \equiv x$ for $x \in W(T)$ and such that

(i) $VT_i = M_{z_i}V$ for $i = 1, ..., d$,

(ii) $\ker V = \bigcap_{m=0}^{\infty} \sum_{|\alpha|=m} T^\alpha H = \bigcap_{z \in \Omega_T} \operatorname{ran}(z - T)$,

(iii) the vector space $\hat{H} = \operatorname{ran} V \subseteq \mathcal{O}(\Omega_T, W(T))$ equipped with the norm $\|Vx\| = \|x + \ker V\|$ is a functional Hilbert space and up to unitary equivalence $T \cong M_z \in L(\hat{H})^d$.

Let us denote by

$$K^\bullet(T,H): 0 \to \Lambda^0 H \xrightarrow{\delta_T^0} \Lambda^1 H \xrightarrow{\delta_T^1} \dots \xrightarrow{\delta_T^{d-1}} \Lambda^d H \to 0$$

the Koszul complex of a commuting tuple T. Each commuting tuple $T \in L(H)^d$, for which the Koszul complex is exact in degree $d-1$ and the operator $\delta_T^{d-1} : \Lambda^{d-1}H \to \Lambda^d H$ has closed range, is regular at 0. Thus our results contain the one-variable result of Giselsson and Olofsson from [GO12] and a one-variable model theorem proved by Shimorin in [Shi01] for single left-invertible Hilbert space operators as special cases. Each Cowen-Douglas tuple on a connected open zero neighbourhood $\Omega \subseteq \mathbb{C}^d$ is regular at 0. In the particular case of Cowen-Douglas tuples we prove even a Banach-space version of Theorem 1.0.4.

Theorem 1.0.5 (Theorem 4.1.9)**.** Let $T \in L(X)^d$ be a weak dual Cowen-Douglas tuple of rank N on a domain $\Omega \subseteq \mathbb{C}^d$, that is, a commuting tuple such that

$$\dim X / \sum_{i=1}^{d} (\lambda_i - T_i)X = N$$

for every $\lambda \in \Omega$. Then for each $\lambda_0 \in \Omega$, there is a continuous linear map $\rho : X \to \mathcal{O}(\Omega_0, D)$, where $\Omega_0 \subseteq \Omega$ is a connected open neighborhood of λ_0 in Ω, such that

(i) $\rho T_i = M_{z_i}\rho$ for $i = 1, ..., d$,

(ii) $\ker \rho = \bigcap_{z \in \Omega} \operatorname{ran}(z - T)$,

(iii) $\hat{X} = \rho(X)$ equipped with the norm $\|\rho(x)\| = \|x + \ker \rho\|$ $(x \in X)$ is a divisible holomorphic model space of rank N on Ω_0.

Here a divisible holomorphic model space $\hat{X}$ of rank N on Ω_0 is a continuously embedded Banach space $\hat{X} \subseteq \mathcal{O}(\Omega_0, \mathbb{C}^N)$ such that $M_z \in L(\hat{X})^d$ and such that the sequence

$$\hat{X}^d \xrightarrow{M_z - \lambda} \hat{X} \xrightarrow{\epsilon_\lambda} \mathbb{C}^N \to 0 \quad (\epsilon_\lambda(f) = f(\lambda))$$

are exact for every $\lambda \in \Omega_0$. Let D be a finite-dimensional vector space and let $M \subseteq \mathcal{O}(\Omega, D)$ be a linear subspace. The fiber dimension of M is defined as

$$\operatorname{fd}(M) = \max_{\lambda \in \Omega} M_\lambda,$$

where $M_\lambda = \epsilon_\lambda M \subseteq D$ is the image of M under the point evaluation $\epsilon_\lambda : M \to D, f \mapsto f(\lambda)$. In light of Theorem 1.0.5, the question becomes apparent whether such an invariant can be defined for invariant (or even arbitrary) subspaces $Y \subseteq X$ with respect to a weak Cowen-Douglas tuple $T \in L(X)^d$ using the local representations $\rho : X \to \mathcal{O}(\Omega_0, D)$. To ensure that the resulting fiber dimensions $\mathrm{fd}(\rho(Y))$ are independent of the choice of ρ, we only admit representations of a more restricted type. In the one-variable case the following definition can be found in [CCF15].

Definition 1.0.6. Let $\emptyset \neq \Omega_0 \subseteq \Omega$ be a connected open subset. A CF-representation of T on Ω_0 is a $\mathbb{C}[z]$-module homomorphism

$$\rho : X \to \mathcal{O}(\Omega_0, D)$$

with a finite-dimensional complex vector space D such that

(i) $\ker \rho = \bigcap_{z \in \Omega} \mathrm{ran}(z - T)$,

(ii) the submodule $\hat{X} = \rho X \subseteq \mathcal{O}(\Omega_0, D)$ satisfies

$$\mathrm{fd}(\hat{X}) = \dim \hat{X} / \sum_{i=1}^{d} (\lambda_i - M_{z_i}) \hat{X}$$

for all $\lambda \in \Omega_0$.

We show that the fiber dimension of a subspace $Y \subseteq X$ with respect to a weak dual Cowen-Douglas tuple $T \in L(X)^d$ can be defined as

$$\mathrm{fd}_T(Y) = \mathrm{fd}(\rho Y),$$

where $\rho : X \to \mathcal{O}(\Omega_0, D)$ is an arbitrary CF-representation. As in the case of single Hilbert space operators carried out by Chen, Cheng and Fang in [CCF15], we proceed to show that the fiber dimension of an invariant subspace of a weak dual Cowen-Douglas tuple can be calculated using suitable limit formulas from commutative algebra.

Theorem 1.0.7 (Corollary 6.2.3)**.** Let $T \in L(X)^d$ be a weak dual Cowen-Douglas tuple of rank N on a domain $\Omega \subseteq \mathbb{C}^d$ with $0 \in \Omega$, and let $Y \in \mathrm{Lat}(T)$ be a closed invariant subspace for T. Then the fiber dimension of Y with respect to T is given by

$$\mathrm{fd}(Y) = d! \lim_{k \to \infty} \frac{\dim(Y + M_k(T))/M_k(T)}{k^d},$$

where $M_k(T) = \sum_{|\alpha|=k} T^\alpha X$ for $k \in \mathbb{N}$.

Alternatively, the fiber dimension of Y can be calculated as an analytic Samuel multiplicity

$$\mathrm{fd}(Y) = e_{\mathcal{O}_0}(\mathcal{M}_{T,0}),$$

where $\mathcal{M}_T \subseteq \mathcal{O}_\Omega^X/(z-T)\mathcal{O}_\Omega^{X^d}$ is a suitable coherent subsheaf.
Given a weak dual Cowen-Douglas tuple $T \in L(X)^d$ and closed invariant subspaces $Y_1, Y_2 \subseteq X$ of T, it is a natural question to ask under which conditions the dimension formula

$$\mathrm{fd}(Y_1) + \mathrm{fd}(Y_2) = \mathrm{fd}(Y_1 \vee Y_2) + \mathrm{fd}(Y_1 \cap Y_2)$$

holds. In the final part of the thesis we describe several cases in which the above dimension formula holds and thus extend one-variable results from [CCF15].
Let $\Omega \subseteq \mathbb{C}^d$ be a connected open zero neighbourhood. Suppose that H admits an orthogonal decomposition $H = \bigoplus_{k=0}^{\infty} H_k$ with closed subspaces H_k. We call a commuting tuple $T \in L(H)^d$ γ-graded, for some given tuple $\gamma = (\gamma_1, ..., \gamma_d)$ of positive integers, if

$$T_j H_k \subseteq H_{k+\gamma_j} \quad (k \in \mathbb{N}, j = 1, ..., d).$$

As a typical result we show that this formula holds for all homogeneous invariant subspaces of a γ-graded dual Cowen-Douglas tuple T on Ω.

Corollary 1.0.8. Let $T \in L(H)^d$ be a γ-graded dual Cowen-Douglas tuple on a domain $\Omega \subseteq \mathbb{C}^d$ with $0 \in \Omega$. Then the fiber dimension formula

$$\mathrm{fd}(M_1 \vee M_2) + \mathrm{fd}(M_1 \cap M_2) = \mathrm{fd}(M_1) + \mathrm{fd}(M_2)$$

holds for any pair of homogeneous closed invariant subspaces M_1, M_2 of T.

2 Unitarily invariant spaces

Let $m \in \mathbb{N}_{\geq 1}$. On the generalized Bergman space $A^2_m(\mathbb{B}_d)$, that is, the functional Hilbert space with reproducing kernel

$$K_m : \mathbb{B}_d \times \mathbb{B}_d \to \mathbb{C}, K_m(z,w) = \frac{1}{(1-\langle z,w\rangle)^m},$$

the row operator

$$M_z : A^2_m(\mathbb{B}_d)^d \to A^2_m(\mathbb{B}_d), M_z(f_i)_{i=1}^d = \sum_{i=1}^d z_i f_i$$

has closed range

$$M_z A^2_m(\mathbb{B}_d)^d = \{f \in A^2_m(\mathbb{B}_d); f(0) = 0\}$$

by Satz 2.5 in [Wer08]. Thus the operator

$$M_z^* M_z : \operatorname{ran} M_z^* \to \operatorname{ran} M_z^*$$

is invertible. In [Esc18], it was shown that its inverse $(M_z^* M_z)^{-1}$ can be extended continuously to the whole of $A^2_m(\mathbb{B}_d)^d$ by the operator δM_z where $\delta : A^2_m(\mathbb{B}_d) \to A^2_m(\mathbb{B}_d)$ is a diagonal operator with respect to the orthogonal decomposition

$$A^2_m(\mathbb{B}_d) = \bigoplus_{k=0}^{\infty} \mathbb{H}_k$$

into the subspaces $\mathbb{H}_k$ consisting of all homogeneous polynomials of degree k. In this chapter, we will extend this and corresponding results to a more general class of functional Hilbert spaces.

2.1 Unitarily invariant spaces

Definition 2.1.1. A functional Hilbert space H_K with reproducing kernel

$$K : \mathbb{B}_d \times \mathbb{B}_d \to \mathbb{C}$$

is called unitarily invariant if $K(0,\cdot) \equiv 1$, K is analytic in the first component and we have

$$K(Uz, Uw) = K(z, w)$$

for all $z, w \in \mathbb{B}_d$ and all unitary maps $U : \mathbb{C}^d \to \mathbb{C}^d$.

It is well-known that the reproducing kernels of unitarily invariant spaces can be characterized by their power series representation (cf. e.g. Lemma 2.2 in [Har17]).

Lemma 2.1.2. Let H_K be a functional Hilbert space with reproducing kernel $K : \mathbb{B}_d \times \mathbb{B}_d \to \mathbb{C}$. Then H_K is unitarily invariant if and only if there is a sequence $(a_k)_{k\in\mathbb{N}}$ of non-negative numbers with $a_0 = 1$ such that

$$K(z, w) = \sum_{k=0}^{\infty} a_k \langle z, w\rangle^k$$

for all $z, w \in \mathbb{B}_d$.

On unitarily invariant spaces, the multiplication operators

$$M_{z_i} : H_K \to H_K, M_{z_i} f = z_i f \quad (i = 1, ..., d),$$

with the coordinate functions need not be bounded. However, one can characterize those unitarily invariant spaces for which the operators M_{z_i} $(i = 1, ..., d)$ and the induced row operator $M_z : H_K^d \to H_K, M_z(f_i)_{i=1}^d = \sum_{i=1}^d M_{z_i} f_i$, behave nicely by considering the sequence $(a_k)_{k\in\mathbb{N}}$ from the preceding lemma.

Lemma 2.1.3 (Theorems 2.3 and Theorem 2.5 in [Wer08])**.** Let H_K be a unitarily invariant space with reproducing kernel

$$K : \mathbb{B}_d \times \mathbb{B}_d \to \mathbb{C}, K(z, w) = \sum_{k=0}^{\infty} a_k \langle z, w\rangle^k$$

such that $a_k > 0$ for all $k \in \mathbb{N}$. The operators $M_{z_i} : H_K \to H_K$ $(i = 1, ..., d)$ are bounded if and only if $\sup_{k\in\mathbb{N}} \frac{a_k}{a_{k+1}} < \infty$. In this case, the row operator

$$M_z : H_K^d \to H_K$$

has closed range $M_z H_K^d = \{f \in H_K; f(0) = 0\}$ if and only if $\inf_{k\in\mathbb{N}} \frac{a_k}{a_{k+1}} > 0$.

All generalized Bergman spaces fulfill the conditions from Lemma 2.1.3.

Example 2.1.4. For $\nu \in]0,\infty[$, the generalized Bergman space $A^2_\nu(\mathbb{B}_d)$ is defined as the functional Hilbert space with reproducing kernel

$$K_\nu : \mathbb{B}_d \times \mathbb{B}_d \to \mathbb{C}, K_\nu(z,w) = \frac{1}{(1-\langle z,w\rangle)^\nu}.$$

Due to the binomial theorem, we have

$$K_\nu(z,w) = \sum_{k=0}^\infty (-1)^k \binom{-\nu}{k} \langle z,w\rangle^k$$

for all $z, w \in \mathbb{B}_d$ where

$$\binom{\nu}{k} = \prod_{j=1}^k \frac{\nu - j + 1}{j} \quad (\nu \in \mathbb{R}, k \in \mathbb{N})$$

denote the generalized binomial coefficients. We conclude that

$$\sup_{k\in\mathbb{N}} \frac{(-1)^k\binom{-\nu}{k}}{(-1)^{k+1}\binom{-\nu}{k+1}} = \sup_{k\in\mathbb{N}} \frac{k+1}{\nu+k} = \left\{ \begin{array}{ll} \frac{1}{\nu}, & \nu < 1 \\ 1, & \nu \geq 1 \end{array} \right\} < \infty$$

and

$$\inf_{k\in\mathbb{N}} \frac{(-1)^k\binom{-\nu}{k}}{(-1)^{k+1}\binom{-\nu}{k+1}} = \inf_{k\in\mathbb{N}} \frac{k+1}{\nu+k} = \left\{ \begin{array}{ll} 1, & \nu < 1 \\ \frac{1}{\nu}, & \nu \geq 1 \end{array} \right\} > 0.$$

2.2 The diagonal operators δ and Δ

In the following, we will consider a unitarily invariant space H_K given by a reproducing kernel

$$K : \mathbb{B}_d \times \mathbb{B}_d \to \mathbb{C}, K(z,w) = \sum_{k=0}^\infty a_k \langle z,w\rangle^k$$

with $a_0 = 1$, $a_k > 0$ for all $k \in \mathbb{N}$ such that $\sup_{k\in\mathbb{N}} \frac{a_k}{a_{k+1}} < \infty$ and $\inf_{k\in\mathbb{N}} \frac{a_k}{a_{k+1}} > 0$.

One can show that

$$H_K = \{ \sum_{\alpha\in\mathbb{N}^d} f_\alpha z^\alpha \in \mathcal{O}(\mathbb{B}_d); \sum_{\alpha\in\mathbb{N}^d} \frac{|f_\alpha|^2}{a_{|\alpha|}\gamma_\alpha} < \infty\}$$

and that

$$\|f\| = (\sum_{\alpha\in\mathbb{N}^d} \frac{|f_\alpha|^2}{a_{|\alpha|}\gamma_\alpha})^{\frac{1}{2}}$$

holds for $f = \sum_{\alpha\in\mathbb{N}^d} f_\alpha z^\alpha \in H_K$ with $\gamma_\alpha = \frac{|\alpha|!}{\alpha!}$ for $\alpha \in \mathbb{N}^d$.
The space H_K admits the orthogonal decomposition

$$H_K = \bigoplus_{k=0}^{\infty} \mathbb{H}_k$$

into the spaces

$$\mathbb{H}_k = \left\{ \sum_{|\alpha|=k} f_\alpha z^\alpha ; f_\alpha \in \mathbb{C} \text{ for all } \alpha \in \mathbb{N}^d \text{ with } |\alpha| = k \right\} \subseteq \mathbb{C}[z]$$

consisting of all homogeneous polynomials of degree k (c.f Theorem 1.15 in [Wer08]). With respect to this decomposition, we consider the diagonal operators

$$\delta : H_K \to H_K, \delta(\sum_{k=0}^{\infty}\sum_{|\alpha|=k} f_\alpha z^\alpha) = f_0 + \sum_{k=1}^{\infty} \frac{a_k}{a_{k-1}} \sum_{|\alpha|=k} f_\alpha z^\alpha$$

and

$$\Delta : H_K \to H_K, \Delta(\sum_{k=0}^{\infty}\sum_{|\alpha|=k} f_\alpha z^\alpha) = \sum_{k=0}^{\infty} \frac{a_{k+1}}{a_k} \sum_{|\alpha|=k} f_\alpha z^\alpha.$$

Obviously δ and Δ are invertible positive operators on H_K. Note that

$$\delta M_{z_i} = M_{z_i}\Delta$$

holds for $i = 1, ..., d$. This also yields $\delta M_z = M_z(\oplus\Delta)$.

Since $M_z H_K^d \subseteq H_K$ is closed, the operator $M_z^* M_z : \operatorname{ran} M_z^* \to \operatorname{ran} M_z^*$ is invertible. We denote its inverse by $(M_z^* M_z)^{-1}$. Then, the following lemma shows that the operator $M_z(M_z^* M_z)^{-1}$ on $\operatorname{ran} M_z^*$ can be extended to the whole space using the diagonal operator δ.

Lemma 2.2.1. For $f \in H_K$, we have

$$(M_z^* M_z)^{-1}(M_z^* f) = M_z^* \delta f = (\oplus\Delta) M_z^* f.$$

In particular, the row operator

$$\delta M_z : H_K^d \to H_K$$

is the trivial extension of

$$M_z(M_z^*M_z)^{-1} : \operatorname{ran} M_z^* \to H_K.$$

Proof. By Lemma 2.4 (i) in [Wer08], we have

$$M_{z_i}^* g = \sum_{\alpha\in\mathbb{N}^d} \frac{a_{|\alpha|}}{a_{|\alpha|+1}} \frac{\alpha_i+1}{|\alpha|+1} g_{\alpha+e_i} z^\alpha$$

for all $g = \sum_{\alpha\in\mathbb{N}^d} g_\alpha z^\alpha \in H_K$ and $i = 1, ..., d$. Now, fix $f \in H_K$. To prove the first assertion, we may suppose that $f(0) = 0$. In this case, we can infer from Lemma 2.4 in [Wer08] that $f = \sum_{i=1}^d z_i f_i$ with

$$f_i = M_{z_i}^* \delta f = \Delta M_{z_i}^* f = \sum_{\alpha\in\mathbb{N}^d} \frac{\alpha_i+1}{|\alpha|+1} f_{\alpha+e_i} z^\alpha$$

for $i = 1, ..., d$. It follows that

$$(M_z^*M_z)^{-1}M_z^*f = (M_z^*M_z)^{-1}M_z^*(M_zM_z^*\delta f) = M_z^*\delta f$$

and that

$$M_z(M_z^*M_z)^{-1}M_z^* = M_zM_z^*\delta.$$

Since $M_zM_z^*$ and δ are diagonal operators, this yields $M_z(M_z^*M_z)^{-1}M_z^* = \delta M_zM_z^*$ and thus the second part of the lemma. □

On the generalized Bergman space $A_m^2(\mathbb{B}_d)$ ($m \in \mathbb{N}^*$), Lemma 3 in [Esc18] yields that the diagonal operator $\Delta : A_m^2(\mathbb{B}_d) \to A_m^2(\mathbb{B}_d)$ admits the representation

$$\Delta = \sum_{k=0}^{m-1} (-1)^k \binom{m}{k+1} \sum_{|\alpha|=k} \gamma_\alpha M_z^\alpha M_z^{*\alpha} = \sum_{k=0}^{m-1} (-1)^k \binom{m}{k+1} \sigma_{M_z}^k (1_{A_m^2(\mathbb{B}_d)}),$$

where $\sigma_{M_z} : L(A_m^2(\mathbb{B}_d)) \to L(A_m^2(\mathbb{B}_d)), X \mapsto \sum_{i=1}^d M_{z_i} X M_{z_i}^*$, and $\gamma_\alpha = \frac{|\alpha|!}{\alpha!}$ for every $\alpha \in \mathbb{N}^d$. The sum on the right-hand side is closely related to the operator $\frac{1}{K_m}(M_z, M_z^*)$ formally obtained by replacing z and $\overline{w}$ in the reciprocal kernel

$$\frac{1}{K_m}(z,w) = (1 - \langle z, w\rangle)^m = \sum_{k=0}^m (-1)^k \binom{m}{k} \sum_{|\alpha|=k} \gamma_\alpha z^\alpha \overline{w}^\alpha$$

by the tuples $(M_{z_1}, ..., M_{z_d})$ and $(M_{z_1}^*, ..., M_{z_d}^*)$. In the more general setting we are considering in this chapter, it is not immediately clear how $\frac{1}{K}(M_z, M_z^*)$

should be interpreted and thus how a representation for Δ should look like. In the sequel we suppose in addition that the holomorphic function

$$k : \mathbb{D} \to \mathbb{C}, z \mapsto \sum_{k=0}^{\infty} a_k z^k$$

has no zeroes and we denote the Taylor coefficients of $\frac{1}{k} : \mathbb{D} \to \mathbb{C}$ by $(c_k)_{k\in\mathbb{N}}$. Following ideas from [Sch18] (see also [CH18]), we set

$$(\frac{1}{K})_N(M_z, M_z^*) = \sum_{k=0}^{N} c_k \sigma_{M_z}^k(1_{H_K})$$

for all $N \in \mathbb{N}$ and we write

$$\frac{1}{K}(M_z, M_z^*) = \mathrm{SOT} - \lim_{N\to\infty} (\frac{1}{K})_N(T) = \mathrm{SOT} - \sum_{k=0}^{\infty} c_k \sigma_{M_z}^k(1_{H_K})$$

if the latter limit exists. In Proposition 2.10 in [Sch18], it is shown that this is the case if almost all the c_k have the same sign. We shall use the following slight modification. For the convenience of the reader, we give a complete proof of this result.

Theorem 2.2.2. Suppose that there exists a natural number $p \in \mathbb{N}$ such that

$$c_k \geq 0 \text{ for all } k \geq p \text{ or } c_k \leq 0 \text{ for all } k \geq p.$$

Then the series

$$\mathrm{SOT} - \sum_{j=0}^{\infty} c_{j+1} \sigma_{M_z}^j(1_{H_K})$$

converges.

Let us fix some notations before we give the proof.

Notation 2.2.3. For a bounded sequence $b = (b_k)_{k\in\mathbb{N}}$ of real numbers, we denote by

$$[b_k]_{k\in\mathbb{N}} : H_K \to H_K, \sum_{k=0}^{\infty} f_k \mapsto \sum_{k=0}^{\infty} b_k f_k$$

the induced diagonal operator with respect to the orthogonal decomposition $H_K = \bigoplus_{k=0}^{\infty} \mathbb{H}_k$. It is easy to see that such diagonal operators satisfy

$$M_{z_i}[b_k]_{k\in\mathbb{N}} = [b_{k-1}]_{k\in\mathbb{N}} M_{z_i} \qquad \text{(setting } b_{-1} = 0\text{)}$$

for $i = 1, ..., d$. This further yields

$$\sigma_{M_z}([b_k]_{k\in\mathbb{N}}) = \sum_{i=1}^{d} M_{z_i}[b_k]_{k\in\mathbb{N}} M_{z_i}^* = [b_{k-1}]_{k\in\mathbb{N}} \sigma_{M_z}(1_{H_K}).$$

Proof of Theorem 1.2.2. For $\alpha \in \mathbb{N}^d$, using Lemma 2.4 (ii) in [Wer08] we have

$$\sigma_{M_z}(1_{H_K}) z^\alpha = \sum_{i=1}^{d} M_{z_i} M_{z_i}^* z^\alpha = \frac{a_{|\alpha|-1}}{a_{|\alpha|}} z^\alpha.$$

Thus, we have $\sigma_{M_z}(1_{H_K}) = [\frac{a_{k-1}}{a_k}]_{k\in\mathbb{N}}$ and using the remarks from Notation 2.2.3 we conclude that

$$\begin{aligned}
\sigma_{M_z}^j(1_{H_K}) &= \sigma_{M_z}^{j-1}([\frac{a_{k-1}}{a_k}]_{k\in\mathbb{N}}) \\
&= \sigma_{M_z}^{j-2}([\frac{a_{k-2}}{a_{k-1}}]_{k\in\mathbb{N}}[\frac{a_{k-1}}{a_k}]_{k\in\mathbb{N}}) \\
&= \sigma_{M_z}^{j-2}([\frac{a_{k-2}}{a_k}]_{k\in\mathbb{N}}) \\
&= ... \\
&= [\frac{a_{k-j}}{a_k}]_{k\in\mathbb{N}}
\end{aligned}$$

for $j \in \mathbb{N}$. Here we set $a_k = 0$ if $k < 0$. For $N \geq p$, we find that

$$\begin{aligned}
\|\sum_{j=p}^{N} c_{j+1}\sigma_{M_z}^j(1_{H_K})\| &= \|\sum_{j=p}^{N} c_{j+1}[\frac{a_{k-j}}{a_k}]_{k\in\mathbb{N}}\| = \|[\sum_{j=p}^{N} c_{j+1}\frac{a_{k-j}}{a_k}]_{k\in\mathbb{N}}\| \\
&= \sup_{k\geq p} |\sum_{j=p}^{N} c_{j+1}\frac{a_{k-j}}{a_k}| = \sup_{k\geq p} \sum_{j=p}^{N} |c_{j+1}|\frac{a_{k-j}}{a_k}.
\end{aligned}$$

Since $(c_k)_{k\in\mathbb{N}}$ is the sequence of Taylor coefficients of $\frac{1}{k}$, we have

$$\sum_{j=0}^{k} c_j a_{k-j} = 0$$

for all $k \geq 1$. This yields

$$\begin{aligned}
\sum_{j=p}^{k} c_{j+1}\frac{a_{k-j}}{a_k} &= \sum_{j=p+1}^{k+1} c_j \frac{a_{k+1-j}}{a_k} \\
&= -\sum_{j=0}^{p} c_j \frac{a_{k+1-j}}{a_k}
\end{aligned}$$

for all $k \geq p$. Setting $s = \sup_{k\in\mathbb{N}} \frac{a_k}{a_{k+1}}$ and $t = (\inf_{k\in\mathbb{N}} \frac{a_k}{a_{k+1}})^{-1}$, we have

$$\frac{a_{k+1}}{a_k} \leq t \text{ and } \frac{a_{k+1-j}}{a_k} = \frac{a_{k+1-j}}{a_{k-j+2}} \cdot \ldots \cdot \frac{a_{k-1}}{a_k} \leq s^{j-1} \text{ for } j \geq 1.$$

Using those observations, we obtain

$$\begin{aligned}
\sum_{j=p}^{N} |c_{j+1}| \frac{a_{k-j}}{a_k} &\leq \sum_{j=p}^{k} |c_{j+1}| \frac{a_{k-j}}{a_k} \\
&= |\sum_{j=p}^{k} c_{j+1} \frac{a_{k-j}}{a_k}| \\
&= |-\sum_{j=0}^{p} c_j \frac{a_{k+1-j}}{a_k}| \\
&\leq \sum_{j=0}^{p} |c_j| \frac{a_{k+1-j}}{a_k} \\
&\leq (\sum_{j=0}^{p} |c_j|) \max(1, s^{p-1}, t)
\end{aligned}$$

for $k, N \geq p$. But then

$$\|\sum_{j=p}^{N} c_{j+1} \sigma_{M_z}^{j}(1_{H_K})\| \leq (\sum_{j=0}^{p} |c_j|) \max(1, s^{p-1}, t)$$

for all $N \geq p$ and in particular

$$\sup_{N\in\mathbb{N}} \|\sum_{j=0}^{N} c_{j+1} \sigma_{M_z}^{j}(1_{H_K})\| < \infty.$$

Since $(\sum_{j=0}^{N} c_{j+1} \sigma_{M_z}^{j}(1_{H_K}))_{N\geq p}$ is a decreasing or increasing sequence of selfadjoint operators, the norm-boundedness of this sequence implies its convergence in the strong operator topology. □

The hypothesis of Theorem 2.2.2 is fulfilled for all generalized Bergman spaces $A^2_\nu(\mathbb{B}_d)$ $(\nu > 0)$ and for all unitarily invariant spaces which are complete Nevanlinna-Pick spaces.

Example 2.2.4. (a) Let $\nu \in]0, \infty[$ and let $A^2_\nu(\mathbb{B}_d)$ be the corresponding generalized Bergman space. Due to the binomial theorem, we have

$$\frac{1}{K_\nu}(z,w) = \sum_{k=0}^{\infty} (-1)^k \binom{\nu}{k} \langle z, w\rangle^k$$

for all $z, w \in \mathbb{B}_d$. Then,

$$c_k = (-1)^{\lceil \nu \rceil} \prod_{j=1}^{\lceil \nu \rceil} \frac{\nu + 1 - j}{j} \prod_{j=\lceil \nu \rceil}^{k} \frac{j - \nu - 1}{j}$$

is ≥ 0 for all $k \geq \lceil \nu \rceil$ if $\lceil \nu \rceil$ is even and ≤ 0 for all $k \geq \lceil \nu \rceil$ if $\lceil \nu \rceil$ is odd.

(b) A functional Hilbert space H with reproducing kernel $K : X \times X \to \mathbb{C}$ is called a complete Nevanlinna-Pick space if the following matrix-valued interpolation problem can be solved: For all $n \in \mathbb{N}$, $\lambda_1, ..., \lambda_n \in X$ and $W_1, ..., W_n \in L(l^2(\mathbb{N}))$ such that

$$((1_{l^2(\mathbb{N})} - W_i W_j^*) K(\lambda_i, \lambda_j))_{i,j=1}^n \in L(l^2(\mathbb{N})^n)$$

is positive, there is a multiplier $\varphi \in \mathrm{Mult}(H_K(l^2(\mathbb{N})))$ such that $\|M_\varphi\| \leq 1$ and $\varphi(\lambda_i) = W_i$ for $i = 1, ..., n$. By Lemma 2.3 in [Har17], a unitarily invariant space with $a_1 > 0$ is a complete Nevanlinna-Pick space if and only if $c_n \leq 0$ for all $n \in \mathbb{N}$.

Before we use Theorem 2.2.2 to give a representation of Δ, we note how the coefficient sequences $(a_k)_{k\in\mathbb{N}}$ and $(c_k)_{k\in\mathbb{N}}$ are related. As before, we write

$$\gamma_\alpha = \frac{|\alpha|!}{\alpha!}$$

for all $\alpha \in \mathbb{N}^d$ and k for the analytic function $k : \mathbb{D} \to \mathbb{C}, k(z) = \sum_{k=0}^\infty a_k z^k$.

Lemma 2.2.5. For $\alpha \in \mathbb{N}^d$, we have

$$a_{|\alpha|+1} = \sum_{\beta \leq \alpha} (-c_{|\beta|+1} a_{|\alpha-\beta|} \frac{\gamma_\beta \gamma_{\alpha-\beta}}{\gamma_\alpha}).$$

Proof. For $z \in G = \{z \in \mathbb{C}^d; \sum_{i=1}^d |z_i| < 1\}$, we have

$$k(\sum_{i=1}^d z_i) = \sum_{k=0}^\infty a_k (\sum_{|\alpha|=k} \gamma_\alpha z^\alpha) = \sum_{\alpha \in \mathbb{N}^d} a_{|\alpha|} \gamma_\alpha z^\alpha$$

and, for $z \in G$ with $\sum_{i=1}^d z_i \neq 0$, we have

$$\frac{1-\frac{1}{k}(\sum_{i=1}^d z_i)}{\sum_{i=1}^d z_i} = \sum_{k=1}^{\infty}(-c_k)(\sum_{i=1}^d z_i)^{k-1} = \sum_{k=0}^{\infty}(-c_{k+1})\sum_{|\alpha|=k}\gamma_\alpha z^\alpha$$
$$= \sum_{\alpha\in\mathbb{N}^d}(-c_{|\alpha|+1})\gamma_\alpha z^\alpha$$

as well as

$$\frac{1-\frac{1}{k}(\sum_{i=1}^d z_i)}{\sum_{i=1}^d z_i} k\left(\sum_{i=1}^d z_i\right) = \frac{1}{\sum_{i=1}^d z_i}[k\left(\sum_{i=1}^d z_i\right)-1] = \sum_{\alpha\in\mathbb{N}^d} a_{|\alpha|+1}\gamma_\alpha z^\alpha.$$

Since all multivariable power series occurring above converge absolutely in each point of the domain G, the Cauchy product formula implies that

$$\sum_{\alpha\in\mathbb{N}^d}(\sum_{\beta\leq\alpha} -c_{|\beta|+1}a_{|\alpha-\beta|}\gamma_\beta\gamma_{\alpha-\beta})z^\alpha = \sum_{\alpha\in\mathbb{N}^d} a_{|\alpha|+1}\gamma_\alpha z^\alpha$$

for $z \in G$. A comparison of the coefficients yields the result. □

Theorem 2.2.6. Suppose that there exists a natural number $p \in \mathbb{N}$ such that

$$c_k \geq 0 \text{ for all } k \geq p \text{ or } c_k \leq 0 \text{ for all } k \geq p.$$

Then the diagonal operator Δ admits the representation

$$\Delta = \mathrm{SOT} - \sum_{k=0}^{\infty}(-c_{k+1})\sigma_{M_z}^k(1_{H_K}).$$

Proof. For all $\alpha \in \mathbb{N}^d$, we obtain using Lemma 1.29 in [Sch18], Theorem 2.2.2 and Lemma 2.2.5

$$\sum_{k=0}^{\infty}(-c_{k+1})\sigma_{M_z}^k(1_{H_K})z^\alpha = \sum_{k=0}^{\infty}(-c_{k+1})\sum_{|\beta|=k}\gamma_\beta M_z^\beta M_z^{*\beta} z^\alpha$$
$$= \sum_{k=0}^{|\alpha|}(-c_{k+1})\sum_{|\beta|=k;\beta\leq\alpha}(\frac{\gamma_\beta\gamma_{\alpha-\beta}}{\gamma_\alpha}\frac{a_{|\alpha-\beta|}}{a_{|\alpha|}})z^\alpha$$
$$= \frac{a_{|\alpha|+1}}{a_{|\alpha|}}z^\alpha$$
$$= \Delta z^\alpha.$$

Since the polynomials are dense in H_K, the claim follows. □

3 A characterization of Toeplitz operators with pluriharmonic symbol

A classical result by Brown and Halmos (cf. [BH64]) characterizes Toeplitz operators on the Hardy space

$$H^2(\mathbb{T}) = \{f \in L^2(\mathbb{T}); \hat{f}(n) = 0 \text{ for all } n < 0\}$$

on the unit circle as those operators $T \in L(H^2(\mathbb{T}))$ which satisfy the algebraic identity

$$M_z^* T M_z = T.$$

In [Eng92], Englis observed that such an algebraic characterization is not possible for Toeplitz operators on the Bergman space $A^2(\mathbb{D})$, since the Toeplitz operators with L^∞-symbols form an SOT-dense subset of $L(A^2(\mathbb{D}))$. In the notation introduced in Example 2.1.4 the Bergman space is just the space $A^2(\mathbb{D}) = A^2_2(\mathbb{D})$. However, Louhichi and Olofsson proved in [LO08] that an operator T on the generalized Bergman space $A^2_m(\mathbb{D})$ is a Toeplitz operator with bounded harmonic symbol if and only if it satisfies the algebraic identity

$$M_z'^* T M_z' = \sum_{k=0}^{m-1} (-1)^k \binom{m}{k+1} M_z^k T M_z^{*k}$$

where $M_z' = M_z(M_z^* M_z)^{-1}$ is the Cauchy dual of the multiplication operator M_z.

In the joint paper [EL19] this result was extended to the multidimensional setting of all generalized Bergman spaces $A^2_m(\mathbb{B}_d)$ ($m \in \mathbb{N}_{\geq 1}$). In this chaper we will extend it to the still more general setting of a class of unitarily invariant spaces which in particular includes all generalized Bergman spaces $A^2_\nu(\mathbb{B}_d)$ ($\nu \in]0, \infty[$) (cf. Example 2.1.4 and 2.2.4).

As in [LO08], an essential tool will be the homogeneous decomposition of operators $T \in L(H_K)$.

3.1 Homogeneous decompositions

In the following chapters, let H_K be a unitarily invariant space with reproducing kernel

$$K : \mathbb{B}_d \times \mathbb{B}_d \to \mathbb{C}, K(z,w) = \sum_{k=0}^{\infty} a_k \langle z, w \rangle^k$$

such that $a_0 = 1$, $a_k > 0$ for all k, $\sup \frac{a_k}{a_{k+1}} < \infty$ and $\inf \frac{a_k}{a_{k+1}} > 0$. Furthermore, suppose that the analytic function $k : \mathbb{D} \to \mathbb{C}, z \mapsto \sum_{k=0}^{\infty} a_k z^k$ has no zeroes and, denoting the sequence of Taylor coefficients of $\frac{1}{k}$ by $(c_k)_{k \in \mathbb{N}}$, that there exists a natural number $p \in \mathbb{N}$ such that

$$c_k \geq 0 \text{ for all } k \geq p \text{ or } c_k \leq 0 \text{ for all } k \geq p.$$

Lemma 3.1.1. The map

$$U : \mathbb{R} \to L(H_K), (U(t)f)(z) = f(e^{it}z)$$

defines a strongly continuous unitary operator group, that is, a strongly continuous operator group such that $U(t)^* = U(-t)$ for all $t \in \mathbb{R}$.

Proof. Using that the norm of the functional Hilbert space H_K is given by

$$\|f\| = \left(\sum_{\alpha \in \mathbb{N}^d} \frac{|f_\alpha|^2}{a_{|\alpha|}\gamma_\alpha} \right)^{\frac{1}{2}} \quad \text{for } f = \sum_{\alpha \in \mathbb{N}^d} f_\alpha z^\alpha \in H_K,$$

we see that the operators $U(t)$ $(t \in \mathbb{R})$ are well-defined isometries. Due to $U(t)U(-t) = 1_{H_K}$ for $t \in \mathbb{R}$, they are also unitary. It is easy to see that $U(0) = 1_{H_K}$ and $U(s+t) = U(s)U(t)$ for all $s, t \in \mathbb{R}$ hold.
By this functional equality, it is enough to prove continuity in 0. Thus, let $(t_k)_{k \in \mathbb{N}}$ be a sequence in $\mathbb{R}$ with $t_k \xrightarrow{k \to \infty} 0$. For $f = \sum_{\alpha \in \mathbb{N}^d} f_\alpha z^\alpha \in H_K$, we have

$$\frac{|f_\alpha|^2 |e^{it_k|\alpha|} - 1|^2}{a_{|\alpha|}\gamma_\alpha} \leq \frac{4|f_\alpha|^2}{a_{|\alpha|}\gamma_\alpha}$$

for all $\alpha \in \mathbb{N}^d$ and $k \in \mathbb{N}$. By the dominated convergence theorem, we conclude that

$$\|U(t_k)f - f\|_{H_K}^2 = \sum_{\alpha \in \mathbb{N}^d} \frac{|f_\alpha|^2 |e^{it_k|\alpha|} - 1|^2}{a_{|\alpha|}\gamma_\alpha} \xrightarrow{k \to \infty} 0.$$

$\square$

In the following, we use the orthogonal decomposition

$$H_K = \bigoplus_{k=0}^{\infty} \mathbb{H}_k$$

into the spaces of all homogeneous polynomials of degree k from Chapter 2.

Definition 3.1.2. (a) An operator $T \in L(H_K)$ is called homogeneous of degree $k \in \mathbb{Z}$ if

$$T\mathbb{H}_r \subseteq \mathbb{H}_{r+k}$$

for all $r \in \mathbb{N}$. Here, we set $\mathbb{H}_k = 0$ for $k < 0$.

(b) For $T \in L(H_K)$ and $k \in \mathbb{Z}$, the operator

$$T_k = \frac{1}{2\pi} \int_0^{2\pi} e^{-ikt} U(t) T U(t)^* dt$$

is called the kth homogeneous component of T. Here, the integrand is regarded as a continuous function with values in the locally convex space $(L(H_K), \tau_{SOT})$ and the integral is a weak integral in the sense of Definition 3.26 in [Rud80]. Note that the weak integral exists by Theorem 3.27 in [Rud80] and Section 20.6(3) in [Köt69]. All operator-valued integrals appearing in the following should be understood in this sense.

To see that the homogeneous components of an operator $T \in L(H_K)$ are indeed homogeneous, we need to understand the orthogonal projections P_k of H_K onto the subspaces $\mathbb{H}_k$.

Lemma 3.1.3. The operators

$$P_k = \frac{1}{2\pi} \int_0^{2\pi} e^{-ikt} U(t) dt \in L(H_K) \quad (k \in \mathbb{Z})$$

are orthogonal projections with $\operatorname{ran} P_k = \mathbb{H}_k$ for all $k \in \mathbb{N}$ and we have $P_k = 0$ for all $k < 0$. Furthermore, we have $U(t)P_k = e^{ikt} P_k$ for all $t \in \mathbb{R}$ and $k \in \mathbb{N}$.

Proof. For every $k \in \mathbb{Z}$ and $\alpha \in \mathbb{N}^d$, we have

$$\begin{aligned} P_k(z^\alpha) &= \frac{1}{2\pi} \int_0^{2\pi} e^{-ikt} U(t) z^\alpha dt \\ &= \frac{1}{2\pi} \left(\int_0^{2\pi} e^{it(|\alpha|-k)} dt \right) z^\alpha \\ &= \begin{cases} z^\alpha, & |\alpha| = k \\ 0, & |\alpha| \neq k \end{cases}. \end{aligned}$$

This immediately yields the first part and due to

$$U(t)P_k(z^\alpha) = \begin{cases} e^{ikt}z^\alpha, & |\alpha| = k \\ 0, & |\alpha| \neq k \end{cases}$$

for every $t \in \mathbb{R}$, $k \in \mathbb{N}$ and $\alpha \in \mathbb{N}^d$ also the second part of the lemma. □

We gather some properties of the homogeneous components of an operator.

Lemma 3.1.4. Let $T \in L(H_K)$ and $k \in \mathbb{Z}$.

(a) The kth homogeneous component of T is homogeneous of degree k.

(b) We have

$$(T^*)_k = (T_{-k})^*.$$

Proof. (a) Let $r \in \mathbb{N}$. We have $U(t)^*|_{\mathbb{H}_r} = U(-t)|_{\operatorname{ran} P_r} = e^{-irt}1_{\operatorname{ran} P_r}$ due to Lemma 3.1.3 and thus

$$T_k f = \frac{1}{2\pi}\int_0^{2\pi} e^{-i(k+r)t}U(t)Tf\,dt = P_{k+r}Tf \in \mathbb{H}_{k+r}$$

for all $f \in \mathbb{H}_r$. Note that $P_k = 0$ as well as $\mathbb{H}_k = 0$ for $k < 0$.

(b) For all $f, g \in H_K$, we have

$$\begin{aligned}\langle (T^*)_k f, g\rangle &= \frac{1}{2\pi}\int_0^{2\pi} \langle e^{-ikt}U(t)T^*U(t)^* f, g\rangle dt \\ &= \frac{1}{2\pi}\int_0^{2\pi} \langle f, e^{ikt}U(t)TU(t)^* g\rangle dt \\ &= \langle f, T_{-k}g\rangle.\end{aligned}$$

□

If we know the homogeneous components of an operator $T \in L(H_K)$, we can recover Tf for every $f \in H_K$. This is particularly easy for the images Tp of polynomials $p \in \mathbb{C}[z]$.

Lemma 3.1.5. Let $T \in L(H_K)$ be arbitrary. Using the Fejér kernel

$$K_N : \mathbb{R} \to \mathbb{R}, K_N(t) = \sum_{|k|\leq N} (1 - \frac{|k|}{N+1})e^{ikt} \quad (N \in \mathbb{N}),$$

we have

$$Tf = \lim_{N\to\infty} \frac{1}{2\pi} \int_0^{2\pi} K_N(t)U(t)TU(t)^* f dt = \lim_{N\to\infty} \sum_{|k|\le N} \left(1 - \frac{|k|}{N+1}\right) T_k f$$

for every $f \in H_K$.
For a polynomial $p \in \mathbb{C}[z]$ of degree at most N, we have

$$Tp = \sum_{k=-N}^{\infty} T_k p = \sum_{k=-\infty}^{\infty} T_k p.$$

Proof. The first part follows from Lemma I.2.2 in [Kat04]. For the second part, first recall from Lemma 3.1.3 that $U(t)^*|_{\mathbb{H}_r} = e^{-irt} 1_{\operatorname{ran} P_r}$ and then observe that for a homogeneous polyonomial $p \in \mathbb{H}_r$ $(r \in \mathbb{N})$, we have

$$\begin{aligned} Tp &= \sum_{k=0}^{\infty} P_k T p = \sum_{k=0}^{\infty} \frac{1}{2\pi} \int_0^{2\pi} e^{-ikt} U(t) T p dt \\ &= \sum_{k=-r}^{\infty} \frac{1}{2\pi} \int_0^{2\pi} e^{-ikt} U(t) T U(t)^* p dt = \sum_{k=-r}^{\infty} T_k p = \sum_{k=-\infty}^{\infty} T_k p, \end{aligned}$$

where the last identity follows since $T_k|_{\mathbb{H}_r} = 0$ for $k < -r$ by Lemma 3.1.4. The claim follows by writing an arbitrary polynomial p as a sum of homogeneous polynomials. $\square$

3.2 Toeplitz operators with pluriharmonic symbol

The classical definition of Toeplitz operators as compressions of multiplication operators does not make sense in our more general setting of unitarily invariant spaces since these spaces need not be contained in a larger space on which multiplication operators with L^∞-symbols are well-defined. However, if we restrict ourselves to pluriharmonic symbols $f = g + \overline{h}$ with $g, h \in H_K$, it is still possible to give a definition of Toeplitz operators $T_f \in L(H_K)$.

First, let us define multiplication operators with symbol $f \in H_K$.

Definition 3.2.1. For $f \in H_K$, set

$$D_f = \{u \in H_K; fu \in H_K\} \subseteq H_K$$

and define $T_f : D_f \to H_K, u \mapsto fu$. Note that T_f is densely defined, since we have $\mathbb{C}[z] \subseteq D_f$, and closed, since convergence in H_K implies pointwise convergence.

In the following, we will denote the αth Taylor coefficient of a function $g \in H_K$ at $z = 0$ by

$$g_\alpha = \frac{g^{(\alpha)}(0)}{\alpha!} = \frac{\langle g, z^\alpha \rangle}{\|z^\alpha\|^2} = a_{|\alpha|}\gamma_\alpha \langle g, z^\alpha \rangle.$$

Pluriharmonic functions $f : \mathbb{B}_d \to \mathbb{C}$ can be characterized as those functions which can be written as $f = g + \overline{h}$ where $g, h : \mathbb{B}_d \to \mathbb{C}$ are holomorphic functions (see for instance Chapter 4.4 in [Rud80]). The next result shows that also the adjoints T_f^* of the operators T_f $(f \in H_K)$ are densely defined and closed.

Lemma 3.2.2. Let $f \in H_K$. The domain of T_f^* contains the polynomials and, for a fixed polynomial $p \in \mathbb{C}[z]$, the mapping

$$H_K \to H_K, f \mapsto T_f^* p$$

is conjugate linear and continuous. Furthermore, we have

$$T_f^* z^\alpha = \sum_{\beta \leq \alpha} \frac{a_{|\alpha-\beta|}\gamma_{\alpha-\beta}}{a_{|\alpha|}\gamma_\alpha} \overline{f_\beta} z^{\alpha-\beta}$$

for all $\alpha \in \mathbb{N}^d$.

Proof. For $\alpha \in \mathbb{N}^d$ and $u \in D_f$, we have

$$\begin{aligned}\langle fu, z^\alpha \rangle_{H_K} &= \frac{(fu)_\alpha}{a_{|\alpha|}\gamma_\alpha} = \sum_{\beta \leq \alpha} \frac{f_\beta u_{\alpha-\beta}}{a_{|\alpha|}\gamma_\alpha} \\ &= \langle u, \sum_{\beta \leq \alpha} \frac{a_{|\alpha-\beta|}\gamma_{\alpha-\beta}}{a_{|\alpha|}\gamma_\alpha} \overline{f_\beta} z^{\alpha-\beta} \rangle_{H_K}.\end{aligned}$$

In particular, the function

$$D_f \to \mathbb{C}, u \mapsto \langle T_f u, z^\alpha \rangle_{H_K}$$

is continuous for all $\alpha \in \mathbb{N}^d$ and the domain of the adjoint T_f^* of T_f contains the polynomials. We also conclude that

$$T_f^* z^\alpha = \sum_{\beta \leq \alpha} \frac{a_{|\alpha-\beta|}\gamma_{\alpha-\beta}}{a_{|\alpha|}\gamma_\alpha} \overline{f_\beta} z^{\alpha-\beta}$$

holds for all $\alpha \in \mathbb{N}^d$. Clearly the right-hand side is conjugate linear as a function of f. Since convergence in H_K implies uniform convergence on all compact subsets of $\mathbb{B}_d$, also the middle parts of Lemma 3.2.2 follow. $\square$

In order to define Toeplitz operators with pluriharmonic symbol $f = g + \overline{h}$ where $g, h \in H_K$, we need the following lemma.

Lemma 3.2.3. Let $g_1, h_1, g_2, h_2 \in H_K$ be such that $g_1 + \overline{h_1} = g_2 + \overline{h_2}$. Then we have

$$T_{g_1} p + T^*_{h_1} p = T_{g_2} p + T^*_{h_2} p$$

for all $p \in \mathbb{C}[z]$.

Proof. For g_1, g_2, h_1, h_2 as above, the function $\overline{h_1 - h_2} = g_2 - g_1$ is analytic. Since $h_1 - h_2$ is also analytic, there is $c \in \mathbb{C}$ with $h_1 - h_2 = c$ and thus $g_2 - g_1 = \overline{c}$. This shows that $h_{1,0} = h_{2,0} + c$, $g_{2,0} = g_{1,0} + \overline{c}$ and $h_{1,\alpha} = h_{2,\alpha}$ as well as $g_{1,\alpha} = g_{2,\alpha}$ for all $\alpha \in \mathbb{N}^d \backslash \{0\}$. For $\alpha \in \mathbb{N}^d$, we conclude that

$$\begin{aligned} T_{g_1} z^\alpha + T^*_{h_1} z^\alpha &= \sum_{\beta \in \mathbb{N}^d} g_{1,\beta} z^{\alpha+\beta} + \sum_{\beta \le \alpha} \frac{a_{|\alpha-\beta|}\gamma_{\alpha-\beta}}{a_{|\alpha|}\gamma_\alpha} \overline{h_{1,\beta}} z^{\alpha-\beta} \\ &= \sum_{\beta > \alpha} g_{1,\beta-\alpha} z^\beta + g_{1,0} z^\alpha + \overline{h_{1,0}} z^\alpha + \sum_{\beta < \alpha} \frac{a_{|\beta|}\gamma_\beta}{a_{|\alpha|}\gamma_\alpha} \overline{h_{1,\alpha-\beta}} z^\beta \\ &= \sum_{\beta > \alpha} g_{2,\beta-\alpha} z^\beta + g_{2,0} z^\alpha + \overline{h_{2,0}} z^\alpha + \sum_{\beta < \alpha} \frac{a_{|\beta|}\gamma_\beta}{a_{|\alpha|}\gamma_\alpha} \overline{h_{2,\alpha-\beta}} z^\beta \\ &= T_{g_2} z^\alpha + T^*_{h_2} z^\alpha. \end{aligned}$$

By linearity, the assertion follows. □

Thus, we can define Toeplitz operators with pluriharmonic symbol in the following way.

Definition 3.2.4. For $g, h \in H_K$, we call an operator $T \in L(H_K)$ a Toeplitz operator with pluriharmonic symbol $f = g + \overline{h}$ if

$$Tp = T_g p + T^*_h p$$

holds for all $p \in \mathbb{C}[z]$.

In the setting of Definition 3.2.4 the symbol f is uniquely determined by the operator $T \in L(H_K)$.

Lemma 3.2.5. Let $T \in L(H_K)$ be a Toeplitz operator with pluriharmonic symbols $f_1 = g_1 + \overline{h_1}$ and $f_2 = g_2 + \overline{h_2}$ where $g_1, g_2, h_1, h_2 \in H_K$. Then, we have $f_1 = f_2$.

Proof. By Lemma 3.2.2 we know that $T^*_{h_1-h_2}p = T^*_{h_1}p - T^*_{h_2}p$ for $p \in \mathbb{C}[z]$. Hence, under the hypothesis of the Lemma, $0 \in L(H_K)$ is a Toeplitz operator with symbol $f = g + \overline{h}$ where $g = g_1 - g_2$ and $h = h_1 - h_2$.
This yields

$$g + \overline{h}(0) = (T_g + T^*_h)(1) = 0.$$

But then $T^*_h z^\alpha = -T_g z^\alpha = \overline{h}(0)z^\alpha$ for all $\alpha \in \mathbb{N}^d$. Using the formula for $T^*_h z^\alpha$ proved in Lemma 3.2.2 we conclude that $h_\beta = 0$ for all $\beta \in \mathbb{N}^d \backslash \{0\}$. Thus $h = h(0)$ and $g = -\overline{h(0)}$. We conclude that $f = 0$ or equivalently $f_1 = f_2$. □

As the final result of this chapter we note that our definition of Toeplitz operators with pluriharmonic symbol coincides with the usual definition on the Hardy space and the generalized Bergman spaces if the symbol is bounded. At the end of the next chapter, we will see that the boundedness of the symbol is actually no further restriction, since for every Toeplitz operator with pluriharmonic symbol f, the function f is automatically bounded.

Remark 3.2.6. Consider the special case

$$K = K_m : \mathbb{B}_d \times \mathbb{B}_d \to \mathbb{C}, (z, w) \mapsto \frac{1}{(1 - \langle z, w \rangle)^m}$$

where $m \geq d$ is an integer and write $A^2_m(\mathbb{B}_d) = H_{K_m}$ as in Example 2.1.4. Note that, for $m = d$, the space $A^2_d(\mathbb{B}_d)$ is the Hardy space

$$A^2_d(\mathbb{B}_d) = \{f \in \mathcal{O}(\mathbb{B}_d); \|f\|^2 = \sup_{0<r<1} \int_{\partial\mathbb{B}_d} |f(r\xi)|^2 d\sigma(\xi) < \infty\},$$

where σ is the canonical probability measure on $\partial\mathbb{B}_d$, while for $m \geq d+1$, the space $A^2_m(\mathbb{B}_d)$ is the weighted Bergman space

$$A^2_m(\mathbb{B}_d) = \{f \in \mathcal{O}(\mathbb{B}_d); \|f\|^2 = \int_{\mathbb{B}_d} |f|^2 d\mu_m < \infty\},$$

of all analytic functions that are square integrable with respect to the measure $\mu_m = \frac{(m-1)!}{(m-d-1)!\pi^d}(1 - |z|^2)^{m-d-1}dz$ which is absolutely continuous with respect to the Lebesque measure dz on $\mathbb{C}^d$.
Let $f : \mathbb{B}_d \to \mathbb{C}$ be a bounded pluriharmonic function. Then there are functions $g, h \in A^2_m(\mathbb{B}_d)$ with $f = g + \overline{h}$ (see for instance Proposition 6.1 in [Zhe98]). Suppose first that $m \geq d+1$. Let $\mathcal{T}_f = P_{A^2_m(\mathbb{B}_d)} M_f|_{A^2_m(\mathbb{B}_d)}$, where $P_{A^2_m(\mathbb{B}_d)}$ denotes the orthogonal projection of $L^2(\mathbb{B}_d, \mu_m)$ onto $A^2_m(\mathbb{B}_d)$ and M_f is the operator of multiplication with f on $L^2(\mathbb{B}_d, \mu_m)$. Then

$$\begin{aligned}\langle \mathcal{T}_f p, q\rangle_{A^2_m(\mathbb{B}_d)} &= \langle fp, q\rangle_{L^2(\mathbb{B}_d,\mu_m)} = \langle gp, q\rangle_{L^2(\mathbb{B}_d,\mu_m)} + \langle p, hq\rangle_{L^2(\mathbb{B}_d,\mu_m)} \\ &= \langle T_g p + T^*_h p, q\rangle_{A^2_m(\mathbb{B}_d)}\end{aligned}$$

for all polynomials $p, q \in \mathbb{C}[z]$. Next let us consider the case $m = d$. Let $h^\infty(\mathbb{B}_d)$ be the Banach space of all bounded $\mathcal{M}$-harmonic functions $f : \mathbb{B}_d \to \mathbb{C}$ equipped with the supremum norm $\|f\|_\infty = \sup_{z \in \mathbb{B}_d} |f(z)|$. By Theorem 3.3.4 and Theorem 4.3.3 (as well as its proof) in [Rud80] it follows that the Poisson Transform defines an isometric isomorphism

$$L^\infty(\sigma) \to h^\infty(\mathbb{B}_d), \varphi \mapsto P[\varphi]$$

between Banach spaces. By Theorem 5.4.9 and Remark 5.3.3 in [Rud80] the inverse of the above isomorphism is given by the boundary map

$$h^\infty(\mathbb{B}_d) \to L^\infty(\sigma), \varphi \mapsto \varphi^*$$

which associates with each function $\varphi \in h^\infty(\mathbb{B}_d)$ its Koranyi limit φ^*. For $\varphi \in h^\infty(\mathbb{B}_d)$, the Toeplitz operator $\mathcal{T}_\varphi : A^2(\mathbb{B}_d) \to A^2(\mathbb{B}_d)$ is defined by

$$\mathcal{T}_\varphi(u) = C[\varphi^* u^*]$$

where the right-hand side denote the Cauchy integral of $\varphi^* u^* \in L^2(S)$ (cf. Theorem 5.6.8 and Corollary 6.3.1 in [Rud80]) with $S = \partial\mathbb{B}_d$. For f, g, h as above and any pair of polynomials $p, q \in \mathbb{C}[z]$, we obtain

$$\langle \mathcal{T}_f p, q\rangle_{H^2(\mathbb{B}_d)} = \langle C[(gp)^*], q\rangle_{H^2(\mathbb{B}_d)} + \langle C[(\overline{h}p)^*], q\rangle_{H^2(\mathbb{B}_d)}.$$

By Theorem 5.6.8 in [Rud80], we have

$$\langle C[(gp)^*], q\rangle_{H^2(\mathbb{B}_d)} = \langle gp, q\rangle_{H^2(\mathbb{B}_d)}$$

and as an application of Theorem 5.6.9 in [Rud80] we obtain

$$\begin{aligned}
&\langle C[(\overline{h}p)^*], q\rangle_{H^2(\mathbb{B}_d)} = \langle C[(\overline{h}p)^*], q^*\rangle_{L^2(S)} \\
&= \langle P_{H^2(S)}(\overline{h}p)^*, q^*\rangle_{L^2(S)} = \langle (\overline{h}p)^*, q^*\rangle_{L^2(S)} \\
&= \langle p^*, (hq)^*\rangle_{L^2(S)} = \langle p, hq\rangle_{H^2(\mathbb{B}_d)}.
\end{aligned}$$

Thus for $m \geq d$, it follows that

$$\langle \mathcal{T}_f p, q\rangle = \langle T_g p + T_h^* p, q\rangle$$

for all polynomials $p, q \in \mathbb{C}[z]$. Hence $T_f = \mathcal{T}_f$ on $A^2_m(\mathbb{B}_d)$ for $m \geq d$.

3.3 The characterisation

As noted before, Louhichi and Olofsson characterized Toeplitz operators with harmonic symbol on the weighted Bergman spaces $A^2_m(\mathbb{D})$ $(m \in \mathbb{N}_{\geq 1})$ by the algebraic identity

$$M_z'^* T M_z' = \sum_{k=0}^{m-1} (-1)^k \binom{m}{k+1} M_z^k T M_z^{*k},$$

where $M_z' = M_z(M_z^* M_z)^{-1}$ is the Cauchy dual of the multiplication operator $M_z : A^2_m(\mathbb{D}) \to A^2_m(\mathbb{D}), g \mapsto zg$.
In the recent joint paper [EL19], this result was extended to a multidimensional setting using the diagonal operator

$$\delta : A^2_m(\mathbb{B}_d) \to A^2_m(\mathbb{B}_d), \delta(\sum_{k=0}^{\infty} \sum_{|\alpha|=k} f_\alpha z^\alpha) = f_0 + \sum_{k=1}^{\infty} \frac{m+k-1}{k} \sum_{|\alpha|=k} f_\alpha z^\alpha.$$

More precisely, it was shown that an operator $T \in L(A^2_m(\mathbb{B}_d))$ is a Toeplitz operator with pluriharmonic symbol if and only if

$$M_z^* \delta T \delta M_z = P_{\operatorname{ran} M_z^*} (\oplus \sum_{k=0}^{m-1} (-1)^k \binom{m}{k+1} \sigma^k_{M_z}(T)) P_{\operatorname{ran} M_z^*}.$$

Here $\delta M_z : A^2_m(\mathbb{B}_d)^d \to A^2_m(\mathbb{B}_d)$ is a continuous linear extension of the operator $M_z(M_z^* M_z)^{-1} : \operatorname{ran} M_z^* \to A^2_m(\mathbb{B}_d)$ and thus coincides with the Cauchy dual of M_z in dimension $d = 1$. Analyzing the proof, one notices that the sum on the right-hand side originates from the representation

$$\Delta = \sum_{k=0}^{m-1} (-1)^k \binom{m}{k+1} \sigma^k_{M_z}(1_{A^2_m(\mathbb{B}_d)})$$

of the diagonal operator

$$\Delta : A^2_m(\mathbb{B}_d) \to A^2_m(\mathbb{B}_d),$$
$$\Delta(\sum_{k=0}^{\infty} \sum_{|\alpha|=k} f_\alpha z^\alpha) = \sum_{k=0}^{\infty} \frac{m+k}{k+1} \sum_{|\alpha|=k} f_\alpha z^\alpha = \sum_{k=0}^{\infty} \frac{a_{k+1}}{a_k} \sum_{|\alpha|=k} f_\alpha z^\alpha$$

with $a_k = (-1)^k \binom{-m}{k}$ for all $k \in \mathbb{N}$ as in Example 2.1.4.
In Theorem 2.2.6, this representation was generalized to the setting of unitarily

invariant spaces H_K with the properties specified at the beginning of Chapter 3.1 as

$$\Delta = \text{SOT} - \sum_{k=0}^{\infty} (-c_{k+1}) \sigma_{M_z}^k (1_{H_K})$$

where $(c_k)_{k \in \mathbb{N}}$ denotes the sequence of Taylor coefficients of the reciprocal $\frac{1}{k}$ of the holomorphic function $k : \mathbb{D} \to \mathbb{C}, z \mapsto \sum_{k=0}^{\infty} a_k z^k$ associated with the reproducing kernel $K : \mathbb{B}_d \times \mathbb{B}_d \to \mathbb{C}, K(z, w) = \sum_{k=0}^{\infty} a_k \langle z, w \rangle^k$.
Before we can write down the identity characterizing Toeplitz operators in this setting, we therefore want to establish that the limit

$$\Delta_{M_z;T} = \text{SOT} - \sum_{k=0}^{\infty} (-c_{k+1}) \sigma_{M_z}^k (T) \in L(H_K)$$

exists for every $T \in L(H_K)$. For this purpose, fix a number $p \in \mathbb{N}$ such that all the coefficients c_k $(k \geq p)$ have the same sign. By Theorem 2.2.2 the strong limit

$$A = \text{SOT} - \sum_{k=p}^{\infty} |c_{k+1}| \sigma_{M_z}^k (1_{H_K}) \in L(H_K)$$

exists.

Lemma 3.3.1. For every operator $T \in L(H_K)$, the sequence of partial sums

$$\left(\sum_{k=0}^{N} c_{k+1} \sigma_{M_z}^k (T) \right)_{N \in \mathbb{N}}$$

is norm-bounded.

Proof. It suffices to show that

$$\left(\sum_{k=p}^{N} |c_{k+1}| \sigma_{M_z}^k (T) \right)_{N \in \mathbb{N}}$$

is norm-bounded for every $T \in L(H_K)$. For $N \geq p$, we define a positive operator $\sigma_N : L(H_K) \to L(H_K)$ by

$$\sigma_N(T) = \sum_{k=p}^{N} |c_{k+1}| \sigma_{M_z}^k (T) \quad (T \in L(H_K)).$$

The sequence $(\sigma_N)_{N\geq p}$ is norm-bounded since

$$\|\sigma_N\| = \|\sigma_N(1_{H_K})\| = \|\sum_{k=p}^{N} |c_{k+1}|\sigma_{M_z}^k(1_{H_K})\| \leq \|A\|$$

for all $N \geq p$. As a norm-bounded sequence the sequence $(\sigma_N)_{N\geq p}$ is also pointwise bounded. □

Lemma 3.3.2. For every operator $T \in L(H_K)$ the strong operator limit

$$\Delta_{M_z;T} = \mathrm{SOT} - \sum_{k=0}^{\infty}(-c_{k+1})\sigma_{M_z}^k(T) \in L(H_K)$$

exists.

Proof. Let $T \in L(H_K)$. Since we can pass to the positive and negative parts of $\operatorname{Re} T$ and $\operatorname{Im} T$, we may suppose that T is a positive operator. In this case the sequence $\left(\sum_{k=0}^{N} c_{k+1}\sigma_{M_z}^k(T)\right)_{N\geq p}$ consists of self-adjoint operators and is either increasing or decreasing. Since by Lemma 3.3.1 this sequence is norm-bounded, the assertion follows. □

Let us define $\mathcal{T}_{BH}(K) \subseteq L(H_K)$ as the set of all operators $T \in L(H_K)$ such that

$$M_z^*\delta T\delta M_z = P_{\operatorname{ran} M_z^*}\left(\bigoplus \Delta_{M_z;T}\right)P_{\operatorname{ran} M_z^*}.$$

Note that the mapping $L(H_K) \to L(H_K), T \mapsto \Delta_{M_z;T}$ is continuous linear. The linearity is obvious. The continuity follows from the positivity and thus continuity of the map

$$L(H_K) \to L(H_K), T \mapsto \mathrm{SOT}\text{-}\sum_{k=p}^{\infty}|c_{k+1}|\sigma_{M_z}^k(T).$$

Now, let $B \subseteq L(H_K)$ be the closed unit ball. For $T \in B$ self-adjoint, we have $-\|T\|1_{H_K} \leq T \leq \|T\|1_{H_K}$ and thus $-\sigma_{M_z}^j(1_{H_K}) \leq \sigma_{M_z}^j(T) \leq \sigma_{M_z}^j(1_{H_K})$ for $j \in \mathbb{N}$. We use this to deduce the estimates

$$\left|\left\langle \sum_{j=q}^{\infty}|c_{j+1}|\sigma_{M_z}^j(T)f, f\right\rangle\right| \leq \left\langle \sum_{j=q}^{\infty}|c_{j+1}|\sigma_{M_z}^j(1_{H_K})f, f\right\rangle$$

for every $f \in H_K$ and $q \in \mathbb{N}$. For $f \in H_K$, the right-hand side tends to 0 for $q \to \infty$. Thus, for $\epsilon > 0$, there is a natural number $N = N(\epsilon, f) \geq p$ such that

$$\left| \left\langle \sum_{j=N}^{\infty} |c_{j+1}| \sigma_{M_z}^j(T) f, f \right\rangle \right| < \epsilon$$

for all self-adjoint operators $T \in B$.

Lemma 3.3.3. (a) For each multiindex $\gamma \in \mathbb{N}^d$, we have $M_z^\gamma \in \mathcal{T}_{\mathrm{BH}}(K)$.

(b) The subset $\mathcal{T}_{BH}(K) \subseteq L(H_K)$ is a weak*-closed operator system.

Proof.

(a) Let $\gamma \in \mathbb{N}^d$. The limit

$$\begin{aligned} \mathrm{SOT} - \sum_{k=0}^{\infty} (-c_{k+1}) \sigma_{M_z}^k (M_z^\gamma) &= \mathrm{SOT} - \sum_{k=0}^{\infty} (-c_{k+1}) \sum_{|\alpha|=k} \gamma_\alpha M_z^\alpha M_z^\gamma M_z^{*\alpha} \\ &= M_z^\gamma \mathrm{SOT} - \sum_{k=0}^{\infty} (-c_{k+1}) \sigma_{M_z}^k (1_{H_K}) = M_z^\gamma \Delta \end{aligned}$$

exists by Theorem 2.2.6.
Using Lemma 2.2.1 as well as the identity $(M_z^* M_z)^{-1} (M_z^* M_z) = P_{\operatorname{ran} M_z^*}$ and the fact that $\ker M_z = (\operatorname{ran} M_z^*)^\perp$, we obtain

$$\begin{aligned} M_z^* \delta M_z^\gamma \delta M_z &= P_{\operatorname{ran} M_z^*} (M_z^* \delta M_z^\gamma \delta M_z) P_{\operatorname{ran} M_z^*} \\ &= P_{\operatorname{ran} M_z^*} ((M_z^* M_z)^{-1} M_z^* M_z^\gamma M_z (\oplus \Delta)) P_{\operatorname{ran} M_z^*} \\ &= P_{\operatorname{ran} M_z^*} ((M_z^* M_z)^{-1} M_z^* M_z (\oplus M_z^\gamma \Delta)) P_{\operatorname{ran} M_z^*} \\ &= P_{\operatorname{ran} M_z^*} (\oplus M_z^\gamma \Delta) P_{\operatorname{ran} M_z^*} \end{aligned}$$

and thus $M_z^\gamma \in \mathcal{T}_{\mathrm{BH}}(K)$.

(b) Obviously, $\mathcal{T}_{BH}(K) \subseteq L(H_K)$ is a linear subspace. By part (a), we have that $1_{H_K} \in \mathcal{T}_{BH}(K)$. Let $T \in \mathcal{T}_{BH}(K)$ be arbitrary. Since the involution on $L(H_K)$ is WOT-continuous, we find that

$$\Delta_{M_z;T}^* = \mathrm{WOT} - \sum_{k=0}^{\infty} (-c_{k+1}) \sum_{|\alpha|=k} \gamma_\alpha M_z^\alpha T^* M_z^{\alpha *} = \Delta_{M_z;T^*}$$

and hence that

$$M_z^* \delta T^* \delta M_z = (M_z^* \delta T \delta M_z)^* = \left(P_{\operatorname{ran} M_z^*} \left(\oplus \Delta_{M_z;T}\right) P_{\operatorname{ran} M_z^*}\right)^* \\ = P_{\operatorname{ran} M_z^*} \left(\oplus \Delta_{M_z;T^*}\right) P_{\operatorname{ran} M_z^*}.$$

Thus we have shown that $\mathcal{T}_{BH}(K) \subseteq L(H_K)$ is an operator system.
By the Krein-Smullian Theorem (Theorem IV.6.4 in [SW99]) the weak*-closedness of $\mathcal{T}_{BH}(K) \subseteq L(H_K)$ follows, if we can show that $\mathcal{T}_{BH}(K) \cap B \subseteq B$ is weak*-closed. Here, $B \subseteq L(H_K)$ again denotes the closed unit ball. Let $(T_\alpha)_{\alpha \in A}$ be a net in $\mathcal{T}_{BH}(K) \cap B$ with $w^* - \lim_\alpha T_\alpha = T$ in B. Since

$$\operatorname{Re} T = \frac{T + T^*}{2} = w^* - \lim_\alpha \frac{T_\alpha + T_\alpha^*}{2}$$

and

$$\operatorname{Im} T = \frac{T - T^*}{2i} = w^* - \lim_\alpha \frac{T_\alpha - T_\alpha^*}{2i}$$

are also the w^*-limits of nets in $\mathcal{T}_{BH}(K) \cap B$, we may assume that $T_\alpha = T_\alpha^*$ for all $\alpha \in A$.
Let $f \in H_K$ and $\epsilon > 0$ be arbitrary. Choose $N \geq p$ as in the remarks preceding this Theorem. In particular, we have

$$\left| \left\langle \sum_{j=N}^{\infty} |c_{j+1}| \sigma_{M_z}^j (T_\alpha) f, f \right\rangle \right| < \epsilon$$

for all $\alpha \in A$. Then there is an index α_0 such that

$$\begin{aligned} &\langle (\Delta_{M_z;T_\alpha} - \Delta_{M_z;T}) f, f \rangle \\ &\leq \left| \sum_{j=0}^{N-1} (-c_{j+1}) \langle \sigma_{M_z}^j (T_\alpha - T) f, f \rangle \right| \\ &\quad + \left| \left\langle \sum_{j=N}^{\infty} |c_{j+1}| \sigma_{M_z}^j (T_\alpha) f, f \right\rangle \right| \\ &\quad + \left| \left\langle \sum_{j=N}^{\infty} |c_{j+1}| \sigma_{M_z}^j (T) f, f \right\rangle \right| < 3\epsilon \end{aligned}$$

for all $\alpha \geq \alpha_0$. Since on norm-bounded subsets of $L(H_K)$ the weak operator topology and the weak*-topology coincide, it follows that $w^* -$

$\lim_\alpha \Delta_{M_z;T_\alpha} = \Delta_{M_z;T}$. But then the observation that

$$\begin{aligned} M_z^*\delta T\delta M_z &= w^* - \lim_\alpha M_z^*\delta T_\alpha \delta M_z \\ =& w^* - \lim_\alpha P_{\operatorname{ran} M_z^*}\left(\oplus\Delta_{M_z;T_\alpha}\right) P_{\operatorname{ran} M_z^*} \\ =& P_{\operatorname{ran} M_z^*}\left(\oplus\Delta_{M_z;T}\right) P_{\operatorname{ran} M_z^*} \end{aligned}$$

completes the proof.

□

Next, we show that the homogeneous components of operators in $\mathcal{T}_{\mathrm{BH}}(K)$ also belong to $\mathcal{T}_{\mathrm{BH}}(K)$.

Lemma 3.3.4. Let $T \in \mathcal{T}_{BH}(K)$. Then $T_k \in \mathcal{T}_{BH}(K)$ for all $k \in \mathbb{Z}$.

Proof. Let $T \in \mathcal{T}_{BH}(K)$ and let $k \in \mathbb{Z}$ be a fixed integer. Since $\mathcal{T}_{BH}(K) \subseteq L(H_K)$ is an operator system, we may suppose that $T \geq 0$. Since $T \in \mathcal{T}_{BH}(K)$, we have

$$\begin{aligned} & M_z^*\delta T_k \delta M_z \\ =& \frac{1}{2\pi}\int_0^{2\pi} e^{-ikt}(\oplus U(t))P_{\operatorname{ran} M_z^*}\left(\oplus\sum_{l=0}^{\infty}(-c_{l+1})\sigma_{M_z}^l(T)\right)P_{\operatorname{ran} M_z^*}(\oplus U(t)^*)dt \\ =& P_{\operatorname{ran} M_z^*}\left(\bigoplus\frac{1}{2\pi}\int_0^{2\pi} e^{-ikt}U(t)\left(\sum_{l=0}^{\infty}(-c_{l+1})\sigma_{M_z}^l(T)\right)U(T)^*dt\right)P_{\operatorname{ran} M_z^*}. \end{aligned}$$

We claim that, for $f \in H_K$,

$$\begin{aligned} & \langle\left(\int_0^{2\pi} e^{-ikt}U(t)\left(\sum_{l=0}^{\infty}(-c_{l+1})\sigma_{M_z}^l(T)\right)U(t)^*dt\right)f, f\rangle \\ &= \sum_{l=0}^{\infty}(-c_{l+1})\int_0^{2\pi} e^{-ikt}\langle U(t)\sigma_{M_z}^l(T)U(t)^*f, f\rangle dt. \end{aligned}$$

Choose a natural number $p \in \mathbb{N}$ such that all coefficients c_k $(k \geq p)$ have the same sign. Let us consider the case that $c_k \geq 0$ for all $k \geq p$. Then

$$\left(\sum_{l=p}^{N} c_{l+1}\langle U(t)\sigma_{M_z}^l(T)U(t)^*f, f\rangle\right)_{N\geq p}$$

is an increasing sequence of continuous functions in $t \in [0, 2\pi]$ which converges pointwise to the continuous function

$$\begin{aligned}
&\left\langle U(t) \left(\sum_{l=p}^{\infty} c_{l+1} \sigma^l_{M_z}(T) \right) U(t)^* f, f \right\rangle \\
&= \sum_{l=p}^{\infty} c_{l+1} \langle U(t) \sigma^l_{M_z}(T) U(t)^* f, f \rangle.
\end{aligned}$$

As an application of the monotone convergence theorem it follows that

$$\begin{aligned}
&\left\langle \left(\int_0^{2\pi} e^{-ikt} U(t) \left(\sum_{l=p}^{\infty} (-c_{l+1}) \sigma^l_{M_z}(T) \right) U(t)^* dt \right) f, f \right\rangle \\
&= \lim_{N\to\infty} \int_0^{2\pi} e^{-ikt} \sum_{l=p}^{N} (-c_{l+1}) \langle U(t) \sigma^l_{M_z}(T) U(t)^* f, f \rangle dt \\
&= \sum_{l=p}^{\infty} (-c_{l+1}) \int_0^{2\pi} e^{-ikt} \langle U(t) \sigma^l_{M_z}(T) U(t)^* f, f \rangle dt.
\end{aligned}$$

An obvious modification of the above arguments yields the same result in the case that $c_k \leq 0$ for all $k \geq p$. Thus we have proved the claim.
A polarization argument shows that

$$\begin{aligned}
&\int_0^{2\pi} e^{-ikt} U(t) \left(\sum_{l=0}^{\infty} (-c_{l+1}) \sigma^l_{M_z}(T) \right) U(t)^* dt \\
&= \mathrm{WOT} - \sum_{l=0}^{\infty} (-c_{l+1}) \int_0^{2\pi} e^{-ikt} U(t) \sigma^l_{M_z}(T) U(t)^* dt \\
&= \mathrm{WOT} - \sum_{l=0}^{\infty} (-c_{l+1}) \int_0^{2\pi} e^{-ikt} U(t) \left(\sum_{|\alpha|=l} M_z^\alpha T M_z^{*\alpha} \right) U(t)^* dt \\
&= \mathrm{WOT} - \sum_{l=0}^{\infty} (-c_{l+1}) \sum_{|\alpha|=l} M_z^\alpha \left(\int_0^{2\pi} e^{-ikt} U(t) T U(t)^* dt \right) M_z^{*\alpha} \\
&= \mathrm{WOT} - \sum_{l=0}^{\infty} (-c_{l+1}) \sigma^l_{M_z}(T_k) = \Delta_{M_z; T_k}.
\end{aligned}$$

But then

$$M_z^* \delta T_k \delta M_z = P_{\operatorname{ran} M_z^*} \left(\oplus \Delta_{M_z; T_k} \right) P_{\operatorname{ran} M_z^*}$$

and hence $T_k \in \mathcal{T}_{BH}(K)$. $\square$

All operators $T \in \mathcal{T}_{\mathrm{BH}}(K)$ that are homogeneous of non-negative degree are multiplication operators.

Theorem 3.3.5. Let $T \in \mathcal{T}_{\mathrm{BH}}(K)$ be homogeneous of degree $r \in \mathbb{N}$. Then T acts as the multiplication operator

$$Tf = (T1)f \quad (f \in H_K).$$

Proof. Define $q = T1 \in \mathbb{H}_r$. Since q is a polynomial, the multiplication operator T_q is defined on the whole of H_K. We show by induction on k that $T = T_q$ on $\mathbb{H}_k$ for all $k \in \mathbb{N}$.
For $k = 0$, this is obvious. Suppose that the assertion has been proved for all $j = 0, ..., k$ and fix a polynomial $p \in \mathbb{H}_{k+1}$. By Lemma 2.2.1 and the identity $M_z(M_z^* M_z)^{-1} M_z^* = P_{\operatorname{ran} M_z}$, using $p \in \mathbb{H}_{k+1} \subseteq \mathbb{C}^\perp = \operatorname{ran} M_z$, we have

$$\begin{aligned} M_z^* \delta T \delta M_z(M_z^* p) &= M_z^* \delta T M_z (M_z^* M_z)^{-1} M_z^* p \\ &= M_z^* \delta T P_{\operatorname{ran} M_z} p \\ &= (\oplus \Delta) M_z^* (Tp) \\ &= \frac{a_{k+r+1}}{a_{k+r}} M_z^*(Tp). \end{aligned}$$

Using the induction hypothesis and Theorem 2.2.6, we find that

$$\begin{aligned} &P_{\operatorname{ran} M_z^*} \left(\oplus \mathrm{SOT} - \sum_{l=0}^{\infty} (-c_{l+1}) \sigma_{M_z}^l (T) \right) P_{\operatorname{ran} M_z^*}(M_z^* p) \\ =& P_{\operatorname{ran} M_z^*} \left(\oplus \mathrm{SOT} - \sum_{l=0}^{\infty} (-c_{l+1}) \sum_{|\alpha|=l} \gamma_\alpha M_z^\alpha T M_z^{*\alpha} \right) (M_z^* p) \\ =& P_{\operatorname{ran} M_z^*} \left(\oplus T_q \ \mathrm{SOT} - \sum_{l=0}^{\infty} (-c_{l+1}) \sigma_{M_z}^l (1_{H_K}) \right) (M_z^* p) \\ =& P_{\operatorname{ran} M_z^*} (\oplus T_q \Delta)(M_z^* p) = \frac{a_{k+1}}{a_k} P_{\operatorname{ran} M_z^*} (\oplus T_q)(M_z^* p). \end{aligned}$$

Since $T \in \mathcal{T}_{BH}(K)$, we conclude that

$$\frac{a_{k+r+1}}{a_{k+r}} M_z^*(Tp) = \frac{a_{k+1}}{a_k} P_{\operatorname{ran} M_z^*} (\oplus T_q)(M_z^* p).$$

We apply the operator $M_z(M_z^* M_z)^{-1} = \delta M_z|_{\operatorname{ran} M_z^*}$ (cf. Lemma 2.2.1) to both sides of this equation and further use the identities

$$(M_z^* M_z)^{-1}(M_z^* M_z) = P_{\operatorname{ran} M_z^*}, M_z(M_z^* M_z)^{-1} M_z^* = P_{\operatorname{ran} M_z}$$

from Lemma 4.2.2 a) to find that

$$\begin{aligned}\frac{a_{k+r+1}}{a_{k+r}}Tp &= M_z(M_z^*M_z)^{-1}(\frac{a_{k+r+1}}{a_{k+r}}M_z^*Tp)\\ &= M_z(M_z^*M_z)^{-1}(\frac{a_{k+1}}{a_k}P_{\operatorname{ran} M_z^*}(\oplus T_q)M_z^*p)\\ &= \frac{a_{k+1}}{a_k}\delta M_z(M_z^*M_z)^{-1}(M_z^*M_z)(\oplus T_q)M_z^*p\\ &= \frac{a_{k+1}}{a_k}\delta M_z(\oplus T_q)M_z^*p\\ &= \frac{a_{k+1}}{a_k}\delta T_q M_z M_z^*p\\ &= \frac{a_{k+1}}{a_k}\frac{a_{k+r+1}}{a_{k+r}}T_q M_z M_z^*p.\end{aligned}$$

Since $M_zM_z^* = \mathrm{SOT} - \sum_{k=1}^{\infty}\frac{a_{k-1}}{a_k}P_{\mathbb{H}_k}$ (Lemma 2.4 in [Wer14]), we conclude that

$$Tp = \frac{a_{k+1}}{a_k}T_q\frac{a_k}{a_{k+1}}p = T_qp.$$

This completes the induction and hence the whole proof. □

Now, we can prove, that every operator in $T_{\mathrm{BH}}(K)$ is indeed a Toeplitz operator with pluriharmonic symbol.

Theorem 3.3.6. Let $T \in \mathcal{T}_{\mathrm{BH}}(K)$ be given. Define

$$g = (T - T_0)(1) \text{ and } h = T^*(1).$$

Then $T = T_f$ is a Toeplitz operator with pluriharmonic symbol $f = g + \overline{h}$.

Proof. For $k \in \mathbb{Z}$, let as before

$$T_k = \frac{1}{2\pi}\int_0^{2\pi} e^{-ikt}U(t)TU(t)^*dt$$

denote the kth homogeneous component of T. Define $g, h \in H_K$ and the pluriharmonic function $f = g + \overline{h}$ as in the statement of the theorem. Our aim is to show that $T = T_f$. Set

$$q_k = T_k1 \text{ for } k \geq 0, q_k = \overline{(T_k)^*1} \text{ for } k < 0.$$

Using Lemma 3.1.4 and Lemma 3.1.5, we find that

$$T1 = \sum_{k=0}^{\infty} T_k1 = \sum_{k=0}^{\infty} q_k$$

and

$$T^*1 = \sum_{k=0}^{\infty}(T^*)_k 1 = \sum_{k=-\infty}^{0}(T_k)^*1 = \sum_{k=-\infty}^{0}\overline{q}_k$$

where all sequences converge in H_K. Since $T, T^* \in \mathcal{T}_{BH}(K)$, it follows from Lemma 3.3.4 and Theorem 3.3.5 that $T_k = T_{T_k(1)} = T_{q_k}$ for $k \geq 0$ and that

$$(T^*)_{-k} = T_{(T^*)_{-k}(1)} = T_{(T_k)^*1} = T_{\overline{q_k}}$$

for $k < 0$. Let $p \in \mathbb{C}[z]$ be a polynomial. Because of $T_k = ((T^*)_{-k})^*$ we can use Lemma 3.1.5 to deduce that

$$Tp = \sum_{k=-\infty}^{\infty} T_k p = \sum_{k=1}^{\infty} q_k p + \sum_{k=-\infty}^{0} T^*_{\overline{q}_k} p = T_g p + \sum_{k=-\infty}^{0} T^*_{\overline{q}_k} p.$$

Since the mapping $H_K \to H_K, u \mapsto T^*_u p$, is conjugate linear and continuous by Lemma 3.2.2, we conclude that

$$Tp = T_g p + T^*_h p.$$

Thus we have shown that $T = T_f$ with $f = g + \overline{h}$. $\square$

We will now prove that conversely every Toeplitz operator with pluriharmonic symbol satisfies our Brown-Halmos type condition and thus that these conditions are equivalent. To show the missing implication, we use an approximation argument.

Let $T = T_f \in L(H_K)$ be a Toeplitz operator with pluriharmonic symbol $f = g + \overline{h}$. $g, h \in H_K$, and let $g = \sum_{j=0}^{\infty} g_j$, $h = \sum_{j=0}^{\infty} h_j$ be the homogeneous expansions of g and h. For $k \in \mathbb{Z}$ and any homogeneous polynomial $p \in \mathbb{H}_r$ ($r \geq 0$), an application of Lemma 3.1.5 and Lemma 3.1.4 yields that

$$(T_f)_k p = P_{k+r}(T_f p) = P_{k+r}(\sum_{j=0}^{\infty} T_{g_j} p + \sum_{j=0}^{\infty} T^*_{h_j} p).$$

Since $T_{g_j}(\mathbb{H}_r) \subseteq \mathbb{H}_{j+r}$ and $T^*_{h_j}(\mathbb{H}_r) \subseteq \mathbb{H}_{-j+r}$, we obtain that

$$(T_f)_k\, p = \begin{cases} T_{g_k} p, & \text{if } k > 0 \\ T_{f(0)} p, & \text{if } k = 0 \\ T^*_{h_{-k}} p, & \text{if } k < 0. \end{cases}$$

Since the linear span of the homogeneous polynomials is dense in H_K, it follows that

$$(T_f)_k = \begin{cases} T_{g_k}, & \text{if } k > 0 \\ T_{f(0)}, & \text{if } k = 0 \\ T^*_{h_{-k}}, & \text{if } k < 0. \end{cases}$$

But then Lemma 3.1.5 shows that T_f is the SOT-limit of a sequence of operators in $\{T_p + T_q^*; p, q \in \mathbb{C}[z]\}$.

Theorem 3.3.7. An operator $T \in L(H_K)$ is a Toeplitz operator with pluriharmonic symbol if and only if $T \in \mathcal{T}_{BH}(K)$.

Proof. Let $T = T_f$ be a Toeplitz operator with pluriharmonic symbol $f = g + \overline{h}$ with $g, h \in H_K$. By the remarks preceding this Theorem and by Lemma 3.3.3 (a) there is a sequence $(T_j)_{j\in\mathbb{N}}$ in $\mathcal{T}_{BH}(K)$ such that $T_f = \mathrm{SOT} - \lim_{j\to\infty} T_j$. Since $(T_j)_{j\in\mathbb{N}}$ is normbounded by the uniform boundedness principle and since $\mathcal{T}_{BH}(K) \subseteq L(H_K)$ is weak*-closed by Lemma 3.3.3 (b), it follows that $T_f = w^* - \lim_{j\to\infty} T_j \in \mathcal{T}_{BH}(K)$. The remaining implication was proved in Theorem 3.3.6. □

3.4 An application to the Berezin transform

Given a Toeplitz operator $T \in \mathcal{T}_{\mathrm{BH}}(K)$ with pluriharmonic symbol, we can also recover this symbol by considering the Berezin transform $\tilde{T}$ of T. Let us recall that $\tilde{T}$ is defined by

$$\tilde{T} : \mathbb{B}_d \to \mathbb{C}, \tilde{T}(z) = \langle TK_z, K_z\rangle$$

where $K_z = \frac{K(\cdot,z)}{\|K(\cdot,z)\|}$ is the normalized kernel vector at $z \in \mathbb{B}_d$.
We observe that the mapping which associates with every $T \in L(H_K)$ its Berezin transform $\tilde{T}$ is one-to-one. This is due to the fact that the holomorphic function

$$\mathbb{B}_d \times \mathbb{B}_d \to \mathbb{C}, (z, w) \mapsto \langle TK(\cdot, \overline{z}), K(\cdot, w)\rangle$$

is uniquely determined by the values it takes on the conjugate diagonal defined as $\{(\overline{z}, z); z \in \mathbb{B}_d\}$.

As usual, we denote the essential norm of an operator $T \in L(H_K)$ by $\|T\|_e = \inf\{\|T - K\|; K \in L(H_K) \text{ compact}\}$.

Corollary 3.4.1. Let $T = T_f \in L(H_K)$ be a Toeplitz operator with pluriharmonic symbol f. Then $f = \tilde{T}$ and $\|T\|_e \geq \sup_{z\in\mathbb{B}_d} |f(z)|$.

Proof. Let $T = T_f \in L(H_K)$ be a Toeplitz operator with pluriharmonic symbol. It follows from Lemma 3.2.5 and the proof of Theorem 3.3.6 that $f = g + \overline{h}$, where

$$g = \sum_{k=1}^{\infty} q_k \quad \text{and} \quad h = \sum_{k=-\infty}^{0} \overline{q}_k$$

with $q_k = T_k 1$ for $k \geq 0$, $\overline{q}_k = (T_k)^* 1$ for $k < 0$. Here the series converge in H_K and hence also pointwise on $\mathbb{B}_d$. As seen in the proof of Theorem 3.3.6 we obtain that $T_k = T_{q_k}$ for $k \geq 0$ as well as $T_k = T^*_{\overline{q_k}}$ for $k < 0$. Thus we can use Lemma 3.1.5 to find that

$$\tilde{T}(z) = \lim_{N\to\infty} \sum_{|k|\leq N} (1 - \frac{|k|}{N+1}) \langle T_k K_z, K_z \rangle$$

$$= \lim_{N\to\infty} \Big[\sum_{k=1}^{N} (1 - \frac{k}{N+1}) \langle T_{q_k} K_z, K_z \rangle + \sum_{k=-N}^{0} (1 - \frac{|k|}{N+1}) \langle K_z, T_{\overline{q}_k} K_z \rangle \Big]$$

$$= \lim_{N\to\infty} \sum_{|k|\leq N} (1 - \frac{|k|}{N+1}) q_k(z) = g(z) + \overline{h(z)} = f(z)$$

for every $z \in \mathbb{B}_d$ In particular, the symbol f is a bounded pluriharmonic and hence also a bounded $\mathcal{M}$-harmonic function on $\mathbb{B}_d$ (cf. remark 4.4.2 b) in [Rud80]). But then the radial limits $f^*(\xi) = \lim_{r\uparrow 1} f(r\xi)$ exist for almost every $\xi \in S = \partial\mathbb{B}_d$ and define a function $f^* \in L^\infty(S)$ with $\sup_{z\in\mathbb{B}_d} |f(z)| = \|f^*\|_{L^\infty(S)}$. For each compact operator $K \in L(H_K)$ and every $z \in \mathbb{B}_d$, we obtain

$$|f(z) - \langle KK_z, K_z \rangle| = |\langle (T-K)K_z, K_z \rangle| \leq \|T - K\|.$$

Since weak $- \lim_{\|z\|\uparrow 1} K_z = 0$, it follows that

$$|f^*(\xi)| = \lim_{r\uparrow 1} |f(r\xi) - \langle KK_{r\xi}, K_{r\xi} \rangle| \leq \|T - K\|$$

for almost every $\xi \in S$. Since K was an arbitrary compact operator, the observation that $\sup_{z\in\mathbb{B}_d} |f(z)| = \|f^*\|_{L^\infty(S)} \leq \|T - K\|$ completes the proof. $\square$

In particular the symbol of a Toeplitz operator in $\mathcal{T}_{BH}(K)$ is always bounded and there are no non-zero compact Toeplitz operators in $\mathcal{T}_{BH}(K)$. On the Hardy and weighted Bergman spaces (i.e. if $K = K_m$ from Example 2.1.4 with $m \geq d$) even the equality $\|T_f\| = \|T_f\|_e = \sup_{z\in\mathbb{B}} |f(z)|$ holds for each Toeplitz operator with pluriharmonic symbol f. Indeed, in these cases the missing estimate $\|T_f\| \leq \sup_{z \in \mathbb{B}_d} |f(z)|$ obviously holds. On the other hand, an observation from [FX11] shows that on the Drury-Arveson space $H_1(\mathbb{B})$, there are even multipliers f for which the inequality in Corollary 3.4.1 is strict.

As a final result we give a characterization of Toeplitz operators with pluriharmonic symbol in terms of their Berezin transform.
For a continuous function $f : \mathbb{B}_d \to \mathbb{C}$ and $k \in \mathbb{Z}$, let us define

$$f_k(z) = \frac{1}{2\pi} \int_0^{2\pi} e^{-ikt} f(e^{it}z)\, dt \qquad (z \in \mathbb{B}_d)$$

If $f = g + \overline{h} : \mathbb{B}_d \to \mathbb{C}$ is pluriharmonic with $g, h \in \mathcal{O}(\mathbb{B}_d)$, then $f_k(z) = g_k(z) + \overline{h_{-k}(z)}$ where $g_k = 0 = h_k$ for $k < 0$ and $g(z) = \sum_{k=0}^{\infty} g_k(z)$, $h(z) = \sum_{k=0}^{\infty} h_k(z)$ are the homogeneous expansions of $g, h \in \mathcal{O}(\mathbb{B}_d)$ (cf. Satz 2.16 in [Esc17]).

Corollary 3.4.2. Let $T \in L(H_K)$. Then T is a Toeplitz operator with pluriharmonic symbol f if and only if the Berezin transform $\tilde{T} : \mathbb{B}_d \to \mathbb{C}, \tilde{T}(z) = \langle TK_z, K_z \rangle$, is pluriharmonic. In this case $f = \tilde{T}$.

Proof. Let $T \in L(H_K)$ be given and define $f = \tilde{T}$. First, suppose that f is pluriharmonic and that $f = g + \overline{h}$ with $g, h \in \mathcal{O}(\mathbb{B}_d)$. Define g_k, h_k as in the remarks preceding Corollary 3.4.2. For all $t \in \mathbb{R}$ and $z, w \in \mathbb{B}_d$, we have

$$(U(t)^* K_z)(w) = K_z(e^{-it}w) = \sum_{k=0}^{\infty} a_k \langle e^{-it}w, z \rangle = K_{e^{it}z}(w).$$

Hence for $k > 0$,

$$\begin{aligned} \langle T_k K_z, K_z \rangle &= \frac{1}{2\pi} \int_0^{2\pi} e^{-ikt} \langle TU(t)^* K_z, U(t)^* K_z \rangle\, dt \\ &= \frac{1}{2\pi} \int_0^{2\pi} e^{-ikt} f(e^{it}z)\, dt = g_k(z) \end{aligned}$$

and $\langle T_0 K_z, K_z \rangle \equiv f(0)$. Using Corollary 3.4.1 and the injectivity of the Berezin transform, we find that

$$T_k = T_{g_k} \quad (k > 0) \quad \text{ and } \quad T_0 = f(0) 1_{H_K}.$$

Since the Berezin transform of T^* is given by $(T^*)^{\sim} = \overline{f} = h + \overline{g}$, the above arguments applied to T^* yield that

$$T_{-k} = ((T^*)_k)^* = T_{h_k}^* \quad \text{for } k > 0.$$

But then we obtain that

$$T = \mathrm{SOT} - \lim_{N\to\infty} \sum_{|k|\le N} (1 - \frac{|k|}{N+1}) T_k = \mathrm{SOT} - \lim_{N\to\infty} \sum_{k=0}^{N} (1 - \frac{|k|}{N+1})(T_{g_k} + T^*_{h_k}).$$

Using Lemma 3.3.3, it follows from Theorem 3.3.7 that T is a Toeplitz operator with pluriharmonic symbol. The remaining assertions are a consequence of Corollary 3.4.1. $\square$

4 Analytic models

By results going back to Cowen and Douglas [CD83], Curto and Salinas [CS84], every Cowen-Douglas operator tuple $T \in L(H)^d$ on a Hilbert space H is locally unitarily equivalent to the tuple $M_z = (M_{z_1}, ..., M_{z_d}) \in L(\hat{H})^d$ of multiplication operators with the coordinate functions on a suitable analytic functional Hilbert space $\hat{H}$. In the first part of this chapter, we prove a similar model theorem for Cowen-Douglas tuples on Banach spaces.

4.1 Cowen-Douglas tuples on Banach spaces

In the following, let $\Omega \subseteq \mathbb{C}^d$ be a connected complex submanifold. Furthermore, let X be a Banach space and $T = (T_1, ..., T_d) \in L(X)^d$ a commuting tuple of bounded operators on X. For $z \in \mathbb{C}^d$, we use the notation $z - T$ for the commuting tuple $z - T = (z_1 - T_1, ..., z_d - T_d) \in L(X)^d$ and $Z - T$ for the row operator

$$Z - T : X^d \to X, (x_i)_{i=1}^d \mapsto \sum_{i=1}^{d} (z_i - T_i)x_i.$$

Sometimes, we will slightly abuse this notation and use $z - T$ when we are talking about the row operator $Z - T$.
With this notation, we have $\sum_{i=1}^d (z_i - T_i)X = \operatorname{ran}(Z - T)$.

Definition 4.1.1. A commuting tuple $T \in L(X)^d$ of bounded operators is called a weak dual Cowen-Douglas tuple of rank $N \in \mathbb{N}$ on Ω if

$$\dim(X/\operatorname{ran}(Z - T)) = N$$

for all $z \in \Omega$. If in addition, the condition

$$\bigcap_{z \in \Omega} \operatorname{ran}(Z - T) = \{0\}$$

holds, then T is called a dual Cowen-Douglas tuple of rank N on Ω.

Note that both conditions are preserved under similarity.

Lemma 4.1.2. Let $T \in L(X)^d$, $S \in L(Y)^d$ be commuting tuples of bounded operators on Banach spaces and let $\Pi : X \to Y$ be an invertible bounded linear operator with $\Pi T_i = S_i \Pi$ for $i = 1, .., d$. Then T is a (weak) dual Cowen-Douglas tuple on Ω if and only if S is a (weak) dual Cowen-Douglas tuple on Ω.

Proof. The operator

$$\hat{\Pi} : X/\operatorname{ran}(Z-T) \to Y/\operatorname{ran}(Z-S), x + \operatorname{ran}(Z-T) \mapsto \Pi x + \operatorname{ran}(Z-S)$$

is an isomorphism for every $z \in \Omega$. Thus T is a weak dual Cowen-Douglas tuple if and only if S is. Since

$$\Pi\left(\operatorname{ran}(Z-T)\right) = \operatorname{ran}(Z-S)$$

for every $z \in \Omega$. it follows that $\bigcap_{z\in\Omega} \operatorname{ran}(Z-T) = \{0\}$ if and only if $\bigcap_{z\in\Omega} \operatorname{ran}(Z-S) = \{0\}$. □

If $X = H$ is a Hilbert space and $\Omega \subseteq \mathbb{C}^d$ is open, then a tuple $T \in L(H)^d$ is a dual Cowen-Douglas tuple of rank N on Ω if and only if the adjoint $T^* = (T_1^*, ..., T_d^*)$ is a Cowen-Douglas tuple of rank N on the complex conjugate domain $\Omega^* = \{\overline{z}; z \in \Omega\}$ in the sense of Curto and Salinas [CS84]. A proof of this statemet can be found in [Wer14], Theorem 4.19.
We want to establish next that a dual Cowen-Douglas tuple on Ω is also a dual Cowen-Douglas tuple on each smaller domain $\emptyset \neq \Omega_0 \subseteq \Omega$. For open sets $\Omega \subseteq \mathbb{C}^d$, this follows immediately from Lemma 4.9 in [Wer14]. Analyzing Chapter 4 in [Wer14], one notices that some of its results and in particular Lemma 4.9 remain esentially true for weak dual Cowen-Douglas tuples on connected complex submanifolds $\Omega \subseteq \mathbb{C}^d$. We give the details for the convenience of the reader.
First, we recall the definition of holomorphic vector bundles in our setting.

Definition 4.1.3. (a) Let $N \in \mathbb{N}^*$, $\Omega \subseteq \mathbb{C}^d$ a complex submanifold and $\pi : E \to \Omega$ a continuous map from a topological space E to Ω. The map $\pi : E \to \Omega$ (abbreviatory (E, π) or E) is called a (topological) vector bundle of rank N over Ω if the following conditions hold:

(i) $E_z = \pi^{-1}(\{z\})$ is an N-dimensional vector space for all $z \in \Omega$.

(ii) For every $z_0 \in \Omega$, there exist an open neighbourhood U of z_0 and a homeomorphism $h : E_U = \pi^{-1}(U) \to U \times \mathbb{C}^N$ (equipped with the product topology) such that $P_1 \circ h = \pi$ (where P_1 denotes the projection on the first d components) and such that, for every $z \in U$, the map $h|_{E_z}$ is a vector space isomorphism from E_z to $\{z\} \times \mathbb{C}^N \cong \mathbb{C}^N$. The map h is called a linear chart of E over U, its inverse h^{-1} is called a trivialization of E over U.

If $(U_i)_{i\in I}$ is a family of open sets covering Ω with associated linear charts $h_i : E_{U_i} \to U_i \times \mathbb{C}^N$, then $(h_i)_{i\in I}$ is called an atlas of E. The set Ω is called the base space of the vector bundle.

(b) An atlas $(h_i)_{i\in I}$ of the vector bundle E is holomorphic if, for all $i, j \in I$ with $U_i \cap U_j \neq \emptyset$, the mappings

$$h_i \circ h_j^{-1} : (U_i \cap U_j) \times \mathbb{C}^N \to (U_i \cap U_j) \times \mathbb{C}^N$$

are holomorphic.

(c) Two holomorphic atlases $\mathcal{A} = (h_i)_{i\in I}, \mathcal{A}' = (g_j)_{j\in J}$ of a vector bundle E consisting of linear charts $h_i : E_{U_i} \to U_i \times \mathbb{C}^N$, $g_j : E_{V_j} \to V_j \times \mathbb{C}^N$ are holomorphically equivalent if the mappings

$$h_i \circ g_j^{-1} : (U_i \cap V_j) \times \mathbb{C}^N \to (U_i \cap V_j) \times \mathbb{C}^N$$

are holomorphic for all $i \in I$, $j \in J$ with $U_i \cap V_j \neq \emptyset$. The equivalence class of a holomorphic atlas is called a linear holomorphic structure of E. A vector bundle $\pi : E \to \Omega$ with a linear holomorphic structure is called a holomorphic vector bundle. Every map $h : E_U \to U \times \mathbb{C}^N$ contained in a representing holomorphic atlas is called a holomorphic chart of E and every inverse of such a map is called a trivialization of the holomorphic vector bundle E.

(d) (i) Let $\pi : E \to \Omega$ be a vector bundle of rank $N \in \mathbb{N}^*$, let $U \subseteq \Omega$ be open and let $(h_i)_{i\in I}$ be an atlas of E consisting of linear charts $h_i : E_{U_i} \to U_i \times \mathbb{C}^N$. A continuous function $f : U \to E$ with $\pi \circ f = \mathrm{id}_U$ is called a section in E over U. For each $i \in I$ with $U \cap U_i \neq \emptyset$, the map

$$f_{h_i} : U \cap U_i \to \mathbb{C}^N, f_i = P_2 \circ h_i \circ f|_{U\cap U_i}$$

is called the representation of f in the chart h_i. Here, $P_2 : \mathbb{C}^d \times \mathbb{C}^N \to \mathbb{C}^N$ is the projection onto the last N variables.

(ii) Let $\pi : E \to \Omega$ be a holomorphic vector bundle and suppose that the atlas $\mathcal{A} = (h_i)_{i\in I}$ represents the holomorphic linear structure of E. A section $f : U \to E$ is called holomorphic if its representations $f_{h_i} : U \cap U_i \to \mathbb{C}^N$ in the charts h_i are holomorphic for all $i \in I$ with $U \cap U_i \neq \emptyset$. This definition does not depend on the choice of $\mathcal{A}$. Let $\Gamma_{\mathrm{hol}}(U, E)$ denote the set of all holomorphic sections in E over U.

Note that for $f, g \in \Gamma_{\mathrm{hol}}(\Omega, E)$, $z \in \Omega$, we have $f(z), g(z) \in E_z$ and thus, the set $\Gamma_{\mathrm{hol}}(\Omega, E)$ can be given the structure of a vector space. The neutral element is the zero section $0 = 0_{\Gamma_{\mathrm{hol}}(\Omega,E)} \in \Gamma_{\mathrm{hol}}(\Omega, E)$ acting as $0(z) = 0_{E_z}$ for every $z \in \Omega$. We can prove a variant of the identity theorem for holomorphic sections.

Lemma 4.1.4. Let $\pi : E \to \Omega$ be a holomorphic vector bundle of rank $N \in \mathbb{N}^*$ over a connected submanifold $\Omega \subseteq \mathbb{C}^d$. Further, let $\gamma \in \Gamma_{\mathrm{hol}}(\Omega, E)$ be a holomorphic section such that there is an open set $\emptyset \neq U \subseteq \Omega$ with $\gamma(z) = 0_{E_z}$ for every $z \in U$. Then, we have $\gamma(z) = 0_{E_z}$ for all $z \in \Omega$.

Proof. The set

$$A = \{w \in \Omega; \exists W \subseteq \Omega \text{ open:} w \in W \text{ and } \gamma(z) = 0_{E_z} \text{ for every } z \in W\} \subseteq \Omega$$

is not empty by hypothesis. Since $A \subseteq \Omega$ is obviously open and Ω is connected, it suffices to show that $A \subseteq \Omega$ is closed. Let $(z_k)_{k\in\mathbb{N}}$ be a sequence in A and $z_0 \in \Omega$ with $\lim_{k\to\infty} z_k = z_0$. By definition of a vector bundle, there are a connected open neighbourhood V of z_0 and a linear chart $h : E_V \to V \times \mathbb{C}^N$. There is $k \in \mathbb{N}$ with $z_k \in V$ and since $z_k \in A$, we can choose $W_k \in \mathcal{U}(z_k)$ open in Ω with $W_k \subseteq V$ and $\gamma(z) = 0_{E_z}$ for every $z \in W_k$. The function $P_2 \circ h \circ \gamma|_V : V \to \mathbb{C}^N$ is holomorphic with

$$(P_2 \circ h \circ \gamma|_V)|_{W_k} \equiv 0.$$

By the identity theorem for holomorphic functions on connected open subsets of manifolds, we have $P_2 \circ h \circ \gamma|_V \equiv 0$. Since every restriction $h|_{E_z}$ $(z \in V)$ is an isomorphism, we conclude that $\gamma(z) = 0_{E_z}$ for every $z \in V$. In particular, $z_0 \in A$ and thus $A \subseteq \Omega$ is indeed closed. □

As in the case of open sets $\Omega \subseteq \mathbb{C}^d$ one can associate with each weak dual Cowen-Douglas tuple $T \in L(X)^d$ on our manifold $\Omega \subseteq \mathbb{C}^d$ a holomorphic vector bundle.

Theorem 4.1.5 (Theorem 4.4 in [Wer14])**.** Let $\Omega \subseteq \mathbb{C}^d$ be a connected complex submanifold and $T \in L(X)^d$ a weak dual Cowen-Douglas tuple of rank $N \in \mathbb{N}^*$ on Ω. Then $E_T = \bigcup_{z\in\Omega}\{z\} \times X/\operatorname{ran}(Z - T)$ can be given the structure of a holomorphic vector bundle in a canonical way.

Proof. We equip

$$E_T = \bigcup_{z\in\Omega}\{z\} \times X/\operatorname{ran}(Z - T)$$

with the quotient topology induced by the surjective map

$$q : \Omega \times X \to E_T, (z, x) \mapsto (z, x + \operatorname{ran}(Z - T)).$$

Writing

$$\pi : E_T \to \Omega, (z, x + \operatorname{ran}(Z - T)) \mapsto z$$

for the canonical projection, we first show that (E_T, π) is a topological vector bundle. First note that for $z \in \Omega$, the set $\pi^{-1}(\{z\}) = \{z\} \times X/\operatorname{ran}(Z - T)$ can be given the vector space structure from $X/\operatorname{ran}(Z - T)$ and since T is a weak dual Cowen-Douglas tuple of rank N, condition (i) in Definition 4.1.3 (a) is fulfilled. Since $\pi \circ q : \Omega \times X \to \Omega$ is continuous, the map π is continuous. Next, we will construct an atlas on E_T. For that end, we make some preliminary observations. Fix an arbitrary point $z_0 \in \Omega$ and let $D \subseteq X$ be an N-dimensional subspace with

$$X = \operatorname{ran}(z_0 - T) \oplus D.$$

Define

$$T(z) : X^d \oplus D \to X, ((x_i)_{i=1}^d, y) \mapsto \sum_{i=1}^{d} (z_i - T_i)x + y.$$

Then,

$$X \oplus D \xrightarrow{T(z)} X \to 0$$

is an analytically parametrized complex on $\mathbb{C}^d$ which is exact at $z = z_0$. By Lemma 2.1.5 in [EP96] there is an open neighbourhood $V \subseteq \mathbb{C}^d$ of z_0 in K such that the induced map

$$\mathcal{O}(V, X^d \oplus D) \to \mathcal{O}(V, X), ((g_i)_{i=1}^d, h) \mapsto \sum_{i=1}^{d} (z_i - T_i)g_i + h$$

is onto. Considering constant functions, we in particular have

$$X = \operatorname{ran}(Z - T) + D$$

for all $z \in V$. Define $\Omega_0 = \Omega \cap V$. Since $\dim X/\operatorname{ran}(Z - T) = N$ for all $z \in \Omega$, this ensures that

$$X = \operatorname{ran}(Z - T) \oplus D$$

for all $z \in \Omega_0$. For $z \in \Omega_0$, let $P_{\Omega_0}(z) : X \to D$ denote the projection onto D with kernel $\operatorname{ran}(Z - T)$ given by the decomposition $X = \operatorname{ran}(Z - T) \oplus D$ and let P_{Ω_0} denote the induced operator-valued function $P_{\Omega_0} : \Omega_0 \to L(X), z \mapsto$

$P_{\Omega_0}(z)$.
Fix a basis $e_1, ..., e_N$ of D and let $E_{\Omega_0} = (E_T)_{\Omega_0} = \pi^{-1}(\Omega_0)$. Furthermore, we consider the corresponding homeomorphism

$$\Phi : D \to \mathbb{C}^N, \sum_{i=1}^{N} \alpha_i e_i \mapsto (\alpha_i)_{i=1}^N$$

and the induced homeomorphism

$$\Phi_{\Omega_0} : \Omega_0 \times D \to \Omega_0 \times \mathbb{C}^N, (z, x) \mapsto (z, \Phi(x)).$$

Define $h_{\Omega_0} = \Phi_{\Omega_0} \circ \tilde{h}_{\Omega_0}$ where

$$\tilde{h}_{\Omega_0} : E_{\Omega_0} \to \Omega_0 \times D, (z, x + \operatorname{ran}(Z - T)) \mapsto (z, P_{\Omega_0}(z)x).$$

Additionally, we define

$$\tilde{g}_{\Omega_0} : \Omega_0 \times D \to E_{\Omega_0}, (z, x) \mapsto (z, x + \operatorname{ran}(Z - T))$$

and consider the composition $g_{\Omega_0} = \tilde{g}_{\Omega_0} \circ \Phi_{\Omega_0}^{-1}$.
Obviously, g_{Ω_0} and h_{Ω_0} are inverse to each other. One easily checks that the restrictions $\tilde{h}_{\Omega_0}|_{E_z}$ on the fibres $E_z = \pi^{-1}(\{z\})$ are linear for all $z \in \Omega_0$, thus h_{Ω_0} has the same property. Since $\tilde{g}_{\Omega_0} = q|_{\Omega_0 \times D}$, the map g_{Ω_0} is continuous by the definition of the topology on E_T and the continuity of $\Phi_{\Omega_0}^{-1}$. It remains to be shown that

$$\tilde{h}_{\Omega_0} \circ q : \Omega_0 \times X \to \Omega_0 \times D, (z, x) \mapsto (z, P_{\Omega_0}(z)x)$$

is continuous to establish the vector bundle structure of E_T with respect to the atlas given by the maps h_{Ω_0}. It is sufficient to show that P_{Ω_0} is continuous. We will even show that P_{Ω_0} is analytic which will be needed to establish the holomorphic vector bundle structure on E_T anyway. By Theorem A.2.9 in [Mül07], we only have to show that the map

$$\Omega_0 \to D, z \mapsto P_{\Omega_0}(z)x$$

is analytic for every $x \in X$. Fix $x \in X$. By the choice of Ω_0 there are analytic functions $f_1^x, ..., f_d^x \in \mathcal{O}(\Omega_0, X)$ and $g^x \in \mathcal{O}(\Omega_0, D)$ such that

$$x = \sum_{i=1}^{d} (z_i - T_i) f_i^x(z) + g^x(z)$$

holds for all $z \in \Omega_0$. Since $g^x(z) \in D$ for all $z \in \Omega_0$, we conclude that

$$g^x(z) = P_{\Omega_0}(z)x \text{ for all } z \in \Omega_0.$$

Hence the mapping $P_{\Omega_0}(\cdot)$ is analytic.
Next, we show that E_T is a holomorphic vector bundle. To that end, we consider the atlas given by the open sets Ω_0 and the linear charts h_{Ω_0} defined above. For two such open sets Ω_0, $\tilde{\Omega}_0$ with $\Omega_0 \cap \tilde{\Omega}_0 \neq \emptyset$, the map

$$h_{\tilde{\Omega}_0} \circ h_{\Omega_0}^{-1} : (\Omega_0 \cap \tilde{\Omega}_0) \times \mathbb{C}^N \to (\Omega_0 \cap \tilde{\Omega}_0) \times \mathbb{C}^N$$

acts as

$$\begin{aligned}(h_{\tilde{\Omega}_0} \circ h_{\Omega_0}^{-1})(z, (\alpha_i)_{i=1}^N) &= (\Phi_{\tilde{\Omega}_0} \circ \tilde{h}_{\tilde{\Omega}_0} \circ \tilde{h}_{\Omega_0}^{-1} \circ \Phi_{\Omega_0}^{-1})(z, (\alpha_i)_{i=1}^N) \\ &= (\Phi_{\tilde{\Omega}_0} \circ \tilde{h}_{\tilde{\Omega}_0})(z, \Phi^{-1}(\alpha_i)_{i=1}^N + \operatorname{ran}(Z-T))) \\ &= \Phi_{\tilde{\Omega}_0}(z, P_{\tilde{\Omega}_0}(z)(\Phi^{-1}(\alpha_i)_{i=1}^N)) \\ &= (z, (\Phi \circ P_{\tilde{\Omega}_0}(z) \circ \Phi^{-1})(\alpha_i)_{i=1}^N).\end{aligned}$$

Since the transition function $\Phi \circ P_{\tilde{\Omega}_0}(\cdot) \circ \Phi^{-1}$ is an analytic function on $\Omega_0 \cap \tilde{\Omega}_0$, the atlas is holomorphic. The corresponding holomorphic linear structure turns (E_T, π) into a holomorphic vector bundle. $\square$

This vector bundle structure and the identity theorem for holomorphic sections from Lemma 4.1.4 enable us as in [Wer14] to prove the following useful observation.

Lemma 4.1.6 (Lemma 4.9 in [Wer14])**.** Let $\Omega \subseteq \mathbb{C}^d$ be a connected submanifold and $T \in L(X)^d$ a weak dual Cowen-Douglas tuple. Then the map

$$\rho : X \to \Gamma_{\text{hol}}(\Omega, E_T), x \mapsto \hat{x},$$

where $\hat{x}(z) = (z, x + \operatorname{ran}(Z - T))$ for $z \in \Omega$, is well-defined and linear with

$$\ker \rho = \bigcap_{z \in \Omega} \operatorname{ran}(Z - T).$$

Further, we have

$$\bigcap_{z \in \Omega} \operatorname{ran}(Z - T) = \bigcap_{z \in U} \operatorname{ran}(Z - T)$$

for every open subset $\emptyset \neq U \subseteq \Omega$.

Proof. Let $x \in X$ be arbitrary. Then $\hat{x}$ is the composition of $q : \Omega \times X \to E_T$ from Theorem 4.1.5 and the continuous map $\Omega \to \Omega \times X, z \mapsto (z, x)$. Hence $\hat{x}$ is continuous with $\pi \circ \hat{x} = \mathrm{id}_\Omega$ and therefore a section in E_T over Ω. Consider the linear charts h_{Ω_0} and the map Φ defined in the proof of Theorem 4.1.5. Then we obtain

$$P_2 \circ h_{\Omega_0} \circ \hat{x}|_{\Omega_0} = \Phi \circ P_{\Omega_0}(\cdot)x$$

for the representation of $\hat{x}$ in the chart h_{Ω_0}, which is analytic by the proof of Theorem 4.1.5. Hence $\hat{x}$ is a holomorphic section in E_T over Ω and ρ is well-defined. The linearity of ρ follows by an easy calculation. Note that $x \in \ker \rho$ if and only if $x \in \mathrm{ran}(Z - T)$ for all $z \in \Omega$. Thus the first equality follows. Now let $\emptyset \neq U \subseteq \Omega$ be an arbitrary open subset. If $x \in \bigcap_{z \in U} \mathrm{ran}(Z - T)$, then $\hat{x}|_U = 0_{\Gamma_{\mathrm{hol}}(U, E_T)}$. By Lemam 4.1.4, we obtain $\hat{x} = 0_{\Gamma_{\mathrm{hol}}(\Omega, E_T)}$. □

In particular, we obtain that every dual Cowen-Douglas tuple T of rank N on a connected submanifold $\Omega \subseteq \mathbb{C}^d$ is also a dual Cowen-Douglas tuple of rank N on every smaller domain $\Omega_0 \subseteq \Omega$.
We want to find holomorphic model spaces for dual Cowen-Douglas tuples.

Definition 4.1.7. Let $\Omega \subset \mathbb{C}^d$ be a connected submanifold. A holomorphic model space of rank N over Ω is a Banach space $\hat{X} \subset \mathcal{O}(\Omega, D)$ such that D is an N-dimensional complex vector space and

(i) $M_z \in L(\hat{X})^d$,

(ii) for each $\lambda \in \Omega$, the point evaluation $\epsilon_\lambda : \hat{X} \to D, f \mapsto f(\lambda)$, is continuous and surjective. Here D is equipped with its unique norm topology.

A holomorphic model space $\hat{X}$ on Ω is called divisible if in addition, for $f \in \hat{X}$ and $\lambda \in \Omega$ with $f(\lambda) = 0$, there are functions $g_1, ..., g_d \in \hat{X}$ with

$$f = \sum_{i=1}^{d} (\lambda_i - M_{z_i}) g_i.$$

We quickly check that the multiplication tuple $M_z \in L(\hat{X})^d$ on a divisible holomorphic model space is a dual Cowen-Douglas tuple.

Theorem 4.1.8. Let $\hat{X} \subseteq \mathcal{O}(\Omega, D)$ be a divisible holomorphic model space of rank N. Then $M_z \in L(\hat{X})^d$ is a dual Cowen-Douglas tuple of rank N on Ω.

Proof. For $\lambda \in \Omega$, the point evaluation $\epsilon_\lambda : \hat{X} \to D$ is surjective and thus induces a vector space isomorphism

$$\hat{\epsilon_\lambda} : \hat{X} / \ker \epsilon_\lambda \to D, x + \ker \epsilon_\lambda \mapsto x(\lambda).$$

Since $\hat{X}$ is divisible, we conclude $\dim \hat{X}/\operatorname{ran}(\lambda - M_z) = \dim \hat{X}/\ker \epsilon_\lambda = N$. Obviously, we have

$$\bigcap_{\lambda\in\Omega} \operatorname{ran}(\lambda - M_z) = \bigcap_{\lambda\in\Omega} \ker \epsilon_\lambda = \{0\}.$$

Thus the assertion follows. □

Every weak dual Cowen-Douglas tuple can be modelled as a multiplication tuple on a divisible holomorphic model space. To state the corresponding result, we equip $\mathcal{O}(\Omega_0, D)$ ($\Omega_0 \subseteq \Omega$ open) with its usual Fréchet space topology.

Theorem 4.1.9. Let $T \in L(X)^d$ be a weak dual Cowen-Douglas tuple of rank N on Ω. Then for each $\lambda_0 \in \Omega$, there is a continuous linear map $\rho : X \to \mathcal{O}(\Omega_0, D)$ where $\Omega_0 \subseteq \Omega$ is a connected open neighbourhood of λ_0 in Ω such that

(i) $\rho T_i = M_{z_i}\rho$ for $i = 1, ..., d$,

(ii) $\ker \rho = \bigcap_{z\in\Omega} \operatorname{ran}(Z - T)$,

(iii) $\hat{X} = \rho(X)$ equipped with the norm $\|\rho(x)\| = \|x + \ker \rho\|$ is a divisible holomorphic model space of rank N on Ω_0.

Proof. Let $\lambda_0 \in \Omega$ be arbitrary. Choose a linear subspace $D \subseteq X$ such that

$$X = \operatorname{ran}(\lambda_0 - T) \oplus D.$$

Then $\dim D = N$. As in the proof of Theorem 4.1.5 we find a connected open neighbourhood Ω_0 of λ_0 in Ω such that, for each $x \in X$, there is an analytic function $g^x = (g^x_1, ..., g^x_d, g^x_D) \in \mathcal{O}(\Omega_0, X^d \oplus D)$ with

$$x - g^x_D(z) = \sum_{i=1}^{d} (z_i - T_i) g^x_i(z) \in \operatorname{ran}(Z - T)$$

for all $z \in \Omega_0$. For each $z \in \Omega_0$, we conclude that the linear map

$$D \to X/\sum_{i=1}^{d}(z_i - T_i)X, x \mapsto [x]$$

is surjective between N-dimensional vector spaces. Hence these maps are isomorphisms. But then, for each $x \in X$ and $z \in \Omega_0$, there is a unique vector

$x(z) \in D$ with $x - x(z) \in \sum_{i=1}^d (z_i - T_i)X$. With the notation from above, the map

$$\Omega_0 \to D, z \mapsto x(z) = g_D^x(z)$$

is analytic for every $x \in X$.The induced mapping

$$\rho : X \to \mathcal{O}(\Omega_0, D), x \mapsto x(\cdot)$$

is linear with

$$\ker \rho = \bigcap_{z \in \Omega_0} \sum_{i=1}^d (z_i - T_i)X = \bigcap_{z \in \Omega} \sum_{i=1}^d (z_i - T_i)X$$

by Lemma 4.1.6. In paricular, it follows that $\ker \rho \subseteq X$ is a closed linear subspace. For $z \in \Omega_0$ and $j = 1, .., d$, the image $\sum_{i=1}^d (z_i - T_i)X$ of $z - T$ is invariant for T_j and this yields

$$T_j x - z_j x(z) = T_j(x - x(z)) - (z_j - T_j)x(z) \in \sum_{i=1}^d (z_i - T_i)X$$

for every $x \in X$. Hence ρ fulfills condition (i). Equipped with the norm $\|\rho(x)\| = \|x + \ker \rho\|$, the space $\hat{X} = \rho(X)$ is a Banach space. Writing $\hat{T}_j : X/\ker \rho \to X/\ker \rho, x + \ker \rho \mapsto T_j x + \ker \rho$ for $j = 1, ..., d$, we have

$$\|M_{z_j} \rho x\| = \|\rho T_j x\| = \|T_j x + \ker \rho\| \leq \|\hat{T}_j\| \|\rho(x)\|$$

for all $x \in X$ and $j = 1, .., d$ and thus $M_z \in L(\hat{X})^d$ is a commuting tuple of bounded operators on $\hat{X}$. By definition,

$$\rho(x) \equiv x \quad \text{for all } x \in D.$$

Hence the point evaluations $\epsilon_z : \hat{X} \to D$ $(z \in \Omega_0)$ are surjective. Since the mappings

$$q_z : D \to X/\sum_{i=1}^d (z_i - T_i)X, x \mapsto [x] \quad (z \in \Omega_0)$$

are topological isomorphisms and since the compositions

$$X \to X/\sum_{i=1}^d (z_i - T_i)X, x \mapsto q_z(\epsilon_z(\rho(x))) = [x]$$

are continuous, it follows that the point evaluations $\epsilon_z : \hat{X} \to D(z \in \Omega_0)$ are continuous. Thus we have shown that $\hat{X} \subseteq \mathcal{O}(\Omega_0, D)$ with the norm induced

by ρ is a holomorphic model space.
To see that $\hat{X}$ is divisible, fix a vector $x \in X$ and a point $\lambda \in \Omega_0$ such that $x(\lambda) = 0$. Then there are vectors $x_1, ..., x_d \in X$ with $x = \sum_{i=1}^{d} (\lambda_i - T_i)x_i$. Hence

$$\rho(x) = \sum_{i=1}^{d} (\lambda_i - M_{z_i})\rho(x_i) \in \sum_{i=1}^{d} (\lambda_i - M_{z_i})\hat{X}.$$

Since the compositions

$$X \xrightarrow{\rho} \hat{X} \xrightarrow{\epsilon_z} D \quad (z \in \Omega_0)$$

are continuous, a straightforward application of the closed graph theorem shows that also $\rho : X \to \mathcal{O}(\Omega_0, D)$ as a mapping with values in the Frèchet space $\mathcal{O}(\Omega_0, D)$ is continuous. □

Note that, if T is even a dual Cowen-Douglas tuple, a mapping $\rho : X \to \hat{X}$ as in the proof of Theorem 4.1.9 is injective and thus an isometric isomorphism. In particular, the tuples $T \in L(X)^d$ and $M_z \in L(\hat{X})^d$ with $\hat{X} \subseteq \mathcal{O}(\Omega_0, D)$ are similar in this case. This local model characterizes the class of all dual Cowen-Douglas tuples.

Corollary 4.1.10. A commuting tuple $T \in L(X)^d$ is a dual Cowen-Douglas tuple of rank N on a given connected complex submanifold $\Omega \subseteq \mathbb{C}^d$ if and only if, for each $\lambda \in \Omega$, there exist a connected open neighbourhood $\Omega_0 \subseteq \Omega$ of λ and a divisible holomorphic model space of rank N on Ω_0 such that T and the multiplication tuple $M_z \in L(\hat{X})^d$ are similar.

Proof. The necessity of the stated condition follows from Theorem 4.1.9 and the subsequent remarks. Since the tuple $M_z \in L(\hat{X})^d$ on a divisible holomorphic model space of rank N on Ω_0 is a Cowen-Douglas tuple of rank N on Ω_0 by Theorem 4.1.8 and since similarity preserves this property (cf. Lemma 4.1.2), the sufficiency follows. □

The preceding result should be compared with Corollary 4.39 in [Wer14], where a characterization of Cowen-Douglas tuples on suitable admissible domains in $\mathbb{C}^d$ is obtained.
There is a canonical way to associate with each weak dual Cowen-Douglas tuple of rank N on Ω a dual Cowen-Douglas tuple of rank N.

Corollary 4.1.11. Let $T \in L(X)^d$ be a weak dual Cowen-Douglas tuple of rank N on Ω. Then the quotient tuple

$$T^{\mathrm{CD}} = T / \bigcap_{z \in \Omega} \sum_{i=1}^{d} (z_i - T_i)X$$

defines a dual Cowen-Douglas tuple of rank N on Ω.

Proof. Fix $\lambda_0 \in \Omega$. Choose a map $\rho : X \to \mathcal{O}(\Omega_0, D)$ as in Theorem 4.1.9. Then $\hat{X} = \rho(X) \subseteq \mathcal{O}(\Omega_0, D)$ is a divisible holomorphic model space of rank N on Ω_0. Since

$$\ker \rho = \bigcap_{z\in\Omega} \sum_{i=1}^{d} (z_i - T_i)X,$$

the tuples T^{CD} and $M_z \in L(\hat{X})^d$ are similar via the isometric isomorphism $X/\ker\rho \xrightarrow{\hat{\rho}} \hat{X}$ induced by ρ. By Corollary 4.1.10 the tuple T^{CD} is a Cowen-Douglas tuple of rank N on Ω. $\square$

4.2 Regular tuples on Hilbert spaces

On Hilbert spaces, a model theorem similar to Theorem 4.1.9 holds for a larger class of operator tuples.
For a tuple $T = (T_1, ..., T_d) \in L(H)^d$ of bounded linear operators on a Hilbert space H, we denote its wandering subspace by

$$W(T) = H \ominus \sum_{i=1}^{d} T_i H.$$

In the following, we will denote by $\|z\| = \sqrt{\sum_{i=1}^{d} |z_i|^2}$ the Euclidean norm of an element $z \in \mathbb{C}^d$.

Definition 4.2.1. A tuple $T \in L(H)^d$ of bounded linear operators on a Hilbert space H is called regular at 0 if there is a positive real number $\epsilon > 0$ such that, for $z \in \mathbb{C}^d$ with $\|z\| < \epsilon$, the subspace $\mathrm{ran}(Z - T) \subseteq H$ is closed and H decomposes into the algebraic direct sum

$$H = \mathrm{ran}(Z - T) \oplus W(T).$$

To obtain a model theorem for regular tuples, we need to make some preliminary observations.
As before, let $T = (T_1, ..., T_d) \in L(H)^d$ be a commuting operator tuple with closed range $\sum_{i=1}^{d} T_i H \subseteq H$. We write $T^* : H \to H^d, h \mapsto (T_i^* h)_{i=1}^d$, for the adjoint of the row operator $T : H^d \to H, (h_i)_{i=1}^d \mapsto \sum_{i=1}^d T_i h_i$.
Since T has closed range, the operator $T^*T : \mathrm{ran}\, T^* \to \mathrm{ran}\, T^*$ is invertible. We denote its inverse by $(T^*T)^{-1}$ and consider the column operator $L = (T^*T)^{-1}T^* \in L(H, H^d)$. Further, for $z \in \mathbb{C}^d$ with $\|z\| < \frac{1}{\|L\|}$, we define

$$P(z) = (T - Z)L(1_H - ZL)^{-1} \in L(H),$$

where $Z : H^d \to H, (h_i)_{i=1}^d \mapsto \sum_{i=1}^d z_i h_i$ denotes the row operator induced by z. Note that we use the convention $\frac{1}{\|L\|} = \infty$ if $L = 0$ and also note that the mapping $B_{\frac{1}{\|L\|}}(0) \to L(H), z \mapsto P(z) = (T - Z)L\sum_{k=0}^{\infty}(ZL)^k$ is holomorphic.

Lemma 4.2.2. Let $T \in L(H)^d$ be a commuting tuple of bounded linear operators such that the induced row operator $T : H^d \to H$ has closed range. With the notation introduced above, we have

(a) $LT = P_{\operatorname{ran} T^*}$ and $TL = P_{\operatorname{ran} T}$,

(b) for all $z \in B_{\frac{1}{\|L\|}}(0)$, the operator $P(z)$ is an idempotent, i.e. $P(z) = P(z)^2$,

(c) for all $z \in B_{\frac{1}{\|L\|}}(0)$, we have $1_H - P(z) = P_{W(T)}(1_H - ZL)^{-1}$ and

$$\operatorname{ran}(1_H - P(z)) = W(T).$$

Proof. Using the orthogonal decomposition

$$H^d = \operatorname{ran} T^* \oplus \ker T$$

one easily obtains the first equality in (a). Let $f \in H$ be arbitrary. Since $H = \operatorname{ran} T \oplus \ker T^*$ and $\operatorname{ran} T = T(\operatorname{ran} T^*)$, there are $f_1 \in \operatorname{ran} T^*$, $f_2 \in \ker T^*$ with $f = Tf_1 + f_2$. Using the first equality in (a) we obtain

$$\begin{aligned} &T(T^*T)^{-1}T^*f = T(T^*T)^{-1}T^*Tf_1 \\ =&TP_{\operatorname{ran} T^*}f_1 = Tf_1 = P_{\operatorname{ran} T}f. \end{aligned}$$

For $z \in B_{\frac{1}{\|L\|}}(0)$, we conclude that

$$\begin{aligned} L(1_H - ZL)^{-1}(T - Z) &= LT + L\sum_{k=0}^{\infty}(ZL)^{k+1}T - L\sum_{k=0}^{\infty}(ZL)^k Z \\ &= LT - L\sum_{k=0}^{\infty}(ZL)^k Z(1_{H^d} - LT) \\ &= P_{\operatorname{ran} T^*} - L(1_H - ZL)^{-1}ZP_{\ker T}. \end{aligned}$$

Due to $\operatorname{ran} L \subseteq \operatorname{ran} T^* = (\ker T)^{\perp}$ this yields the identity

$$P(z)^2 = (T - Z)(P_{\operatorname{ran} T^*} - L(1_H - ZL)^{-1}ZP_{\ker T})L(1_H - ZL)^{-1} = P(z)$$

for every $z \in B_{\frac{1}{\|L\|}}(0)$. Finally, we use the second equality from part (a) to see that

$$\begin{aligned} 1_H - P(z) &= 1_H - (T-Z)L(1_H - ZL)^{-1} \\ =&(1_H - TL)(1_H - ZL)^{-1} = P_{W(T)}(1_H - ZL)^{-1} \end{aligned}$$

and hence that $(1_H - P(z))H = W(T)$ for $z \in B_{\frac{1}{\|L\|}}(0)$. $\square$

If $T \in L(H)^d$ is a commuting tuple such that $TH^d \subseteq H$ is closed, then $H = \operatorname{ran}(Z-T) + W(T)$ for $z \in B_{\frac{1}{\|L\|}}(0)$, This follows immediately from Lemma 4.2.2, since $\operatorname{ran} P(z) \subseteq \operatorname{ran}(Z-T)$ for $z \in B_{\frac{1}{\|L\|}}(0)$. For a regular tuple, we can also determine $\operatorname{ran} P(z)$ for every $z \in B_{\frac{1}{\|L\|}}(0)$.

Lemma 4.2.3. Let $T \in L(H)^d$ be commuting. Then T is regular at 0 if and only if $TH^d \subseteq H$ is closed and there is an $\epsilon > 0$ such that

$$\operatorname{ran}(T-Z) \cap W(T) = \{0\} \text{ for } \|z\| < \epsilon.$$

In this case we have, for all $z \in B_{\frac{1}{\|L\|}}(0)$,

$$\operatorname{ran} P(z) = \operatorname{ran}(T-Z).$$

Proof. Suppose that $TH^d \subseteq H$ is closed. By Lemma 4.2.2 (c)

$$H = \operatorname{ran} P(z) \oplus W(T) \text{ for } \|z\| < \frac{1}{\|L\|}.$$

By definition $\operatorname{ran} P(z) \subseteq \operatorname{ran}(T-Z)$ for $\|z\| < \frac{1}{\|L\|}$. If $\operatorname{ran}(T-Z) \cap W(T) = \{0\}$ for $\|z\| < \epsilon$, then the above direct sum decomposition yields that $\operatorname{ran} P(z) = \operatorname{ran}(T-Z)$ for $\|z\| < \min(\epsilon, \frac{1}{\|L\|})$. The identity theorem applied to the holomorphic map

$$B_{\frac{1}{\|L\|}}(0) \to L(H), z \mapsto (1_H - P(z))(T-Z)$$

yields that $\operatorname{ran}(T-Z) \subseteq \operatorname{ran} P(z) \subseteq \operatorname{ran}(T-Z)$ for all $z \in B_{\frac{1}{\|L\|}}(0)$. The remaining implication obviously holds. $\square$

To give some examples of operator tuples which are regular at 0, we recall the definition of the Koszul complex $K^\bullet(T,X)$ of a tuple $T = (T_1, ..., T_d) \in L(X)^d$ of commuting operators on a Banach space X. It is defined as the sequence

$$K^\bullet(T,X): \ 0 \longrightarrow \Lambda^0(X) \xrightarrow{\delta_T^0} \Lambda^1(X) \xrightarrow{\delta_T^1} ... \xrightarrow{\delta_T^{d-1}} \Lambda^d(X) \longrightarrow 0 \ ,$$

where $\Lambda^i(X) = \Lambda^i(\mathbb{C}) \otimes X$ denotes the space of all forms of degree i with coefficients in X and $\delta_T^i : \Lambda^i(X) \to \Lambda^{i+1}(X)$ acts as

$$\delta_T^i(\sum_{|I|=i} x_I e_I) = \sum_{j=1}^{d} \sum_{|I|=i} (T_j x_I) e_j \wedge e_I$$

for $i = 0, ..., d-1$. We can obviously identify $\Lambda^0(X) \cong \Lambda^d(X) \cong X$ as well as $\Lambda^1(X) \cong \Lambda^{d-1}(X) \cong X^d$ as vector spaces. With respect to this identification, the map $\delta_T^{d-1} : X^d \to X$ can be interpreted as the row operator associated with T.

For $i = 0, ..., d$, the vector spaces

$$H^i(T, X) = \ker(\delta_T^i)/\mathrm{Bild}(\delta_T^{i-1})$$

are called the cohomology groups of the Koszul complex. Here, by convention $\delta_T^d = \delta_T^{-1} = 0$.
As above, we use the identification $H^d(T, X) \cong X/\sum_{i=1}^d T_i X$.

Lemma 4.2.4. Let $T \in L(H)^d$ be a commuting tuple of bounded operators on a Hilbert space. Under any of the following three conditions

(a) $\operatorname{ran} T = H$,

(b) $\operatorname{ran} T \subseteq H$ is closed and $H^{d-1}(T, H) = \{0\}$,

(c) T is a weak dual Cowen-Douglas tuple on some open ball $B_r(0) \subseteq \mathbb{C}^d$,

the tuple T is regular at 0.

Proof. First consider $T \in L(H)^d$ with $\operatorname{ran} T = H$. Then $W(T) = \{0\}$ and T is regular at 0 by Lemma 4.2.3. Secondly, suppose that $\operatorname{ran} T \subseteq H$ is closed. Then the condition $H^{d-1}(T, H) = \{0\}$ means that the sequence

$$\Lambda^{d-2}(H) \xrightarrow{\begin{pmatrix} \delta_{z-T}^{d-2} \\ 0 \end{pmatrix}} \Lambda^{d-1}(H) \oplus W(T) \xrightarrow{(\delta_{z-T}^{d-1}, i)} \Lambda^d(H) \to 0,$$

where $i : W(T) \to H$ denotes the inclusion map, is exact at $z = 0$. By Lemma 2.1.3 in [EP96] there is a real number $\epsilon > 0$ such that this sequence remains exact for every $z \in \mathbb{C}^d$ with $\|z\| < \epsilon$. But then

$$(T - Z)H^d \oplus W(T) = H$$

for $\|z\| < \epsilon$ and Lemma 4.2.3 implies that T is regular at 0.
Suppose that T is a weak dual Cowen-Douglas tuple of rank N on some open ball $B_r(0) \subseteq \mathbb{C}^d$. Then $\mathrm{ran}(T) \subseteq H$ is closed and by the proof of Theorem 4.1.5

$$H = \mathrm{ran}(Z - T) \oplus W(T)$$

for $\|z\|$ sufficiently small. Again the regularity of T at 0 follows from Lemma 4.2.3. □

In the following, let $T \in L(H)^d$ be a commuting tuple that is regular at 0. We denote by $L_i \in L(H)$ $(i = 1, ..., d)$ the components of the column operator $L = (T^*T)^{-1}T^* \in L(H, H^d)$ and we use the notation $L_j = L_{j_1}...L_{j_k}$ for arbitrary index tuples $j = (j_1, ..., j_k) \in \{1, ..., d\}^k$. To simplify the notation we write $\Omega_T = B_{\frac{1}{\|L\|}}(0)$ for the open Euclidean ball with radius $\frac{1}{\|L\|}$ at $z = 0$. We equip the space $\mathcal{O}(\Omega_T, W(T))$ of all analytic $W(T)$-valued functions on Ω_T with its usual Fréchet space topology of uniform convergence on all compact subsets.

Theorem 4.2.5. Let $T \in L(H)^d$ be regular at 0. Then the map

$$V : H \to \mathcal{O}(\Omega_T, W(T)), (Vx)(z) = (1_H - P(z))x$$

is continuous linear with $Vx \equiv x$ for $x \in W(T)$ and

(i) $VT_i = M_{z_i}V (i = 1, ..., d)$,

(ii) $\ker V = \bigcap_{m=0}^{\infty} \sum_{|\alpha|=m} T^\alpha H = \bigcap_{z \in \Omega_T} \mathrm{ran}(Z - T)$,

(iii) if we equip the vector space $\hat{H} = \mathrm{ran}\, V \subseteq \mathcal{O}(\Omega_T, W(T))$ with the norm $\|Vx\| = \|x + \ker V\|$, it is a functional Hilbert space and its reproducing kernel is given by

$$K_T : \Omega_T \times \Omega_T \to L(W(T)),$$
$$K_T(z, w) = P_{W(T)}(1_H - ZL)^{-1}(1_H - L^*W^*)^{-1}|_{W(T)}.$$

Proof. By construction and Lemma 4.2.3, for $z \in \Omega_T$ and $x \in H$, the vector

$$x(z) = (1_H - P(z))x = P_{W(T)}(1_H - ZL)^{-1}x$$

is the unique element in $W(T)$ such that $x - x(z) \in \mathrm{ran}(T - Z)$. Using the remarks preceding Lemma 4.2.2, we see that the vector $x(z)$ depends analytically on z. Obviously the map V is linear with $Vx \equiv x$ for $x \in W(T)$. The function $\Omega_T \to L(H), z \mapsto P(z)$ is continuous and thus uniformly bounded on every compact set $K \subseteq \Omega_T$. We conclude that V is continuous. Since $\operatorname{ran}(T-Z) = \sum_{1\leq i\leq d}(T_i - z_i)H$ is invariant for the operators T_j $(j = 1, ..., d)$, it follows that for $z \in \Omega_T$ and $x \in H$,

$$T_j x - z_j x(z) = T_j(x - x(z)) + (T_j - z_j)x(z) \in \operatorname{ran}(T - Z).$$

Hence the map V intertwines the tuples T on H and M_z on $\mathcal{O}(\Omega_T, W(T))$ componentwise. To calculate the kernel of V, note that, for $x \in H$ and $z \in \Omega_T$,

$$(Vx)(z) = \sum_{k=0}^{\infty} P_{W(T)}(ZL)^k x = \sum_{k=0}^{\infty}\sum_{|\alpha|=k}(P_{W(T)} \sum_{j\in J(\alpha)} L_j x)z^\alpha,$$

where for each $k \in \mathbb{N}$ and $\alpha \in \mathbb{N}^d$ with $|\alpha| = k$, the set $J(\alpha)$ consists of all index tuples $j = (j_1, ..., j_k) \in \{1, ..., d\}^k$ such that, for each $i = 1, ..., d$, exactly α_i of the indices $j_1, ..., j_k$ equal i. The map $\Sigma_T : L(H) \to L(H), X \mapsto \sum_{i=1}^d T_i X L_i$, is continuous linear with $P_{W(T)} = 1_H - TL = 1_H - \Sigma_T(1_H)$ and thus

$$\sum_{j=0}^{m-1} \Sigma_T^j(P_{W(T)}) = 1_H - \Sigma_T^m(1_H) \quad (m \geq 0).$$

For $x \in \ker V$, we have $P_{W(T)} \sum_{j\in J(\alpha)} L_j x = 0$ for all $\alpha \in \mathbb{N}^d$. Hence, for $m \geq 0$, we conclude that

$$\begin{aligned} 0 &= \sum_{k=0}^{m-1}\sum_{|\alpha|=k} T^\alpha \left(P_{W(T)} \sum_{j\in J(\alpha)} L_j x\right) \\ &= \sum_{k=0}^{m-1} \Sigma_T^k(P_{W(T)})x = x - \sum_{|\alpha|=m} T^\alpha \left(\sum_{j\in J(\alpha)} L_j x\right). \end{aligned}$$

Thus $\ker V \subseteq \bigcap_{m=0}^{\infty} \sum_{|\alpha|=m} T^\alpha H$. Conversely, if a vector $x \in H$ belongs to the intersection on the right-hand side, then

$$Vx \in \bigcap_{m=0}^{\infty}\sum_{|\alpha|=m} VT^\alpha H \subseteq \bigcap_{m=0}^{\infty}\sum_{|\alpha|=m} M_z^\alpha \mathcal{O}(\Omega_T, W(T)) = \{0\}.$$

Thus the first equality in part (ii) has been shown. The second equality follows immediately from Lemma 4.2.3, since $\ker(1_H - P(z)) = \operatorname{ran} P(z) = \operatorname{ran}(T - Z)$ for all $z \in \Omega_T$.

Define the mapping $K_T : \Omega_T \times \Omega_T \to L(W(T))$ as in part (*iii*) of the theorem. For $f \in \hat{H}$, there is a unique vector $x(f) \in (\ker V)^\perp$ with $f = Vx(f)$.

Since $\lim_{k\to\infty} f_k = f$ in $\hat{H}$ if and only if $\lim_{k\to\infty} x(f_k) = x(f)$ in H, all point evaluations on $\hat{H}$ are continuous. Thus $\hat{H}$ is a functional Hilbert space. For $y \in W(T)$, $\tilde{y} \in \ker V$ and $w \in \Omega_T$, we can use the representation $1 - P(w) = P_{W(T)}(1_H - WL)^{-1}$ from Lemma 4.2.2 to obtain that

$$\langle (1_H - L^*W^*)^{-1}y, \tilde{y}\rangle = \langle y, (V\tilde{y})(w)\rangle = 0.$$

Let $f \in \hat{H}$, $y \in W(T)$ and $w \in \Omega_T$ be given. Define $x = x(f)$. Then

$$\begin{aligned}\langle f(w), y\rangle_{W(T)} &= \langle P_{W(T)}(1_H - WL)^{-1}x, y\rangle_H = \langle x, (1_H - L^*W^*)^{-1}y\rangle_{(\ker V)^\perp} \\ &= \langle Vx, V(1_H - L^*W^*)^{-1}y\rangle_{\hat{H}} = \langle f, K_T(\cdot, w)y\rangle_{\hat{H}}\end{aligned}$$

and hence K_T is the reproducing kernel of the analytic functional Hilbert space $\hat{H}$. □

Very elementary examples even of single operators on finite-dimensional Hilbert spaces show that Theorem 4.2.5 need not be true if instead of the regularity at 0 one only demands that the space $TH^d \subseteq H$ is closed.

Example 4.2.6. Consider the nilpotent partial isometry $T \in L(\mathbb{C}^2)$ acting as $T(x, y) = (y, 0)$. The operator T^*T acts as the identity on $\operatorname{ran} T^* = \{0\} \oplus \mathbb{C}$. In particular, we have $L(x, y) = T^*(x, y) = (0, x)$ for all $(x, y) \in \mathbb{C}^2$ and $\|L\| = 1$. An elementary computation shows that $P(z)(x, y) = (x, -zx)$ for all $(x, y) \in \mathbb{C}^2$ and $z \in \Omega_T = \mathbb{D}$. Thus the mapping V defined in Theorem 4.2.5 is given by $V : \mathbb{C}^2 \to \mathcal{O}(\mathbb{D}, \{0\} \oplus \mathbb{C})$,

$$V(x, y)(z) = (0, zx + y).$$

Hence $(VT)(1, 0)(z) = (0, 0)$, while $(M_z V(1, 0))(z) = (0, z^2)$ for $z \in \mathbb{D}$.

Condition (ii) in Theorem 4.2.5 implies that $W(T) \subseteq (\ker V)^\perp$. An elementary argument shows that $W(T)$ coincides with the wandering subspace of the compression of T to $(\ker V)^\perp$.

Corollary 4.2.7. Let $T \in L(H)^d$ be regular at 0. Writing V for the map from Theorem 4.2.5, we have

$$W(T) = W(P_{(\ker V)^\perp} T|_{(\ker V)^\perp}).$$

Proof. We have

$$\begin{aligned}(\ker V)^\perp \ominus W(T) &= \{x \in (\ker V)^\perp; \langle x, y\rangle = 0 \text{ for all } y \in H \ominus \operatorname{ran} T\} \\ &= \operatorname{ran} T \cap (\ker V)^\perp.\end{aligned}$$

For $(x_i)_{i=1}^d \in H^d$ with $T(x_i)_{i=1}^d \in (\ker V)^\perp$, we have

$$T(x_i)_{i=1}^d = P_{(\ker V)^\perp} T|_{(\ker V)^\perp} (P_{(\ker V)^\perp} x_i)_{i=1}^d,$$

since $\ker V$ is invariant for T. This yields

$$(\ker V)^\perp \ominus W(T) \subseteq \operatorname{ran}(P_{(\ker V)^\perp} T|_{(\ker V)^\perp}).$$

On the other hand, for $(x_i)_{i=1}^d \in ((\ker V)^\perp)^d$, $y \in W(T)$, we have

$$\langle P_{(\ker V)^\perp} T|_{(\ker V)^\perp} (x_i)_{i=1}^d, y \rangle_{(\ker V)^\perp} = \langle T(x_i)_{i=1}^d, y \rangle_H = 0.$$

$\square$

In the following we use the notation $H_\infty = \bigcap_{m=0}^\infty \sum_{|\alpha|=m} T^\alpha H$. We call a commuting tuple $T \in L(H)^d$ analytic if $H_\infty = \{0\}$. If a commuting tuple $T \in L(H)^d$ is unitarily equivalent to the multiplication tuple $M_z \in L(\hat{H})^d$ on a functional Hilbert space $\hat{H} \subseteq \mathcal{O}(\Omega, D)$ on a connected open zero neighbourhood $\Omega \subseteq \mathbb{C}^d$, then T is neccessarily analytic. The next immediate corollary from Theorem 4.2.5 shows that, under the additional hypothesis that T is regular at 0, also the converse implication holds.
Let $H/\ker V \cong (\ker V)^\perp$ be the quotient space of H modulo the kernel of V. As before, we denote the elements of $H/\ker V$ by $x + \ker V$.

Corollary 4.2.8. Let $T \in L(H)^d$ be regular at 0 and let

$$V : H \to \mathcal{O}(\Omega_T, W(T))$$

and $\hat{H} = \operatorname{ran} V$ be defined as in Theorem 4.2.5.
Then the compression $P_{(\ker V)^\perp} T|_{(\ker V)^\perp}$ is unitarily equivalent to $M_z \in L(\hat{H})^d$ via the unitary operator $V : (\ker V)^\perp \to \hat{H}$. If T is analytic, then T is unitarily equivalent to the multiplication tuple $M_z \in L(\hat{H})^d$.

Proof. The first part follows immediately from Theorem 4.2.5. If in addition T is analytic, then by Theorem 4.2.5 we have $\ker V = H_\infty = \{0\}$ and hence T is unitarily equivalent to $M_z \in L(\hat{H})^d$. $\square$

Let $T \in L(H)^d$ be regular at 0 and let

$$V : H \to \hat{H}$$

be the mapping constructed in Theorem 4.2.5. Then V is a surjective partial isometry onto the functional Hilbert space $\hat{H} \subseteq \mathcal{O}(\Omega_T, W(T))$ that intertwines the tuples $T \in L(H)^d$ and $M_z \in L(\hat{H})^d$ componentwise. Fix an operator $S \in L(H)$ with

$$ST_i = T_i S \quad (i = 1, .., d).$$

Since

$$P_{(\ker V)^\perp} T_i V^* = V^* V T_i V^* = V^* M_{z_i} V V^* = V^* M_{z_i}$$

for $i = 1, .., d$ and since the space

$$\ker V = \bigcap_{z \in \Omega_T} \operatorname{ran}(Z - T)$$

is invariant under S, it follows that

$$(VSV^*) M_{z_i} = VSP_{(\ker V)^\perp} T_i V^* = VST_i V^* = M_{z_i}(VSV^*)$$

for $i = 1, ..., d$. It follows easily from the construction of the functional Hilbert space $\hat{H}$ in Theorem 4.2.5 that all point evaluations $\epsilon_\lambda : \hat{H} \to W(T)$ $(\lambda \in \Omega_T)$ are surjective with

$$\ker \epsilon_\lambda = \sum_{i=1}^{d} (\lambda_i - M_{z_i}) \hat{H}.$$

Then Lemma 4.4 from [Sch18] shows that there is a multiplier $\theta \in \mathcal{O}(\Omega_T, L(W(T)))$ of $\hat{H}$ such that

$$(VSV^*) f = \theta f \qquad (f \in \hat{H}).$$

For all $x, y \in W(T)$ and $w \in \Omega_T$, we use the reproducing kernel from Theorem 4.2.5 and Lemma 4.2.2 c) to conclude

$$\begin{aligned}
\langle \theta(w)x, y \rangle_{W(T)} &= \langle \theta x, K_T(\cdot, w) y \rangle_{\hat{H}} \\
&= \langle \theta x, (1_H - P(\cdot))(1_H - L^* W^*)^{-1} y \rangle_{\hat{H}} \\
&= \langle \theta x, V((1_H - L^* W^*)^{-1} y) \rangle_{\hat{H}} \\
&= \langle (1_H - WL)^{-1} V^* M_\theta x, y \rangle_H \\
&= \langle (1_H - WL)^{-1} S V^* x, y \rangle_H \\
&= \langle P_{W(T)} (1_H - WL)^{-1} S x, y \rangle_{W(T)}.
\end{aligned}$$

Here, we slightly abuse the notation and denote the constant function

$$\Omega_T \to W(T), z \mapsto x \quad (x \in W(T))$$

in $\hat{H}$ also by x. From this, we obtain

$$\theta(w) = P_{W(T)} (1_H - WL)^{-1} S|_{W(T)}$$

for every $w \in \Omega_T$. If T is in addition analytic, then it follows that every operator S in the commutant of T is unitarily equivalent via $V : H \to \hat{H}$ to a multiplication operator M_θ induced by a multiplier θ of the analytic functional Hilbert space $\hat{H}$. For the particular case that $T \in L(H)$ is a single pure isometry, this observation is contained in [MS] (Theorem 2.2).

5 A generalized Wold decomposition

Each isometry $T \in L(H)$ on a Hilbert space H is a direct sum

$$T = T^{(0)} \oplus T^{(1)} \in L(H_0 \oplus H_1)$$

of a unitary operator $T^{(0)} \in L(H_0)$ and an operator $T^{(1)} \in L(H_1)$ which is unitarily equivalent to a Hardy space shift $M_z \in L(H^2(\mathbb{D}, D))$ for some Hilbert space D. The Hardy space $H^2(\mathbb{D}, D)$ is the D-valued analytic functional Hilbert space on $\mathbb{D}$ with reproducing kernel

$$K : \mathbb{D} \times \mathbb{D} \to \mathbb{C}, K(z, w) = \frac{1}{1 - z\overline{w}}.$$

Note that isometries are characterized by the operator identity $1_H - T^*T = 0$. The left-hand side of this identity is obtained by replacing z and $\overline{w}$ in the reciprocal $\frac{1}{K}(z, w)$ of the kernel K by the operators T^* and T. In [GO12], Giselsson and Olafsson proved corresponding decomposition theorems for Hilbert space operators $T \in L(H)$ satisfying higher order operator identities related to the reproducing kernel

$$K_m^{\mathbb{D}} : \mathbb{D} \times \mathbb{D} \to \mathbb{C}, K_m(z, w) = \frac{1}{(1 - z\overline{w})^m}$$

on the unit disc. This was further extended in the joint paper [EL18] to a multivariable setting, i.e., to the case of commuting tuples $T \in L(H)^d$ satsifying suitable identities related to the reproducing kernel

$$K_m : \mathbb{B}_d \times \mathbb{B}_d \to \mathbb{C}, K_m(z, w) = \frac{1}{(1 - \langle z, w\rangle)^m}$$

on the unit ball $\mathbb{B}_d \subseteq \mathbb{C}^d$. The aim of this chapter is to show that a similar decomposition theorem holds even in the setting of unitarily invariant functional Hilbert spaces on $\mathbb{B}_d$.

In the following, let H_K be a unitarily invariant space with reproducing kernel

$$K : \mathbb{B}_d \times \mathbb{B}_d \to \mathbb{C}, K(z, w) = \sum_{k=0}^{\infty} a_k \langle z, w\rangle^k$$

such that $a_0 = 1$, $a_k > 0$ for all k, $\sup \frac{a_k}{a_{k+1}} < \infty$ and $\inf \frac{a_k}{a_{k+1}} > 0$. Furthermore, suppose that $k : \mathbb{D} \to \mathbb{C}, z \mapsto \sum_{k=0}^{\infty} a_k z^k$, has no zeroes. As before let us denote the Taylor coefficients of $\frac{1}{k}$ by $(c_k)_{k\in\mathbb{N}}$.

We consider a commuting tuple $T = (T_1, ..., T_d) \in L(H)^d$ of bounded linear operators on a Hilbert space H which is regular at 0 (cf. Definition 4.2.1). In particular, the row operator $T : H^d \to H$ has closed range and thus $T^*T : \operatorname{ran} T^* \to \operatorname{ran} T^*$ is invertible. As in Chapter 4.2, we denote its inverse by $(T^*T)^{-1}$ and consider the column operator $L = (T^*T)^{-1}T^* \in L(H, H^d)$. We define an operator $\delta_T \in L(H)$ by

$$\delta_T = (\operatorname{ran} T \xrightarrow{T^*} \operatorname{ran} T^*)^{-1}(T^*T)^{-1}T^*,$$

and suppose that the limit

$$\Delta_T = \mathrm{SOT} - \sum_{k=0}^{\infty}(-c_{k+1})\sigma_T^k(1_H)$$

exists. Here, $\sigma_T : L(H) \to L(H)$ is the operator given by $\sigma_T(X) = \sum_{i=1}^{d} T_i X T_i^*$. If $K = K_m$ is the kernel of the generalized Bergman space $A_m^2(\mathbb{B}_d)$, then

$$\Delta_T = \sum_{j=0}^{m-1}(-1)^j \binom{m}{j+1}\sigma_T^j(1_H)$$

and we proved in [EL18] that an analytic commuting tuple $T \in L(H)^d$ that is regular at $z = 0$ is unitarily equivalent to a shift $M_z \in L(A_m^2(\mathbb{B}_d) \otimes D)^d$ if and only if T satisfies the operator identity

$$(T^*T)^{-1} = (\oplus\Delta_T)\,|_{\operatorname{ran} T^*}.$$

For a general kernel K as explained above, let us denote by $\mathcal{W}(K)$ the set of all commuting tuples $T \in L(H)^d$ that are regular at $z = 0$ and satisfy the identity

$$(T^*T)^{-1} = (\oplus\Delta_T)|_{\operatorname{ran} T^*}.$$

For $T \in \mathcal{W}(K)$, the column operator

$$L : H \to H^d, x \mapsto (T^*T)^{-1}T^*x = (\Delta_T T_i^* x)_{i=1}^d$$

induces a commuting tuple $(\Delta_T T_i^*)_{i=1}^d \in L(H)^d$ which we also denote by L.

Lemma 5.1. For $T \in \mathcal{W}(K)$, the intertwining relations $T_i^* \delta_T = \Delta_T T_i^*$ $(i = 1, ..., d)$ hold. In particular, the tuple $L = (\Delta_T T_i^*)_{i=1}^d \in L(H)^d$ is commuting.

Proof. For $i = 1, ..., d$, we immediately see that $T_i^* \delta_T$ is the i-th component of $(T^*T)^{-1}T^* = (\oplus \Delta_T)T^*$ and thus equals $\Delta_T T_i^*$. To show that L is indeed commuting, it suffices to observe that

$$(\Delta_T T_i^*)(\Delta_T T_j^*) = \Delta_T T_i^* T_j^* \delta_T = (\Delta_T T_j^*)(\Delta_T T_i^*)$$

for $i, j = 1, ..., d$. □

In the following, let $T \in \mathcal{W}(K)$. Since T is regular at $z = 0$, the results of Section 4.2 can be applied and yield a continuous linear map $V : H \to \mathcal{O}(\Omega_T, W(T))$ on $\Omega_T = B_{\frac{1}{\|L\|}}(0)$ that intertwines the tuples $T \in L(H)^d$ and M_z on $\mathcal{O}(\Omega_T, W(T))$ componentwise. Since L is commuting, the representation of the map V obtained in the proof of Theorem 4.2.5 (see also Lemma 4.2.2) simplifies to

$$(Vx)(z) = \sum_{\alpha \in \mathbb{N}^d} \gamma_\alpha (P_{W(T)} L^\alpha x) z^\alpha \quad (x \in H, z \in \Omega_T),$$

where $\gamma_\alpha = \frac{|\alpha|!}{\alpha!}$ for $\alpha \in \mathbb{N}^d$.

Lemma 5.2. For $\alpha, \beta \in \mathbb{N}^d$, we have

$$\gamma_\alpha P_{W(T)} L^\alpha T^\beta = \gamma_{\alpha-\beta} P_{W(T)} L^{\alpha-\beta},$$

where the right-hand side has to be read as zero whenever $\alpha - \beta$ has negative components.

Proof. For $\beta \in \mathbb{N}^d$, $x \in H$ and $z \in \Omega_T$,

$$\sum_{\alpha \in \mathbb{N}^d} \gamma_\alpha P_{W(T)} (L^\alpha x) z^{\alpha+\beta} = z^\beta (Vx)(z)$$
$$= (VT^\beta x)(z) = \sum_{\alpha \in \mathbb{N}^d} \gamma_\alpha P_{W(T)} (L^\alpha T^\beta x) z^\alpha.$$

The proof follows by comparing coefficients of these convergent power series. □

Lemma 5.3. For $\alpha \in \mathbb{N}^d$, the identity

$$P_{W(T)} L^\alpha = a_{|\alpha|} P_{W(T)} T^{*\alpha}$$

holds.

Proof. We use induction on $|\alpha|$. Suppose that the result holds for $|\alpha| \leq k$. Let $\alpha \in \mathbb{N}^d$ be a multiindex with $|\alpha| = k$ and let $i \in \{1, ..., d\}$ be arbitrary. Using the inductive hypothesis, Lemma 5.2 and Lemma 2.2.5, we obtain

$$\begin{aligned}
P_{W(T)} L^{\alpha+e_i} &= P_{W(T)} L^\alpha \Delta_T T_i^* \\
&= P_{W(T)} L^\alpha (\mathrm{SOT} - \sum_{k=0}^{\infty} (-c_{k+1}) \sum_{|\beta|=k} \gamma_\beta T^\beta T^{*\beta+e_i}) \\
&= \mathrm{SOT} - \sum_{k=0}^{\infty} (-c_{k+1}) \sum_{|\beta|=k} \gamma_\beta P_{W(T)} L^\alpha T^\beta T^{*\beta+e_i} \\
&= \mathrm{SOT} - \sum_{k=0}^{\infty} (-c_{k+1}) \sum_{|\beta|=k, \beta \leq \alpha} \frac{\gamma_{\alpha-\beta}\gamma_\beta}{\gamma_\alpha} P_{W(T)} L^{\alpha-\beta} T^{*\beta+e_i} \\
&= \sum_{k=0}^{|\alpha|} (-c_{k+1}) \sum_{|\beta|=k, \beta \leq \alpha} \frac{\gamma_{\alpha-\beta}\gamma_\beta}{\gamma_\alpha} a_{|\alpha-\beta|} P_{W(T)} T^{*\alpha+e_i} \\
&= (\sum_{\beta \leq \alpha} -c_{|\beta|+1} a_{|\alpha-\beta|} \frac{\gamma_{\alpha-\beta}\gamma_\beta}{\gamma_\alpha}) P_{W(T)} T^{*\alpha+e_i} \\
&= a_{|\alpha|+1} P_{W(T)} T^{*\alpha+e_i}.
\end{aligned}$$

$\square$

As in Chapter 4.2, we write $H_\infty = \bigcap_{k=0}^{\infty} \sum_{|\alpha|=k} T^\alpha H$ and call T analytic if $H_\infty = \{0\}$. By Theorem 4.2.5, the kernel $\ker V = H_\infty$ is a closed invariant subspace for T. For $T \in \mathcal{W}(K)$, much more than this is true.

Lemma 5.4. The kernel of V is reducing for T with

(a) $\ker V = H_\infty = \{x \in H; P_{W(T)} T^{*\alpha} x = 0 \text{ for all } \alpha \in \mathbb{N}^d\}$,

(b) $(\ker V)^\perp = \bigvee_{\alpha \in \mathbb{N}^d} T^\alpha W(T)$.

Proof. The first equality in (a) holds by Theorem 4.2.5. Since

$$(Vx)(z) = \sum_{\alpha \in \mathbb{N}^d} \gamma_\alpha (P_{W(T)} L^\alpha x) z^\alpha$$

for $x \in H$ and $z \in \Omega_T$, the kernel of V consists of all vectors $x \in H$ with $P_{W(T)} L^\alpha x = 0$ for all $\alpha \in \mathbb{N}^d$. Thus the second equality in part (a) follows from Lemma 5.3. In view of the identity

$$\langle x, T^\alpha y \rangle = \langle P_{W(T)} T^{*\alpha} x, y \rangle \qquad (x \in H, y \in W(T), \alpha \in \mathbb{N}^d)$$

part (b) follows from (a). Both parts together imply that $\ker V$ is a reducing subspace for T. $\square$

In the following we write $[M] \subseteq H$ for the smallest closed T-invariant subspace of H which contains a given subset $M \subseteq H$. For a complex Hilbert space $\mathcal{E}$, we denote by $H_K(\mathcal{E})$ the $\mathcal{E}$-valued analytic functional Hilbert space with reproducing kernel

$$K_{\mathcal{E}} : \mathbb{B}_d \times \mathbb{B}_d \to L(\mathcal{E}), K_{\mathcal{E}}(z,w) = K(z,w)1_{\mathcal{E}}$$

on $\mathbb{B}_d$. A well known alternative description of the space $H_K(\mathcal{E})$ is given by

$$H_K(\mathcal{E}) = \{f = \sum_{\alpha \in \mathbb{N}^d} f_\alpha z^\alpha; \|f\|^2 = \sum_{\alpha \in \mathbb{N}^d} \frac{\|f_\alpha\|^2}{a_\alpha \gamma_\alpha} < \infty\}.$$

Theorem 5.5. Let $T \in \mathcal{W}(K)$. Then the map

$$U : [W(T)] \to H_K(W(T)), (Ux)(z) = \sum_{\alpha \in \mathbb{N}^d} \gamma_\alpha (P_{W(T)} L^\alpha x) z^\alpha$$

is a unitary map which componentwise intertwines the tuples $T|_{[W(T)]}$ and $M_z \in L(H_K(W(T)))^d$.

Proof. For $N \in \mathbb{N}$ and elements $x_\alpha \in W(T)$ $(|\alpha| \leq N)$, we have

$$\begin{aligned} \| \sum_{|\alpha| \leq N} T^\alpha x_\alpha \|^2 &= \sum_{|\alpha|,|\beta| \leq N} \langle P_{W(T)} T^{*\beta} T^\alpha x_\alpha, x_\beta \rangle \\ &= \sum_{|\alpha|,|\beta| \leq N} \langle x_\alpha, P_{W(T)} T^{*\alpha} T^\beta x_\beta \rangle. \end{aligned}$$

Using Lemma 5.3 we conclude that

$$\begin{aligned} \| \sum_{|\alpha| \leq N} T^\alpha x_\alpha \|^2 &= \sum_{|\alpha|,|\beta| \leq N} \frac{1}{a_{|\beta|}} \langle P_{W(T)} L^\beta T^\alpha x_\alpha, x_\beta \rangle \\ &= \sum_{|\alpha|,|\beta| \leq N} \frac{1}{a_{|\alpha|}} \langle x_\alpha, P_{W(T)} L^\alpha T^\beta x_\beta \rangle. \end{aligned}$$

Thus Lemma 5.2 yields that

$$\begin{aligned} \| \sum_{|\alpha| \leq N} T^\alpha x_\alpha \|^2 &= \sum_{|\alpha| \leq N} \frac{1}{a_{|\alpha|}} \langle P_{W(T)} L^\alpha T^\alpha x_\alpha, x_\alpha \rangle \\ &= \sum_{|\alpha| \leq N} \frac{1}{a_{|\alpha|} \gamma_\alpha} \|x_\alpha\|^2 = \| \sum_{|\alpha| \leq N} x_\alpha z^\alpha \|^2_{H_K(W(T))}. \end{aligned}$$

Since the polynomials with coefficients in $W(T)$ are dense in $H_K(W(T))$, there is a unique unitary operator $U : [W(T)] \to H_K(W(T))$ with

$$U(\sum_{|\alpha|\leq N} T^\alpha x_\alpha) = \sum_{|\alpha|\leq N} x_\alpha z^\alpha$$

for all finite families $(x_\alpha)_{|\alpha|\leq N}$ in $W(T)$. Obviously, U satisfies the intertwining relations $UT_i = M_{z_i}U$ $(i = 1, ..., d)$ on the dense linear subspace $M = \text{span}\{T^\alpha x; \alpha \in \mathbb{N}^d \text{ and } x \in W(T)\} \subseteq [W(T)]$. Hence the same relations hold on $[W(T)]$. Since the maps $U : [W(T)] \to \mathcal{O}(\mathbb{B}_d, W(T))$ and $V : [W(T)] \to \mathcal{O}(\Omega_T, W(T))$ are continuous and since the functions $Uh \in \mathcal{O}(\mathbb{B}_d, W(T))$ and $Vh \in \mathcal{O}(\Omega_T, W(T))$ have the same Taylor coefficients at $z = 0$ for $h \in M$, it follows that

$$(Ux)(z) = \sum_{\alpha\in\mathbb{N}^d} \gamma_\alpha(P_{W(T)}L^\alpha x)z^\alpha$$

for $x \in [W(T)]$ and $z \in \mathbb{B}_d$. □

For $N \in \mathbb{N}$, let

$$(\frac{1}{K})_N(T) = \sum_{k=0}^{N} c_k\sigma_T^k(1_H).$$

We call T a K-contraction if the SOT-limit

$$\frac{1}{K}(T) = \text{SOT} - \lim_{N\to\infty}(\frac{1}{K})_N(T)$$

exists and defines a positive operator.
If T is a K-contraction, we set

$$\Sigma_{K,N}(T) = 1_H - \sum_{k=0}^{N} a_k\sigma_T^k(\frac{1}{K}(T))$$

for $N \in \mathbb{N}$. We call a K-contraction T pure if the series $(\Sigma_{K,N})_{N\in\mathbb{N}}$ converges to zero in the strong operator topology. We define

$$\Sigma_K(T) = \text{SOT} - \lim_{N\to\infty}\Sigma_{K,N}(T)$$

if this limit exists. Both properties are preserved under unitary equivalence.

Lemma 5.6. Let $T \in L(H)^d$ and $S \in L(K)^d$ be commuting tuples on Hilbert spaces H and K, respectively. Suppose there exists a unitary $\Pi : H \to K$ such that $\Pi T_i = S_i\Pi$ for $i = 1, ..., d$. If S is a (pure) K-contraction, then T is a (pure) K-contraction.

A proof of Lemma 5.6 can be found in [Sch18] (Lemma 2.13). The following lemma shows in particular that every operator in $\mathcal{W}(K)$ is a K-contraction.

Lemma 5.7. Let $T \in L(H)^d$ be a commuting tuple of bounded linear operators. If the SOT-limit

$$\Delta_T = \mathrm{SOT} - \sum_{k=0}^{\infty} (-c_{k+1}) \sigma_T^k(1_H)$$

exists, then also the SOT-limit

$$\frac{1}{K}(T) = \mathrm{SOT} - \sum_{k=0}^{\infty} c_k \sigma_T^k(1_H)$$

exists and $\frac{1}{K}(T) = 1_H - \sigma_T(\Delta_T)$. If we further suppose that $\operatorname{ran} T \subseteq H$ is closed and that T satisfies the identity

$$(T^*T)^{-1} = (\oplus \Delta_T)|_{\operatorname{ran} T^*},$$

then $P_{W(T)} = 1_H - \sigma_T(\Delta_T) = \frac{1}{K}(T)$ and $P_{H_\infty} = \Sigma(T)$.

Proof. Let $T \in L(H)^d$ be a commuting tuple such that the limit $\Delta_T = \mathrm{SOT} - \sum_{k=0}^{\infty} (-c_{k+1}) \sigma_T^k(1_H)$ exists. For every $N \in \mathbb{N}$, we have

$$\begin{aligned} \sum_{k=0}^{N} c_k \sigma_T^k(1_H) &= 1_H + \sum_{k=0}^{N-1} c_{k+1} \sigma_T(\sigma_T^k(1_H)) \\ &= 1_H - \sigma_T\Big(\sum_{k=0}^{N-1} (-c_{k+1}) \sigma_T^k(1_H)\Big). \end{aligned}$$

By taking the SOT-limit for $N \to \infty$, we conclude that $\frac{1}{K}(T)$ exists and

$$\frac{1}{K}(T) = 1_H - \sigma_T(\Delta_T).$$

If $\operatorname{ran} T \subseteq H$ is closed and T satisfies the identity $(T^*T)^{-1} = (\oplus \Delta_T)|_{\operatorname{ran} T^*}$, we can use Lemma 4.2.2 to see that

$$\frac{1}{K}(T) = 1_H - \sigma_T(\Delta_T) = 1_H - T(\oplus \Delta_T)T^* = 1_H - T(T^*T)^{-1}T^* = P_{W(T)}.$$

Let $x = x_0 + x_1 \in H$ with $x_0 \in H_\infty$ and $x_1 \in [W(T)]$ be given. Then

$$\begin{aligned}
&\sum_{k=0}^{N} a_k \langle \sigma_T^k(\frac{1}{K}(T))x, x\rangle = \sum_{|\alpha|\le N} a_{|\alpha|}\gamma_\alpha \langle T^\alpha P_{W(T)} T^{*\alpha} x, x\rangle \\
&= \sum_{|\alpha|\le N} a_{|\alpha|}\gamma_\alpha \|P_{W(T)} T^{*\alpha} x_1\|^2 = \sum_{|\alpha|\le N} \frac{\gamma_\alpha}{a_{|\alpha|}} \|P_{W(T)} L^\alpha x_1\|^2 \\
&= \| \sum_{|\alpha|\le N} \gamma_\alpha \left(P_{W(T)} L^\alpha x_1\right) z^\alpha \|^2_{H_K(W(T))} \\
&\xrightarrow{N\to\infty} \|U x_1\|^2_{H_K(W(T))} = \|x_1\|^2 = \|x\|^2 - \|P_{H^\infty} x\|^2.
\end{aligned}$$

It follows that

$$\Sigma_K(T) = \mathrm{SOT} - \lim_{N\to\infty} \left(1_H - \sum_{k=0}^{N} a_k \sigma_T^k(\frac{1}{K}(T))\right) = P_{H^\infty}.$$

□

In Chapter 4.2, we established that every analytic commuting tuple $T \in L(H)^d$ that is regular at 0 is unitarily equivalent to the multiplication tuple $M_z \in L(\hat{H})^d$ on a suitable analytic functional Hilbert space $\hat{H} \subseteq \mathcal{O}(\Omega_T, W(T))$.

Theorem 5.8. For $T \in \mathcal{W}(K)$, the following conditions on T are equivalent:

(i) T is analytic,

(ii) $\|x\| = \| \sum_{\alpha\in\mathbb{N}^d} \gamma_\alpha (P_{W(T)} L^\alpha x) z^\alpha \|_{H_K(W(T))}$ for all $x \in H$,

(iii) T is a pure K-contraction.

If there is a natural number $p \in \mathbb{N}$ such that

$$c_k \ge 0 \text{ for all } k \ge p \text{ or } c_k \le 0 \text{ for all } k \ge p,$$

then the previous conditions are equivalent to

(iv) T is unitarily equivalent to $M_z \in L(H_K(D))^d$ for some Hilbert space D.

Proof. The equivalence of (i) and (ii) and the implication (i) to (iv) follows from Lemma 5.4 and Theorem 5.5. The equivalence of (i) and (iii) is a consequence of Lemma 5.7.
Suppose that almost all c_k ($k \in \mathbb{N}$) have the same sign. Then Lemma 2.10 and Lemma 2.12 in [Sch18] show that $M_z \in L(H_K(D))^d$ is a pure K-contraction. By Lemma 5.6, these properties are preserved by unitary equivalence and thus the implication (iv) to (iii) holds. □

Following [GR06], we call a commuting tuple $S = (S_1, ..., S_d) \in L(H)^d$ of bounded operators an m-isometry if

$$\sum_{k=0}^{m} (-1)^k \binom{m}{k} \sigma_{S^*}^k(1_H) = 0.$$

We call S an m-coisometry if S^* is an m-isometry. Using the reproducing kernel

$$K_m : \mathbb{B}_d \times \mathbb{B}_d \to \mathbb{C}, K_m(z, w) = \frac{1}{(1 - \langle z, w \rangle)^m},$$

we see that S is an m-coisometry if and only if $\frac{1}{K_m}(S) = 0$.
With this motivation, we call S a K-coisometry if $\frac{1}{K}(S) = 0$.

Lemma 5.9. If $\Delta_T = \mathrm{SOT} - \sum_{k=0}^{\infty}(-c_{k+1})\sigma_T^k(1_H)$ exists and if T is a K-coisometry, then T is surjective.

Proof. By Lemma 5.7, we know that

$$0 = \frac{1}{K}(T) = 1_H - \sigma_T(\Delta_T) = 1_H - T(\oplus \Delta_T)T^*.$$

But then $T : H^d \to H$ admits a right inverse and hence is surjective. □

We obtain an extension of the main result from [GO12] to our setting of unitarily invariant spaces if we suppose that almost all c_k ($k \in \mathbb{N}$) have the same sign.

Theorem 5.10. Let $T \in L(H)^d$ be a commuting tuple. Then $T \in \mathcal{W}(K)$ if and only if $T = T^{(0)} \oplus T^{(1)} \in L(H_0 \oplus H_1)^d$ is the direct sum of commuting tuples $T^{(0)} = T|_{H_0}$ and $T^{(1)} = T|_{H_1}$ such that

(i) $T^{(0)} \in \mathcal{W}(K)$ is surjective and

(ii) $T^{(1)}$ is unitarily equivalent to $M_z \in L(H_K(D))^d$ for some Hilbert space D.

In this case, the spaces H_0 and H_1 are uniquely determined by

$$H_0 = H_\infty \text{ and } H_1 = [W(T)].$$

Proof. Suppose that $T \in \mathcal{W}(K)$. By Lemma 5.4

$$H = H_0 \oplus H_1$$

is the direct sum of the reducing subspaces $H_0 = H_\infty$ and $H_1 = [W(T)]$. An elementary argument using Lemma 4.2.2 (see also the subsequent remark) yields that $T|_{H_0}$ is regular at 0. Obviously

$$\sum_{k=0}^{\infty}(-c_{k+1})\sigma^k_{T^{(0)}}(1_{H_0})x = \Delta_T x$$

for all $x \in H_0$. Thus $\Delta_{T^{(0)}} = \Delta_T|_{H_0}$ exists. Since the invertible operator

$$\operatorname{ran} T^{(0)*} \xrightarrow{T^{(0)*}T^{(0)}} \operatorname{ran} T^{(0)*}$$

is simply the restriction of the invertible operator $T^*T : \operatorname{ran} T^* \to \operatorname{ran} T^*$, it follows that $(T^{(0)*}T^{(0)})^{-1} = (\oplus\Delta_{T^{(0)}})|_{\operatorname{ran} T^{(0)*}}$. Thus $T^{(0)} \in \mathcal{W}(K)$. Since

$$W(T^{(0)}) = W(T) \cap H_\infty = \{0\},$$

we see that $T^{(0)}$ is surjective. By Theorem 5.5 the restriction $T^{(1)} = T|_{H_1}$ is unitarily equivalent to $M_z \in L(H_K(W(T)))^d$.
Conversely, suppose that $T = T^{(0)} \oplus T^{(1)}$ is the direct sum of tuples $T^{(0)}$ and $T^{(1)}$ as in (i) and (ii). An elementary exercise shows that the class of commuting tuples belonging to the class $\mathcal{W}(K)$ is stable under direct sums and unitary equivalence. Thus the reverse implication follows by Lemma 2.2.1 and Theorem 2.2.6.
For the uniqueness part of the Theorem, write $T = T^{(0)} \oplus T^{(1)} \in L(H_0 \oplus H_1)^d$ as the direct sum of a surjective operator $T^{(0)} \in L(H_0)^d$ in $\mathcal{W}(K)$ and a tuple $T^{(1)} \in L(H_1)^d$ which is unitarily equivalent to $M_z \in L(H_{K_\alpha}(D))^d$ for some Hilbert space D. Since $T^{(0)}$ is surjective, we have

$$W(T) = W(T^{(0)}) \oplus W(T^{(1)}) = W(T^{(1)}).$$

Since $T^{(1)}$ is unitarily equivalent to $M_z \in L(H_{K_\alpha}(D))^d$, it follows that

$$H_1 = \bigvee_{\alpha\in\mathbb{N}^d} T^\alpha W(T^{(1)}) = [W(T)].$$

□

The hypothesis that almost all $c_k (k \in \mathbb{N})$ have the same sign is only needed to ensure that $M_z \in L(H_K(D))^d$ belongs to the class $\mathcal{W}(K)$.

Remark 5.11. (a) Let $T \in L(H)^d$ be a commuting tuple. Then T is surjective and belongs to $\mathcal{W}(K)$ if and only if Δ_T exists, $(\oplus\Delta_T)\operatorname{ran} T^* \subseteq \operatorname{ran} T^*$ and

T is a K-coisometry. Indeed, if $T \in \mathcal{W}(K)$ is surjective, then by Lemma 5.7

$$\frac{1}{K}(T) = P_{W(T)} = 0.$$

Conversely, if Δ_T exists and T is a K-coisometry, then T is surjective by Lemma 5.9. Using Lemma 4.2.2 and Lemma 5.7 we obtain that

$$1_H - T(T^*T)^{-1}T^* = P_{W(T)} = 0 = \frac{1}{K}(T) = 1_H - T(\oplus\Delta_T)T^*.$$

Since $\operatorname{ran} T^* = (\ker T)^\perp$, the condition $(\oplus\Delta_T)\operatorname{ran} T^* \subseteq \operatorname{ran} T^*$ then implies that

$$(T^*T)^{-1} = (\oplus\Delta_T)|_{\operatorname{ran} T^*}.$$

(b) Elementary arguments using Abel's limit theorem show that

$$\lim_{r\uparrow 1} \sum_{k=0}^{\infty} c_k r^k = \lim_{r\uparrow 1} \frac{1}{k(r)} = \frac{1}{\sum_{k=0}^{\infty} a_k} \in [0,1[.$$

By definition this means that the series $\sum_{k=0}^{\infty} c_k$ is always Abel summable with Abel limit $\frac{1}{\sum_{k=0}^{\infty} a_k} \in [0,1[$. If almost all c_k have the same sign, then the series

$$\sum_{k=0}^{\infty} c_k = \frac{1}{\sum_{k=0}^{\infty} a_k} \in [0,1[$$

converges (absolutely) in the ordinary sense. If $T \in L(H)^d$ is a row contraction, then

$$1 \geq \|\sigma_T(1_H)\| = \|\sigma_T\| \geq \|\sigma_T^k\| = \|\sigma_T^k(1_H)\|$$

for all $k \in \mathbb{N}$. Thus if almost all c_k have the same sign, then the limit

$$\Delta_T = \sum_{k=0}^{\infty}(-c_{k+1})\sigma_T^k(1_H)$$

exists for every row contraction $T \in L(H)^d$ even in the operator norm.

Next we consider the particular case where $k = k_\alpha : \mathbb{D} \to \mathbb{C}, z \mapsto \frac{1}{(1-z)^\alpha}$ for some $\alpha \geq 1$. Then $K = K_\alpha : \mathbb{B}_d \times \mathbb{B}_d \to \mathbb{C}, (z,w) \mapsto \frac{1}{(1-\langle z,w\rangle)^\alpha}$, and $H_{K_\alpha} = A^2_\alpha(\mathbb{B}_d)$ is the generalized Bergman space introduced in Example 2.1.4. Note that we have

$$\frac{1}{k}(z) = (1-z)^\alpha = \sum_{k=0}^{\infty}(-1)^k \binom{\alpha}{k} z^\alpha$$

and thus $c_k = (-1)^k \binom{\alpha}{k}$ $(k \in \mathbb{N})$ in this case. It is well known that $\sum_{k=0}^{\infty} c_k = \sum_{k=0}^{\infty} (-1)^k \binom{\alpha}{k} = (1-1)^{\alpha} = 0$ where the series converges even absolutely.
In this case, for a row contraction $T \in L(H)^d$, the limit occurring in the definition of Δ_T exists even in the operator norm by Remark 5.11.
Let us recall that a commuting tuple $S \in L(H)^d$ is called a spherical isometry if it is a 1-isometry, that is, if $\sum_{1 \le i \le d} S_i^* S_i = 1_H$ or, equivalently, if $\sum_{1 \le i \le d} \|S_i x\|^2 = \|x\|^2$ for each vector $x \in H$. A spherical coisometry is a commuting tuple $S \in L(H)^d$ such that the adjoint $S^* \in L(H)^d$ is a spherical isometry.
We can now refine Theorem 5.10 in case of the reproducing kernels K_α $(\alpha \ge 1)$.

Theorem 5.12. Let $T \in L(H)^d$ be a commuting row contraction and let $\alpha \in [1, \infty)$. Then $T \in \mathcal{W}(K_\alpha)$ if and only if $T = T^{(0)} \oplus T^{(1)} \in L(H_0 \oplus H_1)^d$ is the direct sum of commuting tuples $T^{(0)} = T|_{H_0}$ and $T^{(1)} = T|_{H_1}$ such that

(i) $T^{(0)}$ is a spherical coisometry and

(ii) $T^{(1)}$ is unitarily equivalent to $M_z \in L(H_{K_\alpha}(D))^d$ for some Hilbert space D.

In this case, the spaces H_0 and H_1 in (i) and (ii) are uniquely determined by

$$H_0 = H_\infty \text{ and } H_1 = [W(T)].$$

Proof. Fix a real number $\alpha \ge 1$. Let

$$\sum_{k=0}^{\infty} a_k z^k = k_\alpha(z) \quad (z \in \mathbb{D})$$

be the power series representation of $k_\alpha : \mathbb{D} \to \mathbb{C}, k_\alpha(z) = \frac{1}{(1-z)^\alpha}$. Then

$$\sum_{k=0}^{\infty} a_k = \lim_{r \uparrow 1} k_\alpha(r) = \infty.$$

If $T \in L(H)^d$ is a spherical coisometry, then

$$\frac{1}{K_\alpha}(T) = \sum_{k=0}^{\infty} c_k \sigma_T^k(1_H) = (\sum_{k=0}^{\infty} c_k) 1_H = 0$$

and

$$\Delta_T = \sum_{k=0}^{\infty} (-c_{k+1}) \sigma_T^k(1_H) = \sum_{k=0}^{\infty} (-c_{k+1}) 1_H = 1_H.$$

By Remark 5.11 (a) it follows that T belongs to $\mathcal{W}(K_\alpha)$ and is surjective. Conversely, by Theorem 3.51 in [Sch18], if $T \in L(H)^d$ is a row contraction and a K_α-contraction, then

$$\mathrm{SOT} - \sum_{k=0}^{\infty} a_k \sigma_T^k(\frac{1}{K_\alpha}(T)) + T_\infty = 1_H,$$

where $T_\infty = \mathrm{SOT} - \lim_{k\to\infty} \sigma_T^k(1_H)$. Thus if $T \in L(H)^d$ is a K_α-coisometry and a row contraction, then

$$1_H = T_\infty \leq \sigma_T(1_H) \leq 1_H$$

and hence T is a spherical coisometry. It follows from Remark 5.11 (a) that a surjective row contraction $T \in \mathcal{W}(K_\alpha)$ is a spherical coisometry. Thus Theorem 5.12 follows as a particular case of 5.10. □

Remark 5.13. (a) If $T \in L(H)$ is a single left invertible operator, T is regular at 0 by Lemma 4.2.4. Theorem 5.10 for a single left invertible operator $T \in L(H)$ and $K = K_m : \mathbb{D} \times \mathbb{D} \to \mathbb{C}, K_m(z,w) = \frac{1}{(1-z\overline{w})^m}$ $(m \in \mathbb{N})$ is contained in [GO12] (Theorems 2.1 and 3.1). Every K_m-coisometric part $T^{(0)}$ of T as in (ii) is surjective by Lemma 5.9. Since T and thus $T^{(0)}$ is also injective in this case, it is even invertible. Thus, the condition

$$(\Delta_{T^{(0)}}) \operatorname{ran} T^{(0)*} \subseteq \operatorname{ran} T^{(0)*}$$

can be omitted in this case.

However, in our more general setting, this condition is essential. To see this, consider the operator

$$T : H^2(\mathbb{D}) \oplus H^2(\mathbb{D}) \to H^2(\mathbb{D}) \oplus H^2(\mathbb{D}), T(f,g) = (M_z^* f, f + M_z^* g).$$

The adjoint of T acts as

$$T^*(f,g) = (M_z f, M_z g) + (g, 0) \quad (f, g \in H^2(\mathbb{D}))$$

and can thus be written as the sum of an isometry and a nilpotent operator of order 2. By Theorem 2.2 in [BMN13] (compare also [GS15]), T is a 3-coisometry. On the other hand, we have

$$\begin{aligned}\Delta_T &= \sum_{j=0}^{2} (-1)^j \binom{3}{j+1} T^j T^{*j} \\ &= 3 \operatorname{Id}_{H^2(\mathbb{D}) \oplus H^2(\mathbb{D})} - 3TT^* + T^2 T^{*2}.\end{aligned}$$

We compute

$$(3TT^* - T^2T^{*2})T^*(f,g) = (2zf + 3g, z^2f + 2zg)$$

for $f, g \in H^2(\mathbb{D})$ and in particular

$$(3TT^* - T^2T^{*2})T^*(0,1) = (3, 2z) \notin \operatorname{ran} T^*.$$

This yields

$$\Delta_T \operatorname{ran} T^* \not\subseteq \operatorname{ran} T^*.$$

(b) As a consequence of Theorem 5.10 and Theorem 5.12 in the case $K = K_m$ ($m \in \mathbb{N}$) each m-coisometry $T \in L(H)^d$ which is a row contraction is a spherical coisometry. For single operators $T \in L(H)$, this phenomenon is well known and follows for instance from Proposition 3.2 in [Shi01].

In the particular case $\alpha = 1$ the result stated in Theorem 5.12 takes the following form.

Corollary 5.14. Let $T \in L(H)^d$ be a commuting tuple that is regular at 0. Then the row operator $T : H^d \to H$ is a partial isometry if and only if $T = T^{(0)} \oplus T^{(1)} \in L(H^{(0)} \oplus H^{(1)})^d$ is the direct sum of a spherical coisometry $T^{(0)} \in L(H^{(0)})^d$ and a tuple $T^{(1)} \in L(H^{(1)})^d$ which is unitarily equivalent to $M_z \in L(H_{K_1}(D))^d$ for some Hilbert space D.

Proof. For $\alpha = 1$, we have $c_0 = 1$, $c_1 = -1$ and $c_n = 0$ for all $n > 2$. We conclude $\Delta_T = 1_H$ and thus the operator identity from the definition of $\mathcal{W}(K_1)$ means precisely that $T : H^d \to H$ is a partial isometry. Hence the assertion follows as an immediate consequence of Theorem 5.12. □

We briefly consider the one-dimensional case of Corollary 5.14. Following Halmos and Wallen [HW70], we call a bounded linear operator $T \in L(H)$ on a Hilbert space a power partial isometry if every power $T^k \in L(H)$ of T ($k \in \mathbb{N}$) is a partial isometry.

Corollary 5.15. Let $T \in L(H)$ be a partial isometry that is regular at 0. Then T is a power partial isometry.

Proof. Let $T \in L(H)$ be a partial isometry and $k \in \mathbb{N}$. By Corollary 5.14, $T = T_0 \oplus T_1 \in L(H_0 \oplus H_1)$ is the direct sum of a coisometry $T_0 \in L(H_0)$ and an operator $T_1 \in L(H_1)$ which is unitarily equivalent to $M_z \in L(H_{K_1}(D))$ for some Hilbert space D. As a direct sum

$$T^k = T_0^k \oplus T_1^k$$

of two partial isometries the operator T^k is a partial isometry again. □

In [HW70] a Wold decomposition theorem for general power partial isometries is proved.

6 Cowen-Douglas operators and fiber dimension

Let $\mathcal{H} \subseteq \mathcal{O}(\Omega, \mathbb{C}^N)$ be a functional Hilbert space of $\mathbb{C}^N$-valued functions on a domain $\Omega \subseteq \mathbb{C}^d$. The number

$$\mathrm{fd}(\mathcal{H}) = \max_{\lambda \in \Omega} \dim \mathcal{H}_\lambda,$$

where $\mathcal{H}_\lambda = \{f(\lambda); f \in \mathcal{H}\}$ is usually referred to as the fiber dimension of $\mathcal{H}$. In this chapter, we will use the model theorem for weak dual Cowen-Douglas tuples $T \in L(X)^d$ on a Banach space X from Chapter 4 to associate with a linear subspace $Y \subseteq X$ an integer $\mathrm{fd}_T(Y)$ called the fiber dimension of Y. These results were first published in [EL17]. For single Cowen-Douglas operators on Hilbert spaces corresponding results were proved by L. Chen, G. Chang and X. Fang in [CCF15].
As in Chapter 4.1, we also slightly generalise our setting by allowing Ω to be a connected complex submanifold of $\mathbb{C}^d$.

6.1 Fiber dimension for linear subspaces

Let $\Omega \subseteq \mathbb{C}^d$ be a connected complex submanifold. Let D be a finite-dimensional vector space and $M \subseteq \mathcal{O}(\Omega, D)$ a linear subspace. We denote the point evaluations on M by

$$\epsilon_\lambda : M \to D, f \mapsto f(\lambda) \quad (\lambda \in \Omega).$$

If $M \subseteq \mathcal{O}(\Omega, D)$ is a $\mathbb{C}[z]$-submodule, we write

$$M_{z_i} : M \to M, (M_{z_i} f)(z) = z_i f(z) \quad (i = 1, ..., d)$$

for the multiplication operators with the coordinate functions. For $\lambda \in \Omega$, the range of ϵ_λ is a linear subspace

$$M_\lambda = \{f(\lambda); f \in M\} \subseteq D.$$

Definition 6.1.1. The number

$$\mathrm{fd}(M) = \max_{\lambda \in \Omega} \dim M_\lambda$$

is called the fiber dimension of M. A point $\lambda_0 \in \Omega$ with $\dim M_{\lambda_0} = \mathrm{fd}(M)$ is called a maximal point for M.

First we prove that the non-maximal points for a linear subspace $M \subseteq \mathcal{O}(\Omega, D)$ form a nowhere dense set. If $f \in \mathcal{O}(\Omega)$ is a holomorphic function, we denote its zero set as usual by

$$Z(f) = \{\lambda \in \Omega; f(\lambda) = 0\}.$$

Lemma 6.1.2. Let $\Omega \subseteq \mathbb{C}^d$ be a connected complex submanifold. Let D be a finite-dimensional vector space and $M \subseteq \mathcal{O}(\Omega, D)$ a linear subspace. Then

$$\{\lambda \in \Omega; \dim M_\lambda < \mathrm{fd}(M)\} \subseteq \Omega$$

is nowhere dense.

Proof. Let $m = \mathrm{fd}(M)$, $\lambda_0 \in \Omega$ a maximal point and $h_1, ..., h_m \in M$ such that $h_1(\lambda_0), ..., h_m(\lambda_0)$ are linearly independent. Furthermore, let $e_1, ..., e_N$ be a basis for D. We consider the analytic function $h_j^i \in \mathcal{O}(\Omega)$ $(j = 1, .., m, i = 1, ..., N)$ uniquely determined by

$$h_j(\lambda) = \sum_{i=1}^{N} h_j^i(\lambda) e_i \qquad (\lambda \in \Omega)$$

for $j = 1, ..., m$. By permuting the given basis of D, we can assume that the matrix-valued analytic function

$$\theta : \Omega \to \mathbb{C}^{m \times m}, \lambda \mapsto (h_j^i(\lambda))_{i,j=1}^m.$$

is invertible at λ_0. We also get

$$\{\lambda \in \Omega; \dim M_\lambda < \mathrm{fd}(M)\} \subseteq Z(\det\theta).$$

Since $\det\theta$ is analytic, the identity theorem implies that

$$\{\lambda \in \Omega; \dim M_\lambda < \mathrm{fd}(M)\} \subseteq \Omega$$

is nowhere dense. $\square$

We will see later that the set

$$\{\lambda \in \Omega; \dim M_\lambda < \mathrm{fd}(M)\} \subseteq \Omega$$

is even analytic in the setting of the preceding lemma (cf. proof of Lemma 6.2.1).
For any $\mathbb{C}[z]$-submodule $M \subseteq \mathcal{O}(\Omega, D)$ and any point $\lambda \in \Omega$, we have

$$\sum_{i=1}^{d}(\lambda_i - M_{z_i})M \subseteq \ker \epsilon_\lambda.$$

Under the condition that the codimension of $\sum_{i=1}^{d}(\lambda_i - M_{z_i})M$ is constant on Ω, the question whether equality holds here is closely related to corresponding properties of the fiber dimension of M.

Lemma 6.1.3. Consider a $\mathbb{C}[z]$-submodule $M \subseteq \mathcal{O}(\Omega, D)$ such that there is an integer N with

$$\dim M / \sum_{i=1}^{d}(\lambda_i - M_{z_i})M = N$$

for all $\lambda \in \Omega$. Then $\mathrm{fd}(M) \leq N$ and

$$\{\lambda \in \Omega; \dim M_\lambda = N\} = \{\lambda \in \Omega; \sum_{i=1}^{d}(\lambda_i - M_{z_i})M = \ker \epsilon_\lambda\}.$$

In particular, if $\mathrm{fd}(M) < N$, then

$$\sum_{i=1}^{d}(\lambda_i - M_{z_i})M \subsetneq \ker \epsilon_\lambda$$

for all $\lambda \in \Omega$.

Proof. Since the linear maps

$$M / \sum_{i=1}^{d}(\lambda_i - M_{z_i})M \to M / \ker \epsilon_\lambda \cong M_\lambda, [m] \mapsto [m]$$

are surjective for $\lambda \in \Omega$, it follows that $\mathrm{fd}(M) \leq N$ and that

$$\{\lambda \in \Omega; \dim M_\lambda = N\} = \{\lambda \in \Omega; \sum_{i=1}^{d}(\lambda_i - M_{z_i})M = \ker \epsilon_\lambda\}.$$

Hence, if $\mathrm{fd}(M) < N$, then $\sum_{i=1}^{d}(\lambda_i - M_{z_i})M \subsetneq \ker \epsilon_\lambda$ for all $\lambda \in \Omega$. □

In the rest of this section we will show that the concept of fiber dimensions defined in [CCF15] for invariant subspaces of Cowen-Douglas operators on Hilbert spaces admits a natural extension to the multivariable Banach space setting. In the following, let X be a Banach space and let $T \in L(X)^d$ be a weak dual Cowen-Douglas tuple of rank N on a connected complex submanifold $\Omega \subseteq \mathbb{C}^d$. We equip X with the $\mathbb{C}[z]$-module structure defined by $\mathbb{C}[z] \times X \to X, (p, x) \mapsto p(T)x$. For single Cowen-Douglas operators on Hilbert spaces, the following notion was defined in [CCF15].

Definition 6.1.4. Let $\emptyset \neq \Omega_0 \subseteq \Omega$ be a connected open subset. A CF-representation of T on Ω_0 is a $\mathbb{C}[z]$-module homomorphism

$$\rho : X \to \mathcal{O}(\Omega_0, D)$$

with a finite-dimensional complex vector space D such that

(i) $\ker \rho = \bigcap_{z \in \Omega} \operatorname{ran}(Z - T)$,

(ii) the submodule $\hat{X} = \rho X \subseteq \mathcal{O}(\Omega_0, D)$ satisfies

$$\operatorname{fd}(\hat{X}) = \dim \hat{X} / \sum_{i=1}^{d} (\lambda_i - M_{z_i}) \hat{X}$$

for all $\lambda \in \Omega_0$.

Weak dual Cowen-Douglas tuples possess sufficiently many CF-representations that are continuous and satisfy certain additional properties. This follows from the model theorem proved in Chapter 4.1 (Theorem 4.1.9). Here, $\mathcal{O}(\Omega_0, D)$ is equipped with its canonical Fréchet space topology as before.

Corollary 6.1.5. For each point $\lambda_0 \in \Omega$, there is a CF-representation $\rho : X \to \mathcal{O}(\Omega_0, D)$ of T on a connected open neighbourhood $\Omega_0 \subseteq \Omega$ of λ_0 such that

(i) $\rho : X \to \mathcal{O}(\Omega_0, D)$ is continuous,

(ii) $\hat{X} = \rho(X)$ equipped with the norm $\rho(x) = \|x + \ker \rho\|$ is a divisible holomorphic model space of rank N on Ω_0.

Proof. We choose a map ρ as constructed in Theorem 4.1.9. Then only condition (ii) in Definition 6.1.4 remains to be shown. Since $\hat{X}$ is divisible and the point evaluations are surjective, we find that

$$\dim(\hat{X} / \sum_{i=1}^{d} (\lambda_i - M_{z_i}) \hat{X}) = \dim(\hat{X} / \ker \epsilon_\lambda) = \dim(\operatorname{ran}(\epsilon_\lambda)) = \dim D = N$$

for all $\lambda \in \Omega_0$. $\square$

In the following, let $T \in L(X)^d$ be a weak dual Cowen-Douglas tuple of rank N on a connected complex submanifold $\Omega \subseteq \mathbb{C}^d$.

Our next aim is to show that, for every linear subspace $Y \subseteq X$, the fiber dimension of Y with respect to T can be defined as

$$\mathrm{fd}_T(Y) = \mathrm{fd}(\rho(Y)),$$

where ρ is an arbitrary CF-representation of T. To show that the number $\mathrm{fd}(\rho(Y))$ is independent of the chosen CF-representation ρ, we first observe that $\mathrm{fd}(\rho_1(Y)) = \mathrm{fd}(\rho_2(Y))$ for each pair of CF-representations ρ_1, ρ_2 over domains $\Omega_1, \Omega_2 \subseteq \Omega$ with non-trivial intersection.

Lemma 6.1.6. Let $\Omega_1, \Omega_2 \subseteq \Omega$ be domains with $\Omega_1 \cap \Omega_2 \neq \emptyset$. Let $M_j \subseteq \mathcal{O}(\Omega_j, D_j)$ be $\mathbb{C}[z]$-submodules with finite-dimensional vector spaces D_j such that

$$\mathrm{fd}(M_j) = \dim M_j / \sum_{i=1}^{d} (\lambda_i - M_{z_i}) M_j \quad (j = 1, 2, \lambda \in \Omega_j).$$

Suppose that there is a $\mathbb{C}[z]$-module isomorphism $U : M_1 \to M_2$. Then for any linear subspace $M \subseteq M_1$, we have

$$\mathrm{fd}(M) = \mathrm{fd}(UM).$$

Proof. By Lemma 6.1.2 and elementary properties of nowhere dense sets, we can choose a nowhere dense subset $A \subseteq \Omega_1 \cap \Omega_2$ such that each point $\lambda \in (\Omega_1 \cap \Omega_2) \backslash A$ is maximal for M, M_1 and UM. More precisely, by Lemma 6.1.2 there are nowhere dense closed subsets $A_M \subseteq \Omega_1$, $A_{M_1} \subseteq \Omega_1$, $A_{UM} \subseteq \Omega_2$ containing each non-maximal point for M, M_1 and UM, respectively. As a locally compact Hausdorff space the set $\Omega_1 \cap \Omega_2$ is a Baire space. Hence the set

$$A = (A_M \cup A_{M_1} \cup A_{UM}) \cap \Omega_1 \cap \Omega_2 = (A_M \cap \Omega_2) \cup (A_{M_1} \cap \Omega_1) \cup (A_{UM} \cap \Omega_1)$$

is a nowhere dense closed subset of $\Omega_1 \cap \Omega_2$ as a union of three such sets. Clearly every point λ in $(\Omega_1 \cap \Omega_2) \backslash A$ is maximal for M, M_1 and UM. Fix such a point λ. For $f, g \in M$ with $f(\lambda) = g(\lambda)$, by Lemma 6.1.3 there are functions $h_1, ..., h_d \in M_1$ such that $f - g = \sum_{i=1}^{d} (\lambda_i - M_{z_i}) h_i$. But then also

$$U(f - g) = \sum_{i=1}^{d} (\lambda_i - M_{z_i}) U h_i.$$

Hence we obtain a well-defined surjective linear map $U_\lambda : M_\lambda \to (UM)_\lambda$ by setting

$$U_\lambda x = (Uf)(\lambda) \text{ if } f \in M \text{ with } f(\lambda) = x.$$

It follows that $\mathrm{fd}(M) = \dim M_\lambda \geq \dim(UM)_\lambda = \mathrm{fd}(UM)$. By applying the same argument to U^{-1} and UM instead of U and M we find that also $\mathrm{fd}(UM) \geq \mathrm{fd}(M)$. □

Corollary 6.1.7. Let $\Omega_1, \Omega_2 \subseteq \Omega$ be domains with $\Omega_1 \cap \Omega_2 \neq \emptyset$ and let

$$\rho_i : X \to \mathcal{O}(\Omega_i, D_i) \quad (i = 1, 2)$$

be CF-representations of T. Then we have

$$\mathrm{fd}(\rho_1 Y) = \mathrm{fd}(\rho_2 Y)$$

for each linear subspace $Y \subseteq X$.

Proof. Since the submodules $\rho_i X \subseteq \mathcal{O}(\Omega_i, D_i)$ $(i = 1, 2)$ are canonically isomorphic

$$\rho_1 X \cong X/\ker \rho_1 = X/\ker \rho_2 \cong \rho_2 X$$

as $\mathbb{C}[z]$-modules, the assertion follows as an application of Lemma 6.1.6. □

We can use this corollary to derive the same result for CF-representatiosn over arbitrary (not neccessarily intersecting) domains.

Theorem 6.1.8. Let $\rho_i : X \to \mathcal{O}(\Omega_i, D_i)$ $(i = 1, 2)$ be CF-representations of T on domains $\Omega_i \subseteq \Omega$. Then we have

$$\mathrm{fd}(\rho_1 Y) = \mathrm{fd}(\rho_2 Y)$$

for each linear subspace $Y \subseteq X$.

Proof. As a locally path-connected connected space Ω is also path-connected. Choose a continuous path $\gamma : [0, 1] \to \Omega$ with $\gamma(0) \in \Omega_1$ and $\gamma(1) \in \Omega_2$. By Corollary 6.1.5 there is a family $(\rho_\lambda)_{\lambda \in \mathrm{ran}\, \gamma}$ of CF-representations $\rho_\lambda : X \to \mathcal{O}(\Omega_\lambda, D_\lambda)$ of T on connected open neighbourhoods $\Omega_\lambda \subseteq \Omega$ of the points $\lambda \in \mathrm{ran}\, \gamma$ such that $\rho_{\gamma(0)} = \rho_1$ and $\rho_{\gamma(1)} = \rho_2$. Let $\delta > 0$ be a positive number such that each set $A \subseteq [0, 1]$ of diameter less than δ is contained in one of the sets $\gamma^{-1}(\Omega_\lambda)$ (see e.g. Lemma 3.7.2 in [Mun75]). Then we can choose points $\lambda_1 = \gamma(0), \lambda_2, ..., \lambda_n = \gamma(1)$ in $\mathrm{ran}\, \gamma$ such that $\Omega_{\lambda_i} \cap \Omega_{\lambda_{i+1}} \neq \emptyset$ for $i = 1, .., n-1$. Let $Y \subseteq X$ be a linear subspace. By Corollary 6.1.7 we obtain that

$$\mathrm{fd}(\rho_1 Y) = \mathrm{fd}(\rho_{\lambda_2} Y) = ... = \mathrm{fd}(\rho_2 Y)$$

as was to be shown. □

In view of Theorem 6.1.8, we can finally define the fiber dimension of a linear subspace $Y \subseteq X$ with respect to our fixed weal dual Cowen-Douglas tuple $T \in L(X)^d$ on Ω.

Definition 6.1.9. Let $T \in L(X)^d$ be a weak dual Cowen-Douglas tuple of rank N on a connected complex submanifold $\Omega \subseteq \mathbb{C}^d$. For a linear subspace $Y \subseteq X$, we define

$$\mathrm{fd}_T(Y) = \mathrm{fd}(\rho Y),$$

where $\rho : X \to \mathcal{O}(\Omega_0, D)$ is a CF-representation of T on an arbitrary domain $\Omega_0 \subseteq \Omega$.

We are mainly interested in the fiber dimension of closed T-invariant subspaces $Y \subseteq X$ and sometimes suppress the dependence on T by writing $\mathrm{fd}(Y) = \mathrm{fd}_T(Y)$ in that case. But as established above, Definition 6.1.9 makes perfect sense for arbitrary linear subspaces $Y \subseteq X$. Since by Corollary 6.1.5 there are always continuous CF-representations $\rho : X \to \mathcal{O}(\Omega_0, D)$ and since

$$\epsilon_\lambda(\rho(\overline{Y})) \subseteq \overline{\epsilon_\lambda(\rho(Y))} = \epsilon_\lambda(\rho(Y))$$

for any such representation and any point $\lambda \in \Omega_0$, it follows that $\mathrm{fd}_T(Y) = \mathrm{fd}_T(\overline{Y})$ for each linear subspace $Y \subseteq X$.
By Corollary 6.1.5, there is always a CF-representation $\rho : X \to \mathcal{O}(\Omega_0, D)$ such that ρX is a divisible holomorphic model space of rank N. In particular, the point evaluations $\epsilon_\lambda : \rho X \to D$ are surjective and it follows that $\mathrm{fd}_T(X) = N$. In general, the fiber dimension $\mathrm{fd}_T(Y)$ of a linear subspace $Y \subseteq X$ is an integer in $\{0, ..., N\}$ which depends on Y in a monotone way. Obviously, $\mathrm{fd}_T(Y) = 0$ if and only if

$$Y \subseteq \ker \rho = \bigcap_{z \in \Omega} \mathrm{ran}(Z - T).$$

We conclude this section with an alternative characterisation of CF-representations.

Corollary 6.1.10. Let as before $T \in L(X)^d$ be a weak dual Cowen-Douglas tuple of rank N on a connected complex submanifold $\Omega \subseteq \mathbb{C}^d$ and let $\rho : X \to \mathcal{O}(\Omega_0, D)$ be a $\mathbb{C}[z]$-module homomorphism on a domain $\emptyset \neq \Omega_0 \subseteq \Omega$ with a finite-dimensional vector space D such that

$$\ker \rho = \bigcap_{z \in \Omega} \mathrm{ran}(Z - T).$$

Then ρ is a CF-representation of T if and only if $\mathrm{fd}(\rho X) = N$.

Proof. Suppose that $\mathrm{fd}(\rho X) = N$. Define $\hat{X} = \rho X$. The maps

$$X/(\lambda - T)X^d \to \hat{X}/(\lambda - M_z)\hat{X}^d, [x] \mapsto [\rho x]$$

and

$$\hat{X}/(\lambda - M_z)\hat{X}^d \to \hat{X}_\lambda, [f] \mapsto f(\lambda)$$

are surjective for every $\lambda \in \Omega_0$. It follows that

$$\dim \hat{X}/(\lambda - M_z)\hat{X}^d \leq N$$

for all $\lambda \in \Omega_0$ and by Lemma 6.1.2, equality holds on $\Omega_0 \backslash A$ with a suitable nowhere dense subset $A \subseteq \Omega_0$. Equipped with the norm $\|\rho(x)\| = \|x + \ker \rho\|$, the space $\hat{X}$ is a Banach space and $M_z \in L(\hat{X})^d$ is a commuting tuple of bounded operators on $\hat{X}$ which is similar to the quotient tuple $T/\ker\rho$. A result of Kaballo (Satz 1.5 in [Kab79]) shows that

$$B = \{\lambda \in \Omega_0; \dim \hat{X}/(\lambda - M_z)\hat{X}^d > \min_{\mu \in \Omega_0} \dim \hat{X}/(\mu - M_z)\hat{X}^d\}$$

is a proper analytic subset of Ω_0. Since the union $A \cup B$ is nowhere dense again, we can combine these results to deduce that

$$\dim \hat{X}/(\lambda - M_z)\hat{X}^d = N$$

for all $\lambda \in \Omega_0$. Hence ρ is a CF-representation of T.
Conversely, if ρ is a CF-representation of T, then $\mathrm{fd}(\rho X) = N$ by the remarks preceding the corollary. □

6.2 A limit formula for the fiber dimension

Let us consider the particular case of a domain $\Omega \subseteq \mathbb{C}^d$. For convenience, we will also assume that $0 \in \Omega$, but we will later see that the results hold for arbitrary domains $\Omega \subseteq \mathbb{C}^d$. As before, let D be a finite-dimensional complex vector space. For $k \in \mathbb{N}$, consider the map $\mathscr{T}_k : \mathcal{O}(\Omega, D) \to \mathcal{O}(\Omega, D)$ which associates with each function $f \in \mathcal{O}(\Omega, D)$ its k-th Taylor polynomial, that is,

$$\mathscr{T}_k(f)(z) = \sum_{|\alpha| \leq k} \frac{f^{(\alpha)}(0)}{\alpha!} z^\alpha.$$

In [Esc07] (Lemma 1.4) it was shown that, for a given $\mathbb{C}[z]$-submodule $M \subseteq \mathcal{O}(\Omega, D)$, there is a proper analytic subset $A \subseteq \Omega$ such that

$$\dim M_\lambda = \max_{\mu \in \Omega} \dim M_\mu = d! \lim_{k \to \infty} \frac{\dim \mathscr{T}_k(M)}{k^d}$$

holds for all $\lambda \in \Omega \backslash A$.
Based on this observation, we will deduce a similar limit formula for the fiber dimension of an invariant subspace Y of a weak dual Cowen-Douglas tuple T on Ω. As mentioned before, we will suppress the dependence on T in this case and simply write $\mathrm{fd}(Y) = \mathrm{fd}_T(Y)$.
For a commuting tuple $T \in L(X)^d$ of bounded linear operators on a Banach space X, we use the notation from Chapter 4.2 and write

$$K^\bullet(T,X) : 0 \to \Lambda^0(X) \xrightarrow{\delta_T^0} \Lambda^1(X) \xrightarrow{\delta_T^1} ... \xrightarrow{\delta_T^{d-1}} \Lambda^d(X) \to 0$$

for the Koszul complex of T as well as

$$H^i(T,X) = \ker(\delta_T^i)/\operatorname{ran}(\delta_T^{i-1}) \quad (i = 0, ..., d)$$

for the cohomology groups of $K^\bullet(T,X)$. There is a canonical isomorphism $H^d(T,X) \cong X/\sum_{i=1}^d T_i X$ of complex vector spaces.

In the following, given a commuting operator tuple $T \in L(X)^d$ and a closed invariant subspace $Y \in \mathrm{Lat}(T) = \bigcap_{i=1}^d \mathrm{Lat}(T_i)$, we denote by

$$R = T|_Y \in L(Y)^d, S = T/Y \in L(Z)^d$$

the restriction of T to Y and the quotient of T modulo Y on $Z = X/Y$. The inclusion $i : X \to Y$ and the quotient map $q : X \to Z$ induce a short exact sequence of complexes

$$0 \longrightarrow K^\bullet(z-R,Y) \xrightarrow{\ i\ } K^\bullet(z-T,X) \xrightarrow{\ q\ } K^\bullet(z-S,Z) \longrightarrow 0\,.$$

By a standard result from homological algebra, an application of the so-called Snake Lemma (cf. Chapter XX, §2 in [Lan02]), there are connecting homomorphisms

$$d_z^i : H^i(z-S,Z) \to H^{i+1}(z-R,Y) \quad (i = 0, ..., n-1)$$

such that the induced sequence of cohomology spaces

$$0 \longrightarrow H^0(z-R,Y) \xrightarrow{i} H^0(z-T,X) \xrightarrow{q} H^0(z-S,Z)$$

$$\xrightarrow{d_z^0} H^1(z-R,Y) \xrightarrow{i} H^1(z-T,X) \xrightarrow{q} H^1(z-S,Z)$$

$$\xrightarrow{d_z^1} H^2(z-R,Y) \longrightarrow \qquad \ldots$$

$$\xrightarrow{d_z^{d-1}} H^d(z-R,Y) \xrightarrow{i} H^d(z-T,X) \xrightarrow{q} H^d(z-S,Z) \longrightarrow 0$$

is exact again. In particular, we obtain

$$\begin{aligned}\operatorname{ran}(d_z^{d-1}) &= \ker(H^d(z-R,Y) \xrightarrow{i} H^d(z-T,X)) \\ &= (Y \cap (z-T)X^d)/(z-R)Y^d.\end{aligned}$$

Lemma 6.2.1. Let $T \in L(X)^d$ be a weak dual Cowen-Douglas tuple of rank N on a domain $\Omega \subseteq \mathbb{C}^d$ and let $Y \in \operatorname{Lat}(T)$ be a closed invariant subspace of T. Then there is a proper analytic subset $A \subseteq \Omega$ such that, for all $\lambda \in \Omega \setminus A$,

$$\dim H^d(\lambda - S, Z) = N - \operatorname{fd}(Y).$$

Proof. Choose a CF-representation $\rho : X \to \mathcal{O}(\Omega_0, D)$ of T on some domain $\Omega_0 \subseteq \Omega$ as in Corollary 6.1.5. Let $Y \in \operatorname{Lat}(T)$ be arbitrary. Define $\hat{X} = \rho(X)$ and $\hat{Y} = \rho(Y)$. Since the compositions

$$Y^d \xrightarrow{\lambda - R} Y \xrightarrow{\rho} \mathcal{O}(\Omega_0, D) \xrightarrow{\epsilon_\lambda} D \quad (\lambda \in \Omega_0)$$

are zero, we obtain well-defined surjective linear maps

$$\delta_\lambda : H^d(\lambda - R, Y) \to \hat{Y}_\lambda,\ [y] \mapsto \rho(y)(\lambda).$$

Obviously, for each $\lambda \in \Omega_0$, the inclusion

$$\operatorname{ran} d_\lambda^{d-1} = (Y \cap (\lambda - T)X^d)/(\lambda - R)Y^d \subseteq \ker \delta_\lambda$$

holds. To prove the reverse inclusion, fix an element $y \in Y$ with $\rho(y)(\lambda) = 0$. Since $\hat{X}$ is divisible, there are vectors $x_1, ..., x_d \in X$ with

$$\rho(y) = \sum_{i=1}^d (\lambda_i - M_{z_i})\rho(x_i) = \rho(\sum_{i=1}^d (\lambda_i - T_i)x_i).$$

But then
$$y - \sum_{i=1}^{d} (\lambda_i - T_i)x_i \in \ker \rho = \bigcap_{z\in\Omega} \operatorname{ran}(Z - T)$$
and hence $y \in Y \cap (\lambda - T)X^d$. Thus, for each $\lambda \in \Omega_0$, we obtain an exact sequence
$$H^{d-1}(\lambda - S, Z) \xrightarrow{d_\lambda^{d-1}} H^d(\lambda - R, Y) \xrightarrow{\delta_\lambda} \hat{Y}_\lambda \to 0.$$
Using the exactness of these sequences and of the long exact cohomology sequences explained in the section leading to Lemma 6.2.1, we find that
$$\begin{aligned}&\dim H^d(\lambda - S, Z)\\ &= \dim H^d(\lambda - T, X) - \dim H^d(\lambda - R, Y)/\operatorname{ran} d_\lambda^{d-1}\\ &= N - \dim \hat{Y}_\lambda\end{aligned}$$
for all $\lambda \in \Omega_0$. By Theorem 1.5 in [Kab79] the set
$$A = \{\lambda \in \Omega;\ \dim H^d(\lambda - S, Z) > \min_{\mu\in\Omega} \dim H^d(\mu - S, Z)\}$$
is a proper analytic subset of Ω. Since the identity $\dim \hat{Y}_\lambda = \operatorname{fd}(Y)$ holds for each point in a non-empty open subset of Ω_0 by Lemma 6.1.2, the assertion follows with A as defined above. □

In the setting of Lemma 6.2.1, the minimum
$$\min_{\mu\in\Omega} \dim H^d(\mu - S, Z)$$
can also be interpreted as a suitable Samuel multiplicity of the tuples $\mu - S$ for $\mu \in \Omega$. We shortly recall the neccessary details.

For an arbitrary tuple $T \in L(X)^d$ of bounded operators on a Banach space X with
$$\dim H^d(T, X) < \infty,$$
we write $M_k(T) = \sum_{|\alpha|=k} T^\alpha X$ for all $k \in \mathbb{N}$. All these spaces are finite-codimensional in X. The algebraic direct sum $M = \bigoplus_{n=0}^{\infty} M_n(T)/M_{n+1}(T)$ can be given the structure of a $\mathbb{C}[z]$-module by
$$\mathbb{C}[z] \times M \to M, (p, x) \mapsto p(\overline{T})x,$$
where $\overline{T} = (\overline{T}_1, ..., \overline{T}_d)$ is the commuting tuple with components
$$\overline{T}_j : M \to M, \overline{T}_j(x_k + M_{k+1}(T))_{k\in\mathbb{N}} = (T_j x_{k-1} + M_{k+1}(T))_{k\in\mathbb{N}}.$$

Here, we set $x_{-1} = 0$. Writing $[x_1], ..., [x_N]$ for a basis of $M_0(T)/M_1(T) \cong H^d(T, X)$ and $\mathcal{V}_k = \{p \in \mathbb{C}[z]; \deg p \leq k\}$ for all $k \in \mathbb{N}$, one can show that

$$\bigoplus_{n=0}^{k} M_n(T)/M_{n+1}(T) \oplus \{0\} = \{\sum_{i=1}^{N} p_i(x_i + M_1(T)), 0, 0, ...); p_1, ..., p_N \in \mathcal{V}_k\}$$

for all $k \in \mathbb{N}$. In particular, M with the $\mathbb{C}[z]$-module structure from above is generated by the elements $(x_i + M_1(T), 0, 0, ...)$ $(i = 1, ..., N)$ and is thus a finitely generated graded $\mathbb{C}[z]$-module. Due to

$$X/M_k(T) = M_0(T)/M_k(T) \cong \bigoplus_{n=0}^{k-1} M_n(T)/M_{n+1}(T)$$

one can use Theorem 11 in Chapter 7.6 of [Nor68] and the subsequent remarks (including the remarks on p. 323) to show that the limit

$$d! \lim_{k\to\infty} \frac{\dim(X/M_k(T))}{k^d}$$

exists and defines a natural number. This number is usually called the Samuel multiplicity $c(T)$ of T and the idea to use this algebraic concept in the context of Fredholm operators goes back to the paper [DY93] by Douglas and Yan. One can show that, for each domain $\Omega \subseteq \mathbb{C}^d$ with $0 \in \Omega$ and $\dim H^d(\lambda - T, X) < \infty$ for all $\lambda \in \Omega$, there is a proper analytic subset $A \subseteq \Omega$ such that

$$c(T) = \dim H^d(\lambda - T, X) < \dim H^d(\mu - T, X)$$

for all $\lambda \in \Omega \backslash A$ and $\mu \in A$ (see Corollary 3.6 in [Esc08]). In particular, if $S \in L(Z)^d$ is as in Lemma 6.2.1 and $0 \in \Omega$, then the formula

$$c(S) = N - \mathrm{fd}(Y)$$

holds. Hence the following result from [Esc07] allows us to deduce the announced limit formula for the fiber dimension.

Lemma 6.2.2. (Lemma 1.6 in [Esc07]) Let $T \in L(X)^d$ be a commuting tuple of bounded operators on a Banach space X, let $Y \in \mathrm{Lat}(T)$ be a closed invariant subspace and let $S = T/Y \in L(Z)^d$ be the induced quotient tuple on $Z = X/Y$. Suppose that

$$\dim H^d(T, X) < \infty.$$

Then the Samuel multiplicities of T and S satsify the relation

$$c(S) = c(T) - d! \lim_{k\to\infty} \frac{\dim(Y + M_k(T))/M_k(T)}{k^d}.$$

As a direct application of the preceding discussion and this lemma we obtain a corresponding formula for the fiber dimension.

Corollary 6.2.3. Let $T \in L(X)^d$ be a weak dual Cowen-Douglas tuple of rank N on a domain $\Omega \subseteq \mathbb{C}^d$ with $0 \in \Omega$, and let $Y \in \mathrm{Lat}(T)$ be a closed invariant subspace for T. Then the formula

$$\mathrm{fd}(Y) = d! \lim_{k \to \infty} \frac{\dim(Y + M_k(T))/M_k(T)}{k^d}$$

holds.

Proof. It suffices to observe that for a weak dual Cowen-Douglas tuple of rank N the identity $c(T) = N$ holds and then to compare the formula from Lemma 6.2.2 with the formula

$$c(S) = N - \mathrm{fd}(Y)$$

deduced in the section leading to Lemma 6.2.2 □

If $T \in L(X)^d$ is a weak dual Cowen-Douglas tuple of rank N on a domain $\Omega \subseteq \mathbb{C}^d$ and $\lambda \in \Omega$, then $T - \lambda$ is a weak dual Cowen-Douglas tuple of rank N on the domain $\Omega - \lambda = \{z - \lambda; z \in \Omega\}$. For a weak dual Cowen-Douglas tuple $T \in L(X)^d$ on a domain $\Omega \subseteq \mathbb{C}^d$ not neccessarily containing 0 we conclude that the above formula for $\mathrm{fd}(Y)$ remains true if on the right-hand side the spaces $M_k(T)$ are replaced by the spaces $M_k(T - \lambda)$ with $\lambda \in \Omega$ arbitrary.

If in Corollary 6.2.3 the space X is a Hilbert space and if we write P_k for the orthogonal projections onto the subspaces $M_k(T)^\perp$, then there are canonical vector space isomorphism

$$(Y + M_k(T))/M_k(T) \to P_k Y, [y] \mapsto P_k y.$$

Thus the resulting formula

$$\mathrm{fd}(Y) = d! \lim_{k \to \infty} \frac{\dim(P_k Y)}{k^d}$$

extends Theorem 19 in [CCF15].

In the final result of this section we show that the fiber dimension $\mathrm{fd}(Y)$ is invariant under sufficiently small changes of the space Y. For given invariant subspaces $Y_1, Y_2 \in \mathrm{Lat}(T)$ with $Y_1 \subseteq Y_2$, we write $\sigma(T, Y_2/Y_1)$ for the Taylor spectrum of the quotient tuple induced by T on Y_2/Y_1.

Corollary 6.2.4. Let $T \in L(X)^d$ be a weak dual Cowen-Douglas tuple of rank N on a domain $\Omega \subseteq \mathbb{C}^d$. If $Y_1, Y_2 \in \operatorname{Lat}(T)$ are closed T-invariant subspaces with $Y_1 \subseteq Y_2$ and $\Omega \cap (\mathbb{C}^d \backslash \sigma(T, Y_2/Y_1)) \neq \emptyset$, then $\operatorname{fd}(Y_1) = \operatorname{fd}(Y_2)$.

Proof. Since $\Omega \cap (\mathbb{C}^d \backslash \sigma(T, Y_2/Y_1))$ is open, we can use Lemma 6.2.1 to find a point $\lambda \in \Omega \cap (\mathbb{C}^d \backslash \sigma(T, Y_2/Y_1))$ with

$$\dim H^d(\lambda - T/Y_i, X/Y_i) = N - \operatorname{fd}(Y_i)$$

for $i = 1, 2$. The canonical exact sequence

$$0 \to Y_2/Y_1 \to X/Y_1 \to X/Y_2 \to 0$$

induces a long exact sequence

$$\begin{aligned} 0 \to \ldots \to H^d(\lambda - (T|_{Y_2}/Y_1), Y_2/Y_1) \\ \to H^d(\lambda - T/Y_1, X/Y_1) \to H^d(\lambda - T/Y_2, X/Y_2) \to 0. \end{aligned}$$

Since $\lambda \notin \sigma(T, Y_1/Y_2)$, we conclude that the d-th cohomology spaces of $\lambda - T/Y_1$ and $\lambda - T/Y_2$ are isomorphic. Hence we can use Lemma 6.2.1 to obtain that $\operatorname{fd}(Y_1) = \operatorname{fd}(Y_2)$. □

To make the above proof work, it suffices that there is a point in Ω which is not contained in the right spectrum of the quotient tuple induced by T on Y_2/Y_1 (cf. Section 2.6 in [EP96]). The hypotheses of Corollory 6.2.4 are satisfied for instance if $\dim(Y_2/Y_1) < \infty$ since $\sigma(T, Y_2/Y_1)$ is finite in this case. Thus Corollary 6.2.4 can be seen as an extension of Proposition 2.5 in [CGW10].

6.3 Fiber dimension and analytic Samuel multiplicity

In this chapter, we will introduce an alternative way to calculate the fiber dimension of invariant subspaces of a weak dual Cowen-Douglas tuple T which uses the so-called analytic Samuel multiplicity of T. First, we recall some basic definitions and constructions from sheaf theory.

Definition 6.3.1. Let $\Omega \subseteq \mathbb{C}^d$ be an open subset and let X be a Banach space. The sheaf $\mathcal{O}_\Omega^X$ of holomorphic functions on Ω with values in X is the sheaf associated with the presheaf

$$U \mapsto \mathcal{O}(U, X) \quad (U \subseteq \Omega \text{ open})$$

with restrictions maps

$$r_U^V : \mathcal{O}(U, X) \to \mathcal{O}(V, X), f \mapsto f|_V \quad (U, V \subseteq \Omega \text{ open with } V \subseteq U).$$

In the particular case $X = \mathbb{C}$, we write $\mathcal{O}_\Omega = \mathcal{O}_\Omega^{\mathbb{C}}$.

The constructions and results from sheaf theory needed in the following can be found for instance in [Kul70].

Remark 6.3.2. The sheaf $\mathcal{O}_\Omega^X$ can be written as the disjoint union

$$\mathcal{O}_\Omega^X = \bigcup_{w\in\Omega} \mathcal{O}_w^X$$

of its stalks

$$\mathcal{O}_w^X = \{[f, D_f]_w; D_f \in \mathcal{U}(w) \text{ open and } f \in \mathcal{O}(D_f, X)\}.$$

Here, $[f, D_f]_w$ denotes the equivalence class of a holomorphic function $f : D_f \to X$ with respect to the equivalence relation $\sim$ where by definition $[f, D_f]_w \sim [g, D_g]_w$ if and only if there is a neighbourhood $U \in \mathcal{U}(w)$ such that $U \subseteq D_f \cap D_g$ and $f|_U = g|_U$.

Let $\Omega \subseteq \mathbb{C}^d$ be open and let X, Y be Banach spaces. Any mapping $\alpha \in \mathcal{O}(\Omega, L(X, Y))$ induces a sheaf homomorphism

$$\widehat{\alpha} : \mathcal{O}_\Omega^X \to \mathcal{O}_\Omega^Y, [f, D_f]_w \mapsto [\alpha f, D_f]_w,$$

where $\alpha f : D_f \to Y, z \mapsto \alpha(z)f(z)$.
For $T \in L(X)^d$ commuting, the analytically parametrized complex

$$K(z-T, X) : 0 \to \Lambda^0(X) \xrightarrow{\delta^0_{z-T}} \Lambda^1(X) \xrightarrow{\delta^1_{z-T}} \ldots \xrightarrow{\delta^{d-1}_{z-T}} \Lambda^d(X) \to 0.$$

induces a complex of analytic sheaves on each open set $\Omega \subseteq \mathbb{C}^d$.
More precisely, the holomorphic maps

$$\delta^j : \Omega \to L(\Lambda^j X, \Lambda^{j+1} X), z \mapsto \delta^j_{z-T} \quad (j = 0, \ldots, d-1)$$

induce a complex of sheaf homomorphisms

$$K^\bullet(z-T, \mathcal{O}_\Omega^X) : 0 \to \mathcal{O}_\Omega^{\Lambda^0(X)} \xrightarrow{\widehat{\delta^0}} \mathcal{O}_\Omega^{\Lambda^1(X)} \xrightarrow{\widehat{\delta^1}} \ldots \xrightarrow{\widehat{\delta^{d-1}}} \mathcal{O}_\Omega^{\Lambda^d(X)} \to 0.$$

The quotient sheaves

$$H^j(z-T, \mathcal{O}_\Omega^X) = \ker(\widehat{\delta^j}) / \operatorname{ran}(\widehat{\delta^{j-1}}) \quad (j = 0, \ldots, d)$$

are called the cohomology sheaves of $K^\bullet(z-T, \mathcal{O}_\Omega^X)$. Here we set as usual $\widehat{\delta^{-1}} = \widehat{\delta^d} = 0$.

Now let $T \in L(X)^d$ be a commuting tuple and let $\Omega \subseteq \mathbb{C}^d$ be a domain such that

$$\dim H^d(\lambda - T, X) < \infty$$

for all $\lambda \in \Omega$. For simplicity, we again assume $0 \in \Omega$ and note that as before, we can pass to a suitable translation of the given tuple if the domain is general and thus a corresponding statement holds. We then consider the holomorphic map

$$\alpha_T : \Omega \to L(X^d, X), z \mapsto Z - T$$

and the induced sheaf homomorphism $\widehat{\alpha_T} : \mathcal{O}_\Omega^{X^d} \to \mathcal{O}_\Omega^X$ defined above. By Corollary 2.2 (ii) in [Esc08] the quotient sheaf

$$\mathcal{H}_T = \mathcal{O}_\Omega^X / \widehat{\alpha} \mathcal{O}_\Omega^{X^d}.$$

is a coherent analytic sheaf. Let $Y \in \mathrm{Lat}(T)$ be a closed invariant subspace for T. As before denote by $R = T|_Y \in L(Y)^d$ the restriction of T to Y and by $S = T/Y \in L(Z)^d$ the quotient induced by T on $Z = X/Y$. Let $i : Y \to X$ and $q : X \to Z$ denote the inclusion and the quotient map, respectively. Using the same symbols for the induced mappings, we have a short exact sequence

$$0 \to K^\bullet(z - R, \mathcal{O}_\Omega^Y) \xrightarrow{i} K^\bullet(z - T, \mathcal{O}_\Omega^X) \xrightarrow{q} K^\bullet(z - S, \mathcal{O}_\Omega^Z) \to 0$$

of complexes of analytic sheaves on Ω. Since the Snake Lemma also holds for complexes of sheaves (check Lemma 1.10.9 in [Bor94] for a very general version), this induces a long exact sequence

$$\ldots \to H^d(z - R, \mathcal{O}_\Omega^Y) \xrightarrow{i} H^d(z - T, \mathcal{O}_\Omega^X) \xrightarrow{q} H^d(z - S, \mathcal{O}_\Omega^Z) \to 0$$

of cohomology groups. Since $\widehat{\delta^{d-1}} \mathcal{O}_\Omega^{\Lambda^{d-1} X} = \widehat{\alpha} \mathcal{O}_\Omega^{X^d}$, the identity

$$H^d(z - T, \mathcal{O}_\Omega^X) = \mathcal{H}_T$$

of sheaves holds.

In particular, the long exact sequence from above yields that the upper horizontal in the commutative diagram

$$\begin{tikzcd}
\mathcal{H}_R \arrow[r, "i"] & \mathcal{H}_T \arrow[r, "q"] & \mathcal{H}_S \arrow[r] & 0 \\
\mathcal{O}_\Omega^Y \arrow[r, "i"] \arrow[u, two heads, "\pi_Y"] & \mathcal{O}_\Omega^X \arrow[u, two heads, "\pi_X"'] & &
\end{tikzcd}$$

is an exact sequence of analytic sheaves. Here π_Y and π_X denote the canonical quotient maps. The sheaf $\mathcal{M}_T = \pi_X(i\mathcal{O}_\Omega^Y) = i(\mathcal{H}_R)$ is the kernel of the surjective sheaf homomorphism

$$\mathcal{H}_T \xrightarrow{q} \mathcal{H}_S.$$

Thus we have a short exact sequence of sheaf homomorphisms

$$0 \to \mathcal{M}_T \xrightarrow{i} \mathcal{H}_T \xrightarrow{q} \mathcal{H}_S \to 0.$$

Since $\mathcal{H}_T$ and $\mathcal{H}_S$ are coherent, also the sheaf $\mathcal{M}_T$ is a coherent analytic sheaf on Ω (Satz 26.13 in [Kul70]). By passing to stalks, we obtain an exact sequence

$$0 \to \mathcal{M}_{T,0} \xrightarrow{i} \mathcal{H}_{T,0} \xrightarrow{q} \mathcal{H}_{S,0} \to 0$$

of Noetherian $\mathcal{O}_0$-modules. For a Noetherian $\mathcal{O}_0$-module E, let us denote by $e_{\mathcal{O}_0}(E)$ its analytic Samuel multiplicity, that is, the multiplicity of E with respect to the multiplicity system $(z_1, ..., z_n)$ on E (see Section 7.4 in [Nor68]). We can now relate the fiber dimension of closed invariant subspaces to a suitable analytic Samuel multiplicity.

Theorem 6.3.3. Let $T \in L(X)^d$ be a weak dual Cowen-Douglas tuple on a domain $\Omega \subseteq \mathbb{C}^d$ with $0 \in \Omega$. Let $Y \in \operatorname{Lat}(T)$ be a closed invariant subspace for T. The fiber dimension of Y can be calculated as

$$\operatorname{fd}(Y) = e_{\mathcal{O}_0}(\mathcal{M}_{T,0}),$$

where $\mathcal{M}_{T,0}$ is the stalk at $z = 0$ of the subsheaf $\mathcal{M}_T = \pi_X(i\mathcal{O}_\Omega^Y) \subseteq \mathcal{H}_T = \mathcal{O}_\Omega^X / \widehat{\alpha}\mathcal{O}_\Omega^{X^d}$.

Proof. By Theorem 7.5 in [Nor68] the analytic Samuel multiplicity is additive with respect to short exact sequences of Noetherian $\mathcal{O}_0$-modules. Thus using the short exact sequences from the remarks leading to Theorem 6.3.3, it follows that

$$e_{\mathcal{O}_0}(\mathcal{H}_{T,0}) = e_{\mathcal{O}_0}(\mathcal{M}_{T,0}) + e_{\mathcal{O}_0}(H_{S,0}).$$

By Corollary 4.1 in [Esc08] the analytic Samuel multiplicities $e_{\mathcal{O}_0}(\mathcal{H}_{T,0})$ and $e_{\mathcal{O}_0}(H_{S,0})$ coincide with the Samuel multiplicities $c(T)$ and $c(S)$ as defined in Section 5.2. Thus we obtain the identitiy

$$c(T) = e_{\mathcal{O}_0}(\mathcal{M}_{T,0}) + c(S).$$

The result follows from Lemma 6.2.2 and Corollary 6.2.3. □

Note that the analytic Samuel multiplicity $e_{\mathcal{O}_0}(\mathcal{M}_{T,0})$ and thus the fiber dimension of Y can also be calculated as the Euler characteristic $\chi(K^\bullet(z, \mathcal{M}_{T,0}))$ of the Koszul complex of the multiplication operators with $z_1, ..., z_n$ on $\mathcal{M}_{T,0}$ (cf. Theorem 8.5 in [Nor68]).

6.4 A lattice formula for the fiber dimension

Let $T \in L(X)^d$ be a weak dual Cowen-Douglas tuple of rank N on a domain $\Omega \subseteq \mathbb{C}^d$ and let $Y_1, Y_2 \in \mathrm{Lat}(T)$ be closed invariant subspaces. A natural problem studied in [CCF15] in the case of a single Cowen-Douglas operator on a Hilbert space is to find conditions under which the dimension formula

$$\mathrm{fd}(Y_1) + \mathrm{fd}(Y_2) = \mathrm{fd}(Y_1 \vee Y_2) + \mathrm{fd}(Y_1 \cap Y_2)$$

holds for the fiber dimensions formed with respect to T. Note that by the remarks following Definition 6.1.9 the fiber dimensions of the algebraic sum $Y_1 + Y_2$ and of its closure $Y_1 \vee Y_2 = \overline{\mathrm{span}}(Y_1 \cup Y_2)$ coincide. In this chapter, we will proceed as in [EL17] to generalize the results from [CCF15] to the slightly more general setting established in this chapter. For a Cowen-Douglas tuple of rank 1, the validity of the above formula for all closed invariant subspaces Y_1, Y_2 is equivalent to the condition that any two non-zero closed invariant subspaces Y_1, Y_2 have a non-trivial intersection. As in the one-variable case basic linear algebra can be used to obtain at least an inequality.

Lemma 6.4.1. Let $T \in L(X)^d$ be a weak dual Cowen-Douglas tuple on a domain $\Omega \subseteq \mathbb{C}^d$ and let $Y_1, Y_2 \subseteq X$ be linear T-invariant subspaces. Then the inequality

$$\mathrm{fd}(Y_1) + \mathrm{fd}(Y_2) \geq \mathrm{fd}(Y_1 + Y_2) + \mathrm{fd}(Y_1 \cap Y_2)$$

holds.

Proof. Let $\rho : X \to \mathcal{O}(\Omega_0, D)$ be a CF-representation of T on a domain $\Omega_0 \subseteq \Omega$. It suffices to observe that, for each point $\lambda \in \Omega_0$, the estimate

$$\begin{aligned}\dim \epsilon_\lambda \rho(Y_1 + Y_2) &= \dim \epsilon_\lambda \rho(Y_1) + \dim \epsilon_\lambda \rho(Y_2) - \dim(\epsilon_\lambda \rho(Y_1) \cap \epsilon_\lambda \rho(Y_2)) \\ &\leq \dim \epsilon_\lambda \rho(Y_1) + \dim \epsilon_\lambda \rho(Y_2) - \dim \epsilon_\lambda \rho(Y_1 \cap Y_2)\end{aligned}$$

holds and then use Lemma 6.1.2 to choose λ as a common maximal point for the submodules $\rho(Y_1 + Y_2)$, $\rho(Y_1)$, $\rho(Y_2)$ and $\rho(Y_1 \cap Y_2)$. □

In the following we prove that in Lemma 6.4.1 also the reverse inequality holds in some particular cases. For this purpose, we closely follow ideas from [CF10] where a corresponding result is proved for analytic functional Hilbert spaces given by a complete Nevanlinna-Pick kernel. We give a shortend proof under weakened hypotheses and obtain further applications. An alternative proof for the Nevanlinna-Pick case can also be found in [Che15].
Let $\Omega \subseteq \mathbb{C}^d$ be a domain and let D be an N-dimensional complex vector space. We shall say that a function $f \in \mathcal{O}(\Omega, D)$ has coefficients in a given subalgebra

$A \subseteq \mathcal{O}(\Omega)$ if the coordinate functions of f with respect to some, or equivalently, every basis of D belong to A. We write $O_A \subseteq \mathcal{O}(\Omega, D)$ for the subspace of functions with coefficients in A. Let $M \subseteq \mathcal{O}(\Omega, D)$ be a $\mathbb{C}[z]$-submodule. We say that A is dense in M if every function $f \in M$ is the pointwise limit of a sequence $(f_k)_{k\in\mathbb{N}}$ of functions in M such that each f_k is in O_A. Note that we have $\epsilon_\lambda(M) \subseteq \overline{\epsilon_\lambda(O_A)} = \epsilon_\lambda(O_A)$ for all $\lambda \in \Omega$ in this case.

Theorem 6.4.2. Let $A \subseteq \mathcal{O}(\Omega)$ be a subalgebra and let $M_1, M_2 \subseteq \mathcal{O}(\Omega, D)$ be $\mathbb{C}[z]$-submodules such that A is dense in M_1 and in M_2 and such that $AM_i \subseteq M_i$ for $i = 1, 2$. Then we have

$$\mathrm{fd}(M_1 + M_2) + \mathrm{fd}(M_1 \cap M_2) = \mathrm{fd}(M_1) + \mathrm{fd}(M_2).$$

Proof. Exactly as in the proof of Lemma 6.4.1 it follows that

$$\mathrm{fd}(M_1 + M_2) + \mathrm{fd}(M_1 \cap M_2) \leq \mathrm{fd}(M_1) + \mathrm{fd}(M_2).$$

To prove the reverse inequality, define $M = M_1 + M_2$ and choose a point $\lambda \in \Omega$ which is maximal for M_1, M_2 and M. Define $E = (M_1)_\lambda \cap (M_2)_\lambda$ and choose direct complements E_1 of E in $(M_1)_\lambda$ and E_2 of E in $(M_2)_\lambda$. Fix bases $(e_1, ..., e_{n_1})$ of E_1, $(e_{n_1+1}, ..., e_{n_1+n_2})$ for E_2 and $(e_{n_1+n_2+1}, ..., e_{n_1+n_2+n'})$ for E, where $n_1, n_2, n' \geq 0$ are non-negative integers. Set $n = n_1 + n_2 + n'$. An elementary argument shows that $(e_1, ..., e_n)$ is a basis of M_λ. Let us complete this basis to a basis $B = (e_1, ..., e_n, e_{n+1}, ..., e_N)$ of D. Since $\mathrm{fd}(M_1) + \mathrm{fd}(M_2) - \mathrm{fd}(M) = n'$, we have to show that

$$\mathrm{fd}(M_1 \cap M_2) \geq n'.$$

We may of course assume that $n' \neq 0$. Since A is dense in M, there are functions $h_1, ..., h_n \in M$ with $h_i(\lambda) = e_i$ for $i = 1, ..., n$ such that each h_i has coefficients in A. Write

$$h_i = \sum_{j=1}^{N} h_{ij} e_j \qquad (i = 1, ..., n).$$

Then $\theta = (h_{ij})_{1\leq i,j\leq n}$ is an $(n\times n)$-matrix with entries in A such that $\theta(\lambda) = E_n$ is the unit matrix. By basic linear algebra there is an $(n \times n)$-matrix (A_{ij}) with entries in A such that $(A_{ij})\theta = \mathrm{diag}(\det\theta)$ is the $(n \times n)$-diagonal matrix with all diagonal terms equal to $\det(\theta)$. Then

$$(A_{ij})_{1\leq i,j\leq n}(h_{ij})_{\substack{1\leq i\leq n \\ 1\leq j\leq N}} = (\mathrm{diag}(\det\theta), (g_{ij})),$$

where (g_{ij}) is a suitable matrix with entries in A. We define functions $H_1, ..., H_n \in M$ by setting

$$H_i = \det(\theta)e_i + \sum_{j=1}^{N-n} g_{ij}e_{n+j} = \sum_{j=1}^{N}(\sum_{\nu=1}^{n} A_{i\nu}h_{\nu j})e_j = \sum_{\nu=1}^{n} A_{i\nu}h_\nu.$$

By construction $H_i(\lambda) = e_i$ and $(H_1(z), ..., H_n(z))$ is a basis of M_z for every $z \in \Omega$ with $\det(\theta(z)) \neq 0$. If $f = f_1e_1 + ... + f_Ne_N \in M$ is arbitrary, then at each point $z \in \Omega$ not contained in the zero set $Z(\det(\theta))$ of the analytic function $\det(\theta) \in \mathcal{O}(\Omega)$, the function f can be written as a linear combination

$$f(z) = \lambda_1(z, f)H_1(z) + ... + \lambda_n(z, f)H_n(z).$$

Using the definition of the functions H_i and a continuity argument, we find that

$$f_1 = \lambda_1(\cdot, f)\det(\theta), ..., f_n = \lambda_n(\cdot, f)\det(\theta).$$

Hence, for $j = n + 1, ..., N$ and $z \in \Omega \backslash Z(\det\theta)$, we obtain that

$$\begin{aligned} f_j(z) &= \lambda_1(z, f)g_{1,j-n}(z) + ... + \lambda_n(z, f)g_{n,j-n}(z) \\ &= \frac{g_{1,j-n}(z)}{\det\theta(z)}f_1(z) + ... + \frac{g_{n,j-n}(z)}{\det\theta(z)}f_n(z). \end{aligned}$$

In particular, each function $f = f_1e_1 + ... + f_Ne_N \in M$ is uniquely determined by its first n coordinate functions $f_1, ..., f_n$.
Since A is dense in M_1 and in M_2, there are functions $F_1, ..., F_{n_1+n'} \in M_1$ and $G_1, ..., G_{n_2+n'} \in M_2$ with coefficients in A such that

$$(F_i(\lambda))_{i=1,...,n_1+n'} = (e_1, ..., e_{n_1}, e_{n_1+n_2+1}, ..., e_{n_1+n_2+n'})$$

and

$$(G_i(\lambda))_{i=1,...,n_2+n'} = (e_{n_1+1}, ..., e_{n_1+n_2+n'}).$$

Write the first n coordinate functions of each of the functions

$$F_1, ..., F_{n_1}, G_1, ..., G_{n_2}, F_{n_1+1}, ..., F_{n_1+n'}, G_{n_2+1}, ..., G_{n_2+n'}$$

with respect to the basis $(e_1, ..., e_N)$ of D as column vectors and arrange these column vectors to a matrix Δ in the indicated order. Then Δ is an $(n \times (n+n'))$-matrix with entries in A. Write $\Delta = (\Delta_0, \Delta_1)$, where Δ_0 is the $(n \times n)$-matrix consisting of the first n columns of Δ and Δ_1 is the $(n \times n')$-matrix consisting

of the last n' columns of Δ.
By construction we have $\det(\Delta_0(\lambda)) = 1$. On $\Omega\backslash Z(\det \Delta_0)$, we can write

$$(\det \Delta_0)\Delta_0^{-1}\Delta = (\operatorname{diag}(\det \Delta_0), \Gamma),$$

where $\operatorname{diag}(\det \Delta_0)$ is the $(n \times n)$-diagonal matrix with all diagonal terms equal to $\det \Delta_0$ and $\Gamma = (\gamma_{ij})$ is an $(n \times n')$-matrix with entries in A. The column vectors

$$r_j = (\gamma_{1j}, ..., \gamma_{nj}, 0, ..., 0, -\det \Delta_0, 0, ..., 0)^t \quad (j = 1, ..., n'),$$

where $-\det \Delta_0$ is the entry in the $(n + j)$-th position, satisfy the equations

$$(\det \Delta_0)\Delta_0^{-1}\Delta r_j = ((\det \Delta_0)\gamma_{ij} - (\det \Delta_0)\gamma_{ij})_{i=1}^n = 0$$

on $\Omega\backslash Z(\det \Delta_0)$. Hence $\Delta r_j = 0$ for $j = 1, ..., n'$, or equivalently, for each $j = 1, ..., n'$, the first n coordinate functions of

$$\gamma_{1j}F_1 + ... + \gamma_{n_1 j}F_{n_1} + \gamma_{n_1+n_2+1,j}F_{n_1+1} + ... + \gamma_{n_1+n_2+n',j}F_{n_1+n'}$$

with respect to $(e_1, ..., e_N)$ coincide with those of

$$s_j = (\det \Delta_0)G_{n_2+j} - \gamma_{n_1+1,j}G_1 - ... - \gamma_{n_1+n_2,j}G_{n_2}.$$

For each j, due to $AM_i \subseteq M_i$ $(i = 1, 2)$ both functions belong to M and thus, by the first part of the proof, they coincide. But then these functions belong to $M_1 \cap M_2$. Since the vectors

$$G_i(\lambda) = e_{n_1+i} \qquad (i = 1, ..., n_2 + n')$$

are linearly independent and $\det(\Delta_0(\lambda)) = 1$, it follows that $s_1(\lambda), ..., s_{n'}(\lambda)$ are linearly independent and thus $\operatorname{fd}(M_1 \cap M_2) \geq \dim(M_1 \cap M_2)_\lambda \geq n'$. $\square$

Recall that a domain $\Omega \subseteq \mathbb{C}^d$ is called polynomially-convex or a Runge domain if the polynomial-convex hull of each compact subset $K \subseteq \Omega$ is contained in Ω. By the Oka-Weil approximation theorem (Satz 7.10 in [Esc17]) on each Runge domain $\Omega \subseteq \mathbb{C}^d$ the polynomials are dense in $\mathcal{O}(\Omega)$ with respect to the Fréchet space topology of uniform convergence on compact subsets, and hence each $\mathbb{C}[z]$-submodule $M \subseteq \mathcal{O}(\Omega, D)$ which is closed with respect to the Fréchet space topology of $\mathcal{O}(\Omega, D)$ is automatically an $\mathcal{O}(\Omega)$-submodule. Thus by applying Theorem 6.4.2 with $A = \mathcal{O}(\Omega)$ we obtain the following general lattice formula for fiber dimensions in the category of Fréchet submodules of $\mathcal{O}(\Omega, D)$. The reader should be aware that this result does not apply to Banach or Hilbert spaces of analytic functions.

Corollary 6.4.3. Let $\Omega \subseteq \mathbb{C}^d$ be a Runge domain and let D be a finite-dimensional complex vector space. Then the fiber dimension formula

$$\mathrm{fd}(M_1 + M_2) + \mathrm{fd}(M_1 \cap M_2) = \mathrm{fd}(M_1) + \mathrm{fd}(M_2)$$

holds for each pair of closed $\mathbb{C}[z]$-submodules $M_1, M_2 \subseteq \mathcal{O}(\Omega, D)$.

Suppose that $T \in L(X)^d$ is a dual Cowen-Douglas tuple of rank N on a domain $\Omega \subseteq \mathbb{C}^d$, that is, a weak dual Cowen-Douglas tuple such that $\bigcap_{z\in\Omega} \mathrm{ran}(Z-T) = \{0\}$. Choose a CF-representation

$$\rho : X \to \mathcal{O}(\Omega_0, D)$$

of T as in the proof of Corollary 6.1.5. Let $M \in \mathrm{Lat}(T)$ be an invariant subspace of T such that each vector $m \in M$ is the limit of a sequence of vectors in

$$M \cap \mathrm{span}\{T^\alpha x; \alpha \in \mathbb{N}^d \text{ and } x \in D\}.$$

Then $\rho(M) \subseteq \mathcal{O}(\Omega_0, D)$ is a $\mathbb{C}[z]$-submodule in which the polynomials are dense in the sense explained in the section leading to Theorem 6.4.2. Hence, for any two invariant subspaces $M_1, M_2 \in \mathrm{Lat}(T)$ of this type, noting that ρ is injective, the fiber dimension formula

$$\begin{aligned}\mathrm{fd}(M_1 + M_2) + \mathrm{fd}(M_1 \cap M_2) &= \mathrm{fd}(\rho(M_1) + \rho(M_2)) + \mathrm{fd}(\rho(M_1) \cap \rho(M_2)) \\ &= \mathrm{fd}(\rho(M_1)) + \mathrm{fd}(\rho(M_2)) = \mathrm{fd}(M_1) + \mathrm{fd}(M_2)\end{aligned}$$

holds. The above density condition on M is trivially fulfilled for every closed T-invariant subspace M which is generated by a subset of D. But there are other situations to which this observation applies.

Let $T \in L(H)^d$ be a commuting tuple of bounded operators on a complex Hilbert space. Suppose that H admits an orthogonal decomposition $H = \bigoplus_{k=0}^\infty H_k$ with closed subspaces H_k. Let $\gamma = (\gamma_1, ..., \gamma_d)$ be a d-tuple of positive integers $\gamma_i > 0$. The tuple T is said to be γ-graded with respect to this decomposition if $T_j H_k \subseteq H_{k+\gamma_j}$ for $k \in \mathbb{N}$ and $j = 1, ..., d$. By definition a closed subspace $M \subseteq H$ is homogeneous if

$$M = \bigoplus_{k=0}^{\infty} M \cap H_k.$$

We shall say that the algebraic direct sum $\tilde{H} = \bigoplus_{k=0}^\infty H_k$ is finitely generated if there is a finite set of vectors $x_1, ..., x_N \in \tilde{H}$ with

$$\tilde{H} = \mathrm{span}\{T^\alpha x_i; \alpha \in \mathbb{N}^d \text{ and } i = 1, ..., N\}.$$

In Theorem 3.3 of [Esc19] it is shown that $\tilde{H}$ is finitely generated if and only if the wandering subspace $W(T) = H \ominus \sum_{i=1}^d T_i H$ of T is finite dimensional, and that in this case $W(T) \subseteq \tilde{H}$ and each linear basis $x_1, ..., x_N$ of $W(T)$ generates $\tilde{H}$ in the above sense.

Corollary 6.4.4. Let $T \in L(H)^d$ be a γ-graded dual Cowen-Douglas tuple on a domain $\Omega \subseteq \mathbb{C}^d$ with $0 \in \Omega$. Then the fiber dimension formula

$$\mathrm{fd}(M_1 + M_2) + \mathrm{fd}(M_1 \cap M_2) = \mathrm{fd}(M_1) + \mathrm{fd}(M_2)$$

holds for any pair of homogeneous invariant subspaces $M_1, M_2 \in \mathrm{Lat}(T)$.

Proof. Let N be the rank of T as a dual Cowen-Douglas tuple on Ω. Then

$$H = \left(\sum_{i=1}^d T_i H \right) \oplus W(T)$$

and $\dim W(T) = N < \infty$. The proof of Corollary 6.1.5 shows that there is a CF-representation $\rho : H \to \mathcal{O}(\Omega_0, W(T))$ of T on a suitable connected open zero neighbourhood $\Omega_0 \subseteq \Omega$. Let $M \in \mathrm{Lat}(T)$ be a homogeneous invariant subspace for T. Then each element $m \in M$ can be written as a sum $m = \sum_{k=0}^{\infty} m_k$ with

$$m_k \in M \cap H_k \subseteq M \cap \mathrm{span}\{T^\alpha x; \alpha \in \mathbb{N}^d \text{ and } x \in W(T)\}.$$

Hence the remarks following Corollary 6.4.3 imply the assertion. $\square$

Typical examples of graded dual Cowen-Douglas tuples are multiplication tuples $M_z = (M_{z_1}, ..., M_{z_d}) \in L(H)^d$ with the coordinate functions on unitarily invariant subspaces. Slightly more general, consider an analytic functional Hilbert space $H = H(K_f, \mathbb{C}^N)$ given by a reproducing kernel

$$K_f : B_r(a) \times B_r(a) \to L(\mathbb{C}^N), K_f(z,w) = f(\langle z, w \rangle) 1_{\mathbb{C}^N},$$

where $f(z) = \sum_{n=0}^\infty a_n z^n$ is a one-variable power series with radius of convergence $R = r^2 > 0$ such that $a_0 = 1$, $a_n > 0$ for all $n \in \mathbb{N}$ and

$$0 < \inf_{n \in \mathbb{N}} \frac{a_n}{a_{n+1}} \leq \sup_{n \in \mathbb{N}} \frac{a_n}{a_{n+1}} < \infty$$

(see [GHX04] and Chapter 2.2 in [Wer14]). Let $\beta = (\beta_1, ..., \beta_d) \in (\mathbb{N}^*)^d$ be a tuple of positive integers. To obtain the grading of H, recall that a polynomial $p = \sum_{\alpha \in \mathbb{N}^d} a_\alpha z^\alpha$ is called β-homogeneous of degree $k \in \mathbb{N}$ if $\sum_{i=1}^d \alpha_i \beta_i = k$

for all $\alpha \in \mathbb{N}^d$ with $a_\alpha \neq 0$. Let us denote by $\mathbb{H}_k = \mathbb{H}_k(\beta)$ the set of all β-homogeneous polynomials of degree k.
By the results in [Wer08], the monomials $(a_\alpha \frac{|\alpha|!}{\alpha!})^{\frac{1}{2}} z^\alpha$ $(\alpha \in \mathbb{N}^d)$ form an orthonormal basis of $H(K_f, \mathbb{C})$ and thus H is the orthogonal sum

$$H = \bigoplus_{k=0}^{\infty} \mathbb{H}_k \otimes \mathbb{C}^N.$$

The operator tuple $M_z = (M_{z_1}, ..., M_{z_d}) \in L(H)^d$ is obviously β-graded. It is well known and follows for instance from Satz 2.5 in [Wer08] that $W(M_z) = \mathbb{H}_0 = \mathbb{C}^N$. Furthermore, every invariant subspace

$$M = \bigvee_{i=1}^{r} \mathbb{C}[z] p_i \in \mathrm{Lat}(M_z)$$

generated by a finite set of β-homogeneous polynomials $p_i \in \mathbb{H}_{k_i} \otimes \mathbb{C}^N$ is homogeneous.

Let $H = H_K \subseteq \mathcal{O}(\Omega)$ be an analytic functional Hilbert space on a domain $\Omega \subseteq \mathbb{C}^d$, or equivalently, a functional Hilbert space given by a sesqui-analytic reproducing kernel $K : \Omega \times \Omega \to \mathbb{C}$. Let D be a finite-dimensional complex Hilbert space. Then the D-valued functional Hilbert space $H(K_D) \subseteq \mathcal{O}(\Omega, D)$ given by the kernel

$$K_D : \Omega \times \Omega \to L(D), K_D(z, w) = K(z, w) 1_D$$

can be identified with the Hilbert space tensor product $H_K \otimes D$. Let us denote by $M(H) = \{\varphi : \Omega \to \mathbb{C}; \varphi H \subseteq H\}$ the multiplier algebra of H.

Corollary 6.4.5. Suppose that $H = H_K$ contains all constant functions and that $z_1, ..., z_n \in M(H)$.

(a) For any pair of closed subspaces $M_1, M_2 \subseteq H(K_D)$ with $M(H) M_i \subseteq M_i$ for $i = 1, 2$ and such that $M(H)$ is dense in M_1 and M_2, the fiber dimension formula

$$\mathrm{fd}(M_1 \vee M_2) + \mathrm{fd}(M_1 \cap M_2) = \mathrm{fd}(M_1) + \mathrm{fd}(M_2)$$

holds.

(b) If in addition K is a complete Nevanlinna-Pick kernel, that is, K has no zeros and also the mapping $1 - \frac{1}{K}$ is positive definite, then the fiber dimension formula holds for all closed subspaces $M_1, M_2 \subseteq H(K_D)$ which are invariant for $M(H)$.

Proof. Part (a) is a direct consequence of Theorem 6.4.2. If K is a complete Nevanlinna-Pick kernel, then the Beurling-Lax-Halmos theorem proved by McCullough and Trent (see Theorem 8.67 in [AM02] or Theorem 3.3.8 in [Bar07]) implies that $M(H)$ is dense in every closed subspace $M \subseteq H(K_D)$ which is invariant for $M(H)$. □

Note that the condition that $M(H)$ is dense in a subspace $M \subseteq H(K_D)$ is satisfied for every closed $M(H)$-invariant subspace $M \subseteq H(K_D)$ that is generated by an arbitrary family of functions $f_i : \Omega \to D$ $(i \in I)$ with coefficients in $M(H)$. Part (b) for domains $\Omega \subseteq \mathbb{C}$ was proved in [CCF15].

List of symbols

Common Notation

ran . . . . Range of a map

$\mathbb{C}[z]$ . . . . Polynomials in several (usually d) complex variables

e_i . . . . ith unit vector in $\mathbb{N}^d$

$L(X,Y)$ . . . . . . Bounded linear operators between Banach spaces X and Y, $L(X) = L(X,X)$

$l^2(\mathbb{N})$ . . . . Square-summable sequences in $\mathbb{C}$

$\hat{f}(n)$ . . . . nth Fourier coefficient of a function f

$H^2(\mathbb{T})$ . . . . Hardy space on the unit circle

$L^2(\mathbb{T})$ . . . . Square-integrable functions w.r.t arc length measure on the unit circle

$\mathcal{O}(\Omega, D)$ . . . . . . Holomorphic functions on a submanifold $\Omega \subseteq \mathbb{C}^d$ with values in a finite-dimensional vector space D

$\mathbb{D}$ . . . . Unit disc in $\mathbb{C}$

$\mathbb{B}_d$ . . . . Unit ball in $\mathbb{C}^d$

$\|\cdot\|_e$ . . . . Essential operator norm

$\operatorname{Re} T$ . . . . Real part of a bounded linear operator T

$\operatorname{Im} T$ . . . . Imaginary part of a bounded linear operator T

τ_{SOT} . . . . Strong operator topology

$H^2(\mathbb{D}, D)$ . . . . . Hardy space on $\mathbb{D}$ with values in a Hilbert space D

$H \ominus M$ . . . . Orthogonal complement of a subspace M in a Hilbert space H

$[M]$ .... Smallest closed subset invariant for a given operator tuple T containing a set M

$W(T)$ .... Wandering subspace of a bounded linear operator T

$L^\infty(\mathbb{D})$ .... Essentially bounded functions w.r.t the Lebesque measure on $\mathbb{D}$

$L^\infty(\sigma)$ .... Essentially bounded functions w.r.t a measure σ

$h^\infty(\mathbb{B}_d)$ .... Bounded $\mathcal{M}$-harmonic functions on $\mathbb{B}_d$

$\mathrm{fd}(M)$ .... Fiber dimension of a $\mathbb{C}[z]$-submodule M of $\mathcal{O}(\Omega, D)$

ϵ_λ .... Point evaluation at a point λ

M_λ .... Range of the point evaluation at a point λ on a set M of functions on Ω

$Z(f)$ .... Set of zeroes of a holomorphic function f

$\sigma(T)$ .... Taylor spectrum of a commuting operator tuple T

$\mathcal{O}_\Omega^X$ .... Sheaf of holomorphic functions on Ω with values in X

$\mathcal{O}_w^X$ .... Stalk of the sheaf $\mathcal{O}_\Omega^X$ at a point w

$\mathrm{span}(M)$ ..... Linear span of a subset M of a vector space

$\bigvee M$ .... Closed linear span of a subset M

$K^\bullet(T, X)$ ..... Koszul complex of a commuting operator tuple T on a Banach space X

$\Lambda^i(X)$ .... Forms of degree i with coefficients in X

δ_T^i .... ith boundary map in the Koszul complex of a commuting operator tuple T

$H^i(T, X)$ ..... ith cohomology group of the Koszul complex of T

Specific Symbols

$A_m^2(\mathbb{B}_d)$ .... Generalized Bergman space with parameter $m \in \mathbb{N}^*$, p.10

$A_\nu^2(\mathbb{B}_d)$ .... Generalized Bergman space with parameter $\nu \in]0, \infty[$, p.12

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