Easy quantum groups: Linear independencies, models and partition quantum spaces

Dissertation

zur Erlangung des Grades des
Doktors der Naturwissenschaften
der Fakultät für Mathematik und Informatik
der Universität des Saarlandes

vorgelegt von

Stefan Jung

Saarbrücken, April 2019

Datum des Promotionskolloquiums: 4. Juni 2019
Dekan der Fakultät: Prof. Dr. Sebastian Hack
Prüfungsausschuss: Vorsitzender:
    Prof Dr. Mark Groves
  Berichterstatter:
    Prof. Dr. Moritz Weber
    Prof. Dr. Adam Skalski
    Prof. Dr. Roland Speicher
  Akademischer Mitarbeiter:
    Dr. Dominik Schillo

Abstract

This work presents results in the context of (unitary) easy quantum groups. These are compact matrix quantum groups featuring a rich combinatorial structure given by partitions (of sets). This thesis reports on three topics within this area.

Topic 1: Linear independence of the intertwiner maps T_p in the free case: Given a suitable collection of partitions p, there exists by definition a connection to easy quantum groups via intertwiner maps T_p. A sufficient condition for this correspondence to be one-to-one are particular linear independences on the level of maps T_p. In the case of non-crossing partitions, a proof of this linear independence can be traced down to a matrix determinant formula, developed by W. Tutte. We present a revised and adapted version of Tutte's work and the link to the problem above, trusting that this self-contained workout will assist others in the field of easy quantum groups. In particular, we fixed some errors in the original work and adapted in this sense notations, definitions, statements and proofs.

Topic 2: A chain of models for $C(S_N^+)$: For any given natural number $N \in \mathbb{N}_{\geq 4}$ we present a chain of models $(B_n, M_n)_{n\in\mathbb{N}}$ for the C^*-algebra $C(S_N^+)$ which allows an inverse limit (B_∞, M_∞). For small n the elements in the chain have a quite concrete and comprehensive structure. In the inverse limit we obtain a compact matrix quantum group $G = (B_\infty, M_\infty)$ that fulfils $S_N \subsetneq G \subseteq S_N^+$. For $N \in \{4, 5\}$ it holds $G = S_N^+$.

Topic 3: Partition quantum spaces: Analogous to the construction of an easy quantum group G from a given set of partitions, we propose a definition for partition quantum spaces X, which are tuples of quantum vectors inspired by the first d columns of matrices in G. However, we define them as universal C^*-algebras, independently of those quantum groups. The central result is the reconstruction of G from X as its quantum symmetry group $\mathrm{QSymG}(X)$, at least if the number d is sufficiently large. In the case of non-crossing partitions, the minimal value for d to permit this reconstruction is proved to be one or two.

Zusammenfassung

Diese Arbeit widmet sich Forschungsergebnissen im Bereich der (unitären) easy Quantengruppen. Dies sind kompakte Matrix-Quantengruppen mit stark kombinatorisch geprägter Struktur, welche durch Partitionen auf Mengen gegeben ist. Die vorliegende Arbeit beschäftigt sich mit drei Themen innerhalb dieses Bereichs.

Thema 1: Lineare Unabhängigkeit von Intertwiner-Abbildungen T_p im freien Fall: Per definitionem existiert ein Zusammenhang zwischen geeigneten Familien von Partitionen p und easy Quantengruppen, der durch Intertwiner-Abbildungen T_p hergestellt wird. Eine hinreichende Bedingung für die Eineindeutigkeit dieses Zusammenhangs sind gewisse lineare Unabhängigkeiten auf Ebene der Abbildungen T_p. Im Falle nichtkreuzender Partitionen können diese linearen Unabhängigkeiten mittels einer Matrixdeterminanten-Formel, wie sie von W. Tutte entwickelt wurde, bewiesen werden. Wir präsentieren eine überarbeitete, an obige Fragestellung angepasste Version der Arbeit Tuttes und ebenso die Argumentationsschritte zwischen ursprünglichem Problem und erwähnter Determinantenformel. Insbesondere korrigiert die vorliegende Arbeit einige Fehler in der ursprünglichen Quelle und ändert in Folge dessen weitere Schreibweisen, Definitionen, Behauptungen und Beweise ab.

Thema 2: Eine Folge von Modellen für $C(S_N^+)$: Für eine gegebene natürliche Zahl $N \in \mathbb{N}_{\geq 4}$ konstruieren wir eine Folge von Modellen (B_n, M_n) der C^*-Algebra $C(S_N^+)$, die die Konstruktion eines inversen Limes (B_∞, M_∞) erlaubt. Für kleine n haben die Folgenglieder sehr konkrete und anschauliche Strukturen. Der inverse Limes dieser Folge liefert eine kompakte Matrixquantengruppe $G = (B_\infty, M_\infty)$, welche $S_N \subsetneq G \subseteq S_N^+$ erfüllt. Im Falle $N \in \{4, 5\}$ gilt $G = S_N^+$.

Thema 3: Partition quantum spaces: Analog zur Konstruktion einer easy Quantengruppe G auf Grundlage einer gegebenen Menge an Partitionen, stellen wir eine Definition für *partition quantum spaces*, Partitionen-Quantenräume, vor. Deren Elemente sind Tupel von Quantenvektoren, angelehnt an die ersten d Spalten der Quantenmatrizen in G. Wir definieren jedoch diese Quantenräume als universelle C^*-Algebren, zunächst ohne direkten Bezug zu diesen Quantenmatrizen. Das Hauptresultat ist die Rekonstruktion der easy Quantengruppe G aus dem Quantenraum X als dessen Quantensymmetriegruppe $\mathrm{QSymG}(X)$, zumindest für hinreichend große Spaltenzahl d. Im Falle nicht-kreuzender Partitionen wird gezeigt, dass der kleinstmögliche Wert für d, der diese Rekonstruktion erlaubt, entweder eins oder zwei ist.

Acknowledgements

There are a lot of people who contributed to this thesis.
My deepest gratitude goes to my supervisor Moritz Weber. I want to thank him for his support and guidance whenever I needed it and the liberty to work on those topics and questions which filled me with enthusiasm. He introduced me to the field of quantum groups by supervising my Bachelor's thesis and, together with Roland Speicher, made it possible for me to continue my work in this research area shortly after my studenthood. It filled me with joy that questions which had already been asked in my Bachelor's thesis finally found their answers at the very end of my doctoral time and so their way into this thesis. It is to a great extent due to Moritz Weber that I can and will keep my time as a PhD student in more than good memory.
I want to thank Roland Speicher for the opportunity to be part of his working group, to spent my time as a PhD student in such a fruitful and at the same time pleasant environment. I was able to take part in many conferences and come in contact with many people inside the quantum group community and I'm grateful for these opportunities.
I want to thank all members of my working group for accompanying me during my time as a PhD student. Representatively, I thank Tobias Mai, whom I owe fruitful discussions as well as helpful ideas and answers.
Finally, I am happy and thankful that Adam Skalski agreed on being the co-referee of this thesis.
Mein tiefster Dank geht auch an meine Familie und Freunde, die mich, jeder auf seine Weise und seit unterschiedlich langer Zeit, immer unterstützt haben.

Contents

Chapter 1

Introduction

1.1 Motivation

The mathematical branch of quantum objects and quantization combines the idea of dualization with non-commutative phenomena. Taking for example a compact group G, its structure can perfectly be described by a Hopf *-algebra $\big(\mathcal{A}, \Delta, \varepsilon, \delta\big)$ that is dense in $C(G)$. Note that the comultiplication Δ describes the semi-group structure of G, the counit ε reflects the existence of a neutral element and δ encodes invertibility of each element in G.

This dualization allows us to generalize the idea of a compact group by considering arbitrary unital C^*-algebras A with the property that they contain a dense Hopf *-algebra $\mathcal{A} \subseteq A$. Given such an object, it is in general not possible to associate a compact group in the background, but, in an abstract sense, we can postulate a corresponding so called *compact quantum group* and look at the elements of A as *non-commutative functions* over this quantum object.

$$\begin{tikzcd}
G \arrow[r, leftrightarrow, "\text{duality}"] \arrow[d, "\text{quantization}"'] & C(G) + \text{further properties} \arrow[d, "\text{allow non-commutativity}"] \\
\text{quantum } G \arrow[r, leftrightarrow, "\text{postulated duality}", "{A \overset{!}{=} \text{``}C(G)\text{''}}"'] & A + \text{further properties}
\end{tikzcd}$$

In the non-commutative case, the only concrete object at hand is the algebra A, so all statements about the quantum G have to be formulated or at least interpreted on this level. The three-step

$$G \longrightarrow C(G) \longrightarrow A \longrightarrow \text{ quantum } G$$

is the above mentioned combination of dualization and non-commutativity. Declaring the diagram above to be commutative, defines the quantization process.

As long as we can encode the properties of a mathematical object on the level of some function algebra over it, we are in principle able to quantize it. Note that the construction of quantum objects from classical ones is not totally canonical: In the situation above there might be other function algebras over G encoding the topological structure of G. Considering the above *-algebra $\mathcal{A}$, existence of the maps Δ, ε, δ on $\mathcal{A}$ is just one of many ways to encode the group structure of G. See for example Definition 2.1.1 for *compact quantum groups* in this thesis. Despite the fact that choosing different descriptions might lead to different notions of a quantum version

of G, all those quantizations have in common that they are a form of generalization: If the considered algebra is commutative, then there is a unique classical object in the background that can be associated to the structure at hand.
We present two concrete examples illustrating quantizations in the context of matrix groups and spaces of vectors.

Example 1.1.1. Consider the matrix group S_4, consisting of the 4×4 permutation matrices. By Stone-Weierstrass, the C^*-algebra $C(S_4)$ is generated by the coordinate functions u_{ij} that map each matrix on its (i,j)-th entry:

$$u_{ij}:\ S_4 \to \mathbb{C};\ \sigma \mapsto \sigma_{ij}$$

Arranging the functions u_{ij} canonically in a matrix $u \in M_4\big(C(S_4)\big)$, it holds, due to the structure of permutation matrices, that the generators u_{ij} are projections that sum up to $\mathbb{1}$ in every row and column of u. A matrix u with these properties is called *magic*, compare for example [BS09].
In the sense of the diagram above and in order to define a quantum version of S_4, we can consider a magic 4×4-matrix u of generators (without assuming commutativity) and the universal C^*-algebra A generated by its entries. We then look at A as the non-commutative functions over the so-called *quantum permutation group* S_4^+ and write $A = C(S_4^+)$.
Note that both $C(S_4)$ and $C(S_4^+)$ allow Hopf *-algebra structures. For example in both cases the mapping

$$\Delta(u_{ij}) := \sum_k u_{ik} \otimes u_{kj}$$

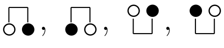
is a comultiplication. The fact that the group elements are (quantum) matrices is encoded by the special structure of Δ together the property that the corresponding C^*-algebra is generated by the respective u_{ij}'s. The object $C(S_4^+)$ is the same as defined in Chapter 2, Section 2.6.1.

Example 1.1.2. Consider for $N \in \mathbb{N}$ the usual Hilbert space $\mathbb{C}^N$ and the complex sphere $X \subseteq \mathbb{C}^N$, i.e. the space of all vectors with norm 1. We have, similar to the situation of $C(S_4)$, that $C(X)$ is generated by the coordinate functions

$$x_i : X \to \mathbb{C}; v \mapsto v_i$$

and it holds $\sum\limits_k x_k x_k^* = \mathbb{1}$. The *complex quantum sphere* X^+ can then be defined by the universal C^*-algebra

$$C(X^+) := C^*\big((x_i)_{1\leq i\leq N} \,|\, \sum_k x_k x_k^* = \sum_k x_k^* x_k = \mathbb{1}\big).$$

The object X^+ corresponds to $X_{N,1}\big(\{$  $\}\big)$ as defined in Chapter 5, Definition 5.2.21.

1.2 Aspects of quantum groups

We should and will not omit to sketch some approaches to quantum groups. As indicated in Section 1.1, different mathematical backgrounds lead to different notions of quantization, although its core, generalisation by allowing non-commutativity, unites them. The overview presented here is a shortened version of [Tim08, pp. xv-xvii].

We already mentioned above that quantum groups are closely related to Hopf algebras. H. Hopf was one of the first to investigate such structures in the 1940s, see for example [Hop41].

Algebraic quantum groups: A huge step towards non-commutativity and away from the classical setting was the introduction of q-deformations of universal enveloping algebras associated to Lie-algebras. L. D. Faddeev presented in the 1970s examples while working on the *quantum inverse scattering method*, [FS63]. Later, in the 1980s, V. Drinfeld and M. Jimbo contributed to the understanding of such deformations by working out in particular its connection to special Hopf algebras, nowadays called Drinfeld-Jimbo Hopf algebras, see [Dri87].

There is also a connection to knot invariants and the Yang-Baxter equation: The structure of so-called *braided* or *triangular* Hopf algebras can be connected to knot invariants and solutions of the quantum Yang-Baxter equation. In some cases, solutions of this equation conversely allow the construction of Hopf algebras. See also [Kas95] on this topic.

In the algebraic setting, A. Van Daele proposed a precise definition for quantum groups, see [Dae94] and [Dae98]. He defined (algebraic) quantum groups as (non-unital) Hopf algebras with an integral, following the idea of a Haar measure on a locally compact group.

Von Neumann- and C^*-algebraic quantum groups: Works of Vainerman and Kac [KV74] as well as Enock and Schwartz [ES92] led to the notion of *Kac algebras* by generalizing Pontrjagin duality: For a locally compact abelian group G its characters $\widehat{G}$ form again a locally compact abelian group and it holds $\widehat{\widehat{G}} \simeq G$. For non-abelian locally compact groups this duality fails within the category of groups and its duals have to be located in the more general setting of quantum groups. Kac algebras A and their duals were defined as von Neumann algebras with structural properties similar to those of Hopf algebras and were proved to fulfil $\widehat{\widehat{A}} \simeq A$. Later, also a C^*-algebraic version of this theory was developed by Enock and Vallin, see [Val85] and [EV93].

S. L. Woronowicz developed a theory of *C^*-algebraic compact quantum groups* including examples not covered by the Kac algebraic setting, see [Wor98]. In addition, his definition of compact quantum groups (see Definition 2.1.1) is quite simple and features the very nice consequence that there is a unique Haar measure on every such

compact quantum group, compare Theorem 2.1.5 and [Tim08, Thm. 5.1.6]. As a special and very important subclass of compact quantum groups, S. L. Woronowicz defined compact matrix quantum groups, initially under the name of compact matrix pseudogroups, see [Wor87]. As the name indicates, they are quantized versions of matrix groups.
S. Baaj and G. Skandalis put in [BS93] the concept of *multiplicative unitaries* in the center of their theory by giving an abstract definition and proving that such a multiplicative unitary encodes the structure of a quantum group.

1.3 The background of easy quantum groups

This work is based on S. L. Woronowicz's definition of *compact matrix quantum groups* in [Wor87]. As mentioned in Section 1.2, they form a subclass of compact quantum groups as defined in [Wor98]. Roughly speaking, such a quantum group G is given by a unital C^*-algebra A generated by the entries of a matrix $u = (u_{ij})$ such that there exists a suitable comultiplication $\Delta : A \to A \otimes A$, see Definition 2.2.1. As important as defining a framework for such quantum objects was Woronowicz's proof of *Tannaka-Krein duality for compact matrix quantum groups* in [Wor88]. A compact matrix quantum group can be reconstructed from its *intertwiner spaces* and one even has that this correspondence is one-to-one: Given any family of maps (fulfilling the properties of a collection of intertwiner spaces), we can construct a compact matrix quantum group that can be associated to this collection. See Theorems 2.3.19 and 2.3.25 for precise formulations.
In [BS09], T. Banica and R. Speicher used this duality to introduce an important class of compact matrix quantum groups, the so-called *(orthogonal) easy quantum groups*, which are quantizations of orthogonal matrix groups. Their structures are encoded via *categories of unicoloured partitions of sets*: Initially, such a category only defines a collection of linear maps but then Tannaka-Krein duality guarantees this collection to be the intertwiner spaces of a unique compact matrix quantum group. In [TW18] and [TW17], these quantum groups were generalized to *unitary easy quantum groups* by introducing *two-coloured partitions*, quantizing the situation of special unitary matrix groups. Boiling down the established theory to a simple construction, one has the following result: Given a partitions p and a matrix of generators $\big(u_{ij}\big)_{1\leq i,j\leq N}$ for some $N \in \mathbb{N}$, we can associate a set of algebraic relations $\mathcal{R}_p^{Gr}(u)$ on the matrix entries u_{ij}, the so-called *quantum group relations*, compare Notation 2.6.6 and Proposition 5.2.5. Given a set of two-coloured partitions Π (fulfilling some mild conditions), the universal C^*-algebra

$$C\big(G_N(\Pi)\big) := C^*\big(\, (u_{ij})_{1\leq i,j\leq N} \mid \forall p \in \Pi : \text{ The relations } \mathcal{R}_p^{Gr}(u) \text{ hold.}\big)$$

defines an easy quantum group $G_N(\Pi)$, compare Lemma 2.6.5, and every easy quantum group can be constructed this way. Using this machinery, many well-known compact matrix quantum groups may be produced, for example the free unitary group U_N^+, the free orthogonal group O_N^+ or the free permutation group S_N^+ as well as the corresponding well known classical matrix groups U_N, O_N and S_N.

1.4 Main questions and results

Topic 1: Linear independence of the intertwiner maps T_p in the free case: In Chapter 3 we take a closer look at the connection between categories of partitions $\mathcal{C}$ and intertwiner spaces $R_N(\mathcal{C})$, given by the construction $p \mapsto T_p$ that maps a partition p to an intertwiner map T_p. Woronowicz's Tannaka-Krein duality guarantees a one-to-one correspondence between these intertwiner spaces and suitable compact matrix quantum groups $G_N(\mathcal{C})$, but, unfortunately, it is possible that different categories of partitions produce the same intertwiner spaces, making the whole construction,

$$\mathcal{C} \mapsto R_N(\mathcal{C}) \mapsto G_N(\mathcal{C}),$$

not injective. See Proposition 4.3.3 for an example. A common way to prove injectivity or the opposite in special situations is to investigate linear (in)dependencies on suitable sets of maps T_p.

In the general situation, a crucial criterion is the size N of the constructed quantum matrices in relation to the number of through-blocks $tb(p)$ of the considered partitions p, compare Propositions 3.2.3 and 3.2.4. In particular, one can deduce from this the well-known fact that for two given categories the associated easy quantum groups differ, at least if we choose the size N of the quantum matrices large enough, see Corollary 3.2.6.

Restricting to so-called *non-crossing* partitions (and matrix sizes $N \geq 4$), the situation becomes much more comfortable: The desired linear independences are always given and different categories produce different easy quantum groups. One often proves this (see for example [FW14, Lemma 4.16]) by tracing down the linear independence question to the invertibility of a special Gram matrix $A(n, 0)$, see Equation 3.3.3. A determinant formula for such a matrix was developed by W. Tutte in his combinatorial work [Tut93].

The main part of chapter 3 is a revision of Tutte's work and a detailed and self-contained exposition of the connection to the linear independence problem in the context of easy quantum groups. Although the final result turned out to remain true we detected some errors in the original work which are eliminated in this recapitulation. Of course, these corrections entailed further changes of notations, definitions, partial results and proofs. If possible or necessary, we adapted formulations to those used in the context of easy quantum groups and we enriched descriptions and proofs with the graphical notations for partitions commonly used when dealing with easy quantum groups. The central results are the recursion formulae 3.3.32 and 3.3.33, from which the desired result is deduced.

Topic 2: A chain of models for $C(S_N^+)$: Chapter 4 deals with the question how the C^*-algebras $C\big(G_N(\Pi)\big)$ can be concretely interpreted as operators on some Hilbert space or at least as elements in other C^*-algebras with comprehensive structures. As displayed in Section 1.3, $C\big(G_N(\Pi)\big)$ has an appealing description as a universal C^*-algebra. The number of algebraic relations involved in the construction of such a C^*-algebra can usually be chosen to be quite small, hence we know the defining relations between its generators u_{ij} and we have an excellent understanding of $C\big(G_N(\Pi)\big)$ as an abstract C^*-algebra.
In contrast to that, we have little knowledge about $C\big(G_N(\Pi)\big)$ when realized as a subset of some $B(H)$. As we can always interpret a C^*-algebra as operators on a suitable Hilbert space (see the Appendix), we can see this lack of knowledge as a lack of "enlightening" models for $C\big(G_N(\Pi)\big)$, i.e. *-homomorphisms φ from $C\big(G_N(\Pi)\big)$ into C^*-algebras B, that map the matrix of generators u canonically onto some matrix $M \in M_N(B)$. The question for models of $C(S_N^+)$ is for example also asked in [BN17], but under specific constraints on those models.
Of course the best scenario to wish for is a faithful model, i.e. an injective *-homomorphism φ, where there exist comprehensive descriptions of the images $\varphi(u_{ij})$ as operators on a Hilbert space. In many situation, however, one faces the following problem: If the kernel of a *-homomorphism $\varphi : C\big(G_N(\Pi)\big) \to B$ is large, then the amount of information in the image about the original easy quantum group is small and if the kernel is small then the image often appears to be quite abstract.
In order to escape this dilemma, our aim is, roughly speaking, to construct a chain of models $\varphi_n : C\big(G_N(\Pi)\big) \to B_n$ allowing a commuting diagram

$$\begin{tikzcd}
 & & \Big(C\big(G_N(\Pi)\big), u\Big) \arrow[ddll, "\varphi_1"'] \arrow[dd, "\varphi_n"'] \arrow[ddr, "\varphi_{n+1}"] & & \\
 & \cdots & & & \cdots \\
(B_1, M_1) & \cdots \arrow[l, "\pi_{2,1}"'] & (B_n, M_n) & (B_{n+1}, M_{n+1}) \arrow[l, "\pi_{n+1,n}"'] & \cdots \arrow[l, "\pi_{n+2,n+1}"']
\end{tikzcd} \tag{1.4.1}$$

where the $\pi_{n+1,n}$ are *-homomorphisms. For small n the C^*-algebras B_n should be "well-understood" and, ideally, the kernel of φ_n should vanish for $n \to \infty$ (in the sense of an inverse limit, see Definition 4.5.1).
We concentrate on the special situation of the easy quantum groups $\big(S_N^+\big)_{N\geq 4}$, a

series of (non-classical) easy quantum groups with one of the most comprehensive structures. We define the step $B_n \to B_{n+1}$ by a so-called $\oplus$ -product $M_{n+1} := M_n \oplus M_1$ and the inverse step $B_{n+1} \to B_n$ is given by a quotient map $\pi_{n+1,n}$, see Section 4.4 and Theorem 4.7.9 . This well-understood connection $B_n \rightleftarrows B_{n+1}$ is the advantage of Diagram 1.4.1 compared to a simple collection of maps φ_n.

We further show, compare Proposition 4.5.2, that such a chain allows the construction of a so-called *inverse limit*

$$\lim_{\infty \leftarrow n} (B_n, M_n) =: \big(B_\infty, M_\infty\big),$$

which in addition defines a compact matrix quantum group, see Theorems 4.6.7 and 4.7.9. For $N \in \{4, 5\}$ this compact matrix group turns out to be isomorphic to S_N^+, see Corollaries 4.6.9 and 4.7.11.

Topic 3: Partition quantum spaces: Chapter 5 is based on the article [JW18] written by the author together with M. Weber. Given a compact matrix quantum group G with associated C^*-algebra $C(G)$ and matrix of generators $u = (u_{ij})$, the C^*-subalgebra generated by the first d columns of u can be interpreted as a quantum space of d-tuples of vectors, the "first d columns space" of G. The comultiplication of G guarantees that G acts on these vector tuples by entrywise matrix-vector multiplication.

In this part of the thesis we present another approach to construct quantum spaces of vector tuples, namely via partitions and universal C^*-algebras: As in the case of easy quantum groups, we start with a set of partitions Π and define a *compact quantum space of* d *vectors* $X_{N,d}(\Pi)$ and call it a *partition quantum space.* This is a universal C^*-algebra generated by the entries x_{ij} of a tuple of vectors

$$x := (x_{ij}) := \left(\begin{pmatrix} x_{11} \\ \vdots \\ x_{N1} \end{pmatrix}, \dots, \begin{pmatrix} x_{1d} \\ \vdots \\ x_{Nd} \end{pmatrix} \right)$$

and so-called *quantum space relations* $\big(\mathcal{R}_p^{Sp}(x)\big)_{p\in\Pi}$. Although defined independently of the easy quantum group $G_N(\Pi)$ and its matrix of generators u, the quantum space relations imposed on the x_{ij} are nonetheless motivated by the first d columns of the matrices in $G_N(\Pi)$. More precisely, the first d columns of matrices in $G_N(\Pi)$ turn out to form a subspace of the partition quantum space $X_{N,d}(\Pi)$, see Theorem 5.2.23. We further prove that there are faithful left and right matrix-vector actions of the easy quantum group $G_N(\Pi)$ on the partition quantum space $X_{N,d}(\Pi)$, see Definition 2.7.8 and Theorem 5.3.1.

The main part of Chapter 5 deals with finding so-called *quantum symmetry groups* of partition quantum spaces: Given a classical space, one often wants to understand its symmetries and find the symmetry group acting on it, i.e the maximal object in the category of groups acting on the given space. In the situation of a partition quantum space one is interested in its quantum symmetries or, equivalently, one wants to determine its (quantum) symmetry group, i.e. the maximal compact matrix quantum group acting on these tuples of quantum vectors. The central result of this part of the thesis is Corollary 5.4.8, saying that for sufficiently large d the easy quantum group $G_N(\Pi)$ can be reconstructed from the quantum space $X_{N,d}(\Pi)$ as its quantum symmetry group $\mathrm{QSymG}\left(X_{d,N}(\Pi)\right)$.

The canonical next step following this result is to ask for the minimal d in $X_{N,d}(\Pi)$, such that this reconstruction works. We investigate this question in the *free case*, i.e. all considered partitions are non-crossing. At least for special choices of sets of partitions Π, we are able to bound the minimal d with a case-by-case computation by 2 and in the so-called *blockstable* situation we prove that even $d = 1$ suffices, compare Theorem 5.5.12.

1.5 Notations

We fix some notations and agree on the following assumptions:

For $n \in \mathbb{N}$ we denote by $[n]$ the set $\{1, \dots, n\}$. All topological spaces appearing in this work are assumed to be Hausdorff and all groups are assumed to be associative. Inner products $\langle \cdot, \cdot \rangle$ on $\mathbb{C}$-vector spaces or C^*-modules are conjugate linear in its first argument, compare the Appendix.

For a Hilbert space H we denote by $B(H)$ the C^*-algebra of bounded operators on H.

If for some $N \in \mathbb{N}$ the complex vector space $\mathbb{C}^N$ is considered as a Hilbert space, then the underlying scalar product should always be the canonical one:

$$\left\langle (x_i)_{i \in [N]}, (y_j)_{j \in [N]} \right\rangle := \sum_{k=1}^{N} \overline{x_k} y_k.$$

In this context the tuple $(e_1, \dots, e_N)$ always denotes the canonical orthonormal basis of $\mathbb{C}^N$, i.e. in e_i all entries are zero but the i-th entry is 1. We identify $M_N(\mathbb{C})$ with $B(\mathbb{C}^N)$ via left matrix-vector multiplication.

Given two C^*-algebras A and B, we denote by $A \otimes B$ its minimal tensor product, see the Appendix.

If it exists, the unit of an algebra A is denoted by $\mathbb{1}_A$ or simply by $\mathbb{1}$ if the considered algebra is clear. In addition we can always assume $\mathbb{C}$ to be a subalgebra of a complex algebra A via the embedding $\iota : \mathbb{C} \hookrightarrow A : a \mapsto a\mathbb{1}_A$ i.e. the identification $\mathbb{1}_A = 1$.

Chapter 2

Preliminaries

The aim of this chapter is to provide sufficient knowledge in the theory of easy quantum groups and its background. It was written with the intention to serve as an introduction to this topic for readers who are not familiar with the concept of quantum groups or quantization. Following this purpose, we tried to keep this section self-contained and proofs have been omitted as rarely as possible.
As easy quantum groups are special cases of compact matrix quantum groups and these, in turn, are C^*-algebraic compact quantum groups, we will develop the presented theory out of this context. Many properties, results and proofs can already be stated in this general context. In addition, some basic theory in compact quantum groups (and their (co)representation theory) justifies and motivates the focus on compact matrix quantum groups and on easy quantum groups.

After giving an overview on compact quantum groups and their corepresentation theory (Section 2.1) we define and motivate compact matrix quantum groups as compact quantum groups with a special and easily accessible category of corepresentations (Section 2.2). We then state Tannaka-Krein duality for compact matrix quantum groups (Section 2.3): Loosely speaking, there is a one-to-one correspondence between compact matrix quantum groups and the collections of intertwiners of their corepresentations. Using this duality, we can define and prove existence of (unitary) easy quantum groups (Section 2.6). These are, by definition, compact matrix quantum groups whose intertwiners can be described by (two-coloured) partitions of sets, see Section 2.4 and 2.5. We finish this chapter with an overview on actions of compact (matrix) quantum groups on quantum space (of vectors) and a definition of *quantum symmetry groups* in this context (Section 2.7.

Displaying the theory of compact quantum groups and their corepresentation theory, we were to great extent guided by the book [Tim08]. The definition of compact matrix quantum groups and Tannaka-Krein duality is based on the works [Wor87] and [Wor88] by S. L. Woronowicz. We note at this point that we present a slight modification of Tannaka-Krein duality as defined in [Wor88]: While Woronowicz considers so-called *concrete monoidal W^*-categories* (and connects them to compact matrix quantum groups), we define and consider *essential* versions of those categories, see Definition 2.3.6 and Remark 2.3.7. Concerning easy quantum groups and their classification, our standard references are the initial work [BS09] as well as [TW18, TW17].
In order to highlight the chain of arguments in (omitted) proofs, we sometimes merge results from different sources or enrich lemmata and theorems with helpful partial results.

2.1 Compact quantum groups and their representation theory

For an overview on (C^*-algebraic) compact quantum groups and their representation theory we refer to the books [Tim08] and [NT13] and the surveys [KT00a, KT00b] and [MD98]. In this thesis, we mainly follow [Tim08].

2.1.1 Compact quantum groups

As described in Section 1.1, the structure of a compact group is encoded in the pair (A, Δ), where $A = C(G)$. Allowing A to be non-commutative, it is in general not possible any more to associate a group with a pair (A, Δ). Motivated by the commutative situation, one, however, interprets the elements in A as *non-commutative functions* on an abstractly given object, a so called *compact quantum group.*

In [Wor98] S. L. Woronowicz defined a compact quantum group in the C^*-algebraic setting. This is the starting point for our exposition.

Definition 2.1.1. Let (A, Δ) be a unital C^*-algebra A together with a unital *-homomorphism $\Delta : A \to A \otimes A$ such that:

(1) Δ is a so-called comultiplication, i.e. it holds

$$(\Delta \otimes \mathrm{id}_A) \circ \Delta = (\mathrm{id}_A \otimes \Delta) \circ \Delta.$$

(2) The linear spans of the sets $\Delta(A)(\mathbb{1}_A \otimes A)$ and of $\Delta(A)(A \otimes \mathbb{1}_A)$ are both linearly dense in $A \otimes A$.

Then we call the elements of A the *non-commutative continuous functions on a C^*-algebraic compact quantum group G* and denote A also by $C(G)$.

We usually omit the phrase 'C^*-algebraic' and we often abuse notation by speaking of A or (A, Δ) as the compact quantum group. In this virtue we write $G = (A, \Delta)$. To shorten notation, we often use the acronym 'CQG' for 'compact quantum group'. As displayed in the introduction, a 'good' definition of a quantum object should be consistent with the classical one when the commutative situation is considered:

Proposition 2.1.2. *Let $G = (A, \Delta)$ be a compact quantum group where A is commutative. Then A is isomorphic to the C^*-algebra of complex-valued, continuous functions $C(G)$ over some compact group G and Δ, seen as a mapping $C(G) \to C(G \times G)$, is given by*

$$\Delta(f)(g, h) = f(gh) \quad \forall f \in C(G), g, h \in G.$$

The correspondence of compact quantum groups $G = (A, \Delta)$ with A commutative and compact groups is (up to equivalence) one-to-one.

Proof. Compare [Tim08, Prop 5.1.3]. By the Gelfand theorem (see Appendix) a commutative A is isomorphic to the C^*-algebra of complex-valued, continuous functions $C(\Sigma)$ on the compact space Σ of characters on A. Now the mapping

$$m : \Sigma \times \Sigma \mapsto \Sigma \ ; \ (x, y) \mapsto xy := \big((x \otimes y) \circ \Delta\big)(\,\cdot\,)$$

defines a multiplication on Σ. It can directly be proven associative as Δ is a comultiplication, so it establishes a semi-group structure on Σ. By definition we have

$$\Delta(f)(x, y) = (x \otimes y)\big(\Delta(f)\big) = (xy)(f) = f(xy).$$

Note now that for $f, g \in A$ and $x, y \in \Sigma$ we have

$$\begin{aligned} \big(\Delta(f)(\mathbb{1} \otimes g)\big)(x, y) &= \Delta(f)(x, y) \cdot (\mathbb{1} \otimes g)(x, y) \\ &= (xy)(f) \cdot g(y) \\ &= f(xy) \cdot g(y) \\ &= (f \otimes g)\big(xy, y\big). \end{aligned} \tag{2.1.1}$$

To prove that Σ is in fact a group, one shows equivalently that it has the so-called cancellation property (compare [MD98, Prop. 3.2]), which states that the maps defined by

$$(x, y) \mapsto (xy, y) \qquad \text{and} \qquad (x, y) \mapsto (x, xy) \tag{2.1.2}$$

are injective:
Consider arbitrary $x_1, x_2, y \in G$ and $x_1 \neq x_2$. As Σ is compact, we have $C(\Sigma) \otimes C(\Sigma) = C(\Sigma \times \Sigma)$, so, by the (first) denseness condition in Definition 2.1.1, we find some $\sum_i \Delta(f_i)(\mathbb{1} \otimes g_i)$ in the linear span of $\Delta(A)(\mathbb{1} \otimes A)$ that seperates the points (x_1, y) and (x_2, y). By Equation 2.1.1 the function $\sum_i (f_i \otimes g_i)$ then separates $(x_1 y, y)$ and $(x_2 y, y)$ so the first map defined in Equation 2.1.2 is injective. Likewise we prove injectivity for the second mapping (using the second denenesss condition). This shows that Σ is a group.
If, conversely, a compact group G is given, then the unital C^*-algebra $C(G)$ admits a comultiplication $\Delta : C(G) \to C(G) \otimes C(G) = C(G \times G)$, defined by $\Delta(f)(x, y) := f(xy)$. As Δ is multiplicative and $C(G)$ is commutative, we have

$$\big(\Delta(f_1)(\mathbb{1} \otimes g_1)\big) \cdot \big(\Delta(f_2)(\mathbb{1} \otimes g_2)\big) = \Delta(f_1 f_2)(\mathbb{1} \otimes g_1 g_2),$$

so the linear span of $\Delta(A)(\mathbb{1} \otimes A)$ is equal to the algebra it generates and we have to prove this algebra to be dense in $C(G) \otimes C(G) = C(G \times G)$.
It is closed under complex conjugation, contains the unit and it is easy to see that it separates the points of $G \times G$, likewise for $\Delta(A)(A \otimes \mathbb{1})$. Hence, by Stone-Weierstrass, also the denseness conditions in Definition 2.1.1 are fulfilled and $\big(C(G), \Delta\big)$ is a compact quantum group. □

Motivated by the subgroup relations between compact groups, we can also define a subobject relation between CQGs.

Definition 2.1.3. Let $G = (A, \Delta_A)$ and $H = (B, \Delta_B)$ be two compact quantum groups. Then we define the subgroup relation $G \subseteq H$, if there is a surjective (so unital) *-homomorphism $\varphi : B \to A$ that intertwines the comultiplications, i.e.

$$\Delta_B \circ \varphi = (\varphi \otimes \varphi) \circ \Delta_A.$$

We call A and B equivalent, if φ is bijective (i.e. an *-isomorphism).

Remark 2.1.4. We note that the definition of equivalence of CQGs as stated above is imprecise in the following sense: Given a compact quantum group $G = (A, \Delta_A)$, it is possible that there are other pairs (B, Δ_B) describing the same G although there is no *-isomorphism φ connecting A and B. See also Section 2.1.3, where the reduced form $(A_{red}, \Delta_{A_{red}})$ and the universal form (A_u, Δ_{A_u}) of a fixed compact quantum group $G = (A, \Delta_A)$ are defined. In order to make Definition 2.1.3 precise, we need to assume both G and H to be given in their universal forms (A_u, Δ_{A_u}) and (B_u, Δ_{B_u}), respectively.
This fact should also be kept in mind when the notion $C(G)$ is used: Strictly speaking, the C^*-algebra $C(G)$ associated to some compact quantum group G might not be unique.

In contrast to an algebraic setting, the conditions on (A, Δ) in Definition 2.1.1 guarantee the existence of a Haar state on A, compare [Tim08, Thm 5.1.6].

Theorem 2.1.5. *Every C^*-algebraic compact quantum group $G = (A, \Delta) = (C(G), \Delta)$ admits a Haar state h, i.e. a state that is left and right invariant with respect to the comultiplication:*

$$(h \otimes \mathrm{id}_{C(G)})\Delta(a) = (\mathrm{id}_{C(G)} \otimes h)\Delta(a) = h(a)\mathbb{1}_{C(G)} \quad , \forall a \in C(G).$$

Every state that is left or right invariant is equal to h.

Remark 2.1.6. In the commutative case, see [Tim08, Ex. 5.1.10], the Haar state h on $C(G)$ is given by integration with respect to the normalized Haar measure λ on G. Left and right invariance of h are a consequence of the left and right invariance of λ:

$$\lambda(gX) = \lambda(Xg) = \lambda(X)$$

for every $g \in G$ and every measurable set $X \subseteq G$.

2.1.2 Corepresentations

We now turn towards non-degenerated corepresentations of compact quantum groups, the analogue of representations of groups as invertible operators on Hilbert spaces. Most of the ideas, observations and definitions can be found in [Tim08, Chpt. 5]. We will see that every corepresentation decomposes into a direct sum of (finite-dimensional) corepresentation matrices. These matrices are $C(G)$-valued, they encode the behaviour of Δ on its entries and it turns out that the entries of all these matrices together are linearly dense in $C(G)$. In this sense the corepresentation matrices of a compact quantum group – and therefore its corepresentation theory – carries (nearly) all the information about the pair $\big(C(G), \Delta\big)$.

As mentioned in Section 1.5, we identify a finite-dimensional Hilbert space H with $\mathbb{C}^{\dim(H)}$. See Observation 2.1.12, where this identification will be discussed in the form of equivalent corepresentations.
Note further (see also the Appendix) that for a given Hilbert space H and C^*-algebra A the algebraic tensor product $H \odot A$ is a right pre-Hilbert A-module via

$$\langle v \odot a, w \odot b \rangle := \langle v, w \rangle_H \cdot a^* b$$

and its completion $H \otimes A$ with respect to the inner product norm

$$\|\cdot\|_{H \odot A} : x \mapsto \|\langle x, x \rangle\|_A^{\frac{1}{2}}$$

is a right Hilbert A-module. Recall that we assume inner products to be conjugate linear in the first argument.

Definition 2.1.7. Let G be a compact group, $N \in \mathbb{N}$.

(1) An N-dimensional unitary representation of G is a continuous group homomorphism

$$\pi : G \to U(H) \subseteq B(H),$$

where H is an N-dimensional Hilbert space and $U(H)$ are the unitary operators in $B(H)$.

(2) An N-dimensional unitary corepresentation matrix of G is a unitary element

$$M = (m_{ij}) \in M_N\big(C(G)\big) = M_N(\mathbb{C}) \otimes C(G)$$

such that the comultiplication of G satisfies

$$\Delta(m_{ij}) = \sum_{k=1}^{N} m_{ik} \otimes m_{kj} \quad \forall 1 \leq i, j \leq N.$$

(3) An N-dimensional unitary corepresentation of G is a linear map

$$\delta : H \to H \otimes C(G)$$

from an N-dimensional Hilbert space H into the right Hilbert $C(G)$-module $H \otimes C(G)$ such that the following hold:

  (i) $(\delta \otimes \mathbb{1}_{C(G)}) \circ \Delta = (\mathbb{1}_{B(H)} \otimes \Delta) \circ \delta$.
  (ii) $\langle \delta(v), \delta(w) \rangle = \langle v, w \rangle \mathbb{1}_{C(G)} \quad \forall v, w \in H$.
  (iii) $\delta(H)C(G)$ is linearly dense in $H \otimes C(G)$.

We now prove that there is a one-to-one correspondence between the objects defined in (1), (2) and (3). This is probably well-known, but we did not find a reference.

Proposition 2.1.8. *Let G be a compact group and H an N-dimensional Hilbert space. There is a one-to-one correspondence between representations π on H, N-dimensional corepresentation matrices M and corepresentations δ of G on H. Fixing an orthonormal basis (e_i) of H, it can be established by the identifications*

$$m_{ij} \qquad \longleftrightarrow \qquad \langle e_i, \pi(\cdot) e_j \rangle$$

and

$$M = (m_{ij})_{1 \leq i,j \leq N} \qquad \longleftrightarrow \qquad \delta : v \mapsto M(v \otimes \mathbb{1}).$$

Proof. Throughout the proof let $e_i' \in H'$ be the i-th coordinate function on $H = \mathbb{C}^N$.
Step 1: From representations to corepresentation matrices: Starting with a representation π, it is obvious that $m_{ij} := \langle e_i, \pi(\cdot) e_j \rangle$ is an element of $C(G)$ and π can be reconstructed out of the matrix $M := (m_{ij})_{1 \leq i,j \leq N}$. The special form of the comultipication Δ on the m_{ij} follows from multiplicativity of π and the way matrix-matrix multiplication is performed: For $1 \leq i, j \leq N$ and group elements $g, h \in G$ it holds

$$\begin{aligned} \Delta(m_{ij})(g,h) = m_{ij}(gh) &= \langle e_i, \pi(gh) e_j \rangle \\ &= \langle e_i, \pi(g)\pi(h) e_j \rangle \\ &= \sum_{k=1}^{N} \langle e_i, \pi(g) e_k \rangle \langle e_k, \pi(h) e_j \rangle \\ &= \sum_{k=1}^{N} (m_{ik} \otimes m_{kj})(g,h). \end{aligned}$$

In order to prove invertibility of M, note that for every $g \in G$ the matrix $\pi(g)$ is unitary, so

$$m_{ij}(g^{-1}) = \langle e_i, \pi(g^{-1}) e_j \rangle = \langle \pi(g) e_i, e_j \rangle = \overline{\langle e_j, \pi(g) e_i \rangle} = m_{ji}^*(g).$$

As $\pi(e)$ must be the identity, we have for all $g \in G$

$$\begin{aligned}(M^*M)_{ij}(g) = \sum_{k=1}^{N} m_{ki}^*(g)m_{kj}(g) &= \sum_{k=1}^{N} \langle e_i, (\pi(g^{-1})e_k\rangle \langle e_k, \pi(g)e_j\rangle \\ &= \langle e_i, \pi(g^{-1})\pi(g)e_j, \rangle \\ &= \langle e_i, \pi(g^{-1}g)e_j\rangle \\ &= \delta_{ij}\mathbb{1}_{C(G)}(g),\end{aligned}$$

showing $M^*M = \mathbb{1}_{M_N(\mathbb{C})\otimes C(G)}$. Likewise one proves that M^* is also the right inverse of M.

Step 2: From corepresentation matrices to representations: Consider conversely a corepresentation matrix M. Define a map

$$\pi : G \to B(H) \; ; \; g \mapsto \mathrm{ev}_g(M),$$

where $\mathrm{ev}_g(M)$ is the evaluation of M in its second leg at the point $g \in G$. It is continuous as each m_{ij} is continuous. The special structure of Δ on the entries m_{ij} guarantees that π is a homomorphism: For each $1 \leq i, j \leq N$ and $g, h \in G$ it holds

$$\begin{aligned}\langle e_i, \pi(gh)e_j\rangle = m_{ij}(gh) = \Delta(m_{ij})(g,h) &= \sum_{k=1}^{N}(m_{ik} \otimes m_{kj})(g,h) \\ &= \sum_{k=1}^{N}\langle e_i, \pi(g)e_k\rangle\langle e_k, \pi(h)e_j\rangle \\ &= \langle e_i, \pi(g)\pi(h)e_j\rangle.\end{aligned}$$

It remains to show for all $g \in G$ that $\pi(g)$ is a unitary. As above we compute

$$\langle e_i, \pi(g)\pi(g)^*e_j\rangle = \Big(\sum_{k=1}^{N}(m_{ik}m_{jk}^*\Big)(g) = \delta_{ij},$$

proving the claim.

Step 3: From corepresentation matrices to corepresentations: Given a corepresentation matrix, the mapping

$$\delta : H \to H \otimes C(G) \; ; \; v \mapsto M(v \otimes \mathbb{1}_{C(G)})$$

is linear and satisfies

$$\begin{aligned}((\delta\otimes\mathbb{1}_{C(G)})\circ\delta)(e_j) &= (\delta\otimes\mathbb{1}_{C(G)})(\sum_{k=1}^{N} e_k\otimes m_{kj})\\ &= \sum_{i=1}^{N}\sum_{k=1}^{N} e_i\otimes m_{ik}\otimes m_{kj}\\ &= (\mathbb{1}_{B(H)}\otimes\Delta)(\sum_{i=1}^{N} e_i\otimes m_{ij})\\ &= ((\mathbb{1}_{B(H)}\otimes\Delta)\circ\delta)(e_j).\end{aligned}$$

The density condition for δ follows from the invertibility of M and the definition of δ:

$$\overline{\text{span}}\big(\delta(H)C(G)\big) = \overline{\text{span}}\big(\underbrace{M(H\otimes\mathbb{1}_{C(G)})C(G)}_{=M(H\otimes C(G))}\big) = \overline{\text{span}}\big(H\otimes C(G)\big) = H\otimes C(G)$$

For the unitarity of δ we compute

$$\langle\delta(e_i),\delta(e_j)\rangle = \sum_{k=1}^{N} m_{ki}^{*}m_{kj} = (M^{*}M)_{ij} = \delta_{ij}\mathbb{1}_{C(G)}.$$

Step 4: From corepresentations to corepresentation matrices: For the last part of the proof we start with a corepresentation δ and associate to it the matrix

$$M = (m_{ij})_{1\le i,j\le N} = \Big((e_i'\otimes\mathbb{1}_{C(G)})\delta(e_j)\Big)_{1\le i,j\le N}.$$

We first investigate the behaviour of Δ on the m_{ij}. Exploiting the relations between δ and Δ it holds for $1\le i,j\le N$:

$$\begin{aligned}\Delta(m_{ij}) &= \Delta\big((e_i'\otimes\mathbb{1}_{C(G)})\delta(e_j)\big)\\ &= (e_i'\otimes\mathbb{1}_{C(G)}\otimes\mathbb{1}_{C(G)})\big((\mathbb{1}_{B(H)}\otimes\Delta)\delta(e_j)\big)\\ &= (e_i'\otimes\mathbb{1}_{C(G)}\otimes\mathbb{1}_{C(G)})(\delta\otimes\mathbb{1}_{C(G)})\delta(e_j)\\ &= (e_i'\otimes\mathbb{1}_{C(G)}\otimes\mathbb{1}_{C(G)})(\delta\otimes\mathbb{1}_{C(G)})(\sum_{k=1}^{N} e_k\otimes m_{kj})\\ &= (e_i'\otimes\mathbb{1}_{C(G)}\otimes\mathbb{1}_{C(G)})(\sum_{k=1}^{N}\sum_{l=1}^{N} e_l\otimes m_{lk}\otimes m_{kj})\\ &= \sum_{k=1}^{N} m_{ik}\otimes m_{kj}.\end{aligned}$$

Using the unitarity of δ we directly prove that M^* is a left inverse of M: For all $1 \leq i, j \leq N$ it holds

$$\begin{aligned}(M^*M)_{ij} &= \sum_{k=1}^{N} m_{ki}^* m_{kj} \\ &= \langle \sum_{k=1}^{N} e_k \otimes m_{ki}, \sum_{l=1}^{N} e_l \otimes m_{lj} \rangle = \langle \delta(e_i), \delta(e_j) \rangle = \delta_{ij} \mathbb{1}_{C(G)}.\end{aligned}$$

To show invertibility of M we have to use the denseness condition of δ: Consider any state τ on $C(G)$ and the GNS-construction $(\pi_\tau, H_{\pi_\tau}, \Lambda_\tau)$ of $C(G)$ with respect to τ. Then $(\mathrm{id}_{B(H)} \otimes \pi_\tau)(M)$ is a linear and bounded operator on the Hilbert space $H \otimes H_{\pi_\tau}$. We know by assumption that $\delta(H)C(G) = M(H \otimes C(G))$ is dense in $H \otimes C(G)$, so $(\mathrm{id}_{B(H)} \otimes \pi_\tau)M$ also has dense image. As the universal representation (π, H_π) of $C(G)$ is given by the direct sum

$$\bigoplus_{\tau \text{ state}} (\pi_\tau, H_{\pi_\tau}),$$

also $(\mathrm{id}_{B(H)} \otimes \pi)(M)$ has dense image. Now $(\mathrm{id}_{B(H)} \otimes \pi)(M^*)$ is a left inverse of it, so $(\mathbb{1}_{B(H)} \otimes \pi)(M)$ is isometric and an isometry on a Hilbert space with dense image is invertible. As π is faithful also M is invertible and M^* must be its inverse. $\square$

Remark 2.1.9. Whenever considering one of the objects in Definition 2.1.7, we always have by Proposition 2.1.8 a triple (π, M, δ) of a representation, a corepresentation matrix and a corepresentation at hand. In particular, considering a corepresentation matrix M, vectors $w \in H$ and group elements $g \in G$, we have that $\delta(w)$ is the evaluation of M (in its first leg) at w and $\pi(g)$ is the evalutation of M (in its second leg) at g .

If we want to adapt the given definitions to the setting of compact quantum groups, we just have to note that part (2) and (3) of Definition 2.1.7 uses only $C(G)$ in its formulation and we proved the respective one-to-one correspondence in Proposition 2.1.8 without making use if the commutativity of $C(G)$. Hence, we can directly state the quantum version of the above:

Definition 2.1.10. Let G be a compact quantum group.

(1) A finite-dimensional, unitary corepresentation matrix of G is a unitary element

$$M = (m_{ij})_{1 \leq i,j \leq N} \in M_N\big(C(G)\big)$$

where $N \in \mathbb{N}$ and such that

$$\Delta(m_{ij}) = \sum_{k=1}^{N} m_{ik} \otimes m_{kj} \quad , \forall 1 \leq i, j \leq N.$$

(2) A finite-dimensional, unitary corepresentation of G is a linear map

$$\delta : H \to H \otimes C(G),$$

where H is a finite-dimensional Hilbert space and δ satisfies the following:

(i) $(\delta \otimes \mathbb{1}_{C(G)}) \circ \Delta = (\mathbb{1}_{B(H)} \otimes \Delta) \circ \delta$

(ii) $\langle \delta(v), \delta(w) \rangle = \langle v, w \rangle \quad \forall v, w \in H$

(iii) $\delta(H)C(G)$ is linearly dense in $H \otimes C(G)$.

The correspondence between the objects in (1) and (2) is one-to-one by the identification

$$m_{ij} \longleftrightarrow (e_i' \otimes \mathbb{1}_{C(G)})\delta(e_j),$$

where (e_i) is a fixed orthonormal basis of H.

As important as the definition of a corepresentation itself is the notion of an intertwiner (space).

Definition 2.1.11. Let G be a compact quantum group and δ_1, δ_2 be two (finite-dimensional, unitary) corepresentations of $C(G)$ on Hilbert spaces H_1 and H_2. An intertwiner from δ_1 to δ_2 is a linear map $T \in B(H_1, H_2)$ such that

$$(T \otimes \mathbb{1}_{C(G)})\delta_1 = \delta_2 T.$$

The vector space of all intertwiners from δ_1 to δ_2 is denoted by

$$\mathrm{Hom}(\delta_1, \delta_2)$$

and called the intertwiner space or Hom space from H_1 to H_2. We also write

$$\mathrm{Hom}(\delta_1) := \mathrm{Hom}(\delta_1, \delta_1)$$

and say that a linear map intertwines δ_1 if it is an element of $\mathrm{Hom}(\delta_1)$.

The corepresentation theory and the structure of intertwiner spaces of a compact quantum group are closely related. We list some observations (and definitions) that are important for this work. Most observations can be found in [Tim08, Section 5.2 and 5.3]:

Observation 2.1.12. Let G be a compact quantum group. Let δ_1, δ_1, $\tilde{\delta}_1, \tilde{\delta}_2$ be (finite-dimensional, unitary) corepresentations of $C(G)$ on Hilbert spaces H_1, H_2, $\tilde{H}_1$ and $\tilde{H}_2$, respectively. Let these four corepresentations correspond to the corepresentation matrices $M^{(1)}$, $M^{(2)}$, $\tilde{M}^{(1)}$ and $\tilde{M}^{(2)}$ of size N_1, N_2, $\tilde{N}_1$ and $\tilde{N}_2$.

(i) The direct orthogonal sum $\delta_1 \oplus \delta_2$ is a unitary corepresentation on the orthogonal sum $H_1 \oplus H_2$ defined by

$$(\delta_1 \oplus \delta_2)(v \oplus w) := \delta_1(v) \oplus \delta_2(w).$$

It corresponds to the direct sum of matrices

$$M^{(1)} \oplus M^{(2)} \in M_{N_1}(C(G)) \oplus M_{N_2}(C(G)) \subseteq M_{N_1+N_2}(C(G)).$$

Given two intertwiners $T_1 \in \operatorname{Hom}(\delta_1, \tilde{\delta}_1)$ and $T_2 \in \operatorname{Hom}(\delta_2, \tilde{\delta}_2)$, their direct sum $T_1 \oplus T_2$ is an intertwiner from $\delta_1 \oplus \delta_2$ to $\tilde{\delta}_1 \oplus \tilde{\delta}_2$.
Obviously, the orthogonal projections p_{H_1} and $p_{H_2} = \mathbb{1}_{B(H_1 \oplus H_2)} - p_{H_1}$ are elements in $\operatorname{Hom}(\delta_1 \oplus \delta_2)$.
Conversely, given a corepresentation δ on some H, each orthogonal projection p in $\operatorname{Hom}(\delta)$ decomposes δ into $\delta_p \oplus \delta_{1-p}$, the orthogonal direct sum of two corepresentations on Hilbert spaces pH and $(pH)^\perp$, respectively.

(ii) The tensor product $\delta_1 \otimes \delta_2$ is a unitary corepresentation on $H_1 \otimes H_2$ and defined by

$$(\delta_1 \otimes \delta_2)(v \otimes w) := m_{2\to 4}\big(\delta_1(v) \otimes \delta_2(w)\big)$$

where $m_{2\to 4}(v \otimes a \otimes w \otimes b) := v \otimes w \otimes ab$ collapses the two $C(G)$-legs and places them last.
The corepresentation $\delta_1 \otimes \delta_2$ is associated to the $\mathbin{\textcircled{\scriptsize T}}$-product of matrices defined by

$$M^{(1)} \mathbin{\textcircled{\scriptsize T}} M^{(2)} := \sum_{i,j=1}^{N_1} \sum_{k,l=1}^{N_2} E_{ij} \otimes E_{kl} \otimes m_{ij}^{(1)} m_{kl}^{(2)}, \tag{2.1.3}$$

where E_{ij} are the rank-one matrices $\big(\delta_{ik}\delta_{jl}\big)_{1 \leq k,l \leq N} \in M_N(\mathbb{C})$.
Given two intertwiners $T_1 \in \operatorname{Hom}(\delta_1, \tilde{\delta}_1)$ and $T_2 \in \operatorname{Hom}(\delta_2, \tilde{\delta}_2)$, their tensor product $T_1 \otimes T_2$ is an intertwiner from $\delta_1 \otimes \delta_2$ to $\tilde{\delta}_1 \otimes \tilde{\delta}_2$.

(iii) The conjugate $\delta_1^{(*)}$ of a corepresentation δ_1 is defined to be the map

$$\delta_1^{(*)} : \overline{H_1} \to \overline{H_1} \otimes C(G)\,;\, \overline{v} \mapsto (\overline{(\cdot)} \otimes (\cdot)^*)\delta(v)$$

on $\overline{H}_1$, the conjugate Hilbert space of H_1. Recall that, as a set, $\overline{H}_1$ is defined by

$$\overline{H}_1 := \{\overline{v} \mid v \in H_1\}$$

and the Hilbert space structure of $\overline{H}_1$ is given by the requirement that

$$\overline{(\cdot)} : H_1 \to \overline{H}_1\,;\, v \mapsto \overline{v}$$

is a conjugate-linear isomorphism, i.e.

$$\overline{\alpha}\cdot\overline{v} := \overline{\alpha v} \quad , \quad \overline{v}+\overline{w} = \overline{v+w} \quad \text{and} \quad \langle\overline{v},\overline{w}\rangle = \langle v, w\rangle.$$

It is a non-trivial fact (see [Tim08, Cor. 5.3.10]) that $\delta_1^{(*)}$ is again a corepresentation but possibly a non-unitary one, compare Definition 2.1.15. If δ_1 corresponds to the corepresentation matrix $M^{(1)}$ via some orthonormal basis (e_i) of H_1, then, with respect to the basis $(\overline{e}_i)$ of $\overline{H}_1$, the corepresentation $\delta_1^{(*)}$ corresponds to the matrix

$$\big(M^{(1)}\big)^{(*)} := \Big(\big(m_{ij}^{(1)}\big)^*\Big)_{1\leq i,j\leq N_1}.$$

Given an intertwiner $S \in \mathrm{Hom}(\delta_1, \delta_2)$, the conjugated intertwiner $\overline{S}$ defined by

$$\langle\overline{w}, \overline{S}\overline{v}\rangle := \overline{\langle w, Sv\rangle}$$

is an element in $\mathrm{Hom}(\delta_1^{(*)}, \delta_2^{(*)})$. Seen as an $N_2 \times N_1$-matrix, $\overline{S}$ is obtained out of S by complex conjugation of its entries, i.e. $\overline{S} = S^{(*)} = \big((S)^*\big)^T$.

(iv) Two corepresentations δ_1 and δ_2 are equivalent if they admit an invertible intertwiner T (from δ_1 to δ_2), i.e.

$$(T \otimes \mathbb{1}_{C(G)})\delta_1 T^{-1} = \delta_2.$$

Two corepresentation matrices $\tilde{M}^{(1)}$ and $\tilde{M}^{(2)}$ are equivalent, if they are equivalent via conjugation with an invertible $\tilde{T} \in M_N(\mathbb{C})$. i.e.

$$(\tilde{T} \otimes \mathbb{1}_{C(G)})\tilde{M}^{(1)}(\tilde{T}^{-1} \otimes \mathbb{1}_{C(G)}) = \tilde{M}^{(2)}.$$

(v) A corepresentation δ is irreducible if it is not the direct sum of two non-trivial corepresentations, i.e. if there is no non-trivial projection inside $\mathrm{Hom}(\delta)$.

(vi) If $T \in \mathrm{Hom}(\delta_1, \delta_2)$ and $S \in \mathrm{Hom}(\delta_2, \delta_3)$, then $S \circ T = ST$ obviously intertwines δ_1 and δ_3:

$$(ST \otimes \mathrm{id}_{C(G)})\delta_1 = (S \otimes \mathrm{id}_{C(G)}) \circ \delta_2 \circ T = \delta_3 \circ ST.$$

(vii) The unitarity of two corepresentations δ_1 and δ_2 implies the identity

$$\mathrm{Hom}(\delta_1, \delta_2)^* = \mathrm{Hom}(\delta_2, \delta_1\big).$$

This can easily be seen on the level of corepresentation matrices: If T is an intertwiner from $M^{(1)}$ to $M^{(2)}$, then it holds

$$\begin{aligned}(T^* \otimes \mathrm{id}_{C(G)})M^{(2)} &= M^{(1)}\left(M^{(1)}\right)^* (T^* \otimes \mathrm{id}_{C(G)})M^{(2)}\\ &= M^{(1)}\left((T \otimes \mathrm{id}_{C(G)})M^{(1)}\right)^* M^{(2)}\\ &= M^{(1)}\left(M^{(2)}(T \otimes \mathrm{id}_{C(G)})\right)^* M^{(2)}\\ &= M^{(1)}(T^* \otimes \mathrm{id}_{C(G)})\left(M^{(2)}\right)^* M^{(2)}\\ &= M^{(1)}(T^* \otimes \mathrm{id}_{C(G)}),\end{aligned}$$

so we have $\mathrm{Hom}(\delta_1, \delta_2)^* \subseteq \mathrm{Hom}(\delta_2, \delta_1)$. Switching roles shows equality.

Remark 2.1.13. Recall that the concrete form of the correspondence between (unitary, finite-dimensional) corepresentations and corepresentation matrices unfortunately required a fixed orthonormal basis for every considered Hilbert space H. This problem is solved by the notion of equivalence in point (v) of Observation 2.1.12, as now the following two results hold:

(i) Consider two corepresentations δ_1, δ_2 on Hilbert spaces H_1, H_2 that are equivalent via some invertible intertwiner T. Note that the situation $\delta_1 = \delta_2$ and $T = \mathbb{1}$ is allowed. Via two orthonormal bases, we can identify δ_1 and δ_2 with two corepresentation matrices and T corresponds to an (invertible) matrix V. In this situation the two constructed matrices are equivalent via conjugation with V.

(ii) Consider conversely two corepresentation matrices M_1, M_2 of size N that are equivalent via some (invertible) matrix V. Fixing two orthonormal bases of respective Hilbert spaces H_1 and H_2 of dimension N, we can identify these corepresentation matrices with two corepresentations and the matrix V with an invertible map T from H_1 to H_2. Note that the situation $M_1 = M_2$ and/or $H_1 = H_2$ is allowed as well as equality of the two orthonormal bases in the case $H_1 = H_2$. In this situation T intertwines the two constructed corepresentations, i.e they are equivalent.

Remark 2.1.14. Replacing corepresentations and corepresentation matrices by their equivalence classes, the one-to-one correspondence described in Definition 2.1.10 reaches a precise and canonical form: When constructing corepresentations out of corepresentation matrices or the other way around, the freedom of choice in the orthonormal bases indeed affects the constructed representative of an equivalence class but not the constructed class itself and the result does not depend on the chosen representative in the equivalence class we start with.

With the definitions above at hand we can now comment on the restriction of corepresentations to the finite-dimensional and unitary case. In the classical situation one can define a representation of a compact group even if it maps into the (not necessarily unitary) bounded operators on an arbitrary Hilbert space. If we require non-degeneracy, then the images still have to be invertible, i.e. the neutral element of the group has to be mapped to the identity.

Definition 2.1.15. Consider the situation as in Definition 2.1.7. Let H be a Hilbert space and $H \otimes C(G)$ the corresponding Hilbert $C(G)$-module (see Appendix). We call a linear map $\delta : H \to H \otimes C(G)$ a (possibly non-unitary and infinite-dimensional) corepresentation of $C(G)$ if the following holds:

(1) $(\delta \otimes \mathbb{1}_{C(G)}) \circ \delta = (\mathbb{1}_{B(H)} \otimes \Delta) \circ \delta$

(2) The linear map

$$\delta' : H \otimes C(G) \to H \otimes C(G) \ ; \ v \otimes a \mapsto \delta(v)a$$

is invertible.

Remark 2.1.16. (a) In [Tim08, Def. 5.2.3] the notion of corepresentation operators is introduced to describe the corepresentation theory of (C^*-algebraic) compact quantum groups. Definition 2.1.15 slightly differs from this notion, but they are equivalent due to property (2) above.

(b) In the finite-dimensional case, property (2) is exactly invertibility of the corepresentation matrix M_δ: so, in order to define arbitrary corepresentation matrices, we just replace in Definition 2.1.10 'unitarity' by 'invertibility'.

(c) Property (2) is what one could expect when trying to generalize the unitary case: In Definition 2.1.7 we have that property (ii) guarantees some kind of isometry, so boundedness from below, and property (iii) resembles image denseness.

(d) It is worth to note that Observation 2.1.12 can canonically be adapted to this generalized situation. Of course the statements about corepresentation matrices only make sense in the finite-dimensional case.

The following theorem justifies our restriction to the unitary and finite-dimensional case. The structure of an arbitrary (non-degenerated) corepresentation can be expressed by finite-dimensional unitary corepresentations, compare [Tim08, Thm. 5.3.3].

Theorem 2.1.17. *Let G be a compact quantum group.*

(1) Every corepresentation is equivalent to a unitary corepresentation.

(2) Every unitary corepresentation admits a finite-dimensional projective intertwiner. In particular, every irreducible corepresentation is finite-dimensional.

(3) Combining (1) and (2), every corepresentation is by Zorn's lemma equivalent to a direct sum of finite-dimensional, unitary corepresentations.

Remark 2.1.18. (i) Given an arbitrary corepresentation δ on some Hilbert space H, it is equivalent to a unitary corepresentation δ' on H' via some invertible intertwiner T. Having a decomposition $\oplus_{i\in I}\delta'_i$ of δ' means that δ' acts separately on Hilbert subspaces $(H'_i)_{i\in I}$ of H' and it holds $H' = \oplus_{i\in I} H'_i$ in the sense of an orthogonal sum.

(ii) Given the decomposition $\delta' = \oplus_{i\in I}\delta'_i$ as above, what happens if we go back to the decomposition of δ via conjugation with T? We obtain a (possibly not orthogonal) decomposition of H into direct summands TH'_i in the sense that each vector in H has a unique representation $(v_i)_{i\in I}$ via the requirements $v = \sum_{i\in I} v_i$ and $v_i \in T^{-1}H'_i$. Now $\delta_i := (T^{-1} \otimes \mathbb{1}_{C(G)})\delta'_i T$ is a corepresentation on $T^{-1}H'_i =: H_i$. We have $\delta = \oplus_{i\in I}\delta_i$ in the sense of a (possibly non-orthogonal) decomposition, i.e. for $v = (v_i)_{i\in I}$ it holds

$$\delta(v) = \delta\big((v_i)_{i\in I}\big) = \big(\delta_i(v_i)\big)_{i\in I}.$$

The maps

$$p_{H_i} : \ v = (v_i)_{i\in I} \mapsto v_i$$

are (possibly non-orthogonal) projections in $B(H)$ and intertwiners of δ.

When not mentioned otherwise, direct sums of corepresentations and decompositions are always assumed to be orthogonal.

The next proposition shows that the structure of intertwiner spaces becomes quite trivial in the irreducible case, compare also [Tim08, Prop. 5.3.4].

Proposition 2.1.19. *The intertwiner space of two irreducible corepresentations δ_1 and δ_2 is either trivial, i.e.* $\mathrm{Hom}(\delta_1, \delta_2) = \{0\}$, *or one-dimensional. If it is one-dimensional, then the two corepresentations are equivalent, so* $\mathrm{Hom}(\delta_1, \delta_2)$ *is spanned by an invertible operator. If both corepresentations are in addition unitary, then they are unitarily equivalent and* $\mathrm{Hom}(\delta_1, \delta_2)$ *is spanned by a unitary operator.*

Proof. We first consider the case of unitary corepresentations.
The unitary case: By items (vi) and (vii) in Observation 2.1.12, the operator space $\mathrm{Hom}(\delta_1)$ is a finite-dimensional von Neumann algebra, so it is linearly spanned by its projections. Hence, by irreducibility, $\mathrm{Hom}(\delta_1)$ only contains multiples of the identity operator.

Assume now that $\mathrm{Hom}(\delta_1, \delta_2)$ contains a nonzero T. Again by items (vi) and (vii) in Observation 2.1.12, the operator T^*T is a positive and non-zero intertwiner in $\mathrm{Hom}(\delta_1)$, so, by the observation above, T^*T is a multiple of the identity and T is a multiple of a partial isometry. Repeating the argument with T^* shows that T is the multiple of a unitary, so δ_1 and δ_2 are unitarily equivalent.
It remains to show that $\mathrm{Hom}(\delta_1, \delta_2)$ is one-dimensional. Considering a second intertwiner T' in $\mathrm{Hom}(\delta_1, \delta_2)$, we have linear dependence of T^*T and T^*T' as $\mathrm{Hom}(\delta_1)$ is one-dimensional, so multiplying with T from the left shows linear dependence of T and T'.
The general case: If δ_1 and δ_2 are not necessarily unitary, then by item (1) in Theorem 2.1.17 they are equivalent to unitary corepresentations δ_1' and δ_2', respectively, via invertible intertwiners $S_1 \in \mathrm{Hom}(\delta_1, \delta_1')$ and $S_2 \in \mathrm{Hom}(\delta_2, \delta_2')$. Now it holds

$$Hom(\delta_1, \delta_2) = S_2^{-1} \mathrm{Hom}(\delta_1', \delta_2') S_1$$

and the results about unitary corepresentations carry over to the general situation. Evidently, the unitarity of an intertwiner T might not survive the mapping $T \mapsto S_2^{-1} T S_1$. ☐

Remark 2.1.20. Note that a unitary corepresentation δ_1 and a non-unitary corepresentation δ_2 (on Hilbert spaces H_1 and H_2) cannot admit a unitary intertwiner $U \in \mathrm{Hom}(\delta_1, \delta_2)$. Using the unitarity of δ_1, we would conclude for all $v, w \in H_2$

$$\begin{aligned}\langle \delta_2(v), \delta_2(w) \rangle &= \langle (U \otimes \mathrm{id}_{C(G)}) \circ \delta_1 \circ U^{-1})(v), (U \otimes \mathrm{id}_{C(G)}) \circ \delta_1 \circ U^{-1})(w) \rangle \\ &= \langle \big(\delta_1 \circ U^{-1}\big)(v), \big(\delta_1 \circ U^{-1}\big)(w) \rangle \\ &= \langle U^{-1}(v), U^{-1}(w) \rangle \mathbb{1}_{C(G)} \\ &= \langle v, w \rangle \mathbb{1}_{C(G)},\end{aligned}$$

i.e. δ_2 was a unitary corepresentation, a contradiction.

By Theorem 2.1.17, finite dimensional, unitary corepresentation are the central building blocks of the whole corepresentation theory of compact quantum groups. In the following we show that they even encode (nearly) the whole structure of the quantum group itself, see Theorem 2.1.26 as well as Theorems 2.1.31 and 2.1.29.

Observation 2.1.21. Given a corepresentation matrix, we know by its definition, how Δ acts on its entries and by the constructions above (tensor product and conjugation) also all *-products appear as entries in corepresentation matrices. Consequently, the linear span of all these entries is a *-algebra inside $C(G)$.

We will see that the collection of all such entries is linearly dense in $C(G)$ (see Theorem 2.1.26). To prove this, we need some definitions and preparatory results. As a first step, we describe the linear span of matrix entries m_{ij} of a corepresentation matrix in a way more suitable for the general (infinite-dimensional) case.

Definition 2.1.22. Given a corepresentation operator δ on a Hilbert space H, the vector space $\mathcal{C}(\delta)$ is defined as

$$\mathcal{C}(\delta) := \overline{\operatorname{span}}\big(\big\{(f \otimes \mathbb{1}_{C(G)})(\delta(v)) \mid f \in H', v \in H\big\}\big)$$

In the finite-dimensional case, $\mathcal{C}(\delta)$ is the linear span of entries m_{ij} in the corresponding corepresentation matrix M_δ.
When it comes to the irreducible case, the subspaces $\mathcal{C}(\delta)$ have nice geometric properties with respect to the Haar state, compare also [Tim08, Prop. 5.3.4 (ii)]:

Lemma 2.1.23. *Let G be a compact quantum group.*

(1) If corepresentations δ_1 and δ_2 of G are equivalent, then $\mathcal{C}(\delta_1) = \mathcal{C}(\delta_2)$.

(2) If $\bigoplus_{i\in I} \delta_i$ is equivalent to some δ, then the subspaces $\mathcal{C}(\delta_i)$ span a dense subspace of $\mathcal{C}(\delta)$. If only finitely many δ_i are non-zero, then equality holds. In particular, this is the case if δ is finite-dimensional.

Proof. Item (1) follows directly from the fact that two equivalent corepresentations on respective Hilbert spaces H_1 and H_2 are connected via an invertible intertwiner $T \in \operatorname{Hom}(\delta_1, \delta_2)$: Given $f \in H_1'$ and $v \in H_1$, it holds

$$(f \otimes \operatorname{id} C(G))\big(\delta_1(v)\big) = \big((f \circ T^{-1}) \otimes \operatorname{id}_{C(G)}\big)\big(\delta_2(Tv)\big),$$

proving $\mathcal{C}(\delta_1) \subseteq \mathcal{C}(\delta_2)$. Switching roles shows equality.
For item (2) we can assume by (1) that $\oplus_{i\in I}\delta_i$ is a decomposition of δ. By definition, each δ_i corresponds to an orthogonal projection $p_i \in B(H)$ and the decomposition of δ corresponds to a decomposition of H into pairwise orthogonal Hilbert subspaces p_iH, each of them invariant under δ. Now, for every $f \in H'$ and $v \in H$, it holds

$$(f \otimes \operatorname{id}_{C(G)})\big(\delta(v)\big) = \lim_{J \subseteq I \text{ finite}} \underbrace{\sum_{j\in J} (f|_{p_jH} \otimes \operatorname{id}_{C(G)})\big(\delta_j(p_jv)\big)}_{\in \sum_{j\in J} \mathcal{C}(\delta_j)}, \tag{2.1.4}$$

so the linear span of all $\mathcal{C}(\delta_i)$ is dense in $\mathcal{C}(\delta)$. If I is finite, then we can omit in Equation 2.1.4 the net-limit and directly consider the summation over all $i \in I$. If δ is finite-dimensional then only finitely many δ_i can be non-zero and again the summation above is essentially finite. ☐

Theorem 2.1.24 (Schur's orthogonality, see [Tim08, Prop. 5.3.7 and 5.3.8])**.** *Let $(H_{\pi_h}, \pi_h, \Lambda_h)$ be the GNS-construction of $C(G)$ with respect to its Haar state h and let δ_1 and δ_2 be two irreducible corepresentations of G on Hilbert spaces H_1 and H_2, respectively.*

(1) If δ_1 and δ_2 are not equivalent, then it holds $\Lambda_h\big(\mathcal{C}(\delta_1)\big) \perp \Lambda_h\big(\mathcal{C}(\delta_2)\big)$, i.e

$$h(a^*b) = 0 \quad \forall a \in \mathcal{C}(\delta_1), b \in \mathcal{C}(\delta_2).$$

(2) If δ_1 is N-dimensional, then the N^2 entries of a fixed associated corepresentation matrix, and so of every associated corepresentation matrix, are linearly independent.

Proof. Due to Lemma 2.1.23, we can consider δ_1 and δ_2 to be unitary. Assume for item (1) that $h\big(\big(m_{ij}^{(1)}\big)^* m_{ij}^{(2)}\big) \neq 0$ holds for some entries $m_{ij}^{(1)}$ and $m_{ij}^{(2)}$ in the respective corepresentation matrices M_{δ_1} and M_{δ_2}. It can be proved that this guarantees a non-zero intertwiner $S \in \operatorname{Hom}(\delta_2, \delta_1)$, compare the proof of [Tim08, Prop. 3.2.6]. By Proposition 2.1.19 this contradicts the fact that δ_1 and δ_2 are not equivalent. For statement (2) see for example [Tim08, Prop. 5.3.8 (iv)]. □

As a last ingredient, we need to define the so-called regular corepresentation of a compact quantum group, see also [Tim08, Thm. 5.2.9].

Definition and Lemma 2.1.25. Let G be a compact quantum group, h its Haar state and $(\pi_h, H_{\pi_h}, \Lambda_h)$ the GNS-construction of $C(G)$ with respect to h.

(1) The map

$$\delta_h : H_{\pi_h} \to H_{\pi_h} \otimes C(G) \; ; \; \Lambda_h(a) \mapsto \big((\Lambda_h \otimes \mathrm{id}_{C(G)})(\Delta(a))\big) \quad , \forall a \in C(G)$$

defines a unitary corepresentation, the so-called *regular corepresentation of G*.

(2) The linear subspace $\mathcal{C}(\delta_h)$ is dense in $C(G)$.

Proof. Well-definedness and unitarity of δ_h follow from the computation

$$\begin{aligned} \big\langle \delta_h\big(\Lambda_h(a)\big), \delta_h\big(\Lambda_h(b)\big)\big\rangle &= (h \otimes \mathrm{id}_{C(G)})\big(\Delta(a^*b)\big) \\ &= h(a^*b)\mathbb{1}_{C(G)} \\ &= \langle \Lambda_h(a), \Lambda_h(b)\rangle \mathbb{1}_{C(G)}. \end{aligned} \tag{2.1.5}$$

The denseness condition on δ_h is a consequence of the denseness condition on Δ:

$$\begin{aligned} \overline{\operatorname{span}}\big(\delta_h(H_{\pi_h})C(G)\big) &= \overline{\operatorname{span}}\Big(\big(\Lambda_h \otimes \mathrm{id}_{C(G)}\big)\Big(\Delta\big(C(G)\big)\big(\mathbb{1}_{C(G)} \otimes C(G)\big)\Big)\Big) \\ &= \overline{\operatorname{span}}\Big(\big(\Lambda_h \otimes \mathrm{id}_{C(G)}\big)\big(C(G) \otimes C(G)\big)\Big) \\ &= H_{\pi_h} \otimes C(G) \end{aligned}$$

The relation

$$(\delta_h \otimes \mathbb{1}_{C(G)}) \circ \delta_h = (\mathbb{1}_{B(H)} \otimes \Delta) \circ \delta_h$$

is nothing but the coassociativity of the comultiplication Δ: For all $a \in C(G)$ it holds

$$\begin{aligned}\big((\delta_h \otimes \mathrm{id}_{C(G)}) \circ \delta_h\big)(\Lambda_h(a)) &= \big((\Lambda_h \otimes \mathrm{id}_{C(G)} \otimes \mathrm{id}_{C(G)}) \circ (\Delta \otimes \mathrm{id}_{C(G)}) \circ \Delta\big)(a) \\ &= (\Lambda_h \otimes \mathrm{id}_{C(G)} \otimes \mathrm{id}_{C(G)}) \circ (\mathrm{id}_{C(G)} \otimes \Delta) \circ \Delta\big)(a) \\ &= \big((\mathrm{id}_{B(H_h)} \otimes \Delta) \circ \delta_h\big)(\Lambda_h(a)).\end{aligned}$$

Finally, we have to compute $\mathcal{C}(\delta_h)$. Let $a \in C(G)$ be given. Because of the denseness conditions on Δ we find finite sums

$$s_n := \sum_k (x_k^{(n)} \otimes \mathbb{1}_{C(G)})\Delta(y_k^{(n)})$$

that approximate $\mathbb{1}_{C(G)} \otimes a$ for $n \to \infty$, so $(h \otimes \mathrm{id}_{C(G)})(s_n)$ approximates a. As h can be considered also an element of H'_{π_h} via $h\big(\Lambda_h(b)\big) := h(b)$, we can write

$$\begin{aligned}(h \otimes \mathrm{id}_{C(G)})(s_n) &= (h \otimes \mathrm{id}_{C(G)})\Big(\sum_k (x_k^{(n)} \otimes \mathbb{1}_{C(G)})\Delta(y_k^{(n)})\Big) \\ &= \sum_k \Big(\big(h \circ \pi_h(x_k^{(n)})\big) \otimes \mathrm{id}_{C(G)}\Big)\delta\big(\Lambda_h(y_k^{(n)})\big).\end{aligned}$$

The compositions $h \circ \big(\pi_h(x_k^{(n)})\big)$ are elements in H'_{π_h}, therefore we proved with the last equation that a is in the closure of $\mathcal{C}(\delta_h)$ and $\mathcal{C}(\delta_h)$ is dense in $C(G)$, as claimed. □

The next theorem is an adaption of [Tim08, Thm. 5.3.11].

Theorem 2.1.26. *Let G be a compact quantum group and $(\delta_\lambda)_{\lambda\in\Lambda}$ be a maximal family of pairwise non-equivalent, irreducible, unitary corepresentations. Let $(M_\lambda)_{\lambda\in\Lambda}$ be the family of their respective corepresentation matrices.*

(1) The entries of the matrices M_λ together form a linear generating system of a unital *-algebra $\mathcal{A}$, so in particular $\mathcal{A} = \sum_\lambda \mathcal{C}(\delta_\lambda)$.

(2) $\mathcal{A}$ is dense in $C(G)$.

(3) $\mathcal{A}$ does not depend on the choice of the family $(\delta_\lambda)_{\lambda\in\Lambda}$.

Proof. To prove (1), note that if elements $m_{ij}^{(1)}$ and $m_{kl}^{(2)}$ are entries in the corepresentation matrices M_{λ_1} and M_{λ_2}, respectively, then $\big(m_{ij}^{(1)}\big)^*$ is an entry in the corepresentation matrix $\overline{M_{\lambda_1}}$ and $m_{ij}^{(1)} m_{kl}^{(2)}$ is an entry in $M_{\lambda_1} \mathbin{\bigcirc\hspace{-0.75em}\top} M_{\lambda_2}$. If one of these was not in $\mathcal{A}$, then, by Lemma 2.1.23 (2), there would be an irreducible summand of $\overline{M_\lambda}$ or $M_{\lambda_1} \mathbin{\bigcirc\hspace{-0.75em}\top} M_{\lambda_2}$, respectively, with some entry not in $\mathcal{A}$. Item (1) of Lemma 2.1.23 says that this summand cannot be equivalent to any δ_λ, a contradiction to

the maximality of the family $(\delta_\lambda)_{\lambda\in\Lambda}$.
$\mathcal{A}$ is unital as $(\mathbb{1}_{C(G)}) \in M_1(\mathbb{C}(G))$ is a corepresentation matrix of G.
To prove (2), we use the regular corepresentation δ_h of $C(G)$ from Definition and Lemma 2.1.25. By Theorem 2.1.17, it can be decomposed into irreducible corepresentations:

$$\delta_h = \bigoplus_{i\in I} \hat{\delta}_i$$

Combining Lemma 2.1.23 (2) and Definition and Lemma 2.1.25 (2), we conclude that the linear span of all the $\mathcal{C}(\hat{\delta}_i)$ is dense in $C(G)$.
Now consider the *-algebra $\mathcal{A}$ as defined in the theorem. If its closure was not $C(G)$, then we would find in its (open) complement an element $a \in \sum_{j\in J} \mathcal{C}(\hat{\delta}_j)$ for a suitable finite set $J \subseteq I$. Then at least one irreducible summand $\hat{\delta}_j$ is not equivalent to all δ_λ, as otherwise $a \in \mathcal{A}$. But this non-equivalence contradicts the maximality of the family $(\delta_\lambda)_{\lambda\in\Lambda}$, as we could add $\hat{\delta}_j$ to it.
A variant of this argument shows (3): Consider additionally another maximal family $(\delta_{\lambda'})_{\lambda'\in\Lambda'}$ of pairwise non-equivalent, irreducible, unitary corepresentations. Assume $\mathcal{A}' \neq \mathcal{A}$ and without constraints $\mathcal{A}' \not\subseteq \mathcal{A}$. If every δ'_λ was equivalent to some δ_λ, we would have again by Lemma 2.1.23 (1) that $\mathcal{A}' \subseteq \mathcal{A}$. But having conversely some $\delta_{\lambda'}$ that is not equivalent to any δ_λ contradicts again the maximality of $(\delta_\lambda)_{\lambda\in\Lambda}$. Consequently, our initial assumption, $\mathcal{A}' \neq \mathcal{A}$, was false. □

2.1.3 The connection to algebraic compact quantum groups

In the last section we saw that finite-dimensional unitary corepresentations of a compact quantum group G carry (nearly) the whole structure of the C^*-algebra $C(G)$: The (unique) *-algebra $\mathcal{A}$ from Theorem 2.1.26 is dense in $C(G)$ and encodes the behaviour of the comultiplication Δ on $\mathcal{A}$, and therefore on $C(G)$. We will show that $\mathcal{A}$ is a so-called *algebraic compact quantum group* and exploit this information to gather further results on $\mathcal{A}$ and $C(G)$.
The results of this section can be found in [Tim08, Sec. 5.4].

Definition 2.1.27. An *algebraic compact quantum group* $\mathcal{G}$ is given by a Hopf *-algebra $(\mathcal{A}, \Delta)$ such that there exists a Haar state on $\mathcal{A}$, i.e. a left- and right-invariant positive linear functional h with $h(\mathbb{1}_\mathcal{A}) = 1$.

Analogous to the C^*-algebraic case, we write $\mathcal{G} = (\mathcal{A}, \Delta)$ for an algebraic CQG.

Remark 2.1.28. The representation theory of algebraic CQGs can be developed analogously to that of C^*-algebraic CQGs, see [Tim08, Chpt. 3]. Note that all appearing tensor products have to be replaced by algebraic tensor products.

Theorem 2.1.29 (see [Tim08, Thm. 5.4.1 (i)])**.** *Let $G = (A, \Delta)$ be a C^*-algebraic compact quantum group. Then the dense unital *-algebra $\mathcal{A}$ from Theorem 2.1.26* *together with the restriction $\Delta_{\mathcal{A}}$ of Δ to $\mathcal{A}$ defines an algebraic compact quantum group $\mathcal{G} = (\mathcal{A}, \Delta_{\mathcal{A}})$.*

Proof. We sketch the main ingredients for the proof. Consider a maximal family (δ_k) of pairwise non-equivalent, irreducible, unitary corepresentations of $C(G)$, so $\mathcal{A}$ is the linear span of entries in the corepresentation matrices $M_{\delta_k} = (m_{ij}^{(k)})$. By definition, Δ maps matrix entries of M_{δ_i} again to elements in $\mathcal{C}(\delta_k) \odot \mathcal{C}(\delta_k)$, so Δ restricts to a unital *-homomorphism $\Delta_{\mathcal{A}} : \mathcal{A} \to \mathcal{A} \odot \mathcal{A}$. One can check, see [Tim08, Thm. 5.4.1], that

$$\varepsilon(m_{ij}^{(k)}) := \delta_{i,j} \qquad , \qquad S(m_{ij}^{(k)}) := \big(m_{ij}^{(k)}\big)^*$$

defines a counit and an antipode on $\mathcal{A}$. As $\Delta_{\mathcal{A}}$ is just the restriction of Δ to $\mathcal{A}$, the restriction $h|_{\mathcal{A}}$ is still a Haar state. □

In virtue of the result above, we speak of $\mathcal{G}$ as an algebraic CQG that lies dense in G. We prove next that there is no other algebraic CQG with this property, compare [Tim08, Thm. 5.4.1 (ii)].

Proposition 2.1.30. *Let $G = (A, \Delta)$ be a C^*-algebraic compact quantum group. Let $\mathcal{A}'$ be a dense *-subalgebra of A that defines an algebraic compact quantum group $\mathcal{G}' = (\mathcal{A}', \Delta_{\mathcal{A}'})$ such that $\Delta_{\mathcal{A}'}$ is given by restricting Δ. Then $\mathcal{G}'$ must be equal to $\mathcal{G} = (\mathcal{A}, \Delta_{\mathcal{A}})$ as defined in Theorem 2.1.29.*

Proof. It suffices to show that $\mathcal{A} = \mathcal{A}'$ as sets. Analogous to Theorem 2.1.26, one can show in the algebraic setting that the space generated by entries of unitary corepresentation matrices of $\mathcal{G}'$ is the whole $\mathcal{A}'$, see [Tim08, Thm. 3.2.13 (iii)]. As every corepresentation matrix of $\mathcal{G}'$ is a corepresentation matrix of G and $\mathcal{A}$ is generated by the entries of those matrices by definition, we have $\mathcal{A}' \subseteq \mathcal{A}$.
Let H_{π_h} be the Hilbert space in the triple $\big(\pi_h, H_{\pi_h}, \Lambda_h\big)$ obtained by the GNS-construction on $C(G)$ with respect to the Haar state h.
Assume $\mathcal{A}' \subsetneq \mathcal{A}$. Then there exists some irreducible corepresentation δ of $C(G)$ such that $\mathcal{C}(\delta) \not\subseteq \mathcal{A}'$. By Schur's orthogonality relations (Theorem 2.1.24) it holds $\Lambda_h\big(\mathcal{C}(\delta)\big) \perp \Lambda_h\big(\mathcal{A}'\big)$, so $\Lambda_h\big(\mathcal{A}'\big)$ is not dense in $\Lambda_h\big(C(G)\big)$. But then $\mathcal{A}'$ is not dense in $C(G)$, a contradiction. □

If we conversely start with an algebraic compact quantum group $\mathcal{G} = (\mathcal{A}, \Delta_{\mathcal{A}})$, we can always construct an enveloping C^*-algebra A associated to an enveloping C^*-algebraic compact quantum group G, compare [Tim08, Thm. 5.4.3]:

Theorem 2.1.31. *Let $\mathcal{G} = (\mathcal{A}, \Delta_{\mathcal{A}})$ be an algebraic compact quantum group. Then the map*

$$\|\cdot\|_u \ : \ a \mapsto \sup\{\, s(a) \mid s(\cdot) \text{ is a } C^*\text{-seminorm on } \mathcal{A}\}$$

is a norm on $\mathcal{A}$. If A_u denotes the completion of $\mathcal{A}$ with respect to $\|\cdot\|_u$, then $\Delta_{\mathcal{A}}$ extends to a comultiplication Δ_{A_u} on A_u and (A_u, Δ_u) is a C^*-algebraic compact quantum group.

Proof. For a non-zero $a \in \mathcal{A}$ the supremum in the definition of $\|\cdot\|_u$ is greater than zero because the (unique) Haar state on an algebraic compact quantum group is faithful, see in [Tim08] Corollary 2.2.5, Definition. 2.2.8 and the definition of algebraic CQGs on page 65. So the GNS-construction $(\pi_h, H_{\pi_h}, \Lambda_h)$ with respects to the Haar state on $\mathcal{A}$ defines a C^*-norm $\|\pi_h(\cdot)\|_{B(H_{\pi_h})}$ on $\mathcal{A}$.
Note that C^*-seminorms are bounded by 1 on entries of unitary corepresentation matrices of $\mathcal{G}$. As $\mathcal{A}$ is linearly generated by such matrix-entries, we have that C^*-seminorms are pointwise bounded on the whole $\mathcal{A}$. This proves that $\|\cdot\|_u$ as described in the theorem is well-defined.
We remark further that we obtain from the norm $\|\cdot\|_{A_u \otimes A_u}$ a C^*-seminorm $\|\Delta_{\mathcal{A}}(\cdot)\|_{A_u \otimes A_u}$ on $\mathcal{A}$, so, after establishing the norm $\|\cdot\|_u$ on $\mathcal{A}$, we see that $\Delta_{\mathcal{A}}$ is norm-decreasing. This guarantees that $\Delta_{\mathcal{A}}$ can be extended to a *-homomorphism $\Delta_{A_u} : A_u \to A_u \otimes A_u$. Finally, it can be proven – compare [Tim08, Lemma 1.3.21]) – that it holds

$$\Delta_{\mathcal{A}}(\mathcal{A})(\mathbb{1}_{\mathcal{A}} \odot \mathcal{A}) = \mathcal{A} \odot \mathcal{A} = \Delta_{\mathcal{A}}(\mathcal{A})(\mathcal{A} \odot \mathbb{1}_{\mathcal{A}}),$$

so the required denseness conditions on Δ_{A_u} are fulfilled. □

Remark 2.1.32. After the last two theorems we can now comment on the question why the finite-dimensional corepresentations of a compact quantum group G only *nearly* encode the structure of the C^*-algebra $C(G)$: Starting with a C^*-algebraic compact quantum group $G = (A, \Delta)$, we can first apply the GNS-construction with respect to the Haar state to A and one can show that this gives again a pair $(A_{red}, \Delta_{A_{red}})$ as described in the definition of a compact matrix quantum group, the so-called *reduced form of G*. As the Haar state commutes with the canonical map $A \to A_{red}$, the dense algebraic compact quantum group inside is the same. Summarizing, the difference between (A, Δ) and (A_{red}, Δ_{red}) vanishes when considering the respective dense algebraic CQG inside.
Starting with an algebraic compact quantum group $\mathcal{G} = (\mathcal{A}, \Delta_{\mathcal{A}})$, we can conversely say that there might be more than one C^*-norm on $\mathcal{A}$ such that its completion A is a C^*-algebraic compact quantum group that extends $\mathcal{G}$. While we chose in Theorem 2.1.31 the maximal C^*-norm on $\mathcal{A}$, $\|\cdot\|_u$, resulting in some kind of universal object (A_u, Δ_{A_u}), the GNS-construction with respect to the Haar-state on $\mathcal{A}$, for example, gives another norm $\|\cdot\|_{red}$ on $\mathcal{A}$ and, after completion, a potentially different pair $(A_{red}, \Delta_{A_{red}})$.

Definition 2.1.33. Let $G = (A, \Delta)$ be a C^*-algebraic compact quantum group and $\mathcal{G} \subseteq G$ be the unique algebraic compact quantum group whose associated *-algebra $\mathcal{A}$ is a dense subset of A and whose comultiplication is given by restricting Δ to $\mathcal{A}$.

Let $\|\cdot\|_u$ be the universal C^*-norm on $\mathcal{A}$ and $\|\cdot\|_{red}$ be the C^*-norm on $\mathcal{A}$ obtained from the GNS-construction with respect to the Haar state $h_{\mathcal{A}}$. Let (A_u, Δ_u) and (A_{red}, Δ_{red}) be the pairs obtained from $(\mathcal{A}, \Delta)$ by completion of A with respect to $\|\cdot\|_u$ and $\|\cdot\|_{red}$, respectively, and extension of Δ. Then we call (A_u, Δ_u) the *universal form of* G and (A_{red}, Δ_{red}) the *reduced form of* G. We also write $A_u =: C_u(G)$ and $A_{red} =: C_{red}(G)$. We say that G is given in its universal form if $A, \Delta)$ is equal to (A_u, Δ_u). Likewise, G is given in its reduced form if (A, Δ) is equal to (A_{red}, δ_{red}).

Remark 2.1.34. Obviously, A_{red} is obtained from A by applying the GNS-construction with respect to the Haar state h_A and it holds $A_{red} = A$ if and only if h_A is faithful.

Remark 2.1.35. Considering the three pairs (A, Δ), (A_u, Δ_{A_u}) and $(A_{red}, \Delta_{A_{red}})$, all C^*-algebras A ,A_u and A_{red} might be different, but we point out again that they are all associated to the same C^*-algebraic compact quantum group. In this sense these pairs are just different ways to describe one and the same quantum group via a C^*-algebra (and a comultiplication on it).

The following proposition justifies the labels *universal* and *reduced*, compare [Tim08, Prop. 5.4.8].

Proposition 2.1.36. *Let $G = (A, \Delta)$ be a C^*-algebraic compact quantum group and consider its universal and reduced form as described in Definition 2.1.33. Then there are surjective *-homomorphisms $A_u \to A \to A_{red}$ which intertwine the respective comultiplications.*

Proof. Denote again with $\mathcal{G} = (\mathcal{A}, \Delta_{\mathcal{A}})$ the algebraic compact quantum group as defined in Definition 2.1.33. Recall that $\mathcal{A}$ is a dense *-algebra of all three associated C^*-algebras A, A_{red} and A_u. Note further that these three C^*-algebras are given by the completion of $\mathcal{A}$ with respect to $\|\cdot\|_A$, $\|\cdot\|_{A_{red}}$ and $\|\cdot\|_{A_u}$, respectively and their comultiplications are extensions of $\Delta_{\mathcal{A}}$.
In virtue of Definition 2.1.3 we can prove the proposition by showing $\|\cdot\|_{A_{red}} \leq \|\cdot\|_A \leq \|\cdot\|_{A_u}$ on $\mathcal{A}$, as this guarantees the existence of the *-homomorphisms in the statement.
$\|\cdot\|_u$ is maximal amongst all possible C^*-norms on $\mathcal{A}$ by definition; hence, we have a *-homomorphism from A_u to A, as claimed.
Concerning $\|\cdot\|_{red}$, we observe that the GNS-construction of A with respects to its Haar state h gives a (norm decreasing) *-homomorphism π_h on A. Now $h_{\mathcal{A}}$ is the restriction of h to $\mathcal{A}$, so restricting π_h to $\mathcal{A}$ coincides with $\pi_{h_{\mathcal{A}}}$. This shows $\|\cdot\|_{red} \leq \|\cdot\|_A$ on $\mathcal{A}$ and π_h coincides with the desired *-homomorphism from A to A_{red}. □

The precise statement about the connection between C^*-algebraic compact quantum groups and their irreducible corepresentations can now be presented.

Corollary 2.1.37. *(1) The pair $\big(\mathcal{A}, \Delta_{\mathcal{A}}\big)$, associated to an algebraic compact quantum group $\mathcal{G}$ is uniquely defined by the family of irreducible unitary corepresentations of $\mathcal{G}$.*

(2) Given an algebraic compact quantum group $\mathcal{G} = \big(\mathcal{A}, \Delta_{\mathcal{A}}\big)$, there exist unique universal and reduced versions $(A_{red}, \Delta_{A_{red}})$ and (A_u, Δ_{A_u}), respectively, of a unique enveloping C^*-algebraic compact quantum groups G.

(3) There is a one-to-one correspondence between universal C^*-algebraic quantum groups, reduced C^*-algebraic compact quantum groups and algebraic compact quantum groups given by identification of their respective families of irreducible, unitary corepresentations.

2.2 Compact matrix quantum groups

In this section we define compact matrix quantum groups (in the sense of S. L. Woronowicz, see [Wor87]) and present two ways to motivate them as special compact quantum groups.

Definition 2.2.1. Let A be a unital C^*-algebra, $N \in \mathbb{N}$ and $u_G = (u_{ij}) \in M_N(A)$ an $N \times N$-matrix over A. Assume that the following holds:

(i) The entries of u generate A as a C^*-algebra.

(ii) The matrices u and $u^{(*)} = \big(u_{ij}^*\big)$ are invertible.

(iii) There is a unital *-homomorphism $\Delta : A \to A \otimes A$ (called comultiplication on A) that fulfils

$$\Delta(u_{ij}) = \sum_{k=1}^{N} u_{ik} \otimes u_{kj} \qquad \forall 1 \leq i, j \leq N.$$

Then we denote A also by $C(G)$ and call it the *non-commutative functions over a compact matrix quantum group G of size N.*

We often use the acronym ‘CMQG’ for ‘compact matrix quantum group’. As done before, we usually do not distinguish between the quantum group and the function algebra over it and write in this virtue $G = (A, u_G) = (C(G), u_G)$.

Proposition 2.2.2 (compare [Tim08, Prop. 6.1.4])**.** *Every compact matrix quantum group is a compact quantum group.*

Proof. We need to proof that the map Δ fulfils the density conditions described in Definition 2.1.1. Consider first the set $\Delta(A)(A \otimes \mathbb{1}_{C(G)})$ and a product

$$p := u^{\varepsilon_1}_{i_1 j_1} \cdots u^{\varepsilon_K}_{i_K j_K}.$$

Here $K \in \mathbb{N}$, $1 \leq i_1, j_1, \cdots, i_K, j_K \leq N$ and $\varepsilon_1, \ldots, \varepsilon_K \in \{-1, 1\}$. We identify the exponent $\varepsilon = -1$ with the symbol * (not to be mistaken for the inverse operation on $C(G)$). If $\mathbb{1}_{C(G)} \otimes p$ is an element in the linear span of $\Delta(A)(A \otimes \mathbb{1}_{C(G)})$, we are done. Denote $u^{(1)} := u$ and let $v^{(\varepsilon)}$ be the inverse of $u^{(\varepsilon)}$. Then it holds

$$\begin{aligned}
&\sum_{l_1, \ldots, l_K = 1}^{N} \Delta(u^{(\varepsilon_1)}_{l_1 j_1} \cdots u^{(\varepsilon_K)}_{l_K j_K})\big((v^{(-\varepsilon_K)}_{i_K l_K})^* \cdots (v^{(-\varepsilon_1)}_{i_1 l_1})^* \otimes \mathbb{1}_{C(G)}\big) \\
&= \sum_{s_1, \ldots, s_K = 1}^{N} \left(\sum_{l_1 = 1}^{N} u^{(\varepsilon_1)}_{l_1 s_1} \left(\cdots \left(\sum_{l_K = 1}^{N} u^{(\varepsilon_K)}_{l_K s_K} (v^{(-\varepsilon_K)}_{i_K l_K})^* \right) \cdots \right) (v^{(-\varepsilon_1)}_{i_1 l_1})^* \right) \otimes u^{(\varepsilon_1)}_{s_1 j_1} \cdots u^{(\varepsilon_K)}_{s_K j_K} \\
&= \sum_{s_1, \ldots, s_K = 1}^{N} \left(\sum_{l_1 = 1}^{N} u^{(\varepsilon_1)}_{l_1 s_1} \left(\cdots \underbrace{\left(\sum_{l_K = 1}^{N} (v^{(-\varepsilon_K)}_{i_K l_K}) u^{(-\varepsilon_K)}_{l_K s_K} \right)}_{= \delta_{i_K s_K}}^{*} \cdots \right) (v^{(-\varepsilon_1)}_{i_1 l_1})^* \right) \otimes u^{(\varepsilon_1)}_{s_1 j_1} \cdots u^{(\varepsilon_K)}_{s_K j_K} \\
&= \mathbb{1}_{C(G)} \otimes u^{\varepsilon_1}_{i_1 j_1} \cdots u^{\varepsilon_1}_{i_K j_K}.
\end{aligned}$$

Similarly, one proves the second denseness condition. □

Remark 2.2.3. There are two perspectives on compact matrix quantum groups as special cases of compact quantum groups.

- In the same way as we started with compact groups and constructed their quantum analoga, we could have started with compact groups of matrices and we would have ended up with Definition 2.2.1: Given a compact matrix group G, the coordinate maps $u_{ij} : M \mapsto m_{ij}$ generate by Stone-Weierstrass the continuous ($\mathbb{C}$-valued) functions over G. The comultiplication $\Delta(f)(M^{(1)}, M^{(2)}) := f(M^{(1)} M^{(2)})$ is given by the formula in Definition 2.2.1 (iii) and the inverses $v^{(\varepsilon)}$ of the matrices $u^{(\varepsilon)}$ are given, as in the case of compact groups, by composition with the inverse map on G, i.e.

  $$v^{(\varepsilon)}_{ij}(M) := u^{\varepsilon}_{ij}(M^{-1}).$$

- Comparing the definition of CMQGs with the Definition of a corepresentation matrix in Definition 2.1.7, we see that CMQGs are exactly those compact quantum groups G such that there exists a corepresentation matrix whose entries generate the whole $C(G)$. In other words: The whole representation theory of a CMQG is already determined by one corepresentation matrix u, every irreducible corepresentation matrix is equivalent to a direct summand in a $\textcircled{\top}$-product of u's and $u^{(*)}$'s. Note that $u^{\textcircled{\top} 0}$ is the trivial corepresentation matrix $(1) \in M_1(C(G)) = C(G)$.

The first perspective allows a definition of CMQGs without embedding it into the theory of compact quantum groups. On the other hand, only the second perspective shows that CMQGs are more than just special cases of CQGs. They are the building blocks of CQGs, in the sense that from every corepresentation matrix u of a compact quantum group G we obtain a compact matrix quantum group $A, u)$ with $A \subseteq C(G)$. By Theorem 2.1.26 and 2.1.29, all these CMQGs together encode the structure of the dense algebraic CQG inside G, hence the structure of G itself (modulo different forms).

Notation 2.2.4. Due to its important rôle in the corepresentation theory of a CMQG, the matrix u_G in Definition 2.2.1 is called the fundamental corepresentation matrix of G.

Note that a CMQG is defined by a pair (A, u_G) and not only by a C^*-algebra A that allows the definition of a suitable matrix u. This is important when comparing CMQGs:

Definition 2.2.5. Let $G = (C(G), u_G)$ and $G' = (C(G'), u_{G'})$ be compact matrix quantum groups in their universal forms. We say that G' is equivalent to G and write $G' \simeq G$ if there is an invertible complex-valued matrix T (of suitable size) and a*-isomorphism

$$\varphi : C(G) \to C(G')$$

mapping the matrix entries of u_G canonically to the entries of $(T \otimes \mathbb{1})u_{G'}(T^{-1} \otimes \mathbb{1})$. The compact matrix quantum group G' is a subgroup of G, written $G' \subseteq G$, if φ is a *-homomorphism.

Note that, in order to be invertible, T has to be a square matrix and existence of φ in particular requires the matrices u and u' to have the same size.

Remark 2.2.6. (a) In [Wor87], Woronowicz distinguishes in the situation of equivalent quantum groups the situation of $T = 1$ (*identical* CMQGs) and $T \neq 1$ (*similar* CMQGs). In this virtue we can speak of a *strong* and a *weak* notion of equivalence and of a *strong* and a *weak* subgroup relation. For the rest of this thesis, subgroup relation and equivalence are, if not specified further, considered in their weak forms, so as described in Definition 2.2.5.

(b) In Definition 2.2.5, weak equivalence of two CMQGs is exactly what one needs in order to guarantee that their corepresentation theory (with fixed fundamental corepresentation) is the same, i.e. that the associated essential concrete monoidal W^*-categories (and distinguished objects) coincide.

(c) Considering the situation of Definition 2.2.5 and the relation $G' \subseteq G$ in the weak sense, we can observe the following:

  (i) The matrices $u_{G'}$ and $Tu_{G'}T^{-1}$ are both corepresentation matrices of A'. In particular, T is an intertwiner from $u_{G'}$ to $Tu_{G'}T^{-1}$.

  (ii) The compact matrix quantum groups $(A', u_{G'})$ and $(A', Tu_{G'}T^{-1})$ are (weakly) equivalent.

  (iii) It holds $(A', Tu_{G'}T^{-1}) \subseteq (A, u)$ in the strong sense.

(d) By Theorem 2.1.17, item (1), the notion of (weak) equivalence allows us to concentrate on CMQGs $G = (A, u)$ with unitary fundamental corepresentation matrices u.

Definition 2.2.5 makes it possible for one C^*-algebra to correspond to different compact matrix quantum groups.
Note further that a map $\varphi : C(G) \to C(G')$ as in Definition 2.2.5 automatically respects the comultiplication,

$$\Delta_{A'} \circ \varphi = (\varphi \otimes \varphi) \circ \Delta_A,$$

so equivalent CMQGs are also equivalent as CQGs and the subgroup relation in the category of CMQGs implies the subgroup relation in the corresponding subcategory of CQGs.

2.3 Tannaka-Krein duality for compact matrix quantum groups

We already saw in Observation 2.1.12 that there are connections between the structure of (finite-dimensional) corepresentations and the structure of intertwiner maps between them: Many operations on the level of corepresentations correspond to operations on the level of intertwiners. The aim of this section is to show that no information is lost when switching from the family of corepresentations to the family of intertwiners. The precise formulation will be Theorem 2.3.25, the Tannaka-Krein duality for compact matrix quantum groups as proved by S. L. Woronowicz in [Wor88].
Throughout this section we follow the brilliant ideas presented in [Wor88] but, as a slight modification, we consider so-called *essential* versions of concrete monoidal W^*-categories, compare item (vi) in Definition 2.3.6. This exactly reflects the idea of identifying corepresentations if they are unitarily equivalent, compare Observation 2.3.1, (vii). See also Remark 2.3.7, where this difference is commented.

2.3.1 Corepresentation theory for compact matrix quantum groups

We start with a collection of observations with respect to corepresentations and corresponding intertwiner spaces. Most of them can be found in [Wor88, Chpt. 1]. In parts these have already been stated in Observation 2.1.12 and we omit the corresponding proofs.

Observation 2.3.1. Consider the family $(\delta_i)_{i\in I}$ of finite dimensional unitary corepresentations of a compact quantum group G. Let H_i be the finite-dimensional Hilbert space the corepresentation δ_i lives on. Define $\mathrm{Hom}(i,j) := \mathrm{Hom}(\delta_i, \delta_j) \subseteq B(H_i, H_j)$ and write $\delta_{ij} := \delta_i \otimes \delta_j$. Then it holds:

(i) Every $\mathrm{Hom}(i,i)$ contains the identity $\mathbb{1}_{B(H_i)}$.

(i) If $T \in \mathrm{Hom}(i,j)$ and $S \in \mathrm{Hom}(j,k)$, then $S \circ T \in \mathrm{Hom}(i,k)$.

(iii) If $T \in \mathrm{Hom}(i,j)$ then $T^* \in \mathrm{Hom}(j,i)$.

(iv) The spaces $\mathrm{Hom}(i,j)$ are weakly closed. This observation is trivial, as we only consider finite-dimensional corepresentations.

(v) The tensor product of corepresentations is associative, $\delta_{(ij)k} = \delta_{i(jk)}$, i.e.

$$(\delta_i \otimes \delta_j) \otimes \delta_k = \delta_i \otimes (\delta_j \otimes \delta_k).$$

(vi) The trivial corepresentation

$$\mathbb{1} : \mathbb{C} \to \mathbb{C} \otimes C(G) \; ; \; \alpha \mapsto \alpha \otimes \mathbb{1}_{C(G)}$$

is of course an object in the family $(\delta_i)_{i\in I}$. Without restriction we can assume that the symbol 1 is contained in the index family I and that δ_1 is the trivial corepresentation.

(vii) We can establish an equivalence relation of unitary corepresentations with respect to unitary equivalence (see Remark 2.3.2): Two corepresentations are unitarily equivalent if their intertwiner space admits a unitary element.
Passing from actual (unitary, finite-dimensional) corepresentations to their equivalence classes guarantees the following property:
If there is a unitary $U \in \mathrm{Hom}(i,j)$, then i is equal to j.

(viii) If $T \in \mathrm{Hom}(i,j)$ and $S \in \mathrm{Hom}(k,l)$, then $T \otimes S \in \mathrm{Hom}(ik, jl)$.

Remark 2.3.2. Note that passing to equivalence classes replaces each δ_i by a collection $\big((U\otimes\mathrm{id})\circ\delta_i\circ U^{-1}\big)_{U\in\mathcal{U}_{H_i\to}}$ on respective Hilbert spaces UH_i, where $\mathcal{U}_{H_i\to}$ is the collection of all unitary operators from H_i to other Hilbert spaces. Likewise, every Hilbert space H_i is replaced by the collection $(UH_i)_{U\in\mathcal{U}_{H\to}}$ and an intertwiner $T\in\mathrm{Hom}(i,j)$ is replaced by $\big(VTU^{-1}\big)_{U\in\mathcal{U}_{H_i\to},V\in\mathcal{U}_{H_j\to}}$. The statements in Observation 2.3.1 have to be interpreted pointwise in this setting:
Each $(U\otimes\mathrm{id})\delta_iU^{-1}$ is a unitary corepresentation on UH_i. The statement $T\in\mathrm{Hom}(i,j)$ now reads that for all $U\in\mathcal{U}_{H_i\to}$ and $V\in\mathcal{U}_{H_j\to}$ we have

$$VTU^{-1}\in\mathrm{Hom}\,\big(U\otimes\mathrm{id})\circ\delta_i\circ U^{-1},(V\otimes\mathrm{id})\circ\delta_j\circ V^{-1}\big).$$

Likewise for $T\in\mathrm{Hom}(\delta_i,\delta_j)$ and $S\in\mathrm{Hom}(H_k,H_l)$ the former statement

$$T\otimes S\in\mathrm{Hom}(\delta_i\otimes\delta_k,\delta_j\otimes\delta_l)$$

now says that for all $U_i\in\mathcal{U}_{H_i\to}$, $U_j\in\mathcal{U}_{H_j\to}$, $U_k\in\mathcal{U}_{H_k\to}$ and $U_l\in\mathcal{U}_{H_l\to}$ we have that

$$U_jTU_i^{-1}\otimes U_lSU_k^{-1}$$

is an element in

$$\mathrm{Hom}\,\Big((U_i\otimes\mathrm{id})\circ\delta_i\circ U_i^{-1}\otimes(U_k\otimes\mathrm{id})\circ\delta_k\circ U_k^{-1}\;,\;(U_j\otimes\mathrm{id})\circ\delta_j\circ U_j^{-1}\otimes(U_l\otimes\mathrm{id})\circ\delta_l\circ U_l^{-1}\Big).$$

Notation 2.3.3. For the sake of readability we keep even in the situation of equivalence classes the notations 'δ_i', 'H_i', 'T' and '$T\in\mathrm{Hom}(i,j)$'.

Given a unitary corepresentation or corepresentation matrix, its conjugate does not have to be unitary again, but at least it is equivalent to a unitary one, see Theorem 2.1.17. This motivates the following observation, see also [Wor88, p. 39].

Observation 2.3.4. Consider the situation of Observation 2.3.1 and a unitary corepresentation δ_k on a Hilbert space H_k of dimension N_k. Let $\delta_{\overline{k}}$ be a fixed unitary corepresentation equivalent to $\delta_k^{(*)}$. Note that we can assume $\delta_{\overline{k}}$ to live on $\overline{H_k}$.
On the level of corepresentation matrices it is easy to see that the unitarity of δ_k is equivalent to the fact that

$$\tau:\mathbb{C}\to H_k\otimes\overline{H_k}\ ;\ 1\mapsto\sum_{i=1}^{N}e_i\otimes\overline{e_i}$$

and

$$\overline{\tau}:\overline{H_k}\otimes H_k\to\mathbb{C}\ ;\ \overline{v}\otimes w\mapsto\langle v,w\rangle$$

fulfill $\tau \in \mathrm{Hom}(\mathbb{1}, \delta_k \otimes \delta_k^{(*)})$ and $\overline{\tau} \in \mathrm{Hom}(\delta_k^{(*)} \otimes \delta_k, \mathbb{1})$.
Let $J : \overline{H_k} \to \overline{H_k}$ be the invertible intertwiner connecting $\delta_k^{(*)}$ and $\delta_{\overline{k}}$, i.e.

$$(J^{-1} \otimes \mathrm{id}_{C(G)}) \circ \delta_{\overline{k}} \circ J = \delta_k^{(*)}.$$

Then we have by items (ii) and (iii) in Observation 2.3.1 that

$$t := (\mathrm{id}_{H_k} \otimes J) \circ \tau \; ; \; 1 \mapsto \sum_{i=1}^{N_f} e_i \otimes J\big(\overline{e_i}\big)$$

is an element in the intertwiner space $\mathrm{Hom}(\mathbb{1}, \delta_k \otimes \delta_{\overline{k}})$.
Likewise it holds that

$$\overline{t} := \overline{\tau} \circ (J^{-1} \otimes \mathrm{id}_{H_k}) : \overline{H_k} \otimes H_k \to \mathbb{C} \; ; \; \overline{v} \otimes w \mapsto \big\langle \overline{w}, J^{-1}(\overline{v}) \big\rangle$$

is an intertwiner in $\mathrm{Hom}(\delta_{\overline{k}} \otimes \delta_k, \mathbb{1})$.

Observation 2.3.5. In the situation of a compact matrix quantum group $G = (C(G), u_G)$ with a unitary u_G we only need the corepresentations δ_f and $\delta_{\overline{f}}$ associated to u_G and $\overline{u_G}$ to construct all unitary corepresentations of G: Given an arbitrary finite-dimensional unitary corepresentation δ_i, we find finitely many corepresentations $r_1, \ldots, r_k$ such that the following holds:

- Each r_k is a $\bigcirc\!\!\!\!\top$-product of δ_f's and $\delta_{\overline{f}}$'s.
- We find decompositions $H_i = H_{i_1} \oplus \ldots \oplus H_{i_K}$ and $\delta_i = \delta_{i_1} \oplus \ldots \oplus \delta_{i_K}$ such that δ_{i_k} lives on H_{i_k} and is equivalent to a direct summand in r_k.

The intertwiners $b_k \in \mathrm{Hom}(\delta_i, r_k)$, establishing the equivalences between the δ_{i_k}'s and summands of r_k's, fulfil $\sum_k b_k b_k^* = \mathrm{id}_{H_i}$.

2.3.2 Categories associated to intertwiner spaces

While in the observations of Section 2.3.1 the Hilbert spaces H_i and the linear spaces $\mathrm{Hom}(i, j)$ of maps between these spaces originated in the corepresentation theory of a considered compact (matrix) quantum group, we define now such a collection in its own right. Note again, that the supplement *essential* is added in comparison to S. L. Woronowicz's work [Wor88].

Definition 2.3.6. An essential concrete monoidal W^*-category is a triple

$$R = \Big(I \; , \; \big(H_i\big)_{i\in I} \; , \; \big(\mathrm{Mor}(i,j)\big)_{i,j\in I}\Big)$$

where I is a collection of index objects, each H_i is a finite-dimensional Hilbert space and each $\mathrm{Mor}(i, j)$ is a linear subset of $B(H_i, H_j)$, called morphisms from i to j, such that the following holds:

(i) For all index objects i from I we have $\mathrm{id}_{H_i} \in \mathrm{Mor}(i,i)$.

(ii) If $T \in \mathrm{Mor}(i,j)$ and $S \in \mathrm{Mor}(j,k)$ then $S \circ T \in \mathrm{Mor}(i,k)$.

(iii) If $T \in \mathrm{Mor}(i,j)$ then $T^* \in \mathrm{Mor}(j,i)$.

(iv) Each $\mathrm{Mor}(i,j)$ is weakly closed.

(v) There is an associative semi-group structure on the collection I: For all index objects i,j,k from I it holds $(ij)k = i(jk) \in I$.

(vi) If $\mathrm{Mor}(i,j)$ contains a unitary element, then $i = j$.

(vii) There is an object 1 in the collection I such that $1i = i1 = i$ for all i from I.

(viii) If $T \in \mathrm{Mor}(i,j)$ and $S \in \mathrm{Mor}(k,l)$ then $T \otimes S \in \mathrm{Mor}(ik, jl)$. In particular $H_{ik} = H_i \otimes H_k$ and $H_{jl} = H_j \otimes H_l$.

Remark 2.3.7. (a) Definition 2.3.6 is due to S. L. Woronowicz in [Wor88], but we added property (vi) and the prefix 'essential'. It reflects identification of objects that are unitarily equivalent. See also (b) below for further reasons to consider equivalence classes. In [Wor88] this property is weakened to absence of duplicates in the collection I.

(b) The properties in Definition 2.3.6 are motivated by Observations 2.3.1. Properties (i) and (ii) are due to the definition of a category and the label 'concrete' comes from the fact that each object $i \in I$ is associated to a concrete Hilbert space H_i. Property (iii) makes it a *-category and (v)-(vii) establish a monoidal structure on I which is compatible with the tensor product structure on the Hilbert spaces and morphisms by (viii). Note that (vi) is not just an assumption to reduce redundant information, but, strictly speaking, it is necessary to establish the monoidal structure on I: Considering again the situation of corepresentations, we see that $\mathbb{1} \otimes \delta_i$ is not equal to δ_i but just equivalent to it by the identification $\mathbb{C} \otimes H_i = H_i$. In the same virtue the neutral element in the monoid would not be unique, as without establishing unitary equivalence, there is more than one Hilbert space of dimension 1.
Property (iv) is always fulfilled because only finite-dimensional Hilbert spaces are considered. The condition on $\mathrm{Mor}(i,j)$ to be weakly closed and the name 'W^*-category' are just relics from a more general setting where infinite-dimensional Hilbert spaces are considered, see [Wor88, p. 39].

Remark 2.3.8. Note that every H_i should actually be a collection of isomorphic Hilbert spaces $(UH_i)_{U \in \mathcal{U}_{H_i \to}}$ and every $\mathrm{Mor}(i,j)$ is a collection of operator spaces

$(U \circ \mathrm{Mor}(i,j) \circ V^{-1})_{U \in \mathcal{U}_{H_{j\to}}, V \in \mathcal{U}_{H_{i\to}}}$, compare Remark 2.3.2. For the sake of readability we already omitted this fact in the definition above and we formulate our statements in the sequel by considering representatives. Note that we could conversely always take a category as above and, with the help of the collections $\big(\mathcal{U}_{H_i\to}\big)_{i\in I}$, enlarge every H_i and $\mathrm{Mor}(i,j)$ as described in Remark 2.3.2.

Remark 2.3.9. We speak of equivalent categories if we obtain one category from the other by relabelling the objects I (in a bijective way). This is the standard notion of equivalence for categories.

We turn now towards the abstract versions of Observation 2.3.4 and 2.3.5 in form of definitions in the setting of an essential concrete monoidal W^*-category.

Definition 2.3.10. Let k be an object from I in the essential concrete monoidal W^*-category R. If it exists, the conjugate $\overline{k}$ of k is the object in I with $H_{\overline{k}} = \overline{H_k}$ and such that there exists an invertible J in $B\big(\overline{H_k}\big)$ such that

$$t : 1 \mapsto \sum_i e_i \otimes J(\overline{e_i}) \qquad ; \qquad \bar{t} : \overline{v} \otimes w \mapsto \langle \overline{w}, J^{-1}(\overline{v}) \rangle$$

are elements in $\mathrm{Mor}(1, k\overline{k})$ and $\mathrm{Mor}(\overline{k}k, 1)$, respectively.

Remark 2.3.11. It is not completely trivial to check that a conjugate is unique in the essential setting. Given a second conjugate $\overline{k}^{'}$ of k it can be shown, see [Wor88, p. 39], that $\mathrm{Mor}(\overline{k}, \overline{k}^{'})$ contains an invertible intertwiner T but it takes some effort to show that $\mathrm{Mor}(\overline{k}, \overline{k}^{'})$ contains a unitary: We can decompose $\mathrm{id}_{H_{\overline{k}}}$ in $\mathrm{Mor}(\overline{k}, \overline{k})$ into a sum of irreducible orthogonal projections $p_1 + \ldots, p_L$. Conjugation with T then gives a corresponding decomposition $\mathrm{id}_{H_{\overline{k}^{'}}} = q_1 + \ldots + q_L$ and each $Tp_l = q_l T p_l$ is an element in $\mathrm{Mor}(\overline{k}, \overline{k}^{'})$. As in the proof of Proposition 2.1.19 one can show that the $q_l T p_l$ are multiples of partial isometries, as otherwise the decomposition of $\mathrm{id}_{H_{\overline{k}}}$ from above could be refined further. We conclude that the sum

$$\sum_{l=1}^{L} \frac{q_l T p_l}{\| q_l T p_l \|}$$

is a unitary in $\mathrm{Mor}(\overline{k}, \overline{k}^{'})$.

The next definition gives a quite technical impression but is motivated by Observation 2.3.5, saying that a proper subset of all corepresentations suffices in order to construct all of them by building tensor-products, sub-corepresentations and direct sums.

Definition 2.3.12. Let $R = \big(I, (H_i)_{i\in I}, \big(\operatorname{Mor}(i,j)\big)_{i,j\in I}\big)$ be an essential concrete monoidal W^*-category. A subfamily I' of I is called a generating family for R if for every $i \in I$ the following holds: There exist (finitely many) $b_k \in \operatorname{Mor}(i, r_k)$ such that $\sum_k b_k b_k^* = \operatorname{id}_{H_i}$ and each r_k is a (finite) product of objects in I'.

There is also a notion of completion in [Wor88, p. 38] for categories as above, but it leaves the unitary case. Again we propose a similar definition adapted to our setting.

Definition 2.3.13. An essential concrete monoidal W^*-category

$$R = \big(I, (H_i)_{i\in I}, \big(\operatorname{Mor}(i,j)\big)_{i,j\in I}\big)$$

is called complete if the following holds:

(i) R is closed under taking subobjects: For every orthogonal projection p in some $\operatorname{Mor}(i,i)$ there is an $i_p \in I$ such that $H_{i_p} = pH_i$ and the embedding $\iota_p : H_{i_p} \to H_i$ is an element in $\operatorname{Mor}(i_p, i)$.

(ii) R is closed under taking direct sums: For every two objects j and k in I there is some l in I such that $H_l = H_j \oplus H_k$ and the embeddings $\iota_j : H_j \to H_l$ and $\iota_k : H_k \to H_l$ are elements in $\operatorname{Mor}(j,l)$ and $\operatorname{Mor}(k,l)$ respectively. Note that if $j = l$ then this says in particular that there are two different embeddings of H_j into H_l.

Remark 2.3.14. (a) Considering in condition (i) the projection $p = 0$, we obtain the zero element $0 \in I$ associated to the Hilbert space $H_0 = \{0\}$. For every $i \in I$ we have $\operatorname{Mor}(0,i) = B(\{0\}, H_i)$ and $\operatorname{Mor}(i,0) = B(H_i, \{0\})$ and it holds $i \oplus 0 = i$.

(b) In the situation of part (ii), the map $\iota_j \iota_j^* =: p$ is an orthogonal projection in $\operatorname{Mor}(l,l)$ and defining l_p and ι_p as in part (i) it holds $l_p = j$ and $\iota_p = \iota_j$. In other words, i and j are subobjects of $i \oplus j$.

Remark 2.3.15. (a) Note that in part (i) of Definition 2.3.13 the requirement $\iota_p \in \operatorname{Mor}(i_p, i)$ completely defines the morphisms from or to i_p in terms of morphisms from or to i: Let i, i' be two elements in I and p, p' be two projections in $\operatorname{Mor}(i,i)$ and $\operatorname{Mor}(i',i')$, respectively. Using the notation as above we easily show

$$\operatorname{Mor}(i_p, i_{p'}) = (\iota_{p'})^* \circ \operatorname{Mor}(i, i') \circ \iota_p.$$

Obviously, every map on the right side is an element in $\operatorname{Mor}(i_p, i_{p'})$ and every $T \in \operatorname{Mor}(i_p, i_{p'})$ can be written as

$$T = (\iota_{p'})^* \circ \Big(\iota_{p'} \circ T \circ (\iota_p)^*\Big) \circ \iota_p$$

where $\iota_{p'} \circ T \circ (\iota_p)^* \in \operatorname{Mor}(i, i')$.

(b) Analogously, we have that in part (ii) of Definition 2.3.13 the requirements $\iota_j \in \mathrm{Mor}(j,l)$ and $\iota_k \in \mathrm{Mor}(k,l)$ completely determine the spaces of morphisms to or from l in terms of morphisms from or to j and k. Considering $l = j \oplus k$ and $l' = j' \oplus k'$ and using the notation as above we have

$$\mathrm{Mor}(l,l') = \mathrm{span}\left(\left\{\iota_{b'} \circ Mor(a,b') \circ (\iota_a)^* \ \mid a,b \in \{j,k\}\right\}\right).$$

The justification is as in (a): Evidently, the maps on the right are morphisms from l to l' and for $T \in \mathrm{Mor}(l,l')$ it holds

$$T = \sum_{a,b \in \{i,j\}} \iota_{b'} \circ \left(\iota_{b'}^* \circ T \circ \iota_a\right) \circ (\iota_a)^*$$

Note that in a complete category every $l' \in I$ can be written $0 \oplus l'$, so it is not a restriction for l' to be a direct sum of two objects.

The following two lemmata are preparatory results for Proposition 2.3.18.

Lemma 2.3.16. *Consider an essential concrete monoidal W^*-category R and $l \in I$. Then a decomposition of* $\mathrm{id}_{H_l} \in \mathrm{Mor}(l,l)$ *into minimal, pairwise orthogonal projections* $\mathrm{id}_{h_l} = p_1 + \ldots + p_N$ *is unique up to equivalence, i.e. up to conjugation with a unitary* $U \in \mathrm{Mor}(l,l)$.

Proof. The proof is similar to its analogue in the situation of corepresentations which has been sketched in Remark 2.3.11:
Consider two decompositions $p_1 + \ldots + p_N = \mathrm{id}_{H_l} = q_1 + \ldots + q_M$ into minimal, pairwise orthogonal projections in $\mathrm{Mor}(l,l)$. As in the situation of corepresentations, one can show that for any n,m the space $p_n \mathrm{Mor}(l,l) q_m$ is either trivial or one dimensional and contains a unitary: If it contains a non-zero element T, then T^*T is a non-zero element in $q_m \mathrm{Mor}(l,l) q_m$. If T^*T is not a multiple of q_m then T^*T would have two eigenvalues $0 \leq \lambda_1 < \lambda_2$ and

$$\lim_{n\to\infty} \left(\frac{T^*T}{\|T^*T\|}\right)^n$$

is a non-trivial projection in the von Neumann algebra $q_m \mathrm{Mor}(l,l) q_m$, a contradiction.
So by scaling it with a suitable factor, T is a partial isometry with cokernel $q_m H_l$. Analogously, we find that T^* is a partial isometry with cokernel $p_n H_l$, i.e. T is a unitary in $p_n \mathrm{Mor}(l,l) q_m \subseteq B(q_m H_l, p_n H_l)$. Now consider q_1 and because

$$\sum_{1 \leq n \leq N} p_n q_1 = q_1 \neq 0$$

we can without restriction assume p_1q_1 to be non-zero. hence, $T_1 := \frac{p_1q_1}{\|p_1q_1\|}$ is a partial isometry with cokernel q_1H_l and image p_1H_l. The observations above in particular tell us that $\operatorname{rank}(p_1) = \operatorname{rank}(q_1)$ and $\ker(q_1) \cap p_1H_l$ is trivial.
Consider next the projection q_2 and assume $\sum_{2 \leq n \leq N} p_nq_2 = 0$. Then $p_1q_2 = q_2$ and so $q_2H_l \subseteq p_1H_l$. On the other side $\ker(q_1)$ and p_1H_l intersect trivially, so $p_1q_2 \neq 0$ implies $q_1p_1q_2 \neq 0$ and we deduce

$$0 \neq q_1p_1q_2 = q_1(p_1q_2) = q_1q_2 = 0,$$

a contradiction. So the assumption was false and it holds $\sum_{2 \leq n \leq N} p_nq_2 \neq 0$. Now we can do the construction above a second time and without restriction we find an isometry from q_2H_l to p_2H_l.
Repeating this arguments until q_N we observe two things: Firstly, having $N \neq M$ would contradict that both the p_n and the q_m sum up to the identity on H_l. Secondly, the sum $U := T_1 + \ldots T_N$ is a unitary that fulfils $U^{-1}p_nU = q_n$ for all $1 \leq n \leq N$. In this sense the two decompositions of id_{H_l} into minimal orthogonal projections are equivalent, i.e. they are equal up to conjugation with a unitary $U \in \operatorname{Mor}(l, l)$. ☐

Lemma 2.3.17. *Let R be an essential concrete monoidal W^*-category and assume an object $l \in I$ to fulfil the properties desribed in part (i) of Definition 2.3.13, i.e. R contains all subobjects of l. If p, q are arbitrary projections in $\operatorname{Mor}(l, l)$, then q is unitarily equivalent to a projection q' which commutes with p, i.e. q' can be written as a direct sum $r + s$, where $r \leq p$ and $s \leq \mathbb{1} - p$.*

Proof. Define $H_i := pH_l$ and $H_j := (\mathrm{id}_{H_l} - p)H_l$, so we can write $H_l = H_i \oplus H_j$. We have to find a projection equivalent to q which respects the decomposition $\mathrm{id}_{H_l} = p + (\mathrm{id}_{H_l} - p)$.
We can refine the decompositions $\mathrm{id}_{H_l} = q + (\mathrm{id}_{H_l} - q)$ and $\mathrm{id}_{H_l} = p + (\mathrm{id}_{H_l} - p)$ into decompositions

$$q_1 + \ldots + q_N = \mathrm{id}_{H_l} = p_1 + \ldots + p_N$$

consisting of minimal projections. By Lemma 2.3.16 we find a unitary U connecting those decompositions, i.e. without restriction $U^{-1}p_nU = q_n$ for all $1 \leq n \leq N$. In particular, we can write $U^{-1}qU = r + s$ with $r \leq p$ and $s \leq \mathrm{id}_{H_l} - p$ as $p_1 + \ldots + p_N$ is a refinement of $p + (\mathrm{id}_{H_l} - p)$. ☐

The following statement is the essential version of [Wor88, prop 2.7], but we chose to present an own (and more detailed) proof.

Proposition 2.3.18. *Every essential concrete monoidal W^*-category $R = \big(I, (H_i)_{i \in I}, \big(\operatorname{Mor}(i, j)\big)_{i,j \in I}\big)$ has a completion, i.e. there exists a smallest complete essential concrete monoidal W^*-category*

$$\widehat{R} = \big(\widehat{I}, (\widehat{H}_{\widehat{i}})_{\widehat{i} \in \widehat{I}}, \big(\widehat{\operatorname{Mor}}(\widehat{i}, \widehat{j})\big)_{\widehat{i}, \widehat{j} \in \widehat{I}}\big)$$

such that R is a subcategory of it.

Note that R is a subcategory of $\widehat{R}$ if all objects in I are objects in $\widehat{I}$ and for all $i, j \in I$ it holds $\widehat{H}_i = H_i$ and $\widehat{\mathrm{Mor}}(i,j) = \mathrm{Mor}(i,j)$.

Proof. The proof of the statement is lengthy although not difficult in each of its steps. The idea of building subcorepresentations and direct sums of corepresentations perfectly guides us through the construction in this abstract setting.
It is always a trivial task to add the zero object to R if it is not already there, see part (i) of Definition 2.3.13. So assume in the following that R has a zero object.
Step 1: Construction of subobjects: Given an object $i \in I$, we consider a projection p in $\mathrm{Mor}(i,i)$. We associate with the symbol $\widehat{i}_p$ the Hilbert space $\widehat{H}_{\widehat{i}_p} = pH_i$ and add the symbol $\widehat{i}_p$ and the Hilbert space $\widehat{H}_{\widehat{i}_p}$ to the collections I and $\big(H_i\big)_{i \in I}$. Denote with ι_p the embedding of pH_i into H_i. We repeat this construction for every object in I and every projection in the corresponding space of morphisms. Doing so, we obtain larger collections $\widehat{I} \supseteq I$ and $\big(\widehat{H}_{\widehat{i}}\big)_{\widehat{i} \in \widehat{I}}$. Note that considering the projection $p = \mathrm{id}_{H_i}$ in the construction above gives again the object i and the Hilbert space H_i, so every $i \in I$ and every H_i appears again in this procedure. We then replace the collection $(\mathrm{Mor}(i,j))_{i,j \in I}$ by the following operator spaces:

$$\widehat{\mathrm{Mor}}(\widehat{i}_p, \widehat{j}_q) := \{(\iota_q)^* \, T \circ \iota_p \mid T \in \mathrm{Mor}(i,j)\} \tag{2.3.1}$$

where i, j are objects in the original I and $\widehat{i}_p, \widehat{j}_q$ are objects obtained from projections p, q in $\mathrm{Mor}(i,i)$ and $\mathrm{Mor}(j,j)$, respectively. In a last step we identify symbols $\widehat{i}, \widehat{j} \in \widehat{I}$ whenever $\widehat{\mathrm{Mor}}(\widehat{i}, \widehat{j})$ contains a unitary element.
The result is a structure $\widehat{R}$ that fulfils condition (i). It is essential by construction, it holds $I \subseteq \widehat{I}$ and for every $i \in I$ we have $\widehat{H}_i = H_i$. Having a look at Equations 2.3.1, we further see that the morphisms between objects in I are left untouched, $\widehat{\mathrm{Mor}}(i,j) = \mathrm{Mor}(i,j)$, and the monoidal structure on the subobject I of $\widehat{I}$ coincides with the old one. Hence, R is a subobject of $\widehat{R}$. In addition, every $\mathrm{Mor}(\widehat{i}, \widehat{j})$ is a linear subset of $B(\widehat{H}_{\widehat{i}}, \widehat{H}_{\widehat{j}})$, as desired. Note that a repeated construction of subobjects would not produce new objects as a projection on a subspace pH_i can be identified with a projection on the whole space H_i.
It remains to show that $\widehat{R}$ is a monoidal W^*-category, compare Definition 2.3.6.
Proof of the category properties of $\widehat{R}$: Firstly, by Equation 2.3.1, every $\mathrm{Mor}(\widehat{i}, \widehat{i})$ contains the identity and it holds $\left(\widehat{\mathrm{Mor}}(\widehat{i}, \widehat{j})\right)^* = \widehat{\mathrm{Mor}}(\widehat{j}, \widehat{i})$. The same equation shows that every $\widehat{\mathrm{Mor}}(\widehat{i}, \widehat{j})$ is a finite-dimensional vector space, so it is weakly closed and it just remains to show that the composition of (composable) morphisms is a morphism again.
We fix subobjects $\widehat{i}_p, \widehat{i'}_q, \widehat{i''_r}$ of $i, i', i'' \in I$ and morphisms T and S in $\widehat{\mathrm{Mor}}(\widehat{i}_p, \widehat{i'_q})$ and

$\widehat{\mathrm{Mor}}(\widehat{i'_q}, \widehat{i''_r})$, respectively. By definition of $\widehat{\mathrm{Mor}}(\widehat{i_p}, \widehat{i'_q})$ and $\widehat{\mathrm{Mor}}(\widehat{i'_q}, \widehat{i''_r})$ we can write

$$T = (\iota_q)^* \circ \tilde{T} \circ \iota_p \qquad , \qquad S = (\iota_r)^* \circ \tilde{S} \circ \iota_q \tag{2.3.2}$$

for suitable $\tilde{T} \in \mathrm{Mor}(i, i')$ and $\tilde{S} \in \mathrm{Mor}(i', i'')$. But then it holds

$$S \circ T = (\iota_r)^* \circ \left(\tilde{S} \circ \iota_q \circ (\iota_q)^* \circ \tilde{T} \right) \circ \iota_p$$

and the term in the middle is a well-defined element in $\mathrm{Mor}(i, i'')$ as $\iota_q \circ (\iota_q)^* = q \in \mathrm{Mor}(i', i')$.

Proof of the monoidal structure of $\widehat{R}$: We have a monoidal structure on the whole $\widehat{I}$ (and not only on the subobject I) as the product of subobjects i_p and j_q (for i, j from the original I) can be defined to be the subobject of ij with respect to the tensor-product $p \otimes q \in \widehat{\mathrm{Mor}}(i, j) = \mathrm{Mor}(i, j)$:

$$i_p j_q := (ij)_{p \otimes q} \in \widehat{I}$$

We have to prove that the multiplication on $\widehat{I}$ is compatible with the tensor product structure on Hilbert spaces and spaces of morphisms, compare item (viii) of Definition 2.3.6.

Consider for objects i, i', j, j' in I and projections p, p', q, q' in $\mathrm{Mor}(i, i)$, $\mathrm{Mor}(i', i')$, $\mathrm{Mor}(j, j)$ and $\mathrm{Mor}(j', j')$, respectively. If $T \in \widehat{\mathrm{Mor}}(i_p, i'_{p'})$ and $S \in \widehat{\mathrm{Mor}}(j_q, j'_{q'})$, then we can write as in Equation 2.3.2

$$T = (\iota_{p'})^* \circ \tilde{T} \circ \iota_p \qquad , \qquad S = (\iota_{q'})^* \circ \tilde{S} \circ \iota_q$$

for suitable $\tilde{T} \in \mathrm{Mor}(i, i')$ and $\tilde{S} \in \mathrm{Mor}(j, j')$. We deduce

$$T \otimes S = (\iota_{p'} \otimes \iota_p)^* \circ (\tilde{T} \otimes \tilde{S}) \circ (\iota_p \otimes \iota_q)$$

Comparing this with Equation 2.3.1, we see that $T \otimes S \in \widehat{\mathrm{Mor}}\big((ii')_{p \otimes p'}, (jj')_{q \otimes q'}\big)$, as desired.

All together, we proved that $\widehat{R}$ is again an essential concrete monoidal W^*-category. By construction, it is closed under taking subobjects and the original R is a subcategory of it.

Step 2: Construction of direct sums: To keep notation simple, we assume R to coincide with the category constructed in Step 1, i.e. R is assumed to be closed under taking subobjects. We consider two symbols $j, k \in I$ and associate to it the symbol $\widehat{l} = j \oplus k$ and the Hilbert space $\widehat{H}_l := H_j \oplus H_k$ and denote with $\iota_j : H_j \to \widehat{H}_l$ and $\iota_k : H_k \to \widehat{H}_l$ the canonical embeddings of H_j and H_k into $\widehat{H}_l$. We repeat this construction for all pairs of elements in I and add the symbols $\widehat{l}$ and Hilbert spaces $\widehat{H}_l$ to the collections I and $(H_i)_{i \in I}$, respectively. Note that, as mentioned in Remark

2.3.14, it holds $0 \oplus i = i$ and $\{0\} \oplus H_i = H_i$ for all $i \in I$, so every $i \in I$ and every H_i appears again in the construction above. Finally, we define for given $j, j', k, k' \in I$ and $\widehat{l} = j \oplus k$, $\widehat{l'} = j' \oplus k'$ the operator space

$$\widehat{\mathrm{Mor}}(\widehat{l}, \widehat{l'}) = \mathrm{span}\left(\iota_{b'} \circ \mathrm{Mor}(a, b') \circ (\iota_a)^* \ \mid a, b \in \{k, l\}\right). \tag{2.3.3}$$

The construction for more than two summands is completely analogous and it is associative so it does not matter if we construct direct sums of arbitrary length or if we repeat the construction with only two summands. Doing so, we obtain direct sums of elements in I with arbitrary length. Again, we identify in the end symbols $\widehat{l}$ and $\widehat{l'}$ whenever $\widehat{\mathrm{Mor}}(\widehat{l}, \widehat{l'})$ contains a unitary element. We finally obtain a triple

$$\widehat{R} = \left(\widehat{I}, (\widehat{H}_{\widehat{i}})_{\widehat{i} \in \widehat{I}}, \left(\widehat{\mathrm{Mor}}(\widehat{i}, \widehat{j})\right)_{\widehat{i}, \widehat{j} \in \widehat{I}}\right)$$

where each $\widehat{\mathrm{Mor}}(\widehat{i}, \widehat{j})$ is a linear subspace of $B(\widehat{H}_{\widehat{i}}, \widehat{H}_{\widehat{j}})$.
The proof of the properties of an essential monoidal W^*-category are analogous to Step 1. Equation 2.3.3 shows that the morphisms between objects in I are left untouched, $\widehat{\mathrm{Mor}}(i, j) = \mathrm{Mor}(i, j)$, and the monoidal structure on the subobject I of $\widehat{I}$ coincides with the old one. Hence, R is a subobject of $\widehat{R}$.
It remains to show that $\widehat{R}$ is a monoidal W^*-category, compare Definition 2.3.6.
Proof of the category properties of $\widehat{R}$: As in Step 1, we first observe that, by Equation 2.3.3, every $\widehat{\mathrm{Mor}}(\widehat{i}, \widehat{j})$ contains the identity and it holds $\left(\widehat{\mathrm{Mor}}(\widehat{i}, \widehat{j})\right)^* = \widehat{\mathrm{Mor}}(\widehat{j}, \widehat{i})$. The same equation shows that every $\widehat{\mathrm{Mor}}(\widehat{i}, \widehat{j})$ is a finite-dimensional vector space, so it is weakly closed and it just remains to show that the composition of (composable) morphisms is a morphism again.
We consider $\hat{l} = i_1 \oplus \ldots \oplus i_K$, $\hat{l}' = i'_1 \oplus \ldots \oplus i'_K$ and $\hat{l}'' = i''_1 \oplus \ldots \oplus i''_K$, three direct sums of elements in I. Recall that it is not a restriction to have the same number of summand in all three cases, as we can always add the zero-object as a summand. Given $T \in \widehat{\mathrm{Mor}}(\widehat{l}, \widehat{l'})$ and $S \in \widehat{\mathrm{Mor}}(\widehat{l'}, \widehat{l''})$, we can write by definition of $\widehat{\mathrm{Mor}}(\widehat{l}, \widehat{l'})$ and $\widehat{\mathrm{Mor}}(\widehat{l'}, \widehat{l''})$

$$T = \sum_{a,b \in \{1,\ldots,K\}} \iota_{i'_b} \circ \tilde{T}_{i'_b i_a} \circ (\iota_{i_a})^* \qquad , \qquad S = \sum_{c,d \in \{1,\ldots,K\}} \iota_{i''_d} \circ \tilde{S}_{i''_d i'_c} \circ \left(\iota_{i'_c}\right)^* \tag{2.3.4}$$

for suitable $\tilde{T}_{i'_b i_a} \in \mathrm{Mor}(i_a, i'_b)$ and $\tilde{S}_{i''_d i'_c} \in \mathrm{Mor}(i'_c, i''_d)$. But then it holds

$$S \circ T = \sum_{a,b,c,d \in \{1,\ldots,K\}} \iota_{i''_d} \circ \left(\tilde{S}_{i''_d i'_c} \circ \left(\iota_{i'_c}\right)^* \circ \iota_{i'_b} \circ \tilde{T}_{i'_b i_a} \circ \right) (\iota_{i_a})^*$$

and again the terms in the middle are well-defined elements in $\mathrm{Mor}(i_a, i''_d)$.
Proof of the monoidal structure of $\widehat{R}$: We have a monoidal structure on the

whole $\widehat{I}$ (and not only on the subobject I), as the product of direct sums (of elements in the original I) can be defined to be the direct sum of the products:

$$(i_1 \oplus \ldots \oplus i_K)(j_1 \oplus \ldots \oplus j_L) := \bigoplus_{1 \leq k \leq K} \bigoplus_{1 \leq l \leq L} i_k j_l \in \widehat{I}$$

We have to prove that the multiplication on $\widehat{I}$ is compatible with the tensor product structure on Hilbert spaces and spaces of morphisms, compare item (viii) of Definition 2.3.6.
Consider four direct sums $\widehat{i} := i_1 \oplus \ldots \oplus i_K$, $\widehat{i'} := i'_1 \oplus \ldots \oplus i'_K$, $\widehat{j} := j_1 \oplus \ldots \oplus j_L$ and $\widehat{j'} := j'_1 \oplus \ldots \oplus j'_L$ of elements in I. If $T \in \mathrm{Mor}(\widehat{i}, \widehat{i'})$ and $S \in \mathrm{Mor}(\widehat{j}, \widehat{j'})$, then we can write as in Equation 2.3.4

$$T = \sum_{a,b \in \{1,\ldots,K\}} \iota_{i'_b} \circ \tilde{T}_{i'_b i_a} \circ (\iota_{i_a})^* \qquad , \qquad S = \sum_{c,d \in \{1,\ldots,K\}} \iota_{j'_d} \circ \tilde{S}_{j_d i'_c} \circ \left(\iota_{i'_c}\right)^*$$

and it holds

$$T \otimes S = \sum_{a,b,c,d \in \{1,\ldots,K\}} (\iota_{i'_b} \otimes \iota_{j'_d}) \circ (\tilde{T}_{i'_b i_a} \otimes \tilde{S}_{j_d i'_c}) \circ (\iota_{i'_a} \otimes \iota_{j'_c})^*.$$

Comparing this with Equation 2.3.3, we see that $T \otimes S \in \widehat{\mathrm{Mor}}(\widehat{\widehat{i}\widehat{j}} \otimes \widehat{i'}\widehat{j'})$, as desired. All together, we proved that $\widehat{R}$ is again an essential concrete monoidal W^*-category. By construction, it it closed under taking direct sums and the original R is a subcategory of it.
Step 3: Compatibility of Step 1 and 2: With the help of Lemma 2.3.17 we show next that the category $\widehat{R}$ constructed in Step 2 is still closed under taking subobjects. Consider two objects $\widehat{i}, \widehat{j}$ such that $\widehat{R}$ contains all their subobjects and assume their direct sum $\widehat{l} = \widehat{i} \oplus \widehat{j}$ would not do so. So we find a projection $p \in \widehat{\mathrm{Mor}}(l, l)$ without an associated object $\widehat{l_p}$ in the sense of Definition 2.3.13, (i). Then we could enlarge our constructed category by adding the object $\widehat{l_p}$ as described in Step 1. Doing so for all projections in $\widehat{\mathrm{Mor}}(l, l)$, we could afterwards apply Lemma 2.3.17, stating that $\widehat{l_p}$ is unitarily equivalent to some $\widehat{i_r} \oplus \widehat{j_s}$ for suitable projections $r \in \widehat{\mathrm{Mor}}(i, i)$ and $s \in \widehat{\mathrm{Mor}}(j, j)$. Repeating this for all objects in $\widehat{I}$, we would obtain a larger category than $\widehat{R}$, closed under taking subobjects; but we just showed that the newly added objects are unitarily equivalent to already existing ones, so $\widehat{R}$ has in fact not been enlarged and already $\widehat{R}$ is closed under taking subobjects. As $\widehat{R}$ is by construction closed under taking direct sums, this proves completeness of $\widehat{R}$.
Final Step: Minimal property of $\widehat{R}$: The object $\widehat{R}$ is the completion of R as we enriched R with only those objects and spaces of morphisms that have to be added, compare Remark 2.3.15. □

The definitions above are motivated by the corepresentation theory of a compact matrix quantum group in the sense that the following theorem holds.

Theorem 2.3.19. *Let $G = \big(C(G), u_G\big)$ be a compact matrix quantum group. Consider all its finite-dimensional unitary representations and let $(\delta_i)_{i\in I}$ be the collection of equivalence classes of those corepresentations (with respect to unitary equivalence) on Hilbert spaces $\big(H_i\big)_{i\in I}$, compare Remark 2.3.2 and Notation 2.3.3. Denote with δ_f and $\delta_{\overline{f}}$ the equivalence classes corresponding to the fundamental corepresentation of G and its conjugate, respectively. Then the triple*

$$R := \Big(I, \big(H_i\big)_{i\in I}, \big(\operatorname{Hom}(i,j)\big)_{i,j\in I}\Big)$$

is a complete essential concrete monoidal W^*-category and it is generated by the two (not necessarily different) objects f and $\overline{f}$. To every element $j \in I$ there exists a conjugate object $\overline{j}$ in the sense of Definition 2.3.10.

Remark 2.3.20. Considering only tensor products of the fundamental corepresentation δ_f and its conjugate $\delta_{\overline{f}}$, we obtain an (essential) concrete monoidal W^*-category, but it is in general not complete as we did not consider all subcorepresentations and direct sums.

The more interesting part is the reverse statement of Theorem 2.3.19 and 2.3.20, namely that we obtain from every category as above a CMQG (and it only depends on the completion of the category). To state this in a precise way we need one further definition, compare [Wor88, pp. 40].

Definition 2.3.21. Let $(R, f) = \Big(I, (H_i)_{i\in I}, \big(\operatorname{Mor}(i,j)\big)_{i,j\in I}, f\Big)$ be an essential concrete monoidal W^*-category with distinguished object $f \in I$ such that its conjugate $\overline{f}$ exists and such that $\{f, \overline{f}\}$ generates R.

(1) Consider a pair $\Big(A, \big(u^{(i)}\big)_{i\in I}\Big)$, where A is a unital C^*-algebra and the $u^{(i)}$ are unitary matrices of size $N_i := \dim(H_i)$ over A. We call $\Big(A, \big(u^{(i)}\big)_{i\in I}\Big)$ a *model for R* if the following are fulfilled:

   (i) For given $k, l \in I$ every linear mapping $T \in \operatorname{Mor}(k,l)$ is an intertwiner from $u^{(k)}$ to $u^{(l)}$, i.e.

$$(T \otimes \mathbb{1}_A)u^{(k)} = u^{(l)}(T \otimes \mathbb{1}_A).$$

   (ii) For every $k, l \in I$ it holds $u^{(kl)} = u^{(k)} \mathbin{⦶} u^{(l)}$ where the $\mathbin{⦶}$-product is defined as in the context of corepresentation matrices, see item (ii) of Observation 2.1.12:

$$u^{(k)} \mathbin{⦶} u^{(l)} := \sum E_{ij} \otimes E_{kl} \otimes u_{ij}u_{kl} \in M_{N_k}(\mathbb{C}) \otimes M_{N_l}(\mathbb{C}) \otimes A$$

(2) An *R-admissible pair* (A, u) consists of a C^*-algebra A and a matrix u of size $\dim(H_f)$ over A such that there exists a model $\left(A, \left(u^{(i)}\right)_{i \in I}\right)$ with $u^{(f)} = u$.

(3) A *universal R-admissible pair* is an R-admissible pair (A, u) such that A is generated by the entries of u and for every other R-admissible pair (B, v) there exists a *-homomorphism $\varphi : A \to B$ fulfilling $(\mathbb{1} \otimes \varphi)u = v$.

Proposition 2.3.22. *Consider the situation as described in Definition 2.3.21. Given an R-admissible pair* (B, v)*, the model* $\left(B, \left(v^{(i)}\right)_{i \in I}\right)$ *fulfilling* $v^f = v$ *is uniquely defined*

Proof. It can be shown, compare [Wor88, Equation 2.12 and 2.8], that for a model $\left(B, \left(v^{(i)}\right)_{i \in I}\right)$ of (R, f) the matrix $v^{(\overline{f})}$ is uniquely determined by $v^{(f)}$ with the help of the bijective mapping J from Definition 2.3.10: It holds $v^{(\overline{f})} = (J \otimes \mathbb{1}_B) \left(v^{(f)}\right)^{(*)} (J^{-1} \otimes \mathbb{1}_B)$. Hence, for a product k in the symbols f and $\overline{f}$ the matrix $v^{(k)}$ is uniquely determined by property (ii) in Definition 2.3.21. If i is a subobject of l, then by property (i) in Definition 2.3.21 the matrix $v^{(i)}$, seen as an element in $B(H_i) \otimes B$ is given by $(\iota_i^* \otimes \mathbb{1})v^{(l)}(\iota_i \otimes \mathbb{1})$, where ι_i is the embedding $H_i \hookrightarrow H_l$. If l is given by the direct sum $i \oplus j$, then the same property says

$$v^{(l)} = (\iota_i \otimes \mathbb{1})v^{(i)}(\iota_i^* \otimes \mathbb{1}) + (\iota_j \otimes \mathbb{1})v^{(j)}(\iota_j^* \otimes \mathbb{1}).$$

As R is generated by $\{f, \overline{f}\}$, this proves the statement. □

Lemma 2.3.23. *Let* (R, f) *be an essential concrete monoidal* W^**-category with distinguished object* f *such that* $\overline{f}$ *exists and* $\{f, \overline{f}\}$ *generates* R*. Then* (R, f) *admits a universal R-admissible pair* (A, u)*.*

Proof. Compare [Wor88, p. 41]. Let $N := \dim(H_f)$. Consider the *-algebra $\mathscr{A}$ consisting of *-polynomials in indeterminants u_{ij} where $1 \leq i, j \leq N$ and arrange the u_{ij} canonically in a matrix u. The family of R-admissible pairs (B, v) is not empty as it always contains the trivial R-admissible pair $\left(\mathbb{C}, \mathbb{1}_{M_N(\mathbb{C})}\right)$. The family of homomorphisms mapping the u_{ij} canonically onto the entries of the matrices v defines a family of C^*-seminorms on $\mathscr{A}$. It is pointwise bounded as all v are unitaries by the definition of a model for R, so the supremum over all these seminorms is a well-defined C^*-seminorm again. Dividing out the kernel of this new seminorm and taking completion produces a C^*-algebra A. By construction, (A, u) is an R-admissible pair. It is a universal R-admissible pair as A is generated by the entries of u and for every given R-admissible pair (B, v) we have by construction of the norm on A a homomorphism from A to B mapping the entries of u canonically to the entries of v. □

Remark 2.3.24. (a) It is clear that a universal R-admissible pair is unique up to isomorphisms φ as described in item (3) of 2.3.21. In this sense we can speak of *the* universal R-admissible pair.

(b) There is an alternative construction for the universal R-admissible pair via universal C^*-algebras: We start again with a matrix of generators $u' = (u'_{ij})$ of size $\dim(H_f) = N$. Define $\overline{u}' := (J \otimes \mathbb{1}) u'^{(*)} (J^{-1} \otimes \mathbb{1})$. Every product i of the symbols f and $\overline{f}$ corresponds canonically to a $\textcircled{\scriptsize T}$-product $u'^{(i)}$ of the matrices u' and $\overline{u}'$. Given now two such products $i, j \in I$, every morphism $T \in \mathrm{Mor}(i, j)$ gives a set of *-algebraic relations $\mathcal{R}_T(u')$ between the u'_{ij}'s, namely

$$\mathcal{R}_T(u') : \quad u'^{(i)}(T \otimes \mathbb{1}) = (T \otimes \mathbb{1}) u'^{(j)}.$$

We finally define A' to be the universal C^*-algebra generated by the entries of the matrix u' and the relations $\mathcal{R}_T(u')$ as described above. Note that this is well-defined as the morphisms t and $\overline{t}$ from Definition 2.3.10 guarantee that u' is a unitary.

It is now easy to see that (A', u') is equal to the universal R-admissible pair (A, u): The constructions of the $u'^{(i)}$ together with the relations $\mathcal{R}_T(u')$ guarantee that (A', u') is an R-admissible pair. By the property of the universal R-admissible pair we have a *-homomorphism $\varphi : A \to A'$ such that $(\mathbb{1} \otimes \varphi) u = u'$. On the other hand, (A, u) itself is an R-admissible pair, so the relations $\mathcal{R}_T(u)$ are fulfilled. The *-homomorphism whose existence is guaranteed by the universal property of the universal C^*-algebra A' (see Appendix) is the inverse map of φ.

The central point in Tannaka-Krein duality and the central result of this section is now that the universal R-admissible pair is a CMQG and its theory of finite dimensional unitary corepresentations is completely determined by R, compare [Wor88, Thm. 1.3].

Theorem 2.3.25. *Let (R, f) be an essential concrete monoidal W^*-category with distinguished object f such that $\overline{f}$ exists and $\{f, \overline{f}\}$ generates R. Let (A, u) be the universal R-admissible pair of (R, f) and $\left(A, \left(u^{(i)}\right)_{i \in I}\right)$ the corresponding model of R fulfilling $u^{(f)} = u$. Then it holds*

(1) $(A, u) =: G$ is a compact matrix quantum group.

(2) For every $i \in I$ the matrix $u^{(i)}$ is a unitary corepresentation matrix of G. If R is complete, then every finite-dimensional unitary corepresentation matrix of G is equivalent to some $u^{(i)}$.

(3) For every linear mapT $\in B(H_i, H_j)$ *the equation*

$$(T \otimes \mathbb{1}_A)u^{(i)} = u^{(j)}(T \otimes \mathbb{1}_A)$$

holds if and only if $T \in \operatorname{Mor}(i, j)$.

Proof. A proof of the theorem takes a lot of effort and is given by chapter 3 in [Wor88]. We will only comment on its main ideas. Again, many constructions are motivated by the concrete situation of corepresentations (or corepresentation matrices) of a CMQG. As mentioned in [Wor88], the construction of the universal R-admissible pair as done in the proof of Lemma 2.3.23 is not suitable in order to show the statements above, so it has to be obtained in a different way.

- Given an essential concrete monoidal W^*-category, one first replaces it by its completion and then considers the set I_{irr} of all irreducible elements in I, i.e. objects without non-trivial subobjects. The direct sum

  $$\widehat{\mathscr{A}} := \bigoplus_{i \in I_{irr}} B(H_i)'$$

  is a vector space of linear functionals on $\bigoplus_{i \in I_{irr}} H_i$. This will turn out to be the dense algebraic CQG $\mathscr{A}$ inside (A, u).

- For every $i \in I_{irr}$ we identify H_f with $\mathbb{C}^{\dim(H_f)}$ and $B(H_f)$ with $M_{\dim(H_f)}(\mathbb{C})$ and then define

  $$\widehat{u}^{(i)} := \sum_{1 \leq k, l \leq \dim(H_f)} E_{kl} \otimes E'_{kl} \in M_{\dim(H_f)}(\widehat{\mathscr{A}}),$$

  so the entries of $\widehat{u}^{(i)}$ are the elements of the canonical dual basis. There is a unique extension of this notion of matrices to the whole I by asking the intertwiners from $\widehat{u}^{(i)}$ to $\widehat{u}^{(j)}$ to be *exactly* given by the spaces of morphisms $\operatorname{Mor}(i, j)$. The matrices $\widehat{u}^{(i)}$ will turn out to be the corepresentation matrices $u^{(i)}$ associated to the objects in I.

- Given a model $M := \left(B, \left(v^{(i)}\right)_{i \in I}\right)$ we can directly compute for $i \in I_{irr}$:

  $$v^{(i)}_{ab} = \left(E'_{ab} \otimes \mathrm{id}_B\right)\left(E_{ab} \otimes v^{(i)}_{ab}\right) = \left(\widehat{u}^{(i)}_{ab} \otimes \mathrm{id}_B\right)\left(E_{ab} \otimes v^{(i)}_{ab}\right)$$

  and it can be deduced to be true for every $i \in I$. The map

  $$\varphi_M : \widehat{\mathscr{A}} \to B \; ; \; \widehat{u}^{(i)}_{ab} \mapsto \left(\widehat{u}^{(i)}_{ab} \otimes \mathrm{id}_B\right)\left(E_{ab} \otimes v^{(i)}_{ab}\right) \tag{2.3.5}$$

  is linear. This will turn out to be the map whose existence is required in the definition of a universal R-admissible pair.

- The monoidal structure on R enables us to define a multiplication on $\widehat{\mathscr{A}}$: We define the product $\widehat{u}^{(i)}_{ab}\widehat{u}^{(j)}_{cd}$ to be the entry of $\widehat{u}^{(ij)}$ at position (ac, bd). This in particular says $\widehat{u}^{(ij)} = \widehat{u}^{(i)} \mathbin{\bigcirc\!\!\!\!\!\top} \widehat{u}^{(j)}$ for all $i, j \in I$.
  The multiplication above turns $\widehat{\mathscr{A}}$ into a unital algebra and the map φ_M from Equation 2.3.5 is a homomorphism. If $1 \in I$ is the neutral element with respect to the monoidal structure on I, then the entry of the 1×1-matrix $\widehat{u}^{(1)}$ is the neutral element $\mathbb{1}_{\widehat{\mathscr{A}}}$ of the multiplication on $\widehat{\mathscr{A}}$.

- As the distinguished element f admits a conjugation and $\{f, \overline{f}\}$ generates R, one can prove that every element in I admits a conjugation. If J is the linear map on $H_{\overline{k}}$ as described in Definition 2.3.10, then we define

  $$\left(\widehat{u}^{(k)}_{ab}\right)^* := \left((J^{-1} \otimes \mathbb{1}_{\widehat{\mathscr{A}}})\widehat{u}^{(\overline{k})}(J \otimes \mathbb{1}_{\widehat{\mathscr{A}}})\right)_{ab},$$

  the (a, b)-th entry in the matrix $(J^{-1} \otimes \mathbb{1}_{\widehat{\mathscr{A}}})\widehat{u}^{(\overline{k})}(J \otimes \mathbb{1}_{\widehat{\mathscr{A}}})$. This way $\widehat{\mathscr{A}}$ becomes a *-algebra and the map φ_M from Equation 2.3.5 turns out to be a *-homomorphism. Using the intertwiners t and $\overline{t}$ from Definition 2.3.10, one can deduce that every $\widehat{u}^{(k)}$ is a unitary matrix.

- We define a linear mapping $h : \widehat{\mathscr{A}} \to \mathbb{C}$ by its behaviour on the entries of the matrices $\widehat{u}^{(i)}$ for $i \in I_{irr}$:

  $$h\left(\widehat{u}^{(i)}_{ab}\right) := \begin{cases} 1 & , \ i = 1 \\ 0 & , \ \text{else.} \end{cases}$$

  In particular $h(\mathbb{1}_{\widehat{\mathscr{A}}}) = 1$. It can be shown that h is a faithful positive linear functional (which will turn out to be the Haar state). Its faithfulness guarantees two things: Firstly, the supremum over all C^*-seminorms on $\widehat{\mathscr{A}}$ defines a C^*-norm $\|\cdot\|_{\widehat{\mathscr{A}}}$. Note that this supremum is again a welldefined semi-norm because the unitarity of the matrices $\widehat{u}^{(i)}$ bounds any seminorm pointwise by 1 on these matrix entries. Secondly, the entries of all $\left(\widehat{u}^{(i)}\right)_{i \in I_{irr}}$ together form a linear basis of $(\widehat{\mathscr{A}}, \|\cdot\|_{\widehat{\mathscr{A}}})$.

- Let $\widehat{A}$ be the completion of $\widehat{\mathscr{A}}$ with respect to $\|\cdot\|_{\widehat{\mathscr{A}}}$ as defined in the last step and define $\widehat{u} := \widehat{u}^{(f)}$. Then $(\widehat{A}, \widehat{u})$ is an R-admissible pair (more or less by construction). It is the universal one because $\widehat{A}$ is generated by the entries of $\widehat{u}$ and for every other R-admissible pair (B, v) with associated model M the mapping φ_M from Equation 2.3.5 fulfils by construction $(\mathbb{1} \otimes \varphi_M)\widehat{u} = v$. This proves in particular that $(\widehat{A}, \widehat{u})$ coincides with (A, u) in the theorem.

- One can observe that $(\widehat{A} \otimes \widehat{A}, \widehat{u} \oplus \widehat{u})$, where

$$\widehat{u} \oplus \widehat{u} := \sum_{i,j=1}^{\dim(H_f)} E_{ij} \otimes \sum_{k=1}^{\dim(H_f)} \widehat{u}_{ik} \otimes \widehat{u}_{kj},$$

is another R-admissible pair. The *-homomorphism $\Delta : \widehat{A} \to \widehat{A} \otimes \widehat{A}$, that exists by the property of the universal R-admissible pair, obviously fulfils

$$\Delta(\widehat{u}_{ij}) = \sum_k \widehat{u}_{ik} \otimes \widehat{u}_{kj}.$$

- The linear map $S : \widehat{u}_{ab}^{(k)} \mapsto \left(\widehat{u}_{ba}^{(k)}\right)^*$ on $\widehat{\mathscr{A}}$ is well-defined, antimultiplicative and by unitarity of the $\widehat{u}^{(k)}$ fulfils

$$\left(\widehat{u}^{(k)}\right)^{-1} = (\mathbb{1} \otimes S)\widehat{u}^{(k)}.$$

- We have proved that $(\widehat{A}, \widehat{u}) =: G$ is a CMQG because one characterization of them is as follows:
  - The *-algebra $\widehat{\mathscr{A}}$ generated by the entries of $\widehat{u}$ is dense in $\widehat{A}$.
  - There is a *-homomorphism $\Delta : \widehat{A} \to \widehat{A} \otimes \widehat{A}$ fulfilling

$$\Delta(\widehat{u}_{ij}) = \sum_k \widehat{u}_{ik} \otimes \widehat{u}_{kj}.$$

  - There is a linear antimultiplicative map S on $\widehat{\mathscr{A}}$ such that

$$\widehat{u}^{-1} = (\mathbb{1} \otimes S)\widehat{u}.$$

- Obviously, $\widehat{u}$ is a corepresentation matrix of G. By definition of a conjugate element and the way every $\widehat{u}^{(i)}$ can be obtained from $\widehat{u}^{(f)}$ and $\widehat{u}^{(\overline{f})}$, one can show that every $\widehat{u}^{(i)}$ is a corepresentation matrix of G. By construction, the intertwiners between some $\widehat{u}^{(i)}$ and $\widehat{u}^{(j)}$ are *exactly* given by the spaces $\mathrm{Mor}(i,j)$. Note that for this purpose the existence of the faithful positive state h was important as otherwise the supremum over all C^*-seminorms on $\widehat{\mathscr{A}}$ might not have been a norm. Dividing out its kernel could have led to additional intertwiners between corepresentations apart from the spaces $\mathrm{Mor}(i,j)$ and to additional finite-dimensional unitary corepresentations apart from the $\widehat{u}^{(i)}$. This proves in particular that the $\widehat{u}^{(i)}$ coincide with the matrices $u^{(i)}$ in the theorem.

- The completeness of R translates directly to the fact that $\left(\widehat{u}^{(i)}\right)_{i\in I}$ is closed under taking sub-corepresentation matrices and direct sums. As this collection contains the fundamental corepresentation matrix and its conjugate, we have that every unitary finite-dimensional corepresentation matrix appears as some $\widehat{u}^{(i)}$, see Observation 2.3.5.

□

Remark 2.3.26. (a) Theorem 2.3.25 in particular says that the CMQG mentioned there only depends on the completion of (R, f), but not on (R, f) itself.

(b) The 'only if'-part in statement (3) of Theorem 2.3.25 is important to guarantee that non-equivalent pairs (R, f) and (R', f') of complete categories and distinguished objects produce non-equivalent CMQGs: Consider two different complete categories. Then the corepresentation theories of the associated compact matrix quantum groups are different as they are exactly given by the categories we started with. Consequently, the CMQGs differ (and even are non-equivalent).

(c) Note further that completions of essential concrete monoidal W^*-categories (R, f) and (R', f') are different if and only if they differ on the subcategories given by products in f and $\overline{f}$ (and f' and $\overline{f'}$, respectively). This is clear from the perspective of a compact matrix quantum group G: To determine the whole corepresentation theory of G we only need to understand products of the fundamental corepresentation and its conjugate (and their intertwines).

(d) Considering $G = (A, u)$ from Theorem 2.3.25 as a compact quantum group (A, Δ), it is easy to see that G is in its universal form in the sense of Definition 2.1.33:
Denote the universal form of G by (B, Δ_B). Looking at u as a matrix v over B, we directly see that (B, v) is an R-admissible pair, as the dense algebraic compact quantum group inside (B, v) and (A, u) is the same. By the property of the universal R-admissible pair (A, u), see Definition 2.3.21, it follows that the mapping

$$u_{ij} \mapsto v_{ij}$$

defines a *-homomorphism from A to B. Conversely, by the universal property of the universal form (B, v), see Proposition 2.1.36, we have a *-homomorphism $B \to A$ given by $v_{ij} \mapsto u_{ij}$. We conclude that both constructed maps are *-isomorphisms, i.e. $G = (A, u)$ is given in its universal form.

Summary of Tannaka-Krein duality: Let $G = (A, u)$ be a CMQG and $\left(u^{(i)}\right)_{i\in I}$ be a collection of unitary, finite-dimensional corepresentation matrices that contains some u^f equivalent to u and that is closed under tensor products and conjugation. Extending Theorem 2.3.19 by some observations from above we can say that from every such collection $\left(u^{(i)}\right)_{i\in I}$ we obtain a pair (R, f), where R is an essential concrete monoidal W^*-category, f and $\overline{f}$ exist in R and $\{f, \overline{f}\}$ generates the category. Theorem 2.3.25 says in its core that every such pair (R, f) can be obtained in this way. In particular, we mention again part (c) of Remark 2.3.24, saying that the CMQG associated to an essential concrete monoidal W^*-category can be constructed as a universal C^*-algebra. The algebraic relations imposed on the generators are the intertwiner relations

$$\mathcal{R}_T(u): \quad u^{(i)}(T \otimes \mathbb{1}) = (T \otimes \mathbb{1})u^{(j)} \quad \forall\, T \in \mathrm{Mor}(i, j).$$

The correspondence between compact matrix quantum groups G and pairs (R, f) becomes one-to-one if we identify every G with its universal form and every R with the subcategory containing only products of f and $\overline{f}$.

2.4 Categories of Partitions

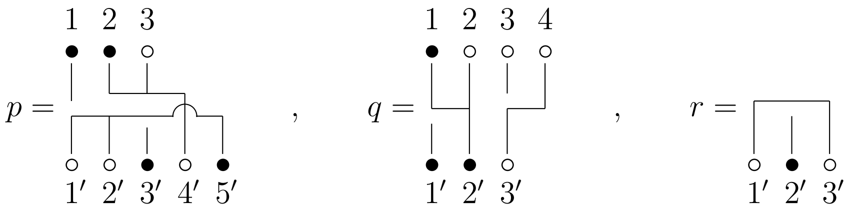
We will define the notion of partitions and introduce so-called tensor categories of partitions. The idea of such objects can already be found in [Bra37] in the form of Brauer diagrams to describe the representation theory of the orthogonal group O_N. In particular, the works by T. Banica [Ban96, Ban97, Ban99, Ban02] displayed the link of Brauer diagrams (and Brauer algebras) to CQGs, compare [FW14]. See also [BS09] and [TW18, TW17] for definitions in the context of easy quantum groups.

Definition 2.4.1. Let $k, l \in \mathbb{N}_0$. A *two-coloured partition on k upper and l lower points* is a collection of non-empty disjoint subsets of $[k]\dot{\cup}[l]$ such that their union is $[k]\dot{\cup}[l]$. Every element of $[k]$ is called an *upper point* and every element of $[l]$ is called a *lower point*. In addition there is a colouring $[k]\dot{\cup}[l] \to \{1,^*\}$, associating to each point either the colour white (and the symbol 1) or the colour black (and the symbol *). It can be described by words $\omega \in \{1,^*\}^k$ and $\omega' \in \{1,^*\}^l$ via the maps

$$[k] \to \{1,^*\}\ ;\ i \mapsto \omega_i \qquad , \qquad [l] \to \{1,^*\}\ ;\ j' \mapsto \omega'_j.$$

Define $|\omega| := k$ and $|\omega'| := l$, the lengths of the words ω and ω'.
The subsets of $[k]\dot{\cup}[l]$ describing the partition are called *blocks*. The number of blocks of a partition p is denoted by $b(p)$.
A block is called a *through-block* if it contains both upper and lower points. The number of through-blocks of a partition p is denoted by $tb(p)$.

In the following we will only speak of partitions when dealing with the objects above. We illustrate partitions by drawing the elements of $[k]$ and $[l]$ as actual black and white points into two horizontal lines on top of each other. Points that lie in the same block are connected via a grid of lines. Here are three examples p, q and r of such partitions:



(2.4.1)

As indicated again with the partition r, the numbers k and/or l are allowed to be zero.
A block with only one element is called a *singleton.* To emphasize this fact in the illustrations we sometimes write  instead of  and likewise for upper and/or white points.
We have primed the lower points in the pictures above just to distinguish the set $[l]$ from $[k]$ in the disjoint union $[k]\dot{\cup}[l]$.
The blocks forming the partition p above for are

$$\{1\}, \{2, 3, 4'\}, \{1', 2', 5'\}, \{3'\}$$

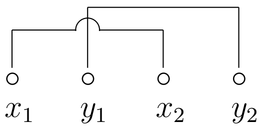
and it is a partition of three upper and five lower points.
Note that in p the points 1', 2' and 5' are not connected to the points 2, 3 and 4'. The corresponding connecting lines need to cross (if we are only allowed to draw them between the upper and lower row of points). We will call a partition *crossing* whenever in its illustration at least two lines belonging to different blocks cross each other. More precisely, we have the following definition.

Definition 2.4.2. Given a partition on k upper and l lower points we can order the $k + l$ points totally by their clockwise appearance in an illustration, for example

$$1 < 2 < \ldots < k < l' < \ldots < 2' < 1'.$$

The partition is called *crossing* if we find points $x_1 < x_2$ in some block b and points $y_1 < y_2$ in a block $b' \neq b$ such that $x_1 < y_1 < x_2 < y_2$. Otherwise it is called *non-crossing.*

The definition becomes clear by considering the following picture of a generic crossing:



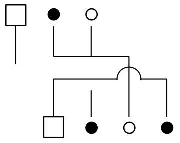
We will denote the set of all partitions on k upper and l lower points by $\mathcal{P}(k,l)$ and define $\mathcal{P} := \bigcup_{k,l\in\mathbb{N}_0}\mathcal{P}(k,l)$. Analogously we define the notions $\mathcal{NC}(k,l)$ and $\mathcal{NC}$ if we restrict to non-crossing partitions. For fixed $k,l \in \mathbb{N}_0$ we can additionally fix colourings $\omega \in \{1,^*\}^k$ and $\omega' \in \{1,^*\}^l$ of the upper and lower points, respectively, and define $\mathcal{P}(\omega,\omega')$ and $\mathcal{NC}(\omega,\omega')$ to be the sets of partitions with upper point labelling ω and lower point labelling ω'. For example if $\omega = (^*,1,1,1)$ and $\omega' = (^*,^*,1)$, the partition q from above fulfils

$$q \in \mathcal{NC}(\omega,\omega') \subseteq \mathcal{NC}(4,3) \subseteq \mathcal{NC}.$$

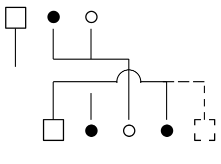
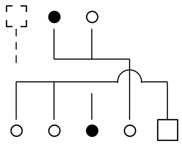
Note that an empty row has only one possible labelling, namely the empty word ε, so for $\omega' = (1,^*,1)$ it holds for the partition r from above

$$r \in \mathcal{NC}(\varepsilon,\omega') \subseteq \mathcal{NC}(0,3) \subseteq \mathcal{NC}.$$

We already introduce one further notion that will appear repeatedly in chapter 3: If we are not interested in the detailed structure of some non-empty (!) parts of the partition, we sometimes replace them by the symbol □. Taking for example the partition p above we could write

$p =$

.

The square in the upper row represents an arbitrary (non-empty) subpartition that is not connected to any of the other points. In this case it is actually just a black singleton. The square in the lower row represents a subpartition such that there is at least one point that is connected to the rightmost point in the lower row. In our case this square represents two white points connected to each other.
If we want to allow the subpartition to be empty, we use dashed lines to draw the corresponding square and connecting lines. Although there is no point in doing so for the moment, it would be correct to write

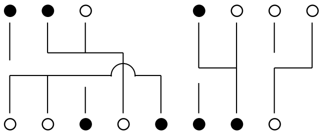
$p =$

$=$

.

In this case the dashed structure in the lower row is indeed empty and the one on the upper row represents again a black singleton.

We consider now some uni- and bivariate operations on partitions that will allow us to define so called *categories of partitions*. We use the partitions from Equations

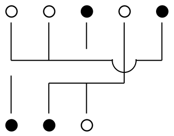
2.4.1 to give examples.
The *tensor product* of two partitions $p \in \mathcal{P}(\omega,\omega')$ and $q \in \mathcal{P}(\psi,\psi')$ is defined by horizontal concatenation, i.e. by placing their pictures side by side and considering this a partition in $\mathcal{P}(\omega\psi,\omega'\psi')$:

$$p \otimes q =$$



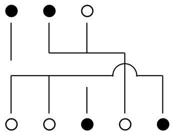
The *involution* is an operator on $\mathcal{P}$ that maps a partition $r \in \mathcal{P}(\omega,\omega') \subseteq \mathcal{P}(k,l)$ to an element $r^* \in (\mathcal{P}(\omega',\omega) \subseteq \mathcal{P}(l,k)$ given by mirroring r at some horizontal axis:

$$p^* =$$

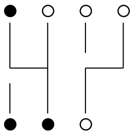


If $s \in \mathcal{P}(\omega,\omega')$ and $t \in \mathcal{P}(\omega',\omega'')$, we can construct the *composition* $ts = t \circ s$. It is defined by vertical concatenation: We place the partition t below s and connect each lower points of s with the corresponding upper points of t. This way we might obtain blocks in the middle that are connected neither to any of the very upper nor lower points. We denote these blocks as *remaining loops* and their number as $\text{rl}(t,s)$. Finally we erase all remaining loops and middle points:

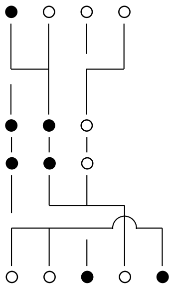
$$pq = p \circ q =$$



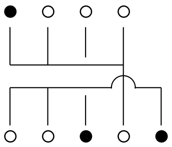
$$\circ$$



$$=$$

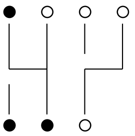


$$=$$

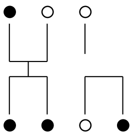


In the example above it holds $\text{rl}(p,q) = 1$ as only the pair of leftmost middle points was not connected to any of the points above or below. Note that the involution deserves its name as it holds $(s^*)^* = s$ and $(ts)^* = s^*t^*$ whenever t and s are composable.
The operator $\text{rot}_{\curvearrowright}$ is defined to take the rightmost point in the upper row of a partition and move it to the right end of the lower row without changing the connections to other points. At the same time the colour of that point is inverted.

$$\text{rot}_{\curvearrowright}(q) = \text{rot}_{\curvearrowright}($$

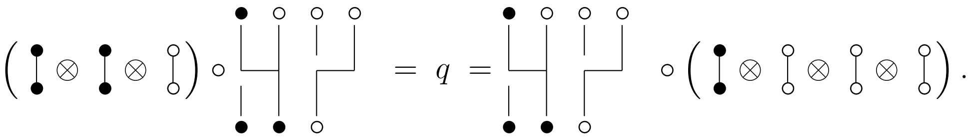


$$) =$$



Analogously, we define the operation $\mathrm{rot}_{\circlearrowright}$ into the opposite direction and the corresponding operations on the left side of the partition, $\mathrm{rot}_{\curvearrowleft}$ and $\mathrm{rot}_{\circlearrowleft}$. A partition q' is called a *rotated version of* q if it is obtained from q by repeated application of these four maps. Note that a rotation operator is only defined on partitions where there exists a point with whom the rotation can be performed. On the partition r from above only the rotations from the lower to the upper row are well-defined.

Remark 2.4.3. The partitions  and  are called *(white and black) identity partitions.* Taking tensor powers of these partitions due to a given colouring ω, we obtain the neutral element with respect to composition from the left on $\mathcal{P}(\omega, \omega')$ as well as the neutral element with respect to composition from the right on $\mathcal{P}(\omega'', \omega)$. For example



Rotating the white and black identity partition onto one line we obtain the four so-called *mixed-coloured pair partitions* , , , . Its unicoloured analoga are the *unicoloured pair partitions* , , , .

We come now to the main definition inside this section, that of categories of partitions.

Definition 2.4.4. A *category of partitions* $\mathcal{C}$ is a subset of $\mathcal{P}$ such that

(i) it contains the black and white identity partition,  and ,

(ii) it contains the four mixed-coloured pair partitions , , ,  and

(iii) it is closed under composition, involution and taking tensor products.

A category of partitions is called *non-crossing* if all its elements are non-crossing. We define for all $k, l \in \mathbb{N}_0$ the subsets

$$\mathcal{C}(k,l) := \mathcal{C} \cap \mathcal{P}(k,l)$$

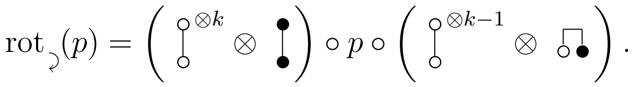
and likewise for all colourings ω, ω'

$$\mathcal{C}(\omega,\omega') := \mathcal{C} \cap \mathcal{P}(\omega,\omega').$$

The four mixed-coloured pair partitions mentioned above guarantee that a category of partitions is rotation-invariant, compare [TW18, Lemma 1.1].

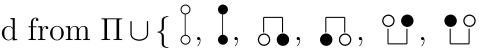
Proposition 2.4.5. *A category of partitions is closed under taking rotated versions of elements.*

Proof. Consider for example the operator $\mathrm{rot}_{\circlearrowright}$ and for $k \in \mathbb{N}$ a partition $p \in \mathcal{P}(k,l)$. Just to simplify the notation in the proof assume all points of p to be white. Then it holds



Analogously, one proves the statement for the other three rotations and for arbitrary colourings. ☐

Given a set of partitions Π, we can define the category of partitions generated by it.

Definition 2.4.6. Let Π be a set of partitions. We define $\langle \Pi \rangle$ to be the intersection of all categories of partitions containing Π, i.e. the set of all partitions that can be constructed from $\Pi \cup \{$  $\}$ by repeated application of tensor products, composition and involution.

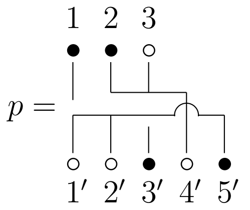
2.5 Linear maps associated to partitions

In the following we associate with a given partition p a linear map T_p on some finite-dimensional Hilbert space. In fact, this will result in a family of maps $(T_p(N))_{N\in\mathbb{N}}$ as we can vary the dimension N of the considered Hilbert space. The results in this section are well-known and can be found in [BS09] (for the uni-coloured situation) and in [TW18] (for the two-coloured case).

Definition 2.5.1. Consider a partition $p \in \mathcal{P}(k,l)$ and two multi-indices $i = (i_1, \ldots, i_k) \in \mathbb{N}^k$ and $j = (j_1, \ldots, j_l) \in \mathbb{N}^l$. The pair (i,j) defines a labelling of p by mapping the upper points of p from left to right to the numbers $i_1, \ldots, i_k$ and the lower points to $j_1, \ldots, j_l$.
We call (i,j) a *valid labelling* for p if connected points of the partitions are mapped to the same numbers. Otherwise it is called an *invalid labelling.*
Analogously, we can only consider i (or only j) and call it a valid labelling for the upper (or lower) points of p if all upper (or all lower) points that are connected get the same labels.

Of course (i,j) can only be a valid labelling for p if both i and j are valid labellings for their respective row of points, but we need in addition, that upper and lower points in a common through-blocks get the same label.

Example 2.5.2. Consider again the partition p from Equation 2.4.1:



Then the following holds:

- $\omega_1 = (5, 6, 6)$ is a valid labelling for the upper row but $\omega_2 = (5, 6, 5)$ is not.
- $\omega_1' = (3, 3, 7, 6, 3)$ is a valid labelling for the lower row but $\omega_2' = (3, 3, 7, 2, 8)$ is not.
- (ω_1, ω_1') is a valid labelling for p because both ω_1 and ω_1' are valid for their respective row of p and, in addition, the points 2 and 3 and $4'$ are all labelled by '6'.

Definition 2.5.3. Let $k, l \in \mathbb{N}_0$ and $p \in \mathcal{P}(k, l)$. Then we define

$$\delta_p : \mathbb{N}^k \times \mathbb{N}^l \to \{0, 1\} \; ; \; (i, j) \mapsto \begin{cases} 1 & , (i, j) \text{ is a valid labelling for } p \\ 0 & , otherwise. \end{cases}$$

Definition 2.5.4. Let $N \in \mathbb{N}$ and $p \in \mathcal{P}(k, l)$ for some $k, l \in \mathbb{N}_0$. Consider the Hilbert space $\mathbb{C}^N$ with canonical orthonormal basis $(e_i)_{i \in [N]}$. Then we define a linear map T_p as follows:

$$T_p : \left(\mathbb{C}^N\right)^{\otimes k} \to \left(\mathbb{C}^N\right)^{\otimes l} \; ; \; e_{i_1} \otimes \ldots \otimes e_{i_k} \mapsto \sum_{j \in [N]^l} \delta_p(i, j) \, (e_{j_1} \otimes \ldots \otimes e_{j_l})$$

The important observation at this point is, that the operations on partitions from the last section translate in a nice way to operations on the maps T_p, compare [BS09, Prop. 1.9].

Lemma 2.5.5. *Let $N \in \mathbb{N}$ and p, q be partitions. Then it holds:*

(1) $T_{q \otimes p} = T_q \otimes T_p$.

(2) $T_{p^*} = (T_p)^*$

(3) $T_{qp} = N^{-\mathrm{rl(q,p)}} \, (T_q \circ T_p)$

where for the last statement p must be composable with q from the left and $\mathrm{rl}(q, p)$ *denotes the number of remaining loops when writing p on top of q and connecting the middle points.*

Proof. Statement (1) and (2) follow directly from the definition of T_p. Consider for (3) some $p \in \mathcal{P}(k,l)$ and $q \in \mathcal{P}(l,m)$. Given a multi-index $i = (i_1, \ldots, i_k)$, we write $e_i := e_{i_1} \otimes \ldots \otimes e_{i_k}$. Then it holds

$$(T_q \circ T_p)(e_i) = T_q\left(\sum_{j\in[N]^l} \delta_p(i,j)e_j\right) = \sum_{s\in[N]^m}\sum_{j\in[N]^l} \delta_q(j,s)\delta_p(i,j)e_s$$

Now observe that $\delta_q(j,s)\delta_p(i,j)$ is zero if (i,s) is invalid for qp. If conversely (i,s) is valid for qp, then we investigate how many possibilities we have for j to make $\delta_q(j,s)\delta_p(i,j)$ equal to one:
Consider the vertical concatenation qp before(!) erasing the remaining loops. We can look at i as a labelling for the very upper points of qp and s a labelling of the very lower points of qp. Doing so and asking $\delta_q(j,s)\delta_p(i,j)$ to be one, j can be considered a labelling of the two rows of middle points that makes (i,j,s) "a valid labelling for qp".
We observe that, for given i and s, the condition $\delta_q(j,s)\delta_p(i,j) \overset{!}{=} 1$ determines the values of j at least on the middle points that are connected to the upper or lower row. For each other block that contains a point in the middle – these are by definition the remaining loops – we can independently choose a label in $[N]$. There are altogether $N^{\mathrm{rl}(q,p)}$ possibilities to do so and we conclude

$$(T_q \circ T_p)(e_i) = \sum_{s\in[N]^m} \underbrace{\left(\sum_{j\in[N]^l} \delta_q(j,s)\delta_p(i,j)\right)}_{=\delta_{qp}(i,s)N^{\mathrm{rl}(q,p)}} e_s = N^{\mathrm{rl}(p,q)} \sum_{s\in[N]^m} \delta_{qp}(i,s)e_s = N^{\mathrm{rl}(q,p)}T_{qp}(e_i).$$

□

Remark 2.5.6. Note that the definition of T_p is independent of the colouring of the points. Those will become relevant in the next section.

We finish our observations by connecting the theory of partitions with the previous section: Using the mapping $p \mapsto T_p$, every category of partitions defines an (essential) concrete monoidal W^*-category. Compare also [TW17, Prop. 3.11].

Lemma 2.5.7. *Let $N \in \mathbb{N}$ and let $\mathcal{C}$ be a category of partitions. Define*

$$I := \bigcup_{n\in\mathbb{N}_0} \{1,^*\}^n,$$

the collection of possible colourings for rows of points in partitions. Define on I the monoidal structure given by concatenation of words. For $\omega \in I$ let

$$H_\omega := \left(\mathbb{C}^N\right)^{\otimes|\omega|}$$

and for $\omega, \omega' \in I$ *we set*

$$\mathrm{Mor}(\omega, \omega') := \mathrm{span}\left(\{T_p \mid p \in \mathcal{C}(\omega, \omega')\}\right).$$

Define further on I *the conjugation* $\omega \mapsto \overline{\omega}$ *by pointwise colour inversion, i.e.* $1 \mapsto {}^*$ *and* ${}^* \mapsto 1$. *Identify elements* ω, ω' *in* I*, if* $\mathrm{Mor}(\omega, \omega')$ *contains a unitary element. Then*

$$R_N(\mathcal{C}) = \Big(I\ ,\ \big(H_i\big)_{i \in I}\ ,\ \big(\mathrm{Mor}(i,j)\big)_{i,j \in I}\Big)$$

is an essential monoidal W^**-category with distinguished object* $f = (1)$ *such that* $\{f, \overline{f}\}$*, i.e.* $\{(1), ({}^*)\}$*, generates* R*. The linear map* J *in the Definition of a conjugate object, see Definition 2.3.10, is given by the identity map on* $\overline{\mathbb{C}^N}$*.*

Note that in the sense of Remark 2.3.2 we have to consider for each H_ω a whole equivalence class of Hilbert spaces, namely all Hilbert spaces with the same dimension. Likewise, each $\mathrm{Mor}(\omega, \omega')$ is a collection of operator spaces. Recall further that the map J mentioned above is defined on a special representative of $H_{(1)}$, namely on $\overline{\mathbb{C}^N}$.

Proof. The category properties follow directly from our observations on the maps T_p. $R_N(\mathcal{C})$ is closed with respect to composition because $T_q \circ T_p$ is a multiple of T_{qp}. Fixing some $\mathrm{Mor}(\omega, \omega)$ we know by the closure property with respect to tensor products and by $\{\overset{\circ}{\underset{\circ}{|}}, \overset{\bullet}{\underset{\bullet}{|}}\} \subseteq \mathcal{C}$ that there is a tensor product $\mathbb{1}_\omega$ of black and white identities inside $\mathcal{C}(\omega, \omega)$. We have already mentioned in Remark 2.4.3 that this is the neutral element with respect to composition on $\mathcal{C}(\omega, \omega)$. Hence, by part (3) of Lemma 2.5.5, the corresponding operator $T_{\mathbb{1}_\omega}$ is the identity on $\mathrm{Mor}(\omega, \omega)$. $R_N(\mathcal{C})$ is an involutive category because of statement (2) of Lemma 2.5.5. The monoidal structure is given because the concatenation of words is assumed to be associative and the empty word is its neutral element. In addition, it is compatible with the tensor product structure on morphisms: If $p \in \mathcal{C}(\omega, \omega')$ and $q \in \mathcal{C}(\psi, \psi')$, then $T_p \otimes T_q = T_{p \otimes q} \in \mathrm{Mor}(\omega\psi, \omega'\psi')$. In particular, $H_{\omega\psi} = H_\omega \otimes H_\psi$. Finally, we observe that the maps t and $\bar{t}$ that are demanded to exist by Definition 2.3.10 coincide with $T_{\underset{\circ\bullet}{\sqcap}}$ and $T_{\overset{\bullet\circ}{\sqcup}}$, respectively. The map J in the background is given by $J := \mathrm{id}_{\overline{C^N}}$. The word $({}^*) \in I$ is the conjugate element of $(1) \in I$ and the pair $\{(1), ({}^*)\}$ generates $R_N(\mathcal{C})$ by definition: Every $\omega \in I$ is a product of letters 1 and *, so we even do not need to consider subobjects when constructing I from the generating set $\{(1), ({}^*)\}$. □

2.6 Easy quantum groups

In this section we show how a category of partitions defines a CMQG. This will lead to the definition of easy quantum groups. At the end of this section we will exploit our (abstract) results so far in this concrete situation. A reformulation of the definition of easy quantum groups will put a concrete way to construct easy quantum groups from a given set of partitions at its heart. Finally, we outline the classification of easy quantum groups and introduce some notations that will be used in the chapters to come.

By Proposition 2.3.18 and Theorem 2.3.25, the essential concrete monoidal W^*-category $R_N(\mathcal{C})$ (with distinguished objects (1)) from Lemma 2.5.7 defines a unique CMQG (in its universal form). This is the definition of easy quantum groups. They were first defined in [BS09] by T. Banica and R. Speicher in the orthogonal case and later generalized in [TW18] and [TW17] by P. Tarrago and M. Weber to the unitary case. If not mentioned otherwise, our exposition in this section follows these references.

Definition 2.6.1. Let $G = (A, u)$ be a compact matrix quantum group. Consider the (complete) essential monoidal W^*-category given by finite-dimensional unitary corepresentations

$$R := \Big(I, \big(H_i\big)_{i\in I}, \big(\mathrm{Hom}(i,j)\big)_{i,j\in I}\Big)$$

as defined in Theorem 2.3.19. Let again f denote the equivalence class of the fundamental corepresentation. If the category R is given by the completion of $R_N(\mathcal{C})$ for some $N \in \mathbb{N}$ and some category of partitions, see Lemma 2.5.7, then we call G a (unitary) easy quantum group of size N.

Remark 2.6.2. (i) Starting conversely with a category of partitions $\mathcal{C}$, we can consider the associated essential concrete monoidal W^*-category $R_N(\mathcal{C})$ from Lemma 2.5.7 and construct a compact matrix quantum group $G_N(\mathcal{C}) = (A, u)$ in its universal form, see Theorem 2.3.25. By definition, any easy quantum group can be constructed this way in the following sense: Given any easy quantum group $G' = (C(G'), v')$, we can replace it by an equivalent compact matrix quantum group $G = (C(G), v)$ where v is a unitary matrix. Writing G in its universal form, $G = (C_u(G), v)$, this pair appears as one of the constructed $G_N(\mathcal{C}) = (A, u)$.

(ii) Note that, given a category $\mathcal{C}$ of partitions, we always associate to this a series of easy quantum groups $(G_N(\mathcal{C}))_{N\in\mathbb{N}}$. As the sizes of their fundamental corepresentation matrices are given by the respective N, they are pairwise non-equivalent. In particular, they pairwisely differ.

Up to now, we have not proved that the category of partitions associated to a given easy quantum group G is unique. This will be discussed in detail in Chapter 3, see Proposition 3.2.3 and Corollary 3.1.4. At this point, we just formulate one consequence of it (see also for example [TW17, Cor. 3.13]):

Proposition 2.6.3. *Consider two different categories $\mathcal{C}_1$ and $\mathcal{C}_2$ of partitions. Let $\big(G_N(\mathcal{C}_1)\big)_{N\in\mathbb{N}}$ and $\big(G_N(\mathcal{C}_2)\big)_{N\in\mathbb{N}}$ be the sequences of easy quantum groups associated to them, i.e. the sequences of universal $R_N(\mathcal{C}_1)$-admissible and $R_N(\mathcal{C}_2)$-admissible pairs, respectively. Then there is some $N \in \mathbb{N}$ such that $G_N(\mathcal{C}_1) \neq G_N(\mathcal{C}_2)$.*

One even has that the easy quantum groups are different for all $N \in \mathbb{N}$ up to finitely many. Conversely spoken, we can only identify easy quantum groups with unique categories of partitions if we consider a whole series of easy quantum groups (obtained from a given category of partitions by considering different $N \in \mathbb{N}$).
Whenever we identify in the following an easy quantum group with a category of partitions, the observation above should be kept in mind, i.e. the correspondence is only one-to-one, and therefore makes sense, if we consider a series of easy quantum groups $G_N(\mathcal{C})$.

By part (c) of Remark 2.3.24 the compact matrix quantum group $G_N(\mathcal{C}) = (A, u)$ associated to a category of partitions and a natural number N can be constructed by considering the matrix of generators u and imposing on its entries the relations $\mathcal{R}_T(u)$ coming from morphisms T in the completion of $R_N(\mathcal{C})$. We recapitulate this construction and investigate it further.

Observation 2.6.4. As the linear map $J \in B\left(\overline{\mathbb{C}^N}\right)$ from Definition 2.3.10 is the identity, it holds that $\overline{u}$ is given by entrywise application of the involution on $u = u^{(1)}$, i.e $\overline{u} = u^{(*)}$. Given a word $\omega = (\omega_1, \ldots, \omega_k)$ over $\{1,^*\}$, we define

$$u^{\omega} := u^{(\omega_1)} \mathbin{\bigcirc\mkern-15mu\top} \ldots \mathbin{\bigcirc\mkern-15mu\top} u^{(\omega_k)}.$$

With every $T \in \operatorname{Mor}(\omega, \omega')$ we associate the relation

$$\mathcal{R}_T^{Gr}(u): \quad (T \otimes \mathbb{1})u^{\omega} = u^{\omega'}(T \otimes \mathbb{1})$$

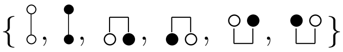
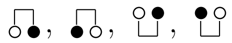
and we have

$$A = C^*\big((u_{ij})_{1\leq i,j\leq N} \mid \forall\, \omega, \omega' \in I, \forall\, T \in \operatorname{Mor}(\omega, \omega'): \text{ The relations } \mathcal{R}_T^{Gr}(u) \text{ hold. }\big).$$

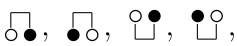
As the linear combination of intertwiners (between the same corepresentation matrices) is always an intertwiner again and $\operatorname{Mor}(\omega, \omega')$ is the span of $\big(T_p\big)_{p\in\mathcal{C}(\omega,\omega')}$, we only need to consider the relations $\mathcal{R}_{T_p}^{Gr}(u)$ for $p \in \mathcal{C}$:

$$A = C^*\big((u_{ij})_{1\leq i,j\leq N} \mid \forall\, \omega, \omega' \in I, \forall\, p \in \mathcal{C}(\omega, \omega'): \text{ The relations } \mathcal{R}_{T_p}^{Gr}(u) \text{ hold. }\big).$$

Likewise we obtain from Observation 2.1.12 that, given two intertwiners, we can build their tensor product and (if they are composable) build their composition and we always obtain again an intertwiner between the corresponding corepresentation

matrices. Knowing that u and $u^{(*)}$ are unitaries, we can in addition apply the involution to an intertwiner to obtain an intertwiner again, so in this case the only relevant relations are those coming from a generating set Π of $\mathcal{C}$ and the partitions $\{$  $\}$.
Now it can easily be seen that the relations associated to the white and black identity partitions are trivial and the relations associated to the mixed-coloured pair partitions  are equivalent to the fact that u and $u^{(*)}$ are unitaries. Therefore, we conclude

$$A = C^*\big((u_{ij})_{1\leq i,j\leq N} \quad | \quad \mathcal{R}^{Gr}_{T_p}(u) \text{ holds for all } p\in\Pi \text{ ; } u \text{ and } u^{(*)} \text{ are unitaries}\big)$$

and if we assume that the generating set Π contains the four mixed-coloured pair partitions, , we have

$$A = C^*\big((u_{ij})_{1\leq i,j\leq N} \quad | \quad \forall\, p\in\Pi: \text{ The relations } \mathcal{R}^{Gr}_{T_p}(u) \text{ hold. }\big).$$

It is worth to write down this construction as a reformulation of the definition of easy quantum groups.

Lemma 2.6.5. *Let $N\in\mathbb{N}$ and A be a C^*-algebra. Let $u' = (u'_{ij})$ be an element in $M_N(A)$ and define the relations $\mathcal{R}^{Gr}_{T_p}(\cdot)$ as in Observation 2.6.4. Then the pair (A, u') is an easy quantum group if and only if the following holds:*

(i) There is a set of partitions Π containing the mixed-coloured pair partitions such that

$$A = C^*\big((u_{ij})_{1\leq i,j\leq N} \quad | \quad \forall\, p\in\Pi: \text{ The relations } \mathcal{R}^{Gr}_{T_p}(u) \text{ hold. }\big).$$

(ii) There is an invertible $V\in M_N(\mathbb{C})\otimes\mathbb{1}_A$ such that $VuV^{-1} = u'$.

In this case $G' = (A, u')$ and $G = (A, u)$ are compact matrix quantum groups and it holds $G' = G$ in the category of CQGs. The set Π can be replaced by any other set Π' of partitions containing the mixed coloured pair partitions, at least as long as $\langle\Pi'\rangle = \langle\Pi\rangle$.

Notation 2.6.6. In the sense of Lemma 2.6.5 we denote an easy quantum group associated to a natural number N and a set of partitions Π as above by $G_N(\Pi)$. The relations $\mathcal{R}^{Gr}_{T_p}(u)$ associated to a partition p and a matrix u will be denoted by $\mathcal{R}^{Gr}_p(u)$ and called the *quantum group relations associated to p and u*. The corresponding C^*-algebra A will be denoted, as usual, by $C\big(G_N(\Pi)\big)$ and called the *non-commutative functions over the easy quantum group $G_N(\Pi)$*.

Remark 2.6.7. Writing out Equation 5.2.1 in the sense of entrywise comparison of matrix entries, we end up with

$$\sum_{t\in[N]^k} \delta_p(t,\gamma')u_{t_1\gamma_1}^{\omega_1}\cdots u_{t_k\gamma_k}^{\omega_k} = \sum_{t'\in[N]^l} \delta_p(\gamma,t')u_{\gamma_1't_1'}^{\omega_1'}\cdots u_{\gamma_l't_l'}^{\omega_l} \tag{2.6.1}$$

for all $\gamma\in[N]^k$ and $\gamma'\in[N]^l$. Recall from Definition 2.5.3 that $\delta_p(a,b)$ is non-zero if and only if (a,b) is a valid labelling for p.

As described in Definition 2.2.5, the relation $G'\subseteq G$ for two compact matrix quantum groups $\big(C(G'),u'\big)$ and $\big(C(G),u\big)$ translates to the existence of a *-homomorphism $\varphi : C(G)\to C(G')$ mapping the matrix u canonically onto u'. As easy quantum groups are given by universal C^*-algebras whose relations come from (categories of) partitions, the existence of such a φ can be guaranteed by the subset relation $\mathcal{C}\subseteq\mathcal{C}'$ between the corresponding categories of partitions. In this sense the largest possible category of partitions, $\mathcal{C}=\mathcal{P}$, corresponds to the smallest easy quantum group, i.e. the minimal object in the category of easy quantum groups. Conversely, the largest easy quantum group corresponds to the smallest category which is by definition the category generated by the empty set.

2.6.1 Classification of easy quantum groups and examples

Orthogonal easy quantum groups

Easy quantum groups were first defined by T. Banica and R. Speicher in [BS09] as a generalization of orthogonal Lie groups. Their definition considered a quantum group $G=(A,u)$ where u was an orthogonal matrix, i.e. its entries were self-adjoint, so $u=u^{(*)}$. The considered partitions were uni-coloured as one did not have to distinguish between u and $u^{(*)}$. In our setting we could express this with the following observation and definition, compare [TW17, Sec. 1.4]:

Proposition 2.6.8. *Let $G_N(\Pi)=(A,u)$ be an easy quantum group. Then $u=u^{(*)}$ holds if and only if $\langle\Pi\rangle$ contains the bi-coloured identity partition $\overset{\circ}{\underset{\bullet}{|}}$. In this case, $\langle\Pi\rangle$ has closure under pointwise colour switches.*

Proof. The definition of the relation $\mathcal{R}^{Gr}_{\overset{\circ}{\underset{\bullet}{|}}}(u)$ is exactly $u=u^{(*)}$. Given a partition p inside $\langle\Pi\rangle$, we can build the tensor product out of $\overset{\circ}{\underset{\bullet}{|}}$ or $\overset{\circ}{\underset{\bullet}{|}}^{\,*}=\overset{\bullet}{\underset{\circ}{|}}$ with suitable black and white identity-partitions and compose it with p to switch the colour of one arbitrary fixed point of p. □

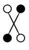
Definition 2.6.9. (i) A uni-coloured partition is a partition on white points. A uni-coloured category of partitions is a set of uni-coloured partitions obtained from a category of two-coloured partitions by identifying all points with white points.

(ii) An *orthogonal easy quantum group* is a compact matrix quantum group $G = (A, u)$ with $u = (u_{ij}) \in M_N(A)$ such that there is a uni-coloured category of partitions $\mathcal{C}$ and

$$A = C^*\big(\ (u_{ij})_{1\leq i,j\leq N} \mid u = u^{(*)} \text{ and } \mathcal{R}_p^{Gr}(u) \text{ holds for all } p \in \mathcal{C}\big)$$

Easy groups

As mentioned before, the notion of a quantum object should be the generalization of a classical situation, so it is not surprising that there are easy quantum groups that are actually groups. In the sense of Proposition 2.1.2 this corresponds to the case of commutative C^*-algebras A. The following result can be found in [TW17, Chpt. 2,3] and it is a generalization of the results in [BS09, Chpt. 2].

Proposition 2.6.10. *Let $G_N(\Pi)$ be an easy quantum group. If $\langle\Pi\rangle$ contains the crossing partition* *, then $G_N(\Pi)$ is a group.*

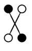
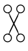
Note that  consists of two blocks, each connecting the points of same colour.

Proof. The relation $\mathcal{R}^{Gr}$(u) is by definition given by

$(u^{(*)} \mathbin{\bigcirc\hspace{-0.6em}\top} u)(T$$\otimes \mathbb{1}_A) = (T$$\otimes \mathbb{1}_A)(u \mathbin{\bigcirc\hspace{-0.6em}\top} u^{(*)}).$

Writing this out, we obtain $N^2 \times N^2$ matrices on both sides and comparing the $\big((j_1, j_2), (i_1, i_2)\big)$-th entries reads

$$u^*_{j_1i_2}u_{j_2i_1} = u_{j_2i_1}u^*_{j_1i_2} \tag{2.6.2}$$

Note that the existence of  in $\langle\Pi\rangle$ is equivalent to the existence of any of ,  and  by Proposition 2.4.5. For example, $\mathcal{R}^{Gr}$(u) reads

$$u_{j_1i_2}u_{j_2i_1} = u_{j_2i_1}u_{j_1i_2}$$

for all $\big((j_1, j_2), (i_1, i_2)\big)$, so A is commutative, as claimed. □

Remark 2.6.11. Taking a closer look at Proposition 2.1.2 and applying it to the situation of an easy group, we see that the group elements are characters on A and are uniquely defined by their effect on the generators u_{ij}. This way each group element g corresponds to a unique complex valued matrix $\big(g(u_{ij})\big)_{1\leq i,j\leq N}$. As u is unitary, every g is a unitary matrix and easy groups are subgroups of the unitary matrices.

Free easy quantum groups

An important class of easy quantum groups originates in categories of non-crossing partitions, compare [BS09, Def. 3.10] and [TW17, Def. 4.1].

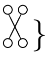
Definition 2.6.12. An easy quantum group $G_N(\Pi)$ is called *free* if Π, and so $\langle\Pi\rangle$, is a subset of the non-crossing partitions $\mathcal{NC}$.

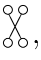
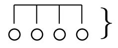
One motivation to consider non-crossing (categories of) partitions comes from free probability theory: In its combinatorial approach the transition from classical probability theory to free probability theory is marked by considering non-crossing partitions instead of all (in particular crossing) partitions, compare for example [BS09, p. 1481]. Once discovered that categories of partitions produce (well-known) matrix groups, the definition of quantum groups based on non-crossing partitions was a natural way to obtain free (or so-called liberated) versions of those groups. The free easy quantum groups presented in [BS09] are fully liberated in the sense that one can consider the commutative C^*-algebras associated to easy groups and obtains their free versions by restricting the corresponding categories of partitions to its non-crossing elements. As we will see in the next section, there are many steps in between, i.e. partial commutative relations on the free easy quantum groups that do not imply commutativity and so producing partially liberated versions of easy groups. Note that also in [BS09] examples of easy groups in between the group case and the free case are given, for example in form of the quantum group O_N^*.

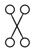
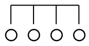
Classification results: The orthogonal case

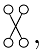
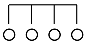
Easy quantum groups are completely classified. The starting point was in [BS09] where orthogonal easy groups and (except one case) all free orthogonal easy quantum groups where classified with the help of uni-coloured partitions. The missing free quantum group was found in [Web13]. In [RW16] also all partially liberated orthogonal quantum groups are classified.
In the following we describe (series) of easy quantum groups by generators of the associated category.

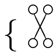
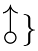
Theorem 2.6.13. *There are six categories corresponding to orthogonal easy groups:*

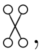
1. *The orthogonal groups O_N correspond to $\Pi' = \{$  $\}$ and the category of all uni-coloured pair partitions, i.e. partitions where each block consists of exactly two points. O_N consists of all orthogonal $N \times N$ matrices.*

2. *The hyperoctahedral groups H_N correspond to $\Pi' = \{$ ,  $\}$ and the category of uni-coloured partitions with even block size. H_N consists of all $N\times N$-matrices with one entry ± 1 in every row and column and all other entries vanishing.*

(3) The alternating permutation groups S'_N *correspond to* $\Pi' = \{$ , ,  $\otimes$ $\}$ *and the category of uni-coloured partitions on even numbers of points.* S'_N *consists of all* $N \times N$ *permutation matrices, additionally multiplied with* ± 1, *i.e.* $S'_N = S_N \times \mathbb{Z}_2$.

(4) The permutation groups S_N *correspond to* $\Pi' = \{$ , , $\}$ *and the category of all uni-coloured partitions.* S_N *consists of all* $N \times N$ *permutation matrices.*

(5) The alternating bistochastic groups B'_N *correspond to* $\Pi' = \{$ ,  $\otimes$ $\}$ *and the category of uni-coloured partions on even numbers of points and blocksizes one or two (so the number of singletons is even).* B'_N *consists of all orthogonal* $N \times N$ *matrices with sum* ± 1 *in each row and column. It is the group of orthogonal matrices that map the canonical diagonal* $\mathbb{C}(e_1 + \ldots + e_N)$ *of* $\mathbb{C}^N$ *onto itself.*

(6) The bistochastic groups B_N *correspond to* $\Pi' = \{$ , $\}$ *and the category of uni-coloured partitions with block sizes one or two.* B_N *consists of all orthogonal* $N \times N$ *matrices with sum 1 in each row and column. It is the group of orthogonal matrices that fix the canonical diagonal* $\mathbb{C}(e_1 + \ldots + e_N)$ *of* $\mathbb{C}^N$.

The meaning of the set Π' *is in every case that*

$\Pi := \Pi' \cup \{$ $\}$

generates the respective category of (two-coloured) partitions.

The following diagram describes the subgroup relations between these groups:

$$\begin{array}{ccccc} B_N & \subseteq & B'_N & \subseteq & O_N \\ \cup\!\!\mid & & \cup\!\!\mid & & \cup\!\!\mid \\ S_N & \subseteq & S'_N & \subseteq & H_N \end{array}$$

If we invert the subset relations we obtain the subset relations for the corresponding categories of partitions. Motivated by this classical situation, the letters O, H, B, S will always be used whenever some categories of partitions should be divided into four cases:

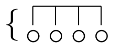
- The case O refers to the situation of only pair partition.
- The case H is the hyperoctahedral situation, i.e. we have no singletons but a block of size four.

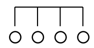
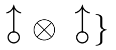
- The Case B is to the bistochastic situation, i.e. blocks of both size one and two but no larger blocks.
- Finally, in the case S we collect all other categories, i.e. where we have blocks of size one, two and four.

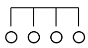
For the analogue of Theorem 2.6.13 in the free, i.e. non-crossing, case we have to invoke again the ordering of points in a partition due to their clockwise appearance in their illustration. We say that a point k lies in between two other points i and j if $i \leq k \leq j$.

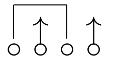
Theorem 2.6.14. *There are seven categories corresponding to free orthogonal quantum groups:*

(1) The free orthogonal groups O_N^+ correspond to $\Pi'^+ = \emptyset$ and the category of all non-crossing, uni-coloured pair partitions.

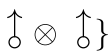
(2) The free hyperoctahedral groups H_N^+ correspond to $\Pi'^+ = \{$  $\}$ and the category of non-crossing, uni-coloured partitions with even block size.

(3) The free alternating permutation groups ${S'_N}^+$ correspond to $\Pi'^+ = \{$  ,  $\}$ and the category of non-crossing, uni-coloured partitions on even numbers of points.

(4) The free permutation groups S_N^+ correspond to $\Pi'^+ = \{$  ,  $\}$ and the category of all non-crossing, uni-coloured partitions.

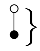
(5) The free alternating bistochastic groups ${B'_N}^+$ correspond to $\Pi'^+ = \{$  $\}$ and the category of all non-crossing, uni-coloured partitions with block sizes one or two and an even number of points. In particular every partition has an even number of singletons.

(6) The free bistochastic groups B_N^+ correspond to $\Pi'^+ = \{$  $\}$ and the category of non-crossing, uni-coloured partitions with block sizes one or two.

(7) The modified free alternating bistochastic groups $B_N^{\#+}$ correspond to $\Pi'^+ = \{$  $\}$ and the category of non-crossing, uni-coloured partitions with block sizes one or two such that the number of points in between two connected points is always even. The last property can be replaced by the requirement, that the number of singletons in between two connected points is even or that the parity

of connected points is always different. In particular, all these properties imply that the number of points is even.

Analogous to Theorem 2.6.13, the meaning of Π' *is in each case that*

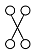
$$\Pi := \Pi'^{+} \cup \{$$  $$\}$$

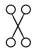
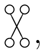
generates the respective category of (two-coloured) partitions.

The subgroup relations between the free easy quantum groups are given by the following diagram:

$$\begin{array}{ccccccc}
B_N^+ & \subseteq & {B'_N}^+ & \subseteq & B_N^{\#+} & \subseteq & O_N^+ \\
\rotatebox[origin=c]{90}{$\subseteq$} & & \rotatebox[origin=c]{90}{$\subseteq$} & & & & \rotatebox[origin=c]{90}{$\subseteq$} \\
S_N^+ & \subseteq & {S'_N}^+ & & \subseteq & & H_N^+
\end{array}$$

Again, the subset relations for the corresponding categories can be obtained by inverting the relations above.

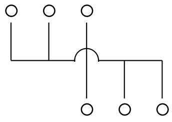
Remark 2.6.15. (i) The free easy quantum groups are quantum versions of their classical analoga but the the term *free* is more specific as it stresses the fact that the defining categories are non-crossing. An orthogonal easy quantum group given by a category of partitions that includes some crossings but not  also deserves to be called a quantum version of a suitable easy group, but it is not a free version of it.

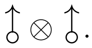
(ii) Comparing Theorem 2.6.13 and Theorem 2.6.14, we see that every orthogonal easy group has exactly one free analogue, but B'_N seems to have two:
On the one hand, $B_N^{\#+}$ is obtained from B'_N by erasing  from the corresponding generating set of partitions Π above, so by erasing the commutation relation for the generators u_{ij} of the corresponding C^*-algebra.
On the other hand, the category that defines ${B'_N}^+$ is obtained from the category defining B'_N by restriction to its non-crossing subset. The reason for this split-up is the fact that in a category of non-crossing, uni-coloured partitions consisting of singletons and pairs we have two possibilities to specify which positions of the singletons are allowed. If we add the partition , these two possibilities collapse again to one category as the crossing in particular enables us to arbitrarily change the position of singletons. As indicated by the notation, only ${B'_N}^+$ can be called the free version of B'_N as it is the one obtained by (full) liberation on the level of the category of partitions and not just on a generating set.

A classification of all remaining orthogonal easy quantum groups can be found in [RW16]. Each is given by a uni-coloured category of partitions and those can be divided into three cases:

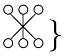
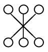
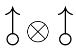
(1) The categories Π_k in the so-called interpolating series $\big(\Pi_k\big)_{k\in\mathbb{N}}$. See [RW16] for a precise definition.

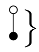
(2) The categories containing the so-called pair-partitioner

 ,

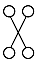
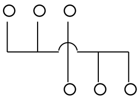
but not the partition . These are uncountably many, but they are uniquely classified by a one-to-one correspondence to strongly symmetric reflection groups on N generators.

(3) The two categories generated by the sets

$\Pi' = \{$  $\}$ or $\Pi' = \{$ ,  $\}$

together with $\{$  $\}$.

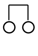
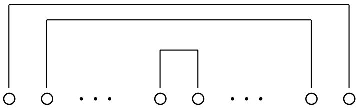
Note that the orthogonal easy group H_N and its free analogue H_N^+ appear again in this setting: H_N^+ corresponds to Π_1 and H_N is associated to

$\Pi' = \{$ ,  $\}$

Classification results: The unitary case

In [TW17] and [TW18] the setting of orthogonal easy quantum groups and uni-coloured partitions was generalized to the two-coloured case. On the level of quantum groups this means that the self-adjointness of the generators u_{ij} was given up and the orthogonality of the matrix u was replaced by unitarity. The equality of the u_{ij} and their adjoints can be replaced by many other (suitable) relations.

In the general situation of two-coloured points, classification results can be found in [TW18], [Gro18] and [MW18a, MW18b]. We concentrate on the classification in the group case and in the free case as done in [TW18]. Before doing so, we introduce a few notations. Denote for $k \in \mathbb{N}$ with b_k (and $\overline{b}_k$) the one-block partition with k white (respectively black) lower points. For $k \in 2\mathbb{N}_0$ let  $^{\mathrm{nest}(\frac{k}{2})}$ be the partition where the uni-coloured pair partition is put $\frac{k}{2}$-times into itself:

 $^{\mathrm{nest}(\frac{k}{2})} :=$  .

As mentioned before, a series of easy quantum groups $(G_N)_{N\in\mathbb{N}}$ associated to a fixed category of partitions is a series of easy groups if and only if the category contains the crossing-partition . In contrast to the orthogonal case, where the number of possible categories including the crossing partition was quite limited, we have infinitely many in the two-coloured case. They can be divided into 7 cases. The following table summarizes the classification by presenting a generating set of partitions Π for every possible category of partitions. For further explanations and motivation for this classification, we refer to [TW18].

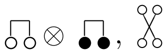
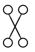
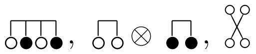
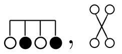
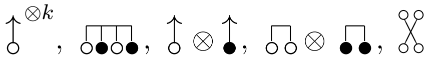
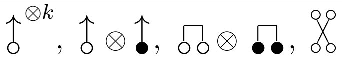
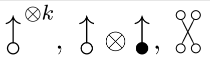
Case	Elements in Π	Parameter range
$\mathcal{O}_{\text{grp,glob}}(k)$		$k \in 2\mathbb{N}_0$
$\mathcal{O}_{\text{grp,loc}}$		–
$\mathcal{H}_{\text{grp,glob}}(k)$	b_k, 	$k \in 2\mathbb{N}_0$
$\mathcal{H}_{\text{grp,loc}}(k,l)$	$b_k, b_l \otimes \bar{b}_l$, 	$k,l \in \mathbb{N}_0 \backslash \{1,2\}, l\vert k$
$\mathcal{S}_{\text{grp,glob}}(k)$		$k \in \mathbb{N}_0$
$\mathcal{B}_{\text{grp,glob}}(k)$		$k \in 2\mathbb{N}_0$
$\mathcal{B}_{\text{grp,loc}}(k)$		$k \in \mathbb{N}_0$

Table 2.1: Classification of categories of two-coloured partitions associated to (series) of unitary easy groups.

If a category contains the partition , we speak of a globally colourized partition as this partition enables us to permute the colouring of partitions in an arbitrary way. In the opposite case we speak of a locally colourized category of partitions.

The orthogonal easy groups, in the order they appear in Theorem 2.6.13, are given by the cases $\mathcal{O}_{\text{grg,glob}}(2), \mathcal{H}_{\text{grg,glob}}(2), \mathcal{S}_{\text{grg,glob}}(2), \mathcal{S}_{\text{grg,glob}}(1), \mathcal{B}_{\text{grg,glob}}(2)$ and $\mathcal{B}_{\text{grg,loc}}(1)$. As it is the smallest category, and therefore the largest easy group, we mention the case $\mathcal{O}_{\text{grp,loc}}$. The associated easy group is the unitary group U_N consisting of all unitary $N \times N$-matrices.

A similar result is obtained in the free case: The following table contains all non-crossing categories of partitions. Again Π denotes a possible set of generators for the respective category of partitions.

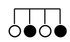
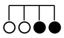
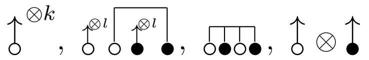
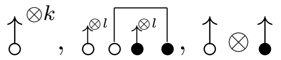
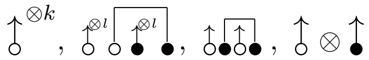
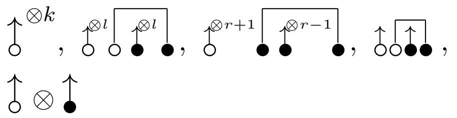
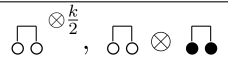
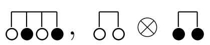
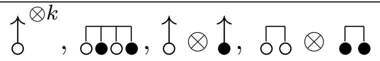
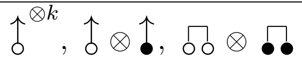
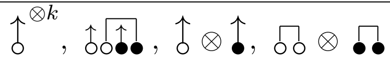
Case	Elements in Π	Parameter range
$\mathcal{O}_{\text{loc}}$	$\emptyset$	–
$\mathcal{H}'_{\text{loc}}$		–
$\mathcal{H}_{\text{loc}}(k,l)$	$b_k, b_l \otimes \bar{b}_l$, 	$k,l\in\mathbb{N}_0\backslash\{1,2\}, l\|k$
$\mathcal{S}_{\text{loc}}(k,l)$		$k,l\in\mathbb{N}_0\backslash\{1\}, l\|k$
$\mathcal{B}_{\text{loc}}(k,l)$		$k,l\in\mathbb{N}_0, l\|k$
$\mathcal{B}'_{\text{loc}}(k,l,0)$		$k,l\in\mathbb{N}_0\backslash\{1\}, l\|k$
$\mathcal{B}'_{\text{loc}}(k,l,\frac{l}{2})$		$k\in\mathbb{N}_0\backslash\{1\}$, $l\in 2\mathbb{N}_0\backslash\{0,2\}$, $l\|k$, $r=\frac{l}{2}$
$\mathcal{O}_{\text{glob}}(k)$		$k\in 2\mathbb{N}_0$
$\mathcal{H}_{\text{glob}}(k)$	b_k, 	$k\in 2\mathbb{N}_0$
$\mathcal{S}_{\text{glob}}(k)$		$k\in\mathbb{N}_0$
$\mathcal{B}_{\text{glob}}(k)$		$k\in 2\mathbb{N}_0$
$\mathcal{B}'_{\text{glob}}(k)$		$k\in\mathbb{N}_0$

Table 2.2: Classification of categories of non-crossing two-coloured partitions.

2.7 Actions of quantum matrices on quantum vectors

2.7.1 Quantum spaces and quantum vectors

In the same way we introduced quantum groups as non-commutative versions of the function algebras over groups we can define quantum spaces. By Gelfand-Naimark every commutative unital C^*-algebra is isomorphic to the continuous complex valued functions on a compact space. Hence, a compact quantum space is just another perspective on an arbitrary (so potentially non-commutative) unital C^*-algebra.

Definition 2.7.1. Let A be a unital C^*-algebra. Then we call its elements the *non-commutative functions on a compact quantum space* X and write $A = C(X)$.

Likewise we could consider arbitrary C^*-algebras as (non-commutative) functions on locally compact quantum spaces.
Considering for $N \in \mathbb{N}$ a compact set of vectors inside $\mathbb{C}^N$, the continuous functions on this set are generated by the coordinate functions $x_1, \ldots, x_N$, mapping each vector to one of its entries. Thus, we have

Definition 2.7.2. Let $N \in \mathbb{N}$ and $X := (A, x)$ be a pair consisting of a unital C^*-algebra A and an N-vector

$$x = \begin{pmatrix} x_1 \\ \vdots \\ x_N \end{pmatrix} \in \mathbb{C}^N \otimes A.$$

If A is generated by the entries of x then we call X a compact quantum space of vectors and write $A = C(X)$.

Analogous to the notation $G = (B, v)$ for a CMQG, we write $X = (A, x)$ for a compact quantum space of vectors.
With this definition at hand and the idea of unitary compact matrix quantum groups in mind it is quite canonical what the complex quantum sphere and a subset of it should be:

Definition 2.7.3. Let $N \in \mathbb{N}$ and consider the (unital) C^*-algebra A generated by the entries of a vector

$$x = \begin{pmatrix} x_1 \\ \vdots \\ x_N \end{pmatrix} \in \mathbb{C}^N \otimes A.$$

and the relations

$$\sum_{k=1}^{N} x_k^* x_k = \mathbb{1} = \sum_{k=1}^{N} x_k x_k^*.$$

Then we call the compact quantum space $X := (A, x)$ the complex quantum sphere and write $A = C(X)$.
A compact quantum space of vectors $X' = (A', x')$ is called a subspace of the complex quantum sphere if there is a *-homomorphism $\varphi : A \to A'$ such that $(\mathbb{1} \otimes \varphi)x = x'$.

Likewise we speak of the (real) quantum sphere and a compact subspace of it if the generators x_k and x'_k are all self-adjoint.

2.7.2 Quantum group actions on quantum spaces

A left action $G \curvearrowright X$ of a compact group G on a compact space X is given by a group homomorphism φ from G into the group of (continuous) mappings on X. In this case we write $g(x) := \varphi(g)(x)$ for given $g \in G$ and $x \in X$. It is called faithful if the neutral element of G is mapped to the identity on X, i.e. $\varphi(G)(X) = X$. Likewise we speak of a right action $X \curvearrowleft G$ if φ is an anti-homomorphism.
The description of actions in the quantized setting has to be given again on the level of functions over the objects:

Definition 2.7.4. Let $G = (A, \Delta)$ be a compact quantum group and X a compact quantum space.

(i) A faithful left action $G \curvearrowright X$ is a *-homomorphism $\alpha : C(X) \to C(G) \otimes C(X)$ such that the following holds:

  (a) $(\mathrm{id}_{C(G)} \otimes \alpha) \circ \alpha = (\Delta \otimes \mathrm{id}_{C(X)}) \circ \delta$

  (b) The linear span of the set $\alpha(X)(C(G) \otimes \mathbb{1}_{C(X)})$ is dense in $C(G) \otimes C(X)$.

(ii) A faithful right action $X \curvearrowleft G$ is a *-homomorphism $\beta : C(X) \to C(G) \otimes C(X)$ such that the following holds:

  (a') $(\Sigma \otimes \mathrm{id}_{C(X)}) \circ (\mathrm{id}_{C(G)} \otimes \beta) \circ \beta = (\Delta \otimes \mathrm{id}_{C(X)} \circ \beta)$

  (b) The linear span of the set $\alpha(X)(C(G) \otimes \mathbb{1}_{C(X)})$ is dense in $C(G) \otimes C(X)$.

Here Σ denotes the linear flip map on $C(G) \otimes C(G)$ defined by $a \otimes b \mapsto b \otimes a$.

Again, the definition is justified by the fact that in the commutative case we end up with classical faithful group actions on compact spaces:

Proposition 2.7.5. *Let $G = (A, \Delta)$ be a compact quantum group and $X = (B, x)$ be a compact quantum space. Let α be a faithful left action $G \curvearrowright X$. If A and B are commutative, then there is a faithful left action φ of the compact group G on the compact space X such that $\alpha : C(X) \to C(G) \otimes C(X) = C(G \times X)$ fulfils*

$$\alpha(f) : G \times X \to \mathbb{C} \; ; \; (g, x) \mapsto f\big(\varphi(g)(x)\big) = f\big(g(x)\big)$$

for all $f \in C(G)$.

Proof. WE use techniques similar to the proof of Proposition 2.1.2 and we just sketch the main idea.
We can define a (classical) action of the group G on the space X with the help of α: Having in mind that X and G are the spaces of characters on A and B, respectively, we define for $g \in G$, $x \in X$ and arbitrary $f \in B$:

$$\big(g(x)\big)(f) := (g \otimes x)\alpha(f),$$

i.e. $g(x)$ is a map $B \to \mathbb{C}$. It is a character as α is a *-homomorphism and both g and x are characters. Condition (a) from Definition 2.7.4 guarantees that the action is associative, i.e. the map φ that identifies group elements with mappings on X is a group homomorphism. Finally, condition (b) is used to show that the action is faithful. By associativity we already know that the neutral element of G acts as an idempotent on X and if it is not the identity, then condition (b) could not be fulfilled. $\square$

The proof for a right action β is nearly the same. The flip map Σ is important for the action from the right, i.e. we can prove in this case (as desired) that $(gh)(x) = h(g(x))$ for all $g, h \in G$ and $x \in X$.

Restricting to compact matrix quantum groups and compact spaces of quantum vectors, we can define quantized matrix-vector actions.

Definition 2.7.6. Let $G = (C(G), u)$ be a compact matrix quantum group and $X = (C(X), x)$ be a compact quantum space of vectors such that u is an $N \times N$-matrix and x an N-vector.

(i) A faithful left matrix-vector action $G \curvearrowright X$ is a unital *-homomorphism $\alpha : C(X) \to C(G) \otimes C(X)$ defined by

$$\alpha(x_i) := \sum_{k=1}^{N} u_{ik} \otimes x_k \qquad \text{for } 1 \leq i \leq N.$$

(ii) A faithful right matrix-vector action $X \curvearrowleft G$ is a unital *-homomorphism $\beta : C(X) \to C(G) \otimes C(X)$ defined by

$$\alpha(x_i) := \sum_{k=1}^{N} u_{ki} \otimes x_k \qquad \text{for } 1 \leq i \leq N.$$

It is not difficult to show that faithful matrix-vector actions are faithful actions in the sense of Definition 2.7.4. The special form of the comultiplication Δ in the setting of CMQGs guarantees condition (a) and (a'), respectively, and the invertibility of u and $u^{(*)}$ assumed for CMQGs finally implies the density condition b.

In the classical situation we can consider tupels $(v_1, \dots, v_d)$ of vectors and let matrices M act on them entrywise:

$$\big((v_1, \dots, v_d\big) \mapsto \big(Mv_1, \dots, Mv_d\big)$$

Analogously, we can define a generalization of quantized matrix-vector actions:

Definition 2.7.7. Let $d, N \in \mathbb{N}$. Consider a unital C^*-algebra A and a d-tupel of N-vectors over A

$$x := (\begin{pmatrix} x_{11} \\ \vdots \\ x_{N1} \end{pmatrix}, \dots, \begin{pmatrix} x_{1d} \\ \vdots \\ x_{Nd} \end{pmatrix}).$$

If A is generates by the vector entries $x_{11}, \dots, x_{Nd}$ then the pair $X := (A, x)$ is called a compact quantum space of d vectors. The elements of A are called non-commutative function over the compact quantum space of d vectors and we also write $A =: C(X)$.

Definition 2.7.8. Let $d, N \in \mathbb{N}$. Consider a compact matrix quantum group $G = (A, u)$ and a compact quantum space $X = (B, x)$ of d vectors such that both the matrix u and the vectors in the tupel x have size N.

(i) A faithful left matrix-vector action $G \curvearrowright X$ is a unital *-homomorphism $\alpha : C(X) \to C(G) \otimes C(X)$ such that

$$\alpha(x_{ij}) = \sum_{k=1}^{N} u_{ik} \otimes x_{kj} \qquad \text{for all } 1 \leq i \leq N, 1 \leq j \leq d.$$

(ii) A faithful right matrix-vector action $X \curvearrowleft G$ is a unital *-homomorphism $\alpha : C(X) \to C(G) \otimes C(X)$ such that

$$\alpha(x_{ij}) = \sum_{k=1}^{N} u_{ki} \otimes x_{kj} \qquad \text{for all } 1 \leq i \leq N, 1 \leq j \leq d.$$

2.7.3 Quantum symmetry groups

In Chapter 5 we will consider compact quantum spaces of d vectors and ask for the maximal objects in the category of CMQGs acting on these quantum spaces by matrix-vector actions. This will define *quantum symmetry groups* of those quantum spaces.

Definition 2.7.9. Let $G = (C(G), u)$ be a compact matrix quantum group and $X = (C(X), x)$ be a compact quantum space of d vectors such that x is of the form

$$x := (\begin{pmatrix} x_{11} \\ \vdots \\ x_{N1} \end{pmatrix}, \dots, \begin{pmatrix} x_{1d} \\ \vdots \\ x_{Nd} \end{pmatrix}).$$

We call G the quantum symmetry group of X and write $G = \mathrm{QSymG}(X)$ if the following are fulfilled:

(i) G consists of $N \times N$-matrices, i.e. u is an $N \times N$-matrix of generating elements for $C(G)$.

(ii) There is a faithful left and a faithful right matrix-vector action of G on X.

(iii) G is the universal compact matrix quantum group fulfilling (i) and (ii), i.e. if $G' = (A', u')$ is another compact matrix quantum group fulfilling (i) and (ii), then it holds $G' \subseteq G$, compare Definition 2.2.5.

Obviously, the quantum symmetry group of a compact quantum space of vectors is unique up to equivalence.

Chapter 3

Linear independence of the maps T_p

In the previous chapter we described how, for a given natural number N, a (suitable) category of partitions $\mathcal{C}$ defines a compact matrix quantum group. It heavily uses Tannaka-Krein duality in the sense that in the two-step construction

$$\mathcal{C} \overset{\Psi}{\mapsto} R_N(\mathcal{C}) \overset{\Phi}{\mapsto} G_N(\mathcal{C}) \tag{3.0.1}$$

the second construction, Φ, that associates a compact matrix quantum group $G_N(\mathcal{C})$ to a given essential concrete monoidal W^*-category $R_N(\mathcal{C})$, is well-defined and injective.

Recall that the objects in $R_N(\mathcal{C})$ are given by words ω over the alphabet $\{1,^*\} = \{\circ, \bullet\}$, the Hilbert space H_ω associated to ω is $\left(\mathbb{C}^N\right)^{\otimes|\omega|}$ and the spaces of morphisms are given by

$$\operatorname{Mor}(\omega, \omega') := \operatorname{span}\left(\{T_p \mid p \in \mathcal{C}(\omega, \omega')\}\right) \subseteq B(H_\omega, H_{\omega'}).$$

See Definition 2.5.4 for a description of the maps T_p.

The whole construction in Equation 3.0.1 is surjective by definition of easy quantum groups, but the question we left open up to now is, if we also have injectivity for the first mapping Ψ. The main result of this chapter concentrates on the case of non-crossing categories of partitions and is given by Theorem 3.3.1: Consider a non-crossing category of partitions $\mathcal{C} \subseteq \mathcal{NC}$ an let ε denote the empty word. Then for $N \geq 4$ and any fixed word ω' the collection of maps $\left(T_p\right)_{p\in\mathcal{C}(\varepsilon,\omega')}$ is linearly independent.

From this it can easily be deduced that the functor $\mathcal{C} \overset{\Psi}{\mapsto} R_N(\mathcal{C})$ is injective for the considered cases and so does the whole construction in Equation 3.0.1, see Corollary 3.1.4. In order to prove the above, we recapture W. Tutte's work [Tut93], see below.

The organization of this chapter is as follows: In Section 3.1 we reformulate the question of injectivity of the functor Ψ as a linear independence problem on the level of maps T_p associated to partitions p. The subsequent Section 3.2 deals with this problem in the general situation (of arbitrary partitions). We present, see Proposition 3.2.3 and Corollary 3.2.6, a more detailed version of Proposition 2.6.3, saying in particular that for two different given categories of partitions $\mathcal{C}_1$ and $\mathcal{C}_2$ we find at least some $N \in \mathbb{N}$ such that the respective easy quantum groups $G_N(\mathcal{C}_1)$ and $G_N(\mathcal{C}_2)$ differ.

With Proposition 3.2.4 we also state some kind of converse result: Considering the category $\mathcal{P}$ of all two-coloured partitions, a linear basis for the space $\operatorname{Hom}(\varepsilon, \omega')$ is already given by the collection of those T_p where $p \in \mathcal{C}(\varepsilon, \omega')$ has at most N blocks.

The main part of the chapter is Section 3.3. It focusses on the non-crossing case, i.e. free easy quantum groups. It is known that different categories of non-crossing partitions produce different easy quantum groups, at least if $N \geq 4$, i.e. the fundamental

corepresentation matrix has at least 4 rows and columns. A common reference from which this can be deduced is W. Tutte's work [Tut93] on the matrix of chromatic joints. It is a purely combinatorial work on the level of (uni-coloured) partitions on one line. One of its advantages is, that it does not use any deep results from other theories. It is straightforward to follow, once the relevant objects have been properly named. As the consequence of Tutte's work in the context of easy quantum groups is well-known, it is worth to motivate this part of the thesis:
The aim of Section 3.3 is to present a self-contained proof of the linear independence of maps T_p in the non-crossing case, without leaving greater parts to other sources or a potential reader. Some of Tutte's arguments turned out to be wrong and in this virtue we adapted the original work. Other parts are changed to fit common notations in the context of easy quantum groups or they are extended to justify results where proofs have been omitted in Tutte's article. In addition, we linked definitions and arguments to the graphical presentations of partitions as introduced in Section 2.4. In this sense the present work contributes to the theory of free easy quantum groups by ensuring known results to stand on solid ground. See also Section 3.3.7, the purpose of which is to make clear for which ideas and results credit is due to Tutte and his work.

3.1 Reformulating the injectivity of Ψ as a problem of linear independence

By definition, CMQGs can only be equal (in the sense of equivalent universal forms) if their fundamental corepresentation matrices have the same size. By Theorem 2.3.25, two easy quantum groups G_1 and G_2, given by categories of partitions $\mathcal{C}_1$ and $\mathcal{C}_2$ and the same natural number $N \in \mathbb{N}$, are equal if and only if all intertwiner spaces $\mathrm{Hom}(\omega, \omega') \subseteq B\big(\big(\mathbb{C}^N\big)^{\otimes|\omega|}, \big(\mathbb{C}^N\big)^{\otimes|\omega'|}\big)$ coincide. So the question of (in)equality of G_1 and G_2 can be formulated on the level of intertwiner maps:

Question 3.1.1. Consider two categories of two-coloured partitions $\mathcal{C}_1$ and $\mathcal{C}_1$ and fix some $N \in \mathbb{N}$.
Do we have

$$(\mathrm{span}\left(\{T_p \,|\, p \in \mathcal{C}_1(\omega, \omega')\}\right) \neq \mathrm{span}\left(\{T_p \,|\, p \in \mathcal{C}_2(\omega, \omega')\}\right)$$

for at least one pair of words ω and ω' over the alphabet $\{1,^*\}$? In that case the associated easy quantum groups $G_N(\mathcal{C}_1)$ and $G_N(\mathcal{C}_2)$ differ.

Our first simplification is the restriction to the case $\omega = \varepsilon$, the empty word.

Proposition 3.1.2. *Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be two categories of partitions. Consider for $k, l \in \mathbb{N}_0$ two words $\omega \in \{1,^*\}^k$ and $\omega' \in \{1,^*\}^l$. Then the sets*

$$\{T_p \mid p \in \mathcal{C}_1(\omega, \omega')\} \qquad \text{and} \qquad \{T_q \mid q \in \mathcal{C}_2(\omega, \omega')\}$$

span the same space of operators in $B\big(\big(\mathbb{C}^N\big)^{\otimes k}, \big(\mathbb{C}^N\big)^{\otimes l}\big)$ if and only if their rotated versions span the same space of operators.

Proof. Without restriction assume $k > 0$ and consider the rotation $\mathrm{rot}_{\curvearrowright}$. Note that by Proposition 2.4.5 the categories above are closed under rotations.
The statement above is just the observation that an equation $\sum_p \alpha_p T_p = \sum_q \beta_q T_q$ implies

$$\sum_p \alpha_q \mathrm{rot}_{\curvearrowright}(T_p) = \sum_q \beta_q \mathrm{rot}_{\curvearrowright}(T_q).$$

Of course the analogous argument works for all other rotations, showing the ‘only if’ part of the claim. The ‘if’ part is the observation that rotation operations are invertible. ☐

Corollary 3.1.3. *Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be two categories of partitions. Fix $N \in \mathbb{N}$ and let $G_N(\mathcal{C}_1) = (A_1, u_1)$ and $G_N(\mathcal{C}_2) = (A_2, u_2)$ be the respective easy quantum groups. Then the universal forms of $G_N(\mathcal{C}_1)$ and $G_N(\mathcal{C}_2)$ are equal if and only if for all words ω' over the alphabet $\{1,^*\}$ the spaces*

$$\operatorname{span}\big(\{T_p \mid p \in \mathcal{C}_1(\varepsilon, \omega')\}\big) \qquad \text{and} \qquad \operatorname{span}\big(\{T_q \mid q \in \mathcal{C}_2(\varepsilon, \omega')\}\big)$$

coincide.

This result enables us to formulate a sufficient condition for different categories to produce different easy quantum groups:

Corollary 3.1.4. *Consider the situation of Corollary 3.1.3 with two different categories $\mathcal{C}_1$ and $\mathcal{C}_2$. If there is a word ω' over the alphabet $\{1,^*\}$ such that $\mathcal{C}_1(\varepsilon, \omega') \neq \mathcal{C}_2(\varepsilon, \omega')$ and the collection*

$$\Big(T_p\Big)_{p \in \mathcal{C}_1(\varepsilon,\omega') \cup \mathcal{C}_2(\varepsilon,\omega')}$$

is linear independent, then $G_N(\mathcal{C}_1)$ and $G_N(\mathcal{C}_2)$ have different universal forms.

Proof. The corollary describes a sufficient condition for the operator spaces $\mathrm{Mor}_1(\varepsilon, \omega')$ and $\mathrm{Mor}_2(\varepsilon, \omega')$ in the respective essential concrete monoidal W^*-categories $R_N(\mathcal{C}_1)$ and $R_N(\mathcal{C}_2)$ to be different. Using Tannaka-Krein duality, we conclude in this case that the associated easy quantum groups cannot be equal. ☐

In order to use Corollary 3.1.4 (finally to answer Question 3.1.1) we aim to solve the following problem:

Question 3.1.5. Consider a category of partitions $\mathcal{C}$ and $N \in \mathbb{N}$. Under which conditions, on $\mathcal{C}$ and/or N, is the collection

$$\left(T_p\right)_{p \in \mathcal{C}(\varepsilon,\omega')}$$

for a given word ω' over $\{1,^*\}$ linearly independent?

Given linear independence as described above, we have that different subcategories of $\mathcal{C}$ produce different easy quantum groups.

Remark 3.1.6. Apart from the question of one-to-one correspondence between categories of partitions and easy quantum groups, we have other problems and theories depending a lot on the question of linear independence of the maps T_p. For example the fusion rules of easy quantum groups are well-established in this linear independent situation but not in the general one, see [Fre14] and [FW14].

3.2 Linear (in)dependencies in the general situation

In virtue of Corollary 3.1.4 it is worth to investigate linear (in)dependencies of the maps T_p where p is from some fixed $\mathcal{C}(\varepsilon,\omega')$. Recall that in this case T_p is a map from $\left(\mathbb{C}^N\right)^{\otimes 0} = \mathbb{C}$ to $\left(\mathbb{C}^N\right)^{\otimes|\omega'|}$. It turns out that the crucial point is the relation between the natural number $N \in \mathbb{N}$ and the number of blocks in the partitions $p \in \mathcal{C}(\varepsilon,\omega')$.
In order to state the results and their proofs we need two further notations. They are well known, see for example [MS17, p. 35] and [NS06, Def. 9.14], respectively.

Definition 3.2.1. Let ω' be a word of length $|\omega'| = l$ over $\{1,^*\}$ and $j = (j_1,\dots,j_l) \in \mathbb{N}^l$ a multi-index. Then we denote with $\ker(j) \in \mathcal{P}(\varepsilon,\omega')$ the partition on l lower points where j numbers its points from left to right such that points are connected by $\ker(j)$ if and only if they have the same number.

Note that the word ω' in the definition above is only necessary to define the colours of the points in the partition $\ker(j)$. If we are not interested in the colouring, we only speak of $\ker(j) \in \mathcal{P}(0,l)$ without mentioning any ω'.

Definition 3.2.2. Let $p, q \in \mathcal{P}(\omega,\omega')$ for some colouring (ω,ω'). We write $p \preceq q$ if and only if each valid labelling (i,j) of q is also a valid labelling of p.

In other words, $p \preceq q$ holds if we can obtain p from q by refining the blocks in the partition. This is obviously a partial ordering, the partition that has only singletons as blocks is the minimal element in $\mathcal{P}(\omega,\omega)$ and the one-block partition, where all points are connected to each other, is the maximal element. The following result can also be found in [Maa18, Lem. 3.4], for example.

Proposition 3.2.3. *Let $N \in \mathbb{N}$ and ω' be a word of length $|\omega'| = l$ over $\{1,^*\}$. Consider for every partition $p \in \mathcal{P}(\varepsilon, \omega')$ the linear map $T_p : \left(\mathbb{C}^N\right)^{\otimes 0} = \mathbb{C} \to \left(\mathbb{C}^N\right)^{\otimes l}$ as defined in Definition 2.5.4. Denote further*

$$\mathcal{P}_N(\varepsilon, \omega') := \{p \in \mathcal{P}(\varepsilon, \omega') \mid p \text{ has at most } N \text{ blocks}\}.$$

Then the collection of maps

$$\left(T_p\right)_{p \in \mathcal{P}_N(0,\omega')}$$

is linearly independent.

Proof. As the colours of the points will not play a role, we can assume all colours $\omega'_1, \ldots, \omega'_l$ to be white. Let $e_1, \ldots, e_N$ be the standard orthonormal basis of $\mathbb{C}^N$ that has already been used to define the maps T_p. Given a multi-index $j = (j_1, \ldots, j_l)$ we write $e_j := e_{j_1} \otimes \ldots \otimes e_{j_l} \in \left(\mathbb{C}^N\right)^{\otimes l}$.
Consider now a linear combination

$$0 = \sum_{p \in \mathcal{P}_N(\varepsilon,\omega')} \alpha_p T_p. \tag{3.2.1}$$

We prove $\alpha_p = 0$ by induction on the number of blocks of p. The base case is the largest possible number of blocks, say M, so consider an arbitrary partition p with exactly M blocks. The idea is to show that there is a direction $\langle v \rangle$ in $\left(\mathbb{C}^N\right)^{\otimes l}$ such that all $T_q(1)$ are orthogonal to $\langle v \rangle$ except $T_p(1)$.
As M does not exceed N by assumption, we find a multi-index $j \in [N]^l$ such that $\ker(j)$ coincides with p. We now claim the following:

$$\langle T_q(1), e_j \rangle = \begin{cases} 1 & , q = p \\ 0 & , q \neq p \end{cases}. \tag{3.2.2}$$

To prove this, recall that for $q \in \mathcal{P}(\varepsilon, \omega')$ the map T_q is uniquely defined by the image $T_q(1)$ and it holds

$$T_q(1) = \sum_{\substack{i \in [N]^l \\ q \preceq \ker(i)}} e_i. \tag{3.2.3}$$

Therefore, the case $q = p$ in the claimed Equation 3.2.2 is clear. Now assume $q \neq p$ and consider a multi-index $i \in [N]^l$ with $q \preceq \ker(i)$ as in the summation in Equation 3.2.3. As q has at most as many blocks as $p = \ker(j)$, there are two points in the same block of q which are in different blocks of $\ker(j)$. Together with $\ker(i) \succeq q$ we deduce that these two points are in the same block of $\ker(i)$, i.e. i and j must differ at least at one entry. But then we have $\langle e_i, e_j \rangle = 0$ for all $\ker(i) \succeq q$ and this shows

the claimed identity $\langle T_q(1), e_j\rangle = 0$ for $q \neq p$.
Combining Equation 3.2.2 with the assumption, Equation 3.2.1, we deduce

$$\alpha_p = \langle \sum_{p\in\mathcal{P}_N(\varepsilon,\omega')} \alpha_p T_p(1), e_j\rangle = 0.$$

Of course this argument holds separately for all p with exactly M blocks, finishing the base case.
The induction step is just the observation that we can repeat the arguments above for coefficients α_q not yet proven to be zero. After at most M steps we have proved $\alpha_p = 0$ for all p, so linear independence holds as claimed. □

We now prove the converse result of Proposition 3.2.3: Whilst for given $N \in \mathbb{N}$ and a fixed pair of colourings (ω, ω') the partitions $\mathcal{P}_N(\omega, \omega')$ give rise to a linear independent set of maps T_p, we prove now that the remaining maps T_q do not enlarge the generated space of linear maps.

Proposition 3.2.4. *In the situation of Proposition 3.2.3 we have*

$$\operatorname{span}\big(\{T_p \mid p \in \mathcal{P}_N(\varepsilon, \omega')\}\big) = \operatorname{span}\big(T_p \mid p \in \mathcal{P}(\varepsilon, \omega')\}\big).$$

Hence, by Proposition 3.2.3, the collection $\big(T_p\big)_{p\in P_N(\varepsilon,\omega')}$ *is a basis for* $\operatorname{Hom}(\varepsilon, \omega')$.

Proof. We use the same notations as in the proof of Propostion 3.2.3. Recall that $b(p)$ denotes the number of blocks of a partition p.
For a partition q with more than N blocks we have to prove

$$T_q \in \operatorname{span}\big(\{T_p \mid p \in \mathcal{P}_N(\varepsilon, \omega')\}\big).$$

To do so, we recursively construct linear combinations $L_N, L_{N-1}, \ldots, L_1$ of T_p's fulfilling the following:

$$b\big(\ker(j)\big) \geq n \quad \Rightarrow \quad \langle L_n(1), e_j\rangle = \langle T_q(1), e_j\rangle \quad ,$$

for $1 \leq n \leq N$. Roughly speaking, with decreasing n, $L_n(1)$ will coincide with $T_q(1)$ in more and more directions until finally $L_1(1)$ coincides with $T_q(1)$, so $L_1 = T_q$.
Base case: Construction of L_N: Consider the linear combination

$$L_N := \sum_{\substack{q\preceq p \\ b(p)=N}} T_p \tag{3.2.4}$$

and we have to show that

$$\langle L_N(1), e_j\rangle = \langle T_q(1), e_j\rangle$$

whenever $\ker(j)$ has N blocks. Using the definitions of L_N and T_q, this Equation reads

$$\langle \sum_{\substack{q \preceq p \\ b(p)=N}} \sum_{\substack{i \in [N]^l \\ p \preceq \ker(i)}} e_i, e_j \rangle = \langle \sum_{\substack{i \in [N]^l \\ q \preceq \ker(i)}} e_i, e_j \rangle. \tag{3.2.5}$$

We have to prove that this is true whenever $b\big(\ker(j)\big) = N$.

Case 1: $q \not\preceq \ker(j)$: On both the left and the right side of Equation 3.2.5 it holds $q \preceq \ker(i)$. Together with $q \not\preceq \ker(j)$ and transitivity of $\preceq$ this implies $\ker(i) \not\preceq \ker(j)$, so $i \neq j$ in all cases and both sides vanish.

Case 2: $q \preceq \ker(j)$: As $\ker(j)$ has N blocks, $\ker(j)$ is equal to exactly one partition $\hat{p} \in P_N(\varepsilon, \omega')$ and for all other partitions $p \in P_N(\varepsilon, \omega')$ it holds $p \not\preceq \ker(j)$. As above, we deduce for $p \not\preceq \ker(j)$ and $p \preceq \ker(i)$ that $i \neq j$. Consequently, Equation 3.2.5 reads

$$\langle \sum_{\substack{i \in [N]^l \\ \hat{p} \preceq \ker(i)}} e_i, e_j \rangle = \langle \sum_{\substack{i \in [N]^l \\ q \preceq \ker(i)}} e_i, e_j \rangle.$$

As $\hat{p} := \ker(j)$ and $q \preceq \ker(j)$, we have that e_j appears on both sides as one of the summands e_i. So we end up with a true statement, finishing the base case.

Induction step: Constructing L_M from L_{M+1}:

Assume that there is for some $1 \leq M < N$ a linear combination

$$L_{M+1} = \sum_{\substack{q \preceq p \\ M+1 \leq b(p) \leq N}} \alpha_p T_p$$

such that

$$b\big(\ker(j)\big) \geq M+1 \quad \Rightarrow \quad \langle L_{M+1}(1), e_j \rangle = \langle T_q(1), e_j \rangle. \tag{3.2.6}$$

We want to construct some L_M such that the analogue of Equation 3.2.6 holds for all j with $b\big(\ker(j)\big) \geq M$.

For a given $p \succeq q$ with $b(p) = M$ consider any $\hat{k} \in [N]^l$ with $\ker(\hat{k}) = p$ and define

$$\alpha_p := 1 - \langle L_{M+1}(1), e_{\hat{k}} \rangle.$$

Note that this is independent of the chosen $\hat{k}$.

Now enlarge the sum L_{M+1} in the following way:

$$L_M := L_{M+1} + \sum_{\substack{q \preceq p \\ b(p)=M}} \alpha_p T_p. \tag{3.2.7}$$

We have to show that L_M fulfils

$$b\big(\ker(j)\big) \geq M \quad \Rightarrow \quad \langle L_M(1), e_j\rangle = \langle T_q(1), e_j\rangle. \tag{3.2.8}$$

Case 1: $b\big(\ker(j)\big) \geq M+1$**:** Considering in Equation 3.2.7 the added summands $\alpha_p T_p$ (with $b(p) = M$), we see

$$\langle T_p(1), e_j\rangle = 0$$

because there are at least $M+1$ different entries in the multi-index j. We conclude that L_M fulfils the properties assumed on L_{M+1}:

$$b(\ker(j)\big) \geq M+1 \quad \Rightarrow \quad \langle L_M(1), e_j\rangle = \langle T_q(1), e_j\rangle$$

Case 2: $b\big(\ker(j)\big) = M$**:** If $\ker(j) \not\succeq q$, then both $\langle L_M(1), e_j\rangle$ and $\langle T_q(1), e_j\rangle$ are zero. The arguments are the same as in the base case.
If $\ker(j) \succeq q$, then $\ker(j)$ coincides with one partition $\hat{p} \in \{p \mid p \succeq q\,,\, b(p) = M\}$ and as in the base case one proves for p with $b(p) = M$:

$$\langle T_p(1), e_j\rangle = \begin{cases} 1 & , p = \hat{p} \\ 0 & , p \neq \hat{p} \end{cases}$$

Hence, by definition of α_p, it holds

$$\langle L_M(1), e_j\rangle = \langle L_{M+1}(1), e_j\rangle + \alpha_{\hat{p}} = 1,$$

which is equal to $\langle T_q(1), e_j\rangle = \langle e_j, e_j\rangle$, proving Implication 3.2.8 also in the case $b\big(\ker(j)\big) = M$. □

Remark 3.2.5. While Proposition 3.2.3 obviously holds if we replace $\mathcal{P}$ by any smaller category of partitions, the proof of Proposition 3.2.4 used the fact that, with every partition q, the set $\mathcal{P}(\varepsilon, \omega')$ also contains all partitions $p \succeq q$, i.e. any partition obtained from q by fusing different blocks.

We use Proposition 3.2.3 to state and prove a more precise version of Proposition 2.6.3.

Corollary 3.2.6. *Consider two different categories of partitions* $\mathcal{C}_1$ *and* $\mathcal{C}_2$*. Let* $M \in \mathbb{N}$ *be the smallest integer such that* $\mathcal{C}_1(0, M) \neq \mathcal{C}_2(0, M)$*. Then* $G_N(\mathcal{C}_1) \neq G_N(\mathcal{C}_2)$ *holds for all* $N \geq M$*.*
In particular, the sequences

$$\big(G_N(\mathcal{C}_1)\big)_{N\in\mathbb{N}} \qquad \text{and} \qquad \big(G_N(\mathcal{C}_1)\big)_{N\in\mathbb{N}}$$

differ.

Note that M might not be the smallest value for N such that $G_N(\mathcal{C}_1) \neq G_N(\mathcal{C}_2)$. If we find a labelling ω' of lower points such that $\mathcal{C}_1(\varepsilon, \omega') \neq \mathcal{C}_2(\varepsilon, \omega')$ and both $\mathcal{C}_1(\varepsilon, \omega')$ and $\mathcal{C}_2(\varepsilon, \omega')$ only contain partitions with at most M' blocks then we even have the result above for all $N \geq M'$. In particular, we can choose M' to be the maximal number of blocks of a partition p inside $\mathcal{C}_1(0, M) \cup \mathcal{C}_2(0, M)$ (and this is at most M).

3.3 Linear independence in the free case

For general categories of partitions we did not find in Section 3.2 a universal $N \in \mathbb{N}$ such that different categories produce different easy quantum groups. In the free case, where only non-crossing partitions are considered, the situation is much more comfortable, at least if we assume $N \geq 4$: For each colouring ω' of lower points the collection of maps $\big(T_p\big)_{p \in \mathcal{NC}(\varepsilon, \omega')}$ is linearly independent. Hence, by Corollary 3.1.4, every free easy quantum group with fundamental corepresentation matrix of size $N \geq 4$ corresponds to a unique category of non-crossing partitions. Conversely, for $N \geq 4$, two different non-crossing categories give rise to different free easy quantum groups.

The structure of this section is as follows: We first boil down the problem of linear independence to the question for a determinant of a special Gram matrix $A(n, 0)$. The main part of this section is an adapted version of W. Tutte's work [Tut93], where a formula for such a determinant is developed. Afterwards, see Section 3.3.6, some easy observations show that the mentioned determinant is non-zero.
We recapitulate Tutte's work [Tut93] up to a recursion formula for the determinant $\det\big(A(n, 0)\big)$. We do not change the principal ideas presented there, but we fixed errors in some of Tutte's arguments. In further consequence, more definitions and partial results have been changed. In addition, the graphical notations for partitions as introduced in Section 2.4 are integrated in definitions and proofs as well as further explanations and more detailed arguments to justify partial results. In the end, this section presents a self-contained proof of the linear independence described above, starting with the initial problem and guiding the reader without gaps through the relevant steps of the proof.

3.3.1 Boiling down the problem to the invertiblity of a matrix

The question of injectivity of the construction $\mathcal{C} \mapsto R_N(\mathcal{C})$ has been tracked down (in the sense of a sufficient condition) to the question of linear independence of the collections

$$\big(T_p\big)_{p\in\mathcal{C}(\varepsilon,\omega)}, \tag{3.3.1}$$

see Corollary 3.1.4 and its proof. For the rest of the chapter, we fix some $N \geq 4$ and we consider $\mathcal{C} = \mathcal{NC}$, the category of all non-crossing partitions. As the colours will not play a role, we can assume all colours to be white, so we fix an arbitrary $n \in \mathbb{N}_0$ and consider the collection

$$\big(T_p\big)_{p\in\mathcal{NC}(0,n)}. \tag{3.3.2}$$

In this situation, $\mathcal{NC}(0,n)$ denotes the set of all uni-coloured non-crossing partitions on n lower points. If the Collection 3.3.2 is linearly independent, so does every collection of the form 3.3.1 with $\mathcal{C} \subseteq \mathcal{NC}$, proving the claim.

As shown in the proof of Proposition 3.2.3, every map T_p in Equation 3.3.2 is uniquely determined by the vector $T_p(1) \in \big(\mathbb{C}^N\big)^{\otimes n}$. Hence, to prove linear independence of the T_p's, we can show linear independence of the vectors $\big(T_p(1)\big)_{p\in\mathcal{NC}(0,n)}$. In other words, we have to prove that the determinant of the Gram matrix

$$\begin{aligned} A(n,0) := \big(\langle T_p(1), T_q(1)\rangle\big)_{p,q\in\mathcal{NC}(0,n)} &= \big(\langle (T_q^*T_p)(1), 1\rangle\big)_{p,q\in\mathcal{NC}(0,n)} \\ &= \big((T_q^*T_p)(1)\big)_{p,q\in\mathcal{NC}(0,n)} \\ &= \big(N^{\mathrm{rl}(q^*,p)}\big)_{p,q\in\mathcal{NC}(0,n)} \end{aligned} \tag{3.3.3}$$

is non-zero. Note that the last equality is due to Lemma 2.5.5 and recall that $\mathrm{rl}(q^*,p)$ are the numbers of remaining loops in the construction of q^*p, see Section 2.4. As both p and q have no upper points, we have $q^*p \in \mathcal{NC}(0,0)$. Hence, the number of remaining loops $\mathrm{rl}(q^*,p)$ is the number of blocks after concatenating p and q^* vertically but before erasing all the blocks around the middle points, compare Section 2.4.

For $n = 0$ we only have to consider the empty partition $\emptyset \in \mathcal{NC}(0,0)$ and as $T_\emptyset = \mathrm{id}_\mathbb{C} \neq 0$, the desired linear independence is given, so we can assume from now on that $n \geq 1$. Nonetheless we could check in all situations if our results also cover the special case $n = 0$.
Summing up these observations, our aim is to prove invertibility of the matrix $A(n,0)$.

3.3.2 Other proofs of the linear independence

At this point, other works dealing with this (or similar) problems should be mentioned.

In [BS09], where (orthogonal) easy quantum groups are introduced, the linear independence of maps T_p for non-crossing partitions and for $N \geq 4$ is already mentioned. The authors refer to [Ban99] and [BC07] where the basic idea of a proof is as follows:

- Using deep results from V. Jones's work on subfactors, [Jon83], the dimension of $\operatorname{Hom}(k,k)$ for $k \in \mathbb{N}_0$ (and given $N \geq 4$) is proved to be C_{2k}, the $2k$-th Catalan number.
- It can easily be shown by induction that C_{2k} is also the number of non-crossing partitions on $2k$ points, see for example [NS06, Prop. 9.4].
- Combining both results, we have that the maps
$$\big(T_p\big)_{p \in \mathcal{NC}(k,k)},$$
that linearly generate by definition the intertwiner space $\operatorname{Hom}(k,k)$, are linearly independent.
- By Frobenius reciprocity, see [Tim08, Prop. 3.1.11], or simply by the fact that the rotation operator $\mathrm{rot}_{\curvearrowleft}$ is bijective, it also holds that the maps $\big(T_p\big)_{p \in \mathcal{NC}(0,2k)}$ are linearly independent.
- For $\big(T_p\big)_{p \in \mathcal{NC}(0,2k+1)}$, the result follows from $\operatorname{Hom}(0,2k+1) \otimes \mathrm{id} \subseteq \operatorname{Hom}(0,2k+2)$.

In [KS91], a matrix $A_{\mathcal{NC}_2}(2n,0)$ as above is investigated, however it is in the case of $\mathcal{NC}_2$, the category of all non-crossing pair-partitions and the investigation methods are different. The main result there is a recursion formula for the determinant of this matrix and its zeros are identified. Adapted to our purposes, it reads as follows:

Theorem (see [KS91, Cor. 2.10]). *Let $(U_k(X))_{k \in \mathbb{N}}$ be the delated Chebyshev polynomials of the second kind from Definition 3.3.3. Then it holds*

$$\det\big(A_{\mathcal{NC}_2}(2n,0)\big) = \prod_{i=1}^{\lfloor (n+1)/2 \rfloor} \big(\det(A_{\mathcal{NC}_2}(n-i,0))\big)^{(-1)^i \binom{n-i+1}{i}} \prod_{i=1}^{\lfloor (n-1)/2 \rfloor} \frac{U_{n-i}(N)}{U_i(N)}^{b(n-i,n-2i)}$$

with

$$b(n,k) = \frac{k}{n}\binom{2n-k-1}{n-1}$$

As the Chebyshev polynomials have no roots greater or equal 2, this determinant is non-zero for $N \geq 2$. A direct formula for this determinant is established in [FCG97], where the central idea is to identify elements in $\mathcal{NC}_2(n,n)$ with generating elements in a Temperley-Lieb algebra. Using a suitable basis exchange matrix E, the matrix $A(n,0)$ is diagonalizable, i.e. it holds

$$A_{\mathcal{NC}_2}(2n,0) = EDE^*$$

for a suitable diagonal matrix D. Evidently, the structure of E, and so its determinant, is encoded in the way the two bases are linked to each other. Combining the results for both the determinants of D and E, one obtains the following result.

Theorem (see [FCG97, Eqn. 5.6])**.** *Let $(U_k(X))_{k\in\mathbb{N}}$ be the delated Chebyshev polynomials of the second kind from Definition 3.3.3. Then it holds*

$$\det\left(A_{\mathcal{NC}_2}(2n,0)\right) = \prod_{i=1}^{n} U_i(N)^{a_{n,i}}$$

with

$$a_{n,i} = \binom{2n}{n-i} - 2\binom{2n}{n-i-1} + \binom{2n}{n-i-2}.$$

See also [BC10], where the same formula is proved by other means but not without using results and arguments from other works. In [BC10], also a formula for the determinant of $A(n,0)$, i.e. the case of $\mathcal{NC}$, is proved and the formula follows from the result for $\mathcal{NC}_2$. In this virtue, the results for $A_{\mathcal{NC}_2}(n,0)$ really contribute to the question in this thesis.

In contrast to that, Tutte's determinant formula is proved by elementary (combinatorial) arguments and these do not rely on results or theories from other sources.

3.3.3 Main result and idea of its proof

We state again the initial problem of Section 3.3 in form of the following theorem to be proved:

Theorem 3.3.1. *Consider $N \geq 4$ and for a given partition $q \in \mathcal{P}(k,l)$ the map*

$$T_q : \left(\mathbb{C}^N\right)^{\otimes k} \to \left(\mathbb{C}^N\right)^{\otimes l},$$

as defined in Definition 2.5.4. Then the collection $\left(T_p\right)_{p\in\mathcal{NC}(0,n)}$ is linearly independent.

By the previous consideration this answers the aforementioned question of this chapter in the non-crossing case: The map $\Psi : \mathcal{C} \mapsto R_N(\mathcal{C})$ from Equation 3.0.1 is injective for non-crossing categories $\mathcal{C}$ of partitions of sets and $N \geq 4$.

As displayed in Section 3.3.1, we prove this by showing the determinant of the Gram matrix $A(n,0)$, see Equation 3.3.3, to be non-zero.

In the following we will also need to consider matrices $A(n,r)$ for $1 \leq r < n$, compare Definition 3.3.17. These matrices are obtained from $A(n,0)$ by deleting certain rows and columns and by putting certain entries to zero.
Recall that the rows and columns of $A(n,0)$ are indexed by the elements in $\mathcal{NC}(0,n)$, the non-crossing partitions on n lower points. The first step towards the main result is to divide $\mathcal{NC}(0,n)$ into disjoint subsets, compare Definitions 3.3.10 and 3.3.8:

$$\mathcal{NC}(0,n) = Y(n,0)\,\dot\cup\, Y(n,1)\,\dot\cup\, \ldots\, \dot\cup\, Y(n,n-1)$$

We arrange the rows and columns of $A(n,0)$ simultaneously such that the first $\#Y(n,0)$ rows and columns are indexed by the elements in $Y(n,0) \subseteq \mathcal{NC}(0,n)$:

$$\begin{array}{cc}
\begin{array}{r} {\scriptstyle Y(n,0)}\ \Big\{ \\ {\scriptstyle Y(n,1)}\ \Big\{ \\ \vdots \\ {\scriptstyle Y(n,n-1)}\ \Big\{ \end{array} &
\begin{pmatrix} B(n,0) & * & \cdots & * \\ * & * & & \\ \vdots & & \ddots & \\ * & & & * \end{pmatrix} = A(n,0) \\
 & \underbrace{\qquad}_{Y(n,0)} \quad \underbrace{\quad}_{Y(n,1)} \quad \cdots \quad \underbrace{\quad}_{Y(n,n-1)}
\end{array}$$

It turns out that multiplying the columns with suitable factors and adding the first $\#Y(n,0)$ columns to the remaining ones in a suitable way puts the first block-row apart from $B(n,0)$ to zero, compare Section 3.3.5:

$$\begin{pmatrix} B(n,0) & 0 & \cdots & 0 \\ * & & & \\ \vdots & & \alpha\tilde{M} & \\ * & & & \end{pmatrix}.$$

Here the factor α is proved to be non-zero. Having traced down the problem to the matrices $B(n,0)$ and $\tilde{M}$ would not be of any usage if we could not control these two submatrices. Fortunately, we are able to show that both $B(n,0)$ and $\tilde{M}$ have structures similar to $A(n,0)$: They are given by the aforementioned $A(n,r)$. We can repeat the procedure for these submatrices and by induction we finally prove the determinant of $A(n,0)$ to be non-zero. More precisely, we prove the following, compare Section 3.3.5.

Theorem. *Consider the Gram marix $A(n,0)$ as defined in Equation 3.3.3 as well as the matrices $A(n,r)$ and $B(n,r)$, see Definition 3.3.17 and 3.3.19, respectively. Let further $(\beta_n)_{n\in\mathbb{N}_0}$ be the reversed Beraha polynomials from Definition 3.3.2. Then the following holds.*

(1) For $r\in\mathbb{N}$ and $N\geq 4$, the number $\beta_r(1/N)$ is non-zero.

(2) For $0\leq r<n-1$ we have

$$\det(A(n,r)) = \alpha_r \cdot \det \begin{pmatrix} B(n,r) & 0 \\ D & A(n,r+1) \end{pmatrix}$$

where

$$\alpha_r = \left(\frac{\beta_{r+3}(1/N)}{\beta_{r+2}(1/N)}\right)^c \neq 0$$

with a suitable positive integer c.

(3) For $0\leq r<n$ we have

$$B(n,r) = \begin{cases} A(n-1,r-1) & , \ r = 2s+1. \\ N\cdot A(n-1,r-1) & , \ r=2s \text{ and } r>0 \\ N\cdot A(n-1,0) & , \ r=0. \end{cases}$$

(4) For $n\in\mathbb{N}$ we have

$$A(n,n-1) = N^{\lceil \frac{n}{2} \rceil} \in M_1(\mathbb{C}) = \mathbb{C}$$

A combination of these four results shows that the determinant of the Gram matrix $A(n,0)$ is a product of non-zero quotients α_r and a suitable power of N, proving Theorem 3.3.1.

3.3.4 Definitions and preparatory results

Sections 3.3.4 and 3.3.5 are based on Tutte's article [Tut93]. See also Section 3.3.7, where we collect the structure of Tutte's arguments and ideas as well as the differences to this thesis.

Reversed Beraha polynomials and Chebyshev polynomials

At the end of this chapter we deal with the so-called *reversed Beraha polynomials*. We define them here and show that they are closely related to the dilated Chebyshev polynomials of the second kind. The definition of reversed Beraha polynomials can be found in [Tut93]. Chebyshev polynomials and their properties presented here are standard knowledge, see for example [Riv74].

Definition 3.3.2. The series of reversed Beraha poynomials $\big(\beta_n(X)\big)_{n\in\mathbb{N}_0}$ is defined by the recursion formula

$$\beta_0(X) = 0 \quad , \quad \beta_1(X) = 1$$

$$\beta_{n+1}(X) = \beta(n, X) - X\beta_{n-1}(X) \quad , \forall n \geq 1.$$

Definition 3.3.3. The series of dilated Chebyshev polynomials of the second kind $\big(U_n(X)\big)_{n\in\mathbb{N}_0}$ is defined by the recursion formula

$$U_0(X) = 1 \quad , \quad U_1(X) = X$$

$$U_{n+1}(X) = XU_n(X) - U_{n-1}(X) \quad , \forall n \geq 1.$$

Definition 3.3.4. The (undilated) Chebyshev polynomial of the second kind $\mathscr{U}_n(X)$ of degree $n \in \mathbb{N}_0$ is defined as the unique polynomial of degree n that fulfils

$$\mathscr{U}_n\big(\cos(t)\big) = \frac{\sin\big((n+1)t\big)}{\sin(t)} \quad , \forall t \in (0, \pi). \tag{3.3.4}$$

Uniqueness follows from the fact that a polynomial (of degree n) that satisfies Equation 3.3.4 must have the (simple) roots $\frac{\pi}{n+1}, \frac{2\pi}{n+1}, \ldots, \frac{n\pi}{n+1}$. Existence is easily proved by induction: The polynomials defined by the following recursion formula fulfil Equation 3.3.4.

$$\mathscr{U}_0(X) = 1 \quad , \quad \mathscr{U}_1(X) = 2X$$

$$\mathscr{U}_{n+1}(X) = X\mathscr{U}_n(X) - \mathscr{U}_{n-1}(X) \quad , \forall n \geq 1.$$

With a recursion formula for both series of Chebyshev polynomials at hand, one proves without any effort

$$\mathscr{U}_n(X) = U_n(\frac{1}{2}X).$$

In particular, we can locate the roots of the dilated Chebyshev polynomials.

Lemma 3.3.5. *For every $n \in \mathbb{N}_0$, the roots of the dilated Chebyshev polynomial of the second kind $U_n(X)$ are in the open interval $(-2, 2)$.*

The reversed Beraha polynomials are closely related to the dilated Chebyshev polynomials of the second kind, compare for example [CSS02, p. 454].

Lemma 3.3.6. *Let $N, j \in \mathbb{N}$ and write $\frac{1}{N} =: z$. Then we have the following relations between the reversed Beraha polynomials and the dilated Chebyshev polynomials of the second kind:*

(i) If $j = 2s$ is even, then it holds

$$N^s \beta_j(z) = \sqrt{N} U_{j-1}(\sqrt{N}).$$

(ii) If $j = 2s{+}1$ is odd, then it holds

$$N^s \beta_j(z) = U_{j-1}(\sqrt{N}).$$

Proof. We use induction on $j \in \mathbb{N}$. To keep the notations short, we write $\beta_i := \beta_i(z)$ and $U_i := U_i(\sqrt{N})$. For $j = 1$ we have $\beta_1 = 1 = U_0$, so the statement is true in the base case. Assume the lemma to be true for all $j' \in \mathbb{N}$ smaller than some $j \in \mathbb{N} + 1$. In case (i), i.e. an even j, we compute with the help of the induction assumption

$$\begin{aligned} N^s \beta_j = N^s \left(\beta_{j-1} - \frac{1}{N} \beta_{j-2} \right) &= N \cdot N^{s-1} \beta_{j-1} - N^{s-1} \beta_{j-2} \\ &= N U_{j-2} - \sqrt{N} U_{j-3} \\ &= \sqrt{N} \left(\sqrt{N} U_{j-2} - U_{j-3} \right) \\ &= \sqrt{N} U_{j-1}. \end{aligned}$$

Analogously, we find in the case of an odd j

$$\begin{aligned} N^s \beta_j = N^s \left(\beta_{j-1} - \frac{1}{N} \beta_{j-2} \right) &= N^s \beta_{j-1} - N^{s-1} \beta_{j-2} \\ &= \sqrt{N} U_{j-2} - U_{j-3} \\ &= U_{j-1}. \end{aligned}$$

□

Combining Lemmata 3.3.5 and 3.3.6 gives us the result that is used later on in order to prove Theorem 3.3.1.

Lemma 3.3.7. *Writing $z := \frac{1}{N}$ it holds*

$$\beta_n(z) \neq 0$$

for all $n \in \mathbb{N}$ and $N \in \mathbb{N}_{\geq 4}$.

The sets of partitions $W(n, r)$ and $Y(n, r)$

Related to $n \in \mathbb{N}$, the number of (lower) points in the partitions $\mathcal{NC}(0, n)$, we fix for all the remaining part of the chapter a few other numbers:
Let r be a number with $0 \leq r < n$ and let $s \in \mathbb{N}_0$ be such that either $2s = r$ or $2s + 1 = r$. We start by defining subsets $W(n, r) \subseteq \mathcal{NC}(0, n)$.

Definition 3.3.8. Let $0 \leq r < n$ and $s \in \mathbb{N}_0$ such that $r = 2s$ or $r = 2s + 1$. We define $W(n, r) \subseteq \mathcal{NC}(0, n)$ by the following properties:

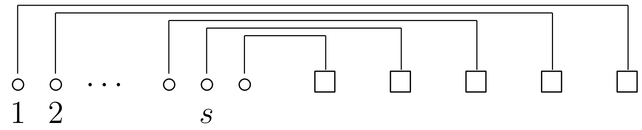
(i) If $r = 2s$, then the s leftmost points of a partition in $W(n, r)$ are no singletons and the $s + 1$ leftmost points belong to pairwise different blocks.

(ii) If $r = 2s + 1$, then the $s + 1$ leftmost points of a partition in $W(n, r)$ are no singletons and belong to pairwise different blocks.

We further define $W(n, n) := \emptyset$.

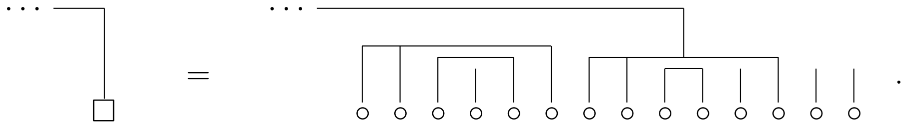
Note that for $r = n$ the conditions (i) and (ii) from above cannot be fulfilled, so we can extend Definition 3.3.8 by $W(n, n) := \emptyset \subseteq \mathcal{NC}(0, n)$.
For $r = 0$ there is no condition at all, so $W(n, 0) = \mathcal{NC}(0, n)$. Obviously, it holds $W(n, n) \subseteq W(n, n-1) \subseteq \ldots \subseteq W(n, 1) \subseteq W(n, 0)$.
If $r = 2s+1$, the basic structure of an element $p \in W(n, r)$ is the following:

$r = 2s + 1$:



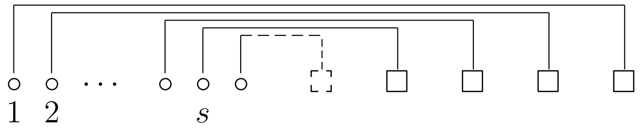
(3.3.5)

Recall from Section 2.4 that the □'s symbolize arbitrary non-empty substructures that we cannot or at least do not specify. For example, the rightmost square might actually be given by



We only know that one of the blocks inside a square is connected to exactly one of the $s+1$ leftmost points. By non-crossingness and the properties of $p \in W(n, r)$, this point amongst the $s+1$ leftmost points is uniquely determined and it is of course the one indicated in picture 3.3.5. Note that there is some ambiguity in the definition of the substructures □ as we might have different possibilities where one square ends and the next square begins.
If $r = 2s$, the only difference to the structure above is the possibility that the point $s+1$ is allowed to be a singleton:

$r = 2s$:



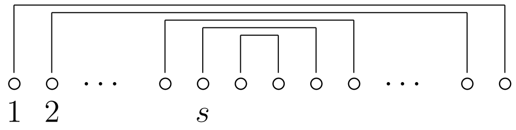
(3.3.6)

Note that the dashed structure might be empty, which is the case of $s+1$ being a singleton.

It is easy to check that the cardinality of $W(n, n-1)$ is one:

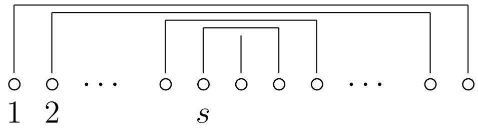
Lemma 3.3.9. *For every $n \in \mathbb{N}$ and $r = n-1$ the set $W(n,r)$ contains only the following element (depending on the parity of r):*

$r = 2s+1:$



(3.3.7)

$r = 2s:$



(3.3.8)

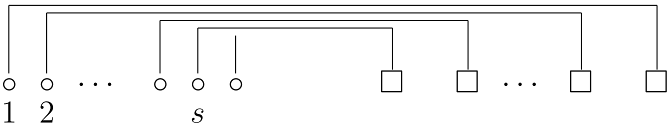
Definition 3.3.10. For $0 \leq r < n$ we define

$$Y(n,r) := W(n,r) \backslash W(n,r+1). \tag{3.3.9}$$

Comparing the definitions and illustrating pictures for partitions in $W(n,r)$ and $W(n,r+1)$, we can easily describe the partitions inside the sets $Y(n,r)$:

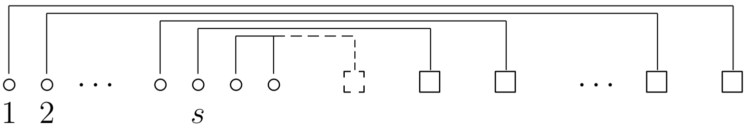
Lemma 3.3.11. *Let $0 \leq r < n$.*

(i) If $r = 2s$, then $Y(n,r)$ contains all elements of $W(n,r)$ such that $s+1$ is a singleton:



(3.3.10)

(ii) If $r=2s+1$, then $Y(n,r)$ contains all elements of $W(n,r)$ such that $s+1$ and $s+2$ are connected:



(3.3.11)

Furthermore, we have $Y(n, n-1) = W(n, n-1)$, as $W(n,n)$ is empty.

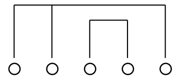
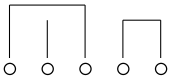
The graphs $G(p,q)$, $H_r(p,q)$ and r-flaws

In the following it will be helpful to look at the vertical concatenation of p and q^* (determining $\langle T_p(1), T_q(1)\rangle = N^{\mathrm{rl}(q^*,p)}$) as an (undirected) graph with $2n$ points:

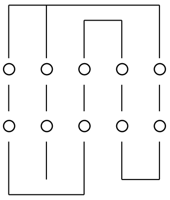
Definition 3.3.12. Let $p, q \in \mathcal{NC}(0,n)$ be two partitions. Considering p, q as (undirected) graphs, i.e. the vertices are given by the points and two vertices are adjacent if they are in the same block of the partition, we define the graph $G(p,q)$ as the direct sum of p and q where we additionally add edges between the i-th point of p and the i-th point of q.

The graph $G(p,q)$ exactly describes the situation of the concatenated partitions p and q^* *before* erasing the points in the middle and *before* erasing remaining loops, compare Section 2.4. The number $\mathrm{rl}(q^*,p)$ is the number of components of $G(p,q)$. We usually label the points of p from left to right by $1, 2, \ldots, n$ and the ones of q by $1', 2', \ldots, n'$.

As an example, consider the partitions

$p =$  , $q =$  .

Identifying $G(p,q)$ with its illustrations, we write

$G(p,q) =$  . (3.3.12)

Remark 3.3.13. Recall that two points in an (undirected) graph are adjacent if there is an edge between them and two points are connected if they are in the same component of the graph, i.e. there is a path starting at one of the points and ending at the other.

Throughout this chapter we are only interested in the connected components of a graphs and not the actual adjacencies of its points. Hence, in many cases we do not distinguish precisely if a line drawn in an illustrating picture is an actual edge in the graph or if it just symbolizes the property that the corresponding points are connected.

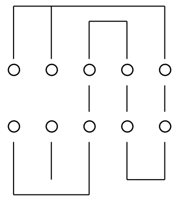
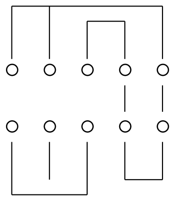
In this virtue we can say that the lines drawn in a picture like 3.3.12 only define the connectivity properties of a graph but not their realizations via actual edges. This way each such picture describes an equivalence class of graphs (on the same points), equivalent by the property to have the same components.

Although we can always refer to Definition 3.3.12 for a precise definition of the graph $G(p,q)$, it does not cause any problems in the following to interpret any drawn line with the equivalence relation "is connected to".

Remark 3.3.14. Note that the illustration of $G(p,q)$ for non-crossing p and q can be drawn in a non-crossing way as well.

Definition 3.3.15. Considering the situation and the graph $G(p,q)$ from Definition 3.3.12, we define the graphs $\big(H_r(p,q)\big)_{1\leq r<n}$ by starting with $G(p,q)$ and erasing the edges (i,i') for $1\leq i\leq s{+}1$.

Considering p and q as above, we have for example the illustrations

$H_3(p,q) =$  , $H_4(p,q) =$  .

Note that in the case $H_3(p,q)$ we have $r=3$ and $s=1$, therefore we removed the $s+1=2$ edges $(1,1')$ and $(2,2')$. Likewise we have in the case $H_4(p,q)$ the values $r=4$ and $s=2$, so $s+1=3$ edges are removed.

To define the matrices $A(n,r)$ from Section 3.3.3, we need the notion of a so called r-*flaw*.

Definition 3.3.16. Let $p,q\in\mathcal{NC}(0,n)$ and let $G(p,q)$ be the graph associated to this pair of partitions as defined in Definition 3.3.15. We define $G(p,q)$ to be r-*flawless* if the following properties are fulfilled:

(1) In $H_r(p,q)$, the points $1,\ldots,(s+1)$ are disconnected.

(2) In $H_r(p,q)$, the points $1',\ldots,(s+1)'$ are disconnected.

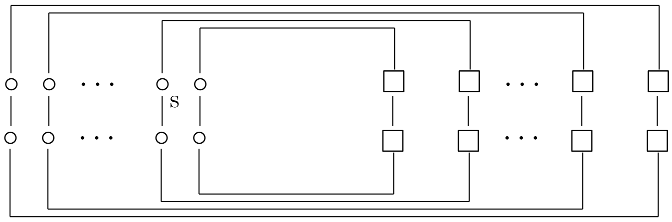
(3) In $H_r(p,q)$, the points i and i' are connected for all $1\leq i\leq s$.

(4) If r is odd, then we also have in $H_r(p,q)$ a path between $(s+1)$ and $(s+1)'$.

In all other cases we call $G(p,q)$ to have an r-*flaw*.

A point in the graph $G(p,q)$ is called a witness for an r-flaw if existence or absence of paths in $H_r(p,q)$ between this point and other points contradicts one of the properties (1) – (4).

Informally, it is easy to describe what an r-flawless graph should be: Consider the picture

$G(p,q) =$ 

.

Firstly, no point $1 \leq i \leq (s{+}1)$ is connected to any of the other points $1 \leq j \leq (s{+}1)$. It is easy to see that this property is equivalent to items (1) and (2) above. Secondly, we have a "path" between i and i' that does not use the line (i, i'), proving item (3). Even for odd r the graph above is r-flawless, compare item (4), as we also have such a path between $(s+1)$ and $(s+1)'$.

Note that 0-flaws do not exist in the sense that conditions (1) – (4) from Definition 3.3.16 are always fulfilled if $r = s = 0$; excluding graphs with 0-flaws is just a way to consider all possible $G(p,q)$.

The matrices $A(n,r)$ and $B(n,r)$

Definition 3.3.17. Let $p, q \in \mathcal{NC}(0,n)$. We define the matrix $A(n,r)$ by

$$A(n,r) := \Big(e_r(p,q)\Big)_{p,q \in W(n,r)}$$

where

$$e_r(p,q) := \begin{cases} 0 & , G(p,q) \text{ has an } r\text{-flaw} \\ N^{\mathrm{rl}(q^*,p)} & , \text{else.} \end{cases}$$

Remark 3.3.18. (a) Note that for the definition of the matrix $A(n,r)$ and its entries $e_r(p,q)$ the basis N above is just a parameter. It could be replaced by (more or less) every other complex number, but, in order to establish the connection to our main result, Theorem 3.3.1, we have to choose this parameter to be the natural number $N \geq 4$ fixed at the beginning of Section 3.3.

(b) Note that $G(p,q)$ would have an r-flaw if $\{p,q\} \not\subseteq W(n,r)$. If conversely $\{p,q\} \subseteq W(n,r)$, then, by definition, the points $1, \ldots, s+1$ are pairwise disconnected in p and $1', \ldots, (s{+}1)'$ are pairwise disconnected in q. In general, it is not clear whether the same holds after constructing $G(p,q)$, but it is guaranteed by r-flawlessness.

Definition 3.3.19. Consider the set of partitions $Y(n,r)$ as defined in Equation 3.3.9. We define the matrix $B(n,r)$ to be the submatrix of $A(n,r)$ obtained by reducing $A(n,r)$ to the rows and columns indexed by elements in $Y(n,r)$.

We now prove a connection between the matrices $B(n,r)$ and suitable matrices $A(n-1,r')$.

Lemma 3.3.20 (odd case). *Let* $0<r=2s+1<n-1$. *Then, modulo row and column permutations, the matrix* $B(n,r)$ *is equal to the matrix* $A(n-1,r-1)$.

Proof. Recall that the rows and columns of $A(n-1,r-1)$ are labelled by elements in $W(n-1,r-1)$, while those of $B(n,r)$ are labelled by elements in $Y(n,r) = W(n,r+1)\backslash W(n,r)$. See also Definition 3.3.8 and Lemma 3.3.11, where the elements in $W(n,r)$ and $Y(n,r)$, respectively, are characterised.
We show the desired equality in three steps.

(1) In Step 1 we establish a bijection α between the sets $Y(n,r)$ and $W(n-1,r-1)$ by removing suitable points in the partitions $p \in Y(n,r)$. Doing so, we can assume the rows and columns of both matrices to be labelled equally.

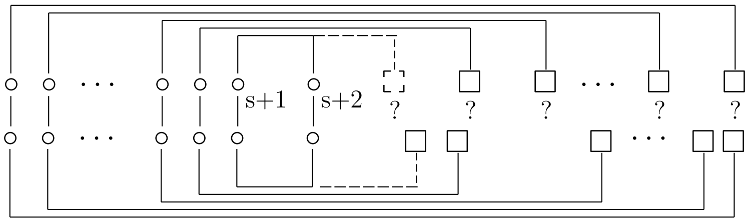
(2) In Step 2 we investigate how the replacement $p \mapsto \alpha(p)$ affects the number of components $\mathrm{rl}(p,q)$ in the graph $G(p,q)$.

(3) In Step 3 we finally check that $A(n-1,r-1)$ and $B(n,r)$ have the same zero-entries, i.e. we prove that the application of α replaces every Graph $G(p,q)$ which has an r-flaw with a graph $G(\alpha(p),\alpha(q))$ which has an $(r-1)$-flaw.

Step 1: $Y(n,r) \simeq W(n-1,r-1)$: Consider $p,q \in Y(n,r)$. For odd r they have a structure as displayed in Picture 3.3.11. In particular, we have in p that $s+1$ is connected to $s+2$ and in q we have that $(s+1)'$ and $(s+2)'$ are connected. Deleting $s+2$ and $(s+2)'$ from p and q, respectively, defines two elements $\alpha(p), \alpha(q) \in W(n-1,r-1)$. Evidently, the map $\alpha : Y(n,r) \to W(n-1,r-1)$ is bijective: We can start with any partition $p' \in W(n-1,r-1)$, add a point after $s+1$ and connect it to $s+1$. The resulting partition as a preimage of p' under α.
By definition, rows and columns of $B(n,r)$ and $A(n-1,r-1)$ are labelled by elements in $Y(n,r)$ and $W(n-1,r-1)$, respectively. Identifying $Y(n,r)$ and $W(n-1,r-1)$ via the above α, we can, after a rearrangement, assume that the rows and columns of $B(n,r)$ and $A(n-1,r-1)$ are labelled equally. It remains to show that their entries are the same, i.e. $e_r(p,q) = e_{r-1}\big(\alpha(p),\alpha(q)\big)$, compare Definition 3.3.17.

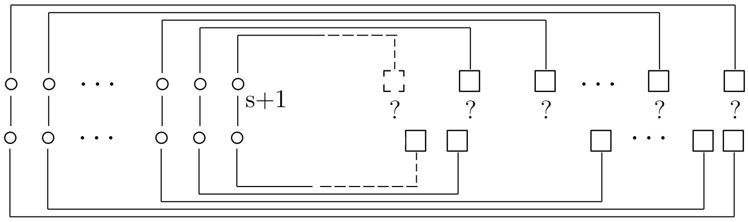
Step 2: $\mathrm{rl}(q^*, p) = \mathrm{rl}\,\big(\alpha(q)^*, \alpha(p)\big)$**:** We compare the graphs $G(p,q)$ and $G\big(\alpha(p), \alpha(q)\big)$:

$G(p,q) =$



(3.3.13)

$G\big(\alpha(p), \alpha(q)\big) =$



(3.3.14)

The symbols ? and the different positions of the □-structures indicate that we do not know and do not care how the block structures of p and q actually look like in these parts of the graphs.
It is clear that $G(p,q)$ and $G\big(\alpha(p), \alpha(q)\big)$ have the same number of components as in p the point $s+2$ was connected to $s+1$ and in q the point $(s+2)'$ was connected to $(s+1)'$. We conclude that the values $\mathrm{rl}(q^*, p)$ and $\mathrm{rl}\,\big(\alpha(q)^*, \alpha(p)\big)$ are the same.
Step 3: r-flaws vs. $(r-1)$-flaws: The proof is finished if we can show

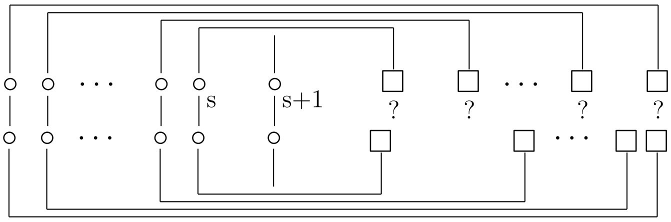
$$e_r(p,q) = e_{r-1}\big(\alpha(p), \alpha(q)\big),$$

but, despite Step 2, this is not yet guaranteed: On the left side we have to check whether $G(p,q)$ has an r-flaw and on the right side we have to check whether $G\big(\alpha(p), \alpha(q)\big)$ has an $(r-1)$-flaw, see Definition 3.3.17. We have to prove these two conditions to be equivalent. The only problematic part of the above equivalence is the statement that an r-flaw in $G(p,q)$ implies an $(r-1)$-flaw in $G\big(\alpha(p), \alpha(q)\big)$.
Consider a graph $G(p,q)$ with an r-flaw and assume $G\big(\alpha(p), \alpha(q)\big)$ to be $(r-1)$-flawless. Let $i \in \{1, \ldots, s+1\} \cup \{1', \ldots, (s+1)'\}$ be a point in $G(p,q)$ that is a witness of the r-flaw of $G(p,q)$. The only possibility for i to be *not* a witness of an $(r-1)$-flaw in $G\big(\alpha(p), \alpha(q)\big)$ is to be equal to $(s+1)$ and to be disconnected from $(s+1)'$ in $H_r(p,q)$ (or the other way around). But this is a contradiction, as by assumption $p, q \in Y(n,r)$, so $(s+1)$ and $(s+1)'$ are connected via the points $(s+2)$ and $(s+2)'$, see Picture 3.3.13. WE conclude that the assumption above was false and $G\big(\alpha(p), \alpha(q)\big)$ has an $(r-1)$-flaw. □

Lemma 3.3.21 (even case)**.** *Let $0 < r = 2s < n-1$. Then, modulo row and column permutations, the matrix $B(n,r)$ is equal to the matrix $N \cdot A(n-1, r-1)$.*

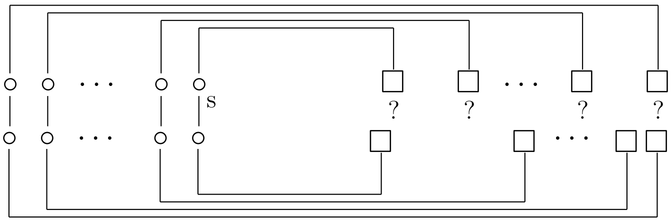
Proof. We have to go through the same steps as in the proof of Lemma 3.3.20.
Step 1: $Y(n,r) \simeq W(n-1,r-1)$**:** As r is even, the point $(s+1)$ in $p \in Y(n,r)$ is a singleton, compare Lemma 3.3.11. Erasing this point establishes the correspondence $\alpha : Y(n,r) \to W(n-1,r-1)$. It is bijective as obviously a preimage of $p' \in W(n-1,r-1)$ can be constructed by adding a singleton in p' after the point s. As in the proof of Lemma 3.3.20, Step 1, this allows us to label the rows and columns of $B(n,r)$ and $A(n-1,r-1)$ equally. It remains to show that their entries are the same, i.e. $e_r(p,q) = e_{r-1}\big(\alpha(p),\alpha(q)\big)$, compare Definition 3.3.17.
Step 2: $\mathrm{rl}(q^*,p) = 1 + \mathrm{rl}\left(\alpha(q)^*,\alpha(p)\right)$**:** We compare the graphs $G(p,q)$ and $G\big(\alpha(p),\alpha(q)\big)$:

$G(p,q) =$



(3.3.15)

$G\big(\alpha(p),\alpha(q)\big) =$



(3.3.16)

It is clear that $G(p,q)$ has one component more than $G\big(\alpha(p),\alpha(q)\big)$, namely the one containing exactly the two points $(s+1)$ and $(s+1)'$, see Equation 3.3.15 and 3.3.16. We conclude that the values $N^{\mathrm{rl}(q^*,p)}$ and $N \cdot N^{\mathrm{rl}(\alpha(q)^*,\alpha(p))}$ are the same.
Step 3: r**-flaws vs.** $(r-1)$**-flaws:** Again, it remains to prove the equality

$$e_r(p,q) = N \cdot e_{r-1}\big(\alpha(p),\alpha(q)\big)$$

and due to Step 2 we only need to show that r-flaws in $G(p,q)$ are equivalent to $(r-1)$-flaws in $G\big(\alpha(p),\alpha(q)\big)$. Note that $r = 2s$ is even, so $r-1 = 2(s-1)+1$ is associated to the increment of s.
As above, the only non-trivial implication is that an r-flaw in $G(p,q)$ implies an $(r-1)$-flaw in $G\big(\alpha(p),\alpha(q)\big)$. To prove this, consider a witness $i \in \{1,\dots,s+1\} \cup \{1',\dots,(s+1)'\}$ of an r-flaw in $G(p,q)$. Assuming $G\big(\alpha(p),\alpha(q)\big)$ to be $(r-1)$-flawless requires i to be the point $s+1$ (or $(s+1)'$) and it must be connected in $H_r(p,q)$ to another point amongst $1,\dots,s$ (or $1',\dots,s'$). This is again a contradiction, as both $(s+1)$ and $(s+1)'$ are singletons in p and q, respectively, so they are singletons in $H_r(p,q)$, too. Hence, the assumption above was false and $G\big(\alpha(p),\alpha(q)\big)$ has an $(r-1)$-flaw. ☐

Lemma 3.3.22. *Let $2 \leq n \in \mathbb{N}$. Modulo row and column permutations, the matrix $B(n,0)$ is equal to the matrix $N \cdot A(n-1,0)$.*

Proof. We consider the map α as in the proof of Lemma 3.3.21 and observe that we establish the same one-to-one correspondence as before between $Y(n,0)$ and $W(n-1,0)$ by deleting the singletons 1 and $1'$ from p and q, respectively. Again the graph $G(\alpha(p),\alpha(q))$ has one component less then $G(p,q)$, justifying the additional factor N. As there are no graphs with zero-flaws we finally have for all $p,q \in Y(n,0)$:

$$\begin{aligned}\big(B(n,0)\big)_{p,q} =&(A(n,0))_{p,q}\\ =&e_0(p,q)\\ =&N^{\mathrm{rl}(q^*,p)}\\ =&N \cdot N^{\mathrm{rl}(\alpha(q)^*,\alpha(p))}\\ =&N \cdot e_0(\alpha(p),\alpha(q))\\ =&N \cdot \big(A(n-1,0)\big)_{\alpha(p),\alpha(q)}.\end{aligned}$$

□

We summarize the results of the last three lemmata in one proposition.

Proposition 3.3.23. *It holds*

$$B(n,r) = \begin{cases} A(n-1,r-1) & ,0 < r = 2s+1 < n-1 \\ N \cdot A(n-1,r-1) & ,0 < r = 2s < n-1 \\ N \cdot A(n-1,0) & ,r = 0, n \geq 2. \end{cases}$$

In the case $r = n-1$ the set $Y(n,r) = Y(n,n-1) = W(n,n-1)$ contains only one element, namely the one displayed in Lemma 3.3.9. So we directly get the following result:

Proposition 3.3.24. *For $n \in \mathbb{N}$ we have*

$$B(n,n-1) = A(n,n-1) = N^{\lceil \frac{n}{2} \rceil} \in M_1(\mathbb{C}).$$

In particular, it holds

$$\det\big(B(n,n-1)\big) = \det\big(A(n,n-1)\big) = N^{\lceil \frac{n}{2} \rceil}.$$

Propositions 3.3.23 and 3.3.24 are ingredients to prove a recursion formula for the determinant of $A(n,0)$, see Section 3.3.5.

Partition and graph manipulations

Definition 3.3.25. For a fixed partition $p \in W(n, r)$ we denote by $X(i)$ the block of p containing i and we define $K(i) := X(i)\backslash\{i\}$.

Notation 3.3.26. In order to display connections between points or sets of points in a graph we will use the following scheme: We list the relevant (sets of) points and draw lines between them to indicate whether they are connected or not. For example by Definition 3.3.25 a partition $p \in W(n, r)$ fulfils the following:

$$\begin{array}{rcl} 1 & — & K(1) \\ & \vdots & \\ i & — & K(i) \\ & \vdots & \\ s & — & K(s) \\ (s{+}1) & — & K(s{+}1) \end{array} \tag{3.3.17}$$

Note that $K(s{+}1)$ might be empty if r is even.

We define now the following partition manipulations:

Definition 3.3.27. Let $0 \leq r < n{-}1$ and $q \in W(n, r{+}1)$. For $1 \leq i \leq (s+1)$ we define the partition $f(i, q)$ by performing the following changes on q:

(i) A point $i \leq j \leq s$ is not connected to $K(j)$ any more, but to $K(j{+}1)$.

(ii) If r is even, then the point $(s{+}1)$ is not connected to $K(s{+}1)$ any more.

(iii) If r is odd, then the point $(s{+}1)$ is not connected to $K(s{+}1)$ any more, but to $X(s{+}2)$.

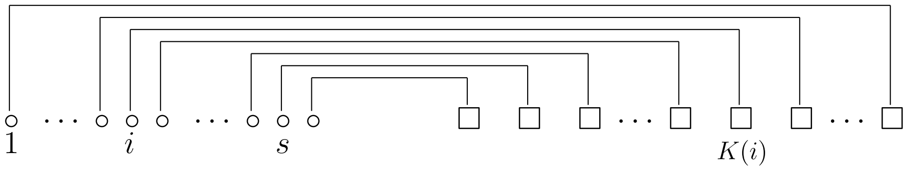
Note that the mentioned sets $K(1), \ldots, K(s+1)$ and $K(1), \ldots, K(s+1), X(s+2)$, respectively, are disjoint and non-empty as $q \in W(n, r+1)$.
Using Notation 3.3.26, we could have defined $f(i, q)$ by starting with q and defining the following changes in the connectivities, depending on the parity of r:

r even: $\qquad\qquad$ r odd:

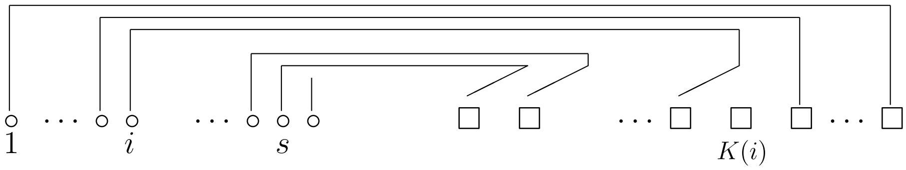
$$\begin{tikzcd}[row sep=small] i \arrow[dr, dash] & K(i) \\ \vdots & K(i{+}1) \\ & \vdots \\ s \arrow[dr, dash] & \\ (s{+}1) & K(s{+}1) \end{tikzcd} \qquad\qquad \begin{tikzcd}[row sep=small] i \arrow[dr, dash] & K(i) \\ \vdots & K(i{+}1) \\ & \vdots \\ s \arrow[dr, dash] & \\ (s{+}1) \arrow[dr, dash] & K(s{+}1) \\ & X(s{+}2) \end{tikzcd} \tag{3.3.18}$$

Here are the picture for this manipulation, once for a partition q in the case $r = 2s$ and once for a partition q' in the case $r = 2s + 1$ (Recall that $q, q' \in W(n, r + 1)$):

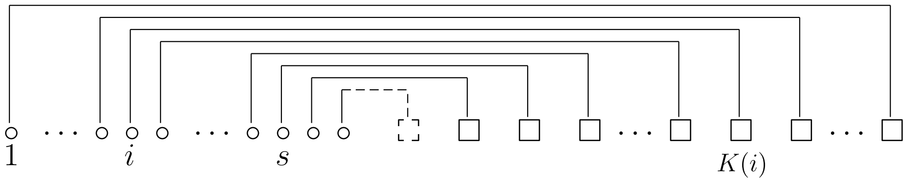
$q =$



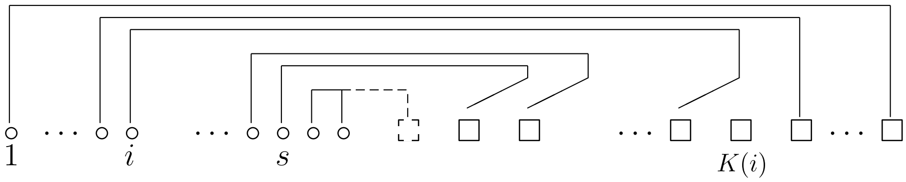
$f(i, q) =$



$q' =$



$f(i, q') =$



Note that in the case $i = s + 1$ we only separate $s + 1$ from $K(s + 1)$ and when $r = 2s + 1$ we additionally join $s+1$ with $X(s+2)$.

Similar to the construction of $f(i, p)$ we define partitions $g(i, p)$. The only difference is that the point i stays connected to $K(i)$.

Definition 3.3.28. Let $0 \leq r < n-1$ and $q \in W(n, r+1)$. For $1 \leq i \leq s$ we define the partition $g(i, q)$ by performing the following changes on q:

(i) The point i is connected additionally to $K(i+1)$.

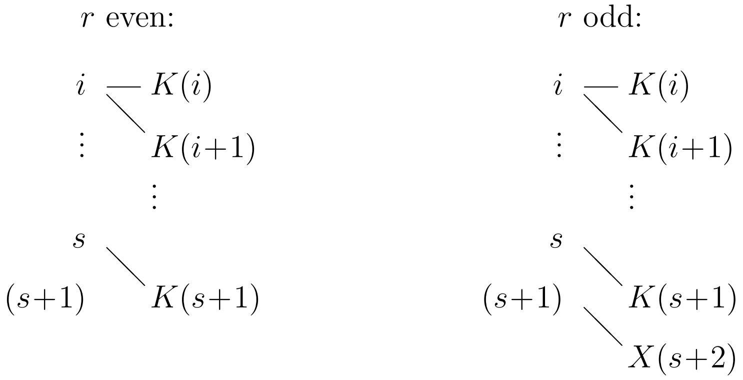
(ii) A point $i < j \leq s$ is not connected to $K(j)$ any more, but to $K(j+1)$

(iii) If r is even, then the point $(s+1)$ is not connected to $K(s+1)$ any more.

(iv) If r is odd, then the point $(s+1)$ is not connected to $K(s+1)$ any more, but to $X(s+2)$.

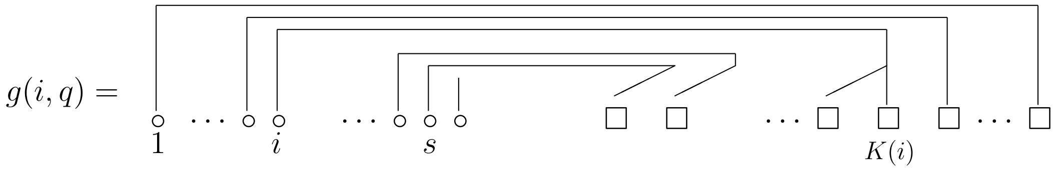
In the case $r = 2s+1$ we define in addition the case $i = s+1$: $g(s+1, q)$ is constructed out of q by joining $X(s+1)$ with $X(s+2)$, so the above changes are reduced to item (i).

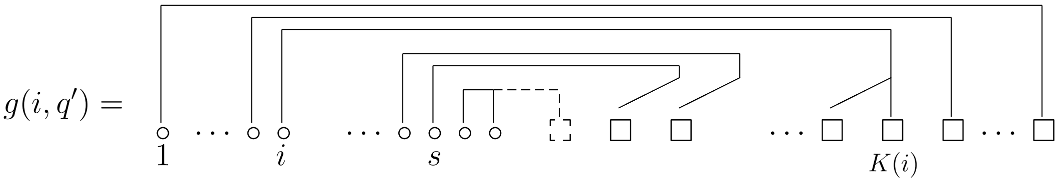
Using the Notations from 3.3.26,we could have defined $g(i,q)$ by starting with q and defining the following changes in the connectivities, depending on the parity of r:



(3.3.19)

Considering q and q' as above we end up with the following partitions:





Taking a closer look at the illustrations above, the following can directly be checked:

Lemma 3.3.29. *For $q \in W(n,r+1)$ the partitions $f(i,q)$ and $g(i,q)$ as defined in Definition 3.3.27 and 3.3.28 are elements in $W(n,r)$.*

Having $p \in W(n,r)$ and $q \in W(n,r+1)$, we know that in each partition the first $s+1$ points are not connected to each other (and in the case $r=2s{+}1$ even the first $s{+}2$ points of q). In contrast to that, we cannot in general guarantee this property in the graph $G(p,q)$, as we do not know enough about the part of $G(p,q)$ right of $s{+}1$ and $(s{+}1)'$. The next definition describes some special structures that might occur. Again the descriptions themselves are quite technical, but see below for comprehensive illustrations of the structures.

Definition 3.3.30. Let $0 \leq r < n-1$, $p \in W(n,r)$ and $q \in W(n,r{+}1)$. For even r we define on the graph $H_r(p,q)$

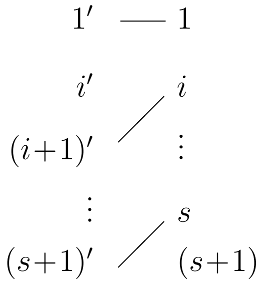
(1) a structure $[i]$, for $1 \leq i \leq s+1$,

(2) a structure $[i, i{+}1]$ for $1 \leq i \leq s$ and

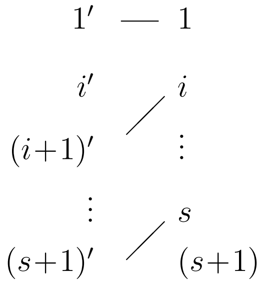
(3) a structure $[0]$

if and only if the following respective connections are given:

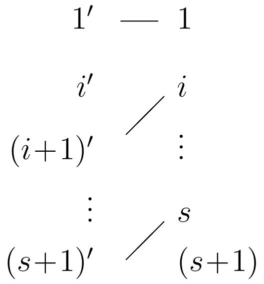
structure $[i]$:



structure $[i, i{+}1]$:



structure $[0]$:



For odd r we define on the graph $H_r(p,q)$

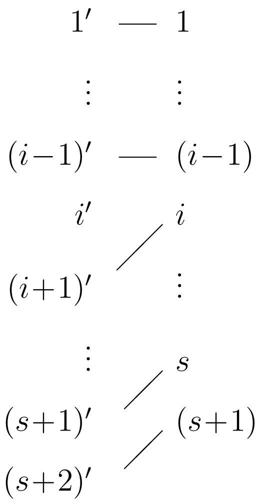
(1) a structure $[i]$, for $1 \leq i \leq s{+}1$,

(2) a structure $[i, i{+}1]$ for $1 \leq i \leq s{+}1$ and

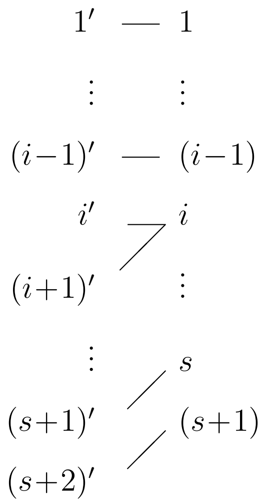
(3) a structure $[0]$

if and only if the following connections are given:

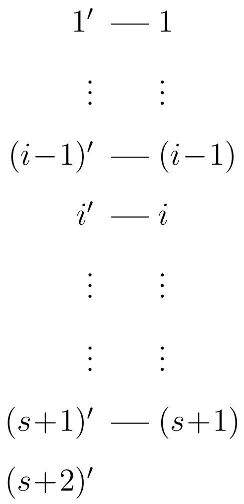
structure $[i]$:



structure $[i, i+1]$:



structure $[0]$:



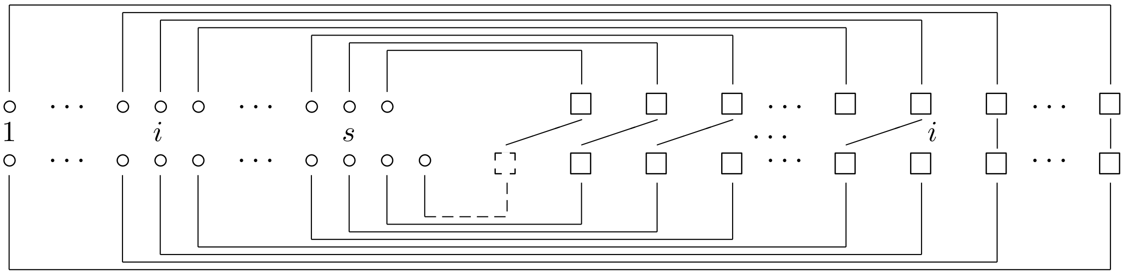
We note again that the schemes above also give information which points are *not* connected in $H_r(p,q)$: Two points that are mentioned in the scheme and that are not connected there should not be connected in the graph $H_r(p,q)$. Conversely, the schemes say nothing about connection to points that are not mentioned. Hence, in structure $[i]$, for example, the point i' is not allowed to be connected to any of the other drawn points but there are no conditions concerning links to points that are not explicitly mentioned.

Remark 3.3.31. (a) The structures described in Definition 3.3.30 are incompatible, i.e. no Graph $H_r(p,q)$ as above can be of more than one of the described structures.

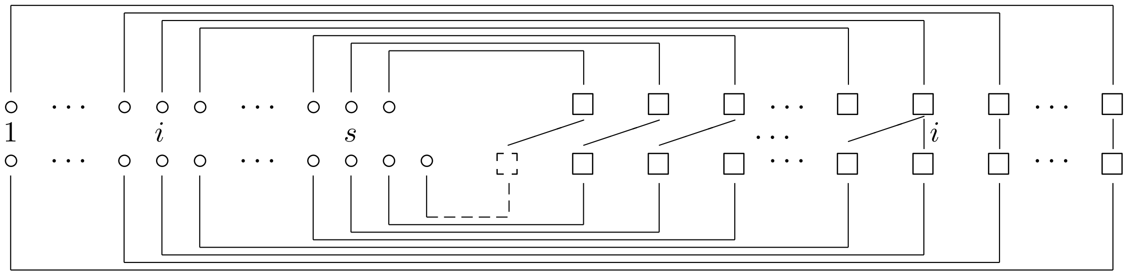
(b) Note that this definition is different from the corresponding structures defined in [Tut93]. There was an overlap in the structures defined there, resulting in false statements in the sequel.

In virtue of the previous examples, the graphs as described in Definition 3.3.30 look as follows:

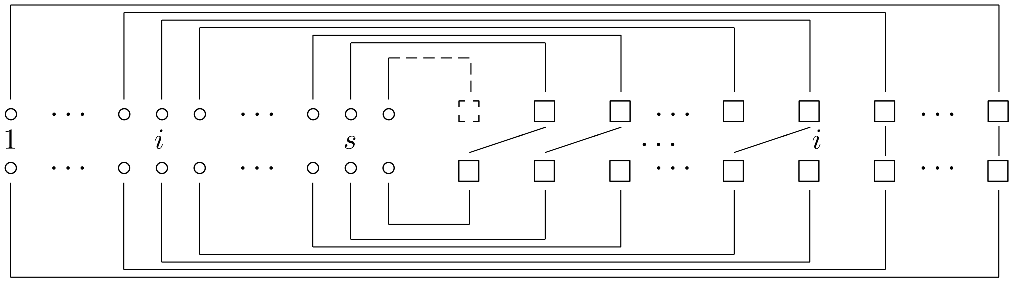
For $r = 2s+1$ and structure $[i]$:



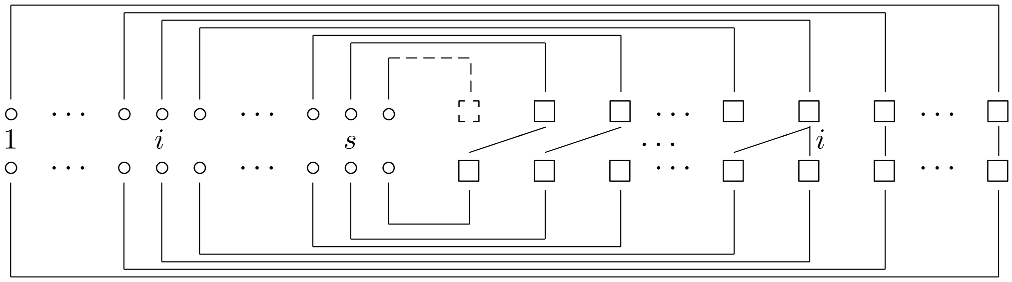
For $r = 2s+1$ and structure $[i, i+1]$:



For $r = 2s$ and structure $[i]$:



For $r = 2s$ and structure $[i, i+1]$:



Note that the pictures above are a little bit imprecise with respect to the meaning of a line between to squares □. Such a line does not mean that there is just *any* connection between these structures but there is a connection between the corresponding $K(i)$ and $K(j')$. For example, the vertical line between the right-most squares says that the points 1 and $1'$ are connected.
Note further that in the first two pictures the diagonal line to the dashed square just means that $s+1$ is in the same block as $(s+2)'$, so if the dashed structure is empty then this diagonal line has to end directly at point $(s+2)'$.

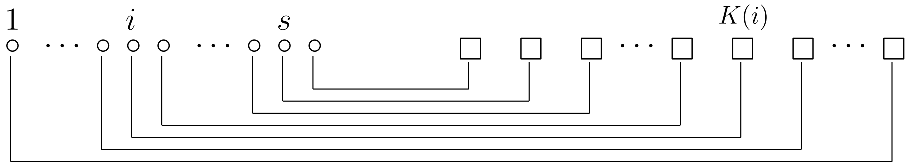
In the next lemma we take a closer look at the matrix $A(n, r)$ and ask under

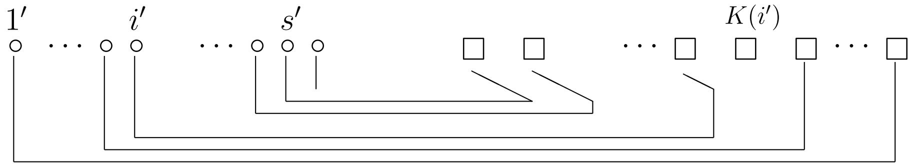
which conditions certain entries are non-zero. Recall, see Remark 3.3.29, that for $q \in W(n, r+1)$ the partitions $f(i, q)$ and $g(i, q)$ are elements in $W(n, r)$. Hence, there are rows and columns in $A(n, r)$ labelled by $f(i, q)$ and $g(i, q)$.
Note further, compare Definition 3.3.16, that a graph $G(p, q)$ is called r-flawless if the Graph $H_r(p, q)$ has the following structure:

$$\begin{array}{ccccccc} & r \text{ even:} & & \qquad & & r \text{ odd:} & \\ 1' & — & 1 & & 1' & — & 1 \\ \vdots & & \vdots & & \vdots & & \vdots \\ s' & — & s & & s' & — & s \\ (s{+}1)' & \text{- - -} & (s{+}1) & & (s{+}1)' & — & (s{+}1) \end{array} \tag{3.3.20}$$

The dashed line between $(s{+}1)'$ and $(s{+}1)$ means that a connection between these two points is allowed but not necessary. The following observation will be used in the proof of Proposition 3.3.33.

Observation 3.3.32. Consider $q \in W(n, r+1)$ and the illustrations of q^* and $f(i, q^*)$:

$q =$ 

$f(i, q^*) =$ 

We observe that in the illustration of $f(i, q^*)$ the space below $K(i')$ is only crossed by the lines which connect j' and $K(j')$ for $1 \leq j < i$.

Proposition 3.3.33. *Consider* $1 \leq r < n-1$ *and partitions* $p \in W(n, r)$ *and* $q \in W(n, r+1)$.

(i) The entry $e_r(p, q)$ *of* $A(n, r)$ *is non-zero exactly in the following cases:*

(1) If $r = 2s$, *then* $H_r(p, q)$ *must have structure* $[s+1]$ *or* $[0]$.

(2) If $r = 2s+1$, *then* $H_r(p, q)$ *must have structure* $[s+1, s+2]$ *or* $[0]$.

(ii) *For* $1 \leq i \leq s+1$ *the entry* $e_r\big(p, f(i,q)\big)$ *in* $A(n,r)$ *is non-zero if and only if* $H_r(p,q)$ *is of structure* $[i]$, $[i,i+1]$ *or* $[i-1,i]$. *Note that* $[i-1,i]$ *is excluded if* $i=1$ *and* $[i,i+1]$ *is excluded if* $i=s+1$ *and* $r=2s$, *so in these cases there are only two possible structures if* $e_r\big(p, g(i,q)\big)$ *should be non-zero.*

(iii) *For* $1 \leq i \leq s$ *the entry* $e_r\big(p, g(i,q)\big)$ *in* $A(n,r)$ *is non-zero if and only if* $H_r(p,q)$ *has structure* $[i]$, $[i+1]$ *or* $[i,i+1]$. *If* $i=s+1$ *(only allowed if* $r=2s+1$*), then* $H_r(p,q)$ *must be of structure* $[i]$, $[i,i+1]$ *or* $[0]$.

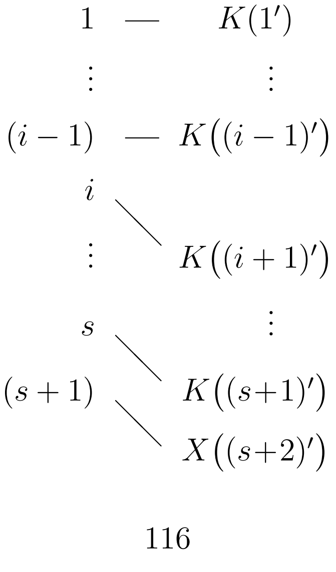
Proof. By definition of $e_r(\cdot,\cdot)$, see Definition 3.3.17, we have to show that absence of r-flaws is equivalent to the respectively mentioned structures.
A detailed proof is lengthy because many different scenarios have to be considered, but each of them can be checked by comparing three kind of schemes:

(1) The Picture 3.3.20 describes absence of r-flaws.

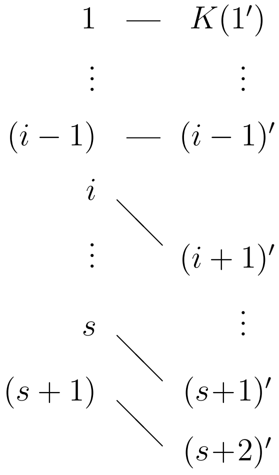
(2) A comparison of the Picture 3.3.17 with the Pictures 3.3.18 and 3.3.19 tells us how to construct the partitions $f(i,q)$ and $g(i,q)$ from q (and vice versa).

(3) The schemes in Definition 3.3.30 define and illustrate the structures $[i]$, $[i,i+1]$ and $[0]$.

Going from (1) to (3), one sees that assuming absence of r-flaws in $G_r(p,q)$, $G_r(p,f(i,q))$ or $G_r(p,g(i,q))$, respectively, implies that $H_r(p,q)$ has one of the respectively claimed structures. Going backwards from (3) to (1) shows the conversion, and hence the desired equivalence.
To convince the reader, we consider the situation of an r-flawless $G\big(p,f(i,q)\big)$ for odd r and show that $H_r(p,q)$ must be of structure $[i]$, $[i,i+1]$ or $[i-1,i]$.
Picture 3.3.20 describes the relevant structure of $H_r\big(p,f(i,q)\big)$ and Picture 3.3.18 the relevant structure of $f(i,q)$. Combining them, we obtain the following scheme:



Note that there are no other links between the mentioned (sets of) points because the points $1, \ldots, (s{+}1)$ are pairwisely disconnected by assumption on $G\big(p, f(i,q)\big)$. Replacing $f(i,q)$ by q, we obtain by definition of the sets $K(j')$ and $X\big((s{+}2)'\big)$ the following structure for $H_r(p,q)$:



(3.3.21)

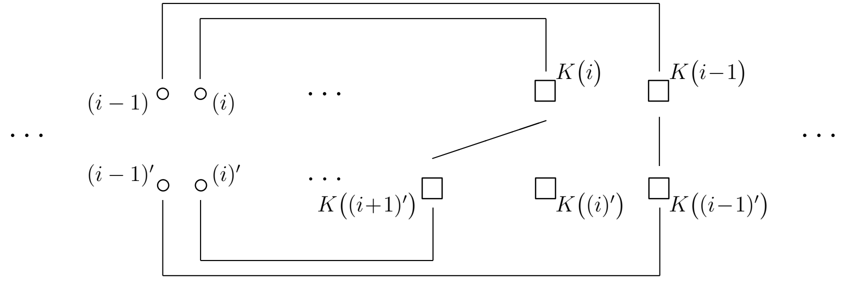
This scheme is compatible with the desired structures, but we have to prove that also the point i' is connected in the proper way:

(1) If i' is connected to none of the points $1 \ldots, (s{+}1)$, then we have structure $[i]$.

(2) If i' is only connected to i but to no other point amongst $1, \ldots, (s{+}1)$, then we have structure $[i, i{+}1]$.

(3) If i' is only connected to $(i{-}1)$ but to no other point amongst $1, \ldots, (s{+}1)$, then we have structure $[i, i{+}1]$.

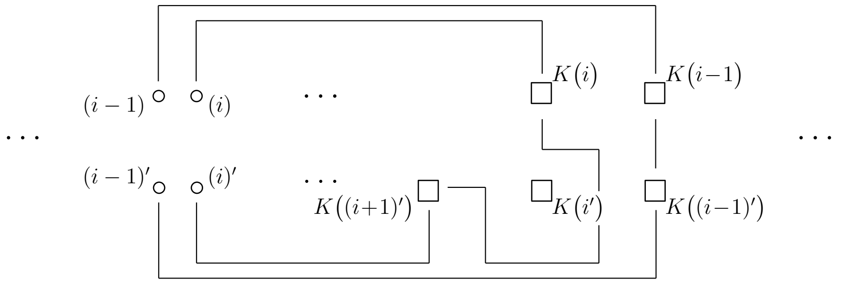
The prove is finished, once we have shown that no other situations occur.
It is clear that i' cannot be connected to more than one point of $1, \ldots, (s+1)$ as this would contradict Scheme 3.3.21, so having another structure than the three situations above means that i' or, equivalently, $K(i')$ is connected to any of the points $1, \ldots, (s{+}1)$ other than i or $(i{-}1)$. We assume this to be true and lead it to a contradiction.
Drawing the illustration of $H_r\big(p, f(i,q)\big)$ in a non-crossing way, the relevant part of $H_r\big(p, f(i,q)\big)$ looks as follows:

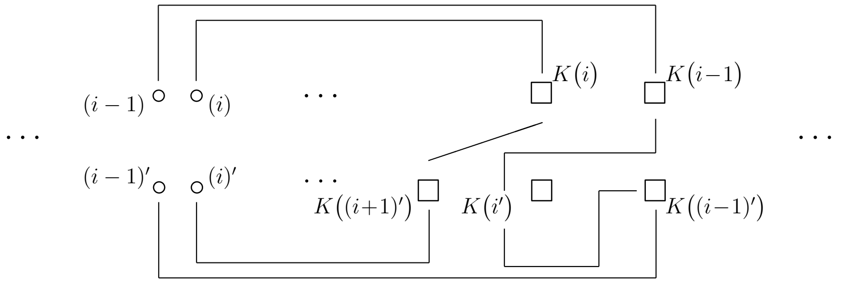


(3.3.22)

See below, why the illustration in deed has to be of this structure. Note that the diagonal line in the picture just symbolizes any path from a point inside $K\big((i+1)'$ to a point inside $K(i)$. This path might cross the area between the two rows of points several times and also points "left of $K\big((i+1)'$" and "right of $K(i)$" might be visited. The analogous remark holds for the vertical line between $K\big((i-1)'\big)$ and $K\big((i-1)\big)$.

The components containing the points i and i' form "circles" around $K(i')$, so, by non-crossingness, a path starting inside $K(i')$ and ending at one of the points $1, \ldots, (i–2), (i+2), \ldots, (s+2)$ has to visit at least one of the circles. We conclude that $K\big((i)'\big)$ would in addition be connected to $(i-1)$ or i, contradicting our assumption. In order to see that Illustration 3.3.22 does not contain inappropriate assumptions, we observe the following: Upon first sight, it seems that the points $K\big((i')$ to not have to be between the drawn "circles" but also the following constellations would be possible:





However, by Observation 3.3.32, the only lines drawn below the parts $\square_{K(i')}$ and ${}_{K(i')}\square$, respectively, can be chosen to be the lines between j' and $K\big(j'\big)$ for $1 \le j < i$. Hence, the two pictures above do not occur. $\square$

Given a structure as defined in 3.3.30, Proposition 3.3.33 tells us when an expression $e_{(\cdot)}(\cdot, \cdot)$ is non-zero. This will be used in the proof of Lemma 3.3.36.

Corollary 3.3.34. *Let $1 \le r \le n-1$, $p \in W(n, r)$ and $q \in W(n, r+1)$. For the statements below let $i, j \in \mathbb{N}$ be such that the respective objects are well-defined.*

(1) If $H_r(p, q)$ has structure $[i, i+1]$, then we have

  - *(1.1)* $e_r(p, f(j,q)) \neq 0$ *only if* $j \in \{i, i{+}1\}$.
  - *(1.2)* $e_r(p, g(j,q)) \neq 0$ *only if* $j{=}i$.
  - *(1.3)* $e_r(p,q) \neq 0$ *only if* $i{=}s{+}1$ *(and* $r{=}2s{+}1$*).*

- *(2) If* $H_r(p,q)$ *has structure* $[i]$*, then we have*
  - *(2.1)* $e_r(p, f(j,q)) \neq 0$ *only if* $j{=}i$.
  - *(2.2)* $e_r(p, g(j,q)) \neq 0$ *only if* $j \in \{i{-}1, i\}$.
  - *(3.3)* $e_r(p,q) \neq 0$ *only if* $i{=}s{+}1$ *(and* $r{=}2s$*).*

- *(3) If* $H_r(p,q)$ *has structure* $[0]$*, then we have*
  - *(3.1)* $e_r(p, f(j,q)) = 0$ *for all* j.
  - *(3.2)* $e_r(p, g(j,q)) \neq 0$ *only if* $j{=}s{+}1$ *(and* $r{=}2s{+}1$*).*
  - *(3.3)* $e_r(p,q) \neq 0$.

We compare now the number of components in $G\big(p, f(i,q)\big)$ and $G\big(p, g(i,q)\big)$ with those of $G(p,q)$.

Lemma 3.3.35. *Let* $1{\leq}r{\leq}n{-}1$, $p \in W(n,r)$ *and* $q \in W(n, r{+}1)$. *If* $e_r\big(p, f(i,q)\big)$ *is non-zero, then we have*

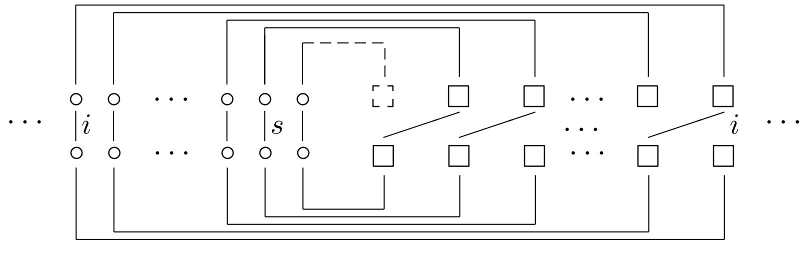
$$\mathrm{rl}\left(p, f(i,q)\right) = \begin{cases} \mathrm{rl}(p,q) + s - i + 2 & \text{if } H_r(p,q) \text{ is of structure } [i] \text{ or } [i{-}1, i] \\ \mathrm{rl}(p,q) + s - i + 1 & \text{if } H_r(p,q) \text{ is of structure } [i, i{+}1]. \end{cases}$$

If $e_r\big(p, g(i,q)\big)$ *is non-zero, then we have*

$$\mathrm{rl}\left(p, g(i,q)\right) = \begin{cases} \mathrm{rl}(p,q) + s - i + 1 & \text{if } H_r(p,q) \text{ is of structure } [i] \text{ or } [i, i{+}1] \\ \mathrm{rl}(p,q) + s - i & \text{if } H_r(p,q) \text{ is of structure } [i{+}1] \\ \mathrm{rl}(p,q) - 1 & \text{if } H_r(p,q) \text{ is of structure } [0]. \end{cases}$$

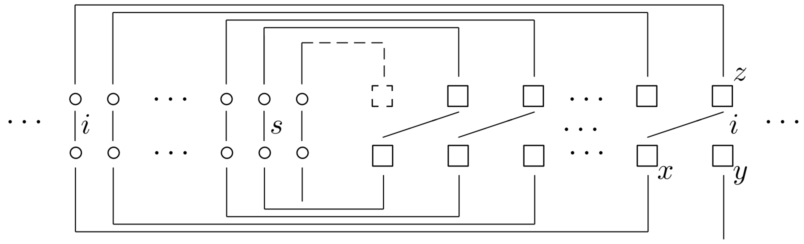
Proof. By Proposition 3.3.33 the cases mentioned here are indeed all of the relevant ones. We start with the situation of $r = 2s$, $f(i,q)$ and structure $[i]$. The relevant part of $G(p,q)$ influenced by the application of $f(i, \cdot)$ is the one "surrounded by the component containing i":

$G(p,q) =$



As mentioned before, the squares symbolize a structure consisting possibly of more than one point and more than one component, but the points not connected to any of the points $i, \dots, (s+2)$ are not affected by the manipulations. In this sense and as long as we are only interested in the alteration of the number of components, we treat the squares in the same way as points. Having this in mind, we see that in the picture above there is just one component drawn. Applying $f(i, \cdot)$ to q, we end up with

$$G\big(p, f(i,q)\big) =$$

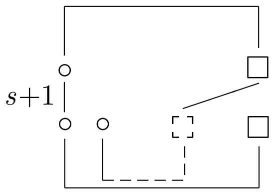


Comparing the two situations, we see that $G\big(p, f(i,q)\big)$ has $s - i + 2$ components more than $G(p,q)$. Starting with structure $[i, i+1]$, we directly see that the only difference is a connection between the squares y and z, reducing the number of components in $G\big(p, f(i,q)\big)$ by one.

Considering $g(i,q)$ instead of $f(i,q)$ just connects in the second picture the squares x and y in the lower row. For structure $[i]$, this decreases the number of components by 1 and in the case $[i, i+1]$ it leaves the components unchanged. Of course, with the cases $[i]$ and $[i, i+1]$ we also proved the corresponding statements for $[i+1]$ and $[i-1, i]$ by shifting i by ± 1.

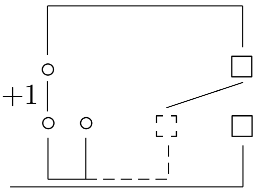
In the case $r = 2s+1$ we get the same results: The component that contains $(s+1)$ looks a bit different, but that does not affect the difference in numbers of components when we replace q by $f(i,q)$ or $g(i,q)$:

$G(p,q):\quad\cdots$



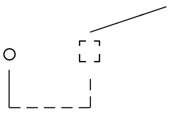
$\cdots$

$G\big(p,f(i,q)\big),G\big(p,g(i,q)\big):\quad\cdots$



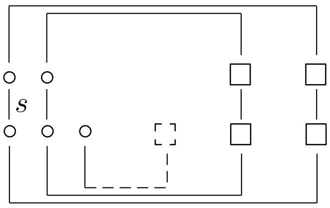
$\cdots$

Comparing this with the situation $r=2s$, one sees that we just have to delete in the lower row



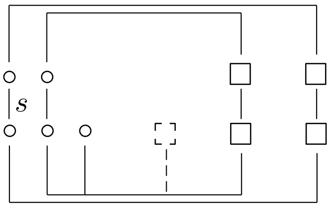
right beside $(s+1)'$. However, this does not change the number of components, so the formulae do not change.
Note that in the case of $g(s+1,q)$ and structure [0] the graphs to compare are given by

$G(p,q)=\quad\cdots$



$\cdots$

$G\big(p,g(s+1,q)\big)=\quad\cdots$



$\cdots$

so $\mathrm{rl}\,\big(p,g(s+1,q)\big)=\mathrm{rl}(p,q)-1$, as desired. $\square$

If $1 \leq r < n-1$ as well as $p \in W(n,r)$ and $q \in W(n, r+1)$ are given, we define now the following terms:

$$\begin{aligned} F_r(p,q) = & \sum_{j=1}^{s+1} N^{-(s-j+2)} \beta_{2j-1}(z) e_r\big(p, f(j,q)\big) \\ & - \sum_{j=1}^{t} N^{-(s-j+1)} \beta_{2j}(z) e_r\big(p, g(j,q)\big) \end{aligned} \tag{3.3.23}$$

The index bound t above is either s or $s{+}1$, depending on whether $r = 2s$ or $r = 2s{+}1$. For the sake of readability we used the abbreviation $z := \frac{1}{N}$. Here β are the reversed Beraha polynomials from Definition 3.3.2, given by the recursion

$$\begin{aligned} \beta_0(X) = 0 \quad , \quad \beta_1(X) = 1 \\ \beta_{n+1}(X) = \beta_n(X) - X\beta_{n-1}(X) \quad , \forall n \geq 1. \end{aligned} \tag{3.3.24}$$

Corollary 3.3.34 and Lemma 3.3.35 are preparatory for the following result about the expressions $F_r(p,q)$.

Lemma 3.3.36. *Let $1 \leq r < n-1$, $p \in W(n,r)$ and $q \in W(n,r+1)$. Using the abbreviation $z := \frac{1}{N}$, it holds*

$$(-1)^r F_r(p,q) = \beta_{r+2}(z)\, e_r(p,q) - \beta_{r+3}(z)\, e_{r+1}(p,q) \tag{3.3.25}$$

Proof. By definition of $F_r(p,q)$, see Equation 3.3.23, we can write out Equation 3.3.25 as

$$\begin{aligned} & \sum_{j=1}^{s+1} N^{-(s-j+2)} \beta_{2j-1}(z) e_r\big(p, f(j,q)\big) \\ & - \sum_{j=1}^{t} N^{-(s-j+1)} \beta_{2j}(z) e_r\big(p, g(j,q)\big) \\ = & (-1)^r \left(\beta_{r+2}(z)\, e_r(p,q) - \beta_{r+3}(z)\, e_{r+1}(p,q)\right) \end{aligned} \tag{3.3.26}$$

We prove this formula for all different structures of $H_r(p,q)$ which might appear. It will turn out that, in order to have at least one non-vanishing $e_\cdot(\cdot,\cdot)$, the graph $H_r(p,q)$ has to be of one of the structures from Definition 3.3.30 and, even in these cases, most summands on the left side turn out to be zero.

Case 1: $H_r(p,q)$ has structure $[i, i{+}1]$: If $i \neq s{+}1$, then by Corollary 3.3.34 only three summands survive in the sums in Equation 3.3.26: The summands for $j = i$ and $j = i{+}1$ in the first sum and the summand for $j = i$ in the second sum. Hence, Equation 3.3.26 reads

$$\begin{aligned} &N^{-(s-i+2)}\beta_{2i-1}(z)e_r\big(p, f(i,q)\big) + N^{-(s-i+1)}\beta_{2i+1}(z)e_r\big(p, f(i+1,q)\big) \\ &\quad - N^{-(s-i+1)}\beta_{2i}(z)e_r\big(p, g(i,q)\big) \\ &= (-1)^r \left(\beta_{r+2}(z)\, e_r(p,q) - \beta_{r+3}(z)\, e_{r+1}(p,q)\right) \end{aligned} \tag{3.3.27}$$

By Lemma 3.3.35 it holds

$$e_r\big(p, f(i,q) = e_r\big(p, f(i{+}1), q\big) = e_r\big(p, g(i,q)\big) = N^{\mathrm{rl}(q^*,p)+s-i+1},$$

so we can rewrite the left side of Equation 3.3.27 to obtain

$$N^{\mathrm{rl}(q^*,p)} \underbrace{\Big(z\beta_{2i-1}(z) - \beta_{2i}(z) + \beta_{2i+1}(z)\Big)}_{=0} = (-1)^r \left(\beta_{r+2}(z)\, e_r(p,q) - \beta_{r+3}(z)\, e_{r+1}(p,q)\right).$$

The bracket vanishes by the recursion formula for the reversed Beraha polynomials, see Equation 3.3.24. Because of structure $[i, i{+}1]$ it holds

$$e_r(p,q) = e_{r+1}(p,q) = 0,$$

and also the right side vanishes, as desired.
The situation $i = s{+}1$ is only defined and relevant if $r = 2s{+}1$ and is odd. In this case we have to omit in Equation 3.3.27 the term with $\beta_{2i+1}(z)$ as the summations over j in Equation 3.3.26 both end at $j = s+1$. Hence, Equation 3.3.26 reads

$$N^{\mathrm{rl}(q^*,p)}\Big(z\beta_{2i-1}(z){-}\beta_{2i}(z)\Big) = (-1)^r \left(\beta_{r+2}(z)\, e_r(p,q) - \beta_{r+3}(z)\, e_{r+1}(p,q)\right). \tag{3.3.28}$$

We compute for the left side

$$\begin{aligned} &N^{\mathrm{rl}(q^*,p)}\Big(z\beta_{2i-1}(z) - \beta_{2i}(z)\Big) \\ =&N^{\mathrm{rl}(q^*,p)}\Big(- \beta_{2i+1}(z)\Big) \\ =&N^{\mathrm{rl}(q^*,p)}\Big(- z\beta_{r+2}(z)\Big) \\ =&(-1)^r \Big(\beta_{r+2}(z)\Big) e_r(p,q) \end{aligned}$$

as $-1 = (-1)^r$ and $e_r(p,q) = N^{\mathrm{rl}(q^*,p)}$ by Proposition 3.3.33. This is equal to the right side of Equation 3.3.28 because $e_{r+1}(p,q)$ is zero: Structure $[s{+}1, s{+}2]$ in particular tells us that $(s+1)'$ and $(s+2)'$ are connected in $H_r(p,q)$, so there is an $(r{+}1)$-flaw in $G(p,q)$ and $e_{r+1}(p,q) = 0$. Again this shows the claim.
Case 2: $H_r(p,q)$ **has structure** $[i]$**:** We exclude at first the situation $i{=}1$ as well

as $i = s+1$ in the case $r = 2s$. By Corollary 3.3.34 only three summands survive in the sums in Equation 3.3.26: The summand for $j = i$ in the first sum and the summands for $j=i-1$ and $j=i$ in the second sum. Hence, Equation 3.3.26 reads in this situation

$$\begin{aligned}&N^{-(s-i+2)}\beta_{2i-1}(z)e_r\big(p, f(i,q)\big)\\&\quad - N^{-(s-i+2)}\beta_{2i-2}(z)e_r\big(p, g(i-1,q)\big) - N^{-(s-i+1)}\beta_{2i}(z)e_r\big(p, g(i,q)\big)\\&= (-1)^r\left(\beta_{r+2}(z)\, e_r(p,q) - \beta_{r+3}(z)\, e_{r+1}(p,q)\right)\end{aligned} \tag{3.3.29}$$

By Lemma 3.3.35 it holds

$$e_r\big(p, f(i,q) = e_r\big(p, g(i,q)\big) = N^{\mathrm{rl}(q^*,p)+s-i+1} \quad , \quad e_r\big(p, g(i-1,q)\big) = N^{\mathrm{rl}(q^*,p)+s-i+2},$$

so we can rewrite the left side of Equation 3.3.27 to obtain

$$N^{\mathrm{rl}(q^*,p)}\underbrace{\Big(-z\beta_{2i-2}(z) + \beta_{2i-1}(z) - \beta_{2i}(z)\Big)}_{=0} = (-1)^r\big(\beta_{r+2}(z)\, e_r(p,q) - \beta_{r+3}(z)\, e_{r+1}(p,q)\big).$$

The bracket vanishes again by the recursion formula for the reversed Beraha polynomials, see Equation 3.3.24. If $i=1$, then in Equation 3.3.23 we do not have a summation index $j = 0$ and so we have to omit at first sight the term $z\beta_{2i-2}(z) = z\beta_0(z)$. However, as $\beta_0(z) = 0$, we still get the same result. It remains to show that also the right side vanished:

We note that having structure $[i]$ (and not the situation of $i = s+1$ in combination with $r = 2s$) says that the point i is the witness of an r-flaw of $G(p,q)$ as i is not connected to i' in $H_r(p,q)$. Hence both $e_r(p,q)$ and $e_{r+1}(p,q)$ are zero.

Finally, we consider the case of $i=s+1$ and $r=2s$. The second sum on the left side of Equation 3.3.26 ends at $t=s$, so there is no term $\beta_{2i}(z)$ to consider and Equation 3.3.29 reads

$$N^{\mathrm{rl}(q^*,p)}\Big(-z\beta_{2i-2}(z) + \beta_{2i-1}(z)\Big) = (-1)^r\big(\beta_{r+2}(z)\, e_r(p,q) - \beta_{r+3}(z)\, e_{r+1}(p,q)\big). \tag{3.3.30}$$

For the left side we compute

$$\begin{aligned}N^{\mathrm{rl}(q^*,p)}\Big(-z\beta_{2i-2}(z) + \beta_{2i-1}(z)\Big) &= N^{\mathrm{rl}(q^*,p)}\Big(\beta_{2i}(z)\Big)\\&= N^{\mathrm{rl}(q^*,p)}\Big(\beta_{r+2}(z)\Big)\\&= (-1)^r\Big(\beta_{r+2}(z)\Big)e_r(p,q)\end{aligned}$$

as $(-1)^r = (-1)^{2s} = 1$ and $e_r(p,q) = N^{\mathrm{rl}(q^*,p)}$ by Proposition 3.3.33. For the right side of Equation 3.3.30 we see again that $e_{r+1}(p,q)$ vanishes: Structure $[s+1]$ tells us that the point $(s+1)$ witnesses an $(r{+}1)$-flaw of $G(p,q)$ as it is not connected to $(s{+}1)'$ in $H_r(p,q)$. Hence, we proved the claim also in this case.

Case 3: $H_r(p,q)$ has structure $[0]$: If we assume structure $[0]$, then there is only one possibility for a non-trivial summand on the left side of Equation 3.3.26, namely if $r{=}2s{+}1$ and then only the summand for $j{=}s{+}1$ in the second sum survives. It can directly be checked by the definition of structure $[0]$, that $e_{r+1}(p,q)$ is non-zero and so Equation 3.3.26 reads in this case

$$\beta_{2s+2}(z)e_r\big(p,g(s+1,q)\big) = (-1)^r\big(\beta_{r+2}(z)\,e_r(p,q) - \beta_{r+3}(z)\,e_{r+1}(p,q)\big) \tag{3.3.31}$$

By Lemma 3.3.35 it holds

$$e_r\big(p,g(s+1,q) = N^{\mathrm{rl}(q^*,p)-1}$$

and we compute

$$\begin{aligned}\beta_{2s+2}(z)e_r\big(p,g(s+1,q)\big)&\\ &=N^{\mathrm{rl}(q^*,p)}z\beta_{r+1}(z)\\ &=N^{\mathrm{rl}(q^*,p)}\Big(\beta_{r+2}(z)-\beta_{r+3}(z)\Big)\\ &=\beta_{r+2}(z)e_r(p,q)-\beta_{r+3}(z)e_{r+1}(p,q),\end{aligned}$$

proving Equation 3.3.31.

Case 4: All other structures: For all other structures apart from the three cases above we have that $e_r\big(p,f(i,q)\big)$, $e_r\big(p,g(i,q)\big)$ and $e_r\big(p,q)\big)$ are zero by Proposition 3.3.33. We have not proved this for $e_{r+1}(p,q)$ yet, but it can easily be seen that structure $[0]$ on $H_r(p,q)$ is a necessary condition for $e_{r+1}(p,q)$ to be non-zero. So also $e_{r+1}(p,q)$ vanishes for all structures not explicitly considered above. Hence, both sides of Equation 3.3.25 are zero. This finishes the proof. □

3.3.5 A recursion formula

Consider the matrix $A(n,r)$ for $0 \le r < n-1$. Assume that the first $\#Y(n,r)$ rows and columns are labelled by the elements in $Y(n,r)$. In this case, $A(n,r)$ has a block structure given by

$$A(n,r) = \begin{pmatrix} B(n,r) & C \\ D & E \end{pmatrix}$$

We denote the column of $A(n,r)$ associated to a partition q by $\mathrm{Col}(q)$.

Now for every $q \in W(n,r+1)$, we consider the linear combination of columns

$$F_r(q) := \sum_{j=1}^{s+1} N^{-(s-j+2)}\beta_{2j-1}(z)\,\mathrm{Col}\,\big(f(j,q)\big) - \sum_{j=1}^{t} N^{-(s-j+2)}\beta 2j(z)\,\mathrm{Col}\,\big(g(j,q)\big)$$

where we again defined $z := \frac{1}{N}$ and the summation bound t depends on the parity of r:

$$t := \begin{cases} s & , r \text{ even} \\ s+1 & , r \text{ odd} \end{cases}$$

Note that $F_r(q)$ is just the column vector $\big(F_r(p,q)\big)_{p \in W(n,r)}$. Multiplying now $\mathrm{Col}(q)$ by $\beta_{r+2}(z)$ and adding $(-1)^r F_r(q)$ to it changes $\mathrm{Col}(q)$ by Equation 3.3.25 to

$$\beta_{r+3}(z)\Big(e_{r+1}(p,q)\Big)_{p \in W(n,r)}.$$

For the first $\#Y(n,r)$ rows this is zero, as for $p \in Y(n,r)$ the graph $G(p,q)$ has an $(r+1)$-flaw, so $e_{r+1}(p,q) = 0$. Hence we have

$$\begin{aligned} &\det \begin{pmatrix} B(n,r) & C \\ D & E \end{pmatrix} \\ =& \left(\frac{\beta_{r+3}(z)}{\beta_{r+2}(z)}\right)^{\#(W(n,r+1))} \det \begin{pmatrix} B(n,r) & 0 \\ D & A(n,r+1) \end{pmatrix} \\ =& \left(\frac{\beta_{r+3}(z)}{\beta_{r+2}(z)}\right)^{\#(W(n,r+1))} \det\big(B(n,r)\big)\det(A(n,r+1)) \end{aligned}$$

By Proposition 3.3.23, the matrix $B(n,r)$ is again given by a matrix of type $A(n-1,r')$, so we can summarize the observations above in the following Proposition:

Proposition 3.3.37. *For $0 \le r < n-1$ we have the following recursion formula for the determinant of $A(n,r)$:*

$$\det\big(A(n,r)\big) = \left(\frac{\beta_{r+3}(z)}{\beta_{r+2}(z)}\right)^{\#(W(n,r+1))} \det\big(B(n,r)\big)\det(A(n,r+1)) \tag{3.3.32}$$

where

$$\det\big(B(n,r)\big) = \begin{cases} \det\big(A(n-1,r-1)\big) & , \ r = 2s+1. \\ N \cdot \det\big(A(n-1,r-1)\big) & , \ r = 2s \text{ and } r > 0 \\ N \cdot \det\big(A(n-1,0)\big) & , \ r = 0. \end{cases} \tag{3.3.33}$$

Remark 3.3.38. Note that by Lemma 3.3.7 the expression $\beta_n(z)$ is non-zero for $n \in \mathbb{N}$, $z = \frac{1}{N}$ and $N \in \mathbb{N}_{\ge 4}$. Hence, also the fractions

$$\frac{\beta_{r+3}(z)}{\beta_{r+2}(z)}$$

are non-zero and, in addition, the whole recursion above is guaranteed to be well-defined.

3.3.6 Conclusion: Linear independence of the maps T_p

With the results of Section 3.3.5, we can finally prove invertibility of the matrix $A(n,0)$ and so Theorem 3.3.1.

Theorem 3.3.39. *Let $N \in \mathbb{N}_{\geq 4}, n \in \mathbb{N}$ and consider the Gram matrix*

$$A(n,0) = \left(\langle T_p(1), T_q(1)\right)_{p,q \in \mathcal{NC}(0,n)} .$$

Then $A(n,0)$ is invertible.

Proof. By the considerations in Section 3.3.1 and Section 3.3.4, the Gram matrix above coincides with the matrix $A(n,0)$ as defined in Definition 3.3.17. Using repeatedly the recursion formula from Proposition 3.3.37, we see that the determinant of $A(n,0)$ is given by a product whose factors are

(1) quotients $\dfrac{\beta_{r+3}(z)}{\beta_{r+2}(z)}$,

(2) powers of N or

(3) determinants of matrices $A(n, n-1)$ with $1 \leq n \leq N$.

The quotients in (1) are non-zero as the reversed Beraha polynomials have no roots at $z = \frac{1}{N}$, see Remark 3.3.38 and Lemma 3.3.7. Concerning the determinants in (3), it holds

$$\det\big(A(n, n-1)\big) = N^{\lceil \frac{n}{2} \rceil} \neq 0$$

by Proposition 3.3.24. Hence, we conclude for $N \in \mathbb{N}_{\geq 4}$ and all $n \in \mathbb{N}$ that $\det\big(A(n,0)\big)$ is non-zero, i.e $A(n,0)$ is invertible. □

Theorem 3.3.1 is a direct consequence of this result:

Corollary (compare Theorem 3.3.1)**.** *For any given $N \in \mathbb{N}_{\geq 4}$ and $n \in \mathbb{N}$, the collection of linear maps*

$$\left(T_p\right)_{p \in \mathcal{NC}(0,n)} ,$$

as defined in Definition 2.5.4, is linearly independent.

Proof. Given the linear independence of the vectors $\left(T_p(1)\right)_{p \in \mathcal{NC}(0,n)}$, this result follows directly from the fact that the linear maps T_p above are maps from the complex numbers (into some Hilbert space). □

As displayed in Section 3.1, the linear independence of maps $\big(T_p\big)_{p \in (0,n)}$ (for $N \geq 4$) guarantees the following result:

Corollary 3.3.40. *Let $N \geq 4$ and consider the functor Ψ from Equation 3.0.1, given by*

$$\mathcal{C} \overset{\Psi}{\longmapsto} R_N(\mathcal{C}).$$

Restricted to non-crossing categories of partitions, Ψ is injective.

3.3.7 A comparison to Tutte's work

Sections 3.3.4 and 3.3.5 are heavily based on W. Tutte's work "The matrix of chromatic joints", [Tut93]. This section is used to stress again the changes and corrections we performed compared to the ideas in the original work.
The aforementioned two sections are an adaption of [Tut93] in the following sense: All objects defined in this work appear (in similar or the same form) in [Tut93] and the logical steps in order to establish a recursion formula for the determinant of the matrix $A(n,0)$ are adopted from [Tut93]:

(1) The sets of partitions $W(n,r)$ and $Y(n,r)$ as well as the partition manipulations $f(i,q)$ and $g(i,q)$.

(2) The Graphs $G(p,q)$ and $H_r(p,q)$ and r-flaws.

(3) The matrices $A(n,r)$ and $B(n,r)$.

(4) The structures $[i]$, $[i,i{+}1]$ and $[0]$.

(5) The expression $F_r(p,q)$ that describes the column manipulations to be performed in the matrix $A(n,r)$.

(6) The comparison of matrices $B(n,r)$ and $A(n{-}1,r')$.

(7) The statements about absence of r-flaws in the graphs $G(p,q)$, $G\big(p,f(i,q)\big)$ and $G\big(p,g(i,q)\big)$, respectively.

(8) The statement about the number of components in the graphs $G(p,q)$, $G\big(p,f(i,q)\big)$ and $G\big(p,g(i,q)\big)$.

(9) The statement about $(-1)^r F_r(p,q)$, describing the outcome of the column maniplations in Section 3.3.5.

(10) The recursion formula from Section 3.3.5 itself.

In [Tut93], item (7) and (8), two of the most important ingredients in order to establish in the end the recursion formula from Section 3.3.5, were detected to have errors. More precisely, the statement of item (7) is still the same, compare Proposition 3.3.33 and [Tut93, p. 278], but in Tutte's work it was not compatible with the definition of the structures $[i]$, $[i,i{+}1]$ and $[0]$, see [Tut93, p. 277].

(a) In order to keep Tutte's ideas usable (and the statement behind item (7), Proposition 3.3.33, true), we changed the definition of the structures $[i]$, $[i,i{+}1]$ and $[0]$, compare Definition 3.3.30 and [Tut93, pp. 277], and the statement referring to item (8), compare Lemma 3.3.35 and [Tut93, Thm. 5.1].

(b) Although the result behind item (9) itself was not wrong in Tutte's work, compare Lemma 3.3.36 and [Tut93, Thm. 5.2], its proof needed to be adapted as it heavily relied on item (7) and (8).

In addition to the corrective adaptation above, we differ in the following aspects from [Tut93]:

(i) The definitions of the graphs $G(p,q)$ and $H_r(p,q)$ in this thesis, Definitions 3.3.12 and 3.3.15, are equivalent to those in [Tut93, p. 270, p. 277], but our definitions of vertices and edges are different.

(ii) As a consequence, also our definition of an r-flawless graph, Definition 3.3.16, uses other formulations.

(iii) The pictures of partitions as used in Equation 3.3.5 and thereafter do not appear in [Tut93] and neither do the illustrations of graphs as in Equation 3.3.12 or Equation 3.3.13.

(iv) The schemes as introduced in Notation 3.3.26, describing or defining connections in a graph, are added in order to improve the readability of definitions and proofs.

Chapter 4

Models for $C(S_N^+)$

4.1 Motivation

As described in Observation 2.6.4 and Notation 2.6.6, we can construct for a given $N \in \mathbb{N}$ an easy quantum groups from a set of partitions Π if it contains the mixed-coloured pair partitions: Given an $N \times N$-matrix of generators $u = (u_{ij})_{1 \leq i,j \leq N}$, the easy quantum group in its universal form associated to the category $\langle \Pi \rangle$ is given by the universal C^*-algebra

$$A := C^*\big(\, (u_{ij})_{1 \leq i,j \leq N} \quad | \quad \forall p \in \Pi : \text{ The relations } \mathcal{R}_p^{Gr}(u) \text{ hold. } \big)$$

together with the matrix $u \in M_N(A)$. In many cases, see for example Section 2.6.1, the cardinalities for generating sets Π can be chosen quite small. Hence, the structure of the corresponding quantum group $G_N(\Pi)$ becomes quite easy to handle in the sense that we have a good understanding of the algebraic relations that define the universal C^*-algebra $A = C\big(G_N(\Pi)\big)$. Of course, this is advantageous with respect to all statements that can be formulated and/or proved on the level of generators i.e. where we make use of the universal property of our given C^*-algebra.

While the description of a C^*-algebra as a universal C^*-algebra reflects the abstract definition of a C^*-algebra, there is also the concrete perspective on C^*-algebras, namely as bounded operators on a Hilbert space.

The starting question for this chapter is, how to interpret a C^*-algebra $C\big(G_N(\Pi)\big)$ as concrete operators on a Hilbert space, i.e. how to improve the second perspective on the C^*-algebra at hand. Looking for representations (π, H_π) of a C^*-algebra, one often (if not necessarily always) faces the following dilemma:

- If the *-homomorphism π has a large kernel, then one usually has a good understanding of its image but the prize is, evidently, that a lot of information about the preimage gets lost when applying π.
- If the *-homomorphism π has a small kernel, or if it is in fact faithful, then the image often remains a quite complicated and abstract object. Considering for example the universal representation (π_u, H_{π_u}), no information about the preimage is lost after having applied π_u. On the backside, this representation does not really provide us with any additional information. The structure of H_{π_u} is, more or less, just the structure of the original C^*-algebra.

A solution for this problem can be to find a sequence of representations

$$\big(\pi_n, H_{\pi_n}\big)_{n \in \mathbb{N}}$$

where the kernel of the π_n is large for small n but vanishes as n tends to infinity. If there is an appealing description of the connection

$$\big(\pi_n, H_{\pi_n}\big) \leftrightarrow \big(\pi_{n+1}, H_{\pi_{n+1}}\big),$$

this provides us with an enlightening description of $\big(\pi_n, H_{\pi_n}\big)$, for large n and also in the limit $n \to \infty$.

4.2 The goal

In this Chapter we concentrate on the easy quantum groups S_N^+ and we exclude the cases $N \leq 3$, as then the free permutation group coincides with its group analogon, S_N, and its structure is already perfectly understood. At the first, we concentrate on the case $N=4$ and then develop from the given observations results for the general case of S_N^+ for $N \geq 4$; so, in the initial step, we strive for a better understanding of the C^*-algebra $C(S_4^+)$. Instead of looking for actual representations of $C(S_4^+)$, we are satisfied with models for $C(S_4^+)$, i.e. *-homomorphisms

$$\varphi_n : C(S_4^+) \to B_n,$$

from $C(S_4^+)$ into C^*-algebras B_n, where B_n has a well-understood structure. As in the situation of representations, we call such a model *faithful* if the corresponding map is injective. Note that we could compose φ_n with the universal representation of B_n to obtain an actual representation of $C(S_4^+)$.
Another work asking for (specific) models of $C(S_N^+)$, so called flat matrix models, is [BN17], but this chapter does not share with it more than this initial question.
Replacing B_n by $\varphi_n(\mathbb{1})B_n\varphi_n(\mathbb{1})$ makes φ_n unital, so we demand from the beginning that B_n, and also φ_n, are unital.
Every such *-homomorphism φ_n corresponds to a pair

$$(B_n, M_n) := \big(B_n, (\mathbb{1} \otimes \varphi_n)u_{S_4^+}\big),$$

where the 4×4-matrix $M_n \in M_4\big(C(S_4^+)\big)$ contains the images of the generators u_{ij} under φ_n and, doing so, carries the whole information about $\varphi_n : C(S_4^+) \to B_n$.

Summarizing, we are interested in the following situation and object:
Consider the category given by pairs (A, M) where A is a unital C^*-algebra , $M \in M_4(A)$ and the entries of M generate A as a C^*-algebra. Arrows ψ between two objects (A, M) and (A', M') are given by *-homomorphisms $\psi : A \to A'$ fulfilling $(\mathbb{1} \otimes \psi)(A) = A'$, i.e. the entries of M are mapped by ψ canonically to the entries of M'. Note that such an arrow ψ is automatically unital and surjective and there exists at most one arrow between two objects because ψ is uniquely defined by the matrices M and M'.
In this setting we are looking for commutative diagrams

$$\begin{tikzcd}
 & & \Big(C(S_4^+), u_{S_4^+}\Big) \arrow[ddll, "\varphi_1"'] \arrow[dd, "\varphi_n"'] \arrow[ddr, "\varphi_{n+1}"] & & \\
 & \cdots & & & \cdots \\
(B_1, M_1) & \cdots \arrow[l, "\pi_{2,1}"'] & (B_n, M_n) & (B_{n+1}, M_{n+1}) \arrow[l, "\pi_{n+1,n}"'] & \cdots \arrow[l, "\pi_{n+2,n+1}"']
\end{tikzcd}$$

In Section 4.4 we present such a diagram. We prove it to have the following properties:

(1) The sequence $\big(B_n, M_n\big)_{n\in\mathbb{N}}$ allows an inverse limit construction (B_∞, M_∞), see Section 4.5, which is again a compact matrix quantum group G, see Theorem 4.6.7.

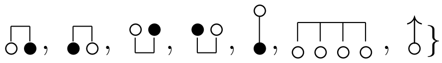
(2) Using results from [Ban18a], we deduce that G equals S_4^+, see Corollary 4.6.9. In particular, we find an inverse limit construction of a C^*-algebra $C_{red}(S_4^+) \subseteq A \subseteq C_u(S_4^+)$ based on a sequence of well-understood C^*-algebras.

While (1) can be generalized to all relevant S_N^+, see Theorem 4.7.9, the equality of G and S_N^+, or, more precisely, its proof, turns out to depend on the maximality of the inclusion $S_N \subseteq S_N^+$, see [Ban18a].

4.3 The easy quantum groups S_N^+

We collect some properties of the sequence $\big(S_N^+\big)_{N\in\mathbb{N}}$. All of them are well-known, see for example [Wan98], where these quantum groups have been discovered and [BS09], where they have been put in the context of easy quantum groups.

As described in Section 2.6.1, the category associated to this sequence of easy quantum groups is generated by

$\Pi = \{$  $\}$

and we have

$$C\big(S_N^+\big) = C^*\big(\ (u_{ij})_{1\leq i,j\leq N} \quad | \quad \forall p \in \Pi : \text{ The relations } \mathcal{R}_p^{Gr} \text{ hold. } \big).$$

As described before, the mixed-coloured pair partitions exactly say that u and $u^{(*)}$ are unitaries and the mixed-coloured identity exactly says that all entries of u are self-adjoint. The uni-coloured four-block above corresponds to the relations

$$\sum_{j=1}^{N} u_{i_1 j} u_{i_2 j} u_{i_3 j} u_{i_4 j} = \begin{cases} \mathbb{1} & , \text{if } i_1 = i_2 = i_3 = i_4 \\ 0 & , \text{else.} \end{cases} \tag{4.3.1}$$

for all $(i_1, \ldots, i_4) \in [N]^4$.
By orthogonality of u we already know

$$\sum_j u_{ij} u_{ij} = \mathbb{1} = \sum_j u_{ji} u_{ji}. \tag{4.3.2}$$

Combining this with Equation 4.3.1 for $i_1 = i_2 = i_3 = i_4 = i$, we see

$$\mathbb{1} = \sum_{j=1}^{N} u_{ij}u_{ij}u_{ij}u_{ij} \leq \sum_{j} u_{ij}u_{ij} = \mathbb{1}$$

which is only possible if $u_{ij}^2 = u_{ij}^4$, i.e. each entry of u is a partial isometry. Considering the u_{ij} as operators on some Hilbert space and using again Equation 4.3.2, we see that the cokernels of the u_{ij} in one row or column of u are orthogonal to each other. Now the quantum group relations associated to the singleton partition $\overset{\uparrow}{\circ}$ are exactly

$$\sum_{j=1}^{N} u_{ij} = \mathbb{1}$$

for all $i \in [N]$. So each u_{ij} is a projection. Using a third time Equation 4.3.2, we find that each row and column of u sums up to $\mathbb{1}$.
If we conversely consider a matrix u such that its entries are projections summing up to $\mathbb{1}$ in every row and column, one can check that the quantum group relations

$$\left(\mathcal{R}_p^{Gr}(u)\right)_{p\in\Pi}$$

are fulfilled. Hence we found an alternative description of $C(S_N^+)$ as a universal C^*-algebra:

Proposition 4.3.1. *It holds*

$$C\big(S_N^+\big) = C^*\Big(\ (u_{ij})_{1\leq i,j\leq N} \mid u_{ij} = u_{ij}^2 = u_{ij}^* \ ; \ \sum_{j=1}^{N} u_{ij} = \sum_{j=1}^{N} u_{ji} = \mathbb{1} \ \forall i \in [N]\ \Big).$$

Note that the difference between $C\big(S_N\big)$ and $C\big(S_N^+\big)$ is indeed just the commutativity of the generators u_{ij}:

Proposition 4.3.2. *The C^*-algebra $C\big(S_N\big)$ is obtained from $C\big(S_N^+\big)$ by dividing out the commutativity relations for the generators u_{ij}.*

Proof. The canonical generators u_{ij} of $C\big(S_N^+\big)$ are self-adjoint, so adding the relations $\mathcal{R}^{Gr}_{\times}(u)$ exactly adds commutativity of the u_{ij}'s to the relations listed in Proposition 4.3.1, see the proof of Proposition 2.6.10. Now for every generating set Π of $\mathcal{NC}$ the set $\Pi \cup \{\times\}$ generates $\mathcal{P}$. We conclude that one obtains $C\big(S_N\big)$ as in Proposition 4.3.1 but, in addition, one has to impose commutativity on the generators u_{ij}. □

It is now easy to prove that S_1^+, S_2^+ and S_3^+ coincide with their classical versions:

Proposition 4.3.3. *For $1 \leq N \leq 3$ the easy quantum groups S_N^+ are actually easy groups, i.e. it holds $S_N^+ = S_N$.*

Proof. By the observations above we just have to prove that the generators u_{ij} of $C\big(S_N^+\big)$ commute in the mentioned cases. For $N=1$ this is trivial and for $N=2$ this is clear as the matrix u must be of the form

$$u = \begin{pmatrix} u_{11} & \mathbb{1} - u_{11} \\ \mathbb{1} - u_{11} & u_{11} \end{pmatrix}.$$

In the case $N=3$ only the commutativity of u_{ij}'s in one common row or column is clear. We consider exemplarily u_{11} and u_{22}, for all other such pairs the arguments are the same.
It holds

$$\begin{aligned} u_{11}u_{22} &= (\mathbb{1} - u_{12} - u_{13})u_{22}(u_{11} + u_{21} + u_{31}) \\ &= u_{22}u_{11} + u_{22}u_{31} - u_{13}u_{22}u_{11} - u_{13}u_{22}u_{31} \\ &= u_{22}u_{11} + u_{22}u_{31} - u_{13}\big(\mathbb{1} - u_{21} - u_{23}\big)u_{11} - u_{13}\big(\mathbb{1} - u_{12} - u_{32}\big)u_{31} \\ &= u_{22}u_{11} + u_{22}u_{31} - u_{13}u_{11} - u_{13}u_{31} \\ &= u_{22}u_{11} + (\mathbb{1} - u_{21} - u_{23})u_{31} - (\mathbb{1} - u_{23} - u_{33})u_{31} \\ &= u_{22}u_{11} + u_{31} - u_{23}u_{31} - u_{31} + u_{23}u_{31} \\ &= u_{22}u_{11}. \end{aligned}$$

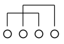
We repeatedly used the fact that each row and column of u sums up to one and inside one row or column the product of different u_{ij}'s is zero.
Alternatively, one computes

$$\begin{aligned} u_{11}u_{22}u_{11}u_{22} &= u_{11}u_{22}(\mathbb{1} - u_{12} - u_{13})u_{22} \\ &= u_{11}u_{22} - u_{11}u_{22}u_{13}u_{22} \\ &= u_{11}u_{22} - u_{11}(\mathbb{1} - u_{21} - u_{23})u_{13}u_{22} \\ &= u_{11}u_{22}. \end{aligned} \tag{4.3.3}$$

Considering u_{11} and u_{22} as operators on a Hilbert space, Equation 4.3.3 tells us that $u_{11}u_{22}$ is equal to the weak limit of the sequence $\big((u_{11}u_{22})^n\big)_{n\in\mathbb{N}}$ but this is obviously the infimum $u_{11} \wedge u_{22} = u_{22} \wedge u_{11}$ of these two projections, compare for example [Hal67, Problem 96]. Switching roles shows $u_{22}u_{11} = u_{22} \wedge u_{11}$, i.e the desired commutativity. $\square$

Remark 4.3.4. We can also prove the commutativity of $C(S_3^+)$ on the level of maps T_p. Going through the proof of Proposition 3.2.4 for $N=3$, one can check that the coefficient in front of T in L_2, and so in L_1, is non-zero. Hence, we have

T $\in \operatorname{span}(\{\, T_p \mid p \in \mathcal{NC}(\varepsilon, 4)\})$.

Thus, the quantum group relations $\mathcal{R}^{Gr}$ (u) are already fulfilled in $C(S_3)$. As $\{$  $\}$ together with $\mathcal{NC}$ generates $\mathcal{P}$, the category of all partitions, we have by Observation 2.6.4 that $C(S_3^+) = C(S_3)$.

In contrast to the above situations, S_4^+ is not a group.

Proposition 4.3.5. *The easy quantum group S_4^+ is not a group, i.e. $C\big(S_4^+\big)$ is non-commutative.*

Proof. There are two ways to prove the claim. One possibility is to have a look at the categories of partitions P and NC associated to S_4 and S_4^+, see Section 2.6.1. The sets $\mathcal{P}(2,2)$ and $\mathcal{NC}(2,2)$ differ as $\mathcal{P}(2,2)$ contains the crossing partition , so the corresponding quantum groups S_4^+ and S_4 are not the same by Proposition 3.1.2, Corollary 3.1.4 and Corollary 3.2.6. Hence, $C\big(S_4^+\big)$ is non-commutative as otherwise it would be equal to $C\big(S_4\big)$ by Proposition 4.3.2.
Alternatively, one can show that there is a *-homomorphism from $C\big(S_4^+\big)$ to another C^*-algebra with non-commutative image. This is easily done, as by the universal property of $C\big(S_4^+\big)$ and Proposition 4.3.1 every pair (p,q) of non-commutative projections in a C^*-algebra A defines a *-homomorphism $\varphi : C(S_4^+) \to A$ via the matrix

$$\widetilde{R} = \begin{pmatrix} p & 1-p & 0 & 0 \\ 1-p & p & 0 & 0 \\ 0 & 0 & q & 1-q \\ 0 & 0 & 1-q & q \end{pmatrix} \in M_4\big(A\big).$$

Hence, $C\big(S_4^+\big)$ cannot be commutative as already its image under φ is not. $\square$

4.4 A chain of matrices

Combining our observations in Section 4.2 with Proposition 4.3.1, we observe the following.

Observation 4.4.1. Consider for some unital C^*-algebra B a matrix $M \in M_4(B)$ where the entries of M are projections and sum up in each row and column to the identity in B. Then the pair (B, M) defines in the sense of Section 4.2 a model $\varphi : C\big(S_4^+\big) \to B$ sending the entries of u canonically to the entries of M.

Given two such matrices $M_1 \in M_4(B_1)$ and $M_2 \in M_4(B_2)$, we can use the operator $\oplus$ that has already been mentioned in the proof of Theorem 2.3.25 to define a new matrix associated to a new *-homomorphism from $C\big(S_4^+\big)$ to $B_1 \otimes B_2$. Note again that this operator already appears in S. L. Woronowicz's work [Wor88].

Definition 4.4.2. Let $N \in \mathbb{N}$ and B_1 and B_2 be C^*-algebras. Let $M_1 = \left(m_{ij}^{(1)}\right)$ and $M_2 = \left(m_{ij}^{(2)}\right)$ be elements in $M_N(B_1)$ and $M_N(B_2)$, respectively. Then we define the $N \times N$-matrix

$$M_1 \mathbin{\bigcirc\hspace{-0.75em}\perp} M_2 := \sum_{i,j=1}^{N} E_{ij} \otimes \sum_{k=1}^{N} m_{ik}^{(1)} \otimes m_{kj}^{(2)} \quad \in M_N(B_1 \otimes B_2),$$

where E_{ij} are the rank-one matrices $\left(\delta_{ik}\delta_{jl}\right)_{1 \le k,l \le N} \in M_N(\mathbb{C})$.

Note that this operator is different from the operator $\mathbin{\bigcirc\hspace{-0.75em}\top}$ as defined in part (ii) of Observation 2.1.12: $M_1 \mathbin{\bigcirc\hspace{-0.75em}\perp} M_2$ is, like M_1 and M_2 themselves, an $N \times N$-matrix, but now over the C^*-algebra $B_1 \otimes B_2$. In contrast to that, $M_1 \mathbin{\bigcirc\hspace{-0.75em}\top} M_2$ is only defined for $B_1 = B_2$ and it is an $N^2 \times N^2$-matrix, again over $B_1 = B_2$. Its entries are the products of $m_{ij}^{(1)}$'s and $m_{kl}^{(2)}$'s. In addition, the matrix arguments of $\mathbin{\bigcirc\hspace{-0.75em}\top}$ do not need to have the same size.

Remark 4.4.3. As the (minimal) tensor product of C^*-algebras is associative, it can easily be seen that also the $\mathbin{\bigcirc\hspace{-0.75em}\perp}$-product is associative.

Lemma 4.4.4. *Let B_1 and B_2 be unital C^*-algebras and consider two matrices $M_1 \in M_4(B_1)$ and $M_2 \in M_4(B_2)$ both defining unital *-homomorphisms $\varphi_1 : C(S_4^+) \to B_1$ and $\varphi_2 : C(S_4^+) \to B_2$, respectively. Then also $M_1 \mathbin{\bigcirc\hspace{-0.75em}\perp} M_2 \in M_4(B_1 \otimes B_2)$ defines a unital *-homomorphism $\tilde{\varphi} : C(S_4^+) \to B_1 \otimes B_2$.*

Proof. Due to Proposition 4.3.1, we only have to prove that the entries of $M_1 \mathbin{\bigcirc\hspace{-0.75em}\perp} M_2 = (\widetilde{m}_{ij})_{1 \le ij \le N}$ are again projections summing up to $\mathbb{1}_{B_1 \otimes B_2}$ in every row and column. Selfadjointness and idempotence follow directly from the corresponding properties in each leg and also the remaining property is straightforward to show:

$$\sum_{k=1}^{N} \widetilde{m}_{ik} = \sum_{k=1}^{N} \sum_{l=1}^{N} m_{il} \otimes m_{lk} = \sum_{l=1}^{N} m_{il} \otimes \mathbb{1}_{B_2} = \mathbb{1}_{B_1} \otimes \mathbb{1}_{B_2} = \mathbb{1}_{B_1 \otimes B_2}$$

and likewise for summations over columns. □

We can now define a chain of matrices as described at the end of Section 4.2. Consider the C^*-algebra

$$A := C^*\big(p, q, 1 \quad | \quad p, q, 1 \text{ projections},\ 1p = p1 = p\,,\ q1 = 1q = q\big),$$

the universal unital C^*-algebra generated by two projections.

Then the pair (A, R) with

$$R := \begin{pmatrix} p & 0 & 1-p & 0 \\ 1-p & 0 & p & 0 \\ 0 & q & 0 & 1-q \\ 0 & 1-q & 0 & q \end{pmatrix} \in M_4(A)$$

defines in the sense of Section 4.2 a unital *-homomorphism $\varphi_1 : C(S_4^+) \to A$ by Observation 4.4.1. See Remark 4.4.7 for a comment on this special structure of R and why we do not consider the more convenient matrix $\widetilde{R}$ from Proposition 4.3.5. By Lemma 4.4.4 also the matrix

$$R \oplus R = \begin{pmatrix} p \otimes p & (1-p) \otimes q & p \otimes (1-p) & (1-p) \otimes (1-q) \\ (1-p) \otimes p & p \otimes q & (1-p) \otimes (1-p) & p \otimes (1-q) \\ q \otimes (1-p) & (1-q) \otimes (1-q) & q \otimes p & (1-q) \otimes q \\ (1-q) \otimes (1-p) & q \otimes (1-q) & (1-q) \otimes p & q \otimes q \end{pmatrix}$$

defines a unital *-homomorphism $\varphi_2 : C\big(S_4^+\big) \to A \otimes A$.
Note that the quotient map

$$\nu : A \otimes A \to \mathbb{C}$$

given by dividing out the relation $p{=}q{=}1$ in both legs of $A{\otimes}A$ gives us the identity matrix

$$(1 \otimes \nu)(R \oplus R) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \in M_4(\mathbb{C})$$

and a *-homomorphism $\varphi_0 : C\big(S_4^+\big) \to A^{\otimes 0} = \mathbb{C}$.

Definition 4.4.5. For $n \in \mathbb{N}$ we define the matrices

$$M_n := \left(m_{ij}^{(n)}\right)_{1 \le i,j \le 4} := (R \oplus R)^{\oplus\, n} \in M_4\left((A \otimes A)^{\otimes n}\right) = M_4\left(A^{\otimes 2n}\right)$$

and the C^*-algebras

$$B_n := C^*\big(\, m_{ij}^{(n)} \mid 1 \le i, j \le 4\big) \subseteq (A \otimes A)^{\otimes n},$$

the C^*-subalgebras of $(A \otimes A)^{\otimes n}$ generated by the entries of M_n. Let

$$\pi_{n+1,n} : B_{n+1} \to B_n$$

be the quotient map given by dividing out the relations $p = q = 1$ in the last two legs of $B_{n+1} \subseteq (A \otimes A)^{\otimes n+1}$, i.e. $\pi_{n+1,n}$ is a suitable restriction of $\left(\mathrm{id}_{B_1}\right)^{\otimes n} \otimes \nu$. By our remark above it holds

$$(1 \otimes \pi_{n+1,n})M_{n+1} = M_n.$$

In addition we define for $n < m \in \mathbb{N}$

$$\pi_{m,n} := \pi_{n+1,n} \circ \cdots \circ \pi_{m,m-1},$$

and

$$\pi_{n,n} := \mathrm{id}_{B_n}$$

.

This construction yields a commutative diagram as described at the end of Section 4.2:

$$\begin{tikzcd}
 & & \left(C(S_4^+), u_{S_4^+}\right) \arrow[dddll, "\varphi_1"'] \arrow[ddd, "\varphi_n"'] \arrow[dddr, "\varphi_{n+1}"] & & \\
 & & & & \\
 & \cdots & & & \cdots \\
(B_1, M_1) & \cdots \arrow[l, "\pi_{2,1}"'] & (B_n, M_n) & (B_{n+1}, M_{n+1}) \arrow[l, "\pi_{n+1,n}"'] & \cdots \arrow[l, "\pi_{n+2,n+1}"']
\end{tikzcd} \tag{4.4.1}$$

where φ_i are the (unital and surjective)*-homomorphisms from $C(S_4^+)$ to B_i mapping the entries of u onto the corresponding entries of M_i.

Remark 4.4.6. Note that we can also consider odd $\oplus$-products $R^{\oplus 2n-1}$. For example, $R^{\oplus 3}$ is given by

$$\begin{pmatrix}
\substack{p\otimes p\otimes p \\ +(1-p)\otimes q\otimes(1-p)} & \substack{p\otimes p\otimes(1-p) \\ +(1-p)\otimes q\otimes p} & \substack{p\otimes(1-p)\otimes q \\ +(1-p)\otimes(1-q)\otimes(1-q)} & \substack{p\otimes(1-p)\otimes(1-q) \\ +(1-p)\otimes(1-q)\otimes q} \\[2em]
\substack{(1-p)\otimes p\otimes p \\ +p\otimes q\otimes(1-p)} & \substack{(1-p)\otimes p\otimes(1-p) \\ +p\otimes q\otimes p} & \substack{(1-p)\otimes(1-p)\otimes q \\ +p\otimes(1-q)\otimes(1-q)} & \substack{(1-p)\otimes(1-p)\otimes(1-q) \\ +p\otimes(1-q)\otimes q} \\[2em]
\substack{q\otimes(1-p)\otimes p \\ +(1-q)\otimes(1-q)\otimes(1-p)} & \substack{q\otimes(1-p)\otimes(1-p) \\ +(1-q)\otimes(1-q)\otimes p} & \substack{q\otimes p\otimes q \\ +(1-q)\otimes q\otimes(1-q)} & \substack{q\otimes p\otimes(1-q) \\ +(1-q)\otimes q\otimes q} \\[2em]
\substack{(1-q)\otimes(1-p)\otimes p \\ +q\otimes(1-q)\otimes(1-p)} & \substack{(1-q)\otimes(1-p)\otimes(1-p) \\ +q\otimes(1-q)\otimes p} & \substack{(1-q)\otimes p\otimes q \\ +q\otimes q\otimes(1-q)} & \substack{(1-q)\otimes p\otimes(1-q) \\ +q\otimes q\otimes q}
\end{pmatrix}$$

From every such odd $\oplus$-product we obtain a model for $C(S_4^+)$. However, we can not (easily) integrate those objects into the chain of objects (B_n, M_n) as we do not have a canonical arrow ν' fulfilling

$$(1 \otimes \nu')R^{\oplus(n+1)} = R^{\oplus n}.$$

Remark 4.4.7. In comparison to the matrix R defined above, the matrix $\widetilde{R}$ from Proposition 4.3.5 would have been at first site much more canonical to consider, as it is a standard example when looking for a non-commutative model for $C(S_4^+)$. It even gives a compact matrix quantum group $G = (A, \widetilde{R})$ as existence of the comultiplication Δ can easily been proved by the universal property of A. However, any $\oplus$-product of matrices $\widetilde{R}$ would give us an object $(\widetilde{B}_n, \widetilde{M}_n)$ equivalent to $(A, \widetilde{R})$, i.e. the sequence of models $\varphi_n : C(S_4^+) \to B_n$ would be trivial. Retrospectively, it is clear why each of them (and in particular $(A, \widetilde{R})$) defines a CMQG: Loosely speaking, for every $n \in \mathbb{N}$ the (i,j)-th entry of $M_n \oplus M_n$ has by definition of $\oplus$ the structure of $\Delta_n\big(m_{ij}^{(n)}\big)$, so equivalence of all (B_n, M_n) guarantees existence of the comultiplications.

4.5 The limit object

We are now interested in some kind of limit (B_∞, M_∞) of the pairs (B_n, M_n). The entries of $M_\infty = \big(m_{ij}^{(\infty)}\big)_{1\le i,j\le 4} \in M_4(B_\infty)$ should generate B_∞ and for every $n \in \mathbb{N}$ we require a *-homomorphism $\phi_n : B_\infty \to B_n$ with $\phi_n(m_{ij}^{(\infty)}) = m_{ij}^{(n)}$. B_∞ becomes uniquely defined up to isomorphism by imposing on it the universal property that for all other pairs $(B, M) = \big(B, (m_{ij})_{1\le i,j\le 4}\big)$ with the same properties like (B_∞, M_∞), the maps

$$\psi_n : B \to B_i : \psi_n(m_{ij}) = m_{ij}^{(n)}$$

factor through the *-homomorphism $\psi : B \to B_\infty : m_{ij} \mapsto m_{ij}^{(\infty)}$, i.e. we have $\psi_n = \phi_n \circ \psi$ for all $n \in \mathbb{N}$.

We can describe this situation in an appealing way from the categorical point of view, in form of a so-called *inverse-limit* of an *inverse system*, compare [Phi88].

Definition 4.5.1. Consider as in Section 4.2 the category $\mathcal{C}$ whose objects are pairs (D, M) where D is a C^*-algebra and $M \in M_4(D)$ is a 4×4-matrix such that its entries generate D. Arrows between objects (D, M) and (D', M') are *-homomorphisms between the C^*-algebras sending canonically the entries of M onto those of M'. Note again that all these arrows are surjective (and hence unital) maps as we assume the D's to be generated by the entries of M. There is at most one arrow between two objects as such an arrow is uniquely defined by the corresponding matrices. An *inverse system* in $\mathcal{C}$ is a diagram of the form

$$(D_1, M_1) \xleftarrow{\pi_{1,2}} \cdots \xleftarrow{\pi_{n-1,n}} (D_n, M_n) \xleftarrow{\pi_{n,n+1}} (D_{n+1}, M_{n+1}) \xleftarrow{\pi_{n+1,n+2}} \cdots \quad . \tag{4.5.1}$$

If it exists, we denote its limit by (D_∞, M_∞) and call it the *inverse limit* of Diagram 4.5.1. We also write

$$\lim_{\infty \leftarrow n} (D_n, M_n) := (D_\infty, M_\infty).$$

Recall, see for example [Mac71], that the limit of a diagram as above is (if it exists) an object (D_∞, M_∞) such that the following property is fulfilled:
There exists a diagram

$$\begin{tikzcd}
 & & (D_\infty, M_\infty) \arrow[dll, "\phi_1"'] \arrow[d, "\phi_n"'] \arrow[dr, "\phi_{n+1}"] & & \\
(D_1, M_1) & \cdots \arrow[l, "\pi_{2,1}"'] & (D_n, M_n) & (D_{n+1}, M_{n+1}) \arrow[l, "\pi_{n+1,n}"'] & \cdots \arrow[l, "\pi_{n+2,n+1}"']
\end{tikzcd} \tag{4.5.2}$$

and for each object $(B, M) = \big(B, (m_{ij})_{1\leq i,j\leq 4}\big)$ that also admits a diagram

$$\begin{tikzcd}
 & & (B, M) \arrow[dll, "\psi_1"'] \arrow[d, "\psi_n"'] \arrow[dr, "\psi_{n+1}"] & & \\
(D_1, M_1) & \cdots \arrow[l, "\pi_{2,1}"'] & (D_n, M_n) & (D_{n+1}, M_{n+1}) \arrow[l, "\pi_{n+1,n}"'] & \cdots \arrow[l, "\pi_{n+2,n+1}"']
\end{tikzcd} \tag{4.5.3}$$

there exists a unique arrow ψ from (B, M) to (D_∞, M_∞), such that for every $n \in \mathbb{N}$ the following diagram commutes:

$$\begin{tikzcd}
 & (B\, ,\, M) \arrow[d, "\psi"] \arrow[ddl, "\psi_n"'] \arrow[ddr, "\psi_{n+1}"] & \\
 & (D_\infty\, ,\, M_\infty) \arrow[dl, "\phi_n"'] \arrow[dr, "\phi_{n+1}"] & \\
(D_n\, ,\, M_n) & & (D_{n+1}, M_{n+1}) \arrow[ll, "\pi_{n+1,n}"']
\end{tikzcd} \tag{4.5.4}$$

As the considered category $\mathcal{C}$ only admits at most one arrow from an object to another one, we only need existence of an arrow ψ from (B, M) to (D_∞, M_∞). i.e. we can drop the uniqueness condition on ψ as well as the commutativity condition on all diagrams above. We prove now that inverse limits as described in Definition 4.5.1 always exist. This result is well-known and it is not difficult to accomplish, compare [Phi88]. To keep this work self-contained, we present an own proof.

Proposition 4.5.2. *Consider the situation of Definition 4.5.1 and a diagram as in Picture 4.5.1. Then the inverse limit* $(D_\infty, M_\infty) = \lim_{\infty \leftarrow n} (D_n, M_n)$ *exists and it is unique (up to isomorphism).*

Proof. We start with the proof of existence.
Step 1: Construction of (D_∞, M_∞)**:** Consider the free *-algebra $\mathcal{D}$ generated by 16 symbols $m_{ij}^{(\infty)}$ with $1 \leq i, j \leq 4$ and let $\phi'_n : \mathcal{D} \to D_n$ be the *-homomorphism given by the mapping $\phi'_n(m_{ij}^{(\infty)}) = m_{ij}^{(n)}$. Impose on $\mathcal{D}$ the C^*-semi norms

$$f_n := \|\phi'_n(\cdot)\|_{D_n}$$

as well as

$$f := \sup_{n \in \mathbb{N}} f_n.$$

Note that the $m_{ij}^{(n)}$ are projections, so $(f_n)_{n\in\mathbb{N}}$ is pointwise bounded and f exists. We have $\phi'_n = \pi_{n+1,n} \circ \phi'_{n+1}$ and $\pi_{n+1,n}$ is norm-decreasing as it is a *-homomorphism. Therefore, the sequence $\big(f_n\big)_{n\in\mathbb{N}}$ is increasing and the supremum that defines f is in fact a limit. Evidently, f gives a C^*-norm on the quotient $\mathcal{D}_\infty := \mathcal{D}/\ker(f)$ and we define D_∞ to be its completion. Considering the $m_{ij}^{(\infty)}$ as elements in D_∞ and defining $M_\infty := \big(m_{ij}^{(\infty)}\big)_{1\leq i,j\leq 4}$, the pair (D_∞, M_∞) is an object in our category $\mathcal{C}$. Existence of the arrows

$$(D_\infty, M_\infty) \xrightarrow{\phi_n} (D_n, M_n)$$

for every $n \in \mathbb{N}$ can now be proved as follows: Firstly, we have $\ker(f) \subseteq \ker(f_n)$ because $f := \sup f_n$. Secondly, it holds $\ker(f_n) = \ker(\phi'_n)$ because $f_n := \|\phi'_n(\cdot)\|_{D_n}$. We conclude

$$\left(\mathcal{D}\big/\ker(f)\right)\Big/\ker(f_n) = \mathcal{D}\big/\ker(f_n) = \mathcal{D}\big/\ker(\phi'_n) = D_n$$

and the last equality holds because $\phi'_n : \mathcal{D} \to D_n$ is by definition a surjective *-homomorphism. Finally, the quotient map

$$\kappa_n : \mathcal{D}_\infty = \mathcal{D}\big/\ker(f) \longrightarrow \left(\mathcal{D}\big/\ker(f)\right)\Big/\ker(f_n) = D_n$$

is a (norm-decreasing) *-homomorphism. Its extension $\phi_n : D_\infty \to D_n$ is the desired arrow from (D_∞, M_∞) to (D_n, M_n) as it holds $\phi_n(m_{ij}^{(\infty)}) = m_{ij}^{(n)}$ by construction. We conclude that a diagram as in Picture 4.5.2 exists, so we can turn towards the universal property of (D_∞, M_∞), described by Diagram 4.5.4.
Step 2: Universal property of (D_∞, M_∞)**:** Consider an object (B, M) as described in Diagram 4.5.3. Denote with $\mathcal{B} \subseteq B$ the *-subalgebra generated by the

entries of M. By the definition of a limit we need to prove the existence of the commuting diagrams 4.5.4. As mentioned before, it suffices to prove existence of an arrow ψ from (B, M) to (D_∞, M_∞). To do so, we consider first a *-algebraic expression b in the letters m_{ij} and we let $\tilde{b}$ be the expression b but every letter m_{ij} is replaced by $m_{ij}^{(\infty)}$. By the properties of our considered category it holds

$$\psi_n(b) = \phi_n(\tilde{b})$$

for every $n \in \mathbb{N}$. We deduce

$$\|b\|_B \geq \sup_{n\in\mathbb{N}} \|\psi_n(b)\|_{D_n} = \sup_{n\in\mathbb{N}} \|\phi_n(\tilde{b})\|_{D_n} = \|\tilde{b}\|_{D_\infty}.$$

i.e. the mapping $m_{ij} \mapsto m_{ij}^{(\infty)}$ defines a norm-decreasing *-homomorphism from $\mathcal{B}$ to D_∞ and it can be extended to a *-homomorphism $\psi : B \to D_\infty$. This finishes the proof of existence.

Step 3: Uniqueness of (D_∞, M_∞) Uniqueness up to isomorphism is clear by the universal property of a limit. In the case of two limit objects we could switch roles to construct invertible arrows between them. ☐

Consider now again the diagram consisting of the pairs (B_n, M_n) from Definition 4.4.5,

$$\begin{tikzcd}
(B_1, M_1) & \cdots \arrow[l, "\pi_{1,2}"'] & (B_n, M_n) \arrow[l, "\pi_{n-1,n}"'] & (B_{n+1}, M_{n+1}) \arrow[l, "\pi_{n,n+1}"'] & \cdots \arrow[l, "\pi_{n+1,n+2}"']
\end{tikzcd},$$

and recall that we have the commuting Diagram 4.4.1. By the universal property of the inverse limit (B_∞, M_∞) we obtain for a suitable $\varphi : C(S_4^+) \to B_\infty$ the following commuting diagram:

$$\begin{tikzcd}
 & & \Big(C(S_4^+), u_{S_4^+}\Big) \arrow[d, "\varphi"] & & \\
 & & \big(B_\infty, M_\infty\big) \arrow[dll, "\phi_1"'] \arrow[d, "\phi_n"] \arrow[dr, "\phi_{n+1}"] & & \\
(B_1, M_1) & \cdots \arrow[l, "\pi_{2,1}"'] & (B_n, M_n) & (B_{n+1}, M_{n+1}) \arrow[l, "\pi_{n+1,n}"'] & \cdots \arrow[l, "\pi_{n+2,n+1}"']
\end{tikzcd} \tag{4.5.5}$$

The *-homomorphisms φ_n from Diagram 4.4.1 fulfil $\varphi_n = \phi_n \circ \varphi$, so the kernel of φ is smaller or equal than the kernel of every φ_n. Nonetheless we have up to now not answered the following questions.

(a) Does every map $\pi_{n+1,n}$ have a non-trivial kernel, i.e. do the kernels $\ker(\varphi_n)$ become smaller and smaller?

(b) Is the kernel of φ smaller than any kernel $\ker(\varphi_n)$? Of course this is true if question (a) is answered to the positive.

(c) Is φ injective or even any of the φ_n? If any φ_n is injective, then of course for any $m \geq n$ also the maps φ_m and $\pi_{m+1,m}$ are injective as well as φ.

(d) Does the pair (B_n, M_n) for any $n \in \mathbb{N} \cup \{\infty\}$ provide a CMQG? This is the case if and only if there exists a *-homomorphism $\Delta_n : B_n \to B_n \otimes B_n$ fulfilling

$$\Delta(m_{ij}^{(n)}) = \sum_{k=1}^{4} m_{ik}^{(n)} \otimes m_{kj}^{(n)}$$

for all $1 \leq i, j \leq 4$, compare Definition 2.2.1.

In the remaining part of this section we will answer question (d) positively for the limit object (B_∞, M_∞). Together with some results to be presented in the sequel we will show that this CMQG is equal to S_4^+. In particular, the model φ turns out to be injective, hence an isomorphism between C^*-algebras. A composition with a faithful representation of (B_∞, M_∞) thus yields a faithful representation of $C(S_4^+)$.

4.6 A faithful model for $C(S_4^+)$

In this section we focus on the pair (B_∞, M_∞) as constructed in Section 4.5. If (B_∞, M_∞) defines a CMQG denoted by G, then we obviously have $G \subseteq S_4^+$ as the *-homomorphism φ from Diagram 4.5.5 describes by Definition 2.2.5 the subgroup relation $G \subseteq S_4^+$. The following lemmata show that we also have $S_4 \subsetneq G$. Note that the C^*-algebra $C(S_4)$ is finite dimensional, so $C_{red}(S_4) = C_u(S_4)$, i.e. all forms of S_4 coincide.

Lemma 4.6.1. *Consider the easy group $S_4 = (C(S_4), u_{S_4})$. There exists an arrow $\varphi : B_\infty \to C_u(S_4)$ between the category objects (B_∞, M_∞) and $(C(S_4), u_{S_4})$.*

Proof. We don't write down all details, but this is what one can check by having a closer look at the respective objects: We can compose the *-homomorphism $\phi_2 : B_\infty \to B_2$ in Diagram 4.5.5 with the quotient map on B_2 that divides out the relation $p = q = \mathbb{1}_A$ in the last leg of the tensor product $(A \otimes A)^{\otimes 2}$. Furthermore, we can divide out the commutation relations $pq = qp$ in all legs. Denote this qotient map by ϕ'. Doing so, we obtain an arrow $\phi' \circ \phi_2$ from (B_∞, M_∞) to some object (B', M'). B' is a commutative C^*-algebra and the matrix M' looks as follows:

$$\begin{pmatrix}
\begin{array}{c} p \otimes p \otimes p \\ +(1-p) \otimes q \otimes (1-p) \end{array} & \begin{array}{c} p \otimes (1-p) \otimes q \\ +(1-p) \otimes (1-q) \otimes (1-q) \end{array} & \begin{array}{c} p \otimes p \otimes (1-p) \\ +(1-p) \otimes q \otimes p \end{array} & \begin{array}{c} p \otimes (1-p) \otimes (1-q) \\ +(1-p) \otimes (1-q) \otimes q \end{array} \\[3ex]
\begin{array}{c} (1-p) \otimes p \otimes p \\ +p \otimes q \otimes (1-p) \end{array} & \begin{array}{c} (1-p) \otimes (1-p) \otimes q \\ +p \otimes (1-q) \otimes (1-q)) \end{array} & \begin{array}{c} (1-p) \otimes p \otimes (1-p) \\ +p \otimes q \otimes p) \end{array} & \begin{array}{c} (1-p) \otimes (1-p) \otimes (1-q) \\ +p \otimes (1-q) \otimes q \end{array} \\[3ex]
\begin{array}{c} q \otimes (1-p) \otimes p \\ +(1-q) \otimes (1-q) \otimes (1-p) \end{array} & \begin{array}{c} q \otimes p \otimes q \\ +(1-q) \otimes q \otimes (1-q) \end{array} & \begin{array}{c} q \otimes (1-p) \otimes (1-p) \\ +(1-q) \otimes (1-q) \otimes p \end{array} & \begin{array}{c} q \otimes p \otimes (1-q) \\ +(1-q) \otimes q \otimes q \end{array} \\[3ex]
\begin{array}{c} (1-q) \otimes (1-p) \otimes p \\ +q \otimes (1-q) \otimes (1-p) \end{array} & \begin{array}{c} (1-q) \otimes p \otimes q \\ +q \otimes q \otimes (1-q) \end{array} & \begin{array}{c} (1-q) \otimes (1-p) \otimes (1-p) \\ +q \otimes (1-q) \otimes p \end{array} & \begin{array}{c} (1-q) \otimes p \otimes (1-q) \\ +q \otimes q \otimes q) \end{array}
\end{pmatrix}$$

The structure resembles $R^{\obot 3}$ but note that this matrix here describes a commutative situation.
We show now that B' is equivalent to $C(S_4)$: Recall from Proposition 4.3.2 that we obtain $C(S_4)$ from $C(S_4^+)$ by dividing out the commutativity relations. As the entries of M' are by construction commutative projections summing up to $\mathbb{1}_{B'}$ in every row and column, we have an arrow ρ from $\big(C(S_4), u_{S_4}\big)$ to (B', M'). Summarizing, we have a Diagram as follows:

$$(B_\infty, M_\infty) \xrightarrow{\phi_2} (B_2, M_2) \xrightarrow{\phi'} (B', M') \xleftarrow{\rho} (C(S_4), u_{S_4})$$

If ρ is injective, then ρ is an *-isomorphism and we are done. Obviously, $C(S_4)$ has vector space dimension 24, as $\#S_4 = 4! = 24$. On the other hand, one can directly check (in all 24 cases) that for $1 \leq i, j, k \leq 4$ pairwise different the product $m'_{i1} m'_{j2} m'_{k3}$ of entries in the matrix above is non-zero. By commutativity, these products are again projections and because all 4 entries in one common row or column of M' are orthogonal to each other, those 24 products are orthogonal to each other, and thus linearly independent. So B' is a C^*-algebra with vector space dimension 24, like $C(S_4)$. As ρ is a surjective linear map, it has to be bijective, so ρ is a *-isomorphism. The composition $\varphi := \rho^{-1} \circ \phi' \circ \phi_2$ is the desired arrow from (B_∞, M_∞) to $(C(S_4), u_{S_4})$. $\square$

Lemma 4.6.2. *The arrow φ from (B_∞, M_∞) to $(C(S_4), u_{S_4})$, that exists by Lemma 4.6.1, is not injective.*

Proof. The matrix $M_1 = R \obot R$ is given by

$$\begin{pmatrix}
p \otimes p & (1-p) \otimes q & p \otimes (1-p) & (1-p) \otimes (1-q) \\
(1-p) \otimes p & p \otimes q & (1-p) \otimes (1-p) & p \otimes (1-q) \\
q \otimes (1-p) & (1-q) \otimes (1-q) & q \otimes p & (1-q) \otimes q \\
(1-q) \otimes (1-p) & q \otimes (1-q) & (1-q) \otimes p & q \otimes q
\end{pmatrix}$$

and as p and q do not commute in A, the entries of M_1 do not all commute, for example

$$m_{11}^{(1)} m_{22}^{(1)} = p \otimes pq \neq p \otimes qp = m_{22}^{(1)} m_{11}^{(1)}.$$

Hence, the image of the arrow ϕ_1 in Diagram 4.5.5 is non-commutative and so does B_∞. Thus, an arrow from (B_∞, M_∞) to $(C(S_4), u_{S_4})$ cannot be injective. □

The following results will be crucial in order to prove the main result of this chapter, Theorem 4.6.7, saying that (B_∞, M_∞) yields a CMQG. The logical structure is as follows: Lemma 4.6.3 is preparatory for Lemma 4.6.4 which in turn entails Lemma 4.6.5. Eventually, Lemma 4.6.5 and Lemma 4.6.6 are used in Theorem 4.6.7.

Lemma 4.6.3. *Consider the arrows*

$$(B_\infty, M_\infty) \xrightarrow{\phi_k} (B_k, M_k)$$

from Diagram 4.5.5 Let $a_1, \dots, a_N \in B_\infty$ be linearly independent. Then there is a $K \in \mathbb{N}$ such that $\phi_k(a_1), \dots, \phi_k(a_N) \in B_k$ are linearly independent for all $k \geq K$. In particular, we find for any non-zero a_i some $K \in N$ such that $\phi_k(a_i) \neq 0$ for all $k \geq K$.

Proof. Recall that the sequence of C^*-semi norms f_n is increasing on $\mathcal{B}$ so it holds $f = \lim_{n\to\infty} f_n$. Consequently, also the sequence $\left(\|\phi_n(\cdot)\|_{B_n}\right)_{n\in\mathbb{N}}$ is increasing and its limit is the norm $\|\cdot\|_{B_\infty}$.
We now use induction on $N \in \mathbb{N}$ to prove our claim. For $N = 1$ we observe that a collection with only one element a_1 is linearly independent if its element is non-zero, so we have $0 \neq \|a_1\|_{B_\infty} = \lim_{k\to\infty} \|\phi_k(a_1)\|_{B_k}$. In particular $\phi_k(a_1)$ is non-zero for all up to finitely many $k \in \mathbb{N}$.
Now let the statement be proved for some $N \in \mathbb{N}$ and consider linear independent $a_1, \dots, a_{N+1} \in B_\infty$. We assume the opposite of our claim i.e. we find arbitrary large $k \in \mathbb{N}$ such that $\phi_k(a_1), \dots, \phi_k(a_{N+1})$ are linearly dependent. By the induction hypothesis we find $K \in \mathbb{N}$ such that $\phi_k(a_1), \dots, \phi_k(a_N)$ are linearly independent for all $k \geq K$. So we find some $L_1 \geq K$ such that

$$\phi_{L_1}(a_{N+1}) = \sum_{i=1}^{N} \alpha_i \phi_{L_1}(a_i)$$

for suitable coefficients α_i. As $a_{N+1} - \sum_{i=1}^{N} \alpha_i a_i$ is non-zero by linear independence of $a_1, \dots, a_{N+1}$, we find by the induction base case $L_2 \geq L_1$ such that

$$\phi_{l_2}(a_{N+1}) \neq \sum_{i=1}^{N} \alpha_i \phi_{l_2}(a_i)$$

for all $l_2 \geq L_2$. With the same arguments as before we find some $L_3 \geq L_2 \geq K$ such that

$$\phi_{L_3}(a_{N+1}) = \sum_{i=1}^{N} \beta_i \phi_{L_3}(a_i).$$

It holds $(\beta_1, \ldots, \beta_N) \neq (\alpha_1, \ldots, \alpha_N)$ because

$$\phi_{L_3}(a_{N+1}) \neq \sum_{i=1}^{N} \alpha_i \phi_{L_3}(a_i).$$

We conclude that

$$\begin{aligned} 0 &= \phi_K(a_{N+1}) - \phi_K(a_{N+1}) \\ &= \big(\pi_{L_3,K} \circ \phi_{L_3}\big)(a_{N+1}) - \big(\pi_{L_1,K} \circ \phi_{L_1}\big)(a_{N+1}) \\ &= \sum_{i=1}^{N} (\beta_i - \alpha_i)\phi_K(a_i), \end{aligned}$$

a contradiction to the linear independence of $\phi_K(a_1), \ldots, \phi_K(a_N)$. $\square$

Lemma 4.6.4. *The C^*-seminorm*

$$g := \sup_{n \to \infty} \|\big(\phi_n \otimes \phi_n\big)(\cdot)\|_{B_n \otimes B_n} \tag{4.6.1}$$

is a C^*-norm on the algebraic tensor product $B_\infty \odot B_\infty$.

Proof. Recall that the algebraic tensor product $B_\infty \odot B_\infty$ is linearly spanned by elements $x \otimes y$ with $x, y \in B_\infty$. Fix $x = \sum_{i=1}^{N} a_i \otimes b_i$ with $a_i, b_i \in B_\infty$, all $b_i \neq 0$ and $a_1, \ldots, a_N$ linearly independent. Note that the sequence $\big(\|\big(\phi_n \otimes \phi_n\big)(x)\|_{B_n \otimes B_n}\big)_{n \in \mathbb{N}}$ is increasing so the supremum above is in fact a limit. The statement is proved if we find an $L \in \mathbb{N}$ such that $\|\big(\phi_L \otimes \phi_L\big)(x)\|_{B_L \otimes B_L}$ is nonzero.
By Lemma 4.6.3 we find $K \in \mathbb{N}$ such that for all $k \geq K$ the elements $\phi_k(a_1), \ldots, \phi_k(a_{N+1})$ are linearly independent. As all b_i are non-zero, we find by Lemma 4.6.3 some $L \geq K$ such that $\phi_L(b_i) \neq 0$ for all $1 \leq i \leq N$. But then we obviously have

$$\sum_{i=1}^{N} \phi_L(a_i) \otimes \phi_L(b_i) \neq 0$$

as the first legs are linearly independent and the second ones are non-zero. In particular, it holds

$$\|\sum_{i=1}^{N} \phi_L(a_i) \otimes \phi_L(b_i)\|_{B_L \otimes B_L} \neq 0.$$

$\square$

We even have that g defines a norm on $B_\infty \otimes B_\infty$ and it is equal to the norm on the minimal tensor product.

Lemma 4.6.5. *The mapping g from Lemma 4.6.4 is equal to the norm on $B_\infty \otimes B_\infty$.*

Proof. Recall that the norm of a minimal tensor product $\|\cdot\|_{B\otimes C}$ of two C^*-algebras is by construction the smallest C^*-norm on $B \odot C$ and it is defined by the supremum of the C^*-seminorms $\|(\xi_1 \otimes \xi_2)(\cdot)\|_{B(H_1)\otimes B(H_2)}$ where ξ_1 and ξ_2 are representations of B on H_1 and and C on H_2, respectively and $\xi_1 \otimes \xi_2$ is the product representation of $B \odot C$ on $H_1 \otimes H_2$. Furthermore, $\xi_1 \otimes \xi_2$ is faithful if both ξ_1 and ξ_2 are and in this case it holds $\|\cdot\|_{B\otimes C} = \|(\xi_1 \otimes \xi_2)(\cdot)\|_{B(H_1)\otimes B(H_2)}$.
It holds $g \leq \|\cdot\|_{B_\infty \otimes B_\infty}$because the C^*-semi norms $\|(\phi_n \otimes \phi_n)(\cdot)\|_{B_n \otimes B_n}$ all appear in the collection of semi-norms $\|(\xi_1 \otimes \xi_2)(\cdot)\|_{B(H_1)\otimes B(H_2)}$ as we can combine ϕ_n with a faithful representation of B_n.
Conversely, we have $g \geq \|\cdot\|_{B_\infty \otimes B_\infty}$ because g defines by Lemma 4.6.4 a C^*-norm on $B_\infty \odot B_\infty$. As $\|\cdot\|_{B_\infty \otimes B_\infty}$ is by construction the smallest possible C^*-norm on $B_\infty \odot B_\infty$, we have $g \geq \|\cdot\|_{B_\infty \otimes B_\infty}$
Combing both inequalities, we conclude that g equals the minimal tensor product norm on $B_\infty \odot B_\infty$, and therefore on the whole $B_\infty \otimes B_\infty$. ☐

Consider the map Δ' defined by the symbolwise replacement

$$m_{ij}^{(\infty)} \xmapsto{\Delta'} \sum_{k=1}^{4} m_{ik}^{(\infty)} \otimes m_{kj}^{(\infty)} \in B_\infty \otimes B_\infty. \tag{4.6.2}$$

Up to now it is not clear if this gives a well-defined map on the *-algebra

$$^*\operatorname{alg}\big(\, m_{ij}^{(\infty)} \in B_\infty \mid 1 \leq i, j \leq 4\big)$$

and a well-defined *-homomorphism on B_∞. The image of some x under Δ' might depend on the way we write the pre-image using the symbols $m_{ij}^{(\infty)}$. Before proving that Δ' indeed provides us with a *-homomorphism from B_∞ to $B_\infty \otimes B_\infty$, i.e. a comultiplication as desired for a CMQG, we prove one preparatory result.

Lemma 4.6.6. *Consider the map*

$$\Delta' : m_{ij}^{(\infty)} \xmapsto{\Delta_\infty} \sum_{k=1}^{4} m_{ik}^{(\infty)} \otimes m_{kj}^{(\infty)} \in B_\infty \otimes B_\infty$$

as defined in Equation 4.6.2. Let x be a *-algebraic expression in the symbols $\big(m_{ij}^{(\infty)}\big)_{1\leq i,j\leq 4}$. Then for any $n \in \mathbb{N}$ it holds

$$\phi_{2n}(x) = (\phi_n \otimes \phi_n)\big(\Delta'(x)\big)$$

as an equation in $A^{\otimes(2n)} = \big(C^*(p, q, 1)\big)^{\otimes(2n)}$ (or any suitable C^*-subalgebra).

Proof. This result follows from the associativity of the $\operp$ -product which in turn follows from the associativity of the tensor product: The map $(\phi_n \otimes \phi_n) \circ \Delta'$ replaces each symbol $m_{ij}^{(\infty)}$ by

$$\sum_{k=1}^{4} m_{ik}^{(n)} \otimes m_{kj}^{(n)}.$$

and the map ϕ_{2n} replaces $m_{ij}^{(\infty)}$ by $m_{ij}^{(2n)}$. Now it holds for $1 \leq i, j \leq 4$

$$\begin{aligned}
m_{ij}^{(2n)} &= \left(M_1^{\operp\, 2n}\right)_{ij} \\
& \sum_{t_1,\ldots,t_{2n-1}=1}^{4} m_{it_1}^{(1)} \otimes \ldots \otimes m_{t_{2n-1}j}^{(1)} \\
&= \sum_{t_n=1}^{4} \left(\left(\sum_{t_1,\ldots t_{n-1}=1}^{4} m_{it_1}^{(1)} \otimes \ldots \otimes m_{t_{n-1}t_n}^{(1)} \right) \otimes \left(\sum_{t_{n+1},\ldots,t_{2n-1}=1}^{4} m_{t_n t_{n+1}}^{(1)} \otimes \ldots \otimes m_{t_{2n-1}j}^{(1)} \right) \right) \\
&= \sum_{k=1}^{4} \left(M_1^{\operp\, n}\right)_{ik} \otimes \left(M_1^{\operp\, n}\right)_{kj} \\
&= \sum_{k=1}^{4} m_{ik}^{(n)} \otimes m_{kj}^{(n)}.
\end{aligned}$$

□

Theorem 4.6.7. *The C^*-algebra B_∞ together with its matrix of generators $M_\infty = \left(m_{ij}^{(\infty)}\right)_{1 \leq i,j \leq 4}$ defines a compact matrix quantum group $G = (B_\infty, M_\infty)$.*

Proof. As mentioned after question (d) on page 144, the only thing left to prove is the existence of a *-homomorphism $\Delta : B_\infty \to B_\infty \otimes B_\infty$ fulfilling

$$\Delta(m_{ij}^{(\infty)}) = \sum_{k=1}^{4} m_{ik}^{(\infty)} \otimes m_{kj}^{(\infty)}.$$

It is sufficient to show that the symbolwise replacement Δ' as defined in Equation 4.6.2 is a well-defined norm-decreasing mapping on the *-subalgebra

$$^*\operatorname{alg}\left(\, (m_{ij}^{(\infty)} \in B_\infty \mid 1 \leq i, j \leq 4\right) \subseteq B_\infty$$

In this case this map is by construction a *-homomorphism and it extends to a *-homomorphism on B_∞ as it is norm-decreasing.
Let now x be a *-algebraic expression in the symbols $\left(m_{ij}^{(\infty)}\right)_{1 \leq i,j \leq 4}$. Due to Lemma 4.6.5 and the fact that the supremum in Equation 4.6.1 is in fact a limit it holds

$$\|x\|_{B_\infty} = \lim_{n\to\infty} \|\phi_n(x)\|_{B_n} = \lim_{n\to\infty} \|\phi_{2n}(x)\|_{B_{2n}}.$$

Due to Lemma 4.6.6 and the fact that B_n is a C^*-subalgebra of $A^{\otimes 2n}$ it holds

$$\begin{aligned}
\lim_{n\to\infty} \|\phi_{2n}(x)\|_{B_{2n}} &= \lim_{n\to\infty} \|\phi_{2n}(x)\|_{A^{\otimes 4n}} \\
&= \lim_{n\to\infty} \big\|\big(\phi_n \otimes \phi_n\big)(\Delta'(x))\big\|_{A^{\otimes 2n}\otimes A^{\otimes 2n}} \\
&= \lim_{n\to\infty} \big\|\big(\phi_n \otimes \phi_n\big)(\Delta'(x))\big\|_{B_n\otimes B_n} \\
&= \|\Delta'(x)\|_{B_\infty \otimes B_\infty}
\end{aligned}$$

Hence, Δ' is a well-defined, isometric *-homomorphism from $^*\operatorname{alg}\big((m_{ij}^{(\infty)} \in B_\infty \mid 1 \leq i, j \leq 4\big)$ to $B_\infty \otimes B_\infty$. In particular, it is norm-decreasing. Its (unique) extension Δ to B_∞ is an (isometric) *-homorphism and it fulfils for all $1 \leq i, j \leq 4$ the identity

$$\Delta(m_{ij}^{(\infty)}) = \sum_{k=1}^{4} m_{ik}^{(\infty)} \otimes m_{kj}^{(\infty)}.$$

Hence $G := (B_\infty, M_\infty)$ is a CMQG. □

Remark 4.6.8. While Theorem 4.6.7 answers Question d on page 144 in the case $n = \infty$, we do not know if there is an $n \in \mathbb{N}$ such that (B_n, M_n) is a CMQG. A comultiplication Δ on (B_n, M_n) would be exactly the inverse arrow of $\pi_{2n,n}$, so the questions (a), (b), (c) and (d) (for $n<\infty$) are closely related.
Conversely speaking, while existence of an arrow

$$(B_n, M_n) \xrightarrow{\Delta \,=\, \pi_{2n,n}} (B_{2n}, M_{2n}) \tag{4.6.3}$$

is nothing but clear (and we conjecture that it does not exist), this problem vanishes for $n=\infty$ because both (B_n, M_n) and (B_{2n}, M_{2n}) give for $n \to \infty$ the same inverse limit (B_∞, M_∞).

Using a result from [Ban18a], it is now easy to prove that the CMQG from Theorem 4.6.7 is equal to S_4^+.

Corollary 4.6.9. *Let* $S_4^+ = \big(C(S_4^+), u\big)$. *Let* $G = \big(B_\infty, M_\infty\big)$ *be the inverse limit of the diagram*

$$\begin{tikzcd}
(B_1, M_1) & \cdots \arrow[l, "\pi_{1,2}"'] & (B_n, M_n) \arrow[l, "\pi_{n-1,n}"'] & (B_{n+1}, M_{n+1}) \arrow[l, "\pi_{n,n+1}"'] & \cdots . \arrow[l, "\pi_{n+1,n+2}"']
\end{tikzcd}$$

as constructed in Section 4.5, which is a compact matrix quantum group by Theorem 4.6.7. Then it holds $G = S_4^+$. *In particular, the* **-homomorphism* φ *from Diagram 4.5.5 defines a faithful model for* $C(S_4^+)$.

Remark 4.6.10. More precisely, we have to say in the situation of Corollary 4.6.9 that there exists a form $(C(S_4^+), u_{S_4^+})$ of S_4^+ such that φ is a faithful model of $C(S_4^+)$.

Proof of Corollary 4.6.9. We have $S_4 \subseteq G$ by Lemmata 4.6.1 and 4.6.2 and due to the universal property of the limit (B_∞, M_∞) we also have $G \subseteq S_4^+$, established by the map ψ in Diagram 4.5.5.
By [Ban18a, Thm. 7.4], there is no CMQG strictly in between S_4 and S_4^+. As $G \neq S_4$ by Lemma 4.6.2, it follows $G = S_4^+$. □

Corollary 4.6.9 finishes our efforts to find "useful" models of $C(S_4^+)$. With the pair $G = (B_\infty, M_\infty)$ we found a CMQG that is equivalent to S_4^+. The object (B_∞, M_∞) lies in between the reduced and universal form of S_N^+ and the structure of (B_∞, M_∞) can be approximated by the inverse limit process

$$\lim_{\infty \leftarrow n} (B_n, M_n) = (B_\infty, M_\infty).$$

In particular, the connection between the objects (B_n, M_n) and (B_{n+1}, M_{n+1}) is easily accessible via the construction $M_n \rightsquigarrow M_n \oplus M_1 =: M_{n+1}$ on one side and the *-homomorphisms $\pi_{n+1,n}$ on the other side.

4.7 Generalization to $C(S_N^+)$ ... and beyond

In this section we move from the special case of S_4^+ to the general situation of S_N^+ for $N \geq 4$. Given any such N, the symbol $\mathcal{C}$ denotes the category of objects (A, M) as defined in Section 4.2, but the matrix size is N.
Having a closer look at the previous part of this chapter, we observe the following:

Observation 4.7.1. (i) The analogues of Propositions 4.5.2 and 4.6.7 hold for all $N \in \mathbb{N}$ in the sense that every diagram of the form

$$\begin{tikzcd}(B_1, M_1) & \cdots \arrow[l, "\pi_{2,1}"'] & (B_n, M_n) & (B_{n+1}, M_{n+1}) \arrow[l, "\pi_{n+1,n}"'] & \cdots \arrow[l, "\pi_{n+2,n+1}"']\end{tikzcd}$$

has an inverse limit in $\mathcal{C}$ and this limit defines a CMQG. The reason is that all relevant arguments given in the respective proofs do not depend on the special situation of $N = 4$.

(ii) Assume that we have in the category $\mathcal{C}$ an object (B_1, M_1) and arrows

$$(C(S_N^+), u_{S_N^+}) \xrightarrow{\varphi_1} (B_1, M_1) \xrightarrow{\nu} \big(\mathbb{C}, \mathbb{1}_{M_N(\mathbb{C})}\big). \tag{4.7.1}$$

Then we can construct, similar to Section 4.4, a chain as in (i) by defining $M_{n+1} := M_n \oplus M_1 \in M_N(B_1^{\otimes(n+1)})$. Each $\pi_{n+1,n}$ is a suitable restriction of $\big(\mathrm{id}_{B_1}\big)^{\otimes n} \otimes \nu$.

(iii) If we find in the situation of (i) or (ii) some $n \in \mathbb{N}$ that allows an arrow

$$(B_n, M_n) \xrightarrow{\phi} (C(S_N), u_{S_N}) \tag{4.7.2}$$

that is not injective, then the compact matrix quantum group $G := (B_\infty, M_\infty)$ fulfils $S_N \subsetneq G \subseteq S_N^+$. In the case where there is no CMQG strictly in between S_N and S_N^+, we have again with (B_∞, M_∞) a very good description of S_N^+, in the sense that $G = S_N^+$ and (B_∞, M_∞) is in between the reduced and the universal form of S_N^+.

For the rest of Section 4.7 we fix some $N \in \mathbb{N}_{\geq 4}$. In the following we construct a pair (B_1, M_1) that fulfils the conditions described in item (ii) and (iii) of Observation 4.7.1, namely Equation 4.7.1 and 4.7.2. Our starting point are matrices similar to the matrix R from Section 4.4.

Definition 4.7.2. Consider the C^*-algebra

$$A := C^*\big(p, q, 1 \quad | \quad p, q, 1 \text{ projections, } 1p = p1 = p\,,\ q1 = 1q = q\big),$$

the universal unital C^*-algebra generated by two projections. Let $1 \leq a, b, c, d \leq N$ be pairwise different. We define the matrix

$$R_{(a,b),(c,d)} \in M_N(A)$$

by the following properties:

(1) Entries (a, a) and (b, b) are given by p.

(2) Entries (c, c) and (d, d) are given by q.

(3) All other diagonal entries are equal to 1.

(4) Entries (a, b) and (b, a) are given by $1-p$.

(5) Entries (c, d) and (d, c) are given by $1-q$.

(6) All other off-diagonal entries are zero.

Note that in order to make the following constructions work, the C^*-algebra A, in fact, does not have to be exactly from the form as described in Definition 4.7.2. See Remark 4.7.13 for more details.

Example 4.7.3. In the case $N=5$ only one entry in a matrix $R_{(a,b),(c,d)}$ is equal to 1. For example we have

$$R_{(1,4),(3,5)} = \begin{pmatrix} p & 0 & 0 & 1-p & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & q & 0 & 1-q \\ 1-p & 0 & 0 & p & 0 \\ 0 & 0 & 1-q & 0 & q \end{pmatrix} \in M_5(A).$$

Notation 4.7.4. In the following we often not interested in the values c and d. In this virtue we write $R_{(a,b),*}$ to denote a matrix as above with any fixed allowed values for c and d.

Remark 4.7.5. As in Section 4.4, any $\bigcirc\!\!\!\!\perp$-product M of k such matrices defines a pair (B, M) where $B \subseteq A^{\otimes k}$ is the C^*-subalgebra generated by the entries of M. Evidently, this is a model for $C(S_N^+)$, i.e. we have an arrow from $\big(C(S_N^+), u_{S_N^+}\big)$ to (B, M). The argument is the same as in the case of Lemma 4.4.4.

The crucial point is now that there is a suitable $\bigcirc\!\!\!\!\perp$-product M_1 of matrices $R_{(a,b),(c,d)}$ such that the associated pair (B_1, M_1) has all the desired properties described in Observation 4.7.1.

Definition 4.7.6. Consider the C^*-algebra

$$A := C^*\big(p, q, 1 \quad | \quad p, q, 1 \text{ projections}, \; 1p = p1 = p\,,\; q1 = 1q = q\big),$$

the universal unital C^*-algebra generated by two projections. Let

$$R_{(a,b),(c,d)} \in M_N(A)$$

be the matrices as defined in Definition 4.7.2 and let L be an upper bound for the number of transpositions needed to obtain a permutation $\sigma \in S_N$.
We define the category object (B_1, M_1) by

$$M_1 = \Big(m_{ij}^{(1)}\Big)_{1 \le i,j \le N} := \left(\bigoplus_{1 \le a < b \le N} R_{(a,b),*} \right)^{\bigcirc\!\!\!\!\perp\, L}$$

and

$$B_1 := C^*\big(\, m_{ij}^{(1)} \mid 1 \le i,j \le N\big) \subseteq A^{\otimes \frac{LN(N-1)}{2}}.$$

Lemma 4.7.7. *Consider the object (B_1, M_1) in $\mathcal{C}$ from Definition 4.7.6.*

(a) There exists an arrow $(C(S_N^+), u_{S_N^+}) \xrightarrow{\varphi_1} (B_1, M_1)$.

(b) There exists an arrow $(B_1, M_1) \xrightarrow{\nu} (\mathbb{C}, \mathbb{1}_{M_N(\mathbb{C})})$.

(c) There exists an arrow $(B_1, M_1) \xrightarrow{\phi} (C(S_N), u_{S_N})$ that is not injective.

Proof. Item (a) is Remark 4.7.5, so there is nothing to do. For item (b) we consider the quotient map

$$\nu_0 : A \to \mathbb{C}$$

that divides out the relations $p=q=1$. Define ν to be the restriction of $\nu_0^{\otimes \frac{LN(N-1)}{2}}$ to B_1, i.e. the quotient map on B_1 that divides out the relation $p=q=1$ in all legs. Now each $\oplus$-factor $R_{(a,b),*}$ fulfils

$$(\mathrm{id}_{M_N(\mathbb{C})} \otimes \nu_0) R_{(a,b),*} = \mathbb{1}_{M_N(\mathbb{C})}.$$

Thus, we have

$$(\mathrm{id}_{M_N(\mathbb{C})} \otimes \nu) M_1 = \mathbb{1}_{M_N(\mathbb{C})}$$

and ν is the arrow described in item (b).

For (c), existence of the arrow $(B_1, M_1) \xrightarrow{\phi} (C(S_N), u_{S_N})$ takes a little more effort. The first steps are the same as in the proof of Lemma 4.6.1. We start with the matrices $R_{(a,b),*}$, the matrix M_1 and the C^*-algebra B_1. Dividing out in all appearing legs the commutativity relations $pq = qp$, we obtain matrices $R'_{(a,b),*}$, a matrix M'_1 and a commutative C^*-algebra B'_1. The corresponding quotient map $\phi' : B_1 \to B'_1$ is an arrow

$$(B_1, M_1) \xrightarrow{\phi'} (B'_1, M'_1)$$

and the entries of M'_1 are pairwisly commuting projections summing up to $\mathbb{1}_{B'_1}$ in every row and column. Consequently, we have an arrow

$$(C(S_N), u_{S_N}) \xrightarrow{\rho} (B'_1, M'_1).$$

In Lemma 4.7.8 we show that in B'_1 all products of matrix entries from different rows and columns are non-zero. As in the proof of Lemma 4.6.1, this shows the vector space dimension of B'_1 to be equal to $N! = \dim\big(C(S_N)\big)$ and hence the arrow ρ to be a *-isomorphism. The composition $\phi := \rho^{-1} \circ \phi'$ is the desired arrow from (B_1, M_1) to $\big(C(S_N), u_{S_N}\big)$. Obviously, ϕ is not injective as we can divide out in all but one leg of B_1 the relations $p=q=1$ and obtain a *-homomorphism with image A, which is non-commutative. $\square$

The following result finishes the proof of Lemma 4.7.7.

Lemma 4.7.8. *Consider the situation and notations as in the proof of Lemma 4.7.7. Let $\sigma \in S_N$ be a permutation and define*

$$m := m^{(1)}_{1,\sigma(1)} \cdot \ldots \cdot m^{(1)}_{N,\sigma(N)} \in M_1.$$

Then $m' := \phi'(m) \in M'_1$ is non-zero.
In particular, we can multiply arbitrary many matrix entries of $M'_1 = (\mathrm{id}_{M_N(\mathbb{C})} \otimes \phi')(M_1)$ and the result is non-zero as long as the factors are from different rows and columns.

Proof. Given m and m' as described in the lemma, it suffices to show that there is a *-homomorphism

$$\mu : B'_1 \longrightarrow \mathbb{C}$$

that fulfils $\mu(m') = 1$.
To do so, write σ^{-1} as a product of l transpositions $\tau_{\alpha,\beta} = (\alpha, \beta) \in S_N$ with $\alpha < \beta$ and such that l is as small as possible:

$$\sigma^{-1} = \tau_{\alpha_1,\beta_1}\tau_{\alpha_2,\beta_2} \cdots \tau_{\alpha_l,\beta_l} = (\alpha_1, \beta_1)(\alpha_2, \beta_2) \cdots (\alpha_l, \beta_l) \tag{4.7.3}$$

By definition of M_1 and M_1' it holds

$$M_1' = \left(\bigoplus_{1 \leq a < b \leq N} R'_{(a,b),*} \right)^{\oplus L}$$

with $L \geq l$. Writing out this $\oplus$-product, it holds that M_1' is of the form

$$M_1' := \ldots \oplus R'_{(\alpha_1,\beta_1),*} \oplus \ldots \oplus R'_{(\alpha_2,\beta_2),*} \oplus \ldots \quad \ldots \oplus R'_{(\alpha_l,\beta_l),*} \oplus \ldots, \tag{4.7.4}$$

i.e. we find, among other $\oplus$-factors, matrices $R'_{(\alpha_1,\beta_1),*}$, $R'_{(\alpha_2,\beta_2),*}$, …,$R'_{(\alpha_l,\beta_l),*}$ that appear from left to right in this order. Let's say these matrices appear in the $\oplus$-product from Equation 4.7.4 at positions $k_1, \ldots, k_l$.
Now define a quotient map μ on B_1' in the following way:

(i) In each of the legs $k_1, \ldots, k_l$ we apply a quotient map μ_1 that divides out exactly the relation $1 - p = q = \mathbb{1}$. Note that we have

$$(1 \otimes \mu_1)(R'_{(a,b),*}) = \tau_{a,b}.$$

(ii) In each of the remaining legs we apply a quotient map μ_0 that divides out exactly the relations $p = q = 1$. Note that we have

$$(1 \otimes \mu_0)(R'_{(a,b),*}) = \mathbb{1}_{S_N}.$$

From these observations we directly deduce

$$(1 \otimes \mu)M_1' = \tau_{\alpha_1,\beta_1}\tau_{\alpha_2,\beta_2} \cdots \tau_{\alpha_l,\beta_l} = \sigma^{-1}.$$

Recall that a permutation matrix σ fulfils

$$\sigma_{ij} = \delta_{i,\sigma(j)},$$

so it holds

$$(\sigma^{-1})_{i\sigma(i)} = \delta_{i,(\sigma^{-1}\circ\sigma)(i)} = 1.$$

We conclude

$$\mu\big(m'\big) = (\sigma^{-1})_{1\sigma(1)} \cdots (\sigma^{-1})_{N\sigma(N)} = 1,$$

and thus $m' = \phi'(m)$ is non-zero. $\square$

As displayed in Observation 4.7.1, Lemma 4.7.7 guarantees that we obtain from the matrix M_1 as defined in Definition 4.7.6 a compact matrix quantum group $S_N \nsubseteq G \subseteq S_N^+$.

Theorem 4.7.9. *Consider the category object* (B_1, M_1) *as defined in Definition 4.7.6 and the arrow* ν *as defined in item (b) of Lemma 4.7.7. For* $n \in \mathbb{N}$ *define further*

$$M_{n+1} = \big(m_{ij}^{(n+1)}\big)_{1\leq i,j\leq N} := M_n \oplus M_1$$

and

$$B_{n+1} := C^*(\, m_{ij}^{(n+1)} \mid 1 \leq i, j \leq N) \subseteq B_1^{\otimes(n+1)}.$$

Let $\pi_{n+1,n}$ *be the arrow in* $\mathcal{C}$ *from* (B_{n+1}, M_{n+1}) *to* (B_n, M_n) *given by the restriction of* $(\mathrm{id}_{B_1})^{\otimes n} \otimes \nu$ *to* B_{n+1}*. Then the limit* $G := (B_\infty, M_\infty)$ *of the diagram*

$$\begin{tikzcd} (B_1, M_1) & \cdots \arrow[l, "\pi_{2,1}"'] & (B_n, M_n) & (B_{n+1}, M_{n+1}) \arrow[l, "\pi_{n+1,n}"'] & \cdots \arrow[l, "\pi_{n+2,n+1}"'] \end{tikzcd}$$

is a compact matrix quantum group and fulfils $S_N \subsetneq G \subseteq S_N^+$*.*

Remark 4.7.10. (i) Considering items (a), (b) and (c) from Lemma 4.7.7, which are the ingredients to prove Theorem 4.7.9, we see that (a) is not the crucial point, i.e. it is not difficult to construct the models

$$\big(C(S_N^+), u_{S_N^+}\big) \xrightarrow{\varphi_n} (B_n, M_n).$$

Neither is item (b), which guarantees existence of the arrows

$$(B_n, M_n) \xleftarrow{\pi_{n+1,n}} (B_{n+1}, M_{n+1})$$

and so the construction of the inverse limit and compact matrix quantum group $G = (B_\infty, M_\infty)$.
The crucial point is item (c), or, more precisely, existence of the arrow

$$(B_1, M_1) \xrightarrow{\phi} (C(S_N), u_{S_N})$$

that finally guarantees $S_N \subseteq G$. The main arguments to prove item (c) are established in Lemma 4.7.8.

(ii) Concerning item (c) from Lemma 4.7.7 and Lemma 4.7.8, it is worth to compare the situation above with the special case of $N = 4$ from the previous sections. To prove $S_4 \subseteq G$, we investigated the matrix $M_2 = R^{\oplus 4}$, compare Lemma 4.6.1. In the sense of the Lemmata above, we can obtain from $R^{\oplus 4}$ every $\sigma \in S_4$ by dividing out in each leg suitable relations (commutativity,

$p = 1$ or $p = 0$, $q = 1$ or $q = 0$).
Again, we see why the choice

$$M_1 := \widetilde{R} = \begin{pmatrix} p & 1-p & 0 & 0 \\ 1-p & p & 0 & 0 \\ 0 & 0 & q & 1-q \\ 0 & 0 & 1-q & q \end{pmatrix} \in M_4(B)$$

would not have worked to obtain $S_4 \subseteq G$: The only transpositions that could be obtained in the sense above would have been $(1, 2)$ and $(3, 4)$, but they do not generate the whole S_4.

As in the case $N = 4$, there is no CMQG strictly in between S_5 and S_5^+, see [Ban18a, Thm. 7.10]. Extending Corollary 4.6.9, Theorem 4.7.9 has the following consequence.

Corollary 4.7.11. *Consider the compact matrix quantum group* $G = (B_\infty, M_\infty)$ *as defined in Theorem 4.7.9. For* $N \in \{4, 5\}$ *it holds* $G = S_N^+$.

In [Ban18a] it is conjectured that there is for all $N \in \mathbb{N}$ no CMQG strictly in between S_N and S_N^+. A direct consequence would be the equality $G = S_N^+$ for all $N \geq 4$.

Conjecture 4.7.12. (1) For all $N \in \mathbb{N}$, there is no compact matrix quantum group strictly in between S_N and S_N^+.

(2) For all $N \geq 4$, the quantum group G from Theorem 4.7.9, given by the inverse limit

$$\lim_{\infty \leftarrow n} (B_n, M_n) = (B_\infty, M_\infty),$$

is equal to S_N^+.

Remark 4.7.13. (i) Note that the constructions in Section 4.7 do not require the C^*-algebra A to be exactly of the form as described in Section 4.4 and in Definition 4.7.2. In fact, it is sufficient to consider an A that is generated by two projections p and q such that $pq \neq qp$ and $\|pq\| = \|(1-p)q)\| = 1$. Note that $\|pq\| = 1$ exactly says that p and q, viewed as operators on a Hilbert space, project on subspaces that intersect non-trivially. While $pq \neq qp$ implies that B_1 is non-commutative, i.e. finally guarantees $S_N \neq G$, the second condition guarantees existence of the arrow $(B_1, M_1) \xrightarrow{\nu} (\mathbb{C}, M_N(\mathbb{C}))$.
To define A, we can, for example, consider any (at least four-dimensional) Hilbert space H and in $B(H)$ two con-commuting projections p and q with $p \wedge q \neq 0 \neq (1-p) \wedge q$.

(ii) Considering different C^*-algebras A and A', or at least different suitable pairs of projections in the same C^*-algebra, the respectively constructed compact matrix quantum groups G and G' might be different. However, if Conjecture 4.7.12 is true, then all of them coincide with S_N^+.

(iv) Note that, in general, the universal and reduced form of a CMQG are non-equivalent objects in $\mathcal{C}$. Therefore, even if two constructions give the same quantum group $G = G'$, the category objects (B_∞, M_∞) and (B'_∞, M'_∞) do not have to be equivalent. Hence, the result on the level of $\mathcal{C}$ might depend on A we started with, even in the case where the resulting CMQG is not affected.

We finish this chapter with a comment on the generalization to arbitrary easy quantum groups. Recall from 2.6.6 that any easy quantum group is given by some $G_N(\Pi)$, where $N \in \mathbb{N}$ and Π is a set of partitions that includes the four mixed-coloured pair partitions.

Proposition 4.7.14. *Consider any easy quantum group* $\widetilde{G}_N(\Pi) := \big(C(\widetilde{G}_N(\Pi)), u\big)$. *Assume that we have in the category* $\mathcal{C}$ *an object* (B_1, M_1) *and a chain*

$$(C(\widetilde{G}_N(\Pi)), u) \xrightarrow{\varphi_1} (B_1, M_1) \xrightarrow{\nu} (\mathbb{C}, \mathbb{1}_{M_N(\mathbb{C})}).$$

Then the construction of the compact matrix quantum group $G := (B_\infty, M_\infty)$ *analogous to Theorem 4.7.9 is well-defined and it holds* $G \subseteq \widetilde{G}_N(\Pi)$.

Proof. Investigating the aforementioned constructions and results of this chapter, we see that existence of the arrow ν guarantees existence of the arrows

$$(B_{n+1}, M_{n+1}) = (B_{n+1}, M_n \oplus M_1)) \xrightarrow{\pi_{n+1,n}} (B_n, M_n)$$

and $G := (B_\infty, M_\infty)$ to be well-defined. Once all $M_n = (M_1)^{\oplus n}$ fulfil the quantum group relations

$$\big(\mathcal{R}_p^{Gr}(M_n)\big)_{p\in\Pi},$$

also $G \subseteq \widetilde{G}_N(\Pi)$ is proved. We show this by induction on $n \in \mathbb{N}$.
For $n=1$ this is true by assumption on (B_1, M_1), or, more precisely, by existence of the arrow φ_1 above. Assume that the relations $\big(\mathcal{R}_p^{Gr}(M_n)\big)$ are in addition fulfilled for some $n \in \mathbb{N}$. To simplify notation, we write

$$M_n = (x_{ij})_{1\le i,j\le N} \qquad \text{and} \qquad M_1 = (y_{ij})_{1\le i,j\le N}.$$

Given $p \in \Pi$, we recall from Remark 2.6.7 that the relations $\mathcal{R}_p^{Gr}(u)$ read

$$\sum_{t\in[N]^k} \delta_p(t, \gamma') u_{t_1\gamma_1}^{\omega_1} \cdots u_{t_k\gamma_k}^{\omega_k} = \sum_{t'\in[N]^l} \delta_p(\gamma, t') u_{\gamma'_1 t'_1}^{\omega'_1} \cdots u_{\gamma'_l t'_l}^{\omega_l}$$

for all $\gamma \in [N]^k$ and $\gamma' \in [N]^l$. Using repeatedly that these relations are fulfilled for the x_{ij}'s and the y_{ij}'s, we can directly check the respective quantum group relations for the matrix

$$M_{n+1} = M_n \oplus M_1 = \left(\sum_{s=1}^{N} x_{is} \otimes y_{sj} \right)_{1 \leq i,j \leq N}.$$

Given $\gamma \in [N]^k$ and $\gamma' \in [N]^l$, we compute

$$\begin{aligned}
&\sum_{t\in[N]^k}\sum_{s\in[N]^k} \delta_p(t,\gamma') x_{t_1 s_1}^{\omega_1}\cdots x_{t_k s_k}^{\omega_k} \otimes y_{s_1\gamma_1}^{\omega_1}\cdots y_{s_k\gamma_k}^{\omega_k}\\
&= \sum_{s\in[N]^k}\left(\sum_{t\in[N]^k} \delta_p(t,\gamma') x_{t_1 s_1}^{\omega_1}\cdots x_{t_k s_k}^{\omega_k}\right) \otimes y_{s_1\gamma_1}^{\omega_1}\cdots y_{s_k\gamma_k}^{\omega_k}\\
&= \sum_{s\in[N]^k}\left(\sum_{t'\in[N]^l} \delta_p(s,t') x_{\gamma_1' t_1'}^{\omega_1'}\cdots x_{\gamma_l' t_l'}^{\omega_l'}\right) \otimes y_{s_1\gamma_1}^{\omega_1}\cdots y_{s_k\gamma_k}^{\omega_k}\\
&= \sum_{t'\in[N]^l} x_{\gamma_1' t_1'}^{\omega_1'}\cdots x_{\gamma_l' t_l'}^{\omega_l'} \otimes \left(\sum_{s\in[N]^k} \delta_p(s,t') y_{s_1\gamma_1}^{\omega_1}\cdots y_{s_k\gamma_k}^{\omega_k}\right)\\
&= \sum_{t'\in[N]^l} x_{\gamma_1' t_1'}^{\omega_1'}\cdots x_{\gamma_l' t_l'}^{\omega_l'} \otimes \left(\sum_{s'\in[N]^l} \delta_p(\gamma,s') y_{t_1' s_1'}^{\omega_1'}\cdots y_{t_l' s_l'}^{\omega_l'}\right)\\
&\overset{s'\leftrightarrow t'}{=} \sum_{t'\in[N]^l}\sum_{s'\in[N]^l} \delta_p(\gamma,t') x_{\gamma_1' s_1'}^{\omega_1'}\cdots x_{\gamma_l' s_l'}^{\omega_l'} \otimes y_{s_1' t_1'}^{\omega_1'}\cdots y_{s_l' t_l'}^{\omega_l'}.
\end{aligned}$$

These are the relations $\mathcal{R}_p^{Gr}(M_{n+1})$ and the proof is finished. □

Chapter 5

Partition quantum spaces

Groups were originally introduced to describe and understand the symmetries of a space. In modern mathematics however, the notion of quantum spaces appeared, for example modelled as possibly non-commutative C^*-algebras. Asking for the quantum symmetries of such topological quantum spaces is one possibility to motivate quantum groups. Now there are two fundamental questions regarding quantum symmetries:

- Given a quantum space X – what is its quantum symmetry group?
- Conversely, given a quantum group G – can we find a quantum space, such that its quantum symmetry group is precisely G?

In this chapter, we mainly deal with the second kind of questions and concentrate again on the context of partitions and easy quantum groups. Similar to the construction of an easy quantum group $G_N(\Pi)$ from a set of partitions Π, we will define so called *partition quantum spaces* $X_{N,d}(\Pi)$, universal C^*-algebras motivated by the first d columns of $G_N(\Pi)$. Just as a matrix acts on the vector tuple given by its first d columns, our definitions will guarantee the existence of a matrix-vector action of $G_N(\Pi)$ on the corresponding partition quantum spaces $X_{N,d}(\Pi)$. Conversely, we will be able to reconstruct the easy quantum group $G_N(\Pi)$ from the partition quantum space $X_{N,d}(\Pi)$ as its *quantum symmetry group*, at least if the number d – i.e. the amount of information about the quantum matrices – is large enough. In the free case – i.e. all considered partitions are non-crossing – the minimal d to make this reconstruction work, takes the values 1 or 2.

This chapter is based on the article [JW18], where the notion of a partition quantum space has been introduced and the results above have been established.

At last, we want to mention some other works touching our topic. P. Podleś's definition of *quantum spheres* in [Pod87] was a first but important step in quantizing the notion of a classical space. Authors like T. Banica, J. Bhowmick, D. Goswami, P. Podleś, A. Skalski and Sh. Wang investigated various quantum spaces and actions of quantum groups on them and asked (for example under the name of *quantum isometry groups*) for the universal objects acting on these spaces (see [Pod87, Pod95, Wan98, Gos08, BG09b, BGS11, BG10, BG09a]). The idea of a quantum space inspired by one or several rows/columns of a compact matrix quantum group G can be found for example in [BSS12], but note that the spaces there are defined via C^*-subalgebras of $C(G)$, whereas we introduce them as universal C^*-algebras. At last we mention the recent work [Ban18b] by T. Banica, where partition induced relations similar to those in our article are used to describe certain quantum subspaces of the free complex sphere. In contrast to the setting presented there, where it is part of the assumptions that an easy quantum group is the quantum symmetry group of a suitable quantum space, this is the central question in our work. Additionally, as mentioned above, we generalize the idea of quantum vectors to tuples of quantum vectors.

5.1 Motivating example

We start our investigations with an example from the classical world, i.e. from the context of easy groups. Although the quantized situation is more complicated – in particular the quantum object exist in general only in an abstract sense – this example illustrates perfectly the initial questions, the obstacles one faces while answering them as well as the solution to these problems.

Consider the easy group H_N given by all $N \times N$ matrices with exactly one entry ± 1 in each row and column and all other entries vanishing, compare Section 2.6.1:

$$H_3 = \{ \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \ldots \}.$$

Reducing all these matrices to its first (or any other fixed) column, we end up with the following space of vectors:

$$X = \{ \pm \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \pm \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \pm \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \}$$

and we observe the following:

(a) The matrix group H_3 acts on X by left as well as right matrix-vector multiplication.

(b) The matrix group H_3 is the symmetry group of X, i.e. it is the maximal group of 3×3-matrices that fulfils (a).

Consider now the easy group $S'_N = S_N \times \mathbb{Z}_2$:

$$S'_3 = \{ \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \ldots \}.$$

Looking at one fixed column, we end up with the space of vectors X, just as in the situation of H_3. So the information about one matrix column does not distinguish these two easy groups. In other words, asking for the symmetry group of such a space of vectors, we do not necessarily end up with the matrix group we have started with. We can solve this problem by looking at several columns of the matrices above at once, for example the first two columns. This way we obtain for each H_3 and S'_3

a space of vector pairs:

$$X_{H_3} = \{(\begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix} , \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}) , (\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} , \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}), \ldots\}$$

$$X_{S_3'} = \{(\begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix} , \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}) , (\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} , \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}), \ldots\}$$

Now the two matrix groups H_3 and S_3' become distinguishable by the spaces X_{H_3} and $X_{S_3'}$ in the sense that S_3' acts on $X_{S_3'}$ but H_3 does not: If we consider

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix} \in H_3$$

and let it act by left multiplication on the pair of vectors

$$(\begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix} , \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}) \in X_{S_3'}$$

we end up with

$$(\begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix} , \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}) \notin X_{S_3'}.$$

Hence, H_3 does not act on $X_{S_3'}$. The reason is, obviously, that, in contrast to S_3', different columns in a matrix in H_3 are allowed to have different signs.
There is always a number d of columns we can consider, such that we can reconstruct from the corresponding space the easy group we have started with. At least if we choose $d = N$ this is the case. This is our third and our last observation in this example:

(c) Considering only $d = 1$ column of the matrices in an easy group at once, it might be that for different easy groups we end up with the same space of vectors X. Conversely, the symmetry group of X does not necessarily give us back the matrices we have started with.

(d) However, considering sufficiently many columns d at once, we obtain a space of vector tuples whose symmetry group is exactly the easy group we have started with.

The examples and observations above reflect our attempts in the generalized situation of easy quantum groups and compact quantum spaces (of quantum vectors). Observations (a) and (b) describe some kind of optimal situation. The problem in (c) will force us to generalize the idea of a quantum vector to d-tuples of quantum vectors in order to prove an analogue of statement (d).

5.2 Definition of partition quantum spaces

In order to quantize the classical situation as described in Section 1.1, we have to generalize the idea of a column space of a matrix group. Given an easy quantum group $G_N(\Pi) = \big(C\big(G_N(\Pi)\big), u\big)$, one possibility would have been to consider the (unital) C^*-subalgebra of $C\big(G_N(\Pi)\big)$ generated by one or several columns of u. In the case of one column this approach has been for example followed by T. Banica, A. Skalski and P. Soltan in [BSS12]. Doing so, the considered C^*-subalgebra is not given by a universal C^*-algebra:

In this thesis we follow another approach: For given $N \in \mathbb{N}$ and $1 \leq d \leq N$ we consider the object

$$x := (x_{ij}) := \left(\begin{pmatrix} x_{11} \\ \vdots \\ x_{N1} \end{pmatrix}, \dots, \begin{pmatrix} x_{1d} \\ \vdots \\ x_{Nd} \end{pmatrix} \right),$$

a collection of generators x_{ij}. We associate to a partition $p \in \Pi$ so-called *quantum space relations* $\big(\mathcal{R}_p^{Sp}(x)\big)_{p\in\Pi}$ on the generators x_{ij}, see Definition 5.2.17. Inspired by the construction of an easy quantum group $G_N(\Pi)$ from a set of partitions Π, we define a so-called *partition quantum space* $X_{N,d}(\Pi)$ via the universal C^*-algebra

$$C\big(X_{N,d}(\Pi)\big) := C^*\Big(\ (x_{ij})_{1\leq i\leq N, 1\leq j\leq d} \mid \forall p \in \Pi : \text{ The relations } \mathcal{R}_p^{Sp}(x) \text{ hold. }\Big),$$

see Definition 5.2.21. This way we will keep the benefits of a universal C^*-algebra.

5.2.1 Decomposition of labellings

In order to find suitable relations for a quantum space of vectors, we have to take a closer look at the quantum group relations

$$\mathcal{R}_p^{Gr}(u) : \quad u^{\omega'}(T_p \otimes \mathbb{1}) = (T_p \otimes \mathbb{1})u^{\omega} \tag{5.2.1}$$

associated to a partition $p \in \mathcal{P}(\omega, \omega') \subseteq \mathcal{P}(k,l)$ and an $N \times N$- matrix of generators u. By Remark 2.6.7, these relations read as

$$\sum_{t\in[N]^k} \delta_p(t,\gamma') u_{t_1\gamma_1}^{\omega_1} \cdots u_{t_k\gamma_k}^{\omega_k} = \sum_{t'\in[N]^l} \delta_p(\gamma,t') u_{\gamma_1' t_1'}^{\omega_1'} \cdots u_{\gamma_l' t_l'}^{\omega_l} \tag{5.2.2}$$

for all $\gamma \in [N]^k$ and $\gamma' \in [N]^l$. Recall from Definition 2.5.3 that $\delta_p(a, b)$ is non-zero if and only if (a, b) is a valid labelling for p. With respect to validity we can distinguish the following situations for the pair (γ, γ') in Equation 5.2.2.

(i) If both γ and γ' are invalid labellings for the upper and lower row of p, respectively, than all δ_p-values are zero and the corresponding relation is trivial.

(ii) If only one of γ, γ' is invalid, then the δ_p-values on the corresponding side of Equation 5.2.2 vanish.

(iii) If γ is a valid labelling of the upper row of p, then the summation over $t' \in [N]^l$ is in fact a summation only over those t' such that (γ, t') is a valid labelling of p. In particular, $\delta_p(\gamma, t')$ can only be non-zero if t' labels the through-block points p as specified by γ. The same holds for valid labellings γ' of the lower row and the summation over $t \in [N]^k$.

In this sense we decompose given sets of multi-indices into disjoint subsets. See also Example 5.2.4 for such a decomposition.

Notation 5.2.1. Given $p \in \mathcal{P}(k, l)$ and $N \in \mathbb{N}$ we can decompose the sets $[N]^k$ and $[N]^l$ in the following way:

$$[N]^k = T_0 \mathbin{\dot\cup} T_1 \mathbin{\dot\cup} \ldots \mathbin{\dot\cup} T_r \quad ; \quad [N]^l = T'_0 \mathbin{\dot\cup} T'_1 \mathbin{\dot\cup} \ldots \mathbin{\dot\cup} T'_r,$$

such that
(i) $r = N^{tb(p)}$, where $tb(p)$ denotes the number of through-blocks of p,
(ii) T_0 and T'_0 are the invalid labellings of the upper (respectively lower) row,
(iii) for every $1 \leq i \leq r$ every labelling $(t, t') \in T_i \times T'_i$ is valid,
(iv) for every $1 \leq i \leq r$ the sets T_i and T'_i are non-empty.
(v) if $(t, t') \in [N]^k \times [N]^l$ is a valid labelling, then $(t, t') \in T_i \times T'_i$ for some $1 \leq i \leq r$,
(vi) for every $1 \leq i \leq r$ and $(t, t'), (s, s') \in T_i \times T'_i$ we have that (t, t') labels the through-block points of p the same way as (s, s') does.
The listed properties above are partially redundant. For example (iv)–(vi) follow from (i)–(iii).

Remark 5.2.2. Note the special case of $k = 0$ or $l = 0$: An empty row has only one possible labelling (which is valid), namely the empty word $\varepsilon \in [N]^0$. So if for example a partition has only lower points, then $r = 1$, $T_1 = \{\varepsilon\}$ and T_0 is empty.
In addition, it holds that $|T_i| = |T_j|$ and $|T'_i| = |T'_j|$ for all $1 \leq i, j \leq r$ because the possibilities to extend a valid labelling of through-block points to a valid labelling of the whole row does not depend on the given through-block labelling.

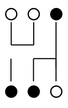
Lemma 5.2.3. *Decompositions of* $[N]^k$ *and* $[N]^l$ *as in Notation 5.2.1 exist and they are unique up to permutations of the index set* $\{i \mid 1 \leq i \leq r\}$.

Proof. **Existence:** Define T_0 and T'_0 as described in (ii). As p has $tb(p)$ through-blocks we have $r = N^{tb(p)}$ possibilities to label the through-block points in a valid way. Numbering these possibilities from 1 to r we can take any $1 \leq i \leq r$ and extend it to labellings of the whole partition. This defines the sets T_i and T'_i:

$$T_i := \{\text{all valid labellings of the upper row of } p \text{ with through-block labelling } i\}$$
$$T'_i := \{\text{all valid labellings of the lower row of } p \text{ with through-block labelling } i\}$$

It is now easy to check that the properties (i)–(vi) are fulfilled: Items (i) and (ii) hold by construction. Item (iii) is true as given labellings $t \in T_i$ and $t' \in T'_i$ are valid for their respective row and the through-block labellings fit together as both t and t' arise from a common through-block labelling i. Obviously (iv) is true, as we can always extend a valid through-block labelling by labelling all remaining points the same. For property (v) note, that a valid labelling (t, t') of a partition always restricts to a valid labelling of the through-block points. If this through-block labelling corresponds to $i \in [r]$, then t and t' appear in the construction of T_i and T'_i, respectively. Property (vi) is fulfilled because, by construction, (t, t') and (s, s') arise from the same through-block labelling. Finally we check $[N]^k = T_0 \dot\cup \ldots \dot\cup T_r$ (the proof of $[N]^l = T'_0 \,\dot\cup\, T'_1 \,\dot\cup\, \ldots \,\dot\cup\, T'_r$ is analogous): By construction, only T_0 contains non-valid labellings of the upper row and it contains all of them. Every valid upper row labelling appears as it restricts to a valid labelling of the upper through-block points, which can be extended to a valid through-block labelling $i \in [r]$ for the whole partition. On the other side, the $T_1, \ldots, T_r$ are disjoint as different through-block labellings $1 \leq i \neq j \leq r$ always differ when restricted to only one row, so $T_i \cap T_j = \emptyset$.
Uniqueness: Of course T_0 and T'_0 are uniquely defined. Consider now two valid labellings t and s of the upper row. Assume that they do not restrict to the same labelling of upper through-block points but are contained in the same T_i. Then for any $t' \in T'_i$ – we have $T'_i \neq \emptyset$ by (iv) – the labellings (t, t') and (s, t') were valid for the whole partition by (iii). This is a contradiction, as t' uniquely determines the upper through-block labellings both of t and s. As there are $N^{tb(p)}$ pairwise different valid labellings of the upper through-block points and $r = N^{tb(p)}$ by property (i), the sets $T_1, \ldots, T_r$ must be the (pairwise different) equivalence classes of valid upper row labellings with respect to the relation "equality on through-block points". Having the sets $T_1 \ldots, T_r$ (and likewise $T'_1, \ldots, T'_r$) at hand, property (iii) says that T'_i must correspond to the same through-block labelling as T_i. hence, up to (simultaneous) permutations of the index set $\{i \mid 1 \leq i \leq r\}$, we have uniqueness as claimed. $\square$

Example 5.2.4. Consider the partition

$p =$  , so $r = N^{tb(p)} = N$. A pair (T_i, T_i') for $1 \leq i \leq N$ corresponds to a distinct valid labelling of the through-block points (i.e. $t_3 = t_2' = t_3'$). So we have (up to permutation of indices)

$$\begin{array}{llll} T_0 = \{(t_1, t_2, t_3) \in [N]^3 \mid t_1 \neq t_2\} & , & T_0' = \{(t_1', t_2', t_3') \in [N]^3 \mid t_2' \neq t_3'\} & \\ T_i = \{(t, t, i) \in [N]^3 \mid t \in [N]\} & , & T_i' = \{(t', i, i) \in [N]^3 \mid t' \in [N]\} & \text{for } 1 \leq i \leq N. \end{array}$$

With this decomposition at hand, we can reformulate the quantum group relations $\mathcal{R}_p^{Gr}(u)$. Recall that for $p \in \mathcal{P}(\omega, \omega') \subseteq \mathcal{P}(k, l)$ (and fixed $N \in \mathbb{N}$) they read as

$$\sum_{t \in [N]^k} \delta_p(t, \gamma') u_{t_1 \gamma_1}^{\omega_1} \cdots u_{t_k \gamma_k}^{\omega_k} = \sum_{t' \in [N]^l} \delta_p(\gamma, t') u_{\gamma_1' t_1'}^{\omega_1'} \cdots u_{\gamma_l' t_l'}^{\omega_l}$$

for all $\gamma \in [N]^k$ and $\gamma' \in [N]^l$, see Equation 5.2.2.

Proposition 5.2.5. *Let $N \in \mathbb{N}$, $u := (u_{ij})$ be an $N \times N$-matrix of generators and $p \in \mathcal{P}(\omega, \omega') \subseteq \mathcal{P}(k, l)$ be a partition. Using Notation 5.2.1, the quantum group relations $\mathcal{R}_p^{Gr}(u)$ from Equation 5.2.2 can be expressed in the following way:*

(i) $\displaystyle\sum_{t \in T_i} u_{t_1 \gamma_1}^{\omega_1} \cdots u_{t_k \gamma_k}^{\omega_k} = \sum_{t' \in T_j'} u_{\gamma_1' t_1'}^{\omega_1'} \cdots u_{\gamma_l' t_l'}^{\omega_l'} \quad , \quad 1 \leq i, j \leq r,\ \gamma \in T_j \text{ and } \gamma' \in T_i'.$

(ii) $\displaystyle\sum_{t \in T_i} u_{t_1 \gamma_1}^{\omega_1} \cdots u_{t_k \gamma_k}^{\omega_k} = 0 \quad , \quad 1 \leq i \leq r \text{ and } \gamma \in T_0.$

(iii) $\displaystyle\sum_{t' \in T_j'} u_{\gamma_1' t_1'}^{\omega_1'} \cdots u_{\gamma_l' t_l'}^{\omega_l'} = 0 \quad , \quad 1 \leq j \leq r \text{ and } \gamma' \in T_0'.$

Note for (i) (compare Remark 5.2.2), that in the case of $k = 0$ or $l = 0$ the sum on the corresponding side is equal to $\mathbb{1}$, corresponding to the empty word ε.

Remark 5.2.6. Item (i) in Proposition 5.2.5 corresponds in the sense of Equation 5.2.2 to the situations where both γ and γ' are valid. Item (ii) describes the situation where only γ' is valid and (iii) reflects the case where only γ is valid.

Inspired by the quantum group relations, we now establish relations between the u_{ij}'s with $j < d$, i.e. all appearing u_{ij} are from the first d columns of u. Those will be the quantum space relations mentioned at the beginning of Section 5.2. We first stick to the case of one column as notations are easier in this situation. The results will be generalized afterwards to the case of d-tuples of vectors.

5.2.2 The case of one vector

Lemma 5.2.7. *Let $N \in \mathbb{N}$ and $p \in P(\omega,\omega') \subseteq P(k,l)$ be a partition. Let $G = \big(C(G), u_G\big)$ be an easy quantum group such that the relations $R_p^{Gr}(u_G)$ are fulfilled. Using Notation 5.2.1, we have for all $1 \leq i \leq r$*

$$\sum_{t\in T_i} u_{t_1 1}^{\omega_1} \dots u_{t_k 1}^{\omega_k} = \sum_{t'\in T_i'} u_{t_1' 1}^{\omega_1'} \dots u_{t_l' 1}^{\omega_l'}. \tag{5.2.3}$$

Proof. As displayed in Observation 2.6.4, also the relations $\mathcal{R}_{p^*}^{Gr}(u_G)$ and $\mathcal{R}_{pp^*}^{Gr}(u_G)$ are fulfilled. Additionally, the decomposition $T_0' \dot\cup \dots \dot\cup T_r'$ of $[N]^l$ for the lower row of p coincides with the decomposition of $[N]^l$ for both the upper and lower row of pp^* because pp^* is self-adjoint, i.e. its picture is symmetric with respect to a horizontal axis.
Assume that $(\underbrace{1,\dots,1}_{k \text{ entries}}) \in T_1$ and $(\underbrace{1,\dots,1}_{l \text{ entries}}) \in T_1'$ hold. The relations $R_{pp^*}^{Gr}(u_G)$ in particular say (see Proposition 5.2.5) that

$$\sum_{t'\in T_i'} u_{t_1' 1}^{\omega_1'} \dots u_{t_l' 1}^{\omega_l'} = \sum_{t'\in T_1'} u_{\beta_1' t_1'}^{\omega_1'} \dots u_{\beta_l' t_l'}^{\omega_l'}$$

for every $1 \leq i \leq r$ and $\beta' \in T_i'$. Using this, for any $\beta' \in T_i'$ we have

$$\sum_{t\in T_i} u_{t_1 1}^{\omega_1} \dots u_{t_k 1}^{\omega_k} \overset{R_p^{Gr}(u_G)}{=} \sum_{t'\in T_1'} u_{\beta_1' t_1'}^{\omega_1'} \dots u_{\beta_l' t_l'}^{\omega_l'} \overset{R_{pp^*}^{Gr}(u_G)}{=} \sum_{t'\in T_i'} u_{t_1' 1}^{\omega_1'} \dots u_{t_l' 1}^{\omega_l'}.$$

Note that this argument is valid even if $k=0$ or $l=0$ as we then have $\varepsilon \in T_1$ or $\varepsilon \in T_1'$, respectively and the corresponding side in Equation 5.2.3 is $\mathbb{1}$ (see Remark 5.2.2). □

We are now ready to formulate the definition of partition quantum spaces in the one vector case.

Definition 5.2.8. Let $N \in \mathbb{N}$ and $x := (x_1, \dots, x_N)^T$ be a vector of generators. Let $p \in \mathcal{P}(\omega,\omega') \subseteq \mathcal{P}(k,l)$ be a partition. Using Notation 5.2.1, we associate with p the following relations on x, denoted by $\mathcal{R}_p^{Sp}(x)$:

$$\mathcal{R}_p^{Sp}(x): \quad \sum_{t\in T_i} x_{t_1}^{\omega_1} \cdots x_{t_k}^{\omega_k} = \sum_{t'\in T_i'} x_{t_1'}^{\omega_1'} \cdots x_{t_l'}^{\omega_l'} \qquad \forall\, 1 \leq i \leq r.$$

We call the relations $\mathcal{R}_p^{Sp}(x)$ the *quantum space relations* associated to the partition p and the vector x.

Recall that for $k=0$ or $l=0$ the corresponding side of the equation above is equal to $\mathbb{1}$ (see Remark 5.2.2).

Example 5.2.9. Consider the partition $\underset{\circ\bullet}{\sqcap}$ (i.e. $r=1$). The relations $\mathcal{R}^{Sp}_{\underset{\circ\bullet}{\sqcap}}(x)$ read as

$$\mathcal{R}^{Sp}_{\underset{\circ\bullet}{\sqcap}}(x): \quad \mathbb{1} = \sum_{i=1}^{N} x_i x_i^* \tag{5.2.4}$$

Definition 5.2.10. Let $N \in \mathbb{N}$ and $\Pi \supseteq \{ \underset{\circ\bullet}{\sqcap}, \underset{\bullet\circ}{\sqcap}, \overset{\circ\bullet}{\sqcup}, \overset{\bullet\circ}{\sqcup} \}$ be a set of partitions. Then we define the universal C^*-algebra

$$C\big(X_N(\Pi)\big) := C^*(x_1, \dots, x_N \mid \forall p \in \Pi: \text{ The relations } \mathcal{R}^{Sp}_p(x) \text{ hold. })$$

and call it the *non-commutative functions* on the *partition quantum space (PQS)* $X_N(\Pi)$ *of one vector.*

Remark 5.2.11. As seen above (Equation 5.2.4), the relations $\mathcal{R}^{Sp}_{\underset{\circ\bullet}{\sqcap}}(x)$ guarantee $0 \leq x_i x_i^* \leq 1$, so the universal C^*-algebra $C\big(X_N(\Pi)\big)$ exists.

Note that by Lemma 5.2.7 the relations $\mathcal{R}^{Sp}_p(x)$ are fulfilled if we replace x by a column of u, so the first column space of an easy quantum group is a subspace of our PQS.

Theorem 5.2.12. *Let $N \in \mathbb{N}$ and $\Pi \supseteq \{ \underset{\circ\bullet}{\sqcap}, \underset{\bullet\circ}{\sqcap}, \overset{\circ\bullet}{\sqcup}, \overset{\bullet\circ}{\sqcup} \}$ be a set of partitions. Let $X_N(\Pi)$ be the corresponding PQS with vector of generators $x=(x_i)$ and $G_N(\Pi)$ the corresponding easy quantum group with matrix of generators $u_{G_N(\Pi)} = (u_{ij})$. Then the map*

$$\phi : x_i \mapsto u_{i1} \qquad , \qquad 1 \leq i \leq N$$

defines a unital *-homomorphism from $C\big(X_N(\Pi)\big)$ to $C\big(G_N(\Pi)\big)$.

5.2.3 The case of d vectors

As a first step, we extract analogous to Lemma 5.2.7 for a given CMQG relations for the entries u_{ij} of u_G, which are more suitable for our purposes.

Definition 5.2.13. Let $p \in \mathcal{P}(\omega, \omega') \subseteq \mathcal{P}(k,l)$ be a partition and G a CMQG with matrix of generators $u_G = \big(u_{ij}\big)_{1 \leq i,j \leq N}$. Using Notation 5.2.1 we associate with the partition p the following relations, denoted by $\mathcal{R}^{Sp}_p(u_G)$:

(i) $\displaystyle\sum_{t \in T_i} u^{\omega_1}_{t_1 \gamma_1} \dots u^{\omega_k}_{t_k \gamma_k} = \sum_{t' \in T'_i} u^{\omega'_1}_{t'_1 \gamma'_1} \dots u^{\omega'_l}_{t'_l \gamma'_l} \quad , \quad 1 \leq i,j \leq r, \gamma \in T_j$ and $\gamma' \in T'_j$.

(ii) $\sum_{t\in T_i} u^{\omega_1}_{t_1\gamma_1}\dots u^{\omega_k}_{t_k\gamma_k} = 0 \quad , \quad 1\leq i\leq r$ and $\gamma\in T_0$.

(iii) $\sum_{t'\in T'_i} u^{\omega'_1}_{t'_1\gamma'_1}\dots u^{\omega'_l}_{t'_l\gamma'_l} = 0 \quad , \quad 1\leq i\leq r$ and $\gamma'\in T'_0$.

Remark 5.2.14. Note, that if (i) is fulfilled, then the left side of equation (i) does not depend on our choice of $\gamma\in T_j$ and likewise the right side does not depend on $\gamma'\in T'_j$:

$$\sum_{t\in T_i} u^{\omega_1}_{t_1\gamma_1}\dots u^{\omega_k}_{t_k\gamma_k} = \sum_{t\in T_i} u^{\omega_1}_{t_1\tilde{\gamma}_1}\dots u^{\omega_k}_{t_k\tilde{\gamma}_k} \quad , \quad \forall\gamma,\tilde{\gamma}\in T_j$$

and

$$\sum_{t'\in T'_i} u^{\omega'_1}_{t'_1\gamma'_1}\dots u^{\omega'_l}_{t'_l\gamma'_l} = \sum_{t'\in T'_i} u^{\omega'_1}_{t'_1\tilde{\gamma}'_1}\dots u^{\omega'_l}_{t'_l\tilde{\gamma}'_l} \quad , \quad \forall\gamma',\tilde{\gamma}'\in T'_j.$$

Comparing these relations $\mathcal{R}^{Sp}_p(u_G)$ with the quantum group relations $\mathcal{R}^{Gr}_p(u_G)$ from Proposition 5.2.5, we obtain the following result:

Lemma 5.2.15. *Let $G = \big(C(G), u_G\big)$ be a CMQG and $p\in P(\omega,\omega')\subseteq P(k,l)$ be a partition. Then it holds*

(1) $\mathcal{R}^{Gr}_p(u_G),\ \mathcal{R}^{Gr}_{pp^*}(u_G),\ \mathcal{R}^{Gr}_{p^*}(u_G) \quad\Rightarrow\quad \mathcal{R}^{Sp}_p(u_G),$

(2) $\mathcal{R}^{Sp}_p(u_G),\ \mathcal{R}^{Sp}_p(u_G^T) \quad\Rightarrow\quad \mathcal{R}^{Gr}_p(u_G).$

Proof. For (1) assume that the quantum group relations of p, pp^* and p^* are fulfilled. Equation (i) in Definition 5.2.13 is proved with the same arguments as in Lemma 5.2.7, we just replace the multi-indices $(1,\dots,1)$ by $\gamma\in T_j$ and $\gamma'\in T'_j$, respectively: For any $\beta'\in T'_i$ it holds

$$\sum_{t\in T_i} u^{\omega_1}_{t_1\gamma_1}\dots u^{\omega_k}_{t_k\gamma_k} \overset{\mathcal{R}^{Gr}_p(u_G)}{=} \sum_{t'\in T'_j} u^{\omega'_1}_{\beta'_1t'_1}\dots u^{\omega'_l}_{\beta'_lt'_l} \overset{\mathcal{R}^{Gr}_{pp^*}(u_G)}{=} \sum_{t'\in T'_i} u^{\omega'_1}_{t'_1\gamma'_1}\dots u^{\omega'_l}_{t'_l\gamma'_l}.$$

Equations (ii) and (iii) in Definition 5.2.13 follow directly from the relations $\mathcal{R}^{Gr}_p(u_G)$ and $\mathcal{R}^{Gr}_{p^*}(u_G)$, see Proposition 5.2.5.
For (2) assume that $\mathcal{R}^{Sp}_p(u_G)$ and $\mathcal{R}^{Sp}_p(u_G^T)$ are fulfilled. We have to prove the quantum group relations of the partition p. Note that $\mathcal{R}^{Sp}_p(u_G)$ and $\mathcal{R}^{Sp}_p(u_G^T)$ already include the quantum group relations (ii) and (iii) in Proposition 5.2.5, so there is only (i) left to prove. As $\mathcal{R}^{Sp}_p(u_G)$ holds, we know for every $1\leq i,j\leq r$, $\gamma\in T_j$ and $n'\in T'_j$ that

$$\sum_{t\in T_i} u^{\omega_1}_{t_1\gamma_1}\cdots u^{\omega_k}_{t_k\gamma_k} = \sum_{m'\in T'_i} u^{\omega'_1}_{m'_1n'_1}\cdots u^{\omega'_l}_{m'_ln'_l}.$$

Out of this we can straightforwardly deduce the desired relation, as for any $\gamma' \in T'_i$ the right side can now be written as

$$\begin{aligned}\sum_{m'\in T'_i} u^{\omega'_1}_{m'_1n'_1}\cdots u^{\omega'_l}_{m'_ln'_l} &\overset{(*)}{=} \frac{1}{|T'_j|}\sum_{t'\in T'_j}\sum_{m'\in T'_i} u^{\omega'_1}_{m'_1t'_1}\cdots u^{\omega'_l}_{m'_lt'_l}\\ &\overset{(*)}{=} \frac{|T'_i|}{|T'_j|}\sum_{t'\in T'_j} u^{\omega'_1}_{\gamma'_1t'_1}\cdots u^{\omega'_l}_{\gamma'_lt'_l}\\ &= \sum_{t'\in T'_j} u^{\omega'_1}_{\gamma'_1t'_1}\cdots u^{\omega'_l}_{\gamma'_lt'_l}\ ,\end{aligned}$$

where we used Remark 5.2.14 in $(*)$ – once for u_G and once for u_G^T – and $|T'_i| = |T'_j|$, see Remark 5.2.2. ☐

Remark 5.2.16. Note that by Observation 2.6.4 the left side of implication (1) in Lemma 5.2.15 is always fulfilled if we consider an easy quantum group $G_N(\Pi)$ with $p \in \langle\Pi\rangle$. As these relations also hold for $u^T_{G_N(\Pi)}$ we altogether have in the case $p \in \langle\Pi\rangle$:

$$\mathcal{R}^{Gr}_p(u_{G_N(\Pi)}) \Leftrightarrow \mathcal{R}^{Sp}_p(u_{G_N(\Pi)}), \mathcal{R}^{Sp}_p(u^T_{G_N(\Pi)})$$

Next, we define relations similar to Definition 5.2.8, but now in the case where x is a d-tuple of vectors. They are inspired by the relations $R^{Sp}_p(u_G)$ which hold for an easy quantum group $G_N(\Pi)$ with $p \in \langle\Pi\rangle$ by Lemma 5.2.15.

Definition 5.2.17. Let $d, N \in \mathbb{N}$ with $d \leq N$ and

$$x := (x_{ij}) := \left(\begin{pmatrix} x_{11}\\ \vdots \\ x_{N1}\end{pmatrix}, \ldots, \begin{pmatrix} x_{1d}\\ \vdots \\ x_{Nd}\end{pmatrix} \right)$$

be a tuple of vectors of generators x_{ij}. Let $p \in \mathcal{P}(\omega, \omega') \subseteq \mathcal{P}(k, l)$ be a partition. Using Notation 5.2.1, we associate with p the following relations on x, denoted by $\mathcal{R}^{Sp}_p(x)$:

(i) $\displaystyle\sum_{t\in T_i} x^{\omega_1}_{t_1\gamma_1}\ldots x^{\omega_k}_{t_k\gamma_k} = \sum_{t'\in T'_i} x^{\omega'_1}_{t'_1\gamma'_1}\ldots x^{\omega'_l}_{t'_l\gamma'_l} \quad , \quad 1 \leq i, j \leq r,\ \gamma \in T_j \cap [d]^k,\ \gamma' \in T'_j \cap [d]^l.$

(ii) $\displaystyle\sum_{t\in T_i} x^{\omega_1}_{t_1\gamma_1}\ldots x^{\omega_k}_{t_k\gamma_k} = 0 \quad , \quad 1 \leq i \leq r,\ \gamma \in T_0 \cap [d]^k.$

(iii) $\displaystyle\sum_{t'\in T'_i} x^{\omega'_1}_{t'_1\gamma'_1}\ldots x^{\omega'_l}_{t'_l\gamma'_l} = 0 \quad , \quad 1 \leq i \leq r,\ \gamma' \in T'_0 \cap [d]^l.$

We call the relations $\mathcal{R}_p^{Sp}(x)$ the *quantum space relations* associated to the partition p and the d-tuple of vectors x. See also Remark 5.2.22 on the (different but compatible) meanings of $\mathcal{R}_p^{Sp}(\cdot)$.

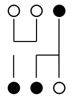
Remark 5.2.18. If the corresponding relations are fulfilled, we have again (compare Remark 5.2.14) that the sums appearing in Definition 5.2.17 do not depend on the chosen $\gamma\in T_j\cap[d]^k$ and $\gamma'\in T'_j\cap[d]^l$, respectively:

$$\sum_{t\in T_i} x^{\omega_1}_{t_1\gamma_1}\dots x^{\omega_k}_{t_k\gamma_k} = \sum_{t\in T_i} x^{\omega_1}_{t_1\tilde{\gamma}_1}\dots x^{\omega_k}_{t_k\tilde{\gamma}_k} \quad , \quad \forall\gamma,\tilde{\gamma}\in T_j\cap[d]^k.$$

and

$$\sum_{t'\in T'_i} x^{\omega'1}_{t'_1\gamma'_1}\dots x^{\omega'_l}_{t'_l\gamma'_l} = \sum_{t'\in T'_i} x^{\omega'_1}_{t'_1\tilde{\gamma}'_1}\dots x^{\omega'_l}_{t'_l\tilde{\gamma}'_l} \quad , \quad \forall\gamma',\tilde{\gamma}'\in T'_j\cap[d]^l$$

Example 5.2.19. Consider any d-tuple x in the sense above and again the partition

$p =$  .

Recall the decomposition of labellings induced by p onto sets T_i and T'_i as in Example 5.2.4. In virtue of the cases (i)-(iii) in Definition 5.2.17, the relations $\mathcal{R}_p^{Sp}(x)$ read as:

(i) $$\left(\sum_{t=1}^N x_{tj_1}x_{tj_1}\right)x^*_{ij_2} = \left(\sum_{t'=1}^N x^*_{t'j_3}\right)x^*_{ij_2}x_{ij_2} \qquad \forall i\in[N], j_1,j_2,j_3\in[d]$$

(ii) $$\left(\sum_{t=1}^N x_{tj_1}x_{tj_2}\right)x^*_{ij_3} = 0 \qquad \forall i\in[N], j_1,j_2,j_3\in[d], j_1\neq j_2$$

(iii) $$\left(\sum_{t'=1}^N x^*_{t'j_1}\right)x^*_{ij_2}x_{ij_3} = 0 \qquad \forall i\in[N], j_1,j_2,j_3\in[d], j_2\neq j_3$$

Example 5.2.20. To see how the number d of vectors affects the relations $\mathcal{R}_p^{Sp}(x)$, consider the partition $p=$ . Whilst for $d=1$ it only holds

$$\sum_{t'=1}^N x_{t'1}x^*_{t'1} = \mathbb{1}$$

we have for $d\geq 2$

$$\sum_{t'=1}^N x_{t'j_1}x^*_{t'j_2} = \delta_{j_1j_2} \qquad , \qquad \forall j_1,j_2\in[d].$$

Definition 5.2.21. Let $d, N \in \mathbb{N}$ with $d \leq N$ and $x := (x_{ij})$ be a d-tuple of vectors of generators. Let $\Pi \supseteq \{$ , , , $\}$ be a set of partitions. Then we define the universal C^*-algebra

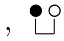
$$C\big(X_{N,d}(\Pi)\big) := C^*(x_{11}, \ldots, x_{Nd} \mid \forall p \in \Pi : \text{ The relations } \mathcal{R}_p^{Sp}(x) \text{ hold. })$$

and call it the *non-commutative functions* on the *partition quantum space (PQS)* $X_{N,d}(\Pi)$ *of d vectors.*

This definiton extends Definition 5.2.10.

Remark 5.2.22. Note that the definition of quantum space relations $\mathcal{R}_p^{Sp}(x)$, where x is a d-tuple of vectors, Definition 5.2.17), reduces in the case $d=1$ to the quantum space relations $\mathcal{R}_p^{Sp}(x)$ as defined in the case of one vector, see Definition 5.2.8; hence, both definitions of quantum space relations $\mathcal{R}_p^{Sp}(x)$ are compatible.
Evidently, also the notions $\mathcal{R}_p^{Sp}(x)$ from Definition 5.2.17 and $\mathcal{R}_p^{Sp}(u_G)$ from Definition 5.2.13 are compatible if we consider the matrix u_G as an N-tuple of vectors.
Summarizing, we can see all three situations as special cases of one definition for $\mathcal{R}_p^{Sp}(\cdot)$, but note that the associated relations depend on the number of columns in the argument.

By Lemma 5.2.15 and Remark 5.2.16 we have in particular the analogue of Theorem 5.2.12 in the case of d vectors.

Theorem 5.2.23. *Let $d, N \in \mathbb{N}$, $d \leq N$ and $\Pi \supseteq \{$ , , , $\}$ be a set of partitions. Then the map*

$$\phi : x_{ij} \mapsto u_{ij} \quad , \quad 1 \leq i \leq N,\ 1 \leq j \leq d$$

defines a *-homomorphism from $C\big(X_{N,d}(\Pi)\big)$ to $C\big(G_N(\Pi)\big)$.

Remark 5.2.24. For every easy quantum group, permutations on rows and/or columns as well as the mapping $u_{ij} \mapsto u_{ji}$ define *-isomorphisms on it. Therefore, we actually have a lot of freedom where to map each x_{ij}. Namely, for every two permutations $\sigma_1, \sigma_2 \in S_N$ we have that

$$\begin{aligned} \phi_1 &: x_{ij} \mapsto u_{\sigma_1(i)\sigma_2(j)} \\ \phi_2 &: x_{ij} \mapsto u_{\sigma_2(j)\sigma_1(i)} \end{aligned}$$

both define alternatives to the *-homomorphism ϕ in Theorem 5.2.23.

Remark 5.2.25. Theorem 5.2.23 is the precise version of what we said in the introductory part of this section. Partition quantum spaces $X_{N,d}(\Pi)\big)$ are inspired by " d columns spaces" of easy quantum groups in the following sense: For $p \in \Pi$ the relations $\mathcal{R}_p^{Sp}(u_G)$ are fulfilled in $C\big(G_N(\Pi)\big)$ by Lemma 5.2.15 and the relations $\mathcal{R}_p^{Sp}(x)$ hold in $X_{N,d}(\Pi)$ by definition. We note again that the definition of $X_{N,d}(\Pi)$, however, does not depend on $G_N(\Pi)$.

Remark 5.2.26. We could define PQSs without requiring $\{\sqcap_{\circ\bullet}, \sqcap_{\bullet\circ}, \sqcup^{\circ\bullet}, \sqcup^{\bullet\circ}\} \subseteq \Pi$. However the existence of $C\big(X_{N,d}(\Pi)\big)$ would not be guaranteed, see Remark 5.2.11.

In any case, for our purposes we only need to consider PQSs with $\{\sqcap_{\circ\bullet}, \sqcap_{\bullet\circ}, \sqcup^{\circ\bullet}, \sqcup^{\bullet\circ}\} \subseteq \Pi$. See [Web17b] for more on issues related to $\{\sqcap_{\circ\bullet}, \sqcap_{\bullet\circ}, \sqcup^{\circ\bullet}, \sqcup^{\bullet\circ}\} \not\subseteq \Pi$.
In principle, also $d > N$ is possible, but as some of our results only hold for $d \leq N$, we excluded all other situations in this work.

5.3 Easy quantum groups act on partition quantum spaces

For integers $d \leq N$ and a set of partitions $\Pi \supseteq \{\sqcap_{\circ\bullet}, \sqcap_{\bullet\circ}, \sqcup^{\circ\bullet}, \sqcup^{\bullet\circ}\}$ we prove in this section that the corresponding easy quantum group always acts on the corresponding PQS of d vectors. In this sense, observation (a) from Section 5.1 turns out to be a true statement in the generalized situation of any easy quantum groups $G_N(\Pi)$ and any corresponding PQS $X_{N,d}(\Pi)$.

Theorem 5.3.1. *Let $d, N \in \mathbb{N}$, $d \leq N$ and $\Pi \supseteq \{\sqcap_{\circ\bullet}, \sqcap_{\bullet\circ}, \sqcup^{\circ\bullet}, \sqcup^{\bullet\circ}\}$ be a set of partitions. Let $u_G := (u_{ij})$ be the matrix of generators associated to the easy quantum group $G_N(\Pi)$ and $x := (x_{ij})$ be the d vectors of generators associated to the partition quantum space $X_{N,d}(\Pi)$. In the sense of Definition 2.7.8 the following holds:*

(i) The map

$$\alpha : x_{ij} \mapsto \sum_{k=1}^{N} u_{ik} \otimes x_{kj} \qquad \text{for } 1 \leq i \leq n,\ 1 \leq j \leq d$$

defines a faithful left matrix-vector action $G_N(\Pi) \curvearrowright X_{N,d}(\Pi)$.

(ii) The map

$$\beta : x_{ij} \mapsto \sum_{k=1}^{N} u_{ki} \otimes x_{kj} \qquad \text{for } 1 \leq i \leq n,\ 1 \leq j \leq d$$

defines a faithful right matrix-vector action $X_{N,d}(\Pi) \curvearrowleft G_N(\Pi)$.

Proof. We have to prove that α and β are well-defined unital *-homomorphisms. We only consider α, the proof for β can be done analogously. By the universal property of $C\big(X_{N,d}(\Pi)\big)$ it suffices to prove that the relations (i)-(iii) from Definition 5.2.17 are fulfilled for $\tilde{x}_{ij} := \alpha(x_{ij}) = \sum_{s=1}^{N} u_{is} \otimes x_{sj}$. This will show in particular unitality of α.

We start with relation (i) from Definition 5.2.17. We use Notation 5.2.1 and fix $1 \le i, j \le r$, $\gamma \in T_j \cap [d]^k$ and $\gamma' \in T'_j \cap [d]^l$. By definition we have for all $1 \le m \le r$

$$\sum_{s \in T_m} x_{s_1\gamma_1}^{\omega_1} \cdots x_{s_k\gamma_k}^{\omega_k} = \sum_{s' \in T'_m} x_{s'_1\gamma'_1}^{\omega'_1} \cdots x_{s'_l\gamma'_l}^{\omega'_l} \tag{5.3.1}$$

and due to Remarks 5.2.16 and 5.2.14 we have that

$$c_m := \sum_{t \in T_i} u_{t_1s_1}^{\omega_1} \cdots u_{t_ks_k}^{\omega_k} = \sum_{t' \in T'_i} u_{t'_1s'_1}^{\omega'_1} \cdots u_{t'_ls'_l}^{\omega'_l} \tag{5.3.2}$$

only depends on $m \in \{0, 1, \ldots, r\}$ but not on $(s, s') \in T_m \times T'_m$. In particular, $c_0 = 0$. Using Equations 5.3.1 and 5.3.2 we infer:

$$\begin{aligned}
\sum_{t \in T_i} (\tilde{x}_{t_1\gamma_1})^{\omega_1} \cdots (\tilde{x}_{t_k\gamma_k})^{\omega_k} &= \sum_{s \in [N]^k} \sum_{t \in T_i} u_{t_1s_1}^{\omega_1} \cdots u_{t_ks_k}^{\omega_k} \otimes x_{s_1\gamma_1}^{\omega_1} \cdots x_{s_k\gamma_k}^{\omega_k} \\
&= \sum_{m=0}^{r} \sum_{s \in T_m} \left(\sum_{t \in T_i} u_{t_1s_1}^{\omega_1} \cdots u_{t_ks_k}^{\omega_k} \right) \otimes x_{s_1\gamma_1}^{\omega_1} \cdots x_{s_k\gamma_k}^{\omega_k} \\
&= 0 + \sum_{m=1}^{r} \sum_{s \in T_m} c_m \otimes x_{s_1\gamma_1}^{\omega_1} \cdots x_{s_k\gamma_k}^{\omega_k} \\
&= 0 + \sum_{m=1}^{r} c_m \otimes \left(\sum_{s \in T_m} x_{s_1\gamma_1}^{\omega_1} \cdots x_{s_k\gamma_k}^{\omega_k} \right) \\
&= 0 + \sum_{m=1}^{r} c_m \otimes \left(\sum_{s' \in T'_m} x_{s'_1\gamma'_1}^{\omega'_1} \cdots x_{s'_l\gamma'_l}^{\omega'_l} \right) \\
&= 0 + \sum_{m=1}^{r} \sum_{s' \in T'_m} c_m \otimes x_{s'_1\gamma'_1}^{\omega'_1} \cdots x_{s'_l\gamma'_l}^{\omega'_l} \\
&= 0 + \sum_{m=1}^{r} \sum_{s' \in T'_m} \sum_{t' \in T'_i} u_{t'_1s'_1}^{\omega'_1} \cdots u_{t'_ls'_l}^{\omega'_l} \otimes x_{s'_1\gamma'_1}^{\omega'_1} \cdots x_{s'_l\gamma'_l}^{\omega'_l} \\
&= \sum_{m=0}^{r} \sum_{s' \in T'_m} \sum_{t' \in T'_i} u_{t'_1s'_1}^{\omega'_1} \cdots u_{t'_ls'_l}^{\omega'_l} \otimes x_{s'_1\gamma'_1}^{\omega'_1} \cdots x_{s'_l\gamma'_l}^{\omega'_l} \\
&= \sum_{t \in T'_i} \left(\tilde{x}_{t'_1\gamma'_1}\right)^{\omega'_1} \cdots \left(\tilde{x}_{t'_l\gamma'_l}\right)^{\omega'_l}.
\end{aligned}$$

For relation (ii) in Definition 5.2.17 we similarly compute for all $\gamma \in T_0$

$$\sum_{t\in T_i} (\tilde{x}_{t_1\gamma_1})^{\omega_1} \cdots (\tilde{x}_{t_k\gamma_k})^{\omega_k} = 0 + \sum_{m=1}^{r} c_m \otimes \underbrace{\sum_{s\in T_m} x_{s_1\gamma_1}^{\omega_1} \cdots x_{s_k\gamma_k}^{\omega_k}}_{=0} = 0.$$

Analogously we prove (iii): For all $\gamma' \in T_0'$ we have

$$\sum_{t\in T_i'} \left(\tilde{x}_{t_1'\gamma_1'}\right)^{\omega_1'} \cdots \left(\tilde{x}_{t_l'\gamma_l'}\right)^{\omega_l'} = 0 + \sum_{m=1}^{r} c_m \otimes \underbrace{\sum_{s'\in T_m'} x_{s_1'\gamma_1'}^{\omega_1'} \cdots x_{s_l'\gamma_l'}^{\omega_l'}}_{=0} = 0.$$

□

5.4 Quantum symmetry groups of partition quantum spaces

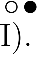
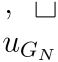
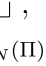
In the following we want to take a closer look at the connection between a PQS and its quantum symmetry group. We recall the definition of quantum symmetry groups in our setting, compare Section 2.7.3.

Definition 5.4.1. Let $X_{N,d}(\Pi)$ be a PQS of d vectors. We call a compact matrix quantum group $G = (C(G), u_G)$ the *quantum symmetry group of* $X_{N,d}(\Pi)$ and write $G = \mathrm{QSymG}\left(X_{N,d}(\Pi)\right)$, if the following are fulfilled:

(i) u_G is an $N \times N$-matrix of generators.

(ii) There are faithful left and right matrix-vector actions α and β of G on $X_{N,d}(\Pi)$ in the sense of Definition 2.7.8 and

(iii) G is maximal with this property, i.e. all G' fulfilling the above satisfy $G' \subseteq G$, compare Definition 2.2.5.

Notation 5.4.2. For the rest of this chapter we fix the following situation/notation: Let $d, N \in \mathbb{N}$ and $d \leq N$. Let Π be a set of partitions containing the set of all mixed coloured pair partitions $\{$ , , ,  $\}$. Let $G = \big(C(G), v_G\big)$ be the quantum symmetry group of $X_{N,d}(\Pi)$. Let $u_{G_N(\Pi)} = (u_{ij})$ be the matrix of generators associated to the easy quantum group $G_N(\Pi)$. For a fixed partition $p \in P(\omega, \omega') \subseteq P(k,l)$ we always use the decompositions $[N]^k = T_0 \,\dot{\cup}\, \ldots \,\dot{\cup}\, T_r$ and $[N]^l = T_0' \,\dot{\cup}\, \ldots \,\dot{\cup}\, T_r'$ as in Notation 5.2.1.

As G is the quantum symmetry group of $X_{N,d}(\Pi)$, we know that *-homomorphisms α and β as decribed in Theorem 5.3.1 exist. The question is now: What does this tell us about $C(G)$ and how may we explicitly compute relations holding in $C(G)$?

Example 5.4.3. Consider $\Pi = \{$ , , , $\}$. In the PQS $X_{N,1}(\Pi)$ we have for example $\mathcal{R}^{Sp}$(x), i.e. $\sum_i x_{i1}x_{i1}^* = \mathbb{1}$. Applying α to this equation gives

$$\sum_{i=1}^{N} \sum_{k_1,k_2=1}^{N} v_{ik_1}v_{ik_2}^* \otimes x_{k_11}x_{k_21}^* = \mathbb{1}. \tag{5.4.1}$$

This encodes some information about the v_{ij}, but it is not always easy to read, as the C^*-algebra $C\big(X_{N,d}(\Pi)\big)$ might be quite complicated.

Notation 5.4.4. Consider any permutation matrix $\sigma \in S_N$ and the following chain of unital *-homomorphisms:

$$\begin{array}{rcccccccc} \mathrm{ev}_\sigma\colon & C\big(X_{N,d}(\Pi)\big) & \xrightarrow{\phi_1} & C\big(G_N(\Pi)\big) & \xrightarrow{\phi_2} & C\big(S_N\big) & \xrightarrow{\phi_3} & \mathbb{C} \\ & x_{ij} & \longmapsto & u_{ij} & \longmapsto & \tilde{u}_{ij} & \longmapsto & \delta_{i\sigma(j)} \end{array}$$

Here, $(\tilde{u}_{ij}) = u_{S_N}$ is the matrix of generators corresponding to $C(S_N)$, i.e.

$$\tilde{u}_{ij} :\ C\big(S_N\big) \to \mathbb{C};\ \tilde{\sigma} \mapsto \tilde{u}_{ij}(\tilde{\sigma}) = \tilde{\sigma}_{ij} = \delta_{i\tilde{\sigma}(j)}$$

is the coordinate function for the entry (i,j). Recall that the $\tilde{u}_{ij}$ are projections summing up to one in each row and column, i.e. $\sum_{k=1}^N \tilde{u}_{ik} = \sum_{k=1}^N \tilde{u}_{ki} = \mathbb{1}$ for all $i \in [N]$. The existence of ϕ_1 is by Theorem 5.2.23. The map ϕ_2 exists as described as we have $S_N \subseteq G_N(\Pi)$ for every easy quantum group $G_N(\Pi)$, see Section 2.6 or [BS09, Web16, Web17a]. For ϕ_3 observe, that the point evaluation $f \mapsto f(\sigma)$ is a character on $C(S_N)$.

Applying $\mathbb{1} \otimes \mathrm{ev}_\sigma$ for some σ with $\sigma(1) = k$ to Equation 5.4.1 results in

$$\mathbb{1} = \sum_{k_1,k_2=1}^{N} \sum_{i=1}^{N} v_{ik}v_{ik}^* \otimes \delta_{k_1\sigma(1)}\delta_{k_2\sigma(1)} = \sum_{i=1}^{N} v_{ik}v_{ik}^* \otimes 1.$$

As $k \in [N]$ was arbitrary, we proved the relations

$$\sum_i v_{ik}v_{ik}^* = \mathbb{1} \qquad \forall k.$$

With the strategy presented above, we can prove (see Theorem 5.4.6) that the relations $\mathcal{R}_p^{Sp}(x)$ hold for any choice of d columns of (v_{ij}). We first present a preparatory result, keeping the consecutive theorem and its proof compact.

Lemma 5.4.5. *In the situation of Notation 5.4.2 it holds that for any* $p \in \Pi$ *the relations* $\mathcal{R}^{Sp}_p(v)$ *of Definition 5.2.17 hold for the first* d *columns of* v, *i.e.:*

(i) $$\sum_{t\in T_i} v^{\omega_1}_{t_1\gamma_1}\cdots v^{\omega_k}_{t_k\gamma_k} = \sum_{t'\in T'_i} v^{\omega'_1}_{t'_1\gamma'_1}\cdots v^{\omega'_l}_{t'_l\gamma'_l} \quad , \quad 1\leq i,j\leq r,\ \gamma\in T_j\cap[d]^k, \gamma'\in T'_j\cap[d]^l.$$

(ii) $$\sum_{t\in T_i} v^{\omega_1}_{t_1\gamma_1}\cdots v^{\omega_k}_{t_k\gamma_k} = 0 \quad , \quad 1\leq i\leq r, \gamma\in T_0\cap[d]^k.$$

(iii) $$\sum_{t'\in T'_i} v^{\omega'_1}_{t'_1\gamma'_1}\cdots v^{\omega'_l}_{t'_l\gamma'_l} = 0 \quad , \quad 1\leq i\leq r, \gamma\in T'_0\cap[d]^l.$$

Proof. We start with the relations in (i). In virtue of Definition 5.2.17 we have for i, j, γ, γ' as defined above

$$\sum_{t\in T_i} x^{\omega_1}_{t_1\gamma_1}\cdots x^{\omega_k}_{t_k\gamma_k} = \sum_{t'\in T'_i} x^{\omega'_1}_{t'_1\gamma'_1}\cdots x^{\omega'_l}_{t'_l\gamma'_l}.$$

We consider $\sigma\in S_N$ and apply $(\mathbb{1}\otimes \mathrm{ev}_\sigma)\circ\alpha$ to receive

$$\sum_{t\in T_i} v^{\omega_1}_{t_1\sigma(\gamma_1)}\cdots v^{\omega_k}_{t_k\sigma(\gamma_k)}\otimes 1 = \sum_{t'\in T'_i} v^{\omega'_1}_{t'_1\sigma(\gamma'_1)}\cdots v^{\omega'_l}_{t'_l\sigma(\gamma'_l)}\otimes 1. \tag{5.4.2}$$

For $\sigma = \mathrm{id}$ this is our claim.
We pass now to the relations (ii) and (iii). Starting with

$$\sum_{t\in T_i} x^{\omega_1}_{t_1\gamma_1}\cdots x^{\omega_k}_{t_k\gamma_k} = 0 \qquad \text{and} \qquad \sum_{t'\in T'_i} x^{\omega'_1}_{t'_1\gamma'_1}\cdots x^{\omega'_l}_{t'_l\gamma'_l} = 0$$

we deduce similarly to the case of Equation (i) with the help of $(\mathbb{1}\otimes \mathrm{ev}_\sigma)\circ\alpha$:

$$\sum_{t\in T_i} v^{\omega_1}_{t_1\sigma(\gamma_1)}\cdots v^{\omega_k}_{t_k\sigma(\gamma_k)}\otimes 1 = 0 \qquad \text{and} \qquad \sum_{t'\in T'_i} x^{\omega'_1}_{t_1\sigma(\gamma'_1)}\cdots x^{\omega'_l}_{t'_l\sigma(\gamma'_l)}\otimes 1 = 0 \tag{5.4.3}$$

In both cases this includes (with the choice $\sigma = \mathrm{id}$) the desired relation. □

Without any further work we can now prove the existence of *-homomorphisms from $C\big(X_{N,d}(\Pi)\big)$ to $C(G)$, where G is the quantum symmetry group of $X_{N,d}(\Pi)$.

Theorem 5.4.6. *Consider the situation of Notation 5.4.2. For any* $\sigma_1,\sigma_2\in S_N$ *the following maps define* **-homomorphisms from* $C\big(X_{N,d}(\Pi)\big)$ *to* $C(G)$:

$$\phi:\quad x_{ij}\mapsto v_{\sigma_1(i)\sigma_2(j)} \qquad ; \qquad \phi^T:\quad x_{ij}\mapsto v_{\sigma_2(j)\sigma_1(i)}$$

In particular (for $\sigma_1 = \sigma_2 = \mathrm{id}$*) the map*

$$\phi : \quad x_{ij} \mapsto v_{ij}$$

defines a **-homomorphism from* $C\big(X_{N,d}(\Pi)\big)$ *to* $C(G)$*.*

Proof. By the universal property of $C\big(X_{N,d}(\Pi)\big)$ we only have to show that the relations $\big(\mathcal{R}_p^{Sp}(x)\big)_{p\in\Pi}$ are fulfilled when we replace every x_{ij} by $v_{\sigma_1(i)\sigma_2(j)}$ or $v_{\sigma_2(j)\sigma_1(i)}$, respectively. Thanks to our results so far there is not much left to do.
Consider first the map ϕ. If $\sigma_1 = \mathrm{id}$, then ϕ exists, because we have with Equations 5.4.2 and 5.4.3 from Lemma 5.4.5 all required relations for the v_{ij} at hand. We can additionally have $\sigma_1 \neq \mathrm{id}$ as the mapping $x_{ij} \mapsto x_{\sigma_1(i)j}$ defines a *-isomorphism on $C\big(X_{N,d}(\Pi)\big)$: Considering the quantum space relations $\mathcal{R}_p^{Sp}(x)$ from Definition 5.2.17, we see that applying $i \mapsto \sigma(i)$ only permutes summands.
The existence of ϕ^T is proved by starting again at Lemma 5.4.5 and replacing α with β. We obtain all the ensuing results with v_{ij} replaced by v_{ji}. □

Remark 5.4.7. Having a closer look at Equation 5.4.2, one can even deduce that for arbitrary $\sigma, \sigma' \in S_N$ we have

$$\sum_{t\in T_i} v_{t_1\sigma(\gamma_1)}^{\omega_1} \cdots v_{t_k\sigma(\gamma_k)}^{\omega_k} = \sum_{t'\in T_i'} v_{t_1'\sigma'(\gamma_1')}^{\omega_1'} \cdots v_{t_l'\sigma'(\gamma_l')}^{\omega_l'},$$

as long as σ and σ' coincide on the through-block labelings of γ (or γ', which is the same condition). In this sense we can ignore in Lemma 5.4.5, Equation (i) the restrictions $\gamma \in [d]^k$ and $\gamma' \in [d]^l$ as long as each tuple has at most d different entries. For the relations (ii) and (iii) this follows directly from the Equations 5.4.3.

In the case $d = N$ we can now prove that observation (d) from Section 1.1 stays true in the situation of easy quantum groups $G_N(\Pi)$ and PQSs $X_{N,N}(\Pi)$.

Corollary 5.4.8. *Consider the situation of Notation 5.4.2. If* $d = N$*, then* $G = G_N(\Pi)$*, i.e.* $G_N(\Pi)$ *is the quantum symmetry group of* $X_{N,N}(\Pi)$*.*

Proof. Due to Theorem 5.3.1 we already know $G_N(\Pi) \subseteq G$ so there is only "$\supseteq$" left to prove, i.e. it remains to show that the quantum group relations $\big(\mathcal{R}_p^{Gr}(v_G)\big)_{p\in\Pi}$ are fulfilled. In the case $d = N$ Theorem 5.4.6 reads as

$$\mathcal{R}_p^{Sp}(x) \Rightarrow \mathcal{R}_p^{Sp}(v_G), \mathcal{R}_p^{Sp}(v_G^T)$$

and by part (2) of Lemma 5.2.15 the quantum space relations for v_G and v_G^T together imply $\mathcal{R}_p^{Gr}(v_G)$. □

There is one further consequence of the theorem above, coming from the fact, that $G_N(\Pi)$ only depends on the category $\langle\Pi\rangle$ and not Π itself.

Corollary 5.4.9. *In the situation of Notation 5.4.2, the quantum symmetry group of $X_{N,N}(\Pi)$ only depends on $\langle\Pi\rangle$, not on Π itself.*

Remark 5.4.10. Having a closer look at the proof of Corollary 5.4.8, we see that it is just an application of the relations proved in Lemma 5.4.5. Of course it heavily uses the fact $d = N$ in the way that $\gamma \in [d]^k$ and $\gamma' \in [d]^l$ are actually no further restrictions. In the following we will consider the situation as in Corollary 5.4.8, but with $d < N$. In the sense of Lemma 5.4.5 and Remark 5.4.7 the only thing to prove is, that the equations there are fulfilled for $\gamma \in T_j$ and $\gamma' \in T'_j$, potentially each with more than d different entries. Up to now, we are only able to do this for concretely given sets Π, so further results will depend not only on $\langle\Pi\rangle$, but on Π itself.

5.5 The free case

In this section we assume that Π defines a free easy quantum group, i.e. Π is a set of non-crossing partitions. In the sense of Remark 5.4.10 we want to find situations where $G_N(\Pi)$ is the quantum symmetry group of $X_{N,d}(\Pi)$ for some $d < N$. Note that this implies the same result for d' with $d \le d' \le N$, so (for fixed Π) we want to show this for d as small as possible.

We first gather some results about relations on the v_{ij} (see Notation 5.4.2) which are implied by special partitions $p \in \Pi$ (and special choices of Π).

Lemma 5.5.1. *Consider the situation as in Notation 5.4.2 with $d \le N$ arbitrary. Assume $\circ\bullet\circ\bullet \in \Pi$. Then the relations $\mathcal{R}^{Gr}_{\circ\bullet\circ\bullet}(v_G)$ are fulfilled, i.e.*

$$\sum_{t'=1}^{N} v_{\gamma'_1 t'} v^*_{\gamma'_2 t'} v_{\gamma'_3 t'} v^*_{\gamma'_4 t'} = \delta_{\gamma'_1 \gamma'_2} \delta_{\gamma'_2 \gamma'_3} \delta_{\gamma'_3 \gamma'_4} \qquad \forall\ \gamma' \in [N]^4.$$

The analogous result holds for $\circ\circ\bullet\bullet \in \Pi$ and the relations $\mathcal{R}^{Gr}_{\circ\circ\bullet\bullet}(v_G)$, i.e.

$$\sum_{t'=1}^{N} v_{\gamma'_1 t'} v_{\gamma'_2 t'} v^*_{\gamma'_3 t'} v^*_{\gamma'_4 t'} = \delta_{\gamma'_1 \gamma'_2} \delta_{\gamma'_2 \gamma'_3} \delta_{\gamma'_3 \gamma'_4} \qquad \forall\ \gamma' \in [N]^4.$$

Proof. We prove the claim for $d = 1$, then it holds for all $d \le N$. We start with the partition $\circ\bullet\circ\bullet$. Together with $\circ\bullet \in \Pi$ we already know by Theorem 5.4.6 that

$$\sum_{t'=1}^{N} \underbrace{v_{st'} v^*_{st'} v_{st'} v^*_{st'}}_{\le v_{st'} v^*_{st'}} = \mathbb{1} = \sum_{t'=1}^{N} v_{st'} v^*_{st'} \quad , \quad \forall s \in [N].$$

However, this is only possible if $v_{st'}v^*_{st'}v_{st'}v^*_{st'} = v_{st'}v^*_{st'}$ for all $s, t' \in [N]$. Hence, all generators $v_{st'}$ are partial isometries. Now, we also know $\sum\limits_s v^*_{st'}v_{st'} = \sum\limits_s v_{st'}v^*_{st'} = \mathbb{1}$ by Theorem 5.4.6. Hence $v^*_{s_2t'}v_{s_1t'} = v_{s_2t'}v^*_{s_1t'} = 0$ for $s_2 \neq s_1$, since projections summing up to one are mutually orthogonal. This implies

$$\sum_{t'=1}^{N} v_{\gamma'_1t'}v^*_{\gamma'_2t'}v_{\gamma'_3t'}v^*_{\gamma'_4t'} = 0$$

for all $\gamma' \in T'_0$, which is the only relation in $\mathcal{R}^{Gr}_{\overset{\sqcap\sqcap\sqcap}{\circ\bullet\circ\bullet}}(v_G)$ not already covered by the results in Theorem 5.4.6.

For the situation $\overset{\sqcap\sqcap\sqcap}{\circ\circ\bullet\bullet} \in \Pi$ we may assume that $C(G)$ is represented faithfully on some Hilbert space H. As in the previous considerations, we get $v_{st'}v_{st'}v^*_{st'}v^*_{st'} = v_{st'}v^*_{st'}$. Thus, for any $w \in H$

$$\|v^*_{s_1t'}v^*_{s_1t'}w\|^2 = \langle v^*_{s_1t'}w, v^*_{s_1t'}w\rangle = \langle \sum_{s_2=1}^{N} v_{s_2t'}v^*_{s_2t'}v^*_{s_1t'}w, v^*_{s_1t'}w\rangle = \sum_{t_2=1}^{N} \|v^*_{s_2t'}v^*_{s_1t'}w\|^2,$$

which implies $v^*_{s_2t'}v^*_{s_1t'} = v_{s_2t'}v_{s_1t'} = 0$ for $s_1 \neq s_2$. But then we also have

$$v_{s_2t'}v^*_{s_1t'} = \sum_{s=1}^{N} v_{s_2t'}v_{st'}v^*_{st'}v^*_{s_1t'} = 0$$

and likewise $v^*_{s_2t'}v_{s_1t'} = 0$ for all t' and $s_2 \neq s_1$. With this result it holds

$$\sum_{t'=1}^{N} v_{\gamma'_1t'}v_{\gamma'_2t'}v^*_{\gamma'_3t'}v^*_{\gamma'_4t'} = 0$$

for all $\gamma' \in T'_0$, which is the only relation in $R'_{\overset{\sqcap\sqcap\sqcap}{\circ\circ\bullet\bullet}}(v_G)$ to be proved. $\square$

Remark 5.5.2. Note that we were able to deduce $v_{s_2t'}v^*_{s_1t'} = v^*_{s_2t'}v_{s_1t'} = 0$ for $s_2 \neq s_1$ in the situation of both four-block partitions. As we can repeat the whole proof with v_{ij} replaced by v_{ji}, see Theorem 5.4.6, we also have $v_{t's_2}v^*_{t's_1} = v^*_{t's_2}v_{t's_1} = 0$. So the non-diagonal entries of $v_Gv^*_G$, $v^*_Gv_G$, $\bar{v}_G\bar{v}^*_G$ and $\bar{v}^*_G\bar{v}_G$ vanish. This is insofar a nontrivial result, as Theorem 5.4.6 together with the fact $\{\overset{\sqcap}{\circ\bullet}, \overset{\sqcap}{\bullet\circ}, \underset{\sqcup}{\circ\bullet}, \underset{\sqcup}{\bullet\circ}\} \in \Pi$ only implies that the diagonals are equal to $\mathbb{1}$. At first side we did not know anything about the off-diagonals, i.e. it was unclear if v_G and v^T_G are unitaries.

Note further, that if we additionally assume $\overset{\sqcap}{\circ\circ} \in \Pi$, then also $\mathcal{R}^{Gr}_{\overset{\sqcap}{\circ\circ}}(v_G)$ are fulfilled, i.e. v_G is orthogonal, $v^T_Gv_G = v_Gv^T_G = \mathbb{1}$: The diagonals are $\mathbb{1}$ by Theorem 5.4.6 and for the off-diagonals we can compute for $k_1 \neq k_2$

$$\sum_{i=1}^{N} v_{ik_1} v_{ik_2} = \sum_{i=1}^{N} \sum_{k=1}^{N} v_{ik_1} v_{ik}^* v_{ik}^* v_{ik_2} = 0$$

and likewise for the off-diagonals of $v_G v_G^T$.

The next four lemmata are just preparatory results. Recall (see [BS09, Web16, Web17a]) that the easy quantum groups O_N^+ and B_N^+ can be associated to the following orthogonal $N \times N$-matrices of generators and partitions:

$O_N^+ : \quad u_{O_N^+} = (o_{ij}) \quad ; \qquad \Pi = \{\text{non-crossing partitions with blocks of size } 2\} =: \mathcal{NC}_2$

$B_N^+ : \quad u_{B_N^+} = (b_{ij}) \quad ; \qquad \Pi = \{\text{non-crossing partitions with blocks of size 1 or 2}\}$

Lemma 5.5.3. *Consider* $N = 2$ *and the easy quantum group* $O_N^+ = O_2^+$. *The entries of the corepresentation matrix*

$$u_{O_2^+}^{\otimes 2} = \sum_{m,n,s,t=1}^{2} o_{mn} o_{st} \otimes E_{mn} \otimes E_{st}$$

linearly generate a vector space W *of dimension 10.*

Proof. One can deduce these results from the fusion rules for O_N^+ as described in [FW14] and [Fre14]. We use the notation introduced there.
By Proposition 3.7 in [Fre14] it holds

$$u_{o_2^+}^{\otimes 2} \simeq u_{|\,|} \oplus u_{\substack{\sqcup \\ \sqcap}} \tag{5.5.1}$$

in the sense that there is an invertible matrix $S \in M_4(\mathbb{C})$ such that $(S \otimes \mathbb{1}) u_{o_2^+}^{\otimes 2} (S^{-1} \otimes \mathbb{1})$ is of the form

$$\begin{pmatrix} u_{|\,|} & 0 \\ 0 & u_{\substack{\sqcup \\ \sqcap}} \end{pmatrix}$$

Note that in [Fre14] it is required $N \geq 4$ but this is just a condition ensuring that all the maps T_p are linearly independent. With respect to the special situation here we can omit this condition as the maps

$$T_{|\,|}(e_i \otimes e_j) = e_i \otimes e_j \qquad \text{and} \qquad T_{\substack{\sqcup \\ \sqcap}}(e_i \otimes e_j) = \delta_{ij} \sum_{k=1}^{2} e_k \otimes e_k$$

are clearly linearly independent. Note further that both $T_{|\,|}$ and $\frac{1}{2} T_{\substack{\sqcup \\ \sqcap}}$ are projections

and $\frac{1}{2}T_{\substack{\sqcup\\\sqcap}} \leq T_{|\,|}$. We remark that in [Fre14] and [FW14] the map $\frac{1}{2}T_{\substack{\sqcup\\\sqcap}}$ above is just called $T_{\substack{\sqcup\\\sqcap}}$ as the definition of the linear maps T_p in [FW14] and [Fre14] include suitable scaling factors.
By Definition 3.2 in [Fre14] the corepresentation matrix $u_{|\,|}$ is a matrix over $C(O_2^+)$ of size

$$\mathrm{rank}(T_{|\,|} - \frac{1}{2}T_{\substack{\sqcup\\\sqcap}}) = \mathrm{rank}(T_{|\,|}) - \mathrm{rank}(T_{\substack{\sqcup\\\sqcap}})$$

and $u_{\substack{\sqcup\\\sqcap}}$ is a matrix over $C(O_2^+)$ of size equal to

$$\mathrm{rank}(\frac{1}{2}T_{\substack{\sqcup\\\sqcap}}) = \mathrm{rank}(T_{\substack{\sqcup\\\sqcap}}).$$

By Proposition 2.16 in [FW14] the ranks of the maps T_p as above just depend on the considered $N \in \mathbb{N}$ and the number of through-blocks $tb(p)$:

$$\mathrm{rank}(T_{\substack{\sqcup\\\sqcap}}) = N^{tb(\substack{\sqcup\\\sqcap})} = 2^0 = 1 \qquad , \qquad \mathrm{rank}(T_{|\,|}) = N^{tb(|\,|)} = 2^2 = 4$$

We now combine the results above in the following way: By equation 5.5.1 the vector space W is also linearly generated by the entries of $Su_{O_2^+}^{\otimes 2}S^{-1}$, i.e. by the entries of $u_{|\,|}$ and $u_{\substack{\sqcup\\\sqcap}}$. The sizes of the matrices $u_{|\,|}$ and $u_{\substack{\sqcup\\\sqcap}}$ can be computed as above and according to the remarks following Definition 3.2 in [Fre14] these two corepresentation matrices are irreducible so its entries are linearly independent, see for example [Tim08, Proposition 5.3.8 (iv)]. Hence, we deduce

$$\begin{aligned}\dim(W) &= \big(\mathrm{rank}(T_{|\,|}) - \mathrm{rank}(T_{\substack{\sqcup\\\sqcap}})\big)^2 + \big(\mathrm{rank}(T_{\substack{\sqcup\\\sqcap}})\big)^2 \\ &= 10.\end{aligned}$$

□

Lemma 5.5.4. *Consider the matrix $u_{O_2^+} = (o_{ij})$ of the canonical generators of $C(O_2^+)$. Then $o_{11}o_{21}$ and $o_{21}o_{11}$ are linearly independent.*

Proof. Using the orthogonality of $u_{O_2^+}$, i.e. $\sum_{i=1}^2 o_{ik_1}o_{ik_2} = \sum_{i=1}^2 o_{k_1i}o_{k_2i} = \delta_{k_1k_2}$ for all $k_1, k_2 \in \{1, 2\}$, we have that the vector space W from Lemma 5.5.3 is generated by the entries of

$$\big(o_{11}o_{21} \,,\, o_{21}o_{11} \,,\, o_{12}o_{11} \,,\, o_{11}o_{12} \,,\, o_{11}o_{11} \,,\, o_{12}o_{12} \,,\, o_{11}o_{22} \,,\, o_{22}o_{11} \,,\, o_{12}o_{21} \,,\, o_{21}o_{12}\big)$$

which must be linear independent as $\dim(W) = 10$. □

The next lemma gives us a tool to trace some problems in $C(B_3^+)$ back to problems in $C\big(O_2^+\big)$.

Lemma 5.5.5. *Consider the matrix w given by*

$$w := \begin{pmatrix} \sqrt{\frac{2}{3}} & 0 & \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \end{pmatrix} \begin{pmatrix} o_{11} & o_{12} & 0 \\ o_{21} & o_{22} & 0 \\ 0 & 0 & \mathbb{1} \end{pmatrix} \begin{pmatrix} \sqrt{\frac{2}{3}} & 0 & \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \end{pmatrix}^T \in M_3\big(C(O_2^+)\big)$$

Then the map $b_{ij} \mapsto w_{ij}$ defines a *-homomorphism $C(B_3^+) \to C(O_2^+)$.

Proof. It is a straightforward computation that the entries of w fulfil the relations of the b_{ij}. See also [Rau12] or [Web13],[WG18]. □

Remark 5.5.6. Although we do not need this here, one can even show that this is an isomorphism and that we can replace the matrix we conjugate with by any other orthogonal matrix as long as we still have the last column like above. We even have this result for all pairs $\big(B_{N+1}^+, O_N^+\big)$ if we replace $\frac{1}{\sqrt{3}}$ by $\frac{1}{\sqrt{N+1}}$, see again [Rau12] or [Web13],[WG18].
Note further that this correspondence is completely analogous to the group case, i.e. when comparing B_N and O_{N-1}. The matrices in B_N are all orthogonal matrices which, considered in the canonical way as elements in $B\big(\mathbb{C}^N\big)$, fix the diagonal $\mathbb{C}(1, \ldots, 1)^T$. So they correspond to orthogonal rotations around this diagonal and these can be described by the elements of O_{N-1}.

Lemma 5.5.7. *Consider the matrix $u_{B_3^+} = (b_{ij})$ of the canonical generators of $C(B_3^+)$. Then $b_{11}b_{21} - b_{21}b_{11} \neq 0$.*

Proof. Using Lemma 5.5.5 and the notations there it suffices to show that $w_{11}w_{21} - w_{21}w_{11} \neq 0$. Writing out this difference, one computes

$$w_{11}w_{21} - w_{21}w_{11} = \frac{2}{3\sqrt{3}}(o_{11}o_{21} - o_{21}o_{11})$$

which, by Lemma 5.5.4, yields the result. □

Notation 5.5.8. One can show, that for $d \leq M \leq N$ we can always map a partition quantum space $X_{N,d}(\Pi)$ on "its shorter M-version" $X_{M,d}(\Pi)$:

$$\psi_1 : C\big(X_{N,d}(\Pi)\big) \to C\big(X_{M,d}(\Pi)\big);\ x_{ij} \mapsto \begin{cases} x'_{ij} & , j \leq M \\ 0 & , j > M \end{cases}$$

Additionally, by Theorem 5.2.23, we always have a unital *-homomorphism

$$\psi_2 : C\big(X_{M,d}(\Pi)\big) \to C\big(G_M(\Pi')\big); x_{ij} \mapsto u_{ij}$$

whenever $G_M(\Pi')$ is a subgroup of $G_M(\Pi)$. Composing ψ_1 and ψ_2 gives a unital *-homomorphism $\psi_{G_M(\Pi')} : C\big(X_{N,d}(\Pi)\big) \to C\big(G_M(\Pi')\big)$.

Lemma 5.5.9. *Consider the situation of Notation 5.4.2 with* $d \leq N$ *arbitrary. If* Π *contains only non-crossing partitions with blocks of size two, then* $\mathcal{R}_p^{Gr}(v_G)$ *is fulfilled for every* $p \in \{ \overset{\sqcap}{\circ\bullet}, \overset{\sqcap}{\bullet\circ}, \underset{\sqcup}{\circ\bullet}, \underset{\sqcup}{\bullet\circ} \}$*. If* Π *contains* $\overset{\sqcap}{\circ\circ}$*, then also* $\mathcal{R}^{Gr}_{\overset{\sqcap}{\circ\circ}}(v_G)$ *holds.*

Proof. We only prove the case $p = \underset{\sqcup}{\circ\bullet}$. The other relations may be proved similarly. For the case $\overset{\sqcap}{\circ\circ}$ we replace every appearing v_{ij}^* and x_{ij}^* by v_{ij} and x_{ij}, respectively. If $N = 1$ or $d \geq 2$ then the result follows from Theorem 5.4.6, so let $d = 1 < N$. By Theorem 5.4.6, $\underset{\sqcup}{\circ\bullet} \in \Pi$ implies that the diagonals of $\bar{v}_G^* \bar{v}_G$ are equal to $\mathbb{1}$. It remains to show, that the off-diagonals are zero.

Applying α to $\sum_{s=1}^{N} x_{s1} x_{s1}^* = \mathbb{1}$ yields $\sum_{t_1,t_2=1}^{N} \sum_{s=1}^{N} v_{st_1} v_{st_2}^* \otimes x_{t_1 1} x_{t_2 1}^* = \mathbb{1}$. As Π contains only non-crossing pairings, we have $O_N^+ \subseteq G_N(\Pi)$ so we have a mapping $\psi_{O_2^+}$ as described in Notation 5.5.8. Applying $\mathbb{1} \otimes \psi_{O_2^+}$ to this relation, we obtain

$$\sum_{t_1,t_2=1}^{2} \sum_{s=1}^{N} v_{st_1} v_{st_2}^* \otimes o_{t_1 1} o_{t_2 1} = \mathbb{1}.$$

Using $\sum_{s=1}^{N} v_{st_1} v_{st_1}^* = \mathbb{1} = \sum_{t_1=1}^{2} o_{t_1 1} o_{t_1 1}$, this implies

$$\sum_{s=1}^{N} v_{s1} v_{s2}^* \otimes o_{11} o_{21} + \sum_{s=1}^{N} v_{s2} v_{s1}^* \otimes o_{21} o_{11} = 0.$$

By linear independence of the right legs (see Lemma 5.5.4) the left legs must be zero.
As the choice of x_{11} and x_{21} – as those rows of x not being sent to zero by $\psi_{O_2^+}$ – was arbitrary, we have for all $\gamma_1 \neq \gamma_2$ the result

$$\sum_{s=1}^{N} v_{s\gamma_1} v_{s\gamma_2}^* = 0.$$

□

Lemma 5.5.10. *Consider the situation of Notation 5.4.2 with* $d \leq N$ *arbitrary. If* Π *only contains non-crossing partitions with blocks of size at most two and if every row and column of* v_G *sums up to* $\mathbb{1}$*, then* $\mathcal{R}_p^{Gr}(v_G)$ *is fulfilled for every* $p \in \{ \overset{\sqcap}{\circ\bullet}, \overset{\sqcap}{\bullet\circ}, \underset{\sqcup}{\circ\bullet}, \underset{\sqcup}{\bullet\circ} \}$*. If* Π *contains* $\overset{\sqcap}{\circ\circ}$*, then also* $\mathcal{R}^{Gr}_{\overset{\sqcap}{\circ\circ}}(v_G)$ *holds.*

Proof. As in Lemma 5.5.9 we only care about the case $1 = d < N$ and we only consider $p = \underset{\sqcup}{\circ\bullet}$. Again, the only thing left to prove is that the off-diagonals of $\bar{v}_G^* \bar{v}_G$ vanish.

First, assume $N=2$. We have by Theorem 5.4.6

$$\underbrace{\left(v_{s1}+v_{s2}\right)}_{=\mathbb{1}}\underbrace{\left(v_{s1}^*+v_{s2}^*\right)}_{=\mathbb{1}}=\mathbb{1}=v_{s1}v_{s1}^*+v_{s2}v_{s2}^*,$$

so $v_{s1}v_{s2}^*=-v_{s2}v_{s1}^*$. Furthermore from

$$v_{11}+v_{12}=\mathbb{1}=v_{11}+v_{21}\quad\text{and}\quad v_{11}+v_{12}=\mathbb{1}=v_{22}+v_{12}$$

we deduce $v_{12}=v_{21}$ and $v_{11}=v_{22}$. Combining these relations yields for $t_1\neq t_2$:

$$\sum_{s=1}^N v_{st_1}v_{st_2}^*=v_{1t_1}v_{1t_2}^*+v_{2t_1}v_{2t_2}^*=v_{1t_1}v_{1t_2}^*+v_{1t_2}v_{1t_1}^*=v_{1t_1}v_{1t_2}^*-v_{1t_1}v_{1t_2}^*=0.$$

For the rest of the proof, let $N\geq 3$.
Step 1. We first prove

$$\sum_{s=1}^N v_{st_1}v_{st_2}^*=-\sum_{s=1}^N v_{st_2}v_{st_1}^*\quad,\quad\forall t_1\neq t_2.\tag{5.5.2}$$

Starting with the equation $\sum\limits_{s=1}^N x_{s1}x_{s1}^*=\mathbb{1}$ we can apply α to it and get

$$\sum_{s=1}^N\sum_{t_1,t_2=1}^N v_{st_1}v_{st_2}^*\otimes x_{t_11}x_{t_21}^*=\mathbb{1}.$$

Using $\sum_{s=1}^N v_{st_1}v_{st_1}^*=\mathbb{1}=\sum_{t_1=1}^N x_{t_11}x_{t_11}^*$, we have

$$\sum_{s=1}^N\sum_{t_1\neq t_2}v_{st_1}v_{st_2}^*\otimes x_{t_11}x_{t_21}^*=0.\tag{5.5.3}$$

Consider now the two *-homomorphisms ϕ_1 and ϕ_2 given by the maps

$$\begin{pmatrix}x_{11}\\x_{21}\\x_{31}\end{pmatrix}\xmapsto{\phi_1}\begin{pmatrix}-\frac{1}{3}\\\frac{2}{3}\\\frac{2}{3}\end{pmatrix}\qquad\text{and}\qquad\begin{pmatrix}x_{11}\\x_{21}\\x_{31}\end{pmatrix}\xmapsto{\phi_2}\begin{pmatrix}\frac{2}{3}\\\frac{2}{3}\\-\frac{1}{3}\end{pmatrix}.$$

and all other x_{i1} sent to zero. To prove the existence of these maps, we observe the following: By the conditions on Π we have $B_N^+\subseteq G_N(\Pi)$. Recall that the matrix $u_{B_N^+}=(b_{ij})$ is orthogonal and each row and column sums up to $\mathbb{1}$. In the sense of

Notation 5.5.8 we can send x_{11}, x_{21} and x_{31} by a map $\psi_{B_3^+}$ to the first column of $u_{B_3^+}$ and the rest to zero. Finally, note that the complex vectors on the right appear as columns of matrices in $B_3 \subseteq B_3^+$ so in a second step we can map the $\psi_{B_3^+}(x_{i1})$ to the complex numbers on the right sides. Applying $\mathbb{1} \otimes \phi_1$ and $\mathbb{1} \otimes \phi_2$ to Equation 5.5.3 leads to

$$\frac{4}{9}\sum_{s=1}^{N}(v_{s2}v_{s3}^* + v_{s3}v_{s2}^*) - \frac{2}{9}\sum_{s=1}^{N}(v_{s1}v_{s2}^* + v_{s2}v_{s1}^*) - \frac{2}{9}\sum_{s=1}^{N}(v_{s1}v_{s3}^* + v_{s3}v_{s1}^*) = 0$$

and

$$-\frac{2}{9}\sum_{s=1}^{N}(v_{s2}v_{s3}^* + v_{s3}v_{s2}^*) + \frac{4}{9}\sum_{s=1}^{N}(v_{s1}v_{s2}^* + v_{s2}v_{s1}^*) - \frac{2}{9}\sum_{s=1}^{N}(v_{s1}v_{s3}^* + v_{s3}v_{s1}^*) = 0,$$

which gives us in the end

$$\sum_{s=1}^{N}(v_{s1}v_{s2}^* + v_{s2}v_{s1}^*) = \sum_{s=1}^{N}(v_{s1}v_{s3}^* + v_{s3}v_{s1}^*).$$

As the choice of $(1,2,3)$ for the non-zero rows in the mappings ϕ_1 and ϕ_2 was arbitrary, we have this result for all pairwise different indices $(1,2,t)$. Accordingly, it holds

$$\sum_{s=1}^{N} v_{s1}v_{s2}^* + v_{s2}v_{s1}^* = \sum_{s=1}^{N} v_{s1}v_{st}^* + v_{st}v_{s1}^*$$

for all t. In particular, since $\sum_{t=1}^{N} v_{st}^* = \mathbb{1} = \sum_{s=1}^{N} v_{s1}$:

$$(N-1)\sum_{s=1}^{N} v_{s1}v_{s2}^* + v_{s2}v_{s1}^* = \sum_{t=2}^{N}\sum_{s=1}^{N} v_{s1}v_{st}^* + v_{st}v_{s1}^* = \mathbb{1} - \sum_{s=1}^{N} v_{s1}v_{s1}^* + \mathbb{1} - \sum_{s=1}^{N} v_{s1}v_{s1}^* = 0$$

Finally, because also the indices $(1,2)$ were arbitrary, this means for all $t_1 \neq t_2$

$$\sum_{s=1}^{N} v_{st_1}v_{st_2}^* = -\sum_{s=1}^{N} v_{st_2}v_{st_1}^*.$$

Step 2: We consider again Equation 5.5.3. As in step 1 we can apply $\psi_{B_3^+}$ to the second legs. Using additionally Equation 5.5.2, we find

$$\sum_{s=1}^{N}\sum_{t_1 < t_2 \leq 3} v_{st_1}v_{st_2}^* \otimes (b_{t_1 1}b_{t_2 1} - b_{t_2 1}b_{t_1 1}) = 0.$$

As $b_{21} = \mathbb{1} - b_{11} - b_{31}$, we easily see $b_{11}b_{21} - b_{21}b_{11} = b_{21}b_{31} - b_{31}b_{21} = b_{31}b_{11} - b_{11}b_{31}$ and so we have

$$\left(\sum_{s=1}^{N}(v_{s1}v_{s2}^* + v_{s2}v_{s3}^* + v_{s3}v_{s1}^*)\right) \otimes (b_{11}b_{21} - b_{21}b_{11}) = 0.$$

By Lemma 5.5.7 the right leg of the tensor product does not vanish, so the left one must be zero. The pairwise different indices $(1,2,3)$ were arbitrary, so using $(1,2,t)$ we have by Step 1

$$\begin{aligned}
0 &= \sum_{t=3}^{N}\sum_{s=1}^{N}(v_{s1}v_{s2}^* + v_{s2}v_{st}^* + v_{st}v_{s1}^*) \\
&= \left((N-2)\sum_{s=1}^{N} v_{s1}v_{s2}^*\right) + \left(\mathbb{1} - \sum_{s=1}^{N} v_{s2}(v_{s1}^* + v_{s2}^*)\right) + \left(\mathbb{1} - \sum_{s=1}^{N}(v_{s1} + v_{s2})v_{s1}^*\right) \\
&= (N-2)\sum_{s=1}^{N} v_{s1}v_{s2}^* - \sum_{s=1}^{N} v_{s2}v_{s1}^* - \sum_{s=1}^{N} v_{s1}v_{s1}^* \\
&= N\sum_{s=1}^{N} v_{s1}v_{s2}^*,
\end{aligned}$$

giving us the desired relation for $t_1 = 1$ and $t_2 = 2$. As the choice of $(1,2)$ was arbitrary we proved the statement for general $t_1 \neq t_2$. □

Before continuing, we need the notion of a blockstable category of partitions:

Definition 5.5.11. We call a category $\mathcal{C}$ of partitions *blockstable*, if for every $p \in \mathcal{C}$ and every block b of p we have $b \in \mathcal{C}$. In other words: By erasing all points (and lines) not belonging to b, we obtain again a partition contained in $\mathcal{C}$.

We recall the classification of free easy quantum groups in the sense that the sets Π in Table 2.6.1 generate all possible (and pairwise different) non-crossing categories of partitions (see [TW18, Thm. 7.1 and 7.2]).

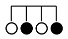
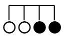
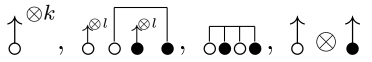
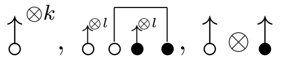
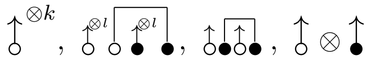
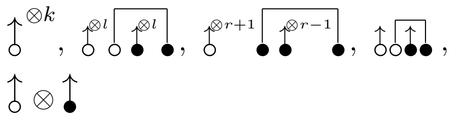
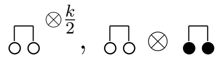
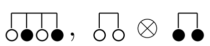
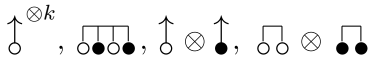
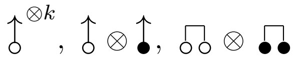
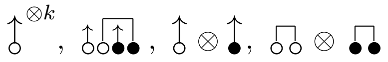
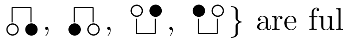
Case	Elements in Π	Parameter range	Blockstable cases
$\mathcal{O}_{\mathrm{loc}}$	$\emptyset$	–	blockstable
$\mathcal{H'}_{\mathrm{loc}}$		–	blockstable
$\mathcal{H}_{\mathrm{loc}}(k,l)$	$b_k, b_l \otimes \bar{b}_l$, 	$k,l\in\mathbb{N}_0\backslash\{1,2\}, l\|k$	$k=l$
$\mathcal{S}_{\mathrm{loc}}(k,l)$		$k,l\in\mathbb{N}_0\backslash\{1\}, l\|k$	not blockstable
$\mathcal{B}_{\mathrm{loc}}(k,l)$		$k,l\in\mathbb{N}_0, l\|k$	$k=l=1$
$\mathcal{B'}_{\mathrm{loc}}(k,l,0)$		$k,l\in\mathbb{N}_0\backslash\{1\}, l\|k$	not blockstable
$\mathcal{B'}_{\mathrm{loc}}(k,l,\frac{l}{2})$		$k\in\mathbb{N}_0\backslash\{1\}$, $l\in 2\mathbb{N}_0\backslash\{0,2\}$, $l\|k$, $r=\frac{l}{2}$	not blockstable
$\mathcal{O}_{\mathrm{glob}}(k)$		$k\in 2\mathbb{N}_0$	$k=2$
$\mathcal{H}_{\mathrm{glob}}(k)$	b_k, 	$k\in 2\mathbb{N}_0$	$k=2$
$\mathcal{S}_{\mathrm{glob}}(k)$		$k\in\mathbb{N}_0$	$k=1$
$\mathcal{B}_{\mathrm{glob}}(k)$		$k\in 2\mathbb{N}_0$	not blockstable
$\mathcal{B}'_{\mathrm{glob}}(k)$		$k\in\mathbb{N}_0$	$k=1$

Table 5.1: Classification of categories of non-crossing two-coloured partitions. Here, $b_k/\bar{b}_k$ is the one-block partition in $P(0,k)$ with only white/black points.

Theorem 5.5.12. *Let* $N\in\mathbb{N}\backslash\{1\}$ *and fix any of the sets* Π *presented in Table 2.6.1. In the case* $d=2$, $G_N(\Pi)$ *is the quantum symmetry group of* $X_{N,2}(\Pi)$.
If the category $\langle\Pi\rangle$ *is blockstable, or if* $N=1$, *then this results even holds for* $d=1$, *i.e. for* $X_{N,1}(\Pi)$.

Proof. We consider again the situation as in Notation 5.4.2. By Theorem 5.3.1 we know $G_N(\Pi)\subseteq G$, so we only need to show that $\mathcal{R}_p^{Gr}(v_G)$ is fulfilled for all $p\in\Pi$. As the case $N=d=1$ is by Corollary 5.4.8, we assume $N\geq 2$.

We have that the relations coming from the partitions  are fulfilled. For $d=2$ this is Theorem 5.4.6 and for $d=1$ see Remark 5.5.2 and Lemmata

5.5.9 and 5.5.10. The same holds for $\sqcap_{\circ\circ}$, where required. Most of the remaining parts of the proof are by Theorem 5.4.6 and Lemma 5.5.1 but in virtue of Remark 5.4.10 we often have to perform some algebraic operations to see that the desired relations are really fulfilled for all relevant multi-indices γ and γ'. We prove two cases, the other ones are handled with similar arguments.

Case $\mathcal{S}_{\text{loc}}(k,l)$: By the arguments above the relations due to *mcpp* and $\sqcap\!\sqcap\!\sqcap_{\circ\bullet\circ\bullet}$ are fulfilled. As $d=2$ also the relations $\mathcal{R}^{Gr}_{\overset{\uparrow}{\circ}\otimes\overset{\uparrow}{\bullet}}(v_G)$ are guaranteed by Theorem 5.4.6. In the case $k=l=0$ this is everything to be proved. From the fact that v_G and v_G^T are unitaries and Observation 2.6.4 we deduce that also the relations $\mathcal{R}^{Gr}_{\overset{\uparrow}{\bullet}\otimes\overset{\uparrow}{\circ}}(v_G)$ are fulfilled because $\overset{\uparrow}{\bullet}\otimes\overset{\uparrow}{\circ}$ lies in the category generated by $\overset{\uparrow}{\circ}\otimes\overset{\uparrow}{\bullet}$. This guarantees now that each row and column of v_G sums up to the same (unitary) element. Using this result, we can consider now the relations $\mathcal{R}^{Gr}_{\overset{\uparrow}{\circ}^{\otimes k}}(v_G)$. They read as

$$\sum_{t_1,\ldots,t_k} v_{\gamma'_1 t_1}\cdots v_{\gamma'_k t_k} = \mathbb{1}$$

which are now proved to be true not only for γ' with at most two different entries (see Theorem 5.4.6) but for all $\gamma\in[N]^k$. The same argument secures all the quantum group relations associated to $p=\overset{\uparrow}{\circ}^{\otimes l}\,\overset{\sqcap}{\circ\;\overset{\uparrow}{\bullet}^{\otimes l}\;\bullet}$ to be fulfilled: They read as

$$\sum_{t'_1,\ldots,t'_{2l+1}} \big(v_{\gamma_1 t'_1}\cdots v_{\gamma'_l t'_l}\big)v_{\gamma'_{l+1}t'_{l+1}}\big(v^*_{\gamma'_{l+2}t'_{l+2}}\cdots v^*_{\gamma'_{2l+1}t'_{2l+1}}\big)v^*_{\gamma'_{2l+2}t'_{l+1}} = \delta_{\gamma'_{l+1},\gamma'_{2l+2}}.$$

and at first site these are true only if $\gamma'=(\gamma'_1,\ldots,\gamma'_{2l+2})$ has at most two different entries. Exploiting again the fact that each row and column of v_G sums up to the same element, we can replace all entries $\gamma'_1,\ldots,\gamma_l,\gamma'_{l+2},\ldots,\gamma'_{2l+1}$ by γ'_{2l+2}, proving the claim.

Case $\mathcal{O}_{\text{glob}}(k)$: For $k=2$ we only need to prove $\mathcal{R}^{Gr}_p(v_G)$ for $p\in\Pi' := \{\sqcap_{\circ\circ}, mcpp\}$ which is Lemma 5.5.9. For $k\in 2\mathbb{N}\backslash\{2\}$ this Lemma only guarantees the relations according to $p\in\{mcpp\}$. We start with the partition $p=\sqcap_{\circ\circ}\otimes\sqcap_{\bullet\bullet}$. The corresponding quantum group relations read as

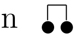
$$\left(\sum_{t'_1} v_{\gamma'_1 t'_1}v_{\gamma'_2 t'_1}\right)\left(\sum_{t'_2} v^*_{\gamma'_3 t'_2}v^*_{\gamma'_4 t'_2}\right) = \delta_{\gamma'_1\gamma'_2}\delta_{\gamma'_3\gamma'_4}$$

and Theorem 5.4.6 only guarantees this result for γ' with at most two different entries. Choosing $\gamma'_1=\gamma'_4\neq\gamma'_2=\gamma'_3$, this reads

$$\left(\sum_{t'_1} v_{\gamma'_1 t_1}v_{\gamma'_2 t_1}\right)\left(\sum_{t'_2} v^*_{\gamma'_2 t'_2}v^*_{\gamma'_1 t'_2}\right) = \left(\sum_{t'_1} v_{\gamma'_1 t_1}v_{\gamma'_2 t_1}\right)\left(\sum_{t'_1} v_{\gamma'_1 t_1}v_{\gamma'_2 t_1}\right)^* = 0,$$

so

$$\sum_{t_1'} v_{\gamma_1' t_1} v_{\gamma_2' t_1} = 0 \qquad \forall \gamma_1' \neq \gamma_2'.$$

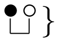
This proves $\mathcal{R}^{Gr}$$\otimes$(v_G). Together with the fact that v_G and v_G^T are unitaries we also have that the quantum group relations for the rotated partition  $\otimes$  are fulfilled, so the sums $\sum_{t_1} v_{\gamma_1' t_1'} v_{\gamma_1' t_1'}$ are unitaries, thus invertible. Therefore, $\mathcal{R}^{Gr}$$\otimes$(v_G) in particular says that $\sum_{t_1'} v_{\gamma_1' t_1'} v_{\gamma_1' t_1'}$ is independent of $\gamma_1' \in [N]$. We finally use all these results in the situation of $p=$ $^{\otimes k}$ to show that the corresponding relations,

$$\left(\sum_{t_1'} v_{\gamma_1' t_1'} v_{\gamma_2' t_1'}\right) \cdots \left(\sum_{t_k'} v_{\gamma_{2k-1}' t_k'} v_{\gamma_{2k}' t_k'}\right) = \delta_{\gamma_1' \gamma_2'} \cdots \delta_{\gamma_{2k-1}' \gamma_{2k}'},$$

are true for all $\gamma' \in [N]^{2k}$, as we can now make the replacement

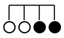
$$(\gamma_{2m+1}', \gamma_{2m+2}') \mapsto \begin{cases} (1,1) & , \gamma_{2m+1}' = \gamma_{2m+2}' \\ (1,2) & , \gamma_{2m+1}' \neq \gamma_{2m+1}' \end{cases}.$$

□

Remark 5.5.13. Adding the crossing partition  to the sets Π in Table 2.6.1 produces all categories for all unitary easy groups, see [TW17]. It obviously guarantees commutativity of the x_{ij}'s and for $d=2$ we have $\mathcal{R}^{Gr}$(v_G) fulfilled by Theorem 5.4.6. So the (quantum) symmetry groups of these partition (quantum) spaces are given by the corresponding easy groups. Note that for $d=2$ we can directly deduce from $\{$ , , , $\} \subseteq \Pi$ that v_G and $\bar{v}_G$ are unitaries, so we do not need to use Lemmata 5.5.9 and 5.5.10. We finally remark that it is unclear, if $d=1$ works in the blockstable cases.

5.6 Open questions and further remarks

Question 5.6.1. Are there situations or conditions (apart from $d=N$) such that the quantum symmetry group (or even the PQS) only depends on $\langle \Pi \rangle$ and not Π itself?

Regarding Corollary 5.4.9, there is a simple counterexample for the analogous statement with $d = 1$: The free hyperoctahedral group H_N^+ corresponds to the case $\mathcal{H}_{\text{loc}}(2,2)$, i.e. $\Pi = \{$ , , *mcpp*$\}$, see Table 2.6.1. The category of partitions

$\langle\Pi\rangle$ is also generated by $\Pi' := \{\ ,\ , mcpp\}$ but obviously $\mathcal{R}^{Sp}(x)$ is just the trivial relation. We thus have $X_{N,1}(\Pi) \neq X_{N,1}(\Pi')$ and the quantum symmetry group of $X_{N,1}(\Pi')$ is O_N^+ (i.e. case $\mathcal{O}_{\text{glob}}(2)$), whereas the one of $X_{N,1}(\Pi)$ is H_N^+.

Question 5.6.2. Can we produce results similar to Theorem 5.5.12 (free case) or Remark 5.5.13 (group case) for other classes of partitions/easy quantum groups?

Question 5.6.3. Is there a way to read off from Π the minimal d such that $G_N(\Pi)$ is the quantum symmetry group of $X_{N,d}(\Pi)$? In the situation of Theorem 5.5.12, is $d=1$ equivalent to $\langle\Pi\rangle$ being blockstable?

Due to Theorem 5.5.12, we have $d=1$ in the free blockstable cases, at least for the choices of Π presented in Table 2.6.1. But the counterexample after Question 5.6.1 already shows that there are other choices for Π, even in the non-crossing situation, where this is not true. Another example from the commutative case is the partition set $\Pi = \{\ ,\ ,\ , mcpp\}$ corresponding to the hyperoctahedral group H_N. For $d=1$ we have $X_{N,1}(\Pi) = X_{N,1}\big(\Pi\backslash\{\ \}\big)$ as commutativity already follows from $\mathcal{R}^{Sp}(v_G)$, see Lemma 5.5.1, so the corresponding quantum symmetry group is H_N^+ by Theorem 5.5.12.
In the cases presented in Table 2.6.1 we have some sets Π where the quantum symmetry group of $X_{N,1}(\Pi)$ is not given by $G_N(\Pi)$, supporting our conjecture, that the case $d=1$ is linked to blockstability. Consider for example $\mathcal{H}_{\text{loc}}(k,l)$ with $k \neq l$ and let $\Pi(k,l)$ be the corresponding set of partitions from Table 2.6.1. We have $X_{N,1}(\Pi(k,l)) = X_{N,1}(\Pi(k,k))$ as the quantum space relations $\mathcal{R}^{Sp}_{b_l\otimes\bar{b}_l}(x)$ are redundant. Hence the quantum symmetry group of $X_{N,1}\big(\Pi(k,l)\big)$ is $G_N\big(\Pi(k,k)\big)$ which is in general larger than $G_N\big(\Pi(k,l)\big)$. Similar results hold in the cases $\mathcal{S}_{\text{loc}}(0,0)$, $\mathcal{H}_{\text{glob}}(k)$ for $k \in 2\mathbb{N}+4$ and $\mathcal{S}_{\text{glob}}(0)$, where respectively $\mathcal{R}^{Sp}_{\otimes}(x)$, $\mathcal{R}^{Sp}_{\otimes}(x)$ and again $\mathcal{R}^{Sp}_{\otimes}(x)$ are redundant.
On the other hand, though, we cannot guarantee that $d=1$ fails in all non-blockstable cases. Our standard method to deduce relations for the v_{ij} was to start with a quantum space relation $\mathcal{R}^{Sp}_p(x)$, apply α or β to it and finally $\mathbb{1}\otimes\text{ev}_\sigma$. But of course by this procedure we might have lost some information as ev_G is far from being an isomorphism. In principle we would have to stay inside $X_{N,d}(\Pi)$ or at least $G_N(\Pi)$. In $G_N(\Pi)$ we could deduce many (in)dependencies by the fusion rules established in [FW14] and [Fre14] as done in Lemma 5.5.3 and the ones following thereafter. Hence, although we expect that for non-blockstable categories we always need $d \geq 2$ in order to reconstruct $G_N(\Pi)$ as the quantum symmetry group of $X_{N,d}(\Pi)$, we have to leave this question open.

Bibliography

[Ban96] T. Banica. The representation theory of free orthogonal quantum groups. *C. R. Acad. Sci. Paris Sér. I Math.*, 322(3):241–244, 1996.

[Ban97] T. Banica. The free unitary compact quantum group. *Comm. Math. Phys.*, 190(1):143–172, 1997.

[Ban99] T. Banica. Symmetries of a generic coaction. *Math. Ann*, 314(4):763–780, 1999.

[Ban02] T. Banica. Quantum groups and Fuss-Catalan algebras. *Comm. Math. Phys.*, 226(1):221–232, 2002.

[Ban18a] T. Banica. Homogeneous quantum groups and their easiness level. *arxiv:1806.06368*, 2018. to appear in *Kyoto J. Math.*

[Ban18b] T. Banica. Unitary easy quantum groups: geometric aspects. *J. Geom. Phys.*, 126:127–147, 2018.

[BC07] T. Banica and B. Collins. Integration over quantum permutation groups. *J. Funct. Anal.*, 242:641–657, 2007.

[BC10] T. Banica and S. Curran. Decomposition results for Gram matrix determinants. *J. Math. Phys.*, 51(11):113503, 2010.

[BG09a] J. Bhowmick and D. Goswami. Quantum group of orientation-preserving Riemannian isometries. *J. Funct. Anal.*, 257(8):2530–2572, 2009.

[BG09b] J. Bhowmick and D. Goswami. Quantum isometry groups: examples and computations. *Comm. Math. Phys.*, 285(2):421–444, Jan 2009.

[BG10] T. Banica and D. Goswami. Quantum isometries and noncommutative spheres. *Comm. Math. Phys.*, 298(2):343–356, Sep 2010.

[BGS11] J. Bhowmick, D. Goswami, and A. Skalski. Quantum isometry groups of 0-dimensional manifolds. *Trans. Amer. Math. Soc.*, 363(2):901–921, 2011.

[Bla06] B. Blackadar. *Operator algebras. Theory of C^*-algebras and von Neumann algebras*, volume 122 of *Encyclopaedia Math. Sci.* Springer-Verlag, Berlin, 2006.

[BN17] T. Banica and I. Nechita. Flat matrix models for quantum permutation groups. *Adv. Appl. Math.*, 83:24–46, 2017.

[Bra37] R. Brauer. On algebras which are connected with the semisimple continuous groups. *Ann. of Math.*, 38(4):857–872, 1937.

[BS93] S. Baaj and G. Skandalis. Unitaires multiplicatifs et dualité pour les produits croisés de C^*-algèbres. *Ann. Sci. École Norm. Sup.*, 26(4):425–488, 1993.

[BS09] T. Banica and R. Speicher. Liberation of orthogonal Lie groups. *Adv. Math.*, 222(4):1461–1501, 2009.

[BSS12] T. Banica, A. Skalski, and P. Sołtan. Noncommutative homogeneous spaces: the matrix case. *J. Geom. Phys.*, 62(6):1451–1466, 2012.

[CSS02] A. Copeland, F. Schmidt, and R. Simio. On two determinants with interesting factorizations. *Discrete Math.*, 256(1):449–458, 2002.

[Dae94] A. Van Daele. Multiplier Hopf algebras. *Trans. Amer. Math. Soc.*, 342(2):917–932, 1994.

[Dae98] A. Van Daele. An algebraic framwork for group duality. *Adv. Math.*, 140(2):342–366, 1998.

[Dix69] J. Diximier. *Les C^*-algèbres et leurs représentations.* Gauthiers-Villars, Paris, 1969.

[Dri87] V. G. Drinfeld. Quantum groups. In *Proc. Internat. Congr. Math., Vol. 1 (Berkley, California, 1986)*, volume 1, pages 798–820, Providence, RI, 1987. Amer. Math. Soc.

[ES92] M. Enock and J.-M. Schwartz. *Kac algebras and duality of locally compact groups.* Springer-Verlag, 1992.

[EV93] M. Enock and J.-M. Vallin. C*-algèbres de Kac et algèbres de Kac. *Proc. London Math. Soc.*, 66(3):619–650, 1993.

[FCG97] P. Di Francesco, O. Colinelli, and E. Guitter. Meanders and the Temperley-Lieb algebra. *Commun. Math. Phys.*, 186(1):1–59, 1997.

[Fre14] A. Freslon. Fusion (semi)rings arising from quantum groups. *J. Algebra*, 417(Supplement C):161–197, 2014.

[FS63] L. D. Faddeyev and B. Seckler. The inverse problem in the quantum theory of scattering. *J. Math. Phys.*, 4(1):72–104, 1963.

[FW14] A. Freslon and M. Weber. On the representation theory of partition (easy) quantum groups. *J. Reine Angew. Math.*, 2016:155, 2014.

[Gos08] D. Goswami. Quantum group of isometries in classical and noncommutative geometry. *Comm. Math. Phys.*, 285(1):141, 2008.

[Gro18] D. Gromada. Classification of globally colorized categories of partitions. *arXiv:1805.10800*, 2018.

[Hal67] P. R. Halmos. *A Hilbert space problem book*. American Book Company, New York, 1967.

[Hop41] H. Hopf. Über die Topologie der Gruppen-Mannigfaltigkeiten und ihre Verallgemeinerungen. *Ann. of Math.*, 42(1):22–52, 1941.

[Jon83] V. F. R. Jones. Index for subfactors. *Invent. Math.*, 72:1–25, 1983.

[JW18] S. Jung and M. Weber. Partition quantum spaces. *arxiv:1801.06376*, 2018. to appear in *J. Noncommut. Geom.*

[Kas95] C. Kassel. *Quantum groups*, volume 155 of *Grad. Texts in Math.* Springer-Verlag, 1995.

[KS91] K. H. Ko and L. Smolinsky. A combinatorial matrix in 3-manifold theory. *Pacific. J. Math*, 149:319–336, 1991.

[KT00a] J. Kustermans and L. Tuset. A survey of C^*-algebraic quantum groups. Part I. *Irish Math. Soc. Bull.*, 43:8–63, 2000.

[KT00b] J. Kustermans and L. Tuset. A survey of C^*-algebraic quantum groups. Part II. *Irish Math. Soc. Bull.*, 44:6–54, 2000.

[KV74] G. I. Kac and L. Vainerman. Nonunimodular ring-groups and Hopf-von Neumann algebras. *Math. USSR Sb.*, 23:185–214, 1974.

[Lan95] E. C. Lance. *Hilbert C^*-modules: A toolkit for operator algebraists*, volume 210 of *London Math. Soc. Lecture Note Ser.* Cambridge University Press, Cambridge, 1995.

[Maa18] L. Maaßen. The intertwiner spaces of non-easy group-theoretical quantum groups. *arXiv:1808.07693*, 2018.

[Mac71] S. MacLane. *Categories for the working mathematician*, volume 5 of *Graduate Texts in Mathematics.* Springer, New York, 1971.

[MD98] A. Maes and A. Van Daele. Notes on compact quantum groups. *Nieuw Arch. Wisk.*, 16(4):73–112, 1998.

[MS17] J. A. Mingo and R. Speicher. *Free Probability and Random Matrices.* Fields Institute Monographs. Springer, 2017.

[Mur90] G. J. Murphy. *C^*-algebras and operator theory.* Academic Press Inc., San Diego, 1990.

[MW18a] A. Mang and M. Weber. Categories of two-colored pair partitions, Part I: Categories indexed by cyclic groups. *arXiv:1809.06948*, 2018.

[MW18b] A. Mang and M. Weber. Categories of two-colored pair partitions, Part II: Categories indexed by semigroups. *arXiv:1901.03266*, 2018.

[NS06] A. Nica and R. Speicher. *Lectures on the Combinatorics of Free Probability.* Number 335 in London Mathematical Society Lecture Note Series. Cambridge Univ. Press, 2006.

[NT13] S. Neshveyev and L. Tuset. *Compact quantum groups and their representation categories.* Collection SMF, courses specialisés, 20. Société Mathématique de France, Paris, 2013.

[Phi88] N. C. Philipps. Inverse limits of C^*-algebras. *J. Operator Theory*, 19:159–195, May 1988.

[Pod87] P. Podleś. Quantum spheres. *Lett. Math. Phys.*, 14(3):193–202, Oct 1987.

[Pod95] P. Podleś. Symmetries of quantum spaces. Subgroups and quotient spaces of quantum SU(2) and SO(3) groups. *Comm. Math. Phys.*, 170(1):1–20, May 1995.

[Rau12] S. Raum. Isomorphisms and fusion rules of orthogonal free quantum groups and their free complexifications. *Proc. Amer. Math. Soc.*, 140(9):3207–3218, 2012.

[Riv74] T. J. Rivlin. *The Chebyshev polynomials.* Wiley, New York, 1974.

[RW16] S. Raum and M. Weber. The full classification of orthogonal easy quantum groups. *Comm. Math. Phys.*, 341(3):751–779, 2016.

[Swe69] M. E. Sweedler. *Hopf algebras.* W. A. Benjamin, New York, 1969.

[Tim08] T. Timmermann. *An invitation to quantum groups and duality.* EMS Textbk. Math. European Mathematical Society, Zürich, 2008.

[Tut93] W. T. Tutte. The matrix of chromatic joins. *J. Combin. Theory, Ser. B*, 57(2):269 – 288, 1993.

[TW17] P. Tarrago and M. Weber. Unitary easy quantum groups: the free case and the group case. *Int. Math. Res. Not.*, 2017(18):5710–5750, 2017.

[TW18] P. Tarrago and M. Weber. The classification of tensor categories of two-coloured noncrossing partitions. *J. Combin. Theory Ser. A*, 154:464–506, 2018.

[Val85] J.-M. Vallin. C^*-algèbres de Hopf et C^*-algèbres de Kac. *Proc. London Math. Soc.*, 50(3):131–174, 1985.

[Wan98] Sh. Wang. Quantum symmetry groups of finite spaces. *Comm. Math. Phys.*, 195(1):195–211, 1998.

[Web13] M. Weber. On the classification of easy quantum groups. *Adv. Math.*, 245:500–533, 2013.

[Web16] M. Weber. Easy quantum groups. In *Free probability and operator algebras*, Münster Lecture Notes in Mathematics. European Mathematical Society, Zürich, 2016.

[Web17a] M. Weber. Introduction to compact (matrix) quantum groups and Banica–Speicher (easy) quantum groups. *Proc. Indian Acad. Sci. Math. Sci.*, 127(5):881–933, 2017.

[Web17b] M. Weber. Partition C^*-algebras. *arXiv:1710.06199v1*, 2017.

[WG18] M. Weber and D. Gromada. Intertwiner spaces of quantum group subrepresentations. *arXiv:1811.02821*, 2018.

[Wor87] S. L. Woronowicz. Compact matrix pseudogroups. *Comm. Math. Phys.*, 111(4):613–665, 1987.

[Wor88] S. L. Woronowicz. Tannaka-Krein duality for compact matrix pseudogroups. Twisted SU(N) groups. *Invent. Math.*, 93(1):35–76, 1988.

[Wor98] S. L. Woronowicz. Compact quantum groups. In *Quantum Symmetries (Les Houches, 1995)*, volume 64 of *Les Houches Summer School proceedings*, pages 845–884, Amsterdam, 1998. North-Holland.

Chapter 6

Appendix

Standard references for C^*-algebras are, for example, [Dix69], [Mur90] and [Bla06]. For Hilbert C^*-modules, we refer to [Lan95]. Regarding Hopf algebras, a standard reference is [Swe69].

6.1 C^*-algebras

In this chapter we collect standard result from the context of C^*-algebras.

Definition 6.1.1 (*-algebras and *-homomorphisms)**.** A **-algebra* is a complex algebra $\mathcal{A}$ together with a conjugate-linear antiautomorphism $* : \mathcal{A} \to \mathcal{A}$, i.e. a bijective map satisfying for all $\alpha \in \mathbb{C}$ and $a, b \in \mathcal{A}$

$$\begin{aligned} \left(a^*\right)^* &= a \\ (\alpha a + b)^* &= \overline{\alpha} a^* + b^* \\ (ab)^* &= b^* a^* \end{aligned}$$

A *-homomorphism φ between two *-algebras $\mathcal{A}$ and $\mathcal{B}$ is an algebra homomorphism between the corresponding algebras that respects the respective involutions, i.e.

$$\varphi(a^*) = \varphi(a)^* \quad \forall a \in \mathcal{A}.$$

It is called an isomorphism if there exists an inverse *-homomorphism from $\mathcal{B}$ to $\mathcal{A}$.

Definition 6.1.2 (C^*-algebras)**.** A C^*-algebra A is a *-algebra which is a Banach space such that its norm satisfies for all $a, b \in A$

$$\begin{aligned} \|ab\| &\leq \|a\| \cdot \|b\| \\ \|a^*\| &= \|a\| \\ \|aa^*\| &= \|a\|^2. \end{aligned}$$

A *-homomorphism between C^*-algebras is a *-homomorphism between the corresponding *-algebras.

It is not necessary to impose on a *-homomorphism between C^*-algebras a condition with respect to the norms as such a map always turns out to be norm-decreasing:

Theorem 6.1.3. *A *-homomorphism $\varphi : A \to B$ between two C^*-algebras is norm-decreasing. If ϕ is injective, then φ is isometric.*

Considering commutative C^*-algebras, examples are given by the continuous complex-valued functions on locally compact spaces. By the Gelfand theorem we have that every commutative C^*-algebra can be obtained this way:

Theorem 6.1.4 (Gelfand)**.** *Let A be a commutative C^*-algebra and $X := \operatorname{Spec}(A)$ be the spectrum of A, i.e. the space of non-zero *-homomorphisms from A to $\mathbb{C}$. Then there is a topology on X that makes X a locally compact Hausdorff space and such that the map*

$$\chi : A \to C(X) = C_0\big(\operatorname{Spec}(X)\big); \chi(a)(\varphi) := \varphi(a)$$

is a *-isomorphism between C^*-algebras.

Here $C_0(X)$ are the continuous complex-valued functions on X vanishing at infinity. X is compact if and only if A is unital and in this case $C_0(X)$ of course coincides with $C(X)$, the unital C^*-algebra of continuous complex valued functions on X.

In the non-commutative situation examples of C^*-algebras are given by suitable sets of bounded linear operators on Hilbert spaces. It turns out that every (abstractly given) C^*-algebra can be realized in such a way. Recall, that a representation of a $*$-algebra A is a pair (π, H_π) where H_π is a Hilbert space and $\pi : A \to B(H)$ is a *-homomorphism. A representation (π, H_π) is called faithful, if π is injective.

Theorem 6.1.5 (GNS-construction)**.** *Let τ be a state on a C^*-algebra A, i.e. a positive functional on A of norm 1. Then there exists a representation $\big(\pi_\tau, H_{\pi_\tau}\big)$ of A and a linear map $\Lambda_\tau : A \to H_{\pi_\tau}$ such that the following holds:*

- *(i) The image of Λ_τ is dense in H_{π_τ}*
- *(ii) The map Λ_τ respects the state τ in the sense*

$$\tau(a^*b) = \langle \Lambda_\tau(a), \Lambda_\tau(b) \rangle \quad \forall a, b \in A.$$

- *(iii) The action of $\pi_\tau(A)$ on H_{π_τ} is defined by left multiplication on A:*

$$\pi_\tau(a)\Lambda_\tau(b) = \Lambda_\tau(ab) \quad \forall a, b \in A$$

We call the triple $(\pi_\tau, H_{\pi_\tau}, \Lambda_\tau)$ the GNS-construction of A with respect to τ and likewise the pair $(\pi_\tau, H_{\pi_\tau}$ the GNS-representation with respect to τ.

H_{π_τ} is obtained by imposing on A the sesquilinear-form $\langle a, b \rangle := \tau(a^*b)$, dividing out the left ideal $\{\tau(a^*a) = \langle a, a \rangle = 0\}$ and completing the resulting pre-Hilbert space.
With the help of the Hahn-Banach theorem one can show that for every $0 \neq a \in A$ we find some state τ such that $\tau(a^*a) = \|a^*a\|$. In this sense the GNS-construction with respect to just one state τ does not necessarily give an injective map π. The collection of all these states and representations, however, turns out to do so. Recall that a direct sum of representations $(\pi, H_\pi) := \bigoplus_{i \in I} (\pi_i, H_{\pi_i})$ is defines by

$$\pi(a)v_i := \pi_i(a)v_i \in H_{\pi_i} \subseteq \bigoplus_{i \in I} H_{\pi_i} \quad \forall a \in A, v_i \in H_i, i \in I$$

Theorem 6.1.6 (universal GNS-representation of A)**.** *Let A be a C^*-algebra. The direct sum of all its GNS-representations*

$$(\pi, H_\pi) := \bigoplus_{\tau \text{ state}} \big(\pi_\tau, H_{\pi_\tau}\big),$$

is a faithful representation, the so-called universal GNS-representation, *of A. It is a faithful representation of A, i.e π is injective.*

In particular, we can realize A as operators on the Hilbert space $\bigoplus_{\tau \text{ state}} H_{\pi_\tau}$.

A common way to construct abstract C^*-algebras is via universal C^*-algebras:

Construction 6.1.7 (universal C^*-algebras)**.** Let $E := \{a_i)_i \in I\}$ be a set of mutually different symbols a_i. Let R be a set of *-polynomials over $\mathbb{C}$ in the indeterminants a_i. In the following we will call the elements in E generators and the elements in R (algebraic) relations (for the generators). Denote with $P^*(E)$ the set of *-polynomials in the indeterminants a_i. This can be seen as a *-algebra, compare Definition 6.1.1. Let $\langle R\rangle \subseteq P^*(E)$ be the *-ideal in $P^*(E)$ generated by R and define the quotient-*-algebra

$$\tilde{A} := {}^{P^*(E)}\!\big/_{\langle R\rangle}.$$

If the family of C^*-seminorms g on $\tilde{A}$ is pointwise bounded then its pointwise supremum

$$s : \tilde{A} \to \mathbb{R}_0^+; a \mapsto \sup(\{g(a) \,|\, g \text{ is } C^*\text{-seminorm on } \tilde{A}\})$$

is a C^*-seminorm again, namely the maximal one on $\tilde{A}$. On the quotient

$$\mathcal{A} := {}^{\tilde{A}}\!\big/_{\langle\{a \mid g(a) = 0\}\rangle}$$

the representative-wise defined mapping

$$\|\cdot\| : a \mapsto g(a)$$

is by construction a C^*-norm and its completion A with respect to $\|\cdot\|$ is a C^*-algebra. We call A the universal C^*-algebra generated by E and the relations R. We also use the notations

$$A =: C^*\big(\, E \mid R \,\big)$$

or

$$A =: C^*\big(\, E \mid \forall r \in R: \text{ It holds } r = 0\,\big).$$

Note that not every pair (E, R) of generators and relations allows the construction of a universal C^*-algebra. The crucial point is the pointwise boundedness of the seminorms on $\tilde{A}$, which cannot be guaranteed in general.
Given a collection of elements $(b_i)_{i\in I}$ in some C^*-algebra B, we denote for $r \in R$ with $r\big((b_i)_{i\in I}\big)$ the expression r after applying the symbolwise replacement $a_i \rightsquigarrow b_i$ and identify it with an element in B. The C^*-algebra A then features the following universal property.

Proposition 6.1.8 (Universal property of universal C^*-algebra)**.** *Consider a universal C^*-algebra $A := C^*\big(\, E \mid R \,\big)$ as defined in Construction 6.1.7. Let $(b_i)_{i\in I}$ be a collection of elements in some C^*-algebra B. If it holds*

$$r\big((b_i)_{i\in I}\big) = 0 \quad \forall r \in R,$$

then there exists a unique *-homomorphism $\varphi : A \to B$ that satisfies

$$\varphi(a_i) = b_i \quad \forall i \in I.$$

6.2 Tensor products of C^*-algebras

Given two C^*algebras A and B we can build the algebraic tensor product $A \odot B$ of vector spaces which becomes a *-algebra via the definitions

$$(a \otimes b)^* := a^* \otimes b^* \qquad , \qquad (a_1 \otimes b_1)(a_2 \otimes b_2) := a_1 a_2 \otimes b_1 b_2.$$

Definition 6.2.1 (C^*-seminorms associated to product representations)**.** Every pair of representations (π_A, H_{π_A}) and (π_B, H_{π_B}) of A and B, respectively, defines a product representation $(\pi_A \otimes \pi_B, H_{\pi_A} \otimes H_{\pi_B})$ on the Hilbert space tensor product $H_{\pi_A} \otimes H_{\pi_B}$ via

$$\big((\pi_A \otimes \pi_B)(a \otimes b)\big)(v_A \otimes v_B) := \pi_A(a)(v_A) \otimes \pi_B(b)(v_B).$$

Each such pair further defines a C^*-seminorm on $A \odot B$ via

$$x \mapsto \|(\pi_A \otimes \pi_B)(x)\|_{B(H_{\pi_A} \otimes H_{\pi_B})}.$$

Each pair of faithful representations (π_A, H_{π_A}) and (π_B, H_{π_B}) defines the same C^*-seminorm which is in fact a norm.

Definition 6.2.2 (Minimal tensor product of C^*-algebras)**.** Let A and B be C^*-algebras. The mapping

$a \mapsto \|a\|_{min} := \sup(\{\|a\|_{B(H_{\pi_A} \otimes H_{\pi_B})} \mid (\pi_A \otimes \pi_B, H_{\pi_A} \otimes H_{\pi_B})$ product representation$\})$

defines on the *-algebra $A \odot B$ a C^*-norm, the so-called minimal C^*-norm. The completion of $(A \odot B, \|\cdot\|_{min})$ is called the minimal tensor product of the C^*-algebras A and B and denoted by $A \underline{\otimes} B$ or simply by $A \otimes B$ (compare Section 1.5).
The norm $\|\cdot\|_{min}$ is the smallest C^*-norm on $A \odot B$. It coincides with the C^*-seminorm associated to an arbitrary pair of faithful representations (π_A, H_{ϕ_A}) and (π_B, H_{π_B}) of A and B, respectively, see Definition 6.2.1.

Note that there might exist more than one C^*-norm on $A \odot B$ and, thus, other tensor products $A \otimes B$ of two given C^*-algebras. For example, the maximal tensor product $A \otimes_{max} B$ is obtained by establishing on $A \odot B$ the norm

$$\|\cdot\|_{max} := \sup\big(\{g \,|\, g \text{ is } C^*\text{-seminorm on } A \odot B\}\big).$$

This norm possibly differs from the minimal tensor norm as there might be C^*-seminorms on $A \odot B$ not coming from product representations $(\pi_A \otimes \pi_B, H_{\pi_A} \otimes H_{\pi_B})$. We remark that, amongst all possible C^*-norms on $A \odot B$, the norms $\|\cdot\|_{min}$ and $\|\cdot\|_{max}$ are the respectively smallest and largest C^*-norms on $A \odot B$. In particular, every C^*-tensor product $A \otimes B$ allows two *-homomorphisms

$$A \otimes_{max} B \xrightarrow{\varphi_1} A \otimes B \xrightarrow{\varphi_2} A \otimes_{min} B$$

each defined by fixing the dense *-algebra $A \odot B$ inside the respective C^*-tensor product.

6.3 Hilbert C^*-modules

Definition 6.3.1 (C^*-module)**.** Let A be a C^*-algebra and let V be a vector space that is a right A-module. Assume that there is a sesquilinear map $\langle\cdot,\cdot\rangle : V \times V \to A$ satisfying for $v, w \in V$ and $a \in A$

$$\langle v, w\rangle^* = \langle w, v\rangle \quad , \quad \langle v, wa\rangle = \langle v, w\rangle a \quad , \quad \langle v, v\rangle \geq 0.$$

Then we call V a *(right) semi-inner product A-module.* If it holds

$$\langle v, v\rangle = 0 \Rightarrow v = 0$$

then $\|v\|^2 := \|\langle v, v\rangle\|_A$ defines a norm on V and we call V a *pre-Hilbert A-module.* If V is in addition complete with respect to this norm we call it a *(right) Hilbert A-module.*

In consistence with Section 1.5 the sesquilinear form above turns out to be conjugate-linear in its first argument.
Note that every C^*-algebra A is a Hilbert A-module via $\langle a, b\rangle := a^*b$.

Definition 6.3.2 (The Hilbert A-module $H \otimes A$)**.** Given a Hilbert space H and a C^*-algebra A, we can define on the algebraic tensor product $H \odot A$ the pre-Hilbert A-module structure

$$\langle v \odot a, w \odot b \rangle := \langle v, w \rangle_H \cdot \langle a, b \rangle_A \quad ; \quad \langle v \odot a, w \odot b \rangle c = \langle v \odot a, w \odot bc \rangle$$

for $v, w \in H$ and $a, b, c \in A$.

Its completion with respect to the norm $\|v \odot a\|^2 := \|\langle v \odot a, v \odot a \rangle\|_A$ is again a (right) Hilbert A-module.

6.4 Hopf *-algebras

Definition 6.4.1. A Hopf *-algebra over $\mathbb{C}$ is given by a tuple $(\mathcal{A}, \Delta, \varepsilon, \delta)$ with the following properties:

(1) $\mathcal{A}$ is a unital *-algebra over $\mathbb{C}$.

(2) $\Delta : \mathcal{A} \to \mathcal{A} \odot \mathcal{A}$ is a *-homomorphism fulfilling

$$(\Delta \odot \mathrm{id}_{\mathcal{A}}) \circ \Delta = (\mathrm{id}_{\mathcal{A}} \odot \Delta) \circ \Delta$$

and called a *comultiplication* or *coproduct.*

(3) $\varepsilon : A \to \mathbb{C}$ is an algebra homomorphism fulfilling

$$(\varepsilon \odot \mathrm{id}_{\mathcal{A}}) \circ \Delta = \mathrm{id}_{\mathcal{A}} = (\mathrm{id}_{\mathcal{A}} \odot \varepsilon) \circ \Delta \tag{6.4.1}$$

and called the *counit.*

(4) $S : \mathcal{A} \to \mathcal{A}$ is a linear map fulfilling

$$m \circ (S \odot \mathrm{id}_{\mathcal{A}}) \circ \Delta = \iota \circ \varepsilon = m \circ (\mathrm{id}_{\mathcal{A}} \odot S) \circ \Delta) \tag{6.4.2}$$

and called the *antipode.*

The definition of a Hopf algebra is nearly the same. The only difference is that $\mathcal{A}$ is just an algebra and Δ just an algebra homomorphism as there is no involution to be respected. In particular, every Hopf *-algebra is a Hopf algebra.
The counit and the antipode of a Hopf algebra are uniquely defined by the pair $(\mathcal{A}, \Delta)$ and Equations 6.4.1 and 6.4.2, compare [Tim08, Rem. 1.3.2 and 1.3.7 (iv)], so we can speak of the pair $(\mathcal{A}, \Delta)$ as a Hopf algebra (or Hopf *-algebra, respectively).