Unirationality of moduli spaces of curves with pencils

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To my parents

Abstract

The main subject of this thesis is the study of the unirationality of moduli spaces of smooth curves admitting several pencils. Using liaison techniques, we prove that the Hurwitz scheme $\mathscr{H}_{g,d}$, parameterizing d-sheeted simply branched covers of the projective line by smooth curves of genus g, up to isomorphism, is unirational for $(g,d) = (10,8)$ and $(13,7)$. We turn the liaison construction to the multiprojective space $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$, which results in the unirationality of an irreducible component of the locus of smooth curves of genus g, carrying three pencils of degree d, for certain values of g and d.

In the second part we study the existing possible numbers of pencils of degree 6 on a smooth hexagonal curve of genus 11. Inside the moduli space of genus 11 curves, we describe a unirational irreducible component of the locus of smooth curves possessing k mutually independent linear systems g^1_6's of type I, for the values $k = 5, \dots, 10$.

Zusamenfassung

Das Hauptthema dieser Arbeit ist das Studim der Unirationalität von Modulräumen von glatten Kurven, welche mehrere Büschel besitzen. Mit Hilfe von sogenannten Liaison-Techniken beweisen wir zunächst, dass das Hurwitz-Schema $\mathscr{H}_{g,d}$ – der Parameterraum von d-blättrigen einfach-verzweigten Überlagerungen der projektiven Geraden von glatten Kurven vom Geschlecht g – unirational ist für $(g,d) = (10,8)$ und $(13,7)$. Wir wenden die Liaison-Konstruktion außerdem auf den multiprojektiven Raum $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ an. Für bestimmte Werte von g und d resultiert dies in der Unirationaliät einer irreduziblen Komponente des Orten von glatten Kurven vom Geschlecht g, welche drei Büschel vom Grad d besitzen.

Im zweiten Teil der Arbeit wird die mögliche Anzahl von Büscheln vom Grad 6 auf glatten hexagonalen Kurven vom Geschlecht 11 untersucht. Im Modulraum von Geschecht 11 Kurven beschreiben wir eine unirationale irreduzible Komponente des Ortes von glatten Kurven mit k unabhängigen Linearsystemen g^1_6 vom Typ I für die Werte $k = 5, \dots, 10$.

Acknowledgement

While my name may be alone on the front cover of this thesis, I'm not its sole contributor. Rather, there are a number of people behind this piece of work who deserve to be acknowledged here:

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Contents

Chapter 1

Introduction and overview of the results

"The first test is beauty: the mathematician's pattern, like painter's or poet's must be beautiful. There is no permanent place in the world for ugly mathematics. "
«G.H. Hardy»

The main focus of this thesis is to study the birational geometry of the moduli spaces of curves, in particular, the unirationality of moduli spaces of curves possessing several pencils. Recall that an algebraic variety X is called unirational, if there exists a rational dominant map $\mathbb{P}^n \dashrightarrow X$ from a projective space $\mathbb{P}^n$ to X. It is a pleasant property, which allows an explicit parametrization of a moduli space in terms of rational maps. This gives the main reason for the historical motivation to study the unirationality of moduli spaces of curves equipped with additional data such as line bundles and marked point.

As the most classical and important moduli space, we consider the Hurwitz schemes

$$\mathscr{H}_{g,d} := \{ C \xrightarrow{d:1} \mathbb{P}^1 \text{ simply branched cover} \mid C \text{ smooth of genus } g\}/\sim$$

parametrizing d-sheeted simply branched covers of the projective line by smooth curves of genus g, up to isomorphism. The natural forgetful map $\pi : \mathscr{H}_{g,d} \longrightarrow \mathscr{M}_g$ relates the geometry of the Hurwitz spaces to that of the moduli spaces $\mathscr{M}_g$ of curves of genus g. It was through Hurwitz spaces that Riemann [Rie57] computed the dimension of $\mathscr{M}_g$, and Severi [Sev68], building on works of Clebsch and Lüroth [Cle73], and Hurwitz [Hur91], showed that $\mathscr{M}_g$ is irreducible. This way the Hurwitz space plays important role in configuration of the geometry of the moduli space of curves $\mathscr{M}_g$.
By classical results of Petri [Pet23], Segre [Seg28], and Arbarello and Cornalba [AC81],

it has been known for a long time that $\mathscr{H}_{g,d}$ is unirational in the range $2 \leq d \leq 5$ and $g \geq 2$. For $g \leq 9$ and $d \geq g$, the unirationality has been proved by Mukai [Muk95]. The most recent contributions, concerning the birational geometry of the Hurwitz spaces has been given by [Ver05, Gei12, Gei13, Sch13, ST16, DS17]. Hurwitz spaces and their Kodaira dimension are also considered in the very recent paper [Far18].

In the second chapter of this thesis, following the paper [KT17], we will settle the unirationality of Hurwitz schemes for two cases as follow.

Theorem 1.0.1. *The Hurwitz space $\mathscr{H}_{10,8}$ is unirational.*

The key point of the proof is that for a general element (C,L) of $\mathscr{W}^1_{10,8}$, the Serre dual bundle $\omega_C \otimes L^{-1}$ gives a g^2_{10}. Using this linear system and a general pencil g^1_6 existing on C, we obtain a model of C in $\mathbb{P}^1 \times \mathbb{P}^2$, which can be linked in two steps to the union of a rational curve and five lines. We show that this process can be reversed and yields a unirational parametrization of the Serre dual space $\mathscr{W}^2_{10,10}$. This proves the unirationality of both $\mathscr{W}^1_{10,8}$ and $\mathscr{H}_{10,8}$. We continue this chapter that by proving the following theorem.

Theorem 1.0.2. *The Hurwitz space $\mathscr{H}_{13,7}$ is unirational.*

This result is obtained by the observation that a general 7-gonal curve of genus 13 can be embedded in $\mathbb{P}^6$ as a curve of degree 17, which is linked to a curve D of genus 10 and degree 15. To exhibit a unirational parametrization of such D's, we prove the unirationality of the moduli space $\mathscr{M}_{10,n}$, of genus 10 curves with up to five marked points. We use a general curve together with 3 marked points to produce a degree 15 curve of genus 10 in $\mathbb{P}^6$.

In the third chapter of this thesis, we study the locus $\mathscr{M}_{g,d}(3)$ of smooth curves of genus g, possessing three mutually independent pencils of degree d. We use the birational model of such a curve induced from the three mutually independent pencils, and turn the liaison construction to the multiprojective space $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$, to link this curve to a rational curve. By reversing the construction, we are able to produce a rational family of curves in $\mathscr{M}_{g,d}(3)$. This leads to the following main result of this chapter.

Theorem 1.0.3. *For $5 \leq d \leq 9$, and all g as in the table 4.1, the moduli space $\mathscr{M}_{g,d}(3)$ of genus g curves possessing three mutually independent pencils of degree d has a unirational irreducible component of expected dimension.*

We remark that in many given unirationality proofs, to show that the rational constructed family yields a dominant map to the moduli space, we only need to carry out the construction for a single test example over a finite field, which satisfies all the needed properties. We make use of the computer algebra system *Macualay2* [GS] to construct the curve with desired properties. Semicontinuity then provides that all assumptions we made actually holds for an open dense subset of the corresponding moduli space in characteristic zero.

In the last chapter of this thesis, motivated by some questions of Michael Kemeny regarding the existence of curves of genus 11 carrying exactly a certain number of pencils of degree 6, we study 6−gonal curves of genus 11.
Let $\mathscr{M}_{11,6}(k) \subset \mathscr{M}_{11}$ be the locus of smooth 6−gonal curves of genus 11, equipped with exactly k mutually independent linear systems g^1_6's of type I. We first investigate the possible values for k, where $\mathscr{M}_{11,6}(k)$ is non-empty. In [Sch02], Schreyer gave a list of conjectural Betti tables for canonical curves of genus 11. Related to our question, and interesting for us, is the plausible Betti table

1	.	.	.	.	.	.	.	.	.
.	36	160	315	288	$5k$	.	.	.	.
.	.	.	.	$5k$	288	315	160	36	.
.	.	.	.	.	.	.	.	.	1

where k is expected to have the values $k = 1, 2, \ldots, 10, 12, 20$. It is known by Hirschowitz–Ramanan [HR98] and [Voi05], that the Green´s conjecture [Gr84] holds for a general curve of genus 11. On the other hand, it is not clear that for a smooth canonical curve of genus 11 with Betti table as above, the number k can be always interpreted as the multiple number of pencils g^1_6's on the curve, however our experiments respect this expectation. Nonetheless, for $k = 1, 2, \ldots, 10, 12$, we can provide families of curves whose generic element carries exactly k mutually independent pencils of type I, and the Betti number $\beta_{5,6} = \beta_{4,6} = 5k$ as expected. Therefore, in this range the locus $\mathscr{M}_{11,6}(k)$ is non-empty.

Once knowing that $\mathscr{M}_{11,6}(k)$ is non-empty for a certain value k, we go further by answering the first natural question concerning the geometry of this locus, in particular its unirationality.

For $k = 1$, the corresponding locus is the famous Brill-Noether divisor $\mathscr{M}_{11,6}$ of 6−gonal curves [HM82], which is irreducible, and known to be unirational [Gei12].

The moduli space $\mathscr{M}_{11,6}(2)$ is irreducible [Ty07], and unirational such that a general element of $\mathscr{M}_{11,6}(2)$ can be obtained from a model of bidegree $(6,6)$ in $\mathbb{P}^1 \times \mathbb{P}^1$ with $\delta = 14$ ordinary double points. Also, by the previous theorem it follows that $\mathscr{M}_{11,6}(3)$ has a unirational irreducible component of expected dimension. A general curves lying on this component can be constructed via liaison in two steps, from a rational curve in multiprojective space $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$.

For $k \geq 4$, we mainly use the model of plane curves of degree 8 and 9 with only ordinary multiple points as singularities. Then, the pencil of lines, conics or cubics through the certain number of singular points cut out the desired number of pencils g^1_6's on the canonical model of these plane curves. As the main results of this chapter, we prove the following theorems.

Theorem 1.0.4. *For $5 \leq k \leq 9$, the moduli space $\mathscr{M}_{11,6}(k)$ has a unirational irreducible component of expected dimension. A general element of this component arises from a degree 9 plane curve with 4 ordinary triple and 5 ordinary double points which contains $k-5$ points among the ninth fixed point of the pencil of cubics passing through the 4 triple and 4 chosen double points. Moreover, at a general point of this component, $\mathscr{M}_{11,6}$ is a simple normal crossing divisor.*

Theorem 1.0.5. *The moduli space $\mathscr{M}_{11,6}(10)$ has a unirational irreducible component of excess dimension 26, where the curves arise from degree 8 plane models with 10 ordinary double points. More precisely, the locus $\mathscr{M}^2_{11,8}$ of curves possessing a linear system g^2_8 is a unirational irreducible component of $\mathscr{M}_{11,6}(10)$ of expected dimension 26.*

The key technique of the proof is to study the space of first order equisingular deformations of the plane curves with prescribed singularities, as well as that of the first order embedded deformations of their canonical model. In fact, let M denote the $5k \times 5k$ submatrix in the deformed minimal resolution corresponding to the general first order deformation family of a canonical curve C with the previous Betti table. We use the condition $M = 0$ to determine the subspace of the deformations with extra syzygies of rank $5k$, and to prove that this space coincide with the tangent space $T_C\mathscr{M}_{11,6}(k)$. It turns out that for $5 \leq k \leq 9$, and respectively k linearly independent linear forms $l_1, \ldots, l_k$ in the free deformation parameters corresponding to a basis of $T_C\mathscr{M}_{11}$, we have $\det M = l_1^5 \cdot \ldots \cdot l_k^5$. This implies that $\mathscr{M}_{11,6}(k)$ has an irreducible component of exactly codimension k inside the moduli space $\mathscr{M}_{11}$. Furthermore, considering $\mathscr{K}_{11}$ to be the locus of the curves $C \in \mathscr{M}_{11}$ with extra syzygies, that is $\beta_{5,6} \neq 0$, it is known by Hirschowitz and Ramanan [HR98] that $\mathscr{K}_{11}$ is a divisor, called the Koszul divisor, such

that $\mathscr{K}_{11} = 5\mathscr{M}_{11,6}$. Thus, $\mathscr{M}_{11,6}$ at the point C is locally analytically union of k smooth transversal branches.
We will then compute the kernel of the Kodaira-Spencer map, and from that the rank of the induced differential maps, to show that the rational families of plane curves dominate this component.

Imposing some further conditions on the model of plane curves, or the choice of singular points, provides a model of genus 11 curves for the two missing cases $k = 4, 12$. A plane curve of degree 9 with 3 triple and 8 ordinary double points, where exactly one of the double points is the ninth fixed point of the pencil of cubics through the 8 singular points by omitting two other double points, gives the model for $k = 4$. By dimension count reasons, the rational family of curves obtained from this model, does not cover any component of the locus $\mathscr{M}_{11,6}(4)$. For 10 general points $P_1, \ldots, P_{10}$ in the projective plane, let $V_1 \subset |L|$ be a pencil inside the linear system of quartics passing through the points, and let $q_1, \ldots, q_6$ be the further fixed points of this pencil. A degree 8 plane curve Γ with 10 ordinary double points $P_1, \ldots, P_{10}$ passing through $q_1, \ldots, q_6$ is a model a genus 11 curve with 12 pencils of degree 6. In fact, considering the rational map associated to $|L|$, the image of Γ under this map is cut out by a unique rank 4 quadric Q on the determinantal image surface of $\mathbb{P}^2$. It turns out that the six points $q_1, \ldots, q_6$ do not lie on a conic, and rather they span a projective plane $\mathbb{P}^2 \subset Q$. As Q is the cone over $\mathbb{P}^1 \times \mathbb{P}^1$, thus the projections to each projective line naturally give two extra pencils of degree 6.

Having described an irreducible unirational component of the loci $\mathscr{M}_{11,6}(k)$ for $k = 5, \ldots, 10$, the first natural question is to ask about the irreducibility of these loci. If the answer is negative, then one can ask how the other irreducible components arise.

Our attempt to utilize the model of plane curves of higher degree with singular points of higher multiplicity, indicates that the models of higher degree are usually a Cremona transformation of our original degree 9 plane model with respect to three singular points. Therefore, considering models of different degrees and singularities, we have not found new elements in these loci. On the other hand, the study of syzygy schemes of curves lying on these loci leads to the following theorem which states the existence of further irreducible components.

Theorem 1.0.6. *For $5 \leq k \leq 8$ and $g = 11$, the locus $\mathscr{M}_{11,6}(k)$ has at least two irreducible components both of expected dimension $3g - 3 - k$, along which $\mathscr{M}_{11,6}$ is generically a simple* *normal crossing divisor.*

This way, the natural remaining question is that whether the moduli spaces $\mathscr{M}_{11,6}(9)$ and $\mathscr{M}_{11,6}(10)$ are irreducible.

Hanieh Keneshlou
Summer 2018

Publication and Software packages

- Hanieh Keneshlou and Fabio Tanturri, *The unirationality of the Hurwitz Schemes $\mathscr{H}_{10,8}$ and $\mathscr{H}_{13,7}$*, accepted for publication in Rendiconti Lincei Matematica e Applicazioni.
- Hanieh Keneshlou and Frank-Olaf Schreyer, *Unirational components of moduli spaces of $6-$gonal genus 11 curves*, in preparation.
- Hanieh Keneshlou and Fabio Tanturri. *SomeUnirationalHurwitzSpaces.m2*, a *Macaulay2* supporting package for the paper "The unirationality of the Hurwitz schemes $\mathscr{H}_{10,8}$ and $\mathscr{H}_{13,7}$", 2017, available at `http://tanturri.perso.math.cnrs.fr/UnirationalHurwitzSchemes/html/`.
- Hanieh Keneshlou. *UnirationalBNSchemes.m2*, Unirationality of Brill-Noether locus of the curves with three pencils, a *Macaulay2* package, 2017, available at: `https://www.math.uni-sb.de/ag/schreyer/index.php/people/researchers/119-hanieh-keneshlou`.
- Hanieh Keneshlou and Frank-Olaf Schreyer, *UnirationalComponentOf6gonalgenus11 curves.m2*, a *Macaulay2* supporting package for the paper "Unirational components of moduli spaces of $6-$gonal genus 11 curves", 2018, available at: `https://www.math.uni-sb.de/ag/schreyer/index.php/people/researchers/119-hanieh-keneshlou`.
- Hanieh Keneshlou and Frank-Olaf Schreyer, *FirstOrderDeformations.m2*, a *Macaulay2* package, 2018, available at: `https://www.math.uni-sb.de/ag/schreyer/index.php/people/researchers/119-hanieh-keneshlou`.

Notation 1.0.7. Throughout this thesis, $\mathbb{K}$ denotes an algebraically closed field, unless otherwise mentioned. For our computational approach, we always reduce the computations to a finite field, which then the semicontinuity gives rise the analogues of the results over a field of characteristic 0.

Chapter 2

Preliminaries

In this chapter, we introduce some notions and background facts on some important or main objects of this thesis. Most of the facts stated here can be found in standard textbooks. Fo sections 2.1 and 2.2, we mainly refer to [ACGH85] and [ACG10]. The last section follows [Sch86] and [vB07].

2.1 Brill-Noether theory

We resume some central facts of Brill-Noether theory, concerning the linear series on algebraic curves. Throughout this section, d, r are non-negative integers.

Let C be a smooth curve of genus g, and $\mathrm{Pic}^d(C)$ be the Picard variety parametrizing the isomorphism classes of degree d line bundles over C (or equivalently, the divisors of degree d modulo linear equivalence).

Definition 2.1.1. A pair (L,V) consisting of a line bundle $L \in \mathrm{Pic}^d(C)$ and an $(r+1)$-dimensional vector space $V \subset \mathrm{H}^0(C,L)$ of global sections of L is called a linear series of degree d and dimension r. A linear series is called base point free if the sections in V have no common zeros. Moreover, it is called complete when $V = \mathrm{H}^0(C,L)$.

In standard notations, a linear series of dimension r and degree d is usually referred to as a g^r_d. In particular, a g^1_d is called a *pencil.*
The set of complete linear series of degree d and dimension at least r is parametrized by the so-called *Brill-Noether locus* $W^r_d(C) \subset \mathrm{Pic}^d(C)$. Indeed,

$$\mathrm{Supp}(W^r_d(C)) = \{L \in \mathrm{Pic}^d(C) \ : \ h^0(C,L) \geq r+1\}.$$

A determinantal description of $W^r_d(C)$ equips this with a natural scheme structure. In fact, let $\mathscr{L}$ be a Poincáre line bundle of degree d on C, that is a line bundle on $C \times \mathrm{Pic}^d(C)$ which restricts to L on $C \times \{L\}$ for each $L \in \mathrm{Pic}^d(C)$. We further choose an effective divisor E on C of degree

$$m := \deg(E) \geq 2g - d - 1.$$

For the second projection map

$$\nu : C \times \mathrm{Pic}^d(C) \longrightarrow \mathrm{Pic}^d(C),$$

and the product divisor $\Gamma = E \times \mathrm{Pic}^d(C)$, the direct image sequence of

$$0 \longrightarrow \mathscr{L} \longrightarrow \mathscr{L}(\Gamma) \longrightarrow \frac{\mathscr{L}(\Gamma)}{\mathscr{L}} \longrightarrow 0$$

is

$$0 \longrightarrow \nu_*\mathscr{L} \longrightarrow \nu_*\mathscr{L}(\Gamma) \xrightarrow{\gamma} \nu_*(\frac{\mathscr{L}(\Gamma)}{\mathscr{L}}) \longrightarrow R^1\nu_*\mathscr{L} \longrightarrow 0.$$

The Brill-Noether locus $W^r_d(C)$ can be realized as degeneracy locus of the map γ, where the middle terms are locally free sheaves of ranks $d + m - g + 1$ and m, respectively. Therefore, setting $X := \mathrm{Pic}^d(C)$ we have the natural scheme structure

$$W^r_d(C) = X_{m+d-g-r}(\gamma).$$

From this definition, the "expected dimension" of $W^r_d(C)$ is the *Brill-Noether number*

$$\rho(g,d,r) := g - (r+1)(g-d+r),$$

and therefore $\dim W^r_d(C) \geq \rho$.

Important results regarding the non-emptiness and the dimension of the Brill-Noether loci were proven by Kempf, Kleiman-Laksov, Griffiths and Harris. More precisely, as we mentioned above, if $W^r_d(C)$ is non-empty, then it has dimension at least equal to $\rho(g,d,r)$. The first natural question is whether the condition $\rho(g,d,r) \geqslant 0$ suffices to conclude that $W^r_d(C)$ is non-empty. This was answered by the following existence theorem, independently proved by Kempf and Kleiman–Laksov.

Theorem 2.1.2 (Existence Theorem). *Let C be a smooth curve of genus g, and d,r be non-negative integers such that $d \geq 1$. If*

$$\rho(g,d,r) := g - (r+1)(g-d+r) \geqslant 0,$$

then $W^r_d(C)$ is non-empty. Moreover, if $r \geqslant d - g$, every component of $W^r_d(C)$ has dimension at least equal to $\rho(g,d,r)$.

Proof. See [Kem71], [KL72], [KL74]. □

Fulton and Lazarsfeld proved the following Lefschetz (or Bertini) type of result.

Theorem 2.1.3 (Connectedness Theorem). *Let C be a smooth curve of genus g, and d, r be non-negative integers such that $d \geq 1$. If*

$$\rho(g,d,r) := g - (r+1)(g-d+r) \geqslant 1,$$

then $W^r_d(C)$ is connected.

Proof. See [FL81]. □

Describing the Zariski tangent space at a point of $W^r_d(C)$ as the set of isomorphism classes of first order deformations of the corresponding line bundle determines the smooth locus of the Brill-Noether variety as follow.

Proposition 2.1.4. *(i) Let L be a point in $W^r_d(C) \setminus W^{r+1}_d(C)$. Then the tangent space to $W^r_d(C)$ at L is*

$$T_L(W^r_d(C)) = (\operatorname{Im}\mu_0)^{\perp},$$

where

$$\mu_0 : H^0(C,L) \otimes H^0(C,K_C \otimes L^{-1}) \longrightarrow H^0(C,K_C),$$

is the Petri map. Thus $W^r_d(C)$ is smooth of dimension ρ at L if and only if μ_0 is injective.
(ii) If $L \in W^{r+1}_d(C)$, then

$$T_L(W^r_d(C)) = T_L(\operatorname{Pic}^d(C)).$$

In particular, if $W^r_d(C)$ has the expected dimension ρ, and $r > d-g$, then L is a singular point of $W^r_d(C)$.

Proof. See [ACGH85], Chapter IV, Proposition 4.2. □

It was conjectured by Brill and Noether that a generic curve C in the moduli space $\mathscr{M}_g$ of curves of genus g, is (so-called now) Brill-Noether general. In fact, the variety $W^r_d(C)$ is empty, whenever $\rho(g,d,r) < 0$, and in case $\rho(g,d,r) \geqslant 0$, it is non-empty of pure dimension $\rho(g,d,r)$. This was first proved by Griffiths and Harris in the following dimension theorem.

Theorem 2.1.5 (Dimension Theorem). *Let C be a general curve of genus g, and d, r be non-negative integers such that $d \geq 1$. If*

$$\rho(g,d,r) := g - (r+1)(g-d+r) < 0,$$

then $W^r_d(C)$ is empty. If $\rho(g,d,r) \geqslant 0$, then $W^r_d(C)$ is reduced and of pure dimension $\rho(g,d,r)$.

Proof. See [GH80]. □

2.2 Hurwitz schemes

In this section, we introduce the Hurwitz scheme, and recall some fundamental properties of it due to Lüroth, Clebsch, and Hurwitz.

Definition 2.2.1. Let C be a smooth curve of genus g. A ramified d-sheeted covering of the projective line $\mathbb{P}^1$

$$f : C \longrightarrow \mathbb{P}^1$$

(or equivalently the corresponding pencil g^1_d) is called simply branched (resp. simple), if for every ramification point $p \in C$, we have the ramification index

$$e_p = \mathrm{Length}(\Omega_{C/\mathbb{P}^1})_p + 1 = 2,$$

and no two ramification points of f lie over the same point of $\mathbb{P}^1$.

For a simply branched covering f, let

$$R_f := \sum_{p \in C} (e_p - 1).p \in \mathrm{Div}(C)$$

denote the ramification divisor of f, and

$$\Lambda_f = f_*(R_f) \in \mathrm{Div}(\mathbb{P}^1)$$

denote the branch divisor. By the Riemann-Hurwitz formula

$$\omega := 2d + 2g - 2$$

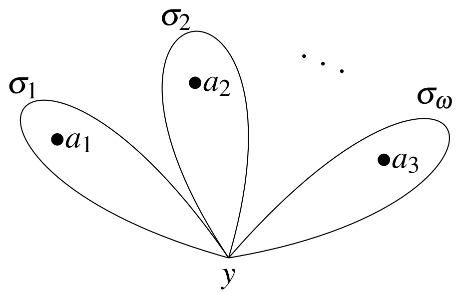
is the degree of branch locus of f. Let $\mathbb{P}_\omega = \mathrm{Sym}^\omega(\mathbb{P}^1) \setminus \Delta$ be the open subset of the ω-th symmetric product of $\mathbb{P}^1$ consisting of unordered ω-tuples of distinct points in $\mathbb{P}^1$, where Δ is the closed subscheme of points with at least two identical summands.
For any $A = (a_1, \dots, a_\omega) \in \mathbb{P}_\omega$, let $\mathscr{H}_{g,d}(A)$ denote the set

$$\mathscr{H}_{g,d}(A) := \{\, f : C \xrightarrow{d:1} \mathbb{P}^1 \ \text{ simply branched covering with } \Lambda_f = A\} / \sim,$$

where two simply branched covering $f \sim f'$ are said to be equivalent, if there exists an isomorphism $\alpha : C \longrightarrow C'$, such that the following diagram is commutative:

$$\begin{array}{ccc} C & \xrightarrow{\ \alpha\ } & C' \\ & {\scriptstyle f}\searrow \quad \swarrow{\scriptstyle f'} & \\ & \mathbb{P}^1 & \end{array}$$

For such A, choose a base point $y \in \mathbb{P}^1 \setminus A$, and let $\sigma_1, \dots, \sigma_\omega$ be a system of generators for $\pi_1(\mathbb{P}^1 \setminus A, y)$ pictured as follow.



The *monodromy representation* is defined by

$$\begin{aligned} \phi(f) : \pi_1(\mathbb{P}^1 \setminus A, y) &\longrightarrow (\text{Permutation group of } f^{-1}(y)) \\ \sigma &\mapsto (p \mapsto p_\sigma) \end{aligned}$$

where p_σ is the endpoint of the unique path lifting of σ to $C \setminus R_f$. This will induce a well-defined map

$$\begin{aligned} \phi : \mathscr{H}_{g,d}(A) &\longrightarrow \mathrm{Hom}(\pi_1(\mathbb{P}^1 \setminus A, y), S_d)^{ext} \\ [f] &\mapsto [\phi(f)] \end{aligned}$$

where S_d is the symmetric group of d letters. The right hand side is the group of homomorphisms $\pi_1(\mathbb{P}^1 \setminus A, y) \longrightarrow S_d$ up to conjugation with elements in S_d.

Theorem 2.2.2 (Riemann's existence theorem). *The map ϕ above is injective. Moreover, its image consists of those classes which are induced by irreducible representations ξ, such that*

$$\xi(\sigma_i) = t_i, \quad i = 1, \ldots, \omega$$

is a transposition and $\prod t_i = 1$.

In conclusion, via ϕ, $\mathscr{H}_{g,d}(A)$ can be identified with the set $\mathscr{C}_{g,d}$ of conjugacy classes of ω-tuples $[t_1, \ldots, t_\omega]$ of transpositions t_i, which generates a transitive subgroup of S_d, and satisfy $\prod t_i = 1$. We define the Hurwitz space as a set to be

$$\mathscr{H}_{g,d} := \bigsqcup_{A \in \mathbb{P}_\omega} \mathscr{H}_{g,d}(A).$$

Given an element $[f] \in \mathscr{H}_{g,d}(A)$, we say that $[t_1, \ldots, t_\omega]$ is the symbol of $[f]$ with respect to basis $\sigma_1, \ldots, \sigma_\omega$. There is a natural map

$$\begin{aligned} \Lambda : \mathscr{H}_{g,d} &\longrightarrow \mathbb{P}_\omega \\ [f] &\mapsto \Lambda_f. \end{aligned}$$

Since each fibre of Λ can be identified with the finite set $\mathscr{C}_{g,d}$, the Hurwitz space $\mathscr{H}_{g,d}$ can be equipped with a unique complex structure, which makes it into a ω-dimensional complex manifold, and Λ is a topological covering.

Theorem 2.2.3 (Lüroth,Clebsch, Hurwitz). *$\mathscr{H}_{g,d}$ is connected.*

Proof. Consider the topological covering

$$\Lambda : \mathscr{H}_{g,d} \longrightarrow \mathbb{P}_\omega.$$

The idea of the prove is to show that that for any point $A = (a_1, \dots, a_\omega) \in \mathbb{P}_\omega$, the fundamental group $\pi_1(\mathbb{P}_\omega, A)$ acts transitively on the corresponding fibre $\mathscr{H}_{g,d}(A)$. To this end, consider the loops

$$\Gamma_i : [0,1] \longrightarrow \mathbb{P}_\omega, \quad i = 1, \dots, \omega,$$

with endpoints at A of the form

$$\Gamma_i(t) = (a_1, \dots, a_{i-1}, \gamma_i(t), \gamma_i'(t), a_{i+2} \dots, a_\omega),$$

where

$$\gamma_i, \gamma_i' : [0,1] \longrightarrow \mathbb{P}^1 \setminus \{a_1, \dots, a_{i-1}, a_{i+2} \dots, a_\omega\}$$

are paths with $\gamma_i(0) = \gamma_i'(1) = a_i, \gamma_i(1) = \gamma_i'(0) = a_{i+1}$.
To show that $\pi_1(\mathbb{P}_\omega, A)$ acts transitively on $\mathscr{H}_{g,d}(A)$, it suffices to show that the subgroup Γ generated by paths Γ_i acts transitively on $\mathscr{H}_{g,d}(A)$. Γ_i acts on elements of $\mathscr{C}_{g,d}$ as following:

$$\Gamma_i.[t_1, \dots, t_\omega] = [t_1, \dots, t_{i-1},\ t_i t_{i+1} t_i,\ t_i,\ t_{i+2}, \dots, t_\omega].$$

A combinatorial argument then shows that the orbit of any element of $\mathscr{C}_{g,d}$ under the action of Γ contains the element

$$[\underbrace{(1\ 2), \dots, (1\ 2)}_{2g+2\text{-times}}, (2\ 3), (2\ 3), (3\ 4), (3\ 4), \dots, (d-1\ d), (d-1\ d)].$$

Therefore, there is only one orbit, and the action is transitive. □

Let $\mathscr{W}^r_{g,d}$ denote the so-called *Universal Brill-Noether scheme*, given set-theoretically by

$$\mathscr{W}^r_{g,d} := \{(C, L)\ :\ C \in \mathscr{M}_g,\ L \in W^r_d(C)\}.$$

There is a natural dominant morphism $\alpha : \mathscr{H}_{g,d} \longrightarrow \mathscr{W}^1_{g,d}$, which is a PGL(2)−bundle over a dense open subset of $\mathscr{W}^1_{g,d}$. Therefore, from the above theorem, the following is concluded:

Corollary 2.2.4. *The universal Brill-Noether scheme $\mathscr{W}^1_{g,d}$ is an irreducible quasi-projective variety of dimension $\omega - 3 = 2g + 2d - 5$.*

2.3 Syzygies of canonical curves, and special linear series

In this section, we introduce some important notions, and review some facts about the relation between the minimal free resolution of the canonical rings and the Brill-Noether theory of canonical curves.

Let C be a smooth non-hyperelliptic curve of genus $g \geq 2$, and ω_C be its canonical line bundle. The associated *canonical map*

$$\phi_{|K_C|} : C \longrightarrow \mathbb{P}^{g-1} = \mathbb{P}(\mathrm{H}^0(C, \omega_C)^*),$$

determined by the canonical linear series $|K_C|$ is an embedding. We usually identify C with its image in $\mathbb{P}^{g-1}$, and we refer to it as a *canonical curve.* Let

$$I_C \subset S := \mathbb{K}[x_0, \ldots, x_{g-1}],$$

be the homogeneous ideal of C in the homogeneous coordinate ring S of $\mathbb{P}^{g-1}$. By a famous theorem of Hilbert (see [Eis05], Theorem 1.1), the homogeneous coordinate ring $S_C := S/I_C$ of C, a finitely generated graded $S-$module, admits a *minimal free resolution* of finite length. More precisely, there is a chain complex (unique up to isomorphisms of chain complexes inducing identity on S_C (see [Eis05], Theorem 1.6)) of graded $S-$modules

$$\mathbf{F}: \ 0 \longleftarrow S_C \longleftarrow F_0 \xleftarrow{\varphi_1} F_1 \xleftarrow{\varphi_2} \cdots \xleftarrow{\varphi_m} F_m \longleftarrow 0$$

with free modules $F_i = \oplus_j S(-j)^{\beta_{i,j}}$, such that the image of the boundary map φ_i is contained in the submodule $\mathfrak{m}F_{i-1}$, where $\mathfrak{m} = \langle x_0, \ldots, x_{g-1} \rangle \subset S$ is the irrelevant maximal ideal of S. In particular, the numbers $\beta_{i,j}$ are independent of the choice of minimal free resolution.

Definition 2.3.1 (Betti numbers). With the above notations, the numbers $\beta_{i,j}$ are called the *graded Betti numbers* of S_C, or rather of the curve C.

By definition, $\beta_{i,j}$ gives the number of the generators of the graded module F_i in degree j. More precisely, if we tensor the above complex with the $S-$module $\mathbb{K}$, since $\mathbf{F}$ is minimal, all maps in $\mathbf{F} \otimes_S \mathbb{K}$ are zero, and so $\mathrm{Tor}_i^S(S_C, \mathbb{K}) = F_i \otimes_S \mathbb{K}$. Thus, we obtain

$$\beta_{i,j} = \dim_{\mathbb{K}} \mathrm{Tor}_i^S(S_C, \mathbb{K})_j.$$

We usually collect all the Betti numbers of the minimal free resolution $\mathbf{F}$ in a so-called *Betti table*, and we use the *Macaulay2* notation to write them down as follows

	0	1	$\cdots$	$m-1$	m
0	$\beta_{0,0}$	$\beta_{1,1}$	$\cdots$	$\beta_{m-1,m-1}$	$\beta_{m,m}$
1	$\beta_{0,1}$	$\beta_{1,2}$	$\cdots$	$\beta_{m-1,m}$	$\beta_{m,m+1}$
$\vdots$	$\vdots$	$\vdots$	$\cdots$	$\vdots$	$\vdots$
s	$\beta_{0,s}$	$\beta_{1,s}$	$\cdots$	$\beta_{m-1,m-1+s}$	$\beta_{s,m+s}$

where the i-th column corresponds to generators of the graded module F_i in the free resolution. For better readability we usually write " ." when $\beta_{i,j} = 0$. We note that the entry in the $j-$row and the $i-$th column is $\beta_{i,i+j}$ rather than $\beta_{i,j}$.

Further properties of the canonical ring reveal a better configuration of the Betti table of a canonical curve. Due to the classical theorem of Max Noether [Noe80], the coordinate ring of a canonical curve is projectively normal, that is the maps

$$\mathrm{H}^0(\mathbb{P}^{g-1}, \mathscr{O}_{\mathbb{P}^{g-1}}(n)) \longrightarrow \mathrm{H}^0(C, \omega_C^{\otimes n})$$

are surjective for every n. Therefore,

$$S_C \cong \bigoplus_{n\geq 0} \mathrm{H}^0(C, \omega_C^{\otimes n}),$$

and S_C is a Cohen-Macaulay S-module. This already shows that the length of the minimal free resolution of C is equal to $g-2$, thanks to the Auslander-Buchsbaum formula ([Eis05], Theorem A2.15). By classical works of Petri [Pet23], the homogeneous ideal of a smooth canonical curve is generated by quadrics, unless it is a trigonal, or a plane quintic. In the exceptional cases, $\beta_{1,3} = g-3$. Thus, for all $i \geq 1$ and $j \geq 3$, we have $\beta_{i,i} = \beta_{1,j+1} = 0$. In particular, a canonical curve admits only linear and quadratic syzygies

$$F_i = S(-i-1)^{\beta_{i,i+1}} \oplus S(-i-2)^{\beta_{i,i+2}},$$

except for the last syzygy module $F_{g-2} = S(-g-1)$.

Summarizing all above, from the nice symmetric property, obeyed by the Betti numbers, the Betti tables of canonical curves have the following shape.

Proposition 2.3.2. *Let C be a non-hyperelliptic canonical curve of genus $g \geq 3$. Then*

$$\omega_C \cong \mathscr{E}xt^{g-2}(\mathscr{O}_C, \mathscr{O}_{\mathbb{P}^{g-1}}(-g)) \cong \mathscr{O}_C(1)$$

The minimal free resolution of S_C is up a shift, self dual, with the symmetric property

$$\beta_{i,j} = \beta_{g-2-i,g+1-j}$$

and has the following Betti table

	0	1	2	$\cdots$	$g-4$	$g-3$	$g-2$
0	1	.	.	$\cdots$	.	.	.
1	.	$\beta_{1,2}$	$\beta_{2,3}$	$\cdots$	$\beta_{g-4,g-3}$	$\beta_{g-3,g-2}$	.
2	.	$\beta_{g-3,g-2}$	$\beta_{g-4,g-3}$	$\cdots$	$\beta_{2,3}$	$\beta_{1,2}$	.
3	.	.	.	$\cdots$	.	.	1

where $\beta_{1,2} = \binom{g-2}{2}$.

Proof. See [Eis05], Proposition 9.6. □

A projective Cohen-Macaulay variety $X \subset \mathbb{P}^r$ is called Gorenstein if there exist an integer $n \in \mathbb{Z}$ such that

$$\operatorname{Ext}^{\operatorname{codim} X}(S_X, S) \cong S_X(n)$$

In this sense, a canonical curve is Gorenstein, and has the self dual minimal free resolution.

Definition 2.3.3. The linear colength $l(C)$ of C is the smallest integer i such that

$$\beta_{g-2-i,g-1-i} = \beta_{i,i+2} \neq 0.$$

Definition 2.3.4. The Clifford index of a line bundle $\mathscr{L}$ on C is defined as

$$\operatorname{Cliff}(\mathscr{L}) := \deg(\mathscr{L}) - 2(\mathrm{h}^0(\mathscr{L}) - 1)$$

The Clifford index $\operatorname{Cliff}(C)$ of C, is the minimum Clifford index of all line bundles $\mathscr{L}$ on C satisfying $\mathrm{h}^i(\mathscr{L}) \geq 2$ for $i = 0, 1$.

We note that by Serre duality, we have $\operatorname{Cliff}(\mathscr{L}) = \operatorname{Cliff}(\omega_C \otimes \mathscr{L}^{-1})$. By Clifford's theorem ([ACGH85], chapter III) the dimension of the complete linear series associated to a special line bundle, that is a line bundle with $\mathrm{h}^1(\mathscr{L}) > 0$, cannot be very big compared to its degree. More precisely, for any special line bundle $\operatorname{Cliff}(\mathscr{L}) \geq 0$ with equality for trivial divisors as the zero divisor, or the canonical divisor. Therefore, the Clifford index is the smallest non-negative integer c such that C has a complete linear series g^n_{2n+c} for some positive integer n.
In [Gr84] Green conjectured that the Clifford index of a canonical curve C is related to some vanishing properties of the Betti numbers, namely to the linear colength of C.

Conjecture 2.3.5 (Green's conjecture). *Let C be a canonical curve over a field of characteristic* 0. *Then* $l(C) = \operatorname{Cliff}(C)$.

The inequality $l(C) \leq \mathrm{Cliff}(C)$ has been proved by Green and Lazarsfeld over any field ([Gr84], Appendix). Although it is known that the obvious extension of the Green's conjecture to positive characteristic fails in charecteristic 2 for curves of genus 7 [Sch86] and 9 [Muk95], in a very recent work [BS18], Bopp and Schreyer have posed a refined version of Green's conjecture, which holds conjecturally in positive characteristic.
Although Green's conjecture is still open in complete context, it is settled for many curves defined over field of characteristic 0. For general curves in [Voi02] and [Voi05], general $d-$gonal curves with $2 < d < \lceil \frac{g-1}{2} \rceil$ in [Apr05], curves of odd genus with $l(C) = (g-1)/2$ in [HR98] and [Voi05], smooth curves lying on any K_3 surface in [AF11], and curves with linear colength $l(C) \leq 2$ in [Noe80], [Pet23] and [Sch91].

Definition 2.3.6. Let C be a smooth curve. The gonality $\mathrm{gon}(C)$ of C is the smallest integer d, for which there exist a $d:1$ morphism to $\mathbb{P}^1$. A curve of gonality d is usually referred to as a $d-$gonal curve.

It turns out that in general the Clifford index of a curve can be computed by complete linear series of dimension one.

Theorem 2.3.7 ([CM91]). *We have* $\mathrm{Cliff}(C)+2 \leq \mathrm{gon}(C) \leq \mathrm{Cliff}(C)+3$.

The equality $\mathrm{gon}(C) = \mathrm{Cliff}(C)+3$ holds if and only if C is isomorphic to a smooth plane curve, or some other "rare" cases (see [ELMS89]). Therefore, except for those exceptional cases, if Green's conjecture holds, then the gonality is $\mathrm{gon}(C) = l(C)+2$, and can be read off directly from the shape of the Betti table.

2.4 Rational normal scrolls and scrollar syzygies

In the last section of this chapter, we resume the definition and some important facts on rational normal scrolls. Almost everything we recall here is treated in the original papers of Schreyer [Sch86] and Bothmer [vB07].

Definition 2.4.1. Let $e_1, \ldots, e_d \geq 0$ be non-negative integers, $\mathscr{E} = \mathscr{O}_{\mathbb{P}^1}(e_1) \oplus \cdots \oplus \mathscr{O}_{\mathbb{P}^1}(e_d)$ be a locally free sheaf of rank d on projective line $\mathbb{P}^1$, and

$$\pi : \mathbb{P}(\mathscr{E}) \longrightarrow \mathbb{P}^1$$

be the corresponding $\mathbb{P}^{d-1}$-bundle. For $f := e_1 + \ldots + e_2 \geq 2$ the image of $\mathbb{P}(\mathscr{E})$ under the map

$$j : \mathbb{P}(\mathscr{E}) \longrightarrow X \subset \mathbb{P}^r = \mathbb{P}(\mathrm{H}^0(\mathbb{P}(\mathscr{E}), \mathscr{O}_{\mathbb{P}(\mathscr{E})}(1))^*),$$

associated to the tautological bundle, is called a rational normal scroll of type $S(e_1, \ldots, e_d)$, where $r = f + d - 1$.

It is known by Harris [Har81], that X is a non-degenerate, irreducible variety of minimal degree

$$\deg X = f = r - d + 1 = \operatorname{codim} X + 1.$$

If all $e_i > 0$, then X is smooth, and $j : \mathbb{P}(\mathscr{E}) \longrightarrow X$ is an isomorphism, otherwise X is singular, and j is a resolution of singularities of X. The singularities of X are rational, that is

$$j_*\mathscr{O}_{\mathbb{P}(\mathscr{E})} = \mathscr{O}_X, \;\; R^i j_*\mathscr{O}_{\mathbb{P}(\mathscr{E})} = 0 \;\; \forall i > 0,$$

and therefore one can usually replace X by $\mathbb{P}(\mathscr{E})$ for cohomological computation.

In [Har81] it is first shown that the Picard group $\operatorname{Pic}(\mathbb{P}(\mathscr{E}))$ is generated by hyperplane class $H = [j^*\mathscr{O}_{\mathbb{P}^r}(1)]$ and the class of the ruling $R = [\pi^*\mathscr{O}_{\mathbb{P}^1}(1)]$, such that the intersection products are given by

$$H^d = f, \;\; H^{d-1}.R = 1, \;\; R^2 = 0.$$

Furthermore, following [Sch86], there exist basic sections $\varphi_i \in \mathrm{H}^0(\mathbb{P}(\mathscr{E}), \mathscr{O}_{\mathbb{P}(\mathscr{E})}(H - e_iR))$, and $s,t \in \mathrm{H}^0(\mathbb{P}(\mathscr{E}), \mathscr{O}_{\mathbb{P}(\mathscr{E})}(R))$ such that every section of $\psi \in \mathrm{H}^0(\mathbb{P}(\mathscr{E}), \mathscr{O}_{\mathbb{P}(\mathscr{E})}(aH + bR))$ can be identified with a homogeneous polynomial

$$\psi = \sum_{\alpha} p_\alpha(s,t)\varphi_1^{\alpha_1}\dots\varphi_d^{\alpha_d}$$

of degree $a = \alpha_1 + \ldots + \alpha_d$ in φ_i's, and coefficients homogeneous polynomials p_α are of degree

$$\deg p_\alpha = \alpha_1 e_1 + \ldots + \alpha_d e_d + b.$$

This gives in fact a description of the coordinate ring of $\mathbb{P}(\mathscr{E})$,

$$R_{\mathbb{P}(\mathscr{E})} = \bigoplus_{a,b\in\mathbb{Z}} \mathrm{H}^0(\mathbb{P}(\mathscr{E}), \mathscr{O}_{\mathbb{P}(\mathscr{E})}(aH + bR)),$$

as the bigraded Cox ring $\mathbb{K}[s,t,\varphi_1,\ldots,\varphi_d]$ with $\deg(s) = \deg(t) = (0,1)$ and $\deg(\varphi_i) = (1,-e_i)$.

Rational normal scrolls can also be defined as determinantal varieties. The resolution of a scroll is an Eagon-Northcott complex.

Theorem 2.4.2. *Let X be a scroll of type $S(e_1,\ldots,e_d)$, and*

$$\Phi = \begin{pmatrix} x_{10} & x_{11} & \cdots & x_{1e_1-1} & \cdots x_{d0} & \cdots & x_{de_d-1} \\ x_{11} & x_{12} & \cdots & x_{1e_1} & \cdots x_{d1} & \cdots & x_{de_d} \end{pmatrix}$$

be the $2 \times f$ *matrix given by multiplication map*

$$\mathrm{H}^0(\mathbb{P}(\mathscr{E}), \mathscr{O}_{\mathbb{P}(\mathscr{E})}(R)) \otimes \mathrm{H}^0(\mathbb{P}(\mathscr{E}), \mathscr{O}_{\mathbb{P}(\mathscr{E})}(H-R)) \longrightarrow \mathrm{H}^0(\mathbb{P}(\mathscr{E}), \mathscr{O}_{\mathbb{P}(\mathscr{E})}(H))$$

where

$$x_{ij} = t^j s^{e_i - j} \varphi_i \text{ with } i = 1, \dots, d, \text{ and } j = 0, \dots, e_i$$

form a basis of $\mathrm{H}^0(\mathbb{P}(\mathscr{E}), \mathscr{O}_{\mathbb{P}(\mathscr{E})}(H)) \cong \mathrm{H}^0(\mathbb{P}^r, \mathscr{O}_{\mathbb{P}^r}(1))$. *Then the vanishing ideal* I_X *of* X *is generated by the* 2×2 *minors of* Φ. *The minimal free resolution of* X *is the Eagon-Northcott complex, with the Betti table*

	0	1	2	$\cdots$	$f-1$
0	1	.	.	$\cdots$	.
1	.	$\binom{f}{2}$	$2\binom{f}{3}$	$\cdots$	$(f-1)\binom{f}{f}$

Proof. See [Sch86], section 1.6. □

Now we turn to the construction of scrolls from pencil on varieties. In fact, for a linearly normal embedded smooth variety $V \subset \mathbb{P}^r = \mathbb{P}(\mathrm{H}^0(V, \mathscr{O}_V(H))$ equipped with a pencil $(D_\lambda)_{\lambda \in \mathbb{P}^1}$ of divisors on V, we can construct a rational normal scroll X such that $V \subset X$, and furthermore the pencil is cut out by the class of the ruling R on X. Suppose D is a divisor on V such that $\mathrm{h}^0(V, \mathscr{O}_V(D)) \geq 2$ and $\mathrm{h}^0(V, \mathscr{O}_V(H-D)) = f$. Let $G \subset \mathrm{H}^0(V, \mathscr{O}_V(D))$ be the 2-dimensional subvector space that defines the pencil of divisors $(D_\lambda)_{\lambda \in \mathbb{P}^1}$. Then from the multiplication map

$$G \otimes \mathrm{H}^0(V, \mathscr{O}_V(H-D)) \longrightarrow \mathrm{H}^0(V, \mathscr{O}_V(H)),$$

we obtain a $2 \times f$ matrix with linear entries whose minors vanish on V. By [EH87], it turns out that the variety defined by these minors is a scroll of dimension $r - f + 1$ and degree f such that the pencil $(D_\lambda)_{\lambda \in \mathbb{P}^1}$ is cut out by the class of the ruling R on X. Geometrically, X can be realized as union of the linear spans of the divisors D_λ, that is

$$X = \bigcup_{\lambda \in \mathbb{P}^1} \overline{D_\lambda}$$

where $\overline{D_\lambda} := \bigcap_{D_\lambda \subset H} H \subset \mathbb{P}^r$. Conversely, if X is a scroll of degree f containing V, the ruling R on X cuts out a pencil of divisors $(D_\lambda)_{\lambda \in \mathbb{P}^1} \subset |D|$ with $\mathrm{h}^0(V, \mathscr{O}_V(H-D)) = f$.

Another way in which rational normal scrolls show up, is syzygy schemes associated to the scrollar syzygies.

Let $C \subset \mathbb{P}^{g-1}$ be a smooth and irreducible canonical curve with homogeneous ideal $I_C \subset S$ and

$$\mathbf{L}_\bullet : \; 0 \longleftarrow S \longleftarrow S(-2)^{\beta_{1,2}} \longleftarrow S(-3)^{\beta_{2,3}} \longleftarrow \cdots$$

be *the linear strand* of a minimal free resolution of $S_C = S/I_C$. For a $p-$th linear syzygy $s \in L_p$, let V_s be the smallest vector space such that there is a commutative diagram

$$\begin{array}{ccc} L_{p-1} & \longleftarrow & L_p \\ \uparrow & & \uparrow \\ V_s \otimes S(-p) & \longleftarrow & S(-p-1). \end{array}$$

The rank of the syzygy s is defined to be $\mathrm{rk}(s) := \dim V_s$.
This diagram can be extended to a map from the Koszul complex to the linear strand of C. Namely, since the dual complex $\mathrm{Hom}(\mathbf{L}_\bullet, S)$ is a free complex, and the dual Koszul complex is exact, the maps of the dual diagram extend to a morphism of complexes, that we dualize again to obtain

$$\begin{array}{ccccccccc} S & \longleftarrow & L_1 & \longleftarrow & \cdots \longleftarrow & L_{p-1} & \longleftarrow & L_p \\ \uparrow{\scriptstyle \varphi_2} & & \uparrow & & & \uparrow & & \uparrow \\ \bigwedge^p V \otimes S(-1) & \longleftarrow & \bigwedge^{p-1} V \otimes S(-2) & \longleftarrow & \cdots \longleftarrow & V \otimes S(-p) & \longleftarrow & S(-p-1) \end{array}$$

By degree reason, there are no non-trivial homotopies, and therefore all the vertical maps except for φ_2 are unique. The *syzygy scheme* $Syz(s)$ of the syzygy $s \in L_p$ is the subscheme defined by the ideal of quadric forms

$$I_s := \mathrm{Im}(\bigwedge^{p-1} V \otimes S(-2) \longrightarrow S).$$

The $p-$th syzygy scheme $Syz_p(C)$ of C is defined by the intersection $\bigcap_{s \in L_p} Syz(s)$.
A $p-$th linear syzygy of a canonical curve has rank $\geq p+1$, and syzygies of minimal possible rank $p+1$ are called *scrollar syzygies.* The following theorem due to Bothmer explains the scrollar terminology.

Theorem 2.4.3 ([vB07], Corollary 5.2). *The syzygy scheme $Syz(s)$ of a scrollar syzygy $s \in L_p$ is a scroll of degree $p+1$ and codimension p that contains the curve C.*

In particular, let C be a $d-$gonal canonical curve lying on the rational normal scroll X, swept out by a pencil g^1_d. The Eagon-Northcott type minimal resolution E of X with length $g-d$ injects into the linear stand of a minimal resolution of the curve. Thus, any $(g-d)-$th linear syzygies of X can be realized as a $(g-d)-$th linear syzygies of C. Furthermore, the space

$$Y_{g-d} \subset \mathbb{P}(\mathrm{Tor}^S_{g-d}(S_X, S)_{g-d+1}) \cong \mathbb{P}^{g-d-1}$$

of scrollar syzygies is a rational normal curve of degree $g-d-1$ (see [SSW13] Proposition 4.4). In this case, the corresponding syzygy schemes are the scroll X such that $\bigcap_{s\in E_{g-d}} Syz(s) = X$.

Definition 2.4.4. For a $d-$gonal canonical curve equipped with several pencils $g_1,\dots,g_t$ of degree d, we can naturally define the syzygy scheme induced by the pencils $g_1,\dots,g_t$ to be

$$Syz(g_1,\dots,g_t) := \bigcap_{s\in (E_i)_{g-d}} Syz(s) = \bigcap_{i=1}^{t} X_i,$$

where X_i is the rational normal scroll swept out by the pencil g_i.

Chapter 3

The unirationality of the Hurwitz schemes $\mathscr{H}_{10,8}$ and $\mathscr{H}_{13,7}$

In this chapter, we show that the Hurwitz scheme $\mathscr{H}_{g,d}$ parametrizing d-sheeted simply branched covers of the projective line by smooth curves of genus g, up to isomorphism, is unirational for $(g,d) = (10,8)$ and $(13,7)$. The unirationality is settled by using liaison constructions in $\mathbb{P}^1 \times \mathbb{P}^2$ and $\mathbb{P}^6$ respectively, and an explicit computation over a finite field. We mainly follow [KT17], a joint work of the author with Fabio Tanturri.

3.1 Introduction

The study of the birational geometry of the moduli spaces of curves together with additional data such as marked points or line bundles is a central subject in modern algebraic geometry. For instance, understanding the geometry of the Hurwitz schemes

$$\mathscr{H}_{g,d} := \{ C \xrightarrow{d:1} \mathbb{P}^1 \text{ simply branched cover} \mid C \text{ smooth of genus } g\} / \sim$$

parametrizing d-sheeted simply branched covers of the projective line by smooth curves of genus g, up to isomorphism, has an important role in shedding light on the geometry of the moduli spaces of curves $\mathscr{M}_g$. It was through Hurwitz spaces that Riemann [Rie57] computed the dimension of $\mathscr{M}_g$, and Severi [Sev68], building on works of Clebsch and Lüroth [Cle73], and Hurwitz [Hur91], showed that $\mathscr{M}_g$ is irreducible.

Recently, the birational geometry of Hurwitz schemes has gained increasing interest, especially concerning their unirationality. By classical results of Petri [Pet23], Segre [Seg28], and Arbarello and Cornalba [AC81], it has been known for a long time that $\mathscr{H}_{g,d}$ is unirational in the range $2 \leq d \leq 5$ and $g \geq 2$. For $g \leq 9$ and $d \geq g$, the unirationality has been proved by Mukai [Muk95]. The most recent contributions have been given by [Ver05, Gei12, Gei13, Sch13, ST16, DS17]. For a more complete picture on the unirationality of Hurwitz spaces, the related speculations and open questions, we refer to [ST16]. Hurwitz spaces and their Kodaira dimension are also considered in the very recent paper [Far18].

The main contribution of this chapter is the proof of the unirationality of the Hurwitz schemes $\mathscr{H}_{10,8}$ and $\mathscr{H}_{13,7}$ (Theorem 3.3.2 and Theorem 3.4.2). In [ST16, §1] it is speculated that $\mathscr{H}_{g,d}$ is unirational for pairs (g,d) lying in a certain range: we remark that our two cases lie in that range, and respect perfectly this speculation.

The key ingredient for both results is the construction of dominant rational families of curves constructed via liaison in $\mathbb{P}^1 \times \mathbb{P}^2$ and $\mathbb{P}^6$ respectively. The proof of the unirationality of $\mathscr{H}_{10,8}$ is based on the observation that a general 8-gonal curve of genus 10 admits a model in $\mathbb{P}^1 \times \mathbb{P}^2$ of bidegree $(6,10)$, which can be linked in two steps to the union of a rational curve and five lines. We show that this process can be reversed and yields a unirational parametrization of $\mathscr{H}_{10,8}$.

For $\mathscr{H}_{13,7}$, we use the fact that a general 7-gonal curve of genus 13 can be embedded in $\mathbb{P}^6$ as a curve of degree 17, which is linked to a curve D of genus 10 and degree 15. We show that this process can also be reversed; to exhibit a unirational parametrization of such D's, we prove the unirationality of $\mathscr{M}_{10,n}$ for $n \leq 5$ (Theorem 3.4.1), a result of independent interest, and we use a general curve together with 3 marked points to produce a degree 15 curve in $\mathbb{P}^6$. A similar approach yields the unirationality of $\mathscr{H}_{12,8}$, already proven in [ST16]. This result is outlined at the end of Section 3.4.

The reversibility of the above constructions corresponds to open conditions on suitable moduli spaces or Hilbert schemes. To show that the constructed families of covers of $\mathbb{P}^1$ are dominant on the Hurwitz schemes it is then sufficient to exhibit single explicit examples of the constructions over a finite field. A computer aided verification with the computer algebra software *Macaulay2* [GS] is implemented in the package [KT17a], whose documentation illustrates the basic commands needed to check the truthfulness of our claims. A ready-to-read compiled execution of our code is also provided.

A priori, it might be possible to mimic these ideas for other pairs (g,d) for which no unirationality result is currently known. However, a case-by-case analysis suggests that, in order to apply the liaison techniques as above, one needs to construct particular curves, which are at the same time far from being general and not easy to realize.

We recall that there is a natural dominant morphism $\alpha : \mathscr{H}_{g,d} \longrightarrow \mathscr{W}^1_{g,d}$, which is a $\mathrm{PGL}(2)-$bundle over a dense open subset of $\mathscr{W}^1_{g,d}$. Therefore, both spaces are irreducible, and the unirationality of $\mathscr{H}_{g,d}$ is equivalent to that of $\mathscr{W}^1_{g,d}$. This allows us to turn our proof to the unirationality of universal Brill-Noether spaces $\mathscr{W}^1_{g,d}$.

The chapter is structured as follows. We start by introducing some notation and some background on liaison theory. In sections 3.3 and 3.4, we continue by proving the unirationality of $\mathscr{H}_{10,8}$ and $\mathscr{H}_{13,7}$ respectively.

3.2 Liaison

In this section, we introduce some notation and background facts on liaison theory, which will be needed later on.

Definition 3.2.1. Let C and C' be two curves in a smooth projective variety X of dimension r with no common components, contained in $r-1$ mutually independent hypersurfaces $Y_i \subset X$ meeting each other transversally. Let $Y = \cap Y_i$ be the complete intersection curve. C and C' are said to be geometrically linked via Y if $C \cup C' = Y$ scheme-theoretically.

If we assume that the curves are locally complete intersections and that they meet only in ordinary double points, then $\omega_Y|_C = \omega_C(C \cap C')$ and the arithmetic genera of the curves are related by

$$2(p_a(C) - p_a(C')) = \deg(\omega_C) - \deg(\omega_{C'}) = \omega_X(Y_1 + \cdots + Y_{r-1}).(C - C'). \tag{3.1}$$

The relation above and the obvious relation $\deg C + \deg C' = \deg Y$ can be used to deduce the genus and degree of C' from the genus and degree of C.

Let $X = \mathbb{P}^1 \times \mathbb{P}^2$ and C be a curve of arithmetic genus $p_a(C)$ and bidegree (d_1, d_2). With the above hypotheses, let Y_1, Y_2 be two hypersurfaces of bidegree (a_1, b_1) and (a_2, b_2). Then the genus and the bidegree of C' are

$$\begin{array}{rcl} (d_1', d_2') & = & (b_1 b_2 - d_1, a_1 b_2 + a_2 b_1 - d_2), \\ p_a(C') & = & p_a(C) - \frac{1}{2}\left((a_1 + a_2 - 2)(d_1 - d_1') + (b_1 + b_2 - 3)(d_2 - d_2')\right). \end{array} \tag{3.2}$$

For curves embedded in a projective space $\mathbb{P}^r$, the invariants $p_a(C')$ and d' of the curve C' can be computed via

$$\begin{array}{rcl} d' & = & \prod d_i - d, \\ p_a(C') & = & p_a(C) - \frac{1}{2}\left(\sum d_i - (r+1)\right)(d - d'), \end{array} \tag{3.3}$$

where the d_i's are the degrees of the $r-1$ hypersurfaces Y_i cutting out Y.

3.3 Unirationality of $\mathscr{H}_{10,8}$

In this section we prove the unirationality of $\mathscr{H}_{10,8}$. To simplify the notation, the multiprojective space $\mathbb{P}^1 \times \mathbb{P}^2$ will be denoted by $\mathbb{P}$.

3.3.1 The double liaison construction

Let (C,L) be a general element of $\mathscr{W}^1_{10,8}$. As $\rho(10,8,2) < 0 \leq \rho(10,8,1)$, $\mathrm{h}^0(L) = 2$ and by Riemann–Roch $|K_C - L|$ is a 2-dimensional linear series of degree 10. For a general 6-gonal pencil $|D_1|$ of divisors on C, let

$$\phi : C \xrightarrow{|D_1|\times|K_C-L|} \mathbb{P}$$

be the associated map. We assume that ϕ is an embedding. In fact this is the case if the plane model of C inside $\mathbb{P}^2$ has only ordinary double points and no other singularities, and the points in the preimage of each node under $|K_C - L|$ are not identified under the map to $\mathbb{P}^1$. This way we can identify C with its image under ϕ, a curve of bidegree $(6,10)$ in $\mathbb{P}$.

Moreover, assume C satisfies the maximal rank condition in bidegrees $(a,3)$ for all $a \geq 1$, that is the maps $\mathrm{H}^0(\mathscr{O}_{\mathbb{P}}(a,3)) \longrightarrow \mathrm{H}^0(\mathscr{O}_C(a,3))$ are of maximal rank. Let a_3 be the minimum degree such that C lies on a hypersurface of bidegree $(a_3,3)$. Then by Riemann–Roch the maximal rank condition gives $a_3 = 3$ and C is expected to be contained in only one hypersurface of bidegree $(3,3)$. Let Y be a complete intersection curve containing C defined by two forms of bidegrees $(3,3)$ and $(4,3)$, and let C' be the curve linked to C via Y. By formula (3.2), C' is expected to be a curve of genus 4 and bidegree $d' = (3,11)$.

Thinking of C' as a family of three points in $\mathbb{P}^2$ parametrized by the projective line $\mathbb{P}^1$, we expect a finite number l' of distinguished fibers where the three points are collinear. In fact, this is the case when the six planar points of C lie on a (possibly reducible) conic. We claim that $l' = 5$.

To compute l', we need to understand the geometry of C'. Let D_2' be the divisor of degree 11 such that the projection of C' to $\mathbb{P}^2$ is defined by a linear subspace of $\mathrm{H}^0(\mathscr{O}(D_2'))$, and let $|D_1'|$ be the 3-gonal pencil of divisors defining the map $C' \longrightarrow \mathbb{P}^1$. Since $\deg(K_{C'} - D_2') < 0$, by Riemann–Roch we have $\mathrm{h}^0(\mathscr{O}(D_2')) = 11 + 1 - 4 = 8$. We consider the map induced by the complete linear system

$$\psi_2 : C' \xrightarrow{|D_2'|} \mathbb{P}^7; \tag{3.4}$$

as shown in [Sch86], the 3-dimensional rational normal scroll S of degree 5 swept out by $|D_1'|$ contains the image of ψ_2. Hence, the image of the map

$$\psi : C' \longrightarrow \mathbb{P}^1 \times \mathbb{P}^7$$

is contained in the graph of the natural projection map from S to $\mathbb{P}^1$, that is

$$\psi(C') \subseteq \mathbb{P}^1 \times S = \bigcup_{D_\lambda \in |D_1'|} ([\lambda] \times \overline{D}_\lambda),$$

where $\overline{D}_\lambda$ is the linear span of $\psi_2(D_\lambda)$ in $\mathbb{P}^7$.

As $\psi(C')$ is a family of three points in $\mathbb{P}^7$ parametrized by $\mathbb{P}^1$, $C' \subset \mathbb{P}^1 \times \mathbb{P}^2$ is obtained by projection of $\psi(C')$ from a linear subspace $\mathbb{P}^1 \times V \subset \mathbb{P}^1 \times \mathbb{P}^7$ of codimension 3. Fix a $\lambda \in \mathbb{P}^1$; by Riemann–Roch, $\dim |D_2' - D_1'| = 4$ and hence $\psi_2(D_\lambda)$ spans a 2-dimensional projective space inside $\mathbb{P}^7$. It is clear that the three points corresponding to λ are distinct and collinear if and only if $V \cap \overline{D_\lambda}$ is a point, and the three points coincide if and only if $V \cap \overline{D_\lambda}$ is a projective line. The latter case does not occur in general, as the plane model of C' has only double points. The former case occurs in $l' = \deg S = 5$ points if S and V intersects transversally. This is an open condition which holds in general.

Now, suppose that for all $b \geq 1$ the maps

$$\mathrm{H}^0(\mathscr{O}_{\mathbb{P}}(b,2)) \longrightarrow \mathrm{H}^0(\mathscr{O}_{C'}(b,2))$$

are of maximal rank, and set

$$b_2 := \min\{b \ : \ \mathrm{h}^0(\mathscr{I}_{C'}(b,2)) \neq 0\}.$$

Under the maximal rank assumption, $b_2 = 5$ and $\mathrm{h}^0(\mathscr{I}_{C'}(5,2)) = 2$. Let Y' be a complete intersection of two hypersurfaces of bidegree $(5,2)$ containing C' and let C'' be the curves linked to C' via Y'.

Interpreting again C' and C'' as families of points parametrized by $\mathbb{P}^1$, we observe that a general fibre of C'' consists of a single point. In the 5 distinguished fibres of C', the two conics of the complete intersection Y' turn out to be reducible with the line spanned by the three points of C' as a common factor. Thus, the curve C'' is the union of a rational curve R of bidegree $(1,4)$ and 5 lines.

3.3.2 A unirational parametrization

The double liaison construction described in the previous section can be reversed and implemented in a computer algebra system. We note that all the assumptions on C and C' correspond to open conditions in suitable moduli spaces or Hilbert schemes, so that it is sufficient to check them on a single example. We can work over a finite field as explained in Remark 3.3.1 here below.

Remark 3.3.1. Here, we will often need to exhibit an explicit example satisfying some open conditions. A priori we could perform our computations directly on $\mathbb{Q}$, but this can increase dramatically the required time of execution. Instead, we can view our choice of the initial parameters in a finite field F_p as the reduction modulo p of some choices of parameters in $\mathbb{Z}$. Then, the so-obtained example E_p can be seen as the reduction modulo p of a family of examples defined over a neighbourhood $\operatorname{Spec}\mathbb{Z}[\frac{1}{b}]$ of $(p) \in \operatorname{Spec}\mathbb{Z}$ for a suitable $b \in \mathbb{Z}$ with $p \nmid b$. If our example E_p satisfies some open conditions, then by semicontinuity the generic fibre E satisfies the same open conditions, and so does the general element of the family over $\mathbb{Q}$ or $\mathbb{C}$.

Our construction depends on a suitable number of free parameters corresponding to the choices we made. Picking 5 lines in $\mathbb{P}^1 \times \mathbb{P}^2$ requires $5 \cdot 3 = 15$ parameters. Choosing 2 forms of bidegree $(2,1)$ to define the rational curve R corresponds to the choice of $\dim \mathbf{Gr}(2,9) = 14$ parameters. By Riemann–Roch we expect $\mathrm{h}^0(\mathscr{I}_{C''}(5,2)) = 7$, so we need $\dim \mathbf{Gr}(2,7) = 10$ parameters to define the complete intersection Y'. Similarly, as $\mathrm{h}^0(\mathscr{I}_{C'}(3,3)) = 1$ and $\mathrm{h}^0(\mathscr{I}_{C'}(4,3)) = 8$, we require $\dim \mathbf{Gr}(1,8) - 2 = 5$ further parameters for the complete intersection Y. This amounts to $15+14+10+5 = 44$ parameters in total.

Theorem 3.3.2. *The Hurwitz space $\mathscr{H}_{10,8}$ is unirational.*

Proof. Let $\mathbb{A}^{44}$ be our parameter space. With the code provided by the function `verify-AssertionsOfThePaper(1)` in [KT17a] and following construction of Section 3.3.1 backwards, we are able to produce an example of a curve $C \subset \mathbb{P}$ and to check that all the assumptions we made are satisfied, that is:

- for a general choice of a curve C'', a union of a rational curve of bidegree $(1,4)$ and 5 lines and for a general choice of two hypersurfaces of bidegree $(5,2)$ containing C'', the residual curve C' is a smooth curve of genus 4 and bidegree $(3,11)$ which intersects C'' only in ordinary double points;

- C' satisfies the maximal rank condition in bidegrees $(b,2)$ for all $b \geq 1$ and its planar model has only ordinary double points as singularities;

- for a general choice of two hypersurfaces of bidegree $(3,3),(4,3)$ containing C', the residual curve C is a smooth curve of genus 10 and bidegree $(6,10)$ that intersects C' only in ordinary double points;

- C satisfies the maximal rank condition in bidegrees $(a,3)$ for all $a \geq 1$ and its planar model is non-degenerate.

This means that our construction produces a rational family of elements in $\mathscr{W}^2_{10,10}$, the Serre dual space to $\mathscr{W}^1_{10,8}$. As all the above conditions are open and $\mathscr{W}^1_{10,8}$ is irreducible, this family is dominant which proves the unirationality of both $\mathscr{W}^1_{10,8}$ and $\mathscr{H}_{10,8}$. □

3.4 Unirationality of $\mathscr{H}_{13,7}$

In this section we will prove the unirationality of the Hurwitz space $\mathscr{H}_{13,7}$. As a preliminary result of independent interest, let us prove the following.

Theorem 3.4.1. *The moduli space $\mathscr{M}_{10,n}$ of curves of genus* 10 *with n marked points is unirational for* $1 \leq n \leq 5$.

Proof. This result is achieved by linkage in $\mathbb{P} := \mathbb{P}^1 \times \mathbb{P}^2$. We start with a reducible curve C of arithmetic genus -3. This curve is a union of 3 general lines and the graph of a rational plane curve of degree 4. On the one hand, the space of such curves is clearly unirational; on the other hand, C will in general be contained in at least two independent hypersurfaces of bidegree $(4,2)$. The linkage with respect to 2 general such hypersurfaces produces a curve C' of expected bidegree $(3,9)$ and genus 4, which will in general be contained in exactly 7 independent hypersurfaces of bidegree $(3,3)$.

For the choice of 5 general points $\{P_1,\ldots,P_5\}$ in $\mathbb{P}$, let I_P be their ideal. In general, the space of bihomogeneous polynomials $(3,3)$ contained in $I_{C'} \cap I_P$ will be generated by two independent polynomials f_1, f_2, defining two hypersurfaces X_1, X_2. The complete intersection of these hypersurfaces link C' to a curve C'' passing through each P_i. The curve C'' turns out to be a curve of genus 10 and bidegree $(6,9)$. The projection onto $\mathbb{P}^1$ yields an element of $\mathscr{H}_{10,6}$.

In [Gei12], Geiß proved that this construction yields a rational dominant family in $\mathscr{H}_{10,6}$. Moreover, the Brill–Noether number $\rho(6,1,10) = 10-(1+1)(10-6+1) = 0$ is non-negative, which implies that this rational family dominates $\mathscr{M}_{10}$ as well. Therefore, as the choice of $\{P_1,\ldots,P_5\}$ is unirational, we get a rational dominant family of curves of genus 10 together with (up to) five marked points. □

Theorem 3.4.2. *The Hurwitz space $\mathscr{H}_{13,7}$ is unirational.*

Proof. Let $(D,L) \in \mathscr{W}^1_{13,7}$ be a general element. By Riemann–Roch, $\omega_D \otimes L^{-1}$ is a general $g^6_{13,17}$ and therefore the linear system $|K_D - L|$ embeds D in $\mathbb{P}^6$ as a curve of genus 13 and degree 17. Conversely, if D is a general curve of genus 13 and degree 17 in $\mathbb{P}^6$, by Riemann–Roch the line bundle $\omega_D \otimes \mathscr{O}_D(-1)$ is a general g^1_7. Hence, in order to prove the unirationality of $\mathscr{H}_{13,7}$, it will be sufficient to exhibit a rational family of projective curves of genus 13 and degree 17 in $\mathbb{P}^6$ which dominates $\mathscr{W}^6_{13,17}$.

Let C be a general curve of genus 13 and degree 17 in $\mathbb{P}^6$. Since $\mathscr{O}_C(2)$ is non-special, C is contained in at least $\binom{6+2}{2} - (17\cdot 2+1-13) = 6$ independent quadric hypersurfaces. Consider five general such hypersurfaces X_i and suppose that the residual curve C' is smooth and that C and C' intersect transversally; these are open conditions on the choice of $(C,\mathscr{O}_C(1)) \in \mathscr{W}^6_{13,17}$. By (3.3), C' has genus $g' = 10$ and degree

$d' = 15$. By Riemann–Roch, the Serre residual divisor $\omega_{C'} \otimes \mathscr{O}_{C'}(-1)$ has degree 3 and one-dimensional space of global sections. Hence, it corresponds to the class of three points on C'. Conversely, by geometric Riemann–Roch three general points on C' form a divisor P with $\mathrm{h}^0(P) = 1$, such that the linear series $|K_C - P|$ embeds C' in $\mathbb{P}^6$ as a curve of degree 15. Hence, the unirationality of $\mathscr{W}^6_{10,15}$ can be deduced from the unirationality of $\mathscr{M}_{10,3}$, proved in Theorem 3.4.1 above.

By means of the implemented code `verifyAssertionsOfThePaper(2)` in [KT17a], we can show with an explicit example that

- for a general curve C' of genus 10 and degree 15 in $\mathbb{P}^6$ and for a general choice of five quadric hypersurfaces containing it, the residual curve C is smooth and intersects C' only in ordinary double points;
- C is not contained in any hyperplane.

This way we get a rational family of curves C of genus 13 and degree 17 in $\mathbb{P}^6$. Since all the assumptions we made correspond to open conditions on $\mathscr{W}^6_{13,17}$ and are satisfied by our explicit examples, such a family dominates $\mathscr{W}^6_{13,17}$. □

Remark 3.4.3. The same argument of Theorem 3.4.2 holds for a general element in $\mathscr{H}_{12,8}$, so that the above proof yields an alternative proof of the unirationality of $\mathscr{H}_{12,8}$ proved in [ST16]. In this case, the Serre dual model is a curve of genus 12 and degree 14 in $\mathbb{P}^4$. The liaison is taken with respect to 3 general cubic hypersurfaces and yields a curve of genus 10 and degree 13, which can be constructed from a curve of genus 10 with 5 marked points. The strategy is the same as above. An implementation of this unirational parametrization of $\mathscr{H}_{12,8}$ via linkage can be found in the package [KT17a].

The package [KT17a] including the implementation of the unirational parametrizations exhibited in this chapter, together with all the necessary and supporting documentation, is available online.

Chapter 4

Brill-Noether locus of curves with three pencils

In this chapter, we mainly deal with the Brill-Noether locus $\mathscr{M}_{g,d}(3)$ of genus g curves, possessing three mutually independent pencils of degree d. The main outcome of this chapter is the theorem 4.5.1, in which we prove that $\mathscr{M}_{g,d}(3)$ has a unirational irreducible component of expected dimension, for $5 \leq d \leq 9$ and g as in the table 4.1.

4.1 Introduction

Let C be a smooth curve of genus g carrying three pencils g^1_d. We further assume that the map to $\mathbb{P}^1 \times \mathbb{P}^1$ defined by any of the two pencils is a birational morphism. In [Aco79], Accola proved the genus of C cannot exceed the bound

$$[6s^2 + (6s+q-2)(q-1)]/2,$$

where $d = 2s+q$ for some integers s, q satisfying $0 \leq q \leq 1$.

The extra assumption on the pencils already provides a birational model of C in the ambient space $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$. For curves with genera as in the table 4.1 we use this model to link it often to a rational curve via a complete intersection of two hypersurfaces of certain degrees. We will show for an open subset of $\mathscr{M}_{g,d}(3)$, this construction is reversible and yields a rational family of curves in $\mathscr{M}_{g,d}(3)$. To indicate that these so-constructed rational families of curves dominate an irreducible component of $\mathscr{M}_{g,d}(3)$, it is then sufficient to exhibit a single example of the constructions over a finite field. We remark that the range of genera in table 4.1 respects Accola's upper bound.

A computer aided verification with the computer algebra software *Macaulay2* [GS] is implemented in the package [K17], which provides examples and basic commands needed to check the main statement of this chapter.

4.2 Accola's genus bound

We briefly recall the upper bound provided by Accola for the genus of curves possessing several pencils.

Definition 4.2.1. Let C be a smooth curve. Two base point free pencils of degree d on C, say g_1 and g_2, are called independent if there is no non-trivial morphism $f : C \longrightarrow C'$ of some degree $a \geq 2$ and two linear system on C', denoted by g'_1, g'_2 such that $f^*(g'_i) = g_i$ for $i = 1,2$. Otherwise, they are called dependent. If $g_1, \dots, g_m$ are m different base point free pencils of degree d on C, then they are called mutually independent if for each $1 \leq i < j \leq m$ the linear systems g_i and g_j are independent.

Remark 4.2.2. For a g^1_d on a curve C and a point $x \in C$, let $E_x \in g^1_d$ denote a divisor containing the point x. Then, it is clear that two pencils g_1, g_2 of degree d are independent if and only if for a general point $x \in C$, $(E_x, F_x) = x$, where (E_x, F_x) denotes the greatest common divisor of the two divisors $E_x \in g_1$ and $F_x \in g_2$. Hence, two linear systems g_1 and g_2 are independent if and only if the corresponding map gives a birational model of C in $\mathbb{P}^1 \times \mathbb{P}^1$.

As a generalization of Castelnouvo's inequality for the genus of curves having a simple linear series (see [Cas93]), the following theorem of Accola provides an upper bound for the genus of a curve possessing a certain number of linear series.

Theorem 4.2.3 (**Accola's genus bound**). *Let C be a smooth curve of genus g possessing m mutually independent linear series g^1_d. Write $d = s(m-1)+q$ for some integers s, q satisfying $-m+3 \leq q \leq 1$. Then*

$$g \leq [s^2(m^2-m) + (2ms+q-2)(q-1)]/2.$$

Proof. See [Aco79], Theorem 4.3. □

4.3 Construction via liaison

In this section we describe the liaison construction for curves with genera and degrees as in the table 4.1, which mostly leads to a rational curve. To simplify our notations, we denote the multiprojective space $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ by $\mathbb{P}$.

Let C be an element of $\mathscr{M}_{g,d}(3)$, where C is a smooth curve of genus g equipped with three mutually independent and base point free pencils of degree d, say g_1, g_2, g_3. By Accola's theorem 4.2.3, we require the genus of C to be bounded by

$$g \leq [6s^2 + (6s+q-2)(q-1)]/2,$$

where for some integers s,q, satisfying $0 \leq q \leq 1$, we have $d = 2s+q$. Let

$$\varphi : C \xrightarrow{g_1 \times g_2 \times g_3} \mathbb{P}$$

be the corresponding map. As the three pencils are assumed to be mutually independent, the map φ gives a birational model of C in $\mathbb{P}$. We further assume that φ is an embedding. This way we can identify C with its image in $\mathbb{P}$. Suppose for a suitable choice of the degrees (a_i,b_i,c_i), $i=1,2$, the maps

$$\mathrm{H}^0(\mathbb{P},\mathscr{O}_{\mathbb{P}}(a_i,b_i,c_i)) \longrightarrow \mathrm{H}^0(C,\mathscr{O}_C(a_i,b_i,c_i))$$

are of maximal rank and $\mathrm{h}^1\,\mathscr{O}_C(a_i,b_i,c_i) = 0$. Let $X = V(f_1,f_2)$ be the complete intersection defined by two general hypersufaces $f_i \in \mathrm{H}^0\,\mathscr{I}_C(a_i,b_i,c_i)$ and let C' be the curve linked to C via X. Under the assumption that the curves are locally complete intersection and they meet only in ordinary double points, by 3.1 if follows that the genus and the degree of C' are given by

$$\begin{aligned} (d_1',d_2',d_3') &= (b_1c_2+c_1b_2-d, a_1c_2+a_2c_1-d, a_1b_2+a_2b_1-d), \\ \rho(C)-\rho(C') &= \tfrac{1}{2}\left((a_1+a_2-2)(d-d_1')+(b_1+b_2-2)(d-d_2')+(c_1+c_2-2)(d-d_3')\right). \end{aligned} \tag{4.1}$$

4.4 Computational construction

Starting from a curve with genus and degree as in table 4.1 that admits three mutually independent pencils of degree d and following the above construction, we are able to link it (in probably several steps) to a rational curve. We can show that this construction is reversible and it provides a unirational family of curves, which parametrizes an irreducible component of $\mathscr{M}_{g,d}(3)$.
To explain the explicit construction, we go through the construction for the case $g=17$ with three mutually independent pencils of degree 7 as a test example.

Example 4.4.1. Let C be a general element of the Brill-locus $\mathscr{M}_{17,7}(3)$. In fact, C is a smooth curve of genus 17 that has a model of degree $(7,7,7)$ in $\mathbb{P}$. Let

$$S = \mathbb{K}[x_0,x_1,y_0,y_1,z_0,z_1]$$

be the Cox ring of the multiprojective space $\mathbb{P}$. As $\mathrm{h}^1\,\mathscr{O}_C(2,2,2)=0$, by Riemann–Roch theorem we deduce $\mathrm{h}^0\,\mathscr{O}_C(2,2,2) = 6\cdot 7+1-17 = 26$. Therefore, it follows from the maximal rank condition in degree $(2,2,2)$ that C lies on only one hypersurface of degree $(2,2,2)$. Similarly, C is contained in a 3-dimensional family of hypersurfaces of

degree $(3,2,2)$. Let X be the complete intersection of the two linearly independent hypersurfaces of degrees $(2,2,2)$ and $(3,2,2)$. By the formulas (4.1), C is linked via X to a rational curve R of degree $(1,3,3)$.
To be able to reverse the construction, we require that a general rational curve R satisfies the maximal rank condition and that R is contained in independent hypersurfaces of degrees $(2,2,2)$ and $(3,2,2)$. As $h^1\,\mathcal{O}_R(2,2,2)=0$, from Riemann–Roch we get $h^0\,\mathcal{O}_R(2,2,2)=2+6\cdot 2+1=15$ and under the maximal rank condition R is contained in 12 independent hypersurfaces of degree $(2,2,2)$. Similarly, R lies on a 20-dimensional family of hypersurfaces of degree $(3,2,2)$. Therefore, we can reverse the above process.

Now, we turn to the actual construction of the curve. We will briefly explain the methods of each steps and we mainly make use of the functions in our *Macualay2* package [K17].

Step 1. First we define the vanishing ideal of the rational curve R. One can easily define the rational curve R as the image curve of a map $\mathbb{P}^1 \longrightarrow \mathbb{P}$ defined by general forms of the given degrees. To speed up our computations, and as an alternative way, we define the ideal of rational curve simply by saturation of the ideal generated by two form of degrees $(3,1,0)$ and $(3,0,1)$. Then, we choose general forms of degrees $(2,2,2),(3,2,2)$ in the ideal of the rational curve to define the complete intersection X. Next, we compute the vanishing ideal of the linked curve C given by the quotient ideal. For all cases in table 4.1, we have unified the construction in the implemented function `curveViaLiaison`. In particular, in this special case we compute the ideal of the curve C by

```
i1: loadPackage"UnirationalBNSchemes"
i2: p=101; -- charachterstic of the finite field
13: IC=time(curveViaLiaison(p,{1,3,3},{{0,0,0},{0,0,0}},{{2,2,2},{3,2,2}}))_1;
     -- used 35.4307 seconds
               ZZ
o3: Ideal of ---[x , x , y , y , z , z ]
               101  0   1   0   1   0   1

i4: g=genusP1P1P1(IC)
o4: 17
```

Step 2. We check that C is smooth and irreducible. To do so, we use the birational space models of C given by compactification of different affine charts $\mathbb{P}\setminus V(x_iy_jz_k)$, $0\leq$

$i,j,k \leq 1$ in $\mathbb{P}^3$. More precisely, considering the map

$$\begin{aligned} \mathbb{P}\setminus V(x_1y_1z_1) &\longrightarrow \mathbb{P}^3 \\ ((x_0:x_1),(y_0:y_1),(z_0:z_1)) &\mapsto (x_1y_1z_1 : x_0y_0z_1 : x_0y_1z_0 : x_1y_0z_0) \end{aligned}$$

for the affine chart $\mathbb{P}\setminus V(x_1y_1z_1)$ (similarly for the other charts), we show that the image curve is smooth outside the hyperplane $V(r_0)$, where $R_3 = \mathbb{K}[r_0,\ldots,r_3]$ is the coordinate ring of $\mathbb{P}^3$. Having C to be smooth, we go further to show $\mathrm{h}^0(\mathscr{O}_C) = 1$, which then implies that C is connected and hence irreducible. In [K17], this method is implemented in the function `isSmoothAndIrreducible`, which tests smoothness and irreducibility in parallel.

```
i5:  time isSmoothAndIrreducible(IC)
     -- used 336.51 seconds
o5:  true
```

Step 4. Finally, we check that C satisfies the maximal rank condition in degrees $(2,2,2)$ and $(3,2,2)$.

```
i6:  d={7,7,7}
i7:  maxRankCondition(IC,g,d,{2,2,2})
o7:  true
i8:  maxRankCondition(IC,g,d,{3,2,2})
o8:  true
```

Therefore, starting from a general rational curve and following the construction backward, we can produce a rational family of genus 17 curves carrying three pencils of degree 7. The general element of this family satisfies all the assumption we made above.

It remains to show that this construction depends on the correct number of parameters for all choices we made. In fact, for the coefficients of the homogeneous forms defining the rational curve, and the complete intersection. As the vector spaces of homogeneous polynomials of S in degrees $(3,0,1)$ and $(3,1,0)$ are 8-dimensional, we need $2\cdot \dim \mathbf{Gr}(1,8) = 14$ parameters to define the ideal of the rational curve. Also, as under the maximal rank assumption, we have $\mathrm{h}^0\mathscr{I}_R(2,2,2) = 12$ and $\mathrm{h}^0\mathscr{I}_R(3,2,2) = 20$, we need $\dim \mathbf{Gr}(1,12) + \dim \mathbf{Gr}(1,20-2) = 28$ more parameters to define the complete intersection X. This amount to $N = 42$ parameters in total.

As the curve C lies on exactly two independent hypersurfaces of degrees $(2,2,2)$ and $(3,2,2)$, it can be linked to a rational curve R in a unique way. Therefore, identifying all the rational curves isomorphic to R arised from the automorphisms of $\mathbb{P}$, we achieve the expected dimension of the moduli space $\mathscr{M}_{17,7}(3)$, given by

$$3 \cdot 17 - 3 + 3 \cdot \rho(17, 7, 1) = 33$$

4.5 Proof of the dominance

The liaison construction described above can be modified for all the cases in table 4.1. We note that all the assumptions we made on C and C', correspond to open conditions in suitable moduli spaces, so that it is sufficient to test them in a single example. In view of 3.3.1, we can even work over a finite field.

Theorem 4.5.1. *For $5 \leq d \leq 9$ and all g as in the table 4.1, the moduli space $\mathscr{M}_{g,d}(3)$ of genus g curves possessing three mutually independent pencils of degree d has a unirational irreducible component of expected dimension.*

Proof. Let $\mathbb{A}^N$ be the parameter space for the construction above and

$$\psi : \mathbb{A}^N \dashrightarrow \mathscr{M}_{g,d}(3)$$

be the induced map. Modifying the steps of example 4.4.1 for each case in 4.1, we can produce an example of a smooth irreducible curve of genus g with three pencils of degree d which satisfy the maximal rank condition in desired degrees and the two linked curves intersect in only ordinary double points.
Thus, by semicontinuity the locus of such curves is a non-empty open subset of $\mathscr{M}_{g,d}(3)$, and therefore the constructed rational family of curves dominates an irreducible component $H \subset \mathscr{M}_{g,d}(3)$ containing the single example. Hence, H is unirational of expected dimension. □

	g	$(a_1,a_2,a_3),(b_1,b_2,b_3)$	g'	d'	$(a'_1,a'_2,a'_3),(b'_1,b'_2,b'_3)$	g''	d''
$3g^1_5$	5	(2,2,2),(3,2,2)	2	(3,5,5)	(2,2,2),(2,2,2)	0	(5,3,3)
	6	(2,2,2),(2,2,2)	0	(3,3,3)			
	7	(2,2,2),(2,3,3)	9	(7,5,5)	(3,2,2),(2,2,2)	0	(1,5,5)
	8	(2,2,2),(2,3,3)	10	(7,5,5)	(2,2,2),(2,2,2)	0	(1,3,3)
	9	(2,2,1),(2,2,2)	0	(1,1,3)			
$3g^1_6$	9	(2,2,3),(3,3,2)	12	(7,7,6)	(4,2,2),(4,2,2)	0	(1,9,10)
	10	(2,2,3),(3,3,2)	13	(7,7,6)	(3,2,2),(4,2,2)	0	(1,7,8)
	11	(2,2,3),(2,3,2)	6	(7,4,4)	(2,2,2),(2,2,2)	0	(1,4,4)
	12	(2,2,3),(2,3,2)	7	(7,4,4)	(1,2,2),(2,2,2)	0	(1,2,2)
	13	(2,2,3),(2,3,2)	8	(7,4,4)	(1,2,2),(1,2,2)	0	(1,0,0)
$3g^1_7$	15	(3,2,2),(4,2,2)	0	(1,7,7)			
	16	(3,2,2),(3,2,2)	0	(1,5,5)			
	17	(2,2,2),(3,2,2)	0	(1,3,3)			
	18	(2,2,2),(2,2,2)	0	(1,1,1)			
$3g^1_8$	21	(3,3,2),(2,3,3)	12	(7,5,7)	(3,2,2),(3,2,2)	0	(1,7,5)
	22	(3,3,2),(2,3,3)	13	(7,5,7)	(2,2,2),(3,2,2)	0	(1,5,3)
	23	(3,3,2),(2,3,3)	14	(7,5,7)	(2,2,2),(2,2,2)	0	(1,3,1)
$3g^1_9$	27	(3,3,2),(2,3,3)	8	(6,4,6)	(2,2,2),(2,2,2)	0	(2,4,2)
	28	(3,3,2),(4,2,4)	26	(7,11,9)	(4,2,2),(4,2,2)	0	(1,5,7)
	29	(3,2,2),(2,3,2)	0	(1,1,4)			

Table 4.1: Numerical data for all cases of theorem 4.5.1.

Remark 4.5.2. We remark that for $d \geq 10$, one might be able to link a curve of genus g with $3g^1_d$'s to one of the above constructed curves. This then provides the same unirationality statement for $d \geq 10$, and complete the above table for curves with pencils of higher degree.

Chapter 5

Unirational components of moduli spaces of genus 11 curves with several pencils of degree 6

In this chapter, we show that the moduli space $\mathscr{M}_{11,6}(k)$ of $6-$gonal curves of genus 11, equipped with k mutually independent and type I pencils of degree 6, has a unirational irreducible component for $5 \leq k \leq 9$. The unirational families arise from degree 9 plane curves with 4 ordinary triple and 5 ordinary double points that dominate an irreducible component of expected dimension. We will further show that the family of degree 8 plane curves with 10 ordinary double points covers an irreducible component of excess dimension in $\mathscr{M}_{11,6}(10)$.

5.1 Introduction

Let C be a smooth irreducible $d-$gonal curve of genus g defined over an algebraically closed field $\mathbb{K}$. Recall that by definition of gonality, there exists a g^1_d but no g^1_{d-1} on C. It is well-known that $d \leq [\frac{g+3}{2}]$ with equality for general curves. In a series of papers ([Cop97],[Cop98],[Cop99], [Cop00], [Cop05]) Coppens studied the number of pencils of degree d on C, for various d and g. For low gonalities up to $d = 5$, the problem is intensively studied for almost all possible genera. For $6-$gonal curves, Coppens has settled the problem only for genera $g \geq 15$.

In this chapter, we focus on $6-$gonal curves of genus $g = 11$. The motivation for our choice of genus 11 was the question asked by Michael Kemeny, whether any smooth curve of genus 11 carrying at least 6 pencils g^1_6's, comes from a degree 8 plane curve with 10 ordinary double points, where the pencils are cut out by the pencil of lines through each of the singular points. More precisely, there exists no smooth curve of genus 11 possessing exactly $6,7,8$ or 9 pencils of degree 6. We will show the answer to this question is negative.

Let $\mathscr{M}_{11,6}(k) \subset \mathscr{M}_{11}$ be the moduli space of smooth 6−gonal curves of genus 11, equipped with exactly k mutually independent g^1_6's of type I. In section 5.2, we first investigate the possible number of g^1_6's on a 6−gonal curve of genus 11, and therefore the possible values of k for which $\mathscr{M}_{11,6}(k)$ is non-empty. In [Sch02], Schreyer gave a list of conjectural Betti tables for canonical curves of genus 11. Related to our question, and interesting for us, is the plausible Betti table of the following form

1	.	.	.	.	.	.	.	.	.
.	36	160	315	288	$5k$	.	.	.	.
.	.	.	.	$5k$	288	315	160	36	.
.	.	.	.	.	.	.	.	.	1

where k is expected to have the values $k = 1,2,\dots,10,12,20$. Although, in view of Green's conjecture 2.3.5, it is not clear that for a smooth canonical curve of genus 11 with Betti table as above, the number k can always be interpreted as the multiple number of pencils of degree 6 existing on the curve. Nonetheless, for $k = 1,2,\dots,10,12$ we can provide families of curves, whose generic element carries exactly k mutually independent pencils of type I. The critical Betti number in this case is $\beta_{5,6} = \beta_{4,6} = 5k$ as expected. Therefore, in this range the locus $\mathscr{M}_{11,6}(k)$ is non-empty.

The first natural question is then to ask about the geometry of the locus $\mathscr{M}_{11,6}(k)$ inside the moduli of curves $\mathscr{M}_{11}$, in particular about its unirationality.

For $k = 1$, the corresponding locus is the famous Brill-Noether divisor $\mathscr{M}_{11,6}$ of 6−gonal curves [HM82], which is irreducible and furthermore known to be unirational [Gei12]. The moduli space $\mathscr{M}_{11,6}(2)$ is irreducible [Ty07], and unirational such that a general element of $\mathscr{M}_{11,6}(2)$ can be obtained from a model of bidegree $(6,6)$ in $\mathbb{P}^1 \times \mathbb{P}^1$ with $\delta = 14$ ordinary double points. For $k = 3$, by theorem 4.5.1 $\mathscr{M}_{11,6}(3)$ has a unirational irreducible component of expected dimension. A general curves lying in this component can be constructed via liaison in two steps from a rational curve in multiprojective space $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$.

Here we construct rational families of curves with additional pencils from plane curves of suitable degrees with only ordinary multiple points, as singularities. As the first significant result, we will prove that for $5 \le k \le 9$ the moduli space $\mathscr{M}_{11,6}(k)$ has a unirational irreducible component of expected dimension. A general curve lying on this component arises from a degree 9 plane model with 4 ordinary triple and 5 ordinary double points which contains $k-5$ points among the ninth fixed point of the pencil of cubics passing through the 4 triple and 4 chosen double points (Theorem 5.5.1).

The key technique of the proof is to study the space of first order equisingular deformations of plane curves with prescribed singularities, as well as that of the first order embedded deformations of their canonical model. In fact, denoting by M the $5k \times 5k$ submatrix in the deformed minimal resolution corresponding to the general first order deformation family of a canonical curve C with Betti table as above, we use the condition $M = 0$ to determine the subspace of the deformations with extra syzygies of rank $5k$. It turns out that for $5 \leq k \leq 9$, and respectively k linearly independent linear forms $l_1, \dots, l_k$ in the free deformation parameters corresponding to a basis of $T_C\mathscr{M}_{11}$, we have $\det M = l_1^5 \cdot \dots \cdot l_k^5$. This implies that $\mathscr{M}_{11,6}(k)$ has an irreducible component of exactly codimension k inside the moduli space $\mathscr{M}_{11}$. Furthermore, let $\mathscr{K}_{11}$ to be the locus of the curves $C \in \mathscr{M}_{11}$ with extra syzygies, that is $\beta_{5,6} \neq 0$. It is known by Hirschowitz and Ramanan [HR98] that $\mathscr{K}_{11}$ is a divisor, called the Koszul divisor, such that $\mathscr{K}_{11} = 5\mathscr{M}_{11,6}$. Thus, $\mathscr{M}_{11,6}$ at the point C is locally analytically the union of k smooth transversal branches.
We will then compute the kernel of the Kodaira-Spencer map and from that the rank of the induced differential maps, in order to show that the rational families of plane curves dominate this component.

By following the similar approach, we obtain our second main result. We show that the family of degree 8 plane curves with 10 ordinary double points covers an irreducible component of excess dimension in $\mathscr{M}_{11,6}(10)$ (Theorem 5.5.2).

This chapter is structured as follow. In section 5.3 we recall some basics of deformation theory for smooth and singular plane curves. In section 5.4 we deal with the computation of the tangent spaces to our parameter spaces and we continue by proving the main theorems on unirationality in section 5.5 and 5.6.

Our results and conjectures rely on the computations and experiments, performed by the computer algebra system *Macualay2* [GS] and using the supporting functions in the packages [KS18a] and [KS18b].

5.2 Planar model description

In this section, we describe families of plane curves of genus 11 carrying $k = 4, \dots, 10, 12$ pencils of degree 6. Throughout this chapter, several pencils on a curve are supposed to be mutually independent (see Definition 4.2.1) of type I. We make the second restriction more precise by the following definition.

Definition 5.2.1. A base point free pencil g^1_d on a smooth curve C is called of type I if $\dim|2g^1_d| = 2$.

This is an important notion in counting the number of linear series on a smooth curve C. Let g be a base point free g^1_d. The pencil g is said to be the limit of two different linear systems g^1_d in a family of curves if there exists a 1-parameter family of curves $\pi : \mathscr{C} \longrightarrow \Delta$ with $\Delta = \{z \in \mathbb{C} \ : \ |z| < 1\}$, and two families G_1 and G_2 of linear systems g^1_d on this family such that $(G_1)_0 = (G_2)_0 = g$ on $C = \pi^{-1}(0)$, and $(G_1)_t \neq (G_2)_t$ on C_t for $t \neq 0$. In this case we count g with a certain multiplicity. In [Cop83], Coppens proved that a linear series is not of type I if and only if it is a limit of two different pencils in a family of curves. Therefore, type I pencils are exactly those that we should count with multiplicity 1.

Remark 5.2.2. Considering a g^1_d as a point x of the Brill-Noether scheme $W^1_d(C) \subset \mathrm{Pic}^d(C)$, then the g^1_d is of type I if and only if x is a reduced point of $W^1_d(C)$.

Remark 5.2.3. There is an easy way to construct $d-$gonal curves having more than one pencil g^1_d, when d is not a prime number. In fact, starting with a curve C' of some smaller gonality d', and taking a finite covering $\pi : C \longrightarrow C'$ of some degree $a \geq 2$ such that $d = ad'$, then it can be shown that C is curve of gonality d that has more than one pencil g^1_d (see [Ce83]). In this case the pencils are not mutually independent. For the rest of this chapter, we exclude such cases.

We first deal with the construction of plane model for smooth curves of genus 11 with $k = 5, \dots, 9$ pencils of degree 6. Clearly, smooth curves of genus 11 with 10 pencils g^1_6's can be constructed from a plane model of degree 8 with 10 ordinary double points in general position. The code provided by the function `random6gonalGenus11Curve10pencil` in [KS18a], uses this plane model to produce a random canonical curve of genus 11 with exactly $10g^1_6$'s. We remark that, although we further provide a method to produce curves with $k = 4, 12$ pencils g^1_6's, by dimension reasons the rational family obtained from these models may not cover any component of the corresponding locus.

MODEL OF CURVES WITH $5 \leq k \leq 9$ PENCILS.

Let $P_1, \dots, P_4, Q_1, \dots, Q_5$ be 9 general points in the projective plane $\mathbb{P}^2$ and let $\Gamma \subset \mathbb{P}^2$ be a plane curve of degree 9 with 4 ordinary triple points $P_1, \dots, P_4$, and 5 ordinary double points $Q_1,\ldots,Q_5$. We note that, since an ordinary triple (resp. double) point in general position imposes 6 (resp. 3) linear conditions, such a plane curve with these singular points exists as

$$\binom{9+2}{2} - 6\cdot 4 - 3\cdot 5 > 0.$$

Blowing up these singular points

$$\sigma : \widetilde{\mathbb{P}}^2 = \mathbb{P}^2(\ldots,P_i,\ldots,Q_j,\ldots) \longrightarrow \mathbb{P}^2,$$

let $C \subset \widetilde{\mathbb{P}}^2$ be the strict transformation of Γ on the blown up surface of $\mathbb{P}^2$. Hence,

$$C \sim 9H - \sum_{i=1}^{4} 3E_{P_i} - \sum_{j=1}^{5} 2E_{Q_j},$$

where H is the pullback of the class of a line in $\mathbb{P}^2$, and E_{P_i} and E_{Q_j} denote the exceptional divisors of the blow up at the points P_i and Q_j, respectively. By the genus-degree formula, C is a smooth curve of genus $11 = \binom{9-1}{2} - 4.3 - 5$. Moreover, C admits 5 mutually independent g^1_6's of type I. Indeed, for $i = 1,\ldots,4$ the linear series $|H - E_{P_i}|$, identified with the pencil of lines through the triple point P_i induces a base point free pencil G_i of degree 6 on C. As by adjunction, the canonical system $|K_C|$ is cut out by the complete linear series

$$|C + K_{\widetilde{\mathbb{P}}^2}| = |6H - \sum_{i=1}^{4} 2E_{P_i} - \sum_{j=1}^{5} E_{Q_j}|,$$

the linear series $|K_C - 2G_i|$ is cut out by

$$|4H - \sum_{i=1}^{4} 2E_{P_i} - \sum_{j=1}^{5} E_{Q_j} + 2E_{P_i}|.$$

Therefore, we have $\dim|K_C - 2G_i| = 0$ and by Riemann–Roch $\dim|2G_i| = 2$. Thus, the induced pencils from linear system of lines through each of the triple points are of type I. Furthermore, the linear series $|2H - \sum_{i=1}^{4} E_{P_i}|$ identified with the the pencil of conics through the four triple points induces an extra pencil G_5 of degree 6 on C. Similarly by adjunction, the corresponding linear system $|K_C - 2G_5|$ can be identified with the linear system of quadrics containing the double points. We obtain $\dim|K_C - 2G_5| = 0$, which then Riemann–Roch implies that $\dim|2G_5| = 2$. Hence, this gives another pencil of type I. In this way we obtain smooth curves of genus 11 having 5 pencils of degree 6.

In order to get the model of curves with further pencils of degree 6, we impose certain one dimensional conditions on the plane curve of degree 9 such that each condition gives exactly one extra g^1_6.

For $j = 1, \ldots, 5$, let R_j be the ninth fix point of the pencil of cubics through the 8 residual singular points by omitting Q_j. The condition that R_j lies on the plane curves imposes exactly one condition on linear series of degree 9 plane curves with 4 ordinary triple points at P_i's and 5 ordinary double points at Q_j's. On the other hand, the linear

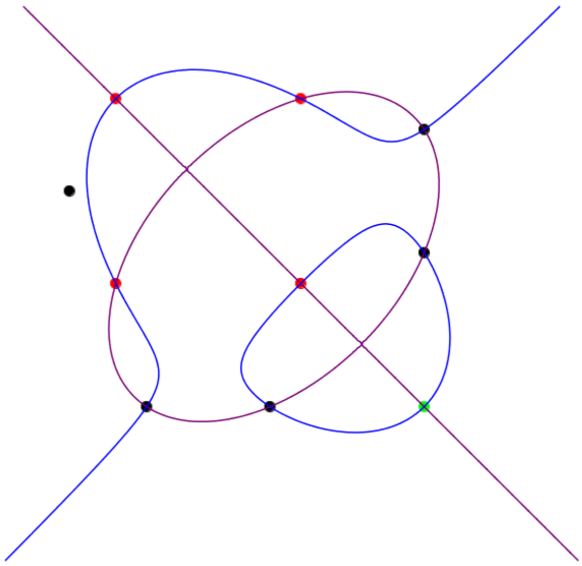


Figure 4.1: Two cubics through 8 ponits by omitting one of the double points

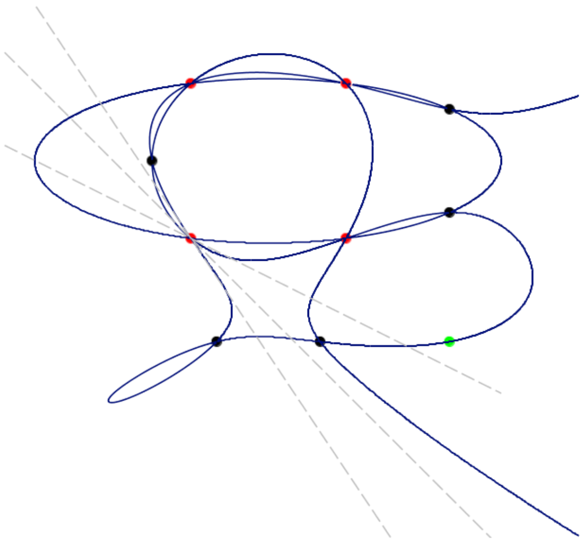


Figure 4.2: Plane curve of degree 9 with six pencils passing through the ninth fixed point

series

$$|3H-\sum_{i=1}^{4}E_{P_i}-\sum_{j=1}^{5}E_{Q_j}+E_{Q_j}|$$

induces a pencil G'_j of degree 7 with a fix point at R_j. Therefore, by forcing the degree 9 plane curves to pass additionally through each R_j, we obtain one further pencil of type I, given by $G'_j - R_j$. This way, by choosing $0 \le m \le 4$ points among $R_1,\dots,R_5$, we get families of smooth curves of genus 11 possessing up to 9 linear series of degree 6. The function `random6gonalGenus11Curvekpencil` in [KS18a] is an implementation of the above construction which produces a random canonical curve of genus 11 possessing $5 \le k \le 9$ pencils of degree 6.

Remark 5.2.4. Although we expect that plane curves of degree 9 with singular points as above, passing through all the five fixed points $R_1,\dots,R_5$, lead to the model of curves of genus 11 with 10 pencils of degree 6, our experimental computations indiactes that such a curve is in general reducible. It is a union of a sextic and the unique cubic through the five double points and $R_1,\dots,R_5$, which has further singular points than expected. Thus, our pattern fails to cover the case $k = 10$.

Our families of plane curves depend on expected number of parameters as desired. In fact, let

$$\mathscr{V}_9^{4,5,m} := \{(\Gamma; P_1,\dots,P_4,Q_1,\dots,Q_5)\} \subset \mathbb{P}^N \times (\mathbb{P}^2)^9$$

denote the variety, where $N = \binom{9+2}{2} - 1$ and $\Gamma \subset \mathbb{P}^2$ is a plane curve of degree 9 with prescribed singular points passing through $0 \le m \le 4$ points among $R_1,\dots,R_5$ as above. As an ordinary triple (resp. double) point in general position imposes 6 (resp. 3) linear conditions, we expect naively that each irreducible component of $\mathscr{V}_9^{4,5,m}$ has dimension

$$\frac{9(9+3)}{2} + 2\cdot 9 - 3\cdot 5 - 6\cdot 4 - m = 33 - m.$$

Identifying the plane curves under automorphisms of $\mathbb{P}^2$ reduces this dimension by $8 = \dim \mathrm{PGL}(2)$. From Brill-Noether theory this fits to the expected dimension of the locus $\mathscr{M}_{11,6}(k) \subset \mathscr{M}_{11}$, of curves possessing $k = m+5$ pencils. In fact, denoting by ρ the Brill-Noether number, we have

$$\dim \mathscr{M}_{11,6}(k) \ge 3\cdot 11 - 3 + (m+5)\rho(11,6,1) = 25 - m.$$

MODELS OF CURVES WITH $k = 4$ PENCILS.

Let $P_1,P_2,P_3,Q_1,\dots,Q_7$ be 10 general points in the projective plane and R be the ninth fix point of a pencil of cubics through 8 points, obtained by omitting two of Q_i's.

Then, the normalization of a general degree 9 plane curve with ordinary triple points at P_1, P_2, P_3 and ordinary double points at $Q_1, \dots, Q_7, R$ is a smooth curve of genus 11 that carries exactly $k = 4$ pencils of degree 6. In fact, the three pencils are induced from the pencil of lines through each of the triple points and the pencil of cubics through the 8 points gives the extra g^1_6. In [KS18a], this construction is implemented in the function `random6gonalGenus11Curve4pencil`.

Remark 5.2.5. The number of parameters for the choice of 10 points in the plane as above plus the dimension of the linear system of plane curves of degree 9 with ordinary triple points at P_1, P_2, P_3 and ordinary double points at $Q_1, \dots, Q_7, R$ amounts to 32 parameters. Therefore, modulo the isomorphisms of the projective plane, we obtain a family of smooth curves of genus 11 with exactly $k = 4$ pencils and smaller dimension than 26, which is the expected dimension of $\mathscr{M}_{11,6}(4)$. Thus, the rational family of curves obtained from this model cannot cover any component of $\mathscr{M}_{11,6}(4)$.

MODELS OF CURVES WITH $k = 12$ PENCILS

Let $P_1, \dots, P_{10}$ be 10 general points in the projective plane and $V_1 \subset |L| = |4H - \sum_{i=1}^{10} E_{P_i}|$ be a pencil inside the linear system of quartics passing through these points. Let $q_1, \dots, q_6$ be the further fixed points of this pencil. Then, normalization of a degree 8 plane curve Γ with 10 ordinary double points $P_1, \dots, P_{10}$ and passing through $q_1, \dots, q_6$, carries exactly 12 pencils of degree 6. On the one hand, considering $Q_1, \dots, Q_6$ to be the 6 moving points of a divisor in V_1, our experiments show that $Q_1, \dots, Q_6$ are the extra fixed points of an another pencil V_2 inside $|L|$. Moreover, let $\mathbb{P}^2 \dashrightarrow \mathbb{P}^4$ be the rational map associated to $|L|$. The image of Γ under this map is a curve C of degree 12, which is cut out by a unique rank 4 quadric hypersuface Q on the determinantal image surface of $\mathbb{P}^2$. As the divisors of the linear series $|L|$ are cut out by the linear system of hyperplanes on C, and the 6 fixed points impose exactly 3 linearly independent conditions on this linear series, they span a projective plane $\mathbb{P}^2 \subset Q$ and they do not lie on a conic. As Q is isomorphic to the cone over $\mathbb{P}^1 \times \mathbb{P}^1$, the projections to each projective line naturally give two extra pencils of degree 6. In [KS18a], the function `random6gonalGenus11Curve12pencil` uses this method to produce a random canonical curve of genus 11 carrying exactly 12 pencils g^1_6's.

5.3 Families of curves and their deformation

To study the local geometry of parameter spaces introduced in the previous section, and also the locus of the smooth curves with several pencils, we study the space of the first order deformation of curves. This leads to the computation of the tangent space at the corresponding points in the moduli space. We recall some basics on deformation theory for smooth and singular plane curves and quickly sketch the typical stages of an application of deformation theory. Most of the statements can be found in the standard textbook [Ser06], and [Art79].

Definition 5.3.1. Suppose $C \subset \mathbb{P}^n$ is a smooth embedded curve. An embedded first order deformation of C is an embedded flat family of curves

$$\begin{array}{c} \mathscr{C} \subset \mathbb{P}^n_A \\ \big\downarrow \pi \\ \operatorname{Spec}(A) \end{array}$$

with $\mathscr{C} \times_{\operatorname{Spec}(A)} \operatorname{Spec}(\mathbb{K}) = C$, where $A = \mathbb{K}[\varepsilon]/(\varepsilon^2)$ denotes the ring of dual numbers.

Let $\mathscr{N}_{C/\mathbb{P}^n} = \mathscr{H}om_{\mathscr{O}_C}(\mathscr{I}/\mathscr{I}^2, \mathscr{O}_C)$ denote the normal bundle of C in $\mathbb{P}^n$. The space of global sections of $\mathscr{N}_{C/\mathbb{P}^n}$ parametrizes the set of first order embedded deformations of C in $\mathbb{P}^n$. This is exactly the tangent space to the Hilbert scheme $\mathscr{H}_{C/\mathbb{P}^n}$ of C inside $\mathbb{P}^n$ (see [Ser06], Theorem 3.2.12).

Theorem 5.3.2. *Let $C \subset \mathbb{P}^n$ be a smooth embedded curve. The set of first order embedded deformations of $C \subset \mathbb{P}^n$ is canonically in one-to-one correspondence with elements in* $\mathrm{H}^0(C, \mathscr{N}_{C/\mathbb{P}^n})$.

Proof. For our computational purpose later on, we will sketch a proof of this theorem in computational context.
Let $\mathscr{I} \subset \mathscr{O}_{\mathbb{P}^n}$ be the ideal sheaf of C and $\mathscr{C} \subset \mathbb{P}^n_A$ be an embedded first order deformation of C given by the ideal sheaf $\mathscr{I}_\varepsilon \subset \mathscr{O}_{\mathbb{P}^n_A}$. We note that these data are obtained by gluing together their restriction to an affine open cover. On an affine open subset $U = \operatorname{Spec}(P) \subset \mathbb{P}^n$ let $C \cap U = \operatorname{Spec}(B)$, where $B = P/I$ for an ideal $I = (f_1, \ldots, f_m) \subset P$. Consider the exact sequence of $P-$modules

$$P^l \xrightarrow{r} P^m \xrightarrow{(f_1,\ldots,f_m)} I \longrightarrow 0,$$

where $r = (r_{ij})_{m \times l}$ is the matrix whose columns generate all the relations among $f_1, \ldots, f_m$. Applying the functor $\operatorname{Hom}_P(-, B)$, we obtain the exact sequence

$$0 \longrightarrow \operatorname{Hom}_P(I, B) \longrightarrow \operatorname{Hom}_P(P^m, B) \xrightarrow{r^\vee} \operatorname{Hom}_P(P^l, B)$$

which identifies $\operatorname{Hom}_P(I,B) \cong \operatorname{Hom}_B(I/I^2, B)$ with $\ker(r^\vee)$.
Let $h = (h_1, \ldots, h_m)$ be an element of $\ker(r^\vee)$, which interpreted as an element of

$\mathrm{Hom}_P(P^m, B)$ is given by scalar product

$$h(p_1, \ldots, p_m) = \sum_{i=1}^{m} h_i p_i.$$

Hence, for each column vector $r_j = (r_{1j}, \ldots, r_{mj})$, we have $\sum_{i=1}^{m} h_i r_{ij} \in I$. This means that for a vector $s_j = (s_{1j}, \ldots, s_{mj}) \in P^m$, we have

$$\sum_{i=1}^{m} h_i r_{ij} = -\sum_{i=1}^{m} f_i s_{ij},$$

or equivalently,

$$(f + \varepsilon h)(r_j + \varepsilon s_j)^t = 0$$

in $P \otimes_{\mathbb{K}} A$. Therefore, there exists a matrix $s = (s_{ij})_{m \times l}$ such that

$$hr = -fs, \tag{5.1}$$

and it extends every relation among $f_1, \ldots, f_m$ to a relation among $f_1 + \varepsilon h_1, \ldots, f_m + \varepsilon h_m$. Now, by Artin's criterion for flatness (see [Art79], Proposition 3.1) it follows that $I_\varepsilon = (f_1', \ldots, f_m')$ with $f_i' = f_i + \varepsilon h_i$ is a perturbation of the generators of I, which defines a first order embedded deformation of $\mathrm{Spec}(B)$ in $\mathrm{Spec}(P)$. Conversely, for a first order deformation of $\mathrm{Spec}(B)$ in $\mathrm{Spec}(P)$ defined by $f_1 + \varepsilon h_1, \ldots, f_m + \varepsilon h_m$, $h = (h_1, \ldots, h_m)$ satisfies the condition 5.1 and hence induces a homomorphism

$$\bar{h} : P^m / \mathrm{Im}(r) = I \longrightarrow B,$$

which is an element of

$$\mathrm{Hom}_P(I, B) \cong \mathrm{Hom}_B(I/I^2, B) \cong \mathrm{Hom}_{\mathscr{O}_{\mathrm{Spec}(B)}}(\mathscr{I}/\mathscr{I}^2, \mathscr{O}_{\mathrm{Spec}(B)}).$$

It turns out that at the global level, this gives a canonical correspondence between first order embedded deformations of C in $\mathbb{P}^n$ and elements in $\mathrm{H}^0(C, \mathscr{N}_{C/\mathbb{P}^n})$. □

Remark 5.3.3. We notice that, up to this point, all the notions and facts on deformation of curves can be simply modified and applied equally well to any smooth variety.

5.3.1 Equisingular infinitesimal deformation of plane curves

In this subsection, we only concentrate on the specific case of families of plane curves with assigned singularities. We will briefly study the space parametrizing the first order deformation of singular plane curves. In fact, an important refinement of the embedded deformation of a smooth curve is the consideration of families of curves with prescribed singularities inside a projective space. These are of families whose members have the same type of singularities in some specified sense. This leads to the notion of equisingularity.

Let $\Gamma \subset \mathbb{P}^2$ be a singular plane curve, and

$$0 \longrightarrow \mathscr{I}/\mathscr{I}^2 \longrightarrow \Omega_{\mathbb{P}^2}|_\Gamma \longrightarrow \Omega_\Gamma \longrightarrow 0$$

be the conormal sequence. By dualization, we obtain

$$0 \longrightarrow T_\Gamma \longrightarrow T_{\mathbb{P}^2}|_\Gamma \longrightarrow \mathscr{N}_{\Gamma/\mathbb{P}^2}$$

where the two middle sheaves are locally free, whereas the first one is not and it jumps at the singular locus of Γ. The first cotangent sheaf is defined as

$$T^1_\Gamma := \mathrm{Coker}[T_{\mathbb{P}^2}|_\Gamma \longrightarrow \mathscr{N}_{\Gamma/\mathbb{P}^2}].$$

More precisely, when Γ is reduced, we have that

$$T^1_\Gamma = \mathscr{E}xt^1_{\mathscr{O}_\Gamma}(\Omega_\Gamma, \mathscr{O}_\Gamma),$$

which is a coherent sheaf supported on the singular locus of Γ (see [Ser06], Proposition 1.1.9). The *equisingular normal sheaf* of Γ in $\mathbb{P}^2$ is defined to be

$$\mathscr{N}' := \ker[\mathscr{N}_{\Gamma/\mathbb{P}^2} \longrightarrow T^1_\Gamma].$$

This describes deformations preserving the singularities of Γ. In fact, $\mathrm{H}^0(\Gamma, \mathscr{N}'_{\Gamma/\mathbb{P}^2})$ parametrizes the first order deformations of Γ in $\mathbb{P}^2$. The following short exact sequence computes the space of global sections of $\mathscr{N}'_{\Gamma/\mathbb{P}^2}$ explicitly. The corresponding code is implemented in the function `equisingularDefOfPlaneCurve` of [KS18a], which for a given plane curve Γ returns $\mathrm{H}^0(\Gamma, \mathscr{N}'_{\Gamma/\mathbb{P}^2})$.

Proposition 5.3.4. *Let Γ be a plane curve of degree d. There exists a short exact sequence*

$$0 \longrightarrow \mathscr{O}_{\mathbb{P}^2} \longrightarrow \mathscr{I}_\Delta(d) \longrightarrow \mathscr{N}'_{\Gamma/\mathbb{P}^2} \longrightarrow 0,$$

where $\mathscr{I}_\Delta$ is the ideal sheaf locally generated by the partial derivatives of a local equation of Γ and the first injective map is defined by multiplication by an equation of Γ.

Proof. See [Ser06], page 255. □

5.4 The tangent space computation

In this section, we compute the tangent space to the parameter space $\mathscr{V}_9^{4,5,m}$ as well as that to the locus $\mathscr{M}_{11,6}(k) \subset \mathscr{M}_{11}$. We further prove the existence of a component with expected dimension on both spaces.

Proposition 5.4.1. *For $m = 0,\dots,4$, the parameter space $\mathscr{V}_9^{4,5,m}$ has an irreducible component of expected dimension.*

Proof. Let $(\Gamma; P_1,\dots,P_4,Q_1,\dots,Q_5) \in \mathscr{V}_9^{4,5,m}$ be a point corresponding to a plane curve $\Gamma : (f = 0) \subset \mathbb{P}^2$ with prescribed singular points and passing through $R_1,\dots,R_m$. Assume x,y,z are the coordinates of the projective plane. Considering Γ as a point in the parameter space $\mathbb{P}^{\binom{9+2}{2}-1}$ of degree 9 plane curves, without loss of generality we can assume it lies in the affine chart, which does not contain the point $(1:0:0)$. Moreover, to simplify our notations, we can assume all the distinguished points of Γ are in the open affine subset of $\mathbb{P}^2$ defined by $z = 1$. Thus, Γ is locally defined by $f = \sum_{u,v} a_{uv}x^u y^v$ such that $a_{9,0} = 1$, (x_i, y_i) for $1 \le i \le 9$ are the affine coordinates of the singular points and (x'_l, y'_l) is the affine coordinate of R_l.
Therefore, in a neighbourhood of Γ, the space $\mathscr{V}_9^{4,5,m}$ is the set of pairs $(\bar{h}; S_1,\dots,S_9)$ with $\bar{h} = \sum_{u,v} b_{uv}x^u y^v$, $b_{9,0} = 1$ and $S_i = (X_i,Y_i)$ for $1 \le i \le 9$, satisfying the following equations:

$$R_{i,s,t}(\dots,b_{uv},\dots,X_j,Y_j,\dots) := \frac{\partial \bar{h}}{\partial^t x \partial^{s-t} y}(X_i,Y_i) = 0,$$

for $1 \le i \le 4,\ s = 0,1,2,\ t \in \{0,\dots,s\}$,

$$R'_{i,s,t}(\dots,b_{uv},\dots,X_j,Y_j,\dots) := \frac{\partial \bar{h}}{\partial^t x \partial^{s-t} y}(X_i,Y_i) = 0,$$

for $5 \le i \le 9,\ s = 0,1,\ t \in \{0,\dots,s\}$ and

$$F_l := (\sum_{u,v} b_{uv}x^u y^v)(X'_l,Y'_l) = 0, \quad \forall\ 1 \le l \le m,$$

where (X'_l,Y'_l) are the coordinates of m points among the fixed points. Then, the tangent space at Γ is the set of points $(\bar{g}; T_1,\dots,T_9)$ with $\bar{g} = \sum_{u,v} c_{uv}x^u y^v$, $c_{9,0} = 1, c_{uv} = a_{uv} + b_{uv}$ for $u \ne 9$ and $T_i = (x_i + X_i, y_i + Y_i)$ for $1 \le i \le 9$, satisfying the following equations with indeterminate in $\dots,b_{uv},\dots,X_j,Y_j,\dots$:

$$\sum_{\substack{u,v\ge 0\\ u+v\le 9\\ u\ne 9}} b_{uv}\frac{\partial R_{i,s,t}}{\partial b_{uv}}(\dots,a_{uv},\dots,x_i,y_i,\dots)$$

$$+\sum_{\alpha=0}^{9}[X_\alpha \frac{\partial R_{i,s,t}}{\partial X_\alpha}(\dots,a_{uv},\dots,x_i,y_i,\dots) + Y_\alpha \frac{\partial R_{i,s,t}}{\partial Y_\alpha}(\dots,a_{uv},\dots,x_i,y_i,\dots)] = 0$$

for all $1 \le i \le 4,\ s = 0,1,2,\ t \in \{0,\dots,s\}$, the same relation with $R'_{i,s,t}$, for all $5 \le i \le 9,\ s = 0,1,\ t \in \{0,\dots,s\}$ and

$$\sum_{\substack{u,v\geq 0\\ u+v\leq 9\\ u\neq 9}} b_{uv}\frac{\partial \bar{h}}{\partial b_{uv}}(x'_l,y'_l) = 0, \quad \forall\ 1 \le l \le m.$$

In [KS18a], the code provided by the implemented function `verifyAssertions(1)` uses this method to compute the tangent space as the space of solutions to the above equations. Our computation of an explicit example (see also 5.6, step 2) for a randomly chosen point on $\mathscr{V}_9^{4,5,m}$ shows that this space is of dimension $33-m$. Therefore, the irreducible component of $\mathscr{V}_9^{4,5,m}$ containing that point is of expected dimension. $\square$

Remark 5.4.2. Let $(\Gamma; P_1,\dots,P_4,Q_1,\dots,Q_5) \in \mathscr{V}_9^{4,5,0}$ be a point and let Δ denote the singular locus of the corresponding plane curve with prescribed number of double and triple points. Via the first projection map

$$p_1 : \mathscr{V}_9^{4,5,0} \longrightarrow \mathscr{V}_9^{4,5} \subset \mathbb{P}^N,$$

the variety $\mathscr{V}_9^{4,5,0}$ maps one-to-one to the Severi variety $\mathscr{V}_9^{4,5}$, parametrizing the degree 9 plane curves with 4 ordinary triple points and 5 ordinary double points. More precisely, $\mathscr{V}_9^{4,5}$ can be realized as a locally closed subscheme of $\mathbb{P}^{\binom{9+2}{2}-1}$ representing the functor $\mathbf{V}_9^{4,5}$, which takes any $\mathbb{K}-$scheme S to the set

$$\mathbf{V}_9^{4,5}(S) = \left\{ \begin{array}{c} \text{flat families } \mathscr{C} \subset \mathbb{P}^2 \times S \text{ of plane curves of degree 9 whose} \\ \text{geometric fibres are curves with 4 triple and 5 double points} \end{array} \right\}.$$

This way, we can naturally denote $\mathscr{V}_9^{4,5,0}$ by $\mathscr{V}_9^{4,5}$ and identify the tangent space to $\mathscr{V}_9^{4,5,0}$ at Γ with the space of the first order deformation of $\Gamma \in \mathscr{V}_9^{4,5}$. Thus, from the Proposition 5.3.4 we obtain

$$T_\Gamma \mathscr{V}_9^{4,5} \cong \mathrm{H}^0(\mathbb{P}^2, \mathscr{I}_\Delta(9))/\langle f\rangle,$$

where $\langle f\rangle$ is the one-dimensional vector space generated by the defining equation of Γ. Moreover, for $m>0$ the computed tangent space to $\mathscr{V}_9^{4,5,m}$ at a random point as in Proposition 5.4.1, can be regarded as a subspace of such a vector space.

Now we turn to the computation of the tangent space to the locus $\mathscr{M}_{11,6}(k)$.

Let $C \subset \mathbb{P}^{10}$ be a smooth canonically embedded curve of genus $g=11$ and

$$0 \longleftarrow S/I_C \longleftarrow S \xleftarrow{f} S(-2)^{36} \xleftarrow{\varphi_1} S(-3)^{160} \xleftarrow{\varphi_2} S(-4)^{315} \xleftarrow{\varphi_3} \begin{array}{c} S(-5)^{288} \\ \oplus \\ S(-6)^{5k} \end{array} \xleftarrow{\varphi_4} \begin{array}{c} S(-6)^{5k} \\ \oplus \\ S(-7)^{288} \end{array}$$

be the part of a minimal free resolution of C of length 5, where $S = \mathbb{K}[x_0, \dots, x_{10}]$ is the coordinate ring of $\mathbb{P}^{10}$, and $f = (f_1, \dots, f_{36})$ is the minimal set of generators of the ideal $I_C \subset S$. Consider the pullback to C of the Euler sequence

$$0 \longrightarrow \mathscr{O}_C \longrightarrow \mathscr{O}_C(1)^{\oplus g} \longrightarrow T_{\mathbb{P}^{10}}|_C \longrightarrow 0. \tag{5.2}$$

From the long exact sequence of cohomologies, the dual vector space $\mathrm{H}^1(C, T_{\mathbb{P}^{10}}|_C)^\vee$ can be identified with the kernel of the Petri map

$$\mu_0 : \mathrm{H}^0(C, L) \otimes \mathrm{H}^0(C, \omega_C \otimes L^{-1}) \longrightarrow \mathrm{H}^0(C, \omega_C)$$

where $L = \mathscr{O}_C(1)$. Therefore, we get

$$\mathrm{H}^1(C, T_{\mathbb{P}^{10}}|_C) = 0$$

and from that, the induced long exact sequence of the normal exact sequence

$$0 \longrightarrow T_C \longrightarrow T_{\mathbb{P}^{10}}|_C \longrightarrow \mathscr{N}_{C/\mathbb{P}^{10}} \longrightarrow 0,$$

reduces to the following short exact sequence

$$0 \longrightarrow \mathrm{H}^0(C, T_{\mathbb{P}^{10}}|_C) \longrightarrow \mathrm{H}^0(C, \mathscr{N}_{C/\mathbb{P}^{10}}) \xrightarrow{\kappa} \mathrm{H}^1(C, T_C) \longrightarrow 0, \tag{5.3}$$

where κ is the so-called Kodaira-Spencer map. More precisely, here we realize $\mathrm{H}^1(C, T_C)$ as the tangent space to the moduli space $\mathscr{M}_{11}$ at the point corresponding to C, and κ as the induced map between the tangent spaces from the natural map $\mathscr{H}_{C/\mathbb{P}^{10}} \longrightarrow \mathscr{M}_{11}$. We observe that by Serre duality

$$\mathrm{H}^1(C, T_C) \cong \mathrm{H}^0(C, \omega_C^{\otimes 2})^\vee.$$

Since we assume that the curve is canonically embedded, the sheaf $\omega_C^{\otimes 2}$ is just the twisted sheaf $\mathscr{O}_C(2)$. Hence, the cohomology group above will be given by the quotient $S_2/(I_C)_2$ and thus $\mathrm{h}^1(C, T_C) = 30$. On the other hand, from the long exact sequence of cohomologies associated to the Euler sequence (5.2), we deduce $\mathrm{h}^0(C, T_{\mathbb{P}^{10}}|_C) = g^2 - 1 = 120$ and therefore

$$\mathrm{h}^0(C, \mathscr{N}_{C/\mathbb{P}^{10}}) = 150.$$

As I_C is minimally generated by 36 generators, we can identify a basis of $\mathrm{H}^1(C, T_C)$ with columns of a matrix T of size 36×30 with entries in $S_2/(I_C)_2$, introducing 30 free deformation parameters $b_0, \dots, b_{29}$ (see 5.6, step 3). Let $\bar{f} = f + f^{(1)}$ be the general first order family perturbing f defined by the general element of $\mathrm{H}^1(C, T_C)$ and let

$$\bar{S} \xleftarrow{\bar{f}} \bar{S}(-2)^{36}$$

be the corresponding morphism, where

$$\bar{S} = \mathbb{K}[b_0,\dots,b_{29}]/(b_0,\dots,b_{29})^2 \otimes_{\mathbb{K}} S.$$

To find a lift $\bar{\varphi}_1 = \varphi_1 + \varphi_1^{(1)}$ of φ_1, we apply the necessary condition $\bar{f} \circ \bar{\varphi}_1 \equiv 0 \bmod (b_0,\dots,b_{29})^2$, and we solve for an unknown $\varphi_1^{(1)}$ the equation:

$$0 \equiv \bar{f} \circ \bar{\varphi}_1 = (f + f^{(1)})(\varphi_1 + \varphi_1^{(1)}) = f \circ \varphi_1 + (f \circ \varphi_1^{(1)} + f^{(1)} \circ \varphi_1) \bmod (b_0,\dots,b_{29})^2.$$

This leads to $f \circ \varphi_1^{(1)} = -f^{(1)} \circ \varphi_1$, such that solving it for $\varphi_1^{(1)}$ by matrix quotient gives the required perturbation of the first syzygy matrix φ_1. Continuing through the remaining resolution maps, we can lift the entire resolution to first order in the same way. In [KS18b], an implementation of this algorithm is provided by the function `liftDeformationToFreeResolution`, which lifts a resolution to the first order deformed resolution.

Theorem 5.4.3. *Let $0 \le m \le 4$, and set $k := m+5$. The locus $\mathscr{M}_{11,6}(k) \subset \mathscr{M}_{11}$ has an irreducible component H_k of expected dimension $30-k$. Moreover, at a general point $P \in H_k$, $\mathscr{M}_{11,6}$ is locally analytically a union of k smooth transversal branches. In other words, $\mathscr{M}_{11,6}$ is a normal crossing divisor around the point P.*

Proof. Consider the natural commutative diagram

$$\begin{array}{ccc} \mathscr{V}_9^{4,5,m} & \xrightarrow{\ \psi\ } & \mathscr{H}_{C/\mathbb{P}^{10}} \\ \downarrow{\scriptstyle\phi} & & \downarrow \\ \mathscr{M}_{11,6}(k) & \hookrightarrow & \mathscr{M}_{11} \end{array}$$

where ϕ takes the plane curve to its canonical model forgetting the embedding. Let $H_k \subset \mathscr{M}_{11,6}(k)$ be the irreducible component containing the image points of curves lying in an irreducible component $H \subset \mathscr{V}_9^{4,5,m}$ with expected dimension (see Proposition 5.4.1). We show that H_k is of expected dimension.

For the image curve $C \in H_k$ of a plane curve Γ and the general first order deformation family of C, let M denote the $5k \times 5k$ submatrix of $\overline{\varphi}_4$ in the deformed free resolution with linear entries in free deformation parameters $b_0,\dots,b_{29}$. The condition $M = 0$ determines the space of the first order deformations with extra syzygies of rank $5k$.

By means of the implemented function `verfiyAssertion(2)` in [KS18a], we can compute an explicit single example (see 5.6, step 4) which shows that for exactly k linearly independent linear forms

$$l_1,\dots,l_k \in \mathbb{K}[b_0,\dots,b_{29}],$$

we have

$$\det\, M = l_1^5 \cdot \ldots \cdot l_k^5.$$

This already proves that the tangent space $T_C\mathscr{M}_{11,6}(k)$ is defined by the zero locus of these linear forms, and is of codimension exactly k inside $T_C\mathscr{M}_{11}$. Hence, H_k is an irreducible component of expected dimension $25-m$. On the other hand, by Hirschowitz–Ramanan [HR98] the Koszul divisor of curves with extra syzygies satisfies $\mathscr{K}_{11} = 5\mathscr{M}_{11,6}$ and the single polynomial det M defines the tangent space to the Koszul divisor at C. Therefore, we obtain that $\mathscr{M}_{11,6}$ at the point C is locally analytically union of k smooth branches. □

Remark 5.4.4. With the notation as above, under a change of basis, we can turn the matrix M to a block (or even a diagonal) matrix

$$M' = \begin{pmatrix} B_1 & 0 & \ldots & 0 \\ 0 & B_2 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & B_k \end{pmatrix}$$

such that for $i = 1, \ldots, k$ the non-zero block is $B_i = A_i L_i$, where A_i is an invertible 5×5 matrix with constant entries and L_i is the diagonal matrix with diagonal entries equal to l_i. In fact, for $i = 1, \ldots, k$, let X_i be the scroll swept out by the pencil g_i of degree 6 on C. Let $M_i = (MV_i)^t$ be the $5\times 5k$ matrix, where V_i is the constant matrix defining the last map $\varphi_i = S(-6)^5 \longrightarrow S(-6)^{5k}$ in the injective morphism of chain complexes from the resolution of X_i to the linear strand of a minimal resolution of C. Set $W_i := \ker M_i$ and for $j \in \{1, \ldots, k\}$, let $\overline{W}_j$ be the intersection of the modules W_i's by omitting W_j. Clearly, we have that $\operatorname{rank} W_i = 5(k-1)$, and a basis of W_i can be identified by columns of a constant matrix of size $5k \times 5(k-1)$. Moreover, we have $\operatorname{rank}\overline{W}_j = 5$ such that a basis of the module $\overline{W}_1 \oplus \ldots \oplus \overline{W}_k$ determines a $5k\times 5k$ invertible constant matrix. Using this invertible matrix for changing the basis of the space $S(-6)^{5k}$ turns the matrix M to a block matrix as above. To speed up our computations, we will use this presentation of M later on to compute its determinant (see 5.6, step 4).

5.5 Unirational irreducible components

In this section, we prove that the so-constructed rational families of plane curves dominate an irreducible component in the locus $\mathscr{M}_{11,6}(k)$ for $k = 5, \ldots, 10$. To this end, we count the number of moduli for these families, by computing the rank of the differential map between the tangent spaces.

Theorem 5.5.1. *For $5 \leq k \leq 9$, the moduli space $\mathscr{M}_{11,6}(k)$ has a unirational irreducible component of expected dimension. A general curve lying on this component arises from a degree 9 plane model with 4 ordinary triple and 5 ordinary double points which contains $k-5$ points among the ninth fixed point of the pencil of cubics passing through the 4 triple and 4 chosen double points.*

Proof. With notations as in Theorem 5.4.3 let $\phi_{|H} : H \longrightarrow H_k$ be the natural map between the irreducible components of expected dimensions. To compute the dimension of $\overline{\phi(H)}$, one has to compute the rank of the differential map

$$d\phi_\Gamma : T_\Gamma H \longrightarrow T_C H_k,$$

at a smooth point $C \in \phi(H)$. We recall that for $m > 0$ the tangent space to $\mathscr{V}_9^{4,5,m}$ at a point Γ is a subspace of $T_\Gamma \mathscr{V}_9^{4,5}$. Therefore, it suffices to show that dim(ker $d\phi_\Gamma$) $= 8$ for the case $k = 5$. Considering the following commutative diagram of tangent maps

$$\begin{array}{ccc}
 & & 0 \\
 & & \downarrow \\
 & & \mathrm{H}^0(C, T_{\mathbb{P}^{10}}|_C) \\
 & & \downarrow \\
T_\Gamma H & \xrightarrow{d\psi_\Gamma} & \mathrm{H}^0(C, \mathscr{N}_{C/\mathbb{P}^{10}}) \\
{\scriptstyle d\phi_\Gamma}\downarrow & & \downarrow \\
T_C H_k & \hookrightarrow & \mathrm{H}^1(C, T_C) \\
 & & \downarrow \\
 & & 0
\end{array}$$

our explicit computation of a single example (see 5.6, step 5 and `VerfiyAssertion(3)` in [KS18a]) shows that the image of the map $d\psi_\Gamma$ has exactly $8-$dimensional intersection with the image of $\mathrm{H}^0(C, T_{\mathbb{P}^{10}}|_C)$ inside $\mathrm{H}^0(C, \mathscr{N}_{C/\mathbb{P}^{10}})$, which corresponds to the automorphisms of the projective plane. Therefore, the rational family of plane curves lying on the irreducible component H dominates an irreducible component of $\mathscr{M}_{11,6}(k)$ with expected dimension. □

Theorem 5.5.2. *The moduli space $\mathscr{M}_{11,6}(10)$ has a unirational irreducible component of excess dimension 26, where the curves arise from degree 8 plane models with 10 ordinary double points. More precisely, the locus $\mathscr{M}^2_{11,8}$ of curves possessing a linear system g^2_8 is a unirational irreducible component of $\mathscr{M}_{11,6}(10)$ of expected dimension 26.*

Proof. Let $\mathscr{V}_8^{10}$ be the Severi variety of degree 8 plane curves with 10 ordinary double points. By classical results [Har86], it is known that $\mathscr{V}_8^{10}$ is smooth at each point and of pure dimension 34. Let Γ be a plane curve of degree 8 with 10 ordinary double points, and let $C \in \mathscr{M}^2_{11,8} \subset \mathscr{M}_{11,6}(10)$ be its normalization. With the same argument as in the proof of 5.4.3 and 5.5.1, the theorem follows from the computation of an example which shows that for linear forms $l_1,\ldots,l_{10}$ we have $\dim T_C\mathscr{M}_{11}(10) = \dim V(l_1,\ldots,l_{10}) = 26$ and furthermore the induced differential map is of full rank 26. The verification of this statement is implemented in the function `verifyAssertion(4)` in [KS18a]. $\square$

Corollary 5.5.3. *Let Γ be a general plane curve of degree 8 with 10 ordinary double points, and let $C \in \mathscr{M}_{11}$ be its normalization. Consider a deformation of C which preserves at least four pencils g^1_6's of the 10 existing pencils. Then, the deformation of C preserves the g^2_8. In other words, a deformation of C which keeps at least four pencils g^1_6's lies still on the locus $\mathscr{M}^2_{11,8}$.*

Proof. By the above theorem, around a general point $C \in \mathscr{M}^2_{11,8}$, the Brill–Noether divisor $\mathscr{M}_{11,6}$ is locally a union of 10 branches defined by $l_1 \cdot \ldots \cdot l_{10} = 0$. On the other hand, $\operatorname{codim} T_C\mathscr{M}^2_{11,8} = \operatorname{codim} V(l_1,\ldots,l_{10}) = 4$, such that any four of the linear forms are independent defining $\mathscr{M}^2_{11,8}$ locally around C. Therefore, a deformation of C which keeps at least four of g^1_6's lies still on the locus $\mathscr{M}^2_{11,8}$. $\square$

5.6 A standard test example

This section is devoted to the computations and the proof of the claims which we made in section 5.4 and 5.5. Having already made clear the reasons why the results obtained here are relevant, in what follows we describe how we are able to construct an explicit example, and do the relavent computations for it. The computer algebra package which the code refers to is *Macaulay2* [GS] We will make use of the implemented functions in the packages [KS18a] and [KS18b].
We note that most of the construction steps invoke only Gröbner basis and linear algebra computations, therefore it suffices to run our computations over a finite field. To speed up this process, and reduce the required memory, we focus on the case $k = 5$. We carry the computations over a ground field of small characteristic $p = 1009$. Although it is not difficult to modify these codes as it stands, and obtain the similar results for the case $k = 6,\ldots,10$, we will slightly explain how to modify some steps for the case $k > 5$.

Step 1. We first fix the coordinate ring of the projective plane $\mathbb{P}^2$. We pick 9 points in $\mathbb{P}^2$, say $P_1,\ldots,P_4,Q_1,\ldots,Q_5$ and we choose a random plane curve of degree 9 with triple points at $P_1,\ldots,P_4$ and ordinary double points at $Q_1,\ldots,Q_5$.

```
i1:  loadPackage"FirstOrderDeformations";
i2:  loadPackage"UnirationalComponentOf6GonalGenus11Curves";
i3:  p=1009;
i4:  L=random6gonalGenus11Curvekpencil(p,5);
       -- used 2.54078 seconds
i5:  Gama=L_3;--plane curve
i6:  Ican=L_4;--canonical model
i7:  genus Ican, degree Ican
o7:  (11, 20)
i8:  F=res(Ican, FastNonminimal => true);
i9:  betti(F, Minimize=>true)

               0  1   2   3   4   5   6   7  8 9
o9  =   total: 1 36 160 315 313 313 315 160 36 1
            0: 1  .   .   .   .   .   .   .  . .
            1: . 36 160 315 288  25   .   .  . .
            2: .  .   .   .  25 288 315 160 36 .
            3: .  .   .   .   .   .   .   .  . 1
```

Step 2. Let Δ be the singular locus of the plane curve Γ. Considering Γ as a point in $\mathscr{V}_9^{4,5}$, we use the Proposition 5.4.2 to identify the tangent space at Γ with the space of the equisingular deformation of it.

```
i10:  H=equisingularDefOfPlaneCurve(Gama);
               ZZ            1          ZZ             33
o10 : Matrix (---[x , x , x ])  <--- (---[x , x , x ])
               101  0   1   2           101  0   1   2
```

This proves the existence of a component $H \subset \mathscr{V}_9^{4,5}$ with expected dimension (see Proposition 5.4.1).

For $k > 5$, one can use explicit coordinates of the nine singular points, and the chosen $k-5$ points among the fix points to define the equations cutting out the tangent space, as in the 5.4.1. In [KS18a], the implemented function `verifyAssertion(0)` computes these equations of the tangent space explicitly. We note that, although we reduced to the affine coordinate, one can modify those equation to the projective coordinate. In both ways we end up with the same dimension of the tangent space defined by those equations.

Step 3. Having computed the canonical model $C \subset \mathbb{P}^{10}$ of the plane curve, we use the flatness condition 5.1 to compute the 36×30 matrix T, whose columns give a basis of $\mathrm{H}^1(C, T_C)$ resulting in 30 vectors with entries in $S_2/(I_C)_2$. In fact, since the Kodaira-Spencer map is surjective, here we realize this basis as quotient of a basis of $\mathrm{H}^0(C, \mathscr{N}_{C/\mathbb{P}^{10}})$, modulo the space of the first order deformations trivially induced by automorphisms of the ambient projective space, that is $\mathrm{H}^0(C, T_{\mathbb{P}^{10}}|_C)$.

```
i11:  time Fres=res(Ican, LengthLimit=>5,DegreeLimit=>2);
          -- used 10781.6 seconds
i12: T=firstOrderDeformation Fres;
                36      30
o12: Matrix R  <--- R
```

We note that the computation of this basis depends on the choice of the free resolution, or more precisely on the first syzygy matrix.

Step 4. Now we are able to define the general family of the first order deformation of C in free deformation parameters $b_0, \dots, b_{29}$, and the corresponding deformed minimal free resolution. Identifying the tangent space $T_C\mathscr{M}_{11,6}(5) \subset T_C\mathscr{M}_{11}$ with the space parametrizing the first order deformations of C with fibres having extra syzygies of rank 25, we need to compute 25×25 submatrix M of the syzygy matrix $\overline{\varphi}_4$ in the deformed resolution 5.4.3. For this purpose, one may simply make use of the implemented function `liftDeformationToFreeResolution` in [KS18b], however in our case of $g = 11$, the computations require a lot of memory, such that the running this computation for the 30 basis vectors in one *Macaulay2* session is out of reach. To tackle this problem, one can save the data of the matrix T and the first five syzygy matrices in a free resolution of the canonical curve and split this computation by running the same procedure for the basis vectors in different *Macaulay2* sessions. For a single basis vector we obtain the lifting by running the commands:

```
i13:  Df0=liftDeformationToFreeResolution(Fres,T_{0});
     --used 35404.607 seconds
i14:  m0=submatrix(Df0.dd_5,{288..312},{0..24})
```

The matrix m_0 depends on the choice of minimal free resolution and the matrix T. We have collected the output matrices $m_0, \dots, m_{29}$ by running the procedure in several *Macaulay2* sessions such that $M = \sum_{i=0}^{29} m_i$ (up to a change of indices of the b_i's). Now, let $T_e = \mathbb{Z}_p[b_0, \dots, b_{29}]$ be the ring of deformation parameters. We use the condition $M = 0$ to define the tangent space to $\mathscr{M}_{11,6}(5)$ at the point corresponding to C:

```
i15:  Te=ZZ/p[b_0..b_29];
i16:  tngSpaceToM1165=ideal flatten entries M;
i17:  dim tngSpaceToM1165
o17:  25
```

This proves the existence of an irreducible component $H_5 \subset \mathscr{M}_{11,6}(5)$ of expected dimension 25. To compute $\det M$, we compute the presentation of M as a block matrix. Each non-zero block is a linear multiple of an invertible constant matrix.

```
i18:  S4=scrollsPencilOfLinesTriple(Gama, Ican);
   --scrolls swept out by pencils induced by lines through each triple point
i19:  S5=scrollsPencilOfConics4Triple (Gama,Ican);
   --scroll swept out by the pencil induced by conics through 4 triple points
i20:  FiveScrolls=append(S4,S5);--list of 5 scrolls
   --the linear strand of a resolution of canonical curve
i21:  time MB=turnMatrixToBlocks(FiveScrolls,M,Ican);
       -- used 124.981 seconds
```

Now, computing the determinant of the each constant matrices, we obtain

```
i22:  detproducts=product apply(MB,j->det j_0)
o22:  48
```

Furthermore, the linear forms $l_1,\ldots,l_5$ are linearly independent:

```
i31:  linearforms=linearforms=apply(MB,j-> j_1);
i32:  dim ideal linearforms
o32:  25
```

This way, we have naturally

$$\det M = 48 l_1^5 \cdot \ldots \cdot l_5^5,$$

which proves that $\mathscr{M}_{11,6}$ is locally the union of 5 smooth branches around the point C (see 5.4.3).

Step 5. Now we show that the induced map on the tangent space has an $8-$dimensional kernel corresponding to the automorphisms of the projective plane (see 5.5.1 for a diagram). To do so, we first compute the image of the vector space $\mathrm{H}^0(C, T_{\mathbb{P}^{10}}|_C)$ inside $\mathrm{H}^0(C, \mathscr{N}_{C/\mathbb{P}^{10}})$, a 36×120 matrix whose columns are the embedded deformations of C induced by automorphisms of the projective space.

```
i33:  time auts=deformationByAutomorphisms(Ican);
      --used 4.183 seconds
```

Now, we turn to the computation of the image of the map $d\psi_\Gamma : T_\Gamma H \longrightarrow \mathrm{H}^0(C, \mathscr{N}_{C/\mathbb{P}^{10}})$. In fact, let f be the defining equation of Γ. For each basis element $h \in T_\Gamma H$ of the tangent space, $d\psi_\Gamma(h)$ is the family of deformation of C arised from the normalization of the equisingular deformation of Γ given by $f + \varepsilon h$.

```
i34: time equiDef=equisingularDefCurveTriples(G,Ican,H);
      -- used 2275.06 seconds
```

We compute the intersection of this image with the kernel of the Kodaira-Spencer map computed above:

```
i35:  time syz(equiDef|auts,DegreeLimit=>2)
       --used 5.36 seconds
              153     8
o35:  Matrix R  <--- R
```

Therefore, the map of tangent spaces has exactly an 8−dimensional kernel (see 5.5.1).

5.7 Further components

Having already described an irreducible unirational component of the moduli space $\mathscr{M}_{11,6}(k)$ for $k = 5, \ldots, 10$, the first natural question is to ask about the irreducibility of these loci. If the answer is negative, then the question is how the other irreducible components arise.

Although one may mimic our pattern to find model of plane curves of higher degree with singular points of higher multiplicity, considering the degree 9 plane curves with 4 ordinary triple and 5 ordinary double points as our original model, our simple computations show that the models of higher degree are usually a Cremona transformation of this model with respect to three singular points. Therefore, considering models of different degrees and singularities, we have not found new elements in these loci. On the other hand, the study of syzygy schemes of curves lying on these loci leads to the following theorem which states the existence of further irreducible components.

Theorem 5.7.1. *For $5 \leq k \leq 8$, the locus $\mathscr{M}_{11,6}(k)$ has at least two irreducible components both of expected dimension, along which $\mathscr{M}_{11,6}$ is generically a simple normal crossing divisor.*

Proof. The proof relies on the syzygy schemes and our computation of tangent cone at a point C in H_k.

Consider $\eta : \mathscr{W}^1_{11,6} \longrightarrow \mathscr{M}_{11,6} \subset \mathscr{M}_{11}$ and let C be a point in our unirational component $H_k \subset \mathscr{M}_{11,6}(k)$ for $6 \leq k \leq 9$. Then, by the Theorem 5.4.3, the tangent cone of the Brill-Noether divisor $\mathscr{M}_{11,6}$ is defined by a product $l_1 \cdot \ldots \cdot l_k$ of k linearly independent linear forms, and $\mathscr{W}^1_{11,6} \longrightarrow \mathscr{M}_{11,6}$ is locally around C the normalization of $\mathscr{M}_{11,6}$. Let $f_1, \ldots, f_k$ be power series which define the k branches of $\mathscr{M}_{11,6}$ in an analytic or étale neighbourhood U of $C \in \mathscr{M}_{11}$. Then

$$f_i = l_i + \text{ higher order terms}$$

and the zero locus $V(f_i) \subset U$ has the following interpretation:

$$V(f_i) \cong \{(C', L') : \ (C', L') \in \ U_i\},$$

where $\eta^{-1}(U) = \bigcup_{i=1}^{k} U_i$ is the disjoint union of smooth $3g-4$ dimensional manifolds with $(C, L_i) \in U_i$ such that L_i denotes line bundle corresponding to the the $i-$th pencil g^1_6 on C in some enumeration of the pencils $L_1, \ldots, L_k$ that we fix.
The submanifold $B_i = \{f_i = 0\}$ then consists of deformations of C induced by deformation of pair (C, L_i), and for any family $\Delta \subset B_i$ the Kuranishi family restricted to Δ extends to a deformation of the pair (C, L_i)

$$\begin{array}{ccccccc} C & \subset & \mathscr{C} & \qquad & (C, L_i) & \subset & (\mathscr{C}, \mathscr{L}_i) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ 0 & \in & \Delta & & 0 & \in & \Delta \end{array}$$

Let $I \subset \{1, \ldots, k\}$ be any subset of cardinality $\ell \geq 5$ and $C' \in U$ be a point such that

$$C' \in \bigcap_{i \in I} V(f_i) \setminus \bigcup_{j \notin I} V(f_j).$$

Then, by Theorem 5.4.3

$$C' \in \mathscr{M}_{11,6}(\ell) \setminus \mathscr{M}_{11,6}(\ell+1)$$

since the l_i with $i \in I$ are linearly independent, $\mathscr{M}_{11,6}(\ell)$ is of codimension ℓ and $\mathscr{M}_{11,6}$ is a normal crossing divisor around C'.

Now, we examine that whether or not C' lies in our component H_ℓ. For this purpose, we deform the L_i for $i \in I$ in a one-dimensional family of curves

$$\Delta = \{C''\} \subset \bigcap_{i \in I} V(f_i)$$

through C and C', which intersects $\bigcup_{j \notin I} V(f_j)$ only in the point C. The syzygy schemes of the $C'' \in \Delta$ forms an algebraic family defined by the intersection of the deformed scrolls

X_i'' swept out by the deformed line bundle L_i''. Thus by semicontinuity, the dimension of the syzygy scheme of C'' near $C \in \Delta$ is smaller or equal than the dimension of the syzygy scheme $\bigcap X_i$, and in case of equality we should have $\deg(\bigcap X_i'') \leq \deg(\bigcap X_i)$. If we take special syzygy scheme of C'' corresponding to the syzygies of $\bigcap_{j\in J} X_j''$ then likewise we have the semicontinuity compare to $\bigcap_{j\in J} X_j$. Therefore, for C'' to lie on H_l we need a subset $J \subset I$ of cardinality 5 such that the syzygy scheme is a surface of degree 15 (see table 5.4). By the Remark 5.7.2, this occurs only if we have $a = 5$ and $b = 0$. Thus, taking I to be a subset of $\{2,\ldots,5\} \cup \{6,\ldots,k\}$ we obtain a point $C'' \in \mathscr{M}_{11,6}(\ell) \setminus H_\ell$. This proves that for $5 \leq \ell \leq 8$ the moduli space $\mathscr{M}_{11,6}(\ell)$ has at least two components, one of which H_ℓ and the other a component containing C''.

□

Remark 5.7.2. For the model of plane curve of degree 9 with nine pencils described in 5.2, we have computed the dimension, degree and the Betti table of the syzyzgy schemes associated to different number $2 \leq l \leq 9$ of pencils g^1_6's. We recall that for a number of pencils indexed by a subset $I \subset \{1,\ldots,9\}$, the associated syzygy scheme is the intersection $\bigcap_{i\in I} X_i$ of the scrolls swept out by each of the pencils. Let $1 \leq a \leq 5$ be the number of chosen pencils which are induced by projection from the triple points or the pencil of conics. Likewise, let $1 \leq b \leq 4$ be the number of chosen pencils arised from the pencil of cubics through the certain number of points. In the following tables, and for a specific genus 11 curve possessing nine pencils of degree 6, we have listed the numerical data of the plausible syzygy schemes arised form different number $l = a + b \geq 2$ of the existing pencils g^1_6's. In [KS18a], one can compute an example of such a curve over a finite field of characteristic p, by running the function `random6gonalGenus11Curvekpencil(p,9)`. In particular, the function `verifyAsserion(5)` provides the explicit equation of our specific curve and the collection of the nine scrolls. In the columns "dim" and "deg" we have marked the possible dimension and the degree of the corresponding syzygy schemes for this specific curve. Based on our experiments, it turns out that the values only depend on the numbers a and b of the chosen pencils.

		dim	deg	genus	Betti table
$a=0$	$b=2$	2	18		1 . . . . 27 96 127 48 10 . . . . . 1 48 220 288 189 64 9
$a=1$	$b=1$	2	18		1 . . . . 27 96 127 48 10 . . . . . 1 48 220 288 189 64 9
$a=2$	$b=0$	2	18		1 . . . . 27 96 127 48 10 . . . . . 1 48 220 288 189 64 9

Table 5.1: Numerical data of possible syzygy schemes with $a+b=2$.

		dim	deg	genus	Betti table
$a=0$	$b=3$	1	21	12	1 . . . . 35 151 279 207 15 . . . . 3 141 414 399 196 45 1 1
$a=1$	$b=2$	1	20	11	1 . . . . 36 160 315 288 45 . . . . 45 288 315 160 36 1
$a=2$	$b=1$	1	21	12	1 . . . . 35 151 279 210 30 . . . . 6 156 414 399 45 1 1
$a=3$	$b=0$	2	16		1 . . . . 29 112 182 113 15 . . . . . . 1 85 176 133 48 7

Table 5.2: Numerical data of possible syzygy schemes with $a+b=3$.

		dim	deg	genus	Betti table
$a=0$	$b=4$	1	20	11	1 . . . . 36 160 315 288 45 . . . . 45 288 315 160 36 1
$a=1$	$b=3$	1	20	11	1 . . . . 36 160 315 288 45 . . . . 45 288 315 160 36 1
$a=2$	$b=2$	1	20	11	1 . . . . 36 160 315 288 45 . . . . 45 288 315 160 36 1
$a=3$	$b=1$	1	21	12	1 . . . . 35 151 279 210 30 . . . . 6 156 414 399 45 1 1
$a=4$	$b=0$	2	15		1 . . . . 30 120 210 169 25 . . . . . . 1 25 120 105 40 6

Table 5.3: Numerical data of possible syzygy schemes with $a+b=4$.

		dim	deg	genus	Betti table
$a=1$	$b=4$	1	20	11	1 . . . . 36 160 315 288 45 . . . . 45 288 315 160 36 1
$a=2$	$b=3$	1	20	11	1 . . . . 36 160 315 288 45 . . . . 45 288 315 160 36 1
$a=3$	$b=2$	1	20	11	1 . . . . 36 160 315 288 45 . . . . 45 288 315 160 36 1
$a=4$	$b=1$	1	21	12	1 . . . . 35 151 279 210 30 . . . . 6 156 414 399 45 1 1
$a=5$	$b=0$	2	15		1 . . . . 30 120 210 169 25 . . . . . . 1 25 120 105 40 6

Table 5.4: Numerical data of possible syzygy schemes with $a+b=5$.

		dim	deg	genus	Betti table
$a=2$	$b=4$	1	20	11	1 . . . . 36 160 315 288 45 . . . . 45 288 315 160 36 1
$a=3$	$b=3$	1	20	11	1 . . . . 36 160 315 288 45 . . . . 45 288 315 160 36 1
$a=4$	$b=2$	1	20	11	1 . . . . 36 160 315 288 45 . . . . 45 288 315 160 36 1
$a=5$	$b=1$	1	21	12	1 . . . . 35 151 279 210 30 . . . . 6 156 414 399 45 1 1

Table 5.5: Numerical data of possible syzygy schemes with $a+b=6$.

		dim	deg	genus	Betti table
a	b	1	20	11	1 . . . . 36 160 315 288 45 . . . . 45 288 315 160 36 1

Table 5.6: Numerical data of possible syzygy schemes with $a+b \geq 7$.

The End

I do the very best I know how, the very best I can; and I mean to keep on doing so until the end...

«A. Lincoln»

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