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Canonical Curves, Scrolls and K3 surfaces

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Abstract

In this thesis we study minimal free resolutions of canonical curves $C \subset \mathbb{P}^{g-1}$ as well as so-called relative canonical resolutions of canonical curves inside scrolls swept out by a pencil of divisors on C. For general 5-gonal canonical curves we will show that all bundles in the relative canonical resolution are balanced. Furthermore, we will give a necessary and sufficient criterion for balancedness of the first syzygy bundle appearing in the relative canonical resolution of Brill–Noether general curves. For general genus 9 curves we will relate the unbalancedness of the second syzygy bundle to the existence of certain K3 surfaces of higher Picard rank which contain the curves. This yields an unirationality result for certain moduli spaces of lattice polarized K3 surfaces.

Another subject of this thesis is the study of the homologies occurring in a linearized free resolution of (k-gonal) canonical curves. Based on computer algebra experiments we will also suggest a refinement of the classical Green-conjecture, which conjecturally also holds in positive characteristic.

Zusammenfassung

In dieser Arbeit studieren wir minimale freie Auflösungen von kanonischen Kurven $C \subset \mathbb{P}^{g-1}$ und sogenannte relative kanonische Auflösungen von kanonischen Kurven auf rationalen Regelvarietäten, die von einem Büschel von Divisoren auf C ausgeschnitten werden. Für eine allgemeine 5-gonale kanonische Kurve zeigen wir, dass alle Bündel in der relativen kanonischen Auflösung balanciert sind. Des Weiteren geben wir ein notwendiges und hinreichendes Kriterium für die Balanciertheit des ersten Syzgygienbündel in der relativen kanonischen Auflösung für Brill-Noether allgemeine Kurven an. Für Geschlecht 9 Kurven setzen wir die Unbalanciertheit des zweiten Syzygienbündels mit der Existenz von K3 Flächen von höherem Picardrang, welche die Kurve enthalten, in Verbindung. Dies führt zu Unirationalitätsresultaten für bestimmte Modulräume von gitterpolarisierten K3 Flächen.

Ein weiteres Thema dieser Arbeit ist das Studium der Homologien in linearisierten freien Auflösungen von (k-gonalen) kanonischen Kurven. Auf Grund von Computeralgebra experimenten werden wir zudem eine Erweiterung der klassischen Greenschen Vermutung vorschlagen, welche mutmaßlich auch in positiver Charakteristik gilt.

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Contents

Chapter 1

Introduction and outline of the results

Let $C \subset \mathbb{P}^{g-1}$ be a canonically embedded curve of genus g which has a complete base point free g^1_k. The g^1_k on C sweeps out a rational normal scroll

$$X = \bigcup_{D \in g^1_k} \overline{D} \subset \mathbb{P}^{g-1}$$

of dimension $d = k-1$ and degree $f = g-k+1$. On the other hand, the scroll X is the image $\mathbb{P}(\mathscr{E}) \to \mathbb{P}\big(H^0(\mathbb{P}(\mathscr{E}), \mathscr{O}_{\mathbb{P}(\mathscr{E})}(1))\big)$ of a projective bundle $\pi : \mathbb{P}(\mathscr{E}) \to \mathbb{P}^1$, where $\mathscr{E}$ is a degree f bundle of the form $\mathscr{E} = \mathscr{O}_{\mathbb{P}^1}(e_1) \oplus \cdots \oplus \mathscr{O}_{\mathbb{P}^1}(e_d)$. If all the e_i are non-negative, then $\mathbb{P}(\mathscr{E})$ is isomorphic to the scroll X. However, even if some of the e_i are zero, it often convenient to consider $\mathbb{P}(\mathscr{E})$ instead of X.

In [Sch86] Schreyer showed, that one can resolve $C \subset \mathbb{P}(\mathscr{E})$ in terms of $\mathscr{O}_{\mathbb{P}(\mathscr{E})}$-modules. The resolution obtained this way is called the *relative canonical resolution* of C (with respect to the g^1_k) and has the form:

$$0 \leftarrow \mathscr{O}_C \leftarrow \mathscr{O}_{\mathbb{P}(\mathscr{E})} \leftarrow \pi^* N_1(-2) \leftarrow \cdots \leftarrow \pi^* N_{k-3}(-k+2) \leftarrow \pi^* N_{k-2}(-k) \leftarrow 0$$

where $\pi^* N_i = \sum_{j=1}^{\beta_i} \mathscr{O}_{\mathbb{P}(\mathscr{E})}(a_j^{(i)} R)$, $\beta_i = \frac{i(k-2-i)}{k-1}\binom{k}{i+1}$ and $R = \pi^* \mathscr{O}_{\mathbb{P}^1}(1)$ denotes the ruling of the scroll.

For a general curve C together with a g^1_k one can show that the generic splitting type of the bundle $\mathscr{E}$ associated to the scroll swept out by the g^1_k is balanced, i.e. $\max_{i,j} |e_i - e_j| \leq 1$ (see e.g. Corollary 2.3.6). On the other hand the generic splitting types of the bundles N_i in the relative canonical resolution are only known in very few cases. Even the degree of the generators of $C \subset \mathbb{P}(\mathscr{E})$, i.e. the splitting type of the bundle N_1, is not known in general.

One of the main subjects of this thesis is the study of the structure of the relative canonical resolution for general curves admitting a g^1_k. In particular the study of the (un-) balancedness of the bundles N_i.

Including the results presented in this thesis, the following is known.

- For $g \leq 9$ and $k < \lceil \frac{g+2}{2} \rceil$ all N_i are generically balanced by [Sch86] and [Sag06].
- More generally, all N_i are generically balanced for $k = 3, 4, 5$ (see [DP15] and Proposition 3.2.3)
- If $g = n \cdot k + 1$ for some integer positive n, then all N_i are generically balanced by [BP15].
- The first bundle N_1 is generically balanced if $g \geq (k-1)(k-3)$ (see [BP15]).
- For a general canonical curve of genus g let k be a positive integer such that Brill-Noether number $\rho := \rho(g, k, 1) \geq 0$ is non-negative. Then the bundle N_1 in the relative canonical resolution associated to a general point in the Brill-Noether variety $W^1_k(C)$ is balanced if and only if $(k - \rho - \frac{7}{2})^2 - 2k + \frac{23}{4} > 0$ and $\rho > 0$ (see Theorem 4.1.1).
- The second syzygy bundle N_2 is unbalanced for hexagonal curves of genus 9 (see Corollary 5.3.8).

Knowing the structure of the relative canonical resolution or the splitting type of the bundles N_i has various applications. In [Sch86] Schreyer made use of the structure of the relative canonical resolution in order to classify all possible Betti tables for canonical curves $C \subset \mathbb{P}^{g-1}$ of genus $g \leq 8$. This classification has been continued for genus 9 curves by Sagraloff (see [Sag06]). On the other hand Patel and Deopurkar-Patel studied the loci inside the Hurwitz space $\mathscr{H}_{g,k}$ consisting of k-gonal curves for which the relative canonical resolution does not have the generic shape (see [Pat13] and [DP15]). In some cases these loci define divisors on the Hurwitz space $\mathscr{H}_{g,k}$ which then can be used to study the geometry of the Hurwitz space itself. For example using the generic balancedness of the bundles N_i for $k=3,4,5$, Deopurkar-Patel showed that the rational Picard group of the Hurwitz space $\mathrm{Pic}_{\mathbb{Q}}\mathscr{H}_{g,k}$ is trivial in these cases.

Further applications of the generic (un-) balancedness of the relative canonical resolution will be discussed later on. We will briefly outline the main results presented in this thesis.

Syzygies of 5-gonal canonical curves

Chapter 3 of this thesis follows the article [Bop15] and extends some of the results of the authors master thesis [Bop13].

In Chapter 3 we study the minimal free resolutions of general 5-gonal canonical curves. Recall that the gonality of a curve is defined as $\mathrm{gon}(C)=\min\{k \mid \exists\ g^1_k \text{ on } C\}$. For a k-gonal curve, the scroll swept out by the g^1_k on C contributes with an Eagon-Northcott complex of length $g-k$ to the linear strand of the curve C and it is natural to ask how much the Betti numbers of C and X differ. It is not hard to see that $\beta_{i,i+1}(C)>\beta_{i,i+1}(X)$ for $i=1,\dots,\lceil\frac{g-3}{2}\rceil$ and therefore, the first "critical" Betti number is $\beta_{n,n+1}(C)$ for $n=\lceil\frac{g-1}{2}\rceil$.

For general curves, the gonality is precisely $\lceil\frac{g+2}{2}\rceil$ and therefore $\beta_{n,n+1}(X)=0$ where X is the scroll associated to a pencil of minimal degree on C. If the genus is odd, then by results of Voisin and Hirschowitz-Ramanan (see [Voi05] and [HR98]) the locus

$$\mathscr{K}_g=\left\{C\in\mathscr{M}_g \mid \beta_{n,n+1}(C)=0\right\}$$

defines a divisor on the moduli space of curves , called the *Koszul divisor*. On the Hurwitz-space, $\mathscr{H}_{g,k}$ parametrizing k-gonal genus g curves, one can consider the natural analogue of the Koszul divisor

$$\mathscr{K}_{g,k}=\left\{C\in\mathscr{M}_g \mid \beta_{n,n+1}(C)>\beta_{n,n+1}(X)\right\}$$

which, by arguments similar to [HR98], is a divisor for odd genus precisely if $\beta_{n,n+1}(C)=\beta_{n,n+1}(X)$ holds for general k-gonal curves. The approach above does no longer work for large genus g. We will show the following.

Theorem. *Let* C *be a* 5*-gonal canonical curve of genus* g *and let* $n=\lceil\frac{g-1}{2}\rceil$. *Then*

$$\beta_{n,n+1}(C)>\beta_{n,n+1}(X)$$

for odd genus $g\geq 13$ *or even genus* $g\geq 28$.

One key ingredient in the proof of the theorem above is the following proposition.

Proposition. *Let* $C \subset \mathbb{P}^{g-1}$ *be a general* 5*-gonal canonical curve of genus* g *and let* $\mathbb{P}(\mathscr{E})$ *be the projective bundle associated to the scroll swept out by the pencil of degree* 5*. Then* $C \subset \mathbb{P}(\mathscr{E})$ *has a balanced relative canonical resolution.*

Knowing the shape of the relative canonical resolution, one can resolve the terms appearing in the resolution by Eagon-Northcott type complexes. An iterated mapping cone then gives a non-minimal free resolution of $C \subset \mathbb{P}^{g-1}$. By determining the ranks of the maps, which give non-minimal parts, one can check whether the curve has extra syzygies. This step is already contained in [Bop13].

Relative canonical resolutions for general curves

Chapter 4 follows the article [BH15b].

We study the structure of relative canonical resolutions for Brill-Noether general curves. The main result in this chapter gives a necessary and sufficient condition for the balancedness of the first syzygy bundle N_1.

Theorem. *Let* $C \subset \mathbb{P}^{g-1}$ *be a general canonical curve and let* k *be a positive integer such that* $\rho := \rho(g,k,1) \geq 0$ *and* $g > k+1$*. Let* $L \in W^1_k(C)$ *be a general point inducing a* $g^1_k = |L|$*. Then the bundle* N_1 *in the relative canonical resolution of* C *is unbalanced if and only if* $(k-\rho-\frac{7}{2})^2 - 2k + \frac{23}{4} > 0$ *and* $\rho > 0$.

The idea for the proof is to consider the birational image C' of C under the residual mapping $|\omega_C \otimes L^{-1}|$. Certain generators of $C \subset \mathbb{P}(\mathscr{E})$, which force N_1 to be unbalanced, correspond to quadric generators of $C' \subset \mathbb{P}^{g-1}$. Using the proof of the maximal rank conjecture in the range of quadrics, we obtain a sharp bound on pairs (k,ρ), for which the curve C' has a quadric generator.

Another minor result in Chapter 4 is a closed formula for the degrees of the bundles N_i in the relative canonical resolution.

Proposition. *The degree of the bundle* N_i *of rank* $\beta_i = \frac{k}{i+1}(k-2-i)\binom{k-2}{i-1}$ *in the relative canonical resolution is*

$$\deg(N_i) = \sum_{j=1}^{\beta_i} a_j^{(i)} = (g-k-1)(k-2-i)\binom{k-2}{i-1}.$$

Moduli of lattice polarized K3 surfaces via relative canonical resolutions

Chapter 5 follows the article [BH17a].

For K3 surfaces, the best understood moduli spaces are those parametrizing H-polarized K3 surfaces, where $\mathscr{O}_S(H)$ defines an ample class with self-intersection $2g-2$. These moduli spaces are usually denoted $\mathscr{F}_g^H$. It is known that $\mathscr{F}_g^H$ is unirational for $g \leq 14$ and $g = 16, 18, 20, 33$ (see [Muk88], [Muk96], [Muk06], [Muk12], [Muk92] [Nue16] and [Kar16]). On the other hand, $\mathscr{F}_g^H$ is known to be of general type for $g = 47, 51, 55, 58, 61$ and $g > 62$ (see [GHS07], [Kon93] and [Kon99]). A generalization of polarized K3 surfaces are the so-called lattice polarized K3 surfaces, introduced in [Dol96]. Instead of fixing an ample polarization on a K3 surface S, one fixes a primitive lattice embedding $\varphi : M \to \mathrm{Pic}(S)$ such that the image of M contains an ample class. By [Dol96] there exisis a quasi-projective $20 - \mathrm{rk}(M)$ dimensional moduli space $\mathscr{F}^M$ parametrizing M-polarized K3 surfaces.

Much less is known about the unirationality of the moduli spaces for lattice polarized K3 surfaces. However, for some lattices M , the spaces $\mathscr{F}^M$ are known to be unirational (see [BHK16], [FV12], [FV16] and [Ver16]). In Chapter 5 we study the moduli space $\mathscr{F}^{\mathfrak{h}}$ parametrizing $\mathfrak{h}$-polarized K3 surfaces for a rank 3 lattice $\mathfrak{h}$ which has the following form

$$\mathfrak{h} \sim \begin{pmatrix} 14 & 16 & 5 \\ 16 & 16 & 6 \\ 5 & 6 & 0 \end{pmatrix}$$

with respect to a fixed ordered basis $\{h_1, h_1, h_3\}$.

For a $\mathfrak{h}$-polarized K3 surface (S, φ) we denote

$$\mathscr{O}_S(H) = \varphi(h_1),\ \ \mathscr{O}_S(C) = \varphi(h_2) \text{ and } \mathscr{O}_S(N) = \varphi(h_3).$$

We note that $H \in |\mathscr{O}_S(H)|$ has genus 8, $C \in |\mathscr{O}_S(C)|$ has genus 9 and $N \in |\mathscr{O}_S(N)|$ cuts out a g^1_6 on C. We will consider the open subset

$$\mathscr{F}_8^{\mathfrak{h}} = \left\{ (S, \varphi) \;\middle|\; (S, \varphi) \in \mathscr{F}^{\mathfrak{h}} \text{ and } \mathscr{O}_S(H) = \varphi(h_1) \text{ ample } \right\}$$

of the moduli space $\mathscr{F}^{\mathfrak{h}}$ and the open subset

$$\mathscr{P}_8^{\mathfrak{h}} = \left\{ (S, \varphi, C) \;\middle|\; (S, \varphi) \in \mathscr{F}_8^{\mathfrak{h}} \text{ and } C \in |\mathscr{O}_S(C)| \text{ smooth} \right\}$$

of the tautological $\mathbb{P}^9$-bundle over $\mathscr{F}_8^{\mathfrak{h}}$. We connect the space $\mathscr{P}_8^{\mathfrak{h}}$ to the universal Brill–Noether variety $\mathscr{W}^1_{9,6}$, parametrizing genus 9 curves together with a pencil of degree 6, via the natural restriction map

$$\phi : \mathscr{P}_8^{\mathfrak{h}} \to \mathscr{W}^1_{9,6}, \ \left(S, \varphi, C\right) \mapsto \left(C, \mathscr{O}_S(N) \otimes \mathscr{O}_C\right).$$

The main result in this chapter is the following

Theorem. *The map $\phi : \mathscr{P}_8^{\mathfrak{h}} \to \mathscr{W}^1_{9,6}$ defined above is dominant. Moreover, $\mathscr{P}_8^{\mathfrak{h}}$ is birational to a $\mathbb{P}^1$-bundle over an open subset of $\mathscr{W}^1_{9,6}$. In particular $\mathscr{P}_8^{\mathfrak{h}}$ and hence $\mathscr{F}^{\mathfrak{h}}$ are unirational.*

The key idea is that elements in the fiber $\phi^{-1}(C, L)$ are parametrized by the syzygy schemes of syzygies in certain degree in the relative canonical resolution of C inside the scroll defined by L. The existence of such syzygies forces the second syzygy bundle in the relative canonical resolution to be unbalanced. Therefore we obtain the following theorem as a consequence of our main theorem,

Theorem. *For any $(C, L) \in \mathscr{W}^1_{9,6}$ the relative canonical resolution has an unbalanced second syzygy bundle.*

For a general element $(C, L) \in \mathscr{W}^1_{9,6}$, the image C' under the embedding defined by the Serre dual line bundle $\omega_C \otimes L^{-1} \in W^3_{10}(C)$ lies on a net of quartics. We will show that the fiber $\phi^{-1}(C, L)$ defines a plane cubic curve inside this net of quartics. We will see that the K3 surface corresponding to the singular point of the plane cubic has a Picard lattice $\mathfrak{h}'$ of the form

$$\mathfrak{h}' \sim \begin{pmatrix} 4 & 10 & 1 & 1 \\ 10 & 16 & 0 & 0 \\ 1 & 0 & -2 & 0 \\ 1 & 0 & 0 & -2 \end{pmatrix}$$

with respect to some ordered basis $\{h'_1, h'_2, h'_3, h'_4\}$. This yields the following the following theorem.

Theorem. *Let*

$$\mathscr{P}_3^{\mathfrak{h}'} = \left\{(S, \varphi, C) \mid (S, \varphi) \in \mathscr{F}^{\mathfrak{h}'}, \varphi(h'_1) \textit{ ample and } C \in |\varphi(h'_2)| \textit{ smooth}\right\}$$

be the open subset of the tautological $\mathbb{P}^9$-bundle over the moduli space $\mathscr{F}_3^{\mathfrak{h}'}$. Then the morphism

$$\phi' : \mathscr{P}_3^{\mathfrak{h}'} \to \mathscr{W}^3_{9,10},\ (S, C) \mapsto (C, \mathscr{O}_S(H') \otimes \mathscr{O}_C)$$

defines a birational equivalence.

The BGG-correspondence for canonical curves and Green's conjecture in positive characteristic

In Chapter 6 we use the The Bernšteĭn–Gel'fand–Gel'fand correspondence to study the homologies in linearized free resolutions of (k-gonal) canonical curves. Let V be an $n+1$ dimensional vector space, $E = \bigwedge V$ the exterior algebra and $S = \mathrm{Sym} V^*$ over the dual vector space. The BGG correspondence consists of a pair of adjoint functors $\mathbf{R}$ and $\mathbf{L}$ from the category of complexes of graded S-modules to the category of complexes of graded E-modules. In [EFS03] the construction of these two functors is made explicit and it is furthermore shown that they define an equivalence of derived categories

$$D^b(S-\mathrm{mod}) \underset{\mathbf{L}}{\overset{\mathbf{R}}{\rightleftarrows}} D^b(E-\mathrm{mod}).$$

This allows us to study properties of an S-module M by studying the complex $\mathbf{R}(M)$ over the exterior algebra. For example, by [EFS03], the Betti numbers of a finitely generated graded S-module M are the vector space dimensions of the graded pieces of the cohomologies in $\mathbf{R}(S_C)$:

$$H^i(\mathbf{R}(M))_j = \mathrm{Tor}^S_{j-i}(\mathbb{k}, M)_j.$$

For general k-gonal canonical curves C, we study the homologies of the linear strands of the coordinate ring S_C, as well as the cohomologies in $\mathbf{R}(S_C)$. Using the generic balancedness of the relative canonical resolution for 4-gonal curves we will give an explicit description of all homologies appearing in the linear strands of a minimal free resolution for general 4-gonal canonical curves.

For a finitely generated module P over the exterior algebra, we denote by lin(P) cokernel of the linearized presentation matrix of P, i.e. the matrix obtained by erasing all terms in degree > 1. We show that the Betti numbers of S_C are already encoded in $\mathrm{lin}(H^1\mathbf{R}(S_C))$ for general 4-gonal canonical curves C:

Theorem. *Let* C *be a general* 4*-gonal canonical curve, then*

$$\dim_{\Bbbk} \operatorname{lin}\big(\mathrm{H}^1\mathbf{R}(\mathrm{S}_\mathrm{C})\big)_j = \dim_{\Bbbk} \mathrm{H}^1\mathbf{R}(\mathrm{S}_\mathrm{C})_j = \beta_{j-1,j}(\mathrm{C})$$

Many of the results in this chapter originate in experiments using computeralgebra. We have implemented the construction of random canonical curves of genus $g \leq 15$ in a *Macaulay2*-Package [BS17]. Based on our experiments we suggest a refinement of the classical Green conjecture.

Conjecture. *Let* $\mathrm{C} \subset \mathbb{P}^{g-1}$ *a canonically embedded curve defined over an algebraically closed field (of arbitrary characteristic) and let*

$$0 \leftarrow \mathrm{S}^{\beta_{1,3}}(-3) \xleftarrow{\varphi_2} \mathrm{S}^{\beta_{2,4}}(-4) \xleftarrow{\varphi_3} \dots \xleftarrow{\varphi_{g-3}} \mathrm{S}^{\beta_{g-3,g-1}}(-(g-1)) \leftarrow 0$$

be the second linear strand of a minimal free resolution of the coordinate ring S_C *(here* $\mathrm{S}^{\beta_{i,i+2}}(-(i+2))$ *sits in homological degree* i*). Then*

(a) $\mathrm{H}_i(\mathrm{strand}_2(\mathrm{S}_\mathrm{C}))$ *is a module of finite length for all* $i \leq p$ *if and only if* $\mathrm{Cliff}(\mathrm{C}) > p$.

(b) If C *is general inside the gonality stratum* $\mathscr{M}^1_{g,k} \subset \mathscr{M}_g$ *with* $2 < k < \lceil \frac{g+2}{2} \rceil$ *then* $\mathrm{H}_{k-2}(\mathrm{strand}_2(\mathrm{S}_\mathrm{C}))$ *is supported on the rational normal scroll swept out by the unique* g^1_k *on* C.

The complete data of the experiments, which led to the conjecture above can be found in Chapter 7.

Publications and Software packages

The chapters 3, 4 and 5 follow the articles listed below.

- Christian Bopp. *Syzygies of 5-gonal canonical curves.* Doc. Math., 20 : 1055 – 1069, 2015.

- Christian Bopp and Michael Hoff. *Resolutions of general canonical curves on rational normal scrolls.* Arch. Math. (Basel), 105(3) : 239 – 249, 2015.

- Christian Bopp and Michael Hoff. *Moduli of lattice polarized K3 surfaces via relative canonical resolutions.* preprint, 2017. Available at https://arxiv.org/pdf/1704.02753.pdf.

The following software packages have been developed during the course of this work.

- Christian Bopp and Michael Hoff. *RelativeCanonicalResolution.m2* - construction of relative canonical resolutions and Eagon-Northcott type complexes, 2015, a Macaulay2 package. Available at https://www.math.uni-sb.de/ag/schreyer/index.php/computeralgebra.

- Christian Bopp and Frank-Olaf Schreyer. *RandomCurvesOverVerySmallFiniteFields.m2* - construction of random curves of genus $g \leq 15$, 2017, a Macaulay2 package. https://www.math.uni-sb.de/ag/schreyer/index.php/computeralgebra.

An examples using each of the two packages can be found in Chapter 7.

Chapter 2

Preliminaries

This chapter is devoted to the introduction of some of the main objects concerned in this thesis. The results stated in this chapter are mostly well-known and can be found in various textbooks. Most of section 2.1 and 2.2 can be found in [Eis05] and Section 2.3 follows Schreyers article [Sch86].

2.1 Free resolutions and syzygies

Throughout this thesis, let $\mathbb{k} = \overline{\mathbb{k}}$ be an algebraically closed field. We denote by $\mathbb{P}^n = \mathbb{P}^n(\mathbb{k})$ the projective n-space and by $\mathrm{S} = \mathbb{k}[x_0, ..., x_n]$ the coordinate ring of $\mathbb{P}^n$. For a projective variety $\mathrm{X} \subset \mathbb{P}^n$, we denote by $\mathrm{I_X} \subset \mathrm{S}$ the vanishing ideal and by $\mathrm{S_X} := \mathrm{S}/\mathrm{I_X}$ the coordinate ring of X. Since S is noetherian, every ideal $\mathrm{I} \subset \mathrm{S}$ is finitely generated. Now, the generators of I can have relations which generate a finitely generated S-module. The generators of this module can also have have relations and so on. Hilbert's famous Syzygy Theorem states that this process terminates after finitely many steps.

Theorem 2.1.1 (Hilbert's Syzygy Theorem). *Let* M *be a finitely generated module over the polynomialring* $\mathrm{S} = \mathbb{k}[x_0, \ldots, x_n]$. *Then* M *has a finite graded free resolution*

$$0 \longleftarrow \mathrm{M} \longleftarrow \mathrm{F}_0 \xleftarrow{\varphi_1} \mathrm{F}_1 \longleftarrow \cdots \longleftarrow \mathrm{F}_{m-1} \xleftarrow{\varphi_m} \mathrm{F}_m \longleftarrow 0$$

Moreover we may take $m \leq n+1$, *the number of variables of* S.

Proof. See [Eis05, Theorem 1.1]. □

Definition 2.1.2. A complex of graded S-modules

$$\dots \longleftarrow F_{i-1} \xleftarrow{\varphi_i} F_i \xleftarrow{\varphi_{i+1}} F_{i+1} \longleftarrow \cdots$$

is called *minimal* if the image of φ_i is contained in $\mathfrak{m}F_{i-1}$ for each i, where $\mathfrak{m} = \langle x_0, ..., x_n \rangle$ is the irrelevant ideal of S.

By choosing a minimal set of generators in each step in Theorem 2.1.1 we obtain a minimal free resolution. On the other hand a free resolution can be minimized, by cancelling trivial sub complexes.

Theorem 2.1.3. *A minimal free resolution of a finitely generated graded* S*-module* M *is unique up to an isomorphism of complexes that induces the identity map on* M.

Proof. See [Eis05, Theorem 1.6]. □

The theorem above implies that the number of generators of F_i in degree j is independent of the minimal free resolution. This leads to the following definition.

Definition 2.1.4 (Betti Numbers). Let M be a finitely generated graded S-module and let

$$\mathbf{F}: \quad 0 \longleftarrow M \longleftarrow F_0 \longleftarrow F_1 \longleftarrow \cdots \longleftarrow F_m \longleftarrow 0$$

be a minimal free resolution of M. The free modules F_i are of the form $F_i = \bigoplus_j S(-j)^{\beta_{ij}}$. The numbers β_{ij} are called the *graded Betti numbers* of M. If $X \subset \mathbb{P}^n$ is a projective variety, then the *Betti numbers* of X are those of the coordinate ring S_X.

If we tensor a minimal free resolution as above with the ground field $\Bbbk$, then all the maps in $\mathbf{F} \otimes \Bbbk$ become zero, because of the minimality of $\mathbf{F}$. It follows, that the i-th homology $Tor_i^S(M, \Bbbk)$ of $\mathbf{F} \otimes \Bbbk$ equals $F_i \otimes \Bbbk$. Hence we obtain:

Proposition 2.1.5. *The graded Betti numbers of a finitely generated graded* S*-module* M *can be expressed as*

$$\beta_{ij} = \dim_{\Bbbk} \operatorname{Tor}_i^S(M, \Bbbk)_j .$$

Betti numbers are usually summarized in a *Betti table* of the form

	0	1	...	$m-1$	m
0	$\beta_{0,0}$	$\beta_{1,1}$	...	$\beta_{m-1,m-1}$	$\beta_{m,m}$
1	$\beta_{0,1}$	$\beta_{1,2}$	...	$\beta_{m-1,m}$	$\beta_{m,m+1}$
$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$
s	$\beta_{0,s}$	$\beta_{1,s+1}$	...	$\beta_{m-1,m+s-1}$	$\beta_{m,m+s}$

where the i^{th} column corresponds to the module F_i in the minimal free resolution of M and the Betti numbers in the i-th column specify the degrees of the generators of F_i. If $\{\beta_{ij}\}$ are the graded Betti numbers of a finitely generated S-module such that for a given number i there is a number d such that $\beta_{ij} = 0$ for all $j < d$, then $\beta_{i+1,j+1} = 0$ for all $j < d$ (see [Eis05, Proposition 1.9]). In particular, if $M = S/I$ for some ideal $I \subset S$ without linear generator, then $\beta_{0,0} = 1$ and $\beta_{1,1} = \beta_{1,1} \ldots = \beta_{m,m} = 0$. We define the 1*-linear strand* (or 1^{st} linear strand) of S/I to be the sub-complex

$$S(-2)^{\beta_{1,2}} \longleftarrow S(-3)^{\beta_{2,3}} \longleftarrow \cdots \longleftarrow S(-m-1)^{\beta_{m,m+1}} \longleftarrow 0$$

of the minimal free resolution of S/I. The *length* of the 1-linear strand is the largest number n such that $\beta_{n,n+1} \neq 0$.

If $I \subset J$ are ideals containing no linear forms, then the 1-linear strand of the resolution of S/I is a sub complex of the minimal free resolution of S/J. In particular the length of the 1-linear strand of S/J is greater or equal than that of the 1-linear strand of S/I (see e.g. [Eis92, Lemma 1]).

Example 2.1.6. The homogeneous ideal of the twisted cubic curve $C \subset \mathbb{P}^3$ is given by $I_C = (x_1x_3 - x_2^2,\ x_1x_2 - x_0x_3,\ x_0x_2 - x_1^2)$ and the minimal free resolution of S_C is of the form

$$0 \longleftarrow S/I_C \longleftarrow S \longleftarrow S(-2)^3 \xleftarrow{\varphi^t} S(-3)^2 \longleftarrow 0$$

where $\varphi = \begin{pmatrix} x_0 & x_1 & x_2 \\ x_1 & x_2 & x_3 \end{pmatrix}$. We read off the Betti table

	0	1	2
0	1	-	-
1	-	3	2

.

2.2 Canonical curves

For non-hyperelliptic curves C let $|\omega_C|$ be the linear series associated to the canonical bundle ω_C on C. This linear series defines an embedding, such that the properties of the coordinate ring S_C depend on C alone. We refer to such curves as *canonical curves,* and we refer to the embedding

$$\phi_{|\omega_C|} : C \hookrightarrow \mathbb{P}\big(H^0(C, \omega_C)^*\big) = \mathbb{P}^{g-1}$$

defined by $|\omega_C|$ as the *canonical embedding.* By a classical theorem of Noether (see e.g [ACGH85, §2 Chapter 3])

$$S_C = \sum_{n \geq 0} H^0(C, \omega_C^{\otimes n}).$$

for any non-hyperelliptic canonical curve $C \subset \mathbb{P}^{g-1}$. In particular, in this situation $H^1(C, \mathscr{I}_C(n)) = 0$ for all $n \geq 0$.

Proposition 2.2.1. *Let* C *be a non-hyperelliptic canonical curve of genus* $g \geq 3$ *then*

$$\omega_C = \mathscr{E}xt^{g-2}(\mathscr{O}_C, \mathscr{O}_{\mathbb{P}^{g-1}}) \cong \mathscr{O}_C(1).$$

The minimal free resolution of S_C *is therefore, up to shift, self dual with*

$$\beta_{i,j} = \beta_{g-2-i, g+1-j}$$

and has a Betti table of the following form

	0	1	2	…	$g-4$	$g-3$	$g-2$
0	1	-	-	…	-	-	-
1	-	$\beta_{1,2}$	$\beta_{2,3}$	…	$\beta_{g-4,g-3}$	$\beta_{g-3,g-2}$	-
2	-	$\beta_{g-3,g-2}$	$\beta_{g-4,g-3}$	…	$\beta_{2,3}$	$\beta_{1,2}$	-
3	-	-	-	…	-	-	1

where $\beta_{1,2} = \binom{g-2}{2}$.

Proof. See [Eis05, Proposition 9.5]. □

Arithmetically Cohen-Macaulay schemes $X \subset \mathbb{P}^r$ (i.e. $S_X \to \bigoplus_n H^0(\mathbb{P}, \mathcal{O}_X(n))$ is an isomorphism) with the property, that

$$\mathcal{E}xt^{\operatorname{codim}(X)}(\mathcal{O}_X, \mathcal{O}_{\mathbb{P}^r}) \cong \mathcal{O}_X(n)$$

for some n, are called ***arithmetically Gorenstein***. Note that arithmetically Gorenstein schemes have a self dual resolution (up to shift).

Definition 2.2.2. The *Clifford index* of a line bundle $\mathcal{L}$ on a curve $C \subset \mathbb{P}^r$ is defined as

$$\operatorname{Cliff}(\mathcal{L}) = \deg \mathcal{L} - 2(h^0(C,\mathcal{L}) - 1) = g + 1 - h^0(C,\mathcal{L}) - h^1(C,\mathcal{L}).$$

Note that $\operatorname{Cliff}(\mathcal{L}) = \operatorname{Cliff}(\mathcal{L}^{-1} \otimes \omega_C)$ by Serre duality. The Clifford index of a curve $C \subset \mathbb{P}^r$ is defined by taking the minimal Clifford index of all "relevant" line bundles on C. To be more precise we only want to consider line bundles with at least 2 independent sections. Hence Cliff(C) is defined as

$$\operatorname{Cliff}(C) = \min\left\{\operatorname{Cliff}(\mathcal{L}) \;\middle|\; h^0(C,\mathcal{L}) \geq 2 \text{ and } h^1(C,\mathcal{L}) \geq 2)\right\}$$

By Clifford's Theorem (see e.g. [ACGH85, §1 Chapter 3]) $\operatorname{Cliff}(\mathcal{L}) \geq 0$ for all special line bundles $\mathcal{L}$ (i.e. $h^1(\mathcal{L}) \neq 0$). Moreover $\operatorname{Cliff}(C) = 0$ if and only if C is hyperelliptic. On the other hand it follows from the Brill-Noether theorem [ACGH85, Ch. 5], that $\operatorname{Cliff}(C) \leq \lceil \frac{g-2}{2} \rceil$ where equality holds for general curves.

The Clifford index can be thought of, as a refinement of another classical invariant, namely the gonality.

Notation 2.2.3. *Throughout the rest of this thesis a g^r_k denotes an r-dimensional linear series of degree k on a curve* C.

Definition 2.2.4. Let $C \subset \mathbb{P}^r$ be a curve. The *gonality* of C is defined as

$$\operatorname{gon}(C) = \min\left\{k \;\middle|\; \exists\, \phi : C \to \mathbb{P}^1 \text{ non-constant map of degree } k\right\} = \min\left\{k \;\middle|\; C \text{ has a } g^1_k\right\}.$$

We will refer to curves of gonality k as *k-gonal curves.*

The Clifford index and the gonality of a curve satisfy

$$\operatorname{gon}(C) - 3 \leq \operatorname{Cliff}(C) \leq \operatorname{gon}(C) - 2$$

where the right hand side is an equality for general (k-gonal) curves (see [CM91] and [ELMS89]). For a canonical curve $C \subset \mathbb{P}^{g-1}$ it is conjectured in [Gre84] that that the length of the linear strand of C determines the Clifford index of C and vice versa.

Conjecture 2.2.5 (Green's Conjecture). *Let $C \subset \mathbb{P}^{g-1}$ be a canonical curve defined over a field $\Bbbk$ of characteristic 0. Then Then the following is conjectured to be equivalent*

$$\beta_{i,i+2}(C) = 0 \textit{ for all } i \leq p \Longleftrightarrow \mathrm{Cliff}(C) > p$$

The direction "⇒" is proved by Green and Lazarsfeld in the appendix to [Gre84] and the other direction was proved for general curves in [Voi02] and [Voi05]. By now, several other cases of Green's conjecture have been established, although the conjecture is still open in full generality. For instance, Aprodu showed in [Apr05] that Green's conjecture holds for general k-gonal curves (where $2 < k < \lceil \frac{g-1}{2} \rceil$) and more recently Aprodu and Farkas showed in [AF11] that Green's conjecture holds for every smooth curve on an arbitrary K3 surface.

It is known that Green's conjecture fails for some cases in positive characteristic [Sch86]. Based on experiments using computer algebra we will suggest a refined version of Green's conjecture in Section 6.4, which conjecturally also holds in positive characteristic.

2.3 Scrolls and relative canonical resolutions

Most of this section follows Schreyers article [Sch86].

2.3.1 Rational normal scrolls

Definition 2.3.1. Let $e_1 \geq e_2 \geq \ldots \geq e_d \geq 0$ be integers and $\mathscr{E} = \mathscr{O}_{\mathbb{P}^1}(e_1) \oplus \ldots \oplus \mathscr{O}_{\mathbb{P}^1}(e_d)$ be a locally free sheaf of rank d on $\mathbb{P}^1$. We denote by $\pi : \mathbb{P}(\mathscr{E}) \longrightarrow \mathbb{P}^1$ the corresponding $\mathbb{P}^{d-1}$-bundle.

A *rational normal scroll* X of type $(e_1, \ldots, e_d)$ is the image of $\mathbb{P}(\mathscr{E})$ in $\mathbb{P}^r = H^0(\mathbb{P}(\mathscr{E}), \mathscr{O}(1))$, where $r = f + d - 1$ with $f = e_1 + \ldots + e_d \geq 2$.

In [Har81, §3] it is shown that the variety X defined above is a non-degenerate d-dimensional variety of minimal degree

$$\deg(\mathrm{X}) = f = r - d + 1 = \operatorname{codim}\mathrm{X} + 1.$$

If $e_1, \dots, e_d > 0$, then $j : \mathbb{P}(\mathscr{E}) \to \mathrm{X} \subset \mathbb{P}\big(\mathrm{H}^0(\mathbb{P}(\mathscr{E}), \mathscr{O}_{\mathbb{P}(\mathscr{E})}(1))^*\big) = \mathbb{P}^r$ is an isomorphism. Otherwise it is a resolution of singularities and since the singularities of X are rational, we can consider $\mathbb{P}(\mathscr{E})$ instead of X for most cohomological considerations.

In [Har81] it is furthermore shown that the Picard group $\operatorname{Pic}(\mathbb{P}(\mathscr{E}))$ is generated by the ruling $\mathrm{R} = [\pi^*\mathscr{O}_{\mathbb{P}^1}(1)]$ and the hyperplane class $\mathrm{H} = [j^*\mathscr{O}_{\mathbb{P}^r}(1)]$ with intersection products

$$\mathrm{H}^d = f, \quad \mathrm{H}^{d-1}\cdot\mathrm{R} = 1, \quad \mathrm{R}^2 = 0.$$

Remark 2.3.2 ([Sch86, (1.3)]). For $a \geq 0$ we have an isomorphism $\mathrm{H}^0(\mathbb{P}(\mathscr{E}), \mathscr{O}_{\mathbb{P}(\mathscr{E})}(a\mathrm{H} + b\mathrm{R})) \cong \mathrm{H}^0(\mathbb{P}^1, \mathrm{S}_a(\mathscr{E})(b))$, where $\mathrm{S}_a(\mathscr{E})$ denotes the a-th symmetric power of $\mathscr{E}$. Thus we can compute the cohomology of the line bundle $\mathscr{O}_{\mathbb{P}(\mathscr{E})}(a\mathrm{H}+b\mathrm{R})$ explicitly.

If we denote by $\Bbbk[s,t]$ the coordinate ring of $\mathbb{P}^1$ and by $\varphi_i \in \mathrm{H}^0(\mathbb{P}(\mathscr{E}), \mathscr{O}_{\mathbb{P}(\mathscr{E})}(\mathrm{H}-e_i\mathrm{R}))$ the basic sections, then we can identify sections $\psi \in \mathrm{H}^0(\mathscr{O}_{\mathbb{P}(\mathscr{E})}, \mathscr{O}_{\mathbb{P}(\mathscr{E})}(a\mathrm{H} + b\mathrm{R}))$ with homogeneous polynomials

$$\psi = \sum_\alpha \mathrm{P}_\alpha(s,t)\varphi_1^{\alpha_1}\dots\varphi_d^{\alpha_d}$$

of degree $a = \alpha_1 + \dots + \alpha_d$ in φ_i's and with polynomial coefficients $\mathrm{P}_\alpha \in \Bbbk[s,t]$ of degree $\deg \mathrm{P}_\alpha = \alpha_1 e_1 + \dots + \alpha_d e_d + b$. Thus for $a \cdot \min\{e_i\} + b \geq -1$ we get

$$h^0(\mathbb{P}(\mathscr{E}), \mathscr{O}_{\mathbb{P}(\mathscr{E})}(a\mathrm{H}+b\mathrm{R})) = f\binom{a+d-1}{d} + (b+1)\binom{a+d-1}{d-1}.$$

Moreover, $\mathscr{O}_{\mathbb{P}(\mathscr{E})}(a\mathrm{H}+b\mathrm{R})$ is globally generated if and only if $a \geq 0$ and $a\cdot\min\{e_i\}+b \geq 0$.

Rational normal scrolls also admit a determinantal representation. If we choose a basis

$$x_{i,j} = s^j t^{e_i - j}\varphi_i \quad \text{with } j = 0, \dots, e_i \text{ and } i = 1, \dots, d \tag{2.1}$$

for $\mathrm{H}^0(\mathbb{P}(\mathscr{E}), \mathscr{O}_{\mathbb{P}(\mathscr{E})}(\mathrm{H})) \cong \mathrm{H}^0\mathscr{O}_{\mathbb{P}^r}(1)$, then the homogeneous ideal of X is defined by the 2×2 minors of the $2 \times f$ matrix

$$\Phi = \begin{pmatrix} x_{1,0} & \cdots & x_{1,e_1-1} & x_{2,0} & \cdots & \cdots & x_{d,e_d-1} \\ x_{1,1} & \cdots & x_{1,e_1} & x_{2,1} & \cdots & \cdots & x_{d,e_d} \end{pmatrix}. \tag{2.2}$$

The matrix Φ can also be obtained from the multiplication map

$$H^0(\mathbb{P}(\mathscr{E}),\mathscr{O}_{\mathbb{P}(\mathscr{E})}(R))\otimes H^0(\mathbb{P}(\mathscr{E}),\mathscr{O}_{\mathbb{P}(\mathscr{E})}(H-R))\longrightarrow H^0(\mathbb{P}(\mathscr{E}),\mathscr{O}_{\mathbb{P}(\mathscr{E})}(H)).$$

2.3.2 Relative canonical resolutions

Let $C\subset\mathbb{P}^{g-1}$ be a canonically embedded curve of genus g and let further

$$g^1_k=\{D_\lambda\}_{\lambda\in\mathbb{P}^1}\subset|D|$$

be a complete base point free pencil of divisors of degree k on C. If we denote by $\overline{D_\lambda}\subset\mathbb{P}^{g-1}$ the linear span of the divisor then the variety

$$X=\bigcup_{\lambda\in\mathbb{P}^1}\overline{D_\lambda}$$

swept out by the g^1_k on C is a $d=(k-1)$ dimensional rational normal scroll of degree $f=g-k+1$ (see [EH87]). Conversely if X is a rational normal scroll of degree f containing a canonical curve, then the ruling on X cuts out a pencil of divisors $\{D_\lambda\}\subset|D|$ such that $h^0(C,\omega_C\otimes\mathscr{O}_C(D)^{-1})=f$.

In a more intrinsic way one can recover the projective bundle $\mathscr{E}$ associated to the scroll X as follows. Let

$$\pi:C\to\mathbb{P}^1$$

be the map induced by the g^1_k on C then, following [CE96], there is a short exact sequence of the form

$$0\to\mathscr{O}_{\mathbb{P}^1}\to\pi_*\mathscr{O}_C\to\mathscr{E}_T^\vee\to 0.$$

for some bundle $\mathscr{E}_T$ of rank $k-1$, called the Tschirnhausen bundle. By Riemann-Roch

$$\chi(\mathscr{O}_C)=1-g=\chi(\pi_*\mathscr{O}_C)=k+\deg(\mathscr{E}_T^\vee)$$

and one can recover the bundle $\mathscr{E}$ associated to the scroll X as $\mathscr{E}=\mathscr{E}_T\otimes\omega_{\mathbb{P}^1}$.

In particular the type of the scroll is uniquely determined by the g^1_k.

Remark 2.3.3. With notation as above it follows from [Har81] that the type $(e_1,\dots,e_d)$ (with $d=k-1$) of the scroll swept out by a complete base point free $g^1_k=|D|$ on a canonically embedded curve can be computed as $e_i=\#\{j\mid d_j\geq i\}-1$, where

$$d_j:=h^0\big(C,K_C-j\cdot D\big)-h^0\big(C,K_C-(j+1)\cdot D\big)$$

Definition 2.3.4. A vectorbundle $\mathscr{E} = \mathscr{O}_{\mathbb{P}^1}(e_1) \oplus \cdots \oplus \mathscr{O}_{\mathbb{P}^1}(e_d)$ on $\mathbb{P}^1$ is called *balanced* if $h^1(\mathscr{E}nd(\mathscr{E})) = 0$ or eqivalently if $\max_{i,j} |e_i - e_j| \leq 1$.

For general k-gonal curves having a unique g^1_k, the balancedness of the scroll follows from the following theorem.

Theorem 2.3.5 ([Bal89]). *Let $\rho(g,k,1) < 0$ and let further* C *be a generic k-gonal curve of genus g and let g^1_k be the unique one dimensional linear series of degree k on* C. *Then*

$$h^0(\mathrm{C}, n \cdot g^1_k) = \begin{cases} n+1 & \text{if } n < \frac{g}{k-1} \\ nk - g + 1 & \text{if } n \geq \frac{g}{k-1}\ . \end{cases}$$

Corollary 2.3.6. *Let* C *be a general k-gonal curve with a unique g^1_k then the bundle $\mathscr{E}$ associated to the g^1_k is balanced. If* C *is a Petri-general curve and admitting a g^1_k, then the scroll associated to the g^1_k is also balanced and the bundle $\mathscr{E}$ has the form $\mathscr{E} = \mathscr{O}_{\mathbb{P}^1}(1)^{\oplus(k-1-\rho)} \oplus \mathscr{O}_{\mathbb{P}^1}^{\oplus\rho}$.*

Proof. Let C be a general k-gonal curve with a unique g^1_k. Let further $s = \lceil \frac{g}{k-1} \rceil$ and $d_j := h^0\big(\mathrm{C}, \mathrm{K_C} - j \cdot \mathrm{D}\big) - h^0\big(\mathrm{C}, \mathrm{K_C} - (j+1) \cdot \mathrm{D}\big)$. Then it follows from the above theorem that

$$d_j = \begin{cases} k-1 & \text{for } j < s \\ g - s(k-1) & \text{for } j = s \\ 0 & \text{for } j > s\ . \end{cases}$$

Therefore the partition defined by

$$e_i = \#\{j \mid d_j \geq i\} - 1$$

is balanced.

Now, if C is a Petri-general curve, then the map

$$\mu_{0,\mathrm{L}} : \mathrm{H}^0(\mathrm{C}, \mathrm{L}) \otimes \mathrm{H}^0(\mathrm{C}, \omega_\mathrm{C} \otimes \mathrm{L}^{-1}) \to \mathrm{H}^0(\mathrm{C}, \omega_\mathrm{C})$$

is injective for any line bundle L on C. The kernel of $\mu_{0,\mathrm{L}}$ is given as $\ker(\mu_{0,\mathrm{L}}) = \mathrm{H}^0(\mathrm{C}, \omega_\mathrm{C} \otimes \mathrm{L}^{-2})$ (see e.g. [ACGH85, Ch. III §3]). Thus, by Remark 2.3.3 the scroll associated to a g^1_k on a Petri general curve is balanced and has the desired form. □

Let $C \subset \mathbb{P}^{g-1}$ be a canonical curve of genus g with a base point free g^1_k and let $\mathbb{P}(\mathscr{E})$ be the projective bundle associated to the scroll swept out by the g^1_k. Then one can resolve the curve C in terms of $\mathscr{O}_{\mathbb{P}(\mathscr{E})}$-modules. The free resolution obtained this way is called the *relative canonical resolution* of $C \subset \mathbb{P}(\mathscr{E})$.

Theorem 2.3.7 ([Sch86, Corollary 4.4]). *Let C be a curve with a base point free g^1_k and let $\pi : \mathbb{P}(\mathscr{E}) \to \mathbb{P}^1$ be the corresponding projective bundle associated to the scroll swept out by the g^1_k. Then*

(a) $C \subset \mathbb{P}(\mathscr{E})$ *has a resolution* $F_\bullet$ *of the form*

$$0 \to \pi^* N_{k-2}(-k) \to \pi^* N_{k-3}(-k+2) \to \cdots \to \pi^* N_1(-2) \to \mathscr{O}_{\mathbb{P}(\mathscr{E})} \to \mathscr{O}_C \to 0$$

with $\pi^* N_i = \sum_{j=1}^{\beta_i} \mathscr{O}_{\mathbb{P}(\mathscr{E})}(a_j^{(i)} R)$ *and* $\beta_i = \frac{i(k-2-i)}{k-1}\binom{k}{i+1}$.

(b) The complex $F_\bullet$ *is self-dual, i.e.* $\mathscr{H}om(F_\bullet, \mathscr{O}_{\mathbb{P}(\mathscr{E})}(-kH + (g-k-1)R)) \cong F_\bullet$

Remark 2.3.8. (a) The degrees of the bundles N_i in the theorem above can be computed, by taking the Euler characteristic of the complex $F_\bullet$. For instance, for $k = 5$ Schreyer showed that $\deg(N_1) = 2g - 12$ and by duality $a_j^{(1)} + a_j^{(2)} = g - 6$ (see [Sch86, § 6]). The general computation of the degrees of the bundles N_i is carried out in Proposition 4.2.3.

(b) A generalization of theorem 2.3.7 for degree k covers $\pi : Y \to Y'$ of equal dimensional varieties of can be found in [CE96], where the authors consider resolutions of $C \subset \mathbb{P}(\mathscr{E}_T)$ where $\mathscr{E}_T$ is the Tschirnhausen bundle, defined by the short exact sequence

$$0 \to \mathscr{O}_{Y'} \to \pi_* \mathscr{O}_Y \to \mathscr{E}_T^\vee \to 0.$$

Definition 2.3.9. Similar to Definition 2.3.4 we say that a curve $C \subset \mathbb{P}^{g-1}$ has a balanced relative canonical resolution if all bundles N_i appearing in the relative canonical resolution are balanced.

Given a canonical curve $C \subset \mathbb{P}^{g-1}$ with a complete base point free g^1_k, it is an interesting question whether the relative canonical resolution is balanced or not. We will discuss this problem further in Proposition 3.2.3, Chapter 4 and Chapter 5.

2.3.3 Iterated mapping cones

The content of this section was originally published in [Sch86, §1 and §4] and the appendix in [MS86]. Throughout this section let $C \subset \mathbb{P}^{g-1}$ be a canonically embedded curve with a base point free g^1_k and let $\pi : \mathbb{P}(\mathscr{E}) \to \mathbb{P}^1$ be the projective bundle associated to the scroll X swept out by the g^1_k on C. We further denote by $f = g - k + 1 = \deg(X)$ the degree of the scroll.

The aim of this section is to describe, how we can obtain a free resolution of $C \subset \mathbb{P}^{g-1}$ from the relative canonical resolution of $C \subset \mathbb{P}(\mathscr{E})$.

Recall from Section 2.3.2 that $C \subset \mathbb{P}(\mathscr{E})$ has a relative canonical resolution $F_\bullet = \pi^* N_\bullet$ of the form

$$0 \to \pi^* N_{k-2}(-k) \to \pi^* N_{k-3}(-k+2) \to \cdots \to \pi^* N_1(-2) \to \mathscr{O}_{\mathbb{P}(\mathscr{E})} \to \mathscr{O}_C \to 0$$

with $\pi^* N_i = \sum_{j=1}^{\beta_i} \mathscr{O}_{\mathbb{P}(\mathscr{E})}(a_j^{(i)} R)$ and $\beta_i = \frac{i(k-2-i)}{k-1}\binom{k}{i+1}$.

We start by explaining how to resolve line bundles $\mathscr{O}_{\mathbb{P}(\mathscr{E})}(aH + bR)$ by $\mathscr{O}_{\mathbb{P}^{g-1}}$-modules. Therefore let Φ denote the $2 \times f$ with entries in $H^0(\mathbb{P}(\mathscr{E}), \mathscr{O}_{\mathbb{P}(\mathscr{E})}(H))$ obtained by the multiplication matrix

$$H^0(\mathbb{P}(\mathscr{E}), \mathscr{O}_{\mathbb{P}(\mathscr{E})}(R)) \otimes H^0(\mathbb{P}(\mathscr{E}), \mathscr{O}_{\mathbb{P}(\mathscr{E})}(H - R)) \longrightarrow H^0(\mathbb{P}(\mathscr{E}), \mathscr{O}_{\mathbb{P}(\mathscr{E})}(H)).$$

Recall, that the ideal defining X is given by the 2×2 minors of Φ. We define

$$F := H^0(\mathbb{P}(\mathscr{E}), \mathscr{O}_{\mathbb{P}(\mathscr{E})}(H - R)) \otimes \mathscr{O}_{\mathbb{P}^r} = \mathscr{O}^f_{\mathbb{P}^r}$$

and

$$G := H^0(\mathbb{P}(\mathscr{E}), \mathscr{O}_{\mathbb{P}(\mathscr{E})}(R)) \otimes \mathscr{O}_{\mathbb{P}^r} = \mathscr{O}^2_{\mathbb{P}^r}$$

and regard Φ as a map $\Phi : F(-1) \to G$. For $b \geq -1$, we consider the *Eagon-Northcott type* complex $\mathscr{C}^b$ (see [Sch86, §1] or [Eis95, Appendix A2.6]), whose j-th term is defined by

$$\mathscr{C}^b_j = \begin{cases} \bigwedge^j F \otimes S_{b-j} G \otimes \mathscr{O}_{\mathbb{P}^r}(-j), & \text{for } 0 \leq j \leq b \\ \bigwedge^{j+1} F \otimes D_{j-b-1} G^* \otimes \mathscr{O}_{\mathbb{P}^r}(-j-1), & \text{for } j \geq b+1\ . \end{cases}$$

Here $S_j G$ denotes the j-th symmetric power and $D_j G^*$ denotes the j-th divided power of G. The differentials $\delta_j : \mathscr{C}^b_j \to \mathscr{C}^b_{j-1}$ are given by the multiplication with Φ for $j \neq b+1$ and by $\bigwedge^2 \Phi$ for $j = b+1$.

Theorem 2.3.10. *The Eagon-Northcott type complex* $\mathscr{C}^b(a) := \mathscr{C}^b \otimes \mathscr{O}_{\mathbb{P}}^{g-1}(a)$*, defined above, gives a minimal free resolution of* $\mathscr{O}_{\mathbb{P}(\mathscr{E})}(a\mathrm{H} + b\mathrm{R}) = \left(\pi^* \mathscr{O}_{\mathbb{P}^1}(b)\right)(a)$ *as an* $\mathscr{O}_{\mathbb{P}}^{g-1}$*-module if* $b \geq -1$.

Proof. See [Sch86, §1]. □

Schreyer showed, that one can obtain a free resolution of $\mathrm{C} \subset \mathbb{P}^{g-1}$ by an iterated mapping cone.

Theorem 2.3.11. *With notation as above let*

$$0 \longrightarrow \mathscr{O}_{\mathbb{P}(\mathscr{E})}(-k\mathrm{H} + (f-2)\mathrm{R}) \longrightarrow \sum_{j=1}^{\beta_{k-3}} \mathscr{O}_{\mathbb{P}(\mathscr{E})}(-(k-2)\mathrm{H} + a_j^{(k-3)}\mathrm{R}) \longrightarrow$$
$$\ldots \longrightarrow \sum_{j=1}^{\beta_1} \mathscr{O}_{\mathbb{P}(\mathscr{E})}(-2\mathrm{H} + a_j^{(1)}\mathrm{R}) \longrightarrow \mathscr{O}_{\mathbb{P}(\mathscr{E})} \longrightarrow \mathscr{O}_{\mathrm{C}} \longrightarrow 0$$

with $\beta_i = \frac{i(k-2-i)}{k-1}\binom{k}{i+1}$ *be the relative canonical resolution of* $\mathrm{C} \subset \mathbb{P}(\mathscr{E})$*. If all* $a_k^{(j)} \geq -1$*, then an iterated mapping cone*

$$\left[\left[\ldots\left[\mathscr{C}^{(f-2)}(-k) \longrightarrow \sum_{j=1}^{\beta_{k-3}} \mathscr{C}^{(a_j^{(k-3)})}(-k+2)\right] \longrightarrow \ldots\right] \longrightarrow \mathscr{C}^0\right]$$

gives a, not necessarily minimal, resolution of C *as an* $\mathscr{O}_{\mathbb{P}^{g-1}}$*-module.*

Proof. See [Sch86, Corollary 4.4]. □

For a section $\Psi : \mathscr{O}_{\mathrm{X}}(-\mathrm{H} + b\mathrm{R}) \longrightarrow \mathscr{O}_{\mathrm{X}}(a\mathrm{R})$ in $\mathrm{H}^0(\mathrm{X}, \mathscr{O}_{\mathrm{X}}(\mathrm{H} - (b-a)\mathrm{R}))$ the induced comparison maps $\psi_\bullet : \mathscr{C}_\bullet^b(-1) \longrightarrow \mathscr{C}_\bullet^a$ between the corresponding Eagon-Northcott type complexes are determined by Ψ up to homotopy. Now since by degree reasons

$$\mathrm{Hom}(\mathscr{C}_{a+1}^b(-1), \mathscr{C}_{a+2}^a) = \mathrm{Hom}(\mathscr{C}_a^b(-1), \mathscr{C}_{a+1}^a) = 0,$$

the $(a+1)^{st}$–comparison map ψ_{a+1} is uniquely determined by Ψ (not only up to homotopy). The following lemma is due to Martens and Schreyer.

Lemma 2.3.12. *If* $\mathrm{Hom}(\mathscr{C}^b_j(-1),\mathscr{C}^a_{j+1}) = \mathrm{Hom}(\mathscr{C}^b_{j-1}(-1),\mathscr{C}^a_j) = 0$*, then the j-th-comparison map $\psi_j : \mathscr{C}^b_j \to \mathscr{C}^a_j$ is given (up to a scalar factor) by the composition*

$$\psi_j : \bigwedge^j \mathrm{F}\otimes \mathrm{S}_{b-j}\mathrm{G} \longrightarrow \bigwedge^j \mathrm{F}\otimes \mathrm{S}_{b-j}\mathrm{G}\otimes \mathrm{S}_{j-a-1}\mathrm{G}\otimes \mathrm{D}_{j-a-1}\mathrm{G}^*$$

$$\xrightarrow{id\otimes mult\otimes id} \bigwedge^j \mathrm{F}\otimes \mathrm{S}_{b-a-1}\mathrm{G}\otimes \mathrm{D}_{j-a-1}\mathrm{G}^* \xrightarrow{id\otimes\Psi\otimes id} \bigwedge^j \mathrm{F}\otimes \mathrm{F}\otimes \mathrm{D}_{j-a-1}\mathrm{G}^*$$

$$\xrightarrow{\wedge\otimes id} \bigwedge^{j+1} \mathrm{F}\otimes \mathrm{D}_{j-a-1}\mathrm{G}^*.$$

Proof. In the appendix of [MS86] this is shown for the $(a+1)^{st}$ comparison map but the proof immediately generalizes as long as

$$\mathrm{Hom}(\mathscr{C}^b_j(-1),\mathscr{C}^a_{j+1}) = \mathrm{Hom}(\mathscr{C}^b_{j-1}(-1),\mathscr{C}^a_j) = 0.$$

□

Chapter 3

Syzygies of 5-gonal canonical curves

This chapter follows the article [Bop15] and extends some results of the authors master thesis [Bop13]. Theorem 3.2.1, Remark 3.2.2 and Proposition 3.3.1 are already contained in a similar form in [Bop13]. The new contributions compared to the master thesis [Bop13] are Proposition 3.2.3, which shows that the hypothesis of the main theorem in [Bop13] are satisfied in a non-empty open subset of the Hurwitz scheme $\mathscr{H}_{g,k}$ and Proposition 3.3.4 in which the syzygy scheme for general 5-gonal canonical curves of genus 13 is described.

3.1 Introduction

We study minimal free resolutions of the coordinate ring S_C of 5-gonal canonically embedded curves $C \subset \mathbb{P}^{g-1}$. Recall, that the ***gonality*** of a curve C is defined as the minimal degree of a nonconstant map $C \longrightarrow \mathbb{P}^1$.

A pencil of degree k on a canonically embedded curve C of genus g, defining a degree k map $C \to \mathbb{P}^1$ sweeps out a rational normal scroll X of dimension $d = k-1$ and degree $f = g-k+1$. It follows that the linear strand of X is a subcomplex of the linear strand of the curve C. To be more precise the scroll contributes with an Eagon-Northcott complex of length $g-k$ to the linear strand of the curve. This means in particular, that the Betti numbers of X give a lower bound for the Betti numbers of C.

The main focus of this chapter lies on the relation between the Betti numbers of the canonical curve $C \subset \mathbb{P}^{g-1}$ and the Betti numbers of the scroll X defined by a pencil of minimal degree on C.

From the values of the Hilbert function H_{S_C} and the relation with Betti numbers (see [Eis05, Corollary. 9.4 and Corollary. 1.10]) one obtains the following relation for the Betti numbers of a canonical curve $C \subset \mathbb{P}^{g-1}$:

$$\beta_{i,i+1}(C) = i \cdot \binom{g-2}{i+1} - (g-i-1) \cdot \binom{g-2}{i-2} + \beta_{i-1,i+1}(C).$$

Since the minimal free resolution of a canonical curve is self-dual, we have

$$\beta_{i-1,i+1}(C) = \beta_{g-i-1,g-i}(C) \geq \beta_{g-i-1,g-i}(X)$$

and a direct computation for the case $i = \lceil \frac{g-3}{2} \rceil$ shows that if $k > 3$ and $g \geq 5$, then

$$\beta_{i,i+1}(C) \geq i \cdot \binom{g-2}{i+1} - (g-i-1) \cdot \binom{g-2}{i-2} + \beta_{g-i-1,g-i}(X) > \beta_{i,i+1}(X)$$

for all $i = 1, \ldots, \lceil \frac{g-3}{2} \rceil$. We are interested in the Betti numbers $\beta_{i,i+1}(C)$ for $i \geq \lceil \frac{g-1}{2} \rceil$.

The gonality of a general canonical curve of genus g is precisely $\lceil \frac{g+2}{2} \rceil$ and therefore $\beta_{n,n+1}(X) = 0$, where $n = \lceil \frac{g-1}{2} \rceil$. For odd genus g and ground field of characteristic 0, Voisin and Hirschowitz-Ramanan (see [Voi05] and [HR98]) showed that the locus

$$\mathcal{K}_g := \{C \in \mathcal{M}_g \mid \beta_{n,n+1}(C) \neq 0\}$$

defines an effective divisor in the moduli space of curves, the so-called *Koszul divisor*.

On the Hurwitz-scheme $\mathcal{H}_{g,k}$ a natural analogue of the Koszul divisor could be the following

$$\mathcal{K}_{g,k} := \{C \in \mathcal{H}_{g,k} \mid \beta_{n,n+1}(C) > \beta_{n,n+1}(X)\}.$$

Similar to the arguments in [HR98] one can describe $\mathcal{K}_{g,k}$ as the degeneracy locus between vectorbundles of same rank over $\mathcal{H}_{g,k}$. If the genus is odd, it follows that $\mathcal{K}_{g,k}$ is a divisor on the Hurwitz-scheme as long as $\beta_{n,n+1}(C) = \beta_{n,n+1}(X)$ holds for a general curve $C \in \mathcal{H}_{g,k}$ (see [HR98, §3]). For gonality $k = 3,4$ it is known from [Sch86, §6] that the so-called iterated mapping cone construction, which was introduced in Section 2.3.3 always gives a minimal free resolution of $C \subset \mathbb{P}^{g-1}$. In particular $\beta_{n,n+1}(C) =$

$\beta_{n,n+1}(X)$ holds for general 3-gonal and 4-gonal canonical curves. For 5-gonal curves we used *Macaulay2* (see [GS]) to verify computationally, that $\beta_{n,n+1}(C) = \beta_{n,n+1}(X)$ holds for general 5-gonal canonical curves of genus $g < 13$.

We will show that $\mathcal{K}_{g,5}$ is no longer a divisor for odd $g \geq 13$ by proving the following theorem.

Theorem 3.1.1 ([Bop15]). *Let* C *be a* 5*-gonal canonical curve of genus* g *and* $n = \lceil \frac{g-1}{2} \rceil$. *Then*

$$\beta_{n,n+1}(C) > \beta_{n,n+1}(X)$$

for odd genus $g \geq 13$ *and even genus* $g \geq 28$.

The proof is based on the techniques introduced in [Sch86]. First we resolve the curve C as an $\mathcal{O}_{\mathbb{P}(\mathscr{E})}$-module, where $\mathbb{P}(\mathscr{E})$ is the bundle associated to the rational normal scroll swept out by the g^1_5. We will show that these relative canonical resolutions are generically balanced for 5-gonal curves. In the next step we resolve the $\mathcal{O}_{\mathbb{P}(\mathscr{E})}$-modules occurring in this resolution by Eagon-Northcott type complexes. An iterated mapping cone construction then gives a non-minimal resolution of $C \subset \mathbb{P}^{g-1}$. By determining the ranks of the maps which give rise to non-minimal parts in the iterated mapping cone we can decide whether the curve has extra syzygies. In the last section we discuss the genus 13 case in detail.

Remark 3.1.2. The proof of Theorem 3.1.1 does not depend on the characteristic of the ground field $\Bbbk$. However for $\mathrm{char}(\Bbbk) > 0$ it is possible that $\beta_{n,n+1}(C) > \beta_{n,n+1}(X)$ for general 5-gonal curves of genus $g < 13$. This happens, for example, for 5-gonal curves of genus 7 over a field of characteristic 2 [Sch86, §7].

Some of the statements in this Chapter rely on computations done with *Macaulay2* [GS]. The *Macaulay2* code which verifies these statements can be found here:
http://www.math.uni-sb.de/ag-schreyer/images/data/computeralgebra/fiveGonalFile.m2

3.2 Extra syzygies for 5-gonal curves of large genus

In order to prove Theorem 3.1.1, we proceed in two steps. In the first step, which is already contained in the authors master thesis [Bop13], we show that Theorem 3.1.1

holds for curves having a balanced relative canonical resolution. In the second step we prove that a general 5-gonal curve has a balanced relative canonical resolution. The main theorem then follows by semi-continuity on the Betti numbers.

Throughout this section let $C \subset \mathbb{P}^{g-1}$ be a 5-gonal canonical curve of genus g. In this case X is a $d = 4$ dimensional rational normal scroll of degree $f = g-4$. Recall from Theorem 2.3.11 and Remark 2.3.8 that $C \subset \mathbb{P}(\mathscr{E})$ has a resolution of the form

$$\mathscr{O}_{\mathbb{P}(\mathscr{E})}(-5H+(f-2)R) \to \sum_{i=1}^{5} \mathscr{O}_{\mathbb{P}(\mathscr{E})}(-3H+b_iR) \xrightarrow{\Psi} \sum_{i=1}^{5} \mathscr{O}_{\mathbb{P}(\mathscr{E})}(-2H+a_iR) \to \mathscr{O}_{\mathbb{P}(\mathscr{E})} \to \mathscr{O}_C$$

where $\sum_{i=1}^5 a_i = 2g-12,\ a_i + b_i = f-2$.
The matrix Ψ is skew symmetric by the structure theorem for Gorenstein ideals in codimension 3 and the 5 Pfaffians of Ψ generate the ideal of C (see [BE77, Theorem 2.1]).

As in Theorem 2.3.10, we denote by $F = H^0(\mathbb{P}(\mathscr{E}), \mathscr{O}_{\mathbb{P}(\mathscr{E})}(H-R)) \otimes \mathscr{O}_{\mathbb{P}^{g-1}} \cong \mathscr{O}^f_{\mathbb{P}^{g-1}}$ and by $G = H^0(\mathbb{P}(\mathscr{E}), \mathscr{O}_{\mathbb{P}(\mathscr{E})}(R)) \otimes \mathscr{O}_{\mathbb{P}^{g-1}} \cong \mathscr{O}^2_{\mathbb{P}^{g-1}}$. By abuse of notation, we will also refer to the vector spaces $H^0(\mathbb{P}(\mathscr{E}), \mathscr{O}_{\mathbb{P}(\mathscr{E})}(H-R))$ and $H^0(\mathbb{P}(\mathscr{E}), \mathscr{O}_{\mathbb{P}(\mathscr{E})}(R))$ by F and G, respectively.
If a_i and b_i are non-negative for $i = 1, \dots, 5$, we can resolve the $\mathscr{O}_{\mathbb{P}(\mathscr{E})}$-modules occurring in the minimal resolution of $C \subset \mathbb{P}(\mathscr{E})$, and get

$$\begin{tikzcd}
\sum_{i=1}^5 \mathscr{O}_{\mathbb{P}(\mathscr{E})}(-3H+b_iR) \arrow[r, "\Psi"] & \sum_{i=1}^5 \mathscr{O}_{\mathbb{P}(\mathscr{E})}(-2H+a_iR) \\
\vdots \arrow[u] & \vdots \arrow[u] \\
\sum_{i=1}^5 \mathscr{C}^{b_i}_{n-2}(-3) \arrow[r, "\Psi_{n-2}"] \arrow[u] & \sum_{i=1}^5 \mathscr{C}^{a_i}_{n-2}(-2) \arrow[u] \\
\sum_{i=1}^5 \mathscr{C}^{b_i}_{n-1}(-3) \arrow[r, "\Psi_{n-1}"] \arrow[u] & \sum_{i=1}^5 \mathscr{C}^{a_i}_{n-1}(-2) \arrow[u] \\
\vdots \arrow[u] & \vdots \arrow[u]
\end{tikzcd}$$

where $\mathrm{rk}(\sum_{i=1}^5 \mathscr{C}^{b_i}_{n-2}) \le \mathrm{rk}(\sum_{i=1}^5 \mathscr{C}^{a_i}_{n-2})$ (with equality for odd genus).

Next note that if C is a canonical curve of odd genus $g = 2n+1$ having a balanced relative canonical resolution, then we obtain the following inequality

$$\min\{b_i\} = \left\lfloor \frac{\sum_{i=1}^5 b_i}{5} \right\rfloor = \left\lfloor \frac{6n-15}{5} \right\rfloor \ge n-2 \ge \left\lceil \frac{4n-10}{5} \right\rceil = \max\{a_i\} \tag{3.1}$$

if $n \geq 5$. Similarly, for even genus $g = 2n$ and $n \geq 8$ we get

$$\min\{b_i\} = \left\lfloor \frac{6n-18}{5} \right\rfloor \geq n-2 \geq \left\lceil \frac{4n-12}{5} \right\rceil = \max\{a_i\}.$$

Therefore, in these cases

$$\mathscr{C}^{b_i}_{n-2}(-3) = \bigwedge^{n-2} \mathrm{F} \otimes \mathrm{S}_{b_i-n+2}\mathrm{G}(-n-1) \quad \text{and} \quad \mathscr{C}^{a_i}_{n-2}(-2) = \bigwedge^{n-1} \mathrm{F} \otimes \mathrm{D}_{n-a_i-3}\mathrm{G}^*(-n-1).$$

Thus, if C is a canonical curve of odd genus $g \geq 11$ or even genus $g \geq 16$ which has a balanced relative canonical resolution, then the $(n-2)$-th comparison map in the diagram above

$$\psi := \psi_{n-2} : \sum_{i=1}^{5} \mathscr{C}^{b_i}_{n-2}(-3) \longrightarrow \sum_{i=1}^{5} \mathscr{C}^{a_i}_{n-2}(-2)$$

has entries in the ground field $\Bbbk$ and its rank determines the Betti number $\beta_{n,n+1}(\mathrm{C})$. To be more precise, we have

$$\beta_{n,n+1}(\mathrm{C}) = \beta_{n,n+1}(\mathrm{X}) + \dim\ker(\psi),$$

where the Betti number $\beta_{n,n+1}(\mathrm{X})$ is given by $\mathrm{rk}(\mathscr{C}^0_n) = n \cdot \binom{f}{n+1}$.

We are now able to complete the first step in the proof the main theorem, by showing the following theorem.

Theorem 3.2.1 ([Bop13]). *Let* $\mathrm{C} \subset \mathbb{P}^{g-1}$ *be a general* 5*-gonal canonical curve of genus* g *with a balanced relative canonical resolution. If* X *is the scroll swept out by the* g^1_5 *and* $n = \lceil \frac{g-1}{2} \rceil$, *then*

$$\beta_{n,n+1}(\mathrm{C}) > \beta_{n,n+1}(\mathrm{X}) \textit{ for odd } g \geq 13 \textit{ and even } g \geq 28.$$

For the proof of Theorem 3.2.1 we restrict ourselves to curves of odd genus since the theorem is proved in exactly the same way for even genus. We distinguish 5 different types of curves having a balanced relative canonical resolution. These types depend on the congruence class of n modulo 5, i.e., on the block structure of the skew symmetric matrix Ψ. Setting $r := \lfloor \frac{n}{5} \rfloor$ we have the following five possibilities.

Type I $(a_1, \dots, a_5) = (a,a,a,a,a), (b_1, \dots, b_5) = (b,b,b,b,b) \Leftrightarrow a = 4r+2,\ b = 6r+3$ and $n = 5r+5$.

Type II $(a_1,\dots,a_5)=(a-1,a,a,a,a),(b_1,\dots,b_5)=(b+1,b,b,b,b) \Leftrightarrow a=4r-1,\ b=6r-2$ and $n=5r+1$.

Type III $(a_1,\dots,a_5)=(a-1,a-1,a,a,a),(b_1,\dots,b_5)=(b+1,b+1,b,b,b) \Leftrightarrow a=4r$, $b=6r-1$ and $n=5r+2$.

Type IV $(a_1,\dots,a_5)=(a,a,a,a+1,a+1),(b_1,\dots,b_5)=(b,b,b,b-1,b-1) \Leftrightarrow a=4r$, $b=6r+1$ and $n=5r+3$.

Type V $(a_1,\dots,a_5)=(a,a,a,a,a+1),(b_1,\dots,b_5)=(b,b,b,b,b-1) \Leftrightarrow a=4r+1,\ b=6r+2$ and $n=5r+4$.

Proof of Theorem 3.2.1.
Since the proof of the theorem is similar for all different types above, we will only carry out the proof for curves of type II, leaving the other cases to the reader. We show that the map

$$\psi:\Big(\mathscr{C}_{n-2}^{(b+1)}\oplus\sum_{i=1}^{4}\mathscr{C}_{n-2}^{b}\Big)(-3)\longrightarrow\Big(\mathscr{C}_{n-2}^{(a-1)}\oplus\sum_{i=1}^{4}\mathscr{C}_{n-2}^{a}\Big)(-2)$$

induced by the skew-symmetric matrix

$$\Psi:\begin{matrix}\mathscr{O}_{\mathbb{P}(\mathscr{E})}(-3\mathrm{H}+(b+1)\mathrm{R})\\ \oplus\\ \mathscr{O}_{\mathbb{P}(\mathscr{E})}(-3\mathrm{H}+b\mathrm{R})^{\oplus 4}\end{matrix}\longrightarrow\begin{matrix}\mathscr{O}_{\mathbb{P}(\mathscr{E})}(-2\mathrm{H}+(a-1)\mathrm{R})\\ \oplus\\ \mathscr{O}_{\mathbb{P}(\mathscr{E})}(-2\mathrm{H}+a\mathrm{R})^{\oplus 4}\end{matrix}$$

has a non-trivial decomposable element in the kernel. Note that the map

$$\Psi_{(11)}:\mathscr{O}_{\mathbb{P}(\mathscr{E})}(-3\mathrm{H}+(b+1)\mathrm{R})\longrightarrow\mathscr{O}_{\mathbb{P}(\mathscr{E})}(-3\mathrm{H}+(a-1)\mathrm{R})$$

is zero by the skew-symmetry of Ψ. Thus it is sufficient to find an element in the kernel of the map $\psi_{(41)}:\mathscr{C}_{n-2}^{(b+1)}(-3)\longrightarrow\sum_{i=1}^{4}\mathscr{C}_{n-2}^{(a)}(-2)$, induced by the submatrix

$$\Psi_{(41)}:\mathscr{O}_{\mathbb{P}(\mathscr{E})}(-3\mathrm{H}+(b+1)\mathrm{R})\longrightarrow\mathscr{O}_{\mathbb{P}(\mathscr{E})}(-2\mathrm{H}+a\mathrm{R})^{\oplus 4}$$

of the matrix Ψ. By Lemma 2.3.12, the map $\psi_{(41)}$ is uniquely determined and is given as the composition

$$\begin{aligned}
\mathscr{C}^{b+1}_{n-2}(-3) = \bigwedge^{n-2} F \otimes S_{b-n+3}G = \bigwedge^{n-2} F \otimes S_{n-a-2}G \\
\hookrightarrow \bigwedge^{n-2} F \otimes S_{n-a-2}G \otimes S_{n-a-3}G \otimes D_{n-a-3}G^* \\
\longrightarrow \bigwedge^{n-2} F \otimes S_{2n-2a-5}G \otimes D_{n-a-3}G^* \xrightarrow{id\otimes\Psi_{(41)}\otimes id} \bigwedge^{n-2} F \otimes F^{\oplus 4} \otimes D_{n-a-3}G^* \\
\xrightarrow{\wedge\otimes id} \Big(\bigwedge^{n-1} F\Big)^{\oplus 4} \otimes D_{n-a-3}G^* = \sum_{i=1}^{4} \mathscr{C}^{a}_{n-2}(-2)\ .
\end{aligned}$$

Since the multiplication map $S_{n-a-2}G \otimes S_{n-a-3}G \longrightarrow S_{2n-2a-5}G$ is not injective, we show that the existence of an $f \in \bigwedge^{n-2} F$ and a $g \in S_{n-a-2}G$ such that $f \wedge \Psi_{(41)}(g \cdot g') = 0$ for all $g' \in S_{n-a-3}G$.

To this end, let $g \in S_{n-a-2}G$ be an arbitrary element and let $\{g'_1, \dots, g'_{n-a-2}\}$ be a basis of $S_{n-a-3}G$. For $i = 1, \dots, (n-a-2)$, we define

$$(f^{(i)}_1, f^{(i)}_2, f^{(i)}_3, f^{(i)}_4)^t := \Psi_{(41)}(g \cdot g'_i) \in F^4$$

and choose a maximal linearly independent subset $\{f_k\}_{k=1,\dots,p}$ of $\{f^{(i)}_j\} \subset F$. Since

$$\begin{aligned}
n-2 = 5r-1 \geq \#\{f^{(i)}_j\} = 4 \cdot \dim_{\Bbbk}(S_{n-a-3}G) = 4(n-a-2) = 4r \\
\geq \#\{f_k\} = p
\end{aligned}$$

holds for all $r \geq 1$ (i.e. $g \geq 13$), we find a non-zero element f of the form

$$f = f_1 \wedge f_2 \wedge \cdots \wedge f_p \wedge \tilde{f} \in \bigwedge^{n-2} F$$

for some $\tilde{f} \in \bigwedge^{n-p-2} F$. By construction, $f \otimes g$ is in the kernel of $\psi_{(41)}$, and hence $(f \otimes g, 0, 0, 0, 0)^t$ lies in the kernel of ψ. □

Remark 3.2.2 ([Bop13]). Using the same argument as in Theorem 3.2.1, one can actually show a stronger statement. Let us again assume that C is a 5-gonal canonical curve, which has a balanced relative canonical resolution. Then with notation as before

$$\mathrm{rk}\Big(\sum_{i=1}^{5} \mathscr{C}^{b_i}_{n-2+c}\Big) \leq \mathrm{rk}\Big(\sum_{i=1}^{5} \mathscr{C}^{a_i}_{n-2+c}\Big).$$

and $\min\{b_i\} \geq n-2+c \geq \max\{a_i\}$ for odd genus $g = 2n+1$ if $n \geq 5c+5$ or even genus $g = 2n$ if $n \geq 5c+8$. Thus, the Betti number $\beta_{n+c,n+c+1}(C)$ is determined by the rank of the matrix ψ_{n-2+c}, which has constant entries. Repeating the argument in Theorem 3.2.1, one can find elements in the kernel of ψ_{n-2+c} for $r = \lfloor \frac{n}{5} \rfloor \geq 3c+1$ (for odd genus) or $r = \lfloor \frac{n}{5} \rfloor \geq 3c+3$ (for even genus). This gives

$$\beta_{n+c,n+c+1}(C) > \beta_{n+c,n+c+1}(X)$$

for odd genus $g = 2n+1 \geq 30c+13$ and even genus $g = 2n \geq 30c+28$.

In the next step we show that a general 5-gonal canonical curve has a balanced relative canonical resolution which then completes the proof of Theorem 3.1.1 by semi-continuity on the Betti numbers.

Therefore note that having a balanced relative canonical resolution is an open condition. Thus it is sufficient to find a single example of a curve having a balanced relative canonical resolution for each genus.

Proposition 3.2.3. *For any odd $g \geq 13$ (and even $g \geq 28$), there exists a smooth and irreducible 5-gonal canonical genus g curve which has a balanced relative canonical resolution.*

Proof. We illustrate the proof for odd genus curves of type II:

To this end, let $X = S(e_1, \ldots, e_4) \cong \mathbb{P}(\mathscr{E})$ with $e_1 \geq \cdots \geq e_4$ be a fixed 4-dimensional balanced rational normal scroll of degree $f = g-4$. Let further

$$(a_1, \ldots, a_5) = (a-1, a, a, a, a) \quad \text{and} \quad (b_1, \ldots, b_5) = (b+1, b, b, b, b)$$

be balanced partitions such that $\sum_{i=1}^5 a_i = 2g-12$, $g = 2n+1$, $n = 5r+1$, $a = 4r-1$ and $a_i + b_i = g-6$. We consider a general skew-symmetric morphism

$$\underbrace{\begin{matrix} \mathscr{O}_{\mathbb{P}(\mathscr{E})}(-3H+(b+1)R) \\ \oplus \\ \mathscr{O}_{\mathbb{P}(\mathscr{E})}(-3H+bR)^{\oplus 4} \end{matrix}}_{=:\mathscr{F}} \xrightarrow{\Psi} \underbrace{\begin{matrix} \mathscr{O}_{\mathbb{P}(\mathscr{E})}(-2H+(a-1)R) \\ \oplus \\ \mathscr{O}_{\mathbb{P}(\mathscr{E})}(-2H+aR)^{\oplus 4} \end{matrix}}_{=:\mathscr{F}^* \otimes \mathscr{L}}$$

If $\bigwedge^2 \mathscr{F}^* \otimes \mathscr{L}$ is globally generated, then it follows by a Bertini type theorem (see e.g. [Oko94, §3]) that the scheme $\mathrm{Pf}(\Psi)$ cut out by the Pfaffians of Ψ is smooth of codimension 3 or empty. Recall from Remark 2.3.2, that a line bundle $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(\mathrm{H}+c\mathrm{R})$ is globally generated if and only if $e_4+c\geq 0$. Thus, since for $r\geq 1$

$$\min\{e_i\} = \left\lfloor \frac{g-4}{4} \right\rfloor = \left\lfloor \frac{10r-1}{4} \right\rfloor = 2r + \left\lfloor \frac{2r-1}{4} \right\rfloor \geq (b-a+1) = 2r,$$

we conclude that in this case

$$\bigwedge^2 \mathcal{F}^* \otimes \mathcal{L} = \mathcal{O}_{\mathbb{P}(\mathcal{E})}(\mathrm{H}-(b-a+1)\mathrm{R})^{\oplus 4} \oplus \mathcal{O}_{\mathbb{P}(\mathcal{E})}(\mathrm{H}-(b-a)\mathrm{R})^{\oplus 6}$$

is globally generated. It follows that $\mathrm{C}=\mathrm{Pf}(\Psi)$ is smooth of codimension 3 or empty. The iterated mapping cone construction gives a free resolution of $\mathrm{C}\subset\mathbb{P}^{g-1}$ of length $g-2$ which is not null-homotopic. Therefore it follows from the Auslander-Buchsbaum formula that C is a non-empty (and therefore smooth) arithmetically Cohen-Macaulay scheme. In particular we have a surjective map

$$\underbrace{\mathrm{H}^0(\mathbb{P}^{g-1},\mathcal{O}_{\mathbb{P}^{g-1}})}_{\cong\Bbbk} \to \mathrm{H}^0(\mathrm{C},\mathcal{O}_{\mathrm{C}}) \to 0$$

and therefore C is smooth and connected and hence an irreducible curve.

Doing the same for curves of type I, III, IV and V and the even genus cases the result follows for all genera except for $g=15$ (in this case $\bigwedge^2\mathcal{F}^*\otimes\mathcal{L}$ is not globally generated). For the $g=15$ case one can verify the statement by using *Macaulay2* (see [GS]). □

Remark 3.2.4. The finitely many cases of 5-gonal curves which are not covered by the above theorem (because $\bigwedge^2\mathcal{F}^*\otimes\mathcal{L}$ is not globally generated) can easily be check similar to the genus 15 case, using computer algebra. Doing this it follows that the general 5-gonal curve has a balanced relative resolution.

The Strategy of the proof above can also be applied to 4-gonal curves. For 3-gonal curves the relative canonical resolution consists of a map between line bundles on $\mathbb{P}(\mathcal{E})$ and therefore is trivially balanced.

In [DP15], Deopurkar and Patel have also shown that general k-gonal curves ($k=4,5$) have a balanced relative canonical resolution using very different methods.

The Question whether a general canonical curve together with a g^1_k has a balanced relative canonical resolution is widely open and will be further discussed in Chapter 4 and Chapter 5.

Corollary 3.2.5. *$\mathscr{K}_{g,5} = \{C \in \mathscr{H}_{g,5} \mid \beta_{n,n+1}(C) > \beta_{n,n+1}(X)\}$ equals $\mathscr{H}_{g,5}$ for $n = \lceil \frac{g-1}{2} \rceil$ and odd genus $g \geq 13$ or even genus $g \geq 28$.*

Proof. For odd genus $g \geq 13$ and even genus $g \geq 28$ it follows by Theorem 3.2.1 and Proposition 3.2.3 that $\mathscr{K}_{g,5}$ is a non-empty and dense subset of $\mathscr{H}_{g,5}$. The conclusion now follows by upper semi-continuity on the Betti numbers. □

3.3 5-gonal curves of genus 13

In this section, we discuss the case of a general 5-gonal canonical curve of genus 13. The rational normal scroll X swept out by the g^1_5 on C is therefore a 4-dimensional scroll of type $S(3,2,2,2)$ and degree $f = 9$. The curve $C \subset \mathbb{P}(\mathscr{E})$ has a resolution of the form

$$\mathscr{O}_{\mathbb{P}(\mathscr{E})}(-5H+7R) \to \begin{array}{c} \mathscr{O}_{\mathbb{P}(\mathscr{E})}(-3H+5R) \\ \oplus \\ \mathscr{O}_{\mathbb{P}(\mathscr{E})}(-3H+4R)^{\oplus 4} \end{array} \xrightarrow{\Psi} \begin{array}{c} \mathscr{O}_{\mathbb{P}(\mathscr{E})}(-2H+2R) \\ \oplus \\ \mathscr{O}_{\mathbb{P}(\mathscr{E})}(-2H+3R)^{\oplus 4} \end{array} \to \mathscr{O}_{\mathbb{P}(\mathscr{E})} \to \mathscr{O}_C$$

where Ψ is a skew-symmetric matrix with entries as indicated below

$$(\Psi) \sim \begin{pmatrix} 0 & (H-2R) & (H-2R) & (H-2R) & (H-2R) \\ & 0 & (H-R) & (H-R) & (H-R) \\ & & 0 & (H-R) & (H-R) \\ & & & 0 & (H-R) \\ & & & & 0 \end{pmatrix} \tag{3.2}$$

We resolve the $\mathscr{O}_{\mathbb{P}(\mathscr{E})}$-modules in the resolution above by Eagon-Northcott type complexes and determine the rank of the maps which give rise to non-minimal parts in the iterated mapping cone. As in Section 2.3.3, we denote by $F = H^0(\mathbb{P}(\mathscr{E}), \mathscr{O}_{\mathbb{P}(\mathscr{E})}(H-R))$

and by $G = H^0(\mathbb{P}(\mathscr{E}), \mathscr{O}_{\mathbb{P}(\mathscr{E})}(R))$.

$$\begin{tikzcd}
{} & {} \\
\sum_{i=1}^5 \mathscr{C}_3^{b_i}(-3) = \begin{matrix} \bigwedge^3 F\otimes S_2G(-6) \\ \oplus \\ (\bigwedge^3 F\otimes S_1G(-6))^{\oplus 4}\end{matrix} \arrow[u] \arrow[r, "\psi_3"] & \sum_{i=1}^5 \mathscr{C}_3^{a_i}(-2) = \begin{matrix} \bigwedge^4 F(-6) \\ \oplus \\ (\bigwedge^4 F(-5))^{\oplus 4}\end{matrix} \arrow[u] \\
\sum_{i=1}^5 \mathscr{C}_4^{b_i}(-3) = \begin{matrix} \bigwedge^4 F\otimes S_1G(-7) \\ \oplus \\ (\bigwedge^4 F(-7))^{\oplus 4}\end{matrix} \arrow[u] \arrow[r, "\psi_4"] & \sum_{i=1}^5 \mathscr{C}_4^{a_i}(-2) = \begin{matrix} \bigwedge^5 F\otimes D_1G^*(-7) \\ \oplus \\ (\bigwedge^5 F(-7))^{\oplus 4}\end{matrix} \arrow[u] \\
\sum_{i=1}^5 \mathscr{C}_5^{b_i}(-3) = \begin{matrix} \bigwedge^5 F(-8) \\ \oplus \\ (\bigwedge^5 F(-9))^{\oplus 4}\end{matrix} \arrow[u] \arrow[r, "\psi_5"] & \sum_{i=1}^5 \mathscr{C}_5^{a_i}(-2) = \begin{matrix} \bigwedge^6 F\otimes D_2G^*(-8) \\ \oplus \\ (\bigwedge^6 F\otimes D_1G^*(-8))^{\oplus 4}\end{matrix} \arrow[u] \\
{} \arrow[u] & {} \arrow[u]
\end{tikzcd}$$

Note that by degree reasons, the maps indicated above are the only ones which give rise to possibly non-minimal parts in the iterated mapping cone.

By the Gorenstein property of canonical curves, it follows that the maps

$$\psi_3' : (\bigwedge^3 F \otimes S_1G)^{\oplus 4}(-6) \to \bigwedge^4 F(-6)$$

and

$$\psi_5' : \bigwedge^5 F(-8) \to (\bigwedge^6 F \otimes D_1G^*)^{\oplus 4}(-8)$$

are dual to each other and one can easily check the surjectivity of ψ_3'. It remains to compute the rank of ψ_4. The following proposition is already contained in the authors master thesis.

Proposition 3.3.1 ([Bop13]). *Let Ψ be a general skew symmetric matrix with entries as indicated above. Then the induced matrix $\psi_4 : \sum_{i=1}^5 \mathscr{C}_4^{b_i}(-3) \to \sum_{i=1}^5 \mathscr{C}_4^{a_i}(-2)$ has a six dimensional kernel.*

Proof. According to Section 2.3.1, we can write down the relevant cohomology groups. Let $\{s,t\}$ be a basis of $G = H^0(\mathbb{P}(\mathscr{E}), \mathscr{O}_{\mathbb{P}(\mathscr{E})}(R))$ and $\{\varphi_1\}$ be a basis of $H^0(\mathbb{P}(\mathscr{E}), \mathscr{O}_{\mathbb{P}(\mathscr{E})}(H - 3R))$ then a basis of $H^0(\mathbb{P}(\mathscr{E}), \mathscr{O}_{\mathbb{P}(\mathscr{E})}(H-2R))$ is given by $\{s\varphi_1, t\varphi_1, \varphi_2, \varphi_3, \varphi_4\}$. We consider the submatrix $\psi_{(41)} : \bigwedge^4 F \otimes S_1G(-7) \to \left(\bigwedge^5 F(-7)\right)^4$ of ψ_4 induced by the first column of Ψ. As in the proof of Theorem 3.2.1, the map $\psi_{(41)}$ is given as the composition

$$\bigwedge^4 F \otimes S_1 G \cong \bigwedge^4 F \otimes H^0(\mathcal{O}_{\mathbb{P}(\mathscr{E})}(R)) \xrightarrow{id \otimes \Psi_{(41)}} \bigwedge^4 F \otimes H^0(\mathcal{O}_{\mathbb{P}(\mathscr{E})}(H-R))^{\oplus 4} \cong \bigwedge^4 F \otimes F^{\oplus 4} \xrightarrow{\wedge} (\bigwedge^5 F)^{\oplus 4}$$

By our generality assumption on C, we can assume that the 4 entries of $\Psi_{(41)}$ are independent and after acting with an element in Aut(X), we can furthermore assume that $\Psi_{(41)} = (s\varphi_1, \varphi_2, \varphi_3, \varphi_4)^t$. It follows that elements of the form

$$(\lambda s + \mu t)s\varphi_1 \wedge (\lambda s + \mu t)\varphi_2 \wedge (\lambda s + \mu t)\varphi_3 \wedge (\lambda s + \mu t)\varphi_4 \otimes (\lambda s + \mu t), \text{with } \lambda, \mu \in \mathbb{k}$$

lie in the kernel of $\psi_{(41)}$. Expanding those elements we get

$$\lambda^5 s^2\varphi_1 \wedge s\varphi_2 \wedge s\varphi_3 \wedge s\varphi_4 \otimes s + \cdots + \mu^5 st\varphi_1 \wedge t\varphi_2 \wedge t\varphi_3 \wedge t\varphi_4 \otimes t$$

and conclude that a rational normal curve of degree 5 lies in $\mathbb{P}(\mathrm{Syz})$ where $\mathrm{Syz} \subset \mathrm{Tor}_6^T(T/I_C)_7$ is the subspace of the 6-th syzygy module spanned by the extra syzygies and $I_C \subset T$ denotes the ideal of the canonical curve C. We get $\beta_{6,7}(C) \geq \beta_{6,7}(X) + 6 = 222$ and by computing one example using *Macaulay2*, it follows that ψ_4 has a 6-dimensional kernel in general. □

In general, as already observed in [Bop13], none of the entries of the skew symmetric matrix Ψ can be made zero by suitable row and column operations respecting the block structure of Ψ. By [Sch86, §5] this implies that the 6 extra syzygies are not induced by an additional linear series on C.

The question arises of how the extra syzygies of a 5-gonal canonical curve $C \subset \mathbb{P}^{12}$ differ from the syzygies induced by the scroll swept out by the g^1_5 on C. At least in the genus $g = 13$ case we can give an answer in this direction by considering so-called *syzygy schemes*, originally introduced in [Ehb94].

Definition and Remark 3.3.2. *Let* $C \subset \mathbb{P}^{g-1}$ *be a smooth and irreducible canonical curve and let* $I_C \subset S$ *be the ideal of* C. *Let further*

$$\mathbf{F}_\bullet : \quad S \leftarrow S(-2)^{\beta_{1,2}} \leftarrow S(-3)^{\beta_{2,3}} \leftarrow \cdots$$

be the linear strand of a minimal free resolution of $S_C = S/I_C$. *For a* p*-th linear syzygy* $s \in F_p$, *let* V_s *be the smallest vector space such that the following diagram commutes*

$$\begin{array}{ccc} F_{p-1} & \longleftarrow & F_p \\ \cup & & \cup \\ V_s \otimes S(-p) & \longleftarrow & S(-p-1) \cong \langle s \rangle \end{array} .$$

The rank of the syzygy s *is defined to be* $\mathrm{rk}(s) := \dim V_s$.

Since $\mathrm{Hom}(\mathbf{F}_\bullet, S)$ *is a free complex and the Koszul complex is exact, it follows that the maps of the dual diagram extend to a morphism of complexes. Dualizing again we get*

$$\begin{tikzcd} S & F_1 \arrow[l] & \cdots \arrow[l] & F_{p-1} \arrow[l] & F_p \arrow[l] \\ \bigwedge^p V_s \otimes S(-1) \arrow[u, "\varphi_p"] & \bigwedge^{p-1} V_s \otimes S(-2) \arrow[l] \arrow[u] & \cdots \arrow[l] & V_s \otimes S(-p) \arrow[l] \arrow[u] & S(-p-1) \arrow[l] \arrow[u] \end{tikzcd} .$$

By degree reasons there are only trivial homotopies and therefore all the vertical maps except φ_p *are unique. The* syzygy scheme $\mathrm{Syz}(s)$ *of the syzygy* $s \in F_p$ *is the scheme defined by the ideal*

$$I_s = \mathrm{Im}(S \longleftarrow \bigwedge^{p-1} V_s \otimes S(-2)).$$

The p*-th* syzygy scheme $\mathrm{Syz}_p(C)$ *of a curve* C *is defined as the intersection* $\bigcap_{s \in F_p} \mathrm{Syz}(s)$.

Any p-th syzygy of a canonical curve has rank $\geq p+1$ and the syzygies of rank $p+1$ are called scrollar syzygies. The name is justified by a theorem due to von Bothmer:

Theorem 3.3.3 ([GvB07, Corollary 5.2]). *Let* $s \in F_p$ *be a* p*-th scrollar syzygy. Then* $\mathrm{Syz}(s)$ *is a rational normal scroll of degree* $p+1$ *and codimension* p *that contains the curve* C.

We can now come back to our example of a 5-gonal genus 13 curve.

Proposition 3.3.4. *Let* $C \subset \mathbb{P}^{12}$ *be a general* 5*-gonal canonical curve. Then* $\mathrm{Syz}_6(C)$ *is the scheme cut out by the* 4 *Pfaffians of* Ψ *involving the first column. In particular* $\mathrm{Syz}_6(C) = C \cup p$ *for some point* $p \in X$.

Proof. Recall from Proposition 3.3.1 that the space of extra syzygies can be identified with the kernel of the map

$$\mathscr{C}_4^5(-3) \to (\mathscr{C}_4^3)^{\oplus 4}(-2)$$

which is induced by the first column of the skew symmetric matrix Ψ.

We denote by $\mathrm{Pf}_1,\dots,\mathrm{Pf}_4 \in H^0(\mathcal{O}_{\mathbb{P}(\mathcal{E})},\mathcal{O}_{\mathbb{P}(\mathcal{E})}(2H-3R))$ the 4 Pfaffians of the matrix Ψ that involve the first column and consider the iterated mapping cone

$$[[\mathscr{C}^5 \to (\mathscr{C}^3)^{\oplus 4}] \to \mathscr{C}^0],$$

where $\sum_{i=1}^4 \mathscr{C}^3 \to \mathscr{C}^0$ is induced by the multiplication with $(\mathrm{Pf}_1,\dots,\mathrm{Pf}_4)$. This complex is a resolution of the ideal J generated by the 4 Pfaffians as an $\mathcal{O}_{\mathbb{P}^{12}}$-module. In particular the minimized resolution is a subcomplex of the minimal free resolution of S_C.

Since all extra syzygies are induced by the first column of Ψ, it follows that the 6-th syzygy modules in the linear strand of these minimal resolutions are canonically isomorphic. Therefore $\mathrm{Syz}_6(V(J))$ and $\mathrm{Syz}_6(C)$ coincide and $V(J) \subset \mathrm{Syz}_6(C)$.

Now, computing one example using *Macaulay2* shows that

$$V(J) = C \cup p \supset \mathrm{Syz}_6(C).$$

□

Remark 3.3.5. For 5-gonal canonical curves of higher genus, the extra syzygies are no longer induced by a single column. Thus a similar description of the syzygy scheme (as for genus 13) seems not possible for higher genus. An additional difficulty for higher genus arises because we no longer have a full description of the space of extra syzygies.

Chapter 4

Relative canonical resolutions for general curves

This chapter follows the article [BH15b] by Michael Hoff and the author of this thesis.

4.1 Introduction

Let $C \subset \mathbb{P}^{g-1}$ be a canonical curve of genus g that admits a complete base point free g^1_k, then the g^1_k sweeps out a rational normal scroll X of dimension $d = k-1$ and degree $f = g-k+1$. One can resolve the curve $C \subset \mathbb{P}(\mathscr{E})$, where $\mathbb{P}(\mathscr{E})$ is the $\mathbb{P}^{d-1}$-bundle associated to the scroll X. Schreyer showed in [Sch86] that this so-called *relative canonical resolution* is of the form

$$0 \to \pi^*\mathrm{N}_{k-2}(-k) \to \pi^*\mathrm{N}_{k-3}(-k+2) \to \cdots \to \pi^*\mathrm{N}_1(-2) \to \mathscr{O}_{\mathbb{P}(\mathscr{E})} \to \mathscr{O}_{\mathrm{C}} \to 0$$

where $\pi : \mathrm{C} \to \mathbb{P}^1$ is the map induced by the g^1_k and $\mathrm{N}_i = \bigoplus_{j=1}^{\beta_i} \mathscr{O}_{\mathbb{P}^1}(a_j^{(i)})$.

To determine the splitting type of these N_i is an open problem. If C is a general canonical curve with a g^1_k such that the genus g is large compared to k, it is conjectured that the bundles N_i are balanced, which means that $\max|a_j^{(i)} - a_l^{(i)}| \leq 1$. This is known to hold for $k \leq 5$ (see e.g. [DP15] and [Bop15] or Proposition 3.2.3). Gabriel Bujokas and Anand Patel [BP15] gave further evidence to the conjecture by showing that all N_i are balanced if $g = n \cdot k + 1$ for $n \geq 1$ and the bundle N_1 is balanced if $g \geq (k-1)(k-3)$.
The aim of this chapter is to provide a range in which the first syzygy bundle N_1,

hence the relative canonical resolution, is unbalanced for a general pair (C, g^1_k) with non-negative Brill-Noether number $\rho(g,k,1)$. Our main theorem is the following.

Theorem 4.1.1. *Let $C \subset \mathbb{P}^{g-1}$ be a general canonical curve and let k be a positive integer such that $\rho := \rho(g,k,1) \geq 0$ and $g > k+1$. Let $L \in W^1_k(C)$ be a general point inducing a $g^1_k = |L|$. Then the bundle N_1 in the relative canonical resolution of C is unbalanced if and only if $(k-\rho-\frac{7}{2})^2 - 2k + \frac{23}{4} > 0$ and $\rho > 0$.*

The strategy for the proof is to study the birational image C' of C under the residual mapping $|\omega_C \otimes L^{-1}|$. Quadratic generators of C' correspond to special generators of $C \subset \mathbb{P}(\mathscr{E})$ whose existence forces N_1 to be unbalanced in the case $\rho > 0$. Under the generality assumptions on C and L, one obtains a sharp bound for which pairs (k,ρ), the curve C' has quadratic generators. Finally in section 4.4, we state a more precise conjecture about the splitting type of the bundles in the relative canonical resolution.

Our theorem and conjecture are motivated by experiments using the computer algebra software *Macaulay2* ([GS]) and the package `RelativeCanonicalResolution.m2` [BH15a].

4.2 Relative canonical resolutions revisited

Let $C \subset \mathbb{P}^{g-1}$ be a canonically embedded curve of genus g with a base point free g^1_k. Recall from Section 2.3.2, that we can associate a projective bundle $\pi : \mathbb{P}(\mathscr{E}) \to \mathbb{P}^1$ to the g^1_k and resolve $C \subset \mathbb{P}(\mathscr{E})$. By Theorem 2.3.7 this *relative canonical resolution* $F_\bullet$ has the form

$$0 \to \pi^*N_{k-2}(-k) \to \pi^*N_{k-3}(-k+2) \to \cdots \to \pi^*N_1(-2) \to \mathscr{O}_{\mathbb{P}(\mathscr{E})} \to \mathscr{O}_C \to 0$$

with $\pi^*N_i = \sum_{j=1}^{\beta_i} \mathscr{O}_{\mathbb{P}(\mathscr{E})}(a_j^{(i)}R)$ and $\beta_i = \frac{i(k-2-i)}{k-1}\binom{k}{i+1}$. The complex is furthermore self-dual, i.e. $\mathscr{H}om(F_\bullet, \mathscr{O}_{\mathbb{P}(\mathscr{E})}(-kH + (g-k-1)R)) \cong F_\bullet$

Also recall that a bundle of the form $\sum_{j=1}^{\beta} \mathscr{O}_{\mathbb{P}(\mathscr{E})}(nH + a_jR)$ is called *balanced* if $\max_{i,j} |a_i - a_j| \leq 1$. Consequently, we call the relative canonical resolution balanced, if all bundles appearing in the resolution are balanced.

By Corollary 2.3.6, the bundle $\mathscr{E}$ associated to the scroll swept out by a g^1_k on a curve C is always balanced for general (k-gonal) curves.

We will give a lower bound on the integers $a_j^{(1)}$ appearing in the resolution $F_\bullet$.

Proposition 4.2.1. *Let* C *be a general canonically embedded curve of genus* g *and let* $k \geq 4$ *be an integer such that* $\rho(g,k,1) \geq 0$ *and* $g > k+1$. *Let further* $L \in W^1_k(C)$ *be a general point inducing a complete base point free* g^1_k. *Then with notation as in Theorem 2.3.7, all twists* $a_j^{(1)}$ *of the bundle* N_1 *are non-negative.*

Proof. As usual, we denote by $\mathbb{P}(\mathscr{E})$ the projective bundle induced by the g^1_k. We consider the relative canonical resolution of $C \subset \mathbb{P}(\mathscr{E})$. Twisting of the relative canonical resolution by 2H and pushing forward to $\mathbb{P}^1$, we get an isomorphism

$$\pi_*(\mathscr{I}_{C/\mathbb{P}(\mathscr{E})}(2H)) \cong N_1 = \bigoplus_{j=1}^{\beta_1} \mathscr{O}_{\mathbb{P}^1}(a_j^{(1)}).$$

Then, all twists $a_j^{(1)}$ are non-negative if and only if

$$h^1(\mathbb{P}^1, N_1(-1)) = h^1(\mathbb{P}^1, \pi_*(\mathscr{I}_{C/\mathbb{P}(\mathscr{E})}(2H-R))) = h^1(\mathbb{P}(\mathscr{E}), \mathscr{I}_{C/\mathbb{P}(\mathscr{E})}(2H-R)) = 0.$$

We consider the long exact cohomology sequence

$$\begin{aligned} 0 \to H^0(\mathbb{P}(\mathscr{E}), \mathscr{I}_{C/\mathbb{P}(\mathscr{E})}(2H-R)) &\to H^0(\mathbb{P}(\mathscr{E}), \mathscr{O}_{\mathbb{P}(\mathscr{E})}(2H-R)) \to H^0(\mathbb{P}(\mathscr{E}), \mathscr{O}_C(2H-R)) \to \\ \to H^1(\mathbb{P}(\mathscr{E}), \mathscr{I}_{C/\mathbb{P}(\mathscr{E})}(2H-R)) &\to \dots \end{aligned}$$

obtained from the standard short exact sequence.

The vanishing of $H^1(\mathbb{P}(\mathscr{E}), \mathscr{I}_{C/\mathbb{P}(\mathscr{E})}(2H-R))$ is equivalent to the surjectivity of the map

$$H^0(\mathbb{P}(\mathscr{E}), \mathscr{O}_{\mathbb{P}(\mathscr{E})}(2H-R)) \longrightarrow H^0(C, \mathscr{O}_C(2H-R)).$$

From the commutative diagram

$$\begin{tikzcd} H^0(\mathbb{P}(\mathscr{E}), \mathscr{O}_{\mathbb{P}(\mathscr{E})}(2H-R)) \arrow[r] & H^0(C, \mathscr{O}_C(2H-R)) \\ H^0(\mathbb{P}(\mathscr{E}), \mathscr{O}_{\mathbb{P}(\mathscr{E})}(H)) \otimes H^0(\mathbb{P}(\mathscr{E}), \mathscr{O}_{\mathbb{P}(\mathscr{E})}(H-R)) \arrow[u] \arrow[r, "\cong"] & H^0(C, \mathscr{O}_C(H)) \otimes H^0(C, \mathscr{O}_C(H-R)) \arrow[u, "\eta"'] \end{tikzcd}$$

we see that it suffices to show the surjectivity of η.

Note that the system $|H-R|$ on C is $\omega_C \otimes L^{-1}$ and the residual line bundle $\omega_C \otimes L^{-1} \in W^{g-k}_{2g-2-k}(C)$ is general since L is general. Hence, the residual morphism induced by $|\omega_C \otimes L^{-1}|$ is birational for $g-k \geq 2$ by [GH80, Section 0.b (4)].

We may apply [AS78, Theorem 1.6] and get a surjection

$$\bigoplus_{q\geq 0} \mathrm{Sym}_q(H^0(C,\omega_C\otimes L^{-1}))\otimes H^0(C,\omega_C) \longrightarrow \bigoplus_{q\geq 0} H^0(C,\omega_C\otimes(\omega_C\otimes L^{-1})^q),$$

i.e., the $\mathrm{Sym}(H^0(C,\omega_C\otimes L^{-1}))$-module $\bigoplus_{q\in\mathbb{Z}} H^0(C,\omega_C\otimes(\omega_C\otimes L^{-1})^q)$ is generated in degree 0. In particular, this implies the surjectivity of η. □

Remark 4.2.2. Using the projective normality of $C\subset\mathbb{P}(\mathscr{E})$, one can show that all twists $a_j^{(1)}$ of N_1 are greater or equal to -1. There exist several examples where N_1 has negative twists (see [Sch86]). We conjecture that all $a_j^{(i)}\geq -1$ and for general curves $a_j^{(i)}\geq 0$.

It is known that the degrees of the bundles N_i can be computed recursively (see e.g [Sch86]). However, we did not find a closed formula for the degrees in the literature.

Proposition 4.2.3. *The degree of the bundle* N_i *of rank* $\beta_i=\frac{k}{i+1}(k-2-i)\binom{k-2}{i-1}$ *in the relative canonical resolution* $F_\bullet$ *is*

$$\deg(N_i)=\sum_{j=1}^{\beta_i} a_j^{(i)} = (g-k-1)(k-2-i)\binom{k-2}{i-1}.$$

In particular, for $i=1,2$ *one obtains* $\deg(N_1)=(k-3)(g-k-1)$ *and* $\deg(N_2)=(k-4)(k-2)(g-k-1)$.

Proof. The degrees of the bundles N_i can be computed by considering the identity

$$\chi(\mathscr{O}_C(\nu))=\sum_{i=0}^{k-2}(-1)^i\chi(F_i(\nu)). \tag{4.1}$$

If $b\geq -1$, we have

$$h^i(\mathbb{P}(\mathscr{E}),\mathscr{O}_{\mathbb{P}(\mathscr{E})}(aH+bR))=\begin{cases} h^i(\mathbb{P}^1,S_a(\mathscr{E})(b)), & \text{for } a\geq 0\\ 0, & \text{for } -k<a<0\\ h^{k-i}(\mathbb{P}^1,S_{-a-k}(\mathscr{E})(f-2-b)), & \text{for } a\leq -k\end{cases}$$

where $f=\deg(\mathscr{E})=g-k+1$. As in the construction of the bundles in [CE96, Proof of Step B, Theorem 2.1], one obtains that the degree of N_i is independent of the splitting type of the bundle. Hence, we assume that $a_j^{(i)}\geq -1$ and therefore, we can apply the above formula to all terms in $F_\bullet$.

We compute the degree of N_n by induction. The base case is straightforward. We twist the relative canonical resolution by $\mathscr{O}_{\mathbb{P}(\mathscr{E})}(n+1)$ and compute the Euler characteristic of each term. By the Riemann-Roch Theorem, $\chi(\mathscr{O}_C(n+1)) = (2n+1)g-(2n+1)$. Applying the above formula yields

$$\chi(F_i(n+1)) = \begin{cases} \binom{k-1+n}{k-2} + f\binom{k-1+n}{k-1}, & \text{for } i=0 \\ (\deg(N_i)+\beta_i)\binom{k-2+n-i}{k-2} + \beta_i f\binom{k-2+n-i}{k-1}, & \text{for } n \geq i \geq 1 \\ 0, & \text{for } i \geq n+1 \end{cases}$$

Substituting all formulas in (4.1), we get

$$\begin{aligned}(2n+1)g-(2n+1) = &\binom{k-1+n}{k-2} + f\binom{k-1+n}{k-1} \\ &+ \sum_{i=1}^{n-1}(-1)^i\left((\deg(N_i)+\beta_i)\binom{k-2+n-i}{k-2} + \beta_i f\binom{k-2+n-i}{k-1}\right) \\ &+ (-1)^n(\deg(N_n)+\beta_n).\end{aligned}$$

Using the induction step, the alternating sums simplify to

$$\begin{aligned}\sum_{i=1}^{n-1}(-1)^i\deg(N_i)\binom{k-2+n-i}{k-2} &= (f-2)(2n+1-nk)+(-1)^{n+1}(f-2)(k-2-n)\binom{k-2}{n-1} \\ \sum_{i=1}^{n-1}(-1)^i\beta_i\binom{k-2+n-i}{k-2} &= k-\binom{k-1+n}{k-2}+(-1)^{n+1}\frac{k}{n+1}(k-2-n)\binom{k-2}{n-1} \\ \sum_{i=1}^{n-1}(-1)^i\beta_i f\binom{k-2+n-i}{k-1} &= nkf - f\binom{k-1+n}{k-1}\end{aligned}$$

and we get the desired formula for $\deg(N_n)$. □

4.3 The bundle of quadrics

Let $C \subset \mathbb{P}^{g-1}$ be a general canonically embedded genus g curve and let k be a positive integer such that the Brill-Noether number $\rho := \rho(g,k,1)$ is non-negative and $g > k+1$. Let further $L \in W^1_k(C)$ be general. We denote by X the rational normal scroll swept out by the $g^1_k = |L|$ and by $\mathbb{P}(\mathcal{E}) \to X$ the projective bundle associated to X. By Corollary 2.3.6, the bundle $\mathcal{E}$ on $\mathbb{P}^1$ is of the form

$$\mathcal{E} = \bigoplus_{i=1}^{k-1-\rho} \mathcal{O}_{\mathbb{P}^1}(1) \oplus \bigoplus_{i=1}^{\rho} \mathcal{O}_{\mathbb{P}^1}.$$

Furthermore, by Theorem 2.3.11, the resolution of the ideal sheaf $\mathcal{I}_{C/\mathbb{P}(\mathcal{E})}$ is of the form

$$0 \longleftarrow \mathcal{I}_{C/\mathbb{P}(\mathcal{E})} \longleftarrow Q := \sum_{j=1}^{\beta_1} \mathcal{O}_{\mathbb{P}(\mathcal{E})}(-2H + a_j^{(1)}R) \longleftarrow \dots$$

where $\beta_1 = \frac{1}{2}k(k-3)$. We denote Q the *bundle of quadrics*. By Proposition 4.2.3, the degree of $N_1 = \pi_*(Q)$ is precisely

$$\deg(N_1) = \sum_{j=1}^{\beta_1} a_j^{(1)} = (k-3)(g-k-1).$$

and by Proposition 4.2.1, all a_i are non-negative. Since each summand of Q corresponds to a non-zero global section of $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(2H - a_j^{(1)}R)$, we get $2 \cdot e_1 - a_j^{(1)} \geq 0$. Hence $a_j^{(1)} \leq 2$ for all j. It follows that the bundle of quadrics Q is of the following form

$$Q = \mathcal{O}_{\mathbb{P}(\mathcal{E})}(-2H)^{\oplus l_0} \oplus \mathcal{O}_{\mathbb{P}(\mathcal{E})}(-2H+R)^{\oplus l_1} \oplus \mathcal{O}_{\mathbb{P}(\mathcal{E})}(-2H+2R)^{\oplus l_2}.$$

We will describe the possible generators of $\mathcal{I}_{C/\mathbb{P}(\mathcal{E})}$ in $H^0(\mathbb{P}(\mathcal{E}), \mathcal{O}_{\mathbb{P}(\mathcal{E})}(2H-2R))$. Therefore, we consider the residual line bundle $\omega_C \otimes L^{-1}$ with

$$h^0(C, \omega_C \otimes L^{-1}) = f = g-k+1 \text{ and } \deg(\omega_C \otimes L^{-1}) = 2g-k-2.$$

By [GH80, Section 0.b (4)], $|\omega_C \otimes L^{-1}|$ induces a birational map for $g > k+1$.

Lemma 4.3.1. *Let $C' \subset \mathbb{P}^{g-k}$ be the birational image of C under the residual linear system $|\omega_C \otimes L^{-1}|$. There is a one-to-one correspondence between quadratic generators of $C' \subset \mathbb{P}^{g-k}$ and quadratic generators of $C \subset \mathbb{P}(\mathcal{E})$ contained in $H^0(\mathbb{P}(\mathcal{E}), \mathcal{O}_{\mathbb{P}(\mathcal{E})}(2H-2R))$.*

Proof. Since $\rho \geq 0$, the scroll X is a cone over the Segre variety $\mathbb{P}^1 \times \mathbb{P}^{g-k}$. Let $p : \mathbb{P}(\mathcal{E}) \longrightarrow \mathbb{P}^{g-k}$ be the projection on the second factor. An element q of $H^0(\mathbb{P}(\mathcal{E}), \mathcal{O}_{\mathbb{P}(\mathcal{E})}(2H-2R))$ corresponds to a global section of $H^0(\mathbb{P}^1, S_2(\mathcal{E}) \otimes \mathcal{O}_{\mathbb{P}^1}(-2))$ which does not depend on the fiber over $\mathbb{P}^1$. Hence, the image of $V(q)$ under the projection yields a quadric containing C'. Conversely, the pullback under the projection p of a quadratic generator of $C' \subset \mathbb{P}^{g-k}$ does not depend on the fiber and has therefore to be contained in $H^0(\mathbb{P}(\mathcal{E}), \mathcal{O}_{\mathbb{P}(\mathcal{E})}(2H-2R))$. □

We are now interested in a bound on k and ρ such that the curve C' lies on a quadric.

Lemma 4.3.2. *For a general curve* C *and a general line bundle* $L \in W^1_k(C)$, *the curve* $C' \subset \mathbb{P}^{g-k}$ *lies on a quadric if and only if the pair* (k,ρ) *satisfies the inequality*

$$(k-\rho-\frac{7}{2})^2-2k+\frac{23}{4}>0.$$

Proof. By [JP16], the map

$$H^0(\mathbb{P}^{g-k},\mathscr{O}_{\mathbb{P}^{g-k}}(2)) \to H^0(C',\mathscr{O}_{C'}(2))$$

has maximal rank for a general curve C and a general line bundle $\omega_C \otimes L^{-1}$. Using the long exact cohomology sequence associated to the short exact sequence

$$0 \to \mathscr{I}_{C'}(2) \to \mathscr{O}_{\mathbb{P}^{g-k}}(2) \to \mathscr{O}_{C'}(2) \to 0,$$

we see that C' lies on a quadric if and only if

$$h^0(\mathbb{P}^{g-k},\mathscr{O}_{\mathbb{P}^{g-k}}(2))-h^0(C',\mathscr{O}_{C'}(2))>0.$$

We compute the Hilbert polynomial of C': $h_{C'}(n)=(2g-k-2)n+1-g$ and get $h_{C'}(2)=3g-2k-3$. The dimension of the space of quadrics in $\mathbb{P}^{g-k}$ is $\binom{g-k+2}{2}$. Hence,

$$h^0(\mathbb{P}^{g-k},\mathscr{O}_{\mathbb{P}^{g-k}}(2))-h^0(C',\mathscr{O}_{C'}(2))=\binom{g-k+2}{2}-3g+2k+3>0. \tag{4.2}$$

Expressing g in terms of k and ρ, the inequality (4.2) is equivalent to

$$(k-\rho-\frac{7}{2})^2-2k+\frac{23}{4}>0.$$

□

Proof of Theorem 4.1.1. As mentioned before, the bundle $Q=\pi^*N_1(-2H)$ is of the form $Q=\mathscr{O}_{\mathbb{P}(\mathscr{E})}(-2H)^{\oplus l_0}\oplus\mathscr{O}_{\mathbb{P}(\mathscr{E})}(-2H+R)^{\oplus l_1}\oplus\mathscr{O}_{\mathbb{P}(\mathscr{E})}(-2H+2R)^{\oplus l_2}$ (see also Proposition 4.2.1). By Lemma 4.3.1, the bundle of quadrics is balanced if no quadratic generator of $C' \subset \mathbb{P}^{g-k}$ exists. So, we are done for pairs (k,ρ) with $(k-\rho-\frac{7}{2})^2-2k+\frac{23}{4}\le 0$.

It remains to show that the bundle of quadrics is unbalanced in the case $\rho > 0$ for pairs (k,ρ) satisfying the inequality in Lemma 4.3.2.

Let k and ρ be non-negative integers satisfying the above inequality and let $l_2 = h^0(C', \mathscr{I}_{C'}(2)) = (k-\rho-\frac{7}{2})^2 - 2k + \frac{23}{4}$ be the positive dimension of quadratic generators of the ideal of C'. Now, by Lemma 4.3.1, the bundle Q is unbalanced if a summand of the type $\mathscr{O}_{\mathbb{P}(\mathscr{E})}(-2H)$ exists. Such a summand exists if and only if the following inequality holds

$$l_0 = \beta_1 - l_2 - l_1 = \beta_1 - l_2 - (\sum_{i=1}^{\beta_1} a_i - 2\cdot l_2) > 0. \tag{4.3}$$

An easy calculation shows that the inequality (4.3) is equivalent to

$$l_0 = \binom{\rho+1}{2} > 0.$$

□

For pairs (k,ρ) in the following marked region, the bundle Q is unbalanced.

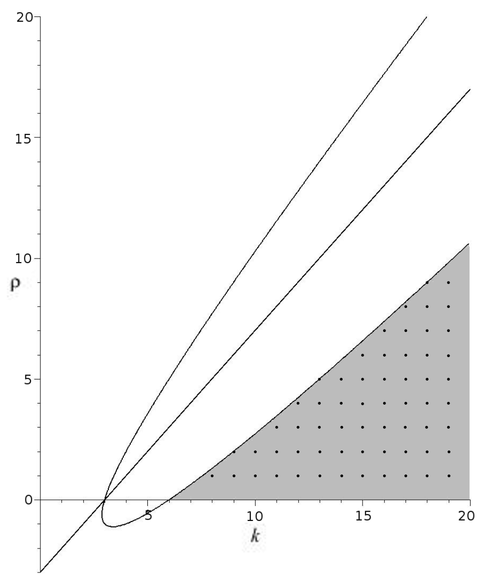

Figure 4.1: The conic: $(k-\rho-\frac{7}{2})^2 - 2k + \frac{23}{4}=0$ and the line: $k-\rho-3=0 \Leftrightarrow g=k+1$.

Remark 4.3.3. With our presented method, the whole first linear strand of the resolution of $C' \subset \mathbb{P}^{g-k}$ lifts to the resolution of $C \subset \mathbb{P}(\mathscr{E})$ (see also Example 4.4.1).

4.4 Example and open problems

Example 4.4.1. Let $C \subset \mathbb{P}^{g-1}$ be a general canonical curve of genus 19 together with a general line bundle $L \in W^1_{11}(C)$ and let $C' \subset \mathbb{P}^8$ be the birational image of C under the map $|\omega_C \otimes L^{-1}|$. By Lemma 4.3.2 $h^0(\mathbb{P}^8, \mathscr{I}_{C'}(2)) = 13$ and assuming the maximal rank conjecture in the range of cubics we further get $h^0(\mathbb{P}^8, \mathscr{I}_{C'}(3)) = 108$. Thus, we expect that C' has $h^0(\mathbb{P}^8, \mathscr{O}_{\mathbb{P}^8}(1)) \cdot h^0(\mathbb{P}^8, \mathscr{I}_{C'}(2)) - h^0(\mathbb{P}^8, \mathscr{I}_{C'}(3)) = 9$ linear syzygies.

Using [BH15a], we construct a nodal curve $C \subset \mathbb{P}^{18}$ of genus 19 with a concrete realization of $L \in W^1_{11}(C)$. The ideal of the scroll X swept out by $|L|$ is given by the 2×2 minors of the matrix

$$\begin{pmatrix} x_0 & x_2 & \dots & x_{16} \\ x_1 & x_3 & \dots & x_{17} \end{pmatrix}.$$

The resolution of the birational image C' of C under the map $|\omega_C \otimes L^{-1}|$ has the following Betti table

	0	1	2	3	4	5	6	7
0	1	-	-	-	-	-	-	-
1	-	13	9	-	-	-	-	-
2	-	-	91	259	315	197	56	1
3	-	-	-	-	-	-	-	2

Assuming that the relative canonical resolution is as balanced as possible, the first part of the relative canonical resolution is of the following form

$$0 \leftarrow \mathscr{I}_{C/\mathbb{P}(\mathscr{E})} \leftarrow \begin{matrix} \mathscr{O}_{\mathbb{P}(\mathscr{E})}(-2H+2R)^{\oplus 13} \\ \oplus \\ \mathscr{O}_{\mathbb{P}(\mathscr{E})}(-2H+R)^{\oplus 30} \oplus \mathscr{O}_{\mathbb{P}(\mathscr{E})}(-2H) \end{matrix} \leftarrow \begin{matrix} \mathscr{O}_{\mathbb{P}(\mathscr{E})}(-3H+3R)^{\oplus 9} \\ \oplus \\ \mathscr{O}_{\mathbb{P}(\mathscr{E})}(-3H+2R)^{\oplus 192} \oplus \mathscr{O}_{\mathbb{P}(\mathscr{E})}(-3H+R)^{\oplus 30} \end{matrix} \leftarrow \dots$$

However, in any case neither the bundle N_1 nor the bundle N_2 are balanced for a general curve $C \in \mathscr{M}_{19}$ and a general line bundle $L \in W^1_{11}(C)$.

Using the *Macaulay2*-Package [BH15a], our experiments lead to conjecture the following:

Conjecture 4.4.2. *(a) Let $C \subset \mathbb{P}^{g-1}$ be a general canonical curve and let k be a positive integer such that $\rho := \rho(g,k,1) \geq 0$. Let $L \in W^1_k(C)$ be a general point inducing a $g^1_k = |L|$. Then for bundles $N_i = \bigoplus \mathscr{O}_{\mathbb{P}^1}(a_j^{(i)})$, $i = 2, \dots, \lceil \frac{k-3}{2} \rceil$ there is the following bound*

$$\max_{j,l} |a_j^{(i)} - a_l^{(i)}| \leq \min\{g-k-1, i+1\}.$$

This bound is furthermore sharp in the following sense. Given two integers $k \geq 3$ *and* $2 \leq i \leq \lceil \frac{k-3}{2} \rceil$*, there exists an integer* g *such that the general canonical curve* C *of genus* g *has an* i*-th syzygy bundle* N_i *in the relative canonical resolution, associated to a general* $L \in W^1_k(C)$*, which satisfies* $max_{j,l}|a_j^{(i)} - a_l^{(i)}| = \min\{g-k-1, i+1\}$*. In particular, if* $g - k = 2$*, the relative canonical resolution is balanced.*

(b) For general pairs (C, g^1_k) *with* $\rho(g,k,1) \leq 0$*, the bundle* N_1 *is balanced.*

Remark 4.4.3. Recall, that conjecture (b) is known for $3 \leq k \leq 5$ by [DP15] and [Bop15]. In order to verify Conjecture (b), it is enough to show the existence of one curve with these properties. With the help of [BH15a], we construct a g-nodal curve on a normalized scroll swept out by a g^1_k and compute the relative canonical resolution. Our experiments show that Conjecture (b) is true for

k	g
6	$10 \leq g \leq 24$
7	$12 \leq g \leq 24$
8	$14 \leq g \leq 24$
9	$16 \leq g \leq 24$

We found several examples (e.g. $(g,k) = (17,7), (19,8), \dots$) of g-nodal k-gonal curves where some of the higher syzygy modules N_i, $i \geq 2$ are unbalanced. We believe that the generic relative canonical resolution is unbalanced in these cases.

In the next Chapter we will show, that a genus 9 curve, together with a line bundle $L \in W^1_6(C)$ has an unbalanced second syzygy bundle in the relative canonical resolution.

Chapter 5

Moduli of lattice polarized K3 surfaces via relative canonical resolutions

This chapter follows the article [BH17a] by Michael Hoff and the author of this thesis.

5.1 Introduction

Lattice polarized K3 surfaces were introduced in [Dol96] and provide a direct generalization of polarized K3 surfaces. Instead of fixing a polarization, that is, an ample class on a K3 surface S, one fixes a primitive lattice embedding $\varphi : M \to \mathrm{Pic}(S)$ such that the image of M contains an ample class. Furthermore, in [Dol96] Dolgachev showed that there exists a quasi-projective $20 - \mathrm{rk}(M)$ dimensional moduli space $\mathscr{F}^M$ parametrizing isomorphism classes of M-polarized K3 surfaces (S, φ). It is a natural question to ask about the geometry of the moduli space $\mathscr{F}^M$, in particular about its unirationality. Quite often, to prove unirationality, each particular case requires individual treatment (even for the case of curves).

For K3 surfaces, the best studied moduli spaces are those parametrizing polarized K3 surfaces (i.e. K3 surfaces polarized by a primitive lattice of rank one). In a series of papers ([Muk88], [Muk96], [Muk06], [Muk12], [Muk92]) Mukai showed that the moduli space $\mathscr{F}_g^H$ parametrizing H-polarized K3 surfaces, where H is an ample class with $H^2 = 2g - 2$, is unirational for $g \leq 13$ and $g = 16, 18, 20$. In [Muk09], he also announced the case $g = 17$. The unirationality was recently shown for $g = 14$ and $g = 33$ in [Nue16] and [Kar16], respectively.

On the other hand Gritsenko–Hulek–Sankaran [GHS07] showed that $\mathscr{F}^{\mathrm{H}}$ is of general type for $g = 47, 51, 55, 58, 61$ and $g > 62$ (see also [Kon93] and [Kon99]).

For M-polarized K3 surfaces where M is a lattice of higher rank much less is known. For certain higher rank lattices M the unirationality of $\mathscr{F}^{\mathrm{M}}$ was recently proved by Bhargava–Ho–Kumar ([BHK16]) and for the Nikulin lattice $\mathfrak{N}$ Farkas–Verra and Verra showed the unirationality of $\mathscr{F}_g^{\mathfrak{N}}$ for $g \leq 8$ (see [FV12], [FV16] and [Ver16]). Here g refers to the self intersection $\mathrm{H}^2 = 2g-2$ of the ample class H in $\mathfrak{N}$.

In this chapter we prove the unirationality of the moduli space $\mathscr{F}^{\mathfrak{h}}$, where $\mathfrak{h}$ is the rank 3 lattice defined by the following intersection matrix with respect to an ordered basis $\{h_1, h_2, h_3\}$

$$\mathfrak{h} \sim \begin{pmatrix} 14 & 16 & 5 \\ 16 & 16 & 6 \\ 5 & 6 & 0 \end{pmatrix}.$$

If $(\mathrm{S}, \varphi) \in \mathscr{F}^{\mathfrak{h}}$ is an $\mathfrak{h}$-polarized K3 surface, then we denote

$$\varphi(h_1) = \mathscr{O}_{\mathrm{S}}(\mathrm{H}), \varphi(h_2) = \mathscr{O}_{\mathrm{S}}(\mathrm{C}) \text{ and } \varphi(h_3) = \mathscr{O}_{\mathrm{S}}(\mathrm{N}).$$

We relate the moduli space $\mathscr{F}^{\mathfrak{h}}$ to the universal Brill–Noether variety parametrizing genus 9 curves together with a pencil of degree 6. To be more precise, we consider the open subset

$$\mathscr{F}_8^{\mathfrak{h}} = \left\{(\mathrm{S}, \varphi) \;\middle|\; (\mathrm{S}, \varphi) \in \mathscr{F}^{\mathfrak{h}} \text{ and } \mathscr{O}_{\mathrm{S}}(\mathrm{H}) = \varphi(h_1) \text{ ample}\right\}$$

of the moduli space $\mathscr{F}^{\mathfrak{h}}$ and the open subset

$$\mathscr{P}_8^{\mathfrak{h}} = \left\{(\mathrm{S}, \varphi, \mathrm{C}) \;\middle|\; (\mathrm{S}, \varphi) \in \mathscr{F}_8^{\mathfrak{h}} \text{ and } \mathrm{C} \in |\mathscr{O}_{\mathrm{S}}(\mathrm{C})| \text{ smooth}\right\}$$

of the tautological $\mathbb{P}^9$-bundle over $\mathscr{F}_8^{\mathfrak{h}}$. The natural restriction map

$$\phi : \mathscr{P}_8^{\mathfrak{h}} \to \mathscr{W}_{9,6}^1, \; \left(\mathrm{S}, \varphi, \mathrm{C}\right) \mapsto \left(\mathrm{C}, \mathscr{O}_{\mathrm{S}}(\mathrm{N}) \otimes \mathscr{O}_{\mathrm{C}}\right)$$

connects $\mathscr{P}_8^{\mathfrak{h}}$ with the universal Brill–Noether variety $\mathscr{W}_{9,6}^1 = \left\{(\mathrm{C}, \mathrm{L}) \;\middle|\; \mathrm{C} \in \mathscr{M}_9 \text{ and } \mathrm{L} \in \mathrm{W}_6^1(\mathrm{C})\right\}$. Note that $\dim \mathscr{P}_8^{\mathfrak{h}} = \dim \mathscr{W}_{9,6}^1 + 1 = 26$. Our main theorem is the following.

Theorem (see Theorem 5.3.6 and Corollary 5.3.7). *The map* $\phi : \mathscr{P}_8^{\mathfrak{h}} \to \mathscr{W}_{9,6}^1$ *defined above is dominant. Moreover,* $\mathscr{P}_8^{\mathfrak{h}}$ *is birational to a* $\mathbb{P}^1$*-bundle over an open subset of* $\mathscr{W}_{9,6}^1$*. In particular* $\mathscr{P}_8^{\mathfrak{h}}$ *and hence* $\mathscr{F}^{\mathfrak{h}}$ *are unirational.*

The key idea behind the proof of our main theorem is the following: A base point free $|L| = g_6^1$ on a canonically embedded genus 9 curve $C \subset \mathbb{P}^8$ sweeps out a rational normal scroll $\mathbb{P}(\mathscr{E})$. One can consider the minimal free resolution of $C \subset \mathbb{P}(\mathscr{E})$ in terms of $\mathscr{O}_{\mathbb{P}(\mathscr{E})}$-modules (see Section 2.3 for the desired background). We show that the elements in the fiber $\phi^{-1}(C, L)$ are in one-to-one correspondence with syzygies of certain degree in this so-called relative canonical resolution of $C \subset \mathbb{P}(\mathscr{E})$. To a syzygy we associate a K3 surface in the fiber of ϕ via the syzygy scheme. We will deduce the theorem by computing one example with the desired properties in *Macaulay2* [GS] and a semicontinuity argument.

Mukai's work [Muk88] shows that the moduli space of genus 9 curves $\mathscr{M}_9$ is dominated by a projective bundle over the moduli space of polarized K3 surfaces $\mathscr{F}_9^{H}$. He describes (Brill–Noether general) K3 surfaces containing a general curve $C \in \mathscr{M}_9$. In contrast, we will show the existence of a unique K3 surface of Picard rank 4 containing C for a general point $(C, \omega_C \otimes L^{-1}) \in \mathscr{W}_{9,10}^3$. The idea is the following: Let $(C, L) \in \mathscr{W}_{9,6}^1$ be a general point. Then the image C' under the residual embedding $\omega_C \otimes L^{-1}$ lies on a net of quartics. We will show that the fiber $\phi^{-1}(C, L)$ defines a plane cubic inside this net of quartics. By studying the geometry of the quartic corresponding to the singular point, it follows that its Picard lattice with respect to an ordered basis $\{h_1', h_2', h_3', h_4'\}$ has the form

$$\mathfrak{h}' \sim \begin{pmatrix} 4 & 10 & 1 & 1 \\ 10 & 16 & 0 & 0 \\ 1 & 0 & -2 & 0 \\ 1 & 0 & 0 & -2 \end{pmatrix}$$

This yields the following theorem.

Theorem (see Theorem 5.4.3). *Let*

$$\mathscr{P}_3^{\mathfrak{h}'} = \left\{(S, \varphi, C) \;\middle|\; (S, \varphi) \in \mathscr{F}^{\mathfrak{h}'}, \mathscr{O}_S(H') = \varphi(h_1') \textit{ ample and } C \in |\varphi(h_2')| \textit{ smooth}\right\}$$

be the open subset of the tautological $\mathbb{P}^9$*-bundle over the moduli space* $\mathscr{F}_3^{\mathfrak{h}'}$. *Then the morphism*

$$\phi' : \mathscr{P}_3^{\mathfrak{h}'} \to \mathscr{W}^3_{9,10}, \ (S,\varphi,C) \mapsto (C, \mathscr{O}_S(H') \otimes \mathscr{O}_C)$$

defines a birational equivalence.

The initial intention of the work presented in this chapter was to study the structure of the relative canonical resolution for pairs $(C,L) \in \mathscr{W}^1_{9,6}$. By upper-semicontinuity it suffices to show the balancedness of the relative canonical resolution for a single example. However, the relative canonical resolution was unbalanced (see Section 5.2.1) in all examples which we computed (see Proposition 5.3.1). Thus the usual upper-semicontinuity argument failed to describe the resolution. As a direct consequence of the main theorem we obtain the following result.

Theorem (see Corollary 5.3.8). *For any* $(C,L) \in \mathscr{W}^1_{9,6}$ *the relative canonical resolution has an unbalanced second syzygy bundle.*

In Section 5.2 we recall the definition and basic results of relative canonical resolutions and lattice polarized K3 surfaces. Section 5.3 is devoted to the proof of our main theorem. In Section 5.4 we deduce the birationality of $\mathscr{P}_3^{\mathfrak{h}'}$ and $\mathscr{W}^3_{9,10}$.

5.2 Preliminaries

In this section we briefly recall the definition of relative canonical resolutions (see Section 2.3 for more details) and summarize the construction of the moduli space for lattice polarized K3-surfaces.

5.2.1 Relative canonical resolutions

Let $C \subset \mathbb{P}^{g-1}$ be a canonically embedded curve of genus g with a base point free g^1_k. We can associate a projective bundle $\pi : \mathbb{P}(\mathscr{E}) \to \mathbb{P}^1$ to the scroll swept out by the g^1_k and resolve C in terms of $\mathscr{O}_{\mathbb{P}(\mathscr{E})}$-modules.

By Theorem 2.3.7 $F_\bullet$ this *relative canonical resolution* has the form

$$0 \to \pi^*N_{k-2}(-kH) \to \pi^*N_{k-3}\big((-k+2)H\big) \to \cdots \to \pi^*N_1(-2H) \to \mathscr{O}_{\mathbb{P}(\mathscr{E})} \to \mathscr{O}_C \to 0$$

where $N_i = \sum_{j=1}^{\beta_i} \mathcal{O}_{\mathbb{P}^1}(a_j^{(i)})$, $\beta_i = \frac{i(k-2-i)}{k-1}\binom{k}{i+1}$ and $R = \pi^* \mathcal{O}_{\mathbb{P}^1}(1)$ denotes the ruling of the scroll. The complex is furthermore self-dual, i.e. $\mathcal{H}om(F_\bullet, \mathcal{O}_{\mathbb{P}(\mathcal{E})}(-kH + (g-k-1)R)) \cong F_\bullet$. Furthermore, by Proposition 4.2.3 the slopes of the bundles N_i are known to be $\mu(N_i) = \frac{(g-k-1)(i+1)}{k}$.

Also recall that a bundle $N = \mathcal{O}_{\mathbb{P}^1}(a_1) \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^1}(a_d)$ on $\mathbb{P}^1$ is called balanced if $\max_{i,j} |a_i - a_j| \leq 1$ (or equivalently $h^1(\mathbb{P}(\mathcal{E}), \mathcal{E}nd(N)) = 0$). Consequently, we call the relative canonical resolution balanced, if all bundles appearing in the resolution are balanced.

For example, as we have seen in Section 2.3 the bundle $\mathcal{E}$ defining a scroll swept out by a pencil on a Petri-general canonical curve C is always balanced.

The splitting types of the bundles N_i in the relative canonical resolution are only known in a few cases which we sum up below. The relative canonical resolution is generically balanced if $k \leq 5$ (see [DP15] and [Bop15] or Proposition 3.2.3) or if $g = nk+1$ for some $n > 1$ (see [BP15]). Furthermore the first bundle N_1 is known to be generically balanced for $g \geq (k-1)(k-3)$ (see [BP15]) or if the Brill-Noether number $\rho = \rho(g,k,1)$ is non-negative and $(k - \rho - \frac{7}{2})^2 - 2k + \frac{23}{4} \leq 0$ (see [BH15b] or Theorem 4.1.1).

In the next section we will show that the unbalancedness of the second syzygy module of the relative canonical resolution of a general point $(C, L) \in \mathscr{W}^1_{9,6}$ corresponds to the existence of K3-surfaces $S \subset \mathbb{P}(\mathcal{E})$ of Picard rank 3 containing the curve C. In this case the bundle $\mathcal{E}$ is generically of the form $\mathcal{E} = \mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}$. Hence, the corresponding scroll X has a zero-dimensional vertex. But since the curve generically will not pass through the vertex, we will consider $\mathbb{P}(\mathcal{E})$ instead of X.

5.2.2 Lattice polarized K3 surfaces

In this section we recall the construction of the moduli space of so-called lattice polarized K3-surfaces due to Dolgachev [Dol96], i.e. given a lattice M we want to construct a parameter space for K3 surfaces whose Picard lattice is an over-lattice of M. We start over by recalling some well known facts on K3 surfaces which can be found in various textbooks (see e.g. [Huy15] or [BHPVdV04]).

By [Siu83] every K3 surface is Kähler and therefore admits a Hodge decomposition. The only interesting cohomology group appearing in this decomposition is $H^2(S, \mathbb{C})$.

It has a Hodge decomposition of the form $H^2(S,\mathbb{C}) = H^{2,0}(S) \oplus H^{1,1}(S) \oplus H^{0,2}(S)$, where $H^{2,0}(S) = H^0(S,\Omega_S^2) = H^0(S,\omega_S) =< w >$ and $H^{0,2}(S) =< \overline{w} >$ are one dimensional and $h^{1,1}(20) = 20$. As we will see later on, the cohomology group $H^2(S,\mathbb{C})$ plays an important role in the construction of a moduli spaces for K3 surfaces.

The second cohomology group $H^2(S,\mathbb{Z})$ endowed with the bilinear symmetric form $(\cdot,\cdot)$ induced by the cup-product pairing has the structure of a lattice. This lattice turns out to be isometric to the *K3-lattice*

$$\Lambda_{K3} = U^{\oplus 3} \oplus E_8(-1)^{\oplus 2}$$

where U the unique even unimodular lattice of signature $(1,1)$ and E_8 is the unique even unimodular positive definite lattice of rank 8 (see e.g. [Huy15, Ch. 1 Prop 3.5]). Hence Λ_{K3} is an even unimodular lattice of signature $(3,19)$.

Another important lattice associated to a K3 surface S is the Néron-Severi lattice $NS(S)$. For K3 surfaces the Néron-Severi lattice is isomorphic to the Picard lattice $\mathrm{Pic}(S)$ (see e.g. [Huy15, Ch. 1 Prop. 2.4]) and by the Lefschetz Theorem on $(1,1)$ classes (see e.g. [BHPVdV04, IV Thm 2.13]) it follows furthermore that

$$NS(S) = H^{1,1}(S) \cap H^2(S,\mathbb{Z}),$$

where the cohomology group $H^2(S,\mathbb{Z})$ is identified with its image in $H^2(S,\mathbb{C})$ under the natural inclusion.

As we have already remarked, the second cohomology groups of K3 surfaces play an important role in the construction of their moduli spaces. The following fundamental results shed a little light on how this works.

Theorem 5.2.1. *(a) (Weak Torelli) Let* S *and* S' *be two* K3 *surfaces. Then* S *and* S' *are isomorphic if and only if there is a Hodge isometry* $H^2(S,\mathbb{Z}) \to H^2(S',\mathbb{Z})$ *(i.e. a lattice isometry* $H^2(S,\mathbb{Z}) \to H^2(S',\mathbb{Z})$ *whose* $\mathbb{C}$*-linear* $H^2(S,\mathbb{C}) \to H^2(S',\mathbb{C})$ *extension preserves the Hodge decomposition).*

(b) (Strong Torelli) Let S *and* S' *be two* K3 *surfaces and let* $\psi : H^2(S,\mathbb{Z}) \to H^2(S',\mathbb{Z})$ *be a Hodge isometry such that* $\psi(Amp(S)) \cap Amp(S') \neq \emptyset$*. Then there exists a unique isomorphism* $f : S' \to S$ *such that* $f^* = \psi$*.*

Proof. See [Huy15, Ch. 7, Thm. 5.3] □

Definition 5.2.2. A *marking* on a K3 surface S is an isometry $\Phi : \mathrm{H}^2(\mathrm{S},\mathbb{Z}) \to \Lambda_{\mathrm{K3}}$. We refer to pairs (S,Φ) as *marked* K3 surfaces.

An isomorphism between marked K3 surfaces (S,Φ) and (S',Φ') is an isomorphism $f : \mathrm{S} \to \mathrm{S}'$ such that $\Phi' = \Phi \circ f^*$.

Let (S,Φ) be a marked K3 surface and let $\Phi_{\mathbb{C}} : \mathrm{H}^2(\mathrm{S},\mathbb{C}) \to \Lambda_{\mathrm{K3}} \otimes \mathbb{C}$ be the $\mathbb{C}$-linear extension of the marking. By abuse of notation, we will also denote the $\mathbb{C}$-linear extension of the cup product pairing by $(\cdot,\cdot)$. If $w \in \mathrm{H}^{2,0}(\mathrm{S}) \subset \mathrm{H}^2(\mathrm{S},\mathbb{C})$ is a nowhere vanishing 2-form then $(w,w) = 0$, $(w,\overline{w}) > 0$. Now $\Phi(\mathrm{H}^{2,0}(\mathrm{S})) = \mathbb{C}\Phi(w)$ defines a line through the origin in the vector space $\Lambda_{\mathrm{K3}} \otimes \mathbb{C}$ and thus a point in the *period domain*

$$\Omega_{\Lambda_{\mathrm{K3}}} = \left\{[x] \in \mathbb{P}(\Lambda_{\mathrm{K3}} \otimes \mathbb{C}) \;\middle|\; (x,x) = 0,\ (x,\overline{x}) > 0\right\}.$$

The point $[\Phi(w)]$ inside the complex manifold $\Omega_{\Lambda_{\mathrm{K3}}}$ is called the *period point* of the marked K3 surface (S,Φ).

If $p : \mathscr{S} \to \mathrm{U}$ is a flat family of K3 surfaces over a small contractible open set U, then a marking $\Phi_0 : \mathrm{H}^0(\mathrm{S},\mathbb{Z}) \to \Lambda_{\mathrm{K3}}$ can be extended (uniquely) to a marking $\Phi_{\mathrm{U}} : \mathrm{R}^2 p_* \mathbb{Z}_{\mathrm{U}} \to (\Lambda_{\mathrm{K3}})_{\mathrm{U}}$ on the family. Here $(\Lambda_{\mathrm{K3}})_{\mathrm{U}}$ denotes the constant sheaf with fibers Λ_{K3} over U. This allows us to define a holomorphic map $\pi_{\mathrm{U}} : \mathrm{U} \to \Omega_{\Lambda_{\mathrm{K3}}}$ called the *period map* for the family $p : \mathscr{S} \to \mathrm{U}$ (see [BHPVdV04, VIII.19]).

Theorem 5.2.3 (Local Torelli). *The period map $\pi_{\mathrm{U}} : \mathrm{U} \to \Omega_{\Lambda_{\mathrm{K3}}}$ associated to a versal deformation $p : \mathscr{S} \to \mathrm{U}$ of a marked* K3 *surface* S *is a local isomorphism.*

Proof. See [Huy15, Ch. 6 Prop 2.8] □

Moreover, we have the following theorem, which was first proved by Todorov [Tod80].

Theorem 5.2.4 (Surjectivity of the period map). *Every point in $\Omega_{\Lambda_{\mathrm{K3}}}$ occurs as the period point of some marked* K3 *surface.*

Let $\mathrm{O}(\Lambda_{\mathrm{K3}})$ denote the group of isometries of the K3 lattice. Combining the above theorems and the weak Torelli theorem, it follows that the elements in the set

$$\mathrm{O}(\mathrm{K3}) \setminus \Omega_{\Lambda_{\mathrm{K3}}}$$

are in one to one correspondence with isomorphism classes of K3 surfaces. However, the group action of $\mathrm{O}(\Lambda_{\mathrm{K3}})$ on $\Omega_{\Lambda_{\mathrm{K3}}}$ is not well-behaved (the action is not proper discontinuous) and therefore, the resulting quotient has no nice structure.

In order to fix this issue, we will consider (lattice-) polarized K3 surfaces.

Definition 5.2.5. A *(pseudo-) polarized* K3 surface is a pair (S,L) where S is a K3 surface and $\mathrm{L} \in \mathrm{Pic}(\mathrm{S})$ is a (pseudo-) ample line bundle. The number L^2 is called the degree of the (pseudo-) polarized K3 surface.

Two (pseudo-) polarized K3 surfaces (S,L) and (S,L') are isomorphic if there is an isomorphism $f : \mathrm{S} \to \mathrm{S}'$ such that $f^*(\mathrm{L}') = \mathrm{L}$.

A *marked (pseudo-) polarized* K3 surface is marked K3 surface (S,Φ) such that $\Phi^{-1}(h)$ is (pseudo-) ample for some fixed class $h \in \Lambda_{\mathrm{K3}}$ with $(h,h) = 2k > 0$.

Definition 5.2.6. Let M be an even non-degenerate lattice of signature $(1, r-1)$ which can be primitively embedded into Λ_{K3}. Let $\mathrm{V}(\mathrm{M})^+$ be one of the two components of the cone $\mathrm{V}(\mathrm{M}) = \left\{x \in \mathrm{M} \otimes \mathbb{R} \;\middle|\; (x,x) > 0\right\}$. Let further $\mathrm{M}_{-2} = \left\{\delta \in \mathrm{M} \;\middle|\; (\delta,\delta) = -2\right\}$ denote the set of roots of M and let $\mathrm{M}_{-2} = \mathrm{M}^+_{-2} \cup (-\mathrm{M}^+_{-2})$ be a disjoint decomposition. We define

$$\mathrm{C}(\mathrm{M})^+ = \left\{x \in \mathrm{V}(\mathrm{M})^+ \cap \mathrm{M} \;\middle|\; (x,\delta) > 0 \text{ for all } \delta \in \mathrm{M}^+_{-2}\right\}.$$

Let S be a K3 surface such that there is a primitive embedding $\varphi : \mathrm{M} \to \mathrm{Pic}(\mathrm{S})$. The pair (S,φ) is called a *(pseudo-) ample M-polarized* K3 surface if $\varphi(\mathrm{C}(\mathrm{M})^+)$ contains a (pseudo-) ample line bundle.

An isomorphism between two (pseudo-) ample M-polarized K3 surfaces (S,φ) and (S',φ') is an isomorphism $f : \mathrm{S} \to \mathrm{S}'$ such that $f^* \circ \varphi' = \varphi$.

A marked (pseudo-) ample M-polarized K3 surface is a marked K3 surface (S,Φ) such that $(\mathrm{S},\Phi^{-1}|_{\mathrm{M}})$ is a (pseudo-) ample M-polarized K3 surface.

In what follows we will often just write M-polarized K3 surface instead of pseudo-ample M-polarized K3 surface.

Lattice polarized K3 surfaces were introduced by Nikulin and Dolgachev (see [Nik79a] and [Dol96]). The construction of a moduli space for lattice polarized K3 surfaces which we present here follows [Dol96]. Note that the concept of lattice polarization is a natural generalization of the notion of polarization. Both notions coincide if we take $\mathrm{M} =< h >$ to be a lattice spanned by a single class h with positive self-intersection.

We furthermore remark, that the assumptions on the lattice M in the definition above are necessary in order to obtain a primitive embedding into some Picard lattice. Indeed, for a K3 surface S the restriction of the cup-product pairing to $H^{1,1}(S)$ has signature $(1,19)$ (see e.g. [BHPVdV04, Thm IV.2.5]). Furthermore, by [Mor84, Cor. 2.9] (see also [Huy15, Ch. 14 Cor. 3.1]) every even lattice of signature $(1,r-1)$ occurs as the Picard lattice of some K3 surface and can be uniquely embedded into Λ_{K3} (up to automorphisms of Λ_{K3}) if $r \leq 10$.

In the next step, we want to construct a moduli space for lattice polarized K3 surfaces. Therefore we fix an even lattice M of signature $(1,r-1)$ with $r \leq 10$ and regard M as a primitive sub-lattice of Λ_{K3}.

Let (S,Φ) be a marked M-polarized K3 surface, let $\varphi = \Phi^{-1}|_M : M \to \mathrm{Pic}(S)$ be the induced primitive lattice embedding and let $w \in H^{2,0}$ be a nowhere vanishing 2-form. Since $H^{2,0}(S)$ (as well as $H^{0,2}(S)$) is orthogonal to $H^{1,1}(S)$ it follows that $(w,m) = 0$ for all $m \in M$. Hence the period point of (S,Φ) lies in the domain

$$\begin{aligned}\Omega_M &= \left\{[x] \in \mathbb{P}(\Lambda_{K3} \otimes \mathbb{C}) \;\middle|\; (x,x) = 0,\ (x,\overline{x}) > 0,\ (x,M) = 0\right\}\\ &= \left\{[x] \in \mathbb{P}(M^\perp \otimes \mathbb{C}) \;\middle|\; (x,x) = 0,\ (x,\overline{x}) > 0\right\}\end{aligned}$$

to which we refer as the *period domain for* M*-polarized* K3 *surfaces*. The signature of $M^\perp$ is $(2,19-r)$ and therefore, the space Ω_M has the structure of a $(20-r)$-dimensional complex manifold which is biholomorphic to two disjoint copies of a bounded symmetric domain of type IV (see [BHPVdV04, VIII Sec. 20]). Furthermore, the two components of $\Omega_{\Lambda_{K3}}$ are interchanged by complex conjugation.

By the surjectivity of the period map, every point in Ω_M occurs as the period point of some marked M-polarized K3 surface.

Remark 5.2.7. Let $h \in M$ be a distinguished element with positive self-intersection. Let (S,Φ) be a marked M-polarized K3 surface with period point $[w] \in \Omega_M$ and let $L = \Phi^{-1}|_M(h)$ be the line bundle in Pic(S) corresponding to the distinguished element $h \in M$. By changing the marking of S, we can assume that L is nef. Indeed, for a (-2)-curve δ we define the Picard-Lefschetz reflection

$$s_\delta : \mathrm{Pic}(S) \to \mathrm{Pic}(S),\ L \mapsto L + (L,\delta)\cdot L.$$

By construction $(s_\delta(L),\delta) = -(L,\delta)$ and $(s_\delta(L), s_\delta(L')) = (L,L')$. We will regard s_δ as an element in $O(H^2(S,\mathbb{Z}))$. Now for any class L with positive self-intersection on S there exist finitely many (-2)-curves $\delta_1,\dots,\delta_n$ on S such that $s_{\delta_1}\circ\cdots\circ s_{\delta_n}(L)$ is nef (see e.g. [Huy15, Ch. 8 Cor. 2.9]).

If we denote $\Phi' = \Phi\circ s_{\delta_1}\circ\cdots\circ s_{\delta_n}$, then (S,Φ') is a marked K3 surface with period point $[w]$ such that $(\Phi')^{-1}|_M(h)$ is nef.

Furthermore, if no (-2)-curve δ exists with $(L,\delta) = 0$, then by the Nakai–Moishezon–Kleiman criterion for ampleness (see [Huy15, Ch. 8 Thm 2.1 and Rem. 2.7]) the line bundle $s_{\delta_1}\circ\cdots\circ s_{\delta_n}(L)$ is in fact ample.

In the next step we want to get rid of the markings, in order to obtain a moduli space for M-polarized K3 surfaces.

To this end let again $O(\Lambda_{K3})$ denote the automorphism group of the K3 lattice and

$$\Gamma(M) := \{g\in O(\Lambda_{K3}) \mid g(m) = m \ \forall m\in M\}.$$

Let $M^\perp$ be the orthogonal complement of M in Λ_{K3} then every $g\in\Gamma(M)$ induces naturally to an automorphism of $M^\perp$. Thus we have a natural injective homomorphism

$$\Gamma(M)\to O(M^\perp)$$

and we denote the image of $\Gamma(M)$ by Γ_M. We will give an alternative description of the group Γ_M below.

The lattice $M^\perp$ can naturally be embedded into $(M^\perp)^* = \mathrm{Hom}(M^\perp,\mathbb{Z})$ via the mapping $x\mapsto (x,\cdot)$. The group $A(M^\perp) = (M^\perp)^*/M^\perp$ is finite and called the discriminant group of $M^\perp$. For more information about discriminant groups we refer to [Huy15, Ch. 14]). An element $g\in O(M^\perp)$ induces an element $g^*\in O((M^\perp)^*)$ via $g^*\varphi : x\mapsto \varphi(g^{-1}x)$. Thus, there is a natural homomorphism

$$O(M^\perp)\longrightarrow \mathrm{Aut}\big(A(M^\perp)\big)$$

and the kernel of this homomorphism is called the stable orthogonal group of $M^\perp$ and is denoted by $\widetilde{O}(M^\perp)$. As a direct consequence of [Nik79b, Cor. 1.5.2] (see also [Huy15, Ch. 14 Prop. 2.6]) it follows that

$$\Gamma_M = \widetilde{O}(M^\perp).$$

This shows in particular, that Γ_{M} is a finite index subgroup of $\mathrm{O}(\mathrm{M}^{\perp})$ (i.e. an arithmetic subgroup). Therefore, it follows from a classical result due to Baily and Borel [BB66] (see also [Huy15, Ch. 8 Sec. 1]) that the quotient

$$\mathscr{F}_{pa}^{\mathrm{M}} := \Gamma_{\mathrm{M}} \setminus \Omega_{\mathrm{M}}$$

can be endowed with the structure of a quasi-projective variety.

Theorem 5.2.8 ([Dol96, Sec. 1.3]). *Let* M *be an even non-degenerate lattice of signature* $(1, r-1)$ *with* $r \leq 10$. *The quasi-projective variety* $\mathscr{F}_{pa}^{\mathrm{M}}$ *is a coarse moduli space for pseudo-ample* M*-polarized* K3 *surfaces.*

We skip the proof of the above theorem, since it is similar to the ample M-polarized case (c.f. Theorem 5.2.11).

In the next step we modify the space Ω_{M} by removing certain hyperplanes in order to ensure that the period points correspond to marked ample M-polarized K3 surfaces. To this end let

$$\mathrm{M}^{\perp}_{-2} := \left\{\delta \in \mathrm{M}^{\perp} \;\middle|\; (\delta, \delta) = -2\right\}$$

be the set of roots of $\mathrm{M}^{\perp}$. For each $\delta \in \mathrm{M}^{\perp}_{-2}$ let

$$\mathrm{H}_{\delta} = \left\{x \in \mathrm{M}^{\perp} \otimes \mathbb{C} \;\middle|\; (x, \delta) = 0\right\} \cap \Omega_{\mathrm{M}}$$

be the hyperplane in Ω_{M} which is fixed by the Picard-Lefschetz reflexion defined by δ. We further denote by

$$\Delta_{\mathrm{M}} = \bigcup_{\delta \in \mathrm{M}^{\perp}_{-2}} \mathrm{H}_{\delta} \subset \Omega_{\mathrm{M}}$$

the union of all those hyperplanes and by

$$\Omega^{\circ}_{\mathrm{M}} = \Omega_{\mathrm{M}} \setminus \Delta_{\mathrm{M}}$$

the complement of Δ_{M} in Ω_{M}.

Remark 5.2.9. Recall from Remark 5.2.7 that, by changing the marking of a surface (S, Φ) by finitely many Picard-Lefschetz reflexions, we can assume that a given $\mathrm{L} \in \mathrm{Pic}(\mathrm{S})$ with positive self-intersection is ample unless there exists a (-2)-curve δ on

S with $(L, \delta) = 0$. This shows that elements in Ω_M° correspond to marked ample M-polarized K3 surfaces. On the other hand assume that (S, Φ) is a marked pseudo-ample M-polarized K3 surface with period point $[w] \in H_\delta$ for some $\delta \in M_{-2}^\perp$. Recall that a marked pseudo-ample M-polarized K3 surface is a K3 surface S together with a lattice isometry $\Phi : H^2(S, \mathbb{Z}) \to \Lambda_{K3}$ such that $(S, \Phi^{-1}|_M \to \mathrm{Pic}(S))$ is a (pseudo-) ample M-polarized K3 surface. Then $\pm\Phi^{-1}(\delta)$ is a (-2)-curve on S because $(w, \delta) = 0$ and therefore $\Phi^{-1}(\delta) \in H^{1,1}(S)$. Furthermore, $(\Phi^{-1}|_M(M), \delta) = 0$ and thus $\Phi^{-1}|_M(M)$ can not contain an ample class.

Remark 5.2.10. By [Dol96, Thm 3.1] Ω_M does not define a moduli space for marked pseudo-ample M-polarized K3 surfaces. If (S, Φ) is a marked pseudo-ample M-polarized K3 surface with period point $[w] \in \Omega_M$, then changing the marking by a reflexions defined by elements in $\{\delta \in w^\perp \cap M^\perp \mid (\delta, \delta) = -2\}$ gives a marked K3 surface with period point $[w]$ which is non-isomorphic to (S, Φ) (as marked K3 surfaces). However, by [Dol96, Cor. 3.2] the space Ω_M° is a (fine) moduli space for marked ample M-polarized K3 surfaces since in this case $\Phi^{-1}|_M(M)^\perp \cap H^{1,1}(S)$ can not contain elements with self-intersection -2.

We can again get rid of the markings by taking the quotient

$$\mathscr{F}_a^M = \Gamma_M \setminus \Omega_M^\circ.$$

By [Kne02, Satz 30.2] the group $O(M^\perp)$ (and hence the finite index subgroup Γ_M) acts on the set $M_{-2}^\perp$ with finitely many orbits. Therefore, the complement of $\mathscr{F}_a^M$ in $\mathscr{F}_{pa}^M$ consists of finitely many divisors. In particular, $\mathscr{F}_a^M$ is a Zariski open subset of $\mathscr{F}_{pa}^M$.

Theorem 5.2.11 ([Dol96, Sec. 1.3]). *Let* M *be an even lattice of signature* $(1, r-1)$ *with* $r \le 10$. *The quasi projective variety* $\mathscr{F}_a^M$ *is a (coarse) moduli space for ample* M*-polarized* K3 *surfaces.*

Proof. Let (S, φ) be an ample M-polarized K3 surface and let $\Phi : H^2(S, \mathbb{Z}) \to \Lambda_{K3}$ be a marking of (S, φ). Recall that $\Phi^{-1}|_M = \varphi$. Let $[w] = \Phi_{\mathbb{C}}(H^{2,0}(S))$ be the corresponding period point. Then $(w, \varphi(M)) = 0$ since $\varphi : M \to \mathrm{Pic}(S)$ is a primitive embedding and therefore $[w] \in \Omega_M$. By Remark 5.2.9 it follows furthermore, that $[w] \in \Omega_M^\circ$. Recall that by [Mor84, Cor. 2.9] an embedding $\mathrm{M} \hookrightarrow \Lambda_{\mathrm{K3}}$ is unique up to automorphisms of Λ_{K3}. Therefore we can consider M as a primitive sub-lattice of Λ_{K3} and it then follows that two markings of (S, φ) only differ by an element in $\Gamma(\mathrm{M}) \subset \mathrm{O}(\Lambda_{\mathrm{K3}})$. This gives rise to a map which associates a point in $\mathscr{F}_a^{\mathrm{M}}$ to an isomorphism class of an ample M-polarized K3 surface. This map is surjective by the surjectivity of the period map and it therefore remains to check the injectivity.

Assume that two ample M-polarized K3 surfaces (S, φ) and (S', φ') are represented by the same point in $\mathscr{F}_a^{\mathrm{M}}$. Thus there exists a Hodge isometry $\Psi : \mathrm{H}^2(\mathrm{S}', \mathbb{Z}) \to \mathrm{H}^2(\mathrm{S}, \mathbb{Z})$ which maps an ample class in $\mathrm{Pic}(\mathrm{S}')$ to an ample class in $\mathrm{Pic}(\mathrm{S})$. By the strong Torelli theorem it follows that $\Psi = f^*$ for a unique isomorphism $f : \mathrm{S} \to \mathrm{S}'$. Now if Φ (resp. Φ') are markings for (S, φ) (resp. (S', φ')) then there exists a $g \in \Gamma(\mathrm{M})$ such that the diagram

$$\begin{tikzcd}
\mathrm{H}^2(\mathrm{S}', \mathbb{Z}) \arrow[r, "f^*"] \arrow[d, "\Phi'"'] & \mathrm{H}^2(\mathrm{S}, \mathbb{Z}) \arrow[d, "\Phi"] \\
\Lambda_{\mathrm{K3}} \arrow[r, "g"] & \Lambda_{\mathrm{K3}}
\end{tikzcd}$$

commutes. Changing the marking of (S', φ') to $g \circ \Phi'$ it follows that $(\mathrm{S}', g \circ \Phi')$ and (S, Φ) are isomorphic as marked ample M-polarized K3 surfaces and hence (S, φ) and (S', φ') are isomorphic as ample M-polarized K3 surfaces. □

Let (S, ϕ) be a pseudo-ample M-polarized K3 surface and let $h \in \mathrm{M}$ be a distinguished element with positive self-intersection. By Remark 5.2.7, we can assume that $\varphi(h)$ is pseudo-ample. We want to explain, that we similarly can assume that $\varphi(h)$ is ample, if we restrict to an Zariski open subset inside $\mathscr{F}_a^{\mathrm{M}}$ (see [Bea04]).

Therefore we fix an even non-degenerate lattice M of signature $(1, r-1)$ (with $r \le 10$) which we consider as a sub-lattice of Λ_{K3}. We furthermore fix an element $h \in \mathrm{M}$ which is formally ample, i.e. $(h, \delta) \neq 0$ for all $\delta \in \mathrm{M}_{-2}$. A very general element in $\mathscr{F}_a^{\mathrm{M}}$ corresponds to the isomorphism class of an ample M-polarized K3 surface $(\mathrm{S}_0, \varphi_0)$, such that $\varphi_0 : \mathrm{M} \to \mathrm{Pic}(\mathrm{S}_0)$ is a lattice isomorphism. Hence by Remark 5.2.7 we can assume that $\varphi_0(h) \in \mathrm{Pic}(\mathrm{S}_0)$ is ample.

Let $\Phi_0 : \mathrm{H}^2(\mathrm{S}_0, \mathbb{Z}) \to \Lambda_{\mathrm{K3}}$ be a marking for $(\mathrm{S}_0, \varphi_0)$ and let $p : \mathscr{S} \to \mathrm{U}$ be a local deformation of the marked M-polarized K3 surface (see [Dol96, Sec. 1.2] or [Bea04]).

We denote

$$\Phi_U : R^2 p_* \mathbb{Z}_U \to (\Lambda_{K3})_U$$

the (unique) induced marking on the family. Note that Φ_U is an isomorphism of sheaves. Let

$$\varphi_U : M_U \to \mathrm{Pic}_{\mathscr{S}/U} \subset R^2 p_* \mathbb{Z}_U$$

be the induced map, where $\mathrm{Pic}_{\mathscr{S}/U}$ denotes the relative Picard scheme. Thus we get a sheaf $\varphi_U(h_U) \subset R^2 p_* \mathbb{Z}_U$ such that $\varphi_U(h_U)\big|_0 = \varphi_0(h) \in \mathrm{Pic}(S_0)$ is ample (here again h_U denotes the constant sub-sheaf of the constant sheaf $M_U \subset (\Lambda_{K3})_U$).

Now by [Gro61, 4.7.1] (see also [Laz04, Thm. 1.2.13]) it follows that (possibly after shrinking U) $\varphi_U(h_U)\big|_t$ is an ample line bundle on $S_t = p^{-1}(t)$ for all $t \in U$. This shows that there is a Zariski open subset of $\mathscr{F}_a^M$ parametrizing M-polarized K3-surfaces (S, φ) such that $\varphi(h) \in \mathrm{Pic}(S)$ is ample.

Remark 5.2.12. Alternatively, one could argue the following way. By [SD74] the third power of an ample line bundle on a K3 surface is very ample. Let $H_t := \varphi_U(h_U)\big|_t$ be the distinguished line bundle on $S_t = p^{-1}(t)$. Then $h^0(S_t, H_t^{\otimes 3}) = N$ for all $t \in U$. Thus (modulo a choice of basis for $H^0(S_t, H_t^{\otimes 3})$) we get a flat family of K3 surfaces in $\mathbb{P}^{N-1}$ which has a smooth member. Therefore the general member of the family has to be smooth by [Gro66, 12.2.4], which implies that H_t is ample in an open subset of U.

Definition 5.2.13. We denote the Zariski open subset of $\mathscr{F}_a^M$ which parametrizes M-polarized K3 surfaces (S, φ) such that $\varphi(h) \in \mathrm{Pic}(S)$ is ample by $\mathscr{F}^{M,h}$. If the choice of $h \in M$ is clear from the context we will use the same notation as in [Bea04] and write $\mathscr{F}_g^M$, where $h^2 = 2g - 2$.

Remark 5.2.14. The space $\mathscr{F}^{M,h}$ can also be described as a quotient of a subset of Ω_M. By similar arguments as in Theorem 5.2.11 we see that there is a one to one correspondence between points in $\mathscr{F}^{M,h}$ and points in the quotient

$$\Gamma_M \setminus \Omega_M^h$$

where $\Omega_M^h = \Omega_M \setminus \Delta_h$ and Δ_h is the union of all hyperplanes defined by reflections for elements in $(h^\perp)_{-2}$. However, we do not have a lattice theoretic argument which shows that Γ_M acts on $(h^\perp)_{-2}$ with finitely many orbits. Thus, it is not clear that $\mathscr{F}^{M,h}$ is open in $\mathscr{F}_a^{\mathrm{M}}$ without using that ampleness is an open condition. We remark furthermore, that if we pick an element $h \in \mathrm{M}$ which is not formally ample, then there exists a $\delta \in (\mathrm{M} \cap h^{\perp})_{-2}$ and Ω_{M} is contained in the hyperplane H_{δ} defined by δ. Thus Ω_{M}^{h} would be empty.

5.2.3 The lattice $\mathfrak{h}$

For the rest of this Chapter we will denote by $\mathfrak{h}$ the rank 3 lattice which has the following intersection matrix with respect to a fixed ordered basis $\{h_1, h_2, h_3\}$

$$\mathfrak{h} = \begin{pmatrix} 14 & 16 & 5 \\ 16 & 16 & 6 \\ 5 & 6 & 0 \end{pmatrix}$$

and consider the moduli space $\mathscr{F}^{\mathfrak{h}}$. If $(\mathrm{S}, \varphi) \in \mathscr{F}^{\mathfrak{h}}$ then we denote

$$\varphi(h_1) = \mathscr{O}_{\mathrm{S}}(\mathrm{H}),\ \varphi(h_2) = \mathscr{O}_{\mathrm{S}}(\mathrm{C}) \text{ and } \varphi(h_3) = \mathscr{O}_{\mathrm{S}}(\mathrm{N}).$$

By abuse of notation we will sometimes also say, that $\{\mathscr{O}_{\mathrm{S}}(\mathrm{H}), \mathscr{O}_{\mathrm{S}}(\mathrm{C}), \mathscr{O}_{\mathrm{S}}(\mathrm{N})\}$ forms a basis of $\mathfrak{h}$.

After suitable sign changes and Picard-Lefschetz reflections we may assume that $\mathscr{O}_{\mathrm{S}}(\mathrm{H})$ is big and nef (see Remark 5.2.7 or [BHPVdV04, VIII, Prop 3.10]). To check the ampleness of a class, it is sufficient to compute the intersection with all smooth rational curves, that is, curves with self-intersection -2 (see [Huy15, Ch. 2 Prop. 1.4]). A Maple computation (see [BH17c]) shows that there are in fact many smooth rational curves on S and if $(\mathrm{S}, \varphi) \in \mathscr{F}^{\mathfrak{h}}$ such that $\varphi(\mathfrak{h}) = \mathrm{Pic}(\mathrm{S})$, then $\mathscr{O}_{\mathrm{S}}(\mathrm{H})$ intersects all of them positive. Hence $\mathscr{O}_{\mathrm{S}}(\mathrm{H})$ is ample. We summarize several properties of the other relevant classes All the statements in the follwoing remark follow from classical results in [SD74] (see also [Huy15, Ch. 2]) and lattice computations which are done in [BH17c].

Remark 5.2.15. We may assume that all basis elements of the lattice $\mathfrak{h}$ are effective. For a K3 surface $\mathrm{S} \in \mathscr{F}^{\mathfrak{h}}$ with $\mathrm{Pic}(\mathrm{S}) = \mathfrak{h}$, such that $\mathscr{O}_{\mathrm{S}}(\mathrm{H})$ is ample, one can check that

- $\mathscr{O}_{\mathrm{S}}(\mathrm{H})$ and $\mathscr{O}_{\mathrm{S}}(\mathrm{H}-\mathrm{N})$ are ample, base point free and the generic elements in the linear systems are smooth.

- $\mathcal{O}_S(C)$ is big and nef, base point free and the generic element in the linear system is smooth.
- $\mathcal{O}_S(N)$ is nef and base point free and can be represented by a smooth and irreducible elliptic curve.

Although the assumption $\mathrm{Pic}(S) = \mathfrak{h}$ is only satisfied for very general K3 surfaces in $\mathcal{F}^{\mathfrak{h}}$, all conditions above are open in the moduli space.

We remark furthermore, that for the lattice $\mathfrak{h}$ it can be checked that the ample class $\mathcal{O}_S(H)$ determines the classes $\mathcal{O}_S(C)$ and $\mathcal{O}_S(N)$ (with desired intersection numbers) uniquely.

For the rest of this Chapter we denote $\mathcal{F}_8^{\mathfrak{h}}$ the moduli space

$$\mathcal{F}_8^{\mathfrak{h}} = \left\{(S,\varphi) \;\middle|\; (S,\varphi) \in \mathcal{F}^{\mathfrak{h}} \text{ and } \mathcal{O}_S(H) \text{ ample}\right\}.$$

Recall from Section 5.2.2 that $\mathcal{F}_8^{\mathfrak{h}}$ is a Zariski open subset of $\mathcal{F}^{\mathfrak{h}}$. In particular, $\mathcal{F}_8^{\mathfrak{h}}$ is again a quasi-projective irreducible variety of dimension 17. Moreover, $\mathcal{F}_8^{\mathfrak{h}}$ is irreducible by [Dol96, Prop 5.9] for this particular lattice. In what follows, we will omit referring to the primitive lattice embedding $\varphi : \mathfrak{h} \to \mathrm{Pic}(S)$ for elements in $(S,\varphi) \in \mathcal{F}_8^{\mathfrak{h}}$ most of the time. K3 surfaces in $\mathcal{F}_8^{\mathfrak{h}}$ come with a distinguished polarization $S \to \mathbb{P}(H^0(S,\mathcal{O}_S(H))^*)$. Whenever we will consider the projective model $S \subset \mathbb{P}^8$ of a K3 surface $S \in \mathcal{F}_8^{\mathfrak{h}}$ we identify S with its image in $\mathbb{P}(H^0(S,\mathcal{O}_S(H))^*)$.

Since for generic $S \in \mathcal{F}_8^{\mathfrak{h}}$ the general element in the linear system $|\mathcal{O}_S(C)|$ is a smooth curve of genus 9, we may consider the open subset of the tautological $\mathbb{P}^9$-bundle over the moduli space $\mathcal{F}_8^{\mathfrak{h}}$

$$\mathcal{P}_8^{\mathfrak{h}} = \left\{(S,C) \mid S \in \mathcal{F}_8^{\mathfrak{h}} \text{ and } C \in |\mathcal{O}_S(C)| \text{ smooth}\right\}.$$

In the next section we prove that $\mathcal{P}_8^{\mathfrak{h}}$ is a $\mathbb{P}^1$-bundle over the universal Brill-Noether variety $\mathcal{W}_{9,6}^1$.

5.3 The space $\mathcal{P}_8^{\mathfrak{h}}$ as a $\mathbb{P}^1$-bundle over $\mathcal{W}_{9,6}^1$

In this section we prove the dominance of the morphism

$$\phi : \mathcal{P}_8^{\mathfrak{h}} \to \mathcal{W}_{9,6}^1,\ (S,C) \mapsto (C, \mathcal{O}_S(N) \otimes \mathcal{O}_C)$$

and conclude that $\mathscr{P}_8^{\mathfrak{h}}$ as well as $\mathscr{F}_8^{\mathfrak{h}}$ and $\mathscr{F}^{\mathfrak{h}}$ are unirational.

Some of the statements rely on a computational verification using *Macaulay2* [GS]. The *Macaulay2*-script, which verifies all these statements, can be found in [BH17b]. We start over by showing that there exist K3-surfaces with the desired properties.

Proposition 5.3.1.

(a) *There exists a smooth canonical genus* 9 *curve* C *together with a line bundle* $L \in W^1_6(C)$ *such that the relative canonical resolution has the form*

$$\begin{array}{ccccccccc}
 & & \mathscr{O}_{\mathbb{P}(\mathscr{E})}(-2H+R)^{\oplus 6} & & \mathscr{O}_{\mathbb{P}(\mathscr{E})}(-3H+2R)^{\oplus 2} & & & & \\
 & & \oplus & & \oplus & & & & \\
\mathscr{I}_{C/\mathbb{P}(\mathscr{E})} & \leftarrow & \mathscr{O}_{\mathbb{P}(\mathscr{E})}(-2H)^{\oplus 3} & \leftarrow & \mathscr{O}_{\mathbb{P}(\mathscr{E})}(-3H+R)^{\oplus 12} & \leftarrow & \mathscr{O}_{\mathbb{P}(\mathscr{E})}(-4H+2R)^{\oplus 3} & \leftarrow & \mathscr{O}_{\mathbb{P}(\mathscr{E})}(-6H+2R) \leftarrow 0. \\
 & & & & \oplus & & \oplus & & \\
 & & & & \mathscr{O}_{\mathbb{P}(\mathscr{E})}(-3H)^{\oplus 2} & & \mathscr{O}_{\mathbb{P}(\mathscr{E})}(-4H+R)^{\oplus 6} & &
\end{array}$$

(b) *There exists a syzygy* $s : \mathscr{O}_{\mathbb{P}(\mathscr{E})}(-3H+2R) \to \mathscr{O}_{\mathbb{P}(\mathscr{E})}(-2H+R)^{\oplus 6}$ *whose syzygy scheme defines a* K3 *surface* $S \in \mathscr{F}_8^{\mathfrak{h}}$. *In particular, the general elements in the linear series* $|\mathscr{O}_S(H)|, |\mathscr{O}_S(C)|, |\mathscr{O}_S(N)|$ *are smooth, irreducible and Clifford general.*

Proof. Using *Macaulay2*, we have implemented the construction of such curves together with the relative canonical resolution in [BH17b]. In our example the relative canonical resolution is of the form as stated in (a).

A syzygy $s \in \mathscr{O}_{\mathbb{P}(\mathscr{E})}(-3H+2R)^{\oplus 2}$ is a generalized column

$$s : \mathscr{O}_{\mathbb{P}(\mathscr{E})}(-3H+2R) \to \mathscr{O}_{\mathbb{P}(\mathscr{E})}(-2H+R)^{\oplus 6}$$

of the 6×2 submatrix of the relative canonical resolution of $C \subset \mathbb{P}(\mathscr{E})$. The entries of s span the four-dimensional vector space $H^0(\mathbb{P}(\mathscr{E}), \mathscr{O}_{\mathbb{P}(\mathscr{E})}(H-R))$. Let $f_1, \dots, f_6$ be the generators of $\mathscr{I}_{C/\mathbb{P}(\mathscr{E})}$ corresponding to $\mathscr{O}_{\mathbb{P}(\mathscr{E})}(-2H+R)^{\oplus 6}$. By definition of s we have $(f_1, \dots, f_6) \cdot s = 0$. After a base change we may assume that s is of the form

$$s = (s_1, s_2, s_3, s_4, 0, 0)^t.$$

Applying this base change to $f_1, \dots, f_6$, we get new generators $f_1', \dots, f_6'$ such that

$$(f_1', \dots, f_6') \cdot (s_1, s_2, s_3, s_4, 0, 0)^t = 0.$$

In this case the syzygy scheme associated to s is given by $\mathscr{I}_{syz(s)} = \langle f_1', \dots, f_4' \rangle$. For the general definition of a syzygy scheme see [GvB07].

Again by [BH17b], the image of the syzygy scheme in the scroll X, swept out by $|L|$, is the union of its vertex and a K3 surface $S \subset X \subset \mathbb{P}^8$. Hence, after saturating with the vertex, we obtain a K3-surface $S \subset \mathbb{P}^8$ of degree 14 such that the ruling on X defines an elliptic curve N on S and the hyperplane section H is a canonical curve of genus 8. The intersection products of the classes $\{\mathscr{O}_S(H), \mathscr{O}_S(C), \mathscr{O}_S(N)\}$ define the lattice $\mathfrak{h}$. □

Lemma 5.3.2. *Let* $(S,C) \in \mathscr{P}_8^{\mathfrak{h}}$ *be general. Then* $L = \mathscr{O}_S(N) \otimes \mathscr{O}_C$ *defines a* g^1_6 *on* C *such that* S *is contained in the scroll* $X = \bigcup_{D \in |L|} \overline{D}$ *swept out by* $|L|$.

Proof. Let $H \in |\mathscr{O}_S(H)|$ be a general element and let $N \in |\mathscr{O}_S(N)|$ be an elliptic curve of degree 5. Assume that the span $\overline{N} \cong \mathbb{P}^3$ is three-dimensional. Then the intersection $N \cap H$ consists of 5 points and the span $\overline{N \cap H}$ is a $\mathbb{P}^2$. But this would give a g^2_5 by the geometric version of Riemann-Roch. Because of the genus formula we have $W^2_5(H) = \emptyset$, a contradiction. Thus, N is an elliptic normal curve and $\overline{N} \cong \mathbb{P}^4$.

Now since $S \subset \bigcup_{N \in |\mathscr{O}_S(N)|} \overline{N}$ it remains to show that $\overline{N \cap C} \cong \mathbb{P}^4$. The intersection $N \cap C$ consists of 6 points. Assume that these 6 points only span a hyperplane $h \cong \mathbb{P}^3 \subset \mathbb{P}^4$. Then $\deg(h \cap N) > \deg(N)$ which means, by Bézout, that $h \cap N$ is a component of N. Thus, the general N is reducible, a contradiction by Remark 5.2.15. □

Lemma 5.3.3. *Let* $(S,C) \in \mathscr{P}_8^{\mathfrak{h}}$ *be general and* $L = \mathscr{O}_S(N) \otimes \mathscr{O}_C$ *such that the relative canonical resolution of* $C \subset \mathbb{P}(\mathscr{E})$ *has a balanced first syzygy bundle. If we further assume that* $S \subset \mathbb{P}(\mathscr{E})$*, where* $\mathbb{P}(\mathscr{E})$ *is the scroll associated to* L*, then* $S \subset \mathbb{P}(\mathscr{E})$ *has a resolution of the form*

$$0 \leftarrow \mathscr{O}_{S/\mathbb{P}(\mathscr{E})} \leftarrow \begin{matrix} \mathscr{O}_{\mathbb{P}(\mathscr{E})}(-2H+R)^{\oplus 4} \\ \oplus \\ \mathscr{O}_{\mathbb{P}(\mathscr{E})}(-2H)^{\oplus 1} \end{matrix} \xleftarrow{\Psi} \begin{matrix} \mathscr{O}_{\mathbb{P}(\mathscr{E})}(-3H+2R)^{\oplus 1} \\ \oplus \\ \mathscr{O}_{\mathbb{P}(\mathscr{E})}(-3H+R)^{\oplus 4} \end{matrix} \leftarrow \mathscr{O}_{\mathbb{P}(\mathscr{E})}(-5H+2R) \leftarrow 0$$

for a skewsymmetric matrix Ψ *and is generated by the* 5 *Pfaffians of the matrix* Ψ.

Proof. The surface $S \subset \mathbb{P}(\mathscr{E})$ is Gorenstein of codimension 3, and therefore it follows by the Buchsbaum-Eisenbud structure theorem [BE77], that S is generated by the Pfaffians of a skew-symmetric matrix Ψ and has (up to twist) a self-dual resolution. The shape of the resolution of $S \subset \mathbb{P}(\mathscr{E})$ is the same as the shape of the resolution of $S \cap H \subset \mathbb{P}(\mathscr{E}) \cap H$

for a general hyperplane H. Since we assume $(S,C) \in \mathscr{P}_8^{\mathfrak{h}}$ to be general, $S \cap H$ is a 5-gonal genus 8 curve (as in Proposition 5.3.1) and $\mathbb{P}(\mathscr{E}) \cap H$ is a 4 dimensional variety of degree 4, hence isomorphic to a scroll $\mathbb{P}(\mathscr{E}')$. By [Sch86] we know that $S \cap H \subset \mathbb{P}(\mathscr{E}')$ is generated by the 5 Pfaffians of a skew-symmetric 5×5 matrix and therefore also Ψ needs to be a 5×5 matrix. It remains to determine the balancing type. By our assumption $C \subset \mathbb{P}(\mathscr{E})$ has a balanced first syzygy bundle as in Proposition 5.3.1. Since the relative linear strand of the resolution of $S \subset \mathbb{P}(\mathscr{E})$ is a subcomplex of the relative linear strand in the resolution of $C \subset \mathbb{P}(\mathscr{E})$, we obtain that the resolution of $S \subset \mathbb{P}(\mathscr{E})$ has the following form

$$0 \leftarrow \mathscr{I}_{S/\mathbb{P}(\mathscr{E})} \leftarrow \begin{matrix} \mathscr{O}_{\mathbb{P}(\mathscr{E})}(-2H+R)^{\oplus a_1} \\ \oplus \\ \mathscr{O}_{\mathbb{P}(\mathscr{E})}(-2H)^{\oplus a_2} \end{matrix} \xleftarrow{\Psi} \begin{matrix} \mathscr{O}_{\mathbb{P}(\mathscr{E})}(-3H+2R)^{\oplus b_2} \\ \oplus \\ \mathscr{O}_{\mathbb{P}(\mathscr{E})}(-3H+R)^{\oplus b_1} \end{matrix} \leftarrow \mathscr{O}_{\mathbb{P}(\mathscr{E})}(-5H+2R) \leftarrow 0$$

with $a_i = b_i$ for $i = 1,2$ and $a_1 + a_2 = 5$. By taking the first Chern classes of the bundles in the resolution above we get

$$\big(b_2 \cdot (-3H+2R) + b_1 \cdot (-3H+R)\big) - \big(a_1 \cdot (-2H+R) + a_2 \cdot (-2H)\big) = (-5H+2R).$$

and hence, $2b_2 + b_1 - a_1 = 2$. Therefore, $b_2 = 1$ and the only possible shape for the resolution $S \subset \mathbb{P}(\mathscr{E})$ is the one in the lemma. □

Corollary 5.3.4. *Let* $(S,C) \in \mathscr{P}_8^{\mathfrak{h}}$ *be a general element and let* $L = \mathscr{O}_S(N) \otimes \mathscr{O}_C$. *Then the relative canonical resolution of* $C \subset \mathbb{P}(\mathscr{E})$ *has an unbalanced second syzygy bundle where* $\mathbb{P}(\mathscr{E})$ *is the scroll associated to* L.

Proof. For a general pair $(S,C) \in \mathscr{P}_8^{\mathfrak{h}}$ the class $\mathscr{O}_S(N)$ is nef. Thus by Lemma 5.3.2 it follows that S is contained in the scroll $\mathbb{P}(\mathscr{E})$ defined by $L = \mathscr{O}_S(N) \otimes \mathscr{O}_C$. Note that having a balanced first syzygy bundle in the relative canonical resolution is an open condition. Therefore, by Proposition 5.3.1 $C \subset \mathbb{P}(\mathscr{E})$ has a balanced first syzygy bundle and we can apply the previous lemma.

Since the relative linear strand of $S \subset \mathbb{P}(\mathscr{E})$ is a subcomplex of the relative linear strand of the resolution of $C \subset \mathbb{P}(\mathscr{E})$, it follows from Lemma 5.3.3 that the resolution of $C \subset \mathbb{P}(\mathscr{E})$ has an unbalanced second syzygy bundle. □

By the above corollary it follows for $(S,C) \in \mathscr{P}_8^{\mathfrak{h}}$ general that $C \subset \mathbb{P}(\mathscr{E})$ has a second syzygy bundle of the form

$$\mathscr{O}_{\mathbb{P}(\mathscr{E})}(-3H+2R)^{\oplus a} \oplus \mathscr{O}_{\mathbb{P}(\mathscr{E})}(-3H+R)^{\oplus(16-2a)} \oplus \mathscr{O}_{\mathbb{P}(\mathscr{E})}(-3H)^{\oplus a},$$

for some $a \geq 1$. The next lemma relates the balancing type of the second syzygy bundle to the fiberdimension of the morphism $\phi : \mathscr{P}_8^{\mathfrak{h}} \to \mathscr{W}_{9,6}$.

Lemma 5.3.5. *Let* $(S,C) \in \mathscr{P}_8^{\mathfrak{h}}$ *and* $L = \mathscr{O}_S(N) \otimes \mathscr{O}_C$ *such that the relative resolution of* $S \in \mathbb{P}(\mathscr{E})$ *is of the form as in Lemma 5.3.3. Then the* K3 *surface* S *is uniquely determined by subcomplex*

$$0 \leftarrow \mathscr{O}_{S/\mathbb{P}(\mathscr{E})} \leftarrow \mathscr{O}_{\mathbb{P}(\mathscr{E})}(-2H+R)^{\oplus 4} \leftarrow \mathscr{O}_{\mathbb{P}(\mathscr{E})}(-3H+2R)^{\oplus 1}$$

of the relative canonical resolution of $C \subset \mathbb{P}(\mathscr{E})$. *In particular, the fiber dimension of* ϕ *is bounded by* $a-1$ *where* a *is the rank of the sub bundle* $\mathscr{O}_{\mathbb{P}(\mathscr{E})}(-3H+2R)^{\oplus a}$ *in the relative canonical resolution of* $C \subset \mathbb{P}(\mathscr{E})$.

Proof. Let $q_1,\dots,q_4 \in H^0(\mathbb{P}(\mathscr{E}), \mathscr{O}_{\mathbb{P}(\mathscr{E})}(2H-R))$ be the entries of the matrix $\mathscr{O}_{S/\mathbb{P}(\mathscr{E})} \leftarrow \mathscr{O}_{\mathbb{P}(\mathscr{E})}(-2H+R)^{\oplus 4}$ and $l_1,\dots,l_4 \in H^0(\mathbb{P}(\mathscr{E}), \mathscr{O}_{\mathbb{P}(\mathscr{E})}(H-R))$ be the entries of $\mathscr{O}_{\mathbb{P}(\mathscr{E})}(-2H+R)^{\oplus 4} \leftarrow \mathscr{O}_{\mathbb{P}(\mathscr{E})}(-3H+2R)^{\oplus 1}$. Then by [Sch91, Lemma 4.2] there exists a skew-symmetric 4×4 matrix $A = (a_{i,j})_{i,j=1,\dots,4}$ such that

$$q_i = \sum_{j=1}^{4} a_{i,j} l_i.$$

and the 5th Pfaffian $q_5 \in H^0(\mathbb{P}(\mathscr{E}), \mathscr{O}_{\mathbb{P}(\mathscr{E})}(2H))$ defining the surface S is given as Pf(A). So $q_1,\dots q_5$ are the Pfaffians of the 5×5 matrix

$$\psi = \begin{pmatrix} 0 & -l_1 & -l_2 & -l_3 & -l_4 \\ l_1 & 0 & -a_{3,4} & a_{2,4} & -a_{2,3} \\ l_2 & a_{3,4} & 0 & a_{1,4} & -a_{1,3} \\ l_3 & -a_{2,4} & -a_{1,4} & 0 & a_{1,2} \\ l_4 & a_{2,3} & a_{1,3} & -a_{1,2} & 0 \end{pmatrix}$$

Considering the Koszul resolution associated to the section $(l_1,\dots,l_4) \in H^0(\mathscr{O}_{\mathbb{P}(\mathscr{E})}(H-R))^4$ we get

$$\bigwedge^3 \mathscr{O}(-H+R)^4 \to \bigwedge^2 \mathscr{O}(-H+R)^4 \to \mathscr{O}(-H+R)^4 \to \mathscr{O}$$

with $\bigwedge^2 \mathscr{O}(-H+R)^4 = \mathscr{O}(-2H+2R)^6$ and $\bigwedge^3 \mathscr{O}(-H+R)^4 = \mathscr{O}(-3H+3R)^4$. Tensoring the whole sequence with $\mathscr{O}(3H-2R)$ we get

$$\begin{tikzcd}
 & & \mathscr{O}_{\mathbb{P}(\mathscr{E})} \arrow[dll] \arrow[d, "\exists!"] \arrow[dr, "(q_1,\dots,q_4)^t"] & & \\
\bigwedge^3 \mathscr{O}^4_{\mathbb{P}(\mathscr{E})} \otimes \mathscr{O}_{\mathbb{P}(\mathscr{E})}(R) \arrow[r, "\varphi"] & \bigwedge^2 \mathscr{O}^4_{\mathbb{P}(\mathscr{E})} \otimes \mathscr{O}_{\mathbb{P}(\mathscr{E})}(H) \arrow[r] & \mathscr{O}_{\mathbb{P}(\mathscr{E})}(2H-R)^4 \arrow[r, "(l_1,\dots,l_4)"] & \mathscr{O}_{\mathbb{P}(\mathscr{E})}(3H-2R) \arrow[r] & 0
\end{tikzcd}$$

The space $H^0(\mathbb{P}(\mathscr{E}), \bigwedge^2 \mathscr{O}^4_{\mathbb{P}(\mathscr{E})} \otimes \mathscr{O}_{\mathbb{P}(\mathscr{E})}(H))$ parametrizes skew-symmetric 4×4 matrices with entries in $H^0(\mathbb{P}(\mathscr{E}), \mathscr{O}_{\mathbb{P}(\mathscr{E})}(H))$. Fixing the 4 Pfaffians $q_1, \dots, q_4$ together with their syzygy $(l_1, \dots, l_4)$ we see that the matrix A and hence the matrix Ψ is unique up to the image of φ. We identify an element $e_i \wedge e_j \in H^0(\mathbb{P}(\mathscr{E}), \bigwedge^2 \mathscr{O}^4_{\mathbb{P}(\mathscr{E})} \otimes \mathscr{O}_{\mathbb{P}(\mathscr{E})}(H))$ with the skew-symmetric matrix where the index of the only non-zero entries is precisely $\{k, l\} = \{1, \dots, 4\} \setminus \{i, j\}$. The image of φ consists of those matrices which are obtained by the operation of the first column (resp. row) of Ψ on A which respects the skew-symmetric structure. □

Theorem 5.3.6. *The morphism*

$$\phi : \mathscr{P}_8^{\mathfrak{h}} \to \mathscr{W}_{9,6}^1, \ (S, C) \mapsto \big(C, \mathscr{O}_S(N) \otimes \mathscr{O}_C\big)$$

is dominant.

Proof. The morphism $\phi : \mathscr{P}_8^{\mathfrak{h}} \to \mathscr{W}_{9,6}^1$ is locally of finite type since $\mathscr{P}_8^{\mathfrak{h}}$ and $\mathscr{W}_{9,6}^1$ are algebraic quasi-projective varieties (and hence schemes of finite type over $\Bbbk$). Therefore by Chevalley's Theorem [Gro66, Thm. 13.1.3] the map $\mathscr{P}_8^{\mathfrak{h}} \to \mathbb{Z}, \ x \mapsto \dim_x \phi^{-1}(\phi(x))$ is upper semicontinuous.

By Proposition 5.3.1 we obtain a point $(C, L) \in \mathscr{W}_{9,6}^1$ in the image of ϕ. The preimage in part (b) of Proposition 5.3.1, constructed via syzygy schemes, satisfies all generality assumptions in the previous lemmata. Now Lemma 5.3.2 implies that a general K3 surface in the fiber over (C, L) is contained in the 5-dimensional scroll $\mathbb{P}(\mathscr{E})$, defined by the pencil L on C.

By Corollary 5.3.4 it follows that such K3 surfaces $S \subset \mathbb{P}(\mathscr{E})$ in the fiber are defined by the Pfaffians of a skew symmetric 5×5 matrix

$$\begin{matrix} \mathscr{O}_{\mathbb{P}(\mathscr{E})}(-2H+R)^{\oplus 4} & & \mathscr{O}_{\mathbb{P}(\mathscr{E})}(-3H+2R)^{\oplus 1} \\ \oplus & \xleftarrow{\Psi} & \oplus \\ \mathscr{O}_{\mathbb{P}(\mathscr{E})}(-2H)^{\oplus 1} & & \mathscr{O}_{\mathbb{P}(\mathscr{E})}(-3H+R)^{\oplus 4}. \end{matrix}$$

Since the relative linear strand of $S \subset \mathbb{P}(\mathscr{E})$ is a subcomplex of the relative linear strand of $C \in \mathbb{P}(\mathscr{E})$, it follows from the shape of resolution and Lemma 5.3.5 that the fiber over (C, L) is at most 1-dimensional. By semicontinuity it follows that $\dim_x \phi^{-1}(\phi(x)) \leq 1$ for all x in some open subset $U \subset \mathscr{P}_8^{\mathfrak{h}}$. Now since ϕ is a morphism of algebraic quasi-projective varieties, we have a dominant map $\phi : \mathscr{P}_8^{\mathfrak{h}} \to \overline{\mathrm{Im}(\phi)}$. The space $\mathscr{P}_8^{\mathfrak{h}}$ is equidimensional and we get

$$\dim \mathscr{P}_8^{\mathfrak{h}} = \dim \overline{\mathrm{Im}(\phi)} + \dim_x \phi^{-1}(\phi(x)) \leq \dim \mathscr{W}_{9,6}^1 + 1.$$

Since $\dim \mathscr{P}_8^{\mathfrak{h}} = 26$ and $\dim \mathscr{W}_{9,6}^1 = 25$, we obtain $\dim \overline{\mathrm{Im}(\phi)} = \dim \mathscr{W}_{9,6}^1$. The universal Brill–Noether variety $\mathscr{W}_{9,6}^1$ is irreducible and therefore it follows that the image of ϕ and hence $\phi(U)$ is also dense in $\mathscr{W}_{9,6}^1$. □

Corollary 5.3.7. *The general fiber of ϕ is a rational curve parametrized by syzygy schemes as in part (b) of Proposition 5.3.1. The moduli space $\mathscr{P}_8^{\mathfrak{h}}$ is birational to a $\mathbb{P}^1$-bundle over an open subset of $\mathscr{W}_{9,6}^1$. In particular $\mathscr{P}_8^{\mathfrak{h}}$, $\mathscr{F}_8^{\mathfrak{h}}$ and hence $\mathscr{F}^{\mathfrak{h}}$ are unirational.*

Proof. By Theorem 5.3.6 the map $\phi : \mathscr{P}_8^{\mathfrak{h}} \to \mathscr{W}_{9,6}^1$ is dominant. Thus by Proposition 5.3.1 the general element in $\mathscr{W}_{9,6}^1$ has a relative canonical resolution with second syzygy bundle of the form

$$\mathscr{O}_{\mathbb{P}(\mathscr{E})}(-3H + 2R)^{\oplus 2} \oplus \mathscr{O}_{\mathbb{P}(\mathscr{E})}(-3H + R)^{\oplus 12} \oplus \mathscr{O}_{\mathbb{P}(\mathscr{E})}(-3H)^{\oplus 2}$$

and therefore, by the dominance of ϕ and Lemma 5.3.5, the construction in Proposition 5.3.1 holds in an open set. To be more precise, over an open subset of of $\mathscr{W}_{9,6}^1$ the syzygy schemes defined by syzygies in the free $\mathscr{O}_{\mathbb{P}(\mathscr{E})}$-module $\mathscr{O}_{\mathbb{P}(\mathscr{E})}(-3H + 2R)^{\oplus 2}$ correspond (after saturating with the vertex of the scroll) to the K3 surfaces in the fiber of ϕ. Therefore we obtain a birational map

$$\widetilde{\phi} : \mathscr{P}_8^{\mathfrak{h}} \to \widetilde{\mathscr{W}_{9,6}^1}$$

where

$$\widetilde{\mathscr{W}_{9,6}^1} = \left\{ (C, L, s) \;\middle|\; (C, L) \in \mathscr{W}_{9,6}^1,\ s \in \mathscr{O}_{\mathbb{P}(\mathscr{E})}(-3H + 2R)^{\oplus 2} \right\}$$

is a $\mathbb{P}^1$-bundle over an open dense subset of $\mathscr{W}_{9,6}^1$. Now $\mathscr{W}_{9,6}^1$ is classically known to be unirational (see [Seg28] and [AC81]) and hence $\mathscr{P}_8^{\mathfrak{h}}$ is unirational as well. □

Corollary 5.3.8. *For any* $(C,L) \in \mathscr{W}^1_{9,6}$, *the relative canonical resolution has an unbalanced second syzygy bundle.*

Proof. Having a balanced second syzygy bundle is an open condition in $\mathscr{W}^1_{9,6}$ (by semicontinuity of $h^1(\mathbb{P}^1, \mathscr{E}nd(N_2))$). The claim follows from the fact that the general point in $\mathscr{W}^1_{9,6}$ has an unbalanced second syzygy bundle by Corollary 5.3.7. □

Remark 5.3.9. There exists an unirational codimension 4 subvariety $V \subset \mathscr{W}^1_{9,6}$, parametrizing pairs (C,L) such that C is the rank one locus of a certain 3×3 matrix defined on the scroll $\mathbb{P}(\mathscr{E})$ swept out by $|L|$ (see [Gei13, Section 4.3]).

Although there is in general no structure theorem for resolutions of Gorenstein subschemes of codimension ≥ 4, the relative canonical resolution of elements parametrized by V is given by a so-called Gulliksen–Negard complex.

It is easy to check that the splitting type of the bundles in the Gulliksen–Negard complex are the same as in Proposition 5.3.1. However, the subvariety V does not lie in the image of the map $\phi : \mathscr{P}^{\mathfrak{h}}_8 \to \mathscr{W}_{9,6}$. Indeed, since curves parametrized by V are degeneracy loci of 3×3 matrices, all linear syzygies (as in Proposition 5.3.1 (b)) have rank 3. Therefore, the corresponding syzygy schemes do not define K3 surfaces.

5.4 A birational description of $\mathscr{W}^3_{9,10}$

The Serre dual of a g^1_6 on a general genus 9 curve C is a g^3_{10} defining an embedding into $\mathbb{P}^3$. Let C' be the image of a general genus 9 curve C under the residual map

$$C \xrightarrow{|\omega\otimes L^{-1}|} C' \subset \mathbb{P}^3.$$

Then all maps in the long exact cohomology sequence induced by the sequence

$$0 \to \mathscr{I}_{C'/\mathbb{P}^3}(n) \to \mathscr{O}_{\mathbb{P}^3}(n) \to \mathscr{O}_{C'}(n) \to 0$$

have maximal rank and C' is contained in a net of quartics whose general element is smooth (see [BH17b]). Let $\mathfrak{n}$ be the rank $r \geq 2$ Picard lattice of a very general quartic in this family. We fix a basis $\{n_1, n_2, \dots\}$ for $\mathfrak{n}$ such that $n_1^2 = 4, n_2^2 = 16$ and $n_1 n_2 = 10$ and consider the moduli space

$$\mathscr{F}^{\mathfrak{n}}_3 = \left\{(S,\varphi) \;\middle|\; (S,\varphi) \in \mathscr{F}^{\mathfrak{n}} \text{ and } \mathscr{O}_S(H') = \varphi(n_1) \text{ ample } \right\}$$

and the open subset of the tautological bundle

$$\mathscr{P}_3^{\mathfrak{n}} = \left\{(S,\varphi,C) \;\middle|\; (S,\varphi) \in \mathscr{F}_3^{\mathfrak{n}} \text{ and } C \in |\mathscr{O}_S(C)| \text{ smooth } \right\}$$

where $\mathscr{O}_S(C) = \varphi(n_2)$. We get a dominant map

$$\mathscr{P}_3^{\mathfrak{n}} \to \mathscr{W}^3_{9,10} \cong \mathscr{W}^1_{9,6}$$

whose general fiber has dimension 2. Now, since

$$\dim \mathscr{P}_3^{\mathfrak{n}} = \dim \mathscr{F}_3^{\mathfrak{n}} + \dim |C'| = (20 - r) + 9 = \dim \mathscr{W}^1_{9,6} + 2 = 27,$$

we see that $\mathfrak{n}$ is a rank 2 lattice and hence the Picard lattice of a very general K3 surface in $\mathscr{F}_3^{\mathfrak{n}}$ is generated by the class of a plane quartic and the class of C'. As a consequence of Theorem 5.3.6 and Corollary 5.3.7 we now obtain:

Corollary 5.4.1. *With notation as above $\mathscr{P}_3^{\mathfrak{n}} \to \mathscr{W}^3_{9,10}$ is a $\mathbb{P}^2$-bundle over an open subset of $\mathscr{W}^3_{9,10}$. The general fiber contains a rational curve parametrizing* K3 *surface contained in $\mathscr{F}_8^{\mathfrak{h}}$.*

Proposition 5.4.2. *There exists a pair $(C', \omega_C \otimes L^{-1}) \in \mathscr{W}^3_{9,10}$ such that*

(1) $V = H^0(\mathbb{P}^3, \mathscr{I}_{C'}(4))$ *is* 3*-dimensional,*

(2) he plane rational curve Γ in $\mathbb{P}(V)$, whose points correspond to K3 *surfaces given as syzygy schemes as in Proposition 5.3.1, has degree* 3 *and*

(3) the abstract K3 *surface S_p corresponding to the unique singular point p of the rational curve Γ has a smooth model in $\mathbb{P}^3$.*

Proof. We verify the above statement in our *Macaulay2*-script [BH17b]. □

In the following we describe the Picard lattice of S_p. Recall that the linear syzygy in the relative canonical resolution of a surface S in $\mathscr{F}_8^{\mathfrak{h}}$ determines the polarized K3 surface $(S, \mathscr{O}_S(H))$ uniquely by Proposition 5.3.5. Hence, all K3 surfaces $(S, \mathscr{O}_S(H))$ given as syzygy schemes as in Proposition 5.3.1 are non-isomorphic (as polarized K3 surfaces). Therefore, to be a singular point of the rational curve Γ means that there are two K3 surfaces in $\mathbb{P}^8$ mapping to the same quartic in $\mathbb{P}^3$. In other words, the Picard group $\mathrm{Pic}(S_p)$ contains two (pseudo-) polarizations $\mathscr{O}_{S_p}(H_1)$ and $\mathscr{O}_{S_p}(H_2)$ (and corresponding elliptic classes $\mathscr{O}_{S_p}(N_i)$, $i = 1,2$) such that $H_i^2 = 14$, $H_i.C = 16$ and $|\mathscr{O}_{S_p}(H_1 - N_1)| = |\mathscr{O}_{S_p}(H_2 - N_2)|$. Note that the image of S_p in $\mathbb{P}^3$ is given by $|\mathscr{O}_{S_p}(H_i - N_i)|$.

Since by Section 5.2.3 fixing two basis elements C and N (or equivalently $(H-N)$) for the lattice $\mathfrak{h}$ determines the third class H uniquely, it follows that $\mathrm{Pic}(S_p)$ has rank at least 4.

Thus, $\mathrm{Pic}(S_p)$ contains a lattice of the following form

$$\begin{pmatrix} 14 & 16 & 5 & a \\ 16 & 16 & 6 & 16 \\ 5 & 6 & 0 & b \\ a & 16 & b & 14 \end{pmatrix}$$

with respect to an ordered basis $\{\mathscr{O}_{S_p}(H_1), \mathscr{O}_{S_p}(C), \mathscr{O}_{S_p}(N_1), \mathscr{O}_{S_p}(H_2)\}$ and a, b integers. Using that $(C - H_i)$ is a (-2)-curve for $i = 1,2$, an easy computation yields $a = 16$ and $b = 6$. Note that $H_1.(C - H_2) = 0$ and therefore $\mathscr{O}_{S_p}(H_i)$ does not define an ample class on S_p. Hence, the surface S_p lies in the boundary of $\mathscr{F}_8^{\mathfrak{h}}$.

If we change the basis of the above lattice to

$$\{\mathscr{O}_{S_p}(H') = \mathscr{O}_{S_p}(H_i - N_i), \mathscr{O}_{S_p}(C), \mathscr{O}_{S_p}(Q_1) = \mathscr{O}_{S_p}(C - H_1), \mathscr{O}_{S_p}(Q_2) = \mathscr{O}_{S_p}(C - H_2)\},$$

then the corresponding intersection matrix has the following form

$$\begin{pmatrix} 4 & 10 & 1 & 1 \\ 10 & 16 & 0 & 0 \\ 1 & 0 & -2 & 0 \\ 1 & 0 & 0 & -2 \end{pmatrix}.$$

.

We denote by $\mathfrak{h}'$ be the (abstract) rank 4 lattice which is defined by the intersection matrix above with respect to some fixed basis $\{h_1', h_2', h_3', h_4'\}$. For a lattice polarized K3 surface $(S, \varphi) \in \mathscr{F}^{\mathfrak{h}'}$ we denote

$$\varphi(h_1') = \mathscr{O}_S(H'),\ \varphi(h_2') = \mathscr{O}_S(C), \varphi(h_3') = \mathscr{O}_S(Q_1) \text{ and } \varphi(h_4') = \mathscr{O}_S(Q_2).$$

Again, we will omit referring to the primitive lattice embedding $\varphi : \mathfrak{h}' \to \mathrm{Pic}(S)$ for elements $(S, \varphi) \in \mathscr{F}^{\mathfrak{h}'}$ and we will say that $\{\mathscr{O}_S(H'), \mathscr{O}_S(C), \mathscr{O}_S(Q_1), \mathscr{O}_S(Q_2)\}$ forms a basis of $\mathfrak{h}'$.

As for the lattice $\mathfrak{h}$ one can check using *Maple* (see [BH17c]) that for a surface $S \in \mathscr{F}^{\mathfrak{h}'}$ with $\mathrm{Pic}(S) = \mathfrak{h}'$ the class $\mathscr{O}_S(H')$ is ample. We consider again the open subset

$$\mathscr{F}_3^{\mathfrak{h}'} := \left\{ S \;\middle|\; S \in \mathscr{F}^{\mathfrak{h}'} \text{ and } \mathscr{O}_S(H') \text{ ample} \right\}$$

of the moduli space $\mathscr{F}^{\mathfrak{h}'}$ and the open subset of the tautological $\mathbb{P}^9$-bundle over $\mathscr{F}_8^{\mathfrak{h}'}$

$$\mathscr{P}_3^{\mathfrak{h}'} = \left\{ (S, C) \;\middle|\; S \in \mathscr{F}_3^{\mathfrak{h}'} \text{ and } C \in |\mathscr{O}_S(C)| \text{ smooth} \right\}.$$

Furthermore, the class $\mathscr{O}_S(H')$ determines the classes $\mathscr{O}_S(C), \mathscr{O}_S(Q_1)$ and $\mathscr{O}_S(Q_2)$ (with desired intersection numbers) uniquely. Hence, we get generic injections

$$\mathscr{F}_3^{\mathfrak{h}'} \hookrightarrow \mathscr{F}_3^{\mathfrak{h}} \hookrightarrow \mathscr{F}_3^{\mathfrak{n}} \hookrightarrow \mathscr{F}_3$$

into the moduli space of polarized K3 surfaces of genus 3.

Theorem 5.4.3. *The morphism*

$$\phi' : \mathscr{P}_3^{\mathfrak{h}'} \to \mathscr{W}^3_{9,10}, \; (S, C) \mapsto (C, \mathscr{O}_S(H') \otimes \mathscr{O}_C)$$

defines a birational equivalence. In particular $\mathscr{P}_3^{\mathfrak{h}'}$, $\mathscr{F}_3^{\mathfrak{h}'}$ and $\mathscr{F}^{\mathfrak{h}'}$ are unirational.

Proof. We proceed as in the proof of Theorem 5.3.6. By Proposition 5.4.2 and the preceeding discussion there exists a pair $(C, \omega_C \otimes L^{-1}) \in \mathscr{W}^3_{9,10}$ in the image of the map ϕ'. Furthermore, every point in the fiber corresponds to a singular point of rational curve Γ as in Proposition 5.4.2. Indeed, the spaces $\mathscr{F}_8^{\mathfrak{h}}$ and $\mathscr{F}_3^{\mathfrak{h}} := \{S \mid S \in \mathscr{F}^{\mathfrak{h}} \text{ and } \mathscr{O}_S(H') = \varphi(h_1 - h_3) \text{ ample}\}$ are birational (the mapping $\mathscr{O}_S(H) \mapsto \mathscr{O}_S(H - N)$ is defined on an open subset and for a very general K3 surface $S \in \mathscr{F}^{\mathfrak{h}}$, it is equivalent to choose a polarization $\mathscr{O}_S(H)$ or a polarization $\mathscr{O}_S(H - N)$). Therefore, by Theorem 5.3.6 and Corollary 5.3.7 we get a dominant morphism $\widetilde{\phi} : \mathscr{P}_3^{\mathfrak{h}} \to \mathscr{W}^3_{9,10}$ whose fibers are rational curves which we identify with Γ. Hence, the fiber of ϕ' is contained in the fiber of the map $\widetilde{\phi}$ and we get the following diagram

$$
\begin{tikzcd}
\mathrm{Sing}(\Gamma) \arrow[r, hook] \arrow[ddr, maps to, "\phi'"'] & \Gamma \arrow[r, hook] \arrow[dd, maps to, "\widetilde{\phi}"] & \mathbb{P}^2 \arrow[ddl, maps to] & & \mathscr{P}_3^{\mathfrak{h}'} \arrow[r, hook, dashed] \arrow[ddr, "\phi'"'] & \mathscr{P}_3^{\mathfrak{h}} \arrow[r, hook, dashed] \arrow[dd, "\widetilde{\phi}"] & \mathscr{P}_3^{\mathfrak{n}} \arrow[ddl, "\mathbb{P}^2\text{-bundle}"] \\
& & & \arrow[r, hook, dashed] & {} & & \\
& (C, \omega_C \otimes L^{-1}) & & \in & & \mathscr{W}_{10,9}^3 &
\end{tikzcd}
$$

The dimension of $\mathscr{P}_3^{\mathfrak{h}'}$ is

$$\dim \mathscr{P}_3^{\mathfrak{h}'} = 20 - \mathrm{rk}\mathfrak{h}' + g(C) = 16 + 9 = 25 = \dim \mathscr{W}_{9,10}^3$$

and both spaces are irreducible. Thus, by upper-semicontinuity on the fiber dimension the map ϕ' is generically finite and dominant.

Recall that in the example of Proposition 5.4.2 the fiber of $\widetilde{\phi}$ is a rational plane cubic, and hence, has a unique singular point. In the last part of our ***Macaulay2***-file [BH17b] we verify that this is the generic behaviour: A general pair $(C, \omega_C \otimes L^{-1})$ in the image of $\widetilde{\phi}$ gives rise to an unbalanced relative canonical resolution as in Proposition 5.3.1. The rational curve of K3 surfaces given as the fiber of $\widetilde{\phi}$ corresponds to a one-dimensional family of generic syzygy schemes cut out by the maximal Pfaffians of 5×5 skew-symmetric matrices. We show that the one-dimensional family of such matrices with indeterminant coefficients always induces a rational cubic. We conclude that the dominant morphism $\phi' : \mathscr{P}_3^{\mathfrak{h}'} \to \mathscr{W}_{9,10}^3$ is generically injective and therefore defines a birational equivalence. □

5.5 Outlook and open problems

In [Muk02] Mukai showed that a transversal linear section $\mathbb{P}^8 \cap G(2,6) \subset \mathbb{P}^{14}$ of the embedded Grassmannian $G(2,6) \subset \mathbb{P}^{14}$ is a Brill-Noether general K3 surface and that every Brill-Noether general K3 surface arises in this way.

One can show that a very general surface $S \in \mathscr{F}_8^{\mathfrak{h}}$ is indeed Brill-Noether general and therefore arises as a transversal linear section of $G(2,6)$. In ongoing work with Michael Hoff we try to show that the generators of the Picard group $\mathrm{Pic}(S)$ can also be obtained by taking linear sections of sub-varieties inside $G(2,6)$. To be more precise we expect the following to hold. Changing the basis of the lattice $\mathfrak{h}$ to $\{\mathcal{O}_S(H), \mathcal{O}_S(Q) = \mathcal{O}_S(C-H), \mathcal{O}_S(N)\}$, we have $Q = \mathbb{P}^8 \cap G(2,4)$ and $N = \mathbb{P}^8 \cap G(2,5)$ for Grassmannians $G(2,4), G(2,5) \subset G(2,6) \subset \mathbb{P}^{14}$ not containing each other.

Another interesting problem is the following. Computer algebra experiments indicate that a similar behavior, as for canonical genus 9 curves together with a g^1_6, can also be observed for general genus g curves together with a g^1_{g-3}. Based on experiments for small genus, we make the following conjecture.

Conjecture 5.5.1. *Let $g \geq 9$ be an integer, $(C,L) \in \mathscr{W}^1_{g,g-3}$ be general an let $\mathbb{P}(\mathscr{E})$ be the scroll defined by* L.

(a) The relative canonical resolution of $C \subset \mathbb{P}(\mathscr{E})$ has an unbalanced second syzygy bundle and the relative linear strand has the following form

$$\mathcal{O}_{\mathbb{P}(\mathscr{E})} \longleftarrow \mathcal{O}_{\mathbb{P}(\mathscr{E})}(-2H+R)^{\oplus(2g-12)} \longleftarrow \mathcal{O}_{\mathbb{P}(\mathscr{E})}(-3H+2R)^{\oplus(g-7)}.$$

(b) If we pick a syzygy $s \in \mathcal{O}_{\mathbb{P}(\mathscr{E})}(-3H+2R)^{\oplus(g-7)}$ then the corresponding syzygy scheme (after saturating with the vertex of the scroll) is a $(g-4)$ dimensional variety $Y \subset \mathbb{P}^{g-1}$, such that $Y \cap \mathbb{P}^8$ is a K3 surface of degree 14. Similar to the genus 9 case, the syzygies in $\mathcal{O}_{\mathbb{P}(\mathscr{E})}(-3H+2R)^{\oplus(g-7)}$ parametrize a $(g-8)$ dimensional family of such varieties Y containing the curve C.

Since for $g > 9$ the curve is no longer a divisor on the variety Y, we expect that a proof of the above conjecture requires different techniques than the genus 9 case.

Chapter 6

The BGG-correspondence for canonical curves and Green's conjecture in positive characteristic

6.1 Introduction

Let V be an $n+1$ dimensional vector space and let further $E = \bigwedge V$ be the exterior algebra and $S = \mathrm{Sym} V^*$ the symmetric algebra over the dual vector space V^*. The Bernšteĭn-Gel'fand-Gel'fand (BGG) correspondence [BGG78] consists of a pair of adjoint functors **R** and **L** from the category of complexes of graded S-modules to the category of complexes of graded E-modules. In [EFS03], the authors made the construction of these two functors explicit and furthermore showed that **R** and **L** define an equivalence of derived categories

$$D^b(S-\mathrm{mod}) \underset{\mathbf{L}}{\overset{\mathbf{R}}{\rightleftarrows}} D^b(E-\mathrm{mod}).$$

This allows us to study complexes of S- or E-modules by studying the image of these objects under the functor **R**, resp. **L**. For instance, for a finitely generated graded S-module M, which is considered as a 1-term complex, the Betti numbers of M are given as the vector space dimensions of the graded parts of the cohomologies of $\mathbf{R}(M)$ (see [EFS03, Prop. 2.3]).

In this chapter we apply the theory developed in [EFS03] to (k-gonal) canonical curves $C \subset \mathbb{P}^{g-1}$. Our main focus lies in the study of the cohomologies of $\mathbf{R}(S_C)$ as well as the homologies in the linear strands of the coordinate ring S_C.

Our main result, Theorem 6.3.6, concerns the homologies in the linear strands of general 4-gonal canonical curves. For general 4-gonal canonical curves we give an explicit description of the homologies in the linear strands in terms of maps appearing in the iterated mapping cone, obtained by resolving the terms in the relative canonical resolution of C by Eagon-Northcott type complexes (see Section 2.3 for relative canonical resolutions and iterated mapping cones).

For a matrix φ over E the linear part $\mathrm{lin}(\varphi)$ of φ is defined as the matrix obtained from φ by erasing all entries of degree > 1. Accordingly, for a finitely generated graded E-module P with presentation matrix φ we define the linearized module as $\mathrm{lin}(P) := \mathrm{coker}(\mathrm{lin}(\varphi))$. Using Theorem 6.3.6 we will show that the Betti numbers of S_C are already encoded in the linearized cohomologies of the complex $\mathbf{R}(S_C)$.

Most of the results in this chapter were originally motivated by computer algebra experiments using the *Macaulay2*-package [ADE+12]. One particular class of interesting examples are canonical curves which have unexpected extra syzygies. In particular canonical curves defined over a field of positive characteristic which (possibly) violate the statement of Green's conjecture.

We have implemented the construction of random curves of genus up to 15 over fields of very small characteristic in the *Macaulay2*-package *RandomCurvesOverVerySmallFiniteFields* [BS17] (see also Section 7.2). In Section 6.4 we summarize the results of our experiments and give a conjectural list of all exceptional cases for the generic Green conjecture in positive characteristic for genus up to 15. Based on our experiments, we suggest a refinement of Green's conjecture which conjecturally also holds for curves defined over fields of positive characteristic.

6.2 The Bernšteĭn-Gel'fand-Gel'fand correspondence

We begin by introducing the relevant terminology. Most of this section follows [EFS03].

6.2.1 Terminology

For a field $\mathbb{k}$ let V be an $n+1$ dimensional $\mathbb{k}$-vector space and let $\mathrm{W} = \mathrm{Hom}_{\mathbb{k}}(\mathrm{V}, \mathbb{k})$ be the dual vector space. We define the $n+1$ generators $\{x_i\}$ of W to have degree 1, so that the elements in the dual basis $\{e_i\}$ of W have degree -1. We furthermore denote $\mathrm{S} = \mathrm{Sym}(\mathrm{W}) = \mathbb{k}[x_0, \ldots, x_n]$ the symmetric algebra over W and $\mathrm{E} = \bigwedge \mathrm{V}$ the exterior algebra over V and regard S and E as graded algebras, where elements in $\mathrm{Sym}_j(\mathrm{W})$ have degree j and elements in $\bigwedge^j \mathrm{V}$ have degree $-j$. For a graded module M over E (resp. S) we write as usual M_i for the component of degree i and $\mathrm{M}(a)$ for the shifted module, i.e. $\mathrm{M}(a)_b = \mathrm{M}_{a+b}$. Often a complex $\mathbf{F}$ is written cohomologically with upper indices

$$\mathbf{F}: \quad \ldots \longrightarrow \mathrm{F}^i \longrightarrow \mathrm{F}^{i+1} \longrightarrow \ldots$$

and differentials in degree 1. We write $\mathbf{F}[a]$ for the shifted complex whose term in cohomological degree j is given by F^{a+j}.

Furthermore, we denote by $\omega_{\mathrm{S}} = \mathrm{S} \otimes_{\mathbb{k}} \bigwedge^{n+1} \mathrm{W}$ the module associated to the canonical bundle $\omega_{\mathbb{P}(\mathrm{W})}$ on $\mathbb{P}(\mathrm{W})$ and by $\omega_{\mathrm{E}} = \mathrm{E} \otimes_{\mathbb{k}} \bigwedge^{n+1} \mathrm{W}$. Note that ω_{S} (resp. ω_{E}) is (non-canonically) isomorphic to $\mathrm{S}(-n-1)$ (resp. $\mathrm{E}(-n-1)$). It can be furthermore shown that $\mathrm{Hom}_{\mathbb{k}}(\mathrm{E}, \mathbb{k}) = \omega_{\mathrm{E}}$ and $\mathrm{Hom}_{\mathbb{k}}(\mathrm{E}, \mathrm{D}) = \omega_{\mathrm{E}} \otimes_{\mathbb{k}} \mathrm{D}$ for any graded vector space D (see e.g. Section [Eis05, Section 7E]).

6.2.2 The adjoint functors R and L

In this subsection we introduce the *Bernšteĭn-Gel'fand-Gel'fand* correspondence, i.e. there is a pair of adjoint functors from the category of complexes over E and over S. This section follows [EFS03, section 2].

Let $\mathrm{M} = \bigoplus \mathrm{M}_d$ be a graded modules over S, which we consider as a complex of S-modules with only one term in cohomological degree 0. We define the complex

$$\mathbf{R}(\mathrm{M}): \quad \ldots \xrightarrow{\phi} \mathrm{Hom}_{\mathbb{k}}(\mathrm{E}, \mathrm{M}_d) \xrightarrow{\phi} \mathrm{Hom}_{\mathbb{k}}(\mathrm{E}, \mathrm{M}_{d+1}) \xrightarrow{\phi} \ldots$$
$$\phi: \alpha \mapsto \{e \mapsto \sum_i x_i \alpha(e_i e)\}$$

where $\mathrm{Hom}_{\mathbb{k}}(\mathrm{E}, \mathrm{M}_d)$ has cohomological degree d and an element $\alpha \in \mathrm{Hom}_{\mathbb{k}}(\mathrm{E}, \mathrm{M}_d)$ has degree t if it factors through E_{d-t}. Therefore, the graded parts are $(\mathbf{R}(\mathrm{M}))^k_j =$

$\mathrm{Hom}_{\Bbbk}(\mathrm{E}_{k-j}, \mathrm{M}_k)$. Note that all differentials of $\mathbf{R}(\mathrm{M})$ are given by matrices with linear entries.

In [EFS03, section 2] it is shown that the functor $\mathbf{R}$ defines an equivalence between the category of graded S-modules and the category of linear free complexes over E, for which the module in cohomological degree d has socle in degree d. Recall that the socle of a module is its unique maximal semi-simple sub-module.

The functor $\mathbf{R}$ can be extended to complexes over S in the following way. Again, let M be a graded S-module. We regard the shifted module $\mathrm{M}[a]$ as a complex with only one term in cohomological degree a and set $\mathbf{R}(\mathrm{M}[a]) = \mathbf{R}(\mathrm{M})[a]$. Now, if

$$\mathbf{M}: \quad \ldots \longrightarrow \mathrm{M}^i \longrightarrow \mathrm{M}^{i+1} \longrightarrow \ldots$$

is a complex of S-modules we can apply the functor $\mathbf{R}$ to each term M^i of the complex $\mathbf{M}$, regarded as a complex concentrated in cohomological degree i. We obtain a double complex

$$\begin{tikzcd}
 & & & \\
\ldots \arrow[r] & \mathrm{Hom}_{\Bbbk}(\mathrm{E}, (\mathrm{M}^{i+1})_j) \arrow[r] \arrow[u] & \mathrm{Hom}_{\Bbbk}(\mathrm{E}, (\mathrm{M}^{i+1})_{j+1}) \arrow[r] \arrow[u] & \ldots \\
\ldots \arrow[r] & \mathrm{Hom}_{\Bbbk}(\mathrm{E}, (\mathrm{M}^{i})_j) \arrow[r] \arrow[u] & \mathrm{Hom}_{\Bbbk}(\mathrm{E}, (\mathrm{M}^{i})_{j+1}) \arrow[r] \arrow[u] & \ldots \\
 & \arrow[u] & \arrow[u] &
\end{tikzcd}$$

whose vertical differentials are induced by the differentials in $\mathbf{M}$ and define $\mathbf{R}(\mathbf{M})$ to be the total complex of this double complex. Note that the k-th term in the complex $\mathbf{R}(\mathbf{M})$ is given by

$$(\mathbf{R}(\mathbf{M}))^k = \sum_{i+j=k} \mathrm{Hom}_{\Bbbk}(\mathrm{E}, (\mathrm{M}^i)_j)$$

with graded parts

$$(\mathbf{R}(\mathbf{M}))^k_j = \sum_m \mathrm{Hom}_{\Bbbk}(\mathrm{E}_{m-j}, (\mathrm{M}^{k-m})_m).$$

A similar construction can be made for graded modules over the exterior algebra E. For a graded E-module $P = \bigoplus P_j$ we define the complex

$$\mathbf{L}(P) : \ldots \longrightarrow S \otimes_{\Bbbk} P_j \longrightarrow S \otimes_{\Bbbk} P_{j-1} \longrightarrow \ldots$$
$$s \otimes p \mapsto \sum_i x_i s \otimes e_i p$$

where the term $S \otimes_{\Bbbk} P_j$ has cohomological degree $-j$ and the graded parts are given by $(\mathbf{L}(P))^k_j = S_{k+j} \otimes_{\Bbbk} P_{-k}$. Similarly to the functor $\mathbf{R}$, we can extend $\mathbf{L}$ to complexes $\mathbf{P}$ of graded E-modules by applying $\mathbf{L}$ to each term in the complex and defining $\mathbf{L}(\mathbf{P})$ to be the total complex of the double complex obtained this way. In this case the k-th term of $\mathbf{L}(\mathbf{P})$ is given by

$$(\mathbf{L}(\mathbf{P}))^k = \sum_{i-j=k} S \otimes_{\Bbbk} (P^i)_j$$

with graded parts

$$(\mathbf{L}(\mathbf{P}))^k_j = \sum_m S_{j-m} \otimes_{\Bbbk} (P^{k+m})_m.$$

Theorem 6.2.1 ([BGG78], [EFS03, Thm 2.2]). *The functor* $\mathbf{L}$*, from the category of complexes of graded* E*-modules to the category of complexes of graded* S*-modules, is a left adjoint functor to* $\mathbf{R}$.

Let M be a graded S-module and let P be a graded E-module, then the graded parts of the cohomologies of $\mathbf{R}(M)$ and $\mathbf{L}(P)$ can be expressed as follows.

Proposition 6.2.2 ([EFS03, Prop. 2.3]). *With notation as above*

(a) $H^i(\mathbf{R}(M))_j = \mathrm{Tor}^S_{j-i}(\Bbbk, M)_j$

(b) $H^i(\mathbf{L}(P))_j = \mathrm{Ext}^{j-i}_E(\Bbbk, P)_j$

Recall from Proposition 2.1.5 that $\dim_{\Bbbk} \mathrm{Tor}^S_i(\Bbbk, M)_j = \beta_{i,j}(M)$ are the Betti numbers of a minimal projective resolution of M. A similar statement which will be made precise in Section 6.2.4 holds for injective resolutions and the numbers

$$\mu_{i,j}(P) = \dim_{\Bbbk} \mathrm{Ext}^i_E(\Bbbk, P)_j.$$

If M is a finitely generated graded S-module such that $d \geq \mathrm{reg}(M)$ then the truncated module $M_{\geq d}$ has a linear free resolution (see e.g. [EG84, Theorem 1.2]). Thus, applying the above proposition to the truncated module $M_{\geq d}$ we obtain:

Corollary 6.2.3 ([EFS03, Cor. 2.4]). *Let* M *be a finitely generated graded* S*-module, then the truncated complex*

$$\mathbf{R}(\mathrm{M})_{\geq d}:\quad \mathrm{Hom}_{\Bbbk}(\mathrm{E},\mathrm{M}_d) \longrightarrow \mathrm{Hom}_{\Bbbk}(\mathrm{E},\mathrm{M}_{d+1}) \longrightarrow \dots$$

is acyclic if and only if M *is* d*-regular.*

The functors **R** and **L** give a general method to construct resolutions of graded modules or more generally resolutions of complexes of graded modules.

Theorem 6.2.4 ([EFS03, Thm. 2.6 and Cor. 2.7]). *Let* **M** *be a complex of graded* S*-modules and let* **P** *be a complex of graded* E*-modules. Then*

- **LR**(**M**) *is a free resolution of* **M** *which surjects onto* **M**, *and*
- **RL**(**P**) *is an injective resolution of* **P** *into which* **P** *injects.*

Moreover, the functors **R** *and* **L** *define an equivalence of derived categories*

$$\mathrm{D}^b(\mathrm{S}\text{-}mod) \cong \mathrm{D}^b(\mathrm{E}\text{-}mod).$$

6.2.3 The linear part of a complex

For a matrix A over E or S we define the linear part lin(A) to be the matrix obtained by erasing all terms of degree > 1 (resp. < -1). Note that for two matrices A, B such that $\mathrm{AB} = 0$ it is in general false that $\mathrm{lin(A)lin(B)} = 0$ (e.g. consider an element x of degree 1, $\mathrm{A} = (x, x^2)$ and $\mathrm{B} = \binom{-x}{1}$). However if the entries of A and B are contained in an appropriate maximal ideal this does follows. So, for a minimal free complex **F** we can define the complex lin(**F**) to be the complex obtained from **F** by replacing each differential by its linear part.

More generally for any free complex **F** we define lin(**F**) to be the linear part of a minimal free complex homotopic to **F**.

Proposition 6.2.5 ([EFS03, Cor. 3.6]). *Let* **M** *be a left-bounded complex of graded* S*-modules and let* **P** *be a left-bounded complex of graded* E*-modules then*

$$\mathrm{lin}(\mathbf{R}(\mathbf{M})) = \bigoplus_i \mathbf{R}(\mathrm{H}^i\mathbf{M}) \quad \textit{and} \quad \mathrm{lin}(\mathbf{L}(\mathbf{P})) = \bigoplus_i \mathbf{L}(\mathrm{H}^i\mathbf{P})$$

where the cohomology groups above are regarded as complexes with only one term concentrated in cohomological degree i.

Applying the above proposition together with Theorem 6.2.4 one can show the following.

Theorem 6.2.6 ([EFS03, Thm. 3.7]). *(a) Let M be a finitely generated S-module and let P be a finitely generated E-module. Then $\mathbf{L}(\mathrm{P})$ is a free resolution of M if and only if $\mathbf{R}(\mathrm{M})$ is an injective resolution of P.*

(b) Let $\mathbf{M}$ be a bounded complex of finitely generated graded S-modules and let $\mathbf{F}$ be a minimal free resolution of $\mathbf{M}$. Then

$$\operatorname{lin}(\mathbf{F}) = \bigoplus_i \mathbf{L}\big(\mathrm{H}^i(\mathbf{R}(\mathbf{M}))\big)$$

If $\mathbf{P}$ is a bounded complex of finitely generated graded E-modules and $\mathbf{G}$ is an injective resolution of $\mathbf{P}$ then

$$\operatorname{lin}(\mathbf{G}) = \bigoplus_i \mathbf{R}\big(\mathrm{H}^i(\mathbf{L}(\mathbf{P}))\big).$$

The cohomology groups above are regarded as complexes with only one term concentrated in cohomological degree i.

Tate resolutions As one of the main applications in [EFS03], the authors develop an efficient machinery to compute cohomology of coherent sheaves on $\mathbb{P}^n$. This is done by computing a so-called *Tate resolution.* Although this plays a minor role for this thesis, we recall their result.

A *Tate resolution* over E is a double infinite complex

$$\mathbf{T}: \quad \cdots \to \mathrm{T}^{d-1} \to \mathrm{T}^{d} \to \mathrm{T}^{d+1} \to \dots$$

which is exact at every position.

Using the functor $\mathbf{R}$ one can construct Tate resolutions of finitely generated modules or more generally coherent sheaves. Let $\mathscr{F}$ be a coherent sheaf over $\mathbb{P}(\mathrm{W})$ and let $\mathrm{M} = \bigoplus_{n\geq 0} \mathrm{H}^0(\mathscr{F}(n))$ be a graded S-module associated to $\mathscr{F}$. Fixing an integer $d > \operatorname{reg}(\mathrm{M})$ it follows from Corollary 6.2.3 that $\mathbf{R}(\mathrm{M}_{\geq d})$ is a minimal acyclic complex. This complex can be completed to a Tate resolution by adjoining a free resolution of

$$\ker\Big(\mathrm{Hom}_{\Bbbk}(\mathrm{E},\mathrm{M}_d) \to \mathrm{Hom}_{\Bbbk}(\mathrm{E},\mathrm{M}_{d+1})\Big).$$

The Tate resolution obtained this way, will be denoted by $\mathbf{T}(\mathscr{F})$. Note that $\mathbf{T}(\mathscr{F})$ is independent of the chosen M and d and has the form

$$\mathbf{T}(\mathscr{F}): \quad \cdots \to \mathrm{T}^{d-1} \to \mathrm{Hom}_{\Bbbk}(\mathrm{E},\mathrm{H}^0(\mathbb{P}(\mathrm{W}),\mathscr{F}(d))) \to \mathrm{Hom}_{\Bbbk}(\mathrm{E},\mathrm{H}^0(\mathbb{P}(\mathrm{W}),\mathscr{F}(d+1))) \to \dots$$

In [EFS03] the authors prove the following theorem.

Theorem 6.2.7 ([EFS03, Thm. 4.1 and Cor. 4.2]). *If $\mathscr{F}$ is a coherent sheaf on $\mathbb{P}(\mathrm{W})$, then*

$$\mathrm{lin}(\mathbf{T}(\mathscr{F})) = \bigoplus_j \mathbf{R}\Big(\bigoplus_e \mathrm{H}^j\big(\mathbb{P}(\mathrm{W}),\mathscr{F}(e-j)\big)\Big).$$

In particular

$$\mathrm{T}^e = \bigoplus_j \mathrm{Hom}_{\Bbbk}\big(\mathrm{E},\mathrm{H}^j\big(\mathbb{P}(\mathrm{W}),\mathscr{F}(e-j)\big)\big) = \bigoplus_j \mathrm{H}^j\big(\mathbb{P}(\mathrm{W}),\mathscr{F}(e-j)\big)\otimes\omega_{\mathrm{E}},$$

where $\mathrm{H}^j\big(\mathbb{P}(\mathrm{W}),\mathscr{F}(e-j)\big)$ is regarded as a vector space in degree $(e-j)$. Moreover, for all $j,l\in\mathbb{Z}$ we have

$$\mathrm{H}^j\big(\mathbb{P}(\mathrm{W}),\mathscr{F}(l)\big) = \mathrm{Hom}_{\mathrm{E}}\big(\Bbbk,\mathrm{T}^{j+l}\big)_{-l}.$$

6.2.4 Resolutions and duality

We fix some further notation which we will use throughout the next sections. Let M be a graded S-module and let

$$\mathbf{F}: \quad 0 \leftarrow \mathrm{M} \leftarrow \mathrm{F}_0 \leftarrow \mathrm{F}_1 \leftarrow \cdots \leftarrow \mathrm{F}_{i-1} \leftarrow \mathrm{F}_i \leftarrow \dots$$

be a minimal free resolution of M, where $\mathrm{F}_i = \bigoplus_j \mathrm{S}(-j)^{\beta_{ij}}$. We will refer to a minimal free resolution of M by res(M) and denote by

$$\beta(\mathrm{M}) = \begin{array}{c|cccc} & i & i+1 & i+2 & \dots \\ \hline j & \beta_{i,i+j} & \beta_{i+1,i+j+1} & \beta_{i+2,i+j+2} & \dots \\ j+1 & \beta_{i,i+j+1} & \beta_{i+1,i+j+2} & \beta_{i+2,i+j+3} & \dots \\ \vdots & \vdots & \vdots & \vdots & \end{array}$$

the Betti table of the minimal resolution res(M). Furthermore, we denote by $\mathrm{strand}_j(\mathrm{M})$ the linear sub-complex of $\mathbf{F}$ given by

$$\cdots \leftarrow \mathrm{S}(-i-j)^{\beta_{i,i+j}} \leftarrow \mathrm{S}(-i-j-1)^{\beta_{i+1,i+j+1}} \leftarrow \mathrm{S}(i-j-2)^{\beta_{i+2,i+j+2}} \leftarrow \dots$$

and refer to this complex as the j-th linear strand of the minimal resolution res(M). If M is a finitely generated module over S, we denote a minimal presentation matrix of M by pres(M). For graded modules over E we will use the same notation to refer to minimal free resolutions, linear strands and the presentation matrix.

Similar to modules over S, one can show (see e.g. [Eis05, Section 7B]), that

$$\beta_{i,j}(\mathrm{P}) = \dim_{\Bbbk} \mathrm{Tor}_i^{\mathrm{E}}(\Bbbk, \mathrm{P})_j$$

for any finitely generated graded E-module P.

For the rest of this chapter, we write $\mathrm{M}^\vee$ (resp. $\mathrm{P}^\vee$) for the S-dual (resp. E-dual) of a graded module M over S or P over E and M^* (resp. P^*) for the $\Bbbk$-dual. The functor $\mathrm{Hom}_{\Bbbk}(\_, \Bbbk)$ is exact on the category of graded S-modules (resp. E-modules) and takes projective modules to injective ones and vice versa. Now, since the $\Bbbk$-dual of a free module is again a free module it follows that every finitely generated graded S- or E-module has a graded injective resolution. We say that an injective resolution

$$\mathbf{I}: \quad 0 \to \mathrm{P} \to \mathrm{I}^0 \to \mathrm{I}^1 \to \dots$$

is minimal, if $\mathrm{Hom}_{\Bbbk}(\mathbf{I}, \Bbbk)$ is a minimal free resolution of P^*. In this case we can write the i-th term as

$$\mathrm{I}^i = \bigoplus_j \mathrm{E}(n+1-j)^{\mu_{i,j}}.$$

The numbers $\mu_{i,j}$ are called the *Bass numbers* of a minimal injective resolution of P and it can be shown (see e.g. [AHH97, section 5]), that

$$\mu_{i,j}(\mathrm{P}) = \dim_{\Bbbk} \mathrm{Ext}_{\mathrm{E}}^i(\Bbbk, \mathrm{P})_j$$

and

$$\mu_{i,j}(\mathrm{P}) = \beta_{i,n+1-j}(\mathrm{P}^\vee).$$

Similar to the case of free resolutions, we will refer by $\mathrm{res}^{inj}(\ \_\ )$ to a minimal injective resolution of a module over S (resp. E).

In [Eis05, Section 7E] it is shown, that also $\mathrm{Hom}_{\mathrm{E}}(\_,\mathrm{E})$ is an exact functor. We repeat the argument. By the tensor-hom adjunction (see e.g. [Bou89, II §4]) we have

$$\mathrm{P}^* = \mathrm{Hom}_{\Bbbk}(\mathrm{P},\Bbbk) = \mathrm{Hom}_{\Bbbk}(\mathrm{P}\otimes_{\mathrm{E}}\mathrm{E},\Bbbk) = \mathrm{Hom}_{\mathrm{E}}(\mathrm{P},\mathrm{Hom}_{\Bbbk}(\mathrm{E},\Bbbk)) = \mathrm{Hom}_{\mathrm{E}}(\mathrm{P},\mathrm{E}^*)$$

for a graded E-module P. Now as we already remarked in Section 6.2.1, $\mathrm{E}^* = \mathrm{E}\otimes_{\Bbbk} \bigwedge^{n+1}\mathrm{W} = \omega_{\mathrm{E}}$ and therefore $\mathrm{P}^* = \mathrm{Hom}_{\mathrm{E}}(\mathrm{P},\mathrm{E})\otimes\omega_{\mathrm{E}}$.

Lemma 6.2.8. *Let* P *be a graded* E*-module, then* $\mathrm{Hom}_{\mathrm{S}}(\mathbf{L}(\mathrm{P}),\mathrm{S}) = \mathbf{L}(\mathrm{P}^*)$.

Proof. Since

$$(\mathrm{P}^*)_i = \big(\mathrm{Hom}_{\Bbbk}(\mathrm{P},\Bbbk)\big)_i \cong \mathrm{Hom}_{\Bbbk}(\mathrm{P}_{-i},\Bbbk) = (\mathrm{P}_{-i})^*$$

and $\mathrm{S}\otimes_{\Bbbk}\mathrm{P}_i \cong \mathrm{S}^{\dim \mathrm{P}_i}(-i)$ we get

$$\mathrm{Hom}_{\mathrm{S}}(\mathrm{S}\otimes_{\Bbbk}\mathrm{P}_i,\mathrm{S}) \cong \mathrm{S}\otimes_{\Bbbk}(\mathrm{P}_i)^* \cong \mathrm{S}\otimes_{\Bbbk}(\mathrm{P}^*)_{-i} \tag{6.1}$$

Therefore, it remains to identify the differentials in $\mathbf{L}(\mathrm{P}^*)$ and $\mathrm{Hom}_{\mathrm{S}}(\mathbf{L}(\mathrm{P}),\mathrm{S})$. For $\mathbf{L}(\mathrm{P}^*)$ we have

$$\begin{aligned}\mathbf{L}(\mathrm{P}^*) = \quad & \cdots \leftarrow \mathrm{S}\otimes_{\Bbbk}\mathrm{P}^*_{-i} \longleftarrow \mathrm{S}\otimes_{\Bbbk}\mathrm{P}^*_{-i+1} \leftarrow \dots \\ & \textstyle\sum x_i s\otimes e_i\varphi \leftarrow\!\shortmid\; s\otimes\varphi\end{aligned}$$

and for $\mathrm{Hom}_{\mathrm{S}}(\mathbf{L}(\mathrm{P}),\mathrm{S})$ we have

$$\begin{aligned}\mathrm{Hom}_{\mathrm{S}}(\mathbf{L}(\mathrm{P}),\mathrm{S}) = \quad & \cdots \leftarrow \mathrm{Hom}_{\mathrm{S}}(\mathrm{S}\otimes_{\Bbbk}\mathrm{P}_i,\mathrm{S}) \longleftarrow \mathrm{Hom}_{\mathrm{S}}(\mathrm{S}\otimes_{\Bbbk}\mathrm{P}_{i-1},\mathrm{S}) \leftarrow \dots \\ & \Big\{s\otimes p \mapsto \alpha\big(\textstyle\sum x_i s\otimes e_i p\big) = \sum x_i s\otimes\alpha(e_i p)\Big\} \leftarrow\!\shortmid\; \alpha\end{aligned}$$

Using the isomorphism (6.1) we identify $1\otimes\varphi \in \mathrm{S}\otimes_{\Bbbk}(\mathrm{P}^*)_{-i+1}$ with the corresponding $\alpha \in \mathrm{Hom}_{\mathrm{S}}(\mathrm{S}\otimes_{\Bbbk}\mathrm{P}_{i-1},\mathrm{S})$ and regard $\sum x_i\otimes e_i\varphi \in \mathrm{S}\otimes_{\Bbbk}\mathrm{P}^*_{-i}$ as an element in $\mathrm{Hom}_{\mathrm{S}}(\mathrm{S}\otimes_{\Bbbk}\mathrm{P}_i,\mathrm{S})$ via

$$\begin{aligned}s\otimes p \mapsto \Big(\textstyle\sum x_i\otimes e_i\varphi\Big)(s\otimes p) &= \textstyle\sum x_i s\otimes\varphi(e_i p) \\ &= \textstyle\sum x_i s\cdot\big(1\otimes\varphi\big)(1\otimes e_i p) = \sum x_i s\cdot\alpha(1\otimes e_i p) \\ &= \alpha\Big(\textstyle\sum x_i s\otimes e_i p\Big).\end{aligned}$$

We conclude that we can identify the differentials in $\mathrm{Hom}_{\mathrm{S}}(\mathbf{L}(\mathrm{P}),\mathrm{S})$ with the differentials in $\mathbf{L}(\mathrm{P}^*)$. □

6.3 BGG for canonical curves

In this section we want to apply the previous sections to coordinate rings of (k-gonal) canonical curves. For free resolutions of canonical curves we will use homological indexing, as it is done in all previous chapters. The homological index of the modules in the linear strands will be the homological index induced by the free resolution.

Recall that the minimal free resolution of a canonical curve $C \subset \mathbb{P}^{g-1}$ has the following shape

	0	1	2	…	$g-4$	$g-3$	$g-2$
0	1	-	-	…	-	-	-
1	-	$\beta_{1,2}$	$\beta_{2,3}$	…	$\beta_{g-4,g-3}$	$\beta_{g-3,g-2}$	-
2	-	$\beta_{g-3,g-2}$	$\beta_{g-4,g-3}$	…	$\beta_{2,3}$	$\beta_{1,2}$	-
3	-	-	-	…	-	-	1

In particular S_C is 3-regular. Applying the functor $\mathbf{R}$ to the truncated coordinate ring $(S_C)_{>3}$ and completing $\mathbf{R}((S_C)_{>3})$ to a Tate-resolution, we obtain (after tensoring with $\omega_E^\vee$):

$$\begin{array}{cccccccccccccc}
\mathbf{T}: \dots \longleftarrow & T^3 & \longleftarrow & T^2 & \longleftarrow & T^1 & \longleftarrow & T^0 & \longleftarrow & T^{-1} & \longleftarrow & T^{-2} & \longleftarrow \dots \\
& \parallel & & \parallel & & \parallel & & \parallel & & \parallel & & \parallel & \\
& H^0(\mathcal{O}_C(3)) & & H^0(\mathcal{O}_C(2)) & & H^0(\mathcal{O}_C(1)) & & H^0(\mathcal{O}_C) & & & & & \\
\dots \longleftarrow & & \longleftarrow & \oplus & \longleftarrow & \oplus & \longleftarrow & \oplus & \longleftarrow & & \longleftarrow & & \longleftarrow \dots \\
& & & H^1(\mathcal{O}_C(1)) & & H^1(\mathcal{O}_C) & & H^1(\mathcal{O}_C(-1)) & & H^1(\mathcal{O}_C(-2)) & & H^1(\mathcal{O}_C(-3)) &
\end{array}$$

Here, by abuse of notation, we write $H^i(\mathcal{O}_C(j))$ instead of $H^i(\mathbb{P}^{g-1}, \mathcal{O}_C(j)) \otimes E$. Note that by Theorem 6.2.5 and 6.2.7 the first linear strand can be identified with $\mathbf{R}(S_C)$. Now, by Theorem 6.2.6

$$\operatorname{lin}(\operatorname{res}(S_C)) = \bigoplus_{i=0}^{3} \mathbf{L}(H^i(\mathbf{R}(S_C)))$$

where $H^i(\mathbf{R}(S_C))$ is considered as a 1-Term complex concentrated in cohomological degree i and $\mathbf{L}(H^i(\mathbf{R}(S_C)))$ corresponds to the i-th linear strand $\operatorname{strand}_i(S_C)$. From the self-duality of the canonical resolution and Lemma 6.2.8 it follows that

$$H^{3-i}(\mathbf{R}(S_C)) \cong H^i(\mathbf{R}(S_C))^*.$$

After twisting, the cokernel of the multiplication map

$$\mathrm{H}^0\mathscr{O}_\mathrm{C}(1)\otimes\mathrm{E}\to\mathrm{H}^0\mathscr{O}_\mathrm{C}(2)\otimes\mathrm{E}$$

has a resolution of the following form

	2	1	0	−1	−2	−3	−4	…
1	$3g-3$	g	1	−	−	−	−	…
0	−	−	−	−	−	−	−	…
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
$-p$	−	−	∗	∗	∗	∗	∗	…
$-(p+1)$	−	−	∗	∗	∗	∗	∗	…
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
$-(g-4)$	−	−	∗	∗	∗	∗	∗	…
$-(g-3)$	−	−	−	−	1	g	$\binom{g+1}{2}$	…
$-(g-2)$	−	−	−	−	−	−	−	…

for some $p\geq 0$. Equivalently, $\mathrm{P}=\mathrm{H}^1(\mathbf{R}(\mathrm{S_C}))$ has a resolution of the form

	0	−1	−2	−3	…
$-p$	∗	∗	∗	∗	…
$-(p+1)$	∗	∗	∗	∗	…
⋮	⋮	⋮	⋮	⋮	⋮
$-(g-4)$	∗	∗	∗	∗	…
$-(g-3)$	−	−	1	g	…
$-(g-2)$	−	−	−	−	…

Indeed, the Betti numbers of the resolution of P over E are the bass numbers of an injective resolution of $\mathrm{P}^*\cong\mathrm{H}^2(\mathbf{R}(\mathrm{S_C}))$ over E. Now $\mathbf{L}(\mathrm{P}^*)=\mathrm{Hom_S}(\mathbf{L}(\mathrm{P}),\mathrm{S})$ by Lemma 6.2.8, which is on the other hand isomorphic to $\mathrm{strand}_2(\mathrm{S_C})$ by the self-duality of canonical resolutions. By Theorem 6.2.7 the strands in an injective resolution of P^* in degree $i=0,\ldots,-(g-3)$ correspond to $\mathbf{R}(\mathrm{H}_i(\mathbf{L}(\mathrm{P}^*)))$. This means, that the strands of $\mathrm{res}^{inj}(\mathrm{P}^*)$ are determined by the homologies in the second linear strand of $\mathrm{res}(\mathrm{S_C})$. In particular, since $\mathrm{H}_{g-3}(\mathrm{strand}_2(\mathrm{S_C}))\cong\mathrm{S}$, the very last linear strand in degree $-(g-3)$ of $\mathrm{res}^{inj}(\mathrm{P}^*)$

is (up to shift ant twist) $\mathbf{R}(S)$, the Cartan resolution resolving the ground field $\Bbbk$ over E (see [Eis05, Cor.2.7] for details about the Cartan complex).

We will discuss the different strands occurring in res(P) and res(P*) later on for the case of general 4-gonal canonical curves. Before doing a first example we remark the following.

Remark 6.3.1. With notation as above the number p in the table above is the smallest number such that $H_p(\mathrm{strand}_2(S_C)) \neq 0$. This implies that $\beta_{p,p+2} \neq 0$ and $\beta_{i,i+2} = 0$ for $i < p$. Thus Green's conjecture holds for a canonical curve C if and only if $p = \mathrm{Cliff}(C)$.

Moreover, if $C \subset \mathbb{P}^{g-1}$ is a canonically embedded curve of genus g which satisfies Green's conjecture, then by Proposition 6.2.2

$$\beta_{0,-p}(\mathrm{res}(P)) = \beta_{p,p+2}(\mathrm{res}(S_C))$$

where again $P = H^1(\mathbf{R}(S_C))$ and $p = \mathrm{Cliff}(C)$.

We continue with a first example, in which we apply the previous sections to general 4-gonal canonical curves of genus 8. The example relies on experiments done with the *Macaulay2* packages [ADE+12] and [BH15a] and its purpose is to exhibit several interesting observations related to the BGG-correspondence and canonical curves which will be proved afterwards.

Example 6.3.2. A general 4-gonal canonical curve of genus 8 has a minimal free resolution of the following form

	0	1	2	3	4	5	6
0 :	1	–	–	–	–	–	–
1 :	–	15	35	25	4	–	–
2 :	–	–	4	25	35	15	–
3 :	–	–	–	–	–	–	1

The linear part $\mathbf{R}(S_C)$ of the Tate resolution (tensorized with $\omega_E^\vee$) has the form

$$\ldots \longleftarrow H^0(\omega_C^3) \otimes E \longleftarrow H^0(\omega_C^2) \otimes E \xleftarrow{\varphi} H^0(\omega_C) \otimes E \longleftarrow H^0(\mathscr{O}_C) \otimes E \longleftarrow 0$$

and (after twisting) the Betti tables of coker(φ) and therefore $P = H^1\big(\mathbf{R}(S_C)\big)$ are given by

	2	1	0	−1	−2	...
1:	21	8	1	-	-	...
0:	-	-	-	-	-	...
−1:	-	-	-	-	-	...
−2:	-	-	4	17	44	...
−3:	-	-	10	70	283	...
−4:	-	-	5	38	164	...
−5:	-	-	-	-	1	...

and

	0	−1	−2	...
−2:	4	17	44	...
−3:	10	70	283	...
−4:	5	38	164	...
−5:	-	-	1	...

Recall that by Theorem 6.2.6

$$\mathrm{lin}(\mathrm{res}(S_C)) = \bigoplus_{i=0}^{3} \mathbf{L}(H^i(\mathbf{R}(S_C)))$$

and $P^* \cong H^2\big(\mathbf{R}(S_C)\big)$ by self-duality of $\mathrm{res}(S_C)$ and Lemma 6.2.8. Therefore, applying the functor $\mathbf{L}$ to P (resp. P^*) we recover the strands of the canonical curve. In particular, we have

$$\beta(\mathbf{L}(P)) \sim \begin{array}{|cccc} \hline 15 & 35 & 25 & 4 \end{array} \quad \text{and} \quad \beta(\mathbf{L}(P^*)) \sim \begin{array}{|cccc} \hline 4 & 25 & 35 & 15 \end{array}.$$

On the other hand, the strands in a projective resolution of P correspond to the strands of an injective resolution of P^*, which in turn are given by $\mathbf{R}H_i(\mathbf{L}(P^*)) = \mathbf{R}H_i(\mathrm{strand}_2(S_C))$. Thus, the second linear strand of a general 4-gonal canonical curve of genus 8 has 4 non-vanishing homologies. This is no coincidence and will be explained in Theorem 6.3.6.

Now, if we replace the module P with the linearized module, i.e. the module presented by the linearized presentation matrix $\mathrm{lin}(\mathrm{pres}(P))$, then our experiment shows, that the Hilbert functions of P and $\mathrm{lin}(P)$ coincide. In particular

$$\beta(\mathbf{L}(\mathrm{lin}(P))) \sim \begin{array}{|cccc} \hline 15 & 35 & 25 & 4 \end{array}$$

although $\mathrm{strand}_1(S_C) = \mathbf{L}(P)$ is not homotopic to $\mathbf{L}(\mathrm{lin}(P))$. This observation is topic of Proposition 6.3.9.

If $X \subset \mathbb{P}^{g-1}$ is the scroll swept out by the unique g^1_4 on C then X has a minimal free resolution of the form

$$\beta(X) \sim \begin{array}{|ccccc} \hline 1 & - & - & - & - \\ - & 10 & 20 & 15 & 4 \end{array}$$

If we pick the linear 4×17 submatrix p_1 of pres(P) then

$$\beta\big(\mathbf{L}(\mathrm{coker}(p_1))\big) \sim \begin{array}{|cccc} \hline 4 & 15 & 20 & 10 \end{array}$$

and $\mathbf{L}(\mathrm{coker}(p_1))$ is indeed homotopic the the S-dual of the first strand of X. This holds more generally and will be explained in Proposition 6.3.5.

Before proving the last observation in the example above, we state the following definition.

Definition 6.3.3 ([SSW13]). Let $C \subset \mathbb{P}^{g-1}$ be a k-gonal canonical curve of genus g which is contained in a $(k-1)$-dimensional rational normal scroll X. The curve C is called *goneric* (or *k-goneric*) if the first linear strand has length $g-k$ and

$$\beta_{g-k,g-k+1}(C) = \beta_{g-k,g-k+1}(X) = g-k.$$

Remark 6.3.4. In [SSW13] it is conjectured that a curve C of linear colength l is $(l+2)$ goneric if and only if $\beta_{g-2-l,g-1-l}(C) = g-2-l$ unless $(g,l) = (6,1)$ and C is isomorphic to a plane quintic.

There are plenty of examples of goneric curves.

(1) For general k-gonal curves (with $k = 3,4,5$) it follows from the generic balancedness of the relative canonical resolution and the iterated mapping cone construction (see Section 2.3.3 and Remark 3.2.4), that these curves are k-goneric.

(2) All 4-gonal curves which do not have a g^r_{2+2r} are goneric by [Sch86].

(3) If the characteristic of the ground field is zero it follows from [HR98] and [Voi05] that general curves in the biggest gonality stratum $\mathscr{M}^1_{g,\lceil g/2\rceil} \subset \mathscr{M}_g$ are goneric. This was recently generalized by Farkas and Kemeny [FK16], who showed that general curves of non-maximal gonality k are goneric.

Note that general curves of genus g are not goneric.

For k-goneric curves, we can recover the scroll from the first strand of a resolution of $P = H^1(\mathbf{R}(S_C))$.

Proposition 6.3.5. *Let* $C \subset \mathbb{P}^{g-1}$ *be a k-goneric curve and let* X *be the* $(k-1)$*-dimensional scroll containing* C*. Then, with notation as before, the first strand appearing in an injective resolution of* $P^* = H^1(\mathbf{R}(S_C))^*$ *is given by* $\mathbf{R}(\omega_X)$ *where* ω_X *denotes the module associated to the canonical bundle on* X.

Proof. The statement is very similar to [SSW13, Prop 4.11]. We repeat their argument. First note that the first strand in an injective resolution of P^* corresponds to the first non-vanishing homology in $\mathrm{strand}_2(S_C) = \mathrm{Hom}_S(\mathrm{strand}_1(S_C), S)$. Let $\mathrm{strand}_1(X)$ be the first linear strand of a minimal free resolution of X. Then, by adjunction, the cokernel of the last (non-zero) map $\varphi^\vee$ in the dual complex $\mathrm{strand}_1(X)^\vee$ is precisely ω_X. Recall that $\mathrm{strand}_1(X)$ is a sub-complex of $\mathrm{strand}_1(C)$. Now C being k-goneric means, that we have a commutative diagram

$$\begin{tikzcd} \mathrm{strand}_1(C)_{g-k-1} & \mathrm{strand}_1(C)_{g-k} \arrow[l] \arrow[d, equal] \\ \mathrm{strand}_1(X)_{g-k-1} \arrow[u, hook] & \mathrm{strand}_1(X)_{g-k} \arrow[l, "\varphi"'] \end{tikzcd}$$

Dualizing the above diagram we get a commutative diagram

$$\begin{tikzcd} \mathrm{strand}_1(C)^\vee_{g-k-1} \arrow[r] \arrow[d] & \mathrm{strand}^\vee_1(C)_{g-k} \arrow[d, equal] \\ \mathrm{strand}_1(X)^\vee_{g-k-1} \arrow[r, "\varphi^\vee"] & \mathrm{strand}^\vee_1(X)_{g-k} \end{tikzcd}$$

and conclude that $\omega_X = \mathrm{coker}(\varphi^\vee) = \mathrm{coker}(\mathrm{strand}_1(C)^\vee_{g-k-1} \to \mathrm{strand}^\vee_1(C)_{g-k})$ □

Now that we understand the first strand in an injective resolution of $H^1(\mathbf{R}(S_C))^*$, we can ask where the other strands come from. Alternatively, one could ask which homologies in the linear strands of minimal free resolutions of canonical curves do not vanish and whether there is also a geometric interpretation for the non-vanishing.

At least in the case of general 4-gonal canonical curves we are able to give a complete description of the homologies appearing in the first and second linear strand. This will be topic of the next section.

6.3.1 Linear strands of general 4-gonal curves

Homologies in the linear strands. The aim of this paragraph is to describe all homologies appearing in the linear strands of a general 4-gonal canonical curves, or equivalently, with notation as in the previous sections, we want to describe the strands appearing in injective resolutions of $\mathrm{P} = \mathrm{H}^1(\mathbf{R}(\mathrm{S_C}))$ and $\mathrm{P}^* = \mathrm{H}^2(\mathbf{R}(\mathrm{S_C}))$. The main tool for this purpose will be the iterated mapping cone construction, which was introduced in Section 2.3.3.

We will give an explicit description of the homologies in the linear strands in terms of the maps in the iterated mapping cone, obtained by resolving the $\mathscr{O}_{\mathbb{P}(\mathscr{E})}$-modules in a relative canonical resolution of C by Eagon-Northcott type complexes. The main reason why we restrict to general 4-gonal curves is that in this case the iterated mapping cone always gives a minimal free resolution.

Throughout this section we denote by $\mathrm{C} \subset \mathbb{P}^{g-1}$ a general 4-gonal canonical curve of genus $g > 6$. We further denote by X the rational normal scroll swept out by the unique g^1_4 on C and $\mathbb{P}(\mathscr{E})$ will denote the projective bundle associated to X. Recall from Remark 3.2.4 that general 4-gonal curves have a balanced relative canonical resolution of the form

$$0 \to \mathscr{O}_{\mathbb{P}(\mathscr{E})}(-4\mathrm{H}+(g-5)\mathrm{R}) \to \mathscr{O}_{\mathbb{P}(\mathscr{E})}(-2\mathrm{H}+b_1\mathrm{R}) \oplus \mathscr{O}_{\mathbb{P}(\mathscr{E})}(-2\mathrm{H}+b_2\mathrm{R}) \to \mathscr{O}_{\mathbb{P}(\mathscr{E})} \to \mathscr{O}_\mathrm{C} \to 0,$$

with $b_1 + b_2 = g-5$.

We briefly recall the iterated mapping cone construction from Section 2.3.3. From the multiplication map

$$\mathrm{H}^0(\mathbb{P}(\mathscr{E}), \mathscr{O}_{\mathbb{P}(\mathscr{E})}(\mathrm{R})) \otimes \mathrm{H}^0(\mathbb{P}(\mathscr{E}), \mathscr{O}_{\mathbb{P}(\mathscr{E})}(\mathrm{H}-\mathrm{R})) \longrightarrow \mathrm{H}^0(\mathbb{P}(\mathscr{E}), \mathscr{O}_{\mathbb{P}(\mathscr{E})}(\mathrm{H}))$$

we obtain a $2 \times (g-3)$ matrix Φ, whose minors define the homogeneous ideal of X. We regard Φ as a map $\Phi : \mathrm{F}(-1) \to \mathrm{G}$, where

$$\mathrm{F} := \mathrm{H}^0(\mathbb{P}(\mathscr{E}), \mathscr{O}_{\mathbb{P}(\mathscr{E})}(\mathrm{H}-\mathrm{R})) \otimes \mathscr{O}_{\mathbb{P}^{g-1}} = \mathscr{O}^{g-3}_{\mathbb{P}^{g-1}} \text{ and } \mathrm{G} := \mathrm{H}^0(\mathbb{P}(\mathscr{E}), \mathscr{O}_{\mathbb{P}(\mathscr{E})}(\mathrm{R})) \otimes \mathscr{O}_{\mathbb{P}^{g-1}} = \mathscr{O}^2_{\mathbb{P}^{g-1}}.$$

For $b \in \{0,\ b_1\ , b_2,\ g-5\}$, the Eagon-Northcott type complex $\mathscr{C}^b$, whose j-th term is defined by

$$\mathscr{C}^b_j = \begin{cases} \bigwedge^j \mathrm{F} \otimes \mathrm{S}_{b-j}\mathrm{G} \otimes \mathscr{O}_{\mathbb{P}^r}(-j), & \text{for } 0 \le j \le b \\ \bigwedge^{j+1} \mathrm{F} \otimes \mathrm{D}_{j-b-1}\mathrm{G}^* \otimes \mathscr{O}_{\mathbb{P}^r}(-j-1), & \text{for } j \ge b+1 \end{cases},$$

(with differentials induced by the multiplication with Φ) is a minimal free resolution by $\mathscr{O}_{\mathbb{P}^{g-1}}$-modules of $\mathscr{O}_{\mathbb{P}(\mathscr{E})}(b\mathrm{R})$. Furthermore, for general 4-gonal canonical curves, the iterated mapping cone

$$\left[\left[\mathscr{C}^{g-5}(-4) \longrightarrow \mathscr{C}^{b_1}(-k+2) \oplus \mathscr{C}^{b_2}(-k+2)\right] \longrightarrow \mathscr{C}^0\right]$$

is a minimal free resolution of $\mathrm{C} \subset \mathbb{P}^{g-1}$.

In what follows, we restrict to curves of odd genus, since in this case $b := b_1 = b_2 = \frac{g-5}{2}$. The even genus case can be treated similarly. The iterated mapping cone has the form as indicated on the next page. Here arrows marked in red have linear entries and contribute to the first linear strand whereas blue arrows are linear maps contributing to the second linear strand. Black arrows as well as dotted arrows indicate maps with quadratic entries.

$$
\begin{tikzcd}
\mathscr{C}_0^{g-5}(-4) \arrow[r, "\rho_0"] \arrow[drr, dotted] & \mathscr{C}_0^{b}(-2)^{\oplus 2} \arrow[r, "\delta_0"] & \mathscr{C}_0^{0} \\
\mathscr{C}_1^{g-5}(-4) \arrow[r, "\rho_1"] \arrow[u, "\alpha_1"] & \mathscr{C}_1^{b}(-2)^{\oplus 2} \arrow[r, "\delta_1"] \arrow[u, "\beta_1"] & \mathscr{C}_1^{0} \arrow[u, "\gamma_1"] \\
\vdots \arrow[u] & \vdots \arrow[u] & \vdots \arrow[u] \\
\mathscr{C}_{b-1}^{g-5}(-4) \arrow[r, "\rho_{b-1}"] \arrow[u, "\alpha_{b-1}"] \arrow[drr, dotted] & \mathscr{C}_{b-1}^{b}(-2)^{\oplus 2} \arrow[r, "\delta_{b-1}"] \arrow[u, "\beta_{b-1}"] & \mathscr{C}_{b-1}^{0} \arrow[u, "\gamma_{b-1}"] \\
\mathscr{C}_{b}^{g-5}(-4) \arrow[r, "\rho_{b}"] \arrow[u, "\alpha_{b}"] \arrow[drr, dotted] & \mathscr{C}_{b}^{b}(-2)^{\oplus 2} \arrow[r, "\delta_{b}"] \arrow[u, "\beta_{b}"] & \mathscr{C}_{b}^{0} \arrow[u, "\gamma_{b}"] \\
\mathscr{C}_{b+1}^{g-5}(-4) \arrow[r, "\rho_{b+1}"] \arrow[u, "\alpha_{b+1}"] \arrow[drr, dotted] & \mathscr{C}_{b+1}^{b}(-2)^{\oplus 2} \arrow[r, "\delta_{b+1}"] \arrow[u, "\beta_{b+1}"] & \mathscr{C}_{b+1}^{0} \arrow[u, "\gamma_{b+1}"] \\
\mathscr{C}_{b+2}^{g-5}(-4) \arrow[r, "\rho_{b+2}"] \arrow[u, "\alpha_{b+2}"] & \mathscr{C}_{b+2}^{b}(-2)^{\oplus 2} \arrow[r, "\delta_{b+2}"] \arrow[u, "\beta_{b+2}"] & \mathscr{C}_{b+2}^{0} \arrow[u, "\gamma_{b+2}"] \\
\vdots \arrow[u] & \vdots \arrow[u] & \vdots \arrow[u] \\
\mathscr{C}_{g-5}^{g-5}(-4) \arrow[r, "\rho_{g-5}"] \arrow[u, "\alpha_{g-5}"] \arrow[drr, dotted] & \mathscr{C}_{g-5}^{b}(-2)^{\oplus 2} \arrow[r, "\delta_{g-5}"] \arrow[u, "\beta_{g-5}"] & \mathscr{C}_{g-5}^{0} \arrow[u, "\gamma_{g-5}"] \\
\mathscr{C}_{g-4}^{g-5}(-4) \arrow[r, "\rho_{g-4}"] \arrow[u, "\alpha_{g-4}"] & \mathscr{C}_{g-4}^{b}(-2)^{\oplus 2} \arrow[r, "\delta_{g-4}"] \arrow[u, "\beta_{g-4}"] & \mathscr{C}_{g-4}^{0} \arrow[u, "\gamma_{g-4}"] \\
0 \arrow[u] & 0 \arrow[u] & 0 \arrow[u]
\end{tikzcd}
$$

The main result in this paragraph is the following.

Theorem 6.3.6. *Let* $C \subset \mathbb{P}^{g-1}$ *be a general* 4*-gonal canonical curve of odd genus* $g > 6$*. Let further* S_C *denote the coordinate ring of* $C \subset \mathbb{P}^{g-1}$ *and let* $b = \frac{g-5}{2}$*. Then the only non-vanishing homologies in the first and second linear strand of a minimal free resolution of* C *are*

- $H_1(\mathrm{strand}_1(S_C)) \cong \mathrm{coker}\begin{pmatrix} -\beta_1 & 0 \\ \delta_1 & \gamma_2 \end{pmatrix}$
- $H_{b+1}(\mathrm{strand}_1(S_C)) \cong \mathrm{coker}(\beta_{b+2})$
- $H_2(\mathrm{strand}_2(S_C)) \cong \mathrm{coker}(\alpha_1) \cong \omega_X$
- $H_{b+2}(\mathrm{strand}_2(S_C)) \cong \mathrm{coker}(\beta_{b+2})$
- $H_{g-3}(\mathrm{strand}_2(S_C)) \cong \ker(\alpha_{g-5}) \cong S$

Proof. First note that the vanishing of all other homologies follows from the exactness of the Eagon-Northcott type complexes and the corresponding mapping cones. The statements for $H_2(\mathrm{strand}_2(S_C))$ and $H_{g-3}(\mathrm{strand}_2(S_C))$ follow from the discussion at the beginning of this section and Proposition 6.3.5. Now, the identification $H_1(\mathrm{strand}_1(S_C)) \cong \mathrm{coker}\begin{pmatrix} -\beta_1 & 0 \\ \delta_1 & \gamma_2 \end{pmatrix}$ is obvious from the diagram and it therefore remains to show that

$$H_{b+1}(\mathrm{strand}_1(S_C)) \cong H_{b+2}(\mathrm{strand}_2(S_C)) \cong \mathrm{coker}(\beta_{b+2}).$$

We compute a presentation of both homologies, starting with $H_{b+2}(\mathrm{strand}_2(S_C))$. To this end note the following. If M is an S-module which is given by generators and relations, i.e. there are matrices A and B over S with the same target and $M = (\mathrm{Im}(A) + \mathrm{Im}(B))/\mathrm{Im}(B)$, then a matrix Q is a presentation of M if the image of Q is the same as the preimage under A of the image of B. Now

$$H_{b+2}(\mathrm{strand}_2(S_C)) = \ker\left[\mathscr{C}_b^{g-5}(-4) \oplus \mathscr{C}_{b+1}^{b}(-2)^{\oplus 2} \xrightarrow{(\alpha_b,\ 0)} \mathscr{C}_{b-1}^{g-5}(-4)\right] \Big/ \mathrm{Im}\begin{pmatrix} -\alpha_{b+1} & 0 \\ \rho_{b+1} & \beta_{b+2} \end{pmatrix}$$

and $\ker(\alpha_b,\ 0) = \mathrm{Im}\begin{pmatrix} -\alpha_{b+1} & 0 \\ 0 & id_{\mathscr{C}_{b+1}^{b}(-2)^{\oplus 2}} \end{pmatrix} = \mathrm{Im}\begin{pmatrix} -\alpha_{b+1} & 0 \\ \rho_{b+1} & id_{\mathscr{C}_{b+1}^{b}(-2)^{\oplus 2}} \end{pmatrix}$. Therefore we conclude, that a presentation of $H_{b+2}(\mathrm{strand}_2(S_C))$ is given by

$$\begin{pmatrix} id_{\mathscr{C}_b^{g-5}(-4)} & 0 \\ 0 & \beta_{b+2} \end{pmatrix}.$$

The homology $H_{b+1}(\text{strand}_1(S_C))$ is given as

$$H_{b+1}(\text{strand}_1(S_C)) = \ker\begin{pmatrix} -\beta_b & 0 \\ \delta_b & \gamma_{b+1} \end{pmatrix} \Big/ \operatorname{Im}\left[\mathscr{C}^0_{b+2} \xrightarrow{\begin{pmatrix} 0 \\ \gamma_{b+2} \end{pmatrix}} \mathscr{C}^b_b(-2)^{\oplus 2} \oplus \mathscr{C}^0_{b+1}\right]$$

Since the mapping cone $\left[\mathscr{C}^b(-2)^{\oplus 2} \longrightarrow \mathscr{C}^0\right]$ is exact, we have

$$\ker\begin{pmatrix} -\beta_b & 0 \\ \delta_b & \gamma_{b+1} \end{pmatrix} = \operatorname{Im}\begin{pmatrix} -\beta_{b+1} & 0 \\ \delta_{b+1} & \gamma_{b+2} \end{pmatrix}.$$

In order to compute a presentation matrix of $H_{b+1}(\text{strand}_1(S_C))$, we compute the relations of the surjection

$$\mathscr{C}^b_{b+1}(-2) \oplus \mathscr{C}^0_{b+2} \xrightarrow{\begin{pmatrix} -\beta_{b+1} & 0 \\ \delta_{b+1} & \gamma_{b+2} \end{pmatrix}} H_{b+1}(\text{strand}_2(S_C)).$$

If $\begin{pmatrix} -\beta_{b+1}x \\ \delta_{b+1}x + \gamma_{b+2}y \end{pmatrix} = 0 \in H_{b+1}(\text{strand}_1(S_C))$, then $x = \beta_{b+2}z$ for some $z \in \mathscr{C}^b_{b+2}(-2)$. Now

$$\delta_{b+1}x + \gamma_{b+2}y = \delta_{b+1}\beta_{b+2}z + \gamma_{b+2}y = \gamma_{b+2}\delta_{b+2}z + \gamma_{b+2}y = \gamma_{b+2}(\delta_{b+2}z + y) \in \operatorname{Im}(\gamma_{b+2})$$

and therefore, a presentation matrix for $H_{b+1}(\text{strand}_1(S_C))$ is given by

$$\begin{pmatrix} -\beta_{b+2} & 0 \\ 0 & id_{\mathscr{C}^0_{b+2}} \end{pmatrix}.$$

Hence $H_{b+1}(\text{strand}_1(S_C))$ and $H_{b+2}(\text{strand}_2(S_C))$ are both minimally presented by β_{b+2}. $\square$

Remark 6.3.7. The same approach yields a similar result for general 4-gonal canonical curves of even genus. For even genus the homologies $H_{b_i+1}(\text{strand}_1(S_C))$ and $H_{b_i+2}(\text{strand}_2(S_C))$ with $i = 1,2$ do not vanish and are presented by the corresponding matrices in $\mathscr{C}^{b_1}(-2)$ and $\mathscr{C}^{b_2}(-2)$.

The linearized presentation of $H^1(\mathbf{R}(S_C))$ for 4-gonal canonical curves. Let $C \subset \mathbb{P}^{g-1}$ be a general 4-gonal canonical curve of genus $g > 6$. Let further $P = H^1(\mathbf{R}(S_C))$ be the first cohomology in $\mathbf{R}(S_C)$ as before and let lin(P) be the module presented by the linearized presentation matrix lin(pres(P)).

In Example 6.3.2 we observed that the Hilbert functions $H_{\mathrm{lin}(P)}(i)$ and $H_P(i)$ coincide. This means that all the information about the Betti numbers $\beta_{i,j}(S_C)$ is already encoded in the linearized module lin(P). This paragraph is devoted to the study of the linearized modules lin(P) and lin(P^*).

Proposition 6.3.8. *With notation as above* $\beta(H^1(\mathbf{R}(S_C))) = \beta(\mathrm{lin}\ H^1(\mathbf{R}(S_C)))$.

Proof. Again, we restrict to the odd genus case, since the even genus case is treated similarly. Identifying $P^* \cong H^2(\mathbf{R}(S_C))$, it follows from Theorem 6.2.6 and Theorem 6.3.6 that

$$\mu(P^*) = \bigoplus_{j \in \{2, b+2, g-3\}} \mu\big(\mathbf{R}\big(H_j(\mathbf{L}H^2\mathbf{R}(S_C))\big)\big) = \bigoplus_{j \in \{2, b+2, g-3\}} \mu\big(\mathbf{R}\big(H_j(\mathrm{strand}_2(S_C))\big)\big), \quad (6.2)$$

where $b = \frac{g-5}{2}$. For odd genus, the presentation matrix pres(P) of P has a block structure as indicated below

$$\mathrm{pres}(P) \sim \begin{pmatrix} p_1 & * \\ 0 & p_2 \end{pmatrix}.$$

By Theorem 6.3.6 we know that $\mathrm{res}^{inj}(\mathrm{coker}(p_1)^*) = \mathbf{R}(\omega_X)$, where X is the scroll swept out by the unique g^1_4 on C. Now, $\mathbf{L}(H^0\mathbf{R}(\omega_X))$ has two non-vanishing homologies, namely $H_2\big(\mathbf{L}H^2\mathbf{R}(S_C)\big) \cong \omega_X$ and $H_{g-3}\big(\mathbf{L}H^2\mathbf{R}(S_C)\big) \cong S$. We conclude that

$$\mu(\mathrm{coker}(p_1)^*) = \mu\Big(\bigoplus_{i=2,g-3} \mathbf{R}H_j(\mathbf{L}H^2\mathbf{R}(S_C))\Big).$$

The second non-zero strand in an injective resolution of P^* is given by

$$\mathbf{R}\big(H_{b+2}\big(\mathbf{L}H^2\mathbf{R}(S_C)\big)\big) = \mathbf{R}\big(H_{b+2}(\mathrm{strand}_2(S_C)\big).$$

By Theorem 6.3.6 the homology $H_{b+2}(\mathrm{strand}_2(S_C)$ is resolved by a linear part of an Eagon-Northcott type complex,

$$0 \leftarrow H_{b+2}(\mathrm{strand}_2(S_C)) \leftarrow \mathscr{C}^b_{b+1}(-2)^{\oplus 2} \leftarrow \mathscr{C}^b_{b+2}(-2)^{\oplus 2} \leftarrow \cdots \leftarrow \mathscr{C}^b_{g-4}(-2)^{\oplus 2} \leftarrow 0.$$

In particular $H_{b+2}(\mathrm{strand}_2(S_C))$ is 0-regular and therefore $\mathbf{R}\big(H_{b+2}(\mathrm{strand}_2(S_C))\big)$ is an injective resolution of $\mathrm{coker}(p_2)^*$ by Corollary 6.2.3. Putting this together, we obtain

$$\begin{aligned}\mu\big(\mathrm{lin}(P)^*\big) &= \mu\big(\mathrm{coker}(p_1 \oplus \mathrm{coker}\, p_2)^*\big) = \mu\big(\mathrm{coker}(p_1)^* \oplus \mathrm{coker}(p_2)^*\big)\\ &= \mu\Big(\bigoplus_{j=2,b+2,g-3} \mathbf{R}H_j(\mathrm{strand}_2(S_C))\Big) \overset{(6.2)}{=} \mu(P^*).\end{aligned}$$

The statement of the proposition now follows by dualizing. □

Similarly we can proof the following.

Proposition 6.3.9. *With notation as above* $\beta(\mathbf{L}(P^*)) = \beta(\mathbf{L}(\mathrm{lin}(P)^*))$ *and (up to shift)* $\beta(\mathbf{L}(P)) = \beta(\mathbf{L}(\mathrm{lin}(P)))$.

Proof. Since $\mathbf{R}\bigoplus_j H_j(\mathrm{strand}_2(S_C))$ is the linearized injective resolution of P^* it follows that

$$\mathrm{lin}(P^*) = H^0\Big(\mathbf{R}\bigoplus_j H_j(\mathrm{strand}_2(S_C))\Big)$$

By Corollary 6.2.5 we have

$$\mathrm{lin}\ \mathbf{L}\Big(\bigoplus_j \mathbf{R}H_j(\mathrm{strand}_2(S_C))\Big) = \bigoplus_i \mathbf{L}\Big(H^i\Big(\bigoplus_j \mathbf{R}H_j\big(\mathrm{strand}_2(S_C)\big)\Big)\Big). \tag{6.3}$$

We analyze both sides of the above equation, starting with the left hand side. We have

$$\mathrm{lin}\ \mathbf{L}\Big(\bigoplus_j \mathbf{R}H_j(\mathrm{strand}_2(S_C))\Big) = \mathrm{lin}\ \mathbf{LR}\Big(\bigoplus_j H_j(\mathrm{strand}_2(S_C))\Big)$$

and by Theorem 6.2.4 $\mathbf{LR}\big(\bigoplus_j H_j(\mathrm{strand}_2(S_C))\big)$ is a (non-minimal) free resolution of $\bigoplus_j H_j(\mathrm{strand}_2(S_C))$, where the homologies are considered as 1-term complexes in homological degree j. Now, by Theorem 6.3.6,

$$\mathrm{lin}\ \mathbf{L}\Big(\bigoplus_j \mathbf{R}H_j(\mathrm{strand}_2(S_C))\Big) = \mathrm{lin}\ \mathbf{LR}\Big(\bigoplus_{j\in\{2,b+2,g-3\}} H_j(\mathrm{strand}_2(S_C))\Big)$$

where $H_2(\mathrm{strand}_2(S_C)) \cong \omega_X$, $H_{b+2}(\mathrm{strand}_2(S_C)) \cong \mathrm{coker}\big(\mathscr{C}^b_{b+1}(-2)^{\oplus 2} \leftarrow \mathscr{C}^b_{b+2}(-2)^{\oplus 2}\big)$ and $H_{g-3}(\mathrm{strand}_2(S_C)) \cong S$. In particular

$$\beta\Big(\mathrm{lin}\ \mathbf{L}\Big(\bigoplus_j \mathbf{R}H_j\big(\mathrm{strand}_2(S_C)\big)\Big)\Big) \sim \begin{array}{|c|c|}\hline \mathrm{strand}_2(S_C) & - \\ \hline - \ \ \dots \ \ - & 2 \\ \hline\end{array}$$

For the right hand side of (6.3) note that $\mathbf{R}\bigoplus_j \mathrm{H}_j(\mathrm{strand}_2(\mathrm{S}_\mathrm{C}))$ has three non-vanishing cohomologies. The cohomologies $\mathrm{H}^1\big(\mathbf{R}\bigoplus_j \mathrm{H}_j(\mathrm{strand}_2(\mathrm{S}_\mathrm{C}))\big)$ and $\mathrm{H}^2\big(\mathbf{R}\bigoplus_j \mathrm{H}_j(\mathrm{strand}_2(\mathrm{S}_\mathrm{C}))\big)$ are both isomorphic to the groundfield $\Bbbk$ over the exterior algebra E and therefore applying the functor $\mathbf{L}$ to them gives a complex homotopic to S. By comparing both sides of (6.3) we conclude that

$$\beta(\mathrm{lin}(\mathrm{P}^*)) = \beta(\mathrm{strand}_2(\mathrm{S}_\mathrm{C})) = \beta\mathbf{L}(\mathrm{P}^*)$$

Now since $\mathrm{lin}(\mathrm{P})^* = \mathrm{lin}(\mathrm{P}^*)$ it follows that

$$\beta\big(\mathbf{L}\big(\mathrm{lin}(\mathrm{P})^*\big)\big) = \beta\big(\mathbf{L}\big(\mathrm{P}^*\big)\big).$$

and since $(\mathrm{P}^*)_i = \mathrm{Hom}_\Bbbk(\mathrm{P},\Bbbk)_i \cong \mathrm{Hom}_\Bbbk(\mathrm{P}_{-i},\Bbbk) = (\mathrm{P}_{-i})^*$ we see that up to shift $\beta(\mathbf{L}(\mathrm{P})) = \beta(\mathbf{L}(\mathrm{lin}(\mathrm{P})))$. □

Remark 6.3.10. The above proposition can also be proved by using Proposition 6.3.8 together with the fact, that (similarly to the case of resolutions over S) the Hilbert function is determined by the Betti (resp. Bass) numbers of a minimal free (resp. injective) resolution.

Another prove can be obtained as follows. Using Theorem 6.3.6 and carefully analyzing the degrees of the homologies in $\mathrm{strand}_1(\mathrm{S}_\mathrm{C})$ one can show that P^* has a linear presentation matrix. Therefore the above proposition trivially holds for P^* and hence for its $\Bbbk$-dual P.

Canonical curves of higher gonality. Before summarizing the situation for canonical curves of higher gonality, we make a remark on the situation for 3-gonal canonical curves and general canonical curves.

Remark 6.3.11. (a) Similar to the case of 4-gonal curves of genus $g \geq 7$, the mapping cone

$$\left[\mathscr{C}^{g-4} \longrightarrow \mathscr{C}^0\right]$$

gives a minimal free resolution for a general 3-gonal canonical curve C of genus $g \geq 5$. Hence, the two linear strands of C are the linear strands of $\mathscr{C}^0$ and $\mathscr{C}^{g-4}$.

In particular,

$$\mathrm{H}^i(\mathrm{strand}_1(\mathrm{S_C})) \cong \begin{cases} \mathrm{S_X} & \text{if } i = 1 \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad \mathrm{H}^i(\mathrm{strand}_2(\mathrm{S_C})) \cong \begin{cases} \omega_\mathrm{X} & \text{if } i = 1 \\ \mathrm{S} & \text{if } i = g-3 \\ 0 & \text{otherwise} \end{cases}$$

where $\mathrm{S_X}$ is the coordinate ring of the scroll X swept out by the g^1_k on C. In particular $\mathrm{H}^1\mathbf{R}(\mathrm{S_C})$ and $\mathrm{H}^2\mathbf{R}(\mathrm{S_C})$ both have a linear presentation matrix.

(b) Let $\mathrm{C} \subset \mathbb{P}^{g-1}$ be a general canonically embedded curve of odd genus defined over an algebraically closed field. Then, by Voisins proof of the generic Green conjecture (see [Voi05]), $\mathrm{strand}_1(\mathrm{S_C})$ and $\mathrm{strand}_2(\mathrm{S_C})$ have precisely two non-vanishing homologies. Hence, setting $m = \frac{g-3}{2}$ and denoting by φ_{m+2} the first non-zero map in $\mathrm{strand}_2(\mathrm{S_C})$ we get

$$\mathrm{H}_1(\mathrm{strand}_1(\mathrm{S_C})) \cong \mathrm{I_C}$$
$$\mathrm{H}_m(\mathrm{strand}_1(\mathrm{S_C})) \cong \mathrm{H}_{m+1}(\mathrm{strand}_2(\mathrm{S_C})) \cong \mathrm{coker}(\varphi_{m+2})$$
$$\mathrm{H}_{g-3}(\mathrm{strand}_2(\mathrm{S_C})) \cong \mathrm{S}$$

For even genus, the number of non-vanishing homologies in the linear strand is less obvious. However, setting $m = \lfloor \frac{g-3}{2} \rfloor$, experiments for small genus indicate, that the statement is the same as above for odd genus (see also Conjecture 6.3.12).

For higher gonality the situation becomes much more difficult, since in these cases the iterated mapping cone is no longer minimal. The next case, which has the best chances to be understood are 5-gonal canonical curves. In this case, because of the Buchsbaum-Eisenbud structure theorem for Gorenstein ideals in codimension 3, we have a precise understanding of the maps in the relative canonical resolution. This allows us in some cases to gain control on the non-minimal parts in the iterated mapping cone (see Chapter 3 for details).

However, as we have seen in Section 3.2, the horizontal maps, which give rise to non-minimal parts in the iterated mapping cone, no longer have maximal rank for large genus. As pointed out at the end of Section 3.3, we also do not have a complete description of the space of extra syzygies (as in the genus 13 case by Proposition 3.3.1).

Altogether it seems very hard to gain control on the homologies in the linear strands of 5-gonal curves by using similar methods as for 4-gonal curves.

Experiments using the *Macaulay2* package [ADE+12] show that, for 5-gonal canonical curves, the middle homologies have in general no linear resolution, which was the key point in our proof of Proposition 6.3.8 and Proposition 6.3.9. This occurs for example for general 5-gonal canonical curve of genus 13. Moreover Proposition 6.3.8 and Proposition 6.3.9 do not hold in this case.

Higher gonality curves share all the difficulties of the 5-gonal case. In addition to that, because of the non-existence of structure theorems for Gorenstein ideals in higher codimension, we no longer have a good understanding of the horizontal maps in the relative canonical resolution. The number of homologies also depends on the balancing type of the bundles in the relative canonical resolution, which is not known in general for curves of higher gonality (c.f. Chapter 4 and 5).

However, motivated by experiments using the *Macaulay2*, we make the following conjecture concerning the validity of Proposition 6.3.8 and Proposition 6.3.9 for curves of higher gonality.

Conjecture 6.3.12. *Let* C *be a non-hyperelliptic* k*-gonal canonical curve without unexpected extra syzygies, in the sense of Chapter 3 (i.e.* $\beta_{p,p+1}(C) = \beta_{p,p+1}(X)$ *for all* $p \geq \lceil \frac{g-1}{2} \rceil$*), then*

$$\beta\big(H^1\mathbf{R}(S_C)\big) = \beta\big(\mathrm{lin}H^1\mathbf{R}(S_C)\big) \quad \textit{and} \quad \beta\big(\mathbf{L}\big(H^1\mathbf{R}(S_C)^*\big)\big) = \beta\big(\mathbf{L}\big(\mathrm{lin}H^1\mathbf{R}(S_C)^*\big)\big)$$

Using the *Macaulay2*-packages [BH17b] and [ADE+12] we tested the above conjecture for k-gonal g-nodal canonical curves for for $g \leq 12$ and $2 < k < \lceil \frac{g+4}{2} \rceil$.

6.4 Green's conjecture in positive characteristic

As we have seen in Remark 6.3.1 we can rephrase the statement of Green's conjecture in terms of the vanishing of the homologies in the linear strands:

$$H_i(\mathrm{strand}_2(S_C)) = 0 \text{ for all } i \leq p \text{ if and only if } \mathrm{Cliff}(C) > p.$$

The purpose of this section is to give experimental evidence for a refined version of Green's conjecture, which includes fields of positive characteristic. Recall that for an algebraically closed field $\Bbbk$ with $\mathrm{char}(\Bbbk) > 0$ it is known that the statement of Green's conjecture does not hold in some cases. For example, by [Sch86] and [Muk10], the Betti tables of a general genus 7 curve over a field of characteristic 2 and a general genus 9 curve over a field of characteristic 3 are of the form:

1	.	.	.	.	.
.	10	16	1	.	.
.	.	1	16	10	.
.	.	.	.	.	1

and

1	.	.	.	.	.	.	.
.	21	64	70	6	.	.	.
.	.	.	6	70	64	21	.
.	.	.	.	.	.	.	1

It arises the question how the homologies in the linear strands behave. An experiment using the *Macaulay2* packages [ADE⁺12] and [BS17] shows in these cases, that $P = H^1\mathbf{R}(S_C)$ has a resolution of the following form

	0	−1	−2	−3	...
−2 :	1	−	−	−	...
−3 :	9	74	294	855	...
−4 :	−	−	1	7	...

and

	0	−1	−2	−3	...
−3 :	6	−	−	...	
−4 :	16	269	1605	...	
−5 :	−	−	−	...	
−6 :	−	−	1	...	

Thus, if we denote by φ the first non-zero map in $\mathrm{strand}_2(S_C)$, then $\mathrm{coker}(\varphi)$ is an S-module of finite length.

Conjecturally these are not the only exceptional cases, for which (the generic) Green's conjecture fails (see also [Sch03]). Along the lines of the *Macaulay2* packages [BGS11] and [Sch13], we have implemented the construction of random curves of genus $g \leq 15$ over very small finite fields in the *Macaulay2*-package *RandomCurvesOverVerySmallFiniteFields* [BS17]. Based on our experiments, we conjecture that the exceptional cases for the generic Green's conjecture for genus $g \leq 15$ are as follows.

genus	char($\Bbbk$)	extra syzygies
7	2	$\beta_{2,4} = 1$
9	3	$\beta_{3,5} = 6$
11	2, 3	$\beta_{4,6} = 28,\ 10$
12	2	$\beta_{4,6} = 1$
13	2, 5	$\beta_{5,7} = 64,\ 120$
15	2, 3, 5	$\beta_{6,8} = 299,\ 390,\ 315$

For the complete data of our experiments see Section 7.3. Note that our experiments neither do show that the generic Green's conjecture fails for the cases listed in the table above nor that there are no further exceptions in higher characteristic. Our evidence is based on our belief, that such exceptions can only occur for small characteristic and that it is very unlikely that we always hit the locus inside $\mathscr{M}_g$ consisting of curves with extra syzygies.

In all our examples for curves in the above table, the homology in the second linear strand at the critical position is a module of finite length. We note that therefore, the extra syzygies can not be induced by a linear series on the curve. Thus, the examples in the table above do not satisfy the statement of the classical Green's conjecture.

For genus 12 this can be seen from the fact, that the one extra syzygy is too small to be induced by a linear series. More generally, by Green and Lazarsfeld's proof of the "easier" direction in Green's conjecture (see [Gre84, Appendix]), this is also true for the other cases. Given a g^r_d on a canonical curve $C \subset \mathbb{P}^{g-1}$ Green and Lazarsfeld explicitly construct a syzygy (induced by this linear series) which does not involve all variables. Hence if one of the first syzygies in the second linear strand is induced by a linear series on C, then the homology at this position can not be a module of finite length.

It is worth mentioning, that general genus 10 or genus 14 curves defined over a field of characteristic 2 do in general not have extra syzygies. However, there exist smooth curves of genus 10 and 14 which have precisely one extra syzygy and therefore also violate the statement in the classical Green's conjecture for characteristic 2.

Based on experiments using the *Macaulay2* package *RandomCurvesOverVerySmall-FiniteFields* [BS17] (see Section 7.3 for the complete data of our experiments) we make the following conjecture.

Conjecture 6.4.1 (Refined Green Conjecture). *Let* $C \subset \mathbb{P}^{g-1}$ *a canonically embedded curve defined over an algebraically closed field and let*

$$\mathrm{strand}_2(S_C):\ 0 \leftarrow S(-3)^{\beta_{1,3}} \xleftarrow{\varphi_2} S(-4)^{\beta_{2,4}} \xleftarrow{\varphi_3} \dots \xleftarrow{\varphi_{g-3}} S(-(g-1))^{\beta_{g-3,g-1}} \leftarrow 0$$

be the second linear strand of a minimal free resolution of the coordinate ring S_C *(here* $S(-(i+2))^{\beta_{i,i+2}}$ *sits in homological degree* i*). Then*

(a) $H_i(\mathrm{strand}_2(S_C))$ *is a module of finite length for all* $i \le p$ *if and only if* $\mathrm{Cliff}(C) > p$.

(b) If C *is general inside the gonality stratum* $\mathcal{M}^1_{g,k} \subset \mathcal{M}_g$ *with* $2 < k < \lceil \frac{g+2}{2} \rceil$ *then* $H_{k-2}(\mathrm{strand}_2(S_C))$ *is supported on the rational normal scroll swept out by the unique* g^1_k *on* C.

Remark 6.4.2. (a) Note that the generality assumption in part (b) of the above conjecture is necessary. For instance the statement will not hold for a curve which has several g^1_k's.

(b) Recall from Section 2.3 that if $C \subset \mathbb{P}^{g-1}$ is a canonical curve of genus g which is contained in a rational normal scroll X of dimension $k-1$ and degree $g-k+1$, then the ruling on X cuts out a g^1_k on C. Thus if the first non-vanishing homology $H_{k-2}(\mathrm{strand}_2(S_C))$ in the second linear strand of C is supported on a scroll with the above invariants, then C admits a g^1_k.

Chapter 7

Computeralgebra and experimental results

In this chapter we briefly present the two *Macaulay2*-packages *RelativeCanonicalResolutions* and *RandomCurvesOverVerySmallFiniteFields* (see [BH17b] and [BS17]). Both packages were intensively used for experiments which motivated many of the results or conjectures presented in this thesis (see e.g Conjecture, 4.4.2 Proposition 5.3.1 or Conjecture 6.4.1). We will show an exemplary computation for each of the two packages. For a documentation of both packages see [BH15a] and [BS17]. In Section 7.3 we present our experimental data which gives us some evidence on the refined Green conjecture (see Conjecture 6.4.1).

7.1 The package RelativeCanonicalResolutions

This package contains methods for the construction of g-nodal k-gonal canonical curves, such that the scroll swept out by the g^1_k has a nice determinantal representation. For the theoretical background of the construction of g-nodal k-gonal curves see [Bop13]. The package also provides methods to compute the relative canonical resolution as well as the iterated mapping cone (see Section 2.3 for details on relative canonical resolutions and iterated mapping cones). For a documentation of the package see [BH15a].

Example 7.1.1. We compute a nodal 5-gonal canonical curve of genus 8.

```
i1 : loadPackage("RelativeCanonicalResolution")
i2 : g=8; -- the genus
i3 : k=5; -- the degree of the pencil
i4 : n=10000; -- nextPrime(n) will define the characteristic of the groundfield
i5 : Ican=canCurveWithFixedScroll(g,k,n); -- the desired canonical curve
i6 : (dim Ican,genus Ican, degree Ican)
o6 = (2, 8, 14)
i7 : betti(res(Ican,DegreeLimit=>1))
            0  1  2  3
o7 = total: 1 15 35 21
         0: 1  .  .  .
         1: . 15 35 21
```

Next, we compute the ideal of the curve on the scroll X and relative canonical resolution.

```
i8 : Jcan=curveOnScroll(Ican,g,k); -- the curve on the scroll
i9 : RX=ring Jcan; -- the bigraded coordinate ring of the scroll
```

By Remark 2.3.2, the coordinate ring R_X of a scroll X of type $(e_1,\ldots,e_d)$ can be described as $\Bbbk[s,t,\varphi_1,\ldots,\varphi_d]$ with grading $\deg(s)=\deg(t)=(1,0)$ and $\deg(\varphi_j)=(e_1-e_j,1)$.

```
i10 : T=ring Ican; -- the canonical ring
i11 : H=basis({1,1},RX); -- a basis of H^0(PE, OO_PE(H))
i12 : phi=map(RX,T,H)
i13 : Ican==preimage_phi(Jcan)
o13 = true
i14 : lengthRes=2; -- a lengthlimit for the resolution on the scroll
```

With respect to the grading given by the total degree, the relative canonical resolution has the following form.

```
   -- the relative canonical resolution:
i15 : betti(resX=resCurveOnScroll(Jcan,g,lengthRes))
             0 1 2 3
o15 = total: 1 5 5 1
          0: 1 . . .
          1: . . . .
          2: . 4 1 .
          3: . 1 4 .
          4: . . . .
          5: . . . 1
```

The scroll has the following determinantal representation.

```
i16 : X=preimage_phi(ideal 0_RX); -- the ideal of the scroll
i17 : repX=matrix{{t_0,t_2,t_4,t_6},{t_1,t_3,t_5,t_7}}
o17 = | t_0 t_2 t_4 t_6 |
      | t_1 t_3 t_5 t_7 |
i18 : minors(2,repX)==X
o18 = true
```

It remains to compute the iterated mapping cone.

```
i19 : e={1,1,1,1}; -- the type of the scroll
    -- the (non-minimal) iterated mapping cone:
i20 : betti(resC=iteratedMC(resX,e))
             0  1  2  3  4  5 6
o20 = total: 1 15 41 54 41 15 1
          0: 1  .  .  .  .  . .
          1: . 15 35 27  6  . .
          2: .  .  6 27 35 15 .
          3: .  .  .  .  .  . 1
    -- we check if the minimized resolution has extra syzygies:
```

By computing the rank of the matrix with constant entries, we can check whether the curve has extra syzygies.

```
i21 : rank submatrixByDegrees(resC.dd_4,5,5)==6
o21 = true
```

7.2 The package RandomCurvesOverVerySmallFiniteFields

This package is based on the two *Macaulay2* packages [BGS11] and [Sch13] and provides methods to compute random canonical curves for genus $g \leq 15$. By catching all possible missteps in the construction, the methods contained in this package also work over finite fields of arbitrary small characteristic. For the theoretical background of the constructions we refer to [ST02] and [Sch15]. For a documentation of the package see [BS17].

Example 7.2.1. We compute a genus 13 curve over a field of characteristic 2 and test the refined Green conjecture for this example.

```
i1 : loadPackage("RandomCurvesOverVerySmallFiniteFields")
i2 : g=13; -- the genus
i3 : p=2; -- the characteristic of the groundfield
i4 : time Ican=smoothCanonicalCurve(g,p);
     -- used 9.31392 seconds
```

We compute a non-minimal resolution of the coordinate ring using the newly implemented methods for fast syzygy computations and minimize the last part of the linear strand.

```
i5 :   betti(sresIcan:=res(Ican,FastNonminimal=>true))
            0  1   2    3    4    5    6    7    8   9  10 11
o5 = total: 1 65 434 1475 3184 4718 4948 3689 1920 665 138 13
         0: 1  .   .    .    .    .    .    .    .   .   .  .
         1: . 55 330  994 1879 2415 2183 1400  629 190  35  3
         2: . 10 103  472 1269 2219 2639 2163 1207 439  94  9
         3: .  .   1    9   36   84  126  126   84  36   9  1
```

The Betti table of a minimal resolution has the following form.

```
i6 :   B:=betti(sresIcan,Minimize=>true) -- the minimal Betti numbers
```

```
           0  1   2   3    4    5    6    7   8   9 10 11
o6 = total: 1 55 320 891 1408 1219 1219 1408 891 320 55  1
         0: 1  .   .   .    .    .    .    .   .   .  .  .
         1: . 55 320 891 1408 1155   68    .   .   .  .  .
         2: .  .   .   .    .   68 1155 1408 891 320 55  .
         3: .  .   .   .    .    .    .    .   .   .  .  1
```

Note that Macaulay does not compute the actual maps in a minimal free resolution. We will compute a "partial minimalization" of the non-minimal resolution above (see `o14` and `o15`).

```
i7 :   strand=select(apply(1..(g-2),i->B_(i,{i+1},i+1)),b->b!=0)
o7 = (55, 320, 891, 1408, 1155, 68)
i8 :    mm:=#strand; -- the critical homological degree
 -- we pick the constant submatrix and compute the kernel
i9 :   betti (M1:= submatrixByDegrees(sresIcan.dd_(mm),mm+1,mm+1))
                0    1
o9 = total: 2219 2183
         6:    . 2183
         7: 2219    .
i10 :   M1kk:= lift( M1, coefficientRing ring M1);
i11 :   ns:= nullSpace mutableMatrix(M1kk);
i12 :   mns:=matrix( ns );
i13 :   syzM:=map(T^{rank target mns:-mm-1},T^{rank source mns:-mm-1},sub(mns,T))
i14 :   betti syzM
                0  1
o14 = total: 2183 68
          6:    . 68
          7: 2183  .
i15 :    betti (M:=submatrixByDegrees(sresIcan.dd_(mm),mm,mm+1)*syzM)
                0  1
o15 = total: 2415 68
          6: 2415 68
```

The homology we are interested in is given by coker(M^t). The computations of the annihilator of coker(M^t) is the bottleneck of the whole computation.

```
i16 :   time annCokMt:= ann(coker transpose M,Strategy => Quotient);
     -- used 143.405 seconds
  -- the computation above takes about 1.5h if we do not change the strategy
  -- If we are not interested in the ideal of the annihilator we can test if
  -- CokMt is a module of finite length (or supported on a scroll) much faster,
  -- by checking if dim CokMt==0 (or codim CokMt+1==degree CokMt)
i17 :   dim annCokMt==0 -- is it of finite length?
o17 = false
i18 :   (codim annCokMt, degree annCokMt) -- is a scroll if codim+1==degree
o18 = (6, 7)
i19 :   Ican+annCokMt==Ican
o19 = true
```

Hence, the curve C is contained in a 6 dimensional scroll X of degree 7 and therefore the ruling on X cuts out a pencil of degree 7 on C.

7.3 Experimental data

This section contains the data obtained from our experiments using the *Macaulay2*-package [BS17].

We computed resolutions of the coordinate ring of canonical curves over a finite field $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$ for $p \leq 101$. We think of the corresponding curve, as a curve over the closure $\overline{\mathbb{F}}_p$. The table below lists the (possibly) exceptional cases for the classical Green conjecture for generic curves.

genus	char($\mathbb{F}_p$)	extra syzygies
7	2	$\beta_{2,4} = 1$
9	3	$\beta_{3,5} = 6$
11	2, 3	$\beta_{4,6} = 28,\ 10$
12	2	$\beta_{4,6} = 1$
13	2, 5	$\beta_{5,7} = 64,\ 120$
15	2, 3, 5	$\beta_{6,8} = 299,\ 390,\ 315$

In all other cases for $g \leq 15$ and $p \leq 101$ we found examples, for which the critical Betti number $\beta_{m,m+1}(C)$ (for $m = \lceil \frac{g-1}{2} \rceil$) is zero.

On the following pages we list the complete data of our experiments. Let $\beta_{n,n+2}$ be the first non-zero Betti number in the second linear strand and let $\mathrm{H}_n(\mathrm{strand}_2(\mathrm{S}_C))$ be the homology at this position. In the column "RGC", we mark cases for which $\mathrm{H}_n(\mathrm{strand}_2(\mathrm{S}_C))$ is a module of finite length with an "(a)". Cases in which $\mathrm{H}_n(\mathrm{strand}_2(\mathrm{S}_C))$ is supported on a $n+1$ dimensional scroll of degree $g-n+3$ which contains the curve, are marked with "(b)". In all other cases, we expect that the curve has several g^1_k's and therefore we expect $\mathrm{H}_n(\mathrm{strand}_2(\mathrm{S}_C))$ to be supported on a union of scrolls. For some cases we give an evidence for this by computing the degree and the dimension of the support of $\mathrm{H}_n(\mathrm{strand}_2(\mathrm{S}_C))$. This data is also tracked in the column "RGC" as a tuple (degree,dimension).

Remark 7.3.1. Computing several random examples of genus g curves over $\mathbb{F}_p$ for odd g one expects to get a curve inside $\mathscr{M}^1_{g,k}$ for $k = \lceil \frac{g}{2} \rceil$ with a chance of roughly $\frac{1}{p}$ (c.f. [vBS05]).

genus	char($\mathbb{F}_p$)	#	RGC	Betti table
$g=7$	$p=2$	272	(a) (1,0)	$\begin{array}{\|cccccc} \hline 1 & . & . & . & . & . \\ . & 10 & 16 & 1 & . & . \\ . & . & 1 & 16 & 10 & . \\ . & . & . & . & . & 1 \end{array}$
		228	(b) (4,3)	$\begin{array}{\|cccccc} \hline 1 & . & . & . & . & . \\ . & 10 & 16 & 3 & . & . \\ . & . & 3 & 16 & 10 & . \\ . & . & . & . & . & 1 \end{array}$

Table 7.1: Betti tables of 500 random examples of genus 7 curves over $\mathbb{F}_2$

genus	char($\mathbb{F}_p$)	#	RGC	Betti table
$g=9$	$p=3$	350	(a) (6,0)	$\begin{array}{\|cccccccc} \hline 1 & . & . & . & . & . & . & . \\ . & 21 & 64 & 70 & 6 & . & . & . \\ . & . & . & 6 & 70 & 64 & 21 & . \\ . & . & . & . & . & . & . & 1 \end{array}$
		103	(b) (5,4)	$\begin{array}{\|cccccccc} \hline 1 & . & . & . & . & . & . & . \\ . & 21 & 64 & 70 & 8 & . & . & . \\ . & . & . & 8 & 70 & 64 & 21 & . \\ . & . & . & . & . & . & . & 1 \end{array}$
		31	(10,4)	$\begin{array}{\|cccccccc} \hline 1 & . & . & . & . & . & . & . \\ . & 21 & 64 & 70 & 10 & . & . & . \\ . & . & . & 10 & 70 & 64 & 21 & . \\ . & . & . & . & . & . & . & 1 \end{array}$
		16	(b) (6,3)	$\begin{array}{\|cccccccc} \hline 1 & . & . & . & . & . & . & . \\ . & 21 & 64 & 75 & 24 & 5 & . & . \\ . & . & 5 & 24 & 75 & 64 & 21 & . \\ . & . & . & . & . & . & . & 1 \end{array}$

Table 7.2: Betti tables of 500 random examples of genus 9 curves over $\mathbb{F}_3$

genus	char($\mathbb{F}_p$)	#	RGC	Betti table
$g=11$	$p=2$	230	(a) $(60,0)$	$\begin{matrix} 1 & . & . & . & . & . & . & . & . & . \\ . & 36 & 160 & 315 & 288 & 28 & . & . & . & . \\ . & . & . & . & 28 & 288 & 315 & 160 & 36 & . \\ . & . & . & . & . & . & . & . & . & 1 \end{matrix}$
		76	(b) $(6,5)$	$\begin{matrix} 1 & . & . & . & . & . & . & . & . & . \\ . & 36 & 160 & 315 & 288 & 30 & . & . & . & . \\ . & . & . & . & 30 & 288 & 315 & 160 & 36 & . \\ . & . & . & . & . & . & . & . & . & 1 \end{matrix}$
		82	$(12,5)$	$\begin{matrix} 1 & . & . & . & . & . & . & . & . & . \\ . & 36 & 160 & 315 & 288 & 32 & . & . & . & . \\ . & . & . & . & 32 & 288 & 315 & 160 & 36 & . \\ . & . & . & . & . & . & . & . & . & 1 \end{matrix}$
		55	$(18,5)$	$\begin{matrix} 1 & . & . & . & . & . & . & . & . & . \\ . & 36 & 160 & 315 & 288 & 34 & . & . & . & . \\ . & . & . & . & 34 & 288 & 315 & 160 & 36 & . \\ . & . & . & . & . & . & . & . & . & 1 \end{matrix}$
		24	$(24,5)$	$\begin{matrix} 1 & . & . & . & . & . & . & . & . & . \\ . & 36 & 160 & 315 & 288 & 36 & . & . & . & . \\ . & . & . & . & 36 & 288 & 315 & 160 & 36 & . \\ . & . & . & . & . & . & . & . & . & 1 \end{matrix}$
		10	$(30,5)$	$\begin{matrix} 1 & . & . & . & . & . & . & . & . & . \\ . & 36 & 160 & 315 & 288 & 38 & . & . & . & . \\ . & . & . & . & 38 & 288 & 315 & 160 & 36 & . \\ . & . & . & . & . & . & . & . & . & 1 \end{matrix}$
		14	$(36,5)$	$\begin{matrix} 1 & . & . & . & . & . & . & . & . & . \\ . & 36 & 160 & 315 & 288 & 40 & . & . & . & . \\ . & . & . & . & 40 & 288 & 315 & 160 & 36 & . \\ . & . & . & . & . & . & . & . & . & 1 \end{matrix}$
		6	$(42,5)$	$\begin{matrix} 1 & . & . & . & . & . & . & . & . & . \\ . & 36 & 160 & 315 & 288 & 42 & . & . & . & . \\ . & . & . & . & 42 & 288 & 315 & 160 & 36 & . \\ . & . & . & . & . & . & . & . & . & 1 \end{matrix}$
		2	$(48,5)$	$\begin{matrix} 1 & . & . & . & . & . & . & . & . & . \\ . & 36 & 160 & 315 & 288 & 44 & . & . & . & . \\ . & . & . & . & 44 & 288 & 315 & 160 & 36 & . \\ . & . & . & . & . & . & . & . & . & 1 \end{matrix}$
		1	$(60,5)$	$\begin{matrix} 1 & . & . & . & . & . & . & . & . & . \\ . & 36 & 160 & 315 & 288 & 50 & . & . & . & . \\ . & . & . & . & 50 & 288 & 315 & 160 & 36 & . \\ . & . & . & . & . & . & . & . & . & 1 \end{matrix}$

Table 7.3: Betti tables of 500 random examples of genus 11 curves over $\mathbb{F}_2$

genus	char($\mathbb{F}_p$)	#	RGC	Betti table
$g = 11$	$p = 3$	311	(a) (12,0)	$\begin{array}{\|cccccccccc} 1 & . & . & . & . & . & . & . & . & . \\ . & 36 & 160 & 315 & 288 & 10 & . & . & . & . \\ . & . & . & . & 10 & 288 & 315 & 160 & 36 & . \\ . & . & . & . & . & . & . & . & . & 1 \end{array}$
		95	(b) (6,5)	$\begin{array}{\|cccccccccc} 1 & . & . & . & . & . & . & . & . & . \\ . & 36 & 160 & 315 & 288 & 14 & . & . & . & . \\ . & . & . & . & 14 & 288 & 315 & 160 & 36 & . \\ . & . & . & . & . & . & . & . & . & 1 \end{array}$
		57	(12,5)	$\begin{array}{\|cccccccccc} 1 & . & . & . & . & . & . & . & . & . \\ . & 36 & 160 & 315 & 288 & 18 & . & . & . & . \\ . & . & . & . & 18 & 288 & 315 & 160 & 36 & . \\ . & . & . & . & . & . & . & . & . & 1 \end{array}$
		24	(18,5)	$\begin{array}{\|cccccccccc} 1 & . & . & . & . & . & . & . & . & . \\ . & 36 & 160 & 315 & 288 & 22 & . & . & . & . \\ . & . & . & . & 22 & 288 & 315 & 160 & 36 & . \\ . & . & . & . & . & . & . & . & . & 1 \end{array}$
		5	(24,5)	$\begin{array}{\|cccccccccc} 1 & . & . & . & . & . & . & . & . & . \\ . & 36 & 160 & 315 & 288 & 26 & . & . & . & . \\ . & . & . & . & 26 & 288 & 315 & 160 & 36 & . \\ . & . & . & . & . & . & . & . & . & 1 \end{array}$
		4	(30,5)	$\begin{array}{\|cccccccccc} 1 & . & . & . & . & . & . & . & . & . \\ . & 36 & 160 & 315 & 288 & 30 & . & . & . & . \\ . & . & . & . & 30 & 288 & 315 & 160 & 36 & . \\ . & . & . & . & . & . & . & . & . & 1 \end{array}$
		4	(36,5)	$\begin{array}{\|cccccccccc} 1 & . & . & . & . & . & . & . & . & . \\ . & 36 & 160 & 315 & 288 & 34 & . & . & . & . \\ . & . & . & . & 34 & 288 & 315 & 160 & 36 & . \\ . & . & . & . & . & . & . & . & . & 1 \end{array}$

Table 7.4: Betti tables of 500 random examples of genus 11 curves over $\mathbb{F}_3$

genus	char($\mathbb{F}_p$)	#	RGC	Betti table
$g=12$	$p=2$	386	(a) $(1,0)$	$\begin{array}{\|ccccccccccc} 1 & . & . & . & . & . & . & . & . & . & . \\ . & 45 & 231 & 550 & 693 & 331 & 1 & . & . & . & . \\ . & . & . & . & 1 & 331 & 693 & 550 & 231 & 45 & . \\ . & . & . & . & . & . & . & . & . & . & 1 \end{array}$
		8	(a) $(3,0)$	$\begin{array}{\|ccccccccccc} 1 & . & . & . & . & . & . & . & . & . & . \\ . & 45 & 231 & 550 & 693 & 333 & 3 & . & . & . & . \\ . & . & . & . & 3 & 333 & 693 & 550 & 231 & 45 & . \\ . & . & . & . & . & . & . & . & . & . & 1 \end{array}$
		92	(b) $(7,5)$	$\begin{array}{\|ccccccccccc} 1 & . & . & . & . & . & . & . & . & . & . \\ . & 45 & 231 & 550 & 693 & 336 & 6 & . & . & . & . \\ . & . & . & . & 6 & 336 & 693 & 550 & 231 & 45 & . \\ . & . & . & . & . & . & . & . & . & . & 1 \end{array}$
		14	(b) $(7,5)$	$\begin{array}{\|ccccccccccc} 1 & . & . & . & . & . & . & . & . & . & . \\ . & 45 & 231 & 550 & 693 & 337 & 7 & . & . & . & . \\ . & . & . & . & 7 & 337 & 693 & 550 & 231 & 45 & . \\ . & . & . & . & . & . & . & . & . & . & 1 \end{array}$

Table 7.5: Betti tables of 500 random examples of genus 12 curves over $\mathbb{F}_2$

genus	char($\mathbb{F}_p$)	#	RGC	Betti table
$g=13$	$p=2$	112	(a) $(200,0)$	1 . . . . 55 320 891 1408 1155 64 . . . . 64 1155 1408 891 320 55 1
		77	(b) $(7,6)$	1 . . . . 55 320 891 1408 1155 68 . . . . 68 1155 1408 891 320 55 1
		67	$(14,6)$	1 . . . . 55 320 891 1408 1155 72 . . . . 72 1155 1408 891 320 55 1
		51	$(21,6)$	1 . . . . 55 320 891 1408 1155 76 . . . . 76 1155 1408 891 320 55 1
		44	$(28,6)$	1 . . . . 55 320 891 1408 1155 80 . . . . 80 1155 1408 891 320 55 1
		38		1 . . . . 55 320 891 1408 1155 84 . . . . 84 1155 1408 891 320 55 1
		26		1 . . . . 55 320 891 1408 1155 88 . . . . 88 1155 1408 891 320 55 1
		15		1 . . . . 55 320 891 1408 1155 92 . . . . 92 1155 1408 891 320 55 1
		11		1 . . . . 55 320 891 1408 1155 96 . . . . 96 1155 1408 891 320 55 1
		5		1 . . . . 55 320 891 1408 1155 100 . . . . 100 1155 1408 891 320 55 1

Table 7.6: Betti tables of 500 random examples of genus 13 curves over $\mathbb{F}_2$ (part 1 of 3)

genus	char($\mathbb{F}_p$)	#	RGC	Betti table
$g = 13$	$p = 2$	4		$\begin{array}{cccccccccccc} 1 & . & . & . & . & . & . & . & . & . & . & . \\ . & 55 & 320 & 891 & 1408 & 1155 & 104 & . & . & . & . & . \\ . & . & . & . & . & 104 & 1155 & 1408 & 891 & 320 & 55 & . \\ . & . & . & . & . & . & . & . & . & . & . & 1 \end{array}$
		1		$\begin{array}{cccccccccccc} 1 & . & . & . & . & . & . & . & . & . & . & . \\ . & 55 & 320 & 891 & 1408 & 1155 & 112 & . & . & . & . & . \\ . & . & . & . & . & 112 & 1155 & 1408 & 891 & 320 & 55 & . \\ . & . & . & . & . & . & . & . & . & . & . & 1 \end{array}$
		3		$\begin{array}{cccccccccccc} 1 & . & . & . & . & . & . & . & . & . & . & . \\ . & 55 & 320 & 891 & 1408 & 1155 & 120 & . & . & . & . & . \\ . & . & . & . & . & 120 & 1155 & 1408 & 891 & 320 & 55 & . \\ . & . & . & . & . & . & . & . & . & . & . & 1 \end{array}$
		1		$\begin{array}{cccccccccccc} 1 & . & . & . & . & . & . & . & . & . & . & . \\ . & 55 & 320 & 891 & 1408 & 1155 & 124 & . & . & . & . & . \\ . & . & . & . & . & 124 & 1155 & 1408 & 891 & 320 & 55 & . \\ . & . & . & . & . & . & . & . & . & . & . & 1 \end{array}$
		1		$\begin{array}{cccccccccccc} 1 & . & . & . & . & . & . & . & . & . & . & . \\ . & 55 & 320 & 891 & 1408 & 1155 & 128 & . & . & . & . & . \\ . & . & . & . & . & 128 & 1155 & 1408 & 891 & 320 & 55 & . \\ . & . & . & . & . & . & . & . & . & . & . & 1 \end{array}$
		1		$\begin{array}{cccccccccccc} 1 & . & . & . & . & . & . & . & . & . & . & . \\ . & 55 & 320 & 891 & 1408 & 1155 & 132 & . & . & . & . & . \\ . & . & . & . & . & 132 & 1155 & 1408 & 891 & 320 & 55 & . \\ . & . & . & . & . & . & . & . & . & . & . & 1 \end{array}$
		5	(b) (8,5)	$\begin{array}{cccccccccccc} 1 & . & . & . & . & . & . & . & . & . & . & . \\ . & 55 & 320 & 891 & 1408 & 1162 & 96 & 7 & . & . & . & . \\ . & . & . & . & 7 & 96 & 1162 & 1408 & 891 & 320 & 55 & . \\ . & . & . & . & . & . & . & . & . & . & . & 1 \end{array}$
		6	(b) (8,5)	$\begin{array}{cccccccccccc} 1 & . & . & . & . & . & . & . & . & . & . & . \\ . & 55 & 320 & 891 & 1408 & 1162 & 100 & 7 & . & . & . & . \\ . & . & . & . & 7 & 100 & 1162 & 1408 & 891 & 320 & 55 & . \\ . & . & . & . & . & . & . & . & . & . & . & 1 \end{array}$
		7	(b) (8,5)	$\begin{array}{cccccccccccc} 1 & . & . & . & . & . & . & . & . & . & . & . \\ . & 55 & 320 & 891 & 1408 & 1162 & 104 & 7 & . & . & . & . \\ . & . & . & . & 7 & 104 & 1162 & 1408 & 891 & 320 & 55 & . \\ . & . & . & . & . & . & . & . & . & . & . & 1 \end{array}$
		11	(b) (8,5)	$\begin{array}{cccccccccccc} 1 & . & . & . & . & . & . & . & . & . & . & . \\ . & 55 & 320 & 891 & 1408 & 1162 & 108 & 7 & . & . & . & . \\ . & . & . & . & 7 & 108 & 1162 & 1408 & 891 & 320 & 55 & . \\ . & . & . & . & . & . & . & . & . & . & . & 1 \end{array}$

Table 7.7: Betti tables of 500 random examples of genus 13 curves over $\mathbb{F}_2$ (part 2 of 3)

genus	char($\mathbb{F}_p$)	#	RGC	Betti table
$g = 13$	$p = 2$	3	(b) (8,5)	1 . . . . 55 320 891 1408 1162 112 7 . . . . 7 112 1162 1408 891 320 55 1
		4	(b) (8,5)	1 . . . . 55 320 891 1408 1162 120 7 . . . . 7 120 1162 1408 891 320 55 1
		3	(b) (8,5)	1 . . . . 55 320 891 1408 1162 124 7 . . . . 7 124 1162 1408 891 320 55 1
		2	(b) (8,5)	1 . . . . 55 320 891 1408 1162 128 7 . . . . 7 128 1162 1408 891 320 55 1
		1	(b) (8,5)	1 . . . . 55 320 891 1408 1162 132 7 . . . . 7 132 1162 1408 891 320 55 1
		1	(b) (8,5)	1 . . . . 55 320 891 1408 1162 156 7 . . . . 7 156 1162 1408 891 320 55 1

Table 7.8: Betti tables of 500 random examples of genus 13 curves over $\mathbb{F}_2$ (part 3 of 3)

genus	char($\mathbb{F}_p$)	#	RGC	Betti table
$g=13$	$p=5$	331	(a) $(525,0)$	$\begin{array}{l\|cccccccccccc} & & & & & & & & & & & & \\ \hline & 1 & . & . & . & . & . & . & . & . & . & . & . \\ & . & 55 & 320 & 891 & 1408 & 1155 & 120 & . & . & . & . & . \\ & . & . & . & . & . & 120 & 1155 & 1408 & 891 & 320 & 55 & . \\ & . & . & . & . & . & . & . & . & . & . & . & 1 \end{array}$
		116	(b) $(7,6)$	$\begin{array}{l\|cccccccccccc} & & & & & & & & & & & & \\ \hline & 1 & . & . & . & . & . & . & . & . & . & . & . \\ & . & 55 & 320 & 891 & 1408 & 1155 & 122 & . & . & . & . & . \\ & . & . & . & . & . & 122 & 1155 & 1408 & 891 & 320 & 55 & . \\ & . & . & . & . & . & . & . & . & . & . & . & 1 \end{array}$
		41	$(14,6)$	$\begin{array}{l\|cccccccccccc} & & & & & & & & & & & & \\ \hline & 1 & . & . & . & . & . & . & . & . & . & . & . \\ & . & 55 & 320 & 891 & 1408 & 1155 & 124 & . & . & . & . & . \\ & . & . & . & . & . & 124 & 1155 & 1408 & 891 & 320 & 55 & . \\ & . & . & . & . & . & . & . & . & . & . & . & 1 \end{array}$
		8		$\begin{array}{l\|cccccccccccc} & & & & & & & & & & & & \\ \hline & 1 & . & . & . & . & . & . & . & . & . & . & . \\ & . & 55 & 320 & 891 & 1408 & 1155 & 126 & . & . & . & . & . \\ & . & . & . & . & . & 126 & 1155 & 1408 & 891 & 320 & 55 & . \\ & . & . & . & . & . & . & . & . & . & . & . & 1 \end{array}$
		2		$\begin{array}{l\|cccccccccccc} & & & & & & & & & & & & \\ \hline & 1 & . & . & . & . & . & . & . & . & . & . & . \\ & . & 55 & 320 & 891 & 1408 & 1155 & 128 & . & . & . & . & . \\ & . & . & . & . & . & 128 & 1155 & 1408 & 891 & 320 & 55 & . \\ & . & . & . & . & . & . & . & . & . & . & . & 1 \end{array}$
		1		$\begin{array}{l\|cccccccccccc} & & & & & & & & & & & & \\ \hline & 1 & . & . & . & . & . & . & . & . & . & . & . \\ & . & 55 & 320 & 891 & 1408 & 1155 & 130 & . & . & . & . & . \\ & . & . & . & . & . & 130 & 1155 & 1408 & 891 & 320 & 55 & . \\ & . & . & . & . & . & . & . & . & . & . & . & 1 \end{array}$
		1	(b) $(8,5)$	$\begin{array}{l\|cccccccccccc} & & & & & & & & & & & & \\ \hline & 1 & . & . & . & . & . & . & . & . & . & . & . \\ & . & 55 & 320 & 891 & 1408 & 1162 & 148 & 7 & . & . & . & . \\ & . & . & . & . & 7 & 148 & 1162 & 1408 & 891 & 320 & 55 & . \\ & . & . & . & . & . & . & . & . & . & . & . & 1 \end{array}$

Table 7.9: Betti tables of 500 random examples of genus 13 curves over $\mathbb{F}_5$

genus	char($\mathbb{F}_p$)	#	RGC	Betti table
$g=15$	$p=2$	8	(a) (404, 0)	$\begin{array}{cccccccccccccc} 1 & . & . & . & . & . & . & . & . & . & . & . & . & . \\ . & 78 & 560 & 2002 & 4368 & 6006 & 4576 & 299 & . & . & . & . & . & . \\ . & . & . & . & . & . & 299 & 4576 & 6006 & 4368 & 2002 & 560 & 78 & . \\ . & . & . & . & . & . & . & . & . & . & . & . & . & 1 \end{array}$
		4	(b) (8, 7)	$\begin{array}{cccccccccccccc} 1 & . & . & . & . & . & . & . & . & . & . & . & . & . \\ . & 78 & 560 & 2002 & 4368 & 6006 & 4576 & 303 & . & . & . & . & . & . \\ . & . & . & . & . & . & 303 & 4576 & 6006 & 4368 & 2002 & 560 & 78 & . \\ . & . & . & . & . & . & . & . & . & . & . & . & . & 1 \end{array}$
		3	(16, 7)	$\begin{array}{cccccccccccccc} 1 & . & . & . & . & . & . & . & . & . & . & . & . & . \\ . & 78 & 560 & 2002 & 4368 & 6006 & 4576 & 307 & . & . & . & . & . & . \\ . & . & . & . & . & . & 307 & 4576 & 6006 & 4368 & 2002 & 560 & 78 & . \\ . & . & . & . & . & . & . & . & . & . & . & . & . & 1 \end{array}$
		9		$\begin{array}{cccccccccccccc} 1 & . & . & . & . & . & . & . & . & . & . & . & . & . \\ . & 78 & 560 & 2002 & 4368 & 6006 & 4576 & 311 & . & . & . & . & . & . \\ . & . & . & . & . & . & 311 & 4576 & 6006 & 4368 & 2002 & 560 & 78 & . \\ . & . & . & . & . & . & . & . & . & . & . & . & . & 1 \end{array}$
		7		$\begin{array}{cccccccccccccc} 1 & . & . & . & . & . & . & . & . & . & . & . & . & . \\ . & 78 & 560 & 2002 & 4368 & 6006 & 4576 & 315 & . & . & . & . & . & . \\ . & . & . & . & . & . & 315 & 4576 & 6006 & 4368 & 2002 & 560 & 78 & . \\ . & . & . & . & . & . & . & . & . & . & . & . & . & 1 \end{array}$
		4		$\begin{array}{cccccccccccccc} 1 & . & . & . & . & . & . & . & . & . & . & . & . & . \\ . & 78 & 560 & 2002 & 4368 & 6006 & 4576 & 319 & . & . & . & . & . & . \\ . & . & . & . & . & . & 319 & 4576 & 6006 & 4368 & 2002 & 560 & 78 & . \\ . & . & . & . & . & . & . & . & . & . & . & . & . & 1 \end{array}$
		2		$\begin{array}{cccccccccccccc} 1 & . & . & . & . & . & . & . & . & . & . & . & . & . \\ . & 78 & 560 & 2002 & 4368 & 6006 & 4576 & 323 & . & . & . & . & . & . \\ . & . & . & . & . & . & 323 & 4576 & 6006 & 4368 & 2002 & 560 & 78 & . \\ . & . & . & . & . & . & . & . & . & . & . & . & . & 1 \end{array}$
		1		$\begin{array}{cccccccccccccc} 1 & . & . & . & . & . & . & . & . & . & . & . & . & . \\ . & 78 & 560 & 2002 & 4368 & 6006 & 4576 & 327 & . & . & . & . & . & . \\ . & . & . & . & . & . & 327 & 4576 & 6006 & 4368 & 2002 & 560 & 78 & . \\ . & . & . & . & . & . & . & . & . & . & . & . & . & 1 \end{array}$
		2		$\begin{array}{cccccccccccccc} 1 & . & . & . & . & . & . & . & . & . & . & . & . & . \\ . & 78 & 560 & 2002 & 4368 & 6006 & 4576 & 331 & . & . & . & . & . & . \\ . & . & . & . & . & . & 331 & 4576 & 6006 & 4368 & 2002 & 560 & 78 & . \\ . & . & . & . & . & . & . & . & . & . & . & . & . & 1 \end{array}$
		1		$\begin{array}{cccccccccccccc} 1 & . & . & . & . & . & . & . & . & . & . & . & . & . \\ . & 78 & 560 & 2002 & 4368 & 6006 & 4576 & 351 & . & . & . & . & . & . \\ . & . & . & . & . & . & 351 & 4576 & 6006 & 4368 & 2002 & 560 & 78 & . \\ . & . & . & . & . & . & . & . & . & . & . & . & . & 1 \end{array}$

Table 7.10: Betti tables of 50 random examples of genus 15 curves over $\mathbb{F}_2$ (part 1 of 2)

genus	char($\mathbb{F}_p$)	#	RGC	Betti table
$g=15$	$p=2$	2	(b) (9,6)	$\begin{array}{l\|cccccccccccccc} & 1 & . & . & . & . & . & . & . & . & . & . & . & . & . \\ & . & 78 & 560 & 2002 & 4368 & 6006 & 4584 & 343 & 8 & . & . & . & . & . \\ & . & . & . & . & . & 8 & 343 & 4584 & 6006 & 4368 & 2002 & 560 & 78 & . \\ & . & . & . & . & . & . & . & . & . & . & . & . & . & 1 \end{array}$
		1	(b) (9,6)	$\begin{array}{l\|cccccccccccccc} & 1 & . & . & . & . & . & . & . & . & . & . & . & . & . \\ & . & 78 & 560 & 2002 & 4368 & 6006 & 4584 & 359 & 8 & . & . & . & . & . \\ & . & . & . & . & . & 8 & 359 & 4584 & 6006 & 4368 & 2002 & 560 & 78 & . \\ & . & . & . & . & . & . & . & . & . & . & . & . & . & 1 \end{array}$
		1	(b) (9,6)	$\begin{array}{l\|cccccccccccccc} & 1 & . & . & . & . & . & . & . & . & . & . & . & . & . \\ & . & 78 & 560 & 2002 & 4368 & 6006 & 4584 & 363 & 8 & . & . & . & . & . \\ & . & . & . & . & . & 8 & 363 & 4584 & 6006 & 4368 & 2002 & 560 & 78 & . \\ & . & . & . & . & . & . & . & . & . & . & . & . & . & 1 \end{array}$
		1	(b) (9,6)	$\begin{array}{l\|cccccccccccccc} & 1 & . & . & . & . & . & . & . & . & . & . & . & . & . \\ & . & 78 & 560 & 2002 & 4368 & 6006 & 4584 & 375 & 8 & . & . & . & . & . \\ & . & . & . & . & . & 8 & 375 & 4584 & 6006 & 4368 & 2002 & 560 & 78 & . \\ & . & . & . & . & . & . & . & . & . & . & . & . & . & 1 \end{array}$
		1	(b) (10,5)	$\begin{array}{l\|cccccccccccccc} & 1 & . & . & . & . & . & . & . & . & . & . & . & . & . \\ & . & 78 & 560 & 2002 & 4368 & 6015 & 4656 & 563 & 80 & 9 & . & . & . & . \\ & . & . & . & . & 9 & 80 & 563 & 4656 & 6015 & 4368 & 2002 & 560 & 78 & . \\ & . & . & . & . & . & . & . & . & . & . & . & . & . & 1 \end{array}$
		1	(b) (10,5)	$\begin{array}{l\|cccccccccccccc} & 1 & . & . & . & . & . & . & . & . & . & . & . & . & . \\ & . & 78 & 560 & 2002 & 4368 & 6015 & 4656 & 567 & 80 & 9 & . & . & . & . \\ & . & . & . & . & 9 & 80 & 567 & 4656 & 6015 & 4368 & 2002 & 560 & 78 & . \\ & . & . & . & . & . & . & . & . & . & . & . & . & . & 1 \end{array}$
		1	(b) (10,5)	$\begin{array}{l\|cccccccccccccc} & 1 & . & . & . & . & . & . & . & . & . & . & . & . & . \\ & . & 78 & 560 & 2002 & 4368 & 6015 & 4656 & 575 & 80 & 9 & . & . & . & . \\ & . & . & . & . & 9 & 80 & 575 & 4656 & 6015 & 4368 & 2002 & 560 & 78 & . \\ & . & . & . & . & . & . & . & . & . & . & . & . & . & 1 \end{array}$
		1	(b) (10,5)	$\begin{array}{l\|cccccccccccccc} & 1 & . & . & . & . & . & . & . & . & . & . & . & . & . \\ & . & 78 & 560 & 2002 & 4368 & 6015 & 4656 & 591 & 80 & 9 & . & . & . & . \\ & . & . & . & . & 9 & 80 & 591 & 4656 & 6015 & 4368 & 2002 & 560 & 78 & . \\ & . & . & . & . & . & . & . & . & . & . & . & . & . & 1 \end{array}$

Table 7.11: Betti tables of 50 random examples of genus 15 curves over $\mathbb{F}_2$ (part 2 of 2)

genus	char($\mathbb{F}_p$)	#	RGC	Betti table
$g=15$	$p=3$	38	(a)	$\begin{array}{cccccccccccccc} 1 & . & . & . & . & . & . & . & . & . & . & . & . & . \\ . & 78 & 560 & 2002 & 4368 & 6006 & 4576 & 390 & . & . & . & . & . & . \\ . & . & . & . & . & . & 390 & 4576 & 6006 & 4368 & 2002 & 560 & 78 & . \\ . & . & . & . & . & . & . & . & . & . & . & . & . & 1 \end{array}$
		28	(b) (8, 7)	$\begin{array}{cccccccccccccc} 1 & . & . & . & . & . & . & . & . & . & . & . & . & . \\ . & 78 & 560 & 2002 & 4368 & 6006 & 4576 & 393 & . & . & . & . & . & . \\ . & . & . & . & . & . & 393 & 4576 & 6006 & 4368 & 2002 & 560 & 78 & . \\ . & . & . & . & . & . & . & . & . & . & . & . & . & 1 \end{array}$
		10		$\begin{array}{cccccccccccccc} 1 & . & . & . & . & . & . & . & . & . & . & . & . & . \\ . & 78 & 560 & 2002 & 4368 & 6006 & 4576 & 396 & . & . & . & . & . & . \\ . & . & . & . & . & . & 396 & 4576 & 6006 & 4368 & 2002 & 560 & 78 & . \\ . & . & . & . & . & . & . & . & . & . & . & . & . & 1 \end{array}$
		5		$\begin{array}{cccccccccccccc} 1 & . & . & . & . & . & . & . & . & . & . & . & . & . \\ . & 78 & 560 & 2002 & 4368 & 6006 & 4576 & 399 & . & . & . & . & . & . \\ . & . & . & . & . & . & 399 & 4576 & 6006 & 4368 & 2002 & 560 & 78 & . \\ . & . & . & . & . & . & . & . & . & . & . & . & . & 1 \end{array}$
		3		$\begin{array}{cccccccccccccc} 1 & . & . & . & . & . & . & . & . & . & . & . & . & . \\ . & 78 & 560 & 2002 & 4368 & 6006 & 4576 & 402 & . & . & . & . & . & . \\ . & . & . & . & . & . & 402 & 4576 & 6006 & 4368 & 2002 & 560 & 78 & . \\ . & . & . & . & . & . & . & . & . & . & . & . & . & 1 \end{array}$
		2		$\begin{array}{cccccccccccccc} 1 & . & . & . & . & . & . & . & . & . & . & . & . & . \\ . & 78 & 560 & 2002 & 4368 & 6006 & 4576 & 405 & . & . & . & . & . & . \\ . & . & . & . & . & . & 405 & 4576 & 6006 & 4368 & 2002 & 560 & 78 & . \\ . & . & . & . & . & . & . & . & . & . & . & . & . & 1 \end{array}$
		1		$\begin{array}{cccccccccccccc} 1 & . & . & . & . & . & . & . & . & . & . & . & . & . \\ . & 78 & 560 & 2002 & 4368 & 6006 & 4576 & 417 & . & . & . & . & . & . \\ . & . & . & . & . & . & 417 & 4576 & 6006 & 4368 & 2002 & 560 & 78 & . \\ . & . & . & . & . & . & . & . & . & . & . & . & . & 1 \end{array}$
		4	(b) (9, 6)	$\begin{array}{cccccccccccccc} 1 & . & . & . & . & . & . & . & . & . & . & . & . & . \\ . & 78 & 560 & 2002 & 4368 & 6006 & 4584 & 429 & 8 & . & . & . & . & . \\ . & . & . & . & . & 8 & 429 & 4584 & 6006 & 4368 & 2002 & 560 & 78 & . \\ . & . & . & . & . & . & . & . & . & . & . & . & . & 1 \end{array}$
		4	(b) (9, 6)	$\begin{array}{cccccccccccccc} 1 & . & . & . & . & . & . & . & . & . & . & . & . & . \\ . & 78 & 560 & 2002 & 4368 & 6006 & 4584 & 432 & 8 & . & . & . & . & . \\ . & . & . & . & . & 8 & 432 & 4584 & 6006 & 4368 & 2002 & 560 & 78 & . \\ . & . & . & . & . & . & . & . & . & . & . & . & . & 1 \end{array}$
		1	(b) (9, 6)	$\begin{array}{cccccccccccccc} 1 & . & . & . & . & . & . & . & . & . & . & . & . & . \\ . & 78 & 560 & 2002 & 4368 & 6006 & 4584 & 435 & 8 & . & . & . & . & . \\ . & . & . & . & . & 8 & 435 & 4584 & 6006 & 4368 & 2002 & 560 & 78 & . \\ . & . & . & . & . & . & . & . & . & . & . & . & . & 1 \end{array}$

Table 7.12: Betti tables of 100 random examples of genus 15 curves over $\mathbb{F}_3$ (part 1 of 2)

genus	char($\mathbb{F}_p$)	#	RGC	Betti table
$g=15$	$p=3$	1	(b) $(9,6)$	$\begin{array}{cccccccccccccc} 1 & . & . & . & . & . & . & . & . & . & . & . & . & . \\ . & 78 & 560 & 2002 & 4368 & 6006 & 4584 & 441 & 8 & . & . & . & . & . \\ . & . & . & . & . & 8 & 441 & 4584 & 6006 & 4368 & 2002 & 560 & 78 & . \\ . & . & . & . & . & . & . & . & . & . & . & . & . & 1 \end{array}$
		2	(b) $(9,6)$	$\begin{array}{cccccccccccccc} 1 & . & . & . & . & . & . & . & . & . & . & . & . & . \\ . & 78 & 560 & 2002 & 4368 & 6006 & 4584 & 444 & 8 & . & . & . & . & . \\ . & . & . & . & . & 8 & 444 & 4584 & 6006 & 4368 & 2002 & 560 & 78 & . \\ . & . & . & . & . & . & . & . & . & . & . & . & . & 1 \end{array}$
		1		$\begin{array}{cccccccccccccc} 1 & . & . & . & . & . & . & . & . & . & . & . & . & . \\ . & 78 & 560 & 2002 & 4368 & 6006 & 4592 & 441 & 16 & . & . & . & . & . \\ . & . & . & . & . & 16 & 441 & 4592 & 6006 & 4368 & 2002 & 560 & 78 & . \\ . & . & . & . & . & . & . & . & . & . & . & . & . & 1 \end{array}$

Table 7.13: Betti tables of 100 random examples of genus 15 curves over $\mathbb{F}_3$ (part 2 of 2)

genus	char($\mathbb{F}_p$)	#	RGC	Betti table
$g=15$	$p=5$	63	(a)	$\begin{array}{cccccccccccccc} 1 & . & . & . & . & . & . & . & . & . & . & . & . & . \\ . & 78 & 560 & 2002 & 4368 & 6006 & 4576 & 315 & . & . & . & . & . & . \\ . & . & . & . & . & . & 315 & 4576 & 6006 & 4368 & 2002 & 560 & 78 & . \\ . & . & . & . & . & . & . & . & . & . & . & . & . & 1 \end{array}$
		22	(b) $(8,7)$	$\begin{array}{cccccccccccccc} 1 & . & . & . & . & . & . & . & . & . & . & . & . & . \\ . & 78 & 560 & 2002 & 4368 & 6006 & 4576 & 319 & . & . & . & . & . & . \\ . & . & . & . & . & . & 319 & 4576 & 6006 & 4368 & 2002 & 560 & 78 & . \\ . & . & . & . & . & . & . & . & . & . & . & . & . & 1 \end{array}$
		11		$\begin{array}{cccccccccccccc} 1 & . & . & . & . & . & . & . & . & . & . & . & . & . \\ . & 78 & 560 & 2002 & 4368 & 6006 & 4576 & 323 & . & . & . & . & . & . \\ . & . & . & . & . & . & 323 & 4576 & 6006 & 4368 & 2002 & 560 & 78 & . \\ . & . & . & . & . & . & . & . & . & . & . & . & . & 1 \end{array}$
		2		$\begin{array}{cccccccccccccc} 1 & . & . & . & . & . & . & . & . & . & . & . & . & . \\ . & 78 & 560 & 2002 & 4368 & 6006 & 4576 & 327 & . & . & . & . & . & . \\ . & . & . & . & . & . & 327 & 4576 & 6006 & 4368 & 2002 & 560 & 78 & . \\ . & . & . & . & . & . & . & . & . & . & . & . & . & 1 \end{array}$
		2	(b) $(9,6)$	$\begin{array}{cccccccccccccc} 1 & . & . & . & . & . & . & . & . & . & . & . & . & . \\ . & 78 & 560 & 2002 & 4368 & 6006 & 4584 & 351 & 8 & . & . & . & . & . \\ . & . & . & . & . & 8 & 351 & 4584 & 6006 & 4368 & 2002 & 560 & 78 & . \\ . & . & . & . & . & . & . & . & . & . & . & . & . & 1 \end{array}$

Table 7.14: Betti tables of 100 random examples of genus 15 curves over $\mathbb{F}_5$

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